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Weak and Measure-valued Solutions to Evolutionary PDEs

J. Málek, J. Nečas, M. Rokyta
and M. Růžička

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Contents

Preface

This book deals with evolution partial differential equations of both hyperbolic and parabolic type with particular emphasis on problems that arise in nonlinear fluid mechanics. If an alternative title were to be given to the book, it could be 'on the passage to the limit within nonlinearities'. Fortunately enough, the preface is usually longer than one sentence, which gives us the opportunity to describe briefly the contents of the book.

After presenting some preliminary results, we devote the second chapter to the study of scalar hyperbolic equations of first order (or scalar hyperbolic conservation laws) in arbitrary spatial dimensions. In the first part we treat the usual Cauchy problem, following the presentation of GODLEWSKI AND RAVIART [1991]. The second part focuses on recent results of OTTO [1992] concerning the solvability of a scalar hyperbolic conservation law in a bounded smooth domain. In both cases, we prove the existence and uniqueness of the entropy weak solution via the method of parabolic perturbation. This method, together with the Galerkin method, are the basic means for constructing convenient approximations of the original problems.

In the third chapter we introduce the concept of the Young measure. This is a very effective tool to describe the behaviour of weakly convergent sequences under superpositions of nonlinearities. As an application, we prove again the existence of an entropy weak solution to a scalar hyperbolic conservation law in one space dimension exploiting the reduction of the support of a corresponding Young measure.

The last two chapters deal with problems where nonlinearities depend on gradients of the solution. In the fourth chapter we study the nonlinear scalar hyperbolic equation of the second order. Chapter 5 is devoted to a class of non-Newtonian fluids, sometimes called fluids with shear-dependent viscosity or generalized Newtonian fluids. Both compressible and incompressible models are studied here.

Using the fundamental theorem on Young measures, we prove the global-in-time existence of measure-valued solutions to the above problems. Although the measure-valued solution can be subject to further investigation we want to emphasize that more attention is paid to the questions of existence, uniqueness and regularity of weak solutions. We have addressed these questions for incompressible fluids with shear dependent viscosity, studied in Sections 5.3 and 5.4. For the nonlinear hyperbolic equation of second order (studied in Chapter 4) as well as for the compressible fluid with shear dependent viscosity (studied in Section 5.5), the existence of weak solutions is still open. Nevertheless, the question of existence of a weak solution to approximating equations, shown here, is an interesting problem on its own.

This monograph is one of the few attempts to carry out a detailed analysis for a class of evolution equations for non-linear fluids (essentially in Section 1.1 and Chapter 5). Although we have tried to provide a systematic investigation, the text should be considered as an introduction to the topic, since many problems remain to be studied and a lot of interesting questions are still unanswered. We feel that the reader can easily find interesting issues for further investigation, here.

In order to make the book self-contained, we give in Section 1.2 an overview of the definitions and basic properties of the function spaces needed. The Appendix contains some useful assertions concerning the linear theory. For the benefit of the reader we have included some references that are not cited in the main text but are related to the subject of the book.

For readers interested in particular problems, we indicate the main topics together with the sections where they are discussed.

- *Non-Newtonian fluids:* Sections 1.1.1, 1.1.4–1.1.5, Section 4.2, Chapter 5.
- *Hyperbolic conservation laws:* Sections 1.1.1–1.1.2, Chapter 2, Sections 3.3–3.4.
- *Young measures:* Sections 3.1–3.2 and Sections 4.1–4.2 with applications in Sections 3.3–3.4, 4.3, 5.2 and 5.5.
- *Hyperbolic equations of second order:* Chapter 4, see also Section 1.1.2.

We are thankful to many people for their help, advice and time spent in discussions, as well as for their support and interest. First of all, we would like to thank Professor G.P. Galdi and Professor

K.R. Rajagopal. Since there are several authors to this book, the number of reasons for expressing gratitude is rather extensive. Nevertheless, G.P. Galdi's essential support to organize periodically the Winter School on Mathematical Theory in Fluid Mechanics as well as K.R. Rajagopal's permanent effort to provide us with new views on continuum mechanics are common for all of the authors.

A large part of this monograph has been written at the Department of Applied Analysis, University of Bonn, headed by Professor Jens Frehse. We wish to thank him for his permanent support as well as for the pleasant atmosphere that we found there.

We are also very obliged to Felix Otto, who agreed that we use results of his thesis (see OTTO [1992, 1993]) on scalar hyperbolic conservation laws in bounded domains and provided us with a preliminary version of the text.

Our thanks go as well to numerous colleagues who have pointed out misprints and imprecise statements. Of these, special thanks go to Mária Lukáčová, Antonín Novotný, Luboš Pick, Ondřej Pokluda and Jan Seidler, and especially to Milan Pokorný and Mark Steinhauer for reading a large portion of the manuscript and for valuable comments. Very special thanks for many valuable remarks go to Endre Süli, University of Oxford, who read the final draft of the whole manuscript.

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Writing of the book has some unpleasant side-effects. These include evenings and weekends devoted to the preparation of the manuscript instead of spending that time with wives and children. Very special thanks for their enduring support and understanding are, therefore, expressed to Jana and Markéta.

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Prague
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CHAPTER 1

Introduction

Throughout the book the usual Einstein summation convention is used: whenever an index appears twice in one expression, the summation over that index is performed. The term **domain** is reserved for an open set in a euclidean space.

1.1 Examples of evolution systems

Most of the equations and systems studied in this book have physical origin. They can be derived from basic balance laws by taking particular forms for constitutive relations and/or by considering particular kind of materials and processes.

Let $\Omega = \bigcup_{t=0}^{T} \Omega_t$, where $\Omega_t \subset \mathbb{R}^d$ is a domain occupied by the material at an instant of time $t \in [0,T)$, $T > 0$. Then the following system of equations:

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho v_j)}{\partial x_j} = 0\,, \tag{1.1}$$

$$\frac{\partial(\rho v_i)}{\partial t} + \frac{\partial(\rho v_i v_j)}{\partial x_j} = \frac{\partial T_{ij}}{\partial x_j} + \rho f_i\,, \quad 1 \le i \le d\,, \tag{1.2}$$

$$\frac{\partial(\rho e)}{\partial t} + \frac{\partial(\rho e v_j)}{\partial x_j} = \frac{\partial(T_{kj} v_k)}{\partial x_j} - \frac{\partial q_j}{\partial x_j} + \rho r + \rho f_j v_j \tag{1.3}$$

represents the local forms of the law of conservation of mass, the law of balance of momentum and balance of energy, respectively. See for example TRUESDELL [1991] or CHADWICK [1976]. Here, ρ is the density, $\mathbf{v}$ is the velocity field, E is the specific internal energy of the material, $\mathbf{T}$ is the symmetric stress tensor, $\mathbf{q}$ is the spatial heat flux vector, r is the rate of external communication of heat to the body through radiation, $\mathbf{f}$ represents the specific external body forces and e denotes $E + |\mathbf{v}|^2/2$. All quantities are evaluated at $(t,x) \in [0,T) \times \Omega_t$. In the sequel we consider $\Omega_t = \Omega$ for all $t \ge 0$.

We note that (1.1)–(1.3) can have various forms, for example when the notion of material derivative is used. Let us recall that the **material derivative** of a scalar- (or vector- or tensor-) valued function $\xi : (0,T) \times \Omega \longrightarrow \mathbb{R}$ (or $\mathbb{R}^d$ or $\mathbb{R}^{d^2}$, respectively) is defined by

$$\frac{d\xi}{dt} \equiv \frac{\partial \xi}{\partial t} + v_j \frac{\partial \xi}{\partial x_j}\,.$$

In view of this, (1.1)–(1.3) can be rewritten as

$$\frac{d\rho}{dt} + \rho \frac{\partial v_j}{\partial x_j} = 0\,, \tag{1.4}$$

$$\frac{d(\rho v_i)}{dt} + \rho v_i \frac{\partial v_j}{\partial x_j} = \frac{\partial T_{ij}}{\partial x_j} + \rho f_i\,, \quad 1 \le i \le d\,, \tag{1.5}$$

$$\frac{d(\rho e)}{dt} + \rho e \frac{\partial v_j}{\partial x_j} = \frac{\partial (T_{kj} v_k)}{\partial x_j} - \frac{\partial q_j}{\partial x_j} + \rho r + \rho f_j v_j\,. \tag{1.6}$$

Now, we define hyperbolic systems of partial differential equations, an important example of a large class of evolution systems.

Definition 1.7 *Let $\mathcal{O} \subseteq \mathbb{R}^s$ be a domain and let $\mathbf{f}_j, 1 \le j \le d$, be smooth functions from $\mathcal{O}$ into $\mathbb{R}^s$. The system of partial differential equations*

$$\frac{\partial \mathbf{u}}{\partial t} + \frac{\partial \mathbf{f}_j(\mathbf{u})}{\partial x_j} = \mathbf{0} \qquad in \quad \mathbb{R}^+ \times \mathbb{R}^d$$

is said to be **hyperbolic** *if for every $\mathbf{u} \in \mathcal{O}$ and every vector $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_d)$, $\alpha_i \in \mathbb{R}$, the matrix*

$$\mathbf{J}(\mathbf{u}, \boldsymbol{\alpha}) \equiv \alpha_j \, \mathbf{J}_j(\mathbf{u})$$

has s real eigenvalues and is diagonalizable. Here we denote by $\mathbf{J}_j(\mathbf{u})$ the Jacobian matrix of the function $\mathbf{f}_j(\mathbf{u})$, $j = 1, \dots, d$, with the components

$$[\mathbf{J}_j(\mathbf{u})]_{ik} = \frac{\partial f_{ji}}{\partial u_k}(\mathbf{u})\,, \qquad 1 \le i, k \le s\,.$$

1.1.1 Euler equations in 2D

Let us consider an inviscid (ideal) non-heat-conductive fluid for which both heat sources and external forces are neglected, i.e.,

$$\begin{gathered}\mathbf{T} = -p\,\mathbf{I}\,, \\ \frac{\partial q_j}{\partial x_j} = 0\,, \quad r = 0\,, \quad \mathbf{f} = \mathbf{0}\,, \end{gathered} \tag{1.8}$$

respectively.

Under the assumptions (1.8) the right-hand side of (1.2) is equal to $-\frac{\partial p}{\partial x_i}$, while (1.3) takes the form of

$$\frac{\partial(\rho e)}{\partial t} + \frac{\partial(\rho e v_j)}{\partial x_j} = -\frac{\partial(p v_j)}{\partial x_j}\,. \tag{1.9}$$

In such a way the system (1.1)–(1.3) turns into the so-called **Euler system** $(1 \leq i \leq d)$

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho v_j)}{\partial x_j} = 0\,, \tag{1.10}$$

$$\frac{\partial(\rho v_i)}{\partial t} + \frac{\partial(\rho v_i v_j + p\delta_{ij})}{\partial x_j} = 0\,, \tag{1.11}$$

$$\frac{\partial(\rho e)}{\partial t} + \frac{\partial\left((\rho e + p)v_j\right)}{\partial x_j} = 0\,, \tag{1.12}$$

where δ_{ij} stands for the Kronecker delta. As before, all the functions are defined on $[0,T) \times \Omega$; $T \in (0,\infty]$, $\Omega \subseteq \mathbb{R}^d$ being a domain.

According to classical literature (PIPPARD [1957], SOMMERFELD [1964], COURANT AND FRIEDRICHS [1948]) only two of the quantities describing the thermodynamic state of a fluid (as the pressure p, the temperature θ, the density ρ, the internal energy E, the entropy η) are independent. Considering ρ, θ to be independent, one obtains

$$p = p(\rho,\theta)\,, \tag{1.13}$$

$$E = E(\rho,\theta)\,, \tag{1.14}$$

$$\eta = \eta(\rho,\theta)\,, \tag{1.15}$$

and

$$e = e(\rho,\theta,\mathbf{v})\,. \tag{1.16}$$

Therefore, in general, the Euler system (1.10)–(1.12) together with

(1.13)–(1.16) is considered as the system of $(d+2)$ equations for $(d+2)$ unknown functions ρ, θ, v_j. However, the knowledge of explicit relations of the type (1.13), (1.14) allows us to consider the Euler system as a system for another set of unknown functions.

As an example, let us consider the state equation of a **perfect** gas

$$p = R\rho\theta\,, \tag{1.17}$$

where R is the universal gas constant. Further, let us consider a **polytropic** perfect gas, for which we have

$$E = c_V\theta\,, \tag{1.18}$$

where the constant c_V is the specific heat at constant volume. Then we have $p = \frac{R}{c_V}\rho E$ and, setting $\gamma \equiv 1 + R/c_V > 1$ (usually called the **Poisson constant**), we arrive at

$$p = (\gamma - 1)\rho\left(e - \frac{|\mathbf{v}|^2}{2}\right)\,. \tag{1.19}$$

Hence, in this situation, the Euler system (1.10)–(1.12) together with the state equation (1.19) can be viewed as a system of $(d+2)$ equations for $(d+2)$ unknown functions ρ, e, v_j.

If we now set

$$\mathbf{u} = (\rho, \rho v_1, \ldots, \rho v_d, \rho e)^T \in \mathbb{R}^s, \quad s = d+2\,,$$

we can rewrite (1.10)–(1.12) (under an additional assumption that $\rho > 0$) as a quasilinear system of partial differential equations

$$\frac{\partial \mathbf{u}}{\partial t} + \frac{\partial \mathbf{f}_j(\mathbf{u})}{\partial x_j} = \mathbf{0}\,, \tag{1.20}$$

where

$$\mathbf{f}_j(\mathbf{u}) \equiv \left(\rho v_j, \rho v_1 v_j + \delta_{1j}p, \ldots, \rho v_d v_j + \delta_{dj}p, (\rho e + p)\, v_j\right)^T\,,$$

$j = 1, \ldots, d$ and $\mathbf{f}_j \in C^1(\mathcal{O}; \mathbb{R}^s)$.

In particular, for $d = 2$, we have $\mathbf{u} = \left(\rho, \rho v_1, \rho v_2, \rho e\right)^T$ and

$$\mathbf{f}_1(\mathbf{u}) = \begin{pmatrix} \rho v_1 \\ \rho v_1^2 + p \\ \rho v_1 v_2 \\ (\rho e + p)v_1 \end{pmatrix}, \quad \mathbf{f}_2(\mathbf{u}) = \begin{pmatrix} \rho v_2 \\ \rho v_1 v_2 \\ \rho v_2^2 + p \\ (\rho e + p)v_2 \end{pmatrix}. \tag{1.21}$$

Setting for $1 \leq i, k \leq s$

$$[\mathbf{J}_1(\mathbf{u})]_{ik} \equiv \frac{\partial f_{1i}}{\partial u_k}(\mathbf{u}), \qquad [\mathbf{J}_2(\mathbf{u})]_{ik} \equiv \frac{\partial f_{2i}}{\partial u_k}(\mathbf{u}), \tag{1.22}$$

it can be easily shown that

$$\mathbf{J}_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ \frac{\gamma-3}{2}v_1^2 + \frac{\kappa}{2}v_2^2 & (3-\gamma)v_1 & -\kappa v_2 & \kappa \\ -v_1 v_2 & v_2 & v_1 & 0 \\ -\gamma e v_1 + \kappa v_1 |\mathbf{v}|^2 & \gamma e - \frac{\kappa}{2}(3v_1^2 + v_2^2) & -\kappa v_1 v_2 & \gamma v_1 \end{pmatrix}$$

and

$$\mathbf{J}_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ -v_1 v_2 & v_2 & v_1 & 0 \\ \frac{\gamma-3}{2}v_2^2 + \frac{\kappa}{2}v_1^2 & -\kappa v_1 & (3-\gamma)v_2 & \kappa \\ -\gamma e v_2 + \kappa v_2 |\mathbf{v}|^2 & -\kappa v_1 v_2 & \gamma e - \frac{\kappa}{2}(3v_2^2 + v_1^2) & \gamma v_2 \end{pmatrix}$$

where $\kappa \equiv \gamma - 1$. Now, for any $\boldsymbol{\alpha} = (\alpha_1, \alpha_2) \in \mathbb{R}^2$ it can be shown (FERNANDEZ [1988]) that the matrix

$$\mathbf{P}(\mathbf{u}, \boldsymbol{\alpha}) \equiv \alpha_1 \mathbf{J}_1(\mathbf{u}) + \alpha_2 \mathbf{J}_2(\mathbf{u}) \tag{1.23}$$

has for all physically relevant values (i.e. for $\rho > 0$) four real eigenvalues, namely,

$$\begin{aligned} \lambda_1 = \lambda_2 &= \alpha_1 v_1 + \alpha_2 v_2, \\ \lambda_3 &= \lambda_1 + a\sqrt{\alpha_1^2 + \alpha_2^2}, \\ \lambda_4 &= \lambda_1 - a\sqrt{\alpha_1^2 + \alpha_2^2}, \end{aligned} \tag{1.24}$$

where a is the local sound speed in isentropic processes defined by

$$a^2 \equiv \gamma \frac{p}{\rho}. \tag{1.25}$$

Then $\mathbf{P}$ can be shown to be a diagonalizable matrix:

$$\mathbf{P} = \mathbf{C}\,\boldsymbol{\Lambda}\,\mathbf{C}^{-1}, \tag{1.26}$$

where $\mathbf{\Lambda} = \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ and (see WADA ET AL. [1988], FERNANDEZ [1988])

$$\mathbf{C} = \begin{pmatrix} 1 & 0 & \dfrac{1}{2a^2} & \dfrac{1}{2a^2} \\ v_1 & \eta_2 & \dfrac{v_1 + a\eta_1}{2a^2} & \dfrac{v_1 - a\eta_1}{2a^2} \\ v_2 & -\eta_1 & \dfrac{v_2 + a\eta_2}{2a^2} & \dfrac{v_2 - a\eta_2}{2a^2} \\ \dfrac{|\mathbf{v}|^2}{2} & \eta_2 v_1 - \eta_1 v_2 & \dfrac{H + a\boldsymbol{\eta}\cdot\mathbf{v}}{2a^2} & \dfrac{H - a\boldsymbol{\eta}\cdot\mathbf{v}}{2a^2} \end{pmatrix}$$

while

$$\mathbf{C}^{-1} = \begin{pmatrix} 1 - \dfrac{\kappa|\mathbf{v}|^2}{2\,a^2} & \dfrac{\kappa}{a^2}v_1 & \dfrac{\kappa}{a^2}v_2 & -\dfrac{\kappa}{a^2} \\ \eta_1 v_2 - \eta_2 v_1 & \eta_2 & -\eta_1 & 0 \\ -a\boldsymbol{\eta}\cdot\mathbf{v} + \dfrac{\kappa|\mathbf{v}|^2}{2} & a\eta_1 - \kappa v_1 & a\eta_2 - \kappa v_2 & \kappa \\ a\boldsymbol{\eta}\cdot\mathbf{v} + \dfrac{\kappa|\mathbf{v}|^2}{2} & -a\eta_1 - \kappa v_1 & -a\eta_2 - \kappa v_2 & \kappa \end{pmatrix}.$$

Here,

$$H \equiv \frac{|\mathbf{v}|^2}{2} + \frac{a}{\kappa} \tag{1.27}$$

is the so-called **enthalpy** and

$$\eta_1 \equiv \frac{\alpha_1}{\sqrt{\alpha_1^2 + \alpha_2^2}}, \qquad \eta_2 \equiv \frac{\alpha_2}{\sqrt{\alpha_1^2 + \alpha_2^2}}, \qquad \kappa \equiv \gamma - 1. \tag{1.28}$$

Hence the system of Euler equations is a hyperbolic one (see also Section 2.1). The same result holds also for $d = 3$ (see WADA ET AL. [1988]).

1.1.2 The p-system

Consider the easiest model of one-dimensional gas dynamics for isentropic processes in Lagrangian coordinates[†] in the form of the

[†] Due to different coordinates, the operators $\frac{\partial}{\partial t}$, $\frac{\partial}{\partial x}$ do not coincide with the operators denoted by the same symbols in Euler coordinates.

so-called p-**system** (see for example LANDAU AND LIFSHITZ [1959] and KRÖNER [1996]):

$$\begin{aligned} \frac{\partial \sigma}{\partial t} - \frac{\partial v}{\partial x} &= 0\,, \\ \frac{\partial v}{\partial t} + \frac{\partial p(\sigma)}{\partial x} &= 0\,. \end{aligned} \tag{1.29}$$

Here, $\sigma = \sigma(t,x)$, $\sigma \equiv 1/\rho > 0$, is the specific volume, $v = v(t,x)$ is the velocity and $p = p(\sigma)$ is the pressure.

Denoting

$$\mathbf{w} = \begin{pmatrix} \sigma \\ v \end{pmatrix}, \qquad \mathbf{f}(\mathbf{w}) = \begin{pmatrix} -v \\ p(\sigma) \end{pmatrix}, \tag{1.30}$$

one can write (1.29) as a system of two equations

$$\frac{\partial \mathbf{w}}{\partial t} + \frac{\partial \mathbf{f}(\mathbf{w})}{\partial x} = \mathbf{0}\,. \tag{1.31}$$

The system (1.31) is hyperbolic provided that $p'(\sigma) < 0$. Indeed, the Jacobian matrix $\mathbf{J} = \left(\frac{\partial f_i}{\partial w_j}\right)$ is of the form

$$\mathbf{J} = \begin{pmatrix} 0 & -1 \\ p'(\sigma) & 0 \end{pmatrix}$$

and therefore, under the assumption $p'(\sigma) < 0$,[‡] $\mathbf{J}$ has two distinct real eigenvalues:

$$\lambda_1 = \sqrt{-p'(\sigma')}\,, \qquad \lambda_2 = -\sqrt{-p'(\sigma)}\,.$$

Hence, the system (1.31) is (strictly) hyperbolic.

Note that under the setting (justified by $(1.29)_1$)

$$v \equiv \frac{\partial u}{\partial t}\,, \quad \sigma \equiv \frac{\partial u}{\partial x}\,, \quad p(\sigma) \equiv -a(\sigma)\,,$$

the p-system (1.29) can be considered as a non-linear wave equation

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial}{\partial x} a\left(\frac{\partial u}{\partial x}\right) = 0\,, \tag{1.32}$$

with $a'(\sigma) > 0$. In the next example we will show that also in d space dimensions there is a connection between scalar hyperbolic equations of second order of the type analogous to (1.32) and some hyperbolic systems.

‡ For example, $p(\sigma) = c\sigma^{-\gamma}$ for perfect isentropic gas ($c > 0$, $\gamma > 1$).

1.1.3 Scalar hyperbolic equation of second order

Let $\Omega \subseteq \mathbb{R}^d$, $d \geq 2$, be a bounded domain, $\partial\Omega \in C^{0,1}$. Let $T > 0$, $I \equiv (0,T)$ and $Q_T \equiv I \times \Omega$. We consider the scalar hyperbolic equation of second order

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial}{\partial x_i}\big(a_i(\nabla u)\big) = 0 \qquad \text{in } Q_T\,, \tag{1.33}$$

supposing that there exists a function $\vartheta \in C^2(\mathbb{R}^d)$, called **potential** to $\boldsymbol{a} \equiv (a_1, \ldots, a_d)$, such that

$$\frac{\partial \vartheta}{\partial \xi_i}(\boldsymbol{\xi}) = a_i(\boldsymbol{\xi}) \qquad \forall\, \boldsymbol{\xi} \in \mathbb{R}^d\,, \tag{1.34}$$

for $i = 1, \ldots, d$, and

$$\frac{\partial^2 \vartheta(\boldsymbol{\xi})}{\partial \xi_i \partial \xi_j}\eta_i\eta_j \geq \alpha|\boldsymbol{\eta}|^2 \qquad \forall\, \boldsymbol{\xi}, \boldsymbol{\eta} \in \mathbb{R}^d \tag{1.35}$$

for a certain positive constant $\alpha \in \mathbb{R}$.

Chapter 4 is devoted to the detailed study of an initial-boundary value problem for the equation (1.33) (with a right-hand side f). Here we just show that under the assumptions (1.34)–(1.35) the equation (1.33) can be considered as a hyperbolic system of $(d+1)$ equations.

To this end, let us put $v_0 \equiv \frac{\partial u}{\partial t}$ and $v_j \equiv \frac{\partial u}{\partial x_j}$. We will also use the notation $\widehat{\mathbf{v}} \equiv (v_1, \ldots, v_d)$ and $\mathbf{v} \equiv (v_0, \widehat{\mathbf{v}})$. Then, (1.33) can be written as

$$\begin{aligned} \frac{\partial v_0}{\partial t} - \frac{\partial}{\partial x_j} a_j(\widehat{\mathbf{v}}) &= 0\,, \\ \frac{\partial v_j}{\partial t} - \frac{\partial v_0}{\partial x_j} &= 0\,, \quad 1 \leq j \leq d\,. \end{aligned} \tag{1.36}$$

Applying the chain rule to $\frac{\partial a_j}{\partial x_j}$, one can write the system (1.36) [of $(d+1)$ equations] in a matrix form:

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{J}_k \frac{\partial \mathbf{v}}{\partial x_k} = 0\,, \tag{1.37}$$

where

$$\mathbf{J}_k \equiv \begin{pmatrix} 0 & -\frac{\partial a_k}{\partial \xi_1} & -\frac{\partial a_k}{\partial \xi_2} & \dots & -\frac{\partial a_k}{\partial \xi_d} \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix} \begin{matrix} \\ \\ \\ \leftarrow (k+1)^{\mathrm{st}}, \\ \\ \\ \end{matrix} \tag{1.38}$$

$k = 1, \dots, d$. Recall that the hyperbolicity of the system (1.37) will be proved as soon as we show that for any constant $\boldsymbol{\eta} = (\eta_1, \dots, \eta_d) \in \mathbb{R}^d$ all the eigenvalues of the matrix

$$\mathbf{J} \equiv \eta_k \mathbf{J}_k \tag{1.39}$$

are real and the matrix itself is diagonalizable.

Denoting $S_i \equiv \eta_k \frac{\partial a_k}{\partial \xi_i}$ we see that

$$\mathbf{J} = \begin{pmatrix} 0 & -S_1 & -S_2 & \dots & -S_d \\ -\eta_1 & 0 & 0 & \dots & 0 \\ -\eta_2 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\eta_d & 0 & 0 & \dots & 0 \end{pmatrix}. \tag{1.40}$$

Now it is not difficult to see (e.g. by induction) that

$$\det(\lambda \mathbf{I} - \mathbf{J}) = \lambda^{d+1} - \lambda^{d-1}(\eta_i S_i) \tag{1.41}$$

and therefore the eigenvalues λ_i of $\mathbf{J}$ are of the form

$$\begin{aligned} \lambda_1 = \dots = \lambda_{d-1} = 0\,, \\ \lambda_d^2 = \lambda_{d+1}^2 = \eta_i S_i\,. \end{aligned} \tag{1.42}$$

Lemma 1.43 *Under the assumptions (1.34)–(1.35) all the eigenvalues λ_i are real and the matrix $\mathbf{J}$ is diagonalizable, i.e., the system (1.37) is hyperbolic.*

PROOF : If $\eta_i = 0$ for all $i = 1, \dots, d$, then $\lambda_1 = \dots = \lambda_{d+1} = 0$ and $\mathbf{J} = \mathbf{0}$, i.e. $\mathbf{J}$ is diagonal.

Therefore we assume without loss of generality that $|\boldsymbol{\eta}| > 0$. Then, using the definition of S_i, (1.34) and (1.35), we get

$$\eta_i S_i = \eta_i \eta_k \frac{\partial^2 \vartheta}{\partial \xi_i \partial \xi_k} \geq \alpha |\boldsymbol{\eta}|^2 > 0\,. \tag{1.44}$$

Hence, all eigenvalues λ_i are real, $\lambda_d = \sqrt{\eta_i S_i} \neq -\sqrt{\eta_i S_i} = \lambda_{d+1}$.

To prove that $\mathbf{J}$ is diagonalizable we recall that $\mathbf{J}$ is diagonalizable if and only if $\dim N_\lambda = m(\lambda)$, where $N_\lambda \equiv \{\mathbf{x}; \mathbf{J}\mathbf{x} = \lambda\mathbf{x}\}$ and $m(\lambda)$ is an algebraic multiplicity of λ as a root of the characteristic polynomial $\det(\lambda\mathbf{I} - \mathbf{J})$ (cf. (1.41)). Since trivially $\dim N_\lambda \leq m(\lambda)$ and $\dim N_\lambda \geq 1$ for any eigenvalue λ, it remains to prove $\dim N_0 = d - 1$.

Now, if $\mathbf{x} = (x_0, \dots, x_d)$ then from the form of the matrix $\mathbf{J}$ it is clear that $\mathbf{x} \in N_0$ if and only if $x_0 = 0$ and $\sum_{i=1}^{d} x_i S_i = 0$. Since there is j_0 such that $S_{j_0} \neq 0$ (otherwise we get a contradiction with (1.44)), the latter equation reduces to

$$x_{j_0} = -\frac{1}{S_{j_0}} \sum_{i \neq j_0} x_i S_i \,,$$

giving us the possibility to choose arbitrarily $(d-1)$ coordinates of $\mathbf{x}$ on the right-hand side of this expression. Thus, $\dim N_0 = d - 1$ and the proof is complete. ∎

1.1.4 Compressible non-Newtonian liquids undergoing isothermal processes

In Section 1.1.1 the considered stress tensor $\mathbf{T}$ was determined by pressure, i.e.,

$$\mathbf{T} = -p(\rho, \theta)\,\mathbf{I}\,,$$

(cf. $(1.8)_1$, (1.13)). If the viscous effects taken into consideration are substantial, the dependence of $\mathbf{T}$ on other quantities, say $\nabla\mathbf{v}$, $\nabla\theta$, is supposed. In this case,[§]

$$\mathbf{T} = \widehat{\mathbf{T}}(\rho, \theta, \nabla\theta, \nabla\mathbf{v})\,.$$

However, in case the motion of the material is **isothermal**, i.e. the temperature $\theta = \theta_0 > 0$ is constant, the tensor function $\widehat{\mathbf{T}}$ does not depend on θ and $\nabla\theta$. Thus,

$$\mathbf{T} = \overline{\mathbf{T}}(\rho, \nabla\mathbf{v})\,. \tag{1.45}$$

Consequently, the equations (1.1), (1.2) are not coupled with (1.3) and can be considered separately. In other words: once having ρ,

[§] For simplicity, we suppose that $\mathbf{T}$ at each point (t, x) depends only on the values of ρ, θ, $\nabla\theta$ and $\nabla\mathbf{v}$ at the same point (t, x). Consequently, any impact of history or non-local effects are not allowed.

$\mathbf{v}$ determined from (1.1), (1.2), one can use (1.3) to calculate the remaining thermodynamical quantities.

Now, taking into account the **principle of material frame indifference** (cf. for example TRUESDELL [1991]), one can show that (1.45) reduces to

$$\mathbf{T} = -p(\rho)\,\mathbf{I} + \widetilde{\mathbf{T}}(\rho, \mathbf{e})\,, \tag{1.46}$$

where $2\,\mathbf{e} = 2\,\mathbf{e}(\mathbf{v}) \equiv \nabla\mathbf{v} + (\nabla\mathbf{v})^T$ is the symmetric part of the velocity gradient $\nabla\mathbf{v}$.

In this book we will study a special form of (1.46), namely

$$\mathbf{T} = -p(\rho)\,\mathbf{I} + \boldsymbol{\tau}^E\,, \tag{1.47}$$

with $\boldsymbol{\tau}^E$ given by

$$\boldsymbol{\tau}^E = \boldsymbol{\tau}(\mathbf{e})\,. \tag{1.48}$$

Here, $\boldsymbol{\tau} : \mathbb{R}^{d^2}_{\text{sym}} \longrightarrow \mathbb{R}^{d^2}_{\text{sym}}$ is a given continuous function and $\mathbb{R}^{d^2}_{\text{sym}} \equiv \{\mathbf{M} \in \mathbb{R}^d \times \mathbb{R}^d; M_{ij} = M_{ji},\, i,j = 1,\dots,d\}$.

Assuming that a liquid (gas) obeys the state equation (1.17), we obtain that (in the isothermal case) the pressure is a linear function of ρ, i.e.,

$$p(\rho) = \beta\rho\,, \qquad \beta = R\theta_0 > 0\,. \tag{1.49}$$

Under the assumptions (1.47)–(1.49), the system (1.1), (1.2) reads

$$\frac{\partial\rho}{\partial t} + \frac{\partial(\rho v_j)}{\partial x_j} = 0\,, \tag{1.50}$$

$$\frac{\partial(\rho v_i)}{\partial t} + \frac{\partial(\rho v_i v_j)}{\partial x_j} = -\beta\frac{\partial\rho}{\partial x_i} + \frac{\partial\tau_{ij}(\mathbf{e})}{\partial x_j} + \rho f_i\,, \tag{1.51}$$

for $i = 1,\dots,d$. The left-hand side of (1.51) is equal to $\rho\frac{\partial v_i}{\partial t} + \rho v_j\frac{\partial v_i}{\partial x_j}$ due to (1.50).

In order to develop a mathematical theory (see Section 5.5), we will assume that there exist constants C_1, $C_2 > 0$ and parameters $p > 1$ and $q \in [p-1, p)$ such that for all $\boldsymbol{\eta} \in \mathbb{R}^{d^2}_{\text{sym}}$ the p-**coercivity** condition

$$\boldsymbol{\tau}(\boldsymbol{\eta}) \cdot \boldsymbol{\eta} \geq C_1|\boldsymbol{\eta}|^p\,, \tag{1.52}$$

and the q-**growth** condition

$$|\boldsymbol{\tau}(\boldsymbol{\eta})| \leq C_2(1 + |\boldsymbol{\eta}|)^q \tag{1.53}$$

hold. As usual, $|\boldsymbol{\eta}| = (\eta_{ij}\eta_{ij})^{1/2}$ and $\boldsymbol{\tau} \cdot \boldsymbol{\eta} = \tau_{ij}\eta_{ij}$ for $\boldsymbol{\tau}$, $\boldsymbol{\eta} \in \mathbb{R}^{d^2}_{\text{sym}}$.

1.1.5 Incompressible non-Newtonian fluids undergoing isothermal processes

If a material is incompressible, i.e. $\rho(t,x) = \rho_0 > 0$ for all $(t,x) \in [0,T) \times \Omega$, and undergoes an isothermal process, then

- we get from (1.45) that

$$\mathbf{T} = -\pi \mathbf{I} + \boldsymbol{\tau}^E, \tag{1.54}$$

  where π is the so-called **undetermined pressure** and $\boldsymbol{\tau}^E$ is the **extra stress** tensor;
- the system (1.50)–(1.51) reads

$$\operatorname{div} \mathbf{v} = \frac{\partial v_i}{\partial x_i} = 0, \tag{1.55}$$

$$\rho_0 \frac{\partial v_i}{\partial t} + \rho_0 v_j \frac{\partial v_i}{\partial x_j} = -\frac{\partial \pi}{\partial x_i} + \frac{\partial \tau_{ij}^E}{\partial x_j} + \rho_0 f_i, \tag{1.56}$$

  for $i = 1, \ldots, d$.

We will assume that the extra stress $\boldsymbol{\tau}^E$ is given by the sum of two symmetric tensor functions of $\mathbf{e} = \mathbf{e}(\mathbf{v})$. This means that we have for all $(t,x) \in [0,T) \times \Omega$,

$$\boldsymbol{\tau}^E(t,x) = \boldsymbol{\tau}\big(\mathbf{e}(\mathbf{v}(t,x))\big) + \boldsymbol{\sigma}\big(\mathbf{e}(\mathbf{v}(t,x))\big), \tag{1.57}$$

or equivalently

$$\boldsymbol{\tau}^E = \boldsymbol{\tau}(\mathbf{e}) + \boldsymbol{\sigma}(\mathbf{e}). \tag{1.58}$$

Similarly to Section 1.1.4, the following assumptions are imposed on $\boldsymbol{\tau}$ and $\boldsymbol{\sigma}$: for certain $p > 1$, $q \in [p-1, p)$, C_1, $C_2 > 0$, we have

$$\boldsymbol{\tau}(\boldsymbol{\eta}) \cdot \boldsymbol{\eta} \geq C_1 |\boldsymbol{\eta}|^p, \tag{1.59}$$

$$\boldsymbol{\sigma}(\boldsymbol{\eta}) \cdot \boldsymbol{\eta} \geq 0, \tag{1.60}$$

$$|\boldsymbol{\tau}(\boldsymbol{\eta}) + \boldsymbol{\sigma}(\boldsymbol{\eta})| \leq C_2 \big(1 + |\boldsymbol{\eta}|\big)^q \tag{1.61}$$

for all $\boldsymbol{\eta} \in \mathbb{R}^{d^2}_{\mathrm{sym}}$.

Notice that the tensor $\boldsymbol{\tau} + \boldsymbol{\sigma}$ satisfies (1.52)–(1.53) and it might not be clear why the decomposition (1.58) is introduced. Actually, we will use (1.58) only in such parts of the mathematical theory where the assumptions on $\boldsymbol{\tau}$ and $\boldsymbol{\sigma}$ will be strengthened. Roughly speaking, we will suppose that $\boldsymbol{\tau}$ is well behaved and $\boldsymbol{\sigma}$ is controlled by $\boldsymbol{\tau}$. By 'well behaved' we mean the existence of a strictly convex potential $U : \mathbb{R}^{d^2}_{\mathrm{sym}} \longrightarrow \mathbb{R}$ of $\boldsymbol{\tau}$, such that

$$\frac{\partial U(\mathbf{e})}{\partial e_{ij}} = \tau_{ij}(\mathbf{e}) \,, \qquad i,j = 1, \dots, d \,.$$

Appropriate assumptions put on U will in particular imply (1.59). For more details see Section 5.1.

Some interesting examples of $\boldsymbol{\tau}^E$ of the form (1.58) can be found in what follows.

Example 1.62 (Stokes' law) If the dependence of $\boldsymbol{\tau}$ on $\mathbf{e}$ is linear, i.e.,

$$\boldsymbol{\tau}(\mathbf{e}) = 2\,\nu\,\mathbf{e} \,, \quad \nu > 0 \,, \tag{1.63}$$

and $\boldsymbol{\sigma} \equiv \mathbf{0}$, then the system (1.55)–(1.56) turns into the well-known **Navier-Stokes system**

$$\operatorname{div} \mathbf{v} = 0 \,, \tag{1.64}$$

$$\rho_0 \frac{\partial v_i}{\partial t} + \rho_0 v_j \frac{\partial v_i}{\partial x_j} = -\frac{\partial \pi}{\partial x_i} + \nu \Delta v_i + \rho_0 f_i \,, \tag{1.65}$$

for $i = 1, 2, \dots, d$. Let us notice that in this case

$$\boldsymbol{\tau}(\boldsymbol{\eta}) \cdot \boldsymbol{\eta} = 2\,\nu |\boldsymbol{\eta}|^2 \,.$$

Therefore, the condition (1.59) is satisfied with $p = 2$.

Definition 1.66 *An incompressible fluid, the behaviour of which is characterized by Stokes' law (1.63), is called* **Newtonian fluid***. Fluids that cannot be adequately described by (1.63) are usually called* **non-Newtonian fluids**.

Due to the negative character of the definition of non-Newtonian fluids, it would be useful:

- to characterize main points of deviance from Newtonian behaviour,
- to classify the models of some non-Newtonian fluids.

This book is not intended to be an introduction to the mechanics of non-Newtonian fluids. For this purpose, we refer the reader to SCHOWALTER [1978], HUILGOL [1975] and RAJAGOPAL [1993]. Nevertheless, we would like at least to specify which of the basic properties of non-Newtonian behaviour can be captured by the model characterized by (1.58).

Let us recall (see e.g. RAJAGOPAL [1993]) that the main points of non-Newtonian behaviour are:

1. the ability of the fluid to shear thin or shear thicken in shear flows;
2. the presence of non-zero normal stress differences in shear flows;
3. the ability of the fluid to yield stress;
4. the ability of the fluid to exhibit stress relaxation;
5. the ability of the fluid to creep.

A non-Newtonian fluid can possess just one or all of the above listed characteristics.

The model of fluid described by (1.58) exhibits mainly the first property, while it cannot predict 'elastic' phenomena 2–4. This means that for a fluid given by (1.58) the dominant departure from the Newtonian behaviour is shear thinning or shear thickening, while the other features are not captured. For this reason we will discuss just the first phenomenon in more detail.

Consider a steady shear flow for which

$$\mathbf{v} = \big(v_1(x_2), 0, 0\big)\,.$$

Setting $\kappa \equiv |\frac{d}{dx_2} v_1(x_2)|$ we can define the so-called **generalized** (or **apparent**) **viscosity** μ_g by

$$\mu_g(\kappa) \equiv \frac{\tau_{12}(\kappa) + \sigma_{12}(\kappa)}{\kappa}\,. \tag{1.67}$$

Definition 1.68 *If the generalized viscosity μ_g defined in (1.67) is an increasing function of κ, then the fluid described by (1.58) is called* **shear thickening fluid**. *If μ_g is a decreasing function of κ, the fluid is called* **shear thinning fluid**.

Note that for Stokes' law (1.63) we have $\mu_g(\kappa) = \nu$ for all κ.

Example 1.69 (Generalized Newtonian fluids and power-law fluids) Let $\boldsymbol{\tau}$ be given by

$$\boldsymbol{\tau}(\mathbf{e}) = 2\mu\big(|\mathbf{e}|^2\big)\,\mathbf{e} = 2\widetilde{\mu}(\mathbf{e})\,\mathbf{e} \tag{1.70}$$

and $\boldsymbol{\sigma} \equiv 0$. Then, the potential U is defined by

$$U(\mathbf{e}) = \int_0^{|\mathbf{e}|^2} \mu(s)\,ds\,. \tag{1.71}$$

If, in particular, we take

$$\mu(s) = \nu_0 s^{\frac{r}{2}}\,, \quad \nu_0 > 0\,, \tag{1.72}$$

then

$$U(\mathbf{e}) = \frac{2\nu_0}{r+2}|\mathbf{e}|^{r+2},$$
$$\boldsymbol{\tau}(\boldsymbol{\eta}) \cdot \boldsymbol{\eta} = 2\,\nu_0|\boldsymbol{\eta}|^r\boldsymbol{\eta} \cdot \boldsymbol{\eta} = 2\,\nu_0|\boldsymbol{\eta}|^{r+2},$$

and we see from (1.59) that $p = r + 2$. Therefore $p \in (1, +\infty)$ if and only if $r \in (-1, +\infty)$. Further, for the shear flow $\mu_g(\kappa) = \mu(\kappa)$, $\frac{d\mu}{d\kappa} < 0$ if $r \in (-1, 0)$ and $\frac{d\mu}{d\kappa} > 0$ if $r > 0$. In other words, the model (1.70) with (1.72) captures the shear thinning fluid if $r \in (-1, 0)$ (or $p \in (1, 2)$), and captures the shear thickening fluid if $r > 0$ (or $p > 2$). The case $r = 0$ (or $p = 2$) corresponds to the Newtonian fluid.

The fluids characterized by (1.70) are called **generalized Newtonian fluids** (even if they are non-Newtonian ones). The fluids described by (1.70) and (1.72) are called **power-law fluids**.

We refer to MÁLEK, RAJAGOPAL AND RŮŽIČKA [1995] for an exhaustive, but not complete list of literature, where the power-law fluids (1.70), (1.72) are used in several fields of chemistry, glaciology, biology, geology, etc.

Example 1.73 (Various variants of power-law fluids) Despite its simple structure, the model (1.70) includes submodels with a great deal of disparity. To illustrate this fact, we will investigate apparently almost identical types of $\boldsymbol{\tau}$ in a shear flow. Let us consider

$$\begin{aligned}
&\text{(a)} \qquad \boldsymbol{\tau}^{(1)}(\mathbf{e}) = 2\,\nu_0|\mathbf{e}|^r\mathbf{e}\,,\\
&\text{(b)} \qquad \boldsymbol{\tau}^{(2)}(\mathbf{e}) = 2\,\nu_0\big(1 + |\mathbf{e}|\big)^r\mathbf{e}\,,\\
&\text{(c)} \qquad \boldsymbol{\tau}^{(3)}(\mathbf{e}) = 2\,\nu_0\big(1 + |\mathbf{e}|^2\big)^{r/2}\mathbf{e}\,,\\
&\text{(d)} \qquad \boldsymbol{\tau}^{(3+i)}(\mathbf{e}) = 2\,\nu_\infty\mathbf{e} + \boldsymbol{\tau}^{(i)}(\mathbf{e})\,, \quad i = 1, 2, 3\,,
\end{aligned} \tag{1.74}$$

where ν_0 and ν_∞ are positive constants related to the limits of $\mu_g(\kappa)$ when $\kappa \to 0$ and $\kappa \to \infty$, respectively.

Let first $r \in (-1, 0)$. Figure 1.1 depicts the graphs of μ_g corresponding to the tensors $\boldsymbol{\tau}^{(i)}$, $i = 1, \ldots, 6$, defined in (1.74).

It is worth observing that the value of parameter p in (1.59) varies for different models in (1.74); while $p = r + 2$ for the cases (a), (b), (c), we see that $p = 2$ for all cases in (d).

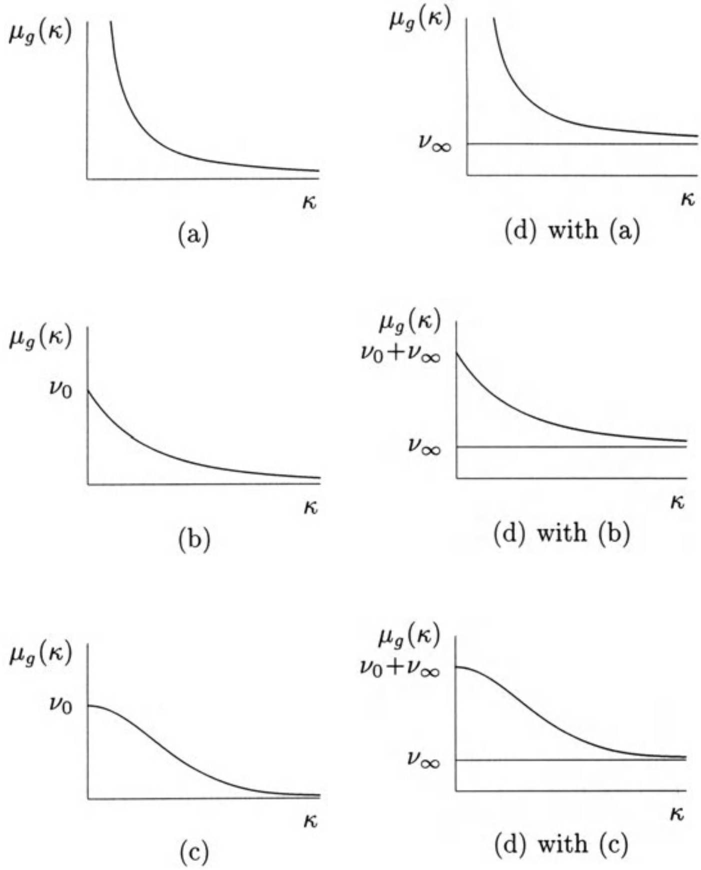

Figure 1.1 *The graphs of* μ_g *for* $\boldsymbol{\tau}^{(i)}$, $i = 1, \ldots, 6$, *with* $r \in (-1, 0)$.

We will see in Chapter 5 that some results cannot be proved for all values of $p > 1$. More often, we will get some restriction p_0 on the value of p from below such that the corresponding result is valid for all $p > p_0$. It can happen that $r + 2 < p_0 < 2$. Then the result is valid for the fluid described by (1.74)(d), while our approach fails for any of the models (a)–(c).

In fact, the model (1.74)(d) can be understood in the form of (1.58). For example, we can decompose $\boldsymbol{\tau}^{(4)}(\mathbf{e}) = \boldsymbol{\tau}(\mathbf{e}) + \boldsymbol{\sigma}(\mathbf{e})$, where

$$\boldsymbol{\tau}(\mathbf{e}) = 2\,\nu_\infty \mathbf{e}\,, \qquad \nu_\infty > 0\,, \tag{1.75}$$

and

$$\boldsymbol{\sigma}(\mathbf{e}) = 2\,\nu_0|\mathbf{e}|^r\mathbf{e}\,, \qquad \nu_0 > 0\,. \tag{1.76}$$

It can be easily verified that in this case $r \geq -1$ can even attain the value -1. In particular, the case $r = -1$ is of special interest from the point of view of applications (see (1.82) for example).

Let us now consider the case $r > 0$. Then we can draw the pictures for the models (1.74a, b, c), see Figure 1.2.

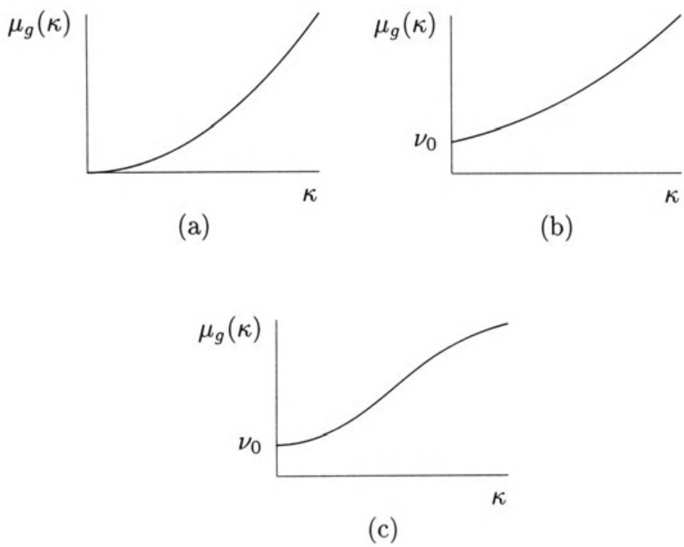

Figure 1.2 *The graphs of* μ_g *for* $\boldsymbol{\tau}^{(i)}$, $i = 1,\ 2,\ 3$, *with* $r \in (0,1)$.

We are primarily interested in fluids that have a non-zero "zero shear rate viscosity". Figure 1.2 (a) reveals that the viscosity function vanishes with the shear rate (when $\kappa \to 0$) and thus we shall not consider this possibility any further. The particular case of (d) ($r > 0$), namely

$$\boldsymbol{\tau}(\mathbf{e}) = 2\,\nu_\infty\mathbf{e} + 2\,\nu_0|\mathbf{e}|^r\mathbf{e}\,, \qquad \nu_0, \nu_\infty > 0\,, \tag{1.77}$$

was introduced in the mathematical literature by LADYZHENSKAYA [1969] and the corresponding system (1.55), (1.56) and (1.77) is sometimes called the **modified Navier-Stokes system**.

Using (1.70) and (1.71) it is easy to observe that the potentials $U^{(i)}$ corresponding to the tensors $\boldsymbol{\tau}^{(i)}$ defined in (1.74) for $r > -1$ and $i = 1, \dots, 6$, are as follows:

$$
\begin{aligned}
&\text{(a)} \qquad && U^{(1)}(\mathbf{e}) = \frac{2\,\nu_0}{r+2}|\mathbf{e}|^{r+2}\,,\\
&\text{(b)} \qquad && U^{(2)}(\mathbf{e}) = \frac{2\,\nu_0}{r+2}\big(1+|\mathbf{e}|\big)^{r+2} - \frac{2\,\nu_0}{r+1}\big(1+|\mathbf{e}|\big)^{r+1}\\
& && \qquad\qquad + \frac{2\,\nu_0}{(r+1)(r+2)}\,,\\
&\text{(c)} \qquad && U^{(3)}(\mathbf{e}) = \frac{2\,\nu_0}{r+2}\Big[\big(1+|\mathbf{e}|^2\big)^{\frac{r+2}{2}} - 1\Big]\,,\\
&\text{(d)} \qquad && U^{(3+i)}(\mathbf{e}) = \nu_\infty|\mathbf{e}|^2 + U^{(i)}(\mathbf{e})\,, \quad i = 1,2,3\,.
\end{aligned}
$$

In the last part of this section we will give three examples of models which are widely used in recent years as up-to-date ones in various fields of applied sciences. The first example presents the latest model for the flow of glacier; the second example has been proposed in blood rheology for modelling the flow of blood through arteries. Finally, the last model is used in geology for describing the dynamics of tectonic plates in the Earth's mantle.

In connection with the introduction of these models we wish to emphasize one more point here. As we shall see below, all three models mentioned above belong to the class described by (1.55)–(1.61) only for a special choice of values of parameters. It is presumed, however, that the mathematical theory developed in Chapter 5 could serve as a starting point of mathematical study of the models, presented in the following examples.

Example 1.78 (Glacier ice in creeping flow) Based on experimental tests performed by KJARTANSON [1986], KJARTANSON ET AL. [1988] and in agreement with experimental results of VAN DER VEEN AND WHILLANS [1990], Man together with his co-workers (see MAN AND SUN [1987] and references therein) proposed that the system (1.55)–(1.56) with the extra stress $\boldsymbol{\tau}^E$ given by

$$\boldsymbol{\tau}^E = \mu|\mathbf{A}_1|^r\mathbf{A}_1 + \alpha_1\mathbf{A}_2 + \alpha_2\mathbf{A}_1^2 \tag{1.79}$$

is a reasonable model for the creeping flow of ice. The tensors $\mathbf{A}_1$, $\mathbf{A}_2$ are the first two of the so-called **Rivlin-Ericksen tensors**, r, μ, α_1, α_2 are material constants, $\mu > 0$, $r \approx -\frac{2}{3}$. Let us recall (see for example TRUESDELL [1991]) that

$$\mathbf{A}_1 = 2\,\mathbf{e}$$

and

$$\mathbf{A}_2 = \frac{d}{dt}\mathbf{A}_1 + \mathbf{A}_1(\nabla\mathbf{v}) + (\nabla\mathbf{v})^T\mathbf{A}_1\,.$$

Notice that on putting $\alpha_1 = \alpha_2 = 0$ in (1.79) we obtain (1.70), (1.72). On the other hand, (1.79) with general α_1, α_2 cannot be included into the class described by (1.58)–(1.61) due to the dependence of $\boldsymbol{\tau}^E$ not only on $\mathbf{A}_1$ but also on $\frac{d}{dt}\mathbf{A}_1$. In fact, the model (1.79) with $r = 0$ describes what is called the **second grade fluid**. We refer to DUNN AND RAJAGOPAL [1995] for an exposition on n-**grade** fluids and for further references.

Example 1.80 (Blood flow) Experimental tests reveal that blood exhibits non-Newtonian phenomena such as shear thinning, creep and stress relaxation. In order to include all these features in the model, YELESWARAPU ET AL. [1994] proposed the so-called **generalized Oldroyd-B model**. The constitutive equation of that model has the form

$$\begin{aligned}\boldsymbol{\tau}^E + \lambda_1 \left[\frac{d}{dt}\boldsymbol{\tau}^E - (\nabla\mathbf{v})\,\boldsymbol{\tau}^E - \boldsymbol{\tau}^E(\nabla\mathbf{v})^T\right] \\ = \mu(\mathbf{e})\,\mathbf{e} + \lambda_2\left(\frac{d\mathbf{e}}{dt} - (\nabla\mathbf{v})\,\mathbf{e} - \mathbf{e}\,(\nabla\mathbf{v})^T\right)\end{aligned} \tag{1.81}$$

with

$$\mu(\mathbf{e}) = \nu_\infty + (\nu_0 - \nu_\infty)\left[\frac{1 + \ln(1 + \lambda|\mathbf{e}|)}{1 + \lambda|\mathbf{e}|}\right]. \tag{1.82}$$

Here $\lambda > 0$ is a material constant, λ_1, λ_2 are the relaxation and retardation times, respectively, and $\nu_0 > \nu_\infty > 0$ are the limits of $\mu(\mathbf{e})$ when $|\mathbf{e}| \to 0$ and $|\mathbf{e}| \to \infty$, respectively. (Compare with Figure 1.1.) The model (1.81)–(1.82) can be included into the class (1.55)–(1.61) only when $\lambda_1 = \lambda_2 = 0$.

Example 1.83 (Dynamics in the Earth's mantle) The study of flows in the Earth's mantle consists of thermal convection in a highly viscous fluid. As pointed out in MALEVSKY AND YUEN [1991], laboratory tests of the creep of mantle materials show a non-linear dependence of $\boldsymbol{\tau}^E$ on $\mathbf{e}$. For a description of dynamics in the planet's mantle the so-called **Boussinesq approximation** for the power-law fluid is used. For example, MALEVSKY AND YUEN [1991] investigate

$$\operatorname{div} \mathbf{v} = 0\,, \tag{1.84}$$

$$-\frac{\partial \tau_{ij}^E}{\partial x_j} + \frac{\partial \pi}{\partial x_i} = \mathrm{Ra}\,\theta \mathbf{e}_d\,, \quad 1 \le i \le d\,, \tag{1.85}$$

$$\frac{\partial \theta}{\partial t} + v_j \frac{\partial \theta}{\partial x_j} - \Delta \theta = 0\,, \tag{1.86}$$

where θ is the temperature, Ra is the so-called non-dimensional Rayleigh number and $\mathbf{e}_d \equiv (0, 0, \dots, 1)$. Further,

$$\boldsymbol{\tau}^E(\mathbf{e}) = A|\mathbf{e}|^r \mathbf{e}\,, \qquad A > 0\,,\ r \approx -\frac{2}{3} \text{ or } -\frac{4}{5}\,. \tag{1.87}$$

More precisely, the Boussinesq approximation reads

$$\operatorname{div} \mathbf{v} = 0\,, \tag{1.88}$$

$$\rho \frac{\partial v_i}{\partial t} + \rho v_j \frac{\partial v_i}{\partial x_j} = \frac{\partial \tau_{ij}^E}{\partial x_j} - \frac{\partial \pi}{\partial x_i} - \rho \theta \mathbf{e}_d\,, \tag{1.89}$$

$$\rho \frac{\partial \theta}{\partial t} + \rho v_j \frac{\partial \theta}{\partial x_j} - \Delta \theta = 0\,, \tag{1.90}$$

which converts to (1.55)–(1.61) for $\theta = \text{const}$. We refer the reader to PADULA [1994] and HILLS AND ROBERTS [1991] for a derivation of Boussinesq approximation.

In (1.85), the term $\frac{\partial v_i}{\partial t} + v_j \frac{\partial v_i}{\partial x_j}$ is neglected since the changes in velocity are substantially smaller with respect to the changes of temperature.

1.2 Function spaces

This book is devoted to the mathematical analysis of systems of equations described in Section 1.1. The fundamental problem of such analysis is to show the existence of solutions in appropriate function spaces. Thus, in the next section, we give a survey of their definitions and basic properties.

1.2.1 Basic elements of Banach spaces

Let X be a Banach space equipped with a norm $\|\cdot\|_X$. By X^* we denote the dual space to X consisting of all linear continuous functionals

$$\varphi : X \to \mathbb{R}.$$

Let $\varphi \in X^*$ and $x \in X$. Then $\langle \varphi, x \rangle_X$ denotes the value of φ at the point x or we say that the brackets $\langle \cdot, \cdot \rangle_X$ denote the duality between X and X^*. The natural norm on X^* is defined as

$$\|\varphi\|_{X^*} \equiv \sup_{\|x\|_X \leq 1} |\langle \varphi, x \rangle_X| .$$

Let $\{x_n\} \subset X$ be a sequence of elements from X. Different types of convergence can be introduced. Let $x \in X$.

1. A sequence $\{x_n\}$ converges **strongly** to x,

$$x_n \to x \quad \text{in } X$$

if and only if

$$\|x_n - x\|_X \to 0 \quad \text{as} \quad n \to \infty .$$

2. A sequence $\{x_n\}$ converges **weakly** to x,

$$x_n \rightharpoonup x \quad \text{in } X$$

if and only if for all $\varphi \in X^*$

$$\langle \varphi, x_n \rangle_X \to \langle \varphi, x \rangle_X \quad \text{as} \quad n \to \infty .$$

3. Let Z be a predual space to X, i.e. Z is a Banach space satisfying $Z^* = X$. Then a sequence $\{x_n\}$ converges **weakly-*** to x,

$$x_n \overset{*}{\rightharpoonup} x \quad \text{in } X$$

if and only if for all $\xi \in Z$

$$\langle x_n, \xi \rangle_Z \to \langle x, \xi \rangle_Z \quad \text{as} \quad n \to \infty .$$

Theorem 2.1 (Alaoglu) *Let X have a separable predual Z and assume the sequence $\{x_n\} \subset X$ is bounded in X. Then there exists a subsequence $\{x_{n_k}\} \subset \{x_n\}$ and $x \in X$ with*

$$x_{n_k} \overset{*}{\rightharpoonup} x \qquad \text{in } X .$$

PROOF : See, e.g., YOSIDA [1965, Appendix to Chapter 5]. ∎

As far as X is a reflexive separable space, the weak and weak-* convergences coincide. Consequently, for a reflexive separable Banach space X, $\overset{*}{\rightharpoonup}$ can be replaced by $\rightharpoonup$ in the previous theorem.

Let us recall that a Banach space is **reflexive** if and only if $J(X) = (X^*)^*$, where $J : X \to (X^*)^*$ is the canonical isomorphism defined by

$$\langle J(x), \varphi \rangle_{X^*} \equiv \langle \varphi, x \rangle_X \qquad \forall \varphi \in X^* .$$

Theorem 2.2 *Let $x_n \rightharpoonup x$ (or $x_n \overset{*}{\rightharpoonup} x$) in X. Then*

- *$\{x_n\}$ is bounded in X,*
- *$\|x\|_X \le \liminf_{n\to\infty} \|x_n\|_X$.*

PROOF : See for example YOSIDA [1965, Chapter 5]. ∎

Let X, Y be two Banach spaces. The space X is **(continuously) imbedded** into Y,

$$X \hookrightarrow Y ,$$

if and only if

- $X \subset Y$,
- there exists $c > 0$ such that $\|x\|_Y \le c\,\|x\|_X$ for all $x \in X$.

The space X is **compactly imbedded** into Y,

$$X \hookrightarrow\hookrightarrow Y ,$$

if and only if

- $X \hookrightarrow Y$,
- the identity map $I : X \to Y$ is compact, i.e. $\overline{I(B)}$ is compact in Y for every bounded subset B of X.

Recall that if $X \hookrightarrow Y$ then $Y^* \hookrightarrow X^*$ and if $X \hookrightarrow\hookrightarrow Y$ then $Y^* \hookrightarrow\hookrightarrow X^*$.

1.2.2 Spaces of continuous functions

Many examples of Banach spaces will be introduced in the sequel. Most of them consist of functions (scalar or vector) defined on a domain Ω, i.e. on an open set in the Euclidean space $\mathbb{R}^d$, $d \in \mathbb{N}$. Hereafter, we will give definitions of function spaces only for scalar functions, but the definitions and also all corresponding properties of these spaces can be easily extended to vector functions. The following convention for the notation of spaces of vector functions is used in the whole book:

1. If $X(\Omega)$ denotes some space of scalar functions $u : \Omega \to \mathbb{R}$ then the space of vector functions $\mathbf{u} : \Omega \to \mathbb{R}^s$, $s \in \mathbb{N}$, for which

each component u_j, $j = 1, \dots, s$, belongs to $X(\Omega)$, is denoted by $X(\Omega)^s$.

2. Vector functions are printed in boldface (as $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$) in contrast to scalar functions which are printed in italic mode (as u, v, w).

In the case when a domain Ω is not the entire space $\mathbb{R}^d$, some information about the smoothness of boundary $\partial\Omega$ is useful. Let us therefore recall the definition of spaces of **smooth functions**.

Let Ω be a domain in $\mathbb{R}^d$. Denote $\partial\Omega \equiv \overline{\Omega} \setminus \Omega$ where, as usual, $\overline{\mathbb{R}^d} = \mathbb{R}^d$. The space of all real continuous functions $u : \overline{\Omega} \to \mathbb{R}$ for which

$$\|u\|_\infty = \|u\|_{C(\overline{\Omega})} \equiv \sup_{x \in \overline{\Omega}} |u(x)|,$$

is finite, is denoted by $C(\overline{\Omega})$. Equipped with the norm $\|\cdot\|_\infty$, $C(\overline{\Omega})$ is a Banach space.

Let $\alpha = (\alpha_1, \dots, \alpha_d)$, $\alpha_i \in \mathbb{N} \cup \{0\}$, be a **multi-index** and put $|\alpha| = \sum_{i=1}^d \alpha_i$. For a function $u : \Omega \to \mathbb{R}$, the symbol $D^\alpha u$ denotes partial derivatives of the order $|\alpha|$,

$$D^\alpha u \equiv \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \dots \partial x_d^{\alpha_d}}.$$

For $k \in \mathbb{N}$ we denote by $C^k(\overline{\Omega})$ the space of all functions u which together with their derivatives $D^\alpha u$, $|\alpha| \le k$, belong to $C(\overline{\Omega})$. The norm in $C^k(\overline{\Omega})$ is defined by

$$\|u\|_{C^k(\overline{\Omega})} \equiv \sum_{|\alpha| \le k} \|D^\alpha u\|_{C(\overline{\Omega})}.$$

Let us put

$$C^\infty(\overline{\Omega}) \equiv \bigcap_{k=0}^{\infty} C^k(\overline{\Omega}).$$

The subspaces of $C(\overline{\Omega})$, $C^k(\overline{\Omega})$ of functions having compact support in Ω will be denoted by $C_C(\Omega)$, $C_C^k(\Omega)$, respectively, except for the space of functions from $C^\infty(\overline{\Omega})$ with compact support in Ω for which we use the notation $\mathcal{D}(\Omega)$.

Let us denote all functions from $C^\infty(\mathbb{R}^d)$ which are periodic in all directions with some period $L > 0$ by $C^\infty_{\mathrm{per}}(\Omega)$, where $\Omega = (0, L)^d$.

The space of **distributions** on Ω denoted by $\mathcal{D}'(\Omega)$, consists of all continuous linear functionals on $\mathcal{D}(\Omega)$. If $G \in \mathcal{D}'(\Omega)$ then the

distributional derivative $D^\alpha G \in \mathcal{D}'(\Omega)$ (or the derivative in the sense of distributions) of G is understood as

$$\langle D^\alpha G, \varphi\rangle_{\mathcal{D}(\Omega)} = (-1)^{|\alpha|} \langle G, D^\alpha \varphi\rangle_{\mathcal{D}(\Omega)} \qquad \forall \varphi \in \mathcal{D}(\Omega)\,.$$

Further, let $0 < \beta \leq 1$. For $u : \overline{\Omega} \to \mathbb{R}$, put

$$[u]_\beta \equiv \sup_{\substack{x,y\in\overline{\Omega} \\ x\neq y}} \frac{|u(x) - u(y)|}{|x-y|^\beta}$$

and

$$\|u\|_{C^{k,\beta}(\overline{\Omega})} \equiv \|u\|_{C^k(\overline{\Omega})} + \sum_{|\alpha|=k} [D^\alpha u]_\beta\,.$$

Then the space of **Hölder continuous functions** $C^{k,\beta}(\overline{\Omega})$ is defined as

$$C^{k,\beta}(\overline{\Omega}) \equiv \{u \in C^k(\overline{\Omega}); \|u\|_{C^{k,\beta}(\overline{\Omega})} < \infty\}\,.$$

Now we can give the 'definition' of smooth domains. Let $\Omega \subset \mathbb{R}^d$ be an open set. Roughly speaking, the boundary $\partial\Omega$ is of class $C^{k,\beta}$, $k + \beta \geq 1$,

$$\partial\Omega \in C^{k,\beta}\,,$$

if and only if '$\partial\Omega$ is $(d-1)$-dimensional $C^{k,\beta}$-manifold with Ω lying locally on one side of $\partial\Omega$'. Precise definition can be found in NEČAS [1967, Chapter 1]. In WLOKA [1987], the domains possessing weaker regularity properties, as segment or cone property, are introduced and discussed.

Theorem 2.3 (Imbeddings) *Let $\partial\Omega \in C^{0,1}$.*

- *Let $m \geq k \geq 0$ and $m + \beta_1 > k + \beta_2$ with $0 < \beta_i \leq 1$, $i = 1, 2$. Then*

$$C^{m,\beta_1}(\overline{\Omega}) \hookrightarrow\hookrightarrow C^{k,\beta_2}(\overline{\Omega})\,. \tag{2.4}$$

- *Let $m \geq 1$. Then*

$$C^m(\overline{\Omega}) \hookrightarrow C^{m-1,1}(\overline{\Omega})\,. \tag{2.5}$$

PROOF : See KUFNER, JOHN AND FUČÍK [1977, Section 1.5]. ∎

1.2.3 Lebesgue spaces

Let Ω be a domain in $\mathbb{R}^d$, $d \geq 1$, $1 \leq p \leq \infty$. We denote by $L^p(\Omega)$ the space of all measurable functions $u : \Omega \to \mathbb{R}$ for which the norm

$$\|u\|_p \equiv \left\{ \int_\Omega |u(x)|^p \, dx \right\}^{1/p}, \quad (1 \leq p < \infty)$$

or

$$\|u\|_\infty \equiv \operatorname*{ess\,sup}_{x \in \Omega} |u(x)|, \quad (p = \infty),$$

respectively, is finite. We recall that

$$\operatorname*{ess\,sup}_{x \in \Omega} |u(x)| \equiv \inf_{\substack{N \subset \Omega \\ |N|=0}} \sup_{x \in \Omega \setminus N} |u(x)|,$$

where $|N|$ denotes the Lebesgue measure of the set N.

Lemma 2.6 (Hölder's inequality) *Assume $1/p + 1/q = 1$, $1 < p, q < \infty$ or $p = 1$, $q = \infty$. Then for $u \in L^p(\Omega)$ and $v \in L^q(\Omega)$*

- $u\, v \in L^1(\Omega)$,
- $\|u\, v\|_1 \leq \|u\|_p \, \|v\|_q$.

PROOF : We omit trivial cases $p = 1$, $q = \infty$ and $\|u\|_p = 0$ or $\|v\|_q = 0$. Then Hölder's inequality follows from **Young's inequality**: for all a, $b \geq 0$ we have

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}. \tag{2.7}$$

Indeed, setting $a = \frac{|u(x)|}{\|u\|_p}$ and $b = \frac{|v(x)|}{\|v\|_q}$, we get

$$\begin{aligned} \frac{\|u\, v\|_1}{\|u\|_p \, \|v\|_q} &= \int_\Omega \frac{|u(x)|}{\|u\|_p} \frac{|v(x)|}{\|v\|_q} \, dx \\ &\overset{(2.7)}{\leq} \frac{1}{p} \frac{\int_\Omega |u(x)|^p \, dx}{\|u\|_p^p} + \frac{1}{q} \frac{\int_\Omega |v(x)|^q \, dx}{\|v\|_q^q} = \frac{1}{p} + \frac{1}{q} = 1, \end{aligned}$$

which gives the assertion. ∎

Let us note that an elementary proof of Young's inequality follows from the concavity of the logarithmic function: if a, $b > 0$ then

$$\begin{aligned} \log(ab) = \log a + \log b &= \tfrac{1}{p} \log a^p + \tfrac{1}{q} \log b^q \\ &\leq \log\left(\tfrac{a^p}{p} + \tfrac{b^q}{q}\right). \end{aligned}$$

We will frequently use an ε-version of (2.7), i.e.,

$$ab \leq \varepsilon a^p + \frac{b^q}{q(\varepsilon p)^{q/p}} = \varepsilon a^p + C(\varepsilon) b^q \,. \tag{2.8}$$

Corollary 2.9 *Let Ω be a bounded domain in $\mathbb{R}^d$ and $\infty \geq p \geq q \geq 1$. Then*

- $\|u\|_q \leq |\Omega|^{1/q-1/p}\|u\|_p$,
- $L^p(\Omega) \hookrightarrow L^q(\Omega)$.

Corollary 2.10 (Interpolation in p) *Assume $\infty \geq p_1 \geq p \geq p_2 \geq 1$ and $u \in L^{p_1}(\Omega) \cap L^{p_2}(\Omega)$. Then*

$$\|u\|_p \leq \|u\|_{p_1}^{\alpha} \|u\|_{p_2}^{1-\alpha} \,,$$

where $\frac{1}{p} = \frac{\alpha}{p_1} + \frac{1-\alpha}{p_2}$, $\alpha \in [0,1]$.

PROOF : Using Hölder's inequality we have

$$\|u\|_p = \left\{ \int_\Omega |u(x)|^{p\alpha} |u(x)|^{(1-\alpha)p}\, dx \right\}^{1/p} \leq \|u\|_{\alpha p \delta}^{\alpha} \|u\|_{(1-\alpha)p\delta'}^{1-\alpha} \,,$$

where $\frac{1}{\delta} + \frac{1}{\delta'} = 1$. The requirements $\alpha p \delta = p_1$ and $(1-\alpha)p\delta' = p_2$ imply $\frac{(1-\alpha)p}{p_2} + \frac{\alpha p}{p_1} = 1$, and the assertion follows. ∎

Lemma 2.11 (Vitali) *Let Ω be a bounded domain in $\mathbb{R}^d$ and $f^n : \Omega \to \mathbb{R}$ be integrable for every $n \in \mathbb{N}$. Assume that*

- $\lim_{n\to\infty} f^n(y)$ *exists and is finite for almost all $y \in \Omega$;*
- *for every $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$\sup_{n\in\mathbb{N}} \int_H |f^n(y)|\, dy < \varepsilon \qquad \forall\, H \subset \Omega,\ |H| < \delta \,.$$

Then

$$\lim_{n\to\infty} \int_\Omega f^n(y)\, dy = \int_\Omega \lim_{n\to\infty} f^n(y)\, dy \,.$$

PROOF : See ALT [1992, p. 63] or DUNFORD AND SCHWARTZ [1958]. ∎

1.2.4 Sobolev spaces

By $W^{k,p}(\Omega)$, $k \in \mathbb{N}$, $1 \le p \le \infty$, we mean the **Sobolev space** of all functions $u : \Omega \to \mathbb{R}$ having all distributional derivatives up to order k in $L^p(\Omega)$. The space $W^{k,p}(\Omega)$, equipped with the norm

$$\|u\|_{k,p} = \|u\|_{k,p;\Omega} \equiv \Big\{ \sum_{|\alpha| \le k} \|D^\alpha u\|_p^p \Big\}^{1/p},$$

is a Banach space. We identify $W^{0,p}(\Omega)$ with $L^p(\Omega)$. For k non-integer, we denote by $[k]$ the integer part of k. Then $W^{k,p}(\Omega)$ is a subspace of $W^{[k],p}(\Omega)$ consisting of functions $u \in W^{[k],p}(\Omega)$ for which

$$[D^\alpha u]^p_{k-[k],p} \equiv \int_\Omega \int_\Omega \frac{|D^\alpha u(x) - D^\alpha u(y)|^p}{|x-y|^{d+p(k-[k])}} dx\, dy$$

is finite for all α, $|\alpha| = [k]$. Then $W^{k,p}(\Omega)$ is a Banach space with the norm

$$\|u\|_{k,p} = \|u\|_{k,p;\Omega} \equiv \Big\{ \|u\|^p_{[k],p} + [D^\alpha u]^p_{k-[k],p} \Big\}^{1/p}, \quad k \notin \mathbb{N}.$$

For details see NEČAS [1967], WLOKA [1987] or TRIEBEL [1978, 1992].

If no assumptions on Ω are made, it is understood that we consider two types of domains: either

$$\Omega \subset \mathbb{R}^d \quad \text{is open and bounded with } \partial\Omega \in C^{0,1} \tag{2.12}$$

or

$$\Omega = \mathbb{R}^d . \tag{2.13}$$

The space $W_0^{k,p}(\Omega)$ is defined as the closure of $\mathcal{D}(\Omega)$ with respect to the $W^{k,p}(\Omega)$-norm. The dual space of $W_0^{k,p}(\Omega)$ is denoted by $W^{-k,q}(\Omega)$, where $\frac{1}{q} + \frac{1}{p} = 1$.

Let us recall, see NEČAS [1967], that for Ω satisfying (2.12) there exists a linear continuous operator $\gamma : W^{1,p}(\Omega) \to L^p(\partial\Omega)$ called a **trace operator** such that

$$\begin{aligned} \gamma(u) &= u|_{\partial\Omega} \qquad \text{for } u \in C^1(\overline{\Omega}), \\ \operatorname{Ker}\gamma &= W_0^{1,p}(\Omega), \\ \operatorname{Range}\gamma &= W_0^{1-\frac{1}{p},p}(\partial\Omega) \subset L^p(\partial\Omega) \end{aligned} \tag{2.14}$$

and there exists $C > 0$ such that for all $u \in W^{1,p}(\Omega)$

$$\|\gamma(u)\|_{L^p(\partial\Omega)} \le \|\gamma(u)\|_{1-\frac{1}{p},p;\partial\Omega} \le C\|u\|_{1,p}\,. \tag{2.15}$$

If $\Omega = (0, L)^d$, then

$$W^{k,p}_{\mathrm{per}}(\Omega) \equiv \overline{\{u \in C^\infty_{\mathrm{per}}(\Omega), \int_\Omega u(x)\,dx = 0\}}^{W^{k,p}(\Omega)}$$

denotes the **Sobolev space of periodic functions**.

Lemma 2.16 *Assume $1 < q < \infty$, $k \in \mathbb{N}$ and $G \in W^{-k,q}(\Omega)$. Then there exist functions $\{g_\alpha\}_{|\alpha|\le k} \subset L^q(\Omega)$ such that*

$$G = \sum_{|\alpha|\le k} (-1)^{|\alpha|} D^\alpha g_\alpha\,.$$

PROOF : See NEČAS [1967, Chapter 2] or ADAMS [1975]. Let us note that g_α are in general not uniquely determined. ■

Theorem 2.17 (Imbeddings) *Let Ω satisfy (2.12) and let $0 \le j < k$, $1 \le p, q < \infty$. Put*

$$m_0 \equiv \frac{1}{p} - \frac{k-j}{d} \qquad \textit{and} \qquad m \equiv \frac{1}{m_0} \quad \textit{if} \quad m_0 \ne 0\,.$$

- *Assume $m_0 > 0$. Then*

$$\begin{aligned} &W^{k,p}(\Omega) \hookrightarrow W^{j,m}(\Omega)\,,\\ &W^{k,p}(\Omega) \hookrightarrow\hookrightarrow W^{j,m_1}(\Omega)\,, \qquad m_1 < m\,,\\ &W^{k,p}(\mathbb{R}^d) \hookrightarrow W^{j,m}(\mathbb{R}^d)\,. \end{aligned}$$

- *Assume $m_0 < 0$. Then for $\alpha \in [0,1)$*

$$\begin{aligned} m_0 + \frac{\alpha}{d} = 0 &\Rightarrow \begin{cases} W^{k,p}(\Omega) \hookrightarrow C^{j,\alpha}(\overline{\Omega})\,,\\ W^{k,p}(\mathbb{R}^d) \hookrightarrow C^{j,\alpha}(\mathbb{R}^d)\,,\end{cases}\\ m_0 + \frac{\alpha}{d} < 0 &\Rightarrow \quad W^{k,p}(\Omega) \hookrightarrow\hookrightarrow C^{j,\alpha}(\overline{\Omega})\,. \end{aligned}$$

- *Assume $m_0 = 0$. Then*

$$W^{k,p}(\Omega) \hookrightarrow\hookrightarrow W^{j,q}(\Omega)\,, \qquad q \in [1,\infty)\,.$$

PROOF : If Ω satisfies (2.12) then the proofs can be found for example in NEČAS [1967, Chapter 2] or KUFNER, JOHN AND FUČÍK

[1977, Chapter 5]; if $\Omega = \mathbb{R}^d$ then the imbeddings are proved in NIKOLSKIJ [1975, Chapter 9] or ADAMS [1975], BERGH AND LÖFSTRÖM [1976], TRIEBEL [1978]. ■

Lemma 2.18 (Interpolation in k**)** *Let $k_1 \geq k_2 > 0$ with k_1, k_2 not necessarily integer. Then there exists a constant c such that for all $u \in W^{k_1,p}(\Omega)$*

$$\|u\|_{k_2,p} \leq c\|u\|_{k_1,p}^{\frac{k_2}{k_1}} \|u\|_p^{1-\frac{k_2}{k_1}} \quad . \tag{2.19}$$

PROOF : See for example ADAMS [1975, Chapter 7], BERGH AND LÖFSTRÖM [1976]. An elementary proof for $p = 2$ can be done by using Fourier transform, see for example ZEIDLER [1990a]. ■

Lemma 2.20 (Green's theorem) *Let $\Omega \subseteq \mathbb{R}^d$ satisfy (2.12) and $\mathbf{n} \equiv (n_1, \dots, n_d)$ be the outward normal vector. Then for $u \in W^{1,1}(\Omega)$ we have*

$$\int_\Omega \frac{\partial u(x)}{\partial x_i}\, dx = \int_{\partial\Omega} u\, n_i\, ds\,, \qquad i = 1, \dots, d\,,$$

where the values of u on $\partial\Omega$ are understood in the sense of traces.

PROOF : See for example NEČAS [1967, Section 3.1]. ■

1.2.5 Orlicz spaces

Let $\varphi : [0, \infty) \to \mathbb{R}$ be a non-negative nondecreasing right continuous function satisfying $\varphi(0) = 0$, $\varphi(\infty) \equiv \lim_{s\to\infty} \varphi(s) = \infty$. Then the function

$$\Phi(t) \equiv \int_0^{|t|} \varphi(s)\, ds\,, \tag{2.21}$$

usually called the **Young function** corresponding to φ, is even, continuous, convex, and satisfies

$$\lim_{t\to 0} \frac{\Phi(t)}{t} = 0\,, \qquad \lim_{t\to\infty} \frac{\Phi(t)}{t} = \infty\,.$$

Define $\psi(t) \equiv \sup_{\varphi(s)\leq t} s$. Let Ψ be the Young function corresponding to ψ. Then the functions Φ and Ψ are called **complementary Young functions**.

Let Φ and Ψ be complementary Young functions. Let us give some basic definitions:

1. The **Orlicz class** $\tilde{L}_\Phi(\Omega)$ consists of all measurable functions $u : \Omega \to \mathbb{R}$ such that

$$d(\Phi; u) \equiv \int_\Omega \Phi\big(|u(x)|\big)\, dx \tag{2.22}$$

is finite. It is known that in general $\tilde{L}_\Phi(\Omega)$ is not a linear space (see Kufner, John and Fučík [1977, Section 3.1]).
2. Let us call the number

$$\|u\|_{L_\Phi(\Omega)} = \|u\|_\Phi \equiv \sup_{\substack{v \in \tilde{L}_\Psi(\Omega) \\ d(\Psi; v) \le 1}} \int_\Omega |u(x)v(x)|\, dx \tag{2.23}$$

the **Orlicz norm** of $u : \Omega \to \mathbb{R}$. The **Orlicz space** $L_\Phi(\Omega)$ is defined as the set of all measurable functions $u : \Omega \to \mathbb{R}$ for which the Orlicz norm (2.23) is finite.
3. By $C_\Phi(\Omega)$, we denote the closure of all bounded measurable compactly supported functions $u : \Omega \to \mathbb{R}$, with respect to the norm $\|\cdot\|_\Phi$. If $u \in C_\Phi(\Omega)$ then $\|u\|_{C_\Phi(\Omega)} \equiv \|u\|_{L_\Phi(\Omega)}$.

Example 2.24 The functions

1. t^p/p and t^q/q,
2. $\Phi(t) = e^t - t - 1$ and $\Psi(t) = (1+t)\log(1+t) - t$,
3. $\Phi_2(t) = e^{t^2} - 1$ and $\Psi_{1/2}$

are pairs of complementary Young functions. The explicit expression for function $\Psi_{1/2}$ is not known, nevertheless its asymptotic behaviour can be shown to be $t(\ln t)^{\frac{1}{2}}$ (Kufner, John and Fučík [1977, Chapter 3]).

The Orlicz norm (2.23) requires knowledge of the complementary function. Sometimes, it is more convenient to use the so-called **Luxemburg norm** defined by

$$|||u|||_\Phi \equiv \inf\left\{ \xi > 0\ ;\ \int_\Omega \Phi\left(\frac{|u(x)|}{\xi}\right) dx \le 1 \right\}. \tag{2.25}$$

Both norms are equivalent (Kufner, John and Fučík [1977, Section 3.8]). More precisely, for all $u \in L_\Phi(\Omega)$ we have

$$|||u|||_\Phi \le \|u\|_\Phi \le 2|||u|||_\Phi\ .$$

Lemma 2.26 *Let Φ, Ψ be two complementary Young functions. Then*

- *for all $u \in \tilde{L}_\Phi(\Omega)$, $v \in \tilde{L}_\Psi(\Omega)$*
$$\|u\,v\|_1 \le d(\Phi; u) + d(\Psi; v)\,; \tag{2.27}$$

- *for all $v \in L_\Psi(\Omega)$*
$$\|v\|_\Psi \le 1 + \int_\Omega \Psi(|v(x)|)\,dx\,; \tag{2.28}$$

- **(Hölder's inequality)** *for all $u \in L_\Phi(\Omega)$, $v \in L_\Psi(\Omega)$*
$$\|u\,v\|_1 \le \|u\|_\Phi \|v\|_\Psi\,. \tag{2.29}$$

PROOF : See KUFNER, JOHN AND FUČÍK [1977, Chapter 3]; note that the second assertion is a direct consequence of the first one. ∎

Lemma 2.30 *Let Φ be a Young function. Then*

- *$L_\Phi(\Omega)$ is a Banach space;*
- *$C_\Phi(\Omega)$ is a separable Banach space and $\mathcal{D}(\Omega)$ is dense in $C_\Phi(\Omega)$;*
- *$C_\Phi(\Omega) \hookrightarrow \tilde{L}_\Phi(\Omega) \hookrightarrow L_\Phi(\Omega)$;*
- *$\left[C_\Phi(\Omega)\right]^* = L_\Psi(\Omega)$ provided that Ψ, Φ are complementary Young functions.*

PROOF : See KUFNER, JOHN AND FUČÍK [1977, Chapter 3]. ∎

A Young function Φ satisfies the Δ_2**-condition** (in brief $\Phi \in \Delta_2$) if and only if there exist $c > 0$ and $t_0 \ge 0$ such that

$$\Phi(2t) \le c\,\Phi(t)$$

for every $t > t_0$. If Ω is unbounded, we require $t_0 = 0$. The following lemma shows the importance of Young functions satisfying the Δ_2-condition (cf. Lemma 2.30).

Lemma 2.31 *Let $\Phi \in \Delta_2$. Then*

- *$C_\Phi(\Omega) = \tilde{L}_\Phi(\Omega) = L_\Phi(\Omega)$;*
- *$\left[L_\Phi(\Omega)\right]^* = L_\Psi(\Omega)$, where Ψ is the complementary Young function to Φ;*
- *$L_\Phi(\Omega)$ is separable.*

PROOF : See KUFNER, JOHN AND FUČÍK [1977, Chapter 3]. ∎

Let Φ_1, Φ_2 be two Young functions. We introduce the following orderings:

- $\Phi_1 < \Phi_2$ if and only if there exist $c > 0$ and $t_0 \geq 0$ such that $\Phi_1(t) \leq \Phi_2(ct)$ for all $t \geq t_0$. If Ω is unbounded, we require $t_0 = 0$.
- $\Phi_1 \ll \Phi_2$ if and only if $\lim\limits_{t\to\infty} \frac{\Phi_1(t)}{\Phi_2(\lambda t)} = 0$ for all $\lambda > 0$.

Lemma 2.32 *Let Φ_i, Ψ_i, $i = 1, 2$, be two pairs of complementary Young functions. If $\Phi_1 \ll \Phi_2$ then $\Psi_2 \ll \Psi_1$. If $\Phi_1 < \Phi_2$ then $\Psi_2 < \Psi_1$.*

PROOF : See KUFNER, JOHN AND FUČÍK [1977, Chapter 3]. ∎

Lemma 2.33 *Let Φ_1, Φ_2 be two Young functions. Then*

- *$\tilde{L}_{\Phi_2}(\Omega) \subset \tilde{L}_{\Phi_1}(\Omega)$ if and only if $\Phi_1 < \Phi_2$;*
- *$L_{\Phi_2}(\Omega) \hookrightarrow L_{\Phi_1}(\Omega)$ if and only if $\Phi_1 < \Phi_2$;*
- *If $\Phi_1 \ll \Phi_2$ then $L_{\Phi_2}(\Omega) \hookrightarrow C_{\Phi_1}(\Omega)$.*

PROOF : See KUFNER, JOHN AND FUČÍK [1977, Chapter 3]. ∎

Remark 2.34 Let $\Omega \subset \mathbb{R}^d$ satisfy (2.12) and $k, p \in \mathbb{R}$ be such that $kp > d$. Then due to Lemma 2.17

$$W^{k,p}(\Omega) \hookrightarrow C(\overline{\Omega}) \,. \tag{2.35}$$

Since $C(\overline{\Omega})$ is a subset of the set of all bounded measurable functions on $\overline{\Omega}$, we obviously have

$$W^{k,p}(\Omega) \subset C_\Phi(\Omega) \tag{2.36}$$

for any Young function Φ. Further, for $u \in W^{k,p}(\Omega)$

$$\begin{aligned} \|u\|_\Phi = \sup_{d(\Psi;v)\leq 1} \int_\Omega |u(x)v(x)|\,dx &\overset{(2.28)}{\leq} \|u\|_{C(\overline{\Omega})}\left[1 + \Phi(1)\mu(\Omega)\right] \\ &\equiv K\,\|u\|_{C(\overline{\Omega})} \,. \end{aligned}$$

By (2.35),

$$\|u\|_\Phi \leq c\,\|u\|_{k,p} \qquad \forall\, u \in W^{k,p}(\Omega) \,.$$

Thus, for any Young function Φ,

$$W^{k,p}(\Omega) \hookrightarrow C_\Phi(\Omega) \qquad \text{if } kp > d \,. \tag{2.37}$$

We will use this imbedding in Section 5.5.

Theorem 2.38 *Let $\Omega \subset \mathbb{R}^d$ satisfy (2.12). Let $kp = d$. Then*

$$W^{k,p}(\Omega) \hookrightarrow L_{\hat{\Phi}}(\Omega)$$

where $\hat{\Phi}(t) = \exp(|t|^{d/(d-1)}) - 1$. If $\Phi \ll \hat{\Phi}$ then the imbedding $W^{k,p}(\Omega)$ into $L_\Phi(\Omega)$ is compact.

PROOF : See for example KUFNER, JOHN AND FUČÍK [1977, Section 5.7] or TRUDINGER [1967]. ∎

Example 2.39 The Young functions Ψ and $\Psi_{1/2}$ defined in Example 2.24 satisfy the Δ_2-condition. Thus (see Lemma 2.31)

$$L_{\Psi_{1/2}}(\Omega) = C_{\Psi_{1/2}}(\Omega) \ , \quad L_\Psi(\Omega) = C_\Psi(\Omega) \ ,$$

and

$$\Big[L_\Psi(\Omega)\Big]^* = L_\Phi(\Omega) \ , \quad \Big[L_{\Psi_{1/2}}(\Omega)\Big]^* = L_{\Phi_2}(\Omega) \ .$$

Because $\Phi \ll \Phi_2$, we have $L_{\Phi_2}(\Omega) \hookrightarrow L_\Phi(\Omega)$. Moreover, from Lemma 2.30 we obtain $C_{\Phi_2}(\Omega) \hookrightarrow C_\Phi(\Omega)$ and due to Lemma 2.32 we also have $L_\Psi(\Omega) \hookrightarrow L_{\Psi_{1/2}}(\Omega)$.

Theorem 2.40 (Absolute continuity of the Orlicz norm) *Let $\Phi \in \Delta_2$ and $g \in L_\Phi(\Omega)$. Then for every $\varepsilon > 0$ there exists $\delta > 0$ such that $\|g\|_{L_\Phi(\Omega')} \le \varepsilon$ provided that $|\Omega'| < \delta$, $\Omega' \subset \Omega$.*

PROOF : See KRASNOSELSKII AND RUTICKII [1958, Chapter 2] or RAO AND REN [1991]. ∎

1.2.6 Bochner spaces

The main part of this book is devoted to evolution problems. In this case, one of the variables—time t—can be understood in a different way to the other ones, mostly denoted by $x = (x_1, \dots, x_d)$, $y = (y_1, \dots, y_d)$.

Let $\Omega \subset \mathbb{R}^d$ and $T > 0$. By Q_T we denote a time-space cylinder $Q_T \equiv I \times \Omega$, where $I \equiv (0, T)$. For $u : Q_T \to \mathbb{R}$ the map

$$u(t) : x \longmapsto u(t, x)$$

is an element of some function space (for instance Lebesgue, Hölder, Sobolev, Orlicz space). Then the function

$$t \longmapsto u(t)$$

maps the time interval I into that function space. Spaces of functions which map time interval I into some Banach space are called **Bochner spaces**.

Let X be a Banach space. The space $C^k(I;X)$, $k \in \mathbb{N}$, contains all continuous functions $u : \overline{I} \to X$ for which all (time-)derivatives, up to order k can be continuously extended to $\overline{I}$. In particular,

$$\|u\|_{C^k(I;X)} \equiv \sum_{s=0}^{k} \sup_{[0,T)} \left\| \frac{\partial^s u(t)}{\partial t^s} \right\|_X$$

is finite and $\|\cdot\|_{C^k(I;X)}$ defines the norm in $C^k(I;X)$.

Similarly, by $L^p(I;X)$, $1 \le p \le \infty$, we denote the space of all measurable functions¶ $u : I \to X$ for which the norm

$$\|u\|_{L^p(I;X)} \equiv \left\{ \int_0^T \|u(t)\|_X^p \, dt \right\}^{\frac{1}{p}}, \qquad p < \infty$$

or

$$\|u\|_{L^\infty(I;X)} \equiv \operatorname*{ess\,sup}_{t \in I} \|u(t)\|_X$$

respectively, is finite.

Let us summarize some basic properties of Bochner spaces.

Lemma 2.41 (Hölder's inequality) *Let X be a Banach space and $\frac{1}{p} + \frac{1}{q} = 1$. Assume $u \in L^p(I;X)$, $v \in L^q(I;X^*)$. Then $\langle v(\cdot), u(\cdot)\rangle_X \in L^1(I)$ and*

$$\int_0^T |\langle v(s), u(s)\rangle_X| \, ds \le \|u\|_{L^p(I;X)} \, \|v\|_{L^q(I;X^*)} \, .$$

PROOF : See ZEIDLER [1990b, Chapter 23]. ∎

Lemma 2.42 *Let X, Y be Banach spaces. Then*

- *$X \hookrightarrow Y$ implies $L^q(I;X) \hookrightarrow L^p(I;Y)$ if $1 \le p \le q \le \infty$;*
- *if $1 \le p < \infty$ then $C(I;X)$ is dense in $L^p(I;X)$ and*

$$C(I;X) \hookrightarrow L^p(I;X)\, .$$

PROOF : See ZEIDLER [1990b, Chapter 23]. ∎

¶ A function $u : I \to X$ is (strongly) measurable if and only if there exist step functions $u_n : I \to X$ such that $\|u_n(t) - u(t)\|_X \to 0$ for almost all $t \in I$.

Now let us clarify in what way we understand the time derivative of a function $u : I \to X$ from a Bochner space. We consider the following situation:

Let H be a Hilbert space with a scalar product $(\cdot,\cdot)_H$ and let X be a Banach space such that

$$X \hookrightarrow H \simeq H^* \hookrightarrow X^* \tag{2.43}$$

and

$$X \text{ is dense in } H\,. \tag{2.44}$$

Then, if $u \in L^p(I;X)$, we denote by $\frac{du}{dt}$ an element of the space $L^{p'}(I;X^*)$ (where $1/p + 1/p' = 1$) such that

$$\int_0^T \left\langle \frac{du(t)}{dt}, v \right\rangle_X \varphi(t)\,dt = -\int_0^T (u(t),v)_H \varphi'(t)\,dt$$

for all $v \in X$ and $\varphi \in \mathcal{D}(I)$. Note that the scalar product $(\cdot,\cdot)_H$ in the above formula must be appropriately chosen. For example, for $W^{2,2}(\Omega) \hookrightarrow W^{1,2}(\Omega) \hookrightarrow \left(W^{2,2}(\Omega)\right)^*$ the appropriate one is

$$(u,v)_H = (u,v)_{L^2} + (\nabla u, \nabla v)_{L^2}\,.$$

Lemma 2.45 *Let (2.43), (2.44) be satisfied and let $p \in (1,\infty)$. Then*

- $W \equiv \{u \in L^p(I;X);\ \frac{du}{dt} \in L^{p'}(I;X^*)\} \hookrightarrow C(I;H)$*;*
- **(Partial integration)** *for all $u, v \in W$ and all $s, t \in I$*

$$\begin{aligned}(u(t),v(t))_H &- (u(s),v(s))_H \\ &= \int_s^t \left\langle \frac{du(\tau)}{dt}, v(\tau)\right\rangle_X + \left\langle \frac{dv(\tau)}{dt}, u(\tau)\right\rangle_X d\tau\,.\end{aligned} \tag{2.46}$$

In particular for $u = v$,

$$\frac{1}{2}\|u(t)\|_H^2 - \frac{1}{2}\|u(s)\|_H^2 = \int_s^t \left\langle \frac{du(\tau)}{dt}, u(\tau)\right\rangle_X d\tau \tag{2.47}$$

for all $s, t \in I$.

PROOF : See GAJEWSKI, GRÖGER AND ZACHARIAS [1974, Section 4.1]. ∎

The following lemma on the compact imbedding in Bochner spaces is very important:

Lemma 2.48 (Aubin-Lions) *Let $1 < \alpha, \beta < +\infty$. Let X be a Banach space, and let X_0, X_1 be separable and reflexive Banach spaces. Provided that $X_0 \hookrightarrow\hookrightarrow X \hookrightarrow X_1$ we have*

$$\left\{v \in L^\alpha(I; X_0);\ \frac{dv}{dt} \in L^\beta(I; X_1)\right\} \hookrightarrow\hookrightarrow L^\alpha(I; X).$$

PROOF : See LIONS [1969, Section 1.5] or the survey paper SIMON [1987]. A generalized form of this lemma for locally convex spaces and $\beta = 1$ can be found in ROUBÍČEK [1990]. ■

1.2.7 The space of functions with bounded variation

Let $u \in L^1(\Omega)$. The **total variation** of u is defined as

$$TV_\Omega(u) \equiv \sup_{\substack{\varphi \in \mathcal{D}(\Omega)^d \\ \|\varphi\|_\infty \le 1}} \int_\Omega u \operatorname{div} \varphi \, dx .$$

By $BV(\Omega)$ we denote the subspace of functions $u \in L^1(\Omega)$ with $TV_\Omega(u) < \infty$,

$$BV(\Omega) \equiv \left\{u \in L^1(\Omega); TV_\Omega(u) < \infty\right\}.$$

Let $u \in BV(\Omega)$ and put

$$\|u\|_{BV(\Omega)} \equiv \|u\|_1 + TV_\Omega(u). \tag{2.49}$$

Lemma 2.50

- *$\|\cdot\|_{BV(\Omega)}$ is a norm on $BV(\Omega)$, and $BV(\Omega)$ with $\|\cdot\|_{BV(\Omega)}$ is a Banach space.*
- *$W^{1,1}(\Omega) \hookrightarrow BV(\Omega) \hookrightarrow L^1(\Omega)$.*

PROOF : It is not difficult to see that $\|\cdot\|_{BV(\Omega)}$ is a norm. The following lower semicontinuity property plays the key role in the proof of completeness of $BV(\Omega)$:

$$\left[u_n \to u \quad \text{in } L^1(\Omega)\right] \Longrightarrow TV_\Omega(u) \le \liminf_{n\to\infty} TV_\Omega(u_n). \tag{2.51}$$

Assuming (2.51) we can prove that $BV(\Omega)$ is complete. Indeed, if u_n is a Cauchy sequence in $BV(\Omega)$, then there exists some $u \in L^1(\Omega)$ such that

$$u_n \to u \qquad \text{in } L^1(\Omega). \tag{2.52}$$

According to (2.51), $TV_\Omega(u) < \infty$ and $u \in BV(\Omega)$. It remains to prove

$$TV_\Omega(u_n - u) \overset{n\to\infty}{\to} 0.$$

However, for all $\varepsilon > 0$

$$TV_\Omega(u_n - u_k) < \varepsilon \qquad \text{for } n, k \text{ large enough.}$$

Using (2.52) we get

$$(u_n - u_k) \overset{k\to\infty}{\to} (u_n - u) \qquad \text{in } L^1(\Omega)\,.$$

This together with (2.51) gives

$$TV_\Omega(u_n - u) \le \liminf_{k\to\infty} TV_\Omega(u_n - u_k) \le \varepsilon \quad \text{for } n \text{ large enough,}$$

which proves the completeness of $BV(\Omega)$.

It remains to verify the validity of (2.51). Let $\boldsymbol{\varphi} \in \mathcal{D}(\Omega)^d$ with $\|\boldsymbol{\varphi}\|_\infty \le 1$. It follows from the assumption $u_n \to u$ in $L^1(\Omega)$ that

$$\lim_{n\to\infty} \int_\Omega u_n \operatorname{div} \boldsymbol{\varphi}\, dx = \int_\Omega u \operatorname{div} \boldsymbol{\varphi}\, dx\,. \tag{2.53}$$

But for each $n \in \mathbb{N}$

$$\int_\Omega u_n \operatorname{div} \boldsymbol{\varphi}\, dx \le TV_\Omega(u_n)\,.$$

Passing to the liminf in the previous inequality as $n \to \infty$ and using (2.53) we get the assertion.

The second statement follows from the definition of the corresponding spaces. ∎

Lemma 2.54 (Imbeddings) *Let Ω satisfy (2.12). If $d > 1$ then*

$$BV(\Omega) \hookrightarrow L^{\frac{d}{d-1}}(\Omega)\,,$$

$$BV(\Omega) \hookrightarrow\hookrightarrow L^q(\Omega)\,, \quad \forall\, 1 \le q < \frac{d}{d-1}\,.$$

PROOF : See FEDERER [1969] or GIUSTI [1984, Chapter 1]. ∎

1.2.8 Radon measures

Let Ω be a bounded domain. We denote by $M(\Omega)$ the space of the so-called **Radon measures** defined as the dual space of $C(\overline{\Omega})$.

Clearly $L^1(\Omega) \hookrightarrow M(\Omega)$ since for $f \in L^1(\Omega)$

$$F(\varphi) \equiv \int_\Omega f(x)\varphi(x)\, dx \qquad \forall\, \varphi \in C(\overline{\Omega})$$

defines a continuous linear functional on $C(\overline{\Omega})$ and consequently $F \in M(\Omega)$. Moreover,

$$\|F\|_{M(\Omega)} \le \|f\|_1\,,$$

where $\|\cdot\|_{M(\Omega)}$ is defined as a dual norm. Thus, the space of (Radon) measures represents a natural extension of the space of integrable functions.

Lemma 2.55 (Imbeddings) *Let Ω satisfy (2.12) and let $\{\nu_k\} \subset M(\Omega)$ be a bounded sequence in $M(\Omega)$. Then for every $1 \le q < \frac{d}{d-1}$ there exist $\nu \in W^{-1,q}(\Omega)$ and a subsequence $\{\nu_{k_j}\} \subset \{\nu_k\}$ such that*

$$\nu_{k_j} \to \nu \qquad \textit{in } W^{-1,q}(\Omega) \quad \textit{as } j \to \infty\,.$$

PROOF : (See also EVANS [1990]). Since $C(\overline{\Omega})$ is separable, it follows from Theorem 2.1 that there exist $\nu \in M(\Omega)$ and $\{\nu_{k_j}\} \subset \{\nu_k\}$ such that

$$\nu_{k_j} \stackrel{*}{\rightharpoonup} \nu \qquad \text{in } M(\Omega)\,,$$

i.e.,

$$\int_\Omega \Phi\, d\nu_{k_j} \to \int_\Omega \Phi\, d\nu \qquad \forall\, \Phi \in C(\overline{\Omega}) \text{ as } j \to \infty\,. \tag{2.56}$$

Let $1 \le q < \frac{d}{d-1}$ and let q' be the dual index to q, i.e. $\frac{1}{q} + \frac{1}{q'} = 1$. Our aim is to show

$$\sup_{\substack{\Phi \in W_0^{1,q'}(\Omega)\\ \|\Phi\|_{1,q'} \le 1}} \left| \int_\Omega \Phi\, d\nu_{k_j} - \int_\Omega \Phi\, d\nu \right| \to 0 \qquad \text{as } j \to \infty\,.$$

Since $q' > d$, Theorem 2.17 implies that the unit ball in $W_0^{1,q'}(\Omega)$ is precompact in $C(\overline{\Omega})$. Consequently, for every $\Phi \in W_0^{1,q'}(\Omega)$, $\|\Phi\|_{1,q'} \le 1$, and for all $\varepsilon > 0$ there exists a net $\{\Phi_i\}_{i=1}^{N(\varepsilon)} \subset W_0^{1,q'}(\Omega)$ such that

$$\min_{1 \le i \le N(\varepsilon)} \|\Phi - \Phi_i\|_\infty \le \varepsilon\,.$$

Thus the following calculation finishes the proof:

$$\begin{aligned} &\left| \int_\Omega \Phi\, d\nu_{k_j} - \int_\Omega \Phi\, d\nu \right| \\ &\qquad \le \left| \int_\Omega (\Phi - \Phi_i) d\nu_{k_j} \right| + \left| \int_\Omega \Phi_i d\nu_{k_j} - \int_\Omega \Phi_i d\nu \right| \\ &\qquad\qquad + \left| \int_\Omega (\Phi_i - \Phi) d\nu \right| \\ &\qquad \le 2\varepsilon \sup_j |\nu_{k_j}|(\Omega) + \left| \int_\Omega \Phi_i d\nu_{k_j} - \int_\Omega \Phi_i d\nu \right|. \end{aligned} \tag{2.57}$$

Due to (2.56), the last term in (2.57) tends to zero as $j \to \infty$. ■

If $\Omega = \mathbb{R}^d$ (a locally compact set which is not compact) we define

$$C_0(\mathbb{R}^d) \equiv \left\{ u \in C(\mathbb{R}^d);\ \lim_{|x|\to\infty} u(x) = 0 \right\} .$$

Note that

$$C_0(\mathbb{R}^d) = \overline{\mathcal{D}(\mathbb{R}^d)}^{\|\cdot\|_\infty} .$$

The space of Radon measures is defined as

$$\begin{aligned} M(\mathbb{R}^d) \equiv \{\mu : C_0(\mathbb{R}^d) \to \mathbb{R}; \mu \text{ linear s. t. } \exists\, c > 0 \\ |\mu(f)| \le c\|f\|_\infty \ \forall f \in \mathcal{D}(\mathbb{R}^d)\} . \end{aligned} \tag{2.58}$$

Let us further define

$$\|\mu\|_{M(\mathbb{R}^d)} \equiv \sup_{\substack{f\in\mathcal{D}(\mathbb{R}^d)\\ \|f\|_\infty \le 1}} |\mu(f)| . \tag{2.59}$$

If $\mu \in M(\mathbb{R}^d)$, $\mu(f) \ge 0$ for all $f \in \mathcal{D}(\mathbb{R}^d)$, $f \ge 0$ we say that μ is a non-negative bounded Radon measure.

The space of **probability measures** is then defined as follows:

$$\mathrm{Prob}(\mathbb{R}^d) \equiv \{\mu \in M(\mathbb{R}^d),\ \mu \text{ non-negative}, \|\mu\|_{M(\mathbb{R}^d)} = 1\} . \tag{2.60}$$

Lemma 2.61

- *The space $(M(\mathbb{R}^d), \|\cdot\|_{M(\mathbb{R}^d)})$ is a Banach space.*
- *Any $\mu \in M(\mathbb{R}^d)$ can be uniquely extended to the element of dual space to $C_0(\mathbb{R}^d)$. In this sense,*

$$\big(C_0(\mathbb{R}^d), \|\cdot\|_\infty\big)^* \equiv M(\mathbb{R}^d) . \tag{2.62}$$

PROOF : See DUNFORD AND SCHWARTZ [1958], BOURBAKI [1965]. ■

Remark 2.63 The functionals $\mu \in M(\mathbb{R}^d)$ are called (Radon) measures, since there is a one-to-one correspondence between elements of $M(\mathbb{R}^d)$ and a class of (Borel) measures $\widetilde{\mu}$ on $\mathbb{R}^d$ with finite total mass $\widetilde{\mu}(\mathbb{R}^d) < \infty$, such that

$$\mu(f) = \int_{\mathbb{R}^k} f\, d\widetilde{\mu} \qquad \forall f \in C_0(\mathbb{R}^d) . \tag{2.64}$$

As usual, we do not distinguish between μ and $\widetilde{\mu}$. Finally, instead of $\mu(f)$ we use the standard duality notation

$$\langle \mu, f \rangle \equiv \mu(f) = \int_{\mathbb{R}^d} f \, d\mu \,, \quad \mu \in M(\mathbb{R}^d),\ f \in C_0(\mathbb{R}^d)\,. \tag{2.65}$$

For more details see BOURBAKI [1965].

CHAPTER 2

Scalar conservation laws

2.1 Introduction

If one looks at the basic equations of physics, one sees that a lot of them can be written in the form of conservation law, i.e.,

$$\frac{\partial \mathbf{u}}{\partial t} + \frac{\partial}{\partial x_j}\,\mathbf{f}_j(\mathbf{u}) = \mathbf{0} \qquad \text{in } \mathbb{R}^+ \times \mathbb{R}^d, \tag{1.1}$$

where $x \in \mathbb{R}^d, t > 0$. Here $\mathbf{u} = (u_1, \ldots, u_s)$ stands for a density of the investigated quantities and $\mathbf{f} = (\mathbf{f}_1, \ldots, \mathbf{f}_d)$ is the flux vector. Since the equation (1.1) is the subject of many mathematical considerations, we will state precisely in which sense this equation is understood.

Let $\mathcal{O} \subseteq \mathbb{R}^s$ be a domain and let $\mathbf{f}_j, 1 \le j \le d$, be smooth functions mapping $\mathcal{O}$ into $\mathbb{R}^s$. We look for solutions $\mathbf{u} : \mathbb{R}^+ \times \mathbb{R}^d \to \mathcal{O}$ of the system (1.1). In the sequel we restrict ourselves to hyperbolic systems, which were defined in Chapter 1. For the sake of completeness, we repeat here the definition: for every $j = 1, \ldots, d$ we denote by $\mathbf{J}_j$ the Jacobian matrix of the function $\mathbf{f}_j$, i.e.,

$$[\mathbf{J}_j(\mathbf{u})]_{ik} = \frac{\partial f_{ji}}{\partial u_k}(\mathbf{u}), \qquad i, k = 1, \ldots, s.$$

The system (1.1) is called **hyperbolic** if for every $\mathbf{u} \in \mathcal{O}$ and every vector $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_d) \in \mathbb{R}^d$ the matrix

$$\mathbf{J}(\mathbf{u}, \boldsymbol{\alpha}) = \alpha_j\, \mathbf{J}_j(\mathbf{u})$$

has s real eigenvalues

$$\lambda_1(\mathbf{u}, \boldsymbol{\alpha}) \le \cdots \le \lambda_s(\mathbf{u}, \boldsymbol{\alpha})$$

and is diagonalizable. Recall that $\lambda_1 < \cdots < \lambda_s$ implies that the corresponding matrix $\mathbf{J}$ is diagonalizable. Such systems are sometimes called **strictly hyperbolic**.

In this chapter we will investigate **scalar conservation laws**, i.e., the case $s = 1$. Note that in this case the equation (1.1) is trivially hyperbolic since the Jacobian matrices $\mathbf{J}_j(\mathbf{u})$ are of dimension 1×1. For scalar conservation laws we consider the Cauchy problem

$$\begin{aligned} \frac{\partial u}{\partial t} + \operatorname{div} \mathbf{f}(u) = 0 \qquad & \text{in } \mathbb{R}^+ \times \mathbb{R}^d\,, \\ u(0,\cdot) = u_0 \qquad & \text{in } \mathbb{R}^d, \end{aligned} \tag{1.2}$$

where $\mathbf{f} = (f_1, \ldots, f_d)$, $f_j \in C^1(\mathbb{R})$ and $u_0 : \mathbb{R}^d \to \mathbb{R}$ are given functions. To solve this problem we use the **method of vanishing viscosity**, the basic idea of which is as follows: for $\varepsilon > 0$ we study the parabolic perturbation of the Cauchy problem (1.2) given by

$$\begin{aligned} \frac{\partial u^\varepsilon}{\partial t} + \operatorname{div} \mathbf{f}(u^\varepsilon) - \varepsilon \Delta u^\varepsilon = 0 \qquad & \text{in } \mathbb{R}^+ \times \mathbb{R}^d\,, \\ u^\varepsilon(0,\cdot) = u_0 \qquad & \text{in } \mathbb{R}^d. \end{aligned} \tag{1.3}$$

After having obtained the classical solutions u^ε of (1.3) for $\varepsilon > 0$, which turn out to be smooth for smooth data, one tries to find a function u solving (1.2) as a limit of u^ε as $\varepsilon \to 0+$.

Let us note that the vanishing viscosity method is not the only one used to study the solutions of the Cauchy problem (1.2). Another interesting approach was presented in the recent preprint by Cockburn, Gripenberg and Londen [1995], where the evolution integral equation

$$\frac{\partial}{\partial t}\big(k * (u - u_0)\big) + \operatorname{div} \mathbf{f}(u) = 0 \tag{1.4}$$

was studied instead of (1.2). Here,

$$(k * u)(t,x) \equiv \int_0^t k(t-s) u(s,x)\, ds\,.$$

Formally, for $k(t)\,dt = \delta_0$ (the Dirac measure at the origin), (1.4) reduces to (1.2), while for $k \in L^1_{\mathrm{loc}}(\mathbb{R}^+)$, a positive non-increasing function, one obtains solutions u^k and tries to find a function u solving (1.2) as a limit of u^k when k approaches δ_0. For more details see Cockburn, Gripenberg and Londen [1995].

In the first part of this chapter we want to prove the existence, uniqueness and also some qualitative properties of the weak solution of the parabolic equation (1.3). Employing the vanishing viscosity method we prove existence and uniqueness of the so-called weak entropy solution of the Cauchy problem (1.2). All main ideas and concepts presented in the first part of this chapter go back to the fundamental paper of KRUŽKOV [1970]. However, our approach is very close to GODLEWSKI AND RAVIART [1991]; in only a few points do we differ or are more explicit.

The second part contains an overview of analogous results for bounded domains proved recently by OTTO [1992, 1993].

2.2 Parabolic perturbation to scalar conservation laws

The aim of this section is to study the existence, uniqueness and further properties of a weak solution to the Cauchy problem (1.3). After introducing the notion of a weak solution, we prove in Lemma 2.3 its existence and uniqueness under the assumptions that $u_0 \in L^2(\mathbb{R}^d)$ and that $\mathbf{f} \in C^1(\mathbb{R})^d$ is globally Lipschitz continuous. If moreover $u_0 \in L^\infty(\mathbb{R}^d)$ then we show in Theorem 2.9 a uniform L^∞-estimate of u^ε which will allow us to consider general nonlinearities $\mathbf{f} \in C^1(\mathbb{R})^d$. Using then the regularity theorems for linear parabolic equations, we get for smoother u_0 and $\mathbf{f}$ in Lemma 2.16 the corresponding regularity results for parabolic perturbation (1.3). This will help us to derive in Theorem 2.29 the estimates of u^ε and its spatial and time derivatives in L^1 norms. These will play the key role in the limiting process as $\varepsilon \to 0+$ in Section 2.4.

Let us start with the definition of weak solutions to the Cauchy problem (1.3). Define for $T \leq \infty$ the space

$$W(T) \equiv \left\{ u \in L^2(0,T; W^{1,2}(\mathbb{R}^d)), \frac{\partial u}{\partial t} \in L^2(0,T; W^{-1,2}(\mathbb{R}^d)) \right\}.$$

Recall that by Lemma 2.45 in Chapter 1 the space $W(T)$ is continuously imbedded into the space $C(0,T; L^2(\mathbb{R}^d))$. In particular, $\lim_{t\to 0+} v(t) = v(0)$ is a well-defined element of the space $L^2(\mathbb{R}^d)$ for all $v \in W(T)$.

Definition 2.1 *Let $u_0 \in L^2(\mathbb{R}^d)$. A function $u^\varepsilon : \mathbb{R}^+ \times \mathbb{R}^d \to \mathbb{R}$ is called a weak solution to (1.3) if and only if $u^\varepsilon \in W(T)$ for all $T < \infty$, $u^\varepsilon(0) = u_0$, and the identity*

$$\left\langle \frac{\partial u^\varepsilon}{\partial t}(t), \varphi \right\rangle + \varepsilon \int_{\mathbb{R}^d} \nabla u^\varepsilon(t) \nabla \varphi \, dx = \int_{\mathbb{R}^d} \mathbf{f}(u^\varepsilon(t)) \nabla \varphi \, dx \quad (2.2)$$

is fulfilled for almost all $t > 0$ and all $\varphi \in W^{1,2}(\mathbb{R}^d)$. Here $\langle \cdot, \cdot \rangle \equiv \langle \cdot, \cdot \rangle_{W^{1,2}(\mathbb{R}^d)}$ denotes the duality between $W^{1,2}(\mathbb{R}^d)$ and its dual space, $W^{-1,2}(\mathbb{R}^d)$.

Note that for general $u_0 \in L^2(\mathbb{R}^d)$ and $\mathbf{f} \in C^1(\mathbb{R})^d$ the integral on the right-hand side of (2.2) need not exist. However, this integral is finite if, e.g., $\mathbf{f}$ is globally Lipschitz continuous and $\mathbf{f}(0) = \mathbf{0}$. The following lemma shows that the assumption of global Lipschitz continuity of $\mathbf{f}$ is strong enough to ensure the existence of a unique weak solution u^ε to (1.3).

In order to simplify the notation, we drop the superscript ε in all remaining proofs of this section.

Lemma 2.3 *Let $u_0 \in L^2(\mathbb{R}^d)$ and let $\mathbf{f} \in C^1(\mathbb{R})^d$ be globally Lipschitz continuous, i.e., there exists $M > 0$ such that*

$$|\mathbf{f}(\xi_1) - \mathbf{f}(\xi_2)| \leq M|\xi_1 - \xi_2| \qquad \forall\, \xi_1, \xi_2 \in \mathbb{R}. \tag{2.4}$$

Then there exists exactly one $u^\varepsilon \in W(T)$ for all $T > 0$, satisfying (2.2) and $u^\varepsilon(0) = u_0$.

PROOF : Without loss of generality we can assume that $\mathbf{f}(0) = \mathbf{0}$ (otherwise we define $\widetilde{\mathbf{f}}(u) \equiv \mathbf{f}(u) - \mathbf{f}(0)$, which gives $\widetilde{\mathbf{f}}(0) = \mathbf{0}$, $\operatorname{div} \widetilde{\mathbf{f}}(u) = \operatorname{div} \mathbf{f}(u)$). We seek $u \in W(T)$ satisfying

$$\left\langle \frac{\partial u}{\partial t}(t), \varphi \right\rangle + \varepsilon \int_{\mathbb{R}^d} \nabla u(t) \nabla \varphi \, dx = \int_{\mathbb{R}^d} \mathbf{f}(u(t)) \nabla \varphi \, dx \tag{2.5}$$

for almost every $t > 0$ and all $\varphi \in W^{1,2}(\mathbb{R}^d)$. Let $\lambda > 0$ be a fixed parameter. Then using $\varphi = \psi e^{-\lambda t}$ in (2.5) and setting $u = v e^{\lambda t}$ we can rewrite (2.5) as

$$\begin{aligned} \left\langle \frac{\partial v}{\partial t}(t), \psi \right\rangle &+ \varepsilon \int_{\mathbb{R}^d} \nabla v(t) \nabla \psi \, dx \\ &+ \int_{\mathbb{R}^d} \lambda v(t) \psi \, dx = \int_{\mathbb{R}^d} e^{-\lambda t} \mathbf{f}(v(t) e^{\lambda t}) \nabla \psi \, dx, \end{aligned} \tag{2.6}$$

where we used the fact that $e^{\lambda t}$ is a multiplicator in $\mathcal{D}'(\mathbb{R}^+ \times \mathbb{R}^d)$. Now we want to use the Banach fixed point theorem. Thus, let $v \in L^2(0, \infty; L^2(\mathbb{R}^d))$ be a given function and let us look for $w \in W(\infty)$ solving the linearized problem

$$\begin{aligned} \left\langle \frac{\partial w}{\partial t}(t), \psi \right\rangle &+ \varepsilon \int_{\mathbb{R}^d} \nabla w(t) \nabla \psi dx \\ &+ \int_{\mathbb{R}^d} \lambda w(t) \psi dx = \int_{\mathbb{R}^d} e^{-\lambda t} \mathbf{f}(v(t) e^{\lambda t}) \nabla \psi \, dx \,, \\ w(0) &= v(0)\,. \end{aligned} \tag{2.7}$$

We know that $e^{-\lambda t}\operatorname{div}\mathbf{f}(ve^{\lambda t})$ belongs to $L^2(0,\infty;W^{-1,2}(\mathbb{R}^d))$, as

$$\begin{aligned}\Big|\int_0^\infty\int_{\mathbb{R}^d} e^{-\lambda t}\mathbf{f}(ve^{\lambda t})\nabla\psi\,dx\,dt\Big|\\ &\le \int_0^\infty\int_{\mathbb{R}^d} e^{-\lambda t}|\mathbf{f}(ve^{\lambda t})-\mathbf{f}(0)||\nabla\psi|\,dx\,dt\\ &\le M\int_0^\infty\int_{\mathbb{R}^d}|v||\nabla\psi|\,dx\,dt\end{aligned}$$

holds for all $\psi\in L^2(0,\infty;W^{1,2}(\mathbb{R}^d))$. Thus we get the existence of a $w\in W(\infty)$ solving (2.7) from Theorem 2.2 in the Appendix. Let us define the mapping $F_\lambda : L^2(0,\infty;L^2(\mathbb{R}^d))\to W(\infty)$ by

$$F_\lambda(v)\equiv w.$$

We want to show that there is a $\lambda_0>0$ such that F_{λ_0} is a contraction in the space $L^2(0,\infty;L^2(\mathbb{R}^d))$. Let $v_i, i=1,2$, be two elements of $L^2(0,\infty;L^2(\mathbb{R}^d))$ and let $w_i\equiv F_\lambda(v_i)$. Setting $w\equiv w_1-w_2$ we obtain for almost every $t>0$ and all $\psi\in W^{1,2}(\mathbb{R}^d)$ the identity

$$\begin{aligned}\Big\langle\frac{\partial w}{\partial t}(t),\psi\Big\rangle + \varepsilon\int_{\mathbb{R}^d}\nabla w(t)\nabla\psi\,dx + \lambda\int_{\mathbb{R}^d} w(t)\psi\,dx\\ = e^{-\lambda t}\int_{\mathbb{R}^d}\big[\mathbf{f}(e^{\lambda t}v_1(t))-\mathbf{f}(e^{\lambda t}v_2(t))\big]\nabla\psi\,dx.\end{aligned}\tag{2.8}$$

At this point we choose $\psi=w(t)$ and integrate (2.8) over an interval $(0,t)$. Using $w(0)=0$ and (2.4) we obtain

$$\begin{aligned}\frac12\|w(t)\|^2_{L^2(\mathbb{R}^d)} + \varepsilon\int_0^t\|\nabla w(s)\|^2_{L^2(\mathbb{R}^d)}\,ds + \lambda\int_0^t\|w(s)\|^2_{L^2(\mathbb{R}^d)}\,ds\\ \le M\int_0^t\|v_1(s)-v_2(s)\|_{L^2(\mathbb{R}^d)}\|\nabla w(s)\|_{L^2(\mathbb{R}^d)}\,ds.\end{aligned}$$

Dropping the first term and applying Young's inequality, we get

$$\int_0^t\|w(s)\|^2_{L^2(\mathbb{R}^d)}ds \le \frac{M^2}{4\lambda\varepsilon}\int_0^t\|v_1(s)-v_2(s)\|^2_{L^2(\mathbb{R}^d)}ds.$$

Notice that $v_1-v_2\in L^2(0,\infty;L^2(\mathbb{R}^d))$ and the left-hand side is a monotonous continuous function of t. Therefore, letting $t\to\infty$ we obtain

$$\|w_1-w_2\|_{L^2(0,\infty;L^2(\mathbb{R}^d))} \le \frac{M}{2\sqrt{\lambda\varepsilon}}\|v_1-v_2\|_{L^2(0,\infty;L^2(\mathbb{R}^d))}\,.$$

We see that F_{λ_0} is a contraction for λ_0 large enough and therefore, has a unique fixed point $v \in L^2(0,\infty; L^2(\mathbb{R}^d))$. Consequently, v solves (2.6) and belongs to the space $W(\infty)$. Now, setting

$$u \equiv v\, e^{\lambda t},$$

we obtain a unique weak solution to our problem (1.3) belonging to $W(T)$ for all $T > 0$. Note that $e^{\lambda t} \notin L^\infty(0,\infty)$ and therefore in general $u \notin W(\infty)$. ■

The next theorem shows that under an additional assumption on u_0, namely $u_0 \in L^2(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$, we can avoid the assumption of global Lipschitz continuity of $\mathbf{f}$ since a uniform L^∞-estimate of u^ε will be derived. Note that also in this case the integral on the right-hand side of (2.2) is finite.

Theorem 2.9 *Let $u_0 \in L^2(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ and $\mathbf{f} \in C^1(\mathbb{R})^d$. Then the Cauchy problem (1.3) has a unique weak solution $u^\epsilon \in W(T) \cap L^\infty((0,T) \times \mathbb{R}^d)$ for every $T > 0$. This solution satisfies*

$$\|u^\varepsilon(t)\|_{L^\infty(\mathbb{R}^d)} \le \|u_0\|_{L^\infty(\mathbb{R}^d)} \qquad \text{for a.a. } t \in (0,T) \tag{2.10}$$

and for any bounded open set $Q \subset \mathbb{R}^+ \times \mathbb{R}^d$,

$$\|\sqrt{\varepsilon}\, \nabla u^\varepsilon\|_{L^2(Q)} \le c(Q)\,, \tag{2.11}$$

where the constant $c(Q)$ is independent of ε.

PROOF : We will use Lemma 2.3 and therefore we cut off the nonlinearity $\mathbf{f}$. Let $\zeta \in \mathcal{D}(\mathbb{R})$ be such that

$$\zeta(r) \equiv \begin{cases} 1\,, & r \le \|u_0\|_{L^\infty(\mathbb{R}^d)}\,, \\ 0\,, & r \ge \|u_0\|_{L^\infty(\mathbb{R}^d)} + 1. \end{cases}$$

Define

$$\begin{aligned} g_j &\equiv \zeta\, f_j\,, \qquad 1 \le j \le d\,, \\ \mathbf{g} &\equiv (g_1, \ldots, g_d)\,. \end{aligned}$$

Thus, $\mathbf{g}$ is a C^1-function with compact support and the assumption (2.4) is fulfilled for $\mathbf{g}$ with some constant M. Then Lemma 2.3 gives the existence of a unique $u \in W(T)$ for all $T > 0$, satisfying

$u(0) = u_0$ and

$$\left\langle \frac{\partial u}{\partial t}(t), \varphi \right\rangle + \varepsilon \int_{\mathbb{R}^d} \nabla u(t) \nabla \varphi \, dx = - \int_{\mathbb{R}^d} g_j'(u(t)) \frac{\partial u}{\partial x_j} \varphi \, dx \tag{2.12}$$

for almost all $t > 0$ and all $\varphi \in W^{1,2}(\mathbb{R}^d)$. We want to show that u satisfies also (2.2). We define

$$v \equiv u - \|u_0\|_{L^\infty(\mathbb{R}^d)} \tag{2.13}$$

and will show that $v^+ = 0$. Since $v^+ \in W^{1,2}(\mathbb{R}^d)$ due to Lemma 1.7 in the Appendix, we can use v^+ as a test function in (2.12). Using (1.16) from the Appendix and the fact that $\nabla v^+ = 0$ on the set $\{v \le 0\}$, while $\nabla v^+ = \nabla v = \nabla u$ and $v^+ = v$ elsewhere, we obtain

$$\frac{1}{2}\|v^+(t)\|^2_{L^2(\mathbb{R}^d)} - \frac{1}{2}\|v^+(0)\|^2_{L^2(\mathbb{R}^d)} + \varepsilon \int_0^t \|\nabla v^+(s)\|^2_{L^2(\mathbb{R}^d)} \, ds \le M \int_0^t \|\nabla v^+(s)\|_{L^2(\mathbb{R}^d)} \|v^+(s)\|_{L^2(\mathbb{R}^d)} \, ds.$$

Further, Young's inequality and $v^+(0) = 0$ imply

$$\|v^+(t)\|^2_{L^2(\mathbb{R}^d)} \le \frac{M^2}{2\varepsilon} \int_0^t \|v^+(s)\|^2_{L^2(\mathbb{R}^d)} \, ds.$$

Now, Gronwall's lemma implies

$$v^+(t) = \left(u(t) - \|u_0\|_{L^\infty(\mathbb{R}^d)}\right)^+ = 0 \tag{2.14}$$

for almost all $t \in (0, T)$. In the same way it can also be shown that

$$\left(- u(t) - \|u_0\|_{L^\infty(\mathbb{R}^d)}\right)^+ = 0. \tag{2.15}$$

However, (2.14) and (2.15) mean nothing other than

$$\|u(t)\|_{L^\infty(\mathbb{R}^d)} \le \|u_0\|_{L^\infty(\mathbb{R}^d)} \qquad \text{a.e. in } (0, T).$$

Hence, the definition of $\mathbf{g}$ implies that for our particular u

$$\mathbf{f}(u) = \mathbf{g}(u),$$

and therefore u satisfies (2.2) and belongs to $W(T) \cap L^\infty((0,T) \times \mathbb{R}^d)$ for all $T > 0$.

Now we show that our solution is unique. Let $u_1, u_2 \in W(T) \cap L^\infty((0,T) \times \mathbb{R}^d)$ be two solutions satisfying (2.2) and $u_i(0) = u_0$. Cutting off the nonlinearity $\mathbf{f}$ outside the ball

$$B_{1,2} \equiv \left\{u \in \mathbb{R},\ |u| \le \max_{i=1,2} \left(\|u_i\|_{L^\infty((0,T)\times\mathbb{R}^d)}\right)\right\}$$

we get a function $\mathbf{g}$, which is globally Lipschitz continuous and therefore u_1 and u_2 satisfy (2.12) while $u_1(0) = u_2(0) = u_0$. But Lemma 2.3 gives the uniqueness of such solutions in the class $W(T)$. Thus $u_1 = u_2$.

It remains to show (2.11). Let Q be any bounded open set in $\mathbb{R}^+ \times \mathbb{R}^d$. Define $\chi \in \mathcal{D}(\mathbb{R}^+ \times \mathbb{R}^d)$ such that $\chi \ge 0$ on $\mathbb{R}^+ \times \mathbb{R}^d$, $\chi = 1$ on Q and $\chi(t,x) = 0$ for $\operatorname{dist}((t,x), Q) > 1$. Using $u\chi^2$ as a test function in (2.2), we get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^d} (u\chi)^2\,dx + \int_{\mathbb{R}^d} |\sqrt{\varepsilon}\,\nabla(u\chi)|^2\,dx &= -\int_{\mathbb{R}^d} \frac{\partial f_i(u)}{\partial x_i} u\chi^2\,dx \\ &+ \int_{\mathbb{R}^d} u^2\chi\frac{\partial\chi}{\partial t}\,dx + \varepsilon\int_{\mathbb{R}^d} u^2|\nabla\chi|^2\,dx.\end{aligned}$$

Using the fact that $u \in W(T) \cap L^\infty((0,T) \times \mathbb{R}^d)$ for all $T > 0$ and that χ has compact support, one easily obtains (2.11). ■

After this general existence result we will show which qualitative properties the solution u^ε has if the data $\mathbf{f}$ and u_0 are more regular. The results are based on the regularity theory for linear parabolic equations (see, e.g., the Appendix).

Lemma 2.16 *Let $\mathbf{f} \in C^m(\mathbb{R})^d$ and let $u_0 \in W^{m,2}(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$, $m \in \mathbb{N}$. Then the weak solution u^ε of the problem (1.3) satisfies, for all $T > 0$,*

$$u^\varepsilon \in L^2(0,T; W^{m+1,2}(\mathbb{R}^d)) \cap C(0,T; W^{m,2}(\mathbb{R}^d)). \quad (2.17)$$

Further, for $k \in \mathbb{N}$, $2k \le m$ we have

$$\frac{\partial^k u^\varepsilon}{\partial t^k} \in L^2(0,T; W^{m+1-2k,2}(\mathbb{R}^d)) \cap C(0,T; W^{m-2k,2}(\mathbb{R}^d))$$

while for $2k = m+1$ we have

$$\frac{\partial^k u^\varepsilon}{\partial t^k} \in L^2(0,T; L^2(\mathbb{R}^d)).$$

PROOF : By Theorem 2.9, a weak solution u solves for almost all $t \in (0,T)$ the equation

$$\frac{\partial u(t)}{\partial t} - \varepsilon \Delta u(t) = -\operatorname{div} \mathbf{f}\big(u(t)\big), \tag{2.18}$$

understood as the equation in $\big(W^{1,2}(\mathbb{R}^d)\big)^*$. However, since $u \in L^2\big(0,T;W^{1,2}(\mathbb{R}^d)\big) \cap L^\infty\big((0,T)\times\mathbb{R}^d\big)$, one sees that

$$\operatorname{div} \mathbf{f}(u) = f_j(u)\frac{\partial u}{\partial x_j} \in L^2\big(0,T;L^2(\mathbb{R}^d)\big),$$

and the regularity theory for linear parabolic equations can be applied. Thus, putting $m = 1$ in Theorem 2.4 in the Appendix, we obtain

$$\begin{aligned} &u \in L^2(0,T;W^{2,2}(\mathbb{R}^d)) \cap C(0,T;W^{1,2}(\mathbb{R}^d)), \\ &\frac{\partial u}{\partial t} \in L^2(0,T;L^2(\mathbb{R}^d)), \end{aligned} \tag{2.19}$$

which is the statement of Lemma 2.16 for $m = 1$. Moreover, u satisfies the equation

$$\frac{\partial u}{\partial t} - \varepsilon \Delta u = -\operatorname{div} \mathbf{f}(u) \tag{2.20}$$

for almost all $(t,x) \in (0,T)\times\mathbb{R}^d$.

Let us prove (2.17) for $m = 2$. We differentiate formally (2.20) with respect to t and denote $v \equiv \frac{\partial u}{\partial t}$. We get

$$\frac{\partial v}{\partial t} - \varepsilon \Delta v = -\operatorname{div}\left(\mathbf{f}'(u)\frac{\partial u}{\partial t}\right). \tag{2.21}$$

Since $u \in L^\infty\big((0,T)\times\mathbb{R}^d\big)$ and $\frac{\partial u}{\partial t} \in L^2\big((0,T)\times\mathbb{R}^d\big)$, the right-hand side of (2.21) belongs to $L^2(0,T;W^{-1,2}(\mathbb{R}^d))$. In agreement with (2.20), let us demand

$$v(0) = \varepsilon\Delta u_0 - \operatorname{div}\mathbf{f}(u_0). \tag{2.22}$$

Because $u_0 \in W^{2,2}(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$, we see that $v(0) \in L^2(\mathbb{R}^d)$. Therefore, we can use Theorem 2.2 from the Appendix on the existence of a weak solution to the Cauchy problem for linear parabolic equations. We obtain

$$v \in L^2\big(0,T;W^{1,2}(\mathbb{R}^d)\big) \cap C\big(0,T;L^2(\mathbb{R}^d)\big). \tag{2.23}$$

The standard argument implies that $v = \frac{\partial u}{\partial t}$ and we obtain from (2.19) and (2.23) that

$$\frac{\partial u}{\partial x_i} \in W^{1,2}((0,T) \times \mathbb{R}^d)\,, \qquad i = 1, \ldots, d\,.$$

The Sobolev imbedding theorem (see Theorem 2.17 in Chapter 1) implies

$$\frac{\partial u}{\partial x_i} \in L^{\frac{2(d+1)}{d-1}}\big((0,T) \times \mathbb{R}^d\big)\,, \qquad i = 1, \ldots, d\,, \tag{2.24}$$

for $d > 1$ and

$$\frac{\partial u}{\partial x} \in L^r\big((0,T) \times \mathbb{R}\big)\,, \qquad \forall\, r < \infty\,, \tag{2.25}$$

if $d = 1$.

Now, we compute formally the derivative of (2.20) with respect to x_i and obtain

$$\begin{aligned} \frac{\partial}{\partial t}\Big(\frac{\partial u}{\partial x_i}\Big) - \varepsilon\Delta\frac{\partial u}{\partial x_i} &= -f_j''(u)\frac{\partial u}{\partial x_j}\frac{\partial u}{\partial x_i}\,, \\ \frac{\partial u}{\partial x_i}(0) &= \frac{\partial u_0}{\partial x_i}. \end{aligned} \tag{2.26}$$

Again, we will use the regularity result for linear parabolic equations, this time with a view to $\frac{\partial u}{\partial x_i}$. The assertion will follow immediately, once we prove that the right-hand side of (2.26) belongs to $L^2\big((0,T) \times \mathbb{R}^d\big)$, i.e., if

$$\frac{\partial u}{\partial x_i} \in L^4\big((0,T) \times \mathbb{R}^d\big)\,, \qquad i = 1, \ldots, d\,. \tag{2.27}$$

Let us consider two cases. If $d \le 3$, then (2.27) is a consequence of (2.24), (2.25). Indeed, if $d = 3$, then $\frac{2(d+1)}{d-1} = 4$ and if $1 < d < 3$, we get (2.27) by interpolation between $L^{\frac{2(d+1)}{d-1}}\big((0,T) \times \mathbb{R}^d\big)$ and $L^2\big((0,T) \times \mathbb{R}^d\big)$. The case $d = 1$ is trivial.

If $d > 3$, then we use the L^q-theory of linear parabolic equations applied to the equation (2.18). Using Theorem 2.3 in Giga and Sohr [1991] and (2.24), we obtain for $p_0 \equiv \frac{2(d+1)}{d-1}$,

$$\begin{aligned} u &\in L^{p_0}\big(0,T; W^{2,p_0}(\mathbb{R}^d)\big)\,, \\ \frac{\partial u}{\partial t} &\in L^{p_0}\big(0,T; L^{p_0}(\mathbb{R}^d)\big)\,. \end{aligned} \tag{2.28}$$

It is known (see SOLONNIKOV [1977]) that if a function u satisfies (2.28), then

$$\nabla u \in L^{p_1}\big(0,T;L^{p_1}(\mathbb{R}^d)^d\big)\,,$$

where

$$p_1=\frac{(d+2)p_0}{(d+2)-p_0}\,.$$

Iterating this process, we can conclude that

$$\nabla u \in L^{p_k}\big(0,T;L^{p_k}(\mathbb{R}^d)^d\big)\,,$$

with

$$p_k=\frac{(d+2)p_0}{(d+2)-kp_0}\,.$$

Obviously, there exists a $k_0 \in \mathbb{N}$ such that $\tilde p \equiv p_{k_0} > 4$ (if not, then $p_0 < \frac{4(d+2)}{(d+2)+4k} \to 0$ as $k \to \infty$, which is a contradiction). Interpolating between $L^{\tilde p}\big(0,T;L^{\tilde p}(\mathbb{R}^d)\big)$ and $L^2\big(0,T;L^2(\mathbb{R}^d)\big)$ we get (2.27) and consequently (2.17) for $m=2$.

For $m>2$ one uses the same procedure. The proof of Lemma 2.16 is complete. ∎

Theorem 2.29 *Let* $\mathbf{f} \in C^m(\mathbb{R})^d$ *and let* $u_0 \in W^{m,2}(\mathbb{R}^d) \cap W^{2,1}(\mathbb{R}^d)$ *for some* $m > \max\{\frac{d}{2}+2, 3\}$. *Put*

$$M \equiv \sup\{|\mathbf{f}'(\zeta)|, |\zeta| \le \|u_0\|_{L^\infty(\mathbb{R}^d)}\}. \tag{2.30}$$

Then for all $t>0$ *the solution* u^ε *of the problem (1.3) satisfies*

$$\|u^\varepsilon(t)\|_{L^\infty(\mathbb{R}^d)} \le \|u_0\|_{L^\infty(\mathbb{R}^d)}\,, \tag{2.31}$$

$$\int_{\mathbb{R}^d} u^\varepsilon(t,x)\,dx = \int_{\mathbb{R}^d} u_0(x)\,dx\,, \tag{2.32}$$

$$\Big\|\frac{\partial u^\varepsilon}{\partial x_i}(t)\Big\|_{L^1(\mathbb{R}^d)} \le \Big\|\frac{\partial u_0}{\partial x_i}\Big\|_{L^1(\mathbb{R}^d)}\,, \tag{2.33}$$

$$\Big\|\frac{\partial u^\varepsilon}{\partial t}(t)\Big\|_{L^1(\mathbb{R}^d)} \le M\|\nabla u_0\|_{L^1(\mathbb{R}^d)^d} + \varepsilon\|\Delta u_0\|_{L^1(\mathbb{R}^d)} \tag{2.34}$$

and for $t \in [0,T]$ *we have*

$$\|u^\varepsilon(t)\|_{L^1(\mathbb{R}^d)} \le \|u_0\|_{L^1(\mathbb{R}^d)} + MT\|\nabla u_0\|_{L^1(\mathbb{R}^d)^d}. \tag{2.35}$$

To prove this theorem we need some technical results. Let $\eta : \mathbb{R}^+ \to \mathbb{R}$ be a non-increasing C^2 function satisfying

$$\eta(s) = \begin{cases} 1 & \text{for } s \in [0,1/2]\,, \\ \text{non-negative polynomial} & \text{for } s \in [1/2,1]\,, \\ \exp(-s) & \text{for } s \geq 1. \end{cases} \tag{2.36}$$

For $R > 0$ we put

$$\eta_R(x) = \eta\left(\frac{|x|}{R}\right), \qquad x \in \mathbb{R}^d. \tag{2.37}$$

Lemma 2.38 *There is a constant c such that*

$$\begin{aligned} |\nabla \eta_R(x)| &\leq \frac{c}{R}\eta_R(x)\,, \\ |\Delta \eta_R(x)| &\leq \frac{c}{R^2}\eta_R(x)\,. \end{aligned} \tag{2.39}$$

PROOF : From (2.37) it follows that

$$\begin{aligned} \frac{\partial}{\partial x_i}\eta_R(x) &= \frac{1}{R}\eta'\left(\frac{|x|}{R}\right)\frac{x_i}{|x|}\,, \\ \Delta \eta_R(x) &= \frac{1}{R^2}\eta''\left(\frac{|x|}{R}\right) + \frac{d-1}{R\,|x|}\eta'\left(\frac{|x|}{R}\right). \end{aligned} \tag{2.40}$$

Further, from (2.36) it is clear that there is some constant c_0 such that

$$|\eta'(s)| \leq c_0\eta(s)\,, \qquad |\eta''(s)| \leq c_0\eta(s).$$

Thus $(2.39)_1$ is proved. Since $\eta'(|x|/R)$ is non-zero only if $\frac{1}{|x|} < \frac{2}{R}$, we get $(2.39)_2$ as well. ■

Lemma 2.41 *Let η_R be defined by (2.36)–(2.37). Then we have for all $v \in W^{2,2}(\mathbb{R}^d)$,*

$$\int_{\mathbb{R}^d} \Delta v \;\mathrm{sgn}\,(v)\eta_R\,dx \leq \int_{\mathbb{R}^d} |v|\Delta\eta_R\,dx. \tag{2.42}$$

PROOF : For $\theta > 0$ we put

$$j_\theta(s) = \begin{cases} -1 & \text{if } s \leq -\theta\,, \\ s/\theta & \text{if } -\theta \leq s \leq \theta\,, \\ 1 & \text{if } s \geq \theta. \end{cases} \tag{2.43}$$

Thus we get

$$\int_{\mathbb{R}^d} j_\theta(v)\Delta v\, \eta_R\, dx = -\int_{\mathbb{R}^d} j_\theta'(v)|\nabla v|^2 \eta_R\, dx - \int_{\mathbb{R}^d} j_\theta(v)\nabla v \nabla \eta_R\, dx$$

and due to the fact that $j_\theta'(s) \geq 0$ almost everywhere,

$$\int_{\mathbb{R}^d} \Delta v\, j_\theta(v)\eta_R\, dx \leq -\int_{\mathbb{R}^d} j_\theta(v)\nabla v \nabla \eta_R\, dx\,.$$

The Lebesgue dominated convergence theorem enables us to pass with θ to $0+$. Thus,

$$\begin{aligned}\int_{\mathbb{R}^d} \Delta v\ \mathrm{sgn}(v)\eta_R\, dx &\leq -\int_{\mathbb{R}^d} \mathrm{sgn}\,(v)\nabla v \nabla \eta_R\, dx \\ &= -\int_{\mathbb{R}^d} \nabla |v| \nabla \eta_R\, dx \\ &= \int_{\mathbb{R}^d} |v| \Delta \eta_R\, dx,\end{aligned}$$

where we also used Remark 1.12 from the Appendix. ■

Proof (of Theorem 2.29): First we note that in our situation $W^{m,2}(\mathbb{R}^d) \hookrightarrow L^\infty(\mathbb{R}^d)$. Therefore the assumptions of Theorem 2.9 and Lemma 2.16 are satisfied. Thus we have the existence of a unique weak solution to problem (1.3) satisfying for all $T > 0$

$$\begin{aligned} u &\in L^2(0,T; W^{m+1,2}(\mathbb{R}^d)) \cap C(0,T; W^{m,2}(\mathbb{R}^d))\,, \\ \frac{\partial u}{\partial t} &\in L^2(0,T; W^{m-1,2}(\mathbb{R}^d)) \cap C(0,T; W^{m-2,2}(\mathbb{R}^d))\,, \\ \frac{\partial^2 u}{\partial t^2} &\in L^2(0,T; L^2(\mathbb{R}^d)) \end{aligned} \tag{2.44}$$

and inequality (2.31). Let us differentiate (1.3) with respect to x_i. We obtain

$$\frac{\partial}{\partial t}\Big(\frac{\partial u}{\partial x_i}\Big) + \frac{\partial^2 f_j(u)}{\partial x_i \partial x_j} - \varepsilon \Delta \frac{\partial u}{\partial x_i} = 0 \tag{2.45}$$

almost everywhere in $(0,T) \times \mathbb{R}^d$. The following calculations are motivated by the fact that we are going to multiply (2.45) by $\mathrm{sgn}\,\big(\frac{\partial u}{\partial x_i}\big)$. Due to (2.44), ∇u is an element of $W^{1,2}((0,T) \times \mathbb{R}^d)$

and we can use formula (1.13) from the Appendix, which gives†

$$\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x_i}\right)\operatorname{sgn}\left(\frac{\partial u}{\partial x_i}\right) = \frac{\partial}{\partial t}\left|\frac{\partial u}{\partial x_i}\right|,$$

$$\begin{aligned}\frac{\partial^2 f_j(u)}{\partial x_i\,\partial x_j}\operatorname{sgn}\left(\frac{\partial u}{\partial x_i}\right) &= f_j''(u)\frac{\partial u}{\partial x_j}\left|\frac{\partial u}{\partial x_i}\right| + f_j'(u)\frac{\partial}{\partial x_j}\left|\frac{\partial u}{\partial x_i}\right| \\ &= \frac{\partial}{\partial x_j}\left(f_j'(u)\left|\frac{\partial u}{\partial x_i}\right|\right).\end{aligned}$$

Therefore, multiplying (2.45) by $\operatorname{sgn}\left(\frac{\partial u}{\partial x_i}\right)$ we get

$$\frac{\partial}{\partial t}\left|\frac{\partial u}{\partial x_i}\right| + \frac{\partial}{\partial x_j}\left(f_j'(u)\left|\frac{\partial u}{\partial x_i}\right|\right) - \varepsilon\Delta\frac{\partial u}{\partial x_i}\operatorname{sgn}\left(\frac{\partial u}{\partial x_i}\right) = 0.$$

This we multiply by $\eta_R(x)$, integrate over $\mathbb{R}^d$ and obtain

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}^d}\left|\frac{\partial u}{\partial x_i}\right|\eta_R\,dx &= \int_{\mathbb{R}^d} f_j'(u)\left|\frac{\partial u}{\partial x_i}\right|\frac{\partial \eta_R}{\partial x_j}\,dx \\ &\quad + \varepsilon\int_{\mathbb{R}^d}\Delta\frac{\partial u}{\partial x_i}\operatorname{sgn}\left(\frac{\partial u}{\partial x_i}\right)\eta_R\,dx \\ &\le \int_{\mathbb{R}^d} f_j'(u)\left|\frac{\partial u}{\partial x_i}\right|\frac{\partial \eta_R}{\partial x_j}\,dx \\ &\quad + \varepsilon\int_{\mathbb{R}^d}\left|\frac{\partial u}{\partial x_i}\right|\Delta\eta_R\,dx\,,\end{aligned} \tag{2.46}$$

where we used Lemma 2.41. But Lemma 2.38 with $R > 1$ gives

$$\frac{d}{dt}\int_{\mathbb{R}^d}\left|\frac{\partial u}{\partial x_i}\right|\eta_R\,dx \le \frac{c}{R}\int_{\mathbb{R}^d}\left|\frac{\partial u}{\partial x_i}\right|\eta_R\,dx.$$

Using Gronwall's lemma we obtain

$$\int_{\mathbb{R}^d}\left|\frac{\partial u}{\partial x_i}(t,x)\right|\eta_R(x)\,dx \le \exp\left(\frac{ct}{R}\right)\int_{\mathbb{R}^d}\left|\frac{\partial u_0}{\partial x_i}(x)\right|\eta_R(x)\,dx\,.$$

The function $\eta(s)$ is non-increasing and therefore the Lebesgue monotone convergence theorem gives, as $R \to \infty$,

$$\int_{\mathbb{R}^d}\left|\frac{\partial u}{\partial x_i}(t,x)\right|dx \le \int_{\mathbb{R}^d}\left|\frac{\partial u_0}{\partial x_i}(x)\right|dx\,.$$

This proves (2.33).

† In the following two formulae we do not sum over an index i.

In order to prove (2.34) we differentiate (1.3) with respect to t and multiply by $\operatorname{sgn}\left(\frac{\partial u}{\partial t}\right)\eta_R$. Since $\frac{\partial u}{\partial t} \in W^{1,2}((0,T)\times\mathbb{R}^d)$, we can proceed in a similar way as before to obtain

$$\left\|\frac{\partial u}{\partial t}(t)\right\|_{L^1(\mathbb{R}^d)} \le \left\|\frac{\partial u}{\partial t}(0)\right\|_{L^1(\mathbb{R}^d)}. \tag{2.47}$$

On the other hand the regularity result (2.44) allows us to rewrite equation (1.3) as

$$\frac{\partial u}{\partial t}(0) = \varepsilon\Delta u_0 - f_i'(u_0)\frac{\partial u_0}{\partial x_i}.$$

Substituting this equation into (2.47) and using the assumption (2.30), one gets (2.34). Finally, we multiply (1.3) by $\operatorname{sgn}(u)\,\eta_R$ and integrate over $\mathbb{R}^d$. Thus,

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}^d}|u|\eta_R\,dx + \int_{\mathbb{R}^d} f_j'(u)\frac{\partial u}{\partial x_j}\operatorname{sgn}(u)\,\eta_R\,dx \\ = \varepsilon\int_{\mathbb{R}^d}\Delta u\,\operatorname{sgn}(u)\,\eta_R\,dx.\end{aligned}$$

In the same way as before, using (2.42), (2.39) and (2.30), one obtains

$$\frac{d}{dt}\int_{\mathbb{R}^d}|u|\eta_R\,dx \le M\int_{\mathbb{R}^d}|\nabla u|\eta_R\,dx + \frac{c\,\varepsilon}{R^2}\int_{\mathbb{R}^d}|u|\eta_R\,dx.$$

Integrating the last inequality between 0 and $t \in (0,T]$, employing the Lebesgue monotone convergence theorem as $R \to \infty$ and using (2.33), one gets (2.35).

It remains to show (2.32). We multiply (1.3) by η_R, integrate over $\mathbb{R}^d$, and after partial integration we arrive at

$$\frac{d}{dt}\int_{\mathbb{R}^d}u\eta_R\,dx = \int_{\mathbb{R}^d} f_j(u)\frac{\partial\eta_R}{\partial x_j}\,dx + \varepsilon\int_{\mathbb{R}^d}u\Delta\eta_R\,dx. \tag{2.48}$$

Now, we integrate this over $(0,t)$ and use relation (2.40). Letting $R \to \infty$ and using again the Lebesgue monotone convergence theorem, one gets (2.32). The proof of Theorem 2.29 is complete. ■

2.3 The concept of entropy

In the previous section we have shown that for any $\varepsilon > 0$ there exists a unique solution $u^\varepsilon \in W(T)\cap L^\infty(\mathbb{R}^+\times\mathbb{R}^d)$ to the parabolic perturbation (1.3) of the original hyperbolic problem (1.2). We are interested in the limiting process as $\varepsilon \to 0+$.

It is well known that problem (1.2) cannot, in general, have a classical solution, i.e. $u \in C^1(\mathbb{R}^+ \times \mathbb{R}^d)$, even for infinitely smooth data. This is a consequence of the fact that a classical solution has to be constant along the characteristics (see for example SMOLLER [1983] or ZACHMANOGLOU AND THOE [1986]). In the nonlinear case, the characteristics can intersect in finite time for suitable chosen smooth initial data u_0. At the intersection point the solution need not be continuous (for details see, e.g., SMOLLER [1983, Chapter 15]). In such a way, one is led to introduce the concept of a weak solution.

Let us denote by $C^n_C(\mathbb{R} \times \mathbb{R}^d)$ the set of n-times continuously differentiable functions with compact support (note that $\varphi(0,x)$ is in general non-zero for $\varphi \in C^n_C(\mathbb{R} \times \mathbb{R}^d)$). This choice of test functions allows us to include the initial condition u_0 into the weak formulation of the problem (see (3.2)).

Definition 3.1 *Let $u_0 \in L^\infty(\mathbb{R}^d)$. A function $u \in L^\infty_{\mathrm{loc}}(\mathbb{R}^+ \times \mathbb{R}^d)$ is called a weak solution to the problem (1.2) if, for all $\varphi \in C^1_C(\mathbb{R} \times \mathbb{R}^d)$, the following integral identity is fulfilled:*

$$\int_0^\infty \int_{\mathbb{R}^d} \left(u \frac{\partial \varphi}{\partial t} + f_j(u) \frac{\partial \varphi}{\partial x_j} \right) dx\, dt + \int_{\mathbb{R}^d} u_0(x)\, \varphi(0,x)\, dx = 0. \tag{3.2}$$

Remark 3.3

1. From Definition 3.1 it is obvious that a weak solution of (1.2) is the distributional solution on $\mathbb{R}^+ \times \mathbb{R}^d$.
2. We say that $u : \mathbb{R}^+ \times \mathbb{R}^d \to \mathbb{R}$ is a **piecewise C^1 function**, if there exists a finite number of smooth oriented d-dimensional surfaces in $\mathbb{R}^+ \times \mathbb{R}^d$ outside of which u is a C^1 function and across which u has a jump discontinuity. If Γ is such a surface and $\mathbf{n} = (n_t, n_1, \ldots, n_d)$ a normal vector to Γ, we define for $(t,x) \in \Gamma$: $u_\pm(t,x) \equiv \lim_{\delta \to 0+} u\left((t,x) \pm \delta \mathbf{n}\right)$. Then, $[u] \equiv u_+ - u_-$ and $[f_j(u)] = f_j(u_+) - f_j(u_-)$.

   If $u \in L^\infty_{\mathrm{loc}}(\mathbb{R}^+ \times \mathbb{R}^d)$ is a piecewise C^1 function, then it can be shown using (3.2) (see SMOLLER [1983]) that the following two statements are equivalent:
   - u is a weak solution to (1.2);
   - u is a classical solution of (1.2) in domains where u is a C^1

function, and

$$n_t[u] + n_j[f_j(u)] = 0 \tag{3.4}$$

holds for all points which lie on exactly one discontinuity surface.

The condition (3.4) is often referred to as the **Rankine-Hugoniot condition** and can be easily obtained if one uses (3.2) with φ compactly supported in a small ball centred at a point belonging to just one discontinuity surface. Then, employing the smoothness of u on both sides of the discontinuity surface, (3.4) follows from Green's theorem.

3. Note that the validity of (3.4) is not required at points in which the surfaces of discontinuity intersect. However, it can be supposed that the $(d+1)$-dimensional Lebesgue measure of the set of such points is zero.
4. Without loss of generality, one can assume that $(n_1, \ldots, n_d) \neq 0$ on the discontinuity surface Γ. Indeed, $(n_1, \ldots, n_d) = 0$ would imply $n_t \neq 0$ and then (3.4) would give $[u] = 0$ on Γ, which contradicts the assumptions. Therefore, $\mathbf{n}$ can be rescaled in such a way that $|\boldsymbol{\nu}| = 1$ for $\boldsymbol{\nu} = (n_1, \ldots, n_d)$. Further, denoting $s \equiv -n_t$, we have $\mathbf{n} = (-s, \boldsymbol{\nu})$, $|\boldsymbol{\nu}| = 1$ and (3.4) reads

$$s[u] = \nu_j[f_j(u)]\,. \tag{3.5}$$

Particularly, in one space dimension, $\mathbf{n} = (-s, 1)$ on the curve of discontinuity and the Rankine-Hugoniot condition takes the form of

$$s[u] = [f(u)]\,. \tag{3.6}$$

In this case, if Γ is a smooth oriented discontinuity curve parametrized by $\big(t, x(t)\big)$, we see that $\big(1, \frac{dx}{dt}\big)$ is the tangent and $\big(-\frac{dx}{dt}, 1\big)$ the normal vector to Γ. By comparison, we have for s in (3.6) that $s = \frac{dx}{dt} = 1/\frac{dt}{dx}$. Therefore, s represents the reciprocal of the slope of the discontinuity curve Γ considered in the x–t plane rather than in the t–x one.

Unfortunately, Definition 3.1 permits nonuniqueness of weak solutions even in simple cases, as shown by the following example.

Example 3.7 (inviscid Burgers equation) Let us consider the equation

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(\frac{u^2}{2}\right) = 0 \tag{3.8}$$

with initial condition

$$u_0(x) = \begin{cases} u_\ell\,, & x < 0\,, \\ u_r\,, & x > 0, \end{cases} \tag{3.9}$$

where $u_\ell \neq u_r$. The Rankine-Hugoniot condition (3.6) for $s \in \mathbb{R}$ reads

$$s = \frac{u_+ + u_-}{2} \tag{3.10}$$

whenever u jumps from the value u_- to the value u_+ over a curve in the x–t plane with slope $1/s$. Therefore, defining

$$u(t,x) \equiv \begin{cases} u_\ell\,, & x < \frac{u_\ell+u_r}{2}t\,, \\ u_r\,, & x > \frac{u_\ell+u_r}{2}t\,, \end{cases} \tag{3.11}$$

we see that u satisfies the Rankine-Hugoniot condition on the line of discontinuity $x = \frac{u_\ell+u_r}{2}t$. Outside the line, u is constant and therefore satisfies (1.2) in the classical sense. By Remark 3.3, u is a weak solution to (3.8), (3.9). On the other hand, there is a one-parameter family of weak solutions given by

$$u_a(t,x) = \begin{cases} u_\ell\,, & x < s_1 t\,, \\ -a\,, & s_1 t < x < 0\,, \\ a\,, & 0 < x < s_2 t\,, \\ u_r\,, & x > s_2 t\,, \end{cases} \tag{3.12}$$

where

$$s_1 = \frac{u_\ell - a}{2}\,, \qquad s_2 = \frac{u_r + a}{2}\,.$$

Note that $s_1 < 0$ and $s_2 > 0$ if one chooses

$$a > \max\left(u_\ell, -u_r\right)$$

and the whole construction makes sense. All the solutions u_a satisfy the Rankine-Hugoniot condition on the lines of discontinuity and by the same argument as above, all of them are weak solutions to (3.8)–(3.9) in the sense of Definition 3.1.

Further, if $u_\ell < u_r$, then the characteristics $t = \frac{x-x_0}{u_\ell}$ emanating from $(0, x_0)$, $x_0 < 0$, do not intersect the characteristics $t = \frac{x-x_0}{u_r}$ emanating from $(0, x_0)$, $x_0 > 0$. Moreover, there is a domain $G \equiv \{(t,x), u_\ell t < x < u_r t\}$—an angle in the x–t plane—which does not contain any of the characteristics. This 'gap' G can be 'filled' with a smooth function in such a way that the resulting function u_c is a weak solution to (3.8)–(3.9), which is continuous for $t > 0$. Namely, defining

$$u_c(t,x) = \begin{cases} u_\ell\,, & x \leq u_\ell t\,, \\ \frac{x}{t}\,, & u_\ell t < x < u_r t\,, \\ u_r\,, & x \geq u_r t\,, \end{cases} \tag{3.13}$$

we see that inside G, $\frac{\partial u_c}{\partial t} + u_c \frac{\partial u_c}{\partial x} = -\frac{x}{t^2} + \frac{x}{t}\frac{1}{t} = 0$ and u_c solves (3.8) in a classical sense inside G. The same holds outside G, where u_c is constant. The Rankine-Hugoniot condition on border lines of G is satisfied trivially, since u_c exhibits no jump there. Consequently, u_c is a weak solution to (3.8)–(3.9), continuous for $t > 0$.

Note that in the case $u_\ell > u_r$ the characteristics carrying the value of u_ℓ intersect with those carrying the value of u_r and no continuous solution exists.

Now, it is natural to ask if there is a criterion which would, from the mathematical viewpoint, ensure the uniqueness of a weak solution, and, from the physical viewpoint, select the 'correct' physical solution among all weak ones. One of the possibilities is to consider an additional conservation law which on the one hand would be automatically satisfied by any smooth solution u of (1.2), but on the other would play the role of a selector for weak solutions. Hence, we want to find conditions for η, $\mathbf{q}$, $\mathbf{q} = (q_1, \ldots, q_d)$, $\eta, q_j \in C^1(\mathbb{R})$, such that a smooth solution u will automatically satisfy an additional conservation law

$$\frac{\partial}{\partial t}\eta(u) + \operatorname{div}\mathbf{q}(u) = 0\,. \tag{3.14}$$

Let us multiply (1.2) by $\eta'(u)$ and use the chain rule. We obtain

$$\frac{\partial}{\partial t}\eta(u) + \eta'(u) f_j'(u) \frac{\partial u}{\partial x_j} = 0. \tag{3.15}$$

We see that if $\eta, q_j \in C^1(\mathbb{R})$ satisfy the following compatibility conditions

$$\eta'(u) f_j'(u) = q_j'(u) \qquad \forall\, u \in \mathbb{R},\ 1 \leq j \leq d, \tag{3.16}$$

then (3.14) is automatically fulfilled for any smooth solution u of (1.2).

Suppose now that u is only a piecewise C^1 solution to (1.2) in the weak sense. Then (3.14) is satisfied only in domains where u is C^1. Proceeding in the same way as in Remark 3.3 it can be shown that the weak form of equation (3.14) implies the corresponding Rankine-Hugoniot condition across discontinuities,

$$s[\eta(u)] = \nu_j[q_j(u)] \tag{3.17}$$

for all $\eta, q_j \in C^1(\mathbb{R})$ satisfying (3.16).

However, the condition (3.17) is in general not compatible with (3.5). Indeed, let us consider again the problem (3.8)–(3.9) and put $u_\ell = 1$, $u_r = 0$ for simplicity. Then (3.10) implies $s = \frac{1}{2}$ for the solution of the type (3.11). On the other hand, taking $\eta(u) = u^k$ and $q(u) = \frac{k}{k+1}u^{k+1}$ (satisfying (3.16)), one obtains $s = \frac{k}{k+1} \neq \frac{1}{2}$ for all $k \neq 1$. In this way, there is no weak solution to our problem of the type (3.11) satisfying an additional conservation law (3.14) for polynomial η in a weak sense. The same conclusion can be drawn for solutions of the type (3.12).

Therefore, it turns out that the demand that a solution $u \in L^\infty(\mathbb{R}^+ \times \mathbb{R}^d)$ should satisfy an additional conservation law (3.14) in a weak sense is too restrictive. We will see that replacing the equality in (3.14) by a proper inequality will result in a so-called entropy inequality which will play the role of a proper selector. In the rest of this section, we present a heuristic derivation of this inequality.

Let us return to the vanishing viscosity method. If we multiply the parabolic perturbation (1.3) of (1.2) by $\eta'(u^\varepsilon)$ and use the chain rule—recall that u^ε are smooth enough—we get

$$\frac{\partial}{\partial t}\eta(u^\varepsilon) + \eta'(u^\varepsilon)f_j'(u^\varepsilon)\frac{\partial u^\varepsilon}{\partial x_j} = \varepsilon\eta'(u^\varepsilon)\Delta u^\varepsilon\,. \tag{3.18}$$

Now, using the compatibility conditions (3.16), we obtain for $\eta \in C^2(\mathbb{R})$,

$$\frac{\partial}{\partial t}\eta(u^\varepsilon) + \operatorname{div}\mathbf{q}(u^\varepsilon) = \varepsilon\Delta\eta(u^\varepsilon) - \varepsilon\eta''(u^\varepsilon)|\nabla u^\varepsilon|^2.$$

At this point, to be able to control the sign on the right, we introduce the additional requirement that η has to be convex, and we obtain

$$\frac{\partial}{\partial t}\eta(u^\varepsilon) + \operatorname{div}\mathbf{q}(u^\varepsilon) \le \varepsilon\Delta\eta(u^\varepsilon) \tag{3.19}$$

for every smooth solution u^ε of (1.3). Now, (3.19) can be viewed as a parabolic perturbation of an additional conservation inequality

$$\frac{\partial}{\partial t}\eta(u) + \operatorname{div} \mathbf{q}(u) \le 0\,; \tag{3.20}$$

compare (1.2), (1.3) with (3.19), (3.20). Indeed, if we assume for a moment that some kind of convergence of u^ε to u has already been proved, for example, that

$$\begin{aligned} u^\varepsilon \to u \qquad &\text{a.e. in } \mathbb{R}^+ \times \mathbb{R}^d\,, \\ \|u^\varepsilon\|_{L^\infty(\mathbb{R}^+\times\mathbb{R}^d)} &\le c\,, \end{aligned} \tag{3.21}$$

we will be able to show that (3.19) implies (3.20) in the sense of distributions (see Theorem 4.22 below). Moreover, it will turn out that there is just one weak solution to (1.2) in the sense of Definition 3.1 satisfying the conservation inequality (3.20) in a weak sense (see Theorem 5.1).

Our considerations motivate the following definitions.

Definition 3.22 *Let $\eta \in C^1(\mathbb{R})$ be a convex function. If there exist functions $q_j \in C^1(\mathbb{R}), 1 \le j \le d$, such that for all $u \in \mathbb{R}$*

$$\eta'(u) f_j'(u) = q_j'(u)\,, \qquad 1 \le j \le d\,, \tag{3.23}$$

then $(\eta, \mathbf{q})$ is called an **entropy-entropy flux pair** *of the conservation law (1.2).*

Definition 3.24 *A weak solution u to the problem (1.2) is called* **entropy solution** *if for every entropy-entropy flux pair $\eta, \mathbf{q}$ of (1.2) the so-called* **entropy inequality**

$$\frac{\partial}{\partial t}\eta(u) + \operatorname{div} \mathbf{q}(u) \le 0 \tag{3.25}$$

is fulfilled in the following sense: for all $\varphi \in \mathcal{D}(\mathbb{R} \times \mathbb{R}^d), \varphi \ge 0$, it holds

$$\int_0^\infty \int_{\mathbb{R}^d} \big(\eta(u) - \eta(u_0)\big)\frac{\partial \varphi}{\partial t} + q_j(u)\frac{\partial \varphi}{\partial x_j}\, dx\, dt \ge 0. \tag{3.26}$$

Note that

$$\begin{aligned} -\int_0^\infty \int_{\mathbb{R}^d} \eta(u_0)\frac{\partial \varphi}{\partial t}\, dx\, dt &= -\int_{\mathbb{R}^d} \eta(u_0) \int_0^\infty \frac{\partial \varphi}{\partial t}\, dt\, dx \\ &= \int_{\mathbb{R}^d} \eta(u_0)\varphi(0)\, dx\,, \end{aligned}$$

which shows the equivalence of (3.25) and (3.26) for smooth u.

In the following sections we will show that the solution constructed with the help of the parabolic perturbation (1.3) is a unique entropy solution to the problem (1.2) in the sense of Definition 3.24.

Example 3.27 Let us show that the entropy inequality (3.25) selects just one among all weak solutions obtained in Example 3.7. We have observed that if a weak solution to (1.2) is a piecewise C^1 function, (3.25) is satisfied with the equality sign in domains where u is C^1. On a discontinuity surface Γ, one can use again the Rankine-Hugoniot approach to obtain

$$s[\eta(u)] \geq \nu_j[q_j(u)] \tag{3.28}$$

for all η, $q_j \in C^1(\mathbb{R})$ satisfying (3.16), η convex. Since we are dealing with an inequality now, it is important to recall that on Γ the normal vector $\mathbf{n} = (-s, \boldsymbol{\nu})$ points in 'the domain of u_+', i.e., $u_+(t,x) = \lim_{\delta\to 0+} u((t,x) + \delta\mathbf{n})$ on Γ.

For the particular case of an inviscid Burgers equation in one space dimension (3.8), we will show the following fact: if a piecewise C^1 solution u to (3.8) jumps from the value U_L to the value U_R over a smooth curve Γ for some fixed time t, as x increases, then (3.28) implies $U_L \geq U_R$. Indeed, considering (3.10) together with (3.28), and using the fact that $q'(u) = u\eta'(u)$ in the case of the equation (3.8), one obtains that

$$\frac{1}{2}(U_L + U_R)\big(\eta(U_R) - \eta(U_L)\big) \geq \int_{U_L}^{U_R} \lambda\,\eta'(\lambda)\,d\lambda$$

should hold for all convex $\eta \in C^1(\mathbb{R})$. Now, if $U_L < U_R$, one chooses $\eta(u) \equiv (u - c_0)^2$, where $c_0 = \frac{U_L+U_R}{2}$ so that $\eta(U_L) = \eta(U_R)$. On the other hand, the integral on the right-hand side equals $\frac{1}{6}(U_R - U_L)^3 > 0$ which is a contradiction.

This fact can be formulated as follows: 'For the particular case of Burgers equation, (3.28) implies that no jump from U_L to U_R, $U_L < U_R$, as x increases, is allowed for piecewise C^1 weak solutions.' In such a way we see that (3.28) selects a uniquely determined weak solution within the class of solutions constructed in Example 3.7. More precisely, for $u_\ell > u_r$ the proper solution is of the type (3.11), while for $u_\ell < u_r$ one gets the continuous solution u_c defined in (3.13). This is of course by no means a uniqueness proof even in the case of Burgers equation, since not all possible weak solutions were taken into account. Nevertheless, these considerations give us a hope that we are on the right track.

Remark 3.29 If we deal only with a scalar conservation law, every convex function $\eta \in C^1(\mathbb{R})$ is an entropy. Indeed, if we define the entropy flux as

$$q_j(u) = \int_0^u \eta'(\lambda) f_j'(\lambda)\, d\lambda\,,$$

relation (3.23) is trivially fulfilled.

Another important entropy-entropy flux pair in the case of one equation is

$$\eta(u) = |u - k|\,, \tag{3.30}$$

$$q_j(u) = \operatorname{sgn}(u-k)(f_j(u) - f_j(k)). \tag{3.31}$$

$k \in \mathbb{R}$. Of course, in this case η and q_j are not smooth enough, but one can overcome this difficulty by an appropriate smoothing—see Lemma 5.3.

For a general system of conservation laws, the entropy is defined by compatibility conditions analogous to (3.23) (see GODLEWSKI AND RAVIART [1991] or SMOLLER [1983]). However, in general it can be difficult to fulfill these conditions and the question arises if there is at least one entropy flux pair for a general system of conservation laws. The situation becomes better if the system is symmetrizable, which is typical for physically motivated models. In that case one can always find an entropy having a physical interpretation, as for example in the case of Euler equations (for details see SCHOCHET [1989], GODLEWSKI AND RAVIART [1991], ROKYTA [1992]).

2.4 Existence of an entropy solution

In this section we will show that an entropy solution to the Cauchy problem (1.2) exists under the only assumptions that $u_0 \in L^\infty(\mathbb{R}^d)$ and $\mathbf{f} \in C^1(\mathbb{R})$. We will use the method of vanishing viscosity, combined here with a regularization of the initial condition and the nonlinearity $\mathbf{f}$.

Let $\rho \in \mathcal{D}(\mathbb{R}^p)$ be a symmetric **mollifier**, i.e.,

$$\begin{aligned} \rho(x) \geq 0\,, \quad & \operatorname{supp}\rho \subseteq B_1(0)\,, \\ \int_{\mathbb{R}^p} \rho(x)\,dx = 1\,, \quad & \rho(-x) = \rho(x)\,. \end{aligned} \tag{4.1}$$

Put

$$\rho_\varepsilon(x) \equiv \frac{1}{\varepsilon^p}\rho\Big(\frac{x}{\varepsilon}\Big) \tag{4.2}$$

and denote for any $g \in L^1_{\text{loc}}(\mathbb{R}^p)$

$$g_\varepsilon(x) \equiv (g * \rho_\varepsilon)(x) \equiv \int_{\mathbb{R}^p} g(x-y)\rho_\varepsilon(y)\,dy\,. \tag{4.3}$$

In particular, we define

$$\begin{aligned} & u_{0\varepsilon} \equiv u_0 * \rho_\varepsilon\,, \\ & f_{j\varepsilon}(\xi) \equiv (f_j * \rho_\varepsilon)(\xi)\,, \qquad \mathbf{f}_\varepsilon \equiv (f_{1\varepsilon}, \ldots, f_{d\varepsilon})\,. \end{aligned} \tag{4.4}$$

We will study the following approximation of the problem (1.2):

$$\begin{aligned} \frac{\partial u^\varepsilon}{\partial t} + \operatorname{div}\mathbf{f}_\varepsilon(u^\varepsilon) - \varepsilon\Delta u^\varepsilon = 0 \qquad & \text{in } \mathbb{R}^+ \times \mathbb{R}^d\,, \\ u^\varepsilon(0,\cdot) = u_{0\varepsilon} \qquad & \text{in } \mathbb{R}^d\,. \end{aligned} \tag{4.5}$$

In what follows, we give estimates of u^ε in various norms by means of nonregularized initial condition u_0. Prior to this, let us recall that for $g \in L^1(\mathbb{R}^d)$, the total variation of g is defined as

$$TV(g) \equiv \sup_{\substack{\varphi \in C^1_{\mathrm{C}}(\mathbb{R}^d)^d \\ \|\varphi\|_\infty \leq 1}} \int_{\mathbb{R}^d} g \operatorname{div}\boldsymbol{\varphi}\,dx\,.$$

Here $C^1_{\mathrm{C}}(\mathbb{R}^d)$ denotes the space of continuously differentiable functions with compact support in $\mathbb{R}^d$. Denote by

$$BV(\mathbb{R}^d) \equiv \{g \in L^1(\mathbb{R}^d)\,, TV(g) < \infty\}\,,$$

the space of functions with bounded variation. See Lemma 2.50 in Chapter 1 for more details on these spaces.

Lemma 4.6 *Let $u_0 \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d) \cap BV(\mathbb{R}^d)$. Then for any $m \in \mathbb{N}$ the regularization $u_{0\varepsilon} \equiv u_0 * \rho_\varepsilon$ belongs to the space $W^{m,2}(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ and satisfies*

$$\|u_{0\varepsilon}\|_{L^1(\mathbb{R}^d)} \le \|u_0\|_{L^1(\mathbb{R}^d)}\,, \tag{4.7}$$

$$\int_{\mathbb{R}^d} u_{0\varepsilon}\,dx = \int_{\mathbb{R}^d} u_0\,dx\,, \tag{4.8}$$

$$\|u_{0\varepsilon}\|_{L^\infty(\mathbb{R}^d)} \le \|u_0\|_{L^\infty(\mathbb{R}^d)}\,, \tag{4.9}$$

$$\|\nabla u_{0\varepsilon}\|_{L^1(\mathbb{R}^d)^d} \le TV(u_0)\,, \tag{4.10}$$

$$\|\Delta u_{0\varepsilon}\|_{L^1(\mathbb{R}^d)} \le \frac{c}{\varepsilon} TV(u_0)\,, \tag{4.11}$$

where the constant c does not depend on ε.

PROOF : The first part of the lemma (up to (4.9)) is standard (see, e.g., KUFNER, JOHN AND FUČÍK [1977, Section 2.5]) and therefore we omit the proofs. We will show (4.10), (4.11). Let $\boldsymbol{\varphi} \in C_C^1(\mathbb{R}^d)^d$ be such that $\|\boldsymbol{\varphi}\|_{L^\infty(\mathbb{R}^d)^d} \le 1$. We have for $\boldsymbol{\varphi}_\varepsilon \equiv \rho_\varepsilon * \boldsymbol{\varphi}$ that $\|\boldsymbol{\varphi}_\varepsilon\|_{L^\infty(\mathbb{R}^d)^d} \le 1$ and

$$\begin{aligned}\int_{\mathbb{R}^d} u_{0\varepsilon} \operatorname{div} \boldsymbol{\varphi}\,dx &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \rho_\varepsilon(y) u_0(x-y) \operatorname{div}_x \boldsymbol{\varphi}(x)\,dy\,dx \\ &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} u_0(x-y) \operatorname{div}_x(\rho_\varepsilon(y)\boldsymbol{\varphi}(x))\,dy\,dx\,.\end{aligned}$$

Now we use the substitution $x' = x - y$, $y' = -y$, the symmetry property $\rho_\varepsilon(-y') = \rho_\varepsilon(y)$ and finally replace x' by x, y' by y. We get that the last integral is equal to

$$\begin{aligned}&\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} u_0(x) \operatorname{div}_x(\rho_\varepsilon(y)\boldsymbol{\varphi}(x-y))\,dy\,dx \\ &\qquad = \int_{\mathbb{R}^d} u_0(x) \operatorname{div}_x\left(\int_{\mathbb{R}^d} \rho_\varepsilon(y)\boldsymbol{\varphi}(x-y)\,dy\right)dx \\ &\qquad = \int_{\mathbb{R}^d} u_0(x) \operatorname{div}_x(\boldsymbol{\varphi}_\varepsilon)(x)\,dx \le TV(u_0)\,.\end{aligned}$$

Therefore, $TV(u_{0\varepsilon}) \le TV(u_0)$. However,

$$\int_{\mathbb{R}^d} |\nabla u_{0\varepsilon}|\,dx = TV(u_{0\varepsilon})$$

and inequality (4.10) follows. Similarly, for $0 \ne \psi \in L^\infty(\mathbb{R}^d)$ we get

$$\int_{\mathbb{R}^d} \Delta u_{0\varepsilon} \psi \, dx = \int_{\mathbb{R}^d} u_0 \operatorname{div}(\nabla \psi_\varepsilon) \, dx$$

and therefore

$$\begin{aligned} \|\Delta u_{0\varepsilon}\|_{L^1(\mathbb{R}^d)} &= \sup_{0 \neq \psi \in L^\infty(\mathbb{R}^d)} \frac{\left| \int_{\mathbb{R}^d} \Delta u_{0\varepsilon} \psi \, dx \right|}{\|\psi\|_{L^\infty(\mathbb{R}^d)}} \\ &= \sup_{0 \neq \psi \in L^\infty(\mathbb{R}^d)} \frac{\left| \int_{\mathbb{R}^d} u_0 \operatorname{div}(\nabla \psi_\varepsilon) \, dx \right|}{\|\psi\|_{L^\infty(\mathbb{R}^d)}} \\ &\leq TV(u_0) \sup_{0 \neq \psi \in L^\infty(\mathbb{R}^d)} \frac{\|\nabla \psi_\varepsilon\|_{L^\infty(\mathbb{R}^d)^d}}{\|\psi\|_{L^\infty(\mathbb{R}^d)}} \; . \end{aligned} \tag{4.12}$$

We know that $\nabla \psi_\varepsilon = \psi * \nabla \rho_\varepsilon$ and

$$\begin{aligned} \|\nabla \psi_\varepsilon\|_{L^\infty(\mathbb{R}^d)^d} &\leq \|\nabla \rho_\varepsilon\|_{L^1(\mathbb{R}^d)^d} \|\psi\|_{L^\infty(\mathbb{R}^d)} \\ &\leq \frac{c}{\varepsilon} \|\psi\|_{L^\infty(\mathbb{R}^d)} \; , \end{aligned}$$

which together with (4.12) implies (4.11). ∎

Lemma 4.13 *Let $u_0 \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d) \cap BV(\mathbb{R}^d)$. Then there exists a unique C^∞ solution u^ε to the problem (4.5), which satisfies for all $t \geq 0$*

$$\|u^\varepsilon(t)\|_{L^\infty(\mathbb{R}^d)} \leq \|u_0\|_{L^\infty(\mathbb{R}^d)} \; , \tag{4.14}$$

$$\int_{\mathbb{R}^d} u^\varepsilon(t, x) \, dx = \int_{\mathbb{R}^d} u_0(x) \, dx \, , \tag{4.15}$$

$$\|\nabla u^\varepsilon(t)\|_{L^1(\mathbb{R}^d)^d} \leq TV(u_0) \, , \tag{4.16}$$

$$\|\frac{\partial u^\varepsilon}{\partial t}(t)\|_{L^1(\mathbb{R}^d)} \leq c TV(u_0) \tag{4.17}$$

and for $t \in [0, T]$

$$\|u^\varepsilon(t)\|_{L^1(\mathbb{R}^d)} \leq \|u_0\|_{L^1(\mathbb{R}^d)} + cT \cdot TV(u_0) \; , \tag{4.18}$$

where c does not depend on ε.

PROOF : According to Lemma 4.6 and (4.4), all assumptions of Lemma 2.16 are satisfied. Therefore, we have the existence of a unique solution u^ε to the problem (4.5), satisfying

$$\frac{\partial^k u^\varepsilon}{\partial t^k} \in C(0,T;W^{n,2}(\mathbb{R}^d)),$$
$$u^\varepsilon \in C(0,T;W^{n,2}(\mathbb{R}^d))$$

for every n, $k \in \mathbb{N}$. However, this means that $u^\varepsilon \in C^\infty((0,T)\times\mathbb{R}^d)$. On the other hand, Lemma 4.6 and Theorem 2.29 imply

$$\begin{aligned}
&\|u^\varepsilon(t)\|_{L^\infty(\mathbb{R}^d)} \le \|u_{0\varepsilon}\|_{L^\infty(\mathbb{R}^d)} \le \|u_0\|_{L^\infty(\mathbb{R}^d)}, \\
&\int_{\mathbb{R}^d} u^\varepsilon(t,x)\,dx = \int_{\mathbb{R}^d} u_{0\varepsilon}(x)\,dx = \int_{\mathbb{R}^d} u_0(x)\,dx, \\
&\|\nabla u^\varepsilon(t)\|_{L^1(\mathbb{R}^d)^d} \le \|\nabla u_{0\varepsilon}\|_{L^1(\mathbb{R}^d)^d} \le TV(u_0), \qquad (4.19)\\
&\left\|\frac{\partial u^\varepsilon}{\partial t}(t)\right\|_{L^1(\mathbb{R}^d)} \le M_\varepsilon\|\nabla u_{0\varepsilon}\|_{L^1(\mathbb{R}^d)^d} + \varepsilon\|\Delta u_{0\varepsilon}\|_{L^1(\mathbb{R}^d)} \\
&\qquad\qquad \le M_\varepsilon TV(u_0) + cTV(u_0),
\end{aligned}$$

where

$$M_\varepsilon = \sup\left\{|\mathbf{f}_\varepsilon'(\xi)|, |\xi| \le \|u_{0\varepsilon}\|_{L^\infty(\mathbb{R}^d)}\right\}.$$

We know that

$$f_{j\varepsilon}'(\xi) = \int_{\mathbb{R}} f_j'(\xi-\theta)\rho_\varepsilon(\theta)\,d\theta,$$

which gives

$$M_\varepsilon \le \sup\left\{|\mathbf{f}'(\xi)|, |\xi| \le \|u_0\|_{L^\infty} + \varepsilon\right\} \le \widetilde{c}.$$

This together with $(4.19)_4$ implies

$$\left\|\frac{\partial u^\varepsilon}{\partial t}(t)\right\|_{L^1(\mathbb{R}^d)} \le (c+\widetilde{c})\,TV(u_0), \qquad (4.20)$$

which is (4.17).

Finally, from (2.35) and Lemma 4.6 we have

$$\begin{aligned}
\|u^\varepsilon(t)\|_{L^1(\mathbb{R}^d)} &\le \|u_{0\varepsilon}\|_{L^1(\mathbb{R}^d)} + M_\varepsilon T\|\nabla u_{0\varepsilon}\|_{L^1(\mathbb{R}^d)^d} \\
&\le \|u_0\|_{L^1(\mathbb{R}^d)} + \widetilde{c}T\cdot TV(u_0).
\end{aligned} \qquad (4.21)$$

Formulae (4.19)–(4.21) prove the lemma. ■

Now we are ready to show the existence of an entropy solution to the Cauchy problem (1.2). The following existence theorem is formulated under slightly stronger assumptions on u_0 than will be finally needed. The general situation is studied in Theorem 4.71 below, where the existence is obtained for $u_0 \in L^\infty(\mathbb{R}^d)$ only.

The key point in proving both theorems is to obtain an L^∞-estimate and almost everywhere convergence of a subsequence of $\{u^\varepsilon\}$ (see (4.26), (4.27), cf. the proof of Theorem 4.71). Both proofs then differ only in the way they establish the above-mentioned properties of $\{u^\varepsilon\}$.

Theorem 4.22 *Let $u_0 \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d) \cap BV(\mathbb{R}^d)$ and let $\mathbf{f} \in C^1(\mathbb{R})^d$. Then the problem (1.2) has an entropy solution u, belonging to the space $L^\infty((0,\infty) \times \mathbb{R}^d) \cap C(0,T; L^1(\mathbb{R}^d))$ for all $T > 0$ and satisfying, for almost every $t > 0$,*

$$\|u(t)\|_{L^\infty(\mathbb{R}^d)} \leq \|u_0\|_{L^\infty(\mathbb{R}^d)} . \tag{4.23}$$

Furthermore, $u(t) \in BV(\mathbb{R}^d)$ for all $t \geq 0$ and

$$TV(u(t)) \leq TV(u_0) , \tag{4.24}$$

$$\int_{\mathbb{R}^d} |u(t_1,x) - u(t_2,x)| \, dx \leq c\,TV(u_0)\,|t_2 - t_1| , \tag{4.25}$$

for all t, t_1, $t_2 \geq 0$.

PROOF : Let $\{u^\varepsilon\}$ be the sequence of solutions to the approximate problem (4.5). Due to Lemma 4.13 this sequence is bounded in the space $L^\infty((0,\infty) \times \mathbb{R}^d) \cap W^{1,1}_{\mathrm{loc}}((0,\infty) \times \mathbb{R}^d)$. Let $\{K_n\}$ be an increasing compact smooth covering of $[0,\infty) \times \mathbb{R}^d$. Due to the compact imbedding of $W^{1,1}(K_n)$ into $L^1(K_n)$ it is possible to select a subsequence, still denoted u^ε, converging strongly in $L^1(K_n)$ and almost everywhere. Using a diagonalization process we obtain a sequence $\{u^\varepsilon\}$ such that

$$\|u^\varepsilon\|_{L^\infty(\mathbb{R}^+ \times \mathbb{R}^d)} \leq \|u_0\|_{L^\infty(\mathbb{R}^d)} \tag{4.26}$$

and

$$\begin{aligned} u^\varepsilon &\to u \quad \text{in } L^1_{\mathrm{loc}}(\mathbb{R}^+ \times \mathbb{R}^d) , \\ u^\varepsilon &\to u \quad \text{a.e. in } \mathbb{R}^+ \times \mathbb{R}^d . \end{aligned} \tag{4.27}$$

In what follows we will show that the limit function u is an entropy solution to the problem (1.2). We have that

$$u_{0\varepsilon} \to u_0 \quad \text{in } L^1(\mathbb{R}^d) , \tag{4.28}$$

$$\mathbf{f}_\varepsilon \overset{\mathrm{loc}}{\rightrightarrows} \mathbf{f} \quad \text{in } \mathbb{R} , \tag{4.29}$$

and due to (4.26) we also get

$$\|\mathbf{f}_\varepsilon(u^\varepsilon) - \mathbf{f}(u^\varepsilon)\|_{L^\infty(\mathbb{R}^+\times\mathbb{R}^d)} \to 0\,,$$
$$\|\mathbf{f}(u^\varepsilon)\|_{L^1_{\mathrm{loc}}(\mathbb{R}^+\times\mathbb{R}^d)} \le c\,.$$

Thus the Lebesgue dominated convergence theorem gives

$$\mathbf{f}(u^\varepsilon) \to \mathbf{f}(u) \quad \text{in } L^1_{\mathrm{loc}}(\mathbb{R}^+ \times \mathbb{R}^d) \tag{4.30}$$

and altogether we have

$$\mathbf{f}_\varepsilon(u^\varepsilon) \to \mathbf{f}(u) \quad \text{in } L^1_{\mathrm{loc}}(\mathbb{R}^+ \times \mathbb{R}^d). \tag{4.31}$$

The weak formulation of (4.5) for $\varphi \in C^1_C(\mathbb{R}\times\mathbb{R}^d)$ reads

$$\begin{aligned}\int_0^\infty \int_{\mathbb{R}^d} u^\varepsilon \frac{\partial\varphi}{\partial t} + f_{j\varepsilon}(u^\varepsilon)\frac{\partial\varphi}{\partial x_j}\,dx\,dt + \int_{\mathbb{R}^d} u_{\varepsilon 0}(x)\varphi(0,x)\,dx& \\ - \varepsilon\int_0^\infty \int_{\mathbb{R}^d} \nabla u^\varepsilon \nabla\varphi\,dx\,dt = 0\,.&\end{aligned} \tag{4.32}$$

From (4.27), (4.28), (4.31) and (4.16), it immediately follows that u is a weak solution to the problem (1.2).

Now, let η be a C^2 entropy. Multiplying (4.5) by $\eta'(u^\varepsilon)$ we get

$$\frac{\partial}{\partial t}\eta(u^\varepsilon) + \frac{\partial}{\partial x_j}q_j^\varepsilon(u^\varepsilon) = \varepsilon\eta'(u^\varepsilon)\Delta u^\varepsilon\ ,$$

where $\mathbf{q}^\varepsilon$ is the entropy flux corresponding to η, $\mathbf{f}_\varepsilon$, and is defined by the compatibility conditions (3.16). The right-hand side can be rewritten as

$$\varepsilon\eta'(u^\varepsilon)\Delta u^\varepsilon = \varepsilon\Delta\eta(u^\varepsilon) - \varepsilon\eta''(u^\varepsilon)|\nabla u^\varepsilon|^2\,.$$

Then, the convexity of η gives

$$\frac{\partial}{\partial t}\eta(u^\varepsilon) + \frac{\partial}{\partial x_j}q_j^\varepsilon(u^\varepsilon) \le \varepsilon\Delta\eta(u^\varepsilon)\,. \tag{4.33}$$

Similarly as in (4.30) we obtain

$$\eta(u^\varepsilon) \to \eta(u) \quad \text{in } L^1_{\mathrm{loc}}(\mathbb{R}^+ \times \mathbb{R}^d)\,. \tag{4.34}$$

Further,

$$\begin{aligned}\Big|\int_0^{u^\varepsilon} \eta'(s) f'_{j\varepsilon}(s)\,ds - \int_0^{u} \eta'(s) f'_{j}(s)\,ds\Big| &\le \int_{-K}^{K} |\eta'(s)||f'_{j\varepsilon}(s) - f'_j(s)|\,ds \\ &\quad + \Big|\int_u^{u^\varepsilon} \eta'(s) f'_{j\varepsilon}(s)\,ds\Big|,\end{aligned}$$

where $K = \|u_0\|_{L^\infty(\mathbb{R}^d)}$, and thus

$$q_j^\varepsilon(u^\varepsilon) = \int_0^{u^\varepsilon} \eta'(s) f'_{j\varepsilon}(s)\,ds \to \int_0^{u} \eta'(s) f'_{j}(s)\,ds = q_j(u)$$

a.e. in $\mathbb{R}^+ \times \mathbb{R}^d$. This implies, again by the Lebesgue dominated convergence theorem,

$$q_j^\varepsilon(u^\varepsilon) \to q_j(u) \quad \text{in } L^1_{\mathrm{loc}}(\mathbb{R}^+ \times \mathbb{R}^d)\,. \tag{4.35}$$

Since

$$\int_0^\infty \int_{\mathbb{R}^d} \varepsilon \Delta \eta(u^\varepsilon)\,\varphi\,dx\,dt = -\varepsilon \int_0^\infty \int_{\mathbb{R}^d} \eta'(u^\varepsilon)\,\nabla u^\varepsilon\,\nabla\varphi\,dx\,dt\,, \tag{4.36}$$

and (4.16) holds, we see that

$$\int_0^\infty \int_{\mathbb{R}^d} \varepsilon \Delta \eta(u^\varepsilon)\,\varphi\,dx\,dt \to 0\,, \qquad \text{as } \varepsilon \to 0+\,, \tag{4.37}$$

for all $\varphi \in C^1_{\mathrm{C}}(\mathbb{R} \times \mathbb{R}^d)$. The limiting process in the weak formulation of (4.33) then follows from (4.34)–(4.37) in the same way as before. Thus, we have proved that u is an entropy solution.

Now we want to show that for all $T > 0$,

$$u^\varepsilon \to u \quad \text{in } C(0,T; L^1_{\mathrm{loc}}(\mathbb{R}^d)). \tag{4.38}$$

The estimates (4.16)–(4.18) give that

$$\begin{aligned}&u^\varepsilon \text{ is bounded in the norm of } C(0,T; W^{1,1}(\mathbb{R}^d))\,,\\ &\frac{\partial u^\varepsilon}{\partial t} \text{ is bounded in the norm of } C(0,T; L^1(\mathbb{R}^d)).\end{aligned}$$

Let K be a bounded smooth set in $\mathbb{R}^d$. For $0 \le t_1 \le t_2 \le T$ and all $x \in K$ we can write

$$u^\varepsilon(t_2,x) - u^\varepsilon(t_1,x) = \int_{t_1}^{t_2} \frac{\partial u^\varepsilon}{\partial t}(s,x)\,ds$$

and due to (4.17) we get

$$\begin{aligned}\|u^\varepsilon(t_2) - u^\varepsilon(t_1)\|_{L^1(K)} &\le \int_{t_1}^{t_2} \|\frac{\partial u^\varepsilon}{\partial t}(s)\|_{L^1(K)}\,ds \\ &\le c\,TV(u_0)(t_2 - t_1)\ .\end{aligned} \tag{4.39}$$

Thanks to the compact imbedding of $W^{1,1}(K)$ into $L^1(K)$, the sequence $\{u^\varepsilon\}$ lies in a compact set of $L^1(K)$ and one can apply the Arzelà-Ascoli theorem to obtain (4.38). Due to (4.18) we have for every bounded K:

$$\|u^\varepsilon(t)\|_{L^1(K)} \le c_0 \equiv \|u_0\|_{L^1(\mathbb{R}^d)} + cT \cdot TV(u_0)$$

and therefore we have $u(t) \in L^1(\mathbb{R}^d)$. Since the right-hand side of (4.39) does not depend on ε and K, we have

$$\|u(t_2) - u(t_1)\|_{L^1(\mathbb{R}^d)} \le c\,TV(u_0)(t_2 - t_1)\ ,$$

which proves on the one hand $u \in C(0,T;L^1(\mathbb{R}^d))$ and on the other hand (4.25).

It remains to show (4.24). Let $\boldsymbol{\varphi} \in C_C^1(\mathbb{R}^d)^d$, $\|\boldsymbol{\varphi}\|_{L^\infty(\mathbb{R}^d)^d} \le 1$. Using (4.16) we obtain

$$\begin{aligned}\int_{\mathbb{R}^d} u^\varepsilon(t)\,\mathrm{div}\,\boldsymbol{\varphi}\,dx &= -\int_{\mathbb{R}^d} \nabla u^\varepsilon(t)\boldsymbol{\varphi}\,dx \\ &\le TV(u_0)\|\boldsymbol{\varphi}\|_{L^\infty(\mathbb{R}^d)^d}\ .\end{aligned}$$

Now (4.38) gives (4.24) and proof of the theorem is complete. ∎

Remark 4.40 In the rest of this section we want to prove the existence of an entropy solution u under a weaker assumption on the initial data, namely that $u_0 \in L^\infty(\mathbb{R}^d)$ only. In this case, the estimate (4.16) cannot be used and therefore the limiting processes in (4.32) and in the integral (4.36) cannot be justified. However, supposing that $\varphi \in \mathcal{D}(\mathbb{R} \times \mathbb{R}^d)$ in (4.32) and (4.36), we can write

$$-\varepsilon \int_0^\infty \int_{\mathbb{R}^d} \nabla u^\varepsilon\, \nabla\varphi\,dx\,dt = \varepsilon \int_0^\infty \int_{\mathbb{R}^d} u^\varepsilon \Delta\varphi\,dx\,dt$$

and

$$\varepsilon \int_0^\infty \int_{\mathbb{R}^d} \Delta\eta(u^\varepsilon)\,\varphi\,dx\,dt = \varepsilon \int_0^\infty \int_{\mathbb{R}^d} \eta(u^\varepsilon)\Delta\varphi\,dx\,dt\,.$$

Using these formulae we see that the limiting processes as $\varepsilon \to 0+$ in both (4.32) and the weak form of (4.33) can be justified in the sense of distributions under the assumptions that $u^\varepsilon \to u$ almost everywhere and $\|u^\varepsilon\|_\infty \le c$.

Our goal in the rest of this section is firstly to prove the existence of solutions u^ε of the parabolic perturbations for $u_0 \in L^\infty(\mathbb{R}^d)$, and secondly to show that u^ε converge to u almost everywhere in $\mathbb{R}^+ \times \mathbb{R}^d$.

To do this, we need some technical results.

Let $B_r \equiv \{x \in \mathbb{R}^d; |x| < r\}$. For $v \in L^1(B_r)$ we denote‡

$$\omega_r(h) = \sup_{|z|\le h} \int_{B_r} |v(x+z) - v(x)|\, dx \ .$$

It is clear that for all $r > 0$

$$\lim_{h\to 0+} \omega_r(h) = 0 \ . \tag{4.41}$$

Lemma 4.42 *Let $v \in L^1(B_{r+2\lambda})$. Then for $0 < \varepsilon < h$ we have*

$$\omega_r^\varepsilon(\lambda) \equiv \sup_{|z|\le\lambda} \int_{B_r} |v_\varepsilon(x+z) - v_\varepsilon(x)|\, dx \le \omega_{r+h}(\lambda)\,, \tag{4.43}$$

$$\int_{B_r} \big|\,|v| - v(\operatorname{sgn} v)_\varepsilon\big|\, dx \le 2\omega_r(\varepsilon)\,. \tag{4.44}$$

PROOF : We have

$$\begin{aligned}
&\int_{B_r} |v_\varepsilon(x+z) - v_\varepsilon(x)|\, dx \\
&\qquad = \int_{B_r} \left| \int_{B_\varepsilon(0)} \big(v(x-y+z) - v(x-y)\big)\rho_\varepsilon(y)\, dy \right| dx \\
&\qquad \le \int_{B_1(0)} \rho(y) \int_{B_r} |v(x-\varepsilon y+z) - v(x-\varepsilon y)|\, dx\, dy \\
&\qquad \le \int_{B_1(0)} \rho(y)\, dy \int_{B_{r+h}} |v(x+z) - v(x)|\, dx
\end{aligned}$$

which immediately gives (4.43). In order to prove (4.44) we note that

$$\begin{aligned}
&\int_{B_r} \big|\,|v| - v(\operatorname{sgn} v)_\varepsilon\big|\, dx \\
&\qquad = \int_{B_r} \int_{B_\varepsilon(0)} \rho_\varepsilon(y) \big|\,|v(x)| - v(x)\operatorname{sgn} v(x-y)\big|\, dy\, dx
\end{aligned}$$

‡ We define $v \equiv 0$ outside B_r.

and use

$$\big||v| - v \operatorname{sgn} w\big| = \big||v| - |w| - (v-w)\operatorname{sgn} w\big|$$
$$\leq 2|v-w|.$$

■

Let $\mathbf{a} = (a_1, \ldots, a_d) \in C([0,t_0] \times \mathbb{R}^d)^d$ and let us denote

$$K \equiv \sup_{[0,t_0]\times\mathbb{R}^d} |\mathbf{a}(t,x)| + 1.$$

For some smooth function q defined in $[0,t_0] \times \mathbb{R}^d$ we can introduce the operator

$$L(q) \equiv \frac{\partial q}{\partial t} + a_i \frac{\partial q}{\partial x_i} + \varepsilon \Delta q. \tag{4.45}$$

Then we have the following version of the maximum principle:

Lemma 4.46 *Let $q \in C^2([0,t_0]\times\mathbb{R}^d)$ for which $L(q) \geq 0$. Assume the existence of q_0 and $r_0 > 0$ such that*

$$\begin{aligned} |q(t,x)| &\leq q_0 && \text{on } [0,t_0] \times \mathbb{R}^d\,, \\ q(t_0,x) &= 0 && \text{if } |x| \geq r_0\,. \end{aligned}$$

Then for $0 \leq t \leq t_0$, $|x| \geq r_0 + K(t_0 - t)$ we have the estimate

$$q(t,x) \leq q_0 \exp\left(\frac{1}{\varepsilon}(K(t_0-t) + r_0 - |x|)\right). \tag{4.47}$$

PROOF : Let us denote the right-hand side of (4.47) by q_ε. Then we have for all $(t,x) \in [0,t_0] \times \mathbb{R}^d$,

$$\begin{aligned} L(q_\varepsilon) &= q_0 \exp\left(\frac{1}{\varepsilon}(K(t_0-t) + r_0 - |x|)\right) \times \\ &\qquad \times \left[-\frac{K}{\varepsilon} - a_i \frac{x_i}{|x|}\frac{1}{\varepsilon} + \varepsilon\left(\frac{1}{\varepsilon^2} - \frac{1}{\varepsilon}\Delta|x|\right)\right] \\ &\leq q_0 \exp\left(\frac{1}{\varepsilon}(K(t_0-t) + r_0 - |x|)\right) \times \\ &\qquad \times \left[\frac{-K+1}{\varepsilon} - \frac{1}{\varepsilon} + \frac{K-1}{\varepsilon} + \frac{1}{\varepsilon} - \Delta|x|)\right] \\ &\leq 0 \end{aligned}$$

and

$$\begin{aligned} q_\varepsilon(t,x) &= q_0 && \forall\, |x| = r_0 + K(t_0 - t) \\ q_\varepsilon(t_0,x) &\geq 0 && \forall\, |x| \geq r_0\,. \end{aligned}$$

Now the statement of the lemma follows from the usual parabolic maximum principle (cf. PROTTER AND WEINBERGER [1967]). ∎

In the following theorem we keep $\varepsilon > 0$ fixed and consider the solution u^ε to the parabolic perturbation (1.3) of the problem (1.2) under the assumptions $\mathbf{f} \in C^1(\mathbb{R})^d$, $u_0 \in L^\infty(\mathbb{R}^d)$ only.

Theorem 4.48 *Let $u_0 \in L^\infty(\mathbb{R}^d)$ and let $\mathbf{f} \in C^1(\mathbb{R})^d$. Then there exists a solution $u^\varepsilon \in L^\infty(\mathbb{R}^+ \times \mathbb{R}^d)$ of (1.3), such that*

$$\|u^\varepsilon(t)\|_{L^\infty(\mathbb{R}^d)} \leq \|u_0\|_{L^\infty(\mathbb{R}^d)} \quad \text{for a.a. } t > 0 \tag{4.49}$$

and

$$\int_0^\infty \int_{\mathbb{R}^d} u^\varepsilon \frac{\partial \varphi}{\partial t} + \varepsilon u^\varepsilon \Delta \varphi + f(u^\varepsilon) \nabla \varphi \, dx \, dt + \int_{\mathbb{R}^d} u_0(x) \varphi(0, x) \, dx = 0 \tag{4.50}$$

for all $\varphi \in \mathcal{D}(\mathbb{R} \times \mathbb{R}^d)$.

PROOF : For simplicity we drop the index ε. Let $n \in \mathbb{N}$ and let χ_n be the characteristic function of the ball $B_n(0) \subset \mathbb{R}^d$. We regularize the initial data as follows:

$$u_0^n \equiv (u_0 \, \chi_n) * \rho_{1/n} \,.$$

From Theorem 2.9 and Lemma 2.16 we now obtain the existence of some $u^n \in L^\infty(\mathbb{R}^+ \times \mathbb{R}^d) \cap L^2(0, T; W^{2,2}(\mathbb{R}^d)) \cap C(0, T; W^{1,2}(\mathbb{R}^d))$ with $\frac{\partial u^n}{\partial t} \in L^2((0, T) \times \mathbb{R}^d)$ for all $T > 0$, solving the problem

$$\begin{aligned} \frac{\partial u^n}{\partial t} - \varepsilon \Delta u^n + \operatorname{div} \mathbf{f}(u^n) &= 0 \,, \\ u^n(0, \cdot) &= u_0^n \,. \end{aligned} \tag{4.51}$$

We denote $w(t,x) \equiv u^n(t,x) - u^k(t,x)$, $k, n \in \mathbb{N}$. We see that $w \in L^\infty(\mathbb{R}^+ \times \mathbb{R}^d)$. For w we have the equation

$$\frac{\partial w}{\partial t} - \varepsilon \Delta w + \operatorname{div}(\mathbf{a}\, w) = 0 \,, \tag{4.52}$$

where

$$a_i(t, x) = \int_0^1 f_i'(u^k(t,x) + \alpha(u^n(t,x) - u^k(t,x))) \, d\alpha \,.$$

We multiply (4.52) by $g \in C^1(\mathbb{R}; C_C^2(\mathbb{R}^d))$ and after integration over $\mathbb{R}^d$ and between 0 and $t_0 > 0$ we obtain (see (4.45))

$$\int_{\mathbb{R}^d} w(t_0)g(t_0)\,dx - \int_0^{t_0}\int_{\mathbb{R}^d} L(g)w\,dx\,dt = \int_{\mathbb{R}^d} w(0)g(0)\,dx\,. \tag{4.53}$$

Now, we will construct a particular test function g. Let $r > 1$ and $0 < h \ll 1$ be arbitrary. Put

$$\beta(x) \equiv \begin{cases} \operatorname{sgn} w(t_0, x) & |x| \le r - h\,, \\ 0 & |x| > r - h\,, \end{cases}$$

and $\beta_h \equiv \beta * \rho_h$. Let q_h be the solution of

$$\begin{aligned} L(q_h) &= 0 \quad \text{on } (0, t_0) \times \mathbb{R}^d\,, \\ q_h(t_0, x) &= \beta_h(x)\,. \end{aligned} \tag{4.54}$$

Let $\widetilde{\rho}$ be the mollifier in $\mathbb{R}$. Set

$$\eta_m(\lambda) \equiv 1 - \int_{-\infty}^{\lambda} \widetilde{\rho}(s - m)\,ds$$

for $m \in \mathbb{N}$, and use

$$g(t, x) \equiv q_h(t, x)\eta_m(|x|)$$

as a test function in (4.53). After partial integration, using (4.54), we obtain

$$\begin{aligned} \int_{\mathbb{R}^d} w(t_0)q_h(t_0)\eta_m\,dx = &- \int_0^{t_0}\int_{\mathbb{R}^d} \Big[\widetilde{\rho}(|x| - m)\frac{x_i}{|x|}a_i w + \varepsilon\Delta\eta_m w \\ &- 2\varepsilon\frac{\partial w}{\partial x_i}\frac{x_i}{|x|}\widetilde{\rho}(|x| - m)\Big] q_h\,dx\,dt \\ &+ \int_{\mathbb{R}^d} w(0)q_h(0)\eta_m\,dx\,. \end{aligned} \tag{4.55}$$

By the usual maximum principle applied to equation (4.54) we get that $|q_h(t, x)| \le 1$ for all $(t, x) \in [0, t_0] \times \mathbb{R}^d$. Moreover, for $|x| > r + K(t_0 - t)$ we have from Lemma 4.46 that

$$|q_h(t, x)| \le e^{\frac{1}{\varepsilon}(K(t_0 - t) + r - |x|)} \le c\,e^{-|x|}\,,$$

where c is independent of $\varepsilon < 1$. Altogether we get

$$|q_h(t,x)| \le c_0 e^{-|x|} \quad \forall\,(t,x) \in [0,t_0] \times \mathbb{R}^d\,. \tag{4.56}$$

Now, letting m tend to infinity in (4.55) and using $\tilde{\rho}(|x|-m) \to 0, \eta_m \to 1$, as $m \to \infty$, we get

$$\begin{aligned}\int_{\mathbb{R}^d} w(t_0) q_h(t_0)\,dx &= \int_{\mathbb{R}^d} w(0) q_h(0)\,dx \\ &\le c_0 \int_{\mathbb{R}^d} e^{-|x|} |w(0)|\,dx\,.\end{aligned} \tag{4.57}$$

Letting $h \to 0+$ and using (4.56) we find:

$$\int_{B_r} |w(t_0)|\,dx \le c_0 \int_{\mathbb{R}^d} e^{-|x|} |w(0)|\,dx\,. \tag{4.58}$$

From the definition of u_0^n it is clear that

$$u_0^n \to u_0 \quad \text{a.e. in } \mathbb{R}^d$$

which gives, together with (4.58), that for all $r > 1$ and $t_0 \in [0,T]$, $u^n(t_0,x)$ forms a Cauchy sequence in $L^1(\mathbb{R}^d)$. Therefore,

$$u^n(t,x) \to u(t,x) \quad \text{a.e. in } \mathbb{R}^+ \times \mathbb{R}^d. \tag{4.59}$$

Moreover, Theorem 2.9 gives

$$\|u^n(t)\|_{L^\infty(\mathbb{R}^d)} \le \|u_0^n\|_{L^\infty(\mathbb{R}^d)} \le \|u_0\|_{L^\infty(\mathbb{R}^d)}, \tag{4.60}$$

which implies (4.49). If we now multiply (4.51) by some test function $\varphi \in \mathcal{D}(\mathbb{R} \times \mathbb{R}^d)$, we get

$$\begin{aligned}\int_0^\infty \int_{\mathbb{R}^d} u^n \frac{\partial \varphi}{\partial t} + \varepsilon u^n \Delta\varphi + f(u^n)\nabla\varphi\,dx\,dt \\ = -\int_{\mathbb{R}^d} u_0^n(x)\varphi(0,x)\,dx\,.\end{aligned} \tag{4.61}$$

The limiting process in (4.61) based on (4.59) and (4.60) uses the same ideas as in the proof of Theorem 4.22 and is therefore left to the reader. ■

The following theorem gives us the desired almost everywhere convergence of the sequence u^ε under the assumption $u_0 \in L^\infty(\mathbb{R}^d)$ only.

Theorem 4.62 *Let $u_0 \in L^\infty(\mathbb{R}^d)$, $\mathbf{f} \in C^1(\mathbb{R})^d$ and let $u^\varepsilon \in L^\infty(\mathbb{R}^+ \times \mathbb{R}^d)$ be the solutions of (1.3) constructed in Theorem 4.48. Then there is a subsequence still denoted u^ε and a function $u \in L^\infty(\mathbb{R}^+ \times \mathbb{R}^d)$ such that*

$$u^\varepsilon \to u \quad \text{a.e. } in\ \mathbb{R}^+ \times \mathbb{R}^d \qquad as\ \varepsilon \to 0+ \,. \tag{4.63}$$

PROOF : Let $T > 0$ and $R > 1$ be arbitrary. In order to prove (4.63), we will use the compactness criterion for $\{u^\varepsilon\}$ on $L^1((0,T)\times B_R(0))$. By (4.49) we know that the set $\{u^\varepsilon\}$ is uniformly bounded in $L^1((0,T)\times B_R(0))$. It remains to show that for any $m \in \mathbb{N}$ there exists a $\delta > 0$ such that for all $|z| \leq \delta$, $|s| \leq \delta$,

$$\int_{B_R} |u^\varepsilon(t+s, x+z) - u^\varepsilon(t,x)|\, dx \leq \frac{1}{m} \tag{4.64}$$

independently of ε and uniformly in $t \in [0,T]$. Due to (4.58) we obtain

$$\int_{B_R} |u^k(t,x) - u^n(t,x)|\, dx \leq c_0 \int_{\mathbb{R}^d} e^{-|x|} |u_0^k(x) - u_0^n(x)|\, dx \,.$$

Letting $k \to \infty$ and using (4.59) we get

$$\int_{B_R} |u^\varepsilon(t,x) - u^n(t,x)|\, dx \leq c_0 \int_{\mathbb{R}^d} e^{-|x|} |u_0(x) - u_0^n(x)|\, dx \,. \tag{4.65}$$

It is clear that the right-hand side of (4.65) tends to zero for $n \to \infty$. Now, for

$$w(t,x) \equiv u^n(t, x+z) - u^n(t,x)$$

one obtains, in the same way as in the proof of Theorem 4.48, that

$$\int_{B_R} |u^n(t,x+z) - u^n(t,x)|\, dx \leq c_0 \int_{\mathbb{R}^d} e^{-|x|} |u_0^n(x+z) - u_0^n(x)|\, dx \,. \tag{4.66}$$

Altogether we have

$$\begin{aligned}\int_{B_R} &|u^\varepsilon(t,x+z) - u^\varepsilon(t,x)|\,dx \\ &\le c_0 \int_{\mathbb{R}^d} \Big(|u_0(x+z) - u_0^n(x+z)|\Big)\, e^{-|x|}\,dx \\ &\quad + \int_{\mathbb{R}^d} \Big(|u_0^n(x+z) - u_0^n(x)|\Big)\, e^{-|x|}\,dx \\ &\quad + \int_{\mathbb{R}^d} \Big(|u_0^n(x) - u_0(x)|\Big)\, e^{-|x|}\,dx\,.\end{aligned}$$

Note that due to (4.65) there is an n_0 such that for all $n \ge n_0$ the first and the third integral can be made less than $\frac{1}{6m}$. In order to treat the second one, we choose an R_0 such that

$$\begin{aligned}c_0 &\int_{\mathbb{R}^d \setminus B_{R_0}} e^{-|x|} |u_0^n(x+z) - u_0^n(x)|\,dx \\ &\le 2c_0 \|u_0\|_{L^\infty(\mathbb{R}^d)} \int_{\mathbb{R}^d \setminus B_{R_0}} e^{-|x|}\,dx \le \frac{1}{12m}\,.\end{aligned}$$

For this R_0, n_0 it follows from (4.43) that

$$c_0 \int_{B_{R_0}} e^{-|x|} |u_0^n(x+z) - u_0^n(x)|\,dx \le c_0\, \omega_{R_0+1/n_0}(|z|)\,.$$

Now we choose a δ_1 such that $|z| \le \delta_1$ implies that the right-hand side of the last inequality is less than $\frac{1}{12m}$. Therefore we have proved that for all $m \in \mathbb{N}$ there is a δ_1 such that for $|z| \le \delta_1$,

$$\int_{B_R} |u^\varepsilon(t,x+z) - u^\varepsilon(t,x)|\,dx \le \frac{1}{2m} \tag{4.67}$$

uniformly in $t \in [0,T]$ and independently of ε.

Let $t_0 < T$ and $s \ll 1$ be fixed. Denote

$$v(x) \equiv u^n(t_0+s,x) - u^n(t_0,x)$$

and put

$$\beta(x) \equiv \begin{cases} \operatorname{sgn} v\,, & |x| \le R\,, \\ 0\,, & |x| > R\,. \end{cases}$$

Clearly, the mollification $\beta_h \equiv \beta * \rho_h$ satisfies

$$\beta_h \le 1\,, \quad |\nabla \beta_h| \le \frac{c}{h}\,, \quad |\Delta \beta_h| \le \frac{c}{h^2}\,.$$

Multiplying (4.51) by β_h we obtain

$$\begin{aligned}\int_{\mathbb{R}^d} v\beta_h\,dx &= \int_{\mathbb{R}^d} \big(u^n(t_0+s,x) - u^n(t_0,x)\big)\beta_h(x)\,dx \\ &= \int_{t_0}^{t_0+s}\int_{\mathbb{R}^d} \varepsilon u^n \Delta\beta_h + \mathbf{f}(u^n)\nabla\beta_h\,dx\,dt \\ &\le s\,c_1\big(\|u_0\|_{L^\infty(\mathbb{R}^d)}\big)\Big(\frac{1}{h}+\frac{1}{h^2}\Big)\,.\end{aligned}$$

For $|x| < R$ we have $v\beta_h = v\,(\operatorname{sgn} v)_h$ and it follows from (4.44) that

$$\begin{aligned}\int_{B_R} |v|\,dx \le \int_{B_R} v\beta_h\,dx + 2\omega_R(h) &\le \int_{\mathbb{R}^d} v\beta_h\,dx + 2\omega_R(h) \\ &\le s\,c_1\Big(\frac{1}{h}+\frac{1}{h^2}\Big) + 2\omega_R(h)\,,\end{aligned} \tag{4.68}$$

where

$$\begin{aligned}\omega_R(h) = \sup_{|z|\le h}\int_{B_R}\Big|&\big(u^n(t_0+s,x+z) - u^n(t_0,x+z)\big) \\ &- \big(u^n(t_0+s,x) - u^n(t_0,x)\big)\Big|\,dx\,.\end{aligned}$$

According to (4.66),

$$\omega_R(h) \le 2\,c_0 \sup_{|z|\le h}\int_{\mathbb{R}^d} |u_0^n(x+z) - u_0^n(x)| e^{-|x|}\,dx\,,$$

and therefore $\omega_R(h)$ can be made arbitrarily small for small enough h uniformly in t_0 and s. Thus we have

$$\begin{aligned}\int_{B_R} &|u^\varepsilon(t_0+s,x) - u^\varepsilon(t_0,x)|\,dx \\ &\le \int_{B_R} |u^\varepsilon(t_0+s,x) - u^n(t_0+s,x)|\,dx \\ &\quad + \int_{B_R} |u^n(t_0,x) - u^\varepsilon(t_0,x)|\,dx \\ &\quad + \int_{B_R} |u^n(t_0+s,x) - u^n(t_0,x)|\,dx\,.\end{aligned} \tag{4.69}$$

The first two integrals can be made smaller than $\frac{1}{6m}$ due to (4.65) for all $n \ge n_0$. For this n_0 we find an h_0 such that $2\omega_R(h_0) \le \frac{1}{12m}$.

Further, for this h_0 we find a δ_2 such that for all $|s| < \delta_2$,

$$s\, c_0\Big(\frac{1}{h_0} + \frac{1}{h_0^2}\Big) \le \frac{1}{12m}\,.$$

Therefore we get that for all $|s| \le \delta_2$,

$$\int_{B_R} |u^\varepsilon(t_0+s,x) - u^\varepsilon(t_0,x)|\,dx \le \frac{1}{2m} \tag{4.70}$$

uniformly in ε and t_0. Now for $\delta = \min(\delta_1,\delta_2)$ we immediately get (4.64) from (4.67) and (4.70). Thus, $\{u^\varepsilon\}$ is uniformly 1-mean equicontinuous and a uniformly bounded set in $L^1((0,T)\times B_R(0))$, and (4.63) on $(0,T)\times B_R(0)$ follows at least for a subsequence. Now, since T and R were arbitrary, (4.63) follows. Finally, the limit function u belongs to $L^\infty(\mathbb{R}^+ \times \mathbb{R}^d)$ due to (4.49), (4.63). ■

Theorem 4.71 *Let $u_0 \in L^\infty(\mathbb{R}^d)$ and $\mathbf{f} \in C^1(\mathbb{R})^d$. Then the problem (1.2) has an entropy solution $u \in L^\infty(\mathbb{R}^+ \times \mathbb{R}^d)$ such that, for almost all $t > 0$,*

$$\|u(t)\|_{L^\infty(\mathbb{R}^d)} \le \|u_0\|_{L^\infty(\mathbb{R}^d)}\,. \tag{4.72}$$

PROOF : Theorem 4.48 and Theorem 4.62 imply the existence of weak solutions $u^\varepsilon \in L^\infty(\mathbb{R}^+ \times \mathbb{R}^d)$ of the problem (1.3) and a function $u \in L^\infty(\mathbb{R}^+ \times \mathbb{R}^d)$ such that

$$\|u^\varepsilon(t)\|_{L^\infty(\mathbb{R}^d)} \le \|u_0\|_{L^\infty(\mathbb{R}^d)} \qquad \text{for a.a. } t \in \mathbb{R}^+\,,$$
$$u^\varepsilon(t,x) \to u(t,x) \qquad \text{for a.a. } (t,x) \in \mathbb{R}^+ \times \mathbb{R}^d\,.$$

Using the same argument as in the proof of Theorem 4.22 together with Remark 4.40, we get that u is an entropy weak solution to (1.2). ■

2.5 Uniqueness of the entropy solution

The aim of this section is to prove the following uniqueness theorem:

Theorem 5.1 *Let $u_0 \in L^\infty(\mathbb{R}^d)$ and $\mathbf{f} \in C^1(\mathbb{R}^d)$. Then there exists exactly one entropy solution to the problem (1.2).*

We will prove this theorem in several steps. Firstly, we will show that the validity of the entropy inequality (3.26) for all smooth entropy-entropy flux pairs η, $\mathbf{q}$ is equivalent to the validity of (3.26) for the following family of non-smooth entropy-entropy flux pairs:

$$\begin{aligned} \eta(u) &= |u-k|\,, \\ q_j(u) &= \operatorname{sgn}(u-k)(f_j(u)-f_j(k))\,, \end{aligned} \qquad k \in \mathbb{R}\,, \tag{5.2}$$

see Lemma 5.3 and Remark 5.7. In Lemma 5.9 the local version of entropy inequality for these non-smooth entropy-entropy flux pairs is derived.

Secondly, having in mind the aim to prove the uniqueness, we will find an inequality (5.22) for the difference of two entropy solutions u, v, corresponding to the initial data u_0, v_0, respectively. This and Lemma 5.12 will help us to obtain in Theorem 5.38 the crucial L^1-contraction inequality

$$\int_{B_R} |u(t,x)-v(t,x)|\,dx \le \int_{B_{R+Mt}} |u_0(x)-v_0(x)|\,dx\,,$$

holding for all B_R, $R>0$ and for almost every $t>0$. Here, M denotes the constant of Lipschitz continuity of $\mathbf{f}$. The uniqueness result will then follow almost immediately using $u_0 = v_0$ in the last inequality and can be found at the very end of this section.

Let us start with the following lemma.

Lemma 5.3 *The entropy solution u of (1.2) satisfies for all $k \in \mathbb{R}$ and for all $\varphi \ge 0$, $\varphi \in C^1_C(\mathbb{R}^+ \times \mathbb{R}^d)$, the inequality*

$$\int_0^\infty \int_{\mathbb{R}^d} |u-k|\frac{\partial \varphi}{\partial t} + \operatorname{sgn}(u-k)(f_j(u)-f_j(k))\frac{\partial \varphi}{\partial x_j}\,dx\,dt \ge 0\,, \tag{5.4}$$

and for all $\varphi \ge 0$, $\varphi \in C^1_C(\mathbb{R} \times \mathbb{R}^d)$ the inequality

$$\begin{aligned} \int_0^\infty \int_{\mathbb{R}^d} &\big(|u-k| - |u_0-k|\big)\frac{\partial \varphi}{\partial t} \\ &+ \operatorname{sgn}(u-k)\big(f_j(u)-f_j(k)\big)\frac{\partial \varphi}{\partial x_j}\,dx\,dt \ge 0\,. \end{aligned} \tag{5.5}$$

PROOF : Let $G \in C^\infty(\mathbb{R})$ be such that

$$\begin{aligned} G(x) &= |x| \quad \forall\,|x| \ge 1\,, \\ G'(0) &= 0\,, \quad G'' \ge 0\,. \end{aligned}$$

For $k \in \mathbb{R}$ fixed we put

$$G_\varepsilon(x) \equiv \varepsilon\, G\Big(\frac{x-k}{\varepsilon}\Big)$$

which immediately implies that

$$G_\varepsilon(x) \to |x-k| \qquad \text{as } \varepsilon \to 0+\ .$$

The function G_ε is convex and smooth and therefore $(G_\varepsilon, \mathbf{q}_\varepsilon)$ is an entropy-entropy flux pair provided that $\mathbf{q}_\varepsilon \equiv (q_{1\varepsilon}, \ldots, q_{d\varepsilon})$ is defined by

$$q_{j\varepsilon}(u) \equiv \int_k^u G'_\varepsilon(v) f'_j(v)\, dv\,.$$

Since u is an entropy solution of (1.2) in the sense of distributions on $\mathbb{R}^+ \times \mathbb{R}^d$, we have for all $\varphi \geq 0$, $\varphi \in \mathcal{D}(\mathbb{R} \times \mathbb{R}^d)$,

$$\int_0^\infty \int_{\mathbb{R}^d} (G_\varepsilon(u) - G_\varepsilon(u_0))\frac{\partial \varphi}{\partial t} + q_{j\varepsilon}\frac{\partial \varphi}{\partial x_j}\, dx\, dt \geq 0\,. \tag{5.6}$$

It is a simple matter to check that for $\varepsilon \to 0+$ we get the pointwise convergences

$$\begin{aligned} G'_\varepsilon(v) &\to \operatorname{sgn}(v-k)\,, \\ \mathbf{q}_\varepsilon(v) &\to \mathbf{f}(k) - \mathbf{f}(v) \qquad \text{if } v \geq k\,, \\ \mathbf{q}_\varepsilon(v) &\to \mathbf{f}(v) - \mathbf{f}(k) \qquad \text{if } v \leq k\,. \end{aligned}$$

Since φ has compact support, we obtain (5.5) from the limiting process as $\varepsilon \to 0+$ in (5.6). Note that for $\varphi \in \mathcal{D}(\mathbb{R}^+ \times \mathbb{R}^d)$ the term obtaining $G_\varepsilon(u_0)$ vanishes, which immediately gives (5.4). ∎

Remark 5.7 Let us notice that a weak solution u satisfying (5.5) for all $k \in \mathbb{R}$ is an entropy solution. This follows from the fact that every convex function belongs to the convex hull of the set of all affine functions and all functions of the form

$$x \mapsto |x-k|\,, \quad k \in \mathbb{R}\ .$$

For the proof of uniqueness we will need a local version of (5.5), valid on the characteristic 'cone' of equation (1.2).

Let $M \in \mathbb{R}^+$ be fixed.§ For $R > 0$, $t_0 > 0$ we define

$$\begin{aligned} C_R &\equiv \{(t,x) \in [0,\infty) \times \mathbb{R}^d, |x| + Mt \leq R\}\,, \\ C_{R,t_0} &\equiv \{(t,x) \in C_R\,, t < t_0\}. \end{aligned} \tag{5.8}$$

§ In what follows, M will usually denote the constant of Lipschitz continuity of $\mathbf{f}$, restricted to an appropriate ball, cf. (2.30).

Lemma 5.9 *Let $T > 0$ and $k \in \mathbb{R}$. Let $u_0 \in L^\infty(\mathbb{R}^d)$, $\mathbf{f} \in C^1(\mathbb{R})^d$ and M be given by (2.30). Then for all $t_0 \in (0,T), R > 0$, the entropy solution u of (1.2) satisfies*

$$\begin{aligned}\int_{B_{R-Mt_0}} |u(t_0) - k|\varphi(t_0)\,dx \leq \iint_{C_{R,t_0}} |u-k|\frac{\partial\varphi}{\partial t}\,dx\,dt \\ + \iint_{C_{R,t_0}} \operatorname{sgn}(u-k)(f_j(u) - f_j(k))\frac{\partial\varphi}{\partial x_j}\,dx\,dt \\ + \int_{B_R} |u_0 - k|\varphi(0)\,dx\end{aligned} \tag{5.10}$$

for all $\varphi \geq 0$, $\varphi \in \mathcal{D}(C_R)$.

PROOF : Let ρ_h be the mollifier defined for $p = 1$ in (4.1). Put

$$\psi_h(\lambda) \equiv 1 - \int_{-\infty}^{\lambda} \rho_h(s)\,ds\,.$$

Hence,

$$\lim_{h\to 0+} \psi_h(\lambda) = \begin{cases} 0\,, & \lambda > 0\,, \\ 1\,, & \lambda < 0\,. \end{cases}$$

For given $\varphi \geq 0$, $\varphi \in \mathcal{D}(C_R)$, we define

$$\varphi_h(t,x) \equiv \varphi(t,x)\,\psi_h(t-t_0)\ .$$

Therefore, we have

$$\begin{aligned}\frac{\partial\varphi_h}{\partial t}(t,x) &= \frac{\partial\varphi}{\partial t}(t,x)\,\psi_h(t-t_0) - \varphi(t,x)\,\rho\Big(\frac{t-t_0}{h}\Big)\frac{1}{h}\,, \\ \frac{\partial\varphi_h}{\partial x_j}(t,x) &= \frac{\partial\varphi}{\partial x_j}(t,x)\,\psi_h(t-t_0)\,, \\ \varphi_h(0,x) &= \varphi(0,x) \quad \text{for } h \leq t_0\,.\end{aligned} \tag{5.11}$$

If we use φ_h instead of φ in (5.5), we get

$$\begin{aligned}&\iint_{C_R} |u-k|\frac{\partial\varphi}{\partial t}\,\psi_h(t-t_0)\,dx\,dt - \iint_{C_R} |u_0-k|\frac{\partial\varphi_h}{\partial t}\,dx\,dt \\ &- \iint_{C_R} |u-k|\varphi(t,x)\,\rho\Big(\frac{t-t_0}{h}\Big)\frac{1}{h}\,dx\,dt \\ &+ \iint_{C_R} \operatorname{sgn}(u-k)(f_j(u)-f_j(k))\frac{\partial\varphi}{\partial x_j}\,\psi_h(t-t_0)\,dx\,dt \geq 0\,.\end{aligned}$$

Note that

$$-\iint_{C_R} |u_0 - k| \frac{\partial \varphi_h}{\partial t}\, dx\, dt = \int_{B_R} |u_0 - k| \varphi(0)\, dx \ .$$

Letting $h \to 0+$ and using the fact that $\frac{1}{h}\rho(\frac{t-t_0}{h})$ is an approximation of the Dirac distribution, we obtain

$$\iint_{C_{R,t_0}} |u-k|\frac{\partial \varphi}{\partial t} + \operatorname{sgn}(u-k)(f_j(u) - f_j(k))\frac{\partial \varphi}{\partial x_j}\, dx\, dt$$

$$+ \int_{B_R} |u_0 - k|\varphi(0)\, dx \geq \int_{B_{R-Mt_0}} |u(t_0) - k|\varphi(t_0)\, dx \ ,$$

which is (5.10). ∎

Let us now give a further characterization in which the initial condition is fulfilled.

Lemma 5.12 *Let $u \in L^\infty(\mathbb{R}^+ \times \mathbb{R}^d)$ be an entropy solution to the problem (1.2) corresponding to initial data $u_0 \in L^\infty(\mathbb{R}^d)$, $\mathbf{f} \in C^1(\mathbb{R})^d$. Then for all $R > 0$,*

$$\lim_{t \to 0+} \int_{B_{R-Mt}} |u(t) - u_0|\, dx = 0 \ ,$$

where M is given by (2.30).

PROOF : Let $x_0 \in \mathbb{R}^d$, $\ell > 0$ be such that $|x_0| + 2\ell < R$. Further let $\omega \in C^\infty(\mathbb{R})$ be such that

$$\begin{aligned} &\omega' \leq 0\,, \\ &\omega = 1 \text{ on } (-\infty, \ell]\,, \\ &\omega = 0 \text{ on } [2\ell, \infty)\,. \end{aligned} \tag{5.13}$$

Now we define $\varphi \in C(C_R)$, $\varphi \geq 0$ by

$$\varphi(t,x) \equiv \omega(Mt + |x - x_0|) \ , \tag{5.14}$$

where M is given by (2.30). Thus, we have

$$\begin{aligned} \frac{\partial \varphi}{\partial t}(t,x) &= \omega'(Mt + |x-x_0|)M \ , \\ \frac{\partial \varphi}{\partial x_i}(t,x) &= \omega'(Mt + |x-x_0|)\frac{(x-x_0)_i}{|x-x_0|} \end{aligned}$$

and therefore it holds

$$\frac{\partial \varphi}{\partial t}(t,x) + M|\nabla \varphi(t,x)| = 0\,. \tag{5.15}$$

Due to (5.10) and (5.15) we obtain for $k = u_0(x_0)$

$$\begin{aligned}
\int_{B_{R-Mt_0}} & |u(t_0) - u_0(x_0)|\varphi(t_0)\,dx \\
&\leq \int_{B_R} |u_0 - u_0(x_0)|\varphi(0)\,dx + \iint_{C_{R,t_0}} |u - u_0(x_0)|\frac{\partial \varphi}{\partial t} \\
&\quad + \operatorname{sgn}(u - u_0(x_0))(f_j(u) - f_j(u_0(x_0)))\frac{\partial \varphi}{\partial x_j}\,dx\,dt \\
&\leq \int_{B_{2\ell}(x_0)} |u_0 - u_0(x_0)|\,dx \qquad (5.16)\\
&\quad + \iint_{C_{R,t_0}} |u - u_0(x_0)|\frac{\partial \varphi}{\partial t} + M|u - u_0(x_0)|\,|\nabla \varphi|\,dx\,dt \\
&= \int_{B_{2\ell}(x_0)} |u_0(x) - u_0(x_0)|\,dx\,.
\end{aligned}$$

But for the left-hand side of (5.16) we have

$$\begin{aligned}
\int_{B_{R-Mt_0}} & |u(t_0) - u_0(x_0)|\varphi(t_0)\,dx \\
&\geq \int_{B_{\ell-Mt_0}(x_0)} |u(t_0) - u_0(x_0)|\,dx\;,
\end{aligned} \tag{5.17}$$

where we used (5.13). Thus,

$$\begin{aligned}
\int_{B_{\ell-Mt_0}(x_0)} & |u(t_0,x) - u_0(x)|\,dx \\
&\leq \int_{B_{\ell-Mt_0}(x_0)} |u(t_0,x) - u_0(x_0)|\,dx \\
&\quad + \int_{B_{\ell-Mt_0}(x_0)} |u_0(x) - u_0(x_0)|\,dx \qquad (5.18)\\
&\leq 2\int_{B_{2\ell}(x_0)} |u_0(x) - u_0(x_0)|\,dx\,.
\end{aligned}$$

We will integrate the last inequality over $B_{R-2\ell}(0)$ with respect to x_0. Then Fubini's theorem applied to the left-hand side implies

$$\begin{aligned}
&\int_{B_{R-2\ell}(0)} \int_{B_{\ell - Mt_0}(x_0)} |u(t_0, x) - u_0(x)|\, dx\, dx_0 \\
&= \int_{B_{\ell - Mt_0}(0)} \int_{B_{R-2\ell}(x)} |u(t_0, x_0) - u_0(x_0)|\, dx_0\, dx \\
&\geq \int_{B_{\ell - Mt_0}(0)} \int_{B_{R-3\ell+Mt_0}(0)} |u(t_0, x_0) - u_0(x_0)|\, dx_0\, dx \\
&= \mathrm{meas}(B_{\ell - Mt_0}(0)) \int_{B_{R-3\ell+Mt_0}(0)} |u(t_0) - u_0|\, dx \ ,
\end{aligned} \tag{5.19}$$

where we used the fact that $|x| \leq \ell - Mt_0$, $|x_0 - x| \leq R - 2\ell$ implies $|x_0| \leq R - 3\ell + Mt_0$ for all x. Further we have

$$\begin{aligned}
&\int_{B_{R-2\ell}(0)} \int_{B_{2\ell}(x_0)} |u_0(x) - u_0(x_0)|\, dx\, dx_0 \\
&= \int_{B_{2\ell}(0)} \int_{B_{R-2\ell}(0)} |u_0(x_0) - u_0(x_0 + x)|\, dx_0\, dx \\
&\leq \mathrm{meas}(B_{2\ell}(0)) \sup_{|z| \leq 2\ell} \int_{B_{R-2\ell}(0)} |u_0(x) - u_0(x+z)|\, dx
\end{aligned} \tag{5.20}$$

and therefore we obtain

$$\frac{\widehat{c}}{2} \int_{B_{R-3\ell+Mt_0}} |u(t_0) - u_0|\, dx \leq \sup_{|z| \leq 2\ell} \int_{B_{R-2\ell}} |u_0(x) - u_0(x+z)|\, dx$$

for $\widehat{c} = \left(\frac{\ell - Mt_0}{2\ell}\right)^d$. Putting $\ell \equiv 2Mt_0$ we get

$$\begin{aligned}
&\int_{B_{R-5Mt_0}} |u(t_0) - u_0|\, dx \\
&\qquad \leq 2^{2d+1} \sup_{|z| \leq 4Mt_0} \int_{B_{R-4Mt_0}} |u_0(x) - u_0(x+z)|\, dx \ .
\end{aligned}$$

But this implies

$$\int_{B_{R-Mt_0}} |u(t_0,x) - u_0(x)|\,dx$$
$$\le 2^{2d+1} \sup_{|z|\le 4Mt_0} \int_{B_{R-4Mt_0}} |u_0(x) - u_0(x+z)|\,dx$$
$$+ \int_{B_{R-Mt_0}\setminus B_{R-5Mt_0}} |u(t_0,x) - u_0(x)|\,dx\ .$$

Note that the first integral converges to 0 for $t_0 \to 0+$ because of (4.41) and the second integral converges to 0, because the integrand is bounded in L^∞ and $\operatorname{meas}(B_{R-Mt_0}\setminus B_{R-5Mt_0}) \to 0$ for $t_0 \to 0+$.

■

The following lemma is of great importance for the proof of uniqueness of an entropy solution.

Lemma 5.21 *Let $\mathbf{f} \in C^1(\mathbb{R})^d$. Let $u, v \in L^\infty(\mathbb{R}^+ \times \mathbb{R}^d)$ be two entropy solutions of (1.2) corresponding to the initial data u_0, $v_0 \in L^\infty(\mathbb{R}^d)$, respectively. Then we have*

$$\frac{\partial}{\partial t}|u-v| + \operatorname{div}\big(\operatorname{sgn}(u-v)(\mathbf{f}(u) - \mathbf{f}(v))\big) \le 0 \qquad (5.22)$$

in the sense of distributions on $\mathbb{R}^+ \times \mathbb{R}^d$.

PROOF : From (5.4) we get for $k, \ell \in \mathbb{R}$ and all $\varphi = \varphi(s,x), \psi = \psi(t,y) \in \mathcal{D}((0,\infty)\times\mathbb{R}^d)$, $\psi, \varphi \ge 0$,

$$\int_0^\infty \int_{\mathbb{R}^d} |u-k|\frac{\partial\varphi}{\partial s} + \operatorname{sgn}(u-k)(f_j(u) - f_j(k))\frac{\partial\varphi}{\partial x_j}\,dx\,ds \ge 0 \qquad (5.23)$$

and

$$\int_0^\infty \int_{\mathbb{R}^d} |v-\ell|\frac{\partial\psi}{\partial t} + \operatorname{sgn}(v-\ell)(f_j(v) - f_j(\ell))\frac{\partial\psi}{\partial y_j}\,dy\,dt \ge 0\ . \qquad (5.24)$$

Put

$$k = v(t,y)\ , \quad \ell = u(s,x) \qquad (5.25)$$

and choose φ, ψ in the following way. Let $\phi = \phi(\theta,\lambda) \in \mathcal{D}(\mathbb{R}^+\times\mathbb{R}^d)$,

$\phi \geq 0$, and let $\rho_\varepsilon \in C_0^\infty(\mathbb{R} \times \mathbb{R}^d)$ be a symmetric mollifier. Put

$$\begin{aligned} \varphi(s,x) = \varphi_{t,y}(s,x) = \psi(t,y) &= \psi_{s,x}(t,y) \\ &\equiv \phi\Big(\frac{s+t}{2}, \frac{x+y}{2}\Big)\rho_\varepsilon\Big(\frac{s-t}{2}, \frac{x-y}{2}\Big) . \end{aligned} \tag{5.26}$$

For this special choice of φ, ψ we integrate (5.23) with respect to t, y and (5.24) with respect to s, x and add up the results. Thus we obtain

$$\begin{aligned} 0 \leq \int_0^\infty \int_0^\infty \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} & |u(s,x) - v(t,y)| \\ & \times \left(\frac{\partial \varphi_{t,y}}{\partial s}(s,x) + \frac{\partial \psi_{s,x}}{\partial t}(t,y) \right) \\ + \operatorname{sgn} & \big(u(s,x) - v(t,y)\big)\big(f_j(u(s,x)) - f_j(v(t,y))\big) \\ & \times \left(\frac{\partial \varphi_{t,y}}{\partial x_j}(s,x) + \frac{\partial \psi_{s,x}}{\partial y_j}(t,y) \right) dx\, dy\, ds\, dt . \end{aligned} \tag{5.27}$$

From (5.26) it follows that

$$\begin{aligned} & \frac{\partial}{\partial s}\varphi_{t,y}(s,x) + \frac{\partial}{\partial t}\psi_{s,x}(t,y) \\ & \qquad = \frac{\partial \phi}{\partial \theta}\left(\frac{s+t}{2}, \frac{x+y}{2}\right)\rho_\varepsilon\left(\frac{s-t}{2}, \frac{x-y}{2}\right), \\ & \frac{\partial}{\partial x_j}\varphi_{t,y}(s,x) + \frac{\partial}{\partial y_j}\psi_{s,x}(t,y) \\ & \qquad = \frac{\partial \phi}{\partial \lambda_j}\left(\frac{s+t}{2}, \frac{x+y}{2}\right)\rho_\varepsilon\left(\frac{s-t}{2}, \frac{x-y}{2}\right) . \end{aligned}$$

Let us introduce new coordinates

$$\begin{aligned} \xi &= \frac{x+y}{2} , & \eta &= \frac{x-y}{2} , \\ \sigma &= \frac{s+t}{2} , & \tau &= \frac{s-t}{2} , \end{aligned}$$

and therefore, we now integrate in (5.27) over

$$\Omega = \big\{(\sigma, \tau, \xi, \eta) \in \mathbb{R}^2 \times (\mathbb{R}^d)^2 \ , \ \sigma + \tau \geq 0 \ , \ \sigma - \tau \geq 0\big\} .$$

If we denote

$$\begin{aligned} G(\sigma,\tau,\xi,\eta) &\equiv \big|u(\sigma+\tau,\xi+\eta)-v(\sigma-\tau,\xi-\eta)\big|\frac{\partial\phi}{\partial\theta}(\sigma,\xi) \\ &\quad+\operatorname{sgn}(u(\sigma+\tau,\xi+\eta)-v(\sigma-\tau,\xi-\eta)) \\ &\quad\times\big(f_j(u(\sigma+\tau,\xi+\eta))-f_j(v(\sigma-\tau,\xi-\eta))\big)\frac{\partial}{\partial\lambda_j}\phi(\xi,\sigma)\,, \end{aligned} \tag{5.28}$$

we can write (5.27) as

$$J_\varepsilon \equiv \int_\Omega G(\sigma,\tau,\xi,\eta)\rho_\varepsilon(\tau,\eta)\,d\xi\,d\eta\,d\sigma\,d\tau \geq 0\,. \tag{5.29}$$

To prove (5.22), it is enough to show

$$J_\varepsilon \to J_0 \equiv \int_0^\infty\int_{\mathbb{R}^d} G(\sigma,0,\xi,0)\,d\xi\,d\sigma \tag{5.30}$$

for $\varepsilon \to 0+$. Indeed, if (5.30) holds, then one has for all $\psi \in \mathcal{D}((0,\infty)\times\mathbb{R}^d)$, $\psi \geq 0$,

$$J_0 = \int_0^\infty\int_{\mathbb{R}^d} |u-v|\frac{\partial\psi}{\partial\theta}+\operatorname{sgn}(u-v)(f_j(u)-f_j(v))\frac{\partial\psi}{\partial\lambda_j}\,d\xi\,d\sigma \geq 0\,,$$

which is exactly (5.22). Let us show that (5.30) is true. Denote

$$\chi(\sigma,\tau) = \begin{cases} 1 & \text{if } \sigma+\tau\geq 0,\ \sigma-\tau\geq 0\,,\\ 0 & \text{elsewhere.}\end{cases}$$

Then (5.29) can be rewritten as

$$J_\varepsilon = \int_{\mathbb{R}^{2d+2}} G(\sigma,\tau,\xi,\eta)\chi(\sigma,\tau)\rho_\varepsilon(\tau,\eta)\,d\xi\,d\eta\,d\sigma\,d\tau\,.$$

But $\rho_\varepsilon(\tau,\eta)$ is a mollifier (see (4.1)) and therefore we have

$$J_0 = \int_{\mathbb{R}^{2d+2}} G(\sigma,0,\xi,0)\chi(\sigma,0)\rho_\varepsilon(\tau,\eta)\,d\xi\,d\eta\,d\sigma\,d\tau\,.$$

Let us denote $K = \operatorname{supp}\phi$, $C_\varepsilon = \operatorname{supp}\rho_\varepsilon$. Note that

$$C_\varepsilon \subseteq \big\{(\tau,\eta)\ ,\ |\tau|\leq\varepsilon\ ,\ |\eta|\leq\varepsilon\big\}\,.$$

Thus, we have that $|J_\varepsilon - J_0|$ is less than or equal to

$$\begin{aligned} &\int_K\bigg\{\int_{C_\varepsilon} |G(\sigma,\tau,\xi,\eta)\chi(\sigma,\tau)-G(\sigma,0,\xi,0)\chi(\sigma,0)| \\ &\qquad\qquad \times\rho_\varepsilon(\tau,\eta)\,d\eta\,d\tau\bigg\}\,d\xi\,d\sigma \leq A_\varepsilon + B_\varepsilon\,, \end{aligned} \tag{5.31}$$

where A_ε and B_ε are defined by

$$\int_K \left\{ \int_{C_\varepsilon} |G(\sigma,\tau,\xi,\eta) - G(\sigma,0,\xi,0)| \rho_\varepsilon(\tau,\eta)\, d\eta\, d\tau \right\} d\xi\, d\sigma\,,$$

and

$$\int_K \left\{ |G(\sigma,0,\xi,0)| \int_{C_\varepsilon} |\chi(\sigma,\tau) - \chi(\sigma,0)| \rho_\varepsilon(\tau,\eta)\, d\eta\, d\tau \right\} d\xi\, d\sigma\,,$$

respectively.

Let us recall that, for $r_0 \equiv \max(\|u_0\|_\infty, \|v_0\|_\infty)$, $\mathbf{f}$ satisfies the Lipschitz condition

$$|\mathbf{f}(u) - \mathbf{f}(v)| \le M|u - v|\,, \qquad \forall\, |u|, |v| \le r_0\,.$$

Then we have for any $|u_i|, |v_i| \le r_0$, $i = 1, 2$, and $j = 1, \ldots, d$,

$$\begin{aligned} &|\operatorname{sgn}(u_1 - v_1)(f_j(u_1) - f_j(v_1)) - \operatorname{sgn}(u_2 - v_2)(f_j(u_2) - f_j(v_2))| \\ &\qquad \le M(|u_1 - u_2| + |v_1 - v_2|)\,. \end{aligned} \tag{5.32}$$

Indeed, defining

$$g(\zeta,\omega) \equiv \operatorname{sgn}(\zeta - \omega)(f_j(\zeta) - f_j(\omega))\,,$$

one gets for almost all $|\zeta|$, $|\omega| < r_0$

$$\begin{aligned} \frac{\partial}{\partial\zeta} g(\zeta,\omega) &= \operatorname{sgn}(\zeta - \omega)\, f_j'(\zeta)\,, \\ \frac{\partial}{\partial\omega} g(\zeta,\omega) &= -\operatorname{sgn}(\zeta - \omega)\, f_j'(\omega) \end{aligned}$$

and the statement (5.32) follows. In particular, putting $f_j \equiv \mathrm{Id}$ we obtain (for all u_i, $v_i \in \mathbb{R}$)

$$\big|\, |u_1 - v_1| - |u_2 - v_2| \,\big| \le |u_1 - u_2| + |v_1 - v_2|\,. \tag{5.33}$$

Using (5.28), (5.32) and (5.33) we conclude that

$$\begin{aligned} |G(\sigma,\tau,\xi,\eta) - G(\sigma,0,\xi,0)| \le c_1(M,\phi)(&|u(\sigma+\tau,\xi+\eta) - u(\sigma,\xi)| \\ &+ |v(\sigma-\tau,\xi-\eta) - v(\sigma,\xi)|)\,. \end{aligned} \tag{5.34}$$

Note that

$$|\rho_\varepsilon| \le c_2 \varepsilon^{-(d+1)}\,, \tag{5.35}$$

which together with (5.34) gives

$$A_\varepsilon \le c_3 \varepsilon^{-(d+1)} \int_K \left\{ \int_{C_\varepsilon} |u(\sigma+\tau, \xi+\eta) - u(\sigma,\xi)| \right.$$
$$\left. + |v(\sigma-\tau, \xi-\eta) - v(\sigma,\xi)| \, d\eta \, d\tau \right\} d\xi \, d\sigma \; .$$

But $u \in L^1_{\mathrm{loc}}((0,\infty) \times \mathbb{R}^d)$, and thus for almost every $(\sigma, \xi) \in (0,\infty) \times \mathbb{R}^d$ we have

$$\lim_{\varepsilon \to 0+} \frac{1}{\operatorname{meas}(C_\varepsilon)} \int_{C_\varepsilon} |u(\sigma+\tau, \xi+\eta) - u(\sigma,\xi)| \, d\eta \, d\tau = 0$$

and therefore also

$$\lim_{\varepsilon \to 0+} \varepsilon^{-(d+1)} \int_K \int_{C_\varepsilon} |u(\sigma+\tau, \xi+\eta) - u(\sigma,\xi)| \, d\eta \, d\tau \, d\xi \, d\sigma = 0 \; .$$

In the same way we obtain

$$\lim_{\varepsilon \to 0+} \varepsilon^{-(d+1)} \int_K \int_{C_\varepsilon} |v(\sigma-\tau, \xi-\eta) - v(\sigma,\xi)| \, d\eta \, d\tau \, d\xi \, d\sigma = 0 \; .$$

Therefore we proved

$$\lim_{\varepsilon \to 0+} A_\varepsilon = 0 \; . \tag{5.36}$$

Due to (5.35) we have

$$\int_{C_\varepsilon} |\chi(\sigma,\tau) - \chi(\sigma,0)| \, \rho_\varepsilon(\tau,\eta) \, d\eta \, d\tau \le c_4 \frac{1}{\varepsilon} \int_{-\varepsilon}^{\varepsilon} |\chi(\sigma,\tau) - \chi(\sigma,0)| \, d\tau$$

and therefore

$$B_\varepsilon \le \frac{c_5}{\varepsilon} \int_{\mathbb{R}} \int_{-\varepsilon}^{\varepsilon} |\chi(\sigma,\tau) - \chi(\sigma,0)| \, d\tau \, d\sigma \, .$$

From the properties of the function χ it follows that

$$\int_{\mathbb{R}} \int_{-\varepsilon}^{\varepsilon} |\chi(\sigma,\tau) - \chi(\sigma,0)| \, d\tau \, d\sigma \; = \int_{-\varepsilon}^{\varepsilon} \int_0^{|\tau|} d\sigma \, d\tau = \varepsilon^2$$

and thus

$$\lim_{\varepsilon \to 0+} B_\varepsilon = 0 \, . \tag{5.37}$$

The statement of the lemma follows from (5.31), (5.36), (5.37). ∎

The following theorem is due to KRUŽKOV [1970].

Theorem 5.38 (Kružkov) *Let* $\mathbf{f} \in C^1(\mathbb{R})^d$. *Let* u *and* $v \in L^\infty(\mathbb{R}^+ \times \mathbb{R}^d)$ *be two entropy solutions of the problem (1.2) for initial conditions* u_0 *and* $v_0 \in L^\infty(\mathbb{R}^d)$, *respectively. Put*

$$M \equiv \max\{|\mathbf{f}'(\xi)|, |\xi| \le \max\{\|u_0\|_\infty, \|v_0\|_\infty\}\}. \tag{5.39}$$

Then for all $R > 0$ *and almost every* $t > 0$

$$\int_{B_R} |u(t,x) - v(t,x)|\, dx \le \int_{B_{R+Mt}} |u_0(x) - v_0(x)|\, dx. \tag{5.40}$$

PROOF : We define for a given $R, T > 0$

$$D_{R,T} \equiv \{(t,x) \in [0,T] \times \mathbb{R}^d, |x| \le R + M(T-t)\} .$$

Let us start with an approximation of the characteristic function of the set $D_{R,T}$. Again we use the following approximation of the Heaviside function:

$$Y_\varepsilon(\lambda) \equiv \int_{-\infty}^{\lambda} \rho_\varepsilon(s)\, ds ,$$

where ρ_ε is a mollifier with support in $[-\varepsilon, \varepsilon]$. For arbitrary but fixed $\delta > \varepsilon > 0$, $\theta > 0$, $T > \delta$, we put

$$\varphi(t,x) \equiv \big(Y_\varepsilon(t-\delta) - Y_\varepsilon(t-T)\big)\big(1 - Y_\theta(|x| - R - M(T-t))\big) .$$

It is obvious that $\varphi \in \mathcal{D}((0,\infty) \times \mathbb{R}^d)$, $\varphi \ge 0$, and

$$\varphi(t,x) = \begin{cases} 1 & \text{for } |x| \le R + M(T-t) - \theta \\ & \quad \text{and } \varepsilon + \delta \le t \le T - \varepsilon\,, \\ 0 & \text{for } |x| \ge R + M(T-t) + \theta \\ & \quad \text{or } t \le \delta - \varepsilon \text{ or } t \ge T + \varepsilon\,. \end{cases} \tag{5.41}$$

For such a φ the weak form of the inequality (5.22) reads

$$\begin{aligned} &\int_0^\infty \int_{\mathbb{R}^d} |u - v|(\rho_\varepsilon(t-\delta) - \rho_\varepsilon(t-T)) \\ &\qquad\qquad \times (1 - Y_\theta(|x| - R - M(T-t)))\, dx\, dt \\ &- \int_0^\infty \int_{\mathbb{R}^d} (Y_\varepsilon(t-\delta) - Y_\varepsilon(t-T)) \\ &\qquad\qquad \times \left\{ M|u-v| + \operatorname{sgn}(u-v)(f_j(u) - f_j(v)) \frac{x_j}{|x|} \right\} \\ &\qquad\qquad \times \rho_\theta(|x| - R - M(T-t))\, dx\, dt \ge 0\,. \end{aligned} \tag{5.42}$$

We have

$$Y_\varepsilon(t-\delta) - Y_\varepsilon(t-T) \geq 0$$

for $t \geq \delta$ and

$$\left|\operatorname{sgn}(u-v)(f_j(u)-f_j(v))\frac{x_j}{|x|}\right| \leq M|u-v| .$$

Thus the second integral in (5.42) is non-negative and therefore we get

$$\int_0^\infty \int_{\mathbb{R}^d} |u-v|(\rho_\varepsilon(t-\delta) - \rho_\varepsilon(t-T)) \\ \times (1 - Y_\theta(|x| - R - M(T-t)))\, dx\, dt \geq 0 .$$

We now let θ tend to 0+. From the Lebesgue dominated convergence theorem it follows that (see (5.41))

$$\int_{D^\varepsilon_{R,T}} |u-v|(\rho_\varepsilon(t-\delta) - \rho_\varepsilon(t-T))\, dx\, dt \geq 0 , \qquad (5.43)$$

where

$$D^\varepsilon_{R,T} \equiv \{(t,x), |x| \leq R + M(T-t), \delta - \varepsilon < t \leq T + \varepsilon\} .$$

Denoting for fixed $t_0 \in [0,T]$

$$S_{t_0} \equiv \{(t_0,x), |x| \leq R + M(T-t_0)\} = \{(t,x) \in D_{R,T}, t = t_0\} ,$$

we can rewrite (5.43) as

$$\int_0^\infty (\rho_\varepsilon(t-\delta) - \rho_\varepsilon(t-T)) \int_{S_t} |u(t,x) - v(t,x)|\, dx\, dt \geq 0 . \qquad (5.44)$$

Introducing the function $w : \mathbb{R}^+ \to \mathbb{R}$ by

$$w(t) \equiv \int_{S_t} |u(t,x) - v(t,x)|\, dx ,$$

we see that (5.44) is nothing else than

$$w_\varepsilon(\delta) - w_\varepsilon(T) \geq 0 .$$

Now, w belongs to $L^\infty(\mathbb{R}^+)$ and therefore $w_\varepsilon \to w$ in all Lebesgue points, i.e. almost everywhere in $\mathbb{R}^+$. Thus we get in Lebesgue points δ, T:¶

$$\int_{S_\delta} |u(\delta,x) - v(\delta,x)|\,dx \geq \int_{B_R} |u(T,x) - v(T,x)|\,dx\,.$$

The triangle inequality and Lemma 5.12 imply

$$\int_{B_R} |u(T,x) - v(T,x)|\,dx \leq \int_{B_{R+MT}} |u_0(x) - v_0(x)|\,dx$$

for all Lebesgue points. ■

Now we are ready to prove Theorem 5.1.

PROOF (of Theorem 5.1): By applying (5.40) to $u_0(x) = v_0(x)$ one gets $u(t,x) = v(t,x)$ for all Lebesgue points $t \in \mathbb{R}^+$, which gives $u = v$ almost everywhere. ■

Remark 5.45 Uniqueness of the entropy solution to scalar conservation law (1.2) can be shown under weaker assumptions on $\mathbf{f}$. Namely, suppose that $u_0 \in L^\infty(\mathbb{R}^d)$ and that $\mathbf{f}$ satisfies the following **generalized Lipschitz continuity** condition:

$$|f_i(u) - f_i(v)| \leq \omega_i\big(|u - v|\big) \qquad i = 1, \ldots, d\,, \tag{5.46}$$

for all u, $v \in [-K, K]$, where $K \equiv \|u_0\|_\infty$ and $\omega_i \in C\left([0, 2K]\right)$ are convex functions, $\omega_i(0) = 0$, such that

$$\prod_{i=1}^{d} \omega_i(u) \leq c_\omega u^{d-1}\,.$$

Then it is shown in KRUŽKOV AND PANOV [1991] that there exists a unique entropy solution to scalar conservation law (1.2). Easily, any $\mathbf{f} \in C^{0,1}(\mathbb{R})^d$ satisfies (5.46).

Moreover, the condition (5.46) is sharp in the following sense: there exist f_1, $f_2 \in C(\mathbb{R})$ which do not satisfy (5.46) and a function $u_0 \in L^\infty(\mathbb{R}^2)$ such that there exists a one-parameter family of weak entropy solutions to corresponding scalar conservation law (1.2) in two space dimensions. Such an example is explicitly constructed in KRUŽKOV AND PANOV [1991, Example 2].

¶ Note that $S_T = B_R \times \{t = T\}$.

2.6 Conservation laws in bounded domains

In the following sections we would like to show similar results concerning the existence and uniqueness of solutions to the hyperbolic conservation law as in the sections before, but now for bounded smooth domains. Namely, we show that even when boundary and initial data u^D and u^0, respectively, are only in the space L^∞, it is possible to define a well-posed initial-boundary value problem for the scalar hyperbolic conservation law which admits a unique (entropy) solution $u \in L^\infty$. More precisely, uniqueness is established for $\mathbf{f} \in C^1$ (see Theorem 7.28), whereas existence is obtained for $\mathbf{f} \in C^2$ (see Theorem 8.20).

Moreover, here we will also use the **vanishing viscosity** method to obtain the desired result. Our aim is to search for a solution $u \in L^\infty$ even in the case of bounded domains, i.e. in the same class as for the Cauchy problem in the whole space. The main problem in this approach is that in general an L^∞-function does not admit a trace at the boundary. This difficulty was overcome in the elegant paper by OTTO [1992], who introduced the so-called **boundary entropy-entropy flux pairs**, which turned out to be a proper tool to establish the well-posedness of the problem. In a sense, the remainder of this chapter is nothing other than a treatment of a special case from the unpublished PhD thesis of OTTO [1993]. In particular, a part of the exposition and all proofs we give here follow the lines of Otto's thesis.

Let us note that all previous results concerning bounded domains need, to the knowledge of the authors, the solution u of the problem to admit a trace in some sense. For more details we refer to LE ROUX [1977] (he obtained a unique solution $u \in BV \cap L^\infty$, under the assumptions $d = 1$, $u^0, u^D \in BV$, $\mathbf{f} \in C^1$), BARDOS, LE ROUX AND NEDELEC [1979] (they obtained a unique solution $u \in BV$, under the assumptions $u^0, u^D, \mathbf{f} \in C^2$ for $d \geq 1$), LE FLOCH [1988] (he obtained a solution u which is piecewise continuous, under the assumptions $u^0, u^D \in L^\infty$, $\mathbf{f} \in C^1$, $\mathbf{f}$ strictly convex, for u^D depending on $\mathbf{f}$ and $d = 1$. This solution was proved to be unique in the class of piecewise C^1 functions).

In the rest of this chapter, $\Omega \subseteq \mathbb{R}^d$ will always be a bounded smooth domain. Let us denote by $\partial\Omega$ the boundary of Ω and by $\boldsymbol{\nu}(x)$ the outer normal vector to $\partial\Omega$ at a point $x \in \partial\Omega$. As before, we denote $Q_T \equiv (0,T) \times \Omega$ and $\Gamma \equiv (0,T) \times \partial\Omega$.

To denote the closed interval between points a and b, the following notation will be used throughout the rest of this chapter:

$$\mathcal{I}[a,b] \equiv [\min(a,b), \max(a,b)] .$$

In what follows we will look for solutions $u \in L^\infty(Q_T)$ of the initial-boundary value problem for the scalar conservation law

$$\begin{aligned} \frac{\partial u}{\partial t} + \operatorname{div} \mathbf{f}(u) &= 0 \qquad \text{in } Q_T, \\ u(0,\cdot) &= u^0 \qquad \text{in } \Omega , \\ u &= u^D \qquad \text{on } \Gamma . \end{aligned} \tag{6.1}$$

Let $\mathbf{f} \in C^1(\mathbb{R})^d$. We say that $(\eta, \mathbf{q})$, $\mathbf{q} = (q_1, \ldots, q_d)$, η, $q_j \in C^2(\mathbb{R})$, is a convex **entropy-entropy flux pair**, if

$$\eta''(z) \geq 0 \quad \text{and} \quad q_j'(z) = \eta'(z)\, f_j'(z)$$

for all $z \in \mathbb{R}$, $j = 1, \ldots, d$. By an entropy-entropy flux pair we will always mean the convex one, at least in the rest of this chapter.

We will consider weak entropy solutions $u \in L^\infty(Q_T)$ of the scalar conservation law, i.e., bounded measurable functions u satisfying both the integral identity

$$\int_0^T \int_\Omega u \frac{\partial \psi}{\partial t} + f_j(u) \frac{\partial \psi}{\partial x_j}\, dx\, dt = 0 \tag{6.2}$$

for all $\psi \in \mathcal{D}(Q_T)$, and the entropy inequality

$$\int_0^T \int_\Omega \eta(u) \frac{\partial \varphi}{\partial t} + q_j(u) \frac{\partial \varphi}{\partial x_j}\, dx\, dt \geq 0 \tag{6.3}$$

for all $\varphi \in \mathcal{D}(Q_T)$, $\varphi \geq 0$, and all entropy-entropy flux pairs $(\eta, \mathbf{q})$.

Before we formulate the sense in which boundary data on Γ for solutions of (6.2) and (6.3) will be assumed, let us recall some specific difficulties occurring in the boundary-value problem for hyperbolic conservation laws.

Consider for a moment the parabolic perturbation of (6.1) given by

$$\begin{aligned} \frac{\partial u^\varepsilon}{\partial t} + \operatorname{div} \mathbf{f}(u^\varepsilon) - \varepsilon \Delta u^\varepsilon &= 0 \qquad \text{in } Q_T, \\ u^\varepsilon(0,\cdot) &= u^0 \qquad \text{in } \Omega , \\ u^\varepsilon &= u^D \qquad \text{on } \Gamma , \end{aligned} \tag{6.4}$$

$\varepsilon > 0$. It is well known that for sufficiently smooth $\mathbf{f}$ there exists a unique smooth solution u^ε to given sufficiently smooth initial and boundary value data u^0 and u^D, respectively, satisfying suitable compatibility conditions at $\partial\Omega$—see Section 2.8 or the Appendix for more details.

The situation drastically changes for $\varepsilon = 0$. Let us consider for a moment that there exists some smooth solution u of an equation

$$\frac{\partial u}{\partial t} + \operatorname{div} \mathbf{f}(u) = 0 \qquad \text{in } Q_T \, . \tag{6.5}$$

This solution automatically satisfies both (6.2) and (6.3). Let u have the value $u(p)$ at the point $p = (t, x) \in Q_T$. It is well known (from the method of characteristics) that u is constant along the maximal segment of the characteristic line in Q_T containing point p, i.e., along the segment

$$\sigma \mapsto (t + \sigma, x + \sigma \, \mathbf{f}'(u(p))) \, . \tag{6.6}$$

Now, if this segment intersects both $\{0\} \times \Omega$ and Γ, the solution u would in general be overdetermined if both (independently prescribed) initial and boundary conditions were expected to be assumed in a classical sense. In connection with this observation, we note that since the slope of the segment (6.6) depends on $\mathbf{f}$, it would be appropriate to require a formulation of boundary conditions involving in some sense a function $\mathbf{f}$.

Let us therefore modify the notion of a boundary value problem for (6.1) starting again with smooth solutions to (6.5). Let $(t_0, x_0) \in \Gamma$ be a boundary point for which all the characteristics

$$\sigma \mapsto (t_0 + \sigma, x_0 + \sigma \, \mathbf{f}'(k))$$

emanating from (t_0, x_0) enter into Q_T, as σ increases, independently of the value k they transport, i.e., where

$$\mathbf{f}'(k) \cdot \boldsymbol{\nu}(x_0) < 0 \qquad \forall \, k \in \mathbb{R} \, . \tag{6.7}$$

In such a point the prescribed boundary data should be assumed, i.e.,

$$u(t_0, x_0) = u^D(t_0, x_0) \, .$$

If, on the other hand, the opposite takes place, i.e.,

$$\mathbf{f}'(k) \cdot \boldsymbol{\nu}(x_0) \geq 0 \qquad \forall \, k \in \mathbb{R} \, , \tag{6.8}$$

the prescribed boundary data should be ignored. Of course, for nonlinear $\mathbf{f}$ there can be boundary points $(t_0, x_0) \in \Gamma$ for which neither (6.7) nor (6.8) happens. At these points we would like the following condition to hold: if the prescribed value $u^D(t_0, x_0)$ is not assumed, then the characteristic line, carrying the actual boundary value $u(t_0, x_0)$, should emerge from the interior of Q_T, i.e.,

$$u(t_0, x_0) \neq u^D(t_0, x_0) \quad \text{should imply} \quad \mathbf{f}'(u(t_0, x_0)) \cdot \boldsymbol{\nu}(x_0) \geq 0 .$$

Unfortunately, the above listed set of 'boundary conditions', derived from the behaviour of a smooth solution to (6.5), is not suitable for the generally non-smooth solutions. (In fact, with these conditions, the uniqueness of, in general, non-smooth solutions, would not be guaranteed—cf. OTTO [1993].)

Let us therefore consider the one-dimensional **Riemann problem** as a model problem for non-smooth solutions. That is, let $u \in L^\infty((0,\infty) \times \mathbb{R})$ satisfy both

$$\int_0^\infty \int_{\mathbb{R}} u \frac{\partial \varphi}{\partial t} + f(u) \frac{\partial \varphi}{\partial x}\, dx\, dt + \int_{\mathbb{R}} u_0 \varphi(0) = 0 \tag{6.9}$$

for all $\varphi \in \mathcal{D}\big((-\infty,\infty) \times \mathbb{R}\big)$, and

$$\int_0^\infty \int_{\mathbb{R}} \eta(u) \frac{\partial \varphi}{\partial t} + q(u) \frac{\partial \varphi}{\partial x}\, dx\, dt \geq 0 \tag{6.10}$$

for all $\varphi \in \mathcal{D}((0,\infty) \times \mathbb{R})$, $\varphi \geq 0$, and all entropy-entropy flux pairs η, $q \in C^2(\mathbb{R})$. Here, the **Riemann initial data** u_0 are defined as

$$u_0(x) = \begin{cases} u_-^0 & \text{for } x < 0 , \\ u_+^0 & \text{for } x > 0 , \end{cases} \tag{6.11}$$

u_+^0, u_-^0 being given constants. Therefore, $u_0 \in L^\infty(\mathbb{R})$, and there is a unique $u \in L^\infty\big((0,+\infty) \times \mathbb{R}\big)$ satisfying (6.9)–(6.11), cf. Theorem 5.1.

Now, consider a function $\psi(t,x) \equiv \varphi(at, ax)$ for fixed $a > 0$. We see that ψ is a test function for (6.9) (or (6.10)) if and only if φ is a test function, too. Thanks to this and the particular structure of initial data (6.11) it follows that the weak formulation (6.9)–(6.10) is invariant under transformations of the form

$$(t,x) \mapsto (at, ax) , \qquad a > 0 .$$

Therefore, if u is a weak entropy solution to (6.9)–(6.11), so is $v(t,x) \equiv u(at, ax)$, $a > 0$. Then, the uniqueness result gives us for $a > 0$,

$$u(at, ax) = u(t,x) \quad \text{for a.e. } (t,x) \in (0,\infty) \times \mathbb{R} .$$

In other words, u depends on the ratio x/t only. Thus, there exists a $\mu \in L^\infty(\mathbb{R})$ such that

$$u(t,x) = \mu\left(\frac{x}{t}\right). \tag{6.12}$$

Relation (6.12) allows us to construct explicitly a uniquely determined weak entropy solution to the Riemann problem (6.9)–(6.11). We will outline this only for given $u_-^0 < u_+^0$ and $f \in C^2(\mathbb{R})$ with finitely many inflection points. For more general discussion we refer to GODLEWSKI AND RAVIART [1991].

Let us consider $\widetilde{f} : [u_-^0, u_+^0] \to \mathbb{R}$, the lower convex envelope of f on $[u_-^0, u_+^0]$. Then the interval $[u_-^0, u_+^0]$ splits into finitely many subintervals on which either

1. $\widetilde{f} = f$ and f is convex; or
2. $\widetilde{f}$ is affine and $\widetilde{f} < f$ in the interior of such an interval.

We can number the subintervals in such a way that $[u_{2i}, u_{2i+1}]$ always corresponds to an interval of the first type and $[u_{2i+1}, u_{2i+2}]$ always corresponds to an interval of the second type.

Consider the interval $[u_{2i}, u_{2i+1}]$. In the interior of this interval we have $f'' > 0$. Hence, $(f')^{-1}$ is well-defined and we can put

$$\mu(\alpha) \equiv (f')^{-1}(\alpha) \qquad \text{for } \alpha \in \big(f'(u_{2i}), f'(u_{2i+1})\big). \tag{6.13}$$

In such a way, μ is defined on the whole interval $\big(f'(u_-^0), f'(u_+^0)\big)$ except for the points $\alpha_i = f'(u_{2i+1}) = f'(u_{2i+2})$ (to which the remaining intervals $[f'(u_{2i+1}), f'(u_{2i+2})]$ reduce). At those points the function μ has a jump discontinuity such that

$$\begin{aligned} \mu(\alpha_i-) &= u_{2i+1}, \\ \mu(\alpha_i+) &= u_{2i+2}. \end{aligned} \tag{6.14}$$

Since $\widetilde{f}$ is affine on the intervals $[u_{2i+1}, u_{2i+2}]$ of the second type,

$$\alpha_i = f'(u_{2i+1}) = \frac{f(u_{2i+2}) - f(u_{2i+1})}{u_{2i+2} - u_{2i+1}},$$

which means that for the jump (6.14) the Rankine-Hugoniot condition is fulfilled as the solution $u(t,x)$ jumps across $\{x/t = \alpha_i\}$.

Finally, we define on the remainder of the real axis

$$\begin{aligned} \mu(\alpha) &= u_-^0 \qquad \forall\, \alpha \in (-\infty, f'(u_-^0)], \\ \mu(\alpha) &= u_+^0 \qquad \forall\, \alpha \in [f'(u_+^0), \infty). \end{aligned} \tag{6.15}$$

These are the only intervals on which μ is constant.

Now, (6.13) and (6.15) imply that on $\big(f'(u_{2i}), f'(u_{2i+1})\big)$ we have the equality $f'(\mu(\alpha)) - \alpha = 0$, while on $(-\infty, f'(u_-^0))$ and $(f'(u_+^0), \infty)$ we have $\mu'(\alpha) = 0$. Combining both, we obtain that on $\mathbb{R} \setminus \{\alpha_i\}$ it holds

$$f'\big(\mu(\alpha)\big)\mu'(\alpha) - \alpha\mu'(\alpha) = 0\,. \tag{6.16}$$

From (6.12) we deduce that (taking into account $\alpha = x/t$)

$$-\alpha\mu'(\alpha) = t\frac{\partial u}{\partial t}(t,x)\,,$$
$$\mu'(\alpha) = t\frac{\partial u}{\partial x}(t,x)\,,$$

which converts (6.16) into

$$\frac{\partial u}{\partial t} + f'(u)\frac{\partial u}{\partial x} = 0\,.$$

This means that u, defined by (6.12), (6.13), (6.15), is a classical solution on $(\mathbb{R}^+ \times \mathbb{R}) \setminus \{t = \alpha_i x\}$. Since the Rankine-Hugoniot condition is satisfied on all jumps, we get that u is a weak solution to our Riemann problem.

We show that u is also an entropy solution. Since u is a piecewise C^1 function, the entropy inequality is trivially satisfied in domains where $u \in C^1$. If u jumps from the value u_L to the value u_R, $u_L < u_R$, as x increases, the Rankine-Hugoniot condition

$$s(u_R - u_L) = f(u_R) - f(u_L) \tag{6.17}$$

is fulfilled. Moreover, our construction ensures that the graph of f lies above the chord connecting the values $f(u_L)$ and $f(u_R)$, i.e.,

$$f\big(\lambda u_L + (1-\lambda)u_R\big) \geq \lambda f(u_L) + (1-\lambda)f(u_R)\,, \tag{6.18}$$

for $\lambda \in [0,1]$. Setting $k \equiv \lambda u_L + (1-\lambda)u_R$ we obtain

$$\begin{aligned} f(u_R) + f(u_L) - 2f(k) &\overset{(6.18)}{\leq} (2\lambda - 1)\big(f(u_R) - f(u_L)\big) \\ &\overset{(6.17)}{=} s\,(2\lambda - 1)(u_R - u_L) \\ &= s(u_L + u_R - 2k) \\ &= s\,\big(|u_R - k| - |u_L - k|\big)\,. \end{aligned}$$

Since $u_L \leq k \leq u_R$, the left-hand side can be written as

$$\operatorname{sgn}(u_R - k)\big(f(u_R) - f(k)\big) - \operatorname{sgn}(u_L - k)\big(f(u_L) - f(k)\big)\,.$$

Altogether we have shown that

$$s\big(\widetilde{\eta}(u_R) - \widetilde{\eta}(u_L)\big) \geq \widetilde{q}(u_R) - \widetilde{q}(u_L) \tag{6.19}$$

for a non-smooth entropy-entropy flux pair (cf. (5.2))

$$\begin{aligned} \widetilde{\eta}(u) &= |u-k|\,, \\ \widetilde{q}(u) &= \operatorname{sgn}(u-k)\big(f(u)-f(k)\big)\,, \end{aligned} \tag{6.20}$$

with $u_L \leq k \leq u_R$. Further, for $k \leq u_L$ or $u_R \leq k$, the equality in (6.19) follows trivially from (6.17). Therefore, the entropy inequality is satisfied for all $\widetilde{\eta}(u) = |u-k|$, $k \in \mathbb{R}$, and corresponding $\tilde{q}$s. As we have already observed (see e.g. Remark 5.7), this implies that u is an entropy solution. We conclude that we have constructed the only weak entropy solution of the problem (6.9)–(6.11).

From the construction one easily deduces that μ is a monotone function, so that $\lim_{\alpha\to 0-}\mu(\alpha)$ exists. Moreover, this limit is equal to the lowest point at which a global minimum of f on the interval $[u_-^0, u_+^0]$ is achieved. In other words, in the case of $u_-^0 \leq u_+^0$

$$\begin{aligned} \mu(0-) &\equiv \lim_{\alpha\to 0-}\mu(\alpha) \\ &= \min\{z \in \mathcal{I}\big[u_-^0, u_+^0\big] : f(z) \leq f(k)\ \forall\, k \in \mathcal{I}\big[u_-^0, u_+^0\big]\}\,. \end{aligned}$$

Similarly, in the case of $u_-^0 \geq u_+^0$ it can be shown that

$$\mu(0-) = \max\{z \in \mathcal{I}\big[u_+^0, u_-^0\big] : f(k) \leq f(z)\ \forall\, k \in \mathcal{I}\big[u_+^0, u_-^0\big]\}\,.$$

In both cases we particularly have that for all $k \in \mathcal{I}\big[\mu(0-), u_+^0\big]$,

$$\operatorname{sgn}(\mu(0-) - u_+^0)\big(f(\mu(0-)) - f(k)\big) \geq 0\,. \tag{6.21}$$

Let $\widetilde{u}$ be the solution of the Riemann problem above, but now with initial data given by $u_-^0, \widetilde{u}_+^0$. Let us assume that also in this situation a condition analogous to (6.21) is fulfilled, namely that

$$\operatorname{sgn}(\widetilde{\mu}(0-) - u_+^0)\big(f(\widetilde{\mu}(0-)) - f(k)\big) \geq 0 \tag{6.22}$$

for all $k \in \mathcal{I}\big[\widetilde{\mu}(0-), u_+^0\big]$, where

$$\widetilde{\mu}\Big(\frac{x}{t}\Big) = \widetilde{u}(t,x)\,. \tag{6.23}$$

From the definition of $\mu(0-)$ and $\widetilde{\mu}(0-)$ and assumption (6.22) it can be seen that

$$\widetilde{\mu}(0-) = \mu(0-)\,. \tag{6.24}$$

Let us show this equality in the case of $u_-^0 < \widetilde{u}_+^0 < u_+^0$.

Suppose that (6.24) does not hold. Since $\widetilde{\mu}(0-) \in [u^0_-, \widetilde{u}^0_+]$ and $\mu(0-) \in [u^0_-, u^0_+]$ are the lowest points (of the respective intervals) at which a global minimum of f is achieved, the inequality in (6.24) would imply that $\mu(0-) \notin [u^0_-, \widetilde{u}^0_+]$. Consequently, $u^0_- \leq \widetilde{\mu}(0-) \leq \widetilde{u}^0_+ \leq \mu(0-) \leq u^0_+$ and $f(\mu(0-)) < f(\widetilde{\mu}(0-))$. On the other hand, (6.22) implies $f(\widetilde{\mu}(0-)) \leq f(k)$ for all $k \in [\widetilde{\mu}(0-), u^0_+]$, which is a contradiction for $k = \mu(0-)$. All remaining cases can be discussed similarly.

Now, since μ and $\widetilde{\mu}$ are constructed using the same nonlinearity f (cf. (6.13)–(6.15)), and achieving the same value both in $-\infty$ and $0-$, we conclude that

$$\widetilde{\mu}(\alpha) = \mu(\alpha) \qquad \forall\, \alpha \in (-\infty, 0)$$

independently on the difference between u^0_+ and $\widetilde{u}^0_+$.

It follows that the solution u of the Riemann problem corresponding to the initial data u^0_-, u^0_+ coincides in $(0,\infty) \times (-\infty, 0)$ with the solution $\widetilde{u}$ of the Riemann problem with initial data $u^0_-, \widetilde{u}^0_+$ if and only if

$$\operatorname{sgn}\big(\widetilde{u}(t,0-) - u^0_+\big)\big(f(\widetilde{u}(t,0-)) - f(k)\big) \geq 0 \tag{6.25}$$

holds for all $k \in \mathcal{I}\big[\widetilde{u}(t,0-), u^0_+\big]$ and $t \in (0,\infty)$ (cf. (6.23)).

Thus we see that for the 'Riemann' initial-boundary problem

$$\begin{aligned} \frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} &= 0 \qquad \text{in } (0,\infty) \times (-\infty, 0)\,, \\ u(0,x) &= u^0 \qquad \text{on } (-\infty, 0)\,, \\ u(t,0) &= u^D \qquad \text{on } (0,\infty)\,, \end{aligned} \tag{6.26}$$

$u^0 = \text{const.}$, $u^D = \text{const.}$, the boundary condition $u(t,0) = u^D$ considered in the sense of (6.25) ensures the uniqueness of the entropy solution. In this special case there are only two possibilities: if $u(t,0) = u^D$ for all $t \in (0,\infty)$, (6.25) is satisfied trivially. On the other hand, if $u(t,0) \neq u^D$, our construction ensures that $u(t,0)$ is determined by (6.25) uniquely, independently of the prescribed value u^D (cf. Lemma 7.24).

The aim of this section was to convince the reader that the heuristically developed condition (6.25) plays a crucial role in the problem of uniqueness. We will see that in the general situation the formulation of a well-posed boundary value problem will be an infinitesimal copy of the one above.

As it turns out, the problem of uniqueness is more important than the problem of existence in the context of bounded domains. Therefore, we will study the problem of uniqueness in the following section. After that, an existence result will be presented.

2.7 Uniqueness in bounded domains

First of all we need a new notion, namely that of a so-called boundary entropy-entropy flux pair.

Definition 7.1 *The pair $(H, \mathbf{Q})$, $\mathbf{Q} = (Q_1, \ldots, Q_d)$, $H, Q_j \in C^2(\mathbb{R}^2)$ is called a* **boundary entropy-entropy flux pair** *if and only if for all $w \in \mathbb{R}$, $(H(\cdot, w), \mathbf{Q}(\cdot, w))$ is an entropy-entropy flux pair in the sense of Definition 3.22 and $H, \mathbf{Q}$ satisfy*

$$H(w, w) = 0\,, \quad \mathbf{Q}(w, w) = 0\,, \quad \partial_1 H(w, w) = 0\,.$$

Here, $\partial_1 H$ denotes the partial derivative with respect to the first variable.

Next we will give a definition of a weak solution to the initial-boundary value problem (6.1). This definition as well as the interpretation of boundary entropy-entropy flux pairs will be discussed later. In particular, Lemma 7.24 will show that Definition 7.2 is really motivated by the considerations from the previous section.

Definition 7.2 *Let $u^0 \in L^\infty(\Omega)$, $u^D \in L^\infty(\Gamma)$, $\mathbf{f} \in C^1(\mathbb{R})^d$. We say that u is a weak solution of the initial-boundary value problem (6.1) if and only if*

$$u \in L^\infty(Q_T)\,, \tag{7.3}$$

and satisfies:

- *the conservation law and the entropy condition in the sense*

$$\int_{Q_T} \eta(u)\frac{\partial \varphi}{\partial t} + q_i(u)\frac{\partial \varphi}{\partial x_i}\, dx\, dt \geq 0 \tag{7.4}$$

  for all $\varphi \in \mathcal{D}_0^\infty(Q_T)$, $\varphi \geq 0$ and all entropy-flux pairs $(\eta, \mathbf{q})$;
- *the boundary condition $u^D \in L^\infty(\Gamma)$ in the sense*

$$\operatorname*{ess\,lim}_{s \to 0-} \int_\Gamma \mathbf{Q}(u(r + s\,\boldsymbol{\nu}(r)), u^D(r)) \cdot \boldsymbol{\nu}(r)\beta(r)\, dr \geq 0 \tag{7.5}$$

  for all $\beta \in L^1(\Gamma)$, $\beta \geq 0$ almost everywhere, and all boundary entropy-entropy flux pairs $(H, \mathbf{Q})$;

- *the initial condition* $u^0 \in L^\infty(\Omega)$ *in the sense*

$$\operatorname*{ess\,lim}_{t \to 0+} \int_\Omega |u(t,x) - u^0(x)|\, dx = 0 \,. \tag{7.6}$$

Remark 7.7 Note that the weak formulation (6.2) of the conservation law $(6.1)_1$ is included in (7.4). Indeed, for a special choice of $\eta = \pm\,\mathrm{Id}$ one obtains (6.2) for all non-negative test functions. Equation (6.2) then follows by the usual method of splitting any test function into its positive and negative parts which are mollified appropriately. For further details see also Remark 7.22. The condition (7.6) is quite usual (see for example KRUŽKOV [1970, Definition 1]). On the other hand, the boundary condition (7.5) was introduced by OTTO [1992]. The formulation of (7.5) is motivated heuristically (see previous section). However, we will see that it has a precise mathematical interpretation, which corresponds to our understanding of boundary conditions (see Lemma 7.24, Lemma 7.26).

Remark 7.8

- Let $k \in \mathbb{R}$ be arbitrary but fixed. Define, for $\ell \in \mathbb{N}$, the entropy-entropy flux pair $(\eta_\ell, \mathbf{q}_\ell)$ by

$$\eta_\ell(z) \equiv \left((z-k)^2 + \left(\frac{1}{\ell}\right)^2\right)^{1/2} - \frac{1}{\ell}\,,$$

$$\mathbf{q}_\ell(z) \equiv \int_k^z \eta_\ell'(r)\mathbf{f}'(r)\, dr \,.$$

  Obviously, $(\eta_\ell, \mathbf{q}_\ell)$ converges uniformly, as $\ell \to \infty$, to a non-smooth entropy-entropy flux pair

$$(|z-k|, \mathbf{F}(z,k))\,,$$

  where

$$\mathbf{F}(z,k) \equiv \begin{cases} \mathbf{f}(k) - \mathbf{f}(z) & \text{for } z \le k\,, \\ \mathbf{f}(z) - \mathbf{f}(k) & \text{for } z \ge k\,. \end{cases} \tag{7.9}$$

  Define a boundary entropy-entropy flux pair $(H_\ell, \mathbf{Q}_\ell)$ by

$$\begin{aligned} H_\ell(z,w) &\equiv \left(\big(\operatorname{dist}(z, \mathcal{I}[w,k])\big)^2 + \left(\frac{1}{\ell}\right)^2\right)^{1/2} - \frac{1}{\ell}\,, \\ \mathbf{Q}_\ell(z,w) &\equiv \int_w^z \partial_1 H_\ell(\lambda, w)\mathbf{f}'(\lambda)\, d\lambda \,, \end{aligned} \tag{7.10}$$

where $\mathcal{I}[w,k]$ is the closed interval with end points w, k. Obviously, this sequence converges uniformly, as $\ell \to \infty$, to

$$\big(\operatorname{dist}\big(z, \mathcal{I}[w,k]\big), \boldsymbol{\mathcal{F}}(z,w,k)\big)$$

where $\boldsymbol{\mathcal{F}} \in C(\mathbb{R}^3)^d$ is given by

$$\boldsymbol{\mathcal{F}}(z,w,k) \equiv \begin{cases} \left.\begin{array}{ll} \mathbf{f}(w) - \mathbf{f}(z) & \text{for } z \le w \\ \mathbf{0} & \text{for } w \le z \le k \\ \mathbf{f}(z) - \mathbf{f}(k) & \text{for } k \le z \end{array}\right\} & \text{and } w \le k\,, \\ \left.\begin{array}{ll} \mathbf{f}(k) - \mathbf{f}(z) & \text{for } z \le k \\ \mathbf{0} & \text{for } k \le z \le w \\ \mathbf{f}(z) - \mathbf{f}(w) & \text{for } w \le z \end{array}\right\} & \text{and } w \ge k\,. \end{cases} \tag{7.11}$$

- Assume $g \in L^\infty(0,\delta)$. Then we have

$$\liminf_{n\to\infty} n \int_0^{1/n} g(t)\,dt \ge \operatorname*{ess\,lim\,inf}_{t\to 0+} g(t)\,,$$

where $\operatorname{ess\,lim\,inf}_{t\to 0+} g(t) \equiv \lim_{n\to\infty} \operatorname{ess\,inf}_{(0,1/n)} g(t)$. Indeed, we have for almost every $t \in (0,1/n)$

$$g(t) \ge \operatorname*{ess\,inf}_{(0,1/n)} g(t)$$

and thus

$$n \int_0^{1/n} g(t)\,dt \ge \operatorname*{ess\,inf}_{(0,1/n)} g(t)$$

and the assertion follows.

- Similarly we get

$$\limsup_{n\to\infty} n \int_0^{1/n} g(t)\,dt \le \operatorname*{ess\,lim\,sup}_{t\to 0+} g(t).$$

Therefore, combining the above results, we obtain the following assertion: if for $g \in L^\infty(0,\delta)$ there exists $\operatorname{ess\,lim}_{t\to 0+} g(t)$, we have

$$\lim_{n\to\infty} n \int_0^{1/n} g(t)\,dt = \operatorname*{ess\,lim}_{t\to 0+} g(t).$$

Some interesting consequences of (7.4)–(7.6) are collected in the following lemma.

Lemma 7.12 *Let* $\mathbf{F}$ *and* $\mathcal{F}$ *be defined by (7.9) and (7.11), respectively.*

- *Let* $u \in L^\infty(Q_T)$ *satisfy (7.4); then*

$$\int_0^T \int_\Omega |u-k|\frac{\partial \varphi}{\partial t} + F_i(u,k)\frac{\partial \varphi}{\partial x_i}\,dx\,dt \ge 0 \tag{7.13}$$

for all $\varphi \in \mathcal{D}(Q_T)$, $\varphi \ge 0$, *and all* $k \in \mathbb{R}$.

- *Let* $u \in L^\infty(Q_T)$ *satisfy (7.4), (7.6); then*

$$\begin{aligned} -\int_0^T \int_\Omega |u-k|\frac{\partial \varphi}{\partial t} + F_i(u,k)\frac{\partial \varphi}{\partial x_i}\,dx\,dt &\le \int_\Omega |u^0-k|\varphi(0)\,dx \\ &\quad - \operatorname*{ess\,lim\,inf}_{s\to 0-} \int_\Gamma \mathbf{F}(u(r+s\boldsymbol{\nu}),k)\cdot\boldsymbol{\nu}(r)\varphi(r)\,dr \end{aligned} \tag{7.14}$$

for all $\varphi \in \mathcal{D}(\mathbb{R}\times\mathbb{R}^d)$, $\varphi \ge 0$, *and all* $k \in \mathbb{R}$.

- *Let* $u \in L^\infty(Q_T)$ *satisfy (7.5); then*

$$\operatorname*{ess\,lim}_{s\to 0-} \int_\Gamma \mathcal{F}\big(u(r+s\,\boldsymbol{\nu}(r)), u^D(r), k\big)\cdot\boldsymbol{\nu}(r)\,\beta(r)\,dr \ge 0 \tag{7.15}$$

for all $\beta \in L^1(\Gamma)$, $\beta \ge 0$ *almost everywhere, and all* $k \in \mathbb{R}$.

PROOF : The proofs of the first and the third assertion follow from the first part of Remark 7.8.

It remains to show the second statement. In order to make the argument simpler, we restrict ourselves to the case of a half-space, i.e.,

$$\begin{aligned} \Omega &= \{x=(x',x_d) \in \mathbb{R}^{d-1}\times\mathbb{R}; x_d<0\}\,, \\ \boldsymbol{\nu} &= (0,\ldots,0,1)\,, \\ \Gamma &= (0,T)\times\mathbb{R}^{d-1}\,, \qquad r=(t,x')\in\Gamma\,, \\ Q_T &= \{p=(r,x_d); r\in\Gamma, x_d<0\}. \end{aligned}$$

The general case can be obtained by the usual covering argument, i.e., by considering that $\partial\Omega$ can be locally replaced by the border of a half-space. Notice that in this case we get an additional dependence of the nonlinearity on the space variable x, but this implies no additional substantial difficulties and the argument follows the same lines.

Let $(\eta, \mathbf{q})$ be an entropy-entropy flux pair and let us define $\beta_n, n \in \mathbb{N}$, by

$$\beta_n(t,x) \equiv \begin{cases} -nx_d & \text{for } t \geq -x_d, \quad x_d \in (-1/n, 0)\,, \\ nt & \text{for } x_d \leq -t, \quad t \in (0, 1/n)\,, \\ 0 & \text{for } x_d = 0, \quad t \geq 0 \text{ or } t = 0, \quad x_d \leq 0\,, \\ 1 & \text{for } x_d \leq -1/n, \quad t \geq 1/n\,. \end{cases}$$

Let $\varphi \in \mathcal{D}(\mathbb{R} \times \mathbb{R}^d)$, $\varphi \geq 0$ and use $\varphi_n \equiv \varphi\beta_n$ as a test function in (7.4). We obtain:

$$\begin{aligned} 0 &\leq \int_0^T \int_\Omega \eta(u)\Big[\frac{\partial\varphi}{\partial t}\beta_n + \varphi\frac{\partial\beta_n}{\partial t}\Big] + q_i(u)\Big[\frac{\partial\varphi}{\partial x_i}\beta_n + \varphi\frac{\partial\beta_n}{\partial x_i}\Big]\,dx\,dt \\ &\equiv I_1 + \cdots + I_4. \end{aligned} \tag{7.16}$$

Note that β_n is not smooth enough to be a test function in a classical sense. Nevertheless, (7.16) is meaningful, as the derivatives of β_n are defined almost everywhere on Q_T.

Since $\beta_n \overset{\text{loc}}{\rightrightarrows} 1$ (uniformly on compact sets, as $n \to \infty$) and φ has compact support, we have, for the first and third integral in (7.16),

$$\begin{aligned} \lim_{n\to\infty} \int_0^T \int_\Omega \eta(u)\frac{\partial\varphi}{\partial t}\beta_n\,dx\,dt &= \int_0^T \int_\Omega \eta(u)\frac{\partial\varphi}{\partial t}\,dx\,dt\,, \\ \lim_{n\to\infty} \int_0^T \int_\Omega q_i(u)\frac{\partial\varphi}{\partial x_i}\beta_n\,dx\,dt &= \int_0^T \int_\Omega q_i(u)\frac{\partial\varphi}{\partial x_i}\,dx\,dt. \end{aligned} \tag{7.17}$$

From the definition of β_n it is clear that we can consider

$$\frac{\partial\beta_n}{\partial t} = \begin{cases} n & \text{for } x_d \leq -t, t \in (0, 1/n)\,, \\ 0 & \text{elsewhere,} \end{cases}$$

and therefore

$$\begin{aligned} I_2 &= n\int_0^{1/n}\int_{-\infty}^{-t}\int_{\mathbb{R}^{d-1}} \eta(u(t,x))\,\varphi(t,x)\,dx'\,dx_d\,dt \\ &= n\int_0^{1/n}\int_{-\infty}^{0}\int_{\mathbb{R}^{d-1}} \eta(u(t,x))\,\varphi(t,x)\,dx'\,dx_d\,dt \\ &\quad - n\int_0^{1/n}\int_{-t}^{0}\int_{\mathbb{R}^{d-1}} \eta(u(t,x))\,\varphi(t,x)\,dx'\,dx_d\,dt \\ &\equiv I_5 + I_6. \end{aligned}$$

We show that the integral I_6 tends to zero for $n \to \infty$. Indeed,

$$
\begin{aligned}
|I_6| &\le n \int_0^{1/n} \int_{-1/n}^{0} \int_{\mathbb{R}^{d-1}} |\eta(u(t,x))\, \varphi(t,x)|\, dx'\, dx_d\, dt \\
&\le n\, \frac{1}{n^2} c_0 \to 0 \qquad \text{as } n \to \infty,
\end{aligned}
\tag{7.18}
$$

where

$$
c_0 \equiv c_0(\eta, \varphi, \|u\|_\infty) < \infty\,.
$$

Further we have

$$
\begin{aligned}
I_5 &= n \int_0^{1/n} \int_\Omega \eta(u(t,x))\, \varphi(t,x)\, dx\, dt \\
&\equiv n \int_0^{1/n} g(t)\, dt\,.
\end{aligned}
$$

Note that $\operatorname{ess\,lim}_{t\to 0+} g(t) = \int_\Omega \eta(u^0)\, \varphi(0)\, dx$ exists thanks to (7.6) and therefore due to the third part of Remark 7.8 we have

$$
\lim_{n\to\infty} n \int_0^{1/n} \int_\Omega \eta(u(t,x))\, \varphi(t,x)\, dx\, dt = \int_\Omega \eta(u^0)\varphi(0)\, dx. \tag{7.19}
$$

The integral I_4 can be treated in a similar way. Using

$$
\frac{\partial \beta_n}{\partial x_i} = \begin{cases} -n\nu_i & \text{for } t \ge -x_d, \quad x_d \in (-1/n, 0)\,, \\ 0 & \text{elsewhere,} \end{cases}
$$

we find

$$
\begin{aligned}
I_4 &= -n \int_{-1/n}^{0} \int_{-x_d}^{T} \int_{\mathbb{R}^{d-1}} \nu_i q_i(u(t,x))\, \varphi(t,x)\, dx'\, dt\, dx_d \\
&= -n \int_{-1/n}^{0} \int_0^{T} \int_{\mathbb{R}^{d-1}} \nu_i q_i(u(t,x))\, \varphi(t,x)\, dx'\, dt\, dx_d \\
&\quad - n \int_{-1/n}^{0} \int_0^{-x_d} \int_{\mathbb{R}^{d-1}} \nu_i q_i(u(t,x))\, \varphi(t,x)\, dx'\, dt\, dx_d \\
&\equiv I_7 + I_8\,.
\end{aligned}
$$

Again we can show that (cf. (7.18))

$$
\lim_{n\to\infty} I_8 = 0 \tag{7.20}
$$

and rewrite I_7 as

$$I_7 = -n \int_{-1/n}^{0} h(x_d)\, dx_d \,.$$

From the second part of Remark 7.8 and $\Gamma = (0,T) \times \mathbb{R}^{d-1}$ it follows that

$$\begin{aligned}
\liminf_{n\to\infty} \int_{-1/n}^{0} \int_0^T \int_{\mathbb{R}^{d-1}} & n\, \nu_i q_i(u(t,x',x_d))\varphi(t,x',x_d)\, dx'\, dt\, dx_d \\
&\geq \operatorname*{ess\,lim\,inf}_{s\to 0-} \int_\Gamma q_i(u(r+s\boldsymbol{\nu}))\nu_i \varphi(r)\, dr \qquad (7.21) \\
&+ \operatorname*{ess\,lim\,inf}_{s\to 0-} \int_\Gamma q_i(u(r+s\boldsymbol{\nu}))\nu_i \big(\varphi(r+s\boldsymbol{\nu}) - \varphi(r)\big)\, dr \,,
\end{aligned}$$

where the last term is zero due to the smoothness of φ. Hence, from (7.16)–(7.21) we have

$$\begin{aligned}
0 \leq & \int_0^T \int_\Omega \eta(u)\frac{\partial\varphi}{\partial t} + q_i(u)\frac{\partial\varphi}{\partial x_i}\, dx\, dt \\
& + \int_\Omega \eta(u^0)\varphi(0)\, dx - \operatorname*{ess\,lim\,inf}_{s\to 0-} \int_\Gamma q_i(u(r+s\boldsymbol{\nu}))\nu_i \varphi(r)\, dr \,,
\end{aligned}$$

which is (7.14) for smooth entropy flux pairs. Using now the approximation of $(|z-k|, \mathbf{F}(z,k))$ defined in Remark 7.8, we easily obtain the assertion. ∎

Remark 7.22 Note that also in the case of a bounded domain, (7.13) together with $u \in L^\infty(Q_T)$ imply the integral identity

$$\int_0^T \int_\Omega u\frac{\partial\psi}{\partial t} + f_i(u)\frac{\partial\psi}{\partial x_i}\, dx\, dt = 0 \qquad (7.23)$$

for all $\psi \in \mathcal{D}(Q_T)$. Indeed, since the smooth function $\varphi \geq 0$ in (7.13) can be chosen arbitrarily, it is obvious that (7.13) for $k = \pm\|u\|_\infty$ implies (7.23).

The following lemma connects our understanding of the boundary conditions in the sense of (7.5) with the condition (6.21), cf. assertion no. 5 of the following lemma.

Lemma 7.24 *Let $u^D \in L^\infty(\Gamma)$ and $u \in L^\infty(Q_T)$ be given. Further, let u assume some boundary value $u^\tau \in L^\infty(\Gamma)$ in the sense*

$$\operatorname*{ess\,lim}_{s\to 0-} \int_\Gamma |u(r + s\,\boldsymbol{\nu}(r)) - u^\tau(r)|\, dr = 0 \,.$$

Then the following statements are equivalent:

1. (u, u^D) *satisfy (7.5);*
2. (u, u^D) *satisfy (7.15);*
3. *for* $\mathcal{H}^d$*-almost all*[||] $r \in \Gamma$ *and all entropy-entropy flux pairs* $(\eta, \mathbf{q})$ *we have*

$$\mathbf{q}(u^\tau(r)) \cdot \boldsymbol{\nu}(r) \geq 0$$

provided that $\eta'(u^D(r)) = 0$ *and* $\mathbf{q}(u^D(r)) = 0$*;*

4. *for* $\mathcal{H}^d$*-almost all* $r \in \Gamma$ *we have*

$$\mathcal{F}(u^\tau(r), u^D(r), k) \cdot \boldsymbol{\nu}(r) \geq 0$$

for all $k \in \mathbb{R}$*;*

5. *for* $\mathcal{H}^d$*-almost all* $r \in \Gamma$ *we have*

$$\operatorname{sgn}(u^\tau(r) - u^D(r))(\mathbf{f}(u^\tau(r)) \cdot \boldsymbol{\nu}(r) - \mathbf{f}(k) \cdot \boldsymbol{\nu}(r)) \geq 0$$

for all $k \in \mathcal{I}\big[u^\tau(r), u^D(r)\big]$.

PROOF :

The implication '1 ⇒ 2' was proved in Lemma 7.12.

'2 ⇒ 4': From the assumptions and (7.15) it follows for fixed $k \in \mathbb{R}$ that

$$\int_\Gamma \mathcal{F}(u^\tau, u^D(r), k) \cdot \boldsymbol{\nu}(r)\, \beta(r)\, dr$$
$$= \operatorname*{ess\,lim}_{s \to 0-} \int_\Gamma \mathcal{F}(u(r + s\,\boldsymbol{\nu}(r)), u^D(r), k) \cdot \boldsymbol{\nu}(r)\, \beta(r)\, dr \geq 0$$

for all $\beta \in L^1(\Gamma)$, $\beta \geq 0$ almost everywhere. Therefore, there is a set E, $\mathcal{H}^d(E) = 0$, such that

$$\mathcal{F}(u^\tau(r), u^D(r), k) \cdot \boldsymbol{\nu}(r) \geq 0 \quad \forall k \in \mathbb{Q}, \quad \forall r \in \Gamma \setminus E\,.$$

This and properties of $\mathcal{F}$ imply 4.

The implication '4 ⇒ 5' follows immediately from the definition of $\mathcal{F}$.

'5 ⇒ 3': For an entropy-entropy flux pair $(\eta, \mathbf{q})$ with $\eta'(w) = 0$, $\mathbf{q}(w) = 0$ we get

$$\mathbf{q}(z) = \int_w^z \eta''(\lambda)(\mathbf{f}(z) - \mathbf{f}(\lambda))\, d\lambda\,.$$

This formula shows that 5 implies 3.

[||] By $\mathcal{H}^d$ we denote the d-dimensional Hausdorff measure.

'3 ⇒ 1': For all boundary entropy fluxes $\mathbf{Q}$ and $\beta \in L^1(\Gamma)$, $\beta \geq 0$ almost everywhere, we obviously get

$$\begin{aligned} 0 &\leq \int_\Gamma \mathbf{Q}(u^\tau(r), u^D(r)) \cdot \boldsymbol{\nu}(r)\, \beta(r)\, dr \\ &= \operatorname*{ess\,lim}_{s\to 0-} \int_\Gamma \mathbf{Q}(u(r + s\,\boldsymbol{\nu}(r)), u^D(r)) \cdot \boldsymbol{\nu}(r)\, \beta(r)\, dr\,, \end{aligned}$$

which is (7.5). ∎

Remark 7.25 The formulation of statement no. 5 of the previous lemma corresponds to the formulation of the boundary value problem in BARDOS, LE ROUX AND NEDELEC [1979]; compare it also to (6.21). Statement no. 3 is a special case of the 'boundary entropy condition' due to DUBOIS AND LE FLOCH [1988, Theorem 1.1]. Note that both 3 and 5 need the existence of a trace u^τ.

All statements of the previous lemma are equivalent only in the case that the existence of a trace u^τ is assumed, despite the fact that some of the statements do not involve u^τ. In the following two lemmas we will see that there is a subset $\Gamma_D \subset \Gamma$, depending on $\mathbf{f}$, on which the existence of $u^\tau = u^D$ can be deduced from (7.5). Moreover, there is a subset $\Gamma_N \subset \Gamma$, depending on $\mathbf{f}$, on which the boundary condition (7.5) is trivially satisfied (see Lemma 7.27). Compare also the definition of Γ_D and Γ_N to (6.7), (6.8), respectively.

Let $\mathbf{f} \in C^1(\mathbb{R})^d$. Define the Borel sets $\Gamma_D, \Gamma_N \subset \Gamma$ by

$$\Gamma_N = \{r \in \Gamma, \text{the function '}x \mapsto \mathbf{f}(x)\cdot\boldsymbol{\nu}(r)\text{' is non-decreasing}\}\,,$$

$$\Gamma_D = \{r \in \Gamma, \text{the function '}x \mapsto \mathbf{f}(x)\cdot\boldsymbol{\nu}(r)\text{' is decreasing}\}\,.$$

Lemma 7.26 *For every $u \in L^\infty(Q_T)$, $u^D \in L^\infty(\Gamma)$ satisfying (7.5), we have*

$$\operatorname*{ess\,lim}_{s\to 0-} \int_{\Gamma_D} |u(r + s\,\boldsymbol{\nu}(r)) - u^D(r)|\, dr = 0\,,$$

with Γ_D defined above.

PROOF : Let us use (7.5) with the boundary entropy-entropy flux pair $(H_\ell, \mathbf{Q}_\ell)$ defined by

$$H_\ell(z,w) \equiv \left((z-w)^2 + \left(\frac{1}{\ell}\right)^2\right)^{1/2} - \frac{1}{\ell}\,,$$

$$\mathbf{Q}_\ell(z,w) \equiv \int_w^z \partial_1 H_\ell(\lambda, w)\, \mathbf{f}'(\lambda)\, d\lambda.$$

Letting $\ell \to \infty$ we obtain

$$\operatorname*{ess\,lim}_{s\to 0-} \int_\Gamma \mathbf{F}(u(r+s\,\boldsymbol{\nu}(r)), u^D(r)) \cdot \boldsymbol{\nu}(r)\,\beta(r)\,dr \geq 0$$

for all $\beta \in L^1(\Gamma)$, $\beta \geq 0$ almost everywhere. Using the characteristic function of the set Γ_D as a test function β we obtain

$$\mathbf{F}(z,w) \cdot \boldsymbol{\nu}(r) = -|\mathbf{f}(z) \cdot \boldsymbol{\nu}(r) - \mathbf{f}(w) \cdot \boldsymbol{\nu}(r)|$$

for all $(z,w) \in \mathbb{R}^2$ and all $r \in \Gamma_D$. Therefore, there is an $\mathcal{H}^1$-null set E such that

$$\lim_{\substack{s\to 0-\\ s\notin E}} \int_{\Gamma_D} |\mathbf{f}(u(r+s\,\boldsymbol{\nu}(r))) \cdot \boldsymbol{\nu}(r) - \mathbf{f}(u^D(r)) \cdot \boldsymbol{\nu}(r)|\,dr = 0$$

and for a subsequence of $\{s_\ell\}_\ell \subset (-\infty, 0) \setminus E$, still denoted s_ℓ,

$$\lim_{\ell\to\infty} \mathbf{f}(u(r+s_\ell\,\boldsymbol{\nu}(r))) \cdot \boldsymbol{\nu}(r) = \mathbf{f}(u^D(r)) \cdot \boldsymbol{\nu}(r) \quad \text{for a.a. } r \in \Gamma_D\,.$$

Due to the monotonicity of $\mathbf{f} \cdot \boldsymbol{\nu}(r)$ on Γ_D we get

$$\lim_{\ell\to\infty} u(r+s_\ell\,\boldsymbol{\nu}(r)) = u^D(r) \quad \text{for a.a. } r \in \Gamma_D\,.$$

Using the Lebesgue dominated convergence theorem we conclude that

$$\lim_{\ell\to\infty} \int_{\Gamma_D} |u(r+s_\ell\,\boldsymbol{\nu}(r)) - u^D(r)|\,dr = 0.$$

■

Lemma 7.27 *For every $u \in L^\infty(Q_T)$ and $u^D \in L^\infty(\Gamma)$ we have*

$$\operatorname*{ess\,lim}_{s\to 0-} \int_{\Gamma_N} \mathbf{Q}(u(r+s\,\boldsymbol{\nu}(r)), u^D(r)) \cdot \boldsymbol{\nu}(r)\,\beta(r)\,dr \geq 0$$

for all $\beta \in L^1(\Gamma)$, $\beta \geq 0$ almost everywhere, and all boundary entropy-entropy flux pairs $(H, \mathbf{Q})$. Here, Γ_N is defined before Lemma 7.26.

PROOF : Due to the definition of a boundary entropy-entropy flux pair we have

$$\mathbf{Q}(w,w) = 0\,, \quad \partial_1 H(w,w) = 0\,, \quad \partial^2_{11} H(\lambda,w) \geq 0$$

and therefore

$$\mathbf{Q}(z,w) = \int_w^z \partial_1 H(\lambda,w)\,\mathbf{f}'(\lambda)\,d\lambda\,,$$
$$\operatorname{sgn}(\lambda - w)\,\partial_1 H(\lambda,w) \geq 0\,.$$

Finally, if $r \in \Gamma_N$, we have $\mathbf{f}'(\lambda) \cdot \boldsymbol{\nu}(r)$ for all $\lambda \in \mathbb{R}$. Hence,

$$\mathbf{Q}(z,w) \cdot \boldsymbol{\nu}(r) = \int_w^z \partial_1 H(\lambda, w)\, \mathbf{f}'(\lambda) \cdot \boldsymbol{\nu}(r)\, d\lambda \geq 0 \ ,$$

which implies the assertion. ∎

Now we will formulate two theorems which we will prove in the remainder of this section. The L^1-contraction property, proved in the first theorem, implies uniqueness of weak solutions and the second theorem gives an equivalent formulation of (7.4)–(7.6).

Theorem 7.28 (Uniqueness) *Let $\mathbf{f} \in C^1(\mathbb{R})^d$ and let $\mathbf{F}$ be defined by (7.9). Let $(u_i, u_i^D, u_i^0) \in \big(L^\infty(Q_T) \times L^\infty(\Gamma) \times L^\infty(\Omega)\big)$, $i = 1, 2$, satisfy (7.3)–(7.6). Then*

$$\begin{aligned} -\int_0^T \int_\Omega |u_1 - u_2| \frac{\partial \beta}{\partial t} + F_i(u_1, u_2) \frac{\partial \beta}{\partial x_i}\, dx\, dt \\ \leq \int_\Omega |u_1^0 - u_2^0| \beta(0)\, dx + \int_\Gamma \operatorname{diam}\big(\mathbf{f} \cdot \boldsymbol{\nu}, \mathcal{I}[u_1^D, u_2^D]\big) \beta\, dr \end{aligned} \tag{7.29}$$

holds for all $\beta \in \mathcal{D}((-\infty, T) \times \mathbb{R}^d)$, where

$$\begin{aligned} &\operatorname{diam}\big(\mathbf{f} \cdot \boldsymbol{\nu}(r), \mathcal{I}[u_1^D(r), u_2^D(r)]\big) \\ &\quad = \sup\big\{ |\mathbf{f}(z_1) \cdot \boldsymbol{\nu}(r) - \mathbf{f}(z_2) \cdot \boldsymbol{\nu}(r)|; z_1, z_2 \in \mathcal{I}[u_1^D(r), u_2^D(r)] \big\} . \end{aligned}$$

Moreover, for almost all $t \in (0, T)$ we have

$$\begin{aligned} \int_\Omega |u_1(t) - u_2(t)|\, dx \leq \int_\Omega |u_1^0 - u_2^0|\, dx \\ + M \int_0^t \int_{\partial\Omega} |u_1^D - u_2^D|\, dr\, ds , \end{aligned} \tag{7.30}$$

where M is the constant of Lipschitz continuity of $\mathbf{f}$, restricted on the ball with radius $\max\{\|u_i\|_\infty, \|u_i^D\|_\infty, \|u_i^0\|_\infty\}$.

Theorem 7.31 *Provided that $(u, u^D, u^0) \in \big(L^\infty(Q_T) \times L^\infty(\Gamma) \times L^\infty(\Omega)\big)$, the formulation (7.4)–(7.6) is equivalent to the statement that*

$$\begin{aligned} -\int_0^T \int_\Omega H(u,k) \frac{\partial \beta}{\partial t} + Q_i(u,k) \frac{\partial \beta}{\partial x_i}\, dx\, dt \\ \leq \int_\Omega H(u^0, k)\, \beta(0)\, dx + M \int_\Gamma H(u^D, k)\, \beta\, dr \end{aligned} \tag{7.32}$$

holds for all $\beta \in \mathcal{D}((-\infty, T) \times \mathbb{R}^d)$, $\beta \geq 0$, $k \in \mathbb{R}$, *and all boundary entropy-entropy flux pairs* $(H, \mathbf{Q})$.

Again, M *is the constant of Lipschitz continuity of* $\mathbf{f}$, *restricted on the ball with radius* $\max\{\|u\|_\infty, \|u^D\|_\infty, \|u^0\|_\infty\}$.

Remark 7.33

- Let us emphasize that due to Theorem 7.31, the statements (7.4)–(7.6) are formulated weakly enough to be stable under L^1-convergence. That is, if $(u_\ell, u_\ell^D, u_\ell^0)$ is a sequence of solutions which converges in L^1 as $\ell \to \infty$, then the limit (u, u^D, u^0) also satisfies (7.4)–(7.6). On the other hand, the formulation (7.4)–(7.6) is strong enough to ensure uniqueness and even more, continuous dependence on the data (see Theorem 7.28).
- Theorem 7.31 implies the following maximum principle:

$$\sup_{Q_T} u \leq \max(\sup_\Gamma u^D, \sup_\Omega u^0)\,,$$
$$\inf_{Q_T} u \geq \min(\inf_\Gamma u^D, \inf_\Omega u^0)\,.$$

Indeed, using (7.32) with the boundary entropy-entropy flux pair

$$H_\ell(z,k) \equiv \left(\left(\max(z-k,0)\right)^2 + \left(\frac{1}{\ell}\right)^2\right)^{1/2} - \frac{1}{\ell}\,,$$
$$\mathbf{Q}_\ell(z,k) \equiv \int_k^z \partial_1 H_\ell(\lambda,k)\,\mathbf{f}'(\lambda)\,d\lambda$$

where

$$k = \max\left(\sup_\Gamma u^D, \sup_\Omega u^0\right)$$

and letting $\ell \to \infty$, we obtain

$$\int_0^T \int_\Omega \max(u-k,0)\frac{\partial \beta}{\partial t} + F_i^+(u,k)\frac{\partial \beta}{\partial x_i}\,dx\,dt \geq 0,$$

where

$$\mathbf{F}^+(z,k) \equiv \begin{cases} \mathbf{f}(z) - \mathbf{f}(k) & \text{for } z \geq k\,, \\ \mathbf{0} & \text{for } z \leq k\,. \end{cases}$$

In particular, for $\beta(t,x) = \alpha(t)\gamma(x), \gamma = 1$ on Ω, we have

$$\int_0^T \int_\Omega \max(u(t,x)-k,0)\,dx\,\alpha'(t)\,dt \geq 0$$

for all $\alpha \in \mathcal{D}((-\infty, T))$, $\alpha \geq 0$, and thus

$$\int_\Omega \max(u(t,x) - k, 0)\, dx = 0 \quad \text{for a.a. } t \in (0,T).$$

- If the existence of a weak solution to (7.4)–(7.6) for smooth boundary and initial data has already been proved, we will be able to show that the problem is solvable under only the assumption that u^D, u^0 are L^∞-functions. Let us sketch the proof of this statement: for given $u^D \in L^\infty(\Gamma)$, $u^0 \in L^\infty(\Omega)$, let $u^D_\ell \in C_0^\infty(\Gamma)$, $u^0_\ell \in C_0^\infty(\Omega)$ be such that

$$\lim_{\ell\to\infty} u^D_\ell = u^D \qquad \text{in } L^1(\Gamma)\,,$$
$$\lim_{\ell\to\infty} u^0_\ell = u^0 \qquad \text{in } L^1(\Omega)\,.$$

Knowing that our problem is solvable for smooth data, we also know that due to the second part of Theorem 7.28, the solutions $\{u_\ell\}$ form a Cauchy sequence in $L^1(Q_T)$. Moreover, due to the maximum principle, they are bounded in $L^\infty(Q_T)$. Therefore, there is a $u \in L^\infty(Q_T)$ such that

$$\lim_{\ell\to\infty} u_\ell = u \quad \text{in } L^1(Q_T)\,.$$

Due to the first part of Remark 7.33, this u solves (7.4)–(7.6).

In order to prove Theorems 7.28 and 7.31, we need some lemmas.

Lemma 7.34 *Let $(H, \mathbf{Q})$ be a boundary entropy-entropy flux pair and let $u \in L^\infty(Q_T)$ satisfy*

$$\int_0^T \int_\Omega H(u,k)\frac{\partial\gamma}{\partial t} + Q_i(u,k)\frac{\partial\gamma}{\partial x_i}\, dx\, dt \geq 0 \tag{7.35}$$

for all $\gamma \in \mathcal{D}(Q_T)$, $\gamma \geq 0$ and all $k \in \mathbb{R}$. Then there is a set E of Lebesgue measure zero, such that

$$\lim_{\substack{s\to 0- \\ s\notin E}} \int_\Gamma \mathbf{Q}(u(r + s\,\boldsymbol{\nu}(r)), v^D(r)) \cdot \boldsymbol{\nu}(r)\, \beta(r)\, dr$$

exists for all $\beta \in L^1(\Gamma)$, $\beta \geq 0$ almost everywhere, and all $v^D \in L^\infty(\Gamma)$.

Moreover, if u additionally satisfies for some $u^D \in L^\infty(\Gamma)$

$$-\int_0^T \int_\Omega H(u,k)\frac{\partial\gamma}{\partial t} + Q_i(u,k)\frac{\partial\gamma}{\partial x_i}\, dx\, dt \leq M \int_\Gamma H(u^D,k)\,\gamma\, dr \tag{7.36}$$

for all $\gamma \in \mathcal{D}((0,T)\times\mathbb{R}^d)$, $\gamma \geq 0$ *and all* $k \in \mathbb{R}$, *where* M *is constant of Lipschitz continuity of* $\mathbf{f}$, *then*

$$\operatorname*{ess\,lim}_{s\to 0-} \int_\Gamma \mathbf{Q}(u(r+s\,\boldsymbol{\nu}(r)), u^D(r))\cdot\boldsymbol{\nu}(r)\,\beta(r)\,dr \geq 0$$

for all $\beta \in L^1(\Gamma)$, $\beta \geq 0$ *almost everywhere.*

PROOF : In order to simplify the argument, we again restrict ourselves to the case of half-space, i.e.,

$$\Omega = \{x = (x', s) \in \mathbb{R}^{d-1}\times\mathbb{R}; s \equiv x_d < 0\}\ .$$

Then we have (cf. the proof of Lemma 7.12):

$$\begin{aligned}&\boldsymbol{\nu} = (0,\dots,0,1) \in \mathbb{R}^d\ ,\\ &\partial\Omega = \mathbb{R}^{d-1}\times\{0\},\quad \Gamma = (0,T)\times\mathbb{R}^{d-1},\quad r = (t,x') \in \Gamma\,,\\ &Q_T = \{p = (r,s); r \in \Gamma\ ,\ s < 0\}\,.\end{aligned}$$

Let $w \in \mathbb{Q}$ and $\varphi \in \mathcal{D}(\Gamma)$, $\varphi \geq 0$, be arbitrary but fixed. We denote

$$\eta(z) \equiv H(z,w)\ ,\quad \mathbf{q}(z) \equiv \mathbf{Q}(z,w)\ .$$

Then for $\gamma(r,s) = \beta(r)\,\alpha(s)$ we obtain from (7.35):

$$-\int_{-\infty}^0\int_\Gamma \mathbf{q}(u(r,s))\cdot\boldsymbol{\nu}\,\beta(r)\,dr\,\alpha'(s)\,ds \leq C\int_{-\infty}^0 \alpha(s)\,ds \tag{7.37}$$

for all $\alpha \in \mathcal{D}((-\infty,0))$, $\alpha \geq 0$, where

$$C = \operatorname*{ess\,sup}_{Q_T}|\eta(u)|\int_\Gamma\left|\frac{\partial\beta}{\partial t}\right|dr + \operatorname*{ess\,sup}_{Q_T}|\mathbf{q}(u)|\int_\Gamma|\nabla_{x'}\beta|\,dr\,.$$

Using partial integration of the left-hand side of (7.37) and $\alpha \geq 0$ we see that the function

$$s \mapsto \int_\Gamma \mathbf{q}(u(r,s))\cdot\boldsymbol{\nu}\,\beta(r)\,dr - Cs\,, \tag{7.38}$$

after a possible modification on a set of zero measure, is non-increasing on $(-\infty,0)$. On the other hand we have

$$\operatorname*{ess\,lim\,inf}_{s\to 0-}\int_\Gamma \mathbf{q}(u(r,s))\cdot\boldsymbol{\nu}\,\beta(r)\,dr \geq -\operatorname*{ess\,sup}_{Q_T}|\mathbf{q}(u)|\int_\Gamma \beta(r)\,dr\,.$$

From the monotonicity and the boundedness from below we infer that

$$\operatorname*{ess\,lim}_{s\to 0-} \int_\Gamma \mathbf{q}(u(r,s)) \cdot \boldsymbol{\nu}\, \beta(r)\, dr \tag{7.39}$$

exists. Provided that (7.36) is fulfilled, we get

$$\begin{aligned} -\int_{-\infty}^{0} \int_\Gamma \mathbf{q}(u(r,s)) \cdot \boldsymbol{\nu}\, \beta(r)\, dr\, \alpha'(s)\, ds \\ \le C \int_{-\infty}^{0} \alpha(s)\, ds + M \int_\Gamma \eta(u^D(r))\, \beta(r)\, dr\, \alpha(0) \end{aligned}$$

for all $\alpha \in \mathcal{D}(\mathbb{R})$, $\alpha \ge 0$. Now the special choice of $\alpha_n(s) = (ns + 1)\chi_{(1/n,0)}$ (mollified properly) and (7.39) imply for $n \to \infty$ that

$$\operatorname*{ess\,lim}_{s\to 0-} \int_\Gamma \mathbf{q}(u(r,s)) \cdot \boldsymbol{\nu}\, \beta(r)\, dr \ge -M \int_\Gamma \eta(u^D(r))\, \beta(r)\, dr\ .$$

In what follows the consequences of this inequality will appear in parentheses.

Let $J \subset \mathcal{D}(\Gamma)$ be a countable set of non-negative functions such that for all $\beta \in L^1(\Gamma)$, $\beta \ge 0$ almost everywhere, there is a sequence $\{\beta_\ell\}_\ell \subset J$ such that

$$\lim_{\ell\to\infty} \beta_\ell = \beta \quad \text{in } L^1(\Gamma).$$

Thus

$$\lim_{\ell\to\infty} \int_\Gamma \mathbf{q}(u(r,s)) \cdot \boldsymbol{\nu}\, \beta_\ell(r)\, dr = \int_\Gamma \mathbf{q}(u(r,s)) \cdot \boldsymbol{\nu}\, \beta(r)\, dr$$

uniformly in $s \in (-\infty, 0)$ and

$$\lim_{\ell\to\infty} \int_\Gamma \eta(u^D(r))\, \beta_\ell(r)\, dr = \int_\Gamma \eta(u^D(r))\, \beta(r)\, dr\ .$$

Due to (7.39) there is a set E_w of measure zero (recall that η and $\mathbf{q}$ depend on w), such that for all $\beta \in J$

$$\lim_{\substack{s\to 0- \\ s\notin E_w}} \int_\Gamma \mathbf{q}(u(r,s)) \cdot \boldsymbol{\nu}\, \beta(r)\, dr \ \left(\ge -M \int_\Gamma \eta(u^D(r))\, \beta(r)\, dr\right)$$

exists. Of course, this can be extended to functions $\beta \in L^1(\Gamma)$,

$\beta \geq 0$ almost everywhere, and therefore

$$\lim_{\substack{s\to 0- \\ s\notin E}} \int_\Gamma \mathbf{Q}(u(r,s),w)\cdot \boldsymbol{\nu}\,\beta(r)\,dr \qquad\qquad \left(\geq -M\int_\Gamma H(u^D(r),w)\,\beta(r)\,dr\right) \tag{7.40}$$

exists for all $\beta \in L^1(\Gamma)$, $\beta \geq 0$ almost everywhere, and all $w \in \mathbb{Q}$.

Now let $v^D \in L^\infty(\Gamma)$ and $\beta \in L^1(\Gamma)$, $\beta \geq 0$ be given. Further, let $\{v_\ell^D\}$ be a sequence of simple functions with values in $\mathbb{Q}$ such that

$$\lim_{\ell\to\infty} v_\ell^D = v^D \quad \text{a.e. in } \Gamma.$$

Thus (7.40) holds for all $w = v_\ell^D$. On the other hand we have

$$\lim_{\ell\to\infty}\int_\Gamma \mathbf{Q}(u(r,s),v_\ell^D(r))\cdot\boldsymbol{\nu}\,\beta(r)\,dr = \int_\Gamma \mathbf{Q}(u(r,s),v^D(r))\cdot\boldsymbol{\nu}\,\beta(r)\,dr$$

uniformly in $s \in (-\infty, 0)$ and

$$\lim_{\ell\to\infty}\int_\Gamma H(u^D(r),v_\ell^D(r))\,\beta(r)\,dr = \int_\Gamma H(u^D(r),v^D(r))\,\beta(r)\,dr\,.$$

Altogether we get that

$$\lim_{\substack{s\to 0- \\ s\notin E}} \int_\Gamma \mathbf{Q}(u(r,s),v^D(r))\cdot \boldsymbol{\nu}\,\beta(r)\,dr \qquad \left(\geq -M\int_\Gamma H(u^D(r),v^D(r))\beta(r)\,dr\right)$$

exists, which implies the assertions of Lemma 7.34. ∎

Lemma 7.41 *Let $(u,u^0) \in L^\infty(Q_T)\times L^\infty(\Omega)$ satisfy*

$$-\int_0^T\int_\Omega |u-k|\frac{\partial\beta}{\partial t} + F_i(u,k)\frac{\partial\beta}{\partial x_i}\,dx\,dt \leq \int_\Omega |u^0-k|\,\beta(0)\,dx$$

for all $\beta \in \mathcal{D}((-\infty,T)\times\Omega)$, $\beta \geq 0$, and all $k \in \mathbb{R}$. Then we have

$$\operatorname*{ess\,lim}_{t\to 0+}\int_\Omega |u(t,x)-u^0(x)|\,dx = 0\,.$$

PROOF : With the same methods as in the previous lemma we show

$$\operatorname*{ess\,lim\,sup}_{t\to0+}\int_\Omega |u(t,x)-k|\beta(x)\,dx \le \int_\Omega |u^0(x)-k|\beta(x)\,dx$$

for all $\beta\in L^1(\Omega)$, $\beta\ge0$ almost everywhere, and all $k\in\mathbb{Q}$. Thus, we conclude that

$$\operatorname*{ess\,lim\,sup}_{t\to0+}\int_\Omega |u(t,x)-v^0(x)|\,dx \le \int_\Omega |u^0(x)-v^0(x)|\,dx$$

for all $v^0\in L^\infty(\Omega)$ and the assertion follows. ■

Now we are ready to prove both main theorems of this section.

PROOF (of Theorem 7.28): Again we will show the theorem only in the case of a half-space, where $\boldsymbol{\nu}\equiv\boldsymbol{\nu}(r)$ for all $r\in\Gamma$. We use the same notation as in the proof of Lemma 7.34.

Let u be a solution of (7.4) and let

$$\begin{aligned} H_\ell(z,w) &= \left((z-w)^2+\frac{1}{\ell^2}\right)^{1/2}-\frac{1}{\ell},\\ \mathbf{Q}_\ell(z,w) &= \int_w^z \partial_1 H_\ell(\lambda,w)\,\mathbf{f}'(\lambda)\,d\lambda \end{aligned}$$

be a boundary entropy-entropy flux pair. Lemma 7.34 now implies that there is a set E of measure zero such that the limit

$$\lim_{\substack{s\to0-\\ s\notin E}}\int_\Gamma \mathbf{F}(u(r,s),v^D(r))\cdot\boldsymbol{\nu}\,\beta(r)\,dr \tag{7.42}$$

exists for all $\beta\in L^1(\Gamma)$, $\beta\ge0$ almost everywhere, and all $v^D\in L^\infty(\Gamma)$. Now let (u,u^D) satisfy (7.4), (7.5) and therefore, due to Lemma 7.12, also (7.13), (7.15). From the obvious relation

$$2\boldsymbol{\mathcal{F}}(z,w,k)=\mathbf{F}(z,w)-\mathbf{F}(k,w)+\mathbf{F}(z,k)\,, \tag{7.43}$$

and (7.42), (7.15) we get

$$\begin{aligned} &-\operatorname*{ess\,lim}_{s\to0-}\int_\Gamma \mathbf{F}(u(r,s),k)\cdot\boldsymbol{\nu}\beta(r)\,dr\\ &\qquad\le \operatorname*{ess\,lim}_{s\to0-}\int_\Gamma \mathbf{F}(u(r,s),u^D(r))\cdot\boldsymbol{\nu}\,\beta(r)\,dr\\ &\qquad-\int_\Gamma \mathbf{F}(k,u^D(r))\cdot\boldsymbol{\nu}\,\beta(r)\,dr \end{aligned}$$

for all $\beta \in L^1(\Gamma)$, $\beta \geq 0$ almost everywhere, and all $k \in \mathbb{R}$. This and (7.13) give

$$\begin{aligned} -\int_0^T \int_\Omega |u-k|\frac{\partial \beta}{\partial t} + F_i(u,k)\frac{\partial \beta}{\partial x_i}\,dx\,dt \\ \leq \operatorname*{ess\,lim}_{s\to 0-} \int_\Gamma \mathbf{F}(u(r,s),u^D(r))\cdot \boldsymbol{\nu}\,\beta(r)\,dr \qquad (7.44) \\ -\int_\Gamma \mathbf{F}(k,u^D(r))\cdot \boldsymbol{\nu}\beta(r)\,dr \end{aligned}$$

for all $\beta \in \mathcal{D}((0,T)\times\mathbb{R}^d)$, $\beta \geq 0$ and all $k \in \mathbb{R}$.

Now let $(u_i, u_i^D, u_i^0)_{i=1,2}$ be two solutions of (7.3)–(7.6). From (7.42) we infer that there exist $\theta_{i,j} \in L^\infty(\Gamma)$, such that for all $\beta \in L^1(\Gamma)$, $\beta \geq 0$ almost everywhere,

$$\operatorname*{ess\,lim}_{s\to 0-} \int_\Gamma \mathbf{F}(u_i(r,s),u_j^D(r))\cdot \boldsymbol{\nu}\,\beta(r)\,dr = \int_\Gamma \theta_{i,j}(r)\,\beta(r)\,dr\ .$$

Note that we are using the convention $\beta(r) = \beta(r,0)$ on Γ (see also (7.44)). Let $\rho \in \mathcal{D}(\mathbb{R}^{d+1})$ be a symmetric mollifier and set

$$\beta_\varepsilon(p,\tilde p) = \beta\Big(\frac{p+\tilde p}{2}\Big)\rho_\varepsilon(p-\tilde p) \quad \forall\, p,\tilde p \in Q_T$$

for given $\beta \in \mathcal{D}((0,T)\times\mathbb{R}^d)$, $\beta \geq 0$. Hold $\tilde p \in Q_T$ fixed and replace in (7.44) (u,u^D) by (u_1,u_1^D), k by $u_2(\tilde p)$ and β by $\beta_\varepsilon(\cdot,\tilde p)$. After integration over $\tilde p \in Q_T$ we get

$$\begin{aligned} -\frac{1}{2}\int_{Q_T}\int_{Q_T} \Big\{ |u_1(p)-u_2(\tilde p)|\frac{\partial \beta}{\partial t}\Big(\frac{p+\tilde p}{2}\Big) \\ + F_i(u_1(p),u_2(\tilde p))\frac{\partial \beta}{\partial x_i}\Big(\frac{p+\tilde p}{2}\Big)\Big\}\rho_\varepsilon(p-\tilde p)\,dp\,d\tilde p \\ -\int_{Q_T}\int_{Q_T}\Big\{|u_1(p)-u_2(\tilde p)|\frac{\partial \rho_\varepsilon}{\partial t}(p-\tilde p) \\ + F_i(u_1(p),u_2(\tilde p))\frac{\partial \rho_\varepsilon}{\partial x_i}(p-\tilde p)\Big\}\beta\Big(\frac{p+\tilde p}{2}\Big)\,dp\,d\tilde p \\ \leq \int_{Q_T}\int_\Gamma \theta_{1,1}(r)\,\beta\Big(\frac{r+\tilde p}{2}\Big)\rho_\varepsilon(r-\tilde p)\,dr\,d\tilde p \\ -\int_{Q_T}\int_\Gamma \mathbf{F}(u_2(\tilde p),u_1^D(r))\cdot\boldsymbol{\nu}\,\beta\Big(\frac{r+\tilde p}{2}\Big)\rho_\varepsilon(r-\tilde p)\,dr\,d\tilde p\ . \end{aligned}$$

Changing the role of u_1 and u_2, and of p and $\tilde{p}$, we get

$$\begin{aligned}
-\frac{1}{2}\int_{Q_T}\int_{Q_T} &\Big\{|u_2(\tilde{p}) - u_1(p)|\frac{\partial\beta}{\partial t}\Big(\frac{p+\tilde{p}}{2}\Big) \\
&\quad + F_i(u_2(\tilde{p}), u_1(p))\frac{\partial\beta}{\partial x_i}\Big(\frac{p+\tilde{p}}{2}\Big)\Big\}\rho_\varepsilon(p-\tilde{p})\,d\tilde{p}\,dp \\
&+ \int_{Q_T}\int_{Q_T}\Big\{|u_2(\tilde{p}) - u_1(p)|\frac{\partial\rho_\varepsilon}{\partial t}(p-\tilde{p}) \\
&\quad + F_i(u_2(\tilde{p}), u_1(p))\frac{\partial\rho_\varepsilon}{\partial x_i}(p-\tilde{p})\Big\}\beta\Big(\frac{p+\tilde{p}}{2}\Big)\,d\tilde{p}\,dp \\
\le \int_{Q_T}\int_{\Gamma} &\theta_{2,2}(\tilde{r})\,\beta\Big(\frac{p+\tilde{r}}{2}\Big)\rho_\varepsilon(p-\tilde{r})\,d\tilde{r}\,dp \\
&- \int_{Q_T}\int_{\Gamma}\mathbf{F}(u_1(p), u_2^D(\tilde{r}))\cdot\boldsymbol{\nu}\,\beta\Big(\frac{p+\tilde{r}}{2}\Big)\rho_\varepsilon(p-\tilde{r})\,d\tilde{r}\,dp\ .
\end{aligned}$$

Adding these two inequalities we note that the integrals obtaining $\frac{\partial\rho_\varepsilon}{\partial t}$ and $\frac{\partial\rho_\varepsilon}{\partial x_i}$ vanish. Now, we are about to show that

$$\begin{aligned}
\lim_{\varepsilon\to 0+}\int_{Q_T}\int_{Q_T} &\Big\{|u_1(p) - u_2(\tilde{p})|\frac{\partial\beta}{\partial t}\Big(\frac{p+\tilde{p}}{2}\Big) \\
&+ F_i(u_1(p), u_2(\tilde{p}))\frac{\partial\beta}{\partial x_i}\Big(\frac{p+\tilde{p}}{2}\Big)\Big\}\rho_\varepsilon(p-\tilde{p})\,dp\,d\tilde{p} \\
&= \int_{Q_T}|u_1(p) - u_2(p)|\frac{\partial\beta}{\partial t}(p) + F_i(u_1(p), u_2(p))\frac{\partial\beta}{\partial x_i}(p)\,dp\ .
\end{aligned} \tag{7.45}$$

Denote

$$\begin{aligned}
\mathcal{T}_1 \equiv \int_{Q_T}\int_{Q_T} &\Big\{|u_1(p) - u_2(\tilde{p})|\frac{\partial\beta}{\partial t}\Big(\frac{p+\tilde{p}}{2}\Big) \\
&+ F_i(u_1(p), u_2(\tilde{p}))\frac{\partial\beta}{\partial x_i}\Big(\frac{p+\tilde{p}}{2}\Big)\Big\}\rho_\varepsilon(p-\tilde{p})\,dp\,d\tilde{p} \\
&- \int_{Q_T}|u_1(p) - u_2(p)|\frac{\partial\beta}{\partial t}(p) + F_i(u_1(p), u_2(p))\frac{\partial\beta}{\partial x_i}(p)\,dp\,.
\end{aligned}$$

Further, denote

$$\Delta p \equiv p - \tilde{p}$$

and

$$Q^{\Delta p} \equiv \{p \in Q_T : p - \Delta p \in Q_T\}\,.$$

Then

$$
\begin{aligned}
\mathcal{T}_1 = &\int_{\mathbb{R}^{d+1}} \rho_\varepsilon(\Delta p) \int_{Q^{\Delta p}} |u_1(p) - u_2(p - \Delta p)| \\
&\qquad\qquad \times \left\{ \frac{\partial \beta}{\partial t}\Big(p - \frac{1}{2}\Delta p\Big) - \frac{\partial \beta}{\partial t}(p) \right\} dp\, d\Delta p \\
&+ \int_{\mathbb{R}^{d+1}} \rho_\varepsilon(\Delta p) \int_{Q^{\Delta p}} F_i(u_1(p), u_2(p - \Delta p)) \\
&\qquad\qquad \times \left\{ \frac{\partial \beta}{\partial x_i}\Big(p - \frac{1}{2}\Delta p\Big) - \frac{\partial \beta}{\partial x_i}(p) \right\} dp\, d\Delta p \\
&+ \int_{\mathbb{R}^{d+1}} \rho_\varepsilon(\Delta p) \int_{Q^{\Delta p}} \Big\{ |u_1(p) - u_2(p - \Delta p)| \\
&\qquad\qquad - |u_1(p) - u_2(p)| \Big\} \frac{\partial \beta}{\partial t}(p)\, dp\, d\Delta p \\
&+ \int_{\mathbb{R}^{d+1}} \rho_\varepsilon(\Delta p) \int_{Q^{\Delta p}} \Big\{ F_i(u_1(p), u_2(p - \Delta p)) \\
&\qquad\qquad - F_i(u_1(p), u_2(p)) \Big\} \frac{\partial \beta}{\partial x_i}(p)\, dp\, d\Delta p \\
&+ \int_{Q_T} \left\{ |u_1(p) - u_2(p)| \frac{\partial \beta}{\partial t}(p) + F_i(u_1(p), u_2(p)) \frac{\partial \beta}{\partial x_i}(p) \right\} \\
&\qquad\qquad \times \left\{ \int_{Q_T} \rho_\varepsilon(p - \tilde{p})\, d\tilde{p} - 1 \right\} dp\,.
\end{aligned}
$$

Therefore we get the estimate

$$
\begin{aligned}
|\mathcal{T}_1| \leq &\operatorname*{ess\,sup}_{(p,\tilde{p}) \in Q_T^2} |u_1(p) - u_2(\tilde{p})| \sup_{|\Delta p| < \varepsilon} \int_Q \left| \frac{\partial \beta}{\partial t}(p - \frac{1}{2}\Delta p) - \frac{\partial \beta}{\partial t}(p) \right| \\
&+ M \left| \nabla \beta(p - \frac{1}{2}\Delta p) - \nabla \beta(p) \right| dp \\
&+ \sup_{|\Delta p| < \varepsilon} \int_{Q^{\Delta p}} |u_2(p - \Delta p) - u_2(p)| \left\{ \left| \frac{\partial \beta}{\partial t}(p) \right| + M |\nabla \beta(p)| \right\} dp \\
&+ \operatorname*{ess\,sup}_{Q_T} |u_1 - u_2| \int_{Q_\varepsilon} \left| \frac{\partial \beta}{\partial t}(p) \right| + M |\nabla \beta|\, dx\, dt\ ,
\end{aligned}
$$

where

$$
Q_\varepsilon = \{(r, s) \in Q_T : -\varepsilon < s < 0\}\,.
$$

This gives (7.45).

The same argument implies that

$$\begin{aligned}
&\lim_{\varepsilon\to 0+} \int_{Q_T}\int_{Q_T} \Big\{ |u_2(\tilde p) - u_1(p)| \frac{\partial\beta}{\partial t}\Big(\frac{p+\tilde p}{2}\Big) \\
&\quad + F_i(u_2(\tilde p), u_1(p)) \frac{\partial\beta}{\partial x_i}\Big(\frac{p+\tilde p}{2}\Big)\Big\} \rho_\varepsilon(p-\tilde p)\, d\tilde p\, dp \\
&\quad = \int_{Q_T} |u_2(p) - u_1(p)| \frac{\partial\beta}{\partial t}(p) \\
&\quad + F_i(u_2(p), u_1(p)) \frac{\partial\beta}{\partial x_i}(p)\, dp\ .
\end{aligned} \tag{7.46}$$

Next, we have that

$$\lim_{\varepsilon\to 0+} \int_{Q_T} \beta\Big(\frac{r+\tilde p}{2}\Big)\, \rho_\varepsilon(r-\tilde p)\, d\tilde p = \frac{1}{2}\, \beta(r)\,,$$

uniformly in $r \in \Gamma$. Note that the factor $1/2$ on the right-hand side occurs since $r \in \Gamma$ are boundary points of a half-space.

Using this result, we can calculate the following two limits:

$$\begin{aligned}
\lim_{\varepsilon\to 0+} \int_{Q_T}\int_\Gamma \theta_{1,1}(r)\, \beta\Big(\frac{r+\tilde p}{2}\Big)\, \rho_\varepsilon(r-\tilde p)\, dr\, d\tilde p& \\
= \frac{1}{2}\int_\Gamma \theta_{1,1}(r)\, \beta(r)\, dr&
\end{aligned} \tag{7.47}$$

and

$$\begin{aligned}
\lim_{\varepsilon\to 0+} \int_{Q_T}\int_\Gamma \theta_{2,2}(\tilde r)\, \beta\Big(\frac{p+\tilde r}{2}\Big)\, \rho_\varepsilon(p-\tilde r)\, d\tilde r\, dp& \\
= \frac{1}{2}\int_\Gamma \theta_{2,2}(r)\, \beta(r)\, dr&\ .
\end{aligned} \tag{7.48}$$

Further, we show that

$$\begin{aligned}
\lim_{\varepsilon\to 0+} \int_{Q_T}\int_\Gamma \mathbf{F}(u_1(p), u_2^D(\tilde r))\cdot\boldsymbol{\nu}\, \beta\Big(\frac{p+\tilde r}{2}\Big)\, \rho_\varepsilon(p-\tilde r)\, d\tilde r\, dp& \\
= \frac{1}{2}\int_\Gamma \theta_{1,2}(r)\, \beta(r)\, dr&\ .
\end{aligned} \tag{7.49}$$

Indeed, we can write

$$\begin{aligned}
\mathcal{T}_2 \equiv \int_{Q_T} \int_\Gamma & \mathbf{F}(u_1(p), u_2^D(\tilde{r})) \cdot \boldsymbol{\nu}\, \beta\Big(\frac{p+\tilde{r}}{2}\Big)\, \rho_\varepsilon(p-\tilde{r})\, d\tilde{r}\, dp \\
& - \frac{1}{2} \int_\Gamma \theta_{1,2}(r)\, \beta(r)\, dr \\
= \int_{-\infty}^0 \int_{\mathbb{R}^d} & \rho_\varepsilon(s, \Delta r) \int_\Gamma \mathbf{F}(u_1(r,s), u_2^D(r-\Delta r)) \cdot \boldsymbol{\nu} \\
& \times \Big\{ \beta\Big(r - \frac{1}{2}\Delta r, \frac{1}{2}s\Big) - \beta(r,0) \Big\}\, dr\, d\Delta r\, ds \\
+ \int_{-\infty}^0 \int_{\mathbb{R}^d} & \rho_\varepsilon(s, \Delta r) \int_\Gamma \{ \mathbf{F}(u_1(r,s), u_2^D(r-\Delta r)) \\
& - \mathbf{F}(u_1(r,s), u_2^D(r)) \} \cdot \boldsymbol{\nu}\, \beta(r,0)\, dr\, d\Delta r\, ds \\
+ \int_{-\infty}^0 \int_{\mathbb{R}^d} & \rho_\varepsilon(s, \Delta r)\, d\Delta r \Big\{ \int_\Gamma \mathbf{F}(u_1(r,s), u_2^D(r)) \cdot \boldsymbol{\nu} \\
& \times \beta(r,0)\, dr - \int_\Gamma \theta_{1,2}(r) \beta(r)\, dr \Big\}\, ds
\end{aligned}$$

and thus we get that $|\mathcal{T}_2|$ is less than or equal to

$$\begin{aligned}
& \frac{M}{2} \operatorname*{ess\,sup}_{p \in Q_T, r \in \Gamma} |u_1(p) - u_2^D(r)| \sup_{|\Delta r| < \varepsilon} \int_\Gamma \Big|\beta(r - \frac{1}{2}\Delta r) - \beta(r)\Big|\, dr \\
& + \frac{M}{2} \sup_{|\Delta r| < \varepsilon} \int_\Gamma |u_2^D(r-\Delta r) - u_2^D(r)|\, \beta(r)\, dr \\
& + \sup_{s<0} \int_{\mathbb{R}^d} \rho(s, \Delta r)\, d\Delta r \frac{1}{\varepsilon} \int_{-\varepsilon}^0 \Big| \int_\Gamma \mathbf{F}(u_1(r,s), u_2^D(r)) \cdot \boldsymbol{\nu} \\
& \qquad \times \beta(r,0)\, dr - \int_\Gamma \theta_{1,2}(r)\, \beta(r)\, dr \Big|\, ds
\end{aligned}$$

which proves (7.49).

Similarly, we get

$$\begin{aligned}
\lim_{\varepsilon \to 0+} \int_{Q_T} \int_\Gamma \mathbf{F}(u_2(\tilde{p}), u_1^D(r)) \cdot \boldsymbol{\nu}\, \beta\Big(\frac{r+\tilde{p}}{2}\Big) \rho(r-\tilde{p})\, dr\, d\tilde{p} & \\
= \frac{1}{2} \int_\Gamma \theta_{2,1}(r)\, \beta(r)\, dr\ . &
\end{aligned} \tag{7.50}$$

Altogether from (7.45)–(7.50) we obtain

$$-\int_0^T \int_\Omega |u_1 - u_2| \frac{\partial \beta}{\partial t} + F_i(u_1, u_2) \frac{\partial \beta}{\partial x_i} \, dx \, dt \\ \leq \frac{1}{2} \int_\Gamma \big(\theta_{1,1}(r) - \theta_{2,1}(r) + \theta_{2,2}(r) - \theta_{1,2}(r)\big) \beta(r) \, dr \ .$$

If one discusses the cases, one easily sees that for all $z_1, z_2, w_1, w_2 \in \mathbb{R}$ the inequality

$$\left| \sum_{i,j=1}^{2} (-1)^{i+j} \mathbf{F}(z_i, w_j) \cdot \boldsymbol{\nu} \right| \leq 2 \operatorname{diam} \big(\mathbf{f} \cdot \boldsymbol{\nu}, \mathcal{I}[w_1, w_2]\big)$$

holds and thus we have for all $s \in (-\infty, 0)$

$$\left| \sum_{i,j=1}^{2} (-1)^{i+j} \int_\Gamma \mathbf{F}(u_i(r,s), u_j^D(r)) \, \boldsymbol{\nu} \, \beta(r) \, dr \right| \\ \leq 2 \int_\Gamma \operatorname{diam} \big(\mathbf{f} \cdot \boldsymbol{\nu}, \mathcal{I}[u_1^D, u_2^D]\big) \, \beta \, dr \, .$$

Letting $s \to 0-$ we get

$$\left| \sum_{i,j=1}^{2} (-1)^{i+j} \int_\Gamma \theta_{i,j}(r) \, \beta(r) \, dr \right| \leq 2 \int_\Gamma \operatorname{diam} \big(\mathbf{f} \cdot \boldsymbol{\nu}, \mathcal{I}[u_1^D, u_2^D]\big) \, \beta \, dr \, ,$$

and therefore we have

$$-\int_0^T \int_\Omega |u_1 - u_2| \frac{\partial \beta}{\partial t} + F_i(u_1, u_2) \frac{\partial \beta}{\partial x_i} \, dx \, dt \\ \leq \int_\Gamma \operatorname{diam} \big(\mathbf{f} \cdot \boldsymbol{\nu}, \mathcal{I}[u_1^D, u_2^D]\big) \, \beta(r) \, dr$$

for all $\beta \in \mathcal{D}((0,T) \times \mathbb{R}^d)$, $\beta \geq 0$. Assertion (7.29) is obtained by this inequality and (7.6). Note that the integral term with $\beta(0)$ is obtained by the same procedure as in Lemma 7.12.

If we now restrict ourselves in (7.29) to test functions α which are functions of t only and if we use

$$\operatorname{diam} \big(\mathbf{f}(r) \cdot \boldsymbol{\nu}, \mathcal{I}[u_1^D(r), u_2^D(r)]\big) \leq M |u_1^D(r) - u_2^D(r)| \quad \forall r \in \Gamma \, ,$$

we obtain

$$
\begin{aligned}
-\int_0^T \int_\Omega & |u_1(t,x) - u_2(t,x)| \, dx \, \alpha'(t) \, dt \\
&\leq \int_\Omega |u_1^0(x) - u_2^0(x)| \, dx \, \alpha(0) \\
&\quad + M \int_0^T \int_{\partial\Omega} |u_1^D(t,y) - u_2^D(t,y)| \, dy \, \alpha(t) \, dt
\end{aligned}
$$

for all $\alpha \in \mathcal{D}((-\infty, T))$, $\alpha \geq 0$. This obviously gives (7.30), if one uses the sequence of test functions $\alpha_n(0) = \alpha_n(t + 1/n) = 0, \alpha_n(s) \equiv 1$ on $[1/n, t]$, α_n piecewise linear, appropriately mollified. ∎

PROOF (of Theorem 7.31):

The implication '(7.32) $\Rightarrow$ (7.4)–(7.6)': Let $(\eta, \mathbf{q})$ be an entropy-entropy flux pair. The function u is bounded and therefore we can assume that η attains a minimum at some $k_0 \in \mathbb{R}$. Then of course

$$
\begin{aligned}
H(z,k) &= \eta(z - k + k_0) - \eta(k_0) \, , \\
\mathbf{Q}(z,k) &= \mathbf{q}(z - k + k_0) - \mathbf{q}(k_0) \, ,
\end{aligned}
$$

is a boundary entropy-entropy flux pair and thus we get

$$
\begin{aligned}
-\int_0^T \int_\Omega & \eta(u) \frac{\partial \gamma}{\partial t} + q_i(u) \frac{\partial \gamma}{\partial x_i} \, dx \, dt \\
&= -\int_0^T \int_\Omega H(u,k) \frac{\partial \gamma}{\partial t} + Q_i(u,k) \frac{\partial \gamma}{\partial x_i} \, dx \, dt \leq 0
\end{aligned}
$$

for all $\gamma \in \mathcal{D}(Q_T)$, $\gamma \geq 0$. Therefore u satisfies (7.4). Due to Lemma 7.34, (u, u^D) satisfy (7.5). In order to show (7.6) we use

$$
\begin{aligned}
H_\ell(z,k) &\equiv \left((z-k)^2 + \frac{1}{\ell^2} \right)^{1/2} - \frac{1}{\ell} \, , \\
\mathbf{Q}_\ell(z,k) &= \int_k^z \partial_1 H_\ell(\lambda, k) \, \mathbf{f}'(\lambda) \, d\lambda
\end{aligned}
$$

in (7.32) and obtain

$$
-\int_0^T \int_\Omega H_\ell(u,k) \frac{\partial \beta}{\partial t} + Q_{\ell i}(u,k) \frac{\partial \beta}{\partial x_i} \, dx \, dt \leq \int_\Omega H_\ell(u^0, k) \, \beta(0) \, dx
$$

for all $\beta \in \mathcal{D}((-\infty, T) \times \Omega)$, $\beta \geq 0$ and all $k \in \mathbb{R}$. Letting $\ell \to \infty$

we obtain

$$-\int_0^T \int_\Omega |u-k|\frac{\partial \beta}{\partial t} + F_i(u,k)\frac{\partial \beta}{\partial x_i}\, dx\, dt \le \int_\Omega |u^0 - k|\, \beta(0)\, dx\,.$$

This enables us to use Lemma 7.41, which gives that (u, u^0) satisfy (7.6).

The implication '(7.4)–(7.6) $\Rightarrow$ (7.32)': Let (u, u^D, u^0) satisfy (7.4)–(7.6). Let $(H, \mathbf{Q})$ be a boundary entropy-entropy flux pair and $k \in \mathbb{R}$. We put

$$\eta(z) \equiv H(z,k)\ , \quad \mathbf{q}(z) \equiv \mathbf{Q}(z,k)\ .$$

Let us define (notice that $\eta(k) = 0$)

$$\widetilde{H}(z,w) \equiv \begin{cases} \left.\begin{array}{ll} \eta(z)-\eta(w) & \text{if } z \le w \\ 0 & \text{if } w \le z \le k \\ \eta(z) & \text{if } k \le z \end{array}\right\} & \text{and } w \le k\,, \\ \left.\begin{array}{ll} \eta(z) & \text{if } z \le k \\ 0 & \text{if } k \le z \le w \\ \eta(z)-\eta(w) & \text{if } w \le z \end{array}\right\} & \text{and } w \ge k \end{cases}$$

and

$$\widetilde{\mathbf{Q}}(z,w) \equiv \begin{cases} \left.\begin{array}{ll} \mathbf{q}(z)-\mathbf{q}(w) & \text{if } z \le w \\ \mathbf{0} & \text{if } w \le z \le k \\ \mathbf{q}(z) & \text{if } k \le z \end{array}\right\} & \text{and } w \le k\,, \\ \left.\begin{array}{ll} \mathbf{q}(z) & \text{if } z \le k \\ \mathbf{0} & \text{if } k \le z \le w \\ \mathbf{q}(z)-\mathbf{q}(w) & \text{if } w \le z \end{array}\right\} & \text{and } w \ge k\,. \end{cases}$$

Thus $(\widetilde{H}, \widetilde{\mathbf{Q}}) \in C(\mathbb{R}^2) \times C(\mathbb{R}^2)^d$ and $(\widetilde{H}, \widetilde{\mathbf{Q}})$ can be locally uniformly approximated by $(\widetilde{H}_\ell, \widetilde{\mathbf{Q}}_\ell)$ which are defined as

$$\widetilde{H}_\ell(z,w) \equiv \int_{\mathbb{R}} H_\ell(\lambda, w)\, \rho_{1/\ell}(z-\lambda)\, d\lambda\,,$$

$$\widetilde{\mathbf{Q}}_\ell(z,w) \equiv \int_w^z \partial_1\, \widetilde{H}_\ell(z,w)\, \mathbf{f}'(\lambda)\, d\lambda\,,$$

where $\rho_{1/\ell}$ is a usual mollifier, and $H_\ell(z,w)$ is defined as

$$\begin{cases} \left.\begin{array}{ll} \eta(z) - \eta(w - \frac{1}{\ell}) & \text{if } z \le w - \frac{1}{\ell} \\ 0 & \text{if } w - \frac{1}{\ell} \le z \le k + \frac{1}{\ell} \\ \eta(z) - \eta(k + \frac{1}{\ell}) & \text{if } k + \frac{1}{\ell} \le z \end{array}\right\} \text{ and } w \le k, \\ \left.\begin{array}{ll} \eta(z) - \eta(k - \frac{1}{\ell}) & \text{if } z \le k - \frac{1}{\ell} \\ 0 & \text{if } k - \frac{1}{\ell} \le z \le w + \frac{1}{\ell} \\ \eta(z) - \eta(w + \frac{1}{\ell}) & \text{if } w + \frac{1}{\ell} \le z \end{array}\right\} \text{ and } w \ge k. \end{cases}$$

It follows from (7.5) that

$$\operatorname*{ess\,lim}_{s\to 0-} \int_\Gamma \widetilde{\mathbf{Q}}_\ell(u(r + s\,\boldsymbol{\nu}(r)), u^D(r)) \cdot \boldsymbol{\nu}(r)\,\beta(r)\,dr \ge 0$$

for all $\beta \in L^1(\Gamma)$, $\beta \ge 0$ almost everywhere. For $\ell \to \infty$ we have

$$\operatorname*{ess\,lim}_{s\to 0-} \int_\Gamma \widetilde{\mathbf{Q}}(u(r + s\,\boldsymbol{\nu}(r)), u^D(r)) \cdot \boldsymbol{\nu}(r)\,\beta(r)\,dr \ge 0\ .$$

Discussing the cases in the definition of $\widetilde{\mathbf{Q}}$ and using the properties of η we obtain

$$|\widetilde{\mathbf{Q}}(z,w) - \mathbf{q}(z)| \le M\eta(w)$$

and therefore for all $\beta \in C^1(\Gamma)$, $\beta \ge 0$,

$$\operatorname*{ess\,lim\,inf}_{s\to 0-} \int_\Gamma \mathbf{q}(u(r + s\,\boldsymbol{\nu}(r))) \cdot \boldsymbol{\nu}(r)\,\beta(r)\,dr \\ \ge -M \int_\Gamma \eta(u^D(r))\,\beta(r)\,dr\ .$$

Because of (7.6), we have for all $\beta \in L^1(\Omega)$, $\beta \ge 0$ almost everywhere,

$$\operatorname*{ess\,lim}_{t\to 0+} \int_\Omega \eta(u(t,x))\,\beta(x)\,dx = \int_\Omega \eta(u^0(x))\,\beta(x)\,dx\,.$$

It follows from (7.4) that

$$-\int_0^T \int_\Omega \eta(u)\frac{\partial \gamma}{\partial t} + q_i(u)\frac{\partial \gamma}{\partial x_i}\,dx\,dt \le 0$$

for all $\gamma \in \mathcal{D}(Q_T)$, $\gamma \ge 0$. Altogether we have (cf. again proof of

Lemma 7.12)

$$-\int_0^T \int_\Omega \eta(u)\frac{\partial \beta}{\partial t} + q_i(u)\frac{\partial \beta}{\partial x_i}\,dx\,dt \le \int_\Omega \eta(u^0)\,\beta(0)\,dx + M\int_\Gamma \eta(u^D)\,\beta(r)\,dr$$

for all $\beta \in \mathcal{D}((-\infty,T)\times\mathbb{R}^d)$, $\beta \geq 0$, which is (7.32). ∎

2.8 Existence in bounded domains via parabolic approximation

In order to show the existence of the solution $u \in L^\infty(Q_T)$ satisfying (7.4)–(7.6), we again use the parabolic perturbation

$$\begin{aligned} \frac{\partial u^\varepsilon}{\partial t} + \operatorname{div}\mathbf{f}(u^\varepsilon) - \varepsilon\,\Delta u^\varepsilon = 0 &\quad \text{in } Q_T\,, \\ u^\varepsilon(0,\cdot) = u^{\varepsilon 0} &\quad \text{in } \Omega\,, \\ u^\varepsilon = u^{\varepsilon D} &\quad \text{on } \Gamma\,. \end{aligned} \tag{8.1}$$

For smooth $u^{\varepsilon 0}$, $u^{\varepsilon D}$, satisfying compatibility conditions on $\overline{\Gamma}\cap\overline{\Omega}$, and $\varepsilon > 0$ fixed, there exists a unique smooth solution u^ε of (8.1) (to prove this, we can use the same method as in the proof of Lemma 2.3, namely, apply Theorems 2.7 and 2.10 from the Appendix and then proceed to the nonlinear case using $v = ue^{-\lambda t}$). The main goal of this section is to show that if $u^{\varepsilon D}$, $u^{\varepsilon 0}$ approximate $u^D \in L^\infty(\Gamma)$, $u^0 \in L^\infty(\Omega)$, respectively, then the sequence of solutions u^ε of (8.1) approaches a function $u \in L^\infty(Q_T)$, satisfying (7.4)–(7.6).

Let us start with the following construction: for $\delta > 0$ sufficiently small we define

$$s(x) \equiv \begin{cases} \min(\operatorname{dist}(x,\partial\Omega),\delta) & \text{for } x \in \Omega\,, \\ -\min(\operatorname{dist}(x,\partial\Omega),\delta) & \text{for } x \in \mathbb{R}^d\setminus\Omega. \end{cases}$$

This function is Lipschitz continuous in $\mathbb{R}^d$ and smooth on the closure of $\{x \in \mathbb{R}^d : |s(x)| < \delta\}$. For $\varepsilon > 0$ we define ξ_ε by

$$\xi_\varepsilon(x) \equiv 1 - \exp\Big(-\frac{M+\varepsilon L}{\varepsilon}s(x)\Big)\,, \tag{8.2}$$

where $L \equiv \sup_{0<s(x)<\delta}|\Delta s(x)|$ and $M > 0$. This function satisfies the weak differential inequality

$$M\int_\Omega |\nabla\xi_\varepsilon|\,\beta \le \varepsilon\int_\Omega \nabla\xi_\varepsilon\,\nabla\beta + (M+L\,\varepsilon)\int_{\partial\Omega}\beta\,dr \tag{8.3}$$

for all $\beta \in \mathcal{D}(\mathbb{R}^d)$, $\beta \geq 0$. Indeed, define for $\kappa > 0$

$$w_\kappa(x) = 1 - \exp\Big(-\frac{1}{\kappa} s(x)\Big) .$$

Since for $x \in \overline{\Omega}$,

$$\begin{aligned}\nabla w_\kappa(x) &= \frac{1}{\kappa} \exp\Big(-\frac{1}{\kappa} s(x)\Big) \nabla s(x) , \\ |\nabla s(x)| &= \begin{cases} 1 & \text{for } 0 \leq s(x) < \delta , \\ 0 & \text{for } s(x) = \delta \end{cases}\end{aligned}$$

and

$$\Delta w_\kappa = -\frac{1}{\kappa^2} \exp\Big(-\frac{1}{\kappa} s(x)\Big) + \frac{1}{\kappa} \exp\Big(-\frac{1}{\kappa} s(x)\Big) \Delta s(x) ,$$

we obtain $w_\kappa \in W^{1,\infty}(\Omega) \cap C^2(\overline{\{0 < s < \delta\}})$. Moreover, for $\beta \in \mathcal{D}(\mathbb{R}^d)$, $\beta \geq 0$ we have

$$\begin{aligned}\int_\Omega \nabla w_\kappa \nabla \beta \, dx &= \int_{\{0<s<\delta\}} \nabla w_\kappa \nabla \beta \, dx \\ &= - \int_{\{0<s<\delta\}} \Delta w_\kappa \, \beta \, dx - \int_{\{s=0+\}} \nabla w_\kappa \nabla s \, \beta \, dr \\ &\qquad + \int_{\{s=\delta-\}} \nabla w_\kappa \nabla s \, \beta \, dr \\ &= \frac{1}{\kappa^2} \int_{\{0<s<\delta\}} \exp\Big(-\frac{1}{\kappa} s(x)\Big) \beta \, dx \\ &\qquad - \frac{1}{\kappa} \int_{\{0<s<\delta\}} \exp\Big(-\frac{1}{\kappa} s(x)\Big) \Delta s \, \beta \, dx \\ &\qquad - \frac{1}{\kappa} \int_{\{s=0+\}} \beta \, dr + \frac{1}{\kappa} \exp\Big(-\frac{1}{\kappa}\delta\Big) \int_{\{s=\delta-\}} \beta \, dr \\ &\geq \frac{1}{\kappa^2} \int_{\{0<s<\delta\}} \exp\Big(-\frac{1}{\kappa} s(x)\Big) \beta(x) \, dx \\ &\qquad - \frac{L}{\kappa} \int_{\{0<s<\delta\}} \exp\Big(-\frac{1}{\kappa} s(x)\Big) \beta(x) \, dx \\ &\qquad - \frac{1}{\kappa} \int_{\{s=0+\}} \beta \, dr\end{aligned}$$

and therefore we obtain

$$\kappa \int_\Omega \nabla w_\kappa \nabla \beta + \int_{\partial\Omega} \beta \, dr \geq (1 - \kappa L) \int_\Omega |\nabla w_\kappa| \, \beta \, dx$$

and if $1 - \kappa L > 0$,

$$M \int_\Omega |\nabla w_\kappa|\, \beta\, dx \le \frac{M\kappa}{(1-\kappa L)} \int_\Omega \nabla w_\kappa \nabla \beta + \frac{M}{1-\kappa L} \int_{\partial\Omega} \beta\, dr$$

For $\kappa = \frac{\varepsilon}{M+\varepsilon L}$ we obtain (8.3).

The following theorem investigates the properties of the solution u^ε to the problem (8.1) for fixed $\varepsilon > 0$. For simplicity we drop the superscript ε in u^ε, $u^{\varepsilon 0}$, $u^{\varepsilon D}$.

Theorem 8.4 *Let u, u_1, u_2 be solutions of (8.1) corresponding to the smooth initial data u^0, u_1^0, u_2^0, and boundary data u^D, u_1^D, u_2^D, respectively. Let M be a constant of Lipschitz continuity of $\mathbf{f} \in C^1(\mathbb{R})^d$, restricted to the ball with radius defined as the maximum of $\{\|u^0\|_\infty, \|u_i^0\|_\infty, \|u^D\|_\infty, \|u_i^D\|_\infty\}$. Let ξ_ε, L be defined by (8.2). Then we have:*

- $$\begin{aligned} &-\int_0^T \int_\Omega \Big\{ H(u,k)\frac{\partial \beta}{\partial t} + Q_i(u,k)\frac{\partial \beta}{\partial x_i} + \varepsilon\, H(u,k)\, \Delta\beta \Big\} \xi_\varepsilon\, dx\, dt \\ &\le \int_\Omega H(u^0,k)\, \beta(0)\, \xi_\varepsilon\, dx + (M + L\,\varepsilon) \int_\Gamma H(u^D,k)\, \beta\, dr \\ &\quad + 2\varepsilon \int_0^T \int_\Omega H(u,k) \frac{\partial \beta}{\partial x_i} \frac{\partial \xi_\varepsilon}{\partial x_i}\, dx\, dt \end{aligned} \tag{8.5}$$

  holds for all $\beta \in \mathcal{D}((-\infty,T) \times \mathbb{R}^d)$, $\beta \ge 0$, all $k \in \mathbb{R}$ and all boundary entropy-entropy flux pairs $(H, \mathbf{Q})$;

- $$\begin{aligned} \int_\Omega |u_1(t) - u_2(t)|\, \xi_\varepsilon\, dx &\le \int_\Omega |u_1^0 - u_2^0|\, \xi_\varepsilon\, dx \\ &\quad + (M + L\,\varepsilon) \int_\Gamma |u_1^D - u_2^D|\, dr \end{aligned} \tag{8.6}$$

  holds for all $t \in (0,T)$;

- $$\begin{aligned} &\sup_{Q_T} u \le \max\,(\sup_\Omega u^0,\, \sup_\Gamma u^D)\,, \\ &\inf_{Q_T} u \ge \min\,(\inf_\Omega u^0,\, \inf_\Gamma u^D)\,. \end{aligned} \tag{8.7}$$

In particular,

$$\sup_{Q_T} |u| \le \max\,(\sup_\Omega |u^0|,\, \sup_\Gamma |u^D|)\,. \tag{8.8}$$

- *Moreover, if* $\mathbf{f} \in C^2(\mathbb{R})^d$, *we have*

$$\sup_{t\in(0,T)} \int_\Omega \left|\frac{\partial u}{\partial t}(t)\right| + |\nabla u(t)|\, dx \le \lambda\,, \tag{8.9}$$

where $\lambda = \lambda(\|u^0\|_\Omega, \|u^D\|_\Gamma, T, \Omega, \mathbf{f})$ *does not depend on* $\varepsilon \in (0,1)$. *Here, we use the notation*

$$\|u^0\|_\Omega \equiv \int_\Omega |\Delta u^0| + |\nabla u^0| + |u^0|\, dx\,,$$

$$\begin{aligned} \|u^D\|_\Gamma = \sup_{Q_T}\{|\Delta u^D| + \left|\frac{\partial u^D}{\partial t}\right| + |\nabla u^D| + |u^D|\} \\ + \int_0^T \int_\Omega \left|\nabla^2 \frac{\partial u^D}{\partial t}\right| + |\nabla^3 u^D| + \left|\frac{\partial^2 u^D}{\partial t^2}\right| \\ + \left|\nabla \frac{\partial u^D}{\partial t}\right| + |\nabla^2 u^D|\, dx\, dt\,, \end{aligned}$$

where u^D *is identified with its smooth extension to* $\overline{Q_T}$.

PROOF : • Let $(H, \mathbf{Q})$ be a boundary entropy-entropy flux pair and let $k \in \mathbb{R}$ be fixed. We again use the abbreviation

$$\eta(z) \equiv H(z,k)\,, \qquad \mathbf{q}(z) \equiv \mathbf{Q}(z,k)\,.$$

Multiplying (8.1) by $\eta'(u)$ and using the same argument as in Section 2.3, we obtain

$$\frac{\partial \eta(u)}{\partial t} + \operatorname{div} \mathbf{q}(u) \le \varepsilon\, \Delta \eta(u)\,. \tag{8.10}$$

Now we multiply (8.10) by $\xi_\varepsilon\, \beta$, integrate over Q_T, and after partial integration we obtain

$$\begin{aligned} &- \int_0^T \int_\Omega \Big\{ \eta(u) \frac{\partial \beta}{\partial t} + q_i(u) \frac{\partial \beta}{\partial x_i} + \varepsilon\, \eta(u)\, \Delta\beta \Big\} \xi_\varepsilon\, dx\, dt \\ &\le \int_\Omega \eta(u^0)\, \beta(0)\, \xi_\varepsilon\, dx + \int_0^T \int_\Omega q_i(u)\, \beta \frac{\partial \xi_\varepsilon}{\partial x_i}\, dx\, dt \\ &\quad - \varepsilon \int_0^T \int_\Omega \frac{\partial\big(\eta(u)\beta\big)}{\partial x_i} \frac{\partial \xi_\varepsilon}{\partial x_i} - 2\eta(u) \frac{\partial \beta}{\partial x_i} \frac{\partial \xi_\varepsilon}{\partial x_i}\, dx\, dt \\ &\le \int_\Omega \eta(u^0)\, \beta(0)\, \xi_\varepsilon\, dx + M \int_0^T \int_\Omega \eta(u)\, \beta\, |\nabla \xi_\varepsilon|\, dx\, dt \\ &\quad - \varepsilon \int_0^T \int_\Omega \frac{\partial\big(\eta(u)\, \beta\big)}{\partial x_i} \frac{\partial \xi_\varepsilon}{\partial x_i}\, dx\, dt + 2\,\varepsilon \int_0^T \int_\Omega \eta(u) \frac{\partial \beta}{\partial x_i} \frac{\partial \xi_\varepsilon}{\partial x_i}\, dx\, dt\,, \end{aligned}$$

where we used the fact that

$$|\mathbf{q}(z)| \leq M\,\eta(z)\,.$$

Using (8.3) with $\eta(u)\beta$ instead of β in the last but one inequality, we obtain (8.5).

- Let us denote

$$w \equiv u_1 - u_2\,, \qquad w^D \equiv u_1^D - u_2^D\,, \qquad w^0 \equiv u_1^0 - u_2^0\,.$$

Multiplying the equation

$$\frac{\partial w}{\partial t} + \operatorname{div}(\mathbf{f}(u_1) - \mathbf{f}(u_2)) - \varepsilon\,\Delta w = 0$$

by $\varphi_\delta'(w)\xi_\varepsilon$, where $\varphi_\delta(z) \equiv (z^2+\delta^2)^{1/2}$, and integrating over $(0,t)\times\Omega$ we obtain

$$\begin{aligned}
\int_\Omega \varphi_\delta(w(t))\,\xi_\varepsilon\,dx - \int_\Omega \varphi_\delta(w^0)\xi_\varepsilon\,dx \\
&- \int_0^t \int_\Omega (f_i(u_1) - f_i(u_2))\,\varphi_\delta''(w)\frac{\partial w}{\partial x_i}\xi_\varepsilon\,dx\,d\tau \\
&- \int_0^t \int_\Omega (f_i(u_1) - f_i(u_2))\,\varphi_\delta'(w)\frac{\partial \xi_\varepsilon}{\partial x_i}\,dx\,d\tau \\
&+ \varepsilon \int_0^t \int_\Omega |\nabla w|^2\,\varphi_\delta''(w)\,\xi_\varepsilon\,dx\,d\tau \\
&+ \varepsilon \int_0^t \int_\Omega \frac{\partial \varphi_\delta(w)}{\partial x_i}\,\frac{\partial \xi_\varepsilon}{\partial x_i}\,dx = 0\,.
\end{aligned}$$

Using the Lipschitz continuity of $\mathbf{f}$, Young's inequality and the fact that $z^2\varphi_\delta''(z) = z^2\delta^2(z^2+\delta^2)^{-3/2} < \delta$, we get

$$\begin{aligned}
-\big(f_i(u_1) - f_i(u_2)\big)\frac{\partial w}{\partial x_i}\varphi_\delta''(w)\,\xi_\varepsilon + \varepsilon\,|\nabla w|^2\,\varphi_\delta''(w)\xi_\varepsilon \\
&\geq \big\{-M\,|w|\,|\nabla w| + \varepsilon\,|\nabla w|^2\big\}\varphi_\delta''(w)\,\xi_\varepsilon \\
&\geq -\,\frac{M^2}{4\,\varepsilon}w^2\,\varphi_\delta''(w)\,\xi_\varepsilon \\
&\geq -\,\frac{M^2\,\delta}{4\,\varepsilon}\xi_\varepsilon\,.
\end{aligned}$$

Moreover, observing that $|z|\,|\varphi_\delta'(z)| \leq \varphi_\delta(z)$, we obtain

$$\begin{aligned}
-\big(f_i(u_1) - f_i(u_2)\big)\frac{\partial \xi_\varepsilon}{\partial x_i}\,\varphi_\delta'(w) &\geq -\,M\,|w|\,|\varphi_\delta'(w)|\,|\nabla\xi_\varepsilon| \\
&\geq -\,M\,\varphi_\delta(w)\,|\nabla\xi_\varepsilon|\,.
\end{aligned}$$

Finally,

$$\int_\Omega \varphi_\delta(w(t))\xi_\varepsilon\,dx - \int_\Omega \varphi_\delta(w^0)\xi_\varepsilon\,dx - \frac{M^2 T\delta}{4\varepsilon}\int_\Omega \xi_\varepsilon\,dx$$
$$- M\int_0^t\int_\Omega \varphi_\delta(w)\,|\nabla\xi_\varepsilon|\,dx\,d\tau + \varepsilon\int_0^t\int_\Omega \frac{\partial\varphi_\delta(w)}{\partial x_i}\frac{\partial\xi_\varepsilon}{\partial x_i}\,dx\,d\tau \le 0$$

and therefore, taking $\beta = \varphi_\delta$ in (8.3),

$$\begin{aligned}\int_\Omega \varphi_\delta(w(t))\,\xi_\varepsilon\,dx \le &\int_\Omega \varphi_\delta(w^0)\,\xi_\varepsilon\,dx\\ &+ (M+L\varepsilon)\int_0^t\int_{\partial\Omega}\varphi_\delta(w^D)\,dr\,d\tau\\ &+\frac{M^2 T}{4\varepsilon}\delta\int_\Omega \xi_\varepsilon\,dx\,.\end{aligned} \tag{8.11}$$

Letting $\delta \to 0+$ in (8.11) we obtain (8.6).

- Let us multiply (8.1) by $\varphi_\delta'(u)$, where

$$\varphi_\delta(z) = \begin{cases} ((z-m)^2+\delta^2)^{1/2}-\delta & \text{for } z \ge m\,,\\ 0 & \text{for } z \le m,\end{cases}$$
$$m = \max\{\sup_\Gamma u^D, \sup_\Omega u^0\}\,.$$

Using the properties of φ_δ, we obtain

$$\int_\Omega \varphi_\delta(u(t))\,dx - \int_0^t\int_\Omega (f_i(u)-f_i(m))\frac{\partial u}{\partial x_i}\varphi_\delta''(u)\,dx\,d\tau$$
$$+\varepsilon\int_0^t\int_\Omega |\nabla u|^2\varphi_\delta''(u)\,dx\,d\tau = 0\,.$$

Now, the estimate

$$\begin{aligned}-(f_i(u)-f_i(m))&\frac{\partial u}{\partial x_i}\varphi_\delta''(u) + \varepsilon|\nabla u|^2\varphi_\delta''(u)\\ &\ge \{-M\,|u-m|\,|\nabla u| + \varepsilon\,|\nabla u|^2\}\varphi_\delta''(u)\\ &\ge -\frac{M^2}{4\varepsilon}(u-m)^2\varphi_\delta''(u)\\ &\ge -\frac{M^2\delta}{4\varepsilon}\end{aligned}$$

follows by an analogous argument as above, and the limit $\delta \to 0+$

gives

$$\int_\Omega \max\,(u(t) - m, 0)\, dx \le 0$$

which is $(8.7)_1$. Similarly, we get $(8.7)_2$ and therefore we have (8.8).

• Let u^D be the smooth extension of u^D onto $\overline{Q_T}$. Denoting for $\mathbf{f} \in C^2(\mathbb{R})^d$,

$$v \equiv \frac{\partial u}{\partial t} - \frac{\partial u^D}{\partial t}\,,$$
$$e \equiv \frac{\partial^2 u^D}{\partial t^2} + \operatorname{div}\Big(\mathbf{f}'(u)\frac{\partial u^D}{\partial t}\Big) - \varepsilon\,\Delta\frac{\partial u^D}{\partial t}\,,$$

we have

$$\frac{\partial v}{\partial t} + \operatorname{div}(\mathbf{f}'(u)v) - \varepsilon\,\Delta v = -e\,. \tag{8.12}$$

Multiplying (8.12) by $\varphi_\delta'(v)$, where $\varphi_\delta(z) \equiv (z^2+\delta^2)^{1/2}$, and integrating over $(0,t)\times\Omega$, we obtain

$$\begin{aligned}\int_\Omega \varphi_\delta(v(t))\,dx - \int_\Omega \varphi_\delta(v(0))\,dx - \int_0^t\int_\Omega \mathbf{f}'(u)\cdot\nabla v\, v\,\varphi_\delta''(v)\,dx\,d\tau \\ + \varepsilon\int_0^t\int_\Omega |\nabla v|^2\varphi_\delta''(v)\,dx\,d\tau = -\int_0^t\int_\Omega e\,\varphi_\delta'(v)\,dx\,d\tau\,,\end{aligned} \tag{8.13}$$

where we used $\varphi_\delta'(v) = 0$ on Γ. Further we have

$$\begin{aligned}-\mathbf{f}'(u)\cdot\nabla v\, v\,\varphi_\delta''(v) + \varepsilon\,|\nabla v|^2\varphi_\delta''(v) &\ge -\frac{1}{4\,\varepsilon}|\mathbf{f}'(u)|^2\,v^2\,\varphi_\delta''(v) \\ &\ge -\frac{M^2\,\delta}{4\,\varepsilon}\end{aligned}$$

and thus if we let $\delta \to 0+$ in (8.13), we obtain

$$\int_\Omega |v(t)|\,dx \le \int_\Omega |v(0)|\,dx + \int_0^t\int_\Omega |e|\,dx\,d\tau\,.$$

Using

$$v(0) = \frac{\partial u}{\partial t}(0) - \frac{\partial u^D}{\partial t}(0) = -\operatorname{div}\mathbf{f}(u^0) + \varepsilon\,\Delta u^0 - \frac{\partial u^D}{\partial t}(0)$$

we obtain

$$\int_\Omega \Big|\frac{\partial u}{\partial t}\Big|\,dx \le \lambda\Big(1 + \int_0^t\int_\Omega |\nabla u|\,dx\,d\tau\Big) \tag{8.14}$$

for all $t \in [0,T]$, with λ depending on T, Ω, $\mathbf{f}$, $\|u^0\|_\Omega$, $\|u^D\|_\Gamma$. Further we have

$$\frac{\partial w}{\partial t} + \operatorname{div}(\mathbf{f}(u) - \mathbf{f}(u^D)) - \varepsilon\,\Delta w = -g\,, \tag{8.15}$$

where

$$w \equiv u - u^D\,, \qquad g \equiv \frac{\partial u^D}{\partial t} + \operatorname{div}\mathbf{f}(u^D) - \varepsilon\,\Delta u^D\,.$$

Now we multiply (8.15) by $\varphi_\delta'(w)\,\beta$, where $\beta \in \mathcal{D}(\mathbb{R}^d)$, $\beta \geq 0$, depends only on the space variable and where

$$\varphi_\delta(z) \equiv (z^2 + \delta^2)^{1/2} - \delta\,.$$

After integration over $(0,t) \times \Omega$, partial integration and using the fact that on Γ

$$\varphi_\delta'(w) = 0\,, \qquad \varphi_\delta(w) = 0\,, \qquad \nabla\varphi_\delta(w)\cdot\boldsymbol{\nu} = 0\,,$$

we arrive at

$$\begin{aligned}
\int_\Omega \varphi_\delta(w(t))\,\beta\,dx - \int_\Omega \varphi_\delta(w(0))\,\beta\,dx - \varepsilon\int_0^t\int_\Omega \varphi_\delta(w)\,\Delta\beta\,dx\,d\tau \\
- \int_0^t\int_\Omega \varphi_\delta'(w)(f_i(u) - f_i(u^D))\frac{\partial\beta}{\partial x_i}\,dx\,d\tau \\
- \int_0^t\int_\Omega (f_i(u) - f_i(u^D))\frac{\partial w}{\partial x_i}\varphi_\delta''(w)\,\beta\,dx\,d\tau \\
+\varepsilon\int_0^t\int_\Omega |\nabla w|^2\,\varphi_\delta''(w)\,\beta\,dx\,d\tau = -\int_0^t\int_\Omega \varphi_\delta'(w)\,g\,\beta\,dx\,d\tau\,.
\end{aligned}$$

Now we let $\delta \to 0+$ and obtain (F_i are as in Remark 7.8)

$$\begin{aligned}
\int_\Omega |w(t)|\,dx - \int_\Omega |w(0)|\,dx - \int_0^t\int_\Omega F_i(u,u^D)\frac{\partial\beta}{\partial x_i}\,dx\,d\tau \\
-\varepsilon\int_0^t\int_\Omega |w|\,\Delta\beta\,dx\,d\tau \leq -\int_0^t\int_\Omega \operatorname{sgn}(w)\,g\,\beta\,dx\,d\tau\,.
\end{aligned}$$

We put

$$\beta(x) = \beta_\rho(x) \equiv \gamma\Big(\frac{s(x)}{\rho}\Big)\,,$$

where $s(x)$ is as before and $\gamma \in C^\infty(\mathbb{R})$ is a fixed non-negative function such that

$$\gamma(0) = 0\,, \qquad \gamma(\sigma) = 1 \quad \text{for } \sigma \geq 1.$$

Due to $\beta_\rho = 0$ on Γ we have

$$\lim_{\rho\to 0+} \int_0^t \int_\Omega \mathbf{F}(u, u^D)\, \nabla\beta_\rho \, dx\, d\tau$$

$$= \lim_{\rho\to 0+} \int_0^{\frac{\delta}{\rho}} \gamma'(\sigma) \int_0^t \int_{\partial\Omega} \mathbf{F}(u(\tau, r+\rho\sigma), u^D(\tau, r+\rho\sigma)) \cdot \boldsymbol{\nu}(r)\, dr\, d\tau\, d\sigma$$

$$= \int_0^\infty \gamma'(\sigma)\, d\sigma \int_0^t \int_{\partial\Omega} \mathbf{F}(u, u^D) \cdot \boldsymbol{\nu}\, dr\, d\tau$$

$$= \int_0^t \int_{\partial\Omega} \mathbf{F}(u, u^D) \cdot \boldsymbol{\nu}\, dr\, d\tau = 0\,,$$

and if $r(x)$ is the nearest boundary point to x, then similarly

$$\lim_{\rho\to 0+} \int_0^t \int_\Omega |w|\, \Delta\beta_\rho \, dx\, d\tau$$

$$= \lim_{\rho\to 0+} \int_0^t \int_\Omega |w(t,x) - w(t, r(x))|\, \Delta\beta_\rho(x)\, dx\, d\tau$$

$$= \int_0^\infty \sigma\gamma''(\sigma)\, d\sigma \int_0^t \int_{\partial\Omega} |\nabla w \cdot \boldsymbol{\nu}|\, dr\, d\tau$$

$$= -\int_0^t \int_{\partial\Omega} |\nabla w \cdot \boldsymbol{\nu}|\, dr\, d\tau\,.$$

Thus we get

$$\int_\Omega |w(t)|\, dx + \varepsilon \int_0^t \int_{\partial\Omega} |\nabla w \cdot \boldsymbol{\nu}|\, dr\, d\tau \le \int_\Omega |w(0)|\, dx + \int_0^t \int_\Omega |g|\, dx\, d\tau\ . \tag{8.16}$$

Let us denote $w_i \equiv \frac{\partial w}{\partial x_i}$, $i = 1, \dots, d$, $\mathbf{w} \equiv (w_1, \dots, w_d)$. Then we have (again note that $\mathbf{f} \in C^2(\mathbb{R})^d$)

$$\frac{\partial w_i}{\partial t} + \operatorname{div}(\mathbf{f}'(u) w_i) - \varepsilon\, \Delta w_i = -h_i\,, \tag{8.17}$$

$i = 1, \dots, d$, where

$$h_i = \frac{\partial^2 u^D}{\partial x_i \partial t} + \operatorname{div}\Big(\mathbf{f}'(u) \frac{\partial u^D}{\partial x_i}\Big) - \varepsilon\, \Delta \frac{\partial u^D}{\partial x_i}\ .$$

Let us multiply the ith equation of (8.17) by $\frac{\partial}{\partial \xi_i}\phi_\delta(\mathbf{w})$, where

$\phi_\delta(\xi) = (|\xi|^2 + \delta^2)^{1/2}$, add it up and integrate over $(0,t) \times \Omega$. Let us investigate the resulting terms:

$$\int_0^t \int_\Omega \frac{\partial w_i}{\partial t} \frac{\partial \phi_\delta}{\partial \xi_i}(\mathbf{w})\, dx\, d\tau$$
$$= \int_\Omega \phi_\delta(\mathbf{w}(t))\, dx - \int_\Omega \phi_\delta(\mathbf{w}(0))\, dx\,,$$

$$-\varepsilon \int_0^t \int_\Omega \Delta w_i \frac{\partial \phi_\delta}{\partial \xi_i}(\mathbf{w})\, dx\, d\tau$$
$$= \varepsilon \int_0^t \int_\Omega \frac{\partial w_i}{\partial x_j} \frac{\partial^2 \phi_\delta}{\partial \xi_i \partial \xi_k}(\mathbf{w}) \frac{\partial w_k}{\partial x_j}\, dx\, d\tau$$
$$- \varepsilon \int_0^t \int_{\partial\Omega} \frac{\partial w_i}{\partial x_j} \nu_j \frac{\partial \phi_\delta}{\partial \xi_i}(\mathbf{w})\, dr\, d\tau\,,$$

$$\int_0^t \int_\Omega \operatorname{div}(\mathbf{f}'(u) w_i) \frac{\partial \phi_\delta}{\partial \xi_i}(\mathbf{w})\, dx\, d\tau$$
$$= -\int_0^t \int_\Omega f_j'(u)\, w_i \frac{\partial^2 \phi_\delta}{\partial \xi_i \partial \xi_k} \frac{\partial w_k}{\partial x_j}\, dx\, d\tau$$
$$+ \int_0^t \int_{\partial\Omega} f_j'(u)\, \nu_j\, w_i \frac{\partial \phi_\delta}{\partial \xi_i}(\mathbf{w})\, dr\, d\tau\ .$$

Due to the estimate

$$\varepsilon \frac{\partial w_i}{\partial x_j} \frac{\partial^2 \phi_\delta}{\partial \xi_i \partial \xi_k}(\mathbf{w}) \frac{\partial w_k}{\partial x_j} - f_j'(u)\, w_i \frac{\partial^2 \phi_\delta}{\partial \xi_i \partial \xi_k}(\mathbf{w}) \frac{\partial w_k}{\partial x_j}$$
$$= \frac{\delta^2}{(|\mathbf{w}|^2 + \delta^2)^{3/2}} \left[\varepsilon |\nabla \mathbf{w}|^2 - f_j'(u)\, w_i \frac{\partial w_i}{\partial x_j}\right]$$
$$\geq -\frac{1}{4\,\varepsilon} |\mathbf{f}'(u)|^2 \frac{\delta^2 |\mathbf{w}|^2}{(|\mathbf{w}|^2 + \delta^2)^{3/2}} \geq -\frac{M^2\, \delta}{4\,\varepsilon}$$

we obtain for $\delta \to 0+$

$$\int_\Omega |\mathbf{w}(t)|\, dx \leq \int_\Omega |\mathbf{w}(0)|\, dx + \int_0^t \int_\Omega |\mathbf{h}|\, dx\, d\tau$$
$$+ \limsup_{\delta \to 0+} \int_0^t \int_{\partial\Omega} \Big| \mathbf{f}'(u) \cdot \boldsymbol{\nu}\, w_j \frac{\partial \phi_\delta}{\partial \xi_j}(\mathbf{w}) \qquad (8.18)$$
$$- \varepsilon\, \nabla w_i \cdot \boldsymbol{\nu} \frac{\partial \phi_\delta}{\partial \xi_i}(\mathbf{w}) \Big|\, dr\, d\tau\ .$$

Due to $w = 0$ on Γ, we have on Γ

$$\mathbf{w} = \nabla w = (\nabla w \cdot \boldsymbol{\nu})\boldsymbol{\nu}\,, \qquad \Delta w = D^2 w(\boldsymbol{\nu}, \boldsymbol{\nu}) + \Delta s \, \nabla w \cdot \boldsymbol{\nu}\ ,$$

where $D^2 w$ is the bilinear form of the second differential of w. Therefore, the integrand can be rewritten as

$$\begin{aligned}
&\mathbf{f}'(u) \cdot \boldsymbol{\nu}\, w_j \frac{\partial \phi_\delta}{\partial \xi_j}(\mathbf{w}) - \varepsilon \nabla w_i \cdot \boldsymbol{\nu} \frac{\partial \phi_\delta}{\partial \xi_i}(\mathbf{w}) \\
&= \mathbf{f}'(u) \cdot \boldsymbol{\nu} \frac{|\nabla w|^2}{(|\nabla w|^2 + \delta^2)^{1/2}} - \varepsilon D^2 w\Big(\boldsymbol{\nu}, \frac{\nabla w}{(|\nabla w|^2 + \delta^2)^{1/2}}\Big) \\
&= (\mathbf{f}'(u) \cdot \nabla w - \varepsilon D^2 w(\boldsymbol{\nu}, \boldsymbol{\nu})) \frac{\nabla w \cdot \boldsymbol{\nu}}{(|\nabla w|^2 + \delta^2)^{1/2}} \\
&= \Big(\frac{\partial w}{\partial t} + \operatorname{div}(\mathbf{f}(u) - \mathbf{f}(u^D)) - \varepsilon \Delta w \\
&\qquad + \varepsilon \Delta s \, \nabla w \cdot \boldsymbol{\nu}\Big) \frac{\nabla w \cdot \boldsymbol{\nu}}{(|\nabla w|^2 + \delta^2)^{1/2}} \\
&= (-g + \varepsilon \Delta s \, \nabla w \cdot \boldsymbol{\nu}) \frac{\nabla w \cdot \boldsymbol{\nu}}{(|\nabla w|^2 + \delta^2)^{1/2}}\,,
\end{aligned}$$

where g is defined after (8.15). From this and (8.18) we get

$$\begin{aligned}
\int_\Omega |\nabla w(t)| \, dx \le \int_\Omega |\nabla w(0)| \, dx + \int_0^t \int_\Omega |\mathbf{h}| \, dx \, d\tau \\
+ \int_0^t \int_{\partial\Omega} |g| + \varepsilon L \, |\nabla w \cdot \boldsymbol{\nu}| \, dr \, d\tau\,,
\end{aligned}$$

which, together with (8.16), gives

$$\begin{aligned}
\int_\Omega |\nabla w(t)| \, dx + L \int_\Omega |w(t)| \, dx \le \int_\Omega |\nabla w(0)| \, dx + L \int_\Omega |w(0)| \, dx \\
+ \int_0^t \int_{\partial\Omega} |g| \, dr \, d\tau \\
+ \int_0^t \int_\Omega \big(|\mathbf{h}| + L|g|\big) \, dx \, d\tau\ .
\end{aligned}$$

This implies

$$\int_\Omega |\nabla u(t)| \, dx \le \lambda\Big(1 + \int_0^t \int_\Omega |\nabla u| \, dx \, d\tau\Big) \tag{8.19}$$

for all $t \in [0, T]$, where λ depends on T, Ω, $\mathbf{f}$, $\|u^0\|_\Omega$, $\|u^D\|_\Gamma$.

Assertion (8.9) now follows from Gronwall's lemma, (8.19) and (8.14). ■

Now we are ready to prove the main theorem of this section. We note that since the existence result uses the last assertion of Theorem 8.4, we need the nonlinearity $\mathbf{f}$ to be in the class C^2, whereas the uniqueness can be established for $\mathbf{f} \in C^1$ (see Theorem 7.31).

Theorem 8.20 (Existence) *Let $\mathbf{f} \in C^2(\mathbb{R})^d$. Let u^ε, $\varepsilon > 0$, be the solutions of (8.1) corresponding to the smooth boundary and initial data $u^{\varepsilon D}$, $u^{\varepsilon 0}$, satisfying suitable compatibility conditions on $\partial\Omega$. Let $u^{\varepsilon D}$, $u^{\varepsilon 0}$ be uniformly bounded in the respective L^∞ norms and let*

$$\lim_{\varepsilon\to 0+} u^{\varepsilon D} = u^D \quad \text{in } L^1(\Gamma)\,,$$
$$\lim_{\varepsilon\to 0+} u^{\varepsilon 0} = u^0 \quad \text{in } L^1(\Omega)\,,$$

where $u^D \in L^\infty(\Gamma)$, $u^0 \in L^\infty(\Omega)$. Then the sequence $\{u^\varepsilon\}$ is also uniformly bounded and converges in $C^0([0,T], L^1(\Omega))$ to some function $u \in L^\infty(Q_T)$ which solves (7.4) and satisfies (7.5), (7.6).

PROOF : Let $\varepsilon_\ell > 0$ be an arbitrary sequence which converges to zero as $\ell \to \infty$. Denote

$$(u_\ell, u_\ell^D, u_\ell^0) \equiv (u^{\varepsilon_\ell}, u^{\varepsilon_\ell D}, u^{\varepsilon_\ell 0})\,.$$

In order to use Theorem 8.4 we need a smooth extension of u_ℓ^D and u_ℓ^0 to $\overline{Q_T}$. We define $u_\ell^{D,0}$ by

$$\begin{aligned}
u_\ell^{D,0}(t, r + s\,\boldsymbol{\nu}(r)) &\equiv u_\ell^D(t,r) && t \in (0,T),\ r \in \partial\Omega,\\
& && |s| < \min(t,\delta)\,,\\
u_\ell^{D,0}(t,x) &\equiv u_\ell^0(x) && -\delta < t < \min(\operatorname{dist}(x,\partial\Omega),\delta)\,,\\
& && x \in \Omega,\\
u_\ell^{D,0}(t,x) &\equiv 0 && \text{elsewhere}
\end{aligned}$$

and then we mollify for $h > 0$ to get

$$u_{\ell,h}^{D,0}(p) \equiv \int_{\mathbb{R}^{d+1}} u_\ell^{D,0}(\tilde{p})\,\phi_h(p-\tilde{p})\,d\tilde{p}\,,$$

where ϕ_h is the usual mollifier. Let us now denote by $u_{\ell,h}^D$ and $u_{\ell,h}^0$ the restriction of $u_{\ell,h}^{D,0}$ to Γ and $\{0\} \times \Omega$, respectively. Let $u_{\ell,h}$ be the solution of (8.1) with $\varepsilon = \varepsilon_\ell$ and initial and boundary data $u_{\ell,h}^0$ and $u_{\ell,h}^D$, respectively. The uniform boundedness of u_ℓ^0, u_ℓ^D implies the uniform boundedness of $u^0_{\ell,h}$, $u^D_{\ell,h}$, which (see (8.8)) gives the uniform boundedness of u_ℓ, $u_{\ell,h}$. Obviously, we also get

$$\lim_{h\to 0+} u^D_{\ell,h} = u^D_\ell \quad \text{in } L^1(\Gamma)\,,$$
$$\lim_{h\to 0+} u^0_{\ell,h} = u^0_\ell \quad \text{in } L^1(\Omega)\,,$$

uniformly with respect to ℓ. This, together with (8.6), implies

$$\lim_{h\to 0+} u_{\ell,h} = u_\ell \quad \text{in } C^0([0,T], L^1(\Omega))\,, \tag{8.21}$$

uniformly in $\ell \in \mathbb{N}$. On the other hand, it follows from the boundedness of $u^D_\ell \in L^1(\Gamma)$ and $u^0_\ell \in L^1(\Omega)$, that

$$\|u^D_{\ell,h}\|_\Gamma \le \frac{c}{h^3}\,, \qquad \|u^0_{\ell,h}\|_\Omega \le \frac{c}{h^2}\,,$$

respectively. For fixed $h > 0$ it follows from (8.9) that the sequences

$$\left\{\frac{\partial u_{\ell,h}}{\partial t}\right\}, \left\{\nabla u_{\ell,h}\right\} \text{ are bounded in } C^0([0,T], L^1(\Omega)). \tag{8.22}$$

Let $\alpha > 0$ be given. From (8.21) we get the existence of some $h > 0$ such that

$$2\int_\Omega |u_{\ell,h}(t) - u_\ell(t)|\,dx < \frac{\alpha}{2} \qquad \forall\, t \in [0,T]\,, \quad \ell \in \mathbb{N}$$

and due to (8.22) there is some $\delta > 0$ such that

$$\delta \int_\Omega \left|\frac{\partial u_{\ell,h}}{\partial t}(t)\right|\,dx < \frac{\alpha}{2} \qquad \forall\, t \in [0,T]\,, \quad \ell \in \mathbb{N}\,.$$

Thus for all $\ell \in \mathbb{N}$ and all $t_1, t_2 \in [0,T]$ such that $|t_1 - t_2| \le \delta$, we have

$$\begin{aligned}\int_\Omega |u_\ell(t_1) - u_\ell(t_2)|\,dx &\le \sum_{i=1}^{2} \int_\Omega |u_{\ell,h}(t_i) - u_\ell(t_i)|\,dx \\ + \int_\Omega |u_{\ell,h}(t_1) - u_{\ell,h}(t_2)|\,dx &\le \sum_{i=1}^{2} \int_\Omega |u_{\ell,h}(t_i) - u_\ell(t_i)|\,dx \\ &\quad + |t_1 - t_2| \sup_{[t_1,t_2]} \int_\Omega \left|\frac{\partial u_{\ell,h}}{\partial t}(t)\right|\,dx \le \alpha\,.\end{aligned} \tag{8.23}$$

This means that u_ℓ is uniformly continuous in $C^0([0,T]; L^1(\Omega))$.

On the other hand, it follows from (8.22) that there is a $\delta > 0$ such that

$$\delta \int_\Omega |\nabla u_{\ell,h}(t)|\, dx < \frac{\alpha}{2} \qquad \forall\, t \in [0,T], \quad \ell \in \mathbb{N}\,.$$

Therefore, we get uniformly in $t \in [0,T]$ and $\ell \in \mathbb{N}$,

$$\begin{aligned}\int_{\Omega^{\Delta x}} |u_\ell(t, x+\Delta x) - u_\ell(t,x)|\, dx &\le 2 \int_\Omega |u_{\ell,h}(t,x) - u_\ell(t,x)|\, dx \\ &\quad + |\Delta x| \int_\Omega |\nabla u_{\ell,h}(t,x)|\, dx < \alpha\,,\end{aligned}$$

for $|\Delta x| < \delta$ and

$$\Omega^{\Delta x} \equiv \{x \in \Omega; x + \Delta x \in \Omega\}\,.$$

This, together with the uniform boundedness of u_ℓ, gives that

$$\{u_\ell(t)\}_{t\in[0,T],\ell\in\mathbb{N}} \text{ is precompact in } L^1(\Omega).$$

This and (8.23) imply, according to the Arzelà-Ascoli Theorem, that

$$\{u_\ell\}_\ell \text{ is precompact in } C^0([0,T]; L^1(\Omega))$$

and therefore (for a subsequence)

$$\lim_{\ell\to\infty} u_\ell = u \quad \text{in } C^0([0,T]; L^1(\Omega))\,.$$

Finally, $u \in L^\infty(Q_T)$ (see (8.8)).

Let us now show that u satisfies (7.4)–(7.6). From (8.5) we have for all boundary entropy-entropy flux pairs $(H, \mathbf{Q})$, all $k \in \mathbb{R}$ and $\beta \in \mathcal{D}((-\infty,T)\times\mathbb{R}^d)$, $\beta \ge 0$,

$$\begin{aligned}-\int_0^T \int_\Omega \Big\{ H(u_\ell,k)\frac{\partial\beta}{\partial t} + Q_i(u_\ell,k)\frac{\partial\beta}{\partial x_i} + \varepsilon_\ell H(u_\ell,k)\,\Delta\beta \Big\}\xi_{\varepsilon_\ell}\, dx\, dt \\ \le \int_\Omega H(u_\ell^0,k)\,\beta(0)\, dx + (M + L\,\varepsilon_\ell)\int_\Gamma H(u_\ell^D,k)\,\beta\, dr \\ + 2\,\varepsilon_\ell \int_0^T \int_\Omega H(u_\ell,k)\frac{\partial\beta}{\partial x_i}\frac{\partial\xi_{\varepsilon_\ell}}{\partial x_i}\, dx\, dt\end{aligned} \tag{8.24}$$

for all $\ell \in \mathbb{N}$. Due to

$$\lim_{\ell\to\infty} \int_\Omega |1 - \xi_{\varepsilon_\ell}|\, dx = 0\,,$$

$$\lim_{\ell\to\infty} \varepsilon_\ell \int_\Omega |\nabla \xi_{\varepsilon_\ell}|\, dx = 0\,,$$

where in the last limit (8.3) is used, we can let $\ell \to \infty$ in (8.24) and obtain

$$- \int_0^T \int_\Omega H(u,k)\frac{\partial \beta}{\partial t} + Q_i(u,k)\frac{\partial \beta}{\partial x_i}\, dx\, dt \leq \int_\Omega H(u^0,k)\,\beta(0)\, dx + M \int_\Gamma H(u^D,k)\,\beta\, dr\ .$$

Now from Theorem 7.31 it follows that (u, u^D, u^0) satisfy (7.4)–(7.6). From Theorem 7.28 it follows that there is only one such $u \in L^\infty(Q_T)$ and therefore

$$\lim_{\varepsilon \to 0+} u^\varepsilon = u \quad \text{in } C^0([0,T]; L^1(\Omega)).$$

■

CHAPTER 3

Young measures and scalar conservation laws

3.1 Introduction

In the previous chapter we have seen how the general strategy of vanishing viscosity method (introduced in the 1950s by LAX [1954] and HOPF [1950]) was successfully applied to obtain a unique weak entropy solution to the Cauchy problem

$$\begin{aligned} \frac{\partial u}{\partial t} + \frac{\partial}{\partial x_j} f_j(u) &= 0\,, \qquad \text{in } \mathbb{R}^+ \times \mathbb{R}^d\,, \\ u(0,\cdot) &= u_0\,, \qquad \text{in } \mathbb{R}^d\,, \end{aligned} \tag{1.1}$$

for $u_0 \in L^\infty(\mathbb{R}^d)$, $f_j \in C^1(\mathbb{R})$, $j = 1, \ldots, d$, (see Theorem 5.1 in Chapter 2). In this chapter we want to have a closer look at the Young measure technique in the vanishing viscosity method.

Recall that assuming $u_0 \in L^\infty(\mathbb{R}^d)$, $f_j \in C^1(\mathbb{R})$, $j = 1, \ldots, d$, one knows that there are $u^\varepsilon \in L^\infty(\mathbb{R}^+ \times \mathbb{R}^d)$ solutions to the parabolic perturbation of (1.1),

$$\begin{aligned} \frac{\partial u^\varepsilon}{\partial t} + \frac{\partial}{\partial x_j} f_j(u^\varepsilon) &= \varepsilon \Delta u^\varepsilon, \qquad \varepsilon > 0\,, \\ u^\varepsilon(0,\cdot) &= u_0\,, \end{aligned} \tag{1.2}$$

satisfying

$$\|u^\varepsilon\|_{L^\infty(\mathbb{R}^+ \times \mathbb{R}^d)} \le c \tag{1.3}$$

(see Theorem 4.48 in Chapter 2).

This uniform estimate implies that for a subsequence (here we use $u^k \equiv u^{\varepsilon_k}$, $\varepsilon_k \to 0+$ as $k \to \infty$),

$$u^k \overset{*}{\rightharpoonup} u \qquad \text{in } L^\infty(\mathbb{R}^+ \times \mathbb{R}^d)\,. \tag{1.4}$$

It is not difficult to show that a weak convergence of the type (1.4) is enough to ensure that

$$\begin{aligned} \frac{\partial u^k}{\partial t} &\to \frac{\partial u}{\partial t} \qquad \text{in } \mathcal{D}'(\mathbb{R}^+ \times \mathbb{R}^d)\,. \\ \varepsilon_k \Delta u^k &\to 0 \end{aligned}$$

Thus, there is the second (nonlinear) term in (1.2) which remains to be studied as $\varepsilon_k \to 0+$.

Typically, there are no *a priori* estimates independent of ε on derivatives of u^ε. Hence, the classical compactness argument cannot be applied and one must deal with the weak convergence (1.4). In our situation, we have $f_j \in C^1(\mathbb{R})$, therefore (1.3) implies

$$\|f_j(u^\varepsilon)\|_{L^\infty(\mathbb{R}^+\times\mathbb{R}^d)} \le c\,,$$

and consequently, for a subsequence,

$$f_j(u^k) \overset{*}{\rightharpoonup} \overline{f_j} \qquad \text{in } L^\infty(\mathbb{R}^+ \times \mathbb{R}^d)\,.$$

Unfortunately, for a nonlinear (smooth) function f_j (1.4) is in general not enough to ensure the desired equality

$$\overline{f_j} = f_j(u)\,. \tag{1.5}$$

To see this, let us consider the following example.

Example 1.6 Let $u^n(x) = \sin nx$, $x \in [0, 2\pi]$, $f(y) = y^2$. In this situation we use the following lemma.

Lemma 1.7 *Let $v \in L^2(0, 2\pi)$ be 2π-periodic. Let $v^n(x) \equiv v(nx)$. Then $v^n \rightharpoonup \dfrac{a_0}{2}$ in $L^2(0, 2\pi)$, where $a_0 = \frac{1}{\pi}\int_0^{2\pi} v(x)\,dx$.*

PROOF : We use the Fourier series expansion for v to express v^n as

$$v^n(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} \left(a_k \cos(knx) + b_k \sin(knx)\right)\,.$$

Then for $0 \le a < b \le 2\pi$,

$$\begin{aligned} \left|\int_a^b \sum_{k=1}^{\infty} a_k \cos(knx)\,dx\right| &= \left|\sum_{k=1}^{\infty} a_k \int_a^b \cos(knx)\,dx\right| \\ &\le \sum_{k=1}^{\infty} |a_k| \frac{c}{kn} \le \frac{c}{n} \left(\sum_{k=1}^{\infty} |a_k|^2\right)^{\frac{1}{2}} \left(\sum_{k=1}^{\infty} \frac{1}{k^2}\right)^{\frac{1}{2}} \to 0 \end{aligned}$$

as $n \to \infty$ and the same holds when replacing cos by sin. Therefore

$$\int_0^{2\pi} v^n(x)\chi_{[a,b]}(x)\,dx \longrightarrow \int_0^{2\pi} \frac{a_0}{2}\chi_{[a,b]}(x)\,dx \tag{1.8}$$

as $n \to \infty$. The proof now follows by approximating $\varphi \in L^2(0, 2\pi)$ by step-functions and using (1.8). ■

Applying this lemma to both u^n and $f \circ u^n$, we get:

$$u^n \rightharpoonup \frac{1}{2\pi}\int_0^{2\pi} \sin x\,dx = 0 \qquad \text{in } L^2(0,2\pi)\,,$$

$$(u^n)^2 = f(u^n) \rightharpoonup \frac{1}{2\pi}\int_0^{2\pi} \sin^2 x\,dx = \frac{1}{2} \qquad \text{in } L^2(0,2\pi).$$

Note that also in $L^\infty(0, 2\pi)$ we have $u^n \stackrel{*}{\rightharpoonup} 0$, $(u^n)^2 \stackrel{*}{\rightharpoonup} 1/2$ (at least for some subsequence), since bounded functions on bounded domains are taken into account. Thus we see that

$$f(\text{weak-}^*\lim_{n\to\infty} u^n) < \text{weak-}^*\lim_{n\to\infty} f(u^n)\,. \tag{1.9}$$

This example shows that the lack of strong convergence (here especially the oscillation of the bounded sequence $\{\sin nx\}_{n=1}^{\infty}$) can cause undesired results even in the case of smooth nonlinear f.

In what follows we will see that the composite limits $\overline{f_j}$ can be described by a family of compactly supported probability measures $\{\nu_{t,x}\}_{(t,x)\in\mathbb{R}^+\times\mathbb{R}^d}$, called Young measures. Namely, we will see that

$$\overline{f_j}(t,x) = \langle \nu_{t,x}, f_j\rangle \qquad \text{for a.e. } (t,x) \in \mathbb{R}^+ \times \mathbb{R}^d\,,$$

where $\langle \cdot, \cdot\rangle$ denotes the duality between continuous functions and measures (to be discussed later).

In this way, the desired equality (1.5) can be obtained by showing that the support of Young measures reduces to a point:

$$\nu_{t,x} = \delta_{u(t,x)} \qquad \text{for a.e. } (t,x) \in \mathbb{R}^+ \times \mathbb{R}^d\,. \tag{1.10}$$

In this chapter we will follow the described strategy to prove the existence of a weak entropy solution to scalar conservation laws in one space dimension.

Even if the existence result proved in this chapter (see Theorem 4.15) is weaker than that obtained in the previous chapter (see Theorem 5.1 in Chapter 2), we present it here, since it nicely describes the Young measure approach. Also, in the following chapters the concept of Young measures will be used in a slightly more complicated situation.

3.2 Young measures

Before we formulate the main theorem on the existence of Young measures, we recall some basic notions (see Chapter 1 for more details).

By $C_C(\mathbb{R}^s)$ we denote the space of continuous functions with compact support in $\mathbb{R}^s$ and put $C_0(\mathbb{R}^s) \equiv \overline{C_C(\mathbb{R}^s)}^{\|\cdot\|_\infty}$. Further, $M(\mathbb{R}^s)$ denotes the space of bounded Radon measures on $\mathbb{R}^s$ and

$$\langle \mu, f \rangle \equiv \mu(f) = \int_{\mathbb{R}^s} f \, d\mu \,, \quad \mu \in M(\mathbb{R}^s),\, f \in C_0(\mathbb{R}^s)\,. \tag{2.1}$$

The space of probability measures is then defined as follows:

$$\mathrm{Prob}(\mathbb{R}^s) \equiv \{\mu \in M(\mathbb{R}^s),\ \mu \text{ non-negative, } \mu(\mathbb{R}^s) = 1\}\,. \tag{2.2}$$

The following theorem introduces the concept of Young measures which turns out to be an appropriate tool for describing composite limits of smooth nonlinearities with weakly convergent sequences.

Note that Theorem 2.3 will be proved in a more general setting than will in fact be needed in this chapter. Thus, this theorem indicates that the concept of Young measures as a technical tool for describing the above-mentioned weak composite limits would also make sense in the system case ($s > 1$), if of course L^∞ *a priori* estimates of the type (1.3) were available for solutions of the perturbed parabolic problem. Unfortunately, as far as the authors know, in the general case of systems it is still an open problem to establish uniform control on the amplitude of viscous approximations $\mathbf{u}^\varepsilon$ (see for example DiPerna [1985, p.247]).

Theorem 2.3 (Existence of the Young measures) *Let $\mathbf{u}^n : \mathbb{R}^m \to \mathbb{R}^s$ be an arbitrary sequence of measurable functions for which*

$$\|\mathbf{u}^n\|_{L^\infty(\mathbb{R}^m)^s} \le c\,. \tag{2.4}$$

Then there exists a (weakly- convergent) subsequence $\mathbf{u}^{n_k}$ of $\mathbf{u}^n$ and a family of probability measures $\{\nu_y\}_{y \in \mathbb{R}^m}$, called* **Young measures**, *supported uniformly in a compact set $K \subset \mathbb{R}^s$:*

$$\{\nu_y\}_{y\in\mathbb{R}^m} \subseteq \mathrm{Prob}(\mathbb{R}^s)\,, \quad \mathrm{supp}(\nu_y) \subseteq K \qquad \text{for a.e. } y \in \mathbb{R}^m\,,$$

which represents the subsequence $\mathbf{u}^{n_k}$ in the following sense:
For any $\mathbf{g} \in C(\mathbb{R}^s)^p$ we have

$$\mathbf{g} \circ \mathbf{u}^{n_k} \overset{*}{\rightharpoonup} \overline{\mathbf{g}} \quad \text{in } L^\infty(\mathbb{R}^m)^p\,, \tag{2.5}$$

and

$$\overline{\mathbf{g}}(y) = \int_{\mathbb{R}^s} \mathbf{g}(\lambda)\, d\nu_y(\lambda) = \langle \nu_y, \mathbf{g} \rangle \qquad \text{for a.e. } y \in \mathbb{R}^m \,. \tag{2.6}$$

The proof we give is based on the duality Theorem 2.11. First, we have to introduce some useful notions.

Definition 2.7 *Let $Q \subseteq \mathbb{R}^m$ be an open set. The mapping $\nu : Q \to M(\mathbb{R}^s)$ is said to be weak-* measurable, if for all $F \in L^1(Q; C_0(\mathbb{R}^s))$ the function*†

$$x \mapsto \langle \nu_x, F(x, \cdot) \rangle = \int_{\mathbb{R}^s} F(x, \lambda)\, d\nu_x(\lambda) \tag{2.8}$$

is measurable. Further, let us define‡

$$\|\nu\|_{L^\infty_\omega(Q;M(\mathbb{R}^s))} = \operatorname*{ess\,sup}_{x \in Q} \|\nu_x\|_{M(\mathbb{R}^s)} \,. \tag{2.9}$$

Finally, let

$$\begin{aligned} L^\infty_\omega(Q; M(\mathbb{R}^s)) = \big\{ \nu : Q \to M(\mathbb{R}^s);\, \nu \text{ weak-}^* \text{ measurable}\,, \\ \|\nu\|_{L^\infty_\omega(Q;M(\mathbb{R}^s))} < \infty \big\} \,. \end{aligned} \tag{2.10}$$

Then the following theorem holds.

Theorem 2.11 *Let $Q \subseteq \mathbb{R}^m$ be open. Let $\Phi \in \big(L^1(Q; C_0(\mathbb{R}^s))\big)^*$ be a linear bounded functional. Then there exists a unique $\nu \in L^\infty_\omega(Q; M(\mathbb{R}^s))$ such that*

$$\Phi(F) = \int_Q \langle \nu_x, F(x) \rangle \, dx \qquad \forall\, F \in L^1(Q; C_0(\mathbb{R}^s)) \,,$$

$$\|\Phi\|_{(L^1(Q;C_0(\mathbb{R}^s)))^*} = \|\nu\|_{L^\infty_\omega(Q;M(\mathbb{R}^s))} \,.$$

PROOF : Throughout this proof we will use the notation

$$X = L^1(Q; C_0(\mathbb{R}^s)) \,,$$
$$Y = L^\infty_\omega(Q; M(\mathbb{R}^s)) \,.$$

Uniqueness: Let there exist ν^1, $\nu^2 \in Y$, both representing the functional $\Phi \in X^*$. Define $\widetilde{\nu} \equiv \nu^1 - \nu^2$. Let $R > 0$ and $g \in C_0(\mathbb{R}^s)$.

† Here and in the sequel we use the standard notation $\nu_x \equiv \nu(x)$, as if measures ν_x were 'parametrized' by x.

‡ Note that since $\|\nu_x\|_{M(\mathbb{R}^s)} = \sup_{\|f\|_\infty \le 1} |\langle \nu_x, f \rangle|$, the function $x \mapsto \|\nu_x\|_{M(\mathbb{R}^s)}$ is measurable.

Let $\varphi \in \mathcal{D}(\mathbb{R}^m)$ be such that $\varphi = 1$ on $B_R(0)$, $\varphi = 0$ outside $B_{R+1}(0)$, $\varphi \leq 1$ on $\mathbb{R}^m$. Then, from the inequality

$$\int_Q |\langle \widetilde{\nu}_x, g\rangle\, \varphi(x)|\, dx \leq \text{meas}\,(\text{supp}(\varphi))\, \|\widetilde{\nu}\|_Y\, \|g\|_{C_0(\mathbb{R}^s)} < \infty\,,$$

it follows that the function

$$x \mapsto \langle \widetilde{\nu}_x, g\rangle\, \varphi(x)$$

is an element of the space $L^1(Q)$. Hence, according to the theorem on Lebesgue points, we conclude that for almost all $x_0 \in B_R(0) \cap Q$ we have

$$\lim_{r\to 0} \frac{1}{|B_r(x_0)|} \int_{B_r(x_0)} \langle \widetilde{\nu}_x, g\rangle\; \varphi(x)\, dx = \langle \widetilde{\nu}_{x_0}, g\rangle\,. \tag{2.12}$$

Note that the null set of x_0s for which (2.12) does not hold depends on g, and that for r small enough $B_r(x_0) \subset B_R(0)$ and hence $\varphi = 1$ on $B_r(x_0)$. Now, for such x_0 and r we define $F_r(x) \equiv g$ whenever $x \in B_r(x_0)$ and $F_r(x) \equiv 0$ otherwise. We have

$$\begin{aligned}\int_{B_r(x_0)} \langle \widetilde{\nu}_x, g\rangle\, \varphi(x)\, dx &= \int_Q \big\langle \nu_x^1 - \nu_x^2, F_r(x)\big\rangle\; dx\\ &= \Phi(F_r) - \Phi(F_r) = 0.\end{aligned}$$

This together with (2.12) gives us for $g \in C_0(\mathbb{R}^s)$ fixed, that

$$\langle \widetilde{\nu}_{x_0}, g\rangle = 0 \tag{2.13}$$

for almost all $x_0 \in B_R(0) \cap Q$. At this point, we use the separability of the function space $C_0(\mathbb{R}^s)$. Namely, we choose a countable dense set $S \subset C_0(\mathbb{R}^s)$. Using the same argument as before we obtain (2.13) for every $g \in S$ and for almost all $x_0 \in B_R(0) \cap Q$. Now, since $\widetilde{\nu}_{x_0} \in M(\mathbb{R}^s)$, as an element of dual space to $C_0(\mathbb{R}^s)$ is uniquely determined by the values on the dense set $S \subset C_0(\mathbb{R}^s)$, we have $\widetilde{\nu}_{x_0} = 0 \in M(\mathbb{R}^s)$ for almost all $x_0 \in B_R(0) \cap Q$. Now take $R \to \infty$ to get $\tilde{\nu}_{x_0} = 0 \in M(\mathbb{R}^s)$ for almost all $x_0 \in Q$ and so $\widetilde{\nu} = 0$ as an element of the space Y, which proves uniqueness.

Existence: As before, we will work with a countable dense set $S \subset C_0(\mathbb{R}^s)$. Let $\Phi \in X^*$. Choose $g \in S$, $h \in L^1(Q)$ and define

$$\begin{aligned} &F_{hg} : Q \to C_0(\mathbb{R}^s)\,,\\ &F_{hg} : x \mapsto h(x)g \quad \text{ for a.e. } x \in Q\,.\end{aligned} \tag{2.14}$$

We show that $F_{hg} \in X$. Indeed, for all $\mu \in M(\mathbb{R}^s)$ the function

$$x \mapsto \langle \mu, F_{hg}(x)\rangle = h(x)\, \langle \mu, g\rangle \tag{2.15}$$

is measurable, which means F_{hg} is weakly measurable and hence, since $C_0(\mathbb{R}^s)$ is separable, we conclude, according to the Pettis theorem on weak and strong measurability (see YOSIDA [1965, p. 130]) that F_{hg} is measurable. Moreover,

$$\|F_{hg}\|_X = \int_Q \|h(x)\, g\|_{C_0(\mathbb{R}^s)}\, dx = \|g\|_{C_0(\mathbb{R}^s)} \int_Q |h(x)|\, dx < \infty\,.$$

Consequently, $F_{hg} \in X$. Now,

$$\begin{aligned} |\Phi(F_{hg})| &\le \|\Phi\|_{X^*} \|F_{hg}\|_X \\ &\le \|\Phi\|_{X^*} \|g\|_{C_0(\mathbb{R}^s)} \|h\|_{L^1(Q)} \\ &\le c(g)\, \|h\|_{L^1(Q)}\,. \end{aligned} \tag{2.16}$$

Hence, defining the mapping

$$\psi_g : h \mapsto \Phi(F_{hg})\,,$$

we have from (2.16) that $\psi_g \in \left(L^1(Q)\right)^*$. So, there exists a $u_g \in L^\infty(Q)$ such that

$$\Phi(F_{hg}) = \psi_g(h) = \int_Q u_g(x)\, h(x)\, dx \qquad \forall\, h \in L^1(Q)\,. \tag{2.17}$$

Since $u_g \in L^\infty(Q)$ and S is a countable set, there is a null set N such that for all $x \in Q \setminus N$ and all $g \in S$ it makes sense to consider the value $u_g(x)$ at a point x. For any $x \in Q \setminus N$, the mapping

$$g \mapsto u_g(x)$$

is linear and continuous on $C_0(\mathbb{R}^s)$: indeed, using (2.16) and (2.17) it follows that

$$\begin{aligned} |u_g(x)| \le \|u_g\|_{L^\infty(Q)} &= \sup_{\|h\|_{L^1(Q)} \le 1} \left| \int_Q u_g(x)\, h(x)\, dx \right| \\ &\le \sup_{\|h\|_{L^1(Q)} \le 1} |\Phi(F_{hg})| \le \|\Phi\|_{X^*}\, \|g\|_{C_0(\mathbb{R}^s)}\,. \end{aligned}$$

Hence, for any $x \in Q \setminus N$ there exists a $\nu_x \in M(\mathbb{R}^s) = (C_0(\mathbb{R}^s))^*$ such that

$$\langle \nu_x, g \rangle = u_g(x) \qquad \forall\, g \in S\,.$$

This gives us the family of measures $\{\nu_x\}_{x \in Q}$. Now, employing the countability of the set S we conclude that

$$\langle \nu_x, g \rangle = u_g(x) \qquad \forall\, g \in C_0(\mathbb{R}^s)\,, \quad \text{for a.a. } x \in Q\,. \tag{2.18}$$

It follows that $\Phi(F_{hg})$ equals

$$\int_Q u_g(x)\,h(x)\,dx = \int_Q \langle \nu_x, g\rangle \; h(x)\,dx = \int_Q \langle \nu_x, F_{hg}(x)\rangle \; dx\,, \tag{2.19}$$

which is the desired representation formula for all functions of the type $F_{hg} = h\,g \in X$, $h \in L^1(Q)$, $g \in C_0(\mathbb{R}^s)$. But, finite sums of the type

$$\sum_{i=1}^{m} h_i\, g_i\,, \qquad h_i \in L^1(Q)\,,\ g_i \in C_0(\mathbb{R}^s)$$

are dense in X and consequently the representation formula (2.19) holds true for all $F \in X$.

It remains to verify that the mapping

$$\nu : x \mapsto \nu_x$$

is an element of the space Y. For $F_{hg} \in X$ of the type (2.14) the function

$$x \mapsto \langle \nu_x, F_{hg}(x)\rangle = h(x)\ \langle \nu_x, g\rangle = h(x)\,u_g(x)$$

is measurable and the same holds if we use any $F \in X$, because of the density mentioned above. Consequently, ν is weak-* measurable. Moreover,

$$\|\nu\|_Y = \operatorname*{ess\,sup}_{x\in Q} \|\nu_x\|_{M(\mathbb{R}^s)} = \operatorname*{ess\,sup}_{x\in Q} \sup_{\|g\|_{C_0(\mathbb{R}^s)}\le 1} |\,\langle \nu_x, g\rangle\,|$$

and

$$|\,\langle \nu_x, g\rangle\,| = |u_g(x)| \le \|\Phi\|_{X^*}\,\|g\|_{C_0(\mathbb{R}^s)}\,.$$

This implies

$$\|\nu\|_Y \le \|\Phi\|_{X^*}\,. \tag{2.20}$$

Therefore, $\nu \in Y$. Finally,

$$\begin{aligned}\|\Phi\|_{X^*} &= \sup_{\|F\|_X\le 1} \int_Q |\,\langle \nu_x, F(x)\rangle\,|\,dx \\ &\le \sup_{\|F\|_X\le 1} \int_Q \|\nu_x\|_{M(\mathbb{R}^s)}\|F(x)\|_{C_0(\mathbb{R}^s)}\,dx \le \|\nu\|_Y\,,\end{aligned}$$

which together with (2.20) finishes the proof. ∎

Now we are ready to prove the existence of the Young measure.

PROOF (of Theorem 2.3):

First note that since $\mathbf{g}$ is continuous and the sequence $\{\mathbf{u}^n\}$ is uniformly bounded in $L^\infty(\mathbb{R}^m)^s$, a uniform bound in $L^\infty(\mathbb{R}^m)^p$ holds for the sequence $\{\mathbf{g} \circ \mathbf{u}^n\}$. Therefore, for any $\mathbf{g}$ we always have a subsequence $\mathbf{u}^{n_k}$ of $\mathbf{u}^n$ such that (2.5) holds. In fact, we show that there is a subsequence $\mathbf{u}^{n_k}$ of $\mathbf{u}^n$ such that (2.5) holds for all $\mathbf{g} \in C(\mathbb{R}^s)^p$.

Note also that (2.6) is equivalent to

$$\overline{g_j}(y) = \int_{\mathbb{R}^s} g_j(\lambda)\, d\nu_y(\lambda) \equiv \langle \nu_y, g_j \rangle \qquad \text{for a.e. } y \in \mathbb{R}^m \;, \tag{2.21}$$

$j = 1, \ldots, p$. Therefore we can prove (2.6) componentwise, i.e., consider $p = 1$ and $g \in C(\mathbb{R}^s)$ without loss of generality.

Note finally that since $\|\mathbf{u}^n\|_{L^\infty} \le c$ uniformly in n, we can consider $g \in C_0(\mathbb{R}^s)$ (instead of $g \in C(\mathbb{R}^s)$) without loss of generality.

Now, define the sequence of probability measures

$$\nu_y^n \equiv \delta_{\mathbf{u}^n(y)} \qquad \forall\, y \in \mathbb{R}^m \;,$$

where δ_a stands for the Dirac measure at the point $a \in \mathbb{R}^s$. Then, for any $F \in L^1(\mathbb{R}^m; C_0\,(\mathbb{R}^s))$, we find that

$$\langle \nu_y^n, F(y) \rangle = \int_{\mathbb{R}^s} F(y, \lambda)\, d\nu_y^n(\lambda) = F(y, \mathbf{u}^n(y))$$

is a measurable function of y. Moreover, if we define

$$\nu^n : y \mapsto \nu_y^n \;,$$

we obtain

$$\|\nu^n\|_{L_w^\infty(\mathbb{R}^m; M(\mathbb{R}^s))} = \operatorname*{ess\,sup}_{y \in \mathbb{R}^m} \|\delta_{\mathbf{u}^n(y)}\|_{M(\mathbb{R}^s)} = 1 \,.$$

Therefore, $\{\nu^n\}$ is uniformly bounded in $L_\omega^\infty(\mathbb{R}^m; M(\mathbb{R}^s))$. Hence, there exists a subsequence still labelled ν^n, such that

$$\nu^n \xrightarrow{*} \nu \qquad \text{in } L_\omega^\infty(\mathbb{R}^m; M(\mathbb{R}^s)) \,.$$

This means that for all $F \in L^1(\mathbb{R}^m; C_0\,(\mathbb{R}^s))$

$$\int_{\mathbb{R}^m} \langle \nu_y^n, F(y) \rangle\; dy \to \int_{\mathbb{R}^m} \langle \nu_y, F(y) \rangle\; dy, \qquad \text{as } n \to \infty \,. \tag{2.22}$$

Let us now for $\phi \in L^1(\mathbb{R}^m)$ and $g \in C_0(\mathbb{R}^s)$ define the function F

by $F(y) \equiv \phi(y)g$. Then $F \in L^1(\mathbb{R}^m; C_0(\mathbb{R}^s))$. Using this particular F we obtain from (2.22)

$$\int_{\mathbb{R}^m} \phi(y)\, g(\mathbf{u}^n(y))\, dy \to \int_{\mathbb{R}^m} \phi(y)\ \langle \nu_y, g\rangle\ dy, \qquad \text{as } n \to \infty\,. \tag{2.23}$$

But since the function '$y \mapsto \langle \nu_y, g\rangle$' is an element of the space $L^\infty(\mathbb{R}^m)$ for any $g \in C_0(\mathbb{R}^s)$, (2.23) means nothing other than

$$g \circ \mathbf{u}^n \overset{*}{\rightharpoonup} \langle \nu_{(\cdot)}, g\rangle \equiv \overline{g} \qquad \text{in } L^\infty(\mathbb{R}^m)\,.$$

Furthermore, for $g \in C_0(\mathbb{R}^s)$, $g \geq 0$, $\phi \in L^1(\mathbb{R}^m)$, $\phi \geq 0$ we have, according to (2.23),

$$0 \leq \int_{\mathbb{R}^m} \phi(y)\, g(\mathbf{u}^n(y))\, dy \to \int_{\mathbb{R}^m} \phi(y)\ \langle \nu_y, g\rangle\ dy, \qquad \text{as } n \to \infty \tag{2.24}$$

and consequently $\nu_y \geq 0$ for almost all $y \in \mathbb{R}^m$.

Moreover, thanks to the estimate $\|\mathbf{u}^n\|_{L^\infty} \leq c$, we have $\langle \nu_y, g\rangle = 0$ for all g such that $\operatorname{supp} g \cap \overline{B_{c+1}(0)} = \emptyset$. Hence, $\operatorname{supp} \nu_y \subseteq \overline{B_{c+1}(0)} \equiv K$ (uniformly) for almost all $y \in \mathbb{R}^m$.

Finally, choosing $g_0 \equiv 1$ on $\overline{B_{c+2}(0)}$, g_0 continuous with compact support, $|g_0| \leq 1$, we obtain from (2.6)

$$\overline{g_0}(y) = 1 = \langle \nu_y, g_0\rangle \qquad \text{for a.e. } y \in \mathbb{R}^m\,.$$

Since $\|\nu_y\|_{M(\mathbb{R}^s)}$ is a supremum of expressions of this kind, we see that $\|\nu_y\|_{M(\mathbb{R}^s)} \geq 1$. On the other hand, the weak-* lower semicontinuity of the norm implies that

$$\|\nu_y\|_{M(\mathbb{R}^s)} \leq \liminf \|\nu_y^n\|_{M(\mathbb{R}^s)} = 1\,.$$

In such a way, ν_y is a probability measure on $\mathbb{R}^s$, for almost all $y \in \mathbb{R}^m$, and thus the proof concludes. ■

Remark 2.25 The Young measure ν_y can intuitively be thought of as giving the limit probability distribution of the values $\mathbf{u}^n$ in the neighbourhood of y as $n \to \infty$. More precisely, if $B_\delta(y)$ is a ball centred at y with radius $\delta > 0$, we can define $\nu_y^{n,\delta}$ as

$$\nu_y^{n,\delta}(A) \equiv \operatorname{meas}\big(B_\delta(y)\big)^{-1} \operatorname{meas}\{x \in B_\delta(y)\,,\ \mathbf{u}^n(x) \in A\}\,,$$

i.e., '$\nu_y^{n,\delta}$ is the probability distribution of values $\mathbf{u}^n(x)$ for $x \in B_\delta(y)$'. Then it can be shown (see BALL [1989]) that

$$\nu_y = \lim_{\delta \to 0} \lim_{n \to \infty} \nu_{y,\delta}^n\,,$$

where the convergence is weak-* in the sense of measures. We emphasize the word 'neighbourhood' above: the Young measure is not obtained when limiting $\delta \to 0$ first and then (not taking 'neighbourhood' into account) $n \to \infty$.

Example 2.26 In the case of the sequence $\{\sin nx\}$ (see Example 1.6) it can be shown that the corresponding Young measure can be expressed explicitly as (see DiPerna [1985] for details):

$$d\nu_y(\lambda) = \frac{1}{\pi}\frac{d\lambda}{\sqrt{1-\lambda^2}}\chi_{(-1,1)}(\lambda)\,.$$

Then for all $g \in C(\mathbb{R})$, (2.6) gives

$$\begin{aligned} &g(\sin nx) \overset{*}{\rightharpoonup} \overline{g} \quad \text{in } L^\infty(\mathbb{R}), \qquad \text{where} \\ &\overline{g} = \text{const.} = \frac{1}{\pi}\int_{-1}^{1}\frac{g(\lambda)}{\sqrt{1-\lambda^2}}\,d\lambda\,. \end{aligned}$$

In fact, a simple calculation shows that the integral on the right-hand side is equal to

$$\frac{1}{2\pi}\int_0^{2\pi} g(\sin x)\,dx\,,$$

which corresponds to the result of Lemma 1.7.

Further, we have seen in Example 1.6 that the lack of strong convergence of the sequence $\{\sin nx\}$ implies the inequality (1.9) to occur for $f(u) = u^2$ instead of the desired equality. The following lemma shows that this result holds for any strictly convex function g. Moreover, Lemma 2.27 shows that the inequality of the type (1.9) is typical when strictly convex nonlinearities are superposed to a weak-* converging sequence $\mathbf{u}^n \in L^\infty(\mathbb{R}^m)^s$ which does not converge strongly. More precisely, it will be proved that the equality in (1.9) occurs if and only if the Young measure representing the sequence $\{\mathbf{u}^n\}$ is the family of Dirac ones. As we will see in Theorem 2.31, the latter statement holds if and only if $\mathbf{u}^n$ converges strongly in $L^2_{\text{loc}}(\mathbb{R}^m)^s$.

Lemma 2.27 (generalized Jensen's inequality) *Let g be a strictly convex function, $g : \mathbb{R}^s \to \mathbb{R}$. Let μ be a probability measure on $\mathbb{R}^s$ with compact support. Then*

$$\langle \mu, g\rangle \geq g(\langle \mu, \mathrm{Id}\rangle)\,, \tag{2.28}$$

with equality occurring if and only if μ is a Dirac measure.

PROOF : Note that in our notation $\langle \mu, g\rangle$ is a scalar

$$\int_{\mathbb{R}^s} g(\lambda)\, d\mu(\lambda)\,,$$

while $\langle \mu, \mathrm{Id}\rangle$ denotes a vector

$$\left(\int_{\mathbb{R}^s} \lambda_1\, d\mu(\lambda), \int_{\mathbb{R}^s} \lambda_2\, d\mu(\lambda), \dots, \int_{\mathbb{R}^s} \lambda_s\, d\mu(\lambda)\right) \in \mathbb{R}^s\,.$$

We start the proof by recalling that for any strictly convex $g : \mathbb{R}^s \to \mathbb{R}$ there exists a constant $\beta \in \mathbb{R}^s$ such that

$$g(\lambda) > g(y) + \beta_i(\lambda - y)_i \quad \forall\, \lambda \neq y\,. \tag{2.29}$$

Putting

$$y \equiv \langle \mu, \mathrm{Id}\rangle = \int_{\mathbb{R}^s} \lambda\, d\mu \;\in \mathbb{R}^s$$

we get (since μ is a probability measure):

$$\begin{aligned} \int_{\mathbb{R}^s} \big[g(y) + \beta_i(\lambda_i - y_i)\big]\, d\mu(\lambda) \\ &= g(y) + \beta_i \int_{\mathbb{R}^s} (\lambda_i - y_i)\, d\mu(\lambda) \\ &= g(y) = g(\langle \mu, \mathrm{Id}\rangle)\,. \end{aligned} \tag{2.30}$$

Hence, if $\operatorname{supp}\mu \neq \{y\}$, (2.29) and (2.30) give

$$\langle \mu, g\rangle > g(\langle \mu, \mathrm{Id}\rangle)\,.$$

Otherwise, if $\mu = \delta_y$ then

$$\langle \mu, g\rangle = \int_{\mathbb{R}^s} g(\lambda)\, d\mu(\lambda) = g(y) = g(\langle \mu, \mathrm{Id}\rangle)$$

and equality in (2.28) occurs. ■

We will see in the following theorem that Young measures turn out to be a proper tool in distinguishing between strong and weak convergences of the sequences they represent.

Before the theorem is formulated, let us recall the following fact: if $u^n \overset{*}{\rightharpoonup} u$ weak-* in $L^\infty(\Omega)$, where $\Omega \subseteq \mathbb{R}^m$ is a bounded domain, then clearly $u^n \rightharpoonup u$ weakly in $L^p(\Omega)$, $p \in [1, \infty)$. Moreover, if there exists $r \in [1, \infty)$ such that $u^n \to v$ in $L^r(\Omega)$, then necessarily $u = v$ and $u^n \to u$ in all $L^p(\Omega)$, $p \in [1, \infty)$, cf. EVANS [1990]. We conclude that for uniformly bounded sequences in $L^\infty(\Omega)$ there are just two possibilities: either $u^n \rightharpoonup u$ weakly in $L^p(\Omega)$ for all $p \in [1,\infty)$, or $u^n \to u$ strongly in these spaces. Similarly, if $\Omega \subseteq \mathbb{R}^m$ is an unbounded domain and u^n is a sequence bounded uniformly in $L^\infty(\Omega)$ then a strong convergence of u^n in $L^q_{\text{loc}}(\Omega)$ for some $q \in [1,\infty)$ is equivalent to a strong convergence of u^n in $L^p_{\text{loc}}(\Omega)$ for all $p \in [1,\infty)$. See for example DiPerna [1985, p. 233] or Evans [1990].

The two cases just mentioned are fully characterized by the properties of the corresponding family of Young measures:

Theorem 2.31 *Let $\|\mathbf{u}^n\|_{L^\infty(\mathbb{R}^m)^s} \le c$, suppose (without loss of generality) that $\mathbf{u}^n \overset{*}{\rightharpoonup} \mathbf{u}$ weak-* in $L^\infty(\mathbb{R}^m)^s$. Let a family $\{\nu_y\}_{y\in\mathbb{R}^m} \subset \operatorname{Prob}(\mathbb{R}^s)$ correspond to this weak-* convergent sequence. Then $\mathbf{u}^n$ converges strongly in $L^2_{\text{loc}}(\mathbb{R}^m)^s$ if and only if ν_y reduces to a Dirac measure concentrated at $\mathbf{u}(y)$, i.e.,*

$$\nu_y = \delta_{\mathbf{u}(y)} \qquad \text{for a.e. } y \in \mathbb{R}^m\,. \tag{2.32}$$

Proof : Assume that (2.32) holds. Then (2.5) and (2.6) give

$$\overline{g} \equiv \text{weak-}^* \lim_{n\to\infty} g(\mathbf{u}^n) = g(\mathbf{u})\,, \tag{2.33}$$

for any $g \in C(\mathbb{R}^s)$. Choosing auxiliary functions $g_i(x) = x_i^2$, $i = 1,\ldots,s$, one can write for any compact subset $K \subset \mathbb{R}^m$:

$$\begin{aligned}\int_K |\mathbf{u}^n - \mathbf{u}|^2\,dy &= \sum_{i=1}^s \int_K (u_i^n - u_i)^2\,dy \\ &= \sum_{i=1}^s \left[\int_K (u_i^n)^2\,dy - 2\int_K u_i\,(u_i^n - u_i)\,dy - \int_K (u_i)^2\,dy\right]\,.\end{aligned} \tag{2.34}$$

Now, the first integral in the sum converges to $\int_K (u_i)^2\,dy$, since (2.33) gives $(u_i^n)^2 \overset{*}{\rightharpoonup} (u_i)^2$ in $L^\infty(K)$. The second integral converges to zero, since $u_i \in L^\infty(K) \hookrightarrow L^1(K)$ can be viewed as a test function for the sequence $(u_i^n - u_i)$. Hence $\mathbf{u}^n \to \mathbf{u}$ in $L^2_{\text{loc}}(\mathbb{R}^m)^s$.

Conversely, if $\{\mathbf{u}^n\}$ converges strongly in $L^2_{\text{loc}}(\mathbb{R}^m)^s$ and at the same time is bounded in $L^\infty(\mathbb{R}^m)^s$, one can pass to the limit in the superposition $g \circ \mathbf{u}^n$ for arbitrary $g \in C(\mathbb{R}^s)$. Hence, we obtain from (2.5) and (2.6) that for almost all $y \in \mathbb{R}^m$ it holds

$$\langle \delta_{\mathbf{u}(y)}, g\rangle = g(\mathbf{u}(y)) = \overline{g}(y) = \langle \nu_y, g\rangle\,. \tag{2.35}$$

Repeating this process for all $g \in S$, where S is a countable dense set in $C(\mathbb{R}^s)$, we find that for almost all $y \in \mathbb{R}^m$ measures $\delta_{\mathbf{u}(y)}$ and ν_y coincide on S and (2.32) follows. ∎

Hence, as we have claimed (see (1.10) and earlier claims), the existence proof for hyperbolic equations via the vanishing viscosity method and the Young measure technique consists in showing that the support of the Young measure, representing a weak-* convergent sequence in L^∞, is a point.

At the end of this chapter we show how to prove such a statement using the so-called Murat-Tartar relation, to which we devote the following section.

3.3 The Murat-Tartar relation for non-convex entropies

Up to this point, the systems in more than one space dimension were also permitted. From now on, we restrict ourselves to the case of a scalar equation in one space dimension ($s = d = 1$). We start by proving the following theorem.

Theorem 3.1 (Murat-Tartar relation) *Let $u_0 \in L^\infty(\mathbb{R})$, $f \in C^2(\mathbb{R})$. Let $u^\varepsilon \in L^\infty(\mathbb{R}^+ \times \mathbb{R})$ be solutions to (1.2), satisfying a uniform estimate (1.3). Further, let $\{\nu_{t,x}\}$ be the family of Young measures, corresponding to some weak-* converging subsequence u^k of u^ε, $u^k \equiv u^{\varepsilon_k}$, $\varepsilon_k \to 0+$ as $k \to \infty$. Finally, let η, $q \in C^2(\mathbb{R})$ be any (not necessarily convex) entropy-entropy flux pair, i.e. $q' = f'\eta'$ pointwise. Then the Murat-Tartar relation:*

$$\langle \nu_{t,x}, \mathrm{Id}\; q - f\eta \rangle = \langle \nu_{t,x}, \mathrm{Id} \rangle \, \langle \nu_{t,x}, q \rangle - \langle \nu_{t,x}, f \rangle \, \langle \nu_{t,x}, \eta \rangle \,, \qquad (3.2)$$

holds for almost all $(t,x) \in \mathbb{R}^+ \times \mathbb{R}$.

To prove this theorem, we will need two auxiliary technical lemmas.

Lemma 3.3 (div-curl) *Let $Q \subset \mathbb{R}^2$ be a bounded domain, $Q \in C^{1,1}$. Let*

$$\begin{aligned} w_1^k &\rightharpoonup \overline{w}_1 \,, & w_2^k &\rightharpoonup \overline{w}_2 \,, \\ v_1^k &\rightharpoonup \overline{v}_1 \,, & v_2^k &\rightharpoonup \overline{v}_2 \,, \end{aligned} \qquad (3.4)$$

weakly in $L^2(Q)$, as $k \to \infty$. Suppose that[§]

$$\{\mathrm{div}(v_1^k, v_2^k), \mathrm{curl}(w_1^k, w_2^k)\} \subset E \,,$$

where E is a compact set in $W^{-1,2}(Q)$. Then, for a subsequence,

$$v_1^k\, w_1^k + v_2^k\, w_2^k \to \overline{v}_1\, \overline{w}_1 + \overline{v}_2\, \overline{w}_2 \qquad \text{in } \mathcal{D}'(Q) \text{ as } k \to \infty \,. \qquad (3.5)$$

§ Recall that $\mathrm{curl}(w_1^k, w_2^k) \equiv \frac{\partial w_2^k}{\partial x_1} - \frac{\partial w_1^k}{\partial x_2}$.

PROOF : Since we have $w_i^k \in L^2(Q)$, it follows from the theory of linear elliptic equations that there are uniquely determined solutions $u_i^k \in W^{2,2}(Q) \cap W_0^{1,2}(Q)$ of Dirichlet problems,

$$\begin{aligned} -\Delta u_i^k &= w_i^k \qquad && \text{in } Q\,, \\ u_i^k &= 0 && \text{on } \partial Q\,, \quad i = 1,2\,. \end{aligned} \tag{3.6}$$

Moreover, u_i^k are bounded in $W^{2,2}(Q)$, since weakly convergent sequences w_i^k are bounded in $L^2(Q)$. Consequently,

$$f^k \equiv \frac{\partial u_1^k}{\partial x_1} + \frac{\partial u_2^k}{\partial x_2}$$

is bounded in $W^{1,2}(Q)$ and therefore

$$f^k \rightharpoonup f \qquad \text{in } W^{1,2}(Q) \tag{3.7}$$

for a subsequence still denoted f^k.

Further we set

$$g_i^k \equiv w_i^k + \frac{\partial f^k}{\partial x_i} = -\Delta u_i^k + \frac{\partial}{\partial x_i}\operatorname{div}(u_1^k, u_2^k) \tag{3.8}$$

to obtain, after straightforward calculation,

$$g_1^k = \frac{\partial r^k}{\partial x_2}, \qquad g_2^k = -\frac{\partial r^k}{\partial x_1}$$

for

$$r^k \equiv \frac{\partial u_2^k}{\partial x_1} - \frac{\partial u_1^k}{\partial x_2} = \operatorname{curl}(u_1^k, u_2^k)\,.$$

Now, from (3.6) one can see that $r^k \in W^{1,2}(Q)$ solve

$$\begin{aligned} -\Delta r^k &= \operatorname{curl}(w_1^k, w_2^k) \qquad && \text{in } Q\,, \\ r^k &= s^k && \text{on } \partial Q\,, \end{aligned} \tag{3.9}$$

where s^k can be expressed in terms of traces of partial derivatives of u_i^k. Consequently, due to $W^{1,2}(Q) \hookrightarrow\hookrightarrow L^2(\partial Q)$, s^k lie in a compact set of $L^2(\partial Q)$. But, according to our assumptions, $\operatorname{curl}(w_1^k, w_2^k)$ lie in a compact set of $W^{-1,2}(Q)$. Hence, the linearity of the problem (3.9) together with the continuous dependence on the data

$$\|r^k\|_{1,2} \le c\ \left(\|\operatorname{curl}(w_1^k, w_2^k)\|_{-1,2} + \|s^k\|_{L^2(\partial Q)}\right)$$

gives us that r^k lie in a compact set of $W^{1,2}(Q)$. Finally, it follows that g_i^k lie in a compact set in $L^2(Q)$ and consequently

$$g_i^k \to g_i \qquad \text{in } L^2(Q) \tag{3.10}$$

for a subsequence. It is worth noting at this point that uniqueness of the weak limit in $L^2(Q)$ gives us, together with (3.4), (3.7), (3.8), (3.10), that

$$\overline{w}_i = g_i - \frac{\partial f}{\partial x_i}, \qquad i = 1, 2. \tag{3.11}$$

Now we are ready to prove the convergence (3.5). For $\varphi \in \mathcal{D}(Q)$ we have

$$\int_Q v_i^k w_i^k \varphi \, dx = \int_Q v_i^k (g_i^k - \frac{\partial f^k}{\partial x_i}) \varphi \, dx \equiv I_1 - I_2 ,$$

where due to (3.10), (3.4),

$$I_1 = \int_Q v_i^k g_i^k \varphi \, dx \longrightarrow \int_Q \overline{v}_i g_i \varphi \, dx .$$

Further, because of (3.4), (3.7) together with Rellich's compactness theorem, and the fact that $\operatorname{div}(v_1^k, v_2^k)$ converges strongly in $W^{-1,2}(Q)$ necessarily to $\operatorname{div}(\overline{v}_1, \overline{v}_2)$ (which follows from the assumptions of the theorem), we have

$$\begin{aligned} I_2 &= \int_Q v_i^k \frac{\partial f^k}{\partial x_i} \varphi \, dx \\ &= -\left\langle \operatorname{div}(v_1^k, v_2^k), f^k \varphi \right\rangle_{W_0^{1,2}(Q)} - \int_Q v_i^k f^k \frac{\partial \varphi}{\partial x_i} \, dx \\ &\longrightarrow \; -\langle \operatorname{div}(\overline{v}_1, \overline{v}_2), f\varphi \rangle_{W_0^{1,2}(Q)} - \int_Q \overline{v}_i f \frac{\partial \varphi}{\partial x_i} \, dx \\ &= \int_Q \overline{v}_i \frac{\partial f}{\partial x_i} \varphi \, dx . \end{aligned}$$

Thus, due to (3.11) we have

$$\int_Q v_i^k w_i^k \varphi \, dx \longrightarrow \int_Q \overline{v}_i (g_i - \frac{\partial f}{\partial x_i}) \varphi \, dx = \int_Q \overline{v}_i \overline{w}_i \varphi \, dx ,$$

which proves the assertion. ∎

Lemma 3.12 (Murat) *Let $Q \subset \mathbb{R}^2$ be a bounded domain, $Q \in C^{1,1}$. Let A be a compact set in $W^{-1,2}(Q)$, B be a bounded set in*

$M(Q)$ and C be a bounded set in $W^{-1,p}(Q)$ for some $2 < p \le \infty$. Further, let $D \subset \mathcal{D}'(Q)$ be such that

$$D \subset (A + B) \cap C \,.$$

Then there exists E, a compact set in $W^{-1,2}(Q)$ such that

$$D \subset E \,.$$

Here $M(Q)$ denotes the space of bounded Radon measures on Q.

PROOF : Let $f^k \in D$, then $f^k = g^k + h^k$, $g^k \in A$, $h^k \in B$, $f^k \in C$. Firstly, solving the equations

$$\begin{aligned} -\Delta v^k &= g^k \qquad && \text{in } Q\,, \\ v^k &= 0 && \text{on } \partial Q\,, \end{aligned} \tag{3.13}$$

we get from $g^k \in A$ that v^k lie in a compact set of $W_0^{1,2}(Q)$. Then, using the compactness theorem for measures (cf. Lemma 2.55 in Chapter 1), we deduce from $h^k \in B$ that h^k lie in a compact set of $W^{-1,q}(Q)$ for $1 \le q < 2$. This, and the L^p-theory of elliptic equations (SIMADER [1972, Section 7]), implies that solutions w^k of the equations

$$\begin{aligned} -\Delta w^k &= h^k \qquad && \text{in } Q\,, \\ w^k &= 0 && \text{on } \partial Q \end{aligned} \tag{3.14}$$

lie in a compact set of $W_0^{1,q}(Q)$ for $1 \le q < 2$. In such a way,

$$u^k \equiv v^k + w^k$$

lie in a compact set of $W_0^{1,q}(Q)$ for $1 \le q < 2$. Now, since $-\Delta u^k = f^k$, we have

$$\|f^k\|_{-1,q} = \sup\left\{ |\langle f^k, \varphi\rangle|;\, \varphi \in W_0^{1,q'}(Q), \|\varphi\|_{1,q'} \le 1\right\} \le \|u^k\|_{1,q}$$

and consequently f^k lie in a compact set of $W^{-1,q}(Q)$ for $1 \le q < 2$. But general interpolation theory provides us with an inequality (TRIEBEL [1978, Section 1.11])

$$\|f^k\|_{-1,2} \le c\,\|f^k\|_{-1,p}^{\theta}\,\|f^k\|_{-1,q}^{1-\theta} \qquad \text{for} \quad \frac{1}{2} = \frac{1-\theta}{p'} + \frac{\theta}{q'}\,,$$

which together with $f^k \in C$ (a bounded set in $W^{-1,p}(Q)$) proves the lemma. ∎

Now we are ready to prove the Murat-Tartar relation (3.2).

PROOF (of Theorem 3.1): Recall that by $u^k \equiv u^{\varepsilon_k}$, $\varepsilon_k \to 0+$ as $k \to \infty$, we denoted some weak-* converging subsequence of u^ε. Multiplying (1.2) by $\eta'(u^k)$, we obtain

$$\frac{\partial \eta(u^k)}{\partial t} + \frac{\partial q(u^k)}{\partial x} = \varepsilon_k \frac{\partial^2 \eta(u^k)}{\partial x^2} - \eta''(u^k)\left(\sqrt{\varepsilon_k}\,\frac{\partial u^k}{\partial x}\right)^2 . \tag{3.15}$$

At this point we use the results on parabolic equations of type (1.2) (cf. LADYZHENSKAYA, SOLONNIKOV AND URALTZEVA [1968], see Theorem 2.9 in Chapter 2). Namely, we know that for any open bounded set $Q \subset \mathbb{R}^+ \times \mathbb{R}$ there exists constant c, independent of k such that

$$\|\sqrt{\varepsilon_k}\,\frac{\partial u^k}{\partial x}\|_{L^2(Q)} \le c(Q)\,. \tag{3.16}$$

Now, since $\eta \in C^2$, (1.3) implies $\|\eta''(u^k)\|_{L^\infty(Q)} \le c$, which together with (3.16) yields

$$\|\eta''(u^k)\left(\sqrt{\varepsilon_k}\,\frac{\partial u^k}{\partial x}\right)^2\|_{L^1(Q)} \le c.$$

But the same estimate also holds for the $M(Q)$-norm, since $L^1(Q)$ is imbedded into $M(Q)$ isometrically. Hence,

$$\left\{\eta''(u^k)\left(\sqrt{\varepsilon_k}\,\frac{\partial u^k}{\partial x}\right)^2\right\}_{k\in\mathbb{N}} \subset B\,, \tag{3.17}$$

where B is a bounded set in $M(Q)$.

Further, (3.16) gives

$$\begin{aligned}\|\sqrt{\varepsilon_k}\,\frac{\partial \eta(u^k)}{\partial x}\|_{L^2(Q)} &= \|\sqrt{\varepsilon_k}\,\eta'(u^k)\,\frac{\partial u^k}{\partial x}\|_{L^2(Q)} \\ &\le c\,\|\sqrt{\varepsilon_k}\,\frac{\partial u^k}{\partial x}\|_{L^2(Q)} \le c.\end{aligned} \tag{3.18}$$

Consequently, $\varepsilon_k \dfrac{\partial \eta(u^k)}{\partial x} \to 0$ in $L^2(Q)$ and therefore we have

$$\varepsilon_k \frac{\partial^2 \eta(u^k)}{\partial x^2} \to 0 \text{ in } W^{-1,2}(Q)\,.$$

In particular,

$$\left\{\varepsilon_k \frac{\partial^2 \eta(u^k)}{\partial x^2}\right\}_{k\in\mathbb{N}} \subset A\,, \tag{3.19}$$

where A is a compact set in $W^{-1,2}(Q)$. Note that for $\eta \equiv \mathrm{Id}$ we immediately obtain from (3.19) that

$$\left\{\varepsilon_k \frac{\partial^2 u^k}{\partial x^2}\right\}_{k\in\mathbb{N}} \subset E\,, \tag{3.20}$$

where E is a compact set in $W^{-1,2}(Q)$.

Finally, since (1.3) implies $\|\eta(u^k)\|_\infty \leq c$, $\|q(u^k)\|_\infty \leq c$, the left-hand side of (3.15) lies in a bounded set of $W^{-1,\infty}(Q)$. But this is nothing other than

$$\left\{\varepsilon_k \eta'(u^k) \frac{\partial^2 u^k}{\partial x^2}\right\}_{k\in\mathbb{N}} \subset C\,, \tag{3.21}$$

where C is a bounded set in $W^{-1,\infty}(Q)$.

So, from (3.17), (3.19), (3.21) it follows that we can apply Lemma 3.12 to (3.15). This gives us

$$\begin{aligned}\left\{\varepsilon_k \eta'(u^k) \frac{\partial^2 u^k}{\partial x^2}\right\}_{k\in\mathbb{N}} &= \left\{\frac{\partial \eta(u^k)}{\partial t} + \frac{\partial q(u^k)}{\partial x}\right\}_{k\in\mathbb{N}} \\ &= \left\{\mathrm{curl}_{(t,x)}\left(-q(u^k), \eta(u^k)\right)\right\}_{k\in\mathbb{N}} \subset E\,,\end{aligned} \tag{3.22}$$

where E is some compact set in $W^{-1,2}(Q)$. Eventually we obtain, directly from (3.20), without using Lemma 3.12, that

$$\begin{aligned}\left\{\varepsilon_k \frac{\partial^2 u^k}{\partial x^2}\right\}_{k\in\mathbb{N}} &= \left\{\frac{\partial u^k}{\partial t} + \frac{\partial f(u^k)}{\partial x}\right\}_{k\in\mathbb{N}} \\ &= \left\{\mathrm{div}_{(t,x)}\left(u^k, f(u^k)\right)\right\}_{k\in\mathbb{N}} \subset E\,,\end{aligned} \tag{3.23}$$

where E can be considered to be the same as above. Applying Lemma 3.3 to $w^k = \left(u^k, f(u^k)\right)$, and $v^k = \left(-q(u^k), \eta(u^k)\right)$ we obtain:

$$u^k q(u^k) - f(u^k)\eta(u^k) \to \overline{u}\,\overline{q} - \overline{f}\,\overline{\eta}\,, \tag{3.24}$$

as $k \to \infty$, in the sense of distributions. (Overlines denote the weak limits in $L^2(Q)$.) Since we work with bounded sequences on bounded sets, the left-hand side of (3.24) (at least a subsequence) also converges weak-* in $L^\infty(Q)$ and then necessarily to the right-hand side of (3.24). Using Young measures, this can be expressed as

$$\langle \nu_{t,x}, \mathrm{Id}\; q - f\eta \rangle = \langle \nu_{t,x}, \mathrm{Id} \rangle\ \langle \nu_{t,x}, q \rangle - \langle \nu_{t,x}, f \rangle\ \langle \nu_{t,x}, \eta \rangle\ , \tag{3.25}$$

which is exactly the Murat-Tartar identity (3.2). ∎

In the next section we will see how (3.25) can help us to reduce the support of $\nu_{t,x}$ to a point in the case of scalar equations in one space dimension.

3.4 Scalar hyperbolic equations in one space dimension

In this particular case ($s = d = 1$) we have for $f \in C^1(\mathbb{R})$, $u_0 \in L^\infty(\mathbb{R})$ and $\varepsilon > 0$ a viscous perturbation

$$\begin{aligned} \frac{\partial u^\varepsilon}{\partial t} + \frac{\partial f(u^\varepsilon)}{\partial x} &= \varepsilon \frac{\partial^2 u^\varepsilon}{\partial x^2} \qquad && \text{in } \mathbb{R}^+ \times \mathbb{R}\,, \\ u^\varepsilon(0,\cdot) &= u_0 && \text{in } \mathbb{R}, \end{aligned} \tag{4.1}$$

of the scalar hyperbolic equation

$$\begin{aligned} \frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} &= 0 \qquad && \text{in } \mathbb{R}^+ \times \mathbb{R}\,, \\ u(0,\cdot) &= u_0 && \text{in } \mathbb{R}\,. \end{aligned} \tag{4.2}$$

Recall that if we prove the Young measure $\nu_{t,x}$ corresponding to the sequence $\{u^k\} \subset \{u^\varepsilon\}$, $u^k \overset{*}{\rightharpoonup} u$ in $L^\infty(\mathbb{R}^+ \times \mathbb{R})$ to be a Dirac one supported at $u(t,x)$ for almost all $(t,x) \in \mathbb{R}^+ \times \mathbb{R}$, we obtain two pieces of additional information: firstly, according to Theorem 2.31, u^k converges strongly in $L^2_{\mathrm{loc}}(\mathbb{R}^+ \times \mathbb{R})$ and secondly, due to (2.5) and (2.6), $f(u^k) \overset{*}{\rightharpoonup} f(u)$ for any $f \in C^1(\mathbb{R})$. This in fact implies the existence result for the scalar equation in one space dimension, which is formulated and proved at the end of this section.

Note that, since subsequences of u^ε were taken into account, we do not obtain the uniqueness result via the Young measure technique.

Now we show how to use the Murat-Tartar identity (3.2) to obtain the desired result

$$\nu_{t,x} = \delta_{u(t,x)} \qquad \text{for a.e. } (t,x) \in \mathbb{R}^+ \times \mathbb{R}\,. \tag{4.3}$$

There are various approaches how to prove (4.3) using (3.2). In TARTAR [1983] and VECCHI [1989] it is shown that if there is no interval $[a,b]$, $a < b$, on which f' is constant (we say that f is **genuinely nonlinear**), (4.3) holds. This is done by applying (3.2) to an infinite family of smooth, strictly convex entropy-entropy flux pairs, which in the scalar 1D case can be expressed explicitly.

Moreover, in TARTAR [1983] it is shown that if the scalar flux is a strictly convex function, then it is sufficient to apply (3.2) to a single smooth strictly convex entropy to obtain (4.3).

Another proof is given by Vecchi—see VECCHI [1989, 1990]. He was able to show that even if a genuinely nonlinear function f is not convex, (4.3) holds. This was done by using just one smooth entropy-entropy flux pair. We will follow this approach to obtain the following theorem.

Theorem 4.4 *Let $f \in C^2(\mathbb{R})$ be a genuinely nonlinear function. Let η, $q \in C^2(\mathbb{R})$ satisfy $q' = \eta' f'$ pointwise. Let $\nu \in \mathrm{Prob}(\mathbb{R})$ be a probability measure with compact support $K \subset \mathbb{R}$, such that the Murat-Tartar relation*

$$\langle \nu, \lambda q(\lambda) - f(\lambda)\eta(\lambda)\rangle = \langle \nu, \lambda\rangle \, \langle \nu, q(\lambda)\rangle - \langle \nu, f(\lambda)\rangle \, \langle \nu, \eta(\lambda)\rangle \quad (4.5)$$

holds. Then the support of ν is a point.

PROOF (Vecchi): The basic idea of the proof is as follows: Suppose we have a compactly supported probability measure $\nu \in \mathrm{Prob}(\mathbb{R})$ such that $\operatorname{supp} \nu = K$. Then we define an auxiliary product measure $\nu \otimes \nu$ on $\mathbb{R} \times \mathbb{R}$ (points in $\mathbb{R} \times \mathbb{R}$ being denoted by (λ, τ)), $\operatorname{supp}(\nu \otimes \nu) = K \times K$, a rectangle. Then, if we find an integrable function $F(\lambda, \tau)$ such that

$$\begin{aligned} &F(\lambda, \tau) > 0 \quad (\lambda \neq \tau), \qquad F(\lambda, \lambda) = 0, \\ &\int_{K \times K} F(\lambda, \tau)\, d(\nu(\lambda) \otimes \nu(\tau)) = 0, \end{aligned} \quad (4.6)$$

clearly the support of $\nu \otimes \nu$ must lie on the line $\{\lambda = \tau\}$. Now, if $\operatorname{supp} \nu$ consists of more than one point, then in the set $\operatorname{supp}(\nu \otimes \nu) = \operatorname{supp} \nu \times \operatorname{supp} \nu$ there are points outside the line $\{\lambda = \tau\}$, which is a contradiction. Hence, $\operatorname{supp} \nu$ is a point. The rigorous proof of this idea (slightly generalized) will be given in Lemma 4.13. Now we prove the theorem by constructing the function F, satisfying (4.6).

The right-hand side of (4.5) can be written (using Fubini's theorem—note that $\nu \otimes \nu$ is a product measure) as

$$\begin{aligned} &\frac{1}{2}\Big[\int_K \tau\, d\nu(\tau) \int_K q(\lambda)\, d\nu(\lambda) + \int_K \lambda\, d\nu(\lambda) \int_K q(\tau)\, d\nu(\tau) \\ &- \int_K \eta(\tau)\, d\nu(\tau) \int_K f(\lambda)\, d\nu(\lambda) - \int_K \eta(\lambda)\, d\nu(\lambda) \int_K f(\tau)\, d\nu(\tau)\Big] \\ &= \frac{1}{2} \int_{K \times K} \big(\tau q(\lambda) + \lambda q(\tau) - \eta(\tau) f(\lambda) - \eta(\lambda) f(\tau)\big)\; d(\nu(\lambda) \otimes \nu(\tau)). \end{aligned} \quad (4.7)$$

On the other hand, since $\int_K d\nu(\lambda) = 1$, the left-hand side is equal to

$$\frac{1}{2}\int_{K\times K}\Big(\tau q(\tau) + \lambda q(\lambda) - \eta(\tau)f(\tau) - \eta(\lambda)f(\lambda)\Big)\, d(\nu(\lambda)\otimes\nu(\tau))\,. \tag{4.8}$$

Denote $[a]_\lambda^\tau \equiv a(\tau) - a(\lambda)$. Subtracting (4.7) from (4.8) we can write (4.5) as

$$\int_{K\times K}\left((\tau-\lambda)\,[q]_\lambda^\tau - [\eta]_\lambda^\tau\,[f]_\lambda^\tau\right)\, d(\nu(\lambda)\otimes\nu(\tau)) = 0\,. \tag{4.9}$$

Now, we can choose $\eta \equiv f$ and $q(u) \equiv \int_{u_0}^u [f'(s)]^2\, ds$. Denoting the integrand in (4.9) by $F(\lambda,\tau)$, we obtain:

$$F(\lambda,\tau) = (\tau-\lambda)\left(\int_\lambda^\tau [f'(s)]^2\, ds\right) - \left(f(\tau) - f(\lambda)\right)^2. \tag{4.10}$$

Clearly, $F(\lambda,\lambda) = 0$, while for $\lambda \neq \tau$

$$F(\lambda,\tau) = (\tau-\lambda)\left(\int_\lambda^\tau [f'(s)]^2\, ds\right) - \left(\int_\lambda^\tau f'(s)\, ds\right)^2. \tag{4.11}$$

We recall the classical Jensen's inequality (see e.g. RUDIN [1974]): for any strictly convex function g and $h \in L^1((\lambda,\tau))$ there is:

$$\frac{1}{\tau-\lambda}\int_\lambda^\tau g(h(s))\, ds \geq g\left(\frac{1}{\tau-\lambda}\int_\lambda^\tau h(s)\, ds\right),$$

with equality occurring if and only if $h(s) = \text{const}$ on (λ,τ). Therefore, for $g(s) = s^2$, $h(s) = f'(s) \neq \text{const}$ on (λ,τ), one gets $F(\lambda,\tau) > 0 \quad (\lambda \neq \tau)$. In such a way, the function F satisfies the properties (4.6) and the proof will be finished by proving Lemma 4.13. ■

Remark 4.12 If $f'(s) = \text{const}$ on $[a,b]$, it follows from (4.11) that $F(\lambda,\tau) = 0$ on $[a,b]\times[a,b]$. Consequently, ν can be concentrated either at a point or on the interval, where $f'(s) = 0$, which is in correspondence with the result of TARTAR [1983].

Now we generalize the basic idea used in the proof of Theorem 4.4 for the case of more than one dimension. Note that the expression 'more than one dimension' in this context refers to the dimension s of the space on which the measure μ is defined. Hence, Lemma 4.13 would help to reduce the support of the Young measure to a point in the case of system of s hyperbolic equations, if, of course, there was a function F, satisfying (4.14). However, the problem of finding such a function in general seems to be as difficult as solving the corresponding system of equations. In view of this, the following lemma, even if it holds generally, is of some use only for $s = 1$.

Lemma 4.13 *Let $\mu \in M(\mathbb{R}^s)$ be a non-negative bounded Radon measure,* $\operatorname{supp}\mu = K$*, a compact set. Let $F \in L^1(\mu \otimes \mu)$ satisfy:*

$$\begin{aligned} &F(x,x) = 0 \quad \forall x \in \mathbb{R}^s\,, \\ &F(x,y) > 0 \quad \forall x,y \in \mathbb{R}^s\,, \quad x \neq y\,, \\ &\int_{\mathbb{R}^{2s}} F(\lambda,\tau)\, d(\mu(\lambda) \otimes \mu(\tau)) = 0\,. \end{aligned} \tag{4.14}$$

Then $\operatorname{supp}\mu$ *is at most a point.*

PROOF : Let us assume that there exist $x \neq y$, $x \in K$, $y \in K$ such that $\{x,y\} \subset \operatorname{supp}\mu$. Then there exist compact disjoint neighbourhoods $U(x)$ and $V(y)$ of x and y, respectively. It follows that $F(\lambda,\tau) > 0$ on $U(x) \times V(y)$. But then we get

$$\begin{aligned} 0 &< \int_{U(x)\times V(y)} F(\lambda,\tau)\, d(\mu(\lambda) \otimes \mu(\tau)) \\ &\leq \int_{\mathbb{R}^{2s}} F(\lambda,\tau)\, d(\mu(\lambda) \otimes \mu(\tau)) = 0\,, \end{aligned}$$

which is a contradiction. ■

Finally, we formulate the main theorem of this section.

Theorem 4.15 *Let $f \in C^2(\mathbb{R})$ be genuinely nonlinear, $u_0 \in L^\infty(\mathbb{R})$. Then there exists a weak entropy solution $u \in L^\infty(\mathbb{R}^+ \times \mathbb{R})$ to the scalar conservation law (4.2).*

PROOF : Let u^ε be solutions to the parabolic perturbations (4.1) of (4.2), $u^k \stackrel{*}{\rightharpoonup} u$ in $L^\infty(\mathbb{R}^+ \times \mathbb{R})$, $u^k = u^{\varepsilon_k}$, $\varepsilon_k \to 0+$ as $k \to +\infty$. Denote by $\{\nu_{t,x}\}$ the family of Young measures corresponding to u^k, which exists according to Theorem 2.3. Then we have for all $g \in C(\mathbb{R})$:

$$g \circ u^k \stackrel{*}{\rightharpoonup} \overline{g}\,, \quad \overline{g}(t,x) = \langle \nu_{t,x}, g\rangle \qquad \text{for a.e. } (t,x) \in \mathbb{R}^+ \times \mathbb{R}\,. \tag{4.16}$$

According to Theorem 3.1, the measures $\nu_{t,x}$ satisfy for almost all $(t,x) \in \mathbb{R}^+ \times \mathbb{R}$ the Murat-Tartar relation (3.2) for some entropy-entropy flux pair η, $q \in C^2(\mathbb{R})$ and therefore, according to Theorem

4.4 there exists $a = a(t,x)$ such that

$$\nu_{t,x} = \delta_a \qquad \text{for a.e. } (t,x) \in \mathbb{R}^+ \times \mathbb{R}\,. \tag{4.17}$$

But (2.5) for $g = \mathrm{Id}$ implies

$$u(t,x) = \langle \delta_a, \mathrm{Id} \rangle \qquad \text{for a.e. } (t,x) \in \mathbb{R}^+ \times \mathbb{R}\,,$$

and therefore

$$\nu_{t,x} = \delta_{u(t,x)} \qquad \text{for a.e. } (t,x) \in \mathbb{R}^+ \times \mathbb{R}\,. \tag{4.18}$$

Thus, using (2.5) again, we have

$$f(u^k) \stackrel{*}{\rightharpoonup} f(u)\,, \tag{4.19}$$

which gives that u is a weak solution to (4.2).

Finally, for a convex entropy-entropy flux pair η, $q \in C^2(\mathbb{R})$ we deduce from (3.15) that

$$\frac{\partial \eta(u^k)}{\partial t} + \frac{\partial q(u^k)}{\partial x} \leq \varepsilon_k \frac{\partial^2 \eta(u^k)}{\partial x^2}\,. \tag{4.20}$$

Similarly to (4.19), (4.18) now implies

$$\eta(u^k) \stackrel{*}{\rightharpoonup} \eta(u)\,,$$
$$q(u^k) \stackrel{*}{\rightharpoonup} q(u)\,.$$

Therefore, passing to the limit in (4.20) one finds that u is an entropy solution. Moreover, due to (4.18) and Theorem 2.31, $u^k \to u$ strongly in $L^p_{\mathrm{loc}}(\mathbb{R})$ for all $p \in [1, \infty)$. ■

Remark 4.21 Young measures were given their name after L.C. Young, who in his pioneering works (YOUNG [1937, 1938, 1942]) studied nonlinear problems developing oscillations of solutions. Since the fundamental papers of TARTAR [1979, 1983], see also TARTAR [1990], the theory of measure-valued solutions became important in the context of partial differential equations. For an extensive study of the Young measure technique we also refer to the monograph by ROUBÍČEK [1996].

The Young measure approach can also be successfully used in numerical analysis, namely to prove the convergence of so-called finite volume methods for scalar conservation laws in d space dimensions—for details see, e.g., COCKBURN, COQUEL, LEFLOCH AND SHU [1991] and KRÖNER AND ROKYTA [1994].

CHAPTER 4

Measure-valued solutions and nonlinear hyperbolic equations

4.1 Introduction

Let $Q \subset \mathbb{R}^d$ be a measurable set, on which a sequence of measurable functions $\mathbf{z}^j : Q \longrightarrow \mathbb{R}^s$, $j = 1, 2, \ldots$, is defined. Let $\tau : \mathbb{R}^s \longrightarrow \mathbb{R}$ be a continuous function. We are interested in describing the behaviour of $\tau(\mathbf{z}^j)$ as $j \to \infty$. As already pointed out in Theorem 3.2.3, the behaviour of $\tau(\mathbf{z}^j)$ can be represented by a measure-valued function $\nu : Q \longrightarrow \mathrm{Prob}(\mathbb{R}^s)$ provided that the sequence $\{\mathbf{z}^j\}_{j=1}^{\infty}$ is uniformly bounded in $L^\infty(Q)^s$. More precisely, since $\tau(\mathbf{z}^j)$ is also uniformly bounded in $L^\infty(Q)$, there exists a subsequence still denoted by $\mathbf{z}^j$ and a function $\overline{\tau} \in L^\infty(Q)$ such that

$$\tau(\mathbf{z}^j) \overset{*}{\rightharpoonup} \overline{\tau} \qquad \text{weakly-}^* \text{ in } L^\infty(Q)\,, \tag{1.1}$$

and, according to Theorem 2.3 in Chapter 3, we have

$$\overline{\tau}(y) = \langle \nu_y, \tau \rangle = \int_{\mathbb{R}^s} \tau(\boldsymbol{\lambda})\, d\nu_y(\boldsymbol{\lambda})\,, \tag{1.2}$$

for almost all $y \in Q$. Here we use the notation $\nu_y \equiv \nu(y)$. Moreover, the measures ν_y are uniformly compactly supported. Note that since the values of $\mathbf{z}^j$ belong to some compact set of $\mathbb{R}^s$, we could suppose, without loss of generality, that $\tau \in C_C(\mathbb{R}^s)$.

In this chapter we extend the characterization (1.2) to a broader class of functions τ and $\mathbf{z}^j$. Following Ball [1989], we prove a more general version of Theorem 2.3 in Chapter 3. As a consequence of this generalization, we obtain the following result: If Q is a bounded set and $\mathbf{z}^j$ are uniformly bounded in $L^p(Q)^s$ for some $p \in (1, \infty)$, then there exists a subsequence still denoted by $\mathbf{z}^j$ and a function $\nu : Q \longrightarrow \mathrm{Prob}(\mathbb{R}^s)$ such that for all $\tau \in C(\mathbb{R}^s)$,

$$\begin{aligned} &\tau(\mathbf{z}^j) \rightharpoonup \overline{\tau} \quad \text{weakly in } L^{\frac{p}{p-1}}(Q)\,, \\ &\overline{\tau}(y) = \langle \nu_y, \tau \rangle \quad \text{for a.a. } y \in Q\,, \end{aligned} \tag{1.3}$$

whenever τ satisfies the growth condition

$$\left|\tau(\boldsymbol{\xi})\right| \leq c\left(1+|\boldsymbol{\xi}|\right)^{p-1}, \qquad \boldsymbol{\xi} \in \mathbb{R}^s. \tag{1.4}$$

As we will see, mainly in the next chapter, the nonlinearities satisfying (1.4) occur naturally in problems of continuum mechanics. However, we will use the above-mentioned theorem later in this chapter (see Section 4.3). Using again the method of vanishing viscosity, we prove the global in time existence of a measure-valued solution to the scalar hyperbolic equation of second order:

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial}{\partial x_j} a_j(\nabla u) = f \qquad \text{in } I \times \Omega, \tag{1.5}$$

where $\Omega \subset \mathbb{R}^d$ is a bounded open set and $I = (0,T)$. The equation (1.5) is considered together with Dirichlet boundary conditions and initial conditions for u and $\frac{\partial u}{\partial t}$. From the physical point of view, the unknown function $u : Q_T \to \mathbb{R}$ can be interpreted as a displacement of a vibrating membrane. The functions $a_i : \mathbb{R}^d \longrightarrow \mathbb{R}$ are considered to have linear growth, i.e., $p = 2$ in (1.4). More precisely, we assume that a_i are represented by a scalar quadratic potential.

Recall that equation (1.5) can be rewritten as a hyperbolic system of $(d+1)$ equations. Indeed, putting $v_0 \equiv \frac{\partial u}{\partial t}$ and $v_j \equiv \frac{\partial u}{\partial x_j}$, we obtain for $\widehat{\mathbf{v}} \equiv (v_1, \dots, v_d)$, $\mathbf{v} \equiv (v_0, \widehat{\mathbf{v}})$,

$$\begin{aligned} \frac{\partial v_0}{\partial t} - \frac{\partial}{\partial x_j} a_j(\widehat{\mathbf{v}}) &= f, \\ \frac{\partial v_i}{\partial t} - \frac{\partial v_0}{\partial x_i} &= 0, \quad i = 1, \dots, d. \end{aligned} \tag{1.6}$$

See Section 1.1.3 in Chapter 1 for details.

Finally, let us emphasize that, to the knowledge of the authors, (global in time) existence of a weak solution to (1.5) is still an open problem. This is one of the reasons why the concept of a measure-valued solution to problem (1.5) is introduced. Within this class we obtain an existence result (see Theorem 3.34) and thus the question of the existence of a weak solution to (1.5) can be reformulated as a question of whether the measures ν_y are Dirac ones (compare with Chapter 3).

4.2 A version of the fundamental theorem on Young measures

Following BALL [1989], we prove in this section a version of the theorem about Young measures. This theorem will play a key role in the construction of measure-valued solutions not only in the next section, but also in the following chapter. Let us recall that the symbol $L^\infty_\omega(Q; M(\mathbb{R}^s))$ represents the space of all weakly measurable functions $\nu : Q \longrightarrow M(\mathbb{R}^s)$, for which the norm $\operatorname{ess\,sup}_{y\in Q} \|\nu_y\|_{M(\mathbb{R}^s)}$ is finite (see Chapter 1 and Definition 2.7 in Chapter 3 for more details).

Theorem 2.1 *Let $Q \subset \mathbb{R}^d$ be a measurable set and let $\mathbf{z}^j : Q \to \mathbb{R}^s$, $j = 1, 2, \ldots$, be a sequence of measurable functions. Then there exists a subsequence still denoted by $\mathbf{z}^j$ and a measure-valued function ν with the following properties:*

1. *The function ν satisfies*

$$\nu \in L^\infty_\omega(Q; M(\mathbb{R}^s)), \tag{2.2}$$

$$\|\nu_y\|_{M(\mathbb{R}^s)} \le 1 \qquad \text{for a.e. } y \in Q\,, \tag{2.3}$$

and we have for every $\varphi \in C_0(\mathbb{R}^s)$, as $j \to \infty$,

$$\varphi(\mathbf{z}^j) \stackrel{*}{\rightharpoonup} \overline{\varphi} \quad \text{weakly-* in } L^\infty(Q)\,, \qquad \overline{\varphi}(y) \stackrel{a.e.}{=} \langle \nu_y, \varphi\rangle\,. \tag{2.4}$$

2. *Moreover, if*

$$\lim_{k\to\infty} \sup_{j=1,2,\ldots} \operatorname{meas}\left\{y \in Q \cap B_R\,;\, |\mathbf{z}^j(y)| \ge k\right\} = 0 \tag{2.5}$$

for every $R > 0$, where $B_R \equiv \{y \in Q\,;\, |y| \le R\}$, then

$$\|\nu_y\|_{M(\mathbb{R}^s)} = 1 \qquad \text{for a.e. } y \in Q. \tag{2.6}$$

3. *Let $\Psi : [0, \infty) \longrightarrow \mathbb{R}$ be a Young function satisfying the Δ_2-condition.*† *If condition (2.5) holds and if we have for some continuous function $\tau : \mathbb{R}^s \longrightarrow \mathbb{R}$*

$$\sup_{j=1,2,\ldots} \int_Q \Psi\left(|\tau(\mathbf{z}^j)|\right) dy < \infty\,, \tag{2.7}$$

† Definitions and properties of Young functions as well as Orlicz spaces can be found in Chapter 1.

then

$$\tau(\mathbf{z}^j) \overset{*}{\rightharpoonup} \overline{\tau} \quad \textit{weakly-* in } L_\Psi(Q)\,, \qquad \overline{\tau}(y) \overset{a.e.}{=} \langle \nu_y, \tau\rangle\,. \quad (2.8)$$

Before proving Theorem 2.1 let us make the following remarks.

Remark 2.9 If $\mathbf{z}^j$ are uniformly bounded in $L^p(Q)^s$ for some $p \in [1,\infty)$, the condition (2.5) is satisfied. Indeed, denoting $A_k^j \equiv \{y \in Q \cap B_R; |\mathbf{z}^j(y)| \geq k\}$, we have

$$|A_k^j|\, k^p \leq \int_{A_k^j} |\mathbf{z}^j(y)|^p\, dy \leq \int_Q |\mathbf{z}^j(y)|^p\, dy \leq c\,.$$

Since c is independent of both j and k we obtain

$$\sup_{j=1,2,\dots} |A_k^j| \leq \frac{c}{k^p}\,,$$

which implies (2.5).

Corollary 2.10 *Let $Q \subset \mathbb{R}^d$ be a bounded open set. Let $\mathbf{z}^j$ be uniformly bounded in $L^p(Q)^s$. Then there exists a subsequence still denoted by $\mathbf{z}^j$ and a measure-valued function ν, such that for all $\tau : \mathbb{R}^s \longrightarrow \mathbb{R}$ satisfying for some $q > 0$ the growth condition*

$$\big|\tau(\boldsymbol{\xi})\big| \leq c\,\big(1 + |\boldsymbol{\xi}|\big)^q \qquad \forall \boldsymbol{\xi} \in \mathbb{R}^d\,, \qquad (2.11)$$

we have

$$\begin{aligned} \tau(\mathbf{z}^j) &\rightharpoonup \overline{\tau} \quad \textit{weakly in } L^r(Q)\,,\\ \overline{\tau}(y) &\overset{a.e.}{=} \langle \nu_y, \tau\rangle\,, \end{aligned} \qquad (2.12)$$

provided that

$$1 < r \leq \frac{p}{q}\,. \qquad (2.13)$$

PROOF : Due to Remark 2.9 the condition (2.5) of Theorem 2.1 is satisfied. Let us verify (2.7). We have (for the Young function $\Psi(u) = u^r$)

$$\int_Q \Psi\big(|\tau(\mathbf{z}^j)|\big)\, dy = \int_Q |\tau(\mathbf{z}^j)|^r\, dy \overset{(2.11)}{\leq} c^r \int_Q \big(1 + |\mathbf{z}^j|\big)^{qr}\, dy$$

and the last integral is uniformly bounded (with respect to j) if $qr \leq p$. The lower bound $r > 1$ follows from the properties of Orlicz functions, namely from $\lim_{s\to\infty} \frac{\Psi(s)}{s} = \infty$. ∎

We immediately see from (2.13) that we can easily find an $r \in (1, \frac{p}{q}]$ if $p > q$. The case $p = q$ is not covered by the above result. The investigation of this situation can be found in KINDERLEHRER AND PEDREGAL [1992b].

PROOF (of Theorem 2.1):

Ad 1: With $\mathbf{z}^j$ we associate the mapping $\nu^j : Q \longrightarrow M(\mathbb{R}^s)$ defined for almost all $y \in Q$ by

$$\nu_y^j = \delta_{\mathbf{z}^j(y)}, \tag{2.14}$$

where $\delta_{\mathbf{z}}$ is a Dirac measure supported at $\mathbf{z} \in \mathbb{R}^s$. Hence,

$$\|\nu_y^j\|_{M(\mathbb{R}^s)} = 1 \qquad \text{for a.e. } y \in Q. \tag{2.15}$$

Since $C_0(\mathbb{R}^s)$ and consequently also $L^1(Q; C_0(\mathbb{R}^s))$ are separable, we obtain from Theorem 2.11 in Chapter 3 the existence of a subsequence still denoted by ν^j and an element $\nu \in L_\omega^\infty(Q; M(\mathbb{R}^s))$ such that

$$\nu^j \overset{*}{\rightharpoonup} \nu \qquad \text{weakly-}^* \text{ in } L_\omega^\infty(Q; M(\mathbb{R}^s)). \tag{2.16}$$

Therefore, taking in (2.16) test functions $h \in L^1(Q; C_0(\mathbb{R}^s))$ in the form of $h(y, \boldsymbol{\lambda}) = g(y)\varphi(\boldsymbol{\lambda})$ with $g \in L^1(Q)$ and $\varphi \in C_0(\mathbb{R}^s)$, we see that

$$\langle \nu_y^j, \varphi \rangle \overset{*}{\rightharpoonup} \langle \nu_y, \varphi \rangle \qquad \text{weakly-}^* \text{ in } L^\infty(Q) \tag{2.17}$$

for every $\varphi \in C_0(\mathbb{R}^s)$. Hence, (2.4) is proved, and (2.3) follows from the weak-* lower semicontinuity of the norm $\|\cdot\|_{M(\mathbb{R}^s)}$ and (2.15).

Ad 2: We prove that ν is almost everywhere a probability measure, provided that (2.5) holds. Let us define, for $\boldsymbol{\lambda} \in \mathbb{R}^s$,

$$\vartheta^k(\boldsymbol{\lambda}) \equiv \begin{cases} 1 & |\boldsymbol{\lambda}| \le k\,, \\ 1 + k - |\boldsymbol{\lambda}| & k \le |\boldsymbol{\lambda}| \le k+1\,, \\ 0 & |\boldsymbol{\lambda}| \ge k+1\,. \end{cases} \tag{2.18}$$

Let $E \subset Q$ be an arbitrary bounded measurable set and let $|E|$ denote its Lebesgue measure in $\mathbb{R}^d$. Then, for fixed $k \in \mathbb{N}$, we denote

$$A_k^j \equiv \left\{ y \in E; |\mathbf{z}^j(y)| \ge k \right\}. \tag{2.19}$$

Then

$$\begin{aligned} 0 &\le \frac{1}{|E|}\int_E \left(1-\vartheta^k(\mathbf{z}^j(y))\right)\,dy \\ &\le \frac{|A_k^j|}{|E|} \le \sup_{j=1,2,\dots} \frac{|A_k^j|}{|E|} \equiv \varepsilon_k\,, \end{aligned} \tag{2.20}$$

where $\varepsilon_k \to 0$ if $k \to \infty$ due to (2.5). Therefore

$$1-\varepsilon_k \le \frac{1}{|E|}\int_E \vartheta^k(\mathbf{z}^j(y))\,dy\,. \tag{2.21}$$

Then, letting $j \to \infty$ in (2.21) and using (2.17), we obtain

$$1-\varepsilon_k \le \frac{1}{|E|}\int_E \langle \nu_y, \vartheta^k\rangle\,dy \le \frac{1}{|E|}\int_E \|\nu_y\|_{M(\mathbb{R}^s)}\,dy \overset{(2.3)}{\le} 1. \tag{2.22}$$

Thus,

$$1 = \frac{1}{|E|}\int_E \|\nu_y\|_{M(\mathbb{R}^s)}\,dy, \tag{2.23}$$

which implies that $\|\nu_y\|_{M(\mathbb{R}^s)} = 1$ for almost all $y \in Q$ and (2.6) is proved.

Ad 3: It remains to prove (2.8). Let $\tau : \mathbb{R}^s \to \mathbb{R}$ be a continuous function satisfying (2.7). We can suppose without loss of generality that $\tau \ge 0$. Taking ϑ^k as in (2.18), we put

$$\tau^k(\mathbf{z}) \equiv \tau(\mathbf{z})\vartheta^k(\mathbf{z})\,. \tag{2.24}$$

Choose Φ such that Φ and Ψ are complementary Young functions. According to Lemma 2.30 in Chapter 1 we have $(C_\Phi(\Omega))^* = L_\Psi(\Omega)$. In order to prove (2.8) one must show that

$$\langle \nu_y, \tau\rangle \in L_\Psi(Q) \tag{2.25}$$

and verify the following limiting processes for an arbitrary $g \in C_\Phi(Q)$:

$$\int_Q g(y)\,\tau^k(\mathbf{z}^j(y))\,dy \overset{k\to\infty}{\longrightarrow} \int_Q g(y)\,\tau(\mathbf{z}^j(y))\,dy\,, \tag{2.26}$$

$$\int_Q g(y)\tau^k(\mathbf{z}^j(y))\,dy \overset{j\to\infty}{\longrightarrow} \int_Q g(y)\,\langle \nu_y, \tau^k\rangle\,dy\,, \tag{2.27}$$

$$\int_Q g(y)\langle \nu_y, \tau^k\rangle\,dy \overset{k\to\infty}{\longrightarrow} \int_Q g(y)\langle \nu_y, \tau\rangle\,dy\,, \tag{2.28}$$

where the convergence in (2.26) is meant to be uniform with respect to j. Thus, using the triangle inequality, one can see

$$\int_Q g(y)\,\tau(\mathbf{z}^j(y))\,dy \overset{j\to\infty}{\longrightarrow} \int_Q g(y)\langle \nu_y, \tau\rangle\,dy \qquad \text{for all } \ g\in C_\Phi(Q)\,,$$

which is exactly (2.8). Let us show (2.25)–(2.28). The fact that the convergence in (2.26) is uniform (with respect to j) can be proved directly. Indeed, for A_k^j as in (2.19),

$$\begin{aligned}
&\int_Q |g(y)|\;|\tau^k(\mathbf{z}^j(y)) - \tau(\mathbf{z}^j(y))|\,dy \\
&\qquad \le \int_{A_k^j} |g(y)|\,(1-\vartheta^k(\mathbf{z}^j(y)))\tau(\mathbf{z}^j(y))\,dy \\
&\qquad \overset{(2.29),\ \text{Ch.1}}{\le} \|g\|_{L_\Phi(A_k^j)}\|\tau(\mathbf{z}^j)\|_{L_\Psi(A_k^j)} \\
&\qquad \overset{(2.28),\ \text{Ch.1}}{\le} \|g\|_{L_\Phi(A_k^j)}\left(1+\int_Q \Psi\left(|\tau(\mathbf{z}^j)|\right)dy\right) \\
&\qquad \overset{(2.7)}{\le} C\,\|g\|_{L_\Phi(A_k^j)}\,.
\end{aligned} \tag{2.29}$$

By virtue of (2.5) and the absolute continuity of the Orlicz norm $\|g\|_{L_\Phi(Q')}$ with respect to Q' (see Theorem 2.40 in Chapter 1), one finds that for every $\varepsilon>0$ there exists a k_0 such that for $k>k_0$ the right-hand side of (2.29) is less than ε. Thus (2.26) is proved.

The limiting process in (2.27) follows immediately from (2.4) and the imbedding $C_\Phi(Q)\hookrightarrow L_1(Q)$.

To verify (2.28), note that $\tau^{k+1}\ge\tau^k\ge 0$. Due to the monotone convergence theorem we have for almost all $y\in Q$, as $k\to\infty$,

$$\langle \nu_y, \tau^k\rangle = \int_{\mathbb{R}^N} \tau^k d\nu_y \longrightarrow \int_{\mathbb{R}^N} \tau\, d\nu_y = \langle \nu_y, \tau\rangle\,. \tag{2.30}$$

For $g\ge 0$ we have

$$g(y)\,\langle \nu_y, \tau^{k+1}\rangle \ \ge\ g(y)\,\langle \nu_y, \tau^k\rangle\ \ge 0 \tag{2.31}$$

and from (2.30)

$$\lim_{k\to\infty} g(y)\langle \nu_y, \tau^k\rangle = g(y)\langle \nu_y, \tau\rangle \qquad \text{for a.e. } y\in Q. \tag{2.32}$$

Again, due to the monotone convergence theorem, we obtain (2.28).

Note that $\int_Q g(y)\langle \nu_y, \tau\rangle\,dy$ can still be infinite. Its boundedness is a consequence of (2.25), which will be proved in what follows.

Since $\tau^k \leq \tau^{k+1} \leq \tau$, we have for fixed k,

$$\sup_{j=1,2,\dots} \int_Q \Psi(|\tau^k(\mathbf{z}^j)|)\,dy \leq \sup_{j=1,2,\dots} \int_Q \Psi(|\tau(\mathbf{z}^j)|)\,dy \leq c\,, \quad (2.33)$$

or

$$\|\tau^k(\mathbf{z}^j)\|_{L_\Psi(Q)} = \|\langle \nu^j, \tau^k\rangle\|_{L_\Psi(Q)} \leq c\,, \quad (2.34)$$

where c depends neither on k nor on j. Thus there exist $a_k \in L_\Psi(Q)$ such that, for $j \to \infty$,

$$\langle \nu^j, \tau^k\rangle \overset{*}{\rightharpoonup} a_k \qquad \text{weakly-}^* \text{ in } L_\Psi(Q). \quad (2.35)$$

The comparison of (2.35) with (2.27) gives $a_k(y) = \langle \nu_y, \tau^k\rangle$ for almost every $y \in Q$. Moreover, $a_k = \langle \nu, \tau^k\rangle$ belongs to the ball in $L_\Psi(Q)$ given by (2.34), independently of k. Consequently,

$$\|\langle \nu, \tau^k\rangle\|_{L_\Psi(Q)} \leq \text{ c}. \quad (2.36)$$

Then there exists $a \in L_\Psi(Q)$ such that

$$\int_Q g(y)\,\langle \nu_y, \tau^k\rangle\,dy \longrightarrow \int_Q g(y)a(y)\,dy \quad (2.37)$$

for every $g \in C_\Phi(Q)$, as $k \to \infty$. Comparing (2.37) with (2.28) for smooth test functions g, we obtain

$$a(y) = \langle \nu_y, \tau\rangle \qquad \text{for a.e. } y \in Q \quad (2.38)$$

and (2.25) is proved. The proof of Theorem 2.1 is complete. ■

Remark 2.39 In BALL [1989] the author claims that the theorem can be regarded as a consequence of a general lower semicontinuity theorem proved by BALDER [1984].

The original statement of Theorem 2.1 in BALL [1989] is a bit different. Instead of (2.7) it is assumed that $\{\tau(\mathbf{z}^j)\}_{j=1}^{\infty}$ is sequentially weakly relatively compact in $L^1(Q)$. In fact, this condition is equivalent to (2.7) if Q is bounded and $\Psi : [0,\infty) \longrightarrow \mathbb{R}$ is a continuous function such that $\lim_{|\xi|\to\infty} \frac{\Psi(\xi)}{\xi} = \infty$. In our case, Ψ is moreover a Young function, i.e., a convex one. If we omit this assumption, we only have

$$\tau(\mathbf{z}^j) \rightharpoonup \langle \nu, \tau\rangle \qquad \text{weakly in } L^1(Q)\,,$$

which corresponds to the assertion of the theorem in BALL [1989].

4.3 Measure-valued solutions to a hyperbolic equation of second order

The subject of this section is to prove the existence of measure-valued solutions (global in time) to the following problem.

Let $\Omega \subset \mathbb{R}^d$, $d \geq 2$, be a bounded domain with a smooth boundary $\partial\Omega$. Let $T \in (0, \infty)$, $I \equiv (0, T)$ and $Q_T \equiv I \times \Omega$. Given functions $f : Q_T \longrightarrow \mathbb{R}$, u_0, $u_1 : \Omega \longrightarrow \mathbb{R}$, we look for a function $u : Q_T \longrightarrow \mathbb{R}$, solving

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial}{\partial x_j} a_j(\nabla u) = f \qquad \text{in } Q_T\,, \tag{3.1}$$

$$u(0,\cdot) = u_0\,, \quad \frac{\partial u}{\partial t}(0,\cdot) = u_1 \qquad \text{in } \Omega\,, \tag{3.2}$$

$$u = 0 \qquad \text{on } I \times \partial\Omega\,, \tag{3.3}$$

provided that there exists a function $\vartheta \in C^2(\mathbb{R}^d)$, called **potential** to $\boldsymbol{a} = (a_1, \ldots, a_d)$, and positive constants α, $\beta \in \mathbb{R}$ such that for all $i, j = 1, \ldots, d$,

$$\frac{\partial \vartheta}{\partial \xi_i} = a_i\,, \tag{3.4}$$

$$\vartheta(\mathbf{0}) = \frac{\partial \vartheta}{\partial \xi_i}(\mathbf{0}) = 0\,, \tag{3.5}$$

$$\left| \frac{\partial^2 \vartheta}{\partial \xi_i \partial \xi_j} \right| \leq \beta\,, \tag{3.6}$$

and

$$\alpha |\boldsymbol{\eta}|^2 \leq \frac{\partial^2 \vartheta(\boldsymbol{\xi})}{\partial \xi_k \partial \xi_\ell} \eta_k \eta_\ell \tag{3.7}$$

for all $\boldsymbol{\xi}$, $\boldsymbol{\eta} \in \mathbb{R}^d$.

A simple calculation (see also Section 5.1) shows that (3.4)–(3.7) imply

$$|a_i(\boldsymbol{\xi})| \leq \beta |\boldsymbol{\xi}|\,, \tag{3.8}$$

$$\frac{\alpha}{2} |\boldsymbol{\xi}|^2 \leq \vartheta(\boldsymbol{\xi}) \leq \frac{\beta}{2} |\boldsymbol{\xi}|^2\,, \tag{3.9}$$

$$|a_i(\boldsymbol{\xi}) - a_i(\boldsymbol{\eta})| \leq \beta |\boldsymbol{\xi} - \boldsymbol{\eta}|\,, \tag{3.10}$$

for all $\boldsymbol{\xi}$, $\boldsymbol{\eta} \in \mathbb{R}^d$ and $i = 1, \ldots, d$.

Let us suppose, for a moment, that u^ε are some approximations of the problem (3.1)–(3.7) having the following properties:

$$\nabla u^\varepsilon \text{ is uniformly bounded in } L^2(Q_T)^d\,, \tag{3.11}$$

$$\frac{\partial^2 u^\varepsilon}{\partial t^2} \rightharpoonup \frac{\partial^2 u}{\partial t^2} \text{ weakly in } L^2(I; W^{-1,2}(\Omega))\,. \tag{3.12}$$

Due to (3.11) we can use Theorem 2.1 or rather its Corollary 2.10 with $p = 2$, $q = 1$ and $r = 2$ (for $s = d$) and we obtain the existence of a measure-valued function $\nu : Q_T \longrightarrow \mathrm{Prob}(\mathbb{R}^d)$ such that for each $i = 1, \dots, d$,

$$a_i(\nabla u^\varepsilon) \rightharpoonup \overline{a_i} \quad \text{weakly in } L^2(Q_T)$$

with

$$\overline{a_i}(t, x) \overset{a.e.}{=} \langle \nu_{t,x}, a_i \rangle \,,$$

at least for a subsequence. We see that under assumptions (3.11), (3.12) there is a hope that we will be able to find a limiting process for solutions u^ε of a suitable approximate problem to a measure-valued solution—a couple (u, ν)—of the original problem (3.1)–(3.7).

Now we are in a position to define the notion of a measure-valued solution to the problem (3.1)–(3.7).

Definition 3.13 *Let*

$$f \in L^2(Q_T)\,, \quad u_0 \in W_0^{1,2}(\Omega)\,, \quad u_1 \in L^2(\Omega) \tag{3.14}$$

be given. A couple (u, ν) such that

$$\begin{aligned} u &\in L^2(I; W_0^{1,2}(\Omega))\,, \\ \frac{\partial u}{\partial t} &\in L^2(Q_T)\,, \\ \frac{\partial^2 u}{\partial t^2} &\in L^2(I; W^{-1,2}(\Omega))\,, \\ \nu &\in L^\infty_\omega(Q_T; \mathrm{Prob}\,(\mathbb{R}^d))\,, \end{aligned} \tag{3.15}$$

is called a **measure-valued solution** *to the problem (3.1)–(3.7) if and only if*

$$u(0) = u_0\,, \qquad \frac{\partial u}{\partial t}(0) = u_1\,, \tag{3.16}$$

and the following identities are fulfilled:

$$\frac{\partial u}{\partial x_j}(t, x) = \int_{\mathbb{R}^d} \sigma_j \, d\nu_{t,x}(\boldsymbol{\sigma}) \tag{3.17}$$

for almost all $(t,x) \in Q_T$, *and*

$$\int_0^T \Big(\frac{\partial^2 u}{\partial t^2}(t), \varphi(t)\Big)\, dt + \int_{Q_T} \frac{\partial \varphi}{\partial x_i}(t,x)\, \langle \nu_{t,x}, a_i\rangle \, dx\, dt$$
$$= \int_0^T \big(f(t), \varphi(t)\big)\, dt \tag{3.18}$$

for all $\varphi \in L^2\big(I; W_0^{1,2}(\Omega)\big)$.

Remark 3.19

- We have $u \in C(I; L^2(\Omega))$ and $\frac{\partial u}{\partial t} \in C(I; W^{-1,2}(\Omega))$ due to Lemma 2.45 in Chapter 1. Hence, the initial conditions (3.16) are understood in the sense of the spaces $L^2(\Omega)$, $W^{-1,2}(\Omega)$, respectively, for example

$$\lim_{t \to 0+} \|u(t) - u_0\|_2 = 0\,.$$

- In this section we will denote by $(\cdot,\cdot)$ not only the usual L^2 scalar product, but also the duality between the spaces $W_0^{1,2}(\Omega)$ and $W^{-1,2}(\Omega)$. The reason is that $\langle \cdot,\cdot\rangle$ denotes here the duality between $M(\mathbb{R}^d)$ and $C_0(\mathbb{R}^d)$ (see (3.18)).

In order to prove the existence of a measure-valued solution, we need to find convenient approximations, denoted by u^ε, having (at least) the properties (3.11) and (3.12). For this purpose we perturb the equation (3.1) by the term $-\varepsilon\Delta\frac{\partial u}{\partial t}$ and we will investigate the following problem.

Find, for each $\varepsilon > 0$, a function $u^\varepsilon : Q_T \longrightarrow \mathbb{R}$ satisfying

$$\frac{\partial^2 u^\varepsilon}{\partial t^2} - \frac{\partial}{\partial x_j} a_j(\nabla u^\varepsilon) - \varepsilon\Delta \frac{\partial u^\varepsilon}{\partial t} = f \quad \text{in } Q_T\,, \tag{3.20}$$

$$u^\varepsilon(0,\cdot) = u_0\,, \quad \frac{\partial u^\varepsilon}{\partial t}(0,\cdot) = u_1 \quad \text{in } \Omega\,, \tag{3.21}$$

$$u^\varepsilon = 0 \quad \text{on } I \times \partial\Omega\,. \tag{3.22}$$

Definition 3.23 *Let* f, u_0, u_1 *fulfill (3.14). A function* u^ε, *defined on* Q_T, *is called a weak solution to the problem (3.20)–(3.22),* *(3.4)–(3.7), if and only if*

$$u^\varepsilon \in L^2\big(I; W_0^{1,2}(\Omega)\big)\,, \tag{3.24}$$

$$\frac{\partial u^\varepsilon}{\partial t} \in L^2\big(I; W_0^{1,2}(\Omega)\big)\,, \tag{3.25}$$

$$\frac{\partial^2 u^\varepsilon}{\partial t^2} \in L^2\big(I; W^{-1,2}(\Omega)\big)\,, \tag{3.26}$$

and the following equality

$$\begin{aligned}\int_0^T \Big(\frac{\partial^2 u^\varepsilon}{\partial t^2}(\tau), \varphi(\tau)\Big)\, d\tau + \int_0^T \Big(a_i(\nabla u^\varepsilon(\tau)), \frac{\partial \varphi(\tau)}{\partial x_i}\Big)\, d\tau \\ + \varepsilon \int_0^T \Big(\frac{\partial}{\partial t}\frac{\partial u^\varepsilon}{\partial x_i}(\tau), \frac{\partial \varphi}{\partial x_i}(\tau)\Big)\, d\tau = \int_0^T \big(f(\tau), \varphi(\tau)\big)\, d\tau\end{aligned} \tag{3.27}$$

is fulfilled for all $\varphi \in L^2\big(I; W_0^{1,2}(\Omega)\big)$.

Remark 3.28 The initial conditions (3.21) with assumptions (3.14) are meaningful because of Lemma 2.45 in Chapter 1. In fact, $u^\varepsilon \in C\big(I; W_0^{1,2}(\Omega)\big)$ and $\frac{\partial u^\varepsilon}{\partial t} \in C\big(I; L^2(\Omega)\big)$.

Theorem 3.29 *Let f, u_0 and u_1 satisfy (3.14). Then for every $\varepsilon > 0$ there exists just one weak solution u^ε to the problem (3.20)–(3.22), (3.4)–(3.7). Moreover, u^ε satisfies the following uniform estimates:*

$$\|u^\varepsilon\|_{L^\infty(I; W_0^{1,2}(\Omega))} \le c\,, \tag{3.30}$$

$$\Big\|\frac{\partial u^\varepsilon}{\partial t}\Big\|_{L^\infty(I; L^2(\Omega))} \le c\,, \tag{3.31}$$

$$\varepsilon \int_0^T \Big\|\nabla \frac{\partial u^\varepsilon}{\partial t}(\tau)\Big\|_2^2\, d\tau \le c\,, \tag{3.32}$$

$$\Big\|\frac{\partial^2 u^\varepsilon}{\partial t^2}\Big\|_{L^2(I; W^{-1,2}(\Omega))} \le c\,. \tag{3.33}$$

Although we use the standard Galerkin method to solve the parabolic problem (3.20)–(3.22), we present a detailed proof here, since it shows another method of obtaining the almost everywhere convergence of ∇u^ε, needed for the limiting process in the nonlinearities a_i, $i = 1, \ldots, d$. However, in order to concentrate on our main goal, i.e., to prove the existence of a measure-valued solution to our hyperbolic problem, we postpone the proof of Theorem 3.29 to the end of this section.

Theorem 3.34 *Let f, u_0, u_1 satisfy (3.14). Then there exists a measure-valued solution to the problem (3.1)–(3.7).*

PROOF : Due to the uniform estimates (3.30)–(3.33), we can choose a subsequence $\{u^{\varepsilon_k}\} \subset \{u^\varepsilon\}$ such that

$$u^{\varepsilon_k} \rightharpoonup u \qquad \text{weakly in } L^2(I; W_0^{1,2}(\Omega)), \tag{3.35}$$

$$\frac{\partial u^{\varepsilon_k}}{\partial t} \rightharpoonup \frac{\partial u}{\partial t} \qquad \text{weakly in } L^2(Q_T), \tag{3.36}$$

$$\frac{\partial^2 u^{\varepsilon_k}}{\partial t^2} \rightharpoonup \frac{\partial^2 u}{\partial t^2} \qquad \text{weakly in } L^2(I; W^{-1,2}(\Omega)), \tag{3.37}$$

as $k \to \infty$.

Furthermore, u^{ε_k} are weak solutions to the approximate problem (3.20)–(3.22), (3.4)–(3.7), i.e., they satisfy the equality (3.27) for all $\varphi \in L^2(0, T; W_0^{1,2}(\Omega))$. Let us investigate the limiting process in (3.27).

Due to (3.37),

$$\int_0^T \Big(\frac{\partial^2 u^{\varepsilon_k}}{\partial t^2}(\tau), \varphi(\tau)\Big)\, d\tau \to \int_0^T \Big(\frac{\partial^2 u}{\partial t^2}(\tau), \varphi(\tau)\Big)\, d\tau \tag{3.38}$$

and according to (3.32),

$$\sqrt{\varepsilon_k} \int_0^T \sqrt{\varepsilon_k}\Big(\nabla \frac{\partial u^{\varepsilon_k}}{\partial t}, \nabla\varphi\Big)\, d\tau \to 0\,, \tag{3.39}$$

as $\varepsilon_k \to 0+$. Considering the limit of $\int_0^T \big(a_i(\nabla u^{\varepsilon_k}(t), \frac{\partial \varphi(t)}{\partial x_i}\big)\, dx\, dt$, we apply Corollary 2.10. Since $\|\nabla u^{\varepsilon_k}\|_{L^2(Q_T)} \le c$ due to (3.30), we take $\mathbf{z}^j = \nabla u^{\varepsilon_j}$, $p = 2$, $q = 1$ and $r = 2$, $s = d$ in the above cited corollary. Thus we get the existence of a measure-valued function $\nu \in L^\infty_\omega(Q_T; \mathrm{Prob}(\mathbb{R}^d))$ such that for $\tau = a_i$,

$$a_i(\nabla u^{\varepsilon_k}) \rightharpoonup \overline{a_i} \qquad \text{weakly in } L^2(Q_T),$$

where

$$\overline{a_i}(t, x) \overset{\text{a.e.}}{=} \int_{\mathbb{R}^d} a_i(\boldsymbol{\sigma})\, d\nu_{t,x}(\boldsymbol{\sigma})\,, \quad i = 1, \dots, d.$$

Therefore,

$$\int_{Q_T} a_i(\nabla u^{\varepsilon_k}) \frac{\partial \varphi}{\partial x_i}\, dx\, dt \to \int_{Q_T} \frac{\partial \varphi}{\partial x_i} \langle \nu_{t,x}, a_i\rangle\, dx\, dt\,. \tag{3.40}$$

Using (3.38)–(3.40) in (3.27), we can conclude that the couple (u, ν)

satisfies (3.18) for all $\varphi \in L^2(0,T;W_0^{1,2}(\Omega))$. It remains to verify (3.17). Due to (3.35),

$$\int_{Q_T} \nabla u^{\varepsilon_k} \varphi \, dx \, dt \overset{k\to\infty}{\longrightarrow} \int_{Q_T} \nabla u \, \varphi \, dx \, dt \tag{3.41}$$

for all $\varphi \in \mathcal{D}(Q_T)$. Using again Corollary 2.10 for τ equal to components of **Id**, (the other parameters are the same as above), we obtain

$$\int_{Q_T} \nabla u^{\varepsilon_k} \varphi \, dx \, dt \to \int_{Q_T} \varphi \, \langle \nu_{t,x}, \mathbf{Id} \rangle \, dx \, dt \tag{3.42}$$

for all $\varphi \in \mathcal{D}(Q_T)$. Hence,

$$\int_{Q_T} \varphi(t,x) \langle \nu_{t,x}, \mathbf{Id} \rangle \, dx \, dt = \int_{Q_T} \varphi(t,x) \nabla u(t,x) \, dx \, dt$$

for all $\varphi \in \mathcal{D}(Q_T)$, which implies (3.17). ■

In the rest of this section we will prove Theorem 3.29. For simplicity we drop the superscript ε.

PROOF (of Theorem 3.29):

Uniqueness: Let v, w be two weak solutions of the problem (3.20)–(3.22), (3.4)–(3.7). Putting $u \equiv v - w$, we have for all $t \in (0,T)$ the identity

$$\begin{aligned} &\int_0^t \Big(\frac{\partial^2 u}{\partial t^2}(\tau), \varphi(\tau) \Big) \, d\tau \\ &\qquad + \int_0^t \Big(a_i(\nabla v(\tau)) - a_i(\nabla w(\tau)), \frac{\partial \varphi(\tau)}{\partial x_i} \Big) \, d\tau \\ &\qquad + \varepsilon \int_0^t \Big(\frac{\partial}{\partial x_i} \frac{\partial u}{\partial t}(\tau), \frac{\partial \varphi(\tau)}{\partial x_i} \Big) \, d\tau = 0 \end{aligned} \tag{3.43}$$

for all $\varphi \in L^2\big(0,t;W_0^{1,2}(\Omega)\big)$. Taking $\frac{\partial u}{\partial t}$ as a test function in (3.43), we obtain

$$\begin{aligned} &\frac{1}{2} \Big\| \frac{\partial u}{\partial t}(t) \Big\|_2^2 - \frac{1}{2} \Big\| \frac{\partial u}{\partial t}(0) \Big\|_2^2 + \varepsilon \int_0^t \Big\| \nabla \frac{\partial u}{\partial t}(\tau) \Big\|_2^2 \, d\tau \\ &\quad = \int_0^t \Big(a_i(\nabla w(\tau)) - a_i(\nabla v(\tau)), \frac{\partial}{\partial x_i} \frac{\partial u}{\partial t}(\tau) \Big) \, d\tau \equiv I_1 \,, \end{aligned}$$

or (since $\frac{\partial u}{\partial t}(0) = 0$),

$$\varepsilon \int_0^t \Big\| \nabla \frac{\partial u}{\partial t}(\tau) \Big\|_2^2 \, d\tau \le |I_1| \,. \tag{3.44}$$

Because of (3.10), the right-hand side of (3.44) is estimated by Hölder's and Young's inequalities as follows:

$$|I_1| \le \frac{\varepsilon}{2} \int_0^t \|\nabla \frac{\partial u}{\partial t}(\tau)\|_2^2 \, d\tau + \frac{\beta^2}{2\varepsilon} \int_0^t \|\nabla u(\tau)\|_2^2 \, d\tau \,, \tag{3.45}$$

which gives

$$\int_0^t \|\nabla \frac{\partial u}{\partial t}(\tau)\|_2^2 \, d\tau \le \frac{\beta^2}{\varepsilon^2} \int_0^t \|\nabla u(\tau)\|_2^2 \, d\tau \,. \tag{3.46}$$

Relation $u(\tau) = \int_0^\tau \frac{\partial u}{\partial t}(s) ds$ allows us to rewrite (3.46) in the form

$$\int_0^t \|\nabla \frac{\partial u}{\partial t}(\tau)\|_2^2 \, d\tau \le \frac{\beta^2 T}{\varepsilon^2} \int_0^t \Big(\int_0^\tau \|\nabla \frac{\partial u}{\partial t}(s)\|_2^2 \, ds \Big) \, d\tau \,. \tag{3.47}$$

If we define $y(t) \equiv \int_0^t \|\nabla \frac{\partial u}{\partial t}(\tau)\|_2^2 \, d\tau$ then (3.47) reads

$$y(t) \le \frac{\beta^2 T}{\varepsilon^2} \int_0^t y(\tau) \, d\tau \,,$$

and Gronwall's lemma 3.5 in the Appendix finishes the proof. In fact, we obtain

$$\int_0^t \|\nabla \frac{\partial u}{\partial t}(\tau)\|_2^2 \, d\tau = 0 \quad \forall t \in (0,T] \,.$$

If we again use the relation $u(t) = \int_0^t \frac{\partial u}{\partial t}(s) \, ds$, we obtain

$$\|\nabla u(t)\|_2 = 0 \qquad \forall t \in (0,T] \,, \tag{3.48}$$

which implies $v(t) = w(t)$ for all $t \in [0,T]$.

Existence: Let $\{\omega^k\}$ be a basis in $W_0^{1,2}(\Omega)$ consisting of eigenvectors of the Laplace operator‡ for which λ_k are the corresponding eigenvalues. This means that

$$(\nabla \omega^k, \nabla \varphi) = \lambda_k (\omega^k, \varphi) \qquad \forall \varphi \in W_0^{1,2}(\Omega).$$

If $v(t)$ is an element of $W_0^{1,2}(\Omega)$ then there exist $\gamma_j(t) \in \mathbb{R}$ such

‡ See Remark 4.14 in the Appendix for some comments on the existence of such a basis. Let us recall two advantages of this construction. Firstly, $\{\omega^k\}$ are orthogonal both in $L^2(\Omega)$ and in $W_0^{1,2}(\Omega)$ and they can be orthonormalized either in $W_0^{1,2}(\Omega)$ or in $L^2(\Omega)$. In our case, we assume that the basis is orthonormal in $L^2(\Omega)$. Secondly, $\omega^k \in C^\infty(\overline{\Omega})$ if $\partial\Omega$ is smooth enough, which is a consequence of the regularity theory for elliptic equations.

that $v(t) = \sum_{j=1}^{\infty} \gamma_j(t)\omega^j$. Denoting $v^m(t) = \sum_{j=1}^{m} \gamma_j(t)\omega^j$, we can define the mapping $\mathbb{P}_m$ by

$$\mathbb{P}_m v = v^m \,.$$

Then due to the properties of $\{\omega^k\}$, $\mathbb{P}_m$ is a continuous orthogonal projector both in

$$W_0^{1,2}(\Omega) \to \operatorname{span}\{\omega^1, \omega^2, \dots, \omega^m\},$$

and in

$$L^2(\Omega) \to \operatorname{span}\{\omega^1, \omega^2, \dots, \omega^m\}.$$

Moreover,

$$\|\mathbb{P}_m v\|_{1,2} \le \|v\|_{1,2} \quad \text{and} \quad \|\mathbb{P}_m v\|_2 \le \|v\|_2 \,.$$

Let us look for coefficients $\gamma_j^m : I \to \mathbb{R}$ such that the so-called **Galerkin approximations** u^m in the form

$$u^m(t,x) = \sum_{j=1}^{m} \gamma_j^m(t)\omega^j(x)$$

solve the system of ordinary differential equations

$$\begin{aligned}\Big(\frac{\partial^2 u^m}{\partial t^2}, \omega^j\Big) + \varepsilon\Big(\nabla\frac{\partial u^m}{\partial t}, \nabla\omega^j\Big) + \int_\Omega a_i(\nabla u^m)\frac{\partial \omega^j}{\partial x_i}\,dx \\ = (f, \omega^j)\,, \quad j = 1, \dots, m\,,\end{aligned} \tag{3.49}$$

$$u^m(0,x) = \mathbb{P}_m u_0\,, \quad \frac{\partial u^m}{\partial t}(0,x) = \mathbb{P}_m u_1\,. \tag{3.50}$$

Then the system (3.49), called the **Galerkin system**, can be rewritten in the form (we write $\gamma_j = \gamma_j^m$ for simplicity)

$$\begin{aligned}\frac{d^2}{dt^2}\gamma_j(t) + \varepsilon\,\lambda_j\frac{d}{dt}\gamma_j(t) = (f, \omega^j) + \int_\Omega a_i(\nabla u^m)\frac{\partial \omega^j}{\partial x_i}\,dx\,, \\ \gamma_j(0) = (\mathbb{P}_m(u_0))_j\,, \quad \frac{d}{dt}\gamma_j(0) = (\mathbb{P}_m(u_1))_j\,,\end{aligned} \tag{3.51}$$

where $j = 1, \dots, m$. If we put $z_j(t) \equiv \frac{d}{dt}\gamma_j(t)$ and denote $\mathbf{z} \equiv (z_1, \dots, z_m)$, $\boldsymbol{\lambda} \equiv (\lambda_1, \dots, \lambda_m)$ and $\boldsymbol{\gamma} \equiv (\gamma_1, \dots, \gamma_m)$, then (3.51) takes the form

$$\begin{aligned}\frac{d}{dt}\mathbf{z} + \varepsilon\,\boldsymbol{\lambda}\cdot\mathbf{z} = \mathbf{F}(\boldsymbol{\gamma}, f, \mathbf{a})\,, \\ \frac{d}{dt}\boldsymbol{\gamma} = \mathbf{z}\,,\end{aligned} \tag{3.52}$$

or for $\mathbf{y} \equiv (\mathbf{z}, \boldsymbol{\gamma})^T$ and $\mathbf{H} \equiv (\mathbf{F} - \varepsilon\, \boldsymbol{\lambda} \cdot \mathbf{z}, \mathbf{z})^T$,

$$
\begin{aligned}
&\frac{d}{dt}\mathbf{y} = \mathbf{H}(\mathbf{y})\,, \\
&\mathbf{y}(0) = \mathbf{y}_0\,.
\end{aligned} \tag{3.53}
$$

Since the right-hand side of (3.53) satisfies the Carathéodory conditions, the existence of a solution is ensured due to Theorem 3.4 in the Appendix.

The global existence then follows from the following *a priori* estimates, namely from (3.54) and (3.55). These estimates give uniform bounds on $\mathbf{y}$ on the interval of local existence and allow to prolong the local solution onto the whole interval I.

For all $t \in (0, T]$ we have:

$$\|u^m\|_{L^\infty(0,t;W_0^{1,2}(\Omega))} \le C\,, \tag{3.54}$$

$$\Big\|\frac{\partial u^m}{\partial t}\Big\|_{L^\infty(0,t;L^2(\Omega))} \le C\,, \tag{3.55}$$

$$\varepsilon \int_0^t \Big\|\nabla \frac{\partial u^m}{\partial t}(\tau)\Big\|_2^2 \, d\tau \le C\,, \tag{3.56}$$

$$\Big\|\frac{\partial^2 u^m}{\partial t^2}\Big\|_{L^2(0,t;W^{-1,2}(\Omega))} \le C\,, \tag{3.57}$$

where C is independent of m and ε.

Let us first derive (3.54)–(3.56). Multiplying the jth equation of (3.49) by $\frac{d}{dt}\gamma_j^m(t)$ and adding the resulting equations, we obtain

$$
\begin{aligned}
\Big(\frac{\partial^2 u^m}{\partial t^2}, \frac{\partial u^m}{\partial t}\Big) &+ \varepsilon \Big(\nabla \frac{\partial u^m}{\partial t}, \nabla \frac{\partial u^m}{\partial t}\Big) \\
&+ \int_\Omega a_i(\nabla u^m) \frac{\partial}{\partial x_i} \frac{\partial u^m}{\partial t} \, dx = \Big(f, \frac{\partial u^m}{\partial t}\Big)\,.
\end{aligned} \tag{3.58}
$$

Using (3.4), we obtain

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\Big\|\frac{\partial u^m}{\partial t}\Big\|_2^2 + \varepsilon \Big\|\nabla \frac{\partial u^m}{\partial t}\Big\|_2^2 &+ \frac{d}{dt}\int_\Omega \vartheta(\nabla u^m)\, dx \\
&= \Big(f, \frac{\partial u^m}{\partial t}\Big)\,.
\end{aligned} \tag{3.59}
$$

Let us integrate (3.59) over $(0, t)$, where $t \in (0, T]$. Applying

Hölder's inequality to the right-hand side, we obtain

$$\begin{aligned}
&\frac{1}{2}\left\|\frac{\partial u^m}{\partial t}(t)\right\|_2^2 + \varepsilon\int_0^t \left\|\nabla\frac{\partial u^m}{\partial t}(s)\right\|_2^2 ds + \int_\Omega \vartheta(\nabla u^m(t))\,dx \\
&\le \frac{1}{2}\left\|\frac{\partial u^m}{\partial t}(0)\right\|_2^2 + \int_\Omega \vartheta(\nabla u^m(0))\,dx + \int_0^t \|f(\tau)\|_2 \left\|\frac{\partial u^m}{\partial t}(\tau)\right\|_2 .
\end{aligned}$$

Using (3.9), Young's inequality and the properties of $\mathbb{P}_m$, we obtain

$$\begin{aligned}
&\frac{1}{2}\left\|\frac{\partial u^m}{\partial t}(t)\right\|_2^2 + \varepsilon\int_0^t \left\|\nabla\frac{\partial u^m}{\partial t}(s)\right\|_2^2 ds + \frac{\alpha}{2}\|\nabla u^m(t)\|_2^2 \\
&\le \frac{1}{2}c(u_0,u_1,f) + \frac{1}{2}\int_0^t \left\|\frac{\partial u^m}{\partial t}(\tau)\right\|_2^2 ,
\end{aligned}\tag{3.60}$$

where $c(u_0,u_1,f) \equiv \|u_1\|_2^2 + \beta\|\nabla u_0\|_2^2 + \int_0^t \|f(\tau)\|_2^2\,d\tau$. In particular we have

$$\left\|\frac{\partial u^m}{\partial t}(t)\right\|_2^2 \le c(u_0,u_1,f) + \int_0^t \left\|\frac{\partial u^m}{\partial t}(\tau)\right\|_2^2 . \tag{3.61}$$

Applying Gronwall's lemma 3.5 from the Appendix to (3.61), we obtain

$$\left\|\frac{\partial u^m}{\partial t}(t)\right\|_2^2 \le c(u_0,u_1,f,T) \qquad \forall\, t \le T , \tag{3.62}$$

which is (3.55). Using again the inequality (3.60) together with (3.62) we immediately obtain (3.56) and (3.54), where C depends only on f, u_0, u_1 and T.

In order to obtain (3.57), it is enough to verify that

$$\left|\int_0^t \left(\frac{\partial^2 u^m}{\partial t^2}(\tau), \varphi(\tau)\right) d\tau\right| \le C$$

for all $\varphi \in L^2(0,t;W_0^{1,2}(\Omega))$ satisfying

$$\|\varphi\|_{L^2(0,t;W_0^{1,2}(\Omega))} \le 1 . \tag{3.63}$$

Now,

$$\int_0^t \Big(\frac{\partial^2 u^m}{\partial t^2}, \varphi\Big)\, ds = \int_0^t \Big(\mathbb{P}_m \frac{\partial^2 u^m}{\partial t^2}, \varphi\Big)\, ds = \int_0^t \Big(\frac{\partial^2 u^m}{\partial t^2}, \mathbb{P}_m\varphi\Big)\, ds$$

$$\overset{(3.49)}{=} -\varepsilon \int_0^t \Big(\nabla \frac{\partial u^m}{\partial t}, \nabla \mathbb{P}_m\varphi\Big)\, ds + \int_0^t (f, \mathbb{P}_m\varphi)\, ds$$

$$- \int_0^t \Big(a_i(\nabla u^m), \mathbb{P}_m \frac{\partial \varphi}{\partial x_i}\Big)\, ds\,.$$

Using (3.8), (3.63) and Hölder's inequality we obtain

$$\Big|\int_0^t \Big(\frac{\partial^2 u^m}{\partial t^2}, \varphi\Big)\, ds\Big| \le \varepsilon \int_0^t \|\nabla \frac{\partial u^m}{\partial t}\|_2^2\, ds + \|f\|_{L^2(Q_T)} + c\|u^m\|^2_{L^2(0,t;W_0^{1,2}(\Omega))}\,, \tag{3.64}$$

where we used the continuity of the projector $\mathbb{P}_m$. Since all terms on the right-hand side of (3.64) are bounded (uniformly with respect to m and ε) due to (3.54)–(3.56), we obtain (3.57).

Therefore, there exists a subsequence $\{u^\mu\} \subset \{u^m\}$ such that

$$u^\mu \rightharpoonup u \quad \text{weakly in } L^2(I; W_0^{1,2}(\Omega))\,, \tag{3.65}$$

$$\frac{\partial u^\mu}{\partial t} \overset{*}{\rightharpoonup} z_2 \quad \text{weakly-* in } L^\infty(I; L^2(\Omega))\,, \tag{3.66}$$

$$\frac{\partial u^\mu}{\partial t} \rightharpoonup z_1 \quad \text{weakly in } L^2(I; W_0^{1,2}(\Omega))\,, \tag{3.67}$$

$$\frac{\partial^2 u^\mu}{\partial t^2} \rightharpoonup \omega \quad \text{weakly in } L^2(I; W^{-1,2}(\Omega))\,. \tag{3.68}$$

Since $C^\infty(\Omega)$ is dense both in $W^{1,2}(\Omega)$ and in $L^2(\Omega)$, we have $z_1 = z_2 \equiv z$. Now, from the formula (2.46) in Chapter 1 we obtain for all $\varphi \in C_C(I; W_0^{1,2}(\Omega))$

$$\int_0^t \Big(\frac{\partial u^\mu}{\partial t}(s), \varphi\Big)\, ds = -\int_0^t \Big(u^\mu, \frac{\partial \varphi}{\partial t}\Big)\, dt\,. \tag{3.69}$$

Letting $\mu \to \infty$ in (3.69), we obtain

$$\int_0^t (z, \varphi)\, ds = -\int_0^t \Big(u, \frac{\partial \varphi}{\partial t}\Big)\, dt\,, \tag{3.70}$$

and therefore, $z = \frac{\partial u}{\partial t}$. Analogously, $\omega = \frac{\partial^2 u}{\partial t^2}$.

Next, we want to prove that u is a weak solution to the problem (3.20)–(3.22), (3.4)–(3.7). Immediately from the definition of weak convergence it follows that for $\varphi \in L^2(I; W_0^{1,2}(\Omega))$

$$\int_0^t \left(\frac{\partial^2 u^\mu}{\partial t^2}, \varphi\right) ds \to \int_0^t \left(\frac{\partial^2 u}{\partial t^2}, \varphi\right) ds \tag{3.71}$$

and

$$\varepsilon \int_0^t \left(\nabla \frac{\partial u^\mu}{\partial t}, \nabla\varphi\right) ds \to \varepsilon \int_0^t \left(\nabla \frac{\partial u}{\partial t}, \nabla\varphi\right) ds\,. \tag{3.72}$$

It remains to show

$$\int_0^t \left(a_i(\nabla u^\mu), \frac{\partial \varphi}{\partial x_i}\right) ds \to \int_0^t \left(a_i(\nabla u), \frac{\partial \varphi}{\partial x_i}\right) ds \tag{3.73}$$

for all $\varphi \in L^2(I; W_0^{1,2}(\Omega))$. For this purpose we will prove the strong convergence

$$\nabla u^\mu \to \nabla u \text{ in } L^2(Q_T)\,. \tag{3.74}$$

In fact, we will show that

$$\nabla \frac{\partial u^\mu}{\partial t} \to \nabla \frac{\partial u}{\partial t} \text{ in } L^2(Q_T)\,, \tag{3.75}$$

which implies (3.74) due to (2.47) in Chapter 1. Indeed,

$$\begin{aligned}\|\nabla(u^\mu - u)(t)\|_2^2 &= \|\nabla(u^\mu - u)(0)\|_2^2 \\ &+ 2\int_0^t \left(\nabla\left(\frac{\partial u^\mu}{\partial t} - \frac{\partial u}{\partial t}\right), \nabla\left(u^\mu - u\right)\right) ds\,,\end{aligned} \tag{3.76}$$

and the right-hand side of (3.76) tends to 0 because of assumptions (3.75) and (3.65).

Let $\mathbb{L}_m : L^2(I; W^{1,2}(\Omega)) \longrightarrow L^2(I; \operatorname{span}\{\omega^1, \ldots, \omega^m\})$ be a continuous projector. Then for $\varphi \in L^2(I; W_0^{1,2}(\Omega))$

$$\int_0^T \|\mathbb{L}_m \varphi - \varphi\|_{1,2}^2 \, ds \to 0 \qquad \text{as } m \to \infty\,, \tag{3.77}$$

and the following equality holds (compare with (3.49)):

$$\begin{aligned}&\int_0^t \left(\frac{\partial^2 u^\mu}{\partial t^2}, \mathbb{L}_\mu \varphi\right) ds + \int_0^t \left(a_i(\nabla u^\mu), \frac{\partial \mathbb{L}_\mu \varphi}{\partial x_i}\right) ds \\ &+ \varepsilon \int_0^t \left(\nabla \frac{\partial u^\mu}{\partial t}, \nabla \mathbb{L}_\mu \varphi\right) ds = \int_0^t (f, \mathbb{L}_\mu \varphi)\, ds.\end{aligned} \tag{3.78}$$

We will use $\varphi = \frac{\partial u^\mu}{\partial t} - \frac{\partial u}{\partial t}$ as a test function in (3.78). To make the argument more transparent, let us consider each term in (3.78) separately, starting with the third one.

We have

$$\begin{aligned}
\varepsilon \int_0^t \Big(\nabla \frac{\partial u^\mu}{\partial t}, \nabla \frac{\partial u^\mu}{\partial t} - \nabla \mathbb{L}_\mu \frac{\partial u}{\partial t}\Big)\, ds
&= \varepsilon \int_0^t \Big(\nabla \frac{\partial u^\mu}{\partial t} - \nabla \frac{\partial u}{\partial t}, \nabla \frac{\partial u^\mu}{\partial t} - \nabla \frac{\partial u}{\partial t}\Big)\, ds \\
&\quad + \varepsilon \int_0^t \Big(\nabla \frac{\partial u}{\partial t}, \nabla \frac{\partial u^\mu}{\partial t} - \nabla \frac{\partial u}{\partial t}\Big)\, ds \qquad (3.79)\\
&\quad + \varepsilon \int_0^t \Big(\nabla \frac{\partial u^\mu}{\partial t}, \nabla \frac{\partial u}{\partial t} - \nabla \mathbb{L}_\mu \frac{\partial u}{\partial t}\Big)\, ds \\
&= \varepsilon \int_0^t \Big\|\nabla \Big(\frac{\partial u^\mu}{\partial t} - \frac{\partial u}{\partial t}\Big)\Big\|_2^2\, ds + \delta_\mu^1 + \delta_\mu^2 .
\end{aligned}$$

We see from (3.67) that $\delta_\mu^1 \to 0$ as $\mu \to \infty$. Also, $\delta_\mu^2 \to 0$, since $\frac{\partial u}{\partial t} \in L^2(I; W_0^{1,2}(\Omega))$ and (3.77) and (3.56) hold.

Similarly we will handle the first integral in (3.78). We obtain

$$\begin{aligned}
\int_0^t \Big(\frac{\partial^2 u^\mu}{\partial t^2}, \frac{\partial u^\mu}{\partial t} - \mathbb{L}_\mu \frac{\partial u}{\partial t}\Big)\, ds &= \int_0^t \Big(\frac{\partial^2 u^\mu}{\partial t^2} - \frac{\partial^2 u}{\partial t^2}, \frac{\partial u^\mu}{\partial t} - \frac{\partial u}{\partial t}\Big)\, ds \\
&\quad + \int_0^t \Big(\frac{\partial^2 u}{\partial t^2}, \frac{\partial u^\mu}{\partial t} - \frac{\partial u}{\partial t}\Big)\, ds \\
&\quad + \int_0^t \Big(\frac{\partial^2 u^\mu}{\partial t^2}, \frac{\partial u}{\partial t} - \mathbb{L}_\mu \frac{\partial u}{\partial t}\Big)\, ds \\
&= \frac{1}{2}\Big\|\frac{\partial u^\mu}{\partial t}(t) - \frac{\partial u}{\partial t}(t)\Big\|_2^2 - \frac{1}{2}\Big\|\frac{\partial u^\mu}{\partial t}(0) - \frac{\partial u}{\partial t}(0)\Big\|_2^2 \\
&\quad + \delta_\mu^3 + \delta_\mu^4 ,
\end{aligned}$$

where $\delta_\mu^3 \to 0$ due to (3.67) and $\delta_\mu^4 \to 0$ due to (3.77) and (3.57). Putting

$$\delta_\mu^5 \equiv \frac{1}{2}\Big\|\frac{\partial u^\mu(0)}{\partial t} - \frac{\partial u^{(0)}}{\partial t}\Big\|_2^2 ,$$

we see that $\delta_\mu^5 \to 0$ as $\mu \to \infty$.

Further, the second term in (3.78) can be rewritten as follows:

$$
\begin{aligned}
\int_0^t \Big(a_i(\nabla u^\mu), \frac{\partial}{\partial x_i}\Big(\frac{\partial u^\mu}{\partial t} - \mathbb{L}_\mu \frac{\partial u}{\partial t}\Big)\Big)\, ds
&= \int_0^t \Big(a_i(\nabla u^\mu) - a_i(\nabla u), \frac{\partial}{\partial x_i}\Big(\frac{\partial u^\mu}{\partial t} - \frac{\partial u}{\partial t}\Big)\Big)\, ds \\
&\quad + \int_0^t \Big(a_i(\nabla u), \frac{\partial}{\partial x_i}\Big(\frac{\partial u^\mu}{\partial t} - \frac{\partial u}{\partial t}\Big)\Big)\, ds \\
&\quad + \int_0^t \Big(a_i(\nabla u^\mu), \frac{\partial}{\partial x_i}\Big(\frac{\partial u}{\partial t} - \mathbb{L}_\mu \frac{\partial u}{\partial t}\Big)\Big)\, ds \\
&\equiv I_2 + \delta_\mu^6 + \delta_\mu^7 .
\end{aligned}
$$

From condition (3.8) it follows that $a_i(\nabla u) \in L^2(Q_T)$. This together with (3.67) implies that $\delta_\mu^6 \to 0$. Similarly, $\delta_\mu^7 \to 0$ because of (3.8), (3.54) and (3.77).

We will estimate I_2 by means of (3.10), Hölder's and Young's inequalities. We obtain

$$
\begin{aligned}
|I_2| \le & K(\varepsilon) \int_0^t \|\nabla(u^\mu - u)\|_2^2\, ds \\
& + \frac{\varepsilon}{2} \int_0^t \Big\|\nabla\Big(\frac{\partial u^\mu}{\partial t} - \frac{\partial u}{\partial t}\Big)\Big\|_2^2\, ds .
\end{aligned}
\tag{3.80}
$$

Using the formula

$$
\nabla(u^\mu - u)(s) = \nabla(u^\mu - u)(0) + \int_0^s \nabla \frac{\partial}{\partial t}(u^\mu - u)(\tau)\, d\tau ,
$$

we obtain

$$
\begin{aligned}
\|\nabla(u^\mu - u)(s)\|_2^2 \le & 2\|\nabla u^\mu(0) - \nabla u(0)\|_2^2 \\
& + 2T \int_0^s \Big\|\nabla\Big(\frac{\partial u^\mu}{\partial t} - \frac{\partial u}{\partial t}\Big)(\tau)\Big\|_2^2\, d\tau .
\end{aligned}
$$

Denoting $\delta_\mu^8 \equiv 2TK(\varepsilon)\|\nabla u^\mu(0) - \nabla u(0)\|_2^2$, which clearly converges to 0 as $\mu \to \infty$, we obtain from (3.80)

$$
\begin{aligned}
|I_2| \le \delta_\mu^8 + 2TK(\varepsilon) & \int_0^t \int_0^s \Big\|\nabla\Big(\frac{\partial u^\mu}{\partial t} - \frac{\partial u}{\partial t}\Big)(\tau)\Big\|_2^2\, d\tau\, ds \\
& + \frac{\varepsilon}{2} \int_0^t \Big\|\Big(\frac{\partial u^\mu}{\partial t} - \frac{\partial u}{\partial t}\Big)\Big\|_2^2\, ds .
\end{aligned}
$$

Finally,

$$\delta_\mu^9 \equiv \int_0^t \left(f, \frac{\partial u^\mu}{\partial t} - \mathbb{L}_\mu \frac{\partial u}{\partial t}\right) ds$$

also converges to zero due to (3.77).

Let us put $\delta_\mu \equiv \sum_{j=1}^9 \delta_\mu^j$. Collecting the previous calculations, we obtain from (3.78)

$$\begin{aligned}\frac{\varepsilon}{2} \int_0^t \Big\|\nabla\Big(\frac{\partial u^\mu}{\partial t} - \frac{\partial u}{\partial t}\Big)\Big\|_2^2 ds & \\ \le \delta_\mu + c \int_0^t & \left(\int_0^t \Big\|\nabla\Big(\frac{\partial u^\mu}{\partial \tau} - \frac{\partial u}{\partial t}\Big)\Big\|_2^2 ds\right) d\tau\,.\end{aligned}$$

Since $\delta_\mu \to 0$ when $\mu \to \infty$, the assertion (3.75) is a consequence of Gronwall's lemma 3.5 in the Appendix.

Thus (3.75) and consequently (3.74) is proved. This means that $\nabla u^\mu \to \nabla u$ almost everywhere and also $a_i(\nabla u^\mu) \to a_i(\nabla u)$ almost everywhere, at least for a subsequence. Further, we know that for all measurable $H \subset Q_T$,

$$\int_H |a_i(\nabla u^\mu)|\, dx\, dt \overset{(3.8)}{\le} c \int_H |\nabla u^\mu|\, dx\, dt \le c\|\nabla u^\mu\|_{L^2(Q_T)}\sqrt{|H|}\,,$$

which implies the continuity of $a_i(\nabla u^\mu)$ with respect to the Lebesgue measure. Thus we are in position to apply Vitali's lemma 2.11 in Chapter 1 and we obtain (3.73) for all $\varphi \in L^2(0,T; W_0^{1,2}(\Omega))$. In this way, u satisfies (3.27). Moreover, since u is a weak limit of the sequence u^μ which fulfills the estimates (3.54)–(3.57), u satisfies the same estimates. Therefore, (3.30)–(3.33) hold and the proof of Theorem 3.29 is complete. ∎

CHAPTER 5

Mathematical theory for a class of non-Newtonian fluids

5.1 Introduction

In this chapter we will study solutions (global in time) to systems describing the motion of both incompressible liquids and compressible isothermal gases in a bounded domain $\Omega \subset \mathbb{R}^d$, $d \geq 2$. The equations in question have already been formulated in Section 1.1.4 and Section 1.1.5 in Chapter 1. It is worth recalling the assumptions put on the stress $\boldsymbol{\tau}^E$ for both compressible and incompressible models. As before, $\mathbb{R}^{d^2}_{\text{sym}}$ represents the set of all symmetric $d \times d$ matrices, i.e.,

$$\mathbb{R}^{d^2}_{\text{sym}} \equiv \left\{ \mathbf{M} \in \mathbb{R}^{d^2} ; M_{ij} = M_{ji},\, i,j = 1, \dots, d \right\}.$$

For $\mathbf{u} : (0,T) \times \Omega \longrightarrow \mathbb{R}^d$ we denote, as usual, by $\mathbf{e} = \mathbf{e}(\mathbf{u}) : \mathbb{R}^d \longrightarrow \mathbb{R}^{d^2}_{\text{sym}}$ the symmetric part of the gradient of $\mathbf{u}$. That is, the components of $\mathbf{e}$ are defined by

$$2\, e_{ij} \equiv \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}, \quad i,j = 1, \dots, d.$$

Further, let $\boldsymbol{\tau}, \boldsymbol{\sigma} : \mathbb{R}^{d^2}_{\text{sym}} \longrightarrow \mathbb{R}^{d^2}_{\text{sym}}$ be given continuous functions.

Considering a flow of a **compressible** fluid described by (1.50)–(1.51) in Chapter 1, when $\boldsymbol{\tau}^E = \boldsymbol{\tau}(\mathbf{e})$, we assume that for a certain $p > 1$ and $q \in [p-1, p)$ there exist α, $\beta > 0$ such that for all $\boldsymbol{\eta} \in \mathbb{R}^{d^2}_{\text{sym}}$,

$$\boldsymbol{\tau}(\boldsymbol{\eta}) \cdot \boldsymbol{\eta} \geq \alpha |\boldsymbol{\eta}|^p, \tag{1.1}$$

$$|\boldsymbol{\tau}(\boldsymbol{\eta})| \leq \beta (1 + |\boldsymbol{\eta}|)^q. \tag{1.2}$$

Considering a flow of an **incompressible** fluid described by system (1.55)–(1.56) in Chapter 1, when $\boldsymbol{\tau}^E = \boldsymbol{\tau}(\mathbf{e}) + \boldsymbol{\sigma}(\mathbf{e})$, we assume that there exist $p > 1$, $q \in [p-1, p)$ and α, $\beta > 0$ (usually different from α, β above) such that for all $\boldsymbol{\eta} \in \mathbb{R}^{d^2}_{\mathrm{sym}}$,

$$\boldsymbol{\tau}(\boldsymbol{\eta}) \cdot \boldsymbol{\eta} \geq \alpha|\boldsymbol{\eta}|^p , \tag{1.3}$$

$$\boldsymbol{\sigma}(\boldsymbol{\eta}) \cdot \boldsymbol{\eta} \geq 0, \tag{1.4}$$

$$|\boldsymbol{\tau}(\boldsymbol{\eta}) + \boldsymbol{\sigma}(\boldsymbol{\eta})| \leq \beta(1 + |\boldsymbol{\eta}|)^q . \tag{1.5}$$

Notice that if $\boldsymbol{\tau}$, $\boldsymbol{\sigma}$ fulfil (1.3)–(1.5), $\boldsymbol{\tau} + \boldsymbol{\sigma}$ satisfies (1.1)–(1.2). Unless additional assumptions are imposed on $\boldsymbol{\tau}$ and $\boldsymbol{\sigma}$, the decomposition of $\boldsymbol{\tau}^E$ into $\boldsymbol{\tau}$ and $\boldsymbol{\sigma}$ is not useful. In such cases we will assume that $\boldsymbol{\sigma} \equiv \mathbf{0}$ and $\boldsymbol{\tau}$ satisfies (1.1)–(1.2).

Our aim is to find global solutions to equations of motion for models of both incompressible and compressible fluids. In this chapter we will try to answer the following questions.

1. *For which p do measure-valued solutions exist?*

Let us emphasize that measure-valued functions are expected to appear only as a tool to describe the behaviour of a sequence $\boldsymbol{\tau} \circ (\mathbf{e}(\mathbf{u}^k))$, where $\boldsymbol{\tau}$ is as above and $\mathbf{u}^k$ are some approximate solutions to our problem.† In other words, the limits of the other nonlinearities appearing in the equations will be characterized by integrable functions and will be obtained in the classical weak sense. The restriction on p comes in fact from these limits and not from the limit of $\boldsymbol{\tau} \circ (\mathbf{e}(\mathbf{u}^k))$.

Assumption (1.1) ensures uniform boundedness of the sequence $\mathbf{e}(\mathbf{u}^k)$ in $L^p(Q_T)^{d^2}$. Using the Korn inequality (see Theorem 1.10 below), a uniform bound on $\nabla \mathbf{u}^k$ in $L^p(Q_T)^{d^2}$ can be found. Because $\boldsymbol{\tau}$ satisfies (1.2), we can apply Theorem 2.1 and Corollary 2.10 from Chapter 4 directly in order to obtain the existence of a measure-valued function characterizing the limit of $\boldsymbol{\tau} \circ (\mathbf{e}(\mathbf{u}^k))$, cf. Chapter 4.

The existence of measure-valued solutions will be proved in Section 5.2 (incompressible case) and in Section 5.5 (compressible case).

2. *For which p are measure-valued solutions Dirac ones?*—or equivalently—*For which p do weak solutions exist?*
3. *For which p, if any, do weak solutions have some qualitative properties such as higher regularity, uniqueness, etc.?*

† Let us remark that in this chapter $\mathbf{u}^k$ will denote Galerkin approximations, in contrast to Chapter 4, where $\mathbf{u}^k$ were solutions to perturbed problems.

Satisfactory answers to these questions will be given only in the case of models for incompressible fluids, see Sections 5.3 and 5.4. A complete overview of known results is presented in Section 5.2.3. As will be seen, the most interesting results require some additional assumptions on the tensor function $\boldsymbol{\tau}$. More precisely, for the incompressible model we will assume the existence of a scalar function $U \in C^2(\mathbb{R}^{d^2})$, called **potential of $\boldsymbol{\tau}$**, such that for some $p \in (1,\infty)$, C_1, $C_2 > 0$ we have for all $\boldsymbol{\eta}$, $\boldsymbol{\xi} \in \mathbb{R}^{d^2}_{\mathrm{sym}}$ and i, j, k, $\ell = 1, \ldots, d$,

$$\frac{\partial U(\boldsymbol{\eta})}{\partial \eta_{ij}} = \tau_{ij}(\boldsymbol{\eta}), \tag{1.6}$$

$$U(\mathbf{0}) = \frac{\partial U(\mathbf{0})}{\partial \eta_{ij}} = 0, \tag{1.7}$$

$$\frac{\partial^2 U(\boldsymbol{\eta})}{\partial \eta_{mn} \partial \eta_{rs}} \xi_{mn} \xi_{rs} \geq C_1 \begin{cases} |\boldsymbol{\eta}|^{p-2} |\boldsymbol{\xi}|^2, \\ (1 + |\boldsymbol{\eta}|)^{p-2} |\boldsymbol{\xi}|^2, \end{cases} \tag{1.8}$$

$$\left| \frac{\partial^2 U(\boldsymbol{\eta})}{\partial \eta_{ij} \partial \eta_{k\ell}} \right| \leq C_2 (1 + |\boldsymbol{\eta}|)^{p-2}. \tag{1.9}$$

Let us recall that examples of functions $\boldsymbol{\tau}$ for which a potential U can be found were presented in Chapter 1, Example 1.69.

Notice that $(1.8)_2$ implies $(1.8)_1$ if $p \geq 2$. We understand (1.8) (and the cases in it) in the sense that only one of the two conditions is considered. Let us remark that the typical example of the tensor $\boldsymbol{\tau}$ whose potential satisfies the condition $(1.8)_1$ is

$$\boldsymbol{\tau}\big(\mathbf{e}(\mathbf{u})\big) = |\mathbf{e}(\mathbf{u})|^{p-2} \mathbf{e}(\mathbf{u}),$$

while the tensors

$$\boldsymbol{\tau}\big(\mathbf{e}(\mathbf{u})\big) = \big(1 + |\mathbf{e}(\mathbf{u})|\big)^{p-2} \mathbf{e}(\mathbf{u}),$$
$$\boldsymbol{\tau}\big(\mathbf{e}(\mathbf{u})\big) = \big(1 + |\mathbf{e}(\mathbf{u})^2|\big)^{\frac{p-2}{2}} \mathbf{e}(\mathbf{u}),$$

are standard examples of $\boldsymbol{\tau}$s whose potentials satisfy $(1.8)_2$. Compare with Example 1.73 in Chapter 1.

The properties of $\boldsymbol{\tau}$ which follow from (1.6)–(1.9) are derived at the end of this section. In particular, it will be shown that (1.6)–$(1.8)_1$ imply (1.3), see Lemma 1.19 below.

Finally, let us note that to the knowledge of the authors nothing is known about the existence of weak solutions to the compressible isothermal model even for very large p. This remains, similarly to the case of the nonlinear hyperbolic equation of second order, an interesting open problem (for $d \geq 2$).

Before finishing this section, we first prove the Korn inequality which turns out to be an important mathematical tool in the sequel, and then we will study the properties of $\boldsymbol{\tau}$ which follow from (1.6)–(1.9).

5.1.1 Korn's inequality

Theorem 1.10 *Let $1 < p < \infty$. Then there exists a constant $K_p = K_p(\Omega)$ such that the inequality*

$$K_p \|\mathbf{v}\|_{1,p} \leq \|\mathbf{e}(\mathbf{v})\|_p \tag{1.11}$$

is fulfilled for all $\mathbf{v}$ *satisfying either*

- $\mathbf{v} \in W_0^{1,p}(\Omega)^d$, *where $\Omega \subset \mathbb{R}^d$ is open and bounded, $\partial\Omega \in C^{0,1}$,*

or‡

- $\mathbf{v} \in W_{\text{per}}^{1,p}(\Omega)^d$, *where $\Omega = (0, L)^d$, $L > 0$.*

Remark 1.12 There are several proofs of this theorem if $p = 2$, see for example HLAVÁČEK AND NEČAS [1970], NITSCHE [1981], KONDRATIEV AND OLEINIK [1989], covering more general boundary conditions. For general $p \in (1, \infty)$ see GOBERT [1962, 1971], NEČAS [1966], TEMAM [1985] or FUCHS [1994]. The case of $p = 1$ must be excluded because of the counterexample due to ORNSTEIN [1962]. For $p = 1$ it holds only

$$K_1 \|\mathbf{v}\|_{\frac{d}{d-1}} \leq \|\mathbf{e}(\mathbf{v})\|_1 \,. \tag{1.13}$$

For the proof of (1.13) see ANZELOTTI AND GIAQUINTA [1980].

In order to prove Theorem 1.10 we need one general result concerning distributions with derivatives in $\left(W_0^{1,p}(\Omega)\right)^*$, $\left(W_{\text{per}}^{1,p}(\Omega)\right)^*$, respectively. We restrict ourselves to the first case of Theorem 1.10; the second case is easier.

Theorem 1.14 *Let Ω be an open bounded subset of $\mathbb{R}^d$ with $\partial\Omega \in C^{0,1}$. Let $T \in \mathcal{D}'(\Omega)$. If T, $\frac{\partial T}{\partial x_i} \in \left(W_0^{1,q}(\Omega)\right)^*$ for some $q \in$*

‡ By $W_{\text{per}}^{1,p}(\Omega)^d$ for $\Omega = (0, L)^d$, $L > 0$, we understand the space of space-periodic functions from $W^{1,p}(\Omega)^d$ with mean value zero (see also Section 1.2.4 in Chapter 1).

$(1,\infty)$, *and all* $i = 1,\ldots,d$, *then there exists a function* $u \in L^{q'}(\Omega)$, $q' = \frac{q}{q-1}$, *such that*

$$\langle T, \varphi\rangle = \int_\Omega u\varphi \, dx \qquad \forall\, \varphi \in \mathcal{D}(\Omega)\,.$$

Moreover, there exists $C > 0$ *such that*

$$\|u\|_{q'}^{q'} \le C\Big(\|T\|_{-1,q'}^{q'} + \sum_{i=1}^{d} \|\frac{\partial T}{\partial x_i}\|_{-1,q'}^{q'}\Big)\,. \tag{1.15}$$

PROOF : See NEČAS [1966], where a more general statement is proved. In fact, if $m \in \mathbb{Z}$, $T \in W^{m-1,q'}(\Omega)$ and $\frac{\partial T}{\partial x_i} \in W^{m-1,q'}(\Omega)$ then $T \in W^{m,q'}(\Omega)$ and an inequality corresponding to (1.15) holds. ∎

PROOF (of Theorem 1.10): Let us define the space

$$E(\Omega)^d \equiv \{\mathbf{u} \in L^p(\Omega)^d ; \mathbf{e}(\mathbf{u}) \in L^p(\Omega)^{d^2}\}$$

with $\|\mathbf{u}\|_{E(\Omega)^d} \equiv \|\mathbf{u}\|_p + \|\mathbf{e}(\mathbf{u})\|_p$. Then $E(\Omega)^d$ is a Banach space. Let

$$I : W^{1,p}(\Omega)^d \longrightarrow E(\Omega)^d$$

be the identity mapping. Clearly I is a continuous map. We want to prove the surjectivity of I. Taking $\mathbf{v} \in E(\Omega)^d$ we have, in the sense of distributions, for all i, j, $k = 1,\ldots,d$,

$$\frac{\partial^2 v_i}{\partial x_j \partial x_k} = \frac{\partial e_{ik}(\mathbf{v})}{\partial x_j} + \frac{\partial e_{ij}(\mathbf{v})}{\partial x_k} - \frac{\partial e_{jk}(\mathbf{v})}{\partial x_i}\,. \tag{1.16}$$

Since $\mathbf{e}(\mathbf{v}) \in L^p(\Omega)^{d^2}$, (1.16) implies $\frac{\partial^2 v_i}{\partial x_j \partial x_k} \in \big(W_0^{1,p'}(\Omega)\big)^*$, where p' is the dual exponent to p. Further, we obtain from $\mathbf{v} \in L^p(\Omega)^d$ that $\frac{\partial v_i}{\partial x_j} \in \big(W_0^{1,p'}(\Omega)\big)^*$. Due to Theorem 1.14 we obtain $\frac{\partial v_i}{\partial x_j} \in L^p(\Omega)$ for $i, j = 1,\ldots,d$, therefore $\mathbf{v} \in W^{1,p}(\Omega)^d$ and I is surjective. Hence $W^{1,p}(\Omega)^d$ coincides with $E(\Omega)^d$ and the Open Mapping Theorem (see YOSIDA [1965]) gives the inequality

$$\|\mathbf{v}\|_{1,p} \le c(p,\Omega)\,\big(\|\mathbf{v}\|_p + \|\mathbf{e}(\mathbf{v})\|_p\big)\,. \tag{1.17}$$

It remains to show that for all $\mathbf{v} \in W_0^{1,p}(\Omega)^d$,

$$\|\mathbf{v}\|_p \le \widetilde{c}(p,\Omega)\,\|\mathbf{e}(\mathbf{v})\|_p\,. \tag{1.18}$$

To obtain a contradiction, we will assume that there is a sequence $\{\mathbf{v}^n\}_{n=1}^\infty \subset W_0^{1,p}(\Omega)^d$ such that $\|\mathbf{v}^n\|_p = 1$ and $n\,\|\mathbf{e}(\mathbf{v}^n)\|_p < 1$.

Thus $\mathbf{e}(\mathbf{v}^n) \to \mathbf{0}$ in $L^p(\Omega)^{d^2}$ for $n \to \infty$. Using (1.17), we can choose a subsequence of $\{\mathbf{v}^n\}$ still labelled $\{\mathbf{v}^n\}$ such that $\mathbf{v}^n \rightharpoonup \mathbf{v}$ weakly in $W^{1,p}(\Omega)^d$ and strongly in $L^p(\Omega)^d$. Then $\|\mathbf{v}\|_p = 1$, $\mathbf{v}|_{\partial\Omega} = 0$ and $\mathbf{e}(\mathbf{v}) = \mathbf{0}$. A vector field $\mathbf{v}$ satisfying $\mathbf{e}(\mathbf{v}) = \mathbf{0}$ has an equivalent form $\mathbf{v} = \mathbf{a} + \mathbf{b} \times \mathbf{x}$, see NEČAS AND HLAVÁČEK [1981]. Due to the homogeneous boundary conditions $\mathbf{v}$ has to be identically zero, which contradicts $\|\mathbf{v}\|_p = 1$.

Therefore, inequality (1.18) holds and the proof of Theorem 1.10 is complete. ∎

5.1.2 Two algebraic lemmas

In this section we will derive some inequalities which are consequences of assumptions (1.6)–(1.9).

The cases in (1.20) and (1.25) below are understood to be in agreement with (1.8). This means that $(1.8)_1$ implies $(1.20)_1$ and $(1.25)_1$, and $(1.8)_2$ implies $(1.20)_2$ and $(1.25)_2$.

Lemma 1.19 *Let $p > 1$ and $\boldsymbol{\tau} : \mathbb{R}^{d^2}_{\text{sym}} \longrightarrow \mathbb{R}^{d^2}_{\text{sym}}$, $U : \mathbb{R}^{d^2}_{\text{sym}} \longrightarrow \mathbb{R}$ satisfy (1.6)–(1.9). Then there exist positive constants C_3, C_4, C_5 such that for all $\mathbf{e} \in \mathbb{R}^{d^2}_{\text{sym}}$,*

$$\tau_{ij}(\mathbf{e})e_{ij} \geq C_3 \begin{cases} |\mathbf{e}|^p\,, \\ |\mathbf{e}|(|\mathbf{e}|^{p-1} - 1) \geq C_4(|\mathbf{e}|^p - 1)\,, \end{cases} \tag{1.20}$$

$$|\tau_{ij}(\mathbf{e})| \leq C_5\,(1 + |\mathbf{e}|)^{p-1}\,, \tag{1.21}$$

$i, j = 1, \ldots, d$, and for all $\mathbf{e}$, $\widehat{\mathbf{e}} \in \mathbb{R}^{d^2}_{\text{sym}}$,

$$(\tau_{ij}(\mathbf{e}) - \tau_{ij}(\widehat{\mathbf{e}}))(e_{ij} - \widehat{e}_{ij}) \geq 0\,. \tag{1.22}$$

Further, the inequality $(1.20)_2$ can be replaced by

$$\tau_{ij}(\mathbf{e})e_{ij} \geq C_3 \min(|\mathbf{e}|^2, |\mathbf{e}|^p)\,. \tag{1.23}$$

Moreover, if $p \geq 2$ then

$$\tau_{ij}(\mathbf{e})e_{ij} \geq C_3(1 + |\mathbf{e}|^{p-2})|\mathbf{e}|^2\,, \tag{1.24}$$

and there also exists C_6 such that

$$(\tau_{ij}(\mathbf{e}) - \tau_{ij}(\widehat{\mathbf{e}}))(e_{ij} - \widehat{e}_{ij}) \geq C_6 \begin{cases} |\mathbf{e} - \widehat{\mathbf{e}}|^p\,, \\ (|\mathbf{e} - \widehat{\mathbf{e}}|^2 + |\mathbf{e} - \widehat{\mathbf{e}}|^p)\,. \end{cases} \tag{1.25}$$

If $p \in (1,2)$ and $|\mathbf{e}|$, $|\widehat{\mathbf{e}}| \leq R$ then there exists $C_7 = C_7(R)$ such that

$$(\tau_{ij}(\mathbf{e}) - \tau_{ij}(\widehat{\mathbf{e}}))(e_{ij} - \widehat{e}_{ij}) \geq C_7|\mathbf{e} - \widehat{\mathbf{e}}|^2\,. \tag{1.26}$$

PROOF : Due to (1.6) and (1.7), we have

$$\begin{aligned}\tau_{ij}(\mathbf{e}) = \frac{\partial U(\mathbf{e})}{\partial e_{ij}} - \frac{\partial U(\mathbf{0})}{\partial e_{ij}} &= \int_0^1 \frac{d}{ds}\frac{\partial U(s\mathbf{e})}{\partial e_{ij}}\,ds \\ &= \int_0^1 \frac{\partial^2 U(s\mathbf{e})}{\partial e_{ij}\partial e_{k\ell}} e_{k\ell}\,ds\,.\end{aligned} \tag{1.27}$$

Using (1.9) we obtain

$$\begin{aligned}|\tau_{ij}(\mathbf{e})| &\le C_2 d^2 \int_0^1 (1+s|\mathbf{e}|)^{p-2}|\mathbf{e}|\,ds = \frac{C_2 d^2}{p-1}\left[(1+s|\mathbf{e}|)^{p-1}\right]_0^1 \\ &\le \frac{C_2 d^2}{p-1}(1+|\mathbf{e}|)^{p-1}\,,\end{aligned}$$

which is (1.21) with $C_5 = \frac{C_2 d^2}{p-1}$. From (1.27) and (1.8) we also have

$$\tau_{ij}(\mathbf{e})e_{ij} \ge C_1 \begin{cases} \int_0^1 s^{p-2}\,ds\; |\mathbf{e}|^p = \frac{1}{p-1}|\mathbf{e}|^p\,, \\ \int_0^1 (1+s|\mathbf{e}|)^{p-2}\,ds\; |\mathbf{e}|^2\,. \end{cases} \tag{1.28}$$

Since $(1.28)_1$ proves $(1.20)_1$, we henceforth pay attention to an estimate of $(1.28)_2$ from below.

Let $\alpha > 0$ be arbitrary. Then

$$\max_{x\in[0,\infty)} \frac{1+x^\alpha}{(1+x)^\alpha} < 2\,.$$

This implies

$$\begin{aligned} (1+s|\mathbf{e}|)^{p-2} &\ge \tfrac{1}{2}(1+(s|\mathbf{e}|)^{p-2})\,, &\quad \text{if } p\ge 2\,, \\ (1+|\mathbf{e}|)^{p-1} &\ge \tfrac{1}{2}(1+|\mathbf{e}|^{p-1})\,, &\quad \text{if } p>1\,. \end{aligned} \tag{1.29}$$

Therefore, for $p \ge 2$,

$$\tau_{ij}(\mathbf{e})e_{ij} \ge \frac{C_1}{2}\int_0^1 (1+(s|\mathbf{e}|)^{p-2})\,ds\,|\mathbf{e}|^2 \ge \frac{C_1}{2(p-1)}(1+|\mathbf{e}|^{p-2})\,|\mathbf{e}|^2\,,$$

which proves (1.24) with $C_3 = \frac{C_1}{2(p-1)}$ and also trivially (1.23) for $p \ge 2$.

If $p > 1$, then it follows from $(1.28)_2$ that

$$\begin{aligned}\tau_{ij}(\mathbf{e})e_{ij} &\ge \frac{C_1}{p-1}|\mathbf{e}|\left[(1+s|\mathbf{e}|)^{p-1}\right]_0^1 \\ &= \frac{C_1}{p-1}|\mathbf{e}|\big((1+|\mathbf{e}|)^{p-1}-1\big)\,.\end{aligned} \tag{1.30}$$

Using $(1.29)_2$ we obtain

$$\tau_{ij}(\mathbf{e})e_{ij} \geq \frac{C_1}{2}\frac{|\mathbf{e}|}{p-1}(1+|\mathbf{e}|^{p-1}-2) = \frac{C_1|\mathbf{e}|}{2(p-1)}(|\mathbf{e}|^{p-1}-1)\,,$$

which is the first inequality in $(1.20)_2$. The second inequality in $(1.20)_2$ then follows from the fact that

$$|\mathbf{e}| = 1|\mathbf{e}| \leq \frac{1}{2}|\mathbf{e}|^p + \hat{C}_3 1^{p'}\,.$$

In order to prove (1.23) for $p \in (1,2)$ we start with (1.30) and we will show that

$$\begin{aligned} (1+|\mathbf{e}|)^{p-1} - 1 &\geq c|\mathbf{e}| && \text{for } 0 \leq |\mathbf{e}| \leq 1\,, \\ (1+|\mathbf{e}|)^{p-1} - 1 &\geq c|\mathbf{e}|^{p-1} && \text{for } 1 \leq |\mathbf{e}| < \infty\,. \end{aligned} \tag{1.31}$$

Looking for the minimum of the function $f(y) = \frac{(1+y)^{p-1}-1}{y}$ on the interval $[0,1]$, we find $f(1) = 2^{p-1}-1 > 0$, $f(0) = p-1 > 0$ and $f(y) > 0$ on $(0,1)$. Thus $(1.31)_1$ is proved. Seeking the minimum of the function $g(y) = \frac{(1+y)^{p-1}-1}{y^{p-1}}$ on the interval $[1,\infty)$, we see that

$$\begin{aligned} \min_{y\in[1,\infty)} g(y) &= \min_{y\in[1,\infty)} \left(1+\frac{1}{y}\right)^{p-1} - \left(\frac{1}{y}\right)^{p-1} \\ &= \min_{z\in(0,1]} (1+z)^{p-1} - z^{p-1} \equiv \min_{z\in(0,1]} h(z)\,. \end{aligned}$$

We find $h(0) = 1$, $h(1) = 2^{p-1}-1$ and $h'(z) < 0$ on $[0,1]$. Hence we have $\min g(y) = g(1) > 0$, which shows $(1.31)_2$ and therefore also (1.23).

Next,

$$\begin{aligned} &(\tau_{ij}(\mathbf{e}) - \tau_{ij}(\widehat{\mathbf{e}}))(e_{ij} - \widehat{e}_{ij}) \\ &\quad\overset{(1.6)}{=} \int_0^1 \frac{d}{ds}\Big(\frac{\partial U(\widehat{\mathbf{e}} + s(\mathbf{e}-\widehat{\mathbf{e}}))}{\partial e_{ij}}\Big)\, ds\, (e_{ij} - \widehat{e}_{ij}) \\ &\quad= \int_0^1 \frac{\partial^2 U(\widehat{\mathbf{e}} + s(\mathbf{e}-\widehat{\mathbf{e}}))}{\partial e_{ij}\partial e_{k\ell}}(e_{ij} - \widehat{e}_{ij})(e_{k\ell} - \widehat{e}_{k\ell})\, ds \qquad (1.32) \\ &\quad\overset{(1.8)}{\geq} C_1|\mathbf{e}-\widehat{\mathbf{e}}|^2 \begin{cases} \int_0^1 |\widehat{\mathbf{e}} + s(\mathbf{e}-\widehat{\mathbf{e}})|^{p-2}\, ds\,, \\ \int_0^1 \big(1 + |\widehat{\mathbf{e}} + s(\mathbf{e}-\widehat{\mathbf{e}})|\big)^{p-2}\, ds\,, \end{cases} \end{aligned}$$

which gives (1.22) for general $p > 1$. If $p \in (1,2)$ and $|\mathbf{e}| \leq R$ and

$|\widehat{\mathbf{e}}| \le R$, then

$$\int_0^1 |s\mathbf{e} + (1-s)\widehat{\mathbf{e}}|^{p-2}\, ds \ge \frac{1}{R^{2-p}}\,,$$

and

$$\int_0^1 \big(1 + |s\mathbf{e} + (1-s)\widehat{\mathbf{e}}|\big)^{p-2}\, ds \ge \frac{1}{(1+R)^{2-p}}\,,$$

and the strong monotonicity condition (1.26) holds.

If $p \ge 2$, then according to $(1.29)_1$,

$$\begin{aligned}\int_0^1 \big(1 + |\widehat{\mathbf{e}} + s(\mathbf{e} - \widehat{\mathbf{e}})|\big)^{p-2}\, ds \\ \ge \frac{1}{2}\Big(1 + \int_0^1 |\widehat{\mathbf{e}} + s(\mathbf{e} - \widehat{\mathbf{e}})|^{p-2}\, ds\Big)\,.\end{aligned} \tag{1.33}$$

If we show the existence of a constant C such that

$$\int_0^1 |\widehat{\mathbf{e}} + s(\mathbf{e} - \widehat{\mathbf{e}})|^{p-2}\, ds \ge C\,|\mathbf{e} - \widehat{\mathbf{e}}|^{p-2}\,, \tag{1.34}$$

then assertion (1.25) follows from (1.32)–(1.34).

In order to prove (1.34), let us consider two cases. If $|\widehat{\mathbf{e}}| \ge |\mathbf{e} - \widehat{\mathbf{e}}|$ then $|\widehat{\mathbf{e}} + s(\mathbf{e} - \widehat{\mathbf{e}})| \ge \big||\widehat{\mathbf{e}}| - s|\mathbf{e} - \widehat{\mathbf{e}}|\big| \ge (1-s)|\mathbf{e} - \widehat{\mathbf{e}}|$ and (1.34) holds.

If $|\widehat{\mathbf{e}}| < |\mathbf{e} - \widehat{\mathbf{e}}|$ then

$$\begin{aligned}\int_0^1 |\widehat{\mathbf{e}} + s(\mathbf{e} - \widehat{\mathbf{e}})|^{p-2}\, ds &= \int_0^1 \frac{\big(|\widehat{\mathbf{e}} + s(\mathbf{e} - \widehat{\mathbf{e}})|^2\big)^{\frac{p}{2}}}{|\widehat{\mathbf{e}} + s(\mathbf{e} - \widehat{\mathbf{e}})|^2}\, ds \\ &\ge \frac{1}{2|\mathbf{e} - \widehat{\mathbf{e}}|^2}\Big(\int_0^1 |s\mathbf{e} + (1-s)\widehat{\mathbf{e}}|^2\, ds\Big)^{\frac{p}{2}} \\ &= \frac{1}{2|\mathbf{e} - \widehat{\mathbf{e}}|^2}\frac{1}{3^{\frac{p}{2}}}\big(|\mathbf{e}|^2 + (\mathbf{e}, \widehat{\mathbf{e}}) + |\widehat{\mathbf{e}}|^2\big)^{\frac{p}{2}} \\ &\ge \frac{1}{2|\mathbf{e} - \widehat{\mathbf{e}}|^2}\frac{1}{6^{\frac{p}{2}}}\big(|\mathbf{e}|^2 + |\widehat{\mathbf{e}}|^2\big)^{\frac{p}{2}}\,.\end{aligned}$$

However, $|\mathbf{e} - \widehat{\mathbf{e}}|^2 \le 2\big(|\mathbf{e}|^2 + |\widehat{\mathbf{e}}|^2\big)$ and (1.34) follows. The proof of Lemma 1.19 is complete. ∎

Lemma 1.35 *Let $p > 1$ and the assumptions (1.6)–(1.9) be satisfied. Then there exist C_8, C_9, C_{10} such that*

$$C_9(1 + |\mathbf{e}|)^p \ge U(\mathbf{e}) \ge C_8 \begin{cases} |\mathbf{e}|^p\,, \\ |\mathbf{e}|(|\mathbf{e}|^{p-1} - p)\,. \end{cases} \tag{1.36}$$

If $p \geq 2$ then

$$C_{10}(1 + |\mathbf{e}|^{p-2})|\mathbf{e}|^2 \leq U(\mathbf{e})\,. \tag{1.37}$$

PROOF : Starting from

$$U(\mathbf{e}) \overset{(1.7)}{=} \int_0^1 \frac{d}{ds} U(s\mathbf{e})\, ds = \int_0^1 \frac{\partial U(s\mathbf{e})}{\partial e_{ij}} e_{ij}\, ds$$
$$= \int_0^1 \frac{1}{s} \tau_{ij}(s\mathbf{e})\, s e_{ij}\, ds\,,$$

we use the previous lemma directly and obtain (1.36), (1.37) with $C_8 = C_3/p$, $C_9 = d^2 C_5/p$ and $C_{10} = C_3/p$. ■

5.2 Incompressible non-Newtonian fluids and measure-valued solutions

The introductory part of this section contains the formulation of the basic problem called $(\mathrm{NS})_\mathrm{p}$. We have striven for having Sections 5.2–5.4 consistent in the sense that the existence of measure-valued solutions, the existence of weak solutions and further properties of solutions are studied basically for the same problem under the same assumptions. Later on, the possibilities for how to extend the validity of the results to more general situations are discussed in remarks and theorems and some of the assertions are proved under weaker assumptions or in more general settings in comparison with the basic version. Section 5.2.3 of this section provides an exposition of all the results related to the subject, at least to the knowledge of the authors.

Before presenting this summary, the existence of a measure-valued solution to the problem $(\mathrm{NS})_\mathrm{p}$ is demonstrated with the aim of illustrating some basic techniques and methods used in the evolution theory of incompressible fluids, see Section 5.2.2.

5.2.1 Formulation of the problem

In the following three sections we will study the initial-boundary value problem, called shortly the **problem $(\mathbf{NS})_\mathrm{p}$**.

Let $\Omega = (0, L)^d$, $L \in (0, \infty)$, be a cube in $\mathbb{R}^d$. Let us denote $\Gamma_j = \partial\Omega \cap \{x_j = 0\}$ and $\Gamma_{j+d} = \partial\Omega \cap \{x_j = L\}$ for $j = 1, \ldots, d$. For $T \in (0, \infty)$, we denote by Q_T the time-space cylinder $I \times \Omega$, where $I = (0, T)$ is a time interval. We assume that $d \geq 2$.

Let $\mathbf{f} : Q_T \to \mathbb{R}^d$ and $\mathbf{u}_0 : \Omega \to \mathbb{R}^d$ be given and assume that some tensor function $\boldsymbol{\tau} : \mathbb{R}^{d^2}_{\mathrm{sym}} \to \mathbb{R}^{d^2}_{\mathrm{sym}}$ satisfies for some $p \in (1, \infty)$ the assumptions (1.6), (1.7), $(1.8)_2$ and (1.9). We seek $\mathbf{u} = (u_1, \dots, u_d) : Q_T \to \mathbb{R}^d$ and $\pi : Q_T \to \mathbb{R}$ solving the system of $d + 1$ equations in Q_T,

$$\operatorname{div} \mathbf{u} = 0\,, \tag{2.1}$$

$$\frac{\partial \mathbf{u}}{\partial t} + u_k \frac{\partial \mathbf{u}}{\partial x_k} = -\nabla \pi + \operatorname{div} \boldsymbol{\tau}(\mathbf{e}(\mathbf{u})) + \mathbf{f}\,, \tag{2.2}$$

satisfying the initial condition

$$\mathbf{u}(0, \cdot) = \mathbf{u}_0 \quad \text{in } \Omega \tag{2.3}$$

and the space-periodicity requirements

$$\begin{gathered} \mathbf{u}|_{\Gamma_j} = \mathbf{u}|_{\Gamma_{j+d}}, \qquad \nabla\mathbf{u}|_{\Gamma_j} = \nabla\mathbf{u}|_{\Gamma_{j+d}}, \\ \pi|_{\Gamma_j} = \pi|_{\Gamma_{j+d}}, \end{gathered} \tag{2.4}$$

for $j = 1, \dots, d$.

5.2.2 Measure-valued solutions

The goal of this section is to define the notion of a measure-valued solution to the problem $(\mathrm{NS})_\mathrm{p}$ and to prove its existence. For that purpose, the reader will easily find that (except for the definition of function spaces) no changes in the proof are needed if:

1. we replace the space-periodicity condition (2.4) by the Dirichlet condition

$$\mathbf{u} = \mathbf{0} \qquad \text{on } I \times \partial\Omega\,; \tag{2.5}$$

2. instead of assuming the existence of the potential U for $\boldsymbol{\tau}$, see (1.6), we use the inequalities

$$\boldsymbol{\tau}(\boldsymbol{\eta}) \cdot \boldsymbol{\eta} \ge C_4(|\boldsymbol{\eta}|^p - 1)\,, \tag{2.6}$$

$$|\tau_{ij}(\boldsymbol{\eta})| \le C_5(1 + |\boldsymbol{\eta}|)^{p-1}\,, \tag{2.7}$$

which are (due to Lemma 1.19) consequences of (1.6), (1.7), $(1.8)_2$ and (1.9).

Due to the constraint (2.1) we introduce spaces of **divergence-free** functions. Let us denote for $q \in (1, \infty)$[§]

$$\begin{aligned}
&\mathcal{V} \equiv \{\boldsymbol{\phi} \in C^\infty_{\mathrm{per}}(\Omega)^d ; \operatorname{div} \boldsymbol{\phi} = 0\,, \int_\Omega \boldsymbol{\phi}\, dx = \mathbf{0}\};\\
&H \equiv \text{closure of } \mathcal{V} \text{ in the } L^2(\Omega)^d\text{-norm ;}\\
&V_q \equiv \text{closure of } \mathcal{V} \text{ in the } L^q(\Omega)^{d^2}\text{-norm of gradients.}
\end{aligned} \tag{2.8}$$

The dual space of V_q is denoted by V_q^* and $\langle \cdot , \cdot \rangle_q$ denotes the duality between V_q and V_q^*.

The use of spaces of divergence-free functions has some special consequences; these are summarized in the following lemma.

Lemma 2.9 *Let $\pi : \Omega \to \mathbb{R}$ and $\boldsymbol{\varphi}, \mathbf{u}, \mathbf{v} : \Omega \to \mathbb{R}^d$ be periodic functions. If $\boldsymbol{\varphi}, \mathbf{u}$ are divergence-free, then*

$$\int_\Omega \frac{\partial \pi}{\partial x_i} \varphi_i \, dx = 0\,, \tag{2.10}$$

$$\int_\Omega u_j \frac{\partial u_i}{\partial x_j} \varphi_i \, dx = - \int_\Omega u_j u_i \frac{\partial \varphi_i}{\partial x_j} \, dx\,, \tag{2.11}$$

$$\int_\Omega u_j \frac{\partial v_i}{\partial x_j} v_i \, dx = 0\,, \tag{2.12}$$

whenever the integrals in (2.10)–(2.12) are well defined.

PROOF : The proof is based on the use of Green's Theorem 2.20 in Chapter 1 and the fact that $\mathbf{u}, \boldsymbol{\varphi}$ are divergence-free. Thus (the boundary terms vanish due to periodicity),

$$\int_\Omega \frac{\partial \pi}{\partial x_i} \varphi_i \, dx = - \int_\Omega \pi \operatorname{div} \boldsymbol{\varphi} \, dx = 0\,,$$

$$\begin{aligned}
\int_\Omega u_j \frac{\partial v_i}{\partial x_j} v_i \, dx &= \frac{1}{2} \int_\Omega u_j \frac{\partial |\mathbf{v}|^2}{\partial x_j} \, dx\\
&= -\frac{1}{2} \int_\Omega (\operatorname{div} \mathbf{u}) |\mathbf{v}|^2 \, dx = 0\,,
\end{aligned}$$

[§] For the definition of $C^\infty_{\mathrm{per}}(\Omega)$ see Chapter 1.

and

$$\int_\Omega u_j \frac{\partial u_i}{\partial x_j} \varphi_i \, dx = -\int_\Omega u_j u_i \frac{\partial \varphi_i}{\partial x_j} \, dx - \int_\Omega \operatorname{div} \mathbf{u} \, u_i \varphi_i \, dx$$
$$= -\int_\Omega u_j u_i \frac{\partial \varphi_i}{\partial x_j} \, dx \,,$$

∎

From identity (2.10) it follows that the pressure π disappears from the weak formulation of (2.1), (2.2) if divergence-free test functions are used. Consequently, we end up with a formulation of (2.1), (2.2) where the only unknown is the velocity $\mathbf{u}$. In this way, we will consider only the function $\mathbf{u}$ to be the solution of our problem and will not be interested in the pressure π. However, having solved the problem for $\mathbf{u}$, it is a standard matter to determine the pressure π from the velocity field $\mathbf{u}$ and the weak formulation of (2.1), (2.2) at least in the sense of distributions. For more details we refer to TEMAM [1977], GALDI [1994a, 1994b] and GIRAULT AND RAVIART [1986]. We will not discuss the regularity of the pressure π in this book.

Definition 2.13 *Let $p > 1$, $\mathbf{u}_0 \in H$, $\mathbf{f} \in L^{p'}(I; (W^{1,p}_{\text{per}}(\Omega)^d)^*)$, where $p' = \frac{p}{p-1}$. Then a couple $(\mathbf{u}, \nu)$ is called a* **measure-valued solution** *to the problem (NS)$_\text{p}$ if*

$$\begin{aligned} &\mathbf{u} \in L^\infty(I; H) \cap L^p(I; V_p) \,, \\ &\nu \in L^\infty_\omega (Q_T; \operatorname{Prob}(\mathbb{R}^{d^2})) \,, \end{aligned} \tag{2.14}$$

and if $(\mathbf{u}, \nu)$ satisfy

$$\int_{Q_T} \Big[- u_i \frac{\partial \varphi_i}{\partial t} - u_j u_i \frac{\partial \varphi_i}{\partial x_j} + e_{ij}(\boldsymbol{\varphi}) \int_{\mathbb{R}^{d^2}} \tau_{ij}\big(\tfrac{\boldsymbol{\lambda}+\boldsymbol{\lambda}^T}{2}\big) \, d\nu_{t,x}(\boldsymbol{\lambda}) \Big] \, dx \, dt - \int_{Q_T} f_i \varphi_i \, dx \, dt = \int_\Omega u_{0i} \varphi_i \, dx \tag{2.15}$$

for all $\boldsymbol{\varphi} \in \mathcal{D}((-\infty, T); \mathcal{V})$, and

$$\nabla \mathbf{u}(t,x) = \int_{\mathbb{R}^{d^2}} \boldsymbol{\lambda} \, d\nu_{t,x}(\boldsymbol{\lambda}) \qquad \text{a.e. in } Q_T \,. \tag{2.16}$$

Theorem 2.17 *Let* $\mathbf{u}_0 \in H$ *and* $\mathbf{f} \in L^{p'}(I; (W^{1,p}_{\text{per}}(\Omega)^d)^*)$. *Then there exists a measure-valued solution to the problem* (*NS*)$_{\text{p}}$ *provided that*

$$p > \frac{2d}{d+2} \cdot \tag{2.18}$$

PROOF : The proof is split into several smaller parts.

- *The choice of basis*

We will show the existence of a measure-valued solution to the system (2.1)–(2.4) via Galerkin approximations. For this purpose we can take the set $\{\widehat{\boldsymbol{\omega}}^r\}_{r=1}^{\infty}$ formed by the eigenvectors $\widehat{\boldsymbol{\omega}}^r$, $r = 1, 2, \ldots$, of the Stokes operator. However, as shown in Lemma 4.26 in the Appendix, the projectors $P^N\mathbf{u} \equiv \sum_{r=1}^{N}(\mathbf{u}, \widehat{\boldsymbol{\omega}}^r)\,\widehat{\boldsymbol{\omega}}^r$ are uniformly continuous at most in the norm of $W^{2,2}(\Omega)^d$. Due to imbedding, it follows that, e.g. in three dimensions, we have an upper bound for p, $p < 6$. On the other hand, the use of $\{\widehat{\boldsymbol{\omega}}^r\}$ allows us to exploit the Stokes operator as a test function in an easy way in order to get the so-called second energy estimate. This will be needed in Sections 5.3 and 5.4.

Nevertheless, the second energy estimate is not needed for proving the existence of measure valued solutions. Therefore, a more convenient set $\{\boldsymbol{\omega}^r\}_{r=1}^{\infty}$ is chosen here. For

$$s > \frac{d}{2} + 1, \tag{2.19}$$

we denote by ($\mathcal{V}$ is defined in (2.8))

$$V^s \equiv \text{ the closure of } \mathcal{V} \text{ with respect to the } W^{s,2}(\Omega)^d\text{-norm.}$$

Let us denote the scalar product in V^s by $((\cdot,\cdot))_s$ and let $\{\boldsymbol{\omega}^r\}_{r=1}^{\infty}$ be the set of eigenvectors to the problem

$$((\boldsymbol{\omega}^r, \boldsymbol{\varphi}))_s = \lambda_r(\boldsymbol{\omega}^r, \boldsymbol{\varphi}) \qquad \forall\, \boldsymbol{\varphi} \in V^s\,, \tag{2.20}$$

which are orthonormal in H and orthogonal in V^s, see Theorem 4.11 in the Appendix. The reason why we choose s satisfying (2.19) is the following: if $\mathbf{v} \in W^{s,2}(\Omega)^d$ then $\nabla\mathbf{v} \in W^{s-1,2}(\Omega)^{d^2}$ and

$$W^{s-1,2}(\Omega) \hookrightarrow L^\infty(\Omega) \quad \text{if} \quad \frac{1}{2} - \frac{s-1}{d} < 0\,,$$

which is just (2.19). Consequently, for all $p > 1$ we have $\nabla\mathbf{v} \in L^p(\Omega)^{d^2}$ and $V^s \hookrightarrow V_p$.

• *Galerkin system and a priori estimates*

Let us define $\mathbf{u}^N(t,x) \equiv \sum_{r=1}^N c_r^N(t)\boldsymbol{\omega}^r(x)$, where the coefficients $c_r^N(t)$ solve the so-called **Galerkin system**

$$\begin{aligned}\Big(\frac{d}{dt}\mathbf{u}^N, \boldsymbol{\omega}^r\Big) + \int_\Omega \tau_{ij}(\mathbf{e}(\mathbf{u}^N))e_{ij}(\boldsymbol{\omega}^r)\,dx + \int_\Omega u_j^N \frac{\partial u_i^N}{\partial x_j}\omega_i^r\,dx \\ = (\mathbf{f}, \boldsymbol{\omega}^r)\,, \qquad 1 \le r \le N\,, \qquad (2.21) \\ \mathbf{u}^N(0) = P^N \mathbf{u}_0\,.\end{aligned}$$

Here, P^N is the orthogonal continuous projector of H (resp. V^s) onto the linear hull of the first N eigenvectors $\boldsymbol{\omega}^r$, $r = 1, \dots, N$, see (4.12)–(4.13) in the Appendix. Due to the orthonormality of $\{\boldsymbol{\omega}^r\}$ in H, the system (2.21) can be rewritten as

$$\begin{aligned}\frac{d}{dt}c_r^N &= \mathcal{F}_r(c_1^N, \dots, c_N^N, t)\,, \\ c_r^N(0) &= (\mathbf{u}_0, \boldsymbol{\omega}^r)\,, \end{aligned} \qquad (2.22)$$

where

$$\begin{aligned}\mathcal{F}_r(c_1^N, \dots, c_N^N, t) \equiv (\mathbf{f}, \boldsymbol{\omega}^r) - c_\ell^N c_s^N \int_\Omega \omega_j^\ell \frac{\partial \omega_i^s}{\partial x_j}\omega_i^r\,dx \\ - \int_\Omega \tau_{ij}(\mathbf{e}(c_k^N \boldsymbol{\omega}^k))e_{ij}(\boldsymbol{\omega}^r)\,dx\,,\end{aligned}$$

$r = 1, \dots, N$. Before discussing the solvability (local and global) of (2.21), let us derive the following *a priori* estimates:

There is a constant C depending on $T, \Omega, \|\mathbf{f}\|_{L^{p'}(I;(W^{1,p}_{\mathrm{per}}(\Omega)^d)^*)}$ and $\|\mathbf{u}_0\|_2$ such that for all $N = 1, 2, \dots$,

$$\|\mathbf{u}^N\|_{L^\infty(I;H)} \le C\,, \qquad (2.23)$$

$$\|\mathbf{u}^N\|_{L^p(I;V_p)} \le C\,, \qquad (2.24)$$

$$\Big\|\frac{\partial \mathbf{u}^N}{\partial t}\Big\|_{L^{p'}(I;(V^s)^*)} \le C\,. \qquad (2.25)$$

In order to prove (2.23), (2.24) we multiply the rth equation of the Galerkin system (2.21) by $c_r^N(t)$ and add the equations. The result can be written in the form

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{u}^N\|_2^2 + \int_\Omega \tau_{ij}(\mathbf{e}(\mathbf{u}^N))e_{ij}(\mathbf{u}^N)\,dx = (\mathbf{f}, \mathbf{u}^N)\,, \qquad (2.26)$$

because of $\int_\Omega u_j^N \frac{\partial u_i^N}{\partial x_j} u_i^N\,dx = 0$ (see (2.12)).

Due to assumption $(1.8)_2$ and its consequence $(1.20)_2$,

$$\int_\Omega \tau_{ij}(\mathbf{e}(\mathbf{u}^N))e_{ij}(\mathbf{u}^N)\,dx \geq C_4\big(\|\mathbf{e}(\mathbf{u}^N)\|_p^p - |\Omega|\big)\,.$$

Using Korn's inequality (1.11) we obtain from (2.26),

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\mathbf{u}^N\|_2^2 + K_p^p C_4 \|\nabla \mathbf{u}^N\|_p^p & \\
&\leq |(\mathbf{f},\mathbf{u}^N)| + C_4|\Omega| \\
&\leq \|\mathbf{f}\|_{-1,p'}\|\nabla\mathbf{u}^N\|_p + C_4|\Omega| \\
&\underset{\text{ineq.}}{\overset{\text{Young}}{\leq}} \frac{K_p^p C_4}{2}\|\nabla\mathbf{u}^N\|_p^p + c\,\|\mathbf{f}\|_{-1,p'}^{p'} + C_4|\Omega|
\end{aligned}$$

or finally,

$$\frac{d}{dt}\|\mathbf{u}^N\|_2^2 + K_p^p C_4\|\nabla\mathbf{u}^N\|_p^p \leq c\|\mathbf{f}\|_{-1,p'}^{p'} + C_4|\Omega|\,. \tag{2.27}$$

Integrating (2.27) between 0 and t, $t \in I$, we get

$$\begin{aligned}
\|\mathbf{u}^N(t)\|_2^2 + K_p^p C_4 \int_0^t \|\nabla\mathbf{u}^N(\tau)\|_p^p\,d\tau & \\
&\leq c\int_0^T \|\mathbf{f}(\tau)\|_{-1,p'}^{p'}\,d\tau + \|\mathbf{u}^N(0)\|_2^2 + C_4 T\,|\Omega| \\
&\leq C\,.
\end{aligned} \tag{2.28}$$

Thus (2.23) and (2.24) are proved.

The *a priori* estimate (2.23) implies

$$|\mathbf{c}^N(t)|^2 \leq C \qquad \text{for all } t \in I\,. \tag{2.29}$$

Since $\mathcal{F}_r$, $r = 1,\ldots,N$, in (2.22) satisfy the Carathéodory conditions (see Appendix, Definition 3.2), we obtain by Theorem 3.4 in the Appendix the local existence of a continuous function $\mathbf{c}^N : (0,t^*) \longrightarrow \mathbb{R}^N$ solving (2.22), having $\frac{d\mathbf{c}^N}{dt}$ defined almost everywhere. But due to the continuity of $\mathbf{c}^N$ on $[0,t^*)$ and the uniform boundedness (2.29) we can shift t^* to T. See ZEIDLER [1990b, Chapter 30] for more details.

To prove (2.25), let us take $\boldsymbol{\varphi} \in L^p(I;V^s)$, $\|\boldsymbol{\varphi}\|_{L^p(I;V^s)} \le 1$. Then

$$\Big(\frac{d\mathbf{u}^N}{dt},\boldsymbol{\varphi}\Big) = \Big(\frac{d\mathbf{u}^N}{dt},P^N\boldsymbol{\varphi}\Big) \overset{(2.21)}{=} -\int_\Omega u_j^N \frac{\partial u_i^N}{\partial x_j} P^N\varphi_i\,dx - \int_\Omega \tau_{ij}(\mathbf{e}(\mathbf{u}^N))e_{ij}(P^N\boldsymbol{\varphi})\,dx + (\mathbf{f},P^N\boldsymbol{\varphi})\,.$$

By (4.12) in the Appendix, $\|P^N\boldsymbol{\varphi}\|_{s,2} \le \|\boldsymbol{\varphi}\|_{s,2}$. Thus,

$$\begin{aligned}
\Big|\int_0^T\int_\Omega u_j^N\frac{\partial u_i^N}{\partial x_j}P^N\varphi_i\,dx\,dt\Big|
&= \Big|\int_0^T\int_\Omega u_i^N u_j^N \frac{\partial(P^N\varphi_i)}{\partial x_j}\,dx\,dt\Big| \\
&\le \int_0^T\int_\Omega |\mathbf{u}^N|^2|\nabla(P^N\boldsymbol{\varphi})|\,dx\,dt \\
&\le \int_0^T \|\nabla(P^N\boldsymbol{\varphi})\|_\infty\|\mathbf{u}^N\|_2^2\,dt \\
&\overset{(2.19)}{\le} c\int_0^T \|P^N\boldsymbol{\varphi}\|_{s,2}\|\mathbf{u}^N\|_2^2\,dt \overset{(2.23)}{\le} C\,,
\end{aligned}$$

$$\begin{aligned}
\Big|\int_0^T\int_\Omega \tau_{ij}(e(\mathbf{u}^N))e_{ij}(P^N\boldsymbol{\varphi})\,dx\,dt\Big| \\
&\overset{(1.21)}{\le} C_5 d^2\int_0^T\int_\Omega (1+|\mathbf{e}(\mathbf{u}^N)|)^{p-1}|\mathbf{e}(P^N\boldsymbol{\varphi})|\,dx\,dt \\
&\underset{\text{ineq.}}{\overset{\text{Hölder}}{\le}} c\int_0^T \|\mathbf{e}(P^N\boldsymbol{\varphi})\|_p\|(1+|\mathbf{e}(\mathbf{u}^N)|)\|_p^{p-1}\,dt \\
&\le c\int_0^T \|\mathbf{e}(P^N\boldsymbol{\varphi})\|_p(1+\|\nabla\mathbf{u}^N\|_p)^{p-1}\,dt \\
&\overset{(2.19)}{\le} c\int_0^T \|P^N\boldsymbol{\varphi}\|_{s,2}(1+\|\nabla\mathbf{u}^N\|_p)^{p-1}\,dt \\
&\underset{\text{ineq.}}{\overset{\text{Hölder}}{\le}} c\Big(\int_0^T\|\boldsymbol{\varphi}\|_{s,2}^p\,dt\Big)^{1/p}\Big(\int_0^T(1+\|\nabla\mathbf{u}^N\|_p)^p\,dt\Big)^{\frac{p-1}{p}} \\
&\overset{(2.24)}{\le} C\,,
\end{aligned}$$

$$\int_0^T |(\mathbf{f}, P^N\boldsymbol{\varphi})|\, dt \le \|\mathbf{f}\|_{L^{p'}(I;(W^{1,p}_{\mathrm{per}}(\Omega)^d)^*)} \|P^N\boldsymbol{\varphi}\|_{L^p(I;V_p)} \le C\,.$$

Since

$$\Big\|\frac{\partial \mathbf{u}^N}{\partial t}\Big\|_{L^{p'}(I;(V^s)^*)} = \sup_{\substack{\boldsymbol{\varphi}\in L^p(I;V^s)\\ \|\boldsymbol{\varphi}\|_{L^p(I;V^s)}\le 1}} \Big|\int_0^T \Big(\frac{\partial \mathbf{u}^N}{\partial t}, \boldsymbol{\varphi}\Big)\, dt\Big|,$$

the uniform estimate (2.25) is proved.

- *Limiting process*

From (2.23), (2.24) follows the existence of $\mathbf{u}$ satisfying for all $r > 1$ that

$$\mathbf{u}^N \rightharpoonup \mathbf{u} \qquad \text{weakly in } L^r(I;H)\cap L^p(I;V_p) \tag{2.30}$$

at least for a subsequence. In order to prove

$$\begin{aligned}\int_0^T\int_\Omega u_j^N \frac{\partial u_i^N}{\partial x_j}\varphi_i\, dx\, dt &\overset{(2.11)}{=} -\int_0^T\int_\Omega u_j^N u_i^N \frac{\partial \varphi_i}{\partial x_j}\, dx\, dt\\ &\to -\int_0^T\int_\Omega u_j u_i \frac{\partial \varphi_i}{\partial x_j}\, dx\, dt\end{aligned} \tag{2.31}$$

for all $\boldsymbol{\varphi}\in \mathcal{D}(-\infty, T; \mathcal{V})$, we need the strong convergence

$$\mathbf{u}^N \to \mathbf{u} \qquad \text{in } L^p(I;H)\,. \tag{2.32}$$

Let us first show (2.32) and then we will prove (2.31). By virtue of the compact imbedding $W^{1,p}(\Omega)^d \hookrightarrow\hookrightarrow L^2(\Omega)^d$ if $p > \frac{2d}{d+2}$ (here the lower bound (2.18) for the parameter p appears!), we can apply the Aubin-Lions Lemma 1.2.48 with $X_0 = V_p, X = H, X_1 = (V^s)^*$, $\alpha = p$ and $\beta = p'$. As a conclusion we obtain (2.32) at least for some subsequence still denoted by $\mathbf{u}^N$.¶

¶ Notice that due to (2.23) and (2.32), $\mathbf{u}^N \to \mathbf{u}$ in $L^r(I;H)$ for arbitrary $r\in(1,\infty)$. Indeed, for the interesting case $r > p$ we have

$$\begin{aligned}\int_0^T \|\mathbf{u}^N - \mathbf{u}\|_2^r\, dt &= \int_0^T \|\mathbf{u}^N - \mathbf{u}\|_2^p \|\mathbf{u}^N - \mathbf{u}\|_2^{r-p}\, dt\\ &\overset{(2.23)}{\le} 2C\int_0^T \|\mathbf{u}^N - \mathbf{u}\|_2^p\, dt\,,\end{aligned}$$

and the last integral vanishes due to (2.32).

Let us now prove (2.31). First, for all $\boldsymbol{\omega}^r$, $r \in \mathbb{N}$,

$$\begin{aligned}\int_0^T \int_\Omega (u_j^N u_i^N - u_j u_i) \frac{\partial \omega_i^r}{\partial x_j}\, dx\, dt &= \int_0^T \int_\Omega (u_j^N - u_j) u_i^N \frac{\partial \omega_i^r}{\partial x_j}\, dx\, dt \\ &\quad + \int_0^T \int_\Omega u_j (u_i^N - u_i) \frac{\partial \omega_i^r}{\partial x_j}\, dx\, dt \\ &\equiv I_1 + I_2\,.\end{aligned}$$

Then,

$$\begin{aligned}|I_1| &\le \|\nabla \boldsymbol{\omega}^r\|_\infty \int_0^T \|\mathbf{u}^N - \mathbf{u}\|_2 \|\mathbf{u}^N\|_2\, dt \\ &\overset{(2.23)}{\le} C \|\nabla \boldsymbol{\omega}^r\|_\infty \int_0^T \|\mathbf{u}^N - \mathbf{u}\|_2\, dt\,,\end{aligned}$$

which tends to zero due to (2.32). On the other hand the weak convergence (2.30) implies $|I_2| \to 0$ when $N \to \infty$. Since the functions of the type $g(t)\boldsymbol{\omega}^r(x)$, $g \in \mathcal{D}(-\infty, T)$, are dense in $\mathcal{D}(-\infty, T; \mathcal{V})$, we obtain

$$\int_0^T \int_\Omega u_j^N u_i^N \frac{\partial \psi_i}{\partial x_j}\, dx\, dt \to \int_0^T \int_\Omega u_j u_i \frac{\partial \psi_i}{\partial x_j}\, dx\, dt$$

for all $\boldsymbol{\psi} \in \mathcal{D}(-\infty, T; \mathcal{V})$. Next,

$$\begin{aligned}\int_0^T \Big(\frac{\partial \mathbf{u}^N(t)}{\partial t}, \boldsymbol{\omega}^r\Big) g(t)\, dt &= -\int_0^T (\mathbf{u}^N(t), \boldsymbol{\omega}^r) \frac{d}{dt} g(t)\, dt \\ &\quad - (\mathbf{u}^N(0), \boldsymbol{\omega}^r) g(0)\end{aligned}$$

for $g \in \mathcal{D}(-\infty, T)$, $r = 1, 2, \dots$. Due to the weak convergence (2.30) and $\mathbf{u}^N(0) = P^N \mathbf{u}_0 \to \mathbf{u}_0$ in H, we obtain

$$\begin{aligned}\int_0^T \Big(\frac{\partial \mathbf{u}^N(t)}{\partial t}, \boldsymbol{\varphi}(t)\Big)\, dt &\to -\int_0^T \Big(\mathbf{u}(t), \frac{\partial \boldsymbol{\varphi}(t)}{\partial t}\Big)\, dt \\ &\quad - (\mathbf{u}_0, \boldsymbol{\varphi}(0)) \qquad (2.33)\end{aligned}$$

for all $\boldsymbol{\varphi} \in \mathcal{D}(-\infty, T; \mathcal{V})$.

- *Limiting process in* $\boldsymbol{\tau}$

It remains to find the limit of the nonlinear term given by $\boldsymbol{\tau}$. According to (2.24) the sequence $\{\nabla \mathbf{u}^N\}$ is bounded in $L^p(Q_T)^{d^2}$. Because the components of the nonlinear continuous function $\boldsymbol{\tau}$ have the $(p-1)$-growth due to (1.9) and (1.21), we can use Corollary

2.10 from Chapter 4 with $\mathbf{z}^j = \nabla \mathbf{u}^j$, $\tau = \tau_{ij}$, $q = p-1$, $s = d^2$ and $Q = Q_T$. We obtain

$$\tau_{ij}\left(\tfrac{\nabla \mathbf{u}^N + (\nabla \mathbf{u}^N)^T}{2}\right) \rightharpoonup \overline{\tau}_{ij} \qquad \text{in } L^{\frac{p}{p-1}}(Q_T)\,, \tag{2.34}$$

where

$$\overline{\tau}_{ij} \overset{\text{a.e.}}{=} \int_{\mathbb{R}^{d^2}} \tau_{ij}\left(\tfrac{\boldsymbol{\lambda}+\boldsymbol{\lambda}^T}{2}\right) d\nu_{t,x}(\boldsymbol{\lambda})\,.$$

This means that

$$\begin{aligned}&\int_{Q_T} \tau_{ij}(\mathbf{e}(\mathbf{u}^N)) e_{ij}(\boldsymbol{\varphi})\, dx\, dt \\ &\qquad \to \int_{Q_T} e_{ij}(\boldsymbol{\varphi}) \int_{\mathbb{R}^{d^2}} \tau_{ij}\left(\tfrac{\boldsymbol{\lambda}+\boldsymbol{\lambda}^T}{2}\right) d\nu_{t,x}(\boldsymbol{\lambda})\, dx\, dt\end{aligned} \tag{2.35}$$

for all $\boldsymbol{\varphi} \in \mathcal{D}(-\infty, T; \mathcal{V})$. Taking into account (2.31), (2.33) and (2.35), equation (2.15) is derived.

Using Corollary 2.10 from Chapter 4 with $\tau = \mathrm{Id}$, $r = p$ and $q = 1$ (the other parameters are as above) we obtain for all $\psi \in L^{p'}(Q_T)$, $i, j = 1, \ldots, d$,

$$\int_{Q_T} \frac{\partial u_i^N}{\partial x_j} \psi \, dx\, dt \to \int_{Q_T} \psi \int_{\mathbb{R}^{d^2}} \lambda_{ij}\, d\nu_{t,x}(\boldsymbol{\lambda})\, dx\, dt\,.$$

Since $\nabla \mathbf{u}^N \rightharpoonup \nabla \mathbf{u}$ weakly in $L^p(Q_T)$,

$$\frac{\partial u_i}{\partial x_j}(t,x) \overset{\text{a.e.}}{=} \int_{\mathbb{R}^{d^2}} \lambda_{ij}\, d\nu_{t,x}(\boldsymbol{\lambda})\,,$$

which is (2.16) and the proof of the theorem is complete. ∎

Remark 2.36 The measure-valued function ν satisfies besides (2.16) more restrictive conditions. It can be found in KINDERLEHRER AND PEDREGAL [1992b] that for arbitrary quasiconvex and bounded below φ such that $\lim_{|\mathbf{A}|\to\infty} \frac{\varphi(\mathbf{A})}{1+|\mathbf{A}|^p}$ exists, we have

$$\varphi(\langle \nu_x, \mathbf{Id}\rangle) \leq \langle \nu_x, \varphi\rangle \qquad \text{for a.a. } x \in \Omega\,,$$

whenever ν is generated by gradients $\nabla \mathbf{u}_j$ of a sequence $\mathbf{u}_j$ bounded in $W^{1,p}(\Omega)^d$, $p > 1$. Let us recall a function $\varphi : \mathbb{R}^{d^2} \to \mathbb{R}$ is quasiconvex provided that for all $\boldsymbol{\xi} \in \mathbb{R}^{d^2}$ and $\mathbf{u} \in W_0^{1,\infty}(\Omega)^d$,

$$\varphi(\boldsymbol{\xi}) \leq \frac{1}{|\Omega|} \int_\Omega \varphi(\boldsymbol{\xi} + \nabla \mathbf{u})\, dx\,.$$

Remark 2.37 (to the proof of Theorem 2.17) As can be seen from the proof of Theorem 2.17, we can weaken some of the assumptions immediately. We formulate this in the following two theorems.

Theorem 2.38 *Let $p > \frac{2d}{d+2}$, $\mathbf{u}_0 \in H$, $\mathbf{f} \in L^{p'}\big(I; (W^{1,p}_{\text{per}}(\Omega)^d)^*\big)$, $p' = \frac{p}{p-1}$. Assume that for certain positive constants C_4, C_5 and $q \in [p-1, p)$ we have for all $\boldsymbol{\eta} \in \mathbb{R}^{d^2}_{\text{sym}}$,*

$$\boldsymbol{\tau}(\boldsymbol{\eta}) \cdot \boldsymbol{\eta} \geq C_4(|\boldsymbol{\eta}|^p - 1)\,, \tag{2.39}$$

$$|\tau_{ij}(\boldsymbol{\eta})| \leq C_5(1 + |\boldsymbol{\eta}|)^q\,. \tag{2.40}$$

Then there exists a measure-valued solution to the problem (2.1)–(2.4).

PROOF : If $q = p - 1$ then the proof coincides with the proof of Theorem 2.17. For $q \in (p-1, p)$ two insignificant changes take place. Firstly, instead of (2.25), we obtain

$$\Big\| \frac{\partial \mathbf{u}^N}{\partial t} \Big\|_{L^{\frac{p}{q}}(I;(V^s)^*)} \leq C\,.$$

Secondly, (2.34) is replaced by

$$\tau_{ij}\Big(\frac{\nabla \mathbf{u}^N + (\nabla \mathbf{u}^N)^T}{2}\Big) \rightharpoonup \overline{\tau}_{ij} \qquad \text{in } L^{\frac{p}{q}}(Q_T)\,.$$

The rest of the proof remains as before. ■

Let us now consider the problem (2.1)–(2.3), where $\boldsymbol{\tau}$ fulfills (2.39)–(2.40). Let us study the Dirichlet boundary problem. It means that $\Omega \subset \mathbb{R}^d$ is a bounded domain with $\partial\Omega \in C^{0,1}$, and (2.4) is replaced by (2.5). Then we modify the definition of the space $\mathcal{V}$ in (2.8) to

$$\mathcal{V} \equiv \big\{\boldsymbol{\phi} \in \mathcal{D}(\Omega)^d; \operatorname{div} \boldsymbol{\phi} = 0\big\}\,.$$

Following the proofs of previous theorems, we get

Theorem 2.41 *Let $p > \frac{2d}{d+2}$, $\mathbf{u}_0 \in H$, $\mathbf{f} \in L^{p'}\big(I; (W^{1,p}(\Omega)^d)^*\big)$, $p' = \frac{p}{p-1}$. Then there exists a measure-valued solution to the problem (2.1)–(2.3), (2.5), (2.39)–(2.40).*

5.2.3 Survey of known results related to the problem $(NS)_{\text{p}}$

We will now summarize the results connected with the problem $(\text{NS})_{\text{p}}$ from the points of view of

1. existence theory,
2. uniqueness and its consequences,
3. extensions to other problems.

This survey will be completed by some bibliographical remarks.

1. *The existence theory* includes the discussion on the values of p for which one obtains

- the existence of a measure-valued solution,
- the existence of a weak solution,
- the existence of a strong solution,
- the existence of a strong solution for small data,
- the local existence of a strong solution for arbitrary data.

See Figures 5.1 and 5.2 for the graphic presentation of the results in three and two dimensions, respectively.

The existence mentioned in the first four items above is understood globally in time, i.e., for arbitrary $T \in (0, +\infty)$ the solution of a given type is constructed. A measure-valued solution has been defined in Definition 2.13. A function $\mathbf{u} \in L^\infty(I; H) \cap L^p(I; V_p)$ will be called a **weak solution** if, besides the limiting processes (2.31) and (2.33), the limit

$$\int_0^T \int_\Omega \tau_{ij}(\mathbf{e}(\mathbf{u}^N))e_{ij}(\boldsymbol{\varphi})\, dx\, dt \to \int_0^T \int_\Omega \tau_{ij}(\mathbf{e}(\mathbf{u}))e_{ij}(\boldsymbol{\varphi})\, dx\, dt \tag{2.42}$$

can be justified also for all $\boldsymbol{\varphi} \in \mathcal{D}(-\infty, T; \mathcal{V})$. By a **strong solution** (or **semiregular** one), we mean a function

$$\mathbf{u} \in L^2(I; W^{2,2}_{\mathrm{per}}(\Omega)^d) \cap L^\infty(I; V_p \cap V_2) \cap L^p(I; V_{3p}),$$
$$\frac{\partial \mathbf{u}}{\partial t} \in L^2(Q_T),$$

satisfying the weak formulation (3.7) at least for all $\boldsymbol{\varphi} \in L^p(I; V_p)$.

- *The existence of a measure-valued solution* has been proved both for the space-periodic problem $(\mathrm{NS})_\mathrm{p}$ and for the Dirichlet problem in Theorems 2.17, 2.38 and 2.41 for

$$p > \frac{2d}{d+2}.$$

The nonlinear tensor function $\boldsymbol{\tau}$ is assumed to satisfy the p-coercivity condition (2.39) and the polynomial growth condition of order q, $q < p$, see (2.40). This result was first obtained in MÁLEK, NEČAS AND NOVOTNÝ [1992] and NEČAS [1991], where $\mathbf{u}^N$ were not Galerkin approximations but solutions of the so-called **bipolar model problem**. Let us recall that the bound $p > \frac{2d}{d+2}$ is due to the strong convergence $\mathbf{u}^N \to \mathbf{u}$ in $L^p(I, H)$ needed to pass to the limit in the convective term, cf. (2.31).

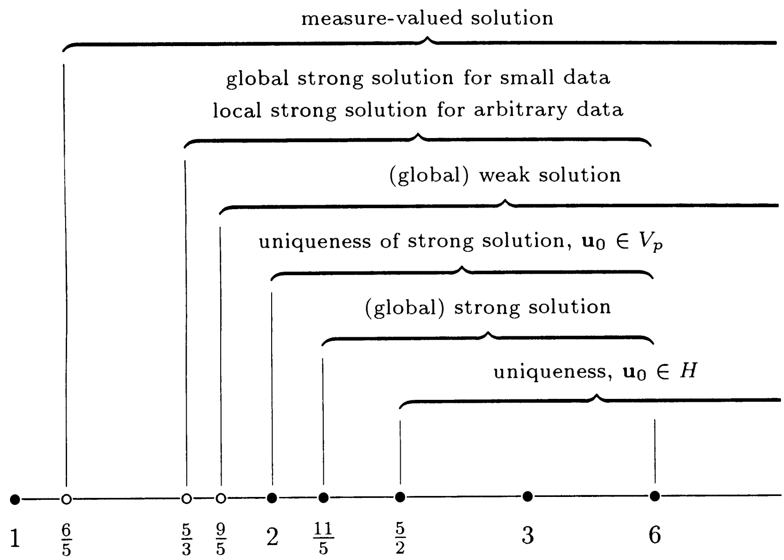

Figure 5.1 *Existence, semiregularity and uniqueness results for problem* $(NS)_{\mathrm{p}}$ *in three dimensions.*

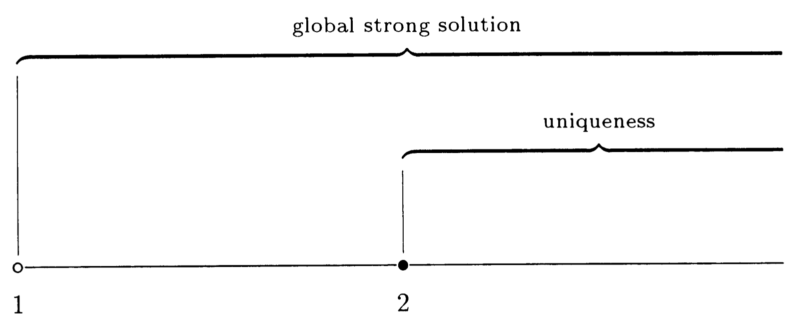

Figure 5.2 *Existence, semiregularity and uniqueness results for problem* $(NS)_{\mathrm{p}}$ *in two dimensions.*

• *The existence of a weak solution* requires in addition (2.42). The first results were obtained at the end of sixties by LADYZHENSKAYA [1969, 1970a, 1970b] and LIONS [1969], and in a different setting by KANIEL [1970]. While Ladyzhenskaya derived the nonlinear $\boldsymbol{\tau} \sim \mathbf{e}$ dependence by kinetic theory argument, Lions used the nonlinear p-Laplace operator.‖ Combining the methods of monotone operators and compactness, they both showed the existence of a weak solution for

$$p \geq 1 + \frac{2d}{d+2} = \frac{3d+2}{d+2}\,.$$

This result holds both for space-periodic and Dirichlet problems, provided that the p-coercivity, the $(p-1)$-growth condition and the monotonicity of nonlinear operator hold.

For the particular case $d = 3$, the results of Ladyzhenskaya and Lions are valid for $p \geq \frac{11}{5}$. Thus, the special subcase of $p = 2$, the Navier-Stokes system (1.64)–(1.65) in Chapter 1, is not covered by these results, although the existence of a weak solution to the Navier-Stokes system is well-known due to LERAY [1934] and many others. Moreover, even in two dimensions, no existence proof follows from the results of Ladyzhenskaya and Lions if $p < 2$, but this case in particular has a lot of applications as shown, e.g., in Examples 1.78, 1.80 and 1.83 in Chapter 1.

This fact has motivated an effort to extend the results of Ladyzhenskaya and Lions. Thus, first the notion of a measure-valued solution to $(\mathrm{NS})_\mathrm{p}$ was defined and after proving its existence, attention was paid to the question of when the measure-valued function reduces to a Dirac one. Note that this is equivalent to (2.42) or to the construction of a weak solution.

For space-periodic problem $(\mathrm{NS})_\mathrm{p}$ BELLOUT, BLOOM AND NEČAS [1994] and MÁLEK, NEČAS AND RŮŽIČKA [1993] have proved the existence of a weak solution whenever

$$p > \frac{3d}{d+2}\,.$$

‖ However, then the stress

$$\boldsymbol{\tau}^E = |\nabla \mathbf{u}|^{p-2}\nabla \mathbf{u} \tag{2.43}$$

does not satisfy the principle of frame indifference. Nevertheless, a modification of (2.43), namely,

$$\boldsymbol{\tau}^E = |\mathbf{e}(\mathbf{u})|^{p-2}\mathbf{e}(\mathbf{u})$$

brings no deeper change into Lions' mathematical approach.

Thus the particular case '$p = 2$ if $d = 3$' is captured. This existence proof is presented in detail in Section 5.3.

Regarding the Dirichlet boundary value problem, the first attempt in MÁLEK, NEČAS AND RŮŽIČKA [1996] gives a proof of the existence of a weak solution for $p \geq 2$ in three space dimensions.

• *The existence of a strong solution* to the problem $(\mathrm{NS})_{\mathrm{p}}$ is demonstrated also in BELLOUT, BLOOM AND NEČAS [1994] as well as in MÁLEK, NEČAS AND RŮŽIČKA [1993], whenever

$$p \geq 1 + \frac{2d}{d+2} .$$

One of the key points in getting the above-mentioned results on weak and strong solutions lies in estimating the term

$$\mathcal{K} \equiv \left| \sum_{i,j,k} \int_0^T \int_\Omega \frac{\partial u_j^N}{\partial x_k} \frac{\partial u_i^N}{\partial x_j} \frac{\partial u_i^N}{\partial x_k} \, dx \, dt \right| .$$

In two dimensions, however, the term $\mathcal{K}$ vanishes due to the condition $\operatorname{div} \mathbf{u}^N = 0$ (or, $\frac{\partial u_1^N}{\partial x_1} = -\frac{\partial u_2^N}{\partial x_2}$). This almost immediately implies the existence of a strong solution for $p > 1$ (if $d = 2$).

• *The existence of a strong solution for small data* was first proved in MÁLEK, RAJAGOPAL AND RŮŽIČKA [1995]. It is shown (see also Section 5.4.2) that whenever

$$p > \frac{3d-4}{d} \qquad \left(\text{and } p < 1 + \frac{2d}{d+2}\right),$$

strong solutions exist provided that the $W^{1,2}$-norm of the initial value $\mathbf{u}_0$ is small enough.

The global existence of a strong solution for small data belonging to $C(I; W^{2,q}(\Omega)^d)$, $q > d$, can be found in AMANN [1994] under the sole assumption that the stress $\boldsymbol{\tau}$ does not differ too much from that one of the Newtonian fluid (given by (1.1.63)) for small values of $\mathbf{e}$.

All of the results mentioned above are global in time, i.e., they remain valid for arbitrarily large finite time interval $(0, T)$.

• As regards *the local existence of a strong solution*, the results are similar to the previous item. Whenever

$$p > \frac{3d-4}{d} ,$$

we can find for arbitrary data a t^* such that a strong solution exists on time interval $(0, t^*)$. See Section 5.4.2 for details.

2. *Uniqueness and its consequences*

In the works of LADYZHENSKAYA [1969] and LIONS [1969], the existence of the unique weak solution is proved if

$$p \geq \frac{d+2}{2} \quad \text{and} \quad \mathbf{u}_0 \in H\,,$$

see also Theorem 4.29.

Let $\mathbf{u}$ and $\mathbf{v}$ be a weak and a strong solution, respectively, corresponding to the same data $\mathbf{u}_0 \in V_2$ and $\mathbf{f}$. Then they must coincide if $p \geq 2$. Since the strong solution exists globally for

$$p \geq 1 + \frac{2d}{d+2} \quad \text{and} \quad \mathbf{u}_0 \in V_2\,,$$

we also get 'global' uniqueness for these ps, see Theorem 4.37. For $p \in [2, 1 + \frac{2d}{d+2})$ the uniqueness result holds only locally or for sufficiently small data. Note that if $d = 2$ all limit bounds coincide.

As a consequence of uniqueness, we see that for $p \geq \frac{d+2}{2}$ the operators

$$S_t : \mathbf{u}_0 \in H \mapsto \mathbf{u}(t) \in H$$

form a semigroup. This allows us, following the approach developed by LADYZHENSKAYA [1972], CONSTANTIN AND FOIAS [1985], TEMAM [1988] and others, to study the existence of a global attractor and to investigate its properties (e.g. finite Hausdorff or fractal dimensions). See MÁLEK AND NEČAS [1994] and MÁLEK, RŮŽIČKA AND THÄTER [1994] for results in this direction.

Considering asymptotic behaviour, there are also results concerning the stability of the rest state. In MÁLEK, RAJAGOPAL AND RŮŽIČKA [1995], the exponential stability is proved for

$$p > \frac{2d}{d+2}\,.$$

Thus, in three space dimensions, for $p \in (\frac{2d}{d+2}, \frac{3d-4}{d}]$, the rest state is exponentially stable without any hypothesis about the local existence of the solution.

The exponential stability of the rest state is also studied in AMANN [1994] within the framework of his setting mentioned earlier.

3. *Extensions of the results to other problems*

Until now, we have discussed mostly the results related to the space-periodic problem $(\mathrm{NS})_{\mathrm{p}}$. These results can be extended in many directions.

Firstly, it is possible to weaken the assumptions on $\mathbf{u}_0$ and to perturb conveniently the potential tensor function $\boldsymbol{\tau}$ by $\boldsymbol{\sigma}$ in the sense of decomposition (1.58) in Chapter 1. Then existence results also hold, and they are discussed in Section 5.3.3. Compare with MÁLEK, RAJAGOPAL AND RŮŽIČKA [1995].

Secondly, we can modify the domain. As already mentioned, some (but not all) results have been extended from the space-periodic case to the Dirichlet problem. Except for the paper of POKORNÝ [1996] ($\Omega = \mathbb{R}^d$), no research has been done for the case of unbounded domains.

Thirdly, the presented approach can be applied to other systems. Thus, for example, MÁLEK, RŮŽIČKA AND THÄTER [1994] studied the properties of solutions of the modified Boussinesq approximation (1.88)–(1.90) in Chapter 1. Since the temperature equation obeys a sort of maximum principle, many of the results are valid for the same range of p as in the problem $(\mathrm{NS})_{\mathrm{p}}$. An analogous procedure can also be used for simple turbulence models, compare with PARÉS [1992].

As for fluids of rate type (such as Oldroyd-B fluid, described by (1.81)–(1.82) in Chapter 1) or n-grade fluids with shear-dependent viscosity, see for example (1.79) in Chapter 1, rigorous mathematical results are rare. We are aware only of the paper of MAN [1992], where a one-dimensional channel flow for a second grade fluid with power-law viscosity is studied, and of the paper of HAKIM [1994] considering the system of type (1.81) in Chapter 1 in two dimensions. We hope, however, that the theory developed here and in the above-mentioned papers could play a positive role in studying questions of existence, uniqueness and regularity of solutions for more complicated problems.

The following lemma shows that the obtained bounds

$$\frac{3d}{d+2} \qquad \text{and} \qquad 1 + \frac{2d}{d+2}$$

are natural and come from the convective term.

Lemma 2.44

1. *Let* $\mathbf{u}, \mathbf{v} \in V_p$ *be arbitrary. Then*

$$\int_\Omega u_j \frac{\partial u_i}{\partial x_j} v_i \, dx$$

is finite if $p \geq \frac{3d}{d+2}$.

2. *Let* $\mathbf{u} \in L^p(I; V_p) \cap L^\infty(I; H)$ *and* $\mathbf{v} \in V_p$ *be arbitrary. Then*

$$\int_0^T \int_\Omega u_j \frac{\partial u_i}{\partial x_j} v_i \, dx \, dt$$

is finite if $p \geq \max(\frac{d+\sqrt{3d^2+4d}}{d+2}, \frac{3d}{d+2})$.

3. *Let* $\mathbf{u} \in L^p(I; V_p) \cap L^\infty(I; H)$ *be arbitrary. Then*

$$\int_0^T \int_\Omega u_j \frac{\partial u_i}{\partial x_j} v_i \, dx \, dt$$

is finite for all $\mathbf{v} \in L^p(I; V_p)$ *if* $p \geq 1 + \frac{2d}{d+2}$.

Remark 2.45 From the third assertion of the previous lemma it follows that $u_j \frac{\partial u_i}{\partial x_j} \in L^{p'}(Q_T)$ just for $p \geq 1 + \frac{2d}{d+2}$, which implies $\frac{\partial \mathbf{u}}{\partial t} \in L^{p'}(I, L^{p'}(\Omega)^d)$. This is a crucial point in applying the theory of monotone operators to the problem $(\mathrm{NS})_\mathrm{p}$, see LIONS [1969].

PROOF (of Lemma 2.44):

Ad 1. Due to the imbedding $W^{1,p}_{\mathrm{per}}(\Omega)^d \hookrightarrow L^{\frac{dp}{d-p}}_{\mathrm{per}}(\Omega)$, we get by Hölder's inequality

$$\left| \int_\Omega u_j \frac{\partial u_i}{\partial x_j} v_i \, dx \right| \leq \|\mathbf{u}\|_{\frac{dp}{d-p}} \|\nabla \mathbf{u}\|_p \|\mathbf{v}\|_{\frac{dp}{d-p}} \, ,$$

provided that

$$\frac{1}{p} + \frac{2(d-p)}{dp} \leq 1 \, .$$

However, the last inequality is equivalent to $p \geq \frac{3d}{d+2}$.

Ad 2. Again by Hölder's inequality

$$\left| \int_0^T \int_\Omega u_j \frac{\partial u_i}{\partial x_j} v_i \, dx \, dt \right| \leq \|\mathbf{v}\|_{\frac{dp}{d-p}} \int_0^T \|\nabla \mathbf{u}\|_p \|\mathbf{u}\|_q \, dt \, , \qquad (2.46)$$

where

$$\frac{1}{q} = 1 - \frac{1}{p} - \frac{d-p}{dp} = \frac{(d+1)p - 2d}{dp} \, .$$

If $p \geq \frac{4d}{d+2}$ then $q \leq 2$ and the assertion follows immediately from the assumptions on $\mathbf{u}$. Whenever $p \in \left[\frac{3d}{d+2}, \frac{4d}{d+2}\right)$ we can use the interpolation inequality

$$\|\mathbf{v}\|_q \leq \|\mathbf{v}\|_2^{1-\alpha} \|\mathbf{v}\|_{\frac{dp}{d-p}}^{\alpha} \quad \text{with } \alpha = \frac{4d-(d+2)p}{(d+2)p-2d} \tag{2.47}$$

and together with (2.46) we obtain

$$\begin{aligned}\left| \int_0^T \int_\Omega u_j \frac{\partial u_i}{\partial x_j} v_i \, dx\, dt \right| &\leq c\, \|\nabla \mathbf{v}\|_p \int_0^T \|\mathbf{u}\|_2^{1-\alpha} \|\nabla \mathbf{u}\|_p^{1+\alpha}\, dt \\ &\leq c\, \|\nabla \mathbf{v}\|_p \|\mathbf{u}\|_{L^\infty(I,H)}^{1-\alpha} \int_0^T \|\nabla \mathbf{u}\|_p^{1+\alpha}\, dt\,.\end{aligned}$$

However,

$$1+\alpha = \frac{2d}{(d+2)p-2d} \leq p$$

whenever

$$p \geq \frac{d+\sqrt{3d^2+4d}}{d+2} \qquad \left(> \frac{3d}{d+2} \text{ for } d < 4 \right).$$

Ad 3. Similarly to before we obtain

$$\begin{aligned}\left| \int_0^T \int_\Omega u_j \frac{\partial u_i}{\partial x_j} v_i \, dx\, dt \right| &\overset{\text{Hölder}}{\underset{\text{ineq.}}{\leq}} \int_0^T \|\mathbf{v}\|_{\frac{dp}{d-p}} \|\nabla \mathbf{u}\|_p \|\mathbf{u}\|_q\, dt \\ &\overset{(2.47)}{\leq} c \int_0^T \|\nabla \mathbf{v}\|_p \|\mathbf{u}\|_2^{1-\alpha} \|\nabla \mathbf{u}\|_p^{1+\alpha}\, dt \\ &\leq c\, \|\mathbf{u}\|_{L^\infty(I;H)}^{1-\alpha} \int_0^T \|\nabla \mathbf{v}\|_p \|\nabla \mathbf{u}\|_p^{1+\alpha}\, dt \\ &\overset{\text{Hölder}}{\underset{\text{ineq.}}{\leq}} c\, \|\mathbf{u}\|_{L^\infty(I;H)}^{1-\alpha} \|\mathbf{v}\|_{L^p(I;V_p)} \left(\int_0^T \|\nabla \mathbf{u}\|_p^{(1+\alpha)\frac{p}{p-1}}\, dt \right)^{\frac{p-1}{p}}.\end{aligned}$$

However,

$$(1+\alpha)\frac{p}{p-1} \leq p \qquad \text{if} \quad p \geq 1 + \frac{2d}{d+2}\,.$$

■

5.3 Incompressible non-Newtonian fluids and weak solutions

5.3.1 *Basic theorem and idea of the proof*

In the previous section we have introduced the notion of measure-valued solutions in order to describe

$$\lim_{N\to\infty} \int_0^T \int_\Omega \tau_{ij}\big(\mathbf{e}(\mathbf{u}^N)\big)e_{ij}(\boldsymbol{\varphi})\,dx\,dt\,. \tag{3.1}$$

Here, $\boldsymbol{\varphi}$ is a smooth function and $\mathbf{u}^N$ are Galerkin approximations.

As was shown, (3.1) can be described by means of a measure-valued function if $p > \frac{2d}{d+2}$. In this section, we show that whenever $p > \frac{3d}{d+2}$, the limit of (3.1) can be characterized classically, i.e., for some $\mathbf{u} \in L^p(I, V_p)$ the limiting process (for all $\boldsymbol{\varphi} \in \mathcal{D}(-\infty, T; \mathcal{V})$)

$$\int_0^T \int_\Omega \tau_{ij}\big(\mathbf{e}(\mathbf{u}^N)\big)e_{ij}(\boldsymbol{\varphi})\,dx\,dt \longrightarrow \int_0^T \int_\Omega \tau_{ij}\big(\mathbf{e}(\mathbf{u})\big)e_{ij}(\boldsymbol{\varphi})\,dx\,dt \tag{3.2}$$

can be justified and we consequently prove the existence of a weak solution.

In order to focus in the proof of the first theorem only on verifying (3.2), we will use the definition of a weak solution which coincides with the definition of a measure-valued one, of course, except for the nonlinear term given by $\boldsymbol{\tau}$. We will show the existence of such weak solutions in Theorem 3.4.

Let us now assume that

$$\mathbf{u}_0 \in V_2 \quad \text{and} \quad \begin{array}{ll} \mathbf{f} \in L^2(Q_T)^d & \text{if } p \geq 2\,, \\ \mathbf{f} \in L^{p'}(Q_T)^d & \text{if } p < 2\,. \end{array} \tag{3.3}$$

The aim of this section is to prove the following theorem.

Theorem 3.4 *Let $\mathbf{u}_0$, $\mathbf{f}$ satisfy (3.3). Let*

$$\begin{array}{ll} \dfrac{3d}{d+2} < p < \dfrac{2d}{d-2} & \text{if } d \leq 4\,, \\ \dfrac{3d-4}{d} < p < \dfrac{2d}{d-2} & \text{if } 5 \leq d \leq 9\,. \end{array} \tag{3.5}$$

Then[††] *there exists a* **weak solution** $\mathbf{u} : Q_T \longrightarrow \mathbb{R}^d$ *to the problem*

[††] Note that for $d > 9$ the interval $\left(\frac{3d-4}{d}, \frac{2d}{d-2}\right)$ is an empty set.

$(NS)_p$. This means

$$\mathbf{u} \in L^\infty(I;H) \cap L^p(I;V_p) \tag{3.6}$$

and the integral identity

$$\int_0^T \int_\Omega \Big(- u_i \frac{\partial \varphi_i}{\partial t} - u_j u_i \frac{\partial \varphi_i}{\partial x_j} + \tau_{ij}\big(\mathbf{e}(\mathbf{u})\big) e_{ij}(\boldsymbol{\varphi}) - f_i \varphi_i \Big)\, dx\, dt$$
$$= (\mathbf{u}_0, \boldsymbol{\varphi}) \tag{3.7}$$

holds for all $\boldsymbol{\varphi} \in \mathcal{D}(-\infty, T; \mathcal{V})$.

Remark 3.8 In Remark 3.66 we show that weak solutions in fact satisfy stronger versions of weak formulations. In Section 5.3.3 we will also extend the existence proof to initial values $\mathbf{u}_0 \in H$ and to broader classes of stress tensors.

Section 5.3.2 will be devoted to the proof of Theorem 3.4, but an idea of the proof together with some preliminaries are given immediately. As pointed out already, the key point is to justify the limiting process in the nonlinearity $\boldsymbol{\tau}$, see (3.2). The basic scheme of our approach is the following:

1. We define for some set $\{\boldsymbol{\omega}^r\}_{r=1}^\infty \subset V_p$ the Galerkin approximations $\mathbf{u}^N(t,x) = \sum_{r=1}^N c_r^N(t)\boldsymbol{\omega}^r(x)$. Because of the independence of the first *a priori* estimates on the choice of $\{\boldsymbol{\omega}^r\}_{r=1}^\infty$, we have (compare with (2.23), (2.24))

$$\|\mathbf{u}^N\|_{L^\infty(I;H)} \le C\,, \tag{3.9}$$
$$\|\mathbf{u}^N\|_{L^p(I;V_p)} \le C\,. \tag{3.10}$$

2. Assume that for some $\tilde{p} \ge 1$, in general less than p,

$$\nabla \mathbf{u}^N \to \nabla \mathbf{u} \qquad \text{strongly in } L^{\tilde{p}}(Q_T)^{d^2}\,. \tag{3.11}$$

Then $\nabla \mathbf{u}^N \to \nabla \mathbf{u}$ almost everywhere in Q_T and also (since $\tau_{ij} \in C^1(\mathbb{R}^{d^2}_{\mathrm{sym}})$)

$$\tau_{ij}(\mathbf{e}(\mathbf{u}^N)) \to \tau_{ij}(\mathbf{e}(\mathbf{u})) \qquad \text{a.e. in } Q_T\,. \tag{3.12}$$

As the growth parameter is less than the coercivity parameter

$(p-1<p)$, we obtain for all measurable $M \subset Q_T$,

$$\begin{aligned}\int_M \left|\tau_{ij}(\mathbf{e}(\mathbf{u}^N))\right| dx\,dt &\overset{(1.21)}{\leq} C_5 \int_M \left(1+|\mathbf{e}(\mathbf{u}^N)|\right)^{p-1} dx\,dt \\ &\leq C_5 \Big(\int_0^T \int_\Omega \left(1+|\mathbf{e}(\mathbf{u}^N)|\right)^p dx\,dt\Big)^{\frac{p-1}{p}} |M|^{\frac{1}{p}} \\ &\overset{(3.10)}{\leq} c\,|M|^{\frac{1}{p}}\,.\end{aligned} \tag{3.13}$$

Taking into account Lemma 2.11 from Chapter 1, we see that (3.12) and (3.13) imply that

$$\int_0^T \int_\Omega \tau_{ij}(\mathbf{e}(\mathbf{u}^N)) e_{ij}(\boldsymbol{\varphi})\,dx\,dt \longrightarrow \int_0^T \int_\Omega \tau_{ij}(\mathbf{e}(\mathbf{u})) e_{ij}(\boldsymbol{\varphi})\,dx\,dt \tag{3.14}$$

for all $\boldsymbol{\varphi} \in \mathcal{D}(-\infty, T; \mathcal{V})$, which is just (3.2).

3. In order to verify (3.11), it is sufficient to find uniform bounds of $\{\nabla \mathbf{u}^N\}$ in some fractional Sobolev space, namely, for some σ positive

$$\|\mathbf{u}^N\|_{L^{\tilde{p}}(I; W^{1+\sigma,\tilde{p}}_{\mathrm{per}}(\Omega)^d)} \leq c\,, \qquad \tilde{p} < p\,. \tag{3.15}$$

Provided that for $\gamma \geq 1$,

$$\Big\|\frac{\partial \mathbf{u}^N}{\partial t}\Big\|_{L^\gamma(I; (W^{2,2}_{\mathrm{per}}(\Omega)^d \cap V_p)^*)} \leq c\,, \tag{3.16}$$

we can use Lemma 2.48 from Chapter 1 with $X_1 = (W^{2,2}_{\mathrm{per}}(\Omega)^d \cap V_p)^*$, $X = W^{1,\tilde{p}}_{\mathrm{per}}(\Omega)^d$, $X_0 = W^{1+\sigma,\tilde{p}}_{\mathrm{per}}(\Omega)^d$, $\alpha = \tilde{p}$ and $\beta = \gamma$, and we obtain (3.11) at least for some subsequence of $\mathbf{u}^N$.

Notice that (3.16), (3.9), (3.10) are sufficient for the limiting process in the terms not containing τ_{ij} (similarly to the existence proof for measure-valued solutions). Moreover, (3.15) and (3.16) imply (3.11) which is enough for the limiting process in (3.14). Thus the existence of weak solutions is demonstrated as soon as the assumptions (3.15), (3.16) are fulfilled.

For the proof of (3.16) see Lemma 3.30 below.

4. To get (3.15), we will use the second energy estimate. For this purpose we choose a special set of $\{\boldsymbol{\omega}^r\}$ consisting of eigenvectors of the Stokes operator denoted by A. Define $P^N \mathbf{u} \equiv \sum_{r=1}^N (\mathbf{u}, \boldsymbol{\omega}^r)\,\boldsymbol{\omega}^r$. By Lemma 4.26 in the Appendix, we know that P^N are continuous uniformly with respect to the norm of

$W^{2,2}(\Omega)^d$. Because of the imbedding $W^{2,2}(\Omega)^d \hookrightarrow W^{1,p}(\Omega)^d$, valid for $p \le \frac{2d}{d-2}$, we consider henceforth the problem $(\mathrm{NS})_\mathrm{p}$ only for such p.

If we use the set $\{\boldsymbol{\omega}^r\}_{r=1}^{\infty}$ in the Galerkin system (2.21), we can multiply the rth equation by $\lambda_r c_r^N(t)$, where λ_r are the corresponding eigenvalues. Using

$$\lambda_r(\boldsymbol{\omega}^r, \mathbf{u}^N) = (A\boldsymbol{\omega}^r, \mathbf{u}^N) = (\nabla\boldsymbol{\omega}^r, \nabla\mathbf{u}^N)\,,$$

(see (4.22) in the Appendix) and summing over $r = 1, \ldots, N$, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\nabla\mathbf{u}^N\|_2^2 &+ \int_\Omega u_j^N \frac{\partial u_i^N}{\partial x_j}(A\mathbf{u}^N)_i\,dx \\ &+ \int_\Omega \tau_{ij}(\mathbf{e}(\mathbf{u}^N))e_{ij}(A\mathbf{u}^N)\,dx = (\mathbf{f}, A\mathbf{u}^N).\end{aligned} \tag{3.17}$$

Because of the periodicity of $\boldsymbol{\omega}^r$,

$$A\mathbf{u}^N = -\Delta\mathbf{u}^N\,.$$

Hence, using $\int_\Omega u_j^N \frac{\partial^2 u_i^N}{\partial x_j \partial x_k}\frac{\partial u_i^N}{\partial x_k} = 0$, we have

$$\begin{aligned}&\int_\Omega u_j^N \frac{\partial u_i^N}{\partial x_j}(A\mathbf{u}^N)_i\,dx = \int_\Omega \frac{\partial u_j^N}{\partial x_k}\frac{\partial u_i^N}{\partial x_j}\frac{\partial u_i^N}{\partial x_k}\,dx\,,\\ &\int_\Omega \tau_{ij}(\mathbf{e}(\mathbf{u}^N))e_{ij}(A\mathbf{u}^N)\,dx \\ &\qquad\overset{(1.6)}{=} \int_\Omega \frac{\partial^2 U(\mathbf{e}(\mathbf{u}^N))}{\partial e_{k\ell}\partial e_{ij}}\frac{\partial e_{k\ell}(\mathbf{u}^N)}{\partial x_s}\frac{\partial e_{ij}(\mathbf{u}^N)}{\partial x_s}\,dx\,.\end{aligned} \tag{3.18}$$

Using (3.18) together with $(1.8)_2$, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\nabla\mathbf{u}^N\|_2^2 + C_1\int_\Omega \left(1+|\mathbf{e}(\mathbf{u}^N)|\right)^{p-2}\frac{\partial e_{ij}(\mathbf{u}^N)}{\partial x_k}\frac{\partial e_{ij}(\mathbf{u}^N)}{\partial x_k}\,dx \\ \le \|\nabla\mathbf{u}^N\|_3^3 + |(\mathbf{f}, \Delta\mathbf{u}^N)|\,.\end{aligned} \tag{3.19}$$

It may be worth commenting, at this moment, on the nice behaviour of the 'convective' term $(3.18)_1$ in the two-dimensional space-periodic setting. Namely, if $d = 2$ then

$$\int_\Omega u_j^N \frac{\partial u_i^N}{\partial x_j}(A\mathbf{u}^N)_i\,dx = 0\,. \tag{3.20}$$

Let us verify this. As above, we have

$$\int_\Omega u_j^N \frac{\partial u_i^N}{\partial x_j}(A\mathbf{u}^N)_i\,dx = -\int_\Omega u_j^N \frac{\partial u_i^N}{\partial x_j}\Delta u_i^N\,dx$$
$$= \int_\Omega \frac{\partial u_j^N}{\partial x_k}\frac{\partial u_i^N}{\partial x_j}\frac{\partial u_i^N}{\partial x_k}\,dx\,,$$

where we sum over all indices from 1 to 2. By a simple comparison, we see that the relation $\frac{\partial u_1^N}{\partial x_1} = -\frac{\partial u_2^N}{\partial x_2}$ implies

$$\sum_{i,j,k=1}^{2} \int_\Omega \frac{\partial u_j^N}{\partial x_k}\frac{\partial u_i^N}{\partial x_j}\frac{\partial u_i^N}{\partial x_k} = 0\,. \tag{3.21}$$

Consequently, the term $\|\nabla\mathbf{u}^N\|_3^3$ does not enter into (3.19). This almost straightforwardly implies the existence of a strong solution for $p > 1$ in two dimensions with space-periodic boundary conditions, as shown in Theorem 4.21 below.

However, mainly for the historical reasons, in the proof of Theorem 3.4 we will not use the fact that the convective term in $(3.18)_1$ vanishes for $d = 2$.

Let us denote by

$$\mathcal{I}_p(\mathbf{u}) \equiv \int_\Omega \big(1 + |\mathbf{e}(\mathbf{u})|\big)^{p-2}\frac{\partial e_{ij}(\mathbf{u})}{\partial x_k}\frac{\partial e_{ij}(\mathbf{u})}{\partial x_k}\,dx\,. \tag{3.22}$$

The main part of the proof of Theorem 3.4, see Section 5.3.2, will be devoted to the following implication: if $p > \frac{3d-4}{d}$ then it is possible to find a $\lambda > 0$ such that

$$\int_0^T \frac{\mathcal{I}_p(\mathbf{u}^N)}{\big(1 + \|\nabla\mathbf{u}^N\|_2^2\big)^\lambda}\,dt \le C\,. \tag{3.23}$$

Thus, roughly speaking, (3.23) provides a uniform estimate for the time integral of the ratio of $W^{2,p}$- and $W^{1,2}$-norms of $\mathbf{u}^N$. Having (3.23), we will prove for $p > \frac{3d}{d+2}$ that some L^s-norm[‡‡] of the second derivatives of $\mathbf{u}^N$ is uniformly summable with some fractional exponent (which vanishes for p approaching $\frac{3d}{d+2}+$). Nevertheless, it reveals that this fractional information is sufficient for proving (3.15).

In order to handle the right-hand side of (3.19), we need estimates

[‡‡] Namely, $s = 2$ if $p \ge 2$ and $s = p$ if $p < 2$.

from below of the integral $\mathcal{I}_p(\mathbf{u})$ by means of some norms of the first and second derivatives. This is provided by the following result.

Lemma 3.24 *Let $\mathbf{u} \in C^2_{\mathrm{per}}(\Omega)^d$. Then there exists a constant c depending only on Ω, p and d such that*

$$\|D^2\mathbf{u}\|_p \le c\big(\mathcal{I}_p(\mathbf{u})\big)^{\frac{1}{2}}\big(1+\|\nabla\mathbf{u}\|_p\big)^{\frac{2-p}{2}} \quad \text{if } p \in (1,2)\,, \tag{3.25}$$

$$\|D^2\mathbf{u}\|_2 \le c\big(\mathcal{I}_p(\mathbf{u})\big)^{\frac{1}{2}} \quad \text{if } p \ge 2\,. \tag{3.26}$$

For $1 \le q \le 2$, $q \ne d$,

$$\|\nabla\mathbf{u}\|_{\frac{dp}{d-q}} \le c\big(\mathcal{I}_p(\mathbf{u})\big)^{\frac{q}{2p}}\big(1+\|\nabla\mathbf{u}\|_p\big)^{\frac{2-q}{2}} \quad \text{if } p > 1\,. \tag{3.27}$$

PROOF : Simple calculations imply

$$\begin{aligned}
\|\nabla\mathbf{e}(\mathbf{u})\|_p^p &= \int_\Omega \left(\frac{\partial e_{ij}(\mathbf{u})}{\partial x_k}\frac{\partial e_{ij}(\mathbf{u})}{\partial x_k}\right)^{\frac{p}{2}} dx \\
&= c\int_\Omega \left\{(1+|\mathbf{e}(\mathbf{u})|)^{p-2}\frac{\partial e_{ij}(\mathbf{u})}{\partial x_k}\frac{\partial e_{ij}(\mathbf{u})}{\partial x_k}\right\}^{\frac{p}{2}} \\
&\qquad \times (1+|\mathbf{e}(\mathbf{u})|)^{-\frac{p}{2}(p-2)}\,dx \\
&\overset{\text{Hölder}}{\underset{\text{ineq.}}{\le}} c\left(\mathcal{I}_p(\mathbf{u})\right)^{\frac{p}{2}}\left(\int_\Omega (1+|\mathbf{e}(\mathbf{u})|)^p\,dx\right)^{\frac{2-p}{2}} \\
&\le c\left(\mathcal{I}_p(\mathbf{u})\right)^{\frac{p}{2}}(1+\|\nabla\mathbf{u}\|_p)^{p\frac{2-p}{2}}\,.
\end{aligned}$$

Assertion (3.25) then follows from (1.16).

If $p \ge 2$ then $\big(1+|\mathbf{e}(\mathbf{u})|\big)^{p-2} \ge 1$, and (3.26) is a direct consequence of Korn's inequality (1.11) and the definition of $\mathcal{I}_p(\mathbf{u})$.

Let $1 \le q \le 2$. From the equality

$$\frac{\partial}{\partial x_s}\big(1+|\mathbf{e}(\mathbf{u})|\big)^{\frac{p}{q}} = \frac{p}{q}\big(1+|\mathbf{e}(\mathbf{u})|\big)^{\frac{p-q}{q}}\frac{e_{ij}(\mathbf{u})\frac{\partial e_{ij}(\mathbf{u})}{\partial x_s}}{|\mathbf{e}(\mathbf{u})|}$$

it follows that

$$\begin{aligned}
&\|\nabla\big(1+|\mathbf{e}(\mathbf{u})|\big)^{\frac{p}{q}}\|_q^q \\
&\quad\le c\left(\frac{p}{q}\right)^q \int_\Omega \big(1+|\mathbf{e}(\mathbf{u})|\big)^{p-q}|\mathbf{e}(\nabla\mathbf{u})|^q\,dx \\
&\quad\le c\int_\Omega \big(1+|\mathbf{e}(\mathbf{u})|\big)^{\frac{p-2}{2}q}|\mathbf{e}(\nabla\mathbf{u})|^q\big(1+|\mathbf{e}(\mathbf{u})|\big)^{\frac{2-q}{2}p}\,dx \\
&\overset{\text{Hölder}}{\underset{\text{ineq.}}{\le}} c\left(\mathcal{I}_p(\mathbf{u})\right)^{\frac{q}{2}}\left(\int_\Omega \big(1+|\mathbf{e}(\mathbf{u})|\big)^p\,dx\right)^{\frac{2-q}{2}}.
\end{aligned}$$

Hence

$$\|\nabla(1+|\mathbf{e}(\mathbf{u})|)^{\frac{p}{q}}\|_q \le c\,(\mathcal{I}_p(\mathbf{u}))^{\frac{1}{2}}\,(1+\|\nabla\mathbf{u}\|_p)^{\frac{2-q}{2q}p}. \tag{3.28}$$

Since $W^{1,q}(\Omega) \hookrightarrow L^{\frac{dq}{d-q}}(\Omega)$ if $q < d$ (this implies that $q < 2$ for $d = 2$), we obtain from (3.28) that

$$\|(1+|\mathbf{e}(\mathbf{u})|)^{\frac{p}{q}}\|_{\frac{dq}{d-q}} \le c\,(\mathcal{I}_p(\mathbf{u}))^{\frac{1}{2}}\,(1+\|\nabla\mathbf{u}\|_p)^{\frac{2-q}{2q}p}.$$

On the other hand,

$$\|(1+|\mathbf{e}(\mathbf{u})|)^{\frac{p}{q}}\|_{\frac{dq}{d-q}} \ge \||\mathbf{e}(\mathbf{u})|^{\frac{p}{q}}\|_{\frac{dq}{d-q}} = \|\mathbf{e}(\mathbf{u})\|^{\frac{p}{q}}_{\frac{dp}{d-q}}.$$

Using again Korn's inequality (1.11) we obtain

$$\|\nabla\mathbf{u}\|_{\frac{dp}{d-q}} \le c\,(\mathcal{I}_p(\mathbf{u}))^{\frac{q}{2p}}\,(1+\|\nabla\mathbf{u}\|_p)^{\frac{2-q}{2}},$$

which is (3.27). ■

Remark 3.29 Let us use Lemma 3.24 in order to estimate the term $\int_0^T |(\mathbf{f},\Delta\mathbf{u}^N)|\,dt$, which appears on the right-hand side of (3.19) after integrating it with respect to time.
If $p < 2$,

$$\begin{aligned}
&\int_0^T |(\mathbf{f},\Delta\mathbf{u}^N)|\,dt \\
&\quad\le \int_0^T \|\mathbf{f}\|_{p'}\|D^2\mathbf{u}^N\|_p\,dt \\
&\quad\overset{(3.25)}{\le} c\int_0^T \|\mathbf{f}\|_{p'}\,(1+\|\nabla\mathbf{u}^N\|_p)^{\frac{2-p}{2}}\,(\mathcal{I}_p(\mathbf{u}^N))^{\frac{1}{2}}\,dt \\
&\quad\le \frac{C_1}{4}\int_0^T \mathcal{I}_p(\mathbf{u}^N)\,dt \\
&\qquad\qquad + c\int_0^T \|\mathbf{f}\|^2_{p'}\,(1+\|\nabla\mathbf{u}^N\|_p)^{2-p}\,dt \\
&\quad\le \frac{C_1}{4}\int_0^T \mathcal{I}_p(\mathbf{u}^N)\,dt \\
&\qquad\qquad + c\Big(\int_0^T \|\mathbf{f}\|^{p'}_{p'}\,dt\Big)^{\frac{2}{p'}}\Big(\int_0^T (1+\|\nabla\mathbf{u}^N\|_p)^p\,dt\Big)^{\frac{2-p}{p}}.
\end{aligned}$$

If $p \geq 2$,

$$\begin{aligned}\int_0^T |(\mathbf{f}, \Delta \mathbf{u}^N)| \, dt & \\ &\leq \int_0^T \|\mathbf{f}\|_2 \|D^2 \mathbf{u}^N\|_2 \, dt \\ &\overset{(3.26)}{\leq} c \int_0^T \|\mathbf{f}\|_2 \left(\mathcal{I}_p(\mathbf{u}^N)\right)^{\frac{1}{2}} \, dt \\ &\leq c\|\mathbf{f}\|^2_{L^2(Q_T)} + \frac{C_1}{4} \int_0^T \mathcal{I}_p(\mathbf{u}^N) \, dt \, .\end{aligned}$$

In both cases the term $\frac{C_1}{4} \int_0^T \mathcal{I}_p(\mathbf{u}^N) \, dt$ can be moved to the left-hand side of (3.19) integrated with respect to time. The remaining terms coming from the estimate of $\int_0^T |(\mathbf{f}, \Delta \mathbf{u}^N)| \, dt$ are uniformly bounded due to (3.10) and (3.3).

Since in this section we work with a different basis in V_p than before, the estimate of $\frac{\partial \mathbf{u}^N}{\partial t}$ is slightly different. In fact, the following lemma holds (compare with (2.25)).

Lemma 3.30 *Let $\mathbf{u}^N(t,x) = \sum_{r=1}^N c_r^N(t) \boldsymbol{\omega}^r(x)$ be solutions to the Galerkin systems (2.21), where $\boldsymbol{\omega}^r$ are eigenvectors of the Stokes operator. Let us denote*

$$Y \equiv \left\{ \mathbf{u} \in W^{2,2}_{\mathrm{per}}(\Omega)^d; \operatorname{div} \mathbf{u} = 0, \int_\Omega \mathbf{u} \, dx = \mathbf{0} \right\}.$$

Then for arbitrary $p \in \left(\frac{3d-2}{d+2}, \frac{2d}{d-2}\right)$ there exist a constant C and a parameter $\gamma \in (1, \infty)$ such that

$$\left\| \frac{\partial \mathbf{u}^N}{\partial t} \right\|_{L^\gamma(I;Y^*)} \leq C \, . \qquad (3.31)$$

PROOF : We proceed similarly to the derivation of (2.25). We look for an appropriate $\gamma \in (1, \infty)$ such that the estimate

$$\sup_{\substack{\boldsymbol{\varphi} \in L^{\gamma'}(I;Y) \\ \|\boldsymbol{\varphi}\|_{L^{\gamma'}(I;Y)} \leq 1}} \int_0^T \left| \left(\frac{\partial \mathbf{u}^N}{\partial t}, \boldsymbol{\varphi} \right) \right| dt \leq C \, , \qquad \gamma' = \frac{\gamma}{\gamma - 1}$$

holds. Now,

$$\begin{aligned}\Big|\Big(\frac{\partial \mathbf{u}^N}{\partial t},\boldsymbol{\varphi}\Big)\Big| &= \Big|\Big(\frac{\partial \mathbf{u}^N}{\partial t},P^N\boldsymbol{\varphi}\Big)\Big| \\ &\le \Big|\int_\Omega u_j^N\frac{\partial u_i^N}{\partial x_j}\big(P^N\boldsymbol{\varphi}\big)_i\,dx\Big| \\ &\quad+\Big|\int_\Omega \tau_{ij}(\mathbf{e}(\mathbf{u}^N))e_{ij}(P^N\boldsymbol{\varphi})\,dx\Big| \\ &\quad+\big|(\mathbf{f},P^N\boldsymbol{\varphi})\big| \equiv I_1+I_2+I_3\,.\end{aligned}$$

Due to our choice of basis, compare with (4.27) in the Appendix,

$$\|P^N\boldsymbol{\varphi}\|_{2,2}\le C\|\boldsymbol{\varphi}\|_{2,2}\qquad \forall\,\boldsymbol{\varphi}\in Y\,, \tag{3.32}$$

where C is independent of N.

Let us estimate I_1, I_2, I_3 separately starting with I_2. It holds that

$$\begin{aligned}\int_0^T I_2\,dt &\overset{(1.21)}{\le} C\int_0^T\int_\Omega \big(1+|\mathbf{e}(\mathbf{u}^N)|\big)^{p-1}|\nabla P^N\boldsymbol{\varphi}|\,dx\,dt \\ &\underset{\text{ineq.}}{\overset{\text{Hölder}}{\le}} C\int_0^T \big(1+\|\nabla\mathbf{u}^N\|_p\big)^{p-1}\|\nabla P^N\boldsymbol{\varphi}\|_p\,dt\,.\end{aligned}$$

Since $W^{2,2}(\Omega)^d\hookrightarrow W^{1,p}(\Omega)^d$ if $p<\frac{2d}{d-2}$, we get not only the upper bound on p but also

$$\begin{aligned}\int_0^T I_2\,dt &\le C\int_0^T \big(1+\|\nabla\mathbf{u}^N\|_p\big)^{p-1}\|P^N\boldsymbol{\varphi}\|_{2,2}\,dt \\ &\overset{(3.32)}{\le} C\int_0^T \big(1+\|\nabla\mathbf{u}^N\|_p\big)^{p-1}\|\boldsymbol{\varphi}\|_{2,2}\,dt \\ &\underset{\text{ineq.}}{\overset{\text{Hölder}}{\le}} C\,\|\boldsymbol{\varphi}\|_{L^p(I;Y)}\big(1+\|\nabla\mathbf{u}^N\|^{p-1}_{L^p(I;V_p)}\big)\overset{(3.10)}{\le} C\,.\end{aligned}$$

Since the estimate of I_3 is trivial, we can focus on estimating I_1. We will distinguish four cases:

- $d>2$ and $p<d$,
- $d>2$ and $p\ge d$,
- $d=2$ and $p\ge 2$,
- $d=2$ and $p<2$.

• The case $d > 2$ and $p < d$: Having

$$
\begin{aligned}
\int_0^T I_1\, dt &= \int_0^T \left| \int_\Omega u_k^N \frac{\partial u_i^N}{\partial x_k} (P^N \boldsymbol{\varphi})_i \, dx \right| dt \\
&= \int_0^T \left| \int_\Omega u_k^N u_i^N \frac{\partial (P^N \boldsymbol{\varphi})_i}{\partial x_k} \, dx \right| dt \\
&\overset{\text{Hölder}}{\underset{\text{ineq.}}{\leq}} \int_0^T \|\nabla P^N \boldsymbol{\varphi}\|_{\frac{2d}{d-2}} \|\mathbf{u}^N\|^2_{\frac{4d}{d+2}} \, dt \,,
\end{aligned}
\tag{3.33}
$$

we use the interpolation inequality

$$
\|\mathbf{v}\|_{\frac{4d}{d+2}} \leq \|\mathbf{v}\|_2^{1-\lambda} \, \|\mathbf{v}\|^{\lambda}_{\frac{dp}{d-p}} \qquad \text{with} \quad \lambda = \frac{(d-2)p}{2(dp - 2d + 2p)} \,,
$$

and we obtain

$$
\begin{aligned}
\int_0^T I_1\, dt &\leq C \int_0^T \|\mathbf{u}^N\|_2^{2(1-\lambda)} \|\nabla \mathbf{u}^N\|_p^{\frac{(d-2)p}{dp-2d+2p}} \|P^N \boldsymbol{\varphi}\|_{2,2}\, dt \\
&\leq C\, \|\mathbf{u}^N\|^{2(1-\lambda)}_{L^\infty(I;H)} \int_0^T \left(\|\nabla \mathbf{u}^N\|_p^p \right)^{\frac{d-2}{dp-2d+2p}} \|\boldsymbol{\varphi}\|_{2,2}\, dt \\
&\overset{\text{Hölder}}{\underset{\text{ineq.}}{\leq}} C\, \|\mathbf{u}^N\|^{2(1-\lambda)}_{L^\infty(I;H)} \|\mathbf{u}^N\|^{\frac{(d-2)p}{dp-2d+2p}}_{L^p(I;V_p)} \left(\int_0^T \|\boldsymbol{\varphi}\|_{2,2}^{\delta'}\, dt \right)^{\frac{1}{\delta'}} \\
&\overset{(3.9)}{\underset{(3.10)}{\leq}} C\,,
\end{aligned}
$$

where $\delta' = \frac{\delta}{\delta-1}$ and $\delta = \frac{dp-2d+2p}{d-2}$. Note that $\delta > 1$ if $p > \frac{3d-2}{d+2}$.

• The case $d > 2$ and $p \geq d$: With help of (3.33) and since $W^{1,d}(\Omega)^d \hookrightarrow L^q(\Omega)^d$, for all $q < \infty$, we get

$$
\begin{aligned}
\int_0^T I_1\, dt &\leq \int_0^T \|\mathbf{u}^N\|^2_{\frac{4d}{d+2}} \|P^N \boldsymbol{\varphi}\|_{2,2}\, dt \\
&\leq C \int_0^T \|\mathbf{u}^N\|^2_{1,d} \|\boldsymbol{\varphi}\|_{2,2}\, dt \\
&\overset{\text{Hölder}}{\underset{\text{ineq.}}{\leq}} C\, \|\mathbf{u}^N\|^2_{L^p(I;V_p)} \|\boldsymbol{\varphi}\|_{L^{\delta'}(I;Y)} \overset{(3.10)}{\leq} C\,,
\end{aligned}
$$

now with $\delta = \frac{p}{2}$.

• The case $d = 2$ and $p \geq 2$: We estimate I_1 as follows:

$$\int_0^T I_1\,dt \leq \int_0^T \int_\Omega |\mathbf{u}^N|\,|\nabla\mathbf{u}^N|\,|P^N\boldsymbol{\varphi}|\,dx\,dt$$

$$\overset{\text{Hölder}}{\underset{\text{ineq.}}{\leq}} \int_0^T \|P^N\boldsymbol{\varphi}\|_\infty \|\nabla\mathbf{u}^N\|_2 \|\mathbf{u}^N\|_2\,dt$$

$$\overset{(3.9)}{\leq} C \int_0^T \|\nabla\mathbf{u}^N\|_p \|\boldsymbol{\varphi}\|_{2,2}\,dt$$

$$\overset{\text{Hölder}}{\underset{\text{ineq.}}{\leq}} C\,\|\mathbf{u}^N\|_{L^p(I;V_p)} \|\boldsymbol{\varphi}\|_{L^{p'}(I;Y)} \overset{(3.10)}{\leq} C\,.$$

Put $\delta = p$ in this case.

• The case $d = 2$ and $p < 2$: Similarly to the previous case we obtain

$$\int_0^T I_1\,dt \leq \int_0^T \|P^N\boldsymbol{\varphi}\|_\infty \|\nabla\mathbf{u}^N\|_p \|\mathbf{u}^N\|_{\frac{p}{p-1}}\,dt.$$

Due to the interpolation inequality

$$\|\mathbf{v}\|_{\frac{p}{p-1}} \leq \|\mathbf{v}\|_2^{\frac{3p-4}{2(p-1)}}\,\|\mathbf{v}\|_{\frac{2p}{2-p}}^{\frac{2-p}{2(p-1)}}\,;$$

and (3.9), we have

$$\int_0^T I_1\,dt \leq C \int_0^T \|\boldsymbol{\varphi}\|_{2,2} \|\nabla\mathbf{u}^N\|_p^{1+\frac{2-p}{2(p-1)}}\,dt$$

$$\overset{\text{Hölder}}{\underset{\text{ineq.}}{\leq}} C\,\|\mathbf{u}^N\|_{L^p(I;V_p)}^{\frac{1}{2(p-1)}} \|\boldsymbol{\varphi}\|_{L^{\delta'}(I;Y)} \overset{(3.10)}{\leq} C\,,$$

where $\delta = \frac{\delta'}{\delta'-1} = 2(p-1)$.

Therefore, $\int_0^T I_1\,dt \leq C$ in all cases. Putting

$$\gamma = \min(p,\delta)\,,$$

we obtain from previous calculations the required estimate (3.31). The proof of Lemma 3.30 is finished. ■

5.3.2 Proof of the basic theorem

PROOF (of Theorem 3.4): First, let us summarize the situation: taking the basis $\{\boldsymbol{\omega}^r\}_{r=1}^\infty$ consisting of eigenvectors of the Stokes operator, we have derived not only the first *a priori* estimates (3.9), (3.10) and (3.31), but also the inequality (3.19), from which we want to find the estimate of the type (3.15): to establish $\sigma > 0$ and $C > 0$ such that

$$\|\mathbf{u}^N\|_{L^{\tilde{p}}(I;W^{1+\sigma,\tilde{p}}_{\mathrm{per}}(\Omega)^d)} \le C \qquad \text{for some } \tilde{p} < p\,. \tag{3.34}$$

Let us recall that the estimate (3.34) is the only one which remains to be proved in order to get the existence of a weak solution to the problem $(\mathrm{NS})_{\mathrm{p}}$.

Finally, it has been also shown in Remark 3.29 that the term $|(\mathbf{f}, \Delta\mathbf{u}^N)|$ can be handled easily. Therefore, we put $\mathbf{f} \equiv \mathbf{0}$ to simplify the following calculations.

Thus, our starting inequality (3.19) has the form of

$$\frac{1}{2}\frac{d}{dt}\|\nabla\mathbf{u}^N\|_2^2 + C_1\mathcal{I}_p(\mathbf{u}^N) \le \|\nabla\mathbf{u}^N\|_3^3\,, \tag{3.35}$$

(see (3.22) for the definition of $\mathcal{I}_p(\mathbf{u})$).

From now on we keep $d \ge 2$ and $p > 1$ arbitrary but fixed. Of course, $p < \frac{2d}{d-2}$.

Let us consider two cases:

1. $p \ge 3$,
2. $p < 3$.

Ad 1. The first case is trivial. Indeed, after integrating (3.35) with respect to time between 0 and t, $t \le T$, we obtain

$$\begin{aligned}\frac{1}{2}\|\nabla\mathbf{u}^N(t)\|_2^2 + C_1\int_0^t \mathcal{I}_p(\mathbf{u}^N)\,dt &\le \frac{1}{2}\|\nabla\mathbf{u}_0\|_2^2 + \|\nabla\mathbf{u}^N\|^3_{L^3(Q_T)} \\ &\le \frac{1}{2}\|\nabla\mathbf{u}_0\|_2^2 + c\|\nabla\mathbf{u}^N\|^3_{L^p(Q_T)} \\ &\le C\,,\end{aligned}$$

due to (3.3) and (3.10). Particularly,

$$\int_0^T \mathcal{I}_p(\mathbf{u}^N)\,dt \le C$$

and we use (3.26) to obtain immediately (3.34) with $\widetilde{p} = 2$ and $\sigma = 1$.

Ad 2. Let us consider $p < 3$ in the sequel. In order to estimate the right-hand side of (3.35), we will have to include into the consideration all possible norms of $\nabla\mathbf{u}^N$; it means

- the L^2-norm, appearing in the first term of (3.35),

- the L^p-norm, for which *a priori* estimates of $\nabla \mathbf{u}^N$ are available, see (3.10),
- the $L^{\frac{dp}{d-q}}$-norm, estimating from below the elliptic term $\mathcal{I}_p(\mathbf{u}^N)$, see (3.27).

To combine all norms mentioned above we use the following two interpolation inequalities which are correct for $q \geq \frac{d(3-p)}{3}$:

$$\|\mathbf{v}\|_3 \leq \|\mathbf{v}\|_2^{\frac{2(dp+3q-3d)}{3(dp+2q-2d)}} \|\mathbf{v}\|_{\frac{dp}{d-q}}^{\frac{dp}{3(dp+2q-2d)}}, \tag{3.36}$$

$$\|\mathbf{v}\|_3 \leq \|\mathbf{v}\|_p^{\frac{dp+3q-3d}{3q}} \|\mathbf{v}\|_{\frac{dp}{d-q}}^{\frac{d(3-p)}{3q}}. \tag{3.37}$$

Since for $\alpha \in (0,1)$ we have

$$\|\nabla \mathbf{u}^N\|_3^3 = \|\nabla \mathbf{u}^N\|_3^{3(1-\alpha)+3\alpha}, \tag{3.38}$$

we obtain from (3.36)–(3.38)

$$\begin{aligned}\|\nabla \mathbf{u}^N\|_3^3 \leq & \|\nabla \mathbf{u}^N\|_2^{2(1-\alpha)\frac{dp+3q-3d}{dp+2q-2d}} \|\nabla \mathbf{u}^N\|_p^{\alpha\frac{dp+3q-3d}{q}} \\ & \times \|\nabla \mathbf{u}^N\|_{\frac{dp}{d-q}}^{\alpha\frac{d(3-p)}{q}+\frac{(1-\alpha)dp}{dp+2q-2d}}.\end{aligned} \tag{3.39}$$

Set

$$\begin{aligned} Q_1 &\equiv (1-\alpha)\frac{dp+3q-3d}{dp+2q-2d}, \\ Q_2 &\equiv \alpha\frac{dp+3q-3d}{q}, \\ Q_3 &\equiv \alpha\frac{d(3-p)}{q} + \frac{(1-\alpha)dp}{dp+2q-2d}. \end{aligned}$$

Then, combining (3.35) and (3.39), we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\nabla \mathbf{u}^N\|_2^2 + C_1\mathcal{I}_p(\mathbf{u}^N) \\ &\quad \leq \left(\|\nabla \mathbf{u}^N\|_2^2\right)^{Q_1}\left(1+\|\nabla \mathbf{u}^N\|_p\right)^{Q_2}\|\nabla \mathbf{u}^N\|_{\frac{dp}{d-q}}^{Q_3}.\end{aligned} \tag{3.40}$$

By the inequality (3.27) we obtain from (3.40) that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\nabla \mathbf{u}^N\|_2^2 + C_1\mathcal{I}_p(\mathbf{u}^N) \\ &\quad \leq C\left(\|\nabla \mathbf{u}^N\|_2^2\right)^{Q_1}\left(1+\|\nabla \mathbf{u}^N\|_p\right)^{Q_2+\frac{2-q}{2}Q_3}\left(\mathcal{I}_p(\mathbf{u}^N)\right)^{\frac{q}{2p}Q_3}.\end{aligned} \tag{3.41}$$

By Young's inequality we obtain

$$\begin{aligned}\frac{d}{dt}\|\nabla \mathbf{u}^N\|_2^2 &+ C_1 \mathcal{I}_p(\mathbf{u}^N) \\ &\leq C \left(\|\nabla \mathbf{u}^N\|_2^2\right)^{Q_1 \cdot \delta'} \left(1 + \|\nabla \mathbf{u}^N\|_p\right)^{\left(Q_2 + \frac{2-q}{2} Q_3\right) \cdot \delta'},\end{aligned} \tag{3.42}$$

provided that

$$\frac{1}{\delta} + \frac{1}{\delta'} = 1 \qquad \text{and} \qquad \frac{q}{2p} Q_3 \, \delta = 1 \,, \quad \delta, \, \delta' > 1 \,. \tag{3.43}$$

Requiring

$$\left(Q_2 + \frac{2-q}{2} Q_3\right) \delta' = p \,, \tag{3.44}$$

we can easily compute α, δ and δ'. Indeed, we have from (3.43) and (3.44),

$$1 = \frac{1}{\delta} + \frac{1}{\delta'} = \frac{Q_2}{p} + \frac{Q_3}{p} = \frac{3\alpha}{p} + \frac{(1-\alpha)d}{dp + 2q - 2d} \,.$$

Therefore,

$$\begin{aligned}\alpha &= \frac{p(dp + 2q - 3d)}{2(dp + 3q - 3d)} \,, \\ 1 - \alpha &= \frac{(3-p)(dp + 2q - 2d)}{2(dp + 3q - 3d)} \,,\end{aligned}$$

and

$$\delta' = \frac{4}{dp - 3d + 4} \,.$$

Notice that the necessary condition for α to be from $(0, 1)$ and δ' to be defined, reads

$$p > \frac{3d - 4}{d} \,; \tag{3.45}$$

compare with (3.5). Notice also that δ' does not depend on the choice of q.

Inserting α, $1 - \alpha$ and δ' into the inequality (3.42), we obtain

$$\frac{d}{dt}\|\nabla \mathbf{u}^N\|_2^2 + C_1 \mathcal{I}_p(\mathbf{u}^N) \leq C \left(\|\nabla \mathbf{u}^N\|_2^2\right)^{\frac{2(3-p)}{dp - 3d + 4}} \left(1 + \|\nabla \mathbf{u}^N\|_p\right)^p$$

or

$$\frac{d}{dt}\left(1 + \|\nabla \mathbf{u}^N\|_2^2\right) + C_1 \mathcal{I}_p(\mathbf{u}^N) \leq C \left(1 + \|\nabla \mathbf{u}^N\|_2^2\right)^{\lambda} \left(1 + \|\nabla \mathbf{u}^N\|_p\right)^p , \tag{3.46}$$

where

$$\lambda = \frac{2(3-p)}{dp - 3d + 4}\,. \tag{3.47}$$

Dividing (3.46) by $\left(1 + \|\nabla \mathbf{u}^N\|_2^2\right)^\lambda$, we obtain for $\lambda \neq 1$

$$\begin{aligned} \frac{1}{1-\lambda}\frac{d}{dt}\left(1 + \|\nabla \mathbf{u}^N\|_2^2\right)^{1-\lambda} + C_1\left(1 + \|\nabla \mathbf{u}^N\|_2^2\right)^{-\lambda}\mathcal{I}_p(\mathbf{u}^N) \\ \leq C\left(1 + \|\nabla \mathbf{u}^N\|_p\right)^p. \end{aligned} \tag{3.48}$$

If $\lambda = 1$ then the first term in (3.46) is replaced by

$$\frac{1}{2}\frac{d}{dt}\ln\left(1 + \|\nabla \mathbf{u}^N\|_2^2\right). \tag{3.49}$$

Let us keep in mind that one should always replace the term with power $1-\lambda$ by the logarithm term whenever $\lambda = 1$.

Integrating (3.48) between 0 and t, $t \in (0,T]$, we get

$$\begin{aligned} &\frac{1}{(1-\lambda)}\left(1 + \|\nabla \mathbf{u}^N(t)\|_2^2\right)^{1-\lambda} \\ &\qquad + C_1\int_0^t \mathcal{I}_p(\mathbf{u}^N(\tau))\left(1 + \|\nabla \mathbf{u}^N(\tau)\|_2^2\right)^{-\lambda} d\tau \\ &\qquad \underset{(3.10)}{\overset{(3.3)}{\leq}} C = C(\mathbf{u}_0)\,. \end{aligned} \tag{3.50}$$

We are interested in getting 'global' estimates. Here, the word 'global' means 'independent of the size of initial values (in an appropriate space) and of the length of the time interval'. Because the first term in (3.50) gives this 'global' information only when $\lambda \leq 1$ and since

$$\lambda \leq 1 \qquad \text{if and only if} \qquad p \geq \frac{3d+2}{d+2} = 1 + \frac{2d}{d+2}\,,$$

we will consider two possibilities:

a) $p \geq 1 + \frac{2d}{d+2}$, i.e., $\lambda \leq 1$,
b) $p < 1 + \frac{2d}{d+2}$, i.e., $\lambda > 1$.

Ad a), $p \geq 1 + \frac{2d}{d+2}$. Taking the supremum over all $t \in (0,T]$ (in (3.50)), we obtain the estimate

$$\|\nabla \mathbf{u}^N\|_{L^\infty(I;L^2_{\mathrm{per}}(\Omega))} \leq C\,. \tag{3.51}$$

Consequently,

$$\int_0^T \mathcal{I}_p(\mathbf{u}^N(t))\,dt \le C\,. \tag{3.52}$$

Let us remark that $1 + \frac{2d}{d+2} \ge 2$ for all $d \ge 2$. Using (3.26) we finally have

$$\int_0^T \|D^2\mathbf{u}^N\|_2^2\,dt \le C \tag{3.53}$$

and we are in the situation of (3.34) with $\widetilde{p} = 2$ and $\sigma = 1$. Case a) is finished.

Ad b), $p < 1 + \frac{2d}{d+2}$. Because $\lambda > 1$, the first term in (3.50) is negative. However, it can be moved across to the right-hand side and estimated by $\frac{1}{\lambda-1}$. Therefore, for $\lambda > 1$ we can dispose of

$$\int_0^T \mathcal{I}_p(\mathbf{u}^N(\tau))\big(1 + \|\nabla\mathbf{u}^N(\tau)\|_2^2\big)^{-\lambda}\,d\tau \le C\,. \tag{3.54}$$

Due to different types of the lower estimates of $\mathcal{I}_p(\mathbf{u}^N)$ by means of some norms of the second derivatives of $\mathbf{u}^N$, compare (3.25) with (3.26), we further divide the proof into two subcases:

(i) $p \ge 2$,
(ii) $p < 2$.

In what follows, we expect to find the lower bound for p.

Ad (i). This case is nontrivial only for $d \ge 3$. Let us first show that (3.54) implies

$$\int_0^T \|D^2\mathbf{u}^N\|_2^{2\beta}\,dt \le C \quad \text{with } \beta = \frac{dp - 3d + 4}{(d-2)p - 3d + 10}\,. \tag{3.55}$$

Indeed,

$$\begin{aligned}
\int_0^T \|D^2\mathbf{u}^N\|_2^{2\beta}\,dt \overset{(3.26)}{\le} {} & C \int_0^T \{\mathcal{I}_p(\mathbf{u}^N)(1 + \|\nabla\mathbf{u}^N\|_2^2)^{-\lambda}\}^\beta \\
& \times \quad (1 + \|\nabla\mathbf{u}^N\|_2^2)^{\lambda\beta}\,dt \\
\underset{\text{ineq.}}{\overset{\text{Hölder}}{\le}} {} & C\Big(\int_0^T \mathcal{I}_p(\mathbf{u}^N)(1 + \|\nabla\mathbf{u}^N\|_2^2)^{-\lambda}\,dt\Big)^\beta \\
& \times \quad \Big(\int_0^T (1 + \|\nabla\mathbf{u}^N\|_2^2)^{\frac{\lambda\beta}{1-\beta}}\,dt\Big)^{1-\beta} \underset{(3.10)}{\overset{(3.54)}{\le}} C\,,
\end{aligned}$$

if

$$\frac{\lambda\beta}{1-\beta} \overset{(3.47)}{=} \frac{2(3-p)}{dp-3d+4}\,\frac{\beta}{1-\beta} = 1 \qquad \Longleftrightarrow$$

$$\beta = \frac{dp-3d+4}{(d-2)p-3d+10} = 1 - \frac{2(3-p)}{(d-2)p-3d+10}\,.$$

Notice that

$$\begin{aligned} &\beta \in \left[\tfrac{1}{3}, \tfrac{1}{2}\right) \qquad &&\text{for } p \in \left[2, \tfrac{11}{5}\right), \qquad && d = 3\,, \\ &\beta \in \left(0, \tfrac{3}{4}\right) \qquad &&\text{for } p \in \left(\tfrac{3d-4}{d}, 1 + \tfrac{2d}{d+2}\right), \qquad && d \geq 4\,. \end{aligned}$$

Thus, we have obtained the lower bound for p if $d \geq 4$.

Having (3.55), we use

$$W^{2,2}(\Omega)^d \hookrightarrow W^{1+s,p}(\Omega)^d \quad \text{with } s = \frac{2d-(d-2)p}{2p} \tag{3.56}$$

and the interpolation inequality (see Lemma 2.18 in Chapter 1)

$$\|\mathbf{v}\|_{1+\sigma,p} \leq c\|\mathbf{v}\|_{1,p}^{1-\frac{\sigma}{s}}\|\mathbf{v}\|_{1+s,p}^{\frac{\sigma}{s}} \tag{3.57}$$

with $0 < \sigma < s$, in order to show that for chosen $r \in (1,p)$,

$$\int_0^T \|\mathbf{u}^N\|_{1+\sigma,p}^r\,dt \leq C \quad \text{with} \quad \sigma = s\,\frac{2\beta(p-r)}{r(p-2\beta)}\,. \tag{3.58}$$

Once we have (3.58), the estimate of type (3.34), with $\tilde{p} = r$, follows immediately. Let us therefore verify (3.58). We have that

$$\begin{aligned} \int_0^T \|\mathbf{u}^N\|_{1+\sigma,p}^r\,dt &\underset{(3.56)}{\overset{(3.57)}{\leq}} C\int_0^T \|\mathbf{u}^N\|_{1,p}^{r(1-\frac{\sigma}{s})}\|\mathbf{u}^N\|_{2,2}^{\frac{\sigma r}{s}}\,dt \\ &\underset{\text{ineq.}}{\overset{\text{Hölder}}{\leq}} C\left(\int_0^T \|\mathbf{u}^N\|_{1,p}^p\right)^{1/\delta'}\left(\int_0^T \|\mathbf{u}^N\|_{2,2}^{2\beta}\right)^{1/\delta} \\ &\underset{(3.55)}{\overset{(3.10)}{\leq}} C\,. \end{aligned}$$

The last inequality holds provided that

$$1 = \frac{1}{\delta} + \frac{1}{\delta'} = \frac{\sigma r}{2\beta s} + \frac{r}{p}\left(1 - \frac{\sigma}{s}\right) = \frac{r}{p} + \sigma\frac{r}{s}\left(\frac{p-2\beta}{2\beta p}\right),$$

which implies that

$$\sigma = \frac{2\beta s}{r}\,\frac{p-r}{p-2\beta}\,.$$

Thus, (3.58) is proved and the subcase (i) is finished. For $d \geq 4$, the proof of Theorem 3.4 is complete.

Ad (ii), $p < 2$ (which in fact reduces to the case $p \in (\frac{3d}{d+2}, 2)$ for $d = 2, 3$). As a simple consequence of (3.54) and (3.25) we obtain

$$\int_0^T \frac{\|D^2\mathbf{u}^N\|_p^2}{\big(1+\|\nabla\mathbf{u}^N\|_2^2\big)^{\lambda}\big(1+\|\nabla\mathbf{u}^N\|_p\big)^{2-p}}\,dt \leq C\,. \tag{3.59}$$

Let us first show that (3.59) implies

$$\begin{gathered}\int_0^T \|D^2\mathbf{u}^N\|_p^{2\beta}\,dt \leq C\\ \text{with}\quad \beta = \frac{p\big((d+2)p-3d\big)}{2\big(-p^2+(d+5)p-3d\big)}\,.\end{gathered} \tag{3.60}$$

Indeed, denoting the integrand in (3.59) by $\mathcal{K}(\mathbf{u}^N)$, we have

$$\begin{aligned}&\int_0^T \|D^2\mathbf{u}^N\|_p^{2\beta}\,dt\\ &= \int_0^T \big(\mathcal{K}(\mathbf{u}^N)\big)^{\beta}\big(1+\|\nabla\mathbf{u}^N\|_2^2\big)^{\lambda\beta}\big(1+\|\nabla\mathbf{u}^N\|_p\big)^{(2-p)\beta}\,dt\\ &\overset{\text{Hölder}}{\underset{\text{ineq.}}{\leq}} \Big(\int_0^T \mathcal{K}(\mathbf{u}^N)\Big)^{\beta}\Big(\int_0^T \big(1+\|\nabla\mathbf{u}^N\|_2^2\big)^{\frac{\lambda\beta}{1-\beta}}\\ &\qquad\qquad \times\ \big(1+\|\nabla\mathbf{u}^N\|_p\big)^{(2-p)\frac{\beta}{1-\beta}}\,dt\Big)^{1-\beta}\\ &\overset{(3.59)}{\leq} C\Big(\int_0^T \big(1+\|\nabla\mathbf{u}^N\|_2^2\big)^{\frac{\lambda\beta}{1-\beta}}\big(1+\|\nabla\mathbf{u}^N\|_p\big)^{(2-p)\frac{\beta}{1-\beta}}\,dt\Big)^{1-\beta}\\ &\leq C\Big(\int_0^T \big(1+\|\nabla\mathbf{u}^N\|_p\big)^{(2-p)\frac{\beta}{1-\beta}}\,dt\Big)^{1-\beta}\\ &\quad + C\Big(\int_0^T \big(\|\nabla\mathbf{u}^N\|_2^2\big)^{\frac{\lambda\beta}{1-\beta}}\big(1+\|\nabla\mathbf{u}^N\|_p\big)^{(2-p)\frac{\beta}{1-\beta}}\,dt\Big)^{1-\beta}\\ &\equiv I_1^{1-\beta} + I_2^{1-\beta}\,.\end{aligned} \tag{3.61}$$

Using the interpolation inequality

$$\|\mathbf{v}\|_2 \leq \|\mathbf{v}\|_p^{\frac{(d+2)p-2d}{2p}}\,\|\mathbf{v}\|_{\frac{dp}{d-p}}^{\frac{d(2-p)}{2p}}$$

and the continuous imbedding $W^{2,p}(\Omega)^d$ into $W^{1,\frac{dp}{d-p}}(\Omega)^d$, we find that I_2 in (3.61) is less than or equal to

$$C\int_0^T \left(1+\|\nabla\mathbf{u}^N\|_p\right)^{\left[(2-p)+\frac{(d+2)p-2d}{p}\lambda\right]\frac{\beta}{1-\beta}} \|D^2\mathbf{u}^N\|_p^{\frac{d(2-p)}{p}\frac{\lambda\beta}{1-\beta}}\,dt$$
$$\overset{\text{Hölder}}{\underset{\text{ineq.}}{\leq}} C\left(\int_0^T \left(1+\|\nabla\mathbf{u}^N\|_p\right)^p dt\right)^{\frac{1}{\delta}}\left(\int_0^T \|D^2\mathbf{u}^N\|_p^{2\beta}\,dt\right)^{\frac{1}{\delta'}}, \tag{3.62}$$

as far as

$$1=\frac{1}{\delta}+\frac{1}{\delta'}=\left(\frac{2-p}{p}+\frac{(d+2)p-2d}{p^2}\lambda\right)\frac{\beta}{1-\beta} + \frac{\lambda}{1-\beta}\frac{d(2-p)}{2p}, \tag{3.63}$$

where λ is defined in (3.47). It follows from (3.63) that

$$\beta=\frac{[(d+2)p-3d]\,p}{2(-p^2+(d+5)p-3d)}=\begin{cases}\frac{(5p-9)p}{2(-p^2+8p-9)} & \text{if } d=3\,,\\ \frac{(2p-3)p}{(p-1)(6-p)} & \text{if } d=2\,.\end{cases} \tag{3.64}$$

Since β must be positive, the lower bound for p, namely

$$p>\frac{3d}{d+2}$$

is obtained. Notice that

$$\begin{aligned}&\beta\in\left(0,\tfrac{1}{2}\right) &&\text{for } p\in\left(\tfrac{3}{2},2\right), && d=2\,,\\ &\beta\in\left(0,\tfrac{1}{3}\right) &&\text{for } p\in\left(\tfrac{9}{5},2\right), && d=3\,.\end{aligned}$$

Integral I_1 is finite, because of $(2-p)\frac{\beta}{1-\beta}\leq p$, and the first *a priori* estimates (3.10).

Thus we can finally conclude using also (3.61) and (3.62),

$$\int_0^T \|D^2\mathbf{u}^N\|_p^{2\beta}\,dt \leq C+\widetilde{C}\left(\int_0^T \|D^2\mathbf{u}^N\|_p^{2\beta}\,dt\right)^{\frac{1-\beta}{\delta'}}.$$

Applying Young's inequality, we obtain (3.60).

Proceeding similarly to the subcase (i), it is possible to choose $r\in(1,p)$ and to find $\sigma>0$ such that (3.60) implies the required estimate

$$\int_0^T \|\mathbf{u}^N\|_{1+\sigma,p}^r\,dt\leq C \qquad \text{with} \qquad \sigma=\frac{2\beta(p-r)}{r(p-2\beta)}\,. \tag{3.65}$$

In order to prove (3.65), we start from the interpolation inequality

$$\|\mathbf{v}\|_{1+\sigma,p}\leq c\,\|\mathbf{v}\|_{1,p}^{1-\sigma}\,\|\mathbf{v}\|_{2,p}^{\sigma}\,.$$

Using (3.60) we obtain

$$\int_0^T \|\mathbf{u}^N\|_{1+\sigma,p}^r \, dt \le \int_0^T \|\mathbf{u}^N\|_{1,p}^{(1-\sigma)r} \|\mathbf{u}^N\|_{2,p}^{\sigma r} \, dt$$
$$\le \Big(\int_0^T \|\mathbf{u}^N\|_{1,p}^p \, dt \Big)^{1/\delta} \Big(\int_0^T \|\mathbf{u}^N\|_{2,p}^{2\beta} \, dt \Big)^{1/\delta'} \underset{(3.10)}{\overset{(3.60)}{\le}} C$$

provided that $1 = \frac{1}{\delta} + \frac{1}{\delta'} = \frac{(1-\sigma)r}{p} + \frac{\sigma r}{2\beta}$. This easily gives the formula for σ. The subcase (b) is finished.

The proof of Theorem 3.4 is now complete. ■

The following remark contains some other weak formulations equivalent to (3.7) which are satisfied by the weak solution $\mathbf{u}$ of the problem $(\mathrm{NS})_\mathrm{p}$. We will also show that whenever $p \ge \frac{3d}{d+2}$, we have $\mathbf{u} \in C_\omega\big(I; H\big)$ and $\frac{\partial \mathbf{u}}{\partial t} \in L^\gamma\big(I; Y^*\big)$ and whenever $p \ge \max\big(\frac{d+\sqrt{3d^2+4d}}{d+2}, \frac{3d}{d+2}\big)$, we have $\frac{\partial \mathbf{u}}{\partial t} \in L^1\big(I; V_p^*\big)$.

Let us recall that $Y \equiv \big\{\mathbf{u} \in W^{2,2}_{\mathrm{per}}(\Omega)^d; \operatorname{div}\mathbf{u} = 0, \int_\Omega \mathbf{u}\, dx = \mathbf{0}\big\}$. By $C_\omega\big(I; H\big)$ we denote the space of weakly continuous functions with values in H, i.e.,

$$C_\omega\big(I; H\big) \equiv \big\{\mathbf{u} \in L^\infty(I; H); \lim_{t \to t_0} \big(\mathbf{u}(t), \mathbf{h}\big) = \big(\mathbf{u}(t_0), \mathbf{h}\big)\,,$$
$$\forall\, \mathbf{h} \in H, \text{ for a.a. } t_0 \in I\big\}\,.$$

Remark 3.66 In the proof of Theorem 3.4, we have found the following *a priori* estimates:

$$\begin{aligned}
\|\mathbf{u}^N\|_{L^p(I;V_p)} &\le C\,, \\
\|\mathbf{u}^N\|_{L^\infty(I;H)} &\le C\,, \\
\Big\|\frac{\partial \mathbf{u}^N}{\partial t}\Big\|_{L^\gamma(I;Y^*)} &\le C\,, \qquad \gamma > 1\,, \\
\|\mathbf{u}^N\|_{L^r(I;W^{1+\sigma,r}_{\mathrm{per}}(\Omega)^d)} &\le C\,,
\end{aligned} \tag{3.67}$$

which allow us to find a subsequence still labelled $\mathbf{u}^N$ such that

$$\begin{aligned}
\mathbf{u}^N &\rightharpoonup \mathbf{u} && \text{weakly in } L^p(I; V_p)\,, \\
\mathbf{u}^N &\overset{*}{\rightharpoonup} \mathbf{u} && \text{weakly-* in } L^\infty(I; H)\,, \\
\mathbf{u}^N &\to \mathbf{u} && \text{strongly in } L^q(I; H)\,, \quad q \ge 1 \text{ arbitrary}\,, \\
\nabla\mathbf{u}^N &\to \nabla\mathbf{u} && \text{strongly in } L^r(I; L^r_{\mathrm{per}}(\Omega)^{d^2})\,,
\end{aligned}$$

$r \in (1,p)$ arbitrary, and

$$\frac{\partial \mathbf{u}^N}{\partial t} \rightharpoonup \frac{\partial \mathbf{u}}{\partial t} \quad \text{weakly in } L^\gamma(I;Y^*)\,.$$

Then an equivalent form of (3.7) is also

$$\int_0^T \left\langle \frac{\partial \mathbf{u}}{\partial t}, \boldsymbol{\varphi} \right\rangle_Y + \int_0^T \int_\Omega \Big[- u_j u_i \frac{\partial \varphi_i}{\partial x_j} + \tau_{ij}\big(\mathbf{e}(\mathbf{u})\big) e_{ij}(\boldsymbol{\varphi}) \Big]\, dx\, dt = \int_0^T (\mathbf{f}, \boldsymbol{\varphi})\, dt \tag{3.68}$$

for all $\boldsymbol{\varphi} \in \mathcal{D}(0,T;\mathcal{V})$. Since $\mathcal{V}$ is dense in Y, and the integrals

$$\int_0^T \int_\Omega u_j \frac{\partial u_i}{\partial x_j} \varphi_i \, dx\, dt \quad \text{and} \quad \int_0^T \int_\Omega \tau_{ij}\big(\mathbf{e}(\mathbf{u})\big) e_{ij}(\boldsymbol{\varphi})\, dx\, dt$$

are finite[§§] for $\mathbf{u} \in L^\infty(I;H) \cap L^p(I;V_p)$ and $\boldsymbol{\varphi} \in \mathcal{D}(0,T;Y)$ if $p \in \left(\frac{3d}{d+2}, \frac{2d}{d-2}\right)$, we obtain another equivalent formulation of (3.7), namely

$$\int_0^T \left\langle \frac{\partial \mathbf{u}}{\partial t}, \boldsymbol{\varphi} \right\rangle_Y + \int_0^T \int_\Omega \Big[u_j \frac{\partial u_i}{\partial x_j} \varphi_i + \tau_{ij}\big(\mathbf{e}(\mathbf{u})\big) e_{ij}(\boldsymbol{\varphi}) \Big]\, dx\, dt = \int_0^T (\mathbf{f}, \boldsymbol{\varphi})\, dt \tag{3.69}$$

for all $\boldsymbol{\varphi} \in \mathcal{D}(0,T;Y)$. Taking in (3.69) $\boldsymbol{\varphi} \in Y$ independent of t, we get

$$\begin{aligned} \int_0^T \left\langle \frac{\partial \mathbf{u}(t)}{\partial t}, \boldsymbol{\varphi} \right\rangle_Y dt + \int_0^T \int_\Omega u_j(t) \frac{\partial u_i(t)}{\partial x_j} \varphi_i \, dx\, dt \\ + \int_0^T \int_\Omega \tau_{ij}\big(\mathbf{e}(\mathbf{u}(t))\big) e_{ij}(\boldsymbol{\varphi})\, dx\, dt = \int_0^T (\mathbf{f}(t), \boldsymbol{\varphi})\, dt \end{aligned} \tag{3.70}$$

and due to integrability of all terms at least in $L^1(I)$, the weak solution $\mathbf{u} \in L^\infty(I;H) \cap L^p(I;V_p)$ with

$$\frac{\partial \mathbf{u}}{\partial t} \in L^\gamma(I;Y^*)$$

[§§] In order to verify the finiteness of $\int_0^T \int_\Omega u_j \frac{\partial u_i}{\partial x_j} \varphi_i \, dx\, dt$, one uses the same approach as in the proof of the second assertion of Lemma 2.44.

satisfies for all $\boldsymbol{\varphi} \in Y$ the identity

$$\begin{aligned}\left\langle \frac{\partial \mathbf{u}(t)}{\partial t}, \boldsymbol{\varphi} \right\rangle_Y &+ \int_\Omega u_j(t) \frac{\partial u_i(t)}{\partial x_j} \varphi_i \, dx \\ &+ \int_\Omega \tau_{ij}\big(\mathbf{e}(\mathbf{u}(t))\big) e_{ij}(\boldsymbol{\varphi}) \, dx = (\mathbf{f}(t), \boldsymbol{\varphi})\end{aligned} \tag{3.71}$$

for almost all $t \in I$.

Now, let us show that $\mathbf{u}$ satisfying (3.71) belongs to $C_\omega\big(I; H\big)$. Since $\mathbf{u} \in L^\infty(I; H)$, we know that

$$\|\mathbf{u}(t)\|_2 \le C \qquad \forall\, t \in I \setminus S\,, \text{ where } |S| = 0\,. \tag{3.72}$$

Integrating (3.71) between $t_0 \in I \setminus S$ and $t \in I \setminus S$, using (2.46) from Chapter 1, we obtain

$$\begin{aligned}(\mathbf{u}(t), \boldsymbol{\varphi}) - (\mathbf{u}(t_0), \boldsymbol{\varphi}) = &\int_{t_0}^t (\mathbf{f}(s), \boldsymbol{\varphi}) \; ds \\ &- \int_{t_0}^t \int_\Omega \tau_{ij}\big(\mathbf{e}(\mathbf{u}(s))\big) e_{ij}(\boldsymbol{\varphi}) \, dx \, ds \\ &- \int_{t_0}^t \int_\Omega u_j(s) \frac{\partial u_i(s)}{\partial x_j} \varphi_i \, dx \, ds\,.\end{aligned} \tag{3.73}$$

However, the right-hand side of (3.72) vanishes as $t \to t_0$. Therefore,

$$(\mathbf{u}(t), \boldsymbol{\varphi}) \underset{t \to t_0}{\longrightarrow} (\mathbf{u}(t_0), \boldsymbol{\varphi}) \quad \forall\, t_0 \in I \setminus S \text{ and } \forall\, \boldsymbol{\varphi} \in Y\,. \tag{3.74}$$

But due to (3.72), we extend (3.74) also for all $\boldsymbol{\varphi} \in H$. Thus,

$$\mathbf{u} \in C_\omega\big(I; H\big)\,. \tag{3.75}$$

Finally, we want to show the validity of the weak formulation (3.7) for test functions $\boldsymbol{\varphi} \in \mathcal{D}(-\infty, T; V_p)$. According to the second part of Lemma 2.44, the integral

$$\int_0^T \int_\Omega u_j \frac{\partial u_i}{\partial x_j} \varphi_i \, dx \, dt < +\infty \tag{3.76}$$

for $\mathbf{u} \in L^\infty(I; H) \cap L^p(I; V_p)$ and $\boldsymbol{\varphi} \in L^\infty(I; V_p)$ whenever $p \ge \max\left(\frac{d+\sqrt{3d^2+4d}}{d+2}, \frac{3d}{d+2}\right)$. Therefore, we have on the one hand the weak formulation

$$\int_0^T \int_\Omega \Big[- u_i \frac{\partial \varphi}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} \varphi_i + \tau_{ij}\big(\mathbf{e}(\mathbf{u})\big) e_{ij}(\varphi) \Big] \, dx \, dt = \int_0^T (\mathbf{f}, \varphi) \, dt + (\mathbf{u}_0, \varphi(0))$$

for all $\varphi \in \mathcal{D}(-\infty, T; V_p)$. On the other hand, using (3.76), we immediately find that

$$\frac{\partial \mathbf{u}}{\partial t} \in L^1(I; V_p^*) \,. \tag{3.77}$$

The same argument as in the first part of this remark implies that $\mathbf{u} \in C_\omega\big(I; H\big)$. Since $\mathbf{u} \in L^p(I; V_p)$ and τ satisfies (1.21), the operator $\mathbf{B}$ defined by

$$\langle \mathbf{B}(\mathbf{u}(t)), \varphi \rangle_{V_p} \equiv \int_\Omega \tau_{ij}\big(\mathbf{e}(\mathbf{u}(t))\big) e_{ij}(\varphi) \, dx \,, \qquad \varphi \in V_p \,,$$

maps V_p onto V_p^*. We conclude: if $p \geq \max\big(\frac{d + \sqrt{3d^2 + 4d}}{d+2}, \frac{3d}{d+2}\big)$, then weak solutions of the problem $(\mathrm{NS})_\mathrm{p}$ satisfy

$$\mathbf{u} \in C_\omega\big(I; H\big) \cap L^p(I; V_p) \,,$$
$$\frac{\partial \mathbf{u}}{\partial t} \in L^1(I; V_p^*) \,,$$

and for almost all $t \in I$ and all $\varphi \in V_p$ the identity

$$\left\langle \frac{\partial \mathbf{u}(t)}{\partial t} + u_j(t) \frac{\partial \mathbf{u}(t)}{\partial x_j} + \mathbf{B}(\mathbf{u}(t)), \varphi \right\rangle_{V_p} = \langle \mathbf{f}(t), \varphi \rangle_{V_p} \tag{3.78}$$

is fulfilled.

5.3.3 Extensions

In this section we will extend the validity of the assertion of Theorem 3.4 to a larger class of initial values and a broader classes of extra stresses. More precisely, we consider two kinds of extensions for the extra stress τ^E. Firstly, we will admit singular nonlinear tensor functions, as, e.g., $\tau(\mathbf{e}(\mathbf{u})) = |\mathbf{e}(\mathbf{u})|^{p-2}\mathbf{e}(\mathbf{u})$, $p \in (1, 2)$, using the assumption $(1.8)_1$ instead of $(1.8)_2$ considered in the problem $(\mathrm{NS})_\mathrm{p}$. Secondly, we will also admit perturbations of the constitutive law, cf. (1.58) in Chapter 1. We will assume that the extra stress τ^E is given by the relation

$$\tau^E(t, x) = \tau\big(\mathbf{e}\big(\mathbf{u}(t, x)\big)\big) + \sigma\big(\mathbf{e}\big(\mathbf{u}(t, x)\big)\big) \,,$$

where $\boldsymbol{\tau}$ satisfies (1.6), (1.7), $(1.8)_2$ and (1.9) and $\boldsymbol{\sigma}$ is, roughly speaking, an arbitrary tensor function controlled by $\boldsymbol{\tau}$. See (3.88)–(3.90) for precise assumptions on $\boldsymbol{\sigma}$. For $p \in (1,2)$, the assumption $(1.8)_1$ is in this case also possible.

The following theorem shows that the same result as in Theorem 3.4 holds under weaker assumptions on $\mathbf{u}_0$, namely $\mathbf{u}_0 \in H$.

Theorem 3.79 *Let $\mathbf{u}_0 \in H$ and $\mathbf{f}$ satisfy (3.3). Further, let p satisfy (3.5). Then there exists a weak solution $\mathbf{u} : Q_T \longrightarrow \mathbb{R}^d$ to the problem $(NS)_p$ such that*

$$\mathbf{u} \in L^\infty(I; H) \cap L^p(I; V_p)$$

and (3.7) holds for all $\boldsymbol{\varphi} \in \mathcal{D}(-\infty, T; \mathcal{V})$.

PROOF : The proof coincides basically with the proof of Theorem 3.4 until the inequality (3.48) is derived. This inequality reads

$$\begin{aligned} \frac{1}{1-\lambda}\frac{d}{dt}\big(1 + \|\nabla\mathbf{u}^N\|_2^2\big)^{1-\lambda} + C_1\mathcal{I}_p(\mathbf{u}^N)\big(1 + \|\nabla\mathbf{u}^N\|_2^2\big)^{-\lambda} \\ \le C\,\big(1 + \|\nabla\mathbf{u}^N\|_p\big)^p \,. \end{aligned} \tag{3.80}$$

Since $\mathbf{u}_0 \notin V_2$ we cannot immediately integrate (3.48) between $(0,t)$, $t \le T$. Let us first define for $\delta > 0$ a cut-off function $\xi \in C^1(I)$ such that $\xi(t) \in [0,1]$ on I and

$$\xi(t) = \begin{cases} 0 & \text{if } t \in [0, \frac{\delta}{2}]\,, \\ 1 & \text{if } t \in [\delta, T]\,, \end{cases}$$

and multiply (3.80) by ξ. We obtain

$$\begin{aligned} \frac{1}{1-\lambda}\frac{d}{dt}\Big(\xi\big(1 + \|\nabla\mathbf{u}^N\|_2^2\big)^{1-\lambda}\Big) + C_1\xi\,\mathcal{I}_p(\mathbf{u}^N)\big(1 + \|\nabla\mathbf{u}^N\|_2^2\big)^{-\lambda} \\ \le C\,\big(1 + \|\nabla\mathbf{u}^N\|_p\big)^p + \frac{1}{1-\lambda}\frac{d\xi}{dt}\big(1 + \|\nabla\mathbf{u}^N\|_2^2\big)^{1-\lambda}\,. \end{aligned} \tag{3.81}$$

Since $\lambda \le 1$ for $p \ge 1 + \frac{2d}{d+2} \ge 2$, the last term in (3.81) is estimated as follows:

$$\frac{1}{1-\lambda}\frac{d\xi}{dt}\big(1 + \|\nabla\mathbf{u}^N\|_2^2\big)^{(1-\lambda)} \le \begin{cases} \frac{1}{1-\lambda}\frac{c}{\delta}\big(1 + \|\nabla\mathbf{u}^N\|_p^p\big)\,, & \text{if } \lambda < 1\,, \\ \frac{1}{\lambda-1}\frac{c}{\delta}\,, & \text{if } \lambda > 1\,. \end{cases}$$

Therefore, integrating (3.81) between 0 and t, $t \in [\delta, T]$, we obtain

$$\frac{1}{1-\lambda}\big(1 + \|\nabla\mathbf{u}^N(t)\|_2^2\big)^{1-\lambda} + \int_\delta^t \mathcal{I}_p(\mathbf{u}^N)\big(1 + \|\mathbf{u}^N\|_2^2\big)^{-\lambda}\,ds \le C\,.$$

Since the last inequality coincides with the inequality (3.50), we can proceed in the same way as in the proof of Theorem 3.4 in order to derive (for chosen $\tilde{p} \in (1,p)$ and appropriate $\sigma > 0$) the estimate

$$\int_\delta^T \|\mathbf{u}^N\|_{1+\sigma,\tilde{p}}^{\tilde{p}} \leq C\,.$$

As we already know, the last estimate is crucial for proving

$$\int_\delta^T \int_\Omega \tau_{ij}\big(\mathbf{e}(\mathbf{u}^N)\big)e_{ij}(\boldsymbol{\varphi})\,dx\,dt \underset{N\to\infty}{\longrightarrow} \int_\delta^T \int_\Omega \tau_{ij}\big(\mathbf{e}(\mathbf{u})\big)e_{ij}(\boldsymbol{\varphi})\,dx\,dt \tag{3.82}$$

for all $\boldsymbol{\varphi} \in \mathcal{D}(-\infty,T;\mathcal{V})$. However, $\nabla\mathbf{u}$ and $\nabla\mathbf{u}^N$ are uniformly bounded in $L^p((0,T)\times\Omega)$. Therefore, for all $\varepsilon > 0$ there is a $\delta > 0$ small enough such that for all $\boldsymbol{\varphi} \in \mathcal{D}(-\infty,T;\mathcal{V})$, $\|\nabla\boldsymbol{\varphi}\|_{\infty,Q_T} \leq 1$,

$$\begin{aligned}
&\left|\int_0^\delta \int_\Omega \big(\tau_{ij}\big(\mathbf{e}(\mathbf{u}^N)\big) - \tau_{ij}\big(\mathbf{e}(\mathbf{u})\big)\big)e_{ij}(\boldsymbol{\varphi})\,dx\,dt\right| \\
&\leq 2\,C_4\|\nabla\boldsymbol{\varphi}\|_{\infty,Q_T}\Big(\int_0^\delta \big(1 + \|\nabla\mathbf{u}^N\|_p\big)^p\,dt\Big)^{\frac{p-1}{p}}\big(|\Omega|\delta\big)^{1/p} \\
&\leq \varepsilon.
\end{aligned} \tag{3.83}$$

Finally, (3.82) and (3.83) imply that the limiting process

$$\int_0^T \int_\Omega \tau_{ij}\big(\mathbf{e}(\mathbf{u}^N)\big)e_{ij}(\boldsymbol{\varphi})\,dx\,dt \longrightarrow \int_0^T \int_\Omega \tau_{ij}\big(\mathbf{e}(\mathbf{u})\big)e_{ij}(\boldsymbol{\varphi})\,dx\,dt$$

can be justified. The proof of Theorem 3.79 is finished. ■

As emphasized in Section 1.1.4, the extra stress $\boldsymbol{\tau}^E$ of the following form:

$$\boldsymbol{\tau}^E = \boldsymbol{\tau}(\mathbf{e}(\mathbf{u})) = |\mathbf{e}(\mathbf{u})|^{p-2}\mathbf{e}(\mathbf{u}) \tag{3.84}$$

with $p \in (1,2)$, plays an important role in modelling flows of non-Newtonian fluids, while the model (3.84) with $p > 2$ is of no interest from the point of view of mechanics of non-Newtonian fluids. However, the assumption $(1.8)_2$ does not allow us to take (3.84) into consideration. Let us therefore replace $(1.8)_2$ by $(1.8)_1$.

Then the following theorem holds.

Theorem 3.85 *Let $\mathbf{u}_0 \in H$ and $\mathbf{f}$ satisfy (3.3). Let us assume that the tensor function $\boldsymbol{\tau}$ satisfies (1.6), (1.7), $(1.8)_1$ and (1.9). Let $p \in \big(\frac{3d}{d+2}, 2\big)$. Then there exists a weak solution $\mathbf{u} : Q_T \longrightarrow \mathbb{R}^d$*

to the problem $(NS)_p$ *such that*

$$\mathbf{u} \in L^\infty(I;H) \cap L^p(I;V_p)\,,$$

and (3.7) holds for all $\boldsymbol{\varphi} \in \mathcal{D}(-\infty,T;\mathcal{V})$.

PROOF : If we change the definition of $\mathcal{I}_p(\mathbf{u})$ such that (instead of (3.22)) $\mathcal{I}_p(\mathbf{u})$ is given by

$$\mathcal{I}_p(\mathbf{u}) \equiv \int_\Omega \big|\mathbf{e}(\mathbf{u})\big|^{p-2} e_{ij}\Big(\frac{\partial \mathbf{u}}{\partial x_k}\Big) e_{ij}\Big(\frac{\partial \mathbf{u}}{\partial x_k}\Big)\,dx\,, \tag{3.86}$$

then the proof is identical with the proof of Theorem 3.4. This comes from the fact that the assertions (3.25) and (3.27) of Lemma 3.24 remain valid also for $\mathcal{I}_p(\mathbf{u})$ defined in (3.86). ∎

In the remainder of this section we will consider incompressible fluids whose extra stress $\boldsymbol{\tau}^E$ is given by

$$\tau^E_{ij}(t,x) = \tau_{ij}\big(\mathbf{e}(\mathbf{u}(t,x))\big) + \sigma_{ij}\big(\mathbf{e}(\mathbf{u}(t,x))\big) \tag{3.87}$$

for all $t \geq 0$, $x \in \Omega$. Here, we assume again that $\boldsymbol{\tau}$ has a potential, i.e., that the assumptions (1.6)–(1.9) are fulfilled (with the convention that $(1.8)_1$ is considered only for $p \in (1,2)$), and $\boldsymbol{\sigma} : \mathbb{R}^{d^2}_{\text{sym}} \longrightarrow \mathbb{R}^{d^2}_{\text{sym}}$ fulfills

$$\sigma_{ij}(\mathbf{e})\,e_{ij} \geq 0 \qquad \forall\, \mathbf{e} \in \mathbb{R}^{d^2}_{\text{sym}}\,. \tag{3.88}$$

We shall also require that for $i,\,j = 1,\dots,d$,

$$|\sigma_{ij}(\mathbf{e})| \leq C_{11}\big(1+|\mathbf{e}|\big)^{p-1}\,. \tag{3.89}$$

Finally, we shall assume that there exists a constant $C_{12} > 0$ such that for $i,\,j,\,k,\,\ell = 1,\dots,d$,

$$\Big|\frac{\partial \sigma_{k\ell}(\mathbf{e})}{\partial e_{ij}}\Big| \leq C_{12}\big(1+|\mathbf{e}|\big)^{p-2}\,. \tag{3.90}$$

Now, we will define the corresponding extended problem and give a definition of weak solutions to it. Let $\mathbf{f} : Q_T \longrightarrow \mathbb{R}^d$ and $\mathbf{u}_0 : \Omega \longrightarrow \mathbb{R}^d$ be given and satisfy for some $p \in (1,\infty)$ the assumptions (1.6), (1.7), (1.9), (3.88)–(3.90) and $(1.8)_2$. (If $p \in (1,2)$, $(1.8)_2$ can be replaced by $(1.8)_1$.)

We look for $\mathbf{u} : Q_T \longrightarrow \mathbb{R}^d$ and $\pi : Q_T \longrightarrow \mathbb{R}$ solving in Q_T

$$\begin{aligned} \frac{\partial \mathbf{u}}{\partial t} + u_k \frac{\partial \mathbf{u}}{\partial x_k} &= -\nabla \pi + \operatorname{div} \boldsymbol{\tau}(\mathbf{e}(\mathbf{u})) + \operatorname{div} \boldsymbol{\sigma}(\mathbf{e}(\mathbf{u})) + \mathbf{f}\,, \\ \operatorname{div} \mathbf{u} &= 0\,, \end{aligned} \tag{3.91}$$

and satisfying the initial condition

$$\mathbf{u}(0,\cdot) = \mathbf{u}_0 \qquad \text{in } \Omega\,, \tag{3.92}$$

and the space-periodicity requirements

$$\begin{aligned} \mathbf{u}|_{\Gamma_j} = \mathbf{u}|_{\Gamma_{j+d}}, \qquad \nabla\mathbf{u}|_{\Gamma_j} &= \nabla\mathbf{u}|_{\Gamma_{j+d}}\,, \\ \pi|_{\Gamma_j} = \pi|_{\Gamma_{j+d}}\,. \end{aligned} \tag{3.93}$$

$j = 1, \dots, d$. We will refer to the problem (3.91)–(3.93) as the **problem** $(\mathbf{NSext})_\mathbf{p}$.

Let us suppose that

$$\mathbf{u}_0 \in H \quad \text{and} \quad \mathbf{f} \in \begin{cases} L^2(Q_T)^d & \text{if } p \geq 2\,, \\ L^{p'}(Q_T)^d & \text{if } p < 2\,. \end{cases} \tag{3.94}$$

Definition 3.95 *Let* $\mathbf{u}_0$, $\mathbf{f}$ *satisfy (3.94). A function*

$$\mathbf{u} \in C_\omega\big(I; H\big) \cap L^p(I; V_p)$$

with

$$\frac{\partial \mathbf{u}}{\partial t} \in L^1(I; Y^*)$$

is called a **weak solution** *to the problem* $(NSext)_p$ *if the identity*

$$\begin{aligned} \Big\langle \frac{\partial \mathbf{u}(t)}{\partial t}, \boldsymbol{\varphi} \Big\rangle_Y + \int_\Omega u_k(t) \frac{\partial u_i(t)}{\partial x_k} \varphi_i + \tau_{ij}\big(\mathbf{e}(\mathbf{u}(t))\big) e_{ij}(\boldsymbol{\varphi})\, dx \\ + \int_\Omega \sigma_{ij}\big(\mathbf{e}(\mathbf{u}(t))\big) e_{ij}(\boldsymbol{\varphi})\, dx = \int_\Omega f_i(t) \varphi_i\, dx \end{aligned} \tag{3.96}$$

is fulfilled almost everywhere in I *and for all* $\boldsymbol{\varphi} \in Y$*. As before,* $Y \equiv \big\{\mathbf{u} \in W^{2,2}_{\mathrm{per}}(\Omega)^d; \operatorname{div} \mathbf{u} = 0, \int_\Omega \mathbf{u}\, dx = \mathbf{0}\big\}$.

A general result is formulated in the following theorem.

Theorem 3.97 *Let* $\mathbf{u}_0$, $\mathbf{f}$ *satisfy (3.94). Let* p *satisfy (3.5). Let also*

$$C_{12} d^2 < C_1\,. \tag{3.98}$$

Then there exists a weak solution to the problem $(NSext)_p$.

PROOF : The proof follows the same lines as the proof of Theorem 3.4 except for small modifications. First let us note that we can assume $\mathbf{u}_0 \in V_2$ because of Theorem 3.79 and its proof. Even if the Galerkin system now contains a new term, namely

$\int_\Omega \sigma_{ij}(\mathbf{e}(\mathbf{u}^N))e_{ij}(\boldsymbol{\omega}^r)\,dx$, we notice that the first *a priori* inequality coincides with (2.27) due to the assumption (3.88). Also, the second energy inequality has the same form as (3.19), except for the coefficient in front of $\mathcal{I}_p(\mathbf{u}^N)$, where instead of C_1 we have $C_1 - C_{12}d^2$. This coefficient is positive due to (3.98).

The proof of Theorem 3.97 is finished. ■

5.4 Incompressible non-Newtonian fluids and strong solutions

We continue in the analysis of the properties of weak solutions $\mathbf{u}$ to the problem $(\mathrm{NS})_\mathrm{p}$; the existence of these has been proved for $p > \frac{3d}{d+2}$ in the previous section, if $d \le 4$. The aim of this section is to present three types of results:

1. To prove the global existence of strong solution (for $p \ge \frac{11}{5}$ if $d = 3$ and for $p > 1$ if $d = 2$) and its uniqueness for $p \ge 1 + \frac{2d}{d+2}$. A function $\mathbf{u}$ is said to be a **strong solution** if $\mathbf{u}$ is at least a weak solution to the problem $(\mathrm{NS})_\mathrm{p}$ and if $\mathbf{u}$ satisfies the following semiregularity properties:

$$\mathbf{u} \in C(I;H) \cap L^\infty(I;V_2) \cap L^2(I;Y)\,, \tag{4.1}$$

$$\mathbf{u} \in L^\infty(I;V_p)\,, \tag{4.2}$$

$$\frac{\partial \mathbf{u}}{\partial t} \in L^2(I;H)\,. \tag{4.3}$$

Recall that $Y \equiv \{\mathbf{u} \in W^{2,2}_{\mathrm{per}}(\Omega)^d; \operatorname{div}\mathbf{u} = 0, \int_\Omega \mathbf{u}\,dx = \mathbf{0}\}$. If $p \in (1,2)$ then (4.1) is replaced by

$$\mathbf{u} \in C(I;H) \cap L^\infty(I;V_2) \cap L^2(I;W^{2,p}_{\mathrm{per}}(\Omega)^d \cap V_p)\,. \tag{4.4}$$

In order to prove (4.1) and (4.4), the assumption $\mathbf{u}_0 \in V_2$ is needed, while for proving (4.2) and (4.3) we will have to assume $\mathbf{u}_0 \in V_p$.

2. To find conditions on the size of data (resp. on the length of time interval) under which the global in time existence of strong solutions for small data (resp. local in time existence of strong solutions for arbitrary data) can be proved. This problem will be studied for $p < \frac{11}{5}$ and $d = 3$ only, since otherwise strong solutions exist without any restriction on data. We will prove both kinds of conditional existence results for $p > \frac{5}{3}$.

3. To find fractional estimates for the L^2-norm of the time derivative of $\mathbf{u}^N$ as well as for the L^p-norm of the second derivatives of $\mathbf{u}^N$ for $p \in (\frac{3d}{d+2}, 1+\frac{2d}{d+2})$.

All assertions and proofs will basically deal with two- or three-dimensional domains.

5.4.1 Global existence of strong solutions and uniqueness

This section consists of four theorems and their proofs. The first one, Theorem 4.5, deals with strong solutions to the problem $(\mathrm{NS})_\mathrm{p}$ if $p \geq 1 + \frac{2d}{d+2}$. Under the assumption $\mathbf{u}_0 \in V_p$ we will prove (4.1)–(4.3) without any restrictions on the size of data.

The next theorem, Theorem 4.21, strengthens the result in two dimensions. It will be shown that a strong solution exists for all $p > 1$. The space-periodic setting is essential for the proof.

The remaining two theorems give an answer to the following question: *When is a weak solution unique?* If $\mathbf{u}_0 \in H$, then Theorem 4.29 gives uniqueness for $p \geq \frac{d+2}{2}$. If $\mathbf{u}_0 \in V_2$, then the weak solution is unique according to Theorem 4.37 for $p \geq 1 + \frac{2d}{d+2}$. Notice that the lower bounds on p are the same in two space dimensions.

Theorem 4.5 *Let $\mathbf{u}_0 \in V_p$, $\mathbf{f} \in L^2(Q_T)$ and $p \geq 1 + \frac{2d}{d+2}$. Then the weak solution $\mathbf{u}$ of the problem $(NS)_p$ has the semiregularity properties (4.1)–(4.3), i.e. $\mathbf{u}$ is a strong solution.*

Remark 4.6 It is worth focusing upon changes of the assertion of Theorem 4.5 if the assumption on $\mathbf{u}_0$ is weakened. Assuming $\mathbf{u}_0 \in H$, we obtain

$$\frac{\partial \mathbf{u}}{\partial t} \in L^{p'}(I; V_p^*) \tag{4.7}$$

and by Lemma 2.45 from Chapter 1 and (3.6),

$$\mathbf{u} \in C(I; H)\,. \tag{4.8}$$

The other regularity properties in (4.1)–(4.3) are still valid provided that the interval I is replaced by $I_\delta \equiv [\delta, T)$, $\delta > 0$ arbitrary.

Finally, as far as $\mathbf{u}_0 \in V_2$ and $\mathbf{u}_0 \notin V_p$, (4.1) holds without any change, while (4.2) and (4.3) requires I to be replaced by I_δ. These modifications of Theorem 4.5 can be proved by means of the cut-off function technique explained in the proof of Theorem 3.79.

PROOF (of Theorem 4.5): Using the Galerkin approximations $\{\mathbf{u}^N\}$ defined in (2.21), we derived the first and second *a priori* estimates, see (3.9), (3.10), (3.51) and (3.53),

$$\|\mathbf{u}^N\|_{L^\infty(I;H)} + \|\mathbf{u}^N\|_{L^p(I;V_p)} \le C\,, \tag{4.9}$$

$$\|\mathbf{u}^N\|_{L^2(I;Y)} + \|\mathbf{u}^N\|_{L^\infty(I;V_2)} \le C\,, \tag{4.10}$$

which have been essential for proving the existence of weak solution $\mathbf{u}$ to the problem $(\mathrm{NS})_\mathrm{p}$. Since $\mathbf{u}$ was found as a weak limit of $\mathbf{u}^N$ and since balls in Banach spaces are weakly closed, we see that the *a priori* estimates (4.9) and (4.10) are valid also for $\mathbf{u}$. Thus,

$$\mathbf{u} \in L^\infty(I;H) \cap L^p(I;V_p) \cap L^\infty(I;V_2) \cap L^2(I;Y)\,. \tag{4.11}$$

According to the third part of Lemma 2.44, $u_j \frac{\partial \mathbf{u}}{\partial x_j} \in L^{p'}(I;V_p^*)$. Consequently, $\frac{\partial \mathbf{u}}{\partial t} \in L^{p'}(I;V_p^*)$ and by Lemma 2.45 in Chapter 1,

$$\Big\{\mathbf{u} \in L^p(I;V_p);\, \frac{\partial \mathbf{u}}{\partial t} \in L^{p'}(I;V_p^*)\Big\} \hookrightarrow C(I;H)\,. \tag{4.12}$$

Thus, (4.1) is proved.

In order to demonstrate (4.2) and (4.3), we will derive the third kind of *a priori* estimate for $\{\mathbf{u}^N\}$. For that purpose, multiply (2.21) by $\frac{d}{dt}c_r^N(t)$ and sum up over $r = 1,\dots,N$. As a result we obtain

$$\begin{aligned} \Big\|\frac{\partial \mathbf{u}^N}{\partial t}\Big\|_2^2 &+ \int_\Omega \tau_{ij}\big(\mathbf{e}(\mathbf{u}^N)\big)\frac{\partial}{\partial t} e_{ij}(\mathbf{u}^N)\,dx \\ &= -\int_\Omega u_j^N \frac{\partial u_i^N}{\partial x_j}\frac{\partial u_i^N}{\partial t}\,dx + \Big(\mathbf{f}, \frac{\partial \mathbf{u}^N}{\partial t}\Big)\,. \end{aligned} \tag{4.13}$$

Applying the Schwarz inequality to both terms at the right-hand side of (4.13) and using (1.6), we get

$$\begin{aligned} \frac{1}{2}\Big\|\frac{\partial \mathbf{u}^N}{\partial t}\Big\|_2^2 &+ \frac{d}{dt}\int_\Omega U\big(\mathbf{e}(\mathbf{u}^N)\big)\,dx \\ &\le c\,\|\mathbf{f}\|_2^2 + c\int_\Omega |\mathbf{u}^N|^2|\nabla \mathbf{u}^N|^2\,dx\,. \end{aligned} \tag{4.14}$$

Let us denote by I the integral $\int_\Omega |\mathbf{u}^N|^2|\nabla \mathbf{u}^N|^2\,dx$.

If $d = p = 2$, then using $\|\mathbf{v}\|_\infty \le c\|\mathbf{v}\|_2^{1/2}\|D^{(2)}\mathbf{v}\|_2^{1/2}$, we obtain

$$I \le \|\mathbf{u}^N\|_\infty^2\|\nabla \mathbf{u}^N\|_2^2 \le c\|\mathbf{u}^N\|_2\|D^{(2)}\mathbf{u}^N\|_2\|\nabla \mathbf{u}^N\|_2^2\,.$$

Thus, due to (4.9) and (4.10), $\int_0^T I\,d\tau \le C$.

For $p \geq d$ ($p > 2$ if $d = 2$) we can estimate I as follows:

$$\begin{aligned}\int_0^T I\,dt &\overset{\text{Hölder}}{\underset{\text{ineq.}}{\leq}} \int_0^T \|\mathbf{u}^N\|^2_{\frac{2p}{p-2}} \|\nabla \mathbf{u}^N\|_p^2\,dt \\ &\leq c \int_0^T \|\nabla \mathbf{u}^N\|_2^2 \|\nabla \mathbf{u}^N\|_p^2\,dt \\ &\overset{(4.10)}{\leq} c \int_0^T \|\nabla \mathbf{u}^N\|_p^2\,dt \overset{(4.9)}{\leq} C\,,\end{aligned} \tag{4.15}$$

where the second inequality expresses the imbedding $W^{1,2}(\Omega)^d \hookrightarrow L^{\frac{2p}{p-2}}(\Omega)^d$ valid for $\frac{2p}{p-2} \leq \frac{2d}{d-2}$, which is $p \geq d$.

It remains to estimate I for $d = 3$ and $p \in [\frac{11}{5}, 3)$. We know (see (3.52)) that

$$\int_0^T \mathcal{I}_p(\mathbf{u}^N)\,dt \leq C\,. \tag{4.16}$$

On the other hand, taking particularly $q = 2$ in (3.27), we have

$$\|\nabla \mathbf{u}^N\|_{3p}^p \leq \mathcal{I}_p(\mathbf{u}^N)\,. \tag{4.17}$$

Hence

$$\begin{aligned}\int_0^T I\,dt &\overset{\text{Hölder}}{\underset{\text{ineq.}}{\leq}} \int_0^T \|\nabla \mathbf{u}^N\|_{3p}^2 \|\mathbf{u}^N\|^2_{\frac{6p}{3p-2}}\,dt \\ &\leq c \int_0^T \|\nabla \mathbf{u}^N\|_2^2 \|\nabla \mathbf{u}^N\|_{3p}^2\,dt \\ &\overset{(4.10)}{\leq} c\|\mathbf{u}^N\|_{L^\infty(I;V_2)} \int_0^T \|\nabla \mathbf{u}^N\|_{3p}^2\,dt \overset{(4.16)}{\underset{(4.17)}{\leq}} C\,.\end{aligned} \tag{4.18}$$

Considering (4.14), (4.15) and (4.18), we obtain for all $t \leq T$

$$\int_0^t \|\frac{\partial \mathbf{u}^N}{\partial t}\|_2^2\,dt + \int_\Omega U\big(\mathbf{e}(\mathbf{u}^N(t))\big)\,dx \leq C\,. \tag{4.19}$$

Due to (1.37) and Korn's inequality (1.11),

$$\|\nabla \mathbf{u}^N(t)\|_p \leq C \qquad \forall\, t \leq T\,.$$

Therefore,

$$\begin{aligned}\|\frac{\partial \mathbf{u}^N}{\partial t}\|_{L^2(I;H)} &\leq C\,, \\ \|\mathbf{u}^N\|_{L^\infty(I;V_p)} &\leq C\,,\end{aligned} \tag{4.20}$$

and the same estimate is valid also for $\mathbf{u}$. Thus (4.2)–(4.3) hold. ∎

Theorem 4.21 *Let $\mathbf{u}_0 \in V_p \cap V_2$, $\mathbf{f} \in L^2(Q_T)$ for $p \geq 2$ and $\mathbf{f} \in L^{p'}(Q_T)$ if $p < 2$. Provided that Ω is a two-dimensional cube, a strong solution to the problem $(NS)_p$ exists for all $p > 1$.*

PROOF : If $p \geq 2$ then the result follows from Theorem 4.5. Let $p \in (1,2)$ in the sequel. According to (3.20)–(3.21),

$$\int_\Omega \frac{\partial u_j^N}{\partial x_k} \frac{\partial u_i^N}{\partial x_j} \frac{\partial u_i^N}{\partial x_k}\, dx = 0\,.$$

Then (3.19) turns into

$$\frac{1}{2}\frac{d}{dt}\|\nabla \mathbf{u}^N\|_2^2 + C_1 \mathcal{I}_p(\mathbf{u}^N) \leq |(\mathbf{f}, \Delta \mathbf{u}^N)|\,. \tag{4.22}$$

Integrating (4.22) between 0 and t and using Remark 3.29, we obtain for almost all $t \in I$,

$$\|\nabla \mathbf{u}^N(t)\|_2^2 + C\int_0^t \mathcal{I}_p(\mathbf{u}^N)\, d\tau \leq \|\nabla \mathbf{u}_0\|_2^2 + C\|\mathbf{f}\|_{L^{p'}(Q_T)}^2\,. \tag{4.23}$$

This implies

$$\|\mathbf{u}^N\|_{L^\infty(I;V_2)} \leq C \tag{4.24}$$

and

$$\int_0^T \mathcal{I}_p(\mathbf{u}^N)\, dt \leq C\,. \tag{4.25}$$

By (3.25), (4.24) and (4.25),

$$\int_0^T \|D^{(2)}\mathbf{u}^N\|_p^2\, dt \leq C\,. \tag{4.26}$$

Next,

$$\begin{aligned}\int_0^T\int_\Omega |\mathbf{u}^N|^2|\nabla \mathbf{u}^N|^2\, dx\, dt &\leq \int_0^T \|\nabla \mathbf{u}^N\|_{\frac{2p}{2-p}}^2 \|\mathbf{u}^N\|_{\frac{p}{p-1}}^2\, dt \\ &\leq \int_0^T \|D^{(2)}\mathbf{u}^N\|_p^2 \|\nabla \mathbf{u}^N\|_2^2\, dt \underset{(4.26)}{\overset{(4.24)}{\leq}} C\,.\end{aligned}$$

Then it follows from (4.14) that

$$\left\|\frac{\partial \mathbf{u}^N}{\partial t}\right\|_{L^2(I;L^2(\Omega)^d)} \leq C\,. \tag{4.27}$$

From Aubin-Lions Lemma 2.48 in Chapter 1, using (4.26), (4.27), we obtain the strong convergence

$$\nabla \mathbf{u}^N \to \nabla \mathbf{u} \quad \text{in } L^p(Q_T)\,.$$

Thus $\mathbf{u}$ is a weak solution. Clearly, by Lemma 2.45 in Chapter 1, we have $\mathbf{u} \in C(I;H)$. Moreover, $\mathbf{u}$ satisfies all estimates (4.24), (4.26), (4.27). The proof of Theorem 4.21 is complete. ■

Now, we shall prove two uniqueness theorems under different assumptions on the regularity of the initial value $\mathbf{u}_0$. If $p \geq 1 + \frac{2d}{d+2}$, then a weak solution $\mathbf{u}$ to the problem $(\mathrm{NS})_\mathrm{p}$ satisfies the weak formulation in the form

$$\begin{aligned}\Big\langle \frac{\partial \mathbf{u}(t)}{\partial t} + u_j(t)\frac{\partial \mathbf{u}(t)}{\partial x_j}, \boldsymbol{\varphi}\Big\rangle_{V_p} + \int_\Omega \tau_{ij}\big(\mathbf{e}(\mathbf{u}(t))\big)e_{ij}(\boldsymbol{\varphi})\,dx \\ = (\mathbf{f}(t), \boldsymbol{\varphi})\end{aligned} \tag{4.28}$$

for almost all $t \in I$ and for all $\boldsymbol{\varphi} \in V_p$, which follows from (3.78) and the semiregularity properties (4.1)–(4.3).

Theorem 4.29 *Let $\mathbf{u}_0 \in H$, $\mathbf{f} \in L^{p'}(Q_T)$. If*

$$p \geq \frac{d+2}{2}\,, \tag{4.30}$$

then there exists a unique weak solution to the problem $(NS)_p$.

PROOF : Let us suppose that there exist two weak solutions $\mathbf{u}$, $\mathbf{v}$ to the problem $(\mathrm{NS})_\mathrm{p}$ with the same initial value $\mathbf{u}_0$. Put $\mathbf{w} \equiv \mathbf{u} - \mathbf{v}$. Subtracting the weak formulation for $\mathbf{v}$ from the one for $\mathbf{u}$ and taking $\mathbf{w}(t)$ as the test function in the resulting equation, we obtain¶¶

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\mathbf{w}\|_2^2 + \int_\Omega \Big(\tau_{ij}\big(\mathbf{e}(\mathbf{u})\big) - \tau_{ij}\big(\mathbf{e}(\mathbf{v})\big)\Big)e_{ij}(\mathbf{w})\,dx \\ = -\int_\Omega \Big(u_j\frac{\partial u_i}{\partial x_j} - v_j\frac{\partial v_i}{\partial x_j}\Big)w_i\,dx \\ = -\int_\Omega w_j\frac{\partial u_i}{\partial x_j}w_i\,dx - \int_\Omega v_j\frac{\partial w_i}{\partial x_j}w_i\,dx \\ = -\int_\Omega w_j\frac{\partial u_i}{\partial x_j}w_i\,dx\,,\end{aligned} \tag{4.31}$$

¶¶ We will not write the explicit dependence of $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ on t.

where we have used the fact (see (2.12)) that

$$\int_\Omega v_j \frac{\partial w_i}{\partial x_j} w_i \, dx = 0.$$

By Hölder's inequality,

$$\left| \int_\Omega w_j \frac{\partial u_i}{\partial x_j} w_i \, dx \right| \le \int_\Omega |\mathbf{w}|^2 |\nabla \mathbf{u}| \, dx \le \|\nabla \mathbf{u}\|_p \|\mathbf{w}\|^2_{\frac{2p}{p-1}} . \tag{4.32}$$

Combining (4.31), (4.32), (1.25) and (1.11), we obtain

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{w}\|_2^2 + K_2^2 C_6 \|\nabla \mathbf{w}\|_2^2 \le \|\nabla \mathbf{u}\|_p \|\mathbf{w}\|^2_{\frac{2p}{p-1}} .$$

Because of the interpolation inequality (4.36) proved in Lemma 4.35 below, we obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\mathbf{w}\|_2^2 + K_2^2 C_6 \|\nabla \mathbf{w}\|_2^2 &\le \|\nabla \mathbf{u}\|_p \|\mathbf{w}\|_2^{\frac{2p-d}{p}} \|\nabla \mathbf{w}\|_2^{\frac{d}{p}} \\ &\underset{\text{ineq.}}{\overset{\text{Young}}{\le}} K_2^2 C_6 \|\nabla \mathbf{w}\|_2^2 \\ &\quad + C\|\nabla \mathbf{u}\|_p^{\frac{2p}{2p-d}} \|\mathbf{w}\|_2^2 . \end{aligned} \tag{4.33}$$

Finally,

$$\frac{d}{dt}\|\mathbf{w}\|_2^2 \le C\|\nabla \mathbf{u}\|_p^{\frac{2p}{2p-d}} \|\mathbf{w}\|_2^2 . \tag{4.34}$$

Since $\mathbf{w}(0) = \mathbf{0}$ and $\mathbf{u} \in L^p(I; V_p)$, we can use Gronwall's lemma 3.5 from the Appendix if $\frac{2p}{2p-d} \le p$, which is equivalent to (4.30). As a conclusion of Gronwall's lemma we obtain

$$\|\mathbf{w}(t)\|_2 = 0 \qquad \forall t \le T .$$

Thus, $\mathbf{u}(t) = \mathbf{v}(t)$ almost everywhere in I.
Theorem 4.31 is proved. ■

Lemma 4.35 *Let $\mathbf{v} \in W^{1,2}_{\text{per}}(\Omega)$ and $q \in [2, \frac{2d}{d-2}]$ for $d \ge 3$ and $q \in (2, +\infty)$ if $d = 2$. Then there exists $c > 0$ such that*

$$\|\mathbf{v}\|_q \le c\|\mathbf{v}\|_2^\alpha \|\nabla \mathbf{v}\|_2^{1-\alpha} , \tag{4.36}$$

with $\alpha = \frac{2d-q(d-2)}{2q}$.

PROOF : First let $d \geq 3$. Then $W^{1,2}(\Omega) \hookrightarrow L^{\frac{2d}{d-2}}(\Omega)$ and therefore there exists $c > 0$ such that $\|\mathbf{v}\|_{\frac{2d}{d-2}} \leq c\|\nabla \mathbf{v}\|_{1,2}$. Combining this with the interpolation inequality (see Corollary 2.10 in Chapter 1)

$$\|\mathbf{v}\|_q \leq c\|\mathbf{v}\|_2^{\alpha}\|\mathbf{v}\|_{\frac{2d}{d-2}}^{1-\alpha}, \quad \alpha = \frac{2d - q(d-2)}{2q},$$

we obtain the assertion (4.36).

If $d = 2$, then $W^{1,2}(\Omega) \not\hookrightarrow L^{\infty}(\Omega)$. Nevertheless, for $s = \frac{q-2}{q}$, $W^{s,2}(\Omega) \hookrightarrow L^q(\Omega)$ and $\|\mathbf{v}\|_q \leq c\|\mathbf{v}\|_{s,2}$. Due to Lemma 2.18 in Chapter 1, we have

$$\|\mathbf{v}\|_{s,2} \leq c\|\mathbf{v}\|_2^{\alpha}\|\mathbf{v}\|_{1,2}^{1-\alpha}, \quad \alpha = \frac{2}{q}.$$

Putting both inequalities together gives (4.36). ∎

Theorem 4.37 *Let $\mathbf{u}_0 \in V_2$, $\mathbf{f} \in L^{p'}(Q_T)$. If*

$$p \geq 1 + \frac{2d}{d+2}, \tag{4.38}$$

then there exists a unique weak solution to the problem $(NS)_p$.

Remark 4.39 In two dimensions, the lower bound $1 + \frac{2d}{d+2}$ coincides with $\frac{d+2}{2}$ and the uniqueness of a weak solution follows from Theorem 4.29. Thus, the assertion of Theorem 4.37 is new only for $d \geq 3$.

PROOF (of Theorem 4.37): Let us consider three-dimensional domains Ω only. Since $\mathbf{u}_0 \in V_2$, we know (see (4.1)) that

$$\nabla \mathbf{u} \in L^{\infty}(I; V_2). \tag{4.40}$$

Following the proof of Theorem 4.29, we can start with the inequality

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\mathbf{w}\|_2^2 + K_2^2 C_6\|\nabla \mathbf{w}\|_2^2 &\overset{(4.36)}{\leq} \|\nabla \mathbf{u}\|_2\|\mathbf{w}\|_2^{\frac{1}{2}}\|\nabla \mathbf{w}\|_2^{\frac{3}{2}} \\ &\underset{\text{ineq.}}{\overset{\text{Young}}{\leq}} K_2^2 C_6\|\nabla \mathbf{w}\|_2^2 + C\|\nabla \mathbf{u}\|_2^4\|\mathbf{w}\|_2^2.\end{aligned}$$

Hence,

$$\frac{d}{dt}\|\mathbf{w}\|_2^2 \leq \|\nabla \mathbf{u}\|_2^4\|\mathbf{w}\|_2^2$$

together with (4.40) and Gronwall's lemma finishes the proof. ∎

5.4.2 Existence of a strong solution under some restriction on data

This section deals with the conditional existence of strong solutions in three dimensions. We will first show the existence of such solutions for $p \in (\frac{5}{3}, \frac{11}{5})$ provided that some norms of $\mathbf{f}$ and $\mathbf{u}_0$ are small enough. Later, we will formulate an assertion saying that it is possible to find a (short) time interval on which strong solutions exist for arbitrarily large $\mathbf{f}$ and $\mathbf{u}_0$. Throughout this section, we will assume that

$$\mathbf{u}_0 \in V_p \cap V_2 \quad \text{and} \quad \mathbf{f} \in \begin{cases} L^2(Q_T) & \text{if } p \geq 2\,, \\ L^{p'}(Q_T) & \text{if } p < 2\,. \end{cases} \tag{4.41}$$

The method of the proof is again based on the second *a priori* estimate which finally reads (compare with (3.48))

$$\begin{aligned} &\frac{1}{1-\lambda}(1 + \|\nabla \mathbf{u}^N(t)\|_2^2)^{1-\lambda} \\ &\qquad + \int_0^t \mathcal{I}_p(\mathbf{u}^N)(1 + \|\mathbf{u}^N\|_2^2)^{-\lambda}\, d\tau \\ &\qquad \leq C \int_0^t \|\nabla \mathbf{u}^N\|_p^p\, d\tau + c_{\mathbf{f}} + \frac{1}{1-\lambda}(1 + \|\nabla \mathbf{u}_0\|_2^2)^{1-\lambda}\,. \end{aligned} \tag{4.42}$$

By (3.47), we know that $\lambda = \frac{2(3-p)}{3p-5} > 1$ for $p < \frac{11}{5}$. For 'global' estimates the negative term

$$\frac{1}{1-\lambda}(1 + \|\nabla \mathbf{u}^N(t)\|_2^2)^{1-\lambda} \tag{4.43}$$

was moved to the left-hand side and estimated by $\frac{1}{\lambda-1}$ there. In order to obtain the results mentioned above, the term (4.43) will enter essentially into the proof.

More precisely, we rewrite (4.42) as follows:

$$\begin{aligned} \frac{1}{\lambda-1}\frac{1}{(1+\|\nabla \mathbf{u}_0\|_2^2)^{\lambda-1}} \leq & c_p \int_0^t \|\nabla \mathbf{u}^N\|_p^p\, d\tau + c_{\mathbf{f}} \\ & + \frac{1}{\lambda-1}\frac{1}{(1+\|\nabla \mathbf{u}^N(t)\|_2^2)^{\lambda-1}}\,. \end{aligned} \tag{4.44}$$

The first *a priori* estimates, see (2.28), give the existence of a constant c_a depending on $\|\mathbf{u}_0\|_2$, $\|\mathbf{f}\|_{L^{p'}(Q_T)}$, T and $|\Omega|$ such that

$$\int_0^T \|\nabla \mathbf{u}^N\|_p^p\, dt \leq c_a\,. \tag{4.45}$$

Then it follows from (4.44) that

$$\|\nabla \mathbf{u}^N(t)\|_2^2 \le \frac{1+\|\nabla \mathbf{u}_0\|_2^2}{\left[1-(c_pc_a+c_{\mathbf{f}})(\lambda-1)(1+\|\nabla \mathbf{u}_0\|_2^2)^{\lambda-1}\right]^{1/(\lambda-1)}}. \tag{4.46}$$

Now, whenever

$$c_pc_a+c_{\mathbf{f}} < \frac{1}{(\lambda-1)(1+\|\nabla \mathbf{u}_0\|_2^2)^{\lambda-1}} \tag{4.47}$$

then we will get an uniform bound of $\{\mathbf{u}^N\}$ in $L^\infty(I;V_2)$, which is a springboard to a proof of the existence of a strong solution. Since this last statement has been clarified for example in the proof of Theorem 4.21, the remaining task is to show that c_a can be made as small as needed and depending only on norms of $\mathbf{u}_0$ and $\mathbf{f}$ (and not on $|\Omega|$!).

The reason why Ω appears at the right-hand side of (2.28) comes from the coercivity condition (1.20). If one uses the condition (1.23) for $p \in (1,2)$ or (1.24) for $p \ge 2$ instead of $(1.20)_2$, the dependence of the right-hand side of (2.28) on $|\Omega|$ can be removed. Let us illustrate it only for the more complicated case, $p \in (1,2)$. Then we have from (2.26) and (1.23) that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\mathbf{u}^N\|_2^2 + c_3\int_\Omega \min\left(|\mathbf{e}(\mathbf{u}^N)|^2, |\mathbf{e}(\mathbf{u}^N)|^p\right)dx \\ \le \int_\Omega |\mathbf{f}\cdot\mathbf{u}^N|\,dx\,.\end{aligned} \tag{4.48}$$

Denoting $B^N \equiv \{x\in\Omega, |\mathbf{e}(\mathbf{u}^N)| \le 1\}$, $C^N \equiv \{x\in\Omega, |\mathbf{e}(\mathbf{u}^N)| > 1\}$ we obtain from (4.48) that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\mathbf{u}^N\|_2^2 + c_3\int_{B^N}|\mathbf{e}(\mathbf{u}^N)|^2\,dx + c_3\int_{C^N}|\mathbf{e}(\mathbf{u}^N)|^p\,dx \\ &\quad\le c\|\mathbf{f}\|_{p'}\left(\|\mathbf{e}(\mathbf{u}^N)\|_{p,B^N} + \|\mathbf{e}(\mathbf{u}^N)\|_{p,C^N}\right) \\ &\quad\le c\|\mathbf{f}\|_{p'}^{p'} + \frac{c_3}{2}\|\mathbf{e}(\mathbf{u}^N)\|_{p,C^N}^p + c\|\mathbf{f}\|_{p'}^2 + \frac{c_3}{2}\|\mathbf{e}(\mathbf{u}^N)\|_{2,B^N}^2\end{aligned}$$

and thus

$$\begin{aligned}\|\mathbf{u}^N(t)\|_2^2 + c_3\int_0^t\left(\|\mathbf{e}(\mathbf{u}^N)\|_{p,C^N}^p + \|\mathbf{e}(\mathbf{u}^N)\|_{2,B^N}^2\right)ds \\ \le \|\mathbf{u}_0\|_2^2 + c\int_0^T \|\mathbf{f}\|_{p'}^{p'} + \|\mathbf{f}\|_{p'}^2\,dt \equiv c(\mathbf{u}_0,\mathbf{f})\,.\end{aligned} \tag{4.49}$$

Since $p < 2$, we have

$$\int_0^T \|\nabla \mathbf{u}^N\|_p^p \, dt \le \int_0^T \|\nabla \mathbf{u}^N\|_{p,C^N}^p \, dt + c\left(\int_0^T \|\nabla \mathbf{u}^N\|_{2,B^N}^2 \, dt\right)^{p/2}$$
$$\stackrel{(4.49)}{\le} \frac{1}{c_3}\Big(c(\mathbf{u}_0,\mathbf{f}) + c\big(c(\mathbf{u}_0,\mathbf{f})\big)^{p/2}\Big) \equiv c_a\,. \tag{4.50}$$

As a conclusion of the above considerations we obtain the following theorem.

Theorem 4.51 *Let $\mathbf{u}_0$ and $\mathbf{f}$ satisfying (4.41) be such that (4.47) holds. Then there exists a strong solution to the problem $(NS)_p$ for $p > \frac{5}{3}$ $(d = 3)$.*

The local existence of a strong solution is treated in the following theorem.

Theorem 4.52 *Let $\mathbf{u}_0$, $\mathbf{f}$ satisfy (4.41). If $p > \frac{5}{3}$ $(d = 3)$ then there exists a $t^* > 0$ such that a strong solution to the problem $(NS)_\mathrm{p}$ exists on $(0, t^*)$. This solution is unique for $p \ge 2$.*

PROOF : We will proceed similarly to before. For simplicity, however, we assume that $\mathbf{f}$ is independent of t and we denote a given $p > \frac{5}{3}$ by p_0. Let us assume that instead of (4.44) we have

$$\begin{aligned}\frac{1}{\lambda_\varepsilon - 1}\frac{1}{(1 + \|\nabla \mathbf{u}_0\|_2^2)^{\lambda_\varepsilon - 1}} &\le c_p \int_0^t \|\nabla \mathbf{u}^N\|_p^{\frac{p}{1+\varepsilon}} \, d\tau + t\, c_{\mathbf{f}} \\ &+ \frac{1}{\lambda_\varepsilon - 1}\frac{1}{(1 + \|\nabla \mathbf{u}^N(t)\|_2^2)^{\lambda_\varepsilon - 1}}\,.\end{aligned} \tag{4.53}$$

where $\lambda_\varepsilon \equiv \lambda_\varepsilon(p) \to \lambda$ as $\varepsilon \to 0$ and $\lambda_\varepsilon(p_0) > 0$. By Hölder's inequality,

$$\int_0^t \|\nabla \mathbf{u}^N\|_p^{\frac{p}{1+\varepsilon}} \, d\tau \le \left(\int_0^t \|\nabla \mathbf{u}^N\|_p^p \, d\tau\right)^{\frac{1}{1+\varepsilon}} t^{\frac{\varepsilon}{1+\varepsilon}} \le c_a t^{\frac{\varepsilon}{1+\varepsilon}}\,.$$

As in (4.46), we have

$$\|\nabla \mathbf{u}^N\|_2^2 \le C \tag{4.54}$$

whenever

$$c_p c_a t^{\frac{\varepsilon}{1+\varepsilon}} + t c_{\mathbf{f}} \le \frac{1}{(\lambda_\varepsilon - 1)(1 + \|\nabla \mathbf{u}_0\|_2^2)^{\lambda_\varepsilon - 1}}\,. \tag{4.55}$$

Thus, for all t^* satisfying (4.55), the existence of a strong solution on $(0, t^*)$ follows.

In order to derive (4.53) we have to return to Section 5.3, and to modify the calculations between (3.42) and (3.48). Instead of (3.44), we require the condition

$$\left(Q_2 + \frac{2-q}{2} Q_3\right) \delta' = \frac{p}{1+\varepsilon}$$

to hold. It is clear that for ε sufficiently small, $\lambda_\varepsilon(p_0) > 0$.

The uniqueness proof coincides with the proof of Theorem 4.37. The restriction $p \geq 2$ is due to (1.25).

The details are left to the reader. ∎

5.4.3 Fractional derivative estimates

In this section we will find exponents β, γ such that

$$\begin{aligned} &\int_0^T \|D^{(2)}\mathbf{u}^N\|_r^{2\beta} < +\infty\,, \\ &\int_0^T \left\|\frac{\partial \mathbf{u}^N}{\partial t}\right\|_2^{2\gamma} < +\infty\,. \end{aligned} \tag{4.56}$$

Here, $\mathbf{u}^N$ denotes Galerkin approximations to the problem $(\mathrm{NS})_\mathrm{p}$.

We have already proved several estimates of the type (4.56) in Section 5.3. Let us recall them:

- If $p \geq 1 + \frac{2d}{d+2}$ then (4.56) holds with $r = 2$ and $\beta = \gamma = 1$.
- If moreover $d = 2$ and $p \in (1, 2)$, then (4.56) holds with $r = p$ and $\beta = \gamma = 1$.
- If $d = 3$, then $(4.56)_1$ holds with

$$\begin{aligned} r = 2 \quad &\text{and} \quad \beta = \frac{3p-5}{p+1} && \text{if} \quad p \in \left[2, \tfrac{11}{5}\right), \\ r = p \quad &\text{and} \quad \beta = \frac{p}{2}\,\frac{5p-9}{(-p^2+8p-9)} && \text{if} \quad p \in \left(\tfrac{9}{5}, 2\right). \end{aligned}$$

  Compare these results with (3.60) for $p < 2$ and (3.55) for $p \in [2, \frac{11}{5})$.

In this section we will show the estimates of the type $(4.56)_2$.

Lemma 4.57 *Let $d = 3$ and $p \in [2, \frac{11}{5})$. Then the Galerkin approximations $\mathbf{u}^N$ to the problem $(NS)_p$ satisfy $(4.56)_2$ with*

$$\gamma = \frac{(3p-5)\,p}{5p^2 - 15p + 16}\,. \tag{4.58}$$

Remark 4.59 Taking $p = 2$ in (3.55) and (4.58), we obtain

$$\int_0^T \|D^{(2)}\mathbf{u}^N\|_2^{2/3} < +\infty\,, \tag{4.60}$$

$$\int_0^T \|\frac{\partial \mathbf{u}^N}{\partial t}\|_2^{2/3} < +\infty\,. \tag{4.61}$$

As emphasized in Example 1.62 in Chapter 1, the classical Navier-Stokes system is a particular subcase of the model $(\mathrm{NS})_\mathrm{p}$, when $p = 2$. For a weak solution of that system, the estimates (4.60)–(4.61) were derived in FOIAS, GUILLOPÉ AND TEMAM [1981] (for the space-periodic problem) and in DUFF [1990b] (for the Dirichlet problem).

PROOF (of Lemma 4.57): In order to prove $(4.56)_2$ with γ defined in (4.58), we multiply the rth member of the Galerkin system by $\frac{d}{dt}c_r^N(t)$ and sum up. Following the same lines as in (4.13)–(4.14), we obtain, applying Hölder's inequality,

$$\begin{aligned} &\frac{1}{2}\|\frac{\partial \mathbf{u}^N}{\partial t}\|_2^2 + \frac{d}{dt}\|U(\mathbf{e}(\mathbf{u}^N))\|_1 \\ &\qquad \le c\|\mathbf{f}\|_2^2 + \|\mathbf{u}^N\|^2_{\frac{3p}{3-p}}\|\nabla\mathbf{u}^N\|^2_{\frac{6p}{5p-6}}\,. \end{aligned} \tag{4.62}$$

By the interpolation lemma (see Corollary 2.10 in Chapter 1), we have

$$\|\nabla\mathbf{u}^N\|_{\frac{6p}{5p-6}} \le \|\nabla\mathbf{u}^N\|_p^{\frac{5p-8}{4}}\|\nabla\mathbf{u}^N\|_{3p}^{\frac{12-5p}{4}}\,. \tag{4.63}$$

Taking $q = 2$ in (3.27), we obtain

$$\|\nabla\mathbf{u}^N\|_{3p}^p \le \mathcal{I}_p(\mathbf{u}^N)\,. \tag{4.64}$$

Then (4.62)–(4.64) imply

$$\begin{aligned} &\frac{1}{2}\|\frac{\partial \mathbf{u}^N}{\partial t}\|_2^2 + \frac{d}{dt}\|U(\mathbf{e}(\mathbf{u}^N))\|_1 \\ &\qquad \le c\|\mathbf{f}\|_2^2 + \|\nabla\mathbf{u}^N\|_p^{\frac{5p-4}{2}}\left\{\mathcal{I}_p(\mathbf{u}^N)\right\}^{\frac{12-5p}{2p}}\,. \end{aligned} \tag{4.65}$$

By (3.54) we know that

$$\int_0^T \mathcal{I}_p(\mathbf{u}^N)(\tau)\big(1 + \|\nabla\mathbf{u}^N(\tau)\|_2^2\big)^{-\lambda}\,d\tau \le C\,, \tag{4.66}$$

where $\lambda = 2\,\frac{3-p}{3p-5}$. According to (1.36) and Korn's inequality (1.11) and after some algebraic manipulation it is possible to find a constant $c > 0$ such that

$$\|\nabla \mathbf{v}\|_p^p \le c\big(1 + \|U\big(\mathbf{e}(\mathbf{v})\big)\|_1\big)\,. \tag{4.67}$$

Using $\big(1 + \|\nabla \mathbf{u}^N\|_2^2\big) \le c\big(1 + \|\nabla \mathbf{u}^N\|_p^p\big)$, (4.66) and (4.67), the inequality (4.65) turns into

$$\begin{aligned}
&\frac{1}{2}\Big\|\frac{\partial \mathbf{u}^N}{\partial t}\Big\|_2^2 + \frac{d}{dt}\big(1 + \|U\big(\mathbf{e}(\mathbf{u}^N)\big)\|_1\big) \\
&\qquad \le c_{\mathbf{f}} + c\left\{\frac{\mathcal{I}_p(\mathbf{u}^N)}{\big(1 + \|\nabla \mathbf{u}^N\|_2^2\big)^{\lambda}}\right\}^{\frac{12-5p}{2p}} \\
&\qquad\qquad \times \big(1 + \|U\big(\mathbf{e}(\mathbf{u}^N)\big)\|_1\big)^{\lambda\frac{12-5p}{2p} + \frac{5p-4}{2p}}\,.
\end{aligned} \tag{4.68}$$

Let $\mu > 0$. Multiplying (4.68) by $\big(1+\|U\big(\mathbf{e}(\mathbf{u}^N)\big)\|_1\big)^{-\mu}$, integrating between 0 and t, $t \le T$, we get

$$\begin{aligned}
&\int_0^t \frac{\|\frac{\partial \mathbf{u}^N}{\partial t}\|_2^2}{\big(1 + \|U\big(\mathbf{e}(\mathbf{u}^N)\big)\|_1\big)^{\mu}}\,d\tau + \frac{1}{1-\mu}\big(1 + \|U\big(\mathbf{e}(\mathbf{u}^N(t))\big)\|_1\big)^{1-\mu} \\
&\qquad \le c_{\mathbf{f},\mathbf{u}_0} + c\int_0^t \left\{\frac{\mathcal{I}_p(\mathbf{u}^N)}{\big(1 + \|\nabla \mathbf{u}^N\|_2^2\big)^{\lambda}}\right\}^{\frac{12-5p}{2p}} \\
&\qquad\qquad \times \big(1 + \|U\big(\mathbf{e}(\mathbf{u}^N)\big)\|_1\big)^{-\mu+\lambda\frac{12-5p}{2p} + \frac{5p-4}{2p}}\,d\tau\,.
\end{aligned} \tag{4.69}$$

Because of the uniform estimates (4.66), we apply Hölder's inequality on the last term in (4.69) together with the requirements

$$\begin{aligned}
\frac{1}{\delta} + \frac{1}{\delta'} &= 1\,, \\
\Big(-\mu + \lambda\frac{12-5p}{2p} + \frac{5p-4}{2p}\Big)\,\delta &= 1\,, \\
\frac{12-5p}{2p}\,\delta' &= 1\,.
\end{aligned} \tag{4.70}$$

Solving (4.70) we get

$$\mu = \frac{2(p^2 - 5p + 8)}{p(3p-5)}\,. \tag{4.71}$$

Since

$$\int_0^t \left(1+\|U(\mathbf{e}(\mathbf{u}^N))\|_1\right) d\tau \overset{(1.36)}{\leq} c\int_0^t \left(1+\|\nabla\mathbf{u}^N\|_p^p\right) d\tau \overset{(3.10)}{\leq} C\,, \tag{4.72}$$

it follows from (4.69)–(4.72) that

$$\int_0^T \frac{\|\frac{\partial \mathbf{u}^N}{\partial t}\|_2^2}{\left(1+\|U(\mathbf{e}(\mathbf{u}^N))\|_1\right)^{\mu}}\, d\tau \leq C\,. \tag{4.73}$$

Next,

$$\int_0^T \|\frac{\partial \mathbf{u}^N}{\partial t}\|_2^{2\gamma}\, dt \leq \int_0^T \left(\|\frac{\partial \mathbf{u}^N}{\partial t}\|_2^2 \left(1+\|U(\mathbf{e}(\mathbf{u}^N))\|_1\right)^{-\mu}\right)^{\gamma} \times \left(1+\|U(\mathbf{e}(\mathbf{u}^N))\|_1\right)^{\mu\gamma} dt\,.$$

By Hölder's inequality, using (4.73) and the same argument as in (4.72), we obtain

$$\int_0^T \|\frac{\partial \mathbf{u}^N}{\partial t}\|_2^{2\gamma} \leq C \qquad \text{with} \qquad \gamma = \frac{p(3p-5)}{5p^2-15p+16}\,. \tag{4.74}$$

Thus, (4.58) is found and Lemma 4.57 is proved. ∎

5.5 Compressible non-Newtonian gases and measure-valued solutions

The objective of this final section is to prove the existence of measure-valued solutions to an initial-boundary value problem for a flow of compressible gases undergoing isothermal processes in bounded domains.

The result that we will present here has been proved by MATUŠŮ AND NOVOTNÝ [1994], using considerations from NEČAS, NOVOTNÝ AND ŠILHAVÝ [1989, 1990] and NOVOTNÝ [1992]. A different kind of measure-valued solution has been studied in NEUSTUPA [1993] for barotropic flows and by KRÖNER AND ZAJACZKOWSKI [1996] for the Euler equations of compressible fluids.

Let us emphasize that the global existence of weak solutions for the problem studied in this section is not known. Very inspiring in this direction is the paper of Padula (see PADULA [1986]); unfortunately some conclusions are not correct as pointed out in PADULA [1988]. For perfect isentropic gases, where the pressure is given by $p = \kappa\rho^{\gamma}, \gamma > 1$, the global existence of weak solutions has been proved very recently by LIONS [1993a, 1993b] for $\gamma \geq \frac{3d}{d+2}$. A promising method in the direction of the investigation of the properties of weak solutions seems to be the method of decomposition proposed in NOVOTNÝ AND PADULA [1994]. This method has already been successfully applied for studying steady flows of the isentropic perfect gases, see NOVOTNÝ [1995, 1996].

Let us now formulate the problem. Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with $\partial\Omega \in C^{0,1}$, $d = 2$ or 3. For $T \in (0,\infty)$, we use the notation $I \equiv (0,T)$ and $Q_T \equiv I \times \Omega$. We also set $\Omega_t \equiv \{(t,x) \in Q_T; x \in \Omega\}$.

Let $\mathbf{f} : Q_T \to \mathbb{R}^d$, $\rho_0 : \Omega \to \mathbb{R}$, $\mathbf{u}_0 : \Omega \to \mathbb{R}^d$ and $\boldsymbol{\tau} : \mathbb{R}^{d^2}_{\mathrm{sym}} \to \mathbb{R}^{d^2}_{\mathrm{sym}}$ be given. Assume that there exists a $p \in (1,\infty)$ and $c_1, c_2 > 0$ such that for all $\boldsymbol{\eta} \in \mathbb{R}^{d^2}_{\mathrm{sym}}$,

$$\boldsymbol{\tau}(\boldsymbol{\eta}) \cdot \boldsymbol{\eta} \geq c_1 |\boldsymbol{\eta}|^p , \tag{5.1}$$

$$|\boldsymbol{\tau}(\boldsymbol{\eta})| \leq c_2 \big(1 + |\boldsymbol{\eta}|\big)^{p-1} . \tag{5.2}$$

By the **problem** $(\mathbf{CF})_{\mathrm{p}}$ we denote the initial-boundary value problem to find $\mathbf{u} = (u_1, \dots, u_d) : Q_T \to \mathbb{R}^d$ and $\rho : Q_T \to \mathbb{R}$ solving the system of $(d+1)$-equations

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_j}(\rho u_j) = 0 , \tag{5.3}$$

$$\frac{\partial}{\partial t}(\rho u_i) + \frac{\partial}{\partial x_j}(\rho u_j u_i) + \frac{\partial p}{\partial x_i} - \frac{\partial}{\partial x_j}\tau_{ij}(\mathbf{e}(\mathbf{u})) = \rho f_i , \tag{5.4}$$

$i = 1, \dots, d$, and satisfying the initial and boundary conditions

$$\mathbf{u}(0,x) = \mathbf{u}_0(x) , \qquad \rho(0,x) = \rho_0(x) , \qquad \forall\, x \in \Omega , \tag{5.5}$$

$$\mathbf{u}(t,x) = \mathbf{0} \qquad \forall\, (t,x) \in I \times \partial\Omega . \tag{5.6}$$

Recall that $2\,\mathbf{e} = \nabla\mathbf{u} + (\nabla\mathbf{u})^T$. We refer to Section 1.1.4 for more comments on the system of equations.

Let

$$\rho_0 \in C^1(\overline{\Omega}),\ \rho_0 > 0 \text{ in } \overline{\Omega} \text{ and } \mathbf{u}_0 \in W^{3,2}_0(\Omega)^d , \tag{5.7}$$

$$\mathbf{f} \in C(I; L^\infty(\Omega)^d) . \tag{5.8}$$

It is possible to weaken the assumptions on ρ_0, $\mathbf{u}_0$ and $\mathbf{f}$, if we regularize the data in appropriate parts of proofs. Nevertheless, we prefer to put stronger assumptions on ρ_0, $\mathbf{u}_0$ and $\mathbf{f}$ in order to retain the main ideas.

Besides Sobolev, Bochner, Lebesgue spaces, and spaces of Radon measures, we will use also Orlicz spaces here. See Section 1.2.5 for definition and an exposition of basic properties of these spaces.

Let $\Psi(t) \equiv (1+t)\ln(1+t) - t$ be a Young function.

Definition 5.9 *Let $\rho_0, \mathbf{u}_0$ and $\mathbf{f}$ satisfy the assumptions (5.7), (5.8). The triple $(\rho, \mathbf{u}, \nu)$ is called a* **measure-valued solution** *to the problem (CF)$_{\mathrm{p}}$ if*

$$\rho \in L^\infty(I; L_\Psi(\Omega)), \quad \rho \geq 0, \tag{5.10}$$

$$\mathbf{u} \in L^p(I; W_0^{1,p}(\Omega)^d), \tag{5.11}$$

$$\nu \in L^\infty_w(Q_T; \mathrm{Prob}(\mathbb{R}^{d^2})), \tag{5.12}$$

satisfy

$$\int_{\mathbb{R}^{d^2}} \boldsymbol{\lambda}\, d\nu_{t,x}(\boldsymbol{\lambda}) = \nabla\mathbf{u}(t,x) \quad \text{for a.a. } (t,x) \in Q_T, \tag{5.13}$$

$$\begin{gathered}\int_0^T \int_\Omega \rho \frac{\partial\psi}{\partial t}\, dx\, dt + \int_\Omega \rho u_j \frac{\partial\psi}{\partial x_j}\, dx\, dt + \int_\Omega \rho_0 \psi(0)\, dx = 0, \\ \forall \psi \in \mathcal{D}(-\infty, T; \mathcal{D}(\Omega)),\end{gathered} \tag{5.14}$$

and

$$\begin{aligned}&- \int_0^T \left(\int_\Omega \rho u_i \frac{\partial\varphi_i}{\partial t}\, dx \right) dt - \int_0^T \left(\int_\Omega \rho u_j u_i \frac{\partial \varphi_i}{\partial x_j}\, dx \right) dt \\ &+ \int_0^T \int_\Omega e_{ij}(\boldsymbol{\varphi}) \int_{\mathbb{R}^{d^2}} \tau_{ij}(\mathbf{e}(\boldsymbol{\lambda}))\, d\nu_{t,x}(\boldsymbol{\lambda})\, dx\, dt - \int_0^T \int_\Omega \rho \frac{\partial\varphi_i}{\partial x_i}\, dx\, dt \\ &= \int_0^T \int_\Omega \rho f_i \varphi_i\, dx\, dt + \int_\Omega \rho_0 u_{0i} \varphi_i(0)\, dx,\end{aligned} \tag{5.15}$$

for all $\boldsymbol{\varphi} \in \mathcal{D}(-\infty, T, \mathcal{D}(\Omega)^d)$.

Theorem 5.16 *Let $\rho_0, \mathbf{u}_0, \mathbf{f}$ satisfy (5.7), (5.8). If $p > d$ then there exists a measure-valued solution in the sense of Definition 5.9.*

The proof of Theorem 5.16 will be presented in the rest of this section. For $\mu > 0$ we will first define a singular perturbation to the problem (CF)$_{\mathrm{p}}$, denoted by (CFpert)$_{\mathrm{p}}^{\mu}$. Then we will construct an approximation of the problem (CFpert)$_{\mathrm{p}}^{\mu}$ and we will show the existence of a solution by a fixed point argument combined with the method of characteristics and the Galerkin method. By passing to the limit, we will prove the existence of a weak solution (global in time) to the problem $(\mathrm{CFpert})_\mathrm{p}^\mu$. Finally, letting $\mu \to 0+$, we will demonstrate the existence of a measure-valued solution to the problem $(\mathrm{CF})_\mathrm{p}$ and the proof will be finished.

We will split the exposition of the proof into the following four steps:

- Definition of the problem $(\mathrm{CFpert})_\mathrm{p}^\mu$ and its approximations.
- The solvability of the approximate problems.
- The existence of a weak solution to the weak problem $(\mathrm{CFpert})_\mathrm{p}^\mu$.
- The existence of a measure-valued solution.

- *Definition of the problem $(CFpert)_\mathrm{p}^\mu$ and its approximations*

Let $\mu > 0$ and let $((\cdot,\cdot))_3$ denote a scalar product in $W_0^{3,2}(\Omega)^d$. Then $\|\cdot\|_{3,2} = ((\cdot,\cdot))_3^{\frac{1}{2}}$. We say that a couple $(\rho^\mu, \mathbf{u}^\mu)$ is a **weak solution to the problem $(\mathbf{CFpert})_\mathrm{p}^\mu$**, if

$$\mathbf{u}^\mu \in L^\infty(I; W_0^{3,2}(\Omega)^d) \cap L^p(I; W_0^{1,p}(\Omega)^d)\,, \tag{5.17}$$

$$\frac{\partial \mathbf{u}^\mu}{\partial t} \in L^{p'}(I, L^{p'}(\Omega)^d)\,, \tag{5.18}$$

$$\rho^\mu \in L^\infty(Q_T) \cap L^2(I; W^{1,2}(\Omega))\,, \tag{5.19}$$

$$\frac{\partial \rho^\mu}{\partial t} \in L^2(Q_T)\,, \tag{5.20}$$

and if the following identities are satisfied:

$$\int_{Q_T} \rho^\mu \frac{\partial \psi}{\partial t}\, dx\, dt + \int_{Q_T} \rho^\mu u_j^\mu \frac{\partial \psi}{\partial x_j}\, dx\, dt + \int_\Omega \rho_0^\mu \psi(0,x)\, dx = 0\,, \tag{5.21}$$

for all $\psi \in \mathcal{D}\big(-\infty, T, \mathcal{D}(\Omega)\big)$, and

$$\begin{aligned} -\int_0^T \left(\int_\Omega \rho^\mu u_i^\mu \frac{\partial \varphi_i}{\partial t}\, dx \right) dt - \int_0^T \left(\int_\Omega \rho^\mu u_j^\mu u_i^\mu \frac{\partial \varphi_i}{\partial x_j}\, dx \right) dt \\ + \mu((\mathbf{u}^\mu, \boldsymbol{\varphi}))_3 \\ + \int_0^T \int_\Omega \tau_{ij}(\mathbf{e}(\mathbf{u}^\mu)) e_{ij}(\boldsymbol{\varphi}) dx\, dt - \int_0^T \int_\Omega \rho^\mu \frac{\partial \varphi_i}{\partial x_i}\, dx\, dt \\ = \int_0^T \int_\Omega \rho^\mu f_i \varphi_i\, dx\, dt + \int_\Omega \rho_0^\mu u_{0i}^\mu \varphi_i(0)\, dx\,, \end{aligned} \tag{5.22}$$

for all $\boldsymbol{\varphi} \in \mathcal{D}\big(-\infty, T, \mathcal{D}(\Omega)^d\big)$.

Remark 5.23 There are many possibilities for defining the perturbation $(\text{CFpert})^{\mu}_{\text{p}}$. Basically, we add to the term $\frac{\partial}{\partial x_j}\tau_{ij}(\mathbf{e}(\mathbf{u}))$ another linear operator of sixth order and of course we have to increase the number of boundary conditions. For our presentation the precise form of the boundary conditions (which disappear when $\mu \to 0+$) and the form of the elliptic operator are not important. We refer to NEČAS AND ŠILHAVÝ [1991] for introducing **multipolar fluids** that could be convenient for that purpose. See also NEČAS, NOVOTNÝ AND ŠILHAVÝ [1989, 1990] or NOVOTNÝ [1992].

Let us fix $\mu > 0$ and for simplicity, let us denote $(\rho^{\mu}, \mathbf{u}^{\mu})$ by $(\rho, \mathbf{u})$. Then $(\rho^N, \mathbf{u}^N)$ will denote the solution of the approximation to the problem $(\text{CFpert})^{\mu}_{\text{p}}$. As presented in Theorem 4.11 and Remark 4.14 of the Appendix, we can construct a basis $\{\boldsymbol{\omega}^r\}_{r=1}^{\infty} \subset W_0^{3,2}(\Omega)^d$ consisting of solutions to the eigenvalue problem

$$((\boldsymbol{\omega}^r, \boldsymbol{\xi}))_3 = \lambda_r(\boldsymbol{\omega}^r, \boldsymbol{\xi}), \quad \forall \boldsymbol{\xi} \in W_0^{3,2}(\Omega)^d .$$

Then $\boldsymbol{\omega}^r$ are orthonormal in $L^2(\Omega)^d$, and due to the regularity of elliptic systems we can assume that $\boldsymbol{\omega}^r \in C^{\infty}(\Omega)^d$, if $\partial\Omega$ is smooth enough. Defining $P^N\mathbf{u} \equiv \sum_{k=1}^{N}(\boldsymbol{\omega}^r, \mathbf{u})\boldsymbol{\omega}^r$, we know, from (6.4.12)–(6.4.13), that

$$\begin{aligned} \|P^N\mathbf{u}\|_{3,2} &\le \|\mathbf{u}\|_{3,2} , \\ \|P^N\mathbf{u}\|_2 &\le \|\mathbf{u}\|_2 . \end{aligned} \tag{5.24}$$

Let us put $\mathbf{u}^N(t,x) \equiv \sum_{k=1}^{N} c_k^N(t)\boldsymbol{\omega}^k(x)$. Then functions $(\rho^N, \mathbf{u}^N)$ are called **approximations of the problem $(\mathbf{CFpert})^{\mu}_{\text{p}}$** if they satisfy

$$\frac{\partial \rho^N}{\partial t} + \frac{\partial}{\partial x_i}(\rho^N u_i^N) = 0 \quad \text{in } \Omega , \tag{5.25}$$

$$\begin{aligned} &\int_{\Omega}\Big(\rho^N\frac{\partial u_i^N}{\partial t} + \rho^N u_j^N\frac{\partial u_i^N}{\partial x_j} + \frac{\partial \rho^N}{\partial x_i} - \rho^N f_i\Big)w_i^r\,dx \\ &+ \int_{\Omega}\tau_{ij}(\mathbf{e}(\mathbf{u}^N))e_{ij}(\boldsymbol{\omega}^r)\,dx + \mu((\mathbf{u}^N, \boldsymbol{\omega}^r))_3 = 0 , \end{aligned} \tag{5.26}$$

$r = 1, \ldots, N$, with initial conditions

$$\rho^N(0) = \rho_0 , \tag{5.27}$$

and

$$\mathbf{u}^N(0) = P^N \mathbf{u}_0 , \tag{5.28}$$

respectively.

• *The solvability of the approximate problems*

Lemma 5.29 *Let $(\rho_0, \mathbf{u}_0, \mathbf{f})$ satisfy (5.7), (5.8). Then, for fixed N, the system (5.25)–(5.28) has a solution $(\rho^N, \mathbf{u}^N)$, where*

$$\begin{aligned} \rho^N &\in C^1(\overline{Q}_T) , \\ \mathbf{u}^N &\in C^1(I; W_0^{3,2}(\Omega)^d) . \end{aligned} \tag{5.30}$$

In order to prove the existence of $(\rho^N, \mathbf{u}^N)$ solving (5.25)–(5.28) we will proceed as follows: For fixed $N \in \mathbb{N}$ and $L > 0$, we define

$$B_L(0) \equiv \{\mathbf{c} \in C(\overline{I})^N ; \|\mathbf{c}\|_{C(\overline{I})^N} \leq L\} . \tag{5.31}$$

Let $\overline{\mathbf{c}} \in B_L(0)$ be such that

$$\overline{c}_k(0) \equiv (\mathbf{u}_0, \boldsymbol{\omega}^k) \ , \quad k = 1, 2, \dots, N . \tag{5.32}$$

Setting

$$\overline{\mathbf{u}}^N(t,x) \equiv \sum_{i=1}^{N} \overline{c}_k(t) \boldsymbol{\omega}^k(x) , \tag{5.33}$$

we will seek ρ^N, by the method of characteristics, as a solution of

$$\frac{\partial \rho^N}{\partial t} + \frac{\partial}{\partial x_i}(\rho^N \overline{u}_i^N) = 0 \ , \quad \rho^N(0) = \rho_0 \ . \tag{5.34}$$

Having ρ^N, we will finally look for $\mathbf{u}^N \in C^1(I; C^\infty(\overline{\Omega})^d)$ in the form

$$\mathbf{u}^N(t,x) = \sum_{k=1}^{N} c_k(t) \boldsymbol{\omega}^k(x) , \tag{5.35}$$

solving the Galerkin system

$$\begin{aligned} &\int_\Omega \Big(\rho^N \frac{\partial u_i^N}{\partial t} + \rho^N \overline{u}_j^N \frac{\partial u_i^N}{\partial x_j} + \frac{\partial \rho^N}{\partial x_i} - \rho^N f_i\Big) w_i^r \\ &+ \int_\Omega \tau_{ij}(\mathbf{e}(\mathbf{u}^N)) e_{ij}(\boldsymbol{\omega}^r)\, dx + \mu((\mathbf{u}^N, \boldsymbol{\omega}^r))_3 = 0 , \qquad r = 1, \dots, N . \end{aligned} \tag{5.36}$$

Getting $\mathbf{u}^N$ as a solution of (5.36), we define

$$S : \overline{\mathbf{c}}^N \in B_L(0) \mapsto \mathbf{c}^N .$$

If we verify that S satisfies the assumptions of the Schauder Fixed Point Theorem, we will finally find a solution of (5.25)–(5.28).

Let us denote

$$k(N) \equiv \sup_{k=1,\dots,N} \max_{x\in\overline{\Omega}} |\nabla \boldsymbol{\omega}^k(x)| . \tag{5.37}$$

Further, let $\overline{\mathbf{u}}^N \in C^1(I; C^\infty(\overline{\Omega})^d)$ be given by (5.33), $\overline{\mathbf{c}} \in B_L(0)$. Then the characteristics $\mathbf{z}^N$ to (5.34) are determined as a solution of

$$\begin{aligned} \frac{d\mathbf{z}^N(\tau; t, x)}{d\tau} &= \overline{\mathbf{u}}^N(\tau, \mathbf{z}^N(\tau; t, x)) , \\ \mathbf{z}^N(0; t, x) &= x , \end{aligned} \tag{5.38}$$

or in the equivalent integral form

$$\mathbf{z}^N(\tau; t, x) = x + \int_\tau^t \overline{\mathbf{u}}^N(s, \mathbf{z}^N(s; t, x))\, ds . \tag{5.39}$$

Since $\|\overline{\mathbf{c}}\|_{C(I)} \le L$, the method of successive approximations will provide the existence of a solution for all $t \in I$; see any book on ordinary differential equations for details. Taking (5.38) into consideration, equation (5.34) becomes

$$\frac{d}{d\tau}\big(\ln \rho^N(\tau, \mathbf{z}^N(\tau; t, x))\big) = -\operatorname{div} \overline{\mathbf{u}}^N(\tau, \mathbf{z}^N(\tau; , t, x)) . \tag{5.40}$$

Hence, for $t \le \tau$,

$$\begin{aligned} \rho^N(t, x) =& \rho_0(\mathbf{z}^N(0; t, x)) \\ & \times \exp\left(- \int_0^t \operatorname{div} \overline{\mathbf{u}}^N(\tau, \mathbf{z}^N(\tau; t, x))\, d\tau \right) . \end{aligned} \tag{5.41}$$

Thus $\rho^N \in C^1(Q_T)$ is the uniquely defined solution of (5.34). Moreover,

$$0 < \rho_1 \le \rho^N(t, x) \le \rho_2 , \tag{5.42}$$

where

$$\begin{aligned} \rho_1 &\equiv \min_{y\in\overline{\Omega}} \rho_0(y) \exp(-L\, k(N) T) , \\ \rho_2 &\equiv \max_{y\in\overline{\Omega}} \rho_0(y) \exp(L\, k(N) T) . \end{aligned}$$

Taking $(\rho^N, \overline{\mathbf{u}}^N)$, we will find the coefficients $(c_k^N)_{k=1}^N$, such that (5.36) holds. The system (5.36) is a system of ordinary differential equations, that can be rewritten as

$$\mathbf{A}\frac{d\mathbf{c}^N}{dt} = \mathbf{F}(\mathbf{c}^N)\,, \qquad \mathbf{c}^N(0) = \mathbf{c}_0\,, \tag{5.43}$$

where $\mathbf{A} = (a_{ij})_{i,j=1}^N$ has components

$$a_{ij} = \int_\Omega \rho^N(t,x)\boldsymbol{\omega}^i(x)\boldsymbol{\omega}^j(x)\,dx$$

and the vectors $\mathbf{F}(\mathbf{c}^N) = (F_r(\mathbf{c}^N))_{r=1}^N$ and $\mathbf{c}_0 = (c_{0r})_{r=1}^N$ are determined by

$$\begin{aligned}
F_r(\mathbf{c}^N) = & - c_k^N(t)\overline{c}_s^N(t)\int_\Omega \rho^N\frac{\partial w_i^k}{\partial x_j}w_j^s w_i^r\,dx - \int_\Omega \frac{\partial \rho^N}{\partial x_i}w_i^r\,dx \\
& + \int_\Omega \rho^N f_i w_i^r\,dx - \int_\Omega \tau_{ij}(\mathbf{e}(\mathbf{u}^N))e_{ij}(\boldsymbol{\omega}^r)\,dx \\
& - \mu c_r^N((\boldsymbol{\omega}^r,\boldsymbol{\omega}^r))_3\,, \\
c_{0r} = & (\mathbf{u}_0,\boldsymbol{\omega}^r)\,.
\end{aligned}$$

Since $\mathbf{A}$ is invertible, we get, from (5.43)

$$\frac{d\mathbf{c}^N}{dt} = \mathbf{A}^{-1}\mathbf{F}(\mathbf{c}^N)\,, \qquad \mathbf{c}(0) = \mathbf{c}_0\,. \tag{5.44}$$

The local solvability follows from Theorem 3.4 in the Appendix, the global solvability is obtained exactly in the same way as in Section 5.2, where we proved the solvability of the Galerkin system (2.21). This means that global *a priori* estimates for $\mathbf{c}^N$ have to be derived. We will find them first for the equations (5.25)–(5.26) and then for the equations (5.36). Let us recall that $\Psi(\xi) \equiv (1+\xi)\ln(1+\xi) - \xi$.

Lemma 5.45 *Let $(\rho^N, \mathbf{u}^N)$ be a solution of (5.25)–(5.28). Then*

- *there exists a constant $C > 0$ independent of μ and N such that*

$$\int_{\Omega_t} \rho^N\,dx = \int_{\Omega_0} \rho_0\,dx, \tag{5.46}$$

$$\begin{aligned}
&\int_{\Omega_t} \rho^N\,|\mathbf{u}^N|^2\,dx + \int_{\Omega_t} \Psi(\rho^N)\,dx \le C\,, \\
&\mu\int_0^t \|\mathbf{u}^N\|_{3,2}^2\,d\tau + K_p^p\int_0^t \|\nabla\mathbf{u}^N\|_p^p\,d\tau \le C\,;
\end{aligned} \tag{5.47}$$

• *there exists a constant* $C(\frac{1}{\mu}) > 0$ *and* $\rho^* = \rho^*(\frac{1}{\mu}) > 0$ *such that for* $t \leq T$

$$\|\rho^N\|_{\infty,Q_t} + \int_0^t \|\frac{\partial \rho^N}{\partial t}\|_2^2 + \int_0^t \|\nabla\rho^N\|_2^2 \leq C(\tfrac{1}{\mu}), \quad (5.48)$$

$$\rho^N(t,x) \geq \rho^*, \qquad \forall\,(t,x) \in Q_T, \quad (5.49)$$

$$\int_0^T \|\frac{\partial \mathbf{u}^N}{\partial t}\|_2^2 + \mu\|\mathbf{u}^N\|_{L^\infty(I;W_0^{3,2}(\Omega)^d)} \leq C(\tfrac{1}{\mu}). \quad (5.50)$$

PROOF : Let g be any function (scalar, vector or tensor) defined on Q_T. Then, $\frac{dg}{dt} = \frac{\partial g}{\partial t} + u_k \frac{\partial g}{\partial x_k}$ and by the Transport Theorem (see CHORIN AND MARSDEN [1992] or FEISTAUER [1993]),

$$\frac{d}{dt}\int_{\Omega_t} g\,dx = \int_{\Omega_t}\left(\frac{dg}{dt} + g\ \mathrm{div}\,\mathbf{u}\right)dx. \quad (5.51)$$

Thus, from (5.25),

$$\frac{d}{dt}\int_{\Omega_t} \rho^N\,dx = 0, \quad (5.52)$$

and (5.46) follows.
Also,

$$\begin{aligned}\frac{d}{dt}\int_{\Omega_t} &(1+\rho^N)\ln(1+\rho^N)\,dx \\ &= -\int_{\Omega_0} \rho^N \frac{\partial u_i^N}{\partial x_i}\,dx + \int_{\Omega_t} \ln(1+\rho^N)\frac{\partial u_j^N}{\partial x_j}\,dx.\end{aligned} \quad (5.53)$$

Now, if we multiply (5.26) by $c_r^N(t)$, sum up over $r = 1,\ldots,N$, use (5.1) and the generalized Korn inequality (1.11), we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_{\Omega_t} \rho^N|\mathbf{u}^N|^2\,dx &+ K_p^p\int_{\Omega_t}|\nabla\mathbf{u}^N|^p\,dx + \mu((\mathbf{u}^N,\mathbf{u}^N))_3 \\ &\leq \int_{\Omega_t}\rho^N\frac{\partial u_i^N}{\partial x_i}\,dx + \int_{\Omega_t}\rho^N f_i u_i^N\,dx.\end{aligned} \quad (5.54)$$

Let us denote

$$E^N(t) \equiv \frac{1}{2}\int_{\Omega_t}(\rho^N|\mathbf{u}^N|^2 + \Psi(\rho^N))\,dx,$$

$$D^N(t) \equiv \frac{K_p^p}{2}\|\nabla\mathbf{u}^N\|_p^p + \mu\|\mathbf{u}^N\|_{3,2}^2.$$

If we add (5.53) and (5.54) and subtract (5.52), we obtain

$$\begin{aligned}\frac{d}{dt}E^N(t) + D^N(t) \le \int_{\Omega_t} \rho^N f_i u_i^N \, dx \\ + \int_{\Omega_t} \ln(1+\rho^N) \frac{\partial u_i^N}{\partial x_i} \, dx \, .\end{aligned} \tag{5.55}$$

Since

$$\begin{aligned}\Big| \int_{\Omega_t} \rho^N f_i u_i^N \, dx \Big| &\le \|\mathbf{f}\|_\infty \Big(\int_{\Omega_t} \rho^N |\mathbf{u}^N|^2 \, dx \Big)^{\frac{1}{2}} \Big(\int_{\Omega_t} \rho^N \, dx \Big)^{\frac{1}{2}} \\ &\overset{(5.46)}{\le} \frac{1}{2}\|\mathbf{f}\|_\infty^2 \int_{\Omega_t} \rho_0 \, dx + \frac{1}{2} \int_{\Omega_t} \rho^N |\mathbf{u}^N|^2 \, dx\end{aligned}$$

and

$$\begin{aligned}&\Big| \int_{\Omega_t} \ln(1+\rho^N) \frac{\partial u_i^N}{\partial x_i} \, dx \Big| \\ &\quad = \Big| \int_{\Omega_t} (\ln(1+\rho^N))^{\frac{1}{2}} (\ln(1+\rho^N))^{\frac{1}{2}} \frac{\partial u_i^N}{\partial x_i} \, dx \Big| \\ &\quad \le \frac{1}{e} \int_{\Omega_t} \big[(1+\rho^N)\ln(1+\rho^N)\big]^{\frac{1}{2}} |\nabla \mathbf{u}^N| \, dx \\ &\quad \le \int_{\Omega_t} \Psi(\rho^N) \, dx + \int_{\Omega} \rho_0 \, dx + \frac{K_p^p}{2} \|\nabla \mathbf{u}^N\|_p^p + C \, .\end{aligned}$$

We finally obtain, for all $t \le T$,

$$E^N(t) + \int_0^t D^N(\tau) \, d\tau \le c_{\mathbf{f},\rho_0} + \int_0^t E^N(\tau) \, d\tau + E^N(0) \, .$$

The Gronwall lemma 3.5 in the Appendix finishes the proof of (5.47).

In order to prove (5.48)–(5.50), let us first observe, from (5.47), that

$$\int_0^t \|\mathbf{u}^N\|_{3,2}^2 \, d\tau \le \frac{C}{\mu} \, , \qquad t \le T \, . \tag{5.56}$$

By (5.41),

$$\begin{aligned}&\|\rho_0\|_\infty \exp\Big(- \int_0^T \|\operatorname{div} \mathbf{u}^N\|_\infty \, dt \Big) \\ &\qquad \le \|\rho^N\|_\infty \le \|\rho_0\|_\infty \exp\Big(\int_0^T \|\operatorname{div} \mathbf{u}^N\|_\infty \, dt \Big) \, .\end{aligned}$$

Since for $t \le T$,

$$\int_0^T \|\operatorname{div}\mathbf{u}^N\|_\infty\,dt \le \int_0^T \|\mathbf{u}^N\|_{3,2}\,dt \le \Big(\frac{C}{\mu}\Big)^{1/2}\sqrt{T}\,,$$

we obtain

$$\begin{aligned}\rho^* \equiv \|\rho_0\|_\infty \exp\left(-\Big(\frac{C}{\mu}\Big)^{1/2}\sqrt{T}\right)&\\ \le \|\rho^N\|_\infty \le \|\rho_0\|_\infty \exp\left(\Big(\frac{C}{\mu}\Big)^{1/2}\sqrt{T}\right)&\,,\end{aligned} \tag{5.57}$$

which is (5.49) and a part of (5.48). The further estimates in (5.48) follow from the formulae (5.41) differentiated with respect to t and x_i, (5.25), and the estimates (5.56) and (5.57). We will skip the details.

Note that, due to (5.49),

$$\rho^*\Big\|\frac{\partial\mathbf{u}^N}{\partial t}\Big\|_2^2 \le \int_{\Omega_t}\rho^N\Big|\frac{\partial\mathbf{u}^N}{\partial t}\Big|^2\,dx\,. \tag{5.58}$$

Let us finally multiply the rth equation in (5.26) by $\frac{dc_r^N}{dt}$ and sum up over $r = 1,\dots,N$. One of the terms we obtain reads as

$$\int_\Omega \tau_{ij}(\mathbf{e}(\mathbf{u}^N))e_{ij}\Big(\frac{\partial\mathbf{u}^N}{\partial t}\Big)\,dx\,.$$

In order to avoid estimating this integral, we will strengthen the assumptions on $\boldsymbol{\tau}$ slightly. Namely, we will assume that there exists a non-negative potential $U : \mathbb{R}^{d^2}_{\mathrm{sym}} \longrightarrow \mathbb{R}$ for $\boldsymbol{\tau}$, i.e., that $\frac{\partial U}{\partial e_{ij}} = \tau_{ij}$ holds. See Section 1.1.5 for some examples of such $\boldsymbol{\tau}$. Then,

$$\begin{aligned}\int_\Omega \tau_{ij}(\mathbf{e}(\mathbf{u}^N))e_{ij}\Big(\frac{\partial\mathbf{u}^N}{\partial t}\Big)\,dx &= \frac{d}{dt}\int_\Omega U(\mathbf{e}(\mathbf{u}^N))\,dx\\ &= \frac{d}{dt}\|U(\mathbf{e}(\mathbf{u}^N))\|_1\,.\end{aligned}$$

Therefore, the inequality which is derived from (5.26) has the form:

$$\begin{aligned}\frac{\rho^*}{2}\Big\|\frac{\partial\mathbf{u}^N}{\partial t}\Big\|_2^2 + \frac{\mu}{2}\frac{d}{dt}\|\mathbf{u}^N\|_{3,2}^2 + \frac{d}{dt}\|U(\mathbf{e}(\mathbf{u}^N))\|_1&\\ \le C\Big(\|\nabla\rho^N\|_2^2 + \int_{\Omega_t}|\rho^N|^2|\mathbf{f}|^2\,dx&\\ + \int_{\Omega_t}|\rho^N|^2|\mathbf{u}^N|^2|\nabla\mathbf{u}^N|^2\,dx\Big)&\,,\end{aligned} \tag{5.59}$$

where we have used (5.58) and Young's inequality many times. Since all terms on the right-hand side of (5.59) can be estimated by means of (5.56) and (5.48), the proof of Lemma 5.45 is complete. ■

Lemma 5.60 *Let N be fixed. Let $\overline{\mathbf{u}}^N$ be in the form (5.33) with $\overline{\mathbf{c}}^N \in B_L(0)$. Then for L big enough there exists a t^* such that the coefficients $\mathbf{c}^N$ satisfy*

$$\|\mathbf{c}^N\|_{C([0,t^*])} \leq L\,, \tag{5.61}$$

$$\|\frac{d\mathbf{c}^N}{dt}\|_{C([0,t^*])} \leq C\,. \tag{5.62}$$

PROOF : Due to the orthonormality of $\boldsymbol{\omega}^r$ in $L^2(\Omega)^d$,

$$|\mathbf{c}^N|^2 = (\mathbf{u}^N, \mathbf{u}^N) \quad \text{and} \quad \left|\frac{d\mathbf{c}^N}{dt}\right|^2 = \left(\frac{\partial \mathbf{u}^N}{\partial t}, \frac{\partial \mathbf{u}^N}{\partial t}\right).$$

From (5.49) we have

$$|\mathbf{c}^N|^2 \leq \frac{1}{\rho^*}\int_{\Omega_t} \rho^N |\mathbf{u}^N|^2\, dx$$

and

$$\left|\frac{d\mathbf{c}^N}{dt}\right|^2 \leq \frac{1}{\rho^*}\int_{\Omega_t} \rho^N \left|\frac{\partial \mathbf{u}^N}{\partial t}\right|^2 dx\,.$$

Since the proof of (5.61)–(5.62) is based on multiplying (5.36) by $c_r^N(t)$ and $\frac{dc_r^N(t)}{dt}$, respectively, we proceed in the same way as in the proof of Lemma 5.45. Moreover, we can use the fact that N is fixed, i.e., $k(N)$ defined in (5.37) is finite, and that $\|\overline{\mathbf{c}}^N\|_{C([0,T])} \leq L$. Essentially, it is possible to obtain an estimate

$$|\mathbf{c}^N(t)|^2 \leq c(\mathbf{u}_0, \rho^*) + c(T, \mathbf{f})\,,$$

where $c(T,\mathbf{f}) \to 0$ if $T \to 0$. Thus, taking $L > c(\mathbf{u}_0, \rho^*)$, we see that it is possible to find a t^* such that (5.61) holds. We leave the details to the reader. ■

It follows from (5.61)–(5.62) that

$$S : \overline{\mathbf{c}}^N \mapsto \mathbf{c}^N$$

satisfies all assumptions of the Schauder Fixed Point Theorem. Thus (5.25)–(5.28) has a solution at least on $(0,t^*)$. But the *a priori* estimates (5.46)–(5.50) hold on the whole interval $(0,T)$.

Let us therefore assume that the maximal solution to (5.25)–(5.28) exists on $[0, t_0)$. We want to show that t_0 must be T. Let us assume that $t_0 < T$. Then there exist a solution $\mathbf{c}^N : C([0,t_0)) \to C([0,t_0))$ and some $L > 0$ such that $|\mathbf{c}^N(t)| \le L$ for all $t < t_0$. But then also $\mathbf{c}^N(t_0) \equiv \lim_{t \to t_0} \mathbf{c}^N(t)$ satisfies

$$|\mathbf{c}^N(t_0)| \le L\,,$$

and we extend the above procedure to some interval $(t_0, t_0 + t^*)$, which gives the contradiction.

• *The existence a of weak solution to the problem* $(CFpert)^\mu_{\mathrm{p}}$

Now, $\mu > 0$ is fixed and we will first look for $(\rho^\mu, \mathbf{u}^\mu)$ solving (5.21)–(5.22). Then we will derive other estimates on $(\rho^\mu, \mathbf{u}^\mu)$ independent of μ.

For simplicity we will drop the index μ.

Lemma 5.63 *Let $p > 1$ and $\mu > 0$. Let $(\rho_0, \mathbf{u}_0)$ satisfy (5.7), (5.8). Then there exists a weak solution to the problem $(CFpert)^\mu_{\mathrm{p}}$.*

PROOF : Since approximations $(\rho^N, \mathbf{u}^N)$ satisfy the estimates (5.48)–(5.50), we can extract a subsequence, labelled again by $(\rho^N, \mathbf{u}^N)$, such that

$$\rho^N \rightharpoonup \rho \qquad \text{weakly}^* \text{ in } L^\infty(Q_T)\,, \tag{5.64}$$

$$\frac{\partial \rho^N}{\partial t} \rightharpoonup \frac{\partial \rho}{\partial t} \qquad \text{weakly in } L^2(Q_T)\,, \tag{5.65}$$

$$\nabla \rho^N \rightharpoonup \nabla \rho \qquad \text{weakly in } L^2(I; L^2(\Omega)^d)\,, \tag{5.66}$$

$$\mathbf{u}^N \rightharpoonup \mathbf{u} \qquad \text{weakly in } L^2(I; W^{3,2}_0(\Omega)^d)\,, \tag{5.67}$$

$$\frac{\partial \mathbf{u}^N}{\partial t} \rightharpoonup \frac{\partial \mathbf{u}}{\partial t} \qquad \text{weakly in } L^2(I; L^2(\Omega)^d)\,, \tag{5.68}$$

and by Aubin-Lions Lemma 2.48 in Chapter 1,

$$\rho^N \to \rho \qquad \text{strongly in } L^2(Q_T)\,, \tag{5.69}$$

$$\nabla \mathbf{u}^N \to \nabla \mathbf{u} \qquad \text{strongly in } L^p(Q_T)\,. \tag{5.70}$$

Let $\varphi \in \mathcal{D}(-\infty, T)$. By Vitali's theorem 2.11 in Chapter 1, see also (3.11)–(3.14), we obtain

$$\begin{aligned} &\int_0^T \int_{\Omega_t} \tau_{ij}(\mathbf{e}(\mathbf{u}^N)) e_{ij}(\boldsymbol{\omega}^r)\, dx\, \varphi\, d\tau \\ &\qquad \to \int_0^T \int_{\Omega_t} \tau_{ij}(\mathbf{e}(\mathbf{u})) e_{ij}(\boldsymbol{\omega}^r)\, dx\, \varphi\, d\tau. \end{aligned} \tag{5.71}$$

Also,

$$\mu \int_0^T ((\mathbf{u}^N, \boldsymbol{\omega}^r))_3 \varphi \, dt \overset{(5.67)}{\rightarrow} \mu \int_0^T ((\mathbf{u}, \boldsymbol{\omega}^r))_3 \varphi \, dt \,. \tag{5.72}$$

The other terms are treated by combining strong and weak convergences. Since these limiting processes will be also studied later (even under worse circumstances), when we will prove the existence of measure-valued solution to the problem $(\mathrm{CF})_{\mathrm{p}}$, we will skip these limiting processes now. ■

Let us set $\Phi(\xi) \equiv e^{\xi} - \xi - 1$, $\Phi_2(\xi) \equiv e^{\xi^2} - 1$, $\Psi(\xi) \equiv (1+\xi)\ln(1+\xi) - \xi$, and let $\Psi_{1/2}$ denote the complementary Young function to Φ_2. Since $p > d$, there exist $\gamma < 1$ and $\varepsilon > 0$ such that $\gamma(p-\varepsilon) > d$. Then by Remark 2.34 and Example 2.39 from Chapter 1,

$$\begin{aligned} W_0^{3,2}(\Omega) \hookrightarrow\hookrightarrow W_0^{1,p}(\Omega) \hookrightarrow\hookrightarrow W_0^{\gamma,p-\varepsilon}(\Omega) \\ \hookrightarrow L_{\Phi_2}(\Omega) \hookrightarrow L_{\Phi}(\Omega) \,, \end{aligned} \tag{5.73}$$

and

$$\begin{aligned} L_{\Psi}(\Omega) \hookrightarrow L_{\Psi_{1/2}}(\Omega) \hookrightarrow \left[W^{\gamma,p-\varepsilon}(\Omega)\right]^* \\ \hookrightarrow\hookrightarrow W^{-1,p}(\Omega) \hookrightarrow\hookrightarrow W^{-3,2}(\Omega) \,. \end{aligned} \tag{5.74}$$

Lemma 5.75 *Let $(\rho^\mu, \mathbf{u}^\mu)$ be weak solutions to the problem $(\mathrm{CFpert})_{\mathrm{p}}^{\mu}$. Then there exists a constant $C > 0$ independent of μ such that*

$$\begin{aligned} \int_{\Omega_t} \rho^\mu \, |\mathbf{u}^\mu|^2 \, dx + \|\rho^\mu\|_{L^\infty(I;L_\Psi(\Omega))} \le C \,, \\ \mu \int_0^t \|\mathbf{u}^\mu\|_{3,2}^2 \, d\tau + K_p^p \int_0^t \|\nabla \mathbf{u}^\mu\|_p^p \, d\tau \le C \,, \end{aligned} \tag{5.76}$$

$$\|\rho^\mu \mathbf{u}^\mu\|_{L^\infty(I;L_{\Psi_{1/2}}(\Omega)^d)} \le C \,, \tag{5.77}$$

$$\|\rho^\mu u_i^\mu u_j^\mu\|_{L^2(I;L_{\Psi_{1/2}}(\Omega))} \le C \,, \qquad i,j = 1,\ldots,d \,, \tag{5.78}$$

$$\left\|\frac{\partial \rho^\mu}{\partial t}\right\|_{L^2(I;W^{-3,2}(\Omega))} \le C \,, \tag{5.79}$$

$$\left\|\frac{\partial (\rho^\mu \mathbf{u}^\mu)}{\partial t}\right\|_{L^2(I;W^{-3,2}(\Omega)^d)} \le C \,. \tag{5.80}$$

Moreover,

$$\rho^\mu \ge 0 \quad \text{a.e. in } Q_T \,. \tag{5.81}$$

PROOF : Because $(\rho^\mu, \mathbf{u}^\mu)$ are obtained as limits of approximations $(\rho^N, \mathbf{u}^N)$, (more precisely $(\rho^{N_\mu}, \mathbf{u}^{N_\mu})$), the estimate (5.76) follows immediately from (5.47), and (5.81) follows from (5.49). Further, let us take $\varphi \in L_{\Phi_2}(\Omega)$, $d(\Phi_2, \varphi) \leq 1$. Then from the definition of Orlicz spaces it follows that $\varphi^2 \in L_\Phi(\Omega)$, $d(\Phi, \varphi^2) \leq 1$. Therefore, for $i = 1, \ldots, d$,

$$\begin{aligned}\int_\Omega |\rho^\mu u_i^\mu \varphi| \, dx &\leq \left(\int_\Omega \rho^\mu |u_i^\mu|^2 \, dx\right)^{1/2} \left(\int_\Omega \rho^\mu |\varphi|^2 \, dx\right)^{1/2} \\ &\overset{(5.76)}{\leq} C \left(\int_\Omega \rho^\mu |\varphi|^2 \, dx\right)^{1/2} \\ &\overset{(2.27),\ \mathrm{Ch.1}}{\leq} C\big(d(\Psi; \rho^\mu) + d(\Phi; \varphi^2)\big) \overset{(5.76)}{\leq} C \,.\end{aligned}$$

Thus,

$$\|\rho^\mu u_i^\mu\|_{L^\infty(I; L_{\Psi_{1/2}}(\Omega))} = \operatorname*{ess\,sup}_{t\in I} \sup_{\substack{\varphi \in L_{\Phi_2}(\Omega) \\ d(\Phi_2;\varphi)\leq 1}} \int_\Omega |\rho^\mu u_i^\mu \varphi| \, dx \leq C \,,$$

and (5.77) is proved. Next,

$$\begin{aligned}&\sup_{\substack{v \in L^2(I; L_{\Phi_2}(\Omega)) \\ d(\Phi_2, v(t)) \leq 1}} \int_0^T \int_\Omega |\rho^\mu u_i^\mu u_j^\mu v| \, dx \, dt \\ &\qquad \overset{(2.27),\ \mathrm{Ch.1}}{\leq} \int_0^T \|\mathbf{u}^\mu\|_\infty \left(\|\rho^\mu \mathbf{u}^\mu\|_{L_{\Psi_{1/2}}(\Omega)} + 1\right) \\ &\qquad \overset{(5.77)}{\leq} C \left(\|\mathbf{u}^\mu\|_{L^2(I; W^{1,p}(\Omega)^d)} + 1\right) \overset{(5.76)}{\leq} C \,,\end{aligned}$$

where we have used $W^{1,p}(\Omega)^d \hookrightarrow L^\infty(\Omega)^d$, $p > d$. So (5.78) holds. In order to prove (5.79) and (5.80), let us take $\psi \in L^2(I; W_0^{3,2}(\Omega))$, $\|\psi\|_{L^2(I; W_0^{3,2}(\Omega))} \leq 1$. Then it follows from equation (5.21) that

$$\begin{aligned}\left|\int_0^T \int_\Omega \frac{\partial \rho^\mu}{\partial t} \psi \, dx \, dt\right| &\leq c(\rho_0) + \int_0^T \int_\Omega \rho^\mu |\mathbf{u}^\mu| \, |\nabla \psi| \, dx \\ &\leq c(\rho_0) + \int_0^T \|\rho^\mu \mathbf{u}^\mu\|_{L_{\Psi_{1/2}}} \|\nabla \psi\|_{L_{\Phi_2}} \\ &\overset{(5.73)}{\leq} c(\rho_0) + \|\rho^\mu \mathbf{u}^\mu\|_{L^2(I; L_{\Psi_{1/2}}(\Omega)^d)} \|\psi\|_{L^2(I; W_0^{3,2}(\Omega))} \\ &\overset{(5.77)}{\leq} C \,,\end{aligned}$$

which implies (5.79).

Let us consider $\boldsymbol{\varphi} \in L^2(I; W_0^{3,2}(\Omega)^d)$, $\|\boldsymbol{\varphi}\|_{L^2(I;W_0^{3,2}(\Omega)^d)} \leq 1$. Taking (5.22) into consideration, we have

$$\begin{aligned}
\Big|\int_0^T \int_\Omega \frac{\partial \rho^\mu u_i^\mu}{\partial t} \varphi_i \, dx\, dt\Big| &= \Big|\int_0^T \int_\Omega \rho^\mu u_i^\mu \frac{\partial \varphi_i}{\partial t} \, dx\, dt\Big| \\
&\overset{(5.22)}{\leq} \int_0^T \int_\Omega |\rho^\mu u_j^\mu u_i^\mu \frac{\partial \varphi_i}{\partial x_j}| + |\rho^\mu \frac{\partial \varphi_i}{\partial x_i}| + |\rho^\mu f_i \varphi_i| \, dx\, dt \\
&+ \int_0^T \int_\Omega |\tau_{ij}(\mathbf{e}(\mathbf{u}^\mu)) e_{ij}(\boldsymbol{\varphi})| \, dx\, dt + \mu \int_0^T ((\mathbf{u}^\mu, \boldsymbol{\varphi}))_3 dt + c(\rho_0, \mathbf{u}_0) \\
&\leq C \int_0^T \left(\|\rho^\mu |\mathbf{u}^\mu|^2\|_1 + \|\rho^\mu\|_1 + (1 + \|\nabla \mathbf{u}^\mu\|_p)^{p-1} \right) \|\nabla \boldsymbol{\varphi}\|_\infty \, dt \\
&+ \int_0^T \|\mathbf{f}\|_\infty \|\boldsymbol{\varphi}\|_\infty \|\rho^\mu\|_1 \, dt + \mu \int_0^T \|\mathbf{u}^\mu\|_{3,2} \, \|\boldsymbol{\varphi}\|_{3,2} \, dt + c(\rho_0, \mathbf{u}_0) \\
&\overset{(5.76)}{\leq} C \int_0^T \|\boldsymbol{\varphi}\|_{3,2}^2 \, dt \leq C.
\end{aligned}$$

Thus, (5.80) is valid and the proof of Lemma 5.75 is complete. ■

• *The existence of a measure-valued solution*

Let $p > d$. Letting $\mu \to 0+$, and using the estimates (5.76)–(5.80), we will finish the proof of Theorem 5.16. We will first find a subsequence of $(\rho^\mu, \mathbf{u}^\mu)$, denoted again by $(\rho^\mu, \mathbf{u}^\mu)$ such that

$$\begin{aligned}
\rho^\mu &\rightharpoonup \rho && \text{weakly-* in } L^\infty(I; L_\Psi(\Omega)), && (5.82)\\
\mathbf{u}^\mu &\rightharpoonup \mathbf{u} && \text{weakly in } L^p(I; W_0^{1,p}(\Omega)^d), && (5.83)\\
\rho^\mu &\to \rho && \text{strongly in } L^2(I; W^{-1,p}(\Omega)), && (5.84)\\
\rho^\mu \mathbf{u}^\mu &\to \rho \mathbf{u} && \text{strongly in } L^2(I; W^{-1,p}(\Omega)^d), && (5.85)\\
\rho^\mu u_i^\mu u_j^\mu &\rightharpoonup \rho u_i u_j && \text{weakly-* in } L^2(I; L_{\Psi_{1/2}}(\Omega)), && (5.86)
\end{aligned}$$

as $\mu \to 0+$.

Let us prove (5.82)–(5.86). The first two assertions follow immediately from (5.76) and the fact that $(L_\Phi(\Omega))^* = L_\Psi(\Omega)$. By (5.74),

$$L_\Psi(\Omega) \hookrightarrow\hookrightarrow W^{-1,p}(\Omega) \hookrightarrow W^{-3,2}(\Omega)\,.$$

Then the Aubin-Lions lemma 2.48 in Chapter 1, (5.76) and (5.79) imply (5.84). Similarly, because of (5.77), (5.80) and

$$L_{\Psi_{1/2}}(\Omega)^d \hookrightarrow\hookrightarrow W^{-1,p}(\Omega)^d \hookrightarrow W^{-3,2}(\Omega)^d\,,$$

we obtain

$$\rho^\mu \mathbf{u}^\mu \to \rho\mathbf{u} \quad \text{strongly in } L^2(I;W^{-1,p}(\Omega)^d)\,.$$

In general we obtain only the existence of $\mathbf{z} \in L^2(I;W^{-1,p}(\Omega)^d)$ such that $\rho^\mu \mathbf{u}^\mu \to \mathbf{z}$. However, for all $\varphi \in \mathcal{D}(I;W_0^{1,p}(\Omega)^d)$,

$$\begin{aligned}\int_0^T \int_\Omega (\rho^\mu u_i^\mu - \rho u_i)\varphi_i\,dx &= \int_0^T \int_\Omega (\rho^\mu - \rho)u_i^\mu \varphi_i\,dx \\ &\quad + \int_0^T \int_\Omega \rho\big(u_i^\mu - u_i^\mu\big)\varphi_i\,dx = I_1 + I_2\,.\end{aligned} \tag{5.87}$$

Letting $\mu \to 0+$, $|I_1| \to 0$ due to (5.84) and (5.76), while $I_2 \to 0$ owing to (5.83). Thus, (5.85) holds.

Finally, $\big(L_{\Psi_{1/2}}(\Omega)\big)^* = L_{\Phi_{1/2}}(\Omega)$ and (5.78) gives for $i,j = 1,\dots,d$,

$$\rho^\mu u_i^\mu u_j^\mu \rightharpoonup c_{ij} \text{ weakly-}^* \text{ in } L^2(I;L_{\Psi_{1/2}}(\Omega))\,.$$

Let us show that $c_{ij} = \rho u_i u_j$. Taking $\varphi \in C^\infty(I;\mathcal{D}(\Omega))$, we have

$$\begin{aligned}&\int_0^T \int_\Omega (\rho^\mu u_i^\mu u_j^\mu - \rho u_i u_j)\varphi\,dx \\ &\qquad = \int_0^T \int_\Omega (\rho^\mu u_i^\mu - \rho u_i)u_j^\mu \varphi\,dx \\ &\qquad\quad + \int_0^T \int_\Omega \rho u_i(u_j^\mu - u_j)\varphi\,dx \\ &\qquad \equiv I_1 + I_2\,.\end{aligned} \tag{5.88}$$

Again, $|I_1| \to 0$ due to (5.85) and $\|\mathbf{u}^\mu\|_{L^p(I;W_0^{1,p}(\Omega)^d)} \le c$, and $I_2 \to 0$ due to (5.83) and $\rho u_i \varphi \in W^{-1,p}(\Omega)$. Thus (5.86) holds.

Take $\boldsymbol{\varphi} \in \mathcal{D}(-\infty,T;\mathcal{D}(\Omega)^d)$, and consider the system (5.21)–(5.22) under the limit as $\mu \to 0+$. Then

$$\begin{aligned}\left|\mu \int_0^T ((\mathbf{u}^\mu,\boldsymbol{\varphi}))_3\,dt\right| &\le \sqrt{\mu}\left(\mu \int_0^T \|\mathbf{u}^\mu\|_{3,2}^2\,dt\right)^{1/2} \|\boldsymbol{\varphi}\|_{L^2(I;W_0^{3,2}(\Omega)^d)} \\ &\overset{(5.76)}{\le} c\sqrt{\mu}\,.\end{aligned}$$

Thus,

$$\left|\mu \int_0^T ((\mathbf{u}^\mu,\boldsymbol{\varphi}))_3\,dt\right| \to 0 \qquad \text{as } \mu \to 0+\ .$$

In order to characterize the limit of the nonlinear term given by $\boldsymbol{\tau}$, we will apply Corollary 2.10 from Chapter 4. Because $\{\nabla \mathbf{u}^\mu\}$ is bounded uniformly in $L^p(Q_T)^{d^2}$ and the components of $\boldsymbol{\tau}$ satisfy (5.2), we put in Corollary 2.10 from Chapter 4 $z^j = \nabla \mathbf{u}^\mu$, $\boldsymbol{\tau} = \tau_{ij}$, $q = p - 1$, $s = d^2$ and $Q = Q_T$, and we obtain the existence of

$$\nu \in L^\infty(Q_T; M(\mathbb{R}^{d^2}))$$

such that

$$\lim_{N\to\infty} \int_0^T \int_\Omega \tau_{ij}(\mathbf{e}(\mathbf{u}^\mu)) e_{ij}(\boldsymbol{\varphi})\, dx\, d\tau$$
$$= \int_0^T \int_\Omega e_{ij}(\boldsymbol{\varphi}) \int_{\mathbb{R}^{n^2}} \tau_{ij}(\mathbf{e}(\boldsymbol{\lambda}))\, d\nu_{t,x}(\boldsymbol{\lambda})\, dx\, d\tau$$

and

$$\nabla \mathbf{u}(t,x) \overset{\text{a.e.}}{=} \int_{\mathbb{R}^{d^2}} \boldsymbol{\lambda}\, d\nu_{t,x}(\boldsymbol{\lambda})\,.$$

The passage to the limit in the other terms appearing in (5.21)–(5.22) follow from (5.82)–(5.86). Justifications of these limiting processes in fact use only modifications of (5.87)–(5.88) and we will leave this to the reader. The existence of a measure-valued solution to the problem $(\mathrm{CF})_\mathrm{p}$ is proved.

The proof of Theorem (5.16) is complete.

Appendix

A.1 Some properties of Sobolev spaces

Lemma 1.1 *Let $\Omega \subseteq \mathbb{R}^d$ be an open set. Let $a_0 \geq 0$ be given and let $G \in C^1(\mathbb{R})$. We assume that there are constants $M, k = k(a_0)$, such that*

$$\begin{aligned} |G'(\xi)| &\leq M\,, \\ |G(\xi - a_0)| &\leq k|\xi|\,. \end{aligned} \tag{1.2}$$

Let $u \in W^{1,p}(\Omega)$, $1 \leq p < \infty$, and denote $v \equiv u - a_0$. Then $G \circ v$ belongs to $W^{1,p}(\Omega)$ and

$$\nabla(G \circ v) = G'(v)\nabla v \qquad \text{a.e. in } \Omega\,. \tag{1.3}$$

PROOF : The following proof is only a slight modification of a similar result by KINDERLEHRER AND STAMPACCHIA [1980]. Since $u \in W^{1,p}(\Omega)$, there exists a sequence $u^n \in C^1(\Omega)$ such that $u^n \to u$ in $W^{1,p}(\Omega)$ and almost everywhere in Ω. Let us denote $v^n \equiv u^n - a_0$, then obviously $\nabla v^n \to \nabla v$ in $L^p(\Omega)^d$ and $v^n \to v$ almost everywhere in Ω (but in general not $v^n \to v$ in $L^p(\Omega)$). Further we have $G \circ v^n \in C^1(\Omega)$ and due to $(1.2)_2$, $G \circ v^n$, $G \circ v \in L^p(\Omega)$. Now, from

$$|G(v^n) - G(v)| \leq M\,|v^n - v| \leq |u^n - u|$$

we obtain that

$$G \circ v^n \to G \circ v \qquad \text{in } L^p(\Omega)\,.$$

We will show that also

$$\frac{\partial}{\partial x_i} G(v^n) = G'(v^n)\frac{\partial v^n}{\partial x_i} \to G'(v)\frac{\partial v}{\partial x_i} \qquad \text{in } L^p(\Omega)\,. \tag{1.4}$$

Indeed,

$$\begin{aligned}G'(v^n)\frac{\partial v^n}{\partial x_i} - G'(v)\frac{\partial v}{\partial x_i} \\ &= G'(v^n)\left(\frac{\partial v^n}{\partial x_i} - \frac{\partial v}{\partial x_i}\right) + \left(G'(v^n) - G'(v)\right)\frac{\partial v}{\partial x_i} \\ &\equiv A_n + B_n \,.\end{aligned}$$

Now, obviously, A_n converges to zero in $L^p(\Omega)$ and B_n converges to zero almost everywhere in Ω. Since

$$|B_n|^p \leq (2M)^p\left|\frac{\partial v}{\partial x_i}\right|^p,$$

we conclude from the Lebesgue dominated convergence theorem that B_n converges to zero in $L^p(\Omega)$. Since the derivatives of $G \circ v$ in the sense of distributions are the limits of $\frac{\partial}{\partial x_i}\left(G \circ v^n\right)$ in $L^p(\Omega)$, we obtain immediately (1.3). ∎

Remark 1.5 Note that if $v \in W^{1,p}(\Omega)$ is such that $v : \Omega' \to (\alpha, \beta)$, for some $\Omega' \subset \Omega$, and $G \in C^1(\alpha, \beta)$, then (1.3) holds in Ω'.

Lemma 1.6 *Let $\Omega \subseteq \mathbb{R}^d$ be an open set and let $u \in W^{1,p}(\Omega)$, $1 \leq p < \infty$, and again set $v \equiv u - a_0$ for given $a_0 \geq 0$. Then for all $1 \leq i \leq d$ we have*

$$\frac{\partial v}{\partial x_i} = 0 \qquad \text{a.e. in } E = \{x \in \Omega; v(x) = 0\}\,.$$

PROOF : see for example KINDERLEHRER AND STAMPACCHIA [1980, Lemma A.4, page 53]. ∎

Note that this lemma also holds for $E = \{x \in \Omega; v(x) = a\}$, $a \in \mathbb{R}$ fixed constant.

Now we are able to give a slight generalization of Lemma 1.1.

Lemma 1.7 *Let $\Omega \subseteq \mathbb{R}^d$ be an open set. For $u \in W^{1,p}(\Omega)$, $1 \leq p < \infty$, and a given constant $a_0 \geq 0$ put $v \equiv u - a_0$. Then $v^+ \equiv \max(v, 0)$ belongs to $W^{1,p}(\Omega)$ and*

$$\nabla v^+ = H(v)\nabla v \qquad \text{a.e. in } \Omega\,, \tag{1.8}$$

where $H(\xi) = 1$ if $\xi > 0$, $H(\xi) = 0$ if $\xi \leq 0$. In (1.8) the convention that both sides are zero on the set $\{x \in \Omega; v(x) = 0\}$ is used. Moreover, the mapping $u \mapsto v^+$ is continuous in $W^{1,p}(\Omega)$.

PROOF : The first part follows immediately from Lemma 1.1 and Remark 1.5 with $\Omega' = \Omega \setminus \{x \in \Omega; v(x) = 0\}$. Lemma 1.6 justifies the convention used in (1.8). It remains to show the continuity of the mapping $u \mapsto v^+$. Put $F(\xi) \equiv \max(\xi, 0)$. From (1.8) and $|F(\xi - a_0)| \leq |\xi|$ we get

$$\|v^+\|_{W^{1,p}(\Omega)} \leq \|u\|_{W^{1,p}(\Omega)} , \tag{1.9}$$

which is the boundedness of the (nonlinear) mapping $u \mapsto v^+$. From (1.9) follows that for a given sequence $u_n \to u$ in $W^{1,p}(\Omega)$ we can extract from $v_n^+ \equiv F \circ (u_n - a_0)$ a subsequence still denoted $\{v_n^+\}$ converging weakly in $W^{1,p}(\Omega)$ and having then necessarily the limit v^+. Indeed,

$$\int_\Omega |v_n^+ - v^+|^p \, dx \leq \int_\Omega |v_n - v|^p \, dx = \int_\Omega |u_n - u|^p \, dx , \tag{1.10}$$

where we used the fact that the constant of Lipschitz continuity of F is equal to 1. Relation (1.10) is nothing other than the strong continuity of $u \mapsto v^+$ in $L^p(\Omega)$. Further we have,

$$\begin{aligned}
&\int_\Omega |\nabla v_n^+ - \nabla v^+|^p \, dx \\
&\quad = \int_\Omega |H(v_n)\nabla v_n - H(v)\nabla v|^p \, dx \\
&\quad \leq c \int_\Omega |H(v_n)|^p |\nabla v_n - \nabla v|^p + |H(v_n) - H(v)|^p |\nabla v|^p \, dx \\
&\quad \leq c \int_\Omega |\nabla u_n - \nabla u|^p \, dx + c \int_\Omega |H(v_n) - H(v)|^p |\nabla u|^p \, dx \\
&\quad = A_n + B_n ,
\end{aligned} \tag{1.11}$$

where we used $|H(\xi)| \leq 1$ and $\nabla v_n = \nabla u_n$, $\nabla v = \nabla u$, respectively. The sequence A_n converges to zero due to $u_n \to u$ in $W^{1,p}(\Omega)$, while B_n converges to zero due to the Lebesgue dominated convergence theorem. Therefore we have $F(v_n) \to F(v)$ in $W^{1,p}(\Omega)$ which completes the proof. ■

Remark 1.12 Note that for the special case of $a_0 = 0$ we can extend the previous lemma also for $G(\xi) \equiv |\xi|$. Thus we have for $v \in W^{1,p}(\Omega)$,

$$\frac{\partial}{\partial x_i}|v| = \operatorname{sgn}(v)\frac{\partial v}{\partial x_i} \qquad \text{a.e. in } \Omega . \tag{1.13}$$

Lemma 1.14 *Let $T > 0$ and let $a_0 \geq 0$ be a given constant. Let $u \in W(T) \equiv \{u \in L^2(0,T;W^{1,2}(\mathbb{R}^d)), \frac{\partial u}{\partial t} \in L^2(0,T;W^{-1,2}(\mathbb{R}^d))\}$. Set $v \equiv u - a_0$ again. Then*

$$v^+ \in L^2(0,T;W^{1,2}(\mathbb{R}^d)) \cap C(0,T;L^2(\mathbb{R}^d)), \tag{1.15}$$

and, for almost all $0 \leq t_1 < t_2 \leq T$,

$$2\int_{t_1}^{t_2} \left\langle \frac{\partial v}{\partial t}(s), v^+(s) \right\rangle ds = \|v^+(t_2)\|^2_{L^2(\mathbb{R}^d)} - \|v^+(t_1)\|^2_{L^2(\mathbb{R}^d)}. \tag{1.16}$$

PROOF : The assumption $u \in W(T)$ implies for almost all $t \in (0,T)$ that $u(t) \in W^{1,2}(\mathbb{R}^d)$ and thus Lemma 1.1 gives $v^+(t) \in W^{1,2}(\mathbb{R}^d)$. Further we have for $F(\xi) \equiv \max(\xi, 0)$ that

$$|F(\xi - a_0)| \leq |\xi|, \quad |F'(\xi)| \leq 1. \tag{1.17}$$

Formulae (1.8) and (1.17) imply

$$\|v^+(t)\|_{W^{1,2}(\mathbb{R}^d)} \leq \|u(t)\|_{W^{1,2}(\mathbb{R}^d)} \quad \text{a.e. in } (0,T). \tag{1.18}$$

Thus we obtain $v^+ \in L^2(0,T;W^{1,2}(\mathbb{R}^d))$ and

$$\|v^+\|_{L^2(0,T;W^{1,2}(\mathbb{R}^d))} \leq \|u\|_{L^2(0,T;W^{1,2}(\mathbb{R}^d))}.$$

Let us first show (1.16) for $u \in W^{1,2}((0,T) \times \mathbb{R}^d)$: Lemma 1.1 implies $v^+ \in W^{1,2}((0,T) \times \mathbb{R}^d)$ and since $v^+ F'(v) = v^+$ (almost everywhere), we obtain

$$v^+ \frac{\partial v^+}{\partial t} = v^+ F'(v) \frac{\partial v}{\partial t} = v^+ \frac{\partial v}{\partial t} \quad \text{a.e. in } (0,T) \times \mathbb{R}^d. \tag{1.19}$$

From (1.19) it follows that

$$\left\langle \frac{\partial v}{\partial t}, v^+ \right\rangle = \int_{\mathbb{R}^d} \frac{\partial v}{\partial t} v^+ \, dx = \int_{\mathbb{R}^d} \frac{\partial v^+}{\partial t} v^+ \, dx,$$

which, together with the partial integration formula (2.46) from Chapter 1 for $u = v = v^+$, implies (1.16) for $u \in W^{1,2}((0,T) \times \mathbb{R}^d)$. Now we use the density of $W^{1,2}((0,T) \times \mathbb{R}^d)$ in $W(T)$. Let $u \in W(T)$ and let $\{u^n\} \subseteq W^{1,2}((0,T) \times \mathbb{R}^d)$ converge strongly in $W(T)$ to u. Let $v^n \equiv u^n - a_0$. We want to show that

$$(v^n)^+ \to v^+ \text{ in } L^2(0,T;W^{1,2}(\Omega)) \cap C(0,T;L^2(\mathbb{R}^d)). \tag{1.20}$$

We know that $W(T) \hookrightarrow C(0,T;L^2(\mathbb{R}^d))$ and therefore it follows (see (1.17)) that

$$\|(v^n)^+(t)\|_{L^2(\mathbb{R}^d)} \leq \|u^n(t)\|_{L^2(\mathbb{R}^d)} \qquad \forall\, t \in (0,T)\,, \tag{1.21}$$

$$\|(v^n)^+(t) - (v^n)^+(s)\|_{L^2(\mathbb{R}^d)} \leq \|u^n(t) - u^n(s)\|_{L^2(\mathbb{R}^d)}. \tag{1.22}$$

From (1.21) and (1.22) we find that $(v^n)^+ \in C(0,T;L^2(\mathbb{R}^d))$. In the same way we obtain

$$\max_{[0,T]} \|(v^n)^+(t) - v^+(t)\|_{L^2(\mathbb{R}^d)} \leq \max_{[0,T]} \|u^n(t) - u(t)\|_{L^2(\mathbb{R}^d)}$$

which implies

$$(v^n)^+ \to v^+ \qquad \text{in} \qquad C(0,T;L^2(\mathbb{R}^d)). \tag{1.23}$$

However, Lemma 1.7 shows that $u(t) \mapsto v^+(t)$ is a continuous mapping in the space $W^{1,2}(\mathbb{R}^d)$, which implies

$$\|(v^n)^+(t) - v^+(t)\|_{W^{1,2}(\mathbb{R}^d)} \to 0 \qquad \text{for a.a. } t \in [0,T]\,.$$

From this and (1.18) we obtain

$$(v^n)^+ \to v^+ \qquad \text{in } L^2(0,T;W^{1,2}(\mathbb{R}^d))\,. \tag{1.24}$$

In order to finish the proof we must justify the limiting process in

$$2\int_{t_1}^{t_2} \Big\langle \frac{\partial v^n}{\partial t}(s),(v^n)^+(s)\Big\rangle\, ds = \|(v^n)^+(t_2)\|^2_{L^2(\mathbb{R}^d)} - \|(v^n)^+(t_1)\|^2_{L^2(\mathbb{R}^d)}.$$

This is possible in the first term due to the boundedness of $\frac{\partial v^n}{\partial t}$ $(= \frac{\partial u^n}{\partial t})$ in the space $L^2(0,T;W^{-1,2}(\mathbb{R}^d))$ and due to (1.24). On the right-hand side we use (1.23). ■

A.2 Parabolic theory

Let us consider the following Cauchy problem

$$\begin{aligned} \frac{\partial u}{\partial t} - \varepsilon \Delta u + \lambda u = f \qquad & \text{in } (0,T) \times \mathbb{R}^d\,, \\ u(0,\cdot) = u^0 \qquad & \text{in } \mathbb{R}^d\,, \end{aligned} \tag{2.1}$$

where $T \in (0,\infty]$, $\lambda \geq 0$, $\varepsilon > 0$. We have the following existence and regularity result.

Theorem 2.2 *Let $u^0 \in L^2(\mathbb{R}^d)$, $f \in L^2\big(0,T;W^{-1,2}(\mathbb{R}^d)\big)$. Then (2.1) has a unique solution $u \in W(T) \equiv \big\{u \in L^2(0,T;W^{1,2}(\mathbb{R}^d));\ \frac{\partial u}{\partial t} \in L^2(0,T;W^{-1,2}(\mathbb{R}^d))\big\}$, satisfying (2.1) in the following sense: for almost all $t \in (0,T)$ we have*

$$\Big\langle \frac{\partial u}{\partial t}(t),\varphi\Big\rangle + \varepsilon\int_{\mathbb{R}^d} \nabla u(t)\,\nabla\varphi\,dx + \int_{\mathbb{R}^d}\lambda u\varphi\,dx = \langle f(t),\varphi\rangle$$

for all $\varphi \in W^{1,2}(\mathbb{R}^d)$.

PROOF : See LIONS AND MAGENES [1972a, Chapter 3]. ∎

Remark 2.3 Recall that $W(T) \hookrightarrow C(0,T;L^2(\mathbb{R}^d))$ and therefore the initial condition $(2.1)_2$ is meaningful in the classical sense.

Theorem 2.4 *Assume that for some $m \geq 0$, $m \in \mathbb{N}$, we have $u^0 \in W^{m,2}(\mathbb{R}^d)$, $f \in L^2(0,T;W^{m-1,2}(\mathbb{R}^d))$. Then the solution of (2.1) satisfies*

$$\begin{aligned} &u \in L^2(0,T;W^{m+1,2}(\mathbb{R}^d)) \cap C(0,T;W^{m,2}(\mathbb{R}^d))\,,\\ &\frac{\partial u}{\partial t} \in L^2(0,T;W^{m-1,2}(\mathbb{R}^d))\,. \end{aligned} \tag{2.5}$$

PROOF : See LIONS AND MAGENES [1972b, Chapter 4]. ∎

In the case of a smooth bounded domain $\Omega \subset \mathbb{R}^d$ and $T \in (0,\infty)$ we consider

$$\begin{aligned} \frac{\partial u}{\partial t} - \varepsilon\Delta u + \lambda u = f \qquad & \text{in } (0,T)\times\Omega\,,\\ u(0,\cdot) = u^0 \qquad & \text{in } \Omega\,,\\ u = u^D \qquad & \text{on } (0,T)\times\partial\Omega\,, \end{aligned} \tag{2.6}$$

where u^0 and u^D are supposed to satisfy appropriate compatibility conditions. Then we have the following theorem.

Theorem 2.7 *Let $u^0 \in L^2(\Omega)$, $u^D \in L^2(0,T;W^{1/2,2}(\partial\Omega)) \cap W^{1/4,2}(0,T;L^2(\partial\Omega))$, $f \in L^2(0,T;W^{-1,2}(\Omega))$. Then there exists a unique solution $u \in W(T) \equiv \big\{u \in L^2(0,T;W^{1,2}(\Omega));\ \frac{\partial u}{\partial t} \in L^2(0,T;W^{-1,2}(\Omega))\big\}$, satisfying (2.6) in the following sense: for almost all $t \in (0,T)$ we have*

$$\Big\langle \frac{\partial u}{\partial t}(t),\varphi\Big\rangle + \varepsilon\int_{\Omega} \nabla u(t)\,\nabla\varphi\,dx + \int_{\Omega}\lambda u\varphi\,dx = \langle f(t),\varphi\rangle$$

for all $\varphi \in W_0^{1,2}(\Omega)$. *Moreover, for almost every* $t \in (0,T)$,†

$$u(t) - \widetilde{u}^D(t) \in W_0^{1,2}(\Omega)\,. \tag{2.8}$$

PROOF : See LIONS AND MAGENES [1972b, Chapter 4]. ■

Remark 2.9 For the initial condition $(2.6)_2$ we argue in the same way as in Remark 2.3.

Theorem 2.10 *Let* $f \in L^2(0,T;W^{m-1,2}(\Omega))$, $u^0 \in W^{m,2}(\Omega)$ *and* $u^D \in L^2(0,T;W^{m+1/2,2}(\partial\Omega)) \cap W^{m+1/4,2}(0,T;L^2(\partial\Omega))$ *for some* $m \in \mathbb{N} \cup \{0\}$. *Then the solution of (2.6) satisfies*

$$\begin{aligned} &u \in L^2(0,T;W^{m+1,2}(\Omega)) \cap C(0,T;W^{m,2}(\Omega))\,,\\ &\frac{\partial u}{\partial t} \in L^2(0,T;W^{m-1,2}(\Omega))\,. \end{aligned} \tag{2.11}$$

PROOF : See LIONS AND MAGENES [1972b, Chapter 4]. ■

A.3 Ordinary differential equations

Let us consider for $\mathbf{c} : I_\delta \equiv (t_0 - \delta, t_0 + \delta) \longrightarrow \mathbb{R}^N$, the system of ordinary differential equations

$$\begin{aligned} \frac{d}{dt}\mathbf{c}(t) &= \mathbf{F}(t, \mathbf{c}(t))\,, \qquad t \in I_\delta\,,\\ \mathbf{c}(t_0) &= \mathbf{c}_0 \in \mathbb{R}^N\,. \end{aligned} \tag{3.1}$$

Assume $\mathbf{F} : I_\delta \times K \longrightarrow \mathbb{R}^N$, where $K \equiv \{\mathbf{c} \in \mathbb{R}^N, |\mathbf{c} - \mathbf{c}_0| < \Delta\}$ for some $\Delta > 0$.

Definition 3.2 *A function* $\mathbf{F} : I_\delta \times K \longrightarrow \mathbb{R}^N$ *is said to satisfy the* **Carathéodory conditions** *if*

- $t \mapsto F_i(t, \mathbf{c})$ *is measurable for all* $i = 1, \ldots, N$ *and for all* $\mathbf{c} \in K$,
- $\mathbf{c} \mapsto F_i(t, \mathbf{c})$ *is continuous for almost all* $t \in I_\delta$,
- *there exists an integrable function* $G : I_\delta \longrightarrow \mathbb{R}$ *such that*

$$|F_i(t, \mathbf{c})| \le G(t) \quad \forall\, (t, \mathbf{c}) \in I_\delta \times K\,, \quad \forall i = 1, \ldots, N\,. \tag{3.3}$$

Theorem 3.4 *Let* $\mathbf{F}$ *satisfy the Carathéodory conditions. Then there exist* $\delta' \in (0, \delta)$ *and a continuous function* $\mathbf{c} : I_{\delta'} \longrightarrow \mathbb{R}^N$ *such that*

† From the assumption on u^D it follows that there is a $\widetilde{u}^D(t) \in W^{1,2}(\Omega)$ such that $u^D(t) = \widetilde{u}^D(t)$ on $\partial\Omega$ for almost all $t \in (0,T)$ in the sense of traces. This observation makes (2.8) meaningful.

- $\frac{d\mathbf{c}}{dt}$ *exists for almost all* $t \in I_{\delta'}$,
- $\mathbf{c}$ *solves (3.1).*

PROOF : See for example CODDINGTON AND LEVINSON [1955, Chapter 2] or WALTER [1970, Chapter 1]. ∎

Lemma 3.5 (Gronwall) *Let* $y : (0,T) \to \mathbb{R}$ *and* $g : (0,T) \to \mathbb{R}$ *be non-negative functions,* $g \in L^1(0,T)$. *Let the inequality*

$$y(t) \le C + \int_0^t g(s)\, y(s)\, ds \tag{3.6}$$

hold for $t \in (0,T)$ *with* $C \in \mathbb{R}$. *Then*

$$y(t) \le C \exp \int_0^t g(s)\, ds\,, \quad t \in (0,T)\,. \tag{3.7}$$

PROOF : See for example WALTER [1970, Chapter 1]. ∎

A.4 Bases consisting of eigenfunctions of an elliptic operator

Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with a Lipschitz boundary $\partial\Omega$. We will investigate (construct) bases of spaces of divergence-free functions. For $s \ge 1$ and $p > 1$ let us define

$$\mathcal{V} \equiv \{\boldsymbol{\varphi} \in \mathcal{D}(\Omega)^d ; \operatorname{div} \boldsymbol{\varphi} = 0\}; \tag{4.1}$$

$$H \equiv \text{the closure of } \mathcal{V} \text{ in the } L^2(\Omega)^d\text{-norm}; \tag{4.2}$$

$$V_p \equiv \text{the closure of } \mathcal{V} \text{ in } W^{1,p}(\Omega)^d \text{ with } \|\nabla(\cdot)\|_p\text{-norm}; \tag{4.3}$$

$$V^s \equiv \text{the closure of } \mathcal{V} \text{ in } W^{s,2}(\Omega)^d\text{-norm}. \tag{4.4}$$

If $s = 1$ or $p = 2$, then V will denote the spaces V_2, V^1, respectively. The scalar product in H is marked by $(\cdot,\cdot)$ while the scalar product in V^s is marked by $((\cdot,\cdot))_s$.

Remark 4.5 The spaces V_p and V^s can be characterized as follows:

$$V_p = \{\mathbf{u} \in W_0^{1,p}(\Omega)^d ;\ \operatorname{div} \mathbf{u} = 0\}\,,$$
$$V^s = \{\mathbf{u} \in W^{s,2}(\Omega)^d ;\ \gamma(\mathbf{u}) = \mathbf{0} \text{ at } \partial\Omega,\ \operatorname{div} \mathbf{u} = 0\}\,,$$

where γ is a trace operator defined in (2.14)–(2.15) in Chapter 1.

The characterization of H requires a definition of an appropriate trace operator. Let

$$\gamma : W^{1,2}(\Omega) \hookrightarrow H^{1/2}(\partial\Omega)$$

denote an usual trace operator introduced in (2.14)–(2.15) in Chapter 1. By $H^{-1/2}(\partial\Omega)^d$ we mean the dual space of $H^{1/2}(\partial\Omega)^d$, i.e., the space $\left(H^{1/2}(\partial\Omega)^d\right)^*$. Defining

$$E(\Omega) \equiv \left\{\mathbf{u} \in L^2(\Omega)^d; \operatorname{div}\mathbf{u} \in L^2(\Omega)\right\},$$

it is possible to construct a trace operator

$$\widehat{\gamma} : E(\Omega) \hookrightarrow H^{-1/2}(\partial\Omega)^d,$$

such that $\widehat{\gamma}(\mathbf{u}) = \mathbf{u}\cdot\mathbf{n}$ for $\mathbf{u} \in C^1(\overline{\Omega})$, $\mathbf{n}$ being an outer normal vector. Then it holds that

$$H = \left\{\mathbf{u} \in L^2(\Omega)^d, \widehat{\gamma}(\mathbf{u}) = \mathbf{0}, \operatorname{div}\mathbf{u} = 0 \text{ in } \mathcal{D}'(\Omega)\right\}. \tag{4.6}$$

We refer to CONSTANTIN AND FOIAS [1988], TEMAM [1977] and GALDI [1994a, 1994b] for proofs of these and other detailed results about the space H.

In some parts of the book, we consider spaces of space-periodic functions. For this purpose, it will be necessary to change slightly the definition of $\mathcal{V}$ in (4.1) to

$$\mathcal{V} = \{\varphi \in C^\infty_{\text{per}}(\Omega)^d; \operatorname{div}\varphi = 0, \int_\Omega \varphi\, dx = \mathbf{0}\}, \tag{4.7}$$

where Ω is a cube in $\mathbb{R}^d$.

Let us introduce the projector

$$\mathbb{P} : L^2(\Omega)^d \longrightarrow H,$$

sometimes called **Leray's operator**.

Definition 4.8 *The operator $A \equiv -\mathbb{P}\Delta\mathbf{u} : H \longrightarrow H$ with the domain of definition $D(A) = W^{2,2}(\Omega)^d \cap V$ is called the* **Stokes operator**.

It is worth recalling that in general (for Dirichlet boundary conditions) the operators $\mathbb{P}$ and Δ do not commute. But if the space periodic problem is taken into consideration, then

$$A\mathbf{u} = -\mathbb{P}\Delta\mathbf{u} = -\Delta\mathbf{u}. \tag{4.9}$$

We will study the following spectral problem: find $\boldsymbol{\omega}^r \in V^s$ and $\lambda_r \in \mathbb{R}$ satisfying

$$((\boldsymbol{\omega}^r, \boldsymbol{\varphi}))_s = \lambda_r(\boldsymbol{\omega}^r, \boldsymbol{\varphi}), \quad \forall \boldsymbol{\varphi} \in V^s. \tag{4.10}$$

Theorem 4.11 *There exists a countable set $\{\lambda_r\}_{r=1}^{\infty}$ and a corresponding family of eigenvectors $\{\boldsymbol{\omega}^r\}_{r=1}^{\infty}$ solving the problem (4.10) such that*

- $(\boldsymbol{\omega}^r, \boldsymbol{\omega}^s) = \delta_{rs} \qquad \forall r, s \in \mathbb{N}$,
- $1 \le \lambda_1 \le \lambda_2 \le \ldots$ *and* $\lambda_r \to \infty$ *as* $r \to \infty$,
- $((\frac{\boldsymbol{\omega}^r}{\sqrt{\lambda_r}}, \frac{\boldsymbol{\omega}^s}{\sqrt{\lambda_s}}))_s = \delta_{rs} \qquad \forall r, s \in \mathbb{N}$,
- $\{\boldsymbol{\omega}^r\}_{r=1}^{\infty}$ *forms a basis in* V^s.

Moreover, defining $H^N \equiv \operatorname{span}\{\boldsymbol{\omega}^1, \ldots, \boldsymbol{\omega}^N\}$ *(a linear hull) and* $P^N(\mathbf{v}) \equiv \sum_{i=1}^{N} (\mathbf{v}, \boldsymbol{\omega}^r)\, \boldsymbol{\omega}^r : V^s \longrightarrow H^N$, *we obtain*

$$\|P^N\|_{\mathcal{L}(V^s, V^s)} \le 1, \qquad \|P^N\|_{\mathcal{L}((V^s)^*, (V^s)^*)} \le 1, \tag{4.12}$$

$$\|P^N\|_{\mathcal{L}(H,H)} \le 1. \tag{4.13}$$

Remark 4.14 If we consider instead of (4.10) the problem of finding $\boldsymbol{\omega}^r \in W_0^{s,2}(\Omega)^d$ and $\lambda_r \in \mathbb{R}$ satisfying

$$((\boldsymbol{\omega}^r, \boldsymbol{\varphi}))_s = \lambda_r(\boldsymbol{\omega}^r, \boldsymbol{\varphi}), \quad \forall \boldsymbol{\varphi} \in W_0^{s,2}(\Omega)^d, \tag{4.15}$$

all conditions of Theorem 4.11 hold, too. A particular case, $s = 1$, is used in Chapter 4 of this book.

PROOF (of Theorem 4.11):

- *The existence of $\boldsymbol{\omega}^1$*

  Define

$$\frac{1}{\lambda_1} \equiv \sup_{\|\mathbf{v}\|_{s,2} \le 1} (\mathbf{v}, \mathbf{v}). \tag{4.16}$$

(Note that $\frac{1}{\lambda_1} \le 1$). Then there exists a sequence $\{\mathbf{v}^k\}_{k=1}^{\infty}$ such that

$$(\mathbf{v}^k, \mathbf{v}^k) \to \frac{1}{\lambda_1} \quad \text{and} \quad \|\mathbf{v}^k\|_{s,2} = 1.$$

Consequently, there exists $\boldsymbol{\omega}^1 \in V^s$ and a subsequence $\{\mathbf{v}^{k'}\} \subset \{\mathbf{v}^k\}$ such that

$$\mathbf{v}^{k'} \rightharpoonup \boldsymbol{\omega}^1 \quad \text{in } V^s \quad \text{and } \mathbf{v}^{k'} \to \boldsymbol{\omega}^1 \quad \text{in } H.$$

Clearly, $\|\boldsymbol{\omega}^1\|_{s,2} \le 1$. In fact, $\|\boldsymbol{\omega}^1\|_{s,2} = 1$. Indeed, if $\|\boldsymbol{\omega}^1\|_{s,2} < 1$ then $\boldsymbol{\omega} \equiv \boldsymbol{\omega}^1/\|\boldsymbol{\omega}^1\|_{s,2}$ fulfills

$$\|\boldsymbol{\omega}\|_{s,2} = 1 \quad \text{and} \quad (\boldsymbol{\omega}, \boldsymbol{\omega}) = \frac{(\boldsymbol{\omega}^1, \boldsymbol{\omega}^1)}{\|\boldsymbol{\omega}^1\|_{s,2}} > \frac{1}{\lambda_1},$$

which contradicts the definition of λ_1, see (4.16). It remains to show that $\boldsymbol{\omega}^1$ is an eigenvector.

Let $\mathbf{h} \in V^s$. Defining

$$\Phi(t) \equiv \frac{(\boldsymbol{\omega}^1 + t\mathbf{h}, \boldsymbol{\omega}^1 + t\mathbf{h})}{((\boldsymbol{\omega}^1 + t\mathbf{h}, \boldsymbol{\omega}^1 + t\mathbf{h}))_s},$$

we obtain

$$\begin{aligned} 0 = \frac{d}{dt}\,\Phi(t)|_{t=0} &= \frac{2\,(\boldsymbol{\omega}^1, \mathbf{h})\,((\boldsymbol{\omega}^1, \boldsymbol{\omega}^1))_s - 2\,(\boldsymbol{\omega}^1, \boldsymbol{\omega}^1)\,((\boldsymbol{\omega}^1, \mathbf{h}))_s}{((\boldsymbol{\omega}^1, \boldsymbol{\omega}^1))_s^2} \\ &= \frac{2\,(\boldsymbol{\omega}^1, \mathbf{h}) - \frac{2}{\lambda_1}\,((\boldsymbol{\omega}^1, \mathbf{h}))_s}{((\boldsymbol{\omega}^1, \boldsymbol{\omega}^1))_s^2}. \end{aligned}$$

This implies

$$\lambda_1\,(\boldsymbol{\omega}^1, \mathbf{h}) = ((\boldsymbol{\omega}^1, \mathbf{h}))_s \qquad \forall h \in V^s.$$

- *Iterative construction*

Let us assume that the existence of the first N, $N \ge 1$, eigenvectors $\{\boldsymbol{\omega}^i\}_{i=1}^N$ and corresponding eigenvalues $\{\lambda_i\}_{i=1}^N$ has already been proved. Define

$$W^N \equiv \{\mathbf{v} \in V^s; ((\mathbf{v}, \boldsymbol{\omega}^i))_s = 0,\ i = 1, \dots, N\}.$$

Then the same construction as in the first part of the proof allows to find $\boldsymbol{\omega}^{N+1}$ such that

$$(\boldsymbol{\omega}^{N+1}, \boldsymbol{\omega}^{N+1}) = \sup_{\substack{\|\mathbf{v}\|_{s,2}=1 \\ \mathbf{v} \in W^N}} (\mathbf{v}, \mathbf{v}) \equiv \frac{1}{\lambda_{N+1}}. \tag{4.17}$$

From this construction we see that

$$1 \le \lambda_1 \le \lambda_2 \le \dots \le \lambda_{N+1} \le \dots, \tag{4.18}$$

$$(\boldsymbol{\omega}^r, \boldsymbol{\omega}^s) = 0 \quad \text{if } r \ne s, \tag{4.19}$$

$$((\boldsymbol{\omega}^r, \boldsymbol{\omega}^s))_s = \delta_{rs} \qquad \forall r, s \in \mathbb{N}. \tag{4.20}$$

• *$\lambda_r \to \infty$ as $r \to \infty$*

Let $\lim\limits_{r\to\infty} \lambda_r = \lambda < \infty$. Since $\|\boldsymbol{\omega}^r\|_{s,2} = 1$, we have

$$\boldsymbol{\omega}^{r_k} \to \boldsymbol{\omega} \quad \text{in } H, \qquad \text{as } k \to \infty. \tag{4.21}$$

Simple calculations give

$$2 = ((\boldsymbol{\omega}^{r_k} - \boldsymbol{\omega}^{r_\ell}, \boldsymbol{\omega}^{r_k} - \boldsymbol{\omega}^{r_\ell}))_s = \lambda_{r_k}(\boldsymbol{\omega}^{r_k}, \boldsymbol{\omega}^{r_k} - \boldsymbol{\omega}^{r_\ell}) - \lambda_{r_\ell}(\boldsymbol{\omega}^{r_\ell}, \boldsymbol{\omega}^{r_k} - \boldsymbol{\omega}^{r_\ell}),$$

and due to boundedness of λ_{r_k} and (4.21), we can make the right-hand side arbitrarily small letting k, $\ell \to \infty$.

• *$\{\lambda_r\}_{r=1}^{\infty}$ are all eigenvalues of (4.10)*

Let us assume for contradiction the existence of $\lambda \in \mathbb{R}$ and $\boldsymbol{\omega} \in V^s$ such that $\lambda \neq \lambda_r$ for all $r \in \mathbb{N}$, $\|\boldsymbol{\omega}\|_{s,2} = 1$ and $((\boldsymbol{\omega}, \boldsymbol{\varphi}))_s = \lambda(\boldsymbol{\omega}, \boldsymbol{\varphi})$ for all $\boldsymbol{\varphi} \in V^s$. Clearly, there is an $i \in \mathbb{N}$ such that $\lambda_i < \lambda < \lambda_{i+1}$. Since $((\boldsymbol{\omega}^k, \boldsymbol{\omega}))_s = \lambda_k(\boldsymbol{\omega}^k, \boldsymbol{\omega})$ and $((\boldsymbol{\omega}, \boldsymbol{\omega}^k)) = \lambda(\boldsymbol{\omega}, \boldsymbol{\omega}^k)$ for all $k = 1, \dots, i$, we see that $(\boldsymbol{\omega}, \boldsymbol{\omega}^k) = 0$ for all $k = 1, \dots, i$. Therefore $\boldsymbol{\omega} \in W^i$. However,

$$(\boldsymbol{\omega}, \boldsymbol{\omega}) = \frac{1}{\lambda} > \frac{1}{\lambda_{i+1}} = \sup_{\substack{\|\mathbf{v}\|_{s,2}=1 \\ \mathbf{v}\in W^i}} (\mathbf{v}, \mathbf{v}),$$

which is a contradiction.

• *$\{\boldsymbol{\omega}^r\}_{r=1}^{\infty}$ is a basis in V^s*

Put $X \equiv \operatorname{span}\{\boldsymbol{\omega}^1, \dots, \boldsymbol{\omega}^N, \dots\}$. Assume that $X \neq V^s$. Therefore there exists $\boldsymbol{\Psi} \in V^s$, $\|\boldsymbol{\Psi}\|_{s,2} = 1$ such that $((\boldsymbol{\Psi}, \boldsymbol{\omega}^r))_s = 0$ for all $r \in \mathbb{N}$. Then we have for all $r \in \mathbb{N}$,

$$(\boldsymbol{\Psi}, \boldsymbol{\Psi}) \leq \sup_{\substack{\mathbf{v}\in W^r \\ \|\mathbf{v}\|_{s,2}=1}} (\mathbf{v}, \mathbf{v}) = \frac{1}{\lambda_r}.$$

Since $\frac{1}{\lambda_r} \to 0$, $\boldsymbol{\Psi}$ has to be zero element of V^s. This contradicts the fact that $\|\boldsymbol{\Psi}\|_{s,2} = 1$.

• *Renormalization of basis*

We have already proved that $\{\boldsymbol{\omega}^r\}_{r=1}^{\infty}$ satisfies (4.19), (4.20) and (4.10). Setting $\widehat{\boldsymbol{\omega}}^r \equiv \boldsymbol{\omega}^r/\sqrt{\lambda_r}$ we see that $\widehat{\boldsymbol{\omega}}^r$ solves again (4.10) with eigenvalues λ_r. Moreover, $\widehat{\boldsymbol{\omega}}^r$ are orthonormal in H.

Replacing $\{\boldsymbol{\omega}^r\}_{r=1}^{\infty}$ by $\{\widehat{\boldsymbol{\omega}}^r\}_{r=1}^{\infty}$ the proof of the first four assertions is finished.

• *The continuity of P^N*

Consider now $\mathbf{v} \in V^s$. Then

$$\|P^N(\mathbf{v})\|_{s,2}^2 = \sum_{i=1}^{N} (\mathbf{v}, \boldsymbol{\omega}^i)^2 ((\boldsymbol{\omega}^i, \boldsymbol{\omega}^i))_s = \sum_{i=1}^{N} \frac{((\mathbf{v}, \boldsymbol{\omega}^i))_s^2}{\lambda_i^2} ((\boldsymbol{\omega}^i, \boldsymbol{\omega}^i))_s$$
$$\leq \sum_{i=1}^{N} ((\mathbf{v}, \tfrac{\boldsymbol{\omega}^i}{\sqrt{\lambda_i}}))_s^2 \leq \|\mathbf{v}\|_{s,2}^2 .$$

Thus (4.12) is proved, since P^N is a selfadjoint operator. Finally, (4.13) follows immediately.

The proof of Theorem 4.11 is complete. ■

We will consider henceforth the eigenvalue problem (4.10) for the special case $s = 1$, i.e. $V^1 = V$. Then the eigenvalue problem (4.10) is equivalent to the problem of finding eigenvalues (and eigenvectors) for the Stokes operator A defined in (4.8), if $\partial\Omega$ is smooth (C^2, for example), and we have

$$\lambda_r(\boldsymbol{\omega}^r, \mathbf{u}^N) = (A\boldsymbol{\omega}^r, \mathbf{u}^N) = (\nabla\boldsymbol{\omega}^r, \nabla\mathbf{u}^N) . \tag{4.22}$$

Let us consider the so-called Stokes system, studied in detail in GALDI [1994a, 1994b]:

$$\begin{aligned} -\Delta\mathbf{u} + \nabla p = \mathbf{F} \quad & \text{in } \Omega , \\ \operatorname{div}\mathbf{u} = 0 \quad & \text{in } \Omega , \\ \mathbf{u} = 0 \quad & \text{at } \partial\Omega . \end{aligned} \tag{4.23}$$

If $\partial\Omega \in C^2$, then from the regularity result for the Stokes system follows (let us refer again to GALDI [1994a, 1994b] and references therein) that for all $\mathbf{F} \in L^2(\Omega)^d$,

$$\|\mathbf{u}\|_{2,2} + \|p\|_{1,2} \leq c\,\|\mathbf{F}\|_2 , \tag{4.24}$$

where $c = c(\Omega, d)$. Taking $\mathbf{F} = A\mathbf{u} \in H$, we have in particular (see also Proposition 4.7 in CONSTANTIN AND FOIAS [1988]),

$$\|\mathbf{u}\|_{2,2} \leq c\|A\mathbf{u}\|_2 \qquad \forall\, \mathbf{u} \in D(A) . \tag{4.25}$$

From Theorem 4.11 it follows that the operators P^N are continuous in $V = V^1$. The inequality (4.25) allows us to prove the continuity of P^N also in $D(A)$.

Lemma 4.26 *Let $A = -\mathbb{P}\Delta\mathbf{u}$, $D(A)$ and P^N be as above. Then there exists $C = C(\Omega, d) > 0$ such that*

$$\|P^N\mathbf{u}\|_{2,2} \leq C\,\|\mathbf{u}\|_{2,2} \quad \forall\, \mathbf{u} \in D(A) . \tag{4.27}$$

PROOF : Let us first prove that $AP^N(\mathbf{u}) = P^N A\mathbf{u}$ a.e. in Ω. It suffices to show that

$$(AP^N\mathbf{u}, \mathbf{v}) = (P^N A\mathbf{u}, \mathbf{v}) \qquad \forall\, \mathbf{v} \in D(A)\,. \tag{4.28}$$

However, $(AP^N\mathbf{u}, \mathbf{v}) = ((P^N\mathbf{u}, \mathbf{v}))_1$ and

$$\begin{aligned}(P^N A\mathbf{u}, \mathbf{v}) &= \sum_{r=1}^{N} (A\mathbf{u}, \boldsymbol{\omega}^r)\,(\boldsymbol{\omega}^r, \mathbf{v}) = \sum_{r=1}^{N} \frac{1}{\lambda_r}(A\mathbf{u}, \boldsymbol{\omega}^r)\,((\boldsymbol{\omega}^r, \mathbf{v}))_1 \\ &= \sum_{r=1}^{N} \frac{1}{\lambda_r}((\mathbf{u}, \boldsymbol{\omega}^r))_1\,((\boldsymbol{\omega}^r, \mathbf{v}))_1 = \sum_{r=1}^{N} (\boldsymbol{\omega}^r, \mathbf{u})\,((\boldsymbol{\omega}^r, \mathbf{v}))_1 \\ &= ((P^N\mathbf{u}, \mathbf{v}))_1\,.\end{aligned}$$

So (4.28) holds.

Now we have

$$\|P^N\mathbf{u}\|_{2,2} \overset{(4.25)}{\leq} c\,\|AP^N\mathbf{u}\|_2 = c\,\|P^N A\mathbf{u}\|_2 \overset{(4.13)}{\leq} c\,\|A\mathbf{u}\|_2\,.$$

But $\|A\mathbf{u}\|_2 \leq \|\mathbf{u}\|_{2,2}$, and (4.27) follows. ∎

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