George Grätzer
Friedrich Wehrung
Editors

Lattice Theory: Special Topics and Applications

Volume 1

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Introduction

George Grätzer started writing his *General Lattice Theory* in 1968. It was published in 1978. It set out "to discuss in depth the basics of general lattice theory." Almost 900 exercises, 193 research problems, and a detailed Further Topics and References for each chapter completed the picture.

As T.S. Blyth wrote in the Mathematical Reviews: "*General Lattice Theory* has become the lattice theorist's bible. Now, two decades on, we have the second edition, in which the old testament is augmented by a new testament that is epistolic. The new testament gospel is provided by leading and acknowledged experts in their fields."

Another decade later, Grätzer considered updating the second edition to reflect some exciting and deep developments. "When I started on this project, it did not take me very long to realize that what I attempted to accomplish in 1968–1978, I cannot even try in 2009. To lay the foundation, to survey the contemporary field, to pose research problems, would require more than one volume or more than one person. So I decided to cut back and concentrate in this volume on the *foundation*."

So *Lattice Theory: Foundation* (referenced in this volume as LTF) provides the foundation. Now we complete this project with *Lattice Theory: Special Topics and Applications*, written by a distinguished group of experts, to cover some of the vast areas not in LTF.

As in LTF, Theorems (lemmas) presented without proofs are often marked by the diamond symbol $\diamondsuit$.

This Volume 1 is divided into three parts.

Volume 2 will follow with contributions by K. Adaricheva, N. Caspard, R. Freese, P. Jipsen, J.B. Nation, H. Priestley, N. Reading, L. Santocanale, and F. Wehrung.

George Grätzer and Friedrich Wehrung, editors

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Part I

Topology and Lattices

Introduction

There are at least two quite distinctive ways in which lattice theory and topology interact. On the one hand, one may consider lattices equipped with a topology that is related to the lattice structure in some way. For example, we may require that the lattice operations be continuous with respect to the topology, from which one arrives at the concept of a topological lattice. We may relax this requirement and ask that only one of the lattice operations, for example, the meet, be continuous and then we get the concept of a topological semilattice. A further relaxation is to ask only that the (graph of the) order be closed, which leads to the concept of an ordered topological space. Of particular interest in this setting is the study of such topologies on lattices that are intrinsic, in the sense that they may be defined directly in terms of the lattice operations and/or the order relation.

On the other hand, by considering those lattices that arise as the lattice of open subsets of a topological space, we arrive at a certain class of distributive complete lattices, those that are "frames" or "locales" in the standard terminology. Such lattices may serve as "pointless" versions of topological spaces. Thus concepts of topology, indeed a whole variant topological theory, can be studied in this general lattice theoretical or "localic" framework.

The two chapters in this part on Topology and Lattices are devoted to these two aspects. The first aspect is too vast to be treated in all its breadth. We chose to put the emphasis on continuous lattices, a field that has gained prominence because of its applications in semantics and theoretical computer science. We included a parallel treatment of completely distributive lattices, since there are strong similarities between the theories of these two classes of lattices.

Chapter 1

Continuous and Completely Distributive Lattices

by Klaus Keimel and Jimmie Lawson

1-1. Introduction

The study of continuous lattices was initiated by Dana Scott in the late 1960s in order to build mathematical models for certain constructs in theoretical computer science ([638] in LTF), and computational notions and motivations have continued to play a key role in the theory. Early successes included construction of a denotational semantics for certain programming languages where programs were semantically interpreted as functions between appropriate input and output domains (see, e.g., [271]) and construction of a specific domain of computation that provided a model for the untyped lambda calculus (see, [639] in LTF), no concrete model of the untyped lambda calculus having hitherto been given. One idea that rather quickly emerged was to introduce a special class of ordered structures called continuous lattices, or more generally domains, the elements of which were viewed as states of information ordered by the *information order*, larger states representing states of increased information.

As a simple example, suppose that one programmed a computer to evaluate some computable function on the natural numbers and the computer was constantly spewing out the functional values for larger and larger numbers.

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At any stage one would have only partial information about the function, a partial function that represented the function for only finite many values. We may think of this partial function as approximating the computable function in the sense that it gives correct partial information about the function and that any computational scheme that computes, over time, all values of the function must at some finite stage yield the information in the partial function. Hence in the information order the original function is the supremum of the directed sequence of its finite approximations.

The prominent role of an auxiliary order, the order of approximation (or way-below relation), is a distinctive feature that the theory of continuous lattices (and its generalization, *domain theory*) brings to lattice theory. A second is a focus on a general type of morphism, namely, those functions, called *Scott-continuous* functions, that preserve joins of directed sets.

By considering two-element chains as directed sets, one observes that Scott-continuous functions must be order-preserving, hence carry directed sets to directed sets. From a computational point of view, we may view directed sets as consisting of stages of a computation and are thus requiring that morphisms preserve outcomes, joins in the information order, of computations.

Algebraic lattices constitute a special class of continuous lattices. Historically, they have been considered much earlier, as they occur as lattices of subalgebras and lattices of congruence relations in universal algebra and specific algebraic structures. Continuous and algebraic lattices have both been introduced briefly in LTF, Chapter I, Sections 3.15, 3.16, 3.17.

Another special class of continuous lattices is the class of completely distributive lattices. There are some remarkable similarities between the theory of continuous lattices and that of completely distributive lattices, which we seek to develop in a more systematic fashion than has appeared in earlier literature. It is a natural idea to develop these similarities in a unified framework which covers both continuous and completely distributive lattices as special cases. We have not adopted this strategy in order to fully exploit the similarities and to keep matters more directly accessible. The interested reader may consult, for example, [80] and the references therein.

While algebraic lattices occur pervasively in connection with algebraic structures, continuous lattices occur in topological situations as well. Relating topology and order is a main concern of the present chapter. Another aspect along these lines is the characterization of topological properties by properties of the lattice of open subsets. This aspect of continuous lattices is treated in detail in Chapter 2 "Frames: Topology Without Points", in particular in Section 2-7, by A. Pultr and J. Sichler in this volume.

For a more detailed treatment of the theory of continuous lattices and domains and for extensive bibliographies we refer to the monograph *Continuous Lattices and Domains* by G. Gierz, K.H. Hofmann, K. Keimel, J.D. Lawson, M. Mislove and D.S. Scott [104] and to the chapter *Domain Theory* by S. Abramsky and A. Jung in the Handbook of Logic in Computer Science [2].

We have not included detailed references to the original sources of the material presented here, but refer to [104] for detailed comments on these developments. Decisive ideas and results on completely distributive lattices go back to G.N. Raney ([276, 277] and [604] of LTF). Ground breaking papers on continuous lattices are due to D.S. Scott [299, 300], and [639] in LTF. J.D. Lawson's characterization of compact semilattices with small subsemilattices [241] and A. Day's characterization of continuous lattices as the algebras of the filter monad [59] have also been cornerstones for the developments presented here.

1-1.1 Some terminology

– *Continuous lattices* in the sense of this chapter should be distinguished from the *continuous geometries* of J. von Neumann ([552] of LTF), which, by definition, are complete complemented modular lattices that are meet- and join-continuous.

– *Directed sets* are ordered sets in which every finite subset has an upper bound. The empty set being finite, directed sets are nonempty.

– We use $\bigvee^{\uparrow} D$ for denoting the least upper bound of D provided that D is a directed set for which the least upper bound exists in the poset.

– Semilattices are meet-semilattices in this chapter. When we want to work with join-semilattices, we state this explicitly.

– For a map d from a poset T to a poset S we use the standard convention for the meaning of "d preserves a certain type of joins or meets," namely that for every subset X of T of a specified type that possesses a supremum u, it is the case that $d(X)$ is a subset of S of the same type and $d(u)$ is its supremum. For example, we say that *d preserves arbitrary joins* if, for every subset X of T which has a least upper bound $\bigvee X$ in T, the image $d(X)$ has a least upper bound $\bigvee d(X)$ in S and $d(\bigvee X) = \bigvee d(X)$.

1-2. Basics

Significant portions of the theory of continuous lattices can be extended to more general ordered sets, and there is good theoretical and practical motivation for doing so. Hence in the early parts of this chapter we work primarily in the more general setting of ordered sets. Ordered sets are typically referred to as "posets" in the domain theory literature, and we adopt this terminology in order to build bridges with that literature, even though it is at variance with LTF. The theory of continuous lattices and domains is quite asymmetric in its treatment of the order relation, so one should bear in mind that there are order dual notions and approaches for what we present. We include semilattices heavily in our treatment, among other things, to emphasize this asymmetry.

The basic completeness conditions that we require are twofold. Firstly we require the existence of a least or bottom element $\perp$, which represents a state of total ignorance in an information order. (This requirement is convenient, but not absolutely necessary, in the general setting of posets.) Secondly, and more importantly, we require that every directed set possess a supremum; an ordered set with these properties is called *directed complete.* For us a *dcpo* will be a **d**irected **c**omplete **po**set; in particular, it will have a least element $\perp$ (or sometimes 0 in the case of lattices or join-semilattices).

Besides directed completeness the other key notion in the theory of continuous lattices, semilattices, and posets is that of "approximation." Ordered structures in which the approximation relation is sufficiently rich are then called "continuous" and the study of such ordered structures is referred to as "domain theory."

Definition 1-2.1.

(i) Let P be a poset and let a, b be elements of P. Then *a approximates b*, in symbols, $a \ll b$, if $b \leq \bigvee^{\uparrow} D$, for any directed $D \subseteq P$, implies that $a \leq d$ for some $d \in D$ (and hence $a \leq e$ for all $d \leq e \in D$). The relation $\ll$ is called the *order of approximation*, or more suggestively, the *way-below relation.*

(ii) A lattice is called a *continuous lattice* if it is a complete lattice and if every element is the join of the (typically infinite) set of elements approximating it.

(iii) A poset is a *continuous poset* if every element is the join of a directed set of elements approximating it and is a *continuous domain* if it is a continuous dcpo, i.e., a dcpo that is a continuous poset.

Remark 1-2.2. We note that $\perp \ll c$ for any c and that if $a \ll c$ and $b \ll c$, then $a \vee b \ll c$, provided $a \vee b$ exists, so that the set of elements approximating c is a directed (in particular, nonempty) set in a lattice or join-semilattice with $\perp$ (and hence a continuous lattice is a continuous poset). The directedness property is crucial in generalizations of continuity to general orders.

The following proposition gives basic properties of the approximation relation. The interpolation property (v) is a key one for the theory.

Proposition 1-2.3. *Let P be a poset.*

(i) *$x \ll y$ implies $x \leq y$.*

(ii) *$w \leq x \ll y \leq z$ implies $w \ll z$.*

(iii) *In a continuous poset, the set $\twoheaddownarrow y := \{x : x \ll y\}$ is directed with supremum y for all elements y.*

(iv) *In a continuous poset, if $x \ll z$ and if $z \leq \bigvee^{\uparrow} D$ for some directed set D, then $x \ll d$ for some $d \in D$.*

(v) (*Interpolation property*) *In a continuous poset,* $x \ll z$ *implies* $x \ll w \ll z$ *for some* w.

(vi) *If* S *is a lattice or join-semilattice, then* $x \ll y$ *if and only if whenever* $\sup A \geq y$ *for some nonempty set* A *with supremum, then* $\sup F \geq x$ *for some finite subset* F *of* A.

Proof. For (i) consider the directed set $D = \{y\}$. Item (ii) is straightforward.

For (iii), let D be a directed set contained in $\{x : x \ll y\}$ with supremum y. Given $x_1, x_2 \ll y$, there exist $w_1, w_2 \in D$ such that $x_i \leq w_i$ for $i = 1, 2$. Pick $w_3 \in D$ such that $w_1, w_2 \leq w_3$. Then $x_1, x_2 \leq w_3 \in \mathord{\downarrow\!\!\!\downarrow} x$. Finally note that $y = \bigvee^{\uparrow} D \leq \sup\{x : x \ll y\} \leq y$.

For (iv), let D be a directed set with $z \leq \bigvee^{\uparrow} D$, and let I be the set of all y such that $y \ll d$ for some $d \in D$. From (ii) and (iii) we conclude that I is directed and $\bigvee^{\uparrow} I = \bigvee^{\uparrow} D$. Hence, if $x \ll z$, there is a $y \in I$ such that $x \leq y$. As $y \ll d$ for some $d \in D$, we conclude that $x \ll d$. Now (v) follows from (iii) and (iv) by choosing $D = \{y : y \ll z\}$.

Item (vi) follows from the observations that the set D of all sups of nonempty finite subsets of A is a directed set with the same supremum as A and that if F is a nonempty finite subset of a directed set D, then its supremum is bounded above by a member of D. $\square$

Remark 1-2.4. We note from Proposition 1-2.3(iii) that we may alternatively define a continuous poset to be one for which every element is the directed join of *all* elements approximating it, i.e., $x = \bigvee^{\uparrow} \mathord{\downarrow\!\!\!\downarrow} x$ for all x.

It is sometimes the case that one singles out some particular or distinguished subset of approximating elements.

Definition 1-2.5. A subset B of a continuous poset P is called a *basis* if for each $x \in P$, there exists a directed set $B_x \subseteq B \cap \mathord{\downarrow\!\!\!\downarrow} x$ such that $x = \bigvee^{\uparrow} B_x$.

Remark 1-2.6. In complete analogy to Remark 1-2.4, for any basis B the set $\{b \in B : b \ll x\}$ is directed with supremum x. Using Proposition 1-2.3(iv) one then easily sees that both the sets B_x and $\{b \in B : b \ll x\}$ are directed with respect to the relation $\ll$.

An important class of continuous posets are those that have a countable basis. For one thing, one can develop a theory of computability for such continuous posets. One can also replace directed sets by increasing sequences.

Proposition 1-2.7. *Let* P *be a continuous poset with a countable basis* B. *Then one can choose for each* x *an increasing sequence*

$$c_0 \ll c_1 \ll c_2 \ll \cdots$$

in B *such that* $x = \bigvee^{\uparrow}{}_{n \in \mathbb{N}} c_n$.

Proof. Enumerate B_x (see Definition 1-2.5) as $\{b_0, b_1, \ldots\}$, set $c_0 = b_0$, and inductively choose $c_n \in B_x$ so that $c_{n-1} \ll c_n$ and $b_n \ll c_n$, which is possible since B_x is directed with respect to $\ll$ (Remark 1-2.6). Then

$$x = \bigvee{}^{\uparrow} B_x \leq \bigvee_{n\in\mathbb{N}}{}^{\uparrow} c_n \leq x$$

which implies $x = \bigvee^{\uparrow}_{n\in\mathbb{N}} c_n$. □

Recall that a meet-semilattice S is *meet-continuous* if for any directed set D with supremum x and for any $y \in S$, $\bigvee^{\uparrow}(D \wedge y) = x \wedge y$ (LTF, Section 3.15). *Join-continuity* is defined dually.

Proposition 1-2.8. *A meet-semilattice S that is also a continuous poset is meet-continuous.*

Proof. For $x = \bigvee^{\uparrow} D$ and $y \in S$, we first note that $x \wedge y$ is an upper bound for $D \wedge y$. Let u be another upper bound for $D \wedge y$. For any $z \ll x \wedge y \leq x$, there exists $w \in D$ such that $z \leq w$. Then $z \leq w \wedge y \leq u$. We conclude that $x \wedge y = \sup\{z : z \ll x \wedge y\} \leq u$. □

Meet-continuous lattices and semilattices are much more general than continuous ones, but they do allow a relaxed definition of the approximation relation.

Lemma 1-2.9. *If S is a meet-continuous semilattice, then $x \ll y$ provided for every directed set D with $y = \bigvee^{\uparrow} D$, it follows that $x \leq d$ for some $d \in D$.*

Proof. Suppose x and y satisfy the condition of the lemma and D is a directed set with $y \leq \bigvee^{\uparrow} D$. Then $y \wedge D$ is a directed set, and $y = \bigvee^{\uparrow} y \wedge D$ by meet-continuity. By hypothesis $x \leq y \wedge d \leq d$ for some $d \in D$, so that $x \ll y$. □

Definition 1-2.10. We define an element k of a poset to be a *compact element* if $k \ll k$. We define a poset to be *compactly generated* if every element is the join of a directed set of compact elements. An *algebraic lattice* is a complete lattice in which every element is a supremum of compact elements.

It follows from Proposition 1-2.3(vi) for the case of lattices and join-semilattices that the above definition of a compact element agrees with a common alternative one: if $k \leq \bigvee A$, then $k \leq \bigvee F$ for some finite subset of A (the directedness is automatic since the join to two compact elements is again compact); see Section 3.15 of LTF. The algebraic lattices form an important subclass of the continuous lattices, and thus provide a host of examples of the latter. On the other hand, continuous lattices may be viewed as natural generalizations of algebraic lattices. The proof of the following is an easy consequence of the definitions and Proposition 1-2.3(ii).

Proposition 1-2.11. *A compactly generated poset is a continuous poset. In particular, every algebraic lattice is a continuous lattice.*

We note that Proposition 1-2.8 generalizes the long known result that algebraic lattices are meet-continuous; see Section 3.15 of LTF.

A special class of semilattices that are slightly more general than complete lattices has proven useful in domain theory. A *bounded complete semilattice* is a meet-semilattice that is a dcpo with the property that every subset that is bounded above has a least upper bound. A *bounded complete domain* is a bounded complete semilattice that is also a continuous domain.

Proposition 1-2.12. *Let S be an ordered set with $\perp$, and let $S^\top$ denote S with a new element $\top$ adjoined as the largest element. The following are equivalent:*

(i) *S is a bounded complete semilattice.*

(ii) *$S^\top$ is a complete lattice with compact element $\top$.*

(iii) *S is a dcpo in which every nonempty subset of S has an infimum (in particular, S is a meet-semilattice).*

(iv) *S is a dcpo in which any two elements bounded above have a supremum.*

Proof. Noting that S is directed complete if and only if $\top$ is a compact element of $S^\top$, one deduces directly the equivalence of (ii) and (iii) and the implication (ii) implies (i). That (i) implies (iv) is immediate.

(iv) implies (ii): For $A \subseteq S^\top$, we have $\bigvee A = \perp$ if $A = \emptyset$ and $\bigvee A = \top$ if A is not bounded above in S. For $A \neq \emptyset$ bounded above by $b \in S$, every finite subset of F has a supremum by hypothesis (and induction). The set of all such finite sups form a directed set, which thus has a supremum in S, and this supremum is a supremum for A. $\square$

We close this section with two important examples of bounded complete domains that are not lattices, examples where a top element would have no natural interpretation.

Example 1-2.13. (Strings) Consider the set consisting of all strings of zeros and ones (finite, infinite, and the empty string). We order this set with the "prefix order": for strings w_1, w_2, $w_1 \leq w_2$ if and only if w_1 is a finite string and w_1 and w_2 agree in the first n places, where n is the length of the first string w_1. This domain is actually an algebraic domain, the finite strings being the compact elements. Given an infinite string, its finite prefixes are approximations, which we may think of as giving partial information about the given string. The infinite strings give maximal or total information and form the maximal elements in the order. Of course, we could build a bounded complete domain of strings starting from any finite alphabet A.

Bounded complete domains that are compactly generated, such as the preceding example, are frequently referred to as *Scott domains.*

Example 1-2.14. (The Interval Domain) Let $\mathbb{I}$ consist of all closed subintervals $[a, b]$ of the unit interval $[0, 1]$ including the degenerate one-point intervals $[a, a]$ ordered by reverse inclusion: $[a, b] \leq [c, d]$ if and only if $[a, b] \supseteq [c, d]$. Given $t \in [0, 1]$, the subintervals containing t in their interior are approximations to $t = [t, t]$, the smaller intervals being better approximations and hence larger in the information order. The approximation or way-below relation is given by $[a, b] \ll [c, d]$ if and only if $a < c \leq d < b$ and $\perp = [0, 1]$ (if $c = 0$, resp. $d = 1$, we allow $a = 0$, resp. $b = 1$). Two subintervals bounded above have a join, their intersection, and the meet is given by $[a, b] \wedge [c, d] = [\min\{a, c\}, \max\{b, d\}]$. Given this information it is straightforward to verify that $\mathbb{I}$ is a bounded complete domain. The degenerate intervals are the maximal elements and represent states of total knowledge.

1-3. The equational theory of continuous lattices and completely distributive lattices

We give an equational description of continuous lattices, which extends to bounded complete semilattices.

Theorem 1-3.1. *For a bounded complete semilattice, resp. complete lattice L, the following conditions are equivalent.*

(1) *L is a bounded complete domain, resp. continuous lattice.*

(2) *For any nonempty family $(D_i)_{i \in I}$ of directed subsets, L satisfies the identity*

$$\text{(DD1)} \qquad \bigwedge_{i \in I} {\bigvee}^{\uparrow} D_i = {\bigvee}^{\uparrow}_{(x_i) \in \prod_i D_i} \bigwedge_{i \in I} x_i.$$

Proof. (1) $\Longrightarrow$ (2): Let $s = \bigwedge_{i \in I} {\bigvee}^{\uparrow} D_i$ and $t = {\bigvee}^{\uparrow}_{(x_i) \in \prod_i D_i} \bigwedge_{i \in I} x_i$. The inequality $s \geq t$ holds in any bounded complete semilattice. For the converse, choose any $z \ll s$. For every i, we have $s \leq {\bigvee}^{\uparrow} D_i$. Hence, there is an element $x_i \in D_i$ such that $z \leq x_i$. It follows that $z \leq \bigwedge_i x_i$ and consequently $z \leq t$. Under hypothesis (1), s is the join of the elements $z \ll s$, and thus $s \leq t$.

(2) $\Longrightarrow$ (1): Let $x \in L$. Consider the family $(D_i)_{i \in I}$ of all directed subsets D_i of L such that $x \leq {\bigvee}^{\uparrow} D_i$. Let $(x_i)_{i \in I} \in \prod_i D_i$. Then $\bigwedge_i x_i \ll x$ from the choice of the family $\{D_i\}$. But

$${\bigvee}^{\uparrow}_{(x_i) \in \prod_i D_i} \bigwedge_i x_i = \bigwedge_i {\bigvee}^{\uparrow} D_i = x$$

by (2) and the fact that the singleton set $\{x\}$ is one of the D_i. $\square$

One can strengthen the way-below relation on a complete lattice L by replacing directed sets by arbitrary nonempty sets.

Definition 1-3.2. On a complete lattice L we define $a \lll b$ if every nonempty subset X of L with $\bigvee X \geq b$ contains an element $x \geq a$.

The proof of the following theorem mimics that of Theorem 1-3.1 with the directed sets D_i replaced by arbitrary nonempty subsets X_i and the relation $x \ll y$ replaced by $x \lll y$.

Theorem 1-3.3. *For a complete lattice L the following are equivalent:*

(1) *Every element of $x \in L$ is the supremum of the set $\{y : y \lll x\}$.*

(2) *For any nonempty family $(X_i)_{i \in I}$ of nonempty subsets, L satisfies the infinite distributive law:*

(DD2)
$$\bigwedge_{i \in I} \bigvee X_i = \bigvee_{(x_i) \in \prod_i X_i} \bigwedge_{i \in I} x_i.$$

Remark 1-3.4. As we shall see later in this section (Corollary 1-3.13), property (DD2) is equivalent to its dual property:

(3) For any nonempty family $(X_i)_{i \in I}$ of nonempty subsets, L satisfies the infinite distributive law:

(DD3)
$$\bigvee_{i \in I} \bigwedge X_i = \bigwedge_{(x_i) \in \prod_i X_i} \bigvee_{i \in I} x_i.$$

Definition 1-3.5. A complete lattice L is called *completely distributive*, if it satisfies one, hence all, of the three equivalent conditions of the previous theorem and remark.

Remark 1-3.6. Because of the use of choice functions, constructivists will not want to use (DD2) or (DD3) for defining complete distributivity. So they use condition (1) or a constructively equivalent condition for defining *constructive complete distributivity* (see, for example, [85]).

Note that, in contrast to the situation for continuous lattices, complete distributivity is a self-dual notion in the sense that a complete lattice is completely distributive if and only if its order dual L^{op} is also completely distributive. Clearly, every completely distributive lattice is continuous, since condition (DD1) of Theorem 1-3.1 is replaced by a more stringent condition. Hence, also the dual L^{op} is continuous. Also if in (DD2), resp. (DD3), we choose $X_1 = \{a, b\}$ and $X_2 = \{a, c\}$, then these laws reduce to the usual laws for a distributive lattice. In particular, a completely distributive lattice is a distributive lattice.

We use the equational characterizations for showing that the classes of continuous, resp. completely distributive, lattices are closed for 'direct products', 'subobjects' and 'homomorphic images':

Corollary 1-3.7. (i) *A direct product $\prod_{i\in I} L_i$ of complete lattices is continuous, resp. completely distributive, resp. algebraic, if and only if all the factors are continuous, resp. completely distributive, resp. algebraic lattices.*

(ii) *A nonempty subset of bounded complete domain (continuous lattice) will again be a bounded complete domain (continuous lattice) if it is closed with respect to nonempty infima and directed sups (and contains a largest element). Similarly a nonempty subset of a completely distributive lattice will be a completely distributive sublattice if it is closed under sups and infs of arbitrary nonempty subsets.*

(iii) *A nonempty subset of an algebraic lattice is again an algebraic lattice if it is closed under nonempty infs and directed sups and contains a largest element.*

Proof. (i) The respective laws given by (DD1) and (DD2) will hold in the product if and only if they hold in each coordinate. The algebraic case is left as an exercise (Exercise 1.10).

(ii) This is because the characterizing equational law (DD1), resp. (DD2), will continue to hold in such subsets.

(iii) Let M be a nonempty subset of an algebraic lattice L closed under nonempty infima and directed suprema and containing a largest element. It is easily seen that every subset of M has an infimum and hence M is a complete lattice. Let $x \in M$ and let $k \in L$ be a compact element with $k \leq x$. Then $\hat{k} = \inf\{w \in M : k \leq w\} \in M$ and $k \leq \hat{k} \leq x$. If $\hat{k} \leq \bigvee^{\uparrow} D$ for some directed set $D \subseteq M$, then $k \leq d$ for some $d \in D$ and hence $\hat{k} \leq d$. Thus $\hat{k}$ is a compact element of M and x is the directed supremum of the elements formed in this way since L is algebraic. □

Proposition 1-3.8. *Let L, M be complete lattices and $f \colon L \to M$ a surjective map.*

(i) *If L is continuous and f preserves nonempty meets and directed joins, then M is also a continuous lattice.*

(ii) *If L is completely distributive and f preserves nonempty joins and meets, then M is also completely distributive.*

Proof. (i) For $y \in M$, the fact that f preserves infs of nonempty sets and is surjective implies that $f^{-1}(y)$ is nonempty and $\bigwedge f^{-1}(y) = \bigwedge f^{-1}(\uparrow y)$ maps to y. Define $g \colon M \to L$ by $g(y) = \bigwedge f^{-1}(y)$. The map g is easily seen to be order-preserving and is injective since $f \circ g = \mathrm{id}_M$. In particular, g carries directed sets to directed sets. By continuity of L, property (DD1) of Theorem 1-3.1 holds for the family $(g(D_i))_{i\in I}$ for any nonempty family $(D_i)_{i\in I}$ of directed subsets of M, and hence holds for $(D_i)_{i\in I}$ by the hypothesized properties of f. By Theorem 1-3.1 M is continuous.

(ii) Follows along the same lines as part (i) from Theorem 1-3.3. □

The relation $\lll$ satisfies properties analogous to $\ll$. For example, we have the following variant of Proposition 1-2.3. The proof is similar – one simply replaces directed sups with sups of nonempty sets.

Proposition 1-3.9. *Let P be a complete lattice.*

(i) *$x \lll y$ implies $x \le y$.*

(ii) *$w \le x \lll y \le z$ implies $w \lll z$.*

(iii) *In a completely distributive lattice, the set $\{x : x \lll y\}$ is nonempty with supremum y for all elements y.*

(iv) *In a completely distributive lattice, if $x \lll z$ and if $z \le \bigvee A$ for some nonempty set A, then $x \lll d$ for some $d \in A$.*

(v) (*Interpolation Property*) *In a completely distributive lattice, $x \lll z$ implies $x \lll w \lll z$ for some w.*

The two element lattice $\mathbf{2} = \{0, 1\}$ and the unit interval $[0, 1]$ with the usual linear order are completely distributive, and, more generally, every complete chain is completely distributive. A finite lattice is completely distributive if and only if it is distributive.

By Corollary 1-3.7, the powers $\mathbf{2}^X$ and $[0, 1]^X$ are completely distributive for any set X and appropriate subobjects of these powers are continuous, resp. completely distributive, lattices. We are heading towards a representation theorem that tells us that all algebraic, continuous and completely distributive lattices arise as appropriate subobjects of 2^X or $[0, 1]^X$ à la Corollary 1-3.7. Let us begin with the simplest algebraic case.

For any element k in a complete lattice L we define a map $g_k \colon L \to \mathbf{2}$ by

$$g_k(x) = \begin{cases} 1, & \text{if } k \le x, \\ 0, & \text{if } k \not\le x. \end{cases}$$

Clearly, g_k preserves arbitrary meets. Further, g_k preserves directed joins if and only if k is a compact element of L, and g_k preserves arbitrary joins if and only if $k \lll k$. An element with this latter property will be called *completely coprime*, since it is characterized by the property that, whenever $k \le \bigvee A$ for a nonempty subset A of L, then $k \le a$ for some $a \in A$. Let us say that a lattice is *prime algebraic* if it is complete and each of its elements is a join of completely coprime elements. Clearly, a prime algebraic lattice is completely distributive, and $\mathbf{2}^X$ is prime algebraic.

Proposition 1-3.10. *A lattice L is algebraic* (*resp. prime algebraic*) *if and only if it can be embedded in some power $\mathbf{2}^X$ under preservation of arbitrary meets and directed* (*resp. arbitrary*) *joins.*

Proof. Suppose that L is algebraic and let K denote the set of all compact elements of L. The map $g\colon L \to \mathbf{2}^K$ defined by $g(x) = (g_k(x))_{k\in K}$ preserves arbitrary meets and directed joins, as this holds coordinatewise. This map is injective, since for any two different elements, say $a \not\leq b$, there is a compact element $k \leq a$ with $k \not\leq b$ whence $g_k(a) = 1$ but $g_k(b) = 0$.

The converse follows from Corollary 1-3.7(iii) in the algebraic case. In the prime algebraic case one replaces the compact by the completely prime elements and, for the converse, one mimics the proof of Corollary 1-3.7(iii). $\square$

Since $\mathbf{2}^X$ can be identified with the lattice of all subsets of X, the preceding proposition can be rephrased as follows: A lattice is algebraic (resp. prime algebraic) if and only if it can be represented as a collection of subsets of a set X closed under arbitrary intersections and directed (resp. arbitrary) unions (see also Lemma 398 of LTF for the algebraic case).

The following lemma allows us to extend the representation of (prime) algebraic lattices to continuous and completely distributive lattices, where we replace the two element lattice $\mathbf{2}$ by the unit interval $[0, 1]$ with the usual total order.

Lemma 1-3.11. *Consider a pair of elements of a continuous* (*resp. completely distributive*) *lattice L such that $x_0 \ll x_1$* (*resp. $x_0 \lll x_1$*)*. Then there is a map $g\colon L \to [0, 1]$ preserving arbitrary meets and directed* (*resp. arbitrary*) *joins such that $g(x_1) = 1$ and $g(b) = 0$ for every element $b \in L$ such that $x_0 \not\leq b$.*

Proof. We first consider the continuous case. Let $x_0 \ll x_1$. For every dyadic rational number $r = m/2^n$ ($n \in \mathbb{N},\ m = 0, 1, \ldots, 2^n$) let us define an element x_r in such a way that $r < s \implies x_r \ll x_s$.

We proceed by recursion over n using successive interpolation: If x_r is defined for $r = m/2^n$, $m = 0, 1, 2, \ldots, 2^n$ such that $x_{m/2^n} \ll x_{(m+1)/2^n}$, then we can find elements $x_{2m+1/2^{n+1}}$ such that $x_{m/2^n} \ll x_{2m+1/2^{n+1}} \ll x_{(m+1)/2^n}$.

We now can define $g\colon L \to [0, 1]$ by

$$g(x) = \bigvee\nolimits^{\uparrow}\{r \in [0, 1] : r \text{ is dyadic},\ x_r \leq x\}.$$

In particular, $g(x_1) = 1$, while $g(b) = 0$ whenever $x_0 \not\leq b$.

Clearly, g is order-preserving. To see that g preserves directed suprema, let $w = \bigvee^{\uparrow} D$. By order preservation, $g(d) \leq g(w)$ for all $d \in D$. Suppose that $t < g(w)$. Then there exist dyadic rational numbers s, r such that $t < s < r < g(w)$. By construction $x_s \ll x_r \leq w$, so $x_s \leq d$ for some $d \in D$, and then $t < s \leq g(d)$. We conclude that $t < \bigvee^{\uparrow} g(D)$. Since this holds for all $t < g(w)$, it follows that $\bigvee^{\uparrow} g(D) = g(w)$. On the other hand, let A be any subset of L. As g is order-preserving, we have $g(\bigwedge A) \leq \bigwedge g(A)$. For the converse inequality, suppose that s is a dyadic rational such that $s < \bigwedge g(A)$. There is another dyadic rational number r such that $s < r < \bigwedge g(A)$. Then $r < g(a)$, whence $x_r \leq a$, for all $a \in A$. We conclude that $x_r \leq \bigwedge A$, whence

$r \leq g(\bigwedge A)$. Thus, $s < g(\bigwedge A)$. As this holds for all $s < \bigwedge g(A)$, we derive $\bigwedge g(A) \leq g(\bigwedge A)$.

In the completely distributive case one replaces the relation $\ll$ by $\lll$ in the above proof, and then proceeds in the same way. $\square$

The following theorem is due to Raney ([605] in LTF, [276] in the bibliography to Part I) in the completely distributive case.

Theorem 1-3.12. *A complete lattice L is continuous* (*resp. completely distributive*) *if and only if there is an injection of L into some power $[0,1]^I$ of the unit interval preserving arbitrary meets and directed* (*resp. arbitrary*) *joins.*

Proof. One implication follows from Corollary 1-3.7. For the converse, we use the preceding lemma. For every pair of elements a, b in L such that $a \not\leq b$ one can find an element $x_0 \ll a$ (resp. $x_0 \lll x_1$) such that $x_0 \not\leq b$. We let $x_1 = a$ and we obtain a map $g_{a,b}\colon L \to [0,1]$ preserving arbitrary meets and directed (resp. arbitrary) joins such that $g_{a,b}(b) = 0, g_{a,b}(a) = 1$. This allows us to embed L into $[0,1]^I$ under preservation of arbitrary meets and directed (resp. arbitrary) joins by defining $g(x) = (g_{a,b}(x))_{a \not\leq b}$, where I ranges over all pairs $a \not\leq b$ in L. $\square$

Corollary 1-3.13. *In a complete lattice, the distributive law* (DD2) *is equivalent to* (DD3).

Proof. Let L be a complete lattice satisfying (DD2). Since by Corollary 1-3.7, $[0,1]^I$ satisfies (DD2) (since $[0,1]$ does) and since it is self-dual (under the map $(x_i)_{i\in I} \mapsto (1-x_i)_{i \in I}$), $[0,1]^I$ also satisfies the dual identity (DD3). Since by the preceding theorem L is order-isomorphic to some sublattice of $[0,1]^I$ closed under arbitrary sups and infs, it also satisfies (DD3). By a dual argument (DD3) implies (DD2). $\square$

Continuous lattices are standardly defined in terms of the approximation relation $\ll$, while completely distributive lattices are typically defined in terms of the equations (DD2) and (DD3). We close this section with an example where it is more convenient, however, to use the definition of complete distributivity in terms of the relation $\lll$. We first recall the analogous situation for continuous lattices.

Definition 1-3.14. A topological space X is said to be *core compact* if given x in U open, there exists an open set V such that $x \in V \subseteq U$ and every open cover of U admits finitely many members that cover V. The latter is equivalent to saying that $V \ll U$ in the lattice $\mathcal{O}(X)$ of open sets of X.

Recall that a topological space is *locally compact* if given x in U open, there exist an open set V and a compact set K such that $x \in V \subseteq K \subseteq U$. It is easy to see that a locally compact space is core compact.

Proposition 1-3.15. *A topological space X is core compact if and only if the lattice of open sets $\mathcal{O}(X)$ is a continuous lattice.*

Proof. If X is core compact, then it is immediate from the definition that any nonempty open set U is the union of all V such that $V \ll U$. Noting also that $\emptyset \ll \emptyset$, we conclude that $\mathcal{O}(X)$ is continuous.

Conversely, suppose that $\mathcal{O}(X)$ is a continuous lattice, and let x be in U open. Since $U = \bigcup\{V \in \mathcal{O}(X) : V \ll U\}$, we conclude that $x \in V \ll U$ for some V. Then by Proposition 1-2.3(vi) for every open cover of U, V is covered by finitely many members of the cover. Hence X is core compact. $\square$

Given a topological space X, the *preorder of specialization* on X is defined by $x \leq y$ if $x \in \mathrm{cl}\{y\}$, the closure of the singleton set $\{y\}$. It is straightforward to show that a topological space X is a T_0-space (any two points may be separated by some open set) if and only if the preorder of specialization is an order (see Exercise 1.5). In this case we refer to it as the *order of specialization.*

Definition 1-3.16. A space X is called a *c-space* if for every x in U open, there exist an open set V and a $y \in U$ such that $x \in V \subseteq \uparrow y \subseteq U$, where $\uparrow y$ is taken in the preorder of specialization. A space X is a *C-space* if it is both a c-space and T_0.

We shall see in the next section how C-spaces arise naturally from continuous posets. Our current motivation for introducing them is the following completely distributive analogue of Proposition 1-3.15. It is due independently to M. Erné [79] and Yu.L. Ershov [82].

Proposition 1-3.17. *A topological space X is a c-space if and only if the lattice $\mathcal{O}(X)$ of open sets is completely distributive.*

Proof. Suppose that X is a c-space and x is in U, an open set. There exist an open set V and a $y \in U$ such that $x \in V \subseteq \uparrow y \subseteq U$. Then for any open cover of U, there exists some V in the cover that contains y, and hence $\uparrow y \subseteq \uparrow V = V$, since V is open. It follows that $V \lll U$. Since x was arbitrary in U, we see that $U = \bigcup\{V \in \mathcal{O}(X) : V \lll U\}$. By Theorem 1-3.3(i) $\mathcal{O}(X)$ is completely distributive.

Conversely suppose that $\mathcal{O}(X)$ is completely distributive and that x is in U, an open set. Then there exists V open such that $x \in V$ and $V \lll U$. Suppose for every $y \in U$, V is not contained in $\uparrow y$. Then pick the open set $W_y = U \setminus \downarrow z_y = U \setminus \overline{\{z_y\}}$, where $z_y \in V \setminus \uparrow y$. Then $y \in W_y$, so $\{W_y : y \in U\}$ covers U, but there is no single one that covers V, since $z_y \in V \setminus W_y$. This contradicts $V \lll U$. Hence there does exist $y \in U$ such that $V \subseteq \uparrow y$, and thus X is a c-space. $\square$

1-4. The Scott topology

Suppose that we are able to order the possible states of a system by an "information order": $x \leq y$ means all the information of state x is contained in state y. If x_n is an increasing sequence of states with supremum x, we interpret this to mean the information of x is the union of the information in all x_n. It is natural to consider this as a type of convergence, and if the x_n arise as the stages of a computation, we interpret the computation as yielding x. Further, it is useful to assert that the sequence converges to any state $z \leq x$ (alternatively that the computation yields z), since the sequence $\{x_n\}$ also yields the information content of z.

The preceding intuition motivates the requirement that directed sets D should converge to any element $y \leq \sup D$. (We replace increasing sequences by more general directed sets, since sequences do not in general suffice to describe convergence.) From our given notion of convergence we describe a topology in the standard way.

Definition 1-4.1. For a poset P, define a subset A to be *Scott-closed* if
(i) A is a lower set, i.e., $A = \downarrow A = \{x : x \leq y \text{ for some } y \in A\}$, and
(ii) $\sup D \in A$ for every directed set $D \subseteq A$ for which $\sup D$ exists.
Dually a set U is *Scott-open* if
(i) U is an upper set, i.e., $U = \uparrow U$, and
(ii) for a directed set D with $\sup D \in U$, we have $d \in U$ for some $d \in D$ (and hence for all $e \in D$ such that $d \leq e$).

It is not difficult to see that the Scott-open sets satisfy the axioms of a topology, which we call the *Scott topology* (Exercise 1.6).

Remark 1-4.2. All principal ideals $\downarrow x$ are Scott-closed subsets, and so their complements are Scott-open.

Call a topology on a poset P *order consistent* if its order of specialization agrees with the given order. We note that this is equivalent to requiring $\downarrow x = \mathrm{cl}\{x\}$ for every $x \in P$ (see Exercise 1.7). The Scott topology is an example of an order consistent topology (see Exercise 1.8).

The definition of the Scott topology does not give direct constructions for Scott-open sets outside the complements of principal ideals. For this purpose the next proposition is important.

Proposition 1-4.3. *Let P be a continuous poset.*

(i) *All sets of the form*
$$\twoheaduparrow x = \{y : x \ll y\}$$
are Scott-open, and given $y \in U$, a Scott-open set, there exists $x \in U$ such that $y \in \twoheaduparrow x \subseteq \uparrow x \subseteq U$.

(ii) *The Scott-open filters form a basis for the Scott topology.*

Proof. (i) From Proposition 1-2.3(ii) we conclude that $\uparrow(\twoheaduparrow x) = \twoheaduparrow x$, i.e., $\twoheaduparrow x$ is an upper set. Suppose that D is a directed set with $\bigvee^{\uparrow} D \in \twoheaduparrow x$. Then $x \ll z = \bigvee^{\uparrow} D$. By Proposition 1-2.3(iv) there exists $d \in D$ such that $x \ll d$, i.e., $d \in \twoheaduparrow x$. Hence $\twoheaduparrow x$ is Scott-open.

Let $y \in U$, a Scott-open set. Since $y = \bigvee^{\uparrow}\{x : x \ll y\}$, there exists $x \ll y$ such that $x \in U$. Then $y \in \twoheaduparrow x \subseteq \uparrow x \subseteq \uparrow U = U$.

(ii) Let $y \in U$, a Scott-open set. As in the preceding paragraph, pick $x \in U$ with $x \ll y$. By an inductive application of the interpolation property, we may pick a sequence $\{x_n\}$ such that $x \ll x_1 \ll y$ and for all n, $x \ll x_{n+1} \ll x_n$. Let $F = \bigcup_n \twoheaduparrow x_n = \bigcup_n \uparrow x_n$. The first union is a Scott-open set by part (i) and the second union is the union of an increasing sequence of (principal) filters, hence a filter. By construction, $y \in F \subseteq \twoheaduparrow x \subseteq U$. □

The preceding yields the following topological description of continuous posets equipped with the Scott topology.

Corollary 1-4.4. *A continuous poset equipped with the Scott topology is a C-space.*

Proof. We have already remarked that the Scott topology has for its order of specialization the given order and is hence T_0. Let $y \in U$, a Scott-open set. By Proposition 1-4.3(i), there exists $x \ll y$ such that $y \in \twoheaduparrow x \subseteq \uparrow x \subseteq U$ and $\twoheaduparrow x$ is Scott-open. Hence the requirements for a C-space are satisfied. □

A function $f \colon S \to T$ between ordered sets S and T *preserves directed sups* if whenever D is a directed subset of S for which the supremum $\bigvee^{\uparrow} D$ exists, then $f(\bigvee^{\uparrow} D)$ is the supremum of $f(D)$. We note that if f preserves directed sups, then it must be order-preserving, since if $a \leq b$ in S, then b is the supremum of the directed set $\{a, b\}$, and so $f(b)$ must be the supremum of $\{f(a), f(b)\}$, i.e., $f(a) \leq f(b)$.

Lemma 1-4.5. *For a function $f \colon S \to T$ between posets S and T, the following are equivalent:*

(i) *f preserves directed sups;*

(ii) *f is order-preserving and preserves directed sups;*

(iii) *f is a* Scott-continuous map, *that is, it is continuous with respect to the Scott topologies on S and T.*

Proof. The equivalence of (i) and (ii) follows from the remarks just before the statement of the lemma. It follows easily from (ii) that the inverse $f^{-1}(A)$ of a Scott-closed subset of T is Scott-closed in S, and thus (ii) implies (iii). Let D be a directed subset of S with supremum a. If f is Scott-continuous, then $f^{-1}(\downarrow f(a))$ is a Scott-closed subset of S containing a, hence also $\downarrow a$, and thus D. So $f(a)$ is an upper bound of $f(D)$. Let b be another upper bound of $f(D)$. Then $f^{-1}(\downarrow b)$ is a Scott-closed set containing D and hence $a = \bigvee^{\uparrow} D$. Thus $f(a) \leq b$, which establishes that $f(a) = \bigvee^{\uparrow} f(D)$, yielding (i). □

Lemma 1-4.5 establishes that a Scott-continuous function has an alternative order theoretic characterization as a map preserving directed sups. We thus have both an order theoretic and a topological characterization of this class of maps. The Scott topology thus suggests a useful class of maps that has been historically ignored in the theory of lattices and ordered sets: order-preserving maps that preserve suprema of directed sets.

According to the general definition of continuity, a function f from a topological space X into a poset S is called Scott-continuous, if the inverse image of every Scott-open subset of S is open in X. We denote by

$$[X \to S]$$

the set of all Scott-continuous functions from X to S. We always endow this function space with the pointwise order: $f \leq g$ if $f(x) \leq g(x)$ for all $x \in X$. A common construction is the formation of the pointwise supremum of a directed family of Scott-continuous maps. These are again Scott-continuous:

Lemma 1-4.6. *Let X be a topological space or, more specifically, a poset endowed with the Scott topology. Let S be a poset and $\{f_i : i \in I\}$ be a directed family of Scott-continuous functions from X to S. If $f(x) = \bigvee^{\uparrow}{}_{i \in I} f_i(x)$ for all $x \in X$, then f is also Scott-continuous. In particular, if S is a dcpo, then $[X \to S]$ is a dcpo, where directed joins are formed pointwise.*

Proof. Using the definition of a Scott-open set U, it is a straightforward computation to see that $f^{-1}(U) = \bigcup \{f_i^{-1}(U) : i \in I\}$, and hence $f^{-1}(U)$ is open in X. □

We have already seen (Corollary 1-3.7) that a product of continuous lattices is again a continuous lattice and the same is true for other classes of continuous objects.

Proposition 1-4.7. *The product $\Pi\{P_i : i \in I\}$ is a continuous lattice if and only if each factor P_i is, and the analogue is true for dcpos, continuous dcpos, continuous posets with bottom element, and continuous bounded complete semilattices.*

Proof. The proof follows easily from the observations that directed sups are computed from the projections in each of the coordinates and that in $\Pi_{i \in I} P_i$, $x \ll y$ if and only if $x_i = \perp$ for all but finitely many i and $x_i \ll y_i$ in the other coordinates. □

We turn next to the behavior of the Scott topology with respect to products. Unfortunately, there are complete lattices P and Q such that the Scott topology on $P \times Q$ is strictly finer than the product of the respective Scott topologies on P and Q. This defect is cured by restricting to continuous posets:

Proposition 1-4.8. *Let $\{P_i : i \in I\}$ be a family of posets, each equipped with the Scott topology.*

(i) *The Scott topology on the product $\Pi_{i\in I} P_i$ contains the product topology of the individual Scott topologies on the P_i.*

(ii) *If each P_i is a continuous poset with bottom element, then the Scott topology on the product agrees with the product of the Scott topologies.*

Proof. (i) One sees easily that the subbasic open sets $U_i \times \Pi_{j\neq i} P_j$ of the product topology are Scott-open subsets of the product, from which item (i) follows.

(ii) Let U be a Scott-open subset of $\Pi_{i\in I} P_i$, and let $x \in U$. Consider the subset D of $\Pi_{i\in I} P_i$ consisting of all points y for which there is some finite $J \subseteq I$ such that $y_i \ll x_i$ for $i \in J$ and $y_i = \perp$ for $i \in I \setminus J$. The set D is directed with supremum x, and hence $y \in U$ for some $y \in D$. Then $\Uparrow y = \Pi_{i\in J} \Uparrow y_i \times \Pi_{i\in I\setminus J} P_i$ is open in the product topology, contains x and is contained in $U = \uparrow U$. Thus the Scott topology is contained in the product topology and the other inclusion was shown in (i). □

An *order retraction* is an order-preserving map $r\colon P \to Q$ between posets which admits an order-preserving section, that is, an order-preserving $j\colon Q \to P$ such that $r \circ j = \mathrm{id}_Q$, the identity on Q. In this case Q is called an *order retract* of P.

Proposition 1-4.9. *Let $r\colon P \to Q$ be an order retraction with an order-preserving section $j\colon Q \to P$.*

(i) *Let $A \subseteq Q$. If $\bigvee j(A)$ exists in P, then $\bigvee A$ exists in Q and $\bigvee A = r(\bigvee j(A))$. A dual statement holds for meets.*

(ii) *The properties of being a dcpo, a lattice, a semilattice, a bounded complete semilattice, and a complete lattice are all preserved by order retractions.*

(iii) *A retract of a continuous poset under a Scott-continuous retraction r with a Scott-continuous section j is again a continuous poset.*

Proof. (i) Let q be an upper bound of A in Q. Then $j(q)$ is an upper bound of $j(A)$ in P, whence $j(q) \geq \bigvee j(A)$. We conclude that $q = r(j(q)) \geq r(\bigvee j(A))$, which shows that $r(\bigvee j(A))$ is the least upper bound of A in Q.

(ii) is an immediate consequence of (i).

(iii) Let $r\colon P \to Q$ be a Scott-continuous retraction with a Scott-continuous section $j\colon Q \to P$. Consider any $y \in Q$. We first claim: If $x \ll j(y)$ in P, then $r(x) \ll y$ in Q. Let indeed D be a directed subset of Q with $y \leq \bigvee^{\uparrow} D$. Then $j(y) \leq \bigvee^{\uparrow} j(D)$ by the Scott continuity of j. As $x \ll j(y)$, there is a $d \in D$ such that $x \leq j(d)$. We conclude that $r(x) \leq r(j(d)) = d$. Since P is a continuous poset, $j(y) = \bigvee^{\uparrow}\{x \in P \mid x \ll j(y)\}$. Applying the Scott-continuous r, we obtain that $y = r(j(y))$ is the join of the directed set of elements $r(x)$, $x \ll j(y)$, which are all way-below y. □

A particularly nice class of order retractions are the *projections*. These are order retractions $g\colon P \to Q$ for which there is an order-preserving section $d\colon Q \to P$ satisfying $d \circ g \leq \mathrm{id}_P$. Note that these conditions are equivalent to g being a projection in the sense of Definition 1-9.1, that is, g is a surjective order-preserving map having a lower Galois adjoint d (see Section 1-9 for details on Galois adjoints). A lower adjoint d is always Scott-continuous by Lemma 1-9.3. Thus, all of Proposition 1-4.9 applies to Scott-continuous projections.

Corollary 1-4.10. *The properties of being a continuous poset, a domain, a bounded complete domain, and a continuous lattice, are preserved under Scott-continuous projections.*

For complete lattices (resp. bounded complete dcpos), the projections are characterized as those maps that preserve arbitrary (resp. nonempty) meets (see Proposition 1-9.7). Thus, the image of a continuous lattice (resp. bounded complete domain) under a surjective map preserving arbitrary (resp. nonempty) meets and directed joins is a continuous lattice (resp. bounded complete domain) again, a result we derived previously (Proposition 1-3.8) from the equational characterization of continuous lattices.

Example 1-4.11. The image of an algebraic lattice under a Scott-continuous retraction or even projection need not be algebraic. On the contrary, every continuous lattice L is the image of the algebraic lattice $\mathrm{Id}(L)$ of all ideals of L under the map p defined by $p(J) = \bigvee^{\uparrow} J$ for every ideal J of L, a map that is easily verified to be Scott-continuous. This map p has a lower adjoint $y \mapsto \twoheaddownarrow y = \{x \in L \mid x \ll y\}$ and, thus, p is a Scott-continuous projection.

1-5. Function spaces and Cartesian closed categories

Let us consider the category DCPO with objects dcpos and morphisms Scott-continuous maps. Recall that a category C possessing a terminal object and finite products is *Cartesian closed* if there are an *internal hom functor* $(Y,Z) \mapsto Z^Y\colon \mathsf{C}^{\mathrm{op}} \times \mathsf{C} \to \mathsf{C}$ and a natural isomorphism $\mathrm{Hom}(X \times Y, Z) \simeq \mathrm{Hom}(X, Z^Y)$. In DCPO the terminal object is the one point ordered set and products are the Cartesian set products with the coordinatewise order. We take for Z^Y the set $[Y \to Z]$ of Scott-continuous maps from Y to Z. It follows from Lemma 1-4.6 that $[Y \to Z]$ is again a dcpo.

It is standard that the category of sets and functions is Cartesian closed with the exponential object Z^Y given by the set of all functions from Y to Z. By the law of exponents $Z^{X\times Y} \simeq (Z^Y)^X$, where $F\colon X \times Y \to Z$ corresponds to $\widetilde{F}\colon X \to Y^Z$ if and only if for all $x \in X$, $y \in Y$, $F(x,y) = \widetilde{F}(x)(y)$. One shows directly for dcpos X, Y, Z that the function F is Scott-continuous if and only if $\widetilde{F}$ is Scott-continuous (see Exercise 1.11). Hence the natural bijection $F \leftrightarrow \widetilde{F}$ restricts to one between $[X \times Y \to Z]$ and $\big[X \to [Y \to Z]\big]$. It thus follows that the Cartesian closeness of the category of sets and functions restricts to the category of dcpos and Scott-continuous maps in such a way that

Proposition 1-5.1. *The category* DCPO *with objects all dcpos and morphisms all Scott-continuous maps is Cartesian closed.*

Suppose that we consider some full subcategory C of DCPO containing the singleton terminal object and closed under finite products. Then we can conclude that C is also Cartesian closed (inheriting the Cartesian closedness of DCPO) if we determine that the function space $[Y \to Z]$ is again an object in the category C for all objects Y, Z in C. This leads us to investigate the structure of the function space $[Y \to Z]$ of Scott-continuous maps in various contexts.

The following theorem characterizes those situations in which the function spaces are continuous (resp. completely distributive) lattices.

Theorem 1-5.2. *For a topological space X and a non-singleton complete lattice L, the directed complete poset $[X \to L]$ is a continuous* (*resp. completely distributive*) *lattice if and only if both the lattice $\mathcal{O}(X)$ of open subsets of X and the lattice L are continuous* (*resp. completely distributive*).

For the completely distributive case, this theorem is due to M. Erné [81]. Before proving this theorem, let us derive some consequences. In view of the characterization in Proposition 1-3.15 (resp. Proposition 1-3.17) of those spaces for which the lattice of open subsets is continuous (resp. completely distributive) we can reformulate this theorem as follows:

Corollary 1-5.3. *For a topological space X and a non-singleton complete lattice L, the directed complete poset $[X \to L]$ is a continuous* (*resp. completely distributive*) *lattice if and only if X is core compact and L a continuous lattice* (*resp. X is a c-space and L completely distributive*).

As a continuous poset P is a C-space for the Scott topology by Corollary 1-4.4, the function spaces $[P \to L]$ are continuous (resp. completely distributive) lattices if and only if L is. In view of Proposition 1-5.1 and the subsequent remarks we can state:

Corollary 1-5.4. *The category of continuous lattices* (*resp. completely distributive lattices*) *and Scott-continuous maps is Cartesian closed.*

Cartesian closedness is a basic requirement for categories to be appropriate for semantics of functional programming languages. Besides the categories mentioned above, the following categories are Cartesian closed: the categories of algebraic lattices, of continuous lattices, of continuous lattices with countable bases, of algebraic lattices with countably many compact elements, the morphisms being the Scott-continuous maps in all cases. Unfortunately, the category of continuous dcpos and Scott-maps and the full subcategory of compactly generated dcpos are not Cartesian closed. For an in depth investigation on Cartesian closed categories of continuous dcpos one should consult A. Jung's monograph [216].

In preparation for the proof of Theorem 1-5.2, let X be a topological space and L a complete lattice.

For any subspace Y of X and any map $g\colon Y \to L$, define $\widetilde{g}\colon X \to L$ by

$$\text{(Ext)} \qquad \widetilde{g}(x) = \bigvee^{\uparrow}_{U \in \mathcal{U}(x)} \bigwedge_{u \in U \cap Y} g(u) \text{ for all } x \in X,$$

where $\mathcal{U}(x)$ denotes the filter of open neighborhoods of x. Note that the elements $\bigwedge_{u\in U\cap Y} g(u)$, $U \in \mathcal{U}(x)$, do indeed form a directed set.

Lemma 1-5.5. *Suppose that L is a continuous lattice, Y a subspace of a topological space X, and $g\colon Y \to L$ an arbitrary function. We claim:*

(i) *$\widetilde{g}$ is Scott-continuous;*

(ii) *$\widetilde{g}$ is the greatest among the Scott-continuous functions $f\colon X \to L$ such that $f(y) \le g(y)$ for all $y \in Y$;*

(iii) *$\widetilde{g}$ agrees with g on Y if and only if g is Scott-continuous on Y.*

Proof. (i) Let V be any Scott-open subset of L and suppose that x is an element of X such that $\widetilde{g}(x) \in V$. As the family $\bigwedge_{u\in U\cap Y} g(u)$, $U \in \mathcal{U}(x)$, is directed, there is an open neighborhood U of x such that $\bigwedge_{u\in U\cap Y} g(u) \in V$. For any $z \in U$ one has $\bigwedge_{u\in U\cap Y} g(u) \le \widetilde{g}(z)$, which implies that $\widetilde{g}(z) \in V$.

(ii) Clearly, $\widetilde{g}(y) \le g(y)$ for all $y \in Y$. Let $f\colon X \to L$ be Scott-continuous and $f(y) \le g(y)$ for all $y \in Y$. Let $x \in X$. For $z \ll f(x)$, the set $U = f^{-1}(\twoheaduparrow z)$ is an open neighborhood of x. It follows that $z \ll f(u) \le g(u)$ for all $u \in U \cap Y$ so that $z \le \bigwedge_{u\in U\cap Y} g(u) \le \widetilde{g}(x)$. As this holds for all $z \ll f(x)$, we conclude from the continuity of L that $f(x) \le \widetilde{g}(x)$.

(iii) For $y \in Y$ and $z \ll g(y)$, by continuity of g, $V = g^{-1}(\twoheaduparrow z)$ is an open neighborhood of y in Y. Pick U open in X such that $U \cap Y = V$. Then $z \le \bigwedge_{u\in U\cap Y} g(u) \le \widetilde{g}(y)$. We conclude that $\widetilde{g}(y) \le g(y)$ by continuity of L, which establishes the needed inequality. $\square$

Corollary 1-5.6. *Every continuous lattice L endowed with the Scott topology is an* injective *space in the sense that, for any subspace Y of a topological space X, every Scott-continuous function $f\colon Y \to L$ has a Scott-continuous extension $\widetilde{f}\colon X \to L$.*

It can be shown conversely that all injective T_0-spaces arise in this way: Every injective T_0-space is a continuous lattice with respect to its order of specialization and its topology is the Scott topology (see [104, Proposition II-3.7]).

In the case that X is a continuous poset with the Scott topology the extension of an order-preserving function can be described in a simpler way than by (Ext):

Lemma 1-5.7. *Let X be a continuous poset with the Scott topology and B a* basis *of X in the sense of Definition* (1-2.5)*. Let L be a continuous lattice and g an order-preserving function defined on B with values in L. Then the function $\widetilde{g}\colon X \to L$ defined by*

$$\text{(Ext}')\qquad \widetilde{g}(x) = \bigvee_{y \ll x, y \in B}^{\uparrow} g(y) \text{ for all } x \in X$$

agrees with the function defined by (Ext) *and has all the properties described in Lemma* (1-5.5)*. In particular, $\widetilde{g}$ extends g if and only if for all $x \in B$ one has $g(x) = \bigvee^{\uparrow}_{y \ll x, y \in B} g(y)$.*

Proof. Appropriately modify the proof of Lemma 1-5.5. □

By Lemma 1-4.6, the pointwise join of a directed family of Scott-continuous maps f_i from a space X to a complete lattice L is Scott-continuous. Suppose now that f and g are two Scott-continuous maps from X to L. We would like to conclude that the pointwise join and meet $(f \vee g)(x) = f(x) \vee g(x)$ and $(f \wedge g)(x) = f(x) \vee g(x)$ are also Scott-continuous. One checks directly that the join operation $(a, b) \mapsto a \vee b\colon L \times L \to L$ is Scott-continuous, and the same holds for the meet operation on L if and only if L is meet-continuous. The map $f \vee g$ is the composition of the maps $x \mapsto (f(x), g(x))\colon X \to L \times L$ and $(a, b) \mapsto a \vee b\colon L \times L \to L$, and similarly for $f \wedge g$. But in general, we cannot conclude that $f \vee g$ or $f \wedge g$ are Scott-continuous. Indeed, $x \mapsto (f(x), g(x))\colon X \to L \times L$ is continuous for the product topology $\sigma(L) \times \sigma(L)$ on $L \times L$, where $\sigma(L)$ is the Scott topology on L, but this product topology may be strictly coarser than the Scott topology $\sigma(L \times L)$ of the lattice $L \times L$. However, in the case that L is a continuous lattice, by Proposition 1-4.8 the product topology $\sigma(L) \times \sigma(L)$ agrees with the Scott topology $\sigma(L \times L)$ and we conclude that the pointwise join $f \vee g$ and the pointwise meet $f \wedge g$ are Scott-continuous (for the latter we also use Proposition 1-2.8, which tells us that continuous lattices are meet-continuous). Noting that the join of an arbitrary family of Scott-continuous functions is the directed join of the joins of finite subfamilies, we conclude that the pointwise join $\bigvee_i f_i$ of any family of Scott-continuous functions $f_i\colon X \to L$ is Scott-continuous. We have shown:

Lemma 1-5.8. *Let X be a topological space and L a continuous lattice. Then the function space $[X \to L]$ is a complete lattice. Arbitrary joins and finite meets are formed pointwise.*

While the pointwise meet of a finite family of Scott-continuous functions $g_i\colon X \to L$ is Scott-continuous, this need not be true for infinite families. The meet of a family (g_i) in the complete lattice $[X \to L]$ is given by the greatest Scott-continuous function below the pointwise meet $g(x) = \bigwedge_i g_i(x)$, which by Lemma 1-5.5 is

$$\widetilde{g}(x) = \bigvee_{U \in \mathcal{U}(x)}^{\uparrow} \bigwedge_{u \in U} g(x).$$

We will write $\widetilde{g} = \widetilde{\bigwedge}_i g_i$ for this meet of the g_i in the lattice $[X \to L]$.

Now we are ready for the proof of Theorem 1-5.2:

Proof. Let X be a topological space and L a complete lattice. The constant functions form a complete sublattice of the whole function space isomorphic to L. Thus, if $[X \to L]$ is continuous (resp. completely distributive), L is continuous (resp. completely distributive), too. The lattice $\mathcal{O}(X)$ of open subsets of X is isomorphic to the lattice of Scott-continuous functions with $\perp$ and $\top$ as their only values and these functions form a complete sublattice of $[X \to L]$. Thus, if $[X \to L]$ is continuous (resp. completely distributive), then L and $\mathcal{O}(X)$ are continuous (resp. completely distributive) also.

For the converse we first suppose that both the lattice $\mathcal{O}(X)$ of open subsets of X and the lattice L are continuous. We show that the complete lattice $[L \to X]$ satisfies the equation characterizing continuous lattices (see Theorem 1-3.1): for every nonempty family of directed subsets $D_i \subseteq [X \to L]$

(DD1) $$\widetilde{\bigwedge_i} \bigvee^{\uparrow} D_i = \bigvee_{(g_i) \in \prod_i D_i}^{\uparrow} \widetilde{\bigwedge_i} g_i,$$

where the directed joins are pointwise but the meets are the intrinsic meets in the complete lattice $[X \to L]$. As the inequality $\geq$ is always satisfied, we concentrate on the proof of the inequality $\leq$. Let l denote the function on the left-hand side and r the function on the right-hand side. As L is a continuous lattice, it suffices to take an arbitrary x, any $z \ll l(x)$, and prove that $z \leq r(x)$.

Let $l_i = \bigvee^{\uparrow} D_i$. Then $l(x) = \bigvee^{\uparrow}_{U \in \mathcal{U}(x)} \bigwedge_{u \in U} l_i(u)$. For $z \ll l(x)$, there is a $U \in \mathcal{U}(x)$ such that $z \ll \bigwedge_{u \in U} l_i(u)$, that is, $U \subseteq l_i^{-1}(\twoheaduparrow z)$ for any i. Using the continuity of the lattice $\mathcal{O}(X)$, we may choose an open neighborhood V of x with $V \ll U$. As l_i is the pointwise join of the functions g in the directed set D_i, $l_i^{-1}(\twoheaduparrow z)$ is the directed union of the sets $g^{-1}(\twoheaduparrow z)$ $(g \in D_i)$, which are open, since $\twoheaduparrow z$ is open. Thus, for every i, there is a function $g_i \in D_i$ such that $V \subseteq g_i^{-1}(\twoheaduparrow z)$ which implies that $z \leq \bigwedge_{v \in V} g_i(v)$. Thus, $z \leq \bigvee^{\uparrow}_{U \in \mathcal{U}(x)} \bigwedge_{u \in U} g_i(u) = (\widetilde{\bigwedge}_i g_i)(x)$. We conclude that $z \leq \bigvee^{\uparrow}_{(g_i) \in \prod_i D_i} (\widetilde{\bigwedge}_i g_i)(x) = r(x)$.

In order to obtain a proof for the completely distributive case, one uses the equational characterization (DD2) (see Theorem 1-3.3) for completely distributive lattices and one replaces the relation $\ll$ by $\lll$ and directed joins by arbitrary joins in the above proof. $\square$

Example 1-5.9. A function f from a topological space X into the real unit interval $[0, 1]$ is Scott-continuous if and only if it is lower semicontinuous in the sense of classical analysis. Lemma 1-5.8 generalizes the well-known fact that pointwise suprema of arbitrary families of lower semicontinuous functions are lower semicontinuous while for pointwise infima this holds only for finite families.

If X is core compact, in particular, if X is locally compact, the lattice $\big[X \to [0,1]\big]$ of lower semicontinuous functions is continuous and, if X is a c-space, it is completely distributive. Every continuous dcpo, and in particular the unit interval, is a C-space when endowed with the Scott topology. The order-preserving lower continuous functions $f\colon [0,1] \to [0,1]$ agree with the Scott-continuous ones, so that the space $\big[[0,1] \to [0,1]\big]$ of order-preserving lower semicontinuous functions is completely distributive.

1-6. The Lawson topology

The first topologies defined on a lattice directly from the lattice ordering, Birkhoff's order topology and Frink's interval topology, involved "symmetrical" definitions – the topologies assigned to L and to L^{op} were identical. A guiding example was the unit interval $[0,1]$ in its natural order, which is of course a highly symmetrical lattice. The initial interest was in such questions as which lattices became compact and/or Hausdorff in these topologies. The Scott topology stands in strong contrast to such an approach. Indeed it is a "unidirectional" topology, since, for example, all the open sets are always upper sets; thus, for nontrivial lattices, the T_0-separation axiom is the strongest it satisfies. Nevertheless, it was Dana Scott's important insight that this topology captured many important aspects of continuous lattices and was useful in considering computational connections.

In this section we introduce a new topology, called the Lawson topology, which is crucial in linking continuous lattices and domains to topological algebra. Its definition is more in the spirit of the interval and order topologies, and indeed it may be viewed as a mixture of the two. However, it remains asymmetrical – the Lawson topologies on L and L^{op} often do not agree. But if one is seeking an appropriate Hausdorff topology for continuous lattices, this asymmetry is not at all surprising; indeed it is just another aspect of the asymmetry exhibited by continuous lattices.

Given a poset P, there is a coarsest or weakest topology $\nu(P)$ on P for which the given order is the order of specialization, namely the topology which has a subbase of closed sets given by all $\downarrow x$, $x \in P$. Clearly, this topology is coarser than the Scott topology. This topology has no widely accepted name in the literature. It is has been called the *upper topology* in [104], but has also been called the *weak topology* or *weak upper topology* in other places in light of its characteristic property. One might also call it the *lower half-interval topology* in light of the fact that it together with its dual form a subbasis for the classical *interval topology* on lattices . Let us in what follows adopt the terminology "weak upper topology" and call the order dual of this topology, the one with a subbasis of closed sets given by all $\uparrow x$, $x \in P$, the *weak lower topology*. (Upper and lower indicate that the open sets are upper resp. lower sets.) We denote the weak lower topology by $\omega(P)$.

Definition 1-6.1. The *Lawson topology* on a poset P is the common refinement $\sigma(L) \vee \omega(L)$ of the Scott topology and weak lower topology and is denoted $\lambda(L)$.

An important relationship between the Lawson and Scott topologies is that the Scott-open sets are precisely the Lawson-open upper sets. But although the Scott topology determines the underlying partial order, the Lawson topology does not do so.

Proposition 1-6.2. *For a poset P, the Lawson-open upper sets are precisely the Scott-open sets and the Lawson-closed upper sets are closed in the Scott topology of L^{op}.*

Proof. By definition all Scott-open sets are Lawson-open upper sets. For the converse, let us define a subset A of P to have property (S) if whenever $\bigvee^{\uparrow} D \in A$ for some directed set D, then there exists $e \in D$ such that $d \in A$ for $e \leq d \in D$. Members of both $\sigma(P)$ and $\omega(P)$ have property (S), and property S is preserved by finite intersections and arbitrary unions. Hence all members of $\lambda(L)$ have property (S). But it is immediate that an upper set with property (S) is Scott-open. The second claim follows from an order dual argument with filtered sets. $\square$

A map $g\colon L \to M$ between bounded complete semilattices preserving meets for all nonempty sets is continuous for the respective weak lower topologies, since the inverse image of any principal filter $\uparrow t$ in M is empty or is the principal filter $\uparrow s$ in L, where $s = \bigwedge\{x \in L \mid g(x) \geq t\}$. Thus, if g preserves directed joins and meets of nonempty sets, then it is Lawson-continuous. More generally, if S and T are posets and $g\colon S \to T$ is a Scott-continuous function that has a lower adjoint (see Section 1-9), then g is Lawson-continuous. Conversely:

Proposition 1-6.3. *Let L and M be bounded complete semilattices and $g\colon L \to M$ a function preserving binary meets. Then g is Lawson-continuous if and only if g preserves directed joins and meets of nonempty sets.*

Proof. Because of the remarks preceding this proposition, it suffices to consider a Lawson-continuous function $g\colon L \to M$ preserving binary meets.

We first show $\bigwedge g(A) = g(\bigwedge A)$ for a nonempty subset A. As g is order-preserving, $g(\bigwedge A) \leq \bigwedge g(A)$. For the converse, note that the inverse image $g^{-1}(\uparrow \bigwedge g(A))$ of the principal filter $\uparrow \bigwedge g(A)$ is firstly Lawson-closed (since g is Lawson continuous) and secondly a filter of L (since g preserves binary meets). By the second part of Proposition 1-6.2 this filter is closed for the

Scott-topology on L^{op} and hence is a principal filter (dual of Exercise 1.8): $g^{-1}(\uparrow \bigwedge g(A)) = \uparrow s$ for some $s \in L$. Since $A \subseteq g^{-1}(\uparrow \bigwedge g(A)) = \uparrow s$, we have $\bigwedge A \geq s$, whence $g(\bigwedge A) \geq g(s) \geq \bigwedge g(A)$.

Secondly, let us show that g is Scott-continuous. Indeed. Let U be a Scott-open subset of M. The inverse image $g^{-1}(U)$ is firstly Lawson-open (as g is Lawson-continuous) and secondly an upper set (as g is order-preserving). Thus U is Scott-open by the first part of Proposition 1-6.2. □

It is an old result that for a complete lattice the interval topology (the join of the weak upper and weak lower topologies) is compact. Since the Lawson topology always refines the interval topology, the next result is a strengthening of this result.

Theorem 1-6.4. *For a complete lattice L the Lawson topology $\lambda(L)$ is a compact T_1-topology.*

Proof. Firstly, for $x \in L$ we have $\{x\} = {\downarrow} x \cap {\uparrow} x$. Since ${\downarrow} x$ is Scott-closed, while ${\uparrow} x$ is closed in the weak lower topology, the intersection $\{x\}$ is Lawson closed, that is, $\lambda(L)$ is a T_1-topology.

To prove that $\lambda(L)$ is compact, we use the Alexander Subbasis Lemma: a space is compact if every open cover consisting of *subbasic* open sets contains a finite subcover.

Thus assume $\{U_j \in \sigma(L)\colon j \in J\}$ and $\{L \setminus {\uparrow} x_k \colon k \in K\}$ together form a cover of L. Let $x = \sup\{x_k \colon k \in K\}$. Then

$$\bigcup\{L \setminus {\uparrow} x_k \colon k \in K\} = L \setminus \bigcap\{{\uparrow} x_k \colon k \in K\} = L \setminus {\uparrow} x.$$

But $x \notin L \setminus {\uparrow} x$; therefore, there is a j such that $x \in U_j$. Since U_j is Scott-open, there are indices $k_1, \ldots, k_n$ such that $x_{k_1} \vee \cdots \vee x_{k_n} \in U_j$. Then

$$U_j \cup (L \setminus {\uparrow} x_{k_1}) \cup \cdots \cup (L \setminus {\uparrow} x_{k_n}) = L,$$

and we are finished. □

Remark 1-6.5. If L is a bounded complete semilattice, then $L^{\top}$ is a complete lattice, hence compact in the Lawson topology. Since $\{\top\}$ is Scott, hence Lawson, open, L is closed, hence compact, in the relative topology. But it is straightforward to see that the relative Lawson topology of a Scott-closed lower set L is the Lawson topology of L.

It is also important to understand when the Lawson topology is Hausdorff. In this regard the next lemma shows the suitability of the Lawson topology for continuous lattices, even continuous posets.

Lemma 1-6.6. *For a continuous poset P, the Lawson topology $\lambda(P)$ is a Hausdorff topology.*

Proof. Suppose that $x \not\leq y$ in P. Then there exists $u \ll x$ with $u \not\leq y$. Then $\twoheaduparrow u$ is a Scott- (hence, Lawson-)open neighborhood of x, and $L \setminus \uparrow u$ is an $\omega(L)$- (hence, Lawson-)open neighborhood of y. Clearly these two neighborhoods are disjoint. $\square$

Theorem 1-6.4, Remark 1-6.5, and Lemma 1-6.6 imply immediately the following:

Corollary 1-6.7. *For continuous lattices, indeed for any bounded complete domain, the Lawson topology is compact and Hausdorff.*

Definition 1-6.8. A *topological semilattice* is a semilattice S equipped with a topology for which the semilattice operation as a function from $S \times S$ (with the product topology) into S is continuous. We say that S has a *basis of subsemilattices* if for any $x \in U$ open, there exists an open set V and a subsemilattice T such that $x \in V \subseteq T \subseteq U$.

In what follows we restrict our attention to meet-semilattices, but note the theory is equally applicable to the dual notion of join-semilattices.

Theorem 1-6.9 (Fundamental theorem of compact semilattices I)**.** *Let S be a meet-semilattice that is also a continuous poset. Then with respect to the Lawson topology S is a Hausdorff topological semilattice with a basis of open subsemilattices. If additionally S is a bounded complete domain or continuous lattice, then it is compact Hausdorff.*

Proof. By Lemma 1-6.6 S is Hausdorff. By Proposition 1-4.3(ii) the Scott topology has a basis of open filters, which are then open subsemilattices. The open sets of the weak lower topology are lower sets. Thus the intersection of a Scott-open filter with open sets for the weak lower topology is still a subsemilattice, and these intersections form a basis for the Lawson topology.

For continuity of the meet operation, it is enough to see that the inverses of the subbasic open sets are open. For a Scott open filter U, $U \times U = \{(x, y) \in S \times S : x \wedge y \in U\}$, which is open in $S \times S$. For $z \in S$, $\uparrow z \times \uparrow z = \{(x, y) \in S \times S : x \wedge y \in \uparrow z\}$ is closed, and hence the inverse image of $S \setminus \uparrow z$ is open in $S \times S$.

The last assertion follows from Corollary 1-6.7. $\square$

Theorem 1-6.10 (Fundamental theorem of compact semilattices II)**.** *Let S be a compact Hausdorff topological semilattice with a basis of neighborhoods at each point that are subsemilattices. Then S is a bounded complete domain and its topology is the Lawson topology. In addition, if S has a largest element $\top$, then it is a continuous lattice.*

Proof. We carry out the proof in a series of steps.

Step 1. *For each $a \in S$, $\downarrow a$ and $\uparrow a$ are closed.* By continuity of the meet operation $\downarrow a = a \wedge S$ is compact, hence closed. The inverse image of $\{a\}$ under the map $x \mapsto x \wedge a$ is $\uparrow a$, so that is also closed.

Step 2. *Each directed set D (viewed as a net indexed by itself) has a supremum to which it converges.* For each $x \in D$, set $A_d = \mathrm{cl}\{e \in D : e \geq d\}$. The family $\{A_d : d \in D\}$ is a descending family of compact sets and hence must have a nonempty intersection. Let x be in the intersection. Since A_d must be contained in the closed set $\uparrow d$ for each $d \in D$, we have that x is an upper bound for D. Let y be another upper bound. Then the closed set $\downarrow y$ must contain each A_d and hence x. Hence $x = \bigvee^{\uparrow} D$. Since x was an arbitrary point in $\bigcap_d A_d$, we conclude that this intersection is the singleton $\{x\}$. Since x is the only cluster point of the net D in the compact Hausdorff space S, the net D must converge to x.

Step 3. *For $x \in U$ open, there exists $z \in U$ such that $\uparrow z$ is a neighborhood of x.* By regularity of S, pick an open set V containing x with closure $\mathrm{cl}(V) \subseteq U$. By hypothesis there exists a neighborhood $W \subseteq V$ of x such that W is a subsemilattice. By continuity $\mathrm{cl}(W)$ is again a subsemilattice which is contained in U and compact since closed. Since $\mathrm{cl}(W)$ is a semilattice, the sets $\downarrow s \cap \mathrm{cl}(W)$, $s \in \mathrm{cl}(W)$ form a descending family of compact sets. The nonempty intersection must be a singleton set consisting of a bottom element z for $\mathrm{cl}(W)$, and then $W \subseteq \uparrow z$.

Step 4. *S is a domain, i.e., a continuous dcpo.* For $x \in S$, consider the set $D_x = \{z : \uparrow z \text{ is a neighborhood of } x\}$. It follows from step 3 that this set is directed and clusters to x. Applying step 2 we see that D_x converges to $x = \bigvee^{\uparrow} D_x$. Let D be a directed set with $x \leq y = \bigvee^{\uparrow} D$. By step 2, D converges to y, so by continuity of meet, $x \wedge D$ converges to $x \wedge y = x$. Hence for any $z \in D_x$ the neighborhood $\uparrow z$ must contain some $x \wedge d$, $d \in D$. Then $z \leq x \wedge d \leq d$, so $z \ll x$. We have thus established that S is a continuous poset, and it follows from step 2 that it is directed complete. As we have seen in the proof of step 3, any compact subsemilattice has a bottom element, in particular, S itself.

Step 5. *S is a bounded complete domain.* We need only show that any nonempty set A has a greatest lower bound, which can be easily shown by taking the bottom element of the closure of the subsemilattice generated by A.

Step 6. *The topology of S is the Lawson topology.* We consider the identity map from S with the given topology to S with the Lawson topology. The inverse image of $\uparrow a$ is closed for any a by Step 1. Let A be a nonempty Scott-closed set, and let $x \in \mathrm{cl}(A)$, the closure being taken in the given topology. By step 4 and its proof, for any $z \in D_x$, $\uparrow z$ is a neighborhood of x, and hence must contain some member of A. Since Scott-closed sets are lower, $z \in A$. Since A is Scott-closed, $x = \bigvee^{\uparrow} D_x \in A$. Hence A is equal to its closure, hence closed in S with the given topology. The identity is thus continuous from S with the given compact topology to S with the Hausdorff Lawson topology and is hence a homeomorphism. □

Theorem 1-6.11. *In a completely distributive lattice the Lawson topology, its dual, and the interval topology all agree, and with respect to this topology the lattice is a compact Hausdorff topological lattice with a basis of open sublattices.*

Proof. For $x \neq y$ in the completely distributive lattice L, by Theorem 1-3.12 (and its proof) there exists a map $\alpha\colon L \to [0,1]$ preserving arbitrary sups and infs and separating x and y, say $\alpha(x) < \alpha(y)$. Let $\alpha(x) < t < \alpha(y)$. Since α preserves arbitrary sups, $w = \bigvee \alpha^{-1}([0,t])$ satisfies $\alpha^{-1}([0,t]) = {\downarrow}\, w$. Thus $L \setminus {\downarrow}\, w$ is an open set in the interval topology containing y. Similarly $v = \bigwedge \alpha^{-1}([t,1])$ satisfies ${\uparrow}\, v = \bigwedge \alpha^{-1}([t,1])$ and $L \setminus {\uparrow}\, v = \alpha^{-1}([0,t)\,)$ is an open set in the interval topology containing x and disjoint from $L \setminus {\downarrow}\, w$. This shows that L is Hausdorff in the interval topology. Since the interval topology is a Hausdorff topology coarser than the compact Lawson topology, the two topologies must agree. Dually the interval topology agrees with the opposite of the Lawson topology.

If follows from Theorem 1-6.9 and its dual that L is a topological lattice with respect to the previous topologies. By Theorem 1-3.12 L embeds as a lattice and, by the previous paragraph, simultaneously topologically in $[0,1]^X$ for some X. Since $[0,1]^X$ has a basis of open sublattices, so does L. $\square$

Example 1-6.12. On the unit interval $[0,1]$ (with its usual order) the Scott topology agrees with the upper topology, $\sigma([0,1]) = \nu([0,1])$; the proper open subsets are the half-open intervals $(t,1], 0 \leq t \leq 1$. The Lawson topology agrees with the interval topology which is the usual compact Hausdorff topology on the unit interval. Using Proposition 1-4.9(ii) one can see that the Scott topology on the power lattice $[0,1]^X$ is the product topology of the Scott topology on the factors, and the Lawson topology is the usual product topology, and hence agrees with the interval topology.

In the following theorem, powers $[0,1]^X$ of the unit interval are considered with their interval(=product) topology.

Theorem 1-6.13.

(a) *A lattice L is continuous if and only if L is isomorphic* (*as a meet-semilattice*) *to a closed meet-subsemilattice containing the top element of some power $[0,1]^X$. The Lawson topology on L is the topology induced by the interval topology on $[0,1]^X$.*

(b) *A lattice L is completely distributive if and only if L is isomorphic* (*as a lattice*) *to a closed sublattice containing the top and bottom element of some power $[0,1]^X$ of the unit interval. And if this is the case, the Lawson topology $\lambda(L)$, the dual Lawson topology $\lambda(L^{\mathrm{op}})$ and the interval topology on L all agree and they are induced by the interval topology on $[0,1]^X$.*

Proof. Let L be a closed meet-subsemilattice of $[0,1]^X$ (up to an isomorphism). Then L is compact Hausdorff in the relative topology and inherits the property of being a topological semilattice with a base of subsemilattices.

By Theorem 1-6.10 the relative topology is the Lawson topology, and S is a bounded complete domain, and a continuous lattice if it has a largest element. Hence the inclusion map $j\colon L \to [0,1]^X$ is continuous in the Lawson topologies, so by Proposition 1-6.3 it preserves directed sups and nonempty infs. It follows that S is closed in $[0,1]^X$ with respect to nonempty infs and directed sups, and thus closed with respect to all infs if it contains the top element. If S is also a sup-subsemilattice containing the bottom element, then dually it will be closed under arbitrary sups, and hence be completely distributive by 1-3.7(ii). Hence by Theorem 1-6.11 the interval and dual Lawson topologies agree with the Lawson topology, which is the relative topology.

Conversely, let L be a continuous (resp. completely distributive) lattice. By Theorem 1-3.12, there is an injective map ι from L into some power $[0,1]^X$ of the unit interval preserving directed (resp. arbitrary) joins and arbitrary meets, and hence the bottom (resp. bottom and top) elements. This map is Lawson-continuous by Proposition 1-6.3, hence $\iota(L)$ is compact since the Lawson topology is compact on L, and thus $\iota(L)$ is closed in the Hausdorff space $[0,1]^X$. $\square$

1-7. Generation by irreducibles and primes

An element $a \neq \top$ of a lattice is *meet-irreducible* if $a = b \wedge c$ implies $a = b$ or $a = c$; dually $a \neq \bot$ is *join-irreducible* if $a = b \vee c$ implies $a = b$ or $a = c$ (Section I.6.3 of LTF). Similarly $a \neq \top$ is *meet-prime* or simply *prime* if $b \wedge c \leq a$ implies $b \leq a$ or $c \leq a$. *Join-primes* a.k.a. *coprimes* are defined dually. We note that every meet- resp. join-prime is meet- resp. join-irreducible. We note also that in the preceding definitions the conditions involving pairs can be replaced by finite sets, for example, p is coprime iff $p \leq \bigvee F$ for F finite implies $p \leq a$ for some $a \in F$. We can also replace finite sets by infinite sets, but this is a more restrictive property: an element $a \neq \top$ of a lattice is *completely meet-irreducible* if $a = \bigwedge A$ for any nonempty set A implies $a \in A$.

Remark 1-7.1. In a distributive lattice a is meet-irreducible iff it is prime and join-irreducible if and only if it is coprime. Indeed $b \wedge c \leq a$ implies $a = a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$. Hence if a is meet-irreducible, $a = a \vee b$ or $a = a \vee c$, i.e., $b \leq a$ or $c \leq a$.

Meet-irreducibles order generate in a continuous lattice. (See Exercise I.6.15 of LTF for the dual statement for finite lattices.)

Proposition 1-7.2. *Every element of a continuous lattice, resp. algebraic lattice, is the greatest lower bound of the meet-irreducible, resp. completely meet-irreducible, elements above it.*

Proof. The proposition is vacuously true for $\top$. Let $x \in L$, a continuous lattice and let $x < y$. By Proposition 1-4.3(ii) there exists an open filter V containing y and contained in the Scott open set $L \setminus \mathop{\downarrow} x$. Since $L \setminus V$ is

Scott-closed, by Zorn's lemma there exists w maximal in $L \setminus V$ such that $x \leq w$. Since F is a filter, in particular a meet-subsemilattice, it follows that w is a meet-irreducible element that is not above y. Since y was an arbitrary element strictly larger than x, the conclusion of the proposition follows.

In the case of an algebraic lattice we modify the proof by picking a compact element $k \leq y$, $k \not\leq x$, choosing $\Uparrow k = \uparrow k$ for the Scott-open filter, and choosing w maximal above x in the complement. If $w = \bigwedge A$ for $A \neq \emptyset$, then $w \in A$, for otherwise by maximality of w, $A \subseteq \uparrow k$ and hence k is a lower bound for A that is not below w. As previously, x is the infimum of such elements. □

From Remark 1-7.1 and Proposition 1-7.2 we immediately deduce a corollary:

Corollary 1-7.3. *In a distributive continuous lattice every element is a meet of prime elements.*

We have seen (Proposition 1-3.15) that every core compact space has as a lattice of open sets a distributive continuous lattice. The previous corollary provides the means for a converse construction. Each prime element p uniquely gives rise to a frame homomorphism (a map preserving arbitrary sups and finite infs) into the two-element lattice $\mathbf{2}$, namely the map sending $\downarrow p$ to 0 and its complement to 1. Furthermore, every frame homomorphism into $\mathbf{2}$ arises in this way (Exercise 1.23). Thus we may identify the prime elements of a distributive continuous lattice L with its *spectrum*, the $\mathbf{2}$-valued frame homomorphisms appropriately topologized. This turns out to be a locally compact sober space, and if one starts with a core compact space, then one obtains its sobrification as the spectrum of its lattice of open sets. All this leads to the Hofmann–Lawson duality between category of locally compact sober spaces with continuous maps on the one hand and category of distributive continuous lattices and frame homomorphisms on the other, the connecting functors being $X \to \mathcal{O}(X)$ and $L \to \mathrm{Spec}(L)$. See Sections 2 and 7 of Chapter 2 "Frames: Topology Without Points" by A. Pultr and J. Sichler for further details.

The following is another important consequence of Proposition 1-7.2.

Proposition 1-7.4. *The dual lattice of an algebraic lattice L is again an algebraic lattice if and only if L is join-continuous.*

Proof. Suppose L is an algebraic lattice and assume that its dual L^{op} is also an algebraic lattice. By Propositions 1-2.8 and 1-2.11, L^{op} is meet-continuous, hence L is join-continuous.

Conversely suppose that L is join-continuous. Then by Lemma 1-2.9 applied to the meet-continuous dual L^{op} every completely meet-irreducible element of L is compact in L^{op}. By Proposition 1-7.2 every element of L^{op} is a join of compact elements, i.e., L^{op} is an algebraic lattice. □

Lemma 1-7.5. *For a coprime element q of a complete lattice, $q \ll x$ implies $q \lll x$.*

Proof. Let $q \ll x$. For any nonempty A such that $x \leq \bigvee A$, we have $q \leq \bigvee F$ for some finite subset F of A, since such finite suprema form a directed set with supremum $\bigvee A$. But q coprime implies $q \leq a$ for some $a \in F \subseteq A$. Hence $q \lll x$. □

Proposition 1-7.6. *A continuous lattice is completely distributive if and only if every element is a supremum of coprimes.*

Proof. If L is completely distributive, then so is L^{op}, and one implication then follows from Corollary 1-7.3 (recalling that a completely distributive lattice is, in particular, a distributive continuous lattice).

Assume that L is a continuous lattice in which every element is a supremum of coprimes. Then any $x \neq \perp$ is the supremum of all $y \ll x$, and each such y is in turn a sup of coprimes. It follows that x is a sup of coprimes way-below it. But $q \lll x$ for every coprime $q \ll x$ by the preceding lemma, so $x = \bigvee\{z : z \lll x\}$. Hence L is completely distributive by Theorem 1-3.3. □

Corollary 1-7.7. *A distributive algebraic lattice is completely distributive if and only if it is join-continuous.*

Proof. If L is completely distributive, then it is a continuous lattice with respect to both the meet and join operations, and thus, in particular, join-continuous.

If L is join-continuous, then by Proposition 1-7.4, L^{op} is a distributive algebraic lattice, and hence meet-generated by primes (Corollary 1-7.3), i.e., L is join-generated by coprimes. Then by Proposition 1-7.6 L is completely distributive. □

We come now to a version of what is sometimes referred as the "Lemma on Primes," a basic result about prime elements in continuous lattices; see Section V-1, particularly Corollary V-1.2, of [104]. It is another illustration of the observation that compact sets behave like finite sets.

Lemma 1-7.8. *Let L be a continuous lattice, $p \in L$ a prime, and suppose that $K \subseteq L$ is compact in the Scott topology. Then $\bigwedge K \leq p$ implies $a \leq p$ for some $a \in K$ and, in particular, $p \in K$ provided that K is saturated.*

Proof. Suppose that $x \not\leq p$ for each $x \in K$. Then we may pick for each x an element $y_x \ll x$ such that $y_x \not\leq p$. The Scott-open sets $\twoheaduparrow y_x$, $x \in K$, cover K, and so by compactness $K \subseteq \uparrow F$ for some finite subset F of $\{y_x : x \in K\}$. Thus $\bigwedge F \leq \bigwedge K \leq p$ and, since p is a prime element, $a \leq p$ for some $a \in F$, a contradiction to the choice of the y_x. □

We denote by $\mathrm{CoPrime}(L)$ the set of coprime elements of L considered as a poset with the order induced from L.

Proposition 1-7.9. *For a completely distributive lattice L, the coprime elements form a continuous domain for the order induced from L and L is isomorphic to the lattice of Scott-closed subsets of* $\mathrm{CoPrime}(L)$.

Proof. It is straightforward to verify that a directed supremum of coprime elements is again coprime, so $\mathrm{CoPrime}(L)$ is a dcpo, indeed a sub-dcpo of L. We showed that $p = \bigvee\{q \in \mathrm{CoPrime}(L)\colon q \ll p\}$ for $p \in \mathrm{CoPrime}(L)$ in the proof of Proposition 1-7.6. Thus to show $\mathrm{CoPrime}(L)$ is a continuous domain, we need only show that this set of coprimes is directed.

For this, consider the closure A of the set $\{q \in \mathrm{CoPrime}(L)\colon q \ll p\}$ with respect to the Scott topology in L. It suffices to show $p \in A$, for then for any $q_1, q_2 \ll p$ the set $\{q \in \mathrm{CoPrime}(L)\colon q \ll p\}$ must meet the Scott-open neighborhood $\twoheaduparrow q_1 \cap \twoheaduparrow q_2$ of p. The set A is closed, hence compact, for the Lawson topology which equals the dual Lawson topology by Theorem 1-6.11. Hence, A is compact for the weaker dual Scott topology. Since $p \le \bigvee A$ according to the preceding paragraph, the dual version of the Lemma on Primes (1-7.8) implies that $p \in A$.

Since $\mathrm{CoPrime}(L)$ sup-generates and is a sub-dcpo, $x \mapsto {\downarrow} x \cap \mathrm{CoPrime}(L)$ is an injective map from L to the lattice of Scott-closed subsets of $\mathrm{CoPrime}(L)$. Let A be a Scott-closed subset of $\mathrm{CoPrime}(L)$ and $x = \bigvee A$. Clearly $A \subseteq {\downarrow} x$. Let $p \in {\downarrow} x \cap \mathrm{CoPrime}(L)$ and let $q \ll p$, $q \in \mathrm{CoPrime}(L)$. Then $q \lll p$ by Lemma 1-7.5, and hence $q \in {\downarrow} A = A$. We now use that A is Scott-closed and conclude that $p = \bigvee^{\uparrow}\{q \in \mathrm{CoPrime}(L)\colon q \ll p\} \in A$. This establishes that $x \mapsto {\downarrow} x \cap \mathrm{CoPrime}(L)$ is a bijection onto the lattice of Scott-closed sets, which is clearly order-preserving, hence a lattice isomorphism. □

The next proposition is a variant of the preceding one.

Proposition 1-7.10. *For any poset P, the lattice $\mathcal{A}(P)$ of all lower sets ordered by inclusion is prime algebraic and every prime algebraic lattice is isomorphic to the lattice $\mathcal{A}(P)$ of some poset P, namely the poset of all completely coprime elements.*

Proof. The principal ideals ${\downarrow} x$, $x \in P$, are easily seen to be completely coprime elements of the lattice $\mathcal{A}(P)$. As every lower set is a union of principal ideals, $\mathcal{A}(P)$ is prime algebraic. The proof of the converse follows along the lines of the proof of the preceding proposition. □

The lower sets of a poset form the closed sets of a topology, called the *Alexandroff topology*. One can restate Proposition 1-7.10 in terms of this topology so that it closely parallels Proposition 1-7.9.

Let us collect the various characterizations of completely distributive and prime algebraic lattices.

Remark 1-7.11. For a complete lattice L the following, along with their order duals, are all equivalent:

(a) L is completely distributive, that is, it satisfies the law (DD2).

(b) Every element x of L is a supremum of the elements $y \lll x$.

(c) L is continuous and each of its elements is a supremum of coprimes.

(d) L is isomorphic (as a lattice) to a closed sublattice of some power $[0,1]^X$ of the unit interval with the usual topology.

(e) L is isomorphic to the lattice of all closed subsets of some continuous domain P.

Similarly, the following are equivalent among themselves and to their order duals:

(a) L is algebraic and completely distributive.

(b) L is prime algebraic.

(c) L is distributive, algebraic, and join-continuous.

(d) L is isomorphic to a collection of subsets of some set X closed under arbitrary intersections and arbitrary unions.

(e) L is isomorphic (as a lattice) to a closed sublattice of some power $\mathbf{2}^X$ of the two element lattice $\mathbf{2}$ with the discrete topology.

(f) L is isomorphic to the collection $\mathcal{A}(X)$ of all lower sets of some partially ordered set X.

1-8. Fixed point theorems and domain equations

In this section we present two kinds of 'fixed point theorems' that are of great importance in the semantics of programming languages, although the first one is quite simple. Both theorems require continuity hypotheses.

Remember that a fixed point of a map f of a set X into itself is an element $x \in X$ such that $f(x) = x$. Tarski's fixed point theorem (Chapter I, Theorem 40 of LTF) asserts that every order-preserving map f from a complete lattice L into itself has a least fixed point.

Alternatively to the proof indicated in Chapter I, Exercise 3.76 of LTF, one can proceed by transfinite induction: Let $x_0 = \bot$ and $x_{\alpha+1} = f(x_\alpha)$ for successor ordinals, $x_\alpha = \sup_{\beta<\alpha} x_\beta$ for limit ordinals. Since $\bot \le f(\bot)$ and since f is order-preserving, this transfinite sequence is increasing. As the cardinality of this sequence is bounded by the cardinality of L, there is an ordinal α such that $x_\alpha = x_{\alpha+1} = f(x_\alpha)$. Thus, x_α is a fixed point of f.

It is the least fixed point. For if $f(x) = x$, then $x \geq \perp = x_0$ whence $x = f(x) \geq f(\perp) = x_1$ and, by transfinite induction, $x \geq x_\alpha$ for every ordinal α.

The above proof of Tarski's fixed point theorem uses transfinite induction, hence is non-constructive. If the endofunction f is Scott-continuous, then the above proof becomes constructive: Indeed, if we consider

$$x_\omega = \bigvee_{n\in\mathbb{N}}^{\uparrow} x_n = \bigvee_{n\in\mathbb{N}}^{\uparrow} f^n(\perp),$$

then $f(x_\omega) = f(\bigvee^{\uparrow}_{n\in\mathbb{N}} x_n) = \bigvee^{\uparrow}_{n\in\mathbb{N}} f(x_n) = \bigvee^{\uparrow}_{n\in\mathbb{N}} x_{n+1} = x_\omega$. We have Scott's Fixed Point Theorem:

Theorem 1-8.1. *Every Scott-continuous map f from a complete lattice L into itself has a least fixed point x_ω obtained as the supremum of the increasing sequence $x_0 = \perp,\ x_1 = f(x_0),\ x_2 = f(x_1), \dots$.*

Remark 1-8.2. One may notice that both Tarski's and Scott's fixed point theorems hold for arbitrary dcpos with bottom instead of complete lattices.

One can generalize both results in the following way: If f is an endofunction on a complete lattice (resp. dcpo) L and x_0 is any element of L such that $x_0 \leq f(x_0)$, then f has a least fixed point x above x_0. If f is Scott-continuous, this fixed point is the least upper bound of the increasing sequence $x_n = f^n(x_0)$, $n \in \mathbb{N}$.

There is another important 'fixed point theorem' on a higher level, where we replace the lattice L above by a category CLatt: The objects of this category are the complete lattices and the morphisms are the order-preserving functions between them.

Remark 1-8.3. Instead of the category CLatt of all complete lattices and order-preserving maps one may restrict to the category $\mathsf{ContLatt}$ of all continuous lattices and Scott-continuous maps. All the subsequent developments can be carried through in this subcategory. Similarly, one can use algebraic lattices or bounded complete domains and Scott-continuous maps.

We replace the endofunction f of L by an endofunctor F of the category CLatt. Such a functor assigns to every complete lattice D a complete lattice $F(D)$ and to every order-preserving function $g\colon D_1 \to D_2$ between complete lattices D_1 and D_2 an order-preserving function $F(g)\colon F(D_1) \to F(D_2)$ in such a way that

$$F(\mathrm{id}_D) = \mathrm{id}_{F(D)} \text{ and } F(h \circ g) = F(h) \circ F(g)$$

for arbitrary complete lattices D, D_1, D_2 and arbitrary order-preserving functions $g\colon D_1 \to D_2, h\colon D_2 \to D_3$.

We ask the question whether such an endofunctor F has a 'fixed point' in the sense that there is a complete lattice D such that $F(D)$ is isomorphic to D. We then say that D is a solution of the *domain equation*

$$F(D) \cong D.$$

In order to find a solution to the domain equation we try to mimic the procedure applied for the fixed point theorems of endofunctions on complete lattices. For a given endofunctor F we form the sequence of complete lattices

$$D_0 = \{\bot\},\ D_1 = F(D_0),\ D_2 = F(D_1),\ \ldots.$$

In order to proceed, we need a substitute for the order relation on L and for the $\bigvee^{\uparrow}$-operation on increasing sequences.

According to the Appendix 1-9.9, 1-9.10 a function $p\colon D_1 \to D_0$ between complete lattices D_1 and D_0 is a *projection* if p is surjective and preserves arbitrary meets. Every projection p has a lower adjoint $p_*\colon D_0 \to D_1$ defined by $p_*(y) = \min p^{-1}(\uparrow y)$ for all $y \in D_0$ and this lower adjoint is injective and preserves arbitrary joins. Conversely, every injective map $e\colon D_0 \to D_1$ preserving arbitrary joins has an upper adjoint $e^*\colon D_1 \to D_0$ defined by $e^*(x) = \max e^{-1}(\downarrow x)$ which is a projection. We say that (p, e) is a *projection-embedding pair*, if $e = p_*$ or, equivalently, if $p = e^*$. According to the remarks following 1-9.9, a projection-embedding pair (p, e) is characterized to be a pair of order-preserving functions satisfying

$$p \circ e \leq \mathrm{id}_{D_0}, \quad \text{and} \quad e \circ p = \mathrm{id}_{D_1}.$$

The notion of projection-embedding pairs gives rise to a preorder on the collection of all complete lattices: $D_0 \sqsubseteq D_1$ if there is a projection $p\colon D_1 \to D_0$ or, equivalently, if there is a meet preserving injection $e\colon D_0 \to D_1$.

An endofunctor on the category CLatt induces a map $g \mapsto F(g)$ from the set $[D_0 \to_o D_1]$ of order-preserving functions $f\colon D_0 \to D_1$ to the set $[F(D_0) \to_o F(D_1)]$ of order-preserving functions $h\colon F(D_1) \to F(D_2)$. These hom-sets are complete lattices with respect to the pointwise order $f \leq g$ if $f(x) \leq g(x)$ for all x. Meets and joins are formed pointwise.

Definition 1-8.4. An endofunctor F on CLatt is said to be *locally order-preserving* (resp. *locally Scott-continuous*), if the map

$$g \mapsto F(g)\colon [D_0 \to_o D_1] \to [F(D_0) \to_o F(D_1)]$$

is order-preserving (resp. Scott-continuous) for all complete lattices D_0, D_1.

A locally order-preserving endofunctor F preserves the preorder $\sqsubseteq$:

Lemma 1-8.5. *Let F be a locally order-preserving endofunctor on the category* CLatt*. Applying F to a projection-embedding pair*

$$D_0 \underset{e}{\overset{p}{\rightleftarrows}} D_1$$

yields a projection-embedding pair

$$F(D_0) \underset{F(e)}{\overset{F(p)}{\rightleftarrows}} F(D_1).$$

Proof. Indeed, since $p \circ e = \mathrm{id}_{D_0}$, we conclude that $F(p) \circ F(e) = F(p \circ e) = F(\mathrm{id}_{D_0}) = \mathrm{id}_{F(D_0)}$ and, since $e \circ p \leq \mathrm{id}_{D_1}$, we also have $F(e) \circ F(p) = F(e \circ p) \leq F(\mathrm{id}_{D_1}) = \mathrm{id}_{F(D_1)}$ where we have used that F is locally order-preserving. □

We now construct a 'join' for a sequence of complete lattices that is increasing for the preorder $\sqsubseteq$. For the general case of a 'join' for a family of dcpos, directed under the relation $\sqsubseteq$, one may consult [104, Chapter IV].

Consider a sequence of complete lattices and projection-embedding pairs:

$$D_0 \underset{e_0}{\overset{p_0}{\rightleftarrows}} D_1 \underset{e_1}{\overset{p_1}{\rightleftarrows}} D_2 \underset{e_2}{\overset{p_2}{\rightleftarrows}} D_3 \underset{e_3}{\overset{p_3}{\rightleftarrows}} \cdots$$

We define D_ω as a subset of the direct product of the D_n:

$$D_\omega = \Big\{ x = (x_n)_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} D_n \mid p_n(x_{n+1}) = x_n \text{ for all } n \in \mathbb{N} \Big\}.$$

We also define maps $p_n^\omega \colon D_\omega \to D_n$ and $e_n^\omega \colon D_n \to D_\omega$ by

$$p_n^\omega(x) = x_n,$$

$$e_n^\omega(x_n) = (x_i)_{i\in\mathbb{N}}, \text{ where } x_i = \begin{cases} p_i \circ \cdots \circ p_{n-1}(x_n) & \text{for } i < n, \\ x_n & \text{for } i = n, \\ e_{i-1} \circ \cdots \circ e_n(x_n) & \text{for } i > n. \end{cases}$$

The subset D_ω is closed in $\prod_{n\in\mathbb{N}} D_n$ under arbitrary meets, since the p_n preserve arbitrary meets. It follows that D_ω is a complete lattice. From the definitions one easily deduces the following properties:

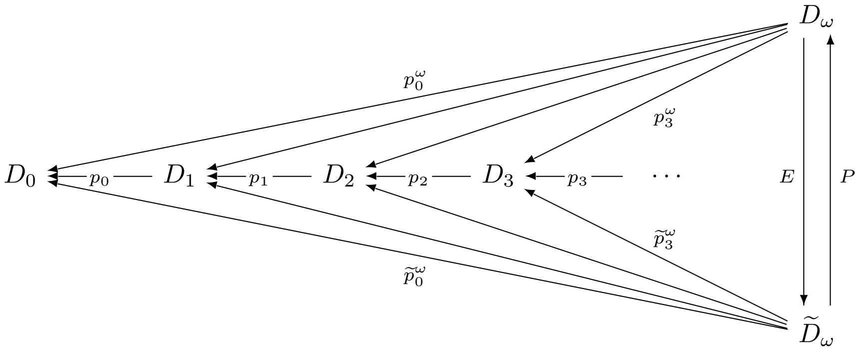
(A) $$p_n^\omega \circ e_n^\omega = \mathrm{id}_{D_n} \quad \text{and} \quad e_n^\omega \circ p_n^\omega \leq \mathrm{id}_{D_\omega},$$

that is, p_n^ω and e_n^ω form a projection-embedding pair. Moreover,

(B) $$p_n^\omega = p_n \circ p_{n+1}^\omega, \quad e_n^\omega = e_{n+1}^\omega \circ e_n.$$

This means that the upper half of the diagram below commutes. (In this diagram only projections are presented. But remember that to each projection there corresponds the adjoint embedding e in the other direction.) We deduce $e^\omega_n \circ p^\omega_n = e^\omega_{n+1} \circ (e_n \circ p_n) \circ p^\omega_{n+1} \leq e^\omega_{n+1} \circ \mathrm{id}_{D_{n+1}} p^\omega_{n+1} = e^\omega_{n+1} \circ^\omega_{n+1}$, that is, the functions $e^\omega_n \circ p^\omega_n \colon D_\omega \to D_\omega$ form an increasing sequence, and moreover

$$\text{(C)} \qquad \bigvee_{n\in\mathbb{N}}^{\uparrow} e^\omega_n \circ p^\omega_n = \mathrm{id}_{D_\omega}.$$



We now justify that D_ω is a 'least upper bound' of the sequence $(D_n)_n$. For this we suppose that $\widetilde{D}_\omega$ is a complete lattice together with projection-embedding pairs $(\widetilde{p}^\omega_n, \widetilde{e}^\omega_n)$ between $\widetilde{D}_\omega$ and D_n such that (B) holds for $(\widetilde{p}^\omega_n, \widetilde{e}^\omega_n)$ instead of (p^ω_n, e^ω_n). This means that the lower half of the diagram above commutes. We claim:

Lemma 1-8.6. *There is a natural projection-embedding pair (P, E) between $\widetilde{D}_\omega$ and D_ω such that $p^\omega_n \circ P = \widetilde{p}^\omega_n$ and dually $E \circ e^\omega_n = \widetilde{e}^\omega_n$. Moreover, P and E are mutually inverse lattice isomorphisms if and only if we have equation*

$$\text{(C}'\text{)} \qquad \bigvee_{n\in\mathbb{N}}^{\uparrow} \widetilde{e}^\omega_n \circ \widetilde{p}^\omega_n = \mathrm{id}_{\widetilde{D}_\omega}.$$

Proof. Consider the maps $E_n = \widetilde{e}^\omega_n \circ p^\omega_n$ from D_ω to $\widetilde{D}_\omega$ and $P_n = e^\omega_n \circ \widetilde{p}^\omega_n$ from $\widetilde{D}_\omega$ to D_ω.

We first check that $E_n \leq E_{n+1}$ and $P_n \leq P_{n+1}$. Indeed, using (B) and the fact that (p_n, e_n) is a projection-embedding pair, we have $E_n = \widetilde{e}^\omega_n \circ p^\omega_n = \widetilde{e}^\omega_{n+1} \circ e_n \circ p_n \circ p^\omega_{n+1} \leq \widetilde{e}_{n+1} \circ \mathrm{id}_{D_{n+1}} \circ p^\omega_{n+1} = E_{n+1}$. The proof for the second inequality is similar.

We form the joins $E = \bigvee^{\uparrow}_n E_n \colon D_\omega \to \widetilde{D}_\omega$ and $P = \bigvee^{\uparrow}_n P_n \colon \widetilde{D}_\omega \to D_\omega$ of the increasing sequences of functions (E_n) and (P_n). Clearly, E and P are order-preserving. We show that $E \circ P \leq \mathrm{id}_{\widetilde{D}_\omega}$:

$$\begin{aligned} g \circ h &= \Big(\bigvee_{n\in\mathbb{N}}^{\uparrow} \widetilde{e}^\omega_n \circ p^\omega_n\Big) \circ \Big(\bigvee_{n\in\mathbb{N}}^{\uparrow} e^\omega_n) \circ \widetilde{p}^\omega_n\Big) = \bigvee_{n\in\mathbb{N}}^{\uparrow} \big(\widetilde{e}^\omega_n \circ p^\omega_n \circ e^\omega_n \circ \widetilde{p}^\omega_n\big) \\ &= \bigvee_{n\in\mathbb{N}}^{\uparrow} \widetilde{e}^\omega_n \circ \mathrm{id}_{D_n} \circ \widetilde{p}^\omega_n = \bigvee_{n\in\mathbb{N}}^{\uparrow} \widetilde{e}^\omega_n \circ \widetilde{p}^\omega_n \leq \mathrm{id}_{\widetilde{D}_\omega} \end{aligned}$$

Equality holds if and only if equation (C') is satisfied. An analogous argument shows that $P \circ E = \mathrm{id}_{D_\omega}$. (Because equation (C) holds, we always have equality in this second case!). Thus, P and E form a projection-embedding pair. They are mutually inverse lattice isomorphisms if and only if equation (C') holds. □

Remark 1-8.7. The preceding lemma in the language of category theory states that D_ω is the limit of the inverse system $(D_n, p_n)_n$ and, at the same time, the colimit of the direct system $(D_n, e_n)_n$ in the category with objects complete lattices and morphisms projection-embedding pairs. This fact is known under the name of limit-colimit-coincidence. This justifies the notation

$$D_\omega = \lim_n (D_n, p_n, e_n).$$

Remark 1-8.8. (a) Suppose that all the lattices D_n are continuous (resp. algebraic) and all the projections p_n are Scott-continuous. Then the limit D_ω is a continuous (resp. an algebraic lattice), too. Indeed, by Corollary 1-3.7(i), the product lattice $\prod_n D_n$ is continuous (resp. algebraic) and D_ω is closed in this product for arbitrary meets and directed joins (use 1-3.7(ii)). Moreover the projections p_n^ω are Scott-continuous and, hence, the embeddings e_n^ω preserve the way-below relation (resp. compact elements) by Proposition 1-9.11 and Corollary 1-9.12.

(b) The elements $e_n^\omega(x_n), x_n \in D_n, n \in \mathbb{N}$, form a basis of D_ω. In the algebraic case, the compact elements of D_ω are the images $e_n^\omega(k_n)$, where the k_n range over the compact elements of D_n and n ranges over $\mathbb{N}$.

(c) If all the lattices D_n are completely distributive, then the product $\prod_n D_n$ is completely distributive, too, and if in addition the projections p_n preserve arbitrary joins, then the limit D_ω is closed in $\prod_n D_n$ for arbitrary meets and arbitrary joins, hence a completely distributive lattice, too.

Proposition 1-8.9. *Let F be an endofunctor of the category* CLatt. *If F is locally Scott-continuous, then F is a continuous functor in the sense that it preserves limits of sequences of projection-embedding pairs:*

$$F(\lim_n (D_n, p_n, e_n)) \cong \lim_n (F(D_n), F(p_n), F(e_n)).$$

Proof. For the limit D_ω we have the upper half of the above commuting diagram of projection-embedding pairs (p_n, e_n) and (p_n^ω, e_n^ω) satisfying (C). We apply the functor F to this diagram. As we suppose F to be locally order-preserving, we obtain a commuting diagram of projection-embedding pairs $(F(p_n), F(e_n))$ between $F(D_n)$ and $F(D_{n+1})$ and $(F(p_n^\omega), F(e_n^\omega))$ between $F(D_n)$ and $F(D_\omega)$ by Lemma 1-8.5. Moreover, equation (C) remains valid, since F is supposed to be locally Scott-continuous. The preceding Lemma 1-8.6 allows us to conclude that $F(D_\omega)$ is (isomorphic to) the limit of the sequence $(F(D_n), F(p_n), F(e_n))_n$. □

We apply the proposition to the following situation, where F is supposed to be a locally Scott-continuous endofunctor of the category CLatt:

Let $D_0 = \{\perp\}$ and $D_1 = F(D_0)$. There is an embedding $e_0 \colon D_0 \to D_1$ mapping $\perp$ to the bottom element of D_1. Together with the unique map $p_0 \colon D_1 \to D_0$, we have a (Scott-continuous) projection-embedding pair

$$D_0 = \{\perp\} \underset{e_0}{\overset{p_0}{\rightleftarrows}} D_1 = F(\{\perp\}).$$

By defining recursively

$$D_n = F(D_{n-1}), \quad p_n = F(p_{n-1}), \quad e_n = F(e_{n-1})$$

for $n > 0$ we obtain a sequence of projection-embedding pairs

$$\{\perp\} \underset{e_0}{\overset{p_0}{\rightleftarrows}} F(\{\perp\}) \underset{e_1}{\overset{p_1}{\rightleftarrows}} F^2(\{\perp\}) D_2 \underset{e_2}{\overset{p_2}{\rightleftarrows}} F^3(\{\perp\}) \underset{e_3}{\overset{p_3}{\rightleftarrows}} \quad \ldots\ldots$$

We form the limit D_ω of this sequence of projection-embedding pairs. Under the hypothesis that F is locally Scott-continuous, Proposition 1-8.9 tells us that

$$\begin{aligned} F(D_\omega) &= F(\lim_n(F^n(\{\perp\}), F^n(p_0), F^n(e_0))) \\ &\cong \lim_n(F^{n+1}(\{\perp\}), F^{n+1}(p_0), F^{n+1}(e_0)) = D_\omega. \end{aligned}$$

We have shown:

Theorem 1-8.10. *If F is a locally Scott-continuous endofunctor on the category CLatt of complete lattices and order-preserving maps, the domain equation $F(D) \cong D$ has a solution D_ω constructed as the limit of the sequence $\{\perp\} \sqsubseteq F(\{\perp\}) \sqsubseteq F^2(\{\perp\}) \sqsubseteq \cdots$ as above.*

One should observe the similarity of the proof of this theorem with the proof of the fixed point Theorem 1-8.1.

Remark 1-8.11. (a) The solution D_ω of the domain equation $F(D) \cong D$ given in the preceding theorem is minimal (or *initial*) in the sense that, for every other solution D there is a projection $p \colon D \to D_\omega$; see Exercise 1.21.

(b) If F is a locally Scott-continuous endofunction on the full subcategory of continuous lattices and Scott-continuous maps, then the solution D_ω of the domain equation $F(D) \cong D$ is a continuous lattice, too (use Remark 1-8.8).

(c) The solution D_ω may be trivial, for example, if $F(\{\perp\}) = \{\perp\}$. We may find other solutions by choosing any complete lattice D_0 with the property that there is a projection-embedding pair

$$D_0 \underset{e_0}{\overset{p_0}{\rightleftarrows}} F(D_0)$$

and apply the construction above to this projection-embedding pair instead of $D_0 = \{\perp\}, D_1 = F(\{\perp\})$. Compare Remark 1-8.2.

The first example of a domain equation considered in the literature was

$$[D \to D] \cong D.$$

This domain equation only makes sense in a category with an internal Hom-functor, that is, in a category in which the set of all homomorphisms from one object to another can be enriched in such a way that it becomes itself an object of the category. Nontrivial solutions for this equation yield models for the untyped lambda-calculus, provided that one works in a Cartesian closed category. It was D. S. Scott (see [639] in LTF) who constructed the first model for the untyped lambda-calculus by constructing nontrivial solutions of the above domain equation in the category of continuous lattices, which is Cartesian closed by 1-5.4. In fact, he introduced continuous lattices for this purpose.

Let us pause for a moment in order to realize that the domain equation $[D \to D] \cong D$ has no non-singleton solution in all of the standard categories that one encounters in mathematics – even when they have an internal Hom-functor – as, for example, (1) the category of sets and arbitrary functions (the cardinality of the set of functions from D to D is always bigger than the cardinality of D except for the singleton set); (2) the category of partially ordered sets and order-preserving functions (see [58]); (3) the category of compact Hausdorff spaces and continuous functions (see [198]), (4) the category of Abelian groups and group homomorphisms. It was Scott's merit to exhibit appropriate categories in the world of directed complete posets.

The general construction leading to Theorem 1-8.10 cannot be used for solving the domain equation $[D \to D] \cong D$, since the Hom-functor is covariant in the second, but contravariant in the first argument. One can extend the procedure leading to Theorem 1-8.10 to bifunctors that are contravariant in the first and covariant in the second argument (see, e.g., [104, Chapter IV]). In Exercise 1.22 we indicate how to modify the above procedure in order to solve the domain equation $[D \to D] \cong D$.

1-9. Appendix: Galois adjunctions

A basic notion in category theory is that of a pair of *adjoint functors* between two categories. A partially ordered set S can be considered to be a category: The objects are the elements of S and there is one (and only one) morphism between two elements x, y if and only if $x \leq y$. If we specialize the notion of adjoint functors to this situation we arrive at the following notion:

Definition 1-9.1. A pair (g, d) of maps $g\colon S \to T$, $d\colon T \to S$ is called a *Galois adjunction* between the posets S and T provided that

(1) for all $s \in S$ and $t \in T$, we have $d(t) \leq s \Leftrightarrow t \leq g(s)$.

In an adjunction (g, d), we say that g is the *upper adjoint* of d and d the *lower adjoint* of g. We also say that a map g from a poset S to a poset T is an upper adjoint if there is a map $d\colon T \to S$ such that (g, d) is a Galois adjunction. In a similar way we speak of d as a *lower adjoint*.

Remark 1-9.2. Note that if the orders in S and T are reversed, then the upper adjoint g becomes a lower adjoint and the lower adjoint d an upper adjoint. This observation gives rise to a duality for the theory of Galois adjunctions.

The following observation is crucial.

Lemma 1-9.3. *An upper adjoint preserves arbitrary meets and a lower adjoint preserves arbitrary joins. In particular, lower and upper adjoints are order-preserving.*

Proof. Let (g, d) be a Galois adjunction between posets S and T. Note first that g is order-preserving since for $b \le a$

$$g(b) \le g(b) \Rightarrow dg(b) \le b \le a \Rightarrow g(b) \le g(a).$$

Thus for any subset X of S with greatest lower bound $\bigwedge X$, $g(\bigwedge X) \le g(x)$ for all $x \in X$. That $g(\bigwedge X)$ is the least upper bound of $g(X)$ then follows from the observation

$$\begin{array}{rcll} t \le g(\bigwedge X) & \iff & d(t) \le \bigwedge X & \text{by (1)} \\ & \iff & d(t) \le x \text{ for all } x \in X & \\ & \iff & t \le g(x) \text{ for all } x \in X & \text{by (1).} \end{array}$$

Applying these results to the dual situation, d preserves arbitrary meets if the orders are reversed, i.e., d preserves arbitrary joins and, hence, is order-preserving, too. □

Often Galois adjunctions are called *isotone Galois connections.* In fact, the concept of Galois adjunctions is much older than category theory. In the contravariant form of a pair of functions (g, d) with the property that $g(s) \ge t \iff d(t) \ge s$ they have been known under the name of *Galois connections* for a long time. In order to distinguish clearly the contravariant from the covariant case, we prefer to use the word *adjunction* and we reserve the word *connection* for the classical contravariant situation.

Galois adjunctions are used in several contexts in this volume, in particular in Section 1-8 on domain equations in this chapter. Since they are not directly treated elsewhere in this volume, we collect some relevant information about Galois adjunctions in this appendix.

As is the case for pairs of adjoint functors, there are other characterizations of adjoint pairs of maps.

Proposition 1-9.4. *Let S and T be two posets. A pair (g, d) of functions $g\colon S \to T$ and $d\colon T \to S$ is a Galois adjunction if and only if*

(2) *both g and d are order-preserving and*

(3) $d \circ g \leq \mathrm{id}_S$ *and* $\mathrm{id}_T \leq g \circ d$.

Proof. For a Galois adjunction (g, d), both maps g and d are order-preserving by Lemma 1-9.3 and, by (1), $d(t) \leq d(t)$ implies $t \leq gd(t)$, and $g(s) \leq g(s)$ implies $dg(s) \leq s$. Conversely if $d(t) \leq s$, then $t \leq gd(t) \leq g(s)$, where the first inequality holds by (3) and the second by (2). Similarly $t \leq g(s)$ implies $d(t) \leq dg(s) \leq s$. □

In Chapter I, Definition 31 of LTF, a *Galois connection* between posets S, T is a pair of order-reversing maps $g\colon S \to T$, $d\colon T \to S$ such that $g \circ d \geq \mathrm{id}_T$ and $d \circ g \geq \mathrm{id}_S$. The approaches via connections and adjunctions are equivalent in the sense that if the order on S is reversed, then the Galois connection (g, d) satisfies (2) and (3) and hence is converted to a Galois adjunction and vice-versa.

Remark 1-9.5. Let (g, d) be a Galois adjunction between posets S and T. Using (3), one has $d = \mathrm{id}_S \circ d \geq (d \circ g) \circ d = d \circ (g \circ d) \geq d \circ \mathrm{id}_T = d$, whence

(4) $d \circ g \circ d = d$ and, similarly, $g \circ d \circ g = g$.

It follows that

(5) $(d \circ g) \circ (d \circ g) = d \circ g$ and $(g \circ d) \circ (g \circ d) = g \circ d$,

that is, $d \circ g$ and $g \circ d$ are order retractions. As $\mathrm{id}_T \leq g \circ d$, the latter map is a closure operator on T (see Chapter I, Definition 26 of LTF) and, similarly, $d \circ g$ is a "kernel operator" on S.

We want to characterize those functions between posets that have a lower or an upper adjoint.

Proposition 1-9.6. *Let S and T be posets.*

(a) *If (g, d) is a Galois adjunction between S and T, then $d(t)$ is the least element of the preimage $g^{-1}(\uparrow t) = \{s \in S \mid g(s) \geq t\}$ for every $t \in T$ and $g(s)$ is the greatest element of $d^{-1}(\downarrow s)$ for every $s \in S$. In particular, the lower adjoint d is uniquely determined by the upper adjoint g and vice-versa.*

(b) *If, conversely, a map $g\colon S \to T$ has the property that, for all $t \in T$, the preimage $g^{-1}(\uparrow t) = \{s \in S \mid g(s) \geq t\}$ has a least element, then g has a lower adjoint $d\colon T \to S$, given by*

$$d(t) = \min g^{-1}(\uparrow t).$$

Similarly, if $d\colon T \to S$ is a map such that $d^{-1}(\downarrow s)$ has a greatest element for every $s \in S$, then d has an upper adjoint $g\colon S \to T$, given by

$$g(s) = \max d^{-1}(\downarrow s).$$

Proof. (a) Suppose that (g, d) is a Galois adjunction between S and T. By Definition 1-9.1, $g(s) \geq t$ iff $s \geq d(t)$. This is equivalent to stating that $d(t)$ is the greatest lower bound of $g^{-1}(\uparrow t)$.

(b) If $g^{-1}(\uparrow t)$ has a least element, call it $d(t)$, then $g(s) \geq t$ iff $s \geq d(t)$. Thus, if $g^{-1}(\uparrow t)$ has a least element for every $t \in T$, then g has a lower adjoint d given by $d(t) = \min g^{-1}(\uparrow t)$. □

In the presence of completeness the existence of a lower adjoint is equivalent to the preservation of arbitrary meets.

Proposition 1-9.7. *Let S and T be complete lattices. A function $g\colon S \to T$ has a lower adjoint $d\colon T \to S$ if and only if g preserves arbitrary meets, and in this case the lower adjoint d is given by*

$$d(t) = \bigwedge g^{-1}(\uparrow t) = \min g^{-1}(\uparrow t).$$

Similarly, a function $d\colon T \to S$ has an upper adjoint g if and only if d preserves arbitrary joins, and in this case the upper adjoint g is given by

$$g(s) = \bigvee d^{-1}(\downarrow s) = \max d^{-1}(\downarrow s).$$

Proof. If g has a lower adjoint, g preserves meets by Lemma 1-9.3. Suppose conversely that g preserves meets. Then, for every $t \in T$, we have $g(\bigwedge g^{-1}(\uparrow t)) = \bigwedge g(g^{-1}(\uparrow t)) \geq t$, since $g(g^{-1}(\uparrow t)) \subseteq \uparrow t$. Thus, the set $g^{-1}(\uparrow t)$ has a least element. By Proposition 1-9.6, g has a lower adjoint d defined by $d(t) = \min g^{-1}(\uparrow t)$. □

In adjunctions, injective and surjective maps are paired off as follows:

Proposition 1-9.8. *For an adjunction (g, d) between posets S and T, the following conditions are equivalent:*

(a) *g is surjective,* (b) $g \circ d = \mathrm{id}_T$, (c) *$d$ is injective.*

Likewise, the following statements are equivalent:

(a*) *g is injective,* (b*) $d \circ g = \mathrm{id}_S$, (c*) *$d$ is surjective.*

Proof. It is clear that (b) implies (a) and (c). From property (4) in Remark 1-9.5 it follows that both (c) and (a) imply (b). The equivalence of (a*), (b*), (c*) is proved in the same way. □

Definition 1-9.9. An order-preserving map $p\colon S \to T$ of posets is called a *projection* if it is surjective and has a lower adjoint e.

The lower adjoint e of a projection p is order-preserving and injective. The pair (p, e) is also called a *projection-embedding pair*. As such it is also characterized as a pair of order-preserving functions such that $p \circ e = \mathrm{id}_S$ and $e \circ p \leq \mathrm{id}_T$. As a map between complete lattices has a lower adjoint if and only if it preserves arbitrary meets, we have:

Corollary 1-9.10. *A map p from a complete lattice S to a complete lattice T is a projection if and only if p is surjective and preserves arbitrary meets. A map e from T to S is the lower adjoint of a projection if and only if e is injective and preserves arbitrary sups.*

Warning. This terminology deviates from that used in [104], where 'projection' is used for 'retraction operators' and 'kernel operator' for 'projection'.

Let us now consider adjunctions with respect to Scott continuity. As a lower adjoint preserves joins (see Lemma 1-9.3, in particular directed joins), it is always Scott-continuous. Let us add Scott continuity to the properties of an upper adjoint.

Proposition 1-9.11. *Let (g, d) be a Galois adjunction between posets S and T. Suppose that T is a continuous poset. Then g is Scott-continuous if and only if d preserves the way-below relation in the sense that $t \ll t' \implies d(t) \ll d(t')$.*

Proof. Consider first the situation where (g, d) is a Galois adjunction for which the upper adjoint $g\colon S \to T$ is Scott-continuous. Let $t \ll t'$ for elements t, t' in T. In order to show that $d(t) \ll d(t')$, consider any directed family of elements s_i in S such that $d(t') \le \bigvee^{\uparrow}_i s_i$. Then $t' \le g(d(t')) \le g(\bigvee^{\uparrow}_i s_i) = \bigvee^{\uparrow}_i g(s_i)$ by the Scott continuity of g. As $t \ll t'$, there is an index i such that $t \le g(s_i)$. Then $d(t) \le d(g(s_i)) \le s_i$.

Conversely, let us suppose that d preserves the way-below relation. Take any directed family s_i in S which has a least upper bound s in S. We show that $g(s)$ is the least upper bound of the family $g(s_i)$. Since g is order-preserving, we have $g(s_i) \le g(s)$ for all i. Choose any element $t \ll g(s)$ in T. Then $d(t) \ll dg(s)$, since d preserves the way-below relation. Since $d(g(s)) \le s = \bigvee^{\uparrow}_i s_i$, there is an index i such that $d(t) \le s_i$, whence $t \le g(s_i)$. Since $g(s)$ is the join of the $t \ll g(s)$, we conclude that $g(s) = \bigvee_i g(s_i)$. □

As the compact elements are those satisfying $k \ll k$ we have the following special case of the previous theorem:

Corollary 1-9.12. *Let (g, d) be an adjunction between posets S and T. Suppose that T is an algebraic poset. Then the map g is Scott-continuous if and only if $d(k)$ is a compact element of S for every compact element k of T.*

1-10. Exercises

1.1. In an algebraic lattice L (or even compactly generated poset) the subset K of compact elements is the 'canonical' basis in the sense that K is a basis and any basis of L contains K.

1.2. Recall that an ideal of a poset Q is a nonempty directed lower set. The *ideal completion* $\mathrm{Id}(Q)$ of Q consists of all ideals of Q ordered by inclusion. Show that $\mathrm{Id}(Q)$ is a compactly generated directed complete poset, that $j\colon Q \to \mathrm{Id}(Q)$ defined by $j(x) = {\downarrow}x$ is an order embedding, and that $\mathrm{Id}(Q)$ is an algebraic lattice if and only if Q is a join-semilattice with least element.

1.3. (i) In a lattice L show that if $y \ll x$ and p is a coprime below y, then $p \lll x$. (ii) If L is distributive and meet-continuous and p is completely join-irreducible, then $p \lll p$.

1.4. For a complete lattice M, show that the following properties are equivalent:

(i) M is a continuous, resp. completely distributive, lattice.

(ii) For every nonempty family of ideals, resp. lower sets, L_i, $i \in I$,

$$\sup \bigcap_i L_i = \inf_i \sup L_i.$$

(iii) For every nonempty family of directed, resp. arbitrary, subsets,

$$\sup \bigcap_i {\downarrow} A_i = \inf_i \sup A_i.$$

(iv) The following map preserves arbitrary meets:

$$L \mapsto \sup L\colon\ \mathrm{Id}(M) \to M \qquad \text{resp. } L \mapsto \sup L\colon \mathcal{A}(M) \to M.$$

1.5. Let X be a topological space. A subset S of X is called *saturated* if it is the intersection of a family of open sets.

(i) Show that the preorder of specialization (see Definition 1-3.16) is indeed reflexive and transitive, i.e., a preorder.

(ii) Show that the space X is T_0 if and only if the preorder of specialization is an order.

(iii) Show that $\mathrm{cl}\{x\} = {\downarrow}x$, that latter taken in the (pre)order of specialization.

(iv) Show that ${\downarrow}A = A$ if and only if A is a union of closed sets and ${\uparrow}B = B$ if and only if B is a saturated set.

(v) For $A \subseteq X$, show that $\mathrm{sat}(A)$, by definition the intersection of all open sets containing A, is the smallest saturated set containing A, called its *saturation*.

(vi) Show ${\uparrow}x = \mathrm{sat}\{x\}$.

1.6. Show that the union of two Scott-closed sets is again closed. Since the Scott-closed sets are immediately seen to be closed under arbitrary intersection, it follows that they form the closed sets for a topology.

1.7. Show that a topology on an ordered set P is order consistent if and only if $\downarrow x = \mathrm{cl}\{x\}$ for all x (see the observation following Remark 1-4.2).

1.8. Show that if D is a directed subset of an ordered set P with supremum e, then $\downarrow e$ is the closure of D in the Scott topology, or, more generally, in any order consistent topology contained in the Scott topology.

1.9. Show that the Scott topology is order consistent and is the finest order consistent topology on an ordered set such that every directed set having a supremum converges to its supremum.

1.10. Show that an element in $\prod_{i\in I} L_i$ of algebraic lattices is compact if and only if all but finitely many coordinates are $\perp$ and the remaining coordinates are compact. From this deduce that the product is algebraic. (Note that the result generalizes directly to compactly generated dcpos.)

1.11. For dcpos X, Y, Z show that $F\colon X \times Y \to Z$ is Scott-continuous if and only if $\widetilde{F}\colon X \to Y^Z$ is. (Hint: See the developments preceding Proposition 1-5.1.)

1.12. Let X be a topological space and L a bounded complete domain. Show that every Scott-continuous function $f\colon Y \to L$ defined on a subspace Y of X can be extended to a Scott-continuous function $g\colon \mathrm{cl}(Y) \to L$, defined on the closure of Y in X. (Hint: See Lemma 1-5.5.)

1.13. Let X be a core compact space and L a continuous lattice. Let $K \subseteq [X \to L]$ be a Scott-compact set of Scott-continuous functions from X to L. Show that the pointwise meet $f(x) = \bigwedge_{g\in K} g(x)$ is Scott-continuous.

1.14. Show that the category of continuous meet-semilattices and Scott-continuous functions is Cartesian closed.

1.15. For a topological space X and a complete lattice L, show that the directed complete poset $[X \to L]$ of Scott-continuous functions $f\colon X \to L$ is an algebraic lattice if and only if both the lattice $\mathcal{O}(X)$ and the lattice L are algebraic. Conclude that the category of algebraic lattices and Scott-continuous functions is Cartesian closed.

1.16. Characterize those topological spaces for which the lattice of open subsets is algebraic. (Hint: Compare with Theorem 1-5.2.)

1.17. Supply the details of the proof of Lemma 1-5.7.

1.18. Fill in the missing details of the proof of Proposition 1-7.10 and give a reformulation of it in terms of the Alexandroff topology that parallels Proposition 1-7.9.

1.19. Let F be the *lifting functor* on CLatt, that is, $F(L) = L_\perp$, L with a new bottom element added, and, for $g\colon L \to M$, defining $F(g)\colon F(L) \to F(M)$ by $F(g)(\perp) = \perp$ and $F(g) = g$ otherwise.

Show that F is locally Scott-continuous. Show that the "least fixed point" of F starting with $D_0 = \{\bot\}$ can be identified with $\mathbb{N}^\infty = \mathbb{N} \cup \{\infty\}$, where $\mathbb{N}$ are the natural numbers with the usual order augmented by ∞ as a top element.

1.20. Show that the limit of a sequence of Scott-continuous projection-embedding pairs between algebraic lattices is an algebraic lattice. If F is a locally continuous endofunctor on the category of algebraic lattices and Scott-continuous maps, conclude that the domain equation $F(D) \cong D$ has an algebraic lattice solution.

1.21. Let F be a locally continuous endofunctor on the category CLatt. Show that the solution D_ω of the domain equation $F(D) \cong D$ according to Theorem 1-8.10, is the 'least' solution in the following sense: For every solution D of the domain equation $F(D) \cong D$, there is a projection p from D onto D_ω. (Hint: Let $\widetilde{p}_0$ be the unique map from D onto $D_0 = \{\bot\}$. Show firstly that $\widetilde{p}_n = F^n(\widetilde{p}_0)$ is a projection from D onto $D_n = F^n(D_0)$ and secondly that $p = e_n^\omega \circ \bigvee^\uparrow \widetilde{p}_n$ is a projection from D onto D_ω.)

1.22. In order to solve the domain equation $[D \to D] \cong D$ in the category of continuous lattices and Scott-continuous functions, we start with an arbitrary continuous lattice D_0 with at least two elements. (Beginning with $D_0 = \{\bot\}$ leads to the trivial one element solution.) We define functions

$$p_0 \colon [D_0 \to D_0] \to D_0, \quad e_0 \colon D_0 \to [D_0 \to D_0]$$

by $p_0(f) = f(\bot)$ for all $f \in [D_0 \to D_0]$ and $e_0(x) = c_x$, the constant function from D_0 to D_0 with value x, for all $x \in D_0$.

(a) Check that p_0, e_0 is a Scott-continuous projection-embedding pair. We now define recursively for $n \geq 1$:

$$D_n = [D_{n-1} \to D_{n-1}],$$
$$p_n \colon D_{n+1} \to D_n \text{ by } p_n(f) = p_{n-1} \circ f \circ e_{n-1} \text{ for all } \in D_{n+1},$$
$$e_n \colon D_n \to D_{n+1} \text{ by } e_n(g) = e_{n-1} \circ g \circ p_{n-1} \text{ for all } g \in D_n.$$

(b) Check that p_n, e_n is a projection-embedding pair for each $n \geq 1$.

(c) Show that

$$D_\omega = \Big\{ f = (f_n)_n \in \prod_n D_n \mid p_n(f_n) = f_{n-1} \text{ for all } n \geq 1 \Big\}$$

is a solution for the domain equation $[D \to D] \cong D$.

1.23. Let $\alpha \colon L \to \mathbf{2}$ be a frame homomorphism from a distributive continuous lattice L onto the two-element lattice $\mathbf{2} = \{0, 1\}$. Then there exists a prime element p such that $\alpha^{-1}(0) = {\downarrow} p$ and $\alpha^{-1}(1) = L \setminus {\downarrow} p$.

1.24. Let S and T be completely distributive lattices. Let $g\colon S \to T$ be a map preserving arbitrary meets and $d\colon T \to S$ its lower adjoint. Show that g also preserves arbitrary joins if and only if d preserves the relation $\lll$. If this is the case, then g has also an upper adjoint.

Chapter 2

Frames: Topology Without Points

by Aleš Pultr and Jiří Sichler[1]

2-1. Introduction

In classical (synthetic) geometry, lines and planes are not sets of points. They are entities in their own right, and the geometry is based on relations between them (and points, the other entities present). It is only in analytic geometry that one starts with a set and imposes on it the geometric structure by defining specific subsets.

In topology – generalized geometry – we have from the very beginning an "analytic version". We are given a set X of points, and a structure on X specifies some particular subsets (say, open sets). But what about a synthetic version? Can we develop topology starting with a concept of "location" and with some natural structure on the system of these locations? We would certainly wish to be able to amalgamate, or join, smaller locations to form larger ones; we would also wish to be able to tell whether two locations meet or not. This calls for utilizing lattice theory and its concepts. Thus, the system of locations should be a complete lattice (with the amalgam of a system of smaller

[1] Passed away November 26, 2013. See the Fried–Sichler memorial issue of Algebra Universalis for an appreciation of his work.

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locations the supremum of such system). And it is natural to assume that if a location a meets an amalgam $\bigvee b_i$, then it meets some of its constituents b_i, suggesting the distributive law

$$a \wedge \bigvee b_i = \bigvee (a \wedge b_i).$$

This is, in fact, somewhat stronger than the requirement we started with (which is the implication $a \wedge \bigvee b_i \neq 0 \;\; \Rightarrow \bigvee (a \wedge b_i) \neq 0$). It is, however, technically useful, and, above all, this distributivity formula holds in the lattice of open sets of a topological space (and the concept of open set agrees with our idea of a "location"). Does this suffice? Surprisingly enough, the concept of a *frame* thus obtained, together with a suitably defined concept of a frame homomorphism, forms a good basis of a theory that can in many respects match the classical pointy one, and in some respects produce even better results, e.g., constructivity of classically non-constructive facts, or better behaviour of some concepts.

In this introduction we will not go into the history of the subject. The interested reader can learn about it elsewhere ([208], [210], [255], [272]).

In Section 2-2 we start with the relation of the pointy and the point-free theories, and show that for an important class of spaces all the information can be recovered from the algebraic structure of lattices of open sets. The algebraic (lattice) approach leads to a representation of subspaces that may not be quite intuitive at first. Therefore we discuss it in some detail in Section 2-3. Another notion is the product of spaces. How can one, at least to some small extent, mimic the situation in classical topology where – perhaps as a product of two spaces – the open sets are unions of open rectangles, and two systems of open rectangles are equivalent if they constitute the same set of points? The answer is an interesting algebraic construction which we present in Section 2-4. In Section 2-5 we discuss point-free variants of separation axioms of classical topology, and show that their algebraic form is easy to work with. Section 2-6 is devoted to compactness; in particular, we present a very easy counterpart of the Stone–Čech compactification that does not require the Axiom of Choice. In Section 2-7 we meet an old acquaintance, the continuous lattice. We easily see that distributive continuous lattices are point-free models of local compactness. But not only that; it turns out that in this case we, in fact, obtain a precise (contravariant) representation of locally compact spaces as well (Hofmann–Lawson duality). Thus, in this case (albeit modulo the Axiom of Choice), the point-free and the classical theories are fully in parallel. Section 2-8 is a brief introduction to some basic concepts of point-free uniformity theory, presented without proofs. Among other topics, we introduce the basic principles behind the pleasing behaviour of paracompact frames, contrasting unsatisfactory properties of constructions with paracompact spaces.

2-2. Topological spaces and lattices of open sets: frames

2-2.1 Spaces and frames

All our spaces are T_0.

The lattice of open sets of a space X will be denoted by $\Omega(X)$. In this (complete) lattice the joins are the unions and the finite meets are the intersections, and they distribute over the joins. This important feature leads to the following definition.

A *frame* is a complete lattice L satisfying the Frame Distributive Law

(FDL) $$(\bigvee A) \wedge b = \bigvee\{a \wedge b \mid a \in A\}$$

for all $A \subseteq L$ and $b \in L$. This law is called the Infinite Distributive Law (IDL) elsewhere in this volume.

If $f\colon X \to Y$ is a continuous map, then the preimages of open sets are open and we have a map

$$\Omega(f) = (U \mapsto f^{-1}[U])\colon \Omega(Y) \to \Omega(X).$$

The behaviour of such maps is mimicked in the definition of a *frame homomorphism* $h\colon L \to M$ as a map preserving all joins and finite meets.

2-2.2 Sobriety

The natural question arises of how much information about the space X the lattice $\Omega(X)$ retains, and how much a frame homomorphism tells us about the map f. The answers are very satisfactory for the so-called sober spaces. For any space X, the open sets $W = X \smallsetminus \overline{\{x\}}$ are prime (that is, meet irreducible) in the lattice $\Omega(X)$. A space is *sober* (Dieudonné and Grothendieck [189]) if there are no other primes (apart from the trivial X). Note that this is not a rare property; for instance, every Hausdorff space is sober: let $x_1, x_2 \notin U$, $x_1 \neq x_2$; separate x_i by disjoint open $U_i \ni x_i$ to obtain $U = (U \cup U_1) \cap (U \cup U_2)$.

The following equivalent formulation of sobriety is often convenient. For a point $x \in X$, set

$$\mathcal{U}(x) = \{U \in \Omega(X) \mid x \in U\}.$$

For trivial reasons, $\mathcal{U}(x)$ is a completely prime filter in $\Omega(X)$ (that is, if $\bigcup U_i \in \mathcal{U}(x)$ for any system of the U_i, then there is a j such that $U_j \in \mathcal{U}(x)$).

Proposition 2-2.1. *X is sober iff each completely prime filter in $\Omega(X)$ is of the form $\mathcal{U}(x)$.*

Proof. $\Rightarrow$: Let X be sober and let $F \subseteq \Omega(X)$ be a completely prime filter. Then $V_F = \bigvee\{U \mid U \notin F\} \notin F$ and hence $U \in F$ iff $U \nsubseteq V_F$. Now if $V_F = U_1 \cap U_2$, we cannot have $U_i \nsubseteq V_F$ for both i (it would make V_F an element of F). Thus, say, $U_1 \subseteq V_F \subseteq U_1 \cap U_2 \subseteq U_1$, and V_F is prime and hence equal to an $X \smallsetminus \overline{\{x\}}$ for some $x \in X$. Thus,

$$U \in F \quad \text{iff} \quad U \nsubseteq X \smallsetminus \overline{\{x\}} \quad \text{that is, iff} \quad x \in U.$$

$\Leftarrow$: Let the condition hold and let U be a prime in $\Omega(X)$. Then

$$\{V \mid V \nsubseteq U\}$$

is a filter ($V_1 \cap V_2$ cannot be a subset of U if $V_i \nsubseteq U$ for both $i = 1, 2$). It is obviously completely prime, and hence there is an x such that $V \nsubseteq U$ iff $x \in V$ which makes $U = X \smallsetminus \overline{\{x\}}$. □

Theorem 2-2.2. *Let Y be sober and X arbitrary and let $h\colon \Omega(Y) \to \Omega(X)$ be a frame homomorphism. Then $h = \Omega(f)$ for precisely one continuous $f\colon X \to Y$.*

Proof. Since all our spaces are T_0, the uniqueness is evident.

If $x \in X$ then $h^{-1}[\mathcal{U}(x)]$ is obviously a completely prime filter. Hence by sobriety, $h^{-1}[\mathcal{U}(x)] = \mathcal{U}(y)$ for some y. Setting $y = f(x)$, we obtain

$$x \in h(U) \quad \text{iff} \quad h(U) \in \mathcal{U}(x) \quad \text{iff} \quad U \ni f(x) \quad \text{iff} \quad x \in f^{-1}[U],$$

so that $h = \Omega(f)$. □

Thus, if we denote by **Top** the category of topological spaces and continuous maps, by **Sob** its full subcategory of sober spaces, and by **Frm** the category of frames and frame homomorphisms, we then have a contravariant functor

$$\Omega\colon \mathbf{Top} \to \mathbf{Frm}$$

the restriction of which to **Sob** is a contravariant full embedding. To conform with the spaces, we introduce the *category of locales* **Loc** as the opposite of **Frm** and we have covariant

$$\Omega\colon \mathbf{Top} \to \mathbf{Loc} \quad \text{and a full embedding} \quad \mathbf{Sob} \to \mathbf{Loc}.$$

Spatiality. A general frame is not necessarily *spatial*, that is, isomorphic with an $\Omega(X)$. In Exercises 2.6 and 2.7 we present simple examples of non-spatial frames.

2-2.3 Spectrum

We can easily reconstruct a sober space Y from $\Omega(Y)$: we have the points $y \in Y$ in a one-to-one correspondence with the (continuous) maps $f\colon P \to Y$ where P is a one-point space and hence with the frame homomorphisms

$$\alpha\colon \Omega(Y) \to \mathbf{2} = \{0, 1\} \cong \Omega(P).$$

Now we can endow the set $\{\alpha \mid \alpha\colon \Omega(Y) \to \mathbf{2}\}$ with the open sets $\widetilde{U} = \{\alpha \mid \alpha(U) = 1\}$ and obtain a space homeomorphic with the original one.

This is a special case of a more general construction. Let L be a frame. Define the *spectrum* of L as the space

$$\Sigma L = (\{\alpha \mid \alpha\colon L \to \mathbf{2}\}, \{\Sigma_a \mid a \in L\})$$

where $\Sigma_a = \{\alpha \mid \alpha(a) = 1\}$. We obviously have

$$(2.3.1) \qquad \Sigma_0 = \varnothing, \quad \Sigma_1 = \Sigma L, \quad \Sigma_{a\wedge b} = \Sigma_a \cap \Sigma_b, \quad \text{and} \quad \Sigma_{\bigvee a_i} = \bigcup \Sigma_{a_i}$$

and hence $\{\Sigma_a \mid a \in L\}$ is indeed a topology on ΣL.

Further, for a frame homomorphism $h\colon L \to M$, set

$$\Sigma h = (\alpha \mapsto \alpha h)\colon \Sigma M \to \Sigma L.$$

Since we have

$$(2.3.2) \qquad (\Sigma h)^{-1}[\Sigma_a] = \Sigma_{h(a)}$$

(as $\alpha \in (\Sigma h)^{-1}[\Sigma_a]$ iff $\alpha h \in \Sigma_a$ iff $\alpha(h(a)) = 1$), Σh is a continuous map and we have a contravariant functor

$$\Sigma\colon \mathbf{Frm} \to \mathbf{Top} \quad (\text{a covariant functor } \ \Sigma\colon \mathbf{Loc} \to \mathbf{Top}).$$

2-2.4 The spectrum adjunction

Recall that two (covariant) functors $F\colon \mathcal{A} \to \mathcal{B}$ and $G\colon \mathcal{B} \to \mathcal{A}$ are adjoint, F to the left and G to the right, if there are natural transformations $\varepsilon\colon FG \to \mathrm{Id}$ and $\eta\colon \mathrm{Id} \to GF$ such that the compositions

$$F \xrightarrow{F\eta} FGF \xrightarrow{\varepsilon_F} F \quad \text{and} \quad G \xrightarrow{\eta_G} GFG \xrightarrow{G\varepsilon} G$$

are identities; this is equivalent with the existence of a natural one-to-one correspondence between the morphisms $F(A) \to B$ in $\mathcal{B}$ and $A \to G(B)$ in $\mathcal{A}$.

Theorem 2-2.3. $\Sigma\colon \mathbf{Loc} \to \mathbf{Top}$ *is a right adjoint to* $\Omega\colon \mathbf{Top} \to \mathbf{Loc}$.

Proof. We have formulated the statement in the language of locales to make clear which of the functors is to the left and which of them is to the right. In the computations below we will, however, adopt comfortable frame reasoning. Thus we define, for a space X,

$$\eta_X\colon X \to \Sigma\Omega X \ \text{ by setting } \ \eta_X(x)(U) = 1 \text{ iff } x \in U,$$

but also

$$\varepsilon\colon L \to \Omega\Sigma L \ \text{ in } \mathbf{Frm} \text{ instead of “localic” } \ \Omega\Sigma L \to L$$

by setting $\varepsilon_L(a) = \Sigma_a$. This ε_L is a frame homomorphism; further,

$$(*) \qquad \eta_X^{-1}[\Sigma_U] = \{x \mid \eta_X(x)(U) = 1\} = U$$

and hence η_X is continuous. It is easy to check that the $(\eta_X)_X$ and $(\varepsilon_L)_L$ constitute natural transformations.

Now we have (working in **Frm**, not in **Loc**), by $(*)$,

$$\Omega\eta_X(\varepsilon_{\Omega(X)}(U)) = \eta_X^{-1}[\Sigma_U] = U,$$

and

$$(\Sigma\varepsilon_L(\eta_{\Sigma L}(\alpha)))(a) = 1, \quad \text{that is,} \quad (\eta_{\Sigma L}(\alpha)\cdot\varepsilon_L)(a) = \eta_{\Sigma L}(\alpha)(\Sigma_a) = 1$$
$$\text{iff} \quad \alpha \in \Sigma_a \quad \text{iff} \quad \alpha(a) = 1. \quad \square$$

2-3. Sublocales (generalized subspaces)

2-3.1 On the definition

A *sublocale map*, or, a *sublocale embedding*, is an onto frame homomorphism $h\colon L \to M$.

This definition calls for an explanation.

Suppose we have a map $f\colon Y \to X$ between structured objects, respecting the structures. To be an embedding of a subobject it does not suffice to be just a one-to-one map, even if it respects the structure: there may be a "stronger" structure replacing that of Y still respected by the same map.

In category theory, one-to-one maps are, roughly speaking, represented by monomorphisms. Preventing the existence of a "stronger structure" on the smaller object is modeled by the concept of *extremal monomorphism*, a monomorphism m such that in every decomposition $m = m' \cdot e$, where e is an epimorphism, the e must be an isomorphism. Now the extremal monomorphisms in the category **Loc** are the extremal epimorphisms in **Frm**, and these happen to be precisely the onto homomorphisms.

If this sounds too formalistic, consider a space X and a subspace Y. The embedding $j\colon Y \to X$ is expressed as $\Omega(j) = (U \mapsto U \cap Y)$. Recall the standard definition of a subspace $Y \subseteq X$ of a space (X, τ) with the topology $\tau\rceil Y = \{U \cap Y \mid U \in \tau\}$.

2-3.2 Frame congruences

Up to isomorphism, the onto homomorphisms $h\colon X \to Y$ are represented by (*frame*) *congruences* (respecting finite meets and all joins). The system $\operatorname{Con} L$ of all congruences on L is a complete lattice (with the inclusion order dual to the natural order of sublocale embeddings – the bigger is the congruence the smaller is the M in $h\colon L \to M$).

In a frame congruence C, the congruence classes can be identified by their largest elements. This leads to the concept of *nucleus*. Define a map $\nu\colon L \to L$ by setting

$$(*) \qquad \nu(a) = \bigvee\{x \mid x\,C\,a\} = \bigvee\{x \mid h(x) = h(a)\}$$

(h is an onto homomorphism associated with C). This formula yields a monotone map such that

$$(\mathrm{N}) \qquad a \le \nu(a), \quad \nu(a) = \nu\nu(a), \quad \text{and} \quad \nu(a \wedge b) = \nu(a) \wedge \nu(b)$$

(the first two formulas and $\nu(a)\wedge\nu(b) \ge \nu(a\wedge b)$ are obvious; as $h(\nu(a)\wedge\nu(b)) = h\nu(a) \wedge h\nu(b) = h(a) \wedge h(b) = h(a \wedge b)$, we also have $\nu(a) \wedge \nu(b) \le \nu(a \wedge b)$).

Frame congruences are in a one-to-one correspondence with their nuclei, the inverse to $(*)$ being given by $C_\nu = \{(x, y) \mid \nu(x) = \nu(y)\}$.

2-3.3 Sublocales

Before proceeding further, let us recall the Galois adjunction $f(x) \le y$ iff $x \le g(y)$ between the monotone maps $f\colon (X, \le) \to (Y, \le)$ and $g\colon (Y, \le) \to (X, \le)$. In such an adjunction, the map f preserves suprema and the map g preserves infima, and if $(X, \le)$ and $(Y, \le)$ are complete lattices, each suprema preserving f has a right adjoint. In particular, the distributive law in a frame makes the map $x \mapsto x \wedge a$ preserve suprema and creates a (unique) Heyting operation $\to$ on L with

$$(\mathrm{H}) \qquad a \wedge b \le c \quad \text{iff} \quad a \le b \to c.$$

This operation will prove to be useful. Note that each frame is therefore a Heyting algebra, but that frame homomorphisms need not preserve the Heyting operation.

Define a *sublocale* S of L as a subset $S \subseteq L$ such that

(S1) $\bigwedge A \in S$ for each $A \subseteq S$ (in particular, $1 = \bigwedge \varnothing \in S$);

(S2) $a \to b \in S$ for all $a \in L$ and $b \in S$.

Note that, by (S1), S is a complete lattice, and from (S2) we see that it is a frame. But it is not a subframe of L, for the suprema may differ.

Fact. *$\nu[L]$ is a sublocale for every nucleus ν.*

Proof. We have $a \in \nu[L]$ iff $\nu(a) = a$, from which it immediately follows that $\nu[L]$ is closed under meets. Now for every Heyting algebra we have $(a \to b) \wedge a \le b$ (in fact, $(a \to b) \wedge a = a \wedge b$), hence

$$\nu(a \to \nu(b)) \wedge a \le \nu(a \to \nu(b)) \wedge \nu(a) \le \nu((a \to \nu(b)) \wedge a) \le \nu\nu(b) = \nu(b),$$

so that $\nu(a \to \nu(b)) \le a \to \nu(b) \le \nu(a \to \nu(b))$. $\square$

On the other hand, we easily see that defining $\nu_S(x) = \bigwedge\{s \mid x \leq s \in S\}$ for a sublocale S we obtain a nucleus, and that a one-to-one correspondence between nuclei and sublocales (and hence between congruences and sublocales, and between sublocale embeddings and sublocales) is established. From now on, we will consider the sublocales $S \subseteq L$ as representations of generalized subspaces of a locale (frame) L.

Proposition 2-3.1. *The collection $\mathcal{S}(L)$ of all sublocales of a frame L constitutes a co-frame (that is, the order dual of the complete lattice $\mathcal{S}(L)$ is a frame).*

Proof. $\mathcal{S}(L)$ is a complete lattice since obviously any intersection of sublocales is a sublocale. Further, the join in $\mathcal{S}(L)$ is given by

$$\bigvee S_i = \{\bigwedge A \mid A \subseteq \bigcup S_i\}$$

($x \to \bigwedge A = \bigwedge\{x \to a \mid a \in A\}$, since the map $y \mapsto (x \to y)$ is a right Galois adjoint; hence, $\bigvee S_i$ is a sublocale, obviously contained in any sublocale containing all the S_i).

Now about the co-frame distributivity. We trivially have $(\bigcap A_i) \vee B \subseteq \bigcap(A_i \vee B)$. Let $x \in \bigcap(A_i \vee B)$. Then we have $a_i \in A_i$ and $b_i \in B$ such that $x = a_i \wedge b_i$ for all i. Set $b = \bigwedge_i b_i$. Then

$$x = \bigwedge_j a_j \wedge \bigwedge_j b_j = \bigwedge a_j \wedge b \leq a_i \wedge b = x.$$

In a Heyting algebra, we have $u \to v = u \to (u \wedge v)$. Thus, since the $a_i \wedge b$ coincide we also have a common value $a = b \to a_i$ for all i, and

$$x = b \wedge a_i = b \wedge (b \to a_i) = b \wedge a \in B \vee \bigcap A_i. \qquad \square$$

Corollary. *The congruence lattice* $\operatorname{Con} L$ *is a frame.*

2-3.4 Open and closed sublocales

With any $a \in L$, we associate the *open* and *closed* congruences

$$\Delta_a = \{(x, y) \mid x \wedge a = y \wedge a\} \quad \text{and} \quad \nabla_a = \{(x, y) \mid x \vee a = y \vee a\}$$

induced by the sublocale embeddings

$$\widehat{a} = (x \mapsto x \wedge a) \colon L \to {\downarrow} a \quad \text{and} \quad \check{a} = (x \mapsto x \vee a) \colon L \to {\uparrow} a$$

(where ${\downarrow} a = \{x \mid x \leq a\}$ and ${\uparrow} a = \{x \mid x \geq a\}$). The corresponding *open* and *closed sublocales* are defined as

$$\mathfrak{o}(a) = \{a \to x \mid x \in L\} = \{x \mid a \to x = x\} \quad \text{and} \quad \mathfrak{c}(a) = {\uparrow} a.$$

The reader might expect the open sublocale to be $\downarrow a$ but this subset is not a sublocale at all, and the nucleus associated with Δ_a is

$$\begin{aligned}\nu(x) &= \bigvee\{y \mid y \wedge a = x \wedge a\} = \bigvee\{y \mid y \wedge a \leq x \wedge a\} \\ &= \bigvee\{y \mid y \wedge a \leq x\} = a \to x.\end{aligned}$$

Proposition 2-3.2. *The sublocales $\mathfrak{o}(a)$ and $\mathfrak{c}(a)$ are complements of each other.*

Proof. If y is in $\mathfrak{c}(a) \cap \mathfrak{o}(a)$, then $a \leq a \to x = y$ for some $x \in L$. Then by (H), $a = a \wedge a \leq x$ and hence $1 \leq a \to x = y$. Thus, $\mathfrak{o}(a) \cap \mathfrak{c}(a) = \{1\}$, the smallest sublocale. Now in each Heyting algebra we have $x = (x \vee a) \wedge (a \to x)$ (indeed, $x \leq a \to x$ and hence $x \leq (x \vee a) \wedge (a \to x)$, and on the other hand $(x \vee a) \wedge (a \to x) = (x \wedge (a \to x)) \vee (a \wedge (a \to x)) \leq x \vee x = x$), and hence x is in $\mathfrak{c}(a) \vee \mathfrak{o}(a)$. $\square$

Proposition 2-3.3. *For every frame congruence C (resp., sublocale S), we have*

$$C = \bigvee\{\nabla_a \cap \Delta_b \mid a\,C\,b\} \quad (\textit{resp.,} \quad S = \bigcap_{a\,C\,b} (\mathfrak{c}(a) \vee \mathfrak{o}(b))\,).$$

Proof. If $a\,C\,b$ and $(x, y) \in \nabla_a \cap \Delta_b$, we have

$$\begin{aligned}x &= x \wedge (x \vee a) = x \wedge (y \vee a) = (x \wedge y) \vee (x \wedge a)\,C\,(x \wedge y) \vee (x \wedge b) \\ &= (x \wedge y) \vee (y \wedge b) = (x \vee b) \wedge y\,C\,(x \vee a) \wedge y = (y \vee a) \wedge y = y,\end{aligned}$$

so that $\nabla_a \cap \Delta_b \subseteq C$. On the other hand, if $E \supseteq \nabla_a \cap \Delta_b$ for all $(a, b) \in C$, then for $(a, b) \in C$ we obtain $(b, a \vee b), (a, a \vee b) \in E$ and $a\,E\,(a \vee b)\,E\,b$. $\square$

Closure. The smallest closed sublocale containing S is obviously

$$\overline{S} = \uparrow \bigwedge S.$$

From this extremely simple formula, we immediately obtain the standard properties

$$S \subseteq \overline{S}, \quad \overline{\overline{S}} = \overline{S}, \quad \text{and} \quad \overline{S \vee T} = \overline{S} \vee \overline{T}$$

and easily see that S *is dense in* L *iff* $\bigwedge S = 0$.

What follows is radically different from the situation in classical topology.

Proposition 2-3.4. (Isbell's Density Theorem) *Each frame L has a smallest dense sublocale, namely*

$$B_L = \{x \to 0 \mid x \in L\}.$$

Proof. A dense sublocale has to contain 0 and, as it is a sublocale, also all the $x \to 0$. It is easy to check that the resulting B_L is a sublocale. $\square$

Note that B_L is the Booleanization of the Heyting algebra L.

2-3.5 How to construct the congruence generated by a relation

Suppose we have a relation $R \subseteq L \times L$ and wish to construct the smallest congruence containing R. This sounds complicated but it turns out to be very easy as we will see. In fact, we will easily construct the associated nucleus.

An element $s \in L$ is called *R-saturated* (in short, *saturated*) if

$$\forall a, b, c \;\; a \, R \, b \;\;\Rightarrow\;\; (a \wedge c \leq s \text{ iff } b \wedge c \leq s);$$

note that if R is such that $a \, R \, b$ implies $a \wedge c \, R \, b \wedge c$, then the last condition reads:

$$\ldots (a \leq s \text{ iff } b \leq s).$$

Fact. *The set S of all saturated elements is a sublocale and consequently*

$$\nu(x) = \bigwedge\{s \text{ saturated} \mid x \leq s\}$$

is a nucleus.

Proof. The intersection of any system of saturated elements is obviously saturated. Now let s be saturated and let x be arbitrary. We have

$$a \wedge c \leq x \to s \quad \text{iff} \quad a \wedge c \wedge x \leq s \quad \text{iff} \quad b \wedge c \wedge x \leq s \quad \text{iff} \quad b \wedge c \leq x \to s. \; \square$$

Proposition 2-3.5. *Set $L/R = \nu[L]$ and define $\mu\colon L \to M$ by setting $\mu(x) = \nu(x)$. Then L/R is a frame and $\mu\colon L \to L/M$ is a frame homomorphism such that $a \, R \, b \;\Rightarrow\; \mu(a) = \mu(b)$.*

For every frame homomorphism $h\colon L \to M$ such that $a \, R \, b \;\Rightarrow\; h(a) = h(b)$ there is a frame homomorphism $\overline{h}\colon L/R \to M$ such that $\overline{h}\mu = h$. Moreover, $\overline{h}$ is the restriction of h to L/R.

Proof. In L/R we have the supremum $\bigsqcup a_i = \nu(\bigvee a_i)$. Indeed, if $b \in L/R$ and $b \geq a_i$ for all i, then $b = \nu(b) \geq \nu(\bigvee a_i)$. For a system of $a_i \in L$ we have $\mu(\bigvee a_i) \leq \nu(\bigvee \nu(a_i)) = \bigsqcup \mu(a_i) \leq \mu(\bigvee a_i)$ and μ also obviously preserves finite meets. Thus, L/R, as a homeomorphic image of a frame, is a frame, and μ is a frame homomorphism. If aRb then, as $a \leq \mu(a) = \nu(a)$, also $b \leq \nu(a)$ and $\nu(b) \leq \nu\nu(a) = \nu(a)$; by symmetry, $\nu(b) = \nu(a)$.

Now let $h\colon L \to M$ be such that $a \, R \, b$ implies $h(a) = h(b)$. Set

$$\sigma(x) = \bigvee\{y \mid h(y) \leq h(x)\}.$$

We obviously have

$$x \leq \sigma(x) \quad \text{and} \quad h(\sigma(x)) \leq h(x) \; (\leq h(\sigma(x))).$$

If $a \wedge c \leq \sigma(x)$, then $h(b \wedge c) = h(b) \wedge h(c) \leq h(\sigma(x)) \leq h(x)$ and hence $b \wedge c \leq \sigma(x)$ and by symmetry also $b \wedge c \leq \sigma(x)$ implies $a \wedge c \leq \sigma(x)$. Thus, $\sigma(x)$ is saturated and hence $\nu(x) \leq \sigma(x)$. Consequently,

$$h(x) \leq h(\nu(x)) \leq h(\sigma(x)) \leq h(x)$$

and we have $h\mu(x) = h(x)$. Now we see that we can define $\overline{h}(x) = h(x)$ for $x \in L/R$. $\square$

2-4. Free frames. Coproduct

2-4.1 Free frames: the down-set functor

In this section the underlying meet-semilattice structure of a frame will play a fundamental role. Therefore, we will be, first of all, interested in free frames built on meet-semilattices. Free frames over sets are easily constructed by combining our procedure with the (easy) construction of free meet-semilattices, but we will not make use of it here.

More precisely, we have in mind the category of meet-semilattice *with top* 1, and homomorphisms preserving meets $\wedge$ and 1. It will be denoted by

$$\mathbf{SLat}_1.$$

Consider the *down-set functor* $\mathfrak{D}\colon \mathbf{SLat}_1 \to \mathbf{Frm}$ defined by

$$\mathfrak{D}S = (\{X \subseteq S \mid {\downarrow}X = X\}, \subseteq), \quad \mathfrak{D}h(X) = {\downarrow}h[X].$$

$\mathfrak{D}S$ is indeed a frame, with unions for joins and finite intersections for finite meets – hence they properly distribute; and it is easy to check that each $\mathfrak{D}h$ is a frame homomorphism.

Proposition 2-4.1. *For a semilattice S with 1 define*

$$\alpha_S = (x \mapsto {\downarrow}x)\colon S \to \mathfrak{D}S.$$

Then α_S is a $(\wedge, 1)$-homomorphism such that for every frame L and every $(\wedge, 1)$-homomorphism $h\colon S \to L$ there is exactly one frame homomorphism $\overline{h}\colon \mathfrak{D}S \to L$ such that $\overline{h} \cdot \alpha = h$.

Proof. Since $\overline{h}$ should preserve joins,

$$\overline{h}(X) = \overline{h}(\bigcup\{{\downarrow}x \mid x \in X\}) = \bigvee_{x\in X} \overline{h}({\downarrow}x) = \bigvee_{x\in X} h(x),$$

hence the uniqueness.

Now define $\overline{h}\colon \mathfrak{D}S \to L$ by $\overline{h}(X) = \bigvee_{x\in X} h(x)$. Obviously this $\overline{h}$ preserves $0, 1$ and all joins, and we have

$$\begin{aligned}\overline{h}(X) \wedge \overline{h}(Y) &= \bigvee_{x\in X} h(x) \wedge \bigvee_{y\in Y} h(y) = \bigvee\{h(x \wedge y) \mid x \in X, y \in Y\}\\ &\le \bigvee\{h(z) \mid z \in X \cap Y\} = \overline{h}(X \cap Y) \le \overline{h}(X) \wedge \overline{h}(Y);\end{aligned}$$

thus, $\overline{h}$ is a frame homomorphism. $\square$

2-4.2 The construction

We do the construction below in parallel for semilattices and frames.

Let L_i, $i \in J$, be a collection of semilattices or of frames. Set

$$\prod_i' L_i = \{(x_i)_i \in \prod_i (L_i \mid x_i \neq 1 \text{ for finitely many } i)\},$$

where $\prod_i L_i$ is the Cartesian product.

For a fixed $j \in J$, $x \in L_j$ and $u \in \prod' L_i$, set

$$x *_j u = v, \quad v_i = \begin{cases} x \text{ for } i = j, \\ u_i \text{ for } i \neq j. \end{cases}$$

Then set $\overline{1} = (1_{L_i})_i$ and define homomorphisms $\kappa_j \colon L_j \to \prod_{i \in J}' L_i$ by

$$\kappa_j(x) = x *_j \overline{1}.$$

Recall that a coproduct of a collection A_i, $i \in J$, of objects in a category $\mathcal{A}$ is a collection $(\iota_i \colon A_i \to A)_{i \in J}$ of morphisms such that for any $(\varphi_i \colon A_i \to B)_{i \in J}$ in $\mathcal{A}$ there is precisely one $\varphi \colon A \to B$ such that $\varphi \cdot \iota_i = \varphi_i$ for all i.

Proposition 2-4.2. *$(\kappa_j \colon L_j \to \prod_{i \in J}' L_i)_j$ is a coproduct in* $\mathbf{SLat}_1$*.*

Proof. Let $h_j \colon L_j \to M$ be $(\wedge, 1)$-homomorphisms. If $h\kappa_j = h_j$ and $(x_i)_i \in \prod_i' L_i$, let $x_{j_1}, \dots, x_{j_n}$ be the coordinates that are not 1. Then we have

$$h((x_i)_i) = h(\bigwedge_{k=1}^{n} (x_{j_k} *_{j_k} \overline{1})) = \bigwedge_{k=1}^{n} h(x_{j_k} *_{j_k} \overline{1}) = \bigwedge_{k=1}^{n} h_{j_k}(x_{j_k}),$$

hence the uniqueness. On the other hand, define

$$h((x_i)_i) = \bigwedge_{i \in J} h_i(x_i)$$

and note that this is a finite meet. Then $h\kappa_j = h_j$, and for $(x_i)_i$ and $(y_i)_i$ we have

$$\begin{aligned} h(x \wedge y) &= h((x_i \wedge y_i)_i) = \bigwedge_i h_i(x_i \wedge y_i) \\ &= \bigwedge_i (h_i(x_i) \wedge h_i(y_i)) = \bigwedge_i h_i(x_i) \wedge \bigwedge_i h_i(y_i) = h(x) \wedge h(y). \quad \square \end{aligned}$$

2-4.3 Coproducts of frames

Now let the L_i be frames. Recall Section 2-3.5. Define a relation R on the frame $\mathfrak{D}(\prod_i' L_i)$ by setting

$$R = \{(\bigcup_k {\downarrow}(x_k *_j u), {\downarrow}((\bigvee_k x_k) *_j u)) \mid u \in \prod_i' L_i,\ j \in J,\ \{x_k \mid k \in K\} \subseteq L_j\},$$

where the index sets K are arbitrary, possibly void. Thus, in particular, R contains all the pairs

$$(\varnothing, \downarrow(0 *_j u)), \quad j \in J,\ u \in \prod\nolimits_i' L_i. \tag{4.3.1}$$

It is easy to check that the R-saturated elements are precisely the down-sets $U \subseteq \prod' L_i$ (that is, $U \in \mathfrak{D}(\prod' L_i)$) such that

$$\{x_k *_j u \mid k \in K\} \subseteq U \quad \Rightarrow \quad (\bigvee_k x_k) *_j u \in U \tag{4.3.2}$$

for all j, $\{x_k \mid k \in K\} \subseteq L_j$ and $u \in \prod_i' L_i$. Set

$$\bigoplus_i L_i = \mathfrak{D}(\prod\nolimits_i' L_i)/R$$

and set $\iota_j = \mu\alpha\kappa_j$ (μ from Proposition 2-3.5, α from Proposition 2-4.1).

Proposition 2-4.3. *$(\iota_j \colon L_j \to \bigoplus_i L_i)_j$ is a coproduct in* **Frm**.

Proof. Since μ, α, and κ_j preserve finite meets, to prove that ι_j are frame homomorphisms we only need to show that they preserve joins. To prove this, we use the relation R as follows. We have

$$\iota_j(\bigvee_{k\in K} x_k) = \mu(\downarrow\bigvee(x_k *_j \overline{1})) = \mu(\bigcup_k \downarrow(x_k *_j \overline{1})) = \bigvee_k \mu(\downarrow(x_k *_j \overline{1})) = \bigvee_k \iota_j(x_k).$$

Now let $h_j \colon L_j \to M$ be frame homomorphisms. Regarding, for a moment, L_j and M as semilattices, we obtain from Proposition 2-4.2, a $(\wedge, 1)$-homomorphism $f \colon \prod_i' L_i \to M$, defined by $f((x_i)_i) = \bigwedge f_i(x_i)$, such that $f\kappa_i = h_i$ for all i. This f then yields by Proposition 2-4.1 a frame homomorphism $g \colon \mathfrak{D}(\prod' L_i) \to M$, defined by $g(X) = \bigvee\{f(x) \mid x \in X\}$, such that $g \cdot \alpha = f$. Now take a $j \in J$, $u \in \prod_i' L_i$, and $\{x_k \mid k \in K\}$. We obtain (note that $x *_j u = (x *_j \overline{1}) \wedge (1 *_j u)$)

$$\begin{aligned}
g(\bigcup_k \downarrow(x_k *_j u)) &= \bigvee_k f(x_k *_j u) = \bigvee_k f(x_k *_j \overline{1}) \wedge f(1 *_j u) \\
&= \bigvee_k f_j(x_k) \wedge f(1 *_j u)) = (\bigvee_k f_j(x_k)) \wedge f(1 *_j u) \\
&= f_j(\bigvee_k x_k) \wedge f(1 *_j u) = f((\bigvee_k x_k) *_j \overline{1}) \wedge f(1 *_j u) \\
&= f(\bigvee_k x_k) *_j u) = g(\downarrow(\bigvee_k x_k) *_j u).
\end{aligned}$$

Thus, g respects the relation R and hence there is a frame homomorphism $h \colon \bigoplus_i L_i \to M$ such that $h\mu = g$. Now we have

$$h \cdot \iota_j = h\mu\alpha\kappa_j = f\kappa_j = h_j'.$$

The uniqueness is clear since $\bigoplus L_i$ is join-generated by the images $\iota_j[L_j]$. $\square$

2-4.4 The basic elements $\oplus_i a_i$

Consider the set

$$N = \{u \in \prod{}' L_i \mid \exists i,\ u_i = 0\}.$$

This down-set is obviously saturated and by (4.3.1) it is contained in every saturated set. Now take an arbitrary $a = (a_i)_i \in \prod' L_i$. It is easy to check that

$$\downarrow a \cup N$$

is saturated. This element of $\bigoplus L_i$ is usually denoted by

$$\oplus_i a_i \quad \text{(in finite coproducts,} \quad a \oplus b,\ a_1 \oplus \cdots \oplus a_n,\ \text{etc.)}$$

For any $U \in \bigoplus_i L_i$ we have

$$U = \bigvee \{\oplus a_i \mid \oplus a_i \leq U\} = \bigcup \{\oplus a_i \mid \oplus a_i \leq U\}.$$

Working with products one often uses an important fact which follows from the definition:

$$\textit{if } N \neq \oplus a_i \leq \oplus b_i \textit{ then } a_i \leq b_i \textit{ for all } i.$$

2-4.5 Products on Loc compared with topological products

Coproducts in **Frm** are, of course, products in **Loc**. We remember from Section 2-2.4 that the functor Ω is a left, not a right, adjoint, and cannot be expected to preserve products. We have a natural connection, the canonical morphism (in **Frm**)

$$\pi\colon \bigoplus_i \Omega(X_i) \to \Omega(\prod_i X_i) \quad \text{defined by} \quad \pi\iota_i = \Omega(p_i).$$

It is always onto and dense, and surprisingly enough (see Section 2-7) it can be an isomorphism in some important cases.

Note that the elements $\oplus a_i$ play the role of the boxes $\prod U_i$ (with U_i open in X_i and equal to X_i for all but finitely many i).

Remark 2-4.4. Let us concentrate, for a moment, on a coproduct of two frames. We take the product $L_1 \times L_2$; think of it as a meet semilattice, form a free frame $\mathfrak{D}(L_1 \times L_2)$, and finally factorize it by the relation

$$(\bigvee x_k, y) \sim \bigvee (x_k, y) \quad \text{and} \quad (x, \bigvee y_k) \sim \bigvee (x, y_k).$$

This is reminiscent of the construction of a tensor product of Abelian groups: take $A_1 \times A_2$, forget the structure, form the free Abelian group $F(A_1 \times A_2)$, and then factorize it by

$$(a_1 + a_2, b) \sim (a_1, b) + (a_2, b) \quad \text{and} \quad (a, b_1 + b_2) \sim (a, b_1) + (a, b_2).$$

This similarity is not fortuitous. In fact, the tensor product of the Abelian parts can be turned into a coproduct of commutative rings, and there is a general pattern connecting tensor product and coproducts in structures enriching an "additive base" in a distributive way, see [13, 14].

2-5. Separation axioms

As in classical topology we are often not interested in quite general spaces, but restrict their scope by imposing specific properties such as various separation axioms. These may seem to be hard to use in the point-free context; yet regularity, complete regularity, and normality can be carried over in a quite satisfactory fashion. There are also other useful conditions that are not often used in the classical context.

2-5.1 Normal, regular, and completely regular frames

The translation of normality is straightforward. A frame is *normal* if

> whenever $a \vee b = 1$, there exist u, v such that $u \wedge v = 0$, $u \vee b = 1$, and $a \vee v = 1$.

Somewhat less obvious is a correct translation of regularity and complete regularity. For classical spaces, regularity can be expressed by the requirement that each open U is the union $\bigcup\{V \text{ open} \mid \overline{V} \subseteq U\}$. Thus, we can declare a frame to be *regular* if

$$\text{for every } a \in L,\ a = \bigvee\{b \mid b^* \vee a = 1\},$$

where $x^* = \bigvee\{y \mid y \wedge x = 0\}$ is the pseudocomplement of x in L. One defines a relation *rather below* by setting

$$x \prec y \quad \equiv_{\text{def}} \quad x^* \vee y = 1 \quad (\text{compare with: } \overline{V} \subseteq U).$$

In this notation, the formula above becomes $a = \bigvee\{b \mid b \prec a\}$.

In general, the relation $\prec$ is not interpolative (that is, it may happen that $a \prec b$ and there is no c with $a \prec c \prec b$). We define *completely below*, $x \prec\!\!\prec y$, as the largest interpolative subrelation of $\prec$, and express *complete regularity* by requiring that

$$\text{for every } a \in L,\ \ a = \bigvee\{b \mid b \prec\!\!\prec a\}.$$

Again, this definition is correct in the sense that X is completely regular iff $\Omega(X)$ is a completely regular frame. To see this observe that the maximal interpolative subrelation of $\prec$ can be constructed as

(CR) $a_0 \prec\!\!\prec a_1$ iff there exist a_r for all rational r, $0 \le r \le 1$, such that $r < s$ implies $a_r \prec a_s$.

Then it is quite easy to show the correctness by mimicking the proof of the well-known Urysohn Lemma.

Proposition 2-5.1. *In a normal frame, the relation $\prec$ interpolates. Consequently, a regular normal frame is completely regular.*

Proof. Let $a \prec b$. Then $a^* \vee b = 1$. We have u, v such that $u \wedge v = 0$, $u \vee b = 1$ and $a^* \vee v = 1$. Thus, $u \leq v^*$, $v^* \vee b = 1$, and $a^* \vee v = 1$, that is, $a \prec v \prec b$. □

2-5.2 Subfitness

We do not have a suitable counterpart of the T_1 axiom, but there is a weaker one that is fairly useful. L is said to be *subfit* if

whenever $a \not\leq b$, there is a c such that $a \vee c = 1 \neq b \vee c$.

Obviously,

every regular frame is subfit

(if $a = \bigvee\{x \mid x \prec a\} \not\leq b$, then there is an $x \prec a$ such that $x \not\leq b$; set $c = x^*$).

As in classical topology, normality alone does not imply complete regularity (or regularity). Usually one adds T_1, but subfitness suffices.

Proposition 2-5.2. *A subfit normal frame L is completely regular.*

Proof. By Proposition 2-5.1 it suffices to prove that L is regular. Suppose $b = \bigvee\{x \mid x \prec a\} \neq a$. Then $a \not\leq b$ and we have a c such that $a \vee c = 1 \neq b \vee c$. Take u, v such that $u \wedge v = 0$, $u \vee c = 1$, and $a \vee v = 1$. Then $u^* \vee a = 1$, hence $u \prec a$ and $u \leq b$. Consequently, $b \vee c = 1$, a contradiction. □

An interesting feature of subfit frames is that a congruence E such that $E\,1 = \{x \mid x\,E\,1\} = \{1\}$ is necessarily trivial. That is, we have the following statement

Proposition 2-5.3. *Let L be subfit and let $h\colon L \to M$ be a frame homomorphism such that $h(a) = 1$ implies $a = 1$. Then h is one-to-one.*

Proof. Let $h(a) = h(b)$ with $a \not\leq b$. Then for the c from the definition, $h(b \vee c) = h(a) \vee h(c) = 1$ and hence $b \vee c = 1$, a contradiction. □

Remark 2-5.4. In fact, this proposition characterizes subfitness. A stronger property, the *fitness*, will be briefly discussed in the exercises. Let us mention here two of characterizations of fitness: a frame is fit

– iff each of its sublocales is subfit;

– iff any two congruences agreeing on the top agree everywhere.

2-5.3 Hausdorff axiom

For frames, we have no entirely satisfactory parallel with the classical Hausdorff axiom. Mimicking the classical fact, we can declare L to be *strongly Hausdorff* (or *Isbell–Hausdorff*) if the codiagonal in $L \oplus L$ is closed (as in spaces, X is Hausdorff iff the diagonal is closed in $X \times X$). As we have stated before, however, the coproduct $\Omega(X) \oplus \Omega(X)$ does not generally reflect the product $X \times X$, and the property does not fully correspond to the classical one. It is, however, a useful concept.

See Exercises 2.32–2.34.

2-5.4 More about regular frames

As for spaces, we have

Proposition 2-5.5. *Regularity and complete regularity are hereditary* (*that is, they carry over to sublocales*).

Proof. It suffices to prove that $a \prec b$, resp. $a \prec\!\!\prec b$, implies $h(a) \prec h(b)$ resp. $h(a) \prec\!\!\prec h(b)$; in view of (CR) in Section 2-5.1 we, in fact, need only the former. This is then immediately obtained from the obvious inequality $h(x^*) \leq h(x)^*$. □

Compare the following statement to Proposition 2-5.3 and Remark 2-5.4.

Proposition 2-5.6. *Let E_1, E_2 be congruences on a regular frame and let $E_1\, 1 = E_2\, 1$. Then $E_1 = E_2$.*

Proof. We will prove that if some homomorphisms $h_i \colon L \to M_i$ are such that $h_1(c) = 1$ iff $h_2(c) = 1$ then $h_1(a) = h_1(b)$ iff $h_2(a) = h_2(b)$.

Thus, let $h_1(a) = h_1(b)$ and $h_2(a) \not\leq h_2(b)$. Then $h_2(\bigvee\{x \mid x \prec a\}) = \bigvee\{h_2(x) \mid x \prec a\} \not\leq h_2(b)$ and there is an $x \prec a$ such that $h_2(x) \not\leq h_2(b)$. Now $x^* \vee a = 1$, hence $h_1(x^* \vee a) = h_1(x^* \vee b) = 1$ and hence $h_2(x^* \vee b) = 1$. Thus, $h_2(x)^* \vee h_2(b) \geq h_2(x^*) \vee h_2(b) = 1$ and we have a contradiction $h_2(x) \prec h_2(b)$. □

In classical topology, if Z is a regular space (Hausdorff suffices, but we will not go into that) and if $f \colon X \to Y$ is such that $\overline{f[X]} = Y$ then for any continuous $g, h \colon Y \to Z$ such that $gf = hf$ one has $g = h$. This holds in the point-free context as well.

A homomorphism $h \colon L \to M$ is said to be *dense* if $h(a) = 0$ implies $a = 0$. This agrees with the notion of density discussed after Proposition 2-3.3: h is dense iff the sublocale corresponding to its restriction $L \to h[L]$ contains 0. Indeed, the associated nucleus is $\nu(a) = \bigvee\{x \mid h(x) = h(a)\}$ so that $\nu(x) = 0$ iff h is dense as above. We have the following

Proposition 2-5.7. *Let L be regular, $f, g \colon L \to M$ homomorphisms, and let $h \colon M \to N$ be dense. If $hf = hg$ then $f = g$.*

Proof. Let $x \prec a$ in L. We have $h(g(x) \wedge f(x^*)) = h(f(x) \wedge f(x^*)) = h(0) = 0$ and hence $g(x) \wedge f(x^*) = 0$. From $f(x^*) \vee f(a) = 1$ we now obtain $g(x) \wedge f(a) = g(x)$ and hence $g(x) \leq f(a)$. Thus,

$$g(a) = \bigvee \{g(x) \mid x \prec a\} \leq f(a),$$

and by symmetry also $f(a) \leq g(a)$. □

2-6. Compactness and compactification

2-6.1 A few concepts

Here are a few concepts that do not need any modification: in fact, they are "point-free" already in the classical setting.

A *cover* of L is a subset $A \subseteq L$ such that $\bigvee A = 1$, and a subcover B of A is (of course) a subset $B \subseteq A$ that is still a cover. L is *compact* if each cover of L has a finite subcover (similarly, Lindelöf, α-compact, etc.).

2-6.2 Properties

We will need some basic properties of the relations $\prec$ and $\prec\!\!\prec$.

Lemma 2-6.1. 1. *If $a \leq x \prec y \leq b$, then $a \prec b$.*

2. *If $x_1, x_2 \prec y$, then $x_1 \vee x_2 \prec y$, and if $x \prec y_1, y_2$, then $x \prec y_1 \wedge y_2$. Similarly for $\prec\!\!\prec$.*

Proof. 1 is obvious. 2: If $x_i^* \vee y = 1$, we have

$$1 = (x_1^* \vee y) \wedge (x_2^* \vee y) = (x_1^* \wedge x_2^*) \vee y = (x_1 \vee x_2)^* \vee y.$$

If $x^* \vee y_i = 1$ then $1 = (x^* \vee y_1) \wedge (x^* \vee y_2) = x^* \vee (y_i \wedge y_2)$. The statement on $\prec\!\!\prec$ now follows from (CR) in Section 2-5.1. □

Proposition 2-6.2. 1. *Each subframe of a compact frame is compact.*

2. *Each closed sublocale of a compact frame is compact.*

3. *Each regular compact frame is normal (and consequently completely regular).*

Proof. 1: This is obvious.

2: The suprema in $\uparrow a$ coincide with those in L. Hence, if A is a cover of $\uparrow a$, it is a cover of L.

3: If $a \vee b = 1$, then $\bigvee\{x \mid x \prec a\} \vee b = 1$ and there are $x_1, \ldots, x_n \prec a$ such that $\bigvee_i x_i \vee a = 1$. By Lemma 2-6.1, $x = \bigvee_i x_i \prec a$ and we have $x^* \vee a = 1 = b \vee x$ and $x \wedge x^* = 1$. □

Notes. Statement 1 is a counterpart to the classical fact that a continuous image of a compact space is compact, 2 does not need any comment, and 3 is a weaker counterpart of the statement that Hausdorff compact spaces are normal. In fact, using the standard procedure one can prove that a Lindelöf regular frame is normal.

2-6.3 Two more counterparts of classical Hausdorff facts

Again, we will prove these statements for regular frames only.

Proposition 2-6.3. *Let L be regular and let M be compact. Then each dense $h\colon L \to M$ is one-to-one.*

Proof. Let $h(a) = 1$. Since $a = \bigvee\{x \mid x \prec a\}$, the set $\{h(x) \mid x \prec a\}$ is a cover and there are $x_1, \dots, x_n \prec a$ such that $h(x_1 \vee \cdots \vee x_n) = 1$. By Lemma 2-6.1, setting $x = x_1 \vee \cdots \vee x_n$ we obtain an $x \prec a$ such that $h(x) = 1$. Now $h(x^*) \leq h(x)^* = 0$ and by density $x^* = 0$. Thus, $a = x^* \vee a = 1$. Use Proposition 2-5.3. □

Proposition 2-6.4. *A compact sublocale of a regular frame is closed.*

Proof. We will work with the closure in the language of sublocale embeddings. If h_1, h_2 are sublocale embeddings, then h_2 represents a smaller sublocale iff there is a g such that $gh_1 = h_2$ (compare the associated congruences). Thus $h\colon L \to M$ is smaller than a closed sublocale embedding $\check{c} = (x \mapsto x \vee c)\colon L \to {\uparrow}c$ iff there is a $g\colon {\uparrow}c \to M$ such that $g(x \vee c) = h(x)$. Since c is the bottom of ${\uparrow}c$ this amounts precisely to $h(c) = 0$. Consequently, the closure, viewed as a sublocale embedding is the $\check{c}\colon L \to {\uparrow}c$ with $c = \bigvee\{x \mid h(x) = 0\}$.

Now the g such that $g \cdot \check{c} = h$ is dense: indeed, if $g(x) = 0$ then (since $x \geq c$) $h(x) = g(x \vee c) = 0$ and hence $x \leq c$, that is, $x = c$. By Proposition 2-6.3, g is one-to-one, and since it is obviously onto, it is an isomorphism, and h represents the same sublocale as $\check{c}$. □

2-6.4 A simple but not very satisfactory compactification

For a frame L denote by

$$\mathrm{Id}\, L$$

the lattice of all (nonempty) ideals in L. It is easy to check that the join in $\mathrm{Id}\, L$ is given by the formula

$$\bigvee_{i \in I} J_i = \{x_1 \vee \cdots \vee x_k \mid \{x_1, \dots, x_k\} \subseteq \bigcup_{i \in I} J_i\}.$$

Using this formula it is easily seen that

$$\mathrm{Id}\, L \textit{ is a frame.}$$

Lemma 2-6.5. $\mathrm{Id}\, L$ *is a compact frame.*

Proof. If $\bigvee_{i \in I} J_i = L$, the top of $\mathrm{Id}\, L$, then $1 = x_1 \vee \cdots \vee x_n$ with $x_j \in J_{i_j}$ for some i_j. But then $1 \in \bigvee_{j=1}^n J_{i_j}$ and hence $\bigvee_{j=1}^n J_{i_j} = L$. □

Consider the maps

$$v_L = (J \mapsto \bigvee J)\colon \operatorname{Id} L \to L \quad \text{and} \quad \alpha_L = (x \mapsto {\downarrow} x)\colon L \to \operatorname{Id} L.$$

Obviously

$$v_L \alpha_L = \mathrm{id} \quad \text{and} \quad \alpha_L v_L(J) \supseteq J$$

which makes v_L a left Galois adjoint and hence shows that it preserves all joins.

Proposition 2-6.6. $v_L\colon \operatorname{Id} L \to L$ *is a dense sublocale embedding.*

Proof. If $\bigvee J = 0$ then $J = \{0\}$, the bottom of $\operatorname{Id} L$. Thus, it remains to be proved that v_L preserves finite meets. Trivially, $v_L(L) = 1$ and

$$\begin{aligned} v(J_1) \wedge v(J_2) &= \bigvee \{x \wedge y \mid x \in J_1, y \in J_2\} \le \bigvee \{z \mid z \in J_1 \cap J_2\} \\ &= v(J_1 \cap J_2) \le v(J_1) \wedge v(J_2). \quad \square \end{aligned}$$

Of course, the v_L are not isomorphisms for all compact L. This will be taken care of shortly.

An ideal $J \in \operatorname{Id} L$ is said to be *regular* if for each $a \in J$ there is a $b \in J$ such that $a \prec\!\!\prec b$. The set of all regular ideals will be denoted by

$$\mathrm{R}\operatorname{Id} L.$$

Proposition 2-6.7. $\mathrm{R}\operatorname{Id} L$ *is a subframe of* $\operatorname{Id} L$. *Consequently, it is a compact frame.*

Proof. The intersection of two regular ideals is regular, by Section 2-6.2, and similarly for the joins of arbitrary systems of regular ideals (if $x_i \prec\!\!\prec x_i'$ then $x_j \prec\!\!\prec \bigvee_{i=1}^n x_i'$ and hence $\bigvee_{j=1}^n x_j \prec\!\!\prec \bigvee_{i=1}^n x_i'$). $\square$

For an element a of a completely regular L, set

$$\sigma_L(a) = \{x \mid x \prec\!\!\prec a\}.$$

Since $\prec\!\!\prec$ interpolates, $\sigma_L(a)$ is a regular ideal (also recall Lemma 2-6.1).

We will now use the symbol v_L for the restriction of the original v_L from Proposition 2-6.6. Thus we have

– a dense homomorphism $v_L\colon \mathrm{R}\operatorname{Id} L \to L$ and

– a map $\sigma_L\colon L \to \mathrm{R}\operatorname{Id} L$

such that

$$(*) \qquad v_L \sigma_L = \mathrm{id} \quad \text{and} \quad \sigma_l v_l(J) \supseteq J.$$

Lemma 2-6.8. $\mathrm{R\,Id}\,L$ *is completely regular.*

Proof. Since $\mathrm{R\,Id}\,L$ is compact it suffices to prove it is regular. For a regular ideal J, we obviously have $J = \bigcup\{\sigma(a) \mid a \in J\}$ and hence, by the first statement in Lemma 2-6.1, it suffices to show that

$$\text{if } b \prec\!\!\prec a \text{ in } L \text{ then } \sigma(b) \prec \sigma(a) \text{ in } \mathrm{R\,Id}\,L.$$

Interpolate $b \prec\!\!\prec x \prec\!\!\prec y \prec\!\!\prec a$. Since $\sigma(b^*) \cap \sigma(b) = \{0\}$, we have $\sigma(b^*) \subseteq \sigma(b)^*$. If $b \prec\!\!\prec x$ then obviously $x^* \prec\!\!\prec b^*$, and $x^* \in \sigma(b^*) \subseteq \sigma(b)^*$. Thus, $1 = x^* \vee y \in \sigma(b)^* \vee \sigma(a)$ and hence $\sigma(b)^* \vee \sigma(a) = L$, the top of $\mathrm{R\,Id}\,K$. □

The constructions of Id and R Id are easily extended to functors by setting $\mathrm{Id}\,h(J) = \mathrm{R\,Id}\,h(J) = \downarrow h[J]$. It is easy to check that $(v_L)_L$ is then a natural transformation.

This yields a point-free counterpart of the Stone–Čech compactification.

Theorem 2-6.9. (Banaschewski and Mulvey [12]) *The functor* R Id *together with the natural transformation $v = (v_L)_L$ is a compactification of completely regular frames in the sense that*

- *all the v_L are dense embeddings of L into a compact regular frame;*
- *if L is compact then v_L is an isomorphism.*

Proof. It remains to prove the last statement. Let L be compact. In view of the formula $(*)$ in Proposition 2-6.7 we have to prove only that $\sigma_L v_L(J) \subseteq J$. Let $x \in \sigma_L v_L(J) \subseteq J$. Then $x \prec \bigvee J$, that is, $x^* \vee \bigvee L = 1$, and we have $a_1, \dots, a_n \in J$ such that $x^* \vee a_1 \vee \cdots \vee a_n = 1$. Then $a = a_1 \vee \cdots \vee a_n \in J$, $x \prec a \in J$ and finally $x \in J$. Thus, v_L and σ_L are inverses of each other. □

Note. In the constructions and the proofs the choice principle or the excluded middle are not used (if we take (CR) from Section 2-5.1 for the definition of complete regularity; but we can also adopt the approach of the largest interpolative subrelation, see [17]). Thus we have a fully constructive proof that the subcategory of compact regular frames is coreflexive in the category of completely regular frames. The standard fact that a coreflexive category is closed under coproducts is also constructive; so in the point-free setting the fact that products of compact regular locales are compact is fully constructive (by much more involved methods one can prove that this holds without the regularity, see [209]). This strongly contrasts with classical topology, where preserving compactness under products is equivalent with the Axiom of Choice.

2-7. Continuous frames; locally compact spaces. Hofmann–Lawson duality

2-7.1 Continuous lattices

Continuous lattices constitute a very important and broad subject. In this chapter we discuss only the fact that continuous frames represent locally compact spaces. For more information see, e.g., Chapter 1.

Recall that a *directed set* D in a partial order $(X, \leq)$ is a nonempty $D \subseteq X$ such that for any two $x, y \in D$ there is a $z \in D$ with $x, y \leq z$. In other words, it is a set in which every finite subset has an upper bound.

We will work with complete lattices L (although much would make sense in more general types of orders).

The relation *well-below* in L, denoted

$$x \ll y,$$

is defined by requiring that for any directed $D \subseteq L$, if $y \leq \bigvee D$, then $x \leq d$ for some $d \in D$. We have an immediate

Observation. $0 \ll a$ *for all* $a \in L$ *and* $a \leq x \ll y \leq b$ *imply* $a \ll b$*, and if* $a_1, a_2 \ll b$*, then* $a_1 \vee a_2 \ll b$*.*

A complete lattice L is said to be *continuous* if

$$\text{for each } a \in L,\ a = \bigvee\{x \mid x \ll a\}.$$

Note that by the Observation the set $\{x \mid x \ll a\}$ is directed.

Proposition 2-7.1. *In a continuous lattice the relation* $\ll$ *interpolates.*

Proof. Let $a \ll b$. We have

$$b = \bigvee\{x \mid x \ll b\} = \bigvee\{\bigvee\{x \mid x \ll y\} \mid y \ll b\} = \bigvee\{y \mid \exists x,\ y \ll x \ll b\}$$

and the last joined set is directed. Thus there is a y and an x such that $a \leq y \ll x \ll b$. □

A continuous lattice is generally not required to be distributive. If it is, we have more.

Proposition 2-7.2. *Any distributive continuous lattice is a frame.*

Proof. If $x \ll (\bigvee A) \wedge b$, then in particular

$$x \ll \bigvee A = \bigvee\{\bigvee C \mid C \subseteq A \text{ finite}\}.$$

The set $\{\bigvee C \mid C \subseteq A \text{ finite}\}$ is directed and hence $x \leq \bigvee C$ for some finite $C \subseteq A$ so that $x \leq (\bigvee C) \wedge b = \bigvee\{c \wedge b \mid c \in C\} \leq \bigvee\{a \wedge b \mid a \in A\}$. Thus, $(\bigvee A) \wedge b \leq \bigvee\{a \wedge b \mid a \in A\}$; the other inequality is trivial. □

2-7.2 Locally compact spaces and continuous frames

If a space X is locally compact, then the frame $\Omega(X)$ is continuous. For an open $U \subseteq X$ and an $x \in U$, take an open $V(x)$ and a compact $K(x)$ such that $x \in V(x) \subseteq K(x) \subseteq U$. Then $V(x) \ll U$ because if $U \subseteq \bigcup U_i$ and $\{U_i \mid i \in J\}$ is directed then $K(x) \subseteq U_i$, and hence $V(x) \subseteq U_i$ for some i. Obviously $U = \bigcup\{V(x) \mid x \in X\}$.

It should be noted that a compact space is not necessarily locally compact. But we have the following

Proposition 2-7.3. *Each regular compact space is locally compact.*

Proof. For each x in an open U, there is a closed – and hence compact – neighborhood K such that $x \in K \subseteq U$. □

Similarly there holds that

Proposition 2-7.4. *Every locally compact frame is continuous.*

Proof. We will show that in a compact frame $x \prec y$ implies $x \ll y$. Indeed, if $y \leq \bigvee D$ and $x \prec y$ then $x^* \vee \bigvee D = 1$ and if D is directed there is a $d \in D$ such that $x^* \vee d = 1$ and $x \prec d$ so that $x \leq d$. □

2-7.3 Adjustment of the spectrum construction

There is an obvious one-to-one correspondence between the *frame homomorphisms* $\alpha\colon L \to \mathbf{2}$ and the *completely prime filters* $P \subseteq L$ given by

$$P_\alpha = \{a \mid \alpha(a) = 1\} \quad \text{and} \quad \alpha_P(a) = 1 \text{ iff } a \in P.$$

For the purposes of this section it will be advantageous to modify the spectrum by this correspondence. Thus, we will have

$$\Sigma L = \{P \mid P \text{ completely prime filter in } L\},$$
$$\Sigma_a = \{P \mid a \in P\} \quad \text{and} \quad \Sigma h(p) = h^{-1}[P].$$

2-7.4 Scott topology

The Scott topology which is used here just to characterize completely prime filters is a very important tool in the theory of continuous posets and continuous lattices; see, e.g., Section 1-4 in the previous chapter.

A subset U of a lattice L is said to be *Scott-open* if $\uparrow U = U$, and $U \cap D \neq \varnothing$ whenever $\bigvee D \in U$ for a directed set D. Roughly speaking, the topology thus obtained is that in which suprema of directed sets appear as limits; the continuous maps are precisely the (monotone) maps preserving suprema of directed sets.

Lemma 2-7.5. *A filter P in a frame is completely prime iff it is prime and Scott-open.*

Proof. $\Rightarrow$ is trivial. Now let F be prime and Scott-open, and let $\bigvee_i a_i \in F$. Since F is open there are $a_{i_1}, \ldots, a_{i_n}$ with $a_{i_1} \vee \cdots \vee a_{i_n} \in F$, and since F is prime, some of the a_{i_k} are in F. $\square$

Proposition 2-7.6. *Let F be a Scott-open filter in a frame L such that $a \in F$ and $b \notin F$. Then there is a completely prime filter $P \supseteq F$ such that $a \in P$ and $b \notin P$.*

Consequently, each Scott-open filter in a frame is an intersection of completely prime filters.

Proof. This is the famous Birkhoff theorem modified by (Scott)-openness. Using Zorn's lemma in the standard way and taking into account that unions of open sets are open, we obtain an open $P \supseteq F$ maximal with respect to the condition $b \notin F \ni a$. We will prove that it is prime and hence, by Lemma 2-7.5, completely prime. Suppose there are $u, v \notin P$ such that $u \vee v \in P$. Set

$$G = \{x \mid x \vee v \in P\}.$$

G is obviously a Scott-open filter and $G \supsetneq P$ since $u \in G$. Hence, $b \in G$ and $b \vee v \in P$. We can repeat the procedure with $H = \{x \mid x \vee b \in P\}$ to obtain a contradiction $b = b \vee b \in P$. $\square$

Proposition 2-7.7. *Every continuous frame is spatial.*

Proof. Recall Section 2-2.4. Since each ε_L is onto, it suffices to prove it is one-to-one if L is continuous. This reduces to finding for $a \not\leq b$ a completely prime filter P such that $b \notin P \ni a$, and by Proposition 2-7.6 it suffices to find a Scott-open filter F such that $b \notin F \ni a$.

Since L is continuous there is a $c \ll a$ such that $c \not\leq b$. Interpolate inductively (recall Proposition 2-7.1)

$$c \ll \cdots \ll x_n \ll \cdots \ll x_2 \ll x_1 \ll a$$

and set

$$(*) \qquad F = \{x \mid x \geq x_k \text{ for some } k\}.$$

Then F is obviously a Scott-open filter and $b \notin F \ni a$. $\square$

Lemma 2-7.8. *Let L be a frame. A subset $K \subseteq \Sigma L$ of the spectrum is compact iff $\bigcap\{P \mid P \in K\}$ is Scott-open.*

Proof. Let $\bigcap\{P \mid P \in K\}$ be Scott-open and let $K \subseteq \bigcup\{\Sigma_a \mid a \in A\}$. Then $\bigvee A \in \bigcap\{P \mid P \in K\}$: indeed, for each $P \in K$ there is an $a \in A$ with $a \in P$ and hence $\bigvee A \in P$. By the openness there are $a_1, \ldots, a_n \in A$ such that $a_1 \vee \cdots \vee a_n \in \bigcap\{P \mid P \in K\}$, resulting in $K \subseteq \Sigma_{a_1 \vee \cdots \vee a_n} = \bigcup_{i=1}^n \Sigma_{a_i}$.

If K is compact and $\bigvee A \in \bigcap\{P \mid P \in K\}$, then

$$K \subseteq \Sigma_{\bigvee A} = \bigcup\{\Sigma_a \mid a \in A\}$$

and there are $a_1, \ldots, a_n \in A$ such that $K \subseteq \Sigma_{a_1 \vee \cdots \vee a_n} = \bigcup_{i=1}^n \Sigma_{a_i}$ and we have $a_1 \vee \cdots \vee a_n \in \bigcap\{P \mid P \in K\}$. □

Theorem 2-7.9. (Hofmann–Lawson duality) *The spectrum adjunction* (recall Section 2-2.4) *restricts to a dual equivalence of the category of sober locally compact spaces and the category of continuous frames.*

Proof. After Proposition 2-7.6 it remains to be proved that if L is a continuous frame then ΣL is locally compact.

Let $P \in \Sigma_a \subseteq \Sigma L$. Then $a = \bigvee\{x \mid x \ll a\} \in P$ and hence there is a $c \ll a$ such that $c \in P$. Construct an open filter F as in Proposition 2-7.7 $(*)$ and set

$$K = \{Q \in \Sigma L \mid F \subseteq Q\}.$$

By Proposition 2-7.6, $\bigcap K = F$, hence it is open, and by Lemma 2-7.8, K is compact. Now if $c \in Q \in \Sigma L$, then $x_k \in Q$ for all k and hence $\subseteq Q$. If $F \subseteq Q$, then $a \in Q$. Thus, $P \in \Sigma_c \subseteq K \subseteq \Sigma_a$. □

Note. Thus, by Section 2-7.2, we have, in particular, that every regular compact frame is spatial, which together with the compactification in Section 2-6 yields that a product of compact regular spaces is compact. But not without the Axiom of Choice: the duality in Lemma 2-7.8 is choice dependent (Zorn's lemma is used in Proposition 2-7.6. One can say that it is not the compactness of the product of compact spaces that needs the choice principle; rather, the problem is the existence of points in the product.

2-8. Notes on uniform frames

Unlike the previous sections, this one does not contain the crucial proofs (although we go into details in definitions): the proofs are somewhat lengthy for the size of this chapter. The reader can find them, e.g., in [272] or [255].

2-8.1 Covers and systems of covers

If A, B are covers (recall Section 2-6.1) we say that A *refines* B and write $A \leq B$ if

for every $a \in A$ there is a $b \in B$ such that $a \leq b$.

We set

$$A \wedge B = \{a \wedge b \mid a \in A, b \in B\},$$

and this will be used also for general $A, B \subseteq L$. From the frame distributivity it follows that if A, B are covers, then $A \wedge B$ is a cover as well.

For a cover A of L and an element $x \in L$, set

$$Ax = \bigvee\{a \in A \mid a \wedge x \neq 0\}.$$

We have a Galois adjunction

$$Ax \leq y \quad \text{iff} \quad x \leq y/A \quad \text{where} \quad y/A = \bigvee\{z \mid Az \leq y\}. \tag{8.1.1}$$

Further we set $AB = \{Ab \mid b \in B\}$ and say that B is a *star-refinement* of A if $BB \leq A$.

For a system $\mathcal{A}$ of covers of L, we write

$$x \lhd_{\mathcal{A}} y$$

if there is an $A \in \mathcal{A}$ such that $Ax \leq y$.

A nonempty system $\mathcal{A}$ of covers of a frame L is said to be *admissible* if

$$\forall x \in L, \quad x = \bigvee\{y \mid y \lhd_{\mathcal{A}} x\}.$$

Fact. *For any cover A we have $x \prec Ax$. Consequently, for any non-void system of covers, $x \lhd_{\mathcal{A}} y \ \Rightarrow \ x \prec y$.*

Proof. $1 = \bigvee A = \mathcal{A}x \vee \bigvee\{a \mid a \in A,\ a \wedge x = 0\} \leq Ax \vee x^*$. □

2-8.2 Uniformities

A *uniformity* on a frame L is an admissible system $\mathcal{A}$ of covers such that

(1) $A \in \mathcal{A}$ and $A \leq B$ imply $B \in \mathcal{A}$, and if $A, B \in \mathcal{A}$, then $A \wedge B \in \mathcal{A}$;

(2) for each $A \in \mathcal{A}$ there is a $B \in \mathcal{A}$ such that $BB \leq A$.

Note. This definition follows Tukey's approach to uniformities via covers. The admissibility is a counterpart of the requirement that the topology induced by the uniformity coincides with the original topology of the space the structure of which we enrich. If X is a space, then the uniformities on $\Omega(X)$ just defined coincide with the classical ones.

For a treatment of uniformities, mimicking Weil systems of entourages (specified "neighborhoods of the diagonal") see, e.g., [255].

Fact. *If $\mathcal{A}$ is a uniformity then $\lhd_{\mathcal{A}}$ is interpolative. Consequently, a frame that admits a uniformity is completely regular.*

Proof. If $a \lhd_{\mathcal{A}} y$, we have an $A \in \mathcal{A}$ with $Ax \leq y$. Choose a $B \in \mathcal{A}$ such that $BB \leq A$. Then $x \leq Bx$ and $B(Bx) \leq (BB)x \leq Ax$ and hence $x \lhd_{\mathcal{A}} Bx \lhd_{\mathcal{A}} y$. □

Note. It is easy to prove that, on the other hand, each completely regular frame admits a uniformity. See Exercise 2.44.

Uniform homomorphisms. If $\mathcal{A}$ is a uniformity on L, we speak of the pair $(L, \mathcal{A})$ as a *uniform frame*. A *uniform homomorphism* $h\colon (L, \mathcal{A}) \to (M.\mathcal{B})$ is a frame homomorphism $h\colon L \to M$ such that $h[A] \in \mathcal{B}$ for all $A \in \mathcal{A}$. It is said to be a *uniform embedding* if it is onto (hence, a sublocale embedding) and if $\{h[A] \mid A \in \mathcal{A}\} = \mathcal{B}$.

Note that in the spatial case uniform homomorphisms correspond to uniformly continuous maps, and uniform embeddings correspond to classical ones.

Uniqueness. A completely regular frame typically admits various uniformities. A compact regular frame, however, admits precisely one uniformity, namely the set of all covers (see Exercise 2.46). Consequently, if M is compact, then each frame homomorphism $h\colon L \to M$ is uniform with respect to any uniformity on L (as in spaces where each continuous map defined on a compact Hausdorff space is uniformly continuous).

Fine uniformities. The union of a nonempty system of uniformities on a given frame is obviously a uniformity. Consequently, each completely regular frame possesses a largest uniformity, the so-called *fine uniformity*. It is not always the system of all covers (as in compact frames). That is, it is not always the case that every cover of a completely regular frame has a star refinement. The frames in which this holds are called *fully normal* (alluding to the fact that in normal frames this holds for finite covers). This property coincides with another very important one: see Section 2-8.4 below.

2-8.3 Completeness and completion

The characteristic of complete uniform spaces that is easy to imitate is that

a uniform space X is complete iff each uniform embedding $j\colon X \to Y$ into another uniform space results in a closed subset $j[X]$ of Y.

Thus, we say that a uniform frame $(L, \mathcal{A})$ is *complete* if each dense uniform embedding $h\colon (M, \mathcal{B}) \to (L, \mathcal{A})$ is an isomorphism.

It should be noted that for a classical space this is a stronger property than the standard completeness: we confront the space with a much larger class of generalized (uniform) spaces. For countably generated uniformities, however, the two properties of completeness coincide.

A *completion* of a uniform frame $(L, \mathcal{A})$ is a dense uniform embedding $(M, \mathcal{B}) \to (L, \mathcal{A})$ such that $(M, \mathcal{B})$ is complete.

A completion exists, it is unique and functorial, and can be constructed as a frame of specific down-sets, namely as the $\mathbf{C}(L, \mathcal{A})$ consisting of all the $U = \downarrow U \subseteq L$ such that

(C1) if $\{x \mid x \lhd_{\mathcal{A}} y\} \subseteq U$ then $y \in U$;

(C2) if $\{a \wedge \{x\} \mid a \in A\} \subseteq U$ for some $A \in \mathcal{A}$, then $x \in U$.

This $\mathbf{C}(L, \mathcal{A})$ is endowed with the uniformity generated by

$$\{\{\downarrow a \mid a \in A\} \mid A \in \mathcal{A}\}$$

and we have dense embeddings $\gamma_{(L,\mathcal{A})} \colon \mathbf{C}(L, \mathcal{A}) \to (L, \mathcal{A})$ (compare with the compactification in Theorem 2-6.9). To prove that these simple formulas do the job takes, of course, a few pages (see, e.g., [15], [255]) but it can be argued that it is simpler than the classical completion, and, above all, similarly to the point-free Stone–Čech compactification, it is fully constructive.

2-8.4 An application. Behaviour of paracompact frames

We have already encountered paracompactness under another name. The condition below is equivalent – both for spaces and frames – with full normality (see the preceding subsection). The standard definition is as follows. A (regular) frame (or space) is *paracompact* if every cover A has a locally finite refinement B (that is, a $B \leq A$ such that for some cover W the set $\{b \in B \mid b \wedge w \neq 0\}$ is finite).

Besides the full normality, this property is equivalent with all the properties usually considered as its variants in classical topology. But one has also another, very useful, and a very pretty characteristic that does not hold in classical theory. Namely we have

Theorem 2-8.1 (Isbell [204]). *A frame is paracompact iff it admits a complete uniformity.*

See also [18].

It is well known that in classical topology, the paracompactness, although a very useful property, is not very well behaved under standard constructions (even a product of a paracompact space with a metric one is not always paracompact). Not so in the point-free context. Here,

> *the category of paracompact locales is reflexive in the category of completely regular locales* (*and consequently in* **Loc** *itself*).

Hence, all the limit constructions (in particular the product) inherit, and the colimit ones reflect. This is obtained as an application of Theorem 2-8.1 – the trick is to take the underlying frame of the completion of the given one endowed with the fine uniformity.

2-9. Exercises

2.1. Find a simple example of a T_1 space that is not sober, and an example of a sober space that is not T_1.

2.2. A space X is sober iff the homomorphism $\Omega(j)$ induced by an embedding $j\colon X \subsetneq Y$ as a subspace is never an isomorphism.

2.3. A space X is said to be T_D (see [11]) if for each $x \in X$ there is an open $U \ni x$ such that $U \smallsetminus \{x\}$ is open. We have the implications $T_1 \;\Rightarrow\; T_D \;\Rightarrow\; T_0$ none of which can be reversed.

2.4. A space X is T_D iff the homomorphism $\Omega(j)$ induced by an embedding $j\colon Y \subsetneq X$ of a subspace is never an isomorphism.

2.5. An element a of a frame is *prime* if $x \wedge y \le a$ implies that either $x \le a$ or $y \le a$. Show that there is a natural correspondence between prime elements $a \in L$ and frame homomorphisms $h\colon L \to \mathbf{2}$. Reformulate the definition of spectrum accordingly.

2.6. An element a in a Boolean frame is prime iff it is a co-atom. Consequently there are arbitrarily large frames that have no (spectral) points.

2.7. A frame L is spatial iff the unit ε_L from Section 2-2.4 is an isomorphism. Thus, if L is not isomorphic to $\Omega(\Sigma L)$ it is not isomorphic to any $\Omega(X)$ whatsoever.

2.8. A space X is sober iff the unit η_X from Section 2-2.4 is a homeomorphism.

2.9. The subcategory **Sob** is reflexive in **Top**.

2.10. A *localic map* $M \to L$ is a right Galois adjoint to a frame homomorphism $L \to M$.

Let $f\colon M \to L$ preserve all meets (and hence have a left Galois adjoint $f^*\colon L \to M$). Then it is a localic map iff

(a) $f(a) = 1 \;\Rightarrow\; a = 1$, and

(b) $f(f^*(a) \to b) = a \to f(b)$ (where $\to$ is the Heyting operation).

2.11. Let L be a frame. A subset $S \subseteq L$ is a sublocale iff the embedding $j\colon S \subseteq L$ is a localic map.

2.12. The image $f[S]$ of a sublocale $S \subseteq M$ under a localic map $f\colon M \to L$ is a sublocale of L.

2.13. A localic map sends prime elements to prime elements. Reformulate spectrum as a covariant functor from the category of frames and localic maps into **Top**.

2.14. Let L be a frame and $A \subseteq L$ a subset closed under all meets. Then there is a largest sublocale $A_{\mathrm{sl}} \subseteq A$.

2.15. We can define a preimage $f_{-1}[S]$ of a sublocale $S \subseteq L$ under a localic map $f\colon M \to L$ as $(f^{-1}[S])_{\mathrm{sl}}$. There is a Galois adjunction

$$f[S] \subseteq T \quad \text{iff} \quad S \subseteq f_{-1}[T].$$

2.16. Preimages of open (resp. closed) sublocales are open (resp. closed).

2.17. A frame homomorphism $h\colon L \to M$ is said to be open if the images of open sublocales under the associated localic map are open.

Prove that $h\colon L \to M$ is open iff for each $a \in M$ there is a $b \in L$ such that

$$\forall x, y \in L, \quad x \wedge b = y \wedge b \;\text{ iff }\; h(x) \wedge a = h(y) \wedge a.$$

2.18. (Joyal and Tierney [215]) A frame homomorphism $h\colon L \to M$ is open iff it is a complete Heyting homomorphism.

(Hint: replace the formula $x \wedge b = y \wedge b$ iff $h(x) \wedge a = h(y) \wedge a$ by $x \wedge b \leq y \wedge b$ iff $h(x) \wedge a \leq h(y) \wedge a$ and consider the associated Galois adjunctions.)

2.19. Recall Section 2-3.4. We have a one-to-one frame homomorphism

$$\nabla = (a \mapsto \nabla_a)\colon L \to \operatorname{Con} L.$$

This ∇ is an epimorphism. It is onto iff L is Boolean.

2.20. Formulate a definition of closure in terms of sublocale embeddings.

2.21. Construct explicitly a free frame over a set.

2.22. Recall Section 2-4.4. Prove in detail that $\downarrow a \cup \mathbb{O}$ is saturated.

2.23. Prove that the subset

$$\downarrow(a_1, 1) \cup \downarrow(1, a_2) \cup N$$

of $L_1 \times L_2$ is saturated. Generalize.

2.24. For $a_i \in L_1$, $i = 1, 2$, consider $\downarrow a_i$ as frames. We have $\downarrow a_1 \oplus \downarrow a_2 = \downarrow(a_1 \oplus a_2)$; that is, the homomorphisms

$$(x \mapsto \iota_i(x) \wedge (a_1 \oplus a_2))\colon\ \downarrow a_i \to \downarrow(a_1 \oplus a_2)$$

constitute a coproduct in **Frm**.

2.25. For the codiagonal $\nabla\colon L \oplus L \to L$ (that is, the homomorphism satisfying $\nabla \cdot \iota_i = \mathrm{id}$) we have $\nabla(a \oplus b) = a \wedge b$, and its closure (do Exercise 2.20 first) is the sublocale embedding

$$\check{d}_L\colon L \oplus L \to \uparrow d_L$$

where $d_L = \bigvee\{x \oplus y \mid x \wedge y = 0\}$.

2.26. A frame is said to be *fit* if

$$a \not\leq b \quad \Rightarrow \quad \exists c,\ a \vee c = 1 \text{ and } c \to b \neq b.$$

Prove that fitness is a hereditary property, that is, that each sublocale of a fit frame is fit.

2.27. A frame is regular iff

$$a \not\leq b \quad \Rightarrow \quad \exists c,\ a \vee c = 1 \text{ and } c \to 0 = c^* \neq b.$$

Consequently, a regular frame is fit.

2.28. Use the formula $x = (x \vee c) \wedge (c \to x)$ from the proof of Proposition 2-3.2 and prove that a fit frame is subfit.

2.29. Prove that regularity and complete regularity are hereditary properties.

2.30. A coproduct of regular (resp., completely regular) frames is regular (resp., completely regular).

2.31. A frame L is normal iff for any a_i, $i = 1, \dots, n$, with $a_1 \vee \cdots \vee a_n = 1$ there are $b_i \prec a_i$ such that $b_1 \vee \cdots \vee b_n = 1$.

2.32. A frame L is Hausdorff iff there exists a frame homomorphism $\alpha\colon L \to \uparrow d_L$ such that $\alpha \cdot \nabla = \check{d}_L$.

Prove that a frame L is Hausdorff iff for any $a, b \in L$,

$$a \oplus b \leq ((a \wedge b) \oplus (a \wedge b)) \vee d_L.$$

(Hint: set $\alpha(x) = (x \oplus x) \vee d_L$.)

2.33. Prove that each regular frame is Hausdorff.

(Hint: for $x \prec a$ and $y \prec b$ we have $x \oplus y = (x \wedge (y^* \vee b)) \oplus (y \wedge (x^* \vee a)) \leq ((a \wedge b) \oplus (a \wedge b)) \vee d_L$.)

2.34. Let L be Hausdorff. Then the coequalizer of frame homomorphisms $h_1, h_2\colon L \to M$ is the sublocale embedding $\check{c}\colon M \to \uparrow c$ where $c = \bigvee\{h_1(x) \oplus h_2(x) \mid x \wedge y = 0\}$.

(Hint: compare c with d_L.)

2.35. Prove the previous statement for regular frames L directly.

2.36. Use Proposition 2-5.6 to prove that if the frame M is regular then $h : M \to L$ is open iff it is complete.

2.37. Check the functor and transformation properties of $\mathrm{R\,Id}$ and the system $v = (v_L)_L$.

2.38. A frame L is *Lindelöf* if each of its covers has a countable subcover. Each closed sublocale of a Lindelöf frame is Lindelöf. Generalize.

2.39. A regular Lindelöf frame is normal.

(Imitate the procedure from classical spaces taking advantage of pseudocomplements.)

2.40. For any covers A, B, we have $A(Bx) \leq (AB)x = A(B(Ax))$.

2.41. Recall the Galois adjunction (8.1.1). A system of covers $\mathcal{A}$ on L is admissible iff

$$\forall x \in L, \quad x = \bigvee \{x/A \mid A \in \mathcal{A}\}.$$

2.42. The system of all covers of a frame L is admissible iff L is regular.

2.43. An admissible system of covers of L satisfying just the condition (1) from Section 2-8.2 is called a *nearness* on L.

A frame L admits a nearness iff it is regular.

2.44. A frame L admits a uniformity iff it is completely regular.

Hint: For a sequence $a_1 \prec\!\!\prec a_2 \prec\!\!\prec \cdots \prec\!\!\prec a_n$ set

$$A(a_1, a_2, \ldots, a_n) = \{a_2, a_1^* \wedge a_3, a_2^* \wedge a_4, \ldots, a_{n-2}^* \wedge a_n, a_{n-1}^*\}.$$

Show that for the sequence

$$B = A(a_1, u_1, v_1, a_2, u_2, v_2, a_3, \ldots, v_{n-1}a_n)$$

one has $BB \leq A(a_1, a_2, \ldots, a_n)$.

2.45. Each finite cover of a normal frame has a finite star refinement.

Hint: Recall Exercise 2.31. For a cover $\{a_1, \ldots, a_n\}$ take a cover $\{b_1, \ldots, b_n\}$ such that $b_i \prec a_i$. Consider the cover

$$\{b_1, \ldots, b_n\} \wedge \{b_1^*, a_1\} \wedge \cdots \wedge \{b_n^*, a_n\}.$$

2.46. A compact regular frame admits precisely one nearness, namely the system of all covers, and this nearness is a uniformity.

(Hint: it admits a uniformity $\mathcal{A}$; prove that each cover is refined by a cover from $\mathcal{A}$.)

2-10. Problems

We mention just a few open problems, chosen so that their formulation is not particularly technical. All of them are probably rather difficult.

Problem 2.1 (Endomorphism monoids)**.** Using the deep result of C. Cook [40] one can prove that each monoid can be represented by the monoid of non-constant continuous maps of a topological space into itself (see [273]). Including the constants, of course, disqualifies spaces for such a universal result. Now in point-free topology one has generalized spaces that have no maps corresponding to the constant ones and hence we can ask whether there is for every monoid M a frame L such that the monoid of ALL homomorphisms $h\colon L \to L$ is isomorphic to M.

For uniform frames and uniform homomorphisms this has been answered in the positive in [16], but the role of the enriched structure is very essential there. In fact, one rather doubts one can have a universal result for plain frames. Any answer, positive or negative, would be of interest.

Problem 2.2. Recall Exercise 2.19. The construction can be repeated to obtain embeddings

$$L \to \operatorname{Con} L \to \operatorname{Con}^2 L \to \cdots \to \operatorname{Con}^n L \to \cdots ,$$

and this can be even extended transitively to $\operatorname{Con}^\alpha L$ for arbitrary ordinals α. This procedure is sometimes called the "tower construction".

There exist frames L for which the tower never stops growing, producing rather paradoxical epimorphisms with fixed size of the domain and arbitrarily large codomains. The construction also can stop after the first, second or third step. And this is all that is known. Are there frames L with the tower stopping precisely at the αth step? Even an example with a concrete finite $n > 3$ would be a break.

Problem 2.3 (c-subfitness)**.** We have mentioned in Section 2-5.2 that subfit frames are characterized by the property that every *frame congruence* E on L such that $E1 = \{1\}$ is trivial. In connection with openness and similar questions one encounters the property

(c-subfit) every *complete congruence* E on L such that $E1 = \{1\}$ is trivial.

This seems to be formally a much weaker condition, but the question whether it is not equivalent with subfitness is open.

Problem 2.4 (Localic groups)**.** In analogy with topological groups (and, more generally, other algebras in categories), one studies the groups in the category of locales. A topological group is not necessarily a localic one, because the functor Ω does not in general preserve products (or, in the more comfortable frame setting, does not send products to coproducts). Thus, the multiplication $m\colon X \times X \to X$ transforms into $\Omega(m)\colon \Omega(X) \to \Omega(X \times X)$ while we need a co-multiplication $\mu\colon \Omega(X) \to \Omega(X) \oplus \Omega(X)$. Now there is the obvious canonical morphism $\pi\colon \Omega(X) \oplus \Omega(X) \to \Omega(X \times X)$ given by $\pi \cdot \iota_i = \Omega(p_i)$, and if we can lift $\Omega(m)$ to a μ such that $\pi\mu = \Omega(m)$ then we really obtain a localic group. In all the known cases this is so for the trivial reason that the π in question happens to be an isomorphism. Is there a case of a nontrivial lifting?

Problem 2.5 (Completeness of localic groups)**.** A localic group, similarly to a topological one admits natural uniformities, in the non-commutative case the right one, the left one, and the two-sided one. It was proved by Banaschewski and Vermeulen (see [19]) that the two-sided uniformity is always complete, and it was claimed by Isbell that there exists a non-complete one-sided one. The latter was never published. It would be nice to see an example.

Part II

Special Classes of Finite Lattices

Chapter 3

Planar Semimodular Lattices: Structure and Diagrams

by Gábor Czédli[1] *and George Grätzer*

3-1. Introduction

While the study of *planar* lattices goes back to the 1970s (K.A. Baker, P.C. Fishburn, and F.S. Roberts [20] and D. Kelly and I. Rival [223]), a systematic study of *planar semimodular* lattices began only in 2007 (G. Grätzer and E. Knapp [140]–[144] and G. Grätzer and T. Wares [182]). This was followed by G. Czédli and E.T. Schmidt [55]–[57], and G. Czédli [44]. This chapter presents an overview of these papers.

Many properties of planar semimodular lattices are properties of their planar diagrams; we emphasize this point of view in this chapter.

We start in Section 3-2 by discussing some results on semimodular lattices not requiring planarity. In Section 3-3, we develop the basic concepts of planarity of lattices and diagrams. Slim lattices are introduced in Section 3-4. In Section 3-5, we introduce a construction of planar semimodular lattices

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from planar distributive lattices by inserting forks. The twin construction, using resections, is presented in Section 3-6. Rectangular lattices form an important subclass of planar semimodular lattices; we specialize the results of the previous two sections to this subclass in Section 3-7. Slim semimodular lattices can be described by 0-1-matrices, as described in Section 3-8. They can also be described by permutations, see Section 3-9. Finally, in Section 3-10, we present variants of the Jordan–Hölder Theorem.

Conventions

A planar lattice or a planar diagram is finite by definition. Hence, unless otherwise stated, all lattices and diagrams are assumed to be *finite*. A planar diagram of a planar lattice is a *planar diagram*. Lattice *properties* are also used for diagrams in a self-explanatory way. For example, for "a planar diagram D of a semimodular lattice", we write "a planar semimodular diagram D". If D is a planar diagram of a planar lattice L, then $x \in D$ and $x \in L$ have the same meaning. We use $x \in D$ to indicate that the context is D, for instance, for the left boundary.

3-2. ◊ Some related results

We briefly survey some concepts and results for semimodular lattices that do not require planarity.

A map of a subset of a lattice into another lattice is called *cover-preserving* if it preserves the $\preceq$ ($\prec$ or $=$) relation. A subset X of a lattice L is a *cover-preserving* subset if the natural embedding of X into L is cover-preserving. By Exercise 3.1 (note that semimodularity is a property of join-semilattice reducts rather than of lattices), cover-preserving embeddings and cover-preserving join-homomorphism are natural morphisms in the category of semimodular lattices of finite length.

The following theorem is due to M. Stern [308].

◊ **Theorem 3-2.1.** *Finite semimodular lattices can be characterized as cover-preserving join-homomorphic images of finite distributive lattices.*

A new approach to this result for the planar case started a systematic study of planar semimodular lattices, see G. Grätzer and E. Knapp [140] and Exercises V.2.26 and 27 in LTF.

A stronger form of the Grätzer-Knapp result was stated in G. Czédli and E.T. Schmidt [51]:

◊ **Theorem 3-2.2.** *Let L be a finite semimodular lattice.*

(i) *Let $C_1, \ldots, C_n$ be maximal chains of L such that* $\mathrm{Ji}\, L \subseteq C_1 \cup \cdots \cup C_n$. *Then the map $(c_1, \ldots, c_n) \mapsto c_1 \vee \cdots \vee c_n$ of $C_1 \times \cdots \times C_n$ into L is a surjective cover-preserving join-homomorphism.*

(ii) *There is a unique finite distributive lattice* D *and a unique surjective cover-preserving join-homomorphism* $\varphi\colon D \to L$ *such that* $\operatorname{Ji} D = \operatorname{Ji} L$ *and* φ *acts identically on* $\operatorname{Ji} D$.

Another way of characterizing finite semimodular lattices is to embed them into lattices with special properties. G. Grätzer and E.W. Kiss [138] proved the following result.

◊ **Theorem 3-2.3.** *Finite semimodular lattices can be embedded as cover-preserving* $\{0,1\}$*-sublattices into finite geometric lattices.*

M. Wild [341] gave this theorem a matroid-theoretic proof. G. Czédli and E.T. Schmidt [52] extended this result from finite lattices to lattices of finite length. B. Skublics [303] even further extended the class of lattices for which this result holds.

3-3. Planarity and diagrams

Just as for planar geometry, our geometric intuition regards many statements about planar lattices as obvious. However, sometimes it is not so easy to provide proofs. In this section, we discuss several such statements; prove them or at least reference them, and provide some hints in the exercises. Most of these statements can be found in D. Kelly and I. Rival [223].

Unless otherwise stated, semimodularity is not assumed in this section.

A finite lattice L is *planar* if it has a *planar diagram*, that is, a diagram in which edges can be incident only at their endpoints; see also Exercises 3.5 and 3.7. For a lattice L, the set of *planar diagrams* of L will be denoted by $\operatorname{Dgr}(L)$; this set is nonempty iff L is a planar lattice. To make the definition of a planar diagram more precise, let $\mathbb{R}$ be the field of real numbers, so $\mathbb{R}^2$ is the plane.

Here is a formal definition of planar diagrams from D. Kelly and I. Rival [223]:

Definition 3-3.1. A *planar diagram* D of a finite lattice L is a pair $D = (\varphi, E)$ with the following three properties:

(i) φ is a one-to-one map of L into $\mathbb{R}^2$ such that if $a < b$ in L and $\varphi(a) = (a_1, a_2)$, $\varphi(b) = (b_1, b_2)$, then $a_2 < b_2$;

(ii) E is the set of line segments between $\varphi(a)$ and $\varphi(b)$ for all $a \prec b$ in L;

(iii) two distinct line segments of E are not incident except possibly at their endpoints.

The elements of E are called the *edges* of the diagram.

Next, we recall some basic concepts from D. Kelly and I. Rival [223]. One could formally define them using Definition 3-3.1.

A planar diagram D of a lattice L has a *left boundary chain* $\mathrm{C_l}(D)$, a *right boundary chain* $\mathrm{C_r}(D)$, and a *boundary* $\mathrm{Bnd}(D) = \mathrm{C_l}(D) \cup \mathrm{C_r}(D)$. The *interior* of D, is defined as $\mathrm{int}(D) = D - \mathrm{Bnd}(D)$.

If C is a maximal chain of L, then it has a *left side*, $\mathrm{LS}(C, D)$ ($\mathrm{LS}(C)$, for short), and a *right side*, $\mathrm{RS}(C, D)$ ($\mathrm{RS}(C)$, for short). Observe that

$$L = \mathrm{LS}(C) \cup \mathrm{RS}(C),$$
$$C = \mathrm{LS}(C) \cap \mathrm{RS}(C).$$

Assume that $a \leq b$ in L and $D \in \mathrm{Dgr}(L)$. Let $D_{a,b}$ be the restriction of the diagram D to $[a, b]$. Let C_1 and C_2 be maximal chains of $[a, b]$ such that $C_1 \subseteq \mathrm{LS}(C_2, D_{a,b})$ and $C_2 \subseteq \mathrm{RS}(C_1, D_{a,b})$. Then

$$R = \mathrm{RS}(C_1) \cap \mathrm{LS}(C_2)$$

is a *region* of D. It is a convex sublattice, see Exercise 3.3, and $\mathrm{C_l}(R, D_{a,b}) = C_1$ and $\mathrm{C_r}(R, D_{a,b}) = C_2$.

A minimal non-chain region is called a *cell*, a four-element cell is a 4-*cell*; it is also a *covering square*, that is, cover-preserving four-element Boolean sublattice of L. A diagram of M_3 has exactly two 4-cells and three covering squares. A 4-cell A of D consists of its bottom, 0_A, top, 1_A, *left corner*, lc(A), and *right corner*, rc(A). (Upper case acronyms define sets, lower case acronyms, elements.)

A planar lattice diagram is called a 4-*cell diagram* if all of its cells are 4-cells. A planar lattice L is a 4-*cell lattice* if it has a 4-cell diagram. Equivalently, see Exercise 3.19, if *all* planar diagrams of L are 4-cell diagrams. For example, M_3 is a 4-cell lattice but N_5 is not.

Recall that $\mathrm{Ji}\, L$ is the order of non-zero join-irreducible elements of L, and $\mathrm{Mi}\, L$ is defined dually. Finally, $\mathrm{Di}\, L$ is the order of doubly irreducible elements of L.

The following statements are in D. Kelly and I. Rival [223, Lemmas 1.2 and 1.5 and Proposition 2.2].

◊ **Lemma 3-3.2.** *For a planar lattice L, let $D \in \mathrm{Dgr}(L)$, and let C be a maximal chain of L.*

(i) *If $x, y \in D$ are on different sides of C and $x \leq y$, then there is an element $z \in C$ with $x \leq z \leq y$. In particular, if $x \prec y$, then they cannot be on different sides of C outside of C.*

(ii) *Every interval of L is a region of D.*

(iii) *If $|L| \geq 3$, then there are doubly irreducible elements in $\mathrm{C_l}(D)$ and $\mathrm{C_r}(D)$.*

The following statement follows easily from Lemma 3-3.2.

◊ **Lemma 3-3.3.** *Let R be a region of $D \in \mathrm{Dgr}(L)$.*

(i) $\mathrm{int}(R) \subseteq \mathrm{int}(L)$.

(ii) *If $u < v$ in L and $|R \cap \{u, v\}| = 1$, then $[u, v] \cap \mathrm{Bnd}(R)$ is nonempty.*

(iii) *If $x \in \mathrm{int}(R)$, then all upper and lower covers of x in L belong to R.*

For $i \in \{1, 2\}$, let L_i be a planar lattice and let $D_i \in \mathrm{Dgr}(L_i)$. A bijective map $\varphi\colon D_1 \to D_2$ is a *diagram isomorphism* if it is a lattice isomorphism $\varphi\colon L_1 \to L_2$. Equivalently, if $x \prec y$ iff $\varphi(x) \prec \varphi(y)$ for any pair of vertices $x, y \in D_1$. A diagram isomorphism $\varphi\colon D_1 \to D_2$ is called a *similarity map* if

$$\text{for all } x, y, z \in D_1 \text{ such that } x \prec y \text{ and } x \prec z, \quad y \text{ is to the left of } z \text{ iff } \varphi(y) \text{ is to the left of } \varphi(z), \tag{3-3.1}$$

and symmetrically, see also Exercise 3.9. Following D. Kelly and I. Rival [223, p. 640], we say that D_1 and D_2 are *similar* lattice diagrams if there exists a similarity map $D_1 \to D_2$.

Similarity is an equivalence relation on $\mathrm{Dgr}(L)$. Since all the concepts we have defined so far are invariant under similarity (see Exercises 3.10, 3.12, and 3.13), we consider lattice diagrams up to similarity.

In addition to similarity, there is left-right similarity. Two lattice diagrams, D_1 and D_2, are *left-right similar* if D_1 is similar to D_2 or D_1 is similar to the mirror image of D_2 over a vertical axis. We say that the diagrams of a planar lattice L are *unique up to left-right symmetry* if D_1 is left-right similar to D_2 for any $D_1, D_2 \in \mathrm{Dgr}(L)$.

Planar semimodular lattices can be characterized by properties of their diagrams, see G. Grätzer and E. Knapp [140, Lemmas 4 and 5].

Lemma 3-3.4. *Let L be a planar lattice.*

(i) *If L is semimodular, then it is a 4-cell lattice. If $D \in \mathrm{Dgr}(L)$ and A, B are 4-cells of D with the same bottom, then these 4-cells have the same top.*

(ii) *If L has a planar 4-cell diagram E in which no two 4-cells with the same bottom have distinct tops, then L is semimodular.*

Proof. Assume that L is semimodular and $D \in \mathrm{Dgr}(L)$. A cell that is not a 4-cell is a non-semimodular cover-preserving sublattice, contradicting Exercise 3.1. Let A and B be 4-cells with $0_A = 0_B$. Among lc(A), rc(A), lc(B), and rc(B), let x be the leftmost one and y be the rightmost one. Then $x \neq y$, and the interval $[0_A, x \vee y]$ is of length 2 by semimodularity. Hence this interval is a region by Lemma 3-3.2(ii). Since lc(A), rc(A), lc(B), and rc(B) all belong to this region, we easily conclude from Lemma 3-3.3 that $1_A = 1_B$, proving (i).

To prove (ii), we verify that if $x \wedge y \prec x$ and $x \wedge y \prec y$, then $x \prec x \vee y$ and $y \prec x \vee y$. It is folklore that this implies semimodularity for finite lattices.

Assume that $z = x \wedge y$ is covered by x and y such that x is to the left of y in E; let $v = x \vee y$. Assume also that $x = a_0, a_1, \dots, a_n = y$ are all the covers of z between x and y, listed from left to right. Let $i \in \{1, \dots, n\}$. We conclude, using Lemma 3-3.2(ii), that the region R_i with

$$\mathrm{C_l}(R_i) = \{z\} \cup \mathrm{C_r}([a_{i-1}, a_{i-1} \vee a_i]),$$
$$\mathrm{C_r}(R_i) = \{z\} \cup \mathrm{C_l}([a_i, a_{i-1} \vee a_i])$$

is a cell. Thus R_i is a 4-cell and so $a_{i-1} \prec a_{i-1} \vee a_i$ and $a_i \prec a_{i-1} \vee a_i$. Furthermore, the 4-cells R_i have the same top element, v. Hence $x = a_0 \prec v$ and $y = a_n \prec v$, completing the proof. $\square$

We also need the following well-known concepts.

Definition 3-3.5. An element a of a lattice L is a *narrows* if a is comparable with all elements of L. The set of narrows of L is denoted by $\mathrm{Nar}(L)$. L is (*glued sum*) *indecomposable* if $|L| \geq 3$ and $\mathrm{Nar}(L) = \{0, 1\}$. For finite lattices L_1 and L_2, we obtain the *glued sum* of L_1 and L_2 (LTF, page 8) by putting L_2 atop L_1 and identifying 1_{L_1} with 0_{L_2}.

Some questions on lattices can be reduced to the indecomposable case.

Let D be a slim semimodular diagram. Two prime intervals of D are *consecutive* if they are opposite sides of a 4-cell (see Section 3-3). As in G. Czédli and E.T. Schmidt [54], maximal sequences of consecutive prime intervals form a C_2*-trajectory.* So a C_2-trajectory is an equivalence class of the transitive reflexive closure of the "consecutive" relation.

Similarly, let A and B be two cover-preserving C_3-chains of D. If they are opposite sides of a cover-preserving $\mathsf{C}_3 \times \mathsf{C}_2$, then A and B are called *consecutive.* An equivalence class of the transitive reflexive closure of this "consecutive" relation is called a C_3*-trajectory.*

We recall the basic properties of C_2-trajectories from [54] and [56]; they also hold for C_3-trajectories. For $i \in \{2, 3\}$, a C_i-trajectory goes from left to right (unless otherwise stated); they do not branch out. A C_i-trajectory is of three types: an *up-trajectory*, which goes up (possibly, in zero steps), a *down-trajectory*, which goes down (possibly, in zero steps), and a *hat-trajectory*, which goes up (at least one step), then turns to the lower right, and finally it goes down (at least one step).

Note that the left and right ends of a C_2-trajectory are on the boundary of L; this may fail for a C_3-trajectory.

The *elements* of a C_i-trajectory are the elements of the C_i-chains forming it. Let A be a cover-preserving C_i-chain in D. By planarity, there is a unique C_i-trajectory through A. The C_i-chains of this trajectory to the left of A and including A form the *left wing of* A. The *right wing of* A is defined analogously.

3-4. Slim lattices, the basics

Slim semimodular lattices were defined in G. Grätzer and E. Knapp [140]; for the original definition, see Exercise 3.40. We use here the definition in G. Czédli and E.T. Schmidt [54], which does not require semimodularity.

A finite lattice L is called *slim* if $\mathrm{Ji}\, L$ contains no three-element antichain. It follows from R.P. Dilworth [63] that L is slim iff $\mathrm{Ji}\, L$ is the union of two chains.

◊ **Lemma 3-4.1.** *Every slim lattice is planar.*

For a hint of the proof, see Exercise 3.23. Some properties of slim semimodular lattices are presented in this section and in Exercises 3.24–3.40. The following result is folklore.

Lemma 3-4.2. *A slim semimodular lattice can be uniquely decomposed into a glued sum of maximal chain intervals and indecomposable slim semimodular lattices.*

The next result is from G. Czédli and E.T. Schmidt [54] and G. Grätzer and E. Knapp [140].

Theorem 3-4.3. *For a finite lattice L, the following seven statements are equivalent.*

(i) *L is a slim semimodular lattice.*

(ii) *L is a slim semimodular lattice and a planar 4-cell lattice.*

(iii) *L is a planar semimodular lattice with no cover-preserving diamond sublattice.*

(iv) *L is a planar semimodular lattice and for all $D \in \mathrm{Dgr}(L)$, the 4-cells of D and the covering squares of L are the same.*

(v) *L is a planar semimodular lattice and there exists a diagram $D \in \mathrm{Dgr}(L)$ such that the 4-cells of D and the covering squares of L are the same.*

(vi) *L has a planar 4-cell diagram in which no two distinct 4-cells have the same bottom.*

(vii) *L is a planar and all $D \in \mathrm{Dgr}(L)$ are 4-cell diagrams with no two distinct 4-cells having the same bottom.*

Proof.

(i) $\Rightarrow$ (ii) by Lemma 3-4.1 and Lemma 3-3.4(i).
(ii) $\Rightarrow$ (iii) by Exercise 3.28.
(iii) $\Rightarrow$ (iv) by Lemma 3-3.3(ii).
(iv) $\Rightarrow$ (v) is trivial.

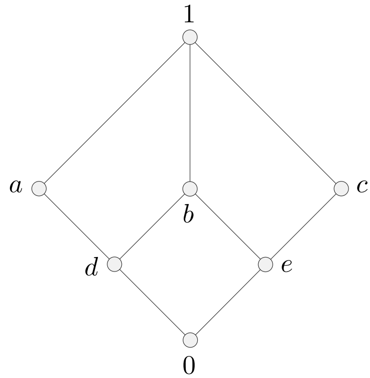

Figure 3-4.1: The lattice N_7.

(v) ⇒ (vi) by Lemma 3-3.4(i) and Exercise 3.28.
(vi) ⇒ (i) by Lemma 3-3.4(ii) and Exercise 3.40.
(ii) ⇒ (vii) by Exercise 3.28.
(vii) ⇒ (vi) is trivial. □

The following lemma is proved in G. Grätzer and E. Knapp [140, Lemma 6] and G. Czédli and E.T. Schmidt [55, Lemma 15]:

◊ **Lemma 3-4.4.** *A slim, planar, semimodular lattice L is distributive iff N_7 (see Figure 3-4.1) is not a cover-preserving sublattice of L.*

Let L be a planar lattice and $D \in \mathrm{Dgr}(L)$. For $u \in L$, the *left support* of u is the largest element of $\mathrm{C}_1(D) \cap \downarrow u$; it is denoted by $\mathrm{lsp}(u, D)$, $\mathrm{lsp}(u)$, for short. We will denote by $\mathrm{lsp}(u)^*$ its unique cover on the left boundary chain. We define the *right support* of u, $\mathrm{rsp}(u, D)$, $\mathrm{rsp}(u)$, symmetrically.

According to Theorem 3-4.3(iv), for a slim semimodular lattice L, we can consider the 4-cells of L. The set of 4-cells is denoted by $\mathrm{Cells}(L)$ or, for $D \in \mathrm{Dgr}(L)$, by $\mathrm{Cells}(D)$. By dropping the assumption of semimodularity, the following theorem generalizes some statements from G. Czédli and E.T. Schmidt [55, Lemma 6] and [56].

Theorem 3-4.5. *Let L be a slim lattice. Then the following statements hold:*

(i) $\mathrm{Bnd}(D) = \mathrm{Bnd}(E)$ *for* $D, E \in \mathrm{Dgr}(L)$ (*that is,* $\mathrm{Bnd}(L)$ *does not depend on the diagram chosen*).

(ii) $\mathrm{Ji}\, L \subseteq \mathrm{Bnd}(L)$.

(iii) *If L is an indecomposable slim lattice, then its planar diagrams are unique up to left-right symmetry.*

Proof. (ii) follows from Exercise 3.26. Using Exercise 3.17, we can decompose L into a glued sum of indecomposable lattices and maximal chain intervals, which are subsets of $\mathrm{Nar}(L)$. Hence it suffices to prove (iii) since (iii) implies (i).

To verify (iii), assume that L is an indecomposable slim lattice and $D, E \in \mathrm{Dgr}(L)$. First, we show that

$$\{\mathrm{C_l}(D), \mathrm{C_r}(D)\} = \{\mathrm{C_l}(E), \mathrm{C_r}(E)\}. \tag{3-4.1}$$

It is clear from Exercise 3.28 that L has exactly two atoms; we denote them by a_1 and b_1. We can assume that $a_1 \in \mathrm{C_l}(D) \cap \mathrm{C_l}(E)$; otherwise, we reflect E vertically. Clearly, $b_1 \in \mathrm{C_r}(D) \cap \mathrm{C_r}(E)$.

Let

$$\mathrm{C_l}(D) = \{0 \prec a_1 \prec \cdots \prec a_n = 1\},$$
$$\mathrm{C_r}(D) = \{0 \prec b_1 \prec \cdots \prec b_m = 1\}.$$

We prove by induction on i and j that $a_i \in \mathrm{C_l}(E)$ for $i \leq n$ and $b_j \in \mathrm{C_r}(E)$ for $j \leq m$; it suffices to deal with the elements a_i. We know that $a_1 \in \mathrm{C_l}(E)$. Assume that $1 < i \leq n$ and $a_{i-1} \in \mathrm{C_l}(E)$. If a_{i-1} is meet-irreducible, then its unique cover, a_i, belongs to $\mathrm{C_l}(E)$ since $\mathrm{C_l}(E)$ is a maximal chain.

Next, assume that $i \geq 2$ and a_{i-1} is meet-reducible. By Exercise 3.28, it has exactly two covers, a_i and x. Exactly one of them belongs to the maximal chain $\mathrm{C_l}(E)$. If $x \in \mathrm{C_l}(E)$. Exercise 3.16, applied to E, yields that $x \in \mathrm{Ji}\, L$. Hence $x \in \mathrm{Bnd}(D)$ by Exercise 3.26. Since $x \parallel a_i$ and $a_i \in \mathrm{C_l}(D)$, we obtain that $x \in \mathrm{C_r}(D)$. The unique lower cover of $x = a_{i-1}$, belongs to $\mathrm{C_r}(D)$ since $\mathrm{C_r}(D)$ is a maximal chain. Hence $a_{i-1} \in \mathrm{C_l}(D) \cap \mathrm{C_r}(D)$ contradicts the indecomposability of L. Thus, $\mathrm{C_l}(D) = \mathrm{C_l}(E)$ and, similarly, $\mathrm{C_r}(D) = \mathrm{C_r}(E)$, whence (3-4.1) follows. Thus, after reflecting E if necessary, we can assume that $\mathrm{C_l}(D) = \mathrm{C_l}(E)$ and $\mathrm{C_r}(D) = \mathrm{C_r}(E)$.

Next, let $x \in L - \{1\}$. According to (3-3.1) and Exercises 3.9 and 3.28, to verify (iii) we have to show that the leftmost upper cover of x is the same in D as it is in E. We can assume that $x \notin \mathrm{Bnd}(D) = \mathrm{Bnd}(E)$. Let $x_0 = \mathrm{lsp}(x)$ and $y_0 = \mathrm{lsp}^*(x)$; for an example, see Figure 3-4.2.

By Exercise 3.36, $C = [x_0, x]$ is a chain. Let x_0^+ denote the cover of x_0 in C; clearly, it is distinct from y_0. Let $z_1 = y_0 \vee x_0^+$, and denote the interval $[x_0, z_1]$ by R_1. Note that R_1 is a region by Lemma 3-3.2(ii). Let

$$x_1 = z_1 \wedge x \in R_1,$$

and note that $x_1 \geq x_0^+$. Note also that R_1 is slim by Exercise 3.22. Furthermore, $y_0 \parallel x_1$, and by Exercise 3.38, $\mathrm{int}(R_1, D) = \mathrm{int}(R_1, E) = \varnothing$. Thus, it follows that y_0 and x_1 belong to opposite boundary chains of R_1. Hence, by Exercise 3.11, we conclude that $x_1 \in \mathrm{C_r}(R_1, D) = \mathrm{C_r}(R_1, E)$.

Let y_1 be the unique cover of x_1 in the chain $[x_1, z_1] \subseteq \mathrm{C_r}(R_1, D) = \mathrm{C_r}(R_1, E)$. We claim that y_1 is the leftmost cover of x_1 in both diagrams.

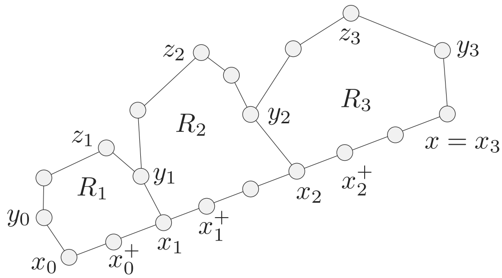

Figure 3-4.2: If x is not on the left boundary chain.

Assume to the contrary that x_1 has a cover t_1 strictly on the left of y_1 with respect to, say, D. Since $\operatorname{int}(R_1, D) = \varnothing$, Lemma 3-3.2(i) implies that $t_1 \in \mathrm{C_l}(R_1, D)$. Thus, $y_0 \leq t_1$, and we obtain that $y_1 \leq z_1 = y_0 \vee x_1 \leq t_1$, contradicting that $y_1 \parallel t_1$.

If $x_1 = x$, then we are ready since the leftmost cover of x_1 is the same with respect to both D and E. Assume that $x_1 \neq x$, that is $x_1 < x$. Let x_1^+ be the unique cover of x_1 in C. If $x_1^+ = y_1$, then $y_1 \leq x \wedge z_1 = x_1 \prec y_1$ is a contradiction. Thus $x_1^+ \neq y_1$. Let $z_2 = y_1 \vee x_1^+$, $R_2 = [x_1, z_2]$, and $x_2 = z_2 \wedge x$. As for R_1, we conclude that R_2 is a slim region and $\operatorname{int}(R_2, D) = \operatorname{int}(R_2, E) = \varnothing$. Clearly, $y_1 \nleq x$ implies that $x_2 \parallel y_1$. This together with the fact that y_1 is the leftmost cover of x_1 yields that $x_2 \in \mathrm{C_r}(R_2, D) = \mathrm{C_r}(R_2, E)$. Let y_2 be the unique cover of x_2 in the chain $[x_2, z_2] \subseteq \mathrm{C_r}(R_2, D) = \mathrm{C_r}(R_2, E)$. The previous argument, with all subscripts increased by one, shows that y_2 is the leftmost cover of x_2 with respect to both diagrams.

If $x = x_2$, then we are ready. If not, then we continue. Finally, we have that $x = x_k \in \mathrm{C_r}(R_k, D) = \mathrm{C_r}(R_k, E)$, and we conclude that y_k is the leftmost cover of x with respect to both diagrams. □

From Theorem 3-4.5(iii), we obtain the following statement immediately.

Theorem 3-4.6. *Let E_1 and E_2 be slim lattice diagrams, and let $\varphi\colon E_1 \to E_2$ be a diagram isomorphism. Then φ is a similarity map iff $\varphi\bigl(\mathrm{C_l}(E_1)\bigr) = \mathrm{C_l}(E_2)$ iff $\varphi\bigl(\mathrm{C_r}(E_1)\bigr) = \mathrm{C_r}(E_2)$.*

This theorem makes it possible to define quotient diagrams for slim lattices. Let L be a slim lattice, $E \in \mathrm{Dgr}(L)$, and let $\boldsymbol{\alpha}$ be a *join-congruence* of E, that is, a congruence of $(L; \vee)$. Then the quotient join-semilattice $L/\boldsymbol{\alpha}$ is a lattice. If there is a diagram $E' \in \mathrm{Dgr}(L/\boldsymbol{\alpha})$ such that

$$\mathrm{C_l}(E') = \{\, x/\boldsymbol{\alpha} \mid x \in \mathrm{C_l}(E) \,\},$$
$$\mathrm{C_r}(E') = \{\, x/\boldsymbol{\alpha} \mid x \in \mathrm{C_r}(E) \,\},$$

then E' is called the *quotient diagram* of E modulo $\boldsymbol{\alpha}$, and it is denoted by $E/\boldsymbol{\alpha}$. By Theorem 3-4.6, $E/\boldsymbol{\alpha}$ is uniquely determined up to similarity. With some additional conditions, we next show that $E/\boldsymbol{\alpha}$ exists.

The kernels of cover-preserving join-homomorphisms are called *cover-preserving join-congruences.* They are characterized in Exercise 3.2. The following theorem generalizes G. Czédli [44, Lemma 11].

Theorem 3-4.7. *If $\boldsymbol{\alpha}$ is a cover-preserving join-congruence of a slim semimodular diagram E, then $E/\boldsymbol{\alpha}$ exists, and it is a slim semimodular diagram.*

Let L be the lattice with $E \in \mathrm{Dgr}(L)$, and let

$$\begin{aligned}\mathrm{C_l}(E) &= \{0 = c_0 \prec c_1 \prec \cdots \prec \cdots \prec c_n = 1\},\\ \mathrm{C_r}(E) &= \{0 = d_0 \prec d_1 \prec \cdots \prec \cdots \prec d_n = 1\}.\end{aligned}$$

Since $\mathrm{Ji}\, L \subseteq \mathrm{Bnd}(E)$ by Theorem 3-4.5(ii),

$$\text{every element of } E \text{ is of the form } c_i \vee d_j. \tag{3-4.2}$$

To verify the Theorem 3-4.7, it suffices to prove the following statement:

Claim. *For every $k \leq \mathrm{len}(L/\boldsymbol{\alpha})$, there exists a diagram $E' \in \mathrm{Dgr}(L/\boldsymbol{\alpha})$ such that whenever $\mathrm{height}(c_i/\boldsymbol{\alpha}) \leq k$ and $\mathrm{height}(d_j/\boldsymbol{\alpha}) \leq k$, then $c_i/\boldsymbol{\alpha} \in \mathrm{C_l}(E')$ and $d_j/\boldsymbol{\alpha} \in \mathrm{C_r}(E')$.*

Proof. Assume that this statement fails. Let k be the smallest integer for which no such E' exists. Let i and j be the smallest integers such that $\mathrm{height}(c_i/\boldsymbol{\alpha}) = k$ and $\mathrm{height}(d_j/\boldsymbol{\alpha}) = k$. Since $\boldsymbol{\alpha}$ is cover-preserving, we have that $c_{i-1}/\boldsymbol{\alpha} \prec c_i/\boldsymbol{\alpha}$ and $d_{j-1}/\boldsymbol{\alpha} \prec d_j/\boldsymbol{\alpha}$. By the minimality of k, we can choose an $E' \in \mathrm{Dgr}(L/\boldsymbol{\alpha})$ such that $c_0/\boldsymbol{\alpha}, \ldots, c_{i-1}/\boldsymbol{\alpha} \in \mathrm{C_l}(E')$ and $d_0/\boldsymbol{\alpha}, \ldots, d_{j-1}/\boldsymbol{\alpha} \in \mathrm{C_r}(E')$. We know that $c_i/\boldsymbol{\alpha} \notin \mathrm{C_l}(E')$ or $d_j/\boldsymbol{\alpha} \notin \mathrm{C_r}(E')$, so we can assume that $c_i/\boldsymbol{\alpha} \notin \mathrm{C_l}(E')$. Clearly, $i > 0$.

If $c_i/\boldsymbol{\alpha}$ is join-irreducible, then $c_i/\boldsymbol{\alpha} \notin \mathrm{C_l}(E')$ and Theorem 3-4.5(ii) yield that $c_i/\boldsymbol{\alpha}$ belongs to $\mathrm{C_r}(E')$. So does its unique lower cover, $c_{i-1}/\boldsymbol{\alpha}$, whence $c_{i-1}/\boldsymbol{\alpha}$ is a narrows by Exercise 3.17. Since

$$\mathrm{height}(c_{i-1}/\boldsymbol{\alpha}) = k - 1 = \mathrm{height}(d_{j-1}/\boldsymbol{\alpha}),$$

we conclude that $c_{i-1}/\boldsymbol{\alpha} = d_{j-1}/\boldsymbol{\alpha}$. Using (3-4.2), we obtain that $c_i/\boldsymbol{\alpha}$ and $d_j/\boldsymbol{\alpha}$ are the only covers of $c_{i-1}/\boldsymbol{\alpha}$, that is, the only atoms of $\uparrow(c_{i-1}/\boldsymbol{\alpha})$. They are distinct since $c_i/\boldsymbol{\alpha} \notin \mathrm{C_l}(E')$. It follows from Theorem 3-4.5(ii) that $d_j/\boldsymbol{\alpha} \in \mathrm{C_l}(E')$ and $c_j/\boldsymbol{\alpha} \in \mathrm{C_r}(E')$. Reflecting the $\uparrow(c_{i-1}/\boldsymbol{\alpha})$ part of the diagram vertically, we obtain a new diagram with $c_i/\boldsymbol{\alpha}$, resp. $d_j/\boldsymbol{\alpha}$, on the left, resp. right, boundary chain. This is a contradiction, so $c_i/\boldsymbol{\alpha}$ is join-reducible.

By (3-4.2) and the join-reducibility of $c_i/\boldsymbol{\alpha}$, we can find indices s and t such that $c_{i-1}/\boldsymbol{\alpha} \parallel (c_s \vee d_t)/\boldsymbol{\alpha} \prec c_i/\boldsymbol{\alpha}$. Clearly, $s < i$ and $0 < t$. Since

$$(c_{i-1} \vee d_t)/\boldsymbol{\alpha} = (c_{i-1} \vee c_s \vee d_t)/\boldsymbol{\alpha} = c_{i-1}/\boldsymbol{\alpha} \vee (c_s \vee d_t)/\boldsymbol{\alpha} = c_i/\boldsymbol{\alpha},$$

there is a smallest j such that $(c_{i-1} \vee d_j)/\boldsymbol{\alpha} = c_i/\boldsymbol{\alpha}$. We have that $0 < j$, because $c_{i-1}/\boldsymbol{\alpha} \neq c_i/\boldsymbol{\alpha}$. Since

$$c_{i-1}/\boldsymbol{\alpha} \leq (c_{i-1} \vee d_{j-1})/\boldsymbol{\alpha} \leq (c_{i-1} \vee d_j)/\boldsymbol{\alpha} = c_i/\boldsymbol{\alpha} \succ c_{i-1}/\boldsymbol{\alpha}$$

and the second inequality is strict by the minimality of j, we conclude the equality $c_{i-1}/\boldsymbol{\alpha} = (c_{i-1} \vee d_{j-1})/\boldsymbol{\alpha}$.

Let us consider an arbitrary $z \in L$ with $c_{i-1}/\boldsymbol{\alpha} < z/\boldsymbol{\alpha}$. Then, for the element $y = z \vee c_{i-1} \vee d_{j-1} \in L$, we have that

$$z/\boldsymbol{\alpha} = z/\boldsymbol{\alpha} \vee c_{i-1}/\boldsymbol{\alpha} = z/\boldsymbol{\alpha} \vee (c_{i-1} \vee d_{j-1})/\boldsymbol{\alpha} = y/\boldsymbol{\alpha}.$$

Since $y = c_{i-1} \vee d_{j-1}$ would imply that $z/\boldsymbol{\alpha} = y/\boldsymbol{\alpha} = (c_{i-1} \vee d_{j-1})/\boldsymbol{\alpha} = c_{i-1}/\boldsymbol{\alpha}$, we obtain that $y \neq c_{i-1} \vee d_{j-1}$. Hence $c_{i-1} \vee d_{j-1} < y$, which together with (3-4.2), implies that $c_{i-1} \vee d_j \leq y$ or $c_i \vee d_{j-1} \leq y$. In the first case, $c_i/\boldsymbol{\alpha} = (c_{i-1} \vee d_j)/\boldsymbol{\alpha} \leq y/\boldsymbol{\alpha} = z/\boldsymbol{\alpha}$, while $c_i/\boldsymbol{\alpha} \leq z/\boldsymbol{\alpha}$ is even more trivial in the second case. This shows that $c_i/\boldsymbol{\alpha}$ is the only cover of $c_{i-1}/\boldsymbol{\alpha}$. Therefore, the unique element covering $c_{i-1}/\boldsymbol{\alpha}$ in $\mathrm{C_l}(E')$ is $c_i/\boldsymbol{\alpha}$, which implies that $c_i/\boldsymbol{\alpha} \in \mathrm{C_l}(E')$, a contradiction.

This proves that $E/\boldsymbol{\alpha}$ exists. It is slim and semimodular by Exercises 3.1 and 3.22. ☐

◊ Slimming and anti-slimming

Let L be a planar semimodular lattice and let $D \in \mathrm{Dgr}(L)$. If we omit the interior elements of D in all intervals of length two, then we obtain a $\{0,1\}$-sublattice, $\mathrm{Slim}\, L$. The elements of $D - \mathrm{Slim}\, L$ are called the *eyes* of D.

$\mathrm{Slim}\, L$, as a subset, depends on D, as illustrated by M_3. However, the following statement – G. Czédli and E.T. Schmidt [56, Lemma 4.1] – establishes that the isomorphism class of $\mathrm{Slim}\, L$ does not depend on D; we call it the *full slimming* (*lattice*) of L.

◊ **Lemma 3-4.8.** *Let L_1 and L_2 be planar semimodular lattices. If L_1 is isomorphic to L_2, then* $\mathrm{Slim}\, L_1$ *is isomorphic to* $\mathrm{Slim}\, L_2$.

The slimming construction has a natural inverse. Let L' be a planar semimodular lattice and $D' \in \mathrm{Dgr}(L')$. Let S be a 4-cell of D'. Replace S by a copy of the diamond M_3 (with a fixed diagram). That is, we insert a new element, which is called an *eye*, into the interior of S. This way we obtain a new diagram, which determines a new lattice. If D is obtained from D' by inserting eyes one-by-one, then D and the corresponding L are called an *anti-slimming* of D', and of L', respectively. Clearly, L is an anti-slimming of $\mathrm{Slim}\, L$. Taking Exercise 3.40 into account, we obtain

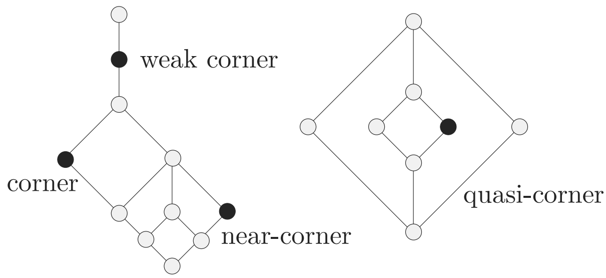

Figure 3-5.1: Corner variants.

◊ **Proposition 3-4.9.** *A planar lattice is semimodular iff some* (*equivalently, all*) *of its full slimming sublattices is slim and semimodular.*

◊ **Corollary 3-4.10.** *Planar semimodular lattices are characterized as anti-slimmings of slim semimodular lattices.*

3-5. Construction with forks

Our goal is to present a construction of all planar semimodular lattices from planar distributive lattices. In view of Corollary 3-4.10, it suffices to deal with *slim* semimodular lattices. This section is based on G. Czédli and E.T. Schmidt [55].

Corners

G. Grätzer and E. Knapp [140] introduced *corners* (corner elements). Several variants of this concept appeared in G. Czédli and E.T. Schmidt [55] and [56], and in G. Czédli [45].

Definition 3-5.1. Let d be an element of a planar lattice L, and let $D \in \mathrm{Dgr}(L)$; see Figure 3-5.1 for an illustration where d is one of the black-filled elements.

(i) If $d \in \mathrm{Di}\, L \cap \mathrm{Bnd}(D)$, then d is called a *weak corner* of D. The elements of $d \in \mathrm{Di}\, L \cap \mathrm{C}_{\mathrm{l}}(D)$ are *left weak corners*. *Right weak corners* are defined similarly.

(ii) A *near corner* is a weak corner d such that d_* has exactly two covers and d^* has at least two lower covers.

(iii) A *corner* is a near corner d such that d_* has exactly two covers and d^* has exactly two lower covers. As in (i), corners and near corners of D are *left* or *right*.

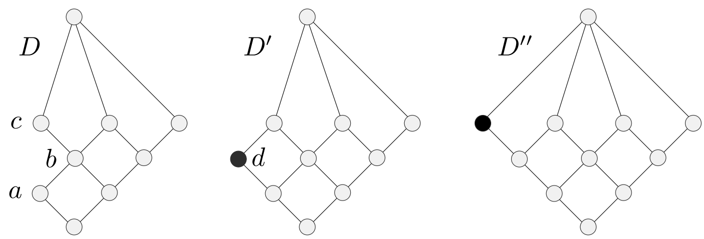

Figure 3-5.2: Adding a corner to D and a near corner to D'.

Assume that L is a *slim* semimodular lattice. Then, since $\mathrm{Bnd}(D)$ does not depend on $D \in \mathrm{Dgr}(L)$ by Theorem 3-4.5(i), we can define weak corners of L. A near corner or a corner of L can be removed to form a cover-preserving sublattice.

Consider the reverse procedure. If $a \prec b \prec c$ is a subchain of $\mathrm{C_l}(D)$ and $a \in \mathrm{Mi}\, L$, then we can *add a near corner* d to D by stipulating that $a \prec d \prec c$ and d be to the left to b. This way, we obtain a new diagram D' with $\mathrm{C_l}(D') = (\mathrm{C_l}(D) - \{b\}) \cup \{d\}$; if $c \in \mathrm{Ji}\, L$, we *add a corner*; see Figure 3-5.2 for examples. Of course, we can add a near corner or a corner to the right boundary chain analogously. If $D \in \mathrm{Dgr}(L)$ is understood, we say that $L' = L \cup \{d\}$, the lattice determined by D', is obtained from L by adding a near corner or a corner.

Proposition 3-5.2.

(i) *Let L' be obtained from a planar lattice L by adding a near corner. Equivalently, let L be obtained from a planar lattice L' by removing a near corner. Then L is semimodular iff L' is semimodular. Similarly, L is slim and semimodular iff L' is slim and semimodular.*

(ii) *Each slim semimodular lattice can be obtained from a chain by adding near corners, one-by-one.*

Proof. (i) follows from Lemma 3-3.4 and Exercise 3.40. To prove (ii), remove near corners, one-by-one, as many as possible. Then by Lemma 3-3.2(iii) and Exercise 3.44, we obtain a chain. The inverse procedure proves the statement. ☐

◊ Forks

Proposition 3-5.2(ii) is our first constructive description of slim semimodular lattices. However, as N_7 of Figure 3-4.1 illustrates, we cannot replace near corners by corners in the proposition. We obtain a deeper result using the following construction.

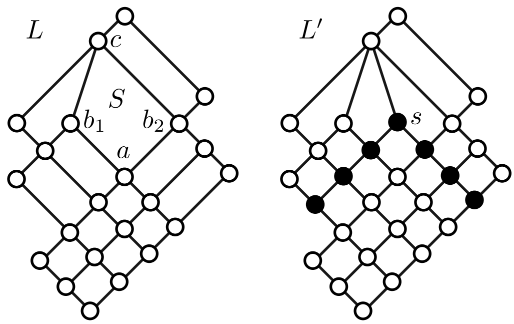

Figure 3-5.3: Adding a fork to L.

Definition 3-5.3.

(i) Let $S = \{a = b_1 \wedge b_2, b_1, b_2, c = b_1 \vee b_2\}$ be a 4-cell of a slim semimodular diagram D. We change D to a new diagram D' as follows.

Firstly, we replace S by a copy of N_7. We get three 4-cells replacing S.

Secondly, we do a series of steps: if there is a chain $u \prec v \prec w$ such that v is a new element and $T = \{x = u \wedge z, z, u, w = z \vee u\}$ is a 4-cell in the original diagram D but $x \prec z$ at the present stage of the construction, see Figure 3-5.3, we insert a new element y such that $x \prec y \prec z$ and $y \prec v$. We get two 4-cells to replace the 4-cell T.

Let D' denote the diagram we obtain when the procedure terminates. (The collection of all new elements, which is an order, is called a *fork*.) We say that D' is obtained from D by *adding a fork to* D at the 4-cell S.

(ii) Let S be a covering square of a slim semimodular lattice L. Choose a diagram $D \in \mathrm{Dgr}(L)$. By Theorem 3-4.3(iv), S is a 4-cell of D. By adding a fork to D at S we obtain a diagram D', which determines a lattice $L' = L[S]$. We say that $L[S]$ is obtained from L by *adding a fork* at S.

(iii) "Adding forks" means adding forks one-by-one.

By Exercises 3.45 and 3.46, $L[S]$ does not depend on the choice of $D \in \mathrm{Dgr}(L)$, and it is a slim semimodular lattice. By G. Czédli and E.T. Schmidt [55], we have

◊ **Theorem 3-5.4.** *A slim semimodular diagram can be obtained from a chain by adding forks and corners.*

A chain with more than one element is a *nontrivial chain*. The direct product of two nontrivial chains is a *grid*. The diagram of a grid is a *grid diagram*.

Now we can state the main result of G. Czédli and E.T. Schmidt [55].

◊ **Theorem 3-5.5.** *A slim semimodular diagram* (*or lattice*) *with at least three elements can be obtained from a grid diagram by*

(i) *first, adding forks,*

(ii) *then, removing corners.*

We can prove Theorems 3-5.4 and 3-5.5 using Exercises 3.30–3.32 and 3.41.

3-6. Construction with resections

We now present a twin of the construction of adding forks, presented in the last section.

We construct slim (planar) semimodular lattices from planar distributive lattices by a series of *resections*. A resection starts with a cover-preserving C_3^2 (the dark gray square of the three-element chain in Figure 3-6.1), and it deletes two elements to get an N_7 (see Figure 3-4.1), and then deletes some more elements (all the black-filled ones), going up and down to the left and to the right, to preserve semimodularity; see Figure 3-6.2 for the result of the resection.

Let B be a cover-preserving $\mathsf{C}_3^2 = \mathsf{C}_3 \times \mathsf{C}_3$ of the diagram D. Let W_l be the left wing of the upper left boundary of B and let W_r be the right wing of the upper right boundary of B. Assume that W_l and W_r terminate on the boundary of D (that is, the last C_3-chains are on the boundary of D). In this case, the collection of elements of $S = B \cup W_l \cup W_r$ is called a C_3-*scheme* of D, see Figure 3-6.1 for an example. The elements of W_l and W_r form the *left wing* and the *right wing* of this C_3-scheme, respectively, while B is the *base*. The middle element of S is the *anchor* of the scheme. A C_3-scheme is uniquely determined by its anchor. Of course, D may have cover-preserving C_3^2's that cannot be extended to C_3-schemes. For example, the slim semimodular diagrams in Figure 3-6.3 have cover-preserving C_3^2 sublattices but no C_3-schemes.

The concept of a C_2-scheme and the related terminology are analogous, see Figure 3-6.2 for an example. The base of a C_2-scheme is a cover-preserving N_7, and its wings are in C_2-trajectories. The middle element of the base is again called the anchor, and it determines the C_2-scheme. Since C_2-trajectories always reach the boundary of D, each cover-preserving N_7 sublattice is the base of a unique C_2-scheme.

For $i \in \{2, 3\}$ and a C_i-scheme S, we define the *upper boundary*, the *lower boundary*, and the *interior* of S as expected.

Let S be a C_3-scheme of a slim semimodular diagram D. By removing all the interior elements of S but its anchor, we obtain a new slim semimodular diagram, D', and S turns into a C_2-scheme of D'. We say that D' is obtained from D by a *resection*; this process is illustrated in Figures 3-6.1 and 3-6.2.

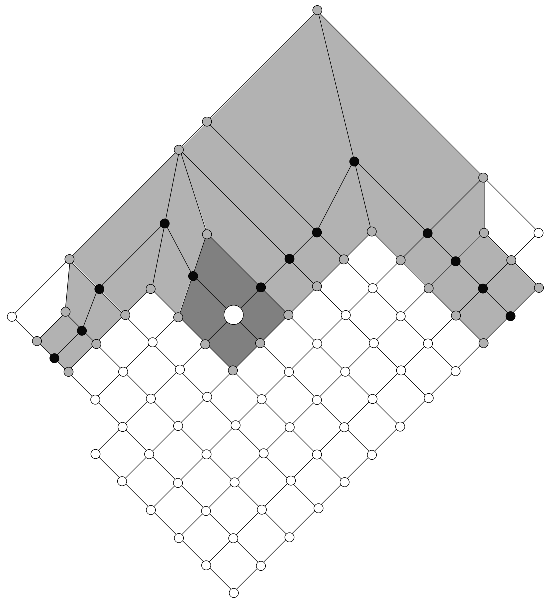

Figure 3-6.1: Resect this diagram at the element marked by the big circle by deleting the black-filled elements ...

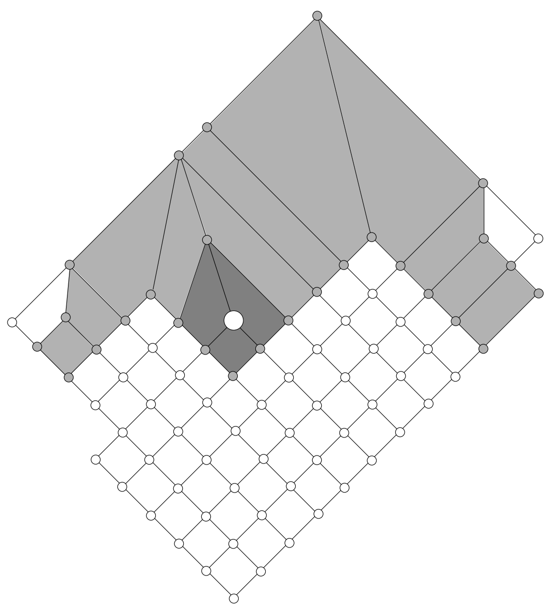

Figure 3-6.2: ... to obtain this diagram.

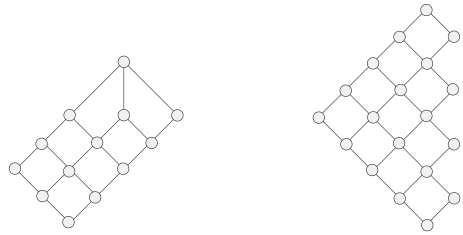

Figure 3-6.3: Two slim semimodular diagrams.

The reverse procedure, transforming a C_2-scheme to a C_3-scheme by adding new interior elements, is called an *insertion*.

We obtain a slim distributive diagram from a grid by a sequence of steps; each step omits a doubly irreducible element from a boundary chain. Our main result generalizes this to slim semimodular lattice diagrams.

◊ **Theorem 3-6.1.** *Slim semimodular lattice diagrams are characterized as diagrams obtained from planar distributive lattice diagrams by a sequence of resections.*

The proof of this theorem appears clear. Let D be a slim semimodular lattice diagram. Find in it a covering N_7 as in Figure 3-6.2. Perform an insertion to obtain the diagram of Figure 3-6.1. The diagram of Figure 3-6.1 has one fewer covering N_7-s. Proceed this way until a diagram is obtained without covering N_7-s.

However, this argument does not necessarily work. Start with D_0, the first diagram in Figure 3-6.4. Apply an insertion at the black-filled element, to obtain the second diagram. Apply an insertion at the gray-filled element of the second diagram, to obtain the third diagram. And so on. It is clear that the number of covering N_7-s is not diminishing.

The proof proceeds by defining the *rank* of an anchor, and performing a resection at an anchor of minimal rank.

3-7. ◊ Rectangular lattices

◊ From the basics to structure theorems

Following G. Grätzer and E. Knapp [143], a semimodular lattice diagram D is *rectangular* if the left boundary chain, $\mathrm{C_l}(D)$, has exactly one weak corner, lc(D) and the right boundary chain, $\mathrm{C_r}(D)$, has exactly one weak corner, rc(D), and these two weak corners are complementary, that is,

$$\mathrm{lc(D)} \vee \mathrm{rc(D)} = 0,$$
$$\mathrm{lc(D)} \wedge \mathrm{rc(D)} = 1.$$

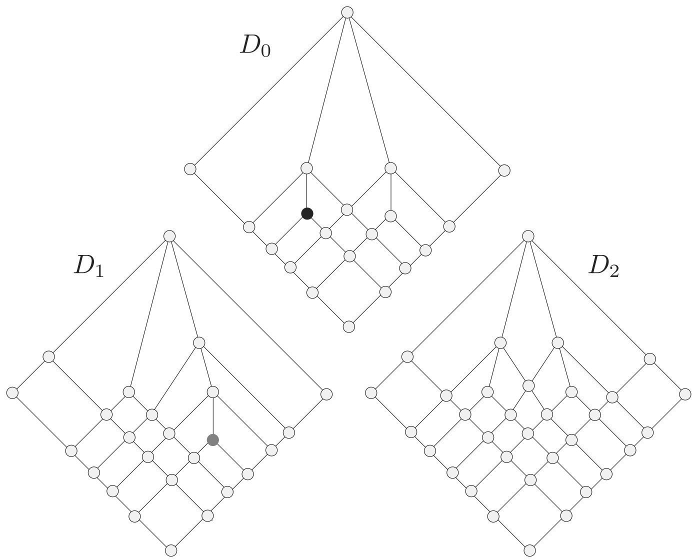

Figure 3-6.4: The process does not stop.

A semimodular lattice L is called a *rectangular lattice* if some $D \in \mathrm{Dgr}(L)$ is rectangular; equivalently, if all $D \in \mathrm{Dgr}(L)$ are rectangular, see Exercise 3.55. Rectangular lattices have nice rectangle-shaped diagrams.

Given a rectangular lattice, for instance, the diamond M_3, its weak corners are not unique. But the rest of the boundary is unique, see Exercise 3.56.

◊ **Lemma 3-7.1** (G. Grätzer and E. Knapp [144]). *Let D be a rectangular diagram. Then the intervals* $[0, \mathrm{lc(D)}]$, $[\mathrm{lc(D)}, 1]$, $[0, \mathrm{rc(D)}]$, *and* $[\mathrm{rc(D)}, 1]$ *are chains.*

So the chains $\mathrm{C_l}(D)$ and $\mathrm{C_r}(D)$ are split into two, a lower and an upper part: $\mathrm{C_{ll}}(D) = [0, \mathrm{lc(D)}]$, $\mathrm{C_{ul}}(D) = [\mathrm{lc(D)}, 1]$, $\mathrm{C_{lr}}(D) = [0, \mathrm{rc(D)}]$, and $\mathrm{C_{ur}}(D) = [\mathrm{rc(D)}, 1]$ ($\mathrm{C_{ll}}$, $\mathrm{C_{ul}}$, $\mathrm{C_{lr}}$, and $\mathrm{C_{ur}}$, for short).

The structure of rectangular lattices is described in the following two statements. The first follows from Theorem 3-5.5 and Exercise 3.54.

◊ **Theorem 3-7.2** (G. Czédli and E.T. Schmidt [55]). *L is a rectangular lattice iff it is an anti-slimming of a lattice that can be obtained from a grid by adding forks.*

◊ **Theorem 3-7.3** (G. Czédli and G. Grätzer [49]). *Every slim rectangular lattice L can be constructed from a grid by a sequence of resections.*

◊ Gluings of rectangular lattices

For a slim rectangular lattice L, let $x \in \mathrm{C}_{\mathrm{ul}}(L) - \{1, \mathrm{lc(L)}\}$ and let $y \in \mathrm{C}_{\mathrm{ur}}(L) - \{1, \mathrm{rc(L)}\}$. We introduce some notation (see Figure 3-7.1):

$$L_{\mathrm{top}}(x, y) = [x \wedge y, 1],$$
$$L_{\mathrm{left}}(x, y) = [\mathrm{lc(L)} \wedge y, x],$$

$$L_{\mathrm{right}}(x, y) = [x \wedge \mathrm{rc(L)}, y],$$
$$L_{\mathrm{bottom}}(x, y) = [0, (\mathrm{lc(L)} \wedge y) \vee (x \wedge \mathrm{rc(L)})].$$

The following result is from G. Grätzer and E. Knapp [144].

◊ Theorem 3-7.4 (Decomposition Theorem). *Let L be a slim rectangular lattice, and let $x \in \mathrm{C}_{\mathrm{ul}}(L) - \{1, \mathrm{lc(L)}\}$, $y \in \mathrm{C}_{\mathrm{ur}}(L) - \{1, \mathrm{rc(L)}\}$. Then L can be decomposed into four slim rectangular lattices $L_{\mathrm{top}}(x, y)$, $L_{\mathrm{left}}(x, y)$, $L_{\mathrm{right}}(x, y)$, $L_{\mathrm{bottom}}(x, y)$, and the lattice L can be reconstructed from these by repeated gluing.*

Let L be a nontrivial lattice. If L cannot be obtained as a gluing of two lattices, we call L *gluing indecomposable.*

◊ Theorem 3-7.5 (G. Czédli and E.T. Schmidt [56])**.** *Let L be a planar semimodular lattice with at least four elements. Then the following six conditions are equivalent.*

(i) *L is gluing indecomposable;*

(ii) *L is gluing indecomposable over chains;*

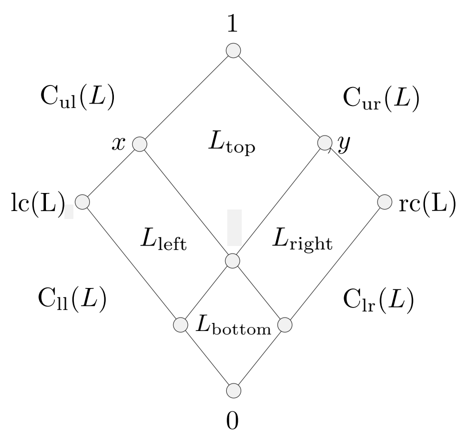

Figure 3-7.1: Decomposing a slim rectangular lattice.

(iii) *L is a rectangular lattice whose weak corners,* $\mathrm{lc}(\mathrm{D})$ *and* $\mathrm{rc}(\mathrm{D})$*, are dual atoms for some rectangular diagram D of L;*

(iv) *L has a planar diagram such that the intersection of the leftmost dual atom and the rightmost dual atom is* 0*;*

(v) *for any planar diagram of L, the intersection of the leftmost dual atom and the rightmost dual atom is* 0*;*

(vi) *L is an anti-slimming of a lattice obtained from the four-element Boolean lattice by adding forks.*

These lattices, along with finite chains, are the "building stones" for constructing all planar semimodular lattices. Instead of the binary operation of gluing, it is possible to construct a planar semimodular lattice in one step with the *patchwork systems* of [56].

The next result trivially follows.

Theorem 3-7.6 (Construction Theorem). *Let L be a rectangular lattice. Then there is a sequence of lattices*

$$K_1, K_2, \ldots, K_n = L$$

such that each K_i, for $i = 1, 2, \ldots, n$, is either a patch lattice or it is the gluing of the lattices K_j and K_k for $j, k < i$.

3-8. ◊ A description by matrices

The main result

The quotient join-semilattice of a grid modulo a cover-preserving join-congruence is a slim semimodular lattice, see Exercise 3.1. By Theorem 3-2.2(i), every slim semimodular lattice can be represented this way. We are going to deal with minimal representations of this kind. Since (finite) chains are trivial as slim semimodular lattices, we only deal with *non-chains*. Remember that, by Definition 3-3.5, an indecomposable lattice is not a chain.

This section is based on G. Czédli [44]. The only change is that instead of dealing with lattices, first we focus on a matrix description for the corresponding diagrams. We consider diagrams up to similarity. Note that, except for symmetric diagrams, we distinguish between a diagram and its vertical mirror image.

Let K be a grid, and let $G \in \mathrm{Dgr}(K)$ be a grid diagram. If $F \subseteq \mathrm{Cells}(G)$, then $A = (G; F)$ is called a *matrix diagram* and the elements of F are called *F-cells*. See Figure 3-8.1 for an illustration, where the F-cells are gray-filled. For an m-by-n matrix diagram $A = (G; F)$, we use the notation

$$\mathrm{C}_{\mathrm{ll}}(G) = \{0 = b_0 \prec \cdots \prec b_m\},$$
$$\mathrm{C}_{\mathrm{lr}}(G) = \{0 = c_0 \prec \cdots \prec c_n\}.$$

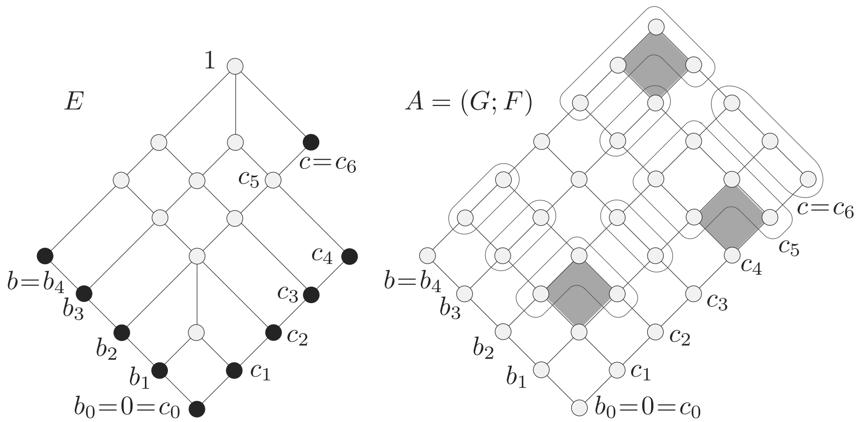

Figure 3-8.1: E and $A = \mathrm{Mtx}(E)$.

The unique 4-cell of G with top $b_i \vee c_j$ is denoted by $\mathrm{cell}(i, j)$. This notation will be used even where we use different symbols for the elements of $\mathrm{C}_{\mathrm{ll}}(G)$ and $\mathrm{C}_{\mathrm{lr}}(G)$.

The ith *row* of $A = (G; F)$ is

$$\{\, \mathrm{cell}(i, j) \in \mathrm{Cells}(G) \mid 0 \le j \le n \,\},$$

and the jth column is defined analogously.

Matrix diagrams are in a bijective correspondence with 0-1-matrices as follows; the m-by-n 0-1-matrix $P = (p_{ij})_{m\times n}$ corresponding to $A = (G; F)$ is defined by the rule $p_{ij} = 1$ if the $\mathrm{cell}(i, j) \in F$ and $p_{ij} = 0$, otherwise.
For example, the 0-1-matrix corresponding to $A = (G; F)$ in Figure 3-8.1 is

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Definition 3-8.1. For a matrix diagram $A = (G; F)$, we define a slim semimodular diagram $\mathrm{QDgr}(A)$ as follows. For $U \in \mathrm{Cells}(G)$, let $\mathrm{con}_{\vee}(U)$ denote the smallest join-congruence of G that collapses $\{\mathrm{lc(U)}, \mathrm{rc(U)}, 1_U\}$. Let $\boldsymbol{\beta} = \bigvee\{\, \mathrm{con}_{\vee}(U) \mid U \in F \,\}$. If $\boldsymbol{\beta}$ is a cover-preserving join-congruence, then $\mathrm{QDgr}(A)$ is the *quotient diagram* $G/\boldsymbol{\beta}$, see Theorem 3-4.7. Otherwise, $\mathrm{QDgr}(A)$ is undefined.

We also need a construction in the opposite direction.

Definition 3-8.2. With an indecomposable, slim, semimodular diagram E, we associate a matrix diagram as follows. Let b and c be the largest elements

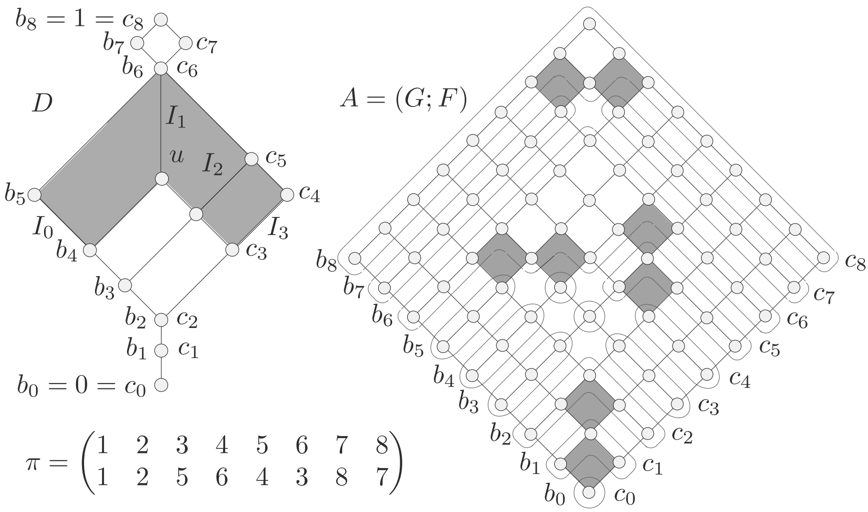

Figure 3-8.2: A diagram D and the corresponding permutation.

of $\mathrm{Ji}\, E \cap \mathrm{C}_{\mathrm{l}}(E)$ and $\mathrm{Ji}\, E \cap \mathrm{C}_{\mathrm{r}}(E)$, respectively. Let

$$\text{(3-8.1)} \qquad \begin{aligned} B &= \{0 = b_0 \prec \cdots \prec b_m = b\} = \mathrm{C}_{\mathrm{l}}(E) \cap {\downarrow} b, \\ C &= \{0 = c_0 \prec \cdots \prec c_n = c\} \,= \mathrm{C}_{\mathrm{r}}(E) \cap {\downarrow} c, \end{aligned}$$

see Theorem 3-4.5(ii) and Figure 3-8.1. Let G be the diagram of $B \times C$ such that

$$\text{(3-8.2)} \qquad \begin{aligned} \mathrm{C}_{\mathrm{ll}}(G) &= \{\, (b_i, 0) \mid 0 \le i \le m \,\}, \\ \mathrm{C}_{\mathrm{lr}}(G) &= \{\, (0, c_j) \mid 0 \le i \le n \,\}. \end{aligned}$$

On G, we define an equivalence $\boldsymbol{\alpha}$ by $((b_i, c_j), (b_h, c_k)) \in \boldsymbol{\alpha}$ iff $b_i \vee c_j = b_h \vee c_k$. In Figure 3-8.2, the blocks of $\boldsymbol{\alpha}$ are represented by dotted lines.

A 4-cell $U \in \mathrm{Cells}(G)$ is called a *source cell* of $\boldsymbol{\alpha}$ if $0_U \notin 1_U/\boldsymbol{\alpha}$ and $\mathrm{lc(U)}, \mathrm{rc(U)} \in 1_U/\boldsymbol{\alpha}$. The set of these source cells is denoted by $\mathrm{SCells}(\boldsymbol{\alpha})$. In Figure 3-8.1, $\mathrm{SCells}(\boldsymbol{\alpha})$ consists of the three gray-filled 4-cells. The matrix diagram we associate with E is $\mathrm{Mtx}(E) = (G, \mathrm{SCells}(\boldsymbol{\alpha}))$.

◊ **Proposition 3-8.3.** *Let $A = (G; F)$ be a matrix diagram. Then the following two conditions are equivalent*

(i) $\mathrm{QDgr}(A)$ *is defined, it is an indecomposable, slim, semimodular diagram and for any matrix diagram $A' = (G'; F')$ such that* $\mathrm{QDgr}(A')$ *is similar to* $\mathrm{QDgr}(A)$*, we have* $|\mathrm{C}_{\mathrm{ll}}(G)| \le |\mathrm{C}_{\mathrm{ll}}(G')|$ *and* $|\mathrm{C}_{\mathrm{lr}}(G)| \le |\mathrm{C}_{\mathrm{lr}}(G')|$.

(ii) *A satisfies the following five conditions:*

(mr1) *every row and every column of A contains at most one F-cell;*

(mr2) $|F| < \min\{m, n\}$*;*

(mr3) $|F \cap \mathrm{Cells}(\downarrow(c_k \vee d_k))| < k$ *for* $k = 1, \dots, \min\{m, n\} - 1$*;*

(mr4) *if* $\mathrm{cell}(i, n) \in F$*, then there is an* i' *such that* $1 \leq i' < i$ *and there is no F-cell in the i'th row;*

(mr5) *if* $\mathrm{cell}(m, j) \in F$*, then there is a* j' *such that* $1 \leq j' < j$ *and there is no F-cell in the j'th column.*

Matrix diagrams satisfying (mr1)–(mr5) are called *regular matrix diagrams*. By Proposition 3-8.3, they are the minimal matrix diagrams to characterize indecomposable slim semimodular diagrams in the following theorem. This theorem was stated in G. Czédli [44] for lattices rather than diagrams in [44]; the proof is similar.

◊ **Theorem 3-8.4.** *Let E be an indecomposable, slim, semimodular diagram, and let A be a regular matrix diagram. Then* $\mathrm{Mtx}(E)$ *is a regular matrix diagram,* $\mathrm{QDgr}(A)$ *is an indecomposable slim semimodular diagram,* $\mathrm{QDgr}(\mathrm{Mtx}(E)) = E$*, and* $\mathrm{Mtx}(\mathrm{QDgr}(A)) = A$.

3-9. Description by permutations

Permutations of slim semimodular lattices were introduced by P. Stanley [306] and H. Abels [1], in a different context.

In this section, we discuss a description of slim semimodular lattices by permutations, see G. Czédli and E.T. Schmidt [57]. We start with a variant of the definition from G. Czédli, L. Ozsvárt, and B. Udvari [50].

Definition 3-9.1. Assume that D is a slim semimodular diagram. Let

$$\begin{aligned} \mathrm{C_l}(D) = B &= \{0 = b_0 \prec b_1 \prec \dots \prec b_h = 1\}, \\ \mathrm{C_r}(D) = C &= \{0 = c_0 \prec c_1 \prec \dots \prec c_h = 1\}. \end{aligned} \tag{3-9.1}$$

We define two maps, $\pi = \pi(D)$ and $\sigma = \sigma(D)$, as follows. For $i, j \in \{1, \dots, h\}$, let

$$\begin{aligned} I(i) &= \{\, j \in \{1, \dots, h\} \mid b_{i-1} \vee c_j = b_i \vee c_j \,\}, \\ \pi(i) &= \text{the smallest element of } I(i), \\ J(j) &= \{\, i \in \{1, \dots, h\} \mid b_i \vee c_{j-1} = b_i \vee c_j \,\}, \\ \sigma(j) &= \text{the smallest element of } J(j). \end{aligned}$$

Then $\pi(D)$ is the *permutation associated with D*.

Of course, we have to prove that $\pi(D)$ is a permutation. The set of permutations acting on $\{1, \ldots, h\}$ is denoted by S_h.

It was proved in H. Abels [1, Remark 2.14] that a slim semimodular lattice is determined by the permutations associated with it. The following statement, due to G. Czédli and E.T. Schmidt [57] and G. Czédli, L. Ozsvárt, and B. Udvari [50], is a stronger version.

Lemma 3-9.2. *If D is a slim semimodular diagram, then $\pi(D)$ and $\sigma(D)$ are permutations, and $\pi(D)^{-1} = \sigma(D)$.*

Proof. Let $\pi = \pi(D)$ and $\sigma = \sigma(D)$. Clearly, $0 \notin I(i) \cup J(j)$ and $h \in I(i) \cap J(j)$. Assume that j belongs to $I(i)$ and $j < h$. Then

$$b_{i-1} \vee c_{j+1} = b_{i-1} \vee c_j \vee c_{j+1} = b_i \vee c_j \vee c_{j+1} = b_i \vee c_{j+1}$$

shows that $j + 1 \in I(i)$. Since the same argument works for $J(j)$, we conclude that, for $i, j \in \{1, \ldots, h\}$, both $I(i)$ and $J(j)$ are (order) filters of $\{1, \ldots, h\}$. For $i \in \{1, \ldots, h\}$, let $j = \pi(i)$. Since $j - 1 \notin I(i)$ and $j \in I(i)$, we obtain that

$$b_{i-1} \vee c_{j-1} < b_i \vee c_{j-1} \leq b_i \vee c_j = b_{i-1} \vee c_j. \tag{3-9.2}$$

Semimodularity implies that $b_{i-1} \vee c_{j-1} \preceq b_{i-1} \vee c_j$. This and (3-9.2) yield that $b_i \vee c_{j-1} = b_i \vee c_j$. Hence $i \in J(j)$, and we obtain that $\sigma(j) \leq i$. If $\sigma(j) < i$, then $i - 1 \in J(j)$ implies that $b_{i-1} \vee c_{j-1} = b_{i-1} \vee c_j$, contradicting (3-9.2). Hence $i = \sigma(j) = \sigma(\pi(i))$, that is, $\sigma \circ \pi$ is the identity map on $\{1, \ldots, h\}$. By symmetry, so is $\pi \circ \sigma$. ☐

G. Czédli and E.T. Schmidt [56] define $\pi(D)$ as follows:

Definition 3-9.3. For $i, j \in \{1, \ldots, k\}$, let $(i, j) \in \pi(D)$ mean that the prime intervals $[b_{i-1}, b_i]$ and $[c_{j-1}, c_j]$ lie in the same trajectory.

This definition is easy to visualize. At this stage, $\pi(D)$ is a binary relation. However, the following statements hold.

Lemma 3-9.4 (G. Czédli and E.T. Schmidt [54] and [56])**.** *Let D be a slim semimodular diagram. Then the following statements hold:*

(i) *Going from left to right, the trajectories depart from the left boundary chain, do not branch out, and arrive at the right boundary chain.*

(ii) *While going from left to right, a trajectory first goes up, possibly in zero steps, then it may take a turn to the lower right, and finally it keeps going down, possibly in zero steps. In particular, once it is going down, there is no further turn.*

(iii) *$\pi(D)$ is a permutation.*

(iv) *$\pi(D)$ is the same as the permutation given in Definition 3-9.1*

Proof. We conclude (i) and (ii) from Theorem 3-4.3(vii). Also, we can obtain them from Theorem 3-5.5. (iii) follows from (i) and its left-right dual. (iv) is Exercise 3.66. ☐

Definition 3-9.5. Given $\pi \in S_h$, we define a matrix diagram $D(\pi)$ as follows. Let $B = \{b_0 \prec \cdots \prec b_h\}$ and $C = \{c_0 \prec \cdots \prec d_h\}$, and let G be the square grid diagram satisfying (3-8.2) with $m = n = h$. Let $F = \{\, \mathrm{cell}(i, \pi(i)) \mid 1 \leq i \leq h \,\}$. This way we obtain a square matrix diagram $A = (G; F)$. With reference to Definition 3-8.1, let

$$\boldsymbol{\beta} = \boldsymbol{\beta}_\pi = \bigvee_{i=1}^{h} \mathrm{con}_\vee(\mathrm{cell}(i, \pi(i))) = \bigvee_{U \in F} \mathrm{con}_\vee(U),$$

and define $D(\pi)$ to be the quotient diagram $G/\boldsymbol{\beta}$.

The proof that $\boldsymbol{\beta}$ is cover-preserving is left to Exercise 3.67. Hence $D(\pi)$ exists and it is a slim semimodular diagram by Theorem 3-4.7. For example, if π is the permutation in Figure 3-8.2, then $A = (G; F)$ and, with dotted lines, the $\boldsymbol{\beta}$-blocks are depicted in the figure. In this case, $D(\pi)$ equals D on the left of the figure.

Let $\mathrm{SSD}(h)^\sim$ be the set of slim semimodular lattice diagrams of length h, where similar diagrams are considered equal.

◊ **Theorem 3-9.6** (G. Czédli and E.T. Schmidt [57]). *For $h \in \mathbb{N}$, the maps*

$$S_h \to \mathrm{SSD}(h)^\sim, \quad \pi \mapsto D(\pi), \qquad \textit{and} \qquad \mathrm{SSD}(h)^\sim \to S_h, \quad D \mapsto \pi(D),$$

are inverse bijections.

Assume that $1 \leq u \leq v \leq h$ and $\pi \in S_h$. Let $I = [u, v] = \{\, i \in \mathbb{N} \mid u \leq i \leq v \,\}$ be nonempty and let $[1, u-1]$, I, and $[v+1, h]$ be closed with respect to π. Then I is called a *section* of π. Sections minimal with respect to set inclusion are called *segments*.

Let $\mathrm{Seg}(\pi)$ denote the set of all segments of π. For example, for the permutation π in Figure 3-8.2, we have $\mathrm{Seg}(\pi) = \{\{1\}, \{2\}, \{3, 4, 5, 6\}, \{7, 8\}\}$. For $\pi, \mu \in S_h$, we say that π and μ are *equal or sectionally inverted* if $\mathrm{Seg}(\pi) = \mathrm{Seg}(\mu)$ and $\mu\rceil_I \in \{\pi\rceil_I, (\pi\rceil_I)^{-1}\}$ for all $I \in \mathrm{Seg}(\pi)$.

We can derive the following statement from Theorem 3-9.6 (see Exercise 3.72).

◊ **Corollary 3-9.7** (G. Czédli and E.T. Schmidt [57]). *Let L_1 and L_2 be slim semimodular lattices of the same length, and let $D_i \in \mathrm{Dgr}(L_i)$ for $i = 1, 2$. Then $L_1 \cong L_2$ iff the permutations $\pi(D_1)$ and $\pi(D_2)$ are equal or sectionally inverted.*

3-10. Variants of the Jordan–Hölder Theorem

Strengthening the Jordan–Hölder Theorem

The classical Jordan–Hölder Theorem goes back to C. Jordan [214] and O. Hölder [199]. Firstly, we deal with its lattice theoretical counterpart. It states that whenever L is a semimodular lattice of finite length, then any two maximal chains of L are of the same length and, in addition, if

$$B = \{b_0 \prec \cdots \prec b_h\}, \quad C = \{c_0 \prec \cdots \prec c_h\} \tag{3-10.1}$$

are maximal chains of L, then there is a permutation $\pi \in S_h$ such that, for all $i \in \{1, \ldots, h\}$, the prime interval $[b_{i-1}, b_i]$ is projective to the prime interval $[c_{\pi(i)-1}, c_{\pi(i)}]$.

For intervals $[a_1, b_1]$ and $[a_2, b_2]$ of a lattice, $[a_1, b_1]$ is *up-perspective* to $[a_2, b_2]$, in notation, $[a_1, b_1] \nearrow [a_2, b_2]$ if $a_2 \vee b_1 = b_2$ and $a_2 \wedge b_1 = a_1$. Dually, $[a_2, b_2] \searrow [a_1, b_1]$ means that $[a_1, b_1] \nearrow [a_2, b_2]$. We say that $[a_1, b_1]$ is *up-and-down projective* to $[a_2, b_2]$, in notation $[a_1, b_1] \mathrel{\nearrow\!\!\searrow} [a_2, b_2]$, if there is an interval $[x, y]$ such that $[a_1, b_1] \nearrow [x, y]$ and $[x, y] \searrow [a_2, b_2]$. This concept was used in the first step of extending the Jordan–Hölder theorem, as follows.

Theorem 3-10.1 (G. Grätzer and J.B. Nation [158]). *Assume that B and C in* (3-10.1) *are maximal chains of a semimodular lattice L. Then there is a permutation $\pi \in S_h$ such that $[b_{i-1}, b_i] \mathrel{\nearrow\!\!\searrow} [c_{\pi(i)-1}, c_{\pi(i)}]$ for all $i \in \{1, \ldots, h\}$.*

Although planarity is not assumed in this theorem, we need the theory of planar semimodular lattices to strengthen it with a uniqueness statement.

Theorem 3-10.2 (G. Czédli and E.T. Schmidt [54]). *The permutation π in Theorem 3-10.1 is uniquely determined. Furthermore, for all $i, j \in \{1, \ldots, h\}$, the up-and-down projectivity $[b_{i-1}, b_i] \mathrel{\nearrow\!\!\searrow} [c_{j-1}, c_j]$ implies that $j \le \pi(i)$.*

To prove these two theorems, we need some notation and lemmas.

The set of *all intervals* and the set of *all prime intervals* of a lattice L are denoted by $\mathrm{Int}(L)$ and $\mathrm{PrInt}(L)$, respectively. As usual, *projectivity* is defined on $\mathrm{Int}(L)$ as the reflexive and transitive closure of perspectivity. If we restrict perspectivity to $\mathrm{PrInt}(L)$ and form its reflexive and transitive closure, then we obtain a relation on $\mathrm{PrInt}(L)$; we call this relation the $\mathrm{PrInt}(L)$-*projectivity*. Note that $\mathrm{PrInt}(L)$-projectivity and the restriction of projectivity to $\mathrm{PrInt}(L)$ are different in general; see Exercise 3.77.

Lemma 3-10.3. *Let L be a semimodular lattice of finite length, and let $[a_0, a_1]$, $[b_0, b_1] \in \mathrm{PrInt}(L)$. Then these two prime intervals are $\mathrm{PrInt}(L)$-projective iff there is $k \in \mathbb{N}_0$ and there are intervals $[x_i, y_i] \in \mathrm{PrInt}(L)$ for $i \le k$ such that $[a_0, a_1] = [x_0, y_0]$, $[b_0, b_1] = [x_k, y_k]$, and $\{x_{i-1}, y_{i-1}, x_i, y_i\}$ is a covering square for all $i \in \{1, \ldots, k\}$.*

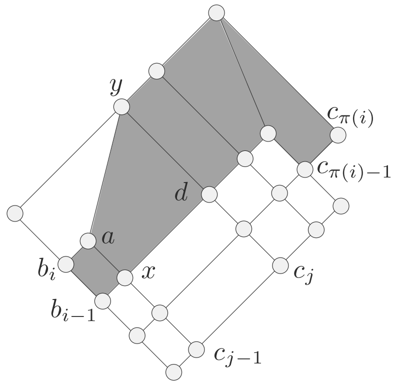

Figure 3-10.1: D with a trajectory.

Proof. Assume that $[a,b],[c,d] \in \mathrm{PrInt}(L)$ such that $[a,b] \nearrow [c,d]$. Take a chain $a = z_0 \prec z_1 \prec \cdots \prec z_t = c$, and define $z_i' = z_i \vee b$. Then the intervals $\{z_{i-1}, z_i, z_{i-1}', z_i'\}$ are covering squares by semimodularity. If $[a,b] \searrow [c,d]$, then $[c,d] \nearrow [a,b]$, and we obtain again covering squares. So each perspectivity defines a covering square, and the collection of these squares verify the "only if" part. The "if" part is evident. $\square$

Lemma 3-10.4. *Consider two prime intervals in a slim semimodular diagram D. Then they are up-and-down projective iff they belong to the same trajectory.*

Proof. Assume that $[a_0,a_1],[b_0,b_1] \in \mathrm{PrInt}(D)$ and $[a_0,a_1] \nearrow\!\searrow [b_0,b_1]$. Then there is an interval $[x,y]$ of D such that $[a_0,a_1] \nearrow [x,y] \searrow [b_0,b_1]$. Since $x \prec y$ by semimodularity, $[a_0,a_1]$ and $[b_0,b_1]$ are $\mathrm{PrInt}(D)$-projective. Hence they belong to the same trajectory by Lemma 3-10.3. The converse follows from Lemma 3-9.4. $\square$

Proof of Theorems 3-10.1 *and* 3-10.2. Let L be the semimodular lattice of Theorem 3-10.1, and let K be the join-semilattice in L generated by $B \cup C$. Since semimodularity depends only on the join operation, K is a cover-preserving join-subsemilattice of L and

(3-10.2) K is a slim semimodular lattice.

Applying Exercise 3.24 with $m = n = h$, we obtain a diagram $D \in \mathrm{Dgr}(K)$ such that $B = \mathrm{C_l}(D)$ and $C = \mathrm{C_r}(D)$.

Observe that several up-perspectivities combine into a single up-perspectivity, and dually. Note also that up-perspectivities and down-perspectivities are also up-and-down projectivities. Therefore, Theorem 3-10.1 follows from Lemma 3-9.4.

Observe that if $\pi, \sigma \in S_h$ and $\pi(i) \leq \sigma(i)$ for all $i \in \{1, \ldots, h\}$, then $\pi = \sigma$. Hence, in order to prove Theorem 3-10.2, it suffices to show its second part. Suppose, for a contradiction, that there exist $i, j \in \{1, \ldots, h\}$ such that $[b_{i-1}, b_i] \nearrow\!\!\searrow [c_{j-1}, c_j]$ holds in L but $j > \pi(i)$. Let $x = b_{i-1} \vee c_{j-1}$, $y = b_i \vee c_j$, $a = b_i \vee x$, and $d = c_j \vee x$, see Figure 3-10.1. (The trajectory that yields $\pi(i)$ by Lemma 3-9.4 is depicted in gray. Naturally, we cannot depict the indirect assumption $j > \pi(i)$.) We assert that

(3-10.3) $\qquad S = \{x, a, d, y\}$ is a 4-cell in D with $\mathrm{lc(S)} = a$ and $\mathrm{rc(S)} = d$.

Clearly, $y = a \vee d$. Since $[b_{i-1}, b_i] \nearrow\!\!\searrow [c_{j-1}, c_j]$ in L, Exercise 3.78 implies that $b_i, c_j \nleq x$. Hence, by semimodularity, $x \prec a \leq y$ and $x \prec d \leq y$. If we had $a = d$, then $[b_{i-1}, b_i] \nearrow [x, a] = [x, d] \searrow [c_{j-1}, c_j]$ would hold in D, which together with Lemma 3-10.4 would yield that $[b_{i-1}, b_i]$ and $[c_{j-1}, c_j]$ would belong to the same trajectory of D. However, then Lemma 3-9.4 would imply that $j = \pi(i)$, a contradiction. This shows that $a \neq d$, and (3-10.3) follows from Exercise 3.37. By (3-10.3), $[d, y] \in \mathrm{PrInt}(D)$.

Since $\pi(i) < j$ gives that $c_{\pi(i)} \leq c_j \leq c_j \vee x = d \leq d \vee c_{\pi(i)-1}$, Exercise 3.78 implies that

(3-10.4) $\qquad [d, y]$ is *not* up-and-down projective to $[c_{\pi(i)-1}, c_{\pi(i)}]$ in D.

We know from Lemma 3-9.4 that there is a unique trajectory T of D such that $[b_{i-1}, b_i] \in T$. Since $[b_{i-1}, b_i] \nearrow [x, a] \nearrow [d, y]$ gives that $[b_{i-1}, b_i] \nearrow [d, y]$ and so $[b_{i-1}, b_i] \nearrow\!\!\searrow [d, y]$ in D, it follows from Lemma 3-10.4 that $[d, y] \in T$. By Lemma 3-9.4, $[c_{\pi(i)-1}, c_{\pi(i)}] \in T$. Hence Lemma 3-10.4 yields that $[d, y] \nearrow\!\!\searrow [c_{\pi(i)-1}, c_{\pi(i)}]$, which contradicts (3-10.4). ☐

Now, we convert Theorems 3-10.1 and 3-10.2 to group theoretic results. As usual, the relation *subnormal subgroup* is the transitive closure of the relation *normal subgroup*. For subnormal subgroups $A \triangleleft B$ and $C \triangleleft D$ of a given group G, the quotient B/A will be called *subnormally down-and-up projective* to D/C if there are subnormal subgroups $X \triangleleft Y$ of G such that $AY = B$, $A \cap Y = X$, $CY = D$ and $C \cap Y = X$. Clearly, $B/A \cong D/C$ in this case, because both are isomorphic to the group Y/X by the Second Isomorphism Theorem.

The well-known concept of a composition series in a group goes back to É. Galois (1831), see J.J. Rotman [281, Thm. 5.9]. The Jordan–Hölder theorem, stating that any two composition series of a finite group have the same length, was also proved in the nineteenth century, see C. Jordan [214] and O. Hölder [199]. The group does not have to be finite; it suffices to assume that there exists a finite composition series.

The first statement of the following theorem is in G. Grätzer and J.B. Nation [158], while the second statement is in G. Czédli and E.T. Schmidt [54].

Theorem 3-10.5. *Let*

$$\begin{aligned}\vec{H} &: \{1\} = H_0 \lhd H_1 \lhd \cdots \lhd H_n = G,\\ \vec{K} &: \{1\} = K_0 \lhd K_1 \lhd \cdots \lhd K_m = G\end{aligned} \tag{3-10.5}$$

be composition series of a group G. Then

(i) *$n = m$, and there exists a permutation π of the set $\{1, \dots, n\}$ such that H_i/H_{i-1} is subnormally down-and-up projective to $K_{\pi(i)}/K_{\pi(i)-1}$ for all i;*

(ii) *this permutation π is* uniquely determined *and it has the following property: if $i, j \in \{1, \dots, n\}$ and H_i/H_{i-1} is subnormally down-and-up projective to K_j/K_{j-1}, then $j \geq \pi(i)$.*

Proof. Let G be a group with a finite composition series. We know from a classical result of H. Wielandt [340] (see also R. Schmidt [298, Theorem 1.1.5] and the remark after its proof or see M. Stern [308, p. 302]) that the subnormal subgroups form a sublattice $\mathrm{SnSub}\, G$ of the lattice $\mathrm{Sub}\, G$ of all subgroups of G. It is not hard to strengthen this result to the following one:

$$\mathrm{SnSub}\, G \text{ is a dually semimodular lattice;} \tag{3-10.6}$$

see [298, Theorem 2.1.8], or the proof of [308, Theorem 8.3.3], or the proof of J.B. Nation [251, Theorem 9.8]. Therefore, the theorem follows from Theorems 3-10.1 and 3-10.2 by the Duality Principle. ☐

How many ways can two composition series intersect?

Assume that $\vec{H}$ and $\vec{K}$ in (3-10.5) are composition series of a group G and $h = m = n$. Let

$$\mathrm{CSL}_h(\vec{H}, \vec{K}) = \{\, H_i \cap K_j \mid i, j \in \{0, \dots, h\} \,\}.$$

Then $\mathrm{CSL}_h(\vec{H}, \vec{K}) = \big(\mathrm{CSL}_h(\vec{H}, \vec{K}); \subseteq\big)$ is an order. Since it has a largest element and it is closed with respect to intersection, $\mathrm{CSL}_h(\vec{H}, \vec{K})$ is a finite lattice; we call it a *composition series lattice.* We are going to determine which lattices are (isomorphic to) composition series lattices and how many there are. The following theorem strengthens G. Czédli and E.T. Schmidt [57, Corollary 3.5] and G. Czédli, L. Ozsvárt, and B. Udvari [50].

Theorem 3-10.6. *Composition series lattices are the duals of slim semimodular lattices. Furthermore, if G is the direct product of h nontrivial simple cyclic groups, then for each slim semimodular lattice L of length h, there exist composition series $\vec{H}$ and $\vec{K}$ of G such that L is isomorphic to the dual of $\mathrm{CSL}_h(\vec{H}, \vec{K})$.*

Proof. It follows from (3-10.2) and (3-10.6) that composition series lattices are duals of slim semimodular lattices.

To prove the converse and the second part of the statement, note that G in the theorem is commutative. Hence $\operatorname{SnSub} G = \operatorname{Sub} G$ is modular. It is of length h. The simple cyclic subgroups are atoms in it. Thus $\operatorname{SnSub} G$ contains an h-element independent set of atoms. Therefore, the Boolean lattice B_h of length h is a sublattice, in fact, a cover-preserving sublattice, of $\operatorname{SnSub} G$. Let $D \in \operatorname{Dgr}(L)$ and let L^δ be the dual of L. By the selfduality of B_h and Exercise 3.50 (or Exercise 3.75), we obtain a meet-embedding $\varphi\colon L^\delta \to B_h \subseteq \operatorname{SnSub} G$. Clearly, $\vec{H} = \varphi\bigl(\mathrm{C}_{\mathrm{l}}(D)\bigr)$ and $\vec{K} = \varphi\bigl(\mathrm{C}_{\mathrm{r}}(D)\bigr)$ are composition series of G, and $L^\delta \cong \mathrm{CSL}_h(\vec{H}, \vec{K})$. ☐

Let $N(h)$ denote the number of isomorphism classes of all composition series lattices $\mathrm{CSL}_h(\vec{H}, \vec{K})$. This number counts how many ways two composition series of length h can intersect. By Theorem 3-10.6, $N(h)$ is also the number of isomorphism classes of slim semimodular lattice of length h. Using matrices and Theorem 3-8.4, G. Czédli, L. Ozsvárt, and B. Udvari [50] gave a recursive method of computing $N(h)$. On a personal computer, it can be used up to $h = 100$.

The following tables, in which $N_0(h)$ and $N(h)$ denote the number of isomorphism classes of indecomposable slim semimodular lattices of length h and of slim semimodular lattices of length h, respectively, was computed in a fraction of a second.

h	0	1	2	3	4	5	6	7	8
$N_0(h)$	0	0	1	2	8	39	242	1,759	14,674
$N(h)$	1	1	2	5	17	73	397	2,623	20,414

h	9	10	11	12
$N_0(h)$	137,127	1,416,430	16,006,403	196,400,810
$N(h)$	181,607	1,809,104	19,886,032	238,723,606

The following result of G. Czédli, L. Ozsvárt, and B. Udvari [50] is based on Section 3-9.

◊ **Theorem 3-10.7.** *The asymptotic value of $N(h)$ is $h!/2$, that is,*

$$\lim_{h\to\infty} \frac{N(h)}{h!} = \frac{1}{2}.$$

3-11. Exercises

3.1. Show that if L is a semimodular lattice of finite length, then so are its cover-preserving join-sublattices and cover-preserving join-homomorphic images. (G. Grätzer and E. Knapp [140, Lemma 16].)

3.2. Let L be a lattice and $\boldsymbol{\alpha}$ be a join-congruence of L. A covering square $\{a \wedge b, a, b, a \vee b\}$ of L is *$\boldsymbol{\alpha}$-forbidden* if the $\boldsymbol{\alpha}$-classes $a/\boldsymbol{\alpha}$, $b/\boldsymbol{\alpha}$, and $(a \wedge b)/\boldsymbol{\alpha}$ are pairwise distinct but $(a \vee b)/\boldsymbol{\alpha}$ equals $a/\boldsymbol{\alpha}$ or $b/\boldsymbol{\alpha}$. Let L and K be semimodular lattices in which all intervals are of finite length. Show that a surjective join-homomorphism $\varphi\colon L \to K$ is cover-preserving iff L does not have $\mathrm{Ker}(\varphi)$-forbidden covering squares. Consequently, a join-congruence $\boldsymbol{\alpha}$ is cover-preserving iff L does not have $\boldsymbol{\alpha}$-forbidden covering squares. (G. Czédli and E.T. Schmidt [53].)

3.3. Let L be a planar lattice and $D \in \mathrm{Dgr}(L)$. Show that every region of D is a cover-preserving sublattice. (D. Kelly and I. Rival [223, Proposition 1.4].)

Hint: Apply Lemmas 3-3.2(i) and 3-3.3(ii).

3.4. Let D be a planar lattice diagram and let a_1, a_2, a_3, a_4 be (not necessarily distinct) lower covers of some $x \in D$ such that a_i is to the right of a_{i-1} for $i = 2, 3, 4$. Prove that $a_1 \wedge a_4 \leq a_2 \wedge a_3$.

Hint: Use Exercise 3.3.

3.5. For $n \in \mathbb{N}$, show that an order $P = (P, \rho)$ is a suborder of the direct product of n chains iff ρ is the intersection of n linear orderings. The least such n is called the *order-dimension* of P.

Hint: If $\rho = \rho_1 \cap \cdots \cap \rho_n$ and the relations ρ_i are linear, then $x \mapsto (x, \ldots, x)$ embeds (P, ρ) into $\prod_{i=1}(P, \rho_i)$. If $(C_i, \leq_i)$, $i = 1, \ldots, n$, are chains and P is order-embedded into their direct product, then let ρ_i be the restriction of the lexicographic ordering of

$$C_i \times C_1 \times \cdots \times C_{i-1} \times C_{i+1} \times \cdots \times C_n$$

to P.

3.6. Let D be a planar lattice diagram and $a, b \in D$ such that $a \parallel b$. Show that either a is strictly on the left of all maximal chains that contain b, or a is strictly on the right of all these chains. (D. Kelly and I. Rival [223, Proposition 1.7]; originally J. Zilber, see G. Birkhoff [29, p. 32, Exercise 7(c)].)

Hint: Otherwise, b would be a narrows by Exercise 3.17.

3.7. Show that a finite lattice is planar iff it is of order-dimension at most two. (It is important that L be a lattice, not just an order, see [LTF, Exercise I.1.21].)

Hint: If $D \in \mathrm{Dgr}(L)$, then let $a \leq_1 b$ mean that $a \leq b$, or $a \parallel b$ and Exercise 3.6 holds with "left". Show that $\leq_1$ is a linear ordering. Conversely, if $L \subseteq C_1 \times C_2$, then depict $C_1 \times C_2$ as a grid in the usual way with slopes $-45°$ and $45°$, and restrict this diagram to L. If the edges $a_1 \prec_L b_1$ and $a_2 \prec_L b_2$ of L give a non-planar intersection, then $a_i \leq b_j$, for $i, j \in \{1, 2\}$, by the slopes, and $a_1 \vee_L a_2 \leq b_1 \wedge_L b_2$

leads to a contradiction. (D. Kelly and I. Rival [223, Proposition 5.2], originally B. Dushnik and E.W. Miller [74] combined with J. Zilber, see G. Birkhoff [29, p. 32, Exercise 7(c)].)

3.8. Show that for every planar lattice diagram D, $\mathrm{C}_{\mathrm{l}}(D)$, and $\mathrm{C}_{\mathrm{r}}(D)$ are maximal chains. (D. Kelly and I. Rival [223, p. 641].)

3.9. Show that (3-3.1) is a selfdual condition.

Hint: Assume that $y_1 \prec x_1$ and $z_1 \prec x_1$. Let $x = y_1 \wedge z_1$, and choose $y \in [x, y_1]$ and $z \in [x, z_1]$ such that $x \prec y$ and $y \prec z$. Apply Lemma 3-3.2(i) to the elements y, y_1 and a maximal chain extending $\{x, z, z_1, x_1\}$.

3.10. Let C be a maximal chain of a planar lattice L. Assume that $D_1, D_2 \in \mathrm{Dgr}(L)$ are similar diagrams. Prove that $\mathrm{LS}(C, D_1) = \mathrm{LS}(C, D_2)$.

Hint: For $x \in L - C$, let $c = \bigvee(C \cap \downarrow x)$ and let y be an atom in $[c, x]$. Apply Lemma 3-3.2(i) to show that x and y are on the same side of C with respect to D_i.

3.11. Let R be a region of a lattice diagram D and let $x \in \mathrm{C}_{\mathrm{l}}(D)$. Prove that if $x \in R$, then $x \in \mathrm{C}_{\mathrm{l}}(R, D)$.

3.12. Assume that L is a planar lattice, $D_1, D_2 \in \mathrm{Dgr}(L)$ are similar diagrams, and R is a region of D_1. Prove that R is a region of D_2 and $\mathrm{C}_{\mathrm{l}}(R, D_1) = \mathrm{C}_{\mathrm{l}}(R, D_2)$, $\mathrm{C}_{\mathrm{r}}(R, D_1) = \mathrm{C}_{\mathrm{r}}(R, D_2)$.

Hint: Extend $\mathrm{C}_{\mathrm{l}}(R, D_1)$ and $\mathrm{C}_{\mathrm{r}}(R, D_1)$ to maximal chains in L and apply Exercise 3.10.

3.13. Let L be a planar lattice. Show that if $D_1, D_2 \in \mathrm{Dgr}(L)$ have the same regions and these regions have the same left and the same right boundary chains with respect to D_1 and D_2, then D_1 and D_2 are similar.

3.14. A *contour* of a planar lattice L is a planar diagram T of the order (in fact, a lattice) $\mathrm{Bnd}(L, D)$ for some $D \in \mathrm{Dgr}(L)$. The contour of L is *arbitrary* if for each contour T of L there exists an $E \in \mathrm{Dgr}(L)$ such that $\mathrm{Bnd}(E)$ is congruent to T in the Euclidean metric.

Prove that if a planar lattice satisfies the Jordan-Hölder Chain Condition (in particular, if L is semimodular, see [LTF, Theorem 374]), then its contour is arbitrary. Give a planar lattice whose contour is not arbitrary. (G. Czédli and E.T. Schmidt [55].)

Hint: Consider the lattice of Figure 3-11.1.

3.15. Let D be a planar 4-cell lattice diagram, and let b and c be neighboring covers of $a \in D$. Prove that all maximal chains of $[a, b \vee c]$ are of length 2.

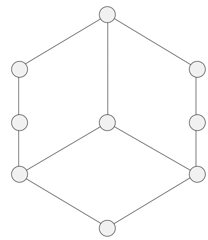

Figure 3-11.1: A lattice for Exercise 3.14.

Hint: Assume that b is to the left of c, and $d \in [a, b \vee c]$ with $a \prec d$ is the right neighbor of c. Then $c \vee d = b \vee c$, and we can proceed to the right.

3.16. Let D be a lattice diagram and $a \prec b \in D$. Prove that if $\{a, b\} \subseteq \mathrm{C}_{\mathrm{l}}(D)$ or $\{a, b\} \subseteq \mathrm{C}_{\mathrm{r}}(D)$, then $a \in \mathrm{Mi}\, D$ or $b \in \mathrm{Ji}\, D$. (G. Czédli and E.T. Schmidt [55, Lemma 4].)

Hint: Apply Lemma 3-3.2(i).

3.17. Show that $\mathrm{C}_{\mathrm{l}}(D) \cap \mathrm{C}_{\mathrm{r}}(D) = \mathrm{Nar}(D)$ holds for every planar lattice diagram D.

3.18. Let L be a planar lattice, $D \in \mathrm{Dgr}(L)$, and $u \in \mathrm{int}(L, D)$. Show that $\mathrm{lsp}^*(u), \mathrm{rsp}^*(u) \in \mathrm{Ji}\, L$, and $\mathrm{lsp}^*(u) \neq \mathrm{rsp}^*(u)$.

Hint: Use Exercise 3.16 and if $\mathrm{lsp}^*(u) = \mathrm{rsp}^*(u)$, use Exercise 3.17.

3.19. Assume that L has a 4-cell diagram. Prove that all $D \in \mathrm{Dgr}(L)$ are 4-cell diagrams.

Hint: Suppose to the contrary that $D, E \in \mathrm{Dgr}(L)$ such that D is a 4-cell diagram but E has a cell R that is not a 4-cell. Let a and b be the left and right atom of R, respectively. We can assume that $0_R = 0_L$ and $1_R = 1_L$, and a is to the left of b in D. Listed from left to right, let $x_0 = a, x_1, \ldots, x_n = b$ be all the atoms of D between a and b. Using Exercises 3.15 and Lemma 3-3.2, show by induction that all the x_i are to the left of a.

3.20. Assume that S is a *diamond sublattice* of a planar semimodular lattice L, that is, a sublattice isomorphic to M_3. Prove that L has a cover-preserving sublattice T such that $T \cong \mathsf{M}_3$ and $0_T = 0_S$.

Hint: Assume that $S = \{0, a, b, c, 1\} \subseteq D \in \mathrm{Dgr}(L)$ with a on the left and c on the right. Let $0 \prec a_1 \leq a$ and $0 \prec c_1 \leq c$, and take a maximal chain C that contains b. We can assume that $a_1 \vee c_1$ is on the left of C. Applying Lemma 3-3.2(i) to C and $c_1 < a_1 \vee c_1$ and using the height function, we conclude that $T = \{0, a_1, b, c_1, a_1 \vee c_1\}$ is a cover-preserving diamond sublattice.

3.21. Prove that every *diamond sublattice* M_3 of a planar modular lattice is a cover-preserving sublattice. Find a diamond in a planar semimodular lattice that is not a cover-preserving sublattice. (G. Grätzer and R.W. Quackenbush [159].)

Hint: Take a one-step anti-slimming of N_7.

3.22. Let L be an interval or a join-homomorphic image of a slim lattice. Prove that L is slim.

3.23. Prove Lemma 3-4.1.

Hint: Let $\mathrm{Ji}\, L = C_1 \cup C_2$, where C_1 and C_2 are chains. Denote $\bigvee(C_i \cap \downarrow x)$ by x_i, and apply Exercises 3.5 and 3.7 to the order-embedding $L \to C_1 \times C_2$, $x \mapsto (x_1, x_2)$.

3.24. If L is a slim lattice and e is a maximal element of $\mathrm{Ji}\, L$, then $\uparrow e$ is a chain, and it is a subset of $\mathrm{Bnd}(D)$ for all $D \in \mathrm{Dgr}(L)$. (G. Czédli and E.T. Schmidt [54, Lemma 2.1].)

3.25. Let

$$E = \{0 = e_0 \prec e_1 \prec \cdots \prec e_n\},$$
$$F = \{0 = f_0 \prec f_1 \prec \cdots \prec f_m\}$$

in a slim lattice L such that $\mathrm{Ji}\, L \subseteq E \cup F$. Then L is planar, and $\mathrm{C_l}(L, D) = E \cup \uparrow e_n$ and $\mathrm{C_r}(L, D) = F \cup \uparrow f_m$ hold for some $D \in \mathrm{Dgr}(L)$. (G. Czédli and E.T. Schmidt [54, Lemma 2.2].)

3.26. If L is a slim lattice, then $\mathrm{Ji}\, L \subseteq \mathrm{Bnd}(D)$ for all $D \in \mathrm{Dgr}(L)$. (G. Czédli and E.T. Schmidt [55, Lemma 6].)

Hint: Use Exercises 3.16 and 3.17 to show that for $p \in \mathrm{Ji}\, L - \mathrm{Bnd}(L, E)$, the set $\{\mathrm{lsp}^*(p), p, \mathrm{rsp}^*(p)\}$ would be a three-element antichain in $\mathrm{Ji}\, L$.

3.27. Derive from Exercise 3.26 that $x = \mathrm{lsp}(x) \vee \mathrm{rsp}(x)$ holds for every slim lattice L, $x \in L$, and $D \in \mathrm{Dgr}(L)$.

3.28. Prove that in a slim lattice, an element has at most two covers. See also Exercise 3.35 for a stronger statement. (G. Grätzer and E. Knapp [140, Lemma 8].)

Hint: Each cover of $x \in L$ is of the form $x \vee a$ for some $a \in \mathrm{Ji}\, L$.

3.29. Show that every slim modular lattice is distributive. (G. Grätzer and E. Knapp [140, Lemma 3].)

Hint: Apply Exercises 3.20 and 3.28.

3.30. Show that a slim semimodular lattice L is distributive iff N_7, see Figure 3-4.1, is not a cover-preserving sublattice of L. (G. Czédli and E.T. Schmidt [55, Lemma 15].)

3.31. Let L be a slim semimodular lattice. Let t be an element of L such that t has at least three lower covers, and assume that t is minimal with respect to this property. Prove that t is the top of a cover-preserving N_7 sublattice. (G. Czédli and E.T. Schmidt [55, Lemma 14].)

3.32. Show that slim modular lattices are dually slim.

Hint: Apply Exercises 3.29 and 3.31 and Theorem 3-4.3.

3.33. Let L be a semimodular slim lattice. If L is also dually slim (that is, $\mathrm{Mi}\, L$ contains no three-element antichain), then L is distributive.

Hint: Apply Exercise 3.30.

3.34. Let L be a slim lattice, $D \in \mathrm{Dgr}(L)$, and $u \in \mathrm{int}(L, D)$. Show that $\mathrm{lsp}(u) \neq \mathrm{rsp}(u)$.

Hint: Use Exercise 3.27.

3.35. Let L be a slim lattice and $D \in \mathrm{Dgr}(L)$. Show that if $u, v \in L$ and $u \prec v$, then

$$v = \mathrm{lsp}^*(u) \vee u = \mathrm{lsp}^*(u) \vee \mathrm{rsp}(u) = \mathrm{lsp}(v) \vee \mathrm{rsp}(u)$$

or

$$v = u \vee \mathrm{rsp}^*(u) = \mathrm{lsp}(u) \vee \mathrm{rsp}^*(u) = \mathrm{lsp}(u) \vee \mathrm{rsp}(v).$$

Hint: We can assume that $\mathrm{lsp}(v) \nleq \mathrm{lsp}(u)$, see Exercise 3.27.

3.36. Show that if L is slim, $D \in \mathrm{Dgr}(L)$ and $x \in L$, then $[\mathrm{lsp}(x), x]$ and $[\mathrm{rsp}(x), x]$ are chains.

3.37. Let D be a slim semimodular lattice diagram. Assume that $a, d, x \in D$ such that $x \neq 1$ and $a = x \vee \mathrm{lsp}^*(x) \neq x \vee \mathrm{rsp}^*(x) = d$. Prove that $\{x, a, d, a \vee d\}$ is a 4-cell with left corner a and right corner d.

Hint: Apply Lemma 3-3.2(i).

3.38. For $u \in L - \{1\}$, let $u^\diamond$ denote the join of the upper covers of u. Prove that if L is slim, then $\mathrm{int}([u, u^\diamond], D) = \varnothing$ for all $D \in \mathrm{Dgr}(L)$.

Hint: Use Exercises 3.22 and 3.26.

3.39. Show that each slim semimodular lattice L is join-distributive (also called, locally upper distributive), that is, $[u, a^\diamond]$ is a distributive lattice for all $u \in L - \{1\}$. (See, for example, K.V. Adaricheva, V.A. Gorbunov, and V.I. Tumanov [4, Section 1.2] for some information on these lattices.)

Hint: Use Exercise 3.28.

3.40. Prove that a planar semimodular lattice is slim iff it contains no cover-preserving diamond sublattice. (G. Czédli and E.T. Schmidt [54, Lemma 2.3]. This was the original definition of slimness for semimodular lattices, see G. Grätzer and E. Knapp [140].)

Hint: If L has no covering diamond, then its covering squares are 4-cells by Lemmas 3-3.2(i) and 3-3.4(i). Hence no element of L has more than two covers. However, if $p \in \mathrm{Ji}\, L \cap \mathrm{int}(L)$ and $q \prec p$, then $q \vee \mathrm{lsp}^*(p)$, p, and $q \vee \mathrm{rsp}^*(p)$ are three distinct covers of q by Exercise 3.18, a contradiction. The converse follows from Exercise 3.28.

3.41. Let x and y be two neighboring lower covers of z in a 4-cell lattice diagram. Prove that $\{x \wedge y, x, y, z\}$ is a 4-cell. (G. Czédli and E.T. Schmidt [55, Lemma 13].)

3.42. An element d of $\mathrm{Di}\, L$ is called a *quasi corner* (see Figure 3-5.1) if d_* has exactly two covers and d^* has exactly two lower covers.

Prove that the quasi corners of a planar semimodular diagram are corners.

Hint: Use Theorem 3-4.5(ii) and Corollary 3-4.10.

3.43. Assume that d is a corner of a slim semimodular diagram. Prove that there is a unique $b \in D - \{d\}$ such that $d_* \prec b \prec d^*$.

3.44. Let d be a weak corner of a slim semimodular lattice diagram D such that d is not a narrows. Prove that d is a near corner.

Hint: Use Theorem 3-4.5(ii), Exercise 3.28, the non-singleton trajectory containing $[d_*, d]$, and the non-singleton trajectory containing $[d, d^*]$.

3.45. Prove that $L[S]$ does not depend on the choice of $D \in \mathrm{Dgr}(L)$ in Definition 3-5.3(ii).

Hint: Apply Theorem 3-4.5 and the fact that the bottom left edge of a 4-cell cannot be the bottom right edge of another 4-cell by Exercise 3.28.

3.46. Assume that $L[S]$ is obtained from a slim semimodular lattice L by adding a fork at S. Prove that $L[S]$ is also a slim semimodular lattice. (G. Czédli and E.T. Schmidt [55, Theorem 11].)

Hint: Use Theorem 3-4.3 (vi)-(vii).

3.47. Use the Decomposition Theorem (Theorem 3-7.4) to prove G. Czédli and E.T. Schmidt's Construction Theorem for Slim Semimodular Lattices (Theorem 3-5.5).

3.48. Show that the order of (i) and (ii) in Theorem 3-5.5 cannot be interchanged.

Hint: See the lattice D in Figure 3-5.2.

3.49. Prove that $|\mathrm{Mi}\, L| = \mathrm{len}(L)$ holds for every slim semimodular lattice L.

Hint: Apply Theorem 3-5.5 or use Exercise 3.39 together with the dual of K.V. Adaricheva, V.A. Gorbunov, and V.I. Tumanov [4, Theorem 1.7(1)].

3.50. Let L be a slim semimodular lattice of length h, and let A be an h-element set. The *powerset lattice* of A is denoted by $\mathrm{Pow}\, A$. Prove that there is a cover-preserving join-embedding of L into $(\operatorname{Pow} A; \cup)$. (G. Czédli, L. Ozsvárt, and B. Udvari [50, Proposition 2.6].)

Hint: Firstly, use Exercise 3.49 to give a dual embedding into $(\operatorname{Pow} A; \cap)$. See also Exercise 3.75 for another approach.

3.51. Prove that if D is a rectangular lattice diagram, then every element of $\operatorname{Bnd}(D) - \operatorname{Bnd}_{\mathrm{l}}(D)$ has at least two lower covers, and every element of $\operatorname{Bnd}(D) - \operatorname{Bnd}_u(D)$ has at least two upper covers. (G. Grätzer and E. Knapp [143].)

3.52. Prove that if D is a rectangular lattice diagram, then

$$(\mathrm{C}_{\mathrm{ul}}(D) \cup \mathrm{C}_{\mathrm{ur}}(D)) - \{1\} \subseteq \operatorname{Mi} D,$$
$$(\mathrm{C}_{\mathrm{ll}}(D) \cup \mathrm{C}_{\mathrm{lr}} D) - \{0\} \subseteq \operatorname{Ji} D.$$

(G. Grätzer and E. Knapp [143].)

3.53. Prove that a planar semimodular lattice diagram with exactly one left weak corner and exactly one right weak corner is rectangular iff 1 is join-reducible and 0 is meet-reducible. (G. Grätzer and E. Knapp [143, Lemma 6].)

3.54. Prove that a planar semimodular lattice L is rectangular iff $\operatorname{Slim} L$ is rectangular. (G. Czédli and E.T. Schmidt [56].)

3.55. Prove that if a planar lattice has a rectangular diagram, then all of its diagrams are rectangular. (G. Czédli and E.T. Schmidt [56].)

3.56. Prove that if L is a rectangular lattice, then $\operatorname{Bnd}(D) - \operatorname{Bnd}_u(D)$ and $\operatorname{Bnd}(D) - \operatorname{Bnd}_{\mathrm{l}}(D)$ do not depend on the choice of $D \in \operatorname{Dgr}(L)$. (G. Czédli and E.T. Schmidt [56].)

3.57. Let L be a slim rectangular lattice. Prove that its weak corners are near corners.

3.58. Let L be a slim semimodular lattice. Prove that L is rectangular iff $\operatorname{Ji} L$ is the union of two chains such that no element in the first chain is comparable with some element of the second chain. (E.T. Schmidt, oral communication.)

Hint: Combine Theorem 1-3.4(ii) with Exercise 1.51 for the "only if" part, and with Exercise 1.36 (where $x = 1$) for the "if" part.

3.59. Prove that a planar semimodular lattice diagram is rectangular iff we cannot add a corner to it. (G. Czédli [45, Lemma 6.4].)

3.60. Prove that Condition (iii) in Theorem 3-7.5 is equivalent to the condition that "every $D \in \operatorname{Dgr}(L)$ is rectangular and its weak corners are dual atoms."

Hint: Use Exercises 3.4, 3.51, and 3.55.

3.61. Prove that the conditions (mr1), ..., (mr5) in Proposition 3-8.3 are independent. (G. Czédli [44].)

3.62. Prove that if $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are cover-preserving join-congruences of a finite semimodular lattice L and $\boldsymbol{\alpha} \subseteq \boldsymbol{\beta}$, then $\boldsymbol{\beta}/\boldsymbol{\alpha}$ is a cover-preserving join-congruence of the quotient lattice $L/\boldsymbol{\alpha}$. (G. Czédli [44].)

3.63. For $i \in I$, let $\boldsymbol{\alpha}_i$ be a congruence of a join-semilattice $(L; \vee)$ and let $\boldsymbol{\beta} = \bigvee(\boldsymbol{\alpha}_i \mid i \in I)$ in the congruence lattice of $(L; \vee)$. Then, for each x, y in L, $(x, y) \in \boldsymbol{\beta}$ iff there is a $k \in \mathbb{N}_0$ and there are elements $x = z_0 \leq z_1 \leq \cdots \leq z_k = v_k \geq v_{k-1} \geq \cdots \geq v_0 = y$ in L such that $\{(z_{j-1}, z_j), (v_{j-1}, v_j)\} \subseteq \bigcup_{i \in I} \boldsymbol{\alpha}_i$ for $j = 1, \ldots, k$. (G. Czédli and E.T. Schmidt [53].)

3.64. Is the join of two cover-preserving join-congruences of a grid also cover-preserving?

Hint: Consider $\mathrm{con}_{\vee}(\mathrm{cell}(2,1)) \vee \mathrm{con}_{\vee}(\mathrm{cell}(2,2))$ in a grid.

3.65. Describe $\mathrm{con}_{\vee}(U)$ in Definition 3-8.1. (G. Czédli [44].)

3.66. Prove Lemma 3-9.4(iv).

Hint: Apply Theorem 3-5.5.

3.67. Prove that $\boldsymbol{\beta}_{\pi}$ in Definition 3-9.5 is a cover-preserving join-congruence. (G. Czédli and E.T. Schmidt [56].)

3.68. Let D be as in Definition 3-9.1. For $i \in \{1, \ldots, h\}$, take a meet-irreducible element $u \in L$ such that b_i is the smallest element of $\mathrm{C}_{\mathrm{l}}(D) - {\downarrow} u$, that is, $b_i = \mathrm{lsp}^*(u)$. Let $c_j = \mathrm{rsp}^*(u)$. Prove that the map $i \mapsto j$ is well defined and equals $\pi(D)$. (G. Czédli and E.T. Schmidt [56].)

Hint: Apply Theorem 3-5.5.

3.69. Let D be as in Definition 3-9.1. Take the grid diagram $G = \mathrm{C}_{\mathrm{l}}(D) \times \mathrm{C}_{\mathrm{r}}(D)$ such that

$$\mathrm{C}_{\mathrm{ll}}(G) = \{(b_0, 0), \ldots, (b_h, 0)\},$$
$$\mathrm{C}_{\mathrm{lr}}(G) = \{(0, c_0), \ldots, (0, c_h)\}.$$

Let $\boldsymbol{\beta}$ be the kernel of the surjective join-homomorphism $G \to D$, $(b_i, c_j) \mapsto b_i \vee c_j$. Prove that the rule "$i \mapsto j$ iff $\mathrm{cell}(i,j) \in \mathrm{SCells}(\boldsymbol{\beta})$" defines a map, which equals $\pi(D)$. (G. Czédli and E.T. Schmidt [56].)

3.70. Prove that the "equal or sectionally inverted" relation is an equivalence on S_h. (G. Czédli and E.T. Schmidt [56].)

3.71. Prove that b_i in (3-9.1) is a narrows iff $i = 0$ or i is the largest element of a segment of $\pi(D)$. (G. Czédli and E.T. Schmidt [56].)

3.72. Derive Corollary 3-9.7 from Theorems 3-4.5(iii) and 3-9.6, and Exercises 3.69, 3.70, and 3.71.

3.73. Let $\boldsymbol{\gamma}$ be a cover-preserving join-congruence of a square grid diagram G such that $\boldsymbol{\gamma}$ collapses neither an edge of $\mathrm{C}_{\mathrm{ll}}(G)$, nor an edge of $\mathrm{C}_{\mathrm{lr}}(G)$. Prove that $\boldsymbol{\gamma} = \bigvee_{U \in \mathrm{SCells}(\boldsymbol{\gamma})} \mathrm{con}_{\vee}(U)$. (G. Czédli and E.T. Schmidt [56].)

3.74. Let $A = (G; F)$ be as in Definition 3-9.5. Prove that

$$\mathrm{SCells}\Big(\bigvee_{U \in F} \mathrm{con}_{\vee}(U)\Big) = F.$$

(G. Czédli and E.T. Schmidt [56].)

3.75. For L from Exercise 3.50, let $A = \{1, \dots, h\}$. Let $D \in \mathrm{Dgr}(L)$ and $\pi = \pi(D)$. With the notation (3-9.1), prove that

$$(L; \vee) \to (\mathrm{Pow}\, A; \cup), \quad x \mapsto \{i : b_i \leq x\} \cup \{i : c_{\pi(i)} \leq x\}$$

is a cover-preserving join-embedding. (G. Czédli, L. Ozsvárt, and B. Udvari [50].)

3.76. Prove (3-10.2). (G. Czédli and E.T. Schmidt [54, Lemma 2.4].)

3.77. Starting from the four-element non-chain lattice, we add three forks, in sequence, always to a 4-cell whose top is the greatest element of the lattice. This way we obtain a patch lattice M. Find two prime intervals in M that are projective but not $\mathrm{PrInt}(M)$-projective. (G. Czédli and E.T. Schmidt [54, Remark 2.6].)

3.78. Let L be an arbitrary lattice, and let $[b_0, b_1]$ and $[c_0, c_1]$ be nontrivial intervals of L such that $[b_0, b_1] \nearrow\!\searrow [c_0, c_1]$. Prove that $b_1 \nleq b_0 \vee c_0$ and $c_1 \nleq b_0 \vee c_0$. (G. Czédli and E.T. Schmidt [54, Lemma 2.10].)

Chapter 4

Planar Semimodular Lattices: Congruences

by George Grätzer

4-1. Introduction

For every result representing a finite distributive lattice D with n join-irreducible elements as the congruence lattice of a finite lattice L in some class $\mathbf{K}$ of lattices, the natural question arises: How small can we make L as a function of n and $\mathbf{K}$?

There are two results of this type in the literature. For the first result, $\mathbf{K}$ is the class of all lattices, that is, there is no restriction on L. The first proof of this representation theorem constructs a lattice of size $O(2^{2n})$, see G. Grätzer and E.T. Schmidt [164].[1] The size $O(2^{2n})$ was improved to $O(n^3)$ by G. Grätzer and H. Lakser [146] and to $O(n^2)$ by G. Grätzer, H. Lakser, and E.T. Schmidt [152]. Finally, it was proved in G. Grätzer, I. Rival, and N. Zaguia [160] that $O(n^2)$ is best possible.

[1] We use the *Landau notation:* for the functions $f(x)$ and $g(x)$, $f(x)$ is of size $O(g(x))$ if there exists a positive constant C such that $f(x) \leq Cg(x)$ for all large enough x and $f(x)$ is of size $\Theta(g(x))$ if there exist positive constants c and C such that $cg(x) \leq f(x) \leq Cg(x)$ for all large enough x.

G. Grätzer, F. Wehrung (eds.), *Lattice Theory: Special Topics and Applications: Volume 1*, DOI 10.1007/978-3-319-06413-0_4,

$\diamondsuit$ **Theorem 4-1.1.**

(i) *Let D be a finite distributive lattice with $n \geq 1$ join-irreducible elements. Then there exists a lattice L of $O(n^2)$ elements with* $\operatorname{Con} L \cong D$. *In fact, there is such a* planar *lattice L.*

(ii) *Let α be a real number satisfying the following condition: Every distributive lattice D with n join-irreducible elements can be represented as the congruence lattice of a lattice L with $O(n^\alpha)$ elements. Then $\alpha \geq 2$.*

In this chapter, we will discuss a similar result for planar semimodular lattices. To describe this result, we start with the result of G. Grätzer, H. Lakser, and E.T. Schmidt [153] stating that for the class of semimodular lattices, we can construct a lattice of size $O(n^3)$. In fact, we prove that this can be achieved with a *finite planar* lattice L. This result was improved in G. Grätzer and E. Knapp [143]: we can construct L as a *rectangular lattice.*

G. Grätzer and E. Knapp [144] proved the converse for rectangular lattices: $O(n^3)$ is the best possible. We now state these results in their equivalent forms for finite orders and their representations as the order of join-irreducible congruences of finite lattices.

We call an order P *bipartite* if every element of P is either minimal or maximal. Let A be the set of minimal elements of P and M the set of maximal elements of P. We call the order P *balanced bipartite* if

(i) $A \cap M = \varnothing$;

(ii) P is complete (that is, every element of A is covered by all elements of M);

(iii) either $|A| = |M|$ or $|A| = |M| + 1$.

Theorem 4-1.2.

(i) *Let P be a finite order with $n \geq 1$ elements. Then P can be represented as the order of join-irreducible congruences of a* rectangular lattice *L satisfying*
$$|L| \leq \frac{2}{3}n^3 + 2n^2 + \frac{4}{3}n + 1.$$

(ii) *Let L_n be a rectangular lattice whose order of join-irreducible congruences is a balanced bipartite order on n elements. Then, for some constant $k > 0$, the inequality $|L_n| \geq kn^3$ holds.*

We prove this theorem in this chapter.

That is, L_n is of size $O(n^3)$. Theorem 4-1.2(i) was proved in G. Grätzer, H. Lakser, and E.T. Schmidt [152] for finite semimodular lattices (of course, with a different cubic polynomial). A rather short proof of Theorem 4-1.2(i) can be found in G. Grätzer and E.T. Schmidt [180].

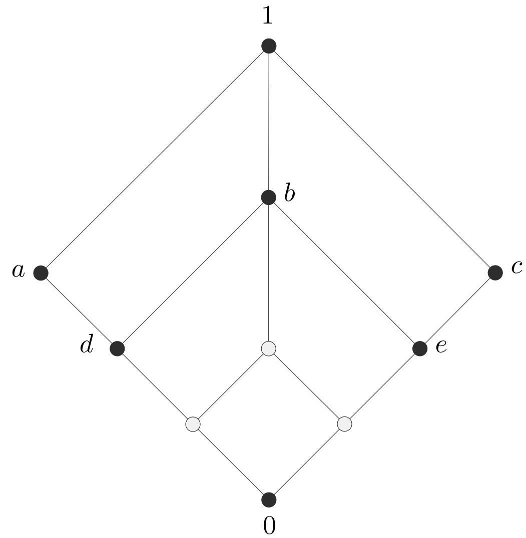

Figure 4-2.1: A tight N_7 that is not a covering N_7.

In Section 4-2, we investigate congruences in lattices, in general, and semimodular lattices, in particular. The main result is the Tight N_7 Theorem. In Section 4-3, we provide the construction for Theorem 4-1.2(i).

To prove Theorem 4-1.2(ii), we have to find lots of elements in L. We construct them from tight N_7-s whose existence was discussed in Section 4-2. In Section 4-4, we further investigate tight N_7-s. The most combinatorial part is Section 4-5, where we prove the Lower Bound Theorem. The proof of Theorem 4-1.2(ii) now easily follows in Section 4-6.

A great deal of research has been recently done on congruences of planar semimodular lattices. We briefly survey this field in Section 4-7.

4-2. Congruence structure and N_7 sublattices

In a finite lattice L, the congruence lattice, $\operatorname{Con} L$, is determined by the order $\operatorname{Ji}(\operatorname{Con} L)$ of join-irreducible congruences. A congruence $\boldsymbol{\alpha}$ is join-irreducible iff it is the congruence, $\operatorname{con}(\mathfrak{p})$, generated by a prime interval $\mathfrak{p}$. The inequality $\operatorname{con}(\mathfrak{p}) \leq \operatorname{con}(\mathfrak{q})$ holds iff $\mathfrak{q} \Rightarrow \mathfrak{p}$. (Recall that $\Rightarrow$ is the transitive extension of $\twoheadrightarrow$, the congruence-perspectivity relation, and $\twoheadrightarrow$ is $\overset{\text{up}}{\twoheadrightarrow}$ or $\overset{\text{dn}}{\twoheadrightarrow}$, see Section III.1 of LTF.)

How does $\operatorname{con}(\mathfrak{p}) \prec \operatorname{con}(\mathfrak{q})$ hold in $\operatorname{Ji}(\operatorname{Con} L)$? In the construction of the paper G. Grätzer, H. Lakser, and E.T. Schmidt [153], this is accomplished as follows.

When we want $\operatorname{con}(\mathfrak{p}) \prec \operatorname{con}(\mathfrak{q})$ to hold in $\operatorname{Ji}(\operatorname{Con} L)$, we insert a cover-preserving sublattice N_7 so that $\operatorname{con}(b, 1) = \operatorname{con}(\mathfrak{p})$ and $\operatorname{con}(a, 1) = \operatorname{con}(\mathfrak{q})$ (using the notation of Figure 3-4.1).

We are interested in the question, whether the covering N_7-s are the *only way* join-irreducible congruences are ordered in a planar semimodular lattice. Clearly not, as witnessed by the lattice in Figure 4-2.1. The black-filled elements of the lattice in Figure 4-2.1 form a sublattice $S \cong \mathsf{N}_7$, where $0 \prec d$ and $0 \prec e$ fail; nevertheless, in this sublattice, the join-irreducible congruence pair ordering is accomplished.

To accommodate this example, we slightly modify the concept of a cover-preserving N_7. We define a *tight* N_7 as a sublattice $S \cong \mathsf{N}_7$ of the lattice L, in which the following coverings in L must hold:

$$a \prec 1,\ b \prec 1,\ c \prec 1,\ d \prec a,\ d \prec b,\ e \prec b,\ e \prec c;$$

however, we do not require that

$$0 \prec d \text{ or } 0 \prec e$$

hold in L.

The following result of G. Grätzer and E. Knapp [142] shows that all orderings of join-irreducible congruences in a finite semimodular lattice are done by tight N_7-s.

Theorem 4-2.1 (Tight N_7 Theorem)**.** *Let L be a finite semimodular lattice. Let $\mathfrak{p}$ and $\mathfrak{q}$ be prime intervals in L such that* $\operatorname{con}(\mathfrak{p}) \prec \operatorname{con}(\mathfrak{q})$ *in* $\operatorname{Ji}(\operatorname{Con} L)$*. Then there exists a sublattice N of L, a tight N_7, such that*

$$\begin{aligned}\operatorname{con}(\mathfrak{p}) &= \operatorname{con}(b, 1),\\ \operatorname{con}(\mathfrak{q}) &= \operatorname{con}(a, 1).\end{aligned}$$

We prove this theorem in this section.

4-2.1 Congruences in finite lattices

To prepare for the proof of the Tight N_7 Theorem, we develop a theorem for general finite lattices. We start with a lemma.

Lemma 4-2.2. *Let L be a finite lattice and let I and J be intervals of L satisfying $I \overset{\text{up}}{\twoheadrightarrow} J$. Let $\mathfrak{p}$ be a prime interval in J. Then there exist in L an interval $J^* \subseteq J$ and prime intervals $\mathfrak{t} \subseteq I$ and $\mathfrak{p}^* \subseteq J^*$ satisfying $\mathfrak{t} \overset{\text{up}}{\twoheadrightarrow} J^*$ and* $\operatorname{con}(\mathfrak{p}^*) = \operatorname{con}(\mathfrak{p})$*.*

Proof. We prove this statement by induction on the length of I.

If I is prime, take $J^* = J$, $\mathfrak{t} = I$, and $\mathfrak{p}^* = \mathfrak{p}$.

Next, assume that I is not prime and that the statement is proved for intervals shorter than I.

Without loss of generality, we can assume that the following three conditions hold:

(i) $J \nsubseteq I$ (otherwise, take $J^* = \mathfrak{p}^* = \mathfrak{t} = \mathfrak{p}$).

(ii) $0_\mathfrak{p} = 0_J$ (otherwise, replace J with $[0_\mathfrak{p}, 1_J]$).

(iii) $0_I = 1_I \wedge 0_\mathfrak{p}$ (otherwise, replace I with $[1_I \wedge 0_J, 1_I]$).

Since I is not a prime interval, there is an element $a \in I$ satisfying $0_I \prec a < 1_I$. If $1_\mathfrak{p} \leq a \vee 0_\mathfrak{p}$, then take $J^* = [0_\mathfrak{p}, a \vee 0_\mathfrak{p}]$, $\mathfrak{t} = [0_I, a]$, and $\mathfrak{p}^* = \mathfrak{p}$.

Therefore, additionally, we can assume that $a \vee 0_\mathfrak{p} \parallel 1_\mathfrak{p}$. We define the prime interval $\mathfrak{p}_1$ so that $a \vee 0_\mathfrak{p} \leq 0_{\mathfrak{p}_1} \prec 1_{\mathfrak{p}_1} = a \vee 1_\mathfrak{p}$. Observe that $\operatorname{con}(\mathfrak{p}_1) = \operatorname{con}(\mathfrak{p})$. Applying the induction hypotheses to the intervals $I_1 = [a, 1_I]$, $J_1 = [0_{\mathfrak{p}_1}, 1_J]$, and the prime interval $\mathfrak{p}_1$, we obtain the interval $J_1^* \subseteq J_1$ and prime intervals $\mathfrak{t} \subseteq I_1$ and $\mathfrak{p}_1^* \subseteq J_1^*$, satisfying $\mathfrak{t} \overset{\text{up}}{\twoheadrightarrow} J_1^*$ and $\operatorname{con}(\mathfrak{p}_1^*) = \operatorname{con}(\mathfrak{p}_1)$. Since $\operatorname{con}(\mathfrak{p}_1^*) = \operatorname{con}(\mathfrak{p}_1) = \operatorname{con}(\mathfrak{p})$, the interval J_1^* and the prime intervals $\mathfrak{t}$ and $\mathfrak{p}_1^*$ also work for I, J, and $\mathfrak{p}$. ☐

Now our result on general lattices states that if $\operatorname{con}(\overline{\mathfrak{p}}) \prec \operatorname{con}(\overline{\mathfrak{q}})$ in $\operatorname{Ji}(\operatorname{Con} L)$, for prime intervals $\overline{\mathfrak{p}}$ and $\overline{\mathfrak{q}}$ of a finite lattice, then this happens in an N_5 sublattice as pictured in Figure 4-2.2. More formally,

Theorem 4-2.3. *Let L be a finite lattice. Let $\mathfrak{p}$ and $\mathfrak{q}$ be prime intervals in L. If $\operatorname{con}(\mathfrak{p}) \prec \operatorname{con}(\mathfrak{q})$ in $\operatorname{Ji}(\operatorname{Con} L)$, then there exist prime intervals $\overline{\mathfrak{p}}$ and $\overline{\mathfrak{q}}$ in L with $\operatorname{con}(\mathfrak{p}) = \operatorname{con}(\overline{\mathfrak{p}})$ and $\operatorname{con}(\mathfrak{q}) = \operatorname{con}(\overline{\mathfrak{q}})$ such that the sublattice generated by $\{0_{\overline{\mathfrak{p}}}, 1_{\overline{\mathfrak{p}}}, 0_{\overline{\mathfrak{q}}}, 1_{\overline{\mathfrak{q}}}\}$ is isomorphic to N_5; specifically, $\{0_{\overline{\mathfrak{p}}} \wedge 0_{\overline{\mathfrak{q}}}, 0_{\overline{\mathfrak{p}}}, 1_{\overline{\mathfrak{p}}}, 0_{\overline{\mathfrak{q}}}, 1_{\overline{\mathfrak{q}}}\} \cong \mathsf{N}_5$ or $\{0_{\overline{\mathfrak{q}}}, 0_{\overline{\mathfrak{p}}}, 1_{\overline{\mathfrak{p}}}, 1_{\overline{\mathfrak{q}}}, 1_{\overline{\mathfrak{p}}} \vee 1_{\overline{\mathfrak{q}}}\} \cong \mathsf{N}_5$* (see Figure 4-2.2).

Proof. Since $\mathfrak{p}$ is prime, from the general description of $\operatorname{con}(\mathfrak{q})$ (see Section II.1.3 of LTF), we conclude that $\mathfrak{q} \Rightarrow \mathfrak{p}$. So for some natural number n, there exists a sequence

$$\mathfrak{q} = [u_0, v_0] \twoheadrightarrow \cdots \twoheadrightarrow [u_n, v_n] \supseteq \mathfrak{p}.$$

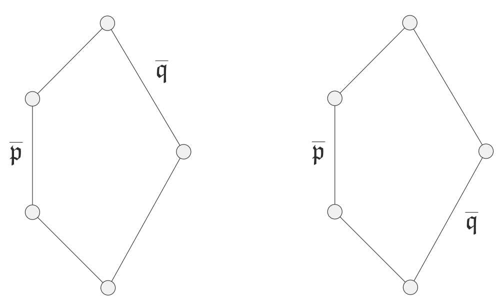

Figure 4-2.2: The conclusion of Theorem 4-2.3: $\operatorname{con}(\overline{\mathfrak{p}}) \prec \operatorname{con}(\overline{\mathfrak{q}})$.

Choose the prime intervals $\overline{\mathfrak{p}}$ and $\overline{\mathfrak{q}}$ so that $\operatorname{con}(\mathfrak{p}) = \operatorname{con}(\overline{\mathfrak{p}})$, $\operatorname{con}(\mathfrak{q}) = \operatorname{con}(\overline{\mathfrak{q}})$, and we can choose a sequence

$$\overline{\mathfrak{q}} = [e_0, f_0] \twoheadrightarrow \cdots \twoheadrightarrow [e_m, f_m] \supseteq \overline{\mathfrak{p}}$$

with m minimal.

Case 1: $m = 1$ and $\overline{\mathfrak{q}} \overset{\mathrm{dn}}{\twoheadrightarrow} [e_1, f_1]$.

Observe that $0_{\overline{\mathfrak{p}}} \wedge 0_{\overline{\mathfrak{q}}} \neq 0_{\overline{\mathfrak{p}}}$, because if $0_{\overline{\mathfrak{p}}} \wedge 0_{\overline{\mathfrak{q}}} = 0_{\overline{\mathfrak{p}}}$, then $\operatorname{con}(\overline{\mathfrak{p}}) = \operatorname{con}(\overline{\mathfrak{q}})$, contradicting that $\operatorname{con}(\mathfrak{p}) = \operatorname{con}(\overline{\mathfrak{p}})$, $\operatorname{con}(\mathfrak{q}) = \operatorname{con}(\overline{\mathfrak{q}})$, and the assumption of the theorem, namely, that $\operatorname{con}(\mathfrak{p}) \prec \operatorname{con}(\mathfrak{q})$. Therefore, $\{0_{\overline{\mathfrak{p}}} \wedge 0_{\overline{\mathfrak{q}}}, 0_{\overline{\mathfrak{p}}}, 1_{\overline{\mathfrak{p}}}, 0_{\overline{\mathfrak{q}}}, 1_{\overline{\mathfrak{q}}}\} \cong \mathsf{N}_5$, as required.

Case 2: $m = 1$ and $\overline{\mathfrak{q}} \overset{\mathrm{up}}{\twoheadrightarrow} [e_1, f_1]$.

This follows by duality from Case 1.

Case 3: $m > 1$ and $[e_{m-1}, f_{m-1}] \overset{\mathrm{up}}{\twoheadrightarrow} [e_m, f_m]$.

We apply Lemma 4-2.2 to $I = [f_{m-1} \wedge e_m, f_{m-1}]$, $J = [e_m, f_m]$, and $\overline{\mathfrak{p}}$ to conclude that there exists a subinterval J^* of J and prime intervals $\mathfrak{t} \subseteq I$ and $\overline{\mathfrak{p}}^* \subseteq J^*$ such that $\mathfrak{t} \overset{\mathrm{up}}{\twoheadrightarrow} J^*$ and $\operatorname{con}(\overline{\mathfrak{p}}^*) = \operatorname{con}(\overline{\mathfrak{p}})$. So $\operatorname{con}(\overline{\mathfrak{p}}^*) = \operatorname{con}(\overline{\mathfrak{p}}) \leq \operatorname{con}(\mathfrak{t}) \leq \operatorname{con}(\overline{\mathfrak{q}})$. Since $\operatorname{con}(\overline{\mathfrak{p}}) \prec \operatorname{con}(\overline{\mathfrak{q}})$ in $\operatorname{Ji}(\operatorname{Con} L)$ and $\operatorname{con}(\mathfrak{t}) \in \operatorname{Ji}(\operatorname{Con} L)$, we have either $\operatorname{con}(\overline{\mathfrak{p}}) = \operatorname{con}(\mathfrak{t})$ or $\operatorname{con}(\mathfrak{t}) = \operatorname{con}(\overline{\mathfrak{q}})$.

If $\operatorname{con}(\overline{\mathfrak{p}}) = \operatorname{con}(\mathfrak{t})$, then substitute $\overline{\mathfrak{p}}$ by $\mathfrak{t}$. We conclude that $\operatorname{con}(\mathfrak{t}) = \operatorname{con}(\mathfrak{p})$ and we have a sequence of $m - 1$ steps between $\overline{\mathfrak{q}}$ and $\mathfrak{t}$, contradicting the minimality of m.

If $\operatorname{con}(\mathfrak{t}) = \operatorname{con}(\overline{\mathfrak{q}})$, then $\operatorname{con}(\mathfrak{t}) = \operatorname{con}(\mathfrak{q})$ and we find a single step between $\mathfrak{t}$ and $\overline{\mathfrak{p}}$, contradicting the minimality of m and $m > 1$.

Case 4: $m > 1$ and $[e_{m-1}, f_{m-1}] \overset{\mathrm{dn}}{\twoheadrightarrow} [e_m, f_m]$.

By duality from Case 3. □

4-2.2 Finite semimodular lattices

We now specialize Theorem 4-2.3 to semimodular lattices.

Theorem 4-2.4. *Let L be a finite semimodular lattice. Let $\mathfrak{p}$ and $\mathfrak{q}$ be prime intervals in L. If $\operatorname{con}(\mathfrak{p}) \prec \operatorname{con}(\mathfrak{q})$ in $\operatorname{Ji}(\operatorname{Con} L)$, then there exist prime intervals $\overline{\mathfrak{p}}$ and $\overline{\mathfrak{q}}$ in L with $\operatorname{con}(\mathfrak{p}) = \operatorname{con}(\overline{\mathfrak{p}})$ and $\operatorname{con}(\mathfrak{q}) = \operatorname{con}(\overline{\mathfrak{q}})$ such that the sublattice generated by $\{0_{\overline{\mathfrak{p}}}, 1_{\overline{\mathfrak{p}}}, 0_{\overline{\mathfrak{q}}}, 1_{\overline{\mathfrak{q}}}\}$ is isomorphic to N_5; specifically, $\{0_{\overline{\mathfrak{p}}} \wedge 0_{\overline{\mathfrak{q}}}, 0_{\overline{\mathfrak{p}}}, 1_{\overline{\mathfrak{p}}}, 0_{\overline{\mathfrak{q}}}, 1_{\overline{\mathfrak{q}}}\} \cong \mathsf{N}_5$.* (See the diagram on the left in Figure 4-2.2.)

Proof. By Theorem 4-2.3, we obtain the prime intervals $\overline{\mathfrak{p}}$ and $\overline{\mathfrak{q}}$ such that the sublattice generated by $\{0_{\overline{\mathfrak{p}}}, 1_{\overline{\mathfrak{p}}}, 0_{\overline{\mathfrak{q}}}, 1_{\overline{\mathfrak{q}}}\}$ is an N_5 as shown in Figure 4-2.2. Semimodularity excludes the possibility of the diagram on the right in Figure 4-2.2, hence the statement. □

At this point, we have established that each cover in $\operatorname{Ji}(\operatorname{Con} L)$ can be represented in an N_5 by prime intervals as shown. The next lemma produces an N_5 with one more cover.

Lemma 4-2.5. *Let L be a finite semimodular lattice. Let $\mathfrak{p}$ and $\mathfrak{q}$ be prime intervals in L such that* $\operatorname{con}(\mathfrak{p}) \prec \operatorname{con}(\mathfrak{q})$ *in* $\operatorname{Ji}(\operatorname{Con} L)$ *and*

$$\{0_\mathfrak{p} \wedge 0_\mathfrak{q}, 0_\mathfrak{p}, 1_\mathfrak{p}, 0_\mathfrak{q}, 1_\mathfrak{q}\} \cong \mathsf{N}_5.$$

Then there exist prime intervals $\overline{\mathfrak{p}}$ and $\overline{\mathfrak{q}}$ in L such that $\operatorname{con}(\mathfrak{p}) = \operatorname{con}(\overline{\mathfrak{p}})$, $\operatorname{con}(\mathfrak{q}) = \operatorname{con}(\overline{\mathfrak{q}})$,

$$\{0_{\overline{\mathfrak{p}}} \wedge 0_{\overline{\mathfrak{q}}}, 0_{\overline{\mathfrak{p}}}, 1_{\overline{\mathfrak{p}}}, 0_{\overline{\mathfrak{q}}}, 1_{\overline{\mathfrak{q}}}\} \cong \mathsf{N}_5,$$

$1_{\overline{\mathfrak{p}}} \prec 1_{\overline{\mathfrak{q}}}$, *and the length of* $[0_{\overline{\mathfrak{p}}} \wedge 0_{\overline{\mathfrak{q}}}, 1_{\overline{\mathfrak{q}}}]$ *is less than or equal to the length of* $[0_\mathfrak{p} \wedge 0_\mathfrak{q}, 1_\mathfrak{q}]$.

Proof. Set $o = 0_\mathfrak{p} \wedge 0_\mathfrak{q}$ and $K = [o, 1_\mathfrak{q}]$. We prove the statement by induction on the length of K.

If K has length 3, then we must have $0_\mathfrak{p} \wedge 0_\mathfrak{q} \prec 0_\mathfrak{p} \prec 1_\mathfrak{p} \prec 1_\mathfrak{q}$ as required.

Now assume that the interval K is of length greater than 3 and that the statement is true for shorter intervals. Choose an element x with $o \prec x < 0_\mathfrak{q}$ and let $y = x \vee 1_\mathfrak{p}$. If $x \leq 1_\mathfrak{p}$, then $x \leq 1_\mathfrak{p} \wedge 0_\mathfrak{q} = o$, a contradiction. So by semimodularity, we obtain that $1_\mathfrak{p} \prec y$.

If $y = 1_\mathfrak{q}$, then $1_\mathfrak{p} \prec 1_\mathfrak{q}$, as required. Otherwise, $1_\mathfrak{p} \prec y < 1_\mathfrak{q}$, so $0_\mathfrak{q} \nleq y$. Therefore, $y \wedge 0_\mathfrak{q} < 0_\mathfrak{q}$. Since $o \prec x \leq y \wedge 0_\mathfrak{q}$, we conclude that $o < y \wedge 0_\mathfrak{q} < 0_\mathfrak{q}$.

Let $I = [y \wedge 0_\mathfrak{q}, y]$ and $J = [o, 1_\mathfrak{p}]$. Since $I \overset{\mathrm{dn}}{\twoheadrightarrow} J \supseteq \mathfrak{p}$, we can apply the dual of Lemma 4-2.2 to I, J, and $\mathfrak{p}$ and we obtain the prime interval $\mathfrak{t} \subseteq I$, the interval $J^* \subseteq J$, and the prime interval $\mathfrak{p}^* \subseteq J^*$ satisfying $\mathfrak{t} \overset{\mathrm{dn}}{\twoheadrightarrow} J^*$ and $\operatorname{con}(\mathfrak{p}) = \operatorname{con}(\mathfrak{p}^*)$.

Since $\operatorname{con}(\mathfrak{p}^*) = \operatorname{con}(\mathfrak{p}) \leq \operatorname{con}(\mathfrak{t}) \leq \operatorname{con}(\mathfrak{q})$, $\operatorname{con}(\mathfrak{p}) \prec \operatorname{con}(\mathfrak{q})$ in $\operatorname{Ji}(\operatorname{Con} L)$ by assumption, and $\operatorname{con}(\mathfrak{t}) \in \operatorname{Ji}(\operatorname{Con} L)$, we conclude that $\operatorname{con}(\mathfrak{p}) = \operatorname{con}(\mathfrak{t})$ or $\operatorname{con}(\mathfrak{t}) = \operatorname{con}(\mathfrak{q})$. Accordingly, we distinguish two cases.

Case 1*:* $\operatorname{con}(\mathfrak{p}) = \operatorname{con}(\mathfrak{t})$.

Then $y \wedge 0_\mathfrak{q} \neq 0_\mathfrak{t}$, because $y \wedge 0_\mathfrak{q} = 0_\mathfrak{t}$ would imply that $\operatorname{con}(\mathfrak{t}) = \operatorname{con}(\mathfrak{q})$ and so $\operatorname{con}(\mathfrak{p}) = \operatorname{con}(\mathfrak{q})$, contradicting the lemma's assumption that $\operatorname{con}(\mathfrak{p}) \prec \operatorname{con}(\mathfrak{q})$. Therefore, we can take the new N_5: $\{y \wedge 0_\mathfrak{q}, 0_\mathfrak{t}, 1_\mathfrak{t}, 0_\mathfrak{q}, 1_\mathfrak{q}\}$ and let $\mathfrak{t}$ play the role of $\mathfrak{p}$. Observe that the interval $[y \wedge 0_\mathfrak{q}, 1_\mathfrak{q}]$ is shorter than K since $o < y \wedge 0_\mathfrak{q}$. We apply our inductive assumption and obtain prime intervals $\overline{\mathfrak{p}}$ and $\overline{\mathfrak{q}}$ in L which satisfy the statement.

Case 2*:* $\operatorname{con}(\mathfrak{t}) = \operatorname{con}(\mathfrak{q})$.

Then we argue as in the previous paragraph that $0_\mathfrak{t} \wedge 0_{\mathfrak{p}^*} \neq 0_{\mathfrak{p}^*}$. So we can take the new N_5: $\{0_\mathfrak{t} \wedge 0_{\mathfrak{p}^*}, 0_{\mathfrak{p}^*}, 1_{\mathfrak{p}^*}, 0_\mathfrak{t}, 1_\mathfrak{t}\}$. Let $\mathfrak{t}$ play the role of $\mathfrak{q}$ and let $\mathfrak{p}^*$ play the role of $\mathfrak{p}$. We proceed, as above, noticing that $[0_\mathfrak{t} \wedge 0_{\mathfrak{p}^*}, 1_\mathfrak{t}]$ is shorter than K, since $1_\mathfrak{t} < 1_\mathfrak{q}$. Applying our inductive assumption, we obtain the prime intervals $\overline{\mathfrak{p}}$ and $\overline{\mathfrak{q}}$ in L that satisfy the statement, proving the lemma. ☐

4-2.3 Proof of the Tight N_7 Theorem

Let L be a finite *semimodular* lattice. Let $\mathfrak{p}$ and $\mathfrak{q}$ be prime intervals in L such that $\operatorname{con}(\mathfrak{p}) \prec \operatorname{con}(\mathfrak{q})$ in $\operatorname{Ji}(\operatorname{Con} L)$. By Theorem 4-2.4, we can assume that the prime intervals $\mathfrak{p}$ and $\mathfrak{q}$ satisfy the condition:

$$\{0_\mathfrak{p} \wedge 0_\mathfrak{q}, 0_\mathfrak{p}, 1_\mathfrak{p}, 0_\mathfrak{q}, 1_\mathfrak{q}\} \cong \mathsf{N}_5.$$

By Lemma 4-2.5, we can assume, additionally, that $\mathfrak{p}$ and $\mathfrak{q}$ satisfy $1_\mathfrak{p} \prec 1_\mathfrak{q}$. Set $o = 0_\mathfrak{p} \wedge 0_\mathfrak{q}$ and $K = [o, 1_\mathfrak{q}]$.

By induction on the length of K, we prove that there exists a tight N_7 sublattice N of K, such that $\operatorname{con}(a, 1) = \operatorname{con}(\mathfrak{p})$ and $\operatorname{con}(b, 1) = \operatorname{con}(\mathfrak{q})$, using the notation of Figure 3-4.1.

Let K be of length 3. The chain $o < 0_\mathfrak{q} \prec 1_\mathfrak{q} \prec 1_\mathfrak{q}$ also has length 3, hence $o \prec 0_\mathfrak{q}$ and there is an element $t \in K$ satisfying $o \prec t \prec 0_\mathfrak{q}$. Set $N = \{o, t, 0_\mathfrak{p}, 1_\mathfrak{p}, t \vee 0_\mathfrak{p}, 0_\mathfrak{q}, 1_\mathfrak{q}\}$. Since $t \parallel 1_\mathfrak{p}$, it follows that $t \vee 0_\mathfrak{p} \parallel 1_\mathfrak{p}$. Symmetrically, $0_\mathfrak{p} \parallel 0_\mathfrak{q}$ and therefore $t \vee 0_\mathfrak{p} \parallel 0_\mathfrak{q}$. By the semimodularity of L, since $o \prec 0_\mathfrak{p}$, t, we obtain that $0_\mathfrak{p}$, $t \prec 0_\mathfrak{p} \vee t$. It follows that $0_\mathfrak{p} \vee t \prec 1_\mathfrak{q}$, since K is of length 3. Clearly, $\operatorname{con}(\mathfrak{p}) = \operatorname{con}(t \vee 0_\mathfrak{p}, 1_\mathfrak{q})$. Therefore, N is a covering N_7 satisfying the conditions of the Tight N_7 Theorem.

Next assume that the interval K is of length greater than 3 and that the statement is true for shorter intervals of L. Since K has length greater than 3, there exist elements s and t of L such that $o \prec s < t \prec 0_\mathfrak{q}$. Similarly, there is an element x of L such that $o \prec x < 0_\mathfrak{p}$. Set $y = x \vee t$. Since L is semimodular, it follows that $t \prec y$ and so the interval $[t, 1_\mathfrak{q}]$ is of length 2. We conclude that $y \prec 1_\mathfrak{q}$. Therefore, $y \parallel 0_\mathfrak{q}$.

Now compare y and $0_\mathfrak{p}$. Clearly, $y \leq 0_\mathfrak{p}$ contradicts that $y \prec 1_\mathfrak{q}$. On the other hand, if $y > 0_\mathfrak{p}$, then $\{o, 0_\mathfrak{p}, 1_\mathfrak{p}, t, 0_\mathfrak{q}, 0_\mathfrak{q}\}$ is a tight N_7 containing $\mathfrak{p}$ and $\mathfrak{q}$, as required, completing the proof in this special case.

So we can also assume that $y \parallel 0_\mathfrak{p}$.

Next we form $y \wedge 0_\mathfrak{p}$ and $y \wedge 1_\mathfrak{p}$. We distinguish two cases as to whether these are two distinct elements or not. Set $z = y \wedge 0_\mathfrak{p}$.

Case 1*:* $z = y \wedge 0_\mathfrak{p} < y \wedge 1_\mathfrak{p}$.

Let $\mathfrak{p}_1$ be a prime interval such that $z = 0_{\mathfrak{p}_1} \prec 1_{\mathfrak{p}_1} \leq y \wedge 1_\mathfrak{p}$ and also satisfying $\operatorname{con}(\mathfrak{p}) = \operatorname{con}(\mathfrak{p}_1)$; set $\mathfrak{q}_1 = [t, y]$, see Figure 4-2.3. Obviously $\mathfrak{q}_1$ is a prime interval and $\operatorname{con}(\mathfrak{q}) = \operatorname{con}(\mathfrak{q}_1)$. It is easy to check that the lattice $\{0_{\mathfrak{p}_1} \wedge 0_{\mathfrak{q}_1}, 0_{\mathfrak{p}_1}, 1_{\mathfrak{p}_1}, 0_{\mathfrak{q}_1}, 1_{\mathfrak{q}_1}\}$ is isomorphic to N_5; for instance, $0_{\mathfrak{p}_1} \wedge 0_{\mathfrak{q}_1} = o$, because $0_{\mathfrak{p}_1} \wedge 0_{\mathfrak{q}_1} \leq 0_\mathfrak{p} \wedge 1_\mathfrak{q} = o$. The corresponding interval $[0_{\mathfrak{p}_1} \wedge 0_{\mathfrak{q}_1}, 1_{\mathfrak{q}_1}]$ has length smaller than the length of $[0_\mathfrak{p} \wedge 0_\mathfrak{q}, 1_\mathfrak{q}]$ since $1_{\mathfrak{q}_1} < 1_\mathfrak{q}$. We may apply Lemma 4-2.5 to obtain yet another pair of prime intervals $\overline{\mathfrak{p}}$ and $\overline{\mathfrak{q}}$ such that $\operatorname{con}(\mathfrak{p}) = \operatorname{con}(\mathfrak{p}_1) = \operatorname{con}(\overline{\mathfrak{p}})$ and $\operatorname{con}(\mathfrak{q}) = \operatorname{con}(\mathfrak{q}_1) = \operatorname{con}(\overline{\mathfrak{q}})$. The resulting lattice, $\{0_{\overline{\mathfrak{p}}} \wedge 0_{\overline{\mathfrak{q}}}, 0_{\overline{\mathfrak{p}}}, 1_{\overline{\mathfrak{p}}}, 0_{\overline{\mathfrak{q}}}, 1_{\overline{\mathfrak{q}}}\} \cong \mathsf{N}_5$, spans an interval of length sufficiently small that we may apply our inductive hypothesis to obtain a sublattice N that will satisfy the statement.

Case 2*:* $z = y \wedge 0_\mathfrak{p} = y \wedge 1_\mathfrak{p}$.

Define the set $U = \{o, z, t, y, 0_\mathfrak{p}, 1_\mathfrak{p}, 0_\mathfrak{q}, 1_\mathfrak{q}\}$. We claim that U is a sublattice of L, whose diagram is in Figure 4-2.4. This follows trivially from

$$\{0_\mathfrak{p} \wedge 0_\mathfrak{q}, 0_\mathfrak{p}, 1_\mathfrak{p}, 0_\mathfrak{q}, 1_\mathfrak{q}\} \cong \mathsf{N}_5$$

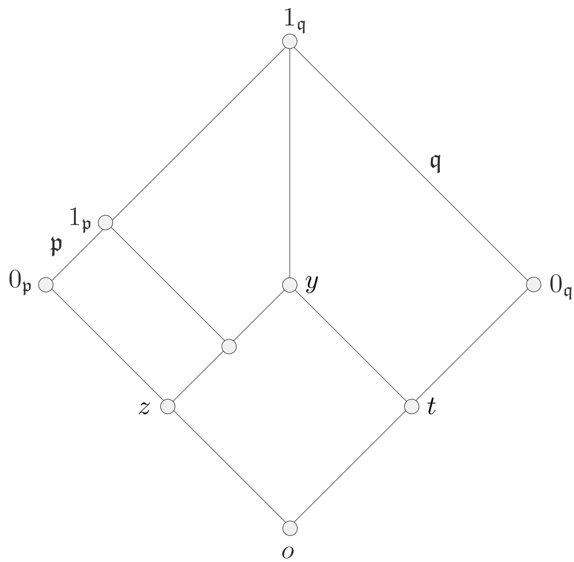

Figure 4-2.3: Case 1.

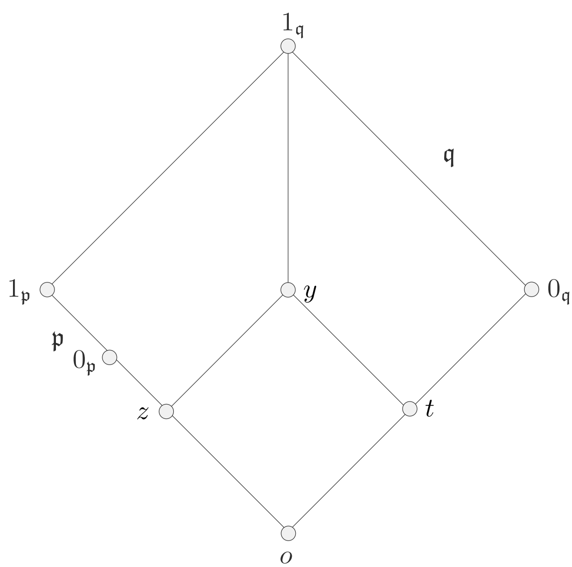

Figure 4-2.4: Case 2: the sublattice U.

and $1_\mathfrak{p} \prec 1_\mathfrak{q}$, along with the coverings we have already verified.

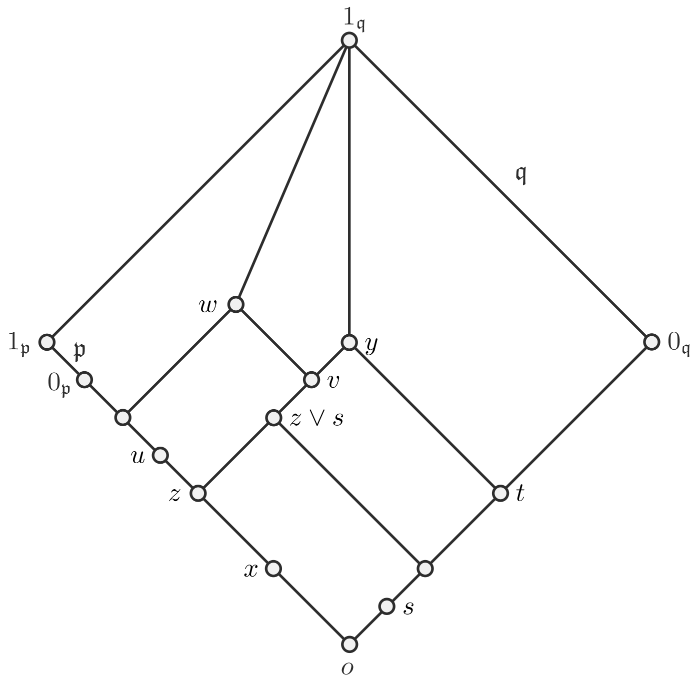

Figure 4-2.5: Assuming $z < 0_{\mathfrak{p}}$ and $z \nprec y$.

If $0_{\mathfrak{p}} = z$, then the interval $[z, 1_{\mathfrak{q}}]$ is of length 2, so $z \prec y$. It follows that U is a tight N_7 that satisfies the theorem.

Next assume that $0_{\mathfrak{p}} \neq z$, that is, $z < 0_{\mathfrak{p}}$. If $z \prec y$, then the interval $[z, 1_{\mathfrak{q}}]$ has a maximal chain: $\{z, y, 1_{\mathfrak{q}}\}$, so $z < 0_{\mathfrak{p}}$ conflicts with semimodularity.

So we can assume that $z < 0_{\mathfrak{p}}$ and $z \nprec y$. Choose elements u and v satisfying $z \prec u \leq 0_{\mathfrak{p}}$ and $z \prec z \vee s \leq v \prec y$. Set $w = u \vee v$, see Figure 4-2.5. Obviously, $v \prec w$ and $w \prec 1_{\mathfrak{q}}$.

Clearly, $\operatorname{con}(\mathfrak{p}) \leq \operatorname{con}(y, 1_{\mathfrak{q}}) = \operatorname{con}(v, w) \leq \operatorname{con}(\mathfrak{q})$. Since $[y, 1_{\mathfrak{q}}]$ is a prime interval, it follows that $\operatorname{con}(y, 1_{\mathfrak{q}}) \in \operatorname{Ji}(\operatorname{Con} L)$. However, $\operatorname{con}(\mathfrak{p}) \prec \operatorname{con}(\mathfrak{q})$ in $\operatorname{Ji}(\operatorname{Con} L)$, therefore, either $\operatorname{con}(\mathfrak{p}) = \operatorname{con}(v, w)$ or $\operatorname{con}(\mathfrak{q}) = \operatorname{con}(y, 1_{\mathfrak{q}})$. So we have to distinguish two subcases.

Case 2a*: $z < 0_{\mathfrak{p}}$, $z \nprec y$, and* $\operatorname{con}(\mathfrak{q}) = \operatorname{con}(y, 1_{\mathfrak{q}})$.

Replace $\mathfrak{q}$ by $\mathfrak{q}^* = [y, 1_{\mathfrak{q}}]$. The prime intervals $\mathfrak{p}$ and $\mathfrak{q}^*$ satisfy all the assumptions we made on $\mathfrak{p}$ and $\mathfrak{q}$ and they are contained in a shorter interval, $[z, 1_{\mathfrak{q}}]$, so by induction $[z, 1_{\mathfrak{q}}]$ contains the desired tight N_7.

Case 2b*: $z < 0_{\mathfrak{p}}$, $z \nprec y$, and* $\operatorname{con}(\mathfrak{p}) = \operatorname{con}(v, w)$.

Set $\mathfrak{p}^* = [v, w]$. Consider the interval $[t \wedge v, 1_{\mathfrak{q}}]$ and the prime intervals therein, $\mathfrak{p}^*$ and $\mathfrak{q}$. They satisfy the assumptions

$$\{0_{\mathfrak{p}^*} \wedge 0_{\mathfrak{q}}, 0_{\mathfrak{p}^*}, 1_{\mathfrak{p}^*}, 0_{\mathfrak{q}}, 1_{\mathfrak{q}}\} \cong \mathsf{N}_5$$

and $1_{\mathfrak{p}^*} \prec 1_{\mathfrak{q}}$.

Finally, the interval $[v \wedge t, 1_{\mathfrak{q}}]$ is shorter than K, because $o \prec s \leq v \wedge t$. We can apply the inductive hypothesis to obtain a tight N_7 that satisfies the theorem.

This completes the proof of the Tight N_7 Theorem.

4-3. Congruence lattices of rectangular lattices

4-3.1 Preliminaries

We work with colorings of a finite lattice.

Definition 4-3.1. Let L be a finite lattice and let P be a finite order. Let $\operatorname{Prime}(L)$ denote the set of prime intervals of L. We say that L can be *colored* with P, if there is a map $\operatorname{col}\colon \operatorname{Prime}(L) \to P$ such that the following conditions are satisfied for $\mathfrak{p}$, $\mathfrak{q} \in \operatorname{Prime}(L)$:

(i) col maps $\operatorname{Prime}(L)$ *onto* P;

(ii) if $\mathfrak{q} \Rightarrow \mathfrak{p}$, then $\operatorname{col}(\mathfrak{p}) \leq \operatorname{col}(\mathfrak{q})$;

(iii) if $\operatorname{col}(\mathfrak{p}) \leq \operatorname{col}(\mathfrak{q})$, then $\mathfrak{q} \Rightarrow \mathfrak{p}$.

◊ **Theorem 4-3.2.** *Let L be a finite lattice. The map $\mathfrak{p} \to \operatorname{con}(\mathfrak{p})$ is a coloring of L with $P = \operatorname{Ji}(\operatorname{Con} L)$. In fact, if L can be* colored *with the order P and* col *is a bijection, then* $\operatorname{Ji}(\operatorname{Con} L) \cong P$.

This is a recasting of a result of J. Jakubik [205]. See Exercises 4.1–4.2.

In addition to N_7, we will also use the lattice N_7^+, a modified version of N_7, the top lattice in Figure 4-3.1. The order $\operatorname{Ji}(\operatorname{Con} \mathsf{N}_7^+)$ is a single covering pair.

Now we introduce the gadget we use to order the join-irreducible congruences in our construction.

Let m be a positive integer. Let U_1 be an N_7^+. Let U_2, ..., U_m be N_7-s. Then $N = U_1 \cup \dots \cup U_m$ is an *m-stacked* N_7^+ (see Figure 4-3.1), where $1 \leq i < m$, if

$$\begin{aligned} 1_{U_i} &= b_{U_{i+1}},\\ a_{U_i} &= d_{U_{i+1}},\\ c_{U_i} &= e_{U_{i+1}},\\ 0_{U_i} &= 0_{U_{i+1}}. \end{aligned}$$

We define the *0-stacked* N_7^+ as M_3.

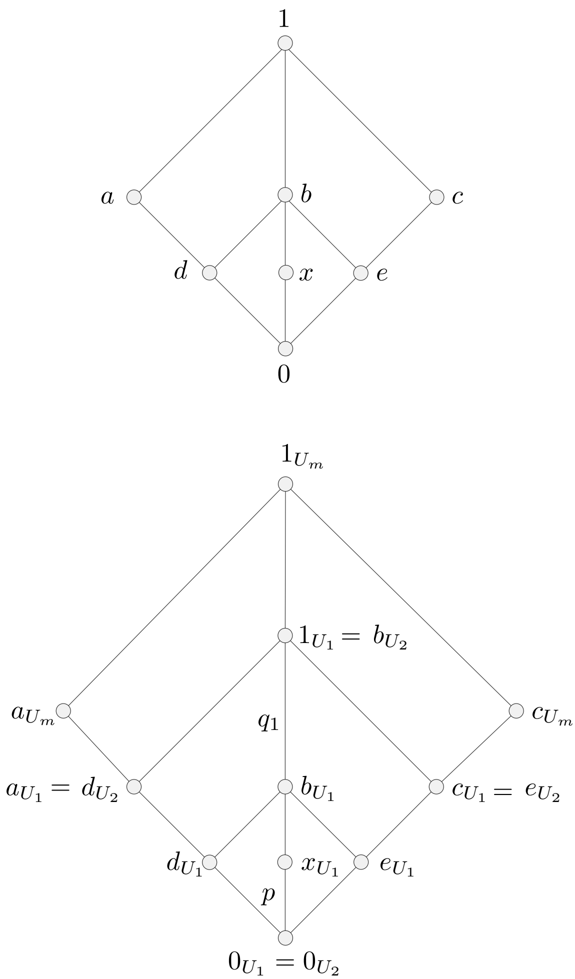

Figure 4-3.1: The lattice N_7^+ and the m-stacked N_7^+.

The order of join-irreducible congruences of an m-stacked N_7^+ is isomorphic to the $(m+1)$-element order $\{q_1, \dots, q_m, p\}$ with the coverings $q_1 \prec p$, $\dots$, $q_m \prec p$. See the "coloring" in Figure 4-3.1. In the next section, we give an example of how m-stacked N_7^+-s can be used to represent more complicated orders.

4-3.2 The construction and proof

We now present the construction for Theorem 4-1.2(i). Let $P = \{p_1, \dots, p_n\}$ be a finite order whose elements are enumerated so that if $p_i < p_j$, then $i < j$.

We prove the following statement by induction on n.

There exists a rectangular lattice, L_n, with

$$\mathrm{C_{ul}}(L_n) = a_0 \prec a_1 \prec \cdots \prec a_n, \tag{4-3.1}$$

$$\mathrm{C_{ur}}(L_n) = b_0 \prec b_1 \prec \cdots \prec b_n \tag{4-3.2}$$

such that

$$p_i \mapsto \mathrm{con}(a_{i-1}, a_i) = \mathrm{con}(b_{i-1}, b_i), \tag{4-3.3}$$

for $i = 1, \ldots, n$, defines an isomorphism $P_n \cong \mathrm{Ji}(\mathrm{Con}\, L)_n$.

Recall that the chains $\mathrm{C_{ul}}(L_n)$ and $\mathrm{C_{ur}}(L_n)$ were defined in Section 3-7.

Let $n = 1$. Then set $L_1 = \mathsf{M}_3$.

Inductive step. Let $p_{i_1}, \ldots, p_{i_m}$ list the elements of P_n covered by p_n, enumerated so that if $p_{i_j} < p_{i_k}$, then $j < k$; of course, the list is empty if p_n is minimal in P (that is, $m = 0$). Let $A = A_n$ be an m-stacked N_7^+. Form $B = B_n \geq C_n \times C_{m+1}$ by adding an element to the interior of the interval $[(k-1, j), (k, j+1)]$ whenever $p_k = p_{i_j}$, so that $[(k-1, j), (k, j+1)]_B \cong \mathsf{M}_3$. Let $K = K_n$ be the gluing of A and B over the bottom right border of A and the top left border of B. The lattice K is rectangular.

Applying the inductive hypothesis to the order $P_{n-1} = \{p_1, \ldots, p_{n-1}\}$, we obtain the rectangular lattice L_{n-1} satisfying (4-3.1)–(4-3.3) for $n-1$.

Let $D = D_n = C_{m+1} \times C_n$. Glue L_{n-1} and D over the top left boundary of L_{n-1} and the bottom right boundary of D to form the lattice K'. The lattice K' is rectangular. Finally, glue K' and K over the top right boundary of K' and the bottom left boundary of K to obtain the lattice $L = L_n$.

To describe the congruences of L, we define a map ψ from $\mathrm{Prime}(L)$ to the order P. We shall utilize the map col_{n-1} of $\mathrm{Prime}(L_{n-1})$ to P_{n-1}, the coloring of L_{n-1}.

For the lattice A, we define the map

$$\psi_A \colon \mathrm{Prime}(A) \to P_n$$

as follows:

(A.i) For all prime intervals $\mathfrak{p}$ in $[0_{U_1}, b_{U_1}]$ (using the notation developed in Section 4-3.1 – see Figure 4-3.1) set $\psi_A(\mathfrak{p}) = p_n$. If $m = 0$, then $A \cong \mathsf{M}_3$ and we set $\psi_A(\mathfrak{p}) = p_n$ for all prime intervals $\mathfrak{p}$ in A.

(A.ii) For all $1 \leq j \leq m$, set $\psi_A[a_{U_j}, 1_{U_j}] = \psi_A[c_{U_j}, 1_{U_j}] = p_n$.

(A.iii) For all $1 \leq j \leq m$, set $\psi_A[d_{U_j}, a_{U_j}] = \psi_A[b_{U_j}, 1_{U_j}] = \psi_A[e_{U_j}, c_{U_j}] = p_{i_j}$.

This is, in fact, a coloring of A with the order $p_{i_1}, \ldots, p_{i_m} \prec p_n$.

For the lattice B, we define the map

$$\psi_B \colon \mathrm{Prime}(B) \to P_n$$

as follows:

(B.i) For all $1 \le k \le n-1$ and $0 \le j \le m+1$, set $\psi_B[(k-1,j),(k,j)] = p_k$.

(B.ii) For all $0 \le k \le n-1$ and $1 \le j \le m$, set $\psi_B[(k,j),(k,j+1)] = p_{i_j}$.

(B.iii) For all $0 \le k \le n$, set $\psi_B[(k,0),(k,1)] = p_n$.

(B.iv) Whenever $p_k = p_{i_j}$, for all prime intervals $\mathfrak{p}$ of $\mathsf{M}_3 \cong [(k-1,j),(k,j+1)]$, set $\psi_B(\mathfrak{p}) = p_k = p_{i_j}$.

Clearly $\psi_B[(j-1,t),(j,t)] = \psi_B[(s,k),(s,k+1)]$ iff $p_j = p_{i_k}$.

For the lattice D, we define the map

$$\psi_D \colon \mathrm{Prime}(B) \to P_n$$

as follows:

(D.i) For all $0 \le k \le m+1$ and $1 \le j \le n-1$, set $\psi_D[(k,j-1),(k,j)] = p_j$.

(D.ii) For all $1 \le k \le m$ and $0 \le j \le n$, set $\psi_D[(k,j),(k+1,j)] = p_{i_k}$.

(D.iii) For all $0 \le k \le n$, set $\psi[(0,k),(1,k)] = p_n$.

Note that this is not a coloring for D. However,

For the lattice L_{n-1}, we already have the coloring col_{n-1}.

Now to define

$$\psi \colon \mathrm{Prime}(L) \to P,$$

let $\mathfrak{p} \in \mathrm{Prime}(L)$. Then $\mathfrak{p} \in \mathrm{Prime}(A)$, or $\mathfrak{p} \in \mathrm{Prime}(B)$, or $\mathfrak{p} \in \mathrm{Prime}(D)$, or $\mathfrak{p} \in \mathrm{Prime}(L_{n-1})$. We then define $\psi(\mathfrak{p}) = \psi_A(\mathfrak{p})$, or $\psi(\mathfrak{p}) = \psi_B(\mathfrak{p})$, or $\psi(\mathfrak{p}) = \psi_D(\mathfrak{p})$, or $\psi(\mathfrak{p}) = \mathrm{col}_{n-1}(\mathfrak{p})$, respectively. Observe that ψ is well defined. Indeed, if $\mathfrak{p}$ belongs to two of $\mathrm{Prime}(A)$, $\mathrm{Prime}(B)$, $\mathrm{Prime}(D)$, $\mathrm{Prime}(L_{n-1})$, say, $\mathfrak{p} \in \mathrm{Prime}(A) \cap \mathrm{Prime}(B)$, then $\psi(\mathfrak{p}) = \psi_A(\mathfrak{p}) = \psi_B(\mathfrak{p})$. There are three other pairs to consider, A, D; B, L_{n-1}; D, L_{n-1}; the arguments are the same for all four possibilities.

By Definition 4-3.1, we have to verify for ψ the three properties listed therein.

Property (i) of Definition 4-3.1 is obvious, in fact, ψ_D already maps *onto* P.

Next, we verify the following statement.

Claim. *Let* I, J *be intervals of* L_n *with* $I \twoheadrightarrow J$. *Then for each prime interval* $\mathfrak{p}$ *in* J, *there exists a prime interval* $\mathfrak{q}$ *in* I *such that* $\psi(\mathfrak{p}) \le \psi(\mathfrak{q})$.

Proof. There are 16 cases:

$$I \in \{A, B, D, L_{n-1}\} \text{ and } J \in \{A, B, D, L_{n-1}\}.$$

We discuss three cases, the others are similar.

Case 1*: I, $J \subseteq A$.*

Since ψ_A is a coloring of A, the statement is true.

Case 2*: I, $J \subseteq D$.*

This obvious, because D is distributive.

Case 3*: $I \subseteq A$ and $J \subseteq B$.*

Then $I \overset{\mathrm{dn}}{\twoheadrightarrow} J$. Set $H = I \wedge 1_B$; then $I \overset{\mathrm{dn}}{\twoheadrightarrow} H \overset{\mathrm{dn}}{\twoheadrightarrow} J$. The interval H is on the bottom right boundary of A and, therefore, on the top left boundary of B. Applying Case 1 to $I \overset{\mathrm{dn}}{\twoheadrightarrow} H$ and the analogue of Case 1 for B to $H \overset{\mathrm{dn}}{\twoheadrightarrow} J$, we get the result. ☐

To verify (ii) of Definition 4-3.1, take the prime intervals $\mathfrak{p}$ and $\mathfrak{q}$ with $\mathfrak{q} \Rightarrow \mathfrak{p}$, that is, with $\mathfrak{q} = I_0 \twoheadrightarrow \cdots \twoheadrightarrow I_k = \mathfrak{p}$. With k applications of the Claim, we conclude that $\psi(\mathfrak{p}) \leq \psi(\mathfrak{q})$.

Finally, we verify (iii) of Definition 4-3.1. Take the prime intervals $\mathfrak{p}$ and $\mathfrak{q}$ with $\psi(\mathfrak{p}) \leq \psi(\mathfrak{q})$. Again, there are 16 cases:

$$\mathfrak{p} \in \{A, B, D, L_{n-1}\} \text{ and } \mathfrak{p} \in \{A, B, D, L_{n-1}\}.$$

Again, we discuss three cases, the others are similar.

Case 1*: $\mathfrak{p}$, $\mathfrak{q} \subseteq A$.*

Then since ψ_A is a coloring of A, we conclude that $\mathfrak{q} \Rightarrow \mathfrak{p}$ in A, and therefore, in L.

Case 2*: $\mathfrak{p} \subseteq A$ and $\mathfrak{q} \subseteq B$.*

By the definition of ψ_A, we can find a prime interval $\mathfrak{p}'$ on the bottom right boundary of A so that $\psi(\mathfrak{p}) = \psi(\mathfrak{p}')$ and $\mathfrak{p} \Leftrightarrow \mathfrak{p}'$. Then $\mathfrak{p}'$ is also in B, so $\psi_B(\mathfrak{p}') \leq \psi_B(\mathfrak{q})$. Since ψ_B is a coloring of B, we conclude that $\mathfrak{q} \Rightarrow \mathfrak{p}'$ in B. So $\mathfrak{q} \Rightarrow \mathfrak{p}$, from which $\mathfrak{q} \Rightarrow \mathfrak{p}$ in L follows.

Case 3*: $\mathfrak{p} = [(x, y), (x + 1, y)] \subseteq D$ and $\mathfrak{q} \subseteq A$.*

Then $\mathfrak{p} \Leftrightarrow [(x, m+1), (x+1, m+1)]$ and $[(x, m+1), (x+1, m+1)]$ is in A and has the same color as $\mathfrak{p}$. Applying Case 1, we have our result.

4-3.3 The size of L_n

Define

$$p(n) = \frac{2}{3}n^3 + 2n^2 + \frac{4}{3}n + 1.$$

We will prove that

$$|L_n| \leq p(n) \tag{4-3.4}$$

by induction on n, from which $L_n = O(n^3)$ follows.

Let $n = 1$. Then $L_1 = \mathsf{M}_3$ and $|L_1| = 5 = p(1)$, so $|L_1| \leq p(1)$, as required.

Inductive step. The lattice $L = L_n$ is made up of the parts A, B, K, L_{n-1}, D, and K', as described in Section 4-3.2. A is an m-stacked N_7^+ so

$$|A| = 3m + 5. \tag{4-3.5}$$

B contains $C_n \times C_{m+2}$ as a sublattice and m elements were added to form M_3-s, so

$$|B| = n(m + 2) + m. \tag{4-3.6}$$

Since

$$|A \cap B| = m + 2, \tag{4-3.7}$$

from (4-3.5) and (4-3.7), we conclude that

$$|K| = 3m + 5 + n(m + 2) + m - (m + 2) = nm + 2n + 3m + 3. \tag{4-3.8}$$

Since $D = C_{m+2} \times C_n$, therefore,

$$|D| = n(m + 2). \tag{4-3.9}$$

Since

$$|D \cap L_{n-1}| = n, \tag{4-3.10}$$

from (4-3.9) and (4-3.10), we obtain that,

$$|K'| = |L_{n-1}| + nm + n. \tag{4-3.11}$$

Finally,

$$|K \cap K'| = n + m + 1. \tag{4-3.12}$$

Observing that $m \leq n - 1$, from (4-3.8), (4-3.11), and (4-3.12), we conclude that

$$\begin{aligned}
|L| &= |K| + |K'| - (n + m + 1) \\
&= nm + 2n + 3m + 3 + |L_{n-1}| + nm + n - (n + m + 1) \\
&= |L_{n-1}| + 2nm + 2n + 2m + 2 \\
&\leq |L_{n-1}| + 2n^2 + 2n = p(n - 1) + 2n^2 + 2n = p(n),
\end{aligned}$$

concluding the proof of (4-3.4), and hence $L_n = O(n^3)$.

4-4. More on tight N_7-s

To prove Theorem 4-1.2(ii), we have to find lots of elements in L. We construct them from tight N_7-s whose existence was discussed in Section 4-2.

For a tight N_7, we use the notation introduced in Figure 3-4.1. For a lattice L, we denote by $\mathrm{TN7}(L)$ the set of all tight N_7-s in L.

We start out by investigating tight N_7-s in *slim* rectangular lattices.

For an element x in a rectangular lattice L, we denote by x^{ll} its projection to the lower left chain, $\mathrm{C}_{\mathrm{l}}(L)$, and by x^{lr} its projection to the lower right chain, $\mathrm{C}_{\mathrm{lr}}(L)$. (Note that $x^{\mathrm{ll}} = \mathrm{lsp}(u)$ and $x^{\mathrm{lr}} = \mathrm{rsp}(u)$; see Section 3-4.) The first lemma states that distinct tight N_7-s have distinct middle elements and these elements have distinct projections onto $\mathrm{C}_{\mathrm{ll}}(L)$ and $\mathrm{C}_{\mathrm{lr}}(L)$.

Lemma 4-4.1. *Let L be a* slim rectangular lattice. *Let $U \neq V \in \mathrm{TN7}(L)$. Then $b_U \neq b_V$, $b_U^{\mathrm{ll}} \neq b_V^{\mathrm{lr}}$, and $b_U^{\mathrm{ll}} \neq b_V^{\mathrm{lr}}$.*

Proof. We assume that $U \neq V \in \mathrm{TN7}(L)$ and $b_U = b_V$. Since $b_U = b_V$ is meet-irreducible in L, it follows that $1_U = (b_U)^* = (b_V)^* = 1_V$. Since U is a tight N_7, the interval $[a_U \wedge b_U, 1_U]$ is of length 2. Therefore, the set $\{1_U = 1_V, a_U, b_U = b_V, a_U \wedge b_U\}$ is a 4-cell. It does not contain a_V because L is slim. So a_V is outside this cell, contradicting that a_V is immediately to the left of b_V.

This proves that $b_U \neq b_V$. We next prove that $b_U^{\mathrm{ll}} \neq b_V^{\mathrm{ll}}$ and $b_U^{\mathrm{lr}} \neq b_V^{\mathrm{lr}}$. Assume, to the contrary, that this fails. If $b_U^{\mathrm{ll}} = b_V^{\mathrm{ll}}$ and $b_U^{\mathrm{lr}} = b_V^{\mathrm{lr}}$, then $b_U = b_V$, a contradiction. So we can assume, by symmetry, that $b_U^{\mathrm{ll}} \neq b_V^{\mathrm{ll}}$ and $b_U^{\mathrm{lr}} = b_V^{\mathrm{lr}}$; and again, by symmetry, we can assume that

$$b_U^{\mathrm{ll}} < b_V^{\mathrm{ll}} \quad \text{and} \quad b_U^{\mathrm{lr}} = b_V^{\mathrm{lr}}.$$

So $b_U < b_V$ and $c_U < 1_U = b_U^* \leq b_V$. Since $c_U \leq b_V$, we have that $c_U^{\mathrm{lr}} = c_U \wedge u_r \leq b_V \wedge u_r \leq b_V^{\mathrm{lr}}$. Since c_U is to the right of b_U, it follows that $b_U^{\mathrm{lr}} < c_U^{\mathrm{lr}}$ and so $b_U^{\mathrm{lr}} < c_U^{\mathrm{lr}} \leq b_V^{\mathrm{lr}}$, contradicting that $b_U^{\mathrm{lr}} = b_V^{\mathrm{lr}}$. ☐

The next lemma will produce more elements for the proof of Theorem 4-1.2(ii). In a slim rectangular lattice, we associate with each pair U, V of tight N_7's satisfying $b_U \parallel b_V$, a single element, $b_U \wedge b_V$. We now show that the element $b_U \wedge b_V$ determines the pair U, V.

Lemma 4-4.2. *Let L be a slim rectangular lattice. Let*

$$U_1, V_1, U_2, V_2 \in \mathrm{TN7}(L)$$

and let $b_{U_1} \parallel b_{V_1}$ and $b_{U_2} \parallel b_{V_2}$.

If $b_{U_1} \wedge b_{V_1} = b_{U_2} \wedge b_{V_2}$, then $\{U_1, V_1\} = \{U_2, V_2\}$.

Proof. Set $x = b_{U_1} \wedge b_{V_1} = b_{U_2} \wedge b_{V_2}$. Let $\{U_1, V_1\} \neq \{U_2, V_2\}$. Since x is meet-reducible in L, it has exactly two covers, y and z. By symmetry, we can assume that $y \leq b_{U_1}$, $y \leq b_{U_2}$, $z \leq b_{V_1}$, $z \leq b_{V_2}$, and $U_1 \neq U_2$.

Case 1: b_{U_1} and b_{U_2} are comparable.

By symmetry, let $b_{U_1} \leq b_{U_2}$. Since $U_1 \neq U_2$, by Lemma 4-4.1, $b_{U_1} < b_{U_2}$ and so $z \nleq 1_{U_1} \leq b_{U_2}$. By semimodularity, $b_{U_1} \prec z \vee b_{U_1} \neq 1_{U_1}$ and therefore, b_{U_1} has two covers, contradicting that it is meet-irreducible.

Case 2: b_{U_1} and b_{U_2} are not comparable, that is, $b_{U_1} \parallel b_{U_2}$.

Let $t = b_{U_2} \wedge b_{U_1}$. Since L is semimodular and $t \wedge z = x$, we conclude that $t \prec z \vee t$. The element t has at most two covers, so either $z \vee t \leq b_{U_1}$ or $z \vee t \leq b_{U_2}$. By symmetry, we can assume that $z \leq b_{U_2}$. So z is a lower bound of b_{U_2} and b_{V_2}, contradicting that $x = b_{U_2} \wedge b_{V_2}$. ☐

Lemma 4-4.3. *Let L be a slim rectangular lattice. Let $U, V \in \mathrm{TN7}(L)$. Then $b_U \wedge b_V$ is not on the boundary of L.*

Proof. Since b_U and b_V are interior elements, we can assume that $b_U \parallel b_V$. By symmetry, we can further assume that b_U is to the left of b_V.

Set $x = b_U \wedge b_V$. If $x \leq 0_U$, then $x = 0_U \wedge b_V$. Choose the element y such that $x \prec y \leq b_V$. By semimodularity, $0_U \prec 0_U \vee y$. The element 0_U has exactly two covers, one in $[0_U, d_U]$ and one in $[0_U, e_U]$. Therefore $0_U \vee y \leq d_U$ or $0_U \vee y \leq e_U$. In either case, y is a lower bound of b_U and b_V, contradicting that $x = b_U \wedge b_V$.

Thus we can assume that $x \wedge 0_U < x$ and symmetrically, that $x \wedge 0_V < x$. Note that $x \wedge 0_U = x \wedge 0_V$ would be an element with three covers, which is a contradiction. So $x \wedge 0_U \neq x \wedge 0_V$, Therefore, 0_U is to the left of x and 0_V is to the right of x; therefore, x is in the interior of L. ☐

In the next three lemmas, we proceed to investigate tight N_7-s in rectangular lattices, in general; we no longer assume that L is slim.

Observe that if L is a rectangular lattice, $S \in \mathrm{TN7}(L)$, and $S \subseteq L^{\mathrm{slim}}$, then $S \in \mathrm{TN7}(L)^{\mathrm{slim}}$ also holds. So for a sublattice N of L, a tight N_7, the statements: "N is a tight N_7 of L^{slim}" and "$N \subseteq L^{\mathrm{slim}}$" are equivalent.

Lemma 4-4.4. *Let L be a rectangular lattice, and let $U \in \mathrm{TN7}(L)$. Then there exists a sublattice V in L^{slim} such that the following conditions hold:*

(i) *$V \in \mathrm{TN7}(L^{\mathrm{slim}})$;*

(ii) *$x_U = x_V$ for $x \in \{0, d, e, b, 1\}$;*

(iii) *the following two equalities hold:*

$$\mathrm{con}(a_V, 1_V) = \mathrm{con}(a_U, 1_U),$$
$$\mathrm{con}(c_V, 1_V) = \mathrm{con}(c_U, 1_U).$$

Proof. Replace a_U with the leftmost atom of the interval $[d_U, 1_U]$; denote it by a_V; define c_V symmetrically. Then $V = (U - \{a_U, c_U\}) \cup \{a_V, c_V\}$ is a tight N_7 in L^{slim}. If $a_V \neq a_U$, then $\mathrm{con}(a_V, 1_V) = \mathrm{con}(a_U, 1_U)$, since the interval $[d_U, 1_U]$ is isomorphic to M_k for some $k \geq 3$. A symmetric argument shows that $\mathrm{con}(c_V, 1_V) = \mathrm{con}(c_U, 1_U)$. $\square$

Lemma 4-4.5. *Let L be a rectangular lattice. Let us assume that $U \in \mathrm{TN7}(L)$ satisfies $U \cap \mathrm{C_{ll}} \neq \varnothing$. Then $\{0_U, d_U\} \subseteq \mathrm{C_{ll}}$. Moreover, if $a_U \in L^{\mathrm{slim}}$, then $\{0_U, d_U, a_U\} \subseteq \mathrm{C_{ll}}$.*

Proof. The element 0_U is in $\mathrm{C_{ll}}$. By Lemma 4-4.4, we can replace U by V, a tight N_7 in L^{slim}, satisfying $0_U = 0_V \in \mathrm{C_{ll}} = \mathrm{C_{ll}}(L^{\mathrm{slim}})$. If $d_U \notin \mathrm{C_{ll}}$, then 0_U has at least three covers in L^{slim}, a contradiction. So $d_U \in \mathrm{C_{ll}}$. Similarly, if $d_U \in \mathrm{C_{ll}}$ and L is slim, then $a_U \in \mathrm{C_{ll}}$. $\square$

Lemma 4-4.6. *Let L be a rectangular lattice. Let U, $V \in \mathrm{TN7}(L)$ and let U satisfy $U \cap \mathrm{C_{ll}} \neq \varnothing$. If $1_V \leq 1_U$, then $\mathrm{con}(a_V, 1_V) \leq \mathrm{con}(a_U, 1_U)$.*

Proof. We can assume that $U \neq V$. It follows from Lemma 4-4.5 that $d_U \in \mathrm{C_{ll}}$ and $a_U \wedge b_V = d_U \wedge b_V \in \mathrm{C_{ll}}$. Since d_V is the leftmost element covered by b_V and a_U is to the left of b_V, it follows that $a_U \wedge b_V \leq d_V$. Now observe that $[a_U, 1_U] \overset{\mathrm{dn}}{\twoheadrightarrow} [a_U \wedge b_V, b_V] \overset{\mathrm{up}}{\twoheadrightarrow} [a_V, 1_V]$; therefore, $\mathrm{con}(a_V, 1_V) \leq \mathrm{con}(a_U, 1_U)$. $\square$

Finally, we switch back to slim rectangular lattices to ascertain the existence of certain left-maximal and right-maximal tight N_7-s.

Let L be a rectangular lattice. Let $U \in \mathrm{TN7}(L^{\mathrm{slim}})$. We call U *left-maximal*, if the following two conditions hold:

(i) $U \cap \mathrm{C_{ll}} \neq \varnothing$;

(ii) $a_W \leq a_U$ holds for any $W \in \mathrm{TN7}(L^{\mathrm{slim}})$ satisfying $W \cap \mathrm{C_{ll}} \neq \varnothing$.

Note that by Lemma 4-4.5, condition (i) is equivalent to $\{0_U, d_U, a_U\} \subseteq \mathrm{C_{ll}}$. By symmetry, we introduce *right-maximal* tight N_7-s.

Lemma 4-4.7. *Let L be a slim rectangular lattice. If the interval $I = [0, u_l^*]$ contains a tight N_7, then there exists a left-maximal tight N_7. Symmetrically, if the interval $J = [0, u_r^*]$ contains a tight N_7, then there exists a right-maximal tight N_7.*

Proof. I is a slim rectangular lattice. We will show that either I is the direct product of two chains or there is a tight N_7 intersecting $\mathrm{C_{ll}} = \mathrm{C_{ll}}(I)$.

If $0 \prec u_l^* \wedge u_r$, then it follows from the semimodularity of L that $I = C_m \times C_2$ for some integer m.

Otherwise, 0 is not covered by $u_l^* \wedge u_r$. Since $\mathrm{C_{ll}}$ is a chain, there is a smallest element $y \in \mathrm{C_{ll}}$ such that $y \prec z$ for some element $z \in [u_l^* \wedge u_r, u_l^*]$. Let $u \prec y$ in $\mathrm{C_{ll}}$ and $v \prec z$ in $[u_l^* \wedge u_r, u_l^*]$. Now $y \wedge v = w \in \mathrm{C_{ll}}$. By the minimality of y, there exists an element $t \in L$ such that $w \prec t < v$. Set $s = u \vee t$. By semimodularity, $u \prec s$. Also by semimodularity, every cell is a 4-cell, so $s \prec z$. Since s is not on the right boundary, there is an element $r \prec z$ immediately to the right of s. Then $s \wedge r \prec s, r$ by the selection of r. Therefore,

$$U = \{y \wedge r, u, s \wedge r, y, s, r, z\},$$

is a tight N_7 intersecting the left boundary of L. By the minimality of y, the interval $[y, u_l^*]$ is isomorphic to $C_m \times C_2$ for some integer m. Therefore, U is a left-maximal tight N_7.

Applying the proof symmetrically, the result also holds for J. ☐

Where are the tight N_7-s in a decomposition of a slim rectangular lattice? We start with the following statement:

Lemma 4-4.8. *For a slim rectangular lattice L and $x \in \mathrm{C}_{\mathrm{ul}} - \{1, u_l\}$, let $S \in \mathrm{TN7}(L)$. Then $S \subseteq [u_r \wedge x, 1]$ or $S \subseteq [0, x]$.*

Proof. If $a \geq u_r \wedge x$, then b, $c \geq u_r \wedge x$ and since the set $\{a, b, c\}$ generates S, we conclude that $S \subseteq [u_r \wedge x, 1]$.

If $a \ngeq u_r \wedge x$, then $y = a \wedge u_r$. There exists an element $z \leq u_r \wedge x$ such that $y \prec z$. By semimodularity, the equality $1 = z \vee a$ holds. Since x is an upper bound of both z and a, we conclude that $q \leq x$ and so $S \subseteq [0, x]$. ☐

Lemma 4-4.8 can be rewritten as follows:

$$\mathrm{TN7}(L) = \mathrm{TN7}[u_r \wedge x, 1] \cup \mathrm{TN7}[0, x], \tag{4-4.1}$$

where, in fact, the union is a disjoint union.

This is the simplest form of the Partition Theorem, which – in its full generality – we now state:

Theorem 4-4.9 (Partition Theorem). *Let L be a slim rectangular lattice, let $x \in \mathrm{C}_{\mathrm{ul}} - \{1, u_l\}$, $y \in \mathrm{C}_{\mathrm{ur}} - \{1, u_r\}$. We decompose L into the lattices L_{top}, L_{left}, L_{right}, L_{bottom}* (see Section 3-7 and Figure 3-7.1)*. This defines a partitioning of* $\mathrm{TN7}(L)$*:*

$$\mathrm{TN7}(L) = \mathrm{TN7}(L_{\mathrm{top}}) \cup \mathrm{TN7}(L_{\mathrm{left}}) \cup \mathrm{TN7}(L_{\mathrm{right}}) \cup \mathrm{TN7}(L_{\mathrm{bottom}}). \tag{4-4.2}$$

4-5. The Lower Bound Theorem

We continue our preparation for the proof of Theorem 4-1.2(ii).

Let L be a rectangular lattice. By the Tight N_7 Theorem and Lemma 4-4.4, we may associate each covering pair in $\mathrm{Ji}(\mathrm{Con}\, L)$ with some tight N_7 in L^{slim}. Form a collection $\mathcal{U}_L$ of tight N_7-s in L^{slim} such that every element in $\mathcal{U}_L$ is associated with a covering pair in $\mathrm{Ji}(\mathrm{Con}\, L)$. Furthermore, we take $\mathcal{U}_L$ to be minimal in size. We refer to $\mathcal{U}_L$ as a *minimal collection* of L. Observe that $2|\mathcal{U}_L|$ is an upper bound for the number of covering pairs in $\mathrm{Ji}(\mathrm{Con}\, L)$.

We introduce one more notation. Let $\mathcal{U}_L$ be a minimal collection. Let N be a sublattice of L. We denote by $\mathcal{U}_L \rceil N$ the collection of those members of $\mathcal{U}_L$ that lie in N.

We start with this crucial result ("top interval" was introduced in Section 4-3.1):

Theorem 4-5.1 (**Lower Bound Theorem**). *Let L be a rectangular lattice with a minimal collection $\hat{\mathcal{U}}$. Let N be a top interval of L^{slim} and $\mathcal{U} = \hat{\mathcal{U}} \rceil N$. Let n and i be positive integers. Let us make the following assumptions on L, n, and i:*

(i) $\mathrm{Ji}(\mathrm{Con}\, L)$ *is a suborder of a balanced bipartite order on $2n$ elements;*

(ii) $|\mathcal{U}| \geq ni$.

Then one of the following two conclusions hold:

(α) $|N| \geq \frac{i^2 n}{2}$;

(β) *there exists a positive integer $j < i$ and a top interval M of N satisfying the following two conditions:*

(a) $|N| \geq |M| + \frac{(i-j)in}{3}$;

(b) $|\mathcal{U} \rceil M| \geq nj$.

Proof. We work in N, so N as a modifier will be omitted; for instance, we write lc for lc(N). Each $U \in \mathcal{U}$ is associated with an element $b_U^{\mathrm{ll}} \in \mathrm{C}_{\mathrm{ll}}$ and an element $b_U^{\mathrm{lr}} \in \mathrm{C}_{\mathrm{lr}}$. By Lemma 4-4.1, either one of these elements determines U. Therefore, by condition (ii),

$$|\mathrm{C}_{\mathrm{ll}}| \geq ni \quad \text{and} \quad |\mathrm{C}_{\mathrm{lr}}| \geq ni. \tag{4-5.1}$$

We claim that $\mathrm{C}_{\mathrm{ll}} \cap [\mathrm{lc}^* \wedge \mathrm{rc}, 1_N] = \varnothing$. Indeed, if $a \in \mathrm{C}_{\mathrm{ll}} \cap [\mathrm{lc}^* \wedge \mathrm{rc}, 1_N]$, then the element $a \wedge \mathrm{rc}$ is in $\mathrm{C}_{\mathrm{ll}} \cap \mathrm{C}_{\mathrm{lr}}$. Since N is rectangular, it follows that $a = 0_N$. Let b be such that $0_N \prec b \leq \mathrm{rc}$. By semimodularity, $\mathrm{lc} \prec b \vee \mathrm{lc} = \mathrm{lc}^*$. Since b is a lower bound of lc^*, we conclude that $0_N \prec b \leq \mathrm{lc}^* \wedge \mathrm{rc} \leq a \wedge \mathrm{rc} = 0_N$ which is a contradiction.

Define $u = \mathrm{lc}^* \wedge \mathrm{rc}^*$ and $W = N_{\mathrm{top}}(\mathrm{lc}^*, \mathrm{rc}^*)$; see Figure 4-5.1.

Intersecting both sides of (4-4.2) with $\hat{\mathcal{U}}$, we obtain:

$$\mathcal{U} = \mathcal{U} \rceil N_{\mathrm{top}} \cup \mathcal{U} \rceil N_{\mathrm{left}} \cup \mathcal{U} \rceil N_{\mathrm{right}} \cup \mathcal{U} \rceil N_{\mathrm{bottom}}. \tag{4-5.2}$$

To prove the Lower Bound Theorem, we distinguish three cases.

Case I: $|\mathcal{U} \rceil (N_{\mathrm{left}} \cup N_{\mathrm{bottom}})| < 3n$.

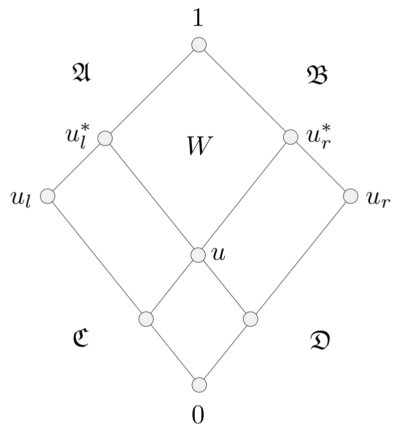

Figure 4-5.1: Some notation for N.

We distinguish two subcases.

Case I.1*:* $i \leq 3$.

Then, by (4-5.1),

$$|N| \geq |\mathrm{C}_{\mathrm{ll}} \cup \mathrm{C}_{\mathrm{lr}} \cup \{1\}| \geq 2ni - 1 + 1 \geq \frac{2i^2 n}{3},$$

and so conclusion (α) holds.

Case I.2*:* $i > 3$.

Set $M = [\mathrm{lc}^* \wedge \mathrm{rc}, 1]$ and $j = i - 3$; see Figure 4-5.2.

By (4-5.2), $|\mathcal{U}| = |\mathcal{U}\rceil M| + |\mathcal{U}\rceil[0, \mathrm{lc}^*]|$. It follows that

$$|\mathcal{U}\rceil M| = |\mathcal{U}| - |\mathcal{U}\rceil[0, \mathrm{lc}^*]| \geq ni - 3n = nj,$$

utilizing the assumptions for Case I and Case I.2, verifying (b). Finally, since $\mathrm{C}_{\mathrm{ll}} \cap M = \varnothing$, it follows that

$$|N| \geq |M| + |\mathrm{C}_{\mathrm{ll}}| \geq |M| + ni = |M| + \frac{(i-j)in}{3},$$

verifying (a). This verifies conclusion (β) and completes the proof for Case I.2 and for Case I.

Case II. $|\mathcal{U}\rceil(N_{\mathrm{right}} \cup N_{\mathrm{bottom}})| < 3n$.

This is symmetric to Case I.

Case III*:* $|\mathcal{U}\rceil(N_{\mathrm{left}} \cup N_{\mathrm{bottom}})| \geq 3n$ *and* $|\mathcal{U}\rceil(N_{\mathrm{right}} \cup N_{\mathrm{bottom}})| \geq 3n$.

Set $M = W$. Applying Lemma 4-4.7 to N, we obtain a left-maximal tight N_7, which we denote by U_{left}, and a right-maximal tight N_7, which we denote by U_{right}.

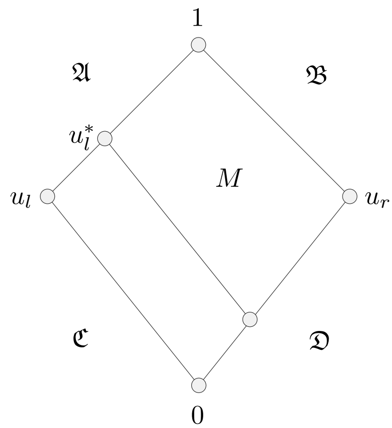

Figure 4-5.2: Some notation for N in Case I.2.

We wish to show that $\mathcal{U}\rceil N_{\mathrm{bottom}}$ contains at most $2n$ elements. If $V \in \mathcal{U}\rceil N_{\mathrm{bottom}}$, then from Lemmas 4-4.6 and 4-4.7, we conclude that

$$\begin{aligned}\operatorname{con}(a_V, 1_V) &\leq \operatorname{con}(a_{U_{\mathrm{left}}}, 1_{U_{\mathrm{left}}}),\\ \operatorname{con}(c_V, 1_V) &\leq \operatorname{con}(c_{U_{\mathrm{right}}}, 1_{U_{\mathrm{right}}}).\end{aligned}$$

Also, $\operatorname{con}(b_V, 1_V) \leq \operatorname{con}(a_V, 1_V)$, $\operatorname{con}(c_V, 1_V)$, so

$$\operatorname{con}(b_V, 1_V) \leq \operatorname{con}(a_V, 1_V) \leq \operatorname{con}(a_{U_{\mathrm{left}}}, 1_{U_{\mathrm{left}}}), \tag{4-5.3}$$

$$\operatorname{con}(b_V, 1_V) \leq \operatorname{con}(c_V, 1_V) \leq \operatorname{con}(c_{U_{\mathrm{right}}}, 1_{U_{\mathrm{right}}}). \tag{4-5.4}$$

Since a chain in $\operatorname{Ji}(\operatorname{Con} L)$ has at most two elements, it follows that

$$\operatorname{con}(b_V, 1_V) \preceq \operatorname{con}(a_{U_{\mathrm{left}}}, 1_{U_{\mathrm{left}}}), \tag{4-5.5}$$

$$\operatorname{con}(b_V, 1_V) \preceq \operatorname{con}(c_{U_{\mathrm{right}}}, 1_{U_{\mathrm{right}}}). \tag{4-5.6}$$

If equality holds, say, in (4-5.5), then by (4-5.3), we conclude that

$$\operatorname{con}(b_V, 1_V) = \operatorname{con}(a_V, 1_V). \tag{4-5.7}$$

Now $V \in \mathcal{U}$ means that V is associated with a covering pair in $\operatorname{Ji}(\operatorname{Con} L)$, namely, with $\operatorname{con}(b_V, 1_V) \prec \operatorname{con}(a_V, 1_V)$ or with $\operatorname{con}(b_V, 1_V) \prec \operatorname{con}(c_V, 1_V)$. So (4-5.7) implies that $\operatorname{con}(b_V, 1_V) \prec \operatorname{con}(c_V, 1_V)$ holds. By (4-5.4), equality cannot hold in (4-5.6).

We conclude that equality holds for at most one of (4-5.5) and (4-5.6). Since the maximal elements of $\operatorname{con}(V)$ are contained in the set

$$\{\operatorname{con}(a_{U_{\mathrm{left}}}, 1_{U_{\mathrm{left}}}), \operatorname{con}(c_{U_{\mathrm{right}}}, 1_{U_{\mathrm{right}}})\},$$

the minimality of $\hat{\mathcal{U}}$ now gives us that $|\mathcal{U}\rceil N_{\mathrm{bottom}}| \leq 2n$.

Define

$$k_1 = \left\lfloor \frac{|\mathcal{U}\rceil N_{\mathrm{left}}|}{n} \right\rfloor,$$
$$k_2 = \left\lfloor \frac{|\mathcal{U}\rceil N_{\mathrm{right}}|}{n} \right\rfloor.$$

Since, in this case, we assumed that

$$|\mathcal{U}\rceil(N_{\mathrm{left}} \cup N_{\mathrm{bottom}})| \geq 3n$$

and

$$|\mathcal{U}\rceil(N_{\mathrm{right}} \cup N_{\mathrm{bottom}})| \geq 3n,$$

and in the previous paragraph we proved that

$$|\mathcal{U}\rceil N_{\mathrm{bottom}}| \leq 2n,$$

it follows that $k_1, k_2 \geq 1$. Form the set

$$S = \{\, b_U \wedge b_V \mid U \in \mathcal{U}\rceil N_{\mathrm{left}},\ V \in \mathcal{U}\rceil N_{\mathrm{right}} \,\} \subseteq N_{\mathrm{bottom}}.$$

By Lemma 4-4.2, the element $b_U \wedge b_V$ uniquely determines the pair U, V, so $|S| \geq n^2 k_1 k_2$. By Lemma 4-4.3, the element $b_U \wedge b_V$ is not in C_{ll} or C_{lr}. So $S \cup (\mathrm{C}_{\mathrm{ll}} - \{0\}) \cup \mathrm{C}_{\mathrm{lr}}$ is a disjoint union in $N - M$. We can easily add an element w to make

$$S \cup (\mathrm{C}_{\mathrm{ll}} - \{0\}) \cup \mathrm{C}_{\mathrm{lr}} \cup \{w\}$$

one bigger in $N - M$; for instance, any $w \in U_{\mathrm{left}} - \mathrm{C}_{\mathrm{ll}}$. The inequality $|S| \geq n^2 k_1 k_2$ combined with (4-5.1) yields

$$|S \cup \mathrm{C}_{\mathrm{ll}} \cup \mathrm{C}_{\mathrm{lr}} \cup \{w\}| \geq n^2 k_1 k_2 + 2ni. \tag{4-5.8}$$

There are two subcases to compute.

Case III.1*:* $i \leq k_1 + k_2 + 4$.

Observe that, by assumption (i), $\mathrm{Ji}(\mathrm{Con}\, L)$ has at most n^2 covering pairs, so $|\mathcal{U}| \leq n^2$. Comparing this with assumption (ii), we conclude that $i \leq n$. Utilizing this and (4-5.8):

$$\begin{aligned} |N| \geq |S \cup \mathrm{C}_{\mathrm{ll}} \cup \mathrm{C}_{\mathrm{lr}} \cup \{w\}| &\geq n^2 k_1 k_2 + 2ni \geq \frac{n}{2}(2ik_1k_2 + 4i) \\ &\geq \frac{n}{2}(ik_1 + ik_2 + 4i) \geq \frac{i^2 n}{2}, \end{aligned}$$

and so conclusion (α) holds.

Case III.2*:* $i > k_1 + k_2 + 4$.

Then setting $j = i - (k_1 + k_2 + 4)$ so that $i - j = k_1 + k_2 + 4$, we obtain that

$$\begin{aligned}|N| - |M| &\geq |S \cup \mathrm{C}_{\mathrm{ll}} \cup \mathrm{C}_{\mathrm{lr}} \cup \{w\}| \geq n^2 k_1 k_2 + 2ni \geq ink_1k_2 + 2in \\ &= (k_1k_2 + 2)in \geq \frac{(k_1 + k_2 + 4)in}{3} = \frac{(i-j)in}{3},\end{aligned}$$

since for $k_1, k_2 \geq 1$, $k_1k_2 + 2 \geq (k_1 + k_2 + 4)/3$. This verifies (a). Since $i - 3 \geq j$,

$$|\mathcal{U} \rceil M| = |\mathcal{U}| - |\mathcal{U} \rceil (N_{\mathrm{left}} \cup N_{\mathrm{bottom}})| \geq ni - 3n = n(i-3) \geq nj,$$

conclusion (b) and so conclusion (β) are verified. □

The Lower Bound Theorem does indeed give a lower bound for the size of some intervals of L.

Corollary 4-5.2. *Let L be a rectangular lattice with a minimal collection $\hat{\mathcal{U}}$. Let N be a top interval of L^{slim} and let $\mathcal{U} = \hat{\mathcal{U}} \rceil N$. Let n and i be positive integers. Let us make the following assumptions on L, n, and i:*

(i) $\mathrm{Ji}(\mathrm{Con}\, L)$ *is a suborder of a balanced bipartite order on $2n$ elements;*

(ii) $|\mathcal{U}| \geq ni$.

Then

$$|N| \geq \frac{1}{6} i^2 n + \frac{1}{6} in. \tag{4-5.9}$$

Proof. We assume that for all $\ell < i$ and for all top intervals $I = [0_I, 1_L]$ of N satisfying $|\mathcal{U} \rceil I| \geq n\ell$, the inequality

$$|I| \geq \frac{1}{6} \ell^2 n + \frac{1}{6} \ell n \tag{4-5.10}$$

holds.

By Theorem 4-5.1, condition (α) or condition (β) holds for N.

If (α) holds for N, then

$$|N| \geq \frac{i^2 n}{2} \geq \frac{1}{6} i^2 n + \frac{1}{6} in,$$

as required.

If (β) holds for N, then there exists a positive integer $j < i$ and a top interval M of N such that $|\mathcal{U} \rceil M| \geq nj$, and

$$|N| \geq |M| + \frac{(i-j)in}{3}.$$

Applying (4-5.10) with $I = M$, we obtain that

$$|M| \geq \frac{1}{6}j^2n + \frac{1}{6}jn.$$

Therefore, by (a),

$$\begin{aligned}
|N| &\geq |M| + \frac{(i-j)in}{3} \geq \frac{1}{6}j^2n + \frac{1}{6}jn + \frac{(i-j)in}{3}\\
&= \frac{1}{6}j^2n + \frac{1}{6}jn + \frac{(i-j)in}{3} - \left(\frac{1}{6}i^2n + \frac{1}{6}in\right) + \left(\frac{1}{6}i^2n + \frac{1}{6}in\right)\\
&= \frac{n}{6}(j^2 + j + 2i^2 - 2ij - i^2 - i) + \left(\frac{1}{6}i^2n + \frac{1}{6}in\right)\\
&= \frac{n}{6}((i-j)^2 - (i-j)) + \left(\frac{1}{6}i^2n + \frac{1}{6}in\right)\\
&\geq \frac{1}{6}i^2n + \frac{1}{6}in,
\end{aligned}$$

proving the corollary. □

We rephrase Corollary 4-5.2, so that it be more straightforward to apply.

Corollary 4-5.3. *Let L be a rectangular lattice with a minimal collection $\hat{\mathcal{U}}$. Let N be a top interval of L^{slim} and let $\mathcal{U} = \hat{\mathcal{U}}\rceil N$. Let p and i be positive integers. Let us make the following assumptions on L, p, and i:*

(i) $\mathrm{Ji}(\mathrm{Con}\,L)$ *is a suborder of a balanced bipartite order on p elements;*

(ii) $|\mathcal{U}| \geq pi$.

Then

$$|N| \geq \frac{1}{12}i^2p + \frac{1}{12}ip. \tag{4-5.11}$$

Proof. We may assume that $p > 1$.

Let $n = \lfloor \frac{p+1}{2} \rfloor$. Since $p \leq 2n$, condition (i) of Corollary 4-5.2 follows from the present condition (i). Since $p \geq n$, condition (ii) of Corollary 4-5.2 follows from the present condition (ii), so we can apply Corollary 4-5.2, to obtain that

$$|N| \geq \frac{1}{6}i^2n + \frac{1}{6}in \geq \frac{1}{12}i^2p + \frac{1}{12}ip.$$ □

4-6. Proof of Theorem 4-1.2(ii)

Let L_n be a rectangular lattice such that $\mathrm{Ji}(\mathrm{Con}\,L)_n$ is a balanced bipartite order on n elements. We define

$$k = \frac{1}{3456} = \frac{1}{2 \times 12^3},$$

and prove that L_n has at least kn^3 elements.

If $n \leq 11$, then $kn^3 \leq 1$, so the inequality $|L_n| > kn^3$ is trivial.

So we can assume that $n \geq 12$. Let $\hat{\mathcal{U}}$ be a minimal collection in L. Since $\mathrm{Ji}(\mathrm{Con}\, L)$ is a balanced bipartite order on n elements, there are at least $\frac{n^2}{6}$ covering pairs in $\mathrm{Ji}(\mathrm{Con}\, L)$. Each covering pair in $\mathrm{Ji}(\mathrm{Con}\, L)$ is associated with a tight N_7 in $\hat{\mathcal{U}}$ and each element of $\hat{\mathcal{U}}$ is associated with at most two covering pairs in $\mathrm{Ji}(\mathrm{Con}\, L)$. It follows that

$$|\hat{\mathcal{U}}| \geq \frac{n^2}{12} \geq n \left\lfloor \frac{n}{12} \right\rfloor. \tag{4-6.1}$$

We apply Corollary 4-5.3 to the lattice L, the minimal collection $\hat{\mathcal{U}}$ in L, the top interval $N = L^{\mathrm{slim}}$ of L^{slim}, and the integers $p = n$ and $i = \lfloor \frac{n}{12} \rfloor$. Let $\mathcal{U} = \hat{\mathcal{U}} \rceil N$. Note that, by definition, $\hat{\mathcal{U}} = \mathcal{U}$.

By (4-5.11) and (4-6.1), we obtain that

$$|L| \geq |L^{\mathrm{slim}}| = |N| \geq \frac{1}{12} \left\lfloor \frac{n}{12} \right\rfloor^2 n + \frac{1}{12} \left\lfloor \frac{n}{12} \right\rfloor n = \frac{n}{12} \left\lfloor \frac{n}{12} \right\rfloor \left(\left\lfloor \frac{n}{12} \right\rfloor + 1 \right).$$

Since $\frac{n}{12} - 1 \leq \lfloor \frac{n}{12} \rfloor$ and $\frac{n}{24} \leq \lfloor \frac{n}{12} \rfloor$, for $n \geq 12$, we obtain that

$$|L| \geq \frac{n}{12} \cdot \frac{n}{24} \cdot \frac{n}{12} = \frac{n^3}{3456} = kn^3,$$

completing the proof Theorem 4-1.2(ii).

4-7. A brief survey of recent results

The most recent papers on congruences of planar semimodular lattices, are G. Czédli [47] and [48], G. Grätzer [136] and [135], and G. Grätzer and E.T. Schmidt [180] and [181]. A number of results are being written up, I will mostly ignore them. For G. Grätzer [135], see Exercise 4.23.

Let $S = \{o, a_l, a_r, t\}$ be a covering square of a slim semimodular lattice L. G. Grätzer [136] starts the investigation of the congruences of the fork extension $L[S]$, see Section 3-5. Let m denote the internal element of N_7 and let $\boldsymbol{\gamma}(S) = \mathrm{con}_{L[S]}(m, t)$, the only candidate for a new join-irreducible congruence in $L[S]$.

We call the covering square of $S = \{o, a_l, a_r, t\}$ a *tight square*, if t covers exactly two elements, namely, a_l and a_r, in L; otherwise, S is a *wide square.*

Theorem 4-7.1. *Let L be an SPS lattice. If S is a wide square, then $\boldsymbol{\gamma}(S)$ is generated by a congruence of L.*

Theorem 4-7.2. *Let L be an SPS lattice. Let $S = \{o, a_l, a_r, t\}$ be a* tight square. *Then $L[S]$ has exactly one join-irreducible congruence, namely $\boldsymbol{\gamma}(S)$, that is not generated by a congruence of L.*

We now state the most important property of $\boldsymbol{\gamma}(S)$.

Theorem 4-7.3. *Let S be a* tight square *in an SPS lattice L. Then in the order of join-irreducible congruence of $L[S]$, the congruence $\boldsymbol{\gamma}(S)$ has at most two covers.*

The main contribution in G. Czédli [47] is a quasiordering of C_2-trajectories (trajectories, for short) of slim rectangular lattices (see page 96).

Let $\mathfrak{p} = [0_\mathfrak{p}, 1_\mathfrak{p}]$ be a prime interval of an SPS lattice L, and let $T(\mathfrak{p})$ denote the trajectory to which $\mathfrak{p}$ belongs.

For a trajectory T of L, the *top edge* $\mathfrak{t}(T)$ of T is defined by the property that $\mathfrak{t}(T) \in T$ and $1_{\mathfrak{t}(T)} > 1_\mathfrak{q}$ holds for all $\mathfrak{q} \in T$ with $\mathfrak{q} \neq \mathfrak{t}(T)$.

On the set $\operatorname{Traj} L$ of all trajectories of L, we define a relation $\preccurlyeq$ as follows. For $Q, R \in \operatorname{Traj} L$, we let $Q \preccurlyeq R$ iff Q is a hat-trajectory, $1_{\mathfrak{t}(Q)} \leq 1_{\mathfrak{t}(R)}$, but $0_{\mathfrak{t}(Q)} \not\leq 0_{\mathfrak{t}(R)}$. (See Exercise 4.24 for an example.) Let $\leq$ be the reflexive and transitive extension of $\preccurlyeq$. For $Q, R \in \operatorname{Traj} L$, we let $Q \equiv R$ iff $Q \leq R$ and $R \leq Q$. Then $\equiv$ is an equivalence relation on Traj L. We denote by $\overline{T}$ the equivalence class $T/{\equiv}$ and by $\overline{\operatorname{Traj}}\, L$ the set of equivalence classes. Then, as usual, we define $\leq$ on $\overline{\operatorname{Traj}}\, L$: let $A \leq B$ iff $Q \leq R$ for some (for all) $Q \in A$, $R \in Q$.

Finally, we define the map ξ. Let $\mathfrak{p}$ be a prime interval of L. Then there is a unique trajectory T with $\mathfrak{p} \in T$. Let $\xi(\mathfrak{p}) = \overline{T}$.

Theorem 4-7.4 (Trajectory Coloring Theorem)**.** *If L is a slim rectangular lattice, then the map ξ is a coloring.*

Actually, Czédli's result is much stronger. We simplified the setup to keep the survey brief.

This deep result has many applications. For instance, Czédli applies Theorem 4-7.4 to give a new proof of Theorem 4-7.3 in [48].

4-8. Exercises

4.1. Let L be a finite lattice. Introduce an equivalence relation $\equiv$ on the set $\operatorname{Prime}(L)$ of prime intervals: $\mathfrak{p} \equiv \mathfrak{q}$ iff $\mathfrak{p} \twoheadrightarrow \mathfrak{q}$ and $\mathfrak{q} \twoheadrightarrow \mathfrak{p}$. Show that there is a natural one-to-one correspondence between $\operatorname{Ji}(\operatorname{Con} L)$ (the join-irreducible congruences of L) and $\operatorname{Prime}(L)/{\equiv}$ (the blocks of the equivalence relation $\equiv$).

4.2. Order $\operatorname{Prime}(L)/{\equiv}$ as follows: let $\mathfrak{p}/{\equiv} \ \leq \ \mathfrak{q}/{\equiv}$ iff $\mathfrak{q} \twoheadrightarrow \mathfrak{p}$. Show that this defines an ordering of $\operatorname{Prime}(L)/{\equiv}$ and with this ordering, the order $\operatorname{Prime}(L)/{\equiv}$ is isomorphic to $\operatorname{Ji}(\operatorname{Con} L)$. (J. Jakubik [205].)

4.3. What is $\operatorname{Ji}(\operatorname{Con} \mathsf{N}_7^+)$? Describe $\operatorname{Con} \mathsf{N}_7^+$.

4.4. Let L be an m-stacked N_7^+. Describe $\operatorname{Ji}(\operatorname{Con} L)$ and $\operatorname{Con} L$.

4.5. For a finite lattice L, construct $\operatorname{Con} L$ from $\operatorname{Ji}(\operatorname{Con} L)$.

4.6. Show that in a slim rectangular lattice, a join-irreducible element is on the boundary. (This exercise, as well as the next three, are from the papers of G. Grätzer and E. Knapp.)

4.7. In a rectangular lattice, verify that every $x \in (\mathrm{C}_{\mathrm{ul}} \cup \mathrm{C}_{\mathrm{ur}}) - \{1\}$ is meet-irreducible and every $x \in (\mathrm{C}_{\mathrm{ll}} \cup \mathrm{C}_{\mathrm{lr}}) - \{0\}$ is join-irreducible.

4.8. Let L be a slim rectangular lattice. Let $y \in \mathrm{C}_{\mathrm{ur}} - \{1, \mathrm{rc}\}$. Prove that $N^t = [\mathrm{lc} \wedge y, 1]$ and $N^b = [0, y]$ are slim rectangular lattices. Similarly, if $x \in \mathrm{C}_{\mathrm{ul}} - \{1, \mathrm{lc}\}$, then $[x \wedge \mathrm{rc}, 1]$ and $[0, x]$ are slim rectangular lattices.

4.9. A *top interval* I of a slim rectangular lattice a L is an interval of the form $[x \wedge y, 1]$, where $x \in \mathrm{C}_{\mathrm{ul}} - \{1\}$ and $y \in \mathrm{C}_{\mathrm{ur}} - \{1\}$. Show that any top interval of L is also a slim rectangular lattice.

****

G. Grätzer, H. Lakser, and E.T. Schmidt [152] prove Theorem 4-1.2(i) for finite semimodular lattices (of course, with a different cubic polynomial) in a different way. To illustrate their method, let P be the order of Figure 4-8.1; we verify that we can represent P as $\mathrm{Ji}(\mathrm{Con}\, L)$ for a finite planar semimodular lattice L. To start out – see Figure 4-8.2 – we construct the lattices A_a and A_b. We obtain A_a by gluing two copies of N_7^+ together; and color it with $\{a, b, d\}$ so that $\mathrm{con}(a) > \mathrm{con}(b)$ is accomplished in the top N_7^+ of A_a and $\mathrm{con}(a) > \mathrm{con}(d)$ is accomplished in the bottom N_7^+ of A_a. The lattice A_b is N_7^+ colored by $\{b, c\}$ so that $\mathrm{con}(b) > \mathrm{con}(c)$ in A_b.

Form the glued sum S of A_a and A_b; all the covers of P are taken care of in S. There is only one problem: S is not a colored lattice; in this example, if $\mathfrak{p}$ is a prime interval of color b in S_b (as in Figure 4-8.2) and $\mathfrak{q}$ is a prime interval of color b in S_b (as in Figure 4-8.2), then in S we have $\mathrm{con}(\mathfrak{p}) \wedge \mathrm{con}(\mathfrak{q}) = \omega$. Of course, we should have $\mathrm{con}(\mathfrak{p}) = \mathrm{con}(\mathfrak{q}) = \mathrm{con}(b)$. We accomplish this by extending S to the lattice L of Figure 4-8.3. In L, the black-filled elements form the sublattice S.

We extend S by adding to it a distributive "grid." The right corner is C_5^2 colored by $\{a, b, c, d\}$; each of the four covering squares colored by the same color twice are made into a covering M_3. This makes the coloring behave properly in the right corner. In the rest of the lattice we do the same: we look for a covering square colored by the same color twice, and make it into a covering M_3. This makes L into a colored lattice: any two prime intervals of the same color generate the same congruence.

Finally, we add a "tail" to the lattice and color it e.

It is easy to see that the resulting lattice L is planar and semimodular and that $\mathrm{Ji}(\mathrm{Con}\, L)$ is isomorphic to P; the isomorphism is $x \mapsto \mathrm{con}(x)$ for $x \in \{a, b, c, d, e\}$.

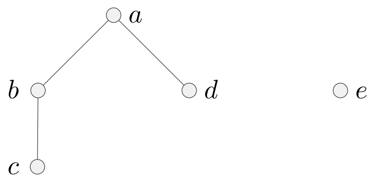

Figure 4-8.1: The order P.

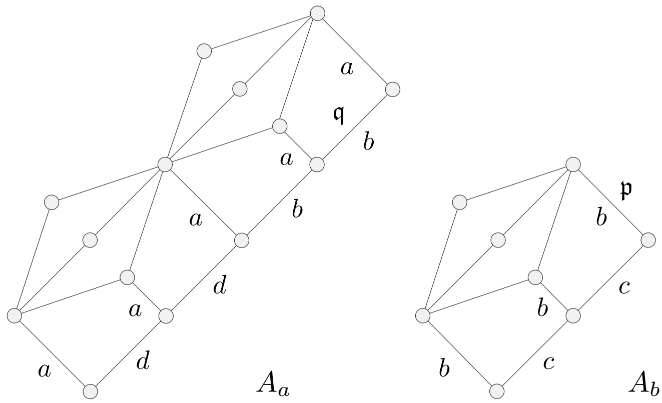

Figure 4-8.2: The lattices A_a and A_b.

4.10. Generalize this example to obtain a proof that every finite distributive lattice can be represented as the congruence lattice of a finite planar semimodular lattice.

4.11. Compute that Theorem 4-1.2(i) hold for finite semimodular lattices with the polynomial $3(n+1)^2$.

****

A rather short proof of Theorem 4-1.2(i) was found in G. Grätzer and E.T. Schmidt [180]. It utilizes an M_3-grid and the lattice S_8, see Figure 4-8.4.

4.12. What is the congruence lattice of an M_3-grid of size $n \times n$?

4.13. Let D be the finite distributive lattice of Theorem 4-1.2. Let $P = \mathrm{Ji}\, D$. Let n be the number of elements in P and e the number of coverings in P. Let $m_i \prec n_i$, for $1 \leq i \leq e$, list all coverings of P.

Construct a rectangular lattice K representing D by induction on e as illustrated by Figure 4-8.5, where in the induction step we color S_8 with $m_i \prec n_i$.

Prove that the lattice K for Theorem 4-1.2(i) is the lattice K_e.

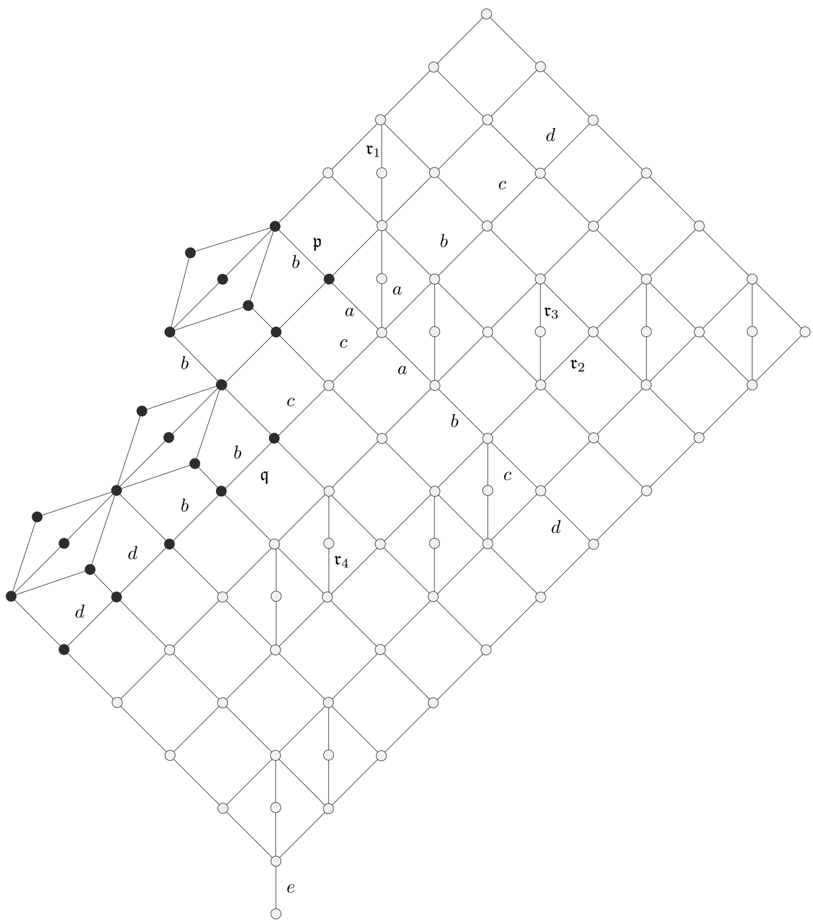

Figure 4-8.3: The lattice L.

4.14. Let L be a lattice. Let $C = \{a \prec b \prec c\} \subseteq L$ with a meet-irreducible in L and c join-irreducible in L.

Order $L[C] = L \cup \{t\}$, $t \notin L$, so that L is a suborder and $a \prec t \prec c$.

Prove that $L[C]$ is a lattice, a congruence-preserving extension of L.

If, in addition, L is finite and semimodular, then $L[C]$ is a semimodular lattice.

If, in addition, L is planar, C is on the left boundary of L, and t is placed to the left of C, above a and below c, then $L[C]$ is a planar semimodular lattice.

4.15. Prove that $L[C]$ is a congruence-preserving extension of L.

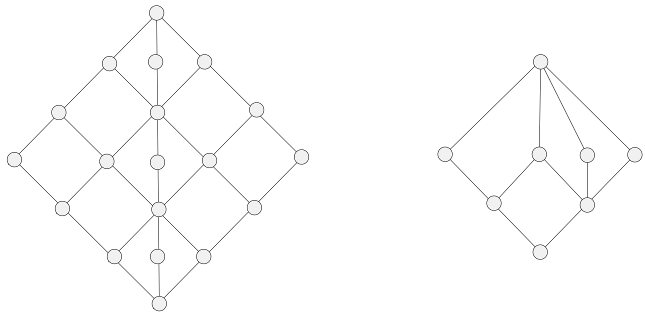

Figure 4-8.4: The M_3-grid and the lattice S_8.

4.16. Let L be a planar semimodular lattice. Prove that applying the previous exercise sufficiently many times, we obtain a rectangular congruence-preserving extension $\widehat{L}$ of L.

4.17. Let P be a balanced bipartite order on $2n$ elements. Let L be the planar semimodular lattice constructed above representing P in the sense of Theorem 4-1.2(i), that is, P is isomorphic to the order of join-irreducible congruences of L. Verify that the lattice $\widehat{L}$ has $O(n^4)$ elements. (This shows that the rectangular lattice of size $O(n^3)$ constructed in Section 4-3 could not be obtained from the construction in G. Grätzer, H. Lakser, and E.T. Schmidt [152] by taking the rectangular lattice "generated" by it.)

****

4.18. Let L be a finite lattice. Let $I = [a, b]$ be a proper non-prime interval of L, that is, $a < b$ and $a \prec b$ fails. Let $c \notin L$ and set $L_I = L \cup \{c\}$. We associate with $x \in L_I$ the elements $\underline{x}$ and $\overline{x}$ of L: for $x \in L$, set $\underline{x} = \overline{x} = x$ and $\underline{c} = a$, $\overline{c} = b$. We then define the relation $\leq_I$ on the set L_I as follows:

$x \leq_I y$ iff $x = y$ or $\overline{x} \leq \underline{y}$.

Prove that L_I is a lattice, an extension of L. Verify that the embedding of L into L_I is cover-preserving. (G. Grätzer and H. Lakser [148].)

4.19. Let L be a finite semimodular lattice and let $I = [a, b]$ be an interval of L. We make the following assumptions:

(i) the interval I is of length two;

(ii) all covers of a are below b.

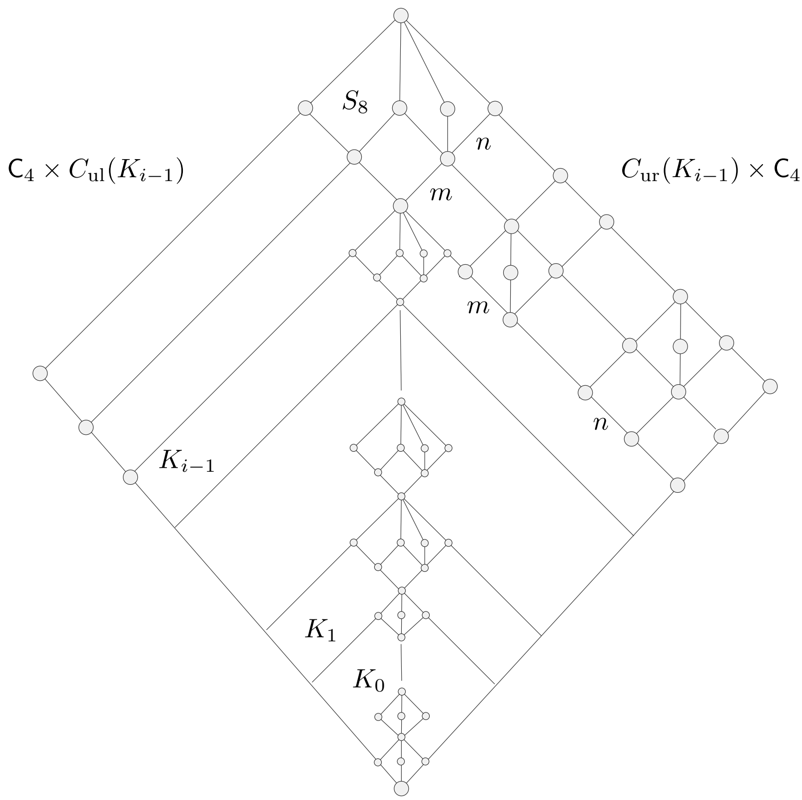

Figure 4-8.5: The lattice K_i for $0 < i \leq e$.

Show that the lattice L_I is also semimodular. (This exercise and the next three are from G. Grätzer and T. Wares [182].)

4.20. Let L be a finite semimodular lattice. Verify that L has a cover-preserving embedding into a finite semimodular lattice L^+ with the following property:

(FI) If $I = [a, b]$ is an interval of length two of L and I has the property that all covers of a in L are below b, then $[a, b]_{L^+} \cong M_n$ for some $n \geq 3$.

4.21. Let L be a finite semimodular lattice. Verify that the extension L^+ of L constructed in the previous exercise is a simple lattice.

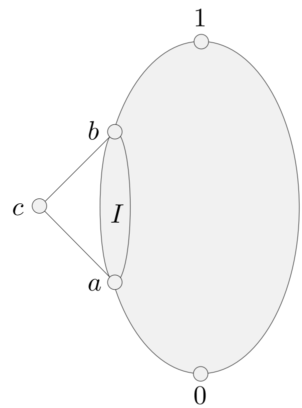

Figure 4-8.6: The L_I construction.

4.22. Prove that every finite semimodular lattice L has a cover-preserving embedding into a finite, simple, semimodular lattice $\overline{L}$.

4.23. Let L be a finite lattice. Let $\boldsymbol{\delta}$ be an equivalence relation on L with intervals as equivalence classes. Then $\boldsymbol{\delta}$ is a congruence relation iff the following condition and its dual hold:

($\mathrm{C}_\vee$) If $x \prec y, z \in L$ and $x \equiv y \pmod{\boldsymbol{\delta}}$, then $z \equiv y \vee z \pmod{\boldsymbol{\delta}}$.

(G. Grätzer [135].)

4.24. Compute that N_7 has three trajectories, and the quasiorder $\leq$ introduced on page 96 is an order, isomorphic to the order of join-irreducible congruencies of N_7.

4.25. Let L be an SPS lattice, let $T, Q \in \operatorname{Traj} L$. Probe that if $T \leq Q$ and $Q \leq T$, then T and Q are hat-trajectories and $1_{\mathfrak{t}(T)} = 1_{\mathfrak{t}(Q)}$.

4-9. Problems

In view of the result of G. Grätzer and T. Wares in Exercise 4-4.22, we ask:

Problem 4.1. Which finite semimodular lattices have cover-preserving embeddings into finite partition lattices?

G. Czédli pointed out that, for instance, M_4 has no such embedding.

Problem 4.2. Is there a lower bound $k_1 n^3$ ($k_1 > 0$) that works for planar semimodular lattices?

Problem 4.3. How close can one bring the constant k in Theorem 4-1.2 to $\frac{2}{3}$?

The next problem raises the question whether the $O(n^2)$ result can be improved.

Problem 4.4. Let L be a lattice such that the order of join-irreducible congruences of L is a balanced bipartite order on n elements. Is it true that L has at least $k_2 n^2$ elements for some constant $k_2 > 0$?

Problem 4.5. Is there an analogue of Theorem 4-1.2 for planar lattices?

Problem 4.6. Is there an analogue of Theorem 4-1.2 for finite semimodular lattices?

Now some problems on the topics in Section 4-7.

Problem 4.7. Characterize the congruence lattices of SPS lattices.

Problem 4.8. Characterize the congruence lattices of slim rectangular lattices.

Problem 4.9. Characterize the congruence lattices of patch lattices.

Problem 4.10. Characterize the congruence lattices of slim patch lattices.

Let L be an SPS lattice. The order $\mathrm{Ji}(\mathrm{Con}\, L)$ consists of two parts:

(i) The maximal elements of $\mathrm{Ji}(\mathrm{Con}\, L)$, at least two in number (for a patch lattice, exactly two). These are the minimal extensions to L of the join-irreducible congruences of D.

(ii) The non-maximal elements of $\mathrm{Ji}(\mathrm{Con}\, L)$, if any. These congruences are covered by exactly one or two congruences in $\mathrm{Ji}(\mathrm{Con}\, L)$.

Problem 4.11. How do $\mathrm{Ji}(\mathrm{Con}\, L)$ and $\mathrm{Ji}(\mathrm{Con}\, L[S])$ interrelate, in general?

G. Grätzer [136] has some interesting diagrams related to this problem.

Chapter 5

Sectionally Complemented Lattices

by George Grätzer

5-1. Introduction

We start this chapter by discussing the representation theorem for finite sectionally complemented lattices; we construct, for a finite distributive lattice D, a finite sectionally complemented lattice L whose congruence lattice is isomorphic to D, in formula, $\operatorname{Con} L \cong D$. We construct such a lattice L by constructing a sectionally complemented *chopped lattice* M with $\operatorname{Con} M \cong D$. Constructing L from M is easy; the lattice L is the ideal lattice of M. It is also straightforward that $\operatorname{Con} L \cong D$.

The rest of the chapter deals with the statement that the lattice L we construct is sectionally complemented. We recap the original proof of G. Grätzer and E.T. Schmidt [164] (as presented in G. Grätzer and H. Lakser [150]) and discuss in detail the algorithm of G. Grätzer and M. Roddy [161] for finding relative complements in L.

This result belongs to the field: representation of finite distributive lattices as congruence lattices of (mostly finite) lattices, discussed in depth in the author's book [131]. Most of the results of this chapter were obtained after the publication of [131].

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In Section 5-2, we review the basic concepts of chopped lattices. Then in Section 5-3, we present the proof of the representation theorem. We define and verify the algorithm of G. Grätzer and M. Roddy [161] that produces the sectional complement in Section 5-4. Finally, in Section 5-5, we prove the result of G. Grätzer, G. Klus, and A. Nguyen [139]: whichever way we carry out the algorithm, the result is the same and it equals the sectional complement constructed in Section 5-3.

5-2. Chopped lattices

5-2.1 Basic definitions

A *chopped lattice* is a finite meet-semilattice $(M; \wedge)$ regarded as a *partial algebra* $(M; \wedge, \vee)$, where $\vee$ is a partial operation defined as follows: for $a, b, c \in M$, let $a \vee b$ be defined and $a \vee b = c$ iff a and b have the least upper bound c.

We denote by Max the set of maximal elements of M.

$\diamondsuit$ **Lemma 5-2.1.** *Let M be a finite order. If M is a meet-semilattice in which $\downarrow m$ is a lattice, for all $m \in \mathrm{Max}$, then M is a chopped lattice.*

Let C and D be lattices such that $J = C \cap D$ is an ideal in both C and D. Then, with the natural ordering, $\mathrm{Merge}(C, D) = C \cup D$, called the *merging* of C and D, is a chopped lattice.

An equivalence relation $\boldsymbol{\alpha}$ on a chopped lattice M is called a *congruence relation*, or *congruence*, iff $a \equiv b \pmod{\boldsymbol{\alpha}}$ and $c \equiv d \pmod{\boldsymbol{\alpha}}$ imply that the following two Substitution Properties hold:

$$(\mathrm{SP}_\wedge) \qquad a \wedge c \equiv b \wedge d \pmod{\boldsymbol{\alpha}};$$

$$(\mathrm{SP}_\vee) \qquad a \vee c \equiv b \vee d \pmod{\boldsymbol{\alpha}}, \text{ provided that } a \vee c \text{ and } b \vee d \text{ exist.}$$

Trivial examples are $\mathbf{0}$ (defined by $x \equiv y \pmod{\mathbf{0}}$ iff $x = y$) and $\mathbf{1}$ (defined by $x \equiv y \pmod{\mathbf{1}}$ for all $x, y \in M$).

The set $\mathrm{Con}\, M$ of all congruence relations of M ordered by set inclusion is a lattice. As for lattices, $\mathrm{Ji}(\mathrm{Con}\, M)$ is the order of join-irreducible congruences.

$\diamondsuit$ **Lemma 5-2.2.** *Let M be a chopped lattice and let $\boldsymbol{\alpha}$ be an equivalence relation on M satisfying the following two conditions for $x, y, z \in M$:*

(1) *If $x \equiv y \pmod{\boldsymbol{\alpha}}$, then $x \wedge z \equiv y \wedge z \pmod{\boldsymbol{\alpha}}$.*

(2) *If $x \equiv y \pmod{\boldsymbol{\alpha}}$ and $x \vee z$ and $y \vee z$ exist, then $x \vee z \equiv y \vee z \pmod{\boldsymbol{\alpha}}$.*

Then $\boldsymbol{\alpha}$ is a congruence relation on M.

A nonempty subset I of the chopped lattice M is an *ideal* iff it is a down-set with the property:

(Id) $a, b \in I$ implies that $a \vee b \in I$, provided that $a \vee b$ exists in M.

The set $\operatorname{Id} M$ of all ideals of M ordered by set inclusion is a lattice. For $I, J \in \operatorname{Id} M$, the meet is $I \cap J$, but the join is a bit more complicated to describe.

◇ **Lemma 5-2.3.** *Let I and J be ideals of the chopped lattice M. Define*

$$\begin{aligned} &U(I,J)_0 = I \cup J, \\ &U(I,J)_i = \{\, x \mid x \leq u \vee v,\ u, v \in U(I,J)_{i-1} \,\} \text{ for } 0 < i. \end{aligned}$$

Then

$$I \vee J = \bigcup_i U(I,J)_i. \tag{5-2.1}$$

5-2.2 Compatible vectors

For a chopped lattice M,

$$M = \bigcup (\, \operatorname{id}(m) \mid m \in \operatorname{Max} \,)$$

and each $\operatorname{id}(m)$ is a (finite) lattice. A *vector* (associated with M) is of the form $(i_m \mid m \in \operatorname{Max})$, where $i_m \in \operatorname{id}(m)$ for all $m \in M$. We order the vectors componentwise.

With every ideal I of M, we can associate the vector $(i_m \mid m \in \operatorname{Max})$ defined by $I \cap \operatorname{id}(m) = \operatorname{id}(i_m)$. Clearly, $I = \bigcup (\, \operatorname{id}(i_m) \mid m \in M \,)$. Such vectors are easy to characterize. Let us call the vector $(j_m \mid m \in \operatorname{Max})$ *compatible* if $j_m \wedge n = j_n \wedge m$ for all $m, n \in \operatorname{Max}$.

◇ **Lemma 5-2.4.** *Let M be a chopped lattice.*

(i) *There is a one-to-one correspondence between ideals and compatible vectors of M.*

(ii) *Given any vector $\mathbf{g} = (g_m \mid m \in \operatorname{Max})$, there is a smallest compatible vector $\overline{\mathbf{g}} = (i_m \mid m \in \operatorname{Max})$ containing $\mathbf{g}$.*

(iii) *Let I and J be ideals of M, with corresponding compatible vectors $(i_m \mid m \in \operatorname{Max})$ and $(j_m \mid m \in \operatorname{Max})$. Then*

  (a) *$I \leq J$ in $\operatorname{Id} M$ iff $i_m \leq j_m$ for all $m \in \operatorname{Max}$.*

  (b) *The compatible vector corresponding to $I \wedge J$ is $(i_m \wedge j_m \mid m \in \operatorname{Max})$.*

  (c) *Let $\mathbf{a} = (i_m \vee j_m \mid m \in \operatorname{Max})$. Then the $\overline{\mathbf{a}}$ is the compatible vector corresponding to $I \vee J$.*

Let M be a chopped lattice. With any congruence $\boldsymbol{\alpha}$ of M, we can associate the *reflection vector* $(\boldsymbol{\alpha}\rceil_m \mid m \in \operatorname{Max})$, where $\boldsymbol{\alpha}\rceil_m$ is the reflection (restriction) of $\boldsymbol{\alpha}$ to $\operatorname{id}(m)$. The reflection $\boldsymbol{\alpha}\rceil_m$ is a congruence of the lattice $\operatorname{id}(m)$.

Let $\boldsymbol{\beta}_m$ be a congruence of the lattice $\mathrm{id}(m)$ for all $m \in \mathrm{Max}$. The *congruence vector* $(\boldsymbol{\beta}_m \mid m \in \mathrm{Max})$ is called *compatible* if $\boldsymbol{\beta}_m$ restricted to $\mathrm{id}(m \wedge n)$ is the same as $\boldsymbol{\beta}_n$ restricted to $\mathrm{id}(m \wedge n)$ for $m, n \in \mathrm{Max}$. Obviously, a reflection vector is compatible. The converse also holds.

◇ **Lemma 5-2.5.** *Let* $(\boldsymbol{\beta}_m \mid m \in \mathrm{Max})$ *be a compatible congruence vector of a chopped lattice* M*. Then there is a unique congruence* $\boldsymbol{\alpha}$ *of* M *such that the reflection vector of* $\boldsymbol{\alpha}$ *agrees with* $(\boldsymbol{\beta}_m \mid m \in \mathrm{Max})$*.*

5-2.3 From the chopped lattice to the ideal lattice

The map $m \mapsto \mathrm{id}(m)$ embeds the chopped lattice M into the lattice $\mathrm{Id}\, M$, so we can regard $\mathrm{Id}\, M$ as an extension of M. It is, in fact, a congruence-preserving extension (G. Grätzer and H. Lakser [145], proof first published in [131]):

◇ **Theorem 5-2.6.** *Let* M *be a chopped lattice. Then* $\mathrm{Id}\, M$ *is a congruence-preserving extension of* M*.*

5-2.4 Sectional complementation

A chopped lattice M is *sectionally complemented* if, for all $a < b \in M$, there exists an element $c \in M$ satisfying $a \wedge c = 0$ and $a \vee c = b$.

We illustrate the use of compatible vectors with the following result of G. Grätzer and E.T. Schmidt [176].

Lemma 5-2.7 (Atom Lemma)**.** *Let* M *be a chopped lattice with two maximal elements* m_1 *and* m_2*. We assume that the lattices* $\mathrm{id}(m_1)$ *and* $\mathrm{id}(m_2)$ *are sectionally complemented. If* $p = m_1 \wedge m_2$ *is an atom, then* $\mathrm{Id}\, M$ *is sectionally complemented.*

Proof. To show that $\mathrm{Id}\, M$ is sectionally complemented, let $I \subseteq J$ be two ideals of M, represented by the compatible vectors (i_1, i_2) and (j_1, j_2), respectively. Let s_1 be the sectional complement of i_1 in j_1 and let s_2 be the sectional complement of i_2 in j_2. If $p \wedge s_1 = p \wedge s_2$, then (s_1, s_2) is a compatible vector, representing an ideal S that is a sectional complement of I in J. Otherwise, without loss of generality, we can assume that $p \wedge s_1 = 0$ and $p \wedge s_2 = p$. Since $\mathrm{id}(m_2)$ is sectionally complemented, there is a sectional complement s_2' of p in s_2. Then (s_1, s_2') satisfies $p \wedge s_1 = p \wedge s_2'$ $(= 0)$, and so it is compatible; therefore, (s_1, s_2') represents an ideal S of M. Obviously, $I \wedge S = \{0\}$.

From $p \wedge s_2 = p$, it follows that $p \leq s_2 \leq j_2$. Since J is an ideal and $j_2 \wedge p = p$, it follows that $j_1 \wedge p = p$, that is, $p \leq j_1$. Obviously, $I \vee S \subseteq J$. So to show that $I \vee S = J$, it is sufficient to verify that $j_1, j_2 \in I \vee S$. Evidently, $j_1 = i_1 \vee s_1 \in I \vee S$. Note that $p \leq j_1 = i_1 \vee s_1 \in I \vee S$. Thus $p, s_2', i_2 \in I \vee S$, and therefore

$$p \vee s_2' \vee i_2 = (p \vee s_2') \vee i_2 = s_2 \vee i_2 = j_2 \in I \vee S. \qquad \square$$

5-3. The representation theorem

For a class $\mathbf{K}$ of lattices, *the congruence representation theorem for* $\mathbf{K}$ states that every finite distributive lattice D can be represented as the congruence lattice of a finite lattice $L \in \mathbf{K}$.

Now we state the congruence representation theorem for sectionally complemented lattices:

Theorem 5-3.1. *Every finite distributive lattice D can be represented as the congruence lattice of a finite sectionally complemented lattice L.*

For general lattices the congruence representation theorem is due to R.P. Dilworth, first published in G. Grätzer and E.T. Schmidt [164]; in the same paper, we prove Theorem 5-3.1.

Using the equivalence of nontrivial finite distributive lattices and finite orders, we can rephrase Theorem 5-3.1 as follows:

Theorem 5-3.2. *Let P be a finite order. Then there exists a sectionally complemented chopped lattice M such that* $\operatorname{Ji}(\operatorname{Con} M)$ *is isomorphic to P.*

We tackle the proof of this theorem in two steps. In Section 5-3.1, we show by example how the chopped lattice M is constructed; by inspecting the congruences of M, we conclude that $\operatorname{Ji}(\operatorname{Con} M) \cong D$. The lattice L is defined as $\operatorname{Id} M$, the ideal lattice of M. Then $\operatorname{Ji}(\operatorname{Con} L) \cong D$ holds by Theorem 5-2.6. In Section 5-3.2, we verify that L is sectionally complemented.

5-3.1 Constructing M, congruences

To convey the idea of how we construct M, we present three small examples in which we construct the chopped lattice M from P and copies of the gadget $\mathsf{N}_6 = N(p, q)$ for $p \succ q$, see Figure 5-3.1.

Example 1*: P is the three-element chain C.* (See Figure 5-3.2.) Let $C = \{p, q, r\}$ with $r \prec q \prec p$. We take two copies of the gadget N_6, $N(p, q)$

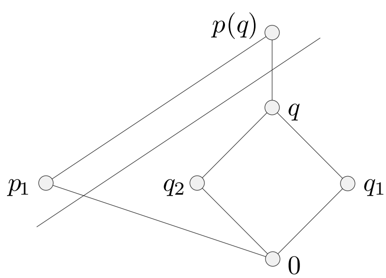

Figure 5-3.1: The gadget $\mathsf{N}_6 = N(p, q)$ for $p \succ q$ and the congruence $\boldsymbol{\alpha}$.

and $N(q,r)$; they share the ideal $I = \{0, q_1\}$; see Figure 5-3.2. So we can merge them and form the chopped lattice

$$M_C = \mathrm{Merge}(N(p,q), N(q,r))$$

as shown in Figure 5-3.2.

The congruences of M_C are easy to find. The isomorphism $C \cong \mathrm{Ji}(\mathrm{Con}\, M)$ is given by $x_1 \mapsto \mathrm{con}(0,x)$ for $x \in C$.

The congruences of M_C can be described by a compatible congruence vector $(\boldsymbol{\alpha}_{p,q}, \boldsymbol{\alpha}_{q,r})$, where $\boldsymbol{\alpha}_{p,q}$ is a congruence of the lattice $N(p,q)$ and $\boldsymbol{\alpha}_{q,r}$ is a congruence of the lattice $N(q,r)$, subject to the condition that $\boldsymbol{\alpha}_{p,q}$ and $\boldsymbol{\alpha}_{q,r}$ agree on I. Looking at Figure 5-3.1, we see that if the shared congruence on I is $\mathbf{0}$ $(= \mathbf{0}_I)$, then we must have $\boldsymbol{\alpha}_{p,q} = \mathbf{0}$ $(= \mathbf{0}_{N(p,q)})$ and $\boldsymbol{\alpha}_{q,r} = \mathbf{0}$ $(= \mathbf{0}_{N(q,r)})$ or $\boldsymbol{\alpha}_{q,r} = \boldsymbol{\alpha}$ on $N(q,r)$. If the shared congruence on I is $\mathbf{1}$ $(= \mathbf{1}_I)$, then we must have $\boldsymbol{\alpha}_{p,q} = \boldsymbol{\alpha}$ or $\boldsymbol{\alpha}_{p,q} = \mathbf{1}$ $(= \mathbf{1}_{N(p,q)})$ on $N(p,q)$ and $\boldsymbol{\alpha}_{q,r} = \mathbf{1}$ $(= \mathbf{1}_{N(q,r)})$ on $N(q,r)$. So there are three congruences distinct from $\mathbf{0}$: $(\mathbf{0}, \boldsymbol{\alpha})$, $(\boldsymbol{\alpha}, \mathbf{1})$, $(\mathbf{1}, \mathbf{1})$. Thus $\mathrm{Ji}(\mathrm{Con}\, M)_C \cong C$.

Example 2: P is the three-element order $V = \{p, q, r\}$ with $r \prec p, q$. (See Figure 5-3.3.) We take two copies of the gadget N_6, $N(p,r)$ and $N(q,r)$; they share the ideal $J = \{0, r_1, r_2, a\}$; we merge them to form the chopped lattice

$$M_V = \mathrm{Merge}(N(p,r), N(q,r)),$$

see Figure 5-3.3. Again, the isomorphism $V \cong \mathrm{Ji}(\mathrm{Con}\, M)_V$ is given by $x_1 \mapsto \mathrm{con}(0,x)$ for $x \in V$.

Example 3: P is the three-element order $H = \{p, q, r\}$ with $q, r \prec p$. (See Figure 5-3.4.) We take two copies of the gadget N_6, $N(p,q)$ and $N(p,r)$; they share the ideal $J = \{0, p_1\}$; we merge them to form the chopped lattice

$$M_H = \mathrm{Merge}(N(p,q), N(p,r)),$$

see Figure 5-3.4. Again, the isomorphism $H \cong \mathrm{Ji}(\mathrm{Con}\, M)_H$ is given by $x_1 \mapsto \mathrm{con}(0,x)$ for $x \in H$.

The reader should now be able to visualize the general construction of the chopped lattice M: instead of the few atoms in these examples, we start with enough atoms to reflect the structure of P, see Figure 5-3.5. Whenever $p \succ q$ in P, we build a copy of $N(p,q)$, see Figure 5-3.6. So $\mathrm{Ji}(\mathrm{Con}\, M) \cong P$ is intuitively clear. Define the lattice $L = \mathrm{Id}\, M$. By Theorem 5-2.6, the isomorphism $\mathrm{Ji}(\mathrm{Con}\, L) \cong P$ holds.

To verify Theorem 5-3.2, we have to prove that L is sectionally complemented. We do this in the next section.

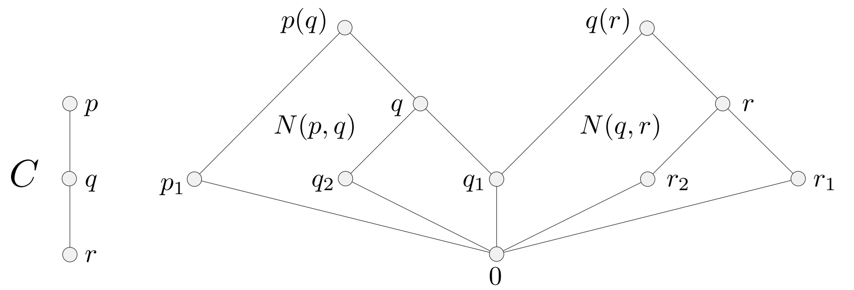

Figure 5-3.2: The chopped lattice M_C.

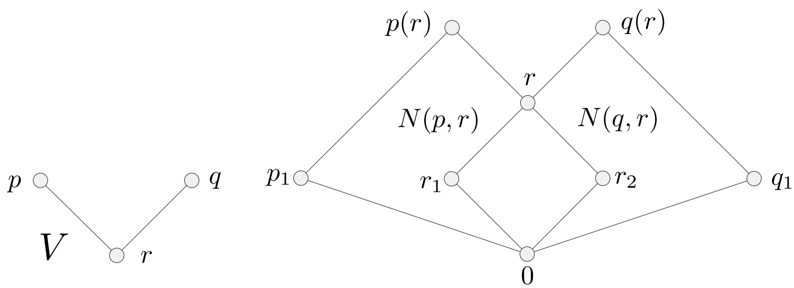

Figure 5-3.3: The chopped lattice M_V.

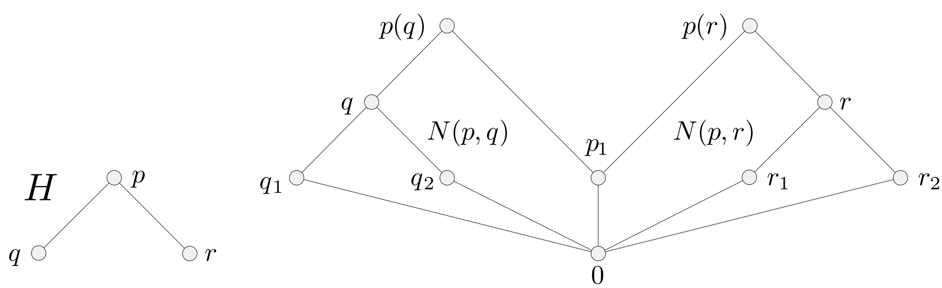

Figure 5-3.4: The chopped lattice M_H.

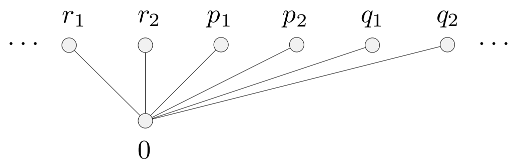

Figure 5-3.5: The chopped lattice M_0.

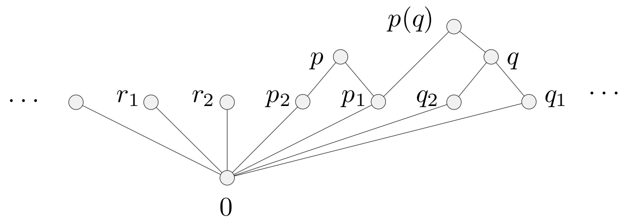

Figure 5-3.6: The chopped lattice M.

5-3.2 *L* is sectionally complemented

Theorem 5-3.3. *The lattice* $L = \operatorname{Id} M$ *is sectionally complemented.*

In this section, we give the proof of G. Grätzer and E.T. Schmidt [164], as presented in G. Grätzer and H. Lakser [150].

We represent $L = \operatorname{Id} M$ as a closure system.

For an ideal U of M, let $\operatorname{Atom}(U)$ be the set of atoms of M in U; the atoms of M are the $\{p_i\}$, where $p \in P$ and $i \in \{1, 2\}$. We start with the following two trivial statements (we compute with the indices modulo 2):

Lemma 5-3.4. *Let A be a set of atoms of M. Then there is an ideal U with* $\operatorname{Atom}(U) = A$ *iff A satisfies the condition:*

(Cl) *For $p \succ q$ in P if $p_1, q_i \in A$, then $q_{i+1} \in A$.*

Let us call a subset A of $\operatorname{Atom}(M)$ *closed* if it satisfies (Cl). It is obvious that every subset A of $\operatorname{Atom}(M)$ has a closure $\overline{A}$.

Lemma 5-3.5. *The assignment $I \mapsto \operatorname{Atom}(I)$ is a bijection between the ideals of M and closed subsets of* $\operatorname{Atom}(M)$, *and*

$$\operatorname{Atom}(I \wedge J) = \operatorname{Atom}(I) \cap \operatorname{Atom}(J),$$
$$\operatorname{Atom}(I \vee J) = \overline{\operatorname{Atom}(I) \cup \operatorname{Atom}(J)}$$

for $I, J \in \operatorname{Id} M$. The inverse map assigns to a closed set X of atoms, the ideal $\operatorname{id}(X)$ *of M generated by X.*

Lemma 5-3.5 allows us to regard L as the lattice of closed sets in $\mathrm{Atom}(M)$, so $I \in L$ is equivalent to I being a closed subset of $\mathrm{Atom}(M)$. Thus $I \wedge J = I \cap J$ and $I \vee J = \overline{I \cup J}$ for $I, J \in L$.

Let $I \subseteq J \in L$. Let us say that $q \in P$ *splits over* (I, J) if there exist elements $p \succ q$ in P, with $p_1, q_i \in J - I$ and $q_{i+1} \in I$. If there is a $q \in P$ that splits over (I, J), then $J - I$ is not closed. Let $X = X(I, J)$ be the set of all elements $q_i \in J - I$ such that q splits over (I, J). Let $\mathrm{NoSplit}(I, J) = (J - I) - X$, that is, $\mathrm{NoSplit}(I, J)$ is the set of all elements q_i in $J - I$ such that q does not split over (I, J).

Lemma 5-3.6. $\mathrm{NoSplit}(I, J) \in L$.

Proof. To prove that $S = \mathrm{NoSplit}(I, J)$ is closed, let $u \succ v$ in P, $u_1 \in S$, and $v_i \in S$. Since, by the definition of S, the element v does not split over (I, J) and $u_1, v_i \in J - I$, it follows that $v_{i+1} \notin I$. Since $u_1 \in J$, $v_i \in J$, and J is closed, we obtain that $v_{i+1} \in J$. Thus $v_{i+1} \in J - I$. Since v does not split over (I, J), we get that $v_{i+1} \in S$ by the definition of S. Thus S is closed. $\square$

Lemma 5-3.7. *S is the sectional complement of I in J.*

Proof. Clearly, $I \cap S = \varnothing$. We have to prove that $\overline{I \cup S} = J$.

Since $I \subseteq J$ and $S \subseteq J$, it is sufficient to show that $\overline{I \cup S} \supseteq J$. Assume, to the contrary, that there is a $q \in P$ and $i \in \{1, 2\}$ such that

$$q_i \in J - \overline{I \cup S}. \tag{5-3.1}$$

We can choose q so that it is maximal with respect to this property, that is, if $p > q$ and $p_j \in J$ for some $j \in \{1, 2\}$, then $p_j \in \overline{I \cup S}$.

It follows from (5-3.1) that $q_i \in J - (I \cup S) = X$. So by the definition of X, there exist elements $p \succ q$ in P with $p_1 \in J - I$ and $q_{i+1} \in I$. Since $p_1 \in J$ and $p \succ q$, by the maximality of q, we conclude that $p_1 \in \overline{I \cup S}$. Also, $q_{i+1} \in I \subseteq \overline{I \cup S}$. So $q_i \in \overline{I \cup S}$ by the definition of closure, contradicting (5-3.1). $\square$

We call this construction of a chopped lattice M the *1960 construction* and denote by $\mathbf{s_{1960}}$ the sectional complement constructed, which we call the *1960 sectional complement*.

5-4. An algorithmic construction of sectional complements

5-4.1 A crude algorithm

Why could we not prove that $L = \mathrm{Id}\, M$ is sectionally complemented following the argument of the Atom Lemma (Lemma 5-2.7)?

Take two compatible vectors

$$\mathbf{i} = (i_1, i_2, \dots, i_n) \leq (j_1, j_2, \dots, j_n) = \mathbf{j}$$

of M. Let s_k be a sectional complement of i_k in j_k and form

$$\mathbf{s} = (s_1, s_2, \dots, s_n).$$

If $\mathbf{s}$ is compatible, then it is the sectional complement of $\mathbf{i}$ in $\mathbf{j}$. What if $\mathbf{s}$ is not compatible? Observe that because of the structure of M, incompatibility occurs in a part M' of M that is an M_C or M_V or M_H. By pushing down one of the three components of $\mathbf{s}$ in M' a notch, we eliminate an instance of the incompatibility.

Considering this as one step, proceed this way in as many steps as necessary, to obtain a compatible vector. This is a crude algorithm to obtain a sectional complement.

In this section, based on G. Grätzer and M. Roddy [161], we show that a variant of this algorithm does indeed work. We have to be careful which sectional complement s_k of i_k in j_k we take and we have to impose restrictions on which incompatibilities we can take in a step.

5-4.2 Incompatibilities and failures

Before we describe the algorithm used to find the sectional complement, we begin with some definitions utilizing the fact that in $N(p, q)$, for every $x \leq y$, there is a unique sectional complement z of x in y, except for $x = p_1$ and $y = p(q)$, in which case, there are three, q, q_1, and q_2; of these, q is the maximal.

Definition 5-4.1. Let $\mathbf{u}$ and $\mathbf{v}$ be compatible vectors of the chopped lattice M with $\mathbf{u} \leq \mathbf{v}$, that is, let $\mathbf{u} = (u_{xy} \mid x \succ y \in P)$ and $\mathbf{v} = (v_{xy} \mid x \succ y \in P)$, with $u_{xy} \leq v_{xy}$ in $N(x, y)$, for all $x \succ y \in P$. Define the vector $\mathbf{s} = (s_{xy} \mid x \succ y \in P)$, where s_{xy} is the *maximal sectional complement* of u_{xy} in v_{xy}.

In this and the next section, we keep the vectors $\mathbf{u}$ and $\mathbf{v}$ and the vector $\mathbf{s}$ defined in Definition 5-4.1 fixed.

Definition 5-4.2. Let $p \succ q \succ r$ in P, that is, let $\{p, q, r\}$ be a cover-preserving suborder C in P.

(i) We call a vector $\mathbf{c} = (c_{xy} \mid x \succ y \in P)$ *C-compatible* at $\{p, q, r\}$ (or *$C(p, q, r)$-compatible*), if $c_{pq} \wedge q_1 = c_{qr} \wedge q_1$ in M – see Figure 5-3.2. Otherwise, $\mathbf{c}$ is *C-incompatible* at $\{p, q, r\}$ (or *$C(p, q, r)$-incompatible*).

(ii) The vector $\mathbf{c}$ is *C-compatible*, if it has no C-incompatibility.

(iii) We say that $\mathbf{c}$ has a *C-failure* at $\{p, q, r\}$ (or *$C(p, q, r)$-failure*), if $\mathbf{c}$ is C-incompatible at $\{p, q, r\}$ and, additionally, $c_{pr} = s_{pr}$ and $c_{qr} = s_{qr}$, that is, $\mathbf{c} = \mathbf{s}$ on $\{p, q, r\}$.

(iv) A $C(p,q,r)$-failure for $\mathbf{c}$ at $p \succ q \succ r$ is *minimal*, iff there is no $C(p',q',r')$-failure for $\mathbf{c}$ with $q' < q$.

Definition 5-4.3. Let $p \succ r \prec q$ and $p \neq q$ in P, that is, let $\{p,q,r\}$ be a cover-preserving suborder V in P.

(i) We call a vector $\mathbf{c} = (c_{xy} \mid x \succ y \in P)$ *V-compatible* at $\{p,q,r\}$ (or *$V(p,q,r)$-compatible*), if $c_{pr} \wedge r = c_{qr} \wedge r$ in M – see Figure 5-3.3. Otherwise, $\mathbf{c}$ is *V-incompatible* at $\{p,q,r\}$ (or *$V(p,q,r)$-incompatible*).

(ii) The vector $\mathbf{c}$ is *V-compatible*, if it has no V-incompatibility.

(iii) We say that $\mathbf{c}$ has a *V-failure* at $\{p,q,r\}$ (or *$V(p,q,r)$-failure*), if $\mathbf{c}$ is V-incompatible at $\{p,q,r\}$ and, additionally, $c_{pr} = s_{pr}$ and $c_{qr} = s_{qr}$, that is, $\mathbf{c} = \mathbf{s}$ on $\{p,q,r\}$.

Definition 5-4.4. Let $q \prec p \succ r$ and $q \neq r$ in P, that is, let $\{p,q,r\}$ be a cover-preserving suborder H (the hat) in P.

(i) We call a vector $\mathbf{c} = (c_{xy} \mid x \succ y \in P)$ *H-compatible* at $\{p,q,r\}$ (or *$H(p,q,r)$-compatible*), if $c_{pq} \wedge p_1 = c_{pr} \wedge p_1$ in M – see Figure 5-3.4. Otherwise, $\mathbf{c}$ is *H-incompatible* at $\{p,q,r\}$ (or *$H(p,q,r)$-incompatible*).

(ii) The vector $\mathbf{c}$ is *H-compatible*, if it has no H-incompatibility.

A vector $\mathbf{c}$ is compatible iff it is C-compatible, V-compatible, and H-compatible. Note that we do not introduce H-failures.

5-4.3 Failures, cuts, and the algorithm

Now we state two lemmas describing the two types of failures.

Lemma 5-4.5. *For a vector $\mathbf{c}$, a $C(p,q,r)$-failure is represented by a row in Table 5.1 with $c_{pq} = s_{pq}$ and $c_{qr} = s_{qr}$.*

Lemma 5-4.6. *For a vector $\mathbf{c}$, a $V(p,q,r)$-failure is represented by a row in Table 5.2 with $c_{pr} = s_{pr}$ and $c_{qr} = s_{qr}$.*

Note that there is no H-failure Table.

Given a vector $\mathbf{c}$, we now define a C-cut and a V-cut, vectors a little smaller than $\mathbf{c}$.

Definition 5-4.7. Let $\mathbf{c}$ be a vector with a $C(p,q,r)$-failure. The *C-cut* (more precisely, *$C(p,q,r)$-cut*) of $\mathbf{c}$ is a vector $R_C(\mathbf{c})$ all but one of whose components are the same as those of $\mathbf{c}$. One component of $\mathbf{c}$ is "cut" (substituted by an element it covers) as shown in Table 5.3.

By Lemma 5-4.5, this definition covers all C-failures. Note that $R_C(\mathbf{c})$ is $C(p,q,r)$-compatible.

u_{pq}	u_{qr}	v_{pq}	v_{qr}	s_{pq}	s_{qr}
q_2	0	$p(q)$	p_1	p_1	q_1
q_2	0	$p(q)$	$q(r)$	p_1	$q(r)$
q_2	r_1	$p(q)$	$q(r)$	p_1	q_1
q_2	r_2	$p(q)$	$q(r)$	p_1	q_1
q_2	r	$p(q)$	$q(r)$	p_1	q_1

Table 5.1: The C-failure Table.

u_{pr}	u_{qr}	v_{pr}	v_{qr}	s_{pr}	s_{qr}
r_2	r_2	r	$q(r)$	r_1	q_1
r_2	r_2	$p(r)$	r	p_1	r_1
r_1	r_1	r	$q(r)$	r_2	q_1
r_1	r_1	$p(r)$	r	p_1	r_2

Table 5.2: The V-failure Table.

u_{pq}	u_{qr}	v_{pq}	v_{qr}	$c_{pq} = s_{pq}$	$c_{qr} = s_{qr}$	
q_2	0	$p(q)$	q_1	p_1	q_1	$c_{qr} \mapsto 0$
q_2	0	$p(q)$	$q(r)$	p_1	$q(r)$	$c_{qr} \mapsto r$
q_2	r_1	$p(q)$	$q(r)$	p_1	q_1	$c_{qr} \mapsto 0$
q_2	r_2	$p(q)$	$q(r)$	p_1	q_1	$c_{qr} \mapsto 0$
q_2	r	$p(q)$	$q(r)$	p_1	q_1	$c_{qr} \mapsto 0$

Table 5.3: The C-cut Table.

u_{pr}	u_{qr}	v_{pr}	v_{qr}	$s_{pr} = c_{pr}$	$s_{qr} = c_{qr}$	
r_2	r_2	r	$q(r)$	r_1	q_1	$c_{pr} \mapsto 0$
r_2	r_2	$p(r)$	r	p_1	r_1	$c_{qr} \mapsto 0$
r_1	r_1	r	$q(r)$	r_2	q_1	$c_{pr} \mapsto 0$
r_1	r_1	$p(r)$	r	p_1	r_2	$c_{qr} \mapsto 0$

Table 5.4: The V-cut Table.

Definition 5-4.8. Let $\mathbf{c}$ be a vector with a $V(p,q,r)$-failure. The V-*cut* (more precisely, $V(p,q,r)$-*cut*) of $\mathbf{c}$ is a vector $R_V(\mathbf{c})$ all but one of whose components are the same as those of $\mathbf{c}$. One component of $\mathbf{c}$ is "cut" (substituted by an element it covers) as shown in Table 5.4.

By Lemma 5-4.6, this definition covers all V-failures. Note that $R_V(\mathbf{c})$ is $V(p,q,r)$-compatible.

Given the compatible vectors $\mathbf{u} \leq \mathbf{v}$ and the vector $\mathbf{s}$ as in Definition 5-4.1, we construct a vector $\mathbf{s}^*$.

The Algorithm.

Step 1. *Set* $\mathbf{c} = \mathbf{s}$.

Step 2. *Look for a V-failure, and perform the corresponding V-cut, obtaining a new* $\mathbf{c} = R_V(\mathbf{c})$.

Step 3. *Repeat Step 2 until there are no more V-failures.*

Step 4. *Look for a* minimal *C-failure, and perform the corresponding C-cut, obtaining a new* $\mathbf{c} = R_C(\mathbf{c})$.

Step 5. *Repeat Step 4 until there are no more C-failures.*

Since M is finite and $R_C(\mathbf{c}), R_V(\mathbf{c}) < \mathbf{c}$, the process must terminate, yielding a vector $\mathbf{s}^*$.

We will refer to this algorithm as the Algorithm.

5-4.4 The result

Given the compatible vectors $\mathbf{u} \leq \mathbf{v}$, the vector $\mathbf{s}$ as in Definition 5-4.1, and a vector $\mathbf{s}^*$, a result of the Algorithm, we have the following result of G. Grätzer and M. Roddy [161]:

Theorem 5-4.9. *The vector* $\mathbf{s}^*$ *is compatible and it is a sectional complement of* $\mathbf{u}$ *in* $\mathbf{v}$ *in* $\operatorname{Id} M$. *Hence the lattice* $\operatorname{Id} M$ *is sectionally complemented. Moreover, for every* $p \succ q$ *in* P, *either* $s^*_{pq} = s_{pq}$ *or* $s^*_{pq} \prec s_{p,q}$ *holds.*

Let m be the number of covering pairs in $\operatorname{Ji} D$, the order of join-irreducible elements of D. The vectors are 6^m in number. The interval $[\mathbf{s}^*, \mathbf{s}]$ in N_6^m is a Boolean lattice of length at most m. So $\mathbf{s}^*$ is at most of distance m from $\mathbf{s}$.

5-4.5 Proving the failure lemmas

In this section, we assume that the vector $\mathbf{c}$ has a $V(p,q,r)$-failure, that is, $(c_{pr}, c_{qr}) = (s_{pr}, s_{qr})$ and $s_{pr} \wedge r \neq s_{qr} \wedge r$.

Proof of Lemma $5-4.6$. There are four cases to consider:

Case 1: $s_{pr} \geq r$ *and* $v_{pr} = s_{pr}$.

Case 2: $s_{pr} \geq r$ *and* $v_{pr} > s_{pr}$.

Case 3: $s_{qr} \geq r$.

Case 4: $s_{pr} = r_1$.

By way of example, we prove Case 4. (Cases 1–3 lead to contradictions.)

Case 4 has two subcases.

Case 4.1*:* $v_{pr} = p(r)$*.*

Since p_1 is the only sectional complement of r_1 in $p(r)$, we have that $u_{pr} = p_1$. But now, by definition, we would have that $s_{pr} = r$, a contradiction.

Case 4.2*:* $v_{pr} = r$*.*

If $v_{pr} = r$, then $a_{pr} = r_1$ yields that $u_{pr} = r_2$. Since $\mathbf{u}$ is compatible, $u_{qr} = r_2$ also holds. If $v_{qr} = r$, then $s_{qr} = r_1 = s_{pr}$, contradicting our incompatibility condition. So we must have $v_{qr} = q(r)$. This gives, uniquely, that

u_{pr}	u_{qr}	s_{pr}	s_{qr}	v_{pr}	v_{qr}
r_2	r_2	r_1	q_1	r	$q(r)$

Thus, this subcase leads to exactly one row of the V-failure Table.

The other three rows of the table follow symmetrically from the other three possibilities for $\{s_{pr}, s_{qr}\} \cap \{r_1, r_2\}$.

Conversely, it is clear that each row of the table leads to a $V(p, q, r)$-failure.

Throughout the proof, we assume that the vector $\mathbf{c}$ has a $V(p, q, r)$-failure, that is, $(c_{pr}, c_{qr}) = (s_{pr}, s_{qr})$ and $s_{pr} \wedge r \neq s_{qr} \wedge r$. $\square$

Proof of Lemma $5-4.5$*.* Again, the vector $\mathbf{c}$ is fixed.

Let us assume that there is a $C(p, q, r)$-failure for $\mathbf{c}$, that is, $(c_{pq}, c_{qr}) = (s_{pq}, s_{qr})$ and $s_{pq} \wedge q_1 \neq s_{qr} \wedge q_1$. There are two cases to consider:

Case 1*:* $s_{pq} \geq q_1$ *(so* $s_{qr} \ngeq q_1$*).*

Case 2*:* $s_{qr} \geq q_1$ *(so* $s_{pq} \ngeq q_1$*).*

We verify Case 2.

The assumption gives that $u_{qr} \in \{0, r_1, r_2, r\}$ and hence (because $\mathbf{u}$ is compatible) $u_{pq} \in \{0, p_1, q_2\}$.

Case 2.1*:* $u_{pq} \in \{0, p_1\}$*.*

If $u_{pq} = 0$, then $s_{pq} = v_{pq} \geq q_1$ (since $\mathbf{v}$ is compatible), a contradiction.

If $u_{pq} = p_1$, then $v_{pq} \geq q_1$ gives that $v_{pq} = p(q)$. Hence $s_{pq} = q_1$, a contradiction.

Case 2.2*:* $u_{pq} = q_2$*.*

$u_{pq} = q_2$, $s_{pq} \ngeq q_1$ and $v_{pq} \geq q_1$ together force $v_{pq} = p(q)$ and $s_{pq} = p_1$. The C-failure Table lists the ways this can happen. Each of these possibilities leads to a C-failure. $\square$

5-4.6 Proving the main result

In this section, we prove Theorem 5-3.2.

Throughout this section, $\mathbf{s}^*$ will be a vector resulting at a termination of the Algorithm starting with $\mathbf{u} \leq \mathbf{v}$ and $\mathbf{s}$.

To prove that $\mathbf{s}^*$ is compatible, we have to prove that it is V-compatible, C-compatible, and H-compatible. Finally, we have to verify that $\mathbf{s}^*$ is a sectional complement.

In proving the three compatibility results, we will distinguish several cases for each. Let $x \succ y$ in P. Then there are three possible cases to arrive at the value s^*_{xy}:

(S) $s^*_{xy} = s_{xy}$;

(V) $s^*_{xy} = 0$ as a result of a V-cut;

(C) $s^*_{xy} \in \{0, y\}$ as a result of a C-cut.

Since each of V-compatibility, C-compatibility, and H-compatibility involves two overlapping covering pairs, we have many cases to cover. A case is described as

Case: (VC)

to indicate that we apply a V-cut to the first covering pair and a C-cut to the second covering pair.

V-compatibility

First, we prove that $\mathbf{s}^*$ is V-compatible.

Assume that $\mathbf{s}^*$ is not $V(p, q, r)$-compatible. There are nine cases to consider. We discuss here two of the nine.

Case (VS): $s^*_{pr} = 0$ *as the result of a* V*-cut, and* $s^*_{qr} = s_{qr}$.

Since $s^*_{pr} = 0$ was obtained from a V-cut, from the V-cut Table we conclude that there are two subcases.

By symmetry, we can assume that $u_{pr} = r_2$, $v_{pr} = r$, $s_{pr} = r_1$.

Then $u_{qr} = r_2$ (since $\mathbf{u}$ is compatible) and $v_{qr} \geq r$ (since $\mathbf{v}$ is compatible).

If $v_{qr} = q(r)$, then $s_{qr} = q_1$ and $s^*_{qr} \wedge r = s_{qr} \wedge r = 0$, making $\mathbf{s}^*$ compatible at $p \succ r \prec q$, a contradiction. Hence, $v_{qr} = r$ and $s_{qr} = r_1$.

Now assume that $s^*_{pr} = 0$ resulted from a $V(p, r, a)$-cut of a vector $\mathbf{c}$ at some point in the Algorithm.

Inspecting the V-cut Table, appropriately relabeled, we see that $c_{ar} = s_{ar} = a_1$, which was the initial value of c_{ar}. But since this value cannot be modified by a V-cut (involving $a \succ r$), we have that $c_{ar} = s_{ar} = a_1$ at the termination of Step 3 of the Algorithm. But also at the termination of Step 3, $c_{pr} = s^*_{pr} = s_{pr} = r_1$. But $r_1 \wedge r = r_1$ and $a_1 \wedge r = 0$ yield the conditions for a V-failure at the termination of Step 3, a contradiction. Thus, this subcase cannot occur.

Case (CC): $s^*_{pr}, s^*_{qr} \in \{0, r\}$*, both as the result of* C*-cuts.*

Since $\mathbf{s}^*$ is $V(p, q, r)$-incompatible, we must have $s^*_{pr} = 0$ and $s^*_{qr} = r$ or we must have $s^*_{pr} = r$ and $s^*_{qr} = 0$.

By symmetry, we can assume that $s^*_{pr} = r$ and $s^*_{qr} = 0$. From the C-cut Table (suitably relabeled), this gives $s_{pr} = p(r)$ and $s_{qr} = q_1$.

But now, since we obtained s^*_{pr} and s^*_{qr} as the result of C-failures, we'd have had $s_{pr} = p(r)$ and $s_{qr} = q_1$ which is a V-failure at the termination of Step 3, a contradiction.

This completes the proof that Case (CC) cannot occur.

C-compatibility

Next we prove that $\mathbf{s}^*$ is C-compatible. There are nine cases to consider. We discuss here only one of the nine.

Let us assume that $\mathbf{s}^*$ is not $C(p,q,r)$-compatible.

Case (VS): $s^*_{pq} = 0$ *as the result of a* V*-cut and* $s^*_{qr} = s_{qr}$.

We assumed that $\mathbf{s}^*$ is not $C(p,q,r)$-compatible, so in this case, $s^*_{qr} = s_{qr} \geq q_1$.

Suppose that $s^*_{pq} = 0$ was obtained from a vector $\mathbf{c}$ as the result of a $V(p,q,a)$-cut.

Inspecting the V-cut Table, suitably relabeled, we see that $c_{aq} = s_{aq} = a_1$ and we observe that $a_1 \wedge q_1 = 0$.

Consider s^*_{aq}.

If $s^*_{aq} = 0$ as the result of a $V(x,a,q)$-cut, then $s_{aq} \in \{q_1, q_2\}$ (see the V-cut Table, suitably relabeled). But, as argued above, $s_{aq} = a_1$, so this cannot occur.

If $s^*_{aq} = s_{aq}$, then, $s^*_{aq} = s_{aq} = a_1$ and $a_1 \wedge q_1 = 0$. But $s^*_{qr} = s_{qr} \geq q_1$, and these together give a $C(a,q,r)$-failure for $\mathbf{s}^*$, a contradiction.

The only other possibility is that s^*_{aq} is the result of a C-cut.

Before considering the details of this situation let us recall some things we have already established,

$$s^*_{qr} = s_{qr} \geq q_1,\ s_{aq} = a_1, \text{ and } s_{aq} \wedge q_1 = 0.$$

Assume that $s^*_{aq} \in \{0, q\}$ as the result of a $C(b,a,q)$-cut of $\mathbf{c}$ at $b \succ a \succ q$ in Step 4.

Consider c_{aq} and c_{qr}.

Since the $C(a,b,q)$-cut of $\mathbf{c}$ was made, we would have, in particular, that $c_{aq} = s_{aq}$. As argued above $c_{aq} = s_{aq} = a_1$ and $s_{aq} \wedge q_1 = 0$.

On the other hand, we have

$$c_{qr} = s^*_{qr} = s_{qr} \geq q_1.$$

This gives a $C(a,q,r)$-failure of $\mathbf{c}$.

But now we have two C-failures for $\mathbf{c}$, one at $b \succ a \succ q$ and one at $a \succ q \succ r$.

The minimality condition of Step 4 would require that the $C(a,q,r)$-cut be done first, so this situation cannot occur.

This completes the proof that Case (VS) cannot occur.

H-compatibility

Finally, we prove that $\mathbf{s}^*$ is H-compatible. There are nine cases to consider. We discuss here only one of the nine.

Assume that $\mathbf{s}^*$ is not $H(p,q,r)$-compatible.

Case (SS): $s^*_{pq} = s_{pq}$ *and* $s^*_{qr} = s_{qr}$.

We may assume (up to symmetry) that $s_{pq} \geq p_1$ and $s_{pr} \ngeq p_1$.

If $v_{pq} = s_{pq}$, then by the definition of $\mathbf{s}$, $u_{pq} = 0$ and since $\mathbf{u}$ is compatible, $u_{pr} \in \{0, r_1, r_2, r\}$.

If $u_{pr} = 0$, then $s_{pr} = v_{pr}$. But $\mathbf{v}$ is compatible and $v_{pq} \geq s_{pq} \geq p_1$, so $s_{pq} = v_{pr} \geq p_1$, contradicting the assumption that $s_{pr} \ngeq p_1$.

If $u_{pr} \in \{r_1, r_2, r\}$, then $v_{pr} \geq p_1$, u_{pr} imply that $v_{pr} = p(q)$. But now the definition of $\mathbf{s}$ gives $s_{pr} = p_1$, contrary to the assumption that $s_{pr} \ngeq p_1$.

This exhausts the possibilities where $v_{pq} = s_{pq}$. So we may assume that $v_{pq} > s_{pq}$. This forces that $v_{pq} = p(q)$ and $s_{pq} = p_1$. This, in turn, gives $u_{pq} \in \{q_1, q_2, q\}$ and since $\mathbf{u}$ is compatible, $u_{pr} \leq r$.

If $u_{pr} = 0$, then $s_{pr} = v_{pr} \geq p_1$, a contradiction. If $u_{pr} \in \{r_1, r_2, r\}$, then $v_{pr} \geq p_1$, u_{pr} gives $v_{pr} = p(r)$ and hence $s_{pr} = p_1$, another contradiction.

This proves that this situation cannot occur, and up to symmetry, proves that Case (SS) cannot occur.

5-4.7 Sectional complement

To show that $\mathbf{s}^*$ is a sectional complement of $\mathbf{u}$ in $\mathbf{v}$, we have to prove that $\mathbf{u} \wedge \mathbf{s}^* = 0$ and $\overline{\mathbf{u} \vee_c \mathbf{s}^*} = \mathbf{v}$.

The first statement is trivial, since $\mathbf{s}^* \leq \mathbf{s}$. To prove the second statement, we prove that the property is preserved by each step of the Algorithm, that is, by both V-cuts and C-cuts.

At Step 1 of the Algorithm $\mathbf{c} = \mathbf{s}$ and $\overline{\mathbf{u} \vee_c \mathbf{s}} = \overline{\mathbf{v}} = \mathbf{v}$.

Assume that at a given stage of the Algorithm, we have the vector $\mathbf{c}$ with $\overline{\mathbf{u} \vee_c \mathbf{c}} = \mathbf{v}$, and let $\mathbf{c}^+$ be the result of a V-cut or a C-cut according to the next step of the Algorithm. To show that $\overline{\mathbf{u} \vee_c \mathbf{c}^+} = \mathbf{v}$, it suffices to show that

$$\overline{\mathbf{u} \vee_c \mathbf{c}^+} \geq \mathbf{c}.$$

Since, $\mathbf{c}^+$ only differs from $\mathbf{c}$ at one of the components determined by a cover in the corresponding V or C, we need only establish that $\overline{\mathbf{u} \vee_c \mathbf{c}^+} \geq \mathbf{c}$ locally.

Assume that $\mathbf{c}^+$ is obtained from $\mathbf{c}$ as the result of a $V(p, r, q)$-cut, say $c^+_{pr} = 0 \prec c_{pr}$.

From the V-cut Table, $c_{pr} = r_1$ or $c_{pr} = r_2$, say $c_{pr} = r_1$. Working locally:

$$\begin{aligned}
&(u_{pr}, u_{qr}) \vee_c (c^+_{pr}, c_{qr}) \\
&\qquad = (r_2, r_2) \vee_c (0, q_1) \qquad \text{(from the } V\text{-cut Table)} \\
&\qquad = (r_2, q(r)),
\end{aligned}$$

and

$$\begin{aligned}\overline{(r_2, q(r))} &\geq (r, q(r))\\ &= (v_{pr}, v_{qr}) \qquad \text{(from the } V\text{-cut Table)}\\ &\geq (c_{pr}, c_{qr}).\end{aligned}$$

All other possible V-cuts are symmetric to this case.

Let us now assume that $c^+_{qr} = r$ is obtained from a $C(p, q, r)$-cut of $\mathbf{c}$. Then

$$\begin{aligned}&(u_{pq}, u_{qr}) \vee_c (c_{pq}, c^+_{qr})\\ &\qquad = (q_2, 0) \vee_c (p_1, r) \qquad \text{(from the } C\text{-cut Table)}\\ &\qquad = (p(q), r).\end{aligned}$$

and

$$\begin{aligned}\overline{(p(q), r)} &\geq (p(q), q(r))\\ &= (v_{pq}, v_{qr}) \qquad \text{(from the } C\text{- failure Table)}\\ &\geq (c_{pq}, c_{qr}).\end{aligned}$$

Now assume that $c^+_{qr} = 0$ is obtained from a $C(p, q, r)$-cut of $\mathbf{c}$.

From the C-cut Table, $u_{qr} \in \{0, r_1, r_2, r\}$.

Assume that $u_{qr} = 0$. Then

$$\begin{aligned}&(u_{pq}, u_{qr}) \vee_c (c_{pq}, c^+_{qr})\\ &\qquad = (q_2, 0) \vee_c (p_1, 0)\\ &\qquad = (p(q), 0),\end{aligned}$$

and

$$\begin{aligned}\overline{(p(q), 0)} &= (p(q), q_1)\\ &= (v_{pq}, v_{qr}) \qquad \text{(from the } C\text{-cut Table)}\\ &\geq (c_{pq}, c_{qr}).\end{aligned}$$

Assume that $u_{qr} = r$. Then

$$\begin{aligned}&(u_{pq}, u_{qr}) \vee_c (c_{pq}, c^+_{qr})\\ &\qquad = (q_2, r) \vee_c (p_1, 0) \qquad \text{(from the } C\text{-cut Table)}\\ &\qquad = (p(q), r),\end{aligned}$$

and

$$\begin{aligned}\overline{(p(q), r)} &\geq (p(q), q(r))\\ &= (v_{pq}, v_{qr}) \qquad \text{(from the } C\text{-cut Table)}\\ &\geq (c_{pq}, c_{qr}).\end{aligned}$$

The other two subcases follow in an almost identical manner.

5-5. Convergence

In Section 5-3.2, we discussed $\mathbf{s_{1960}}$, the sectional complement constructed in G. Grätzer and E.T. Schmidt [164]. In Section 5-4, we introduced the Algorithm that finds sectional complements.

The expectation for the Algorithm was that it would find a large set of sectional complements; we did not know whether $\mathbf{s_{1960}}$ belongs to this set.

Quite unexpectedly, G. Grätzer, G. Klus, and A. Nguyen [139] proved that the set of sectional complements found by the Algorithm is a singleton, in fact, it is $\{\mathbf{s_{1960}}\}$. In this section we prove this result in two steps.

Theorem 5-5.1. *Let Σ be any sequence of cuts in the Algorithm. Then the sectional complement, $\mathbf{s}_\Sigma$, is independent of Σ.*

Theorem 5-5.2. *The unique sectional complement $\mathbf{s}$ produced by the Algorithm is the 1960 sectional complement, that is, $\mathbf{s} = \mathbf{s_{1960}}$.*

5-5.1 Proof of Theorem 5-5.1

In this section we prove that $\mathbf{s}_\Sigma$ does not depend on the choice of Σ.

Let $\mathbf{m}^2$ denote the following vector:

$$\text{(5-5.1)}\quad m^2_{pr} = \begin{cases} 0, & \text{if } m_{pr} = r_i \text{ with a } V(p,q,r)\text{-failure for some } q \succ r; \\ m_{pr}, & \text{otherwise.} \end{cases}$$

Lemma 5-5.3. *At the end of Step 2, we obtain the vector $\mathbf{m}^2$, independent of the sequence of V-cuts performed.*

Proof. Let us look at m_{pr}. If there is a $V(p,q,r)$-failure and the corresponding V-cut was performed, then r_i was replaced by 0.

Now assume that there is a $V(p,q,r)$-failure but the corresponding V-cut was not performed. This can only happen if there is a $V(p',q',r')$-failure, the corresponding V-cut was performed, and after the cut there is no $V(p,q,r)$-failure. Clearly, $p = p'$, $r = r'$, and $q \neq q'$. But then the V-cut at $\{p,q',r\}$ would also replace m_{pr} with 0, so the first line of (5-5.1) is verified.

Of course, if there is a $V(p,q,r)$-failure for $\mathbf{m}^2$, then that would also be a $V(p,q,r)$-failure for $\mathbf{m}$, verifying (5-5.1). □

We get something extra for the vector $\mathbf{m}^2$:

Lemma 5-5.4. *The vector $\mathbf{m}^2$ is V-compatible.*

Proof. By Lemma 5-5.3. □

Next we prove Theorem 5-5.1. We have to prove that any sequence of C-cuts applied to $\mathbf{m}^2$ as specified in Step 3 of the Algorithm, yields a unique vector. This will be done in the next three lemmas.

Let $C(p,q,r)$ be a C-suborder. It will be convenient to call $r \prec q$ the *stem* of C.

Lemma 5-5.5. *Let* $\mathbf{m}^2$ *have a* C*-failure at* $C(p,q,r)$*. Then any* C*-suborder of* P *with the same stem,* $r \prec q$*, also has a* C*-failure. Moreover, all these failures are resolved by the same cut.*

Proof. Since $\mathbf{m}^2$ has a $C(p,q,r)$-failure, by Lemma 5-4.6, $m^2_{pq} \wedge q_1 = 0$ and $m^2_{qr} \wedge q_1 = q_1$. Let $C(t,q,r)$ be a C-suborder; it shares the stem with $C(p,q,r)$. By Lemma 5-5.4, the vector $\mathbf{m}^2$ is V-compatible, in particular, $V(p,t,q)$ is V-compatible and so $m^2_{pq} \wedge q = m^2_{tq} \wedge q$. Therefore, $m^2_{pq} \wedge q_1 = m^2_{tq} \wedge q_1 = 0$. Hence, $m^2_{tq} \wedge q_1 = 0 \neq q_1 = m^2_{qr} \wedge q_1$ and so $C(t,q,r)$ is a C-failure. Since the stem of both $C(p,q,r)$ and $C(t,q,r)$ is $\{q,r\}$, the failures on $C(p,q,r)$ and $C(t,q,r)$ will be corrected (by cutting $m_{q,r}$) the same way. □

Lemma 5-5.6. *Let* C_1 *and* C_2 *be two minimal* C*-failures that do not share a stem. Then, after a* C*-cut at* C_1*, the chain* C_2 *still has a* C*-failure.*

Proof. Since C_1 is a minimal C-failure, the stem of C_1 is not the upper covering pair of C_2. Since the C-cut on C_1 takes place in the $N(p,q)$ corresponding to the stem of C_1, the chain C-failure C_2 is not effected by this cut. □

Lemma 5-5.7. *Let* Σ *be any sequence of* C*-cuts on* $\mathbf{m}^2$ *such that the vector* $\mathbf{m}^2_\Sigma$ *obtained by* Σ *has no* C*-failures. Then* $\mathbf{m}^2_\Sigma$ *does not depend on* Σ*.*

Proof. Let $C(p,q,r)$ be a C-failure. Then $m^2_{pq} = m_{pq}$ and $m^2_{qr} = m_{qr}$. By Lemma 5-5.5, each stem of a C-failure is cut uniquely. By Lemma 5-5.6, a C-failure is not effected by cutting another C-failure, unless they share a stem. Since $C(p,q,r)$ will eventually become a minimal failure, m^2_{qr} will be cut uniquely by the Algorithm. □

So we have proved that $\mathbf{s}_\Sigma$ does not depend on the choice of Σ. Let $\mathbf{s}$ denote this vector. By Theorem 5-4.9, the vector $\mathbf{s}$ is a sectional complement of $\mathbf{u}$ in $\mathbf{v}$.

5-5.2 Proof of Theorem 5-5.2

Let $\mathbf{u} \leq \mathbf{v}$ be vectors in M. Let $\mathbf{m}$ be the vector defined in Definition 5-4.1. Let $\mathbf{s_{1960}}$ denote the vector representing the 1960 sectional complement of $\mathbf{u}$; see also (5-5.3). For a vector $\mathbf{c}$, let $\mathrm{Atom}(\mathbf{c})$ denote the atoms of M (regarded as a compatible vector) contained in $\mathbf{c}$.

Clearly, $\mathrm{Atom}(\mathbf{m}) \subseteq \mathrm{Atom}(\mathbf{v})$, since $\mathbf{m} \leq \mathbf{v}$. Moreover, $\mathrm{Atom}(\mathbf{m}) \cap \mathrm{Atom}(\mathbf{u}) = \varnothing$, because $\mathbf{m} \wedge \mathbf{u} = 0$. Therefore,

$$\mathrm{Atom}(\mathbf{m}) \subseteq \mathrm{Atom}(\mathbf{v}) - \mathrm{Atom}(\mathbf{u}). \tag{5-5.2}$$

We denote by $\mathrm{NoSplit}(\mathbf{u},\mathbf{v})$ the set of all elements q_i in $\mathrm{Atom}(\mathbf{v}) - \mathrm{Atom}(\mathbf{u})$ such that q splits over $(\mathbf{u},\mathbf{v})$ and recall from Section 5-3.2 the formula:

$$\mathbf{s_{1960}} = \bigvee((\mathrm{Atom}(\mathbf{v}) - \mathrm{Atom}(\mathbf{u})) - \mathrm{NoSplit}(\mathbf{u},\mathbf{v})). \tag{5-5.3}$$

Lemma 5-5.8. *The inequality* $\mathbf{s_{1960}} \leq \mathbf{m}$ *holds.*

Proof. Since $\mathbf{u} \wedge \mathbf{s_{1960}} = 0$, it follows that $u_{pq} \wedge (s_{1960})_{pq} = 0$ in $N(p,q)$ for any $p \succ q$ in P. Hence, $(s_{1960})_{pq} \leq m_{pq}$, for all $p \succ q$ in P, therefore, $\mathbf{s_{1960}} \leq \mathbf{m}$. □

Lemma 5-5.9. *Let* $\mathbf{u} \leq \mathbf{v}$ *be vectors in* P. *Let* $\mathbf{c}$ *be a vector obtained in a step of the Algorithm and let* $\mathrm{Cut}(\mathbf{c})$ *be the vector obtained in the next step of the Algorithm. If* $\mathbf{s_{1960}} \leq \mathbf{c}$, *then* $\mathbf{s_{1960}} \leq \mathrm{Cut}(\mathbf{c})$.

Proof. Let us assume that $\mathbf{s_{1960}} \leq \mathbf{c}$. If the Algorithm terminates at $\mathbf{c}$, there is nothing to prove. If the Algorithm does not terminate at $\mathbf{c}$, the next step is a cut of $\mathbf{c}$. We distinguish two cases.

Case 1*: V-cut at* $V = \{p, q, r\}$.

By symmetry, we can assume that $c_{pr} = m_{pr} = r_1$ and $c_{qr} = m_{qr} = q_1$. Since $m_{pr} = r_1$ is the maximal sectional complement of u_{pr} in v_{pr}, it follows that either

(i) $v_{pr} = r_1$ and $u_{pr} = 0$;

or

(ii) $v_{pr} = r$ and $u_{pr} = r_2$.

Since $v_{qr} \geq c_{qr} = q_1$, if (i) holds, then either $v_{qr} = q_1$ or $v_{qr} = q(r)$. In both cases, then,

$$v_{qr} \wedge r \neq v_{pr} \wedge r = r_1. \tag{5-5.4}$$

Hence, $v_{pr} = r$ and $u_{pr} = r_2$. By (5-5.4), since $r_1, q_1 \in \mathrm{Atom}(\mathbf{v}) - \mathrm{Atom}(\mathbf{u})$ but $r_2 \in \mathrm{Atom}(\mathbf{u})$, it follows that r splits over $(\mathbf{u}, \mathbf{v})$. So $\mathbf{s_{1960}} \leq \mathrm{Cut}_V(\mathbf{c})$, when restricted to V. Since $\mathbf{c}$ and $\mathrm{Cut}_V(\mathbf{c})$ are equal outside of V, we conclude that $\mathbf{s_{1960}} \leq \mathrm{Cut}_V(\mathbf{c})$.

Case 2*: C-cut at* $C = \{p, q, r\}$.

We form a C-cut at $C = \{p, q, r\}$. Therefore, we have that $m_{pq} = p_1$ and $m_{qr} \geq q_1$. So $p_1, q_1 \in \mathrm{Atom}(\mathbf{m})$, and by (5-5.2), $p_1, q_1 \in \mathrm{Atom}(\mathbf{v}) - \mathrm{Atom}(\mathbf{u})$. In particular, $p_1, q_1 \notin \mathrm{Atom}(\mathbf{u})$. Now $q_1 \notin \mathrm{Atom}(\mathbf{u})$ implies that $u_{pq} = q_2$ or $u_{pq} = 0$. In view of $m_{pq} \vee u_{pq} = p(q)$, this yields that $u_{pq} = q_2$. Therefore, $q_2 \in \mathrm{Atom}(\mathbf{u})$, and thus, $q_2 \notin \mathrm{Atom}(\mathbf{v}) - \mathrm{Atom}(\mathbf{u})$. Since $p_1, q_1 \in \mathrm{Atom}(\mathbf{v}) - \mathrm{Atom}(\mathbf{u})$ and $q_2 \in \mathrm{Atom}(\mathbf{u})$, we see that q splits over $(\mathbf{u}, \mathbf{v})$. Now $q_1 \notin \mathrm{Atom}(u)$ $\mathbf{s_{1960}} \leq \mathrm{Cut}_V(\mathbf{c})$ when restricted to C. Since $\mathbf{c}$ and $\mathrm{Cut}_C(\mathbf{c})$ are equal outside of C, we conclude that $\mathbf{s_{1960}} \leq \mathrm{Cut}_C(\mathbf{c})$. □

Combining the last two lemmas, we get the inequality $\mathbf{s_{1960}} \leq \mathbf{s}$. We prove the reverse inequality (completing the proof of Theorem 5-5.2) in the following statement.

Lemma 5-5.10. *Let* $\mathbf{c}$ *be a compatible vector for which* $\mathbf{s_{1960}} \leq \mathbf{c} \leq \mathbf{m}$*. Then* $\mathbf{c} = \mathbf{s_{1960}}$*.*

Proof. Let us assume that $\mathbf{c} = \mathbf{s_{1960}}$ fails, that is, $\mathbf{s_{1960}} < \mathbf{c}$. Then $(s_{1960})_{qr} < c_{qr}$, for some $q \succ r$ in P. So $c_{qr} > 0$. We consider three cases:

Case 1*:* $c_{qr} = q_1$*.*

Then $(s_{1960})_{qr} = 0$. Clearly, since $\mathbf{c} \leq \mathbf{m}$, we have that $\mathrm{Atom}(\mathbf{c}) \subseteq \mathrm{Atom}(\mathbf{m})$. Hence, $q_1 \in \mathrm{Atom}(\mathbf{m})$, and by (5-5.2), $q_1 \in \mathrm{Atom}(\mathbf{v}) - \mathrm{Atom}(\mathbf{u})$. However, $q_1 \notin \mathrm{Atom}(\mathbf{s_{1960}})$, so by (5-5.3), $q_1 \in \mathrm{NoSplit}(\mathbf{u}, \mathbf{v})$. Therefore, $p_1 \in \mathrm{Atom}(\mathbf{v}) - \mathrm{Atom}(\mathbf{u})$, for some $p \succ q$, and $q_2 \notin \mathrm{Atom}(\mathbf{v}) - \mathrm{Atom}(\mathbf{u})$.

Since $p_1, q_1 \in \mathrm{Atom}(\mathbf{v})$, clearly, $v_{pq} = p(q)$. Then $q_2 \in \mathrm{Atom}(\mathbf{v})$ and $q_2 \notin \mathrm{Atom}(\mathbf{v}) - \mathrm{Atom}(\mathbf{u})$, so $u_{pq} = q_2$. Therefore, $m_{pq} = p_1$. Since $c_{pq} \leq m_{pq} = p_1$, it follows that $c_{pq} \wedge q_1 = 0$. This contradicts that $\mathbf{c}$ is compatible, indeed, $c_{pq} \wedge q_1 = 0$ and $c_{qr} \wedge q_1 = q_1$ (since $c_{qr} = q_1$).

Case 2*:* $c_{qr} = r_i$*, for* $i = 1$ *or* 2*.*

Observe that if $m_{qr} \geq r$, then $r_1, r_2 \in \mathrm{Atom}(\mathbf{m})$, and it follows from (5-5.2) that $r_1, r_2 \in \mathrm{Atom}(\mathbf{v}) - \mathrm{Atom}(\mathbf{u})$. Hence, $r_1, r_2 \in \mathrm{Atom}(\mathbf{s_{1960}})$ by (5-5.3). Therefore, $(s_{1960})_{qr} \geq r$, which contradicts the assumption that $c_{qr} > (s_{1960})_{qr}$. So

$$m_{qr} = c_{qr} = r_i.$$

Then

$$c_{qr} \wedge r = r_i, \tag{5-5.5}$$

$$(\mathbf{s_{1960}})_{qr} = 0. \tag{5-5.6}$$

Hence, $r_i \in \mathrm{Atom}(\mathbf{m})$ and thus, by (5-5.2), $r_i \in \mathrm{Atom}(\mathbf{v}) - \mathrm{Atom}(\mathbf{u})$. However, $r_i \notin \mathrm{Atom}(\mathbf{s_{1960}})$, so r splits over $(\mathbf{u}, \mathbf{v})$ by (5-5.3). But since $q_1 \notin \mathrm{Atom}(\mathbf{v}) - \mathrm{Atom}(\mathbf{u})$, there exists a covering pair $p \succ r$ in P such that $p_1 \in \mathrm{Atom}(\mathbf{v}) - \mathrm{Atom}(\mathbf{u})$ but $r_{i+1} \notin \mathrm{Atom}(\mathbf{v}) - \mathrm{Atom}(\mathbf{u})$. Since $p_1, r_i \in \mathrm{Atom}(\mathbf{v})$, so $v_{pr} = p(r)$. Then $r_{i+1} \in \mathrm{Atom}(\mathbf{v})$ and $r_{i+1} \notin \mathrm{Atom}(\mathbf{v}) - \mathrm{Atom}(\mathbf{u})$, so $u_{pr} = r_{i+1}$. We conclude that $m_{pr} = p_1$. Therefore, $c_{pr} \leq m_{pr} = p_1$, implying that $c_{pr} \wedge r = 0$, contradicting that $\mathbf{c}$ is compatible and $c_{qr} \wedge r = r_i$ by (5-5.5).

Case 3*:* $c_{qr} \geq r$*.*

Since $r_1, r_2 \in \mathrm{Atom}(\mathbf{m})$, it follows from (5-5.2) that $r_1, r_2 \in \mathrm{Atom}(\mathbf{v}) - \mathrm{Atom}(\mathbf{u})$. So $r_1, r_2 \notin \mathrm{NoSplit}(\mathbf{u}, \mathbf{v})$. Hence $r_1, r_2 \in \mathrm{Atom}(\mathbf{s_{1960}})$, which implies that $(s_{1960})_{qr} \geq r$.

Since $m_{qr} \geq c_{qr} > (s_{1960})_{qr}$, there is only one possibility:

$$c_{qr} = m_{qr} = q(r), \tag{5-5.7}$$

$$(s_{1960})_{qr} = r. \tag{5-5.8}$$

So $q_1 \in \mathrm{Atom}(\mathbf{v}) - \mathrm{Atom}(\mathbf{u})$ and $q_1 \notin \mathrm{Atom}(\mathbf{s_{1960}})$, so we conclude that $q_1 \in \mathrm{NoSplit}(\mathbf{u}, \mathbf{v})$. Then, for some $p \succ q$,

$$p_1 \in \mathrm{Atom}(\mathbf{v}) - \mathrm{Atom}(\mathbf{u}),$$
$$q_2 \notin \mathrm{Atom}(\mathbf{v}) - \mathrm{Atom}(\mathbf{u}).$$

Since $p_1, q_1 \in \mathrm{Atom}(\mathbf{v})$, we conclude that $v_{pq} = p(q)$. Then $q_2 \in \mathrm{Atom}(\mathbf{v})$, but $q_2 \notin \mathrm{Atom}(\mathbf{v}) - \mathrm{Atom}(\mathbf{u})$, so $u_{pq} = q_2$ and $m_{pq} = p_1$. Therefore, $c_{pq} \leq m_{pq} = p_1$, implying that $c_{pq} \wedge q_1 = 0$, contradicting that $\mathbf{c}$ is compatible and $c_{qr} \wedge q_1 = q_1$ by (5-5.7).

Since each case leads to a contradiction and the three cases cover all possibilities, we conclude that $\mathbf{c} = \mathbf{s_{1960}}$. ☐

5-6. Exercises

5.1. Take a finite lattice L with unit, 1, and define $M = L - \{1\}$. Show that M is a chopped lattice.

5.2. Verify the converse of Exercise 5.1.

5.3. Prove Lemma 5-2.1.

5.4. Prove that if $a \vee b = c$ in $\mathrm{Merge}(C, D)$, then either $a, b, c \in C$ and $a \vee b = c$ in C or $a, b, c \in D$ and $a \vee b = c$ in D.

5.5. To prove that a binary relation on a lattice is a congruence is often facilitated by the following lemma (G. Grätzer and E.T. Schmidt [162] and F. Maeda [245]):

Lemma. *A reflexive binary relation $\boldsymbol{\alpha}$ on a lattice L is a congruence relation iff the following three properties are satisfied for any $x, y, z, t \in L$:*

(i) $x \equiv y \pmod{\boldsymbol{\alpha}}$ *iff* $x \wedge y \equiv x \vee y \pmod{\boldsymbol{\alpha}}$.

(ii) *Let $x \leq y \leq z$; then $x \equiv y \pmod{\boldsymbol{\alpha}}$ and $y \equiv z \pmod{\boldsymbol{\alpha}}$ imply that $x \equiv z \pmod{\boldsymbol{\alpha}}$.*

(iii) *$x \leq y$ and $x \equiv y \pmod{\boldsymbol{\alpha}}$ imply that $x \vee t \equiv y \vee t \pmod{\boldsymbol{\alpha}}$ and $x \wedge t \equiv y \wedge t \pmod{\boldsymbol{\alpha}}$.*

Verify this lemma.

5.6. Formulate and prove the lemma of Exercise 5.5 for chopped lattices.

5.7. Use the lemma of Exercise 5.5 to find a formula for the join of two congruences in chopped lattices.

5.8. Use the lemma of Exercise 5.5 to prove Lemma 5-2.2.

5.9. Prove Lemma 5-2.3.

5.10. For every integer n, find a chopped lattice M and ideals I and J of M, such that
$$\bigcup_{i \leq n} U(I, J)_i \subset I \vee J.$$

5.11. Using the notation of Section 5-2.2, let us call the vector $\mathbf{j}$ *balanced* if $j_m \wedge (m \wedge n) = j_n \wedge (m \wedge n)$ for all $m, n \in \mathrm{Max}$. Compare compatible and balanced vectors.

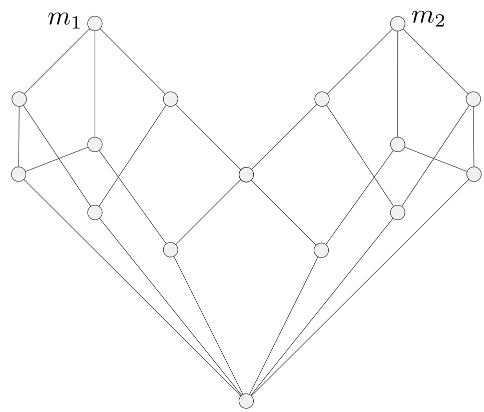

Figure 5-6.1: The chopped lattice G.

5.12. Let M be a chopped lattice. Let $\boldsymbol{\alpha}$ be a congruence relation of M and let $I, J \in \operatorname{Id} M$. Define

$$I \equiv J\,(\overline{\boldsymbol{\alpha}}) \quad \text{iff} \quad I/\boldsymbol{\alpha} = J/\boldsymbol{\alpha}.$$

Prove that $\overline{\boldsymbol{\alpha}}$ is a congruence relation of $\operatorname{Id} M$. (Hint: to prove ($\mathrm{SP}_\vee$), use (5-2.1).)

5.13. Under the conditions of Exercise 5.12, let $\boldsymbol{\beta}$ be a congruence relation of $\operatorname{Id} M$ satisfying $\operatorname{id}(a) \equiv \operatorname{id}(b) \pmod{\boldsymbol{\beta}}$ iff $a \equiv b \pmod{\boldsymbol{\alpha}}$. Show that $\boldsymbol{\beta} = \overline{\boldsymbol{\alpha}}$, thereby verifying Theorem 5-2.6.

5.14. Define the concept of chopped lattices without requiring finiteness. Find a variant for which Theorem 5-2.6 holds.

5.15. Consider the chopped lattice G of Figure 5-6.1. G has two maximal elements m_1 and m_2. The lattices $\operatorname{id}(m_1)$ and $\operatorname{id}(m_2)$ are sectionally complemented (and $\operatorname{id}(m_1) \cong \operatorname{id}(m_2)$). However, unlike in the Atom Lemma, the element $p = m_1 \wedge m_2 \in G$ is not an atom. Verify that $\operatorname{Id} M$ is not sectionally complemented. (G. Grätzer, H. Lakser, and M. Roddy [151].)

5.16. Define formally the chopped lattice M of Figure 5-3.6.

5.17. Describe the congruences of the chopped lattice M of Exercise 5.16. Which congruences are join-irreducible?

5.18. Prove that for the chopped lattice M of Exercise 5.16 the isomorphism $\operatorname{Ji}(\operatorname{Con} M) \cong P$ holds.

5.19. Do we need in Definition 5-4.1 a "maximal sectional complement"? What happens if we drop "maximal"?

5.20. Why is the concept of "H-failure" not needed?

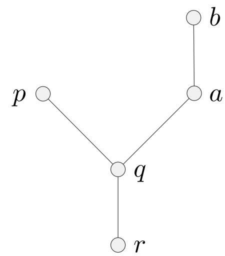

Figure 5-6.2: Case 2 of the C-compatibility proof.

5.21. Verify that the assumption in Case 1 of the proof of Lemma 5-4.5 leads to a contradiction.

5.22. Verify that the assumptions in Cases 1–3 of the proof of Lemma 5-4.6 lead to contradictions.

5.23. In the V-compatibility proof of Theorem 5-3.2, list and verify the other seven cases.

5.24. In the C-compatibility proof of Theorem 5-3.2, list and verify the other eight cases. How many times do we use the "minimal C-failure" condition in Step 4 of the Algorithm in the proofs?

5.25. In the H-compatibility proof of Theorem 5-3.2, list and verify the other eight cases.

5.26. In the Algorithm drop "minimal" in Step 4. Does the Algorithm still work?

5.27. To visualize the many cases in the compatibility proofs, it is useful to draw the diagrams for each subcase.

For instance, in Case 2 of the C-compatibility proof, we can illustrate the clause:

"Assume that $s^*_{aq} \in \{0, q\}$ as the result of a $C(b, a, q)$-cut of $\mathbf{c}$ at $b \succ a \succ q$ at some implementation of Step 4."

with the diagram of Figure 5-6.2. Draw diagrams to illustrate all the steps of the compatibility proofs.

5.28. Is there a congruence representation theorem for finite modular lattices?

5.29. Can Theorem 5-3.1 be improved by requiring that the lattice L be relatively complemented?

5.30. Can Theorem 5-3.1 be improved by requiring that the lattice L be sectionally complemented and dually sectionally complemented? (Hint: No. For a finite sectionally complemented and dually sectionally complemented lattice L, the congruence lattice is always Boolean. See G. Grätzer and E.T. Schmidt [164] and M.F. Janowitz [206].)

5.31. Prove that every finite, semimodular, complemented lattice has a boolean congruence lattice. (Hint: Use Theorem 279 of LTF – see also the discussion in the bottom paragraph on p. 348.)

5-7. Problems

Problem 5.1. Investigate generalizations of Theorem 5-2.6 to the infinite case.

See G. Grätzer and E.T. Schmidt [171] for some related results.

As we note in Section 5-2.2, every finite chopped lattice M decomposes into lattices: $M = \bigcup(\operatorname{id}(m) \mid m \in \mathrm{Max})$.

Problem 5.2. Can chopped lattices, in general, be usefully decomposed into lattices? Could this be utilized in Problem 5.1 by assuming that the chopped lattices decompose into *finitely many* lattices or into lattices with nice properties?

Theorem 5-3.3 is interesting because the property of being sectionally complemented is not inherited, in general, when passing from a chopped lattice M to $\operatorname{Id} M$. (See Exercise 5.15.)

Let **SecComp** denote the class of sectionally complemented lattices and let **SemiMod** denote the class of semimodular lattices.

For a class of lattices $\mathbf{K}$, let $\operatorname{Chop}\mathbf{K}$ denote the class of chopped lattices M with the property that $\operatorname{id}(m) \in \mathbf{K}$ for all $m \in M$. So $\operatorname{Chop}\mathbf{SecComp}$ is what we call the class of sectionally complemented chopped lattices. Similarly, we could look at $\operatorname{Chop}\mathbf{SemiMod}$, and call its members semimodular chopped lattices.

Problem 5.3. Characterize finite semimodular chopped lattices.

Problem 5.4. When is the ideal lattice of a finite semimodular chopped lattice again semimodular?

Let us consider the following property of a class $\mathbf{K}$ of lattices: If $M \in \operatorname{Chop}\mathbf{K}$, then $\operatorname{Id} M \in \mathbf{K}$. Let us call such a class Chop-Id *closed*.

Problem 5.5. Are there any nontrivial Chop-Id closed varieties?

For a natural number n and a class $\mathbf{V}$ of lattices, define $\mathbf{mcr}(n, \mathbf{V})$ (**m**inimal **c**ongruence **r**epresentation) as the smallest integer such that, for any distributive lattice D with n join-irreducible elements, there exists a *finite* lattice $L \in \mathbf{V}$ satisfying $\operatorname{Con} L \cong D$ and $|L| \leq \mathbf{mcr}(n, \mathbf{V})$.

We know that $\mathbf{mcr}(n, \mathbf{SecComp}) \leq 2^{2n}$.

Problem 5.6. Is $\mathbf{mcr}(n, \mathbf{SecComp}) \leq 2^{2n}$ the best possible?

For any class $\mathbf{S}$ of lattices, if the congruence representation theorem holds for $\mathbf{S}$, then theoretically, the function $\mathbf{mcr}(n, \mathbf{S})$ exists, although it may be difficult to compute.

Of course, for any class $\mathbf{S}$ of lattices for which the congruence representation theorem holds, we can raise the question of what is $\mathbf{mcr}(n, \mathbf{S})$, and chances are that we get an interesting problem. In many cases, however, the congruence representation theorem fails for $\mathbf{S}$. Four examples are given in Exercises 5.28–5.31.

Problem 5.7. Is there a natural subclass $\mathbf{S}$ of $\mathbf{SecComp}$ for which Theorem 5-3.1 holds, that is, every finite distributive lattice D can be represented as the congruence lattice of a lattice $L \in \mathbf{S}$?

Since the congruence lattice of every finite modular lattice is Boolean, we do not have for this class the congruence representation theorem. However, there are many results on countably infinite modular lattices, see the full discussion in Chapter 10 of [131].

Problem 5.8. Is there a congruence representation theorem for countably infinite

(i) relatively complemented lattices;

(ii) sectionally complemented and dually sectionally complemented lattices;

(iii) semimodular complemented lattices.

Of course, for (iii), it is not even clear what semimodularity should mean. Even for the Atom Lemma many questions are unanswered.

Problem 5.9. Let M, m_1, and m_2 be as in Lemma 5-2.7. Under what conditions do we get that the method of proof finds all the sectional complements?

Problem 5.10. Let M be a finite sectionally complemented chopped lattice. Find reasonable sufficient conditions under which $\operatorname{Id} M$ is sectionally complemented.

The Algorithm produces a sectional complement under the assumption of the Atom Lemma and the 1960 construction.

Problem 5.11. Is there a natural class of sectionally complemented finite chopped lattices, different from these two classes, for which the Algorithm produces a sectional complement?

Problem 5.12. Is there a direct proof (not utilizing Theorem 5-4.9) that the set of sectional complements found by the Algorithm is a singleton, in fact, it is $\{\mathbf{s_{1960}}\}$?

The Algorithm is different from the crude algorithm (Algorithm) described in Section 5-4.1: it takes an arbitrary sectional complement (not necessarily a maximal one) and it cuts wherever there is a failure.

Problem 5.13. Does the Algorithm always find a sectional complement?

Problem 5.14. How many sectional complements does the Algorithm find?

Problem 5.15. What about the Algorithm that starts with maximal sectional complements?

Problem 5.16. Are there different algorithms that find other sectional complements?

Chapter 6

Combinatorics in finite lattices

by Joseph P.S. Kung

6-1. Introduction

Combinatorial or counting problems in lattices were asked as soon as lattices were discovered. In one of the founding papers of lattice theory, [62], Richard Dedekind asked for the number of elements in the free distributive lattice with n generators. This question has no nice answer in that no closed formula, not even a recursion, seems to exist, although good asymptotic lower and upper bounds are known (see Exercise 6.19). Later, in the 1930's, P. Hall [191] and L. Weisner [322] independently derived formulas for the number of n-tuples of elements generating a finite group. These formulas involved an inclusion-exclusion or Möbius inversion argument over the lattice of subgroups. Their work initiated the theory of Möbius functions in *finite* partially ordered sets. We shall give a *very brief* introduction to this theory in Section 2. This theory is then used in the next four sections. In Section 3, we discuss the existence of complementing permutations, and in Section 4, we obtain matching proofs of rank and covering inequalities in modular and geometric lattices. We discuss Eulerian functions in Section 5 and characteristic polynomials of geometric lattices in Section 6. A major source of combinatorial ideas is Sperner's theorem, that a largest antichain in a finite Boolean algebra occurs at its middle levels. We shall discuss attempts to extend Sperner's theorem to geometric lattices in Section 7.

G. Grätzer, F. Wehrung (eds.), *Lattice Theory: Special Topics and Applications: Volume 1*, DOI 10.1007/978-3-319-06413-0_6,

Lattices and partially ordered sets are indispensable in many areas of algebraic and enumerative combinatorics. An excellent example is the use of lattices in the combinatorics of hyperplane arrangements described in a chapter of volume 2 by Nathan Reading.

6-2. Möbius functions

Let P be a (finite) partially ordered set and $\mathbb{A}$ be a commutative ring with identity. Let Z be the incidence matrix of the order relation, that is, Z is the matrix with rows and columns indexed by P and entries $\zeta(x,y)$ defined by

$$\zeta(x,y) = \begin{cases} 1 & \text{if } x \leq y, \\ 0 & \text{otherwise.} \end{cases}$$

If we extend P to a total or linear order and use the extension to order the row and column indices, then Z is an upper triangular square matrix with all diagonal entries equal to 1. Thus, Z has an inverse M. The entries $\mu(x,y)$ of the inverse M define the Möbius function of P. Using the fact that MZ is the identity matrix, we obtain the following explicit definition: the *Möbius function* $\mu\colon P \times P \to \mathbb{A}$ is the function defined by

$$\begin{aligned} \mu(x,x) &= 1, \\ \mu(x,y) &= 0 \ \text{ if } x \not\leq y, \\ \sum_{z\colon x \leq z \leq y} \mu(x,z) &= 0 \ \text{ if } x < y. \end{aligned}$$

From the definition, we obtain a recursion, going up P: $\mu(x,x) = 1$ and if $x < y$, then

$$\mu(x,y) = -\sum_{z\colon x \leq z < y} \mu(x,z). \tag{UR}$$

An equivalent definition for μ can be obtained using the fact that ZM is the identity matrix. When we do this, the third equation in the initial definition changes to

$$\sum_{z\colon x \leq z \leq y} \mu(z,y) = 0 \ \text{ if } x < y.$$

This yields a recursion going down P: $\mu(x,x) = 1$ and if $x < y$, then

$$\mu(x,y) = -\sum_{z\colon x < z \leq y} \mu(z,y). \tag{DR}$$

Theorem 6-2.1 (The Möbius inversion formula)**.** *Let P be a partially ordered set and f and g be functions from P to $\mathbb{A}$. Then*

$$f(x) = \sum_{y:\, y \leq x} g(y) \text{ for all } x \in P \iff g(x) = \sum_{y:\, y \leq x} f(y)\mu(y,x) \text{ for all } x \in P.$$

Dually,

$$f(x) = \sum_{y:\, y \geq x} g(y) \text{ for all } x \in P \iff g(x) = \sum_{y:\, y \geq x} \mu(x,y)f(y) \text{ for all } x \in P.$$

Proof. Think of f and g as row vectors indexed by P. Then the first part of the theorem says $f = gZ$ if and only if $g = fM$. For the second part, think of f and g as column vectors. □

From recursion (UR) or (DR), it is immediate that if y covers x, then $\mu(x,y) = -1$. Further, if C is the chain $x_0 < x_1 < \cdots < x_n$, then the Möbius function in C is given by

$$\mu(x_0, x_0) = 1,\ \mu(x_0, x_1) = -1, \text{ and } \mu(x_0, x) = 0 \text{ if } x \neq x_0, x_1.$$

Another easy consequence of the recursions is the following lemma.

Lemma 6-2.2 (The product formula)**.** *Let P and Q be partially ordered sets. Then $\mu_{P\times Q}((x,u),(y,v)) = \mu_P(x,y)\mu_Q(u,v)$.*

Finite Boolean algebras are products of chains of length 1. Thus, for subsets A and B of S such that $A \subseteq B$,

$$\mu(A,B) = (-1)^{|B|-|A|}$$

in the Boolean algebra of all subsets of S. If n is a positive integer having prime factorization $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$, then its lattice of divisors under divisibility is isomorphic to a product of k chains, the first having length α_1, the second having length α_2, and so on. Further, if m divides n, then the interval $[m,n]$ is isomorphic to the lattice of divisors of n/m. Hence, if m does not divide n, $\mu(m,n) = 0$ and if m divides n, $\mu(m,n) = \mu(1, n/m) = \mu(n/m)$, where $\mu(k)$ is the single-variable number-theoretic Möbius function defined by

$$\mu(k) = \begin{cases} (-1)^r & \text{if } k \text{ is the product of } r \text{ distinct primes,} \\ 0 & \text{otherwise.} \end{cases}$$

For Boolean algebras and lattices of divisors, Theorem 6-2.1 specializes to two classic inversion formulas. One is the principle of inclusion and exclusion:

$$\begin{aligned} f(A) &= \sum_{B:\, B \subseteq A} g(B) \text{ for all } A \subseteq S \\ &\iff g(A) = \sum_{B:\, B \subseteq A} (-1)^{|A|-|B|} f(B) \text{ for all } A \subseteq S. \end{aligned}$$

The other is the number-theoretic inversion formula due to A.F. Möbius:

$$f(n) = \sum_{d:\, d \text{ divides } n} g(d) \text{ for all } n \iff g(n) = \sum_{d:\, d \text{ divides } n} f(d)\mu(n/d) \text{ for all } n.$$

We next present results useful for calculating Möbius functions in lattices. These results show a surprisingly close relation between values of the Möbius function and the structure of a lattice.

Recall from LTF, p. 47, that a *closure system* on a partially ordered set P is a function or unary operation $P \to P$, $x \mapsto \overline{x}$, satisfying three properties: $x \leq \overline{x}$, $x \leq y \Rightarrow \overline{x} \leq \overline{y}$, and $\overline{x} = \overline{\overline{x}}$. An element x in P is *closed* if $x = \overline{x}$. Under the order restricted from P, the closed elements form a partial order $\operatorname{Cld} P$.

Theorem 6-2.3. *Let $x \mapsto \overline{x}$ be a closure system on a partially ordered set P and* $\operatorname{Cld} P$ *be the suborder of closed elements. If x is an element of P and y is a closed element, then*

$$\sum_{z:\, \overline{z}=y} \mu_P(x,z) = \begin{cases} \mu_{\operatorname{Cld} P}(x,y) & \text{if } x \text{ is closed,} \\ 0 & \text{otherwise.} \end{cases}$$

Proof. We use the fact that the interval $[x,y]$ in P is partitioned into the subsets $\{z : \overline{z} = u\}$, where u ranges over all closed elements in $[x,y]$. Hence,

$$\begin{aligned} 0 &= \sum_{z:\, z \in [x,\overline{y}] \text{ in } P} \mu_P(x,z) \\ &= \sum_{u:\, u \in [x,\overline{y}] \text{ in } \operatorname{Cld} P} \left[\sum_{z:\, z \in P,\, \overline{z}=u} \mu_P(x,z) \right]. \end{aligned}$$

There are two cases. Consider first the case when x is closed. We induct on the length of the longest chain from x to y in $\operatorname{Cld} P$. If the length is zero, then $x = y$, $\overline{z} = x$ if and only if $z = x$, and $\mu_{\operatorname{Cld} P}(x,x) = 1 = \mu_P(x,z)$. By induction, we may assume that the theorem holds for all elements u in $\operatorname{Cld} P$ such that $x \leq u < y$, that is, for all such elements u,

$$\sum_{z:\, z \in [x,y] \text{ in } P,\, \overline{z}=u} \mu_P(x,z) = \mu_{\operatorname{Cld} P}(x,u).$$

Thus,

$$0 = \sum_{z:\, z \in P,\, \overline{z}=y} \mu_P(x,z) \;+ \sum_{u:\, u \in \operatorname{Cld} P,\; x \leq u < y} \mu_{\operatorname{Cld} P}(x,u).$$

Hence the sum

$$\sum_{z:\, z \in P,\, \overline{z}=y} \mu_P(x,z)$$

satisfies the recursion (UR) for $\mu_{\mathrm{Cld}\,P}(x,y)$ and hence the two quantities are equal.

Consider now the case when $x < \overline{x}$. Since every element z in $[x, \overline{x}]$ has closure $\overline{x}$, the base case is

$$\sum_{z\colon z \in P,\, x \leq z \leq \overline{x}} \mu_P(x,z) = \sum_{z\colon z \in P,\, z \in [x,\overline{x}]} \mu_P(x,z) = 0.$$

Induction now yields

$$\sum_{z\colon z \in P,\, \overline{z} = y} \mu_P(x,z) = 0.$$

Since this sum and $\mu_{\mathrm{Cld}\,P}(x,y)$ satisfy the same recursion, we conclude that $\mu_{\mathrm{Cld}\,P}(x,y) = 0$. □

There are two useful special cases of Theorem 6-2.3. If a is a fixed element in a lattice L, then $x \mapsto x \vee a$ is a closure system on L.

Corollary 6-2.4 (Weisner's theorem). *Let L be a lattice and a be an element in L. Then if $a \neq \hat{0}$,*

$$\mu(\hat{0}, \hat{1}) \;=\; - \sum_{x\colon x \vee a = \hat{1},\, x \neq \hat{1}} \mu(\hat{0}, x).$$

Dually, if $a \neq \hat{1}$,

$$\mu(\hat{0}, \hat{1}) \;=\; - \sum_{x\colon x \wedge a = \hat{0},\, x \neq \hat{0}} \mu(x, \hat{1}).$$

Using Weisner's theorem, we calculate the Möbius function of two lattices. The first is the lattice $\mathrm{Part}(n)$ of partitions of the set $\{1, 2, \ldots, n\}$ with n elements ordered by reverse refinement, so that the maximum $\hat{1}$ is the partition with one block and the minimum $\hat{0}$ is the partition with n blocks. We recall several facts about intervals. Let π be the partition into blocks $B_1, B_2, \ldots, B_m$, where $|B_i| = b_i$. If $\sigma \leq \pi$, then σ is obtained by partitioning each block B_i of π further into subblocks. Let $m_i^{\sigma,\pi}$ be the number of subblocks in σ into which the block B_i is partitioned. Then the interval $[\sigma, \pi]$ is isomorphic to the direct product

$$\prod_{i=1}^{m} \mathrm{Part}(m_i^{\sigma,\pi}).$$

For example, the upper interval $[\pi, \hat{1}]$ is isomorphic to $\mathrm{Part}(m)$ and the lower interval $[\hat{0}, \pi]$ is isomorphic to $\prod_{i=1}^{m} \mathrm{Part}(b_i)$.

Choose α to be the partition with 2 blocks $\{1, 2, \ldots, n-1\}, \{n\}$. Then $\pi \neq \hat{1}$ and $\pi \vee \alpha = \hat{1}$ if and only if π has one 2-element block $\{n, i\}$ and $n-2$ 1-element blocks $\{j\}$, where $j \neq i, n$. There are $n-1$ such partitions and for each partition, $[\pi, \hat{1}]$ is isomorphic to a partition lattice on $n-1$ elements.

Hence,

$$\mu_n = \mu(\hat{0}, \hat{1}) = \sum_{\pi} \mu(\pi, \hat{1}) = -(n-1)\mu_{n-1},$$

and by induction, $\mu_n = (-1)^{n-1}(n-1)!$. We can now obtain all the values of the Möbius function from Lemma 6-2.2. For example, if π is a partition of a set of size n with m blocks B_i, with block B_i having size b_i, then

$$\mu(\hat{0}, \pi) = (-1)^{n-m} \prod_{j=1}^{m} (b_j - 1)!\,.$$

We next consider *subspace lattices.* Let $L(n, q)$ be the lattice of subspaces of a vector space V of dimension n over the finite field $\mathrm{GF}(q)$ of order q. The minimum of $L(n, q)$ is the zero subspace 0 containing exactly the zero vector. Choose A to be a subspace of dimension $m-1$ in V. Then a subspace X intersects A at 0 if and only if $\dim X = 1$ and every nonzero vector in X is not in A. As each subspace X contains $q-1$ nonzero vectors and each nonzero vector spans a unique 1-dimensional subspace, the number of such subspaces X equals

$$\frac{|V| - |A|}{q-1} = \frac{q^n - q^{n-1}}{q-1} = q^{n-1}.$$

Hence, by Weisner's theorem,

$$\mu(0, V) = - \sum_{X\colon X \cap A = 0,\, X \neq 0} \mu(X, V) = q^{n-1}\mu(X, V).$$

To finish, we use the fact that the upper interval $[X, V]$ is isomorphic to $L(n-1, q)$. Hence, by induction on n, we conclude that

$$\mu(0, V) = (-1)^m q^{0+1+2+\cdots+(m-1)} = (-1)^m q^{\binom{m}{2}}.$$

The second useful case of Theorem 6-2.3 is the case of a closure system on a set, that is, a closure system on the Boolean algebra of subsets of a set. The closed sets form a lattice.

Corollary 6-2.5. *Let $A \mapsto \overline{A}$ be a closure system on the set S such that the empty set is closed, and L be the lattice of closed sets. If U is a closed set, then*

$$\mu_L(\varnothing, U) = \sum_{A\colon \overline{A} = U} (-1)^{|A|}.$$

The next consequence of Theorem 6-2.3 allows us to conclude quickly that some values of the Möbius function are zero.

Corollary 6-2.6 (P. Hall)**.** *Let L be a lattice. Then $\mu(\hat{0},\hat{1}) = 0$ unless the meet of all the coatoms is $\hat{0}$ and the join of all the atoms is $\hat{1}$.*

Proof. In a lattice, the function

$$x \mapsto \bigwedge\{c : c \text{ is a coatom and } c \geq x\}$$

is a closure system in which the minimum $\hat{0}$ is closed if and only if it is a meet of coatoms. Hence by Theorem 6-2.3, $\mu(\hat{0},\hat{1}) = 0$ if $\hat{0}$ is not a meet of coatoms. To finish the proof, use the dual argument. □

Theorem 6-2.7 (P. Hall)**.** *Let $x < y$ in a partially ordered set P. Then*

$$\mu(x,y) = -c_1 + c_2 - c_3 + c_4 - \cdots,$$

where c_i is the number of length-i chains $x < x_1 < x_2 < \cdots < x_{i-1} < y$ stretched from x to y.

Proof. Let $H = Z - I$, where I is the identity matrix. Then H is an upper triangular matrix with all diagonal entries equal to 0, and hence, H is nilpotent. Thus, M can be expanded as a finite sum:

$$M = Z^{-1} = (I + H)^{-1} = I - H + H^2 - H^3 + H^4 - \cdots.$$

We can now finish the proof by noting that if $x < y$, then the xy-entry of H^i is c_i. □

Theorem 6-2.7 suggests that Möbius functions have homological interpretations. This has led to an intensive area of research. We refer the reader to the survey [31].

The next result is a variation on Theorem 6-2.3. Let $f\colon P \to Q$ be an order-preserving function. Thinking of a as a constant and z as a variable, let $[a,z]$ be an interval in P such that $f(a) < f(z)$. Define $[a,z]_f$ to be the set

$$[a,z]_f = \{x \in [a,z] : f(x) < f(z)\} \cup \{z\},$$

ordered as a subset of $[a,b]$ and $\mu_f(a,z)$ to be the Möbius function evaluated at (a,z) on the partially ordered set $[a,z]_f$.

Theorem 6-2.8. *Let a and b be elements in P such that $a < b$ and $f(a) < f(b)$. Then the following two equations hold:*

(M1) $$\mu(a,b) = \sum_{z\colon z \in [a,b],\, f(z) = f(b)} \mu_f(a,z)\mu(z,b),$$

(M2) $$\mu_P(a,b) = \sum_{y,z\colon y,z \in [a,b],\ f(y) = f(z) = f(b)} \mu(a,y)\zeta(y,z)\mu(z,b).$$

Proof. We prove (M1) using a chain-counting argument. Let R be a partially ordered set with a minimum $\hat{0}$ and a maximum $\hat{1}$. We define the *chain counting polynomial* $C(R;\lambda)$ (in the variable λ) by

$$C(R;\lambda) = \sum_{i:\, i \geq 0} c_i(R)\lambda^i,$$

where $c_i(R)$ is the number of chains of length i stretched from $\hat{0}$ to $\hat{1}$ in R. By Theorem 6-2.7, $C(R;-1) = \mu(\hat{0},\hat{1})$ in R.

Consider an interval $[a,b]$ in P such that $f(a) < f(b)$. Each chain stretched from a to b in $[a,b]$ can be divided into two nonempty segments,

$$a < z_1 < \cdots < z_{i-1} < z_i \text{ and } z_i < z_{i+1} < \cdots < b,$$

where $f(z_{i-1}) \neq f(b)$ and $f(z_i) = f(z_{i+1}) = \cdots = f(b)$. Thus,

$$C([a,b];\lambda) = \sum_{z:\, z \in [a,b],\, f(z)=f(b)} C([a,z]_f;\lambda)C([z,b];\lambda).$$

Setting $\lambda = -1$, we obtain (M1).

Let T be the set $\{z : z \in P, f(z) = f(b)\}$, partially ordered as a subset of P. The proof of (M1) is valid if we replace b by an element z in T, that is, if $z \in T$, then

$$\mu(a,z) = \sum_{y:\, y \in T,\, y \leq z} \mu_f(a,y)\mu(y,z). \tag{C}$$

Regarding $\mu_f(a,z)$ and $\mu(a,z)$ as functions of z defined on T, equation (C) is equivalent by Möbius inversion (Theorem 6-2.1) to

$$\mu_f(a,z) = \sum_{y:\, y \in T,\, y \leq z} \mu(a,y) = \sum_{y:\, y \leq z,\, f(y)=f(z)} \mu(a,y). \tag{D}$$

To finish the proof, we use equation (D) to substitute $\mu_f(a,z)$ into (M1), obtaining the equation

$$\mu(a,b) = \sum_{z:\, z \in [a,b],\, f(z)=f(b)} \left(\sum_{y:\, y \leq z,\, f(y)=f(z)} \mu(a,y) \right) \mu(z,b).$$

This equation can easily be converted to (M2). □

Recall that in a lattice, c is a *complement* of a if $c \wedge a = \hat{0}$ and $c \vee a = \hat{1}$.

Theorem 6-2.9 (Crapo's complementation theorem)**.** *Let L be a lattice, a be any element in L, and $a^{\perp}$ the set of complements of a in L. Then*

$$\mu(\hat{0}, \hat{1}) = \sum_{c,d\colon c,d \in a^{\perp}} \mu(\hat{0}, c)\zeta(c, d)\mu(d, \hat{1}),$$

where the sum is over all ordered pairs (c, d) in $a^{\perp} \times a^{\perp}$ (so that $c = d$ is allowed).

Proof. The theorem holds trivially if $a = \hat{0}$. Thus we can assume $a > \hat{0}$. We apply Theorem 6-2.8 to the order-preserving function $f\colon L \to [a, \hat{1}]$, $x \mapsto x \vee a$, obtaining

$$\mu(\hat{0}, \hat{1}) = \sum_{z\colon z \vee a = \hat{1}} \mu_f(\hat{0}, z)\mu(z, \hat{1}). \tag{N}$$

To finish the proof, we will show that $\mu_f(\hat{0}, z) = 0$ unless $z \wedge a = \hat{0}$. Suppose $a \vee z = \hat{1}$. The infimum of two elements in the partial order $[\hat{0}, z]_f$ is the same as their infimum in the lattice $[\hat{0}, z]$. Thus, $[\hat{0}, z]_f$ is a lattice. Let m be a coatom in $[\hat{0}, z]_f$. Then, as $m \neq z$, $m \vee a < \hat{1}$. Further,

$$[m \vee (z \wedge a)] \vee a = m \vee [(z \wedge a) \vee a] = m \vee a < \hat{1}.$$

Hence, $m \vee (z \wedge a)$ is in $[\hat{0}, z]_f$ and does not equal z. Since m is a coatom, it follows that $m \vee (z \wedge a) = m$, that is to say, $m \geq z \wedge a$. We conclude that $z \wedge a$ is a lower bound for all coatoms in $[\hat{0}, z]_f$. By Corollary 6-2.6, $\mu_f(\hat{0}, z) = 0$ unless the meet of all the coatoms is $\hat{0}$, or $z \wedge a = \hat{0}$. We can now restrict the sum in equation (N) to those z such that $z \wedge a = \hat{0}$ (as well as $z \vee a = \hat{1}$). We can now finish the proof using equation (D) as in the proof of Theorem 6-2.8. $\square$

Corollary 6-2.10. *Let L be a lattice. Suppose there exists an element in L that does not have a complement. Then $\mu(\hat{0}, \hat{1}) = 0$.*

6-3. Complements and determinants

In this section, we discuss determinants involving Möbius functions. A *meet-semilattice* (respectively, *join-semilattice*) is a partially ordered set in which the infimum $x \wedge y$ (respectively, the supremum $x \vee y$) of any two elements exists. The existence of a maximum (respectively, minimum) is not assumed and so P is not necessarily a lattice.

The following elegant theorem is due independently to B. Lindström [243] and H.S. Wilf [342].

Theorem 6-3.1. *Let P be a meet-semilattice, $\mathbb{A}$ be a commutative ring with identity, and $F\colon P \times P \to \mathbb{A}$ be a function. Then*

$$\det[F(x \wedge y, y)]_{x,y \in P} = \prod_{x\colon x \in P} \left(\sum_{z\colon z \leq x} F(z, x)\mu(z, x) \right).$$

Proof. Define $f(x,y)$ by one of the following equivalent conditions:

$$f(x,y) = \sum_{z\colon z \le x} F(z,y)\mu(z,y) \quad \text{or} \quad F(x,y) = \sum_{z\colon z \le x} f(z,y).$$

Let H be the upper-triangular matrix with x,y-entry $f(x,y)\zeta(x,y)$. Then,

$$\det(Z^T H) = (\det Z)(\det H) = \prod_{x\colon x \in P} f(x,x) = \prod_{x\colon x \in P} \left(\sum_{z\colon z \le x} F(z,x)\mu(z,x) \right).$$

Next we calculate the xy-entry of $Z^T H$:

$$\begin{aligned}\sum_{z\colon z \in P} \zeta(z,x) f(z,y)\zeta(z,y) &= \sum_{z\colon z \le x \text{ and } z \le y} f(z,y) \\ &= \sum_{z\colon z \le x \wedge y} f(z,y) \\ &= F(x \wedge y, y).\end{aligned}$$

The determinant formula now follows. □

Corollary 6-3.2. *Let P be a join-semilattice. Then*

$$\det[F(y, x \vee y)]_{x,y \in P} = \prod_{x\colon x \in P} \left(\sum_{z\colon z \ge x} \mu(x,z) F(x,z) \right).$$

Theorem 6-3.1 generalizes a classical number-theoretic identity. The set $\{1, 2, \ldots, n\}$ (of *all* integers between 1 and n) ordered by divisibility is a semilattice because the infimum $i \wedge j$ exists and equals the greatest common divisor $\gcd(i,j)$.

Corollary 6-3.3. *Let $\varphi(n)$ be the number of integers i, $1 \le i \le n$, relatively prime to n. Then*

$$\det[\gcd(i,j)]_{1 \le i,j \le n} = \prod_{i=1}^{n} \varphi(i).$$

Proof. Let $F(i,j) = i$. Then, by Theorem 6-3.1,

$$\det[\gcd(i,j)]_{1 \le i,j \le n} = \prod_{i=1}^{n} \left(\sum_{j\colon j \text{ divides } i} j\mu(i/j) \right).$$

We can now finish the proof using the fact that $n = \sum_{j\colon j \text{ divides } n} \varphi(j)$, and hence, by Möbius inversion, $\varphi(n) = \sum_{j\colon j \text{ divides } n} j\mu(n/j)$. □

Corollary 6-3.4.

(i) *Let P be a meet-semilattice such that $\mu(\hat{0}, x) \neq 0$ for all x in P. Then there exists a permutation $\sigma\colon P \to P$ such that $x \wedge \sigma(x) = \hat{0}$ for all $x \in P$.*

(ii) *Let P be a join-semilattice such that $\mu(x, \hat{1}) \neq 0$ for all x in P. Then there exists a permutation $\tau\colon P \to P$ such that $x \vee \tau(x) = \hat{1}$ for all $x \in P$.*

Proof. In Theorem 6-3.1, let $\mathbb{A}$ be the ring $\mathbb{Z}$ of integers, $F(x, y) = \delta(x, \hat{0})$, the function that equals 1 if $x = \hat{0}$ and 0 if $x \neq \hat{0}$ and c_{xy} be the xy-entry in the matrix in Theorem 6-3.1. Then $c_{xy} = 1$ if $x \wedge y = \hat{0}$ and $c_{xy} = 0$ otherwise. Expanding the determinant and using Theorem 6-3.1, we have

$$\sum_{\pi\colon \pi \text{ is a permutation}} \prod_{x\colon x \in P} c_{x,\pi(x)} = \det[c_{xy}] = \prod_{x\colon x \in P} \mu(\hat{0}, x) \neq 0.$$

We conclude that there is at least one nonzero term in the sum. A permutation σ giving a nonzero term yields a permutation with the required property. The second statement is the order dual of the first. □

Corollary 6-3.4 was sharpened by T.A. Dowling [72].

Theorem 6-3.5. *Let L be a lattice such that $\mu(\hat{0}, x) \neq 0$ and $\mu(x, \hat{1}) \neq 0$ for all x in L. Then there exists a complementing permutation $\sigma\colon P \to P$, that is, a permutation σ such that x and $\sigma(x)$ are complements.*

Proof. We will present a proof of J. van Lint and R.M. Wilson [244]. Let $\mathbb{A} = \mathbb{Z}$, D_0 be the diagonal matrix with xx-entry equal to $\mu(\hat{0}, x)$, and D_1 be the diagonal matrix with xx-entry equal to $\mu(x, \hat{1})$. Consider the matrix $Z^T D_0 Z$. We calculate its xy-entry:

$$\sum_{z\colon z \in L} \zeta(z, x)\mu(\hat{0}, z)\zeta(z, y) = \sum_{z\colon z \leq x \wedge y} \mu(\hat{0}, z) = \delta(\hat{0}, x \wedge y).$$

From this, we conclude that

$$Z^T D_0 Z = [\delta(\hat{0}, x \wedge y)].$$

Using a dual argument, we conclude that

$$Z D_1 Z^T = [\delta(\hat{1}, x \vee y)].$$

Now consider the product $Z D_1 Z^T D_0 Z$. Writing this product as $(Z D_1 Z^T) D_0 Z$, its xy-entry is the sum

$$\sum_{z\colon x \vee z = \hat{1}} \mu(\hat{0}, z)\zeta(z, y).$$

Since $x \vee z \leq x \vee y$ if $z \leq y$, the sum is empty unless $x \vee y = \hat{1}$. Hence, if the xy-entry of $(ZD_1Z^T)D_0Z$ is nonzero, then $x \vee y = \hat{1}$. Next, we write the product as $ZD_1(Z^TD_0Z)$. Applying a dual argument, we conclude that if the xy-entry of $ZD_1Z^TD_0Z$ is nonzero, then $x \wedge y = \hat{0}$.

To finish the proof, we note that

$$\det ZD_1Z^TD_0Z = \prod_{x\colon x \in L} \mu(\hat{0}, x)\mu(x, \hat{1}).$$

By hypothesis, the determinant is nonzero. Hence, by the argument in Corollary 6-3.4, there exists a complementing permutation. □

6-4. Matchings and counting inequalities in lattices

In this section, we discuss matching and counting theorems for modular and geometric lattices. Let J and M be subsets in a partially ordered set P. A *matching* σ from J to M is an injective function $J \to M$ such that for all $j \in J$, $j \leq \sigma(j)$. If a matching exists from J to M, then $|J| \leq |M|$.

The first result in this area is an equality in modular lattices discovered by R.P. Dilworth [64] in 1954.

Theorem 6-4.1 (Dilworth's covering theorem)**.** *Let L be a modular lattice. Then the number of elements covering exactly k elements equals the number of elements covered by k elements.*

A sketch of Dilworth's proof of his theorem can be found in LTF (Section 5.13, p. 401).

Dilworth's covering theorem is a special case of a result about matchings between subsets in lattices [228]. Let J and M be subsets of a lattice L. The subset J is *concordant* with the subset M if for every element x in L, either x is in M or there exists an element $x^{\#}$ such that

CS1. $\mu(x, x^{\#}) \neq 0$, and

CS2. For every element j in J, $x \vee j \neq x^{\#}$.

If H and K are subsets of a partially ordered set, the (*transposed*) *incidence matrix* $\mathcal{I}(H|K)$ is the matrix with rows indexed by H and columns indexed by K with the hk-entry equal to 1 if $h \geq k$ and 0 otherwise.

Theorem 6-4.2. *Let J be concordant with M in a lattice L. Then the incidence matrix $\mathcal{I}(M|J)$ has rank $|J|$.*

Proof. Let $\mathbb{Q}$ be the field of rational numbers, $\operatorname{Fun} L$ be the vector space of functions defined from the set L to $\mathbb{Q}$, and $\operatorname{Fun} J$ be the subspace of functions *supported on* J, that is, functions such that $f(x) = 0$ unless $x \in J$. If $a \in L$, the *delta function* $\delta_a\colon L \to \mathbb{Q}$ is the function defined by $\delta_a(x) = 1$ if $x = a$

and 0 otherwise. The set $\{\delta_a : a \in L\}$ is a basis for $\operatorname{Fun} L$ and the subset $\{\delta_a : a \in J\}$ is a basis for $\operatorname{Fun} J$.

Let $T\colon \operatorname{Fun} J \to \operatorname{Fun} L$ be the linear transformation defined by

$$Tf(y) = \sum_{z\colon z \le y} f(z).$$

Relative to bases of delta functions, the matrix of T is the incidence matrix $\mathcal{I}(L|J)$. We will show that $\mathcal{I}(M|J)$ has rank $|J|$ by showing that the linear transformation $T_M\colon \operatorname{Fun} J \to \operatorname{Fun} M$ obtained by restricting Tf to the elements in M is injective. This will be done by showing that one can reconstruct a function f in $\operatorname{Fun} J$ from the restriction $Tf|_M$ of Tf to M. We need the following lemma.

Lemma 6-4.3.

$$\sum_{y\colon x \le y \le x^{\#}} \mu(y, x^{\#}) Tf(y) = \sum_{z\colon z \vee x = x^{\#}} f(z).$$

Proof. Let $f_x\colon [x, x^{\#}] \to \mathbb{Q}$ be the function defined by

$$f_x(y) = \sum_{z\colon z \vee x = y} f(z).$$

The elements in L are partitioned into equivalence classes by the relation $a \sim b$ if and only if $a \vee x = b \vee x$. In addition, $z \le y$ for an element y in $[x, x^{\#}]$ if and only if $z \vee x \le y$. Hence,

$$Tf(y) = \sum_{z\colon z \le y} f(z) = \sum_{z\colon x \le z \le y} f_x(z).$$

Applying Möbius inversion to f_x on the interval $[x, x^{\#}]$, we obtain

$$\sum_{y\colon x \le y \le x^{\#}} \mu(y, x^{\#}) Tf(y) = f_x(x^{\#}) = \sum_{z\colon z \vee x = x^{\#}} f(z).$$

□

To reconstruct a function $f\colon J \to \mathbb{Q}$, we first reconstruct the (unrestricted) function $Tf\colon L \to \mathbb{Q}$ using as input the restriction $Tf\colon M \to \mathbb{Q}$. Once we have done this, f can be reconstructed using Möbius inversion over L.

To start the reconstruction of Tf, we note that if J is concordant with M, then the maximum $\hat{1}$ must be in M. Hence, $Tf(\hat{1})$ can be read off directly from the input. We now go down the lattice, inductively reconstructing the value $Tf(x)$. If $x \in M$, then $Tf(x)$ is read directly off the input. If $x \notin M$, then by CS2, for all $j \in J$, $x \vee j \ne x^{\#}$. Hence,

$$\sum_{z\colon z \vee x = x^{\#}} f(z) = 0.$$

Rearranging the equation in Lemma 6-4.3, we have

$$\mu(x, x^{\#})Tf(x) = -\sum_{y:\, x<y\le x^{\#}} \mu(y, x^{\#})Tf(y).$$

Since $y > x$, all the values $Tf(y)$ have already been reconstructed. Hence, as $\mu(x, x^{\#}) \neq 0$, the equation yields the value of $Tf(x)$. This completes the proof of Theorem 6-4.2. □

Using the argument in the proof of Corollary 6-3.4, we obtain the following corollary.

Corollary 6-4.4. *If J is concordant with M, then there is a matching σ between J and M. In particular, $|J| \le |M|$.*

We shall apply Theorem 6-4.2 to geometric and modular lattices. We shall freely use notation and results from Chapter 6 of LTF, with one exception: the *rank* (or *height*) of an element x is the length of a maximal chain from the minimum $\hat{0}$ to x. The following "positivity result" of G.-C. Rota [280] will be useful for verifying CS1.

Theorem 6-4.5. *Let L be a geometric lattice, x and y be elements such that $x \le y$, and μ the Möbius function of L. Then $\mu(x, y)$ is nonzero and has sign $(-1)^{\mathrm{rank}(y)-\mathrm{rank}(x)}$.*

Proof. We proceed by induction on the difference $\mathrm{rank}(y) - \mathrm{rank}(x)$. To begin, observe that $\mu(x, x) = 1$ and $\mu(x, y) = -1$ if y covers x. For the induction step, we use Theorem 6-2.4. Choose an element a covering y. Then

$$\mu(x, y) = -\sum_{z:\, z\in[x,y],\, z\vee a=y,\, z\neq y} \mu(x, z).$$

By the submodular inequality,

$$\mathrm{rank}(z) + \mathrm{rank}(a) \ge \mathrm{rank}(y) + \mathrm{rank}(x),$$

and hence,

$$\mathrm{rank}(z) - \mathrm{rank}(x) \ge \mathrm{rank}(y) - \mathrm{rank}(a) = [\mathrm{rank}(y) - \mathrm{rank}(x)] - 1.$$

Since $z < y$, we have $\mathrm{rank}(z) - \mathrm{rank}(x) = [\mathrm{rank}(y) - \mathrm{rank}(x)] - 1$. By induction, $\mu(x, z)$ is nonzero and has sign $(-1)^{\mathrm{rank}(y)-\mathrm{rank}(x)-1}$. We conclude that $\mu(x, y)$ is nonzero and has sign $(-1)^{\mathrm{rank}(y)-\mathrm{rank}(x)}$. □

Our first example of a concordant set yields a set of inequalities due to T.A. Dowling and R.M. Wilson [73].

Theorem 6-4.6. *Let L be a rank-n geometric lattice, and*

$$\begin{aligned} B_k &= \{x : x \in L \ \text{ and } \ \mathrm{rank}(x) \le k\}, \\ T^k &= \{x : x \in L \ \text{ and } \ n - \mathrm{rank}(x) \le k\}. \end{aligned}$$

Then B_k is concordant with T^k with $x^{\#} = \hat{1}$. In particular, if W_i is the number of rank-i elements in L and $k < n/2$, then

$$W_0 + W_1 + W_2 + \cdots + W_k \le W_{n-k} + W_{n-k+1} + \cdots + W_{n-1} + W_n.$$

Equality holds if and only if L is modular.

Proof. CS1 holds by Theorem 6-4.5. If $\mathrm{rank}(j) \le k$ and $x \notin T^k$ (that is, $\mathrm{rank}(x) \le n - k$), then the submodular inequality for rank implies that $\mathrm{rank}(x \vee j) < n$ and $x \vee j \ne \hat{1}$. Thus CS2 holds. For the characterization of those geometric lattices in which equality holds, we refer the reader to [73]. □

Theorem 6-4.1 is another consequence of Theorem 6-4.2. If x is an element in a lattice L, let $x^{\dagger}$ be the join of all the elements covering x. If L is semimodular, then the interval $[x, x^{\dagger}]$ is a geometric lattice. Dually, let $x_{\dagger}$ be the meet of all the elements covered by x. Finally, let

$$\mathrm{Cov}_i\, L = \{x : x \text{ covers exactly } i \text{ elements}\}, \ \mathrm{COV}_k\, L = \bigcup_{i=0}^{k} \mathrm{Cov}_i\, L,$$

$$\mathrm{Cov}^i\, L = \{x : x \text{ is covered by exactly } i \text{ elements}\}, \ \mathrm{COV}^k\, L = \bigcup_{i=0}^{k} \mathrm{Cov}^i\, L,$$

and

$$D_k = \{x : \mathrm{rank}(x) - \mathrm{rank}(x_{\dagger}) \le k\}, \ U_k = \{x : \mathrm{rank}(x^{\dagger}) - \mathrm{rank}(x) \le k\}.$$

Theorem 6-4.7. *Let k be a positive integer and L be a modular lattice. Then*

(i) $\mathrm{COV}_k\, L$ *is concordant with* $\mathrm{COV}^k\, L$ *with* $x^{\#} = x^{\dagger}$.

(ii) D_k *is concordant with* U_k *with* $x^{\#} = x^{\dagger}$.

Proof. CS1 for both parts follow from Theorem 6-4.5. To verify CS2 for (i), we need the following lemma.

Lemma 6-4.8. *If $x \notin \mathrm{COV}^k\, L$, then $x^{\dagger} \notin \mathrm{COV}_k\, L$.*

Proof. If $x \notin \mathrm{COV}^k\, L$, then the interval $[x, x^{\dagger}]$ is a geometric lattice with at least $k+1$ atoms and maximum $x^{\dagger}$. By Theorem 6-4.6, $[x, x^{\dagger}]$ has at least $k+1$ coatoms, and hence $x^{\dagger}$ covers at least $k+1$ elements. □

Suppose that $x \notin \mathrm{COV}^k L$. To verify CS2, let $j \in \mathrm{COV}_k L$, and suppose by way of contradiction that $x \vee j = x^\dagger$. Then by the Dedekind transposition principle (Theorem 348, p. 308, in LTF), the intervals $[x \wedge j, j]$ and $[x, x^\dagger]$ are isomorphic. By Lemma 6-4.8, $x^\dagger$ covers at least $k+1$ elements and hence, by the isomorphism, j covers at least $k+1$ elements, a contradiction. We conclude that for all $j \in \mathrm{COV}_k L$, $x^\dagger \neq x \vee j$.

To prove (ii), we use a similar argument, using the fact that if $[x \wedge j, j]$ and $[x, x^\dagger]$ are isomorphic, then $x_\dagger \leq x \wedge j$. □

Since the order dual of a modular lattice is a modular lattice, Theorem 6-4.7 implies that for all k, $|\mathrm{COV}_k L| = |\mathrm{COV}^k L|$, and hence, the conclusion in Theorem 6-4.1, that $|\mathrm{Cov}_k L| = |\mathrm{Cov}^k L|$.

6-5. Eulerian functions of groups and closure systems

Let $A \to \overline{A}$ be a closure system on a set S and L be the lattice of closed sets. If $\overline{(x_1, x_2, \ldots, x_s)}$ is an ordered s-tuple, we define its *closure* to be the closure $\overline{\{x_1, x_2, \ldots, x_s\}}$ of the underlying set. An s-tuple $(x_1, x_2, \ldots, x_s)$ *spans* or *generates* a subset A of S if $\overline{(x_1, x_2, \ldots, x_s)} = A$. The *Eulerian function* $\varphi(S; s)$ of the closure system $A \to \overline{A}$ is defined by

$$\varphi(S; s) = \sum_{A\colon A \in L} \mu(A, S)|A|^s.$$

The expression on the right-hand side is a *Dirichlet polynomial*, that is, it is a (finite) linear combination of symbols n^s, where n is a nonnegative integer and s is a formal exponent (with the convention that $0^s = 1$). Being finite sums, Dirichlet polynomials can be evaluated by setting s to be a complex number. Like polynomials, two Dirichlet polynomials are identical as formal expressions if and only if they agree when evaluated on a sufficiently large set of numbers.

Theorem 6-5.1. *Let s be a nonnegative integer. Then $\varphi(S; s)$ equals the number of s-tuples spanning S.*

Proof. Let A be a closed subset and let $\varphi(A; s)$ be the number of s-tuples spanning A. An s-tuple $(x_1, x_2, \ldots, x_s)$ with $x_i \in A$ spans a closed subset B of A. Since there are $|A|^s$ s-tuples, we have, for all closed subsets A,

$$|A|^s = \sum_{B\colon B \in L,\ B \subseteq A} \varphi(B; s).$$

By Möbius inversion,

$$\varphi(S; s) = \sum_{B\colon B \in L} \mu(B, S)|B|^s. \qquad \square$$

Eulerian functions were first studied by P. Hall [191] in 1936 for groups. Let G be a finite group. The function sending a subset $A \subseteq G$ to the subgroup generated by A is a closure system on the underlying set G. The lattice of closed sets is $L(G)$, the lattice of subgroups of G. For example, the subgroup lattice of the cyclic group Z_n of order n is isomorphic to the lattice of divisors of n. Hence

$$\varphi(Z_n; s) = \sum_{d:\, d \text{ divides } n} \mu(n/d) d^s.$$

In particular, if p is a prime, $\varphi(Z_p; s) = p^s - 1$. Note that $\varphi(Z_n; 1)$ equals $\varphi(n)$, the "Euler totient function". This explains the terminology.

The *Frattini subgroup* $\mathrm{Frat}\, G$ of the group G is the intersection of all the maximal subgroups of G. The Frattini subgroup is normal. By Corollary 6-2.6, $\mu(H, G) = 0$ unless the subgroup H contains $\mathrm{Frat}\, G$.

Lemma 6-5.2.

$$\varphi(G; s) = \sum_{H:\, \mathrm{Frat}\, G \leq H \leq G} \mu(H, G)|H|^s = |\mathrm{Frat}\, G|^s \, \varphi(G/\mathrm{Frat}\, G; s).$$

Lemma 6-5.2 reduces the problem of calculating the Eulerian function to groups with trivial Frattini subgroups. Another reduction follows from Lemma 6-2.2 and the theorem that if G is the direct product $H \times K$ and the sizes $|H|$ and $|K|$ are relatively prime, then the lattice $L(G)$ is the direct product $L(H) \times L(K)$. From these results, we obtain the following product formula.

Lemma 6-5.3. *Let H and K be groups such that $|H|$ and $|K|$ are relatively prime. Then $\varphi(H \times K; s) = \varphi(H; s)\varphi(K; s)$.*

It follows easily from Lemmas 6-5.2 and 6-5.3 that if $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$, then

$$\varphi(Z_n; s) = \prod_{i=1}^{r} p_i^{s(\alpha_i - 1)}(p_i^s - 1).$$

With a little more work, Lemmas 6-5.2 and 6-5.3 also yield explicit formulas for Eulerian functions of finite Abelian groups. Let G be an Abelian group of order $p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$, where p_i are distinct primes. Then by the structure theory for finite Abelian groups (see, for example, [279]),

$$G \;\cong\; \prod_{i=1}^{r} S_{p_i},$$

where S_{p_i} is the Sylow p_i-subgroup, consisting of elements with order a power of p_i. Since the subgroups S_{p_i} have order $p_i^{\alpha_i}$, Lemma 6-5.3 can be applied, reducing the calculation to Abelian p-groups. Let H be an Abelian group of order p^α. By structure theory again, an Abelian p-group H is a direct product of cyclic groups. If H is the direct product of m cyclic groups, then

$$H/\mathrm{Frat}\, H \;\cong\; Z_p^m,$$

a direct product of m cyclic groups of order p. Since $L(Z_p^m)$ is isomorphic to $L(m,p)$, the lattice of subspaces of the finite vector space $\mathrm{GF}(p)^m$, we have

$$\varphi(H;s) = p^{s(\alpha-m)} \left[\sum_{i=0}^{m} \binom{m}{i}_p (-1)^{m-i} p^{(m-i)(m-i-1)/2} p^{si} \right].$$

Changing the index of summation from i to $m-i$ and using Cauchy's identity (Exercise 6.16), we conclude that

$$\varphi(H;s) = p^{s(\alpha-m)} \prod_{j=0}^{m-1} (p^s - p^j).$$

Summarizing, we have the following theorem.

Theorem 6-5.4. *Let G be an Abelian group of order $p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$ in which the Sylow p_i-subgroup is the direct product of m_i cyclic groups. Then*

$$\varphi(G;s) = \prod_{i=1}^{r} \left[p_i^{s(\alpha_i - m_i)} \prod_{j=0}^{m_i-1} (p_i^s - p_i^j) \right].$$

There seems no easy way to compute the Eulerian function for groups in general. The next theorem (due to W. Gaschütz [103]) shows how Eulerian functions factor in the presence of a normal subgroup. Normal subgroups are modular elements of the subgroup lattice. In this sense, Gaschütz theorem is the precursor of the modular factorization theorem for geometric lattices discussed in the next section.

Theorem 6-5.5 (Gaschütz's factorization theorem)**.** *If N is a normal subgroup of G and G/N is the quotient of G by N, then*

$$\varphi(G;s) = \varphi(G/N;s)\varphi(G \downarrow N;s),$$

where

$$\varphi(G \downarrow N;s) = \sum_{H\colon H \le G,\, NH = G} \mu(H,G)|N \cap H|^s,$$

where the sum on the right ranges over all subgroups H such that the subgroup NH generated by N and H equals G.

Proof. We begin by describing how spanning s-tuples in G/N expand to spanning s-tuples of G.

Lemma 6-5.6. *Let $(Ng_1, Ng_2, \ldots, Ng_s)$ be an s-tuple of cosets and H be a subgroup of G. Let $C(H, s)$ be the number of s-tuples $(x_1, x_2, \ldots, x_s)$ such that $x \in Ng_i$ and $(x_1, x_2, \ldots, x_s)$ generates a subgroup contained in H. Then*

$$C(H,s) = \begin{cases} |H \cap N|^s & \text{if for all } i,\ H \cap Ng_i \neq \varnothing, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. An s-tuple $(x_1, x_2, \ldots, x_s)$ such that $x_i \in Ng_i$ generates a subgroup contained in H if and only if for every i, $x_i \in H \cap Ng_i$. Therefore,

$$C(H, s) = |H \cap Ng_1||H \cap Ng_2| \cdots |H \cap Ng_s|.$$

Next, suppose that the set $H \cap Ng$ is not empty and $x \in H \cap Ng$. Then the function $y \mapsto yx^{-1}$ is a bijection $G \to G$ sending Ng to N and H to H. Hence, it sends $H \cap Ng$ to $H \cap N$ and $|H \cap Ng| = |H \cap N|$ whenever $H \cap Ng$ is nonempty. □

Returning to the proof of Theorem 6-5.5, observe that since the cosets of N partition G, for each s-tuple $(x_1, x_2, \ldots, x_s)$, we can associate with it a unique s-tuple $(Ng_1, Ng_2, \ldots, Ng_s)$ such that for every i, $x_i \in Ng_i$. In addition, if $(x_1, x_2, \ldots, x_s)$ generates G, then the associated s-tuple generates G/N.

Now let $(Ng_1, Ng_2, \ldots, Ng_s)$ be a fixed s-tuple in G/N generating G/N. Then by Möbius inversion, the number of s-tuples $(x_1, x_2, \ldots, x_s)$ generating G associated with $(Ng_1, Ng_2, \ldots, Ng_s)$ equals

$$\sum_{H\colon H \in L(G)} \mu(H, G) C(H, s).$$

By Lemma 6-5.6, $C(H, s) = 0$ unless for all i, $Ng_i \cap H \neq \varnothing$, that is, the subgroup NH contains all the cosets Ng_i. Hence, since $(Ng_1, Ng_2, \ldots, Ng_s)$ generates G/N, $NH = G$. Finally, if $NH = G$, $c(H, s) = |N \cap H|^s$. We conclude that the number of s-tuples $(x_1, x_2, \ldots, x_s)$ associated with an s-tuple $(Ng_1, Ng_2, \ldots, Ng_s)$ generating G/N is independent of that s-tuple and equals the second sum on the right-hand side. □

Two recent papers on Eulerian functions of groups are [246, 247].

6-6. Characteristic polynomials of geometric lattices

6-6.1 Matroids

In the last two sections, we shall use the interpretation of a geometric lattice as the lattice of flats of a matroid. Recall that a *matroid* G on the set S of elements is a closure system $B \mapsto \overline{B}$ satisfying the *exchange condition*:

if $a, b \notin A$, then $b \in \overline{A \cup \{a\}}$ implies $a \in \overline{A \cup \{b\}}$.

The closed sets or *flats* of a matroid form a geometric lattice L. The minimum $\hat{0}$ of L is the closure $\overline{\varnothing}$ of the empty set and the maximum $\hat{1}$ is S. Elements in $\overline{\varnothing}$ are called *loops*. A *parallel class* is the closure $\overline{\{a\}}$ of an element a which is not a loop. Each parallel class determines an atom in L and conversely. The rank function on L gives a rank function on S in the following way: if $A \subseteq S$, then rank(A) is the rank of $\overline{A}$ in L. A matroid G is a *simple matroid* or *(combinatorial) geometry* if $\overline{\varnothing} = \varnothing$ and $\overline{\{a\}} = \{a\}$ for all $a \in S$. Given a geometric lattice L, we can construct a geometry G on the set A of atoms by defining, for $B \subseteq A$,

$$\overline{B} = \bigvee \{b : b \in B\}.$$

This construction gives an equivalence or "cryptomorphism" between geometric lattices and combinatorial geometries. This equivalence is described in detail in Section 5.3.3, p. 349, in LTF.

Let G be a matroid on the set E with closure $B \mapsto \overline{B}$ and lattice L of flats. Given a subset T of S, we can construct two matroids from G. The *restriction* $G|T$ to T is the matroid on T with the closure of a subset A in T defined by $\overline{A} \cap T$. If T is closed, then the lattice of flats of $G|T$ is the lower interval $[\hat{0}, T]$. The *contraction* G/T by T is the matroid on $S \setminus T$ with closure of a set A in $S \setminus T$ defined by $\overline{A \cup T} \setminus T$. The lattice of flats of G/T is isomorphic to the upper interval $[\overline{T}, \hat{1}]$ in L. The loops in G/T are the elements in $\overline{T} \setminus T$. The atoms in $[\overline{T}, \hat{1}]$ are the flats covering $\overline{T}$ and each atom gives a parallel class in G/T.

Let L be a rank-n geometric lattice. The *characteristic polynomial* $\chi(L;\lambda)$ (in the variable λ) and *Whitney numbers* w_i (or $w_i(L)$) *of the first kind* are defined by

$$\chi(L;\lambda) = \sum_{X:\, X \in L} \mu(\hat{0}, X)\lambda^{n-\mathrm{rank}(X)} = \sum_{i=0}^{n} (-1)^i w_i \lambda^{n-i}.$$

If G is a rank-n matroid with lattice L of flats, then its *characteristic polynomial* $\chi(G;\lambda)$ is defined to be $\chi(L;\lambda)$ if G has no loops, and the identically zero polynomial if G has loops. The *Whitney numbers* $w_i(G)$ are defined similarly. Equivalently, we can define $\chi(G;\lambda)$ by

$$\chi(G;\lambda) = \sum_{A:\, A \subseteq S} (-1)^{|A|} \lambda^{n-\mathrm{rank}(A)}.$$

When G has no loops and positive rank, $\chi(G;1) = 0$ and hence $\lambda - 1$ is a factor of $\chi(G;\lambda)$. The *reduced characteristic polynomial* $\bar{\chi}(G;\lambda)$ and the *reduced Whitney numbers* $\bar{w}_i$ are defined by

$$\bar{\chi}(G;\lambda) = \frac{\chi(G;\lambda)}{\lambda - 1} = \sum_{i=0}^{n-1} (-1)^i \bar{w}_i \lambda^{n-1-i}.$$

The following explicit formula for the reduced characteristic polynomial is an easy consequence of Theorem 6-2.4.

Lemma 6-6.1. *Let G be a loopless matroid of positive rank. Then*

$$\bar{\chi}(G;\lambda) = \sum_{U:\, U \wedge b = \varnothing} \mu(\varnothing, U)\lambda^{n-\mathrm{rank}(U)-1},$$

where b is any atom in the lattice of flats of G.

6-6.2 The no-broken-circuit complex

Choose a linear ordering $\leq_A$ on S. (This ordering may be arbitrarily chosen and need not have any relation with the partial order of L.) A set B is a *broken circuit* if there is an element a not in B such that $a \leq_A b$ for all elements $b \in B$ and $B \cup \{a\}$ is a circuit (or minimal dependent set). A subset I of atoms is a *no-broken-circuit set* or *nbc-set* if no subset of I is a broken circuit. Note that a circuit contains a (unique) broken circuit and hence, an nbc-set must be independent. It is immediate from the definition that a subset of an nbc-set is also an nbc-set; thus, the nbc-sets form a simplicial complex on the set S.

To motivate the main result, consider the following example. Let L be the rank-2 geometric lattice with 5 atoms $\{a,b,c,d,e\}$ ordered alphabetically and G be the geometry on the atoms defined by L. The circuits in G are the 3-element subsets of atoms. Thus, $\{b,c\}$ is a broken circuit because $\{a,b,c\}$ is a circuit. Similarly, every 2-element subset of A not containing the alphabetically least element a is a broken circuit. We conclude that there are 4 nbc-sets, $\{a,b\}, \{a,c\}, \{a,d\}$, and $\{a,e\}$ of size 2.

Theorem 6-6.2. *Let X be a flat in a loopless matroid. Then the absolute value $(-1)^{\mathrm{rank}(X)}\mu(\varnothing, X)$ of the Möbius function in its lattice L of flats equals the number of no-broken-circuit sets whose closure equals X.*

Proof. We will prove that the function ν, defined on flats X of L by

$$\nu(\varnothing, X) = (-1)^{\mathrm{rank}(X)}[\text{number of nbc-sets } I \text{ such that } \overline{I} = X],$$

satisfy the same recursion as the Möbius function $\mu(\varnothing, X)$. Since the empty set is an nbc-set, $\nu(\varnothing,\varnothing) = 1 = \mu(\varnothing,\varnothing)$. Next let X be a flat of positive rank and let $\mathcal{N}$ be the collection of all nbc-sets contained in X. Then

$$\sum_{Y:\, Y \leq X,\, Y \in L} \nu(\varnothing, Y) = \sum_{I:\, I \subseteq X,\, I \text{ is an nbc-set, and } \overline{I} = X} (-1)^{|I|}.$$

Lemma 6-6.3. *Let a be the $\leq_A$-least element in X and $J \subset X$ be a set of elements not containing a. Then J is an nbc-set if and only if $J \cup \{a\}$ is an nbc-set.*

Proof. Since a subset of an nbc-set is an nbc-set, the converse is clear. Suppose that J is an nbc-set not containing a. If $J \cup \{a\}$ is not an nbc-set, let C be a circuit with $\leq_A$-least element b such that $C \setminus \{b\} \subseteq J \cup \{a\}$. If $b = a$, then $C \setminus \{a\} \subseteq J$, then J is not nbc, a contradiction. If $b \neq a$, then $a \notin C$ and $C \setminus \{b\} \subseteq J$, also a contradiction. We conclude that $J \cup \{a\}$ is an nbc-set. □

Let $\mathcal{N}^-$ be the subcollection of nbc-sets in $\mathcal{N}$ not containing a and $\mathcal{N}^+$ be the subcollection of nbc-sets in $\mathcal{N}$ containing a. Then by the lemma, the function $I \mapsto I \cup \{a\}$, $\mathcal{N}^- \to \mathcal{N}^+$ is a bijection. Since the nbc-sets I and $I \cup \{a\}$ contribute $+1$ and -1 to the sum $\sum_Y \nu(\varnothing, Y)$, that sum equals 0. Hence ν and the Möbius function μ satisfy the same recursion and $\nu = \mu$. □

6-6.3 A division theorem for characteristic polynomials

The main result in this subsection shows how the characteristic polynomial of a loopless rank-n matroid G on the set S breaks up into smaller parts at a rank-k flat X. To do so, we need two constructions. Let $U \subseteq S \setminus X$. Then we define

$$(G|X)_U = (G|X \cup U)\,/\,U,$$

the matroid on X obtained by adding the elements in U to X and then contracting them. Note that $\mathrm{rank}((G|X)_U) = \mathrm{rank}(X \vee U) - \mathrm{rank}(U)$. The second is the *(complete) principal truncation* $T_X(G)$ of G at X. This is the matroid of rank $n - k + 1$ whose flats are the flats in G containing X together with those flats U in G such that

$$\mathrm{rank}(U \vee X) = \mathrm{rank}(U) + \mathrm{rank}(X) = \mathrm{rank}(U) + k.$$

In $T_X(G)$, $\mathrm{rank}(X)$ drops to 1 and X becomes an atom. The other atoms are those atoms a in L such that $a \not\leq X$. The two constructions are connected: if U is a flat of G and $X \not\subseteq U$, then $(G|X)_U = G|X$ if and only if $U \in T_X(G)$.

Theorem 6-6.4. *Let X be a flat of rank k of a rank-n loopless matroid. Then*

$$\chi(G;\lambda) = \frac{\chi(G|X;\lambda)\chi(T_X(G);\lambda)}{\lambda - 1} + \sum_U \mu(\varnothing, U)\lambda^{n-\mathrm{rank}(X\vee U)}\chi((G|X)_U;\lambda),$$

where the sum on the right-hand side ranges over all flats U in L such that $\mathrm{rank}(U) \geq 2$, $\mathrm{rank}(U \vee X) < \mathrm{rank}(U) + k$, *and* $U \wedge X = \varnothing$.

When $k = n - 1$, Theorem 6-6.4 gives the quotient and remainder when $\chi(G;\lambda)$ is divided by $\chi(G|X;\lambda)$. While this is not necessarily true when $k \leq n - 1$, Theorem 6-6.4 gives good "approximations" with combinatorial meanings to the quotient and remainder.

We sketch the proof of Theorem 6-6.4. Details may be found in J.P.S. Kung [230]. We begin by rewriting the identity in the theorem in the form

$$\chi(G;\lambda) = \chi(G|X;\lambda)\bar{\chi}(T_X(G);\lambda) + \sum_U \mu(\varnothing, U)\lambda^{n-\mathrm{rank}(X\vee U)}\chi((G|X)_U;\lambda).$$

Equating coefficients of λ^{n-m}, the identity is equivalent to the equations: for $0 \le m \le n$,

$$w_m(G) = \sum_{j=0}^{m} [(-1)^{m-j} w_{m-j}(G|X)][(-1)^j \bar{w}_j(T_X(G))] \\ + \sum_U \mu(\hat{0}, U)[(-1)^{m-\mathrm{rank}(U)} w_{m-\mathrm{rank}(U)}((G|X)_U)].$$

Let $\mathcal{I}_m$ be the collection of nbc-sets in G of size m. By Lemma 6-6.2, the left-hand side, $w_m(G)$, equals $|\mathcal{I}_m|$. We shall partition $\mathcal{I}_m$ in a way consistent with the right-hand side.

Choose an ordering $\le_A$ such that the elements in X precede the elements in $S \setminus X$. Let U be a flat. An element a in X is a *least representative* in $(G|X)_U$ if a is not a loop and a is the least element in its parallel class. Let X_U^* be the set of least representatives in $(G|X)_U$ and $(G|X)_U^*$ be the simple matroid obtained from $(G|X)_U$ by restricting to the subset X_U^*.

Lemma 6-6.5. *Let J be an nbc-set contained in $S \setminus X$, $U = \overline{J}$, and $I \subseteq X$. Then*

(i) $U \cap X = \varnothing$.

(ii) *$I \cup J$ is an nbc-set if and only if $I \subseteq {X_U}^*$ and I is an nbc-set in $(G|X)_U^*$.*

Proof. To prove (i), suppose for contradiction that $c \in U \cap X$. Then there is a subset J' in J such that $J' \cup \{c\}$ is a circuit. As $c \in X$, our choice of $\le_A$ puts c before all the elements in J'. Hence J' is a broken circuit, a contradiction. The proof of (ii) uses a similar but more complicated argument. See [230] for details. □

Let $I \in \mathcal{I}_m$. Then $I \cap (S \setminus X)$ is an nbc-set. By Lemma 6-6.5, the closure $\overline{I \cap (S \setminus X)}$ is a flat disjoint from X. Let $\mathcal{I}_m(U)$ be the subcollection in $\mathcal{I}_m$ defined by $I \in \mathcal{I}_m(U)$ if and only if $\overline{I \cap (S \setminus X)} = U$. As U ranges over all flats disjoint from X, the subcollections $\mathcal{I}_m(U)$ form a partition of $\mathcal{I}_m$.

Let U be a flat disjoint from X. By Lemma 6-6.5, every nbc-set in $\mathcal{I}_m(U)$ can be obtained uniquely by choosing

(1) a subset of size $m - \mathrm{rank}(U)$ that is an nbc-set in $(G|X)_U^*$,

(2) a spanning set of U (having size $\mathrm{rank}(U)$) that is an nbc-set in G,

and taking their union. Using Lemma 6-6.2 and the fact that $w_i((G|X)_U^*) = w_i((G|X)_U)$, we conclude that

$$|\mathcal{I}_m(U)| = w_{m-\mathrm{rank}(U)}((G|X)_U)(-1)^{\mathrm{rank}(U)}\mu(\hat{0}, U). \tag{A}$$

There are two kinds of flats U disjoint from X. The first are those flats *skew to* X in the sense that $\mathrm{rank}(U \vee X) = \mathrm{rank}(U) + \mathrm{rank}(X)$. The rank condition implies that these flats must have empty intersection with X. For these flats, $(G|X)^*_U = G|X$ and $U \cap X = \varnothing$ in both G and $T_X(G)$. Therefore,

$$\sum_{U:\, U \text{ skew}} |\mathcal{I}_m(U)| = \sum_{j=0}^{m} w_{m-j}(G|X) \left[\sum_{U:\, U \text{ skew},\, \operatorname{rank}(U)=j} (-1)^{\operatorname{rank}(U)} \mu(\hat{0}, U) \right].$$

By Lemma 6-6.1 and the definition of $T_X(G)$,

$$\sum_{U:\, U \text{ skew},\, \operatorname{rank}(U)=j} (-1)^{\operatorname{rank}(U)} \mu(\hat{0}, U) = \bar{w}_j(T_X(G))$$

and hence,

$$\sum_{U:\, U \text{ skew}} |\mathcal{I}_m(U)| = \sum_{j=0}^{m} w_{m-j}(G|X) \bar{w}_j(T_X(G)). \tag{S}$$

The second kind are flats that are *non-skew*, that is, those flats satisfying $\operatorname{rank}(U \vee X) < \operatorname{rank}(U) + k$. The rank condition implies that $(G|X)_U$ has rank strictly smaller than $G|X$. For these flats, equation (A) holds. We can now complete the proof of Theorem 6-6.4 by adding together equations (A) for all non-skew flats and equation (S).

The identity in Theorem 6-6.4 is *sign-coherent*, in the sense that the coefficient of λ^m in every term on the right-hand side has the same sign $(-1)^{n-m}$. This allows us to derive several consequences. Recall that an element x in a semimodular lattice L is *modular* if for all elements y in L,

$$\operatorname{rank}(x) + \operatorname{rank}(y) = \operatorname{rank}(x \vee y) + \operatorname{rank}(x \wedge y).$$

The first corollary specializes Theorem 6-6.4 to the case when X is modular. It combines results of T. Brylawski [34] and R.P. Stanley [305].

Corollary 6-6.6 (The modular factorization theorem)**.** *Let X be a rank-k flat in a rank-n geometric lattice L or loopless matroid G. Then*

$$\chi(L;\lambda) = \chi([\hat{0}, X];\lambda) \left[\sum_{U: U \wedge X = \hat{0}} \mu(\hat{0}, X) \lambda^{n-k-\operatorname{rank}(U)} \right],$$

$$\chi(G;\lambda) = \frac{\chi(G|X;\lambda)\chi(T_X(G);\lambda)}{\lambda - 1}$$

if and only if X is modular. In particular, if $k = n-1$, then the polynomial $\chi(G|X;\lambda)$ divides the polynomial $\chi(G;\lambda)$ if and only if X is modular.

Another corollary, due to R.P. Stanley [305], is obtained by iterating Corollary 6-6.6. A *flag* (X_i) in a rank-n geometric lattice L is a sequence of flats such that $X_0 < X_1 < X_2 < \cdots < X_{n-1} < X_n$ and $\operatorname{rank}(X_i) = i$. Its *root sequence* (a_i) is the integer sequence with a_i equal to the number of atoms in X_i not in X_{i-1}.

Corollary 6-6.7. *Suppose that there is a flag* (X_i) *with root sequence* (a_i) *in a rank-*n *geometric lattice* L *such that for each* i, X_{i-1} *is modular in* X_i. *Then*

$$\chi(L;\lambda) = \prod_{i=1}^{n}(\lambda - a_i).$$

Theorem 6-6.4 also yields several inequalities between reduced Whitney numbers. An example is the following inequalities for the Möbius function.

Corollary 6-6.8. *Let* μ *be the Möbius function and* X *a rank-*k *flat in a rank-*n *geometric lattice* L. *Then*

$$\mu(\varnothing, S) \geq \mu(\varnothing, X)\left(\sum_{U:\, U\in L, \mathrm{rank}(U\vee X)=n \text{ and } \mathrm{rank}(U)=n-k} \mu(\varnothing, U)\right)$$

with equality if and only if X *is modular.*

Iterating Corollary 6-6.8, we have the following flag inequality.

Corollary 6-6.9. *Let* (X_i) *be a flag with root sequence* (a_i) *in a rank-*n *geometric lattice. Then*

$$\mu(\varnothing, S) \geq a_1 a_2 \cdots a_n,$$

with equality if and only if for all i, X_i *is modular.*

6-7. Antichains and the Sperner property

Let P be a partially ordered set. A *chain* C in P is a subset of P in which every pair x and y such that $x \neq y$ in P are comparable, that is, either $x < y$ or $x > y$. An *antichain* A is a subset in which every pair is incomparable, that is neither $x < y$ nor $x > y$, or equivalently, $|A \cap C| \leq 1$ for any chain C in P. Sperner theory is about maximum-size antichains in partially ordered sets. Sperner theory was founded on two theorems: Dilworth's chain partition theorem [63] and Sperner's theorem on maximum-size antichains in finite Boolean algebras [304]. The emphasis in this section will be on whether Sperner's theorem can be extended to geometric lattices. A good reference for this area is the book of K. Engel [77].

If P is a partially ordered set, then width$\,(P)$ is the maximum size of an antichain in P. If P can be partitioned into m chains, then $m \geq$ width$\,(P)$. Dilworth's chain partition theorem says that equality can be achieved.

Theorem 6-7.1 (Dilworth's chain partition theorem)**.** *Let* P *be a partially ordered set. Then there exists a chain partition of* P *into* width$\,(P)$ *chains. In particular,* width$\,(P)$ *equals* k, *where* k *is the minimum number of chains in a chain partition of* P.

Proof. There are many proofs of the theorem. We give an efficient induction proof due to F. Galvin [101]. Let a be a maximal element of P and suppose that $P \setminus \{a\}$ has width k, so that P has width k or $k+1$. By induction, $P \setminus \{a\}$ has a partition into k chains $C_1, C_2, \ldots, C_k$. If A is an antichain with k elements, then $|A \cap C_i| = 1$. Let a_i be the maximum element in the chain C_i which is in some size-k antichain. Then $\{a_1, a_2, \ldots, a_k\}$ is an antichain. (To see this, suppose that $a_2 > a_1$, say. Let $\{b_1, a_2, b_3, \ldots, b_k\}$ be a size-k antichain containing a_2 so that $b_1 \in C_1$, then $a_2 > a_1 \geq b_1$, a contradiction.)

If $\{a, a_1, a_2, \ldots, a_k\}$ is an antichain, then P has width $k+1$ and $\{a\}, C_1, C_2, \ldots, C_k$ is a partition into $k+1$ chains. If not, $a > a_i$ for some i. Then $\{y : y \in C_i, y \leq a_i\} \cup \{a\}$ is a chain K. Since every size-k antichain in $P \setminus \{a\}$ contains one of the elements in $\{y : y \in C_i, y \leq a_i\}$, $P \setminus K$ does not contain any size-k antichain. By induction, there is a partition $D_1, D_2, \ldots, D_{k-1}$ of $P \setminus K$ into $k-1$ antichains. Adding K to this partition, we obtain a partition of P into k chains. □

Theorem 6-7.2 (Sperner)**.** *Let $\mathcal{A}$ be an antichain in the Boolean algebra $\mathtt{B}_n$. Then*

$$|\mathcal{A}| \leq \binom{n}{\lfloor n/2 \rfloor}.$$

An early result of A. de Moivre and J. Stirling (fundamental in probability and statistics) implies that

$$\binom{n}{\lfloor n/2 \rfloor} \approx \frac{2^n}{\sqrt{\pi n/2}}.$$

Thus, the largest antichain is almost the same order of magnitude as the entire Boolean algebra.

Sperner's theorem says that the largest antichain in $\mathtt{B}_n$ occurs at the biggest levels. Thus, we may ask whether it holds in other partially ordered sets. A partially ordered set P is *ranked* if for every element x, all the maximal chains from a minimal element to x have the same length. This length is the *rank* (or the *height*) of x. The *i-th level* $\mathrm{Lev}(i)$ of P is the set $\{x : \mathrm{rank}(x) = i\}$ and the *Whitney number W_i (of the second kind)* is defined by $W_i = |\mathrm{Lev}(i)|$. Any level of a ranked partial order is an antichain and hence, $\mathrm{width}\, P \geq W_i$. The partially ordered set P is *Sperner* if

$$\mathrm{width}\, P = \max\{W_i\}.$$

The next lemma, due to L.H. Harper and G.-C. Rota [194], extracts two properties of a partial order which would imply that it is Sperner.

Lemma 6-7.3. *Let P be a ranked partially ordered set of rank n satisfying the following two properties:*

Unimodality. *The sequence W_i of Whitney numbers is unimodal, that is, there is an index m, called a peak, such that*

$$W_0 \le W_1 \le W_2 \le \cdots \le W_m \text{ and } W_m \ge W_{m+1} \ge \cdots \ge W_{n-1} \ge W_n.$$

The matching property. *For $1 \le i \le n$, there is a matching between* $\mathrm{Lev}(i-1)$ *and* $\mathrm{Lev}(i)$, *that is, there are functions $\sigma_1, \ldots, \sigma_n$ such that*

- *whenever $1 \le i \le m$, $\sigma_i \colon \mathrm{Lev}(i-1) \to \mathrm{Lev}(i)$ is one-to-one and $x \le \sigma_i(x)$ for all $x \in \mathrm{Lev}(i-1)$;*
- *whenever $m+1 \le i \le n$, $\sigma_i \colon \mathrm{Lev}(i) \to \mathrm{Lev}(i-1)$ is one-to-one and $\sigma_i(x) \le x$ for all $x \in \mathrm{Lev}(i)$.*

Then P is Sperner.

Proof. Let m be a peak of the unimodal sequence W_i. For each element x in $\mathrm{Lev}(m)$, a level with the maximum number of elements, build the chain C_x :

$$\cdots \le \sigma_{m-1}^{-1}\sigma_m^{-1}(x) \le \sigma_m^{-1}(x) \le x \le \sigma_{m+1}^{-1}(x) \le \sigma_{m+2}^{-1}\sigma_{m+1}^{-1}(x) \le \cdots$$

starting at x and adding matched elements until one reaches an unmatched element at either end. The set of chains

$$\{C_x : x \in \mathrm{Lev}(m)\}$$

gives a chain partition of size W_m. Hence $W_m \ge \mathrm{width}(P)$. Since $\mathrm{Lev}(m)$ is an antichain, we also have $\mathrm{width}(P) \ge W_m$. □

We will use Lemma 6-7.3 to prove Sperner's theorem. We note that the Boolean algebra B_n is isomorphic to its order dual and the ith Whitney number equals the binomial coefficient $\binom{n}{i}$. Unimodality now follows from the fact that when $i \le \lfloor n/2 \rfloor$, $n - i \ge i + 1$. Hence

$$\binom{n}{i+1} = \frac{n-i}{i+1}\binom{n}{i} \ge \binom{n}{i}.$$

To prove the matching property, let $i \le \lfloor n/2 \rfloor$ and consider the relation of containment between the collection of all i-subsets and the collection of all $(i+1)$-subsets. For a collection $\mathcal{A}$ of i-subsets, let $\mathcal{A}'$ be the collection of $(i+1)$-subsets B such that B contains an i-subset in $\mathcal{A}$. Since an $(i+1)$-subset contains $i+1$ i-subsets and an i-subset is contained in $n-i$ $(i+1)$-subsets,

$$(i+1)|\mathcal{A}'| \ge (n-i)|\mathcal{A}|.$$

As $n - i \ge i + 1$, we conclude that $|\mathcal{A}'| \ge |\mathcal{A}|$. By the marriage theorem in combinatorics (see, for example, [244], Chapter 5), there exists a matching from the i-subsets to the $(i+1)$-subsets.

This proof of Sperner's theorem extends easily to subspace lattices.

Theorem 6-7.4. *The lattice $L(n,q)$ of subspaces of an n-dimensional vector space over the finite field* $\mathrm{GF}(q)$ *is Sperner.*

Proof. Use the fact that the Whitney numbers of $L(n, q)$ are q-binomial coefficients (see Exercise 6.16). $\square$

Lemma 6-7.3 motivated the optimistic conjecture that all geometric lattices are Sperner. This conjecture was shown to be false. The first counterexamples were found by R.P. Dilworth and C. Greene [66] (see LTF, Exercise 3.29, p. 357). Even worse, E.R. Canfield [37] showed that a partition lattice $\mathrm{Part}(n)$ is not Sperner when n is sufficiently large. Further counterexamples were constructed by J. Kahn [218]. All these counterexamples satisfy unimodality and hence, the matching property fails. A much weaker version of the Sperner property for geometric lattices has been proposed (Exercise 6.23).

We end this section with an instance of Kahn's construction. A rank-n geometric lattice L is a *paving lattice* if for $0 \leq i \leq n-2$, every rank-i flat is above exactly i atoms, or equivalently, for $0 \leq i \leq n-2$ and X a rank-i flat, the lower interval $[\hat{0}, X]$ is a Boolean algebra. Let S be the set of atoms and $\mathcal{B}$ be the collection of rank-$(n-1)$ flats or *coatoms*, considered as subsets of S. The following lemma (due to J.R. Hartmanis [195]) follows immediately from the matroid axioms.

Lemma 6-7.5. *A collection $\mathcal{B}$ of subsets of S is the collection of coatoms of a rank-n paving lattice with S its set of atoms if and only if* (i) $|\mathcal{B}| \geq 2$, (ii) *each $(n-1)$-subset of atoms is in a unique subset in $\mathcal{B}$, and* (iii) *each subset in $\mathcal{B}$ contains at least $n-1$ atoms.*

The set of atoms, the collection of coatoms, and the rank determine a paving lattice L. Let $\mathrm{Pav}(S, \mathcal{B}, n)$ be the rank-n paving lattice with set S of atoms and the collection $\mathcal{B}$ of coatoms. A coatom in a rank-n paving lattice is *nontrivial* if it contains at least n atoms.

Lemma 6-7.6. *Let C be a nontrivial coatom in* $\mathrm{Pav}(S, \mathcal{B}, n)$ *and $\binom{C}{s}$ be the collection of all subsets of s atoms contained in C. Then the collection $\mathcal{B}[C]$, defined by*

$$\mathcal{B}[C] = (\mathcal{B} \setminus \{C\}) \cup \binom{C}{n-1},$$

is the collection of coatoms of a paving lattice.

Recall that a t-(v, k, λ) *design* on the *point set* S with $|S| = v$ is a collection $\mathcal{D}$ of subsets of S called *blocks*, each of size k, such that every t-element subset of S occurs in exactly λ blocks. By Lemma 6-7.5, a t-$(v, k, 1)$ design is the collection of coatoms of a rank-$(t+1)$ paving lattice. To construct a specific non-Sperner paving lattice, let $\mathcal{D}$ be the 3-$(q^n + 1, q + 1, 1)$ design with $q = 11$ and $n = 2$ described in Exercise 6.22 and $L = \mathrm{Pav}(S, \mathcal{D}, 4)$. Then L is a rank-4 paving lattice with Whitney numbers

$$W_0 = 1,\ W_1 = 122,\ W_2 = \binom{122}{2} = 7381,\ W_3 = 1432,\ W_4 = 1.$$

Choose a block C in $\mathcal{D}$ and let $L^\flat = \mathrm{Pav}(S, \mathcal{D}[C], 4)$. Then in $L^\flat$, the Whitney number W_3 changes to $1432 - 1 + \binom{12}{3}$, or 1561. In particular, the maximum Whitney number of $L^\flat$ is W_2. However, the set

$$\left[\mathrm{Lev}(2) \setminus \binom{C}{2}\right] \cup \binom{C}{3}$$

is an antichain in $L^\flat$ with size $7381-66+220$, or 7535, strictly greater than W_2.

6-8. Exercises

6.1. This problem uses the notation in LTF, p. 49. Let (α, β) be a Galois connection between the partially ordered sets K and L, and $C(\alpha, \beta)$ be the partially ordered set of closed elements in K. Let $x \in K$ and $y \in L$. If both x and y are closed, then

$$\sum_{a\colon \alpha(a)=y} \mu_K(a, x) \;=\; \mu_{C(\alpha,\beta)}(y, x) \;=\; \sum_{s\colon \beta(s)=x} \mu_L(y, s).$$

If at least one of x and y is not closed, then

$$\sum_{a\colon \alpha(a)=y} \mu_K(a, x) \;=\; 0 \;=\; \sum_{s\colon \beta(s)=x} \mu_L(y, s)$$

(G.-C. Rota [280]).

6.2. Let $\mathrm{Int}\, P$ be the set of intervals in a partially ordered set P ordered by set containment. Show that when P is finite, $\mathrm{Int}\, P$ is a lattice. Let μ be the Möbius function of $\mathrm{Int}\, P$. Show that if $[x, y]$ is nonempty, then

$$\mu(\varnothing, [x, y]) = -\mu_P(x, y),$$

and if both $[u, v]$ and $[x, y]$ are nonempty, then

$$\mu([u, v], [x, y]) = \mu_P(x, u)\mu_P(v, y)$$

(H.H. Crapo [42]).

6.3. The set $L(\mathcal{C})$ of unions of subsets from a finite collection $\mathcal{C}$ of subsets forms a lattice with join defined by $A \vee B = A \cup B$ and meet defined by

$$A \wedge B = \bigcup_{C\colon C \subseteq A \text{ and } C \subseteq B} C.$$

An *integer interval* is a subset of integers of the form $\{a, a+1, a+2, \ldots, b-1, b\}$ where $b \geq a$. A lattice L is an *integer-interval lattice* if there is a collection $\mathcal{C}$ of integer intervals such that $L \cong L(\mathcal{C})$.

(a) Show that an interval in an integer-interval lattice is an integer-interval lattice.

(b) Show that in an integer-interval lattice, $\mu(A, B) = -1$, 0, or 1 (C. Greene [187]).

6.4. Let L be a lattice in which $\mu(x, \hat{1}) \neq 0$ for all $x \in L$, and f and g be functions from L to a ring $\mathbb{A}$. Show that

$$g(x) = \sum_{y:\, y \vee x = \hat{1}} f(y) \text{ for all } x \iff$$

$$f(x) = \sum_{y:\, y \in L} \left(\sum_{z:\, 0 \leq z \leq x \wedge y} \frac{\mu(z,x)\mu(z,y)}{\mu(z,\hat{1})} \right) g(y) \text{ for all } x$$

(T.A. Dowling and R.M. Wilson [73]).

6.5. There is exactly one complementing permutation in the Boolean algebra B_n. How many complementing permutations are there in the lattice $L(n, q)$ of subspaces of an n-dimensional vector space over the finite field $\mathrm{GF}(q)$? in the partition lattice $\mathrm{Part}(n)$?

6.6. Let x be an element in a rank-n geometric lattice L. Show that x is modular if and only if for all elements u in L, $\mathrm{rank}(u \vee x) = n$ and $u \wedge x = \hat{0}$ imply that $\mathrm{rank}(u) = n - k$ (R.P. Stanley [305]).

6.7. Let L be a rank-n geometric lattice and

$$T^k_{\mathrm{mod}} = \{x :\, x \,\text{is modular and}\, \mathrm{rank}(x) \geq n - k\}.$$

Show that when $k \leq n/2$, T^k_{mod} is concordant with B_k with $x^{\#} = \hat{0}$ in the order dual of L. Conclude that

$$W_0 + W_1 + W_2 + \cdots + W_k \geq M_{n-k} + M_{n-k+1} + \cdots + M_{n-1} + M_n$$

(J.P.S. Kung [228]).

6.8. *The top-heaviness conjecture.* Let L be a rank-n geometric lattice. Is it true that if $k \leq n/2$, then $W_k \leq W_{n-k}$? (T.A. Dowling and R.M. Wilson [73]).

6.9. *Finite consistent lattices.* Let L be a lattice. A join-irreducible j in L is *consistent* if for all elements x in L, $x \vee j$ equals x or is a join-irreducible in the upper interval $[x, \hat{1}]$. A lattice is *consistent* if every join-irreducible is consistent. Let $\mathrm{CJi}\, L$ be the set of consistent join-irreducibles and $\mathrm{Mi}\, L$ the set of meet-irreducibles in L.

(a) Show that $\mathrm{CJi}\, L \cup \{\hat{0}\}$ is concordant $\mathrm{Mi}\, L \cup \{\hat{1}\}$. Conclude that in a consistent lattice L, $|\mathrm{Ji}\, L| \leq |\mathrm{Mi}\, L|$.

(b) Show that a modular lattice is consistent.

(c) Let L be a consistent semimodular lattice. Show that if $|\mathrm{Ji}\, L| = |\mathrm{Mi}\, L|$, then L is modular.

(d) Let L be a consistent dually semimodular lattice. Show that if $|\mathrm{Ji}\, L| = |\mathrm{Mi}\, L|$, then L is modular.

(e) Let L be a lattice in which the number of join-irreducibles equals the number of meet-irreducibles. Is it true that both L and its order dual are consistent? (K.M. Gragg and J.P.S. Kung [124], J.P.S. Kung [227]).

6.10. Let G be a finite group. A subgroup H of G is *subnormal* if there exists a chain $G = N_0 \supset N_1 \supset N_2 \supset \cdots \supset N_{r-1} \supset N_r = H$ such that N_{i+1} is a normal subgroup in the subgroup N_i. Show that the subnormal subgroups form a sublattice $W(G)$ of the lattice of subgroups of G. Show that $W(G)$ is consistent and dually semimodular (H. Wielandt [340]).

6.11. Let D_{2n} be the dihedral group of order $2n$ consisting of the symmetries of a regular n-gon.

(a) Suppose that n has prime factorization $p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$. Show that

$$\varphi(D_{2n}; s) = \left(\frac{n}{p_1 p_2 \cdots p_r} \right)^s \varphi(D_{2p_1 p_2 \cdots p_r}; s).$$

(b) Show that $\varphi(D_{2p_1 p_2 \cdots p_r}; s) = (2^s - 1) \prod_{i=1}^{r} (p_i^s - p_i)$.

6.12. *Quotients of geometric lattices.* Let L be a rank-n geometric lattice. A *modular cut* M in L is a nonempty subset of flats of L satisfying

MC1. If $x \geq y$ and $y \in M$, then $x \in M$.

MC2. If $x, y \in M$, $x \vee y$ covers x and y, and x, y cover $x \wedge y$, then $x \wedge y \in M$.

The *collar* of a modular cut M is the set of flats of L such that $z \notin M$ and z is covered by a flat y in M. The *quotient* L/M is the set obtained from L by removing the flats in the collar of M with the partial order inherited from L. Show that L/M is a geometric lattice of rank $n - 1$.

6.13. *Chromatic polynomials of graphs.* We use the notation in LTF, Section 3-3.4. Note that the edge lattice of a graph is the lattice of flats of the cycle matroid of that graph.

(i) Let λ be a nonnegative integer. A *proper λ-coloring* of a graph $(G; E)$ with vertex set G and edge set E is a function $c \colon G \to \{1, 2, \ldots, \lambda\}$ such that if $\{a, b\}$ is an edge, then $c(a) \neq c(b)$. Let

L be the edge lattice of a graph G with c connected components. Show that the number of proper λ-colorings of $(G; E)$ equals $\lambda^c \chi(L; \lambda)$.

(ii) Let K_n be the edge lattice of the complete graph with vertex set equal to $\{1, 2, \ldots, n\}$ and edge set all possible 2-element subsets of $\{1, 2, \ldots, n\}$. Show that the edge lattice of K_n is isomorphic to the partition lattice $\mathrm{Part}(n)$. Observing that the number of λ-colorings of K_n is $\lambda(\lambda-1)(\lambda-2)\cdots(\lambda-n+1)$, conclude that

$$\chi(\mathrm{Part}(n); \lambda) = \prod_{i=1}^{n-1} (\lambda - i).$$

6.14. Let L be a geometric lattice. Show that

$$\det[\lambda^{\mathrm{rank}(X \vee Y)}]_{X,Y \in L} = \prod_{X:\, X \in L} \chi([X, \hat{1}]; \lambda).$$

Find an analog of this identity for Eulerian functions (H.S. Wilf [342], slightly generalized).

6.15. *A multiplication identity.* Show that if L is a rank-n geometric lattice,

$$\chi(L; \lambda\xi) = \sum_{X:\, X \in L} \lambda^{n - \mathrm{rank}(X)} \chi([\hat{0}, X]; \lambda) \chi([X, \hat{1}]; \xi)$$

(J.P.S. Kung [232]).

6.16. *Characteristic polynomials of $L(n, p)$ and Cauchy's identity.* Let $L(n, q)$ be a subspace lattice.

(i) Show that the number of subspaces of rank i in $L(n, q)$ is the q-binomial coefficient $\binom{n}{i}_q$, defined by

$$\binom{n}{i}_q = \frac{(q^n - 1)(q^{n-1} - 1)\cdots(q^{n-i+1} - 1)}{(q^i - 1)(q^{i-1} - 1)\cdots(q - 1)}.$$

(ii) Calculate the characteristic polynomial $\chi(L(n, q); \lambda)$ in two different ways, from the basic definition and using Corollary 6-6.7, to obtain

$$\sum_{i=0}^{n} (-1)^i \binom{n}{i}_q q^{i(i-1)/2} \lambda^{n-i} = \chi(L(n, q); \lambda) = \prod_{i=0}^{n-1} (\lambda - q^i).$$

The identity we obtain by leaving out the middle term is called *Cauchy's identity.*

6.17. Identify an antichain $\mathcal{A}$ with the order ideal it generates, that is, let

$$\mathcal{A} \longleftrightarrow \{x : x \leq a \text{ for some } a \text{ in } \mathcal{A}\}.$$

Show that under this identification, the maximum-size antichains form a sublattice $\mathcal{S}(P)$ of the lattice $\mathcal{D}(P)$ of all antichains (or ideals). Conclude that $\mathcal{S}(P)$ is distributive. Show that when A and B are maximum-size antichains, $A \wedge B$ is the antichain consisting of the minimal elements in the union $A \cup B$.

6.18. Show that there is a natural bijection between the free distributive lattice $\mathrm{Free}_{\mathbf{D}}(n)$ and the set of proper antichains in the Boolean algebra B_n, where an antichain is *proper* if it is not the 1-element antichains consisting of the minimum $\hat{0}$ or the maximum $\hat{1}$.

6.19. *Dedekind's problem.* The free distributive lattice $\mathrm{Free}_{\mathbf{D}}(n)$ on n generators is finite (LTF, Theorem 126). Show that

$$2^{(1+\alpha_n)\binom{n}{\lfloor n/2 \rfloor}} \leq |\mathrm{Free}_{\mathbf{D}}(n)| \leq 2^{(1+\beta_n)\binom{n}{\lfloor n/2 \rfloor}},$$

where $\alpha_n = ce^{-n/4}$, $\beta_n = c'(\log n)/n)$, and c, c' are constants (D.J. Kleitman and G. Markowsky [225], A.D. Korshunov [226], A.A. Sapozhenko [289], J. Kahn [219]).

6.20. *Explicit chain partitions.* A *symmetric chain* in the Boolean algebras $2^{\{1,2,\ldots,n\}}$ is a chain $E_k, E_{k+1}, \ldots, E_{n-k}$ such that $|E_i| = i$ (and, of course, $E_i \subset E_{i+1}$). Symmetric chains are centered at a set of size $n/2$ if n is even and two sets of size $\lfloor n/2 \rfloor$ and $\lceil n/2 \rceil$ if n is odd. A *symmetric chain partition* is a partition of $2^{\{1,2,\ldots,n\}}$ into symmetric chains.

(i) Show that the following inductive construction produces a symmetric chain partition: When $n = 1$, partition $2^{\{1\}}$ into one chain of size 2. If $2^{\{1,2,\ldots,n-1\}}$ has been partitioned into symmetric chains, let

$$E_k, E_{k+1}, \ldots, E_{n-k}$$

be a length-l symmetric chain in this partition, where $l = n - 2k$. Define two new chains, the first by removing E_{n-k} and adding the element n to each of the remaining sets, giving the length-$(l-1)$ symmetric chain

$$E_k \cup \{n\}, E_{k+1} \cup \{n\}, \ldots, E_{n-k-1} \cup \{n\},$$

and the second by putting the subset $E_{n-k} \cup \{n\}$ on the top of the chain, giving the length-$(l+1)$ symmetric chain

$$E_k, E_{k+1}, \ldots, E_{n-k}, E_{n-k} \cup \{n\}.$$

The new chains which are nonempty yield a symmetric chain partition of $2^{\{1,2,\ldots,n\}}$.

(ii) Find a similar construction for subspace lattices (N.G. de Bruijn, C. van Ebbenhorst Tengbergen, and D. Kruyswijk [33], F. Vogt and B. Voigt [321]).

6.21. Show that the lattice of divisors of a positive integer is Sperner (L.D. Mešalkin [249]).

6.22. *Inversive planes.* Let q be a power of a prime, $\mathrm{GF}(q^n)$ an extension field of the finite field $\mathrm{GF}(q)$, $S = \mathrm{GF}(q^n) \cup \{\infty\}$, and $\mathrm{PGL}(2, q^n)$ be the group of linear fractional transformations $S \to S$ of the form

$$x \mapsto \frac{ax + b}{cx + d},$$

where $a, b, c, d \in \mathrm{GF}(q^n)$, $ad - bc = 1$, and ∞ is a formal symbol obeying the rules: when $a \neq 0$, $a\infty + b = \infty$, $\infty \cdot 0 = 1$, $1/\infty = 0$, and $1/0 = \infty$. If $\sigma \in \mathrm{PGL}(2, q^n)$, let $[\sigma] = \{\sigma x : x \in \mathrm{GF}(q)\}$. Show that the point set S and the collection $\{[\sigma] : \sigma \in \mathrm{PGL}(2, q^n)\}$ is a 3-$(q^n + 1, q + 1, 1)$ design with $q(q^n + 1)$ blocks (see, for example, D.R. Hughes and F.C. Piper [200], Section 4.3).

6.23. *A unimodality conjecture.* Does the unimodality property in Lemma 6-7.3 always hold in a geometric lattice? (G.-C. Rota [194]).

6.24. Let L be a rank-n geometric lattice. Is any of the following statements true?

(i) $W_0 \leq W_1 \leq W_2 \leq \cdots \leq W_{\lfloor n/2 \rfloor}$.

(ii) For $1 \leq i \leq \lfloor \frac{n}{2} \rfloor$, there is a matching from $\mathrm{Lev}(i-1)$ to $\mathrm{Lev}(i)$ (J.P.S. Kung [229]).

6.25. *The FKG inequality.* Let L be a finite distributive lattice, m a "weight function" from L to the positive real numbers, and $M = \sum_{x \in L} m(x)$. The *m-average* $\langle f \rangle$ of a function $f\colon L \to \mathbb{R}$ is the real number defined by

$$\langle f \rangle = \frac{1}{M} \sum_{x\colon x \in L} m(x) f(x).$$

(a) Suppose that the weight function m satisfies the *supermodular inequality*: for all $x, y \in L$,

$$m(x \vee y) m(x \wedge y) \geq m(x) m(y)$$

and f and g are both increasing functions. Show that

$$\langle fg \rangle \geq \langle f \rangle \langle g \rangle.$$

(b) Let S be a finite set. Show that if $\mathcal{U}$ is a collection of subsets of S which is *closed above* (that is, if $A \in \mathcal{U}$ and $B \supset A$, then $B \in \mathcal{U}$) and $\mathcal{L}$ is a collection of subsets which is closed below (that is, if $A \in \mathcal{L}$ and $B \subset A$, then $B \in \mathcal{L}$), then

$$|\mathcal{U} \cap \mathcal{L}| 2^{|S|} \leq |\mathcal{U}||\mathcal{L}|$$

(C.M. Fortuin, P.W. Kasteleyn, and J. Ginibre [86], P.D. Seymour and D.J.A. Welsh [301], D.J. Kleitman [224]; see also I. Anderson [6], Chapter 6).

Part III

Congruence Lattices of Infinite Lattices, and Beyond

Introduction

Chapters 7–9 deal with the *Congruence Lattice Problem* (from now on, CLP):

> Is every algebraic distributive lattice isomorphic to the congruence lattice of a lattice?

The congruence lattice $\operatorname{Con} L$ of any lattice L is distributive. This result appeared first in the 1942 paper Funayama and Nakayama [99] (see also Exercises 7.1 and 7.2). Further, the congruence lattice of any universal algebra (thus, in particular, of any lattice) is algebraic. This result appeared first in the 1948 paper Birkhoff and Frink [30]. CLP seems natural based on these two results.

CLP was first considered, and solved in the finite case, by Dilworth in the ninety forties. This problem was solved, in the negative, in Wehrung [335].

Although CLP is stated here as a problem of lattice theory, Chapters 7–9 will, in many instances, stray away from that initially set framework and thus touch upon other research domains, most notably ring theory and module theory. Hence we intend Chapters 7–9 as a guided tour of the amazing variety of landscapes that were crossed during the 60 year effort for a solution of CLP, together with new research problems, not limited to lattice theory.

For an overview to the solution of CLP, see Section 9-3.

The first published appearance of CLP for finite lattices was an exercise with asterisk (attributed to Dilworth) in the 1948 edition of Birkhoff's lattice theory book [28]. The first published proof of this result appears in Grätzer and Schmidt's 1962 paper [164]. However, it seems that the earliest attempts at CLP were made by Dilworth himself, as hinted, in particular, by pages 455–456 in Bogart, Freese, and Kung [32] and Grätzer [129].

As the finite case of CLP has been known for a long time, and investigated in detail in many other published works (see Grätzer's monograph [131], also the survey paper Grätzer and Schmidt [179]), Chapters 7–9 will deal mainly with *infinite* lattices. Lest they occupy many times the volume of the present book, we will omit most proofs (but not the references), with few exceptions.

Chapter 10 deals with two formally related topics. The first is the characterization problem of *complete congruence lattices of complete lattices.* These are *not* distributive, as pointed out in K. Reuter and R. Wille [278]. In fact, the solution, provided by G. Grätzer [127], is in the affirmative: every complete lattice can be represented as the lattice of complete congruences of a complete lattice.

The second topic is the investigation of $\operatorname{Princ} L$, the *order of principal congruencies of a lattice L*. For a bounded lattice L, the order $\operatorname{Princ} L$ is characterized as a bounded order, G. Grätzer [134].

Chapter 7

Schmidt and Pudlák's Approaches to CLP

by Friedrich Wehrung

7-1. Introduction

Chapter 7 will be mainly focused on approaches to CLP, initiated in 1968 by Schmidt [292] and in 1985 by Pudlák [274], from which most known representation results as congruence lattices of lattices originate.

A central idea of Part III is the concept of *simultaneous representation*: instead of trying to represent a single algebraic distributive lattice A (as the congruence lattice of a lattice), one tries to represent *diagrams* of algebraic distributive lattices, such as, for example, a morphism $\boldsymbol{f}\colon A_0 \to A_1$ between algebraic distributive lattices A_0 and A_1 (try to represent it as the image, under the Con functor, of a homomorphism between lattices).

The correct concept of morphism that has to be used in this context is not completely trivial *a priori*: namely, $\boldsymbol{f}$ needs to be a *complete join-homomorphism*, that is, $\bigvee \boldsymbol{f}(X) = \boldsymbol{f}(\bigvee X)$ whenever X is a (possibly empty, possibly infinite) subset of A_0; furthermore, it needs to send compact elements to compact elements (we say that $\boldsymbol{f}$ is *compactness-preserving*). Therefore, most authors prefer to work in the category of all $(\vee, 0)$-semilattices with $(\vee, 0)$-homomorphisms, known to be equivalent to the category of all algebraic lattices

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with compactness-preserving, complete join-homomorphisms, and formulate CLP (and related problems) with semilattices of compact congruences instead of lattices of congruences. Let us be a bit more precise.

Denote by $\operatorname{Con} A$ the congruence lattice (i.e., the lattice of all congruences) of an algebra (i.e., a universal algebra) A, and by $\operatorname{Con_c} A$ the $(\vee, 0)$-semilattice of all compact (i.e., finitely generated) congruences of A – often called the *congruence semilattice* of A. For algebras A and B over the same similarity type Σ and a homomorphism $f\colon A \to B$, we denote by $\operatorname{Con} f$ the map from $\operatorname{Con} A$ to $\operatorname{Con} B$ sending every $\boldsymbol{\alpha} \in \operatorname{Con} A$ to the congruence of B generated by $\{\, (f(x), f(y)) \mid (x, y) \in \boldsymbol{\alpha} \,\}$. The map $\operatorname{Con} f$ is a complete join-homomorphism from $\operatorname{Con} A$ to $\operatorname{Con} B$, sending compact elements to compact elements. We denote by $\operatorname{Con_c} f$ the restriction of $\operatorname{Con} f$ from $\operatorname{Con_c} A$ to $\operatorname{Con_c} B$.

The assignment $A \mapsto \operatorname{Con} A$, $f \mapsto \operatorname{Con} f$ defines a functor from the category $\mathbf{Alg}_\Sigma$ of all algebras of a given similarity type Σ to the category of all algebraic lattices with compactness-preserving complete join-homomorphisms. Similarly, the assignment $A \mapsto \operatorname{Con_c} A$, $f \mapsto \operatorname{Con_c} f$ defines a functor from $\mathbf{Alg}_\Sigma$ to the category $\mathbf{Sem}_{\vee,0}$ of all $(\vee, 0)$-semilattices with $(\vee, 0)$-homomorphisms.

For $(\vee, 0)$-semilattices S, T and a $(\vee, 0)$-homomorphism $f\colon S \to T$, we denote by $\operatorname{Id} f$ the map from $\operatorname{Id} S$ to $\operatorname{Id} T$ that to an ideal X of S associates the ideal of T generated by $f(X)$. The assignment $S \mapsto \operatorname{Id} S$, $f \mapsto \operatorname{Id} f$ defines a category equivalence, from $(\vee, 0)$-semilattices with $(\vee, 0)$-homomorphisms to algebraic lattices with compactness-preserving complete join-homomorphisms, and it also defines a natural equivalence from $\operatorname{Con_c}$ to Con. This makes the study of the two functors Con and $\operatorname{Con_c}$ essentially equivalent. We will often prefer $\operatorname{Con_c}$ to Con, mainly because of the more convenient expression of directed colimits in $\mathbf{Sem}_{\vee,0}$ given by Lemma 7-2.3. The corresponding formulation of CLP is the following.

Semilattice formulation of CLP. *Is every distributive* $(\vee, 0)$*-semilattice isomorphic to the congruence semilattice of some lattice?*

Recall from LTF that a $(\vee, 0)$-semilattice S is *distributive* if its ideal lattice $\operatorname{Id} S$ is distributive. Equivalently, for all $a_0, a_1, b \in S$, if $b \leq a_0 \vee a_1$, then there are $x_0 \leq a_0$ and $x_1 \leq a_1$ in S such that $b = x_0 \vee x_1$.

We shall mostly use the notation, terminology, and results of LTF, with deviations backed up by suitable restatements. We shall denote by $\mathbf{2} = \{0, 1\}$ the two-element chain, viewed either as a poset, a semilattice, or a lattice, depending on the context. By "countable" we shall always mean "at most countable". If (Φ) is a property of posets, a poset P is *conditionally* (Φ) if every closed interval of P satisfies (Φ).

We shall set

$$\operatorname{Ker}(f) = \{\, (x, y) \in X \times X \mid f(x) = f(y) \,\}\,,$$

for every function f with domain X. If X and Y have distinguished "zero elements" 0_X and 0_Y, we shall say that f *preserves zero* if $f(0_X) = 0_Y$, and *separates zero* if $f^{-1}\{0_Y\} = \{0_X\}$. An equivalence relation $\boldsymbol{\theta}$ on X *separates zero* if $(0_X, x) \in \boldsymbol{\theta}$ iff $x = 0_X$, for each $x \in X$.

For lattices K and L, a lattice homomorphism $f\colon K \to L$ is 0-*lattice homomorphism* if it preserves zero. We can define similarly $(0,1)$-lattice homomorphisms.

We shall set

$$Q \downarrow X = \{\, q \in Q \mid (\exists x \in X)(q \leq x) \,\}\,,$$
$$Q \uparrow X = \{\, q \in Q \mid (\exists x \in X)(q \geq x) \,\}\,,$$

for all subsets Q and X in a poset P, and we shall write $Q \downarrow p$ (resp., $Q \uparrow p$) instead of $Q \downarrow \{p\}$ (resp., $Q \uparrow \{p\}$), for any $p \in P$.

7-2. Categorical background

Many of the results we shall discuss will rely on the representation of any distributive $(\vee, 0)$-semilattice as a suitable directed colimit of finite distributive $(\vee, 0)$-semilattices – essentially Theorems 7-4.2 and 7-4.6. As we will need to consider the directed colimits of objects other than semilattices, we shall state the required concepts in the more general context of universal algebra. Standard references on universal algebra are Burris and [36], Grätzer [125], McKenzie, McNulty, and Taylor [248].

Our categories will mostly be subcategories of the category $\mathbf{Alg}_\Sigma$ of all algebras over a similarity type Σ. For $A, B \in \mathbf{Alg}_\Sigma$, a morphism from A to B is a *homomorphism*, that is, a map $\varphi\colon A \to B$ such that[1]

$$\varphi(f^A(x_1, \dots, x_n)) = f^B(\varphi(x_1), \dots, \varphi(x_n))\,,$$

for each $f \in \Sigma$ of arity n and all $x_1, \dots, x_n \in A$. If, in addition, f is one-to-one, then we shall say that f is an *embedding*. The following categories will be of special importance to us:

- The category $\mathbf{Alg}_\Sigma$ of all algebras over a given similarity type Σ;
- The category $\mathbf{Sem}_{\vee,0}$ of all $(\vee, 0)$-semilattices with $(\vee, 0)$-homomorphisms;
- The category $\mathbf{DSem}_{\vee,0}$ of all distributive $(\vee, 0)$-semilattices with $(\vee, 0)$-homomorphisms;
- The category $\mathbf{DSem}^{\mathrm{emb}}_{\vee,0}$ of all distributive $(\vee, 0)$-semilattices with $(\vee, 0)$-embeddings;
- The category $\mathbf{Lat}$ of all lattices with lattice homomorphisms.

[1]By f^A, we mean the interpretation of the symbol f in A; so $f^A\colon A^n \to A$. We shall omit the superscript A in most cases, where A will be clear from the context.

For a (nonempty) poset P and a category $\mathcal{S}$, a *diagram in $\mathcal{S}$ indexed by P* is a functor from P, viewed the usual way as a category[2], to $\mathcal{S}$. We shall represent such an object as a collection of the form

$$\vec{S} = (S_p, \sigma_p^q \mid p \leq q \text{ in } P)\,, \tag{7-2.1}$$

where the S_p are objects of $\mathcal{S}$, $\sigma_p^q \colon S_p \to S_q$ for $p \leq q$ in P, $\sigma_p^p = \mathrm{id}_{S_p}$, and $\sigma_p^r = \sigma_q^r \circ \sigma_p^q$ whenever $p \leq q \leq r$ in P. The morphisms σ_p^q are called the *transition morphisms* of $\vec{S}$. If P is directed, then we will say that $\vec{S}$ is a *directed diagram.*

A *co-cone above* the diagram $\vec{S}$ of (7-2.1) is a family of the form

$$\overline{S} = (S, \sigma_p \mid p \in P)\,, \tag{7-2.2}$$

where S is an object of $\mathcal{S}$, $\sigma_p \colon S_p \to S$ (the σ_p are called the *limiting morphisms* of $\overline{S}$), and $\sigma_p = \sigma_q \circ \sigma_p^q$ for all $p \leq q$ in P. The "least" such diagram, if it exists, is the *colimit* of $\vec{S}$. Formally, the family $\overline{S}$ of (7-2.2) is a colimit of $\vec{S}$ if it is a co-cone above $\vec{S}$, and for every co-cone $\overline{X} = (X, \xi_p \mid p \in P)$ above $\vec{S}$, there exists a unique morphism $\tau \colon \overline{S} \to \overline{X}$, that is, a morphism $\tau \colon S \to X$ in $\mathcal{S}$ such that $\tau \circ \sigma_p = \xi_p$ for each $p \in P$. We will also say that $\overline{S}$ *is a colimit co-cone above* $\vec{S}$, and we shall use the notation

$$\overline{S} = \varinjlim \vec{S}\,, \tag{7-2.3}$$

or, in expanded form,

$$(S, \sigma_p \mid p \in P) = \varinjlim (S_p, \sigma_p^q \mid p \leq q \text{ in } P)\,,$$

or simply, when all the transition morphisms and limiting morphisms are clear from the context,

$$S = \varinjlim (S_p \mid p \in P)\,.$$

The formulation (7-2.3) is a standardly enforced abuse of notation, as it determines the co-cone $\overline{S}$ only up to isomorphism. In case the poset P is directed, we shall emphasize this by saying "directed colimit" instead of just "colimit".

A subcategory $\mathcal{S}'$ of $\mathcal{S}$ is *closed under directed colimits* if above every directed diagram $\vec{S}$ of $\mathcal{S}'$ admitting a colimit in $\mathcal{S}$, there is a co-cone in $\mathcal{S}'$ which is also a colimit of $\vec{S}$ in $\mathcal{S}$. This is usually summed up by saying that "if a directed diagram $\vec{S}$ from $\mathcal{S}'$ has a colimit, then this colimit belongs to $\mathcal{S}'$" (strictly speaking, the colimit belongs to $\mathcal{S}'$ only up to isomorphism).

For a first-order theory $\mathcal{T}$ in the similarity type Σ (hence without relation symbols), we denote by $\mathbf{Mod}(\mathcal{T})$ the full subcategory of $\mathbf{Alg}_\Sigma$ consisting of

[2] The objects of that category are the elements of P. For $p, q \in P$ there exists at most one arrow from p to q, and this occurs iff $p \leq q$.

the models that satisfy all the axioms of $\mathcal{T}$. It is well known that if $\mathcal{T}$ consists only of axioms of the form

$$(\forall\vec{\mathsf{x}})\big(\mathsf{E}(\vec{\mathsf{x}}) \Rightarrow (\exists\vec{\mathsf{y}})\mathsf{F}(\vec{\mathsf{x}},\vec{\mathsf{y}})\big)\,, \tag{7-2.4}$$

with each of the formulas E and F either a tautology, a negation of a tautology, or a conjunction of atomic formulas, then $\mathbf{Mod}(\mathcal{T})$ is closed under directed colimits (see, for example, the easy direction of [38, Exercise 5.2.24]). While there are more general first-order sentences preserving directed colimits, those of the form (7-2.4) enjoy the additional advantage of preserving direct products, which will be of importance. Most of our interesting algebras, such as distributive semilattices, sectionally complemented (resp., relatively complemented) lattices, or von Neumann regular rings, can be axiomatized with formulas of the form (7-2.4), so the corresponding classes of algebras are closed under direct products and directed colimits.

Example 7-2.1. Every variety (or even every quasivariety) of algebras is closed under directed colimits. The categories $\mathbf{DSem}_{\vee,0}$ and $\mathbf{DSem}_{\vee,0}^{\mathrm{emb}}$ are also closed under directed colimits.

A functor $\Phi\colon \mathcal{S} \to \mathcal{T}$ *preserves directed colimits* if $\overline{S} = \varinjlim \vec{S}$ implies that $\Phi\overline{S} = \varinjlim \Phi\vec{S}$, for any co-cone $\overline{S}$ and any directed diagram $\vec{S}$ in $\mathcal{S}$. The following well-known result is proved, in the more general context of *algebraic systems* (not just algebras – relations are allowed), in Gillibert and Wehrung [114, Theorem 4.4.1].

Lemma 7-2.2 (folklore)**.** *The functor* $\mathrm{Con_c}\colon \mathbf{Alg}_\Sigma \to \mathbf{Sem}_{\vee,0}$ *preserves directed colimits, for any similarity type* Σ.

A typical example of a directed diagram is the following. Let A be an object in a variety $\mathbf{V}$ of algebras and denote by P the poset of all (nonempty) finitely generated subalgebras of A, endowed with set inclusion. Set $A_p = p$, and denote by $\alpha_p^q\colon p \to q$ the inclusion map, for $p \subseteq q$ in P. Then the family $(A_p, \alpha_p^q \mid p \subseteq q \text{ in } P)$ is a directed diagram in $\mathbf{V}$. Moreover, denoting by α_p the inclusion map from p into A, it is trivial to verify that

$$(A, \alpha_p \mid p \in P) = \varinjlim (A_p, \alpha_p^q \mid p \subseteq q \text{ in } P)\,.$$

This is a particular case of directed colimit: all the transition maps are *embeddings*, in which case we shall usually say "directed union" instead of "directed colimit".

There is a well-known and especially simple description of directed colimits of diagrams in $\mathbf{Alg}_\Sigma$ (for a similarity type Σ), see also Gorbunov [123, Section 1.2.5] or Gillibert and Wehrung [114, Section 1.2.5].

Lemma 7-2.3 (folklore)**.** *Every directed diagram* $(A_p, \alpha_p^q \mid p \leq q \text{ in } P)$ *in* $\mathbf{Alg}_\Sigma$ *has a colimit* $(A, \alpha_p \mid p \in P)$*, that is the co-cone characterized by the conditions*

(i) $A = \bigcup(\, \alpha_p(A_p) \mid p \in P\,)$;

(ii) $\operatorname{Ker}(\alpha_p) = \bigcup(\, \operatorname{Ker}(\alpha_p^q) \mid q \in P \uparrow p\,)$ *for each* $p \in P$.

For categories $\mathcal{A}$ and $\mathcal{B}$, two functors $\Phi, \Psi \colon \mathcal{A} \to \mathcal{B}$ are *isomorphic*, in notation $\Phi \cong \Psi$, if there exists a natural transformation $\varepsilon \colon \Phi \to \Psi$ such that $\varepsilon(A)$ is an isomorphism for every object A of $\mathcal{A}$. This means that ε is a map from the objects of $\mathcal{A}$ to the isomorphisms of $\mathcal{B}$ such that $\varepsilon(A) \colon \Phi(A) \to \Psi(A)$ for every object A of $\mathcal{A}$, and, for any morphism $f \colon A_0 \to A_1$ in $\mathcal{A}$, the equation $\Psi(f) \circ \varepsilon(A_0) = \varepsilon(A_1) \circ \Phi(f)$ is satisfied.

For categories $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{S}$ and functors $\Phi \colon \mathcal{A} \to \mathcal{S}$ and $\Psi \colon \mathcal{B} \to \mathcal{S}$, a functor $\Gamma \colon \mathcal{A} \to \mathcal{B}$ *lifts* Φ *with respect to* Ψ if $\Psi \circ \Gamma \cong \Phi$.

We shall, occasionally, deal with diagrams that are no longer indexed by posets, and in fact not even by categories. The proper formulation of the definition of a diagram, required to accommodate this kind of situation, involves the concept of a *quiver* (or *graph*). The interested reader can find such a formulation in Barr and Wells [23, Page 36], see also Wehrung [339, Definition 2.1].

7-3. Distributive homomorphisms

Schmidt [292] introduced *distributive homomorphisms*, one of the most powerful concepts involved in the study of congruence lattices of infinite lattices. In this section we shall review some of the results that can be obtained by applying this tool.

7-3.1 Algebraic closure operators and congruences of semilattices

Join-homomorphisms and join-congruences are ubiquitous in Schmidt's work on congruence lattices of lattices. An important point is the correspondence, formally expressed by Proposition 7-3.3, between join-congruences of a semilattice S and algebraic closure operators on $\operatorname{Id} S$.

Definition 7-3.1. Let L be a poset. A map $f \colon L \to L$ is

- an *inflator* if $\mathrm{id}_L \leq f$ (i.e., $x \leq f(x)$ for each $x \in L$) and f is isotone;
- a *closure operator* if f is idempotent (i.e., $f \circ f = f$) and f is an inflator;
- *algebraic* if for each $a \in L$ and each directed [3] $D \subseteq L$, $a = \bigvee D$ implies that $f(a) = \bigvee f(D)$.

We denote by $\mathbf{Cl_a}\, L$ the poset of all algebraic closure operators on L, endowed with componentwise ordering (i.e., $f \leq g$ if $f(x) \leq g(x)$ for each $x \in L$).

[3] As in Chapter 1, directed subsets are always supposed nonempty.

Algebraic closure operators will be dealt with only for algebraic lattices. The following lemma, whose straightforward proof we leave to the reader, sums up a few basic facts about that case.

Lemma 7-3.2. *The following statements hold, for any algebraic lattice L.*

(i) *A map $f\colon L \to L$ is algebraic iff $f(a) = \bigvee(\, f(x) \mid x \leq a \text{ compact})$ for each $a \in L$.*

(ii) *For any algebraic inflator $f\colon L \to L$, the map $f^\infty\colon L \to L$, defined by $f^\infty(x) = \bigvee(\, f^n(x) \mid n < \omega\,)$ for each $x \in L$, is the least algebraic closure operator g such that $f \leq g$.*

In particular, it is easy to deduce from Lemma 7-3.2 that $\mathbf{Cl_a}\, L$ is a complete lattice; for example, the join of two algebraic closure operators f and g in $\mathbf{Cl_a}\, L$ is h^∞, where $h(x) = f(x) \vee g(x)$ for each $x \in L$. However, we shall now state a much more precise observation, which is implicitly used at many places in Schmidt [292, 293], although it is not stated explicitly in those papers. Recall from [LTF, Section I.3.5] that the ideal lattice of a $(\vee, 0)$-semilattice S is denoted by $\operatorname{Id} S$, and that every algebraic lattice has this form up to isomorphism. For a binary relation $\boldsymbol{\theta}$ on S and $x, y \in S$, let $x \equiv_{\boldsymbol{\theta}} y$ hold if $(x, y) \in \boldsymbol{\theta}$, and $x \leq_{\boldsymbol{\theta}} y$ if $x \vee y \equiv_{\boldsymbol{\theta}} y$.

Proposition 7-3.3 (Schmidt 1968)**.** *Let S be a $(\vee, 0)$-semilattice. Then* $\operatorname{Con} S \cong \mathbf{Cl_a}(\operatorname{Id} S)$.

Proof. For every congruence $\boldsymbol{\theta}$ of $(S, \vee)$, we define a map $\mathrm{cl}_{\boldsymbol{\theta}}\colon \operatorname{Id} S \to \operatorname{Id} S$ by

(7-3.1) $$\mathrm{cl}_{\boldsymbol{\theta}}(\boldsymbol{a}) = \{\, x \in S \mid (\exists y \in \boldsymbol{a})(x \leq_{\boldsymbol{\theta}} y)\,\}\,, \quad \text{for each } \boldsymbol{a} \in \operatorname{Id} S\,.$$

It is straightforward to verify that $\mathrm{cl}_{\boldsymbol{\theta}}$ is an algebraic closure operator on $\operatorname{Id} S$.

For every closure operator f on $\operatorname{Id} S$ and for all $x, y \in S$, $f(S{\downarrow}x) \subseteq f(S{\downarrow}y)$ iff $S{\downarrow}x \subseteq f(S{\downarrow}y)$ iff $x \in f(S{\downarrow}y)$. By using this observation, it is straightforward to verify that the binary relation

(7-3.2) $$\Theta_f = \{\, (x, y) \in S \times S \mid f(S \downarrow x) = f(S \downarrow y)\,\}$$

is a congruence of $(S, \vee)$. It is also straightforward to verify that the assignments $\boldsymbol{\theta} \mapsto \mathrm{cl}_{\boldsymbol{\theta}}$ and $f \mapsto \Theta_f$ define mutually inverse isotone maps between $\operatorname{Con} S$ and $\mathbf{Cl_a}(\operatorname{Id} S)$; hence they are isomorphisms. □

In particular, $\mathbf{Cl_a}(\operatorname{Id} S)$ is an algebraic lattice.

7-3.2 Weakly distributive homomorphisms and congruences

Weakly distributive homomorphisms and congruences were originally defined in Schmidt [292]. Our current definition is tailored to accommodate both Schmidt's work and the subsequent work on CLP (see Chapter 9). For distributive semilattices, our definition and Schmidt's definition of a weakly distributive (resp., distributive) congruence are equivalent.

Definition 7-3.4. Let S and T be $(\vee, 0)$-semilattices.

(i) A $(\vee, 0)$-homomorphism $f\colon S \to T$ is *weakly distributive at an element s of S* if for all $t_0, t_1 \in T$, if $f(s) \leq t_0 \vee t_1$, then there are $s_0, s_1 \in S$ such that $s \leq s_0 \vee s_1$ and $f(s_i) \leq t_i$ for each $i \in \{0, 1\}$.

(ii) A $(\vee, 0)$-homomorphism $f\colon S \to T$ is *weakly distributive* if it is weakly distributive at every element of S.

(iii) A congruence $\boldsymbol{\theta}$ of S is *weakly distributive* if the canonical projection $\theta\colon S \twoheadrightarrow S/\boldsymbol{\theta}$ is weakly distributive.

For an equivalent form of weak distributivity for a homomorphism, see Exercise 7.7. For further examples, see Exercises 7.6–7.10.

The following result is, mainly, established in Schmidt [292, Hilfssatz 4.5].

Proposition 7-3.5. *A congruence $\boldsymbol{\theta}$ of a $(\vee, 0)$-semilattice S is weakly distributive iff the closure operator $\mathrm{cl}_{\boldsymbol{\theta}}$ associated to $\boldsymbol{\theta}$ as in (7-3.1) is a join-homomorphism.*

Proof. Suppose first that $\boldsymbol{\theta}$ is weakly distributive and let $\boldsymbol{a}, \boldsymbol{a}_0, \boldsymbol{a}_1 \in \operatorname{Id} S$ with $\boldsymbol{a} = \boldsymbol{a}_0 \vee \boldsymbol{a}_1$; we must prove that $\mathrm{cl}_{\boldsymbol{\theta}}(\boldsymbol{a}) \subseteq \mathrm{cl}_{\boldsymbol{\theta}}(\boldsymbol{a}_0) \vee \mathrm{cl}_{\boldsymbol{\theta}}(\boldsymbol{a}_1)$. For any $x \in \mathrm{cl}_{\boldsymbol{\theta}}(\boldsymbol{a})$, there exists $y \in \boldsymbol{a}$ such that $x \leq_{\boldsymbol{\theta}} y$. Since $y \in \boldsymbol{a}_0 \vee \boldsymbol{a}_1$, there are $y_0 \in \boldsymbol{a}_0$ and $y_1 \in \boldsymbol{a}_1$ such that $y \leq y_0 \vee y_1$. Since the projection $\theta\colon S \twoheadrightarrow S/\boldsymbol{\theta}$ is weakly distributive and $\theta(x) \leq \theta(y_0) \vee \theta(y_1)$, there are $x_0, x_1 \in S$ such that $x \leq x_0 \vee x_1$ and $\theta(x_i) \leq \theta(y_i)$ for each $i < 2$. From $x_i \leq_{\boldsymbol{\theta}} y_i$ and $y_i \in \boldsymbol{a}_i$ it follows that $x_i \in \mathrm{cl}_{\boldsymbol{\theta}}(\boldsymbol{a}_i)$, for each $i \in \{0, 1\}$. Therefore, $x \in \mathrm{cl}_{\boldsymbol{\theta}}(\boldsymbol{a}_0) \vee \mathrm{cl}_{\boldsymbol{\theta}}(\boldsymbol{a}_1)$.

Suppose, conversely, that $\mathrm{cl}_{\boldsymbol{\theta}}$ is a join-homomorphism and let $x, y_0, y_1 \in S$ such that $x \leq_{\boldsymbol{\theta}} y_0 \vee y_1$. It follows that

$$x \in \mathrm{cl}_{\boldsymbol{\theta}}\bigl(S \downarrow (y_0 \vee y_1)\bigr) = \mathrm{cl}_{\boldsymbol{\theta}}\bigl((S \downarrow y_0) \vee (S \downarrow y_1)\bigr) = \mathrm{cl}_{\boldsymbol{\theta}}(S \downarrow y_0) \vee \mathrm{cl}_{\boldsymbol{\theta}}(S \downarrow y_1)\,,$$

so there are $x_i \in \mathrm{cl}_{\boldsymbol{\theta}}(S \downarrow y_i)$ (that is, $x_i \leq_{\boldsymbol{\theta}} y_i$), for $i \in \{0, 1\}$, such that $x \leq x_0 \vee x_1$. □

Corollary 7-3.6. *The weakly distributive congruences of a $(\vee, 0)$-semilattice form a complete join-subsemilattice of* $\operatorname{Con} S$.

Proof. By Propositions 7-3.3 and 7-3.5, it suffices to prove that the subset J of $\mathbf{Cl_a}(\operatorname{Id} S)$, consisting of all algebraic closure operators that are also join-homomorphisms, is closed under join. By Lemma 7-3.2(ii), the join in $\mathbf{Cl_a}(\operatorname{Id} S)$ of a subset $(f_i \mid i \in I)$ of J is f^∞ where $f(\boldsymbol{a}) = \bigvee(\, f_i(\boldsymbol{a}) \mid i \in I\,)$ for each $\boldsymbol{a} \in \operatorname{Id} S$. Since each f_i is a join-homomorphism, so is f, and thus so is f^∞. □

The following result is established in Schmidt [292, Hilfssatz 4.2].

Proposition 7-3.7. *Let $\boldsymbol{\theta}$ be a weakly distributive congruence of a $(\vee, 0)$-semilattice S. If S is distributive, then so is $S/\boldsymbol{\theta}$.*

Proof. Let $x, y_0, y_1 \in S$ such that $x/\boldsymbol{\theta} \leq (y_0/\boldsymbol{\theta}) \vee (y_1/\boldsymbol{\theta})$, that is, $x \leq_{\boldsymbol{\theta}} y_0 \vee y_1$. Since $\boldsymbol{\theta}$ is weakly distributive, there are $u_0, u_1 \in S$ such that $x \leq u_0 \vee u_1$ and $u_i \leq_{\boldsymbol{\theta}} y_i$ for each $i \in \{0,1\}$. Since S is distributive, there are $x_i \leq u_i$, for $i \in \{0,1\}$, such that $x = x_0 \vee x_1$. It follows that $x/\boldsymbol{\theta} = (x_0/\boldsymbol{\theta}) \vee (x_1/\boldsymbol{\theta})$ and $x_i/\boldsymbol{\theta} \leq y_i/\boldsymbol{\theta}$ for each $i \in \{0,1\}$. □

7-3.3 Distributive congruences

Definition 7-3.8. A congruence $\boldsymbol{\theta}$ of a $(\vee, 0)$-semilattice S is *monomial* if every $\boldsymbol{\theta}$-block has a largest element.

From Propositions 7-3.3 and 7-3.5 we get immediately the following characterization of monomial congruences.

Lemma 7-3.9. *A congruence $\boldsymbol{\theta}$ of a $(\vee, 0)$-semilattice S is weakly distributive and monomial iff there exists a join-homomorphism $s\colon S \to S$ such that*

$$x \leq_{\boldsymbol{\theta}} y \iff x \leq s(y)\,, \quad \textit{for all } x, y \in S\,.$$

In the context of Lemma 7-3.9, s is necessarily a closure operator on S; hence it is what we shall call a *pre-topological closure operator*[4], that is, a closure operator s such that $s(x \vee y) = s(x) \vee s(y)$ for all $x, y \in S$.

The closure operator $\mathrm{cl}_{\boldsymbol{\theta}}$ associated to $\boldsymbol{\theta}$ *via* (7-3.1) is the unique extension of s to an algebraic closure operator on $\mathrm{Id}\, S$, that is,

$$\mathrm{cl}_{\boldsymbol{\theta}}(\boldsymbol{a}) = S \downarrow s(\boldsymbol{a})\,, \quad \text{for each } \boldsymbol{a} \in \mathrm{Id}\, S\,.$$

Definition 7-3.10. A congruence $\boldsymbol{\theta}$ of a $(\vee, 0)$-semilattice S is *distributive* if it is the join, in $\mathrm{Con}\, S$, of a collection of weakly distributive monomial congruences of S.

It follows from Corollary 7-3.6 that every distributive congruence is weakly distributive. In particular, a monomial congruence is distributive iff it is weakly distributive.

For elements a and b in a generalized Boolean lattice B such that $a \leq b$, we denote by $b - a$ the unique $x \in B$ such that $a \wedge x = 0$ and $a \vee x = b$. The following is established in Dobbertin[5] [71, Theorem 3].

Theorem 7-3.11 (Dobbertin 1989)**.** *The following statements are equivalent, for any congruence $\boldsymbol{\theta}$ of a generalized Boolean semilattice B:*

(i) *$\boldsymbol{\theta}$ is distributive.*

[4] *Topological closure operators* are pre-topological closure operators sending 0 to 0.

[5] Dobbertin's preprint [71] was accepted in 1989, modulo revisions, at the J. London Math. Soc. However, at that time Dobbertin's academic position did not get renewed and the revision process stopped. Hans Dobbertin passed away in February 2006.

(ii) *For each* $(a, b) \in \boldsymbol{\theta}$ *with* $a < b$*, there exists a pre-topological closure operator* s *on* B *such that* $\operatorname{Ker}(s) \subseteq \boldsymbol{\theta}$ *and* $s(a) = b$*.*

(iii) *For each* $(a, b) \in \boldsymbol{\theta}$ *with* $a < b$*, there exists a* $(\vee, 0, 1)$*-homomorphism* $\varphi\colon B\downarrow a \to B\downarrow(b - a)$ *such that* $\varphi(z) \leq_{\boldsymbol{\theta}} z$ *for each* $z \in B\downarrow a$*.*

Proof. (iii)⇒(ii). Set $s(x) = x \vee \varphi(a \wedge x)$, for each $x \in B$. Then s is a pre-topological closure operator. Furthermore, s is a closure operator and $s(x) \equiv_{\boldsymbol{\theta}} x$ for every $x \in B$, so $\operatorname{Ker}(s) \subseteq \boldsymbol{\theta}$.

(ii)⇒(i). If (ii) holds, then $\boldsymbol{\theta}$ is the join of all $\operatorname{Ker}(s)$, for pre-topological closure operators s on B such that $\operatorname{Ker}(s) \subseteq \boldsymbol{\theta}$. Hence $\boldsymbol{\theta}$ is distributive.

(i)⇒(iii). Since $\boldsymbol{\theta}$ is distributive, there are a positive integer n, elements $a = a_0 < a_1 < \cdots < a_n$ of B with $b \leq a_n$, and pre-topological closure operators $s_0, \dots, s_{n-1}$ on B such that $s_i(a_i) = s_i(a_{i+1})$ and $\operatorname{Ker}(s_i) \subseteq \boldsymbol{\theta}$ for each $i < n$. The map $\psi_i\colon B\downarrow a_i \to B\downarrow(a_{i+1} - a_i)$, that sends 0 to 0 and any nonzero $x \leq a_i$ to $s_i(x) \wedge (a_{i+1} - a_i)$, is a $(\vee, 0, 1)$-homomorphism. Now we set $\varphi_0 = \psi_0$ and, for all positive $i < n$,

$$\varphi_i\colon B\downarrow a \to B\downarrow(a_{i+1} - a)\,, \quad x \mapsto \varphi_{i-1}(x) \vee \psi_i(x) \vee \psi_i\varphi_{i-1}(x)\,.$$

It is straightforward to verify, by induction on i, that $\varphi_i\colon B\downarrow a \to B\downarrow(a_{i+1} - a)$ is a $(\vee, 0, 1)$-homomorphism, for each $i < n$. Define $\varphi(x) = \varphi_{n-1}(x) \wedge b$, for each $x \in B\downarrow a$. □

For Boolean semilattices A and C, $\varphi\colon A \to C$, and $\Gamma\colon A \to \operatorname{Id} C$, let $\varphi \ll \Gamma$ be the statement that φ and Γ are both $(\vee, 1)$-homomorphisms and $\varphi(x) \in \Gamma(x)$ for each $x \in A$. The following is established in Dobbertin [71, Lemma 4].

Lemma 7-3.12. *Let* A *and* C *be Boolean semilattices and let* $\Gamma\colon A \to \operatorname{Id} C$ *be a* $(\vee, 1)$*-homomorphism. If* A *is countable and if* $\Gamma(1_A) = C$*, then there exists* $\varphi\colon A \to C$ *such that* $\varphi(0_A) = 0_C$ *and* $\varphi \ll \Gamma$*.*

Proof. Represent A as the union of a countable ascending chain of finite Boolean subsemilattices A_n $(n < \omega)$, where $A_0 = \{0, 1\}$. We define φ by induction. We start by setting $\varphi(0_A) = 0_C$ and $\varphi(1_A) = 1_C$. In order to extend φ from A_n to A_{n+1}, suppose that an atom a of A_n splits, within A_{n+1}, into the atoms $a_1, \dots, a_k$. Then $\varphi(a) \in \Gamma(a) = \bigvee(\,\Gamma(a_i) \mid 1 \leq i \leq k\,)$. Consequently, there are $c_i \in \Gamma(a_i)$, for $1 \leq i \leq k$, such that the equality $\varphi(a) = \bigvee(\,c_i \mid 1 \leq i \leq k\,)$ holds. Now set $\varphi(a_i) = c_i$. □

Say that a poset P is *lower countable* if $P\downarrow a$ is countable for every $a \in P$. The following is established in Dobbertin [70, Lemma 26].

Corollary 7-3.13. *Every weakly distributive congruence* $\boldsymbol{\theta}$ *on a lower countable generalized Boolean semilattice* B *is distributive.*

Proof. Let $(a,b) \in \boldsymbol{\theta}$ with $a < b$ and set

$$\Gamma(x) = \{\, y \in B \downarrow (b-a) \mid y \leq_{\boldsymbol{\theta}} x \,\}\,, \quad \text{for each } x \in B \downarrow a\,.$$

Since $\boldsymbol{\theta}$ is weakly distributive, it is straightforward to verify that Γ is a join-homomorphism from $B \downarrow a$ to $\mathrm{Id}(B \downarrow (b-a))$. Obviously, $\Gamma(a) = B \downarrow (b-a)$. Since $B{\downarrow}a$ is countable and by Lemma 7-3.12, there exists $\varphi\colon B{\downarrow}a \to B{\downarrow}(b-a)$ such that $\varphi \ll \Gamma$. The conclusion follows now from Theorem 7-3.11. □

By using his results in [69], Dobbertin proves in [70, Theorem 27] that every lower countable distributive $(\vee, 0)$-semilattice is the image of some lower countable generalized Boolean semilattice under a so-called *V-homomorphism* (cf. Exercises 7.13 to 7.20). Since every V-homomorphism is (trivially) weakly distributive (cf. [70, page 46]), we get the following result of Dobbertin.

♢ **Theorem 7-3.14** (Dobbertin 1986)**.** *Every lower countable distributive $(\vee, 0)$-semilattice is a distributive image of some lower countable generalized Boolean semilattice.*

Schmidt asked in [292, Problem 7] whether every weakly distributive congruence of a distributive $(\vee, 0)$-semilattice is necessarily distributive. By using Theorem 7-3.11, Dobbertin constructed in [71, Theorem 6] the following counterexample, necessarily uncountable by Corollary 7-3.13.

Example 7-3.15 (Dobbertin 1989)**.** *Denote by ω_1 the first uncountable ordinal and identify every ordinal α with the set of all ordinals $< \alpha$. Denote by A the Boolean subalgebra of $\mathrm{Pow}\,\omega_1$ generated by the chain $\omega_1 + 1 = \{\, \alpha \mid \alpha \leq \omega_1 \,\}$ and by F the free Boolean algebra on ω_1 generators e_α $(0 < \alpha < \omega_1)$. There exists a zero-separating, weakly distributive, nondistributive join-congruence $\boldsymbol{\theta}$ of the Boolean semilattice $B = A \times F$.*

Proof. The main point is to construct a $(\vee, 1)$-homomorphism $\Gamma\colon A \to \mathrm{Id}\,F$ witnessing the failure of the uncountable version of Lemma 7-3.12. Denote by I_α the ideal of F generated by $\{\, e_\xi \mid 0 < \xi \leq \alpha \,\}$, for each $\alpha < \omega_1$, and set $I_{\omega_1} = F$. In this way, we obtain an ascending ω_1-sequence of ideals of F. Set $J_0 = F$ and, for $0 < \alpha \leq \omega_1$, denote by J_α the ideal of F generated by $\{\, \neg e_\xi \mid \alpha \leq \xi < \omega_1 \,\}$. In this way, we obtain a descending ω_1-sequence of ideals of F. Since $e_\alpha \in I_\alpha$ and $\neg e_\alpha \in J_\alpha$ for $0 < \alpha < \omega_1$, we get $F = I_\alpha \vee J_\alpha$ in $\mathrm{Id}\,F$. This equation is also valid for $\alpha = 0$ (in which case $J_\alpha = F$) and for $\alpha = \omega_1$ (in which case $I_\alpha = F$), so

$$F = I_\alpha \vee J_\alpha \text{ for each } \alpha \leq \omega_1\,. \tag{7-3.3}$$

Claim 1. *There exists a unique $(\vee, 0)$-homomorphism $\Gamma\colon A \to \mathrm{Id}\,F$ such that $\Gamma\big([\alpha, \beta)\big) = I_\beta \cap J_\alpha$ for all $0 \leq \alpha < \beta \leq \omega_1$.*

Proof. Observe that the members of A are exactly the finite disjoint unions of intervals of the form $[\alpha,\beta)=\beta\setminus\alpha=\{\,\xi\mid\alpha\leq\xi<\beta\,\}$ for $0\leq\alpha<\beta\leq\omega_1$. The uniqueness statement on Γ follows. The verification of the existence statement easily reduces to the assertion that

$$I_\gamma\cap J_\alpha=(I_\beta\cap J_\alpha)\vee(I_\gamma\cap J_\beta)\,,$$

for all $\alpha,\beta,\gamma\leq\omega_1$ with $\alpha<\beta<\gamma$. We compute, using the distributivity of the lattice $\operatorname{Id}F$, together with (7-3.3) and the fact that $(I_\alpha\mid\alpha\leq\omega_1)$ is ascending while $(J_\alpha\mid\alpha\leq\omega_1)$ is descending:

$$\begin{aligned}(I_\beta\cap J_\alpha)\vee(I_\gamma\cap J_\beta)&=(I_\beta\vee I_\gamma)\cap(I_\beta\vee J_\beta)\cap(I_\gamma\vee J_\alpha)\cap(J_\alpha\vee J_\beta)\\&=I_\gamma\cap(I_\gamma\vee J_\alpha)\cap J_\alpha\\&=I_\gamma\cap J_\alpha\,,\end{aligned}$$

as required. $\square$

Observe, in particular, that $\Gamma(1_A)=\Gamma\bigl([0,\omega_1)\bigr)=I_{\omega_1}=F$, so Γ is a $(\vee,0,1)$-homomorphism from A to $\operatorname{Id}F$.

Claim 2. *There is no $\varphi\colon A\to F$ such that $\varphi\ll\Gamma$.*

Proof. Suppose otherwise. Since φ is a join-homomorphism from A to F, the ω_1-sequence $(\varphi(\alpha)\mid\alpha<\omega_1)$ is nondecreasing. Since F is a free Boolean algebra, every chain of F is countable (the argument of Galvin and Jónsson [102, Lemma 5] works for Boolean algebras, see also Freese, Ježek, and Nation [90, Theorem 1.27]), thus there is a nonzero $\overline{\alpha}<\omega_1$ such that $\varphi(\overline{\alpha})=\varphi(\alpha)$ for each $\alpha\in[\overline{\alpha},\omega_1)$. From $\varphi(\overline{\alpha})\in I_{\overline{\alpha}}$ it follows that there are a positive integer m and nonzero ordinals $\gamma_0,\dots,\gamma_{m-1}\leq\overline{\alpha}$ such that $\varphi(\overline{\alpha})\leq\bigvee(\,e_{\gamma_i}\mid i<m\,)$. Set $\delta=\overline{\alpha}+1$. From

$$1_F=\varphi(\omega_1)=\varphi(\delta)\vee\varphi\bigl([\delta,\omega_1)\bigr)=\varphi(\overline{\alpha})\vee\varphi\bigl([\delta,\omega_1)\bigr)$$

it follows that $\bigwedge(\,\neg e_{\gamma_i}\mid i<m\,)\leq\neg\varphi(\overline{\alpha})\leq\varphi\bigl([\delta,\omega_1)\bigr)$. Since $\varphi\bigl([\delta,\omega_1)\bigr)$ belongs to $\Gamma\bigl([\delta,\omega_1)\bigr)=J_\delta$, it follows that $\bigwedge(\,\neg e_{\gamma_i}\mid i<m\,)\in J_\delta$, so there are a positive integer n and ordinals $\delta_j\in[\delta,\omega_1)$, for $j<n$, such that

$$\bigwedge(\,\neg e_{\gamma_i}\mid i<m\,)\leq\bigvee(\,\neg e_{\delta_j}\mid j<n\,)\,.$$

Since $\gamma_i<\delta\leq\delta_j$ for all $i<m$ and all $j<n$, this contradicts the Boolean independence of the e_ξ. $\square$

Denoting by $+$ the operation of symmetric difference on F (i.e., $z_0+z_1=(z_0\wedge\neg z_1)\vee(z_1\wedge\neg z_0)$ for all $z_0,z_1\in F$), define a binary relation $\boldsymbol{\theta}$ on B by

$$(x_0,z_0)\equiv_{\boldsymbol{\theta}}(x_1,z_1)\underset{\text{def.}}{\iff}\bigl(x_0=x_1\text{ and }z_0+z_1\in\Gamma(x_0)\bigr)\,,$$

for all $(x_0, z_0), (x_1, z_1) \in B$. Since Γ is a join-homomorphism, it is straightforward to verify that $\boldsymbol{\theta}$ is a weakly distributive join-congruence of B. From $\Gamma(0_A) = \{0_F\}$ it follows that $\boldsymbol{\theta}$ separates zero.

Suppose that $\boldsymbol{\theta}$ is distributive. By applying Theorem 7-3.11 to $a = (1_A, 0_F)$ and $b = (1_A, 1_F)$, we obtain that there exists a $(\vee, 1)$-homomorphism $\varphi\colon A \to F$ such that $(x, \varphi(x)) \leq_{\boldsymbol{\theta}} (x, 0)$ for each $x \in A$; so $\varphi \ll \Gamma$, in contradiction with Claim 2. $\square$

For an application of Example 7-3.15 to another problem stated by Schmidt, see Exercise 7.11.

7-3.4 Schmidt's Theorem

As in LTF, for any ideal J of a generalized Boolean lattice B, we denote by $\mathrm{con}(J)$ the unique congruence of B for which J is a block. If $J = B \mathbin{\downarrow} a$ is a principal congruence, we shall write $\mathrm{con}(a)$ instead of $\mathrm{con}(J)$. The congruences of B are exactly the $\mathrm{con}(J)$, for $J \in \mathrm{Id}\, B$ (cf. [LTF, Theorem 146]).

The following is established in Schmidt [292, Satz 6.1].

Lemma 7-3.16 (Schmidt 1968)**.** *Let B be a generalized Boolean lattice and let $s\colon B \to B$ be both a closure operator and a $(\vee, 0)$-homomorphism. Setting $S = s(B)$, we define*

$$L = \{\, (x, y, z) \in S \times B \times B \mid x \wedge y = x \wedge z = y \wedge z \,\}, \quad \textit{ordered componentwise}.$$

Then L is a lattice, and the congruences of L are exactly the relations $\boldsymbol{\beta} = \boldsymbol{\alpha}^3\restriction_L = (\boldsymbol{\alpha}^3) \cap (L \times L)$, where $\boldsymbol{\alpha} = \mathrm{con}(J)$, for an ideal J of B such that $s(J) \subseteq J$. In particular, $\mathrm{Con_c}\, L \cong s(B)$.

Outline of proof. It is easy to verify that L is a closure system in B^3. The closure $\overline{(x, y, z)}$ of a triple $(x, y, z) \in B^3$, defined as the least element of L containing (x, y, z), is $\bigl(s(x'), y' \vee (s(x') \wedge z'), z' \vee (s(x') \wedge y')\bigr)$, where we set $x' = x \vee (y \wedge z)$, $y' = y \vee (x \wedge z)$, and $z' = z \vee (x \wedge y)$. It is important to observe at this point that the components of the triple $\overline{(x, y, z)}$ are terms, in the similarity type $(\vee, \wedge, s)$, in the variables x, y, z. By using this fact, together with an argument similar to the one of the proof of [LTF, Theorem 336], it can be proved that the congruences of L are exactly those of the form $\boldsymbol{\alpha}^3\restriction_L$, where $\boldsymbol{\alpha}$ is a congruence of the algebra $B^s = (B, \vee, \wedge, s)$.

Since B is a generalized Boolean lattice, a congruence $\boldsymbol{\alpha}$ of $(B, \vee, \wedge)$ can always be written in the form $\boldsymbol{\alpha} = \mathrm{con}(J)$ for an ideal J of B. By using the assumption that s is a $(\vee, 0)$-homomorphism, it is not hard to verify that $\boldsymbol{\alpha}$ is a congruence of B^s iff $s(J) \subseteq J$. In particular, the compact congruences of L are exactly the congruences $(\mathrm{con}(a))^3\restriction_L$ with $a \in S$; whence $\mathrm{Con_c}\, L \cong S$. $\square$

The following is established in Schmidt [292, Folgerung 6.2]; see also Schmidt [290].

Corollary 7-3.17 (Schmidt 1968). *Let S be a relatively pseudocomplemented lattice (i.e., for all $a, b \in S$, there exists a least $x \in S$ such that $b \leq a \vee x$). Then there exists a lattice L with zero such that* $\operatorname{Con_c} L \cong S$.

Outline of proof. One proves that S is a distributive lattice with zero, and that, if B denotes the generalized Boolean lattice generated by S (i.e., using the notation of LTF, $B = \mathrm{BR}\, S$), every $x \in B$ is contained in a smallest element of S, denoted by $s(x)$. Then s is both a closure operator and a $(\vee, 0)$-homomorphism. Apply Lemma 7-3.16. □

It turns out that the $(\vee, 0)$-semilattices, to which the proof of Corollary 7-3.17 presented above applies, are exactly the relatively pseudocomplemented lattices (cf. Exercise 7.12).

The following result, by Schmidt, yields one of the most powerful sufficient conditions known, for a given distributive $(\vee, 0)$-semilattice, to be the congruence semilattice of some lattice. It is proved in Schmidt[6] [292, Satz 8.1].

♢ **Theorem 7-3.18** (Schmidt 1968). *Let $\boldsymbol{\theta}$ be a distributive congruence of a generalized Boolean semilattice B. Then there exists a lattice L such that* $\operatorname{Con_c} L \cong B/\boldsymbol{\theta}$.

Here is a very rough outline of the proof of Theorem 7-3.18. The subset $J = 0/\boldsymbol{\theta}$ is an ideal of B and $\boldsymbol{\zeta} = \operatorname{con}(J)$ is a congruence of B, with $B/\boldsymbol{\zeta} = B/J$ generalized Boolean. Write $\boldsymbol{\theta} = \bigvee(\, \boldsymbol{\theta}_i \mid i \in I\,)$, for distributive monomial congruences $\boldsymbol{\theta}_i$ of B; denote by $s_i \colon B \to B$ the closure operator associated (via Lemma 7-3.9) to $\boldsymbol{\theta}_i$, for each $i \in I$. Then each congruence $(\boldsymbol{\theta}_i \vee \boldsymbol{\zeta})/\boldsymbol{\zeta}$ is distributive monomial, with associated closure operator $\overline{s}_i \colon B/\boldsymbol{\zeta} \to B/\boldsymbol{\zeta}$, $x/\boldsymbol{\zeta} \mapsto s_i(x)/\boldsymbol{\zeta}$, and these congruences join to $\boldsymbol{\theta}/\boldsymbol{\zeta}$; whence $\boldsymbol{\theta}/\boldsymbol{\zeta}$ is a distributive congruence of $B/\boldsymbol{\zeta}$. Furthermore, $(\boldsymbol{\theta}/\boldsymbol{\zeta})$ separates zero. Since $B/\boldsymbol{\theta} \cong (B/\boldsymbol{\zeta})/(\boldsymbol{\theta}/\boldsymbol{\zeta})$, we have thus reduced the problem to the case where $\boldsymbol{\theta}$ separates zero, which we shall assume from now on.

Write again $\boldsymbol{\theta} = \bigvee(\, \boldsymbol{\theta}_i \mid i \in I\,)$, with all the $\boldsymbol{\theta}_i$ distributive monomial, with associated closure operator $s_i \colon B \to B$. We may further assume that $\operatorname{card} I \geq 3$. Since $\boldsymbol{\theta}$ separates zero, we get $s_i(0) = 0$ for each $i \in I$. Set $\Omega = I \times B$ and denote by M the set of all maps $x \colon \Omega \to B$ with finite range such that the value of $x(p) \wedge x(q)$, for $p \neq q$ in Ω, is constant. Then M, endowed with the componentwise ordering, is a modular lattice. For each $(i, a) \in \Omega$, we define an ideal of M as follows:

$$B_i^a = \{\, x \in M \mid x(i, a) \leq a \text{ and } (x(q) = 0 \text{ for each } q \neq (i, a))\,\}.$$

Since $B_i^a \cong B \mathbin{\downarrow} a$ is a Boolean lattice, it is self-dual; hence the dual ideal B_i^a of the dual lattice M^{op} is isomorphic to $B \mathbin{\downarrow} a$. Applying Lemma 7-3.16 to

[6] Under the assumption that $\boldsymbol{\theta}$ separates zero. However, as we show at the beginning of our outline of the proof of Theorem 7-3.18, that assumption is easy to dispense with.

the Boolean lattice $B \downarrow a$ and the closure operator $x \mapsto s_i(x) \wedge a$, we obtain a 0-lattice L_i^a with an ideal $A_i^a = \{0\} \times (B \downarrow a) \times \{0\}$ isomorphic to $B \downarrow a$. Now we glue together M^{op} and all the lattices L_i^a, identifying A_i^a and B_i^a in the natural way. We obtain a partial lattice P, which is a meet-semilattice but not a join-semilattice. The lattice L of all finitely generated ideals of P satisfies $\operatorname{Con} L \cong \operatorname{Con} P \cong \operatorname{Id}(S/\boldsymbol{\theta})$.

Theorem 7-3.18 is extended to distributive quotients of relatively pseudocomplemented lattices in Schmidt [292, Satz 8.2].

By using Theorem 7-3.14, we obtain immediately the following result, due to Dobbertin [70, Theorem 28].

$\diamondsuit$ **Theorem 7-3.19** (Dobbertin 1986)**.** *Every lower countable distributive* $(\vee, 0)$*-semilattice is isomorphic to the congruence semilattice of a lattice.*

Huhn proves in [203] that every distributive $(\vee, 0)$-semilattice S with at most $\aleph_1$ elements is a distributive image of a generalized Boolean semilattice (see also Tischendorf [310, Corollary 9]), and thus, by Theorem 7-3.18, that $S \cong \operatorname{Con_c} L$ for some lattice L. For a completely different proof of the latter result, see Theorem 7-5.13.

The following consequence of Theorem 7-3.18 was first observed by Dilworth, and printed in Grätzer and Schmidt [164, Theorem 2]; see also Crawley and Dilworth [43, § 10.10] or Dobbertin [70, Theorem 30].

$\diamondsuit$ **Theorem 7-3.20** (Dilworth)**.** *Every algebraic and dually algebraic distributive lattice is isomorphic to the congruence lattice of a lattice.*

Recall that by Remark 1-7.11 in Chapter 1, a distributive lattice is both algebraic and dually algebraic iff it is isomorphic to the lattice $\operatorname{Down} P$ of all lower subsets of a poset P. Dobbertin establishes Theorem 7-3.20 by proving that the semilattice of all finitely generated lower subsets of P is isomorphic to $B/\boldsymbol{\theta}$, for some distributive congruence $\boldsymbol{\theta}$ on some generalized Boolean semilattice B. In contrast, Grätzer and Schmidt's proof is direct; furthermore, it yields a sectionally complemented, lower finite lattice (a poset P is *lower finite* if $P \downarrow a$ is finite for each $a \in P$).

For a strengthening of Theorem 7-3.20, see Theorem 8-4.8.

By using Theorem 7-3.18, Schmidt established in [295] the following deep result.

$\diamondsuit$ **Theorem 7-3.21** (Schmidt 1981)**.** *Every distributive lattice with zero is isomorphic to the congruence semilattice of some lattice.*

For three other proofs of Theorem 7-3.21, involving different ideas, see Theorem 7-4.14, Theorem 7-6.8, and Corollary 8-4.2.

For further discussion around Theorem 7-3.18, see Tischendorf [310].

We should issue a warning right away: *Not every distributive* $(\vee, 0)$*-semilattice is a weakly distributive image of a generalized Boolean semilattice* (see Theorem 9-2.20). However, the path to that observation is tortuous, and we will visit many landscapes before reaching that conclusion.

7-3.5 Further applications of the Boolean triple construction

Recall the definition of the *Boolean triple* construction, introduced in Grätzer and Wehrung [183] (see also [LTF, Section IV.5]):

$$\mathsf{M}_3[L] = \{\, (x \wedge y, x \wedge z, y \wedge z) \mid x, y, z \in L \,\}\,, \quad \text{for any lattice } L\,.$$

This construction extends a construction by Schmidt [292, Section 5]. It provides a positive solution to the problem, stated in Grätzer and Schmidt [172], whether every nontrivial lattice has a proper congruence-preserving extension. It became further extended in Grätzer and Wehrung [184], by the so-called *lattice tensor product*, denoted there by $A \boxtimes B$. In particular, $\mathsf{M}_3[L] \cong \mathsf{M}_3 \boxtimes L$, for any lattice L.

These constructions are convenient tools for modifying a lattice without modifying its congruence lattice, and they found a number of applications. One of them, already discussed in [LTF, Theorem 340], is the *Strong Independence Theorem for automorphism groups and congruence lattices for arbitrary lattices*, established in Grätzer and Wehrung [185].

We shall now present further applications of the Boolean triple construction and of the lattice tensor product.

Say that a lattice L is *regular* if whenever $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are congruences of L and $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ share a block, then $\boldsymbol{\alpha} = \boldsymbol{\beta}$. Every compact congruence of a regular lattice is principal (Grätzer and Schmidt [177, Lemma 6.1]), every sectionally complemented lattice is regular (Grätzer and Schmidt [177, Lemma 5.1]), but not every finite atomistic lattice is regular (Grätzer and Schmidt [177, Section 5]). The following result is contained in Theorems 1.1 and 1.2 from Grätzer and Schmidt [177].

$\diamondsuit$ **Theorem 7-3.22** (Grätzer and Schmidt 2001)**.** *Every lattice K has a congruence-preserving embedding into some regular lattice L. Furthermore, if K has a zero, then L can be taken with the same zero.*

The case where K is finite is settled in the earlier paper Grätzer and Schmidt [176]: it is proved there that L can be taken finite, sectionally complemented, and with the same zero as K.

Let us proceed to the so-called *magic wands*. Say that a partial function φ between subsets of a lattice K is *algebraic* if there is a lattice polynomial $\boldsymbol{p}$ (i.e., a lattice term with parameters), with parameters from K, such that $\varphi(x) = \boldsymbol{p}(x)$ for each x in the domain of φ. The first "magic wand theorem", established in Grätzer and Schmidt [178], is the following.

$\diamondsuit$ **Theorem 7-3.23** (Grätzer and Schmidt 2003)**.** *Let a, b, c, d be elements in a bounded lattice K such that $a \leq b$ and $c \leq d$, and let $\varphi\colon [a,b] \to [c,d]$ be an isomorphism. Then K can be embedded, as a convex sublattice, into a bounded lattice L such that*

(i) *both φ and φ^{-1} are algebraic in L;*

(ii) *a congruence of the lattice K extends to a congruence of L iff it is a congruence of the partial algebra $(K, \vee, \wedge, \varphi, \varphi^{-1})$, and then the extension is unique;*

(iii) *if K is finite, then so is L.*

In particular, the congruence lattice of L is isomorphic to the congruence lattice of $(K, \vee, \wedge, \varphi, \varphi^{-1})$. This observation is related to weak distributivity in Exercises 7.9 and 7.10.

Since K is a convex sublattice of L, the bounds are in general not preserved from K to L. Nevertheless, Theorem 7-3.23 can be extended to the case where K is a lattice with zero, modulo a suitable weakening of the algebraicity requirement on φ and φ^{-1}; see Grätzer and Schmidt [178, Section 8]. These results are further extended to families of intervals and isomorphisms in Sections 6 and 7 of Grätzer and Schmidt [178].

The second "magic wand theorem" is due to Grätzer, Greenberg, and Schmidt [137, Theorem 1].

$\diamondsuit$ **Theorem 7-3.24** (Grätzer, Greenberg, and Schmidt 2005)**.** *Let a, b, c, d be elements in a bounded lattice K such that $a \leq b$ and $c \leq d$, and let $\varphi\colon [a, b] \twoheadrightarrow [c, d]$ be a surjective lattice homomorphism. Then K can be embedded, as a convex sublattice, into a bounded lattice L such that*

(i) *φ is algebraic in L;*

(ii) *a congruence of the lattice K extends to a congruence of L iff it is a congruence of the partial algebra $(K, \vee, \wedge, \varphi)$, and then the extension is unique;*

(iii) *if K is finite, then so is L.*

Again, Theorem 7-3.24 can be extended, under certain conditions, to families of intervals and surjective homomorphisms. For more details, see Grätzer, Greenberg, and Schmidt [137].

Further results, obtained, in the infinite case, *via* the Boolean triple construction and lattice tensor products, are the so-called *independence theorems* (between $\operatorname{Con} L$ and $\operatorname{Aut} L$) discussed in [LTF, IV.4.8] (for finite lattices) and [LTF, IV.5.4] (for infinite lattices). For the "finite results", see Baranskiĭ [21, 22] and Urquhart [319]. For the "infinite results", see Grätzer and Wehrung [185].

7-4. From finite to infinite semilattices

Aside from Schmidt's Theorem (Theorem 7-3.18), another powerful technique, with many applications for congruence lattices of lattices, was introduced by Pudlák [274]. The basic idea is the following. Given a distributive $(\vee, 0)$-semilattice S, express S as a directed colimit of elementary "building blocks" S_i

(in Pudlák's above cited paper, the S_i are the finite distributive $(\vee, 0)$-subsemilattices of S; for an alternative, see Section 7-4.2); find a family of lattices L_i, with $\mathrm{Con_c}\, L_i \cong S_i$, that can be arranged into a directed diagram; define L as the directed colimit of the L_i. Then it follows from Lemma 7-2.2 that $\mathrm{Con_c}\, L \cong S$.

7-4.1 The Ershov-Pudlák Lemma

We begin with a well-known elementary lemma on finite distributive lattices.

Lemma 7-4.1. *For each join-irreducible element p in a finite distributive lattice D, there exists a largest $u \in D$ such that $p \nleq u$.*

Proof. Since D is distributive and p is join-irreducible, p is also *join-prime*, that is, $p \leq x \vee y$ implies that either $p \leq x$ or $p \leq y$, for all $x, y \in D$. Let $u = \bigvee(x \in D \mid p \nleq x)$. □

A key ingredient in the study of congruence lattices of infinite lattices is the following result, often attributed to Pudlák in his 1985 paper [274, Fact 4, p. 100], but already printed as the main theorem in Section 3 of the Introduction of Ershov's 1977 monograph [83].

Theorem 7-4.2 (Ershov 1977, Pudlák 1985)**.** *Every distributive $(\vee, 0)$-semilattice is the directed union of its finite distributive join-subsemilattices.*

Proof. We must prove that every finite subset X of a distributive $(\vee, 0)$-semilattice S is contained in some finite distributive $(\vee, 0)$-subsemilattice of S. The assumption that S is a distributive $(\vee, 0)$-semilattice means that the ideal lattice $\mathrm{Id}\, S$ of S is distributive (cf. [LTF, Lemma 184]). Denote by $\varepsilon \colon S \hookrightarrow \mathrm{Id}\, S$, $x \mapsto S \downarrow x$ the natural embedding. We may assume that $0 \in X$. Since $\varepsilon(X)$ is a finite subset in the distributive lattice $\mathrm{Id}\, S$, the sublattice $\mathbb{D}$ of $\mathrm{Id}\, S$ generated by $\varepsilon(X)$ is finite.

For each $x \in X$, the ideal $\varepsilon(x)$ is the join of all join-irreducible elements of $\mathbb{D}$ below it, thus, by using the distributivity of the $(\vee, 0)$-semilattice S, we obtain $z^0_{x,P} \in P$, for $P \in (\mathrm{Ji}\, \mathbb{D}) \downarrow \varepsilon(x)$, such that

$$x = \bigvee(z^0_{x,P} \mid P \in (\mathrm{Ji}\, \mathbb{D}) \downarrow \varepsilon(x)) . \tag{7-4.1}$$

By Lemma 7-4.1, for each $P \in \mathrm{Ji}\, \mathbb{D}$ there exists a largest $P^\dagger \in \mathbb{D}$ such that $P \nsubseteq P^\dagger$. Pick $z_P \in P \setminus P^\dagger$, for each $P \in \mathrm{Ji}\, \mathbb{D}$. Since $z^0_{x,P} \in P$ for each $x \in X$ and each $P \in (\mathrm{Ji}\, \mathbb{D}) \downarrow \varepsilon(x)$, we may replace z_P by its join with all $z_{x,P}$ for $x \in X$ such that $P \subseteq \varepsilon(x)$, and thus assume that

$$z^0_{x,P} \leq z_P , \quad \text{for all } x \in X \text{ and all } P \in (\mathrm{Ji}\, \mathbb{D}) \downarrow \varepsilon(x) . \tag{7-4.2}$$

Since $z_P \in P \subseteq \varepsilon(x)$ for all (x, P) as in (7-4.2), it follows from (7-4.1) that

$$x = \bigvee(z_P \mid P \in (\mathrm{Ji}\, \mathbb{D}) \downarrow \varepsilon(x)) , \quad \text{for each } x \in X . \tag{7-4.3}$$

Set $\varphi(A) = \bigvee(z_P \mid P \in (\mathrm{Ji}\,\mathbb{D}) \downarrow A)$, for each $A \in \mathbb{D}$. Observe that $\varphi(A) \in A$ for each $A \in \mathbb{D}$. Furthermore, since every join-irreducible element of $\mathbb{D}$ is join-prime, φ is a $(\vee, 0)$-homomorphism from $\mathbb{D}$ to S. Let $A, B \in \mathbb{D}$ such that $A \nsubseteq B$. There exists $P \in \mathrm{Ji}\,\mathbb{D}$ such that $P \subseteq A$ and $P \nsubseteq B$; whence $B \subseteq P^{\dagger}$. Now $P \in (\mathrm{Ji}\,\mathbb{D}) \downarrow A$ so $z_P \leq \varphi(A)$, while $z_P \notin P^{\dagger}$, thus $z_P \notin B$, and thus (since $\varphi(B) \in B$) we get $z_P \nleq \varphi(B)$; whence $\varphi(A) \nsubseteq \varphi(B)$. Therefore, φ is a $(\vee, 0)$-embedding from $\mathbb{D}$ into S, and therefore $\varphi(\mathbb{D}) \cong \mathbb{D}$. In particular, $\varphi(\mathbb{D})$ is a distributive semilattice.

Finally, it follows from (7-4.3) that $\varphi(\varepsilon(x)) = x$ for each $x \in X$; whence $X \subseteq \varphi(\mathbb{D})$. □

7-4.2 Directed colimits of finite Boolean semilattices

In this subsection we shall discuss an alternate directed colimit representation of an arbitrary distributive $(\vee, 0)$-semilattice, now using finite Boolean semilattices instead of finite distributive semilattices. For further applications, we will need a larger class of semilattices than the finite ones.

Definition 7-4.3. A $(\vee, 0)$-semilattice S is *co-Brouwerian* if S is a complete lattice and it satisfies the infinite meet distributive law (MID), that is, the infinitary identity

$$\text{(MID)} \qquad a \vee \bigwedge(x_i \mid i \in I) = \bigwedge(a \vee x_i \mid i \in I),$$

for every $a \in S$ and every family $(x_i \mid i \in I)$ of elements of S with $I \neq \varnothing$.

Lemma 7-4.4 (Injectivity Lemma)**.** *Let A be a $(\vee, 0)$-subsemilattice of a $(\vee, 0)$-semilattice B and let S be a distributive $(\vee, 0)$-semilattice. If either S is co-Brouwerian, or S is conditionally co-Brouwerian and A is cofinal in B, then every $(\vee, 0)$-homomorphism $\varphi\colon A \to S$ extends to a $(\vee, 0)$-homomorphism $\psi\colon B \to S$, largest* (*for the componentwise ordering*) *with that property.*

Proof. We set $\psi(b) = \bigwedge \varphi(A \uparrow b)$, for each $b \in B$; the meet is evaluated in S, and the meet of the empty set is defined as the largest element of S if it exists – which is the case if S is co-Brouwerian. If A is cofinal in B, then $A \uparrow b \neq \varnothing$ for each $b \in B$. Hence, in any case, there is no problem of definition of ψ. Obviously, ψ is isotone and it extends φ. Furthermore, for all $b_0, b_1 \in B$,

$$\begin{aligned}
\psi(b_0) \vee \psi(b_1) &= \bigwedge \varphi(A \uparrow b_0) \vee \bigwedge \varphi(A \uparrow b_1) \\
&= \bigwedge(\varphi(a_0) \vee \varphi(a_1) \mid (a_0, a_1) \in (A \uparrow b_0) \times (A \uparrow b_1)) \\
&\qquad \text{(by the co-Brouwerianity assumption)} \\
&= \bigwedge(\varphi(a_0 \vee a_1) \mid (a_0, a_1) \in (A \uparrow b_0) \times (A \uparrow b_1)) \\
&\geq \psi(b_0 \vee b_1) \qquad\qquad \text{(because } a_0 \vee a_1 \geq b_0 \vee b_1\text{)},
\end{aligned}$$

whence, as ψ is isotone, $\psi(b_0 \vee b_1) = \psi(b_0) \vee \psi(b_1)$.

Let $\psi' \colon B \to S$ be a $(\vee, 0)$-homomorphism extending φ. Then for each $b \in B$ and each $a \in A \mathbin{\uparrow} b$, $\psi'(b) \leq \psi'(a) = \varphi(a)$; whence, meeting over all $a \in A \mathbin{\uparrow} b$, we get $\psi'(b) \leq \psi(b)$. $\square$

An especially important case of application of Lemma 7-4.4 is the one where S is a *finite* distributive $(\vee, 0)$-semilattice (thus co-Brouwerian).

Lemma 7-4.5 (Triangle Lemma)**.** *Let A and S be $(\vee, 0)$-semilattices, with A finite and S distributive, and let $\varphi \colon A \to S$ be a $(\vee, 0)$-homomorphism. Then there are a finite Boolean semilattice B, a $(\vee, 0, 1)$-homomorphism $\tau \colon A \to B$, and a $(\vee, 0)$-homomorphism $\psi \colon B \to S$ such that $\varphi = \psi \circ \tau$ and* $\operatorname{Ker}(\varphi) = \operatorname{Ker}(\tau)$.

Proof. The premise of Lemma 7-4.5 is represented on the left-hand side diagram of Figure 7-4.1, and its conclusion on the middle diagram of Figure 7-4.1. The proof can be followed on the right-hand side diagram of Figure 7-4.1.

$$
\begin{tikzcd}
 & S \\
A \arrow[ur, "\varphi"] &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & S \\
A \arrow[ur, "\varphi"] \arrow[dr, "\tau"'] & \\
 & B \arrow[uu, "\psi"']
\end{tikzcd}
\qquad
\begin{tikzcd}
 & & S \\
A \arrow[urr, "\varphi"] \arrow[r, two heads, "\pi"] \arrow[drr, "\tau"'] & \overline{A} \arrow[ur, hook, "\overline{\varphi}"'] \arrow[dr, hook, "\overline{\tau}"] & \\
 & & B \arrow[uu, "\psi"']
\end{tikzcd}
$$

Figure 7-4.1: Illustrating the Triangle Lemma and its proof.

By Theorem 7-4.2, $\varphi(A)$ is contained in a finite $(\vee, 0)$-subsemilattice S' of S; hence we may replace S by S' and thus assume that S is finite. Set $\overline{A} = A/\operatorname{Ker}(\varphi)$ and denote by $\pi \colon A \twoheadrightarrow \overline{A}$ the canonical projection. By the universal property of the projection, there exists a unique $(\vee, 0)$-homomorphism $\overline{\varphi} \colon \overline{A} \to S$ such that $\varphi = \overline{\varphi} \circ \pi$, and $\overline{\varphi}$ is an embedding. Now the $(\vee, 0)$-semilattice $B = \operatorname{Pow}(\operatorname{Mi} \overline{A})$ is finite Boolean and the map $\overline{\tau} \colon \overline{A} \to B$, $x \mapsto \{\, u \in \operatorname{Mi} \overline{A} \mid x \nleq u \,\}$ is a $(\vee, 0, 1)$-embedding. Since B is finite and by Lemma 7-4.4, there exists a $(\vee, 0)$-homomorphism $\psi \colon B \to S$ such that $\overline{\varphi} = \psi \circ \overline{\tau}$. Setting $\tau = \overline{\tau} \circ \pi$, it follows that $\operatorname{Ker}(\tau) = \operatorname{Ker}(\pi)$ (because $\overline{\tau}$ is one-to-one), whence $\operatorname{Ker}(\tau) = \operatorname{Ker}(\varphi)$. Furthermore, $\varphi = \overline{\varphi} \circ \pi = \psi \circ \overline{\tau} \circ \pi = \psi \circ \tau$, as required. $\square$

The following Theorem 7-4.6 is originally due to Bulman-Fleming and McDowell [35]. The proof in that paper amounts, essentially, to proving a version of the Triangle Lemma (viz. Lemma 7-4.5) called the *Killing Interpolation Property* (*KIP*) (cf. [35, Proposition 2.7]), and then use a categorical result due to Shannon [302], the latter extending module-theoretical work by Lazard [242]. However, a number of people, including the author of the present chapter, found quite offputting, for non category-theorists, the need to first translate Shannon's result to more "concrete categorical" settings (all the papers, cited in the present work, that use Shannon's result, leave such a translation to the reader); hence our proof presented here is direct. Other direct proofs of either Theorem 7-4.6 or its unital version can be found in Goodearl and Wehrung [121, Corollary 6.7] and Růžička [283, Corollary 1.2].

Theorem 7-4.6 (Bulman-Fleming and McDowell 1978). *Every distributive* $(\vee, 0)$*-semilattice* S *is the directed colimit of a directed diagram* $\vec{S}$ *of finite Boolean semilattices and zero-separating* $(\vee, 0)$*-homomorphisms. Furthermore, if* S *has a unit, then all the morphisms in* $\vec{S}$ *can be taken unital.*

Proof. Observe that the set P of all nonempty finite subsets of $S \times \omega$ is directed and has no maximal element. We set

$$\nu(p) = \big\{x \in S \mid (\exists n < \omega)\big((x, n) \in p\big)\big\}\ , \quad \text{for each } p \in P\,.$$

We construct finite Boolean semilattices B_p, together with $(\vee, 0)$-homomorphisms $\varphi_p^q \colon B_p \to B_q$ and $\varphi_p \colon B_p \to S$, for $p \subseteq q$ in P, in such a way that the following conditions are satisfied:

(L1) $\nu(p) \subseteq \varphi_p(B_p)$, for each $p \in P$;

(L2) $\varphi_p^p = \mathrm{id}_{B_p}$, for each $p \in P$;

(L3) $\varphi_p^r = \varphi_q^r \circ \varphi_p^q$, for all $p \subseteq q \subseteq r$ in P;

(L4) $\varphi_p = \varphi_q \circ \varphi_p^q$, for all $p \subseteq q$ in P;

(L5) $\mathrm{Ker}(\varphi_p^q) = \mathrm{Ker}(\varphi_p)$, for all $p \subsetneqq q$ in P.

Observe that once (L2) is satisfied, (L3) and (L4) need to be verified only in case $p \subsetneqq q \subsetneqq r$ and $p \subsetneqq q$, respectively. We proceed by induction on the cardinality of p. If $p = \{(a, n)\}$ is a singleton, set $B_p = \mathbf{2}$, $\varphi_p^p = \mathrm{id}_{B_p}$, and $\varphi_p(1) = a$. Now assume, for some integer $m \geq 2$, that all the appropriate finite Boolean semilattices and $(\vee, 0)$-homomorphisms have been constructed for all elements of P with cardinality smaller than m.

Let $p \in P$ of cardinality m and set $Q = \{\, q \mid \varnothing \subsetneqq q \subsetneqq p \,\}$. Then the direct sum $B = \bigoplus(B_q \mid q \in Q)$ is a finite Boolean semilattice, and, denoting by $\theta_q \colon B_q \hookrightarrow B$ the canonical embedding, there exists a unique $(\vee, 0)$-homomorphism $\psi \colon B \to S$ such that $\psi \circ \theta_q = \varphi_q$ for each $q \in Q$ (so ψ is defined by the rule $\psi(x_q \mid q \in Q) = \bigvee(\,\varphi_q(x_q) \mid q \in Q\,)$). By Lemma 7-4.5, there are a finite Boolean semilattice B_p, a $(\vee, 0, 1)$-homomorphism $\tau \colon B \to B_p$, and a $(\vee, 0)$-homomorphism $\varphi_p \colon B_p \to S$ such that $\psi = \varphi_p \circ \tau$ and $\mathrm{Ker}(\psi) = \mathrm{Ker}(\tau)$.

For each $q \in Q$, the map $\varphi_q^p = \tau \circ \theta_q$ is a $(\vee, 0)$-homomorphism from B_q to B_p. For all $x, y \in B_q$,

$$\begin{aligned}
\varphi_q^p(x) = \varphi_q^p(y) &\Leftrightarrow (\tau \circ \theta_q)(x) = (\tau \circ \theta_q)(y) && \text{(by the definition of } \varphi_q^p\text{)} \\
&\Leftrightarrow (\psi \circ \theta_q)(x) = (\psi \circ \theta_q)(y) && \text{(because } \mathrm{Ker}(\psi) = \mathrm{Ker}(\tau)\text{)} \\
&\Leftrightarrow \varphi_q(x) = \varphi_q(y)\,,
\end{aligned}$$

so $\mathrm{Ker}(\varphi_q^p) = \mathrm{Ker}(\varphi_q)$. This takes care of (L5). We also set $\varphi_p^p = \mathrm{id}_{B_p}$, which takes care of (L2). The construction can be followed on the left-hand side diagram of Figure 7-4.2.

$$\begin{tikzcd}
B_q \arrow[r, "\varphi_q"] \arrow[d, hook, "\theta_q"'] \arrow[dr, "\varphi_q^p" near end] & S \\
B \arrow[ur, "\psi"' near end] \arrow[r, "\tau"'] & B_p \arrow[u, "\varphi_p"']
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
 & & S \\
B_r \arrow[urr, "\varphi_r"] \arrow[r, "\varphi_r^q"'] & B_q \arrow[ur, "\varphi_q"'] \arrow[r, "\varphi_q^p"'] & B_p \arrow[u, "\varphi_p"']
\end{tikzcd}$$

Figure 7-4.2: Illustrating the proof of Theorem 7-4.6.

For each $x \in \nu(p)$, there is $n < \omega$ such that $(x, n) \in p$, thus, observing that $q = \{(x, n)\}$ belongs to Q,

$$\{x\} \subseteq \nu(q) \subseteq \varphi_q(B_q) \subseteq \psi(B) \subseteq \varphi_p(B_p)\,,$$

so $\nu(p) \subseteq \varphi_p(B_p)$. This takes care of (L1). Next, we verify, for each $q \subsetneqq p$,

$$\varphi_p \circ \varphi_q^p = \varphi_p \circ \tau \circ \theta_q = \psi \circ \theta_q = \varphi_q\,,$$

which takes care of (L4). Let $r \subsetneqq q \subsetneqq p$. Then

$$\psi \circ \theta_q \circ \varphi_r^q = \varphi_q \circ \varphi_r^q = \varphi_r = \psi \circ \theta_r\,,$$

hence, since $\mathrm{Ker}(\psi) = \mathrm{Ker}(\tau)$, we get $\tau \circ \theta_q \circ \varphi_r^q = \tau \circ \theta_r$, that is, $\varphi_q^p \circ \varphi_r^q = \varphi_r^p$. This takes care of (L3), and thus completes the inductive step of the construction.

Due to (L2)–(L4), the collection $\vec{B} = (B_p, \varphi_p^q \mid p \subseteq q \text{ in } P)$ is a directed diagram of finite Boolean semilattices and $(\vee, 0)$-homomorphisms, and the co-cone $(S, \varphi_p \mid p \in P)$ is a point at infinity of $\vec{B}$. Furthermore, it follows from (L1) that $B = \bigcup(\,\varphi_p(B_p) \mid p \in P\,)$.

Let $p \in P$ and let $x, y \in B_p$ such that $\varphi_p(x) = \varphi_p(y)$. Since P has no maximal element, there exists $p' \in P$ such that $p \subsetneqq p'$. Then it follows from (L5) that $\varphi_p^{p'}(x) = \varphi_p^{p'}(y)$. Therefore, by the characterization of directed colimits given in Lemma 7-2.3, it follows that $(S, \varphi_p \mid p \in P) = \varinjlim \vec{B}$.

The transition maps $\varphi_p^q\colon B_p \to B_q$ in $\vec{B}$ are not necessarily zero-separating. However, the $(\vee,0)$-semilattice $\overline{B}_p = B_p/\varphi_p^{-1}\{0\}$ is Boolean for each $p \in P$, and if $\pi_p\colon B_p \twoheadrightarrow \overline{B}_p$ denotes the canonical projection, there exists a unique $(\vee,0)$-homomorphism $\overline{\varphi}_p^q\colon \overline{B}_p \to \overline{B}_q$ such that $\overline{\varphi}_p^q \circ \pi_p = \pi_q \circ \varphi_p^q$. Denoting by $\overline{\varphi}_p\colon \overline{B}_p \to S$ the unique $(\vee,0)$-homomorphism such that $\varphi_p = \overline{\varphi}_p \circ \pi_p$, it is straightforward to verify that

$$(S, \overline{\varphi}_p \mid p \in P) = \varinjlim(\overline{B}_p, \overline{\varphi}_p^q \mid p \subseteq q \text{ in } P)\,,$$

and that all the transition maps $\overline{\varphi}_p^q$ are zero-separating. (So are all the limiting maps $\overline{\varphi}_p$, but this is a consequence of the latter fact.)

Suppose from now on that all the φ_p^q are zero-separating, and that in addition, S has a unit element 1. Denoting by 1_p the unit element of B_p, for each $p \in P$, there exists $o \in P$ such that $1 = \varphi_o(1_o)$. The poset $Q = P \uparrow o$ is directed, and the $(\vee,0)$-semilattice $C_p = B_p \downarrow \varphi_o^p(1_o)$ is Boolean for each $p \in P \uparrow o$. Furthermore, the restriction ψ_p^q of φ_p^q from C_p to C_q and the restriction ψ_p of φ_p from C_p to S are both zero-separating and unit-preserving, and it is straightforward to verify that

$$(S, \psi_p \mid p \in Q) = \varinjlim(C_p, \psi_p^q \mid p \subseteq q \text{ in } Q)\,.$$

This concludes the proof. □

Since there are finite non-Boolean distributive $(\vee,0)$-semilattices, the result of Theorem 7-4.6 looks strange even for *finite* distributive $(\vee,0)$-semilattices. However, a more careful look shows that this does not cause any problem, as, for example, the underlying object of a directed colimit of an infinite constant sequence with unique entry B may not be B. For instance, the three-element chain C_3 is the directed colimit of the infinite sequence

$$\mathsf{B}_2 \to \mathsf{B}_2 \to \cdots\,,$$

where B_2 denotes the two-atom Boolean (semi)lattice, C_3 is identified to $\{\,(x,y) \in \mathsf{B}_2 \times \mathsf{B}_2 \mid x \leq y\,\}$, and the unique transition map in the diagram above, as well as the unique limiting map, is the idempotent map $\mathsf{B}_2 \to \mathsf{C}_3$, $(x,y) \mapsto (x, x \vee y)$.

Say that a $(\vee,0)$-semilattice S is *ultraboolean* if it is a directed union of finite Boolean semilattices. Every ultraboolean $(\vee,0)$-semilattice is distributive, but the converse fails, for example for any finite non-Boolean distributive $(\vee,0)$-semilattice. Denote by $\mathbf{UltraBool}^{\mathbf{idp}}$ the category whose objects are the pairs (B,e), where B is an ultraboolean $(\vee,0)$-semilattice and e is an idempotent endomorphism of B with cofinal range, and where an arrow from (A,a) to (B,b) is a $(\vee,0)$-embedding $\varphi\colon A \hookrightarrow B$ such that $\varphi \circ a = b \circ \varphi$. Define $\Pi(A,a) = a(A)$ and define $\Pi(\varphi)$ as the restriction of φ from $a(A)$ into $b(B)$. Then Π is a functor from $\mathbf{UltraBool}^{\mathbf{idp}}$ to $\mathbf{DSem}_{\vee,0}^{\mathrm{emb}}$, the *canonical projection*. The following result, established in Wehrung [332, Theorem 9.5], expresses the abundance of ultraboolean semilattices.

$\diamondsuit$ **Theorem 7-4.7** (Wehrung 2005)**.** *There exists a functor* Φ *from* $\mathbf{DSem}_{\vee,0}^{\mathrm{emb}}$ *to* $\mathbf{UltraBool}^{\mathbf{idp}}$ *such that* $\Pi \circ \Phi$ *is the identity functor. Moreover,* Φ *sends finite distributive* $(\vee, 0)$*-semilattices to structures* (B, e) *where* B *is a finite Boolean semilattice* (*necessarily,* e *is unit-preserving*)*.*

This can be roughly paraphrased by saying that *every distributive* $(\vee, 0)$*-semilattice is, functorially, a retract of an ultraboolean* $(\vee, 0)$*-semilattice.*

Strictly speaking, [332, Theorem 9.5] states the existence of the restriction of Φ to *finite* distributive $(\vee, 0)$-semilattices. However, by using Theorem 7-4.2 together with Gillibert and Wehrung [114, Proposition 1.4.2], this functor can be extended, *via* directed colimits, to all distributive $(\vee, 0)$-semilattices.

To illustrate the difficulty of constructing the functor Φ solving Theorem 7-4.7, if S is a finite distributive $(\vee, 0)$-semilattice with n join-irreducible elements, then the number of elements of the underlying finite Boolean semilattice of $\Phi(S)$ is, very roughly speaking, a tower of exponentials of length $2n$. This is, of course, beyond the reach of any implementation.

Theorem 7-4.7 is used to prove Theorem 7.4 from Wehrung [333], of which a special case is the following.

$\diamondsuit$ **Theorem 7-4.8** (Wehrung 2005)**.** *Let* $\mathbf{V}$ *be a variety of algebras. If every diagram of finite Boolean semilattices and* $(\vee, 0)$*-embeddings* (*resp.,* $(\vee, 0, 1)$*-embeddings*)*, indexed by a lattice, can be lifted with respect to the* $\mathrm{Con_c}$ *functor on* $\mathbf{V}$*, then so can every diagram of finite distributive semilattices and* $(\vee, 0)$*-embeddings* (*resp.* $(\vee, 0, 1)$*-embeddings*)*, indexed by a lattice.*

Theorem 7-4.8 is, in fact, a special case of the far more general categorical statement [333, Theorem 6.3]. However, the latter statement looks far less user-friendly than the one of Theorem 7-4.8 and requires a far deeper look in the categorical structure in question.

7-4.3 Ladders

Ladders are a special type of posets that can be used as index sets in directed colimit representations of objects of cardinality below some $\aleph_n$, in fact, in all cases encountered as to the present writing, at most $\aleph_1$. These posets first appeared in Ditor [67], under the name *k-lattices.* Their use in solving congruence representation problems originates in Dobbertin [70] (under the name *frames* – in conflict with the already ubiquitous lattice-theoretical concept of a *von Neumann frame*), and is pursued in a number of papers such as Huhn [201, 202, 203], Grätzer, Lakser and Wehrung [154], Wehrung [328].

Recall from Section 7-3.4 that a poset P is lower finite if every principal ideal of P is finite.

Definition 7-4.9. For a positive integer k, a lower finite poset P is a *k-ladder* if P is a lattice and every element of P has at most k lower covers.

Observe that every lower finite lattice (thus, in particular, every k-ladder) has a least element.

Every k-ladder has at most $\aleph_{k-1}$ elements (cf. Ditor [67]; see also Dobbertin [70] for the case $k = 2$). That the bound $\aleph_{k-1}$ can be reached is obviously true for $k = 1$ (consider the chain ω of all nonnegative integers). For $k = 2$ this is also true, by the following result of Ditor [67] (see also Dobbertin [70]), and of which we present a proof for convenience (see Grätzer, Lakser, and Wehrung [154, Proposition 2.2]).

Proposition 7-4.10 (Ditor 1984)**.** *There exists a 2-ladder of cardinality $\aleph_1$.*

Proof. For $\xi < \omega_1$, we construct inductively the lattices L_ξ with no largest element, as follows. Put $L_0 = \omega$. If λ is countable limit ordinal and all L_ξ have been constructed, for $\xi < \lambda$, in such a way that $\xi < \eta$ implies that L_ξ is an ideal of L_η, we put $L_\lambda = \bigcup(L_\xi \mid \xi < \lambda)$. Assume that we have constructed L_ξ, a countable 2-ladder with no largest element. Then L_ξ has a strictly increasing, countable, cofinal sequence $(a_n \mid n < \omega)$. Let $(b_n \mid n < \omega)$ be a strictly increasing countable chain, with $b_n \notin L_\xi$, for all n. Set

$$L_{\xi+1} = L_\xi \cup \{ b_n \mid n < \omega \},$$

endowed with the least partial ordering containing the ordering of L_ξ, the natural ordering of $\{ b_n \mid n < \omega \}$, and all pairs $a_n < b_n$, for $n < \omega$. Observe that L_ξ is an ideal of $L_{\xi+1}$. It is easy to verify that $L = \bigcup(L_\xi \mid \xi < \omega_1)$ is a 2-ladder of cardinality $\aleph_1$. □

For $k \geq 3$ the situation becomes far more mysterious. Ditor raised in [67] the problem whether there exists a 3-ladder of cardinality $\aleph_2$. This was proved to be consistent with ZFC in Wehrung [337]. More precisely,

$\diamondsuit$ **Theorem 7-4.11** (Wehrung 2010)**.** *Suppose that either there exists a gap-1 morass or Martin's Axiom* $\mathsf{MA}(\aleph_1;$ *precaliber* $\aleph_1)$ *holds. Then there exists a 3-ladder of cardinality $\aleph_2$.*

In particular, the nonexistence of a 3-ladder of cardinality $\aleph_2$ implies that ω_2 is inaccessible in the constructible universe. More issues related to Theorem 7-4.11 are discussed in far more detail in the survey paper Wehrung [338].

Oddly enough (some would say "fortunately" – I wouldn't), no use has been found so far for 3-ladders of cardinality $\aleph_2$ in tackling congruence representation problems. About the latter, a crucial, though trivial, step is the following lemma.

Lemma 7-4.12. *Let F and P be posets with F lower finite, P directed, and $\operatorname{card} F \geq \operatorname{card} P$. Then there exists an isotone cofinal map from F to P.*

Proof. By assumption, there exists a surjective map $f\colon F \twoheadrightarrow P$. Since F is lower finite, it is well founded and we can define inductively a map $g\colon F \to P$ by defining $g(x)$, for $x \in F$, as any element of P above $f(x)$ and also above each $g(y)$, for $y < x$. Since P is directed, this is possible. $\square$

Corollary 7-4.13. *Let F be a directed, lower finite poset and let S be a distributive $(\vee, 0)$-semilattice. If $\operatorname{card} F \geq \operatorname{card} S + \aleph_0$, then S can be represented as the colimit of a directed diagram, indexed by F, of finite Boolean semilattices and zero-separating $(\vee, 0)$-homomorphisms, which can furthermore be assumed to be unit-preserving in case S has a unit.*

Proof. By Theorem 7-4.6, there is a directed colimit representation

$$(S, \sigma_p \mid p \in P) = \varinjlim(B_p, \sigma_p^q \mid p \leq q \text{ in } P),$$

in $\mathbf{Sem}_{\vee,0}$, for a directed poset P, all the B_p finite Boolean, all the σ_p^q zero-separating, and all the σ_p^q unit-preserving in case S has a unit. Furthermore, the proof of Theorem 7-4.6 yields $\operatorname{card} P \leq \operatorname{card} S + \aleph_0$. Now by Lemma 7-4.12, there exists an isotone cofinal map $f\colon F \to P$. An elementary argument of category theory, using the directedness assumption on F, now shows that

$$(S, \sigma_{f(x)} \mid x \in F) = \varinjlim \left(B_{f(x)}, \sigma_{f(x)}^{f(y)} \mid x \leq y \text{ in } F\right),$$

as required. $\square$

Corollary 7-4.13 will be used in the two following cases: $\operatorname{card} S = \aleph_0$ with $F = \omega$, and $\operatorname{card} S = \aleph_1$ with F any 2-ladder of cardinality $\aleph_1$ (whose existence follows from Proposition 7-4.10).

7-4.4 Pudlák's approach to CLP

Pudlák's idea for solving CLP, introduced in Pudlák [274], consists of looking for a *functor* $\Gamma\colon \mathbf{DSem}_{\vee,0}^{\mathrm{emb}} \to \mathbf{Lat}$ such that $\mathrm{Con_c} \circ \Gamma$ is isomorphic to the identity functor on $\mathbf{DSem}_{\vee,0}^{\mathrm{emb}}$.

The question of the existence of such a functor is stated on Page 100 of Pudlák [274]. If such a functor Γ existed, then S would be isomorphic to $\mathrm{Con_c}\, \Gamma(S)$ for every distributive $(\vee, 0)$-semilattice S, which would imply a positive answer to CLP.

The main result of Pudlák [274] is an important subcase of the question above. Denote by $\mathbf{DLat}_0^{\mathrm{emb}}$ the category of all distributive lattices with zero, with 0-*lattice* embeddings (not just $(\vee, 0)$-embeddings).

$\diamondsuit$ **Theorem 7-4.14** (Pudlák 1985). *There is a functor $\Gamma\colon \mathbf{DLat}_0^{\mathrm{emb}} \to \mathbf{Lat}$ such that $\mathrm{Con_c} \circ \Gamma$ is isomorphic to the identity functor on $\mathbf{DLat}_0^{\mathrm{emb}}$. Furthermore, Γ preserves directed colimits, and it sends finite distributive lattices to finite atomistic lattices.*

In the statement of Theorem 7-4.14, the category $\mathbf{DLat}_0^{\mathrm{emb}}$ cannot be replaced by the category $\mathbf{DLat}_0$ of all distributive 0-lattices and 0-lattice *homomorphisms*, see Exercise 7.31. Also, Tůma proved in [312] that there is no simultaneous representation for the poset of all distributive $(\vee, 0)$-subsemilattices of the Boolean lattice B_4 by congruence semilattices of finite atomistic lattices. However, the question whether "finite atomistic" could be dispensed with remained open more than ten years after the publication of [312] (see Theorem 7-4.15 for the answer to that question).

Back to Theorem 7-4.14 for a moment, observe that $D \cong \mathrm{Con_c}\,\Gamma(D)$ for any distributive 0-lattice D. In particular, Theorem 7-4.14 yields a second proof of Schmidt's Theorem 7-3.21, stating that every distributive lattice with zero is isomorphic to the congruence semilattice of some lattice.

The lattice $\Gamma(D)$ constructed in Pudlák's proof is quite different from the lattice constructed in Schmidt's proof. In particular, as D is the directed union of all its finite distributive 0-sublattices, $\Gamma(D)$ is the directed colimit of all $\Gamma(X)$ for X a finite distributive 0-sublattice of D. Since each $\Gamma(X)$ is finite atomistic, it is congruence-permutable (see Theorem 268, Lemma 271, and Theorem 272 in LTF), hence $\Gamma(D)$ is congruence-permutable.

Pudlák's functor Γ is obtained *via* a very involved direct construction. Theorem 7-4.14 is extended by a later result of Růžička [284] (see Corollary 8-4.4), which implies that $\Gamma(D)$ can also be taken *modular*. Although Růžička's functor Γ sends any distributive 0-lattice to a directed colimit of finite, modular, atomistic lattices, it cannot preserve finiteness: for example, if D is not Boolean, then $\Gamma(D)$ cannot be finite (for the congruence lattice of a finite modular lattice is Boolean).

Pudlák's problem discussed above got solved in the negative (before CLP got finally settled) in Tůma and Wehrung [318]. While the first draft of that paper stated the result for congruence lattices of lattices, it got soon extended to a much wider class of structures by using deep results in commutator theory by Kearnes and Szendrei [221]. The main result of [318] can be stated as follows.

$\diamondsuit$ **Theorem 7-4.15** (Tůma and Wehrung 2006)**.** *There exists a diagram $\mathcal{D}_{\bowtie}$, indexed by a finite poset, of finite Boolean $(0,1)$-subsemilattices of B_4 and inclusion mappings, which cannot be lifted, with respect to the $\mathrm{Con_c}$ functor, in any variety satisfying a nontrivial congruence lattice identity.*

In particular, the diagram $\mathcal{D}_{\bowtie}$ cannot be lifted, with respect to the $\mathrm{Con_c}$ functor, by *lattices*, *majority algebras*, *groups*, *loops*, *modules*, and so on. On the other hand, Lampe [238] proved that every $(\vee, 0, 1)$-semilattice (distributive or not) is isomorphic to $\mathrm{Con_c}\,G$ for some *groupoid* G (a *groupoid* in universal algebra is just a nonempty set with a binary operation). By Gillibert [105, Corollaire 3.6.10] or Gillibert [106, Corollary 7.10], this result can be extended to any diagram of $(\vee, 0, 1)$-semilattices and $(\vee, 0, 1)$-homomorphisms indexed by a finite poset, and in particular to $\mathcal{D}_{\bowtie}$: that is, *$\mathcal{D}_{\bowtie}$ can be lifted by a diagram of groupoids.*

We shall now describe this diagram. We found it convenient to describe it in terms of Boolean semilattices B_k, with $1 \leq k \leq 4$, and $(\vee, 0, 1)$-embeddings, rather than Boolean subsemilattices of B_4 and inclusion mappings. For each positive integer n, identify the Boolean lattice B_n with the powerset of $\{0, 1, \dots, n-1\}$. Consider the $(\vee, 0, 1)$-embeddings $\boldsymbol{e}\colon \mathsf{B}_1 \hookrightarrow \mathsf{B}_2$, $\boldsymbol{f}_i\colon \mathsf{B}_2 \hookrightarrow \mathsf{B}_3$, and $\boldsymbol{u}_i\colon \mathsf{B}_3 \hookrightarrow \mathsf{B}_4$ (for $i < 3$), determined by their values on the atoms of their respective domains:

$$\boldsymbol{e}\colon \{0\} \mapsto \{0,1\};$$

$$\boldsymbol{f}_0\colon \begin{cases} \{0\} & \mapsto \{0,1\} \\ \{1\} & \mapsto \{0,2\} \end{cases}, \qquad \boldsymbol{f}_1\colon \begin{cases} \{0\} & \mapsto \{0,1\} \\ \{1\} & \mapsto \{1,2\} \end{cases}, \qquad \boldsymbol{f}_2\colon \begin{cases} \{0\} & \mapsto \{0,2\} \\ \{1\} & \mapsto \{1,2\} \end{cases},$$

$$\boldsymbol{u}_0\colon \begin{cases} \{0\} & \mapsto \{0\} \\ \{1\} & \mapsto \{1,3\} \\ \{2\} & \mapsto \{2,3\} \end{cases}, \qquad \boldsymbol{u}_1\colon \begin{cases} \{0\} & \mapsto \{0,3\} \\ \{1\} & \mapsto \{1\} \\ \{2\} & \mapsto \{2,3\} \end{cases}, \qquad \boldsymbol{u}_2\colon \begin{cases} \{0\} & \mapsto \{0,3\} \\ \{1\} & \mapsto \{1,3\} \\ \{2\} & \mapsto \{2\} \end{cases}.$$

The diagram $\mathcal{D}_{\bowtie}$ is represented on Figure 7-4.3.

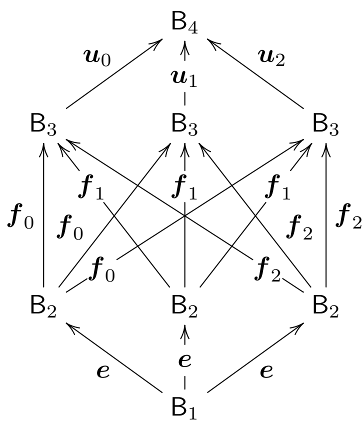

Figure 7-4.3: The diagram $\mathcal{D}_{\bowtie}$.

It is still unknown whether every distributive $(\vee, 0)$-semilattice is isomorphic to the congruence semilattice of some majority algebra (cf. Problem 9.3. A *majority algebra* is a nonempty set M, endowed with a ternary operation m, such that $m(x,x,y) = m(x,y,x) = m(y,x,x) = x$ for all $x, y \in M$. It is well known that the congruence lattice of a majority algebra is distributive, see Exercise 7.2; see also Section 9-3.4). Therefore, the representation problem of distributive $(\vee, 0)$-semilattices as compact congruence semilattices of majority algebras has a diagram counterexample (namely $\mathcal{D}_{\bowtie}$) but no known object counterexample.

The interaction between diagram counterexamples and object counterexamples is quite complex and surprising. We study it in detail in the book Gillibert and Wehrung [114]; see also the survey paper Wehrung [338].

7-4.5 A very simple unliftable triangle

Identify B_n with $\{0,1\}^n$, for each positive integer n. Let $\boldsymbol{e}\colon \mathsf{B}_1 \to \mathsf{B}_2$, $x \mapsto (x,x)$, and let $\boldsymbol{p}\colon \mathsf{B}_2 \to \mathsf{B}_1$, $(x,y) \mapsto x \vee y$. Observe that $\boldsymbol{p} \circ \boldsymbol{e} = \mathrm{id}_{\mathsf{B}_1}$. Consider the diagram $\vec{T}$ of finite Boolean semilattices and $(\vee, 0)$-homomorphisms represented on the left-hand side of Figure 7-4.4.

$$
\begin{tikzcd}
\mathsf{B}_2 \arrow[r, "\boldsymbol{p}"] & \mathsf{B}_1 \\
\mathsf{B}_1 \arrow[u, "\boldsymbol{e}"] \arrow[ur, "\mathrm{id}"'] &
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
F \arrow[r, "p"] & E_1 \\
E_0 \arrow[u, "e"] \arrow[ur, "f"'] \arrow[ur, phantom, shift left=3, "\cong"] &
\end{tikzcd}
$$

Figure 7-4.4: Attempting to lift a triangle of Boolean semilattices.

The following observation was established in Tůma and Wehrung [314, Theorem 8.1].

Proposition 7-4.16. *There is no lifting, with respect to the functor* Con*, of the diagram $\vec{T}$, by a diagram $\vec{E}$ of algebras of any similarity type, such that, labeling $\vec{E}$ as on the right-hand side of Figure 7-4.4, $f = p \circ e$ and f is an isomorphism.*

Proof. Since $p \circ e = f$ is surjective, so is p. But $\operatorname{Con} p \cong \boldsymbol{p}$ separates zero, thus p is one-to-one, hence p is an isomorphism, and hence so is $\boldsymbol{p}$, a contradiction. $\square$

It follows immediately from Proposition 7-4.16 that the congruence lattice functor has no categorical right inverse from finite Boolean semilattices, with $(\vee, 0, 1)$-homomorphisms, to algebras of any similarity type. See also Exercise 7.31.

7-5. Representing semilattices of cardinality up to aleph one

While Pudlák's approach is not sufficient to get a full positive solution of CLP (*for a good reason, see Section* 9-3), it yields an important part of the known representation results of a distributive $(\vee, 0)$-semilattice S as congruence semilattice of a lattice L. In the present section we focus on results obtained by assuming the cardinality of S to be "small" (i.e., at most $\aleph_1$), and we get representation results by *relatively complemented lattices* L satisfying additional properties. Section 7-5 will see the emergence of so-called *d-dimensional congruence amalgamation properties*, mainly for $d \in \{0, 1, 2\}$.

We shall get started with the easier case where S is *countable*.

7-5.1 A one-dimensional amalgamation result for $\mathbb{F}$-lattices

Denote by $\operatorname{Sub} V$ the subspace lattice (i.e., the lattice of all subspaces) of any vector space V. The lattice $\operatorname{Sub} V$ is complemented and modular (and even Arguesian).

Definition 7-5.1. Let $\mathbb{F}$ be a field. An $\mathbb{F}$-*lattice* is an isomorphic copy of

$$\prod(\operatorname{Sub} V_i \mid i < n),$$

where n is a nonnegative integer and the V_i are nontrivial finite-dimensional $\mathbb{F}$-vector spaces.

For a lattice L, put $|\boldsymbol{\theta}| = 1$ if $\boldsymbol{\theta}$ is nonzero, 0 otherwise, for each $\boldsymbol{\theta} \in \operatorname{Con} L$. For an $\mathbb{F}$-lattice $L = \prod_{i<n} \operatorname{Sub} V_i$, with the V_i nontrivial finite-dimensional $\mathbb{F}$-vector spaces, it follows from [LTF, Theorem 25] that every congruence of L can be uniquely written as $\prod_{i<n} \boldsymbol{\theta}_i$, where all $\boldsymbol{\theta}_i \in \operatorname{Con}(\operatorname{Sub} V_i)$, hence, since $\operatorname{Sub} V_i$ is a simple lattice, we get an isomorphism $\operatorname{Con} L \to \mathbf{2}^n$, given by the rule

$$\prod_{i<n} \boldsymbol{\theta}_i \mapsto (|\boldsymbol{\theta}_i| \mid i < n).$$

Now we shall prove a *one-dimensional congruence amalgamation result* for $\mathbb{F}$-lattices.

Theorem 7-5.2. *Let $\mathbb{F}$ be a field, let K be an $\mathbb{F}$-lattice, let B be a finite Boolean lattice, and let $\boldsymbol{f} \colon \operatorname{Con} K \to B$ be a $(\vee, 0)$-homomorphism. Then there are an $\mathbb{F}$-lattice L, a 0-lattice homomorphism $f \colon K \to L$, and an isomorphism $\boldsymbol{e} \colon \operatorname{Con} L \to B$ such that $\boldsymbol{f} = \boldsymbol{e} \circ \operatorname{Con} f$ and such that if $\boldsymbol{f}$ is unit-preserving, then so is f.*

The meaning of Theorem 7-5.2 is illustrated on Figure 7-5.1.

$$\begin{tikzcd} B \\ \operatorname{Con} K \arrow[u, "\boldsymbol{f}"] \end{tikzcd} \qquad \begin{tikzcd} L \\ K \arrow[u, "f"] \end{tikzcd} \qquad \begin{tikzcd} \operatorname{Con} L \arrow[rr, "\boldsymbol{e}", "\cong"'] & & B \\ & \operatorname{Con} K \arrow[ul, "\operatorname{Con} f"] \arrow[ur, "\boldsymbol{f}"'] & \end{tikzcd}$$

Figure 7-5.1: One-dimensional congruence amalgamation for $\mathbb{F}$-lattices.

Proof. We start with the case where $B = \mathbf{2}$. If $\boldsymbol{f} = 0$ then just take $L = \operatorname{Sub}\mathbb{F} = \mathbf{2}$, $f = 0$, and $\boldsymbol{e}$ the unique isomorphism $\operatorname{Con} L \to \mathbf{2}$.

Suppose now that $\boldsymbol{f}$ is nonzero. We can write $K = \prod_{i<m} \operatorname{Sub} V_i$, where $1 \leq m < \omega$ and $1 \leq \dim V_i < \omega$ for each $i < m$. There exists a nonempty subset $I \subseteq \{0, 1, \dots, m-1\}$ such that

$$\boldsymbol{f}\Big(\prod_{i<m} \boldsymbol{\theta}_i\Big) = \bigvee_{i \in I} |\boldsymbol{\theta}_i|\,, \quad \text{for each } (\boldsymbol{\theta}_i \mid i < m) \in \prod(\operatorname{Con}(\operatorname{Sub} V_i) \mid i < m)\,.$$

Set $V = \bigoplus_{i \in I} V_i$ and $L = \operatorname{Sub} V$. Since I is nonempty, L is a simple $\mathbb{F}$-lattice. Define $f \colon K \to L$ as follows:

$$f(X_i \mid i < m) = \bigoplus_{i \in I} X_i\,, \quad \text{for each } (X_i \mid i < m) \in K\,.$$

In particular, f is unit-preserving. We take $\boldsymbol{e} \colon \operatorname{Con} L \to \mathbf{2}$, $\boldsymbol{\theta} \mapsto |\boldsymbol{\theta}|$. Then $\boldsymbol{f} = \boldsymbol{e} \circ \operatorname{Con} f$.

Now consider the general case, say $B = \mathbf{2}^n$, for $n < \omega$. Applying the previous result to the n components $\boldsymbol{f}_j \colon \operatorname{Con} K \to \mathbf{2}$ of $\boldsymbol{f}$, we get simple $\mathbb{F}$-lattices L_j, 0-lattice homomorphisms $f_j \colon K \to L_j$, and isomorphisms $\boldsymbol{e}_j \colon \operatorname{Con} L_j \to \mathbf{2}$, $\boldsymbol{\theta} \mapsto |\boldsymbol{\theta}|$, such that each $\boldsymbol{f}_j = \boldsymbol{e}_j \circ (\operatorname{Con} f_j)$ and f_j is unit-preserving provided $\boldsymbol{f}_j$ is unit-preserving (see Figure 7-5.2). The latter occurs, for each $j < n$, in case $\boldsymbol{f}$ is unit-preserving.

$$\begin{tikzcd} \mathbf{2} \\ \operatorname{Con} K \arrow[u, "\boldsymbol{f}_j"] \end{tikzcd} \qquad \begin{tikzcd} L_j \\ K \arrow[u, "f_j"] \end{tikzcd} \qquad \begin{tikzcd} \operatorname{Con} L_j \arrow[rr, "\boldsymbol{e}_j", "\cong"'] & & \mathbf{2} \\ & \operatorname{Con} K \arrow[ul, "\operatorname{Con} f_i"] \arrow[ur, "\boldsymbol{f}_j"'] & \end{tikzcd}$$

Figure 7-5.2: The case $B = \mathbf{2}^n$ in the proof of Theorem 7-5.2.

Put $L = \prod_{j<n} L_j$, $f(x) = (f_j(x) \mid j < n)$ for each $x \in K$, and define $\boldsymbol{e}\big(\prod_{j<n} \boldsymbol{\theta}_j\big) = (|\boldsymbol{\theta}_j| \mid j < n)$ for each $(\boldsymbol{\theta}_j \mid j < n) \in \prod(\operatorname{Con} L_j \mid j < n)$. $\square$

7-5.2 Representing countable distributive semilattices

Schmidt proved in 1974 that every finite distributive lattice is isomorphic to the congruence lattice of some *modular* lattice (see Schmidt [294]). In 1984, he improved his result from modular to *complemented modular* (see Schmidt [297]). The following result extends Schmidt's results to the case of *countable* distributive semilattices. The origin of Theorem 7-5.3 can be traced back to a famous 1986 unpublished note by Bergman [24], proving that *Every countable distributive* $(\vee, 0)$*-semilattice is the finitely generated two-sided ideal lattice of some locally matricial ring* (cf. Theorem 8-4.5). For a more detailed discussion, see Section 8-4.1.

Theorem 7-5.3. *Let $\mathbb{F}$ be a field. Then every countable distributive* $(\vee, 0)$*-semilattice S is isomorphic to* $\operatorname{Con_c} L$*, for a lattice L with zero which is a countable directed union of $\mathbb{F}$-lattices. Furthermore, if S has a unit, then L can be taken bounded.*

Proof. By Corollary 7-4.13 (with $F = \omega$), S is the directed colimit of a diagram of the form

$$S_0 \xrightarrow{\boldsymbol{f}_0} S_1 \xrightarrow{\boldsymbol{f}_1} S_2 \xrightarrow{\boldsymbol{f}_2} \qquad \cdots \qquad \cdots,$$

with each S_n finite Boolean and each $(\vee, 0)$-homomorphism $\boldsymbol{f}_n \colon S_n \to S_{n+1}$ zero-separating, and also unit-preserving in case S has a unit. (For $m \leq n$, the transition map from S_m to S_n is $\boldsymbol{f}_m^n = \boldsymbol{f}_{n-1} \circ \cdots \circ \boldsymbol{f}_m$.) Pick any $\mathbb{F}$-lattice L_0 with an isomorphism $\boldsymbol{e}_0 \colon \operatorname{Con} L_0 \to S_0$. Apply Theorem 7-5.2 to $\boldsymbol{f}_0 \circ \boldsymbol{e}_0 \colon \operatorname{Con} L_0 \to S_1$. We obtain an $\mathbb{F}$-lattice L_1, a 0-lattice homomorphism $f_0 \colon L_0 \to L_1$, and an isomorphism $\boldsymbol{e}_1 \colon \operatorname{Con} L_1 \to S_1$ such that $\boldsymbol{e}_1 \circ \operatorname{Con} f_0 = \boldsymbol{f}_0 \circ \boldsymbol{e}_0$. Furthermore, if S has a unit, then $\boldsymbol{f}_0$ is unit-preserving, thus f_0 can be taken unit-preserving. Proceed with L_1 instead of L_0, and so on by induction. The resulting infinite commutative diagram is represented in Figure 7-5.3.

$$\begin{tikzcd}
\operatorname{Con} L_0 \arrow[r, "\operatorname{Con} f_0"] \arrow[d, "\boldsymbol{e}_0"', "\cong"] & \operatorname{Con} L_1 \arrow[r, "\operatorname{Con} f_1"] \arrow[d, "\boldsymbol{e}_1"', "\cong"] & \operatorname{Con} L_2 \arrow[r, "\operatorname{Con} f_2"] \arrow[d, "\boldsymbol{e}_2"', "\cong"] & \cdots \\
S_0 \arrow[r, "\boldsymbol{f}_0"] & S_1 \arrow[r, "\boldsymbol{f}_1"] & S_2 \arrow[r, "\boldsymbol{f}_2"] & \cdots
\end{tikzcd}$$

Figure 7-5.3: A commutative diagram of $(\vee, 0)$-homomorphisms.

By using Lemma 7-2.2, we obtain that the directed colimit L of the diagram

$$L_0 \xrightarrow{f_0} L_1 \xrightarrow{f_1} L_2 \xrightarrow{f_2} \qquad \cdots \qquad \cdots,$$

is as required. □

In case the field $\mathbb{F}$ is finite, every $\mathbb{F}$-lattice is finite, so the lattice L constructed in the proof of Theorem 7-5.3 is locally finite. Hence we obtain the following.

Corollary 7-5.4. *Every countable distributive* $(\vee, 0)$*-semilattice is isomorphic to the congruence semilattice of some locally finite, relatively complemented modular* (*and even* Arguesian) *lattice L with zero. In addition, if S has a unit, then L can be taken bounded.*

In a different direction, Freese proved in [88] (see also Schmidt [296, Theorem 3.5.2]), building on a construction by Day, Herrmann, and Wille [60], the following result.

$\diamondsuit$ **Theorem 7-5.5** (Freese 1975)**.** *Every finite distributive lattice is isomorphic to the congruence lattice of some finitely generated, modular lattice of breadth* 2.

7-5.3 Embedding finite lattices into finite equivalence lattices

The following result appears in Crawley and Dilworth [43, § 14.1], and is credited there to Dilworth.

$\diamondsuit$ **Theorem 7-5.6** (Dilworth)**.** *Every finite lattice can be embedded into some finite geometric lattice.*

Denote by $\operatorname{Equ}\Omega$ (the *equivalence lattice of* Ω, see LTF) the lattice of all equivalence relations on a set Ω endowed with set inclusion. Then $\operatorname{Equ}\Omega$ is a geometric lattice, isomorphic to the lattice $\operatorname{Part}\Omega$ of all partitions of Ω under refinement, and simple in case Ω is nonempty (cf. [LTF, Section V.4.1]), so the already difficult Theorem 7-5.6 is superseded by the following even deeper result by Pudlák and Tůma [275].

$\diamondsuit$ **Theorem 7-5.7** (Pudlák and Tůma 1980)**.** *Every finite lattice can be embedded into some finite equivalence lattice.*

The following result is established in Grätzer and Schmidt [176, Lemma 7]. The preservation of the bounds is an easy consequence of their proof, which we outline below.

Lemma 7-5.8. *Every nontrivial*[7] *finite lattice L has a* $(0,1)$*-lattice embedding into some finite, simple, sectionally complemented lattice.*

Outline of proof. We may assume that $\operatorname{card} L \geq 3$. By adding a common complement for all elements of $L \setminus \{0,1\}$, we may assume that 1 is not join-irreducible in L. Denote by N the set of all elements of L that are neither zero nor an atom, and for each $a \in N$, add a new atom p_a with $0 < p_a < a$, with

$$\begin{aligned} x < p_a &\Leftrightarrow x = 0\,, \\ p_a \leq x &\Leftrightarrow a \leq x\,, \\ p_a \leq p_b &\Leftrightarrow a = b\,, \end{aligned}$$

for all $a, b \in N$ and all $x \in L$. Then $L' = L \cup \{\, p_a \mid a \in N \,\}$ (a disjoint union), endowed with the ordering defined above, is as required. $\square$

Corollary 7-5.9. *Every finite lattice has a* $(0,1)$*-lattice embedding into some finite equivalence lattice.*

[7] The assumption of nontriviality is necessary: a trivial lattice has no proper $(0,1)$-extension, and it is not simple (simple lattices have exactly two congruences)!

Proof. The trivial lattice is the equivalence lattice of the empty set, so it is sufficient to prove Corollary 7-5.9 for nontrivial lattices. By Lemma 7-5.8, it suffices to prove Corollary 7-5.9 for a finite, simple, sectionally complemented lattice L. We will actually only use the consequence that L is subdirectly irreducible. By Theorem 7-5.6, L embeds into a finite equivalence lattice P. Since every closed interval in an equivalence lattice is a product of equivalence lattices (this follows from Lemma 403(iv,v) of LTF), we may replace P by the interval $[0_L, 1_L]$ of P and thus obtain L as a $(0,1)$-sublattice of a finite product $P = \prod(P_i \mid i < n)$, where n is a positive integer and each P_i is a finite equivalence lattice.

Denote by $\pi_i \colon P \twoheadrightarrow P_i$ the canonical projection, for each $i < n$. Since L is a subdirectly irreducible sublattice of P, one of the restrictions $\pi_i{\restriction}_L$ is one-to-one, so it defines a $(0,1)$-lattice embedding from L into P_i. $\square$

We obtain the following amalgamation result for finite lattices and 0-lattice embeddings.

Lemma 7-5.10. *Let L_0, L_1, and L_2 be finite lattices with 0-lattice embeddings $f_i \colon L_0 \hookrightarrow L_i$ for $i \in \{1,2\}$. Then there are a finite equivalence lattice L and 0-lattice embeddings $g_i \colon L_i \hookrightarrow L$, for $i \in \{1,2\}$, such that*

(i) $g_1 \circ f_1 = g_2 \circ f_2$;

(ii) *if f_1 and f_2 are both unit-preserving, then so are g_1 and g_2.*

Proof. We may assume that f_1 and f_2 are both inclusion mappings and that $L_0 = L_1 \cap L_2$. On the set $P = L_1 \cup L_2$, we define, as on pages 454 and 455 in LTF, the partial ordering $\leq$ generated by the union of the partial orderings on L_1 and L_2. Then $(P, \leq)$, endowed with its (partial) supremum operation, is a partial lattice in the sense of LTF (cf. [LTF, Lemma 88]), and the ideal-filter construction used in the proof of [LTF, Theorem 84] embeds P into a finite lattice K with the same zero as P (the common zero of L_1 and L_2) and the same unit as P in case f_1 and f_2 are both unit-preserving (in which case $1_P = 1_{L_1} = 1_{L_2}$). Finally, by Corollary 7-5.9, K is a $(0,1)$-sublattice of some finite equivalence lattice L. Define $g_i \colon L_i \hookrightarrow L$ as the inclusion map, for $i \in \{1,2\}$. $\square$

7-5.4 Representing distributive semilattices with at most aleph one elements

Most congruence representation results stated in Subsections 7-5.4 and 7-5.5 will involve the classes of lattices defined as follows.

Definition 7-5.11. A lattice is

- *partitional* if it is isomorphic to a finite product of finite equivalence lattices;

- *locally partitional* if it has a least element and it is isomorphic to a directed union of partitional lattices.

Observe that every locally partitional lattice is both locally finite and relatively complemented with zero.

The following "Two-dimensional finite congruence amalgamation Theorem" is a variant, for finite Boolean lattices, of Grätzer, Lakser, and Wehrung [154, Theorem 1], itself an extension of Tůma [313, Theorem 1]. In those two results, the top semilattice B is an arbitrary finite distributive $(\vee, 0)$-semilattice, but the bounds may not be preserved.

Theorem 7-5.12. *Let L_0, L_1, L_2 be finite lattices and let $f_1\colon L_0 \to L_1$ and $f_2\colon L_0 \to L_2$ be 0-lattice homomorphisms. Let B be a finite Boolean lattice, and, for $i \in \{1, 2\}$, let $\boldsymbol{g}_i\colon \operatorname{Con} L_i \to B$ be $(\vee, 0)$-homomorphisms such that*

$$\boldsymbol{g}_1 \circ \operatorname{Con} f_1 = \boldsymbol{g}_2 \circ \operatorname{Con} f_2\,.$$

Then there are a partitional lattice L, an isomorphism $\boldsymbol{e}\colon \operatorname{Con} L \to B$, and 0-lattice homomorphisms $g_i\colon L_i \to L$, for $i \in \{1, 2\}$, such that

$$\begin{aligned} g_1 \circ f_1 &= g_2 \circ f_2\,, \\ \boldsymbol{e} \circ \operatorname{Con} g_i &= \boldsymbol{g}_i && \text{for each } i \in \{1, 2\}\,, \end{aligned}$$

and such that if, in addition, f_1, f_2, $\boldsymbol{g}_1$, and $\boldsymbol{g}_2$ are all unit-preserving, then so are g_1 and g_2.

The meaning of Theorem 7-5.4 is illustrated on Figure 7-5.4. Dotted arrows are those whose existence is stated by Theorem 7-5.4.

$$\begin{tikzcd} & L & \\ L_1 \arrow[ur, dashed, "g_1"] & & L_2 \arrow[ul, dashed, "g_2"'] \\ & L_0 \arrow[ul, "f_1"] \arrow[ur, "f_2"'] & \end{tikzcd} \qquad \begin{tikzcd} & \operatorname{Con} L \arrow[d, dashed, "\boldsymbol{e}"'] & \\ & B & \\ \operatorname{Con} L_1 \arrow[uur, dashed, "\operatorname{Con} g_1"] \arrow[ur, "\boldsymbol{g}_1"'] & & \operatorname{Con} L_2 \arrow[uul, dashed, "\operatorname{Con} g_2"'] \arrow[ul, "\boldsymbol{g}_2"] \\ & \operatorname{Con} L_0 \arrow[ul, "\operatorname{Con} f_1"] \arrow[ur, "\operatorname{Con} f_2"'] & \end{tikzcd}$$

Figure 7-5.4: Illustrating Theorem 7-5.12.

Proof. We begin with the case where $B \cong \mathbf{2}$.

If $\boldsymbol{g}_1 = 0$ and $\boldsymbol{g}_2 = 0$, then the lattice $L = \mathbf{2}$, the unique isomorphism $\boldsymbol{e}\colon \operatorname{Con}\mathbf{2} \to B$, and the zero maps g_i for $i \in \{1,2\}$, are as required. Now suppose that $\boldsymbol{g}_1$ and $\boldsymbol{g}_2$ are not simultaneously zero.

Set $\boldsymbol{g}_0 = \boldsymbol{g}_1 \circ \operatorname{Con} f_1 = \boldsymbol{g}_2 \circ \operatorname{Con} f_2$ (so that $\boldsymbol{g}_i\colon \operatorname{Con} L_i \to \mathbf{2}$ for each $i \in \{0,1,2\}$). The binary relation

$$\boldsymbol{\theta}_i = \{\, (x,y) \in L_i \times L_i \mid \boldsymbol{g}_i(\operatorname{con}_{L_i}(x,y)) = 0 \,\}\,, \quad \text{for each } i \in \{0,1,2\}\,,$$

is a congruence of L_i. Furthermore, setting $\overline{L}_i = L_i/\boldsymbol{\theta}_i$ and denoting by $p_i\colon L_i \twoheadrightarrow \overline{L}_i$ the canonical projection, for $i \in \{0,1,2\}$, we obtain that whenever $i \in \{1,2\}$, there exists a unique 0-lattice homomorphism $\overline{f}_i\colon \overline{L}_0 \to \overline{L}_i$ such that $\overline{f}_i \circ p_0 = p_i \circ f_i$. In addition, $\overline{f}_i$ is one-to-one, and it is unit-preserving in case f_i is unit-preserving.

Since $\overline{f}_i\colon \overline{L}_0 \hookrightarrow \overline{L}_i$ for $i \in \{1,2\}$, it follows from Lemma 7-5.10 that there are a finite equivalence lattice L and 0-lattice embeddings $\overline{g}_i\colon \overline{L}_i \hookrightarrow L$ such that $\overline{g}_1 \circ \overline{f}_1 = \overline{g}_2 \circ \overline{f}_2$, and also such that if $\overline{f}_1$ and $\overline{f}_2$ are unit-preserving then so are $\overline{g}_1$ and $\overline{g}_2$. Since $\boldsymbol{g}_1$ and $\boldsymbol{g}_2$ are not simultaneously zero, the congruences $\boldsymbol{\theta}_1$ and $\boldsymbol{\theta}_2$ are not simultaneously full, thus either $\overline{L}_1$ or $\overline{L}_2$ is not trivial, and thus L is not trivial.

Denote by $\boldsymbol{e}\colon \operatorname{Con} L \to B$ the unique isomorphism. If $\boldsymbol{g}_1$ and $\boldsymbol{g}_2$ are both unit-preserving, then (since $B \cong \mathbf{2}$) the congruences $\boldsymbol{\theta}_i$ are not full, thus the lattices $\overline{L}_i$ are both nontrivial, and then, if f_1 and f_2 are both unit-preserving, the above-mentioned construction yields the $\overline{g}_i$ unit-preserving.

Set $g_i = \overline{g}_i \circ p_i$, for each $i \in \{1,2\}$. Then $g_i\colon L_i \to L$ and

$$g_i \circ f_i = \overline{g}_i \circ p_i \circ f_i = \overline{g}_i \circ \overline{f}_i \circ p_0\,,$$

hence, since $\overline{g}_1 \circ \overline{f}_1 = \overline{g}_2 \circ \overline{f}_2$, we get $g_i \circ f_1 = g_2 \circ f_2$. If all f_i and $\boldsymbol{g}_i$ are unit-preserving, then so are all $\overline{g}_i$, thus so are all g_i. Furthermore, for each $i \in \{1,2\}$ and all $x, y \in L_i$,

$$\begin{aligned}
(\boldsymbol{e} \circ \operatorname{Con} g_i)(\operatorname{con}_{L_i}(x,y)) = 0 \;&\Leftrightarrow\; (\operatorname{Con} g_i)(\operatorname{con}_{L_i}(x,y)) = 0 \\
&\qquad \text{(because } \boldsymbol{e} \text{ is an isomorphism)} \\
&\Leftrightarrow\; \operatorname{con}_L(g_i(x), g_i(y)) = 0 \\
&\qquad \text{(by the definition of } \operatorname{Con} g_i\text{)} \\
&\Leftrightarrow\; g_i(x) = g_i(y) \\
&\Leftrightarrow\; p_i(x) = p_i(y) \\
&\qquad \text{(because } g_i = \overline{g}_i \circ p_i \text{ and } \overline{g}_i \text{ is one-to-one)} \\
&\Leftrightarrow\; (x,y) \in \boldsymbol{\theta}_i \\
&\Leftrightarrow\; \boldsymbol{g}_i(\operatorname{con}_{L_i}(x,y)) = 0\,.
\end{aligned}$$

Since the congruences $\operatorname{con}_{L_i}(x,y)$ generate $\operatorname{Con} L_i$ as a $(\vee,0)$-semilattice and $B \cong \mathbf{2}$, it follows that $\boldsymbol{e} \circ \operatorname{Con} g_i = \boldsymbol{g}_i$.

In the general case, we can write $B = \prod_{j<n} B_j$, for a natural number n and $B_j \cong \mathbf{2}$ for each $j < n$. Denote by $\beta_j \colon B \twoheadrightarrow B_j$ the canonical projection and set $\boldsymbol{g}_{i,j} = \beta_j \circ \boldsymbol{g}_i$, for all $i \in \{1,2\}$ and all $j < n$. Then $\boldsymbol{g}_{1,j} \circ \operatorname{Con} f_1 = \boldsymbol{g}_{2,j} \circ \operatorname{Con} f_2$ for each $j < n$, hence, since $B_j \cong \mathbf{2}$, there are a finite equivalence lattice $L^{(j)}$ and an isomorphism $\boldsymbol{e}_j \colon \operatorname{Con} L^{(j)} \to B_j$, together with 0-lattice homomorphisms $g_{i,j} \colon L_i \to L^{(j)}$, for $i \in \{1,2\}$, such that

$$g_{1,j} \circ f_1 = g_{2,j} \circ f_2\,, \tag{7-5.1}$$

$$\boldsymbol{e}_j \circ \operatorname{Con} g_{i,j} = \boldsymbol{g}_{i,j} \qquad \text{for each } i \in \{1,2\}\,, \tag{7-5.2}$$

and such that if, in addition, f_1, f_2, $\boldsymbol{g}_1$, and $\boldsymbol{g}_2$ (thus also $\boldsymbol{g}_{1,j}$ and $\boldsymbol{g}_{2,j}$) are all unit-preserving, then so are $g_{1,j}$ and $g_{2,j}$.

The lattice $L = \prod(L^{(j)} \mid j < n)$ is partitional. Furthermore, denoting by $\boldsymbol{e}_j \colon \operatorname{Con} L^{(j)} \to B_j$ the unique isomorphism, then $\boldsymbol{e} = \prod(\boldsymbol{e}_j \mid j < n)$ is an isomorphism from $\operatorname{Con} L = \prod(\operatorname{Con} L^{(j)} \mid j < n)$ (cf. [LTF, Theorem 25]) onto B. The map $g_i = (g_{i,j} \mid j < n)$ is a 0-lattice homomorphism from L_i to L, for each $i \in \{1,2\}$, and g_i is unit-preserving in case f_1, f_2, $\boldsymbol{g}_1$, and $\boldsymbol{g}_2$ are all unit-preserving. Since (modulo the identification of $\prod(\operatorname{Con} L^{(j)} \mid j < n)$ and $\operatorname{Con} L$) $\operatorname{Con} g_i = (\operatorname{Con} g_{i,j} \mid j < n)$, it follows from (7-5.1) that $g_1 \circ f_1 = g_2 \circ f_2$, and from (7-5.2) that $\boldsymbol{e} \circ \operatorname{Con} g_i = \boldsymbol{g}_i$. □

Theorem 7-5.13. *For every distributive $(\vee, 0)$-semilattice S with at most $\aleph_1$ elements, there exists a locally partitional lattice L such that $\operatorname{Con_c} L \cong S$. Furthermore, if S has a unit, then L can be taken bounded.*

Proof. (After the proof of Grätzer, Lakser, and Wehrung [154, Theorem 2].) By Proposition 7-4.10, there exists a 2-ladder I of cardinality $\aleph_1$. By Corollary 7-4.13, there exists a directed colimit representation

$$(S, \boldsymbol{f}_i \mid i \in I) = \varinjlim(S_i, \boldsymbol{f}_i^j \mid i \leq j \text{ in } I)$$

where all the S_i are finite Boolean, all the $\boldsymbol{f}_i^j$ are zero-separating, and all the $\boldsymbol{f}_i^j$ are unit-preserving in case S has a unit.

We construct inductively a family of finite bipartitional lattices L_i, maps $\boldsymbol{e}_i \colon \operatorname{Con} L_i \to S_i$, for $i \in I$, and 0-lattice homomorphisms $f_i^j \colon L_i \to L_j$, for $i \leq j$ in I, satisfying the following properties:

(a) $f_i^i = \mathrm{id}_{L_i}$, for all $i \in I$.

(b) $f_i^k = f_j^k \circ f_i^j$, for all $i, j, k \in I$ with $i \leq j \leq k$.

(c) $\boldsymbol{e}_i$ is an isomorphism from $\operatorname{Con} L_i$ onto S_i, for all $i \in I$.

(d) $\boldsymbol{e}_j \circ \operatorname{Con} f_i^j = \boldsymbol{f}_i^j \circ \boldsymbol{e}_i$, for all $i \leq j$ in I.

For $i = 0$, we just pick any partitional lattice L_0 such that $\operatorname{Con} L_0 \cong S_0$ (for example $L_0 = S_0$, see [LTF, Theorem 145]), then we set $f_0^0 = \mathrm{id}_{L_0}$, and pick any isomorphism $\boldsymbol{e}_0 \colon \operatorname{Con} L_0 \to S_0$. Let us assume that $i > 0$ and that we have performed the construction on all indices smaller than i, so that (a)–(d) above hold on all those indices; we show how to extend the construction to the level i. Since I is a 2-ladder, i has (at most) two lower covers i_1 and i_2 in I. Note that i_1 and i_2 need not be distinct. For $k \in \{1, 2\}$, the map $\boldsymbol{g}_k = \boldsymbol{f}^i_{i_k} \circ \boldsymbol{e}_{i_k}$ is a $(\vee, 0)$-homomorphism from $\operatorname{Con} L_{i_k}$ to S_i, and the equality

$$\boldsymbol{g}_1 \circ \operatorname{Con} f^{i_1}_{i_1 \wedge i_2} = \boldsymbol{g}_2 \circ \operatorname{Con} f^{i_2}_{i_1 \wedge i_2}$$

holds. By Theorem 7-5.12, there are a partitional lattice L_i, 0-lattice homomorphisms $g_k \colon L_{i_k} \to L_i$ for $k \in \{1, 2\}$, and an isomorphism $\boldsymbol{e}_i \colon \operatorname{Con} L_i \to S_i$ such that

$$g_1 \circ f^{i_1}_{i_1 \wedge i_2} = g_2 \circ f^{i_2}_{i_1 \wedge i_2}, \tag{7-5.3}$$

$$\boldsymbol{e}_i \circ \operatorname{Con} g_k = \boldsymbol{g}_k \,, \qquad \text{for } k \in \{1, 2\} \,, \tag{7-5.4}$$

with g_1 and g_2 both unit-preserving in case $\boldsymbol{g}_1$, $\boldsymbol{g}_2$, $f^{i_1}_{i_1 \wedge i_2}$, and $f^{i_2}_{i_1 \wedge i_2}$ are all unit-preserving. Furthermore, if $i_1 = i_2$, then replacing g_2 by g_1 does not affect the validity of (7-5.3) and (7-5.4). Thus we may define $f^i_{i_k} = g_k$, for $k \in \{1, 2\}$, and (7-5.3), (7-5.4) take the following form:

$$f^i_{i_1} \circ f^{i_1}_{i_1 \wedge i_2} = f^i_{i_2} \circ f^{i_2}_{i_1 \wedge i_2} \,, \tag{7-5.5}$$

$$\boldsymbol{e}_i \circ \operatorname{Con} f^i_{i_k} = \boldsymbol{g}_k \,, \qquad \text{for } k \in \{1, 2\} \,. \tag{7-5.6}$$

Now we have defined f^i_j, for every lower cover j of i in I. We extend this definition to arbitrary $j \leq i$. If $j = i$, then we put $f^i_j = \mathrm{id}_{L_i}$. Now assume that $j < i$. There exists an index $k \in \{1, 2\}$ such that $j \leq i_k$. The only possible choice for f^i_j is to define it as

$$f^i_j = f^i_{i_k} \circ f^{i_k}_j \,, \tag{7-5.7}$$

except that this should be independent of k. This means that if $j \leq i_1 \wedge i_2$, then the equality

$$f^i_{i_1} \circ f^{i_1}_j = f^i_{i_2} \circ f^{i_2}_j \tag{7-5.8}$$

should hold. We compute:

$$\begin{aligned} f^i_{i_1} \circ f^{i_1}_j &= f^i_{i_1} \circ f^{i_1}_{i_1 \wedge i_2} \circ f^{i_1 \wedge i_2}_j \\ &= f^i_{i_2} \circ f^{i_2}_{i_1 \wedge i_2} \circ f^{i_1 \wedge i_2}_j && \text{(by (7-5.5))} \\ &= f^i_{i_2} \circ f^{i_2}_j \,, \end{aligned}$$

which establishes (7-5.8).

At this point, the 0-lattice embeddings $f_j^i\colon L_j \to L_i$ are defined for all $j \leq i$. The verification of conditions (a)–(c) above is then straightforward. Let us verify (d). As we already know that this condition holds on all indices smaller than i, the only statement that remains to be proved is

$$\boldsymbol{f}_j^i \circ \boldsymbol{e}_j = \boldsymbol{e}_i \circ \operatorname{Con} f_j^i\,, \quad \text{for each } j < i\,. \tag{7-5.9}$$

It suffices then to verify (7-5.9) for the pairs (j, i') and (i', i), where i' is any lower cover of i such that $j \leq i'$. For the pair (j, i'), this follows from the induction hypothesis, while for the pair (i', i), this follows from (7-5.6).

Hence the construction of the L_i, $\boldsymbol{e}_i$, f_i^j is carried out for the whole poset I. Let $L = \varinjlim_{i \in I} L_i$, with the transition maps f_i^j for $i \leq j$ in I. By Lemma 7-2.2, $\operatorname{Con_c} L$ is the directed colimit of the $\operatorname{Con_c} L_i$, with the transition maps $\operatorname{Con_c} f_j^i$, in the category $\mathbf{DSem}_{\vee,0}$. Thus, by (c) and (d), $\operatorname{Con_c} L$ is isomorphic to the directed colimit of the S_i with the transition maps $\boldsymbol{f}_i^j$, for $i \leq j$ in I. Hence, $\operatorname{Con_c} L \cong S$. Since all $\boldsymbol{f}_i^j$ are zero-separating and $\boldsymbol{f}_i^j \cong \operatorname{Con} f_i^j$, all f_i^j are one-to-one, thus so all the limiting maps $f_i\colon L_i \to L$, whence L is the directed union of its 0-sublattices $f_i(L_i) \cong L_i$. □

As an immediate consequence of Theorem 7-5.13, we obtain the following result, first established in Huhn [203] by using Theorem 7-3.18.

Corollary 7-5.14. *Every distributive* $(\vee, 0)$*-semilattice with at most* $\aleph_1$ *compact elements is isomorphic to the congruence semilattice of some lattice.*

We have seen that the lattice L obtained in the proof of Theorem 7-5.13 satisfies many further properties – for example, it is locally finite, relatively complemented with zero, and it has a unit in case S has a unit.

There is an obvious gap between the result of Corollary 7-5.4 (representing countable distributive semilattices by locally finite relatively complemented *modular* lattices) and Theorem 7-5.13 (representing distributive semilattices with at most $\aleph_1$ elements by locally finite relatively complemented lattices). Part of this gap is filled by the following result of Wehrung [328] (see also Theorem 8-4.16).

♢ **Theorem 7-5.15** (Wehrung 2000)**.** *Every distributive* $(\vee, 0)$*-semilattice* S *with at most* $\aleph_1$ *elements is isomorphic to* $\operatorname{Con_c} L$*, for some relatively complemented, modular* (*and even Arguesian*) *lattice* L *with zero. Furthermore, if* S *has a unit, then* L *can be taken bounded.*

However, the local finiteness is lost in the statement of Theorem 7-5.15! The question of the existence of a "best of two worlds" theorem (getting local finiteness and modularity together in Theorem 7-5.13) had been open for a while, before getting finally settled in Wehrung [331].

$\diamondsuit$ **Theorem 7-5.16** (Wehrung 2004)**.** *There exists a distributive* $(\vee, 0, 1)$*-semilattice* S *of cardinality* $\aleph_1$ *that is not isomorphic to* $\operatorname{Con_c} L$*, for any locally finite modular lattice* L*.*

Of course, by Corollary 7-5.4, the cardinality $\aleph_1$ is optimal in the statement of Theorem 7-5.16.

A very rough outline of the proof of Theorem 7-5.16 runs as follows. It requires the introduction of a natural precursor of $\operatorname{Con_c} L$ called the *dimension monoid* of L, which is a commutative monoid, denoted by $\operatorname{Dim} L$ (see Wehrung [326]). Define binary relations $\leq$, $\propto$, and $\asymp$ on any commutative monoid M by

$$x \leq y \underset{\text{def.}}{\Longleftrightarrow} (\exists z)(x + z = y)\,, \tag{7-5.10}$$

$$x \propto y \underset{\text{def.}}{\Longleftrightarrow} (\exists n \in \omega \setminus \{0\})(x \leq ny)\,, \tag{7-5.11}$$

$$x \asymp y \underset{\text{def.}}{\Longleftrightarrow} (x \propto y \text{ and } y \propto x)\,. \tag{7-5.12}$$

Then $\asymp$ is a monoid congruence on M, and the quotient $M/{\asymp}$ is known as the *maximal semilattice quotient of* M. Further, it turns out (see [326]) that $\operatorname{Con_c} L \cong (\operatorname{Dim} L)/{\asymp}$ (cf. [326, Corollary 2.3]).

If L is locally finite (or, more generally, if every finitely generated sublattice of L has finite length) and modular, then $\operatorname{Dim} L$ is the positive cone of a so-called *dimension group*, which implies that it is cancellative and satisfies the following *refinement property* (see Theorem 5.4 and Proposition 5.5 in [326]):

$$\begin{aligned} x_0 + x_1 = y_0 + y_1 \Rightarrow {} & (\exists z_{0,0}, z_{0,1}, z_{1,0}, z_{1,1}) \\ & (x_i = z_{i,0} + z_{i,1} \text{ and } y_i = z_{0,i} + z_{1,i} \text{ for each } i < 2)\,. \end{aligned} \tag{7-5.13}$$

For any subset A of a $(\vee, 0)$-semilattice S, denote by $A^{[\wedge 2]}$ the set of all finite joins of elements of the set

$$\{\, s \in S \mid (\exists a_0, a_1 \in A)(a_0 \neq a_1 \text{ and } s \leq a_0 \text{ and } s \leq a_1) \,\}\,.$$

The main trick of [331] is to prove that the maximal semilattice quotient S of any cancellative refinement monoid (or, more generally, of any refinement monoid with finite stable rank) satisfies the following infinitary statement:

> $(\mathrm{URP_{sr}})$ For any $e \in S$ and any subsets A and B of S such that A is uncountable, B is countably downward directed, and $a \leq e \leq a \vee b$ for each $(a, b) \in A \times B$, there exists $a \in A^{[\wedge 2]}$ such that $e \leq a \vee b$ for each $b \in B$.

Any distributive $(\vee, 0, 1)$-semilattice not satisfying the property above is thus a witness for Theorem 7-5.16. The construction of such a semilattice, described in Wehrung [331, Section 5], runs as follows.

Let B denote the Boolean algebra generated by all intervals of ω_1. Let I (resp., F) consist of all bounded (resp., unbounded) members of B. Put

$$D = \{\, x \subseteq \omega_1 \mid \text{either } x \text{ is finite or } x = \omega_1 \,\}.$$

The counterexample is

$$S = (\{\varnothing\} \times I) \cup \big((D \setminus \{\varnothing\}) \times F\big)\,.$$

The first example of a distributive $(\vee, 0, 1)$-semilattice not isomorphic to the maximal semilattice quotient of the positive cone of any dimension group (thus not isomorphic to the congruence semilattice of any locally finite, modular lattice) was constructed in Růžička [282]. It has cardinality $\aleph_2$. In Růžička [285], a counterexample to the same property is constructed, which is, in addition, a countable directed colimit of distributive 0-lattices (the transition maps are still $(\vee, 0)$-homomorphisms). However, its cardinality may be larger than $\aleph_2$ (depending on the universe of set theory we are working in).

7-5.5 Lifting tree-indexed diagrams of semilattices

By using recent categorical methods by Gillibert and Wehrung [114], it is possible to extend Theorem 7-5.13 to diagrams of semilattices indexed by well-founded trees subjected to certain size conditions. Recall that a poset I with a bottom element is a *tree* if every principal ideal of I is a chain, and that I is *lower countable* if every principal ideal of I is countable.

Theorem 7-5.17. *Let I be a lower countable well-founded tree such that* $\operatorname{card} I \leq \aleph_1$*, and let $\vec{S} = (S_i, \sigma_i^j \mid i \leq j \text{ in } I)$ be an I-indexed diagram of distributive* $(\vee, 0)$*-semilattices with* $(\vee, 0)$*-homomorphisms. We assume that*

(i) $\operatorname{card} S_i \leq \aleph_1$ *for each $i \in I$;*

(ii) $\operatorname{card} S_i \leq \aleph_0$ *for each non-maximal $i \in I$.*

Then there exists an I-indexed diagram $\vec{L}$, of locally partitional lattices and 0*-lattice homomorphisms, such that* $\operatorname{Con_c} \vec{L} \cong \vec{S}$.

The analogue of the result above, for distributive $(\vee, 0, 1)$*-semilattices, bounded locally partitional lattices, and* $(0, 1)$*-lattice homomorphisms, holds as well.*

Outline of proof. We give an outline for the case of $(\vee, 0)$-semilattices, $(\vee, 0)$-homomorphisms, and locally partitional lattices; the case of $(\vee, 0, 1)$-semilattices, $(\vee, 0, 1)$-homomorphisms, and bounded locally partitional lattices can be treated similarly.

We are using the *Condensate Lifting Lemma* (from now on CLL), established in Gillibert and Wehrung [114, Lemma 3.4.2]. This result makes it possible to reduce the liftability of the *diagram* $\vec{S}$ to the liftability of a larger *object*, called there a *condensate* of $\vec{S}$, and typically denoted there in the form $\mathbf{F}(X) \otimes \vec{S}$. What makes the condensate construction work is that the present categorical data form a so-called *larder*, here an $\aleph_1$-*larder*.

Our larder is an octuple $\Lambda = (\mathcal{A}, \mathcal{B}, \mathcal{S}, \mathcal{A}^\dagger, \mathcal{B}^\dagger, \mathcal{S}^\Rightarrow, \Phi, \Psi)$, the first six objects of which are categories, and the last two functors. In the present case,

- $\mathcal{A} = \mathcal{S}$ is the category of all distributive $(\vee, 0)$-semilattices with $(\vee, 0)$-homomorphisms.
- $\Phi \colon \mathcal{A} \to \mathcal{S}$ is the identity functor.
- $\mathcal{A}^\dagger$ is the full subcategory of $\mathcal{A}$ consisting of all countable members of $\mathcal{A}$.
- $\mathcal{B}$ is the category of all locally partitional lattices, with 0-lattice homomorphisms.
- $\mathcal{B}^\dagger$ is the full subcategory of $\mathcal{B}$ consisting of all countable members of $\mathcal{B}$.
- $\mathcal{S}^\Rightarrow$ is the subcategory of $\mathcal{S}$ consisting of all *ideal-induced* homomorphisms, that is, all surjective $(\vee, 0)$-homomorphisms $\boldsymbol{f} \colon S \twoheadrightarrow T$ such that $\boldsymbol{f}(x) \leq \boldsymbol{f}(y)$ implies that there is $u \in \boldsymbol{f}^{-1}\{0\}$ such that $x \leq y \vee u$ (for all $x, y \in S$).
- Ψ is the functor $\mathrm{Con_c} \colon \mathcal{B} \to \mathcal{S}$.

Then the reduction of Theorem 7-5.17 to Theorem 7-5.13 proceeds essentially *via* the verification of each item of a rather long list of "larder axioms". Most of those verifications are straightforward; we give outlines below.

Left larder axioms

(CLOS($\mathcal{A}$)) $\mathcal{A}$ has all small (i.e., indexed by a set) directed colimits. This is trivial.

(PROD($\mathcal{A}$)) $\mathcal{A}$ has all nonempty finite products. This is trivial.

(CONT(Φ)) Φ preserves all small directed colimits. This is trivial.

(PROJ($\Phi, \mathcal{S}^\Rightarrow$)) Φ sends every directed colimit of projections of $\mathcal{A}$ (i.e., canonical projections of the form $S \times T \twoheadrightarrow S$) to an ideal-induced homomorphism. This is straightforward.

Right larder axioms

(PRES$_{\aleph_1}(\mathcal{B}^\dagger, \Psi)$) $\mathrm{Con_c}\, B$ is weakly $\aleph_1$-presented (i.e., countable, cf. Gillibert and Wehrung [114, Proposition 4.2.3]), for any $B \in \mathcal{B}^\dagger$. This is trivial.

$(\mathrm{LS}^{\mathrm{r}}_{\aleph_1}(B))$ for every object B of $\mathcal{B}$: we must verify that for every countable distributive $(\vee, 0)$-semilattice S, every ideal-induced $\boldsymbol{f}\colon \operatorname{Con_c} B \twoheadrightarrow S$, and every countable sequence $(u_n\colon U_n \to B \mid n < \omega)$ of monomorphisms in $\mathcal{B}$ with all U_n countable, there exists a monomorphism $u\colon U \to B$ in $\mathcal{B}$, with U countable, above all the u_n in the subobject ordering, such that $\boldsymbol{f} \circ (\operatorname{Con_c} u)$ is ideal-induced. Although the verification of this fact is a not completely trivial "Löwenheim-Skolem type" argument, it is not difficult either, and it is proved almost exactly the same way as Claim 2 of the proof of Gillibert and Wehrung [114, Theorem 4.7.2].

Furthermore, [114, Theorem 4.5.2] yields that the larder Λ is "projectable". (For this we need to verify that every homomorphic image of a locally partitional lattice is locally partitional; see Exercise 7.37).

Our assumptions on the cardinalities of the S_i imply that the assumptions required in CLL about the diagram $\vec{S}$ (the $\vec{A}$ of the statement of CLL) are satisfied. In addition, we need to verify that the poset P in question, here the well-founded tree I, has a so-called "$\aleph_1$-lifter" $(X, \boldsymbol{X})$ such that $\operatorname{card} X \leq \aleph_1$. This is a consequence of Gillibert and Wehrung [114, Proposition 3.5.6], itself a reformulation of Gillibert [106, Corollary 4.7].

From $\operatorname{card} X \leq \aleph_1$ it follows that $\mathbf{F}(X)$ is the directed colimit of a directed diagram, indexed by a directed poset of cardinality at most $\aleph_1$, of finite "I-scaled Boolean algebras" (cf. Gillibert and Wehrung [114, Lemma 2.6.6]). It follows from this, and from the inequalities $\operatorname{card} S_i \leq \aleph_1$, that the condensate $S = \mathbf{F}(X) \otimes \vec{S}$ (which is automatically a distributive $(\vee, 0)$-semilattice) has at most $\aleph_1$ elements. By Theorem 7-5.13, there are a locally partitional lattice L and an isomorphism $\chi\colon \operatorname{Con_c} L \twoheadrightarrow S$. By CLL (and the projectability of the larder Λ), there are an I-indexed diagram $\vec{L}$ of $\mathcal{B}$ and a natural transformation $\vec{\chi}\colon \operatorname{Con_c} \vec{L} \to \vec{S}$ whose components χ_i are all isomorphisms. □

7-6. Congruence amalgamation for infinite lattices

In the present section we shall consider problems of the following form. Let $\vec{K}$ be a diagram, indexed by a poset I, of lattices and lattice homomorphisms, let S be a distributive $(\vee, 0)$-semilattice, and let $\vec{\boldsymbol{f}}\colon \operatorname{Con_c} \vec{K} \to S$ be a natural transformation whose components are all $(\vee, 0)$-homomorphisms. Are there a co-cone $(L, f_i \mid i \in I)$ above $\vec{K}$ and an isomorphism $\boldsymbol{e}\colon \operatorname{Con_c} L \to S$ such that, setting $\vec{f} = (f_i \mid i \in I)$, the equation $\vec{\boldsymbol{f}} = \boldsymbol{e} \circ \operatorname{Con_c} \vec{f}$ holds? This situation is illustrated on Figure 7-6.1.

We already met such amalgamation problems in Section 7-5.1 (I was then a one-element poset) and Section 7-5.4 (I was then a "truncated B_2" – that is, B_2 with the top element removed). In all the cases we shall encounter, I will be a truncated d-dimensional cube, that is, $\mathsf{B}_d \setminus \{1\}$, for a positive integer d (mostly $d \in \{0, 1, 2\}$). The problem above will then be called a *d-dimensional congruence amalgamation problem*.

$$
\begin{tikzcd}
S \\
\operatorname{Con_c} \vec{K} \arrow[u, "\vec{\boldsymbol{f}}"]
\end{tikzcd}
\qquad
\begin{tikzcd}
L \\
\vec{K} \arrow[u, dashed, "\vec{f}"]
\end{tikzcd}
\qquad
\begin{tikzcd}
\operatorname{Con_c} L \arrow[rr, dashed, "\boldsymbol{e}", "\cong"'] & & S \\
& \operatorname{Con_c} \vec{K} \arrow[ul, dashed, "\operatorname{Con_c} \vec{f}"] \arrow[ur, "\vec{\boldsymbol{f}}"'] &
\end{tikzcd}
$$

Figure 7-6.1: The general form of a congruence amalgamation problem.

Of course, for $d = 0$ this is just a plain representation problem – find a lattice L such that $\operatorname{Con_c} L \cong S$. For higher values of d, it turns out that in all the interesting cases, the information of $\vec{K}$ can be concentrated in a single structure, namely a *partial lattice*. For the latter we shall use the definition of Dean [61] (see also Freese, Ježek, and Nation [90, Section XI.9]) rather than the one of LTF, which is not equivalent to Dean's.

7-6.1 Stepwise enlargements of partial lattices

The goal of this subsection is mainly to provide some heuristics for the arguments of most proofs underlying the results of Section 7-6. Complete proofs of those results are very long and complex, so we refer the reader to Wehrung [329, 330] for more detail. Basic categorical facts underlying those proofs can be found in Exercises 7.38 and 7.39.

Definition 7-6.1. A *partial prelattice* is a structure $(P, \leq, \bigvee, \bigwedge)$ such that

(i) $\leq$ is a preordering on P;

(ii) both $\bigvee$ and $\bigwedge$ are partial functions from the nonempty finite subsets of P to P;

(iii) $a = \bigvee X$ implies that a is an upper bound of X, and that every upper bound b of X satisfies $a \leq b$ (we say that "a is a least upper bound of X with respect to $\leq$"). Dually, $a = \bigwedge X$ implies that a is a greatest lower bound of X with respect to $\leq$.

We say that P is a *partial lattice*[8] if $\leq$ is antisymmetric. A *congruence* of $(P, \leq)$ is a preordering $\trianglelefteq$ on P containing $\leq$ such that $(P, \trianglelefteq, \bigvee, \bigwedge)$ is a partial prelattice.

Full lattices are identified with partial lattices in which the $\bigvee$ and $\bigwedge$ operations are defined everywhere.

Note: It is important to observe that a congruence of a partial (pre)lattice P is no longer an *equivalence relation* on P, but a *preordering* of P.

[8] This definition of a partial lattice is weaker than the one used in LTF.

A congruence $\boldsymbol{\theta}$ (in the usual sense) on a lattice L is identified with the "partial lattice congruence" of L defined as

$$\leq_{\boldsymbol{\theta}} = \{\, (x, y) \in L \times L \mid (x \vee y, y) \in \boldsymbol{\theta} \,\}.$$

The set $\operatorname{Con} P$ of all congruences of a partial prelattice P is a closure system in $\operatorname{Pow}(P \times P)$, closed under arbitrary directed unions; hence $\operatorname{Con} P$ is an algebraic lattice (cf. Proposition 1-3.10 in Chapter 1). As for lattices, we denote by $\operatorname{Con_c} P$ the $(\vee, 0)$-semilattice of all compact elements of $\operatorname{Con} P$. This semilattice may not be distributive.

For $a, b \in P$, we denote by $\operatorname{con}^+(a, b)$ the least congruence $\boldsymbol{\theta}$ of P such that $(a, b) \in \boldsymbol{\theta}$. We also set $\operatorname{con}(a, b) = \operatorname{con}^+(a, b) \vee \operatorname{con}^+(b, a)$. Observe that

$$\operatorname{con}^+(a, b) = \operatorname{con}(a \wedge b, a) = \operatorname{con}(b, a \vee b) \quad \text{if } P \text{ is a lattice}.$$

The elements of $\operatorname{Con_c} P$ are exactly the finite joins of congruences of the form $\operatorname{con}^+(a, b)$.

For a homomorphism $f\colon P \to Q$ of partial prelattices, we denote by $\operatorname{Con_c} f\colon \operatorname{Con_c} P \to \operatorname{Con_c} Q$ the map that sends a congruence $\boldsymbol{\theta}$ of P to the congruence of Q generated by all pairs $(f(x), f(y))$, where $(x, y) \in \boldsymbol{\theta}$. This way, $\operatorname{Con_c}$ becomes a functor from partial prelattices and their homomorphisms, to the category $\mathbf{Sem}_{\vee,0}$ of all $(\vee, 0)$-semilattices with $(\vee, 0)$-homomorphisms.

The proof of the following lemma is a universal algebraic triviality.

Lemma 7-6.2. *The* $\operatorname{Con_c}$ *functor on partial prelattices preserves directed colimits.*

The standard 1-dimensional congruence amalgamation problem for partial lattices is the following: we are given a $(\vee, 0)$-homomorphism $\boldsymbol{f}\colon \operatorname{Con_c} P \to S$, we want to find a lattice L, a homomorphism $f\colon P \to L$ of partial lattices, and an isomorphism $\boldsymbol{e}\colon \operatorname{Con_c} L \to S$ such that $\boldsymbol{e} \circ \operatorname{Con_c} f = \boldsymbol{f}$. The situation is illustrated on Figure 7-6.2.

$$\begin{tikzcd} S \\ \operatorname{Con_c} P \arrow[u, "\boldsymbol{f}"] \end{tikzcd} \qquad \begin{tikzcd} L \\ P \arrow[u, dashed, "f"] \end{tikzcd} \qquad \begin{tikzcd} \operatorname{Con_c} L \arrow[rr, dashed, "\boldsymbol{e}", "\cong"'] & & S \\ & \operatorname{Con_c} P \arrow[ul, dashed, "\operatorname{Con_c} f"] \arrow[ur, "\boldsymbol{f}"'] & \end{tikzcd}$$

Figure 7-6.2: One-dimensional congruence amalgamation problem for the partial lattice P.

The following result is contained in Wehrung [329, Theorem 4]. The proof presented there depends on an even far more complex result from Wehrung [330]. However, the original (unpublished) proof of Theorem 7-6.3 is easier. We give a very rough sketch of that proof below.

$\diamondsuit$ **Theorem 7-6.3** (Wehrung 2002)**.** *For any conditionally co-Brouwerian* $(\vee, 0)$*-semilattice* S*, any partial lattice* P*, and any* $(\vee, 0)$*-homomorphism* $\boldsymbol{f}\colon \operatorname{Con_c} P \to S$*, there are a relatively complemented lattice* L*, a homomorphism* $f\colon P \to L$ *of partial lattices, and an isomorphism* $\boldsymbol{e}\colon \operatorname{Con_c} L \to S$ *such that* $\boldsymbol{e} \circ \operatorname{Con_c} f = \boldsymbol{f}$*.*

The complete proof of Theorem 7-6.3 is quite involved, and it proceeds by enlarging P step by step, gradually "forcing $\boldsymbol{f}$ to be an isomorphism", occasionally replacing P by the free lattice $\operatorname{Free}(P)$ over P (cf. [LTF, Section VII.3.5], see also Freese, Ježek, and Nation [90, Section XI.9]) as we want to end up with a lattice.

Replacing P by $\operatorname{Free}(P)$

The canonical map $j_P\colon P \hookrightarrow \operatorname{Free}(P)$ induces a cofinal $(\vee, 0)$-embedding $\operatorname{Con_c} j_P\colon \operatorname{Con_c} P \hookrightarrow \operatorname{Con_c} \operatorname{Free}(P)$ (cf. Wehrung [329, Corollary 4.6]). By Lemma 7-4.4, there exists $\boldsymbol{g}\colon \operatorname{Con_c} \operatorname{Free}(P) \to S$ such that $\boldsymbol{g} \circ \operatorname{Con_c} j_P = \boldsymbol{f}$.

Adding a relative complement

We are given $a < b < c$ in P, we find an extension of P with a relative complement x of b in $[a, c]$ that does not destroy the congruence lattice of P.

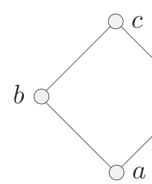

Figure 7-6.3: Freely adding a relative complement.

We do this *freely*, that is, we put $Q = P \cup \{x\}$, where x is free such that $b \vee x = c$ and $b \wedge x = a$ (see Figure 7-6.3). Denote by $\boldsymbol{e}_{P,Q}\colon \operatorname{Con_c} P \to \operatorname{Con_c} Q$ the canonical map. Again by Lemma 7-4.4, there exists $\boldsymbol{g}\colon \operatorname{Con_c} Q \to S$ such that $\boldsymbol{g} \circ \boldsymbol{e}_{P,Q} = \boldsymbol{f}$.

Making two elements perspective

We are given elements $a \leq b$ and $u, v \in [a, b]$ in P such that

$$\boldsymbol{f}(\operatorname{con}_P(a, u)) = \boldsymbol{f}(\operatorname{con}_P(a, v)) \quad \text{and} \quad \boldsymbol{f}(\operatorname{con}_P(u, b)) = \boldsymbol{f}(\operatorname{con}_P(v, b)) \,.$$

We adjoin freely to P a new element x such that

$$x \vee u = x \vee v = b \quad \text{and} \quad x \wedge u = x \wedge v = a \,,$$

(cf. Figure 7-6.4), and we set $Q = P \cup \{x\}$.

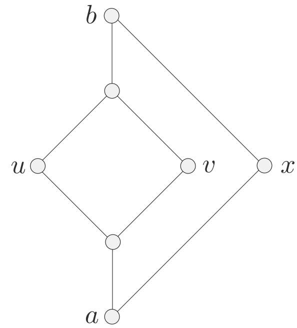

Figure 7-6.4: Forcing the relations $\mathrm{con}_Q(a, u) = \mathrm{con}_Q(a, v)$ and $\mathrm{con}_Q(u, b) = \mathrm{con}_Q(v, b)$.

In particular, this forces the relations

$$\mathrm{con}_Q(a, u) = \mathrm{con}_Q(a, v)\,; \quad \mathrm{con}_Q(u, b) = \mathrm{con}_Q(v, b)\,.$$

Although $\boldsymbol{e}_{P,Q}$ is no longer one-to-one, one can still find $\boldsymbol{g}\colon \mathrm{Con_c}\, Q \to S$ such that $\boldsymbol{g} \circ \boldsymbol{e}_{P,Q} = \boldsymbol{f}$ (and the proof is not easy). A more detailed argument can be found in the proof of Wehrung [330, Lemma 20.3].

Identifying two congruences

We are given elements $u, v \in [a, b]$ in P such that

$$\boldsymbol{f}(\mathrm{con}_P(a, u)) = \boldsymbol{f}(\mathrm{con}_P(a, v))\,.$$

By using the Boolean triple construction $\mathsf{M}_3[L]$ (cf. [LTF, Section IV.5]), we find an extension Q with elements u_i, v_i such that $u = u_0 \oplus u_1$, $v = v_0 \oplus v_1$ (within $[a, b]$), and

$$\boldsymbol{f}(\mathrm{con}_P(a, u_i)) = \boldsymbol{f}(\mathrm{con}_P(a, v_i))\,; \quad \boldsymbol{f}(\mathrm{con}_P(u_i, b)) = \boldsymbol{f}(\mathrm{con}_P(v_i, b))\,,$$

for each $i < 2$. Then apply the step above to u_0, v_0, then to u_1, v_1. We get an extension Q where

$$\mathrm{con}_Q(a, u) = \mathrm{con}_Q(a, v)\,,$$

"free enough" for $\boldsymbol{f}$ to be factored through $\mathrm{Con_c}\, Q$. Again, it is possible to find $\boldsymbol{g}\colon \mathrm{Con_c}\, Q \to S$ such that $\boldsymbol{g} \circ \boldsymbol{e}_{P,Q} = \boldsymbol{f}$. A more detailed argument can be found in the proof of Wehrung [330, Lemma 20.5].

Adding one element to the range of $\boldsymbol{f}$

We are given $\boldsymbol{a} \in S \setminus \{0\}$. Pick $o \in P$, put $Q = P \cup \{x\}$, with x freely adjoined such that $o < x$. Extend $\boldsymbol{f}$ so that $\boldsymbol{g}(\mathrm{con}_Q(o, x)) = \boldsymbol{a}$.

This way, the new element $\boldsymbol{a} \in S$ is forced into the range of $\boldsymbol{f}$.

Further properties of the lattice $\boldsymbol{L}$

The statement of Theorem 7-6.3 tells only part of the story. Indeed, the construction sketched above shows that L can be taken with all sorts of "existential closure" properties. For example (cf. Wehrung [330, Theorem C]),

- The range of f generates L as an ideal (resp., a filter).
- If the range of $\boldsymbol{f}$ is cofinal in S, then the range of f generates L as a convex sublattice.
- L has definable principal congruences.

7-6.2 Two-dimensional congruence amalgamation theorems for partial lattices

In this subsection we list a few consequences of Theorem 7-6.3 and its proof.

The following 2-dimensional congruence amalgamation result is an extension of Theorem 7-5.12 to infinite partial lattices. It is established in Wehrung [329, Theorem 5].

$\diamondsuit$ **Theorem 7-6.4** (Wehrung 2002)**.** *Let K be a lattice, let P_1, P_2 be partial lattices, and let $f_1 \colon K \to P_1$ and $f_2 \colon K \to P_2$ be homomorphisms of partial lattices. Let S be a conditionally co-Brouwerian $(\vee, 0)$-semilattice, and, for $i \in \{1, 2\}$, let $\boldsymbol{g}_i \colon \operatorname{Con_c} P_i \to S$ be $(\vee, 0)$-homomorphisms such that*

$$\boldsymbol{g}_1 \circ \operatorname{Con} f_1 = \boldsymbol{g}_2 \circ \operatorname{Con} f_2\,.$$

Then there are a relatively complemented lattice L, an isomorphism $\boldsymbol{e} \colon \operatorname{Con_c} L \to S$, and homomorphisms $g_i \colon P_i \to L$ of partial lattices, for $i \in \{1, 2\}$, such that

$$\begin{aligned} g_1 \circ f_1 &= g_2 \circ f_2\,, \\ \boldsymbol{e} \circ \operatorname{Con} g_i &= \boldsymbol{g}_i \qquad\qquad \textit{for each } i \in \{1,2\}\,. \end{aligned}$$

Outline of proof. By replacing K and the P_i by suitable homomorphic images, we may assume that f_1 and f_2 are both one-to-one. Hence the problem can be reduced to the case where each f_i is the inclusion map from K into P_i, and then we may also assume that $K = P_1 \cap P_2$. Endow $P = P_1 \cup P_2$ with the partial ordering generated by the union of the partial orderings of P_1 and P_2, and let $a = \bigvee X$ (resp., $a = \bigwedge X$), for $X \subseteq P$ nonempty finite and $a \in P$, if $X \cup \{a\} \subseteq P_i$ and $a = \bigvee X$ (resp., $a = \bigwedge X$) in P_i, for some $i \in \{1, 2\}$. Then P is a partial lattice, and applying Theorem 7-6.3 to P yields the desired result. $\square$

By using 2-ladders and with a proof similar to the one of Theorem 7-5.13, we obtain the following result, established in Wehrung [329, Corollary 6.4].

$\diamondsuit$ **Theorem 7-6.5** (Wehrung 2002). *Every directed colimit S of at most $\aleph_1$ conditionally co-Brouwerian semilattices is isomorphic to* $\mathrm{Con_c}\, L$*, for some relatively complemented lattice L with zero, that may be taken bounded in case S has a unit.*

The methods described above can also be used to get the following result, established in Wehrung [329, Corollary 6.5].

$\diamondsuit$ **Theorem 7-6.6** (Wehrung 2002). *Let $K = \bigcup(K_n \mid n < \omega)$, for a lattice K and an ascending sequence of sublattices K_n, with each* $\mathrm{Con_c}\, K_n$ *conditionally co-Brouwerian. Then K embeds congruence-preservingly into a relatively complemented lattice L which it generates as a convex sublattice.*

7-6.3 One-dimensional congruence amalgamation theorems for partial lattices

For the present subsection we refer the reader to Wehrung [330].

It is proved in Wehrung [329, Section 9] that for any distributive $(\vee, 0)$-semilattice S, if S is not conditionally complete, then there are a partial lattice P and a $(\vee, 0)$-homomorphism $\boldsymbol{f}\colon \mathrm{Con_c}\, P \to S$ that cannot be lifted. However, *if S is a lattice*, then something interesting happens.

The proof of Theorem 7-6.3 outlined above constructs a stepwise enlargement of the partial lattice P we are starting from. If we start with a *lattice* P, the next step (e.g., $Q = P \cup \{x\}$) is, in general, a partial lattice.

However, the partial lattices P and the maps $\boldsymbol{f}\colon \mathrm{Con_c}\, P \to S$ obtained in the process are quite special. View $\boldsymbol{f}$ as a S^{op}*-valued structure* on P, *via*

$$\|x \leq y\| = \boldsymbol{f}(\mathrm{con}^{+}(x, y))\,. \tag{7-6.1}$$

The quantity $\|x \leq y\|$ measures the "truth value" of the statement $x \leq y$. In particular, if $x \leq y$, then $\|x \leq y\| = 1$ (which is why we are talking about S^{op}-valued structures rather than S-valued structures). This truth value function should preserve elementary (intuitionistic!) deduction, so for example

$$\|x \leq y\| \wedge \|y \leq z\| \leq \|x \leq z\|\,.$$

We say that the $(\vee, 0)$-homomorphism $\boldsymbol{f}\colon \mathrm{Con_c}\, P \to S$ is *balanced* if $\|\theta\| = 1$ whenever θ is a sentence "expressing that joins and meets of ideals of P need only finitely many steps to be computed". This concept is quite difficult to capture formally; this is done in Wehrung [330, Definition 13.3]. An important observation is that *balanced* $(\vee, 0)$*-homomorphisms can be defined only in case S is a lattice.*

If either P is a lattice or P is finite and $\bigvee$ and $\bigwedge$ have both nonempty domain, then $\boldsymbol{f}$ is automatically balanced (cf. Wehrung [330, Proposition 12.7]).

A large part of the difficulty is to extend $\boldsymbol{f}$ (or, equivalently, the Boolean value function given by (7-6.1)) to the lattice $\mathrm{Free}(P)$. This is done *via* the description of Free(P) given in Dean [61] (cf. Freese, Ježek, and Nation [90, Section XI.9]), and is initiated in Wehrung[9] [330, Definition 2.6]. Boolean values are evaluated inductively in an intuitive way, for example

$$\|x \leq y_0 \wedge y_1\| = \|x \leq y_0\| \wedge \|x \leq y_1\|\,,$$

for all lattice-theoretical words x, y_0, y_1 over P. There are technical difficulties regarding the evaluation of Boolean values of statements involving ideals and filters of P: namely, we end up with infinite joins and meets *a priori*, while, for lack of any completeness assumption on S, we want only *finite* joins and meets. For precise formulations, see Wehrung[10] [330, Definition 11.4].

If $\boldsymbol{f}\colon \operatorname{Con_c} P \to S$ is balanced, then so are all the intermediary steps of the construction leading to the proof of Theorem 7-6.3, outlined in Section 7-6.1. It follows that Theorem 7-6.3 can be extended to the case where S is only a lattice, but where $\boldsymbol{f}$ is balanced (this is contained in Theorems A and C of Wehrung [330]). The need to check balancedness throughout all steps of the proof of that new result makes it much harder than the one of Theorem 7-6.3.

As observed above, if P is a lattice, then $\boldsymbol{f}$ is balanced. This yields the following result, established in Wehrung [330, Theorem C].

$\diamondsuit$ **Theorem 7-6.7** (Wehrung 2003)**.** *Let K be a lattice, let S be a distributive lattice with zero, and let $\boldsymbol{f}\colon \operatorname{Con_c} K \to S$ be a $(\vee, 0)$-homomorphism. Then there are a relatively complemented lattice L, a lattice homomorphism $f\colon K \to L$, and an isomorphism $\boldsymbol{e}\colon \operatorname{Con_c} L \to S$ such that $\boldsymbol{f} = \boldsymbol{e} \circ \operatorname{Con_c} f$. Furthermore, this can be done in such a way that $f(K)$ is cofinal (resp., dually cofinal) in L, and that if $\boldsymbol{f}$ has cofinal range, then $f(K)$ generates L as a convex sublattice.*

Starting with K being the trivial lattice, we obtain immediately Schmidt's theorem that every distributive lattice with zero is isomorphic to the congruence lattice of a lattice (Theorem 7-3.21). Actually, Theorem 7-6.7, together with an easy imitation of the argument of the proof of Theorem 7-5.3, yields the following extension of Schmidt's Theorem, established in Wehrung [330, Corollary 21.3].

---

[9] There is an obvious misprint in [330, Definition 2.6(v)], which should read

$$\dot{x}_0 \wedge \dot{x}_1 \preceq \dot{y}_0 \vee \dot{y}_1 \text{ iff either } \dot{x}_0 \wedge \dot{x}_1 \ll \dot{y}_0 \vee \dot{y}_1 \text{ or } (\dot{x}_i \preceq \dot{y}_0 \vee \dot{y}_1 \text{ and } \dot{x}_0 \wedge \dot{x}_1 \preceq \dot{y}_i \text{ for some } i < 2)\,.$$

[10] The above-mentioned misprint at [330, Definition 2.6(v)] causes a problem at [330, Definition 11.4(v)], which should read

$$\|\dot{x}_0 \wedge \dot{x}_1 \preceq \dot{y}_0 \vee \dot{y}_1\| = \|\dot{x}_0 \wedge \dot{x}_1 \ll \dot{y}_0 \vee \dot{y}_1\| \vee \bigvee_{i<2} \big(\|\dot{x}_i \preceq \dot{y}_0 \vee \dot{y}_1\| \wedge \|\dot{x}_0 \wedge \dot{x}_1 \preceq \dot{y}_i\|\big)\,.$$

The proofs in [330] are formulated with respect to the correct versions of Definitions 2.6 and 11.4 of that paper, so they are unaffected.

♢ **Theorem 7-6.8** (Wehrung 2003). *Let $S = \varinjlim(S_n \mid n < \omega)$ be a directed colimit representation in the category of all $(\vee, 0)$-semilattices with $(\vee, 0)$-homomorphisms. If each S_n is a lattice, then S is isomorphic to $\mathrm{Con_c}\, L$, for some relatively complemented lattice L with zero. Furthermore, if S has a unit, then L can be taken bounded.*

By applying Theorem 7-6.7 to the case where $\boldsymbol{f}$ is an isomorphism, we obtain the following result, established in Wehrung [330, Corollary 21.1].

♢ **Theorem 7-6.9** (Wehrung 2003). *Every lattice K such that $\mathrm{Con_c}\, K$ is a lattice has a relatively complemented, congruence-preserving extension L. Furthermore, this can be done in such a way that K generates L as a convex sublattice.*

A further application of the methods described above yields the following extension of Theorem 7-6.4 to the case where S is no longer conditionally co-Brouwerian (but still a lattice), and where each P_i is either finite or a full lattice. This result is stated in Theorems B and C of Wehrung [330].

♢ **Theorem 7-6.10** (Wehrung 2002). *Let K be a finite lattice, let P_1, P_2 be partial lattices each of which is either finite or a lattice, and let $f_i \colon K \to P_i$, for $i \in \{1, 2\}$, be homomorphisms of partial lattices. Let S be a distributive lattice with zero, and, for $i \in \{1, 2\}$, let $\boldsymbol{g}_i \colon \mathrm{Con_c}\, L_i \to S$ be $(\vee, 0)$-homomorphisms such that*

$$\boldsymbol{g}_1 \circ \mathrm{Con}\, f_1 = \boldsymbol{g}_2 \circ \mathrm{Con}\, f_2\,.$$

Then there are a relatively complemented lattice L, an isomorphism $\boldsymbol{e} \colon \mathrm{Con_c}\, L \to S$, and homomorphisms $g_i \colon P_i \to L$ of partial lattices, for $i \in \{1, 2\}$, such that

$$\begin{aligned} g_1 \circ f_1 &= g_2 \circ f_2\,, \\ \boldsymbol{e} \circ \mathrm{Con}\, g_i &= \boldsymbol{g}_i && \text{for each } i \in \{1, 2\}\,. \end{aligned}$$

The following strong converse of Theorem 7-6.7 was established in Tůma and Wehrung [315, Theorem A].

♢ **Theorem 7-6.11** (Tůma and Wehrung 2002). *Let S be a distributive lattice with zero. If for every lattice K and every $(\vee, 0)$-homomorphism $\boldsymbol{f} \colon \mathrm{Con_c}\, K \to S$, there are a lattice L, a lattice homomorphism $f \colon K \to L$, and an isomorphism $\boldsymbol{e} \colon \mathrm{Con_c}\, L \to S$ such that $\boldsymbol{f} = \boldsymbol{e} \circ \mathrm{Con_c}\, f$, then S is a lattice.*

In particular, every distributive $(\vee, 0)$-semilattice S satisfying the conclusion of Theorem 7-6.7 is a lattice.

7-6.4 Three-dimensional congruence amalgamation

While we have just seen that one- and two-dimensional congruence amalgamation both give rise to difficult problems, three-dimensional congruence amalgamation is doomed to failure, due to the following example of Wehrung [329, Section 10].

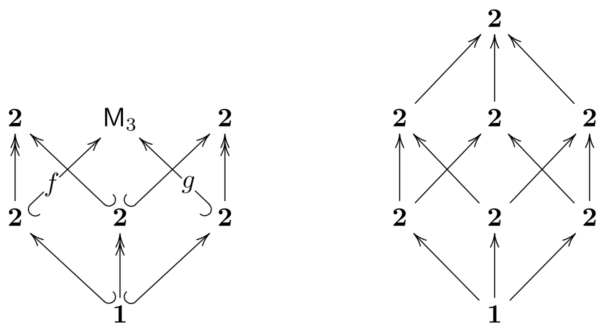

Figure 7-6.5: Attempting to lift a truncated cube of lattices.

Consider the diagram $\vec{D}$ of lattices and 0-preserving lattice embeddings represented on the left-hand side of Figure 7-6.5, where $\mathbf{1} = \{0\}$, $\mathbf{2} = \{0, 1\}$, and $\mathsf{M}_3 = \{0, a, b, c, 1\}$ is the five-element modular nondistributive lattice (with atoms a, b, c), the unlabeled arrows are uniquely determined, $f(1) = a$, and $g(1) = c$.

Then the image of $\vec{D}$ under the $\mathrm{Con_c}$ functor is obtained by truncating the top $\mathbf{2}$ from the diagram, represented on the right-hand side of Figure 7-6.5, of $(\vee, 0)$-semilattices with $(\vee, 0)$-embeddings, which defines a homomorphism $\boldsymbol{f}\colon \mathrm{Con_c}\,\vec{D} \to \mathbf{2}$. Suppose that $\boldsymbol{f}$ can be lifted to a homomorphism from $\vec{D}$ to some lattice L. In particular, L is a simple lattice. Let $u\colon \mathbf{2} \to L$, $w\colon \mathsf{M}_3 \to L$, and $v\colon \mathbf{2} \to L$ be the homomorphisms of partial lattices that correspond to the top part of such a lifting. Chasing through the diagram $\vec{D}$, we obtain

$$w(a) = wf(1) = u(1) = v(1) = wg(1) = w(c)\,.$$

However, $\mathrm{Con_c}\,w$ separates zero, which means that w is an embedding, hence $a = c$, a contradiction.

Therefore, even the simplest nontrivial lattice $\mathbf{2}$ does not satisfy what could be called the "three-dimensional amalgamation property".

For further discussion about three-dimensional congruence amalgamation, see Wehrung [338, Section 5]. For a very deep result, suggesting that three-dimensional congruence amalgamation will never be found anywhere in the universe, see Gillibert [110].

7-7. Exercises

7.1. Let L be a lattice and set

$$m(x,y,z) = (x \wedge y) \vee (x \wedge z) \vee (y \wedge z)\,, \quad \text{for all } x,y,z \in L\,.$$

(1) Verify that $m(x,x,y) = m(x,y,x) = m(y,x,x) = x$ for all $x,y \in L$. (We say that m is a *majority operation* on L.)

(2) Prove that the algebras $(L,\vee,\wedge)$ and (L,m) have the same congruences. (*Hint*: observe that if $o \leq x,y \leq i$, then $x \wedge y = m(x,y,o)$ while $x \vee y = m(x,y,i)$.)

7.2. (Mainly Pixley [257, Theorem 2]) Let $\mathbf{V}$ be a variety of algebras, in a similarity type Σ, and let m be a term of Σ such that all identities

$$\mathsf{m}(\mathsf{x},\mathsf{x},\mathsf{y}) = \mathsf{m}(\mathsf{x},\mathsf{y},\mathsf{x}) = \mathsf{m}(\mathsf{y},\mathsf{x},\mathsf{x}) = \mathsf{x}$$

hold in every member of $\mathbf{V}$ (we say that m is a *majority term* on $\mathbf{V}$).

(1) Prove that $\operatorname{Con} A$ is a distributive lattice, for every algebra $A \in \mathbf{V}$. (*Hint*: if $x \equiv y \pmod{\boldsymbol{\alpha} \wedge (\boldsymbol{\beta} \vee \boldsymbol{\gamma})}$, then there exists a finite sequence $(z_0,\dots,z_n)$ such that $z_0 = x$, $z_n = y$, and $(z_i,z_{i+1}) \in \boldsymbol{\beta} \cup \boldsymbol{\gamma}$ for each $i < n$. Set $t_i = \mathsf{m}(x,y,z_i)$. Observe, in particular, that $t_i \equiv x \pmod{\boldsymbol{\alpha}}$.)

(2) Deduce from this, together with Exercise 7.1(1), that the congruence lattice of a lattice is distributive.

7.3. Let $\boldsymbol{\theta}$ be a congruence of a $(\vee,0)$-semilattice S and consider the closure operator $\mathrm{cl}_{\boldsymbol{\theta}}$ defined in (7-3.1). Prove that the assignment

$$\boldsymbol{a} \mapsto \{\, x \in S \mid x/\boldsymbol{\theta} \in \boldsymbol{a} \,\}\,, \quad \text{for } \boldsymbol{a} \in \operatorname{Id}(S/\boldsymbol{\theta})\,,$$

defines an isomorphism from $\operatorname{Id}(S/\boldsymbol{\theta})$ onto $\mathrm{cl}_{\boldsymbol{\theta}}(\operatorname{Id} S)$.

7.4. Let A, B, and C be $(\vee,0)$-semilattices, let $a \in A$, and let $f\colon A \to B$ and $g\colon B \to C$ be $(\vee,0)$-homomorphisms. Prove that if f is weakly distributive at a and g is weakly distributive at $f(a)$, then $g \circ f$ is weakly distributive at a.

7.5. Let S and T be $(\vee,0)$-semilattices and let $\mu\colon S \to T$ be a $(\vee,0)$-homomorphism. Prove that the set of all $e \in S$, such that μ is weakly distributive at e, is a $(\vee,0)$-subsemilattice of S.

7.6. For an ideal I of a $(\vee,0)$-semilattice S, prove that the binary relation $\equiv_I$ on S defined by

$$x \equiv_I y \underset{\text{def.}}{\iff} (\exists u,v \in I)(x \vee u = y \vee v)\,, \quad \text{for all } x,y \in S$$

is a congruence of S, and that, denoting by S/I the quotient $S/{\equiv_I}$, the canonical projection from S onto S/I is weakly distributive.

7.7. Let S and T be $(\vee,0)$-semilattices and let $f\colon S\to T$ be a $(\vee,0)$-homomorphism. Prove that the inverse map $f^{-1}\colon \operatorname{Id}T\to\operatorname{Id}S$, $\boldsymbol{b}\mapsto f^{-1}(\boldsymbol{b})=\{\,x\in S\mid f(x)\in\boldsymbol{b}\,\}$ is a meet-homomorphism, and that f is weakly distributive iff f^{-1} is a join-homomorphism.

7.8. Let A and B be algebras over the same similarity type and let $f\colon A\twoheadrightarrow B$ be a surjective homomorphism. Prove that

$$\operatorname{Con_c} f\colon \operatorname{Con_c} A\to\operatorname{Con_c} B$$

is a weakly distributive surjective $(\vee,0)$-homomorphism.

7.9. Let K be a convex sublattice of a lattice L and denote by $e\colon K\hookrightarrow L$ the inclusion map. Prove that $\operatorname{Con_c} e\colon \operatorname{Con_c} K\to\operatorname{Con_c} L$ is weakly distributive.

7.10. Let A and B be algebras with the same universe such that every operation of A is *congruence-compatible* with respect to B (i.e., $\operatorname{Con}B\subseteq\operatorname{Con}A$). For example, this occurs in case every operation of A is a polynomial in the operations of B. Prove that the assignment, that to each compact congruence $\boldsymbol{\alpha}$ of A associates the congruence of B generated by $\boldsymbol{\alpha}$, defines a weakly distributive $(\vee,0)$-homomorphism from $\operatorname{Con_c} A$ to $\operatorname{Con_c} B$.

7.11. (Dobbertin [71, Proposition 7].) Let I and J be ideals of a $(\vee,0)$-semilattice S.

(1) Prove that the set $\boldsymbol{\theta}_{I,J}$ of all $(a,b)\in S\times S$ such that

$$\text{either } a=b \text{ or } \bigl(a,b\notin J \text{ and } (\exists x\in I)(a\vee x=b\vee x)\bigr)$$

is a weakly distributive congruence of S.

(2) Prove that if S is a generalized Boolean semilattice, then $\boldsymbol{\theta}_{I,J}$ is a distributive congruence of S. (*Hint*: for $(a,b)\in\boldsymbol{\theta}_{I,J}$ with $a<b$, prove that $\varphi\colon S\mathbin{\downarrow}a\to S\mathbin{\downarrow}(b-a)$, defined by $\varphi(x)=0$ if $x\in J$ and $\varphi(x)=b-a$ if $x\notin J$, is a $(\vee,0,1)$-homomorphism. Use Theorem 7-3.11.)

In particular, item (2) above, together with Dobbertin's counterexample 7-3.15, solves Schmidt's [292, Problem 8] in the negative.

7.12. Let S be a $(\vee,0)$-semilattice and let B be a generalized Boolean lattice containing S as a $(\vee,0)$-subsemilattice such that for each $b\in B$ there exists a least $s\in S$ such that $b\leq s$. Prove that S is a relatively pseudocomplemented lattice.

7.13. Prove that for any finite sequences $(x_i\mid i<m)$ and $(y_j\mid j<n)$, with $m,n>0$, of elements in a refinement monoid M (cf. (7-5.13)), if $\sum_{i<m}x_i=\sum_{j<n}y_j$, then there are elements $z_{i,j}\in M$, for $i<m$ and $j<n$, such that $x_i=\sum_{j<n}z_{i,j}$ for each $i<m$ while $y_j=\sum_{i<m}z_{i,j}$ for each $j<n$.

7.14. (Vaught [320]; see also Hanf [193, Theorem 1.1], Pierce [256, Theorem 1.1.3]) Denote by $\oplus$ the operation of disjoint join in a Boolean algebra (i.e., $z = x \oplus y$ iff $z = x \vee y$ and $x \wedge y = 0$). A *V-relation*, between Boolean algebras A and B, is a binary relation $R \subseteq A \times B$ satisfying the following conditions:

(V1) $1_A \mathrel{R} 1_B$;

(V2) $0_A \mathrel{R} x$ iff $x = 0_B$, for each $x \in B$; and symmetrically, with A and B swapped;

(V3) if $a = a_0 \oplus a_1$ in A and $a \mathrel{R} b$, then there are $b_0, b_1 \in B$ such that $b = b_0 \oplus b_1$, $a_0 \mathrel{R} b_0$, and $a_1 \mathrel{R} b_1$; and symmetrically, with A and B swapped;

(V4) if $a = a_0 \oplus a_1$ in A and $b = b_0 \oplus b_1$ in B, then the conjunction of $a_0 \mathrel{R} b_0$ and $a_1 \mathrel{R} b_1$ implies that $a \mathrel{R} b$.

Let A and B be Boolean algebras.

(1) Let $A_0 \subseteq A_1$, for finite subalgebras A_0 and A_1 of A. Prove that every embedding $f\colon A_0 \hookrightarrow B$ of Boolean algebras, with graph contained in R, extends to an embedding $g\colon A_1 \hookrightarrow B$, with graph contained in R. (*Hint*: it suffices to consider the case where A_1 is generated by $A_0 \cup \{a\}$, for some $a \in A$. Apply (V3) to the relations $u = (u \wedge a) \oplus (u \wedge \neg a)$, for atoms u of A_0.)

(2) (*Vaught's Theorem*) Prove that if A and B are both countable, then every V-relation between A and B contains the graph of an isomorphism from A onto B.

7.15. For a generalized Boolean algebra A and a commutative monoid M, a map $\alpha\colon A \to M$ is a *measure* if the following conditions hold:

(M1) $\alpha(x) = 0$ iff $x = 0_A$, for each $x \in A$;

(M2) $z = x \oplus y$ implies that $\alpha(z) = \alpha(x) + \alpha(y)$, for all $x, y, z \in A$.

We say that a measure α is a *V-measure* if, in addition, the following *V-condition* holds:

(Vcond) For all $a \in A$ and all $x_0, x_1 \in M$ with $\alpha(a) = x_0 + x_1$, there are $a_0, a_1 \in A$ such that $a = a_0 \oplus a_1$, $\alpha(a_0) = x_0$, and $\alpha(a_1) = x_1$.

For $e \in M$ and if A is a Boolean algebra (i.e., A has a unit), we say that a measure $\alpha\colon A \to M$ is a *normalized measure* from A to (M, e), in symbol $\alpha\colon A \to (M, e)$, if $\alpha(1_A) = e$.

Endow a commutative monoid M with its algebraic preordering (i.e., $x \leq y$ iff $y = x + z$ for some z), and let $e \in M$.

(1) Prove that if $\alpha\colon A \to (M, e)$ is a normalized V-measure, then M is conical (i.e., $x + y = 0$ implies $x = y = 0$, for all $x, y \in M$) and it satisfies the implication (7-5.13) for all elements $x_0, x_1, y_0, y_1 \in M$ such that $x_0 + x_1 = y_0 + y_1 \leq e$.

(2) Prove that if $\alpha\colon A \to (M, e)$ and $\beta\colon B \to (M, e)$ are normalized V-measures, with A and B both countable, then there exists an isomorphism $f\colon A \to B$ of Boolean algebras such that $\alpha = \beta \circ f$. (*Hint*: apply Exercise 7.14 to the relation $\alpha(x) = \beta(y)$.)

7.16. (Dobbertin [69, Lemma 3.1]) For an element e in a commutative monoid M, an *(M, e)-measured Boolean algebra* is a pair (A, α), where $\alpha\colon A \to (M, e)$ is a normalized measure. For measures $\alpha\colon A \to M$ and $\beta\colon B \to M$, an *embedding* from (A, α) into (B, β) is an embedding $f\colon A \hookrightarrow B$ of Boolean algebras such that $\alpha = \beta \circ f$.

Prove the following amalgamation property for finite measured Boolean algebras:

> *Let M be a conical (cf. Exercise 7.15) refinement monoid, let $e \in M$, let (A, α), (B_1, β_1), and (B_2, β_2) be finite (M, e)-measured Boolean algebras, and let $f_i\colon (A, \alpha) \to (B_1, \beta_1)$ be an embedding, for each $i \in \{1, 2\}$. Prove that there are a finite (M, e)-measured Boolean algebra (B, β) and embeddings $g_i\colon (B_i, \beta_i) \to (B, \beta)$, for $i \in \{1, 2\}$, such that $g_1 \circ f_1 = g_2 \circ f_2$.*

(*Hint*: let B be the amalgamated sum, in the category of all Boolean algebras, of B_1 and B_2 above A. Reduce the problem to the case where $A = \mathbf{2}$.)

7.17. (Dobbertin [69, Lemma 3.2]) Let A be a countable Boolean algebra, let M be a countable conical (cf. Exercise 7.15) refinement monoid, let $e \in M$, and let $\alpha\colon A \to (M, e)$ be a normalized measure from A to (M, e). Prove that there are a countable (M, e)-measured Boolean algebra (B, β), with β a V-measure, and an embedding $f\colon (A, \alpha) \to (B, \beta)$. (*Hint*: write A as the union of an ascending sequence $(A_n \mid n < \omega)$ of finite subalgebras. For $x_0, x_1 \in M$ and $a \in A_n$ with $\alpha(a) = x_0 + x_1$, use Exercise 7.16 to find a finite extension (B_n, β_n) of $(A_n, \alpha{\restriction}_{A_n})$, and also of (B_{n-1}, β_{n-1}) if $n > 0$, with a decomposition $a = a_0 \oplus a_1$ with each $\beta_n(b_i) = x_i$.)

7.18. (Dobbertin [69, Theorem 3.4]) Let M be a conical (cf. Exercise 7.15) refinement monoid and let $e \in M$ be such that $\operatorname{card}(M \downarrow e) \leq \aleph_1$.

(1) Prove that there are a Boolean algebra A and a normalized V-measure $\alpha\colon A \to (M, e)$. (*Hint*: use Exercise 7.17.)

(2) Verify that if $M{\downarrow}e$ is countable, then A can be taken countable, and then the pair (A, α) is unique up to isomorphism. (*Hint*: use Exercise 7.15.)

(3) Find an example, with $M = \mathbf{2}$, showing that without the countability assumption on A, the uniqueness of (2) above is lost.

7.19. (Dobbertin [70, Theorem 27]) Let M be a conical (cf. Exercise 7.15) refinement monoid such that $\operatorname{card}(M \downarrow e) \leq \aleph_1$ for each $e \in M$.

(1) Prove that there are a generalized Boolean algebra B and a surjective V-measure $\mu\colon B \twoheadrightarrow M$. (*Hint*: for each $e \in M$, there are, by Exercise 7.18, a Boolean algebra B_e and a normalized V-measure $\mu_e\colon B_e \to (M, e)$. Construct a measure on the generalized Boolean algebra $B = \bigoplus_{e \in M} B_e$.)

(2) Prove that if $M \downarrow e$ is countable for each $e \in M$, then B can be taken locally countable. (*Hint*: the B constructed above is locally countable.)

7.20. (Dobbertin [70, Lemma 17]) Prove that for any distributive $(\vee, 0)$-semilattice S and any generalized Boolean semilattice B, every V-measure from B to S is a weakly distributive $(\vee, 0)$-semilattice.

7.21. Set $D = \varphi(\mathbb{D})$ in the proof of Theorem 7-4.2. Prove that the following condition is satisfied:

$$\bigwedge\nolimits^{D}(d_i \mid i < n) \leq x \implies \bigcap(S \downarrow d_i \mid i < n) \subseteq S \downarrow x,$$

for every positive integer n, all $d_0, \dots, d_{n-1} \in D$, and all $x \in X$.

7.22. For a finite distributive $(\vee, 0)$-subsemilattice A of a $(\vee, 0)$-semilattice B, we put

$$\varrho_A^B(b) = \bigwedge\nolimits^{A}(a \in A \mid b \leq a)\,, \tag{7-7.1}$$

for each $b \in B$.

(1) Prove that ϱ_A^B is a $(\vee, 0)$-homomorphism from B onto A – thus a *retraction* from B onto A, and that every retraction $\varrho\colon B \twoheadrightarrow A$ satisfies $\varrho \leq \varrho_A^B$ (i.e., $\varrho(b) \leq \varrho_A^B(b)$ for each $b \in B$). (*Hint*: A is a distributive *lattice*.)

(2) Find an example where $\varrho_A^B(b) \nleq b$ for some $b \in B$.

7.23. Let S be a distributive $(\vee, 0)$-semilattice and denote by $\mathcal{S}$ the set of all finite distributive $(\vee, 0)$-subsemilattices of S. For $A, B \in \mathcal{S}$, let $A \trianglelefteq B$ hold if $A \subseteq B$ and $\varrho_A^S = \varrho_A^B \circ \varrho_B^S$ (cf. Exercise 7.22). Prove that $\trianglelefteq$ is a partial ordering on $\mathcal{S}$.

7.24. Prove that the partial ordering $\trianglelefteq$ defined in Exercise 7.23 is directed. (*Hint*: Set $X = A \cup B$ in the proof of Theorem 7-4.2, and then set $C = \varphi(\mathbb{D})$. By using Exercise 7.21, prove that $A \trianglelefteq C$ and $B \trianglelefteq C$.)

7.25. Say that a directed union $S = \bigcup(S_i \mid i \in I)$ (where I is upward directed) is *strongly reversible* if $\varrho_i^k = \varrho_i^j \circ \varrho_j^k$ for all $i \leq j \leq k$ in I (where we set $\varrho_i^j = \varrho_{S_i}^{S_j}$). By using Exercises 7.23 and 7.24, prove that every distributive $(\vee, 0)$-semilattice is a directed union of a strongly reversible family of finite distributive $(\vee, 0)$-subsemilattices of S, and that the index set can be taken equal to the set of all finite subsets of S.

7.26. Use the notation of Exercise 7.23 and let A be a finite Boolean $(\vee, 0, 1)$-subsemilattice of a distributive $(\vee, 0, 1)$-semilattice S. Prove that if $\{0, 1\} \trianglelefteq A$, then A is a sublattice of S. Deduce from this that if S is not a lattice, then S is not a directed union of any strongly reversible family of finite *Boolean* $(\vee, 0)$-subsemilattices.

7.27. Let S, A, B be $(\vee, 0)$-semilattices with S finite distributive, let $\varphi \colon S \to A$ and $\varrho \colon B \twoheadrightarrow A$ be $(\vee, 0)$-homomorphisms with ϱ surjective. Prove that there exists a $(\vee, 0)$-homomorphism $\psi \colon S \to B$ such that $\varphi = \varrho \circ \psi$. (*Hint*: for each $p \in \operatorname{Ji} S$, pick $b_p \in B$ such that $\varrho(b_p) = \varphi(p)$. Arrange $p \leq q \Rightarrow b_p \leq b_q$. Set $\psi(x) = \bigvee(\, b_p \mid p \leq x\,)$.)

7.28. Let A and B be $(\vee, 0)$-semilattices with A finite distributive, and let $\varrho \colon B \twoheadrightarrow A$ be a surjective $(\vee, 0)$-homomorphism. Prove that there exists a $(\vee, 0)$-embedding $\varepsilon \colon A \hookrightarrow B$ such that $\varrho \circ \varepsilon = \mathrm{id}_A$.

7.29. (Bottom of page 16 in Huhn [203].) Let A, B, and C be $(\vee, 0)$-semilattices such that $A \subseteq C \subseteq B$, B is distributive, and C is finite. Prove that for all $(a, b) \in (\operatorname{Ji} A) \times (\operatorname{Ji} B)$, if $b \leq a$, then there exists $c \in \operatorname{Ji} C$ such that $b \leq c \leq a$. (*Hint*: express a as a join of join-irreducible elements of C; observe that b is join-prime.)

7.30. We say that a cube of distributive $(\vee, 0)$-semilattices and $(\vee, 0)$-embeddings, labeled as on Figure 7-7.1, has the *sandwich property* if for all $(a, b) \in (\operatorname{Ji} A) \times (\operatorname{Ji} B)$ such that $a \geq b$, there are $a_i \in \operatorname{Ji} A_i$ and $b_i \in \operatorname{Ji} B_i$, for $i < 3$, such that $a \geq a_i \geq b_j \geq b$ for all $i \neq j$.

We shall describe a cube of finite Boolean semilattices and $(\vee, 0, 1)$-embeddings without the sandwich property. Put $B = \operatorname{Pow}(7)$ (*with the usual convention* $7 = \{0, 1, 2, 3, 4, 5, 6\}$) and A_i, B_i, A are the

$$\begin{tikzcd}
 & B & \\
B_2 \arrow[ur] & B_1 \arrow[u] & B_0 \arrow[ul] \\
A_0 \arrow[u] \arrow[ur] & A_1 \arrow[ul] \arrow[ur] & A_2 \arrow[ul] \arrow[u] \\
 & A \arrow[ul] \arrow[u] \arrow[ur] &
\end{tikzcd}$$

Figure 7-7.1: A cube of $(\vee, 0, 1)$-semilattices with $(\vee, 0, 1)$-embeddings.

$(\vee, 0)$-subsemilattices of B respectively generated by

$$\begin{aligned}
\operatorname{Ji} B_0 &= \{\{0\}, \{1\}, \{2\}, \{3\}, \{4, 6\}, \{5, 6\}\}, \\
\operatorname{Ji} B_1 &= \{\{0\}, \{1\}, \{2, 6\}, \{3, 6\}, \{4\}, \{5\}\}, \\
\operatorname{Ji} B_2 &= \{\{0, 6\}, \{1, 6\}, \{2\}, \{3\}, \{4\}, \{5\}\}, \\
\operatorname{Ji} A_0 = \operatorname{Ji} A_2 &= \{\{0, 2, 4, 6\}, \{1, 3, 5, 6\}\}, \\
\operatorname{Ji} A_1 &= \{\{1, 2, 4, 6\}, \{0, 3, 5, 6\}\}, \\
\operatorname{Ji} A &= \{\{0, 1, 2, 3, 4, 5, 6\}\}.
\end{aligned}$$

Furthermore, put $a = \{0, 1, 2, 3, 4, 5, 6\}$ and $b = \{6\}$. Prove that there is no family of elements $a_i \in \operatorname{Ji} A_i$ and $b_i \in \operatorname{Ji} B_i$, for $i < 3$, such that $a \geq a_i \geq b_j \geq b$ for all $i \neq j$ in $\{0, 1, 2\}$.

7.31. Prove that in the statement of Theorem 7-4.14, the category $\mathbf{DLat}_0^{\mathrm{emb}}$ cannot be replaced by the category $\mathbf{DLat}_0$ of all distributive 0-lattices and 0-lattice homomorphisms. (*Hint*: for any set X, denote by $\boldsymbol{s}_X \colon \mathbf{2}^X \to \mathbf{2}$ the map that sends 0 to 0 and any nonzero element to 1. Observe that $\Gamma(\boldsymbol{s}_X)$ must be one-to-one.) Compare with Section 7-4.5.

7.32. Prove that a complete distributive semilattice is co-Brouwerian iff it is join-continuous[11].

7.33. Prove that every k-ladder has breadth at most k, for every positive integer k. Find a lower finite lattice of breadth 2 that is a k-ladder for no positive integer k.

7.34. After Kelly and Rival [222], a finite lattice L of cardinality n is *dismantlable* if there exists a chain $L_1 \subset L_2 \subset \cdots \subset L_n = L$ (the

[11] As in Chapter 1, a complete lattice L is *join-continuous* if $a \vee \bigwedge X = \bigwedge (a \vee X)$ whenever $a \in L$ and $X \subseteq L$ is downward directed; as usual, $a \vee X = \{\, a \vee x \mid x \in X \,\}$.

containments are proper, so card $L_k = k$) of sublattices of L. Verify that every finite sublattice of the 2-ladder L constructed in the proof of Proposition 7-4.10 is dismantlable. (*See Exercises* 7.35 *and* 7.36 *for a better result.*)

7.35. Prove that a lower finite lattice L is a 2-ladder iff there are no pairwise incomparable elements $a_0, a_1, a_2 \in L$ such that $a_0 \vee a_1 = a_0 \vee a_2 = a_1 \vee a_2$.

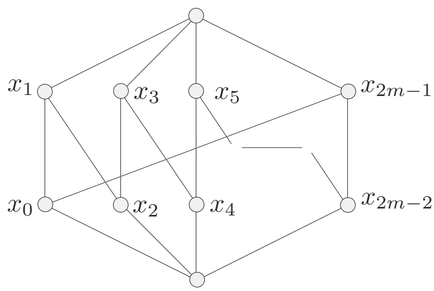

Figure 7-7.2: The lattice C_m.

7.36. In this exercise we assume the main result of Ajtai [5], which is: *A finite lattice is dismantlable iff it contains no sublattice isomorphic to* C_m *for any* $m \geq 3$, where C_m (*the* m*-crown with bounds added*) is the lattice represented in Figure 7-7.2. Prove that every finite 2-ladder is dismantlable. (*Hint*: use Exercise 7.35.)

7.37. (1) Let n be a positive integer and let X_i be a nonempty finite set, for each $i < n$. Prove that the congruence lattice of the lattice $L = \prod_{i<n} \operatorname{Equ} X_i$ is Boolean, and that every homomorphic image of L is partitional (cf. Definition 7-5.11).

(2) Prove that every homomorphic image of a locally partitional lattice is locally partitional.

(3) Prove that every directed colimit, of a diagram of locally partitional lattices and 0-lattice homomorphisms, is locally partitional.

7.38. Let A and B be algebras over the same similarity type and let $f\colon A \to B$ be a homomorphism. Prove that

$$(\operatorname{Res} f)(\boldsymbol{\beta}) = \{ (x,y) \in A \times A \mid (f(x), f(y)) \in \boldsymbol{\beta} \}$$

is a congruence of A, for each $\boldsymbol{\beta} \in \operatorname{Con} B$. Prove that the pair $(\operatorname{Con} f, \operatorname{Res} f)$ is a Galois adjunction (cf. Section 1-9 in Chapter 1) between $\operatorname{Con} A$ and $\operatorname{Con} B$.

7.39. Let K be an algebraic lattice, let L be a complete lattice, and let $f\colon K \to L$ and $g\colon L \to K$ such that (f, g) is a Galois adjunction between K and L (cf. Section 1-9 in Chapter 1). Prove that f is compactness-preserving iff g preserves arbitrary directed joins (see also Corollary 1-9.12 in Chapter 1).

7-8. Problems

In a recent private communication, George Grätzer asked the question *What are the congruence lattices of complete lattices.* Our first problem is a less ambitious, nonetheless certainly very difficult, variant of that question.

Problem 7.1. Let K be a bounded lattice. Does there exist a complete lattice L such that $\operatorname{Con} K \cong \operatorname{Con} L$?

The dimension monoids $\operatorname{Dim} L$, for L a relatively complemented, modular, conditionally complete, and meet-continuous lattice, are characterized, in Goodearl and Wehrung [122, Theorem 5-2.6], as the total members of a class of partial monoids called *continuous dimension scales*. By virtue of the isomorphism $\operatorname{Con_c} L \cong (\operatorname{Dim} L)/{\asymp}$ (cf. Section 7-5.4), this gives in turn a description (of some kind) of $\operatorname{Con_c} L$ for those lattices L. However, the various assumptions added there to completeness (such as modularity or relative complementation) are very strong and it is thus quite unlikely that the methods of [122] could be of any help to solve Problem 7.1.

Problem 7.2. Is every lower countable distributive $(\vee, 0)$-semilattice isomorphic to the congruence semilattice of some locally finite, sectionally complemented, and modular lattice?

For positive evidence on Problem 7.2, see Theorems 7-3.19 and 7-5.3.

Problem 7.3. Let S be a distributive $(\vee, 0)$-semilattice in which every principal ideal has at most $\aleph_1$ elements. Is S isomorphic to the congruence semilattice of some lattice? Can this lattice be taken sectionally complemented and modular?

For positive evidence on Problem 7.3, see Theorems 7-3.19 and 7-5.13.

Problem 7.4. Is every countable distributive $(\vee, 0)$-semilattice isomorphic to $\operatorname{Con_c} L$, for a lattice L generating a locally finite variety?

For a (nondistributive) countable $(\vee, 0)$-semilattice not isomorphic to the congruence semilattice of any locally finite algebra, see Kearnes [220].

Problem 7.5. Let L be a lattice generating a finitely generated variety. Is $\operatorname{Con_c} L$ isomorphic to the maximal semilattice quotient of the positive cone of some dimension group?

The particular instance of Problem 7.5, where L is modular, has a positive solution: indeed, by the results of Wehrung [326], $\mathrm{Con_c}\, L$ is isomorphic to the maximal semilattice quotient of the dimension monoid $\mathrm{Dim}\, L$ of L, and if L is locally finite and modular, then $\mathrm{Dim}\, L$ is the positive cone of a dimension group.

Problem 7.6. If a distributive $(\vee, 0)$-semilattice S is a directed colimit of an ω_1-chain of lattices with zero and $(\vee, 0)$-homomorphisms, does there exist a lattice L such that $S \cong \mathrm{Con_c}\, L$?

The variant of Problem 7.6 with ω_1 replaced by ω is solved positively in Theorem 7-6.8.

Problem 7.7. Let K be a lattice, let S be a distributive $(\vee, 0)$-semilattice, and let $\boldsymbol{f}\colon \mathrm{Con_c}\, K \to S$ be a $(\vee, 0)$-homomorphism. Decide, in each of the following cases, whether $\boldsymbol{f}$ can be lifted, that is, there are a lattice L, a lattice homomorphism $f\colon K \to L$, and an isomorphism $\boldsymbol{e}\colon \mathrm{Con_c}\, L \to S$ such that $\boldsymbol{f} = \boldsymbol{e} \circ \mathrm{Con_c}\, f$:

(1) K and S are both countable;

(2) $\boldsymbol{f}$ is a distributive (resp., weakly distributive) homomorphism.

Problem 7.7 covers a great deal of what is currently not known about unliftable maps. Indeed, the smallest known counterexample of an unliftable map $\boldsymbol{f}\colon \mathrm{Con_c}\, K \to S$ satisfies $\mathrm{card}\, S = \aleph_0$ and $\mathrm{card}\, K = \aleph_1$ (cf. Tůma and Wehrung [315, Section 2]). Furthermore, none of the currently known examples of unliftable maps is weakly distributive.

Problem 7.7(n) is stated as Problem n in Tůma and Wehrung [316], for each $n \in \{1, 2\}$. These problems are related to the following one, stated as part of [316, Problem 4].

Problem 7.8. Does every countable lattice have a relatively complemented, congruence-preserving extension?

While [316, Problem 4] also states Problem 7.8 for lattices of cardinality $\aleph_1$, the latter case was recently solved, in the negative, in Gillibert and Wehrung [114, Chapter 5]. On the positive side, Theorem 7-6.6 implies that every *locally finite* countable lattice has a relatively complemented, congruence-preserving extension L. The latter fact was first established in Grätzer, Lakser, and Wehrung [154, Theorem 3], where it is also proved that the extension L can be taken locally finite as well.

Chapter 8

Congruences of lattices and ideals of rings

by Friedrich Wehrung

8-1. Introduction

While the theory of congruence representations of infinite lattices has developed into a vast topic of its own, a second vast topic kept thriving on its side, mostly unaware of the considerable amount of insight that it would bring to the first one.

Von Neumann regular rings originate in a 1936 paper by von Neumann [253], and since then have become an active field of research, very much intertwined with module theory and operator theory. Our present account of those rings will be much focused on their lattice-theoretical aspects, without touching much upon their module-theoretical or operator-theoretical aspects, or such fundamental questions as coordinatization (which we will not need for our quick congruence-oriented overview).

The connection between ideal lattices of von Neumann regular rings and congruence lattices of lattices is quite straightforward: the principal right ideal lattice $\mathrm{L}^{\mathrm{r}}(R)$ of a von Neumann regular ring R is a sectionally complemented modular lattice (cf. Corollary 8-3.13), and the congruence lattice of $\mathrm{L}^{\mathrm{r}}(R)$

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is isomorphic to the lattice $\mathrm{Id}\, R$ of all two-sided ideals of R (cf. Proposition 8-3.25). These correspondences are functorial.

For a far more complete account of von Neumann regular rings (in the unital case), we refer the reader to Goodearl [117].

8-2. Basic concepts

8-2.1 Ideals and congruences of sectionally complemented modular lattices

Recall from LTF that elements x and y in a lattice L with zero are *perspective*, in notation $x \sim y$, if there exists an element $z \in L$ such that $x \wedge z = y \wedge z = 0$ and $x \vee z = y \vee z$.

By [LTF, Corollary 418], an ideal I in a sectionally complemented modular lattice L is distributive iff it is standard, iff it is neutral, iff it is perspectivity-closed, the last statement meaning that $x \in L$, $y \in I$, and $x \sim y$ implies that $x \in I$.

We denote by $\mathrm{NId}\, L$ the set of all neutral ideals of L, ordered by set inclusion. Denote by $\mathrm{con}(I)$ the least congruence of L containing $\{0\} \times I$, for every $I \subseteq L$. Then the proof of [LTF, Theorem 272] shows that the congruences of L are exactly the $\mathrm{con}(I)$, for neutral ideals I of L, and that the $\mathrm{con}(I)$-block of 0 is I. We sum this up as follows.

Lemma 8-2.1. *Let L be a sectionally complemented modular lattice. Then the assignment $I \mapsto \mathrm{con}(I)$ defines an isomorphism from $\mathrm{NId}\, L$ onto $\mathrm{Con}\, L$.*

8-2.2 Commutative monoids, refinement monoids, dimension groups

We refer to Goodearl [116] for partially ordered Abelian groups and dimension groups. We set $\mathbb{Z}^+ = \{0, 1, 2, \dots\}$ and $\mathbb{N} = \mathbb{Z}^+ \setminus \{0\}$.

We shall write all our commutative monoids additively. A submonoid I of a commutative monoid M is an *o-ideal* of M if $x + y \in I$ implies that $x \in I$ and $y \in I$, for all $x, y \in I$. We say that M is *conical* (cf. Exercise 7.15) if $\{0\}$ is an o-ideal of M. Every commutative monoid M can be endowed with its *algebraic preordering* $\leq$, defined by $x \leq y$ if there exists $z \in M$ such that $y = x + z$. We set

$$M|_e = \{\, x \in M \mid (\exists n \in \mathbb{N})(x \leq ne) \,\}\,, \quad \text{for any } e \in M\,,$$

and we say that e is an *order-unit* of M if $M|_e = M$. This terminology conflicts slightly with the one used for partially ordered Abelian groups: an *order-unit* of a partially ordered Abelian group G is an element e of G such that $0 \leq e$ and for each $x \in G$, there exists $n \in \mathbb{N}$ with $-ne \leq x \leq ne$. Nonetheless, in all the cases we shall meet, no ambiguity should arise about this.

If I is an o-ideal of a commutative monoid M, the binary relation $\equiv_I$ on M defined by

$$a \equiv_I b \iff (\exists x, y \in I)(a + x = b + y)\,, \quad \text{for all } a, b \in M\,, \tag{8-2.1}$$

is a monoid congruence of M, and $M/I = M/{\equiv_I}$ is a conical commutative monoid.

The *positive cone* of a partially preordered Abelian group G is defined as

$$G^+ = \{\, x \in G \mid 0 \leq x \,\}\,.$$

The *Grothendieck group* of a commutative monoid M is the initial object in the category of all monoid homomorphisms from M to a group. It consists of an Abelian group G with a monoid homomorphism $\varepsilon\colon M \to G$, and we shall always endow it with the unique translation-invariant preordering with positive cone $\varepsilon(M)$ (i.e., $x \leq y$ iff $y - x \in \varepsilon(M)$). Hence $G = (M \times M)/{\equiv}$, where $\equiv$ is the monoid congruence of $M \times M$ defined by

$$(x, y) \equiv (x', y') \iff (\exists z \in M)(x+y'+z = x'+y+z)\,, \text{ for all } x, y, x', y' \in M\,,$$

and $\varepsilon(x)$ is the $\equiv$-block of $(x, 0)$, for each $x \in M$.

A commutative monoid M has the *refinement property*, or is a *refinement monoid* (cf. Dobbertin [68]), if it satisfies the implication (7-5.13), that is, for all $a_0, a_1, b_0, b_1 \in M$ such that $a_0 + a_1 = b_0 + b_1$, there are $c_{i,j} \in M$ (for $i, j \in \{0, 1\}$) such that $a_i = c_{i,0} + c_{i,1}$ and $b_i = c_{0,i} + c_{1,i}$ for each $i \in \{0, 1\}$.

We say that a partially ordered Abelian group G is a *dimension group* if it is directed, unperforated (i.e., $mx \geq 0$ implies $x \geq 0$, for each $m \in \mathbb{N}$ and each $x \in G$) and the positive cone G^+ satisfies the refinement property. A *simplicial group* is a group of the form $\mathbb{Z}^n$, for a natural number n, ordered componentwise. Every directed colimit of simplicial groups is a dimension group. The following important result shows that the converse holds.

$\diamondsuit$ **Theorem 8-2.2** (Grillet 1976, Effros, Handelman, and Shen 1980)**.** *Every dimension group is a directed colimit of simplicial groups.*

Theorem 8-2.2 was first stated in Effros, Handelman, and Shen [75]. However, the semigroup analogue of this result established earlier in Grillet [188] (using Shannon's work [302]) is easily seen to be equivalent; for a more detailed discussion about this, see Goodearl and Wehrung [121, Section 3]). A complete proof of Theorem 8-2.2 can also be found in Goodearl [116, Theorem 3.19].

We shall also call a *simplicial monoid* the positive cone of a simplicial group (i.e., $(\mathbb{Z}^+)^n$ for some $n \in \mathbb{Z}^+$). Hence the simplicial monoids are exactly the finitely generated free commutative monoids.

8-2.3 Rings and ideals

All our rings will be associative but not necessarily unital. That is, a *ring* is a structure $(R, +, \cdot, 0)$ such that $(R, +, 0)$ is an Abelian group, $\cdot$ is an associative binary operation on R, and $\cdot$ is distributive on $+$, the latter meaning that

$$x \cdot (y+z) = x \cdot y + x \cdot z \quad \text{and} \quad (x+y) \cdot z = x \cdot z + y \cdot z\,,$$

for all $x, y, z \in R$. Here and elsewhere, we write terms in such a way that the multiplication $\cdot$ has priority over the addition $+$, so, for example, $x \cdot z + y \cdot z = (x \cdot z) + (y \cdot z)$. Also, we will often drop the symbol $\cdot$ from expressions, so, for example, $x \cdot y = xy$. In any ring, $x \cdot 0 = 0 \cdot x = 0$ for each x. As usual, the additive inverse of an element x is denoted by $-x$. We shall denote the additive unit as 0_R in case R needs to be specified.

We shall denote by R^{op} the *opposite ring* of R: so R and R^{op} have the same universe and the same addition (thus the same zero), and multiplication $*$ defined by $x * y = y \cdot x$ for all $x, y \in R$. A property P of rings is *self-dual* if it is preserved by going to the opposite ring, that is, $\mathsf{P}(R)$ implies that $\mathsf{P}(R^{\mathrm{op}})$, for any ring R.

We say that R is *unital* if it has a multiplicative unit, that is, an element 1, or 1_R in case R needs to be specified, such that $x \cdot 1 = 1 \cdot x = x$ for each $x \in R$. Then an element $a \in R$ is *invertible* if there exists $a' \in R$ such that $aa' = a'a = 1$; the element a' is then unique, it is called the *inverse* of a in R and denoted by a^{-1}.

Additive and multiplicative powers are written as

$$nx = xn = \underbrace{x + x + \cdots + x}_{n \text{ times}}\,, \quad x^n = \underbrace{x \cdot x \cdot \cdots \cdot x}_{n \text{ times}}\,,$$

for any $x \in R$ and any positive integer n. This notation is extended as usual to nx for a relative integer n in the general case, $x^0 = 1$ in case R is unital, and x^n for an arbitrary relative integer n in case x is invertible.

An element a in a ring R is *idempotent* if $a^2 = a$.

We shall set

$$X + Y = \{\, x + y \mid x \in X \text{ and } y \in Y \,\}\,, \tag{8-2.2}$$

$$XY = X \cdot Y = \left\{ \sum_{i<n} x_i y_i \mid n < \omega\,,\ \text{all } x_i \in X\,,\ \text{all } y_i \in Y \right\}\,, \tag{8-2.3}$$

for any subsets X and Y of R. Observe the difference of spirit between the notations (8-2.2) and (8-2.3): for example, XY may properly contain the set of all products xy for $(x, y) \in X \times Y$. We nevertheless set

$$aX = \{\, ax \mid x \in X \,\}\,, \tag{8-2.4}$$

$$Xa = \{\, xa \mid x \in X \,\}\,, \tag{8-2.5}$$

for any $a \in R$ and any $X \subseteq R$, and

$$\text{(8-2.6)} \quad RaR = \left\{ \sum_{i<m} x_i a y_i \mid m < \omega \text{ and all } x_i, y_i \in R \right\}, \quad \text{for any } a \in R\,.$$

An additive subgroup I in a ring R is a *left ideal* (*right ideal*, *two-sided ideal*, respectively) if $RI \subseteq I$ ($IR \subseteq I$, $RI + IR \subseteq I$, respectively). We shall often write "ideal" instead of "two-sided ideal".

Observe that IJ is the usual ideal product of I and J, for ideals I and J of R. It is also an ideal, it is contained in the intersection $I \cap J$, and this containment may be proper.

Ring homomorphisms between unital rings need not preserve the unit; we shall call *unital homomorphisms* those ring homomorphisms that do preserve the unit. For a homomorphism $f \colon R \to S$ of rings, the *kernel* $\ker f = f^{-1}\{0\}$ is an ideal of R. Conversely, every ideal I of a ring R defines a ring congruence $\equiv_I$ of R by

$$x \equiv_I y \iff x - y \in I\,, \quad \text{for all } x, y \in R\,,$$

and we write $R/I = R/{\equiv_I}$. In particular, the $\equiv_I$-equivalence class of an element $x \in R$ is $x + I$. The map $R \twoheadrightarrow R/I$, $x \mapsto x + I$ is called the *canonical projection* from R onto R/I.

We denote by $\mathrm{Id}^{\mathrm{l}}\, R$ ($\mathrm{Id}^{\mathrm{r}}\, R$, $\mathrm{Id}\, R$, respectively) the set of all left ideals (right ideals, ideals, respectively) of R, and we order those sets by set inclusion. Any intersection of ideals, and any directed union of ideals, is an ideal; the same holds, obviously, for left ideals and for right ideals. It follows that the subsets $\mathrm{Id}^{\mathrm{l}}\, R$, $\mathrm{Id}^{\mathrm{r}}\, R$, and $\mathrm{Id}\, R$ of the powerset $\mathrm{Pow}\, R$ are all closed under directed unions and arbitrary intersections. In particular, by Proposition 1-3.10 in Chapter 1, we get the following.

Proposition 8-2.3. *The posets* $\mathrm{Id}^{\mathrm{l}}\, R$, $\mathrm{Id}^{\mathrm{r}}\, R$, *and* $\mathrm{Id}\, R$ *are all algebraic lattices.*

The $(\vee, 0)$-semilattices of compact elements (cf. [LTF, Definition 41]) of $\mathrm{Id}^{\mathrm{l}}\, R$, $\mathrm{Id}^{\mathrm{r}}\, R$, and $\mathrm{Id}\, R$ will be denoted by $\mathrm{Id}^{\mathrm{l}}_{\mathrm{c}}\, R$, $\mathrm{Id}^{\mathrm{r}}_{\mathrm{c}}\, R$, and $\mathrm{Id}_{\mathrm{c}}\, R$, respectively. Hence the elements of $\mathrm{Id}^{\mathrm{r}}_{\mathrm{c}}\, R$ are exactly the subsets of R of the form

$$\text{(8-2.7)} \quad \langle A \rangle_R = \Bigg\{ \sum_{i<m} a_i x_i + \sum_{i<m} a_i n_i \mid m < \omega \text{ while } a_i \in A\,,\ x_i \in R\,,\ \text{and } n_i \in \mathbb{Z} \text{ for each } i < m \Bigg\}\,,$$

where $A \subseteq R$ is finite, while the elements of $\mathrm{Id}_{\mathrm{c}}\, R$ are exactly the subsets of R

of the form

$$
\begin{aligned}
{}_R\langle A\rangle_R = \Big\{ &\sum_{i<m} x_i a_i y_i + \sum_{i<m} a_i n_i \mid \\
&m<\omega \text{ while } a_i \in A\,,\ x_i \in R\,,\ y_i \in R\,,\ \text{and } n_i \in \mathbb{Z} \text{ for each } i<m \Big\}\,,
\end{aligned}
$$

where $A \subseteq R$ is finite. If R is unital, then those expressions can be simplified as

$$
\begin{aligned}
\langle A\rangle_R &= AR\,,\\
{}_R\langle A\rangle_R &= RAR\,,
\end{aligned}
$$

so that, using the notations (8-2.4) and (8-2.6),

$$
\begin{aligned}
\langle\{a_0,\dots,a_{m-1}\}\rangle_R &= a_0R+\cdots+a_{m-1}R\,,\\
{}_R\langle\{a_0,\dots,a_{m-1}\}\rangle_R &= Ra_0R+\cdots+Ra_{m-1}R\,.
\end{aligned}
$$

A left ideal (right ideal, ideal, respectively) of R is *principal* if it has the form Ra (aR, RaR, respectively), with $a \in R$. We set

$$\mathrm{L}^{\mathrm{l}}(R) = \{\, Ra \mid a \in R \,\}\,, \tag{8-2.8}$$

$$\mathrm{L}^{\mathrm{r}}(R) = \{\, aR \mid a \in R \,\}\,, \tag{8-2.9}$$

for every ring R. The sets $\mathrm{L}^{\mathrm{l}}(R)$ and $\mathrm{L}^{\mathrm{r}}(R)$ will both be partially ordered under set inclusion. Although it will turn out that these posets are lattices in case R is regular (cf. Corollary 8-3.13), their general behavior is not so nice (cf. Exercise 8.4).

For an Abelian group A and a family $(A_i \mid i \in I)$ of Abelian groups, let $A = \bigoplus_{i\in I} A_i$ hold if each A_i is contained in A and every element of A can be expressed, in a unique way, as $\sum_{i\in I} a_i$ for a family $(a_i \mid i \in I)$ with only finitely many nonzero entries such that $a_i \in A_i$ for each i.

Idempotents a and b in R are *orthogonal* if $ab = ba = 0$. Likewise, a family $(a_i \mid i \in I)$ of idempotents in R is orthogonal if $a_ia_j = 0$ for all distinct $i, j \in I$.

Proposition 8-2.4 (folklore)**.** *The following statements hold, for any ring R and any positive integer n.*

(i) *For every finite sequence $(a_i \mid i < n)$ of pairwise orthogonal idempotents of R, the element $a = \sum_{i<n} a_i$ is idempotent and $aR = \bigoplus_{i<n} a_iR$.*

(ii) *For every idempotent $a \in R$ and every finite sequence $(A_i \mid i < n)$ of right ideals of R such that $aR = \bigoplus_{i<n} A_i$, there is a unique finite sequence $(a_i \mid i < n)$ such that $a_i \in A_i$ for each $i \in I$ while $a = \sum_{i<n} a_i$; furthermore, the sequence $(a_i \mid i < n)$ is orthogonal and $A_i = a_iR$ for each $i < n$.*

Proof. The proof of (i) is trivial. As to (ii), first observe that since $a = a^2 \in aR$, the existence and uniqueness of $(a_i \mid i < n)$ follow from the definition of the statement $aR = \bigoplus_{i<n} A_i$. For each $i < n$, $a_i \in A_i \subseteq aR$ with $a^2 = a$, thus $a_i = aa_i = \sum_{j<n} a_j a_i$. Now $a_i \in A_i$ while $a_j a_i \in A_j$ for each j, thus, since the A_j are in direct sum, $a_i = a_i^2$ while $a_j a_i = 0$ for each $j \neq i$. Hence $(a_i \mid i < n)$ is an orthogonal sequence of idempotents. Finally let $i < n$ and let $x \in A_i$. Then $x \in aR$ (because $A_i \subseteq aR$) thus, since $a^2 = a$, $x = ax = \sum_{j<n} a_j x$ with $x \in A_i$ and $a_j x \in A_j$ for each j, thus, again since the A_j are in direct sum, $x = a_i x \in a_i R$, and thus completing the proof that $A_i \subseteq a_i R$. Since $a_i \in A_i$, it follows that $A_i = a_i R$. $\square$

8-2.4 Premodules and modules

As we are dealing with rings that are not necessarily unital, we need to relax the definition of a module. Accordingly, for a ring R, a *right R-premodule* is an Abelian group M, endowed with a map $M \times R \to M$, $(x, \lambda) \mapsto x \cdot \lambda = x\lambda$ that satisfies the following conditions:

- $x(\alpha + \beta) = x\alpha + x\beta$, for each $x \in M$ and all $\alpha, \beta \in R$;
- $(x + y)\lambda = x\lambda + y\lambda$, for all $x, y \in M$ and each $\lambda \in R$;
- $(x\alpha)\beta = x(\alpha\beta)$, for each $x \in M$ and all $\alpha, \beta \in R$.

A *sub-premodule* of M is an Abelian subgroup N of M such that the subset $N\lambda = \{ x\lambda \mid x \in N \}$ is contained in N, for each $\lambda \in R$.

If R is unital, the action above defines a *module* if $x \cdot 1 = x$ for each $x \in M$. Observe that even in case R is unital, a right R-premodule may not be a right module (set $x\lambda = 0$ for all $(x, \lambda) \in M \times R$). However, the concept of a *submodule* is not affected by that small hiccup: the sub-premodules of a module are exactly its submodules. Likewise, the concept of homomorphism of (pre)modules is not affected.

We denote by $\operatorname{Sub} M$ the poset of all sub-premodules of M.

Proposition 8-2.5. *The poset* $\operatorname{Sub} M$ *is an Arguesian algebraic lattice, for each right premodule M.*

Proof. Observe first that the subset $\operatorname{Sub} M$ of the powerset lattice of M is closed under directed unions and arbitrary intersections, thus, by Proposition 1-3.10 in Chapter 1, it is an algebraic lattice. Furthermore, a premodule can be viewed as a universal algebra, with signature a binary operation (namely, $(x, y) \mapsto x - y$), an R-indexed collection of unary operations (namely, $x \mapsto x\lambda$ for $\lambda \in R$), and a constant (namely, 0), and then the congruences of M are in one-to-one correspondence with the sub-premodules of M. Furthermore, any two congruences of M permute (this means here that $A + B = B + A$ for any sub-premodules A and B). Therefore, by using a result by Jónsson (cf. [LTF, Theorem 410]), we obtain that $\operatorname{Con} M$ is Arguesian. Now $\operatorname{Con} M \cong \operatorname{Sub} M$. $\square$

Everything above can be defined with “right” replaced by “left”: in particular, a left premodule action $R \times M \to M$ now satisfies $\alpha(\beta x) = (\alpha\beta)x$, for all $\alpha, \beta \in R$ and all $x \in M$. The category of left R-premodules is equivalent (and even isomorphic) to the category of all right R^{op}-premodues.

Two particularly important cases of premodules over a ring R are obtained by letting R act on itself by multiplication. We obtain a left R-premodule, usually denoted by ${}_R R$, and a right R-premodule, usually denoted by R_R. More generally, for a subring S of R, multiplication either on the left or on the right by elements of S defines a structure of left S-premodule and right S-premodule on R, respectively denoted by ${}_S R$ and R_S. The right (resp., left) ideals of R are then exactly the sub-premodules of R_R (resp., ${}_R R$), that is,

$$\mathrm{Id}^{\mathrm{r}}\, R = \mathrm{Sub}(R_R)\,, \quad \mathrm{Id}^{\mathrm{l}}\, R = \mathrm{Sub}({}_R R)\,. \tag{8-2.10}$$

We denote by $\mathrm{Sub}_{\mathrm{c}}\, M$ the $(\vee, 0)$-semilattice of all finitely generated sub-premodules of a premodule M.

Definition 8-2.6. Let a and b be elements in a ring R. We say that b is

– a *quasi-inverse* of a if $a = aba$;

– a *generalized inverse* of a if $a = aba$ and $b = bab$.

An element may have more than one generalized inverse (cf. Exercise 8.5).

Definition 8-2.7. Idempotent elements a, b in a ring R are *Murray–von Neumann equivalent*, in notation $a \sim b$, if there are elements $x, y \in R$ such that $a = yx$ and $b = xy$.

Lemma 8-2.8 (folklore)**.** *The following statements are equivalent, for any idempotents a and b in a ring R:*

(i) *aR and bR are isomorphic as right sub-premodules of R_R.*

(ii) *aR and bR are isomorphic as left sub-premodules of ${}_R R$.*

(iii) *There are mutually quasi-inverse elements $x, y \in R$ such that $a = yx$ and $b = xy$.*

(iv) *a and b are Murray–von Neumann equivalent.*

Proof. (i)$\Rightarrow$(iii). Let $\varphi\colon aR \to bR$ be an isomorphism of right R-premodules. The element $x = \varphi(a)$ belongs to bR and $x = \varphi(aa) = \varphi(a)a$ belongs to bRa. Likewise we can prove that the element $y = \varphi^{-1}(b)$ belongs to aRb. Now

$$a = \varphi^{-1}(\varphi(a)) = \varphi^{-1}(x) = \varphi^{-1}(bx) = \varphi^{-1}(b)x = yx\,,$$

and, likewise, $b = xy$. Therefore, by using $x \in Ra$ and $y \in Rb$, we obtain that $xyx = xa = x$ and $yxy = yb = y$.

The implication (iii)⇒(iv) is trivial.

Let us prove the implication (iv)⇒(i). Let $t \in aR$. Observing that the element $xt = xat = xyxt = bxt$ belongs to bR, we can define a homomorphism $\varphi\colon aR \to bR$, $t \mapsto xt$ of right R-premodules. Likewise, we can define a homomorphism $\psi\colon bR \to aR$, $t \mapsto yt$ of right R-premodules. For each $t \in aR$, $\psi(\varphi(t)) = yxt = at = t$, thus $\psi \circ \varphi$ is the identity on aR. Likewise, $\varphi \circ \psi$ is the identity on bR. Therefore, φ and ψ are mutually inverse.

In particular, (i), (iii), and (iv) are equivalent. Since equivalence of idempotents is self-dual, those three statements are also all equivalent to (ii). □

Observe that for idempotents a, b, two elements x, y with $a = yx$ and $b = xy$ are mutually quasi-inverse iff $x \in bRa$ and $y \in aRb$.

Lemma 8-2.9. *Let M be a right premodule over a ring R and let A, B be sub-premodules of M. If A and B are perspective in the lattice* $\operatorname{Sub} M$*, then they are isomorphic. If $A \cap B = \{0\}$, then the converse holds; furthermore, there exists a sub-premodule C of M such that $A \oplus B = A \oplus C = B \oplus C$, and A, B, C are pairwise perspective.*

Proof. The perspectivity of A and B in $\operatorname{Sub} M$ means that there exists a sub-premodule C such that $A \oplus C = B \oplus C$. If this holds, then the map $f\colon A \to B$ that to each $x \in A$ associates the unique $y \in B$ such that $x - y \in C$ is an isomorphism of R-premodules.

Conversely, if $A \cap B = \{0\}$ and $f\colon A \to B$ is an isomorphism, then $A \oplus B = A \oplus C = B \oplus C$ where $C = \{\, x - f(x) \mid x \in A \,\}$. □

Even for subspaces of vector spaces, perspectivity is (properly) stronger than isomorphism as a rule (cf. Exercise 8.7).

Lemma 8-2.10. *Any two Murray–von Neumann equivalent idempotents a and b in a ring R generate the same two-sided ideal of R.*

Proof. By Lemma 8-2.8, there are $x \in bRa$ and $y \in aRb$ such that $a = yx$ and $b = xy$. It follows that $b = xay$ and $a = ybx$. □

8-3. Von Neumann regular rings

Von Neumann regular rings are a special class of rings that provides lots of interesting lattices, not only from the congruence viewpoint. However, the present section will focus on ideals of regular rings, congruences of the associated lattices of right ideals, and their relationship.

8-3.1 Basic properties of regular rings

We refer to Definition 8-2.6 for quasi-inverses and generalized inverses.

Lemma 8-3.1. *Let R be a ring.*

(i) *Let $a, b \in R$. If b is a quasi-inverse of a, then bab is a generalized inverse of a.*

(ii) *Let $a, b \in R$ be idempotent, let $x \in aRb$, and let $y \in R$. If y is a quasi-inverse (resp., a generalized inverse) of x, then bya is also a quasi-inverse (resp., a generalized inverse) of x.*

Proof. (i) Set $c = bab$. From $a = aba$ it follows that $aca = ababa = aba = a$ while $cac = bababab = babab = bab = c$.

(ii) From $x \in aRb$ it follows, using the idempotence of both a and b, that $x = ax = xb$, thus, setting $z = bya$, we get $xzx = xbyax = xyx = x$. If, in addition, y is a generalized inverse of x, then $zxz = byaxbya = byxya = bya = z$, so z is a generalized inverse of x. □

Definition 8-3.2. A ring R is

- *von Neumann regular* – from now on, *regular* – if every element of R has a quasi-inverse;
- *unit-regular* if R is unital and every element of R has a quasi-inverse which is also invertible.

The following result shows that regularity provides a large supply of idempotents.

Proposition 8-3.3. *Let R be a regular ring. Then every principal right (resp., left) ideal of R is generated by an idempotent. Furthermore every right (left, two-sided, respectively) ideal of R is generated by its idempotent elements.*

Proof. Let $a \in R$ and let b be a quasi-inverse of a. Then $a = aba \in abR$ while $ab \in aR$, so $aR = abR$. Observe that ab is idempotent. Thus every principal right (resp., left) ideal of R is generated by an idempotent. Since regularity is self-dual, a similar statement holds for left ideals.

The second statement of Proposition 8-3.3 follows trivially. □

The property of Proposition 8-3.3 characterizes regularity in the *unital* case, but not in the general case (cf. Exercise 8.8).

Lemma 8-3.4. *Let a be an element in a regular ring R. Then a belongs to both aRa (thus also to both aR and Ra) and RaR.*

Proof. Let b be a quasi-inverse of a. Then $a = aba$ belongs to aRa. Moreover, $a = (ab)a(ba)$ belongs to RaR. □

Lemma 8-3.5. *Let R be a regular ring. Then $I \cap J = I \cdot J$, for all two-sided ideals I and J of R. Furthermore,* $\operatorname{Id} R$ *is a distributive lattice.*

Proof. It is trivial that $I \cdot J \subseteq I \cap J$. Conversely, let $x \in I \cap J$ and pick a quasi-inverse x' of x. Since $x = xx'x$ with $x \in I$ and $x'x \in J$, we get that $x \in I \cdot J$. Therefore, $I \cap J = I \cdot J$.

Since the ideal product $\cdot$ is distributive on the ideal sum $+$, the second statement of Lemma 8-3.5 follows. □

An equivalent form of the second part of Lemma 8-3.5 is: $\operatorname{Id_c} R$ *is a distributive* $(\vee, 0)$*-semilattice.*

Proposition 8-3.6.

(i) *Every homomorphic image of a regular ring is regular.*

(ii) *Every product of regular rings is regular.*

(iii) *Every directed colimit of regular rings is regular.*

(iv) *Every two-sided ideal of a regular ring is regular.*

(v) *Let e be an idempotent element in a regular ring R. Then eRe is regular, with unit e.*

Proof. (i) is trivial.

(ii) Let $R = \prod_{i \in I} R_i$ with all R_i regular. If $a = (a_i \mid i \in I) \in R$ and b_i is a quasi-inverse of a_i for each i, then $b = (b_i \mid i \in I)$ is a quasi-inverse of a in R.

(iii) Let $R = \varinjlim_{i \in I} R_i$, with regular rings R_i, a directed poset I, transition maps $f_i^j \colon R_i \to R_j$ and limiting maps $f_i \colon R_i \to R$, for $i \leq j$ in I. Then (cf. Lemma 7-2.3) R is the directed union of all images $f_i(R_i)$ for $i \in I$. By (i) and since each R_i is regular, so is $f_i(R_i)$. Hence R is regular.

(iv) Let I be a two-sided ideal of a regular ring R and let $a \in I$. If b is a quasi-inverse of a in R, then, by Lemma 8-3.1(i), the element $c = bab$ is also a quasi-inverse of a in R. Observe that $c \in I$.

(v) Obviously eRe is a subring of R, and it is unital with unit e. Furthermore, by Lemma 8-3.1(ii), every element of eRe has a quasi-inverse in eRe. □

The result of Proposition 8-3.6(iv) cannot be extended to one-sided ideals, see Exercise 8.6.

Subrings of the form eRe, for e idempotent in R, are called *corner rings* of R. The class of regular rings is not closed under projective limits (cf. Exercise 8.13). Also, the intersection of two regular subrings of a regular ring may not be regular (cf. Exercise 8.14).

The following deep result provides a converse for Proposition 8-3.6(iv). It is established in Fuchs and Halperin [98].

$\diamondsuit$ **Theorem 8-3.7** (Fuchs and Halperin 1964)**.** *Every regular ring is a two-sided ideal in some unital regular ring.*

Example 8-3.8. A unital ring is a *division ring* if every nonzero element is invertible. A *field* is a commutative division ring.

Every division ring is regular.

Example 8-3.9. A ring R is *Boolean* if $x^2 = x$ for each $x \in R$. Every Boolean ring is regular (and commutative as well).

The following is established in Goodearl [117, Page 2].

Lemma 8-3.10. *Let x, y, u be elements in a ring R and let U be a two-sided ideal of R such that $u \in U$. If u is a quasi-inverse of $x - xyx$, then x has a quasi-inverse in $y + U$.*

Proof. Setting $x' = x - xyx$, we compute:

$$\begin{aligned} x &= x' + xyx \\ &= x'ux' + xyx \\ &= (x - xyx)u(x - xyx) + xyx \\ &= xux - xuxyx - xyxux + xyxuxyx + xyx\,. \end{aligned}$$

Hence $x = xvx$ where $v = u - uxy - yxu + yxuxy + y$ belongs to $y + U$. $\square$

The following is established in the unital case in Goodearl [117, Lemma 1.3], nevertheless the unit is not used in the proof.

Corollary 8-3.11. *Let I be a two-sided ideal in a ring R. Then R is regular iff I and R/I are both regular. If this holds, then for every $a \in R$ and every quasi-inverse $\boldsymbol{b}$ of $a + I$ in R/I, there exists a quasi-inverse b of a in R such that $\boldsymbol{b} = b + I$.*

Proof. Assume first that R is regular. By Proposition 8-3.6(i), R/I is also regular. Furthermore, by Proposition 8-3.6(iv), I is regular.

Conversely, suppose that I and R/I are both regular and let $a \in R$. Since R/I is regular, there exists $c \in R$ such that $\boldsymbol{b} = c + I$ is a quasi-inverse of $a + I$ in R/I. In particular, $a - aca \in I$, thus, as I is regular, $a - aca$ has a quasi-inverse $u \in I$. By Lemma 8-3.10, a has a quasi-inverse b in $c + I$, so $\boldsymbol{b} = b + I$. $\square$

8-3.2 Principal right ideals in regular rings

The following result is contained in Fryer and Halperin [96, Section 3.2]. It originates (in the unital case) in von Neumann [253]. We emphasize that the assumptions of Lemma 8-3.12 do not include R being regular: the only required quasi-inverse (viz. u) is already there.

Lemma 8-3.12. *Let a and b be elements in a ring R with $a^2 = a$ and let u be a quasi-inverse of $b - ab$. Then the following statements hold:*

(i) *Set $c = (b - ab)u$. Then $aR + bR = (a + c)R$.*

(ii) *Suppose that $b^2 = b$ and set $d = u(b - ab)$. Then $aR \cap bR = (b - bd)R$.*

(iii) *Suppose that $b^2 = b$ and $aR \subseteq bR$. Then $bR = aR \oplus (b - ab)R$.*

Proof. Observe that $c = (b - ab)u$ and $d = u(b - ab)$ are both idempotent.

(i) From $a^2 = a$ it follows that $ac = a(b - ab)u = (ab - a^2b)u = 0$, thus $a = (a+c)(a-ca) \in (a+c)R$, and thus $aR \subseteq (a+c)R$. Now $b-ab = c(b-ab)$, thus $b = ab+cb-cab = (a+c)(b-cab) \in (a+c)R$ and thus $bR \subseteq (a+c)R$, and so $aR+bR \subseteq (a+c)R$. Conversely, $a+c = a+(b-ab)u = a(a-bu)+bu \in aR+bR$, thus $(a + c)R \subseteq aR + bR$.

(ii) First observe that $b - ab = (b - ab)u(b - ab) = (b - ab)d$, thus $b - bd = ab - abd = a(b - bd) \in aR$. Now $b^2 = b$ implies that $b - bd = b(b - d) \in bR$; thus $(b - bd)R \subseteq aR \cap bR$. Conversely, let $x \in aR \cap bR$. Since a and b are both idempotent, $x = ax = bx$, thus $dx = u(b - ab)x = u(x - x) = 0$, and thus $x = bx = (b - bd)x \in (b - bd)R$. Therefore, $aR \cap bR \subseteq (b - bd)R$.

(iii) First observe that $a = a^2 \in aR \subseteq bR$ and $b^2 = b$, thus $a = ba$. Let $x \in aR \cap (b - ab)R$. Since $a^2 = a$, we get $x = ax \in a(b - ab)R = \{0\}$; whence $aR \cap (b - ab)R = \{0\}$. Now from $b = ab + (b - ab)$ it follows that $bR \subseteq aR + (b - ab)R$. Conversely, from $a = ba$ and $b^2 = b$ it follows that $b - ab = b^2 - bab = b(b - ab) \in bR$, thus $aR + (b - ab)R \subseteq bR$. □

Since every principal right ideal of a regular ring is generated by an idempotent (cf. Proposition 8-3.3) and $\mathrm{Sub}\, M$ is Arguesian for any premodule M (cf. Proposition 8-2.5), we obtain, using (8-2.10), the following result.

Corollary 8-3.13. *Let R be a regular ring. Then $\mathrm{L}^{\mathrm{r}}(R) = \mathrm{Id}^{\mathrm{r}}_{\mathrm{c}} R = \mathrm{Sub}_{\mathrm{c}} R_R$ is a sectionally complemented 0-sublattice of $\mathrm{Id}^{\mathrm{r}} R$. In particular, it is Arguesian.*

Corollary 8-3.14. *Let R and S be regular rings and let $f\colon R \to S$ be a ring homomorphism. Then there exists a unique map $\mathrm{L}^{\mathrm{r}}(f) : \mathrm{L}^{\mathrm{r}}(R) \to \mathrm{L}^{\mathrm{r}}(S)$ such that $\mathrm{L}^{\mathrm{r}}(f)(xR) = f(x)S$ for each $x \in R$. Furthermore,*

(i) *$\mathrm{L}^{\mathrm{r}}(f)$ is a 0-lattice homomorphism.*

(ii) *f is one-to-one iff $\mathrm{L}^{\mathrm{r}}(f)$ is one-to-one.*

(iii) *If f is surjective, then so is $\mathrm{L}^{\mathrm{r}}(f)$.*

Proof. For $x, y \in R$, it follows from the regularity of R that $xR \subseteq yR$ iff $x \in yR$ (cf. Lemma 8-3.4). It follows that $xR \subseteq yR$ implies that $f(x)S \subseteq f(y)S$. The existence and uniqueness statement about $\mathrm{L}^{\mathrm{r}}(f)$ follow. By Lemma 8-3.12, $\mathrm{L}^{\mathrm{r}}(f)$ is a 0-lattice homomorphism.

Suppose that f is one-to-one. Then $\mathrm{L^r}(f)(xR) = \{0\}$ iff $f(x)S = \{0\}$, iff $f(x) = 0$. Since $\mathrm{L^r}(R)$ is a sectionally complemented lattice, it follows that $\mathrm{L^r}(f)$ is one-to-one.

Suppose that $\mathrm{L^r}(f)$ is one-to-one and let $x \in R$ such that $f(x) = 0$. Then $\mathrm{L^r}(f)(xR) = f(x)S = \{0\}$, thus, by assumption, $xR = 0$, so $x = 0$. Hence f is one-to-one.

If f is surjective, then $\mathrm{L^r}(f)$ is obviously surjective. $\square$

The example of the inclusion mapping from a field into a larger field shows that $\mathrm{L^r}(f)$ surjective does not necessarily imply f surjective.

The verification of the following corollary is a straightforward exercise.

Corollary 8-3.15. *The assignment $R \mapsto \mathrm{L^r}(R)$, $f \mapsto \mathrm{L^r}(f)$ defines a functor, from the category of all regular rings and their homomorphisms, to the category of all sectionally complemented Arguesian lattices and* 0*-lattice homomorphisms. This functor preserves directed colimits and direct products.*

The following result is quite effective in reducing problems about regular rings from the general case to the unital case. The argument of its proof is due to Faith and Utumi [84, Lemma 2].

Proposition 8-3.16 (Faith and Utumi 1963)**.** *Every regular ring is the directed union of all its corner rings.*

Proof. Let X be a finite subset in a regular ring R. It follows from Corollary 8-3.13, applied to R^{op} (which is also regular), that there exists an idempotent $f \in R$ such that $X \subseteq Rf$. By applying Corollary 8-3.13 this time to R, we obtain an idempotent $g \in R$ such that $X \cup \{f\} \subseteq gR$. From $f \in gR$ and $g^2 = g$ it follows that $f = gf$. Setting $e = f + g - fg$, we compute

$$\begin{aligned} e^2 &= f^2 + fg - f^2g + gf + g^2 - gfg - fgf - fg^2 + fgfg \\ &= f + fg - fg + f + g - fg - f - fg + fg \\ &= f + g - fg \\ &= e\,. \end{aligned}$$

Since $fe = f$ and $eg = g$, we obtain that $X \subseteq Rf = Rfe$, thus $X = Xe$. Furthermore, $X \subseteq gR = egR$, thus $X = eX$. Therefore, $X = eXe \subseteq eRe$. $\square$

The result of Proposition 8-3.16 is sometimes expressed by saying that the ring R *has local units.*

Say that a subset X in a ring R is *quasi-invertible in* R if every element of X has a quasi-inverse. The following result makes it possible, in the unital case, to reduce the verification of the regularity of R to the one of subrings of the form aRb.

Lemma 8-3.17. *Let $(a_i \mid i < m)$ and $(b_j \mid j < n)$ be finite orthogonal sequences of idempotents in a ring R. Set $a = \sum_{i<m} a_i$ and $b = \sum_{j<n} b_j$. Then aRb is quasi-invertible in R iff a_iRb_j is quasi-invertible in R for each $i < m$ and each $j < n$.*

Proof. An easy induction argument reduces the problem to the case where $m = 2$ and $n = 1$. Let $x \in aRb$. Since a_0x belongs to a_0Rb, it has, by Lemma 8-3.1(ii), a quasi-inverse $y \in bRa_0$. In particular, $ya_1 = 0$. Since $x = ax = a_0x + a_1x$, we obtain

$$\begin{aligned} x - xyx &= a_0x + a_1x - (a_0x + a_1x)y(a_0x + a_1x) \\ &= a_0x + a_1x - (a_0x + a_1x)ya_0x \\ &= \underbrace{(a_0x - a_0xya_0x)}_{=0} + a_1x - a_1xya_0x \\ &\in a_1Rb\,. \end{aligned}$$

Hence, by assumption, $x - xyx$ has a quasi-inverse, and hence, by Lemma 8-3.10, x has a quasi-inverse. $\square$

In the context of Lemma 8-3.17, with $m = n$ and $a_i = b_i$ for each i, the regularity of R does not necessarily follow from the one of each a_iRa_i; see Exercise 8.9.

By using again Lemma 8-3.1(ii) for the easy direction, we obtain the following corollary.

Corollary 8-3.18. *Let R be a unital ring and let $(a_i \mid i < m)$ and $(b_j \mid j < n)$ be finite orthogonal sequences of idempotents such that*

$$1 = \sum_{i<m} a_i = \sum_{j<n} b_j\,.$$

Then R is regular iff a_iRb_j is quasi-invertible in R for each $i < m$ and each $j < n$.

Observe that Lemma 1.6 in Goodearl [117] is the case, where $m = n$ and $a_i = b_i$ for each i, in Corollary 8-3.18.

For a ring R and a positive integer n, denote by $\mathrm{M}_n(R)$ the ring of all $n \times n$ matrices over R. The following result was first observed, in the non-unital case, in Fryer and Halperin [97, Section 3.6]; see also Goodearl [117, Theorem 1.7] for the unital case.

Theorem 8-3.19. *The ring $\mathrm{M}_n(R)$ is regular, for every regular ring R and every positive integer n.*

Proof. By Proposition 8-3.16, R is the directed union of its corner rings; hence $\mathrm{M}_n(R)$ is the directed union of its subrings $\mathrm{M}_n(eRe)$, for $e \in R$ idempotent.

Since each eRe is unital regular (cf. Proposition 8-3.6(v)), this reduces the problem to the case where R is unital.

Suppose that R is unital and set $S = \mathrm{M}_n(R)$. For all $i, j < n$ and each $x \in R$, denote by $x_{(i,j)}$ the matrix with (i, j)-entry x and all other entries 0. Then $(1_{(i,i)} \mid i < m)$ is an orthogonal finite sequence of idempotents of $\mathrm{M}_n(R)$ summing up to 1. Moreover,

$$1_{(i,i)} \cdot S \cdot 1_{(j,j)} = \{\, x_{(i,j)} \mid x \in R \,\},$$

and if y is a quasi-inverse of x in R then $y_{(j,i)}$ is a quasi-inverse of $x_{(i,j)}$ in S. By Corollary 8-3.18, it follows that S is regular. $\square$

Definition 8-3.20. Let $\mathbb{F}$ be a field. An $\mathbb{F}$-*algebra* is a ring R endowed with a structure of left vector space over $\mathbb{F}$ in such a way that the equations $(\lambda x)y = \lambda(xy) = x(\lambda y)$ hold for all $x, y \in R$ and all $\lambda \in \mathbb{F}$. A ring homomorphism $f\colon A \to B$, for $\mathbb{F}$-algebras A and B, is a *homomorphism of* $\mathbb{F}$-*algebras* if $f(\lambda x) = \lambda f(x)$ for each $(\lambda, x) \in \mathbb{F} \times A$.

An $\mathbb{F}$-algebra is

- *full matricial* provided that it is isomorphic to $\mathrm{M}_n(\mathbb{F})$, for some positive integer n;
- *matricial* provided that it is isomorphic to a finite direct product of full matricial rings;
- *locally matricial* provided that it is a directed colimit of matricial rings.

A ring is full matricial (matricial, locally matricial, respectively) if it is a full matricial (matricial, locally matricial, respectively) algebra over some field $\mathbb{F}$.

The combination of Theorem 8-3.19 and Proposition 8-3.6 yields immediately the following.

Corollary 8-3.21. *Every locally matricial ring is regular.*

8-3.3 Neutral ideals of right ideal lattices of regular rings

The two following lemmas, originating in Jónsson [212, Lemma 1.4], are established in the unital case in Lemma 4.2 and Theorem 4.3 of Wehrung [327]. The extensions of those results to the non-unital case, although belonging to the folklore, are not completely trivial to work out, and are difficult to trace back in the literature; hence we provide the corresponding proofs here.

Lemma 8-3.22. *Let R be a regular ring. Then an ideal $\boldsymbol{I}$ of $\mathrm{L}^{\mathrm{r}}(R)$ is neutral iff $\boldsymbol{I}$ is isomorphy-closed, that is, if $\boldsymbol{x} \in \boldsymbol{I}$ and $\boldsymbol{x}$ and $\boldsymbol{y}$ are isomorphic as right R-premodules, then $\boldsymbol{y} \in \boldsymbol{I}$, for any $\boldsymbol{x}, \boldsymbol{y} \in \mathrm{L}^{\mathrm{r}}(R)$.*

Proof. Suppose first that $\boldsymbol{I}$ is isomorphy-closed, and let $\boldsymbol{x} \in \boldsymbol{I}$ and $\boldsymbol{y} \in \mathrm{L}^{\mathrm{r}}(R)$ such that $\boldsymbol{x}$ and $\boldsymbol{y}$ are perspective in $\mathrm{L}^{\mathrm{r}}(R)$. Since $\mathrm{L}^{\mathrm{r}}(R)$ is a 0-sublattice of $\mathrm{Id}^{\mathrm{r}} R = \mathrm{Sub}\, R_R$ (cf. Proposition 8-3.13), $\boldsymbol{x}$ and $\boldsymbol{y}$ are perspective in $\mathrm{Id}^{\mathrm{r}} R$, hence (cf. Lemma 8-2.9) $\boldsymbol{x} \cong \boldsymbol{y}$. By assumption, $\boldsymbol{y} \in \boldsymbol{I}$. Therefore, by [LTF, Corollary 418], $\boldsymbol{I}$ is a neutral ideal of $\mathrm{L}^{\mathrm{r}}(R)$.

Conversely, suppose that $\boldsymbol{I}$ is a neutral ideal of $\mathrm{L}^{\mathrm{r}}(R)$ and let $\boldsymbol{x}, \boldsymbol{y} \in \mathrm{L}^{\mathrm{r}}(R)$ such that $\boldsymbol{x} \cong \boldsymbol{y}$ and $\boldsymbol{x} \in \boldsymbol{I}$. By Corollary 8-3.13, $\boldsymbol{x} \cap \boldsymbol{y} \in \mathrm{L}^{\mathrm{r}}(R)$ and there exists $\boldsymbol{y}' \in \mathrm{L}^{\mathrm{r}}(R)$ such that $\boldsymbol{y} = (\boldsymbol{x} \cap \boldsymbol{y}) \oplus \boldsymbol{y}'$. From $\boldsymbol{x} \cap \boldsymbol{y} \subseteq \boldsymbol{x}$ and $\boldsymbol{x} \in \boldsymbol{I}$ it follows that $\boldsymbol{x} \cap \boldsymbol{y} \in \boldsymbol{I}$. Since $\boldsymbol{x} \cong \boldsymbol{y}$ and $\boldsymbol{y}'$ is a sub-premodule of $\boldsymbol{y}$, there exists $\boldsymbol{x}' \in \mathrm{L}^{\mathrm{r}}(R)$ such that $\boldsymbol{x}' \subseteq \boldsymbol{x}$ and $\boldsymbol{x}' \cong \boldsymbol{y}'$. Now $\boldsymbol{x}' \cap \boldsymbol{y}' \subseteq \boldsymbol{x} \cap \boldsymbol{y} \cap \boldsymbol{y}' = \{0\}$, thus, by Lemma 8-2.9, $\boldsymbol{x}'$ and $\boldsymbol{y}'$ are perspective in $\mathrm{L}^{\mathrm{r}}(R)$. From $\boldsymbol{x}' \subseteq \boldsymbol{x}$ and $\boldsymbol{x} \in \boldsymbol{I}$ it follows that $\boldsymbol{x}' \in \boldsymbol{I}$, hence, as $\boldsymbol{I}$ is neutral and by [LTF, Corollary 418], $\boldsymbol{y}' \in \boldsymbol{I}$. Consequently, $\boldsymbol{y} = (\boldsymbol{x} \cap \boldsymbol{y}) \oplus \boldsymbol{y}'$ belongs to $\boldsymbol{I}$. □

Lemma 8-3.23. *Let R be a regular ring and let $\boldsymbol{I}$ be a neutral ideal of* $\mathrm{L}^{\mathrm{r}}(R)$. *Then $xR \in \boldsymbol{I}$ implies that $yxR \in \boldsymbol{I}$, for all $x, y \in R$.*

Proof. Let y' be a quasi-inverse of y in R. It follows from Proposition 8-3.16 that there exists an idempotent element e of R such that $\{x, y\} \subseteq eRe$. By Corollary 8-3.13, the right ideal $\boldsymbol{y} = xR \cap (e - y'y)R$ belongs to $\mathrm{L}^{\mathrm{r}}(R)$. Since $\boldsymbol{y} \subseteq xR$ and $\mathrm{L}^{\mathrm{r}}(R)$ is sectionally complemented, there exists $\boldsymbol{z} \in \mathrm{L}^{\mathrm{r}}(R)$ such that $xR = \boldsymbol{y} \oplus \boldsymbol{z}$. Denote by $f\colon xR \twoheadrightarrow yxR$ the left multiplication by y. For each $t \in \ker f$, $(e - y'y)t = et - y'yt = et - 0 = t$, thus $t \in \boldsymbol{y}$. Conversely, for each $t \in \boldsymbol{y}$, there exists $t' \in R$ such that $t = (e - y'y)t'$, thus $yt = y(e - y'y)t' = (ye - yy'y)t' = (y - y)t' = 0$, and thus $t \in \ker f$. Therefore, $\ker f = \boldsymbol{y}$, and therefore f induces an isomorphism from $\boldsymbol{z}$ onto yxR. From $\boldsymbol{z} \subseteq xR$ it follows that $\boldsymbol{z} \in \boldsymbol{I}$, hence, by Lemma 8-3.22, $yxR \in \boldsymbol{I}$. □

The unital case of the following result is established in [327, Theorem 4.3].

Theorem 8-3.24. *The following assignments*

$$\varphi(\boldsymbol{I}) = \{\, x \in R \mid xR \in \boldsymbol{I} \,\}\,, \qquad \textit{for each } \boldsymbol{I} \in \mathrm{NId}(\mathrm{L}^{\mathrm{r}}(R))\,,$$
$$\psi(I) = \mathrm{L}^{\mathrm{r}}(R) \downarrow I\,, \qquad \textit{for each } I \in \mathrm{Id}\, R\,,$$

define mutually inverse isomorphisms between $\mathrm{NId}(\mathrm{L}^{\mathrm{r}}(R))$ *and* $\mathrm{Id}\, R$.

Proof. Let $\boldsymbol{I} \in \mathrm{NId}(\mathrm{L}^{\mathrm{r}}(R))$, let $x \in \varphi(\boldsymbol{I})$, and let $y \in R$. From $xyR \subseteq xR$ and $xR \in \boldsymbol{I}$ it follows that $xyR \in \boldsymbol{I}$, thus $xy \in \varphi(\boldsymbol{I})$. Furthermore, it follows from Lemma 8-3.23 that $yxR \in \boldsymbol{I}$, so $yx \in \varphi(\boldsymbol{I})$. For all $x, y \in \varphi(\boldsymbol{I})$, $(x - y)R \subseteq xR + yR \in \boldsymbol{I}$, thus $x - y \in \varphi(\boldsymbol{I})$. Therefore, $\varphi(\boldsymbol{I})$ is a two-sided ideal of R.

Let I be a two-sided ideal of R. By construction, $\psi(I)$ is an ideal of $\mathrm{L}^{\mathrm{r}}(R)$. Let $\boldsymbol{x}, \boldsymbol{y} \in \psi(I)$ such that $\boldsymbol{x} \cong \boldsymbol{y}$, and pick idempotent elements $x, y \in R$ such that $\boldsymbol{x} = xR$ and $\boldsymbol{y} = yR$. From $xR \subseteq \boldsymbol{x} \subseteq I$ it follows that $x \in I$, thus, by Lemmas 8-2.10 and 8-2.8, $y \in I$, whence $\boldsymbol{y} \subseteq I$. This shows that $\psi(I)$ is isomorphy-closed, and so, by Lemma 8-3.22, $\psi(I)$ is a neutral ideal of $\mathrm{L}^{\mathrm{r}}(R)$.

The verification that φ and ψ are mutually inverse is straightforward. $\square$

By putting together Corollary 8-3.13, Lemma 8-2.1, and Theorem 8-3.24, we obtain the following result, the unital case of which is established in Wehrung [327, Corollary 4.4].

Proposition 8-3.25. *The lattices* $\mathrm{Con}(\mathrm{L}^{\mathrm{r}}(R))$*,* $\mathrm{NId}(\mathrm{L}^{\mathrm{r}}(R))$*, and* $\mathrm{Id}\, R$ *are pairwise isomorphic, for any regular ring* R*.*

A trivial application of Corollary 8-3.13 yields the following.

Corollary 8-3.26. *The ideal lattice of any regular ring is isomorphic to the congruence lattice of some sectionally complemented, Arguesian lattice.*

8-3.4 Ideal lattices of ideals and of corner rings

For an idempotent element e in a ring R, the subset ReR is an ideal (namely, the two-sided ideal generated by e) while the subset eRe is a unital subring of R. In the present subsection, we point that ideal-wise, the two structures are essentially equivalent (at least in the regular case), thus making it possible, in certain cases, to transfer non-unital results to unital results.

Lemma 8-3.27. *Let* I *be a two-sided ideal in a regular ring* R*. Then* $\mathrm{Id}\, I = (\mathrm{Id}\, R) \downarrow I$*.*

Proof. It is trivial that $(\mathrm{Id}\, R) \downarrow I$ is contained in $\mathrm{Id}\, I$. Conversely, let $J \in \mathrm{Id}\, I$ and let $(x, y) \in R \times J$. Pick a quasi-inverse y' of y in R. From $y \in J$ it follows that $y \in I$, thus $xyy' \in I$ and $y'yx \in I$. Hence, from $xy = (xyy')y$ it follows that $xy \in J$. Similarly, $yx = y(y'yx)$ belongs to J. Therefore, J is an ideal of R. $\square$

Lemma 8-3.28. *Let* e *be an idempotent element in a ring* R *and consider the maps*

$$\varphi\colon \mathrm{Id}(eRe) \to \mathrm{Id}(ReR)\,, \qquad \boldsymbol{x} \mapsto R\boldsymbol{x}R\,,$$
$$\psi\colon \mathrm{Id}(ReR) \to \mathrm{Id}(eRe)\,, \qquad \boldsymbol{y} \mapsto \boldsymbol{y} \cap eRe\,.$$

Then the following statements hold:

(i) $(\psi \circ \varphi)(\boldsymbol{x}) = \boldsymbol{x}$ *for any ideal* $\boldsymbol{x}$ *of* eRe*.*

(ii) $(\varphi \circ \psi)(\boldsymbol{y}) = \boldsymbol{y}$ *for any idempotent-generated ideal* $\boldsymbol{y}$ *of* ReR*.*

(iii) *If* R *is regular, then* φ *and* ψ *are mutually inverse isomorphisms between* $\mathrm{Id}(eRe)$ *and* $\mathrm{Id}(ReR)$*.*

Proof. (i) The set $\psi\varphi(\boldsymbol{x}) = R\boldsymbol{x}R \cap eRe$ obviously contains $\boldsymbol{x}$. Any element x of $\psi\varphi(\boldsymbol{x})$ has the form $\sum_{i=1}^{n} p_i x_i q_i$, for a positive integer n and elements $x_i \in \boldsymbol{x}$, $p_i \in R$, $q_i \in R$ for each i. From $x_i \in \boldsymbol{x} \subseteq eRe$ it follows that $x_i = ex_ie$, thus

$$x = exe = \sum_{i=1}^{n} ep_i x_i q_i e = \sum_{i=1}^{n} ep_i e x_i e q_i e\,.$$

Since $x_i \in \boldsymbol{x}$ and $ep_ie, eq_ie \in eRe$ for each i, and since $\boldsymbol{x}$ is an ideal of eRe, it follows that $x \in \boldsymbol{x}$.

(ii) The set $\boldsymbol{x} = \boldsymbol{y} \cap eRe$ is an ideal of eRe. For $x \in \boldsymbol{x}$ (so $x = exe$) and $p, q \in R$, the element $pxq = (pee)x(eeq)$ belongs to $(ReR)\boldsymbol{y}(ReR)$, thus to $\boldsymbol{y}$; whence $\varphi\psi(\boldsymbol{y}) = R\boldsymbol{x}R$ is contained in $\boldsymbol{y}$. In order to prove the converse containment, it suffices, as $\boldsymbol{y}$ is idempotent-generated, to prove that every idempotent element y of $\boldsymbol{y}$ belongs to $R\boldsymbol{x}R$. Since $y \in \boldsymbol{y} \subseteq ReR$, there are a positive integer n together with $p_i \in Re$ and $q_i \in eR$, for $1 \leq i \leq n$, such that $y = \sum_{i=1}^{n} p_i q_i$. Working in the full matrix ring $\mathrm{M}_n(R)$, we consider the matrices

$$P = \begin{pmatrix} p_1 & p_2 & \dots & p_n \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix}, \; Q = \begin{pmatrix} q_1 & 0 & \dots & 0 \\ q_2 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ q_n & 0 & \dots & 0 \end{pmatrix}, \; Y = \begin{pmatrix} y & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix}.$$

Denoting by E the $n \times n$ scalar matrix with diagonal entry e, we obtain that $P = PE$, $Q = EQ$, and $Y = PQ$. Since Y belongs to the ideal $\mathrm{M}_n(\boldsymbol{y})$ of $\mathrm{M}_n(R)$, the matrix $Z = QYP$ belongs to $\mathrm{M}_n(\boldsymbol{y})$ as well. Furthermore, $Z = EZE$ belongs to $\mathrm{M}_n(eRe)$, so Z belongs to $\mathrm{M}_n(\boldsymbol{x})$. Since Y is idempotent, we get

$$PZQ = PQYPQ = Y^3 = Y\,,$$

so $Y \in \mathrm{M}_n(R)\,\mathrm{M}_n(\boldsymbol{x})\,\mathrm{M}_n(R)$, that is, $y \in R\boldsymbol{x}R$.

(iii) follows trivially from the combination of (i) and (ii), together with Proposition 8-3.3. □

Corollary 8-3.29. *Let S be a distributive $(\vee, 0)$-semilattice and let $\boldsymbol{e} \in S$. If there exists a regular ring R such that $S \cong \mathrm{Id_c}\, R$, then there exists a unital regular ring R' such that $S \downarrow \boldsymbol{e} \cong \mathrm{Id_c}\, R'$.*

Proof. We may assume that $S = \mathrm{Id_c}\, R$. By Lemma 8-3.12, there exists an idempotent $e \in R$ such that $\boldsymbol{e} = ReR$. By Lemma 8-3.27, $S \downarrow \boldsymbol{e} = \mathrm{Id}(ReR)$. By Lemma 8-3.28, $S \downarrow \boldsymbol{e} \cong \mathrm{Id}(eRe)$. By Proposition 8-3.6, eRe is regular. □

8-4. Representing distributive semilattices by regular rings

While the congruence lattice representation problem for a distributive $(\vee, 0)$-semilattice S aims at representing S as $\mathrm{Con_c}\, L$, for some lattice L, the corresponding problem for regular rings aims at representing S as $\mathrm{Id_c}\, R$, for some regular ring R. The results of Section 8-3, especially Proposition 8-3.25, show that the ring representation implies the lattice representation (and the corresponding lattice is then sectionally complemented Arguesian). We shall take advantage of this observation in Section 8-4.1, by introducing representation results by Bergman and Růžička. Then we shall move on, by introducing the *nonstable K-theory* of rings in Section 8-4.2, making it possible to take advantage of the vast amount of knowledge in that topic to get further congruence representation results, by *locally finite*, sectionally complemented, Arguesian lattices.

8-4.1 Bergman and Růžička's representation results by locally matricial algebras

The main result of Růžička [283], obtained as Theorem 4.7 of that paper, can be stated as follows.

$\diamondsuit$ **Theorem 8-4.1** (Růžička 2004). *Let $\mathbb{F}$ be a field. Then every distributive $(0,1)$-lattice D is isomorphic to $\mathrm{Id_c}\, R$, for some unital locally matricial $\mathbb{F}$-algebra R.*

The proof of Theorem 8-4.1 is achieved *via* a direct, very involved, construction. For an alternate proof, see Ploščica [263]. By Corollary 8-3.26, we thus obtain the following extension of Schmidt's Theorem (Theorem 7-3.21). The local finiteness statement is obtained by taking $\mathbb{F}$ finite in Theorem 8-4.1. Recall that $\mathbb{F}$-*lattices* are particular cases of sectionally complemented Arguesian lattices, introduced in Definition 7-5.1.

Corollary 8-4.2. *Let $\mathbb{F}$ be a field. Then every distributive $(0,1)$-lattice D is isomorphic to the congruence semilattice of some directed colimit L of bounded $\mathbb{F}$-lattices (hence it is isomorphic to the congruence semilattice of some locally finite, complemented, Arguesian lattice).*

In particular, Corollary 8-4.2 would extend the bounded analogue of Pudlák's Theorem 7-4.14, if it could be achieved *via* a functorial construction. This is proved to be the case in Růžička [284, Theorem 5.1].

$\diamondsuit$ **Theorem 8-4.3** (Růžička 2006). *Let $\mathbb{F}$ be a field. There exists a functor Φ, from the category of all bounded distributive lattices with $(0,1)$-lattice embeddings, to the category of all unital locally matricial $\mathbb{F}$-algebras, such that $\mathrm{Id_c} \circ \Phi$ is isomorphic to the identity.*

A simple application of Corollary 8-3.15 yields the following.

Corollary 8-4.4. *There exists a functor Γ, from the category of all bounded distributive lattices with $(0,1)$-lattice embeddings, to the category of all locally finite, complemented, Arguesian lattices with $(0,1)$-lattice embeddings, such that $\mathrm{Con_c} \circ \Gamma$ is isomorphic to the identity.*

In his unpublished note [24], Bergman proved the following two results, of which alternate proofs can be found in Růžička [283].

$\diamondsuit$ **Theorem 8-4.5** (Bergman 1986)**.** *Let* $\mathbb{F}$ *be a field. Then every countable bounded distributive semilattice is isomorphic to* $\mathrm{Id_c}\,R$*, for some unital, locally matricial* $\mathbb{F}$*-algebra* R*.*

Note: Alternatively, Theorem 8-4.5 can be easily obtained from Theorem 7-5.3. See also Exercise 8.43.

$\diamondsuit$ **Theorem 8-4.6** (Bergman 1986)**.** *Let* $\mathbb{F}$ *be a field. Then every bounded distributive semilattice, in which every element is a finite join of join-irreducible elements, is isomorphic to* $\mathrm{Id_c}\,R$*, for some unital, locally matricial* $\mathbb{F}$*-algebra* R*.*

Remark 8-4.7. Simple uses of Lemma 8-3.27 show that all the results above can be extended to the non-unital case (this observation originates in Růžička [283, Section 4]). For example, in Theorem 8-4.1, let D be a distributive lattice with zero. By applying Theorem 8-4.1 to the bounded distributive lattice $D \cup \{1\}$, we obtain a locally matricial $\mathbb{F}$-algebra R such that $D \cup \{1\} \cong \mathrm{Id_c}\,R$. By Lemma 8-3.27, this implies that R has an ideal R' such that $D \cong \mathrm{Id_c}\,R'$.

By using Corollary 8-3.26, we obtain the following consequence of previous results.

Theorem 8-4.8. *Let* $\mathbb{F}$ *be a field. Then every distributive* $(\vee, 0)$*-semilattice* S*, either countable or in which every element is a finite join of join-irreducible elements, is isomorphic to the congruence semilattice of some* $\mathbb{F}$*-lattice, that can be taken bounded in case* S *has a unit.*

The second part of Theorem 8-4.8 strengthens Dilworth's Theorem 7-3.20.

8-4.2 Nonstable K-theory of rings

The nonstable K-theory $\mathrm{V}(R)$ of a ring R is, among many other things, an important precursor of the ideal theory of idempotent-generated two-sided ideals of R. We shall now sketch a few basic facts about $\mathrm{V}(R)$. For more information, see Goodearl [118, Section 4] for the unital case, Ara [8, Section 3] for the general case.

For a ring R and a positive integer n, we shall often identify the ring $\mathrm{M}_n(R)$ of all $n \times n$ matrices over R, with its image, *via* the embedding $x \mapsto \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix}$, in $\mathrm{M}_{n+1}(R)$. Hence the elements of the (non-unital) ring $\mathrm{M}_\infty(R) = \bigcup_{n \in \mathbb{N}} \mathrm{M}_n(R)$ can be identified with the countably infinite matrices with entries from R and only finitely many nonzero entries.

We denote by $[a]_R$, or $[a]$ if R is understood, the Murray–von Neumann (cf. Definition 8-2.7) equivalence class of an idempotent $a \in \mathrm{M}_\infty(R)$. There is a well-defined addition on $\mathrm{V}(R) = \{\, [a] \mid a \in \mathrm{M}_\infty(R) \text{ idempotent} \,\}$ given by

$$[a] + [b] = \left[\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}\right], \quad \text{for all idempotent square matrices } a \text{ and } b \text{ over } R\,.$$

Then $\mathrm{V}(R)$, endowed with this addition, is a *conical commutative monoid*, which encodes the so-called *nonstable K-theory of R*.

If R is unital, then $[1_R]$ is an order-unit of $\mathrm{V}(R)$ and $\mathrm{V}(R)$ is isomorphic to the monoid of all isomorphism types of finitely generated projective right (resp., left) R-modules, with addition defined by $[X] + [Y] = [X \oplus Y]$ (where $[X]$ now denotes the isomorphism class of X), see Goodearl [118], also Exercise 8.26. Furthermore, $\mathrm{K}_0(R)$ is defined as the (preordered) Grothendieck group of $\mathrm{V}(R)$; this definition is no longer valid in the non-unital case.

For a homomorphism $f\colon R \to S$ of rings, there is a unique homomorphism $\mathrm{V}(f)\colon \mathrm{V}(R) \to \mathrm{V}(S)$ of monoids such that

$$\mathrm{V}(f)([a]_R) = [f(a)]_S\,, \quad \text{for every idempotent matrix } a \text{ over } R\,,$$

where $f(a)$ denotes the matrix obtained by applying f to all entries of a. This way, the assignment $R \mapsto \mathrm{V}(R)$, $f \mapsto \mathrm{V}(f)$ is a functor, from the category of all rings with ring homomorphisms, to the category of all commutative monoids with monoid homomorphisms. The functor V preserves all directed colimits and finite products: for instance, $\mathrm{V}(\varinjlim_{j\in I} R_j) \cong \varinjlim_{j\in I} \mathrm{V}(R_j)$ (for directed colimits) and $\mathrm{V}(R \times S) \cong \mathrm{V}(R) \times \mathrm{V}(S)$.

Lemma 8-4.9 (folklore)**.** *Let R be a ring, let $c \in \mathrm{M}_\infty(R)$ be idempotent, and let $\alpha, \beta \in \mathrm{V}(R)$. If $[c] = \alpha + \beta$, then there are orthogonal idempotents $a, b \in \mathrm{M}_\infty(R)$ such that $c = a + b$, $[a] = \alpha$, and $[b] = \beta$.*

Note: Observe, in particular, that if $c \in R$, then $a, b \in R$.

Proof. There are orthogonal idempotents $u, v \in \mathrm{M}_\infty(R)$ such that $[u] = \alpha$ and $[v] = \beta$. Since $[u+v] = \alpha+\beta = [c]$ (cf. Exercise 8.25), there are $x, y \in \mathrm{M}_\infty(R)$ such that $c = xy$ while $u + v = yx$. The matrices $a = xuy$ and $b = xvy$ are as required. □

The following result was first observed, in the unital case, in Goodearl and Handelman [119, Lemma 3.8], see also Goodearl [117, Theorem 2.8]. The general case is reduced to the unital case *via* Proposition 8-3.16.

♢ **Theorem 8-4.10** (Goodearl and Handelman 1975)**.** *The monoid $\mathrm{V}(R)$ has the refinement property, for any regular ring R.*

For locally matricial rings, much more can be said. The proof of Goodearl [117, Lemma 15.22] shows that $\mathrm{V}(R)$ is a simplicial monoid, for any matricial ring R. Since the functor V preserves directed colimits, we obtain the following result.

♢ **Theorem 8-4.11.** *The monoid* $\mathrm{V}(R)$ *is the positive cone of a dimension group, for any locally matricial ring* R.

Unit-regularity can also be read on the nonstable K-theory of a unital regular ring. This is implied by the following result, first established in Handelman [192, Theorem 2], see also Goodearl [117, Theorem 4.5].

♢ **Theorem 8-4.12** (Handelman 1977)**.** *A unital, regular ring* R *is unit-regular iff the monoid* $\mathrm{V}(R)$ *is cancellative.*

8-4.3 Sending the nonstable K-theory to the ideal lattice

The following result introduces a well-known homomorphism from $\mathrm{V}(R)$ to $\mathrm{Id_c}\, R$, for any ring R. Denote by ${}_R\langle x\rangle_R$ the two-sided ideal of R generated by all entries of a matrix x over R.

Proposition 8-4.13. *For any ring* R*, there exists a unique monoid homomorphism* $\nabla_R\colon \mathrm{V}(R) \to \mathrm{Id_c}\, R$ *such that* $\nabla_R([x]_R) = {}_R\langle x\rangle_R$ *for every idempotent matrix* x *over* R.

Proof. Obviously, ${}_R\langle xy\rangle_R$ is contained in both ${}_R\langle x\rangle_R$ and ${}_R\langle y\rangle_R$, for all matrices $x, y \in \mathrm{M}_\infty(R)$. Since two Murray–von Neumann equivalent idempotent matrices $a, b \in \mathrm{M}_\infty(R)$ generate the same two-sided ideal of $\mathrm{M}_\infty(R)$ (cf. Lemma 8-2.10), it follows that $[a]_R = [b]_R$ implies that ${}_R\langle a\rangle_R = {}_R\langle b\rangle_R$. This shows that ∇_R is well defined. For idempotent matrices a and b over R, it is straightforward to verify that

$$_R\left\langle\begin{pmatrix} a & 0 \\ 0 & b\end{pmatrix}\right\rangle_R = {}_R\langle a\rangle_R + {}_R\langle b\rangle_R\,,$$

so ∇_R is a monoid homomorphism. □

Note: It follows easily that the assignment $R \mapsto \nabla_R$ defines a *natural transformation* from the functor V to the functor $\mathrm{Id_c}$.

As in Ara and Goodearl [10], a two-sided ideal of R is a *trace ideal* if it is generated by the entries of all members of a set of idempotent matrices in $\mathrm{M}_\infty(R)$. The range of ∇_R consists exactly of the finitely generated trace ideals of R.

For a two-sided ideal I of a ring R, it is easy to see that the monoid $\mathrm{V}(I)$ is an o-ideal (cf. Section 8-2.2) of $\mathrm{V}(R)$. The following result is observed, for unital exchange rings R and $E \subseteq R$, in the course of the proof of Pardo [254, Teorema 4.1.7]. It is contained, in full generality, in Ara and Goodearl [10, Proposition 10.10].

Lemma 8-4.14. *Let R be a ring, let $E \subseteq \mathrm{M}_\infty(R)$ be a set of idempotent matrices, and denote by I the two-sided ideal of R generated by the entries of all the elements of E. Then $\mathrm{V}(I)$ is the o-ideal of $\mathrm{V}(R)$ generated by $\{\,[x]_R \mid x \in E\,\}$.*

Proof. We prove the nontrivial containment only. Denote by $\overline{R}$ any unital ring containing R as a two-sided ideal (cf. Exercise 8.1) and by A the set of all the entries of all the elements of E. Let $e \in \mathrm{M}_\infty(I)$ be idempotent. There is a decomposition of the form $e = \sum_{j=1}^n x_j^0 a_j y_j^0$, with all $a_j \in A$ and all $x_j^0, y_j^0 \in \mathrm{M}_\infty(\overline{R})$. For each $j \in \{1, \dots, n\}$, there exists $e_j \in E$ such that a_j is an entry of e_j. Using matrix units, we obtain the existence of $x_j^1, y_j^1 \in \mathrm{M}_\infty(\overline{R})$ such that $a_j = x_j^1 e_j y_j^1$. Since e_j is idempotent, this equation remains valid if we replace x_j^1 by $x_j^1 e_j$ and y_j^1 by $e_j y_j^1$, thus we may assume that $x_j^1, y_j^1 \in \mathrm{M}_\infty(R)$. The elements $x_j = x_j^0 x_j^1$ and $y_j = y_j^1 y_j^0$ all belong to $\mathrm{M}_\infty(R)$, and $e = \sum_{j=1}^n x_j e_j y_j$. Pick $m \in \mathbb{N}$ such that all $e_j, x_j, y_j \in \mathrm{M}_m(R)$. Using block matrices, we get $e = XDY$, where e is identified with the $n \times n$ matrix with upper left corner e and all other entries zero, and

$$X = \begin{pmatrix} x_1 & x_2 & \dots & x_n \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix}, Y = \begin{pmatrix} y_1 & 0 & \dots & 0 \\ y_2 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ y_n & 0 & \dots & 0 \end{pmatrix}, D = \begin{pmatrix} e_1 & 0 & \dots & 0 \\ 0 & e_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & e_n \end{pmatrix}$$

Since e is idempotent, $e = eXDYe$ as well. The matrix $Z = DYeXD$ is idempotent and $ZD = DZ = Z$, thus $[Z] \leq [Z] + [D - Z] = [D]$ (cf. Exercise 8.25). Furthermore, $e = (eXD)(DYe)$ and $Z = (DYe)(eXD)$, thus $e \sim Z$. Therefore, $[e] = [Z] \leq [D] = [e_1] + \cdots + [e_n]$. □

After Lemma 8-4.14, it is now easy to describe the kernel of the map ∇_R introduced in Proposition 8-4.13. The following result originates in Goodearl [117, Proposition 2.22]. In the regular case, it is contained in Goodearl and Wehrung [121, Proposition 7.3]. (We defined the maximal semilattice quotient in Section 7-5.4.)

Corollary 8-4.15. *Let R be a ring and let $\alpha, \beta \in \mathrm{V}(R)$. Then*

$$\nabla_R(\alpha) \subseteq \nabla_R(\beta) \iff (\exists n \in \mathbb{N})(\alpha \leq n\beta)\,, \quad \textit{for all } \alpha, \beta \in \mathrm{V}(R)\,, \tag{8-4.1}$$

that is, ∇_R induces a monoid embedding from the maximal semilattice quotient of $\mathrm{V}(R)$ into $\mathrm{Id_c}\, R$. Furthermore, if R is regular, then this embedding is an isomorphism.

Proof. We prove the nontrivial direction of (8-4.1). There are idempotent matrices $x, y \in \mathrm{M}_\infty(R)$ such that $\alpha = [x]$ and $\beta = [y]$. Apply Lemma 8-4.14 to $E = \{y\}$; so I is the two-sided ideal generated by $\{y\}$. Then $\nabla_R(\alpha) \subseteq \nabla_R(\beta)$

means that $x \in \mathrm{M}_\infty(I)$, hence $[x]$ belongs to the o-ideal of $\mathrm{V}(R)$ generated by $\{[y]\}$, which means that $[x] \leq n[y]$ for some $n \in \mathbb{N}$.

If R is regular, then every two-sided ideal of R is generated by its idempotents, thus ∇_R is surjective. $\square$

8-4.4 Semilattices with at most aleph one elements

The following result, established in Wehrung [328, Theorem 5.2], is an analogue, in the world of regular rings, of Theorem 7-5.13.

$\diamondsuit$ **Theorem 8-4.16** (Wehrung 2000). *Let $\mathbb{F}$ be a field. Then every distributive $(\vee, 0)$-semilattice S with at most $\aleph_1$ elements is isomorphic to $\mathrm{Id_c}\, R$, for some regular $\mathbb{F}$-algebra R such that $\mathrm{V}(R)$ is a semilattice. Furthermore, if S has a unit, then R can be taken unital.*

It is actually this way that Theorem 7-5.15 is proved: first establish Theorem 8-4.16, then invoke Proposition 8-3.25.

The proof of Theorem 8-4.16 follows the same lines as the one of Theorem 7-5.13, except that the amalgamation result stated in Lemma 7-5.10 is replaced by an analogue of that lemma, for regular rings, established in Cohn [39, Theorem 4.7].

There is no natural common strengthening of Theorems 8-4.16 and 8-4.5: namely, in the statement of Theorem 8-4.16, the ring R cannot always be taken locally matricial (it cannot even be taken *unit-regular*). This follows from the results of Wehrung [331], especially Theorem 7-5.16 (see also the discussion in Section 7-5.4).

By using the methods from Gillibert and Wehrung [114], it is possible to extend Theorem 8-4.16 to diagrams of semilattices indexed by well-founded trees subjected to the same size conditions as in the statement of Theorem 7-5.17. The proof runs along the same lines as the one of Theorem 7-5.17, thus we shall only give an outline of what needs to be changed, especially the required larder.

Theorem 8-4.17. *Let $\mathbb{F}$ be a field, let I be a lower countable well-founded tree such that $\operatorname{card} I \leq \aleph_1$, and let $\vec{S} = (S_i, \sigma_i^j \mid i \leq j \text{ in } I)$ be an I-indexed diagram of distributive $(\vee, 0)$-semilattices with $(\vee, 0)$-homomorphisms. We assume that*

(i) $\operatorname{card} S_i \leq \aleph_1$ *for each $i \in I$;*

(ii) $\operatorname{card} S_i \leq \aleph_0$ *for each non-maximal $i \in I$.*

Then there exists an I-indexed diagram $\vec{R}$ of regular $\mathbb{F}$-algebras such that $\mathrm{Id_c}\, \vec{R} \cong \vec{S}$.

The analogue of the result above, for distributive $(\vee, 0, 1)$-semilattices and unital regular $\mathbb{F}$-algebras, holds as well.

Outline of proof. We give an outline for the case of $(\vee,0)$-semilattices, $(\vee,0)$-homomorphisms, and regular $\mathbb{F}$-algebras; the case of $(\vee,0,1)$-semilattices, $(\vee,0,1)$-homomorphisms, and unital regular $\mathbb{F}$-algebras can be treated similarly.

Let us describe the $\aleph_1$-larder by which we need to replace the larder Λ used in the proof of Theorem 7-5.17.

We set $\Lambda = (\mathcal{A}, \mathcal{B}, \mathcal{S}, \mathcal{A}^\dagger, \mathcal{B}^\dagger, \mathcal{S}^{\Rightarrow}, \Phi, \Psi)$, with

- $\mathcal{A} = \mathcal{S}$ is the category of all distributive $(\vee,0)$-semilattices with $(\vee,0)$-homomorphisms.
- $\Phi\colon \mathcal{A} \to \mathcal{S}$ is the identity functor.
- $\mathcal{A}^\dagger$ is the full subcategory of $\mathcal{A}$ consisting of all countable members of $\mathcal{A}$.
- $\mathcal{B}$ is the category of all locally matricial $\mathbb{F}$-algebras, with homomorphisms of $\mathbb{F}$-algebras.
- $\mathcal{B}^\dagger$ is the full subcategory of $\mathcal{B}$ consisting of all countable-dimensional members of $\mathcal{B}$.
- $\mathcal{S}^{\Rightarrow}$ is the subcategory of $\mathcal{S}$ consisting of all *ideal-induced* homomorphisms (cf. Section 7-5.5).
- Ψ is the functor $\mathrm{Id_c}\colon \mathcal{B} \to \mathcal{S}$.

The "larder axioms" that need to be verified are now the following:

Left larder axioms

(CLOS($\mathcal{A}$))] $\mathcal{A}$ has all small directed colimits. This is trivial.

(PROD($\mathcal{A}$)) $\mathcal{A}$ has all nonempty finite products. This is trivial.

(CONT(Φ)) Φ preserves all small directed colimits. This is trivial.

(PROJ($\Phi, \mathcal{S}^{\Rightarrow}$)) Φ sends every directed colimit of projections of $\mathcal{A}$ (i.e., canonical projections of the form $S \times T \twoheadrightarrow S$) to an ideal-induced homomorphism. This is straightforward.

Right larder axioms

(PRES$_{\aleph_1}$($\mathcal{B}^\dagger, \Psi$)) $\mathrm{Id_c}\, B$ is weakly $\aleph_1$-presented (i.e., countable, cf. Gillibert and Wehrung [114, Proposition 4.2.3]), for any $B \in \mathcal{B}^\dagger$. This is trivial.

(LS$^{\mathrm{r}}_{\aleph_1}$(B)) for every object B of $\mathcal{B}$: we must verify that for every countable distributive $(\vee,0)$-semilattice S, every ideal-induced $\boldsymbol{f}\colon \mathrm{Id_c}\, B \twoheadrightarrow S$, and every countable sequence $(u_n\colon U_n \to B \mid n < \omega)$ of monomorphisms in $\mathcal{B}$ with all U_n countable-dimensional, there exists a monomorphism $u\colon U \to B$ in $\mathcal{B}$, with U countable-dimensional, above all the u_n in the subobject ordering, such that $\boldsymbol{f} \circ (\mathrm{Id_c}\, u)$ is ideal-induced. Although the verification of this fact is a not completely trivial "Löwenheim-Skolem type" argument, it is not difficult either, and it follows the lines of the proof of Claim 2 of the proof of Gillibert and Wehrung [114, Theorem 4.7.2] (see also the (LS) part in the proof of Wehrung [339, Lemma 13.2]).

The verification of the projectability of Λ runs along the same lines as the one of Gillibert and Wehrung [114, Theorem 4.5.2]. The rest of the proof works as the one of Theorem 7-5.17. □

8-4.5 Lifting arrows of countable distributive semilattices

Tůma and Wehrung [317, Theorem 7.1] establish a *reverse one-dimensional amalgamation theorem*, for the compact ideal semilattice functor, from dimension vector spaces (over the rationals) to distributive $(\vee, 0)$-semilattices. As a consequence of this theorem, the following result is established in [317, Corollary 7.4].

$\diamondsuit$ **Theorem 8-4.18** (Tůma and Wehrung 2003)**.** *Let $\mathbb{F}$ be a field, let S and T be countable distributive $(\vee, 0)$-semilattices, and let $\boldsymbol{f}\colon S \to T$ be a $(\vee, 0)$-homomorphism. Then there are countable-dimensional locally matricial $\mathbb{F}$-algebras A and B, together with a homomorphism $f\colon A \to B$ of $\mathbb{F}$-algebras, such that $\boldsymbol{f} \cong \mathrm{Id_c}\, f$. Furthermore, if S and T are both bounded and $\boldsymbol{f}$ is unit-preserving, then this can be done in such a way that A and B are both unital and $f(1_A) = 1_B$.*

By using Theorem 8-4.18 for finite $\mathbb{F}$, together with Proposition 8-3.25, we obtain the following result, first established as Tůma and Wehrung [317, Corollary 7.5].

Corollary 8-4.19. *Let S and T be countable distributive $(\vee, 0)$-semilattices and let $\boldsymbol{f}\colon S \to T$ be a $(\vee, 0)$-homomorphism. Then there are locally finite, sectionally complemented modular lattices K and L, together with a 0-lattice homomorphism $f\colon K \to L$, such that $\boldsymbol{f} \cong \mathrm{Con_c}\, f$. Furthermore, if S and T are both bounded and $\boldsymbol{f}(1_S) = 1_T$, then this can be done in such a way that K and L are both bounded and $f(1_K) = 1_L$.*

A complete proof of Theorem 8-4.18 is quite involved, and requires the introduction of a special class of dimension vector spaces called *pseudo-simplicial* (cf. Exercise 8.35). This proof shows that in the statement of Theorem 8-4.18, once the $\mathbb{F}$-algebra B is specified with $\mathrm{V}(B)$ divisible (i.e., $m\,\mathrm{V}(B) = \mathrm{V}(B)$ for each positive integer m), then A can be found, in such a way that $\mathrm{V}(A)$ is divisible as well.

This shows that the process can be repeated, making it possible to extend Theorem 8-4.18, to diagrams of countable distributive $(\vee, 0)$-semilattices indexed by *finite dual trees*.

$\diamondsuit$ **Theorem 8-4.20.** *Let $\mathbb{F}$ be a field, let I be a finite dual tree, and let $\vec{S}$ be an I-indexed diagram of countable distributive $(\vee, 0)$-semilattices with $(\vee, 0)$-homomorphisms. Then there exists an I-indexed diagram $\vec{R}$ of regular $\mathbb{F}$-algebras such that $\mathrm{Id_c}\,\vec{R} \cong \vec{S}$.*

The analogue of the result above, for distributive $(\vee, 0, 1)$-semilattices, $(\vee, 0, 1)$-homomorphisms, and unital regular $\mathbb{F}$-algebras with unital homomorphisms, holds as well.

A simple application of Corollary 8-3.15 and Proposition 8-3.25 yields the following.

Corollary 8-4.21. *Let I be a finite dual tree and let $\vec{S}$ be an I-indexed diagram of countable distributive $(\vee, 0)$-semilattices with $(\vee, 0)$-homomorphisms. Then there exists an I-indexed diagram $\vec{L}$ of locally finite, sectionally complemented, Arguesian lattices such that $\mathrm{Con_c}\,\vec{L} \cong \vec{S}$.*

The analogue of the result above, for distributive $(\vee, 0, 1)$-semilattices, $(\vee, 0, 1)$-homomorphisms, locally finite, complemented, Arguesian lattices, and $(0, 1)$-lattice homomorphisms, holds as well.

We propose a proof of Theorem 8-4.20 (only for the very motivated reader!), divided between Exercises 8.34 to 8.41, mainly devoted to showing the slight amendments that need to be brought to the arguments of Tůma and Wehrung [317] in order to get the result. The crucial part of this proof is the *reverse amalgamation theorem* established in Exercise 8.40. The latter is obtained by putting together unital versions of some of the results in [317]. Its complete proof requires a number of nontrivial concepts introduced in [317], mainly *flatness* and *genericity* for homomorphisms of partially ordered vector spaces.

The result of Theorem 8-4.20 is to be put in contrast with Theorem 8-4.17, which makes it possible to lift diagrams of distributive $(\vee, 0)$-semilattices indexed by *trees* (subjected to certain size conditions), as opposed to dual trees.

The results of Tůma and Wehrung [317] are established *via* an in-depth study of the functor that to every partially ordered Abelian group G associates the maximal semilattice quotient of G^+. In particular, that paper contains various examples and counterexamples about that functor, some of which are presented in the exercises.

8-5. Exercises

8.1. Let R be a ring. We endow the product $\overline{R} = \mathbb{Z} \times R$ with componentwise addition, and multiplication defined by

$$(m, x) \cdot (n, y) = (mn, my + nx + xy)\,, \quad \text{for all } (m, x), (n, y) \in \overline{R}\,.$$

Verify that $\overline{R}$ is a unital ring and that the assignment $x \mapsto (0, x)$ defines an isomorphism from R onto a two-sided ideal of $\overline{R}$.

8.2. Let R be a unital ring. An element $s \in R$ is *involutive* if $s^2 = 1$.

(1) Verify that $2e - 1$ is involutive, for each idempotent $e \in R$.

(2) Suppose that $1/2$ exists in R (i.e., the element $2 = 1 + 1$ is invertible). Verify that $\frac{1+s}{2}$ is idempotent, for each involutive $s \in R$.

8.3. Let R be a ring.

(1) Prove that $I(J + K) = IJ + IK$ and $(I + J)K = IK + JK$, for any ideals I, J, K of R.

(2) Find an example where $I \cap (J + K)$ properly contains $(I \cap J) + (I \cap K)$ for ideals I, J, K of R.

(3) Find an example where $IJ \neq JI$, for ideals I and J of R.

(4) Find an example where RaR properly contains the set of all elements of the form xay where $x, y \in R$, for an idempotent element $a \in R$.

8.4. Consider the (commutative, unital) ring $R = \mathbb{Z}[\sqrt{-5}]$, generated by the ring $\mathbb{Z}$ of all integers and an additional square root of -5. Hence the elements of R have the form $x + y\sqrt{-5}$ with $x, y \in \mathbb{Z}$. We set $N(x + y\sqrt{-5}) = x^2 + 5y^2$, for all $x, y \in \mathbb{Z}$.

(1) Verify that $N(uv) = N(u)N(v)$, for all $u, v \in R$.

(2) Set $a = 1 - \sqrt{-5}$ and $b = 1 + \sqrt{-5}$. Verify that $6 = a \cdot b = 2 \cdot 3$, then that $2R$ and aR both contain $6R$ and $2aR$.

(3) Prove that there is no element $X \in \mathrm{L}^{\mathrm{r}}(R)$ such that

$$2R, aR \supseteq X \supseteq 6R, 2aR.$$

(*Hint*: use (i)). Deduce that $\mathrm{L}^{\mathrm{r}}(R)$ is not a lattice.

8.5. Find an example of a regular ring where an element has more than one generalized inverse (*Hint*: use 2×2 matrices over any field).

8.6. Find a unital regular ring R and an idempotent e of R such that eR, viewed as a subring of R, is not regular. (*Hint*: consider the matrices of the form $\begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix}$ within the ring of all 2×2 matrices over any field. Verify that $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ has no quasi-inverse.)

8.7. Let V be a vector space over a division ring K.

(1) Prove that if V is finite-dimensional, then two subspaces A and B of V are isomorphic iff they are perspective.

(2) Prove that if V is infinite-dimensional, then V is isomorphic to a proper subspace U of V. Infer that U and V are not perspective.

(3) Prove that if V is infinite-dimensional, then the relation of perspectivity on subspaces of V is not transitive.

8.8. (1) Prove that the property established in Proposition 8-3.3 characterizes regularity of rings in the *unital* case: that is, if every principal right (resp., left) ideal of a unital ring R is generated by an idempotent, then R is regular.

(2) Find a non-regular, commutative ring where every principal ideal is generated by an idempotent. (*Hint*: set $x \cdot y = 0$ for all $x, y \in R$. We say that R is a *zero ring*.)

8.9. Let $\mathbb{F}$ be a field and denote by R the $\mathbb{F}$-algebra of all 2×2 matrices of the form $\begin{pmatrix} x & y \\ 0 & z \end{pmatrix}$ where $x, y, z \in \mathbb{F}$.

(1) Verify that R is unital but not regular.

(2) Setting $e = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, prove that the corner rings eRe and $(1-e)R(1-e)$ are both isomorphic to $\mathbb{F}$, thus regular.

8.10. (Goodearl [117, Proposition 1.4]) Prove that any subdirect product of a finite collection of regular rings is regular. (*Hint*: it suffices to prove this for a subdirect product $R \subseteq R_0 \times R_1$ with R_0 and R_1 both regular. The kernels I_0 and I_1 of the canonical projections of R onto R_0 and R_1, respectively, are ideals of R with $I_0 \cap I_1 = \{0\}$, while $R/I_0 \cong R_1$ and $R/I_1 \cong R_0$. Now $(I_0 + I_1)/I_0$ is an ideal of the regular ring R/I_0, thus, by Corollary 8-3.11, $(I_0 + I_1)/I_0$ is regular, thus also $I_1 \cong I_1/(I_0 \cap I_1) \cong (I_0 + I_1)/I_0$. Now use again Corollary 8-3.11.)

8.11. (Goodearl [117, Page 3]) Verify that the ring $\mathbb{Z}$ of all integers is not regular, although it is a subdirect product of a countable collection of fields (namely the prime fields $\mathbb{Z}/p\mathbb{Z}$ for prime p).

8.12. Prove that the ring $\mathbb{Z}/4\mathbb{Z}$ cannot be embedded into any regular ring. (*Hint*: prove that 2 has no quasi-inverse).

8.13. (Goodearl [117, Example 1.10]) *The purpose of this exercise is to find a regular ring with a descending sequence of pairwise isomorphic regular subrings whose intersection is not regular.*

We fix a field $\mathbb{F}$ and we define R as the ring of all eventually constant sequences $x = (x_n \mid n < \omega)$ of elements of $\mathrm{M}_2(\mathbb{F})$; then we denote by x_∞ the value of x_n for large enough n.

(1) Prove that the ideal $I = \{\, x \in R \mid x_\infty = 0 \,\}$ is regular and $R/I \cong \mathrm{M}_2(\mathbb{F})$. Deduce that R is regular.

(2) Set $u = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ and set

$$R_n = \{\, x \in R \mid (\forall i < n)(x_i = u x_{i+1} u^{-1}) \,\}\,, \quad \text{for each } n < \omega\,.$$

Prove that $(R_n \mid n < \omega)$ is a descending sequence of subrings of R and that $R_n \cong R$ for each $n < \omega$.

(3) Prove that $\bigcap_{n<\omega} R_n$ is isomorphic to the ring of all triangular matrices of the form $\begin{pmatrix} a & 0 \\ b & a \end{pmatrix}$, for $a, b \in \mathbb{F}$. Verify that this ring is not regular.

8.14. *This exercise constructs a regular ring with two isomorphic regular subrings whose intersection is not regular.*

We fix a field $\mathbb{F}$. We denote by I the identity matrix of $\mathrm{M}_2(\mathbb{F})$ and we set $U = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$.

(1) Verify that $U^2 = I$ and (I, U) is linearly independent over $\mathbb{F}$.

(2) Verify that the inverse of $\begin{pmatrix} I & U \\ 0 & I \end{pmatrix}$ is $\begin{pmatrix} I & -U \\ 0 & I \end{pmatrix}$.

(3) Define unital ring embeddings $u_1, u_2 \colon \mathrm{M}_2(\mathbb{F}) \hookrightarrow \mathrm{M}_4(\mathbb{F}) = \mathrm{M}_2(\mathrm{M}_2(\mathbb{F}))$ by setting

$$u_1 \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} aI & bI \\ cI & dI \end{pmatrix}\,,$$
$$u_2 \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} I & U \\ 0 & I \end{pmatrix} \begin{pmatrix} aI & bI \\ cI & dI \end{pmatrix} \begin{pmatrix} I & -U \\ 0 & I \end{pmatrix}\,,$$

for all $a, b, c, d \in \mathbb{F}$. Denote by R_i the range of u_i, for $i \in \{1, 2\}$. Verify that

$$R_1 \cap R_2 = \left\{ \begin{pmatrix} aI & bI \\ 0 & aI \end{pmatrix} \mid a, b \in \mathbb{F} \right\}.$$

Verify that $R_1 \cap R_2$ is not regular.

8.15. For a subset X in a ring R, we set

$$\mathrm{ann}_{\mathrm{r}}(X) = \{\, \alpha \in R \mid X\alpha = 0 \,\}\,,$$
$$\mathrm{ann}_{\mathrm{l}}(X) = \{\, \alpha \in R \mid \alpha X = 0 \,\}\,.$$

Now fix a *unital* regular ring R.

(1) Prove that $\mathrm{ann}_{\mathrm{l}}(eR) = R(1 - e)$, for each idempotent $e \in R$.

(2) Deduce that the assignments $X \mapsto \mathrm{ann}_{\mathrm{l}}(X)$ and $Y \mapsto \mathrm{ann}_{\mathrm{r}}(Y)$ define mutually inverse dual isomorphisms of lattices between $\mathrm{L}^{\mathrm{r}}(R)$ and $\mathrm{L}^{\mathrm{l}}(R)$.

8.16. Let V be a right vector space over a division ring K and set $R = \mathrm{End}\, V$, the endomorphism ring of V. Prove that R is regular, and that there exists a unique isomorphism $\varepsilon\colon \mathrm{L}^{\mathrm{r}}(R) \to \mathrm{Sub}\, V$ such that $\varepsilon(fR) = \mathrm{im}\, f$ for each $f \in R$ (where $\mathrm{im}\, f$ denotes the image of f).

8.17. (K.R. Goodearl, private communication) A poset is *countably directed* if every (at most) countable subset has an upper bound. *This exercise constructs a regular ring (necessarily without unit, see Exercise 8.15) R such that $\mathrm{L}^{\mathrm{r}}(R)$ is countably directed while $\mathrm{L}^{\mathrm{l}}(R)$ is not.*

Fix a field $\mathbb{F}$ and a countably infinite-dimensional $\mathbb{F}$-vector space V_n, for each $n < \omega$. We set $V = \prod_{n<\omega} V_n$ and

$$R = \{f \in \mathrm{End}\, V \mid \mathrm{im}\, f \text{ has countable dimension}\} .$$

(1) Let $(f_n \mid n < \omega)$ be a countable sequence of elements of R. Observe that there exists a countable-dimensional subspace W of V such that $\mathrm{im}\, f_n \subseteq W$ for each n. Denote by p a projection in V with image W. Prove that $f_n R \subseteq pR$ for each $n < \omega$ (use Exercise 8.16). Deduce that $\mathrm{L}^{\mathrm{r}}(R)$ is countably directed.

(2) Prove that V is not countable-dimensional.

(3) Denote by $p_n\colon V \twoheadrightarrow V_n$ the canonical projection, and let e_n be a projection of V with the same kernel as p_n, for each n. Suppose that there exists $f \in R$ such that $Re_n \subseteq Rf$ for each $n < \omega$. Deduce that $\ker f \subseteq \ker e_n$ for each n, then that f is one-to-one. Deduce that $\mathrm{L}^{\mathrm{l}}(R)$ is not countably directed.

8.18. For a field $\mathbb{F}$, we denote by $\mathrm{B}(\mathbb{F})$ the collection of all infinite matrices $a \in \mathrm{M}_\omega(\mathbb{F})$ such that every row and every column of a have only finitely many nonzero entries. Prove that $\mathrm{B}(\mathbb{F})$ is a non-regular ring. (*Hint*: denoting by $(e_n \mid n < \omega)$ the canonical basis of the free $\mathbb{F}$-vector space V on ω generators, consider the endomorphism s of V defined by $s(e_n) = e_{n+1}$ for each n. Prove that $1 - s$ has no quasi-inverse in $\mathrm{B}(\mathbb{F})$. It may help to first observe that $1 - s$ is one-to-one.)

8.19. ((1) in proof of Giudici [115, Teorema 4.2.1], Herrmann [196, Proposition 9.1]) Let R be a regular ring, let G be an Abelian group, and let f be a ring homomorphism from R to the endomorphism ring of G. Prove that there is a unique 0-lattice homomorphism $\varphi\colon \mathrm{L}^{\mathrm{r}}(R) \to \mathrm{Sub}\, G$ such that $\varphi(xR) = \mathrm{im}\, f(x)$ for each $x \in R$, and that if f is one-to-one, then so is φ. (*Hint*: use Lemma 8-3.12.)

8.20. Let R be a regular ring and let n be a positive integer. Verify that for each element $x \in R_R^n$, there exists an idempotent $e \in R$ such that $x = xe = ex$. (*Hint*: use Proposition 8-3.16.)

8.21. Let R be a regular ring and let n be a positive integer.

(1) Prove that there exists a unique map $\varphi\colon \mathrm{L^r}\left(\mathrm{M}_n(R)\right) \to \mathrm{Sub}(R_R^n)$ such that $\varphi\big(x \cdot \mathrm{M}_n(R)\big)$ is the image space of x (i.e., the sub-premodule of R_R^n generated by the columns of x) for every $x \in \mathrm{M}_n(R)$, and that φ is a 0-lattice embedding. (*Hint*: apply Exercise 8.19.)

(2) Prove that the range of φ consists exactly on the finitely generated sub-premodules of R_R^n. (*Hint*: consider column matrices.)

(3) Deduce from this the following consequences:

(a) The $(\vee, 0)$-semilattice $\mathrm{Sub_c}(R_R^n)$, of all finitely generated sub-premodules of R_R^n, is a sectionally complemented 0-sublattice of $\mathrm{Sub}(R_R^n)$, isomorphic to $\mathrm{L^r}\left(\mathrm{M}_n(R)\right)$.

(b) Every finitely generated sub-premodule of R_R^n is generated by a subset with at most n elements.

(c) If R is unital, then every finitely generated submodule of R_R^n is a direct summand of R_R^n.

8.22. Prove that every ideal of a full matricial (matricial, locally matricial, respectively) ring is full matricial (matricial, locally matricial, respectively).

8.23. Prove that for any idempotent element e in a ring R, if $\varepsilon\colon eRe \hookrightarrow R$ denotes the inclusion map, then $\mathrm{V}(\varepsilon)$ is an isomorphism from $\mathrm{V}(eRe)$ onto $\mathrm{V}(R)|_{[e]}$.

8.24. Prove that $\mathrm{V}(\mathrm{M}_n(R)) \cong \mathrm{V}(R)$, for any ring R. (*Hint*: observe that $\mathrm{M}_\infty\big(\mathrm{M}_n(R)\big) \cong \mathrm{M}_\infty(R)$.)

8.25. Let a and b be orthogonal idempotent matrices over a ring R.

(1) Prove that the matrices $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ and $a + b$ are Murray–von Neumann equivalent.

(2) Prove that $[a] + [b] = [a + b]$.

8.26. For a unital ring R, we denote by $\mathrm{FP}(R)$ the full subcategory of all right R-modules, whose objects are the right R-modules X for which there are a right R-module Y and a positive integer n such that $X \oplus Y \cong R_R^n$ (recall that R_R^n denotes R^n, viewed as a right module over R). It is well known (and it can be easily proved) that $\mathrm{FP}(R)$ consists exactly of all finitely generated projective right R-modules. Denote by $[X]$ the isomorphism type of a member X of $\mathrm{FP}(R)$, and let $\mathrm{V}'(R) = \{\, [X] \mid X \in \mathrm{FP}(R) \,\}$.

(1) Prove that an addition can be defined on $\mathrm{V}'(R)$, by the rule

$$[X] + [Y] = [X \oplus Y]\,, \quad \text{for all } X, Y \in \mathrm{FP}(R)\,.$$

(2) For an idempotent $e \in \mathrm{M}_n(R)$, with n a positive integer, prove that the image space $e(R^n)$ of e is a member of $\mathrm{FP}(R)$, and that the isomorphism type of $e(R^n)$ depends only on the Murray–von Neumann equivalence class $[e]$ (within $\mathrm{M}_\infty(R)$).

(3) Prove that the assignment $[e] \mapsto [e(R^n)]$ introduced above defines a monoid isomorphism from $\mathrm{V}(R)$ onto $\mathrm{V}'(R)$, which sends $[1_R]$ to $[R_R]$.

8.27. (See Theorems 1.11 and 2.3 in Goodearl [117].) Let E be a finitely generated projective right module over a unital regular ring R. Prove that every finitely generated submodule of E belongs to $\mathrm{FP}(R)$, and that the set $\mathrm{Sub_c}\, E$ of all finitely generated submodules of E forms a sectionally complemented 0-sublattice of $\mathrm{Sub}\, E$, isomorphic to $\mathrm{L^r}\,(\mathrm{End}\, E)$. (*Hint*: apply Exercises 8.19 and 8.21.)

8.28. Let R be a unital regular ring. By using Exercise 8.26, we identify $\mathrm{V}(R)$ with the monoid of all isomorphism classes of all members of $\mathrm{FP}(R)$ (i.e., finitely generated projective right R-modules). Let $E \in \mathrm{FP}(R)$.

(1) Prove that $[X] + [Y] = [X \cap Y] + [X + Y]$, for all finitely generated submodules X and Y of E. (*Hint*: apply Exercise 8.27; let X' be a direct summand of $X \cap Y$ in X. Observe that $X + Y = X' \oplus Y$.)

(2) Deduce that $[X \cap Z] + [Y \cap Z] \leq [Z] + [X \cap Y]$, for all finitely generated submodules X, Y, and Z of E.

8.29. For an element x in a (not necessarily unital) ring R, we denote by $\langle x\rangle_R$ (resp., ${}_R\langle x\rangle$ the right ideal (resp., left ideal) of R generated by x. Prove that for all $x, a, b \in R$ with a and b both idempotent, any of the following assumptions implies that a and b are Murray–von Neumann equivalent:

(i) $\langle a\rangle_R = \langle b\rangle_R$;

(ii) ${}_R\langle a\rangle = {}_R\langle b\rangle$;

(iii) $\langle x\rangle_R = \langle a\rangle_R$ and ${}_R\langle x\rangle = {}_R\langle b\rangle$.

(*Hint*: in case (i), prove that $a = ba$ and $b = ab$. In case (iii), prove that $x = ax = xb$ and there are $y, z \in bRa$ such that $a = xy$ and $b = zx$; prove that $y = z$.)

8.30. Let R be a regular ring and set $\overline{R} = \mathrm{M}_\infty(R)$.

(1) For any $x \in \overline{R}$, prove that the value of $[e]$, for $e \in \overline{R}$ idempotent and either $x\overline{R} = e\overline{R}$ or $\overline{R}x = \overline{R}e$, is independent of e. (*Hint*: use Exercise 8.29.) Denote this value by $[x]$.

(2) Prove that if $z\overline{R} = x\overline{R} \oplus y\overline{R}$, then $[z] = [x] + [y]$, for all $x, y, z \in \overline{R}$.

(3) Prove that $[xy] \leq [x]$ and $[xy] \leq [y]$, for all $x, y \in \overline{R}$.

(4) Let $x, y, z \in \overline{R}$ such that $z\overline{R} = x\overline{R} + y\overline{R}$. Prove that $[z] \leq [x] + [y]$. (*Hint*: use a sectional complement of $x\overline{R} \cap y\overline{R}$ in $x\overline{R}$.)

(5) Prove that $[x + y] \leq [x] + [y]$, for all $x, y \in \overline{R}$.

8.31. (Wehrung [328, Example 3.3]) Let $\mathbb{F}$ be a field, set $R = \mathbb{F} \times \mathbb{F}$, $A = \mathrm{M}_2(\mathbb{F})$, and $B = \mathrm{M}_3(\mathbb{F})$. Define embeddings $f\colon R \hookrightarrow A$ and $g\colon R \hookrightarrow B$ of unital algebras by setting

$$f(x, y) = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} \quad \text{and} \quad g(x, y) = \begin{pmatrix} x & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & y \end{pmatrix},$$

for all $x, y \in \mathbb{F}$. Prove that there are no finite-dimensional $\mathbb{F}$-algebra C and no embeddings $f'\colon A \hookrightarrow C$ and $g'\colon B \hookrightarrow C$ such that $f' \circ f = g' \circ g$.

8.32. Prove that a $(\vee, 0)$-semilattice satisfies the refinement property iff it is distributive.

8.33. Prove that the maximal semilattice quotient (cf. Section 7-5.4) of a refinement monoid is a distributive $(\vee, 0)$-semilattice.

8.34. Following the terminology in Goodearl [116], an *ideal* of a partially ordered Abelian group G is a directed, order-convex, additive subgroup of G. Following the notation in Tůma and Wehrung [317], we denote by $\mathrm{Id}\, G$ the (algebraic) lattice of all ideals of G, any by $\mathrm{Id_c}\, G$ the $(\vee, 0)$-semilattice of all finitely generated ideals of G. Verify that $\mathrm{Id_c}$ defines a functor, from partially ordered Abelian groups with positive homomorphisms, to $(\vee, 0)$-semilattices with $(\vee, 0)$-homomorphisms.

Set $G(a) = \{\, x \in G \mid (\exists n \in \mathbb{N})(-na \leq x \leq na) \,\}$, for each $a \in G^+$. Prove that the assignment $a \mapsto G(a)$ induces an isomorphism from the maximal semilattice quotient of G^+ onto $\mathrm{Id_c}\, G$.

The notation $\mathrm{Id_c}\, G$*, introduced in Exercise* 8.34*, will be used in many of the remaining exercises for Chapter* 8.

8.35. (Unital version of Tůma and Wehrung [317, Lemma 5.6]) All our vector spaces will be over the field $\mathbb{Q}$ of all rational numbers. For any nonempty set X, we denote by $\mathbb{Q}_X$ the Abelian group $\mathbb{Q}^X$, endowed with the positive cone consisting of 0 together with all the vectors with positive components. A *pseudo-simplicial* partially ordered vector space is a finite direct sum of partially ordered vector spaces of the form $\mathbb{Q}_X$, for nonempty finite X. Prove the following statement:

> *For every finite Boolean semilattice S, for every pseudo-simplicial partially ordered vector space $(F, 1_F)$ with order-unit, and every $(\vee, 0, 1)$-homomorphism $\boldsymbol{f}\colon S \to \mathrm{Id_c}\, F$, there are a simplicial vector space $(E, 1_E)$ with order-unit, an isomorphism $\boldsymbol{\iota}\colon S \to \mathrm{Id_c}\, E$, and a positive homomorphism $f\colon E \to F$ that is* normalized *(i.e., $f(1_E) = 1_F$) and $\boldsymbol{f} = (\mathrm{Id_c}\, f) \circ \boldsymbol{\iota}$.*

(*Hint*: if $S = \mathbf{2}^m$, take $E = \mathbb{Q}^m$ with the canonical $\boldsymbol{\iota}$. Write $F = \bigoplus_{j<n} F_j$, for simple pseudo-simplicial F_j. Modify the proof of [317, Lemma 5.6], setting $f(e_i) = \sum_{j \in J_i} u_{i,j}$ for suitably chosen order-units $u_{i,j}$ of F_j.)

8.36. (Unital version of Tůma and Wehrung [317, Theorem 7.1]) Prove the following "reverse amalgamation" theorem:

> *For every countable distributive $(\vee, 0, 1)$-semilattice S, every countable partially ordered vector space $(H, 1_H)$ with order-unit, and every $(\vee, 0, 1)$-homomorphism $\boldsymbol{f}\colon S \to \mathrm{Id_c}\, H$, there are a countable partially ordered vector space $(G, 1_G)$ with order-unit, a normalized positive homomorphism $f\colon (G, 1_G) \to (H, 1_H)$, and an isomorphism $\boldsymbol{\alpha}\colon S \to \mathrm{Id_c}\, G$ such that $\boldsymbol{f} = (\mathrm{Id_c}\, f) \circ \boldsymbol{\alpha}$.*

(*Hint*: argue as in the proof of [317, Theorem 7.1], by replacing [317, Lemma 5.6] by Exercise 8.35. Specify the order-unit $1_{G_{i+1}} = s_i(1_{G_i})$. Fix the gap in the proof of [317, Theorem 7.1], where the authors forgot to ensure that $\boldsymbol{f}_{i+1} \circ \boldsymbol{s}_i = (\mathrm{Id_c}\, t_i) \circ \boldsymbol{f}_i$ for each i; this can be easily arranged, by first letting $\boldsymbol{f}_i\colon \mathbf{2}^{m_i} \to \mathrm{Id_c}\, H_n$ for large enough n, then change the indexing accordingly.)

8.37. (Goodearl [117, Lemma 15.23]; Goodearl and Handelman [120, Lemma 1.2]) Let $\mathbb{F}$ be a field and let A and R be unital $\mathbb{F}$-algebras, with A matricial.

(1) Prove that for every monoid homomorphism $\boldsymbol{f}\colon \mathrm{V}(A) \to \mathrm{V}(R)$ such that $\boldsymbol{f}([1_A]) = [1_R]$, there exists a homomorphism $f\colon A \to R$ of unital $\mathbb{F}$-algebras such that $\boldsymbol{f} = \mathrm{V}(f)$.

(2) Prove that for all homomorphisms $f, g\colon A \to R$ of unital $\mathbb{F}$-algebras, $\mathrm{V}(f) = \mathrm{V}(g)$ iff there exists an inner automorphism θ of R such that $g = \theta \circ f$.

8.38. (Elliott [76]; see also Goodearl and Handelman [120, Theorem 1.1], Goodearl [117, Theorem 15.24]) Let $\mathbb{F}$ be a field. Prove that every countable dimension group with order-unit is isomorphic to $(\mathrm{K}_0(A), [1_A])$, for some unital, countable-dimensional, locally matricial $\mathbb{F}$-algebra A. (*Hint*: by Theorem 8-2.2, our dimension group is the directed colimit of a sequence of simplicial groups. Use Exercise 8.37.)

8.39. (Kado [217, Lemma 3], Goodearl and Handelman [120, Lemma 1.3]) Let $\mathbb{F}$ be a field and let A and R be unital $\mathbb{F}$-algebras, with A locally matricial countable-dimensional. Prove that every homomorphism $\boldsymbol{f}\colon \mathrm{V}(A) \to \mathrm{V}(R)$ such that $\boldsymbol{f}([1_A]) = [1_R]$ can be written in the form $\mathrm{V}(f)$, for a homomorphism $f\colon A \to R$ of unital $\mathbb{F}$-algebras. (*Hint*: write A as a directed union of matricial algebras and construct f inductively, by using Exercise 8.37.)

8.40. Say that a locally matricial algebra A is *divisible* if $\mathrm{V}(A)$ is divisible (i.e., if $m\,\mathrm{V}(A) = \mathrm{V}(A)$ for every positive integer m). By using Exercises 8.36, 8.38, and 8.39, prove the following reverse amalgamation theorem for divisible locally matricial algebras:

> *For every countable distributive* $(\vee, 0, 1)$*-semilattice* S*, every unital, countable-dimensional, divisible, locally matricial* $\mathbb{F}$*-algebra* B*, and every* $(\vee, 0, 1)$*-homomorphism* $\boldsymbol{f}\colon S \to \mathrm{Id_c}\, B$*, there are a unital, countable-dimensional, divisible, locally matricial* $\mathbb{F}$*-algebra* A*, a homomorphism* $f\colon A \to B$ *of unital* $\mathbb{F}$*-algebras, and an isomorphism* $\boldsymbol{\alpha}\colon S \to \mathrm{Id_c}\, A$ *such that* $\boldsymbol{f} = (\mathrm{Id_c}\, f) \circ \boldsymbol{\alpha}$*.*

8.41. Deduce Theorem 8-4.20 from the result of Exercise 8.40. (*Hint*: reduce the problem to the unital case, by adjoining a new unit to the semilattices. Argue by induction on the cardinality of I, by proving that any partial lifting of $\vec{S}$, defined on an upper subset J of I, can be extended to a lifting of $\vec{S}$. We may assume that $I = J \cup \{i\}$, for some minimal element i of I. Observe that i has at most one upper cover in I.)

8.42. (Tůma and Wehrung [317, Example 9.1]) Define $\boldsymbol{t}\colon \mathsf{B}_2 \to \mathsf{B}_2$, $(x, y) \mapsto (x \vee y, y)$. Prove that there is no partially ordered Abelian group G, with G^+ satisfying the refinement property, and no idempotent endomorphism t of G such that $\mathrm{Id_c}\, t \cong \boldsymbol{t}$. (We are using the notation of Exercise 8.34.)

8.43. (1) Following the notation of Exercise 8.34, prove that every countable distributive $(\vee, 0)$-semilattice is isomorphic to $\mathrm{Id_c}\, G$, for some countable dimension group G. (*Hint*: use Theorem 7-4.6.)

(2) By using the result of Exercise 8.38 together with Corollary 8-4.15, deduce a new proof of Theorem 8-4.5.

8.44. (1) Prove the following two-dimensional amalgamation theorem for the functor K_0 on unital matricial algebras:

Let $\mathbb{F}$ be a field, let A_0, A_1, and A_2 be matricial $\mathbb{F}$-algebras, let $f_i\colon A_0 \to A_i$ be a homomorphism of unital $\mathbb{F}$-algebras, for $i \in \{1,2\}$. Let (G,u) be a simplicial dimension group with order-unit, and let

$$\boldsymbol{g}_i\colon (\mathrm{K}_0(A_i), [1_{A_i}]) \to (G,u)$$

be a normalized positive homomorphism, for $i \in \{1,2\}$, such that $\boldsymbol{g}_1 \circ \mathrm{K}_0(f_1) = \boldsymbol{g}_2 \circ \mathrm{K}_0(f_2)$. Prove that there are a matricial $\mathbb{F}$-algebra A, an isomorphism

$$\boldsymbol{e}\colon (\mathrm{K}_0(A), [1_A]) \to (G,u)\,,$$

and homomorphisms $g_i\colon A_i \to A$ of unital $\mathbb{F}$-algebras, for $i \in \{1,2\}$, such that $g_1 \circ f_1 = g_2 \circ f_2$ and $\boldsymbol{g}_i = \boldsymbol{e} \circ \mathrm{K}_0(g_i)$ for each $i \in \{1,2\}$.

(*Hint*: any matricial A such that $(\mathrm{K}_0(A), [1_A]) \cong (G,u)$ will do. Use Exercise 8.37.)

(2) Deduce the following result of Goodearl and Handelman [120]: *For every field $\mathbb{F}$, every dimension group (G,u) with order-unit, with* $\operatorname{card} G \leq \aleph_1$, *is isomorphic to* $(\mathrm{K}_0(A), [1_A])$, *for some locally matricial unital $\mathbb{F}$-algebra A.* (*Hint*: argue as in the proof of Theorem 7-5.13.)

Note: The cardinality $\aleph_1$ is optimal in Exercise 8.44(2). This follows from results in Wehrung [325], and will be further discussed in Chapter 9.

8-6. Problems

There are many fascinating open problems about the functor V (nonstable K-theory) on regular rings, discussed in the survey paper Ara [9]. The most challenging of them is undoubtedly the following one, stated by Goodearl, and which has been circulating since the 1990s.

Problem 8.1. Is every countable conical refinement monoid isomorphic to $\mathrm{V}(R)$, for some regular ring R?

Even the extension of Problem 8.1, to monoids with up to $\aleph_1$ elements, is also open.

The problems on ideal lattices of regular rings (resp., congruence lattices of sectionally complemented modular lattices) are often related. We shall now discuss a small sample of the latter.

An element p in a poset P is *doubly irreducible* if p has at most one lower cover and at most one upper cover. A finite poset P of cardinality n is *dismantable* if there is a chain $\varnothing = P_0 \subset P_1 \subset \cdots \subset P_n = P$ such that the unique element of $P_{i+1} \setminus P_i$ is doubly irreducible in P_{i+1} for each $i < n$.

An analogue of the following Problem 8.2, with respect to the maximal semilattice quotient functor, is stated in Tůma and Wehrung [317, Problem 2].

Problem 8.2. Can every diagram, of finite Boolean semilattices and $(\vee, 0)$-homomorphisms, indexed by a finite lattice P, be lifted, with respect to the $\mathrm{Con_c}$ functor, by a diagram of locally finite modular lattices and lattice homomorphisms? Same question in case P is a finite dismantable poset.

The extension of Problem 8.2 to diagrams indexed by arbitrary finite posets has a counterexample, given by the diagram $\mathcal{D}_{\bowtie}$ of Theorem 7-4.15. On the other hand, even in the particular case of diagrams indexed by finite dismantable lattices, we do not know the answer to Problem 8.2 either, although some positive evidence can be found in Tůma and Wehrung [317, Theorem 6.4]. Part of the importance of dismantability for index sets of diagrams comes from the fact that every finite 2-ladder is dismantable, see Exercises 7.34–7.36. For diagrams that are not indexed by posets, see Section 7-4.5, also Exercise 8.42.

Problem 8.3. Let $\vec{S}$ be a diagram, indexed by a finite tree, of countable distributive $(\vee, 0, 1)$-semilattices and $(\vee, 0, 1)$-homomorphisms. Does there exist a diagram $\vec{R}$ of locally matricial algebras such that $\mathrm{Id_c}\,\vec{R} \cong \vec{S}$?

The analogue of Problem 8.3 for dual trees instead of trees has a positive answer, see Theorem 8-4.20. Also, the problem obtained by replacing "locally matricial" by "regular" in Problem 8.3 has a positive answer, see Theorem 8-4.17.

Problem 8.4. Let L be a locally finite, modular lattice. Does there exist a locally matricial ring R such that $\mathrm{Con}\,L \cong \mathrm{Id}\,R$?

Problem 8.5. Let U be a unit-regular ring. Does there exist a locally matricial ring R such that $\mathrm{Id}\,U \cong \mathrm{Id}\,R$?

Problem 8.6. Let L be a sectionally complemented modular lattice. Does there exist a regular ring R such that $\mathrm{Con}\,L \cong \mathrm{Id}\,R$?

Chapter 9

Liftable and Unliftable Diagrams

by Friedrich Wehrung

9-1. Introduction

Due to the positive solution of the countable version of CLP (see, for instance, Theorem 7-3.19), first-order sentences are not sufficient to separate congruence semilattices of lattices from arbitrary distributive semilattices. For infinitary sentences, the situation changes radically, and non-representation results can be proved. The infinitary sentences involved in such approaches often look like infinite extensions of the refinement property (cf. (7-5.13)), thus they are grouped in so-called *uniform refinement properties*.

The first result of this type that we know of appears in Wehrung [325]. It deals mainly with K-theory of regular rings, as opposed to congruence representation theory of lattices. However, the maximal semilattice quotient construction (cf. Section 7-5.4) makes it possible to transfer results from one theory to the other.

9-2. Uniform refinement properties

We shall postpone the discussion of the results of [325] until Section 9-2.5, and we shall start right away with the useful concept of a *V-distance*.

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9-2.1 V-distances of fixed finite type

The idea of the following subsection originates in a combination of Dobbertin's V-measures (cf. Exercises 7.14–7.20) and Jónsson's proof of Whitman's Theorem introduced in Jónsson [211] (see also Theorems 406 and 408 in LTF). We follow the presentation used in Růžička, Tůma, and Wehrung [287].

Definition 9-2.1. Let S be a $(\vee, 0)$-semilattice and let X be a set. A map $\delta\colon X \times X \to S$ is an *S-valued distance* on X if the following statements hold:

(i) $\delta(x, x) = 0$ for each $x \in X$;

(ii) $\delta(x, y) = \delta(y, x)$ for all $x, y \in X$;

(iii) $\delta(x, z) \leq \delta(x, y) \vee \delta(y, z)$ for all $x, y, z \in X$.

The *kernel* of δ is the equivalence relation on X defined as

$$\{\, (x, y) \in X \times X \mid \delta(x, y) = 0 \,\}.$$

The *V-condition* on δ is the following condition:

> For all $x, y \in X$ and all $\boldsymbol{a}, \boldsymbol{b} \in S$ such that $\delta(x, y) \leq \boldsymbol{a} \vee \boldsymbol{b}$, there are a positive integer n and $x = z_0, z_1, \dots, z_{n+1} = y \in X$ such that for each $i \leq n$, $\delta(z_i, z_{i+1}) \leq \boldsymbol{a}$ in case i is even, while $\delta(z_i, z_{i+1}) \leq \boldsymbol{b}$ in case i is odd.

In case n is the same for all x, y, $\boldsymbol{a}$, $\boldsymbol{b}$, we say that the distance δ satisfies the *V-condition of type* n, or is a *V-distance of type* n.

We say that δ satisfies the *V-condition of type* $3/2$, or is a *V-distance of type* $3/2$, if for all $x, y \in X$ and all $\boldsymbol{a}, \boldsymbol{b} \in S$ such that $\delta(x, y) \leq \boldsymbol{a} \vee \boldsymbol{b}$, there exists $z \in X$ such that either ($\delta(x, z) \leq \boldsymbol{a}$ and $\delta(z, y) \leq \boldsymbol{b}$) or ($\delta(x, z) \leq \boldsymbol{b}$ and $\delta(z, y) \leq \boldsymbol{a}$).

A *morphism* from a distance $\lambda\colon X \times X \to A$ to a distance $\mu\colon Y \times Y \to B$ is a pair $(f, \boldsymbol{f})$, where $\boldsymbol{f}\colon A \to B$ is a $(\vee, 0)$-homomorphism and $f\colon X \to Y$ is a map such that $\boldsymbol{f}(\lambda(x, y)) = \mu(f(x), f(y))$, for all x, $y \in X$. The *forgetful functor* sends $\lambda\colon X \times X \to A$ to A and $(f, \boldsymbol{f})$ to $\boldsymbol{f}$.

Recall that $\operatorname{Equ} X$ denotes the lattice of all equivalence relations on a set X, and set

$$\boldsymbol{\alpha\beta} = \bigl\{(x, y) \in X \times X \mid (\exists z \in X)\bigl((x, z) \in \boldsymbol{\alpha} \text{ and } (z, y) \in \boldsymbol{\beta}\bigr)\bigr\}\,,$$

for all binary relations $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ on X. For a positive integer n, we say as usual that two binary relations $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ on X are $(n+1)$-*permutable* if $\boldsymbol{\gamma}_0\boldsymbol{\gamma}_1 \cdots \boldsymbol{\gamma}_n = \boldsymbol{\gamma}_1\boldsymbol{\gamma}_2 \cdots \boldsymbol{\gamma}_{n+1}$, where $\boldsymbol{\gamma}_k$ is defined as $\boldsymbol{\alpha}$ if k is even and as $\boldsymbol{\beta}$ if k is odd, for every natural number k. In particular, 2-permutable is the same as permutable. With every S-valued distance is associated a homomorphism from the ideal lattice of S to some $\operatorname{Equ} X$, as follows.

Proposition 9-2.2. *Let S be a $(\vee, 0)$-semilattice and let $\delta\colon X \times X \to S$ be an S-valued distance. Then one can define a map $\varphi\colon \operatorname{Id} S \to \operatorname{Equ} X$ by the rule*

$$\varphi(\boldsymbol{a}) = \{\, (x, y) \in X \times X \mid \delta(x, y) \in \boldsymbol{a} \,\}, \text{ for each } \boldsymbol{a} \in \operatorname{Id} S\,.$$

Furthermore,

(i) *The map φ preserves all meets and all directed joins.*

(ii) *The distance δ satisfies the V-condition iff φ is a join-homomorphism.*

(iii) *The map φ is an order-embedding iff the range of δ join-generates S (i.e., S is generated, as a $(\vee, 0)$-semilattice, by the range of δ).*

(iv) *The distance δ satisfies the V-condition of type n iff all equivalences in the range of φ are pairwise $(n + 1)$-permutable.*

Proposition 9-2.2 can be viewed as a particular case of duality for lattices (cf. Section 1-9 in Chapter 1). The proof of Proposition 9-2.2 is straightforward and left to the reader as an exercise.

Although we will not need that observation, it is worthwhile pointing out that conversely, any map $\varphi\colon \operatorname{Id} S \to \operatorname{Equ} X$ satisfying (i) (i.e., φ preserves all meets and all directed joins) gives rise to a unique distance $\delta\colon X \times X \to S$ such that $\delta(x, y) \in \boldsymbol{a}$ iff $(x, y) \in \varphi(\boldsymbol{a})$, for all $x, y \in X$ and all $\boldsymbol{a} \in \operatorname{Id} S$: namely, the assumptions on φ imply that for all $x, y \in X$, there is a least $a \in S$ such that $(x, y) \in \varphi(S \downarrow a)$; set $\delta(x, y) = a$.

Any algebra A gives rise to a natural $(\operatorname{Con_c} A)$-valued distance, namely the map con_A that to each $(x, y) \in A \times A$ associates the principal congruence $\operatorname{con}_A(x, y)$ of A generated by (x, y). This assignment defines a functor (cf. Exercise 9.1).

Proposition 9-2.3. *Let n be a positive integer and let A be a congruence $(n + 1)$-permutable algebra (i.e., any two congruences of A are $(n + 1)$-permutable). Then the semilattice $\operatorname{Con_c} A$ of all compact congruences of A is join-generated by the range of a V-distance of type n.*

Proof. Let $\delta\colon A \times A \to \operatorname{Con_c} A$ be defined by $\delta(x, y) = \operatorname{con}_A(x, y)$, the principal congruence generated by (x, y), for all $x, y \in A$. The assumption that A is congruence $(n + 1)$-permutable means exactly that δ is a V-distance of type n. $\square$

In particular, A is congruence-permutable iff the canonical distance con_A satisfies the V-condition of type 1.

We say that A is *almost congruence-permutable* if $\boldsymbol{\alpha} \vee \boldsymbol{\beta} = (\boldsymbol{\alpha}\boldsymbol{\beta}) \cup (\boldsymbol{\beta}\boldsymbol{\alpha})$ for all congruences $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ of A. Of course, A is almost congruence-permutable iff the canonical distance con_A satisfies the V-condition of type $3/2$.

Say that a variety $\mathbf{V}$ of algebras is congruence-permutable if every member of $\mathbf{V}$ is congruence-permutable. We show a few examples of congruence-permutable varieties.

Example 9-2.4. The variety of all right premodules over a given ring R is congruence-permutable. The congruence lattice of a right premodule M is canonically isomorphic to the lattice $\operatorname{Sub} M$ of all sub-premodules of M.

Example 9-2.5. The variety of all groups is congruence-permutable. The congruence lattice of a group G is canonically isomorphic to the normal subgroup lattice $\operatorname{NSub} G$ of G. We shall denote by $\operatorname{NSub_c} G$ the $(\vee, 0)$-semilattice of all finitely generated normal subgroups of G.

Example 9-2.6. The variety of all ℓ-groups (i.e., lattice-ordered groups, see Anderson and Feil [7], or Bigard, Keimel, and Wolfenstein [25]) is congruence-permutable. The congruence lattice of an ℓ-group G is canonically isomorphic to the lattice $\operatorname{Id}^{\ell} G$ of all convex normal subgroups, or *ℓ-ideals*, of G. We shall denote by $\operatorname{Id}^{\ell}_{\mathrm{c}} G$ the $(\vee, 0)$-semilattice of all finitely generated ℓ-ideals of G.

Hence we obtain immediately the following result.

Corollary 9-2.7.

(i) *Let M be a right premodule over a ring R. Then $\operatorname{Sub_c} M$ is join-generated by the range of a V-distance of type 1 on M.*

(ii) *Let G be a group. Then $\operatorname{NSub_c} G$ is join-generated by the range of a V-distance of type 1 on G.*

(iii) *Let G be an ℓ-group. Then $\operatorname{Id}^{\ell}_{\mathrm{c}} G$ is join-generated by the range of a V-distance of type 1 on G.*

The V-distances corresponding to (i), (ii), and (iii) above are, respectively, given by $\delta(x,y) = \langle x - y\rangle_R$ (cf. (8-2.7)), $\delta(x,y) = \langle xy^{-1}\rangle$ (the normal subgroup of G generated by xy^{-1}), and $\delta(x,y) = G(xy^{-1})$ (the ℓ-ideal of G generated by xy^{-1}).

We are interested in $(\vee, 0)$-semilattices representable as congruence semilattices of congruence $(n+1)$-permutable algebras, for fixed n. For $n \geq 2$, a complete answer is given in Grätzer and Schmidt [165] (see also Grätzer and Lampe [157]).

$\diamondsuit$ **Theorem 9-2.8** (Grätzer and Schmidt 1963)**.** *Every $(\vee, 0)$-semilattice S is isomorphic to the congruence semilattice of some congruence 4-permutable algebra A. Furthermore, A can be taken congruence 3-permutable iff the ideal lattice of S is modular.*

The easy direction of the modular side of Theorem 9-2.8 follows from the well-known fact, dating back to Jónsson [211] (see also [LTF, Theorem 408]) that every lattice of pairwise 3-permutable equivalence relations is modular. Hence, by Proposition 9-2.3, we get the following.

Corollary 9-2.9. *Every* $(\vee, 0)$*-semilattice* S *is generated by the range of some V-distance* δ *of type* 3*. Furthermore,* δ *can be taken of type* 2 *iff the ideal lattice of* S *is modular.*

It has been known since Jónsson [211] that modularity is not sufficient to ensure a representation by V-distances of type 1. In the forthcoming subsections, we shall show that even distributivity is not sufficient.

9-2.2 Four uniform refinement properties

In this subsection we shall introduce a few infinitary statements, denoted by the symbol URP_N, for $N \in \{0, 1, 2, 3\}$, and prove that a large array of $(\vee, 0)$-semilattices satisfy those statements. These are not the only possibilities for defining "uniform refinement properties" (see, for example, the property $\mathrm{URP}_{\mathrm{sr}}$ defined in Section 7-5.4, also Exercise 9.7), so we shall regroup such statements informally under the denomination "URP-like statements".

The following URP-like statements are variants of the original "Uniform refinement property" introduced in Wehrung [325] and stated in Wehrung [327, Definition 2.1]. That property implies the property denoted by WURP in Ploščica, Tůma, and Wehrung [270, Definition 1.2], itself equivalent to the statement URP_1 introduced in Definition 9-2.11. Further, URP_3 of Definition 9-2.11 is equivalent to the property $\mathrm{WURP}^=$ introduced in Růžička, Tůma, and Wehrung [287, Definition 2.1].

The strongest URP-like statement of our list, namely URP_0, was introduced in Tůma and Wehrung [316, Definition 2.9], and denoted there by URP^+.

Definition 9-2.10. Let S be a $(\vee, 0)$-semilattice. A pair $\bigl(e, ((a_i, b_i) \mid i \in I)\bigr)$, where e and all the a_i, b_i belong to S, is

- A *constant sum family, centered at* e, if $e = a_i \vee b_i$ for each $i \in I$;
- A *constant subsum family, centered at* e, if $e \leq a_i \vee b_i$ for each $i \in I$.

A family $\gamma = (c_{i,j} \mid (i, j) \in I \times I)$ is a *refinement of type* N, for $N \in \{0, 1, 2, 3\}$, of $\bigl(e, ((a_i, b_i) \mid i \in I)\bigr)$, if one of the following statements holds.

- $N = 0$ and the following conditions hold:
  - (i) $c_{i,j} \leq a_i$ and $c_{i,j} \leq b_j$, for all $i, j \in I$;
  - (ii) $a_i \leq a_j \vee c_{i,j}$ and $b_j \leq b_i \vee c_{i,j}$, for all $i, j \in I$;
  - (iii) $c_{i,k} \leq c_{i,j} \vee c_{j,k}$, for all $i, j, k \in I$.
- $N = 1$ and the following conditions hold:
  - (i) $c_{i,j} \leq a_i$ and $c_{i,j} \leq b_j$, for all $i, j \in I$;
  - (ii) $e \leq a_j \vee b_i \vee c_{i,j}$, for all $i, j \in I$;
  - (iii) $c_{i,k} \leq c_{i,j} \vee c_{j,k}$, for all $i, j, k \in I$.

- $N = 2$ and the following conditions hold:

  (i) $c_{i,j} \leq a_i \vee a_j$ and $c_{i,j} \leq b_i \vee b_j$, for all $i, j \in I$;

  (ii) $e \leq a_j \vee b_i \vee c_{i,j}$, for all $i, j \in I$;

  (iii) $c_{i,k} \leq c_{i,j} \vee c_{j,k}$, for all $i, j, k \in I$.

- $N = 3$ and there exists a finite partition $\mathcal{U}$ of I such that the following conditions hold:

  (i) $c_{i,j} \leq a_i \vee a_j$ and $c_{i,j} \leq b_i \vee b_j$, for all $i, j \in I$ belonging to the same member of $\mathcal{U}$;

  (ii) $e \leq a_j \vee b_i \vee c_{i,j}$, for all $i, j \in I$ belonging to the same member of $\mathcal{U}$;

  (iii) $c_{i,k} \leq c_{i,j} \vee c_{j,k}$, for all $i, j, k \in I$.

Definition 9-2.11. Let e be an element in a $(\vee, 0)$-semilattice S.

(1) We say that S satisfies $\mathrm{URP}_0(e)$ if every constant sum family of S, centered at e, admits a refinement of type 0.

(2) We say that S satisfies $\mathrm{URP}_N(e)$, for $N \in \{1, 2, 3\}$, if every constant subsum family of S, centered at e, admits a refinement of type N.

Further, we say that S satisfies URP_N if it satisfies $\mathrm{URP}_N(e)$ at each $e \in S$.

If S is distributive, then to every constant subsum family

$$\gamma = \big(e, ((a_i, b_i) \mid i \in I)\big)$$

one can associate a constant sum family

$$\gamma' = \big(e, ((a'_i, b'_i) \mid i \in I)\big)\,,$$

with $a'_i \leq a_i$ and $b'_i \leq b_i$ for each $i \in I$, and every refinement of γ' is obviously a refinement of γ. Hence, for $N \in \{1, 2, 3\}$, it is sufficient to verify a given URP_N for constant sum families.

It is trivial that every refinement of type N is also a refinement of type N', whenever $1 \leq N \leq N' \leq 3$. This assumption extends to $0 \leq N \leq N' \leq 3$ in case S is distributive. Hence, if $N = 0$ implies S distributive and if $0 \leq N \leq N' \leq 3$, then $\mathrm{URP}_N(e)$ implies $\mathrm{URP}_{N'}(e)$.

Lemma 9-2.12. *Let e be an element in a distributive $(\vee, 0)$-semilattice S. If $S \downarrow e$ is a lattice, then S satisfies $\mathrm{URP}_0(e)$ (thus, as S is distributive, also $\mathrm{URP}_N(e)$ for each $N \in \{1, 2, 3\}$).*

Proof. Let $e = a_i \vee b_i$ for each $i \in I$. Observe that $a_i \leq e$ and $b_i \leq e$ for each $i \in I$. Set $c_{i,j} = a_i \wedge b_j$ for all $i, j \in I$. It is obvious that $c_{i,j} \leq a_i$ and $c_{i,j} \leq b_j$. Furthermore, arguing in the distributive lattice $S \downarrow e$ and observing that $e = a_j \vee b_j$, we get

$$a_j \vee c_{i,j} = (a_j \vee a_i) \wedge (a_j \vee b_j) = a_j \vee a_i \geq a_i \,.$$

Likewise, we can prove that $b_i \vee c_{i,j} \geq b_j$. Finally, for all $i, j, k \in I$,

$$\begin{aligned} c_{i,j} \vee c_{j,k} &= (a_i \wedge b_j) \vee (a_j \wedge b_k) \\ &= (a_i \vee a_j) \wedge (a_i \vee b_k) \wedge (b_j \vee a_j) \wedge (b_j \vee b_k) \\ &= (a_i \vee a_j) \wedge (a_i \vee b_k) \wedge (b_j \vee b_k) && \text{(because } b_j \vee a_j = e\text{)} \\ &\geq a_i \wedge b_k \\ &= c_{i,k} \,, \end{aligned}$$

which completes the proof. □

Lemma 9-2.13. *Let e_0 and e_1 be elements in a $(\vee, 0)$-semilattice S and let $N \in \{0, 1, 2, 3\}$. If S satisfies both* $\mathrm{URP}_N(e_0)$ *and* $\mathrm{URP}_N(e_1)$*, and if S is distributive in case $N = 0$, then S satisfies* $\mathrm{URP}_N(e_0 \vee e_1)$.

Proof. Set $e = e_0 \vee e_1 \leq a_i \vee b_i$ in S, for each $i \in I$, with S distributive and all $a_i \vee b_i = e$ in case $N = 0$.

Suppose first that $N \neq 0$. Since S satisfies both $\mathrm{URP}_N(e_0)$ and $\mathrm{URP}_N(e_1)$, the family

$$\gamma_l = \big(e_l, ((a_i, b_i) \mid i \in I)\big)$$

has a refinement $(c_{i,j,l} \mid i, j \in I)$ of type N, for each $l < 2$. Set $c_{i,j} = c_{i,j,0} \vee c_{i,j,1}$, for all $i, j \in I$. It is not hard to verify that $(c_{i,j} \mid i, j \in I)$ is a refinement of type N of $\big(e, ((a_i, b_i) \mid i \in I)\big)$. For the verification of points (i) and (ii) in case $N = 3$, we let $\mathcal{U}_l$ be the finite partition of I going together with $(c_{i,j,l} \mid i, j \in I)$, and we define a new finite partition $\mathcal{U}$ of I as the set of all nonempty intersections of a member of $\mathcal{U}_0$ and a member of $\mathcal{U}_1$.

Suppose, next, that $N = 0$. For $i \in I$, it follows from the distributivity of S (cf. Exercise 8.32) together with the equality $e_0 \vee e_1 = a_i \vee b_i$ that there are elements $a_{i,0}, a_{i,1}, b_{i,0}, b_{i,1} \in S$ such that $e_l = a_{i,l} \vee b_{i,l}$ while $a_i = a_{i,0} \vee a_{i,1}$ and $b_i = b_{i,0} \vee b_{i,1}$. Since S satisfies both $\mathrm{URP}_0(e_0)$ and $\mathrm{URP}_0(e_1)$, the family

$$\gamma_l = \big(e_l, ((a_{i,l}, b_{i,l}) \mid i \in I)\big)$$

has a refinement $(c_{i,j,l} \mid i, j \in I)$ of type 0, for each $l < 2$. Set $c_{i,j} = c_{i,j,0} \vee c_{i,j,1}$, for all $i, j \in I$. It is not hard to verify that $(c_{i,j} \mid i, j \in I)$ is a refinement of type 0 of $\big(e, ((a_i, b_i) \mid i \in I)\big)$. □

The following extension of Lemma 9-2.12 was established in Propositions 2.10 and 2.11 of Tůma and Wehrung [316].

$\diamondsuit$ **Theorem 9-2.14** (Tůma and Wehrung 2002)**.**

(i) *Let L be a relatively complemented lattice. Then* $\mathrm{Con_c}\, L$ *satisfies* URP_0*.*

(ii) *Every directed colimit, indexed by a chain, of distributive lattices with zero and* $(\vee, 0)$*-homomorphisms, satisfies* URP_0*.*

Lemma 9-2.15. *Let S and T be $(\vee, 0)$-semilattices, let $\mu\colon S \to T$ be a $(\vee, 0)$-homomorphism, let $e \in S$, and let $N \in \{1, 2, 3\}$. If μ is weakly distributive at e and if S satisfies $\mathrm{URP}_N(e)$, then T satisfies $\mathrm{URP}_N(\mu(e))$.*

Proof. Starting with a constant subsum family $\delta = \big(\mu(e), ((a_i, b_i) \mid i \in I)\big)$ in T, it follows from the weak distributivity of μ at e that there exists a constant subsum family of the form $\gamma = \big(e, ((x_i, y_i) \mid i \in I)\big)$ in S, with $\mu(x_i) \leq a_i$ and $\mu(y_i) \leq b_i$ for each $i \in I$. Then the image under μ of any refinement of type N of γ is a refinement of type N of δ. $\square$

By combining Lemmas 9-2.12 and 9-2.15, we get immediately the following.

Theorem 9-2.16. *Every weakly distributive image of a distributive lattice with zero satisfies* URP_1*.*

The following result is a blend between pieces of various references such as Wehrung [327], Ploščica, Tůma, and Wehrung [270], Růžička, Tůma, and Wehrung [287].

Theorem 9-2.17.

(1) *Let L be a congruence-permutable lattice. Then* $\mathrm{Con_c}\, L$ *satisfies* URP_1*.*

(2) *Every distributive* $(\vee, 0)$*-semilattice join-generated by the range of a V-distance of type* 1 *satisfies* URP_2*.*

(3) *Every distributive* $(\vee, 0)$*-semilattice join-generated by the range of a V-distance of type* $3/2$ *satisfies* URP_3*.*

Proof. (1) It suffices, by Lemma 9-2.13, to prove that $\mathrm{Con_c}\, L$ satisfies $\mathrm{URP}_1(\boldsymbol{e})$ for any principal congruence $\boldsymbol{e}$ of L. Let $\boldsymbol{e} = \mathrm{con}(u, v)$, where $u \leq v$ in L, and let $\boldsymbol{e} \subseteq \boldsymbol{a}_i \vee \boldsymbol{b}_i$ for each $i \in I$. Since L is congruence-permutable, for each $i \in I$ there exists $x_i \in L$ such that $u \equiv_{\boldsymbol{a}_i} x_i$ and $x_i \equiv_{\boldsymbol{b}_i} v$. Replacing x_i by $(u \vee x_i) \wedge v$, we may assume that $u \leq x_i \leq v$. Set $\boldsymbol{c}_{i,j} = \mathrm{con}^+(x_i, x_j) = \mathrm{con}(x_i \wedge x_j, x_i)$. Then $\boldsymbol{c}_{i,j} \subseteq \mathrm{con}(u, x_i) \subseteq \boldsymbol{a}_i$ and $\boldsymbol{c}_{i,j} \subseteq \mathrm{con}(x_j, v) \subseteq \boldsymbol{b}_j$. Furthermore,

$$\boldsymbol{a}_j \vee \boldsymbol{b}_i \vee \boldsymbol{c}_{i,j} \supseteq \mathrm{con}(u, x_j) \vee \mathrm{con}(x_i, v) \vee \mathrm{con}^+(x_i, x_j) \supseteq \mathrm{con}^+(v, u) = \boldsymbol{e}\,.$$

Finally, the containment $\boldsymbol{c}_{i,k} \subseteq \boldsymbol{c}_{i,j} \vee \boldsymbol{c}_{j,k}$ is obvious.

(2) Let S be a $(\vee, 0)$-semilattice and let $\delta\colon X \times X \to S$ be a V-distance of type 1 with range join-generating S, we must prove that S satisfies $\mathrm{URP}_2(\boldsymbol{e})$, for each $\boldsymbol{e} \in S$. Again, it suffices, by Lemma 9-2.13, to deal with the case

where $\boldsymbol{e} = \delta(x, y)$, where $x, y \in X$. Let $((\boldsymbol{a}_i, \boldsymbol{b}_i) \mid i \in I)$ such that $\boldsymbol{e} \leq \boldsymbol{a}_i \vee \boldsymbol{b}_i$ for each $i \in I$. For all $i \in I$, from $\delta(x, y) \leq \boldsymbol{a}_i \vee \boldsymbol{b}_i$ and the assumption on δ it follows that there exists $z_i \in X$ such that

$$\delta(x, z_i) \leq \boldsymbol{a}_i \text{ and } \delta(z_i, y) \leq \boldsymbol{b}_i \,. \tag{9-2.1}$$

Now we set $\boldsymbol{c}_{i,j} = \delta(z_i, z_j)$, for all $i, j \in I$, and we prove that the family $(\boldsymbol{c}_{i,j} \mid (i, j) \in I \times I)$ satisfies the required conditions.

The inequality $\boldsymbol{c}_{i,k} \leq \boldsymbol{c}_{i,j} \vee \boldsymbol{c}_{j,k}$ holds trivially, for all $i, j, k \in I$. Further,

$$\delta(z_i, z_j) \leq \delta(z_i, x) \vee \delta(x, z_j) \leq \boldsymbol{a}_i \vee \boldsymbol{a}_j \,,$$

whence $\boldsymbol{c}_{i,j} \leq \boldsymbol{a}_i \vee \boldsymbol{a}_j$. Replacing x by y in the argument above, we obtain the inequality $\boldsymbol{c}_{i,j} \leq \boldsymbol{b}_i \vee \boldsymbol{b}_j$. Finally,

$$\boldsymbol{a}_j \vee \boldsymbol{b}_i \vee \boldsymbol{c}_{i,j} \geq \delta(x, z_j) \vee \delta(z_i, y) \vee \delta(z_i, z_j) \geq \delta(x, y) = \boldsymbol{e} \,. \tag{9-2.2}$$

(3) Let again $\boldsymbol{e} = \delta(x, y) \leq \boldsymbol{a}_i \vee \boldsymbol{b}_i$ for each $i \in I$. For all $i \in I$, it follows from the assumption on δ that there exists $z_i \in X$ such that

$$\begin{aligned} &\text{either } \delta(x, z_i) \leq \boldsymbol{a}_i \text{ and } \delta(z_i, y) \leq \boldsymbol{b}_i \\ &\quad\text{or } \delta(x, z_i) \leq \boldsymbol{b}_i \text{ and } \delta(z_i, y) \leq \boldsymbol{a}_i \,. \end{aligned} \tag{9-2.3}$$

Set $J = \{\, i \in I \mid \delta(x, z_i) \leq \boldsymbol{a}_i \text{ and } \delta(z_i, y) \leq \boldsymbol{b}_i \,\}$, and set again

$$\boldsymbol{c}_{i,j} = \delta(z_i, z_j) \,, \quad \text{for all } i, j \in I \,.$$

We shall prove that the family $(\boldsymbol{c}_{i,j} \mid (i, j) \in I \times I)$ satisfies the required conditions, with respect to the partition $\{J, I \setminus J\} \setminus \{\varnothing\}$ of I.

Let $i, j, k \in I$. The inequality $\boldsymbol{c}_{i,k} \leq \boldsymbol{c}_{i,j} \vee \boldsymbol{c}_{j,k}$ holds trivially.

If $i, j \in J$, then we compute

$$\begin{aligned} \delta(z_i, z_j) &\leq \delta(z_i, x) \vee \delta(x, z_j) \leq \boldsymbol{a}_i \vee \boldsymbol{a}_j \,, \\ \delta(z_i, z_j) &\leq \delta(z_i, y) \vee \delta(y, z_j) \leq \boldsymbol{b}_i \vee \boldsymbol{b}_j \,, \end{aligned}$$

so $\boldsymbol{c}_{i,j} \leq \boldsymbol{a}_i \vee \boldsymbol{a}_j$ and $\boldsymbol{c}_{i,j} \leq \boldsymbol{b}_i \vee \boldsymbol{b}_j$. Furthermore, we prove as in (9-2.2) that $\boldsymbol{a}_j \vee \boldsymbol{b}_i \vee \boldsymbol{c}_{i,j} \geq \boldsymbol{e}$.

If $i, j \in I \setminus J$, then, by (9-2.3), we get

$$\begin{aligned} \delta(x, z_i) \leq \boldsymbol{b}_i \text{ and } \delta(z_i, y) \leq \boldsymbol{a}_i \,, \\ \delta(x, z_j) \leq \boldsymbol{b}_j \text{ and } \delta(z_j, y) \leq \boldsymbol{a}_j \,, \end{aligned}$$

whence

$$\begin{aligned} \delta(z_i, z_j) &\leq \delta(z_i, x) \vee \delta(x, z_j) \leq \boldsymbol{b}_i \vee \boldsymbol{b}_j \,, \\ \delta(z_i, z_j) &\leq \delta(z_i, y) \vee \delta(y, z_j) \leq \boldsymbol{a}_i \vee \boldsymbol{a}_j \,, \end{aligned}$$

so, again, $\boldsymbol{c}_{i,j} \leq \boldsymbol{a}_i \vee \boldsymbol{a}_j$ and $\boldsymbol{c}_{i,j} \leq \boldsymbol{b}_i \vee \boldsymbol{b}_j$. Finally,

$$\boldsymbol{a}_j \vee \boldsymbol{b}_i \vee \boldsymbol{c}_{i,j} \geq \delta(z_j, y) \vee \delta(x, z_i) \vee \delta(z_i, z_j) \geq \delta(x, y) = \boldsymbol{e} \,,$$

so we are done. □

Note: The elements $\boldsymbol{c}_{i,j}$ constructed in the proofs of (2) and (3) of Theorem 9-2.17 satisfy, in addition, $\boldsymbol{c}_{i,j} = \boldsymbol{c}_{j,i}$, for all $i, j \in I$. Therefore, $\boldsymbol{e} \leq \boldsymbol{a}_i \vee \boldsymbol{b}_j \vee \boldsymbol{c}_{i,j}$.

By applying Proposition 9-2.3, we get immediately the following consequence of Theorem 9-2.17, originating in Růžička, Tůma, and Wehrung [287].

Corollary 9-2.18. *If an algebra A is congruence-permutable (resp. almost congruence-permutable), then* $\operatorname{Con_c} A$ *satisfies* URP_2 *(resp.,* URP_3*).*

9-2.3 Semilattices failing URP-like statements

In many cases, it turns out to be surprisingly hard to find counterexamples to URP-like statements. A typical illustration of this phenomenon is given by the following result.

Proposition 9-2.19. *Every distributive $(\vee, 0)$-semilattice S with at most $\aleph_1$ elements satisfies* URP_0.

Proof. By Theorem 7-5.13, S is isomorphic to $\operatorname{Con_c} L$, for some relatively complemented lattice L. By Theorem 9-2.14, $\operatorname{Con_c} L$ satisfies URP_0. □

The first example of a distributive $(\vee, 0)$-semilattice without URP_1 appears in Wehrung [325, 327]. This example has cardinality $\aleph_2$, and it is defined as the maximal semilattice quotient of the positive cone of a dimension vector space E (whose construction is outlined in Section 9-2.5).

Denote by $\mathrm{Free}_{\mathbf{V}}(X)$ the free object, on the set X, in a variety $\mathbf{V}$. Ploščica, Tůma, and Wehrung [270] established that $\operatorname{Con_c} \mathrm{Free}_{\mathbf{V}}(\aleph_2)$ does not satisfy URP_1, for any nondistributive variety $\mathbf{V}$ of lattices (resp., bounded lattices). This result was slightly strengthened in Růžička, Tůma, and Wehrung [287, Corollary 3.7], as follows.

♢ **Theorem 9-2.20.** *Let L be any lattice that admits a lattice homomorphism onto a free bounded lattice in the variety generated by either* M_3 *or* N_5 *with $\aleph_2$ generators. Then* $\operatorname{Con_c} L$ *does not satisfy* URP_3. *In particular,*

(1) $\operatorname{Con_c} L$ *is not the weakly distributive image of any distributive lattice with zero.*

(2) *There exists no V-distance of type* $3/2$ *with range join-generating* $\operatorname{Con_c} L$. *Hence there is no algebra A with almost permutable congruences such that* $\operatorname{Con} L \cong \operatorname{Con} A$.

The additional items (1) and (2) of Theorem 9-2.20 follow from the main statement, of failure of URP_3, *via* Theorem 9-2.16 and Corollary 9-2.18. One surprising aspect of Theorem 9-2.20 is that failure of representability of a distributive $(\vee, 0)$-semilattice *via* Schmidt's condition (being a weakly distributive image of a generalized Boolean semilattice) may occur for a lattice that we know is already representable (as the congruence semilattice of some lattice).

We shall now briefly discuss the infinite combinatorial facts involved in the proof of Theorem 9-2.20.

Notation 9-2.21. Every cardinal number is identified with the set of all ordinals below it. We set

$$[\kappa]^{\lambda} = \{\, x \subseteq \kappa \mid \operatorname{card} x = \lambda \,\}\,,$$
$$[\kappa]^{<\lambda} = \{\, x \subseteq \kappa \mid \operatorname{card} x < \lambda \,\}\,,$$

for any cardinal numbers κ and λ.

For a subset D of the powerset $\operatorname{Pow}(\kappa)$ of κ and a map $F\colon D \to [\kappa]^{<\lambda}$, we say that a subset H of κ is *free with respect to* F if $F(X) \cap H \subseteq X$ for each $X \in \operatorname{Pow}(H) \cap D$. For cardinals κ, λ, ϱ and a positive integer r, the statement $(\kappa, r, \lambda) \to \varrho$ holds if every map $F\colon [\kappa]^r \to [\kappa]^{<\lambda}$ (we say a *set mapping of order* r) has a ϱ-element free set. We extend this notation to $r = \omega$ and finite subsets of κ: hence $(\kappa, {<}\omega, \lambda) \to \varrho$ means that every map $F\colon [\kappa]^{<\omega} \to [\kappa]^{<\lambda}$ has a ϱ-element free set.

Denote by λ^{+n} the n-th successor of an infinite cardinal λ, and $\lambda^{+} = \lambda^{+1}$, $\lambda^{++} = \lambda^{+2}$.

The infinite combinatorial hard core of Theorem 9-2.20, and also for a large number of failure results of URP-like statements, is the following well-known result, established in 1951 in Kuratowski [233], see also Erdős *et al.* [78, Theorem 46.1].

$\diamondsuit$ **Theorem 9-2.22** (Kuratowski's Free Set Theorem)**.** *The arrow relation* $(\kappa, n, \lambda) \to n+1$ *holds iff* $\kappa \geq \lambda^{+n}$*, for all infinite cardinals* κ *and* λ *and every nonnegative integer* n*.*

Actually, the proofs of most failure results of URP-like statements (including Theorem 9-2.20) require only the following fragment of Theorem 9-2.22.

Corollary 9-2.23. *Let* X *be a set of cardinality at least* $\aleph_2$*. Then every set mapping* $F\colon [X]^2 \to [X]^{<\omega}$ *has a free set with three elements; that is, there are distinct* $x_0, x_1, x_2 \in X$ *such that* $x_k \notin F(\{x_i, x_j\})$ *for all distinct* $i, j, k < 3$*.*

For further discussion about Theorem 9-2.22, see Erdős *et al.* [78], Gillibert and Wehrung [113].

9-2.4 Congruence lattices of congruence-permutable algebras

As an immediate consequence of Theorem 9-2.20 and Corollary 9-2.7, we can state the following result, first established in Růžička, Tůma, and Wehrung [287, Corollary 3.9].

◇ **Theorem 9-2.24.** *Let* $\mathbf{V}$ *be a nondistributive variety of lattices and let* F *be any free* (*resp., free bounded*) *lattice on at least* $\aleph_2$ *generators in* $\mathbf{V}$*. There is no module* M (*resp., no group* G*, no* ℓ*-group* G) *such that* $\operatorname{Sub} M \cong \operatorname{Con} F$ (*resp.,* $\operatorname{NSub} G \cong \operatorname{Con} F$, $\operatorname{Id}^{\ell} G \cong \operatorname{Con} F$).

On the positive side, the following result is established in Theorems 4.1, 5.3, and 6.3 of Růžička, Tůma, and Wehrung [287].

◇ **Theorem 9-2.25** (Růžička, Tůma, and Wehrung 2007)**.** *The following statements hold, for any distributive* $(\vee, 0)$*-semilattice* S*.*

(i) *If* $\operatorname{card} S \leq \aleph_1$*, then there are a locally finite group* G *and a right module* M (*over some ring*) *such that* $S \cong \operatorname{NSub}_{\mathrm{c}} G$ (*resp.,* $S \cong \operatorname{Sub}_{\mathrm{c}} M$).

(ii) *If* S *is countable, then there exists an* ℓ*-group* G *such that* $S \cong \operatorname{Id}^{\ell}_{\mathrm{c}} G$*.*

This shows that in the case of groups and modules, the cardinality $\aleph_2$ is optimal in Theorem 9-2.24. In the case of ℓ-groups, it is still unknown whether every algebraic distributive lattice with $\aleph_1$ compact elements is isomorphic to the lattice of all ℓ-ideals of some ℓ-group (cf. Růžička, Tůma, and Wehrung [287, Problem 1]).

9-2.5 A uniform refinement property for $\mathbf{K_0}$ of a regular ring

Recall from Section 8-4.2 that the Grothendieck group $\mathrm{K}_0(R)$ of $\mathrm{V}(R)$ is a partially preordered Abelian group, for any unital ring R. Denoting by $\mathrm{FP}(R)$ the category of all finitely generated projective right R-modules and by viewing $\mathrm{V}(R)$ as the monoid of isomorphism classes of all members of $\mathrm{FP}(R)$ (cf. Exercise 8.26), it follows that the positive cone of $\mathrm{K}_0(R)$ consists of the *stable isomorphism classes*

$$[X]_{\mathrm{s}} = \{\, Y \in \mathrm{FP}(R) \mid (\exists Z \in \mathrm{FP}(R))(X \oplus Z \cong Y \oplus Z) \,\}\,, \quad \text{for } X \in \mathrm{FP}(R)\,.$$

After Wehrung [324, Section 2], for elements a and b in a commutative monoid with order-unit (M, e) (endowed with its algebraic preordering) and a positive integer m, let $a \ll_m b$ hold if there exists $n \in \mathbb{N}$ such that $mna \leq nb$. The following result introduces a “uniform refinement property” satisfied by $\mathrm{K}_0(R)$ for any unital regular ring R (with the canonical order-unit $[R]_{\mathrm{s}}$). It is contained in the proof of Wehrung [325, Corollary 2.12].

Theorem 9-2.26. *Let* R *be a unital regular ring, let* $\boldsymbol{e} \in \mathrm{K}_0(R)^+$*, and let* $(\boldsymbol{a}_i \mid i \in I)$ *be a family of elements of* $\mathrm{K}_0(R)^+$ *such that* $\boldsymbol{a}_i \leq \boldsymbol{e}$ *for each* $i \in I$*. Then for every positive integer* m*, there are families* $(\boldsymbol{b}_i \mid i \in I)$ *and* $(\boldsymbol{c}_{i,j} \mid i, j \in I)$ *of elements of* $\mathrm{K}_0(R)^+$ *such that*

(i) $\boldsymbol{c}_{i,j} = \boldsymbol{c}_{j,i} \leq \boldsymbol{a}_i$*, for all* $i, j \in I$*;*

(ii) $\boldsymbol{b}_i \ll_m [R]_{\mathrm{s}}$*, for each* $i \in I$*;*

(iii) $\boldsymbol{a}_i + \boldsymbol{a}_j \leq \boldsymbol{c}_{i,j} + \boldsymbol{e} + \boldsymbol{b}_i + \boldsymbol{b}_j$, *for all* $i, j \in I$;

(iv) $\boldsymbol{c}_{i,k} + \boldsymbol{c}_{j,k} \leq \boldsymbol{a}_k + \boldsymbol{c}_{i,j}$, *for all* $i, j, k \in I$.

Outline of proof. Pick $E \in \mathrm{FP}(R)$ such that $\boldsymbol{e} = [E]_{\mathrm{s}}$. For each $i \in I$, there exists $A_i \in \mathrm{FP}(R)$ such that $\boldsymbol{a}_i = [A_i]_{\mathrm{s}}$. The assumption $\boldsymbol{a}_i \leq \boldsymbol{e}$ means that there exists a positive integer n_i such that $[A_i] + n_i[R] \leq [E] + n_i[R]$ in $\mathrm{V}(R)$. Since $\mathrm{V}(R)$ is a refinement monoid (cf. Theorem 8-4.10) and by a standard "approximate additive cancellation" property holding in every refinement monoid (cf. Exercise 9.6), there exists $B_i \in \mathrm{FP}(R)$ such that $[A_i] \leq [E] + [B_i]$ and $mn_i[B_i] \leq n_i[R]$. By the Riesz decomposition property in $\mathrm{V}(R)$ (cf. Exercise 9.5) together with the module-theoretic version of Lemma 8-4.9, there is a decomposition $A_i = A'_i \oplus X_i$ in $\mathrm{FP}(R)$ such that $[A'_i] \leq [E]$ and $[X_i] \leq [B_i]$. Since $[A'_i] \leq [E]$, there exists a submodule U_i of E such that $A'_i \cong U_i$.

By Goodearl [117, Theorem 2.3], $U_i \cap U_j$ and $U_i + U_j$ both belong to $\mathrm{FP}(R)$, for all indices $i, j \in I$ (see also Exercise 8.27). We set $\boldsymbol{b}_i = [X_i]_{\mathrm{s}}$ and $\boldsymbol{c}_{i,j} = [U_i \cap U_j]_{\mathrm{s}}$, for all $i, j \in I$. Obviously, $\boldsymbol{b}_i \ll_m [R]_{\mathrm{s}}$ and $\boldsymbol{c}_{i,j} = \boldsymbol{c}_{j,i} \leq \boldsymbol{a}_i$.

For all $i, j \in I$, it follows from Exercise 8.28 that

$$[U_i]_{\mathrm{s}} + [U_j]_{\mathrm{s}} = \boldsymbol{c}_{i,j} + [U_i + U_j]_{\mathrm{s}} \leq \boldsymbol{c}_{i,j} + \boldsymbol{e}\,,$$

hence

$$\boldsymbol{a}_i + \boldsymbol{a}_j = [U_i]_{\mathrm{s}} + \boldsymbol{b}_i + [U_j]_{\mathrm{s}} + \boldsymbol{b}_j \leq \boldsymbol{c}_{i,j} + \boldsymbol{e} + \boldsymbol{b}_i + \boldsymbol{b}_j\,.$$

Finally, for all $i, j, k \in I$,

$$\begin{aligned} \boldsymbol{c}_{i,k} + \boldsymbol{c}_{j,k} &= [U_i \cap U_k]_{\mathrm{s}} + [U_j \cap U_k]_{\mathrm{s}} \\ &\leq [U_k]_{\mathrm{s}} + [U_i \cap U_j]_{\mathrm{s}} && \text{(see Exercise 8.28)} \\ &\leq \boldsymbol{a}_k + \boldsymbol{c}_{i,j}\,, \end{aligned}$$

which concludes the proof. □

A counterexample to the URP-like statement (now for partially preordered Abelian groups, as opposed to semilattices) described in the statement of Theorem 9-2.26 is introduced in Wehrung [325]. A rough outline of the construction of that counterexample runs as follows. We start with a partially ordered vector space E (over the field $\mathbb{Q}$ of all rational numbers). We form the set $\mathrm{C}(E)$ of all quadruples $x = (x_0, x_1, x_2, x_3)$ of elements of E such that $x_i \leq x_j$ whenever $i \in \{0, 1\}$ and $j \in \{2, 3\}$. We introduce new generators $\bowtie(x)$, for $x \in \mathrm{C}(E)$, and we form the partially ordered vector space $\mathbf{I}(E)$ over the generating set

$$E \cup \{\, \bowtie(x) \mid x \in \mathrm{C}(E)\,\}\,,$$

with all relations from E together with the relations $x_i \leq \bowtie(x) \leq x_j$, for $x \in \mathrm{C}(E)$, $i \in \{0, 1\}$, and $j \in \{2, 3\}$. Then E embeds naturally (in the categorical sense) into $\mathbf{I}(E)$, so we may iterate this construction and form the directed union

$$\mathbf{J}(E) = \bigcup(\, \mathbf{I}^n(E) \mid n < \omega\,)\,.$$

Again, E embeds naturally into $\mathbf{J}(E)$. Furthermore, the partially ordered vector space $\mathbf{J}(E)$ is a dimension vector space, which is, for a suitable choice of Skolem functions, the *free dimension vector space over* E.

Now starting with a set X, we form the partially ordered vector space $\mathbf{E}(X)$ defined by the generators a_x, for $x \in X$, and e, subjected to the relations $0 \leq e$ and

$$0 \leq a_x \leq e\,, \quad \text{for each } x \in X\,.$$

Finally, we form the functor $\mathbf{F} = \mathbf{J} \circ \mathbf{E}$. Observe that $\mathbf{F}(X)$ is a partially ordered vector space with order-unit e, for every set X. The following result is established in Wehrung [325, Theorem 2.8].

$\diamondsuit$ **Theorem 9-2.27** (Wehrung 1998)**.** *Let X be a set of cardinality at least $\aleph_2$. Then there are no families $(b_x \mid x \in X)$ and $(c_{x,y} \mid x, y \in X)$ of elements of $\mathbf{F}(X)^+$ such that*

(i) *$c_{x,y} = c_{y,x} \leq a_x$, for each $x \in X$;*

(ii) *$a_x + a_y \leq c_{x,y} + (11/9)e$, for all $x, y \in X$;*

(iii) *$c_{x,z} + c_{y,z} \leq a_z + c_{x,y} + (1/9)e$, for all $x, y, z \in X$.*

(Further calculations make it possible to improve the bounds 11/9 and 1/9 in the statement above. However, the present statement of Theorem 9-2.27 is sufficient for our purposes.)

By putting together Theorems 9-2.26 and 9-2.27, we obtain the following result, first established in Wehrung [325, Corollary 2.12].

$\diamondsuit$ **Theorem 9-2.28** (Wehrung 1998)**.** *Let X be a set with at least $\aleph_2$ elements. There is no unital regular ring R such that $\mathbf{F}(X) \cong \mathrm{K}_0(R)$.*

The construction $\mathbf{F}(X)$ also gives the first known example of a distributive semilattice not satisfying Schmidt's condition (cf. Wehrung [325, Theorem 2.15]), as follows.

$\diamondsuit$ **Theorem 9-2.29** (Wehrung 1998)**.** *Let X be a set with at least $\aleph_2$ elements. Then the maximal semilattice quotient of $\mathbf{F}(X)^+$ is not a weakly distributive image of any distributive lattice with zero.*

9-3. Non-representable semilattices

The first disproof of CLP, which appears in Wehrung [335], involves a construction that had been known for ten years before that solution came out. It was introduced in Ploščica and Tůma [269], carrying over, to the world of semilattices, the construction of the functor $\mathbf{J}$, introduced in Wehrung [325] and discussed in Section 9-2.5, that embeds any partially ordered $\mathbb{Q}$-vector space E into the "free dimension vector space" $\mathbf{J}(E)$ on E. However, the semilattice analogue of $\mathbf{J}$, denoted here (following Růžička [286]) by $\mathcal{R}^\infty$, seems to be better suited to the study of congruence representation problems.

In Section 9-3.1, we shall discuss, in more detail than in any of its yet published accounts, the basics of the Ploščica-Tůma construction. This should enable the motivated reader to follow the arguments in the main references leading to a solution of CLP, and beyond, which we shall outline in the subsequent subsections.

9-3.1 The Ploščica-Tůma construction

From now on until the end of the proof of Lemma 9-3.6 we shall fix a $(\vee, 0)$-semilattice S, and we shall set

$$\mathcal{C}(S) = \{\, (u, v, w) \in S^3 \mid w \le u \vee v \,\}\,.$$

We set

$$\boldsymbol{x}_* = \boldsymbol{x} \setminus \{\, (s, s, s) \mid s \in S \,\}\,, \quad \text{for each } \boldsymbol{x} \subseteq \mathcal{C}(S)\,.$$

Consider the following properties of a *finite* subset $\boldsymbol{x}$ of $\mathcal{C}(S)$:

(R_0) There exists exactly one $s \in S$ such that $(s, s, s) \in \boldsymbol{x}$; then we write $s = \pi(\boldsymbol{x})$;

(R_1) $(u, v, w) \in \boldsymbol{x}$ and $(v, u, w) \in \boldsymbol{x}$ imply together that $u = v = w$, for all $u, v, w \in S$;

(R_2) $v \nleq \pi(\boldsymbol{x})$, for each $(u, v, w) \in \boldsymbol{x} \setminus \boldsymbol{x}_*$;

(R_3) $u, v, w \nleq \pi(\boldsymbol{x})$, for each $(u, v, w) \in \boldsymbol{x} \setminus \boldsymbol{x}_*$.

Of course, (R_3) implies (R_2). We denote by $\mathcal{R}_I(S)$ the set of all *finite* subsets of $\mathcal{C}(S)$ satisfying (R_i) whenever $i \in I$, for every $I \subseteq \{0, 1, 2, 3\}$. In particular, $\mathcal{R}_\varnothing(S)$ is the set of all finite subsets of $\mathcal{C}(S)$. A finite subset $\boldsymbol{x}$ of $\mathcal{C}(S)$ is *reduced* if it satisfies (R_0)–(R_3) above. Obviously, the singleton $\varepsilon(s) = \{(s, s, s)\}$ is reduced, for each $s \in S$. We denote by $\mathcal{R}(S) = \mathcal{R}_{\{0,1,2,3\}}(S)$ the set of all reduced subsets of $\mathcal{C}(S)$. We set

$$\text{(9-3.1)} \quad \boldsymbol{x} \le \boldsymbol{y} \underset{\text{def.}}{\iff} \Big(\boldsymbol{y} \in \mathcal{R}_{\{0\}}(S) \text{ and } \big(\forall (u, v, w) \in \boldsymbol{x} \setminus \boldsymbol{y}\big) \\ \big(\text{either } u \le \pi(\boldsymbol{y}) \text{ or } w \le \pi(\boldsymbol{y})\big)\Big)\,, \text{ for all } \boldsymbol{x}, \boldsymbol{y} \in \mathcal{R}_{\{0\}}(S)\,.$$

Lemma 9-3.1.

(1) $\boldsymbol{x} \le \boldsymbol{y}$ *implies that* $\pi(\boldsymbol{x}) \le \pi(\boldsymbol{y})$, *for all* $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{R}_{\{0\}}(S)$.

(2) *$\varepsilon(s) \leq \boldsymbol{x}$ iff $s \leq \pi(\boldsymbol{x})$, for each $(s, \boldsymbol{x}) \in S \times \mathcal{R}_{\{0\}}(S)$.*

(3) *The binary relation $\leq$ is a preordering of $\mathcal{R}_{\varnothing}(S)$, and its restriction to $\mathcal{R}_{\{0,3\}}(S)$ is a partial ordering.*

Proof. (1) Set $s = \pi(\boldsymbol{x})$. If $(s, s, s) \notin \boldsymbol{y}$, then (since $(s, s, s) \in \boldsymbol{x}$ and $\boldsymbol{x} \leq \boldsymbol{y}$) $s \leq \pi(\boldsymbol{y})$. If $(s, s, s) \in \boldsymbol{y}$, then (since $\boldsymbol{y} \in \mathcal{R}_{\{0\}}(S)$) $s = \pi(\boldsymbol{y})$.

(2) is trivial.

(3) It is trivial that $\leq$ is reflexive. Let $\boldsymbol{x} \leq \boldsymbol{y}$ and $\boldsymbol{y} \leq \boldsymbol{z}$; in particular, $\boldsymbol{y}, \boldsymbol{z} \in \mathcal{R}_{\{0\}}(S)$, thus, by (1) above, $\pi(\boldsymbol{y}) \leq \pi(\boldsymbol{z})$. Let $(u, v, w) \in \boldsymbol{x} \setminus \boldsymbol{z}$. If $(u, v, w) \notin \boldsymbol{y}$, then (since $\boldsymbol{x} \leq \boldsymbol{y}$) either $u \leq \pi(\boldsymbol{y})$ or $w \leq \pi(\boldsymbol{y})$, thus either $u \leq \pi(\boldsymbol{z})$ or $w \leq \pi(\boldsymbol{z})$. If $(u, v, w) \in \boldsymbol{y}$, then (since $\boldsymbol{y} \leq \boldsymbol{z}$) either $u \leq \pi(\boldsymbol{z})$ or $w \leq \pi(\boldsymbol{z})$. In any case, $\boldsymbol{x} \leq \boldsymbol{z}$.

Suppose now that $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{R}_{\{0,3\}}(S)$ and that $\boldsymbol{x} \leq \boldsymbol{y} \leq \boldsymbol{x}$. Set $s = \pi(\boldsymbol{x}) = \pi(\boldsymbol{y})$, suppose that $\boldsymbol{x} \neq \boldsymbol{y}$, say $\boldsymbol{x} \not\subseteq \boldsymbol{y}$, and pick $(u, v, w) \in \boldsymbol{x} \setminus \boldsymbol{y}$. From $s = \pi(\boldsymbol{x}) = \pi(\boldsymbol{y})$ it follows that (u, v, w) is non-diagonal, thus, by (R_3), $u, v, w \nleq s$. On the other hand, from $\boldsymbol{x} \leq \boldsymbol{y}$ it follows that either $u \leq s$ or $w \leq s$, a contradiction. Therefore, $\boldsymbol{x} = \boldsymbol{y}$. $\square$

Next, we set

$$
\begin{aligned}
&\text{(9-3.2)} \quad \Pi(\boldsymbol{x}) = \{\, s \in S \mid (s, s, s) \in \boldsymbol{x} \,\}, \qquad \text{for each } \boldsymbol{x} \in \mathcal{R}_{\varnothing}(S)\,,\\
&\varphi(\boldsymbol{x}) = \boldsymbol{x}_* \cup \varepsilon\Big(\bigvee \Pi(\boldsymbol{x})\Big)\,, \qquad \text{for each } \boldsymbol{x} \in \mathcal{R}_{\varnothing}(S)\,,\\
&\psi(\boldsymbol{x}) = \boldsymbol{x} \setminus \{\, (a, b, c) \in \boldsymbol{x} \setminus \boldsymbol{x}_* \mid \text{either } a \leq \pi(\boldsymbol{x}) \text{ or } c \leq \pi(\boldsymbol{x}) \,\}\,,\\
&\qquad \text{for each } \boldsymbol{x} \in \mathcal{R}_{\{0\}}(S)\,.
\end{aligned}
$$

Observe that the ranges of φ and ψ are both contained in $\mathcal{R}_{\{0\}}(S)$.

We define binary relations $\to_1$ and $\to_2$ on $\mathcal{R}_{\varnothing}(S)$ as follows:

- $\boldsymbol{x} \to_1 \boldsymbol{y}$ if $\boldsymbol{x} \in \mathcal{R}_{\{0\}}(S)$ and there is $(a, b, c) \in \boldsymbol{x} \setminus \boldsymbol{x}_*$ such that $(b, a, c) \in \boldsymbol{x}$ and

  $$\text{(9-3.3)} \quad \boldsymbol{y} = \big(\boldsymbol{x} \setminus \{(a, b, c), (b, a, c), (\pi(\boldsymbol{x}), \pi(\boldsymbol{x}), \pi(\boldsymbol{x}))\}\big) \cup \varepsilon(c \vee \pi(\boldsymbol{x}))\,,$$

  for all $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{R}_{\varnothing}(S)$.

- $\boldsymbol{x} \to_2 \boldsymbol{y}$ if $\boldsymbol{x} \in \mathcal{R}_{\{0\}}(S)$ and there exists $(a, b, c) \in \boldsymbol{x} \setminus \boldsymbol{x}_*$ such that $b \leq \pi(\boldsymbol{x})$ and

  $$\text{(9-3.4)} \qquad \boldsymbol{y} = \big(\boldsymbol{x} \setminus \{(a, b, c), (\pi(\boldsymbol{x}), \pi(\boldsymbol{x}), \pi(\boldsymbol{x}))\}\big) \cup \varepsilon\big(c \vee \pi(\boldsymbol{x})\big)\,,$$

  for all $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{R}_{\varnothing}(S)$.

Observe that $\boldsymbol{x} \to_i \boldsymbol{y}$, for $i \in \{1, 2\}$, implies that $\boldsymbol{y} \in \mathcal{R}_{\{0\}}(S)$. We denote by $\to_1^*$ and $\to_2^*$ the reflexive and transitive closures of $\to_1$ and $\to_2$, respectively.

The proof of the following lemma is straightforward.

Lemma 9-3.2. *The following statements hold, for all $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{R}_{\varnothing}(S)$.*

(1) *If either $\boldsymbol{y} = \varphi(\boldsymbol{x})$ or $\boldsymbol{y} = \psi(\boldsymbol{x})$ or $\boldsymbol{x} \rightarrow_1^* \boldsymbol{y}$ or $\boldsymbol{x} \rightarrow_2^* \boldsymbol{y}$, then $\boldsymbol{y}_* \subseteq \boldsymbol{x}_*$.*

(2) *If $\boldsymbol{x} \in \mathcal{R}_{\{0\}}(S)$ and either $\boldsymbol{y} = \psi(\boldsymbol{x})$ or $\boldsymbol{x} \rightarrow_1^* \boldsymbol{y}$ or $\boldsymbol{x} \rightarrow_2^* \boldsymbol{y}$, then $\boldsymbol{x} \leq \boldsymbol{y}$.*

Lemma 9-3.3. *Let $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{R}_{\varnothing}(S)$ and let $\boldsymbol{z} \in \mathcal{R}(S)$ such that $\boldsymbol{x} \leq \boldsymbol{z}$. Then each of the following statements implies that $\boldsymbol{y} \leq \boldsymbol{z}$:*

(i) $\boldsymbol{y} = \varphi(\boldsymbol{x})$;

(ii) $\boldsymbol{y} = \psi(\boldsymbol{x})$;

(iii) $\boldsymbol{x} \rightarrow_1^* \boldsymbol{y}$;

(iv) $\boldsymbol{x} \rightarrow_2^* \boldsymbol{y}$.

Proof. (i) Since $\boldsymbol{x} \leq \boldsymbol{z}$, it remains to prove that, setting $s = \bigvee \Pi(\boldsymbol{x})$, the inequality $s \leq \pi(\boldsymbol{z})$ holds, that is, $x \leq \pi(\boldsymbol{z})$ for each $x \in \Pi(\boldsymbol{x})$. If $(x, x, x) \in \boldsymbol{z}$ then $x = \pi(\boldsymbol{z})$. If $(x, x, x) \notin \boldsymbol{z}$, then (since $\boldsymbol{x} \leq \boldsymbol{z}$) $x \leq \pi(\boldsymbol{z})$.

(ii) is trivial, since $\psi(\boldsymbol{x}) \subseteq \boldsymbol{x}$.

(iii) It is sufficient to prove the statement in case $\boldsymbol{x} \rightarrow_1 \boldsymbol{y}$. Let $(a, b, c) \in \boldsymbol{x} \setminus \boldsymbol{x}_*$ such that $(b, a, c) \in \boldsymbol{x}$ and (9-3.3) holds. Since $\boldsymbol{x} \leq \boldsymbol{z}$, it remains to prove that $c \vee \pi(\boldsymbol{x}) \leq \pi(\boldsymbol{z})$, that is, since we already know that $\pi(\boldsymbol{x}) \leq \pi(\boldsymbol{z})$ (because $\boldsymbol{x} \leq \boldsymbol{z}$), we must verify that $c \leq \pi(\boldsymbol{z})$. Suppose, to the contrary, that $c \nleq \pi(\boldsymbol{z})$. If $(a, b, c) \notin \boldsymbol{z}$, then (since $(a, b, c) \in \boldsymbol{x}$ and $\boldsymbol{x} \leq \boldsymbol{z}$) either $a \leq \pi(\boldsymbol{z})$ or $c \leq \pi(\boldsymbol{z})$, so $a \leq \pi(\boldsymbol{z})$. Likewise, $(b, a, c) \notin \boldsymbol{z}$ implies that $b \leq \pi(\boldsymbol{z})$. Hence, if $(a, b, c) \notin \boldsymbol{z}$ and $(b, a, c) \notin \boldsymbol{z}$, then $c \leq a \vee b \leq \pi(\boldsymbol{z})$, a contradiction; so we may assume that $(a, b, c) \in \boldsymbol{z}$. From $\boldsymbol{z} \in \mathcal{R}_{\{1\}}(S)$ we get $(b, a, c) \notin \boldsymbol{z}$, so, as seen above, $b \leq \pi(\boldsymbol{z})$, in contradiction with $(a, b, c) \in \boldsymbol{z}$ and $\boldsymbol{z} \in \mathcal{R}_{\{2\}}(S)$.

(iv) It is sufficient to prove the statement in case $\boldsymbol{x} \rightarrow_2 \boldsymbol{y}$. Let $(a, b, c) \in \boldsymbol{x} \setminus \boldsymbol{x}_*$ such that $b \leq \pi(\boldsymbol{x})$ and (9-3.4) holds. Since $\boldsymbol{x} \leq \boldsymbol{z}$, it remains to prove that $c \vee \pi(\boldsymbol{x}) \leq \pi(\boldsymbol{z})$, that is (since $\pi(\boldsymbol{x}) \leq \pi(\boldsymbol{z})$) $c \leq \pi(\boldsymbol{z})$. Suppose, to the contrary, that $c \nleq \pi(\boldsymbol{z})$. If $(a, b, c) \notin \boldsymbol{z}$, then (since $(a, b, c) \in \boldsymbol{x}$ and $\boldsymbol{x} \leq \boldsymbol{z}$) either $a \leq \pi(\boldsymbol{z})$ or $c \leq \pi(\boldsymbol{z})$, thus $a \leq \pi(\boldsymbol{z})$, and thus (since $b \leq \pi(\boldsymbol{x}) \leq \pi(\boldsymbol{z})$) $c \leq a \vee b \leq \pi(\boldsymbol{z})$, a contradiction. Therefore, $(a, b, c) \in \boldsymbol{z}$, in contradiction with $b \leq \pi(\boldsymbol{z})$ and $\boldsymbol{z} \in \mathcal{R}_{\{2\}}(S)$. □

The following lemma is a reformulation, in terms of φ, ψ, $\rightarrow_1$, and $\rightarrow_2$, of Ploščica and Tůma [269, Lemma 2.1]; see also Wehrung [335, Lemma 3.1].

Lemma 9-3.4. *For all $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{R}(S)$, there is $(\boldsymbol{z}_1, \boldsymbol{z}_2) \in \mathcal{R}_{\{0,1\}}(S) \times \mathcal{R}_{\{0,1,2\}}(S)$ such that $\varphi(\boldsymbol{x} \cup \boldsymbol{y}) \rightarrow_1^* \boldsymbol{z}_1$ and $\boldsymbol{z}_1 \rightarrow_2^* \boldsymbol{z}_2$. Furthermore, for any such pair $(\boldsymbol{z}_1, \boldsymbol{z}_2)$, $\psi(\boldsymbol{z}_2)$ is the join, in $\mathcal{R}(S)$, of $\boldsymbol{x}$ and $\boldsymbol{y}$, and $\psi(\boldsymbol{z}_2)_* \subseteq \boldsymbol{x}_* \cup \boldsymbol{y}_*$.*

Proof. Since $\boldsymbol{p} \rightarrow_1 \boldsymbol{q}$ implies that $\boldsymbol{q}$ has less non-diagonal elements than $\boldsymbol{p}$, there exists $\boldsymbol{z}_1 \in \mathcal{R}_{\varnothing}(S)$ such that $\varphi(\boldsymbol{x} \cup \boldsymbol{y}) \rightarrow_1^* \boldsymbol{z}_1$ and there is no $\boldsymbol{p}$ such that

$\boldsymbol{z}_1 \to_1 \boldsymbol{p}$. The latter condition means, of course, that $\boldsymbol{z}_1 \in \mathcal{R}_{\{1\}}(S)$. Since $\varphi(\boldsymbol{x} \cup \boldsymbol{y}) \in \mathcal{R}_{\{0\}}(S)$, we get $\boldsymbol{z}_1 \in \mathcal{R}_{\{0,1\}}(S)$.

Since $\boldsymbol{p} \to_2 \boldsymbol{q}$ implies that $\boldsymbol{q}$ has less non-diagonal elements than $\boldsymbol{p}$, there exists $\boldsymbol{z}_2 \in \mathcal{R}_{\{0\}}(S)$ such that $\boldsymbol{z}_1 \to_2^* \boldsymbol{z}_2$ and there is no $\boldsymbol{p}$ such that $\boldsymbol{z}_2 \to_2 \boldsymbol{p}$. The latter condition means, of course, that $\boldsymbol{z}_2 \in \mathcal{R}_{\{2\}}(S)$. Furthermore, from $\boldsymbol{z}_1 \to_2^* \boldsymbol{z}_2$ and $\boldsymbol{z}_1 \in \mathcal{R}_{\{0,1\}}(S)$ it follows that $\boldsymbol{z}_2 \in \mathcal{R}_{\{0,1\}}(S)$; whence $\boldsymbol{z}_2 \in \mathcal{R}_{\{0,1,2\}}(S)$. It follows easily from the definition of ψ that $\psi(\boldsymbol{z}_2) \in \mathcal{R}_{\{0,1,2,3\}}(S)$.

Let $(\boldsymbol{z}_1, \boldsymbol{z}_2)$ satisfy the given conditions. It is straightforward to verify that $\boldsymbol{x} \leq \varphi(\boldsymbol{x} \cup \boldsymbol{y})$ and $\boldsymbol{y} \leq \varphi(\boldsymbol{x} \cup \boldsymbol{y})$. Furthermore, by Lemma 9-3.2(2), we get $\varphi(\boldsymbol{x} \cup \boldsymbol{y}) \leq \psi(\boldsymbol{z}_2)$; whence $\boldsymbol{x} \leq \psi(\boldsymbol{z}_2)$ and $\boldsymbol{y} \leq \psi(\boldsymbol{z}_2)$. Let $\boldsymbol{z} \in \mathcal{R}(S)$ such that $\boldsymbol{x} \leq \boldsymbol{z}$ and $\boldsymbol{y} \leq \boldsymbol{z}$. It follows from the definition of $\leq$ that $\boldsymbol{x} \cup \boldsymbol{y} \leq \boldsymbol{z}$, thus, by Lemma 9-3.3, $\psi(\boldsymbol{z}_2) \leq \boldsymbol{z}$.

The containment $\psi(\boldsymbol{z}_2)_* \subseteq \boldsymbol{x}_* \cup \boldsymbol{y}_*$ follows trivially from Lemma 9-3.2(1). □

Observe that Lemma 9-3.4 not only gives the existence of the join of $\{\boldsymbol{x}, \boldsymbol{y}\}$ in $\mathcal{R}(S)$, but also gives an algorithm to calculate that join. In particular, it yields immediately the following.

Theorem 9-3.5. *The poset $(\mathcal{R}(S), \leq)$ is a $(\vee, 0)$-semilattice, and the canonical map $\varepsilon\colon S \hookrightarrow \mathcal{R}(S)$ is a $(\vee, 0)$-embedding. Furthermore,*

$$(\boldsymbol{x} \vee \boldsymbol{y})_* \subseteq \boldsymbol{x}_* \cup \boldsymbol{y}_*\,, \quad \textit{for all } \boldsymbol{x}, \boldsymbol{y} \in \mathcal{R}(S)\,. \tag{9-3.5}$$

The containment (9-3.5) can also be written $\boldsymbol{x} \vee \boldsymbol{y} \subseteq \boldsymbol{x} \cup \boldsymbol{y} \cup \{\pi(\boldsymbol{x} \vee \boldsymbol{y})\}$.

We shall identify s and $\varepsilon(s)$ whenever convenient, for each $s \in S$. Now $\mathcal{R}(S)$ becomes a $(\vee, 0)$-semilattice containing S as a $(\vee, 0)$-subsemilattice. We set

$$\bowtie_S(a, b, c) = \begin{cases} \{(c, c, c)\}\,, & \text{if either } a = b \text{ or } b = 0 \text{ or } c = 0\,, \\ \{(0, 0, 0)\}\,, & \text{if } a = 0\,, \\ \{(0, 0, 0), (a, b, c)\}\,, & \text{otherwise} \end{cases}$$

for each $(a, b, c) \in \mathcal{C}(S)$, and we set $\bowtie(a, b, c) = \bowtie_S(a, b, c)$ in case S is understood from the context. Observe that $\bowtie_S(a, b, c)$ is always an element of $\mathcal{R}(S)$.

Lemma 9-3.6. *The equation $\boldsymbol{x} = \bigvee(\bowtie(a, b, c) \mid (a, b, c) \in \boldsymbol{x})$ holds, in $\mathcal{R}(S)$, for each $\boldsymbol{x} \in \mathcal{R}(S)$.*

Proof. It is straightforward to verify that $\bowtie(a, b, c) \leq \boldsymbol{x}$ for each $(a, b, c) \in \boldsymbol{x}$. It follows that the element $\boldsymbol{y} = \bigvee(\bowtie(a, b, c) \mid (a, b, c) \in \boldsymbol{x})$ lies below $\boldsymbol{x}$, so it suffices to prove that $\boldsymbol{x} \leq \boldsymbol{y}$. Let $(a, b, c) \in \boldsymbol{x} \setminus \boldsymbol{y}$, we must prove that either $a \leq \pi(\boldsymbol{y})$ or $c \leq \pi(\boldsymbol{y})$. It follows from the definition of $\boldsymbol{y}$ that $\bowtie(a, b, c) \leq \boldsymbol{y}$, thus, by the definition of $\leq$, if $(a, b, c) \in \bowtie(a, b, c)$, then either $a \leq \pi(\boldsymbol{y})$ or

$c \leq \pi(\boldsymbol{y})$, as desired. It remains to settle the case where $(a,b,c) \notin \bowtie(a,b,c)$. In that case, either $a = 0$, and then $a \leq \pi(\boldsymbol{y})$, or $\bowtie(a,b,c) = \{(c,c,c)\}$, and then (since $\bowtie(a,b,c) \leq \boldsymbol{y}$) we get $c \leq \pi(\boldsymbol{y})$. $\square$

Definition 9-3.7. Let X be a $(\vee,0)$-subsemilattice of a $(\vee,0)$-semilattice S. An (X,S)-*refiner* is a map $\imath\colon \mathcal{C}(X) \to S$ such that $\imath(x,y,z) \leq x$ and $z = \imath(x,y,z) \vee \imath(y,x,z)$ for each $(x,y,z) \in \mathcal{C}(X)$.

Lemma 9-3.8. *A $(\vee,0)$-semilattice S is distributive iff there exists an (S,S)-refiner.*

Proof. If there exists an (S,S)-refiner, then S is obviously distributive. Conversely, if S is distributive, pick $\imath_0(x,y,z) \leq x$ and $\imath_1(x,y,z) \leq y$ such that $z = \imath_0(x,y,z) \vee \imath_1(x,y,z)$, for each $(x,y,z) \in \mathcal{C}(S)$. Set $\imath(x,y,z) = \imath_0(x,y,z) \vee \imath_1(y,x,z)$. $\square$

We leave to the reader the straightforward proof of the following lemma.

Lemma 9-3.9. *The map $\bowtie_S$ is an $(S,\mathcal{R}(S))$-refiner, for every $(\vee,0)$-semilattice S. That is, $\bowtie_S(x,y,z) \leq x$ and $z = \bowtie_S(x,y,z) \vee \bowtie_S(y,x,z)$, for every $(x,y,z) \in \mathcal{C}(S)$.*

Corollary 9-3.10. *The $(\vee,0)$-semilattice $\mathcal{R}^\infty(S) = \bigcup(\,\mathcal{R}^n(S) \mid n < \omega\,)$ is distributive, for any $(\vee,0)$-semilattice S.*

Lemma 9-3.11. *Let X be a $(\vee,0)$-subsemilattice of a $(\vee,0)$-semilattice S and let $\imath$ be an (X,S)-refiner. Then*

(i) $\imath(a,b,c) \leq a$ *and* $\imath(a,b,c) \leq c$*;*

(ii) $\imath(a,b,c) = c$ *whenever either* $a = b$ *or* $b = 0$ *or* $c = 0$*;*

(iii) $\imath(a,b,c) \vee b = c \vee b$*,*

for each $(a,b,c) \in \mathcal{C}(X)$.

Proof. (i) is trivial.

(ii) If $a = b$, then $c = \imath(a,a,c) \vee \imath(a,a,c) = \imath(a,a,c)$. If $b = 0$, then $\imath(b,a,c) = 0$, thus $c = \imath(a,0,c) \vee \imath(0,a,c) = \imath(a,0,c)$. If $c = 0$, then $\imath(a,b,c) = 0$ follows from $\imath(a,b,c) \leq c$.

(iii) From $\imath(a,b,c) \leq c$ it follows that $\imath(a,b,c) \vee b \leq c \vee b$. Conversely, $c = \imath(a,b,c) \vee \imath(b,a,c) \leq \imath(a,b,c) \vee b$. $\square$

The following lemma extends Ploščica and Tůma [269, Theorem 2.3].

Lemma 9-3.12. *Let S and T be $(\vee,0)$-semilattices, let $f\colon S \to T$ be a $(\vee,0)$-homomorphism, and let $\imath$ be an $(f(S),T)$-refiner. Then f extends to a unique $(\vee,0)$-homomorphism $g\colon \mathcal{R}(S) \to T$ such that*

$$g(\bowtie_S(a,b,c)) = \imath(f(a),f(b),f(c)) \quad \textit{for each } (a,b,c) \in \mathcal{C}(S)\,. \tag{9-3.6}$$

Proof. The uniqueness of g follows immediately from Lemma 9-3.6. As to existence, we set

$$h(\boldsymbol{x}) = \bigvee\bigl(\imath(f(a), f(b), f(c)) \mid (a, b, c) \in \boldsymbol{x}\bigr)\,, \quad \text{for each } \boldsymbol{x} \in \mathcal{R}_{\varnothing}(S)\,.$$

Claim. *Let $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{R}_{\varnothing}(S)$. If either $\boldsymbol{y} = \varphi(\boldsymbol{x})$ or $\boldsymbol{y} = \psi(\boldsymbol{x})$ or $\boldsymbol{x} \to_1^* \boldsymbol{y}$ or $\boldsymbol{x} \to_2^* \boldsymbol{y}$, then $h(\boldsymbol{x}) = h(\boldsymbol{y})$.*

Proof of Claim. Suppose that $\boldsymbol{y} = \varphi(\boldsymbol{x})$ and set $X = \Pi(\boldsymbol{x})$ (cf. (9-3.2)). Since

$$\boldsymbol{x} = \boldsymbol{x}_* \cup \{\, (s, s, s) \mid s \in X \,\} \text{ and } \boldsymbol{y} = \boldsymbol{x}_* \cup \Bigl\{\Bigl(\bigvee X, \bigvee X, \bigvee X\Bigr)\Bigr\}\,,$$

we get $h(\boldsymbol{x}) = h(\boldsymbol{x}_*) \vee \bigvee f(X) = h(\boldsymbol{x}_*) \vee f\bigl(\bigvee X\bigr) = h(\boldsymbol{y})$.

If $\boldsymbol{y} = \psi(\boldsymbol{x})$, then $\boldsymbol{x} \in \mathcal{R}_{\{0\}}(S)$, $\pi(\boldsymbol{x}) = \pi(\boldsymbol{y})$, and $\boldsymbol{x} = \boldsymbol{y} \cup \boldsymbol{z}$, where we set

$$\boldsymbol{z} = \{\, (a, b, c) \in \boldsymbol{x} \setminus \boldsymbol{x}_* \mid \text{either } a \leq \pi(\boldsymbol{x}) \text{ or } c \leq \pi(\boldsymbol{x}) \,\}\,.$$

Let $(a, b, c) \in \boldsymbol{z}$. If $a \leq \pi(\boldsymbol{x})$, then

$$\imath(f(a), f(b), f(c)) \leq f(a) \leq f(\pi(\boldsymbol{x})) = f(\pi(\boldsymbol{y})) \leq h(\boldsymbol{y})\,.$$

Likewise, if $c \leq \pi(\boldsymbol{x})$, then

$$\imath(f(a), f(b), f(c)) \leq f(c) \leq f(\pi(\boldsymbol{x})) = f(\pi(\boldsymbol{y})) \leq h(\boldsymbol{y})\,.$$

In any case, $\imath(f(a), f(b), f(c)) \leq h(\boldsymbol{y})$, so $h(\boldsymbol{z}) \leq h(\boldsymbol{y})$, and so

$$h(\boldsymbol{x}) = h(\boldsymbol{y}) \vee h(\boldsymbol{z}) = h(\boldsymbol{y})\,.$$

The case where $\boldsymbol{x} \to_1^* \boldsymbol{y}$ reduces trivially to the case where $\boldsymbol{x} \to_1 \boldsymbol{y}$ (so $\boldsymbol{x} \in \mathcal{R}_{\{0\}}(S)$). There is $(a, b, c) \in \boldsymbol{x} \setminus \boldsymbol{x}_*$ such that $(b, a, c) \in \boldsymbol{x}$ and, setting

$$\boldsymbol{z} = \boldsymbol{x} \setminus \{(a, b, c), (b, a, c), (\pi(\boldsymbol{x}), \pi(\boldsymbol{x}), \pi(\boldsymbol{x}))\}\,,$$

the equality $\boldsymbol{y} = \boldsymbol{z} \cup \{(c \vee \pi(\boldsymbol{x}), c \vee \pi(\boldsymbol{x}), c \vee \pi(\boldsymbol{x}))\}$ holds (cf. (9-3.3)). Hence,

$$\begin{aligned}
h(\boldsymbol{y}) &= h(\boldsymbol{z}) \vee \imath(f(c \vee \pi(\boldsymbol{x})), f(c \vee \pi(\boldsymbol{x})), f(c \vee \pi(\boldsymbol{x})))\\
&= h(\boldsymbol{z}) \vee f(c \vee \pi(\boldsymbol{x}))\\
&= h(\boldsymbol{z}) \vee f(\pi(\boldsymbol{x})) \vee f(c)\\
&= h(\boldsymbol{z}) \vee f(\pi(\boldsymbol{x})) \vee \imath(f(a), f(b), f(c)) \vee \imath(f(b), f(a), f(c))\\
&= h(\boldsymbol{x})\,.
\end{aligned}$$

The case where $\boldsymbol{x} \to_2^* \boldsymbol{y}$ reduces trivially to the case where $\boldsymbol{x} \to_2 \boldsymbol{y}$ (so again $\boldsymbol{x} \in \mathcal{R}_{\{0\}}(S)$). There is $(a, b, c) \in \boldsymbol{x} \setminus \boldsymbol{x}_*$ such that $b \leq \pi(\boldsymbol{x})$ and, setting

$$\boldsymbol{z} = \boldsymbol{x} \setminus \{(a, b, c), (\pi(\boldsymbol{x}), \pi(\boldsymbol{x}), \pi(\boldsymbol{x}))\}\,,$$

the equality $\boldsymbol{y} = \boldsymbol{z} \cup \bigl\{\bigl(c \vee \pi(\boldsymbol{x}), c \vee \pi(\boldsymbol{x}), c \vee \pi(\boldsymbol{x})\bigr)\bigr\}$ holds (cf. (9-3.4)). We obtain, by using Lemma 9-3.11 together with the inequality $f(b) \leq f(\pi(\boldsymbol{x}))$,

$$\begin{aligned} h(\boldsymbol{x}) &= h(\boldsymbol{z}) \vee \imath(f(a), f(b), f(c)) \vee f(\pi(\boldsymbol{x})) \\ &= h(\boldsymbol{z}) \vee \imath(f(a), f(b), f(c)) \vee f(b) \vee f(\pi(\boldsymbol{x})) \\ &= h(\boldsymbol{z}) \vee f(c) \vee f(b) \vee f(\pi(\boldsymbol{x})) \\ &= h(\boldsymbol{z}) \vee f(c) \vee f(\pi(\boldsymbol{x})) \\ &= h(\boldsymbol{z}) \vee f(c \vee \pi(\boldsymbol{x})) \\ &= h(\boldsymbol{y}), \end{aligned}$$

which concludes the proof of our claim. □ Claim.

Denote by g the restriction of h to $\mathcal{R}(S)$. It follows from Lemma 9-3.4, together with the Claim above, that

$$g(\boldsymbol{x} \vee \boldsymbol{y}) = h(\boldsymbol{x} \vee \boldsymbol{y}) = h(\boldsymbol{x} \cup \boldsymbol{y}) = h(\boldsymbol{x}) \vee h(\boldsymbol{y}) = g(\boldsymbol{x}) \vee g(\boldsymbol{y}),$$

for all $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{R}(S)$; whence g is a $(\vee, 0)$-homomorphism from $\mathcal{R}(S)$ to T.

It remains to prove that (9-3.6) holds. If $a = 0$ then both sides of (9-3.6) are zero so this is trivial. If either $a = b$ or $b = 0$ or $c = 0$, then

$$\begin{aligned} g(\bowtie_S(a, b, c)) &= f(c) && \text{(by the definition of } \bowtie_S) \\ &= \imath(f(a), f(b), f(c)) && \text{(by Lemma 9-3.11).} \end{aligned}$$

In all other cases, $\bowtie_S(a, b, c) = \{(0, 0, 0), (a, b, c)\}$, and the desired conclusion follows. □

Corollary 9-3.13. *Let S and T be $(\vee, 0)$-semilattices. Then every $(\vee, 0)$-homomorphism $f\colon S \to T$ extends to a unique $(\vee, 0)$-homomorphism $\mathcal{R}(f)\colon \mathcal{R}(S) \to \mathcal{R}(T)$ such that*

$$\mathcal{R}(f)(\bowtie_S(a, b, c)) = \bowtie_T(f(a), f(b), f(c)) \quad \textit{for each } (a, b, c) \in \mathcal{C}(S).$$

Proof. By Lemma 9-3.9, the map $\bowtie_T$ is a $(T, \mathcal{R}(T))$-refiner. Apply Lemma 9-3.12 to the composite $\varepsilon_T \circ f\colon S \to \mathcal{R}(T)$ and the restriction $\imath$ of $\bowtie_T$ to $f(S)$. □

Corollary 9-3.13 extends to the assignment $\mathcal{R}^\infty$ introduced in Corollary 9-3.10, as follows. According to Corollary 9-3.13, for all $(\vee, 0)$-semilattices S and T, every $(\vee, 0)$-homomorphism $f\colon S \to T$ extends canonically to a $(\vee, 0)$-homomorphism $\mathcal{R}^\infty(f)\colon \mathcal{R}^\infty(S) \to \mathcal{R}^\infty(T)$, defined as the union of all $\mathcal{R}^n(f)$ for $n < \omega$.

Corollary 9-3.14. *The assignments $\mathcal{R}$ and $\mathcal{R}^\infty$ are both endofunctors of the category $\mathbf{Sem}_{\vee,0}$ of all $(\vee, 0)$-semilattices with $(\vee, 0)$-homomorphisms.*

The $(\vee, 0)$-semilattice $\mathcal{R}^\infty(S)$ behaves like a "free distributive extension" of the $(\vee, 0)$-semilattice S; see also Exercise 9.12.

9-3.2 Free trees, evaporation, erosion

In the present subsection we shall review, mostly without proof, the main technical tools involved in the solution of CLP. We shall also conclude Section 9-3.2 by stating that (negative) solution.

For every set Ω, we denote by $\mathcal{L}(\Omega)$ the $(\vee,0,1)$-semilattice defined by generators $\boldsymbol{a}_0^\xi$ and $\boldsymbol{a}_1^\xi$, for $\xi\in\Omega$, and relations $1=\boldsymbol{a}_0^\xi\vee\boldsymbol{a}_1^\xi$ for $\xi\in\Omega$.

Useful auxiliary elements of $\mathcal{L}(\Omega)$ are defined as follows.

Notation 9-3.15. Let A be a finite subset of Ω and let $\varphi\colon A\to\{0,1\}$. We set

$$\boldsymbol{a}_\varphi^A=\bigvee(\,\boldsymbol{a}_{\varphi(\xi)}^\xi\mid\xi\in A\,)\,.$$

The $(\vee,0,1)$-semilattice $\mathcal{L}(\Omega)$ can be "concretely" represented as the semilattice of all pairs $(X,Y)\in\operatorname{Pow}(\Omega)\times\operatorname{Pow}(\Omega)$ such that either X and Y are finite and disjoint or $X=Y=\Omega$, with

$$\boldsymbol{a}_0^\xi=(\{\xi\},\varnothing)\text{ and }\boldsymbol{a}_1^\xi=(\varnothing,\{\xi\})\quad\text{for each }\xi\in\Omega\,.$$

We shall identify $\mathcal{L}(X)$ with the $(\vee,0,1)$-subsemilattice of $\mathcal{L}(\Omega)$ generated by the subset $\{\,\boldsymbol{a}_i^\xi\mid\xi\in X\text{ and }i<2\,\}$, for each $X\subseteq\Omega$. For sets X and Y, any map $f\colon X\to Y$ gives rise to a unique $(\vee,0,1)$-homomorphism $\mathcal{L}(f)\colon\mathcal{L}(X)\to\mathcal{L}(Y)$ such that $\mathcal{L}(f)(\boldsymbol{a}_i^\xi)=\boldsymbol{a}_i^{f(\xi)}$ for each $(\xi,i)\in X\times\{0,1\}$. Of course, the assignment $X\mapsto\mathcal{L}(X)$, $f\mapsto\mathcal{L}(f)$ is a functor from the category of sets, with maps, to the category of $(\vee,0,1)$-semilattices, with $(\vee,0,1)$-homomorphisms.

The composite $\mathcal{G}=\mathcal{R}^\infty\circ\mathcal{L}$ is a functor from the category of sets, with maps, to the category of distributive $(\vee,0,1)$-semilattices, with $(\vee,0,1)$-homomorphisms.

All currently known proofs of the failure of CLP follow from the conflict between a "structure theorem" and a "non-structure theorem" for congruence semilattices of lattices.

The "non-structure" is expressed by statements called *evaporation lemmas*. The original Evaporation Lemma, established in Wehrung [335, Lemma 4.4], is sufficient to get a counterexample to CLP of cardinality $\aleph_{\omega+1}$. An extension of that lemma, sufficient to get a counterexample of cardinality $\aleph_2$, is established in Růžička [286, Lemma 3.3]. We state the latter version here[1]. The proof of Theorem 9-3.16 makes an essential use of (9-3.5).

$\diamondsuit$ **Theorem 9-3.16** (Evaporation Lemma)**.** *Let Ω be a set, let A_0 and A_1 be disjoint finite subsets of Ω, and let $\delta\in\Omega\setminus(A_0\cup A_1)$. Let $\boldsymbol{v}\in\mathcal{G}(\Omega\setminus\{\delta\})$, let $\varphi_i\colon A_i\to\{0,1\}$, and let $\boldsymbol{u}_i\in\mathcal{G}(\Omega\setminus A_{1-i})$, for $i\in\{0,1\}$. Then*

$$\boldsymbol{v}\leq\boldsymbol{u}_0\vee\boldsymbol{u}_1\ \textit{and}\ \boldsymbol{u}_i\leq\boldsymbol{a}_{\varphi_i}^{A_i},\boldsymbol{a}_i^\delta\quad\textit{for each } i\in\{0,1\}$$

implies that $\boldsymbol{v}=0$.

[1] Růžička [286, Lemma 3.3] is actually stated in a more general form, involving so-called *diluting functors*. Růžička then observes in [286, Corollary 4.2] that the functor $\mathcal{R}^\infty$ is diluting, which is sufficient for proving that $\mathcal{G}(\aleph_2)$ is a counterexample to CLP.

Although the different versions of the Evaporation Lemma are, technically speaking, relatively difficult to prove, they express an intuitively "obvious"[2] fact.

The situation is radically different with the "structure theorem" for congruence lattices of lattices, established in Wehrung [335, Lemma 5.1] and called there the *Erosion Lemma*: the proof of that lemma is easy, but its statement (and thus its intuitive meaning) looks quite impenetrable, probably explaining why CLP had been open for such a long time. Despite the extreme simplicity of the proof of the Erosion Lemma, it carries the gist of the failure of CLP. Also, only one Erosion Lemma has been found so far, in contrast with the multiplicity of the versions of the Evaporation Lemma (three as to the present writing).

Let L be an algebra possessing a congruence-compatible structure (cf. Exercise 7.10) of semilattice $(L, \vee)$. We put

$$U \vee V = \{ u \vee v \mid (u,v) \in U \times V \}, \quad \text{for all } U, V \subseteq L\,,$$

and we denote by $\mathrm{Con}_{\mathrm{c}}^{U} L$ the $(\vee, 0)$-subsemilattice of $\mathrm{Con}_{\mathrm{c}} L$ generated by all principal congruences $\mathrm{con}_L(u,v)$, where $(u,v) \in U \times U$. We also define $\rho(i)$, for any integer i, by letting $\rho(i) = 0$ if i is even, and $\rho(i) = 1$ if i is odd.

Lemma 9-3.17 (The Erosion Lemma)**.** *Let $Z = \{ z_i \mid 0 \leq i \leq n \}$, with $n \in \omega \setminus \{0\}$, be a finite subset of L with $\bigvee_{i<n} z_i \leq z_n$, and let $x_0, x_1 \in L$. Set*

$$\boldsymbol{\alpha}_j = \bigvee (\mathrm{con}_L(z_i, z_{i+1}) \mid i < n,\ \rho(i) = j)\,, \text{ for each } j < 2\,.$$

Then there are congruences $\boldsymbol{\mu}_j \in \mathrm{Con}_{\mathrm{c}}^{\{x_j\} \vee Z} L$, for $j < 2$, such that

$$z_0 \vee x_0 \vee x_1 \equiv z_n \vee x_0 \vee x_1 \pmod{\boldsymbol{\mu}_0 \vee \boldsymbol{\mu}_1} \quad \text{and} \quad \boldsymbol{\mu}_j \subseteq \boldsymbol{\alpha}_j \cap \mathrm{con}_L^{+}(z_n, x_j)$$

for each $j < 2$.

Proof. Set $\boldsymbol{\nu}_i = \mathrm{con}_L(z_i \vee x_{\rho(i)}, z_{i+1} \vee x_{\rho(i)})$, for each $i < n$. Observe that $\boldsymbol{\nu}_i$ belongs to $\mathrm{Con}_{\mathrm{c}}^{\{x_{\rho(i)}\} \cup Z} L$. From $z_n \leq x_{\rho(i)} \pmod{\mathrm{con}_L^{+}(z_n, x_{\rho(i)})}$ and $z_i \equiv z_{i+1} \pmod{\boldsymbol{\alpha}_{\rho(i)}}$ it follows, respectively (and using $z_i \vee z_n = z_{i+1} \vee z_n$ in the first case), that

$$\boldsymbol{\nu}_i \subseteq \mathrm{con}_L^{+}(z_n, x_{\rho(i)}) \quad \text{and} \quad \boldsymbol{\nu}_i \subseteq \boldsymbol{\alpha}_{\rho(i)}\,. \tag{9-3.7}$$

Now we set

$$\boldsymbol{\mu}_j = \bigvee (\boldsymbol{\nu}_i \mid i < n,\ \rho(i) = j)\,, \quad \text{for each } j < 2\,.$$

[2] In the sense that as $\mathcal{G}(\Omega)$ is a "free" structure, any solution of $\boldsymbol{v} \leq \boldsymbol{u}_0 \vee \boldsymbol{u}_1$ with the given conditions, and $\boldsymbol{v} \neq 0$, should be an "obvious" one; but there is no obvious solution!

Hence $\boldsymbol{\mu}_j \in \mathrm{Con}_{\mathrm{c}}^{\{x_j\}\vee Z} L$, for each $j < 2$. Furthermore, from (9-3.7) it follows that $\boldsymbol{\mu}_j \subseteq \boldsymbol{\alpha}_j \cap \mathrm{con}_L^+(z_n, x_j)$. Finally, from $z_i \vee x_{\rho(i)} \equiv z_{i+1} \vee x_{\rho(i)} \pmod{\boldsymbol{\nu}_i}$, for all $i < n$, it follows that $z_i \vee x_0 \vee x_1 \equiv z_{i+1} \vee x_0 \vee x_1 \pmod{\boldsymbol{\mu}_0 \vee \boldsymbol{\mu}_1}$. Therefore, $z_0 \vee x_0 \vee x_1 \equiv z_n \vee x_0 \vee x_1 \pmod{\boldsymbol{\mu}_0 \vee \boldsymbol{\mu}_1}$. □

Now let us move to the infinite combinatorics. While Kuratowski's Free Set Theorem (Theorem 9-2.22) is sufficient, together with the Erosion Lemma and the original Evaporation Lemma, to get the original negative solution of CLP (established in Wehrung [335, Corollary 7.2]), an enhancement of that theorem is required in order to get the optimal cardinality bound. These are the *free trees* introduced in Růžička [286].

As usual, we identify every natural number n with the set $\{0, 1, \dots, n-1\}$. The following notation is introduced in Růžička [286, Section 5].

Notation 9-3.18. Let $k, m, n < \omega$ with $k > 0$ and $m \leq n$. Given a map $g\colon \{m, m+1, \dots, n-1\} \to k$, we set

$$T_{n,k}(g) = \{\, f\colon n \to k \mid f \text{ extends } g \,\}.$$

If $m < n$, then, for every map $g\colon \{m+1, m+2, \dots, n-1\} \to k$ and every $i < k$, we set

$$\begin{aligned} T_{n,k}(g, i) &= \{\, f \in T_{n,k}(g) \mid f(m) = i \,\}, \\ T_{n,k}(g, \neg i) &= \{\, f \in T_{n,k}(g) \mid f(m) \neq i \,\}. \end{aligned}$$

The following definition is stated in Růžička [286, Definition 2].

Definition 9-3.19. Let Ω be a set, let $\Phi\colon [\Omega]^{<\omega} \to [\Omega]^{<\omega}$, and let k be a positive integer. A family $\mathcal{T} = (\alpha(f) \mid f\colon n \to k)$ of distinct elements of Ω is a *free k-tree of height n, with respect to* Φ, if

$$\{\, \alpha(f) \mid f \in T_{n,k}(g, i) \,\} \cap \Phi\bigl(\{\, \alpha(f) \mid f \in T_{n,k}(g, \neg i) \,\}\bigr) = \varnothing,$$

for all $m < n$, all $g\colon \{m+1, m+2, \dots, n-1\} \to k$, and all $i < k$. We will call the set $\{\, \alpha(f) \mid f\colon n \to k \,\}$ the *range* of $\mathcal{T}$.

The following enhancement of the "if" direction of Kuratowski's Free Set Theorem is established in Růžička [286, Lemma 5.1]. Its proof is obtained, very roughly speaking, by an "iteration" of Kuratowski's Free Set Theorem.

$\diamondsuit$ **Theorem 9-3.20** (Růžička's Free Tree Theorem)**.** *Let Ω be a set, let $\Phi\colon [\Omega]^{<\omega} \to [\Omega]^{<\omega}$, and let $n, k < \omega$ with $k > 0$. Then every subset of Ω of cardinality at least $\aleph_{k-1}$ contains the range of a free k-tree of height n.*

Bringing together evaporation, erosion, and (in order to get the optimal cardinality bound $\aleph_2$) free trees, makes it possible to obtain the negative solution to CLP. This solution was first obtained for $\operatorname{card}\Omega \geq \aleph_{\omega+1}$ in Wehrung [335], then improved to the case where $\operatorname{card}\Omega \geq \aleph_2$ in Růžička [286]. In the proof of that improvement, Theorem 9-3.20 is used only for $k = 3$.

$\diamondsuit$ **Theorem 9-3.21.** *Let Ω be a set with at least $\aleph_2$ elements and let A be an algebra with a congruence-compatible structure of either a $(\vee, 1)$-semilattice or a lattice. Then there exists no weakly distributive $(\vee, 0)$-homomorphism from $\operatorname{Con_c} A$ to $\mathcal{G}(\Omega)$ containing 1 in its range. In particular, there is no lattice L such that $\operatorname{Con_c} L \cong \mathcal{G}(\Omega)$.*

Remark 9-3.22. Actually, it is sufficient to state Theorem 9-3.21 with the stronger assumption that A is a $(\vee, 1)$-semilattice. The general form follows easily. Indeed, suppose that this apparently weaker form of Theorem 9-3.21 is proved, and let A be an algebra with a congruence-compatible structure of either a $(\vee, 1)$-semilattice or a lattice. Since the composite of two weakly distributive homomorphisms is weakly distributive (cf. Exercise 7.4), in order to prove Theorem 9-3.21 in its full generality, it is sufficient to prove that for each $\boldsymbol{\alpha} \in \operatorname{Con_c} A$, there are a $(\vee, 1)$-semilattice S, a congruence $\boldsymbol{\sigma} \in \operatorname{Con_c} S$, and a $(\vee, 1)$-homomorphism $f \colon S \to A$ such that $(\operatorname{Con_c} f)(\boldsymbol{\sigma}) = \boldsymbol{\alpha}$.

If A has a congruence-compatible structure of a $(\vee, 1)$-semilattice A_{sem}, define $f \colon A_{\mathrm{sem}} \to A$ as the identity mapping. Then the $(\vee, 0)$-homomorphism $\operatorname{Con_c} f \colon \operatorname{Con_c} A_{\mathrm{sem}} \to \operatorname{Con_c} A$ is both surjective and weakly distributive (cf. Exercise 7.10).

Suppose that A has a congruence-compatible lattice structure A_{lat}, and write $\boldsymbol{\alpha} = \bigvee(\operatorname{con}_A(u_i, v_i) \mid i < n)$, with all $u_i, v_i \in A$. Define u (resp., v) as the meet (resp., join) of all the u_i. We denote by $[u, v]_{\mathrm{sem}}$ (resp., $[u, v]_{\mathrm{lat}}$) the interval $[u, v]$ endowed with its structure of $(\vee, 1)$-semilattice (resp., lattice). We consider the inclusion maps

$$[u, v]_{\mathrm{sem}} \overset{f_1}{\hookrightarrow} [u, v]_{\mathrm{lat}} \overset{f_2}{\hookrightarrow} A_{\mathrm{lat}} \overset{f_3}{\hookrightarrow} A\,.$$

By Exercise 7.10, $\operatorname{Con_c} f_1$ and $\operatorname{Con_c} f_3$ are both weakly distributive. By Exercise 7.9, $\operatorname{Con_c} f_2$ is weakly distributive. Hence, $f = f_3 \circ f_2 \circ f_1$ is weakly distributive. Since f fixes each u_i and each v_i, $\boldsymbol{\alpha}$ belongs to the range of $\operatorname{Con_c} f$.

9-3.3 A uniform refinement property for congruence semilattices of all lattices

Following Wehrung [335, Section 8], we shall discuss briefly, in the present subsection, how the failure of CLP can also be expressed by an URP-like statement.

For a positive integer m and a nonempty set Ω, we denote by $\operatorname{Sem}(m, \Omega)$ the $(\vee, 0, 1)$-semilattice defined by generators $k \cdot \dot{\xi}$, for $0 \leq k \leq m+1$ and $\xi \in \Omega$, subjected to the relations

$$0 = 0 \cdot \dot{\xi} \leq 1 \cdot \dot{\xi} \leq \cdots \leq m \cdot \dot{\xi} \leq (m+1) \cdot \dot{\xi} = 1\,, \quad \text{for } \xi \in \Omega\,. \tag{9-3.8}$$

(*Of course, the* 0 *on the left-hand side of* (9-3.8) *and the* 1 *on the right-hand side of* (9-3.8) *denote the bounds of the semilattice* $\operatorname{Sem}(m, \Omega)$*, as opposed to integers.*)

Definition 9-3.23. For an element $\boldsymbol{e}$ in a $(\vee, 0)$-semilattice S, we say that S satisfies $\mathrm{CLR}(\boldsymbol{e})$ if for every nonempty set Ω and every family $(\boldsymbol{a}_i^\xi \mid (\xi, i) \in \Omega \times \{0,1\})$ with entries in S such that $\boldsymbol{e} \leq \boldsymbol{a}_0^\xi \vee \boldsymbol{a}_1^\xi$ for each $\xi \in \Omega$, there are a decomposition $\Omega = \bigcup(\, \Omega_m \mid m \in \omega \setminus \{0\}\,)$ and mappings $\boldsymbol{c}_m \colon \mathrm{Sem}(m, \Omega_m) \times \mathrm{Sem}(m, \Omega_m) \to S$, for $m \in \omega \setminus \{0\}$, such that the following statements hold for every positive integer m:

(i) $p \leq q$ implies that $\boldsymbol{c}_m(p, q) = 0$, for all $p, q \in \mathrm{Sem}(m, \Omega_m)$;

(ii) $\boldsymbol{c}_m(p, r) \leq \boldsymbol{c}_m(p, q) \vee \boldsymbol{c}_m(q, r)$, for all $p, q, r \in \mathrm{Sem}(m, \Omega_m)$;

(iii) $\boldsymbol{c}_m(p \vee q, r) = \boldsymbol{c}_m(p, r) \vee \boldsymbol{c}_m(q, r)$, for all $p, q, r \in \mathrm{Sem}(m, \Omega_m)$;

(iv) $\boldsymbol{c}_m(1, 0) = \boldsymbol{e}$;

(v) The inequality $\boldsymbol{c}_m((k+1) \cdot \dot{\xi}, k \cdot \dot{\xi}) \leq \boldsymbol{a}_{\rho(k)}^\xi$ holds, for each $\xi \in \Omega_m$ and each $k \leq m$. (*We defined the parity function ρ in Section* 9-3.2.)

Although the proof of the following Theorem 9-3.24 is very complex, it can be obtained by mimicking the proof of Theorem 9-3.21 and changing what needs to be changed. We will omit that proof.

$\diamondsuit$ **Theorem 9-3.24.** *Let L be a lattice and let $\boldsymbol{\varepsilon}$ be a principal congruence of L. Then* $\mathrm{Con_c}\, L$ *satisfies* $\mathrm{CLR}(\boldsymbol{\varepsilon})$. *On the other hand, $\mathcal{G}(\Omega)$ does not satisfy* $\mathrm{CLR}(1)$, *for every set Ω such that* $\operatorname{card}\Omega \geq \aleph_2$.

9-3.4 Further non-representation results for distributive semilattices

The most powerful examples to date, of distributive $(\vee, 0)$-semilattices which are not representable as congruence semilattices of lattices, are due to Ploščica [265]. The descriptions of those examples, which we shall give shortly, are much simpler than the Ploščica-Tůma construction described in Section 9-3.1, at the expense of a noticeably more complex version of the Evaporation Lemma (the same one for the two constructions), which we will not describe here, referring the reader to Ploščica [265] for the details.

The first example described by Ploščica in [265] involves *majority algebras* (cf. Exercises 7.1 and 7.2). A majority algebra is a nonempty set M, endowed with a ternary operation m, such that $m(x, x, y) = m(x, y, x) = m(y, x, x) = x$ for all $x, y \in M$. It is well known that the congruence lattice of a majority algebra is distributive. The majority algebra M is *bounded* if there are elements $0, 1 \in M$ such that

$$m(x, 0, 1) = m(x, 1, 0) = m(0, x, 1) = m(1, x, 0) = m(0, 1, x) = m(1, 0, x) = x$$

for each $x \in M$. Every lattice L gives rise to a majority algebra with the same congruence lattice as $(L, \vee, \wedge)$, by setting

$$m(x, y, z) = (x \vee y) \wedge (x \vee z) \wedge (y \vee z)\,, \quad \text{for all } x, y, z \in L \tag{9-3.9}$$

(cf. Exercise 7.1). The operation m defined in (9-3.9) is called the *upper median* on L. If L is bounded, then the associated majority algebra is also bounded.

In the sequel we shall work with a special five-element majority algebra, obtained by "gluing" together the three lattices represented in Figure 9-3.1.

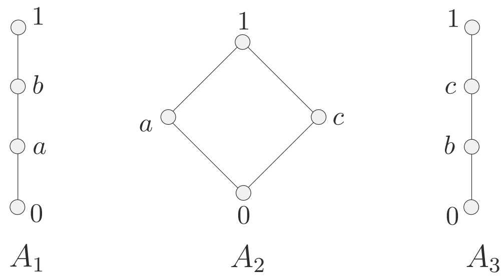

Figure 9-3.1: Lattice pieces of a finite majority algebra.

Precisely, the operation m is defined on the set $W = \{0, 1, a, b, c\}$ by the following rules:

- if $\{x, y, z\} \subseteq A_i$, for $i \in \{1, 2, 3\}$, then $m(x, y, z)$ is the lattice upper median of x, y, z;
- if $\{x, y, z\} = \{a, b, c\}$, then $m(x, y, z) = 0$.

It is straightforward to verify that W is a bounded majority algebra. Denote by $\mathbf{W}$ the variety of bounded majority algebras generated by W. The following result is established in Ploščica [265, Section 4]. Its proof involves the techniques developed in Wehrung [335] (for the algebraic part) and in Růžička [286] (for the optimal cardinality).

$\diamondsuit$ **Theorem 9-3.25** (Ploščica 2008)**.** *Let Ω be a set with at least $\aleph_2$ elements and let A be an algebra with a congruence-compatible structure of either a $(\vee, 1)$-semilattice or a lattice. Then there exists no weakly distributive $(\vee, 0)$-homomorphism from $\operatorname{Con_c} A$ to $\operatorname{Con_c} \operatorname{Free}_{\mathbf{W}}(\Omega)$ containing 1 in its range. In particular, there is no lattice L such that $\operatorname{Con} L \cong \operatorname{Con} \operatorname{Free}_{\mathbf{W}}(\Omega)$.*

This result can be interpreted in terms of congruence classes.

Definition 9-3.26. The *congruence class* (resp., *compact congruence class*) of a class $\mathbf{C}$ of algebras is defined as the class $\operatorname{Con} \mathbf{C}$ (resp., $\operatorname{Con_c} \mathbf{C}$) of all isomorphic copies of congruence lattices (resp., congruence semilattices) of all members of $\mathbf{C}$.

An immediate consequence of Theorem 9-3.25 is the following.

Corollary 9-3.27. *The congruence class of all majority algebras contains properly the congruence class of all lattices.*

Ploščica presents in [265] another construction, now topological, of a distributive $(\vee, 0)$-semilattice not isomorphic to the congruence semilattice of any lattice. This construction was originally defined in Ploščica [264].

Consider the five-element set $M = \{0, 1, a, b, c\}$. For any set Ω, we set

$$T_\Omega = \{\, f \in M^\Omega \mid \text{either } f(\Omega) \subseteq \{0,1\} \text{ or } \{a,b,c\} \subseteq f(\Omega)\,\}\,.$$

For all distinct $u, v \in \{a, b, c\}$, we define maps $p_0^{u,v}, p_1^{u,v} \colon \{0, 1, u, v\} \to \{0, 1\}$ as follows:

$$\begin{aligned}
p_0^{u,v}(0) &= p_1^{u,v}(0) = 0\,, & p_0^{u,v}(1) &= p_1^{u,v}(1) = 1 \quad \text{for all distinct } u, v\,,\\
p_0^{a,b}(a) &= p_0^{a,b}(b) = 0\,, & p_1^{a,b}(a) &= p_1^{a,b}(b) = 1\,,\\
p_0^{b,c}(b) &= p_0^{b,c}(c) = 0\,, & p_1^{b,c}(b) &= p_1^{b,c}(c) = 1\,,\\
p_0^{a,c}(a) &= p_1^{a,c}(c) = 0\,, & p_0^{a,c}(c) &= p_1^{a,c}(a) = 1\,.
\end{aligned}$$

Further, we set

$$\begin{aligned}
S_0 &= \{\, r \colon X \to M \mid X \subseteq \Omega \text{ finite and } r(X) \subseteq \{0,1\}\,\}\,;\\
S_1 &= \{\, r \colon X \to M \mid X \subseteq \Omega \text{ finite and } \{a,b,c\} \subseteq r(X)\,\}\,.
\end{aligned}$$

For each $r \in S_0$, we set

$$K_r = \Big\{ f \in M^\Omega \mid (\exists \text{ distinct } u, v \in \{a,b,c\})\Big(f(\operatorname{dom} r) \subseteq \{0,1,u,v\} \\ \text{and } (\exists i < 2)\big(r = p_i^{u,v} \circ (f{\restriction}_{\operatorname{dom} r})\big)\Big)\Big\}\,.$$

For each $r \in S_1$, we set

$$K_r = \{\, f \in M^\Omega \mid f \text{ extends } r\,\}\,.$$

It is proved in Ploščica [264] that $\mathcal{G} = \{\, K_r \cap T_\Omega \mid r \in S_0 \cup S_1\,\}$ is a basis of a topology on T_Ω, and that the compact open sets in that topology are exactly the finite unions of members of $\mathcal{G}$. In particular, the topological space T_Ω has a basis of compact open sets, hence the collection L_Ω of all open sets of that topology is an algebraic distributive lattice. The compact members of L_Ω form a distributive $(\vee, 0, 1)$-semilattice S_Ω.

The following result is established in Ploščica [265, Section 5].

$\diamondsuit$ **Theorem 9-3.28** (Ploščica 2008)**.** *Let Ω be a set with at least $\aleph_2$ elements and let A be an algebra with a congruence-compatible structure of either a $(\vee, 1)$-semilattice or a lattice. Then there exists no weakly distributive $(\vee, 0)$-homomorphism from $\operatorname{Con_c} A$ to S_Ω containing 1 in its range. In particular, there is no lattice L such that $\operatorname{Con_c} L \cong S_\Omega$.*

Remark 9-3.22 applies to both Theorem 9-3.25 and Theorem 9-3.28: it is sufficient to state each of those theorems in case A is a $(\vee, 1)$-semilattice, then the general case follows easily.

The methods of proof of Theorems 9-3.25 and 9-3.28 are pushed even further by Ploščica, yielding the following wonderful "m-permutable analogue" of Theorem 9-2.20, contained in Theorems 4.4 and 4.5, and the comments following them, in Ploščica [266].

$\diamondsuit$ **Theorem 9-3.29** (Ploščica 2008)**.** *Let $m \geq 2$ be an integer, and let A be a congruence m-permutable algebra, which admits either a congruence-compatible lattice structure or a congruence-compatible $(\vee, 1)$-semilattice structure. Let $\mathbf{V}$ be a nondistributive variety of $(0, 1)$-lattices. Then $\operatorname{Con} A$ is not isomorphic to $\operatorname{Con}\operatorname{Free}_{\mathbf{V}}(\aleph_2)$.*

9-3.5 A representation result for all distributive semilattices

There are very few known positive representation results for a distributive $(\vee, 0)$-semilattice S. Theorem 9-2.8 yields $S \cong \operatorname{Con_c} A$ for some congruence 3-permutable algebra A, but this would already work for *modular* S. So far, the only positive representation result to date that I know of, and that is specific for *distributive* semilattices, is the main theorem of Wehrung [336]. We shall state that theorem now. Although this result was published later than the solution of CLP (i.e., Wehrung [335]), it was proved earlier – and it was a crucial part of the quest for CLP: namely, the first solution of CLP came out of an attempt to extend Theorem 9-3.30 from posets to join-semilattices (this attempt failed, for a good reason).

$\diamondsuit$ **Theorem 9-3.30** (Wehrung 2008)**.** *For every distributive $(\vee, 0)$-semilattice S, there are a meet-semilattice P with zero and a map $\mu\colon P \times P \to S$ such that $\mu(x, z) \leq \mu(x, y) \vee \mu(y, z)$ and $x \leq y$ implies that $\mu(x, y) = 0$, for all $x, y, z \in P$, together with the following conditions:*

(P1) *$\mu(v, u) = 0$ implies that $u = v$, for all $u \leq v$ in P;*

(P2) *for all $u \leq v$ in P and all $\boldsymbol{a}, \boldsymbol{b} \in S$, if $\mu(v, u) \leq \boldsymbol{a} \vee \boldsymbol{b}$, then there are a positive integer n and a decomposition $u = x_0 \leq x_1 \leq \cdots \leq x_n = v$ such that either $\mu(x_{i+1}, x_i) \leq \boldsymbol{a}$ or $\mu(x_{i+1}, x_i) \leq \boldsymbol{b}$, for each $i < n$;*

(P3) *the subset $\{\, \mu(x, 0) \mid x \in P \,\}$ generates the $(\vee, 0)$-semilattice S.*

Furthermore, every finite, bounded subset of P has a join, and P is bounded in case S is bounded. Furthermore, the construction is functorial on lattice-indexed diagrams of finite distributive $(\vee, 0, 1)$-semilattices.

It is observed in Wehrung [336, Proposition 10.5] that for any poset P, any $(\vee, 0)$-semilattice S, and any map $\mu\colon P \times P \to S$, the conditions (P1)–(P3) above imply together that S is distributive. Hence, Theorem 9-3.30 characterizes distributive $(\vee, 0)$-semilattices.

The proof of Theorem 9-3.30 is very long and technical. This seems to be unavoidable, since there is no "sequential" proof of that theorem (i.e., given a "measure" $\mu\colon P \times P \to S$ failing an instance of one of the conditions (P1)–(P3), extend it to a larger measure satisfying that instance). A proof of that negative result ("no sequential proof of Theorem 9-3.30") is given in Wehrung [334].

9-4. Critical points

Critical points are a central concept in the study of congruence classes (cf. Definition 9-3.26) of lattice varieties. The *critical point* between classes $\mathbf{A}$ and $\mathbf{B}$ of algebras, denoted by $\operatorname{crit}(\mathbf{A};\mathbf{B})$, is defined in Gillibert's thesis [105] as the least possible cardinality of a member of $(\operatorname{Con_c}\mathbf{A}) \setminus (\operatorname{Con_c}\mathbf{B})$ if $\operatorname{Con_c}\mathbf{A}$ is not contained in $\operatorname{Con_c}\mathbf{B}$, and ∞ otherwise. (This definition is not equivalent to the definition given in Tůma and Wehrung [316, Section 9]: the "critical point" defined there is equal to $\max\{\operatorname{crit}(\mathbf{A};\mathbf{B}), \operatorname{crit}(\mathbf{B};\mathbf{A})\}$.)

As we will see in the present section, there is a wealth of information on critical points, obtained by quite sophisticated methods. Very roughly speaking, these methods can be divided between *topological methods* and *categorical methods*.

9-4.1 Dual topological spaces

The important papers Ploščica [258, 260] investigate dual topological spaces associated to some congruence lattices. Any algebraic distributive lattice D gives rise to a topological space $\operatorname{Pt} D$, the *dual space* of D. The points of $\operatorname{Pt} D$ are the completely meet-irreducible elements of D and the closed sets of $\operatorname{Pt} D$ are the sets of the form $(\operatorname{Pt} D) \uparrow x$, for $x \in D$. In particular, the distributivity of D means that

$$(\operatorname{Pt} D) \uparrow (x \wedge y) = ((\operatorname{Pt} D) \uparrow x) \cup ((\operatorname{Pt} D) \uparrow y) \quad \text{for all } x, y \in D\,,$$

so the union of two closed sets is, as it should be, closed. The lattice D can be reconstructed from its dual space $\operatorname{Pt} D$ as the lattice of all open subsets of $\operatorname{Pt} D$, ordered by set inclusion.

For any lattice L, the elements of the dual space $\operatorname{Pt}(\operatorname{Con} L)$ of the full congruence lattice of L are exactly the completely meet-irreducible congruences of L, that is, the congruences $\boldsymbol{\theta}$ of L such that the quotient lattice $L/\boldsymbol{\theta}$ is subdirectly irreducible. The dual space $\operatorname{Pt} D$ has a basis of compact open sets, but it is usually not Hausdorff.

Denote by $\mathbf{M}_n$ the lattice variety generated by M_n (the lattice of length 2 with n atoms), for every ordinal $n \geq 3$. Ploščica proves in [258] that the congruence classes (cf. Definition 9-3.26) $\operatorname{Con}\mathbf{M}_n$, for $3 \leq n \leq \omega$, are pairwise distinct. The topological property that distinguishes them is *uniform separability.*

Definition 9-4.1. A subset Q of a topological space T is called *discrete* if every subset of Q is open with respect to the relative topology on Q. The space T is called *uniformly n-separable* (for $n \geq 3$) if for every discrete set $Q \subseteq T$, there exists a family $(U_{p,q} \mid p,\, q \in Q,\ p \neq q)$ of open sets such that $p \in U_{p,q}$ for all distinct $p, q \in Q$, and, for every n-element set $Q_0 \subseteq Q$,

$$\bigcap(U_{p,q} \mid p, q \in Q_0,\ p \neq q) = \varnothing\,.$$

The following two theorems establish the crucial separability properties of the spaces $\mathrm{Pt}(\mathrm{Con}\, L)$, for $L \in \mathbf{M}_n$. The following result is established in Ploščica [258, Theorem 6.3].

$\Diamond$ **Theorem 9-4.2** (Ploščica 2000)**.** *Let $3 \leq n < \omega$. For each $L \in \mathbf{M}_n$, the topological space* $\mathrm{Pt}(\mathrm{Con}\, L)$ *is $(n+1)$-uniformly separable.*

In order to prove Theorem 9-4.2, Ploščica assumes that $(n+1)$-uniform separability fails in $\mathrm{Pt}(\mathrm{Con}\, L)$, and infers, with the help of a clever combinatorial statement, that M_n has $n+1$ distinct atoms, a contradiction.

Following our usual notation, let $\mathrm{Free}_{\mathbf{V}}(X)$ denote the free object on the set X within a variety $\mathbf{V}$. The following result is established in Ploščica [258, Theorem 6.5].

$\Diamond$ **Theorem 9-4.3** (Ploščica 2000)**.** *Let $3 \leq n < \omega$ and* $\mathrm{card}\, \Omega \geq \aleph_2$*. The topological space* $\mathrm{Pt}(\mathrm{Con}\, \mathrm{Free}_{\mathbf{M}_n}(\Omega))$ *is not n-uniformly separable.*

Corollary 9-4.4. *Let $3 \leq n < \omega$. Then* $\mathrm{crit}(\mathbf{M}_{n+1}; \mathbf{M}_n) \leq \aleph_2$.

We will see later that in fact, the equality holds in Corollary 9-4.4: that is, $\mathrm{crit}(\mathbf{M}_{n+1}; \mathbf{M}_n) = \aleph_2$.

As the proofs of many negative congruence representation results, such as Theorem 9-2.24, are based on Kuratowski's Free Set Theorem (Theorem 9-2.22), the proof of Theorem 9-4.3 is based on the following extension of the "if" part of that theorem, due to Hajnal and Máté [190] (see also Erdős *et al.* [78, Theorem 46.2], and Section 9-2.3 for the $(\kappa, r, \lambda) \to \varrho$ notation):

$\Diamond$ **Theorem 9-4.5** (Hajnal and Máté 1975)**.** *The relation $(\lambda^{++}, r, \lambda) \to m$ holds for every infinite cardinal λ and every integer $m > 2$.*

As a corollary, Ploščica gets the following result, established in [258, Theorem 6.6].

$\Diamond$ **Theorem 9-4.6** (Ploščica 2000)**.** *Let $3 \leq n < \omega$ and let Ω be a set of cardinality at least $\aleph_2$. Then there is no lattice $L \in \mathbf{M}_n$ such that* $\mathrm{Con}\, L$ *is isomorphic to* $\mathrm{Con}\, \mathrm{Free}_{\mathbf{M}_{n+1}}(\Omega)$.

In his further paper [260], Ploščica characterizes the dual spaces $\mathrm{Pt}(\mathrm{Con}\, L)$, for lattices L with at most $\aleph_1$ compact elements from the variety $\mathbf{M}_n^{0,1}$ generated by M_n as a bounded lattice, $n \geq 3$. His main result is the following deep theorem, established in Ploščica [260, Theorem 4.2].

$\diamondsuit$ **Theorem 9-4.7** (Ploščica 2003)**.** *Let $3 \leq n < \omega$. Then an algebraic distributive lattice D with at most $\aleph_1$ compact elements is isomorphic to* $\operatorname{Con} L$*, for some $L \in \mathbf{M}_n^{0,1}$, iff the topological space $T = \operatorname{Pt} D$ has a subspace T_0 satisfying the following conditions:*

(i) *T is compact and it has a basis of compact open sets;*

(ii) *both T_0 and $T_n = T \setminus T_0$ are Hausdorff zero-dimensional;*

(iii) *T_0 is a closed subspace of T;*

(iv) *for each $a \in T_n$ and each $b \in T \setminus \{a\}$, there exists a clopen set $V \subseteq T_n$ such that $a \in V$ and $b \notin V$;*

(v) *for all distinct $a, b, c \in T$, there are open sets U, V, W such that $a \in U$, $b \in V$, $c \in W$, and $U \cap V \cap W = \varnothing$.*

In order to establish the harder direction of Theorem 9-4.7, Ploščica embeds directly, *via* an elaborate *ad hoc* construction, the space T as a closed subspace of $\operatorname{Pt}(\operatorname{Con}(\operatorname{Free}_{\mathbf{M}_n^{0,1}}(\omega_1)))$.

Since the conditions on $\operatorname{Pt} D$ do not depend on $n \geq 3$, we get the following corollary, established in Ploščica [260, Theorem 4.3].

$\diamondsuit$ **Theorem 9-4.8** (Ploščica 2003)**.** *Let $3 \leq n < \omega$. For every $K \in \mathbf{M}_n^{0,1}$ with at most $\aleph_1$ elements, there exists $L \in \mathbf{M}_3^{0,1}$ such that* $\operatorname{Con} K \cong \operatorname{Con} L$*.*

Corollary 9-4.9. *Let $3 \leq n < \omega$. Then* $\operatorname{crit}(\mathbf{M}_{n+1}^{0,1}; \mathbf{M}_n^{0,1}) \geq \aleph_2$*.*

Again, it turns out that $\operatorname{crit}(\mathbf{M}_{n+1}^{0,1}; \mathbf{M}_n^{0,1}) = \aleph_2$. Furthermore, Theorem 9-4.8 extends to the case $n = \omega$, but the extension of the argument of Theorem 9-4.8 showing this is unpublished, and not trivial. We will see a different proof of this fact in Section 9-4.3.

9-4.2 Countable critical points

We start this subsection by reviewing a few results from Ploščica [259].

For an algebraic lattice L, a set P of completely meet-irreducible elements of L is *separable within* L if there is a family $(x_p \mid p \in P)$ of elements of L such that $x_p \nleq p$ for each $p \in P$ and $\bigwedge(x_p \mid p \in P) = 0$. Furthermore, denote by $\mathrm{M}_{\mathrm{SI}}(\mathbf{V})$ the class of all finite posets that can be embedded into $\operatorname{Mi}(\operatorname{Con} A)$, for a subalgebra A of a subdirectly irreducible algebra B in a variety $\mathbf{V}$. Finally, denote by $s(\mathbf{V})$ the supremum of the cardinalities of the members of $\mathrm{M}_{\mathrm{SI}}(\mathbf{V})$.

If $\mathbf{V}$ is finitely generated and congruence-distributive over a finite similarity type, then $\mathrm{M}_{\mathrm{SI}}(\mathbf{V})$ is the class of all isomorphic copies of a certain finite set of finite posets; hence $s(\mathbf{V})$ is finite.

The following result is contained in Theorems 2.2 and 2.3 of Ploščica [259].

◇ **Theorem 9-4.10** (Ploščica 2003). *Let* $\mathbf{V}$ *be a finitely generated congruence-distributive variety over a finite similarity type and let* P *be a finite poset. Then the following are equivalent:*

(i) P *belongs to* $\mathrm{M}_{\mathrm{SI}}(\mathbf{V})$*;*

(ii) *there are* $A \in \mathbf{V}$ *and a non-separable isomorphic copy of* P *within* $\mathrm{Con}\, A$*;*

(iii) *there is a non-separable isomorphic copy of* P *within* $\mathrm{Con}\,\mathrm{Free}_{\mathbf{V}}(\aleph_0)$.

The following consequence of Theorem 9-4.10 is immediate.

Corollary 9-4.11. *Let* $\mathbf{V}$ *and* $\mathbf{W}$ *be finitely generated, congruence-distributive varieties of algebras with (not necessarily equal) finite similarity type. If* $\mathrm{crit}(\mathbf{V}; \mathbf{W}) > \aleph_0$*, then* $\mathrm{M}_{\mathrm{SI}}(\mathbf{V}) \subseteq \mathrm{M}_{\mathrm{SI}}(\mathbf{W})$.

By stating the contrapositive of Corollary 9-4.11, we obtain the following observation, contained in Ploščica [259, Consequence 2.5].

Corollary 9-4.12. *Let* $\mathbf{V}$ *and* $\mathbf{W}$ *be finitely generated, congruence-distributive varieties of algebras with (not necessarily equal) finite similarity type. If* $s(\mathbf{V}) > s(\mathbf{W})$*, then* $\mathrm{crit}(\mathbf{V}; \mathbf{W}) \leq \aleph_0$.

Those results are applied in Ploščica [259, Section 4] to some varieties generated by small finite lattices. For example, by following the notation of Jipsen and Rose [207] and denoting the variety generated by a finite lattice K by boldfacing the letter in the name of K,

- $s(\mathsf{M}_{3^2}) = 3$ (*the lattice* M_{3^2} *is represented on the left-hand side of Figure* 9-4.1) while $s(\mathsf{M}_n) = 2$ for $3 \leq n < \omega$; hence $\mathrm{crit}(\mathbf{M}_{3^2}; \mathbf{M}_n) \leq \aleph_0$. Since both congruence classes have the same finite members (namely, the finite Boolean lattices), it follows that $\mathrm{crit}(\mathbf{M}_{3^2}; \mathbf{M}_n) = \aleph_0$.

- $\mathrm{crit}(\mathbf{M}_3; \mathbf{N}_5) = \aleph_0$. This follows easily from the proof of Ploščica [259, Lemma 4.4], see also Exercise 9.20. On the other hand, it is obvious that $\mathrm{crit}(\mathbf{N}_5; \mathbf{V}) = 5$, for any nontrivial finitely generated variety $\mathbf{V}$ of modular lattices (observe that $\mathrm{Con_c}\, \mathsf{N}_5 \notin \mathrm{Con_c}\, \mathbf{V}$).

- $s(\mathsf{L}_3) = 4$ (*see also Figure* 9-5.2 *for the* L_i), thus both $\mathrm{crit}(\mathbf{L}_3; \mathbf{N}_5)$ and $\mathrm{crit}(\mathbf{L}_3; \mathbf{M}_{3^2})$ are smaller than or equal to $\aleph_0$.

- Further tools, introduced in Ploščica [267], make it possible to prove that $\mathrm{crit}(\mathbf{A}_1; \mathbf{M}_{3^3}) = \aleph_0$, where the lattices A_1 and M_{3^3} are represented in Figure 9-4.1.

Some of those results are further extended in Ploščica [261, 262]. For example, the following result is contained in the combination of [261, Theorem 3.1] and [262, Theorem 4.1] (see Exercise 9.27 for a partial explanation).

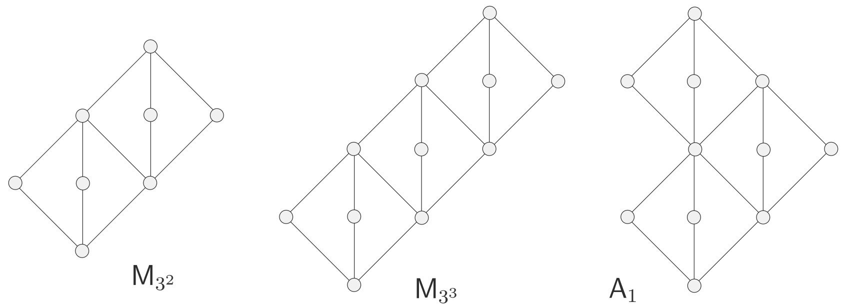

Figure 9-4.1: A few modular lattices.

◇ **Theorem 9-4.13** (Ploščica 2004)**.** *The congruence classes* $\operatorname{Con}\mathbf{N}_5$*,* $\operatorname{Con}\mathbf{L}_1$*,* $\operatorname{Con}\mathbf{L}_2$*, and* $\operatorname{Con}(\mathbf{L}_1 \vee \mathbf{L}_2)$ *all have the same finite members, which are the finite distributive lattices* D *such that* $\operatorname{Mi} D$ *is the disjoint union of two antichains* M_1 *and* M_2 *such that for each* $x_1 \in M_1$ *there are exactly two* $x_2 \in M_2$ *with* $x_1 < x_2$*.*

Since L_2 is the dual lattice of L_1, $\operatorname{Con}\mathbf{L}_1 = \operatorname{Con}\mathbf{L}_2$. Nevertheless, Ploščica proves in [262, Section 4] that the compact congruence classes $\operatorname{Con_c}\mathbf{N}_5$, $\operatorname{Con_c}\mathbf{L}_1$, and $\operatorname{Con_c}(\mathbf{L}_1 \vee \mathbf{L}_2)$ contain different countable members. Therefore,

◇ **Theorem 9-4.14** (Ploščica 2004)**.** $\operatorname{crit}(\mathbf{L}_1 \vee \mathbf{L}_2; \mathbf{L}_1) = \operatorname{crit}(\mathbf{L}_1; \mathbf{N}_5) = \aleph_0$*.*

It is an open problem whether, for finitely generated, congruence-distributive varieties $\mathbf{V}$ and $\mathbf{W}$ of algebras with finite similarity type, the condition $\operatorname{crit}(\mathbf{V}; \mathbf{W}) \geq \aleph_0$ is decidable. The already difficult special case where $\mathbf{V} = \mathbf{N}_5$ is partly settled in Lemma 9 and Theorem 10 of Gillibert and Ploščica [112].

◇ **Theorem 9-4.15** (Gillibert and Ploščica, preprint 2012)**.** *The problem whether* $\operatorname{crit}(\mathbf{N}_5; \mathbf{W}) \geq \aleph_0$*, for* $\mathbf{W}$ *a finitely generated, congruence-distributive variety of algebras with finite similarity type, whose non-simple subdirectly irreducible members all have congruence lattice isomorphic to* $\operatorname{Con}\mathsf{N}_5$*, is decidable.*

The statements of Lemma 9 and Theorem 10 of Gillibert and Ploščica [112] do not require the finiteness of the similarity type of $\mathbf{W}$: the latter is involved in the statement of Theorem 9-4.15 only in order for the decidability statement to make sense.

9-4.3 More critical points

All the examples of critical points that we have seen so far are either countable or equal to $\aleph_2$. It remained open for a few years (see Problem 5 in Tůma and Wehrung [316]) whether $\aleph_1$ could occur as a critical point. This problem was solved in Gillibert [106].

♢ **Theorem 9-4.16** (Gillibert 2009)**.** *Denote by* $\mathbf{A}$ *and* $\mathbf{B}$ *the varieties of (modular) lattices generated by the top lattice of Figure 9-4.2, and the three bottom lattices of Figure 9-4.2, respectively. Then* $\operatorname{crit}(\mathbf{A};\mathbf{B}) = \aleph_1$.

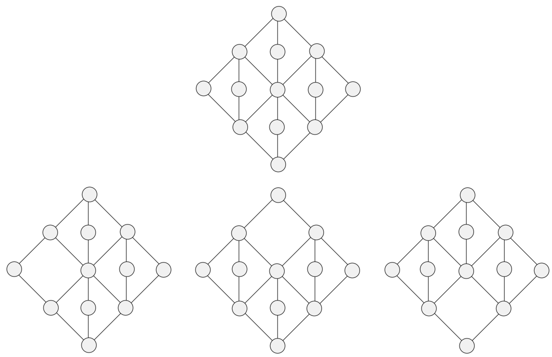

Figure 9-4.2: Four finite modular lattices.

The argument of the proof of Theorem 9-4.16 is radically different from those involved in the earlier subsections, and it involves methods of categorical algebra, in some extent inspired by Gabriel and Ulmer [100], Adámek and Rosický [3], and developed in Gillibert and Wehrung [114].

Those methods are very complex and we can only give a very limited hint of that material here, referring the reader to Gillibert and Wehrung [113, 114] for (much) more detail. For cardinals κ and λ and a poset P, let $(\kappa, {<}\lambda) \rightsquigarrow P$ hold if for every mapping $F\colon \operatorname{Pow}(\kappa) \to [\kappa]^{<\lambda}$, there exists a one-to-one map $f\colon P \hookrightarrow \kappa$ such that

$$F\big(f(P\downarrow x)\big) \cap f(P\downarrow y) \subseteq f(P\downarrow x)\,, \qquad \text{for all } x \leq y \text{ in } P\,.$$

The $\rightsquigarrow$ notation is related to the arrow notation of Section 9-2.3 (cf. Notation 9-2.21) by the following easy (though not completely trivial) result, established in Gillibert and Wehrung [113, Proposition 3.4].

♢ **Theorem 9-4.17** (Gillibert and Wehrung 2011)**.** *The statements* $(\kappa, {<}\lambda) \rightsquigarrow ([\varrho]^{<\omega}, \subseteq)$ *and* $(\kappa, {<}\omega, \lambda) \to \varrho$ *are equivalent, for every cardinal* ϱ *and all infinite cardinals* κ, λ.

The *restricted Kuratowski index* of a finite poset P, denoted by $\operatorname{kur}_0(P)$, is defined as 0, if P is an antichain, and the least positive integer n such that $(\aleph_{n-1}, {<}\aleph_0) \rightsquigarrow P$, otherwise. This number is defined for any finite poset P, and it is related to the *order-dimension*[3] $\dim(P)$ of P by the following result, which is a trivial consequence of Gillibert and Wehrung [113, Proposition 4.7].

[3]That is, the least n such that P order-embeds into a product of n chains.

$\diamondsuit$ **Theorem 9-4.18** (Gillibert and Wehrung 2011)**.** *The inequality* $\mathrm{kur}_0(P) \leq \dim(P)$ *holds for any finite poset* P*.*

The following definition is stated, in a somewhat more general form, in Gillibert and Wehrung [114, Definition 4.8.1].

Definition 9-4.19. A variety $\mathbf{V}$ of algebras is

- *congruence-proper* if $\mathrm{Con}\, A$ finite implies A finite, for any $A \in \mathbf{V}$;
- *strongly congruence-proper* if it is congruence-proper and for every finite $(\vee, 0)$-semilattice S there are only finitely many (up to isomorphism) $A \in \mathbf{V}$ such that $S \cong \mathrm{Con}\, A$.

It is observed in Gillibert and Wehrung [114, Section 4.10], as a consequence of Tame Congruence Theory (see Hobby and McKenzie [197]) that *Every finitely generated variety, of finite similarity type, satisfying a nontrivial congruence identity, is strongly congruence-proper.* This holds, in particular, for congruence-modular varieties, such as varieties of groups (or even *loops*), rings, modules, and, of course, lattices (the latter being congruence-distributive). For those varieties, the results of Freese and McKenzie [92] are sufficient in order to ensure strong congruence-properness.

The proof of the following result is very complex, and it involves the full power of the categorical methods introduced in Gillibert and Wehrung [114]. It is stated in a somewhat more general form[4] in Gillibert and Wehrung [114, Theorem 4.9.2]. It expresses the fact that the existence of unliftable diagrams implies small critical points.

$\diamondsuit$ **Theorem 9-4.20** (Gillibert and Wehrung 2011)**.** *Let* $\mathbf{A}$ *and* $\mathbf{B}$ *be varieties of algebras, with* $\mathbf{B}$ *locally finite and congruence-proper, and let* P *be a finite lattice. We assume that there exists a* P*-indexed diagram* $\vec{A}$ *of finite members of* $\mathbf{A}$ *such that there is no* P*-indexed diagram* $\vec{B}$ *of* $\mathbf{B}$ *with* $\mathrm{Con}\,\vec{A} \cong \mathrm{Con}\,\vec{B}$*. Then* $\mathrm{crit}(\mathbf{A};\mathbf{B}) \leq \aleph_{\mathrm{kur}_0(P)-1}$*.*

In particular, for the proof of Theorem 9-4.16, Gillibert finds a diagram $\vec{A}$ of finite members of $\mathbf{A}$, indexed by the square B_2, such that there is no diagram $\vec{B}$ of $\mathbf{B}$ such that $\mathrm{Con}\,\vec{A} \cong \mathrm{Con}\,\vec{B}$. Since $\dim(\mathsf{B}_2) = 2$ and by Theorem 9-4.20, this implies that $\mathrm{crit}(\mathbf{A};\mathbf{B}) \leq \aleph_1$.

Proving, conversely, that $\mathrm{crit}(\mathbf{A};\mathbf{B})$ is large involves, comparatively, easier (though often technical) arguments. Expressing $A \in \mathbf{A}$ as a directed colimit of a diagram $\vec{A}$ of finite members of $\mathbf{A}$ (hence we need $\mathbf{A}$ to be locally finite), we construct directly a diagram $\vec{B}$ of $\mathbf{B}$ such that $\mathrm{Con}\,\vec{A} \cong \mathrm{Con}\,\vec{B}$, by propagating, *via* a suitable uniqueness result, some pattern on the finite initial segments of $\vec{B}$.

This is done for the proof of Theorem 9-4.16 above, and also for many results of Gillibert [107]. We end Section 9-4.3 with a sample of the latter.

[4] Involving more general types of posets than finite lattices called *almost join-semilattices*, together with *quasivarieties* and *relative congruence lattices*.

Denote by $\mathbf{Var}(\mathbf{C})$ ($\mathbf{Var}_0(\mathbf{C})$, $\mathbf{Var}_{0,1}(\mathbf{C})$, respectively) the variety of lattices (0-lattices, $(0,1)$-lattices, respectively) generated by a class $\mathbf{C}$ of lattices (0-lattices, $(0,1)$-lattices, respectively).

$\diamondsuit$ **Theorem 9-4.21** (Gillibert 2009)**.** *Let F be a field, let $\mathbf{V}$ be a variety of modular lattices (resp., a variety of bounded modular lattices), and let $\ell \leq \omega$ such that $\operatorname{len} K \leq \ell$ for each simple $K \in \mathbf{V}$. Then*

$$\operatorname{crit}(\mathbf{V}; \mathbf{Var}_0(\operatorname{Sub} F^n \mid 0 \leq n < \ell)) \geq \aleph_2$$
$$(\textit{resp.},\ \operatorname{crit}(\mathbf{V}; \mathbf{Var}_{0,1}(\operatorname{Sub} F^n \mid 0 \leq n < \ell)) \geq \aleph_2)\,.$$

For $3 \leq m, n \leq \omega$, set $\mathsf{M}_{m,n} = \{0,1\} \cup \{\, a_i \mid i < m \,\} \cup \{\, b_j \mid j < n \,\}$, ordered in such a way that 0 is the bottom element, 1 is the top element, $a_i < b_0$ for each $i < m$, and $a_0 < b_j$ for each $j < n$ (cf. Figure 9-4.3). Then $\mathsf{M}_{m,n}$ is a modular lattice.

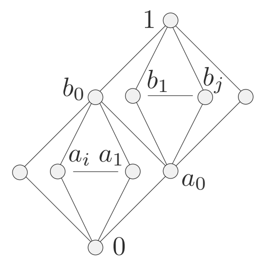

Figure 9-4.3: The lattice $\mathsf{M}_{m,n}$.

$\diamondsuit$ **Theorem 9-4.22** (Gillibert 2009)**.** *The following statements hold, for all m, n with $3 \leq m < n \leq \omega$:*

$$\begin{aligned}
\operatorname{crit}(\mathbf{M}_n; \mathbf{M}_m) &= \operatorname{crit}(\mathbf{M}_n; \mathbf{M}_{m,m}) \\
&= \operatorname{crit}(\mathbf{M}_n^{0,1}; \mathbf{M}_{m,m}) \\
&= \operatorname{crit}(\mathbf{M}_n^{0,1}; \mathbf{M}_{m,m}^{0,1}) \\
&= \operatorname{crit}(\mathbf{M}_n; \mathbf{M}_{m,m}^{0}) \\
&= \operatorname{crit}(\mathbf{M}_n; \mathbf{M}_m^{0}) \\
&= \aleph_2\,.
\end{aligned}$$

$\diamondsuit$ **Theorem 9-4.23** (Gillibert 2009)**.** *The following statements hold, for every finite field F and every integer $n > 1 + \operatorname{card} F$.*

$$\begin{aligned}
\operatorname{crit}(\mathbf{M}_n; \mathbf{Var}(\operatorname{Sub} F^3)) &= \operatorname{crit}(\mathbf{M}_n; \mathbf{Var}_0(\operatorname{Sub} F^3)) \\
&= \operatorname{crit}(\mathbf{M}_n^{0,1}; \mathbf{Var}(\operatorname{Sub} F^3)) \\
&= \operatorname{crit}(\mathbf{M}_n^{0,1}; \mathbf{Var}_{0,1}(\operatorname{Sub} F^3)) \\
&= \aleph_2\,.
\end{aligned}$$

♢ **Theorem 9-4.24** (Gillibert 2009)**.** *The following statements hold, for all finite fields F and K such that* $\operatorname{card} F > \operatorname{card} K$.

$$\begin{aligned}\operatorname{crit}(\mathbf{Var}(\operatorname{Sub} F^3); \mathbf{Var}(\operatorname{Sub} K^3)) &= \operatorname{crit}(\mathbf{Var}(\operatorname{Sub} F^3); \mathbf{Var}_0(\operatorname{Sub} K^3))\\ &= \operatorname{crit}(\mathbf{Var}_{0,1}(\operatorname{Sub} F^3); \mathbf{Var}(\operatorname{Sub} K^3))\\ &= \operatorname{crit}(\mathbf{Var}_{0,1}(\operatorname{Sub} F^3); \mathbf{Var}_{0,1}(\operatorname{Sub} K^3))\\ &= \aleph_2\,.\end{aligned}$$

♢ **Theorem 9-4.25** (Gillibert 2009)**.**

(1) *Let* $\mathbf{V}$ *be a finitely generated variety of lattices* (*resp., of* 0*-lattices*)*. If* $\mathsf{M}_3 \in \mathbf{V}$*, then* $\operatorname{crit}(\mathbf{M}_\omega; \mathbf{V}) = \operatorname{crit}(\mathbf{M}_\omega^0; \mathbf{V}) = \aleph_2$.

(2) *Let* $\mathbf{V}$ *be a finitely generated variety of* $(0,1)$*-lattices. If* $\mathsf{M}_3 \in \mathbf{V}$*, then* $\operatorname{crit}(\mathbf{M}_\omega^{0,1}; \mathbf{V}) = \aleph_2$.

9-4.4 The possible values of critical points

In this subsection we shall discuss the two most tremendous results obtained, to this date, on critical points between varieties; both are due to Gillibert. The first result is proved for varieties of lattices; it is established in Gillibert [108, Section 4]. We denote by $\mathbf{V}^{0,1}$ the variety of $(0,1)$-lattices consisting of all bounded members of a variety $\mathbf{V}$ of lattices.

♢ **Theorem 9-4.26** (Gillibert 2012)**.** *The following statements hold, for all varieties* $\mathbf{V}$ *and* $\mathbf{W}$ *of lattices such that every simple member of* $\mathbf{W}$ *contains a prime interval.*

(1) *If* $\mathbf{V}$ *is contained neither in* $\mathbf{W}$ *nor in its dual, then* $\operatorname{crit}(\mathbf{V}^{0,1}; \mathbf{W}) \leq \aleph_2$*; that is, there exists a bounded lattice* $K \in \mathbf{V}$ *such that* $\operatorname{card} K \leq \aleph_2$ *and* $\operatorname{Con} K \notin \operatorname{Con} \mathbf{W}$*. In particular,* $\operatorname{crit}(\mathbf{V}; \mathbf{W}) \leq \aleph_2$.

(2) *If* $\mathbf{V}$ *has no congruence* n*-permutable member, neither in* $\mathbf{W}$ *nor in its dual, with* $n \geq 4$*, then there exists a congruence* n*-permutable bounded lattice* $K \in \mathbf{V}$ *such that* $\operatorname{card} K \leq \aleph_2$ *and* $\operatorname{Con} K \notin \operatorname{Con} \mathbf{W}$.

The proof of Theorem 9-4.26 is extremely complex, and it involves the full power of the categorical results from Gillibert and Wehrung [114] together with a heavy preparatory work from Gillibert [109].

The second result is established in Gillibert [110].

♢ **Theorem 9-4.27** (Gillibert, 2014)**.** *Let* $\mathbf{V}$ *and* $\mathbf{W}$ *be locally finite varieties of algebras such that for each finite algebra* $A \in \mathbf{V}$ *there are, up to isomorphism, only finitely many* $B \in \mathbf{W}$ *such that* $\operatorname{Con_c} A \cong \operatorname{Con_c} B$*, and every such* B *is finite. If* $\operatorname{Con_c} \mathbf{V} \nsubseteq \operatorname{Con_c} \mathbf{W}$*, then* $\operatorname{crit}(\mathbf{V}; \mathbf{W}) \leq \aleph_2$.

It is observed in Gillibert [110] that Theorem 9-4.27 extends to quasivarieties of first-order structures, with finitely many relation symbols, and relative congruence lattices.

The proof of Theorem 9-4.27 is also extremely complex, and it improves both the algebraic content and the cardinality bound of Gillibert and Wehrung [114, Theorem 4.9.4]. The cardinality bound $\aleph_2$ is sharp. Theorem 9-4.27 applies, in particular, to the case where $\mathbf{V}$ is a locally finite variety and $\mathbf{W}$ is a finitely generated variety of finite similarity type satisfying a nontrivial congruence lattice identity. For the special case of varieties of lattices, Theorem 9-4.26 is a better result, since it does not require any local finiteness assumption on $\mathbf{V}$ and since it states that $\operatorname{Con}\mathbf{V} \subseteq \operatorname{Con}\mathbf{W}$ can happen only for the trivial reason (namely, $\mathbf{V}$ is contained either in $\mathbf{W}$ or its dual).

The problem, whether the assumption that every simple member of $\mathbf{W}$ has a prime interval can be dispensed with in the statement of Theorem 9-4.26, is still open. Nevertheless, Gillibert was able to prove, in [108, Corollary 5.4], the following "functorial" version of Theorem 9-4.26, now without any restriction on $\mathbf{W}$.

$\diamondsuit$ **Theorem 9-4.28** (Gillibert 2012). *Let* $\mathbf{V}$ *and* $\mathbf{W}$ *be varieties of lattices and let* $\Psi\colon \mathbf{V} \to \mathbf{W}$ *be a functor such that* $\operatorname{Con_c}(\Psi(A)) \cong \operatorname{Con_c} A$ *naturally on* $A \in \mathbf{V}$. *Then* $\mathbf{V}$ *is contained either in* $\mathbf{W}$ *or its dual.*

Observe that the assumption $\operatorname{Con_c} \circ \Psi \cong \operatorname{Con_c}$ of Theorem 9-4.28, although it implies that $\mathbf{V}$ is contained in either $\mathbf{W}$ or its dual, does not imply that Ψ is either the inclusion functor or its composition with dualization. For example, take $\mathbf{V} = \mathbf{W} = \mathbf{L}$, the variety of all lattices, and $\Psi(L) = \mathsf{M}_3[L]$ for every lattice L (this assignment is easily seen to extend to a functor).

9-5. Further topics

9-5.1 Congruence-lifting small diagrams of finite Boolean semilattices

Many of the constructions leading to infinite counterexamples to congruence representation problems have a finite counterpart, no longer stated in terms of representing *objects*, but rather in terms of *diagrams*.

Our first example, whose origin can be traced back to Wehrung [325] but which has appeared since then in many places and in many different forms, is related to *permutable congruences*, and it can be viewed as the diagram counterpart of Section 9-2.3. We shall use the notation and terminology of Section 9-2.1 about V-distances. Denote by Π the functor, from the category of all semilattice-valued distances to the category of all $(\vee, 0)$-semilattices, that to a distance $\lambda\colon X \times X \to S$ associates S. Using the maps $\boldsymbol{e}\colon \mathbf{2} \to \mathbf{2}^2$, $\boldsymbol{s}\colon \mathbf{2}^2 \to \mathbf{2}^2$, and $\boldsymbol{p}\colon \mathbf{2}^2 \to \mathbf{2}$ defined by the rules

$$\boldsymbol{e}(x) = (x, x)\,, \quad \boldsymbol{s}(x, y) = (y, x)\,, \quad \boldsymbol{p}(x, y) = x \vee y\,, \quad \text{for all } x, y \in \{0, 1\}\,,$$

we form the (commutative) cube $\mathcal{D}_{\mathrm{c}}$ of finite Boolean $(\vee, 0, 1)$-semilattices and $(\vee, 0, 1)$-homomorphisms represented on the left-hand side of Figure 9-5.1. This cube is the hard core of many examples of unliftable cubes that appear in the literature. It is a blend between the counterexamples underlying Růžička, Tůma, and Wehrung [287, Theorem 7.2] (with a proof inspired from the one of that result) and Wehrung [339, Section 10].

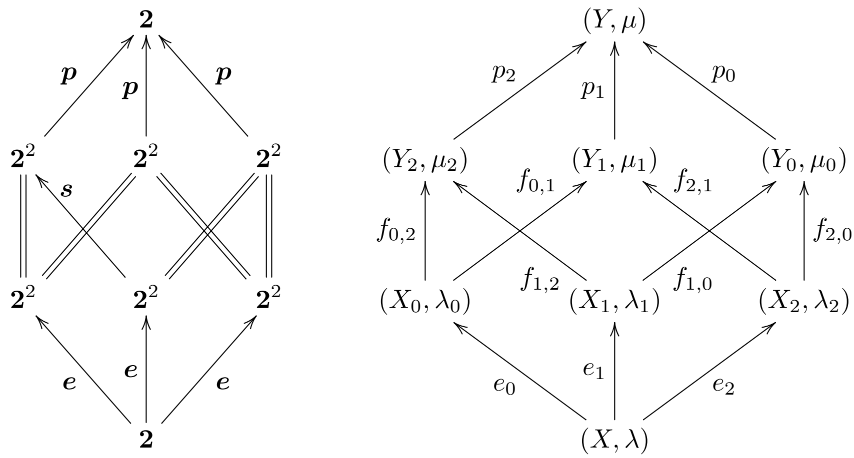

Figure 9-5.1: The cube $\mathcal{D}_{\mathrm{c}}$ and a lifting, with respect to Π, of $\mathcal{D}_{\mathrm{c}}$.

Theorem 9-5.1. *There is no lifting, with respect to the functor* Π*, of the cube* $\mathcal{D}_{\mathrm{c}}$*, by any cube of semilattice-valued distances, such that, indexing the lifting as on the right-hand side of Figure* 9-5.1*,* λ *is surjective and each* λ_i *is a V-distance of type* 1*.*

In particular, $\mathcal{D}_{\mathrm{c}}$ has no lifting, with respect to Π, by join-generating V-distances of type 1.

Proof. Denote by $\mathcal{C}$ a cube, represented on the right-hand side of Figure 9-5.1, lifting $\mathcal{D}_{\mathrm{c}}$. After composing the measures in $\mathcal{C}$ by the components of the given natural equivalence between $\Pi\mathcal{C}$ and $\mathcal{D}_{\mathrm{c}}$, we may assume that $\Pi\mathcal{C} = \mathcal{D}_{\mathrm{c}}$: hence, for example, $\lambda\colon X \times X \to \mathbf{2}$, $\lambda_i\colon X_i \times X_i \to \mathbf{2}^2$, and so on. Furthermore, after replacing each set from $\mathcal{C}$ with its quotient by the kernel of the corresponding measure, we may assume that every measure from $\mathcal{C}$ has kernel zero. Since every morphism from $\mathcal{D}_{\mathrm{c}}$ separates zero, it follows that every morphism from $\mathcal{C}$ is one-to-one. Hence we may replace each set from $\mathcal{C}$ by its canonical image in Y, and thus assume that all maps e_i, $f_{i,j}$, and p_j, for distinct $i, j \in \{0, 1, 2\}$, are inclusion mappings.

Since λ is surjective, there are $x, y \in X$ such that $\lambda(x, y) = 1$. Since e_i is the inclusion mapping from X into X_i, it follows that $\lambda_i(x, y) = \boldsymbol{e}(1) = (1, 1)$, for each $i < 3$. Since λ_i is a V-distance of type 1, there exists $z_i \in X_i$ such that $\lambda_i(x, z_i) = (1, 0)$ and $\lambda_i(z_i, y) = (0, 1)$.

In particular, $\mu_2(x, z_0) = \lambda_0(x, z_0) = (1, 0)$ and $\mu_2(x, z_1) = \boldsymbol{s}\lambda_1(x, z_1) = \boldsymbol{s}(1, 0) = (0, 1)$. Since

$$(1, 0) = \mu_2(x, z_0) \leq \mu_2(x, z_1) \vee \mu_2(z_0, z_1) = (0, 1) \vee \mu_2(z_0, z_1)\,,$$

we get $\mu_2(z_0, z_1) \geq (1, 0)$, so $\mu(z_0, z_1) = \boldsymbol{p}\mu_2(z_0, z_1) = 1$.

Similar calculations, easier to follow because no longer involving the morphism $\boldsymbol{s}$, yield $\mu_1(x, z_i) = \lambda_i(x, z_i) = (1, 0)$, and, likewise, $\mu_1(z_i, y) = (0, 1)$, for each $i \in \{0, 2\}$. It follows that $\mu_1(z_0, z_2) \leq \mu_1(z_0, x) \vee \mu_1(x, z_2) \leq (1, 0)$, and, likewise, $\mu_1(z_0, z_2) \leq \mu_1(z_0, y) \vee \mu_1(y, z_2) \leq (0, 1)$, so $\mu_1(z_0, z_2) = (0, 0)$, and so $\mu(z_0, z_2) = 0$. Likewise, $\mu_0(z_1, z_2) = (0, 0)$, so $\mu(z_1, z_2) = 0$, hence

$$1 = \mu(z_0, z_1) \leq \mu(z_0, z_2) \vee \mu(z_2, z_1) = 0\,,$$

a contradiction. $\square$

By using Proposition 9-2.3 together with Exercise 9.1, we obtain immediately the following non-representation result.

Corollary 9-5.2. *There is no lifting, with respect to the functor* $\mathrm{Con_c}$, *of the cube* $\mathcal{D}_\mathrm{c}$, *by any cube of congruence-permutable algebras.*

Nevertheless, the cube $\mathcal{D}_\mathrm{c}$ can be lifted, with respect to the functor $\mathrm{Con_c}$, by a cube of almost congruence-permutable finite lattices, see Exercise 9.28. For an example of a cube of finite Boolean semilattices, with $(\vee, 0, 1)$-embeddings, that cannot be lifted by any cube of join-generating V-distances of type 3/2 (thus that cannot be lifted by any cube of almost congruence-permutable algebras), see Růžička, Tůma, and Wehrung [287, Theorem 7.2]. For an example of a cube of finite Boolean semilattices, with $(\vee, 0, 1)$-embeddings, that can be lifted by a cube of almost congruence-permutable lattices but not by any cube of congruence-permutable lattices, see Tůma and Wehrung [314, Theorem 6.3]. The cube of that example is the same one as the one of Exercise 9.29, which cannot be lifted by any diagram of join-generating V-distances of type 1 (unlike those in $\mathcal{D}_\mathrm{c}$, all its arrows are *embeddings*).

On the other hand, it is not possible to replace "type 1" by "type 2" in Theorem 9-5.1, due to Exercise 9.29 together with the following result, established in Růžička, Tůma, and Wehrung [287, Theorem 7.1].

$\diamondsuit$ **Theorem 9-5.3** (Růžička, Tůma, and Wehrung 2007)**.** *There exists a functor* Γ, *from the category of all distributive* $(\vee, 0)$*-semilattices with* $(\vee, 0)$*-embeddings, to the category of all semilattice-valued surjective V-distances of type* 2, *such that* $\Pi \circ \Gamma$ *is equivalent to the identity functor.*

In particular, every diagram of distributive $(\vee,0)$-semilattices with $(\vee,0)$-embeddings can be lifted by a diagram of semilattice-valued V-distances of type 2.

As already mentioned, the cubes that we just described were inspired by the counterexamples of cardinality $\aleph_2$ developed in Wehrung [325]. The discovery process could also work the other way around. For example, Gillibert's example, leading to Theorem 9-4.16, of two finitely generated modular lattice varieties with critical point $\aleph_1$, follows from the consideration of a certain commutative *square* of Boolean semilattices, liftable in the big variety but not in the small one.

The direction "from infinite counterexample to finite diagram" can be further illustrated by the following result, inspired by the prior result that $\operatorname{crit}(\mathbf{M}_{n+1}^{0,1};\mathbf{M}_n)=\aleph_2$ together with Gillibert's work on critical points, and established in Ploščica [268, Theorem 4.2]. Recall that C_n denotes the n-element chain, for every positive integer n.

$\diamondsuit$ **Theorem 9-5.4** (Ploščica 2009)**.** *Let* $3\leq n<\omega$*. Then there exists a diagram* $\mathcal{A}_n$ *of finite members of* $\mathbf{M}_{n+1}^{0,1}$*, indexed by the lattice* $\mathsf{C}_n\times\mathsf{C}_n\times\mathsf{C}_n$*, such that* $\operatorname{Con}\mathcal{A}_n$ *has no congruence-lifting in* $\mathbf{M}_n$*.*

The construction and proof leading to Theorem 9-5.4 are both quite sophisticated, with a strong combinatorial flavor, in particular involving the four-color theorem for planar graphs.

9-5.2 Congruence-lifting poset-indexed diagrams of semilattices

Congruence semilattices of arbitrary algebras are completely described by Grätzer and Schmidt's Theorem 9-2.8. One of the difficulties encountered while trying to extend this result from objects (viz. $(\vee,0)$-semilattices) to diagrams (of $(\vee,0)$-semilattices with $(\vee,0)$-homomorphisms) lies in the absence of a cardinality bound on the similarity type of the algebras of that theorem: the algebras of Theorem 9-2.8 require many operations. Due to results in Freese, Lampe, and Taylor [91], this is unavoidable: some algebraic lattices (in particular, subspace lattices of infinite-dimensional vector spaces over uncountable fields) require many operations in any of their congruence representations.

A possible way to get around this difficulty is to extend the category of all algebras over a given similarity type, by allowing variable similarity types. Hence, for algebras A and B, with respective similarity types Σ_A and Σ_B, a homomorphism from A to B is defined, only in case $\Sigma_A\subseteq\Sigma_B$, as a map $\varphi\colon A\to B$ such that

$$\varphi\bigl(f^A(a_1,\dots,a_n)\bigr)=f^B\bigl(\varphi(a_1),\dots,\varphi(a_n)\bigr)\,,$$

for any operation symbol $f\in\Sigma_A$ of arity n and any $a_1,\dots,a_n\in A$. Following the terminology in force in Gillibert and Wehrung [114], we get in this way the category **MAlg** of all *monotone-indexed algebras*: the objects of **MAlg** are just all (universal) algebras, on all similarity types, and the morphisms in **MAlg** are the homomorphisms defined above. By using CLL and larders, the following diagram extension of Grätzer and Schmidt's Theorem is established in Gillibert and Wehrung [114, Theorem 4.7.2]. We refer the reader to Notation 9-2.21 for the symbol $(\kappa, {<}\omega, \lambda) \to \lambda$.

$\diamondsuit$ **Theorem 9-5.5** (Gillibert and Wehrung 2011)**.** *Let P be a poset and let $\vec{S} = (S_p, \sigma_p^q \mid p \leq q \text{ in } P)$ be a P-indexed diagram of $(\vee, 0)$-semilattices with $(\vee, 0)$-homomorphisms. If either P is finite or there are cardinals κ and λ, with λ regular, such that P and all S_p, for $p \in P$, have cardinality smaller than λ and the relation $(\kappa, {<}\omega, \lambda) \to \lambda$ holds, then every P-indexed diagram of $(\vee, 0)$-semilattices with $(\vee, 0)$-homomorphisms can be lifted, with respect to the functor* $\mathrm{Con_c}$*, by some diagram of unary monotone-indexed algebras.*

In particular, we get the following corollary.

Corollary 9-5.6. *Let P be a poset. If either P is finite or there exists a proper class of Erdős cardinals, then every P-indexed diagram of $(\vee, 0)$-semilattices with $(\vee, 0)$-homomorphisms can be lifted by some diagram of unary monotone-indexed algebras.*

Likewise, by using, instead of Grätzer and Schmidt's Theorem 9-2.8, Lampe's Theorem, established in Lampe [238], that every $(\vee, 0, 1)$-semilattice is isomorphic to the congruence semilattice of some *groupoid* (i.e., a nonempty set with a binary operation), we obtain the following result, established in Gillibert and Wehrung [114, Proposition 4.7.4].

$\diamondsuit$ **Theorem 9-5.7** (Gillibert and Wehrung 2011)**.** *Let P be a poset. If either P is finite or there exists a proper class of Erdős cardinals, then every P-indexed diagram of $(\vee, 0, 1)$-semilattices and $(\vee, 0, 1)$-homomorphisms can be lifted by some diagram of groupoids.*

Hence Theorem 9-5.7 extends, to arbitrary poset-indexed diagrams, the result obtained by Lampe [239] for one arrow between $(\vee, 0, 1)$-semilattices.

Theorem 9-5.7 applies, in particular, to the finite diagram $\mathcal{D}_{\bowtie}$ of finite Boolean $(\vee, 0, 1)$-semilattices represented in Figure 7-4.3: hence $\mathcal{D}_{\bowtie}$ has a congruence-lifting by a diagram of groupoids. Recall that this diagram has no congruence-lifting in any variety satisfying a nontrivial congruence lattice identity (cf. Theorem 7-4.15).

9-5.3 Lifting diagrams of semilattices by diagrams of intervals in subgroup lattices of groups

The following deep characterization of algebraic lattices is established in Tůma [311].

♢ **Theorem 9-5.8** (Tůma 1989)**.** *Every algebraic lattice is isomorphic to a closed interval in the subgroup lattice of some group.*

For a (not necessarily normal) subgroup H of a group G, the set $G/H = \{ xH \mid x \in G \}$ of all left H-cosets in G can be endowed with all left translations

$$\overline{g}\colon G/H \to G/H\,,\ \boldsymbol{x} \mapsto g\boldsymbol{x}\,, \quad \text{for } g \in G\,,$$

and the congruences of G/H (endowed with this set of operations) are exactly the relations

$$\boldsymbol{\theta}_K = \{ (xH, yH) \mid x, y \in G \text{ and } x^{-1}y \in K \}\,, \quad \text{for } H \leq K \leq G\,.$$

(By $H \leq K$ we shall mean, in the present subsection, that H is a subgroup of K.) Therefore, $\operatorname{Con}(G/H) \cong [H, G]$, which shows that Tůma's Theorem easily extends Grätzer and Schmidt's Theorem 9-2.8. To paraphrase this, any representation of an algebraic lattice L as an interval in the subgroup lattice of a group yields a congruence representation of L by a universal algebra.

However, while the book Gillibert and Wehrung [114] provides a diagram version of Grätzer and Schmidt's Theorem (viz. Theorem 9-5.5), it does not provide any diagram version of Tůma's Theorem. We shall fill this gap here, *via* an application of CLL (the central result of [114]) to Tůma's Theorem. Hence, in the remainder of the present subsection, we shall use the notation and terminology of Gillibert and Wehrung [114].

Denote by $\mathcal{B}$ the category whose objects are the pairs (G^*, G), where G^* is a subgroup of a group G, and where a morphism of (G^*, G) to (H^*, H) is a group homomorphism $\varphi\colon G \to H$ such that $\varphi(G^*) \subseteq H^*$. Denote by L_{c} the $(\vee, 0)$-semilattice of all compact elements of a lattice L, and set $\Psi(G^*, G) = [G^*, G]_{\mathsf{c}}$, for every object (G^*, G) of $\mathcal{B}$. It is easy to see that Ψ extends naturally to a functor from $\mathcal{B}$ to the category $\mathcal{S} = \mathbf{Sem}_{\vee,0}$ of all $(\vee, 0)$-semilattices with $(\vee, 0)$-homomorphisms, by setting $\Psi(\varphi)(X) = H^* \vee \varphi(X)$, for any $X \in \Psi(G^*, G)$. Furthermore, by using Lemma 7-2.3, it is easy to see that this functor preserves all directed colimits.

As in many applications in Gillibert and Wehrung [114], we define $\mathcal{S}^{\Rightarrow}$ as the subcategory of $\mathcal{S}$ consisting of all *ideal-induced homomorphisms* (cf. Section 7-5.5), and we shall denote the arrows of $\mathcal{S}^{\Rightarrow}$ (the *double arrows*) in the form $\varphi\colon S \Rightarrow T$. Hence, using the notation of [114], $\mathcal{S}^{\Rightarrow} = \mathbf{Sem}^{\mathrm{idl}}_{\vee,0}$. Our first task is to verify the existence of enough *projectability witnesses* (cf. [114, Section 1.5]).

Lemma 9-5.9. *Let (G^*, G) be an object of $\mathcal{B}$ and let S be a $(\vee, 0)$-semilattice. Then any double arrow $\varphi\colon \Psi(G^*, G) \Rightarrow S$ has a projectability witness.*

Proof. Up to isomorphism, φ is the canonical projection from $[G^*, G]_{\mathsf{c}}$ to $[G^*, G]_{\mathsf{c}}/\boldsymbol{I}$, for an ideal $\boldsymbol{I}$ of $[G^*, G]_{\mathsf{c}}$. Necessarily, $\boldsymbol{I} = [G^*, H]_{\mathsf{c}}$ for some $H \in [G^*, G]$. By using the usual category equivalence between algebraic lattices and $(\vee, 0)$-semilattices (cf. [LTF, Section I.3.15]), it follows easily that up to isomorphism, $S = [H, G]_{\mathsf{c}}$ and the map φ has the form

$$\varphi\colon [G^*, G]_{\mathsf{c}} \to [H, G]_{\mathsf{c}}\,, \quad X \mapsto H \vee X\,,$$

where the join $H \vee X$ is evaluated in the subgroup lattice $\operatorname{Sub} G$ of G. The pairs $A = (G^*, G)$ and $\overline{A} = (H, G)$ are both objects of $\mathcal{B}$, and the identity map on G induces an epimorphism (in the categorical sense!) $a\colon A \to \overline{A}$. We define ε as the identity map on $S = [H, G]_{\mathsf{c}}$ and we verify that the pair (a, ε) is a projectability witness for $\varphi\colon \Psi(A) \Rightarrow S$.

The conditions (i)–(iii) of [114, Definition 1.5.1] are trivially satisfied, so it remains to check Condition (iv). Let $f\colon (G^*, G) \to (X^*, X)$ be a morphism in $\mathcal{B}$ and let $\eta\colon [H, G]_{\mathsf{c}} \to [X^*, X]_{\mathsf{c}}$ be a $(\vee, 0)$-homomorphism such that $\Psi(f) = \eta \circ \Psi(a)$. The latter condition means that

$$X^* \vee f(Z) = \eta(H \vee Z)\,, \quad \text{for each } Z \in [G^*, G]_{\mathsf{c}}\,. \tag{9-5.1}$$

In particular, since $f(G^*) \leq X^*$, we get $\eta(H) = X^*$ (apply (9-5.1) to $Z = G^*$), hence, applying (9-5.1) to $Z = H$, we get $f(H) \leq X^*$. Hence f induces a morphism $g\colon (H, G) \to (X^*, X)$ in $\mathcal{B}$. Of course, $f = g \circ a$. We verify that $\eta = \Psi(g)$, that is,

$$\eta(Z) = f(Z) \vee X^*\,, \quad \text{for each } Z \in [H, G]_{\mathsf{c}}\,. \tag{9-5.2}$$

There exists $T \in [G^*, G]_{\mathsf{c}}$ such that $Z = H \vee T$. From $f(H) \leq X^*$ it follows that $f(Z) \vee X^* = f(T) \vee X^*$. Now it follows from (9-5.1) that $\eta(Z) = f(T) \vee X^*$, whence (9-5.2) holds. $\square$

Our next task is to establish the Löwenheim-Skolem Property (LS) for our categorical data. Essentially, this will amount to saying that given a group G, there are arbitrarily large "small" subgroups H of G such that $[G^*, G] \cong [H^*, H]$ canonically.

Definition 9-5.10. Let G^* be a subgroup of a group G. A subgroup H of G is (G^*, G)-*full* if the following conditions are satisfied:

(i) $[G^*, G]_{\mathsf{c}} = \{\, X \vee G^* \mid X \leq H \text{ finitely generated} \,\}$;

(ii) The containment $(G^* \vee X) \cap H \subseteq (G^* \cap H) \vee X$ holds, for any finitely generated subgroup X of H.

Lemma 9-5.11. *Let G^* be a subgroup of a group G and let H be a (G^*, G)-full subgroup of G. Then the mappings*

$$\begin{aligned} \varphi&\colon [G^*, G] \to [G^* \cap H, H]\,, & X &\mapsto X \cap H\,,\\ \psi&\colon [G^* \cap H, H] \to [G^*, G]\,, & Y &\mapsto G^* \vee Y \end{aligned}$$

are mutually inverse isomorphisms between $[G^*, G]$ and $[G^* \cap H, H]$.

Proof. Since φ and ψ are both isotone and preserve directed unions, it suffices to prove that they are mutually inverse maps, that is,

$$(X \cap H) \vee G^* = X\,, \qquad \text{for each } X \in [G^*, G]_{\mathsf{c}}\,, \tag{9-5.3}$$

$$(G^* \vee Y) \cap H = Y\,, \qquad \text{for each } Y \in [G^* \cap H, H]_{\mathsf{c}}\,. \tag{9-5.4}$$

In order to prove (9-5.3), observe that, by Definition 9-5.10(i), there exists a finitely generated subgroup $\overline{X}$ of H such that $X = \overline{X} \vee G^*$. Since $\overline{X} \leq X \cap H$, it follows that $X \leq (X \cap H) \vee G^*$. The converse containment being trivial, (9-5.3) follows.

In order to prove (9-5.4), observe that $Y = (G^* \cap H) \vee Z$ for some finitely generated subgroup Z of H. By using Definition 9-5.10(ii), we obtain

$$(G^* \vee Y) \cap H = (G^* \vee Z) \cap H \leq (G^* \cap H) \vee Z = Y\,.$$

Since the converse containment $Y \leq (G^* \vee Y) \cap H$ is trivial, (9-5.4) follows. $\square$

The following lemma expresses the abundance of small (G^*, G)-full subgroups of G.

Lemma 9-5.12. *Let G^* be a subgroup of a group G, let $X \subseteq G$, and set*

$$\kappa = \operatorname{card} X + \operatorname{card}\bigl([G^*, G]_{\mathsf{c}}\bigr) + \aleph_0\,.$$

Then there exists a (G^*, G)-full subgroup H of G containing X such that $\operatorname{card} H \leq \kappa$.

Proof. There is a subgroup H_0 of G containing X such that $\operatorname{card} H_0 \leq \kappa$ and $[G^*, G]_{\mathsf{c}} = \{\, X \vee G^* \mid X \leq H_0 \text{ finitely generated}\,\}$. If H_n is constructed, there exists $H_{n+1} \leq G$, containing H_n and of cardinality at most κ, such that $(G^* \vee X) \cap H_n \subseteq (G^* \cap H_{n+1}) \vee X$ for every finitely generated subgroup X of H_n. The subgroup $H = \bigcup(\, H_n \mid n < \omega\,)$ is as required. $\square$

Now we are ready to establish the desired Löwenheim–Skolem property for our categorical data.

Lemma 9-5.13. *Let λ be a regular uncountable cardinal, let B^* be a subgroup of a group B, let S be a $(\vee, 0)$-semilattice with* $\operatorname{card} S < \lambda$, *let $\psi\colon \Psi(B^*, B) \Rightarrow S$ be a double arrow in $\mathcal{S}$, let I be a set with* $\operatorname{card} I < \lambda$, *and let $\gamma_i\colon (C_i^*, C_i) \to (B^*, B)$ in $\mathcal{B}$, with* $\operatorname{card} C_i < \lambda$ *for each $i \in I$. Then there are an object (H^*, H) of $\mathcal{B}$, with* $\operatorname{card} H < \lambda$, *and a monomorphism $\gamma\colon (H^*, H) \to (B^*, B)$ in $\mathcal{B}$, above each γ_i for the subobject preordering in $\mathcal{B}\downarrow(B^*, B)$, such that $\psi \circ \Psi(\gamma)$ is a double arrow in $\mathcal{S}$.*

Proof. By the argument presented at the beginning of the proof of Lemma 9-5.9, we may assume that $S = [C, B]_{\mathsf{c}}$, for some subgroup $C \in [B^*, B]$, and $\psi(X) = C \vee X$ for each $X \in [B^*, B]_{\mathsf{c}}$. Since $[C, B]_{\mathsf{c}}$ and all the $\gamma_i(C_i)$ have cardinality smaller than λ, it follows from the assumption on λ, together with Lemma 9-5.12, that there exists a (C, B)-full subgroup H of B containing $\bigcup(\gamma_i(C_i) \mid i \in I)$ with $\operatorname{card} H < \lambda$. Set $H^* = B^* \cap H$. The inclusion map from H into B induces a monomorphism (in the categorical sense!) $\gamma\colon (H^*, H) \to (B^*, B)$. From $\gamma_i(C_i^*) \leq B^*$ and $\gamma_i(C_i^*) \leq \gamma_i(C_i) \leq H$ we get $\gamma_i(C_i^*) \leq H^*$, for each $i \in I$, so γ_i induces a morphism $\delta_i\colon (C_i^*, C_i) \to (H^*, H)$ in $\mathcal{B}$. Obviously, $\gamma_i = \gamma \circ \delta_i$, so γ lies above each γ_i for the subobject preordering in $\mathcal{B} \downarrow (B^*, B)$.

Now observe that

$$(\text{9-5.5}) \qquad (\psi \circ \Psi(\gamma))(Y) = \psi(B^* \vee Y) = C \vee Y\,, \quad \text{for each } Y \in [H^*, H]_{\mathsf{c}}\,.$$

Since H is (C, B)-full and by Lemma 9-5.11, every member of $[C, B]_{\mathsf{c}}$ has the form $C \vee Z$ for some $Z \in [C \cap H, H]_{\mathsf{c}}$; since $Z = Y \vee (C \cap H)$ for some $Y \in [H^*, H]_{\mathsf{c}}$, we get $C \vee Z = C \vee Y$. Hence, $\psi \circ \Psi(\gamma)$ is a surjective $(\vee, 0)$-homomorphism from $[H^*, H]_{\mathsf{c}}$ onto $[C, B]_{\mathsf{c}}$. Now let $Y, Z \in [H^*, H]_{\mathsf{c}}$ such that $(\psi \circ \Psi(\gamma))(Y) \leq (\psi \circ \Psi(\gamma))(Z)$, that is, by (9-5.5), $C \vee Y \leq C \vee Z$. Both $Y' = Y \vee (C \cap H)$ and $Z' = Z \vee (C \cap H)$ belong to $[C \cap H, H]_{\mathsf{c}}$, while $C \vee Y' \leq C \vee Z'$, so, by Lemma 9-5.11, we get $Y' \leq Z'$, and so $Y \leq Z \vee T$ for some $T \in [H^*, C \cap H]_{\mathsf{c}}$. Now $(\psi \circ \Psi(\gamma))(T) = C \vee T = C$, the zero element of $[C, B]_{\mathsf{c}}$. This completes the proof that $\psi \circ \Psi(\gamma)$ is ideal-induced. $\square$

Now we are ready to prove the diagram version of Tůma's Theorem.

Theorem 9-5.14. *Let P be a poset and let $\vec{S} = (S_p, \sigma_p^q \mid p \leq q \text{ in } P)$ be a P-indexed diagram of $(\vee, 0)$-semilattices with $(\vee, 0)$-homomorphisms. If either P is finite or there are cardinals κ and λ, with λ regular, such that P and all S_p, for $p \in P$, have cardinality smaller than λ and the relation $(\kappa, {<}\omega, \lambda) \to \lambda$ holds, then every P-indexed diagram of $(\vee, 0)$-semilattices with $(\vee, 0)$-homomorphisms can be lifted, with respect to the functor Ψ, by some diagram of intervals in subgroup lattices of groups (i.e., by some diagram in $\mathcal{B}$).*

Outline of proof. A great deal of the work has already been done in the proof of Gillibert and Wehrung [114, Theorem 4.7.2], the remaining work being completed in Lemmas 9-5.9–9-5.13. As in the proof of [114, Theorem 4.7.2], the problem is first reduced to the case where P is a $(\vee, 0)$-semilattice. Further, we set $\lambda = \aleph_1$ in case P is finite (in which case we do not require the arrow relation $(\kappa, {<}\omega, \lambda) \to \lambda$ to hold). We set $\mathcal{A} = \mathcal{S} = \mathbf{Sem}_{\vee,0}$, we define Φ as the identity functor on $\mathcal{A}$, and we define $\mathcal{A}^\dagger = \mathcal{S}^\dagger$ as the class of all $(\vee, 0)$-semilattices S such that $\operatorname{card} S < \lambda$. We already established in Claim 1 of the proof of [114, Theorem 4.7.2] that the quadruple

$$(\mathcal{A}, \mathcal{S}, \mathcal{S}^{\Rightarrow}, \Phi)$$

is a left larder. For the right larder part, it remains to define $\mathcal{B}^\dagger$ as the class of all objects (B^*, B) of $\mathcal{B}$ such that $\operatorname{card} B < \lambda$. Now we prove that the 6-tuple

$$(\mathcal{B}, \mathcal{B}^\dagger, \mathcal{S}, \mathcal{S}^\dagger, \mathcal{S}^{\Rightarrow}, \Psi)$$

is a right λ-larder. For each object (B^*, B) of $\mathcal{B}^\dagger$, the semilattice $\Psi(B^*, B) = [B^*, B]_{\mathsf{c}}$ has cardinality smaller than λ, hence the condition $(\mathrm{PRES}_\lambda(\mathcal{B}^\dagger, \Psi)$ is trivially satisfied. Furthermore, the condition $(\mathrm{LS}^{\mathrm{r}}_\lambda(B^*, B))$, for every object (B^*, B) of $\mathcal{B}$, follows trivially from Lemma 9-5.13. Therefore, the 8-tuple

$$\Lambda = (\mathcal{A}, \mathcal{B}, \mathcal{S}, \mathcal{A}^\dagger, \mathcal{B}^\dagger, \mathcal{S}^\Rightarrow, \Phi, \Psi)$$

is a λ-larder. By Lemma 9-5.9, this larder is projectable.

The last part of the proof works *mutatis mutandis* as the one of [114, Theorem 4.7.2], replacing $\mathbf{MAlg}_1$ by $\mathcal{B}$ and $\mathrm{Con_c}$ by Ψ. □

In particular, we get the following corollary.

Corollary 9-5.15. *Let P be a poset. If either P is finite or there exists a proper class of Erdős cardinals, then every P-indexed diagram of $(\vee, 0)$-semilattices with $(\vee, 0)$-homomorphisms can be lifted by some diagram of intervals in subgroup lattices of groups.*

9-5.4 Congruence m-permutable, congruence-preserving extensions

Recall from Grätzer, Lakser, and Wehrung [154, Theorem 3] that every countable, locally finite lattice has a congruence-preserving, locally finite, relatively complemented extension. While it is not known whether this result can be extended to the non locally finite case (cf. Problem 7.8), the case of larger cardinalities is completely solved by the following result, established in Gillibert and Wehrung [114, Theorem 5.5.5].

$\diamondsuit$ **Theorem 9-5.16** (Gillibert and Wehrung 2011)**.** *Let F be a free bounded lattice, on at least $\aleph_1$ generators, in a nondistributive variety $\mathbf{V}$ of lattices. Then F has no congruence-permutable, congruence-preserving extension.*

Since every relatively complemented lattice is congruence-permutable, it follows that F has no relatively complemented, congruence-preserving extension.

Recall that a lattice L is *semidistributive* if

$$x \vee z = y \vee z \implies x \vee z = (x \wedge y) \vee z\,, \quad \text{for all } x, y, z \in L\,,$$

and dually. Further, say that a variety $\mathbf{V}$ of lattices is semidistributive if every member of $\mathbf{V}$ is semidistributive. Theorem 9-5.16 was partially extended, to the case of congruence m-permutable lattices and non-semidistributive varieties, in Gillibert [109, Theorem 10.7].

$\diamondsuit$ **Theorem 9-5.17** (Gillibert, preprint 2010)**.** *Let $\mathbf{V}$ be a non-semidistributive variety of lattices and let m be a positive integer. Then there is a congruence $(m+1)$-permutable bounded lattice in $\mathbf{V}$, of cardinality $\aleph_1$, without any congruence m-permutable, congruence-preserving extension in the variety of all lattices.*

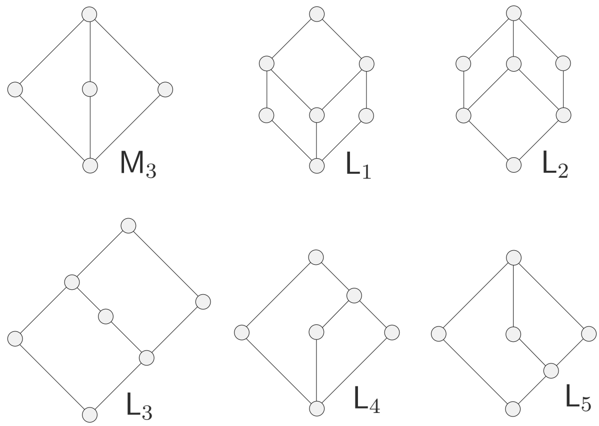

Figure 9-5.2: Finite non-semidistributive lattices.

In the course of the proof of his result, Gillibert used the characterization, obtained in Jónsson and Rival [213, Theorem 1.2] (see also Jipsen and Rose [207, Theorem 4.2], of the semidistributivity of a variety $\mathbf{V}$ by $\mathbf{V}$ not containing any of the lattices M_3, L_1, L_2, L_3, L_4, and L_5, represented in Figure 9-5.2, as a member. (We are following the notation of Jipsen and Rose [207] for the lattices L_i.)

The semidistributive, non-distributive case is still open, see Problem 9.11.

The proofs of Theorems 9-5.16 and 9-5.17 are very complex, and they rely on the larder and CLL machinery developed in Gillibert and Wehrung [114].

9-6. Exercises

9.1. Let Σ be a similarity type. Prove that the assignment that to an algebra A, with similarity type Σ, associates the $(\mathrm{Con_c}\, A)$-valued distance $\mathrm{con}_A\colon A\times A \to \mathrm{Con_c}\, A$, is a functor, from the category $\mathbf{Alg}_\Sigma$ to the category of all semilattice-valued distances (cf. Definition 9-2.1). Prove that this functor preserves directed colimits.

9.2. Find an almost congruence-permutable, non congruence-permutable lattice.

9.3. Let A be an algebra and suppose that there exists a polynomial map $p\colon A^3 \to A$ such that $p(x,y,y) = x$ and $p(x,x,y) = y$ for all $x, y \in A$ (we say that p is a *Mal'cev polynomial* on A). Prove that A is congruence-permutable.

9.4. Prove that the following statements are equivalent, for any variety $\mathbf{V}$ of algebras:

(i) every member of $\mathbf{V}$ is congruence-permutable;

(ii) every member of $\mathbf{V}$ is almost congruence-permutable;

(iii) there exists a ternary term p, in the similarity type of $\mathbf{V}$, such that the identities $\mathsf{p}(\mathsf{x},\mathsf{y},\mathsf{y}) = \mathsf{x}$ and $\mathsf{p}(\mathsf{x},\mathsf{x},\mathsf{y}) = \mathsf{y}$ both hold in every member of $\mathbf{V}$ (we say that p is a *Mal'cev term* of $\mathbf{V}$).

(*Hint*: build on the standard proof for congruence-permutable, see, for example, McKenzie, McNulty, and Taylor [248, Theorem 4.141].)

9.5. (1) Prove that every refinement monoid M satisfies the following *Riesz decomposition property*: *For all $a, b, c \in M$ with $c \leq a+b$, there are $x \leq a$ and $y \leq b$ such that $c = x + y$.*

(2) Find a finite commutative monoid satisfying the Riesz decomposition property but not the refinement property.

9.6. (Wehrung [323, Lemma 1.11]) Endow a refinement monoid M with its algebraic preordering and let $a, b, c \in M$.

(1) Prove that if $a + c = b + c$, then there are $d, a', b', c' \in M$ such that $a = d + a'$, $b = d + b'$, and $c = a' + c' = b' + c'$.

(2) By iterating the process, prove that if $a + c = b + c$, then for every positive integer n, there are $u, v \in M$ such that $nu \leq c$, $nv \leq c$, and $a + u = b + v$.

(3) Prove that if $a + c \leq b + c$, then for every positive integer n, there exists $x \in M$ such that $a \leq b + x$ and $nx \leq c$.

9.7. (After Wehrung [327])

(1) Prove that, for any congruence-permutable lattice L, the $(\vee, 0)$-semilattice $\mathrm{Con_c}\, L$ satisfies the following URP-like statement: *For every system of inequalities of the form $\boldsymbol{e} = \boldsymbol{a}_i \vee \boldsymbol{b}_i$ for each $i \in I$, there are $\boldsymbol{a}_i^*$, $\boldsymbol{b}_i^*$, and $\boldsymbol{c}_{i,j}$, for $i, j \in I$, such that*

(i) $\boldsymbol{a}_i^* \leq \boldsymbol{a}_i$ *and* $\boldsymbol{b}_i^* \leq \boldsymbol{b}_i$, *for each* $i \in I$;

(ii) $\boldsymbol{a}_i^* \leq \boldsymbol{a}_j^* \vee \boldsymbol{c}_{i,j}$ *and* $\boldsymbol{b}_j^* \leq \boldsymbol{b}_i^* \vee \boldsymbol{c}_{i,j}$, *for all* $i, j \in I$;

(iii) $\boldsymbol{c}_{i,k} \leq \boldsymbol{c}_{i,j} \vee \boldsymbol{c}_{j,k}$, *for all* $i, j, k \in I$.

(*Hint*: imitate the proof of Theorem 9-2.17(1), with $\boldsymbol{a}_i^* = \mathrm{con}(u, x_i)$ and $\boldsymbol{b}_i^* = \mathrm{con}(x_i, v)$.)

(2) Prove that the property above implies URP_1.

(3) Find a variant of that property (still involving variables $\boldsymbol{a}_i^*$, $\boldsymbol{b}_i^*$), implied by that property but implying URP_3, which holds in the congruence semilattice of any algebra with almost permutable congruences.

9.8. (Tůma and Wehrung [314]) Find a variant of the uniform refinement property of Exercise 9.7, implying URP_3, and holding in $\mathrm{Con_c}\, L$, for any lattice L with almost permutable congruences.

9.9. (cf. Section 9-3.1) Find an example of a $(\vee, 0)$-semilattice S such that the canonical projection $\pi\colon \mathcal{R}(S) \twoheadrightarrow S$ is not a join-homomorphism.

9.10. Find an example where the preordering on $\mathcal{R}_{\varnothing}(S)$ defined in (9-3.1) is not antisymmetric. (*Hint*: denote by $\overline{\varepsilon}(s)$ the lower subset, with respect to the componentwise ordering on $\mathcal{C}(S)$, generated by $\varepsilon(s) = \{(s, s, s)\}$, for $s \in S$. Prove that $\overline{\varepsilon}(s) \leq \varepsilon(s)$.)

9.11. Let S be a $(\vee, 0)$-semilattice. When does $\mathcal{R}(S) = S$? Find an example where S is both distributive and properly contained in $\mathcal{R}(S)$.

9.12. (1) Verify that $\mathcal{R}(S)$ is the $(\vee, 0)$-semilattice freely generated by S together with symbols $x_{a,b,c}$, where $(a, b, c) \in \mathcal{C}(S)$, subjected to the relations $x_{a,b,c} \leq a$ and $c = x_{a,b,c} \vee x_{b,a,c}$ for every triple $(a, b, c) \in \mathcal{C}(S)$. (*Hint*: use Lemma 9-3.12.)
(2) Prove that the functor $\mathcal{R}$ preserves directed colimits.
(3) Deduce that the functor $\mathcal{R}^{\infty}$ preserves directed colimits.

9.13. (Wehrung [335, Lemma 3.6]) Let $(S_i \mid i \in I)$ be a family of $(\vee, 0)$-subsemilattices of a $(\vee, 0)$-semilattice S. Prove that the following statements hold:

(1) $\mathcal{R}\big(\bigcap_{i\in I} S_i\big) = \bigcap_{i\in I} \mathcal{R}(S_i)$ and $\mathcal{R}^{\infty}\big(\bigcap_{i\in I} S_i\big) = \bigcap_{i\in I} \mathcal{R}^{\infty}(S_i)$;
(2) $\mathcal{R}\big(\bigcup_{i\in I} S_i\big) = \bigcup_{i\in I} \mathcal{R}(S_i)$ and $\mathcal{R}^{\infty}\big(\bigcup_{i\in I} S_i\big) = \bigcup_{i\in I} \mathcal{R}^{\infty}(S_i)$, whenever $\{\, S_i \mid i \in I \,\}$ is directed (so $I \neq \varnothing$).

9.14. Prove that the functors $\mathcal{L}$ and $\mathcal{G}$ (cf. Section 9-3.2) both preserve directed colimits.

9.15. (Lemma 4.1 and Corollary 4.2 in Wehrung [335]) Let $(X_i \mid i \in I)$ be a family of subsets of a set X. Prove that the following statements hold:

(1) $\mathcal{L}\big(\bigcap_{i\in I} X_i\big) = \bigcap_{i\in I} \mathcal{L}(X_i)$ and $\mathcal{G}\big(\bigcap_{i\in I} X_i\big) = \bigcap_{i\in I} \mathcal{G}(X_i)$;
(2) $\mathcal{L}\big(\bigcup_{i\in I} X_i\big) = \bigcup_{i\in I} \mathcal{L}(X_i)$ and $\mathcal{G}\big(\bigcup_{i\in I} X_i\big) = \bigcup_{i\in I} \mathcal{G}(X_i)$, whenever $\{\, X_i \mid i \in I \,\}$ is directed (so $I \neq \varnothing$).

Deduce from this that for each $\boldsymbol{x} \in \mathcal{G}(X)$, there exists a least subset Y of X such that $\boldsymbol{x} \in \mathcal{G}(Y)$, and that Y is finite.

9.16. (cf. Section 9-3.3) Prove that for any $(\vee, 0)$-semilattices S and T, any $\boldsymbol{e} \in S$, and any weakly distributive $(\vee, 0)$-homomorphism $\mu\colon S \to T$, if S satisfies $\mathrm{CLR}(\boldsymbol{e})$, then T satisfies $\mathrm{CLR}(\mu(\boldsymbol{e}))$.

9.17. (Ploščica [259, Lemma 4.1]; see Section 9-4.2) Let $\mathbf{V}$ be a finitely generated variety of lattices and let L be a subdirectly irreducible member of $\mathbf{V}$. Prove that $\operatorname{len} L \leq s(\mathbf{V})$.

9.18. (Ploščica [259, Theorem 4.3]; see Section 9-4.2) Let $\mathbf{V}$ be a finitely generated variety of modular lattices. Prove that

$$s(\mathbf{V}) = \max\{\, \operatorname{len} L \mid L \in \mathbf{V} \text{ subdirectly irreducible} \,\}\,.$$

9.19. Let $\mathbf{V}$ and $\mathbf{W}$ be varieties of algebras and let κ be an infinite cardinal number. We assume that the similarity type of $\mathbf{V}$ has at most κ symbols. Prove that $\operatorname{crit}(\mathbf{V};\mathbf{W}) \leq \kappa$ iff $\operatorname{Con_c}(\operatorname{Free}_{\mathbf{V}}(\kappa)) \notin \operatorname{Con_c}\mathbf{W}$. (*Hint*: for any congruence $\boldsymbol{\theta}$ of an algebra A, $\operatorname{Con}(A/\boldsymbol{\theta})$ is isomorphic to $(\operatorname{Con} A)\uparrow\boldsymbol{\theta}$.)

9.20. (Ploščica [259, Lemma 4.4]) Let $\mathbf{V}$ be a nontrivial finitely generated variety of modular lattices and denote by $\mathbf{D}$ the variety of all distributive lattices. Prove that $\operatorname{Con}\mathbf{D} = \operatorname{Con}\mathbf{V} \cap \operatorname{Con}\mathbf{N}_5$. (*Hint*: let $L \in \mathbf{V}$ and $A \in \operatorname{Con}\mathbf{N}_5$ such that $\operatorname{Con} L \cong \operatorname{Con} A$. If A is not distributive, then there is $\boldsymbol{\alpha} \in \operatorname{Con} A$ such that $\operatorname{Con}(A/\boldsymbol{\alpha}) \cong \operatorname{Con}\mathsf{N}_5$. Get $\boldsymbol{\theta} \in \operatorname{Con} L$ such that $\operatorname{Con}(L/\boldsymbol{\theta}) \cong \operatorname{Con}\mathsf{N}_5$. Observe that $L/\boldsymbol{\theta}$ must be subdirectly irreducible.)

9.21. Prove that there is a locally finite, complemented, modular lattice L such that $\operatorname{Con} L \cong \operatorname{Con}\mathsf{N}_5$. (*Hint*: check out Section 8-4.1.)

9.22. Verify that $\operatorname{crit}(\mathbf{N}_5;\mathbf{D}) = 5$.

9.23. Prove that $\operatorname{crit}(\mathbf{M}_3;\mathbf{D}) = \aleph_0$. (*Hint*: view the two-atom Boolean lattice B_2 as a sublattice of M_3, and let K be the lattice of all eventually constant sequences of elements of M_3 whose limit belongs to B_2.)

9.24. Let F and K be (possibly infinite) fields with $\operatorname{card} F \neq \operatorname{card} K$. Verify that

$$\operatorname{crit}(F\text{-vector spaces}; K\text{-vector spaces}) = \operatorname{card} F + 1\,.$$

9.25. Prove that $\operatorname{crit}(\text{groups};\text{lattices}) = 5$. (*Hint*: consider M_3.)

9.26. Prove that $\operatorname{crit}(\text{lattices};\text{rings}) = \operatorname{crit}(\text{lattices};\text{groups}) = \aleph_2$. (*Hint*: check out Subsections 8-4.4 and 9-2.4.)

9.27. (Beginning of Section 4 in Gillibert and Ploščica [112]) Let $\mathbf{V}$ be a finitely generated congruence-distributive variety of algebras such that the congruence lattice of every subdirectly irreducible member of $\mathbf{V}$ is isomorphic either to $\mathbf{2}$ or to $\operatorname{Con}\mathsf{N}_5$. Prove that for every finite algebra $A \in \mathbf{V}$, there exists a finite lattice $L \in \mathbf{N}_5$ such that $\operatorname{Con} A \cong \operatorname{Con} L$. (*Hint*: use Theorem 9-4.13.)

9.28. Denote by x_0, x_1, and x_2 the atoms of M_3 and set

$$K_i = \{0, 1, x_i\}\,,$$
$$L_i = \mathsf{M}_3 \setminus \{x_i\}\,,$$

for each $i < 3$. Verify that the lattices K_i are all almost congruence-permutable, while $\mathbf{2}$, M_3, and all the L_i are congruence-permutable.

$$\begin{tikzcd}
& \mathsf{M}_3 & \\
L_2 \arrow[ur] & L_1 \arrow[u] & L_0 \arrow[ul] \\
K_0 \arrow[u] \arrow[ur] & K_1 \arrow[ul] \arrow[ur] & K_2 \arrow[ul] \arrow[u] \\
& \mathbf{2} \arrow[ul] \arrow[u] \arrow[ur] &
\end{tikzcd}$$

Figure 9-6.1: The cube $\mathcal{K}$ of almost congruence-permutable lattices.

We consider the cube $\mathcal{K}$ represented in Figure 9-6.1, where all the arrows are inclusion mappings.

Denote by $\alpha\colon \operatorname{Con_c} \mathbf{2} \to \mathbf{2}$ and $\beta\colon \operatorname{Con_c} \mathsf{M}_3 \to \mathbf{2}$ the unique isomorphisms, and, for each $i < 3$, denote by $\alpha_i\colon \operatorname{Con_c} K_i \to \mathbf{2}^2$ and $\beta_i\colon \operatorname{Con_c} L_i \to \mathbf{2}^2$ the isomorphisms defined by

$$\begin{aligned}
\alpha_0(0,x_0) &= (1,0)\,, & \alpha_0(x_0,1) &= (0,1)\,,\\
\alpha_1(0,x_1) &= (1,0)\,, & \alpha_1(x_1,1) &= (0,1)\,,\\
\alpha_2(0,x_2) &= (0,1)\,, & \alpha_2(x_2,1) &= (1,0)\,,\\
\beta_2(0,x_0) &= (1,0)\,, & \beta_2(x_0,1) &= (0,1)\,,
\end{aligned}$$

$$\begin{aligned}
\beta_2(0,x_1) &= (0,1)\,, & \beta_2(x_1,1) &= (1,0)\,,\\
\beta_1(0,x_0) &= (1,0)\,, & \beta_1(x_0,1) &= (0,1)\,,\\
\beta_1(0,x_2) &= (0,1)\,, & \beta_1(x_2,1) &= (1,0)\,,\\
\beta_0(0,x_1) &= (1,0)\,, & \beta_0(x_1,1) &= (0,1)\,,\\
\beta_0(0,x_2) &= (0,1)\,, & \beta_0(x_2,1) &= (1,0)\,.
\end{aligned}$$

Verify that those maps define a natural equivalence from $\operatorname{Con_c} \mathcal{K}$ onto the diagram $\mathcal{D}_c$ introduced in Section 9-5.1.

9.29. Denote by $\boldsymbol{e}\colon \mathbf{2} \to \mathbf{2}^2$, $\boldsymbol{f}, \boldsymbol{g}\colon \mathbf{2}^2 \hookrightarrow \mathbf{2}^4$, and $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}\colon \mathbf{2}^4 \hookrightarrow \mathbf{2}^5$ the maps defined by

$$\begin{aligned}
\boldsymbol{e}(x) &= (x,x)\,,\\
\boldsymbol{f}(x,y) &= (x,x,y,y)\,,\\
\boldsymbol{g}(x,y) &= (x,y,x,y)\,,\\
\boldsymbol{a}(x_1,x_2,x_4,x_4) &= (x_1,x_2,x_3,x_4,x_2 \vee x_3)\,,
\end{aligned}$$

$$\boldsymbol{b}(x_1, x_2, x_4, x_4) = (x_2, x_1, x_3, x_4, x_1 \vee x_4)\,,$$
$$\boldsymbol{c}(x_1, x_2, x_4, x_4) = (x_2, x_3, x_1, x_4, x_1 \vee x_4)\,,$$

for all $x, y, x_1, x_2, x_3, x_4 \in \{0, 1\}$. Further, denote by $\mathcal{D}'_{\mathrm{c}}$ the cube of finite Boolean semilattices and $(\vee, 0, 1)$-embeddings represented in Figure 9-6.2.

$$\begin{tikzcd}
 & \mathbf{2}^5 & \\
\mathbf{2}^4 \arrow[ur, hook, "\boldsymbol{a}"] & \mathbf{2}^4 \arrow[u, hook, "\boldsymbol{b}"] & \mathbf{2}^4 \arrow[ul, hook, "\boldsymbol{c}"'] \\
\mathbf{2}^2 \arrow[u, hook, "\boldsymbol{f}"] \arrow[ur, hook, "\boldsymbol{f}"] & \mathbf{2}^2 \arrow[ul, hook, "\boldsymbol{g}"] \arrow[ur, hook, "\boldsymbol{f}"'] & \mathbf{2}^2 \arrow[ul, hook, "\boldsymbol{g}"'] \arrow[u, hook, "\boldsymbol{g}"'] \\
 & \mathbf{2} \arrow[ul, hook, "\boldsymbol{e}"] \arrow[u, hook, "\boldsymbol{e}"] \arrow[ur, hook, "\boldsymbol{e}"'] &
\end{tikzcd}$$

Figure 9-6.2: The diagram $\mathcal{D}'_{\mathrm{c}}$.

Prove that $\mathcal{D}'_{\mathrm{c}}$ has no lifting, with respect to the functor Π, by any cube of join-generating V-distances of type 1. (*Hint*: prove that there exists a natural transformation from $\mathcal{D}'_{\mathrm{c}}$ to $\mathcal{D}_{\mathrm{c}}$ all of whose components are projections; apply Theorem 9-5.1.)

9-7. Problems

The following problem is stated as Problem 2.11 in Lampe [240].

Problem 9.1. Is every algebraic distributive lattice isomorphic to the congruence lattice of an algebra with finite similarity type?

Further results and problems, about congruence lattices of universal algebras, are developed in Lampe [240]. In particular, since every algebraic lattice with compact unit is isomorphic to the congruence lattice of some groupoid (see Lampe [238]), no counterexample to Problem 9.1 can have a compact unit.

Now that we know the negative solution to CLP, the following problem becomes natural.

Problem 9.2. Is every algebraic distributive lattice isomorphic to the congruence lattice of some algebra generating a congruence-distributive variety?

Even the simplest possible variant of Problem 9.2 is still open:

Problem 9.3. Is every algebraic distributive lattice isomorphic to the congruence lattice of some majority algebra?

Problem 9.4. Prove that any diagram of finite distributive $(\vee, 0)$-semilattices with $(\vee, 0)$-homomorphisms, indexed by a finite lattice, can be lifted, with respect to the $\mathrm{Con_c}$ functor, by a diagram of (finite?) lattices and lattice homomorphisms.

Problem 9.5. Prove that there exists a bounded lattice L such that there is no *modular* lattice M with $\mathrm{Con}\, L \cong \mathrm{Con}\, M$.

More generally, it is natural to ask whether the assumption on prime intervals of simple members of $\mathbf{W}$ can be removed from the statement of Theorem 9-4.26:

Problem 9.6. Let $\mathbf{V}$ and $\mathbf{W}$ be varieties of lattices. Is it the case that if $\mathbf{V}$ is contained neither in $\mathbf{W}$ nor in its dual, then $\mathrm{crit}(\mathbf{V}^{0,1}; \mathbf{W}) \leq \aleph_2$?

To give an idea of the extent of our ignorance, we do not even know whether $\mathrm{Con}\, \mathbf{N}_5$ is contained in $\mathrm{Con}\, \mathbf{M}$ (where $\mathbf{M}$ denotes the variety of all modular lattices).

Problem 9.7. Let $\mathbf{A}$ and $\mathbf{B}$ be varieties of algebras (not necessarily on the same similarity type).

(1) If $\mathrm{Con}\, \mathbf{A} \subseteq \mathrm{Con}\, \mathbf{B}$, can $\mathbf{A}$ be, in some appropriate sense, interpreted in $\mathbf{B}$?

(2) If the similarity types of $\mathbf{A}$ and $\mathbf{B}$ are both finite and $\mathrm{Con}\, \mathbf{A} \not\subseteq \mathrm{Con}\, \mathbf{B}$, does $\mathrm{crit}(\mathbf{A}; \mathbf{B}) \leq \aleph_2$?

Partial positive answers to Problem 9.7(1) and Problem 9.7(2), respectively, are contained in Theorems 9-4.26 and 9-4.27, respectively. Furthermore, Problem 9.7(2) is the restatement, for varieties (not only quasivarieties), of Gillibert and Wehrung [114, Problem 3].

Problem 9.8. Is the ternary relation $\mathrm{crit}(\mathbf{A}; \mathbf{B}) \leq \aleph_n$ decidable, for finitely generated varieties $\mathbf{A}$ and $\mathbf{B}$ over finite similarity types and a nonnegative integer n?

To measure the extent of our ignorance, we do not even know, at this point, the answer to the analogue of Problem 9.8 obtained by replacing "decidable" by "absolute" (say, between inner models of set theory). Problem 9.8 is the restatement, for varieties (not only quasivarieties), of Gillibert and Wehrung [114, Problem 4].

The role of the bounds in the study of congruence lattices of lattices remains quite mysterious. For example, even the following natural question remains open.

Problem 9.9. Let $\mathbf{V}$ be a variety of lattices.

(1) Is every member of $\operatorname{Con}\mathbf{V}$ isomorphic to $\operatorname{Con} L$, for some lattice $L \in \mathbf{V}$ with a least element? That is, does $\operatorname{Con}\mathbf{V} = \operatorname{Con}\mathbf{V}^0$?

(2) Is every bounded member of $\operatorname{Con_c}\mathbf{V}$ isomorphic to $\operatorname{Con_c} L$, for some bounded lattice $L \in \mathbf{V}$?

Both restrictions of Problem 9.9, to the cases where $\mathbf{V}$ is either the trivial variety or the variety $\mathbf{D}$ of all distributive lattices, have a straightforward positive answer. If $\mathbf{V} = \mathbf{M}_n$, with $3 \leq n \leq \omega$, then we also get a positive answer to Problem 9.9(2), established in Gillibert [111], but this is much harder to prove. On the other hand, even the case where $\mathbf{V}$ is the variety of all lattices remains open.

For finite lattices, or even lattices of finite length, being relatively complemented has a strong impact on the congruence structure, as, for example, *the congruence lattice of a relatively complemented lattice of finite length is Boolean* (this follows trivially from [LTF, Corollary 281]). For infinite lattices, the situation is quite different, as, for example, up to cardinality $\aleph_1$, the congruence semilattice of a relatively complemented lattice can be any distributive $(\vee, 0)$-semilattice (cf. Theorem 7-5.13). Nevertheless, Theorem 9-2.14(i) still shows some nontrivial additional structure for congruence semilattices of relatively complemented lattices. This gives some hope for a positive solution to the following problem (we refer to Definition 9-3.26 for congruence classes); see also Tůma and Wehrung [316, Problem 8].

Problem 9.10. Prove that the compact congruence class of the class of all relatively complemented lattices contains none of the following classes of distributive semilattices:

(1) the compact congruence class of the class of all lattices that are both sectionally complemented and dually sectionally complemented;

(2) the class of all quotients of Boolean semilattices by distributive congruences (cf. Section 7-3.3).

Our next problem asks for an extension of Gillibert's Theorem 9-5.17 to all nondistributive varieties.

Problem 9.11. Let $\mathbf{V}$ be a semidistributive, nondistributive variety of lattices and let m be a positive integer. Is there a congruence $(m+1)$-permutable bounded lattice in $\mathbf{V}$, of cardinality $\aleph_1$, without any congruence m-permutable, congruence-preserving extension in the variety of all lattices?

Chapter 10

Two More Topics on Congruence Lattices of Lattices

by George Grätzer

Congruence lattices of lattices are discussed in the first three chapters of this part by F. Wehrung, in Chapter 4 and in the book [131] by myself.

In this chapter, I will discuss two topics, one well-developed, one new, closely related to congruence lattices of lattices.

The well-developed topic is the characterization of *complete congruence lattices of complete lattices*. This will be discussed in Sections 5-1–10-5.

The brand new topic is the investigation of $\operatorname{Princ} L$, the *order of principal congruencies of a lattice* L, started in my paper [134]. This topic is introduced in Section 10-6.

10-1. Introducing complete congruence lattices

The central result of this chapter – Theorem 10-1.3 – is the representation of a complete lattice as the lattice of complete congruences of a complete lattice.

We introduce this result with the historical backdrop of four problems raised and solved in the span of 43 years, 1945–1988.

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10-1.1 Four problems: 1945–1983

A 1945 lecture of G. Birkhoff [27] started the research discussed in this chapter (see also G. Birkhoff [28]). In this lecture, Birkhoff observed that the congruence lattice of an infinitary universal algebra is a complete lattice and he asked:

Problem 1. *Is every complete lattice isomorphic to the congruence lattice of an infinitary algebra?*

Algebraic lattices were introduced in G. Birkhoff and O. Frink [30], where it was observed that the congruence lattice of an algebra is algebraic. They raised the question:

Problem 2. *Is every algebraic lattice isomorphic to the congruence lattice of an algebra?* (This appeared as Problem 50 in G. Birkhoff [28].)

Of course, Birkhoff and Frink knew that the congruence lattice of a lattice is distributive, so they intended to propose in the paper the following (named CLP by F. Wehrung some 50 years later):

Problem 3. *Is every distributive algebraic lattice isomorphic to the congruence lattice of a lattice?*

But they forgot to put it in. The first printed version of this problem I can find is in 1962 (G. Grätzer and E.T. Schmidt [164]).

In 1987, K. Reuter and R. Wille [278] observed that the lattice of complete congruence relations of a complete lattice is not always distributive (see Exercise 10.7) and raised the following question closely connected to the first problem:

Problem 4. *Is every* complete lattice L *isomorphic to the lattice of complete congruence relations of a suitable* complete lattice K*?*

By 1971, a solution of Problem 1 was available (see Section 10-1.2. The algebra $\mathfrak{A}$ constructed to represent a complete lattice L as the congruence lattice of $\mathfrak{A}$ is very complex. Problem 4 asks if the algebra $\mathfrak{A}$ can always be chosen as a complete lattice!

10-1.2 Four solutions: 1960–1988

We present them in chronological order.

Problem 2

In 1960, G. Birkhoff's famous Problem 50 of [28] was solved in the affirmative in G. Grätzer and E.T. Schmidt [165]:

Theorem 10-1.1 (Congruence Lattice Characterization Theorem for Algebras)**.** *Let L be an algebraic lattice. Then there is an algebra $\mathfrak{A}$ such that the congruence lattice of $\mathfrak{A}$ is isomorphic to L.*

See Section I.3.17 of LTF for an overview and Chapter 2 and Appendix 7 in G. Grätzer [125] for proofs.

Problem 1

In 1971, G. Grätzer and W.A. Lampe [156] announced an affirmative answer:

Theorem 10-1.2 (Congruence Lattice Characterization Theorem for Infinitary Algebras)**.** *Every complete lattice L is isomorphic to the congruence lattice of an infinitary algebra $\mathfrak{A}$.*

The proof was published as Appendix 7 of [125]. For stronger forms of this result, see Section 10-1.3.

Problem 4

S.-K. Teo [309] solved this problem for finite lattices. At the 1988 Lisbon Conference on Lattices, Semigroups, and Universal Algebra, I answered this question in the affirmative:

Theorem 10-1.3 (Representation Theorem for Complete Lattices)**.** *Every complete lattice L can be represented as the lattice of complete congruences of a complete lattice K, in formula,* $\operatorname{Com} K \cong L$*.*

See G. Grätzer [127] and [126]. There are a number of stronger results, see Section IV.4.10 of LTF for a brief overview. We mention now only one, the main result of G. Grätzer and E.T. Schmidt [170] (see Section 10-5 for a related topic):

Theorem 10-1.4 (Representation Theorem for Complete Distributive Lattices)**.** *Every complete lattice L is isomorphic to the congruence lattice of a complete distributive lattice K.*

Problem 3

In 2006, F. Wehrung [335] found distributive algebraic lattices that cannot be represented as congruence lattices of lattices:

Theorem 10-1.5 (Counterexample to CLP)**.** *There is a distributive algebraic lattice with $\aleph_{\omega+1}$ compact elements that cannot be represented as the congruence lattice of a lattice.*

In the preceding three chapters F. Wehrung gives an overview of this result and of the many fields that started with his paper.

10-1.3 Related structures of algebras

With an algebra $\mathfrak{A}$, we associate a group, $\operatorname{Aut}\mathfrak{A}$, the *automorphism group* of $\mathfrak{A}$ and two lattices, $\operatorname{Con}\mathfrak{A}$, the *congruence lattice* of $\mathfrak{A}$ and $\operatorname{Sub}\mathfrak{A}$, the *subalgebra lattice* of $\mathfrak{A}$. We call these the *related structures.*

The Independence Theorem for Algebras states that the three related structures of an algebra are independent. We present the result of G. Grätzer and W.A. Lampe (see Appendix 7 of [125] by G. Grätzer and W.A. Lampe) in a form that contains the finite and the infinitary case as well.

Theorem 10-1.6 (Independence Theorem for Algebras)**.** *Let* $\mathfrak{m}$ *be a regular cardinal. Let* L_{sub} *and* L_{con} *be* $\mathfrak{m}$*-algebraic lattices*[1] *and let* G *be a group; assume that* L_{con} *has more than one element. Then there is an algebra* $\mathfrak{A}$ *of characteristic* $\mathfrak{m}$ *such that the subalgebra lattice of* $\mathfrak{A}$ *is isomorphic to* L_{sub}*, the congruence lattice of* $\mathfrak{A}$ *is isomorphic to* L_{con}*, and the automorphism group of* $\mathfrak{A}$ *is isomorphic to* G*, in formula,*

$$\operatorname{Sub}\mathfrak{A} \cong L_{\text{sub}},\ \operatorname{Con}\mathfrak{A} \cong L_{\text{con}},\ \textit{and}\ \operatorname{Aut}\mathfrak{A} \cong G.$$

If $\mathfrak{m} = \aleph_0$, the algebra $\mathfrak{A}$ is finitary, L_{sub} and L_{con} are algebraic lattices, so we get the Independence Theorem for Algebras of W.A. Lampe [234]–[237].

If $\mathfrak{m} > \aleph_0$, then we get the Independence Theorem for Infinitary Algebras, which contains the Congruence Lattice Characterization Theorem for Infinitary Algebras.

The result in Appendix 7 of [125] is actually stronger, but we are not going into any more details here.

For an alternative proof of the infinitary case, see E. Nelson [252].

10-1.4 Related structures of lattices

Closely related to Section 10-1.3 is the topic of related structures of lattices. Since for a lattice L, the lattice $\operatorname{Sub} L$ is not a very nice lattice, we consider only the related structures $\operatorname{Aut} L$ and $\operatorname{Con} L$.

For finite lattices, the independence problem was settled in the late 1970s by V.A. Baranskiĭ [21], [22] and A. Urquhart[319].

Theorem 10-1.7 (Independence Theorem for Finite Lattices)**.** *Let* D *be a finite distributive element with more than one element and let* G *be a group. Then there exists a lattice* K *such that*

$$\operatorname{Con} K \cong D\ \textit{and}\ \operatorname{Aut} K \cong G.$$

There is a much deeper variant in G. Grätzer and E.T. Schmidt [173]:

[1] For the "$\mathfrak{m}$-concepts", see p. 65 of LTF.

Theorem 10-1.8 (Strong Independence Theorem for Finite Lattices)**.** *Let K be a finite lattice with more than one element and let G be a finite group. Then K has a congruence-preserving extension L whose automorphism group is isomorphic to G.*

See Section IV.4.8 of LTF for a detailed discussion and G. Grätzer [131] for a *Proof-by-Picture* and a proof.

In the infinite case, there is no characterization for $\operatorname{Con} L$, so we have to phrase the Independence Theorem as in Theorem 10-1.8:

Theorem 10-1.9 (Independence Theorem for Lattices)**.** *Let L be a lattice with more than one element and let G be a group. Then there exists a lattice K such that*

$$\operatorname{Con} K \cong \operatorname{Con} L \textit{ and } \operatorname{Aut} K \cong G.$$

The lattice K can be chosen as a congruence-preserving extension of L.

G. Grätzer and F. Wehrung [185] prove a stronger form of Theorem 10-1.9 and it would seem to be quite appropriate to include a proof here. Unfortunately, including the results on tensor products (G. Grätzer and F. Wehrung [186] and [184]) that form the foundation for the proof, it would take about 60 pages to present a proof, longer than this chapter.

10-1.5 Related structures of complete lattices

For a complete lattice L, as in Section 10-1.4, we consider only two related structures: the automorphism group, $\operatorname{Aut} L$, and the *complete congruence lattice*, $\operatorname{Com} L$.

We have a characterization of $\operatorname{Com} L$ for a complete lattice L (Theorem 10-1.3), so we can model our result on Theorem 10-1.7. Now we state the independence result, see G. Grätzer [126] and G. Grätzer and H. Lakser [147].

Theorem 10-1.10 (Independence Theorem for Complete Lattices)**.** *Let L be a complete lattice with more than one element and let G be a group. Then there is a complete lattice K such that*

$$\operatorname{Com} K \cong L \textit{ and } \operatorname{Aut} K \cong G.$$

10-2. The Representation Theorem for Complete Lattices

In this section, we shall represent a complete lattice L with 0 and 1 as the lattice of complete congruences of a complete lattice K, as required in the Representation Theorem for Complete Lattices.

If $|L| = 1$, it is trivial to represent L. We shall, henceforth, assume that $|L| \geq 2$.

10-2.1 Preliminary steps

Let $\{ X^\delta \mid \delta < \chi \}$, $0 < \chi$, be the family of all nonempty subsets of L. The elements of X^δ are well ordered:

$$X^\delta = \{ x^\delta_\gamma \mid \gamma < \zeta^\delta \}.$$

For ordinals α, β, the *ordinal product* $\alpha \times \beta$ is regarded as the set

$$\{ (\gamma, \delta) \mid \gamma < \alpha,\ \delta < \beta \}$$

ordered by $(\gamma_1, \delta_1) \leq (\gamma_2, \delta_2)$ iff $\gamma_1 < \gamma_2$ or $\gamma_1 = \gamma_2$ and $\delta_1 \leq \delta_2$.

For a lattice A and $\mathfrak{p} = [u, v] \in \mathrm{PrInt}(A)$ (the set of *prime intervals* of A) and for any lattice B and $b \in B$, we use the notation

$$\mathfrak{p} \times \{b\} = [(u, b), (v, b)] \in \mathrm{PrInt}(A \times B).$$

If $\mathfrak{p}_1 = [x_1, y_1] \in \mathrm{PrInt}(A_1)$ and $\mathfrak{p}_2 = [x_2, y_2] \in \mathrm{PrInt}(A_2)$, then we refer to the elements of $\mathfrak{p}_1 \times \mathfrak{p}_2$ as follows:

$$\begin{aligned} o(\mathfrak{p}_1, \mathfrak{p}_2) &= (x_1, x_2), & a(\mathfrak{p}_1, \mathfrak{p}_2) &= (y_1, x_2),\\ b(\mathfrak{p}_1, \mathfrak{p}_2) &= (x_1, y_2), & i(\mathfrak{p}_1, \mathfrak{p}_2) &= (y_1, y_2). \end{aligned}$$

If $A_1 = A_2 = A$, and $\mathfrak{p}_1 = [x_1, y_1],\ \mathfrak{p}_2 = [x_2, y_2] \in \mathrm{PrInt}(A)$, the notation $o(\mathfrak{p}_1, \mathfrak{p}_2),\ a(\mathfrak{p}_1, \mathfrak{p}_2),\ b(\mathfrak{p}_1, \mathfrak{p}_2),\ i(\mathfrak{p}_1, \mathfrak{p}_2)$ refer to the four elements in A^2, where $\mathfrak{p}_1$ is regarded as a prime interval of the first component, and $\mathfrak{p}_2$ is regarded as a prime interval of the second component.

For a complete lattice A and interval $[u, v]$ in A, we denote by $\mathrm{com}(u, v)$ the *complete congruence relation* generated by $[u, v]$.

For a complete lattice A, the *complete congruence lattice* of A is denoted by $\mathrm{Com}\, A$; the lattice operations in $\mathrm{Com}\, A$ are denoted by $\wedge$, $\vee^c$, and the infinite variants by $\bigwedge$ and $\bigvee^c$.

Let L be a complete lattice. A *coloring* of a chain C is a map

$$\varphi \colon \mathrm{PrInt}(C) \to L - \{0\}.$$

If $\mathfrak{p} \in \mathrm{PrInt}(C)$ and $\varphi(\mathfrak{p}) = a$, we think of $\mathrm{com}_K(\mathfrak{p})$ (the principal complete congruence generated by $\mathfrak{p}$ in K) as the complete congruence representing $a \in L - \{0\}$ in some complete extension K of C.

Following S.-K. Teo [309], for a chain C and coloring φ, we define the lattice $C[\varphi]$ (C extended by φ) as follows: the lattice $C[\varphi]$ is C^2 augmented with the elements $m(\mathfrak{p}_1, \mathfrak{p}_2)$, whenever $\mathfrak{p}_1$, $\mathfrak{p}_2 \in \mathrm{PrInt}(C)$ and $\varphi(\mathfrak{p}_1) = \varphi(\mathfrak{p}_2)$, and we require that the elements

$$o(\mathfrak{p}_1, \mathfrak{p}_2),\ a(\mathfrak{p}_1, \mathfrak{p}_2),\ b(\mathfrak{p}_1, \mathfrak{p}_2),\ m(\mathfrak{p}_1, \mathfrak{p}_2),\ i(\mathfrak{p}_1, \mathfrak{p}_2) \tag{10-2.1}$$

form a sublattice of $C[\varphi]$ isomorphic to M_3, as illustrated by Figure 10-2.1. Obviously, $C[\varphi]$ is an extension of C. If C is complete, so is $C[\varphi]$.

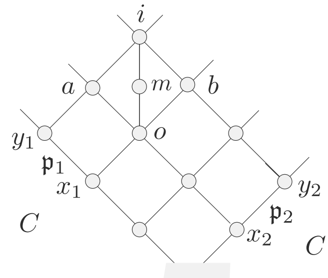

Figure 10-2.1: Adding an element m to form M_3.

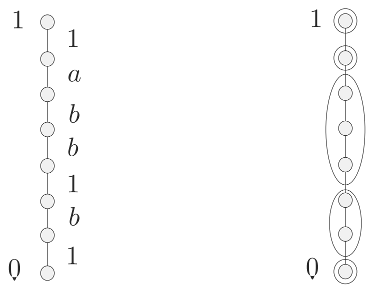

Figure 10-2.2: The chain C and the congruence $\boldsymbol{\alpha}$.

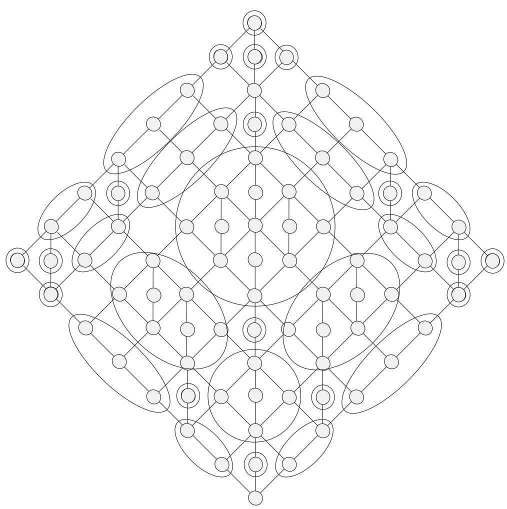

Figure 10-2.3: The lattice $C[\varphi]$ with the congruence $\boldsymbol{\alpha}[\varphi]$.

For instance, let C be the chain of Figure 10-2.2, where we mark the color of prime intervals. Figure 10-2.3 illustrates $C[\varphi]$.

The congruences of C^2 are of the form $\boldsymbol{\alpha}_1 \times \boldsymbol{\alpha}_2$, where $\boldsymbol{\alpha}_1$ and $\boldsymbol{\alpha}_2$ are congruences of C. Now take congruences $\boldsymbol{\alpha}_1$ and $\boldsymbol{\alpha}_2$ with the following property:

If $\mathfrak{p}_1$, $\mathfrak{p}_2 \in \operatorname{PrInt}(C)$ and $\varphi(\mathfrak{p}_1) = \varphi(\mathfrak{p}_2)$, then

$$\operatorname{con}(\mathfrak{p}_1) \leq \boldsymbol{\alpha}_1 \quad \text{iff} \quad \operatorname{con}(\mathfrak{p}_2) \leq \boldsymbol{\alpha}_2. \tag{10-2.2}$$

Now we extend the congruence $\boldsymbol{\alpha}_1 \times \boldsymbol{\alpha}_2$ on C^2 to $C[\varphi]$ as follows: for $\mathfrak{p}_1$, $\mathfrak{p}_2 \in \operatorname{PrInt}(C)$ and $\varphi(\mathfrak{p}_1) = \varphi(\mathfrak{p}_2)$, let the elements in (10-2.1) be in one congruence class. Let $\boldsymbol{\alpha}_1 \times_\varphi \boldsymbol{\alpha}_2$ denote this extension. It is easy to compute that the congruences of $C[\varphi]$ are exactly the congruences of the form $\boldsymbol{\alpha}_1 \times_\varphi \boldsymbol{\alpha}_2$. For $\boldsymbol{\alpha} = \boldsymbol{\alpha}_1 = \boldsymbol{\alpha}_2$, we shall use the notation $\boldsymbol{\alpha}[\varphi] = \boldsymbol{\alpha} \times_\varphi \boldsymbol{\alpha}$.

By taking $\mathfrak{p}_1 = \mathfrak{p}_2$, observe that $\boldsymbol{\alpha}_1$ and $\boldsymbol{\alpha}_2$ collapse the same prime intervals of C. Thus for a finite C, we have $\boldsymbol{\alpha}_1 = \boldsymbol{\alpha}_2$; the congruences of $C[\varphi]$ are of the form $\boldsymbol{\alpha}[\varphi]$, where $\boldsymbol{\alpha}$ is a congruence of C with property (10-2.2).

As an example, take the congruence $\boldsymbol{\alpha}$ of the chain C, see Figure 10-2.2. Then $\boldsymbol{\alpha}[\varphi]$ is the congruence of $C[\varphi]$ as illustrated in Figure 10-2.3.

To handle the infinite case, for a complete lattice A and a complete congruence $\boldsymbol{\alpha}$ on A, we define the *interior* of $\boldsymbol{\alpha}$, $\operatorname{Inter}(\boldsymbol{\alpha})$, as follows:

$$\operatorname{Inter}(\boldsymbol{\alpha}) = \bigvee\nolimits^c (\operatorname{com}(\mathfrak{p}) \mid \mathfrak{p} \in \operatorname{PrInt}(A),\ \operatorname{con}(\mathfrak{p}) \leq \boldsymbol{\alpha}).$$

For the complete congruences $\boldsymbol{\alpha}_1$ and $\boldsymbol{\alpha}_2$ on the complete chain C, the relation $\boldsymbol{\alpha}[\varphi]$, defined above on $C[\varphi]$, is a complete congruence of $C[\varphi]$ iff $\operatorname{Inter}(\boldsymbol{\alpha}_1) = \operatorname{Inter}(\boldsymbol{\alpha}_2)$ and (10-2.2) holds.

Let A be a complete lattice which is *strongly atomic*, that is, for any $w, z \in A$, $w < z$, there is an element $p \in A$ satisfying $w \prec p \leq z$. In a strongly atomic complete lattice, there are very many complete congruences generated by prime intervals:

Lemma 10-2.1. *In a strongly atomic complete lattice A, the equality*

$$\operatorname{Inter}(\boldsymbol{\alpha}) = \boldsymbol{\alpha}$$

holds for any complete congruence $\boldsymbol{\alpha}$ of A.

Proof. The inclusion $\operatorname{Inter}(\boldsymbol{\alpha}) \leq \boldsymbol{\alpha}$ is obvious. Conversely, let $x \equiv y \pmod{\boldsymbol{\alpha}}$ and $x \leq y$; we wish to prove that

$$x \equiv y \pmod{\operatorname{Inter}(\boldsymbol{\alpha})}.$$

Since $\operatorname{Inter}(\boldsymbol{\alpha})$ is a complete congruence, it follows that there exists a maximal element z in $[x, y]$ satisfying $x \equiv z \pmod{\operatorname{Inter}(\boldsymbol{\alpha})}$. If $z = y$, we are done. Otherwise, $z < y$, so by the strong atomicity of A, there exists an element p

with $z \prec p \leq y$. By $x \equiv y \pmod{\boldsymbol{\alpha}}$, it follows that $z \equiv p \pmod{\boldsymbol{\alpha}}$. Since $[z, p]$ is prime, it follows from the definition of $\mathrm{Inter}(\boldsymbol{\alpha})$ that $z \equiv p \pmod{\mathrm{Inter}(\boldsymbol{\alpha})}$. Thus, $x \equiv p \pmod{\mathrm{Inter}(\boldsymbol{\alpha})}$, contradicting the maximality of z. $\square$

Combining Lemma 10-2.1 with the discussion of the congruences of $C[\varphi]$, we obtain:

Lemma 10-2.2. *Let C be a complete strongly atomic chain. Then the complete congruences of $C[\varphi]$ are the congruences of the form $\boldsymbol{\alpha}[\varphi]$, where $\boldsymbol{\alpha}$ is a complete congruence of C satisfying* (10-2.2).

We shall leave the routine, though somewhat tedious, verification of this lemma as an exercise.

10-2.2 The lattice M_X

Let $X = \{\, x_\gamma \mid \gamma < \zeta^X \,\} \subseteq L - \{0\}$ be given as in Section 10-2.1.

First, we construct a chain $X^\dagger$, then the lattice M_X. The chain $X^\dagger$ is defined (see Figure 10-2.5) as the chain

$$\mathsf{C}_1 + (\omega \times X) + \mathsf{C}_1.$$

The elements of $X^\dagger$ are denoted as follows: the zero and unit element is 0^X and 1^X, respectively (for X^δ, we use 0^δ and 1^δ); the other elements are

$$j^X = (0, x_0) < (0, x_1) < \cdots < (i, x_0) < (i, x_1) < \cdots$$

for $i < \omega$.

We define a coloring φ^X (for X^δ, denoted by φ^δ) on $X^\dagger$ as illustrated by Figure 10-2.4:

$$\begin{aligned} \varphi^X[0^X, j^X] &= \bigvee X; \\ \varphi^X[(i, x_\gamma), u] &= x_\gamma \qquad \text{for } i < \omega \text{ and } \gamma < \zeta^X, \end{aligned}$$

where

$$u = \begin{cases} (i, x_{\gamma+1}) & \text{for } \gamma + 1 < \zeta^X; \\ (i+1, x_0) & \text{for } \gamma + 1 = \zeta^X. \end{cases}$$

Note that all the prime intervals are accounted for, so this defines a coloring.

The following observation is trivial but crucial:

Lemma 10-2.3. *The chain $X^\dagger$ is well ordered and 1^X is a limit. In $X^\dagger$, for every $j^X \leq u < 1^X$ and every $\gamma < \zeta^X$, there is a prime interval $\mathfrak{p}$ in $[u, 1^X]$ such that $\varphi(\mathfrak{p}) = x_\gamma$.*

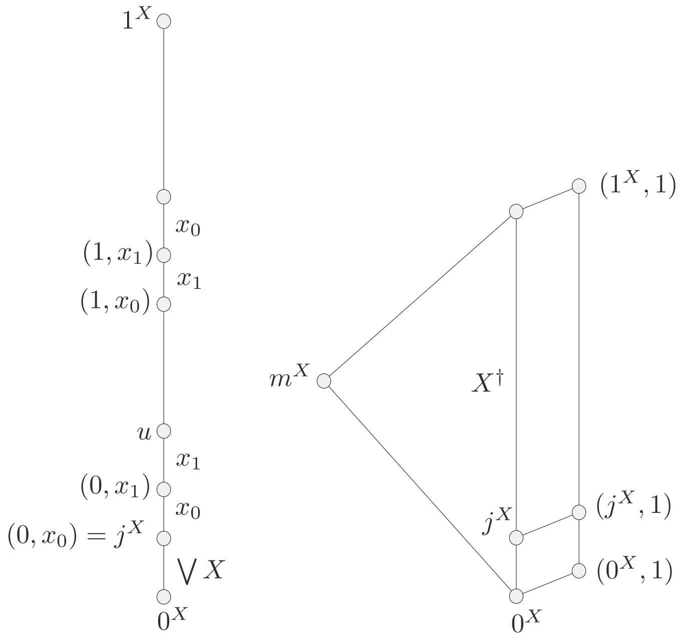

Figure 10-2.4: The chain $X^\dagger$ with coloring φ^X and the lattice M_X.

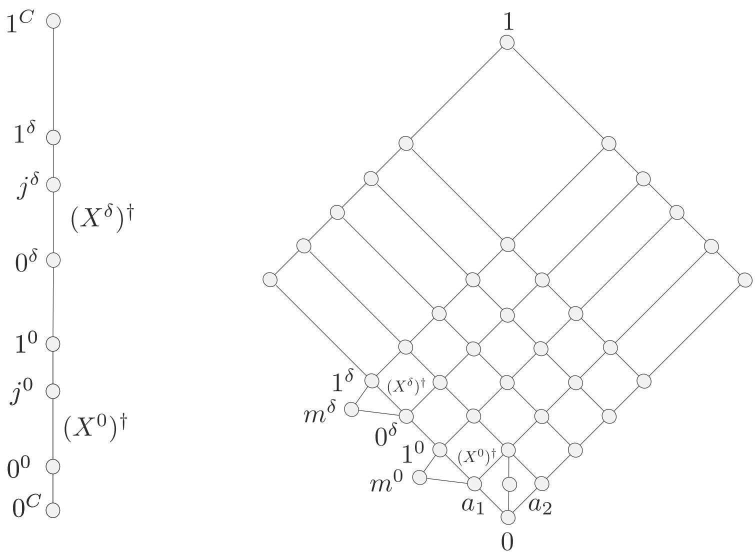

Figure 10-2.5: The chain C and the lattice K.

Now we define the lattice M_X (and M_{X^δ}) – see Figure 10-2.5. First, we form $X^\dagger \times \mathsf{C}_2$. For $x \in X^\dagger$, we identify $(x, 0)$ with x. (We make the same identification in $(X^\delta)^\dagger \times \mathsf{C}_2$ for $\delta < \chi$.) This makes $X^\dagger$ a complete sublattice of $X^\dagger \times \mathsf{C}_2$. Then we form M_X by adding to $X^\dagger \times \mathsf{C}_2$ the element m^X (denoted by m^δ for X^δ) satisfying

$$0^X \prec m^X \prec 1^X.$$

Obviously, M_X is a complete lattice, and $X^\dagger$ is a complete sublattice of M_X.

10-2.3 The complete lattice K

Now we are ready to construct the complete lattice K to verify the Representation Theorem for Complete Lattices. For every X^δ, $\delta < \chi$, we construct the chain $(X^\delta)^\dagger$ and form the ordinal sum:

$$\begin{aligned} C = \mathsf{C}_1 + \Sigma\{\, (X^\delta)^\dagger \mid \delta < \chi \,\} + \mathsf{C}_1 &\quad \text{for } \chi \text{ is limit;}\\ \mathsf{C}_1 + \Sigma\{\, (X^\delta)^\dagger \mid \delta < \chi \,\} &\quad \text{for } \chi \text{ is not limit;} \end{aligned}$$

see Figure 10-2.5. The zero and unit element of C is denoted by 0^C and 1^C, respectively. Observe that if χ is not limit, that is, $\chi = \chi_0 + 1$, then $1^C = 1^{\chi_0}$.

C is a well-ordered chain. In the definition of C, we distinguished the two cases to make sure that the unit element is a limit; this plays a role in the proof that K has no nontrivial automorphism – see Lemma 10-3.2.

Next, we define a coloring φ of C. For a prime interval $\mathfrak{p}$ of C, let

$$\varphi(\mathfrak{p}) = \begin{cases} \varphi^\delta(\mathfrak{p}), & \text{if } \mathfrak{p} \in \operatorname{PrInt}(X^\delta) \text{ for some } \delta < \chi;\\ 1, & \text{if } \mathfrak{p} = [0^C, 0^0];\\ 1, & \text{if } \mathfrak{p} = [1^\delta, 0^{\delta+1}] \text{ for some } \delta < \chi. \end{cases}$$

Finally, we define the lattice K as $C[\varphi]$ augmented with the elements m^δ for $\delta < \chi$, see Figure 10-2.5.

More formally,

$$K = C[\varphi] \cup \{\, m^\delta \mid \delta < \chi \,\}$$

partially ordered as follows, where x, y denote elements of $C[\varphi]$:

$$\begin{aligned} x \leq y &\qquad \text{retains its meaning in } C[\varphi];\\ m^\delta < x &\qquad \text{iff } (1^\delta, 0^C) \leq x \text{ in } C[\varphi] \text{ for } \delta < \chi;\\ x < m^\delta &\qquad \text{iff } x \leq (0^\delta, 0^C) \text{ in } C[\varphi] \text{ for } \delta < \chi. \end{aligned}$$

It is easy to see that K is a complete lattice, $C[\varphi]$ is a complete $\{0,1\}$-sublattice of K, and $(0^\delta, 0^C) \prec m^\delta \prec (1^\delta, 0^C)$ in K.

We name a few elements of K:

$$\begin{aligned} o &= (0^C, 0^C),\\ i &= (1^C, 1^C),\\ a_1 &= (0^0, 0^C),\\ a_2 &= (0^C, 0^0). \end{aligned}$$

o is the zero and i is the unit element of K. The lattice K has three atoms. Every element $x \neq o$ of K contains an atom. In fact, K is strongly atomic since every chain in K is well ordered.

We identify the element $(x, 0^C)$ of K with the element x of C. Thus C becomes a complete sublattice of K. We get that

$$\begin{aligned} m^\delta &\prec 1^\delta &&\text{for } \delta < \chi,\\ 0^\delta &\prec m^\delta &&\text{for } \delta < \chi; \end{aligned}$$

the new named elements become: $o = 0^C$, $a_1 = 0^0$.

Observe that interval $[0^\delta, (1^\delta, 0^0)]$ of K is isomorphic to M_{X^δ} for $\delta < \chi$.

10-3. The Independence Theorem for Complete Lattices

We present here the proof of G. Grätzer and H. Lakser [147].

Based on the results of R. Frucht [94] and G. Sabidussi [288], it is routine to see that we can represent the group G as the automorphism group of a connected undirected graph $\mathsf{G} = (V, E)$ without loops, where V is the set of vertices and E is the set of edges.

Next we represent G by a bounded lattice and lattice automorphisms. As in R. Frucht [94], from G we form the lattice:

$$H = V \mathbin{\dot\cup} E \mathbin{\dot\cup} \{0, 1\},$$

where $0 < v < 1$, $0 < e < 1$ for all $v \in V$ and $e \in E$; let $v < e$ in H iff $v \in e$. Note that H is of length three, and therefore complete.

These graphs have the following property:

(10-3.1) For $v \in V$, there are e_1, $e_2 \in E$ with $v \nleq e_1$, e_2 and $e_1 \cap e_2 = \varnothing$.

It is easy to prove that if the graph G has property (10-3.1), then the associated lattice is simple. Hence, H is a simple lattice.

We attach H to K by identifying the unit element i of K with the zero 0_H of H; we add a complement q of i, see Figure 10-3.1. The next three lemmas show that the resulting lattice K_H will do the job.

Let $\boldsymbol{\alpha}$ be a complete congruence relation of K. We shall define an extension $\overline{\boldsymbol{\alpha}}$ of $\boldsymbol{\alpha}$ to K_H: If $\boldsymbol{\alpha} < \mathbf{1}_K$, let $\overline{\boldsymbol{\alpha}}$ be the congruence relation of K_H that is $\boldsymbol{\alpha}$ on K and trivial outside of K. We define $\overline{\mathbf{1}_K} = \mathbf{1}_{K_H}$.

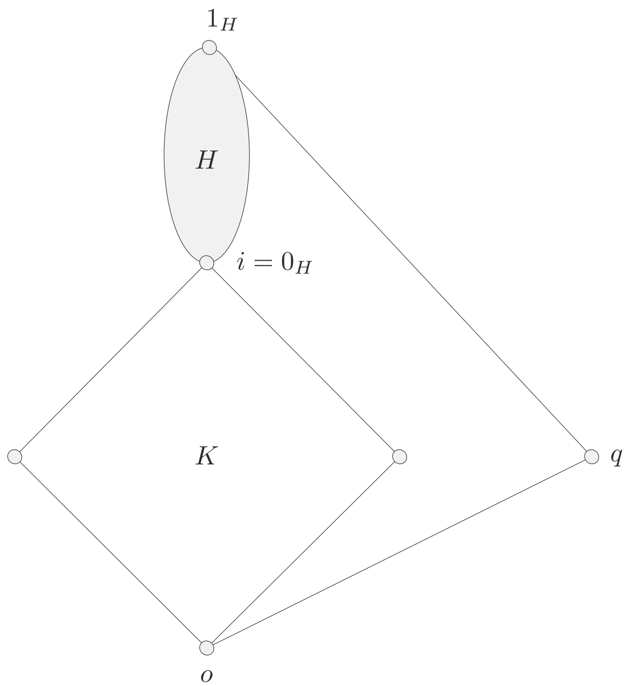

Figure 10-3.1: The lattice K_H.

Lemma 10-3.1. *The complete congruence relations of K_H are the relations of the form $\overline{\boldsymbol{\alpha}}$, where $\boldsymbol{\alpha}$ is a complete congruence of K.*

Proof. By straightforward computation. Note only that if $\boldsymbol{\alpha} < \mathbf{1}$, then, in K, $\{o\}$ is a congruence class of $\boldsymbol{\alpha}$; this is why the extension $\overline{\boldsymbol{\alpha}}$ can be defined to be trivial outside of K. Since H is simple, it follows that all complete congruences of K_H extend from K. □

This lemma immediately implies that the complete congruence lattice of K_H is isomorphic to the complete congruence lattice of K.

Now for the automorphisms.

Lemma 10-3.2. *K has no nontrivial automorphism.*

Proof. Let α be an automorphism of K. Under α, the image of a meet-reducible atom is a meet-reducible atom; therefore, $\alpha(a_1) = a_1$ or $\alpha(a_1) = a_2$. The latter is impossible since, in K, $\operatorname{id}(a_1)$ is nonmodular while $\operatorname{fil}(a_2)$ is modular. Hence,

$$\alpha(a_1) = a_1 \text{ and } \alpha(a_2) = a_2.$$

The elements 1^C $(= (1^C, 0^C))$ and $(0^C, 1^C)$ are the only doubly irreducible and completely join-reducible elements in K; since $\alpha(a_1) = a_1$, it follows that

$$\alpha(1^C) = 1^C \text{ and } \alpha(0^C, 1^C) = (0^C, 1^C).$$

Therefore, the interval $[a_1, 1^C]$ is mapped into itself by α and so the meet-reducible elements of the interval, that is, the elements of the form $(0^C, x)$, $x \in C$, are mapped into themselves. We conclude that α can be regarded as an automorphism of C. Since C is a well-ordered set, it has no nontrivial automorphism and so α is the identity map on C. Arguing similarly, we get that α is the identity map on $\{\, (0^C, x) \mid x \in C \,\}$. Therefore, α is the identity map on $C \times C$. It now easily follows that α is the identity map on $C[\varphi]$ and on K. This completes the proof of the lemma. ☐

Now, let α be an automorphism of H. We extend α to K_H trivially:

$$\overline{\alpha}(x) = \begin{cases} \alpha(x), & \text{if } x \in H; \\ x, & \text{otherwise.} \end{cases} \tag{10-3.2}$$

Lemma 10-3.3. *Let α be an automorphism of H. Then $\overline{\alpha}$ is an automorphism of K_H. Conversely, every automorphism of K_H can be uniquely represented in this form.*

Proof. Let β be an automorphism of K_H. Observe that $0_H = i$ is the only element u of K_H with the property that there is a maximal chain of length three in $[u, 1^H]$. Hence, $\beta(i) = i$. Thus β induces an automorphism β_K of K and an automorphism β_H of H. By Lemma 10-3.2, β_K is the identity map. Define $\alpha = \beta_H$. Then $\beta = \overline{\alpha}$, as defined in (10-3.2), as claimed. This representation is clearly unique, completing the proof of the lemma. ☐

Lemma 10-3.3 obviously implies that the automorphism group of K_H is isomorphic to G. Therefore, Lemmas 10-3.1 and 10-3.3 prove the Independence Theorem for Complete Lattices.

10-4. An application to infinitary algebras

In this section, we show that by applying the Representation Theorem for Complete Lattices, we can get a direct proof of the Independence Theorem for Algebras (G. Grätzer [128]).

10-4.1 The construction of the algebra $\mathfrak{A}$

Let L_{sub} and L_{con} be complete lattices and let G be a group; we assume that L_{con} has more than one element. We shall construct an infinitary algebra $\mathfrak{A}$ such that

$$\operatorname{Sub}\mathfrak{A} \cong L_{\mathrm{sub}}, \ \operatorname{Con}\mathfrak{A} \cong L_{\mathrm{con}}, \text{ and } \operatorname{Aut}\mathfrak{A} \cong G.$$

Let

$$A = K \cup (G \times (L_{\mathrm{sub}} - \{0\})),$$

where K is the lattice of Theorem 10-1.3 for L_{con}, that is, $\mathrm{Com}\, K \cong L_{\mathrm{con}}$.

Let o and i denote the zero and unit element of K, respectively. It will be convenient to use the notation g_a for (g, a) for $g \in G$, $a \in K - \{0\}$. We make A into a complete lattice by extending the partial ordering of K:

$$o \prec g_a \prec i \qquad \text{for } g \in G,\ a \in K - \{0\}.$$

Define

$$G_a = \{g_a \mid g \in G\} \qquad \text{for } a \in L_{\mathrm{sub}} - \{0\}.$$

Figure 10-4.1 should help visualize A.

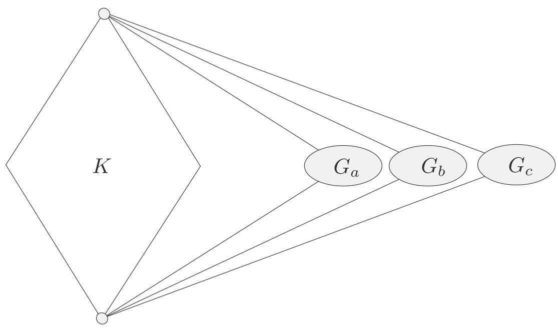

Figure 10-4.1: The lattice A.

The algebra $\mathfrak{A}$ is this complete lattice with the following additional operations:

(i) a nullary operation k for all $k \in K$.

(ii) A unary operation f_h for all $h \in G$, defined by

$$\begin{aligned} f_h(g_a) &= (hg)_a && \text{for } g \in G, a \in L_{\mathrm{sub}} - \{0\}; \\ f_h(k) &= o, && \text{for } k \in K. \end{aligned}$$

(iii) A unary operation $f_{a,b}$, for all a, $b \in L_{\mathrm{sub}} - \{0\}$, $a \leq b$, defined by

$$\begin{aligned} f_{a,b}(g_b) &= g_a && \text{for } g \in G; \\ f_{a,b}(x) &= o, && \text{otherwise.} \end{aligned}$$

(iv) An infinitary operation Σ:

$$\begin{aligned} \Sigma\{g_x \mid x \in X\} &= g_{\bigvee X}, && \text{for } g \in G, X \subseteq L_{\mathrm{sub}} - \{0\}; \\ \Sigma(Y) &= o, && \text{otherwise.} \end{aligned}$$

10-4.2 A proof of the Independence Theorem

In order to prove the Independence Theorem for Algebras, we have to describe the congruences, the subalgebras, and the automorphisms of $\mathfrak{A}$.

Congruences

Let $\boldsymbol{\alpha}$ be a complete congruence relation of the complete lattice K. We define a binary relation $\overline{\boldsymbol{\alpha}}$ on A as follows:

$$\overline{\mathbf{1}_T} = \mathbf{1}_A,$$

and for $\boldsymbol{\alpha} \neq \mathbf{1}$, $\overline{\boldsymbol{\alpha}} = \boldsymbol{\alpha}$ on K and $\overline{\boldsymbol{\alpha}}$ is trivial otherwise.

Claim 3. *$\overline{\boldsymbol{\alpha}}$ is a congruence relation of $\mathfrak{A}$.*

Proof. It is easy to see that $\overline{\boldsymbol{\alpha}}$ is a complete congruence of A. The Substitution Property for the operations (i)–(iv) vacuously holds. ☐

We claim that

$$\psi_c \colon \boldsymbol{\alpha} \to \overline{\boldsymbol{\alpha}}$$

is an isomorphism between the complete congruence lattice of K and the congruence lattice of $\mathfrak{A}$. To verify this, it is sufficient to show the following:

Claim 4. *Every congruence relation $\boldsymbol{\beta}$ of $\mathfrak{A}$ can be represented as $\boldsymbol{\beta} = \overline{\boldsymbol{\alpha}}$ for exactly one complete congruence $\boldsymbol{\alpha}$ of K.*

Proof. For a given $\boldsymbol{\beta}$, define $\boldsymbol{\alpha}$ as the restriction $\boldsymbol{\beta}$ to K. Since K is a complete sublattice of A, we conclude that $\boldsymbol{\alpha}$ is a complete congruence of K; hence, $\overline{\boldsymbol{\alpha}}$ is defined and $\overline{\boldsymbol{\alpha}} \leq \boldsymbol{\beta}$. Let $x \equiv y \pmod{\boldsymbol{\beta}}$; we want to show that $x \equiv y \pmod{\overline{\boldsymbol{\alpha}}}$. This is obvious if $\boldsymbol{\alpha} = \mathbf{1}$. So let $\boldsymbol{\alpha} \neq \mathbf{1}$. We distinguish three subcases:

Case 1. x, $y \in K$. The statement is trivial.

Case 2. $x \in K$ and $y \in G_a$ for some $a \in L_{\mathrm{sub}} - \{0\}$. Firstly, let $x \in \{o, i\}$, for instance, $x = o$ (the case $x = i$ is handled dually). Then $o \equiv y \pmod{\boldsymbol{\beta}}$ implies that $o \vee z \equiv y \vee z \pmod{\boldsymbol{\beta}}$ for any $z \in K - \{o, i\}$; thus, $z \equiv i \pmod{\boldsymbol{\beta}}$ contradicting Exercise 10.4(ii). Secondly, let $x \notin \{o, i\}$. Then $x \equiv y \pmod{\boldsymbol{\beta}}$ implies that $x \vee y = x \wedge y(\boldsymbol{\beta})$, that is, $i \equiv o \pmod{\boldsymbol{\beta}}$; this means that $\boldsymbol{\beta} = \mathbf{1}$, a contradiction.

Case 3. $x \in G_a$ and $y \in G_b$ for some a, $b \in L_{\mathrm{sub}} - \{0\}$. Then $x \equiv y \pmod{\boldsymbol{\beta}}$ implies that $\boldsymbol{\beta} = \mathbf{1}$ as in the last case, a contradiction.

This completes the proof of $\overline{\boldsymbol{\alpha}} \leq \boldsymbol{\beta}$. The uniqueness of $\boldsymbol{\alpha}$ is now obvious.

Since ψ_c is obviously isotone, it follows from Claim 2 that ψ_c is an isomorphism. ☐

Subalgebras

For every $c \in L_{\text{sub}}$, define

$$S_c = K \cup \bigcup (G_b \mid b \leq c).$$

Note that $S_0 = K$. Set

$$\psi_{\text{sub}} \colon c \to S_c \qquad \text{for } c \in L_{\text{sub}}.$$

We shall prove that ψ_{sub} is an isomorphism between L_{sub} and the subalgebra lattice of $\mathfrak{A}$. As a first step we prove:

Claim 5. *S_c is a subalgebra of $\mathfrak{A}$ for every $c \in L_{\text{sub}}$.*

Proof. This is obvious for S_0. If $c \in L_{\text{sub}}$ and $c > 0$, then S_c is a complete sublattice of A. Moreover, S_c is closed under the operations (i), since $K \subseteq S_c$. If $g_a \in S_c$, for $g \in G$, $a \in L_{\text{sub}} - \{0\}$, then $a \leq c$ by the definition of S_c. Hence $f_h(g_a) = (hg)_a$ is also in S_c. Thus S_c is closed under the operations (ii). Let a, $b \in L_{\text{sub}} - \{0\}$, $a \leq b$. Then $f_{a,b}(x) = o \in S_c$ unless $x = g_b$; in this case, $g_b \in S_c$, hence $b \leq c$. Since $a \leq b$ and $b \leq c$, we conclude that $a \leq c$. Thus

$$f_{a,b}(g_b) = g_a \in S_c,$$

proving that S_c is closed under the operations (iii). Finally, $\Sigma(Y) = o \in S_a$ except if $Y = \{g_x \mid x \in X\}$. In this case, $\{g_x \mid x \in X\} \subseteq S_c$, and so $x \leq c$ for all $x \in X$. Thus $\bigvee X \leq c$, implying that

$$\Sigma\{g_x \mid x \in X\} = g_{\bigvee X} \in S_a.$$

Therefore, S_c is closed under the operations (iv), and so S_c is a subalgebra. $\square$

Next, we show that ψ_{sub} is onto.

Claim 6. *Let S be a subalgebra of $\mathfrak{A}$. Then there exists an element $c \in L_{\text{sub}}$ such that $S = S_c$.*

Proof. Let $S \subseteq K$. Then $S = S_0 = K$ because of the operations (i). So let $S \nsubseteq K$. Set

$$Z = \{x \in L_{\text{sub}} \mid x > 0 \text{ and } g_x \in S \text{ for some } g \in G\},$$

and $c = \bigvee Z$ (the join is formed in L_{sub}). We shall verify that $S = S_c$. Obviously, $S \subseteq S_c$. Conversely, let $u \in S_c$. We claim that $u \in S$. This is obvious if $u \in K$. So let $u \notin K$, that is, $u = h_d$ for some $h \in G$ and $d \in L_{\text{sub}} - \{0\}$ satisfying $d \leq c$. If $x \in Z$, then $g_x \in S$, for some $g \in G$. Thus

$$1_x = f_{g^{-1}}(g_x) \in S.$$

It follows that

$$X = \{1_x \mid x \in Z\} \subseteq S.$$

Thus

$$\Sigma(X) = 1_{\bigvee Z} = 1_c \in S.$$

This implies that $G_c \subseteq S$. Indeed, for $g \in G$, $g_c = f_g(1_c)$. Now

$$u = h_d = f_{d,c}(h_c) \in S,$$

as claimed, concluding the proof of the claim. $\square$

Since ψ_{sub} is obviously isotone and one-to-one, ψ_{sub} is an isomorphism between L_{sub} and $\mathrm{Sub}\,\mathfrak{A}$.

Automorphisms

For $h \in G$, define a map α_h of A into itself as follows:

$$\begin{aligned} \alpha_h(k) &= k && \text{for } k \in K; \\ \alpha_h(g_a) &= (gh)_a && \text{for } g \in G \text{ and } a \in L_{\mathrm{sub}} - \{0\}. \end{aligned}$$

Set:

$$\psi_g \colon h \to \alpha_h \qquad \text{for } h \in G.$$

We shall prove that ψ_g is an isomorphism between G and the automorphism group of $\mathfrak{A}$. As a first step we prove:

Claim 7. *α_h is an automorphism of $\mathfrak{A}$.*

Proof. α_h leaves the elements of K fixed, and therefore, permutes the elements of

$$\bigcup(G_a \mid a \in L_{\mathrm{sub}} - \{0\}).$$

Hence α_h is a lattice automorphism of A. It is routine to check that α_h commutes with the operations (i)–(iii). Call a subset Y of A *coordinated* iff

$$Y = \{g_x \mid x \in X\}$$

for some $g \in G$ and $X \subseteq L_{\mathrm{sub}} - \{0\}$. Observe, that Y is coordinated iff $\alpha_h(Y)$ is coordinated. It follows that α_h commutes with Σ. Since α_h is obviously one-to-one and onto, α_h is an automorphism of $\mathfrak{A}$. $\square$

Next we prove that ψ_g is onto.

Claim 8. *Every automorphism α of $\mathfrak{A}$ is of the form α_h for some $h \in G$.*

Proof. Let α be an automorphism of $\mathfrak{A}$. Obviously, $\alpha(k) = k$ for $k \in K$, since every $k \in K$ is a constant.

Let $a \in L_{\mathrm{sub}} - \{0\}$. The solutions of the equation

$$f_{a,a}(x) = x$$

are $x = o$ and $x \in G_a$. Since o is fixed by α, it follows that α maps G_a into itself. Let $\alpha(1_a) = h_a$. Then

$$\alpha(g_a) = \alpha(f_g(1_a)) = f_g(\alpha(1_a)) = f_g(h_a) = (gh)_a = \alpha_h(g_a),$$

hence $\alpha = a_h$ on G_a.

Now let a, $b \in L_{\mathrm{sub}} - \{0\}$; let $\alpha = \alpha_h$ on G_a and $\alpha = \alpha_k$ on G_b for h, $k \in G$. We want to show that $h = k$. Firstly, assume that $a \leq b$ in L_{sub}. Then

$$\begin{aligned} h_a = \alpha_h(1_a) = \alpha(1_a) = \alpha(f_{a,b}(1_b)) &= f_{a,b}(\alpha(1_b)) \\ &= f_{a,b}(\alpha_k(1_b)) = f_{a,b}(k_b) = k_a, \end{aligned}$$

hence $h = k$. Secondly, in the general case, let $\alpha = \alpha_h$ on G_a and $\alpha = \alpha_k$ on G_b for h, $k \in G$. Let $\alpha = \alpha_m$ on $G_{a \vee b}$, for $m \in G$. Apply the previous argument twice, once for a and $a \vee b$, and once for b and $a \vee b$, to conclude that $h = m$ and $k = m$; thus, $h = k$.

Thus, $\alpha = \alpha_h$ on all of A, concluding the proof of the claim.

It is now clear that ψ_g is an isomorphism between G and the automorphism group of $\mathfrak{A}$.

Thus the Independence Theorem for Algebras follows. □

10-5. Complete-simple distributive lattices

In this section, we would have liked to prove the Complete Distributive Lattice Representation Theorem: Every complete lattice L is isomorphic to the congruence lattice of a complete distributive lattice K.

Unfortunately, the proof is too technical to present. So instead, we tackle the following result (G. Grätzer and E.T. Schmidt [166], [168]). But first two definitions.

We call a lattice K *complete-simple* if K is complete and it has only the two trivial complete congruences; in formula, $|\operatorname{Com} K| \leq 2$.

An ICSD lattice is an Infinite Complete-Simple Distributive lattice.

Theorem 10-5.1 (Existence Theorem for ICSD Lattices)**.** *There exists an infinite complete-simple distributive lattice, that is, an ICSD lattice.*

The proof of this result introduces some of the techniques necessary to prove the Complete Distributive Lattice Representation Theorem, but in a much simpler setting.

10-5.1 Why ICSD lattices?

The Existence Theorem for ICSD Lattices appears to be a special case, $L = \mathsf{C}_2$, of the Complete Distributive Lattice Representation Theorem. However, for $L = \mathsf{C}_2$, the lattice $K = \mathsf{C}_2$ satisfies $\operatorname{Com} K = L$. The Existence Theorem claims that there is an *infinite*, complete-simple, distributive lattice K satisfying $\operatorname{Com} K = \mathsf{C}_2$. Why is this relevant? The Representation Theorem for Complete Distributive Lattices does not require the existence of such an infinite lattice.

We claim that any solution of the Representation Theorem for Complete Distributive Lattices must implicitly construct ICDS lattices.

Indeed (G. Grätzer and E.T. Schmidt [171]),

Lemma 10-5.2. *Let L be a complete distributive lattice, and let* $\operatorname{Com} L = \mathsf{C}_3$. *Then L has a complete congruence $\boldsymbol{\alpha}$ such that $L/\boldsymbol{\alpha}$ is an ICSD lattice.*

Proof. Let $\boldsymbol{\alpha}$ be the only nontrivial complete congruence relation of L. Then the quotient lattice, $L/\boldsymbol{\alpha}$, is a complete-simple lattice because $\boldsymbol{\alpha}$ is a maximal proper complete congruence relation. If $L/\boldsymbol{\alpha}$ is infinite, then it is an ICSD lattice, and the lemma is proved.

By way of contradiction, let us assume that $L/\boldsymbol{\alpha}$ is finite; then it is a two-element lattice. Therefore, $\boldsymbol{\alpha}$ has two congruence classes. Since $\boldsymbol{\alpha}$ is complete, the smaller congruence class has a largest element, a, and the larger congruence class has a smallest element, b. It follows that $a \prec a \vee b$.

By Exercise 10.10, $\operatorname{con}(a, a \vee b)$ is a complete congruence. Obviously, $\operatorname{con}(a, a \vee b)$ is the complement of $\boldsymbol{\alpha}$ in $\operatorname{Com} L$, contradicting that $\operatorname{Com} L$ is the three-element chain. $\square$

10-5.2 The $D^{\langle 2\rangle}$ construction

The proof of the Existence Theorem for ICSD Lattices utilizes two constructions. In this section, we tackle the first, the $D^{\langle 2\rangle}$ construction.

We start with two definitions.

Definition 10-5.3. Let V be a lattice with 0 and 1. Then V is called a *JM-lattice* if the lattice is complete, distributive, and the following two conditions are satisfied:

(J) 1 is join-irreducible and completely join-reducible.

(M) 0 is meet-irreducible and completely meet-reducible.

Our first construction is very easy to describe; Figure 10-5.1 shows the construction for $D = \mathcal{I}$, the $[0, 1]$ real interval as a complete chain.

Definition 10-5.4. Let D be a JM-lattice. Then define the following subset of the lattice $D^2 = D \times D$:

$$D^{\langle 2\rangle} = D^2 - ((\{0\} \times (D - \{0\})) \cup ((D - \{1\}) \times \{1\})).$$

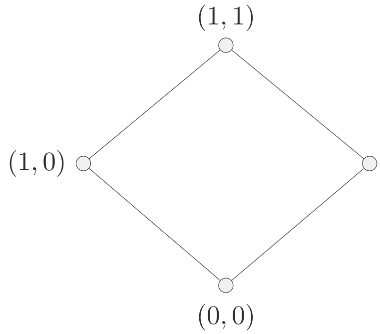

Figure 10-5.1: The lattice $D^{\langle 2\rangle}$.

We shall utilize the following properties of this construct:

Lemma 10-5.5. *Let D be a* JM-*lattice.*

(i) $D^{\langle 2\rangle}$ *is a* JM-*lattice.*

(ii) *Let $\boldsymbol{\alpha}$ be a complete congruence relation of $D^{\langle 2\rangle}$ such that*

$$\langle 1, d\rangle \equiv \langle 1, 1\rangle \pmod{\boldsymbol{\alpha}}$$

for some $d \in D$, $d < 1$. Then $\boldsymbol{\alpha} = \mathbf{1}$.

(iii) *Let $\boldsymbol{\alpha}$ be a complete congruence relation of $D^{\langle 2\rangle}$ such that*

$$\langle d, 0\rangle \equiv \langle 0, 0\rangle \pmod{\boldsymbol{\alpha}},$$

for some $d \in D$, $d > 0$. Then $\boldsymbol{\alpha} = \mathbf{1}$.

Proof.

(i) By the first clause of Condition (J) and by the first clause of Condition (M), $D^{\langle 2\rangle}$ is a sublattice of D^2. Hence, $D^{\langle 2\rangle}$ is a lattice. Since $D^{\langle 2\rangle}$ is a sublattice of a distributive lattice, $D^{\langle 2\rangle}$ is a distributive lattice.

Obviously, $D^{\langle 2\rangle}$ has a zero and a unit element, namely, $\langle 0, 0\rangle$ and $\langle 1, 1\rangle$. To show that $D^{\langle 2\rangle}$ is complete, let $\varnothing \neq A \subseteq D^{\langle 2\rangle}$, and let $a = \bigvee A$ in D^2. If $a \in D^{\langle 2\rangle}$, then $a = \bigvee A$ in $D^{\langle 2\rangle}$. Otherwise, a is of the form $\langle b, 1\rangle$ for some $b \in D$, $b < 1$. Then $\bigvee A = \langle 1, 1\rangle$ in $D^{\langle 2\rangle}$. By duality, we obtain that $\bigwedge A$ exists in $D^{\langle 2\rangle}$. So $D^{\langle 2\rangle}$ is complete.

Conditions (J) and (M) obviously hold for $D^{\langle 2\rangle}$. Hence, $D^{\langle 2\rangle}$ is a JM-lattice.

(ii) Let us assume that $\langle 1, d\rangle \equiv \langle 1, 1\rangle \pmod{\boldsymbol{\alpha}}$. Let a, $c \in D$ with $d \leq c < 1$ and $a > 0$. Compute:

$$\langle a, c\rangle = \langle a, c\rangle \wedge \langle 1, 1\rangle \equiv \langle a, c\rangle \wedge \langle 1, d\rangle = \langle a, d\rangle \pmod{\boldsymbol{\alpha}}.$$

Using the second clause of Condition (J), forming the complete join for all $c < 1$, we obtain that

$$\langle 1, 1\rangle = \bigvee(\, \langle a, c\rangle \mid 1 > c \geq d\,) \equiv \bigvee \langle a, d\rangle = \langle a, d\rangle \pmod{\boldsymbol{\alpha}},$$

that is, $\langle 1, 1\rangle \equiv \langle a, d\rangle \pmod{\boldsymbol{\alpha}}$. Now forming the complete meet for all $a > 0$, using the second clause of Condition (M), we obtain that

$$(1, 1) \equiv \bigwedge(\, \langle a, d\rangle \mid a \in D - \{0\}\,) = \langle 0, 0\rangle \pmod{\boldsymbol{\alpha}};$$

hence $\boldsymbol{\alpha} = \mathbf{1}$.

(iii) By duality.

$\square$

10-5.3 The second construction

The second construction for the Existence Theorem for ICSD Lattices is the formation of a sublattice of the direct product of copies of $\mathcal{I}$.

Let $\mathcal{I}^- = \mathcal{I} - \{1\}$. For every $j \in \mathcal{I}^-$, we take a copy $\mathcal{I}_j$ of $\mathcal{I}$. Let P denote the complete direct product $\Pi(\mathcal{I}_j \mid j \in \mathcal{I}^-)$. Let $\mathbf{O}$ and $\mathbf{I}$ denote the zero and the unit element of P, respectively.

Definition 10-5.6. Let E be the sublattice of P consisting of those $\mathbf{v} \in P$ that satisfy, for all $j \in \mathcal{I}^-$, the following two conditions:

(i) $\mathbf{v}(0) \leq j$ implies that $\mathbf{v}(j) = 0$.

(ii) $\mathbf{v}(j) = 1$ implies that $\mathbf{v}(0) = 1$.

The next three lemmas prove some properties of E.

Lemma 10-5.7. *E is a complete distributive lattice; it is a sublattice of P.*

Proof. Let $A \subseteq E$; form $\mathbf{v} = \bigvee_P A$. Let $j \in \mathcal{I}^-$ and $\mathbf{v}(0) \leq j$. Then $\mathbf{w}(0) \leq j$ for all $\mathbf{w} \in A$, and so applying Definition 10-5.6(i) to $\mathbf{w}$ we obtain that $\mathbf{w}(j) = 0$. Therefore, $\mathbf{v}(j) = 0$, verifying Definition 10-5.6(i) for $\mathbf{v}$.

If $\mathbf{v} = \bigvee_P A$ satisfies Definition 10-5.6(ii), then obviously, $\mathbf{v} = \bigvee_E A$. This is always the case if A is finite. Indeed, if $\mathbf{v}(j) = 1$, then $\bigvee(\, \mathbf{w}(j) \mid \mathbf{w} \in A\,) = 1$ in $\mathcal{I}_j$; the finiteness of A implies that $\mathbf{w}(j) = 1$ for some $\mathbf{w} \in A$. Applying Definition 10-5.6(ii) to $\mathbf{w}$, we obtain that $\mathbf{w}(0) = 1$. Since $\mathbf{v}(0) \geq \mathbf{w}(0) = 1$, it follows that $\mathbf{v}(0) = 1$, verifying Definition 10-5.6(ii) for $\mathbf{v}$. This proves that E is a join-subsemilattice of P.

Now if A is infinite and $\mathbf{v} = \bigvee_P A$ fails Definition 10-5.6(ii), then there exists a $j \in \mathcal{I}^-$ such that $\mathbf{v}(j) = 1$ and $\mathbf{v}(0) < 1$. In this case, define $\overline{\mathbf{v}}$ as follows: $\overline{\mathbf{v}}(0) = 1$ and $\overline{\mathbf{v}}(j) = \mathbf{v}(j)$ for all $j \in (0, 1)$. Obviously, $\overline{\mathbf{v}} \in E$ and $\overline{\mathbf{v}} = \bigvee A$. This proves that E is join-complete.

Now let $\mathbf{v} = \bigwedge_P A$. We claim that $\mathbf{v}$ satisfies Definition 10-5.6(ii). Indeed, if $\mathbf{v}(j) = 1$, then $\mathbf{w}(j) = 1$ for all $\mathbf{w} \in A$. By Definition 10-5.6(ii) applied to $\mathbf{w}$, we conclude that $\mathbf{w}(0) = 1$ for all $\mathbf{w} \in A$; hence, $\mathbf{v}(0) = \bigwedge(\, \mathbf{w}(0) \mid \mathbf{w} \in A\,) = \bigwedge 1 = 1$, which was to be proved.

If $\mathbf{v}$ satisfies Definition 10-5.6(i), then obviously, $\mathbf{v} = \bigwedge_E A$. This is always the case if A is finite. Indeed, let $\mathbf{v}(0) \leq j$, for some $j \in \mathcal{I}^-$. Then $\bigwedge(\, \mathbf{w}(0) \mid \mathbf{w} \in A\,) \leq j$. By the finiteness of A we conclude that $\mathbf{w}(0) \leq j$ for some $\mathbf{w} \in A$. Applying Definition 10-5.6(i) to this $\mathbf{w}$, we conclude that $\mathbf{w}(j) = 0$, and so $\mathbf{v}(j) \leq \mathbf{w}(j) = 0$, proving that $\mathbf{v}(j) = 0$, verifying Definition 10-5.6(i) for $\mathbf{v}$. This proves that E is a meet-subsemilattice of P. This completes the proof of the second statement of this lemma.

E is join-complete and has a zero (namely, $\mathbf{O}$). Therefore, it is a complete lattice. This completes the proof of the lemma. $\square$

We introduce some notation for elements of E (see Figure 10-5.2):

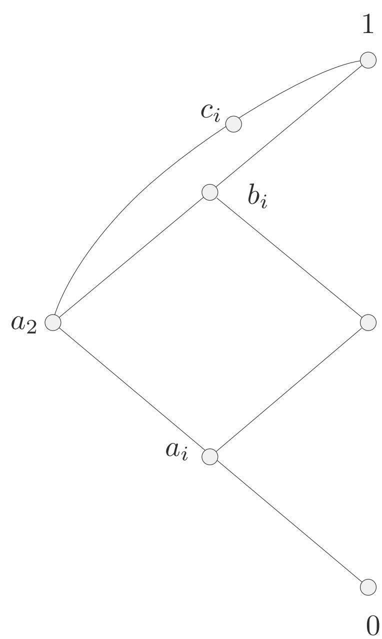

Figure 10-5.2: The lattice E.

Definition 10-5.8. Let i be an element of $\mathcal{I}$.

(i) $\mathbf{a}_i \in E$ is defined by $\mathbf{a}_i(0) = i$ and $\mathbf{a}_i(j) = 0$, for all $j \in (0,1)$.

(ii) If $0 < i$, then let $\mathbf{b}_i \in E$ be defined by $\mathbf{b}_i(0) = \mathbf{b}_i(i) = 1$ and $\mathbf{b}_i(j) = 0$ for all $j \in (0,1)$, $j \neq i$.

(iii) If $0 < i$, then let $\mathbf{c}_i \in E$ be defined by $\mathbf{c}_i(i) = 0$ and $\mathbf{c}_i(j) = 1$ for all $j \in \mathit{Real}^-$, $j \neq i$.

Observe that $\mathbf{c}_i$ is the relative complement of $\mathbf{b}_i$ in the interval $[\mathbf{a}_1, \mathbf{I}]$. We also need to define an ideal and a family of filters of E:

Definition 10-5.9.

(i) Let C_0 denote the interval $[\mathbf{O}, \mathbf{a}_1]$ of E.

(ii) For $j \in (0,1)$, let F_j denote the interval $[\mathbf{c}_j, \mathbf{I}]$ of E.

The most important structural properties of E are stated in the following lemma:

Lemma 10-5.10.

(i) *C_0 is an ideal of E.*

(ii) *F_j is a filter of E for $j \in (0,1)$.*

(iii) *C_0 is isomorphic to Real under the correspondence $\mathbf{a}_i \to i$, $i \in \mathit{Real}$.*

(iv) $[\mathbf{a}_1, \mathbf{b}_j] \cong [\mathbf{c}_j, \mathbf{I}] = F_j \cong \mathit{Real}$, *for $j \in (0,1)$.*

(v) $\mathbf{c}_i \vee \mathbf{c}_j = \mathbf{I}$ *for i, $j \in (0,1)$ and $i \neq j$.*

(vi) $[\mathbf{a}_1, \mathbf{I}] \cong \Pi([\mathbf{a}_1, \mathbf{b}_j] \mid j \in (0,1))$.

(vii) $[\mathbf{a}_i, \mathbf{b}_i] \cong \mathbb{R}^{\langle 2 \rangle}$ *for $i \in (0,1)$.*

Proof. All these statements are trivial except for the last one. To prove the last statement, let α be an isomorphism between the interval $[i, 1]$ of *Real* and *Real*. An element $\mathbf{v}$ of $[\mathbf{a}_i, \mathbf{b}_i]$ satisfies $i \leq \mathbf{v}(0) \leq 1$, $0 \leq \mathbf{v}(i) \leq 1$, and $\mathbf{v}(j) = 0$ for all $j \in (0,1)$. In addition, since $\mathbf{v} \in E$, by Definition 10-5.6, $\mathbf{v}(i) > 0$ implies that $\mathbf{v}(0) > i$; and $\mathbf{v}(0) < 1$ implies that $\mathbf{v}(i) < 1$. So let us assign to $\mathbf{v} \in E$ the ordered pair $(\alpha(\mathbf{v}(0)), \mathbf{v}(i))$; it is easy to see that this is the required isomorphism. $\square$

Finally, we look at complete congruences of E.

Lemma 10-5.11. *Let $\boldsymbol{\alpha}$ be a complete congruence relation of E. Then*

(1) *$\boldsymbol{\alpha}$ is determined by its restrictions $\boldsymbol{\alpha}_{C_0}$ to the ideal C_0 and $\boldsymbol{\alpha}^{F_j}$ to the filters F_j, $j \in (0,1)$.*

(2) *If $\mathbf{0} < \boldsymbol{\alpha} < \mathbf{1}$, then $\boldsymbol{\alpha}_{C_0}$ has exactly one nontrivial congruence class, namely, the class containing $\mathbf{a}_1$, and this class does not contain $\mathbf{O}$.*

(3) *For every $j \in (0,1)$, consider the $\boldsymbol{\alpha}^{F_j}$-class of F_j containing $\mathbf{I}$. There are two possibilities:* (i) *This class is the singleton $\{\mathbf{I}\}$;* (ii) *In F_j, the restriction $\boldsymbol{\alpha}^{F_j}$ collapses all of F_j and in C_0, the element $\mathbf{a}_j$ is in the $\boldsymbol{\alpha}_{C_0}$-class containing $\mathbf{a}_1$.*

Proof.

Ad (1). We need Theorem 141 of LTF:

Let D be a distributive lattice, and let $a \in D$. Then every congruence relation $\boldsymbol{\alpha}$ is determined by its restriction to the ideal $\mathrm{id}(a)$ generated by a and by the restriction to the filter $\mathrm{fil}(a)$ generated by a. In fact,

$$\begin{gathered} x \equiv y \pmod{\boldsymbol{\alpha}} \text{ iff} \\ x \wedge a \equiv y \wedge a \pmod{\boldsymbol{\alpha}} \text{ and } x \vee a \equiv y \vee a \pmod{\boldsymbol{\alpha}}. \end{gathered} \tag{10-5.1}$$

To apply (10-5.1), pick $a = \mathbf{a}_1 \in E$. Then $\mathrm{id}(a) = C_0$ and $\mathrm{fil}(a) = [\mathbf{a}_1, \mathbf{I}]$ in E. Hence by (10-5.1), $\boldsymbol{\alpha}$ is determined by its restriction $\boldsymbol{\alpha}_{C_0}$ to C_0 and its restriction $\boldsymbol{\alpha}_{[\mathbf{a}_1,\mathbf{I}]}$ to $[\mathbf{a}_1, \mathbf{I}]$. By Lemma 10-5.10(vi), $[\mathbf{a}_1, \mathbf{I}]$ is a complete direct product of $[\mathbf{a}_1, \mathbf{b}_j]$, $j \in (0,1)$; since $\boldsymbol{\alpha}$ is a complete congruence, $\boldsymbol{\alpha}_{[\mathbf{a}_1,\mathbf{I}]}$ is determined by its restrictions, $\boldsymbol{\alpha}_{[\mathbf{a}_1,\mathbf{b}_j]}$, $j \in (0,1)$. Finally, $[\mathbf{a}_1, \mathbf{b}_j]$ is perspective to $[\mathbf{c}_j, \mathbf{I}]$, hence $\boldsymbol{\alpha}_{[\mathbf{a}_1,\mathbf{b}_j]}$ is determined by $\boldsymbol{\alpha}_{[\mathbf{c}_j,\mathbf{I}]} = \boldsymbol{\alpha}^{F_j}$.

Ad (2). Let $[u, v]$, $u < v$, be a nontrivial congruence class of $\boldsymbol{\alpha}_{C_0}$. Let $u = \mathbf{a}_j$. Then by Lemma 10-5.10(vii), $[\mathbf{a}_j, \mathbf{b}_j]$ is a sublattice of E isomorphic to $\mathbb{R}^{\langle 2 \rangle}$, hence by Definition 10-5.4(iii) (using $d = v$), $\boldsymbol{\alpha}$ restricted to $[\mathbf{a}_j, \mathbf{b}_j]$ is $\mathbf{1}$, and so $v = \mathbf{a}_1$. Therefore, $\boldsymbol{\alpha}_{C_0}$ has exactly one nontrivial congruence class, and this class contains $\mathbf{a}_1$. Finally, if this class contains $\mathbf{O}$, then $\mathbf{O} \equiv \mathbf{a}_1 \pmod{\boldsymbol{\alpha}}$. Therefore, $\mathbf{a}_j \equiv \mathbf{b}_j \pmod{\boldsymbol{\alpha}}$, for all $j \in (0,1)$. Hence, $\mathbf{a}_1 \equiv \mathbf{b}_j \pmod{\boldsymbol{\alpha}}$ for all $j \in (0,1)$, which implies that $\mathbf{a}_1 \equiv \mathbf{I} \pmod{\boldsymbol{\alpha}}$. Hence $\mathbf{O} \equiv \mathbf{I} \pmod{\boldsymbol{\alpha}}$, contrary to our assumption that $\boldsymbol{\alpha} < \mathbf{1}$.

Ad (3). The interval $[\mathbf{c}_j, \mathbf{I}]$ is projective to $[\mathbf{a}_1, \mathbf{b}_j]$. But a complete congruence restricted to $[\mathbf{a}_1, \mathbf{b}_j]$ must have a single nontrivial congruence class $[\mathbf{a}_1, u]$, by Definition 10-5.4(ii). If $u = \mathbf{b}_j$, we get case (ii) of Lemma 10-5.11(iii), and in this case, $\mathbf{a}_j \equiv \mathbf{a}_1 \pmod{\boldsymbol{\alpha}}$, that is, $\mathbf{a}_j$ is in the $\boldsymbol{\alpha}_{C_0}$-class containing $\mathbf{a}_1$. If $u < \mathbf{b}_j$, then $\mathbf{b}_j$ is a singleton in $[\mathbf{a}_1, \mathbf{b}_j]$, and therefore so is $\mathbf{I}$ in $[\mathbf{c}_j, \mathbf{I}]$, yielding case (ii). □

10-5.4 Support systems

We construct the lattice K for the Existence Theorem for ICSD Lattices by induction. To state the conditions for the inductive step, we need more definitions.

Definition 10-5.12. Let D be a complete distributive lattice with zero, 0_D, and unit, 1_D. Let $\mathcal{C}$ be a family of principal ideals, $C = [0_D, 1_C]$. Let $\mathcal{F}$ be a family of principal filters, $F = [0_F, 1_D]$.

$(\mathcal{C}, \mathcal{F})$ is a *support* of the complete congruences of D if the following conditions are satisfied:

(i) $1_{C_0} \wedge 1_{C_1} = 0_D$ for C_0, $C_1 \in \mathcal{C}$ and $C_0 \neq C_1$.

(ii) $0_{F_0} \vee 0_{F_1} = 1_D$ for F_0, $F_1 \in \mathcal{F}$ and $F_0 \neq F_1$.

(iii) Every complete congruence $\boldsymbol{\alpha}$ of D is determined by its restrictions to all $C \in \mathcal{C}$ and $F \in \mathcal{F}$.

(iv) For every complete congruence $\boldsymbol{\alpha} < \mathbf{1}$ of D, the singletons $\{0_D\}$ and $\{1_D\}$ are $\boldsymbol{\alpha}$-congruence classes.

The base of the induction is provided by the following

Lemma 10-5.13. *Let D be a distributive* JM*-lattice. In $D^{\langle 2 \rangle}$, let*

$$\begin{aligned} C_0 &= [(0,0),(1,0)], \\ \mathcal{C} &= \{C_0\}, \\ F_0 &= [(1,0),(1,1)], \\ \mathcal{F} &= \{F_0\}. \end{aligned}$$

Then $(\mathcal{C}, \mathcal{F})$ is a support for the complete congruences of $D^{\langle 2 \rangle}$.

Proof. The first two conditions of Definition 10-5.12 vacuously hold. The third one holds by Lemma 10-5.1. The last condition easily follows from Definition 10-5.4(ii) and Lemma 10-5.5(iii). □

10-5.5 Multigluing

In this section, let D be a complete distributive lattice with zero, 0_D, and unit, 1_D. Let $(\mathcal{C}, \mathcal{F})$ be a support of the complete congruences of D. We further assume that $C \cong$ *Real* and $F \cong$ *Real* for all $C \in \mathcal{C}$ and $F \in \mathcal{F}$.

We construct a complete distributive lattice G as follows. First, for every $F \in \mathcal{F}$, we apply the construction of Section 10-5.3 to F to obtain the lattice E_F with zero 0_F and unit 1_F. In E_F, we have the family of filters F_j, $j \in (0,1)$; we shall denote this family by $\mathcal{F}_F$. Let B_F denote the filter $[\mathbf{a}_1, 1_F]$ in E_F. Let B be the direct product $\Pi(B_F \mid F \in \mathcal{F})$. Let 0_B and 1_B be the zero and unit element of B, respectively.

We regard B_F as a sublattice of B by identifying $x \in B_F$ with

$$(\dots, 0, \dots, \overset{F}{x}, \dots, 0);$$

that is, $x \in B_F$ is identified with the element of B whose F-component is x and all the other components are zero.

In $D \times B$, we identify $d \in D$ with $(d, 0_B)$; thereby, D becomes a sublattice of $D \times B$. In $D \times B$, we identify $b \in B$ with $(1_D, b)$; thereby, B becomes a sublattice of $D \times B$. Let $F \in \mathcal{F}$; then F is a sublattice of D and B_F is a sublattice of B in $D \times B$. Since E_F is isomorphic to a sublattice of $F \times B_F$, the lattice E_F is now (uniquely) identified with a sublattice of $D \times B$.

Next we define in $D \times B$ the lattice G as the union of D, B, and of all E_F, $F \in \mathcal{F}$. The construction of G is very much like gluing, except that there is a whole family of pairs of lattices D and E_F that have to be glued together simultaneously. Figure 10-5.3 illustrates the construction of G.

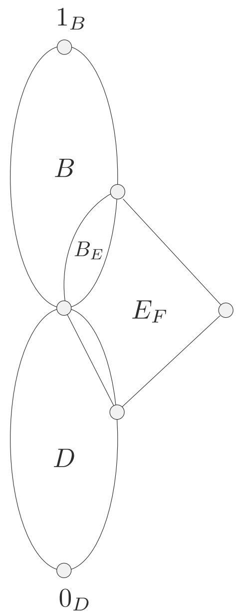

Figure 10-5.3: The lattice G.

Lemma 10-5.14. *G is a sublattice of $D \times B$, and G is a complete distributive lattice.*

Proof. This easy computation is left to the reader. □

Now we define the family $\mathcal{F}_G$ of filters of G. Since E_F is a sublattice of G with its unit element 1_F in B, we can form $1'_F$, the complement of 1_F in B. For every $F \in \mathcal{F}$, we have the family $\mathcal{F}_F$ of filters of E_F; for every $X \in \mathcal{F}_F$, form $1'_F \vee X = \{\, 1'_F \vee x \mid x \in X \,\}$, which is a filter of G. All these filters form the family $\mathcal{F}_G$.

Lemma 10-5.15. $(\mathcal{C}, \mathcal{F}_G)$ *is a support for the complete congruences of* G.

Proof. We have to verify Definition 10-5.12(i) to 10-5.12(iv). Definition 10-5.12(i) is obvious since $\mathcal{C}$ is unchanged.

To prove Definition 10-5.12(ii), let F_0 and F_1 be two distinct elements of $\mathcal{F}_G$. There are two cases to consider: firstly, there is an $F \in \mathcal{F}$ such that $F_0 = 1'_F \vee X_0$ and $F_1 = 1'_F \vee X_1$ for some X_0, $X_1 \in \mathcal{F}_F$. Then Definition 10-5.12(ii) holds for X_0 and X_1 in E_F and $1'_F$ is the complement of 1_F in B, hence Definition 10-5.12(ii) holds for F_0 and F_1. Secondly, let $F_0 = 1'_F \vee X_0$ and $F_1 = 1'_{F'} \vee X_1$, where $F \neq F'$, F, $F' \in \mathcal{F}$. Then Definition 10-5.12(ii) follows from the fact that $B = \Pi(B_F \mid F \in \mathcal{F})$.

Let us assume that $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are complete congruences of G, and $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ agree on all $C \in \mathcal{C}$ and $X \in \mathcal{F}_G$. Then $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ agree on B, since B is isomorphic to the direct product of all $X \in \mathcal{F}_G$. For $F \in \mathcal{F}$, then $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ agree on B_F. By Definition 10-5.11(iii), $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ agree on C_0 in E_F, and so by Lemma 10-5.11(1), $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ agree on E_F. Finally, since $(\mathcal{C}, \mathcal{F})$ is a support for D, it follows that $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ agree on D. We conclude that $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ agree on G, proving Definition 10-5.12(iii).

Let $\boldsymbol{\alpha}$ be a complete congruence of G. Definition 10-5.12(iv) is obvious for $0_G = 0_D$. For $1_G = 1_B$, if $1_B \equiv x \pmod{\boldsymbol{\alpha}}$, $x < 1_B$, then we can choose this x in B. Since $B = \Pi(B_F \mid F \in \mathcal{F})$, we can choose x in some $[1'_F, 1_B]$, $F \in \mathcal{F}$. Then $x \wedge 1_F \equiv 1_F \pmod{\boldsymbol{\alpha}}$, $x \wedge 1_F < 1_F$. So by Lemma 10-5.11(3), Definition 10-5.12(iv) follows, concluding the proof of the lemma. ☐

The following concept will help clarify what we have just proved.

Definition 10-5.16. Let D be a complete lattice, and let $[a, b]$ be an interval of D. The interval $[a, b]$ is called *D-simple* if for any complete congruence $\boldsymbol{\alpha} < \mathbf{1}$ of D, the restriction of $\boldsymbol{\alpha}$ to $[a, b]$ is $\mathbf{0}$.

Lemma 10-5.17. *Let $F \in \mathcal{F}$. Then F is G-simple.*

Proof. Let $\boldsymbol{\alpha} < \mathbf{1}$ be a complete congruence on G. If $\boldsymbol{\alpha}$ restricted to F is not $\mathbf{1}$, then by Lemma 10-5.11(ii), there is only one nontrivial congruence class on F and it contains 1_D. On the other hand, by Definition 10-5.12(iii), the singleton $\{1_D\}$ is a congruence class under the restriction of $\boldsymbol{\alpha}$ to D, a contradiction. Therefore, $\boldsymbol{\alpha}$ restricted to F is $\mathbf{0}$. ☐

10-5.6 The proof of the Existence Theorem for ICSD Lattices

The induction hypotheses are the following (m and n are natural numbers):

(i) There is a complete distributive lattice D_n, with zero 0_{D_n} and unit 1_{D_n}.

(ii) There is a support $(\mathcal{C}_n, \mathcal{F}_n)$ for the complete congruences of D_n.

(iii) Every $X \in \mathcal{C}_n \cup \mathcal{F}_n$ is isomorphic to *Real*.

(iv) For $m < n$, D_m is an interval of D_n.

(v) For $m < n$, every $C \in \mathcal{C}_m$ and every $F \in \mathcal{F}_m$ is D_n-simple.

Let D_0 be $\mathbb{R}^{\langle 2 \rangle}$, and let $\mathcal{C}_0 = \{C_0\}$ and $\mathcal{F}_0 = \{F_0\}$, as defined in Definition 10-5.13. By Definition 10-5.13, D_0 satisfies these five conditions.

Let D_m with support $(\mathcal{C}_m, \mathcal{F}_m)$ be given for all $m \leq n$. We apply the multigluing construction first to D_n and $\mathcal{F}_n$, obtaining the lattice G with support $(\mathcal{C}_n, \mathcal{F}_G)$. Then we apply the dual construction to G and $\mathcal{C}_n$, obtaining the lattice D_{n+1} and the support $(\mathcal{C}_{n+1}, \mathcal{F}_{n+1})$. It is clear by Lemmas 10-5.15 and 10-5.17 that the required conditions hold.

Now we define the lattice K of the Theorem as

$$\bigcup(\, D_n \mid n = 0, 1, \dots\,)$$

with a zero and unit adjoined. Obviously, K is a complete distributive lattice. To show that K is complete-simple, let $\boldsymbol{\alpha} > \mathbf{0}$ be a complete congruence of K. Then there is a natural number n such that $\boldsymbol{\alpha}$ restricted to D_n is $> \mathbf{0}$ on D_n. Since $(\mathcal{C}_n, \mathcal{F}_n)$ is a support for D_n, there exists an $X \in \mathcal{C}_n \cup \mathcal{F}_n$ such that $\boldsymbol{\alpha}$ restricted to X is $> \mathbf{0}$ on X. By assumption, X is D_m-simple for every $m > n$. Hence $\boldsymbol{\alpha} = \mathbf{1}$ on all D_m, $m > n$. Since $\boldsymbol{\alpha}$ is a complete congruence, this implies that $\boldsymbol{\alpha} = \mathbf{1}$ on K, proving that K is complete-simple. This concludes the proof of the Existence Theorem for ICSD Lattices.

10-6. The order of principal congruences

This section and the exercises and problems for it is based on my paper [134].

10-6.1 Principal congruences

As opposed to the *lattice* $\operatorname{Con} L$ of congruences of a lattice L, we deal with the *order* $\operatorname{Princ} L$ of principal congruences of a lattice L. Observe that $\operatorname{Princ} L$ is a directed order with zero.

$\operatorname{Con_c} L$ is the set of compact elements of $\operatorname{Con} L$, a lattice theoretic characterization of this subset.

$\operatorname{Princ} L$ is a directed subset of $\operatorname{Con_c} L$ containing the zero and join-generating $\operatorname{Con_c} L$; there is no lattice theoretic characterization of this subset.

Figure 10-6.1 shows the lattice N_7 and its congruence lattice $\mathsf{B}_2 + 1$. Note that $\operatorname{Princ} \mathsf{N}_7 = \operatorname{Con} \mathsf{N}_7 - \{\boldsymbol{\gamma}\}$. While in the standard representation K of $\mathsf{B}_2 + 1$ as a congruence lattice (G. Grätzer and E.T. Schmidt [164]; see also in my books [131], LTF), we have $\operatorname{Princ} K = \operatorname{Con} K$. This example shows that $\operatorname{Princ} L$ has no lattice theoretic description in $\operatorname{Con} L$.

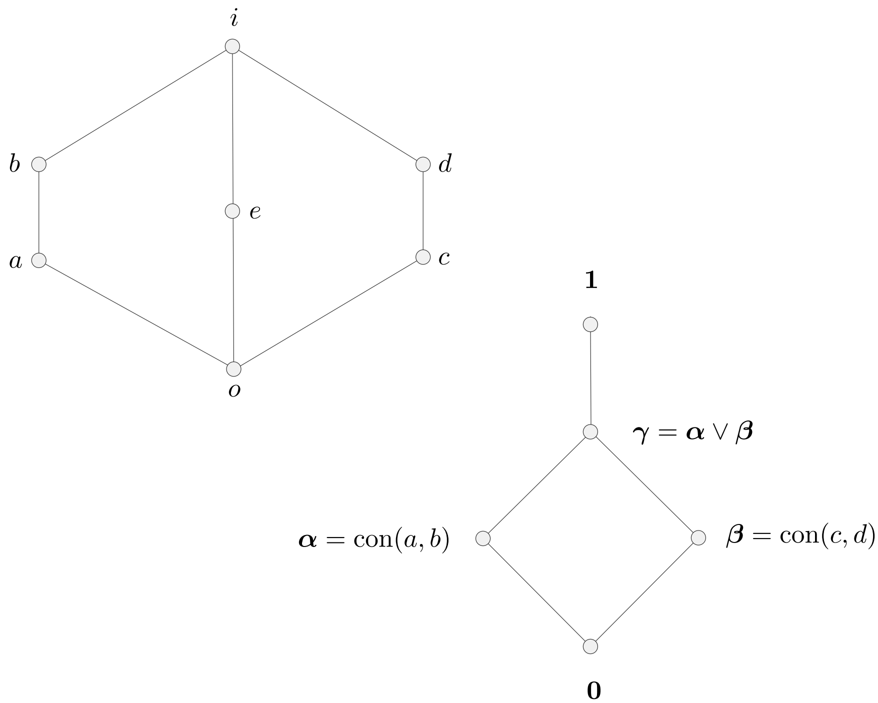

Figure 10-6.1: The lattice N_7 and its congruence lattice.

It was pointed out in G. Grätzer and E.T. Schmidt [163] that for every universal algebra $\mathfrak{A}$ we can construct a universal algebra $\mathfrak{B}$ such that $\operatorname{Con} \mathfrak{A} \cong \operatorname{Con} \mathfrak{B}$ and $\operatorname{Princ} \mathfrak{B} = \operatorname{Con_c} \mathfrak{B}$. (See Exercises 10.24–10.26.)

For a long time, we have tried to prove such a result for lattices but we have been unable to construct even a *proper* congruence-preserving extension for a general lattice; see the discussion in G. Grätzer and E.T. Schmidt [172]. This logjam was broken in G. Grätzer and F. Wehrung [184] by introducing the boolean triple construction (discussed in detail in [131]). G. Grätzer and E.T. Schmidt [177] uses this construction to prove the following result:

Theorem 10-6.1. *Every lattice L has a congruence-preserving extension K satisfying*

$$\operatorname{Princ} K = \operatorname{Con_c} K.$$

So if a distributive join-semilattice with zero S can be represented as $\operatorname{Con_c} L$ for a lattice L, then S can also be represented as $\operatorname{Princ} K$ for a lattice K. This is a further illustration of the fact that $\operatorname{Princ} L$ has no lattice theoretic description in $\operatorname{Con} L$.

10-6.2 The result

For a bounded lattice L, the order $\operatorname{Princ} K$ is bounded. We now state the converse.

Theorem 10-6.2. *Let P be an order with zero and unit. Then there is a bounded lattice K such that*

$$P \cong \operatorname{Princ} K.$$

If P is finite, we can construct K as a finite lattice.

10-6.3 The construction

For a bounded order Q, let Q^- denote the order Q with the bounds removed.

Let P be the order in Theorem 10-6.2. Let 0 and 1 denote the zero and unit of P, respectively. We denote by P^{d} those elements of P^- that are not comparable to any other element of P^-, that is,

$$P^{\mathrm{d}} = \{\, x \in P^- \mid x \parallel y \text{ for all } y \in P^-,\ y \neq x \,\}.$$

The lattice F

We first construct the lattice F consisting of the elements o, i and the elements a_p, b_p for every $p \in P$, where $a_p \neq b_p$ for every $p \in P^-$ and $a_0 = b_0$, $a_1 = b_1$. These elements are ordered and the lattice operations are formed as in Figure 10-6.2.

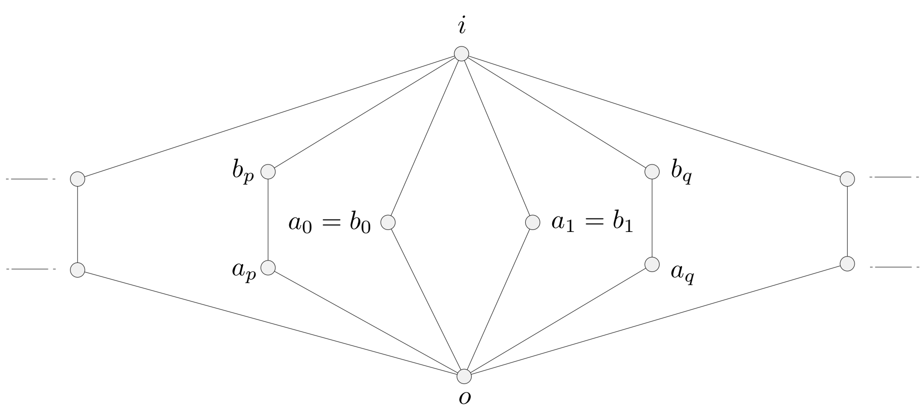

Figure 10-6.2: The lattice F.

The lattice K

We are going to construct the lattice K (of Theorem 10-6.2) as an extension of F.

The principal congruence of K representing $p \in P^-$ will be $\mathrm{con}(a_p, b_p)$.

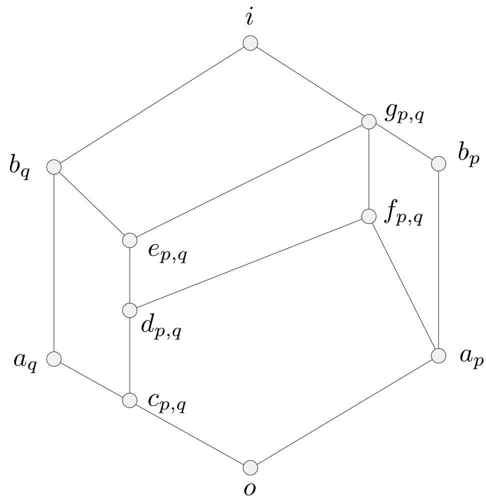

Figure 10-6.3: The lattice $S = S(p, q)$.

We add the set

$$\{c_{p,q}, d_{p,q}, e_{p,q}, f_{p,q}, g_{p,q}\}$$

to the sublattice

$$\{o, a_p, b_p, a_q, b_q, i\}$$

of F for $p < q \in P^-$, as illustrated in Figure 10-6.3, to form the sublattice $S(p, q)$.

For $p \in P^{\mathrm{d}}$, let $C_p = \{o < a_p < b_p < i\}$ be a four-element chain.

We define the set

$$K = \bigcup(\, S(p, q) \mid p < q \in P^- \,) \cup \bigcup(\, C_p \mid p \in P^{\mathrm{d}} \,) \cup \{a_0, a_1\}.$$

Now we are ready to define the lattice K.

We make the set K into a lattice by the following nine rules.

(i) The operations $\vee$ and $\wedge$ are idempotent and commutative and o is the zero and i is the unit of K.

(ii) For $p \in P^{\mathrm{d}}$ and $x, y \in C_p \subseteq K$, we define $x \vee y$, $x \wedge y$ in K as in the chain C_p. (So C_p is a sublattice of K.)

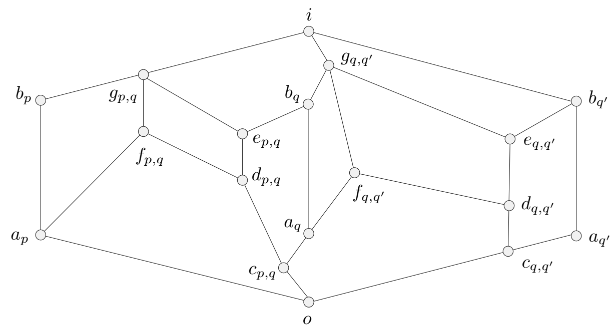

Figure 10-6.4: The lattice $S_{\mathsf{C}} = S(p < q,\ q < q')$.

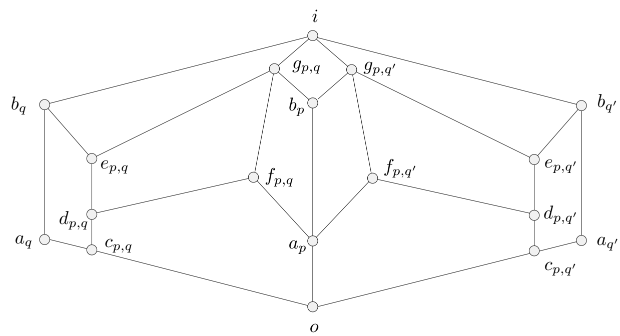

Figure 10-6.5: The lattice $S_{\mathsf{V}} = S(p < q,\ p < q')$ with $q \neq q'$.

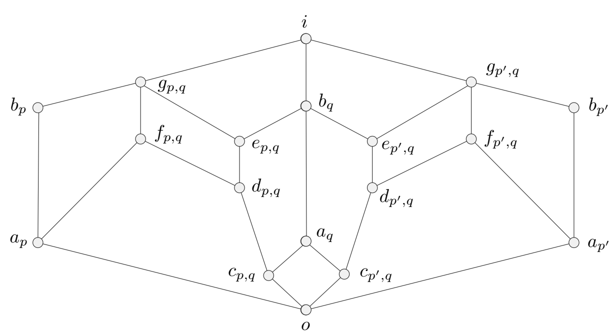

Figure 10-6.6: The lattice $S_{\mathsf{H}} = S(p < q,\ p' < q)$ with $p \neq p'$.

(iii) For $p < q \in P^-$ and $x, y \in S(p,q) \subseteq K$, we define $x \vee y$, $x \wedge y$ in K as in the lattice $S(p,q)$. (So $S(p,q)$ is a sublattice of K.)

(iv) For $p \in P^{\mathrm{d}}$, $x \in C_p^-$, and $y \in K - C_p$, the elements x and y are complementary in K, that is, $x \vee y = i$ and $x \wedge y = o$.

(v) For $x = a_0$ and for $x = a_1$, the element x is complementary to any element $y \neq x \in K^-$.

In the following four rules, let $p < q$, $p' < q' \in P^-$, $x \in S(p,q)^-$, and $y \in S(p',q')^-$. By rule (iii), we can assume that $\{p,q\} \neq \{p',q'\}$.

(vi) If $\{p,q\} \cap \{p',q'\} = \varnothing$, then the elements x and y are complementary in K.

(vii) If $q = p'$, we form $x \vee y$ and $x \wedge y$ in K in the lattice

$$S_{\mathsf{C}} = S(p < q,\ q < q'),$$

illustrated in Figure 10-6.4.

(viii) If $p = p'$ and $q \neq q'$, we form $x \vee y$ and $x \wedge y$ in K in the lattice

$$S_{\mathsf{V}} = S(p < q,\ p < q'),$$

illustrated in Figure 10-6.5.

(ix) If $q = q'$ and $p \neq p'$, we form $x \vee y$ and $x \wedge y$ in K in the lattice

$$S_{\mathsf{H}} = S(p < q,\ p' < q),$$

illustrated in Figure 10-6.6.

In the last three rules, C for chain, V for V-shaped, H for Hat-shaped refer to the shape of the three-element order $\{p,q\} \cup \{p',q'\}$ in P^-.

Observe that Rules (vi)–(ix) exhaust all possibilities under the assumption $\{p,q\} \neq \{p',q'\}$.

Note that

$$\begin{aligned} S &= S(p,q), \\ S_{\mathsf{C}} &= S(p < q,\ q < q'), \\ S_{\mathsf{V}} &= S(p < q,\ p < q'), \\ S_{\mathsf{H}} &= S(p < q,\ p' < q) \end{aligned}$$

are sublattices of K.

Informally, these rules state that to form K, we add elements to F so that we get the sublattices listed in the last paragraph.

10-6.4 The proof

Preliminaries

It is easy, if somewhat tedious, to verify that K is a lattice. Note that all our constructs are bounded planar orders, hence planar lattices.

We have to describe the congruence structure of K.

Let L be a lattice with 0 and 1. We call a singleton congruence block in L *trivial*.

A $\{0,1\}$-*isolating congruence* $\boldsymbol{\alpha}$ of L (an *I-congruence*, for short), is a congruence $\boldsymbol{\alpha} > \mathbf{0}$, such that $\{0\}$ and $\{1\}$ are (trivial) congruence blocks of $\boldsymbol{\alpha}$.

If $|P| \leq 2$, then we can construct K as a one- or two-element chain. So for the proof, we assume that $|P| > 2$, that is, $P^- \neq \varnothing$.

Lemma 10-6.3. *For every $x \in K^-$, there is an $\{o,i\}$-sublattice A of K containing x and isomorphic to M_3.*

Proof. Since $P^- \neq \varnothing$ by assumption, we can choose $p \in P^-$. If $x \in \{a_0, a_1\}$, then

$$A = \{a_p, a_0, a_1, o, i\}$$

is such a sublattice. If $x \notin \{a_0, a_1\}$, then

$$A = \{x, a_0, a_1, o, i\}$$

is such a sublattice. ☐

Lemma 10-6.4. *Let us assume that $\boldsymbol{\alpha}$ is not an I-congruence of K. Then $\boldsymbol{\alpha} = \mathbf{1}$.*

Proof. Indeed, if $\boldsymbol{\alpha}$ is not an I-congruence of K, then there is an $x \in K^-$ such that $x \equiv o \pmod{\boldsymbol{\alpha}}$ or $x \equiv o \pmod{\boldsymbol{\alpha}}$. Using the sublattice A provided by Lemma 10-6.3, we conclude that $\boldsymbol{\alpha} = \mathbf{1}$, since A is a simple $\{o,i\}$-sublattice. ☐

The congruences of S

We start with the congruences of the lattice $S = S(p,q)$ with $p < q \in P^-$, see Figure 10-6.3.

Lemma 10-6.5. *The lattice $S = S(p,q)$ has two I-congruences:*

$$\operatorname{con}(a_p, b_p) < \operatorname{con}(a_q, b_q),$$

see Figure 10-6.7.

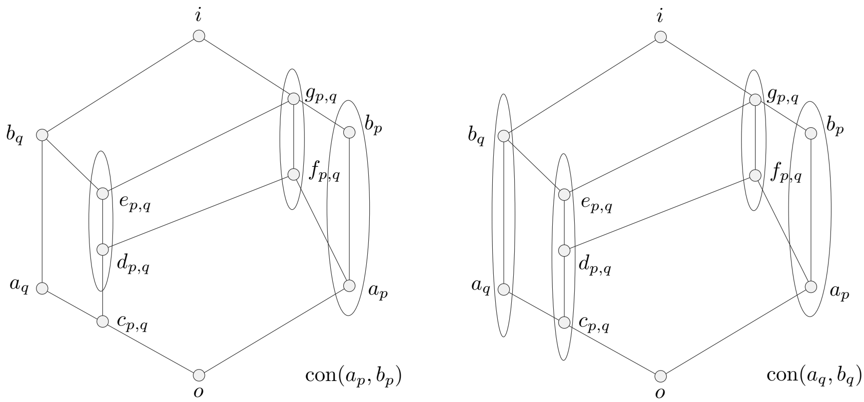

Figure 10-6.7: The I-congruences of $S = S(p, q)$.

Proof. An easy computation.

First, check that Figure 10-6.7 correctly describes the two join-irreducible I-congruences $\mathrm{con}(a_p, b_p)$ and $\mathrm{con}(a_q, b_q)$.

Then, check all 12 prime intervals $[x, y]$ and show that $\mathrm{con}(x, y)$ is either not an I-congruence or equals $\mathrm{con}(a_p, b_p)$ or $\mathrm{con}(a_q, b_q)$.

For instance, $\mathrm{con}(d_{p,q}, e_{p,q}) = \mathrm{con}(a_p, b_p)$ and $[b_p, g_{p,q}]$ is not an I-congruence because $c_{p,q} \equiv o \pmod{\mathrm{con}(b_p, g_{p,q})}$. The other 10 cases are similar.

Finally, note that the two join-irreducible I-congruences we found are comparable, so there are no join-reducible I-congruences. $\square$

Clearly, $S(p, q)/\mathrm{con}(a_q, b_q) \cong \mathsf{C}_2 \times \mathsf{C}_3$.

The congruences of $\boldsymbol{K}$

For $p \in P^{\mathrm{d}}$, let $\boldsymbol{\varepsilon}_p$ denote the congruence $\mathrm{con}(a_p, b_p)$ on K.

Let $H \subseteq P^{\mathrm{d}}$. Let $\boldsymbol{\varepsilon}_H$ denote the equivalence relation

$$\boldsymbol{\varepsilon}_H = \bigvee (\, \boldsymbol{\varepsilon}_p \mid p \in H \,)$$

on K.

Let $\boldsymbol{\beta}$ be an I-congruence of the lattice K. We associate with $\boldsymbol{\beta}$ a subset of the order P^-:

$$\mathrm{Base}(\boldsymbol{\beta}) = \{\, p \in P^- \mid a_p \equiv b_p \pmod{\boldsymbol{\beta}} \,\}.$$

Lemma 10-6.6. *Let $\boldsymbol{\beta}$ be an I-congruence of the lattice K. Then* $\mathrm{Base}(\boldsymbol{\beta})$ *is a down set of P^-.*

Proof. Let $p < q \in P$ and let $q \in \mathrm{Base}(\boldsymbol{\beta})$. Then $a_q \equiv b_q \pmod{\boldsymbol{\beta}}$. By Lemma 10-6.5 (see also Figure 10-6.7), $a_p \equiv b_p \pmod{\boldsymbol{\beta}}$, so $p \in \mathrm{Base}(\boldsymbol{\beta})$, verifying that $\mathrm{Base}(\boldsymbol{\beta})$ is a down set. $\square$

Let H be a down set of P^-. We define the binary relation:

$$\boldsymbol{\beta}_H = \boldsymbol{\varepsilon}_H \cup \bigcup(\, \mathrm{con}_{S(p,q)}(a_q, b_q) \mid q \in H \,) \cup \bigcup(\, \mathrm{con}_{S(p,q)}(a_p, b_p) \mid p \in H \,).$$

Lemma 10-6.7. *$\boldsymbol{\beta}_H$ is an I-congruence on K.*

Note that $\boldsymbol{\beta}_\varnothing = \mathbf{0}$.

Proof. $\boldsymbol{\beta}_H$ is reflexive (because $\boldsymbol{\varepsilon}_H$ is) and symmetric (because it is the union of three symmetric binary relations). It clearly leaves o and i isolated.

It is easy to verify that $\boldsymbol{\beta}_H$ classes are pairwise disjoint two- and three-element chains, so $\boldsymbol{\beta}_H$ is transitive and hence an equivalence relation.

We verify the Substitution Properties. By Lemma I.3.11 of LTF, it is sufficient to verify that if $x < y \in K$, and $x \equiv y \pmod{\boldsymbol{\beta}_H}$, then $x \vee z \equiv y \vee z \pmod{\boldsymbol{\beta}_H}$ and $x \wedge z \equiv y \wedge z \pmod{\boldsymbol{\beta}_H}$ for $z \in K$.

So let $x < y \in K^-$ and $x \equiv y \pmod{\boldsymbol{\beta}_H}$. Then $x < y \in S(p,q)^-$, for some $p < q \in P^-$, and

$$x \equiv y \pmod{\mathrm{con}_{S(p,q)}(a_q, b_q)}$$

with $q \in H$, or

$$x \equiv y \pmod{\mathrm{con}_{S(p,q)}(a_p, b_p)}$$

with $p \in H$.

Let $z \in S(p',q')^-$ with $p' < q' \in P^-$.

If $\{p,q\} = \{p',q'\}$, the Substitution Properties for $\boldsymbol{\beta}_H$ in K follow from the Substitution Properties for $\boldsymbol{\beta}_H$ in $S(p,q)$.

If $\{p,q\} \cap \{p',q'\} = \varnothing$, then by Rule (vi), the elements x, z, and y, z are complementary, so the Substitution Properties are trivial.

Otherwise, $\{p,q\} \cup \{p',q'\}$ has three elements. So we have three cases to consider.

Case C: $p < p' = q < q'$ (or symmetrically, $p' < q' = p < q$).

Case V: $p = p' < q$, $p = p' < q'$, $q \neq q'$.

Case H: $p < q = q'$, $p' < q = q'$, $p \neq p'$.

To verify Case C, utilize Figure 10-6.4. Since

$$x \leq y \in S(p,q)^- \subseteq S(p < q,\ q < q')^-$$

and

$$z \in S(q,q')^- \subseteq S(p < q,\ q < q')^-,$$

there is only way $(\mathrm{SP}_\vee)$ can fail: $x \vee z < y \vee z$.

We can assume that $z \notin S(p < q)$, so $x \vee z, y \vee z \notin S(p < q)$. If $q \in H$, then there is only one possibility for the I-congruence $\boldsymbol{\beta}_H$:

$$(x \vee z, y \vee z) = (f_{q,q'}, g_{q,q'}) \in \operatorname{con}(a_q, b_q) \subseteq \boldsymbol{\beta}_H.$$

If $q \notin H$, then $p \in H$ and $x \vee z < y \vee z$ is impossible. This shows that $\boldsymbol{\beta}_H$ satisfies ($\mathrm{SP}_\vee$). A similar, in fact easier, argument yields ($\mathrm{SP}_\wedge$).

We leave the easier cases, Case V and Case H, to the reader. ☐

Now the following statement is clear:

Lemma 10-6.8. *The correspondence*

$$\varphi\colon \boldsymbol{\beta} \to \operatorname{Base}(\boldsymbol{\beta})$$

is an order-preserving bijection between the order of I-congruences of K and the order of down sets of P^-. We extend φ by $\mathbf{0} \to \{0\}$ and $\mathbf{1} \to P$. Then φ is an isomorphism between $\operatorname{Con} K$ *and* $\operatorname{Down}^- P$*, the order of nonempty down sets of P.*

Lemma 10-6.9. *φ and φ^{-1} both preserve the property of being principal.*

Proof. Indeed, if the I-congruence $\boldsymbol{\beta}$ of K is principal, $\boldsymbol{\beta} = \operatorname{con}(x, y)$ for some $x < y \in K$, then we must have $x, y \in S(p, q)$ for some $p < q \in P^-$ (otherwise, $\boldsymbol{\beta}$ would not be an I-congruence). But in $S(p, q)$ (see Figure 10-6.7), the principal congruences are $\operatorname{con}(a_p, b_p)$ and $\operatorname{con}(a_q, b_q)$. By Lemma 10-6.8, we obtain that $\operatorname{Base}(\boldsymbol{\beta}) = {\downarrow} p$ or $\operatorname{Base}(\boldsymbol{\beta}) = {\downarrow} q$.

Conversely, if $\operatorname{Base}(\boldsymbol{\beta}) = {\downarrow} p$, then $\boldsymbol{\beta} = \operatorname{con}(a_p, b_p)$. ☐

Now Theorem 10-6.2 easily follows. Indeed, by Lemma 10-6.8, φ is an isomorphism between $\operatorname{Con} K$ and $\operatorname{Down}^- P$. Under this isomorphism, by Lemma 10-6.9, principal congruences correspond to principal down sets, so $\operatorname{Princ} K \cong P$, as claimed.

10-7. Exercises

10.1. Why is there no new notation for meets and complete meets in $\operatorname{Com} A$?

10.2. Describe joins of complete congruences.

10.3. What can you say about complete principal congruences?

10.4. Let K be the complete lattice we construct in Section 10-2.3 for the Representation Theorem for Complete Lattices.

(i) Is it true that every complete congruence $\boldsymbol{\alpha} > \mathbf{0}$ of K can be represented as $\boldsymbol{\alpha} = \operatorname{com}(\mathfrak{p})$ for a suitable prime interval $\mathfrak{p}$ in K?

(ii) Verify that for any congruence $\boldsymbol{\alpha} < \mathbf{1}$ of K, the singletons $\{0\}$ and $\{1\}$ are congruence classes.

10.5. The basic building stone for the Representation Theorem for Complete Lattices is the lattice M_X, see Figure 10-2.5. Show that the lattice of Figure 10-7.1 has much the same functionality.

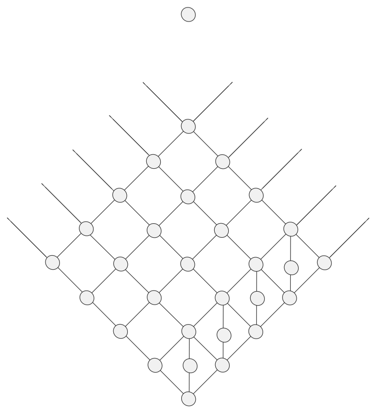

Figure 10-7.1: A modular lattice.

10.6. In Section 10-2.1, we define the interior, $\mathrm{Inter}(\boldsymbol{\alpha})$, of a complete congruence $\boldsymbol{\alpha}$. Can we change $\mathrm{con}(\mathfrak{p}) \leq \boldsymbol{\alpha}$ to $\mathrm{com}(\mathfrak{p}) \leq \boldsymbol{\alpha}$ in the definition?

10.7. Let L be the set of nonnegative integers with the usual partial ordering with two additional elements: a, i. Let 0 be the zero, and i the unit of L. Let $a \wedge n = 0$ and $a \vee n = i$ for all $n \in N$, $n \neq 0$.

We define three complete congruences, $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}$ on L:

nontrivial classes of $\boldsymbol{\alpha}$: $[2n + 1, 2n + 2]$ for $n = 0, 1, 2, \ldots$

nontrivial classes of $\boldsymbol{\beta}$: $[2n + 1, 2n + 2]$ for $n = 1, 2, \ldots$

nontrivial classes of $\boldsymbol{\gamma}$: $[2n, 2n + 1]$ for $n = 1, 2, \ldots$.

Prove that L is a complete lattice and $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}$ generate a sublattice isomorphic to N_5 in $\mathrm{Com}\, L$. (G. Grätzer, H. Lakser, and B. Wolk [155]; an example of a complete lattice for which $\mathrm{Com}\, L$ is not distributive was first published in K. Reuter and R. Wille [278].)

10.8. Show that the example in Exercise 10.7 is “minimal”: If K is a complete lattice for which $\mathrm{Com}\, K$ is not distributive, then K contains L as a suborder.

10.9. Let D be a complete distributive lattice, and let $\mathrm{Com}\, D = \mathsf{N}_5$. Then D has an ICSD lattice as a complete quotient. (G. Grätzer and E.T. Schmidt [170].)

10.10. Let D be a complete distributive lattice. Show that every principal congruence of D is complete. (R. Freese, G. Grätzer, and E.T. Schmidt [89].)

10.11. Let $\boldsymbol{\alpha}$ be a congruence on a complete lattice K. Show that $\boldsymbol{\alpha}$ is a complete congruence iff all congruence blocks of $\boldsymbol{\alpha}$ are intervals. (R. Freese, G. Grätzer, and E.T. Schmidt [89].)

10.12. Let L be a complete distributive lattice satisfying the infinite distributive identities (JID) and (MID).[2] Prove that every principal congruence of L is complete. (G. Grätzer and E.T. Schmidt [174].)

10.13. Construct a complete modular lattice L and $a < b$ in L such that $\operatorname{con}(a, b)$ is not a complete congruence.

10.14. Find a modular complete-simple lattice that is not simple. (Hint: Take a sublattice of Figure 10-7.1.)

10.15. Find a complete lattice L that cannot be represented as the complete congruence lattice of a complete distributive lattice satisfying the infinite distributive identities (JID) and (MID). (Hint: Any complete lattice with more than two elements and with a meet-irreducible zero.) (G. Grätzer and E.T. Schmidt [174].)

10.16. Let L be a finite non-Boolean lattice. Then L cannot be represented as the lattice of complete congruence relations of a complete distributive lattice K satisfying (JID). (G. Grätzer and E.T. Schmidt [174].)

10.17. Construct complete-simple distributive lattices of any cardinality $\geq$ the power of the continuum.

10.18. Verify Lemmas 10-2.2 and 10-2.3.

10.19. Show that any group can be represented as the automorphism group of a connected undirected graph without loops satisfying property (10-3.1).

10.20. Gluing is discussed in detail in Section IV.2 of LTF. Pasting is defined in Section IV.2.3 of LTF.

Find a common generalization of gluing and pasting that includes the construction in Section 10-5.5. (E. Fried, G. Grätzer, and E.T. Schmidt [93].)

10.21. Prove that every complete lattice L can be represented as the congruence lattice of a Scott domain[3] S. (G. Grätzer and E.T. Schmidt [166].)

10.22. Verify that $\operatorname{Princ} L$ is a directed order with zero.

[2]See Section II.4.2 in LTF.

[3]Scott domains are defined in Chapter 1.

10.23. Take the distributive lattice $D = \mathsf{B}_2 + 1$ (see Figure 10-6.1). Find two lattices K_1 and K_2 so that $\operatorname{Con} K_1$ and $\operatorname{Con} K_2$ both represent D, but $\operatorname{Princ} K_1 \neq \operatorname{Princ} K_2$.

10.24. Show that for every universal algebra $\mathfrak{A}$, we can construct a unary algebra $\mathfrak{A}'$ on the same set A so that an equivalence relation $\boldsymbol{\alpha}$ is a congruence relation of $\mathfrak{A}$ iff it is a congruence relation of $\mathfrak{A}'$, so $\operatorname{Con} \mathfrak{A} = \operatorname{Con} \mathfrak{A}'$.

10.25. Let $\mathfrak{A}$ be a unary algebra. Take a disjoint copy $\mathfrak{A}'$ of $\mathfrak{A}$, defined on the set $\{\, x' \mid x \in A \,\}$. Let $a \neq b \in \mathfrak{A}$ and form the unary algebra $\mathfrak{B}$ as the union of $\mathfrak{A}$ and $\mathfrak{A}'$ identifying the a with b'. Define some additional unary operations on $\mathfrak{B}$ so that $\operatorname{con}(a, x) \vee \operatorname{con}(b, y) = \operatorname{con}(x, y)$ for all $x, y \in A$.

10.26. Use Exercises 10.24 and 10.25 to prove: for every universal algebra $\mathfrak{A}$ we can construct a universal algebra $\mathfrak{B}$ such that $\operatorname{Con} \mathfrak{A} \cong \operatorname{Con} \mathfrak{B}$ and $\operatorname{Princ} \mathfrak{B} = \operatorname{Con_c} \mathfrak{B}$. (G. Grätzer and E.T. Schmidt [163].)

10.27. For a bounded lattice L, the order $\operatorname{Princ} K$ is bounded. Is the converse true?

10.28. Describe the congruences of the lattice F (see Figure 10-6.2).

10.29. Give a detailed proof that the operations $\vee$ and $\wedge$ defined (in Section 10-6.3) on the set

$$K = \bigcup(\, S(p, q) \mid p < q \in P^{-} \,) \cup \bigcup(\, C_p \mid p \in P^{\mathrm{d}} \,) \cup \{a_0, a_1\},$$

make K into a lattice.

10.30. Define the ordering on K and use it to verify that K is a lattice.

10.31. Is

$$\bigcup(\, \leq_{S(p,q)} \mid p < q \in P^{-} \,) \cup \bigcup(\, \leq_{C_p} \mid p \in P^{\mathrm{d}} \,)$$

the ordering on K?

10.32. Draw on the diagram of S_{C} (see Figure 10-6.4) all the I-congruences. How many I-congruences does S_{C} have? Same for S_{V} and S_{H}.

10.33. State and prove Theorem 10-6.2 for complete lattices. (There are two ways of interpreting this, use both.)

10.34. Let L be a finite lattice with $\operatorname{Con} L = \mathsf{C}_2^3$. Then $\operatorname{Princ} L$ has seven or eight elements.

10-8. Problems

Problem 10.1. Characterize the lattices of $\mathfrak{m}$-complete congruences of $\mathfrak{m}$-complete distributive lattices satisfying $(\mathrm{JID}_{\mathfrak{m}})$ and/or $(\mathrm{MID}_{\mathfrak{m}})$.

Problem 10.2. Does the Independence Theorem for Complete Lattices hold for distributive lattices?

Problem 10.3. For a complete lattice L, we have the lattices $\operatorname{Com} L$ and $\operatorname{Con} L$, where $\operatorname{Com} L$ is a complete meet subsemilattice of $\operatorname{Con} L$. What can we say about the pair $\operatorname{Com} L$ and $\operatorname{Con} L$?

Problem 10.4. Can we characterize the order $\operatorname{Princ} L$ for a lattice L as a directed order with zero?

A recent preprint of G. Czédli [46] solved this problem for countable lattices.

Even more interesting would be to characterize the pair $P = \operatorname{Princ} L$ in $S = \operatorname{Con_c} L$ by the properties that P is a directed order with zero that join-generates S. We have to rephrase this so it does not require a solution of the congruence lattice characterization problem.

Problem 10.5. Let S be a representable join-semilattice. Let $P \subseteq S$ be a directed order with zero and let P join-generate S. Under what conditions is there a lattice K such that $\operatorname{Con_c} K$ is isomorphic to S and under this isomorphism $\operatorname{Princ} K$ corresponds to P?

For a lattice L, let us define a *valuation* v on $\operatorname{Con_c} L$ as follows.

For a compact congruence $\boldsymbol{\alpha}$ of L, let $v(\boldsymbol{\alpha})$ be the smallest integer n such that the congruence $\boldsymbol{\alpha}$ is the join of n principal congruences.

A valuation v has some obvious properties, for instance, $v(\mathbf{0}) = 0$ and $v(\boldsymbol{\alpha} \vee \boldsymbol{\beta}) \leq v(\boldsymbol{\alpha}) + v(\boldsymbol{\beta})$. Note the connection with $\operatorname{Princ} L$:

$$\operatorname{Princ} L = \{\, \boldsymbol{\alpha} \in \operatorname{Con_c} L \mid v(\boldsymbol{\alpha}) \leq 1 \,\}.$$

Problem 10.6. Let S be a representable join-semilattice. Let v map S to the natural numbers. Under what conditions is there an isomorphism φ of S with $\operatorname{Con_c} K$ for some lattice K so that under φ the map v corresponds to the valuation on $\operatorname{Con_c} K$?

Let D be a finite distributive lattice. In G. Grätzer and E.T. Schmidt [164], we represent D as the congruence lattice of a finite (sectionally complemented) lattice K in which *all congruences are principal* (that is, $\operatorname{Con} K = \operatorname{Princ} K$). See also G. Grätzer and E.T. Schmidt [180], where a planar semimodular lattice K is constructed with $\operatorname{Con} K = \operatorname{Princ} K \cong D$.

Problem 10.7. Let D be a finite distributive lattice. Let Q be a subset of D satisfying $\{0, 1\} \cup \operatorname{Ji} K \subseteq Q \subseteq D$. When is there a finite lattice K such that $\operatorname{Con} K$ is isomorphic to D and under this isomorphism $\operatorname{Princ} K$ corresponds to Q?

In the finite variant of Problem 10.6, we need an additional property.

Problem 10.8. Let S be a finite distributive lattice. Let v be a map of D to the natural numbers satisfying $v(0) = 0$, $v(1) = 1$, and $v(a \vee b) \leq v(a) + v(b)$ for $a, b \in D$. Is there an isomorphism φ of D with $\operatorname{Con} K$ for some finite lattice K such that under φ the map v corresponds to the valuation on $\operatorname{Con} K$?

Problem 10.9. Let K be a bounded lattice. Does there exist a complete lattice L such that $\operatorname{Con} K \cong \operatorname{Con} L$?

Some of these problems seem to be of interest for universal algebras as well.

Problem 10.10. Can we characterize the order $\operatorname{Princ} \mathfrak{A}$ for a universal algebra $\mathfrak{A}$ as an order with zero?

Problem 10.11. For a universal algebra $\mathfrak{A}$, how is the assumption that the unit congruence $\mathbf{1}$ is compact reflected in the order $\operatorname{Princ} \mathfrak{A}$?

Problem 10.12. Let $\mathfrak{A}$ be a universal algebra and let $\operatorname{Princ} \mathfrak{A} \subseteq Q \subseteq \operatorname{Con}_{\mathrm{c}} \mathfrak{A}$. Does there exist a universal algebra $\mathfrak{B}$ such that $\operatorname{Con} \mathfrak{A} \cong \operatorname{Con} \mathfrak{B}$ and under this isomorphism Q corresponds to $\operatorname{Princ} \mathfrak{B}$?

Problem 10.13. Extend the concept of valuation to universal algebras. State and solve Problem 10.6 for universal algebras.

Problem 10.14. Can we sharpen the result of G. Grätzer and E.T. Schmidt [163]: every universal algebra $\mathfrak{A}$ has a congruence-preserving extension $\mathfrak{B}$ such that $\operatorname{Con} \mathfrak{A} \cong \operatorname{Con} \mathfrak{B}$ and $\operatorname{Princ} \mathfrak{B} = \operatorname{Con}_{\mathrm{c}} \mathfrak{B}$.

I do not even know whether every algebra $\mathfrak{A}$ has a proper congruence-preserving extension $\mathfrak{B}$.

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Corrections to LTF

- **Page 102, line** 13:
  $a = \bigvee(\, x \in \operatorname{Ji} L \mid x \leq a\,) = \operatorname{id}(a) \cap \operatorname{Ji} L$
  should be
  $a = \bigvee(\, x \in \operatorname{Ji} L \mid x \leq a\,) = \bigvee\bigl(\operatorname{id}(a) \cap \operatorname{Ji} L\bigr)$
- **Page 318, line** -15:
  $p \wedge \bigvee(I - \{p\}) = 0$
  should be
  $p \wedge \bigvee(I - \{p\}) \neq 0$
- **Page 394, statement of Theorem 447**:
  there is a fourth possibility: $L_i \cong \{0, 1\}$.
- **Page 395, paragraph following Definition 450**:
  interchange the references: "Theorem 447" and "Theorem 449"

Index

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