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Introduction to Nonlinear Dispersive Equations

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Introduction to Nonlinear Dispersive Equations

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ISSN 0172-5939 ISSN 2191-6675 (electronic)
Universitext
ISBN 978-1-4939-2180-5 ISBN 978-1-4939-2181-2 (eBook)
DOI 10.1007/978-1-4939-2181-2

Library of Congress Control Number: 2014958590

Springer New York Heidelberg Dordrecht London

Printed on acid-free paper

Springer is part of Springer Science+Business Media (www.springer.com)

Preface

The goal of this text is to present an introduction to a sampling of ideas and methods from the subject of nonlinear dispersive equations. This subject has been of great interest and has rapidly developed in the last few years. Here we will try to expose some aspects of the recent developments.

The presentation is intended to be self-contained, but we will assume that the reader has knowledge of the material usually taught in courses of theory of one complex variable and integration theory.

This text is the product of lecture notes used for mini-courses and graduate courses taught by the authors. The first version of the lecture notes was written by Gustavo Ponce with Wilfredo Urbina from the Universidad Central de Venezuela and designed to teach a mini-course at the Venezuelan School of Mathematics in Mérida, Venezuela, in 1990. A second version of those notes was presented by Gustavo Ponce at the Colombian School of Mathematics in Cali, Colombia in 1991. These notes comprise a part of the materials covered in the first six chapters of the present text. Most of the original notes were used to teach various graduate courses at IMPA and UNICAMP by Felipe Linares. During these lectures the previous versions were complemented with some new materials presented here. These notes were also used by Hebe Biagioni and Marcia Scialom from UNICAMP in their seminars and graduate courses. The idea to write the present text arose from the need for a more complete treatment of these topics for graduate students.

Before going any further we would first like to give a notion of what a dispersive type of partial differential equation is. We will do this in the one-dimensional frame. We consider a linear partial differential equation

$$F\,(\partial_x, \partial_t)\, u(x,t) = 0, \tag{1}$$

where F is a polynomial in the partial derivatives. We look for plane wave solutions of the form $u(x,t) = Ae^{i(kx-\omega t)}$ where A, k, and ω are constants representing the amplitude, the wavenumber, and the frequency, respectively. Hence u will be a solution if and only if

$$F(ik, -i\omega) = 0. \tag{2}$$

This equation is called the dispersion relation. This relation characterizes the plane wave motion. In several models we can write ω as a real function of k, namely,

$$\omega = \omega(k).$$

The phase and group velocities of the waves are defined by

$$c_p(k) = \frac{\omega}{k} \quad \text{and} \quad c_g = \frac{d\omega}{dk}.$$

The waves are called dispersive if the group velocity $c_g = \omega'(k)$ is not constant, i.e., $\omega''(k) \neq 0$. In the physical context this means that when time evolves, the different waves disperse in the medium, with the result that a single hump breaks into wave-trains.

To present the material we have chosen to study two very well-known models in the class of nonlinear dispersive equations: the Korteweg–de Vries equation

$$\partial_t \nu + \partial_x^3 \nu + \nu \partial_x \nu = 0, \tag{3}$$

where ν is a real-valued function and the nonlinear Schrödinger equation

$$i\partial_t u + \Delta u = f(u, \bar{u}), \tag{4}$$

where u is a complex-valued function.

Before commenting on the theory presented in this text regarding these equations we would like to say a few words concerning the physical models described by these equations in the context of water waves.

The first model (3) goes back to the discovery of Scott Russell in 1835 of what he called a traveling wave. This equation describes the propagation of waves in shallow water and was proposed by Diederik Johannes Korteweg and Gustav de Vries in 1895 [KdV]. In the one-dimensional context, the (cubic) nonlinear Schrödinger equation (4) with $f(u, u) = |u|^2 u$ models the propagation of wave packets in the theory of water waves.

We also have to mention that there is a very well-known strong relationship between these two equations and the theory of completely integrable systems, or Soliton theory.

In many cases, we present the details of simple proof, which may not be that of the strongest result. We give several examples to illustrate the theory. At the end of every chapter we complement the theory described either with a set of exercises or with a section with comments on open problems, extensions, and recent developments.

The first three chapters attempt to review several topics in Fourier analysis and partial differential equations. These are the elementary tools needed to develop the theory in the rest of the notes.

The properties of solutions to the linear problem associated to the Schrödinger equation are discussed in Chapter 4. Then the initial value problem associated to (4) and properties of its solutions are studied in Chaps. 5 and 6. Chapters 7 and 8 are devoted to the study of the initial value problem for the generalized Korteweg–de

Vries equation. A survey of results concerning several nonlinear dispersive equations that generalize (3) and (4) as Davey–Stewartson systems, Ishimori equations, Kadomtsev–Petviashvili equations, Benjamin–Ono equations, and Zakharov systems is presented in Chapter 9. In the last chapter we present the most recent result regarding local well-posedness for the nonlinear Schrödinger equation.

We shall point out that by no means our presentation is completely exhaustive. We refer the reader to the lecture notes by Cazenave [Cz1], [Cz2] and the books by Sulem and Sulem [SS2], Bourgain [Bo2], and Tao [To7]. In these works many topics not covered in these notes are studied in detail.

Acknowledgments

The authors are indebted to several friends who made this project possible. We would like to thank Carlos Kenig, who allowed us to use part of his lecture notes regarding the material in Chapter 10; Luis Vega for useful comments and suggestions; Rafael Iório and Carlos Isnard who are great supporters of the idea of having graduate courses at IMPA in the topics discussed here and the writing of notes concerning; Hebe Biagioni, Marcia Scialom, and Jaime Angulo who gave us feedback about the former lecture notes. We also thank Daniela Bekiranov, Mahendra Panthee, Aniura Milanes, Wee Keong Lim, German Fonseca, Didier Pilod, Aida Gonzalez, José Jiménez, and Luiz Farah for reading the most part of the manuscript and for giving us many corrections and useful comments. The first author is grateful to the Mathematics Department of University of California at Santa Barbara for the support to accomplish this project. The second author was supported by an NSF grant.

Rio de Janeiro and Santa Barbara — Felipe Linares
June 2008 — Gustavo Ponce

Preface to the Second Edition

In this version several errors and typos of the first edition have been corrected thanks to the comments of many friends.

Very few changes were made in the (basic) first four chapters. Some of the material presented in the other chapters has been expanded and updated. In addition, several new exercises have been added.

Despite considerable increase, the bibliography is not intended to be complete. We also thank Cynthia Flores and Derek Smith for reading some parts of the new version of this manuscript.

Rio de Janeiro and Santa Barbara — Felipe Linares
August 2014 — Gustavo Ponce

Contents

Chapter 1
The Fourier Transform

In this chapter, we shall study some basic properties of the Fourier transform. Section 1.1 is concerned with its definition and properties in $L^1(\mathbb{R}^n)$. The case $L^2(\mathbb{R}^n)$ is considered in Section 1.2. The space of tempered distributions is briefly considered in Section 1.3. Finally, Sections 1.4 and 1.5 give an introduction to the study of oscillatory integrals in one dimension and some applications, respectively.

1.1 The Fourier Transform in $L^1(\mathbb{R}^n)$

Definition 1.1. The *Fourier transform* of a function $f \in L^1(\mathbb{R}^n)$, denoted by $\widehat{f}$, is defined as:

$$\widehat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i (x\cdot\xi)} dx, \quad \text{for } \xi \in \mathbb{R}^n, \tag{1.1}$$

where $(x \cdot \xi) = x_1\xi_1 + \cdots + x_n\xi_n$.

We list some basic properties of the Fourier transform in $L^1(\mathbb{R}^n)$.

Theorem 1.1. *Let $f \in L^1(\mathbb{R}^n)$. Then:*

1. *$f \mapsto \widehat{f}$ defines a linear transformation from $L^1(\mathbb{R}^n)$ to $L^\infty(\mathbb{R}^n)$ with*

$$\|\widehat{f}\|_\infty \leq \|f\|_1. \tag{1.2}$$

2. *$\widehat{f}$ is continuous.*
3. *$\widehat{f}(\xi) \to 0$ as $|\xi| \to \infty$ (Riemann–Lebesgue).*
4. *If $\tau_h f(x) = f(x-h)$ denotes the translation by $h \in \mathbb{R}^n$, then*

$$\widehat{(\tau_h f)}(\xi) = e^{-2\pi i (h\cdot\xi)} \widehat{f}(\xi), \tag{1.3}$$

and

$$(\widehat{e^{-2\pi i (x\cdot h)}} f)(\xi) = (\tau_{-h}\widehat{f}\,)(\xi). \tag{1.4}$$

F. Linares, G. Ponce, *Introduction to Nonlinear Dispersive Equations*,
Universitext, DOI 10.1007/978-1-4939-2181-2_1

5. *If* $\delta_a f(x) = f(ax)$ *denotes a dilation by* $a > 0$*, then*

$$\widehat{(\delta_a f)}(\xi) = a^{-n}\widehat{f}(a^{-1}\xi). \tag{1.5}$$

6. *Let* $g \in L^1(\mathbb{R}^n)$ *and* $f * g$ *be the convolution of* f *and* g*. Then,*

$$\widehat{(f * g)}(\xi) = \widehat{f}(\xi)\widehat{g}(\xi). \tag{1.6}$$

7. *Let* $g \in L^1(\mathbb{R}^n)$*. Then,*

$$\int_{\mathbb{R}^n} \widehat{f}(y)g(y)dy = \int_{\mathbb{R}^n} f(y)\widehat{g}(y)dy. \tag{1.7}$$

Notice that the equality in (1.2) holds for $f \geq 0$, i.e., $\widehat{f}(0) = \|\widehat{f}\|_\infty = \|f\|_1$.

Proof. It is left as an exercise. □

Next, we give some examples to illustrate the properties stated in Theorem 1.1.

Example 1.1 Let $n = 1$ and $f(x) = \chi_{(a,b)}(x)$ (the characteristic function of the interval (a, b)). Then,

$$\begin{aligned}\widehat{f}(\xi) &= \int_a^b e^{-2\pi i x\xi}\, dx \\ &= -\frac{e^{-2\pi i b\xi} - e^{-2\pi i a\xi}}{2\pi i\xi} \\ &= -e^{-\pi i(a+b)\xi}\ \frac{\sin(\pi(a-b)\xi)}{\pi\xi}.\end{aligned}$$

Notice that $\widehat{f} \notin L^1(\mathbb{R})$ and that $\widehat{f}(\xi)$ has an analytic extension $\widehat{f}(\xi + i\eta)$ to the whole plane $\xi + i\eta \in \mathbb{C}$. In particular, if $(a, b) = (-k, k)$, $k \in \mathbb{Z}^+$, then we have

$$\widehat{\chi}_{(-k,k)}(\xi) = \frac{\sin(2\pi k\xi)}{\pi\xi}.$$

Example 1.2 Let $n = 1$ and for $k \in \mathbb{Z}^+$ define

$$g_k(x) \ = \begin{cases} k + 1 + x, & \text{if } \ x \in (-k-1, -k+1] \\ 2, & \text{if } \ x \in (-k+1, k-1) \\ k + 1 - x, & \text{if } \ x \in [k-1, k+1) \\ 0, & \text{if } \ x \notin (-k-1, k+1), \end{cases}$$

i.e., $g_k(x) = \chi_{(-1,1)} * \chi_{(-k,k)}(x)$. The identity (1.6) and the previous example show that

$$\widehat{g_k}(\xi) = \frac{\sin(2\pi\xi)\sin(2\pi k\xi)}{(\pi\xi)^2}.$$

Notice that $\widehat{g_k} \in L^1(\mathbb{R})$ and has an analytic extension to the whole plane $\mathbb{C}$.

Example 1.3 Let $n \geq 1$ and $f(x) = e^{-4\pi^2 t|x|^2}$ with $t > 0$. Then, changing variables $x \to x/\sqrt{t}$ and using (1.5), we can restrict ourselves to the case $t = 1$. From Fubini's theorem we write:

$$\begin{aligned}\int_{\mathbb{R}^n} e^{-4\pi^2|x|^2} e^{-2\pi i(x\cdot\xi)}\,dx &= \prod_{j=1}^{n}\int_{-\infty}^{\infty} e^{(-4\pi^2 x_j^2 - 2\pi i\xi_j x_j)}\,dx_j\\ &= \prod_{j=1}^{n}\int_{-\infty}^{\infty} e^{(-4\pi^2 x_j^2 - 2\pi i\xi_j x_j + \xi_j^2/4)} e^{-\xi_j^2/4}\,dx_j\\ &= \prod_{j=1}^{n} e^{-\xi_j^2/4}\int_{-\infty}^{\infty} e^{-(2\pi x_j + i\xi_j/2)^2}\,dx_j\\ &= 2^{-n}\pi^{-n/2} e^{-|\xi|^2/4},\end{aligned}$$

where in the last equality, we have employed the following identities from complex integration and calculus:

$$\int_{-\infty}^{\infty} e^{-(2\pi x + i\xi/2)^2}\,dx = \int_{-\infty}^{\infty} e^{-(2\pi x)^2}\,dx = \int_{-\infty}^{\infty} e^{-x^2}\,\frac{dx}{2\pi} = \frac{1}{2\sqrt{\pi}}.$$

Hence,

$$\widehat{e^{-4\pi^2 t|x|^2}}(\xi) = \frac{e^{-|\xi|^2/4t}}{(4\pi t)^{n/2}}. \tag{1.8}$$

Observe that taking $t = 1/4\pi$ and changing variables $t \to 1/16\pi^2 t$ we get:

$$\widehat{e^{-\pi|x|^2}}(\xi) = e^{-\pi|\xi|^2} \quad \text{and} \quad \frac{\widehat{e^{-|x|^2/4t}}}{(4\pi t)^{n/2}}(\xi) = e^{-4\pi^2 t|\xi|^2},$$

respectively.

Example 1.4 Let $n \geq 1$ and $f(x) = e^{-2\pi|x|}$. Then,

$$\hat{f}(\xi) = \frac{\Gamma[\frac{(n+1)}{2}]}{\pi^{(n+1)/2}}\,\frac{1}{(1+|\xi|^2)^{(n+1)/2}},$$

where $\Gamma(\cdot)$ denotes the Gamma function. See Exercise 1.1 (i).

Example 1.5 Let $n = 1$ and $f(x) = \dfrac{1}{\pi}\dfrac{1}{1+x^2}$. Using complex integration one obtains the identity:

$$\int_{-\infty}^{\infty} \frac{\cos(ax)}{x^2+b^2}\,dx = \frac{\pi}{b}\,e^{-ab}, \quad a,\ b > 0.$$

Hence,

$$\frac{1}{\pi}\widehat{\frac{1}{1+x^2}}(\xi) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{e^{-2\pi i x\xi}}{1+x^2}\,dx$$
$$= \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\cos(2\pi|\xi|x)}{1+x^2}\,dx = e^{-2\pi|\xi|}.$$

One of the most important features of the Fourier transform is its relationship with differentiation. This is described in the following results.

Proposition 1.1. *Suppose* $x_k f \in L^1(\mathbb{R}^n)$*, where* x_k *denotes the kth coordinate of* x*. Then,* $\widehat{f}$ *is differentiable with respect to* ξ_k *and*

$$\frac{\partial \widehat{f}}{\partial \xi_k}(\xi) = (\widehat{-2\pi i x_k f(x)})(\xi). \tag{1.9}$$

In other words, the Fourier transform of the product $x_k f(x)$ is equal to a multiple of the partial derivative of $\widehat{f}(\xi)$ with respect to the kth variable.

To consider the converse result, we need to introduce a definition.

Definition 1.2. Let $1 \le p < \infty$. A function $f \in L^p(\mathbb{R}^n)$ is *differentiable* in $L^p(\mathbb{R}^n)$ with respect to the kth variable, if there exists $g \in L^p(\mathbb{R}^n)$ such that

$$\int_{\mathbb{R}^n}\left|\frac{f(x+he_k)-f(x)}{h} - g(x)\right|^p dx \to 0 \ \text{ as } \ h \to 0,$$

where e_k has kth coordinate equals 1 and 0 in the others. If such a function g exists (in this case it is unique), it is called the partial derivative of f with respect to the kth variable in the L^p-norm.

Theorem 1.2. *Let* $f \in L^1(\mathbb{R}^n)$ *and* g *be its partial derivative with respect to the kth variable in the* L^1*-norm. Then,* $\widehat{g}(\xi) = 2\pi i\xi_k \widehat{f}(\xi)$*.*

Proof. Properties (1.2) and (1.4) in Theorem 1.1 allow us to write

$$\left|\widehat{g}(\xi) - \widehat{f}(\xi)\frac{(1-e^{-2\pi i h(\xi\cdot e_k)})}{h}\right|,$$

then take $h \to 0$ to obtain the result. □

From the previous theorems it is easy to obtain the formulae:

$$\begin{aligned} P(D)\widehat{f}(\xi) &= (P(-2\pi i x)f(x))^\wedge(\xi), \\ (\widehat{P(D)f})(\xi) &= P(2\pi i\xi)\widehat{f}(\xi), \end{aligned} \tag{1.10}$$

where P is a polynomial in n variables and $P(D)$ denotes the differential operator associated to P.

Now we turn our attention to the following question: Given the Fourier transform $\widehat{f}$ of a function in $L^1(\mathbb{R}^n)$, how can one recover f?

Examples 1.3–1.5 suggest the use of the formula

$$f(x) = \int_{\mathbb{R}^n} \widehat{f}(\xi) e^{2\pi i (x\cdot\xi)} d\xi.$$

Unfortunately, $\widehat{f}(\xi)$ may be nonintegrable (see Example 1.1). To avoid this problem, one needs to use the so called method of summability (Abel and Gauss) similar to those used in the study of Fourier series. Combining the ideas behind the Gauss summation method and the identities (1.4), (1.7), (1.8), we obtain the following equalities:

$$\begin{aligned} f(x) &= \lim_{t\to 0} \frac{e^{-|\cdot|^2/4t}}{(4\pi t)^{n/2}} * f(x) = \lim_{t\to 0} \int_{\mathbb{R}^n} \frac{e^{-|x-y|^2/4t}}{(4\pi t)^{n/2}} f(y)dy \\ &= \lim_{t\to 0} \int_{\mathbb{R}^n} \tau_x \frac{e^{-|y|^2/4t}}{(4\pi t)^{n/2}} f(y)dy \\ &= \lim_{t\to 0} \int_{\mathbb{R}^n} (e^{2\pi i (x\cdot\xi)} \widehat{e^{-4\pi^2 t|\xi|^2}})(y) f(y)dy \\ &= \lim_{t\to 0} \int_{\mathbb{R}^n} e^{2\pi i (x\cdot\xi)} e^{-4\pi^2 t|\xi|^2} \widehat{f}(\xi) d\xi, \end{aligned}$$

where the limit is taken in the L^1-norm.

Thus, if f and $\widehat{f}$ are both integrable, the Lebesgue dominated convergence theorem guarantees the point-wise equality. Also, if $f \in L^1(\mathbb{R}^n)$ is continuous at the point x_0, we get:

$$f(x_0) = \lim_{t\to 0} \frac{e^{-|\cdot|^2/4t}}{(4\pi t)^{n/2}} * f(x_0) = \lim_{t\to 0} \int_{\mathbb{R}^n} e^{2\pi i (x_0\cdot\xi)} e^{-4\pi^2 t|\xi|^2} \widehat{f}(\xi) d\xi.$$

Collecting this information, we get the following result.

Proposition 1.2. *Let $f \in L^1(\mathbb{R}^n)$. Then,*

$$f(x) = \lim_{t\to 0} \int_{\mathbb{R}^n} e^{2\pi i (x\cdot\xi)} e^{-4\pi^2 t|\xi|^2} \widehat{f}(\xi) d\xi,$$

where the limit is taken in the L^1-norm. Moreover, if f is continuous at the point x_0, then the following point-wise equality holds:

$$f(x_0) = \lim_{t\to 0} \int_{\mathbb{R}^n} e^{2\pi i (x_0\cdot\xi)} e^{-4\pi^2 t|\xi|^2} \widehat{f}(\xi) d\xi.$$

Let $f, \widehat{f} \in L^1(\mathbb{R}^n)$*. Then,*

$$f(x) = \int_{\mathbb{R}^n} e^{2\pi i(x\cdot\xi)} \widehat{f}(\xi) d\xi, \quad \text{almost everywhere } x \in \mathbb{R}^n.$$

From this result and Theorem 1.1 we can conclude that

$$\wedge \; : \; L^1(\mathbb{R}^n) \longrightarrow C_\infty(\mathbb{R}^n)$$

is a linear, one-to-one (Exercise 1.6 (i)), bounded map. However, it is not surjective (Exercise 1.6 (iii)).

1.2 The Fourier Transform in $L^2(\mathbb{R}^n)$

To define the Fourier transform in $L^2(\mathbb{R}^n)$, we shall first consider that $L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ is a dense subset of $L^1(\mathbb{R}^n)$ and $L^2(\mathbb{R}^n)$.

Theorem 1.3 (Plancherel). *Let* $f \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$*. Then,* $\widehat{f} \in L^2(\mathbb{R}^n)$ *and*

$$\|\widehat{f}\|_2 = \|f\|_2. \tag{1.11}$$

Proof. Let $g(x) = -x$. Using Young's inequality (1.39), (1.6), and Exercise 1.7 (ii), it follows that

$$f * g \in L^1(\mathbb{R}^n) \cap C_\infty(\mathbb{R}^n) \quad \text{and} \quad \widehat{(f * g)}(\xi) = \widehat{f}(\xi)\widehat{g}(\xi).$$

Since $\widehat{g} = \overline{(\widehat{f})}$, we find that $\widehat{(f * g)} = |\widehat{f}|^2 \geq 0$. Hence, $\widehat{(f * g)} \in L^1(\mathbb{R}^n)$ (see Exercise 1.7 (iii)). Proposition 1.2 shows that

$$(f * g)(0) = \int_{\mathbb{R}^n} \widehat{(f * g)}(\xi)\, d\xi,$$

and

$$\begin{aligned} \|\widehat{f}\|_2^2 &= \int_{\mathbb{R}^n} \widehat{(f * g)}(\xi)\, d\xi = (f * g)(0) \\ &= \int_{\mathbb{R}^n} f(x) g(0 - x)\, dx = \int_{\mathbb{R}^n} f(x)\bar{f}(x)\, dx = \|f\|_2^2. \end{aligned}$$

□

This result shows that the Fourier transform defines a linear bounded operator from $L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$. Indeed, this operator is an isometry. Thus, there is a unique bounded extension $\mathcal{F}$ defined in all $L^2(\mathbb{R}^n)$. $\mathcal{F}$ is called the Fourier transform in $L^2(\mathbb{R}^n)$. We shall use the notation $\widehat{f} = \mathcal{F}(f)$ for $f \in L^2(\mathbb{R}^n)$. In general, the definition $\widehat{f}$ is realized as a limit in L^2 of the sequence $\{\widehat{h}_j\}$, where $\{h_j\}$ denotes any sequence in $L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ that converges to f in the L^2-norm. It is convenient to take h_j equals f for $|x| \leq j$ and to have h_j vanishing for $|x| > j$. Then,

$$\widehat{h}_j(\xi) = \int\limits_{|x|<j} f(x)e^{-2\pi i(x\cdot\xi)}\,dx = \int\limits_{\mathbb{R}^n} h_j(x)e^{-2\pi i(x\cdot\xi)}\,dx$$

and so,

$$\widehat{h}_j(\xi) \to \widehat{f}(\xi) \quad \text{in } L^2, \text{ as } j \to \infty.$$

Example 1.6 Let $n = 1$ and $f(x) = \dfrac{1}{\pi}\dfrac{x}{1+x^2}$. Observe that $f \in L^2(\mathbb{R}) \setminus L^1(\mathbb{R})$. Differentiating the identity in the Example 1.5 with respect to a and taking $b = 1$ we get:

$$\int_{-\infty}^{\infty} \frac{x\,\sin(ax)}{1+x^2}\,dx = \pi e^{-a}, \qquad a > 0,$$

which combined with the previous remark gives:

$$\widehat{f}(\xi) = -i\,\mathrm{sgn}(\xi)e^{-2\pi|\xi|}.$$

A surjective isometry defines a "unitary operator." Theorem 1.3 affirms that $\mathcal{F}$ is an isometry. Let us see that $\mathcal{F}$ is also surjective.

Theorem 1.4. *The Fourier transform defines a unitary operator in* $L^2(\mathbb{R}^n)$.

Proof. From the identity (1.11) it follows that $\mathcal{F}$ is an isometry. In particular, its image is a closed subspace of $L^2(\mathbb{R}^n)$. Assume that this is a proper subspace of L^2. Then, there exists $g \neq 0$ such that

$$\int\limits_{\mathbb{R}^n} \widehat{f}(y)g(y)dy = 0, \quad \text{for any } f \in L^2(\mathbb{R}^n).$$

Using formula (1.7; Theorem 1.7), which obviously extends to $f, g \in L^2(\mathbb{R}^n)$, we have that

$$\int\limits_{\mathbb{R}^n} f(y)\widehat{g}(y)dy = \int\limits_{\mathbb{R}^n} \widehat{f}(y)g(y)dy = 0, \text{ for any } f \in L^2.$$

Therefore, $\widehat{g}(\xi) = 0$ almost everywhere, which contradicts

$$\|g\|_2 = \|\widehat{g}\|_2 \neq 0.$$

□

Theorem 1.5. *The inverse of the Fourier transform* $\mathcal{F}^{-1}$ *can be defined by the formula*

$$\mathcal{F}^{-1} f(x) = \mathcal{F} f(-x), \quad \textit{for any} \;\; f \in L^2(\mathbb{R}^n). \tag{1.12}$$

Proof. $\mathcal{F}^{-1}\widehat{f} = \tilde{f}$ is the limit in the L^2-norm of the sequence

$$f_j(x) \;=\; \int\limits_{|\xi|<j} \widehat{f}(\xi) e^{2\pi i(\xi\cdot x)} d\xi.$$

First, we consider the case where $f \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$. It suffices to verify that this agrees with $\mathcal{F}^*\widehat{f}$, where $\mathcal{F}^*$ is the adjoint operator of $\mathcal{F}$ (we recall the fact that for a unitary operator the adjoint and the inverse are equal). This can be checked as follows:

$$\tilde{f}(x) = \int\limits_{\mathbb{R}^n} \widehat{f}(\xi) e^{2\pi i(\xi\cdot x)} d\xi \;=\; \lim_{j\to\infty} f_j(x) \;\text{ in }\; L^2(\mathbb{R}^n),$$

and

$$\begin{aligned}(g, \tilde{f}) &= \int\limits_{\mathbb{R}^n} g(x) \overline{\Big(\int\limits_{\mathbb{R}^n} \widehat{f}(\xi)\, e^{2\pi i(\xi\cdot x)}\, d\xi \Big)} dx \\ &= \int\limits_{\mathbb{R}^n} \Big(\int\limits_{\mathbb{R}^n} g(x) e^{-2\pi i(x\cdot\xi)} dx \Big) \overline{\widehat{f}(\xi)} d\xi = (\mathcal{F}g, \widehat{f})\end{aligned}$$

for any $g \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$. Hence $\tilde{f} = f$.

The general case follows by combining the above result and an argument involving a justification of passing to the limit. □

1.3 Tempered Distributions

From the definitions of the Fourier transform on $L^1(\mathbb{R}^n)$ and on $L^2(\mathbb{R}^n)$, there is a natural extension to $L^1(\mathbb{R}^n) + L^2(\mathbb{R}^n)$. It is not hard to see that $L^1(\mathbb{R}^n) + L^2(\mathbb{R}^n)$ contains the spaces $L^p(\mathbb{R}^n)$ for $1 \le p \le 2$. On the other hand, as we shall prove, any function in $L^p(\mathbb{R}^n)$ for $p > 2$ has a Fourier transform in the distribution sense. However, they may not be function, they are *tempered distributions*. Before studying them, it is convenient to see how far Definition 1.1 can be carried out.

Example 1.7 Let $n \ge 1$ and $f(x) = \delta_0$, the delta function, i.e., the measure of mass one concentrated at the origin. Using (1.1) one finds that

$$\widehat{\delta_0}(\xi) = \int\limits_{\mathbb{R}^n} \delta_0(x)\, e^{-2\pi i(x\cdot\xi)} dx \equiv 1.$$

In fact, Definition 1.1 tells us that if μ is a bounded measure, then $\widehat{\mu}(\xi)$ represents a function in $L^\infty(\mathbb{R}^n)$.

Suppose that given $f(x) \equiv 1$ we want to find $\widehat{f}(\xi)$. In this case, Definition 1.1 cannot be used directly. It is necessary to introduce the notion of tempered distribution. For this purpose, we first need the following family of seminorms.

For each $(\nu, \beta) \in (\mathbb{Z}^+)^{2n}$ we denote the seminorm $|||\cdot|||_{(\nu,\beta)}$ defined as:

$$|||f|||_{(\nu,\beta)} = \|x^\nu \partial_x^\beta f\|_\infty.$$

Now we can define the Schwartz space $\mathcal{S}(\mathbb{R}^n)$, the space of the C^∞-functions decaying at infinity, i.e.,

$$\mathcal{S}(\mathbb{R}^n) = \{\varphi \in C^\infty(\mathbb{R}^n) : |||\varphi|||_{(\nu,\beta)} < \infty \text{ for any } \nu, \beta \in (\mathbb{Z}^+)^n\}.$$

Thus, $C_0^\infty(\mathbb{R}^n) \subsetneq \mathcal{S}(\mathbb{R}^n)$ (consider $f(x)$ as in Example 1.3).

The topology in $\mathcal{S}(\mathbb{R}^n)$ is given by the family of seminorms $|||\cdot|||_{(\nu,\beta)}$, $(\nu, \beta) \in (\mathbb{Z}^+)^{2n}$.

Definition 1.3. Let $\{\varphi_j\} \subset \mathcal{S}(\mathbb{R}^n)$. Then, $\varphi_j \to 0$ as $j \to \infty$, if for any $(\nu, \beta) \in (\mathbb{Z}^+)^{2n}$ one has that

$$|||\varphi_j|||_{(\nu,\beta)} \longrightarrow 0 \text{ as } j \to \infty.$$

The relationship between the Fourier transform and the function space $\mathcal{S}(\mathbb{R}^n)$ is described in the formulae (1.10). More precisely, we have the following result (see Exercise 1.13).

Theorem 1.6. *The map $\varphi \mapsto \widehat{\varphi}$ is an isomorphism from $\mathcal{S}(\mathbb{R}^n)$ into itself.*

Thus, $\mathcal{S}(\mathbb{R}^n)$ appears naturally associated to the Fourier transform. By duality, we can define the tempered distributions $\mathcal{S}'(\mathbb{R}^n)$.

Definition 1.4. We say that $\psi : \mathcal{S}(\mathbb{R}^n) \mapsto \mathbb{C}$ defines a tempered distribution, i.e., $\Psi \in \mathcal{S}'(\mathbb{R}^n)$ if:

1. Ψ is linear.
2. Ψ is continuous, i.e., if for any $\{\varphi_j\} \subseteq \mathcal{S}(\mathbb{R}^n)$ such that $\varphi_j \to 0$ as $j \to \infty$, then the numerical sequence $\Psi(\varphi_j) \to 0$ as $j \to \infty$.

It is easy to check that any bounded function f defines a tempered distribution Ψ_f, where

$$\Psi_f(\varphi) = \int_{\mathbb{R}^n} f(x)\varphi(x)dx, \quad \text{for any } \varphi \in \mathcal{S}(\mathbb{R}^n). \tag{1.13}$$

In fact, this identity allows us to see that any locally integrable function with polynomial growth at infinity defines a tempered distribution. In particular, we have the $L^p(\mathbb{R}^n)$ spaces with $1 \leq p \leq \infty$. The following example gives us a tempered distribution outside these function spaces.

Example 1.8 In $\mathcal{S}'(\mathbb{R})$, define the *principal value function* of $1/x$, denoted by p.v. $\frac{1}{x}$, by the expression

$$\text{p.v. } \frac{1}{x}(\varphi) = \lim_{\epsilon \downarrow 0} \int\limits_{\epsilon<|x|<1/\epsilon} \frac{\varphi(x)}{x} dx,$$

for any $\varphi \in \mathcal{S}(\mathbb{R})$. Since $1/x$ is an odd function,

$$\text{p.v.} \frac{1}{x}(\varphi) = \int\limits_{|x|<1} \frac{\varphi(x) - \varphi(0)}{x} dx + \int\limits_{|x|>1} \frac{\varphi(x)}{x} dx. \tag{1.14}$$

Therefore,

$$\left|\text{p.v. } \frac{1}{x}(\varphi)\right| \leq 2\|\varphi'\|_\infty + 2\|x\varphi\|_\infty, \tag{1.15}$$

and consequently, p.v. $\frac{1}{x} \in \mathcal{S}'(\mathbb{R})$.

Now, given a $\Psi \in \mathcal{S}'(\mathbb{R}^n)$, its Fourier transform can be defined in the following natural form.

Definition 1.5. Given $\Psi \in \mathcal{S}'(\mathbb{R}^n)$, its Fourier transform $\widehat{\Psi} \in \mathcal{S}'(\mathbb{R}^n)$ is defined as:

$$\widehat{\Psi}(\varphi) = \Psi(\widehat{\varphi}), \quad \text{for any } \varphi \in \mathcal{S}(\mathbb{R}^n). \tag{1.16}$$

Observe that for $f \in L^1(\mathbb{R}^n)$ and $\varphi \in \mathcal{S}(\mathbb{R}^n)$, (1.7), (1.13), and (1.16) tell us that

$$\widehat{\Psi}_f(\varphi) = \Psi_f(\widehat{\varphi}) = \int\limits_{\mathbb{R}^n} f(x)\widehat{\varphi}(x)dx = \int\limits_{\mathbb{R}^n} \widehat{f}(x)\varphi(x)dx = \Psi_{\widehat{f}}(\varphi).$$

Therefore, for $f \in L^1(\mathbb{R}^n) + L^2(\mathbb{R}^n)$ one has that $\widehat{\Psi}_f = \Psi_{\widehat{f}}$. Thus, Definition 1.5 is consistent with the theory of the Fourier transform developed in Sects. 1.1 and 1.2.

Example 1.9 Let $f(x) \equiv 1 \in L^\infty(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n)$. Using the previous notation, for any $\varphi \in \mathcal{S}(\mathbb{R}^n)$ it follows that

$$\widehat{\Psi}_1(\varphi) = \Psi_1(\widehat{\varphi}) = \int\limits_{\mathbb{R}^n} 1 \; \widehat{\varphi}(x)dx = \varphi(0) = \int\limits_{\mathbb{R}^n} \delta_0(x) \; \varphi(x)dx = \delta_0(\varphi).$$

Hence $\widehat{1} = \delta_0$. We recall that in Example 1.7 we already saw that $\widehat{\delta_0} = 1$.

Next we compute the Fourier transform of the tempered distribution in Example 1.8.

Example 1.10 Combining Definition 1.5, Fubini's theorem, and the Lebesgue dominated convergence theorem we have that for any $\varphi \in \mathcal{S}(\mathbb{R})$,

$$\begin{aligned} \widehat{\text{p.v. } \frac{1}{x}}(\varphi) \;&=\; \text{p.v. } \frac{1}{x}(\widehat{\varphi}) \;=\; \lim_{\epsilon \downarrow 0} \int\limits_{\epsilon<|x|<1/\epsilon} \frac{\widehat{\varphi}(x)}{x} dx \\ &= \lim_{\epsilon \downarrow 0} \int\limits_{\epsilon<|x|<1/\epsilon} \frac{1}{x}\Big(\int\limits_{-\infty}^{\infty} \varphi(y)e^{-2\pi i x y} \, dy\Big)dx \end{aligned}$$

$$= \lim_{\epsilon\downarrow 0} \int_{-\infty}^{\infty} \varphi(y)\Big(\int_{\epsilon<|x|<1/\epsilon} \frac{e^{-2\pi ixy}}{x} dx\Big)\, dy$$

$$= \int_{-\infty}^{\infty} \varphi(y)\Big(\lim_{\epsilon\downarrow 0} \int_{\epsilon<|x|<1/\epsilon} \frac{e^{-2\pi ixy}}{x} dx\Big) dy$$

$$= -i\pi \int_{-\infty}^{\infty} \operatorname{sgn}(y)\, \varphi(y) dy,$$

where a change of variables and complex integration have been used to conclude that

$$\lim_{\epsilon\downarrow 0} \int_{\epsilon<|x|<1/\epsilon} \frac{e^{-2\pi ixy}}{x} dx = -2i \int_0^{\infty} \frac{\sin(2\pi xy)}{x} dx = -2i \operatorname{sgn}(y) \int_0^{\infty} \frac{\sin(x)}{x} dx$$
$$= -i\pi \operatorname{sgn}(y).$$

This yields the identity:

$$\widehat{\text{p.v.}\, \frac{1}{x}}(\xi) \;=\; -\, i\pi\, \operatorname{sgn}(\xi).$$

The topology in $\mathcal{S}'(\mathbb{R}^n)$ can be described in the following form.

Definition 1.6. Let $\{\Psi_j\} \subset \mathcal{S}'(\mathbb{R}^n)$. Then, $\Psi_j \to 0$ as $j \to \infty$ in $\mathcal{S}'(\mathbb{R}^n)$, if for any $\varphi \in \mathcal{S}(\mathbb{R}^n)$ it follows that $\Psi_j(\varphi) \longrightarrow 0$ as $j \to \infty$.

As a consequence of the Definitions 1.4, 1.6, we get the next extension of Theorem 1.6, whose proof we leave as an exercise.

Theorem 1.7. *The map* $\mathcal{F} : \Psi \mapsto \widehat{\Psi}$ *is an isomorphism from* $\mathcal{S}'(\mathbb{R}^n)$ *into itself.*

Combining the above results with an extension of Example 1.3 (see Exercise 1.2), we can justify the following computation related with the fundamental solution of the time-dependent Schrödinger equation.

Example 1.11 $\widehat{e^{-4\pi^2 it|x|^2}} = \lim_{\epsilon\to 0^+} \widehat{e^{-4\pi^2(\epsilon+it)|x|^2}}$ in $\mathcal{S}'(\mathbb{R}^n)$.

From Exercise 1.2, it follows that

$$(\widehat{e^{-4\pi^2(\epsilon+it)|x|^2}})(\xi) = \frac{e^{-|\xi|^2/4(\epsilon+it)}}{[4\pi(\epsilon+it)]^{n/2}}.$$

Taking the limit $\epsilon \to 0^+$, we obtain:

$$(\widehat{e^{-4\pi^2 it|x|^2}})(\xi) = \frac{e^{i|\xi|^2/4t}}{(4\pi it)^{n/2}}. \tag{1.17}$$

As an application of these ideas, we introduce the Hilbert transform.

Definition 1.7. For $\varphi \in \mathcal{S}(\mathbb{R})$, we define its *Hilbert transform* $\mathsf{H}(\varphi)$ by

$$\mathsf{H}(\varphi)(y) = \frac{1}{\pi}\,\text{p.v.}\frac{1}{x}\,(\varphi(y-\cdot)) = \frac{1}{\pi}\,\text{p.v.}\,\frac{1}{x} * \varphi(y).$$

From (1.14) and (1.15) it is clear that $\mathsf{H}(\varphi)(y)$ is defined for any $y \in \mathbb{R}$ and it is bounded by $g(y) = a|y| + b$, with $a, b > 0$ depending on φ. In particular, we have that $\mathsf{H}(\varphi) \in \mathcal{S}'(\mathbb{R})$. Let us compute its Fourier transform.

Example 1.12 From Example 1.10 and the identity

$$\mathsf{H}(\varphi)(y) = \lim_{\epsilon\to 0}\left(\frac{1}{\pi}\frac{1}{x}\chi_{\{\epsilon<|x|<1/\epsilon\}} * \varphi\right)(y) \quad \text{in} \quad \mathcal{S}'(\mathbb{R})$$

it follows that

$$\lim_{\epsilon\to 0}\left(\frac{1}{\pi}\;\frac{1}{x}\;\widehat{\chi_{\{\epsilon<|x|<1/\epsilon\}} * \varphi}\right)(\xi) = -i\;\text{sgn}(\xi)\,\widehat{\varphi}(\xi).$$

This implies that

$$\widehat{\mathsf{H}(\varphi)}(\xi) = -i\;\text{sgn}(\xi)\,\widehat{\varphi}(\xi), \quad \text{for any} \quad \varphi \in \mathcal{S}(\mathbb{R}). \tag{1.18}$$

The identity (1.18) allows us to extend the Hilbert transform as an isometry in $L^2(\mathbb{R})$. It is not hard to see that

$$\|\mathsf{H}(\varphi)\|_2 = \|\varphi\|_2 \quad \text{and} \quad \mathsf{H}(\mathsf{H}(\varphi)) = -\varphi.$$

Other properties of the Hilbert transform are deduced in the exercises in Chaps. 1 and 2.

In Definition 1.7, we have implicitly utilized the following result, which is employed again in the applications at the end of this chapter.

Proposition 1.3. *Let* $\varphi \in \mathcal{S}(\mathbb{R}^n)$ *and* $\Psi \in \mathcal{S}'(\mathbb{R}^n)$. *Define*

$$\Psi * \varphi(x) = \Psi(\varphi(x-\cdot)). \tag{1.19}$$

Then,

$$\Psi * \varphi \in C^\infty(\mathbb{R}^n) \cap \mathcal{S}'(\mathbb{R}^n)$$

and

$$\widehat{\Psi * \varphi} = \widehat{\Psi}\widehat{\varphi}, \tag{1.20}$$

where $\widehat{\Psi}\widehat{\varphi} \in \mathcal{S}'(\mathbb{R}^n)$ *is defined as* $\widehat{\Psi}\widehat{\varphi}(\phi) = \widehat{\Psi}(\widehat{\varphi}\phi)$ *for any* $\phi \in \mathcal{S}(\mathbb{R}^n)$.

Proof. It is left as an exercise. □

1.4 Oscillatory Integrals in One Dimension

In many problems and applications the following question arises:

What is the asymptotic behavior of $I(\lambda)$ when $\lambda \to \infty$, where

$$I(\lambda) = \int_a^b e^{i\lambda\phi(x)} f(x)dx, \tag{1.21}$$

and ϕ is a smooth real-valued function, called the "phase function," and f is a smooth complex-valued function?

We shall see that this asymptotic behavior is determined by the points $\bar{x}$, where the derivative of ϕ vanishes, i.e., $\phi'(\bar{x}) = 0$.

Proposition 1.4. *Let $f \in C_0^\infty([a,b])$ and $\phi'(x) \neq 0$ for any $x \in [a,b]$. Then*

$$I(\lambda) = \int_a^b e^{i\lambda\phi(x)} f(x)dx = O(\lambda^{-k}), \quad \textit{as } \lambda \to \infty \tag{1.22}$$

for any $k \in \mathbb{Z}^+$.

Proof. Define the differential operator

$$\mathcal{L}(f) = \frac{1}{i\lambda\phi'}\frac{df}{dx},$$

which satisfies

$$\mathcal{L}^t(f) = -\frac{d}{dx}\left(\frac{f}{i\lambda\phi'}\right) \quad \text{and} \quad \mathcal{L}^k(e^{i\lambda\phi}) = e^{i\lambda\phi},$$

where $\mathcal{L}^t$ denotes the adjoint of $\mathcal{L}$. Using integration by parts it follows that

$$\int_a^b e^{i\lambda\phi} f dx = \int_a^b \mathcal{L}^k(e^{i\lambda\phi}) f\, dx$$

$$= (-1)^k \int_a^b e^{i\lambda\phi} (\mathcal{L}^t)^k f\, dx = O(\lambda^{-k}), \text{ as } \lambda \to \infty.$$

□

Proposition 1.5. *Let $k \in \mathbb{Z}^+$ and $|\phi^{(k)}(x)| \geq 1$ for any $x \in [a,b]$ with $\phi'(x)$ monotonic in the case $k = 1$. Then,*

$$\left| \int_a^b e^{i\lambda\phi(x)} dx \right| \leq c_k \lambda^{-1/k}, \tag{1.23}$$

where the constant c_k *is independent of* a, b.

Proof. For $k = 1$, we have that

$$\int_a^b e^{i\lambda\phi} dx = \int_a^b \mathcal{L}(e^{i\lambda\phi}) dx = \frac{1}{i\lambda\phi'} e^{i\lambda\phi}\Big|_a^b - \int_a^b e^{i\lambda\phi} \frac{1}{i\lambda} \frac{d}{dx}\Big(\frac{1}{\phi'}\Big) dx.$$

Clearly, the first term on the right-hand side is bounded by $2\lambda^{-1}$. On the other hand, the hypothesis of monotonicity on ϕ' guarantees that

$$\Big|\int_a^b e^{i\lambda\phi} \frac{1}{i\lambda} \frac{d}{dx}\Big(\frac{1}{\phi'}\Big) dx\Big| \leq \frac{1}{\lambda} \int_a^b \Big|\frac{d}{dx}\Big(\frac{1}{\phi'}\Big)\Big| dx$$

$$= \frac{1}{\lambda}\Big|\frac{1}{\phi'(b)} - \frac{1}{\phi'(a)}\Big| \leq \frac{2}{\lambda}.$$

This yields the proof of the case $k = 1$.

For the proof of the case $k \geq 2$, induction in k is used. Assuming the result for k, we shall prove it for $k + 1$. By hypothesis, $|\phi^{(k+1)}(x)| \geq 1$. Let $x_0 \in [a, b]$ be such that

$$|\phi^{(k)}(x_0)| = \min_{a\leq x\leq b} |\phi^{(k)}(x)|.$$

If $\phi^{(k)}(x_0) = 0$, outside the interval $(x_0 - \delta,\ x_0 + \delta)$, one has that $|\phi^{(k)}(x)| \geq \delta$, with ϕ' monotonic if $k = 1$. Splitting the domain of integration and applying the hypothesis we obtain that

$$\Big|\int_a^{x_0-\delta} e^{i\lambda\phi(x)}\, dx\Big| + \Big|\int_{x_0+\delta}^b e^{i\lambda\phi(x)}\, dx\Big| \leq c_k(\lambda\delta)^{-1/k}.$$

A simple computation shows that

$$\Big|\int_{x_0-\delta}^{x_0+\delta} e^{i\lambda\phi(x)}\, dx\Big| \leq 2\delta.$$

Thus,

$$\Big|\int_a^b e^{i\lambda\phi(x)}\, dx\Big| \leq c_k(\lambda\delta)^{-1/k} + 2\delta.$$

If $\phi^{(k)}(x_0) \neq 0$, then $x_0 = a$ or b and a similar argument provides the same bound. Finally, taking $\delta = \lambda^{-1/(k+1)}$ we complete the proof. □

Corollary 1.1 (van der Corput). *Under the hypotheses of Proposition 1.5,*

$$\Big|\int_a^b e^{i\lambda\phi(x)} f(x)\, dx\Big| \leq c_k \lambda^{-1/k} \left(\|f\|_\infty + \|f'\|_1\right) \tag{1.24}$$

with c_k *independent of* a, b.

Proof. Define

$$G(x) = \int_a^x e^{i\lambda\phi(y)}\, dy.$$

By (1.23) one has that

$$|G(x)| \le c_k \lambda^{-1/k}.$$

Now using integration by parts we obtain:

$$\Big|\int_a^b e^{i\lambda\phi} f\, dx\Big| = \Big|\int_a^b G' f\, dx\Big| \le \Big|\,(Gf)\Big|_a^b + \Big|\int_a^b Gf'\, dx\Big|$$
$$\le c_k \lambda^{-1/k} \left(\|f\|_\infty + \|f'\|_1\right).$$

□

Next, we shall study an application of these results.

Proposition 1.6. *Let* $\beta \in [0, 1/2]$ *and* $I_\beta(x)$ *be the oscillatory integral*

$$I_\beta(x) = \int_{-\infty}^{\infty} e^{i(x\eta+\eta^3)}\, |\eta|^\beta d\eta. \tag{1.25}$$

Then, $I_\beta \in L^\infty(\mathbb{R})$.

Proof. First, we fix $\varphi_0 \in C^\infty(\mathbb{R})$ such that

$$\varphi_0(\eta) = \begin{cases} 1, & \text{if } |\eta| > 2 \\ 0, & \text{if } |\eta| < 1. \end{cases}$$

Observe that $(1-\varphi_0)(\eta)e^{i\eta^3}|\eta|^\beta \in L^1(\mathbb{R})$, therefore its Fourier transform belongs to $L^\infty(\mathbb{R})$. Thus, it suffices to consider

$$\tilde{I}_\beta(x) = \int_{-\infty}^{\infty} e^{i(x\eta+\eta^3)}\, |\eta|^\beta \varphi_0(\eta) d\eta.$$

For $x \ge -3$, the phase function $\phi_x(\eta) = x\eta + \eta^3$, in the support of φ_0, satisfies

$$|\phi_x'(\eta)| = |x + 3\eta^2| \ge (|x| + |\eta|^2).$$

In this case, integration by parts leads to the desired result.

For $x < -3$, we consider the functions $(\varphi_1, \varphi_2) \in C_0^\infty \times C^\infty$ such that $\varphi_1(\eta) + \varphi_2(\eta) = 1$ with

$$\operatorname{supp}\varphi_1 \subset A = \left\{\eta : |x + 3\eta^2| \le \frac{|x|}{2}\right\},$$

and

$$\varphi_2 = 0 \quad \text{in} \quad B = \left\{\eta : |x + 3\eta^2| < \frac{|x|}{3}\right\},$$

and we split the integral $\tilde{I}_\beta(x)$ in two pieces,

$$|\tilde{I}_\beta(x)| \le |\tilde{I}_\beta^1(x)| + |\tilde{I}_\beta^2(x)|,$$

where

$$\tilde{I}_\beta^j(x) = \int_{-\infty}^{\infty} e^{i(x\eta+\eta^3)}\, |\eta|^\beta \varphi_0(\eta)\varphi_j(\eta)d\eta, \quad \text{for} \quad j = 1, 2.$$

When $\varphi_2(\eta) \neq 0$, the triangle inequality shows that

$$|\phi_x'(\eta)| = |x + 3\eta^2| \ge \frac{3}{13}(|x| + |\eta|^2).$$

Integration by parts leads to

$$|\tilde{I}_\beta^2(x)| = \Big|\int_{-\infty}^{\infty} \frac{|\eta|^\beta}{\phi_x'(\eta)}\varphi_0(\eta)\varphi_2(\eta)\frac{d}{d\eta}e^{i(x\eta+\eta^3)}\, d\eta\Big| \ \le\ 100.$$

Now, if $\eta \in A$, we have that

$$\frac{|x|}{2} \le 3\eta^2 \le 3\frac{|x|}{2} \quad \text{and} \quad \Big|\frac{d^2\phi_x}{d\eta^2}(\eta)\Big| = 6|\eta| \ge |x|^{1/2}.$$

Thus (1.24) (van der Corput) and the form of $\varphi_0,\ \varphi_1$ guarantee the existence of a constant c independent of $x < -3$ such that

$$|\tilde{I}_\beta^1(x)| = \Big|\int_{-\infty}^{\infty} e^{i(x\eta+\eta^3)}\, |\eta|^\beta \varphi_0(\eta)\varphi_1(\eta)\, d\eta\Big| \le c\, |x|^{-1/4}|x|^{\beta/2}.$$

□

1.5 Applications

Consider the initial value problem (IVP) for the linear Schrödinger equation:

$$\begin{cases} \partial_t u = i\,\Delta u, \\ u(x,0) = u_0(x), \end{cases} \tag{1.26}$$

$x \in \mathbb{R}^n$, $t \in \mathbb{R}$. Taking the Fourier transform with respect to the space variable x in (1.26) we obtain:

$$\begin{cases} \widehat{\partial_t u}(\xi,t) = \partial_t \widehat{u}(\xi,t) = \widehat{i\Delta u}(\xi,t) = -4\pi^2 i|\xi|^2 \widehat{u}(\xi,t) \\ \widehat{u}(\xi,0) = \widehat{u}_0(\xi). \end{cases}$$

The solution of this family of ordinary differential equations (ODE), with parameter ξ, can be written as:

$$\widehat{u}(\xi,t) = e^{-4\pi^2 it|\xi|^2}\widehat{u}_0(\xi).$$

By Proposition 1.3 it follows that

$$\begin{aligned} u(x,t) &= (e^{-4\pi^2 it|\xi|^2}\widehat{u}_0(\xi))^\vee = (e^{-4\pi^2 it|\xi|^2})^\vee * u_0(x) \\ &= \frac{e^{i|\cdot|^2/4t}}{(4\pi it)^{n/2}} * u_0(x) = e^{it\Delta}u_0(x), \end{aligned} \tag{1.27}$$

where we have introduced the notation $e^{it\Delta}$ which is justified in Chapter 4.

Next, we consider the IVP associated to the linearized Korteweg–de Vries (KdV) equation:

$$\begin{cases} \partial_t v + \partial_x^3 v = 0, \\ v(x,0) = v_0(x) \end{cases} \tag{1.28}$$

for $t, x \in \mathbb{R}$. The previous argument shows that

$$v(x,t) = S_t * v_0(x) = (e^{8\pi^3 it\xi^3}\widehat{v}_0)^\vee = V(t)v_0(x), \tag{1.29}$$

where the kernel $S_t(x)$ is defined by the oscillatory integral:

$$S_t(x) = \int_{-\infty}^{\infty} e^{2\pi ix\xi} e^{8\pi^3 it\xi^3}\, d\xi. \tag{1.30}$$

After changing variables,

$$S_t(x) = \frac{1}{\sqrt[3]{3t}}\, Ai\left(\frac{x}{\sqrt[3]{3t}}\right), \tag{1.31}$$

where $Ai(\cdot)$ denotes the Airy function:

$$Ai(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i(\xi x + \xi^3/3)} d\xi. \tag{1.32}$$

By combining Proposition 1.6 (with $\beta = 0$) and a new change of variable we find that

$$\|S_t\|_\infty \le c|t|^{-1/3}. \tag{1.33}$$

Moreover, if $\beta \in [0, 1/2]$, then

$$\|D_x^\beta S_t\|_\infty \leq c|t|^{-(\beta+1)/3}. \tag{1.34}$$

Hence, using Exercise 1.6 it follows that

$$\|D_x^\beta V(t)v_0\|_\infty = \|D_x^\beta S_t * v_0\|_\infty \leq c|t|^{-(\beta+1)/3}\|v_0\|_1, \tag{1.35}$$

where $D_x^\beta = D^\beta = (-\Delta)^{\beta/2}$ denotes the homogeneous fractional derivative of order β, i.e.,

$$D^\beta f(x) = [(2\pi|\xi|)^\beta \widehat{f}(\xi)]^\vee(x). \tag{1.36}$$

Notice that the derivative of the phase function in (1.32) $\phi(\xi) = \xi x + \xi^3/3$ does not vanish for $x > 0$, i.e., $|\phi'(\xi)| = |x + \xi^2| \geq |x|$, so using Proposition 1.4 one sees that $Ai(x)$ has fast decay for $x > 0$. In fact, one has (see [Ho2] or [SSS]) that

$$|Ai(x)| \leq \frac{1}{(1 + x_-)^{1/4}}\, e^{-cx_+^{3/2}}, \tag{1.37}$$

and

$$|Ai'(x)| \leq (1 + x_-)^{1/4} e^{-cx_+^{3/2}}, \tag{1.38}$$

where $x_+ = \max\{x; 0\}$ and $x_- = \max\{-x; 0\}$.

Hence, (1.34) with $\beta = 1/2$ can be seen as an interpolation between (1.37) and (1.38) and the scaling.

Remark 1.1. The relevant references used in this chapter are the books [SW], [S2], [S3], [Sa], [Du], and [Rd].

1.6 Exercises

1.1 (i) Let $n \geq 1$ and $f(x) = e^{-2\pi|x|}$. Show that

$$\hat{f}(\xi) = \frac{\Gamma[(n+1)/2]}{\pi^{(n+1)/2}} \frac{1}{(1 + |\xi|^2)^{(n+1)/2}}.$$

Hint: From the formula of Example 1.5 with $a = \beta$ and $b = 1$ one sees that

$$e^{-\beta} = \frac{2}{\pi} \int_0^\infty \frac{\cos(\beta x)}{1 + x^2}\, dx,$$

which, combined with the equality:

$$\frac{1}{1 + x^2} = \int_0^\infty e^{-(1+x^2)\rho}\, d\rho, \quad \text{yields} \quad e^{-\beta} = \int_0^\infty \frac{e^{-\rho}}{\sqrt{\rho}}\, e^{-\beta^2/4\rho}\, d\rho.$$

Use this identity to obtain the desired result.

(ii) Let $n = 1$ and $f(x) = \frac{1}{\pi}\frac{1}{(1+x^2)^2}$. Show that

$$\hat{f}(\xi) = \frac{1}{2}e^{-2\pi|\xi|}(2\pi|\xi| + 1).$$

Hint: Differentiate the identity in Example 1.5.

1.2 (i) Prove the following extension in $\mathcal{S}'(\mathbb{R}^n)$ of formula (1.8):

$$\widehat{(e^{-a|x|^2})}(\xi) = \left(\frac{\pi}{a}\right)^{n/2} e^{-\pi^2|\xi|^2/a}, \quad \mathcal{R}e\, a \geq 0, \quad a \neq 0,$$

where $\sqrt{a}$ is defined as the branch with $\mathcal{R}e\, a > 0$.
Hint: Use an analytic continuation argument.

(ii) Show that if $a = 1 + it$, then

$$\left\| \left(\frac{\pi}{a}\right)^{n/2} e^{-\pi^2|x|^2/a} \right\|_p \sim c_p\,(1+t)^{n(\frac{1}{p}-\frac{1}{2})}, \quad 1 \leq p \leq \infty, \quad t > 0,$$

and

$$\|e^{-\pi a|\xi|^2}\|_q \sim c_q, \quad 1 \leq q \leq \infty,$$

where $f(t) \sim g(t)$, for $f, g \geq 0$, means that there exists $c > 1$ such that

$$c^{-1} f(t) \leq g(t) \leq c\, f(t), \quad \forall t > 0.$$

1.3 Prove Young's inequality: Let $f \in L^p(\mathbb{R}^n)$, $1 \leq p \leq \infty$, and $g \in L^1(\mathbb{R}^n)$. Then, $f * g \in L^p(\mathbb{R}^n)$ with

$$\|f * g\|_p \leq \|f\|_p \|g\|_1. \tag{1.39}$$

1.4 Prove the Minkowski integral inequality. If $1 \leq p \leq \infty$, then

$$\left(\int_{\mathbb{R}^n} \left|\int_{\mathbb{R}^n} f(x,y)\,dx\right|^p dy\right)^{1/p} \leq \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} \left|f(x,y)\right|^p dy\right)^{1/p} dx. \tag{1.40}$$

Observe that the proof of the cases $p = 1, \infty$ is immediate.

1.5 Let $f \in L^p((0,\infty))$, $1 < p < \infty$, $f \geq 0$:

(i) Prove Hardy's inequality:

$$\int_0^\infty \left(\frac{1}{x}\int_0^x f(s)\,ds\right)^p dx \leq \left(\frac{p}{p-1}\right)^p \int_0^\infty (f(x))^p\,dx. \tag{1.41}$$

(ii) Prove that equality in (1.41) holds if and only if $f = 0$, a.e., and that the constant $c_p = p/(p-1)$ is optimal in (1.41).

(iii) Prove that (1.41) fails for $p = 1$ and $p = \infty$.
Hint: Assuming $f \in C_0((0,\infty))$ define

$$F(x) = \frac{1}{x} \int_0^\infty f(s)\,ds, \text{so } x F' = f - F.$$

Use integration by parts and the Hölder inequality to obtain (1.41).

1.6 Consider the Fourier transform $\widehat{\ }$ as a map from $L^1(\mathbb{R}^n)$ into $L^\infty(\mathbb{R}^n)$.

(i) Prove that $\widehat{\ }$ is injective.

(ii) Prove that the image of $\widehat{\ }$, i.e., $\widehat{L^1(\mathbb{R}^n)}$, is an algebra with respect to the point-wise multiplication of functions.

(iii) Prove that $\widehat{L^1(\mathbb{R}^n)} \subsetneq C_\infty(\mathbb{R}^n)$, where $C_\infty(\mathbb{R}^n)$ denotes the space of continuous functions vanishing at infinity.
Hint: From Example 1.2 we have that $\|g_k\|_\infty = 2$ and

$$\lim_{k\uparrow\infty} \|\widehat{g}_k\|_1 = \infty.$$

Apply the open mapping theorem to get the desired result.

1.7 (i) Prove the following generalization of (1.6) in Theorem 1.1:
If $f \in L^1(\mathbb{R}^n)$ and $g \in L^p(\mathbb{R}^n)$, $1 \le p \le 2$, then $\widehat{(f * g)}(\xi) = \widehat{f}(\xi)\,\widehat{g}(\xi)$.

(ii) If $f \in L^p(\mathbb{R}^n)$, $g \in L^{p'}(\mathbb{R}^n)$, with $1/p + 1/p' = 1$, $1 < p < \infty$, then $f * g \in C_\infty(\mathbb{R}^n)$. What can you affirm if $p = 1, \infty$?

(iii) If $f \in L^1(\mathbb{R}^n)$, with f continuous at the point 0 and $\widehat{f} \ge 0$, then $\widehat{f} \in L^1(\mathbb{R}^n)$.
Hint: Use Proposition 1.2 and Fatou's lemma.

1.8 Show that

$$\int_0^\infty \frac{\sin^2 x}{x^2}\,dx = \frac{\pi}{2} \quad \text{and} \quad \int_0^\infty \frac{\sin^3 x}{x^3} dx = \frac{3\pi}{8}.$$

Hint: Combine the identities (1.7), (1.11), and Example 1.1.

1.9 For a given $f \in L^2(\mathbb{R}^n)$ prove that the following statements are equivalent:

(i) $g \in L^2(\mathbb{R}^n)$ is the partial derivative of $f \in L^2(\mathbb{R}^n)$ with respect to the kth variable according to Definition 1.2.

(ii) There exists $g \in L^2(\mathbb{R}^n)$ such that

$$\int_{\mathbb{R}^n} f(x)\partial_{x_k}\phi(x)\,dx = -\int_{\mathbb{R}^n} g(x)\phi(x)\,dx \tag{1.42}$$

for any $\phi \in C_0^\infty(\mathbb{R}^n)$. In general, if (1.42) holds for two distributions f, g, then one says that g is the kth partial derivative of f in the distribution sense.

(iii) There exists $\{f_j\} \subset C_0^\infty(\mathbb{R}^n)$ such that

$$\|f_j - f\|_2 \to 0 \quad \text{as} \quad j \to \infty,$$

and $\{\partial_{x_k} f_j\}$ is a Cauchy sequence in $L^2(\mathbb{R}^n)$.

(iv) $\xi_k \widehat{f}(\xi) \in L^2(\mathbb{R}^n)$.

(v)

$$\sup_{h>0} \int_{\mathbb{R}^n} \left| \frac{f(x+he_k) - f(x)}{h} \right|^2 dx < \infty.$$

For $p \neq 2$, which of the above statements are still equivalent?

1.10 (Paley–Wiener theorem) Prove that if $f \in C_0^\infty(\mathbb{R}^n)$ with support in $\{x \in \mathbb{R}^n : |x| \leq M\}$, then $\widehat{f}(\xi)$ can be extended analytically to $\mathbb{C}^n$. Moreover, if $k \in \mathbb{Z}^+$ one has that

$$|\widehat{f}(\xi + i\eta)| \leq c_k \frac{e^{2\pi M|\eta|}}{(1 + |(\xi + i\eta)|)^k} \quad \text{for any } \xi + i\eta \in \mathbb{C}^n. \tag{1.43}$$

Prove the converse, i.e., if $F(\xi + i\eta)$ is an analytic function in $\mathbb{C}^n$ satisfying (1.43), then F is the Fourier transform of some $f \in C_0^\infty(\mathbb{R}^n)$ with support in $\{x \in \mathbb{R}^n : |x| \leq M\}$.

1.11 Show that if $f \in L^1(\mathbb{R}^n)$, $f \not\equiv 0$, with compact support, then for any $\epsilon > 0$, $\hat{f} \notin L^1(e^{\epsilon|x|}dx)$.

1.12 Prove that given $k \in \mathbb{Z}^+$ and $a_\alpha \in \mathbb{R}^k$, with $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$, $|\alpha| = \alpha_1 + \cdots + \alpha_n \leq k$, there exists $f \in C_0^\infty(\mathbb{R}^n)$ such that

$$\int_{\mathbb{R}^n} x^\alpha f(x)\, dx = a_\alpha.$$

Hint: Use Exercise 1.10.

1.13 (i) Prove that if $f, g \in \mathcal{S}$, then $f * g \in \mathcal{S}$.
(ii) Prove that the Fourier transform is an isomorphism from $\mathcal{S}$ into itself.
(iii) Using the results in Section 1.3, find explicitly $\Psi = \widehat{|x|^2} \in \mathcal{S}'(\mathbb{R}^n)$.
(iv) Prove Proposition 1.3.

1.14 In this problem we shall prove that

$$\widehat{\frac{1}{|x|^\alpha}}(\xi) = c_{n,\alpha} \frac{1}{|\xi|^{n-\alpha}} \qquad \text{for } \alpha \in (0, n)$$

as a tempered distribution, i.e., $\forall \varphi \in \mathcal{S}(\mathbb{R}^n)$

$$\int \frac{1}{|x|^\alpha} \widehat{\varphi}(x)\, dx = c_{n,\alpha} \int \frac{1}{|\xi|^{n-\alpha}} \varphi(\xi)\, d\xi, \tag{1.44}$$

where $c_{n,\alpha} = \pi^{\alpha - n/2}\, \Gamma(n/2 - \alpha/2)/\Gamma(\alpha/2)$.

(i) Combining the Parseval identity and Example 1.3 show that for $\delta > 0$

$$\int e^{-\pi\delta|x|^2} \widehat{\varphi}(x)\, dx = \delta^{-n/2} \int e^{-\pi\, |x|^2/\delta} \varphi(x)\, dx. \tag{1.45}$$

(ii) Prove the formula

$$\int_0^\infty e^{-\pi\delta|x|^2} \delta^{\beta-1}\, d\delta = \frac{c_\beta}{|x|^{2\beta}} \quad \text{for any } \beta > 0. \tag{1.46}$$

(iii) Multiply both sides of (1.45) by $\delta^{\frac{n-\alpha}{2}-1}$, integrate on δ, use Fubini's theorem and (1.46) to get (1.44).

1.15 Prove the following identities, where H denotes the Hilbert transform:

(i) $\mathsf{H}(fg) = \mathsf{H}(f)g + f\mathsf{H}(g) + \mathsf{H}(\mathsf{H}(f)\mathsf{H}(g))$.

(ii) $\mathsf{H}(\chi_{(-1,1)})(x) = \dfrac{1}{\pi}\log\left|\dfrac{x+1}{x-1}\right|$.

(iii) $\mathsf{H}\left(\dfrac{a}{x^2+a^2}\right) = \dfrac{x}{x^2+a^2}, \quad a > 0.$

1.16 Prove that if $\varphi \in \mathcal{S}(\mathbb{R})$, then $\mathsf{H}(\varphi) \in L^1(\mathbb{R})$ if and only if $\widehat{\varphi}(0) = 0$.

1.17 Consider the function $f_a(x) = \dfrac{x}{a-x^2}$.

(i) If $a \geq 0$ prove that the principal value function of $f_a(x)$,

$$\text{p.v.}\,\frac{x}{a-x^2}(\varphi) = \lim_{\epsilon\downarrow 0}\int_{\epsilon<|a-x^2|<1/\epsilon} \frac{x}{a-x^2}\,\varphi(x)\,dx,$$

with $\varphi \in \mathcal{S}(\mathbb{R})$ defines a tempered distribution. Moreover, prove that if

$$\widehat{f_a}(\xi) = \lim_{\epsilon\downarrow 0}\int_{\epsilon<|a-x^2|<1/\epsilon} e^{-2\pi i(x\cdot\xi)}\,\frac{x}{a-x^2}\,dx,$$

then

$$\|\widehat{f_a}\|_\infty \leq M, \tag{1.47}$$

where the constant M is independent of a.
Hint: Observe that if $a = 0$, $f_a(x)$ is just a multiple of the kernel $1/x$ of the Hilbert transform H. If $a > 0$, then $f_a(x)$ can be written as sum of translations of the kernel of the Hilbert transform H. Since the Hilbert transform satisfies a similar result, (1.47) follows in both cases. (See Example 1.10).

(ii) Show that (1.47) is also satisfied if $a < 0$.
Hint: Use Example 1.6.

1.18 Consider the IVP associated to the wave equation

$$\begin{cases} \partial_t^2 w - \Delta w = 0, \\ w(x,0) = f(x), \\ \partial_t w(x,0) = g(x), \end{cases} \tag{1.48}$$

$x \in \mathbb{R}^n$, $t \in \mathbb{R}$. Prove that

(i) If $f, g \in C_0^\infty(\mathbb{R}^n)$ are real-valued functions, then using the notation in (1.29), the solution can be described by the following expression:

$$w(x,t) = U'(t)f + U(t)g = \cos(Dt)f + \frac{\sin(Dt)}{D}g, \tag{1.49}$$

with $\widehat{Dh}(\xi) = 2\pi|\xi|\widehat{h}(\xi)$ (see (1.36)).

(ii) If f, g are supported in $\{x \in \mathbb{R}^3 : |x| \leq M\}$, show that $w(\cdot,t)$ is supported in $\{x \in \mathbb{R}^3 : |x| \leq M + t\}$.

(iii) Assuming $n = 3$ and $f \equiv 0$, prove that

$$w(x,t) = \frac{1}{4\pi t}\int_{\{|y|=t\}} g(x+y)\,dS_y.$$

Hint: Derive and apply the following identity:

$$\int\limits_{\{|x|=t\}} e^{2\pi i\xi\cdot x}\,dS_x = 4\pi t\,\frac{\sin(2\pi|\xi|t)}{2\pi|\xi|}.$$

If $g \in C_0^\infty(\mathbb{R}^3)$ is supported in $\{x \in \mathbb{R}^3 : |x| \leq M\}$, where is the support of $w(\cdot,t)$?

(iv) Assuming $n = 3$ and $g \equiv 0$, prove that

$$w(x,t) = \frac{1}{4\pi t^2}\int\limits_{\{|y|=t\}} [f(x+y) + \nabla f(x+y)\cdot y]\,dS_y. \tag{1.50}$$

(v) If $E(t) = \int\limits_{\mathbb{R}^n}((\partial_t w)^2 + |\nabla_x w|^2)(x,t)dx$, then prove that for any $t \in \mathbb{R}$,

$$E(t) = E_0 = \int\limits_{\mathbb{R}^n}(g^2 + |\nabla_x f|^2)(x)\,dx.$$

Hint: Use integration by parts and the equation.

(vi) (Brodsky [Br]) Show that

$$\lim_{t\to\infty}\int\limits_{\mathbb{R}^n}(\partial_t w)^2(x,t)\,dx = \frac{E_0}{2}.$$

Hint: Use the Riemann–Lebesgue lemma (Theorem 1.1(3)).

1.19 Consider the IVP (1.28) with initial data $v_0 \in C_0^\infty(\mathbb{R})$. Prove that for any $t \neq 0$ $v(\cdot,t)$ does not have compact support.

Chapter 2
Interpolation of Operators: A Multiplier Theorem

In this chapter, we shall first study two basic results in interpolation of operators in L^p spaces, the Riesz–Thorin theorem and the Marcinkiewicz interpolation theorem (diagonal case). As a consequence of the former we shall prove the Hardy–Littlewood–Sobolev theorem for Riesz potentials. In this regard, we need to introduce one of the fundamental tools in harmonic analysis, the Hardy–Littlewood maximal function. In Section 2.4, we shall prove the Mihlin multiplier theorem.

The results deduced in this chapter are used frequently in these notes. In particular, in Chapter 4 the proof of Theorem 4.2 is based on the Riesz–Thorin theorem and the Hardy–Littlewood–Sobolev theorem.

2.1 The Riesz–Thorin Convexity Theorem

Let $(X, \mathcal{A}, \mu)$ be a measurable space (i.e., X is a set, $\mathcal{A}$ denotes a σ-algebra of subsets of X, and μ is a measure defined on $\mathcal{A}$). $L^p = L^p(X, \mathcal{A}, \mu)$, $1 \leq p < \infty$ denotes the space of complex-valued functions f that are μ-measurable such that

$$\|f\|_p = \left(\int_X |f(x)|^p \, d\mu \right)^{1/p} < \infty.$$

Functions in $L^p(X, \mathcal{A}, \mu)$ are defined almost everywhere with respect to μ. Similarly, we have $L^\infty(X, \mathcal{A}, \mu)$ the space of functions f that are μ-measurable, complex valued and essentially μ-bounded, with $\|f\|_\infty$ the essential supremum of f. The Riesz–Thorin convexity theorem can be obtained as a consequence of a version of the Hadamard three circles theorem, a result of the Phragmen–Lindelöf theorem, known as the *three lines theorem*.

Lemma 2.1. *Let F be a continuous and bounded function defined on*

$$S = \{z = x + iy : 0 \leq x \leq 1\}$$

which is also analytic in the interior of S. If for each $y \in \mathbb{R}$,

$$|F(iy)| \leq M_0 \quad \text{and} \quad |F(1 + iy)| \leq M_1,$$

F. Linares, G. Ponce, *Introduction to Nonlinear Dispersive Equations,*
Universitext, DOI 10.1007/978-1-4939-2181-2_2

then for any $z = x + iy \in S$

$$|F(x+iy)| \le M_0^{1-x} M_1^x.$$

In other words, the function $\phi(x) = \log k_x$ *is convex, where* $k_x = \sup\{|F(x+iy)| : y \in \mathbb{R}\}$ *for* $x \in [0,1]$.

Proof. Without loss of generality one can assume that $M_0,\ M_1 > 0$. Moreover, considering the function $F(z)/M_0^{1-z} M_1^z$, the proof reduces to the case $M_0 = M_1 = 1$. Thus, we have that

$$|F(iy)| \le 1 \quad \text{and} \quad |F(1+iy)| \le 1 \quad \text{for any } y \in \mathbb{R},$$

and we want to show that $|F(z)| \le 1$ for any $z \in S$. If

$$\lim_{|y|\to\infty} F(x+iy) = 0 \quad \text{uniformly on } 0 \le x \le 1,$$

the result follows from the maximum principle. In this case, there exists $y_0 > 0$ such that $|F(x+iy)| \le 1$ for $|y| \ge y_0$ and $|F(z)| \le 1$ in the boundary of the rectangle with corners

$$iy_0, 1+iy_0, -iy_0, 1-iy_0.$$

The maximum principle guarantees the same estimate in the interior of the rectangle.

In the general case, we consider the function:

$$F_n(z) = F(z)e^{(z^2-1)/n}, \quad n \in \mathbb{Z}^+.$$

Since

$$\begin{aligned} |F_n(z)| &= |F(x+iy)|e^{-y^2/n}\ e^{(x^2-1)/n} \\ &\le |F(x+iy)|e^{-y^2/n} \to 0 \quad \text{as} \quad |y| \to \infty, \end{aligned}$$

uniformly on $0 \le x \le 1$, with $|F_n(iy)| \le 1$ and $|F_n(1+iy)| \le 1$, the previous argument proves that $|F_n(z)| \le 1$ for any $n \in \mathbb{Z}^+$. Letting $n \to \infty$, we obtain the desired estimate. □

Let T be a linear operator from $L^p(X)$ to $L^q(Y)$. If T is continuous or bounded, i.e.,

$$|||T||| = \sup_{f \ne 0} \frac{\|Tf\|_q}{\|f\|_p} < \infty, \tag{2.1}$$

we call the number $|||T|||$ the *norm of the operator* T.

Theorem 2.1 (Riesz–Thorin). *Let* $p_0 \ne p_1,\ q_0 \ne q_1$. *Let* T *be a bounded linear operator from* $L^{p_0}(X, \mathcal{A}, \mu)$ *to* $L^{q_0}(Y, \mathcal{B}, \nu)$ *with norm* M_0 *and from* $L^{p_1}(X, \mathcal{A}, \mu)$ *to* $L^{q_1}(Y, \mathcal{B}, \nu)$ *with norm* M_1. *Then,* T *is bounded from* $L^{p_\theta}(X, \mathcal{A}, \mu)$ *in* $L^{q_\theta}(Y, \mathcal{B}, \nu)$ *with norm* M_θ *such that*

$$M_\theta \le M_0^{1-\theta} M_1^\theta,$$

with

$$\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad \frac{1}{q_\theta} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}, \quad \theta \in (0,1). \tag{2.2}$$

Proof. *(Thorin)*. Combining the notation

$$\langle h, g\rangle = \int_Y h(y)g(y)\,d\nu(y)$$

and a duality argument it follows that

$$\|h\|_q = \sup\{|\langle h, g\rangle| : \|g\|_{q'} = 1\}$$

and

$$M_{pq} \equiv \sup\{|\langle Tf, g\rangle| : \|f\|_p = \|g\|_{q'} = 1\},$$

where $1/p + 1/p' = 1/q + 1/q' = 1$. Since $p < \infty$ and $q' < \infty$, we can assume that $f,\ g$ are simple functions with compact support. Thus,

$$f(x) = \sum_j a_j \chi_{A_j}(x) \ \text{ and } \ g(y) = \sum_k b_k \chi_{B_k}(y).$$

For $0 \le \mathcal{R}e\ z \le 1$, we define

$$\frac{1}{p(z)} = \frac{1-z}{p_0} + \frac{z}{p_1}, \quad \frac{1}{q'(z)} = \frac{1-z}{q_0'} + \frac{z}{q_1'},$$

$$\varphi(z) = \varphi(x,z) = \sum_j |a_j|^{p_\theta/p(z)} e^{i\arg(a_j)} \chi_{A_j}(x),$$

and

$$\psi(z) = \psi(y,z) = \sum_k |b_k|^{q_\theta'/q'(z)} e^{i\arg(b_k)} \chi_{B_k}(y).$$

Thus, $\varphi(z) \in L^{p_j}$, $\psi(z) \in L^{q_j'}$, and $T\varphi(z) \in L^{q_j}, j = 0, 1$. Also, $\varphi'(z) \in L^{p_j}$, $\psi'(z) \in L^{q_j'}$, and $(T\varphi)'(z) \in L^{q_j}$, $j = 0, 1$ for $0 < \mathcal{R}e\ z < 1$. Therefore, the function

$$F(z) = \langle T\varphi(z), \psi(z)\rangle$$

is bounded and continuous on $0 \le \mathcal{R}e\ z \le 1$ and analytic in the interior. Moreover,

$$\|\varphi(it)\|_{p_0} = \||f|^{p_\theta/p_0}\|_{p_0} = \|f\|_{p_\theta}^{p_\theta/p_0} = 1$$

and

$$\|\varphi(1+it)\|_{p_1} = \||f|^{p_\theta/p_1}\|_{p_1} = \|f\|_{p_\theta}^{p_\theta/p_1} = 1.$$

Similarly, $\|\psi(it)\|_{q_0'} = \|\psi(1+it)\|_{q_1'} = 1$.

From the hypotheses it follows that

$$|F(it)| \leq \|T\varphi(it)\|_{q_0} \; \|\psi(it)\|_{q_0'} \leq M_0$$

and

$$|F(1+it)| \leq \|T\varphi(1+it)\|_{q_1} \; \|\psi(1+it)\|_{q_1'} \leq M_1.$$

Since $\varphi(\theta) = f$, $\psi(\theta) = g$, and $F(\theta) = \langle Tf, g\rangle$, by the three lines theorem we obtain $|\langle Tf, g\rangle| \leq M_0^{1-\theta} M_1^{\theta}$. This completes the proof. □

Definition 2.1. An operator T is said to be *sublinear* if $T(f+g)$ is determined by the values of Tf, Tg, and

$$|T(f+g)| \leq |Tf| + |Tg|.$$

We shall say that a linear or sublinear operator T is of (strong) *type* (p,q) with constant M_{pq} if $\|Tf\|_q \leq M_{pq}\|f\|_p$ for any $f \in L^p$.

With this definition we can rephrase the statement of the Riesz–Thorin theorem.

Let $p_0 \neq p_1$, $q_0 \neq q_1$, and T be a linear operator of type (p_0, q_0) with norm M_0 and of type (p_1, q_1) with norm M_1. Then T is of type (p,q) with

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}, \quad \theta \in (0,1),$$

with norm

$$M \leq M_0^{1-\theta} M_1^{\theta}.$$

2.1.1 Applications

Next we use the Riesz–Thorin theorem to establish some properties of the Fourier transform and the convolution operator. We fix $X = Y = \mathbb{R}^n$ and $\mu = \nu = dx$ the Lebesgue measure.

Theorem 2.2 (Young's inequality). *Let $f \in L^p(\mathbb{R}^n)$ and $g \in L^q(\mathbb{R}^n)$, $1 \leq p, q \leq \infty$ with $\frac{1}{p} + \frac{1}{q} \geq 1$. Then $f * g \in L^r(\mathbb{R}^n)$, where $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1$. Moreover,*

$$\|f * g\|_r \leq \|f\|_p \|g\|_q. \tag{2.3}$$

Proof. For $g \in L^q(\mathbb{R}^n)$, we define the operator

$$Tf(x) = \int_{\mathbb{R}^n} f(x-y)g(y)dy = (f * g)(x).$$

The Minkowski integral inequality shows

$$\|Tf\|_q \leq \|g\|_q \|f\|_1.$$

On the other hand, using Hölder's inequality one sees that

$$\|Tf\|_\infty \le \|g\|_q \|f\|_{q'}.$$

Thus, T is of type $(1, q)$ and (q', ∞) with norm bounded by $\|g\|_q$. Hence, Theorem 2.1 (Riesz–Thorin) guarantees that T is of type (p, r), where

$$\frac{1}{p} = \frac{(1-\theta)}{1} + \frac{\theta}{q'} = 1 - \frac{\theta}{q}$$

and

$$\frac{1}{r} = \frac{(1-\theta)}{q} + 0 = \frac{1}{q} + \left(1 - \frac{\theta}{q}\right) - 1 = \frac{1}{q} + \frac{1}{p} - 1,$$

with norm less than $\|g\|_q$. □

Theorem 2.3 (Hausdorff–Young's inequality). *Let $f \in L^p(\mathbb{R}^n)$, $1 \le p \le 2$. Then $\widehat{f} \in L^{p'}(\mathbb{R}^n)$ with $\frac{1}{p} + \frac{1}{p'} = 1$ and*

$$\|\widehat{f}\|_{p'} \le \|f\|_p. \tag{2.4}$$

Proof. From (1.2) and (1.11) it follows that the Fourier transform is of type $(1, \infty)$ and $(2, 2)$ with norm 1. Hence, Theorem 2.1 tells us that it is also of type (p, q) with

$$\frac{1}{p} = \frac{(1-\theta)}{1} + \frac{\theta}{2} = 1 - \frac{\theta}{2} \quad \text{and} \quad \frac{1}{q} = 0 + \frac{\theta}{2} = 1 - \frac{1}{p} = \frac{1}{p'}$$

with norm $M \le 1^{(1-\theta)}\, 1^\theta = 1$. □

This estimate is the best possible when $p = 1$ or 2. This is not the case for $1 < p < 2$. Beckner [B] found the best constant for the Hausdorff–Young inequality. He showed that if $f \in L^p(\mathbb{R}^n)$, $1 \le p \le 2$, then

$$\|\widehat{f}\|_{p'} \le (A_p)^n \|f\|_p, \quad \text{where} \quad A_p = \left(\frac{p^{1/p}}{p'^{1/p'}}\right)^{1/2}.$$

2.2 Marcinkiewicz Interpolation Theorem (Diagonal Case)

Let $(X, \mathcal{A}, \mu)$ be a measurable space.

Definition 2.2. For a measurable function $f : X \to \mathbb{C}$, we define its distribution function as:

$$m(\lambda, f) = \mu(\{x \in X : |f(x)| > \lambda\}) = \mu(E_f^\lambda).$$

Thus, $m(\lambda, f)$ as a function of $\lambda \in [0, \infty]$ is well defined and takes values in $[0, \infty)$. Moreover, it is nonincreasing and continuous from the right.

Proposition 2.1. *For any measurable function* $f : X \to \mathbb{C}$ *and for any* $\lambda \geq 0$ *it follows that*

1. *(Tchebychev)*

$$m(\lambda, f) \leq \lambda^{-p} \int\limits_{E_f^\lambda} |f(x)|^p \, d\mu(x) \leq \lambda^{-p} \|f\|_p^p.$$

2. *If* $1 \leq p < \infty$,

$$\|f\|_p^p = -\int\limits_0^\infty \lambda^p \, dm(\lambda, f) = p \int\limits_0^\infty \lambda^{p-1} m(\lambda, f) \, d\lambda.$$

If $p = \infty$,

$$\|f\|_\infty = \inf \{\lambda : m(\lambda, f) = 0\}.$$

3. $m(\lambda, f + g) \leq m(\lambda/2, f) + m(\lambda/2, g)$.

Proof. It is left as an exercise. □

Definition 2.3. For $1 \leq p < \infty$, we denote by $L^{p*}(X, \mathcal{A}, \mu)$ (*weak* L^p*-spaces*) the space of all measurable functions $f : X \to \mathbb{C}$ such that

$$\|f\|_p^* = \sup_{\lambda > 0} \lambda (m(\lambda, f))^{1/p} < \infty.$$

Observe that $L^{\infty *} = L^\infty$.

Proposition 2.2. *If* $1 \leq p < \infty$, *then*

1. $L^p(\mathbb{R}^n) \subsetneq L^{p*}(\mathbb{R}^n)$.
2. $\|f + g\|_p^* \leq 2(\|f\|_p^* + \|g\|_p^*)$.

Proof. It is left as an exercise. □

Therefore, $L^{p*}(X, \mathcal{A}, \mu)$ is a *quasinormed vector space*

$$\|f + g\| \leq k(\|f\| + \|g\|)$$

with $k = 2$, i.e., it only satisfies a quasitriangular inequality. The spaces L^p and L^{p*} are particular cases of the *Lorentz spaces* $L^{p,q}$ (see [BeL]).

Definition 2.4. Let $(X_j, \mathcal{A}_j, \mu_j)$, $j = 1, 2$, be two measurable spaces. Let $M(X_2)$ be the space of complex-valued, measurable functions defined on X_2. A linear or sublinear operator $T : L^p(X_1) \to M(X_2)$ with $1 \leq p < \infty$ is said to be of *weak type* (p, q) if there exists a constant $c > 0$ such that for any $f \in L^p(X_1)$

$$\|Tf\|_q^* \leq c\|f\|_p.$$

If $q = \infty$, type (p, ∞) and weak type (p, ∞) agree. Tchebychev's inequality shows that if T is of type (p, q), then it is of weak type (p, q).

In the rest of this chapter, we shall consider $X_j = \mathbb{R}^n,\ j = 1, 2$.

Theorem 2.4 (Marcinkiewicz). *Let* $1 < r \le \infty$ *and*

$$T : L^1(\mathbb{R}^n) + L^r(\mathbb{R}^n) \to M(\mathbb{R}^n)$$

be a sublinear operator (see Definition 2.1). If T is of weak type $(1,1)$ and of weak type (r,r), then T is of (strong) type (p,p) for any $p \in (1,r)$.

Proof. First we consider the case $r = \infty$. Changing the operator T by $\|T\|^{-1}T$ one can assume that

$$\|Tf\|_\infty \le \|f\|_\infty.$$

Given $f \in L^1(\mathbb{R}^n) + L^r(\mathbb{R}^n)$, for each $\lambda \in \mathbb{R}^+$ we define

$$f_1^\lambda(x) = \begin{cases} f(x), & \text{if } |f(x)| \ge \lambda/2 \\ 0, & \text{if } |f(x)| < \lambda/2 \end{cases}$$

and $f_2^\lambda(x) = f(x) - f_1^\lambda(x)$. Therefore,

$$|Tf(x)| \le |Tf_1^\lambda(x)| + \lambda/2,$$

and

$$\{x \in \mathbb{R}^n : |Tf(x)| > \lambda\} \subseteq \{x \in \mathbb{R}^n : |Tf_1^\lambda(x)| > \lambda/2\}.$$

Since T is of weak type $(1,1)$, it follows that

$$\begin{aligned} |\{x \in \mathbb{R}^n : |Tf_1^\lambda(x)| > \lambda/2\}| &\le c\left(\frac{\lambda}{2}\right)^{-1} \int_{\mathbb{R}^n} |f_1^\lambda(x)|\,dx \\ &= 2c\lambda^{-1} \int_{|f|>\lambda/2} |f(x)|\,dx, \end{aligned}$$

where $|\cdot|$ denotes the Lebesgue measure. Combining this estimate, part (2) of Proposition 2.1, and a change in the order of integration, one has:

$$\begin{aligned} \int_{\mathbb{R}^n} |Tf(x)|^p\,dx &= p\int_0^\infty \lambda^{p-1}|\{x \in \mathbb{R}^n : |Tf(x)| > \lambda\}|\,d\lambda \\ &\le p\int_0^\infty \lambda^{p-1}\left(2c\lambda^{-1}\int_{|f|>\lambda/2} |f(x)|\,dx\right)d\lambda \\ &= 2cp\int_0^\infty \lambda^{p-2}\left(\int_{|f|>\lambda/2} |f(x)|\,dx\right)d\lambda \\ &= 2cp\int_{\mathbb{R}^n}\left(\int_0^{2|f(x)|} \lambda^{p-2}\,d\lambda\right)|f(x)|\,dx = \frac{2^p cp}{p-1}\|f\|_p^p, \end{aligned}$$

which yields the result for the case $r = \infty$.

In the case $r < \infty$, we have

$$\begin{aligned} m(\lambda, Tf) &= |\{x \in \mathbb{R}^n : |Tf(x)| > \lambda\}| \\ &\le m(\lambda/2, Tf_1^\lambda) + m(\lambda/2, Tf_2^\lambda) \\ &\le c_1 \left(\frac{\lambda}{2}\right)^{-1} \int\limits_{\mathbb{R}^n} |f_1^\lambda(x)|\, dx + c_r^r \left(\frac{\lambda}{2}\right)^{-r} \int\limits_{\mathbb{R}^n} |f_2^\lambda(x)|^r\, dx \\ &= 2c_1\lambda^{-1} \int\limits_{|f|\ge\lambda/2} |f(x)|\, dx + (2c_r)^r \lambda^{-r} \int\limits_{|f|<\lambda/2} |f(x)|^r\, dx. \end{aligned}$$

As in the proof of the case $r = \infty$, we have that

$$\int\limits_0^\infty \lambda^{p-2} \left(\int\limits_{|f|\ge\lambda/2} |f(x)|\, dx \right) d\lambda = \frac{2^{p-1}}{p-1} \|f\|_p^p.$$

A similar argument shows that

$$\int\limits_0^\infty \lambda^{p-1-r} \left(\int\limits_{|f|<\lambda/2} |f(x)|^r\, dx \right) d\lambda = \frac{2^{p-r}}{r-p} \|f\|_p^p.$$

Combining these inequalities and part (2) of Proposition 2.1, we find that

$$\|Tf\|_p \le c_p \|f\|_p, \quad \text{with} \quad c_p = 2\sqrt[p]{p} \left(\frac{c_1}{p-1} + \frac{c_r^r}{r-p} \right)^{1/p}.$$

□

2.2.1 Applications

We shall use the Marcinkiewicz interpolation theorem to study some basic properties of the Hardy–Littlewood maximal function. First, we introduce some notation.

We denote by $L^1_{\text{loc}}(\mathbb{R}^n)$ the spaces of functions $f : \mathbb{R}^n \to \mathbb{C}$ such that $\int_K |f| dx < \infty$ for any compact $K \subseteq \mathbb{R}^n$. The volume of the unit ball in $\mathbb{R}^n$ will be denoted by ω_n and $B_r(x) = \{y \in \mathbb{R}^n : \|x - y\| < r\}$ is the ball of center x and radius r.

Definition 2.5. For a given $f \in L^1_{\text{loc}}(\mathbb{R}^n)$, we define $\mathcal{M}f(x)$, the *Hardy–Littlewood maximal function* associated to f, as:

$$\begin{aligned} \mathcal{M}f(x) &= \sup_{r>0} \frac{1}{|B_r(x)|} \int\limits_{B_r(x)} |f(y)|\, dy = \sup_{r>0} \frac{1}{\omega_n} \int\limits_{B_1(0)} |f(x - ry)|\, dy \\ &= \sup_{r>0} \left(|f| * \frac{1}{|B_r(0)|} \chi_{B_r(0)} \right)(x). \end{aligned}$$

Proposition 2.3.

1. *$\mathcal{M}$ defines a sublinear operator, i.e.,*

$$|\mathcal{M}(f+g)(x)| \le |\mathcal{M}f(x)| + |\mathcal{M}g(x)|, \qquad x \in \mathbb{R}^n.$$

2. *If $f \in L^\infty(\mathbb{R}^n)$, then*

$$\|\mathcal{M}f\|_\infty \le \|f\|_\infty. \tag{2.5}$$

Proof. It is left as an exercise. □

Part (2) of Proposition 2.3 tells us that $\mathcal{M}$ is of type (∞,∞). Next, we show that $\mathcal{M}$ is of weak type $(1,1)$. For this purpose, we need the following result.

Lemma 2.2 (Vitali's covering lemma). *Let $E \subseteq \mathbb{R}^n$ be a measurable set such that $E \subseteq \cup_\alpha B_{r_\alpha}(x_\alpha)$ with the family of open balls $\{B_{r_\alpha}(x_\alpha)\}_\alpha$ satisfying $\sup\limits_\alpha r_\alpha = c_0 < \infty$. Then there exists a subfamily $\{B_{r_j}(x_j)\}_j$ disjoint and numerable such that*

$$|E| \le 5^n \sum_{j=1}^\infty |B_{r_j}(x_j)|.$$

Proof. Choose $B_{r_1}(x_1)$ such that $r_1 \ge c_0/2$. For $j \ge 2$, take $B_{r_j}(x_j)$ such that $B_{r_j}(x_j) \cap \bigcup\limits_{k=1}^{j-1} B_{r_k}(x_k) = \emptyset$ and

$$r_j > \frac{1}{2} \sup\{r_\alpha \;:\; B_{r_\alpha}(x_\alpha) \cap B_{r_k}(x_k) = \emptyset \;\text{ for }\; k = 1, \dots, j-1\}.$$

It is clear that the $B_{r_j}(x_j)$ are disjoint. If $\sum |B_{r_j}(x_j)| = \infty$, we have completed the proof. In the case $\sum |B_{r_j}(x_j)| < \infty$ (hence, $\lim\limits_{j\to\infty} r_j = 0$), it will suffice to show that

$$B_{r_\alpha}(x_\alpha) \subseteq \bigcup_j B_{5r_j}(x_j), \quad \text{for any } \alpha.$$

If $B_{r_\alpha}(x_\alpha) = B_{r_j}(x_j)$ for some j, there is nothing to prove. Thus, we assume that $B_{r_\alpha}(x_\alpha) \ne B_{r_j}(x_j)$ for any j. Define j_α as the smallest j such that $r_j < r_\alpha/2$. By the construction of $B_{r_j}(x_j)$, there exists $j \in \{1, \dots, j_\alpha - 1\}$ such that $B_{r_\alpha}(x_\alpha) \cap B_{r_j}(x_j) \ne \emptyset$. Denoting by j^* this index it follows that $B_{r_\alpha}(x_\alpha) \subseteq B_{5r_{j^*}}(x_{j^*})$ since $r_{j^*} \ge r_\alpha/2$. □

Theorem 2.5 (Hardy–Littlewood). *Let $1 < p \le \infty$. Then $\mathcal{M}$ is a sublinear operator of type (p,p), i.e., there exists c_p such that*

$$\|\mathcal{M}f\|_p \le c_p \|f\|_p, \quad \textit{for any } f \in L^p(\mathbb{R}^n). \tag{2.6}$$

Proof. We first show that $\mathcal{M}$ is of weak type $(1,1)$, that is, there exists a constant c_1 such that for any $f \in L^1(\mathbb{R}^n)$

$$\sup_{\lambda>0} \lambda\, m(\lambda, \mathcal{M}f) \le c_1 \|f\|_1. \tag{2.7}$$

Once (2.7) has been established, a combination of (2.5), (2.7), and the Marcinkiewicz theorem yields (2.6).

To obtain (2.7), we define $E_f^\lambda = \{x \in \mathbb{R}^n : \mathcal{M}f(x) > \lambda\}$ for any $\lambda > 0$. Thus, if $x \in E_f^\lambda$, then there exists $B_{r_x}(x)$ such that

$$\int_{B_{r_x}(x)} |f(y)|dy > \lambda |B_{r_x}(x)|.$$

Clearly, we have that

$$E_f^\lambda \subseteq \bigcup_{x \in E_f^\lambda} B_{r_x}(x),$$

then the Vitali covering lemma guarantees the existence of a countable, disjoint subfamily $\{B_{r_{x_j}}(x_j)\}_{j \in \mathbb{Z}^+}$ such that

$$|E_f^\lambda| \le 5^n \sum_{j=1}^{\infty} |B_{r_{x_j}}(x_j)| \le 5^n \lambda^{-1} \sum_{j=1}^{\infty} \int_{B_{r_{x_j}}(x_j)} |f(y)|\,dy \le 5^n \lambda^{-1} \|f\|_1,$$

which implies (2.7). □

Next, we extend the estimates (2.6) and (2.7) to a large class of kernels.

Proposition 2.4. *Let $\varphi \in L^1(\mathbb{R}^n)$ be a radial, positive, and nonincreasing function of $r = \|x\| \in [0, \infty)$. Then*

$$\sup_{t>0} |\varphi_t * f(x)| = \sup_{t>0} \left| \int_{\mathbb{R}^n} \frac{\varphi(t^{-1}(x-y))}{t^n} f(y)\,dy \right| \le \|\varphi\|_1 \mathcal{M}f(x). \tag{2.8}$$

Proof. First, we assume that, in addition to the hypotheses, φ is a simple function

$$\varphi(x) = \sum_k a_k \chi_{B_{r_k}(0)}(x), \quad \text{with} \quad a_k > 0.$$

Hence,

$$\varphi * f(x) = \sum_k a_k |B_{r_k}(0)| \frac{1}{|B_{r_k}(0)|} \chi_{B_{r_k}(0)} * f(x) \le \|\varphi\|_1 \mathcal{M}f(x).$$

(observe that $\|\varphi\|_1 = \sum_k a_k |B_{r_k}(0)|$).

In the general case, we approximate φ by an increasing sequence of simple functions satisfying the hypotheses. Since dilations of φ satisfy the same hypotheses and preserve the L^1-norm, they verify (2.8). Finally, passing to the limit we obtain the desired result. □

Next, we shall apply these results to deduce some continuity properties of the Riesz potentials. We recall that a fundamental solution of the Laplacian Δ is given by the following formula describing the Newtonian potential

$$Uf(x) = c_n \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-2}}\, dy \quad \text{for } n \geq 3.$$

The Riesz potentials generalize this expression.

Definition 2.6. Let $0 < \alpha < n$. The Riesz potential of order α, denoted by I_α, is defined as:

$$I_\alpha f(x) = c_{\alpha,n} \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}}\, dy = k_\alpha * f(x), \tag{2.9}$$

where $c_{\alpha,n} = \pi^{-n/2} 2^{-\alpha}\, \Gamma(n/2 - \alpha/2)/\Gamma(\alpha/2)$.

Since the Riesz potentials are defined as integral operators, it is natural to study their continuity properties in $L^p(\mathbb{R}^n)$.

Theorem 2.6 (Hardy–Littlewood–Sobolev). *Let* $0 < \alpha < n$, $1 \leq p < q < \infty$, *with* $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$.

1. *If* $f \in L^p(\mathbb{R}^n)$, *then the integral* (2.9) *is absolutely convergent almost every* $x \in \mathbb{R}^n$.
2. *If* $p > 1$, *then* I_α *is of type (p,q), i.e.,*

$$\|I_\alpha(f)\|_q \leq c_{p,\alpha,n} \|f\|_p. \tag{2.10}$$

Proof. We split the kernel

$$k_\alpha(x) = \frac{c_{\alpha,n}}{|x|^{n-\alpha}} = k_\alpha^0(x) + k_\alpha^\infty(x)$$

as

$$k_\alpha^0(x) = \begin{cases} k_\alpha(x) & \text{if } |x| \leq \varepsilon, \\ 0 & \text{if } |x| > \varepsilon \end{cases}$$

and $k_\alpha^\infty(x) = k_\alpha(x) - k_\alpha^0(x)$, where ε is a positive constant to be determined. Thus,

$$|I_\alpha f(x)| \leq |k_\alpha^0 * f(x)| + |k_\alpha^\infty * f(x)| = I + II. \tag{2.11}$$

The integral I represents the convolution of a function $k_\alpha^0 \in L^1(\mathbb{R}^n)$ with $f \in L^p(\mathbb{R}^n)$. The integral II is the convolution of a function $f \in L^p(\mathbb{R}^n)$ with $k_\alpha^\infty \in L^{p'}(\mathbb{R}^n)$. Therefore, both integrals converge absolutely.

Also, using that

$$\int_{|y|<\varepsilon} \frac{dy}{|y|^{n-\alpha}} = c_n \int_0^{\varepsilon} \frac{r^{n-1}}{r^{n-\alpha}}\, dr = c_{\alpha,n}\, \varepsilon^{\alpha},$$

together with (2.8) in Proposition 2.4 we infer that

$$I \leq \varepsilon^{\alpha} \left(\frac{1}{\varepsilon^{\alpha}} \chi_{\{|y/\varepsilon|<1\}}(y) \frac{1}{|y|^{n-\alpha}} * |f| \right)(x) \leq c_{\alpha,n}\, \varepsilon^{\alpha} \mathcal{M} f(x). \tag{2.12}$$

On the other hand, Hölder's inequality implies that

$$\begin{aligned} II &\leq c_{\alpha,n}\, \|f\|_p \left(\int_{|y|\geq\varepsilon} \frac{1}{|y|^{(n-\alpha)p'}}\, dy \right)^{1/p'} \\ &= c_{\alpha,n}\, \|f\|_p \left(\int_{\varepsilon}^{\infty} \frac{r^{n-1}}{r^{(n-\alpha)p'}}\, dr \right)^{1/p'} \\ &= c_{\alpha,n}\, \varepsilon^{n/p'-n+\alpha} \|f\|_p. \end{aligned} \tag{2.13}$$

Next, we minimize the sum of the bounds in (2.12) and (2.13). Hence, we fix $\varepsilon = \varepsilon(x)$ such that

$$c\varepsilon^{\alpha} \mathcal{M} f(x) = c\varepsilon^{n/p'-n+\alpha} \|f\|_p,$$

using $n/p' - n = -n/p$. This is equivalent to

$$c\mathcal{M} f(x) = c\varepsilon^{-n/p} \|f\|_p. \tag{2.14}$$

Combining (2.11)–(2.14) we can write

$$\begin{aligned} |I_\alpha f(x)| &\leq c\, (\|f\|_p\, (\mathcal{M} f(x))^{-1})^{\alpha p/n}\, \mathcal{M} f(x) \\ &= c\, \|f\|_p^{\alpha p/n}\, (\mathcal{M} f(x))^{1-\alpha p/n} \\ &= c\, \|f\|_p^{\theta}\, (\mathcal{M} f(x))^{1-\theta}, \qquad \theta = \alpha p/n \in (0,1). \end{aligned} \tag{2.15}$$

Finally, taking the L^q-norm in (2.15) and using (2.6) we conclude:

$$\|I_\alpha f\|_q \leq c\|f\|_p^{\theta} \|(\mathcal{M} f)^{1-\theta}\|_q = c\|f\|_p^{\theta} \|\mathcal{M} f\|_{(1-\theta)q}^{1-\theta} \leq c\|f\|_p,$$

since $(1-\theta)q = (1-\alpha p/n)q = p$, i.e., $1/q = 1/p - \alpha/n$. This completes the proof. □

2.3 The Stein Interpolation Theorem

So far we have discussed interpolation theorems for fixed linear or sublinear operators. We now have to cover the following situation: Suppose we have linear operators varying together with the indices p and q smoothly. Is it possible to extend the Riesz–Thorin theorem to this case? The answer is affirmative and we shall describe this extension next.

Let S be the strip defined in Lemma 2.1 and $z = x + iy \in S$. Suppose that for each $z \in S$ there corresponds a linear operator T_z defined on the space of simple functions in $L^1(X, \mathcal{A}, \mu)$ into measurable functions on Y in such a way that $(T_z f)g$ is integrable on Y provided f is a simple function in $L^1(X, \mathcal{A}, \mu)$ and g is a simple function in $L^1(Y, \mathcal{B}, \nu)$.

Definition 2.7. The family of operators $\{T_z\}_{z\in S}$ is called *admissible* if the mapping

$$z \mapsto \int_Y (T_z f) g d\nu$$

is analytic in the interior of S, continuous on S and there exists a constant $a < \pi$ such that

$$e^{-a|y|} \log \left| \int_Y (T_z f) g d\nu \right|$$

is uniformly bounded above in the strip S.

Theorem 2.7 (Stein). *Suppose $\{T_z\}, z \in S$, is an admissible family of linear operators satisfying*

$$\|T_{iy} f\|_{q_0} \le M_0(y)\, \|f\|_{p_0} \quad \text{and} \quad \|T_{1+iy} f\|_{q_1} \le M_1(y)\, \|f\|_{p_1}, \quad y \in \mathbb{R}^n,$$

for all simple functions f in $L^1(X, \mathcal{A}, \mu)$, where $1 \le p_j, q_j \le \infty$, $M_j(y)$, $j = 0, 1$, are independent of f and satisfy

$$\sup_{-\infty<y<\infty} e^{-b|y|} \log M_j(y) < \infty$$

for some $b < \pi$. Then, if $0 \le t \le 1$, there exists a constant M_t such that

$$\|T_t f\|_{q_t} \le M_t\, \|f\|_{p_t}$$

for all simple functions f, provided

$$\frac{1}{p_t} = \frac{(1-t)}{p_0} + \frac{t}{p_1} \quad \text{and} \quad \frac{1}{q_t} = \frac{(1-t)}{q_0} + \frac{t}{q_1}.$$

Proof. For the proof of this theorem, we refer the reader to [SW]. □

2.4 A Multiplier Theorem

Let $m(\cdot)$ be a bounded measurable function in $\mathbb{R}^n$. Define the operator

$$T_m f(x) = (m(\cdot)\widehat{f}(\cdot))^{\vee}(x), \qquad f \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n). \tag{2.16}$$

Notice that if $\widehat{m}(x) = K(x)$, then, formally, $T_m f(x) = K * f(x)$. However, $K \in \mathcal{S}'(\mathbb{R}^n)$, i.e., a temperate distribution so $K * f$ is not necessarily defined.

As we have seen in (1.27), (1.29), and (1.49), solutions of the linear evolution equation can be written in this form.

Definition 2.8. An $m(\cdot)$ is said to be an L^p-*multiplier* if

$$\|T_m f\|_p \le c_p \|f\|_p, \quad \text{for all} \quad f \in L^2(\mathbb{R}^n) \cap L^p(\mathbb{R}^n). \tag{2.17}$$

In this case, $T_m(\cdot)$ can be extended to $L^p(\mathbb{R}^n)$. The smallest constant c_p^* in (2.17) is the operator norm of T_m in $L^p(\mathbb{R}^n)$, i.e., $|||T_m|||$ (see 2.1). Notice that if $p = 2$, one has $c_2^* = \|m\|_\infty$. Also, by duality, if $m(\cdot)$ is an L^p-multiplier, $1 < p < \infty$, then $m(\cdot)$ is an $L^{p'}$-multiplier with $\frac{1}{p} + \frac{1}{p'} = 1$, and $c_{p'}^* = c_p^*$.

Theorem 2.8 (Mihlin–Hörmander). *Let $m \in C^k(\mathbb{R}^n \setminus \{0\})$, $k \in Z^+$, $k > n/2$. If for $|\alpha| \le k$*

$$\sup_{R>0} R^{-n+|\alpha|} \int_{R<|\xi|<2R} |\partial_\xi^\alpha m(\xi)|^2 \, d\xi = A_\alpha < \infty, \tag{2.18}$$

then $m(\cdot)$ is an L^p-multiplier for any $p \in (1, \infty)$. Moreover, T_m is of weak type (1,1), i.e., for $\lambda > 0$

$$\lambda\, |\{x \in \mathbb{R}^n : |T_m f(x)| > \lambda\}| \le c\|f\|_1 \quad \textit{for all}\, f \in L^1(\mathbb{R}^n), \tag{2.19}$$

where $|A|$ denotes the Lebesgue measure of the set A.

Notice that if $m \in C^k(\mathbb{R}^n \setminus \{0\})$, $k \in \mathbb{Z}^+$, $k > n/2$ with

$$\sup_{x\neq 0} \sup_{|\alpha|\le k} |x|^{|\alpha|} |\partial_x^\alpha m(x)| = B_\alpha < \infty \quad \text{for} \quad |\alpha| \le k, \tag{2.20}$$

then (2.18) holds. Condition (2.20) is due to Mihlin, the weaker assumptions in (2.18) is due to Hörmander.

Combining a duality argument and the Marcinkiewicz interpolation theorem, it suffices to establish (2.19) to obtain Theorem 2.8. This is done in Appendix A.

2.5 Exercises

2.1 Prove the continuity part of Theorem 2.1 (Riesz–Thorin) in the cases $p_0 = p_1$ and $q_0 = q_1$.

2.2 Prove Proposition 2.1.
2.3 Prove Proposition 2.2.
2.4 Prove Proposition 2.3.
2.5 (i) Prove that the Fourier transform defines a continuous operator from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ only if $1/p + 1/q = 1$ with $q \geq p$.
(ii) Prove that for $1 \leq p < 2$

$$\widehat{L^p(\mathbb{R}^n)} \subsetneqq L^q(\mathbb{R}^n).$$

Hint: Use Exercise 1.2(ii) and the open mapping theorem.
2.6 (i) Prove the Lebesgue differentiation theorem: If $f \in L^1_{\text{loc}}(\mathbb{R}^n)$, then for almost every $x \in \mathbb{R}^n$

$$\lim_{r \to 0} \frac{1}{|B_r(x)|} \int\limits_{B_r(x)} f(y)\, dy = f(x). \tag{2.21}$$

Hint: Without loss of generality take $f \in L^1(\mathbb{R}^n)$. Define $O(f, x)$ the oscillation of f at x as

$$O(f, x) = \left| \limsup_{r \to 0} \frac{1}{|B_r(x)|} \int\limits_{B_r(x)} f(y)\, dy - \liminf_{r \to 0} \frac{1}{|B_r(x)|} \int\limits_{B_r(x)} f(y)\, dy \right|.$$

Prove that (2.21) is equivalent to $O(f, x) = 0$. Use that

$$\lim_{r \to 0} \frac{1}{|B_r(0)|} \chi_{B_r(0)} * f = f \quad \text{in} \quad L^1(\mathbb{R}^n);$$

therefore, there exists a sequence $\{r_j\}$ such that

$$\lim_{j \to 0} \frac{1}{|B_{r_j}(0)|} \chi_{B_{r_j}(0)} * f(x) = f(x) \quad \text{almost everywhere } x \in \mathbb{R}^n.$$

Combine (2.7), the inequality $O(f, x) \leq 2\mathcal{M} f(x)$, and a density argument to obtain the result.
(ii) Let $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ and Q_j be a sequence of closed cubes in $\mathbb{R}^n$ such that $Q_1 \supseteq Q_2 \supseteq \ldots$, $|Q_1| < \infty$ and $|Q_j| = 2^n\, |Q_{j+1}|$. If $x \in \bigcap_{j=1}^{\infty} Q_j$ prove that

$$\lim_{j \to \infty} \frac{1}{|Q_j|} \int\limits_{Q_j} f(y)\, dy = f(x). \tag{2.22}$$

Hint: Define

$$\mathcal{M}^* f(x) = \sup_{\substack{Q \text{ cube} \\ x \in Q}} \frac{1}{|Q|} \int\limits_{Q} |f(y)|\, dy. \tag{2.23}$$

Show that there exist $c_n, d_n > 0$ such that

$$d_n\,\mathcal{M}f(x) \le \mathcal{M}^* f(x) \le c_n\,\mathcal{M}f(x),$$

and reapply the argument in (i).

2.7 Assuming to be true the case $n = 1$ of the Hardy–Littlewood–Sobolev inequality (2.10) prove the general case $n \ge 2$.
Hint: Combine the Hölder, Young, and Minkowski inequalities with the identity

$$\int_{\mathbb{R}^{n-1}} \frac{dy_1 \cdots dy_{n-1}}{|x-y|^n} = \frac{c_n}{|x_n - y_n|}\,.$$

2.8 Prove that the Hilbert transform (see Definition 1.7) is of type (p,q) if and only if $1 < p = q < \infty$.
Hint: (a) The identity (1.18) provides the result for the case $p = 2$. Use the formula deduced in Exercise 1.15 part (i) with $f = g$ to prove the result in the case $p = 4$. Apply the Riesz–Thorin interpolation theorem to extend the result to $2 < p < 4$. Reapply this argument to obtain the proof for $p > 2$. Finally, use duality to complete the proof.
(b) Otherwise use Theorem 2.8.
(c) Use (1.5) and part (ii) of Exercise 1.15.

2.9 Prove that the Riesz potential of order α, I_α, $\alpha \in (0,n)$ defines a bounded operator from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ only if $1 < p < q < \infty$, with $1/q = 1/p - \alpha/n$.
Hint: Prove the formula $\delta_{a^{-1}} I_\alpha \delta_a = a^{-\alpha} I_\alpha$, where $\delta_a f(x) = f(ax)$. Show that the value of the norms of $\delta_a f(x)$ and $\delta_{a^{-1}} I_\alpha \delta_a f$ give the relation $1/q = 1/p - \alpha/n$. To see that the inequality does not hold for the extremal cases $p = 1$ and $q = n/(n-\alpha)$, use an approximation of the identity instead of f (case $p = 1$). For the case $q = n/\alpha$, use duality.

2.10 Prove that the multipliers

$$m_j(\xi) = \frac{i\xi_j}{|\xi|},\ j = 1,\ldots,n, \quad \text{(the } j\text{-Riesz transform)}$$

and

$$m_y(\xi) = |\xi|^{iy}, \quad y \in \mathbb{R},$$

are L^p-multipliers with $1 < p < \infty$.
Hint: Use condition (2.20).

2.11 Let $s > 0$ and $\rho \in (0,s)$:

(i) Prove that for any $p \in (1,\infty)$

$$\|D^\rho f\|_p \le c\,\|f\|_p^{1-\rho/s}\,\|D^s f\|_p^{\rho/s}\ \ f \in \mathcal{S}(\mathbb{R}^n). \tag{2.24}$$

(ii) More general, prove that for any $p,q,r \in (1,\infty)$

$$\|D^\rho f\|_p \le c\,\|f\|_r^{1-\rho/s}\,\|D^s f\|_q^{\rho/s}, \quad f \in \mathcal{S}(\mathbb{R}^n), \tag{2.25}$$

with

$$\frac{1}{p} = \left(1 - \frac{\rho}{s}\right)\frac{1}{r} + \left(\frac{\rho}{s}\right)\frac{1}{q}. \tag{2.26}$$

(iii) Prove that the estimates (2.24) and (2.25) still hold with $\Lambda = (1-\Delta)^{1/2}$ instead of D, and that in both cases the proof for $p = q = r = 2$ is immediate. Prove that $r = \infty$ is allowed in (2.25).

Hint: For (ii) fix $f \in \mathcal{S}(\mathbb{R}^n)$, use that

$$\|D^\rho f\|_p = \sup_{\|g\|_{p'}=1} \Big| \int_{\mathbb{R}^n} D^\rho f(y) g(y) dy \Big|$$

and define

$$F_k(z) = e^{(z^2-1)/k} \int_{\mathbb{R}^n} D^{sz} f(y)\, \Psi(y,z)\, dy, \quad \text{for } z = x + iy \text{ with } 0 \le x \le 1,$$

where

$$\psi(y,z) = |g(y)|^{p'/q(z)} \frac{g(y)}{|g(y)|} \quad \text{and} \quad \frac{1}{q(z)} = \frac{1-z}{r'} + \frac{z}{q'}$$

with $\dfrac{1}{p} + \dfrac{1}{p'} = \dfrac{1}{r} + \dfrac{1}{r'} = \dfrac{1}{q} + \dfrac{1}{q'} = 1$. Verify that $F_k(\cdot)$ satisfies the hypotheses of Lemma 2.1 using Theorem 2.8 (see Exercise 2.10). Let k tend to infinity to get the result.

2.12 [Pi] Pitt's Theorem affirms: if $1 < p \le q < \infty$,

$$0 \le \alpha < n\left(1 - \frac{1}{p}\right), \quad 0 \le \gamma < \frac{n}{q}, \quad \alpha - \gamma = n\left(1 - \frac{1}{q} - \frac{1}{p}\right),$$

then there exists $c > 0$ such that

$$\|\widehat{f}\, |x|^{-\gamma}\|_q \le c\, \|f\, |x|^\alpha\|_p \tag{2.27}$$

with:

(i) Prove (2.27) in the case $\alpha = 0$ and $q \ge 2$.
(ii) Prove (2.27) in the case $\gamma = 0$ and $p \le 2$.

2.13 For the initial value problem associated to the heat equation:

$$\begin{cases} \partial_t u = \Delta u, \\ u(x,0) = f(x), \end{cases}$$

$x \in \mathbb{R}^n$, $t > 0$, prove that the solution $u(x,t) = e^{t\Delta} f(x)$ satisfies the following inequalities:

(i)

$$\|D_x^s u(\cdot,t)\|_p \le c_s\, t^{-(\frac{n}{2r}+\frac{s}{2})}\, \|f\|_q, \tag{2.28}$$

for $s \ge 0$ and

$$\frac{1}{p} = \frac{1}{q} - \frac{1}{r}.$$

(ii)

$$\left(\int_0^\infty \|D_x^\rho u(\cdot,t)\|_p^\sigma\, dt\right)^{1/\sigma} \le c\, \|f\|_q \tag{2.29}$$

with $\rho \in [0,2)$ and

$$0 < \frac{1}{\sigma} = \frac{n}{2}\left(\frac{1}{q} - \frac{1}{p}\right) + \frac{\rho}{2} \le \frac{1}{q}, \quad \text{(see [G1])}.$$

Hint: For (i) use Example 1.3 to deduce that

$$u(x,t) = K_t * f(x) = \frac{e^{-|\cdot|^2/4t}}{(4\pi t)^{n/2}} * f(x).$$

Obtain the identity $\|D_x^s K_t\|_\infty = c_s t^{-(n/2+s/2)}$ for $s > 0$ and combine it with Young's inequality to obtain (2.28).

For (ii) define $(\Omega f)(t) = \|D_x^\rho e^{t\Delta} f\|_p$. Then by (2.28) ,$(\Omega f)(t) \le c\, t^{-1/\sigma}\, \|f\|_q$, $t \in (0,\infty)$. Hence, the sublinear operator Ω is bounded from $L^q(\mathbb{R}^n)$ into $L^{\sigma *}((0,\infty))$, (i.e., L^σ-weak). Use Marcinkiewicz interpolation theorem to get (2.29).

2.14 Consider the initial value problem (IVP) associated to the wave equation:

$$\begin{cases} \partial_t^2 w - \Delta w = 0, \\ w(x,0) = f(x), \\ \partial_t w(x,0) = g(x), \end{cases}$$

$x \in \mathbb{R}^n$, $t \in \mathbb{R}$, prove that

(i) If $n = 1$, then

$$w(x,t) = \frac{f(x+t) + f(x-t)}{2} + \frac{1}{2}\int_{x-t}^{x+t} g(s)\, ds.$$

Hint: Use the formula deduced in Exercise 1.18(i) or the change of variables $\zeta = x + t,\ \eta = x - t$.

(ii) If $n = 3$, $f = 0$ and g is a radial function ($g(\|x\|)$), then

$$w(x,t) = w(\|x\|,t) = \frac{1}{2\|x\|} \int_{|\,\|x\|-t|}^{\|x\|+t} \rho g(\rho)\, d\rho.$$

Hint: Deduce the formula for the Laplacian of radial functions, use the change of variables

$$v(\rho,t) = \rho\, w(\rho,t) = \|x\|\, w(\|x\|,t)$$

and part (i) of this exercise.
(iii) Under the same hypotheses of part (ii) use the Hardy–Littlewood maximal function to show that

$$\left(\int_{-\infty}^{\infty} \|w(\cdot,t)\|_\infty^2\, dt \right)^{1/2} \leq c\, \|g\|_2. \tag{2.30}$$

In [KlM], it was established that (2.30) does not hold for nonradial functions g.

2.15 Let $m_1,\ m_2$ be two L^p-multipliers. Prove

(i) $T_{m_1} \circ T_{m_2} = T_{m_1 \cdot m_2}$.
(ii) $(T_{m_1})^* = T_{\overline{m}_1}$.

2.16 (i) Prove that if $n = 3$, then for any $t \neq 0$

$$m_t(\xi) = \cos(2\pi\, |\xi| t), \quad T_{m_t} f(x) = (m_t(\cdot)\widehat{f}(\cdot))^{\vee}(x) \tag{2.31}$$

is not an L^p-multiplier for $p \neq 2$.

(ii) Prove that if $n = 3$, then (see 3.38)

$$\|T_{m_t} f\|_\infty \leq c t^{-1}\, \|\nabla f\|_{1,2}, \quad \text{for any}\ t \neq 0.$$

(iii) Prove that if $n = 1$, then $m_t(\xi) = \cos(2\pi\, |\xi| t)$ for each $t \in \mathbb{R}$ is an L^p-multiplier for $1 \leq p \leq \infty$.
(Part (i) holds in any dimension $n \geq 2$. See [Lp]).
Hint: Notice that $T_{m_t} f(x) = (m_t(\cdot)\, \widehat{f}\,)^{\vee}(x)$ is the solution $u(x,t)$ of the IVP

$$\begin{cases} \partial_t^2 u - \Delta u = 0, \\ u(x,0) = f(x), \\ \partial_t u(x,0) = 0, \end{cases} \tag{2.32}$$

$x \in \mathbb{R}^3$, $t > 0$. So the formula (1.50) in Exercise 1.18(iv) applies. Take $f(x) = h(|x|)/|x| = h(r)/r$, with $h(\cdot)$ supported in the annulus $\{x \in \mathbb{R}^3 :\ \varepsilon \leq |x| \leq 2\varepsilon\}$. Check that $u(x,t) = (h(r+t) + h(r-t))/2r$, and derive the desired result.

Chapter 3
An Introduction to Sobolev Spaces and Pseudo-Differential Operators

In this chapter, we give a brief introduction to the classical Sobolev spaces $H^s(\mathbb{R}^n)$. Sobolev spaces measure the differentiability (or regularity) of functions in $L^2(\mathbb{R}^n)$ and they are a fundamental tool in the study of partial differential equations. We also list some basic facts of the theory of pseudo-differential operators without proof. This is useful to study smoothness properties of solutions of dispersive equations.

3.1 Basics

We begin by defining Sobolev spaces.

Definition 3.1. Let $s \in \mathbb{R}$. We define the *Sobolev space* of order s, denoted by $H^s(\mathbb{R}^n)$, as:

$$H^s(\mathbb{R}^n)=\left\{f \in \mathcal{S}'(\mathbb{R}^n) : \Lambda^s f(x)=((1+|\xi|^2)^{s/2}\widehat{f}(\xi))^{\vee}(x) \in L^2(\mathbb{R}^n)\right\}, \tag{3.1}$$

with norm $\|\cdot\|_{s,2}$ defined as:

$$\|f\|_{s,2} = \|\Lambda^s f\|_2. \tag{3.2}$$

Example 3.1 Let $n = 1$ and $f(x) = \chi_{[-1,1]}(x)$. From Example 1.1, we have that $\widehat{f}(\xi) = \sin(2\pi\xi)/(\pi\xi)$. Thus, $f \in H^s(\mathbb{R})$ if $s < 1/2$.

Example 3.2 Let $n = 1$ and $g(x) = \chi_{[-1,1]} * \chi_{[-1,1]}(x)$. In Example 1.2, we saw that

$$\widehat{g}(\xi) = \frac{\sin^2(2\pi\,\xi)}{(\pi\,\xi)^2}.$$

Thus, $g \in H^s(\mathbb{R})$ whenever $s < 3/2$.

Example 3.3 Let $n \geq 1$ and $h(x) = e^{-2\pi|x|}$. From Example 1.4, it follows that

$$\widehat{h}(\xi) = \frac{\Gamma[(n+1)/2]}{\pi^{(n+1)/2}}\,\frac{1}{(1+|\xi|^2)^{(n+1)/2}}. \tag{3.3}$$

F. Linares, G. Ponce, *Introduction to Nonlinear Dispersive Equations*,
Universitext, DOI 10.1007/978-1-4939-2181-2_3

Using polar coordinates, it is easy to see that $h \in H^s(\mathbb{R}^n)$ if $s < n/2 + 1$. Notice that in this case s depends on the dimension.

Example 3.4 Let $n \geq 1$ and $f(x) = \delta_0(x)$. From Example 1.9, we have $\widehat{\delta_0}(\xi) = 1$. Thus, $\delta_0 \in H^s(\mathbb{R}^n)$ if $s < -n/2$.

From the definition of Sobolev spaces, we deduce the following properties.

Proposition 3.1.

1. *If* $s < s'$, *then* $H^{s'}(\mathbb{R}^n) \subsetneq H^s(\mathbb{R}^n)$.
2. $H^s(\mathbb{R}^n)$ *is a Hilbert space with respect to the inner product* $\langle \cdot, \cdot \rangle_s$ *defined as follows:*

$$\text{If } f, g \in H^s(\mathbb{R}^n), \text{ then } \langle f, g \rangle_s = \int_{\mathbb{R}^n} \Lambda^s f(\xi) \, \overline{\Lambda^s g(\xi)} \, d\xi.$$

We can see, via the Fourier transform, that $H^s(\mathbb{R}^n)$ *is equal to:*

$$L^2(\mathbb{R}^n; (1 + |\xi|^2)^s \, d\xi).$$

3. *For any* $s \in \mathbb{R}$, *the Schwartz space* $\mathcal{S}(\mathbb{R}^n)$ *is dense in* $H^s(\mathbb{R}^n)$.
4. *If* $s_1 \leq s \leq s_2$, *with* $s = \theta s_1 + (1 - \theta)s_2$, $0 \leq \theta \leq 1$, *then*

$$\|f\|_{s,2} \leq \|f\|_{s_1,2}^{\theta} \|f\|_{s_2,2}^{1-\theta}.$$

Proof. It is left as an exercise. □

To understand the relationship between the spaces $H^s(\mathbb{R}^n)$ and the differentiability of functions in $L^2(\mathbb{R}^n)$, we recall Definition 1.2 in the case $p = 2$.

Definition 3.2. A function f is *differentiable* in $L^2(\mathbb{R}^n)$ with respect to the kth variable, if there exists $g \in L^2(\mathbb{R}^n)$ such that

$$\int_{\mathbb{R}^n} \Big| \frac{f(x + h\, e_k) - f(x)}{h} - g(x) \Big|^2 dx \to 0 \text{ when } h \to 0,$$

where e_k has kth coordinate equal to 1 and 0 in the others.

Equivalently (see Exercise 1.9) $\xi_k \widehat{f}(\xi) \in L^2(\mathbb{R}^n)$, or

$$\int_{\mathbb{R}^n} f(x) \partial_{x_k} \phi(x) \, dx = -\int_{\mathbb{R}^n} g(x) \phi(x) \, dx$$

for every $\phi \in C_0^\infty(\mathbb{R}^n)$ ($C_0^\infty(\mathbb{R}^n)$ being the space of functions infinitely differentiable with compact support).

Example 3.5 Let $n = 1$ and $f(x) = \chi_{(-1,1)}(x)$, then $f' = \delta_{-1} - \delta_1$, where δ_x represents the measure of mass 1 concentrated in x, therefore $f' \notin L^2(\mathbb{R})$.

Example 3.6 Let $n = 1$ and g be as in Example 3.2. Then,

$$\frac{dg}{dx}(x) = \chi_{(-2,0)} - \chi_{(0,2)}, \quad \text{and so} \quad \frac{dg}{dx} \in L^2(\mathbb{R}).$$

With this definition, for $k \in \mathbb{Z}^+$ we can give a description of the space $H^k(\mathbb{R}^n)$ without using the Fourier transform.

Theorem 3.1. *If k is a positive integer, then $H^k(\mathbb{R}^n)$ coincides with the space of functions $f \in L^2(\mathbb{R}^n)$ whose derivatives (in the distribution sense, see (1.42)) $\partial_x^\alpha f$ belong to $L^2(\mathbb{R}^n)$ for every $\alpha \in (\mathbb{Z}^+)^n$ with $|\alpha| = \alpha_1 + \cdots + \alpha_n \leq k$.*

In this case, the norms $\|f\|_{k,2}$ and $\sum_{|\alpha|\leq k} \|\partial_x^\alpha f\|_2$ are equivalent.

Proof. The proof follows by combining the formula $\widehat{\partial_x^\alpha f}(\xi) = (2\pi i \xi)^\alpha \widehat{f}(\xi)$ (see (1.10)) and the inequalities:

$$|\xi^\beta| \leq (1 + |\xi|^2)^{k/2} \leq \sum_{|\alpha|\leq k} |\xi^\alpha|, \qquad \beta \in (\mathbb{Z}^+)^n, \ |\beta| \leq k. \qquad \square$$

Theorem 3.1 allows us to define in a natural manner $H^k(\Omega)$, the Sobolev space of order $k \in \mathbb{Z}^+$ in any subset Ω (open) of $\mathbb{R}^n$. Given $f \in L^2(\Omega)$, we say that $\partial_x^\alpha f$, $\alpha \in (\mathbb{Z}^+)^n$ is the αth partial derivative (in the distribution sense) of f, if for every $\phi \in C_0^\infty(\Omega)$

$$\int_\Omega f \, \partial_x^\alpha \phi \, dx = (-1)^{|\alpha|} \int_\Omega \partial_x^\alpha f \, \phi \, dx.$$

Then,

$$H^k(\Omega) = \{f \in L^2(\Omega) \ : \ \partial_x^\alpha f (\text{in the distribution sense}) \in L^2(\Omega), \ |\alpha| \leq k\}$$

with the norm

$$\|f\|_{H^k(\Omega)} \equiv \Big(\sum_{|\alpha|\leq k} \int_\Omega |\partial_x^\alpha f(x)|^2 \, dx\Big)^{1/2}.$$

Example 3.7 For $n = 1$, $b > 0$, and $f(x) = |x|$, one has that $f \in H^1((-b, b))$ and $f \notin H^2((-b, b))$.

The next result allows us to relate "weak derivatives" with derivatives in the classical sense.

Theorem 3.2 (Embedding). *If $s > n/2 + k$, then $H^s(\mathbb{R}^n)$ is continuously embedded in $C_\infty^k(\mathbb{R}^n)$, the space of functions with k continuous derivatives vanishing at infinity. In other words, if $f \in H^s(\mathbb{R}^n)$, $s > n/2 + k$, then (after a possible modification of f in a set of measure zero) $f \in C_\infty^k(\mathbb{R}^n)$ and*

$$\|f\|_{C^k} \leq c_s \, \|f\|_{s,2}. \tag{3.4}$$

Proof. Case $k = 0$: We first show that if $f \in H^s(\mathbb{R}^n)$, then $\widehat{f} \in L^1(\mathbb{R}^n)$ with

$$\|\widehat{f}\|_1 \le c_s \|f\|_{s,2}, \quad \text{if } s > n/2. \tag{3.5}$$

Using the Cauchy–Schwarz inequality, we deduce:

$$\begin{aligned}\int_{\mathbb{R}^n} |\widehat{f}(\xi)| d\xi &= \int_{\mathbb{R}^n} |\widehat{f}(\xi)|(1+|\xi|^2)^{s/2} \frac{d\xi}{(1+|\xi|^2)^{s/2}} \\ &\le \|\Lambda^s f\|_2 \left(\int_{\mathbb{R}^n} \frac{d\xi}{(1+|\xi|^2)^s} \right)^{1/2} \le c_s \|f\|_{s,2}\end{aligned}$$

if $s > n/2$. Combining (3.5), Proposition 1.2, and Theorem 1.1, we conclude that

$$\|f\|_\infty = \|(\widehat{f})^\vee\|_\infty \le \|\widehat{f}\|_1 \le c_s \|f\|_{s,2}.$$

Case $k \ge 1$: Using the same argument, we have that if $f \in H^s(\mathbb{R}^n)$ with $s > n/2 + k$, then for $\alpha \in (\mathbb{Z}^+)^n$, $|\alpha| \le k$, it follows that $\widehat{\partial_x^\alpha f} \in L^1(\mathbb{R}^n)$ and

$$\|\partial_x^\alpha f\|_\infty \le \|\widehat{\partial_x^\alpha f}\|_1 = \|(2\pi i \xi)^\alpha \widehat{f}\|_1 \le c_s \|f\|_{s,2}.$$

□

Corollary 3.1. *If $s = n/2 + k + \theta$, with $\theta \in (0,1)$, then $H^s(\mathbb{R}^n)$ is continuously embedded in $C^{k+\theta}(\mathbb{R}^n)$, the space of C^k functions with partial derivatives of order k Hölder continuous with index θ.*

Proof. We only prove the case $k = 0$, since the proof of the general case follows the same argument. From the formula of inversion of the Fourier transform and the Cauchy–Schwarz inequality we have:

$$\begin{aligned}|f(x+y) - f(x)| &= \left| \int_{\mathbb{R}^n} e^{2\pi i (x\cdot\xi)} \widehat{f}(\xi)(e^{2\pi i (y\cdot\xi)} - 1)\, d\xi \right| \\ &\le \left(\int_{\mathbb{R}^n} (1+|\xi|^2)^{n/2+\theta} |\widehat{f}(\xi)|^2\, d\xi \right)^{1/2} \left(\int_{\mathbb{R}^n} \frac{|e^{2\pi i (y\cdot\xi)} - 1|^2}{(1+|\xi|^2)^{n/2+\theta}}\, d\xi \right)^{1/2}.\end{aligned}$$

But

$$\begin{aligned}&\int_{\mathbb{R}^n} \frac{|e^{2\pi i (y\cdot\xi)} - 1|^2}{(1+|\xi|^2)^{n/2+\theta}}\, d\xi \\ &\qquad \le c \int_{|\xi|\le|y|^{-1}} |y|^2|\xi|^2 \frac{d\xi}{(1+|\xi|^2)^{n/2+\theta}} + 4 \int_{|\xi|\ge|y|^{-1}} \frac{d\xi}{(1+|\xi|^2)^{n/2+\theta}}\end{aligned}$$

$$\leq c|y|^2 \int_0^{|y|^{-1}} \frac{r^{n+1}}{(1+r)^{n+2\theta}}\,dr + 4\int_{|y|^{-1}}^{\infty} \frac{r^{n-1}}{(1+r)^{n+2\theta}}\,dr \;\leq\; c\,|y|^{2\theta}.$$

If $|y| < 1$, we conclude that $|f(x+y) - f(x)| \leq c\,|y|^{\theta}$. This finishes the proof.□

Theorem 3.3. *If $s \in (0, n/2)$, then $H^s(\mathbb{R}^n)$ is continuously embedded in $L^p(\mathbb{R}^n)$ with $p = 2n/(n-2s)$, i.e., $s = n(1/2 - 1/p)$. Moreover, for $f \in H^s(\mathbb{R}^n)$, $s \in (0, n/2)$,*

$$\|f\|_p \leq c_{n,s}\,\|D^s f\|_2 \leq c\|f\|_{s,2}, \tag{3.6}$$

where

$$D^l f = (-\Delta)^{l/2} f = ((2\pi|\xi|)^l\,\widehat{f}\,)^{\vee}.$$

Proof. The last inequality in (3.6) is immediate, so we just need to show the first one. We define

$$D^s f = g \quad \text{or} \quad f = D^{-s} g = c_{n,s}\left(\frac{1}{|\xi|^s}\,\widehat{g}\right)^{\vee} = \frac{c_{n,s}}{|x|^{n-s}} * g, \tag{3.7}$$

where we have used the result of Exercise 1.14. Thus, by the Hardy–Littlewood–Sobolev estimate (2.10) it follows that

$$\|f\|_p = \|D^{-s} g\|_p = \|\frac{c_{n,s}}{|x|^{n-s}} * g\|_p \leq c_{n,s}\,\|g\|_2 = c\|D^s f\|_2. \tag{3.8}$$

□

We notice from Theorems 3.2 and 3.3, and Corollary 3.1 that the local regularity in H^s, $s > 0$, increases with the parameter s.

Examples 3.1 and 3.3 show that the functions in $H^s(\mathbb{R}^n)$ with $s < n/2$ or $s < n/2 + 1$, respectively, are not necessarily continuous nor C^1. Moreover, let $f \in L^2(\mathbb{R}^n)$ with

$$\widehat{f}(\xi) = \frac{1}{(1+|\xi|)^n\ \log(2+|\xi|)}$$

(which is radial, decreasing, and positive). A simple computation shows that $f \in H^{\frac{n}{2}}(\mathbb{R}^n)$, but $\widehat{f} \notin L^1(\mathbb{R}^n)$ and so $f \notin L^\infty(\mathbb{R}^n)$, since $f(0) = \int \widehat{f}(\xi)d\xi = \infty$ (see also Exercise 3.11(iii)).

To complete the embedding results of the spaces $H^s(\mathbb{R}^n)$, $s > 0$, it remains to consider the case $s = n/2$ (since for $s = k + n/2$, $k \in \mathbb{Z}^+$, the result follows from this one). So, we define the space of functions of the bounded mean oscillation or BMO, introduced by John and Nirenberg [JN].

Definition 3.3. For $f : \mathbb{R}^n \to \mathbb{C}$ with $f \in L^1_{\text{loc}}(\mathbb{R}^n)$, we say that $f \in \text{BMO}(\mathbb{R}^n)$ (f has bounded mean oscillation (BMO)) if

$$\|f\|_{\text{BMO}} = \sup_{\substack{x\in\mathbb{R}^n \\ r>0}} \frac{1}{|B_r(x)|} \int_{B_r(x)} |f(y) - f_{B_r(x)}|\,dy < \infty, \tag{3.9}$$

where

$$f_{B_r(x)} = \frac{1}{|B_r(x)|} \int_{B_r(x)} f(y)dy.$$

Notice that $\|\cdot\|_{\mathrm{BMO}}$ is a semi-norm since it vanishes for constant functions.

$\mathrm{BMO}(\mathbb{R}^n)$ is a vector space with $L^\infty(\mathbb{R}^n) \subsetneq \mathrm{BMO}(\mathbb{R}^n)$ since $\|f\|_{\mathrm{BMO}} \le 2\|f\|_\infty$ and $\log|x| \in \mathrm{BMO}(\mathbb{R}^n)$.

Theorem 3.4. *$H^{n/2}(\mathbb{R}^n)$ is continuously embedded in* $\mathrm{BMO}(\mathbb{R}^n)$. *More precisely, there exists $c = c(n) > 0$ such that*

$$\|f\|_{\mathrm{BMO}} \le c\,\|D^{n/2} f\|_2.$$

Proof. Without loss of generality, we assume f real valued. Consider $x \in \mathbb{R}^n$ and $r > 0$.

Let $\phi_r \in C_0^\infty(\mathbb{R}^n)$ such that $\operatorname{supp}\phi_r \subseteq \{x \mid |x| \le \frac{2}{r}\}$ with $0 \le \phi_r(x) \le 1$ and $\phi_r(x) \equiv 1$ if $|x| < 1/r$, and define

$$f(x) = f_l + f_h = (\widehat{f}\phi_r)^\vee + (\widehat{f}(1-\phi_r))^\vee.$$

We observe that

$$\|f\|_{\mathrm{BMO}} \le \|f_l\|_{\mathrm{BMO}} + \|f_h\|_{\mathrm{BMO}}$$

and $f_l \in H^s(\mathbb{R}^n)$ for any $s > 0$; therefore,

$$f_{l,B_r(x)} = \frac{1}{|B_r(x)|} \int_{B_r(x)} f_l(y)\,dy = f_l(x_0)$$

for some $x_0 \in B_r(x)$, and so for any $y \in B_r(x)$

$$|f_l(y) - f_{l,B_r(x)}| \le 2r\,\|\nabla f_l\|_\infty.$$

Using this estimate we get:

$$\begin{aligned}
\frac{1}{|B_r(x)|} \int_{B_r(x)} \Big|f_l(y) - f_{l,B_r(x)}\Big| dy &\le \frac{1}{|B_r(x)|^{1/2}} \Big(\int_{B_r(x)} |f_l(y) - f_{l,B_r(x)}|^2\,dy\Big)^{1/2} \\
&\le 2r\,\|\nabla f_l\|_\infty \le 2r\,\|\widehat{\nabla f_l}\|_1 \\
&\le 2r \int_{|\xi|\le 1/2r} |\xi|^{1-n/2} |\xi|^{n/2} |\widehat{f}(\xi)|\,d\xi \\
&\le 2r \Big(\int_{|\xi|\le 1/2r} |\xi|^{2-n}\,d\xi\Big)^{1/2} \|D^{n/2} f\|_2 \le c\|D^{n/2} f\|_2.
\end{aligned}$$

Also,

$$\frac{1}{|B_r(x)|} \int_{B_r(x)} |f_h(y) - f_{h,B_r(x)}| dy \le \frac{2}{|B_r(x)|^{1/2}} \|f_h\|_2$$

$$\leq \frac{2}{|B_r(x)|^{1/2}} \left(\int_{|\xi| \geq 1/2r} |\widehat{f}(\xi)|^2 \, d\xi \right)^{1/2}$$
$$= \frac{c_n}{r^{n/2}} \left(\int_{|\xi| \geq 1/2r} r^n |\xi|^n |\widehat{f}(\xi)|^2 \, d\xi \right)^{1/2} \leq \|D^{n/2} f\|_2,$$

which yields the desired result. □

We have shown that $H^s(\mathbb{R}^n)$ with $s > n/2$ is a Hilbert space whose elements are continuous functions. From the point of view of nonlinear analysis, the next property is essential.

Theorem 3.5. *If $s > n/2$, then $H^s(\mathbb{R}^n)$ is an algebra with respect to the product of functions. That is, if $f, g \in H^s(\mathbb{R}^n)$, then $fg \in H^s(\mathbb{R}^n)$ with*

$$\|fg\|_{s,2} \leq c_s \|f\|_{s,2} \|g\|_{s,2}. \tag{3.10}$$

Proof. From the triangle inequality, we have that for every $\xi, \eta \in \mathbb{R}^n$:

$$(1 + |\xi|^2)^{s/2} \leq 2^s [(1 + |\xi - \eta|^2)^{s/2} + (1 + |\eta|^2)^{s/2}].$$

Using this we deduce that

$$\begin{aligned}
|\Lambda^s(fg)| &= |(1 + |\xi|^2)^{s/2} \widehat{(fg)}(\xi)| \\
&= (1 + |\xi|^2)^{s/2} \left| \int_{\mathbb{R}^n} \widehat{f}(\xi - \eta) \widehat{g}(\eta) \, d\eta \right| \\
&\leq 2^s \int_{\mathbb{R}^n} \Big[(1 + |\xi - \eta|^2)^{s/2} \, |\widehat{f}(\xi - \eta) \widehat{g}(\eta)| \\
&\qquad + (1 + |\eta|^2)^{s/2} \, |\widehat{f}(\xi - \eta) \widehat{g}(\eta)| \Big] \, d\eta \\
&\leq 2^s (|\widehat{\Lambda^s f}| * |\widehat{g}| + |\widehat{f}| * |\widehat{\Lambda^s g}|).
\end{aligned}$$

Thus, taking the L^2-norm and using (1.39) it follows that

$$\|fg\|_{s,2} = \|\Lambda^s(fg)\|_2 \leq c(\|\Lambda^s f\|_2 \|\widehat{g}\|_1 + \|\widehat{f}\|_1 \|\Lambda^s g\|_2). \tag{3.11}$$

Finally, (3.5) assures one that if $r > n/2$, then

$$\begin{aligned}
\|fg\|_{s,2} &\leq c_s(\|f\|_{s,2} \|\widehat{g}\|_1 + \|\widehat{f}\|_1 \|g\|_{s,2}) \\
&\leq c_s(\|f\|_{s,2} \|g\|_{r,2} + \|f\|_{r,2} \|g\|_{s,2}).
\end{aligned} \tag{3.12}$$

Choosing $r = s$, we obtain (3.10). □

The inequality (3.12) is not sharp as the following scaling argument shows. Let $\lambda > 0$ and

$$f(x) = f_1(\lambda x), \quad g(x) = g_1(\lambda x), \quad f_1, \, g_1 \in \mathcal{S}(\mathbb{R}^n).$$

Then, as $\lambda \uparrow \infty$ the right-hand side of (3.12) grows as λ^{s+r}, meanwhile the left-hand side grows as λ^s. This will not be the case if we replace $\|\cdot\|_{r,2}$ in (3.12) with the $\|\cdot\|_\infty$-norm to get that

$$\|fg\|_{s,2} \leq c_s(\|f\|_{s,2}\,\|g\|_\infty + \|f\|_\infty\,\|g\|_{s,2}) \tag{3.13}$$

which in particular shows that for any $s > 0$, $H^s(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ is an algebra under the point-wise product.

For $s \in \mathbb{Z}^+$, the inequality (3.13) follows by combining the Leibniz rule for the product of functions and the Gagliardo–Nirenberg inequality:

$$\|\partial_x^\alpha f\|_p \;\leq\; c \sum_{|\beta|=m} \|\partial_x^\beta f\|_q^\theta\;\|f\|_r^{1-\theta} \tag{3.14}$$

with $|\alpha| = j$, $c = c(j,m,p,q,r)$, $1/p - j/n = \theta(1/q - m/n) + (1-\theta)1/r$, $\theta \in [j/m, 1]$. For the proof of this inequality, we refer the reader to the reference [Fm].

For the general case $s > 0$, where the usual point-wise Leibniz rule is not available, the inequality (3.13) still holds (see [KPo]). The inequality (3.13) has several extensions, for instance: Let $s \in (0,1)$, $r \in [1,\infty)$, $1 < p_j, q_j \leq \infty$, $1/r = 1/p_j + 1/q_j$, $j = 1,2$. Then,

$$\|\Phi^s(fg)\|_r \leq c(\|\Phi^s(f)\|_{p_1}\|g\|_{q_1} + \|f\|_{p_2}\|\Phi^s(g)\|_{q_2}),$$

with $\Phi^s = \Lambda^s$ or D^s, (for the proof of this estimate and further generalizations [KPV4], [MPTT], and [GaO]). The extension to the case $r = p_j = q_j = \infty$, $j = 1,2$ was given in [BoLi].

In many applications, the following commutator estimate is often used:

$$\begin{aligned}\sum_{|\alpha|=s} \|[\partial_x^\alpha; g]\,f\|_2 &\equiv \sum_{|\alpha|=s} \|\partial_x^\alpha(gf) - g\partial_x^\alpha\, f\|_2 \\ &\leq c_{n,s}\Big(\|\nabla g\|_\infty \sum_{|\beta|=s-1} \|\partial_x^\beta\, f\|_2 + \|f\|_\infty \sum_{|\beta|=s} \|\partial_x^\beta\, g\|_2\Big),\end{aligned} \tag{3.15}$$

(see [Kl2]). Similarly, for $s \geq 1$ one has

$$\|[\Lambda^s; g]\,f\|_2 \leq c\,(\|\nabla g\|_\infty \|\Lambda^{s-1} f\|_2 + \|f\|_\infty\,\|\Lambda^s g\|_2), \tag{3.16}$$

(see [KPo]).

There are "equivalent" manners to define fractional derivatives without relying on the Fourier transform. For instance:

Definition 3.4 (Stein [S1]). For $b \in (0,1)$ and an appropriate f define

$$\mathcal{D}^b f(x) = \Big(\int \frac{|f(x) - f(y)|^2}{|x-y|^{n+2b}}\,dy\Big)^{1/2}. \tag{3.17}$$

Theorem 3.6 (Stein [S1]). *Let* $b \in (0,1)$ *and* $\frac{2n}{(n+2b)} \leq p < \infty$. *Then* $f,\ D^b f \in L^p(\mathbb{R}^n)$ *if and only if* $f,\ \mathcal{D}^b f \in L^p(\mathbb{R}^n)$.

Moreover,

$$\|f\|_p + \|D^b f\|_p \sim \|f\|_p + \|\mathcal{D}^b f\|_p.$$

The case $p = 2$ was previously considered in [AS].

For other "equivalent" definitions of fractional derivatives see [Str1].

Finally, to complete our study of Sobolev spaces we introduce the localized Sobolev spaces.

Definition 3.5. Given $f : \mathbb{R}^n \to \mathbb{R}$, we say that $f \in H^s_{\text{loc}}(\mathbb{R}^n)$ if for every $\varphi \in C_0^\infty(\mathbb{R}^n)$ we have $\varphi f \in H^s(\mathbb{R}^n)$. In other words, for any $\Omega \subseteq \mathbb{R}^n$ open bounded $f|_\Omega$ coincides with an element of $H^s(\mathbb{R}^n)$.

This means that f has the sufficient regularity, but may not have enough decay to be in $H^s(\mathbb{R}^n)$.

Example 3.8 Let $n = 1$, $f(x) = x$, and $g(x) = |x|$, then $f \in H^s_{\text{loc}}(\mathbb{R})$ for every $s \geq 0$ and $g \in H^s_{\text{loc}}(\mathbb{R})$ for every $s < 3/2$.

3.2 Pseudo-Differential Operators

We recall some results from the theory of pseudo-differential operators that we need to describe the local smoothing effect for linear elliptic systems.

The class $S^m = S^m_{1,0}$ of classical symbols of order $m \in \mathbb{R}$ is defined by

$$S^m = \{p(x,\xi) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n) \ :\ |p|^{(j)}_{S^m} < \infty,\ j \in \mathbb{N}\}, \tag{3.18}$$

where

$$|p|^{(j)}_{S^m} = \sup\{\|\langle\xi\rangle^{-m+|\alpha|}\partial^\alpha_\xi \partial^\beta_x p(\cdot,\cdot)\|_{L^\infty(\mathbb{R}^n\times\mathbb{R}^n)} : |\alpha+\beta| \leq j\} \tag{3.19}$$

and $\langle\xi\rangle = (1+|\xi|^2)^{1/2}$.

The pseudo-differential operator Ψ_p associated to the symbol $p \in S^m$ is defined by

$$\Psi_p f(x) = \int_{\mathbb{R}^n} e^{2\pi i x\cdot\xi}\, p(x,\xi)\hat{f}(\xi)\,d\xi, \qquad f \in \mathcal{S}(\mathbb{R}^n). \tag{3.20}$$

Example 3.9 A partial differential operator

$$P = \sum_{|\alpha|\leq N} a_\alpha(x)\partial^\alpha_x,$$

with $a_\alpha \in \mathcal{S}(\mathbb{R}^n)$ is a pseudo-differential operator $P = \Psi_p$ with symbol

$$p(x,\xi) = \sum_{|\alpha|\leq N} a_\alpha(x)(2\pi i\xi)^\alpha \in S^N.$$

Example 3.10 The fractional differentiation operator defined in (3.1) as $\Lambda^\rho = \Psi_{\langle\xi\rangle^\rho}$ is also a pseudo-differential operator with symbol in S^ρ, $\rho \in \mathbb{R}$.

The collection of symbol classes S^m, $m \in \mathbb{R}$, is in some cases closed under composition, adjointness, division, and square root operations. This is not the case for polynomials in ξ, and sometimes this closure allows one to construct approximate inverses and square roots of pseudo-differential operators.

Next, we list some properties of pseudo-differential operators whose proofs can be found for instance in [Kg].

Theorem 3.7 (Sobolev boundedness). *Let $m \in \mathbb{R}$, $p \in S^m$, and $s \in \mathbb{R}$. Then, Ψ_p extends to a bounded linear operator from $H^{m+s}(\mathbb{R}^n)$ to $H^s(\mathbb{R}^n)$. Moreover, there exist $j = j(n;m;s) \in \mathbb{N}$ and $c = c(n;m;s)$ such that*

$$\|\Psi_p f\|_{H^s} \le c\,|p|^{(j)}_{S^m}\,\|f\|_{H^{m+s}}. \tag{3.21}$$

Theorem 3.8 (Symbolic calculus). *Let $m_1, m_2 \in \mathbb{R}$, $p_1 \in S^{m_1}$, $p_2 \in S^{m_2}$. Then, there exist $p_3 \in S^{m_1+m_2-1}$, $p_4 \in S^{m_1+m_2-2}$, and $p_5 \in S^{m_1-1}$ such that*

$$\begin{aligned} \Psi_{p_1}\Psi_{p_2} &= \Psi_{p_1 p_2} + \Psi_{p_3}, \\ \Psi_{p_1}\Psi_{p_2} - \Psi_{p_2}\Psi_{p_1} &= \Psi_{-i\{p_1,p_2\}} + \Psi_{p_4}, \\ (\Psi_{p_1})^* &= \Psi_{\bar{p}_1} + \Psi_{p_5}, \end{aligned} \tag{3.22}$$

where $\{p_1, p_2\}$ denotes the Poisson bracket, i.e.,

$$\{p_1, p_2\} = \sum_{j=1}^{n} (\partial_{\xi_j} p_1\, \partial_{x_j} p_2 - \partial_{x_j} p_1\, \partial_{\xi_j} p_2), \tag{3.23}$$

and such that for any $j \in \mathbb{N}$ there exist $j' \in \mathbb{N}$ and $c_1 = c_1(n;m_1;m_2;j)$, $c_2 = c_2(n;m_1;j)$ such that

$$\begin{aligned} |p_3|^{(j)}_{S^{m_1+m_2-1}} + |p_4|^{(j)}_{S^{m_1+m_2-2}} &\le c_1\,|p_1|^{(j')}_{S^{m_1}}\,|p_2|^{(j')}_{S^{m_2}} \\ |p_5|^{(j')}_{S^{m_1-1}} &\le c_2\,|p_1|^{(j')}_{S^{m_1}}. \end{aligned}$$

Remark 3.1.

(i) (3.22) tell us that the "principal symbol" of the commutator $[\psi_{p_1};\psi_{p_2}]$ is given by the formula in (3.23).

(ii) It is useful for our purpose to consider the class of symbols $S^{m,N} = S^{m,N}_{1,0}$ defined as $p(x,\xi) \in C^N(\mathbb{R}^n \times \mathbb{R}^n)$ such that

$$|p|^{(N)}_{S^m} < \infty, \quad \text{with } |p|^{(N)}_{S^m} \text{ defined in (3.19).} \tag{3.24}$$

For N sufficiently large the results in Theorem 3.7 extend to the class $S^{m,N}$.

3.3 The Bicharacteristic Flow

In this section, we introduce the notion of bicharacteristic flow. This plays a key role in the study of linear variable coefficients Schrödinger equations and in the well-posedness of the initial value problem (IVP) associated to the quasilinear case as we can see in the next and the last chapters.

Let $\mathcal{L} = \partial_{x_j} a_{jk}(x)\partial_{x_k}$ be an elliptic self-adjoint operator, that is, $(a_{jk}(x))_{jk}$ is a $n \times n$ matrix of functions $a_{jk} \in C_b^{\infty}$, real, symmetric, and positive definite, i.e., $\exists\, \nu > 0$ such that $\forall\, x,\, \xi \in \mathbb{R}^n$,

$$\nu^{-1}\|\xi\|^2 \leq \sum_{j,k=1}^{n} a_{jk}(x)\xi_j\xi_k \leq \nu\|\xi\|^2. \tag{3.25}$$

Let h_2 be the principal symbol of $\mathcal{L}$, i.e.,

$$h_2(x,\xi) = -\sum_{j,k=1}^{n} a_{jk}(x)\xi_j\xi_k. \tag{3.26}$$

The bicharacteristic flow is the flow of the Hamiltonian vector field:

$$H_{h_2} = \sum_{j=1}^{n} [\partial_{\xi_j} h_2 \cdot \partial_{x_j} - \partial_{x_j} h_2 \cdot \partial_{\xi_j}] \tag{3.27}$$

and is denoted by $(X(s;x_0,\xi_0), \Xi(s;x_0,\xi_0))$, i.e.,

$$\begin{cases} \dfrac{d}{ds} X_j(s;x_0,\xi_0) = -2\displaystyle\sum_{k=1}^{n} a_{jk}(X(s;x_0,\xi_0))\, \Xi_k(s;x_0,\xi_0), \\ \dfrac{d}{ds} \Xi_j(s;x_0,\xi_0) = \displaystyle\sum_{k,l=1}^{n} \partial_{x_j} a_{lk}(X(s;x_0,\xi_0))\, \Xi_k(s;x_0,\xi_0)\, \Xi_l(s;x_0,\xi_0) \end{cases} \tag{3.28}$$

for $j = 1, \ldots, n$, with

$$(X(0;x_0,\xi_0), \Xi(0;x_0,\xi_0)) = (x_0,\xi_0). \tag{3.29}$$

The bicharacteristic flow exists in the time interval $s \in (-\delta,\delta)$ with $\delta = \delta(x_0,\xi_0)$, and $\delta(\cdot)$ depending continuously on (x_0,ξ_0).

The bicharacteristic flow preserves h_2, i.e.,

$$\frac{d}{ds} h_2(X(s;x_0,\xi_0), \Xi(s;x_0,\xi_0)) = 0,$$

so the ellipticity hypothesis (3.25) gives

$$\nu^{-2}\|\xi_0\|^2 \leq \|\Xi(s;x_0,\xi_0)\|^2 \leq \nu^2\|\xi_0\|^2, \tag{3.30}$$

and hence $\delta = \infty$.

In the case of constant coefficients, $h_2(x,\xi) = -|\xi|^2$, the bicharacteristic flow is given by $(X,\ \Xi)(\xi, x_0, \xi_0) = (x_0 - 2s\xi_0, \xi_0)$.

For general symbol $h(x,\xi)$, the bicharacteristic flow is defined as:

$$\begin{cases} \dfrac{dX}{ds} = \partial_\xi\, h(X, \Xi) \\ \\ \dfrac{d\,\Xi}{ds} = -\partial_x\, h(X, \Xi). \end{cases} \tag{3.31}$$

In applications, the notion of the bicharacteristic flow

$$t \mapsto (X(t; x_0, \xi_0),\ \Sigma(t; x_0, \xi_0)) \tag{3.32}$$

being nontrapping arises naturally.

Definition 3.6. A point $(x_0,\ \xi_0) \in \mathbb{R}^n \times \mathbb{R}^n \setminus \{0\}$ is nontrapped forward (respectively, backward) by the bicharacteristic flow if

$$\|X(t;\ x_0,\ \xi_0)\| \to \infty \text{ as } t \to \infty \quad (\text{resp}, t \to -\infty). \tag{3.33}$$

If each point $(x_0,\ \xi_0) \in \mathbb{R}^n \times \mathbb{R}^n - \{0\}$ is nontrapped forward, then the bicharacteristic flow is said to be nontrapping.

In particular, if one assumes that the "metric" $(a_{jk}(x))$ in (3.26) possesses an "asymptotic flat property," for example,

$$|\partial_x^\alpha (a_{jk}(x) - \delta_{jk})| \le \frac{c_\alpha}{|x|^{1+\epsilon(\alpha)}}, \quad \epsilon(\alpha) > 0, \quad 0 \le |\alpha| \le m = m(n), \tag{3.34}$$

then it suffices to have that for each $(x_0, \xi_0) \in \mathbb{R}^n \times \mathbb{R}^n \setminus \{0\}$ and for each $\mu > 0$ there exists $\hat{t} = \hat{t}(\mu;\ x_0, \xi_0) > 0$ such that

$$\|X(\hat{t};\ x_0,\ \xi_0)\| \ge \mu$$

to guarantee that the bicharacteristic flow is nontrapping.

The next result shows that the Hamiltonian vector field is differentiation along the bicharacteristics.

Lemma 3.1. *Let* $\phi \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$. *Then,*

$$(H_{h_2}\phi)(x,\xi) = \partial_s[\phi(X(s;x,\xi), \Xi(s;x,\xi))]|_{s=0} = \{h_2, \phi\}. \tag{3.35}$$

Notice that $-i\{h_2, \phi\}$ is the principal symbol of the commutator $[\psi_{h_2}, \psi_\phi]$ (see 3.22).

Proof. By the chain rule,

$$\begin{aligned} \partial_s[\phi(X(s;x,\xi), \Xi(s;x,\xi))] = &\ (\nabla_x\phi)(X(s;x,\xi), \Xi(s;x,\xi)) \cdot \partial_s X(s;x,\xi) \\ &+ (\nabla_\xi\phi)(X(s;x,\xi), \Xi(s;x,\xi)) \cdot \partial_s \Xi(s;x,\xi) \\ = &\ (\nabla_x\phi \cdot \nabla_\xi h_2)(X(s;x,\xi), \Xi(s;x,\xi)) \\ &- (\nabla_\xi\phi \cdot \nabla_x h_2)(X(s;x,\xi), \Xi(s;x,\xi)). \end{aligned}$$

Setting $s = 0$, the lemma follows. □

3.4 Exercises

3.1 Prove that for any $k \in \mathbb{Z}^+$ and any $\theta \in (0,1)$

$$\chi_{(-1,1)} \overset{k \text{ times}}{* \cdots *} \chi_{(-1,1)}(x) \in C_0^{k-1,\theta}(\mathbb{R}) \backslash C^k(\mathbb{R}).$$

3.2 Prove Proposition 3.1.

3.3 Let $f_n : \mathbb{R}^n \to \mathbb{R}$ with $f_n(x) = e^{-2\pi |x|}$.

(i) Prove that $f_1 * f_1(x) = \dfrac{e^{-2\pi |x|}}{2\pi}(1 + 2\pi |x|)$.
Hint: Use an explicit computation or Exercise 1.1(ii).

(ii) Show that $f_1 * f_1(x) \in C^2(\mathbb{R})$, but is not in $C^3(\mathbb{R})$.

(iii) Prove that $f_n * f_n \in C_\infty^{n+1}(\mathbb{R}^n)$.

(iv) More general, prove that if $g \in H^{s_1}(\mathbb{R}^n)$ and $h \in H^{s_2}(\mathbb{R}^n)$, then $g * h \in C_\infty^{[s_1+s_2]}(\mathbb{R}^n)$ (where $[\,\cdot\,]$ denotes the greatest integer function.)

3.4 Let $\phi(x) = e^{-|x|}$, $x \in \mathbb{R}$:

(i) Prove that

$$\phi(x) - \phi''(x) = 2\delta, \tag{3.36}$$

(a) in the distribution sense, i.e., $\forall \varphi \in C_0^\infty(\mathbb{R})$,

$$\int \phi(x)(\varphi(x) - \varphi''(x))\,dx = 2\varphi(0),$$

(b) by taking the Fourier transform in (3.36).

(ii) Prove that given $g \in L^2(\mathbb{R})$ (or $H^s(\mathbb{R})$) the equation:

$$\left(1 - \frac{d^2}{dx^2}\right) f = g$$

has solution $f = \frac{1}{2}\, e^{-|\cdot|} * g \in H^2(\mathbb{R})$ (or $H^{s+2}(\mathbb{R})$).

3.5 Show that if $k \in \mathbb{Z}^+$ and $p \in [1,\infty)$, then

$$F_{k,p}(\mathbb{R}^n) = L_k^p(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$$

is a Banach algebra with respect to point-wise product of functions. Moreover, if $f, g \in F_{k,p}$, then

$$\|fg\|_{k,p} \le c_k(\|f\|_{k,p}\|g\|_\infty + \|f\|_\infty\|g\|_{k,p}). \tag{3.37}$$

Notation:

$$L_k^p(\mathbb{R}^n) = \{f : \mathbb{R}^n \to \mathbb{C} : \partial^\alpha f \text{ (distribution sense) } \in L^p,\ |\alpha| \le k\},$$

whose norm is defined as:

$$\|f\|_{k,p} = \sum_{|\alpha| \le k} \|\partial^\alpha f\|_p.$$

Observe that when $p=2$ one has $L^p_k(\mathbb{R}^n)=H^k(\mathbb{R}^n)$.
More generally, we define

$$L^p_s(\mathbb{R}^n)=(1-\Delta)^{-s/2}L^p(\mathbb{R}^n) \text{ for } s\in\mathbb{R}, \text{ with } \|f\|_{s,p}=\|(1-\Delta)^{s/2}f\|_p. \tag{3.38}$$

Hint: From Leibniz formula and Hölder's inequality it follows that (assume $n=1$ to simplify)

$$\|(fg)^{(k)}\|_p \le \sum_{j=0}^{k} c_j \|f^{(k-j)}\|_{p_{j_1}} \|g^{(j)}\|_{p_{j_2}}, \quad \text{with } \frac{1}{p}=\frac{1}{p_{j_1}}+\frac{1}{p_{j_2}}.$$

Combine the Gagliardo–Nirenberg inequality (3.14):

$$\|h^{(k-j)}\|_{p_j} \le c\|h^{(k)}\|_p^{\theta}\|h\|_\infty^{1-\theta}, \quad \theta=\theta(n,k,j,p_j),$$

with Young's inequality (if $1/p+1/p'=1$ with $p>1$, then $ab\le a^p/p+b^{p'}/p'$) to get the desired result (3.37).

3.6 Extend the result of Theorem 3.3 to the spaces $L^p_s(\mathbb{R}^n)$, i.e., if $f\in L^p_s(\mathbb{R}^n)$, $0<s<n/p$, then $f\in L^r(\mathbb{R}^n)$ with $s=n\big(\frac{1}{p}-\frac{1}{q}\big)$, and

$$\|f\|_r \le c_{n,s}\,\|D^s f\|_p \le c_{n,s}\,\|f\|_{s,p}. \tag{3.39}$$

3.7 (i) Prove that if $1<p<\infty$ and $b\in(0,1)$, then

$$\|\Lambda^b f\|_p \sim \|f\|_p+\|D^b f\|_p.$$

Hint: Use Theorem 2.8.

(ii) Given any $s\in\mathbb{R}$ find $f_s\in H^s(\mathbb{R})$ such that $f_s\notin H^{s'}(\mathbb{R})$ for any $s'>s$.
Hint:

(a) Notice that it suffices to find f_0.
(b) Show that if $g\in L^2(\mathbb{R})$ and $g\notin L^p(\mathbb{R})$ for any $p>2$, then one can take $f_0=g$.
(c) Use (b) to find f_0.

3.8 Show that if $f\in H^s(\mathbb{R}^n)$, $s>n/2$, with $\|f\|_{n/2,2}\le 1$, then

$$\|f\|_\infty \le c\,[1+\log(1+\|f\|_{s,2})]^{1/2}$$

with $c=c(s,n)$, see [BGa].

3.9 Prove the following inequalities:

(i) If $s>n/2$, then

$$\|f\|_\infty \le c_{n,s}\|f\|_2^{1-n/2s}\|D^s f\|_2^{n/2s}.$$

(ii) If $s > n/p$, $1 < p < \infty$, then

$$\|f\|_\infty \leq c_{n,s,p} \|f\|_p^{1-n/ps} \|D^s f\|_p^{n/ps}.$$

(iii) Prove Gagliardo–Nirenberg inequality (3.14) for p even integer, $m = 2$, $j = 2$, and $q, r \in (1, \infty)$ such that $1/q + 1/r = 2/p$.

(iv) Combine Exercises 2.10 and 2.11, and Theorem 2.6 to prove the Gagliardo–Nirenberg inequality in the general case.

3.10 ([AS]). Using Definition 3.4:

(i) Prove that for $b \in (0, 1)$

$$\|D^b f\|_2 = c_n \, \|\mathcal{D}^b f\|_2. \tag{3.40}$$

(ii) Prove that

$$\mathcal{D}^b(fg)(x) \leq \|f\|_\infty \mathcal{D}^b g(x) + |g(x)| \mathcal{D}^b f(x) \tag{3.41}$$

and

$$\|\mathcal{D}^b(fg)\|_2 \leq \|f \mathcal{D}^b g\|_2 + \|g \mathcal{D}^b f\|_2. \tag{3.42}$$

(iii) Let $F \in C_b^1(\mathbb{R} : \mathbb{R})$, $F(0) = 0$. Show that

$$\|D^b(F(f))\|_2 \leq \|F'\|_\infty \|D^b f\|_2.$$

Hint: Apply part (i).

3.11 (i) Let $f \in L^p(\mathbb{R})$, $1 < p < \infty$, be such that $f(x_0^+)$, $f(x_0^-)$ exist and $f(x_0^+) \neq f(x_0^-)$ for some x_0. Prove that $f \notin L^p_{1/p}(\mathbb{R})$.

(ii) Let $\varphi \in C_0^\infty(\mathbb{R})$ with $\varphi(x) = 1$ if $|x| \leq 1$ and $\varphi(x) = 0$ if $|x| > 2$. Let $a, b \in (0, 1)$. Prove that $|x|^a \varphi(x) \in H^b(\mathbb{R})$ if and only if $b < a + 1/2$.

(iii) Let $\alpha \in (0, 1/2)$. Prove that

$$|\log |x||^\alpha \, \chi_{\{|x| \leq 1/10\}} + \frac{10}{9}(1 - |x|) \, \chi_{\{1/10 \leq |x| \leq 1\}} \in H^1(\mathbb{R}^2) - L^\infty(\mathbb{R}^2).$$

3.12 (Sobolev's inequality for radial functions) Let $f : \mathbb{R}^n \to \mathbb{R}$, $n \geq 3$, be a radial function, i.e., $f(x) = f(y)$ if $|x| = |y|$. Show that f satisfies

$$|f(x)| \leq c_n \, |x|^{(2-n)/2} \, \|\nabla f\|_2.$$

3.13 (Hardy's inequalities (see Exercise 1.5))

(i) Let $1 \leq p < \infty$. If $f \in L_1^p(\mathbb{R}^n)$, then

$$\left\| \frac{|f(\cdot)|}{|x|} \right\|_p \leq \frac{p}{n-p} \, \|\nabla f\|_p. \tag{3.43}$$

(ii) Let $1 \le p < \infty$, $q < n$, and $q \in [0, p]$. If $f \in L_1^p(\mathbb{R}^n)$, then

$$\int_{\mathbb{R}^n} \frac{|f(x)|^p}{|x|^q}\, dx \le \Big(\frac{p}{n-q}\Big)^q \|f\|_p^{p-q} \|\nabla f\|_p^q. \tag{3.44}$$

Hint: Assume that $f \in C_0^\infty(\mathbb{R}^n)$. For (i), write $\||\cdot|^{-1} f\|_p^p$ in spherical coordinates, use integration by parts in the radial variable and Hölder inequality to get the result. For (ii), assume $p > q$, and apply (3.43) to $|x|^{-1} g(x)$ with $g(x) = |f(x)|^{p/q}$.

3.14 Prove Heisenberg's inequality. If $f \in H^1(\mathbb{R}^n) \cap L^2(|x|^2\, dx)$, then

$$\|f\|_2^2 \le \frac{2}{n} \|x_j f\|_2 \|\partial_{x_j} f\|_2 = \frac{4\pi}{n} \|x_j f\|_2 \|\xi_j \widehat{f}\|_2 \le \frac{2}{n} \|xf\|_2 \|\nabla f\|_2. \tag{3.45}$$

Hint: Use the density of $\mathcal{S}(\mathbb{R}^n)$ and integration by parts to obtain the identity

$$\|f\|_2^2 = -\frac{1}{n} \int x_j \partial_{x_j}(|f(x)|^2)\, dx.$$

3.15 Denote $u = u(x,t)$, the solution of the IVP associated to the inviscid Burgers' equation:

$$\begin{cases} \partial_t u + u\partial_x u = 0, \\ u(x,0) = u_0(x) \in C_0^\infty(\mathbb{R}), \end{cases} \tag{3.46}$$

$t, x \in \mathbb{R}$. Prove that for every $T > 0$,

$$u \in C^\infty(\mathbb{R} \times [-T, T]) \quad \text{or} \quad u \notin C^1(\mathbb{R} \times [-T, T]).$$

Hint: Combine the commutator estimate (3.16) and integration by parts to obtain the energy estimate

$$\frac{d}{dt} \|u(t)\|_{k,2} \le c_k \|\partial_x u(t)\|_\infty \|u(t)\|_{k,2} \quad \text{for all } k \in \mathbb{Z}^+. \tag{3.47}$$

3.16 Let $P(x, \partial_x) = \sum_{|\alpha| \le m_1} a_\alpha(x) \partial_x^\alpha$ and $Q(x, \partial_x) = \sum_{|\alpha| \le m_2} b_\alpha(x) \partial_x^\alpha$ be two differential operators. Check the properties stated in Theorem 3.8 for P and Q.

3.17 (i) If $\Lambda = (1 - \Delta)^{1/2}$ and $y \in \mathbb{R}$, show that the symbol $p = p(\xi)$ of Λ^{iy}, $p(\xi) = (1 + |\xi|^2)^{iy/2} \in S_0$, and

$$|p|_{S^0}^j \le c_n\, (1 + |y|)^j.$$

(ii) Show that if $p = p(x, \xi) \in S^0 = S_{1,0}^0$, then $e^{p(x,\xi)} \in S^0 = S_{1,0}^0$.

3.18 Prove that the bicharacteristic flow in (3.28) $(X(s; x_0, \xi_0),\ \Xi_k(s; x_0, \xi_0))$ satisfies

(i) $X(s; x_0, \rho\,\xi_0) = X(\rho\,s; x_0, \xi_0)$,
(ii) $\Xi_k(s; x_0, \rho\,\xi_0) = \rho\,\Xi_k(\rho\,s; x_0, \xi_0)$.

Hint: Use the homogeneity of $h_2(x, \xi) = -a_{jk}(x)\,\xi_j \xi_k$.

3.19 Prove that if Ψ_p is a pseudo-differential operator with symbol $p \in S^0$, then for any $b \in \mathbb{R}$,

$$\|\Psi_p f\|_{L^2(\langle x\rangle^b\,dx)} \leq c_{k,n}\,\|f\|_{L^2(\langle x\rangle^b\,dx)}, \tag{3.48}$$

where

$$||g||_{L^2(\langle x\rangle^b\,dx)} = \left(\int |g(x)|^2 \langle x\rangle^b\,dx\right)^{1/2}$$

and

$$\langle x\rangle = (1 + |x|^2)^{1/2}. \tag{3.49}$$

Hint:

(i) Follow an argument similar to that given in the proof of Theorem 2.1 to show that it suffices to establish (3.48) for $b = 4k$, $k \in \mathbb{Z}$.
(ii) Consider the case $b = -4k$, $k \in \mathbb{Z}^+$, and show that (3.48) is equivalent to

$$\left\| \frac{1}{\langle x\rangle^{2k}}\,\Psi_p\,(\langle x\rangle^{2k} g)\right\|_2 \leq c\,\|g\|_2. \tag{3.50}$$

(iii) Obtain (3.50) by combining integration by parts, Theorems 3.7 and 3.8.
(iv) Finally, prove the case $b = 4k$, $k \in \mathbb{Z}^+$, by duality.

3.20 Let $a, b > 0$. Assume that $\Lambda^a f = (1 - \Delta/4\pi^2)^{a/2} f \in L^2(\mathbb{R}^2)$ (i.e., $f \in H^a(\mathbb{R}^n)$) and $\langle x\rangle^b f \in L^2(\mathbb{R}^n)$ (see 3.49). Prove that for any $\theta \in (0, 1)$,

$$||\Lambda^{(1-\theta)a}(\langle x\rangle^{\theta b} f)||_2 \leq c_{a,b,n}\,||\langle x\rangle^b f||_2^{\theta} ||\Lambda^a f||_2^{1-\theta}.$$

Hint: Combine the three lines theorem, Exercises 3.17 part (i) and (3.19).

Chapter 4
The Linear Schrödinger Equation

In this chapter, we study the smoothing properties of solutions of the initial value problem:

$$\begin{cases} \partial_t u = i\Delta u + F(x,t), \\ u(x,0) = u_0(x), \end{cases} \tag{4.1}$$

$x \in \mathbb{R}^n,\ t \in \mathbb{R}$. These properties are fundamental tools in the next chapters. In Section 4.1, we present some general basic results concerning the initial value problem (4.3). The global smoothing properties of solutions of (4.3) described by estimates of the type $L^q(\mathbb{R} : L^p(\mathbb{R}^n))$ are discussed in Section 4.2. In Section 4.3, we derive the local smoothing arising from estimates of type $L^2_{\text{loc}}(\mathbb{R} : H^{1/2}_{\text{loc}}(\mathbb{R}^n))$. We end the chapter with some remarks and comments regarding the issues discussed in the previous sections.

4.1 Basic Results

We begin by recalling the notation (see (1.27))

$$e^{it\Delta}u_0 = \frac{e^{-|x|^2/4it}}{(4\pi it)^{n/2}} * u_0 = \big(e^{-4\pi^2 it|\xi|^2}\widehat{u_0}\big)^{\vee}. \tag{4.2}$$

The identity (4.2) describes the solution $u(x,t)$ of the linear homogeneous initial value problem (IVP)

$$\begin{cases} \partial_t u = i\Delta u, \\ u(x,0) = u_0(x). \end{cases} \tag{4.3}$$

$x \in \mathbb{R}^n,\ t \in \mathbb{R}$. In the following examples, we illustrate some of the properties exhibited by solutions of IVP (4.3).

F. Linares, G. Ponce, *Introduction to Nonlinear Dispersive Equations*,
Universitext, DOI 10.1007/978-1-4939-2181-2_4

Example 4.1 Consider the Gaussian function $u_0(x) = e^{-\pi|x|^2}$. Using Examples 1.3, 1.11, and Exercise 1.2 we find that the solution of the IVP (4.3) is given by

$$\begin{aligned} u(x,t) &= \left(e^{-4\pi^2 it|\xi|^2}\widehat{u}_0(\xi)\right)^{\vee} \\ &= \left(e^{-(1+4\pi it)\pi\,|\xi|^2}\right)^{\vee} \\ &= \frac{1}{(1+4\pi it)^{n/2}}\exp\left(\frac{-\pi|x|^2}{1+4\pi it}\right) \\ &= (1+4\pi it)^{-n/2}\exp\left(-\frac{\pi|x|^2}{1+16\pi^2t^2}\right)\exp\left(\frac{4\pi^2it|x|^2}{1+16\pi^2t^2}\right). \end{aligned} \tag{4.4}$$

Notice that when $t \gg 1$ and $|x| < t$, the absolute value of the solution is bounded below by $c_n\,t^{-n/2}$ and the solution oscillates for $|x| > t^{1/2}$. Furthermore, if $|x| > t$ the absolute value of the solution decays exponentially. Moreover,

$$C\,t^{-n/2}\chi_{\{|x|<t\}}(x) \le |u(x,t)| \le c\,t^{-n/2}, \tag{4.5}$$

which is the expected behavior of the solution in order to have its $L^2(\mathbb{R}^n)$-norm independent of t.

Example 4.2 We can write the solution of the IVP (4.3) as

$$\begin{aligned} u(x,t) &= \left(e^{-4\pi^2it|\xi|^2}\widehat{u}_0\right)^{\vee}(x) = \int_{\mathbb{R}^n}\frac{e^{i|x-y|^2/4t}}{(4\pi it)^{n/2}}\,u_0(y)\,dy \\ &= \frac{e^{i|x|^2/4t}}{(4\pi it)^{n/2}}\int_{\mathbb{R}^n} e^{-2ix\cdot y/4t}\,e^{i|y|^2/4t}\,u_0(y)\,dy \\ &= \frac{e^{i|x|^2/4t}}{(4\pi it)^{n/2}}\left(\widehat{e^{i|\cdot|^2/4t}\,u_0}\right)\left(\frac{x}{4\pi t}\right). \end{aligned} \tag{4.6}$$

Thus, if $c_t = (4\pi it)^{n/2}$,

$$c_t\,e^{-i|x|^2/4t}\,u(x,t) = \left(\widehat{e^{i|\cdot|^2/4t}\,u_0}\right)\left(\frac{x}{4\pi t}\right). \tag{4.7}$$

Notice that if $u_0 \in C_0(\mathbb{R}^n)$ from (4.7) we deduce that for any $t \in \mathbb{R}\setminus\{0\}$ and any $\epsilon > 0$, $u(\cdot,t) \notin L^1(e^{\epsilon|x|}dx)$. In particular, if $t \neq 0$, $u(x,t)$ has an analytic extension to $\mathbb{C}^n$ (see Exercise 4.5).

Example 4.3 This example describes the propagation of oscillatory pulses. Now we take $u_0(x) = e^{ix\cdot x_0}e^{-\pi|x|^2}$, $x_0 \in \mathbb{R}^n$. From Examples 1.3 and 1.4 we have $\widehat{u}_0(\xi) = e^{-\pi|\xi - x_0/2\pi|^2}$. Thus, using Example 4.1 we obtain

$$\begin{aligned} u(x,t) &= \left(e^{-4\pi^2it(|\xi-x_0/2\pi|^2+2(\xi-x_0/2\pi)\cdot x_0/2\pi+|x_0|^2/4\pi^2)}\,e^{-\pi|\xi-x_0/2\pi|^2}\right)^{\vee} \\ &= \left(\tau_{x_0/2\pi}(e^{-4\pi^2it(|\xi|^2+2\xi\cdot x_0/2\pi+|x_0|^2/4\pi^2)}\,e^{-\pi|\xi|^2})\right)^{\vee} \end{aligned}$$

$$= \left(\tau_{x_0/2\pi}(e^{-i2t\xi\cdot x_0}\, e^{-it|x_0|^2}\, e^{-(1+4\pi it)\pi|\xi|^2})\right)^{\vee} \tag{4.8}$$
$$= e^{ix_0\cdot x}\tau_{2x_0t}(\,e^{-it|x_0|^2}\, e^{-(1+4\pi it)\pi|\xi|^2})^{\vee}$$
$$= e^{ix_0\cdot x}\, e^{-it|x_0|^2}(1+4\pi it)^{-n/2}\, e^{\frac{-\pi|x-2tx_0|^2}{(1+4\pi it)}},$$

where τ is the translation operator (see (1.4)). In other words, the solution of the IVP (4.3) with data $u_0(x) = e^{ix_0\cdot x}\, e^{-\pi\,|x|^2}$ is given by

$$u(x,t) = e^{ix\cdot x_0}\, e^{-i|x_0|^2 t}\,\mathbf{u}(x-2t\,x_0,t), \tag{4.9}$$

where $\mathbf{u}$ denotes the solution of the IVP (4.3) given in Example 4.1.

In the next proposition, we list several invariance properties of solutions of the equation in (4.3).

Proposition 4.1. *If $u = u(x,t)$ is a solution of the equation in (4.3), then*

$$u_1(x,t) = e^{i\theta}u(x,t), \quad \theta\in\mathbb{R} \text{ fixed},$$
$$u_2(x,t) = u(x-x_0,t-t_0), \text{with } x_0\in\mathbb{R}^n,\ t_0\in\mathbb{R} \text{ fixed},$$
$$u_3(x,t) = u(Ax,t), \text{with} A \text{ any orthogonal matrix } n\times n,$$
$$u_4(x,t) = u(x-2x_0\,t,t)\,e^{i(x\cdot x_0-|x_0|^2t)}, \text{with } x_0\in\mathbb{R}^n \text{ fixed},$$
$$u_5(x,t) = \lambda^{n/2}u(\lambda x,\lambda^2 t),\ \lambda>0,$$
$$u_6(x,t) = \frac{1}{(\alpha+\omega t)^{n/2}}\,\exp\left[\frac{i\omega|x|^2}{4(\alpha+\omega t)}\right]u\left(\frac{x}{\alpha+\omega t},\frac{\gamma+\theta t}{\alpha+\omega t}\right), \quad \alpha\theta-\omega\gamma=1,$$
$$u_7(x,t) = \overline{u(x,-t)},$$

also satisfy the equation in (4.3).

In (4.2), we have used an exponential formula to describe the solution of the IVP (4.3). To justify this formula, we state next some properties of the family of operators $\{e^{it\Delta}\}_{t=-\infty}^{\infty}$.

Proposition 4.2.

1. *For all $t\in\mathbb{R}$, $e^{it\Delta} : L^2(\mathbb{R}^n)\mapsto L^2(\mathbb{R}^n)$ is an isometry, which implies*
$$\|e^{it\Delta}f\|_2 = \|f\|_2.$$
2. $e^{it\Delta}e^{it'\Delta} = e^{i(t+t')\Delta}$ *with* $(e^{it\Delta})^{-1} = e^{-it\Delta} = (e^{it\Delta})^*$.
3. $e^{i0\Delta} = 1$.
4. *Fixing $f\in L^2(\mathbb{R}^n)$, the function $\Phi_f : \mathbb{R}\mapsto L^2(\mathbb{R}^n)$, where $\Phi_f(t) = e^{it\Delta}f$ is a continuous function, i.e., it describes a curve in $L^2(\mathbb{R}^n)$.*

Proof. The proof is left as an exercise. □

In general, a family of operators $\{T_t\}_{t=-\infty}^{\infty}$ defined on a Hilbert space H which satisfies properties (1)–(4) in Proposition 4.2 is called a *unitary group of operators.*

Example 4.4 Let $L_t : L^2(\mathbb{R}) \mapsto L^2(\mathbb{R})$ be the one parameter family of translation operators $L_t(u_0)(x) = u_0(x+t)$. It is easy to see that $\{L_t\}_{t=-\infty}^{\infty}$ is a unitary group of operators, which describes the solution $u(x,t) = L_t(u_0)(x)$ of the problem

$$\begin{cases} \partial_t u = \partial_x u, \\ u(x,0) = u_0(x), \end{cases}$$

$t,\ x \in \mathbb{R}$.

The next result of M. H. Stone, characterizes the unitary group of operators.

Theorem 4.1 (M. H. Stone). *The family of operators $\{T_t\}_{t=-\infty}^{\infty}$ defined on the Hilbert space H is a unitary group of operators if and only if there exists a self-adjoint operator A (not necessarily bounded) on H such that*

$$T_t = e^{itA} \tag{4.10}$$

in the following sense: Consider $D(A)$ the domain of the operator A, which is a dense subspace of H; if $f \in D(A)$, then we have

$$\lim_{t\to 0} \frac{T_t f - f}{t} = iAf. \tag{4.11}$$

In other words, if $f \in D(A)$, then the curve Φ_f defined in Proposition 4.2 (4) is differentiable at $t = 0$ with derivative iAf.

For a proof of this theorem, we refer the reader to [Yo].

The operator A in Theorem 4.1 is called the *infinitesimal generator* of the unitary group. In (4.2), the operator A is the Laplacian Δ with $D(A) = H^2(\mathbb{R}^n)$. In Example 4.4, we have $A = -i\frac{d}{dx}$ and in this case, formula (4.10) can be interpreted as a generalized Taylor series.

Now we establish the properties of the group $\{e^{it\Delta}\}_{t=-\infty}^{\infty}$ in the $L^p(\mathbb{R}^n)$ spaces.

Lemma 4.1. *If $t \neq 0$, $1/p + 1/p' = 1$ and $p' \in [1,2]$, then $e^{it\Delta}\colon L^{p'}(\mathbb{R}^n) \mapsto L^p(\mathbb{R}^n)$ is continuous and*

$$\left\|e^{it\Delta} f\right\|_p \le c|t|^{-n/2(1/p'-1/p)}\|f\|_{p'}. \tag{4.12}$$

Proof. From Proposition 4.2 it follows that

$$e^{it\Delta}\colon L^2(\mathbb{R}^n) \mapsto L^2(\mathbb{R}^n)$$

is an isometry, that is,

$$\|e^{it\Delta} f\|_2 = \|f\|_2.$$

Using Young's inequality (1.39), we have

$$\|e^{it\Delta} f\|_\infty = \left\|\frac{e^{i|\cdot|^2/4t}}{\sqrt{(4\pi i t)^n}} * f\right\|_\infty$$

$$\leq \left\| \frac{e^{i|\cdot|^2/4t}}{\sqrt{(4\pi i t)^n}} \right\|_\infty \|f\|_1 \leq c|t|^{-n/2}\|f\|_1. \qquad (4.13)$$

A combination of these inequalities with the Riesz–Thorin theorem (Theorem 2.1) lead to

$$e^{it\Delta} : L^{p'}(\mathbb{R}^n) \mapsto L^p(\mathbb{R}^n) \text{ with } \frac{1}{p} + \frac{1}{p'} = 1,$$

and

$$\|e^{it\Delta} f\|_p \leq (c|t|^{-n/2})^{1-\theta} \|f\|_{p'} = c|t|^{-n/2(1/p'-1/p)} \|f\|_{p'},$$

where

$$\frac{1}{p} = \frac{\theta}{2} \quad \text{and} \quad 1-\theta = 1 - \frac{2}{p} = \frac{1}{p'} - \frac{1}{p}, \quad \theta \in [0,1].$$

Thus, the lemma follows. □

This result indicate that if $f \in L^2(\mathbb{R}^n)$ decreases fast enough when $|x| \to \infty$ such that $f \in L^1(\mathbb{R}^n)$, $e^{it\Delta} f$, $t \neq 0$, is bounded (and so more regular than f). In general, decay on the initial data f is translated into smoothing property of the solution $e^{it\Delta} f$ (see Exercise 4.4).

Note that $e^{it\Delta}$ with $t \neq 0$ is not a bounded operator from $L^p(\mathbb{R}^n)$ in $L^p(\mathbb{R}^n)$ if $p \neq 2$, i.e., $m(\xi) = e^{-4\pi^2 it|\xi|^2}$ is not an L^p multiplier for $p \neq 2$ (see Definition 2.8). In fact, if it were bounded for $p \neq 2$ it would be bounded also for p' by duality. Then, without loss of generality, we can assume $p > 2$. Using (4.12), we would have that for all $f \in L^{p'}(\mathbb{R}^n) \cap L^p(\mathbb{R}^n) \subseteq L^2(\mathbb{R}^n)$,

$$\|f\|_p = \|e^{it\Delta} e^{-it\Delta} f\|_p \leq c_0 \|e^{-it\Delta} f\|_p \leq c_0\, c(t) \|f\|_{p'},$$

which is a contradiction.

Next proposition help us to understand the regularizing effects present in the group $\{e^{it\Delta}\}_{t=-\infty}^{\infty}$.

Proposition 4.3.

1. *Given $t_0 \neq 0$ and $p > 2$, there exists $f \in L^2(\mathbb{R}^n)$ such that $e^{it_0\Delta} f \notin L^p(\mathbb{R}^n)$.*

2. *Let $s' > s > 0$ and $f \in H^s(\mathbb{R}^n)$ such that $f \notin H^{s'}(\mathbb{R}^n)$. Then, for all $t \in \mathbb{R}$, $e^{it\Delta} f \in H^s(\mathbb{R}^n)$ and $e^{it\Delta} f \notin H^{s'}(\mathbb{R}^n)$.*

Proof. To show (1), it is enough to choose $g \in L^2(\mathbb{R}^n)$ such that $g \notin L^p(\mathbb{R}^n)$ and take $f = e^{-it_0\Delta} g$.
The statement (2) follows from the fact that $\{e^{it\Delta}\}_{t=-\infty}^{\infty}$ is a unitary group in $H^s(\mathbb{R}^n)$ for all $s \in \mathbb{R}$ since

$$\|e^{it\Delta} f\|_{s,2} = \|\Lambda^s(e^{it\Delta} f)\|_2 = \|e^{it\Delta}(\Lambda^s f)\|_2 = \|\Lambda^s f\|_2 = \|f\|_{s,2}.$$

Therefore, if $e^{it\Delta} f \in H^{s_0}(\mathbb{R}^n)$, then $f = e^{-it\Delta}(e^{it\Delta}) f \in H^{s_0}(\mathbb{R}^n)$. □

4.2 Global Smoothing Effects

The next theorem describes the *global smoothing* property of the group $\{e^{it\Delta}\}_{t=-\infty}^{\infty}$.

Theorem 4.2. *The group* $\{e^{it\Delta}\}_{t=-\infty}^{\infty}$ *satisfies:*

$$\left(\int_{-\infty}^{\infty} \|e^{it\Delta} f\|_p^q \, dt\right)^{1/q} \leq c\,\|f\|_2, \tag{4.14}$$

$$\left(\int_{-\infty}^{\infty} \Big\| \int_{-\infty}^{\infty} e^{i(t-t')\Delta} g(\cdot,t')\,dt' \Big\|_p^q \, dt\right)^{1/q} \leq c \left(\int_{-\infty}^{\infty} \|g(\cdot,t)\|_{p'}^{q'}\,dt\right)^{1/q'}, \tag{4.15}$$

$$\Big\| \int_{-\infty}^{\infty} e^{it\Delta} g(\cdot,t)\,dt \Big\|_2 \leq c \left(\int_{-\infty}^{\infty} \|g(\cdot,t)\|_{p'}^{q'}\,dt\right)^{1/q'}, \tag{4.16}$$

and

$$\left(\int_{-\infty}^{\infty} \Big\| \int_{0}^{t} e^{i(t-t')\Delta} g(\cdot,t')\,dt' \Big\|_p^q \, dt\right)^{1/q} \leq c \left(\int_{-\infty}^{\infty} \|g(\cdot,t)\|_{p'}^{q'}\,dt\right)^{1/q'}, \tag{4.17}$$

with

$$\left.\begin{array}{ll} 2 \leq p < \frac{2n}{n-2} & \text{if } n \geq 3 \\ 2 \leq p < \infty & \text{if } n = 2 \\ 2 \leq p \leq \infty & \text{if } n = 1 \end{array}\right\} \quad \text{and} \quad \frac{2}{q} = \frac{n}{2} - \frac{n}{p}, \tag{4.18}$$

where $c = c(p,n)$ is a constant depending only on p and n.

From here on, we always use the notation

$$\frac{1}{p} + \frac{1}{p'} = \frac{1}{q} + \frac{1}{q'} = 1.$$

Proof. First, we shall prove that (4.14), (4.15), and (4.16) are equivalent.

Fubini's theorem gives us that

$$\int_{-\infty}^{\infty}\int_{\mathbb{R}^n} (e^{it\Delta} f)(x) g(x,t) dx dt = \int_{\mathbb{R}^n} f(x) \left(\int_{-\infty}^{\infty} e^{it\Delta} g(x,t)\,dt\right) dx.$$

Therefore, using duality,

$$\Big(\int_{-\infty}^{\infty} \|h(\cdot,t)\|_p^q \,dt\Big)^{1/q}$$

$$= \sup\left\{\left|\int_{-\infty}^{\infty}\int_{\mathbb{R}^n} h(x,t)w(x,t)\,dx\,dt\right| : \left(\int_{-\infty}^{\infty} \|w(\cdot,t)\|_{p'}^{q'}\,dt\right)^{1/q'} = 1\right\}$$

it follows that (4.14) and (4.16) are equivalent. An argument due to P. Tomas implies that

$$\Big\|\int_{-\infty}^{\infty} e^{it\Delta}g(\cdot,t)\,dt\Big\|_2^2 = \int_{\mathbb{R}^n}\left(\int_{-\infty}^{\infty} e^{it\Delta}g(\cdot,t)\,dt\right)\overline{\left(\int_{-\infty}^{\infty} e^{it'\Delta}g(\cdot,t')dt'\right)}\,dx$$

$$= \int_{\mathbb{R}^n}\int_{-\infty}^{\infty} g(x,t)\left(\int_{-\infty}^{\infty} e^{i(t-t')\Delta}\overline{g(\cdot,t')}\,dt'\right)dt\,dx. \tag{4.19}$$

From these identities we obtain (applying again an argument of duality and Hölder's inequality) the equivalence between (4.15) and (4.16).

Next we shall establish (4.15). Minkowski's inequality (1.40) and Lemma 4.1 give

$$\begin{aligned}\Big\|\int_{-\infty}^{\infty} e^{i(t-t')\Delta}g(\cdot,t')\,dt'\Big\|_p &\le \int_{-\infty}^{\infty} \|e^{i(t-t')\Delta}g(\cdot,t')\|_p\,dt' \\ &\le c\int_{-\infty}^{\infty} \tfrac{1}{|t-t'|^{\alpha}}\|g(\cdot,t')\|_{p'}\,dt'\end{aligned} \tag{4.20}$$

with $\alpha = (n/2)\big(1/p' - 1/p\big)$. Inequality (4.20) and Theorem 2.6 (Hardy–Littlewood–Sobolev) imply

$$\Big(\int_{-\infty}^{\infty}\Big\|\int_{-\infty}^{\infty} e^{i(t-t')\Delta}g(\cdot,t')\,dt'\Big\|_p^q\,dt\Big)^{1/q}$$

$$\le c\,\Big\|\int_{-\infty}^{\infty}\frac{1}{|t-t'|^{\alpha}}\|g(\cdot,t')\|_{p'}\,dt'\Big\|_q \le c\left(\int_{-\infty}^{\infty}\|g(\cdot,t)\|_{p'}^{q'}\,dt\right)^{1/q'}$$

with $1/q' = 1/q + (1-\alpha)$ and $0 < 1-\alpha < 1$, that is, $n/2 = 2/q + n/p$, where

$$\begin{cases} 2 \le p < \dfrac{2n}{n-2} & \text{if } n \ge 3, \\ 2 \le p < \infty & \text{if } n = 2, \\ 2 \le p \le \infty & \text{if } n = 1. \end{cases}$$

Finally, we turn to the proof of (4.17). This is a consequence of (4.15) and the following result due to Christ-Kiselev [CrK].

Lemma 4.2. *Let*

$$Tf(t) = \int_{-\infty}^{\infty} K(t,s)\, f(s)\, ds \tag{4.21}$$

be a bounded map from $L^r(\mathbb{R})$ *to* $L^l(\mathbb{R})$ *with* $1 < r < l < \infty$. *Then the map*

$$\tilde{T} f(t) = \int_{s<t} K(t,s)\, f(s)\, ds \tag{4.22}$$

also maps $L^r(\mathbb{R})$ *into* $L^l(\mathbb{R})$.

For the proof of this lemma, see Appendix B. □

In particular, this theorem tells us that if $f \in L^2(\mathbb{R}^n)$, then $e^{it\Delta} f \in L^p(\mathbb{R}^n)$, for any fixed $p \in (2,\ p(n))$ for almost all time $t \in \mathbb{R}$, with $p(n)$ depending on the dimension. In particular, if $n = 1$, $p(1) = \infty$, and $q = 4$, then for $f \in L^2(\mathbb{R})$ we have

$$\left(\int_{-\infty}^{\infty} \| e^{it\partial_x^2} f \|_\infty^4 \, dt \right)^{1/4} \le c\, \|f\|_2,$$

which implies that $e^{it\partial_x^2} f \in L^\infty(\mathbb{R})$ for almost every t. Indeed, in this case, one has that for almost every $t \in \mathbb{R}$, $e^{it\partial_x^2} f$ is continuous in $\mathbb{R}$ (see Exercise 4.9). Note that this fact does not contradict Proposition 4.3.

Corollary 4.1. *Let* (p_0, q_0), $(p_1, q_1) \in \mathbb{R}^2$ *satisfying the condition* (4.18) *in Theorem 4.2. Then, for all* $T > 0$ *we have*

$$\left(\int_0^T \Big\| \int_0^t e^{i(t-t')\Delta} g(\cdot, t')\, dt' \Big\|_{p_1}^{q_1} dt \right)^{1/q_1} \le c \left(\int_0^T \|g(\cdot,t)\|_{p_0'}^{q_0'} dt \right)^{1/q_0'},$$

with $c = c(n, p_0, p_1)$.

Proof. By hypothesis, the points $(1/p_0, 1/q_0)$ and $(1/p_1, 1/q_1)$ are in the segment of the line connecting $P = (1/2, 0)$ with $Q = (1/p(n), n/4 - n/2\, p(n))$. So $p(n) = \infty$ if $n = 1, 2$, and $p(n) = 2n/(n-2)$ if $n \ge 3$. Therefore, without loss of generality we can assume $p_0 \in [2, p_1)$. An application of the inequalities (4.16) and (4.17) in Theorem 4.2 provides the following estimates:

$$\left(\int_0^T \Big\| \int_0^t e^{i(t-t')\Delta} g(\cdot, t')\, dt' \Big\|_{p_1}^{q_1} dt \right)^{1/q_1} \le c \left(\int_0^T \|g(\cdot,t)\|_{p_1'}^{q_1'} dt \right)^{1/q_1'},$$

and

$$\sup_{[0,T]} \Big\| \int_0^t e^{i(t-t')\Delta} g(\cdot,t')\,dt' \Big\|_2 = \sup_{[0,T]} \Big\| e^{it\Delta} \int_0^t e^{-it'\Delta} g(\cdot,t')\,dt' \Big\|_2$$

$$= \sup_{[0,T]} \Big\| \int_0^t e^{-it'\Delta} g(\cdot,t')\,dt' \Big\|_2 \le c \left(\int_0^T \|g(\cdot,t)\|_{p_1'}^{q_1'}\,dt \right)^{1/q_1'} .$$

These estimates and Hölder's inequality lead to

$$\left(\int_0^T \Big\| \int_0^t e^{i(t-t')\Delta} g(\cdot,t')\,dt' \Big\|_{p_0}^{q_0} dt \right)^{1/q_0} \le c \left(\int_0^T \|g(\cdot,t)\|_{p_1'}^{q_1'}\,dt \right)^{1/q_1'} .$$

To finish the proof, an argument of duality allows us to write the inequality

$$\left(\int_0^T \Big\| \int_0^t e^{i(t-t')\Delta} g(\cdot,t')\,dt' \Big\|_{p_1}^{q_1} dt \right)^{1/q_1} \le c \left(\int_0^T \|g(\cdot,t)\|_{p_0'}^{q_0'}\,dt \right)^{1/q_0'} .$$

This yields the result. □

4.3 Local Smoothing Effects

In this section, we study the local smoothing effects of the group $\{e^{it\Delta}\}_{t=-\infty}^{\infty}$.

Theorem 4.3. *If $n = 1$, then*

$$\sup_x \int_{-\infty}^{\infty} |D_x^{1/2} e^{it\Delta} f(x)|^2\,dt \le c\|f\|_2^2. \tag{4.23}$$

If $n \ge 2$, then for all $j \in \{1,\dots,n\}$

$$\sup_{x_j} \int_{\mathbb{R}^n} |D_{x_j}^{1/2} e^{it\Delta} f(x)|^2 dx_1 \cdots dx_{j-1} dx_{j+1} \cdots dx_n\,dt \le c\|f\|_2^2, \tag{4.24}$$

where $D_{x_j}^{1/2} g(x,t) = ((2\pi|\xi_j|)^{1/2} \widehat{g}(\xi,t))^{\vee}(x,t)$ denotes the homogeneous fractional derivative of order $1/2$ in the variable x_j.

Proof. We begin considering the case $n = 1$. So,

$$\begin{aligned} D_x^{1/2} e^{it\Delta} f &= c\big(|\xi|^{1/2} e^{-4\pi^2 it|\xi|^2} \widehat{f}(\xi)\big)^{\vee} \\ &= c\big(|\xi|^{1/2} e^{-4\pi^2 it|\xi|^2} \widehat{f}_+(\xi)\big)^{\vee} + c\big(|\xi|^{1/2} e^{-4\pi^2 it|\xi|^2} \widehat{f}_-(\xi)\big)^{\vee}, \end{aligned}$$

where $\widehat{f_\pm}(\xi) = \chi_{\mathbb{R}^\pm}\widehat{f}(\xi)$. Thus, it is enough to show (4.23) with f_+ replacing f. A combination of the change of variables $2\pi\xi^2 = r$, Plancherel's theorem (1.11) and the inverse change of variables $\xi = +\sqrt{r/2\pi}$ produce the following identities:

$$\begin{aligned}\int_{-\infty}^{\infty} |D_x^{1/2} e^{it\Delta} f_+|^2(x)\, dt &= c\int_{-\infty}^{\infty}\Big|\int_{-\infty}^{\infty} |\xi|^{1/2} e^{2\pi i x\xi} e^{-4\pi^2 it\xi^2}\widehat{f_+}(\xi)\, d\xi\Big|^2 dt\\ &= c\int_{-\infty}^{\infty}\Big|\int_0^{\infty} r^{1/4} e^{-2\pi itr} e^{ix\sqrt{2\pi r}}\widehat{f_+}\left(\sqrt{\frac{r}{2\pi}}\right)\frac{dr}{r^{1/2}}\Big|^2 dt\\ &= c\int_0^{\infty}\Big|e^{ix\sqrt{2\pi r}}\widehat{f_+}\left(\sqrt{\frac{r}{2\pi}}\right)\frac{1}{r^{1/4}}\Big|^2 dr = c\int_{-\infty}^{\infty}|\widehat{f_+}(\xi)|^2\, d\xi = c\,\|f_+\|_2^2,\end{aligned}$$

which gives (4.23). Moreover, when $\widehat{f}$ has support in $[0,\infty)$ or $(-\infty, 0]$, inequality (4.23) becomes an equality.

To obtain (4.24), we fix $j = 1$ to simplify the notation. We then define $\widehat{f_\pm}(\xi) = \chi_{\mathbb{R}^\pm}(\xi_1)\widehat{f}(\xi)$. Without the loss of generality , we prove (4.24) with f_+ replacing f.

Denote $\bar{x} = (x_2, \ldots, x_n)$ and $\bar{\xi} = (\xi_2, \ldots, \xi_n)$. The change of variables

$$(\xi_1, \xi_2, \ldots, \xi_n) = (\xi_1, \bar{\xi}) \overset{\Phi}{\to} (2\pi(\xi_1^2 + \cdots + \xi_n^2), \bar{\xi}) = (r, \bar{\xi}),$$

$$d\xi_1 d\bar{\xi} = \left|\begin{pmatrix} \frac{\partial r}{\partial \xi_1} & \frac{\partial r}{\partial \xi_2} & \cdots & \frac{\partial r}{\partial \xi_n}\\ 0 & 1 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & 1\end{pmatrix}\right|^{-1} dr d\bar{\xi} = \frac{1}{4\pi|\xi_1|}\, dr\, d\bar{\xi},$$

Plancherel's identity (1.11) and the change of variables Φ^{-1} yield

$$\begin{aligned}\|D_{x_1}^{1/2} e^{it\Delta} f_+\|^2_{L^2_{\bar{x}t}} &= c\,\Big\|\int_{\mathbb{R}^n} e^{2\pi x\cdot\xi}\,|\xi_1|^{1/2} e^{-4\pi^2 it|\xi|^2}\,\widehat{f_+}(\xi) d\xi\,\Big\|^2_{L^2_{\bar{x}t}}\\ &= c\,\Big\|\int_{\mathbb{R}^n} e^{2\pi i(\bar{x}\cdot\bar{\xi}+rt)}\frac{1}{|\xi_1|^{1/2}}\, e^{2\pi x_1\sqrt{\frac{|r-2\pi|\bar{\xi}|^2|}{2\pi}}}\,\widehat{f_+}(r,\bar{\xi})\, dr d\bar{\xi}\,\Big\|_{L^2_{\bar{x}t}}\\ &= c\int_{\mathbb{R}^n}\frac{1}{|\xi_1|}|\widehat{f_+}(r,\bar{\xi})|^2\, dr d\bar{\xi} = c\,\|\widehat{f_+}\|^2_{L^2_\xi} = c\,\|f_+\|^2_{L^2_x},\end{aligned}$$

which leads to (4.24). □

Corollary 4.2.

$$\left(\int_{-\infty}^{\infty}\int_{\{|x|\le R\}} |D_x^{1/2}e^{it\Delta}f|^2(x)\,dx\,dt\right)^{1/2} \le c\,R^{1/2}\,\|f\|_2, \tag{4.25}$$

where $D_x^{1/2}v(x,t) = ((2\pi|\xi|)^{1/2}\widehat{v}(\xi,t))^{\vee}$.

Notice that from this result and the translation invariance property of the solution one gets

$$\sup_{x_0\in\mathbb{R}^n,\ R>0}\left(\frac{1}{R}\int_{-\infty}^{\infty}\int_{B_R(x_0)} |D_x^{1/2}e^{it\Delta}f(x)|^2\,dxdt\right)^{1/2} \le c\|f\|_2.$$

Proof. If $n=1$, inequality (4.25) follows from (4.23).

Consider the case $n\ge 2$. Defining $D_j = \{\xi\in\mathbb{R}^n : |\xi_j| > \frac{1}{\sqrt{2n}}|\xi|\}$, with $j=1,\dots,n$. It is easy to see that $\bigcup_{j=1}^{n} D_j = \mathbb{R}^n - \{0\}$. Let $\{\phi_j\}_{j=1}^n$ be a partition of unity subordinate to the covering $\{D_j\}_{j=1}^n$ (the ϕ_j can be defined in the sphere $\mathbb{S}^{n-1}$ and extended such that they are homogeneous of order zero). Using linearity it suffices to show that

$$\int_{-\infty}^{\infty}\int_{\{|x|\le R\}} |e^{it\Delta}f(x)|^2\,dx\,dt \le c\,R\|D_x^{-1/2}f\|_2^2 = cR\||\xi|^{-1/2}\widehat{f}\|_2^2.$$

From (4.24), we obtain for all $j=1,\dots,n$,

$$\int_{-\infty}^{\infty}\int_{\{|x|\le R\}} |e^{it\Delta}g(x)|^2\,dx\,d\boldsymbol{t} \le c\,R\|D_{x_j}^{-1/2}g\|_2^2.$$

Therefore, using the notation $\widehat{f_j} = \widehat{f}\phi_j$, $j=1,\dots,n$, it follows that

$$\begin{aligned}
\int_{-\infty}^{\infty}\int_{\{|x|\le R\}} |e^{it\Delta}f|^2(x)\,dxdt &\le c\sum_{j=1}^{n}\int_{-\infty}^{\infty}\int_{\{|x|\le R\}} |e^{it\Delta}f_j|^2(x)\,dxdt\\
&\le c\,R\sum_{j=1}^{n}\big\|D_{x_j}^{-1/2}f_j\big\|_2^2 = c\,R\sum_{j=1}^{n}\big\||\xi_j|^{-1/2}\widehat{f_j}\big\|_2^2\\
&= c\,R\sum_{j=1}^{n}\big\||\xi_j|^{-1/2}\widehat{f}\phi_j\big\|_2^2 \le c\,R\big\||\xi|^{-1/2}\widehat{f}\big\|_2^2\\
&= c\,R\,\|D_x^{-1/2}f\|_2^2.
\end{aligned}$$

□

From Corollary 4.2 and the group properties, we deduce that if $f \in L^2(\mathbb{R}^n)$, then $e^{it\Delta} f \in L^2_{loc}(\mathbb{R} : H^{1/2}_{loc}(\mathbb{R}^n))$ and thus $e^{it\Delta} f \in H^{1/2}_{loc}(\mathbb{R}^n)$ for almost every $t \in \mathbb{R}$.

On the other hand, from (4.23) (case $n = 1$) using duality we have

$$\left\| D_x^{1/2} \int_{-\infty}^{\infty} e^{it\Delta} F(\cdot, t)\, dt \right\|_2 \leq c \int_{-\infty}^{\infty} \|F(x, \cdot)\|_2\, dx. \tag{4.26}$$

Similarly, from (4.24) we obtain the corresponding inequality for the case $n \geq 2$.

For solutions of the inhomogeneous problem:

$$\begin{cases} \partial_t u = i\Delta u + F(x,t), \\ u(x,0) = 0, \end{cases} \tag{4.27}$$

$x \in \mathbb{R}^n,\ t \in \mathbb{R}$, we observe that the gain of derivatives doubles that obtained in the homogeneous case.

Theorem 4.4. *If $u(x,t)$ is the solution of problem (4.27), then, when $n = 1$ it satisfies*

$$\sup_x \left(\int_{-\infty}^{\infty} |\partial_x u(x,t)|^2\, dt \right)^{1/2} \leq c \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} |F(x,t)|^2 dt \right)^{1/2} dx, \tag{4.28}$$

and in the case $n \geq 2$

$$\sup_{x_j} \left(\int_{\mathbb{R}^n} |\partial_{x_j} u(x,t)|^2 d\mu_j dt \right)^{1/2} \leq c \int_{-\infty}^{\infty} \left(\int_{\mathbb{R}^n} |F(x,t)|^2 d\mu_j dt \right)^{1/2} dx_j, \tag{4.29}$$

where $d\mu_j = dx_1 \cdots dx_{j-1}\, dx_{j+1} \cdots dx_n$. Therefore in the case $n \geq 2$ we have that

$$\sup_\alpha \left(\int_{Q_\alpha} \int_{-\infty}^{\infty} |\partial_x u(x,t)|^2\, dt\, dx \right)^{1/2} \leq c \sum_\alpha \left(\int_{Q_\alpha} \int_{-\infty}^{\infty} |F(x,t)|^2 dt\, dx \right)^{1/2}, \tag{4.30}$$

where $\{Q_\alpha\}_{\alpha \in \mathbb{Z}^n}$ denotes a family of disjoint unit cubes with sides parallel to the axes and covering $\mathbb{R}^n$.

Proof. We only sketch the proof in the case $n = 1$. Using Exercise 4.16 in this chapter, we deduce that

$$\begin{aligned} \partial_x u(x,t) &= \int_{-\infty}^{\infty} \int_{\mathbb{R}^n} \frac{2\pi i \xi}{4\pi^2 i |\xi|^2 + 2\pi i \tau} (e^{2\pi i \tau t} - e^{-4\pi^2 i |\xi|^2 t}) e^{2\pi i x \cdot \xi} \widehat{F}(\xi, \tau) d\xi d\tau \\ &= \int_{-\infty}^{\infty} \int_{\mathbb{R}^n} \frac{2\pi i \xi\ e^{2\pi i \tau t}}{4\pi^2 i |\xi|^2 + 2\pi i \tau} e^{2\pi i x \cdot \xi} \widehat{F}(\xi, \tau)\, d\xi d\tau \end{aligned} \tag{4.31}$$

$$- \int_{-\infty}^{\infty} \int_{\mathbb{R}^n} \frac{2\pi i\xi \; e^{-4\pi^2 i|\xi|^2 t}}{4\pi^2 i|\xi|^2 + 2\pi i\tau} e^{2\pi i x\cdot\xi} \widehat{F}(\xi,\tau)\, d\xi d\tau$$
$$= \partial_x u_1(x,t) + \partial_x u_2(x,t),$$

where $\widehat{F}(\xi,\eta)$ represents the Fourier transform with respect to the variables x, t. Since the numerator in the first integrand vanishes on the zeros of its denominator, the integrals in the second equality are understood in the principal value sense. From Exercise 1.17, we have that

$$\left(\text{p.v.}\, \frac{2\pi i\xi}{4\pi^2 i|\xi|^2 + 2\pi i\tau}\right)^{\vee(\xi)} = K(x,\tau) \in L^\infty(\mathbb{R}^2).$$

Plancherel's identity (1.11), Young's and Minkowski's inequalities, (1.39) and (1.40), respectively, imply that for all $x \in \mathbb{R}$,

$$\left(\int_{-\infty}^{\infty} |\partial_x u_1(x,t)|^2\, dt\right)^{1/2} = c\left\| \int_{-\infty}^{\infty} e^{2\pi i\tau t} \int_{-\infty}^{\infty} K(x-y,\tau)\widehat{F}^{(t)}(y,\tau)\, dy d\tau \right\|_{2(t)}$$
$$= c\left\| \int_{-\infty}^{\infty} K(x-y,\tau)\widehat{F}^{(t)}(y,\tau)\, dy \right\|_{2(\tau)}$$
$$\leq c\int_{-\infty}^{\infty} \left\|\widehat{F}^{(t)}(y,\cdot)\right\|_{2(\tau)} dy \leq c\int_{-\infty}^{\infty} \left\|F(y,\cdot)\right\|_{2(t)} dy,$$

which proves

$$\sup_x \left(\int_{-\infty}^{\infty} |\partial_x u_1(x,t)|^2\, dt\right)^{1/2} \leq c\int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} |F(x,t)|^2\, dt\right)^{1/2} dx.$$

On the other hand, we have that

$$\partial_x u_2(x,t) = D_x^{1/2} e^{it\Delta} G(x),$$

where

$$\widehat{G}(\xi) = c\int_{-\infty}^{\infty} \frac{\text{sgn}(\xi)\; |\xi|^{1/2}\; \widehat{F}(\xi,\tau)}{4\pi^2 i|\xi|^2 + 2\pi i\tau}\, d\tau.$$

A simple computation and (1.18) shows that

$$\left(\text{p.v.}\, \frac{1}{4\pi^2 i|\xi|^2 + 2\pi i\tau}\right)^{\vee(\tau)} = \int_{-\infty}^{\infty} \frac{e^{-2\pi i\tau t}}{4\pi^2 i|\xi|^2 + 2\pi i\tau}\, d\tau = c\; \text{sgn}(t)\; e^{-4\pi^2 i|\xi|^2 t}.$$

Therefore, using (4.23), (4.26), and Plancherel's identity (1.11), we infer that

$$\sup_x \left(\int_{-\infty}^{\infty} |\partial_x u_2(x,t)|^2 \, dt \right)^{1/2} \leq c \left\| \int_{-\infty}^{\infty} \frac{\operatorname{sgn}(\xi)\ |\xi|^{1/2}\ \widehat{F}(\xi,\tau)}{4\pi^2 i |\xi|^2 + 2\pi i \tau} \, d\tau \right\|_{2(\xi)}$$

$$= c \left\| \int_{-\infty}^{\infty} e^{-4\pi^2 i |\xi|^2 t} \operatorname{sgn}(\xi)\, |\xi|^{1/2} \widehat{F}^{(x)}(\xi,t) \operatorname{sgn}(t)\, dt \right\|_{2(\xi)}$$

$$= c \left\| \left(\int_{-\infty}^{\infty} e^{it\Delta} D_x^{1/2} \mathsf{H} F(\cdot,t) \operatorname{sgn}(t)\, dt \right)^{\vee} \right\|_{2(\xi)}$$

$$= c \left\| D_x^{1/2} \int_{-\infty}^{\infty} e^{it\Delta} \mathsf{H} F(\cdot,t) \operatorname{sgn}(t)\, dt \right\|_2$$

$$\leq c \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} |F(x,t)|^2 \, dt \right)^{1/2} dx,$$

where H denotes the Hilbert transform (see Definition 1.7). This leads to the result.□

4.4 Comments

The first result concerning smoothing effects for the particular group $\{e^{it\Delta}\}_{t=-\infty}^{\infty}$ or for general group of unitary operators was obtained by Kato in [K1]. In this work on theory of operators, Kato introduced the notion of A-regular and A-super regular operators.

Let A be a self-adjoint operator (not necessarily bounded) defined on a Hilbert space H such that the resolvent of A, $R(\lambda) = (\lambda I - A)^{-1}$, exists for all $\lambda \in \mathbb{C}$ with $\mathcal{I}m\, \lambda \neq 0$ and let L be an operator of closed graph with domain $D(L)$ dense in H.

Definition 4.1. We say that the operator L is *A-regular* (respectively, A-super regular) if for all $x \in D(L^*)$ and for all $\lambda \in \mathbb{C}$ with $\mathcal{I}m\ \lambda \neq 0$,

$$|\,\mathcal{I}m\ < R(\lambda) L^* x, L^* x >\,| \leq c\,\pi\, \|x\|^2$$

(respectively, $|\langle R(\lambda) L^* x, L^* x\rangle| \leq c\pi\, \|x\|^2$), where the constant c is independent of x and λ.

The following theorems establish the relationship between the notion of A-regular operator and the type of results described in this chapter .

Theorem 4.5 **([K1]).** *The operator L is A-regular if and only if for all $x \in H$*

$$\int_{-\infty}^{\infty} \|Le^{itA}x\|\,dt \le c\|x\|.$$

In particular, $e^{itA}x \in D(L)$ for almost every $t \in \mathbb{R}$.

Theorem 4.6 **([KY]).** *Let $L = L_h$ be an operator of multiplication by h with $h \in L^n(\mathbb{R}^n)$ and $n \ge 3$. Then, L_h is Δ-super regular.*

Theorem 4.7 **([KY] see also [BKl]).** *Let $\tilde{L}$ be the operator*

$$(1+|x|^2)^{-1/2}\Lambda^{1/2} = (1+|x|^2)^{-1/2}(1-\Delta)^{1/4}$$

with domain $C_0^\infty(\mathbb{R}^n)$ and $n \ge 3$. Then, the closure of $\tilde{L}$ is Δ-super regular.

Combining Theorems 4.5 and 4.6, we have that if $f \in L^2(\mathbb{R}^n)$ with $n \ge 3$, then $e^{it\Delta}f \in D(L_h)$ for almost every $t \in \mathbb{R}$. When $h \notin L^\infty(\mathbb{R}^n)$, then $D(L_h)$ is a set of first category in $L^2(\mathbb{R}^n)$. These results neither imply nor are consequence of the estimate (4.14) in Theorem 4.2.

Later on, Strichartz [Str3], motivated by the work of Segal [Se], studied special properties of the Fourier transform. He proved that

$$\left(\int_{-\infty}^{\infty}\int_{\mathbb{R}^n} |e^{it\Delta}f|^{2(n+2)/n}\,dx\,dt\right)^{n/2(n+2)} \le c\|f\|_2. \tag{4.32}$$

In his proof, he employed previous results of Tomas [Tm] and Stein [S2] regarding restriction theorems (and extension) of the Fourier transform. More precisely, Strichartz used the fact that

$$\begin{aligned} e^{it\Delta}f(x) &= \int_{\mathbb{R}^n} e^{2\pi i x\cdot\xi} e^{-4\pi^2 it|\xi|^2}\widehat{f}(\xi)\,d\xi \\ &= \int_{\mathbb{R}^{n+1}} e^{2\pi i <(x,t);(\xi,\tau)>} g(\xi,\tau)\,d\sigma(\xi,\tau) = \widehat{g d\sigma}, \end{aligned}$$

where g is a measure supported on the hypersurface $M_\sigma \subset \mathbb{R}^{n+1}$, where the symbol $\sigma(\xi,\eta) = \eta + 2\pi|\xi|^2$ vanishes, i.e.,

$$M_\sigma = \{(\xi,\tau) \in \mathbb{R}^n \times \mathbb{R} \ : \ \sigma(\xi,\tau) = 0\} \tag{4.33}$$

(in this case, $\sigma(\xi,\eta) = \eta + 2\pi|\xi|^2$), with density $\widehat{f}(\xi)$ and $d\sigma(\tilde{\xi}) = d\xi$.

Similarly,

$$\widehat{e^{it\Delta}f}(\xi,\tau) = \int_{-\infty}^{\infty} e^{-2\pi it\tau} e^{-4\pi^2 it|\xi|^2}\widehat{f}(\xi)\,dt = \widehat{f}(\xi)\;\delta(\tau + 2\pi|\xi|^2),$$

where $\widehat{\ }$ on the left-hand side denotes the Fourier transform with respect to both variables: space x and time t. In other words, the Fourier transform in the variables (x,t) of the solution $e^{it\Delta}f(x)$ is a distribution with support on the parabola $\tau = -2\pi|\xi|^2$. Thus, inequality (4.14) can be seen as a result on the extension of the Fourier transform of measures with support on this parabola. Similarly, we can see (4.16) as a result of restriction because using the Fubini theorem and the Plancherel identity (1.11) we have,

$$\begin{aligned}\left\| \int_{-\infty}^{\infty} e^{it\Delta} g(\cdot,t)\,dt \right\|_2 &= \left\| \int_{-\infty}^{\infty} \left(\int_{\mathbb{R}^n} e^{2\pi i x\cdot\xi} e^{-4\pi^2 it|\xi|^2} \widehat{g}(\xi,t)\,d\xi \right) dt \right\|_2 \\ &= \left\| \int_{\mathbb{R}^n} e^{2\pi i x\cdot\xi} \left(\int_{-\infty}^{\infty} e^{-4\pi^2 it|\xi|^2} \widehat{g}(\xi,t)\,dt \right) d\xi \right\|_2 = \|\,\widehat{g}\,(\xi,-2\pi|\xi|^2)\|_2.\end{aligned} \tag{4.34}$$

The proof presented in Section 4.2 is due to J. Ginibre and G. Velo [GV1] (see also [M], [P1]).

The main point in the argument is the *curvature* of the hypersurface M_σ defined by the symbol σ as in (4.33) and not the ellipticity of Δ. In particular, the same inequalities (4.14), (4.17) hold when we replace Δ by

$$\mathfrak{L}_j = \partial_{x_1}^2 + \cdots + \partial_{x_j}^2 - \partial_{x_{j+1}}^2 - \cdots \partial_{x_n}^2, \quad \text{for some} \quad j \in \{1,\dots,n\}. \tag{4.35}$$

The curvature of hypersurface M_σ for the symbol $\sigma = \tau + 2\pi|\xi|^2$ is reflected on the decay estimates (4.12) in Lemma 4.1. In fact, the results in Theorem 4.2 are true for any unitary group satisfying decay estimates of the type described in Lemma 4.1. Thus, in particular for the linear problem associated to the KdV equation (1.28), we have that the unitary group $V(t)v_0 = (e^{i8\pi^3\xi^3 t}\widehat{v_0})^\vee$ describing the solutions satisfies for any $(\theta,\alpha) \in [0,1] \times [0,1/2]$

$$\left\| D^{\alpha\theta/2} V(t)v_0 \right\|_{L^{2/1-\theta}} \le c\,|t|^{-\theta(\alpha+1)/3} \|v_0\|_{L^{2/1+\theta}}. \tag{4.36}$$

Therefore, the argument used in Theorem 4.2 shows that for any $(\theta,\alpha) \in [0,1] \times [0,1/2]$,

$$\left\| D^{\alpha\theta/2} V(t)v_0 \right\|_{L^q(\mathbb{R}:\,L^p(\mathbb{R}))} \le c\,\|v_0\|_2, \tag{4.37}$$

where $(q,p) = (6/\theta(\alpha+1), 2/(1-\theta))$. Notice that in (4.37) there is a possible gain of 1/4 derivatives. Roughly speaking, in general this gain is equal to $(m-2)/4$, where m is the order of the dispersive operator (see [KPV2]).

In the case of the IVP associated wave equation:

$$\begin{cases} \partial_t^2 w = \Delta w, \\ w(x,0) = 0, \\ \partial_t w(x,0) = g(x), \end{cases} \tag{4.38}$$

$x \in \mathbb{R}^n$, $t \in \mathbb{R}^+$, whose solution

$$w(x,t) = U(t)g = \Big(\frac{\sin(2\pi|\xi|t)}{2\pi|\xi|}\,\widehat{g}(\xi)\Big)^{\vee}$$

(see (1.49)) is associated to the unitary group $M(t) = (e^{i2\pi|\xi|t}\,\widehat{g})^{\vee}$, we have the decay estimate:

$$\|U(t)g\|_{L^p(\mathbb{R}^n)} \leq c\, t^{(n-1)(\frac{1}{2}-\frac{1}{p'})}\|D^{\alpha}g\|_{L^{p'}(\mathbb{R}^n)}, \tag{4.39}$$

with

$$\alpha = \frac{n-1}{2} - \frac{n+1}{p'}, \quad 2 \leq p < \infty, \quad n \geq 2.$$

From this, we can deduce the equivalent to Theorem 4.2:

$$\big\|(-\Delta)^{(1-b)/4}\, U(t)g\big\|_{L^q(\mathbb{R}:\, L^p(\mathbb{R}^n))} \leq c\,\|g\|_2, \tag{4.40}$$

where

$$2 < q < \infty, \quad \frac{1}{2} - \frac{2}{(n-1)q} = \frac{1}{p}, \quad \text{and} \quad b = \frac{n-1}{2} - \frac{n+1}{p}$$

(see [M], [P1]).

As we mentioned above, the decay estimates (4.12), (4.36), and (4.39) are related to the "curvature" of the hypersurfaces M_{σ_j}, $j = 1, 2, 3$, which described the zero set of the symbols $\sigma_1 = \tau + 2\pi|\xi|^2$, $\sigma_2 = \tau - 4\pi^2\xi^3$, and $\sigma_3 = \tau \pm |\xi|$, respectively. In the case σ_1 and σ_3, we observe that the hypersurfaces M_{σ_1} and M_{σ_3} have nonvanishing curvature in n and $n-1$ directions (rank of the Hessian), respectively.

In the limiting case, the inequality (4.14) in dimension $n = 2$ (i.e., $(q, p) = (2, \infty)$) fails (see [MSm]). Similarly, the limiting case of the estimate (4.40) for the wave equation in dimension $n = 3$ (i.e., $(q, p) = (2, \infty)$) fails (see [KlM]) although both hold in the radial case; see [To1] for the Schrödinger equation and [KlM] for the wave equation. Moreover, in the case of the Schrödinger equation ($(n, p, q) = (2, \infty, 2)$) one has the following generalization of the radial result, see [To1]

$$\|e^{it\Delta}u_0\|_{L^2_t(\mathbb{R}:\, L^\infty_r L^2_\theta(\mathbb{R}^2))} \leq c\,\|u_0\|_2,$$

where

$$\|f\|_{L^\infty_r L^2_\theta} = \sup_{r>0}\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}|f(re^{i\theta})|^2\,d\theta\right)^{1/2}.$$

In [KT1], the limiting cases in higher dimension were shown to hold in both cases, i.e., the Schrödinger equation (4.14) holds for $n \geq 3$, $(q, p) = (2, 2n/(n-2))$, as well as the wave equation in (4.40) hold for $n \geq 4$, $(q, p) = (2, 2(n-1)/(n-3))$.

The problem of finding the best constant for the Strichartz estimate (4.14):

$$c(n;p;q) = \sup_{\|u_0\|_2=1} \left(\int_{-\infty}^{\infty} \|e^{it\Delta}u_0\|_p^q dt \right)^{1/q} \tag{4.41}$$

as well as its maximizers, i.e., the $u_0 \in L^2(\mathbb{R}^n)$ for which the equality (4.41) holds with (p,q) as in (4.18) has been studied in several works. In [Kz], it was proved the existence of a maximizer for $n = 1$ and $p = q = 6$. In [Fs] and [HuZ], it was established that for the case $n = 1,2$ and $p = q = 2 + 4/n$, one has $c(1;6;6) = 12^{-1/12}$ and $c(2;4;4) = 2^{-1/2}$ with the maximizer, up to the invariant of the Schrödinger equation (see Proposition 4.1), equal to $c_n e^{-|x|^2}$, $n = 1,2$. Also in [Fs], the same problem was settled for the case of the wave equation (4.38) in dimension $n = 2,3$ with $p = q = 2 + 4/(n-1)$. The value $c(1;8;4) = 2^{-1/4}$ in (4.41) was computed in [BBCH] and [Car].

Corollary 4.1 was proved in [CzW1]. For further results in this directio, we refer to [Vi1].

Concerning the decay of the free Schrödinger equation, on one hand, one has that if $u_0 \in C_0^\infty(\mathbb{R}^n)$ with $u_0 \not\equiv 0$, then for any $t \neq 0$ and any $\epsilon > 0$, $e^{it\Delta}u_0 \in \mathcal{S}(\mathbb{R}^n)\backslash L^1(e^{\epsilon|x|}\,dx)$ (see Exercises 4.4 and 4.5). On the other hand, Example 4.2 tells us that solutions corresponding to Gaussian data exhibits a global Gaussian decay. In [EKPV1], it was shown that given $u_0 \in \mathcal{S}'(\mathbb{R}^n)$ the following conditions are equivalent:

(i) There are two different real numbers t_1 and t_2, such that $e^{it_j\Delta}u_0 \in L^2(e^{a_j|x|^2}dx)$ for some $a_j > 0$, $j = 1,2$.
(ii) $u_0 \in L^2(e^{b_1|x|^2}dx)$ and $\widehat{u}_0 \in L^2(e^{b_2|x|^2}dx)$, for some $b_j > 0$, $j = 1,2$.
(iii) There is $\nu : [0,+\infty) \longrightarrow (0,+\infty)$, such that $e^{it\Delta}u_0 \in L^2(e^{\nu(t)|x|^2}dx)$, for all $t \geq 0$.
(iv) $u_0(x+iy)$ is an entire function such that $|u_0(x+iy)| \leq N\,e^{-a|x|^2+b|y|^2}$ for some constants N, a, $b > 0$.
(v) There exist δ, $\epsilon > 0$, and $h \in L^2(e^{\epsilon|x|^2}dx)$ such that $u_0(x) = e^{\delta\Delta}h(x)$.

It was also established in [EKPV1] that if one of the above conditions holds then for appropriate values $\alpha, \beta > 0$ the function

$$f(t) = \left\| e^{\frac{|x|^2}{(\alpha t+\beta)^2}} e^{it\Delta}u_0 \right\|_2$$

is logarithmically convex. In particular, one has that

$$f(t) \leq f(0)^{\theta(t)}\, f(T)^{1-\theta(t)},$$

with $\theta(t) = \beta(T-t)/(T(\alpha t+\beta))$ for all $t \in [0,T]$.

In [EKPV1], the constants used above were described in a precise manner as a consequence of (4.7) and the following result due to Hardy for $n = 1$ [H] and its extension to higher dimension given in [SS]: if $f(x) = O(e^{-\pi A|x|^2})$ and $\hat{f}(\xi) = O(e^{-\pi B|\xi|^2})$, with $A > 0$, $B > 0$, and $AB > 1$, then $f \equiv 0$.

Extensions of these results to the case of Schrödinger equation, with potential (in an appropriate class) as (4.42) below depending on x or on (x,t), i.e., $V = V(x,t)$ (as well as application to unique continuation properties of semilinear Schrödinger equations) were given in [EKPV2].

Consider the IVP associated to the Schrödinger equation with a potential V:

$$\begin{cases} i\partial_t u = \Delta u - V(x)\,u, \\ u(x,0) = u_0(x). \end{cases} \tag{4.42}$$

Assume first that the potential $V = V(x)$ is real and regular enough such that $L = -\Delta + V(x)$ is self-adjoint.

A natural question is whether or not the unitary group $e^{itL} = e^{it(-\Delta+V)}$ satisfies the $L^\infty - L^1$ estimate in (4.13) as in the free case $V \equiv 0$, i.e., there exists $c > 0$ for every $f \in L^2(\mathbb{R}^n)$ such that

$$\|e^{itL} f\|_\infty \leq c\, t^{-n/2} \|f\|_1. \tag{4.43}$$

If L has an eigenvalue (with eigenfunction $f \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$), (4.43) fails. Similarly, if zero is a resonance of L. So, one reformulates the inequality (4.43) as

$$\|e^{itL} P_{\mathrm{ac}}(L) f\|_\infty \leq c|t|^{-n/2} \|f\|_1, \tag{4.44}$$

where $P_{\mathrm{ac}}(L)$ defines the projection onto the absolutely continuous spectrum of L.

The following conditions on the decay of V have been shown to be sufficient for (4.43) to hold: $n = 1$ and $(1+|x|)\,V \in L^1(\mathbb{R})$ [GSch], $n = 2$ and $|V(x)| \leq c\,(1+|x|)^{-3-\epsilon}$ [Scl1], $n = 3$ and $V \in L^{3/2-\epsilon}(\mathbb{R}^3)$ [Gb].

In [JSS], for $n \geq 3$ sufficient conditions on the decay and regularity on the potential $V(x)$ which guarantees (4.43) were deduced. In [GVi], it was shown that for $n > 3$ decay assumptions alone do not imply the estimate (4.43). More precisely, it was proved that (4.43) fails for any potential V with compact support such that $\sum_{|\alpha| \leq \frac{n-3}{2}} \|\partial^\alpha V\|_\infty \leq 1$.

The cases of time-dependent potentials have been also studied (see for instance [RS]). Also, decay estimates of the type in (4.43) with electromagnetic potentials were obtained in [FFFP].

For conditions on the potential V that guarantee the extension of the local smoothing effect described in Corollary 4.2 to solutions of the IVP (4.42) see [RV], [BRV].

Local-in-time extensions of Strichartz estimates to the variable coefficients' case, where the Laplacian Δ is replaced by an elliptic operator of the form:

$$L = \partial_{x_k} a_{jk}(x,t)\partial_{x_j} + \partial_{x_l} b_l(x,t) + b_l(x,t)\partial_{x_l} + V(x,t) \tag{4.45}$$

have been considered in several works. In [StTa], Staffilani and Tataru established these estimates under the assumptions: $b_l = V = 0$, $(a_{jk}(x,t))$ a compactly supported perturbation of the Laplacian and a nontrapping condition on the bicharacteric flow. Extensions of this result under appropriate hypotheses on the "asymptotic flatness" and the nontrapping condition of the coefficients a_{jk} were given in [MMTa1], [RZ], [Td]. The one-dimensional case was considered in [Sl].

Next, we briefly treat the periodic case:

$$\begin{cases} i\partial_t u = \partial_x^2 u, \\ u(x,0) = u_0(x), \end{cases} \tag{4.46}$$

$x \in \mathbb{S}^1 \times \cdots \times \mathbb{S}^1$, $t \in \mathbb{S}^1$.

Theorem 4.8 ([Z]).

$$\Big\| \sum_{k=-\infty}^{\infty} a_k e^{i(tk^2+kx)} \Big\|_{L^4(\mathbb{T}^2)} \le c \left(\sum_{k=-\infty}^{\infty} |a_k|^2 \right)^{1/2}, \tag{4.47}$$

where $(x,t) \in \mathbb{S}^1 \times \mathbb{S}^1 = \mathbb{T}^2$.

Note that $u(x,t) = \sum_k a_k e^{i(tk^2+kx)}$ is the solution of the periodic problem (4.46) for $n = 1$ with $u_0(x) = \sum_k a_k e^{ikx}$.

Proof. If $u(x,t) = \sum_k a_k e^{i(tk^2+kx)}$, then $\|u\|^2_{L^4(\mathbb{T}^2)} = \|u \cdot \bar{u}\|_{L^2(\mathbb{T}^2)}$. It is easy to see that

$$u\bar{u} = \sum_k |a_k|^2 + \sum_{k_1 \neq k_2} a_{k_1} \bar{a}_{k_2} e^{i((k_1-k_2)x + (k_1^2-k_2^2)t)}.$$

If we fix $l_1 = k_1 - k_2$ and $l_2 = k_1^2 - k_2^2$ we have at most one pair (k_1, k_2) of solutions of these equations. So, we can conclude that

$$\begin{aligned} \|u \cdot \bar{u}\|_2 &= \sum_k |a_k|^2 + \left(\sum_{k_1 \neq k_2} |a_{k_1} \bar{a}_{k_2}|^2 \right)^{1/2} \\ &\le \sum_k |a_k|^2 + \left(\sum_{k_1} |a_{k_1}|^2 \sum_{k_2} |a_{k_2}|^2 \right)^{1/2} = 2 \sum_k |a_k|^2. \end{aligned}$$

□

We observe that for the case $n = 1$, the corresponding inequality to (4.32) in $\mathbb{R}$ is true with $p = 6$. So, the next question is natural: Is the inequality (4.47) still true if we substitute 4 by 6? The answer is negative. In fact, one has that

$$\Big\| \sum_{k=1}^{N} e^{i(kx+k^2t)} \Big\|_{L^6(\mathbb{T}^2)} \gtrsim (\log N)^{1/6} N^{1/2}. \tag{4.48}$$

So, if $\phi = \sum_{k=1}^{N} e^{ikx}$, then $\|\phi\|_2 = N^{1/2}$, which combined with (4.48) implies that

$$\|e^{it\Delta}\phi\|_{L^{\frac{2(n+2)}{n}}(\mathbb{T}^{n+1})} \le c\,\|\phi\|_2 \tag{4.49}$$

fails for $n = 1$.

Nevertheless, Bourgain [Bo1] proved that there exists a constant $c_0 > 0$ such that for all $\epsilon > 0$ and $N \in \mathbb{Z}^+$ we have

$$\Big\| \sum_{|k|\le N} a_k e^{i(tk^2+kx)} \Big\|_{L^6(\mathbb{T}^2)} \le c_0\, N^{\epsilon} \left(\sum_{|k|\le N} |a_k|^2 \right)^{1/2}. \tag{4.50}$$

It is an *open problem* to determine if the inequality can be obtained in the interval $(4,6)$. More precisely, it was conjectured in [Bo2] that

$$\big\|e^{it\Delta}\phi\big\|_{L^q(\mathbb{T}^{n+1})} \le c\,\|\phi\|_2 \ \text{ if } q < \frac{2(n+2)}{n}, \tag{4.51}$$

and assuming $\operatorname{supp}\widehat{\phi} \subset B(0,N)$

$$\big\|e^{it\Delta}\phi\big\|_{L^q(\mathbb{T}^{n+1})} \ll N^{\frac{n}{2}-\frac{n+2}{q}+\epsilon}\|\phi\|_2 \ \text{ if } \ q \ge \frac{2(n+2)}{n} \tag{4.52}$$

hold. In this direction, some partial results are gathering in the next proposition.

Proposition 4.4 ([Bo2]).

1. *For $n = 1, 2$, inequality (4.52) holds.*
2. *For $n \ge 3$, inequality (4.52) holds for $q \ge 4$.*

For details, see [Bo1] and [Bo2].

The extension of Theorem 4.8 to other compact manifolds (i.e., L^p–L^q estimates for the Schrödinger flow on manifolds) has been studied by Burq, Gerard and Tzvetkov [BGT3].

In the particular case of the two-dimensional sphere $\mathbb{S}^2$, they proved that

$$\left(\int_I \left(\int_{\mathbb{S}^2} \big|e^{it\Delta}u_0(x)\big|^q\, dx \right)^{p/q} dt \right)^{1/q} \le c_I\, \|u_0\|_{1/p,2}, \tag{4.53}$$

where I is a finite time interval and $\|\cdot\|_{1/p,2}$ is defined as in (3.38), for every admissible pair in (4.18) Theorem 4.2 with $n = 2$, i.e.,

$$\frac{1}{p} = \frac{1}{2} - \frac{1}{q}.$$

Roughly, (4.53) gives a gain of $1/2$ derivatives with respect to the Sobolev embedding (Theorem 3.3),

$$\|u_0\|_q \le c\|u_0\|_{1/r,2} \quad \text{with} \quad \frac{1}{r} = n\left(\frac{1}{2} - \frac{1}{q}\right).$$

The local smoothing effect studied in Section 4.3 was first established by T. Kato [K2] for solutions of the Korteweg–de Vries equation:

$$\begin{cases} \partial_t u + \partial_x^3 u + u\partial_x u = 0, \\ u(x,0) = u_0(x). \end{cases} \tag{4.54}$$

$t,\ x \in \mathbb{R}$. More precisely, Kato proved the following inequality:

$$\left(\int_{-T}^{T} \int_{-R}^{R} |\partial_x u(x,t)|^2 \, dx \, dt \right)^{1/2} \leq c(T,R) \|u_0\|_2, \tag{4.55}$$

which was the main ingredient in his proof of existence of the global weak solutions of (4.54) with initial data $u_0 \in L^2(\mathbb{R})$ (see [K2]). In [KF], Kruzhkov and Faminskii independently obtained a similar result to that described in (4.55). Later on and simultaneously, Constantin and Saut [CS], Sjölin [Sj], and Vega [V] showed that the estimates of the type in (4.55) are intrinsic properties of linear dispersive equations. Let $P(\xi)$ be the real symbol associated to the operator $P(D)$. Suppose that at infinity $P(\xi) \sim |\xi|^\alpha$, for α a real positive number, and $u(x,t) = e^{itP(D)} u_0(x)$, then

$$\left(\int_{-T}^{T} \int_{|x|\leq R} \left| (-\Delta)^{(\alpha-1)/4} u(x,t) \right|^2 dx \, dt \right)^{1/2} \leq c\,(T,R) \|u_0\|_2. \tag{4.56}$$

In particular, inequality (4.56) implies that if $u_0 \in L^2(\mathbb{R}^n)$, then the solutions $e^{itP(D)} u_0 \in H^{(\alpha-1)/2}_{\text{loc}}(\mathbb{R}^n)$ for almost all t. Notice that this gain of derivatives is a pure dispersive phenomenon, which cannot hold in hyperbolic problems.

The version of the homogeneous smoothing effect given here (Theorem 4.3) is taken from [KPV3] (see also [LP]). The inhomogeneous smoothing effect version described in Theorem 4.4 was first established in [KPV3]. Observe that the gain of derivatives here doubles from that in the homogeneous case. Also, one has that the result in Theorem 4.4 still holds with $\mathfrak{L}_j$ as in (4.35) instead of the Laplacian.

It is interesting to note that in [CS] the authors extended Kato's result (4.55) to linear dispersive equations. In contrast, in [Sj] and [V] inequality (4.56) with $\alpha = 2$ appears implicitly in the study of the following problem introduced by L. Carleson: Determine the minimum value of s which guarantees that if $u_0 \in H^s(\mathbb{R}^n)$, then

$$\lim_{t \downarrow 0} e^{it\Delta} u_0(x) = u_0(x) \ \text{ for almost every } x \in \mathbb{R}^n. \tag{4.57}$$

In the one-dimensional case $n = 1$, we have that $s \geq 1/4$ implies (4.57) (see [C]) and this is the best possible result (see [DK], [KR]). For the case $n = 2$, the best result asserting (4.57) is $s > 3/8$ obtained in [Le] (improving previous results of [Sj], [V], $s > 1/2$, [Bo3], $s > 1/2 - \epsilon$, [MVV2], $s > (164 + \sqrt{2})/339$, [TV] $s > 15/32$). In [Bo11], it was shown that in any dimension n the statement (4.57) holds if $s > 1/2 - 1/4n$ (improving previous results of [Sj], [V], $s > 1/2$). Moreover, it was also established in [Bo11] that for $n > 4$ the condition $s \geq \frac{n-2}{2n}$ is necessary for (4.57) to hold.

The original Kato's proof of the smoothing effect (4.55) was based on an energy estimate argument. Let us consider the linear problem (4.54) with data $u_0 \in L^2(\mathbb{R})$. Then multiplying the equation by $u(x,t)\varphi(Rx) = u(x,t)\varphi_R(x)$, $\varphi \in C^\infty(\mathbb{R})$, $(\varphi(x) =$

1 for $x > 2$, $\varphi(x) = 0$ for $x < -2$, with $\varphi'(x) > 0$ for $-1 < x < 1$ and $R > 0$), we obtain after integration by parts that

$$\frac{1}{2}\frac{d}{dt}\int u^2\varphi_R\,dx + \frac{3}{2}\int (\partial_x u)^2\,\varphi_R'\,dx - \frac{1}{2}\int u^2\varphi_R^{(3)}\,dx = 0.$$

Thus, integrating in the time interval $[0, T]$ and using that the L^2-norm of the solution is preserved we get (4.55).

The extension of the estimate (4.56) to general dispersive linear models (with constant coefficients) given in [CS] was based on a Fourier transform argument. In nonlinear problems and in linear ones with variable coefficients (where the Fourier transform does not provide the result) it may be useful to obtain the result via "energy estimates."

For example, consider the IVP:

$$\begin{cases} \partial_t u = i\,Au, \\ u(x,0) = u_0(x), \end{cases} \tag{4.58}$$

$x \in \mathbb{R}^n$, $t \in \mathbb{R}$, where A has a real symbol $a = a(x,\xi)$ of order m (for instance, $A = \partial_{x_j}(a_{jk}(x)\partial_{x_k})$, $i\partial_x^3$, Δ, and $i\mathsf{H}\partial_x^2$). By integration by parts, we have that the solutions $u(\cdot,t)$ preserve the L^2-norm, i.e., $\|u(\cdot,t)\|_2 = \|u_0\|_2$. Now to establish the corresponding local smoothing effect (4.55), we follow the argument in [CKS]. First, one applies an operator B of order zero with real symbol $b(x,\xi)$ to our equation to get:

$$\partial_t B\,u = i\,AB\,u + i\,[B;A]\,u. \tag{4.59}$$

By multiplying the equation (4.59) by $\bar{u}$ and the conjugate of equation (4.58) by Bu, adding the results and integrating in the x-variable, and then in the time interval $[0, T]$, it follows that

$$\int_0^T\int_{\mathbb{R}} i\,[B;A]\,u\,\bar{u}\,dx\,dt \le c_0\,(T;B)\|u_0\|_2. \tag{4.60}$$

Let $C = i\,[B;A] = -i\,[A;B]$. The operator C has order $m-1$ and its symbol $c(x,\xi)$ is given by

$$c(x,\xi) = -\{a,b\} = -\frac{d}{ds}b\,(\varphi(s;x,\xi))\big|_{s=0} = H_a(b)(x,\xi), \tag{4.61}$$

(where $\varphi(s;x,\xi)$ denotes the bicharacteristic flow associated to the symbol of A, that is, $a(x,\xi)$, and $H_a(b)$ is defined as in (3.27)). The aim is to find an operator B such that $C > 0$. By quadrature,

$$b(x,\xi) = \int_0^\infty c(\varphi(s;x,\xi))\,ds. \tag{4.62}$$

Thus, if $A = \Delta/8\pi^2$, $a(x,\xi) = |\xi|^2/2$, and $\varphi(s;x,\xi) = (x+s\xi,\xi)$. Taking

$$c(x,\xi) = -\frac{f'(|x_j|)\,\xi_j^2}{\langle\xi\rangle^2}, \tag{4.63}$$

with $f \in L^1([0,\infty):\mathbb{R}^+)$, f decreasing, and $\langle\xi\rangle = (1+|\xi|)^{1/2}$, we have $C>0$ of order 1.

By (4.61) one gets:

$$b(x,\xi) = \frac{f(|x_j|)\,\xi_j}{\langle\xi\rangle} \quad \text{(nonlocal operator of order zero).}$$

Now, from (4.60), (4.61), (4.63) it follows that

$$\begin{aligned}\int_0^T\int_{\mathbb{R}} Cu\,\bar{u}\,dx\,dt &= \int_0^T\int_{\mathbb{R}} -f'(|x_j|)\,\Lambda^{-1}\partial_{x_j}^2 u\,\bar{u}\,dxdt\\ &= \int_0^T\int_{\mathbb{R}} \partial_{x_j}\Lambda^{-1/2}(-f'(|x_j|)\,\Lambda^{-1/2}\partial_{x_j}u)\,\bar{u}\,dxdt\\ &\quad+\int_0^T\int_{\mathbb{R}} \underbrace{[-f'(|x_j|);\partial_{x_j}\Lambda^{-1/2}]\,\Lambda^{-1/2}\partial_{x_j}}_{\text{zero order operator}}u\,\bar{u}\,dxdt.\end{aligned} \tag{4.64}$$

From (4.60) combined with (4.64) and the choice of f, one basically has that

$$\begin{aligned}\int_0^T\int_{|x|\le R} |D^{1/2}u(x,t)|^2\,dx\,dt &\lesssim \int_0^T\int_{\mathbb{R}} \partial_{x_j}\Lambda^{-1/2}(-f'(|x_j|)\,\Lambda^{-1/2}\partial_{x_j}u)\,\bar{u}\,dx\,dt\\ &\le c_0(R;f;T)\|u_0\|_2.\end{aligned} \tag{4.65}$$

Repeating the argument for $A = i\partial_x^3$ and taking $c(x,\xi) = \varphi'(x)\xi^2$ with $\varphi'(x)=1$ if $|x|\le R$ and $\varphi'(x)=0$ and if $|x|\ge 2R$, even, C^∞, nonincreasing for $x>0$, we obtain $b(x,\xi)=\varphi(x)$ (local operator as in Kato's approach). Similarly, for $A = i\mathsf{H}\partial_x^2$ (the dispersive operator associated to the Benjamin–Ono equation) with the same choice of $c(x,\xi) = \varphi'(x)\,\xi^2$, we get the same $b(x,\xi)=\varphi(x)$, again a local operator so the result can be obtained by standard integration by parts.

For the variable coefficients case $A = \partial_{x_j}(a_{jk}(x)\partial_{x_k})$, we need several hypotheses that guarantee the appropriate behavior of the bicharacteristic flow at infinity as well as the integrability of $l(s) = c(\varphi(s;x,\xi))$ in (4.62). In this regard, we find the following result due to Doi [Do1].

Let $A(x) = (a_{jk}(x))$ be a real and symmetric $n\times n$ matrix of functions $a_{jk}\in C_b^\infty$. Assume that

$$|\nabla a_{jk}(x)| = o(|x|^{-1}) \quad \text{as } |x|\to\infty, \quad j,k=1,\ldots,n, \tag{4.66}$$

and that $A(x)$ is positive definite, so the operator $\partial_{x_j}(a_{jk}(x)\partial_{x_k})$ is elliptic as in (3.25). Assume that the bicharacteristic flow is nontrapped in one direction, which means that the set

$$\{X(s;x_0,\xi_0)\ :\ s\in\mathbb{R}\}$$

is unbounded in $\mathbb{R}^n$ for each $(x_0,\xi_0)\in\mathbb{R}^n\times\mathbb{R}^n-\{0\}$.

Lemma 4.3. *Let $A(x)$ and its bicharacteristic flow satisfy the assumptions above. Suppose $\lambda\in L^1([0,\infty))\cap C([0,\infty))$ is strictly positive and nonincreasing. Then, there exist $c>0$ and a real symbol $p\in S^0$, both depending on h_2 and λ, such that*

$$H_{h_2}p=\{h_2,p\}(x,\xi)\geq\lambda(|x|)\,|\xi|-c,\qquad\forall\,(x,\xi)\in\mathbb{R}^n\times\mathbb{R}^n. \tag{4.67}$$

Extensions and refinements as well as different proofs of the estimates in Theorems 4.2 and 4.3 have been deduced in connection with specific problems. To simplify the exposition we shall only mention some of them.

In [Bo5], Bourgain showed that there exists $c_0>0$ such that if $u_1,u_2\in L^2(\mathbb{R}^2)$, $0<M_1\leq M_2$ satisfying that

$$u_j(x)=P_{M_j}\,u_j=\int\limits_{M_j/2\leq|\xi|\leq 2M_j} e^{2\pi x\cdot\xi}\,\widehat{u}_j(\xi)\,d\xi,\quad j=1,2, \tag{4.68}$$

then

$$\big\|(e^{it\Delta}u_1)(e^{-it\Delta}u_2)\big\|_{L^2(\mathbb{R}^2_x\times\mathbb{R}_t)}\leq c_0\Big(\frac{M_1}{M_2}\Big)^{1/2}\|u_1\|_2\|u_2\|_2. \tag{4.69}$$

Inequality (4.69) measures the interaction of a pair of solutions corresponding to data with localized support in the frequency space.

Notice that for $M_1\sim M_2$ (4.69) yields the case $p=q=4=2+2/n$ of Theorem 4.2.

In [OT1], Ozawa and Tsutsumi studying the bilinear form:

$$(u_0,v_0)\to\partial_x(e^{it\partial_x^2}u_0)(e^{-it\partial_x^2}\bar{v}_0)$$

established the following identity: there exists $c_0>0$ such that for any $u_0,v_0\in L^2(\mathbb{R})$

$$\big\|D_x^{1/2}[(e^{it\partial_x^2}u_0)(e^{-it\partial_x^2}\bar{v}_0)]\big\|_{L^2(\mathbb{R}_x\times\mathbb{R}_t)}=c_0\|u_0\|_2\|v_0\|_2. \tag{4.70}$$

The estimate (4.70) resembles the gain of $1/2$ derivative in Theorem 4.3 as well as (after Sobolev embedding) the limit case ($p=\infty$, $q=4$, $n=1$, $u_0=v_0$) of Theorem 4.2.

In higher dimensions, Lions and Perthame [LP] applied the Winger transformation to obtain a different proof of (4.23) in Theorem 4.3. They also showed that for $\alpha\in(0,\infty)$,

$$\Big(\int_{-\infty}^{\infty}\int_{\mathbb{R}^n}\frac{|\nabla e^{it\Delta}u_0(x)|^2}{1+|x|^{1+\alpha}}\,dxdt\Big)^{1/2}\leq c_{n,\alpha}\,\big\|D_x^{1/2}\,u_0\big\|_2. \tag{4.71}$$

Finally, we shall briefly discuss the L^2-well-posedness of the IVP

$$\begin{cases} \partial_t u = i\Delta u + b_j(x)\partial_{x_j} u + d(x)u + f(x,t), \\ u(x,0) = u_0(x), \end{cases} \tag{4.72}$$

where the coefficients b_j and d and their derivatives are assumed to be bounded.

The problem (4.72) is said to be L^2-well-posed if for any $u_0 \in L^2(\mathbb{R}^n)$ and $f \in C_0([0,\infty) : L^2(\mathbb{R}^n))$ (where C_0 stands for the set of continuous functions with compact support) there exist $T > 0$ and a unique solution $u \in C([0,T] : L^2(\mathbb{R}^n))$ of (4.72) such that for $t \in [0,T]$

$$\sup_{[0,t]} \|u(\cdot,s)\|_2 \le c(t)\left\{\|u_0\|_2 + \int_0^t \|f(\cdot,s)\|_2\,ds\right\}.$$

Notice that if the b_j take real values the result follows by integration by parts. Also, if $b_j(x) = b_{0j}$ is a constant then the assumption $\mathcal{I}m\ b_{0j} = 0$ for all j is a necessary and sufficient condition. In the one-dimensional case, Takeushi [Ta1] proved that the condition

$$\sup_{\ell\in\mathbb{R}} \left| \int_0^\ell \mathcal{I}m\, b(s)\,ds \right| < \infty \tag{4.73}$$

is sufficient for the L^2-well-posedness of (4.72). In [Mz] Mizohata showed that in any dimension n the condition

$$\sup_{\widehat{w}\in\mathbb{S}^{n-1}} \sup_{\substack{x\in\mathbb{R}^n \\ \ell\in\mathbb{R}}} \left| \int_0^\ell \mathcal{I}m\, b_j(x + s\cdot\widehat{w})\cdot\widehat{w}_j\,ds \right| < \infty \tag{4.74}$$

is necessary. (4.74) is an integrability condition on the coefficients $b = (b_1,\dots,b_n)$ of the first order term along the bicharacteristic. In fact, Ichinose [I] extended (4.74) to the case where the Laplacian Δ in (4.72) is replaced by the elliptic variable coefficients $A = \partial_{x_j}(a_{jk}(x)\partial_{x_k})$ by deducing that

$$\sup_{\widehat{w}\in\mathbb{S}^{n-1}} \sup_{\substack{x\in\mathbb{R}^n \\ \ell\in\mathbb{R}}} \left| \int_0^\ell \mathcal{I}m\, b_j(X(s;x,\widehat{w}))\cdot\Xi(s;x,\widehat{w})\,ds \right| < \infty \tag{4.75}$$

is a necessary condition for the L^2-well-posedness (to the IVP associated to the equation $\partial_t u = iAu + b_j(x)\partial_{x_j}u + d(x)u + f(x,t)$), where $s \to (X(s;x,\hat{w}),\Sigma(s;x,\hat{w}))$ denotes the bicharacteristic flow associated to A (see 3.28).

Notice that the notion of nontrapping for the bicharacteristic flow associated is essential in the hypothesis (4.75) for $b_j(\cdot)$, even in $C_0^\infty(\mathbb{R}^n)$. We will return to this in Chapter 10, where the above results are further studied.

4.5 Exercises

4.1 Prove Proposition 4.1

4.2 Prove Proposition 4.2.

4.3 Prove that if $1 < p \le q < \infty$, $0 \le \gamma < n/q$, $0 \le \alpha < n(1-1/p)$, and $\alpha - \gamma = n(1 - \frac{1}{q} - \frac{1}{p})$, then there exists $c > 0$ such that for all $t \in \mathbb{R}\backslash\{0\}$

$$\left\| e^{it\Delta} u_0 \, |x|^{-\gamma} \right\|_q \le c\, |t|^{-(\alpha+\gamma)/2 - n/2(1/p-1/q)} \left\| u_0 \, |x|^{\alpha} \right\|_p. \tag{4.76}$$

Notice that the exponent in (4.76) satisfies

$$-\frac{\alpha+\gamma}{2} - \frac{n}{2}\Big(\frac{1}{p} - \frac{1}{q}\Big) = -\frac{n}{2} + 1 - \frac{1}{p} - \alpha = -\frac{n}{2} + \frac{1}{q} - \gamma.$$

Hint: Combine the formula (4.7) and Pitt's theorem (Exercise 2.12).

4.4 Define the operators:

$$\Gamma_j = x_j + 2it\partial_{x_j}, \qquad j = 1, \ldots, n.$$

(i) Prove that for any $\alpha \in (\mathbb{Z}^+)^n$ (with multi-index notation),

$$\Gamma^\alpha f(x,t) = e^{i|x|^2/4t} \, (2it\partial_x)^\alpha \, e^{-i|x|^2/4t} \, f = e^{it\Delta} \, x^\alpha \, e^{-it\Delta} \, f.$$

(ii) Prove that Γ_j commutes with $\mathcal{O}_s = \partial_t - i\Delta$.

(iii) If $u_0 \in L^2$ and $x^\alpha u_0 \in L^2(\mathbb{R}^n)$, show that $\Gamma^\alpha u \in C(\mathbb{R} : L^2(\mathbb{R}^n))$ and so

$$\partial_x^\alpha \big(e^{i|x|^2/4t} \, e^{it\Delta} \, u_0\big) \in C\big(\mathbb{R} \setminus \{0\} : L^2(\mathbb{R}^n)\big).$$

In particular, $\partial_x^\alpha \, e^{it\Delta} \, u_0 \in L^2_{\text{loc}}(\mathbb{R}^n)$ for $t \neq 0$.

(iv) If $u_0 \in H^s(\mathbb{R}^n)$, $s \in \mathbb{Z}^+$, and $x^\alpha u_0 \in L^2$, $|\alpha| \le s$, prove that

$$u = e^{it\Delta} u_0 \in C\left(\mathbb{R} : H^s \cap L^2(|x|^s \, dx)\right).$$

(v) If $u_0 \in \mathcal{S}(\mathbb{R}^n)$ show that $e^{it\Delta} u_0 \in \mathcal{S}(\mathbb{R}^n)$.

4.5 (i) Prove that if u_0, $x^\alpha u_0 \in L^2(\mathbb{R}^n)$, and $\partial_x^\alpha u_0 \notin L^2(\mathbb{R}^n)$, then $x^\alpha e^{it\Delta} u_0 \notin L^2(\mathbb{R}^n)$ for any $t \neq 0$.

(ii) Show that if $u_0 \in C_0(\mathbb{R}^n)$, then for any $t \in \mathbb{R} \setminus \{0\}$ and any $\epsilon > 0$, $e^{it\Delta} u_0 \notin L^1(e^{\epsilon|x|} dx)$, and that $e^{it\Delta} u_0$ has an analytic extension to $\mathbb{C}^n$ for $t \neq 0$.
Hint: Use formula (4.7).

4.6 Using the notation in Definition 3.4.

(i) Prove that for $t > 0$, and $b \in (0,1)$

$$\mathcal{D}^b \, (e^{it|x|^2}) \le c_{n,b} \, (t^{b/2} + t^b \, |x|^b).$$

(ii) Prove that for $b \in (0,1)$

$$\| \, |x|^b \|_2 \le c\big(t^{b/2} \, \|u_0\|_2 + t^b \, \|D^b u_0\|_2 + \| \, |x|^b u_0 \|_2\big).$$

(iii) Prove that if $s \geq b/2$, $b \in (0, 2)$ and

$$u_0 \in H^s(\mathbb{R}^n) \cap L^2(|x|^b\, dx) \equiv \mathcal{F}_b^s,$$

then

(a) $e^{it\Delta}u_0 \in \mathcal{F}_b^s$, for all $t \neq 0$.
(b) Moreover, $e^{it\Delta}u_0 \in C(\mathbb{R} : \mathcal{F}_b^s)$ (see [NhPo1]).

Hint: For (ii) combine part (i) and Exercise 3.10 inequality 3.42.

4.7 Check that for the group of translations

$$L_t : L^2(\mathbb{R}^n) \mapsto L^2(\mathbb{R}^n)$$

defined by $L_t(u_0)(x) = u_0(x + t)$ the inequalities (4.12) and (4.14) are not true.

4.8 Prove that there do not exist p, q, t with $1 \leq q < p < \infty$, $t \in \mathbb{R} \setminus \{0\}$ such that

$$e^{it\Delta} : L^p(\mathbb{R}^n) \mapsto L^q(\mathbb{R}^n) \text{ is continuous.}$$

This is a particular case of Hörmander's theorem in [Ho2].
Hint:

(i) Verify that $e^{it\Delta}$ commutes with translations. That is, if $\tau_h f(x) = f(x - h)$, then $\tau_h(e^{it\Delta} f(x)) = e^{it\Delta}\tau_h f(x)$.
(ii) Show that if $f \in L^{p_0}(\mathbb{R}^n)$, $1 \leq p_0 < \infty$, then

$$\lim_{|h|\to\infty} \|f + \tau_h f\|_{p_0} = 2^{1/p_0}\|f\|_{p_0}.$$

(iii) Using (ii) deduce that if T commutes with translations and $\|Tf\|_q \leq c\|f\|_p$, then

$$\|Tf\|_q \leq c\, 2^{(1/p-1/q)}\|f\|_p,$$

which leads to a contradiction because $q < p$.

4.9 (i) Prove that if $f \in L^2(\mathbb{R})$, then $e^{it\Delta} f$ is continuous in $\mathbb{R}$ for almost every $t \in \mathbb{R}$.
Hint: Combine Strichartz estimate (4.14) with $(p, q) = (\infty, 4)$ and a density argument.
(ii) Prove that inequality (4.14) is not true when the pair (p, q) does not satisfy the condition $2/q = n/2 - n/p$ in (4.18) Theorem 4.2.
Hint: Use the fact that if $u(x, t)$ is a solution of the linear Schrödinger equation, then for all $\lambda > 0$, $\lambda\, u(\lambda x, \lambda^2 t)$ is also a solution.

4.10 Given a sequence of times $A = \{t_j \in \mathbb{R} : j \in \mathbb{Z}^+\}$ converging to t_0, prove that there exists $f \in L^2(\mathbb{R})$ such that $e^{it\Delta} f \notin L^\infty(\mathbb{R})$ if $t \in A$ (compare this result with the inequality (4.14)).
Hint: For all $t \in A$ choose $a_t g_t \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ such that $g_t \notin L^\infty(\mathbb{R})$ and where the constants a_t are fixed and such that if $f_t = e^{-it\Delta} a_t g_t$ then $f = \sum_{t\in A} f_t$ satisfies the statement (use Lemma 4.1).

4.11 Prove that

$$\|e^{it\Delta}u_0\|_{L^{3r}_{xt}(\mathbb{R}^2)} = \left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left|e^{it\Delta}u_0(x)\right|^{3r} dxdt\right)^{1/3r} \leq c\,\|\widehat{u}_0\|_{r'} \qquad (4.77)$$

with $1/r + 1/r' = 1$ and $2 \leq r < \infty$. (The inequality (4.77) holds for $4/3 < r < \infty$ see [Ff]).

4.12 Prove that if $f \in L^2(\mathbb{R}^n)$, then

$$\lim_{t\to\pm\infty} \left\| e^{it\Delta} f - \frac{e^{i|\cdot|^2/4t}}{\sqrt{(4\pi i t)^n}} \widehat{f}(\,\cdot\,/4\pi t)\right\|_2 = 0. \qquad (4.78)$$

Hint:

(i) Verify that for all $t \neq 0$,

$$U(t)f(x) = (4\pi i t)^{-n/2} e^{i|x|^2/4t}\,\widehat{f}(x/4\pi t)$$

defines a unitary operator. Hence, it is enough to prove (4.78) assuming $f \in \mathcal{S}(\mathbb{R}^n)$.

(ii) Prove that

$$e^{it\Delta} f(x) - U(t)f(x) = \frac{e^{i|x|^2/4t}}{\sqrt{(4\pi i t)^{n/2}}}\widehat{F_t}(x/4\pi t),$$

with $F_t(y) = (e^{i|y|^2/4t} - 1)f(y)$.

(iii) Use the estimate $|e^{i|x|^2/4t} - 1| \leq c\frac{|x|^2}{4t}$ to complete the proof (see [Dl]).

4.13 Prove that if $u_0 \in H^1(\mathbb{R}) \cap L^2(|x|^2dx)$, then

$$\|x\, e^{it\partial_x^2}u_0\|_2 \geq 2|t|\,\|\partial_x u_0\|_2 - \|xu_0\|_2.$$

4.14 Show that the initial value problem:

$$\begin{cases} \partial_t u = i\Delta \overline{u}, \\ u(x,0) = u_0(x), \end{cases} \qquad (4.79)$$

$x \in \mathbb{R}^n$, $t > 0$, is ill-posed.

Hint: Differentiate equation (4.79) with respect to the variable t, then use the conjugate of equation (4.79) to obtain an equation in terms of second-order derivatives with respect to t and the bi-Laplacian.

4.15 (Duhamel's principle) Prove that the solution $u(x,t)$ of the inhomogeneous IVP:

$$\begin{cases} \partial_t u = i\Delta u + F(x,t), \\ u(x,0) = u_0(x), \end{cases} \qquad (4.80)$$

$x \in \mathbb{R}^n$, $t \in \mathbb{R}$, with $F \in C(\mathbb{R} : \mathcal{S}(\mathbb{R}^n))$ is given by the formula:

$$u(x,t) = e^{it\Delta}u_0 + \int_0^t e^{i(t-t')\Delta}F(\cdot,t')\,dt'. \tag{4.81}$$

4.16 Prove that if $F \in \mathcal{S}(\mathbb{R}^{n+1})$, then the solution $u(x,t)$ of problem (4.80) can be written as:

$$u(x,t)=e^{it\Delta}u_0+\int_{-\infty}^{\infty}\int_{\mathbb{R}^n} \frac{e^{2\pi i\tau t}-e^{-4\pi^2 i|\xi|^2 t}}{4\pi^2 i|\xi|^2+2\pi i\tau}e^{2\pi i x\cdot\xi}\widehat{F}(\xi,\tau)\,d\xi\,d\tau, \tag{4.82}$$

where $\widehat{F}$ represents the Fourier transform of F with respect to the variables x, t.

4.17 [AvHe]

(i) Show that if $u(x,t)$ is a solution of the IVP for the Schrödinger equation with Stark potential:

$$\begin{cases} \partial_t u = i(\Delta u + (\nu\cdot x)u), \\ u(x,0) = u_0(x) \end{cases} \tag{4.83}$$

with $\nu \in \mathbb{R}^n$ for $(x,t) \in \mathbb{R}^n \times \mathbb{R}$, then

$$w(x,t) = u(x + t^2\nu, t)\,e^{-it\nu\cdot x - it^3|\nu|^2/3}$$

solves the linear Schrödinger equation with the same data, i.e., $w(x,t) = e^{it\Delta}u_0$.

(ii) Do the estimates (4.14) and (4.24) hold for the solution $u(x,t)$ of (4.83)?

4.18 Prove inequality (4.37).

4.19 Prove that $m(\xi) = e^{8\pi^3 it\xi^3}$ is not an L^p-multiplier for $p \neq 2$.

4.20 Using the estimates (4.23), (4.24) from Theorem 4.3, prove that:

(i) If $n > 2$, $\alpha > 1/2$, and $f \in L^2(\mathbb{R}^n)$, then $(1+|x|)^{-\alpha}D_x^{1/2}e^{it\Delta}f \in L^2(\mathbb{R}^n)$, a.e. $t \in \mathbb{R}$.

(ii) If $n = 1$ the result in (i) is not true.

(iii) What can be said in the case $n = 2$? (See [KY]).

4.21 Use the commutator estimates in (3.16) to show that operator defined in (4.64), i.e.,

$$\left[-f'(|x_j|);\partial_{x_j}\Lambda^{-1/2}\right]\Lambda^{-1/2}\partial_{x_j},$$

is in fact of order zero.

Chapter 5
The Nonlinear Schrödinger Equation: Local Theory

In this chapter, we shall study local well-posedness of the nonlinear initial value problem (IVP):

$$\begin{cases} i\,\partial_t u = -\Delta u - \lambda |u|^{\alpha-1}u, \\ u(x,0) = u_0(x), \end{cases} \tag{5.1}$$

$t \in \mathbb{R}$, $x \in \mathbb{R}^n$, where λ and α are real constants with $\alpha > 1$.

The equation (5.1) appears as a model in several physical problems (see references [GV1], [N], [SCMc], [ZS]).

Formally solutions of problem (5.1) satisfy the following *conservation laws,* that is, if $u(x,t)$ is solution of (5.1), then for all $t \in [0,T]$, the L^2-norm

$$M(u_0) = \|u(\cdot,t)\|_2^2 = \|u_0\|_2^2, \tag{5.2}$$

the energy

$$\begin{aligned} E(u_0) \;&=\; \int\limits_{\mathbb{R}^n} \left(|\nabla_x u(x,t)|^2 - \frac{2\lambda}{\alpha+1} |u(x,t)|^{\alpha+1} \right) dx \\ &= \|\nabla u_0\|_2^2 - \frac{2\lambda}{\alpha+1} \|u_0\|_{\alpha+1}^{\alpha+1}, \end{aligned} \tag{5.3}$$

the momentum

$$\mathcal{I}m \int\limits_{\mathbb{R}^n} \nabla u(x,t)\,\bar{u}(x,t)\,dx = \mathcal{I}m \int\limits_{\mathbb{R}^n} \nabla u_0(x)\,\bar{u}_0(x)\,dx, \tag{5.4}$$

and the so-called quasiconformal law [GV1]

$$\begin{aligned} &\|(x+2it\nabla)u(t)\|_2^2 - \frac{8\lambda t^2}{\alpha+1}\|u(t)\|_{\alpha+1}^{\alpha+1} \\ &\qquad = \|xu_0\|_2^2 - 4\lambda \frac{(4-n(\alpha-1))}{\alpha+1} \int\limits_0^t \left(\int\limits_{\mathbb{R}^n} |u(x,s)|^{\alpha+1}\,dx \right) s\,ds. \end{aligned} \tag{5.5}$$

F. Linares, G. Ponce, *Introduction to Nonlinear Dispersive Equations,*
Universitext, DOI 10.1007/978-1-4939-2181-2_5

We will use these identities in the next chapter.

We shall say that equation in (5.1) is focusing if $\lambda > 0$ (attractive nonlinearity) and defocusing if $\lambda < 0$ (repulsive nonlinearity).

In any dimension, the equation in (5.1) in the focusing case $\lambda > 0$ has solutions of the form

$$u(x,t) = e^{it}\,\varphi(x), \tag{5.6}$$

called *standing waves*. The *ground state* φ is closely related to the elliptic problem

$$-\Delta v = f(v), \tag{5.7}$$

which have been extensively studied. In our case, $f(v) = -v + |v|^{\alpha-1}v$, with $\lambda = 1$. Indeed, the problem is to find $\varphi \in H^1(\mathbb{R}^n)$, positive, such that

$$-\Delta\varphi + \varphi = |\varphi|^{\alpha-1}\varphi. \tag{5.8}$$

Hence, for any $\omega > 0$,

$$u_\omega(x,t) = e^{i\omega t}\omega^{1/(\alpha-1)}\varphi(\sqrt{\omega}\,x) = e^{i\omega t}\varphi_\omega(x) \tag{5.9}$$

is a solution of the equation in (5.1) with $\lambda = 1$.

The existence of solutions of the equation (5.8) in dimension $n \geq 3$ was established by Strauss [Sr2] and Berestycki and Lions [BLi] (see also [BLiP]). The bidimensional case was considered in [BGK] by Berestycki, Gallouët and Kavian Regarding the uniqueness of solutions of (5.8), Kwong [Kw1] showed that positive solutions of the problem (5.7) with $f(v) = -v + v^p$ are unique up to translations. We summarize these results in the next theorem.

Theorem 5.1. *Let $n \geq 2$ and $1 < \alpha < (n+2)/(n-2)$ $(1 < \alpha < \infty,\ n = 2)$. Then, there exists a unique positive, spherically symmetric solution of (5.8) $\varphi \in H^1(\mathbb{R}^n)$. Moreover, φ and its derivatives up to order 2 decay exponentially at infinity.*

Remark 5.1. The restriction on α comes from Pohozaev's identity (5.83) since we want to have H^1-solutions of (5.8) (see Exercise 5.3).

Remark 5.2. There are infinitely many radially symmetric solutions under the hypothesis of Theorem 5.1 without the positivity assumption (see [BLi], [E], [JK]).

As given below, once we have a solution of (5.1), we can use the invariance of the equation to generate other solutions. Thus, if $u = u(x,t)$ is a solution of the equation in (5.1), then the following are also solutions:

(i) $$u_\mu(x,t) = \mu^{\frac{2}{\alpha-1}}\,u(\mu x, \mu^2 t),\ \ \mu \in \mathbb{R},\ \text{with initial data given by} \quad u_{0\mu}(x) = \mu^{\frac{2}{\alpha-1}}u_0(\mu x). \tag{5.10}$$

(ii) $$u_\theta(x,t) = e^{i\theta}u(x,t), \theta \in \mathbb{R}.$$

(iii) $$u_A(x,t) = u(Ax,t), \quad A \text{ any } n \times n \text{ orthogonal matrix.}$$

(iv) $$u_{a,b}(x,t) = u(x-a, t-b), \; a \in \mathbb{R}^n, \; b \in \mathbb{R}.$$

(v) $$u_c(x,t) = e^{ic\cdot x} e^{-i|c|^2 t} u(x - 2tc, t) \text{ for } c \in \mathbb{R}^n, \text{ with initial data } u_c(x,0) = e^{ic\cdot x}\, u_0(x). \tag{5.11}$$

(vi) In addition, if $\alpha = 4/n + 1$, then [GV1]

$$u_\omega(x,t) = \frac{1}{(\alpha+\omega t)^{n/2}} \exp\left(\frac{i\omega|x|^2}{4(\alpha+\omega t)}\right) \times u\left(\frac{x}{\alpha+\omega t}, \frac{\gamma+\theta t}{\alpha+\omega t}\right), \quad \alpha\theta - \omega\gamma = 1,$$

(vii) $$u_7(x,t) = \overline{u(x,-t)}.$$

Property (i) is called scaling, property (v) Galilean invariance, and property (vi) pseudo-conformal invariance.

Hence, gathering this information, one gets the multiparametric family of solutions $R = R(\nu, \omega, \theta, x_0)$ with ν, $x_0 \in \mathbb{R}^n$, $\omega > 0$, and $\theta \in \mathbb{R}$.

$$R(x,t) = e^{i(\nu\cdot x - |\nu|^2 t + \omega t + \theta)}\, \varphi_\omega(x - x_0 - 2\nu\, t) \tag{5.12}$$

of (5.1) with $\lambda = 1$ (focusing case), where $\varphi(\cdot)$ is the positive solution of (5.8) and $\varphi_\omega(\cdot)$ is defined in (5.9). Notice that the solitary wave in (5.12) moves on the line $x = x_0 + 2\nu\, t$. In the one-dimensional (1-D) case, the equation (5.8) becomes an ordinary differential equation (ODE) and one has that

$$\varphi_\omega(x) = \left\{\frac{(\alpha+1)}{2}\omega \,\text{sech}^2\left(\frac{\alpha-1}{2}\sqrt{\omega}\, x\right)\right\}^{1/(\alpha-1)}. \tag{5.13}$$

Thus, for all $t \in \mathbb{R}$ and $p \in [1,\infty]$

$$\|u(\cdot,t)\|_p = \|u_0\|_p = K(\alpha,\omega). \tag{5.14}$$

From the nonlinear differential equations point of view, the existence of the solitary wave describes a perfect balance between the nonlinearity and the dispersive character of its linear part. More precisely, although the solutions of the linear problem $e^{it\Delta}u_0$ with $u_0 \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ decay as $t \to \infty$ (see (4.12) for the case $u_0 \in L^1(\mathbb{R}^n)$ and (4.14) for $u_0 \in L^2(\mathbb{R}^n)$), the solutions of (5.1) neither decay nor develop singularities. The latter situation is addressed in the next chapter.

5.1 L^2 Theory

We consider the integral equation (see Exercise 4.15)

$$u(t) = e^{it\Delta}u_0 + i\lambda \int_0^t e^{i(t-t')\Delta}(|u|^{\alpha-1}u)(t')\,dt'. \tag{5.15}$$

The difference between this equation and the one in (5.1) is that (5.15) does not require any differentiability of the solution. Using the properties described in Proposition 4.2, it is easy to see that if u is a solution of the differential equation in (5.1), then it is also a solution of (5.15). We shall prove in Section 5.3 that under some hypotheses on α and n, if $u_0 \in H^2(\mathbb{R}^n)$, the solution of (5.15) also satisfies the differential equation in (5.1).

We say that the integral equation (5.15) is *locally well-posed* in X, where X is a function space, if for every $u_0 \in X$ there exist $T > 0$ and a unique solution $u \in C([0,T) : X) \cap \ldots$ of (5.15) for $(x,t) \in \mathbb{R}^n \times [0,T)$. Moreover, the map data solution, i.e., $u_0 \mapsto u(\cdot,t)$, locally defined from X to $C([0,T) : X)$, is continuous. Therefore, our notion of well-posedness includes existence, uniqueness, and persistence (the solution $u(t)$ belongs to the same space as the initial data and its time trajectory describes a curve on it). Thus, the solution flow of (5.15) defines a dynamical system in X. In the case that T can be taken arbitrarily large, we shall say that (5.15) is *globally well-posed in* X.

As we shall see below in the subcritical case, one has that $T = T(\|u_0\|_X) > 0$ and in the critical case that $T = T(u_0) > 0$. These definitions of local and global well-posedness also apply to the initial value problem (IVP) (5.1).

Our first result indicate that under some restriction on the power of the nonlinearity, $\alpha \in (1, 1+4/n)$, problem (5.15) is locally well-posed in L^2.

Theorem 5.2 (Local theory in L^2). *If $1 < \alpha < 1 + 4/n$, then for each $u_0 \in L^2(\mathbb{R}^n)$ there exist $T = T(\|u_0\|_2, n, \lambda, \alpha) > 0$ and a unique solution u of the integral equation* (5.15) *in the time interval $[-T, T]$ with*

$$u \in C([-T,T]\colon L^2(\mathbb{R}^n)) \cap L^r([-T,T]\colon L^{\alpha+1}(\mathbb{R}^n)), \tag{5.16}$$

where $r = 4(\alpha+1)/n(\alpha-1)$.

Moreover, for all $T' < T$ there exists a neighborhood V of u_0 in $L^2(\mathbb{R}^n)$ such that

$$\mathbb{F} : V \mapsto C([-T',T']\colon L^2(\mathbb{R}^n)) \cap L^r([-T',T']\colon L^{\alpha+1}(\mathbb{R}^n)), \quad \tilde{u}_0 \mapsto \tilde{u}(t),$$

is Lipschitz.

As we shall see in the proof of Theorem 5.2 (see (5.24)) and in Exercise 5.5, one can give a precise estimate for the life span of the solution according to the size of the data in L^2-norm. This fact holds whenever the problem is "subcritical" and the scaling of the norm of the initial data is homogeneous, i.e., in our case, if $u = u(x,t)$ is a solution of (5.1) or (5.15), then

$$u_\mu(x,t) = \mu^{2/(\alpha-1)}\, u(\mu x, \mu^2 t),$$

is also a solution with data $u_\mu(x,0) = \mu^{2/(\alpha-1)}\, u_0(\mu x)$ so that

$$\|u_\mu(0)\|_2 = \mu^{2/(\alpha-1)-n/2}\, \|u_0\|_2.$$

If, in addition to the hypothesis of Theorem 5.2, one has that $u_0 \in H^s(\mathbb{R}^n)$, $s > 0$, and $\alpha \geq [s] + 1$, $[\cdot]$ denoting the greatest integer function, then

$$u \in C([0,T] : H^s(\mathbb{R}^n)) \cap L^r([-T,T] : L_s^{\alpha+1}(\mathbb{R}^n)), \tag{5.17}$$

with T as in the theorem. This fact holds in any subcritical case with a regular enough nonlinearity, since by taking s derivatives the problem becomes linear in this variable.

The proof of Theorem 5.2 is based on the contraction mapping principle. This has the advantage that it also shows that if the nonlinearity is smooth, i.e., α is an odd integer, then the map data-solution $u_0 \mapsto u(t)$ is smooth (see Corollary 5.6).

Corollary 5.1. *The solution u of equation (5.15) obtained in Theorem 5.2 belongs to $L^q([-T,T]\colon L^p(\mathbb{R}^n))$ for all (p,q) defined by condition (4.18) of Theorem 4.2, that is:*

$$\left.\begin{array}{ll} 2 \leq p < \dfrac{2n}{n-2} & \textit{if } n \geq 3 \\ 2 \leq p < \infty & \textit{if } n = 2 \\ 2 \leq p \leq \infty & \textit{if } n = 1 \end{array}\right\} \quad \textit{and} \quad \frac{2}{q} = \frac{n}{2} - \frac{n}{p}. \tag{5.18}$$

In the proof of Theorem 5.2, we use the following notation: For all positive constants T and a, we define

$$\begin{aligned} E(T,a) = \Big\{ v \in C([-T,T]\colon L^2(\mathbb{R}^n)) \cap L^r([-T,T]\colon L^{\alpha+1}(\mathbb{R}^n)) : \\ |||v|||_T \equiv \sup_{[-T,T]} \|v(t)\|_2 + \left(\int_{-T}^{T} \|v(t)\|_{\alpha+1}^r dt \right)^{1/r} \leq a \Big\} \end{aligned} \tag{5.19}$$

with $1 < \alpha < 1 + 4/n$ and $r = 4(\alpha+1)/n(\alpha-1)$. Note that $E(T_0, a)$ is a complete metric space.

Proof of Theorem 5.2 For appropriate values of a and $T > 0$, we shall show that

$$\Phi_{u_0}(u)(t) = \Phi(u)(t) = e^{it\Delta} u_0 + i\lambda \int_0^t e^{i\Delta(t-t')}(|u|^{\alpha-1}u)(t')\, dt' \tag{5.20}$$

defines a contraction map on $E(T,a)$.

Without loss of generality we consider only the case $t > 0$. Using (4.14), (4.17), and Hölder's inequality combined with the definition $\Phi(\cdot)$ in (5.20), we obtain:

$$\begin{aligned}\left(\int_0^T \|\Phi(u)(t)\|_{\alpha+1}^r dt\right)^{1/r} &\leq c\|u_0\|_2 + c|\lambda| \left(\int_0^T \|\,|u(t)|^\alpha\|_{(\alpha+1)/\alpha}^{r'} dt\right)^{1/r'} \\ &\leq c\|u_0\|_2 + c|\lambda| \left(\int_0^T \|u(t)\|_{(\alpha+1)}^{\alpha r'} dt\right)^{1/r'}.\end{aligned} \tag{5.21}$$

By hypothesis ($1 < \alpha < 1 + 4/n$), we have that $\alpha r' < r$, that is,

$$\alpha \frac{r}{r-1} < r \qquad \text{or} \qquad \alpha < r - 1 = \frac{4(\alpha+1)}{n(\alpha-1)} - 1.$$

Therefore, from (5.21) we deduce that

$$\left(\int_0^T \|\Phi(u)(t)\|_{\alpha+1}^r dt\right)^{1/r} \leq c\|u_0\|_2 + c|\lambda| T^\theta \left(\int_0^T \|u\|_{\alpha+1}^r dt\right)^{\alpha/r} \tag{5.22}$$

with $\theta = 1 - n(\alpha-1)/4 > 0$. Then, if $u \in E(T, a)$ we have

$$\left(\int_0^T \|\Phi(u)(t)\|_{\alpha+1}^r dt\right)^{1/r} \leq c\|u_0\|_2 + c|\lambda|\, T^\theta\, a^\alpha.$$

Using 4.16 and the unitary group properties in expression (5.20), we obtain that if $u \in E(T, a)$, then

$$\begin{aligned}\sup_{[0,T]} \|\Phi(u)(t)\|_2 &\leq c\|u_0\|_2 + c|\lambda| \left(\int_0^T \|\,|u|^\alpha\|_{(\alpha+1)/\alpha}^{r'} dt\right)^{1/r'} \\ &\leq c\|u_0\|_2 + c|\lambda|\, T^\theta\, a^\alpha,\end{aligned} \tag{5.23}$$

where the constant c depends only on α and the dimension n. Hence,

$$|||\Phi(u)|||_T \leq c\,\|u_0\|_2 + c|\lambda| T^\theta a^\alpha.$$

If we fix $a = 2c\|u_0\|_2$ and take $T > 0$ such that

$$2^\alpha c^\alpha\, |\lambda|\, T^\theta\, \|u_0\|_2^{\alpha-1} < 1, \tag{5.24}$$

it follows that the application Φ is well defined on $E(T,a)$. Now, if $u, v \in E(T,a)$,

$$(\Phi(v) - \Phi(u))(t) = i\lambda \int_0^t e^{i(t-t')\Delta}(|v|^{\alpha-1}v - |u|^{\alpha-1}u)(t')\,dt'.$$

The same argument as in (5.21) and (5.22) shows that

$$\begin{aligned}
&\left(\int_0^T \|(\Phi(v) - \Phi(u))(t)\|_{\alpha+1}^r\,dt\right)^{1/r} \\
&\le c|\lambda| \left(\int_0^T \|\,|v|^{\alpha-1}v - |u|^{\alpha-1}u\|_{(\alpha+1)/\alpha}^{r'}\,dt\right)^{1/r'} \\
&\le c_\alpha|\lambda| \left(\int_0^T (\|v\|_{\alpha+1}^{\alpha-1} + \|u\|_{\alpha+1}^{\alpha-1})^{r'}\,\|v-u\|_{\alpha+1}^{r'}(t)\,dt\right)^{1/r'} \\
&\le c_\alpha|\lambda|\ T^\theta \left\{ \left(\int_0^T \|v\|_{(\alpha+1)}^r\,dt\right)^{(\alpha-1)/r} + \left(\int_0^T \|u\|_{(\alpha+1)}^r\,dt\right)^{(\alpha-1)/r} \right\} \\
&\quad \times \left(\int_0^T \|v(t) - u(t)\|_{(\alpha+1)}^r\,dt\right)^{1/r} \\
&\le 2c_\alpha\ |\lambda|\ T^\theta\ a^{\alpha-1} \left(\int_0^T \|v(t) - u(t)\|_{(\alpha+1)}^r\,dt\right)^{1/r}.
\end{aligned}$$

Combining (4.16) with the unitary group properties and the arguments used in (5.21) and (5.22), we see as in (5.23) that

$$\sup_{[0,T]} \|(\Phi(v) - \Phi(u))(t)\|_2 \le 2c_\alpha|\lambda|\ T^\theta\ a^{\alpha-1} \left(\int_0^T \|v(t) - u(t)\|_{\alpha+1}^r\,dt\right)^{1/r}.$$

Finally, it follows from the choice of a, $a \le 2c\|u_0\|_2$, and inequality (5.24) that

$$2c\,|\lambda|\ T^\theta\ a^{\alpha-1} \le 2^\alpha c^\alpha |\lambda|\ T^\theta\ \|u_0\|_2^{\alpha-1} < 1.$$

Hence,

$$T \simeq \|u_0\|_2^\beta, \quad \text{with } \beta = \frac{4(1-\alpha)}{4 - n(\alpha-1)}. \tag{5.25}$$

Thus, we have proved the existence and uniqueness in an appropriate class of the solution of equation (5.15). To prove the continuous dependence of $\Phi(u(t)) = \Phi_{u_0}(u(t))$ with respect to u_0, note that if u, v are the corresponding solutions of (5.15) with initial data u_0, v_0, respectively, then

$$u(t) - v(t) = e^{it\Delta}(u_0 - v_0) + i\lambda \int_0^t e^{i(t-t')\Delta}(|u|^{\alpha-1}u - |v|^{\alpha-1}v)(t')dt'.$$

Therefore, the same argument used in (5.21) and (5.22) implies

$$\left(\int_0^T \|u(t) - v(t)\|_{\alpha+1}^r \, dt\right)^{1/r} \le c\|u_0 - v_0\|_2$$
$$+ K_\alpha |\lambda| \, T^\theta \left(\|u_0\|_2^{\alpha-1} + \|v_0\|_2^{\alpha-1}\right) \left(\int_0^T \|u(t) - v(t)\|_{\alpha+1}^r \, dt\right)^{1/r}.$$

As a consequence, if $\|u_0 - v_0\|_2$ is small enough (see (5.24)), then

$$\left(\int_0^T \|u(t) - v(t)\|_{\alpha+1}^r \, dt\right)^{1/r} \le \widetilde{K}\|u_0 - v_0\|_2.$$

Analogously we can prove that

$$\sup_{[0,T]} \|u(t) - v(t)\|_2 \le \widetilde{K}\|u_0 - v_0\|_2,$$

which completes the proof. □

Proof of Corollary 5.1 The proof is obtained by combining Corollary 4.1 with inequality (5.21). That is, taking (p, q) in Corollary 4.1 instead of $(\alpha + 1, r)$ on the left-hand side of (5.21) and then using the argument in the proof of Theorem 5.2. The details of this proof are left as an exercise to the reader. □

Remark 5.3. Observe that in the proof of Theorem 5.2 we only used the hypothesis on the growth of the nonlinear term but not its particular form.

Next, we show how to extend the argument used in the proof of Theorem 5.2 to the critical case $\alpha = 1 + 4/n$.

Proposition 5.1. *Let* (p, q) *be a pair satisfying condition* (5.18) *in Corollary* 5.1. *Given* $u_0 \in L^2(\mathbb{R}^n)$ *and* $\epsilon > 0$, *there exist* $\delta > 0$ *and* $T > 0$ *such that if* $\|v_0 - u_0\|_2 < \delta$, *then*

$$\left(\int_0^T \|e^{it\Delta}v_0\|_p^q \, dt\right)^{1/q} < \epsilon. \tag{5.26}$$

Proof. If we take $\delta < \epsilon/2c$, then it suffices to show that

$$\left(\int_0^T \|e^{it\Delta} u_0\|_p^q \, dt \right)^{1/q} < \epsilon/2. \tag{5.27}$$

We choose $\widetilde{u}_0 \in \mathcal{S}(\mathbb{R}^n)$ such that $\|u_0 - \widetilde{u}_0\|_2 < \epsilon/4c$ and then combining Theorem 4.2 (inequality (4.14)), the fact that $\{e^{it\Delta}\}$ defines a unitary group in $H^s(\mathbb{R}^n)$ and Sobolev's inequality (Theorem 3.3), we have

$$\begin{aligned} \left(\int_0^T \|e^{it\Delta} u_0\|_p^q \, dt \right)^{1/q} &\leq \left(\int_0^T \|e^{it\Delta} (\widetilde{u}_0 - u_0)\|_p^q \, dt \right)^{1/q} + \left(\int_0^T \|e^{it\Delta} \widetilde{u}_0\|_p^q \, dt \right)^{1/q} \\ &\leq c\|\widetilde{u}_0 - u_0\|_2 + cT^{1/q} \, \|\widetilde{u}_0\|_{s,2}, \end{aligned}$$

where $s \geq n(1/2 - 1/p)$. Fixing T such that $cT^{1/q} \, \|\widetilde{u}_0\|_{s,2} < \epsilon/4$, we obtain (5.27). □

Theorem 5.3 (Critical case, $\alpha = 1 + 4/n$ in $L^2(\mathbb{R}^n)$). *If $\alpha = 1 + 4/n$, then for each $u_0 \in L^2(\mathbb{R}^n)$ there exist $T = T(u_0, \lambda, \alpha) > 0$ and a unique solution u of the integral equation (5.15) in the time interval $[-T, T]$ with*

$$u \in C([-T, T] \colon L^2(\mathbb{R}^n)) \cap L^\sigma([-T, T] \colon L^\sigma(\mathbb{R}^n)), \tag{5.28}$$

where $\sigma = 2 + 4/n$.

Moreover, for all $T' < T$ there exists a neighborhood V of u_0 in $L^2(\mathbb{R}^n)$ such that

$$\mathbb{F} : \ V \mapsto C([-T', T'] \colon L^2(\mathbb{R}^n)) \cap L^\sigma([-T', T'] \colon L^\sigma(\mathbb{R}^n)), \quad \tilde{u}_0 \mapsto \tilde{u}(t),$$

is Lipschitz.

Remark 5.4. Notice that the time of existence in Theorem 5.2 depends only on the size of u_0 (that is, on $\|u_0\|_2$); meanwhile, in Theorem 5.3, the time of existence depends on the position of u_0, and not only on its size.

Proof. We shall show that $\Phi_{u_0} = \Phi$ in (5.20) defines a contraction in:

$$\widetilde{E}(T, a) = \Big\{ v \in C([-T, T] \colon L^2(\mathbb{R}^n)) \cap L^\sigma([-T, T] \colon L^\sigma(\mathbb{R}^n)) \ :$$

$$|||v|||_T \equiv \sup_{[-T,T]} \|v(t) - e^{it\Delta} u_0\|_2 + \left(\int_{-T}^T \|v(t)\|_\sigma^\sigma \, dt \right)^{1/\sigma} \leq a \Big\}.$$

First, from (5.20) it follows that

$$\sup_{[0,T]} \|\Phi(u)(t) - e^{it\Delta} u_0\|_2 \leq \sup_{[0,T]} \| \int_0^t e^{i\Delta(t-t')} \lambda |u|^\alpha(t') \, dt' \|_2$$

$$\le c|\lambda| \left(\int_0^T \| |u(t)|^\alpha \|_{\sigma'}^{\sigma'} \, dt \right)^{1/\sigma'} \tag{5.29}$$
$$\le c|\lambda| \left(\int_0^T \| u(t) \|_{\sigma}^{\sigma} \, dt \right)^{\alpha/\sigma}.$$

On the other hand, it is easy to see that the pair (σ, σ) satisfies the condition (5.18) of Corollary 5.1. Then, combining the integral equation (5.20), estimates (4.14), and (5.26) with $(p,q) = (\sigma,\sigma)$, with the argument used on (5.21), we obtain:

$$\left(\int_0^T \| \Phi(u)(t) \|_{\sigma}^{\sigma} \, dt \right)^{1/\sigma} \le c\epsilon + c|\lambda| \left(\int_0^T \| \, |u(t)|^\alpha \|_{\sigma'}^{\sigma'} \, dt \right)^{1/\sigma'}$$
$$\le c\epsilon + c|\lambda| \left(\int_0^T \| u(t) \|_{\sigma}^{\sigma} \, dt \right)^{\alpha/\sigma}, \tag{5.30}$$

because $\alpha\sigma' = (1+4/n)\,((2n+4)/(n+4)) = 2 + 4/n = \sigma.$

From Proposition 5.1, inequalities (5.29) and (5.30), we obtain that given $\epsilon > 0$, there exists $T > 0$ such that if $u \in \widetilde{E}(T,a)$, then

$$|||\Phi(u)|||_T \le c\epsilon + c|\lambda| \, a^\alpha.$$

Therefore, if

$$c\epsilon + c\,|\lambda| \, a^\alpha < a, \tag{5.31}$$

we get that $\Phi(\widetilde{E}(T,a)) \subseteq \widetilde{E}(T,a)$. The argument used in the proof of Theorem 5.2 yields:

$$\left(\int_0^T \| (\Phi(v) - \Phi(u))(t) \|_{\sigma}^{\sigma} \, dt \right)^{1/\sigma} \le 2c|\lambda| \, a^{\alpha-1} \left(\int_0^T \| v(t) - u(t) \|_{\sigma}^{\sigma} \, dt \right)^{1/\sigma}.$$

Thus, for

$$2c \, |\lambda| \, a^{\alpha-1} < 1/2, \tag{5.32}$$

we have that $\Phi(\cdot)$ is a contraction. Now, fixing $\epsilon > 0$ such that

$$c|\lambda| \, \epsilon^{\alpha-1} < 1/2$$

we see that both (5.31) and (5.32) are verified. This basically completes the proof, the remainder of the proof follows using the same argument employed to show Theorem 5.2. □

Corollary 5.2. *There exists* $\epsilon_0 > 0$ *depending on* λ *and* n *such that for all* $u_0 \in L^2(\mathbb{R}^n)$ *with* $\|u_0\|_2 \le \epsilon_0$, *the results of Theorem 5.3 extends to any time interval* $[0, T]$, *i.e,*

$$u \in C(\mathbb{R} : L^2(\mathbb{R}^n)) \cap L^\sigma(\mathbb{R} : L^\sigma(\mathbb{R}^n)), \quad \sigma = 2 + 4/n. \tag{5.33}$$

Proof. It is enough to note that if $\|u_0\|_2$ is sufficiently small, then taking $\epsilon = \|u_0\|_2$ and $a = 2\|u_0\|_2$ (both independent of T), and

$$c\,|\lambda|\,\|u_0\|_2^{\alpha-1} < 1/2,$$

we see that (5.31) and (5.32) hold. □

Combining the results in Corollary 5.2 and those in Exercise 6.2 (concerning the scattering of the solutions obtained in Corollary 5.2), one should expect that the constant ϵ in Corollary 5.2 be given by $\|\varphi\|_2$, where φ is the positive solution of equation (5.8), with $\omega = 1$ and $\alpha = 1 + 4/n$. This has been proved in the radial case and for dimension $n = 2$ in [KTV].

5.2 $\mathbf{H^1}$ Theory

We consider the integral equation (5.15) with $u_0 \in H^1(\mathbb{R}^n)$ with the nonlinearity α satisfying

$$\begin{cases} 1 < \alpha < \dfrac{n+2}{n-2}, & \text{if } n > 2 \\ 1 < \alpha < \infty, & \text{if } n = 1, 2. \end{cases} \tag{5.34}$$

Theorem 5.4 (Local theory in $\boldsymbol{H^1}$). *If* α *satisfies hypothesis* (5.34), *then for all* $u_0 \in H^1(\mathbb{R}^n)$ *there exist* $T = T(\|u_0\|_{1,2}, n, \lambda, \alpha) > 0$ *and a unique solution* u *of the integral equation* (5.15) *in the time interval* $[-T, T]$ *with*

$$u \in C([-T, T] \colon H^1(\mathbb{R}^n)) \cap L^r([-T, T] \colon L_1^\rho(\mathbb{R}^n)), \tag{5.35}$$

where $(\rho, r) = \left(\dfrac{n(\alpha+1)}{n+\alpha-1}, \dfrac{4(\alpha+1)}{(n-2)(\alpha-1)}\right)$ *for* $n \ge 3$, *and* (ρ, r) *satisfies* (5.18) *for* $n = 1, 2$, *and* L_1^ρ *is defined as in (3.38).*

Moreover, for all $T' < T$ *there exists a neighborhood* W *of* u_0 *in* $H^1(\mathbb{R}^n)$ *such that the function*

$$\mathbb{F} \colon W \mapsto C([-T', T'] \colon H^1(\mathbb{R}^n)) \cap L^r([-T', T'] \colon L_1^\rho(\mathbb{R}^n)), \quad \tilde{u}_0 \mapsto \tilde{u}(t),$$

is Lipschitz.

If in addition to the hypothesis of Theorem 5.4 one has that $u_0 \in H^s(\mathbb{R}^n)$, $s > 1$, and $\alpha \ge [s] + 1$, $[\cdot]$ denoting the greatest integer function, then

$$u \in C([0, T] : H^s(\mathbb{R}^n)) \cap L^r([0, T] : L_s^{\alpha+1}(\mathbb{R}^n)), \tag{5.36}$$

where $[0, T]$ is the same time interval given for $s = 1$. As in (5.17), the problem becomes linear in $D_x^s u$ once one takes D_x^s in the equation and the result follows by reapplying the argument in the proof of Theorem 5.4 in this linear equation whose coefficients (depending on u) have sufficient regularity to get the desired result.

As we shall see in the next chapter, in the critical case, a similar result was quite difficult to establish.

Corollary 5.3. *The solution of the integral equation* (5.15) *obtained in Theorem* 5.4 *belongs to* $u \in L^q([-T,T]: L_1^p(\mathbb{R}^n))$ *for all pair* (p,q) *defined by condition* (5.18) *in Corollary* 5.1. *Moreover, in these spaces, the solution depends continuously on the initial data.*

The proof of this theorem is similar to the one given in the previous section for the L^2 case; therefore, we can give only a sketch of it.

Proof of Theorem 5.4 We will show the theorem in the case $n \geq 3$. We first define

$$E^1(T,a)=\Big\{v \in C([-T,T]:\, H^1)\cap L^r([-T,T]:\, L_1^\rho) : |||v|||_T \equiv \sup_{[-T,T]} \|v(t)\|_{1,2} + \left(\int_{-T}^{T} (\|v(t)\|_\rho^r + \|\nabla_x v(t)\|_\rho^r)\, dt\right)^{1/r} \leq a\Big\}. \quad (5.37)$$

Notice that the pair (ρ, r) is an admissible pair (see Corollary 4.1).

We prove that there exist positive constants T and a such that the operator defined in (5.20) is a contraction on $E^1(T,a)$.

Combining Hölder's inequality and the Sobolev inequality (Theorem 3.3) it follows that

$$\||u|^{\alpha-1}\nabla u\|_{\rho'} \leq c\, \||u|^{\alpha-1}\|_l \|\nabla u\|_\rho \leq c\, \|u\|_{(\alpha-1)l}^{\alpha-1} \|\nabla u\|_\rho \leq c\, \|\nabla u\|_\rho^\alpha.$$

Thus,

$$\||u|^{\alpha-1}u\|_{1,\rho'} \leq c\, \|u\|_{1,\rho}^\alpha\,, \quad (5.38)$$

with $1/\rho' = 1/l + 1/\rho$. Then,

$$\frac{1}{l} = 1 - \frac{2}{\rho} \ \text{ and } \ \frac{1}{(\alpha-1)l} = \frac{1}{\rho} - \frac{1}{n}, \ \text{ i.e., } \ \frac{1}{l} = \frac{\alpha-1}{\rho} - \frac{\alpha-1}{n}.$$

Therefore, $(\alpha+1)/\rho = (n+\alpha-1)/n$.

Using Corollary 4.1, (5.20), and (5.38), we have

$$|||\Phi(u)|||_T \leq c\, \|u_0\|_{1,2} + c \left(\int_0^T \||u|^{\alpha-1}u(t)\|_{1,\rho'}^{r'}\, dt\right)^{1/r'}$$

$$\leq c\, \|u_0\|_{1,2} + c \left(\int_0^T \|u(t)\|_{1,\rho}^{\alpha r'}\, dt\right)^{1/r'} \quad (5.39)$$

$$\leq c\, \|u_0\|_{1,2} + c\, T^{\delta} \left(\int_0^T \|u(t)\|_{1,\rho}^r \, dt \right)^{\alpha/r},$$

with $\delta = 1 - (\alpha+1)/r = 1 - (n-2)(\alpha-1)/4$. Hence, taking $a = 2c\, \|u_0\|_{1,2}$ in (5.37), we get from (5.39) that

$$\begin{aligned} \|\!|\Phi(u)|\!\|_T &\leq c\, \|u_0\|_{1,2} + c\, T^{\delta}\, \|\!|u|\!\|_T^{\alpha} \\ &\leq \frac{a}{2} + c\, T^{\delta}\, \frac{a^{\alpha}}{(2c)^{\alpha}} \leq a \end{aligned}$$

if T is sufficiently small, i.e.,

$$\frac{c\, T^{\delta}}{(2c)^{\alpha}}\, a^{\alpha-1} \leq \frac{1}{2}.$$

Thus,

$$T \lesssim a^{(1-\alpha)/\delta}. \tag{5.40}$$

To complete the proof of existence and uniqueness of the solution, it is enough to show that the operator Φ is a contraction. The proof of this as well as the continuous dependence is similar to the one given in the previous section, so it will be omitted. □

Remark 5.5. As we commented in the previous section, in the proof of this (local) result, we did not use the particular structure of the nonlinear term.

Theorem 5.5 (Critical case, $\alpha = (n+2)/(n-2)$, $n > 2$, in $H^1(\mathbb{R}^n)$). *Let $n > 2$ and $\alpha = (n+2)/(n-2)$. Given $u_0 \in H^1(\mathbb{R}^n)$, there exist $T = T(u_0, n, \lambda, \alpha) > 0$ and a unique solution u of the integral equation (5.15) in the time interval $[-T, T]$ with*

$$u \in C([-T,T]\colon H^1(\mathbb{R}^n)) \cap L^r([-T,T]\colon L_1^{\rho}(\mathbb{R}^n)),$$

where $r = 2n/(n-2)$, $\rho = 2n^2/(n^2-2n+4)$ and L_1^p is defined as in (3.38).

Moreover, for all $T' < T$ there exists a neighborhood W of u_0 in $H^1(\mathbb{R}^n)$ such that the function

$$\mathbb{F}\colon W \to C([-T',T']\colon H^1(\mathbb{R}^n)) \cap L^r([-T',T']\colon L_1^{\rho}(\mathbb{R}^n)), \quad \tilde{u}_0 \to \tilde{u}(t),$$

is Lipschitz.

Remark 5.6. We notice that the time of existence depends on the initial data. In Theorem 5.4, it depends only on the size of u_0, that is, on $\|u_0\|_{1,2}$. In Theorem 5.5, the interval of existence depends on the position of u_0, and not only on its size.

Proof. Observe that the pair $(r,\rho) = (2n/(n-2), 2n^2/(n^2-2n+4))$ satisfies condition (5.18) of Corollary 5.1. First, we have that

$$\left(\int_0^T \|\nabla_x(|u|^{\alpha-1}u)\|_{\rho'}^{r'} \right)^{1/r'} \leq c \left(\int_0^T \|\nabla_x u\|_{\rho}^{r} \right)^{1/r} \left(\int_0^T \||u|^{\alpha-1}\|_{\nu}^{l} \right)^{1/l} \tag{5.41}$$

$$\leq c\left(\int_0^T \|\nabla_x u\|_\rho^r\right)^{1/r}\left(\int_0^T \|u\|_{\nu(\alpha-1)}^{l(\alpha-1)}\right)^{1/l},$$

where $1/r + 1/r' = 1/\rho + 1/\rho' = 1$, $1/\rho' = 1/\rho + 1/\nu$, and $1/r' = 1/r + 1/l$. Since $(\alpha, r, \rho) = ((n+2)/(n-2), 2n/(n-2), 2n^2/(n^2-2n+4))$, we have $l(\alpha-1) = r$ and $\nu(\alpha-1) = 2n^2/(n-2)^2$. Then by Gagliardo–Nirenberg's inequality (3.14) it follows:

$$\|u\|_{\nu(\alpha-1)} \leq c\,\|u\|_{1,\rho} = c\,(\|u\|_\rho + \|\nabla_x u\|_\rho). \tag{5.42}$$

Combining (5.41), (5.42), Proposition 5.1, Theorem 4.2, and the notation in the proof of Theorem 5.4, we obtain that for any $\varepsilon > 0$ fixed there exists $T > 0$ such that

$$\begin{aligned}
&\left(\int_0^T \|\Phi(u)(t)\|_{1,\rho}^r\,dt\right)^{1/r}\\
&\leq c\left(\int_0^T \|\Phi(u)(t)\|_\rho^r\,dt\right)^{1/r} + \left(\int_0^T \|\nabla_x\Phi(u)(t)\|_\rho^r\,dt\right)^{1/r}\\
&\leq c\epsilon + c|\lambda|\left(\int_0^T \|u\|_\rho^r\,dt\right)^{1/r}\\
&\quad + c|\lambda|\left(\int_0^T \|\nabla_x u\|_\rho^r\,dt\right)^{1/r}\left(\int_0^T \|u\|_{1,\rho}^r\,dt\right)^{(\alpha-1)/r}\\
&\leq c\epsilon + c|\lambda|\left(\int_0^T \|u\|_{1,\rho}^r\,dt\right)^{\alpha/r}.
\end{aligned} \tag{5.43}$$

On the other hand, we have that

$$\sup_{[0,T_0]}\|\Phi(u)(t) - e^{it\Delta}u_0\|_{1,2} \leq c|\lambda|\left(\int_0^T \|u\|_{1,\rho}^r dt\right)^{\alpha/r}. \tag{5.44}$$

Therefore, defining

$$\begin{aligned}
\widetilde{E}^1(T,a) = \Big\{ & v \in C([0,T] : H^1(\mathbb{R}^n)) \cap L^r([0,T] : L_1^\rho(\mathbb{R}^n)) :\\
& |||v|||_T \equiv \sup_{[0,T_0]}\|v(t) - e^{it\Delta}u_0\|_{1,2} + \left(\int_0^T \|v\|_{1,\rho}^r\,dt\right)^{1/r} \leq a\Big\},
\end{aligned}$$

and applying (5.44) and (5.43), we have that for all $\epsilon > 0$ there exists $T > 0$ such that if $u \in \widetilde{E}^1(T, a)$, then

$$\|\Phi(u)(t)\| \le c\epsilon + c|\lambda|\, a^{\alpha}. \tag{5.45}$$

Once inequality (5.45) is established, the remainder of the proof follows an argument given previously, so it will be omitted. □

Corollary 5.4. *There exists $\epsilon_0 > 0$ depending on λ and n such that for all $u_0 \in H^1(\mathbb{R}^n)$ with $\|u_0\|_{1,2}$ small, the results of Theorem 5.5 extend to all time intervals $[0, T]$, so*

$$u \in C(\mathbb{R} : H^1(\mathbb{R}^n)) \cap L^r(\mathbb{R} : L_1^{\rho}(\mathbb{R}^n)) \tag{5.46}$$

with (r, ρ) as in Theorem 5.5.

Proof. Once Theorem 5.5 is established, we follow the argument used in the proof of Corollary 5.2. □

5.3 H² Theory

Consider again the integral equation (5.15) with $u_0 \in H^2(\mathbb{R}^n)$.

Assume that the nonlinearity α satisfies

$$\begin{cases} 2 \le \alpha < \dfrac{n}{n-4}, & \text{if } n \ge 5 \\ 2 \le \alpha < \infty, & \text{if } n \le 4. \end{cases} \tag{5.47}$$

Theorem 5.6 (Local theory in $H^2(\mathbb{R}^n)$). *If α satisfies (5.47), then for all $u_0 \in H^2(\mathbb{R}^n)$ there exist $T = T(\|u_0\|_{2,2}, n, \lambda, \alpha) > 0$ and a unique solution u of the integral equation (5.15) in the interval of time $[-T, T]$ with*

$$u \in C([-T, T]\colon H^2(\mathbb{R}^n)) \cap L^q([-T, T]\colon L_2^p(\mathbb{R}^n)) \tag{5.48}$$

for all pairs (p, q) defined by condition (4.18) of Corollary 5.1.

Moreover, for all $T' < T$ there exists a neighborhood W of u_0 in $H^2(\mathbb{R}^n)$ such that for all pairs (p, q) in (4.18) the function

$$\mathbb{F} : \; W \mapsto C([-T', T']\colon H^2(\mathbb{R}^n)) \cap L^q([-T', T']\colon L_2^p(\mathbb{R}^n)), \quad \tilde{u}_0 \mapsto \tilde{u}(t),$$

is Lipschitz.

The proof of this result is similar to the one exposed to establish Theorem 5.2 and Corollary 5.1, so it is left to the reader to complete the details.

As a consequence of Theorem 5.5 we obtain the following relation between the differential equation (5.1) and integral equation (5.15).

Corollary 5.5. *If* u *is the solution of equation* (5.15) *obtained in Theorem* 5.6, *then for all pair* (p, q) *which verifies condition* (5.18) *of Corollary* 5.1, *we have*

$$\partial_t u \in L^q([-T, T]: L^p(\mathbb{R}^n)).$$

Moreover, u *is the (unique) solution of the differential equation* (5.1) *in the time interval* $[-T, T]$.

Proof. Using Theorem 3.3 and hypothesis (5.47) on the nonlinearity, it is easy to see that $u \in C([-T, T]: H^2)$ implies that $|u|^{\alpha-1}u \in C([-T, T]: L^2)$. Combining Theorem 5.6, which guarantees $\Delta u \in C([-T, T]: L^2)$ with the previous results and the integral equation (5.15), we see that $\partial_t u \in C([-T, T]: L^2)$, and that the differential equation in (5.1) is realized in the space $C([-T, T]: L^2)$.

The end of the proof is left as an exercise to the reader. □

In the next chapter, we will use the identities (5.2) and (5.3) to establish global solutions. To justify them, we present the following result in H^2.

Theorem 5.7.

1. *Let* $u \in C([-T, T]: L^2(\mathbb{R}^n)) \cap L^q([-T, T]: L^p(\mathbb{R}^n))$ *be the solution of integral equation* (5.15) *obtained in Section 5.1. If* $u_0 \in H^1(\mathbb{R}^n)$, *then*

$$u \in C([-T, T]: H^1(\mathbb{R}^n)) \cap L^q([-T, T]: L^p_1(\mathbb{R}^n)). \tag{5.49}$$

2. *Let* $u \in C([-T, T]: H^1) \cap L^q([-T, T]: L^p_1)$ *be the solution of the integral equation* (5.15) *obtained in Section 5.2. If* $u_0 \in H^2(\mathbb{R}^n)$ *and* $\alpha \geq 2$, *then* $u \in C([-T, T]: H^2)$ *and satisfies the differential equation* (5.1) *and estimates* (5.2) *and* (5.3).

Proof. We prove only part 1 of the theorem. Given $u_0 \in H^1(\mathbb{R}^n)$, we know by Theorem 5.4 that there exists $T' > 0$ such that $u \in C([-T', T']: H^1(\mathbb{R}^n))$. If $T' > T$ it is easy to see that the solution in L^2 can be extended to the interval $[-T', T']$. Thus, we assume that $T' < T$. To get the desired result, it is enough to prove that

$$\sup_{[0,T']} \|\nabla_x u(t)\|_2 \leq K \|u(0)\|_{1,2}$$

with K depending only on T and $M = \sup\{\|u(t)\|_2 : t \in [0, T]\}$.

Differentiate the integral equation (5.15) and use the notation $v_j = \partial_{x_j} u$, $j = 1, \ldots, n$, to have that

$$v_j(t) = e^{it\Delta} v_j(0) + i\lambda\alpha \int_0^t e^{i(t-t')\Delta}(|u|^{\alpha-1} v_j)(t')\, dt', \tag{5.50}$$

which is a linear integral equation, because $u(\cdot)$ is known in the time interval $[0, T]$. With the same method used in the proof of Theorem 5.2, it is easy to see that this new integral equation (5.50) has unique solution on $[0, \Delta T]$, where ΔT depends on α, λ, n, and M, which remains constant in the interval $[0, T]$. Combining this result with an iterative argument, we obtain (5.49), which leads to the result. □

Now, we explain how to use Theorems 5.6 and 5.7 to justify the use of identities (5.2) and (5.3), respectively, in the proof of theorems (global).

Assume that $u_0 \in L^2(\mathbb{R}^n)$ and $\alpha \in (2, 1+4/n)$, we choose $\{u_0^k\}_{k=1}^{\infty}$ in $H^2(\mathbb{R}^n)$ such that $\|u_0^k - u_0\|_2 = o(1)$ when $k \to \infty$. Combining Theorems 5.2, (5.6), and (5.7), we see that for all $T > 0$ there exist $u^k \in C([-T, T] : H^2(\mathbb{R}^n))$, $k = 1, \dots,$ a solution of (5.1) and (5.15) with initial data u_0^k. Since it satisfies the differential equation in (5.1), we infer that for all $t \in [-T, T]$,

$$\|u^k(t)\|_2 = \|u_0^k\|_2,$$

i.e., identity (5.2). From Theorem 5.2 (continuous dependence on the initial data), we have that $\sup\limits_{[-T',T']} \|u^k(t) - u(t)\|_2 = o(1)$ when $k \to \infty$, where $T' < T$. Thus,

$$\|u(t)\|_2 = \|u_0\|_2 \quad \text{for all } t \in [-T', T']. \tag{5.51}$$

This identity allows us to reapply Theorem 5.2 and extend the solution to the interval $[-(T' + \Delta T'),\ T' + \Delta T']$, where (using the same argument) identity (5.51) still holds. By successive applications of this step, we obtain the desired result (identity (5.42) in any time interval).

Finally, the case $\alpha \in (1, 2)$ requires some changes: For initial data $u_0^k \in H^2(\mathbb{R}^n)$ we will have the nonlinear term $\rho_k * (|\rho_k * u|^{\alpha-1} \rho_k * u)$, where $\rho_k(\cdot) = k^n \rho(\cdot / k)$, with $\rho(\cdot)$ an approximation of the identity. In this case it will be necessary to prove the stability of the solution in L^2 with respect to initial data and the nonlinear term.

As we remarked at the end of Theorem 5.2 all the previous existence proofs are based on the contraction principle. This approach has the advantage that it also shows that for smooth nonlinearity the map data-solution is smooth.

This general fact follows from the implicit function theorem. However, to simplify the exposition we will sketch the details in the case of Theorem 5.2.

Corollary 5.6. *Assume the same hypotheses of Theorem 5.2. Suppose $F(u, \bar{u}) = i\lambda|u|^{\alpha-1}u$ is smooth (i.e., $\alpha - 1$ is an even integer). Then there exists a neighborhood $\widetilde{V}$ of $u_0 \in L^2(\mathbb{R}^n)$ such that the map $\mathbb{F} : u_0 \mapsto u(t)$ from $\widetilde{V}$ into $E(T, a)$ is smooth.*

Proof. Define for $F(u, \bar{u}) = i\lambda|u|^{\alpha-1}u$

$$\begin{aligned} H : V \times E(T, a) &\mapsto E(T, a) \\ (v_0, v(t)) &\mapsto v(t) - \Phi_{v_0}(v)(t) \\ &= v(t) - \Big(e^{it\Delta} v_0 + \int_0^t e^{i(t-t')\Delta} F(v, \bar{v})(t')\, dt'\Big). \end{aligned}$$

Thus, H is smooth, $H(u_0,\ u(t)) = 0$, and

$$D_v H(u_0, u(t)) v(t) = v(t) + \int_0^t e^{i(t-t')\Delta} [\partial_v F(u, \bar{u})\, v + \partial_{\bar{v}} F(u, \bar{u}) \bar{v}](t')\, dt'.$$

Hence,

$$D_v H(u_0, u(t)) = I + L.$$

From the proof of Theorem 5.2 it is easy to see that

$$|||Lv||| \leq c|\lambda|T^\theta a^{\alpha-1} < 1$$

for any choice of a in (5.24). Then,

$$D_u H(u_0, u(t)) : E(T, a) \to E(T, a)$$

is invertible, i.e., one-to-one and onto. Thus, by the implicit function theorem there exists $h : \widetilde{V} \to E(T, a)$ smooth ($\widetilde{V} \subset V$ neighborhood of $u_0 \in L^2(\mathbb{R}^n)$) such that

$$H(v_0, h(v_0)) = 0, \quad \forall v_0 \in \widetilde{V},$$

so,

$$h(v_0) = e^{it\Delta} v_0 + \int_0^t e^{i(t-t')\Delta} F(h(v_0), \overline{h(v_0)})(t')\, dt'$$

is a solution of (5.15) with data v_0 (instead of u_0). □

Remark 5.7. The same argument shows that if $F(u, \bar{u}) = i\lambda|u|^{\alpha-1}u$ is $C^{[\alpha]}$ (when $\alpha - 1$ is not an even integer), then the map $\mathbb{F} : u_0 \mapsto u(t)$ from $\widetilde{V}$ into $E(T, a)$ is $C^{[\alpha]}$.

5.4 Comments

The L^2 theory exposed on Section 5.1 was obtained by Y. Tsutsumi [T1] in the case $\alpha \in (1, 1 + 4/n)$. The critical case L^2 $(\alpha = 1 + 4/n)$ was established by Cazenave and Weissler [CzW3]. The results of Section 5.2 were taken from references [CzW2], [GV1], [K1], and [T2]. Finally, the H^2 theory can be found in [K2].

It is important to note that Theorems 5.2, 5.4, and 5.6 prove that under some conditions on the power of the nonlinearity α, the solutions of the integral equation possess, at least locally in time, the same smoothing properties as the Strichartz type (discussed in Section 4.2, Theorem 4.2) that the solution of the associated linear problem.

From the proof of Theorem 5.3, one sees that the conditions on the data u_0 in the existence results can be significantly weaker. To simplify the exposition, let us concentrate on the results in Theorem 5.3: Instead of $u_0 \in L^2(\mathbb{R}^n)$, one can take $u_0 \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\|e^{it\Delta}u_0\|_{L^\sigma(\mathbb{R}^n_x \times \mathbb{R}_t)} < \infty, \quad \sigma = 2 + \frac{4}{n} \tag{5.52}$$

to get the same local result, or

$$\|e^{it\Delta}u_0\|_{L^\sigma(\mathbb{R}^n_x\times\mathbb{R}_t)} \ll 1 \tag{5.53}$$

to obtain a global one in the function space $u(\cdot)$ with

$$u - e^{it\Delta}u_0 \in C([0,T]:L^2(\mathbb{R}^n)) \text{ and } u \in L^\sigma([0,T]:L^\sigma(\mathbb{R}^n)). \tag{5.54}$$

Several methods to construct $u_0 \in \mathcal{S}'(\mathbb{R}^n)$ such that (5.52) or (5.53) are satisfied (or simply, $u_0 \in L^2(\mathbb{R}^n)$ with $\|u_0\|_2 \gg 1$ such that (5.53) holds) have been developed.

Let us consider first the last problem. Without loosing generality, assume the 1-D case. We will use Examples 4.1–4.3 Chapter 4 to obtain $u_0 \in L^2(\mathbb{R})$ with $\|u_0\|_2 \gg 1$ such that (5.53) holds.

Let $\varphi \in C_0^\infty(\mathbb{R})$ with $\operatorname{supp}\varphi \subseteq B_1(0)$ and $\|\varphi\|_2 = 1$. Let $N \in \mathbb{Z}^+$ and define

$$u_0^N(x) \equiv \sum_{j=1}^N \varphi(x-\nu_j)e^{2\pi i\mu_j x} = \sum_{j=1}^N \varphi_j(x), \tag{5.55}$$

where $\nu_1,\dots,\nu_N$ and $\mu_1,\dots,\mu_N$ are numbers chosen such that for $t>0$ the "cones" containing most of the mass of $u_j(x,t) = e^{it\Delta}\varphi_j$, i.e., for $t_0 \gg 1$ fixed

$$c_j = \left\{(x,t): \frac{(2N-1)t_0+1}{t_0}t - 1 \le x \le \frac{(2N+1)t_0-1}{t_0}t + 1,\ 0 \le t \le t_0\right\},$$

do not overlap. Thus,

$$\|u_0^N\|_2 = \sqrt{N} \tag{5.56}$$

and using that outside c_j, $u_j(x,t)$ decays exponentially (for a t fixed and $|x|\to\infty$), one can show (for a similar computation see [Vi2]):

$$\begin{aligned}\|e^{it\Delta}u_0^N\|_{L^6(\mathbb{R}\times\mathbb{R}^+)} &= \Big\|\sum_{j=1}^N e^{it\Delta}\varphi_j\Big\|_{L^6(\mathbb{R}\times\mathbb{R}^+)} \\ &\simeq \left(\sum_{j=1}^N \int_0^\infty\int_{-\infty}^\infty |e^{it\Delta}\varphi_j(x)|^6\,dx dt\right)^{1/6} \simeq N^{1/6}\end{aligned} \tag{5.57}$$

(since $\|e^{it\Delta}\varphi_j\|_{L^6}^6 \le \|\varphi_j\|_{L^2}^6 = 1$). So by taking $v_0^N = u_0^N/\sqrt{N}$, we get a sequence of data with $\|v_0^N\|_{L^2} = 1$ and

$$\|e^{it\Delta}v_0\|_{L^6(\mathbb{R}\times\mathbb{R}^+)} \le cN^{-1/3} \ll 1 \quad \text{for } N \text{ large.}$$

For the same problem, Bourgain [Bo3] introduced the following norm (two-dimensional case, $n=2$)

$$|||u_0|||_{X_p} = \left(\sum_{j=1}^\infty\sum_{k=1}^\infty 2^{-4j}\left(\frac{1}{2^{-2j}}\int_{Q_k^j}|u_0(x)|^p\,dx\right)^{4/p}\right)^{1/4}, \tag{5.58}$$

where $\{Q_k^j\}_{k\in\mathbb{Z}^+}$ denotes a grid of squares with disjoint interior of side 2^{-j} parallel to the axes.

First one notices that the norm $|||\cdot|||_{X_p}$ scales like the $L^2(\mathbb{R}^2)$-norm, i.e., $|||f_\lambda|||_{X_p}$ with $f_\lambda(x)=\lambda f(\lambda x)$ is independent of λ (see Exercise 5.5). In [MVV1], [MVV2] Moyua, Vargas and Vega (improving and extending results in [Bo3]) showed that

$$\|e^{it\Delta}u_0\|_{L^4(\mathbb{R}^2_x\times\mathbb{R}_t)}\leq c|||u_0|||_{X_p} \tag{5.59}$$

for $12/7\leq p\leq 2$ for any $u_0\in L^1_{\text{loc}}(\mathbb{R}^2)$, and for $4(\sqrt{2}-1)\leq p<2$ if u_0 is the characteristic function of a measurable set. Moreover, they showed that $p>4(\sqrt{2}-1)$ is sharp.

Using (5.58) and (5.59), one can find $u_0\in L^1_{\text{loc}}(\mathbb{R}^2)\setminus L^2(\mathbb{R}^2)$ such that (5.53) holds.

Let

$$u_{0j}(x,y)=\chi_{\{[0,2^{-j}]\times[0,2^j]\}}(x,y),\quad j\in\mathbb{Z}^+. \tag{5.60}$$

It is not hard to see that $|||u_{0j}|||_{X_p}\leq 2^{-j/4}$ (Exercise 5.7) while $\|u_{0j}\|_2\equiv 1$. Then taking

$$u_0(x,y)=\epsilon\sum_{j=1}^{\infty}u_{0j}((x,y)-(j,0)),\quad \epsilon>0 \tag{5.61}$$

it follows that $u_0\notin L^2(\mathbb{R}^2)$ and

$$\|e^{it\Delta}u_0\|_{L^4(\mathbb{R}^2\times\mathbb{R})}\leq |||u_{0j}|||_{X_p}\leq c\epsilon.$$

It is not difficult to show that solutions of (5.15) also enjoy the local regularity property described in Section 4.3. For instance, we see that the solution $u(\cdot)$ of (5.15) obtained in Theorem 5.2 satisfies

$$u\in L^2([-T,T]:\,H^{1/2}_{\text{loc}}(\mathbb{R}^n)). \tag{5.62}$$

In fact, writing the equation (5.15) in the form:

$$u(t)=e^{it\Delta}\Big(u_0+\int_0^t e^{-it'\Delta}(|u|^{\alpha-1}u)(t')\,dt'\Big)$$

and using (4.25) (or (4.23) when $n=1$) and (4.16) we have that

$$\begin{aligned}\left(\int_{\{|x|\leq R\}}\int_{-T}^{T}|D_x^{1/2}u(x,t)|^2\,dt\,dx\right)^{1/2}&\leq c\,R\left(\|u_0\|_2+\sup_{[-T,T]}\Big\|\int_0^t e^{-it'\Delta}(|u|^{\alpha-1}u)(t')\,dt'\Big\|_2\right)\\&\leq c\,R\left(\|u_0\|_2+\left(\int_0^T\|\,|u|^{\alpha}(t)\|^{r'}_{(\alpha+1)/\alpha}\,dt\right)^{1/r'}\right),\end{aligned}$$

where $r = 4(\alpha+1)/n(\alpha-1)$. Combining (5.21) and (5.22) with Corollary 5.1, we obtain (5.62).

As we have seen along this chapter, the results concerning local existence are a consequence of the estimates obtained in Theorem 4.2. Thus, the method of proof applied can be extended to any group satisfying Theorem 4.2 (even locally). In particular, we obtain the same local theorems for the nonlinear Schrödinger (NLS) equation with real potential

$$\partial_t u = i\Delta u + V(x)u + \lambda |u|^{\alpha-1}u,$$

under appropriate conditions on V (see the references [C], [Y]).

Theorems 5.6 and 5.7 are concerned with the regularity of solution measured in Sobolev spaces. One can also ask whether the decay properties of the data are preserved by the solution. To simplify the matter, consider the case where α is an odd integer (or, where the nonlinearity has the form $f(|u|^2)u$ with $f(\cdot)$ smooth). In [HNT1], [HNT2], [HNT3], Hayashi, Nakamitsu, and Tsutsumi showed that if $u_0 \in H^m(\mathbb{R}^n) \cap L^2(|x|^k\,dx)$ with $m \geq k$, then there exists $T = T(\|u_0\|_{H^l})$, $l = \min\{m; n/2^+\}$ such that the IVP (5.1) has a unique solution

$$u \in C([0,T] : H^m(\mathbb{R}^n) \cap L^2(|x|^k\,dx)) \cap L^q([0,T] : L^p_k(\mathbb{R}^n) \cap L^p(|x|^k\,dx))$$

with p, q as in Theorem 4.2 and where $L^p_k(\mathbb{R}^n)$ is defined as in (3.38).

In the case $k \geq m$ they showed that the solution u does not belong to $L^2(|x|^k\,dx)$ but possesses a further regularity property, roughly speaking $\partial_x^\alpha u(\cdot,t) \in L^2_{\text{loc}}(\mathbb{R}^n)$, $t \neq 0$, for $|\alpha| \leq k$, (see [HNT1], [HNT2]).

In particular, one has that if $u_0 \in \mathcal{S}(\mathbb{R}^n)$, then the solution $u(\cdot)$ of the IVP (5.1) (with α an odd integer) belongs to $C([0,T] : \mathcal{S}(\mathbb{R}^n))$, and that if $u_0 \in H^1(\mathbb{R}^n)$ with compact support, α an odd integer, and $1 + 4/n < \alpha < 1 + 4/(n-2)$, then $u \in C^\infty(\mathbb{R}^n \times \mathbb{R} - \{0\})$.

The proofs given in [HNT1]–[HNT3] are based on the properties of the operators $\Gamma_j = x_j + 2it\partial_{x_j}$, $j = 1, \dots, n$, deduced there.

In particular, using that for $\Gamma = (\Gamma_1, \dots, \Gamma_n)$,

$$\Gamma^\alpha u = e^{i|x|^2/4t}\,(2it)^{|\alpha|}\partial_x^\alpha(e^{-i|x|^2/4t}u) \quad \text{for } \alpha \in \mathbb{Z}^+ \tag{5.63}$$

and

$$x^\alpha e^{it\Delta}u_0 = e^{it\Delta}\,\Gamma^\alpha u_0 \tag{5.64}$$

(see Exercise 4.4), they developed a calculus of inequalities for the operators Γ_j similar to that in (3.15) for the operators ∂_{x_j}. For instance, for $n = 1$ they showed that

$$\|\Gamma^m(|v|^{2\alpha}v)(t)\|_{L^2} \leq c_m \|v(t)\|^{2\alpha}_{L^\infty}\|\Gamma^m v(t)\|_{L^2}$$

and

$$\|v(t)\|_{L^\infty} \leq t^{-1/2}\,\|\Gamma v(t)\|^{1/2}_{L^2}\|v(t)\|^{1/2}_{L^2}$$

(compare with (3.14), (3.15), and (3.16) in Chapter 3) which have been essential tools in the study of the asymptotic behavior of solution of (5.1). The extension of these weighted results to $L^2(|x|^k\,dx)$ with $k \geq 0$ (not necessarily an integer) was obtained in [NhPo1].

To simplify the exposition, we have presented local well-posedness results in Sobolev spaces with integer indexes, i.e., $H^s(\mathbb{R}^n)$, $s = 0, 1, 2$. Concerning the local existence theory in fractional Sobolev spaces, $H^s(\mathbb{R}^n)$, $s \geq 0$, we have the following result due to Cazenave and Weissler [CzW4].

Theorem 5.8. *Let* $1 + 4/n \leq \alpha < \infty$ *and* $s > s_\alpha = n/2 - 2/(\alpha - 1)$*, with* $[s] < \alpha - 1$ *if* $\alpha - 1$ *is not an even integer. Given* $v_0 \in H^s(\mathbb{R}^n)$*, there exist* $T = T(\|v_0\|_{s,2}; s) > 0$ *and a unique strong solution* $v(\cdot)$ *of the IVP* (5.1) *satisfying*

$$v \in C([-T, T] : H^s(\mathbb{R}^n)) \cap W^T_{s,n}. \tag{5.65}$$

Moreover, given $T' \in (0, T)$ *there exist a constant* $r = r(\|v_0\|_{s,2}; s; T') > 0$ *and a continuous, nondecreasing function* $G(\cdot) = G(\|v_0\|_{s,2})$ *with* $G(0) = 0$ *such that*

$$\sup_{[0,T']} \|(v - \widetilde{v})(t)\|_{s,2} \leq G(\|v_0\|_{s,2})\|v_0 - \widetilde{v}_0\|_{s,2} \tag{5.66}$$

for any $\widetilde{v}_0 \in H^s(\mathbb{R}^n)$ *with* $\|v_0 - \widetilde{v}_0\|_{s,2} < r$*, i.e., the map data-solution is locally Lipschitz.*

The space $W^T_{s,n}$ in (5.65) is related to the Strichartz estimates, and its precise definition will not be needed in the discussion below. We recall that for $1 < \alpha < 1 + 4/n$ the problem is locally well-posed in $L^2(\mathbb{R}^n)$.

From the scaling argument, i.e., if $u(x, t)$ is a solution of the IVP (5.1), then

$$u_\mu(x, t) = \mu^{2/(\alpha-1)} u(\mu x, \mu^2 t), \qquad \mu > 0, \tag{5.67}$$

is also a solution with data $u_\mu(x, 0) = \mu^{2/(\alpha-1)} u_0(\mu x)$, for which one has that

$$\|D^s_x u_\mu(\cdot, 0)\|_2 = c\mu^{2/(\alpha-1)} \mu^{s-n/2} \|u_0\|_2.$$

To have results invariant by rescaling, one needs to consider data $u_0 \in \dot{H}^s(\mathbb{R}^n)$ $(= (-\Delta)^{-s/2} L^2(R^n))$, with $s(\alpha) = s_c = n/2 - 2/(\alpha - 1)$ which is called the *critical case*. The case $s > s_c = n/2 - 2/(\alpha - 1)$ is called *subcritical case*. Notice that Theorem 5.8 above corresponds to the subcritical case and Theorem 5.3 to the critical case in $L^2(\mathbb{R}^n)$ $(s = 0)$.

So the following question arises. Are the results in Theorem 5.8 optimal? This seems to be the case. First, let us consider the "focusing case," i.e., for $\lambda > 0$ in (5.1), the following result was obtained in [BKPSV].

Theorem 5.9. *If* $4/n + 1 \leq \alpha < \infty$*, then the IVP* (5.1) *with* $\lambda > 0$ *is ill-posed in* $H^{s_c}(\mathbb{R}^n)$ *with* $s_c = n/2 - 2/(\alpha - 1)$*, in the sense that the time of existence T and the continuous dependence cannot be expressed in terms of the size of the data in the* H^{s_c}*-norm. More precisely, there exists* $c_0 > 0$ *such that for any* δ*,* $t > 0$ *small there exist data* u_1*,* $u_2 \in \mathcal{S}(\mathbb{R}^n)$ *such that*

$$\|u_1\|_{s,2} + \|u_2\|_{s,2} \leq c_0, \quad \|u_1 - u_2\|_{s,2} \leq \delta, \quad \|u_1(t) - u_2(t)\|_{s,2} > c_0/2,$$

where $u_j(\cdot)$ *denotes the solution of the IVP* (5.1) *with data* u_j*,* $j = 1, 2$*.*

Proof. For simplicity, we shall only consider the case $0 < s_c < 1$ and fix $\lambda = 1$. We consider the one-parameter family of ground states:

$$v_\mu(x,t) = e^{i\mu t}\varphi_\mu(x) = e^{i\mu t}\mu^{1/(\alpha-1)}\varphi(\sqrt{\mu}x),$$

where the function $\varphi(\cdot) = \varphi_1(\cdot)$ solves the nonlinear elliptic eigenvalue problem (5.8) with $0 < \alpha < 4/(n-2)$, if $n > 2$. The idea is to estimate

$$\|D_x^{s_c}(v_{\mu_1} - v_{\mu_2})(t)\|_2^2$$

and

$$\|D_x^{s_c}(\mu_1^{1/(\alpha-1)}\varphi_1(\sqrt{\mu}_1\cdot) - \mu_2^{1/(\alpha-1)}\varphi_1(\sqrt{\mu}_2\cdot))\|_2^2.$$

Choosing $\mu_1 = (N+1)^2$ and $\mu_2 = N^2$ so that $\mu_1 - \mu_2 > 2N$, we have that

$$\begin{aligned}&\|D_x^{s_c}(v_{\mu_1} - v_{\mu_2})(t)\|_2^2\\ &= \|D_x^{s_c}v_{\mu_1}(t)\|_2^2 + \|D_x^{s_c}v_{\mu_2}(t)\|_2^2 - 2\mathcal{R}e\left\{e^{it(\mu_1-\mu_2)}\langle v_{\mu_1}(t), v_{\mu_2}(t)\rangle_{s_\alpha}\right\}\\ &= \Psi(\mu_1,\mu_2)(t).\end{aligned}$$

Given any $T > 0$ there exist $N > c(T)$ and $t \in (0,T)$ such that

$$\mathcal{R}e\ \{e^{it(\mu_1-\mu_2)}\langle v_{\mu_1}(t), v_{\mu_2}(t)\rangle_{s_c}\} = 0,$$

hence,

$$\sup_{[0,T]} \Psi(\mu_1,\mu_2)(t) = 2\|D_x^{s_c}\varphi_1\|_2^2.$$

On the other hand,

$$\begin{aligned}\lim_{N\to\infty} \|D_x^{s_c}(v_{\mu_1} - v_{\mu_2})(0)\|_2^2 &= \|D_x^{s_c}v_{\mu_1}(0)\|_2^2 + \|D_x^{s_c}v_{\mu_2}(0)\|_2^2\\ &\quad - 2\mathcal{R}e\ \{\langle v_{\mu_1}, v_{\mu_2}\rangle_{s_c}\} = 0\end{aligned}$$

by using that $\mu_1/\mu_2 \to 1$ as $N \to \infty$ and so

$$\lim_{N\to\infty} \mathcal{R}e\ \{\langle v_{\mu_1}, v_{\mu_2}\rangle_{s_c}\} = \|D_x^{s_c}\varphi_1\|_2^2.$$

Therefore, for any $T > 0$

$$\lim_{N\to\infty}\sup_{[0,T]} \|D_x^{s_c}(v_{\mu_1} - v_{\mu_2})(t)\|_2 = \sqrt{2}\|D_x^{s_c}\varphi_1\|_2,$$

while

$$\lim_{N\to\infty} \|D_x^{s_c}(\mu_1^{1/(\alpha-1)}\varphi(\sqrt{\mu_1}\cdot) - \mu_2^{1/(\alpha-1)}\varphi(\sqrt{\mu}_2\cdot))\|_2 = 0,$$

which essentially proves the result. □

Christ, Colliander and Tao [CrCT1] have shown that the results in Theorem 5.9 extend to the defocusing case $\lambda < 0$. Moreover, the following stronger ill-posedness result in *norm inflation* concerning the IVP (5.1) in both the focusing and defocusing cases was established in [CrCT3]:

Theorem 5.10. *Given* $s \in (0, s_c)$ $\exists \{u_0^m : m \in \mathbb{Z}^+\} \subset \mathcal{S}(\mathbb{R}^n)$ *and* $\{t_m : t_m > 0\}$ *with* $\|u_0^m\|_{s,2} \to 0,\ t_m \to 0$ *as* $m \uparrow \infty$ *such that the corresponding solution* u^m *of the IVP* (5.1) *with* $\lambda \neq 0$, *and initial data* $u^m(x,0) = u_0^m(x)$ *satisfies that*

$$\|u_m(\cdot, t_m)\|_{s,2} \to \infty, \quad as \quad m \uparrow \infty. \tag{5.68}$$

In the case $\alpha \geq 3$, this result has been strengthened in [AlCa] by showing:

Theorem 5.11. *Given* $\alpha \geq 3$ *and* $s \in (0, s_c)$ *there exist* $\{u_0^m : m \in \mathbb{Z}^+\} \subset \mathcal{S}(\mathbb{R}^n)$ *and* $\{t_m : t_m > 0\}$ *with* $\|u_0^m\|_{s,2} \to 0,\ t_m \to 0$ *as* $m \uparrow \infty$ *such that the corresponding solution* u^m *of the IVP* (5.1), *in the defocusing case* $\lambda < 0$, *with initial data* $u^m(x,0) = u_0^m(x)$, *satisfies that*

$$\|u_m(\cdot, t_m)\|_{l,2} \to \infty, \quad as \quad m \uparrow \infty, \forall l \in \left(\frac{2s}{2 + (\alpha - 1)(s_c - s)}, s\right). \tag{5.69}$$

All the existence results for the IVP (5.1) discussed so far are restricted to Sobolev spaces with nonnegative index, i.e., in $H^s(\mathbb{R}^n)$, $s \geq 0$, even in the cases when the scaling argument tells us that the critical value is negative, that is, $s(\alpha) = s_c = n/2 - 2/(\alpha - 1) < 0$. Thus, for example, we can ask whether for the IVP for the cubic 1-D Schrödinger equation:

$$\begin{cases} i\partial_t v + \partial_x^2 v + \lambda |v|^2 v = 0, \\ v(x,0) = v_0(x), \end{cases} \tag{5.70}$$

$t \in \mathbb{R}$, $x \in \mathbb{R}$, $\lambda \in \mathbb{R}$, for which $s_c = 1/2 - 2/(\alpha - 1) = -1/2$, one can obtain a local existence result in $H^s(\mathbb{R})$, with $s < 0$ (we recall that Theorem 5.2 provides the result in $H^s(\mathbb{R})$, with $s \geq 0$). In this regard, we have the following result found in [KPV4].

Theorem 5.12. *If* $s \in (-1/2, 0)$, *then the mapping data-solution* $u_0 \mapsto u(t)$, *where* $u(t)$ *solves the IVP* (5.70) *with* $\lambda > 0$ *(focusing case), is not uniformly continuous.*

In [VV] and [Gr3], Vargas and Vega, and Grünrock found spaces which scale is below the one from L^2 but above that of $\dot{H}^{-1/2}(\mathbb{R})$, i.e., spaces whose norm is invariant by $\lambda^\theta\, u_0(\lambda x)$ with $\theta \in (-1/2, 0)$, for which the IVP (5.70) is locally and globally well-posed.

Remark 5.8. The result in Theorem 5.12 can be extended to higher dimensions. More precisely, it applies to the IVP

$$\begin{cases} i\partial_t u + \Delta u + |u|^{\rho - 1} u = 0, \\ u(x,0) = u_0(x), \end{cases} \tag{5.71}$$

$t \in \mathbb{R}$, $x \in \mathbb{R}^n$, with $u_0 \in H^s(\mathbb{R}^n)$, $n < 4/(\rho - 1)$, and $s \in (n/2 - 2/(\rho - 1), 0)$.

For the IVP (5.70), on one hand, one has that the map data solution fails to be continuous, i.e., there exist data $u_0 \in \mathcal{S}(\mathbb{R})$ with arbitrary small H^s-norm for $s \leq -1/2$, whose corresponding solution $u(t)$ provided by Theorem 5.2 has arbitrary large H^s-norm at an arbitrary small time (see (5.68)). On the other hand, Theorem 5.12 for the focusing case and the results in [CrCT1] for the defocusing case, shows that the map data-solution is not uniformly continuous in H^s for $s < 0$.

In [KTa3], Koch and Tataru obtained the following a priori estimates for solutions of the IVP (5.70), improving a previous result found in [CrCT2] and [KTa2].

Theorem 5.13 ([KTa3]). *Let $u \in C(\mathbb{R}^+ : L^2(\mathbb{R})) \cap L^4(\mathbb{R} \times [0,T])$ for all $T > 0$ be the global solution of the IVP (5.70) (see Theorem 5.2). Then for all $T > 0$ there exists $\alpha(T) > 0$ such that*

$$\sup_{t\in[0,T]} \|u(t)\|_{H^{-1/4}_{\alpha(T)}} \leq 1,$$

where

$$\|f\|^2_{H^{-1/4}_{\alpha(T)}} = \int_{-\infty}^{\infty} \frac{|\widehat{f}(\xi)|^2}{(\alpha + \xi^2)^{1/4}}\, d\xi.$$

This a priori estimate allows one to establish the existence of an appropriate class of global weak solution of (5.70) (see [CrCT2] and [KTa3]).

Proof of Theorem 5.12 As in the previous proof, consider the one-parameter family of standing wave solutions (with $n = 1$ in this case)

$$v_\omega(x,t) = e^{it\omega^2}\varphi_\omega(x),$$

where $\varphi_\omega(x) = \omega\varphi(\omega x)$ and $\varphi(x) = \varphi_1(x)$ solves the nonlinear equation in (5.8) with $\omega = 1$. Using the Galilean invariance (5.11), we obtain the two-parameter family of solutions:

$$u_{N,\omega}(x,t) = e^{-itN^2+iNx}\, v_\omega(x-2tN,t) = e^{-it(N^2-\omega^2)}e^{iNx}\,\varphi_\omega(x-2tN).$$

We fix s such that $s \in (-1/2, 0)$ and take $\omega = N^{-2s}$ and $N_1, N_2 \simeq N$.

First, we calculate

$$\|u_{N_1,\omega}(0) - u_{N_2,\omega}(0)\|^2_{s,2}\ .$$

Observing that $\widehat{\varphi_\omega}(\xi) = \widehat{\varphi}(\xi/\omega)$ so that $\widehat{\varphi_\omega}(\cdot)$ concentrates in $B_\omega(0)$= $\{\xi \in \mathbb{R} : |\xi| < \omega\}$. From the choice of ω and $s > -1/2$, if $\xi \in B_\omega(\pm N)$, then $|\xi| \simeq N$. Then, a straight calculation yields

$$\begin{aligned}&\|u_{N_1,\omega}(0) - u_{N_2,\omega}(0)\|^2_{s,2}\\ &\qquad \leq cN^{2s}\frac{|N_1-N_2|}{\omega^2}\left(\int_{\eta+N_2}^{\eta+N_1} d\xi\right)\int_{-\infty}^{\infty}|\widehat{\varphi}'_\omega(\eta)|^2 d\eta\end{aligned}$$

$$\leq cN^{2s}(N_1 - N_2)^2 \frac{1}{\omega^2}\ \omega = c(N^{2s}(N_1 - N_2))^2,$$

and that

$$\|u_{N_j,\omega}(0)\|_{s,2}^2 \simeq cN^{2s}\omega = c, \quad j = 1, 2.$$

Now, we consider the solutions $u_{N_1,\omega}(t), u_{N_2,\omega}(t)$ at time $t = T$, and compute

$$\|u_{N_1,\omega}(T) - u_{N_2,\omega}(T)\|_{s,2}.$$

Note first that

$$\|u_{N_j,\omega}(T)\|_{s,2}^2 \simeq c, \quad j = 1, 2.$$

In fact,

$$\|u_{N_j,\omega}(T)\|_{s,2}^2 = \|u_{N_j,\omega}(0)\|_{s,2}^2 \simeq c, \quad j = 1, 2.$$

Note that the frequencies of both $u_{N_j,\omega}(T)$, $j = 1, 2$, are localized around $|\xi| \simeq N$, and hence,

$$\|u_{N_1,\omega}(T) - u_{N_2,\omega}(T)\|_{s,2}^2 \simeq N^{2s}\|u_{N_1,\omega}(T) - u_{N_2,\omega}(T)\|_2^2. \tag{5.72}$$

Next, we observe that

$$u_{N_j,\omega}(x, T) = e^{-i(TN_j^2 - N_j x - T\omega^2)}\omega\varphi(\omega(x - 2TN_j)), \quad j = 1, 2.$$

Thus, the support of $u_{N_j,\omega}(T)$ is concentrated in $B_{\omega^{-1}}(2TN_j)$, $j = 1, 2$. Therefore, if for T fixed, N_1, N_2 are chosen such that

$$T(N_1 - N_2) \gg \omega^{-1} = N^{2s},$$

then there is not interaction and

$$\|u_{N_1,\omega}(T) - u_{N_2,\omega}(T)\|_2^2 \simeq \|u_{N_1,\omega}(T)\|_2^2 + \|u_{N_2,\omega}(T)\|_2^2 \simeq \omega.$$

The above estimate combined with (5.72) yields

$$\|u_{N_1,\omega}(T) - u_{N_2,\omega}(T)\|_{s,2}^2 \geq cN^{2s}\omega = c. \tag{5.73}$$

Take now

$$N_1 = N \quad \text{and} \quad N_2 = N - \frac{\delta}{N^{2s}},$$

so that

$$\begin{cases} c(N^{2s}(N_1 - N_2))^2 = c\delta^2, \\ T(N_1 - N_2) = T\frac{\delta}{N^{2s}} \gg N^{2s}, \quad \text{i.e.,} \quad T \gg \frac{N^{4s}}{\delta}. \end{cases} \tag{5.74}$$

Since $s < 0$, given δ, $T > 0$, we can choose N so large that (5.74) is valid, and from this we see that (5.73) violates the uniform continuity. □

Well-posedness for some particular cases of the IVP (5.71) has been studied in other spaces. In [Pl], the problem was considered in Besov spaces, in [CVV] in L^{p*} (L^p-weak spaces) and in [Gr3] in the spaces $\widehat{H^s_r}(\mathbb{R}^n)$ defined as:

$$f \in \widehat{H^s_r}(\mathbb{R}^n) \quad \text{if} \quad \|f\|_{\widehat{H^s_r}} = \|(1+|\xi|^2)^{s/2}\widehat{f}\|_{L^{r'}_\xi} < \infty. \tag{5.75}$$

The idea was to find larger spaces than $H^s(\mathbb{R}^n)$ or other ones which scale closer to the critical homogeneity given by the equation.

In [KPV12], the study of the IVP:

$$\begin{cases} i\partial_t u \pm \Delta u + N_k(u,\overline{u}) = 0, \\ u(x,0) = u_0(x), \end{cases} \tag{5.76}$$

$x \in \mathbb{R}^n$, $t \in \mathbb{R}$, where

$$N_k(z_1,z_2) = \sum_{a+b=k} C_k\, z_1^a\, z_2^b, \tag{5.77}$$

was first considered. Even though it may not have a physical interpretation in general, the main purpose of this study was motivated to test new local estimates based on $X_{s,b}$ spaces (see Definition 7.1) and variant of them and their relation with the geometry of the nonlinearity N_k.

We summarize next some local results obtained for the IVP (5.76). First, we will consider the 1-D situation. For the nonlinearity $N_2(u,\overline{u}) = u^2$, Bejenaru and Tao [BTo] obtained a sharp local well-posedness result in $H^s(\mathbb{R})$ for $s \geq -1$. In [Ki3], Kishimoto established a similar result for the quadratic nonlinearity $N_2(u,\overline{u}) = (\overline{u})^2$. In [KiT], Kishimoto and Tsugawa showed the local well-posedness for the case $N_2(u,\overline{u}) = u\overline{u} = |u|^2$ in $H^s(\mathbb{R})$ for $s > -1/2$. In each case, these results improve by one fourth the previous ones obtained in [KPV12]. Grünrock [Gr1] has shown that the IVP (5.76) is locally well-posed in $H^s(\mathbb{R})$ with $s > -5/12$ for $N_3(u,\overline{u}) = (\overline{u})^3$ and $N_3(u,\overline{u}) = u^3$ and with $s > -2/5$ for $N_3(u,\overline{u}) = u(\overline{u})^2$. Notice that all these nonlinearities have the same homogeneity, but only $N_3(u,\overline{u}) = |u|^2u$ is Galilean invariant. For higher powers in (5.77), the results known are due to Grünrock [Gr1]. He proved local well-posedness for the IVP (5.76) when the nonlinearity $N_4(u,\overline{u})$ has either of the following forms: $(\overline{u})^4$, u^4, $u^3\overline{u}$, and $\overline{u}^3u$ in $H^s(\mathbb{R})$, $s > -1/6$, and for $N_4(u,\overline{u}) = |u|^4$ in $H^s(\mathbb{R})$, $s > -1/8$.

In dimension $n = 2$, Bejenaru and De Silva [BeDS] showed local well-posedness for the IVP (5.76) in $H^s(\mathbb{R}^2)$, $s > -1$ when $N_2(u,\overline{u}) = u^2$ and a similar result was obtained by Kishimoto [Ki2] for $N_2(u,\overline{u}) = (\overline{u})^2$. These results improved by one fourth the previous ones found in [CDKS]. In the later work, local well-posedness for the nonlinearity $N_2(u,\overline{u}) = u\overline{u}$ was established in $H^s(\mathbb{R}^2)$, $s > -1/4$. In the three-dimensional case, Tao [To3] proved that the IVP (5.76) is locally well-posed in $H^s(\mathbb{R}^3)$, $s > -1/2$ for either $N_2(u,\overline{u}) = u^2$ or $N_2(u,\overline{u}) = \overline{u}^2$, and in $H^s(\mathbb{R}^3)$, $s > -1/4$ for the nonlinearity $N_2(u,\overline{u}) = u\overline{u}$.

Next, we deal with the existence and uniqueness question for the IVP associated to the cubic Schrödinger equation with the delta function as initial datum:

$$\begin{cases} i\partial_t u + \partial_x^2 u \pm |u|^2 u = 0, \\ u(x,0) = \delta(x). \end{cases} \tag{5.78}$$

$t > 0, x \in \mathbb{R}$.

Theorem 5.14 ([KPV5]). *Either there is no weak solution u for the IVP (5.78) in the class*

$$u,\ |u|^2 u \in L^\infty([0,\infty) : \mathcal{S}'(\mathbb{R})) \quad \textit{with} \quad \lim_{t\downarrow 0} u(\cdot,t) = \delta \tag{5.79}$$

or there is more than one.

Consider now the local and global well-posedness of the periodic problem:

$$\begin{cases} i\partial_t u = -\Delta u \pm |u|^{\alpha-1} u, \\ u(x,0) = u_0(x). \end{cases} \tag{5.80}$$

$x \in \mathbb{T}^n, t \in \mathbb{R},\ \alpha > 1$.

For $n = 1$, Bourgain [Bo1] established local well-posedness for (5.80) in $H^s(\mathbb{T})$, $s \in [0, 1/2)$ for $\alpha \in (1, 1 + 4/(1-2s))$. This combined with the conservation law $\|u(t)\|_{L^2} = \|u_0\|_{L^2}$ yields the corresponding global well-posedness result.

In the defocussing cubic NLS case ((+) in (5.80)) it was shown in [BGT2], [CrCT1] that the problem (5.80) is ill-posed (the map data-solution is not uniformly continuous) in $H^s(\mathbb{T})$, $s < 0$.

For $n = 3$, local well-posedness with $\alpha = 3$ was proved in [Bo1] for $u_0 \in H^s(\mathbb{T}^3)$, $s > 1/2$. For $n \geq 2$, local well-posedness was established in [Bo1] for $\alpha \in [3, 4/(n-2s))$ and $s > 3n/n + 4$.

The problem (5.80) in an n-dimensional nonflat compact manifold M^n has been studied by Burq, Gerard and Tzvetkov [BGT2], [BGT3]. Among other results, for the case of the two-dimensional sphere $\mathbb{S}^2$ they have shown that the IVP (5.80) in the cubic defocusing case (i.e., $\alpha = 3$ and positive sign in front of the nonlinearity) is locally well-posed in $H^s(\mathbb{S}^2)$ for $s > 1/4$ and ill-posed for $s < 1/4$.

The IVP problem (5.76) can also be considered in the periodic setting. We list next some results regarding the local well-posedness for this IVP in this situation. In the 1-D case, Bourgain [Bo1] established local well-posedness in $L^2(\mathbb{T})$ for any nonlinearity in (5.76) such that $a + b = k \leq 4$. Kenig, Ponce and Vega [KPV12] established the local well-posedness theory in $H^s(\mathbb{T})$, $s > -1/2$, for $N_2(u,\overline{u}) = u^2$, and for $N_2(u,\overline{u}) = \overline{u}^2$. Also, in the 1-D case Grünrock [Gr1] proved local well-posedness for $N_3(u,\overline{u}) = \overline{u}^3$ and $N_4(u,\overline{u}) = \overline{u}^4$ in $H^s(\mathbb{T})$, with $s > -1/3$ and $s > -1/6$, respectively. In the two-dimensional case, Grünrock [Gr1] showed local well-posedness in $H^s(\mathbb{T}^2)$, $s \geq -1/2$, for $N_2(u,\overline{u}) = \overline{u}^2$, and in dimension three that the IVP (5.76) is locally well-posed in $H^s(\mathbb{T}^3)$, $s \geq -3/10$, for $N_2(u,\overline{u}) = \overline{u}^2$. For further well-posedness results in the spaces

$$\mathcal{H}_{s,p}(\mathbb{T}) = \left(\sum_{n\in\mathbb{Z}}(1+n^2)^{ps/2}\,|\widehat{f}(n)|^p\right)^{1/p},$$

we refer to [Cr] and [Th].

5.5 Exercises

5.1 (i) Prove that if $u = u(x,t)$ satisfies

$$i\partial_t u = -\Delta u + |u|^{4/n}u \tag{5.81}$$

(u is the solution of the equation in (5.1) with $\lambda = 1$ and the critical power $\alpha = 4/n + 1$ in $L^2(\mathbb{R}^n)$) then:

$$\begin{aligned}
&u_1(x,t) = e^{i\theta}u(x,t),\\
&u_2(x,t) = u(x-x_0, t-t_0), \text{with} x_0 \in \mathbb{R}^n,\ t_0 \in \mathbb{R} \text{ fixed},\\
&u_3(x,t) = u(Ax,t),\ \text{with A any orthogonal matrix} n\times n,\\
&u_4(x,t) = u(x-2x_0\,t,t)\,e^{i(x\cdot x_0-|x_0|^2t)},\ \text{with} x_0 \in \mathbb{R}^n \text{ fixed},\\
&u_5(x,t) = \mu^{n/2}u(\mu x,\mu^2 t),\ \mu \in \mathbb{R}\ \text{ fixed},\\
&u_6(x,t) = \frac{1}{(\alpha+\omega t)^{n/2}}\exp\left[\frac{i\omega|x|^2}{4(\alpha+\omega t)}\right]u\left(\frac{x}{\alpha+\omega t},\frac{\gamma+\theta t}{\alpha+\omega t}\right),\\
&\qquad \alpha\,\theta-\omega\,\gamma = 1,\\
&u_7(x,t) = \overline{u(x,-t)},
\end{aligned}$$

also satisfy equation (5.81).

(ii) Prove that u_1, u_2, u_3, u_4, u_5 (with different powers in μ) and u_7 still satisfy the equation (5.81) for general nonlinearity $\pm|u|^{\alpha-1}u$ in (5.81).

5.2 Let $u \in H^1(\mathbb{R}^n)$ solve $-\Delta u + au = b|u|^\alpha u$, where $a > 0$ and $b \in \mathbb{R}$. Show that u satisfies

(i)

$$\int_{\mathbb{R}^n} |\nabla u|^2\,dx + a\int_{\mathbb{R}^n}|u|^2\,dx = b\int_{\mathbb{R}^n}|u|^{\alpha+2}\,dx. \tag{5.82}$$

(ii) Pohozaev's identity:

$$(n-2)\int_{\mathbb{R}^n}|\nabla u|^2\,dx + na\int_{\mathbb{R}^n}|u|^2\,dx = \frac{2nb}{\alpha+2}\int_{\mathbb{R}^n}|u|^{\alpha+2}\,dx. \tag{5.83}$$

5.3 Use Pohozaev's identity to show that a necessary condition to have solution in $H^1(\mathbb{R}^n)$ of problem (5.8) is that the nonlinearity satisfies $1 < \alpha < (n+2)/(n-2)$ $(1 < \alpha < \infty,\ \ n = 1, 2)$.

5.4 (i) Show that a formal scaling argument yields the estimate:

$$T = T(\|u_0\|_2) = c\,\|u_0\|_2^{-\beta}, \quad \beta = \frac{4(\alpha-1)}{4-n(\alpha-1)}, \tag{5.84}$$

for the life span of the L^2-local solution as a function of the size of the data given in Theorem 5.2.

(ii) Review the proof of Theorem 5.2 to obtain the estimate (5.84).

5.5 Consider the IVP for the 1-D NLS equation

$$\begin{cases} \partial_t u = i\,\partial_x^2 u \pm i\lambda\,|u|^{\alpha-1}u, \\ u(x,0) = u_0(x), \end{cases} \tag{5.85}$$

$\lambda \in \mathbb{R}$, $\alpha > 1$.

(i) Prove that if $\alpha \in (1,5)$ and $u_0 \in L^2(\mathbb{R})$, then the solution $u(\cdot,t)$ of the IVP (5.85) provided by Theorem 5.2 satisfies that

$$u(\cdot,t) \in C(\mathbb{R}) \quad \text{a.e.} \quad t \in [-T,T].$$

(ii) Can the result in (i) be extended to the case $\alpha = 5$?
Hint: Combine the idea of the proof of Exercise 4.9(ii) with Theorem 5.2, Corollary 5.1 and Theorem 5.3.

5.6 Let $f_\mu(x) = \mu f(\mu x)$. Show that $|||f_\mu|||_{X_p}$ is independent of μ, where $|||\cdot|||_{X_p}$ was defined in (5.58).

5.7 Let

$$u_{0j}(x,y) = \chi_{\{[0,1/2^j]\times[0,2^j]\}}(x,y) \quad j \in \mathbb{Z}^+.$$

Prove that

$$|||u_{0j}|||_{X_p} \le 2^{-j/4}.$$

5.8 Show that

$$u(x,t) = e^{it}\left\{1 - \frac{4(1+2it)}{1+2x^2+4t^2}\right\}$$

solves the IVP associated to

$$i\partial_t u + \partial_x^2 u + |u|^2 u = 0,$$

with datum

$$u(x,0) = 1 - \frac{4}{1+2x^2}.$$

5.9 (i) Prove that for any $\omega > 0$ the function:

$$u_\omega(x,t) = e^{it\omega}\omega^{1/(\alpha-1)}\varphi(\sqrt{\omega}\,x) = e^{it\omega}\varphi_\omega(x), \quad x \in \mathbb{R}^n,\ t \in \mathbb{R}, \tag{5.86}$$

where $\varphi(\cdot)$ is the unique positive, spherical symmetric solution of (5.8), satisfies the equation in (5.1) with $\lambda = 1$ (focussing case).

(ii) Show that

$$\frac{d}{d\omega}\|u_\omega\|_2 = \frac{d}{d\omega}\|\varphi_\omega\|_2 \begin{cases} > 0, & \text{if } \dfrac{1}{\alpha-1} > \dfrac{n}{4}, \\ = 0, & \text{if } \dfrac{1}{\alpha-1} = \dfrac{n}{4}, \\ < 0, & \text{if } \dfrac{1}{\alpha-1} < \dfrac{n}{4}. \end{cases}$$

5.10 [Generalized pseudo-conformal transformation] Let $u(x,t)$ be a solution of the equation

$$i\partial_t u + \mathfrak{L}_j u \pm |u|^{\alpha-1}\, u = 0, \quad \alpha > 1 \tag{5.87}$$

with

$$\mathfrak{L}_j = \partial_{x_1}^2 + \cdots + \partial_{x_j}^2 - \partial_{x_{j+1}}^2 - \cdots - \partial_{x_n}^2, \quad j \in \{1,\ldots,n\}.$$

Prove that for $\nu, \theta, \omega, \gamma \in \mathbb{R}$ such that $\nu\theta - \omega\gamma = 1$,

$$v(x,t) = \frac{e^{i\omega Q_j(x)/4(\nu+\omega t)}}{(\nu+\omega t)^{n/2}}\, u\left(\frac{x}{\nu+\omega t}, \frac{\gamma+\theta t}{\nu+\omega t}\right)$$

with

$$Q_j(x) = x_1^2 + \cdots + x_j^2 - x_{j+1}^2 - \cdots - x_n^2$$

verifies the equation:

$$i\partial_t v + \mathfrak{L}_j v \pm (\nu+\omega t)^{(\alpha-1)n/2-2}|v|^{\alpha-1}\, v = 0. \tag{5.88}$$

In particular, if $\alpha - 1 = 4/n$ (critical L^2-case) (5.87) and (5.88) are equal, see [GV1].

5.11 Let $u \in C([0,T] : L^2(\mathbb{R})) \cap L^4([0,T] : L^\infty(\mathbb{R}))$ be the local solution of the IVP

$$\begin{cases} \partial_t u = i(\partial_x^2 u \pm |u|^2 u), \\ u(x,0) = u_0(x), \end{cases} \tag{5.89}$$

$x, t \in \mathbb{R}$, provided by Theorem 5.2.

(i) Prove that if $x\, u_0,\, x\, u(\cdot, T) \in L^2(\mathbb{R})$, then

$$u \in C([0,T] : H^1(\mathbb{R}) \cap L^2(|x|^2\, dx)) = C([0,T] : \mathcal{F}_2^1).$$

(ii) Extend the result in (i) for any $m \in \mathbb{Z}^+$, i.e. If $|x|^m\, u_0,\; |x|^m\, u(\cdot, T) \in L^2(\mathbb{R})$, then

$$u \in C([0,T] : H^m(\mathbb{R}) \cap L^2(|x|^{2\,m}\, dx)) = C([0,T] : \mathcal{F}_{2\,m}^m).$$

(iii) Prove that if $u_0 \in H^s(\mathbb{R}) \cap L^2(|x|^{2b}\, dx) = \mathcal{F}_{2b}^s$ with $s \geq b \in \mathbb{Z}^+$, then $u \in C([0,T] : \mathcal{F}_{2b}^s)$.

Chapter 6
Asymptotic Behavior of Solutions for the NLS Equation

In this chapter, we shall study the longtime behavior of the local solutions of the initial value problem (IVP)

$$\begin{cases} i\partial_t u + \Delta u + \lambda\, |u|^{\alpha-1}u = 0, \\ u(x,0) = u_0(x), \end{cases} \tag{6.1}$$

$t \in \mathbb{R},\ \ x \in \mathbb{R}^n$, obtained in the previous chapter.

In the first section, we shall present results that under appropriate conditions involving the dimension n, the nonlinearity α, the sign of λ (focusing $\lambda > 0$, defocusing $\lambda < 0$), and the size of the data u_0 guarantee that these local solutions extend globally in time, i.e., to any time interval $[-T, T]$ for any $T > 0$.

In the second section, we shall see that when these conditions are not satisfied, the local solution should blowup in finite time.

6.1 Global Results

We shall start with the L^2 case. Theorem 5.2 (subcritical case) tells us that the initial value problem (IVP) (6.1) is locally well-posed in $L^2(\mathbb{R}^n)$ for $\alpha \in (1, 1 + 4/n)$ in a time interval $[0, T]$ with $T = T(\|u_0\|_2) > 0$. Multiplying the equation in (6.1) by $\bar{u}$, integrating the result in the space variables, and taking the imaginary part we get that the mass is conserved:

$$M(u) = \|u(t)\|_2^2 = \|u_0\|_2^2, \tag{6.2}$$

(to justify this procedure one needs to use continuous dependence, approximate the data u_0 by a sequence in $H^2(\mathbb{R}^n)$, and take the limit). The conservation law (6.2) allows us to reapply Theorem 5.2 as many times as we wish, preserving the length of the time interval to get a global solution.

Theorem 6.1 (Global L^2-solution, subcritical case). *If the nonlinearity power $\alpha \in (1, 1 + 4/n)$, then for any $u_0 \in L^2(\mathbb{R}^n)$ the local solution $u = u(x,t)$ of the*

F. Linares, G. Ponce, *Introduction to Nonlinear Dispersive Equations*,
Universitext, DOI 10.1007/978-1-4939-2181-2_6

initial value problem (IVP) (6.1) extends globally with

$$u \in C([0,\infty) : L^2(\mathbb{R}^n)) \cap L^q_{\text{loc}}([0,\infty) : L^p(\mathbb{R}^n)),$$

where (p,q) satisfies the condition (4.18) in Theorem 4.2.

The situation for the L^2-critical case $\alpha = 1 + 4/n$,

$$i\partial_t u + \Delta u + \lambda\, |u|^{4/n} u = 0 \tag{6.3}$$

with $u(x,0) = u_0(x) \in L^2(\mathbb{R}^n)$, whose solutions are given by Theorem 5.3, is quite different. In this case, the local result shows the existence of solution in a time interval depending on the data u_0 itself and not on its norm. So, the conservation law (6.2) does not guarantee the existence of a global solution. The problem of the longtime behavior of the L^2-solution of the equation (6.3) has received considerable attention.

The progress on this problem can be roughly described as follows:

(i) $||u_0||_2$ is small enough (Corollary 5.2); and

(ii) For the "defocusing" case, i.e., $\lambda < 0$ in (6.1), with $u_0 \in H^s(\mathbb{R}^n)$, $s > 4/7$, [CKSTT2], and for $s \geq 1/2$ for $n = 2$ [FGr] or under the decay assumption $|x|^l\, u_0 \in L^2(\mathbb{R}^n)$, $l > 3/5$ [Bo4].
In this case, it was also proved [Bo2] that if the local L^2-solution provided by Theorem 6.1 cannot be extended beyond the time interval $[0, T_*)$, then at least in the two-dimensional case ($n = 2$), the following L^2-concentration phenomenon of the L^2 mass occurs: There exists $c > 0$ such that

$$\limsup_{t\uparrow T_*} \sup_{Q\subset\mathbb{R}^2 : |Q|=(T_*-t)^{1/2}} \int_Q |u(x,t)|^2\, dx \geq c, \tag{6.4}$$

where Q denotes a square in $\mathbb{R}^2$ and $|Q|$ the size of its side. The result in (6.4) holds in both the defocusing case $\lambda < 0$ and the focusing case $\lambda > 0$ in which, as we see, blowup takes place but in the H^1-norm.

(iii) If the initial data $u_0(\cdot)$ are assumed to be radial, then
- In the defocusing case ($\lambda = -1$), the global existence and scattering results were established in [TVZ] for dimension $n \geq 3$ and in [KTV] for dimension $n = 2$.
- In the focusing case ($\lambda = 1$) for initial radially symmetric data u_0 satisfying

$$\|u_0\|_2 < \|\varphi\|_2,$$

where φ is the positive solution of the elliptic equation (5.8) with $\alpha = 1+4/n$, it was proved in [KTV] and [KVZ] that the corresponding local solution extends globally and scattering results hold (this is sharp).

(iv) Finally, in [D1]–[D3] Dobson removed the radial assumptions on the result described in (iii). More precisely, in [D1]–[D3] global well-posedness and scattering results were established in the defocussing case for any data $u_0 \in L^2(\mathbb{R}^n)$ and the focussing case for any data $u_0 \in L^2(\mathbb{R}^n)$ with $\|u_0\|_2 < \|\varphi\|_2$, with φ being the positive solution of the elliptic equation (5.8) with $\alpha = 1 + 4/n$.

Definition 6.1. A global solution u of the IVP (6.1) is said to scatter in the space X to a free solution as $t \to \pm\infty$, if there exists $u_\pm \in X$ such that

$$\lim_{t\to\pm\infty} \|e^{it\Delta}u_\pm - u(\cdot,t)\|_X = 0. \tag{6.5}$$

Notice that for the IVP associated to the L^2-critical equation (6.3) focusing case ($\lambda = 1$), scattering cannot occur for all L^2-data in the ball with center as the origin and radius R, $R > \|\varphi\|_2$ (with φ as in (5.8)).

Let us consider now the extension problem of the H^1-local solution proved in Chapter 5. We first examine the subcritical case (Theorem 5.4), i.e., $\alpha \in (1, 1 + 4/(n-2))$, $n \geq 3$, or $1 < \alpha < \infty$, if $n = 1, 2$, where the time of existence T depends on the size of the data, i.e., $T = T(\|u_0\|_{H^1})$. In this case, if u is a solution in the interval $[0, T]$, then multiplying the equation by $-\partial_t \bar{u}$, integrating the result in the space variables, taking its real part and using integration by parts, one gets that for $t \in [0, T]$

$$\frac{d}{dt}E(u(t)) = \frac{d}{dt}\int_{\mathbb{R}^n} \left(|\nabla_x u(x,t)|^2 - \frac{2\lambda}{\alpha+1}|u(x,t)|^{\alpha+1}\right) dx = 0.$$

So, $E(u(t))$ is constant and $E(u(t)) = E(u_0)$ or

$$E(u_0) = \int_{\mathbb{R}^n} \left(|\nabla_x u(x,t)|^2 - \frac{2\lambda}{\alpha+1}|u(x,t)|^{\alpha+1}\right) dx. \tag{6.6}$$

Therefore, if $\lambda < 0$ (defocusing case) it follows that

$$\sup_{[0,T]} \int_{\mathbb{R}^n} |\nabla u(x,t)|^2 dx \leq E(u_0),$$

which combined with (6.2) gives

$$\sup_{[0,T]} \|u(t)\|_{1,2}^2 \leq E(u_0) + \|u_0\|_2^2.$$

This allows us to reapply Theorem 5.4 to extend the local solution u to any time interval.

In the focusing case $\lambda > 0$, using the Gagliardo–Nirenberg inequality, see (3.14), we have that for $t \in [0, T]$

$$\|u(t)\|_{\alpha+1} \leq c\|\nabla_x u(t)\|_2^\theta \ \|u(t)\|_2^{1-\theta} \leq c\ \|\nabla_x u(t)\|_2^\theta \ \|u_0\|_2^{1-\theta}, \tag{6.7}$$

with

$$\frac{1}{\alpha+1} = \theta\Big(\frac{1}{2} - \frac{1}{n}\Big) + \frac{1-\theta}{2} \quad \text{or} \quad \theta = \frac{n(\alpha-1)}{2(\alpha+1)}.$$

Then,

$$\|u(t)\|_{\alpha+1}^{\alpha+1} \leq c\ \|u_0\|_2^{[(\alpha+1)-n(\alpha-1)/2]}\ \|\nabla_x u(t)\|_2^{n(\alpha-1)/2}.$$

This combined with (6.6) proves that if $E(u_0) < \infty$, then

$$\|\nabla_x u(t)\|_2^2 \leq |E(u_0)| + c_\alpha|\lambda|\ \|u_0\|_2^{[(\alpha+1)-n(\alpha-1)/2]}\ \|\nabla_x u(t)\|_2^{n(\alpha-1)/2}. \tag{6.8}$$

Assume first that $\alpha \in (1, 1+4/n)$, so $n(\alpha-1)/2 < 2$. Then, from (6.8) and the notation $y = y(t) = \|\nabla_x u(t)\|_2$, one gets

$$y^2 \le E(u_0) + c\,\|u_0\|_2^{[(\alpha+1)-n(\alpha-1)/2]}\, y^{2-\gamma}, \tag{6.9}$$

with $\gamma = 2-n(\alpha-1)/2 \in (0,2)$. Therefore, there exists $M = M(\|u_0\|_{1,2}; n; \alpha; \lambda) > 0$ independent of T such that

$$\sup_{[0,T]} \|\nabla_x u(t)\|_2 \le M.$$

Thus, the same argument used above allow us to reapply Theorem 5.4 to extend the local solution u to any time interval.

In the case $\alpha = 1+4/n$, the inequality (6.9) becomes

$$y^2 \le E(u_0) + c\,\|u_0\|_2^{4/n}\, y^2. \tag{6.10}$$

Hence, there exists $c_0 > 0$ such that if $\|u_0\|_2 < c_0$, then the local solution u provided by Theorem 5.4 extends to any time interval.

Finally, we consider the case $\alpha \in (1+4/n, (n+2)/(n-2))$. In this case, using the notation $\delta = \|u_0\|_2$, the inequality (6.9) becomes

$$y^2(t) \le E(u_0) + c\,\delta^{[(\alpha+1)-n(\alpha-1)/2]}\, y^{2+\nu}(t), \tag{6.11}$$

with $\nu = n(\alpha-1)/2-2 \ge 0$. For $\|u_0\|_{1,2} = \|u_0\|_2 + \|\nabla u_0\|_2 \le \rho$ sufficiently small, it follows from (6.11), evaluated at $t = 0$, that $E(u_0) > 0$. Also, from (6.11), one gets that there exists $M > 0$ such that $y(t) = \|\nabla_x u(t)\|_2 \le M$, which combined with (6.2) allows us to extend the local solution to any interval of time as in the previous case.

Summarizing, we have the following result:

Theorem 6.2. *Under any of the following set of hypotheses the local solution of the IVP (6.1) with $u_0 \in H^1(\mathbb{R}^n)$ provided by Theorem 5.4 extends globally in time, if*

(i) $\lambda < 0$,
(ii) $\lambda > 0$ *and* $1 < \alpha < 1+4/n$,
(iii) $\lambda > 0$, $\alpha = 1+4/n$, *and* $\|u_0\|_2 < c_0$,
(iv) $\lambda > 0$, $\alpha > 1+4/n$, *and* $\|u_0\|_{1,2} \simeq \|u_0\|_2 + \|\nabla u_0\|_2 \le \rho$, *for* ρ *sufficiently small.*

The size assumption on the data in (iii), i.e., $\alpha = 1+4/n$, can be made precise. In [W3], Weinstein showed that

$$J_n(f) = \inf_{f\in H^1} \frac{\|\nabla f\|_2^2\,\|f\|_2^{4/n}}{\|f\|_{2+4/n}^{2+4/n}} = \frac{\|\varphi\|_2^{4/n}}{1+2/n}, \tag{6.12}$$

and the infimum is attained at φ, where φ is the unique positive solution up to translation of the elliptic problem (5.8) (for details see Exercise 6.6). From (6.12), it follows that $E(\varphi) = 0$ and that if $u_0 \in H^1(\mathbb{R}^n)$ with $\|u_0\|_2 < \|\varphi\|_2$, then the corresponding solution of the IVP (6.1) with $\alpha = 1 + 4/n$ extends globally in time, i.e., $c_0 = \|\varphi\|_2$ in part (iii) of Theorem 6.2. We shall return to this point after Theorem 6.4.

Next, we consider the extension problem of the local solution of the IVP (6.1) with $u_0 \in H^1(\mathbb{R}^n)$ in the critical case $\alpha = (n+2)/(n-2)$. As we shall see in the next section, in the focusing case $\lambda > 0$, local solutions of this problem may blow up. So, we first consider the defocusing case $\lambda < 0$. Under these assumptions, one may ask if the local solution provided by Theorem 5.5 extends to all time and there is scattering. For this problem, the first result known is due to Bourgain [Bo7], who gave a positive answer in the case of dimensions $n = 3, 4$ for radial data, i.e., $u_0(x) = \phi(|x|)$ (see also [Gl1]). In [To5], Tao extended Bourgain's result to any dimension. For any data, not necessarily radial in dimension $n = 3$, Colliander, Keel, Staffilani, Takaoka and Tao [CKSTT7] established global well-posedness and scattering results. Ryckman and Visan [RVi] showed the corresponding result in dimension $n = 4$ and Visan [Vs] obtained it for dimension $n \geq 5$.

A similar problem for the semilinear wave equation:

$$\partial_t^2 w - \Delta w + |w|^{4/(n-2)} w = 0, x \in \mathbb{R}^n,\ t > 0, \tag{6.13}$$

with $(w(0), \partial_t w(0)) = (f, g) \in H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ was previously solved by Struwe [Stw] in the radial case and $n = 3$, and by Grillakis [Gl2], [Gl3] for general data in dimensions $n = 3, 4, 5$ (see [ShS] for a simplified proof and an extension to the cases $n = 6, 7$).

In both cases, one reviews the local existence theory to deduce what happens if the local solution is assumed not to extend beyond the time interval $[0, T^*)$. In this case, a "concentration of energy" in small sets must occur as $t \uparrow T^*$. In the radial case, this should only take place at the origin. Roughly speaking, to exclude this possibility in the case of the wave equation one combines the Morawetz estimate and the finite propagation speed of the solution. The case of the Schrödinger equation is more involved. The corresponding local Morawetz estimate (appropriate truncated version) [LS] (see Exercise 6.3) is significantly more difficult to establish and even in the radial case requires an inductive argument in the accumulation of energy to disprove the possible concentration.

In [KM1], assuming that the $\dot{H}^{1/2}(\mathbb{R}^3)$-norm of the solution of the defocusing

$$i\partial_t u + \Delta u - |u|^2 u = 0$$

remains bounded, Kenig and Merle showed that the above global results apply.

Next, consider the $\dot{H}^1$ critical focusing case ($\lambda = 1$):

$$i\partial_t u + \Delta u + |u|^{4/(n-2)}\, u = 0, \tag{6.14}$$

assuming that the data u_0 are spherically symmetric and $3 \leq n \leq 5$, Kenig and Merle [KM2] established a sharp condition for the global existence and blow-up results. Let Φ be the solution of the elliptic problem:

$$\Delta \Phi + |\Phi|^{4/(n-2)}\, \Phi = 0 \tag{6.15}$$

(so-called Aubin–Talenti solution), where

$$\Phi(x) = \left(1 + \frac{|x|^2}{n(n-2)}\right)^{-(n-2)/2},$$

i.e., the solution of the associated stationary problem to (6.15).

For $u_0 \in H^1(\mathbb{R}^n)$, radial:

(i) If $E(u_0) < E(\Phi)$ and $\|\nabla u_0\|_2 < \|\nabla \Phi\|_2$, then the local solution extends to a global one and scatters as $t \to \pm\infty$.
(ii) If $E(u_0) < E(\Phi)$ and $\|\nabla u_0\|_2 > \|\nabla \Phi\|_2$, then the local solution blows up in finite time in both directions.

In [KV1], Killip and Visan extended the result in (i) to dimension $n \geq 5$ without the radial assumption on the data. They also extended the result in (ii) to all dimension $n \geq 6$ under the radial assumption on the data.

The case $E(u_0) = E(\Phi)$ for radial solutions for the equation (6.14) with $n = 3, 4, 5$ was studied in [DM]. To describe these results, we need to introduce the following notation:

Given $u = u(x,t)$, a radial solution of the IVP associated to (6.14) define Ω_u^{rad} the set of its radial symmetries:

$$\Omega_u^{\text{rad}} = \{e^{i\theta}\, \lambda^{\frac{n-2}{2}}\, u(\lambda x,\, \lambda^2 t) \;:\; \theta \in \mathbb{R}, \lambda > 0\}.$$

Then in [DM] it was shown that if $u_0 \in H^1(\mathbb{R}^n)$, radial, with $E(u_0) = E(\Phi)$ the corresponding radial solutions $u = u(x,t)$ of (6.14) verify the next threshold:

(i) If $\|\nabla u_0\|_2 < \|\nabla \Phi\|_2$, then the local solution extends to a global one in $\mathbb{R}$.
(ii) If $\|\nabla u_0\|_2 = \|\nabla \Phi\|_2$, then $u \in \Omega_\Phi^{\text{rad}}$.
(iii) If $\|\nabla u_0\|_2 > \|\nabla \Phi\|_2$, then either $u \in \Omega_{w^+}^{\text{rad}}$ (for some fixed radial solution w^+) or $u(t)$ blows up in both directions.

Above, we have considered the equation (6.3) critical in $L^2(\mathbb{R}^n)$ and (6.14) critical in $\dot{H}^1(\mathbb{R}^n)$. The critical problem in $\dot{H}^s(\mathbb{R}^n)$ with $s = s_c \in (0,1)$ was studied by Holmer and Roudenko in [HR1]. For the case $n = 3$, $s_c = 1/2$, i.e.,

$$\begin{cases} i\partial_t u + \Delta u + |u|^2 u = 0, \\ u(x,0) = u_0(x), \end{cases} \tag{6.16}$$

(denoting by $\varphi(x)$ the solution of (5.8) with $\omega = 1$) they proved:

Let $u_0 \in H^1(\mathbb{R}^3)$ radial such that

$$M(u_0)\, E(u_0) < M(\varphi)\, E(\varphi).$$

(i) If $\|u_0\|_2\, \|\nabla u_0\|_2 < \|\varphi\|_2\, \|\nabla \varphi\|_2$, then for all t, $u(t)$ satisfies

$$\|u_0\|_2\, \|\nabla u(t)\|_2 < \|\varphi\|_2\, \|\nabla \varphi\|_2$$

and is globally defined and scatters.

(ii) If $\|u_0\|_2\,\|\nabla u_0\|_2 > \|\varphi\|_2\,\|\nabla\varphi\|_2$, then the local solution blows up in finite time.

The radial assumption in (i) was removed by Duyckaerts, Holmer and Roudenko in [DHR]. The method in [HR1] and [DHR] follows some of the ideas introduced in [KM1].

In [NSc], Nakanishi and Schlag obtained the following:

There exists $\epsilon > 0$ such that for any $u_0 \in X_\epsilon$ where

$$X_\epsilon = \{f \in H^1(\mathbb{R}^3) : f \text{ radial}, \; M(f)\,E(f) < M(\varphi)(E(\varphi)+\epsilon^2) \text{ and } \|f\|_2 = \|\varphi\|_2\},$$

the corresponding local solution $u(t)$ of the IVP associated to the equation in (6.14) as $t \to +\infty$ (and $t \to -\infty$) satisfies one of the next three possibilities:

(i) Scatters,
(ii) Finite time blowup,
(iii) After sometime it remains close in H^1 to $B = \{e^{i\theta}\,\varphi(\cdot) \;:\; \theta \in \mathbb{R}\}$.

Considering $t \to \pm\infty$, this gives nine possibilities.

Moreover, the subset of X_ϵ having the behavior in (i) and (ii) (four possibilities) is open and B describes "their boundaries."

The case $\|u_0\|_2\,\|\nabla u_0\|_2 = \|\varphi\|_2\,\|\nabla\varphi\|_2$ for equation (6.16) was studied in [DRu], where they obtained results in the direction of [DM] for the $\dot H^1$ critical case (see above).

The extension of the results in [HR1] and [DHR] to the energy subcritical range $1+\dfrac{4}{n} < \alpha < 1+\dfrac{4}{n-2}$, for $n \geq 3$, and $1+\dfrac{4}{n} < \alpha < \infty$, for $n = 1, 2$ was considered by Fang, Xie and Cazenave [FXC], and Guo [Gq].

The problem of the longtime behavior of the local solution for the supercritical H^1- case, i.e., $\alpha > 1+4/(n-2)$, $n \geq 3$, remains largely open. For some techniques and results in this direction, see [KM1] and [KV2].

6.2 Formation of Singularities

In this section, we prove that the global results in the previous section are optimal. We shall see that if (i)–(iii) in Theorem 6.2 do not hold, then there exists $u_0 \in H^1(\mathbb{R}^n)$ and $T^* < \infty$ such that the corresponding solution u of the IVP (6.1) satisfies

$$\lim_{t\uparrow T^*} \|\nabla u(t)\|_2 = \infty. \tag{6.17}$$

To simplify, the exposition we shall assume $\lambda = 1$. In the proof of (6.17), we need the following identities.

Proposition 6.1. *If $u(t)$ is a solution in $C([-T,T] : H^1(\mathbb{R}^n))$ of the IVP (6.1) with $\lambda = 1$ obtained in Theorems 5.4 and 5.5, then*

$$\frac{d}{dt}\int_{\mathbb{R}^n} |x|^2 |u(x,t)|^2\,dx = 4\,\mathcal{I}m \int_{\mathbb{R}^n} r\,\bar u\,\partial_r u\;dx, \tag{6.18}$$

with $r=|x|$ *and* $\mathcal{I}m\,(\cdot) =$ *imaginary part of* $(\cdot)$, *and*

$$\frac{d}{dt}\mathcal{I}m\int\limits_{\mathbb{R}^n} r\overline{u}\,\partial_r u\,dx = 2\int\limits_{\mathbb{R}^n}|\nabla u(x,t)|^2\,dx + \Big(\frac{2n}{\alpha+1}-n\Big)\int\limits_{\mathbb{R}^n}|u(x,t)|^{\alpha+1}\,dx. \quad (6.19)$$

Proof. To obtain (6.18) we multiply the equation in (6.1) by $2\overline{u}$ and take the imaginary part to get

$$\mathcal{I}m\,(2i\,\partial_t u\,\overline{u}) = \partial_t|u|^2 = -\mathcal{I}m\,(2\Delta u\,\overline{u}) = -2\,\mathrm{div}\,(\mathcal{I}m\,(\nabla u\,\overline{u})).$$

Multiplying this identity by $|x|^2$, integrating in $\mathbb{R}^n$, using integration by parts and that $r\partial_r u = x_j\partial_{x_j}u$ (with summation convention) it follows that

$$\begin{aligned}\frac{d}{dt}\int|x|^2|u|^2\,dx &= \int|x|^2\partial_t|u|^2\,dx = -2\int\mathrm{div}\,(\mathcal{I}m\,(\,\overline{u}\,\nabla u))|x|^2dx\\ &= 2\int\mathcal{I}m\,(\,\overline{u}\,\partial_{x_j}u)\,2x_j\,dx = 4\int\mathcal{I}m\,(r\,\overline{u}\,\partial_r u)\,dx,\end{aligned}$$

which proves (6.18).

For (6.19), we multiply the equation in (6.1) by $2r\partial_r\,\overline{u}$, integrate in $\mathbb{R}^n$, and take the real part of this expression to get

$$\begin{aligned}\mathcal{R}e\,\Big(2i\int r\partial_r\,\overline{u}\,\partial_t u\,dx\Big) &= i\int r(\partial_r\,\overline{u}\,\partial_t u - \partial_r u\partial_t\,\overline{u})\,dx\\ &= -2\mathcal{R}e\int r\partial_r\,\overline{u}\,\Delta u dx - 2\mathcal{R}e\int r\partial_r\,\overline{u}\,|u|^{\alpha-1}u\,dx. \quad (6.20)\end{aligned}$$

By integration by parts and the equation in (6.1) it follows that

$$\begin{aligned} i\int r\,(\partial_r\overline{u}\,\partial_t u - \partial_r u\partial_t\overline{u})\,dx &= i\int x_j\,(\partial_{x_j}\overline{u}\,\partial_t u - \partial_{x_j}u\partial_t\overline{u})\,dx\\ &= i\int x_j\,(\partial_t(\partial_{x_j}\overline{u}\,u) - \partial_{x_j}(u\partial_t\overline{u}))\,dx\\ &= i\frac{d}{dt}\int r\,u\,\partial_r\overline{u}\,dx + n\,i\int u\partial_t\overline{u}\,dx \quad (6.21)\\ &= \frac{d}{dt}\Big(i\int r\,u\,\partial_r\overline{u}\,dx\Big) + n\Big(\int u(\Delta\overline{u} + |u|^{\alpha-1}\overline{u})\,dx\Big)\\ &= \frac{d}{dt}\Big(i\int ru\partial_r\overline{u}\,dx\Big) - n\int|\nabla u|^2\,dx + n\int|u|^{\alpha+1}\,dx.\end{aligned}$$

Similarly, we see that

$$\begin{aligned}
2\,\mathcal{R}e\left(\int r\partial_r\bar{u}\Delta u\,dx\right) &= 2\,\mathcal{R}e\left(\int x_j\,\partial_{x_j}\bar{u}\,\partial^2_{x_k}u\,dx\right)\\
&= 2\,\mathcal{R}e\left(-\int |\nabla u|^2\,dx - \int x_j\partial_{x_k}u\,\partial_{x_k}\partial_{x_j}\bar{u}\,dx\right)\\
&= -2\int |\nabla u|^2\,dx - \int x_j\partial_{x_k}u\,\partial_{x_k}\partial_{x_j}\bar{u}\,dx\\
&\quad -\int x_j\,\partial_{x_k}\bar{u}\,\partial_{x_k}\partial_{x_j}u\,dx \qquad (6.22)\\
&= -2\int |\nabla u|^2\,dx + n\int |\nabla u|^2\,dx\\
&\quad +\int x_j\partial_{x_k}\partial_{x_j}u\,\partial_{x_k}\,\bar{u}\,dx - \int x_j\partial_{x_k}\partial_{x_j}u\,\partial_{x_k}\bar{u}\,dx\\
&= (n-2)\int |\nabla u|^2dx.
\end{aligned}$$

Also,

$$\begin{aligned}
2\,\mathcal{R}e\left(\int |u|^{\alpha-1}ru\,\partial_r\bar{u}\,dx\right) &= 2\,\mathcal{R}e\left(\int |u|^{\alpha-1}u\,x_j\partial_{x_j}\bar{u}\,dx\right)\\
&= \int x_j\,(|u|^2)^{(\alpha-1)/2}(\partial_{x_j}u\,\bar{u} + u\partial_{x_j}\bar{u})\,dx \qquad (6.23)\\
&= \frac{2}{\alpha+1}\int x_j\partial_{x_j}[(|u|^2)^{(\alpha+1)/2}]\,dx\\
&= -\frac{2n}{\alpha+1}\int |u|^{\alpha+1}\,dx.
\end{aligned}$$

Collecting the information in (6.21)–(6.23) we can rewrite (6.20) as:

$$\frac{d}{dt}\mathcal{I}m\left(\int r\bar{u}\partial_r u\,dx\right) = 2\int |\nabla u|^2\,dx + \left(\frac{2n}{\alpha+1} - n\right)\int |u|^{\alpha+1}\,dx,$$

which yields (6.19). □

In the last proof, we used implicitly the following result commented on at the end of Chapter 4.

Proposition 6.2 ([HNT2]). *If u is a solution of the IVP (6.1) in $C([-T,T]: H^1(\mathbb{R}^n))$ provided by Theorems 5.4 and 5.5 such that $x_ju_0 \in L^2(\mathbb{R}^n)$ for some $j = 1,\dots,n$, then*

$$x_j\,u(\cdot,t) \in C([-T,T]:\,L^2(\mathbb{R}^n)).$$

Thus, if $u_0 \in L^2(\mathbb{R}^n, |x|^2\,dx)$, then

$$u(\cdot,t) \in C([-T,T]:\,H^1 \cap L^2(|x|^2\,dx)).$$

Now, we shall prove one of the main results in this section.

6.2.1 Case $\alpha \in (1+4/n, 1+4/(n-2))$

Theorem 6.3. *Let u be a solution in $C([0,T]: H^1(\mathbb{R}^n) \cap L^2(|x|^2\,dx))$ of the IVP (6.1) with $\lambda = 1$ provided by Theorems 5.4 and 5.5 and Proposition 6.2. Assume that the initial data u_0 and the nonlinearity α satisfy the following assumptions:*

$$\text{(i)} \quad \int \Big(|\nabla u_0|^2 - \frac{2}{\alpha+1}|u_0|^{\alpha+1}\Big)\,dx = E(u_0) = E_0 < 0,$$

$$\text{(ii)} \quad \alpha \in (1+4/n, 1+4/(n-2));$$

then there exists $T^* > 0$ such that

$$\lim_{t \uparrow T^*} \|\nabla u(t)\|_2 = \infty. \tag{6.24}$$

We observe that condition (i) implies that $\|u_0\|_{1,2}$ is not arbitrarily small. In particular, for any $u_0 \in H^1(\mathbb{R}^n)$ one has that $E_0(\nu u_0) < 0$ for $\nu > 0$ sufficiently large.

In the proof, we just need $\alpha > 1+4/n$, therefore, the theorem extends to solutions $u \in C([0,T] : H^2(\mathbb{R}^n) \cap L^2(|x|^2\,dx))$, $\alpha < \infty$ for $n \le 4$ and $\alpha \le n/(n-4)$ for $n \ge 5$.

Proof. We first assume that $\mathcal{I}m \left(\int r\bar{u}_0\, \partial_r\, u_0\,dx\right) < 0$. We define

$$f(t) = -\mathcal{I}m \int r(\partial_r u\,\bar{u})(x,t)\,dx.$$

By hypothesis, $f(0) > 0$. Using identities (6.19) and (6.6) it follows that

$$\begin{aligned}
f'(t) &= -2\int |\nabla u(x,t)|^2\,dx - \Big(\frac{2n}{\alpha+1} - n\Big)\int |u(x,t)|^{\alpha+1}\,dx \\
&= -2\int |\nabla u(x,t)|^2\,dx + n\Big(\frac{\alpha+1}{2} - 1\Big)\frac{2}{\alpha+1}\int |u(x,t)|^{\alpha+1}\,dx \\
&= -2\int |\nabla u(x,t)|^2\,dx + n\Big(\frac{\alpha+1}{2} - 1\Big)\Big(\int |\nabla u(x,t)|^2\,dx - E_0\Big) \qquad (6.25) \\
&= -\Big[2 - n\Big(\frac{\alpha+1}{2} - 1\Big)\Big]\int |\nabla u(x,t)|^2\,dx - n\Big(\frac{\alpha+1}{2} - 1\Big)E_0 \\
&\ge M\,\|\nabla u(t)\|_2^2,
\end{aligned}$$

since by hypothesis $E_0 < 0$, $\alpha > 1$ implies that $(\alpha+1)/2 - 1 > 0$, and $\alpha > 1+4/n$ implies that $n((\alpha+1)/2 - 1) - 2 = M > 0$.

From (6.25), $f(t)$ is an increasing function, so $f(t) \ge f(0) > 0$ for all $t > 0$.

Now we use (6.18) to see that

$$\frac{d}{dt}\int |x|^2|u(x,t)|^2\,dx = 4\,\mathcal{I}m\int r(\bar{u}\,\partial_r u)(x,t)\,dx = -4\,f(t) < 0.$$

Thus, $h(t) = \int |x|^2|u(x,t)|^2\,dx$ is a decreasing function with

$$h(t) \le \int |x|^2|u_0(x)|^2\,dx = h(0).$$

The Cauchy–Schwarz inequality tells us that

$$\begin{aligned}
|f(t)| = f(t) &= -\mathcal{I}m\int r(\bar{u}\,\partial_r u)(x,t)\,dx\\
&\le \Big(\int r^2|u|^2(x,t)\,dx\Big)^{1/2}\Big(\int |\partial_r u|^2(x,t)\,dx\Big)^{1/2}\\
&\le (h(0))^{1/2}\|\nabla u(t)\|_2,
\end{aligned}$$

which combined with (6.25) proves that $f(t)$ satisfies the differential inequality:

$$\begin{cases} f'(t) \ge \dfrac{M}{h(0)}\,(f(t))^2,\\ f(0) > 0.\end{cases}$$

Hence,

$$(h(0))^{1/2}\|\nabla u(t)\|_2 \ge f(t) \ge \frac{h(0)f(0)}{h(0) - Mf(0)t}. \tag{6.26}$$

Defining

$$T_0 = \frac{h(0)}{Mf(0)} > 0, \tag{6.27}$$

we obtain (6.24) with $T^* = T_0$.

Next, we consider the case $\mathcal{I}m\,\left(\int r\bar{u}_0\,\partial_r\,u_0\,dx\right) \ge 0$. From (6.25), it follows that

$$\frac{d}{dt}\mathcal{I}m\int r\bar{u}\,\partial_r\,u(x,t)\,dx = 2E_0 + \Big(\frac{2(n+2)}{\alpha+1} - n\Big)\int |u(x,t)|^{\alpha+1}\,dx \le 2\,E_0$$

because $\alpha > 1 + 4/n$. Hence, since $E_0 < 0$ there exists $\hat{t} > 0$ such that

$$\mathcal{I}m\int r\bar{u}\,\partial_r\,u(x,\hat{t})\,dx < 0$$

and we are in the case previously considered. □

The antecedently result gives us an upper bound on the life span of the local solution in H^1 since we have shown that the existence of the interval of time $[0, T^*)$ implies (6.24). This only tells us that the time of life span T^* of the solution is less than or equal to T_0 as above. It is easy to see that the L^p-norm with $p \ge \alpha + 1$ of the solution u, that is $\|u(t)\|_p$, also satisfies an estimate of the type described in (6.24).

6.2.2 *Case* $\alpha = 1 + 4/n$

In this case, $n(1 - 2/(\alpha + 1)) = 4/(\alpha + 1)$, (6.19) can be rewritten as:

$$\frac{d}{dt}\mathcal{I}m \int r\,\bar{u}\,\partial_r u\,dx = 2\Big(\int |\nabla u(x,t)|^2\,dx - \frac{2}{\alpha+1}\int |u(x,t)|^{\alpha+1}\,dx\Big) = 2\,E_0.$$

Integrating this equality we see that

$$\mathcal{I}m \int r\,\bar{u}\,\partial_r u\,dx = \mathcal{I}m \int r\,\bar{u}_0\,\partial_r u_0\,dx + 2t\,E_0,$$

which combined with (6.18) tells us that

$$\frac{d}{dt}\int |x|^2|u(x,t)|^2\,dx = 4\,\mathcal{I}m \int r\,\bar{u}_0\,\partial_r u_0\,dx + 8t\,E_0.$$

Integrating again, we obtain the identity:

$$\int |x|^2|u(x,t)|^2\,dx = \|\,|x|\,u_0\|_2^2 + 4t\,\mathcal{I}m \int r\,\bar{u}_0\,\partial_r u_0\,dx + 4t^2 E_0. \tag{6.28}$$

Assume first that either (i) $E_0 < 0$ or (ii) $E_0 \le 0$ (with E_0 as in Theorem 6.3 (i)) and $\mathcal{I}m \int r\,\bar{u}_0\,\partial_r u_0\,dx < 0$ or (iii) $E_0 > 0$ and $\mathcal{I}m \int r\,\bar{u}_0\,\partial_r u_0\,dx < -\sqrt{E_0}\,\|\,|x|\,u_0\|_2$.

Suppose that the desired result (6.24) does not hold, i.e., the H^1-solution can be extended globally.

Our assumptions and (6.28) allow us to deduce that there exists T^* such that

$$\lim_{t\uparrow T^*} \|\,|x|\,u(\cdot,t)\|_2 = 0. \tag{6.29}$$

Now, we recall Weyl–Heisenberg's inequality (see Exercise 3.14): For any $f \in H^1(\mathbb{R}^n) \cap L^2(|x|^2\,dx)$,

$$\|f\|_2^2 \le \frac{2}{n}\|\,|x|\,f\|_2\,\|\nabla f\|_2. \tag{6.30}$$

Notice that (6.22) still holds when one substitutes x by $x - a$ for any fixed $a \in \mathbb{R}^n$.

Combining (6.2) and (6.30), it follows that

$$0 < \|u_0\|_2^2 = \|u(t)\|_2^2 \le \frac{2}{n}\|\,|x|\,u(\cdot,t)\|_2\,\|\nabla u(\cdot,t)\|_2,$$

which together with (6.29) leads to a contradiction. Therefore, it follows that

$$\lim_{t\uparrow T^*} \|\nabla u(t)\|_2 = \infty.$$

Thus, we have proved the following theorem.

Theorem 6.4. *Let $u \in C([-T,T] : H^1(\mathbb{R}^n) \cap L^2(|x|^2\,dx))$ be the solution of the IVP (6.1) with $\alpha = 1 + 4/n$ obtained in Theorem 5.4 and Proposition 6.2 such that the initial data $u_0 \in H^1(\mathbb{R}^n) \cap L^2(|x|^2\,dx)$ satisfy*

$$(i)\;\; E_0 < 0,$$

$$(ii)\;\; E_0 \leq 0 \;\; and \;\; \mathcal{I}m \int r\,\bar{u}_0\,\partial_r u_0\,dx < 0,$$

or

$$(iii)\;\; E_0 > 0 \; and \; \mathcal{I}m \int r\,\bar{u}_0\,\partial_r u_0\,dx \leq -\sqrt{E_0}\,\|\,|x|\,u_0\|_2,$$

where E_0 was defined in Theorem 6.3(i). Then there exists T^* for which identity (6.24) holds.

It is important to notice that (6.29) has not been proved as part of Theorem 6.3, since the singularity in (6.24) could form before time T^*, i.e., $T_0 < T^*$, T_0 being the time when the inequality (6.24) occurs since we assume the existence in the time interval $[0, T_0]$. However, when T_0 in (6.24) and T^* in (6.29) coincide then (6.24), (6.2), and (6.29) ensure that

$$|u(\cdot,t)|^2 \to c\delta(\cdot) \qquad (\text{“concentration”}), \tag{6.31}$$

when $t \uparrow T^*$ in the distribution sense.

In the critical case $\alpha = 1 + 4/n$, the pseudo-conformal invariance tells us that if $u = u(x,t)$ is a solution of the equation in (6.1) with $\alpha = 1 + 4/n$ and $\lambda = \pm 1$, then

$$v(x,t) = \frac{e^{i|x|^2/4t}}{|t|^{n/2}}\, u\left(\frac{x}{t}, \frac{1}{t}\right) \tag{6.32}$$

solves the same equation for $t \neq 0$, with $v(\cdot,t) \in H^1(\mathbb{R}^n) \cap L^2(|x|^2\,dx)$. In particular, in the focusing case $\lambda = 1$, if $u(x,t) = e^{i\omega t}\varphi(x)$ is the standing wave solution of the equation in (6.1) see Chapter 5 (5.7) and (5.8), to simplify the notation we fix $\omega = 1$. Then,

$$z(x,t) = \frac{e^{i(|x|^2-4)/4t}}{|t|^{n/2}}\, \varphi\left(\frac{x}{t}\right) \tag{6.33}$$

is also a solution in $C(\mathbb{R} - \{0\} : H^1(\mathbb{R}^n) \cap L^2(|x|^2\,dx))$, which blows up at time $t = 0$, i.e.,

$$\lim_{t \uparrow 0} \|\nabla z(t)\|_2 = \infty.$$

Moreover, $\|\nabla z(t)\|_2 \sim c/t$.

The next result tells us that this is the “unique” minimal mass blow up solution. Observe that $\|z(t)\|_2 = \|\varphi\|_2$ and as it was commented before, if $\|u_0\|_2 < \|\varphi\|_2$, then the corresponding H^1-solution extends globally in time.

Theorem 6.5 ([Me3]). *Let u_1 be a solution of the IVP (6.1) with $\lambda = 1$, $\alpha = 1+4/n$, and data $u_{1,0} \in H^1(\mathbb{R}^n)$ with*

$$\|u_{1,0}\|_2 = \|\varphi\|_2,$$

where φ is the unique positive solution of the elliptic problem (5.8). Assume that u_1 blows up at time $T > 0$, i.e.,

$$\lim_{t\uparrow T} \|\nabla u_1(t)\|_2 = \infty. \tag{6.34}$$

Then,

$$u_1(x,t) = \left(\frac{1}{T-t}\right)^{n/2} e^{i(|x|^2-4)/4(T-t)}\, \varphi\left(\frac{x}{T-t}\right)$$

up to the invariance of the equation (see (5.10) and (5.11)).

Next, we consider the IVP (6.1) as in Theorem 6.5, i.e., $\lambda = 1$, $\alpha = 1+4/n$, and $u_0 \in H^1(\mathbb{R}^n)$, $n = 1, 2$ (so that the nonlinearity is smooth). Assuming that for some $\delta > 0$ sufficiently small

$$\|u_0\|_2 = \|\varphi\|_2 + \delta$$

and that (6.34) occurs, Bourgain and Wang [BoW] have shown that the corresponding solution u can be written as:

$$u(x,t) = u_1(x,t) + u_2(x,t),$$

with u_1 as in Theorem 6.5 and where u_2 remains smooth after the blow-up time T, i.e., for some $\rho > 0$,

$$\partial_t u_2 + \Delta u_2 + |u_2|^{4/n} u_2 = 0, \qquad t \in (T-\rho, T+\rho),$$

with $u_2(x,T) = \phi(x)$, where ϕ is smooth, with fast decay at infinity and vanishes at 0 to sufficiently high order.

In particular, this result tells us that at the blow-up time the solution does not need to absorb all the L^2-mass.

The following result is concerned with the concentration phenomenon in the blow up solutions.

Theorem 6.6 ([Me2]). *Given $T > 0$ and a set of points $\{x_1, \ldots, x_k\} \subset \mathbb{R}^n$, there exists an initial datum u_0 such that the corresponding solution of the IVP (6.1) with $\lambda = 1$ and $\alpha = 1 + 4/n$ blows up exactly at time T with the total L^2-mass concentrating at the points $\{x_1, \ldots, x_k\}$.*

Next, we comment on the blow-up rates. As a consequence of the H^1-local existence theorem (Theorem 5.8), we have

Corollary 6.1 ([CzW4]). *If the solution of the IVP (6.1) satisfies*

$$\lim_{t\uparrow T^*} \|\nabla u(t)\|_2 = \infty, \tag{6.35}$$

then

$$\|\nabla u(t)\|_2 \geq c_0 (T^* - t)^{-(1/(\alpha-1)-(n-2)/4)}. \tag{6.36}$$

We recall that (6.35) can only occur in the focusing case $\lambda = 1$ with $\alpha \geq 1+4/n$.

Proof. For $t_0 < T^*$, we consider the IVP (6.1) for time $t > t_0$ with data $u(t_0)$. By hypothesis, the solution cannot be extended in H^1 beyond the interval $[0, T^*)$. From the proof of Theorem 5.4 (estimates (5.38) and (5.39)), it follows that if for some $M > c\,\|u(t_0)\|_{1,2}$ one has that

$$c\,\|u(t_0)\|_{1,2} + c(T-t_0)^{\delta}\,M^{\alpha} \leq M, \quad \delta = 1 - \frac{(n-2)(\alpha-1)}{4}, \tag{6.37}$$

then $T < T^*$. Therefore, for all $M > c\,\|u(t_0)\|_{1,2}$,

$$c\,\|u(t_0)\|_{1,2} + c\,(T^*-t_0)^{\delta}\,M^{\alpha} \geq M. \tag{6.38}$$

Choosing $M = 2c\|u(t_0)\|_{1,2}$, it follows that

$$(T^*-t_0)^{\delta}\,\|u(t_0)\|_{1,2}^{\alpha-1} \geq c_0. \tag{6.39}$$

Since $\|u(t)\|_2 = \|u_0\|_2$, it follows that

$$\|\nabla_x u(t_0)\|_2 \geq c_0(T^*-t_0)^{-\delta/(\alpha-1)} = c_0(T^*-t_0)^{-(1/(\alpha-1)-(n-2)/4)}.$$

□

Thus, on the one hand we have that, in the critical case $\alpha = 1+4/n$, Corollary 6.1 gives the following estimate for the lower bound for the blow-up rate:

$$\|\nabla_x u(t)\|_2 \geq c_0\,(T^*-t)^{-1/2}.$$

On the other hand, numerical simulations in [LPSS] suggested the existence of solutions with blow-up rates as:

$$\|\nabla_x u(t)\|_2 \sim \left(\frac{\ln|\ln|T^*-t||}{T^*-t}\right)^{1/2}. \tag{6.40}$$

The constructions of the two previous blow-up solutions imply the following: there are at least two blow-up dynamics for (6.1) with two different rates, one which is continuation of the explicit $z(x,t)$ blow-up dynamic with the $1/(T-t)$ rate (6.36), and which is expected to be unstable; another one with the log–log rate (6.40), which has been conjectured to be stable.

In the one-dimensional case ($n = 1$), Perelman [Pe1] established the existence of a solution blowing up at the rate described in (6.40).

In [MeRa1], [MeRa2], Merle and Raphael have obtained general upper bound results for the blow up rate. More precisely, they characterize a set of data, i.e.,

$$\mathcal{B}_{\alpha^*} = \{u_0 \in H^1(\mathbb{R}^n) : \int \varphi^2 \leq \int |u_0|^2 \leq \int \varphi^2 + \alpha^*\}, \tag{6.41}$$

where α^* is a small enough parameter and φ is a ground state solution of (6.1), see (5.6)–(5.8), with $\alpha = 1+4/n$ and $\lambda = 1$, satisfying

$$E_G(u) = E(u) - \frac{1}{2}\Big(\frac{\mathcal{I}m\,(\int \nabla_x u\,\overline{u})}{\|u\|_{L^2}}\Big)^2 < 0, \tag{6.42}$$

blow up with an upper rate of the form:

$$\|\nabla_x u(t)\|_2 \lesssim \left(\frac{\ln|\ln|T^* - t||}{T^* - t}\right)^{1/2}. \tag{6.43}$$

Regarding the dynamics of the blow up solutions Raphael [Ra1] established the following result:

Theorem 6.7 ([Ra1]). *Let $n = 1, 2, 3, 4$. There exist universal constants C^*, C_1^* > 0 such that the following is satisfied:*

(i) Rigidity of blow-up rate: Let $u_0 \in \mathcal{B}_{\alpha^*}$ with

$$E_G(u_0) > 0,$$

and assume the corresponding solution $u(t)$ to (6.1) blows up in finite time $T < \infty$; then, there holds for t close to T either

$$\|\nabla_x u(t)\|_2 \le C^* \left(\frac{\log|\log(T - t)|}{T - t}\right)^{1/2} \tag{6.44}$$

or

$$\|\nabla_x u(t)\|_2 \ge \frac{C_1^*}{(T - t)\sqrt{E_G(u_0)}}.$$

(ii) Stability of the log–log law: Moreover, the set of initial data $u_0 \in \mathcal{B}_{\alpha^*}$ such that $u(t)$ blows up in finite time with upper bound (6.44) is open in H^1.

6.3 Comments

The results shown in Section 6.1 are due to Glassey [G2], based on previous ideas of Zakharov and Shabat [ZS]. Section 6.2 was built on the works of Tsutsumi [Ts] and of Nawa and Tsutsumi [NT]. Proposition 6.1, crucial in the proof of Theorem 6.2, is known as "the pseudoconformal invariant property" and was proved by Ginibre and Velo [GV1]. Observe that all these blow up results apply to local solution $u \in C([0, T] : H^1(\mathbb{R}^n) \cap L^2(|x|^2\,dx))$. In [OgT], the one-dimensional case $n = 1$, critical case $\alpha = 5$, the weighted condition $xu_0 \in L^2(\mathbb{R})$ was removed. The formation of singularities in solutions of the problem associated to the equation in (6.1) in the case of boundary and periodic values was studied in [Ka].

We recall that the existence of solutions in L^2 for the critical power $\alpha = 1+4/n$ was established in Theorem 5.2 (see [CzW2]). Using this result, we have that an extension is only possible when the L^2-$\lim_{t\uparrow T_0} u(t)$ exists. The identity (6.2) assures the existence of the limit in the weak topology of L^2. It was proved in [MT] that the strong limit does not exist and moreover that the same extension does not exist. This shows that if a solution that corresponds to radial data and dimension $n \ge 2$ develops singularities, then this solution satisfies (6.24) and (6.29).

In [Ra2], Raphael studied the dynamical structure of the blow up for the quintic nonlinear Schrödinger (NLS) in two dimensions, i.e., the equation in (6.1) with $\alpha = 5$, $\lambda = 1$, and $n = 2$, which is supercritical in L^2. Among other results he showed the existence of H^1 radial initial data for which the associated solutions blow up in finite time on a sphere of strictly positive radius.

In this setting, one has the equation:

$$i\,\partial_t u + \partial_r^2 u + \frac{1}{r}\,\partial_r u + |u|^4 u = 0.$$

Removing the term $\partial_r u/r$ above, one gets the one-dimensional equation with the critical L^2-power ($\alpha = 1 + 4/n$). Roughly proving that the contribution of the term $\partial_r u/r$ is negligible for the analysis and using Theorem 6.6, with one point $x = 1$, one gets the idea of the form of the result in [Ra2].

This approach to get blow up results which concentrate in a surface of $\mathbb{R}^n$ has also been obtained and extended in [HR2], [HR3], [HPR], and [MRS].

Corollary 6.1 was taken from Cazenave and Weissler [CzW4]. There they also showed that the IVP (6.1) with data $u_0 \in H^s(\mathbb{R}^n)$, $s \in (0,1)$, and $\|(-\Delta)^{s/2}u_0\|_2$ sufficiently small and nonlinearity $\alpha = 1 + 4/(n - 2\,s)$ has a unique global solution (notice that the cases $s = 0$, $s = 1$ were covered in Corollaries 5.2, 5.4, respectively). This result holds in both the focusing and defocusing case. In fact, it is just based on the homogeneity of the nonlinearity, so it applies to any nonlinear term of the form $f(u,\overline{u})$ with $f(\lambda u, \lambda\overline{u}) = \lambda^\alpha f(u,\overline{u})$.

Based on a pioneering idea of Bourgain [Bo5], one can obtain a global solution below the "energy norm," which in this case is H^1. The argument in [Bo5] has been significantly refined in a sequence of works of Colliander, Keel, Staffilani, Takaoka and Tao [CKSTT1], [CKSTT2], [CKSTT3]. For the IVP (6.1), with $\lambda < 0$, they have shown that in the cases $(n,\alpha) = (1,5), (2,3), (3,3)$, $u_0 \in H^s(\mathbb{R}^n)$, $s > 1/2,\ 1/2,\ 4/5$, respectively, suffice for the global existence. In the last case $(n,\alpha) = (3,3)$ with radial initial data the condition is lower: $s > 5/7$. Notice that in the above cases, the IVP (6.1) is locally well-posed in $H^s(\mathbb{R}^n)$, with $s > 0,\ 0,\ 1/2$. So, it is unclear whether these results are optimal.

Now, we regard the problem of the rate of growth of the higher Sobolev norm. Consider the IVP (6.1) defocusing case $\lambda < 0$ with $\alpha \in (1 + 4/n, (n+2)/(n-2))$. Theorem 5.4 provides the global solution for data $u_0 \in H^1(\mathbb{R}^n)$ with

$$\sup_{t\in\mathbb{R}} \|u(t)\|_{1,2} < \infty.$$

Assuming that $u_0 \in H^s(\mathbb{R}^n)$, $s > 1$, and the nonlinearity is sufficiently smooth, one can ask what are the best possible bounds for $\|u(t)\|_{s,2} \sim \|(-\Delta)^{s/2}u(t)\|_2$.

Standard energy estimates give an exponential upper bound. In [Bo6], Bourgain showed that if $n = 3$ and $\alpha < 5$, then

$$\|u(t)\|_{s,2} \le c\,|t|^{c(s-1)}$$

for some constant c. In [Sta1], Staffilani refined the arguments seen in [Bo6] and among other results in the case $(n,\alpha) = (1,3)$, she showed that

$$\|u(t)\|_{s,2} \le c|t|^{\mu} \qquad \text{as} \quad t \to \infty,$$

with $\mu = 2/3(s-1)^+$.

Concerning the asymptotic behavior of the H^1-global solution of the IVP (6.1) obtained in Theorem 6.2, one has the following L^p-norm decay result due to Ginibre and Velo [GV2].

Theorem 6.8. *Assume*

$$\lambda < 0, \quad \alpha \in \left(1 + \frac{4}{n},\ 1 + \frac{4}{n-2}\right), \quad \text{and} \ \ n \ge 3. \tag{6.45}$$

Then, for each $u_0 \in H^1(\mathbb{R}^n)$ the corresponding global solution $u(t)$ of the IVP (6.1) provided by Theorem 6.2(i) satisfies

$$\lim_{t\to\pm\infty} \|u(t)\|_p = 0 \quad \text{for } p \in \left(2, \frac{2n}{n-2}\right). \tag{6.46}$$

In addition, in [GV2] and [GV3], Ginibre and Velo proved the following theorems:

Theorem 6.9. *Under assumption (6.45), for each $u_0 \in H^1(\mathbb{R}^n)$ there exists a unique $u_0^\pm \in H^1(\mathbb{R}^n)$ such that*

$$\lim_{t\to\pm\infty} \|e^{it\Delta}u_0^\pm - u(t)\|_{1,2} = 0, \tag{6.47}$$

with

$$\|u_0^\pm\|_2 = \|u_0\|_2, \ \text{and} \ \|\nabla u_0^\pm\|_2^2 = E(u_0).$$

Theorem 6.10. *Under assumption (6.45) for each $u_0^\pm \in H^1(\mathbb{R}^n)$, there exists a unique $u_0 \in H^1(\mathbb{R}^n)$ such that (6.47) holds.*

Theorems 6.9 and 6.10 were used in [GV2] and [GV3] to define (continuous) maps $W^\pm$ (asymptotic states) and $\Omega^\pm$ (wave operators) in $H^1(\mathbb{R}^n)$, respectively, as $W^\pm(u_0) = u_0^\pm$ and $\Omega^\pm(u_0^\pm) = u_0$. Hence, $W^\pm\Omega^\pm = I$ on $H^1(\mathbb{R}^n)$ and one has the scattering operator $S = W^+\Omega^-$, with $S(u_0^-) = u_0^+$. For extensions of these results, see [GV2], [GV3]; for results in the cases $n = 1, 2$, see [Na].

Regarding global well-posedness for the periodic problem:

$$\begin{cases} i\partial_t u = -\Delta u \pm |u|^{\alpha-1}u, \\ u(x,0) = u_0(x), \end{cases} \tag{6.48}$$

$x \in \mathbb{T}^n$, $t \in \mathbb{R}, \alpha > 1$, we have the following results:

For $n = 2$, Bourgain [Bo1] showed that (6.48) with $\alpha \ge 3$ in the defocussing case ((–) sign) is globally well-posed. A similar result holds for the focusing case

((+) sign) under the additional assumption of $\|u_0\|_{L^2}$ being small enough or for $\alpha > 3$ assuming that $\|u_0\|_{1,2}$ is sufficiently small. In [BGT1], Burq, Gerard and Tzvetkov proved the existence of finite time ($T < \infty$) H^1 blow-up solutions, with data close to the ground state, i.e., $\lim_{t\uparrow T} \|u(t)\|_{1,2} = \infty$ for (6.48) with $\alpha = 3$ in the focusing case. Moreover, they found the precise rate of the blowup showing that $(T-t)\,\|u(t)\|_{1,2} \sim c_0$.

Finally, we describe some results concerning the stability and instability of standing waves. Before doing that we introduce the following notation:

$$A = \{\phi \in H^1(\mathbb{R}^n);\ \phi \neq 0 \text{ and } -\Delta\phi + \omega\phi = |\phi|^{\alpha-1}\phi\} \tag{6.49}$$

and

$$G = \{\phi \in A;\ S(\phi) \leq S(v) \text{ for all } v \in A\}, \tag{6.50}$$

where

$$S(\phi) = \frac{1}{2}\int_{\mathbb{R}^n} |\nabla\phi|^2\,dx - \frac{1}{\alpha+2}\int_{\mathbb{R}^n} |\phi|^{\alpha+1}\,dx - \frac{\omega}{2}\int_{\mathbb{R}^n} |\phi|^2\,dx.$$

The functions in the first set are called *ground states* and $u(x,t) = e^{i\omega t}\phi(x)$ bound states or *standing waves* or solitary waves.

If we require the following conditions be satisfied:

(i) $\alpha = 1 + 4/n,\ \omega > 0$ and $\varphi \in A$ or

(ii) $1 + 4/n < \alpha < (n+2)(n-2)$, $(1 + 4/n < \alpha < \infty,\ n = 1, 2)$, $\omega > 0$ and $\varphi \in G$,

then $u(x,t) = e^{i\omega t}\,\varphi(x)$ is an unstable solution of (6.1), in the sense that there is a sequence $\{\varphi_m\}_{m\in\mathbb{N}} \subset H^1(\mathbb{R}^n)$ such that

$$\varphi_m \to \varphi \quad \text{in} \quad H^1(\mathbb{R}^n)$$

and such that the corresponding maximal solution u_m of (6.1) blows up in a finite time for both $t > 0$ and $t < 0$.

The result in case (i) was established by Weinstein [W3] and case (ii) was proved by Berestycki and Cazenave [BC2]. The argument of proof involves variational methods.

On the other hand, if we let $1 < \alpha < 1 + 4/n,\ \omega > 0$, and $\varphi \in G$, then the solution $u(x,t) = e^{i\omega t}\,\varphi(x)$ is a stable solution of (6.1), in the sense that for every $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ such that if $\psi \in H^1(\mathbb{R}^n)$ verifies $\|\varphi - \psi\|_{H^1} < \delta(\epsilon)$, then the corresponding maximal solution v of (6.1) with data ψ verifies

$$\sup_{t\in\mathbb{R}}\ \inf_{\theta\in\mathbb{R}}\ \inf_{y\in\mathbb{R}^n} \|v(\cdot,t) - e^{i\theta}\varphi(\cdot - y)\|_{H^1} \leq \epsilon.$$

This result shows orbital stability in the subcritical case. It was established by Cazenave and Lions [CzL]. Extensions of this result to other dispersive equations can be found in [AL].

The interaction of solitary waves:

$$R(x,t) = e^{i(\nu\cdot x - |\nu|^2 t + \omega t + \theta)}\,\varphi_\omega(x - 2\nu\, t - x_0), \tag{6.51}$$

with $\nu,\ x_0 \in \mathbb{R}^n,\ \omega,\ \theta \in \mathbb{R},\ \omega > 0$, and φ_ω a solution of (5.8), is not yet well understood. For example, the detailed description of solutions of the IVP (6.1) with $\lambda > 0$ and data:

$$u_0(x,0) = \sum_{j=1}^{N} R_j(x,0) = \sum_{j=1}^{N} e^{i(\nu_j\cdot x + \theta_j)}\varphi_\omega(x - x_0), \quad N \geq 2 \tag{6.52}$$

such that $\exists\ j,\ k \in \{1,\dots,N\},\ j \neq k$, and $\widehat{t} > 0$ with

$$|2(\nu_j - \nu_k)\widehat{t} - (x_{0_j} - x_{0_k})| \ll 1$$

(i.e., at $t = \widehat{t}$ the solitary waves $R_j(x,t)$ and $R_k(x,t)$ interact) is quite open. In the integrable case $n = 1,\ \alpha = 3$, the scattering theory [ZS] describes the solution $u(x,t)$ in terms of the data as a nearly perfect *elastic* interaction between solitary waves (see Section 8.3). In the nonintegrable case, numerical simulations predict a similar behavior which has not been rigorously established. However, some results are known: In [MM7], Martel and Merle for the L^2-subcritical case ($1 < \alpha < 1 + 4/n$) proved the existence of multisolitary waves. More precisely, for

$$R(x,t) = \sum_{j=1}^{N} e^{i(\nu_j\cdot x - |\nu_j|^2 t + \omega_j t + \theta_j)}\varphi_\omega(x - 2\,\nu_j\, t - x_0), \tag{6.53}$$

the sum of the N-traveling waves in (6.51) with $\nu_j \neq \nu_k$ if $j \neq k$, they showed that there exists $u \in C([0,\infty) : H^1(\mathbb{R}^n))$ solution of the equation in (6.1) with $\lambda > 0$ such that for all $t > 0$

$$\|R(\cdot,t) - u(\cdot,t)\|_{1,2} \leq c\, e^{-\alpha_0\, t} \quad \text{for some} \ \ c,\ \alpha_0 > 0. \tag{6.54}$$

Notice that in the L^2-subcritical case the solitary waves are stable (see [CzL]), (for other results in this direction see [Pe2].)

In the same vein as in [HoZ], [HMZ], and [DH], the time evolution of the solution of the IVP:

$$\begin{cases} i\,\partial_t u + \partial_x^2 u - q\,\delta_0(x)\,u + |u|^2 u = 0,\\ u(x,0) = e^{i\,\nu\, x}\operatorname{sech}(x - x_0), \qquad x_0 \ll -1, \end{cases} \tag{6.55}$$

$q \in \mathbb{R}$, has been studied. Notice that if $q = 0$, the solution of (6.55) is the soliton:

$$u(x,t) = e^{i\,\nu\, x}\, e^{-i\,\nu^2\, t}\operatorname{sech}(x - 2\nu\, t - x_0), \tag{6.56}$$

and that for $q \neq 0$ the "soliton" should interact with the localized potential at time $t \sim |x_0|/\nu$. In [HoZ], it was shown that for $|q| \ll 1$ the "soliton solution" of

(6.55) remains “intact.” In the repulsive case ($q > 0$), for high velocity ($\nu \gg 1$), it was proven in [HMZ] that the incoming solution split into transmitted and reflected components (traveling with velocity ν to the right and to the left, respectively). The attractive case ($q < 0$) was studied in [DH].

In the one-dimensional L^2-supercritical case:

$$i\partial_t u + \partial_x^2 u + |u|^{\alpha-1}u = 0, \qquad \alpha > 5,$$

it is known that the traveling wave solution in (5.14) is unstable. In [KrS], a kind of “finite dimensional version” of the stable manifold for the ordinary differential equation (ODE) system was constructed. For further developments in this direction see [Scl2] and [Bc].

The “soliton resolution conjecture” claims that any “reasonable” solution to a nonlinear dispersive equation eventually resolve into a radiation component that behaves as a linear solution plus a localized component that behaves as a finite sum of special solutions (traveling waves, standing waves, breathers, . . .). This conjecture is largely open (see Section 8.3). In the case of the NLS (6.1) for the defocusing case ($\lambda < 0$ in (6.1)) (where no nontrivial special solutions exist) is known in some cases. In [To8], a weak form of this conjecture was established for energy-subcritical and mass-supercritical global $H^1(\mathbb{R}^n)$ radial solutions in higher dimension $n \geq 5$.

Finally, we return to the formula (4.7) (and the comment after it). This affirms that if $u_0 \in C_0(\mathbb{R}^n)$, then for any $t \neq 0$ and $\epsilon > 0$, $e^{it\Delta}u_0 \notin L^1(e^{\epsilon|x|}dx)$. So, one can ask if a similar result holds for solutions of the IVP (5.1). The following unique continuation principle established in [KPV15] answers this question:

Consider the equation:

$$\partial_t u = i(\Delta u + \lambda\,|u|^{\alpha-1}), \lambda \in \mathbb{R} - \{0\}, \tag{6.57}$$

with α such that $[\alpha] > n/2$ if α is not an odd integer.

Let $u_1, u_2 \in C([-T,T] : H^k(\mathbb{R}^n))$, $k > n/2$, $T > 0$, be two solutions of the equation (6.57) such that

$$\text{supp}(u_1(\cdot,0) - u_2(\cdot,0)) \subset \{x \in \mathbb{R}^n \,:\, x_1 \leq a_1\}, a_1 < \infty.$$

If for some $t \in (-T,T) - \{0\}$ and for some $\epsilon > 0$,

$$u_1(\cdot,t) - u_2(\cdot,t) \in L^2(e^{\epsilon|x_1|}\,dx),$$

then $u_1 \equiv u_2$.

Notice that fixing $u_2 \equiv 0$ one settles the above question. Moreover, by taking $u_1(x,0) = \varphi(x)$ as in (5.8) and $u_2(x,0) = \varphi(x) + \phi(x)$, with $\phi \in H^s(\mathbb{R}^n)$, $s > n/2$, with compact support, one can see that

$$u_2(\cdot,t) \notin L^2(e^{\epsilon|x|}\,dx) \text{ for any } \epsilon > 0,$$

for any $t \neq 0$ and α as in (6.57). This follows by combining Theorem 5.1 and the above unique continuation result. In other words, regardless of the stability of the standing wave a compact perturbation of it destroys its exponential decay.

6.4 Exercises

6.1 (i) Let $\alpha \in (1, 1 + 4/n]$. Prove that the local L^2-solution of the IVP (5.1) provided by Theorems 5.2 and 5.3 satisfies the identity (5.2).

(ii) Let $\alpha \in (1, 1 + 4/(n-2)]$. Show that the local H^1-solution of the IVP (5.1) provided by Theorems 5.4 and 5.5 satisfies the identities (5.2) and (5.3).

(iii) If in addition to the hypothesis in (ii), assuming that $|x|u_0 \in L^2(\mathbb{R}^n)$, prove (5.5).

(iv) Assume $\lambda < 0$ (defocusing case) and $\alpha \geq 1 + 4/n$ with the hypotheses in (iii), prove the decay estimate:

$$\|u(t)\|_{\alpha+1} \leq c\, t^{-2/(\alpha+1)}.$$

6.2 Consider the IVP (6.1) with $\alpha = 1+4/n$. Let $u(t)$ be its global L^2-solution corresponding to a datum $u_0 \in L^2(\mathbb{R}^n)$ with $\|u_0\|_2 < \epsilon$ provided by Theorem 6.2 (iii).

(i) Prove that there exist $u_0^{\pm} \in L^2(\mathbb{R}^n)$ such that

$$u(t) = e^{\pm it\Delta}\, u_0^{\pm} + R_{\pm}(t) \text{ with } \lim_{t\to\pm\infty} \|R_{\pm}(t)\|_2 = 0. \tag{6.58}$$

(ii) Prove that (6.58) fails for arbitrary $u_0 \in L^2(\mathbb{R}^n)$.
Hint: (i) Using Theorem 4.2, inequality (4.16), and Corollary 5.2 prove that

$$u_0^{\pm} = u_0 + \int_0^{\pm\infty} e^{-it'\Delta}\, |u|^{4/n}\, u(t')\, dt' \in L^2(\mathbb{R}^n).$$

(ii) Use the standing wave solutions in (5.8), (5.9), and (5.13) if $n = 1$.

6.3 (**Morawetz's estimate**) Consider the IVP (6.1) in the defocusing case $\lambda = -1$, with $\alpha < 1 + 4/(n-2)$, and $n \geq 3$. Let $u \in C([-T_0, T_1] : H^1(\mathbb{R}^n))$ be the local solution of this problem provided by Theorem 5.4.

(i) Prove the following estimates:

(a) $\mathcal{R}e \displaystyle\int i\partial_t u\big(\partial_r \bar{u} + \frac{(n-1)}{2r}\bar{u}\big)\, dx = \mathcal{R}e\, \frac{1}{2}\frac{d}{dt}\int iu\partial_r u\, dx.$

(b) $\mathcal{R}e \displaystyle\int \Delta u\, \big(\partial_r \bar{u} + \frac{(n-1)}{r}\bar{u}\big)\, dx \leq 0.$

(c) $\mathcal{R}e \displaystyle\int -|u|^{\alpha-1}u\big(\partial_r \bar{u} + \frac{(n-1)}{r}\bar{u}\big)\, dx = -\frac{\alpha-1}{2(\alpha+1)}\int \frac{(n-1)}{2r}|u|^{\alpha+1}\, dx$

where $r = |x|$.

(ii) Using part (i) and the equation in (6.1) show that

$$\frac{1}{2}\frac{d}{dt}\int iu\partial_r\bar{u}\,dx \geq \frac{(\alpha-1)(n-1)}{4(\alpha+1)}\int\frac{|u(x,t)|^{\alpha+1}}{|x|}\,dx. \tag{6.59}$$

(iii) Integrate (6.59) in the interval $(t_1, t_2) \subset [-T_0, T_1]$ to get

$$\int_{t_1}^{t_2}\int_{\mathbb{R}^n}\frac{|u(x,t)|^{\alpha+1}}{|x|} \leq c\,(\|u(t_1)\|_{1,2} + \|u(t_2)\|_{1,2}). \tag{6.60}$$

(iv) Use Theorem 6.2 to conclude that if in addition $\alpha < 1 + 4/n$, then the global solution $u \in C(\mathbb{R} : H^1(\mathbb{R}^n))$ satisfies

$$\int_{-\infty}^{\infty}\int_{\mathbb{R}^n}\frac{|u(x,t)|^{\alpha+1}}{|x|}\,dxdt < \infty. \tag{6.61}$$

(Notice that $|x|^{-1}$ is not integrable around the origin, so (6.61) gives information over the movement in time of the solution around the origin.)

6.4 For the IVP (6.1) with $\lambda = 1$ (focussing case), with the notation in Exercise 5.9, prove that

$$\begin{aligned} E(u_\omega(x,t)) &= \int_{\mathbb{R}^n}\left(|\nabla_x u_\omega(x,t)|^2 - \frac{2}{\alpha+1}\,|u_\omega(x,t)|^{\alpha+1}\right)dx \\ &= E(\varphi_\omega) = \int_{\mathbb{R}^n}\left(|\nabla_x\varphi_\omega(x)|^2 - \frac{2}{\alpha+1}\,|\varphi_\omega(x)|^{\alpha+1}\right)dx \\ &= \left(\frac{n(\alpha-1)-4}{2(\alpha+1)}\right)\int_{\mathbb{R}^n}|\varphi_\omega(x)|^{\alpha+1}\,dx. \end{aligned} \tag{6.62}$$

Thus, $E(\varphi_\omega) > 0$ if $\alpha - 1 > 4/n$, $E(\varphi_\omega) = 0$ if $\alpha - 1 = 4/n$ and $E(\varphi_\omega) < 0$ if $\alpha - 1 < 4/n$.

Hint: Combine the fact that $\varphi_\omega(x) = \omega^{1/(\alpha-1)}\varphi(\sqrt{\omega}\,x)$ is the solution of (5.8), with the identity (5.83).

6.5 Prove that for any $\alpha > 1$, there exist u_0^+, u_0^-, $u_0^0 \in H^1(\mathbb{R}^n)$ such that if

$$E(u_0) = \int_{\mathbb{R}^n}\left(|\nabla_x u_0(x)|^2 - \frac{2}{\alpha+1}\,|u_0(x)|^{\alpha+1}\right)dx,$$

then $E(u_0^+) > 0$, $E(u_0^-) < 0$ and $E(u_0^0) = 0$.

6.6 [W3] Consider the following case of the Gagliardo–Nirenberg inequality (3.14):

$$\|f\|_{\alpha+1} \le c\|\nabla f\|_2^{\theta}\, \|f\|_2^{1-\theta}, \tag{6.63}$$

$$\theta = n\left(\frac{1}{2} - \frac{1}{\alpha+1}\right) \text{ and } 2 < \alpha+1 < \frac{2n}{n-2}.$$

For $f \in H^1(\mathbb{R}^n)$, define

$$J_{\alpha,n}(f) = \frac{\|\nabla f\|_2^{\theta(\alpha+1)}\|f\|_2^{(1-\theta)(\alpha+1)}}{\|f\|_{\alpha+1}^{\alpha+1}}, \tag{6.64}$$

$c^*_{\alpha,n}$ and $\mu_{\alpha,n}$ as:

$$c^{*\;-(\alpha+1)}_{\alpha,n} = \mu_{\alpha,n} = \operatorname*{Inf}_{\substack{f\in H^1(\mathbb{R}^n)\\ f\neq 0}} J_{\alpha,n}(f),$$

i.e., $c^*_{\alpha,n} = \mu_{\alpha,n}^{-1/(\alpha+1)}$ is the optimal constant in (6.63).

(i) Assuming that $\mu_{\alpha,n}$ is attained, i.e., there exists $f^* \in H^1(\mathbb{R}^n)$ such that $J_{\alpha,n}(f^*) = \mu_{\alpha,n}$, with $f^* \ge 0$, prove that for any $\lambda, \nu > 0$

$$J_{\alpha,n}(\lambda\, f^*(\nu\,\cdot)) = \mu_{\alpha,n}. \tag{6.65}$$

(ii) Choosing $\lambda_0,\ \nu_0$ in (6.65) such that $g(x) = \lambda_0\, f^*(\nu_0\, x)$ satisfies $\|g\|_2 = \|\nabla g\|_2 = 1$, i.e. $\mu_{\alpha,n} = \|g\|_{\alpha+1}^{-(\alpha+1)}$, and using that $D\,J_{\alpha,n}(g) \equiv 0$ (i.e., $\frac{d}{d\epsilon}\, J_{\alpha,n}(g+\epsilon h)|_{\epsilon=0} = 0,\ \ \forall h \in H^1(\mathbb{R}^n)$) prove that

$$-\theta\,\Delta g + (1-\theta)\, g - \mu_{\alpha,n}\, g^{\alpha} \equiv 0, \quad g \ge 0.$$

(iii) Prove that the function $g_{\beta,\rho}(x) = \beta\, g(\rho\, x)$, $\beta, \rho > 0$ satisfies

$$-\frac{\theta}{\rho^2}\,\Delta g_{\beta,\rho} + (1-\theta) g_{\beta,\rho} - \mu_{\alpha,n}\,\beta^{\alpha-1} g_{\beta,\rho} \equiv 0. \tag{6.66}$$

(iv) Choose β_0, ρ_0 in (6.66) such that g_{β_0,ρ_0} solves (5.8) with $\omega \equiv 1$, i.e.,

$$-\Delta\varphi + \varphi - |\varphi|^{\alpha-1}\varphi = 0, \quad \varphi \ge 0,$$

to prove that

$$\mu_{\alpha,n} = \frac{(1-\theta)^{1+\frac{n}{4}(\alpha-1)}}{\theta^{\frac{n}{4}(\alpha-1)}}\,\frac{1}{\|g_{\beta_0,\rho_0}\|_2^{\alpha-1}} = \frac{(1-\theta)^{1+\frac{n}{4}(\alpha-1)}}{\theta^{\frac{n}{4}(\alpha-1)}}\,\frac{1}{\|\varphi\|_2^{\alpha-1}},$$

so,

$$c^*_{\alpha,n} = (\mu_{\alpha,n})^{-1/(\alpha+1)} = \frac{\theta^{\frac{n(\alpha-1)}{4(\alpha+1)}}}{(1-\theta)^{\frac{4+n(\alpha-1)}{4(\alpha+1)}}}\ \|\varphi\|_2^{(\alpha-1)/(\alpha+1)}.$$

(v) Verify that f^* is a “rescaling” of $\varphi(\cdot)$.

6.7 Consider the IVP (6.1) in the focussing case ($\lambda = 1$) and for the L^2-critical power ($\alpha = 1 + 4/n$). Assume that the local H^1-solution provided by Theorem 5.4 blows up in finite time, i.e., there exists $T^* > 0$ such that

$$\lim_{t\uparrow T^*} \|\nabla u(t)\|_2 = \infty. \tag{6.67}$$

(i) Prove that for any $s \in (0, 1]$ it follows that

$$\liminf_{t\uparrow T^*} \|D^s u(t)\|_2 = \infty.$$

(ii) Prove that for any $p \in (2, \infty]$ it follows that

$$\liminf_{t\uparrow T^*} \|u(t)\|_p = \infty.$$

(iii) If instead of $\alpha = 1 + 4/n$ one considers a nonlinearity $\alpha \in (1 + 4/n, 1 + 4/(n-2))$ and assume that (6.67) holds, for which values of s (i) holds, and for which values of p (ii) holds.

6.8 [GHW] Consider the one-dimensional cubic focussing NLS with a δ-potential:

$$i\partial_t v + \partial_x^2 v + \mu\,\delta(\cdot)\,v + |v|^2 v = 0, \quad \mu \in \mathbb{R}. \tag{6.68}$$

(i) Prove that if $\mu = 0$, the equation (6.68) has a one-parameter family of standing wave solutions of the form:

$$v_\omega(x,t) = e^{i\omega t}\sqrt{\omega}\,\phi(\sqrt{\omega}x), \quad \omega > 0,$$

with $\phi(x) = \operatorname{sech}(x)$ (positive, even, and radially decreasing) being the ground state.

(ii) Prove that for $\mu \neq 0$ formally

$$v_{\omega,\mu}(x,t) = e^{i\omega t}\sqrt{\omega}\,\varphi_\mu(\sqrt{\omega}x)$$

is a standing wave solution of (6.68), if

$$\varphi(x) = \varphi_{\mu,\omega}(x) = \sqrt{\omega}\,\varphi_\mu(\sqrt{\omega}x)$$

satisfies the elliptic equation

$$-\omega\,\varphi + \varphi'' + |\varphi|^2\varphi + \mu\,\delta(\cdot)\,\varphi = 0, \tag{6.69}$$

with $\varphi \in H^1(\mathbb{R}) \cap H^2(\mathbb{R}\backslash\{0\})$.

(iii) Prove that if $\omega \le \mu^2/4$, then (6.69) has no even positive radially decreasing nonnull L^2-solution.

(iv) Prove that if $\omega > \mu^2/4$ and $\mu > 0$, then (6.69) has an even, positive, radially decreasing solution of the form:

$$\varphi_{\mu,\omega}(x) = \sqrt{\omega}\operatorname{sech}\left(\sqrt{\omega}|x| + \tanh^{-1}\left(\frac{\mu}{2\sqrt{\omega}}\right)\right). \tag{6.70}$$

(v) Show that if the radially decreasing requirement is removed, then for $\omega > \mu^2/4$ and $\mu < 0$, the formula (6.70) describes another set of solutions of (6.69).
Hint: show that if φ solves (6.69), then

$$\varphi'(0^+) - \varphi'(0^-) + \mu\,\varphi(0) = 0. \tag{6.71}$$

Use that for $x > 0$, one should have

$$\varphi(x) = \sqrt{\omega}\,\mathrm{sech}(\sqrt{\omega}(x + x_0)), \quad x_0 > 0,$$

and a similar argument for $x < 0$. Combine this and (6.71) to obtain the equation:

$$\mu = 2\sqrt{\omega}\,\tanh(\sqrt{\omega}x_0),$$

which yields the desired result.

Chapter 7
Korteweg–de Vries Equation

In this chapter, we study the local well-posedness (LWP) for the initial value problem (IVP):

$$\begin{cases} \partial_t v + \partial_x^3 v + v^k\, \partial_x v = 0, \\ v(x,0) = v_0(x), \end{cases} \tag{7.1}$$

$x, t \in \mathbb{R}$, $k \in \mathbb{Z}^+$. The family of equations above is called the *k-generalized Korteweg–de Vries (k-gKdV) equation*. The case $k = 1$ is known as the Korteweg–de Vries (KdV) equation and is the most famous of the family. It was first derived as a model for unidirectional propagation of nonlinear dispersive long waves [KdV] but it also has been considered in different contexts, namely in its relation with inverse scattering (see Chapter 9, Section 9.6 for a brief introduction to it), in plasma physics, and in algebraic geometry (see [Mu] and references therein). The case $k = 2$ is called the *modified Korteweg–de Vries (mKdV) equation*. Like the KdV equation, it models propagation of weak nonlinear dispersive waves and it also can be solved via inverse scattering, i.e., this is a completely integrable system. There is an important relationship between these two equations given by the Miura transformation [Mu1]. More precisely, if we assume u to be a solution of the mKdV equation, then

$$v = i\,\sqrt{6}\,\partial_x u + u^2 \tag{7.2}$$

is a solution for the KdV equation. This relation was first used to obtain the inverse scattering results for both equations. Below we return to this transformation when we discuss global and ill-posedness results.

The KdV and mKdV equations have an infinite number of conserved quantities (see [MGK]). For $k > 2$, that is not the case. However, real solutions to the k-gKdV equation have the following conserved quantities: total mass

$$I_1(v) = \int_{-\infty}^{\infty} v(x,t)\,dx = \int_{-\infty}^{\infty} v_0(x)\,dx, \tag{7.3}$$

F. Linares, G. Ponce, *Introduction to Nonlinear Dispersive Equations*,
Universitext, DOI 10.1007/978-1-4939-2181-2_7

the L^2-norm

$$I_2(v) = \int_{-\infty}^{\infty} v^2(x,t)\,dx = \int_{-\infty}^{\infty} v_0^2(x)\,dx, \tag{7.4}$$

and the energy

$$I_3(v) = \int_{-\infty}^{\infty} \big((\partial_x v)^2 - c_k\, v^{k+2}\big)(x,t)\,dx = \int_{-\infty}^{\infty} \big((v_0')^2 - c_k\, v_0^{k+2}\big)(x)\,dx, \tag{7.5}$$

where $c_k = 2\{(k+1)(k+2)\}^{-1}$.

The k-gKdV equation admits solitary wave solutions having strong decay at infinity. These solutions are given by $v_{c,k}(x,t) = \phi_{c,k}(x-ct)$, $c > 0$ (c is the propagation speed) where

$$\phi_{c,k}(x) = \left\{\frac{(k+1)(k+2)}{2} c\,\mathrm{sech}^2\Big(\frac{k}{2}\sqrt{c}\,x\Big)\right\}^{1/k} \tag{7.6}$$

are the unique (up to translation) positive solutions decaying at infinity of

$$-c\varphi + \varphi'' + \frac{1}{k+1}\varphi^{k+1} = 0. \tag{7.7}$$

To motivate the local results that we describe in this chapter, we first note that if v solves (7.1), then, for $\lambda > 0$, so does $v_\lambda(x,t) = \lambda^{2/k} v(\lambda x, \lambda^3 t)$, with data $v_\lambda(x,0) = \lambda^{2/k} v(\lambda x, 0)$.

Note that

$$\|v_\lambda(\cdot,0)\|_{\dot{H}^s} = \|D_x^s\, v_\lambda(\cdot,0)\|_2 = \lambda^{2/k+s-1/2}\|v(\cdot,0)\|_{\dot{H}^s}. \tag{7.8}$$

This suggests that the optimal s, for the power k, is $s = s_k = 1/2 - 2/k$. Thus, $s_k \geq 0$ if and only if $k \geq 4$.

A simple computation shows that

$$\|\phi_{c,k}\|_{\dot{H}^{s_k}} = \|D^{s_k}\phi_{c,k}\|_2 = a_k, \text{ independent of } c, \tag{7.9}$$

and if $s \neq s_k$, $\|D^s\phi_{c,k}\|_2 \to 0$ as either $c \to 0$ or $+\infty$. Later on we illustrate the significance of this.

The best LWP results in Sobolev spaces $H^s(\mathbb{R})$ known for the k-gKdV equation can be summarized as follows:

k	Scaling	Result
1	$s = -\frac{3}{2}$	$s \geq -\frac{3}{4}$
2	$s = -\frac{1}{2}$	$s \geq \frac{1}{4}$
3	$s = -\frac{1}{6}$	$s \geq -\frac{1}{6}$
$k \geq 4$	$s = \frac{1}{2} - \frac{2}{k}$	$s \geq \frac{1}{2} - \frac{2}{k}$

In this chapter, the local results apply to both real and complex solutions. In Chapter 8, where we study global well-posedness for the k-gKdV, we will only consider real solutions since they satisfy the conservation laws (7.4) and (7.5).

Here we provide the proofs of the local results for the initial value problem (IVP) associated to the KdV ($k = 1$), mKdV ($k = 2$), and the L^2-critical gKdV ($k = 4$) equations.

The approach we follow for the last two equations is closely related to the previous one described for the nonlinear Schrödinger (NLS) equation. However, we shall remark that the situation faced here is more difficult to deal with due to the presence of derivatives on the nonlinearity that causes the so-called loss of derivatives. The idea is to analyze the special properties of solutions of the associated linear problem, such as smoothing effects like those of Strichartz (4.32) and Kato type (4.55), maximal function estimates combined with interpolated estimates. These along with some commutator estimates for fractional derivatives and the contraction mapping principle are the main ingredients in this method.

On the other hand, to establish LWP for the IVP associated to the KdV equation we use the function spaces $X_{s,b}$ introduced in the context of dispersive equations in [Bo1]. These functions spaces have a norm given in terms of the symbol of the associated linear operator (in this case $\partial_t + \partial_x^3$) and have been very useful in analysis of the interaction between the nonlinear and the dispersive effects. In this point, the so-called bilinear estimates play a main role to obtain sharp results.

In Section 7.3, we also list some results regarding the supercritical case ($k > 4$). There we use (7.6) and (7.9) mentioned above to illustrate ill-posedness results and thus the sharpness of the LWP results for the k-gKdV equation for $k \geq 4$.

7.1 Linear Properties

In this section, we establish a series of estimates for solutions of the linear initial value problem (IVP):

$$\begin{cases} \partial_t v + \partial_x^3 v = 0, \\ v(x,0) = v_0(x), \end{cases} \tag{7.10}$$

$x, t \in \mathbb{R}$. These estimates are useful to show sharp LWP results for the IVP (7.1) for $k = 2$ and $k \geq 4$.

We first recall that the solution of the IVP (7.10) is given by

$$v(x,t) = V(t)v_0(x) = S_t * v_0(x), \tag{7.11}$$

where

$$S_t(x) = \int_{-\infty}^{\infty} e^{2\pi i x\xi} e^{8\pi^3 i t\xi^3}\, d\xi = \frac{1}{\sqrt[3]{t}}\, S_1\Big(\frac{x}{\sqrt[3]{t}}\Big)$$

(see (1.30)).

Notice that $\{V(t)\}_{t=-\infty}^{\infty}$ defines a unitary group operator in $H^s(\mathbb{R})$ (see Proposition 4.2).

We begin by showing a sharpened version of the "local smoothing" effect found by Kato and Kruskov and Faminskii (see (4.55)) for solutions of the linear equation (7.10) and the inhomogeneous problem:

$$\begin{cases} \partial_t v + \partial_x^3 v = f, \\ v(x,0) = 0, \end{cases} \tag{7.12}$$

$x,\ t \in \mathbb{R}$.

Lemma 7.1. *The group* $\{V(t)\}_{t=-\infty}^{\infty}$ *satisfies*

$$\|\partial_x V(t) v_0\|_{L_x^\infty L_t^2} \le c\,\|v_0\|_2, \tag{7.13}$$

$$\left\| \partial_x^2 \int_0^t V(t-t') f(t')\,dt' \right\|_{L_x^\infty L_t^2} \le c\,\|f\|_{L_x^1 L_t^2}. \tag{7.14}$$

Remark 7.1. The proofs show that in (7.13) and (7.14), we can also have $D^{1+i\gamma}$, $D^{2+i\gamma}$, γ real.

Remark 7.2. To simplify the exposition from now on we omit the 2π factor in the Fourier transform. Thus, in particular we write

$$V(t)v_0(x) = \int_{-\infty}^{\infty} e^{i(x\xi + t\xi^3)} \widehat{v_0}(\xi)\,d\xi.$$

Proof. We only give the proof of (7.13), and refer to the proof of Theorem 4.4 estimate (4.28) for an argument similar to that needed to obtain (7.14).

The change of variables $\xi^3 = \eta$ shows that

$$\partial_x V(t) v_0(x) = \frac{1}{3} \int_{-\infty}^{\infty} e^{it\eta} e^{ix\eta^{1/3}} \eta^{-2/3+1/3} \widehat{v_0}(\eta^{1/3})\,d\eta.$$

Using Plancherel's identity (1.11) in the t variable, we get

$$\begin{aligned} \|\partial_x V(t) v_0\|_{L_t^2}^2 &= \frac{1}{9} \int_{-\infty}^{\infty} \left| e^{ix\eta^{1/3}} \eta^{-2/3+1/3} \widehat{v_0}(\eta^{1/3}) \right|^2 d\eta \\ &= c \int_{-\infty}^{\infty} |\widehat{v_0}(\xi)|^2\,d\xi, \end{aligned}$$

using $\eta^{1/3} = \xi$. This proves (7.13). $\square$

A consequence of (7.13) is

Corollary 7.1.

$$\|\partial_x \int_{-\infty}^{\infty} V(t-t')g(\cdot,t')\,dt'\|_{L^2} \le c\,\|g\|_{L^1_x L^2_t}. \tag{7.15}$$

Remark 7.3. The result in (7.15) is equivalent to (7.13) by duality. Note that this corollary implies

$$\sup_{t\in[-T,T]} \|\,\partial_x \int_0^t V(t-t')g(\cdot,t')\,dt'\|_{L^2} \le c\,\|g\|_{L^1_x L^2_t}. \tag{7.16}$$

The next lemma is useful to obtain maximal function estimates.

Lemma 7.2. *For any $x \in \mathbb{R}$,*

$$|I^t(x)| = \Big|\int_{-\infty}^{\infty} e^{i(x\xi+t\xi^3)}\frac{d\xi}{|\xi|^{1/2+i\gamma}}\Big| \le \frac{c\,(1+|\gamma|)}{|x|^{1/2}} \tag{7.17}$$

for γ real.

Proof. Since for $t=0$ the result is obvious (Exercise 1.14), we assume $t \neq 0$ and see that a dilation argument reduces the proof to show

$$|I^1(x)| \le \frac{c\,(1+|\gamma|)}{|x|^{1/2}}.$$

This can be done using a similar argument as the one in the proof of Proposition 1.6, taking into account the following sets:

$$\begin{aligned}
\Omega_1 &= \{\xi \in \mathbb{R} \;:\; |\xi| \le 2\},\\
\Omega_2 &= \{\xi \notin \Omega_1 \;:\; |3\xi^2 + x| \le |x|/2\},\\
\Omega_3 &= \mathbb{R} - (\Omega_1 \cup \Omega_2).
\end{aligned}$$

□

Next, we have maximal function estimates for solutions of (7.10).

Lemma 7.3.

$$\|\sup_{-\infty<t<\infty} |V(t)v_0|\,\|_{L^4_x} = \|V(t)v_0\|_{L^4_x L^\infty_t} \le c\,\|D_x^{1/4}v_0\|_2, \tag{7.18}$$

$$\Big\|D_x^{-1/2+i\gamma}\int_0^t V(t-t')f(t')\,dt'\Big\|_{L^4_x L^\infty_t} \le c_\gamma\,\|f\|_{L^{4/3}_x L^1_t}, \tag{7.19}$$

where γ is real.

Remark 7.4. The estimate (7.18) due to [KR] is sharp in the sense that for any $p \neq 4$ on the left-hand side require even in a finite time interval more than $1/4$ derivatives on the right-hand side of the inequality.

Proof. Showing estimate (7.18) is equivalent to proving

$$\|D_x^{-1/4+i\gamma}\, V(t)v_0\|_{L_x^4 L_t^\infty} \le c_\gamma \|v_0\|_2. \tag{7.20}$$

Hence, we do so for $\gamma = 0$ and prove (7.20).

We see that a version of (7.19) implies (7.20) by a method that follows the proof of the Stein–Tomas L^2-restriction theorem for the Fourier transform. In fact, duality shows that (7.18) is equivalent to

$$\Big\| \int_{-\infty}^{\infty} D_x^{-1/4} V(t) g(\cdot,t)\, dt \Big\|_{L^2} \le c\, \|g\|_{L_x^{4/3} L_t^1}.$$

Squaring the left-hand side of the inequality we obtain

$$\Big\| \int_{-\infty}^{\infty} D_x^{-1/4} V(t) g(\cdot,t)\, dt \Big\|_2^2 = \int\int g(x,t) \int_{-\infty}^{\infty} D_x^{-1/2}\, V(t-t')\overline{g(\cdot,t')}\, dt'\, dx\, dt,$$

so that (7.18) follows from

$$\Big\| \int_{-\infty}^{\infty} D_x^{-1/2}\, V(t-t') g(\cdot,t')\, dt' \Big\|_{L_x^4 L_t^\infty} \le c\, \|g\|_{L_x^{4/3} L_t^1}. \tag{7.21}$$

Next, we observe that (7.17) shows that

$$\Big| \int_{-\infty}^{\infty} D_x^{-1/2}\, V(t-t') g(\cdot,t')\, dt' \Big| \le \frac{c}{|x|^{1/2}} * \int_{-\infty}^{\infty} |g(\cdot,t')|\, dt'.$$

Thus, inequality (7.21) can be deduced from the Hardy–Littlewood–Sobolev theorem (Theorem 2.6), since $\dfrac{c}{|x|^{1/2}} * : L^{4/3} \to L^4$. The estimate (7.19) follows by the same argument. □

Lemma 7.4.

1. *If* $v_0 \in L^2(\mathbb{R})$, *then*

$$\|V(t)v_0\|_{L_x^5 L_t^{10}} \le c \|v_0\|_2. \tag{7.22}$$

2. *If* $g \in L_x^{5/4} L_t^{10/9}$, *then*

$$\Big\| \int_0^t V(t-t') g(t')\, dt' \Big\|_{L_x^5 L_t^{10}} \le c \|g\|_{L_x^{5/4} L_t^{10/9}}. \tag{7.23}$$

Proof. To prove (7.22), we consider the analytic family of operators

$$T_z v_0 = D_x^{-z/4} D_x^{(1-z)} V(t) v_0, \quad \text{with } z \in \mathbb{C},\ 0 \le \mathcal{R}e\ z \le 1.$$

When $z = i\gamma$,

$$T_{i\gamma} v_0 = \frac{\partial}{\partial x} V(t) D_x^{-i5\gamma/4} \mathsf{H} v_0,$$

where H denotes the Hilbert transform (see (1.18)). Hence, estimate (7.13) implies

$$\|T_{i\gamma}\, v_0\|_{L^\infty_x L^2_t} \le c\, \|v_0\|_2,$$

where we used that $\|D_x^{-i5\gamma/4} \mathsf{H} v_0\|_2 = \|v_0\|_2$.

On the other hand, setting $z = 1 + i\gamma$ we get

$$T_{1+i\gamma}\, v_0 = D_x^{-1/4} V(t) D_x^{-i\gamma 5/4}\, v_0.$$

Thus, estimate (7.18) yields

$$\|T_{1+i\gamma}\, v_0\|_{L^4_x L^\infty_t} \le c\, \|v_0\|_2.$$

Hence, from Stein's analytic interpolation (Theorem 2.7), the estimate (7.22) follows by choosing $z = 4/5$.

The proof of part 2 uses a similar, but more delicate arguments (see Corollary 3.8 in [KPV4]). □

Lemma 7.5. *If $v_0 \in L^2(\mathbb{R})$, then*

$$\|D_x V(t) v_0\|_{L^{20}_x L^{5/2}_t} \le \|D_x^{1/4} v_0\|_2. \tag{7.24}$$

Proof. The result follows using the Stein interpolation theorem (see Theorem 2.7) and the estimates (7.13), i.e.,

$$\|D_x^{5/4} D_x^{i\gamma} V(t) v_0\|_{L^\infty_x L^2_t} \le c\, \|D_x^{1/4} v_0\|_2,$$

and (7.18), i.e.,

$$\|D_x^{i\gamma} V(t) v_0\|_{L^4_x L^\infty_t} \le c\, \|D_x^{1/4} v_0\|_2,$$

with $\theta = 4/5$. □

Lemma 7.6 (Leibniz rule).

(i) *Let $\alpha \in (0,1)$. Let $p \in (1,\infty)$, $f = f(x)$, $g = g(x)$, then*

$$\|D^\alpha (fg) - f D^\alpha g\|_p \le c\, \|g\|_\infty \|D^\alpha f\|_p. \tag{7.25}$$

(ii) *Let* $\alpha \in (0,1)$, $\alpha_1, \alpha_2 \in [0,\alpha]$ *with* $\alpha = \alpha_1 + \alpha_2$. *Let* $p, q, p_1, p_2, q_2 \in (1,\infty)$, $q_1 \in (1,\infty]$ *be such that*

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} \qquad \text{and} \qquad \frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}.$$

Let $f = f(x,t)$ *and* $g = g(x,t)$. *Then,*

$$\|D_x^\alpha(fg) - f D_x^\alpha g - g D_x^\alpha f\|_{L_x^p L_T^q} \le c\, \|D_x^{\alpha_1} f\|_{L_x^{p_1} L_T^{q_1}} \|D_x^{\alpha_2} g\|_{L_x^{p_2} L_T^{q_2}}. \tag{7.26}$$

Moreover, for $\alpha_1 = 0$ *the value* $q_1 = \infty$ *is allowed.*

Lemma 7.7 (Chain rule). *Let* $\alpha \in (0,1)$ *and* $p, q, p_1, p_2, q_2 \in (1,\infty)$, $q_1 \in (1,\infty]$ *be such that*

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} \qquad \text{and} \qquad \frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}.$$

Then,

$$\|D_x^\alpha F(f)\|_{L_x^p L_T^q} \le c\, \|F'(f)\|_{L_x^{p_1} L_T^{q_1}} \|D_x^\alpha f\|_{L_x^{p_2} L_T^{q_2}}. \tag{7.27}$$

For the proof of Lemmas 7.6 and 7.7, we refer to [KPV4] (see also [CrW]).

The extra difficulty in obtaining these estimates comes from the fact that one needs to control derivatives in the space variable in a norm depending on the t variable first.

7.2 mKdV Equation

In this section, we establish the LWP theory for the IVP associated to the modified Korteweg–de Vries (mKdV) equation,

$$\begin{cases} \partial_t v + \partial_x^3 v + v^2\, \partial_x v = 0, \\ v(x,0) = v_0(x), \end{cases} \tag{7.28}$$

$x,\, t \in \mathbb{R}$. The idea of the proof is to use the linear estimates that we have obtained in the previous section plus a contraction mapping argument. As in the case of the nonlinear Schrödinger (NLS) equation, we employ the integral equation form of (7.28) for the same reason, i.e., it does not require differentiability of the solution.

Theorem 7.1. *Let* $s \ge 1/4$. *Then, for any* $v_0 \in H^s(\mathbb{R})$ *there exist* $T = T(\|D_x^{1/4} v_0\|_2) = c\, \|D_x^{1/4} v_0\|_2^{-4}$ *and a unique strong solution* $v(t)$ *of the IVP (7.28) such that*

$$v \in C([-T,T] : H^s(\mathbb{R})), \tag{7.29}$$

$$\|D_x^s\, \partial_x v\|_{L_x^\infty L_T^2} = \sup_{-\infty<x<\infty} \left(\int_{-T}^{T} |D_x^s \partial_x v(x,t)|^2\, dt \right)^{1/2} < \infty, \tag{7.30}$$

$$\|D_x^{s-1/4} \partial_x v\|_{L_x^{20} L_T^{5/2}} + \|D_x^s v\|_{L_x^5 L_T^{10}} < \infty, \tag{7.31}$$

and

$$\|v\|_{L_x^4 L_T^\infty} < \infty. \tag{7.32}$$

Moreover, there exists a neighborhood $\mathcal{V}$ *of* v_0 *in* $H^s(\mathbb{R})$ *such that the map* $\widetilde{v}_0 \mapsto \widetilde{v}(t)$ *from* $\mathcal{V}$ *into the class defined by (7.29)–(7.32) is smooth.*

Proof. We define

$$\mathcal{X}_T = \{v \in C([-T,T] : H^s(\mathbb{R})) : \ |||v|||_T < \infty\}$$

and

$$\mathcal{X}_T^a = \{v \in C([-T,T] : H^s(\mathbb{R})) : \ |||v|||_T \le a\},$$

where

$$|||v|||_T = \|v\|_{L_T^\infty H^s} + \|D_x^s\, \partial_x v\|_{L_x^\infty L_T^2} + \|D_x^{s-1/4} \partial_x v\|_{L_x^{20} L_T^{5/2}} + \|D_x^s v\|_{L_x^5 L_T^{10}} + \|v\|_{L_x^4 L_T^\infty}.$$

We shall prove that for appropriate values of a and T the operator

$$\Psi_{v_0}(v)(t) = \Psi(v)(t) = V(t)v_0 - \int_0^t V(t-t')(v^2 \partial_x v)(t')\, dt' \tag{7.33}$$

defines a contraction map on $\mathcal{X}_T^a$.

We only consider the case $s = 1/4$. As the higher derivatives derivatives appear linearly in the norms (7.29)–(7.31), argument below also provide the proof in the general case $s > 1/4$.

Using the operator (7.33), group properties, and the Minkowsky and Cauchy–Schwarz inequalities it follows that

$$\begin{aligned} \|D_x^{1/4} \Psi(v)(t)\|_2 &\le c\, \|D_x^{1/4} v_0\|_2 + \int_0^t \|D_x^{1/4}(v^2\, \partial_x v)\|_2\, dt \\ &\le c\, \|D_x^{1/4} v_0\|_2 + c\, T^{1/2} \|D_x^{1/4}(v^2\, \partial_x v)\|_{L_x^2 L_T^2}. \end{aligned}$$

To estimate the last term we make use of the Leibniz rule for fractional derivatives (7.26) and the chain rule (7.27). Thus,

$$\|D_x^{1/4}(v^2\, \partial_x v)\|_{L_x^2 L_T^2}$$

$$\begin{aligned}&\le \|v^2\|_{L^2_xL^\infty_T}\|D_x^{1/4}\partial_x v\|_{L^\infty_xL^2_T} + \|D_x^{1/4}(v^2)\|_{L_x^{20/9}L_T^{10}}\|\partial_x v\|_{L^{20}_xL^{5/2}_T}\\ &\le \|v\|^2_{L^4_xL^\infty_T}\|D_x^{1/4}\partial_x v\|_{L^\infty_xL^2_T} + \|v\|_{L^4_xL^\infty_T}\|D_x^{1/4}v\|_{L^5_xL^{10}_T}\|\partial_x v\|_{L^{20}_xL^{5/2}_T}\\ &\le c\,|\!|\!|v|\!|\!|^3_T.\end{aligned} \tag{7.34}$$

Hence,

$$\|D_x^{1/4}\Psi(v)(t)\|_2 \le c\,\|D_x^{1/4}\,v_0\|_2 + c\,T^{1/2}\,|\!|\!|v|\!|\!|^3_T. \tag{7.35}$$

From Definition 7.33, Minkowski's inequality, group properties, the smoothing effect (7.13), and the Cauchy–Schwarz inequality it follows that

$$\begin{aligned}&\|D_x^{1/4}\partial_x\Psi(v)(t)\|_{L^\infty_xL^2_T}\\ &\quad\le c\,\|D_x^{1/4}v_0\|_2 + \int_0^t \|\partial_x(V(t)V(-t')D_x^{1/4}(v^2\,\partial_x v))\|_{L^\infty_xL^2_T}\,dt'\\ &\quad\le c\,\|D_x^{1/4}v_0\|_2 + c\,T^{1/2}\|D_x^{1/4}(v^2\,\partial_x v)\|_{L^2_xL^2_T}.\end{aligned} \tag{7.36}$$

The maximal function norm of Ψ can be estimated applying Minkowski's inequality, group properties, (7.18), and the Cauchy–Schwarz inequality:

$$\begin{aligned}\|\Psi(v)(t)\|_{L^4_xL^\infty_T} &\le c\,\|D_x^{1/4}v_0\|_2 + \int_0^t \|V(t)V(-t')(v^2\,\partial_x v)\|_{L^4_xL^\infty_T}\,dt'\\ &\le c\,\|D_x^{1/4}v_0\|_2 + c\,T^{1/2}\|D_x^{1/4}(v^2\,\partial_x v)\|_{L^2_xL^2_T}.\end{aligned} \tag{7.37}$$

A similar argument as the previous one, but using (7.24) instead of (7.18) yields

$$\begin{aligned}\|\partial_x\Psi(v)(t)\|_{L^{20}_xL^{5/2}_T} &\le c\,\|D_x^{1/4}v_0\|_2 + \int_0^t \|\partial_x V(t)V(-t')(v^2\,\partial_x v)\|_{L^{20}_xL^{5/2}_T}\,dt'\\ &\le c\,\|D_x^{1/4}v_0\|_2 + c\,T^{1/2}\|D_x^{1/4}(v^2\,\partial_x v)\|_{L^2_xL^2_T}.\end{aligned} \tag{7.38}$$

Finally, the estimate (7.22) and the above argument gives us

$$\begin{aligned}\|D_x^{1/4}\Psi(v)(t)\|_{L^5_xL^{10}_T} &\le c\|D_x^{1/4}v_0\|_2 + \int_0^t \|V(t)V(-t')D_x^{1/4}(v^2\,\partial_x v)\|_{L^5_xL^{10}_T}\,dt'\\ &\le c\,\|D_x^{1/4}v_0\|_2 + c\,T^{1/2}\|D_x^{1/4}(v^2\,\partial_x v)\|_{L^2_xL^2_T}.\end{aligned} \tag{7.39}$$

Hence, the argument in (7.34) applied in (7.36)–(7.39) yields

$$|\!|\!|\Psi(v)(t)|\!|\!|_T \le c\,\|v_0\|_{1/4,2} + c\,T^{1/2}|\!|\!|v|\!|\!|^3_T. \tag{7.40}$$

Choosing $a = 2c\|v_0\|_{1/4,2}$ and T such that

$$ca^2\, T^{1/2} < \frac{1}{2} \tag{7.41}$$

we obtain that $\Psi_{v_0} : \mathcal{X}_T^a \to \mathcal{X}_T^a$.

A similar argument shows that

$$\|\!|\Psi(v) - \Psi(\tilde{v})\|\!|_T \le cT^{1/2}(\|\!|v\|\!|_T^2 + \|\!|\tilde{v}\|\!|_T^2)\,\|\!|v - \tilde{v}\|\!|_T \le 2c\,T^{1/2}a^2\,\|\!|v - \tilde{v}\|\!|_T.$$

Then, the choice of a and T in (7.41) implies that Ψ is a contraction. Consequently, we have that there exists a unique $v \in \mathcal{X}_T^a$ with $\Psi_{v_0}(v) \equiv v$, i.e.,

$$v(t) = V(t)v_0 - \int_0^t V(t-t')(v^2\partial_x v)(t')\,dt'. \tag{7.42}$$

Using similar arguments as above, we also deduce that for $T_1 \in (0, T)$

$$\|\!|\Psi_{v_0}(v) - \Psi_{\tilde{v}_0}(\tilde{v})\|\!| \le c\,\|v_0 - \tilde{v}_0\|_{s,2} + c\,T_1^{1/2}(\|\!|v\|\!|^2 + \|\!|\tilde{v}\|\!|^2)\,\|\!|v - \tilde{v}\|\!|.$$

This shows that for $T_1 \in (0, T)$, the map $\tilde{v}_0 \mapsto \tilde{v}$ on a neighborhood $\mathcal{W}$ of v_0 depending on T_1 to $\mathcal{X}_T^a$ is Lipschitz. We notice that an argument as the one used in Corollary 5.6 allows one to prove that this map actually is smooth.

Hence, the solution $v(\cdot) \in \mathcal{X}_T^a$ of the integral equation (7.42) is a strong solution of the IVP (7.28). In particular, v satisfies the equation in (7.28) in the distribution sense.

Next, we extend the uniqueness result to the class $\mathcal{X}_T$. Suppose $w \in \mathcal{X}_{T_1}$ for small $T_1 \in (0, T)$ is a strong solution of the IVP (7.28). The argument used in (7.40) shows that for some $T_2 \in (0, T_1)$, $w \in \mathcal{X}_{T_2}^a$. Thus, (7.41) implies $w \equiv v$ in $\mathbb{R} \times [-T_2, T_2]$. By reapplying this argument, the result can be extended to the whole interval $[-T, T]$. This yields the uniqueness result in $\mathcal{X}_T$. □

7.3 Generalized KdV Equation

The local theory for the IVP (7.1) when $k \ge 4$ is discussed in this section. We will prove the local theory for the critical case $k = 4$. For the case $k > 4$, we give the statements of the LWP results without proofs and talk over the sharpness of these results.

We first consider the L^2-critical case (see 7.8), i.e.,

$$\begin{cases} \partial_t v + \partial_x^3 v + v^4 \partial_x v = 0, \\ v(x,0) = v_0(x). \end{cases} \tag{7.43}$$

To show the LWP for (7.43), we follow a similar approach to the one applied for the mKdV equation.

Theorem 7.2. *There exists* $\delta > 0$ *such that for any* $v_0 \in L^2(\mathbb{R})$ *with*

$$\|v_0\|_2 < \delta,$$

there exists a unique strong solution $v(\cdot)$ *of the IVP (7.43) satisfying*

$$v \in C(\mathbb{R} : L^2(\mathbb{R})) \cap L^\infty(\mathbb{R} : L^2(\mathbb{R})), \tag{7.44}$$

$$\|\partial_x v\|_{L_x^\infty L_t^2} < \infty, \tag{7.45}$$

and

$$\|v\|_{L_x^5 L_t^{10}} < \infty. \tag{7.46}$$

Moreover, the map $v_0 \mapsto v(t)$ *from* $\{v_0 \in L^2(\mathbb{R}) : \|v_0\|_2 < \delta\}$ *into the class defined by (7.44)–(7.46) is smooth.*

Remark 7.5. Observe that this global L^2 result is valid for real or complex solutions. This is due to the homogeneity of the equation (scaling argument) and not to the L^2 conserved quantity.

Remark 7.6. It is expected that δ in the theorem be equal to the size of the solitary wave solution in the L^2-norm (7.6) with $k = 4$.

Proof of Theorem 7.2 We now define, for $v_0 \in L^2(\mathbb{R})$, $\|v_0\|_2 < \delta$,

$$\Psi(v)(t) = \Psi_{v_0}(v)(t) = V(t)v_0 - \int_0^t V(t-t')\, v^4 \partial_x v(t')\, dt'. \tag{7.47}$$

We shall show that there is $\delta > 0$ and $a > 0$ such that if $\|v_0\|_2 < \delta$, then

$$\Psi : \mathcal{X}_a \to \mathcal{X}_a$$

is a contraction map, where

$$\mathcal{X}_a = \{w \in C(\mathbb{R} : L^2(\mathbb{R})) : |||w||| \le a\}$$

and

$$|||v||| = \|\partial_x v\|_{L_x^\infty L_t^2} + \|v\|_{L_t^\infty L_x^2} + \|v\|_{L_x^5 L_t^{10}}.$$

From (7.47) and (7.15), we have

$$\begin{aligned} \|\Psi(v)(t)\|_2 &\le \|v_0\|_2 + c\,\|\partial_x \int_0^t V(t-t')\, v^5(t')\, dt'\|_2 \le \|v_0\|_2 + c\,\|v^5\|_{L_x^1 L_t^2} \\ &\le \|v_0\|_2 + c\,\|v\|^5_{L_x^5 L_t^{10}} \le \|v_0\|_2 + c|||v|||^5. \end{aligned} \tag{7.48}$$

Similarly, (7.22) and (7.23) lead to

$$\begin{aligned}\|\Psi(v)(t)\|_{L^5_xL^{10}_t} &\le c\|v_0\|_2 + c\,\Big\|\int_0^t V(t-t')\,\partial_x(v^5)(t')\,dt'\Big\|_{L^5_xL^{10}_t}\\ &\le c\,\|v_0\|_2 + c\,\|v^4\,\partial_x v\|_{L^{5/4}_xL^{10/9}_t}\\ &\le c\,\|v_0\|_2 + c\,\|v\|^4_{L^5_xL^{10}_t}\|\partial_x v\|_{L^\infty_xL^2_t}\\ &\le c\,\|v_0\|_2 + c|\!|\!|v|\!|\!|^5.\end{aligned} \tag{7.49}$$

Finally, we use (7.13) and (7.14) to have

$$\begin{aligned}\|\partial_x\,\Psi(v)(t)\|_{L^\infty_xL^2_t} &\le c\|v_0\|_2 + c\,\Big\|\partial_x^2\int_0^t V(t-t')\,(v^5)(t')\,dt'\Big\|_{L^\infty_xL^2_t}\\ &\le c\,\|v_0\|_2 + c\,\|v^5\|_{L^1_xL^2_t}\\ &\le c\,\|v_0\|_2 + c|\!|\!|v|\!|\!|^5.\end{aligned} \tag{7.50}$$

Using Remark 7.3, it follows that $\Psi(v) \in C(\mathbb{R} : L^2(R))$. Thus, from (7.48) to (7.50) we obtain that

$$|\!|\!|\Psi(v)|\!|\!| \le c\|v_0\|_2 + c|\!|\!|v|\!|\!|^5.$$

Now, choosing δ such that

$$c(4c\delta)^4 < \frac{1}{2} \qquad \text{and} \qquad a \in (2c\delta,\, 3c\delta),$$

we conclude that $\Psi : \mathcal{X}_a \to \mathcal{X}_a$.

A similar argument leads to

$$|\!|\!|\Psi(v) - \Psi(\tilde{v})|\!|\!| \le c\,(|\!|\!|v|\!|\!|^4 + |\!|\!|\tilde{v}|\!|\!|^4)|\!|\!|v-\tilde{v}|\!|\!| \le 2ca^4|\!|\!|v-\tilde{v}|\!|\!| \le \frac{1}{2}|\!|\!|v-\tilde{v}|\!|\!|.$$

As the remainder of the proof follows the argument employed in Theorem 7.1, it is omitted. □

As a corollary of this result, we have the LWP for the L^2-critical case.

Theorem 7.3 (Critical case). *Let $k = 4$. Given any $v_0 \in L^2(\mathbb{R})$ there exist $T = T(v_0) > 0$ and a unique strong solution $v(\cdot)$ of the IVP (7.43) satisfying*

$$v \in C([-T,T] : L^2(\mathbb{R})), \tag{7.51}$$

$$\|v\|_{L^5_xL^{10}_T} < \infty, \tag{7.52}$$

and

$$\|\partial_x v\|_{L^\infty_xL^2_T} < \infty. \tag{7.53}$$

Given $T' \in (0,T)$, there exists a neighborhood $\mathcal{W}$ of v_0 in $L^2(\mathbb{R})$ such that the map $\tilde{v}_0 \mapsto \tilde{v}(t)$ from $\mathcal{W}$ into the class defined by (7.51)–(7.53), with T' instead of T is smooth.

If $v_0 \in H^s(\mathbb{R})$ with $s > 0$, the previous result extends to the class

$$v \in C([-T,T] : H^s(\mathbb{R}))$$

and

$$\|D_x^s \partial_x v\|_{L_x^\infty L_T^2} < \infty,$$

in the above time interval $[-T,T]$.

Remark 7.7. The norm we define to prove this result is as follows:

$$|||v||| = \|v - V(t)v_0\|_{L_T^\infty L_x^2} + \|\partial_x v\|_{L_x^\infty L_T^2} + \|v\|_{L_x^5 L_T^{10}},$$

which is "similar" to the L^2-critical case for the semilinear Schrödinger equation (see Theorem 5.3). Notice that in Theorem 7.3 the time of existence of the local solution depends on v_0 itself and not on its norm.

Next, we have the subcritical local existence result.

Theorem 7.4. *Let $s > 0$. Then, for any $v_0 \in H^s(\mathbb{R})$ there exist $T = T(\|v_0\|_{s,2})$ (with $T(\rho,s) \to \infty$ as $\rho \to 0$) and a unique strong solution $u(\cdot)$ of the IVP (7.43) satisfying*

$$v \in C([-T,T] : H^s(\mathbb{R})), \tag{7.54}$$

$$\|v\|_{L_x^5 L_T^{10}} + \|D_x^s v\|_{L_x^5 L_T^{10}} + \|D_t^{s/3} v\|_{L_x^5 L_T^{10}} < \infty, \tag{7.55}$$

and

$$\|\partial_x v\|_{L_x^\infty L_T^2} + \|D_x^s \partial_x v\|_{L_x^\infty L_T^2} + \|D_t^{s/3} \partial_x v\|_{L_x^\infty L_T^2} < \infty. \tag{7.56}$$

Given $T' \in (0,T)$, there exists a neighborhood $\mathcal{V}$ of v_0 in $H^s(\mathbb{R})$ such that the map $\tilde{v}_0 \to \tilde{v}(t)$ from $\mathcal{V}$ into the class defined by (7.54), (7.55), and (7.56) with T' instead of T is smooth.

Next, we consider the IVP (7.1) in the L^2-supercritical case, i.e., $k > 4$. The results are listed without proof.

Theorem 7.5. *Let $k > 4$ and $s_k = (k-4)/2k$. Then, there exists $\delta_k > 0$ such that for any $v_0 \in \dot{H}^{s_k}(\mathbb{R})$ with*

$$\|D_x^{s_k} v_0\|_2 \le \delta_k,$$

there exists a unique strong solution $v(\cdot)$ of the IVP (7.1) satisfying

$$v \in C(\mathbb{R} : \dot{H}^{s_k}(\mathbb{R})) \cap L^\infty(\mathbb{R} : \dot{H}^{s_k}(\mathbb{R})), \tag{7.57}$$

$$\|D_x^{s_k}\,\partial_x v\|_{L_x^\infty L_t^2} < \infty, \tag{7.58}$$

$$\|D_x^{s_k}\,v\|_{L_x^5 L_t^{10}} < \infty, \tag{7.59}$$

and

$$\big\|D_x^{1/10-2/5k}\,D_t^{3/10-6/5k}\,v\big\|_{L_x^{p_k}L_t^{q_k}} < \infty, \tag{7.60}$$

where

$$\frac{1}{p_k} = \frac{2}{5k} + \frac{1}{10} \quad \text{and} \quad \frac{1}{q_k} = \frac{3}{10} - \frac{4}{5k}.$$

Moreover, the map $v_0 \mapsto v(t)$ from

$$\mathcal{V} = \{v_0 \in \dot{H}^{s_k}(\mathbb{R}) \,:\, \|D_x^{s_k} v_0\|_2 \le \delta_k\}$$

into the class defined by (7.57)–(7.60) is smooth.

Next, we have the result corresponding to any size data.

Theorem 7.6. *Let $k > 4$ and $s_k = (k-4)/2k$. Given $v_0 \in \dot{H}^{s_k}(\mathbb{R})$, there exist $T = T(v_0) > 0$ and a unique strong solution $v(\cdot)$ of the IVP (7.1) satisfying*

$$v \in C([-T,T] \,:\dot{H}^{s_k}(\mathbb{R})), \tag{7.61}$$

$$\|D_x^{s_k}\,\partial_x v\|_{L_x^\infty L_T^2} < \infty, \tag{7.62}$$

$$\|D_x^{s_k}\,v\|_{L_x^5 L_T^{10}} < \infty, \tag{7.63}$$

and

$$\big\|D_x^{1/10-2/5k}\,D_t^{3/10-6/5k}\,v\big\|_{L_x^{p_k}L_T^{q_k}} < \infty \tag{7.64}$$

with p_k and q_k as in (7.60).

Given $T' \in (0,T)$, there exists a neighborhood $\mathcal{W}$ of $v_0 \in \dot{H}^{s_k}(\mathbb{R})$ such that the map $\tilde{v}_0 \to \tilde{v}(t)$ from $\mathcal{W}$ into the class defined by (7.61)–(7.64) is smooth.

If $v_0 \in H^s(\mathbb{R})$ with $s \ge s_k$, the previous results extend to the class

$$v \in C([-T,T] \,:\, H^s(\mathbb{R}))$$

and

$$\|D_x^{s}\,\partial_x v\|_{L_x^\infty L_t^2} < \infty$$

in the above interval $[-T,T]$.

Corollary 7.2. *Let $k > 4$ and $s > s_k = (k-4)/2k$. Then, for any $v_0 \in H^s(\mathbb{R})$ there exist $T = T(\|v_0\|_{s,2}) > 0$ ($T(\rho;s) \to \infty$ as $\rho \to 0$) and a unique strong solution $v(\cdot)$ of the IVP (7.1) satisfying, in addition to (7.62)–(7.64):*

$$v \in C([-T,T] \,:\, H^s(\mathbb{R})), \tag{7.65}$$

$$\|D_x^s\,\partial_x v\|_{L_x^\infty L_T^2} + \|D_t^{s/3}\,\partial_x v\|_{L_x^\infty L_T^2} < \infty, \tag{7.66}$$

$$\|D_x^s\,v\|_{L_x^5 L_T^{10}} + \|D_t^{s/3}\,v\|_{L_x^5 L_T^{10}} < \infty, \tag{7.67}$$

and

$$\|D_x^{s/5}\,D_t^{3s/5}\,v\|_{L_x^{p_k} L_T^{q_k}} < \infty \tag{7.68}$$

with p_k and q_k as in (7.60).

Given $T' \in (0,T)$, there exists a neighborhood $\mathcal{W}$ of $v_0 \in \dot{H}^{s_k}(\mathbb{R})$ such that the map $\tilde{v}_0 \to \tilde{v}(t)$ from $\mathcal{W}$ into the class defined by (7.65)–(7.68) is smooth.

If $v_0 \in H^{s'}(\mathbb{R})$ with $s' > s$, the previous results hold with s' instead of s in the same time interval $[-T,T]$.

To conclude, we discuss the sharpness of the results described in this section.

In [BKPSV], it was proved that if the notion of well-posedness given in Chapter 5 is strengthened, then the IVP (7.1) is ill-posed for $k \geq 4$. More precisely, we have the following.

Theorem 7.7. *The IVP (7.1) with $k \geq 4$ is ill-posed in $H^{s_k}(\mathbb{R})$ with $s_k = 1/2 - 2/k$ in the sense that the time of existence and the continuous dependence cannot be expressed in terms of the size of the data in the H^{s_k}-norm.*

Proof. We only consider the case $k = 4$. We prove that if we assume $T = T(\|v_0\|_{L^2}) > 0$, then the part in the theorem regarding the continuous dependence of the solution upon the data fails. The proof below also establishes the second part of the theorem.

Consider the solitary wave solutions $\phi_{c,4}$ in (7.6) and $v_{c_k,4}(x,t)$ the solution corresponding to initial data $v_0(x) = \phi_{c_k,4}(x)$. We compare

$$\|\phi_{c_1,4} - \phi_{c_2,4}\|_2^2 \qquad \text{and} \qquad \|v_{c_1,4}(\cdot,t) - v_{c_2,4}(\cdot,t)\|_2^2$$

for $t \neq 0$. We show that one can choose c_1 and c_2 so that the first expression tends to 0 while the second one does not. Thus, well-posedness cannot hold for these data.

Let $a_4^2 = \int \phi_{c_j,4}^2$, $j = 1,2$, and note that

$$\|\phi_{c_1,4} - \phi_{c_2,4}\|_2^2 = \|\phi_{c_1,4}\|_2^2 + \|\phi_{c_2,4}\|_2^2 - 2\langle\,\phi_{c_1,4},\,\phi_{c_2,4}\rangle.$$

Writing $\varphi_4(x) = 3^{1/4}\,(\text{sech}^2(2x))^{1/4}$, the inner product equals

$$c_1^{1/4}c_2^{1/4}\int_{-\infty}^{\infty}\varphi_4(\sqrt{c_1}x)\varphi_4(\sqrt{c_2}x)\,dx.$$

If $\sqrt{c_1}x = y$, we get

$$\Big(\frac{c_1}{c_2}\Big)^{1/4}\int_{-\infty}^{\infty}\varphi_4(y)\varphi_4\Big(\sqrt{\frac{c_1}{c_2}}y\Big)\,dy \to a_4^2 \quad \text{if} \quad \frac{c_1}{c_2} \to 1.$$

Thus,

$$\|\phi_{c_1,4} - \phi_{c_2,4}\|_2^2 \to 0. \tag{7.69}$$

Analyzing $\|v_{c_1,4}(\cdot,t) - v_{c_2,4}(\cdot,t)\|_2^2$ similarly, we obtain

$$a_4^2 + a_4^2 - \Big(\frac{c_1}{c_2}\Big)^{1/4} \int_{-\infty}^{\infty} \varphi_4(y - c_1^{3/2}t)\,\varphi_4\Big(\sqrt{\frac{c_1}{c_2}}\,y - c_2^{3/2}t\Big)\,dy$$

$$= a_4^2 + a_4^2 - \Big(\frac{c_1}{c_2}\Big)^{1/4} \int_{-\infty}^{\infty} \varphi_4(z)\,\varphi_4\Big(\sqrt{\frac{c_1}{c_2}}\,z - c_2^{1/2}(c_1 - c_2)t\Big)\,dz.$$

Choose now $c_1/c_2 \to 1$, but $c_2^{1/2}(c_1 - c_2) \to \infty$ (for instance, $c_1 = N + 1$, $c_2 = N$, $N \in Z^+$). The rapid decay of φ_4 shows that the integral approaches 0. Thus,

$$\sup_{[0,T]} \|v_{c_1,4}(\cdot,t) - v_{c_2,4}(\cdot,t)\|_2^2 \to 2\,a_4^2 \quad \text{for any } T > 0 \tag{7.70}$$

as $c_1/c_2 \to 1$.

Finally, (7.8), (7.69), and (7.70) yield the result. □

7.4 KdV Equation

In this section, we establish the local theory of the IVP associated to the KdV equation, that is,

$$\begin{cases} \partial_t v + \partial_x^3 v + v\partial_x v = 0, \\ v(x,0) = v_0(x), \end{cases} \tag{7.71}$$

$x,\ t \in \mathbb{R}$. The method used here is quite different from the one illustrated for the NLS equation in Chapter 5 and in the previous three sections of this chapter for the mKdV and generalized KdV (gKdV) equations.

We start out by defining the function spaces introduced in the context of dispersive equations by Bourgain in [Bo1]:

Definition 7.1. For $s, b \in \mathbb{R}$ and $f \in \mathcal{S}'(\mathbb{R}^2)$, we say that $f \in X_{s,b}$, if

$$\|f\|_{X_{s,b}} = \Big(\int_{\mathbb{R}^2} (1 + |\tau - \xi^3|)^{2b}(1 + |\xi|)^{2s}\,|\widehat{f}(\xi,\tau)|^2\,d\xi d\tau\Big)^{1/2} < \infty\,, \tag{7.72}$$

where $\widehat{\ }$ denotes the Fourier transform in $\mathbb{R}^2$.

We solve (a variant) of the integral equation, namely

$$v(t) = \theta(t)V(t)v_0 + \theta(t)\int_0^t V(t - t')v\partial_x v(t')\,dt', \tag{7.73}$$

where $\theta \in C_0^\infty(\mathbb{R})$, $0 \leq \theta \leq 1$, $\theta \equiv 1$ near 0, supp $\theta \subseteq [-1, 1]$ with $v_0 \in H^s(\mathbb{R})$ and $v \in X_{s,b}$.

Remark 7.8. Let $\widehat{J^s f}(\xi) = (1 + |\xi|^2)^{s/2} \widehat{f}(\xi)$, and $\widehat{\Lambda^b f}(\tau) = (1 + |\tau|^2)^{b/2} \widehat{f}(\tau)$, where $\widehat{\ }$ denotes the Fourier transform in one variable. Then,

$$\|f\|_{X_{s,b}} = \|\Lambda^b J^s V(-t) f\|_{L^2_\xi L^2_\tau}.$$

Corollary 7.3. *If* $b > 1/2$*,*

$$X_{s,b} \subset C((-\infty, \infty) : H^s(\mathbb{R})).$$

This is an easy consequence of Remark 7.8 and the usual Sobolev embedding theorem.

Let us set $\theta_\rho(t) = \theta(\rho^{-1} t)$, $\rho \in (0, 1]$, where θ is as above.

Lemma 7.8. *For any* $b > 1/2$ *and* $s \in \mathbb{R}$*,*

$$\|\theta_\rho V(t) v_0\|_{X_{s,b}} \leq c\, \rho^{(1-2b)/2}\, \|v_0\|_{s,2}. \tag{7.74}$$

Proof.

$$\theta_\rho(t) V(t) v_0 = \theta(\rho^{-1} t) \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{ix\xi}\, e^{it\tau}\, \delta(\tau - \xi^3) \widehat{v}_0\, d\xi\, d\tau,$$

so that $(\theta(\rho^{-1} t) V(t) v_0)^\wedge(\xi, \tau) = \rho\, \widehat{\theta}(\rho(\tau - \xi^3))\, \widehat{v_0}(\xi)$. Thus,

$$\begin{aligned}
&\|\theta_\rho(t) V(t) v_0\|^2_{X_{s,b}} \\
&\quad = c\, \rho^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\widehat{\theta}(\rho(\tau - \xi^3))|^2 (1 + |\tau - \xi^3|)^{2b} (1 + |\xi|)^{2s}\, |\widehat{v_0}(\xi)|^2\, d\xi\, d\tau \\
&\quad = c \int_{-\infty}^{\infty} (1 + |\xi|)^{2s} |\widehat{v_0}(\xi)|^2 \Big(\rho^2 \int_{-\infty}^{\infty} |\widehat{\theta}(\rho(\tau - \xi^3))|^2 (1 + |\tau - \xi^3|)^{2b}\, d\tau\Big)\, d\xi.
\end{aligned}$$

Since $b > 1/2$ and $\rho \in (0, 1)$, the inner integral can be estimated as follows:

$$\begin{aligned}
&\rho^2 \int_{-\infty}^{\infty} |\widehat{\theta}(\rho(\tau - \xi^3))|^2 (1 + |\tau - \xi^3|)^{2b}\, d\tau \\
&\quad \leq c\rho^2 \int_{-\infty}^{\infty} |\widehat{\theta}(\rho(\tau - \xi^3))|^2\, d\tau + c\rho^2 \int_{-\infty}^{\infty} |\widehat{\theta}(\rho(\tau - \xi^3))|^2 |\tau - \xi^3|^{2b}\, d\tau \\
&\quad \leq c\rho + c\rho^{1-2b} \leq c\rho^{1-2b}.
\end{aligned}$$

This completes the proof of the lemma. □

Lemma 7.9. *For all $s \in \mathbb{R}$ and $1/2 < b \le 1$,*

$$\|\theta_\rho\, v\|_{X_{s,b}} \le c\, \rho^{(1-2b)/2}\, \|v\|_{X_{s,b}}. \tag{7.75}$$

Proof. $(\theta_\rho(t)v(x,t))^\wedge = \widehat{v} *_\tau (\rho\,\widehat{\theta}(\rho\ \cdot))$, so that by definition of the $X_{s,b}$-norm, the proof reduces to showing that, for $a \in \mathbb{R}$,

$$\int_{-\infty}^{\infty} |(\rho\widehat{\theta}(\rho\ \cdot)) * \widehat{v}(\tau)|^2\,(1+|\tau-a|)^{2b}\,d\tau \le c\,\rho^{(1-2b)} \int_{-\infty}^{\infty} |\widehat{v}(\tau)|^2\,(1+|\tau-a|)^{2b}\,d\tau.$$

Since

$$\int_{-\infty}^{\infty} |\rho\,\widehat{\theta}(\rho\tau)|\,d\tau < \infty,$$

it follows that

$$\int_{-\infty}^{\infty} |(\rho\widehat{\theta}(\rho\ \cdot)) * \widehat{v}(\tau)|^2\,d\tau \le c \int_{-\infty}^{\infty} |\widehat{v}(\tau)|^2\,d\tau.$$

We turn to

$$\int_{-\infty}^{\infty} |(\rho\widehat{\theta}(\rho\ \cdot)) * \hat{v}(\tau)|^2\,|\tau-a|^{2b}\,d\tau = \int_{-\infty}^{\infty} |D_t^b(e^{iat}\,v(t)\theta(\rho^{-1}t))|^2\,dt.$$

The Leibniz rule (7.25) yields

$$\|D_t^b(e^{iat}v\theta(\rho^{-1}\cdot)) - e^{iat}v\,D_t^b\theta(\rho^{-1}\cdot)\|_{L_t^2} \le c\,\|D_t^b(e^{iat}v)\|_{L_t^2}\|\theta\|_{L_t^\infty}.$$

Note that $\|\theta\|_{L_t^\infty} \le c$, and

$$\|D_t^b(e^{iat}v)\|_{L_t^2}^2 = \int_{-\infty}^{\infty} |\widehat{v}(\tau)|^2|\tau-a|^{2b}\,d\tau.$$

Thus, we only have to bound the term:

$$\int_{-\infty}^{\infty} |e^{iat}v D_t^b\theta(\rho^{-1}t)|^2\,dt.$$

But the Sobolev embedding theorem and the fact that $b > 1/2$ lead to

$$\Big(\int_{-\infty}^{\infty} |e^{iat}v\,D_t^b\theta(\rho^{-1}\cdot)|^2\,dt\Big)$$

$$\leq c\Big(\int_{-\infty}^{\infty} |e^{iat}\, v(t)|^2\, dt + \int_{-\infty}^{\infty} |D_t^b(e^{iat} v)|^2\, dt\Big) \big\| D_t^b \theta(\rho^{-1}\, \cdot\,)\big\|_{L_t^2}^2$$

$$\leq c\Big(\int_{-\infty}^{\infty} |\widehat{v}(\tau)|^2\, d\tau + \int_{-\infty}^{\infty} |\tau - a|^{2b} |\widehat{v}(\tau)|^2\, d\tau\Big)\, \big\| D_t^b \theta(\rho^{-1}\, \cdot\,)\big\|_{L_t^2}^2.$$

By Plancherel's identity (1.11) and since $b > 1/2$ we have

$$\| D_t^b \theta(\rho^{-1}\, \cdot\,)\|_{L_t^2}^2 = \int_{-\infty}^{\infty} |\tau|^{2b} \rho^2\, |\widehat{\theta}(\rho\tau)|^2\, d\tau \leq c\, \rho^{(1-2b)}\, \|\theta\|_{H_t^1}^2.$$

The proof of the lemma then follows. □

Lemma 7.10. *Let* $w(x,t) = \int_0^t V(t-t')h(t')\, dt'$. *If* $1/2 < b \leq 1$*; then*

$$\|\theta_\rho\, w\|_{X_{s,b}} \leq c\, \rho^{(1-2b)/2}\, \|h\|_{X_{s,b-1}}. \tag{7.76}$$

Proof. We write

$$\begin{aligned}
&\theta_\rho(t) \int_0^t V(t-t')h(t')\, dt' \\
&= \theta_\rho(t) \int\!\!\int e^{ix\xi}\, \frac{e^{it\tau} - e^{it\xi^3}}{\tau - \xi^3}\, \widehat{h}(\xi,\tau)\, d\xi\, d\tau \\
&= \theta_\rho(t) \int\!\!\int e^{ix\xi}\, \frac{e^{it\tau} - e^{it\xi^3}}{\tau - \xi^3}\, \theta(\tau - \xi^3)\, \widehat{h}(\xi,\tau)\, d\xi\, d\tau \qquad (7.77)\\
&\quad + \theta_\rho(t) \int\!\!\int e^{ix\xi}\, \frac{e^{it\tau} - e^{it\xi^3}}{\tau - \xi^3}\, (1-\theta)(\tau - \xi^3)\, \widehat{h}(\xi,\tau)\, d\xi\, d\tau \\
&\equiv I + II\, .
\end{aligned}$$

A Taylor expansion gives us

$$I = \sum_{k=1}^{\infty} \frac{i^k}{k!}\, t^k \theta_\rho(t) \int_{-\infty}^{\infty} e^{ix\xi}\, e^{it\xi^3} \Big(\int_{-\infty}^{\infty} \widehat{h}(\xi,\tau)(\tau - \xi^3)^{k-1} \theta(\tau - \xi^3)\, d\tau\Big)\, d\xi. \tag{7.78}$$

Let $t^k \theta_\rho(t) = \rho^k\, (t/\rho)^k\, \theta(\rho^{-1} t) = \varphi_k(t)$. Then,

$$\rho^2 \int_{-\infty}^{\infty} |\widehat{\varphi}_k(\rho\tau)|^2\, (1 + |\tau|)^{2b}\, d\tau$$

$$\le c\rho^2\Big(\int |\widehat{\varphi}_k(\rho\tau)|^2\,d\tau + \int |\tau|^{2b}\,|\widehat{\varphi}_k(\rho\tau)|^2\,d\tau\Big)$$
$$\le c\rho^{(1-2b)}(\|\varphi_k\|^2_{L^2_t} + \|D^b_t\,\varphi_k\|^2_{L^2_t}) \le c\rho^{(1-2b)}\,(1+k)^2.$$

Thus, by the proof of (7.74) and (7.78):

$$\|I\|_{X_{s,b}} \le \sum_{k=1}^{\infty} \frac{1+k^2}{k!}\rho^k\,\rho^{(1-2b)}\,\Big\|\Big(\int_{-\infty}^{\infty} \hat{h}(\xi,\tau)(\tau-\xi^3)^{k-1}\theta(\tau-\xi^3)\,d\tau\Big)^{\vee}\Big\|_{s,2}.$$

But

$$\Big\|\Big(\int_{-\infty}^{\infty} \widehat{h}(\xi,\tau)(\tau-\xi^3)^{k-1}\theta(\tau-\xi^3)\,d\tau\Big)^{\vee}\Big\|^2_{s,2}$$
$$\le \int_{-\infty}^{\infty} (1+|\xi|)^{2s}\Big(\int_{-\infty}^{\infty} |\hat{h}(\xi,\tau)(\tau-\xi^3)^{k-1}\theta(\tau-\xi^3)|\,d\tau\Big)^2 d\xi$$
$$\le \int_{-\infty}^{\infty} (1+|\xi|)^{2s}\Big(\int_{|\tau-\xi^3|<1} |\hat{h}(\xi,\tau)|\,d\tau\Big)^2 d\xi$$
$$\le \int_{-\infty}^{\infty} (1+|\xi|)^{2s}\Big(\int_{-\infty}^{\infty} \frac{|\widehat{h}(\xi,\tau)|}{(1+|\tau-\xi^3|)^{(1-b)}}\frac{1}{(1+|\tau-\xi^3|)^{b}}\,d\tau\Big)^2 d\xi$$
$$\le c\,\|h\|^2_{X_{s,b-1}}$$

since $b > 1/2$.

Next, we estimate II in (7.77). We rewrite it as $II = II_1 + II_2$, where

$$II_1 = -\,\theta_\rho(t)\int_{-\infty}^{\infty} e^{i(x\xi+t\xi^3)}\Big(\int_{-\infty}^{\infty} \frac{(1-\theta)(\tau-\xi^3)}{\tau-\xi^3}\,\widehat{h}(\xi,\tau)\,d\tau\Big)\,d\xi$$

$$II_2 = \theta_\rho(t)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i(x\xi+t\tau)}\,\frac{(1-\theta)(\tau-\xi^3)}{\tau-\xi^3}\,\widehat{h}(\xi,\tau)\,d\xi\,d\tau.$$

Using Lemma 7.8, the Cauchy–Schwarz inequality, and $b > 1/2$, we deduce

$$\|II_1\|_{X_{s,b}} \le c\rho^{(1-2b)/2}\,\Big\|\Big(\int_{-\infty}^{\infty} \frac{(1-\theta)(\tau-\xi^3)}{\tau-\xi^3}\,\widehat{h}(\xi,\tau)\,d\tau\Big)^{\vee}\Big\|_{s,2}$$
$$\le c\rho^{(1-2b)/2}\Big[\int_{-\infty}^{\infty} (1+|\xi|)^{2s}$$

$$\times \Big(\int\limits_{|\tau-\xi^3|\geq 1/2} \frac{1}{1+|\tau-\xi^3|} |\hat{h}(\xi,\tau)|\, d\tau \Big)^2 d\xi \Big]^{1/2}$$

$$\leq c\rho^{(1-2b)/2} \Big[\int\limits_{-\infty}^{\infty} (1+|\xi|)^{2s}$$

$$\times \Big(\int\limits_{|\tau-\xi^3|\geq 1/2} \frac{|\hat{h}(\xi,\tau)|}{(1+|\tau-\xi^3|)^{(1-b)}} \frac{1}{(1+|\tau-\xi^3|)^{b}}\, d\tau \Big)^2 d\xi \Big]^{1/2}$$

$$\leq c\rho^{(1-2b)/2} \|h\|_{X_{s,b-1}}.$$

Finally, by (7.75) and the definition of $X_{s,b-1}$,

$$\|II_2\|_{X_{s,b}} \leq c\,\rho^{(1-2b)/2} \Big\| \int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} e^{i(x\xi+t\tau)} \frac{(1-\theta)(\tau-\xi^3)}{\tau-\xi^3} \widehat{h}(\xi,\tau)\, d\xi\, d\tau \Big\|_{X_{s,b}}$$

$$\leq c\,\rho^{(1-2b)/2} \|h\|_{X_{s,b-1}}.$$

This completes the proof of the lemma. □

Lemma 7.11.

$$\|\theta_\rho(t) \int\limits_0^t V(t-t')h(t')\, dt'\|_{s,2} \leq c\rho^{(1-2b)/2} \|h\|_{X_{s,b-1}}. \tag{7.79}$$

Proof. A similar argument as the one used to show Lemma 7.10 yields (7.79). Thus, we omit it. □

Lemma 7.12. *Let $s \in \mathbb{R}$, $b', b \in (1/2, 7/8)$ with $b < b'$ and $\rho \in (0,1)$, then for $v \in X_{s,b'-1}$, we have*

$$\|\theta_\rho\, v\|_{X_{s,b-1}} \leq c\,\rho^{(b'-b)/8(1-b)} \|v\|_{X_{s,b'-1}}. \tag{7.80}$$

Proof. To prove (7.80), we use duality and prove the estimate

$$\|\theta_\rho\, v\|_{X_{-s,1-b'}} \leq c\,\rho^{(b'-b)/8(1-b)} \|v\|_{X_{-s,1-b}}. \tag{7.81}$$

This result follows by interpolation. To do so, we need to establish the next inequalities:

$$\|\theta_\rho\, v\|_{X_{-s,0}} \leq c\,\rho^{1/8} \|v\|_{X_{-s,1-b}} \tag{7.82}$$

and

$$\|\theta_\rho v\|_{X_{-s,1-b}} \leq c\,\|v\|_{X_{-s,1-b}}. \tag{7.83}$$

Combining Remark 7.8, the Hölder inequality, and the Sobolev inequality (Theorem 3.3), we have

$$\begin{aligned}\|\theta_\rho v\|_{X_{-s,0}} &= \|J^{-s}V(t)(\theta(\rho^{-1}\cdot)v)\|_{L^2_tL^2_x} = \|V(t)\theta(\rho^{-1}\cdot)\,J^{-s}v\|_{L^2_tL^2_x}\\ &= \|\theta(\rho^{-1}\cdot)V(t)\,J^{-s}v\|_{L^2_tL^2_x} \le c\rho^{1/8}\,\|V(t)\,J^{-s}v\|_{L^2_xL^{8/3}_t}\\ &\le c\rho^{1/8}\,\|V(t)\,J^{-s}v\|_{L^2_xH^{1/8}_t} = c\rho^{1/8}\,\|v\|_{X_{-s,1/8}}\\ &\le c\rho^{1/8}\,\|v\|_{X_{-s,1-b}},\end{aligned}$$

where we use that $1-b > 1/8$. This shows (7.82).

On the other hand, to prove (7.83) we use a similar argument to the one applied in the proof of Lemma 7.9. Since $(\theta_\rho(t)\,v(x,t))^\wedge = \widehat{\theta_\rho} *_\tau \widehat{v}$, by the definition of the $X_{s,b}$-space, the proof reduces to showing that, for $a \in \mathbb{R}$,

$$\int_{-\infty}^{\infty} |\widehat{\theta_\rho} *_t \widehat{v}|^2(1+|\tau-a|)^{2(1-b)}d\tau \le c\int_{-\infty}^{\infty} |\widehat{v}|^2(1+|\tau-a|)^{2(1-b)}d\tau. \tag{7.84}$$

Since $\|\rho\,\widehat{\theta}(\rho\ \cdot)\|_{L^1_\tau} < \infty$, we have that

$$\int_{-\infty}^{\infty} |\widehat{\theta_\rho} *_t \widehat{v}|^2\,d\tau \le c\int_{-\infty}^{\infty} |\widehat{v}|^2\,d\tau.$$

Next, we estimate

$$\int_{-\infty}^{\infty} |\widehat{\theta_\rho} *_t \widehat{v}|^2\,|\tau-a|^{2(1-b)}\,d\tau = \int_{-\infty}^{\infty} |D_t^{1-b}(e^{iat}v(t)\,\theta(\rho^{-1}t))|^2\,dt.$$

Using the Leibniz rule (7.25) we have that

$$\|D_t^{1-b}(e^{iat}v\theta_\rho) - e^{iat}vD_t^{1-b}\theta_\rho\|_{L^2_t} \le c\,\|D_t^{1-b}(e^{iat}v)\|_{L^2_t}\|\theta_\rho\|_{L^\infty_t}. \tag{7.85}$$

The first term on the right-hand side of (7.85) can be estimated as follows. We first notice that $\|\theta_\rho\|_{L^\infty} < \infty$. Thus, Plancherel identity (1.11) gives us

$$\|D_t^{1-b}(e^{iat}v)\|_{L^2_t} = \Big(\int_{-\infty}^{\infty} |\widehat{v}(\tau)|^2|\tau-a|^{2(1-b)}\,d\tau\Big)^{1/2}. \tag{7.86}$$

To bound $\|e^{iat}v\,D_t^{1-b}\theta_\rho\|_{L^2_t}$, we use the Hölder inequality to obtain

$$\|e^{iat}v\,D_t^{1-b}\theta_\rho\|_{L^2_t} \le \|e^{iat}v\|_{L^{2p}_t}\|\,D_t^{1-b}\theta_\rho\|_{L^{2q}_t},$$

with $1/p + 1/q = 1$. Then, we choose p such that $1/2 - 1/2p = 1-b$. Using the Sobolev inequality (Theorem 3.3), we have

$$\|e^{iat}v\|_{L^{2p}_t} \le \|e^{iat}v\|_{H^{1-b}_t} = c\Big(\int_{-\infty}^{\infty} (1+|\tau-a|)^{2(1-b)}|\widehat{v}(\tau)|^2\,d\tau\Big)^{1/2}. \tag{7.87}$$

Since the inverse Fourier transform is bounded from $L^{\frac{2q}{(2q-1)}}(\mathbb{R})$ into $L^{2q}(\mathbb{R})$, we have

$$\begin{aligned} \| D_t^{1-b}\theta_\rho\|_{L_t^{2q}} &\le \left(\int_{-\infty}^{\infty} ||\tau|^{1-b}\rho\,\widehat{\theta}(\rho\tau)|^{\frac{2q}{2q-1}}\,d\tau\right)^{\frac{2q-1}{2q}} \\ &= \left(\int_{-\infty}^{\infty} ||\tau|^{1-b}\widehat{\theta}(\tau)|^{\frac{2q}{2q-1}}\,d\tau\right)^{\frac{2q-1}{2q}} < \infty. \end{aligned} \tag{7.88}$$

Combining (7.87) and (7.88), we have

$$\|e^{iat}v\,D_t^{1-b}\theta_\rho\|_{L_t^2} \le c\int_{-\infty}^{\infty}(1+|\tau-a|)^{2(1-b)}|\widehat{v}(\tau)|^2\,d\tau. \tag{7.89}$$

Thus, (7.86) and (7.89) yield (7.84).

The estimates (7.82) and (7.83) and interpolation yield the inequality (7.81). Thus, the lemma follows. □

The next estimate is the key argument to obtain the local result for the IVP (7.71). Notice that when we estimate the $X_{s,b}$-norm of the integral part in Lemma 7.10, we end up in the space $X_{s,b-1}$, we have lost "one derivative," so to apply a contraction mapping argument we need to have an estimate that takes the nonlinear part back to the space $X_{s,b}$.

Lemma 7.13.

1. *If $v \in X_{s,b}$, $s > -3/4$, there exists $b > 1/2$ such that $v\partial_x v \in X_{s,b-1}$ and*

$$\|\partial_x(v^2)\|_{X_{s,b-1}} \le c\,\|v\|^2_{X_{s,b}}.$$

2. *Given $s \le -3/4$ the estimate above fails for any b.*

We restate Lemma 7.13 in an equivalent form:

For $v \in X_{s,b}$, let $f(\xi,\tau) = \hat{v}(\xi,\tau)(1+|\xi|)^s(1+|\tau-\xi^3|)^b$, so that $\|v\|_{X_{s,b}} = \|f\|_2$. In terms of f we can express $v\partial_x v$ in the following way:

$$\begin{aligned} \widehat{\partial_x(v^2)}(\xi,\tau) &= i\xi\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\xi_1,\tau_1)\,f(\xi-\xi_1,\tau-\tau_1) \\ &\times \frac{d\xi_1\,d\tau_1}{(1+|\xi_1|)^s(1+|\tau_1-\xi_1^3|)^b(1+|\xi-\xi_1|)^s(1+|(\tau-\tau_1)-(\xi-\xi_1)^3|)^b}. \end{aligned}$$

Thus, if we let

$$\begin{aligned} B(f,f,s,b) &= \frac{(1+|\xi|)^s}{(1+|\tau-\xi^3|^{1-b}}|\xi| \\ &\times \int_{\mathbb{R}^2} K(\xi,\xi_1,\tau,\tau_1)f(\xi_1,\tau_1)f(\xi-\xi_1,\tau-\tau_1)\,d\xi_1\,d\tau_1, \end{aligned} \tag{7.90}$$

where

$$K(\xi, \xi_1, \tau, \tau_1) = \frac{(1+|\xi_1|)^{-s}(1+|\xi-\xi_1|)^{-s}}{(1+|\tau_1-\xi_1^3|)^b(1+|(\tau-\tau_1)-(\xi-\xi_1)^3|)^b},$$

Lemma 7.13 is equivalent to proving the next result for the bilinear operator $B(\cdot,\cdot)$ defined in (7.90).

Lemma 7.14.

1. *If* $s > -3/4$, *then*

$$\|B(f, f, s, b)\|_2 \le c\,\|f\|_2^2. \tag{7.91}$$

2. *If* $s \le -3/4$, *the above estimate fails for each* b.

We prove (7.91) in detail for $s = 0$. For this purpose, we need some lemmas. The first one is regarding some elementary inequalities.

Lemma 7.15. *If* $b > 1/2$, *there exists* $c > 0$ *such that*

$$\int_{-\infty}^{\infty} \frac{dx}{(1+|x-\alpha|)^{2b}(1+|x-\beta|)^{2b}} \le \frac{c}{(1+|\alpha-\beta|)^{2b}}, \tag{7.92}$$

$$\int_{-\infty}^{\infty} \frac{dx}{(1+|x|)^{2b}\sqrt{|a-x|}} \le \frac{c}{(1+|a|)^{1/2}}. \tag{7.93}$$

Lemma 7.16. *Let*

$$G(\xi,\tau) = \frac{|\xi|}{(1+|\tau-\xi^3|)^{1-b}} \times \left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{d\xi_1 d\tau_1}{(1+|\tau_1-\xi_1^3|)^{2b}(1+|\tau-\tau_1-(\xi-\xi_1)^3|)^{2b}}\right)^{1/2}. \tag{7.94}$$

If $1/2 < b \le 3/4$, *then*

$$|G(\xi,\tau)| \le c.$$

Proof. Let us set $\alpha = \xi_1^3$ and $\beta = \tau - (\xi-\xi_1)^3$ in (7.94). Then by (7.92), we have

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{d\xi_1 d\tau_1}{(1+|\tau_1-\xi_1^3|)^{2b}(1+|\tau-\tau_1-(\xi-\xi_1)^3|)^{2b}} \le \int_{-\infty}^{\infty} \frac{d\xi_1}{(1+|\tau-(\xi-\xi_1)^3-\xi_1^3|)^{2b}}.$$

Next, we use the change of variable

$$\mu = \tau - (\xi - \xi_1)^3 - \xi_1^3 = \tau - \xi^3 + 3\xi\xi_1(\xi - \xi_1), \qquad d\mu = 3\xi(\xi - 2\xi_1)\, d\xi_1,$$

and

$$\xi_1 = \frac{1}{2}\left\{\xi \pm \sqrt{\frac{4\tau - \xi^3 - 4\mu}{3\xi}}\right\}.$$

Thus,

$$|\xi(\xi - 2\xi_1)| \simeq \sqrt{|\xi|}\,\sqrt{4\tau - \xi^3 - 4\mu}$$

and

$$d\xi_1 \simeq \frac{d\mu}{\sqrt{|\xi|}\,\sqrt{4\tau - \xi^3 - 4\mu}}.$$

Substituting this on the right-hand side of the previous inequality and using (7.93), we obtain

$$\frac{1}{\sqrt{|\xi|}} \int_{-\infty}^{\infty} \frac{d\mu}{(1 + |\mu|)^{2b}\sqrt{4\tau - \xi^3 - 4\mu}} \le \frac{1}{\sqrt{|\xi|}} \frac{c}{(1 + |4\tau - \xi^3|)^{1/2}}.$$

Hence, using the hypotheses we conclude that

$$\begin{aligned} |G(\xi, \tau)| &\le \frac{|\xi|}{(1 + |\tau - \xi^3|)^{1-b}} \frac{1}{|\xi|^{1/4}} \frac{c}{(1 + |4\tau - \xi^3|)^{1/4}} \\ &\le \frac{c\,|\xi|^{3/4}}{(1 + |\tau - \xi^3|)^{1-b}(1 + |4\tau - \xi^3|)^{1/4}} \le c. \end{aligned}$$

Thus, the lemma follows. □

Proof of Lemma 7.14 We will prove 1. in the case $s = 0$. Definition (7.90), the Cauchy–Schwarz inequality, Lemma 7.16, Fubinni's theorem, and Young's inequality yield

$$\begin{aligned} \|B(f, f, 0, b)\|_{L^2_\tau L^2_\xi} &= \Big\| \frac{|\xi|}{1 + |\tau - \xi^3|^{1-b}} \\ &\times \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{f(\xi_1, \tau_1)\, f(\xi - \xi_1, \tau - \tau_1)\, d\xi_1\, d\tau_1}{(1 + |\tau_1 - \xi_1^3|)^b (1 + |(\tau - \tau_1) - (\xi - \xi_1)^3|)^b} \Big\|_{L^2_\tau L^2_\xi} \\ &\le \Big\| \frac{|\xi|}{(1 + |\tau - \xi^3|)^{1-b}} \\ &\times \Big(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{d\xi_1 d\tau_1}{(1 + |\tau_1 - \xi_1^3|)^{2b}(1 + |(\tau - \tau_1) - (\xi - \xi_1)^3|)^{2b}}\Big)^{1/2} \Big\|_{L^\infty_\tau L^\infty_\xi} \\ &\quad \times \Big\| \Big(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |f(\xi_1, \tau_1)|^2 |f(\xi - \xi_1, \tau - \tau_1)|^2\, d\xi_1 d\tau_1\Big)^{1/2} \Big\|_{L^2_\tau L^2_\xi} \end{aligned}$$

$$\leq c \, \|f\|^2_{L^2_\tau L^2_\xi}.$$

This proves the lemma. □

As a corollary, we have the next result.

Corollary 7.4. *For $s > -3/4$ and $b \in (1/2, 3/4]$ and $b' \in (1/2, b]$ we have that*

$$\|B(f, f)\|_{X_{s,b-1}} \leq c \, \|f\|^2_{X_{s,b'}}. \tag{7.95}$$

Now, we are in position to establish the LWP result for the IVP (7.71). More precisely, we have the following.

Theorem 7.8. *Let $s \in (-\,3/4, 0]$. Then, there exists $b \in (1/2, 1)$ such that for any $v_0 \in H^s(\mathbb{R})$ there exist $T = T(\|v_0\|_{s,2})$ with $(T(\rho) \to \infty$ as $\rho \to 0)$ and a unique solution $v(t)$ of the IVP (7.71) in the time interval $[-T, T]$ satisfying*

$$v \in C([-T, T] : H^s(\mathbb{R})), \tag{7.96}$$

$$v \in X_{s,b} \subset L^p_{x,\mathrm{loc}}(\mathbb{R} : L^2_t(\mathbb{R})) \quad \text{for } 1 \leq p \leq \infty, \tag{7.97}$$

$$\partial_x(v^2) \in X_{s,b-1} \tag{7.98}$$

and

$$\partial_t v \in X_{s-3,b-1}. \tag{7.99}$$

Moreover, for any $T' \in (0, T)$ there exists $R = R(T') > 0$ such that the map $\tilde{v}_0 \mapsto \tilde{v}(t)$ from $\{\tilde{v}_0 \in H^s(\mathbb{R}) : \|v_0 - \tilde{v}_0\|_{s,2} < R\}$ into the class defined by (7.96)–(7.99) with T' instead of T is smooth.

In addition, if $v_0 \in H^{s'}(\mathbb{R})$ with $s' > s$, the previous results hold with s' instead of s in the same time interval $[-T, T]$.

Proof. We define

$$\mathcal{X}_a = \{v \in X_{s,b} \ : \ \|v\|_{X_{s,b}} < a\}. \tag{7.100}$$

For $v_0 \in H^s(\mathbb{R})$, $s > -3/4$, we define the operator:

$$\Psi_{v_0}(v) = \Psi(v) = \theta_1(t)V(t)v_0 - \frac{\theta_1(t)}{2} \int_0^t V(t - t')\,\theta_\rho(t')\,\partial_x(v^2(t'))\,dt'. \tag{7.101}$$

We see that $\Psi(\cdot)$ defines a contraction on $\mathcal{X}_a$.

Let $\beta = (b - b')/8(1 - b')$. By using (7.74), (7.75), (7.76), and (7.91) in Lemma 7.13 we deduce that

$$\begin{aligned}\|\Psi(v)\|_{X_{s,b}} &\leq c\|v_0\|_{s,2} + c\|\theta_\rho(t)\partial_x v^2(\cdot, t))\|_{X_{s,b-1}} \\ &\leq c\|v_0\|_{s,2} + c\,\rho^\beta\,\|\partial_x v^2(\cdot, t))\|_{X_{s,b'-1}}\end{aligned}$$

$$\begin{aligned} &\le c\|v_0\|_{s,2} + c\,\rho^{\beta}\,\|v\|^2_{X_{s,b}} \\ &\le c\|v_0\|_{s,2} + c\,\rho^{\beta}\,a^2. \end{aligned} \tag{7.102}$$

Setting $a = 2c\|v_0\|_{H^s}$ and ρ such that

$$c\,\rho^{\beta}\,a < \frac{1}{2}$$

we have

$$\|\Psi(v)\|_{X_{s,b}} \le a.$$

A similar argument shows that for $v, \tilde{v} \in \mathcal{X}_a$

$$\begin{aligned} \|\Psi(v) - \Psi(\tilde{v})\|_{X_{s,b}} &= \frac{1}{2}\left\|\theta_1(t)\int_0^t V(t-t')\,\theta_\rho(t')\,\partial_x(v^2 - \tilde{v}^2)(t')\,dt'\right\|_{X_{s,b}} \\ &\le c\rho^{\beta}\,\|v+\tilde{v}\|_{X_{s,b}}\,\|v-\tilde{v}\|_{X_{s,b}} \\ &\le 2\,c\rho^{\beta}\,a\,\|v-\tilde{v}\|_{X_{s,b}} \\ &\le \frac{1}{2}\,\|v-\tilde{v}\|_{X_{s,b}}. \end{aligned}$$

Therefore, $\Psi(\cdot)$ is a contraction from $\mathcal{X}_a$ into itself and we obtain a unique fixed point that solves the equation for $T < \rho$, i.e.,

$$v(t) = \theta_1(t)V(t)v_0 - \frac{\theta_1(t)}{2}\int_0^t V(t-t')\,\theta_\rho(t')\,\partial_x(v^2(t'))\,dt'. \tag{7.103}$$

The additional regularity

$$v \in C([0,T] : H^s(\mathbb{R}))$$

is proved as follows:

Using the integral equation (7.103), Lemmas 7.11, and 7.12, for $0 \le \tilde{t} < t \le 1$ and $t - \tilde{t} \le \Delta t$ it follows that

$$\begin{aligned} \|v(t) - v(\tilde{t})\|_{s,2} &\le \|V(t-\tilde{t})v(\tilde{t}) - v(\tilde{t})\|_{s,2} \\ &\quad + c\left\|\int_{\tilde{t}}^{t} V(t-t')\,\theta^2\Big(\frac{t'-\tilde{t}}{\Delta t}\Big)\,\partial_x(v^2(t'))\,dt'\right\|_{s,2} \\ &\le \|V(t-\tilde{t})v(\tilde{t}) - v(\tilde{t})\|_{s,2} + c\,\Big\|\theta\Big(\frac{\cdot-\tilde{t}}{\Delta t}\Big)\partial_x\,v^2\Big\|_{X_{s,b-1}} \qquad (7.104) \\ &\le \|V(t-\tilde{t})v(\tilde{t}) - v(\tilde{t})\|_{s,2} + c\,(\Delta t)^{\frac{(b-b')}{8(1-b')}}\,\|\partial_x\,v^2\|_{X_{s,b'-1}} \\ &\le \|V(t-\tilde{t})v(\tilde{t}) - v(\tilde{t})\|_{s,2} + c\,(\Delta t)^{\frac{(b-b')}{8(1-b')}}\,\|v\|^2_{X_{s,b'}} = o(1) \end{aligned}$$

as $\Delta t \to 0$. This yields the persistence property. □

7.5 Comments

The well-posedness of the k-gKdV equation has been studied extensively for many years. Improving results in [BS], [BSc], [ST], it was shown in [K5] that the IVP (7.10) is locally well-posed in $H^s(\mathbb{R})$, $s > 3/2$. However, as Kato remarked in [K2], "In fact, local well-posedness has almost nothing to do with the special structure of the KdV equation." In other words, the local result in $H^s(\mathbb{R})$, $s > 3/2$, does not distinguish the powers k and works for any skew-symmetric operator (instead of ∂_x^3) or omitting it (hyperbolic case).

For the study of the stability of solitary wave solutions of the k-gKdV equation, it was important to have LWP in Sobolev spaces $H^s(\mathbb{R})$ with $s \leq 1$, specially for the case $k = 3$.

For the KdV equation the L^2 LWP was established by Bourgain [Bo1]. The proof given here was taken from [KPV6], where LWP was obtained for data in $H^s(\mathbb{R})$, $s > -3/4$ (Lemmas 7.14 and 7.15 were proved in [KPV6] and by Nakanishi, Takaoka and Tsutsumi [NTT1] in the case $s = -3/4$).

Extensions of the bilinear estimates (Lemma 7.14 (1)) were first obtained by Colliander, Staffilani and Takaoka [CST], motivated by the study of global well-posedness below the L^2-norm for the KdV equation. A further extension was given by Tao [To3].

The well-posedness in the limiting case $s = -\frac{3}{4}$ was established in [Ki1] [Gu] using a Besov-like generalization of the $X_{s,b}$ spaces with $(s,b) = (-3/4, 1/2)$ in the low frequency (see also [BTo]).

It is interesting to compare this LWP result for the KdV with those for the viscous Burgers' equation:

$$\begin{cases} \partial_t u = \partial_x^2 u + u\partial_x u, \\ u(x,0) = u_0(x) \in H^s(\mathbb{R}). \end{cases} \tag{7.105}$$

In [Dx], Dix showed that (7.105) is locally well-posed in $H^s(\mathbb{R})$, $s \geq -1/2$ (scaling $s = -1/2$) and uniqueness fails for $s < -1/2$ (a construction based in the Cole–Hopf transformation (see Exercise 9.18). Therefore, from the LWP point of view the KdV equation is better than the viscous Burgers' equation.

The proof of the LWP result for the mKdV was taken from [KPV4]. The estimate (7.13) is the sharp version of the Kato smoothing effect. It was already commented on at the end of Chapter 4 (see (4.54)–(4.67)) which was used to obtain weak L^2 solutions for the KdV equation.

The estimate (7.18) regarding the continuity of the maximal function associated to the group $V(t)$, i.e., $\sup_{t\in[0,T]} |V(t)v_0|$, is due to Kenig and Ruiz [KR] and was obtained in their study of the problem mentioned in Chapter 4 (see (4.57)).

It was shown in [KPV5] that the result $s \geq 1/4$ is optimal, i.e., the map data-solution $v_0 \mapsto v(t)$ cannot be uniformly continuous in $H^s(\mathbb{R})$ for $s < 1/4$. The proof of this assertion follows a close argument to the one provided in Chapter 5 for the cubic (focusing) NLS equation in one dimension. There it was constructed a two-parameter family of solutions for the cubic (focusing) NLS by combining the Galilean and the scaling invariance of the solutions. However, the mKdV equation is not Galilean invariant. So to overcome this, one first considers the complex version of the mKdV equation, namely,

$$\partial_t w + \partial_x^3 w + |w|^2 \partial_x w = 0 \tag{7.106}$$

(see [GO], [KSC]), which has a set of solutions that is Galilean invariant. In fact, we have the two-parameter family

$$w_{N,\omega}(x,t) = \sqrt{3}\, e^{-it(3N\omega^2 - N^3)} e^{ixN} \varphi_\omega(x - t\omega^2 + 3tN^2), \tag{7.107}$$

where φ solves (5.8) (i.e., $-\varphi + \varphi'' + \varphi^3 = 0$ so $\varphi(x) = \sqrt{2}\,\mathrm{sech}(x)$); and $\varphi_\omega(x) = \omega\varphi(\omega x)$ (notice that $\sqrt{3}\,\varphi_\omega(x - t\omega^2)$ solves 7.106). With the two-parameter family, we follow an argument similar to the one employed in Theorem 5.12 to obtain the result for equation (7.106). To pass to the mKdV equation, one uses a special solution called a "breather," see [Wa],

$$\begin{aligned} v_{N,\omega}(x,t) = &-2\sqrt{6}\,\omega\,\mathrm{sech}(\omega x + \gamma t) \\ &\times \left(\frac{\cos(Nx + \delta t) - (\omega/N)\sin(Nx + \delta t)\tanh(\omega x + \gamma t)}{1 + (\omega/N)^2 \sin^2(Nx + \delta t)\,\mathrm{sech}(\omega x + \gamma t)} \right) \end{aligned} \tag{7.108}$$

with $\delta = N(N^2 - 3\omega^2)$ and $\gamma = \omega(3N^2 - \omega^2)$ and observe that for $\omega/N \ll 1$,

$$\begin{aligned} v_{N,\omega}(x,t) \approx &-2\sqrt{6}\,\cos(Nx + N(N^2 - 3\omega^2)t) \\ &\times \omega\,\mathrm{sech}(\omega x + \omega(3N^2 - \omega^2)t), \end{aligned}$$

which is basically a multiple of the real part of (7.107).

As it was remarked above, Bourgain introduced the spaces $X_{s,b}$ in the context of dispersive equations. Previously they were used by Rauch and Reed [RuR] and Beals [Bs] in their respective studies of propagation of singularities for solutions of semilinear wave equation.

In the same spirit that [KTa2], [CrCT2] and [CrHoT] a priori estimates were established for solutions of the modified KdV below the Sobolev index $1/4$ which guarantees the well-posedness. More precisely, it was shown in [CrHoT] that solutions of the IVP associated to the mKdV satisfies for $s \in (-1/4, -1/8)$,

$$\sup_{[0,T]} \|u(t)\|_{s,2} \le c\,(T;\ \|u_0\|_{s,2}).$$

This result does not give control on the difference of two solutions (uniqueness).

In a similar regard, in [BuKo] an a priori estimate in $H^{-1}(\mathbb{R})$ for smooth solutions of the KdV equation was obtained. More precisely, in [BuKo] the following result was established: if $v \in C([0,\infty) : H^s(\mathbb{R}))$, with $s \ge -3/4$, is a solution of the IVP with $k = 1$, then

$$\|v(\cdot,t)\|_{-1,2} \le c\,(\|v_0\|_{-1,2} + \|v_0\|_{-1,2}^3), \quad \text{for any } t \ge 0.$$

This allows to construct global H^{-1}-weak solutions of the associated IVP. In [Mo2], it was shown that the map data-solution associated to the IVP for the KdV equation cannot be continuously extended in $H^s(\mathbb{R})$ for $s < -1$.

In [FaPa], modified proofs of Theorems 7.4, 7.5, and 7.6 and Corollary 7.2 were obtained which simplify the argument by not using norm involving time derivatives D_t of the unknown function.

The LWP result for the case $k = 3$ for data in $H^s(\mathbb{R})$, $s > -1/6$, was obtained by Grünrock [Gr2]. The key tool to prove that result was the following bilinear estimate for solutions of the linear problem. More precisely,

$$\|D_x(V(t)f \cdot V(t)g)\|_{L^2_x L^2_t} \le c\,\|f\|_2 \|g\|_2.$$

Tao [To6] extended Grünrock's result to the critical case by showing LWP for data in $\dot{H}^{-1/6}(\mathbb{R})$. From this result, it follows readily the global one for small data due to the criticality of the space. Thus, the case $k = 3$ exhibits similar properties to the case $k \ge 4$; see Theorems 7.2 and 7.3 and the Remarks 7.15 and 7.16.

The results for $k \ge 4$ were taken from [KPV4] (Theorems 7.2–7.6). Their sharpness was established in [BKPSV] (Theorem 7.7).

In [GrV], LWP was established in the spaces $\widehat{H^s_r}(\mathbb{R})$ (see (5.75)) for the parameters $r \in (1,2)$ and $s \ge 1/2 - 1/2r$.

Results concerning the smoothing effects of solutions of (7.10) due to special decay of the data were first given by Cohen [Co1] and Cohen and Kappeler [CoK] for the KdV equation for step data using the inverse scattering theory.

In [K2], Kato studied the IVP (7.1) in weighted Sobolev spaces and showed that if

$$v_0 \in \mathcal{F}^s_{2k} = H^s(\mathbb{R}) \cap L^2(|x|^{2k}\,dx), \quad k \in \mathbb{Z}^+, \quad s \ge 2k, \tag{7.109}$$

then the local solution describes a continuous curve on $\mathcal{F}^s_{2k}$ as far as it exists. In particular, the solution flow preserves the Schwartz class $\mathcal{S}(\mathbb{R})$. Roughly, this is due to the fact that the operators $L = \partial_t + \partial_x^3$ and $\Gamma = x - 3t\,\partial_x^2$ commute. The extension of this result to solutions of (7.1) with data:

$$v_0 \in \mathcal{F}^s_{2l} \text{ with } s \ge \max\{2l;\ s_k\},\ l > 0, \tag{7.110}$$

$s_k = -3/4$ if $k = 1$, $s_k = 1/2$ if $k = 2$, and $s_k = 1/2 - 2/k$ was established in [Nh] and [FLP4]. In [ILP1], it was shown that this persistence result in $\mathcal{F}^s_l$ with $s \ge l > 0$ is optimal. More precisely, if $u \in C([-T,T] : H^s(\mathbb{R})) \cap \dots$, with $s \ge \max\{s_k;\ 0\}$, is a solution of the IVP (7.1) and there exist $t_1, t_2 \in [-T,T]$, $t_1 \ne t_2$ such that $|x|^\alpha u(\cdot,t_j) \in L^2(\mathbb{R})$ for $2\alpha > s$, then $u \in C([-T,T] : H^{2\alpha}(\mathbb{R}))$. In other words, persistence in $L^2(|x|^{2\alpha}\,dx)$ (decay) can only hold for solutions in $C([-T,T] : H^{2\alpha}(\mathbb{R}))$.

Also in [K2], Kato proved the following result for the KdV equation (which also holds for solutions of (7.1) with $k \in \mathbb{Z}^+$): if $v \in C([0,T] : H^2(\mathbb{R})) \cap \dots$ is a solution of (7.1) with $v_0 \in H^2(\mathbb{R}) \cap L^2(e^{2\beta x}dx)$, $\beta > 0$, then

$$e^{\beta x} v \in C([0,T] : L^2(\mathbb{R})) \cap C((0,T) : H^\infty(\mathbb{R})). \tag{7.111}$$

Formally, one has that the semigroup $\{V(t) = e^{-t\partial_x^3} : t \geq 0\}$ in $L^2(e^{\beta x}dx)$ is equivalent to $\{e^{-t(\partial_x-\beta)^3} : t \geq 0\}$ in $L^2(\mathbb{R})$, i.e., if

$$\partial_t u + \partial_x^3 u = 0,$$

then $v(x,t) = e^{\beta x}u(x,t)$ satisfies

$$\partial_t v + (\partial_x - \beta)^3 v = \partial_t v + \partial_x^3 v - 3\beta\partial_x^2 v + 3\beta^2\partial_x v - \beta^3 v = 0, \tag{7.112}$$

which explain this "parabolic effect." In [KF], Kruskov and Faminskii obtained another version of this result by considering weights of the form $x_+^\alpha = x^\alpha\, \chi_{(0,\infty)}(x)$.

Tarama [Ta2] showed that solutions of the KdV equation with real-valued initial data $v_0(x) \in L^2(\mathbb{R})$ satisfying the condition:

$$\int_{-\infty}^{\infty} e^{\delta|x|^{1/2}}|v_0(x)|^2 dx < \infty$$

with some positive constant δ, become analytic with respect to the variable x for all $t > 0$. The proof of this theorem is based on the inverse scattering method (see Section 9.6), which transforms the KdV equation into a linear dispersive equation for which the analyticity smoothing effect can be established through the analytic properties of the fundamental solutions. However, for higher powers a similar result is unknown even for the integrable case $k = 2$, i.e., for the mKdV.

In [GT], Ginibre and Tsutsumi proved for the mKdV that if $v_0 \in L^2(\mathbb{R})$ and $x_+^{1/8}\, v_0 \in L^2(\mathbb{R})$, then the uniqueness holds (observe that decay corresponds to 1/4 derivatives via the operator Γ above which is the sharp LWP). In the KdV case, this result improves by a factor of 2 the one obtained in [KF].

In (7.111), we have seen that persistence property holds in $L^2(w(x)\,dx)$, $w(x) = e^{\beta x}$, $\beta > 0$ for $t > 0$. The following unique continuation result found in [EKPV3] gives an upper bound on the weight $w(x)$ for which this property remains: there exists $c_k > 0$ such that if

$$v_1, v_2 \in C([0,1] : H^3(\mathbb{R}) \cap L^2(|x|^2\,dx))$$

such that

$$(v_1 - v_2)(\cdot, 0),\ (v_1 - v_2)(\cdot, 1) \in L^2(e^{c_k\, x_+^{3/2}}\,dx), \quad x_+ = \max\{x; 0\}. \tag{7.113}$$

Then,

$$v_1 \equiv v_2 \text{ on } \mathbb{R} \times [0,1].$$

Notice that taking $v_2 \equiv 0$ it follows that persistence in $L^2(w(x)\,dx)$ with $w(x) = e^{c_k\, x_+^{3/2}}$, c_k arbitrarily large cannot hold in the interval $[0,1]$ for a nonnull solution.

In [ILP1], it was shown that (7.113) is optimal. More precisely:

If $v_0 \in H^1(\mathbb{R}) \cap L^2(e^{a_0 x_+^{3/2}}\,dx)$, $a_0 > 0$, then $v = v(x,t)$ the solution of the IVP (7.1) defined in the time interval $[0,T]$ satisfies

$$\sup_{0\le t\le T} \int_{-\infty}^{\infty} e^{a(t)\,x_+^{3/2}}\,|v(x,t)|^2\,dx \le c = c(a_0; \|v_0\|_{1,2}; \|e^{\frac{a_0}{2}\,x_+^{3/2}}\,v_0\|_2; T) \tag{7.114}$$

with

$$a(t) = \frac{a_0}{(1 + 27a_0^2\,t/4)^{1/2}}. \tag{7.115}$$

In other words, the initial decay of $v_0 \in L^2(e^{a_0 x_+^{3/2}}\,dx)$ remains in time but with a constant $a(t)$ decreasing in t as in (7.115).

Concerning the propagation of asymmetric regularity in solutions of the IVP (7.1), one has the following result established in [ILP2]: Let $v \in C([-T,T] : H^{3/4^+}(\mathbb{R})) \cap \ldots$ be a solution of the IVP (7.1). If for some $l \ge 1$

$$\int_0^{\infty} |\partial_x^l v_0(x)|^2\,dx < \infty, \tag{7.116}$$

then for any $\varepsilon > 0$ and any $b > 0$

$$\sup_{0<t<T} \int_{x>\varepsilon - bt} |\partial_x^l v(x,t)|^2\,dx < \infty. \tag{7.117}$$

Roughly speaking, this tells us that the regularity in the right-hand side of the data v_0 propagates with infinite speed to its left as time evolves.

Next, we consider the LWP for the periodic boundary value problem associated to the k-gKdV.

For the case $k = 1$, the KdV equation, local well-posedness was proven in $H^s(\mathbb{T})$, $s \ge -1/2$ by Kenig, Ponce and Vega [KPV6] (improving an earlier result of Bourgain [Bo1] for $s \ge 0$). The proofs are based on the modified version of the $X_{s,b}$ spaces, i.e., the spaces $Y_{s,b}$, which are the completion under the norm:

$$\|f\|_{Y_{s,b}} = \left(\sum_{m\ne 0} \int_{-\infty}^{\infty} (1 + |\tau - m^3|)^{2b}\,|m|^{2s}|\widehat{f}(m,\tau)|^2\,d\tau \right)^{1/2} \tag{7.118}$$

of the space $\mathcal{Y}$ defined as the function space of all f such that

(i) $f : \mathbb{T} \times \mathbb{R} \to \mathbb{C}$,
(ii) $f(x,\cdot) \in \mathcal{S}$ for each $x \in \mathbb{T}$,
(iii) $x \mapsto f(x,\cdot)$ is C^∞,
(iv) $\widehat{f}(0,\tau) = 0$ for all $\tau \in \mathbb{R}$.

Bourgain [Bo8] also showed that below $-1/2$ (for $s < -1/2$) the smoothness of the map data-solution fails. We recall that the smoothness of this map is a by-product of the contraction principle. So this type of result in particular shows that the iteration process by itself does not provide a result in $H^s(\mathbb{T})$, $s < -1/2$. In this regard, using the inverse scattering method Kappeler and Topalov [KpTo] showed that the solution flow of the KdV extends continuously to $H^{-1}(\mathbb{T})$.

For the mKdV equation ($k = 2$), LWP was established in [KPV6] in $H^s(\mathbb{T})$, $s \geq 1/2$. This was proven to be sharp in [CrCT1]. By requiring the dependence of solutions on the initial data be just continuous and considering real solutions, Takaoka and Tsutsumi [TTs] were able to lower the Sobolev index $s > 1/2$ to $s > 3/8$. One of the main new ideas in their approach was the modification of the Bourgain norm (7.118) by

$$\|f\|_{Z_{s,b}} = \Big(\sum_{m\neq 0}\int_{-\infty}^{\infty} (1 + |\tau - m^3 - m|\widehat{u}_0(m)|^2|)^{2b}\,|m|^{2s}\,|\widehat{f}(m,\tau)|^2\,d\tau\Big)^{1/2},$$

where u_0 is the considered initial data. Notice that the definition of the norm $\|\cdot\|_{Z_{s,b}}$ depends on the initial data. In [NTT2], Nakanishi, Takaoka and Tsutsumi extended this LWP result for the mKdV to $H^s(\mathbb{T})$, for $s > 1/3$ (and under some additional hypotheses on the data to $H^s(\mathbb{T})$, $s > 1/4$).

For $k \geq 3$, the best LWP result is in $H^s(\mathbb{T})$, $s \geq 1/2$ (see [CKSTT4]).

7.6 Exercises

7.1 ([BSa2]) Let $A_i(x)$ be defined as in (1.32):

(i) Prove that

$$A_i''(z) - z\,A_i(z) = 0.$$

(ii) Defining

$$v(x,t) = \frac{1}{\sqrt[6]{t}} A_i^2\Big(\frac{1}{2^{2/3}}\frac{x}{\sqrt[3]{3t}}\Big),$$

prove that

$$\partial_t v + \partial_x^3 v = 0, \qquad x \in \mathbb{R},\ t > 0.$$

(iii) Using (1.37), show that for any $t > 0$,

$$\|v(\cdot,t)\|_p < \infty \text{ if and only if } p > 2.$$

(iv) Show that for any $p > 2$,

$$\lim_{t\downarrow 0}\|v(\cdot,t)\|_p = +\infty.$$

7.2 Denote by $(k, \pm 1)$ the equation:

$$\partial_t v + \partial_x^3 v \pm v^k \partial_x v = 0, \quad x, t \in \mathbb{R}, \quad k \in \mathbb{Z}^+,$$

(the cases $(k, -1)$ correspond to the generalized defocusing KdV equation).

(i) Prove that if $v(x,t)$ is a solution of $(k, \pm)$, then

(a) $v(-x, -t)$ also solves $(k, \pm)$
(b) $-v(x,t)$ solves $(k, (-1)^k)$
(c) $\lambda^{2/k} v(\lambda x, \lambda^3 t)$ solves $(k, \pm)$ for $\lambda > 0$

(ii) Prove that for $\lambda > 0$

$$v_\lambda(x,t) = \lambda \sqrt{6} \tanh\left(\lambda(x + 2\lambda^2 t)\right) \text{ (kink solution)}$$

is a solution of $(2, -1)$.

(iii) Prove that if $v(\cdot, \cdot)$ is a solution of $(1, \pm 1)$, then for $h \in \mathbb{R}$, $v(x \pm h\, t, t) \pm h$ also solves $(1, \pm 1)$ (Galilean invariance).

(iv) Let v be a solution of $(2, -1)$. Show that the function (Miura transformation) $w = \sqrt{6}\, \partial_x v + v^2$ solves $(1, -1)$.

(v) Let v be a solution of $(2, 1)$. Show that $w = i\sqrt{6}\, \partial_x v + v^2$ solves $(1, 1)$.

7.3 Consider the IVP associated to the KdV equation (7.71):

(i) Prove that $v(x,t) = \dfrac{x}{1+t}$ is solution of (7.71) with $v(x,0) = x$ for any time $t > 0$.

(ii) Prove that $v(x,t) = \dfrac{-x}{1-t}$ solves (7.71) with $v(x,0) = -x$, but it blows up in finite time.

(iii) Prove that parts (i) and (ii) also hold for the inviscid and viscous Burgers' equation ((3.46) and (7.105), respectively) and for the Benjamin–Ono equation (9.9).

7.4 (Soliton) Let $u(x,t) = \phi_{c,k}(x - ct) = \phi(x - ct)$ be solution of

$$\partial_t u + \partial_x^3 u + u^k \partial_x u = 0$$

with strong decay at infinity.

(i) Show that

(a) $- c\phi' + \phi''' + \phi^k \phi' = 0.$

(b) $- c\phi + \phi'' + \frac{\phi^{k+1}}{k+1} = 0.$

(c) $- c\frac{\phi^2}{2} + \frac{(\phi')^2}{2} + \frac{\phi^{k+2}}{(k+1)(k+2)} = 0,$
(integrating this equation one gets the explicit solution (7.6)).

(ii) Starting in (b) define $x = \phi$ and $y = \phi'$.

(a) Show that the second order ODE can be written as the Hamiltonian system:

$$\begin{cases} \dfrac{dx}{dt} = \partial_y H \\ \dfrac{dy}{dt} = -\partial_x H, \end{cases}$$

where

$$H = \frac{y^2}{2} - c\frac{x^2}{2} + \frac{x^{k+2}}{(k+1)(k+2)}.$$

(b) Using the decay condition at infinity prove that the level set $\{H(x,y)=0\}$ represents the traveling wave.

(c) Prove that $\phi > 0$, symmetric and $\|\phi\|_\infty = [\frac{c}{2}(k+1)(k+2)]^{1/k}$.

7.5 ([Za]) Show that $v(x,t) = \frac{1}{\sqrt{6}}\Big(c - \dfrac{4c}{4c^2(x-6c^2t)^2+1}\Big)$ solves the mKdV equation (7.28).

7.6 (Critical KdV) Show that if $u_0 \in \dot{H}^{-3/2}(\mathbb{R}) \cap \mathcal{S}(\mathbb{R})$, then the solution of the KdV equation (7.71) $u(\cdot,t) \notin \dot{H}^{-3/2}(\mathbb{R})$ for all $t \neq 0$.

7.7 Using a formal scaling argument, obtain the estimate of the life span of the local solutions as a function of the size of the initial data given in Theorem 7.1, i.e., $T(\|D_x^{1/4}v_0\|_2) = c\,\|D^{1/4}v_0\|_2^{-4}$.

7.8 (Two-soliton solution of the KdV) Given the solution of the KdV:

$$v(x,t) = 72\,\frac{3 + 4\cosh(2(x-4t)) + \cosh(4(x-16t))}{[3\cosh(x-28t) + \cosh(3(x-12t))]^2}$$

show that for $\xi_1 = x - 16t$ fixed

$$v(x,t) \sim 48\,\mathrm{sech}^2\Big(2\xi_1 \mp \frac{\log 3}{2}\Big) \qquad \text{as } t \to \pm\infty;$$

show that for $\xi_2 = x - 4t$ fixed

$$v(x,t) \sim 12\,\mathrm{sech}^2\Big(2\xi_2 \pm \frac{\log 3}{2}\Big) \qquad \text{as } t \to \pm\infty.$$

Conclude that

$$v(x,t) \sim 48\,\mathrm{sech}^2\Big(2\xi_1 \mp \frac{\log 3}{2}\Big) + 12\,\mathrm{sech}^2\Big(2\xi_2 \pm \frac{\log 3}{2}\Big) \quad \text{as } t \to \pm\infty.$$

7.9 ([KPV6])

(i) Assuming that the following inequality holds for $s \in (-3/4, -1/2)$ and $b = b(s) \in (1/2, 1)$

$$\Big|\int\!\!\int\!\!\int\!\!\int \frac{|\xi|\,h(\xi,\tau)}{(1+|\tau-\xi^3|)^{1-b}\,(1+|\xi|)^{-s}}\,\frac{(1+|\xi_1|)^{-s}\,f(\xi_1,\tau_1)}{(1+|\tau_1-\xi_1^3|)^b}$$

$$\frac{(1+|\xi-\xi_1|)^{-s}\, g(\xi-\xi_1,\tau-\tau_1)}{(1+|\tau-\tau_1-(\xi-\xi_1)^3|)^b}\, d\tau_1 d\xi_1 d\tau d\xi \Big| \quad (7.119)$$
$$\le c\, \|h\|_{L^2_\xi L^2_\tau} \|f\|_{L^2_\xi L^2_\tau} \|g\|_{L^2_\xi L^2_\tau},$$

prove Corollary 7.4 with $b' = b$. Sketch the LWP result for the IVP associated to the KdV equation (7.71) in $H^s(\mathbb{R})$, $s \in (-3/4, -1/2)$.

(ii) Prove that if either $|\xi_1| \le 1$ or $|\xi - \xi_1| \le 1$, then

$$(1+|\xi_1|)^{-s}(1+|\xi-\xi_1|)^{-s} \le c\,(1+|\xi|)^{-s},$$

so the proof of (7.119) in this domain reduces to the estimate (7.94).

(iii) Show by symmetry that to prove (7.119) it suffices to consider

$$|\tau-\tau_1-(\xi-\xi_1)^3| \le |\tau-\xi_1^3|.$$

(iv) Combine (ii) and (iii) to show that in order to obtain Corollary 7.4 with $b' = b$ it suffices to establish the following inequalities:

$$\sup_{\xi,\tau} \frac{|\xi|}{(1+|\tau-\xi^3|)^{1-b}(1+|\xi|)^{-s}} \times \left(\iint_A \frac{|\xi_1(\xi-\xi_1)|^{-2s} \quad d\tau_1 d\xi_1}{(1+|\tau_1-\xi_1^3|)^{2b}(1+|\tau-\tau_1-(\xi-\xi_1)^3|)^{2b}} \right)^{1/2} < c, \quad (7.120)$$

with $A = A(\xi,\tau)$ as:

$$A = \{(\xi_1,\tau_1) \in \mathbb{R}^2 : \ |\xi_1|,\ |\xi-\xi_1| \ge 1,\ |\tau-\tau_1-(\xi-\xi_1)^3| \le |\tau_1-\xi_1^3| \le |\tau-\xi^3|\}$$

and

$$\sup_{\xi_1,\tau_1} \frac{1}{(1+|\tau_1-\xi_1^3|)^b} \times \left(\iint_B \frac{|\xi|^{2(1+s)}|\xi\xi_1(\xi-\xi_1)|^{-2s}(1+|\xi|)^{2s} \quad d\tau d\xi}{(1+|\tau-\xi^3|)^{2(1-b)}(1+|\tau-\tau_1-(\xi-\xi_1)^3|)^{2b}} \right)^{1/2} < c, \quad (7.121)$$

with $B = B(\xi,\tau)$ as:

$$B = \left\{ \begin{array}{l} (\xi_1,\tau_1) \in \mathbb{R}^2 : \ |\xi_1|,\ |\xi-\xi_1| \ge 1, \ \ |\tau-\tau_1-(\xi-\xi_1)^3| \\ \qquad\qquad \le |\tau_1-\xi_1^3|, |\tau-\xi^3| \le |\tau_1-\xi_1^3|. \end{array} \right\}.$$

(v) Following the argument given in Lemma 7.16 prove the inequality (7.120) (for the proof of (7.121) we refer the reader to [KPV6]).

7.10 Assuming $b > 1/2$ prove the following inequalities:

(i) $$\|g\|_{L^8(\mathbb{R}^2)} \le c\, \|g\|_{X_{0,b}},$$

(ii) $$\|D_x^{1/6} g\|_{L^6(\mathbb{R}^2)} \le c\, \|g\|_{X_{0,b}},$$

(iii) $$\|\partial_x g\|_{L_t^\infty L_x^2} \le c\, \|g\|_{X_{0,b}},$$

(iv) $$\|g\|_{L_x^\infty L_t^2} \le c\, \|g\|_{X_{0,b}}.$$

7.11 Let w be a solution of the IVP:

$$\begin{cases} (\partial_t + a\partial_x^3 + ib\partial_x^2)w = F, \\ w(x,0) = w_0(x). \end{cases} \tag{7.122}$$

Show that

$$z(x,t) = e^{i\frac{b^3}{27a^2}t}\, e^{i\frac{b}{3a}x}\, w\big(x + \frac{b^2}{3a}t, t\big)$$

solves the IVP

$$\begin{cases} \partial_t z + a\partial_x^3 z = \widetilde{F}, \\ z(x,0) = z_0(x). \end{cases} \tag{7.123}$$

where

$$z_0(x) = e^{i\frac{b}{3a}x}\, w_0(x) \quad \text{and} \quad \widetilde{F}(x,t) = e^{i\frac{b^3}{27a^2}t}\, e^{i\frac{b}{3a}x}\, F\big(x + \frac{b^2}{3a}t, t\big). \tag{7.124}$$

7.12 ([ILP2]) Consider the linear IVP (7.10) with $v_0 \in L^2(\mathbb{R})$. Prove that if for some $k \in \mathbb{Z}^+$,

$$v_0|_{(0,\infty)} \in H^k((0,\infty)),$$

then the corresponding solution $v(x,t)$ satisfies that for any $b > 0$,

$$v(\cdot,t)|_{(-b,\infty)} \in H^k((-b,\infty)), \quad \text{for each } t > 0$$

and

$$v(\cdot,t)|_{(-\infty,b)} \in H^k((-\infty,b)) \quad \text{for each } t < 0.$$

Hint: Let $\eta \in C^\infty(\mathbb{R})$, $\eta' \ge 0$, $\eta(x) = 0$ for $x \le 0$ and $\eta(x) = 1$ for $x \ge 1$. Following Kato's argument in [K2] one easily gets the (formal) identity

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int v^2(x,t) \quad \eta(x)\,dx + \frac{3}{2}\int (\partial_x v)^2(x,t)\,\eta'(x)\,dx \\ -\frac{1}{2}\int v^2(x,t)\,\eta^{(3)}(x)\,dx = 0. \end{aligned} \tag{7.125}$$

Modify $\eta(\cdot)$ in each step $j = 1, 2, \ldots, k$ ($\eta_j(\cdot)$) and consider $\eta_j(x + ct)$ with $c > 0$ (for $t > 0$) to obtain the appropriate version of (7.125).

7.13 ([ILP2]) Consider the linear IVP (7.10) with $v_0 \in L^2(\mathbb{R})$. Prove that if for some $m \in \mathbb{Z}^+$,

$$x_+^{m/2} v_0 \in L^2(\mathbb{R}),$$

then for any $t > 0$,

$$x_+^{m/2} v(\cdot, t) \in L^2(\mathbb{R})$$

and for any $b > 0$ and $t > 0$

$$\int_b^\infty (\partial_x^m u(x,t))^2 \, dx < \infty.$$

Hint: Modify the argument in the hint of Exercise 7.12.

7.14 ([ILP2]) Consider the linear IVP (7.10) with $v_0 \in L^2(\mathbb{R})$. Prove that if for some $m \in \mathbb{Z}^+$ and $t_1, t_2 \in \mathbb{R}$ with $t_1 < t_2$ the corresponding solution $v(x,t)$ satisfies

$$x_+^{m/2} v(\cdot, t_1), \; x_-^{m/2} v(\cdot, t_2) \in L^2(\mathbb{R}),$$

then $v_0 \in H^m(\mathbb{R})$.
Hint: Use Exercise 7.13.

7.15 (i) Consider the IVP for the 1-D heat equation

$$\begin{cases} \partial_t u - \partial_x^2 u = 0, \\ u(x,0) = u_0(x), \end{cases} \tag{7.126}$$

$x \in \mathbb{R}$, $t > 0$. Prove that if $u_0 \in L^2(\mathbb{R})$, then for each t positive the solution $u(\cdot, t) = e^{t\partial_x^2} u_0$ of (7.126) has an analytic extension to $\mathbb{C}$. Moreover, if $z = x + iy$, then

$$|u(z,t)| \le \frac{e^{y^2/4t}}{(4\pi t)^{1/4}} \, \|u_0\|_2.$$

(ii) Consider the linear IVP (7.10). Prove that if $v_0 \in L^2(\mathbb{R}) \cap L^2(e^{\beta x} dx)$, $\beta > 0$, then for each t positive the solution $v(\cdot, t)$ has an analytic extension to $\mathbb{C}$.
Hint: Combine the result in part (i) and the formula (7.112).

Chapter 8
Asymptotic Behavior of Solutions for the k-gKdV Equations

This chapter is concerned with the longtime behavior of solutions to the initial value problem (IVP) associated to the k-generalized Korteweg-de Vries equations,

$$\begin{cases} \partial_t v + \partial_x^3 v + v^k \partial_x v = 0, \\ v(x,0) = v_0(x), \end{cases} \tag{8.1}$$

$x,\ t \in \mathbb{R},\ k \in \mathbb{Z}^+$.

We shall restrict ourselves to consider only real solutions of (8.1). In this case, the following quantities are preserved by the solution flow:

$$I_1 = \int_{-\infty}^{\infty} v(x,t)\,dx, \tag{8.2}$$

$$I_2 = \int_{-\infty}^{\infty} v^2(x,t)\,dx, \tag{8.3}$$

$$E(v_0) = I_3 = \int_{-\infty}^{\infty} \Big((\partial_x v)^2 - \frac{2}{(k+1)(k+2)}\, v^{k+2}\Big)(x,t)\,dx. \tag{8.4}$$

Combining them with the local existence theory in Chapter 7, we shall see that for $k = 1, 3$ the IVP (8.1) with $v_0 \in L^2(\mathbb{R})$ has a unique globally bounded solution. For the case $k = 2$, the same holds in $H^1(\mathbb{R})$.

In fact, we shall see in Section 8.1 that a quite stronger set of results has been established in the case $k = 1, 2, 3$.

In Section 8.2, we treat the L^2-critical case $k = 4$. A major set of results due to Martel and Merle as well as extensions and further analysis due to Martel, Merle, and Raphael is discussed. In particular, they have settled a long-standing open problem by proving the finite time blowup of some local H^1-solutions. Similar results for $k \geq 5$ remain as open problems.

In Section 8.3, we add some further comments.

F. Linares, G. Ponce, *Introduction to Nonlinear Dispersive Equations*,
Universitext, DOI 10.1007/978-1-4939-2181-2_8

8.1 Cases k = 1,2,3

We observe that if $v(t)$ is a local real H^1-solution of (8.1), combining Gagliardo–Nirenberg (3.14) and (8.3)–(8.4) gives

$$\begin{aligned} E(v_0) &= \int_{-\infty}^{\infty} \Big[(\partial_x v)^2 - \frac{2}{(k+1)(k+2)}\, v^{k+2}\Big](x,t)\,dx \\ &\geq \|\partial_x v(t)\|_2^2 - \frac{2}{(k+1)(k+2)}\|v(t)\|_{k+2}^{k+2} \\ &\geq \|\partial_x v(t)\|_2^2 - c_k\,\|\partial_x v(t)\|_2^{k/2}\|v(t)\|_2^{2+k/2} \\ &\geq \|\partial_x v(t)\|_2^2 - c_k\,\|\partial_x v(t)\|_2^{k/2}\|v_0\|_2^{2+k/2}. \end{aligned} \tag{8.5}$$

Hence, using the notation $\eta = \eta(t) = \|\partial_x v(t)\|_2$ it follows that

$$E(v_0) \geq \eta^2 - c_k\|v_0\|_2^{2+k/2}\,\eta^{k/2}. \tag{8.6}$$

So for $k < 4$, one obtains the a priori estimate:

$$\eta(t) \leq M(\|v_0\|_2; k). \tag{8.7}$$

In this sense as well as in the scaling argument for the L^2-norm (see (7.8)), the case $k = 4$ is critical.

Thus for $k = 2$, (8.7) allows us to reapply the local existence theory (local well-posedness in $H^s(\mathbb{R})$, $s \geq 1/4$) for data $v_0 \in H^1(\mathbb{R})$.

Theorem 8.1. *For $v_0 \in H^1(\mathbb{R})$ real valued the corresponding local solution of the initial value problem (IVP)* (8.1) *with $k = 2$ given by Theorem* 7.1 *extends in the same class to any time interval with*

$$v \in C(\mathbb{R} : H^1(\mathbb{R})) \cap L^\infty(\mathbb{R} : H^1(\mathbb{R})). \tag{8.8}$$

Moreover, if $v_0 \in H^s(\mathbb{R})$, $s > 1$, then

$$v \in C(\mathbb{R} : H^s(\mathbb{R})). \tag{8.9}$$

In the cases $k = 1, 3$, the local well-posedness was established in $H^s(\mathbb{R})$ for $s \geq -3/4$ and $s > -1/6$, respectively (see Theorem 7.8 and [Gr2]). These cases are subcritical, so the interval of existence in each case $[0, T]$ satisfies that $T = T(\|u_0\|_{s,2}) > 0$. Therefore, if $v_0 \in L^2(\mathbb{R})$ by I_2 (see (8.3)), we can reapply this local theory to obtain the following global result.

Theorem 8.2. *For $v_0 \in L^2(\mathbb{R})$ real the corresponding local solution of the IVP (8.1) with $k = 1$ or 3 extends in the same class to any time interval with*

$$v \in C(\mathbb{R} : L^2(\mathbb{R})) \cap L^\infty(\mathbb{R} : L^2(\mathbb{R})). \tag{8.10}$$

Moreover, if $v_0 \in H^s(\mathbb{R})$, $s > 0$, *then*

$$v \in C(\mathbb{R} : H^s(\mathbb{R})). \tag{8.11}$$

In the cases $k = 1, 2$, due to the form of the infinite conservation laws, one can replace (8.9) and (8.11) by $v \in C(\mathbb{R} : H^s(\mathbb{R})) \cap L^\infty(\mathbb{R} : H^s(\mathbb{R}))$ if $s \in \mathbb{Z}^+$.

These local and global results present the following questions:

Question 1. What happens with the longtime behavior of the solution corresponding to data $v_0 \in H^s(\mathbb{R})$ with $s \in [-3/4, 0)$, $[1/4, 1)$, and $(-1/6, 0)$ in the cases $k = 1, 2$, and 3, respectively?

The first result in this direction was given by Bourgain [Bo5] in his study of the critical two-dimensional nonlinear Schrödinger (NLS) equation. He set up a general argument to obtain global solutions corresponding to data whose regularity is below that required if one is using the conservation law.

To illustrate his approach, we take the mKdV equation, $k = 2$ in (8.1), with $v_0 \in H^s(\mathbb{R})$, $s \in [1/4, 1)$ (see [FLP2]).

First, one splits the data according to the frequency (low–high). For N large to be determined one considers

$$v_0 = (\chi_{\{|\xi|\le N\}}\widehat{v_0})^\vee + (\chi_{\{|\xi|>N\}}\widehat{v_0})^\vee = v_{1,0} + v_{2,0}. \tag{8.12}$$

Thus, $v_{1,0} \in H^\infty(\mathbb{R})$, with $E(v_{1,0}) + \|v_{1,0}\|_{1,2} \le c\,N^{1-s}$ and $\|v_{2,0}\|_{r,2} \le c\,N^{r-s}$ for $r \in [1/4, s)$.

One solves the mKdV with data $v_{1,0}$ as in Theorem 7.1, so the corresponding solution $v_1(t)$ satisfies

$$\|v_1(t)\|_{1,2} \le c\,N^{1-s}, \quad t \in [0, \Delta T], \quad \Delta T \simeq \|D_x^{1/4} v_{1,0}\|_2^{-4}, \tag{8.13}$$

and $v_2(t)$ satisfies the error equation (using that $v = v_1 + v_2$):

$$\partial_t v_2 + \partial_x^3 v_2 + v^2\partial_x v - v_1^2\partial_x v_1 = 0, \qquad t \in [0, \Delta T], \tag{8.14}$$

with data $v_{2,0}$ (small) in $H^r(\mathbb{R})$ for $r \in [0, s)$. The interval $[0, \Delta T]$ is given by the local well-posedness theory. Using that

$$\begin{aligned} v_2(t) &= V(t)v_{2,0} + \int_0^t V(t-t')[(v_1+v_2)^2\,\partial_x(v_1+v_2) - v_1^2\partial_x v_1](t')\,dt' \\ &= V(t)v_{2,0} + z(t), \end{aligned}$$

one observes that $z(t)$ is smoother than $V(t)v_{2,0}$ (see Exercise 8.1 and comments there). Indeed, it belongs to $H^1(\mathbb{R})$ with a "good" estimate for its norm. Define

$$\begin{cases} v_{1,0}(\Delta T) = v_1(\Delta T) + z(\Delta T), \\ v_{2,0}(\Delta T) = V(\Delta T)v_{2,0} \end{cases}$$

and repeat the argument in $[\Delta T, 2\Delta T]$.

Briefly, to reach the time T^* we apply it $T^*/\Delta T$ times. If one proves that

$$E\left(v_{0,1}+\sum_{j=1}^{T^*/\Delta T} z(j\Delta T)\right)+\sum_{j=1}^{T^*/\Delta T}\|z(j\Delta T)\|_{1,2}\le c\,N^{1-s}, \tag{8.15}$$

then all the previous estimates are uniform and one can extend the solution to $[0,T^*]$. It is in (8.15) where the restriction on s appears.

By introducing the I-method (see [KT2]) in this context Colliander, Keel, Staffilani, Takaoka and Tao [CKSTT4], [CKSTT5], [CKSTT6] have improved most of the results obtained by the above argument. By defining

$$If(x)=I_{N,s}f(x)=(m(\xi)\widehat{f}\,)^{\vee}, \tag{8.16}$$

where $m(\xi)$ is a smooth and monotone function given by

$$m(\xi)=\begin{cases}1, & |\xi|\le N,\\ N^{-s}|\xi|^{s}, & |\xi|>2\,N,\end{cases} \tag{8.17}$$

with N to be determined and $s<0$, they obtain a series of "almost conserved quantities."

By using the "cancellations" in the multilinear form working directly with the equation, in this case the KdV, they show that

$$\sup_{t\in[0,T]}\|Iv(t)\|_2\le\|Iv(0)\|_2+cN^{-\beta}\,\|Iv(0)\|_2^3, \tag{8.18}$$

for some small $\beta>0$. So if N is large, the increment in $\|Iv(t)\|_2$ is controlled. In particular for the IVP (8.1) they have shown the following.

Theorem 8.3 ([CKSTT5]).

1. *The local real solutions of the IVP* (8.1) *with* $k=1$ *corresponding to data* $v_0\in H^s(\mathbb{R})$, $s>-3/4$, *extend to any time interval* $[0,T^*]$.
2. *The local real solutions of the IVP* (8.1) *with* $k=2$ *corresponding to data* $v_0\in H^s(\mathbb{R})$, $s>1/4$, *extend to any time interval* $[0,T^*]$.

In [Gu], Guo and [Ki1] Kishimoto have extended these global results to the limiting cases $s=-3/4$ and $s=1/4$ for the KdV and the mKdV equations, respectively.

For the sake of completeness, we explain how the first step of this method works for the IVP associated to the KdV equation (8.1) ($k=1$).

The material described below was essentially taken from the lecture notes given by Staffilani at IMPC (see [Sta3]).

One first notices that the operator defined in (8.16) is the identity operator on low frequencies $\{\xi:\ |\xi|<N\}$ and simply an integral operator in high frequencies. In general, it commutes with differential operators and maps $H^s(\mathbb{R})$ to $L^2(\mathbb{R})$.

As we mentioned before, the goal is to establish an estimate as the one in (8.18). To do so, we first use the fundamental theorem of calculus, the equation and integration by parts to get

$$\begin{aligned}\|Iv(t)\|_2^2 &= \|Iv(0)\|_2^2 + \int_0^t \frac{d}{ds}(Iv(s), Iv(s))\,ds \\ &= \|Iv(0)\|_2^2 + 2\int_0^t \left(\frac{d}{ds}Iv(s), Iv(s)\right)\,ds \\ &= \|Iv(0)\|_2^2 + 2\int_0^t (I(-v_{xxx} - vv_x), Iv(s))\,ds \\ &= \|Iv(0)\|_2^2 + 2\int_0^t (I(-vv_x), Iv(s))\,ds \\ &= \|Iv(0)\|_2^2 + R(t),\end{aligned} \tag{8.19}$$

where

$$R(t) = \int_0^t \int_{\mathbb{R}} \partial_x(-Iv^2)\,Iv\,dxds \tag{8.20}$$

is an error term. Hence,

$$\|Iv(t)\|_2^2 = \|Iv_0\|_2^2 + R(t). \tag{8.21}$$

We shall show then that locally in time $R(t)$ is small. This can be achieved using local well-posedness estimates. Since one introduces the operator I in this analysis, a well-posedness result involving it has to be proved. A similar argument as the one given in the proof of Theorem 7.8 and the bilinear estimates (7.91) obtained by Kenig, Ponce and Vega [KPV6] provide us the local well-posedness result. More precisely:

Theorem 8.4. *For any $v_0 \in H^s(\mathbb{R})$, $s > -3/4$, the IVP (8.1), $k = 1$, is locally well-posed in the Banach space $I^{-1}L^2 = \{\phi \in H^s(\mathbb{R})\}$ furnished with the norm $\|I\phi\|_{L^2}$, with time existence satisfying*

$$T \gtrsim (\|Iv_0\|_2)^{-\alpha}, \qquad \alpha > 0. \tag{8.22}$$

Moreover,

$$\|\theta(\cdot/T)Iv\|_{X_{0,b}} \le C\|Iv_0\|_2, \tag{8.23}$$

where θ was defined in (7.73).

The proof of this theorem follows by using the same procedure to establish Theorem 7.8 once one has the bilinear estimate:

$$\|\partial_x I(uv)\|_{X_{0,-\frac{1}{2}+}} \leq c\,\|Iu\|_{X_{0,\frac{1}{2}+}}\|Iv\|_{X_{0,\frac{1}{2}+}}. \tag{8.24}$$

To prove the bilinear estimate (8.24), one applies the usual bilinear estimate (7.91) due to Kenig, Ponce and Vega [KPV6] combined with the following extra smoothing bilinear estimate whose proof is given in [CKSTT5].

Proposition 8.1. *The bilinear estimate:*

$$\|\partial_x\{IuIv - I(uv)\}\|_{X_{0,-\frac{1}{2}-}} \leq cN^{-\frac{3}{4}+}\,\|Iu\|_{X_{0,\frac{1}{2}+}}\|Iv\|_{X_{0,\frac{1}{2}+}} \tag{8.25}$$

holds.

Proof. Just to give a flavor of the proof we consider the case when u is localized in a very small frequency ($|\xi| \ll 1$) and v localized in a very large one ($|\xi| \gg N$). One notices that in this situation

$$|(\widehat{I(uv) - IuIv})(\xi)| = \int\limits_{\xi=\xi_1+\xi_2} |m(\xi) - m(\xi_2)||\widehat{u}(\xi_1)||\widehat{v}(\xi_2)|.$$

Since m is smooth, the mean value theorem yields

$$|(\widehat{I(uv) - IuIv})(\xi)| \leq \int\limits_{\xi=\xi_1+\xi_2} |m'(\bar{\xi}_2)||\widehat{u}(\xi_1)||\widehat{v}(\xi_2)|,$$

where $|\bar{\xi}_2| \sim |\xi_2| \gg N$. Moreover, it is easy to check that $m'(\bar{\xi}_2) \lesssim N^{-1}m(\xi_2)$. Thus,

$$\|\partial_x(I(uv) - IuIv)\|_{X_{0,-1/2+}} \leq N^{-1}\|\partial_x(I(u)I(v))\|_{X_{0,-1/2+}}. \tag{8.26}$$

In this point, one uses the bilinear estimate (8.24) to get (8.25). For the estimates involving intermediate size frequencies the best gain that one can obtain is $N^{-3/4}$. □

Next we will obtain the so-called almost conserved quantity from (8.21). Note that the cancellation property

$$\int_0^t\int_{-\infty}^{\infty} \partial_x(Iu)^2 Iu\,dx\,dt = 0 \tag{8.27}$$

holds. In what follows this identity play an important role.

Using (8.27) one can write $R(t)$ as:

$$R(t) = \int_0^t\int_{-\infty}^{\infty} \partial_x\{(Iv)^2 - I(v^2)\}Iv\,dx\,ds. \tag{8.28}$$

The Plancherel identity and the Cauchy–Schwarz inequality yield

$$|R(t)| \le c\, \|\partial_x\{(Iv)^2 - I(v^2)\}\|_{X_{0,-\frac{1}{2}-}} \|Iv\|_{X_{0,\frac{1}{2}+}}. \tag{8.29}$$

Now, using (8.29) and Proposition 8.1 the identity (8.21) gives the almost conservation law,

$$\|Iv(t)\|_2^2 \le \|Iv(0)\|_2^2 + c\, N^{-\frac{3}{4}+}\, \|Iv\|^3_{X_{0,\frac{1}{2}+}}. \tag{8.30}$$

From (8.30), it is clear that the contribution of the error term $R(t)$ is very small for large N and therefore one can use (8.30) in the iteration process to extend the local solution.

Now, we are in position to prove the following global well-posedness result.

Theorem 8.5. *The IVP (8.1), $k = 1$, is globally well-posed in $H^s(\mathbb{R})$ for all $s > -3/10$.*

Proof. It is enough to show that the IVP (8.1) can be extended to $[0, T]$ for arbitrary $T > 0$. To make the analysis easy, one uses the scaling (7.8) mentioned in Chapter 7. More precisely, if v solves the IVP (8.1), $k = 1$, with initial data v_0, then for $1 > \lambda > 0$ so does v_λ, where $v_\lambda(x,t) = \lambda^2 v(\lambda x, \lambda^3 t)$, with initial data $v_0^\lambda(x) = \lambda^2 v_0(\lambda x)$. Observe that v exists in $[0, T]$ if and only if v_λ exists in $[0, \lambda^{-3}T]$. So we are interested to extend v_λ in $[0, \lambda^{-3}T]$.

An easy calculation shows that

$$\|Iv_0^\lambda\|_2 \le c\, \lambda^{\frac{3}{2}+s} N^{-s}\, \|v_0\|_{s,2}, \tag{8.31}$$

where $N = N(T)$ is chosen later, but now we pick $\lambda = \lambda(N)$ by demanding

$$c\, \lambda^{\frac{3}{2}+s} N^{-s}\, \|v_0\|_{s,2} = \sqrt{\frac{\varepsilon_0}{2}} \ll 1. \tag{8.32}$$

From (8.32) one deduces that $\lambda \sim N^{\frac{2s}{3+2s}}$ and using (8.32) in (8.31) one gets

$$\|Iv_0^\lambda\|_2^2 \le \frac{\varepsilon_0}{2} \ll 1. \tag{8.33}$$

Therefore, if we choose ε_0 arbitrarily small, then from Theorem 8.4 we see that IVP (8.1), $k = 1$, is well-posed for all $t \in [0, 1]$.

Now, using the almost-conserved quantity (8.30), the identity (8.33), and Theorem 8.4, one gets

$$\|Iv_\lambda(1)\|_2^2 \le \frac{\varepsilon_0}{2} + c\, N^{-\frac{3}{4}+}\left[3\frac{\varepsilon_0}{2}\left(\frac{\varepsilon_0}{2}\right)^{1/2}\right] \le \varepsilon_0 + c\, N^{-\frac{3}{4}+}\varepsilon_0. \tag{8.34}$$

So, one can iterate this process $c^{-1}N^{\frac{3}{4}-}$ times before doubling $\|Iv_\lambda(t)\|_2^2$. Hence, one can extend the solution in the time interval $[0,\, c^{-1}N^{\frac{3}{4}-}]$ by taking $c^{-1}N^{\frac{3}{4}-}$

times steps of size $O(1)$. As one is interested to define the solution in the time interval $[0, \lambda^{-3}T]$, one chooses $N = N(T)$ such that $c^{-1}N^{\frac{3}{4}-} \geq \lambda^{-3}T$. That is,

$$N^{\frac{3}{4}-} \geq c\,\frac{T}{\lambda^3} \sim T N^{\frac{-6s}{3+2s}}.$$

Therefore, for large N, the existence interval is arbitrarily large if we choose s such that $s > -3/10$. This completes the proof of the theorem. □

Question 2. For these global solutions whose regularity is below or between those given by the conservation law, one can ask for upper and lower bounds for the growth of the H^s-norm.

Theorem 8.1 provides some upper bound. In the case $k = 2$, where infinitely many conservation laws are available, one has the upper bound

$$\sup_{t\in[0,T]} \|v(t)\|_{s,2} \leq c\,T^{\theta(s)}, \quad \theta(s) = \min\{s - [s],\, [s+1] - s\} \tag{8.35}$$

(see [Fo], [Sta1]). A similar result for the case $k = 1$ is unknown as well as any lower bound estimate of the growth of the H^s-norm of the solutions.

For the case $k = 3$, the best-known global result for large H^s-data is due to [GPS] for $s > -1/42$. We recall that $s_3 = -1/6$ and the results in [To6] included global well-posedness for small data in $\dot{H}^{-1/6}(\mathbb{R})$.

8.2 Case $k = 4$

In this section, we shall first attempt to describe some of the main results in a series of works by Martel and Merle. Among other conclusions, they proved that blowup in finite time occurs in some H^1 local solutions of the IVP (8.1) with $k = 4$. Later, we shall add some further analysis with a more precise description of the dynamics of this blow-up result given by Martel, Merle, and Raphael.

For convenience sake we shall follow their notation, so we rewrite the equation in (8.1) with $k = 4$ in divergence form to get

$$\begin{cases} \partial_t u + \partial_x(\partial_x^2 u + u^5) = 0, \\ u(x,0) = u_0(x), \end{cases} \tag{8.36}$$

i.e., $v(x,t) = \sqrt[4]{5}\,u(x,t)$. In this setting, the conservation law E (or I_3) becomes

$$E(u_0) = \int_{-\infty}^{\infty} \Big[(\partial_x u)^2 - \frac{2}{6}u^6\Big](x,t)\,dx. \tag{8.37}$$

We shall recall that the "traveling wave" $\varphi(x) = 3^{\frac{1}{4}}\,\mathrm{sech}^{\frac{1}{2}}(2x)$ satisfies

$$\varphi'' + \varphi^5 = \varphi \tag{8.38}$$

and $E(\varphi) = 0$.

In [W3], Weinstein (see Exercise 6.6) obtained the following sharp version of a Gagliardo–Nirenberg inequality,

$$\text{for all } w \in H^1(\mathbb{R}), \qquad \frac{1}{6}\int w^6\,dx \le \frac{1}{2}\left(\frac{\int w^2}{\int \varphi^2}\right)^2 \int (\partial_x w)^2\,dx. \tag{8.39}$$

Thus, if $u_0 \in H^1(\mathbb{R})$ with $\|u_0\|_2 < \|\varphi\|_2$, one has

$$\frac{1}{2}\left(1 - \frac{\int u_0^2}{\int \varphi^2}\right)^2 \int (\partial_x u)^2(x,t)\,dx \le E(u_0) \quad \text{for all} \quad t \in \mathbb{R}. \tag{8.40}$$

This a priori estimate together with I_2 ($\|u(t)\|_2 = \|u_0\|_2$) allows one to extend the local solution of (8.36) globally in time.

Notice that based on homogeneity, Theorem 7.2 guarantees the existence of global solutions for $u_0 \in L^2(\mathbb{R})$ with $\|u_0\|_2 < \delta$ sufficiently small. From these results, it is reasonable to conjecture that $\delta = \|\varphi\|_2$ (see the comments at the end of this chapter).

Also from the proof of Theorem 7.4 with $u_0 \in H^s(\mathbb{R})$, $s \in (0,1]$, and using an idea in [CzW4] one has that if there exists $T^* \in (0,\infty)$ such that

$$\lim_{t\uparrow T^*} \|u(t)\|_{s,2} = \infty \qquad \text{for } s \in [0,1), \tag{8.41}$$

then

$$\|u(t)\|_{s,2} \ge c\,(T^* - t)^{-s/3}, \tag{8.42}$$

and by [W], [Me5] there exist $c_0,\ R_0 > 0$ both depending on $\|u_0\|_2$ such that

$$\lim_{t\uparrow T^*}\inf \int_{|x-x(t)|\le R_0(T^*-t)^{1/3}} |u(x,t)|^2\,dx \ge c_0\,, \tag{8.43}$$

for some function $x(t)$.

The next result by Martel and Merle [MM3] tells us that any global H^1 solution of (8.36) that at $t = 0$ is close to a traveling wave and does not disperse has to be precisely the traveling wave.

Theorem 8.6 ([MM3] of Liouville's type). *Let* $u_0 \in H^1(\mathbb{R})$ *and let*

$$\|u_0 - \varphi\|_{1,2} = \alpha. \tag{8.44}$$

Suppose that the corresponding H^1 *solution* $u = u(x,t)$ *of* (8.36) *satisfies:*

(i) There exist $c_1, c_2 > 0$ *such that*

$$c_1 \le \|u(t)\|_{1,2} \le c_2 \qquad \textit{for all } t \in \mathbb{R}. \tag{8.45}$$

(ii) There exists $x(t)$ such that for every $\varepsilon > 0$ there exists $R_0 > 0$ so that

$$\inf_{x(t)\in\mathbb{R}} \int_{|x-x(t)|>R_0} u^2(x,t)\,dx \le \varepsilon \quad \text{for all } t \in \mathbb{R}. \tag{8.46}$$

Then, there exists $\alpha_0 > 0$ such that for $\alpha \in (0,\alpha_0)$ in (8.44) one has

$$u(x,t) = \lambda_0^{1/2}\,\varphi(\lambda_0(x-x_0) - \lambda_0^3 t) \tag{8.47}$$

for some $\lambda_0 \in \mathbb{R}^+$ and $x_0 \in \mathbb{R}$.

The proof of this theorem is quite interesting.

First, the problem is renormalized by properly fixing the "center of mass" $x(t)$ and the "scaling" $\lambda(t)$, which is possible due to the invariance up to translations and dilations of the equation. Next, the authors establish a uniform-in-time exponential decay in the x-variable by using (8.46). Once this exponential decay is available they reduce the problem in studying which solutions of the associated linearized equation have such decay. They show that the solutions should have nontrivial projection on the singular spectrum of the linearized problem. But this possibility is withdrawn by using the choice of the parameters $x(t)$, $\lambda(t)$. So the solution of the linearized problem has to be the trivial one.

The next theorem complements the result in Theorem 8.6.

Theorem 8.7 ([MM5]). *Under the hypotheses (8.44) and (8.45) in* Theorem 8.6 *there exists α_1 such that if $\alpha \in (0,\alpha_1)$, then there exist $\lambda(t)$, $x(t)$ such that*

$$\lambda^{1/2}(t)\,u(\lambda(t)(x-x(t)),t) = \varphi(x) + u_R(x,t), \tag{8.48}$$

where

$$u_R(t) \underset{\text{(weakly)}}{\rightharpoonup} 0 \qquad \text{in } H^1 \text{ as } t\uparrow\infty. \tag{8.49}$$

In fact, one has that

$$\lambda(t) \in (\lambda_1,\,\lambda_2) \quad \text{for all } t \text{ and } \quad x(t)\uparrow\infty \quad \text{as} \quad t\uparrow\infty. \tag{8.50}$$

In [MM1], Martel and Merle studied the stability of the traveling wave solution of the IVP (8.1) with $k = 4$.

We recall that it was shown in [Be1] and [BSS] that for the IVP (8.1) with $k = 1,2,3$, the corresponding traveling waves were stable and in [BSS] that for $k \ge 5$ they were unstable. Also, we recall that for the IVP (8.36), we have that φ satisfies

$$E(\varphi) = \int \Big((\varphi')^2 - \frac{2}{6}\varphi^6\Big)\,dx = 0$$

and using (8.38) that

$$DE(\varphi)\phi = \frac{d}{d\eta}E(\varphi+\eta\phi)\big|_{\eta=0} = 2\int \big(\varphi'\,\phi' - \varphi^5\,\phi\big)\,dx$$

$$= -2\int (\varphi'' + \varphi^5)\,\phi\,dx = -2\int \varphi\,\phi\,dx = \langle -2\varphi, \phi\rangle.$$

So,

$$DE(\varphi) = -2\varphi.$$

Let $\varepsilon \in H^1(\mathbb{R})$ with $\|\varepsilon\|_{1,2} \ll 1$; thus, $E(\varphi + \varepsilon) \sim \langle -2\varphi, \varepsilon\rangle$.

The next result establishes the instability of the traveling wave in this critical case $k = 4$ in (8.1) (see also (8.36)).

Theorem 8.8 ([MM1]). *There exist* $\alpha_0, a_0, b_0, c_0 > 0$ *such that if* $u_0 = \varphi + \varepsilon$ *with*

$$\varepsilon \in H^1(\mathbb{R}), \qquad \|\varepsilon\|_{1,2} < a_0, \qquad x\varepsilon^2 \in L^1(\mathbb{R}), \tag{8.51}$$

$$|\varepsilon(x)| < b_0(1 + x)^{-2}, \qquad \text{for all } \; x > 0 \tag{8.52}$$

and

$$0 < \int \varepsilon\,\varphi\,dx < c_0 \int \varphi^2\,dx, \tag{8.53}$$

then there exists $t_0 = t_0(u_0)$ *such that*

$$\inf_{y\in\mathbb{R}} \|u(\cdot, t_0) - \varphi(\,\cdot - y)\|_{1,2} \geq \alpha_0. \tag{8.54}$$

In fact, they show that (8.54) holds in $L^2(\mathbb{R})$. Observe that taking $\varepsilon_n = n^{-1}\varphi$ for n large enough, ε_n satisfies the hypotheses (8.51)–(8.53). Similarly, if $\varepsilon = a\varphi + \varepsilon_0$ with $x\varepsilon^2 \in L^1$, $(1 + x)^2\,|\varepsilon_0(x)| \leq c_0$ for all $x \geq 0$ with $\|\varepsilon_0\|_{1,2} \leq b_0\sqrt{a_0}$, then ε also satisfies (8.51)–(8.53).

In [Me4], Merle proved the existence of blow-up solutions of (8.36) in finite or infinite time.

Theorem 8.9 ([Me4]). *There exists* $\alpha_0 > 0$ *such that if* $u_0 \in H^1(\mathbb{R})$ *with*

$$E(u_0) < 0 \qquad \text{and} \qquad \int \varphi^2 < \int u^2 < \int \varphi^2 + \alpha_0, \tag{8.55}$$

then the corresponding solution $u(t)$ *of (8.36) blows up in the* H^1*-norm in finite or infinite time.*

Observe that since $E(\varphi) = 0$ and $DE(\varphi) = -2\varphi$ there is a large class of data u_0 satisfying (8.55) whose corresponding solution blows up.

In [MM5], the authors showed that any blowup solution close to the traveling wave φ behaves asymptotically like it up to rescaling and translation, i.e., for some C^1 functions $x(t)$, $\lambda(t)$,

$$\pm\lambda^{1/2}(t)\,u(\lambda(t)x + x(t), t) \rightharpoonup \varphi \quad \text{in} \;\; H^1(\mathbb{R}) \;\; \text{as} \;\; t \uparrow T, \quad T \leq \infty.$$

(See [ABLS] for a related result.)

As a consequence they established that the blowup at finite time must occur at a rate that in particular excludes the possibility of blowup at the self-similar rate:

$$u(x,t) \sim \frac{1}{(T-t)^{1/6}}\, h\left[\frac{x-x(t)}{(T-t)^{1/3}}\right]$$

since they establish that in this case (finite blow up time T)

$$\lim_{t\uparrow T} (T-t)^{1/3}\,\|\partial_x u(\cdot,t)\|_2 = \infty.$$

Based on these works, Martel and Merle were able to show the blowup in finite time [MM4] for solutions corresponding to data u_0 with negative energy ($E(u_0) < 0$), L^2-norm close to that of the solitary wave, see (8.55), and with sufficient decay at the right, i.e., there exists $\theta > 0$ such that for all $x_0 > 0$

$$\int_{x \geq x_0} u_0^2(x)\, dx \leq \frac{\theta}{x_0^6}. \tag{8.56}$$

Theorem 8.10 ([MM4]). *Under the hypotheses* (8.55) *and* (8.56) *the corresponding solutions of the IVP* (8.36) *blowup in finite time* $T < \infty$*, i.e.,*

$$\lim_{t\uparrow T} \|\partial_x u(t)\|_2 = \infty. \tag{8.57}$$

Moreover, let $t_n \uparrow T$ *be the sequence defined as:*

$$\|\partial_x u(\cdot,t_n)\|_2 = 2^n\, \|\partial_x \varphi\|_2 \tag{8.58}$$

with

$$\|\partial_x u(\cdot,t)\|_2 > 2^n\, \|\partial_x \varphi\|_2, \quad t \in (t_n, T).$$

Then there exists $n_0 = n(u_0)$ *such that for all* $n \geq n_0$*,*

$$\|\partial_x u(\cdot,t_n)\|_2 \leq \frac{c_0}{|E(u_0)|\,(T-t_n)}, \tag{8.59}$$

where $c_0 = 4(\int \varphi)^2\, \|\partial_x \varphi\|_2$.

The proof of this theorem used the results in the previous ones together with some elliptic and oscillatory integral-type estimates.

Finally, we have their following result regarding the nonexistence of minimal mass blow up solutions.

Theorem 8.11 ([MM6]). *Let* $u_0 \in H^1(\mathbb{R})$ *be such that*

$$\|u_0\|_2 = \|\varphi\|_2.$$

Assume that for some $c > 0$ *and* $\theta > 3$

$$\int_{x>x_0} u_0^2(x)\,dx \le \frac{c}{x_0^\theta} \qquad \text{for all } \; x_0 > 0.$$

Then the corresponding solution $u(t)$ *of the IVP (8.36) does not blowup in* $H^1(\mathbb{R})$ *either in finite or in infinite time.*

We recall that for $u_0 \in H^1(\mathbb{R})$ with $\|u_0\|_2 < \|\varphi\|_2$ global existence is known (see 8.40). Also that for the NLS with critical power there exists a unique (up to the invariants of the equation) blow-up solution with minimal mass, i.e., a blow-up solution for

$$\begin{cases} i\partial_t u + \Delta u + |u|^{4/n}u = 0, \\ u(x,0) = u_0(x), \end{cases}$$

$\alpha = 1 + 4/n$, and $\|u_0\|_2 = \|\varphi\|_2$, where φ is a solution of (7.10) (see [Me3]).

The blow-up problem for the local solutions of the IVP (8.36) has been revised in the sequence of works [MMR1], [MMR2], [MMR3]. In these papers, a more concise description of the results in Theorems 8.9, 8.10, and 8.11 was established.

By defining the L^2-tubular neighborhood of the soliton manifold:

$$\mathcal{V}_{\alpha^*} = \{u \in H^1(\mathbb{R}) \; : \; \operatorname*{Inf}_{\lambda_0>0,\, x_0\in\mathbb{R}} \left\| u - \frac{1}{\lambda_0}\varphi((\,\cdot\, - x_0)/\lambda_0)\right\|_2 < \alpha^*\},$$

and the set of data:

$$\mathcal{A}_{\alpha_0} = \{u_0 = \varphi + \varepsilon_0 \; : \; \|\varepsilon_0\|_2 < \alpha_0, \; \int_{y>0} y^{10}\varepsilon_0(y)dy < 1\}$$

with $0 < \alpha_0 << \alpha^* < 1$ and φ the soliton (8.38), it was obtained in [MMR1] the following blow-up scenario near the soliton in $\mathcal{A}_{\alpha_0}$.

Theorem 8.12 ([MMR1]). *There exist universal constants* $\alpha_0,\ \alpha^*$ *with* $0 < \alpha_0 << \alpha^* < 1$ *such that if* $u_0 \in \mathcal{A}_{\alpha_0}$, *with* $E(u_0) \le 0$ *and* $u_0 \neq \varphi$, *then the corresponding solution* $u(t)$ *blows up in finite time* T *and for* $t \in [0,T)$ $u(t) \in \mathcal{V}_{\alpha^*}$. *Moreover, there exists* $l_0 = l_0(u_0) > 0$ *such that*

$$\|\partial_x u(t)\|_2 \sim \frac{\|\varphi'\|_2}{l_0(T-t)}, \text{as } t \uparrow T,$$

and $\exists\ \lambda(t),\ x(t),\ u^* \in H^1,\ u^* \neq 0$ *such that*

$$u(x,t) - \frac{1}{\lambda^{1/2}(t)}\varphi((x-x_0)/\lambda(t)) \to u^* \text{ in } L^2 \text{as } t \uparrow T,$$

with

$$\lambda(t) \sim l_0(T-t) \quad \text{and} \quad x(t) \sim \frac{1}{l_0^2(T-t)}.$$

Also, there exists $\rho_0 = \rho_0(u_0) > 0$ *such that if* $v_0 \in \mathcal{A}_{\alpha_0}$ *with* $\|u_0 - v_0\|_{1,2} < \rho_0$, *then the corresponding solution* $v(t)$ *blows up in finite time* $T(v_0)$ *in the manner described above.*

Notice that $x(t) \to \infty$ as $t \uparrow T$. The next result found in [MMR1] gives a picture of the dynamic of the solution flow in $\mathcal{A}_{\alpha_0}$.

Theorem 8.13 ([MMR1]). *There exist universal constants* α_0, α^* *with* $0 < \alpha_0 \ll \alpha^* < 1$ *such that if* $u_0 \in \mathcal{A}_{\alpha_0}$, *then one of the following three possibilities occurs:*

(i) $\exists\, t^* \in [0, T)$ *such that* $u(t^*) \notin \mathcal{V}_{\alpha^*}$.
(ii) The solution $u(t)$ *blows up in finite time in the regime of the previous theorem.*
(iii) The solution is global, for all t, $u(t) \in \mathcal{V}_{\alpha^*}$ *and there exist* $\lambda_\infty > 0$, $x(t)$ *such that*

$$\lambda_\infty^{1/2} u(\lambda_\infty(\cdot + x(t)), t) \to \varphi \;\; in \;\; H^1_{loc} \quad as \quad t \uparrow \infty$$

$$x(t) \sim \frac{t}{\lambda_\infty} \quad and \quad |\lambda_\infty - 1| = o(1) \quad as \quad \alpha_0 \downarrow 0.$$

Thus, the set of data found in (i) and (ii) are open. Also, results in [MMR3] delineates the exit scenario (i) in Theorem 8.13 and the existence and uniqueness of the minimal mass blowup element.

The following result shows that the decay assumption in the definition of $\mathcal{A}_{\alpha_0}$ is essential in the above theorems. More precisely, H^1-data with slower right decay may produce "exotic" blow-up rates.

Theorem 8.14 ([MMR3]).

(i) $\forall\, \gamma > 11/13 \;\; \exists\, u \in C([0, T) : H^1(\mathbb{R}))$ *solution of the IVP* (8.36) *which blows up at* $t = T$ *with*

$$\|\partial_x u(t)\|_2 \sim \frac{1}{(T - t)^\gamma}.$$

(ii) $\exists\, u \in C([0, \infty) : H^1(\mathbb{R}))$ *solution of the IVP* (8.36) *such that*

$$\|\partial_x u(t)\|_2 \sim e^t.$$

(iii) $\forall \gamma > 0 \, \exists\, u \in C([0, \infty) : H^1(\mathbb{R}))$ *solution of the IVP* (8.36) *such that*

$$\|\partial_x u(t)\|_2 \sim t^\gamma.$$

The possibility of continua blow up rates were first observed in [KST] for solutions of the $H^1(\mathbb{R}^3)$-critical semilinear wave equations.

8.3 Comments

The global solution for the IVP (8.1) with $k \geq 5$ with small data $v_0 \in H^1(\mathbb{R})$ follows by the argument used in (8.5). This tells us that

$$E(v_0) \geq \|\partial_x v(t)\|_2^2 - c_k \,\|\partial_x v(t)\|_2^{k/2}\, \|v_0\|_2^{2+2/k}. \tag{8.60}$$

Since at $t = 0$ we have

$$E(v_0) \geq \|\partial_x v_0\|_2^2 - c_k \|\partial_x v_0\|_2^{k/2} \|v_0\|_2^{2+2/k}$$

then for $\|v_0\|_2 + \|\partial_x v_0\|_2 \ll 1$ one has $E(v_0) > 0$, which inserted into (8.60) provides an a priori estimate for $\|\partial_x v(t)\|_2$ through an argument similar to the one in (6.11). This combined with I_2 gives an a priori estimate for $\|v(t)\|_{1,2}$.

More precisely, in [FaLP] Farah, Linares and Pastor following some arguments in [HR1] proved

Theorem 8.15. *Let $u_0 \in H^1(\mathbb{R})$. Let $k > 4$ and $s_k = (k-4)/2k$. Suppose that*

$$E(u_0)^{s_k} I_2(u_0)^{1-s_k} < E(Q)^{s_k} I_2(Q)^{1-s_k}, \quad E(u_0) \geq 0. \tag{8.61}$$

If

$$\|\partial_x u_0\|_2^{s_k} \|u_0\|_2^{1-s_k} < \|\partial_x Q\|_2^{s_k} \|Q\|_2^{1-s_k}, \tag{8.62}$$

then for any t as long as the solution exists,

$$\|\partial_x u(t)\|_2^{s_k} \|u_0\|_2^{1-s_k} = \|\partial_x u(t)\|_2^{s_k} \|u(t)\|_2^{1-s_k} < \|\partial_x Q\|_2^{s_k} \|Q\|_2^{1-s_k}, \tag{8.63}$$

where $Q(x) = (k+1)^{1/k} \phi_{1,k}(x)$ and $\phi_{1,k}$ is unique positive even solution of the equation (7.7).

This in turn implies that H^1 solutions exist globally in time.

We also recall that in the case $k \geq 5$ global well-posedness based on the homogeneity (scaling argument) was established in Theorem 7.5 for small data in $\dot{H}^{s_k}(\mathbb{R})$, $s_k = 1/2 - 2/k$.

The problem of describing the long time behavior of solutions to the generalized KdV equation corresponding to "small" data was studied by Hayashi and Naumkin [HN1], [HN2].

In [HN1], they answered the following question: what is the smallest power ρ which guarantees that "small" solutions of the generalized KdV equation:

$$\partial_t u + \partial_x^3 u + |u|^{\rho-1} \partial_x u = 0, \quad \rho > 1, \tag{8.64}$$

behave as the solutions of the associated linear problem (7.21) and scatter? They showed that if $\rho > 3$ and the data u_0 satisfies that

$$\|(1+x^2)^{1/2} \Lambda u_0\|_2 \leq \varepsilon \quad \text{(for some } \varepsilon \text{ fixed} \ll 1), \tag{8.65}$$

then the corresponding solution $u(\cdot, t)$ of (8.64) satisfies that for any $t > 0$,

$$\|u(\cdot, t)\|_p \leq c\,(1+t)^{-1/3\ p'}, \quad p \in (4, \infty], \ \frac{1}{p} + \frac{1}{p'} = 1. \tag{8.66}$$

Moreover, there exists $u_+ \in L^2(\mathbb{R})$ such that for $t > 0$

$$\|u(\cdot, t) - V(t) u_+\|_2 \leq c\, t^{-(\rho-3)/3}, \tag{8.67}$$

(see the notations in (3.1) and (7.22)).

In [HN2], they proved that the above result is optimal by establishing that "small" solutions of the mKdV ($\rho = 3$ in (8.64)), although satisfy (8.65), they do not hold (8.66). (The description of their asymptotic behavior involves the self-similar solutions ($= \frac{1}{\sqrt[3]{t}}\,\omega(\frac{x}{\sqrt[3]{t}})$) of the mKdV).

Consider the periodic boundary value problem:

$$\begin{cases} \partial_t v + \partial_x^3 v + v^k \partial_x v = 0, \\ v(x,0) = v_0(x) \in H^s(\mathbb{T}), \end{cases} \tag{8.68}$$

$t \in \mathbb{R}$, $x \in \mathbb{T}$, $k \in Z^+$. Global well-posedness for (8.68) with $k = 1, 2, 3$ has been established in $H^s(\mathbb{T})$ with $s \geq -1/2$, $s \geq 1/2$, $s > 5/6$, respectively, by Colliander, Keel, Staffilani, Takaoka and Tao [CKSTT4], [CKSTT5].

For $k \geq 4$ the best results are due to Staffilani [Sta2] ($s \geq 1$ with a smallness condition on the $\|v_0\|_2$ norm).

In the same regard for the IVP (8.36), global well-posedness was obtained in $H^s(\mathbb{R})$ with $s > 6/13$ (see [MSWX]) for data satisfying $\|u_0\|_2 < \|\varphi\|_2$ (improving previous results in [FLP1] ($s > 3/4$) and in [Fa] ($s > 3/5$)). As it was mentioned this result should hold in L^2, i.e., if $u_0 \in L^2(\mathbb{R})$ and $\|u_0\|_2 < \|\varphi\|_2$, then the local solution extends globally or $\delta = \|\varphi\|_2$ in Theorem 7.2 with φ as in (8.38).

Next, we shall briefly comment on stability for the solitary wave solutions (7.6) for the k-generalized Korteweg–de Vries (k-gKdV) equation. In [Be1] and [Bn2], the stability of the solitary wave solution for the KdV equation was established. The stability is understood in the following sense: Given $\varepsilon > 0$, there exists $\delta > 0$ such that if $\|v_0 - \phi_{c,1}\|_{1,2} < \delta$, then for all $t \in \mathbb{R}$, there is $x(t)$ such that

$$\|v(\cdot + x(t), t) - \phi_{c,1}(\cdot)\|_{1,2} < \varepsilon, \tag{8.69}$$

this is known as orbital stability.

For the k-gKdV, it was proved in [BSS] that for $k < 4$ (subcritical case) the solitary waves are stable, and for $k > 4$ they are unstable (see also [GSS]). Martel and Merle [MM1] have shown the instability of the solitary waves in the critical case $k = 4$. Regarding asymptotic stability of the solitary waves $\phi_{c,k}$, Pego and Weinstein [PW] obtained results for the cases $k = 1$ and $k = 2$ for data decaying exponentially as $x \to \infty$. In [MM2], the following assertion was proved: Given c_0 there exists a δ_{0,c_0} such that for $\|v_0 - \phi_{c_0,k}\|_{1,2} \leq \delta_{0,c_0}$ there exist c_∞ a constant and $x(t)$ a real function so that

$$v(x + x(t), t) \rightharpoonup \phi_{c_\infty,k} \quad \text{in} \quad H^1 \quad \text{as } t \to \infty$$

for $k = 1, 2, 3$, i.e., the subcritical case.

The results listed above were obtained in the H^1-norm. Merle and Vega [MV] have shown the stability and asymptotic stability for the solitary wave solutions of the KdV equation in the L^2-norm. More precisely, in [MV] the following result was proved (see also [MiT]): Let $c_0 > \sigma > 0$. Then, there exist $\hat{c}, \delta > 0$ such that if $u(x,t)$ is a solution of the IVP (8.1) with $k = 1$ (KdV) such that

$$u(x,0) = \phi_{c_0,1}(x) + v_0(x), \quad \text{with} \quad \|v_0\|_2 < \delta,$$

then there exist $c_+ > 0$ and $x : [0,\infty) \mapsto \mathbb{R}$ a C^1 function such that

$$\sup_{t>0} \|u(\cdot,t) - \phi_{c_0,1}(\cdot - x(t))\|_2 \leq \hat{c}\, \|v_0\|_2^{1/2}$$

$$c_+ = \lim_{t\to\infty} x(t)$$

$$|c_+ - c_0| + \sup_{t\geq 0} |\dot{x}(t) - c_0| \leq c\, \|v_0\|_2$$

and

$$\lim_{t\to\infty} \int_{x>\sigma t} |u(x,t) - \phi_{c_+,1}(x - x(t))|^2\, dx = 0.$$

In [KM], the stability of the traveling wave solution for the quartic KdV, i.e., $k = 3$ in (8.1), was studied in the critical space $\dot{H}^{-1/6}(\mathbb{R})$.

In [W4], Weinstein deduced the following variational characterization of the traveling wave $\phi_{c,k}$ in (7.6):

If $u(x,t)$ is a solution of (8.1) with $k = 1,2,3,4$ such that

$$I_3(u(t)) = I_3(\phi_{c,k}) \quad \text{and} \quad I_2(u(t)) = I_2(\phi_{c,k}) \quad \text{for some} \quad c > 0. \tag{8.70}$$

Then, $u(x,t) = \phi_{c,k}(x - x_0 - ct)$ for some $x_0 \in \mathbb{R}$.

In particular, this implies (see Exercise 8.4) that if $u = u(x,t)$ is a solution of (8.1) with $k = 1,2,3,4$ such that

$$\lim_{t\to\infty} \operatorname*{Inf}_{y\in\mathbb{R}} \|u(\cdot,t) - \phi_{c,k}(\cdot - y)\|_{1,2} = 0, \tag{8.71}$$

then $u(x,t) = \phi_{c,k}(x - x_0 - ct)$ for some $x_0 \in \mathbb{R}$.

Based on the previous works [DM], [DRu], concerning related results for the NLS in [Cb] it was shown that (8.71) fails for $n \geq 5$. More precisely, it was proved the existence of a one parameter family of special solutions of (8.1) with $k \geq 5$ $\{U^A(x,t)\}_{A\in\mathbb{R}}$ such that

$$\lim_{t\to\infty} \operatorname*{Inf}_{y\in\mathbb{R}} \|U^A(\cdot,t) - \phi_{c,k}(\cdot - y)\|_{1,2} = 0.$$

Moreover, if $u = u(x,t)$ is a global solution of (8.1) with $k \geq 5$ such that

$$\lim_{t\to\infty} \operatorname*{Inf}_{y\in\mathbb{R}} \|u(\cdot,t) - \phi_{c,k}(\cdot - y)\|_{1,2} = 0,$$

then

$$u(x,t) = U^A(x - x_0, t), \quad t \geq t_0, \qquad \text{for some} \qquad A, t_0, x_0 \in \mathbb{R}.$$

In [AlMn1], it was established that the breather solutions of the mKdV equation (7.108) are orbitally stable in the H^1 topology.

In the introduction to this chapter we mentioned the fact that the KdV and mKdV equations can be solved via the inverse scattering method. Now, we would like to describe some interesting applications deduced from this method. The first one regards the construction of explicit solutions called N-solitons. These solutions generalize the solitary wave solutions or "solitons" (7.6) ($k = 1, 2$) (see [Lb], [Hi1], [Sc]). In particular, they describe the interaction between several solitons with different speeds. In addition, the N-soliton solutions decompose exactly as a sum of N solitons as $t \to +\infty$. In other words, for any given $0 < c_1 < c_2 < \cdots < c_N$, $x_1, \ldots, x_N$, there exists an explicit N-soliton solution $v(t)$ such that

$$\Big\| v(t) - \sum_{j=1}^{N} \phi_{c_j,k}(\cdot - x_j - c_j t) \Big\|_{1,2} \to 0 \ \text{ as } \ t \to +\infty. \tag{8.72}$$

Another interesting result obtained in [ES] for the case $k = 1$ is the following: Any sufficiently smooth and decaying solution v of (7.1) splits into two parts as $t \to \infty$, i.e.,

$$v(x,t) = v_d(x,t) + v_c(x,t),$$

where v_d is an N-soliton solution and $v_c(x,t) \to 0$ uniformly for $x > 0$ as $t \to +\infty$. (see also [Sc]).

Concerning the stability of N-solitons in the sense given in (8.69) for the solitary waves, Martel, Merle and Tsai [MMT] obtained for powers $k = 1, 2$ (integrable cases) and $k = 3$ (nonintegrable) the following result:

Theorem 8.16. *Let* $0 < c_1 < \cdots < c_N$ *and* $k = 1, 2, 3$. *There exists* γ_0, A, L_0, $\alpha_0 > 0$ *such that the following is satisfied. Assume that there exist* $L > L_0$, $\alpha < \alpha_0$, *and* $x_1^0 < \cdots < x_N^0$ *such that*

$$\| v(0) - \sum_{j=1}^{N} \phi_{c_j,k}\left(\cdot - x_j^0\right) \|_{1,2} \le \alpha, \quad \textit{with} \quad x_j^0 > x_{j-1}^0 + L$$

for all $j = 2, \ldots, N$. *Then there exist* $x_1(t), \ldots, x_N(t) \in \mathbb{R}$ *such that for all* $t \ge 0$,

$$\Big\| v(t) - \sum_{j=1}^{N} \phi_{c_j,k}(x - x_j(t)) \Big\|_{1,2} \le A(\alpha + e^{-\gamma_0 L}).$$

The above result tells us that if $v(0)$ is close in the H^1-norm to the sum of N-solitons whose speeds are ordered (so they do not interact for $t > 0$) and whose centers are far apart, then the corresponding solution $v(t)$ remains close in H^1-norm to a translated sum of N-solitons for all $t > 0$.

Using the ideas in [MV] and [MMT], Alejo, Muñoz and Vega [AlMnVe] were able to establish the L^2-stability of the N-solitons solutions.

In [Ma], the following existence and uniqueness result of an asymptotic N-soliton-like solution was established for the subcritical $k = 1, 2, 3$ and critical case $k = 4$ in (8.1).

Theorem 8.17. *Let $N \in \mathbb{Z}^+$, $0 < c_1 < c_2 < \cdots < c_N$, $x_1, \ldots, x_N \in \mathbb{R}$. There exists a unique $v \in C([T_0, \infty) : H^1(\mathbb{R}))$ for some $T_0 > 0$ solution of the equation in (8.1) with $k = 1, 2, 3$, or 4 such that (8.72) holds. Moreover, there exist $A, \gamma > 0$ such that*

$$\|v(t) - \sum_{j=1}^{N} \phi_{c_j,k}(-x_j - c_j\, t)\|_{1,2} \leq A\, e^{-\gamma t}.$$

Notice that Theorem 8.17 extends the estimate (8.72) to the nonintegrable cases $k = 3, 4$.

In [FePaUl], Fermi, Pasta and Ulam and latter in [ZaKr] Zabusky and Kruskal presented numerical evidences describing the remarkable phenomena of the soliton collision. They illustrated the *elastic* character of the collision of two solitons (elastic: the collision preserves the shape of the solitons). So the unique consequence of the collision is a shift translation on each soliton.

For the equation:

$$\partial_t u + \partial_x^3 u + \partial_x(f(u)) = 0,$$

it was established in [Mu] that the collision between two solitons is not elastic in general, except for the KdV equation, for the mKdV equation and for the Gardner equation ($f(u) = u^2 - \mu u^3$) all completely integrable systems. This work was preceded by [MM8], where for the case $f(u) = u^4$ with two solitons of different masses it was shown that the collision is inelastic by proving the nonexistence of a pure two-soliton solution. More precisely, if the solution $u(x,t)$ satisfies that

$$u(x,t) = \phi_{c_1,3}(x - c_1 t) + \phi_{c_2,3}(x - c_2 t) + \eta(x,t), \quad \text{as} \quad t \downarrow -\infty$$

(see 7.6) with

$$\|\eta(t)\|_{1,2} \ll \|\phi_{c_2,3}\|_{1,2} \ll \|\phi_{c_1,3}\|_{1,2},$$

then for $t \gg 1$

$$u(x,t) = \phi_{c_1(t),3}(x - y_1(t)) + \phi_{c_2(t),3}(x - y_2(t)) + \eta(x,t)$$

with

$$\|\eta(t)\|_{1,2} \ll \|\phi_{c_2(t),3}\|_{1,2},$$

and

$$c_1(t) \to c_1^+, \quad c_2(t) \to c_2^+ \quad \text{as} \quad t \uparrow \infty.$$

In the case where $u(x,t)$ is a pure two-soliton as $t \downarrow -\infty$, one has that

$$c_1^+ > c_1, \quad c_2^+ < c_2, \quad \lim_{t \to \infty} \|\eta(t)\|_{1,2} > 0.$$

In the case of the modified KdV equation and the Gardner equation, it is an interesting problem to characterize the initial data which precede to the formation of these special solution solitons or "breathers" (see 7.108). Using the *inverse scattering method* (IST) this question was studied in [SaYa].

Also, it is interesting to describe the interaction between these solutions traveling in opposite directions. In this regard, one has the results concerning the generalized Gardner equation found in [CGD]:

$$\partial_t u + \alpha\, u \partial_x u + \beta\, u^2\, \partial_x u + \delta\, \partial_x^3 u = 0, \quad \alpha,\ \beta,\ \delta \in \mathbb{R}. \tag{8.73}$$

It has been proved that this equation is integrable and also arises in the study of wave propagation (see [GKM]). Notice that in (8.73) the interaction between the dispersion and the nonlinearity cubic and quadratic should be considered. It was shown in [CGD] that (8.73) possesses breather solutions and solitons traveling in both directions when $\beta,\ \delta > 0$. Also based on the Hirota method of constructing multisoliton solution to integrable models (see [Hi2]) explicit expressions describing the interaction of these solutions were deduced. It was proved that these solutions retain their shape after the interaction, except for a phase shift, and numerical simulations were presented to confirm this fact.

In the same regard, one has the special solutions of the modified KdV equation:

$$\partial_t v + \partial_x^3 v + v^2 \partial_x v = 0,$$

given by solitons (described in 7.6) traveling to the right and the breathers (see (7.108)), which travel to the left if $3\, N^2 > \omega^2$. The description of the interaction of these solutions is largely open.

Regarding the "soliton resolution conjecture": any "reasonable" solution of the k-gKdV (7.1) will eventually resolve into a radiation-dispersive wave moving to the right plus a finite number of traveling waves moving to the left. Notice that the breather solution of the mKdV equation (7.108) contradicts this statement. In [ES], Eckhaus and Schuur were able to prove this conjecture for the KdV equation ($k = 1$ in (7.1)). Their proof uses the inverse scattering theory and is based on the relation between properties of the datum $u_0 = q$ and the reflected coefficient $b(k)$ (see (9.59)–(9.64)). More precisely, they proved that if u_0 and its derivatives up to order fourth have an appropriate algebraic decay as $|x| \to \infty$, then $b(k) = b \in C^r(\mathbb{R})$ and

$$\partial^m b(k) = O(|k|^{-5}) \quad \text{as} \quad |k| \to \infty,$$

for $m = 0, 1, \dots, r$. We recall that in the cases when $b(k) \equiv 0$ the solution is the sum of N solitons with N being the number of discrete eigenvalues in (9.62).

8.4 Exercises

8.1 Consider the IVP (8.36) with a real-valued datum $u_0 \in H^1(\mathbb{R})$ such that $\|u_0\|_2 < \|\varphi\|_2$ with φ as in (8.38). As it was shown in this case, $u \in C(\mathbb{R} : H^1(\mathbb{R}))$ is the global solution of the problem.

(i) Prove that for any time interval $(t_0, t_0 + \Delta T)$ with $\Delta T > 0$,

$$\sum_{j=0}^{1} \big(\|\partial_x^j u\|_{L_x^5 L_t^{10}((t_0,t_0+\Delta T))} + \|\partial_x^{j+1} u\|_{L_x^\infty L_t^2((t_0,t_0+\Delta T))} \big) \le c\,(\|u_0\|_{1,2};\ \Delta T). \tag{8.74}$$

Hint: Use Theorem 7.4, and the conservation laws I_2 and I_3 in (7.4) and (7.5). Notice that in this case $s = 1$ one can take ∂_x instead of D_x in (7.55) and (8.38).

(ii) Prove that for any time interval $(t_0, t_0 + \Delta T)$

$$\|u\|_{L_x^4 L_t^\infty((t_0,t_0+\Delta T))} \le c\,(\|u_0\|_{1,2}; \Delta T). \tag{8.75}$$

Hint: Use Lemma 7.3 and the integral equation:

$$u(t) = V(t)u_0 - \int_0^t V(t-t')\,\partial_x(u^5)(t')\,dt' = V(t)u_0 + z(t). \tag{8.76}$$

(iii) Prove that $z(\cdot)$ in (8.76) satisfies

$$z \in C(\mathbb{R} : H^2(\mathbb{R})). \tag{8.77}$$

Hint: First observe that to obtain (8.77) it suffices to show that $\partial_x^2 z \in C(\mathbb{R} : L^2(\mathbb{R}))$. Use (7.16) to reduce the problem to bound $\|\partial_x^2(u^5)\|_{L_x^1 L_t^2((t_0,t_0+\Delta T))}$ with $\Delta T \ll 1$. Now combine parts (i) and (ii) to get the desired result.

Remark 8.1. Roughly speaking, Exercise 8.1 illustrates a general principle, i.e., if $v \in C([0,T] : H^{\hat{s}}(\mathbb{R}))$ is a solution of the k-gKdV (7.10) with $\hat{s} > s_{0,k}$, where $s_{0,k}$ is the smallest Sobolev exponent, where local well-posedness can be established (i.e., $s_{0,1} = -3/4$, $s_{0,2} = 1/4, \ldots$), then the integral term in $z_k(t)$,

$$v(t) = V(t)v_0 - \int_0^t V(t-t')\,v^k \partial_x v(t')\,dt' = V(t)v_0 + z_k(t)$$

is more regular in the $H^s(\mathbb{R})$ scale than both $v(t)$ and the linear part $V(t)v_0$.

8.2 Consider the linear IVP (7.10). Prove:

(i) If $v_0 \in L^2(\mathbb{R}) \cap L^2(|x|^2 dx)$, then $V(t)v_0 \in C^1(\mathbb{R})$ for $t \neq 0$.
Hint: Use the commutative property of the operators $\Gamma = x + 3t\partial_x^2$ and $L = \partial_t + \partial_x^3$.

(ii) Given $\varepsilon > 0$ and the set:

$$A_N = \{(x_j, t_j) : j = 1, \ldots, N\} \subset \mathbb{R}^2,$$

there exists $v_0 \in H^1(\mathbb{R})$ (real valued) with $\|v_0\|_2 < \varepsilon$ such that

(a) If $t \notin \{t_1, \ldots, t_N\}$, then $V(t)v_0 \in C^1(\mathbb{R})$.
(b) If $t = t_j$, then $V(t_j)v_0 \in C^1(\mathbb{R}) - \{x_k : (x_k, t_k) \in A_n\}$, and $\partial_x V(t_j)v_0(x_k)$ does not exist if $(x_k, t_j) \in A_N$.

(iii) If $u = u(x,t)$ is the solution of the IVP (8.36) with data $u(x,0) = v_0(x)$ as in part (ii) with $\varepsilon = \|\varphi\|_2$, then (a) and (b) hold for $u(x,t)$.
Hint: Use Exercise 8.1

Remark 8.2. This is a particular case of the so-called dispersive blow up, studied by Bona and Saut [BSa1], [BSa2].

8.3 Let $v \in C(\mathbb{R} : H^2(\mathbb{R}))$ be a solution of the KdV equation.

(i) Prove that for $t \in \mathbb{R}$,

$$I_4(v)(t) = \int_{-\infty}^{\infty} \left[\frac{9}{5}(\partial_x^2 v)^2 - 3u(\partial_x v)^2 + \frac{1}{4}v^4\right](x,t)\,dx$$
$$= I_4(v)(0) = I_4(v_0). \tag{8.78}$$

(ii) Prove that there exists $c > 0$ such that

$$\sup_{t\in\mathbb{R}} \|v(t)\|_{2,2} \le c\,\|v_0\|_{2,2}.$$

Hint: Combine (8.78) and I_2, I_3 in (7.4) and (7.5).

(iii) If $\tilde{v} \in C(\mathbb{R} : H^1(\mathbb{R}))$ is solution of the IVP associated to the KdV equation and $\tilde{v}_0 \in H^{1+\delta}(\mathbb{R})$, prove that $\tilde{v} \in C(\mathbb{R} : H^{1+\delta}(\mathbb{R}))$ and deduce an upper bound for

$$\Phi(T) = \sup_{0\le t\le T} \|\tilde{v}(t)\|_{1+\delta,2}$$

in terms of T and $\|\tilde{v}_0\|_{1+\delta,2}$ (for the case of the mKdV, see (8.35)).

8.4 (i) Using the notation in (7.6), prove that

$$\frac{d}{dc}\|\phi_{c,k}\|_2 \begin{cases} > 0, & \text{if} \quad k = 1,2,3, \\ = 0, & \text{if} \quad k = 4, \\ < 0, & \text{if} \quad k = 5, \ldots. \end{cases}$$

(ii) Using the notation in (7.6)–(7.6), prove that

$$I_3(\phi_{c,k}) = \frac{k-4}{2(k+1)(k+2)} \int_{-\infty}^{\infty} \phi_{c,k}^{k+2}(x)\,dx.$$

Thus,

$$I_3(\phi_{c,k}) \begin{cases} < 0, & \text{if} \quad k = 1,2,3, \\ = 0, & \text{if} \quad k = 4, \\ > 0, & \text{if} \quad k = 5, \ldots. \end{cases}$$

Hint: Combine the equation (7.7) and the identity (5.83).

8.5 [W2] Defining the functional $B : H^1(\mathbb{R}) \mapsto \mathbb{R}$ as:

$$B(v) = I_3(v) + c\, I_2(v). \tag{8.79}$$

(i) Prove that B is differentiable, and

$$DB(v)w = \frac{d}{d\varepsilon} B(v+\varepsilon w)|_{\varepsilon=0} = 2\int_{-\infty}^{\infty} \left(-\partial_x^2 v + c\, v - \frac{v^{k+1}}{k+1}\right) w\, dx,$$

if $v \in H^2(\mathbb{R})$ and $w \in H^1(\mathbb{R})$.

(ii) Prove that $\phi_{c,k}$ is a critical point of B, i.e. $DB(\phi_{c,k}) \equiv 0$.

(iii) Prove that $DB(\cdot)$ is differentiable, and

$$D^2B(v)(h,w) = \frac{d}{d\varepsilon} DB(v+\varepsilon h)w|_{\varepsilon=0} = 2\int_{-\infty}^{\infty} \left(-\partial_x^2 h - v^k h + ch\right) w\, dx$$

if $v \in H^1(\mathbb{R})$ and $h,\ w \in H^2(\mathbb{R})$.

(iv) Using the notation:

$$\mathcal{L}_{\phi_{c,k}} f(x) = -\frac{d^2}{dx^2} f(x) - \phi_{c,k}^k(x) f(x) + c f(x),$$

show that

(a) $D^2B(\phi_{c,k})(h,w) = 2\int_{-\infty}^{\infty} \mathcal{L}_{\phi_{c,k}} h\, w\, dx = 2\int_{-\infty}^{\infty} h\, \mathcal{L}_{\phi_{c,k}} w\, dx.$

(b) $\mathcal{L}_{\phi_{c,k}} \phi_{c,k}^{\frac{k+2}{2}} = -c^{\frac{k(k+4)}{4}} \phi_{c,k}^{\frac{k+2}{2}}.$

(c) $\mathcal{L}_{\phi_{c,k}} \phi'_{c,k} = 0.$

(d) $\mathcal{L}_{\phi_{c,k}} \left(-\frac{d}{dc}\phi_{c,k}\right) = \phi_{c,k}.$

8.6 Using the notation in (7.6) prove that $h : (0,\infty) \mapsto \mathbb{R}$ defined as:

$$h(c) = I_3(\phi_{c,k}) + c\, I_2(\phi_{c,k})$$

is strictly convex if and only if $k = 1, 2, 3$.

Hint: Use Exercises 8.4(i) and 8.5(ii).

8.7 Assuming the characterization of the traveling wave described in (8.70) for $k = 1, 2, 3, 4$ prove property (8.71).

8.8 Let $u \in C([0,T] : H^4(\mathbb{R}) \cap L^2(|x|^2\, dx))$ be a real solution of the k-gKdV equation (8.1).

(i) Prove the identity:

$$\frac{d}{dt}\int x\, u^2(x,t)\, dx = -3\left[I_3(u_0) + \frac{4-2k}{3(k+1)(k+2)}\int u^{k+2}\, dx\right].$$

(ii) Prove that if $k = 2$ (mKdV) and $a_0 \in \mathbb{R}$ such that

$$\int (x - a_0)\, u_0^2(x)\, dx = 0,$$

then

$$\int (x - a(t))\, u_0^2(x)\, dx = 0,$$

with $a(t) = a_0 + t\,\dfrac{3\, I_3(u_0)}{I_2(u_0)}$.

8.9 Let $u \in C([0, T] : H^1(\mathbb{R}))$ be a solution of the k-gKdV equation (8.1). Prove that if $|x|\, u(0), |x|\, u(1) \in L^2(\mathbb{R})$, then $u \in C([0, T] : H^2(\mathbb{R}))$.

8.10 Consider the equation (8.73) with the parameters $\alpha = \delta = 1$, $\beta = -\gamma$, i.e.,

$$\partial_t u + u\partial_x u - \gamma\, u^2\, \partial_x u + \partial_x^3 u = 0, \quad \gamma \in \mathbb{R}. \tag{8.80}$$

(i) Prove that for $\gamma > 0$ the equation (8.80) has traveling wave solutions of the form $q_{c,\gamma}(x - ct)$ with

$$q_{c,\gamma}(x) = \frac{6c}{1 + \rho\, \cosh(\sqrt{c}x)}, \quad \rho = (1 - 6c\gamma)^{1/2}, \;\; c \in (0, 1/6\gamma). \tag{8.81}$$

(ii) Prove that if $u \in C([0, T] : H^4(\mathbb{R}))$ is a solution of the equation (8.80), then

$$v = v(x, t) = u - \sqrt{6\gamma}\, \partial_x u - \gamma\, u^2 \in C([0, T] : H^3(\mathbb{R}))$$

satisfies the KdV equation.

(iii) Prove that if $\gamma \downarrow 0$, then

$$q_{c,\gamma}(x) \to \phi_{c,1}(x) = 3c\, \mathrm{sech}^2\left(\frac{\sqrt{c}x}{2}\right),$$

the soliton solution of the KdV equation (7.6).

8.11 Let $u \in C([0, T^*) : H^1(\mathbb{R})) \cap \ldots$ be a local solution of the IVP (8.1) with $k = 4$ (L^2-critical case). Assume that

$$\lim_{t \uparrow T^*} \|\partial_x u(t)\|_2 = \infty.$$

(i) Prove that for any $s \in (0, 1]$,

$$\liminf_{t \uparrow T^*} \|D_x^s u(t)\|_2 = \infty.$$

(ii) Prove that for any $p \in (2, \infty]$,

$$\liminf_{t \uparrow T^*} \|u(t)\|_p = \infty.$$

Chapter 9
Other Nonlinear Dispersive Models

In this chapter, we will discuss local and global well-posedness for some nonlinear dispersive models arising in different physical situations. Our goal is to present some relevant results associated to the equations to be contemplated here and it is by no means an exhaustive study of each of them. In Section 9.1 we will treat the Davey–Stewartson systems. The Ishimori equations will be considered in Section 9.2. The Kadomtsev–Petviashvili (KP) equations will be discussed in Section 9.3. The Benjamin–Ono equation will be studied in Section 9.4 and in Section 9.5 we will be examine the Zakharov systems. Finally, in Section 9.6 we will briefly review the inverse scattering method for the KdV equation and well-posedness results regarding higher order KdV equations.

9.1 Davey–Stewartson Systems

The cubic nonlinear Schrödinger equation

$$i\partial_t u + \partial_x^2 u = \pm |u|^2 u, \quad x,\ t \in \mathbb{R},$$

among other phenomena models the propagation of wave packets in the theory of water waves. It is also a complete integrable system. The corresponding bi-dimensional model is called the Davey–Stewartson system, which is given by the nonlinear system of partial differential equations,

$$\begin{cases} i\partial_t u + c_0\partial_x^2 u + \partial_y^2 u = c_1|u|^2 u + c_2 u\partial_x \varphi, \\ \partial_x^2 \varphi + c_3 \partial_y^2 \varphi = \partial_x(|u|^2), \end{cases} \tag{9.1}$$

$x, y \in \mathbb{R},\ t > 0$, where $u = u(x,y,t)$ is a complex-valued function, $\varphi = \varphi(x,y,t)$ is a real-valued function, and c_0, c_3 are real parameters and c_1, c_2 are complex parameters. It was first derived by Davey and Stewartson in [DS] in the case $c_3 > 0$. When capillary effects are important, Djordjevic and Redekopp [DR] showed that c_3 can be negative (see also Benney and Roskes [BnR]). Independently, Ablowitz and Haberman [AH] obtained a particular form of (9.1) as an example of a completely integrable model generalizing the two-dimensional nonlinear Schrödinger

F. Linares, G. Ponce, *Introduction to Nonlinear Dispersive Equations*,
Universitext, DOI 10.1007/978-1-4939-2181-2_9

equation. In the context of inverse scattering theory the system above with parameters $(c_0, c_1, c_2, c_3) = (1, -1, -2, -1), (-1, -2, 1, 1)$, and $(-1, 2, -1, 1)$ are known as DSI, DSII defocusing, and DSII focusing, respectively. For these particular cases several results regarding the existence of solitons and the Cauchy problem have been established by inverse scattering techniques (see [AnF], [BC1], [FS1], [Su1]). For instance, in [FS1] Fokas and Sung proved that for initial data in the Schwartz class $\mathcal{S}(\mathbb{R}^2)$ and boundary data $\partial_x \varphi_1(x,t)$ and $\partial_y \varphi_2(y,t)$ in the Schwartz class in the spatial variable and continuous in t, (9.1) has a unique global solution in time t which, for each t belongs to the Schwartz class in the spatial variable. The same result was obtained in [BC1] for the DSII defocusing.

Ghigladia and Saut [GS] classified the system as elliptic-elliptic, hyperbolic-elliptic, elliptic-hyperbolic, and hyperbolic-hyperbolic according to the signs of the parameters (c_0, c_3), i.e., $(+,+)$, $(-,+)$, $(+,-)$, and $(-,-)$, respectively.

Solutions of (9.1) satisfy the following two conservation laws:

$$M(u_0) = \int\limits_{\mathbb{R}^2} |u(x,y,t)|^2\, dxdy,$$

$$E(u_0) = \int\limits_{\mathbb{R}^2} (c_0\, |\partial_x u(x,y,t)|^2 + |\partial_y u(x,y,t)|^2)\, dxdy$$
$$+ \frac{1}{2} \int\limits_{\mathbb{R}^2} (c_1\, |u(x,y,t)|^4 + c_2\, (\partial_x \varphi)^2(x,y,t) + c_3\, (\partial_y \varphi)^2(x,y,t))\, dxdy.$$

The elliptic-elliptic and hyperbolic-elliptic cases were considered by Ghigladia and Saut [GS]. In these cases they reduced the system (9.1) to the nonlinear cubic Schrödinger equation with a nonlocal nonlinear term, i.e.,

$$i\,\partial_t u + c_0 \partial_x^2 u + \partial_y^2 u = c_1 |u|^2\, u + A(u),$$

where $A(u) = (\Delta^{-1}\, \partial_x^2 |u|^2)u$. They showed local well-posedness for data in $L^2(\mathbb{R}^2)$, $H^1(\mathbb{R}^2)$, and $H^2(\mathbb{R}^2)$ using Strichartz estimates (see 4.23) and the continuity properties of the operator $\Delta^{-1}\partial_x^2$. They also established global well-posedness and blow up results for the elliptic–elliptic case (see also [SG]). Ozawa in [Oz] found exact blow up solutions in the hyperbolic–elliptic case (see Exercise 9.6).

For the elliptic–hyperbolic and hyperbolic–hyperbolic cases the Strichartz estimates by itself does not provide the desired result. To explain this we will consider without loss of generality $c_0 = \pm 1$ and $c_3 = -1$. So using a rotation in the xy-plane and assuming that φ satisfies the radiation condition

$$\lim_{y \to \infty} \varphi(x,y,t) = \varphi_1(x,t) \quad \text{and} \quad \lim_{x \to \infty} \varphi(x,y,t) = \varphi_2(y,t),$$

for some given functions φ_1, φ_2, then the system (9.1) can be written as

$$\begin{cases} i\partial_t u + Hu = d_1\,|u|^2 u + d_2\,u \int\limits_y^\infty \partial_x(|u(x,y',t)|^2)\,dy' \\ \qquad\qquad + d_3\,u \int\limits_x^\infty \partial_y(|u(x',y,t)|^2)\,dx' + d_4 u\partial_x\varphi_1 + d_5 u\partial_y\varphi_2, \\ u(x,y,0) = u_0(x,y), \end{cases} \tag{9.2}$$

where $H = \Delta$ in the elliptic–hyperbolic case and $H = 2\partial_x\partial_y$ in the hyperbolic-hyperbolic case. The difficulty of these problems comes from the fact that the nonlinear terms contain derivatives of the unknown function and that the terms

$$\int\limits_y^\infty \partial_x(|u(x,y',t)|^2)\,dy' \quad \text{and} \quad \int\limits_x^\infty \partial_y(|u(x',y,t)|^2)\,dx'$$

do not decay as $|x| \to \infty$, $|y| \to \infty$, respectively.

To describe the results in these two cases we introduce the weighted Sobolev spaces $\mathcal{F}_l^m$ defined as follows:

$$\mathcal{F}_l^m = H^m(\mathbb{R}^2) \cap L^2(|x|^l\,dx).$$

First we look at the elliptic–hyperbolic case. In [LiPo] Linares and Ponce proved local well-posedness for the IVP (9.2) for sufficiently small data in $\mathcal{F}_{12}^m$, $m \geq 12$, $\varphi_1 = \varphi_2 \equiv 0$. They use the smoothing effect of Kato's type associated to the group $\{e^{iHt}\}$. Chihara [Ch1], using pseudo differential operators, obtained a local result for data in $u_0 \in H^m(\mathbb{R}^2)$ satisfying $\|u_0\|_2 \leq 1/(2\sqrt{\max\{d_1,d_2\}})$, $\varphi_1 = \varphi_2 \equiv 0$, with m sufficiently large. Hayashi in [H2] showed local well-posedness for small data in $\mathcal{F}_{2\,l}^m$, $m, l > 1$. The main tool for accomplishing this was the use of smoothing effects. In [HH2] Hayashi and Hirata proved that one can have local result in the usual Sobolev space $H^{5/2}(\mathbb{R}^2)$ for data with L^2-norm small. The latest updated result is due to Hayashi [H3], where he got local well-posedness for the IVP (9.2) for data of any size in $H^s(\mathbb{R}^2)$, $s \geq 2$. Global results were obtained by Hayashi and Hirota in [HH1] for small data in $\mathcal{F}_6^3$; see also [Ch1]. For analytic function spaces a global result for small data was established by Hayashi and Saut in [HS].

For the hyperbolic–hyperbolic case, using Kato's smoothing effect Linares and Ponce proved local well-posedness for small data in $\mathcal{F}_4^6$, $\varphi_1 = \varphi_2 = 0$ in [LiPo]. Hayashi [H2] showed local well-posedness for small data in $\mathcal{F}_{2\delta}^\delta$, $\delta > 1$. No local well-posedness results are known without restriction on the size of the data.

9.2 Ishimori Equation

In this section we comment on local and global well-posedness results for a two dimensional generalization of the Hesinberg equation, called the Ishimori equation which reads

$$\begin{cases} \partial_t S = S \wedge (\partial_x^2 S \pm \partial_y^2 S) + b(\partial_x \phi \partial_y S + \partial_y \phi \partial_x S), \\ \partial_x^2 \phi \mp \partial_y^2 \phi = \mp 2S \cdot (\partial_x S \wedge \partial_y S), \end{cases} \tag{9.3}$$

$x, y, t \in \mathbb{R}$, where $S(\cdot, t) : \mathbb{R}^2 \to \mathbb{R}^3$ with $\|S\| = 1$, $S \to (0,0,1)$ as $\|(x,y)\| \to \infty$, and $\wedge$ denotes the wedge product in $\mathbb{R}^3$.

This model was proposed by Ishimori in [Is1] as a two-dimensional generalization of the Heisenberg equation in ferromagnetism, which corresponds to the case $b = 0$ and signs $(-,+,+)$ in (9.3) and it was studied in [SSB].

For $b = 1$ the system (9.3) is completely integrable by inverse scattering (see [AH], [BC1], [KMa], [Su2], [ZK], and references therein).

Using the stereographic variable $u : \mathbb{R}^2 \mapsto \mathbb{C}$ one can get rid of the constraint $\|S\| = 1$. Thus, for

$$\begin{aligned} u &= \frac{S_1 + iS_2}{1 + S_3}, \\ S &= (S_1, S_2, S_3) = \frac{1}{1 + |u|^2}(u + \bar{u}, -i(u - \bar{u}), 1 - |u|^2), \end{aligned} \tag{9.4}$$

the IVP for (9.3) can be written as

$$\begin{cases} i\partial_t u + \partial_x^2 u + a\partial_y^2 u = 2u\dfrac{(\partial_x u^2 - \partial_y u^2)}{(1 + |u|^2)} - ib(\partial_x \phi \partial_y u - \partial_y \phi \partial_x u), \\ \partial_x^2 \phi + a'\partial_y^2 \phi = 8\,\mathcal{I}m\,\dfrac{(\partial_x u \partial_y u)}{(1 + |u|^2)^2}, \\ u(x, y, 0) = u_0(x, y), \end{cases} \tag{9.5}$$

with the condition $u(x, y, t) \to 0$ as $\|(x, y)\| \to \infty$, where $a, a' \in \mathbb{R} \setminus \{0\}$.

To discuss the local and global results we will distinguish two cases: case $(-,+)$, i.e., $a < 0$ in the first equation, and $a' > 0$ in the second equation in (9.5) and case $(+,-)$ with similar connotation.

The case $(-,+)$ was studied by Soyeur [Sy]. He obtained local well-posedness for the IVP (9.5) for small data in $H^m(\mathbb{R}^2)$, $m \geq 4$. Assuming additional regularity on the data he extended the local solution globally in $H^m(\mathbb{R}^2)$, $m \geq 6$. The argument used here does not extend to the case $(+,-)$.

The case $(+,-)$ was first studied by Hayashi and Saut [HS]. They considered the problem in a class of analytic functions obtaining local and global existence results for small analytic data. This approach allows them to overcome the loss of derivatives introduced by the nonlinearity.

Hayashi in [H4] removed the analyticity hypotheses used in [HS]. He established local well-posedness for the IVP (9.5) for small data in the weighted Sobolev $\mathcal{F}_8^4$.

In [KPV9] Kenig, Ponce and Vega established a local well-posedness result for data of arbitrary size in the space $\mathcal{F}_{2m}^s = H^s(\mathbb{R}^2) \cap L^2(|x|^{2m}\,dx)$, $s > m$. The method of proof follows closely the method explained in detail in the next chapter.

9.3 KP Equations

Here we shall discuss some well-posedness results for the Kadomtsev–Petviashvili (KP) equations. The KP equations are two-dimensional versions of the KdV equation. They arise in many physical contexts as models for the propagation of weakly nonlinear dispersive long waves, which are essentially one-directional, with weak transverse effects. For instance, in the plasma physics context these models were derived by Kadomtsev and Petviashvili [KP]. Meanwhile, in surface water wave theory, they were deduced by Ablowitz and Segur in [AS1]. It is also one of the classical prototype problems in the field of exactly solvable equations (see [AC] for a complete set of references on this subject).

The equation reads as follows.

$$\begin{cases} \partial_x(\partial_t u + \partial_x^3 u + u\partial_x u) \mp \partial_y^2 u = 0, \\ u(x,y,0) = u_0(x,y), \end{cases} \tag{9.6}$$

$x, y \in \mathbb{R},\ t > 0$. Under some conditions on the initial data, (9.6) can be written as

$$\begin{cases} \partial_t u + \partial_x^3 u + u\partial_x u \mp \partial_x^{-1}\partial_y^2 u = 0, \\ u(x,y,0) = u_0(x,y), \end{cases} \tag{9.7}$$

$x, y \in \mathbb{R},\ t > 0$. When the sign in front of $\partial_x^{-1}\partial_y^2$ in (9.7) is minus we refer to this equation as the KPI equation; otherwise we called it the KPII equation.

The results concerning well-posedness for KPI and KPII equations are quite different. We will first list the results regarding the KPII equation.

Bourgain [Bo10] showed local and global well-posedness for data in $H^s(\mathbb{R}^2)$, $s \geq 0$. The local result was obtained by the Fourier transform restriction method introduced by him to study nonlinear dispersive equations. In [Tz1] Tzvetkov obtained local results in anisotropic Sobolev spaces $H^{s_1,s_2}(\mathbb{R}^2)$ defined as

$$H^{s_1,s_2}(\mathbb{R}^2) = \{f \in \mathcal{S}'(\mathbb{R}^2) \ : \ \|f\|^2_{H^{s_1,s_2}} = \int_{\mathbb{R}^2} (1+|\xi_1|)^{2s_1}(1+|\xi_2|)^{2s_2}\,|\widehat{f}(\xi_1,\xi_2)|^2\,d\xi_1\,d\xi_2 < \infty\},$$

with $s_1 > -1/4$, $s_2 \geq 0$. He combined the ideas in [Bo1] with bilinear estimates in [KPV6] and Strichartz estimates. Improvements of these results were obtained in [Tz2], [Tk2]. Independently, Isaza and Mejia [IM1] and Takaoka and Tzvetkov [TT] established local well-posedness for data in $H^{s_1,s_2}(\mathbb{R}^2)$ for $s_1 > -1/3$ and $s_2 \geq 0$. Global results are also obtained in [IM1], [Tk2] using Bourgain's method in [Bo5]. In [IM2] Isaza and Mejia using the I-method introduced by [CKSTT6] showed global well-posedness for data in $H^{s_1,s_2}(\mathbb{R}^2)$ for $s_1 > -1/14$ and $s_2 \geq 0$. In [HaHK] Hadac, Herr and Koch obtained local well-posedness in the critical space $\dot{H}^{-1/2,0}(\mathbb{R}^2)$ (see Exercise 9.14(i)). These solutions corresponding to small data are global and scattered. They also showed local well-posedness in the inhomogeneous case $H^{1/2,0}(\mathbb{R}^2)$ for arbitrary data.

The problem for the KPI equation is completely different. The techniques used in Bourgain [Bo1] do not work here due to the lack of symmetry of the symbol associated to the equation. In [IN] Iorio and Nunes proved local existence result using the Kato quasilinear theory for data in $H^s(\mathbb{R}^2)$, $s > 2$. Molinet, Saut and Tzvetkov [MST1] showed that the difficulty with respect to the symmetry of the symbol was not at all technical, by proving that a Picard's scheme cannot be applied to study local in well-posedness for that equation in standard Sobolev spaces. However, they obtained [MST2] using the conservation laws for the solution flow of the KPI equation and a compactness argument the global existence of solutions for (9.7).

In [Ke] Kenig showed local well-posedness in

$$Y_s = \{u \in L^2(\mathbb{R}^2) : \|u\|_2 + \|J_x^s u\|_2 + \|\partial_x^{-1}\partial_y u\|_2 < \infty\} \tag{9.8}$$

for the KPI equation, $s > 3/2$. Combining this local result with the results in [MST2] he established global well-posedness in the space

$$Z_0 = \{u \in L^2(\mathbb{R}^2) : \|u\|_2 + \|\partial_x^{-1}\partial_y u\|_2 + \|\partial_x^2 u\|_2 + \|\partial_x^{-2}\partial_y^2 u\|_2 < \infty\}.$$

In [CIKS] Colliander, Ionescu, Kenig and Staffilani obtained local well-posedness in the space $Y_1 \cap L^2(|y|\,dxdy)$.

In [IKT] Ionescu, Kenig and Tataru proved global well-posedness in the energy space Y_1, i.e. $u_0, \partial_x u_0, \partial_y \partial_x^{-1} u_0 \in L^2(\mathbb{R}^2)$.

Regarding the periodic setting there are some results by Bourgain [Bo10], Iorio and Nunes [IN], and Isaza, Mejía and Stallbohm [IMS].

9.4 BO Equation

$$\begin{cases} \partial_t u + \mathsf{H}\partial_x^2 u + u\partial_x u = 0, \\ u(x,0) = u_0(x), \end{cases} \tag{9.9}$$

$x \in \mathbb{R}$, $t > 0$, where H denotes the Hilbert transform (see Definition 1.7).

This integro-differential equation serves as a generic model for the study of weakly nonlinear long waves incorporating the lowest-order effects of nonlinearity and non-local dispersion. In particular, the propagation of internal waves in stratified fluids of great depth is described by the BO equation (see [Be2], [On]) and turns out to be important in other physical situations as well (see [DaR], [Is2], [MK]). Among noticeable properties of this equation we can mention that it defines a Hamiltonian system, can be solved by an analogue of the inverse scattering method (see [AF]), admits (multi)soliton solutions (see [Ca]), and satisfies infinitely many conserved quantities (see [Ca]).

Regarding the IVP associated to the BO equation, local and global results have been obtained by various authors. Iorio [Io1] showed local well-posedness for data in $H^s(\mathbb{R})$, $s > 3/2$, and making use of the conserved quantities he extended globally the result in $H^s(\mathbb{R})$, $s \geq 2$. He also studied the problem in weighted Sobolev spaces. In [Po], Ponce extended the local result for data in $H^{3/2}(\mathbb{R})$ and the global result for any solution in $H^s(\mathbb{R})$, $s \geq 3/2$. The argument of proof combines parabolic regularization, smoothing properties, and energy estimates. In [MST3], Molinet, Saut and Tzvetkov showed that the Picard iteration process cannot be carry out to prove local results for the BO equation in $H^s(\mathbb{R})$ for any $s \in \mathbb{R}$. Koch and Tzvetkov [KTz] established a local result for data in $H^s(\mathbb{R})$, $s > 5/4$, improving the one given in [Po]. In [KeKo] Kenig and Köenig refined the argument in [KTz] to obtain $s > 9/8$. The main idea is the use of the Strichartz estimates to control one derivative of the solution. More precisely, through energy estimates and Kato–Ponce commutator estimates (3.16) a smooth solution of the BO equation satisfies

$$\|D^s u\|_{L^\infty_T L^2_x} \leq \|u(0)\|_{s,2} \exp\left(c\int_0^T \|\partial_x u(t)\|_{L^\infty}\, dt\right). \tag{9.10}$$

Then the Strichartz estimates allow them to establish the existence of a constant c such that

$$\int_0^1 \|\partial_x u(t)\|_{L^\infty}\, dt \leq c \tag{9.11}$$

whenever $u_0 \in H^s(\mathbb{R})$, $s > 5/4$. Thus, a combination of (9.10) and (9.11) and a standard compactness argument yields the result.

Tao in [To4] showed that the IVP associated to the BO equation is globally well-posed in $H^1(\mathbb{R})$. The new tool introduced by him was the following gauge transformation

$$w = P_+(e^{-iF}u), \quad F = F(u) = \int_{-\infty}^x u(y,t)\, dy, \tag{9.12}$$

where $\widehat{P_+ f}(\xi) = \chi_{[0,\infty)}(\xi)\widehat{f}(\xi)$. This is a variant of the Cole–Hopf transformation for viscous Burgers' equation (see Exercise 9.18), which in this setting allows one to remove most of the worst terms involving the derivative.

Tao's gauge transformation idea was further developed by Burq and Planchon [BuPl] to carry the local well-posedness to $H^s(\mathbb{R})$, for $s > 1/4$ and by Ionescu and Kenig [IK1] to extend it to $H^s(\mathbb{R})$, $s \geq 0$. We refer to [MoPi] for further discussion of the latter results.

In the periodic setting, Molinet [Mo1] has shown global well-posedness for data in $L^2(\mathbb{T})$.

In the previous chapters we discussed some decay and smoothness properties for solutions of the NLS and k-gKdV equations and their relationship. In particular, for initial data in the Schwartz class $\mathcal{S}$, the corresponding solutions (for smooth nonlinearity) also belong to this class in their life span. The decay of the data is reflected in the decay of the corresponding solutions of the associated IVP (persistence). Solutions of the BO equation do not share this property, not even mild persistence properties regarding the decay hold. To illustrate this unusual character of solutions of the BO equation we shall recall the following spaces:

$$\mathcal{F}_r^s = H^s(\mathbb{R}) \cap L^2(|x|^r\,dx),$$

and

$$\dot{\mathcal{F}}_r^s = \left\{ f \in \mathcal{F}_r^s \;:\; \int f(x)\,dx = \widehat{f}(0) = 0 \right\}.$$

The following result is due to Iório [Io2].

Theorem 9.1. *Let $u \in C([0,T] : H^2(\mathbb{R}))$, $T > 0$, be the solution of the IVP (9.9).*

(i) If $u_0 \in \mathcal{F}_{2j}^2$, $j = 1, 2$. Then

$$u \in C([0,T] : \mathcal{F}_{2j}^2), \quad j = 1, 2.$$

(ii) If $u_0 \in \mathcal{F}_6^3$ and $\int u_0(x)\,dx = 0$. Then

$$u \in C([0,T] : \dot{\mathcal{F}}_6^3).$$

(iii) If $u \in C([0,T] : \dot{\mathcal{F}}_8^4)$. Then $u \equiv 0$.

In [Io3] Iorio strengthened the result in Theorem 9.1(iii) by proving that if at three different times a solution of the BO equation satisfies $u(\cdot, t_j) \in \mathcal{F}_8^4$, $j = 1, 2, 3$, then $u \equiv 0$. In [FoPo] Iorio's result was extended to non-integer values. In particular it was shown that if $u(\cdot, t_j) \in \mathcal{F}_7^{7/2}$, $j = 1, 2, 3$, then $u \equiv 0$, and that for every $\epsilon > 0$ if $u_0 \in \dot{\mathcal{F}}_{7-\epsilon}^{7/2}$, then the corresponding solution satisfies $u \in C([0,T] : \dot{\mathcal{F}}_{7-\epsilon}^{7/2})$. In [FLP3] it was shown that the uniqueness result of Iorio mentioned above involving a condition a three different times is necessary. More precisely, it was proved that there exist non zero solutions of the BO equation $u \in C([0,T] : \dot{\mathcal{F}}_6^3)$ such that $u(\cdot, t_j) \in \mathcal{F}_8^4$, $j = 1, 2$.

Notice that the above results are mainly a consequence of the lack of smoothness of the symbol $\sigma(\xi) = \xi|\xi|$ modeling the dispersion.

For the sake of completeness we shall explain the parabolic regularization method or artificial viscosity method for the case of the BO equation. This method which is quite general will be used in the next chapter.

The goal is to establish the following local well posedness result for the IVP (9.9) associated to the BO equation.

Theorem 9.2. *Let $s > 3/2$. Given any $u_0 \in H^s(\mathbb{R})$ there exist $T(\|u_0\|_{s,2}) > 0$ and a unique solution u of the IVP (9.9) such that*

$$u \in C([0,T] : H^s(\mathbb{R})) \cap C^1((0,T) : H^{s-2}(\mathbb{R})). \tag{9.13}$$

Moreover, the map data → solution from $H^s(\mathbb{R})$ to $C([0,T]:H^s(\mathbb{R}))$ is locally well defined and continuous.

In addition, if $u_0 \in H^{s'}(\mathbb{R})$ with $s' > s$, then

$$u \in C([0,T]:H^{s'}(\mathbb{R})) \cap C^1((0,T):H^{s'-2}(\mathbb{R})).$$

To simplify the exposition we shall sketch the details in the case $s = 2$. It will be clear from our proof below and the calculus of inequalities in Chapter 3 how to obtain the general result $s > 3/2$.

We consider the IVP associated to the viscous BO equation

$$\begin{cases} \partial_t u + \mathsf{H}\partial_x^2 u + u\partial_x u = \gamma\partial_x^2 u, \\ u(x,0) = u_0(x), \end{cases} \tag{9.14}$$

$t > 0$, $x \in \mathbb{R}$, $\gamma \in (0,1)$.

Step 1 *A priori* estimate for solutions (9.14)

Assume that $u^\gamma \in C([0,T^*]:H^2(\mathbb{R})) \cap C^\infty((0,T^*):H^\infty(\mathbb{R}))$ is a solution of the IVP (9.14), then the standard energy estimate (see 3.12 and 3.13) show that

$$\frac{d}{dt}\|u^\gamma(t)\|_{2,2}^2 + \gamma\|\partial_x^3 u^\gamma(t)\|_2^2 \le c\|\partial_x u^\gamma(t)\|_\infty \|u^\gamma(t)\|_{2,2}^2. \tag{9.15}$$

Thus, from Sobolev Embedding (Theorem 3.2)

$$\frac{d}{dt}\|u^\gamma(t)\|_{2,2} \le c\|u^\gamma(t)\|_{2,2}^2, \tag{9.16}$$

where c here and below will denote a constant whose value may change from line to line but it is independent of the data and the parameters in (9.9) and (9.14) (and later in (9.32)). From (9.16) one has that

$$\|u^\gamma(t)\|_{2,2} \le \frac{\|u_0\|_{2,2}}{1 - ct\|u_0\|_{2,2}}, \tag{9.17}$$

therefore taking T such that

$$cT\|u_0\|_{2,2} = 1/2 \tag{9.18}$$

it follows that

$$\sup_{[0,T]}\|u^\gamma(t)\|_{2,2} \le 2\|u_0\|_{2,2}, \qquad \forall\gamma \in (0,1). \tag{9.19}$$

Now integrating in the t variable in (9.15), using Sobolev embedding and (9.18)–(9.19) one gets that

$$\gamma\int_0^T \|\partial_x^3 u^\gamma(t)\|_2^2 dt \le \|u_0\|_{2,2}^2 + c\int_0^T (2\|u_0\|_{2,2})^3 dt \tag{9.20}$$

$$\leq \|u_0\|_{2,2}^2 + 8cT\|u_0\|_{2,2}^3 \leq c\|u_0\|_{2,2}^2.$$

To complete this step we observe that if $u_0 \in H^{s'}(\mathbb{R})$ with $s' > 2$, then as in (9.15) it follows that

$$\frac{d}{dt}\|u^\gamma(t)\|_{s',2} \leq c\|u^\gamma(t)\|_{2,2}\|u^\gamma(t)\|_{s',2}.$$

Hence,

$$\sup_{[0,T]}\|u^\gamma(t)\|_{s',2} \leq \|u_0\|_{s',2}\, e^{cT\|u_0\|_{2,2}} = K\,\|u_0\|_{s',2}, \qquad K = K(\|u_0\|_{2,2}),$$

i.e. higher derivatives of the solution are also bounded by the data in the same time interval $[0,T]$.

Step 2 Existence of solutions to the IVP (9.14).

We consider the semigroup $\{U^\gamma(t) : t \geq 0\}$ defined as

$$U^\gamma(t)f(x) = \left(e^{4\pi^2 i|\xi|\xi t}e^{-\gamma 4\pi^2\xi^2 t}\widehat{f}\right)^\vee(x).$$

It is easy to see that for any $t \geq 0$

$$\begin{aligned} &(a) \quad \|U^\gamma(t)f\|_2 \leq \|f\|_2,\\ &(b) \quad \|\partial_x U^\gamma(t)f\|_2 \leq \frac{c}{(\gamma t)^{1/2}}\|f\|_2. \end{aligned} \tag{9.21}$$

The solution of the IVP (9.14) is a fixed point of the operator $\Psi = \Psi_{\gamma,u_0}$ with

$$\Psi(v)(t) = U^\gamma(t)u_0 - \int_0^t U^\gamma(t-t')v\partial_x v(t')dt', \tag{9.22}$$

defined on

$$\Omega_{\widehat{T},r} = \{v : \mathbb{R}\times[0,\widehat{T}] \to \mathbb{R} \,:\, v \in C([0,\widehat{T}] : H^2(\mathbb{R})),\ \sup_{[0,\widehat{T}]}\|v(t)\|_{2,2} \leq r\}, \tag{9.23}$$

with $\widehat{T}$ and $r > 0$ to be chosen. From (9.21) it follows that

$$\sup_{[0,T_\gamma]}\|\Psi(v)(t)\|_{2,2} \leq c\|u_0\|_{2,2} + \frac{cT_\gamma^{1/2}}{\gamma^{1/2}}\sup_{[0,T_\gamma]}\|v(t)\|_{2,2}^2,$$

and

$$\sup_{[0,T_\gamma]}\|(\Psi(v) - \Psi(\tilde{v}))(t)\|_{2,2}$$
$$\leq \frac{cT_\gamma^{1/2}}{\gamma^{1/2}}\sup_{[0,T_\gamma]}(\|v(t)\|_{2,2} + \|\tilde{v}(t)\|_{2,2})\sup_{[0,T_\gamma]}\|(v-\tilde{v})(t)\|_{2,2}.$$

Hence, choosing r and T_γ as

$$r = 2c\|u_0\|_{2,2} \quad \text{and} \quad \frac{cT_\gamma^{1/2} r}{\gamma^{1/2}} = \frac{2c^2 T_\gamma^{1/2}\|u_0\|_{2,2}}{\gamma^{1/2}} = \frac{1}{4}, \tag{9.24}$$

it follows that the operator Ψ defines a contraction in $\Omega_{T_\gamma,r}$, and so for any $\gamma > 0$ the IVP (9.14) has a unique solution

$$u^\gamma \in C([0,T_\gamma] : H^2(\mathbb{R})) \cap C^\infty(\mathbb{R} \times (0,T_\gamma)), \quad \text{with} \quad T_\gamma \sim \gamma. \tag{9.25}$$

Now using the *a priori* estimate (step–1) we can reapply the above local existence argument (which only depends on the size of the initial data, see 9.24) to extend for each $\gamma \in (0,1)$ the solution u^γ in the class (9.25) to the whole time interval $[0,T]$ with T as in (9.18). Moreover, we have that

$$\sup_{\gamma>0} \big(\sup_{[0,T]} \|u^\gamma(t)\|_{2,2}^2 + \gamma \int_0^T \|\partial_x^3 u^\gamma(t)\|_2^2 dt\big) \le c\|u_0\|_{2,2}^2. \tag{9.26}$$

Step 3 Convergence of the u^γ's as $\gamma \downarrow 0$.

For $1 > \gamma > \gamma' > 0$ we define

$$\omega(t) = \omega^{\gamma,\gamma'}(t) = u^\gamma(t) - u^{\gamma'}(t), \tag{9.27}$$

which satisfies the equation

$$\partial_t\omega + \mathsf{H}\partial_x^2\omega + \omega\partial_x u^\gamma + u^{\gamma'}\partial_x\omega = \gamma'\partial_x^2\omega + (\gamma-\gamma')\partial_x^2 u^\gamma, \qquad t \in [0,T], \tag{9.28}$$

with data $\omega(x,0) = 0$. Using energy estimates it follows that

$$\frac{d}{dt}\|\omega(t)\|_2 \le c(\|\partial_x u^\gamma(t)\|_\infty + \|\partial_x u^{\gamma'}(t)\|_\infty)\|\omega(t)\|_2 + (\gamma-\gamma')\|\partial_x^2 u^\gamma(t)\|_2.$$

Hence, from (9.26) one has that

$$\begin{aligned} \sup_{[0,T]} \|(u^\gamma - u^{\gamma'})(t)\|_2 &= \sup_{[0,T]} \|\omega(t)\|_2 \\ &\le (\gamma-\gamma')\int_0^T \|\partial_x^2 u^\gamma(t)\|_2 dt \; \exp\{\int_0^T (\|\partial_x u^\gamma\|_\infty + \|\partial_x u^{\gamma'}\|_\infty) dt\} \\ &\le 2(\gamma-\gamma')T\|u_0\|_{2,2} \cdot e^{4cT\|u_0\|_{2,2}} \le (\gamma-\gamma')K, \end{aligned} \tag{9.29}$$

$K = K(\|u_0\|_{2,2})$ which shows that the u^γ's converge as $\gamma \downarrow 0$ in $C([0,T] : L^2(\mathbb{R}))$. Moreover, combining (9.26) and interpolation the u^γ's converge in $C([0,T] : H^{2-\mu}(\mathbb{R}))$ for any $\mu > 0$ to a limit function $\widetilde{u}(x,t)$

$$\widetilde{u} \in C([0,T] : H^{2-\mu}(\mathbb{R})) \cap L^\infty([0,T] : H^2(\mathbb{R})), \qquad \forall\, \mu > 0 \tag{9.30}$$

(using a weak compactness argument) satisfying that

$$\sup_{[0,T]} \|\widetilde{u}(t)\|_{2,2} \leq c\|u_0\|_{2,2}. \tag{9.31}$$

To complete this step we observe that if $u_0 \in H^{s'}(\mathbb{R})$ with $s' > 2$, then the u^{γ}'s converge as $\gamma \downarrow 0$ in $C([0,T] : H^{s'-\mu}(\mathbb{R}))$, $\mu > 0$ and by taking limit with $s' - \mu \geq 2$ one has that

$$\widetilde{u} \in C([0,T] : H^{s'-\mu}(\mathbb{R})) \cap L^{\infty}([0,T] : H^{s'}(\mathbb{R})) \qquad \forall\, \mu > 0$$

is a solution of the IVP (9.9).

Step 4 Persistence property: $u \in C([0,T] : H^{s}(\mathbb{R}))$.

We need some preliminary estimates. Let $\rho \in C_0^{\infty}(\mathbb{R})$ be such that

$$\rho(x) \geq 0 \ \ \forall x \in \mathbb{R}, \quad \int \rho(x)dx = 1, \quad \int x^k\, \rho(x)\, dx = 0, \ \ k = 1,\dots,m, \tag{9.32}$$

for some $m \in \mathbb{Z}^+$. Denote

$$\rho_\epsilon(x) = \frac{1}{\epsilon}\rho\Big(\frac{x}{\epsilon}\Big), \qquad \epsilon > 0.$$

Proposition 9.1. *Let $r > 0$ and $f \in H^r(\mathbb{R})$. Then*

$$\begin{aligned} &\text{(a)} \quad \|\rho_\epsilon * f\|_{r+\alpha,2} \leq c\epsilon^{-\alpha}\|f\|_{r,2}, \qquad \forall \alpha > 0,\\ &\text{(b)} \quad \|f - \rho_\epsilon * f\|_{r-\beta,2} \leq c\epsilon^{\beta}\|f\|_{r,2}, \quad \forall \beta \in [0,r]. \end{aligned} \tag{9.33}$$

Moreover,

$$\begin{aligned} &\text{(a)} \quad \|\rho_\epsilon * f\|_{r+\alpha,2} = O(\epsilon^{-\alpha}) \qquad \textit{as} \qquad \epsilon \downarrow 0 \quad \forall \alpha > 0,\\ &\text{(b)} \quad \|f - \rho_\epsilon * f\|_{r-\beta,2} = o(\epsilon^{\beta}) \qquad \textit{as} \qquad \epsilon \downarrow 0 \quad \forall \beta \in [0,r]. \end{aligned} \tag{9.34}$$

Proof. The proof of part (a) in (9.33) and (9.34) is immediate so we only consider part (b). We shall restrict ourselves to prove the case $r = 1$ and $\beta = 1$. The proof for the case where $r,\ \beta \in \mathbb{Z}^+$ is similar to the argument below. The general case follows by interpolation between the previous cases.

By hypothesis on $\rho(\cdot)$ one has

$$\begin{aligned} f(x) - \rho_\epsilon * f(x) &= \int \rho_\epsilon(y)(f(x) - f(x-y))\,dy\\ &= \int \rho_\epsilon(y)\Big(-\int_0^1 \frac{d}{dt} f(x-ty)dt\Big)dy = \int_0^1\int \rho_\epsilon(y) f'(x-ty)y\,dydt\\ &= \int_0^1\int \rho_\epsilon(y)(f'(x-ty) - f'(x))y\,dydt. \end{aligned}$$

Hence,

$$\|f - \rho_\epsilon * f\|_2 \le \epsilon \int_0^1 \int \frac{|y|}{\epsilon} \rho_\epsilon(y) \|f'(\cdot - ty) - f'(\cdot)\|_2 \, dydt.$$

Since $f \in H^1(\mathbb{R})$ a density argument shows that

$$\lim_{\delta \downarrow 0} \sup_{|y| \le \delta, |t| \le 1} \|f'(\cdot - ty) - f'(\cdot)\|_2 = 0.$$

This together with the fact that for any $\delta > 0$ fixed

$$\lim_{\epsilon \downarrow 0} \int_{|y| \ge \delta} \frac{|y|}{\epsilon} \rho_\epsilon(y)\, dy = \lim_{\epsilon \downarrow 0} \int_{|x| \ge \delta/\epsilon} |x| \rho(x)\, dx = 0$$

yields the desired result.

Next, we turn to the proof of step 3. We consider the IVP

$$\begin{cases} \partial_t u + \mathsf{H}\partial_x^2 u + u\partial_x u = 0, \\ u(x,0) = u_0^\epsilon(x) = \rho_\epsilon * u_0(x), \end{cases} \tag{9.35}$$

$t > 0$, $x \in \mathbb{R}$.

Since the data in (9.35) $u_0^\epsilon \in H^\infty(\mathbb{R})$ the argument in steps 1 and 2 shows for any $\epsilon > 0$ the IVP (9.32) has a solution

$$u^\epsilon \in C([0,T] : H^\infty(\mathbb{R})),$$

with T as in (9.18), i.e.

$$cT\|u_0^\epsilon\|_{2,2} = cT\|u_0\|_{2,2} = 1/2$$

satisfying that

$$\sup_{\epsilon > 0} \sup_{[0,T]} \|u^\epsilon(t)\|_{2,2} \le c\|u_0\|_{2,2} \tag{9.36}$$

with c independent of ε. Also by (9.33) one has that

$$\sup_{[0,T]} \|u^\epsilon(t)\|_{l,2} = O(\epsilon^{-l+2}) \quad \text{as} \quad \epsilon \downarrow 0, \quad \forall\, l > 2. \tag{9.37}$$

Next, for $\epsilon > \epsilon' > 0$ we define

$$v(t) = v^{\epsilon,\epsilon'}(t) = (u^\epsilon - u^{\epsilon'})(t),$$

which solves the IVP

$$\begin{cases} \partial_t v + \mathsf{H}\partial_x^2 v + v\partial_x u^\epsilon + u^{\epsilon'}\partial_x v = 0, \\ v(x,0) = u_0^\epsilon(x) - u_0^{\epsilon'}(x) = (\rho_\epsilon * u_0 - \rho_{\epsilon'} * u_0)(x) = v_0(x), \end{cases} \tag{9.38}$$

$t \in [0, T]$. Using energy estimates it follows that

$$\frac{d}{dt}\|v(t)\|_2 \le c(\|\partial_x u^\epsilon(t)\|_\infty + \|\partial_x u^{\epsilon'}(t)\|_\infty)\|v(t)\|_2, \tag{9.39}$$

which combined with (9.36) and Proposition 9.1 lead to

$$\sup_{[0,T]} \|v(t)\|_2 \le c\|v_0\|_2\, e^{2cT\|u_0\|_{2,2}} \le c\epsilon^2\, K, \qquad K = K(\|u_0\|_{2,2}). \tag{9.40}$$

In the same manner one has

$$\begin{aligned} \frac{d}{dt}\|\partial_x^2 v(t)\|_2 &\le c(\|\partial_x u^\epsilon(t)\|_\infty + \|\partial_x u^{\epsilon'}(t)\|_\infty)\|\partial_x^2 v(t)\|_2 \\ &\quad + c(\|\partial_x^2 u^\epsilon(t)\|_2 + \|\partial_x^2 u^{\epsilon'}(t)\|_2)\|\partial_x v(t)\|_\infty \\ &\quad + \|\partial_x^3 u^\epsilon\|_2\, \|v(t)\|_\infty \\ &\equiv E_1(t) + E_2(t) + E_3(t). \end{aligned} \tag{9.41}$$

Gronwall's inequality will be applied to (9.41) after estimating E_j, $j = 1, 2, 3$. The estimate for E_1 follows from (9.36) and Sobolev Embedding Theorem. Using the Gagliardo–Nirenberg inequality (see (3.13)), (9.36) and (9.40) the contribution of E_2 can be bounded as

$$\sup_{[0,T]} E_2(t) \le c\|u_0\|_{2,2} \sup_{[0,T]} (\|v(t)\|_2^{1/4}\|\partial_x^2 v(t)\|_2^{3/4}) \le c\epsilon^{1/2}K, \qquad K = K(\|u_0\|_{2,2}).$$

Similarly, using (9.37) and (9.40) one controls the contribution of the term E_3 in (9.41)

$$\begin{aligned} \sup_{[0,T]} E_3(t) &\le \sup_{[0,T]} (\|\partial_x^3 u^\epsilon\|_2\|v(t)\|_2^{3/4}\|\partial_x^2 v(t)\|_2^{1/4}) \\ &\le c\epsilon^{-1}\epsilon^{3/2}K = c\epsilon^{1/2}K, \quad K = K(\|u_0\|_{2,2}). \end{aligned}$$

Hence, collecting the above information, using Gronwall's inequality and (9.41), and adding the result to (9.40) we conclude that

$$\sup_{[0,T]} \|v(t)\|_{2,2} = \sup_{[0,T]} \|(u^\epsilon - u^{\epsilon'})(t)\|_{2,2} = o(1) \qquad \text{as} \quad \epsilon \downarrow 0. \tag{9.42}$$

Thus,

$$u^\epsilon \to u \quad \text{in} \quad C([0, T] : H^2(\mathbb{R})), \quad \text{as} \quad \epsilon \downarrow 0,$$

with $u(\cdot)$ solving the IVP (9.9) where the equation holds in $C([0, T] : L^2(\mathbb{R}))$. The uniqueness of the solution $u = u(x, t)$ in the class $C([0, T] : H^2(\mathbb{R}))$ follows by using the argument in (9.39) and (9.40). One can show that our solution u agrees with the function $\widetilde{u}$ found in the step 3, see (9.30).

Step 5 (from [BS]) Proof of the continuous dependence of the solution u upon the data u_0.

We shall show that

$$\forall\, \lambda > 0\ \ \exists\, \delta > 0\ \Big[\|u_0 - z_0\|_{2,2} < \delta\ \Rightarrow \sup_{[0,T/2]} \|(u-z)(t)\|_{2,2} < \lambda\Big], \tag{9.43}$$

where u, z represent the solutions of the IVP (9.9) with data $u_0, z_0 \in H^2(\mathbb{R})$) respectively. Without loss of generality we assume $u_0 \neq 0$.

In (9.43) we take the time interval $[0, T/2]$ with T as in (9.18) to guarantee that if $\|u_0 - z_0\|_{2,2} < \delta$, then the solution $z(t)$ is defined in the time interval $[0, T/2]$.

For $1 > \epsilon > \epsilon' > 0$ we define

$$w(t) = w^{\epsilon,\epsilon'}(t) = (u^{\epsilon} - z^{\epsilon'})(t), \tag{9.44}$$

where u^{ϵ}, $z^{\epsilon'}$ are the solutions of the IVP (9.35) with data $u_0^{\epsilon} = \rho_{\epsilon} * u_0$, $z_0^{\epsilon'} = \rho_{\epsilon'} * z_0$ respectively. Thus, taking $\delta_1 = \|u_0\|_{2,2}/2$ from the above results one has that

$$\sup_{[0,T/2]} \|u^{\epsilon}(t)\|_{2,2} + \sup_{[0,T/2]} \|z^{\epsilon'}(t)\|_{2,2} \leq c\|u_0\|_{2,2} + 2c\|u_0\|_{2,2}. \tag{9.45}$$

Since $w(t)$ satisfies the IVP

$$\begin{cases} \partial_t w + \mathsf{H}\partial_x^2 w + w\partial_x u^{\epsilon} + z^{\epsilon'}\partial_x w = 0, \\ w(x,0) = u_0^{\epsilon}(x) - z_0^{\epsilon'}(x) = (\rho_{\epsilon} * u_0 - \rho_{\epsilon'} * z_0)(x) = v_0(x), \end{cases} \tag{9.46}$$

$t \in [0, T/2]$, one has (combining (9.45) and a familiar argument) that

$$\begin{aligned} \sup_{[0,T/2]} \|w(t)\|_2 &\leq \|u_0^{\epsilon} - z_0^{\epsilon'}\|_2\, K \\ &\leq K(\|u_0 - u_0^{\epsilon}\|_2 + \|z - z_0^{\epsilon'}\|_2 + \delta) \leq K(\epsilon^2 + \delta), \end{aligned} \tag{9.47}$$

$K = K(\|u_0\|_{2,2})$ and

$$\begin{aligned} \frac{d}{dt}\|\partial_x^2 w(t)\|_2 &\leq c(\|\partial_x u^{\epsilon}(t)\|_{\infty} + \|\partial_x z^{\epsilon'}(t)\|_{\infty})\|\partial_x^2 w(t)\|_2 \\ &\quad + c(\|\partial_x^2 u^{\epsilon}(t)\|_2 + \|\partial_x^2 z^{\epsilon'}(t)\|_2)\|\partial_x w(t)\|_{\infty} \\ &\quad + \|\partial_x^3 u^{\epsilon}\|_2\, \|w(t)\|_{\infty} \\ &\equiv G_1(t) + G_2(t) + G_3(t). \end{aligned} \tag{9.48}$$

First using (9.45) and Sobolev embedding one gets that

$$G_1(t) \leq c\|u_0\|_{2,2}\|\partial_x^2 w(t)\|_2, \qquad \forall\, t \in [0, T/2].$$

Next, combining (9.45) and (9.47) one gets the bound

$$\begin{aligned} \sup_{[0,T/2]} G_2(t) &\leq c\|u_0\|_{2,2} \sup_{[0,T/2]} (\|w(t)\|_2^{1/4}\|\partial_x^2 w(t)\|_2^{3/4}) \\ &\leq c\|u_0\|_{2,2}^{7/4} K(\epsilon^2 + \delta)^{1/4} \leq K(\epsilon^2 + \delta)^{1/4} \leq cK(\epsilon^{1/2} + \delta^{1/4}), \end{aligned}$$

$K = K(\|u_0\|_{2,2})$. Finally, from (9.36), (9.37), and (9.47) the term G_3 in (9.48) can bounded as

$$\begin{aligned}\sup_{[0,T/2]} G_3(t) &= \sup_{[0,T/2]} (\|\partial_x^3 u^\epsilon(t)\|_2 \|w(t)\|_\infty)\\ &\le cK\epsilon^{-1} \sup_{[0,T/2]} (\|w(t)\|_2^{3/4} \|\partial_x^2 w(t)\|_2^{1/4})\\ &\le cK\epsilon^{-1}(\epsilon^2+\delta)^{3/4} \le cK'(\epsilon^{1/2}+\epsilon^{-1}\delta^{3/4}).\end{aligned}$$

Combining the estimates for G_j, $j = 1,2,3$, Proposition 9.1, and Gronwall's inequality and adding the result to (9.47) it follows that

$$\begin{aligned}\sup_{[0,T/2]} \|w(t)\|_{2,2} &= \sup_{[0,T/2]} \|(u^\epsilon - z^\epsilon)(t)\|_{2,2}\\ &\le K(\|u_0^\epsilon - z_0^\epsilon\|_{2,2} + T(\epsilon^{1/2}+\delta^{1/4}+\epsilon^{-1}\delta^{3/4})) \qquad (9.49)\\ &\le K(\|u_0^\epsilon - u_0\|_{2,2} + \|z_0^{\epsilon'} - z_0\|_{2,2} + \delta^{1/4} + \epsilon^{1/2} + \epsilon^{-1}\delta^{3/4})\end{aligned}$$

assuming $\delta < 1$. Therefore collecting these results one concludes that

$$\begin{aligned}\sup_{[0,T/2]} \|(u-z)(t)\|_{2,2} &\le \sup_{[0,T/2]} (\|u-u^\epsilon\|_{2,2} + \|z - z^{\epsilon'}\|_{2,2} + \|u^\epsilon - z^{\epsilon'}\|_{2,2})\\ &\le o(1)_\epsilon + o(1)_{\epsilon'}\\ &\quad + K(\|u_0^\epsilon - u_0\|_{2,2} + \|z_0^{\epsilon'} - z_0\|_{2,2} + \delta^{1/4} + \epsilon^{1/2} + \epsilon^{-1}\delta^{3/4})\\ &\le o(1)_\epsilon + o(1)_{\epsilon'} + K(\delta^{1/4} + \epsilon^{-1}\delta^{3/4}).\end{aligned}$$

So we fixed ε small enough such that

$$o(1)_\epsilon \le \lambda/3,$$

then we take $\delta < \min\{1; \|u\|_{2,2}/2\}$ such that

$$K(\delta^{1/4} + \epsilon^{-1}\delta^{3/4}) < \lambda/3,$$

and finally for each $z_0 \in H^2(\mathbb{R})$ such that $\|z_0 - u_0\|_{2,2} < \delta$ we take $\epsilon' = \epsilon'(z_0) > 0$, $\epsilon' \in (0,\epsilon)$ such that

$$o(1)_{\epsilon'} \le \lambda/3,$$

to conclude the proof of Theorem 9.2.

We observe that the only fact used on the operator $\mathsf{H}\partial_x^2$ describing the dispersive relation in the BO equation was that it is skew-symmetric.

Next, consider the IVP associated to the generalized BO equation, that is,

$$\begin{cases}\partial_t u + \mathsf{H}\partial_x^2 u + u^k \partial_x u = 0,\\ u(x,0) = u_0(x),\end{cases} \qquad (9.50)$$

$x \in \mathbb{R}, t > 0, k \in \mathbb{Z}^+, k \geq 2$.

In addition to preserve the L^2-norm, solutions of the IVP (9.50) leave invariant the quantity

$$E(u)(t) = \int_{-\infty}^{\infty} \big(|D_x^{1/2}u(x,t)|^2 - \frac{1}{(k+1)(k+2)}u(x,t)^{k+2}\big)dx. \tag{9.51}$$

These quantities will be useful for extending possible local results globally in the corresponding Sobolev spaces dictated for them.

We also notice that the scaling argument for the equation in (9.50) suggests well-posedness for the IVP in $H^s(\mathbb{R})$ for

$$s > s_k = \frac{1}{2} - \frac{1}{k}. \tag{9.52}$$

Using the oscillatory integral techniques described in Chapters 4 and 7 in [KPV11] local well-posedness for small data was established in Sobolev indices lower than the $3/2$ given by the energy method.

In [MR1] and [MR2] Molinet and Riboud improved the results in [KPV11]. In particular, they showed local well-posedness for small data in $H^s(\mathbb{R})$, $s > 1/3$ for $k = 3$, and $s > s_k$ for $k \geq 4$, and for data in $H^s(\mathbb{R})$ of arbitrary size, $s \geq 3/4$ for $k = 3$, $s > 1/2$ for $k = 4$, and $s \geq 1/2$ for $k \geq 5$. These results can be extended globally using the conserved quantities (9.51) whenever the local well-posedness is realized in $H^{1/2}(\mathbb{R})$. Kenig and Takaoka [KT] has obtained global well-posedness for (9.50) with $k = 2$ for $s \geq 1/2$. One of the main new tool used by these authors was a gauge transformation reminiscent of that introduced by Tao (see (9.12)). In [Ve] Vento established local well-posedness in the critical space $\dot{H}^{s_k}(\mathbb{R})$, $s_k = \frac{1}{2} - \frac{1}{k}$, (and its inhomogeneous version) for $k \geq 4$. It was also proved that for $k = 3$ local well-posedness holds in $H^s(\mathbb{R})$ for $s > 1/3$.

From the ill-posedness results obtained by Biagioni and Linares [BiL] the results in [Ve] for $k \geq 4$ should be optimal.

9.5 Zakharov System

In this section we will give a brief account of some results concerning local and global well-posedness for the Zakharov system,

$$\begin{cases} i\partial_t u + \Delta u = u\,v, \\ \lambda^{-2}\partial_t^2 v - \Delta v = \Delta(|u|^2), \\ u(x,0) = u_0(x),\ v(x,0) = v_0(x),\ \partial_t v(x,0) = v_1(x), \end{cases} \tag{9.53}$$

$x \in \mathbb{R}^n$, $t > 0$, where $u : \mathbb{R}^n \times [0,\infty) \mapsto \mathbb{C}^n$ and $v : \mathbb{R}^n \mapsto \mathbb{R}$.

This model was introduced by Zakharov [Zk] to describe the long wave Langmuir turbulence in a plasma. The function $u = u(x,t)$ represents the slowly varying envelope of the highly oscillatory electric field; $v = v(x,t)$ is the deviation of the ion density from the equilibrium; and λ is proportional to the ionic speed of sound. In the limit when $\lambda \to \infty$ the system (9.53) reduces formally to the cubic (focusing) nonlinear Schrödinger equation,

$$i\,\partial_t u_\infty + \Delta u_\infty = -|u_\infty|^2\, u_\infty. \tag{9.54}$$

Solutions of this system satisfy the following conservation laws:

$$M(u_0) = \int_{\mathbb{R}^2} |u(x,t)|^2\, dx,$$

$$E(u_0, v_0, v_1) = \int_{\mathbb{R}^2} \Big(|\nabla u|^2 + v|u|^2 + \frac{v^2}{2} + \lambda^{-2}((-\Delta)^{-1/2}\partial_t v)^2\Big)(x,t)\, dx. \tag{9.55}$$

The Zakharov system has been studied by several authors. Sulem and Sulem [SS1] showed that for data

$$(u_0, v_0, v_1) \in H^s(\mathbb{R}^n) \times H^{s-1}(\mathbb{R}^n) \times (H^{s-2}(\mathbb{R}^n) \cap \dot{H}^{-1}(\mathbb{R}^n)) \tag{9.56}$$

with $s \geq 3$ and $1 \leq n \leq 3$, the IVP (9.53) has unique local solution

$$(u, v) \in L^\infty([0,T] : H^s(\mathbb{R}^n)) \times L^\infty([0,T] : H^{s-1}(\mathbb{R}^n)).$$

They also proved that in the case $n = 1$ these solutions can be extended globally in time. Later on in [AA2] Added and Added established the global existence for the solutions given in [SS1] in the case $n = 2$ corresponding to data u_0 with $\|u_0\|_2$ sufficiently small. Schochet and Weinstein [SWe] obtained a local existence and uniqueness results for data in (9.56) with time interval $[0,T]$ independent of the parameter λ. This allowed them to show that solutions (u^λ, v^λ) of (9.53) converge to a solution of (9.54) as $\lambda \to \infty$. For small amplitude solutions rates of this convergence were obtained in [AA1]. Latter Ozawa and Tsutsumi [OT3] found optimal rates of convergence of solutions of (9.53) to solutions of (9.54).

In [OT2] Ozawa and Tsutsumi obtained, for a fixed λ, unique local results for the IVP (9.53) for data $(u_0, v_0, v_1) \in H^2(\mathbb{R}^n) \times H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ with $1 \leq n \leq 3$, removing the hypothesis $v_1 \in \dot{H}^{-1}$ in previous works (see (9.55)). Ozawa and Tsutsumi approach relies on the L^p-L^q estimates of Strichartz type.

Kenig, Ponce and Vega [KPV8] proved that an iteration scheme can be used directly to obtain small amplitude solutions. They showed that for $n \geq 1$, there exist $s > 0$, $m \in \mathbb{Z}^+$, and $\delta > 0$ such that for any data

$$(u_0, v_0, v_1) \in \mathcal{X}^{s,m} = H^s(\mathbb{R}^n) \cap H^{s_0}(|x|^m\, dx) \times H^{s-1/2}(\mathbb{R}^n) \times H^{s-3/2}(\mathbb{R}^n), \tag{9.57}$$

$s_0 = [(s+3)/2]$ (where $[r]$ denotes the largest integer $\leq r$) with $\|(u_0, v_0, v_1)\|_{\mathcal{X}^{s,m}} \leq \delta$, there exists a unique solution (u^λ, v^λ) in an interval of time $[0,T]$ independent of $\lambda \geq 1$. They also showed that under some additional hypotheses on v_0 and v_1,

$$\sup_{[0,T]} \|(u^\lambda - u_\infty)(t)\|_{H^{s_0}} = O(\lambda^{-1}) \qquad \text{as } \lambda \to \infty.$$

The main idea used in [KPV8] was to exploit the inhomogeneous n-dimensional version of Kato's smoothing effect (4.30) to overcome the loss of derivatives. This was complemented with maximal function estimates for the group $\{e^{it\Delta}\}$.

In [BoC] Bourgain and Colliander showed local well-posedness of IVP (9.53) in the energy space $(u_0, v_0, v_1) \in H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n) \times H^{-1}(\mathbb{R}^n)$, $n = 2, 3$, by extending the method developed in [Bo1]. Global well-posedness for small data was also established by combining local well-posedness and conservation laws, (see (9.55)).

Ginibre, Tsutsumi and Velo [GTV], using the Fourier restriction method introduced by Bourgain [Bo1], obtained a more complete set of results concerning local well-posedness. Their results are roughly as follows:

For data $(u_0, v_0, v_1) \in H^k(\mathbb{R}^n) \times H^l(\mathbb{R}^n) \times H^{l-1}(\mathbb{R}^n)$ the IVP is locally well-posed provided

(k, l)	Dimension
$-\frac{1}{2} < k - l \leq 1, \quad 2k \geq l + \frac{1}{2}$	$n = 1$
$l \geq 0, \quad 2k - (l+1) \geq 0$	$n = 2, 3$
$l > \frac{n}{2} - 2, \quad 2k - (l+1) > \frac{n}{2} - 2$	$n \geq 4$

The solutions satisfy

$$(u, v, \partial_t v) \in C([0, T] : H^k(\mathbb{R}^n) \times H^l(\mathbb{R}^n) \times H^{l-1}(\mathbb{R}^n)).$$

In [BHHT] for the two-dimensional case, Bejenaru, Herr, Holmer and Tataru obtained local well-posedness in the space $L^2(\mathbb{R}^2) \times H^{-1/2}(\mathbb{R}^2) \times H^{-3/2}(\mathbb{R}^2)$ and showed that this result should be optimal.

Regarding blow up results we shall mention the following. In the two-dimensional case Glangetas and Merle [GM] proved the existence of blow up solutions with radial symmetry and self-similar form:

$$u(x,t) = \frac{\omega}{(T-t)}\, e^{i\Phi(x,t)} P\Big(\frac{\omega|x|}{T-t}\Big),$$
$$v(x,t) = \Big(\frac{\omega}{T-t}\Big)^2 N\Big(\frac{\omega|x|}{T-t}\Big),$$

where $\omega \in \mathbb{R}$ and

$$\Phi(x,t) = \frac{\omega^2}{(T-t)} - \frac{|x|^2}{4(T-t)}.$$

They also showed that concentration happens in L^2 (see (6.4)). In [Me5] Merle found rates for the blow up. In [Me6] he also obtained some extensions of the blow up results.

In the one-dimensional case a global result below the energy space has been proved by Pecher [P2].

The corresponding IVP (9.53) in the periodic setting was treated by Bourgain in [Bo9] and Takaoka in [Tk1].

To end this section we comment on the results obtained by Colin and Métivier [CM] and Linares, Ponce and Saut [LPS] regarding the local theory concerning a system deduced by Zakharov where the Schrödinger linear part has a degenerate Laplacian. In [CM] it was established that the periodic boundary value problem is ill-posed. However, the use of some smoothing properties in [LPS] allow the authors to prove local well-posedness in spaces defined via those regularizing properties. This example illustrates the difference between the nonperiodic and periodic setting.

9.6 Higher Order KdV Equations

In 1967 Gardner, Greene, Kruskal and Miura [GGKM] discovered the remarkable fact that the spectrum of the Sturm–Liouville (or stationary Schrödinger) equation

$$L_q(y) = y'' - q(x)\,y = \frac{d^2y}{dx^2} - q(x)y = \lambda\, y, \quad -\infty < x < \infty, \tag{9.58}$$

does not change when the potential $q(x)$ evolves accordingly to the KdV equation, i.e., if $u(x,t)$ solves the IVP

$$\begin{cases} \partial_t u + \partial_x^3 u + u\partial_x u = 0, \\ u(x,0) = q(x), \end{cases} \tag{9.59}$$

$x, t \in \mathbb{R}$, with $q(\cdot)$ in an appropriate class, then

$$\text{spectrum of} L_q = \sigma(L_q) = \sigma(L_{u(\cdot,t)}) \quad \text{for any } t \in \mathbb{R}. \tag{9.60}$$

This principle allowed them to use results from (direct and inverse) spectral theory to solve the IVP (9.59) through a succession of linear computations. This procedure is called the *inverse scattering method* (ISM) as it was mentioned in previous chapters.

More precisely, to guarantee the validity of the process we will describe next, one assumes that $q(x)$ satisfies the decay assumption

$$\int_{-\infty}^{\infty} (1 + |x|^2)\,|q(x)|^2\,dx < \infty \quad \text{(no optimal condition).} \tag{9.61}$$

The scattering data for the problem (9.58) is the spectral information needed to reconstruct the potential $q(x)$.

First, one has the spectrum $\sigma(L_q)$ where by (9.61)

$$\sigma(L_q) = (-\infty, 0] \cup \{k_j^2\}_{j=1}^N, \quad N \in \mathbb{Z}^+ \cup \{0\}, \tag{9.62}$$

where $(-\infty, 0]$ is the continuous spectrum and $\lambda_j = -k_j^2$, $k_j > 0$, $j = 1, \ldots, N$, are the eigenvalues corresponding to eigenfunctions $\{\psi_j\}_{j=1}^N \subseteq L^2(\mathbb{R})$ normalized, i.e., $\|\psi_j\|_2 = 1$, $j = 1, \ldots, N$. Thus from (9.58) and (9.61)

$$\psi_j(x) \sim c_j\, e^{-k_j x} \quad \text{as } \ x \uparrow \infty, \ j = 1, \ldots, N. \tag{9.63}$$

The $\{c_j\}_{j=1}^N$ are called the "normalizing coefficients."

For $\lambda < 0$ the generalized eigenfunctions can be written as ($k = \sqrt{-\lambda}$)

$$\psi(x) \sim \begin{cases} e^{-ikx} + b(k)\, e^{ikx}, & x \to +\infty \\ a(k)\, e^{-ikx}, & x \to -\infty, \end{cases} \tag{9.64}$$

where $a(k)$ and $b(k)$ are called the *transmitted* and the *reflected coefficients*, respectively, extended to $k \in \mathbb{R}$.

The scattering data are given by the spectrum, the normalizing coefficients, and the reflected coefficients

$$\{\sigma(L_q);\ \{c_j\}_{j=1}^N;\ \{b(k):\ k \in \mathbb{R}\}\}. \tag{9.65}$$

This information permits one to recover the potential $q(x)$ as follows: Define

$$F(x) = \sum_{j=1}^{N} c_j^2\, e^{-k_j x} + \frac{1}{2\pi} \int_{-\infty}^{\infty} b(k)\, e^{ikx}\, dk, \tag{9.66}$$

and let $K(x, z)$ be the solution of the Marchenko (Fredholm integral) equation

$$K(x, z) + F(x + z) + \int_{-x}^{\infty} K(x, x')\, F(x' + z)\, dx' = 0. \tag{9.67}$$

Then the potential is obtained via the formula

$$q(x) = \frac{1}{3} \frac{d}{dx} K(x, z)\big|_{z=x}. \tag{9.68}$$

Assuming now that the potential $q(x)$ evolves accordingly to (9.59), one can show (see [AS2], [DJ] for details of this discussion) that the scattering data change in time, the spectrum as (9.60) and the normalized and reflected coefficients as

$$\begin{cases} c_j(t) = c(0)\, e^{4k_j^3 t} = c_j\, e^{4k_j^3 t}, \\ b(k; t) = b(k; 0)\, e^{8ik^3 t} = b(k)\, e^{8ik^3 t}. \end{cases} \tag{9.69}$$

Hence we know

$$\{\sigma(L_{u(\cdot,t)});\ \{c_j(t)\}_{j=1}^N;\ \{b(k; t):\ k \geq 0\}\}, \tag{9.70}$$

the scattering data for

$$L_{u(\cdot,t)}(y) = y'' - u(\cdot,t)\, y = \lambda\, y,$$

which allows one to recover the potential $u(\cdot,t)$, i.e., the solution of the IVP (9.58) associated to the KdV.

In [Lx2] Lax generalized this principle by finding a class of evolution equations for which the operators

$$L_{u(\cdot,t)} = \frac{d^2}{dx^2} - u(x,t) \tag{9.71}$$

are unitary equivalent whenever $u(\cdot,t)$ is a solution of an equation in this class. One must find a family of unitary operators $\{U(t)\}_{t=-\infty}^{\infty}$ such that

$$U^*(t)\, L_{u(\cdot,t)}\, U(t) = L_{u(\cdot,0)}. \tag{9.72}$$

This family should satisfy an equation of the form

$$\frac{d}{dt} U(t) = B(t)U(t) \tag{9.73}$$

for some $B(t)$ skew-symmetric operator. Combining (9.72) and (9.73) one sees that

$$\frac{d}{dt} L_{u(\cdot,t)} = B(t)\, L_{u(\cdot,t)} - L_{u(\cdot,t)}\, B(t) \equiv [B(t); L_{u(\cdot,t)}]. \tag{9.74}$$

Choosing $B = B_0 = \dfrac{d}{dx}$ one gets

$$\frac{d}{dt} L_{u(\cdot,t)} = \partial_t u = \Big[\frac{d}{dx}; L_{u(\cdot,t)}\Big] = -\partial_x u, \tag{9.75}$$

i.e.,

$$\partial_t u + \partial_x u = 0,$$

whose solution $u(x,t) = u_0(x-t) = q(x-t)$ clearly leaves the spectrum of $L_{u(\cdot,t)}$ in (9.71) independently of t.

The choice

$$B_1 = \alpha \frac{d^2}{dx^2} + \beta\Big(u\,\frac{d}{dx} + \frac{d}{dx}(u\ \cdot)\Big) \tag{9.76}$$

with appropriate values of the constants α and β gives

$$[B_1(t);\ L_{u(\cdot,t)}] = -\partial_x^3 u - u\,\partial_x u. \tag{9.77}$$

Hence, (9.74) becomes the KdV equation.

In general, one has

$$B_j = \alpha_j \frac{d^{2j+1}}{dx^{2j+1}} + \sum_{k=0}^{j-1} \Big[\beta_{jk}(u) \frac{d^{2k+1}}{dx^{2k+1}} + \frac{d^{2k+1}}{dx^{2k+1}}(\beta_{kj}(u) \cdot)\Big] \tag{9.78}$$

with $\beta_{jk}(u)$ selected such that $[B_j;\ L_{u(\cdot,t)}]$ has order zero.

Thus for $B_2(u)$ one obtains (up to rescaling)

$$\partial_t u - \partial_x^5 u + 10\, u\, \partial_x^3 u + 20\, \partial_x u\, \partial_x^2 u - 30\, u^2\, \partial_x u = 0. \tag{9.79}$$

This class can also be described using the conservation laws satisfied by solutions of the KdV [Lx2]

$$F_0(u) = 3 \int u\, dx; \quad F_1(u) = \frac{1}{2} \int u^2\, dx; \quad F_2(u) = \int \Big(\frac{u^3}{6} - \frac{(\partial_x u)^2}{2}\Big)\, dx; \quad \ldots . \tag{9.80}$$

The gradient of these functionals ($\partial F_j = G_j$) are

$$G_0(u) = 3, \quad G_1(u) = u, \quad G_2(u) = \frac{1}{2} u^2 + \partial_x^2 u, \quad \ldots, \tag{9.81}$$

which are related by the formula

$$H\, G_j = \partial G_{j+1}, \quad j = 0, 1, \ldots, \tag{9.82}$$

where

$$H = \frac{d^3}{dx^3} + \frac{2}{3}\, u\, \frac{d}{dx} + \frac{1}{3}\, \frac{du}{dx},$$

and

$$\partial_t u + \frac{d}{dx} G_{j+1} = \partial_t u + [B_j; L_{u(\cdot,t)}] = 0, \quad j = 0, 1, \ldots, \tag{9.83}$$

which is called the jth equation in the KdV hierarchy.

So (9.79) is the second equation in the KdV hierarchy. Related versions of this equation appear as a higher order approximations in the study of water wave problems for long, small amplitude waves over shallow horizontal bottom (see for instance [Ol], [Bn1] and references therein). In 1972 Zhakarov and Shabat [ZS] showed that the ISM used for the KdV and its hierarchy can be extended to other relevant physical equations. More precisely, they proved that the cubic one-dimensional defocusing Schrödinger equation

$$i\, \partial_t u = \partial_x^2 u + \lambda |u|^2 u, \quad \lambda > 0,$$

can be solved by considering an appropriate linear scattering problem and its inverse.

The local and global well-posedness of the IVP and PBVP associated to equation (9.79) was established in [St2]. Also the PBVP for the whole KdV hierarchy was studied in [Sch].

Here we restrict ourselves to consider the IVP for the KdV hierarchy in (9.83).

In a more general framework consider the initial value problem

$$\begin{cases} \partial_t u + \partial_x^{2j+1} u + P(u, \partial_x u, \dots, \partial_x^{2j} u) = 0, \\ u(x,0) = u_0(x), \end{cases} \tag{9.84}$$

$x, t \in \mathbb{R}$, $j \in \mathbb{Z}^+$, where $u = u(x,t)$ is real-(or complex-)valued function and

$$P : \mathbb{R}^{2j+1} \mapsto \mathbb{R} \quad (\text{or} \;\; P : \mathbb{C}^{2j+1} \mapsto \mathbb{C})$$

is a polynomial having no constant or linear terms, i.e.,

$$P(z) = \sum_{|\alpha|=\ell_0}^{\ell_1} a_\alpha \, z^\alpha \quad \text{with} \;\; \ell_0 \geq 2 \tag{9.85}$$

and $z = (z_1, \dots, z_{2j+1})$.

In [KPV13] local well-posedness of the IVP (9.84) in

$$\mathcal{F}_m^s = H^s(\mathbb{R}) \cap L^2(|x|^m \, dx)$$

was established. The proof combines the fact that the results in [HO] extend to diagonal systems and a change of dependent variable, which allows us to write the equation in (9.84) (after a few differentiations with respect to the x-variable) as a diagonal system

$$\partial_t \omega^k + \partial_x^{2j+1} \omega^k + Q_k(\omega^1, \dots, \omega^m, \partial_x \omega^1, \dots, \partial_x^{2j-1} \omega^m) = 0 \tag{9.86}$$

for $k = 1, \dots, m = m(j)$ where the nonlinear terms Q_k are independent of the highest derivatives, i.e., those of order $2j$. In this case some modifications are needed since the Q_k introduced by the change of variable involve nonlocal operators.

More precisely, in [KPV13] the following two results were proven:

Theorem 9.3. *Let $P(\cdot)$ be a polynomial of the type described in (9.85). Then there exist $s, m \in \mathbb{Z}^+$ such that for any $u_0 \in \mathcal{F}_m^s = H^s(\mathbb{R}) \cap L^2(|x|^m \, dx)$ there exist $T = T(\|u_0\|_{\mathcal{F}_m^s}) > 0$ (with $T(\rho) \to \infty$ as $\rho \to 0$) and a unique solution $u(\cdot)$ of the IVP (9.84) satisfying*

$$u \in C([0,T] : \mathcal{F}_m^s), \tag{9.87}$$

$$\sup_x \int_0^T |\partial_x^{s+j} u(x,t)|^2 \, dt < \infty \tag{9.88}$$

and

$$\int_{-\infty}^{\infty} \sup_{[0,T]} |\partial_x^r u(x,t)|\, dx < \infty, \quad r = 0, \dots, [\frac{s+j}{2}]. \tag{9.89}$$

If $u_0 \in \mathcal{F}_m^{s_0}$ *with* $s_0 > s$ *the results above hold with* s_0 *instead of* s *in the same time interval* $[0,T]$.

Moreover, for any $T' \in (0,T)$ *there exists a neighborhood* U_{u_0} *of* u_0 *in* $\mathcal{F}_m^s$ *such that the map* $\widetilde{u}_0 \mapsto \widetilde{u}(t)$ *from* U_{u_0} *into the class defined in (9.87)–(9.89), with* T' *instead of* T*, is smooth.*

Theorem 9.4. *Let* $P(\cdot)$ *be a polynomial of the type described in (9.85) with* $\ell_0 \geq 3$*, or* $P(z) = P(z_1, \dots, z_{j+1})$ *in (9.85). Then the results in* Theorem 9.3 *hold with* $m = 0$ *and* L_x^2*-norm instead of* L_x^1*-norm in (9.89).*

Theorem 9.4 tells us that the IVP for the equation

$$\partial_t u + \partial_x^3 u + (u^2 + (\partial_x u)^2)\, \partial_x^2 u = 0, \tag{9.90}$$

$x, t \in \mathbb{R}$ is locally well-posed in $H^s(\mathbb{R})$, $s \geq s_0$, with s_0 sufficiently large. Roughly speaking Theorems 9.3 and 9.4 establish conditions that guarantee that the local behavior of the solution of (9.84) is controlled by the linear part of the equation. Moreover, it shows that the dispersive structure of the equation is strong enough to overcome nonlinear terms of lower order with arbitrary sign as in (9.90).

However, for a specific model of the kind described in (9.84) the results in Theorems 9.3 and 9.4 can be improved by reducing the index s and m depending on the order $(2j+1)$ considered and the structure of the nonlinear term (see for example [Kw2] and [Ci] for some fifth order cases). In particular in [KePi] Kenig and Pilod have shown that the third equation in the KdV hierarchy (9.79) is globally well posed in the energy space $H^2(\mathbb{R})$.

As it was previously mentioned the existence and uniqueness (in $H^s(\mathbb{T})$) for the periodic boundary value problem (PBVP) for the KdV hierarchy was established in [Sch]. The argument relied heavily on the structure of the equations in the hierarchy. Thus one can ask if a general result can be established for the PBVP associated to the general equation in (9.84). The answer is not, in [Bo12] Bourgain showed that the PBVP for the equation

$$\partial_t u + \partial_x^3 u = u^2\, (\partial_x u)^2$$

is ill-posed in $H^s(\mathbb{T})$ for every $s \in \mathbb{R}$.

9.7 Exercises

9.1 Prove that the Benjamin–Ono equation

$$\partial_t u + \mathsf{H}\partial_x^2 u + u\partial_x u = 0$$

has a traveling wave solution (decaying at infinity) $\phi_c(x+t)$, $c>0$, with

$$\phi(x) = -\frac{4}{1+x^2},$$

and

$$\phi_c(x+t) = c\phi(c(x+t)).$$

Notice that $\phi(x+ct)$ is negative and moves to the left, so $\varphi(x-ct) = -\phi(x-ct)$ is a traveling wave, positive and traveling to the right, of the equation

$$\partial_t v - \mathsf{H}\partial_x^2 v + v\partial_x v = 0.$$

Hint: Integrate the equation for ϕ to get a first order ODE. Take Fourier transform and use Exercise 3.3 to get the result.

9.2 (Camassa–Holm equation [CH]) Consider the equation

$$\partial_t u - \partial_t \partial_x^2 u + 3\,u\partial_x u = 2\partial_x u\,\partial_x^2 u + u\,\partial_x^3 u. \tag{9.91}$$

(i) Prove that (9.91) can be written in the formally equivalent form

$$\partial_t u + u\partial_x u + \frac{1}{2}\,\partial_x\, e^{-|x|} * \big(u^2 + \frac{(\partial_x u)^2}{2}\big) = 0. \tag{9.92}$$

(ii) Prove that for any $c>0$ the equation (9.91) has the nonsmooth traveling wave (peakon)

$$\varphi(x-ct) = c\, e^{-|x-ct|}.$$

Hint: (i) Use Exercise 3.4.
(ii) Notice first that it suffices to consider the ODE for φ with $c=1$. Prove that

$$\big(e^{-|\cdot|} * e^{-2|\cdot|}\big)(x) = \frac{4}{3}\,e^{-|x|} - \frac{2}{3}\,e^{-2|x|}.$$

Integrate the ODE and use that $(\varphi'(x))^2 = \varphi(x)^2$.

9.3 (Benjamin–Bona–Mahony equation [BBM]) Consider the equation

$$\partial_t u + \partial_x u + u\,\partial_x u - \partial_{xxt}^3 u = 0. \tag{9.93}$$

(i) Prove that (9.93) can be written in the following forms

$$\partial_t u - \frac{\operatorname{sgn}(x)}{2}\,e^{-|x|} * \Big(u + \frac{u^2}{2}\Big) = 0 \tag{9.94}$$

and

$$u(x,t) = u_0(x) + \int_0^t \int \frac{\operatorname{sgn}(y)}{2}\,e^{-|y|}\Big(u + \frac{u^2}{2}\Big)(x-y,\tau)\,dy\,d\tau. \tag{9.95}$$

(ii) [BTz] Prove that given $u_0 \in L^2(\mathbb{R})$ there exist $T = T(\|u_0\|_2) > 0$ and a unique solution $u \in C([0,T] : L^2(\mathbb{R}))$ solution of (9.95).
Hint: Given $u_0 \in L^2(\mathbb{R})$ consider the set

$$X_T(\delta) = \{v : \mathbb{R} \times [0,T] \mapsto \mathbb{R} \ : \ \sup_{0 \le t \le T} \|v(t)\|_2 \le 2\delta\}$$

with $\delta = \|u_0\|_2$. Show that

$$\Phi(v)(x,t) = u_0(x) + \int_0^t \int \frac{\text{sgn}(y)}{2}\, e^{-|y|}(u + \frac{u^2}{2})(x-y,\tau)\, dy d\tau$$

defines a contraction map in $X_T(\delta)$ if $T(1+\delta) \le 1/2$. (This result is sharp, see [BTz]).

(iii) Prove that for any $b \in (0,1)$ (9.93) has a traveling wave solution $u_b = u_b(x,t)$ of the form

$$u_b(x,t) = \frac{3b^2}{1-b^2}\, \text{sech}^2\Big(\frac{b}{2}\big(x - \frac{t}{1-b^2}\big)\Big) \tag{9.96}$$

Hint:
(a) For $\alpha > 1$ look for solutions of (9.93) of the form $\eta(x - \alpha\, t)$.
(b) By rescaling the ODE for $\eta(\cdot)$ obtain a relation between $\eta(\cdot)$ and $\phi(\cdot)$ in (7.6) and (7.7) with $k = 2$.

9.4 Consider the sine-Gordon equation

$$\partial_t^2 u - \partial_x^2 u + \sin(u) = 0. \tag{9.97}$$

(i) Show that the function

$$v_{\pm}^{\mu}(x,t) = 4\tan^{-1}(c\, e^{(x-\mu t)/\sigma_{\pm}(\mu)})$$

with $\mu \in (-1,1)$, $\sigma_{\pm}(\mu) = \pm\sqrt{1-\mu^2}$ and $c \in \mathbb{R}$ is a traveling wave solution of the sine-Gordon equation.

(ii) Show that for $\mu \in (0,1)$, $v_+^{\mu}(\cdot)$ (*kink solution*) satisfies:
(a) for each $t_0 \in \mathbb{R}$ fixed $v_+^{\mu}(\cdot, t_0)$ is increasing with

$$\lim_{x \downarrow -\infty} v_+^{\mu}(x,t_0) = 0 \quad \text{and} \quad \lim_{x \uparrow \infty} v_+^{\mu}(x,t_0) = 1.$$

(b) $v_+^{\mu}(x,t)$ moves to the right as t increases.

(iii) Show that for $\mu \in (-1,0)$, $v_-^{\mu}(\cdot)$ (*anti-kink solution*) satisfies
(a) for each $t_0 \in \mathbb{R}$ fixed $v_-^{\mu}(\cdot, t_0)$ is decreasing with

$$\lim_{x \downarrow -\infty} v_-^{\mu}(x,t_0) = -1 \quad \text{and} \quad \lim_{x \uparrow \infty} v_-^{\mu}(x,t_0) = 0.$$

(b) $v_+^{\mu}(x,t)$ moves to the left as t increases.

(iv) Verify that

$$w_\mu(x,t) = 4\tan^{-1}\Big(\frac{\sigma_+(\mu)}{\mu}\frac{\sin(\mu t)}{\cosh(\sigma_+(\mu)x)}\Big)$$

with $0 < |\mu| < 1$ is a (*stationary breather*) solution of the sine-Gordon equation which satisfies

$$\lim_{|x|\to\infty} w_\mu(x,t) = 0 \text{ uniformly in } t \in \mathbb{R}.$$

(v) Show that if $t' = (t - \alpha x)/\sigma_+(\alpha)$, $x' = (x - \alpha t)/\sigma_+(\alpha)$, $\alpha \in (-1,1)$ (Lorentz transformation), then

$$\partial_t^2 u - \partial_x^2 u = \partial_{t'}^2 u - \partial_{x'}^2 u,$$

i.e. the sine-Gordon equation is invariant under the Lorentz transformation.

(vi) Combine (iv) and (v) to conclude that for $|\alpha|,\ |\mu| < 1$ the function

$$Z_{\mu,\alpha}(x,t) = 4\ \tan^{-1}\Big(\frac{\sigma_+(\mu)}{\mu}\frac{\sin(\mu(t-\alpha x)/\sigma_+(\alpha))}{\cosh(\sigma_+(x-\alpha t)/\sigma_+(\alpha))}\Big)$$

is a (*moving breather*) solution of the sine-Gordon equation.

9.5 (Compactons [RH]) Consider the quasilinear equation

$$\partial_t u + \partial_x^3(u^2) + \partial_x(u^2) = 0. \tag{9.98}$$

Show that the C^1-function of compact support

$$\phi(x-ct) = \begin{cases} \dfrac{4c}{3}\cos^2\big(\frac{x-ct}{4}\big), & |x-ct| \le 2\pi, \\ 0, & |x-ct| > 2\pi, \end{cases}$$

$c > 0$, is a traveling wave (classical) solution of (9.98).

9.6 (i) Show that the function

$$u(x,y,t) = 4i\,\frac{e^{(-(x+y+8t)-i(x+y))}}{(1+\exp(-2x-8t))(1+\exp(-2y-8t))+1}$$

is a solution of the elliptic-hyperbolic DS system (9.1) with $(c_0,c_1,c_2,c_3) = (1,2,-1,-1)$ where φ satisfies the boundary conditions

$$\lim_{y\to\infty}\partial_x\varphi(x,y,t) = 4\,\mathrm{sech}^2(x+4t)$$

$$\lim_{x\to\infty}\partial_y\varphi(x,y,t) = 4\,\mathrm{sech}^2(y+4t).$$

(ii) [Oz] For the hyperbolic-elliptic DS system (9.1) with $(c_0,c_1,c_2,c_3) = (-1,16,8,1)$ prove:

(a) $$u(x,y,t) = e^{i(x^2-y^2)/4(1-t)} \frac{1-t}{(1-t)^2+x^2+y^2}$$
$$\varphi(x,y,t) = \frac{x}{2}\frac{1}{(1-t)^2+x^2+y^2}$$
is a solution of (9.1) with $u(x,y,0) \in L^2(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$.

(b) For any $t \in \mathbb{R}$, $u(\cdot,\cdot,t) \notin H^1(\mathbb{R}^2)$ and $(x^2+y^2)^{1/2}u(\cdot,\cdot,t) \notin L^2(\mathbb{R}^2)$ but for $s \in (0,1)$ and $t \in (-\infty,1)$, one has $u(\cdot,\cdot,t) \in H^s(\mathbb{R}^2)$ and $(x^2+y^2)^{s/2}u(\cdot,\cdot,t) \in L^2(\mathbb{R}^2)$.

(c) $$\lim_{t\uparrow 1} \|u(\cdot,\cdot,t)\|_\infty = +\infty.$$

9.7 Show that the function
$$u(x,y,t) = \frac{4i\ \exp(-(x+y+4t)-i(x+y))}{(1+\exp(-2x-4t))(1+\exp(-2y-4t))+1}$$
is a solution of the Davey–Stewartson system (DSI)
$$\begin{cases} i\partial_t u + \frac{1}{2}(\partial_x^2 u + \partial_y^2 u) = |u|^2 u + u\partial_x\varphi, \\ \partial_x^2\varphi - \partial_y^2\varphi = -2\partial_x(|u|^2), \end{cases} \tag{9.99}$$
when ϕ satisfies the following boundary conditions
$$\lim_{y\to\infty} \partial_x\varphi(x,y,t) = -2\text{sech}^2(x+2t), \text{and} \lim_{x\to\infty} \partial_x\varphi(x,y,t) = -2\text{sech}^2(y+2t).$$
This solution is called dromion (see [FSa]).

9.8 Show that the Boussinesq equation
$$\partial_t^2 u - \partial_x^2 u - \partial_x^4 u + \partial_x^2(u^2) = 0$$
has traveling wave solutions of the form
$$u(x,t) = a\,\text{sech}^2(b(x-ct)),$$
with appropriate values of a, b for $c > 0$ and $c < 0$, i.e., the wave can propagate in any direction.

9.9 Consider the linear part of the Benjamin–Ono equation
$$Lu = \partial_t u + \mathsf{H}\partial_x^2 u = 0.$$
Defining
$$\Gamma = x - 2t\,\mathsf{H}\,\partial_x = x - 2t\,D_x.$$
Show that
$$[L;\ \Gamma] = [L;\ \Gamma^2] = 0,$$

$$[L;\,\Gamma^3]\phi = 0 \quad \text{if and only if} \quad \widehat{\phi}(0,t) = 0, \quad \text{for all} \quad t \in \mathbb{R},$$

and

$$[L;\,\Gamma^4] \neq 0.$$

9.10 Consider the IVP

$$\begin{cases} \partial_t v + \mathsf{H}\partial_x^2 v = 0, \\ v(x,0) = v_0(x) \end{cases}$$

and

$$Z_{s,r} = H^s(\mathbb{R}) \cap L^2(|x|^{2r}).$$

Let $k \in \mathbb{Z}^+$, prove that $v(\cdot,t) \in L^2(|x|^{2k}\,dx)$ for all $t > 0$ if and only if $v_0 \in Z_{k,k}$, $k = 1,2$ and for $k \geq 3$,

$$\int_{-\infty}^{\infty} x^j\, v_0(x)\,dx = 0, \quad j = 0,1,\ldots,k-3.$$

9.11 Let H be the Hilbert transform. Prove that $\mathsf{H}:\ L^2(\langle x\rangle^{\theta}\,dx) \to L^2(\langle x\rangle^{\theta}\,dx)$ is continuous if and only if $\theta \in (-\,1/2, 1/2)$.

Remark 9.1. Compare this result with that of Exercise 3.19 in Chapter 3.

9.12 Let $u \in C([0,T] : H^1(\mathbb{R}) \cap L^2(|x|^2\,dx))$ be a "strong" solution of the Benjamin-Ono equation. Assuming that $\int u(x,0)\,dx = 0$, prove the identity

$$\frac{d}{dt}\int x\,(u^2 + (\mathsf{H}u)^2)(x,t)\,dx = 4\,\|D_x^{1/2}u(\cdot,t)\|_2^2.$$

9.13 (i) Show that the following quantities are conserved by the BO solution flow:

$$I_1(u) = \int_{\mathbb{R}} u(x,t)\,dx,$$

$$I_2(u) = \int_{\mathbb{R}} u^2(x,t)\,dx,$$

$$I_3(u) = \int_{\mathbb{R}} \Big(u\partial_x\mathsf{H}u + \frac{u^3}{3}\Big)(x,t)\,dx = \int_{\mathbb{R}} \Big(|D_x^{1/2}u|^2 + \frac{u^3}{3}\Big)(x,t)\,dx$$

and

$$I_4(u) = \int_{\mathbb{R}} \Big(2(\partial_x u)^2 + \frac{3}{2}\,u^2\mathsf{H}\partial_x u + \frac{u^4}{4}\Big)(x,t)\,dx.$$

(ii) Prove that a solution $u \in C([0,T] : H^3(\mathbb{R}) \cap L^2(|x|^4\,dx))$ of the BO equation satisfies

(a) $\displaystyle\int_{\mathbb{R}} x\,u(x,t)\,dx = \int_{\mathbb{R}} x\,u_0(x)\,dx + \frac{t}{2}\,\|u_0\|_2^2.$

(b) $\displaystyle\int_{\mathbb{R}} x\,u^2(x,t)\,dx = \int_{\mathbb{R}} x\,u_0^2(x)\,dx + 2t\,I_3(u_0).$

(c) $\displaystyle\int_{\mathbb{R}} x^2\,u(x,t)\,dx = \int_{\mathbb{R}} x^2\,u_0(x)\,dx + t\int_{\mathbb{R}} x\,u_0^2(x)\,dx + t^2 I_3(u_0).$

9.14 Consider the KPI(−) and KPII(+) equations,

$$\partial_t u + u\partial_x u + \partial_x^3 u \mp \partial_x^{-1}\partial_y^2 u = 0. \tag{9.100}$$

Prove that if $u = u(x,y,t)$ is a solution then

(i) $u_\lambda(x,y,t) = \lambda^2\,u(\lambda x, \lambda^2 y, \lambda^5 t)$, $\lambda > 0$ (scaling) is also a solution.

(ii) $u_c(x,y,t) = u(x - cy \pm c^2 t, y \mp 2ct, t)$ (Galilean invariance) is also a solution of the KPI and KPII, respectively.

9.15 Show that

$$u(x,y,t) = 3c\,\text{sech}^2\Big(\frac{1}{2}(\sqrt{c}x + l\,y - \theta\,t)\Big), \quad \theta = c^{3/2} + lc^{-1/2}, \quad c > 0,\ l \in \mathbb{R},$$

is a solution of the Eqs. (9.100) with + sign, i.e. the KPII equation.

Hint: Use that $u_c(x,t) = 3c\,\text{sech}^2(\frac{\sqrt{c}}{2}(x - ct))$ is the soliton solution of the KdV equation with speed c.

9.16 Prove that the function

$$\phi_c(x - ct, y) = \frac{24c(3 - c(x-ct)^2 + c^2y^2)}{(3 + c(x-ct)^2 + c^2y^2)^2}$$

is a finite energy (ϕ_c, $\partial_x\phi_c$, $\partial_x^{-1}\partial_y\phi_c \in L^2(\mathbb{R}^2)$) solitary wave solution of the KPI.

9.17 Consider the linear IVP associated to (9.84)

$$\begin{cases} \partial_t w + \partial_x^{2j+1} w = 0, \\ w(x,0) = w_0(x), \end{cases} \tag{9.101}$$

$x,\ t \in \mathbb{R}$, $j = 0, 1, \ldots$. Denote by

$$w(x,t) = V_j(t)\,w_0(x) = e^{-t\,\partial_x^{2j+1}}\,w_0(x) \tag{9.102}$$

its solution.

(i) Prove that for any $j = 0, 1, \ldots$ there exists $c_j > 0$ such that for any $x, t \in \mathbb{R}$ one has that

$$c_j \int_{-\infty}^{\infty} |\partial_x^j w(x,s)|^2 \, ds = \int_{-\infty}^{\infty} |w(y,t)|^2 \, dy = \|w_0\|_2^2. \tag{9.103}$$

Hint: Follow the argument given in the proof of Lemma 7.1.

(ii) Prove that for any $j = 0, 1, \ldots$ there exists $c'_j > 0$ such that for any $x, t \in \mathbb{R}$,

$$\|\partial_x^{2j} \int_0^t V_j(t-t')\, F(\cdot, t')\, dt'\|_{L_t^2} \le c'_j \|F\|_{L_x^1 L_t^2}. \tag{9.104}$$

Hint: Follow the argument given in the proof of Theorem 4.4 (estimate (4.28)).

(iii) Show that for any $k = 0, 1, \ldots, j$ there exists $c = c(k; j) > 0$ such that for any $x \in \mathbb{R}$,

$$\Big(\int_0^T |\partial_x^{j+k} \int_0^t V_j(t-t')\, F(\cdot, t')\, dt'|^2 \, dt\Big)^{1/2}$$
$$\le c\, T^{(j-k)/2j} \Big\| \int_0^T |F(\cdot, t)|^2 \, dt \Big\|_{2j/(j+k)}. \tag{9.105}$$

Hint: Combine (i) and Minkowski's integral inequality to obtain (9.105) for $k = 0$. Interpolate between this result and (9.104).

(iv) Combining the identity (9.103) and the (unsharp) estimate

$$\Big\| \sup_{0 \le t \le T} |V_j(t) u_0| \Big\|_2 \le c\,(1+T) \|u_0\|_{2j+1,2} \tag{9.106}$$

to prove Theorem 9.4 with $s \gg 1$ and $m = 0$ (no weight) in the case where

$$P = P(u, \ldots, \partial_x^j u)$$

with $P(\cdot)$ as in (9.85) (nonlinear terms at least quadratic) using a fixed point argument.

Hint: Consider the integral equation equivalent form of the corresponding IVP

$$\Phi(u)(t) = V_j(t) u_0 + \int_0^t V_j(t-t')\, P(u, \ldots, \partial_x^j u)(\cdot, t')\, dt'. \tag{9.107}$$

(v) Combining (9.103), (9.104) and (9.106) prove Theorem 9.4 with $s \gg 1$ and $m = 0$ (no weight) in the case where

$$P = P(u, \dots, \partial_x^{2j-1} u) \tag{9.108}$$

with $P : \mathbb{R}^{2j} \to \mathbb{R}$ being a polynomial such that

$$P(x) = \sum_{|\alpha|=3}^{l_1} a_\alpha x^\alpha. \tag{9.109}$$

i.e. the nonlinear terms are at least cubic.

Note: Parts (iv) and (v) show that the IVP (9.84) can be solved via contraction principle in $H^s(\mathbb{R})$ (without weight) with $s \gg 1$ if either

(i) $P = P(u, \dots, \partial_x^j u)$ with P as in (9.85), i.e., the nonlinear terms are at least quadratic or
(ii) $P = P(u, \dots, \partial_x^{2j-1} u)$ with P as in (9.108) and (9.109).

This is optimal. More precisely, it was established in [Pd] that the IVP (9.84) cannot be solved in $H^s(\mathbb{R})$ for any $s > 0$ with an argument based solely on the contraction principle if $P = \partial_x^{2j-1}(u^2)$.

9.18 (Cole–Hopf transformation) Let $w = w(x,t)$ be a positive C^3-solution of the heat equation

$$\partial_t w = \partial_x^2 w, \tag{9.110}$$

(i) $x \in \mathbb{R},\ t > 0$. Prove that $u(x,t) = -2\,\partial_x(\ln w(x,t))$ satisfies the viscous Burgers' equation (7.105).
(ii) Prove that if $u = u(x,t)$ is a C^2-solution of the viscous Burgers' equation (7.105) with $u \in L^\infty(\mathbb{R}^+ : L^1(\mathbb{R}))$, then $w(x,t) = \exp\big(-\frac{1}{2}\int_{-\infty}^{x} u(s,t)\,ds\big)$ is a positive solution of the heat equation (9.110).

9.19 (i) [BM] Consider the logarithmic Schrödinger equation

$$i\partial_t u + \Delta u + u \ln |u|^2 = 0, x \in \mathbb{R}^n, t \in \mathbb{R}. \tag{9.111}$$

(a) Prove that $\varphi(x) = \pi^{-n/4} e^{-|x|^2/2}$ satisfies the elliptic equation

$$-\Delta\varphi - \varphi \ln \varphi^2 = n(1 + \ln\sqrt{\pi})\varphi. \tag{9.112}$$

(b) Prove that

$$u(x,t) = e^{-i\,n(1+\ln\sqrt{\pi})}\varphi(x)$$

is a (standing wave) solution of (9.111).

(ii) [JP] Consider the logarithmic Korteweg-de Vries equation

$$\partial_t u + \partial_x^3 u + \partial_x(u \ln |u|) = 0, \tag{9.113}$$

$x, t \in \mathbb{R}$.
Prove that for any $c, \alpha \in \mathbb{R}$

$$u(x,t) = e^c \sqrt{e}\, e^{-(x-ct-\alpha)^2/4}$$

is a (Gaussian solitary wave) solution of (9.113).

9.20 [X] Consider the Burgers-Korteweg-de Vries equation

$$\partial_t u + u\partial_x u - \beta\, \partial_x^2 u + \alpha \partial_x^3 u = 0, \tag{9.114}$$

$\alpha,\ \beta > 0, x \in \mathbb{R},\ t > 0$.

(i) Prove that for $\beta = 0$ the equation (9.114) has a traveling wave solution of the form $u_c(x,t) = \phi_c(x - ct)$, $c > 0$, with

$$\phi_c(x) = 3c\, \text{sech}^2\big(\frac{\sqrt{c}}{2\sqrt{\alpha}} x\big).$$

(ii) Prove that for $\alpha = 0$ the equation (9.114) has a traveling wave solution of the form

$$u_c(x,t) = c - \varphi_c(x - ct), \quad c > 0,$$

with

$$\varphi_c(x) = \beta c\ \tanh\left(\frac{c}{2} x\right).$$

(iii) Check that if $\alpha,\ \beta > 0$, then the equation (9.114) has traveling wave solutions of the form

$$u_c(x,t) = \frac{1}{2} c\, \text{sech}^2\big(\frac{1}{\sqrt{24\alpha}} \sqrt{c}(x \mp ct)\big) - c\ \tanh\big(\frac{1}{\sqrt{24\alpha}} \sqrt{c}(x \mp ct)\big) \pm c$$

with $c = \frac{6\beta^2}{25\alpha} > 0$.

Chapter 10
General Quasilinear Schrödinger Equation

10.1 The General Quasilinear Schrödinger Equation

In this chapter, we shall study the local solvability of the initial value problem (IVP) associated with the general quasilinear Schrödinger equation:

$$\begin{cases} \partial_t u = i a_{jk}(x,t,u,\bar{u},\nabla_x u,\nabla_x \bar{u})\,\partial^2_{x_j x_k} u \\ \qquad + b_{jk}(x,t,u,\bar{u},\nabla_x u,\nabla_x \bar{u})\,\partial^2_{x_j x_k} \bar{u} \\ \qquad + \overrightarrow{b_1}(x,t,u,\bar{u},\nabla_x u,\nabla_x \bar{u}) \cdot \nabla_x u \\ \qquad + \overrightarrow{b_2}(x,t,u,\bar{u},\nabla_x u,\nabla_x \bar{u}) \cdot \nabla_x \bar{u} \\ \qquad + c_1(x,t,u,\bar{u})\,u + c_2(x,t,u,\bar{u})\,\bar{u} + f(x,t), \\ u(x,0) = u_0(x) \end{cases} \tag{10.1}$$

(using summation convention).

One may think of this equation as a nonlinear Schrödinger equation, where the operator modeling the dispersion relation is nonisotropic and depends also on the unknown function, its conjugate, and its space gradient.

Equations of this form arise in several fields of physics (plasma, fluids, classical and quantum ferromagnetic, laser theory, etc.)

A well-studied model is

$$\partial_t u = i\Delta u - 2iu\,h'(|u|^2)\,\Delta h(|u|^2) + iu\,g(|u|^2), \tag{10.2}$$

where h and g are given functions, $n \geq 1$. When $n = 1, 2, 3$, Bouard, Hayashi and Saut [BHS] proved local well-posedness of the associated IVP in $H^6(\mathbb{R}^n)$, for small data. This was extended by Colin [Cl] to data of arbitrary size in $H^s(\mathbb{R}^n)$, $s \geq s(n)$ for all n.

Problems of this type also arise in Kähler geometry, where the "Schrödinger flow" is defined as follows:

Let (M, g) be a Riemannian manifold and (N, J, h) be a complete Kähler manifold with complex structure J and Kähler metric h. Then given

$$u_0 : M \mapsto N \tag{10.3}$$

F. Linares, G. Ponce, *Introduction to Nonlinear Dispersive Equations*, Universitext, DOI 10.1007/978-1-4939-2181-2_10

one seeks for

$$u : M \times [0,T] \mapsto N \tag{10.4}$$

such that

$$\begin{cases} \partial_t u = J(u(x,t)) \cdot \tau(u(x,t)), \\ u(x,0) = u_0(x), \end{cases} \tag{10.5}$$

where $\tau(u)$ the tension field of u is given in local coordinates by

$$\tau^\alpha(u) = \Delta_g u^\alpha + g^{jk}\, \Gamma^\alpha_{\beta\gamma}(u)\, \frac{\partial u^\beta}{\partial x_j}\, \frac{\partial u^\gamma}{\partial x_k}, \tag{10.6}$$

where $\Gamma^\alpha_{\beta\gamma}$ represents the Christoffel symbol for the target manifold N. These systems have been studied in [DW], [CSU], [MG], [NSU], [NSVZ], among others. For the minimal regularity problem, i.e., to determinate the minimal Sobolev index that guaranties (local or global) well-posedness see [IK2], [BIK] and references therein.

Before considering nonlinear models, it is convenient to study the IVP for the linear equation involving first-order terms. More precisely, we review the results mentioned at the end of Chapter 4. This will be helpful in understanding the hypotheses and the arguments of the proof of the nonlinear result to be discussed later in this chapter.

Consider the linear IVP,

$$\begin{cases} \partial_t u = iAu + \overrightarrow{b}(x) \cdot \nabla u + d(x)u + f(x,t), \\ u(x,0) = u_0(x) \in L^2(\mathbb{R}^n), \end{cases} \tag{10.7}$$

$x \in \mathbb{R}^n$, $t \in \mathbb{R}$, with $A = \partial_{x_j}(a_{jk}(x)\partial_{x_k})$ a second-order elliptic operator, $\overrightarrow{b} = (b_1, \dots, b_n)$, $b_j : \mathbb{R} \mapsto \mathbb{C}$, $j = 1, 2, \dots, n$, and $f \in C(\mathbb{R} : L^2(\mathbb{R}^n))$. To simplify the exposition, assume $b_j \in C_0^\infty(\mathbb{R}^n)$ and $f \equiv 0$. Concerning the L^2-local well-posedness of (10.7) one has:

(i) If $b = b(x)$ is a real-valued function, the result follows by integrating by parts.
(ii) If $n \geq 1$, $a_{jk}(x) = \delta_{jk}$, i.e., $A = \Delta$ and $b_j(x) = i\, c_0$, $c_0 \in \mathbb{R}$ for some j, then problem (10.7) is ill-posed.
(iii) If $n = 1$ and $A = \partial_x^2$, define $v = \phi\, u$, with ϕ real-valued to be determined (ϕ and $1/\phi$ bounded) so

$$\begin{aligned} \partial_t v = i\, \partial_x^2 v + i \left(2 \frac{\partial_x \phi}{\phi} + \mathcal{I}m\, b(x) \right) \partial_x v + \mathcal{R}e\, b(x) \partial_x v \\ + \text{terms of order zero in} v. \end{aligned} \tag{10.8}$$

Then to eliminate the term which cannot be handled by integration by parts one takes

$$\ln \phi(x) = -\frac{1}{2} \int_0^x \mathcal{I}m\, b(s)\, ds. \tag{10.9}$$

In [Ta1], Takeuchi proved in case, $n = 1$ and $A = \partial_x^2$, and general $b(\cdot)$ in (10.7), that the condition

$$\sup_{l\in\mathbb{R}} \left| \int_0^l \mathcal{I}m\ b(s)\,ds \right| < \infty$$

is sufficient for the L^2-well-posedness of (10.7).

(iv) If $n \geq 2$, and $A = \Delta$ one can reapply the argument above to find $\phi = \phi(x, \widehat{\xi})$, $\widehat{\xi} \in \mathbb{S}^{n-1}$, which should solve the equation

$$2\frac{\nabla\phi}{\phi} + \mathcal{I}m\ \overrightarrow{b}(x) = 0. \tag{10.10}$$

Hence, if $\mu = \ln\phi$,

$$2\partial_{\widehat{\xi}}\mu = -\mathcal{I}m\ \overrightarrow{b}(x)\cdot\widehat{\xi}, \qquad \text{for all } \widehat{\xi} \in \mathbb{S}^{n-1}.$$

Thus,

$$\mu(x,\widehat{\xi}) = -\frac{1}{2}\int_{-\infty}^{0} \mathcal{I}m\ \overrightarrow{b}(x+s\,\widehat{\xi})\cdot\widehat{\xi}\,ds, \quad \widehat{\xi} \in \mathbb{S}^{n-1}, \tag{10.11}$$

and

$$\phi(x,\widehat{\xi}) = e^{-\frac{1}{2}\int_{-\infty}^{0} \mathcal{I}m\ \overrightarrow{b}(x+s\,\widehat{\xi})\cdot\widehat{\xi}\,ds}, \quad \widehat{\xi} \in \mathbb{S}^{n-1}. \tag{10.12}$$

In [Mz], Mizohata showed that if $n \geq 1$ and $A = \Delta$, the condition

$$\sup_{\widehat{\xi}\in\mathbb{S}^{n-1}} \sup_{\substack{x\in\mathbb{R}^n \\ l\in\mathbb{R}}} \left| \int_0^l \mathcal{I}m\ b_j(x+s\,\widehat{\xi})\,ds \right| < \infty \tag{10.13}$$

is necessary for the L^2-local well-posedness (10.7). Notice that (10.11) is an integrability condition on the coefficients $\overrightarrow{b} = (b_1, \ldots, b_n)$ of the first-order term along the bicharacteristics.

(v) Consider now $n \geq 1$ and $A = \partial_{x_j}(a_{jk}(x)\,\partial_{x_k}\cdot)$ a general elliptic operator (see (3.25)). In this case, we apply an invertible pseudo-differential operator $C(x,D)$ with real symbol $c(x,\xi)$ to the equation in (10.7) to get

$$\begin{aligned}\partial_t C &= iACu + i\,[C;A]u + iC(\mathcal{I}m\ \overrightarrow{b}(x)\cdot\nabla u) + \mathcal{R}e\,\overrightarrow{b}(x)\cdot\nabla Cu \\ &\quad + \text{terms of order zero in } u \text{ and } Cu.\end{aligned} \tag{10.14}$$

To cancel the bad first-order term one solves the equation:

$$i[C;A] + i\,C(\mathcal{I}m\ \overrightarrow{b}(x)\cdot\nabla) = 0, \tag{10.15}$$

up to operators of order zero. So, using their symbols one has

$$\{c(x,\xi);a(x,\xi)\}+c(x,\xi)\mathcal{I}m\,\overrightarrow{b}(x)\cdot\xi=-H_a(c)+c(x,\xi)\mathcal{I}m\,\overrightarrow{b}(x)\cdot\xi=0,$$

i.e., (see Lemma 3.1),

$$\frac{d}{ds}c(X(s,x,\xi),\Xi(s,x,\xi))= \\ c(X(s,x,\xi),\Xi(s,x,\xi))\mathcal{I}m\,\overrightarrow{b}(X(s,x,\xi))\cdot\Xi(s,x,\xi). \tag{10.16}$$

Therefore,

$$c(x,\xi)=e^{\int_{-\infty}^{0}\mathcal{I}m\,\overrightarrow{b}(X(s,x,\xi))\cdot\Xi(s,x,\xi)\,ds},$$

where $s\to(X(s,x,\xi)),\Xi(s,x,\xi))$ is the bicharacteristic flow associated to the symbol of A (see (3.28)).
In [I], Ichinose extended the Mizohata condition (10.13) to the case of elliptic variable coefficients deducing that

$$\sup_{\widehat{\xi}\in\mathbb{S}^{n-1}}\ \sup_{\substack{x\in\mathbb{R}^n\\ l\in\mathbb{R}}}\left|\int_0^l \mathcal{I}m\ b_j(X(s,x,\widehat{\xi})\cdot\Xi_j(s,x,\widehat{\xi})\,ds\right|<\infty \tag{10.17}$$

is a necessary condition for the L^2-well-posedness of (10.7).
Notice that the notion of nontrapping for the bicharacteristic flow associated to the symbol of A is essential for (10.17) to hold even for $b_j\in C_0^\infty(\mathbb{R}^n)$. Also, asymptotic flatness conditions in the coefficients $a_{jk}(x)$ (see for instance (4.66)) guarantee an appropriate behavior at infinity of the bicharacteristic flow.

Returning to the nonlinear problem consider the case of the Schrödinger equation, with the constant coefficients semilinear case, i.e.,

$$\partial_t u=i\Delta u+f(u,\bar{u},\nabla_x u,\nabla_x\bar{u}),\qquad x\in\mathbb{R}^n. \tag{10.18}$$

If f is smooth, integration by parts yields the estimate

$$\left|\sum_{|\alpha\leq s}\int_{\mathbb{R}^n}\partial_x^\alpha f(u,\bar{u},\nabla_x u,\nabla_x\bar{u})\partial_x^\alpha u\,dx\right|\leq c\,(1+\|u\|_{s,2}^{\rho})\,\|u\|_{s,2}^2, \tag{10.19}$$

for any $u\in H^{s+1}(\mathbb{R}^n)$, $s>n/2+1$, and $\rho=\rho(f)\in\mathbb{Z}^+$, then energy estimates lead to the desired local well-posedness.

Another technique used to overcome the "loss of derivatives" introduced by the nonlinearity f in (10.18) involving $\nabla_x u$ relies on an analytic function approach (see [H5]).

Local well-posedness for small data and general smooth function $f : \mathbb{C}^{2n+2} \mapsto \mathbb{C}$ was established by Kenig, Ponce and Vega [KPV3]. In [KPV3], the authors consider the integral equation associated to (10.18):

$$u(t) = e^{it\Delta}u_0 + \int_0^t e^{i(t-t')\Delta}\, f(u, \bar{u}, \nabla_x u, \nabla_x \bar{u})(t')\, dt' \tag{10.20}$$

and use the inhomogeneous version of the local smoothing effect (see (4.30)) of the group $\{e^{it\Delta}\}_{t=-\infty}^{\infty}$, i.e.,

$$\Big\| \nabla_x \int_0^t e^{i(t-t')\Delta} g(t')\, dt' \Big\|_{\ell_\alpha^\infty(L^2(Q_\alpha\times[0,T]))} \le c\, \|g\|_{\ell_\alpha^1(L^2(Q_\alpha\times[0,T]))}, \tag{10.21}$$

(where the $\{Q_\alpha\}_\alpha$ is the family of unit cubes with disjoint interiors such that $\underset{\alpha}{\cup} Q_\alpha = \mathbb{R}^n$), to overcome the "loss of derivatives" introduced by the nonlinearity $f(\cdot)$ in (10.18), which depends up to first-order derivatives of the unknown. Briefly, one needs to estimate u in the $\ell_\alpha^\infty(L^2(Q_\alpha \times [0, T]))$-norm, which cannot be made "small" by taking $T \to 0$, so it is here where the conditions on the size of the data appear.

The smallness assumption on the data was removed by Hayashi and Ozawa [HO] in the one-dimensional case ($n = 1$). To do so they introduced a change of variables. To illustrate their argument, let us consider the equation:

$$\partial_t u = i\partial_x^2 u + u\partial_x u + u\partial_x \bar{u}. \tag{10.22}$$

When performing standard energy estimates, one sees that the "bad" term in (10.22) is $u\,\partial_x u$, i.e., the one involving $\partial_x u$. This term cannot be handled by integration by parts except when it has a real coefficient, for instance, $|u|^2\,\partial_x u$. Hence, the idea is to eliminate it. First, take the derivatives of (10.22) up to order 3, and use the notation $\partial_x^j u = u_{j+1}$ to rewrite equation (10.22) as the system:

$$\begin{cases} \partial_t u_1 = i\partial_x^2 u_1 + u_1\, u_2 + u_1 \bar{u}_2, \\ \partial_t u_2 = i\partial_x^2 u_2 + u_2\, u_2 + u_1\, u_3 + u_2\bar{u}_2 + u_1\bar{u}_3, \\ \partial_t u_3 = i\partial_x^2 u_3 + 3u_2 u_3 + u_1\, u_4 + u_3\bar{u}_2 + 2u_2\bar{u}_3 + u_1\bar{u}_4, \\ \partial_t u_4 = i\partial_x^2 u_4 + u_1\partial_x u_4 + u_1\partial_x\bar{u}_4 + Q(u_1, \bar{u}_1, \ldots, u_4, \bar{u}_4). \end{cases} \tag{10.23}$$

The first three equations in (10.23) are semilinear as well as the term $Q(\cdot)$ in the fourth one. One then considers "$u_4\,\phi$" instead of "u_3" with ϕ to be determined.

So we substitute u_4 by $\phi^{-1}\,(u_4\,\phi)$ except in the main part of the fourth equation, i.e.,

$$\partial_t u_4 = i\partial_x^2 u_4 + u_1\,\partial_x u_4 + u_1\,\partial_x \bar{u}_4. \tag{10.24}$$

Here, multiplying by ϕ we rewrite (10.24) as

$$\partial_t(u_4\,\phi) - u_4\partial_t\phi = i\partial_x^2(u_4\,\phi) - 2i\,\partial_x u_4\,\partial_x\phi - i u_4\,\partial_x^2\phi + u_1\,\phi\,\partial_x u_4 \tag{10.25}$$

$$+\,\overline{\phi}^{-1}\phi\, u_1\, \partial_x(\overline{u_4\,\phi}\,) + \phi u_1(\overline{u_4\,\phi}\,)\partial_x\overline{\phi}^{-1}.$$

We now choose ϕ to eliminate the terms involving $\partial_x u_4$, i.e.,

$$-2i\,\partial_x u_4\,\partial_x\phi + u_1\,\phi\,\partial_x u_4 = 0 \qquad \text{or} \qquad -2i\,\partial_x\phi + u_1\,\phi = 0, \tag{10.26}$$

that is,

$$\phi(x,t) = \exp\Big(-\frac{i}{2}\int_0^x u_1(\theta,t)\,d\theta\Big). \tag{10.27}$$

In the new variables $(u_1, u_2, u_3, u_4\,\phi)$ for the system (10.23), the standard energy estimates can be performed to obtain the desired local existence and uniqueness result.

Later, Chihara [Ch2] removed the smallness assumption on the data in any dimension. The change of variables used in [HO] in higher dimensions leads to an "exotic" class of pseudodifferential operators (ψ.d.o.) studied by Craig, Kappeler and Strauss [CKS].

Consider the symbol in (10.11), i.e.,

$$\mu(x,\xi) = -\frac{1}{2}\int_{-\infty}^{0} \mathcal{I}m\ \overrightarrow{b}\Big(x + s\,\frac{\xi}{|\xi|}\Big)\cdot\frac{\xi}{|\xi|}\,ds \tag{10.28}$$

with $\xi \in \mathbb{R}^n$, and $\overrightarrow{b} = (b_1,\dots,b_n)$, $b_j \in C_0^\infty(\mathbb{R}^n)$. One has that for $|\xi| \geq 1$

$$\Big|\partial_x^\alpha\partial_\xi^\beta\mu(x,\xi)\Big| \leq c_{\alpha,\beta}\,\langle x\rangle^{|\alpha|}|\xi|^{-|\beta|} \quad \forall\alpha,\beta \in (\mathbb{Z}^+)^n, \tag{10.29}$$

where $\langle x\rangle^2 = 1 + |x|^2$.

Roughly speaking, the function space for the local well-posedness was $H^s(\mathbb{R}^n)$, $s > s(n)$ in the case where f is at least cubic, and where it was $\mathcal{F}_n^s = H^s(\mathbb{R}^n)\cap L^2(|x|^n\,dx)$, $s \geq s(n)$ when f is just quadratic. This is a clear necessary condition in the light of the integrability (10.12).

In [KPV3], Kenig, Ponce and Vega showed that this local result can be proved by a Picard iteration, so the mapping data solution, $u_0 \mapsto u$, is not only continuous but also analytic. A crucial step in this proof was to establish a "local smoothing" effect (see 4.23) for solutions of (10.18), i.e., if $u_0 \in H^{s_0}(\mathbb{R}^n)$, then

$$\int_0^T\int \frac{1}{\langle x\rangle^2}\,|\Lambda^{s_0+1/2}u(x,t)|^2\,dx\,dt < \infty, \tag{10.30}$$

where $\langle x\rangle^2 = (1+|x|^2)^{1/2}$ and $\Lambda^s = (I-\Delta)^{s/2}$ is the operator with symbol $\langle\xi\rangle^s$.

This might seem like a technical device but Molinet, Saut and Tzvetkov [MST3] showed that for the IVP:

$$\begin{cases}\partial_t u = i\partial_x^2 u + u\partial_x u,\\ u(x,0) = u_0(x)\end{cases} \tag{10.31}$$

the map data-solution, $u_0 \mapsto u$, cannot be C^2 at $u_0 \equiv 0$ for u_0 in any Sobolev space $H^s(\mathbb{R})$. Hence, in order to use Picard iteration, the weights are needed.

Returning to our IVP (10.1), we have that in the one-dimensional case ($n = 1$) Poppenberg [Pp] established local well-posedness for coefficients independent of (x,t) under the following conditions:

Ellipticity. $a(\cdot)$ is real-valued and for $|(z_1,z_2,z_3,z_4)| < R$, there exists $\lambda(R) > 0$ such that

$$a(z_1,z_2,z_3,z_4) - |b(z_1,z_2,z_3,z_4)| \geq \lambda(R). \tag{10.32}$$

Degree of nonlinearity:

$$\begin{cases} \partial_z a(0,0,0,0) = \partial_z b(0,0,0,0) = 0, \\ b_1,\ b_2 \ \text{ vanishing quadratically at } \ (0,0,0,0). \end{cases} \tag{10.33}$$

Poppenberg showed local well-posedness in $H^\infty(\mathbb{R}) = \bigcap_{s\geq 0} H^s(\mathbb{R})$. His proof is based on the Nash–Moser techniques.

In [LmPo], Lim and Ponce showed, in the (x,t)-dependent setting, that under Poppenberg's hypotheses one has local well-posedness in $H^s(\mathbb{R})$, $s \geq s_0$, s_0 large enough, and if b_1, b_2 vanish linearly at $(0,0,0,0)$ and $\partial_z a(0,0,0,0) \neq 0$ or $\partial_z b\,(0,0,0,0) \neq 0$ in the weighted space $\mathcal{F}^s_m = H^s(\mathbb{R}) \cap L^2(|x|^m\,dx)$.

To clarify the elliptic condition notice that when $b \equiv 0$, this is the usual condition and in general, it says that $\partial_x^2 u$ "dominates" $\partial_x^2 \bar{u}$. This is certainly needed, as Exercise 4.14 shows.

Moreover, if the nontrapping condition fails dramatically, i.e., all orbits are periodic, ill-posedness in semilinear problems occurs, as Chihara [Ch3] has shown. He proved that for the IVP,

$$\begin{cases} \partial_t u = i\,\Delta u + \operatorname{div}(\overrightarrow{G}(u)), \\ u(x,0) = u_0(x), \end{cases} \tag{10.34}$$

$x \in \mathbb{T}^n$, $t \in [0,T]$, where $\overrightarrow{G} = (G_1,\dots,G_n) \neq 0$, G_j holomorphic, is ill-posed on any Sobolev space $H^s(\mathbb{T}^n)$.

Now we turn to the positive results in [KPV10] concerning the local well-posedness of the IVP (10.1). To simplify the exposition, we shall consider only the case $b_{jk} \equiv 0$.

We shall assume the following:

(H1) *Ellipticity*. Given $M > 0$ there exists $\gamma_M > 0$ such that

$$\langle a_{jk}(x,t,\overrightarrow{z})\xi,\xi\rangle \ \geq\ \gamma_M \quad \forall\,\xi \in \mathbb{R}^n,\ \text{for all } \overrightarrow{z} \in \mathbb{C}^{2n+2} \tag{10.35}$$

with $|\overrightarrow{z}| \leq M$.

(H2) *Asymptotic flatness*. There exists $c > 0$ such that for any $(x,t) \in \mathbb{R}^n \times \mathbb{R}$,

$$|\partial_{x_l} a_{jk}(x,t,\overrightarrow{0})| + |\partial^2_{x_l x_r} a_{jk}(x,t,\overrightarrow{0})| \leq \frac{c}{\langle x\rangle^2}, \tag{10.36}$$

where $l = 0, 1, \ldots, n$, $r = 1, \ldots, n$ with $\partial_{x_0} = \partial_t$.

(H3) *Growth of the first-order coefficients*. There exist $c, c_1 > 0$ such that for any $x \in \mathbb{R}^n$ and $(x, t) \in \mathbb{R}^n \times \mathbb{R}$,

$$|\overrightarrow{b_m}(x, 0, \overrightarrow{0})| \le \frac{c_1}{\langle x \rangle^2}, \qquad |\partial_t \overrightarrow{b_m}(x, 0, \overrightarrow{0})| \le \frac{c}{\langle x \rangle^2}, \qquad m = 1, 2. \quad (10.37)$$

(H4) *Regularity*. For any $N \in \mathbb{N}$ and $M > 0$ the coefficients $a_{jk}, \overrightarrow{b_1}, \overrightarrow{b_2}, c_1, c_2$ are in

$$C_b^N(\mathbb{R}^n \times \mathbb{R} \times (|\overrightarrow{z}| \le M)).$$

(H5) *Nontrapping condition*. The data $u_0 \in H^s(\mathbb{R}^n)$, $s > n/2 + 2$, are such that the Hamiltonian flow $H_{h(u_0)}$ associated to the symbol

$$h(u_0) = h_{u_0}(x, \xi) = -a_{jk}(x, 0, u_0, \bar{u}_0, \nabla u_0, \nabla \bar{u}_0)\xi_j \xi_k \quad (10.38)$$

is nontrapping.

The main result in this chapter is the following theorem:

Theorem 10.1. *Under the hypotheses (H1) – (H4) there exists* $N = N(n) \in \mathbb{Z}^+$ *such that for any* $u_0 \in H^s(\mathbb{R}^n)$ *with*

$$\langle x \rangle^2 \, \partial_x^\alpha \, u_0 \in L^2(\mathbb{R}^n), \qquad |\alpha| \le s_1,$$

and

$$f \in L^1(\mathbb{R} : H^s(\mathbb{R}^n)) \quad \text{and} \quad \langle x \rangle^2 \partial_x^\alpha f \in L^1(\mathbb{R} : L^2(\mathbb{R}^n)), \quad |\alpha| \le s_1,$$

where s, $s_1 \in \mathbb{Z}^+$ *with* $s_1 \ge n/2 + 7$, $s = \max\{s_1 + 4, N + n + 3\}$ *and* u_0 *satisfying the hypothesis (H5). There exists* $T_0 > 0$ *depending only on*

$$\begin{aligned} &\|u_0\|_{s,2} + \sum_{|\alpha| \le s} \|\langle x \rangle^2 \, \partial_x^\alpha \, u_0\|_2 \\ &\quad + \int_{-\infty}^{\infty} \|f(t)\|_{s,2} dt + \sum_{|\alpha| \le s_1} \int_{-\infty}^{\infty} \|\langle x \rangle^2 \, \partial_x^\alpha \, f(t)\|_2 dt \equiv \lambda, \end{aligned} \quad (10.39)$$

so that the IVP (10.1) is locally well-posed in $[0, T_0)$ *with the solution:*

$$u \in C([0, T_0] : \, H^s(\mathbb{R}^n)), \quad \langle x \rangle^2 \, \partial_x^\alpha \, u \in C([0, T_0] : \, L^2(\mathbb{R}^n))$$

for $|\alpha| \le s_1$.

Remark 10.1.

(i) When $n = 1$, the ellipticity hypothesis (H1) implies the nontrapping one (H5).

(ii) One can also prove that the solution possesses the "local smoothing" effect

$$\Lambda^{s+1/2}u \in L^2(\mathbb{R}^n \times [0, T_0] : \langle x\rangle^{-2}\, dx\, dt).$$

(iii) In the above statements, $\langle x\rangle^2$ can be replaced by $\langle x\rangle^{1+\epsilon},\ \epsilon > 0$.

(iv) Koch and Tataru [KTa1] have noticed that the map data-solution, $u_0 \mapsto u$, is not C^2, hence the result in Theorem 10.1 cannot be established by using only Picard iteration.

(v) The proof of this theorem is based in the so-called artificial viscosity method, which was explained in details in Chapter 9 (Section 9.4).

(vi) The proof sketched below only uses classical pseudodifferential operators.

To apply the "artificial viscosity method," we first consider the IVP:

$$\begin{cases} \partial_t u = -\epsilon\, \Delta^2 u + i a_{jk}(x,t)\, \partial^2_{x_j x_k} u + \overrightarrow{b_1}(x,t)\cdot\nabla u + \overrightarrow{b_2}(x,t)\cdot\nabla\bar{u} \\ \qquad\quad + c_1(x,t)u + c_2(x,t)\bar{u} + f(x,t), \\ u(x,0) = u_0(x) \end{cases} \tag{10.40}$$

under the following assumptions:

(H_l1) *Ellipticity*. $A(x,t) = (a_{jk}(x,t))^n_{j,k=1}$ is a real symmetric matrix and there exists $\gamma \in (0,1)$ such that for any $\xi \in \mathbb{R}^n$ and $(x,t) \in \mathbb{R}^n \times [0,\infty)$,

$$\gamma|\xi|^2 \le \langle A(x,t)\,\xi, \xi\rangle \le \gamma^{-1}|\xi|^2. \tag{10.41}$$

(H_l2) *Asymptotic flatness*. There exists $c > 0$ such that for any $(x,t) \in \mathbb{R}^n \times [0,\infty)$,

$$|\partial_{x_l}\, a_{jk}(x,t)| + |\partial^2_{x_l x_r}\, a_{jk}(x,t)| \le \frac{c}{\langle x\rangle^2} \tag{10.42}$$

with $l = 0, 1, \ldots, n,\ r = 1, \ldots, n$, and $\partial_{x_0} = \partial_t$.

(H_l3) *Growth of the first-order coefficients*. There exists $c > 0$ such that

$$|\mathcal{I}m\, \overrightarrow{b_1}(x,0)| + |\mathcal{I}m\, \partial_t \overrightarrow{b_1}(x,t)| \le \frac{c}{\langle x\rangle^2} \tag{10.43}$$

for all $(x,t) \in \mathbb{R}^n \times [0,\infty)$.

(H_l4) *Regularity*. The coefficients $a_{jk},\ b_{1j},\ b_{2j},\ c_1,\ c_2$ are in $C^N_b(\mathbb{R}^n \times [0,\infty))$ with $\overrightarrow{b_l} = (b_{l1}, \ldots, b_{ln})$, $l = 1, 2$, for $N = N(n)$ sufficiently large.

(H_l5) *Nontrapping condition*. Let $A_0(x) = A(x,0) = (a_{jk}(x,0))^n_{j,k=1}$,

$$h(x,\xi) = -a_{jk}(x,0)\xi_j\xi_k, \tag{10.44}$$

and H_h be the corresponding Hamiltonian flow. Then H_h is nontrapping.

The following a priori estimate for solutions of the linear IVP (10.40) is the key in the proof of the nonlinear result for the IVP (10.1), Theorem 10.1.

Lemma 10.1. *Under the hypotheses (H_l1)–(H_l5) above there exist $N = N(n)$, c_0 and $T_0 > 0$ (depending both c_0, T_0 on the nontrapping condition (H_l5) and on the coefficients at $t = 0$), so that for any $T \in (0, T_0)$ and any $\epsilon \in (0,1)$ we have that the solution of (10.40) satisfies*

$$\begin{aligned} \sup_{0\le t\le T} \ \| u(t)\|_2 + \left(\int_0^T \langle x\rangle^{-2} |\Lambda^{1/2} u|^2 \, dx\, dt\right)^{1/2} \\ \le c_0 \left[\|u_0\|_2 + \left(\int_0^T \langle x\rangle^{2} |\Lambda^{-1/2} f(x,t)|^2 \, dx\, dt\right)^{1/2}\right], \end{aligned} \tag{10.45}$$

and

$$\begin{aligned} \sup_{0\le t\le T} \ \|u(t)\|_2 + \left(\int_0^T \langle x\rangle^{-2} |\Lambda^{1/2} u|^2 \, dx\, dt\right)^{1/2} \\ \le c_0 \left[\|u_0\|_2 + \left(\int_0^T |f(x,t)|^2 \, dx\, dt\right)^{1/2}\right]. \end{aligned} \tag{10.46}$$

In fact, the constant c_0 depends only on the nontrapping condition for $h(x,\xi)$ (H_l5), on the bounds at $t = 0$ of $\langle x\rangle^2\, \overrightarrow{b_j}(x,0)$, $j = 1,2$, and on size estimates for the coefficients and their derivatives at $t = 0$. Thus, in the nonlinear case, c_0 depends only on the data u_0. Assuming the result in Lemma 10.1, we shall prove Theorem 10.1.

We introduce the notations ($v = v(x,t)$, $u = u(x,t)$) for

$$\begin{aligned} \mathcal{L}(v)u = \; & i a_{jk}(x,t,v,\bar v,\nabla v,\nabla \bar v)\, \partial^2_{x_j x_k} u \\ & + \overrightarrow{b_1}(x,t,v,\bar v,\nabla v,\nabla \bar v)\cdot \nabla u + \overrightarrow{b_2}(x,t,v,\bar v,\nabla v,\nabla \bar v)\cdot \nabla \bar u \\ & + c_1(x,t,v,\bar v)\, u + c_2(x,t,v,\bar v)\, \bar u, \end{aligned} \tag{10.47}$$

$$\begin{aligned} X_{T,M_0} = \Big\{ & v : \mathbb{R}^n \times [0,T] \to \mathbb{C} \mid v \in C([0,T] : H^s(\mathbb{R}^n)), \\ & \langle x\rangle^2\, \partial_x^\alpha\, v \in C([0,T] : L^2(\mathbb{R}^n)),\ |\alpha| \le s_1,\ \ v(x,0) = u_0(x)\Big\}, \end{aligned} \tag{10.48}$$

with the norm

$$|||v|||_T = \sup_{[0,T]} \|v(t)\|_{s,2} + \sum_{|\alpha|\le s_1} \sup_{[0,T]} \|\langle x\rangle^2\, \partial_x^\alpha v(t)\|_2 \le M_0. \tag{10.49}$$

For $u \in X_{T,M_0}$, we study the linear IVP:

$$\begin{cases} \partial_t v = -\epsilon\, \Delta^2 v + \mathcal{L}(u)v + f(x,t), & \epsilon \in (0,1) \\ v(x,0) = u_0(x), \end{cases} \tag{10.50}$$

and its integral equation version

$$v(t) = e^{-\epsilon t\, \Delta^2}\, u_0 + \int_0^t e^{-\epsilon(t-t')\Delta^2}\, (\mathcal{L}(u)v + f)(t')\, dt'. \tag{10.51}$$

One defines the operator $\Phi(u)(t)$ as the right-hand side of (10.51). Using that

$$\|e^{-\epsilon t\, \Delta^2} g\|_2 \le \|g\|_2 \qquad \text{and} \qquad \|\Delta\, e^{-\epsilon t\, \Delta^2} g\|_2 \le \frac{1}{\epsilon^{1/2} t^{1/2}}\, \|g\|_2,$$

it is easy to check that the operator $\Phi(\cdot)$ is a contraction on X_{T_ϵ,M_0} with $T_\epsilon = O(\epsilon)$. One needs standard commutator identities to estimate the weighted norms in $X_{T,M}$. Thus, there exists $u^\epsilon \in X_{T_\epsilon,M_0}$ (the fixed point of Φ) solution of the IVP:

$$\begin{cases} \partial_t u = -\epsilon\, \Delta^2 u + \mathcal{L}(u)u + f(x,t), & \epsilon \in (0,1), \\ u(x,0) = u_0(x), \end{cases} \tag{10.52}$$

on the time interval $[0, T_\epsilon]$.

Now we will use Lemma 10.1 to extend all solutions $\{u^\epsilon\}_{\epsilon\in(0,1)}$ to the time interval $[0, T_0]$ with T_0 independent of $\epsilon \in (0,1)$, with $|\!|\!| u^\epsilon |\!|\!|_{T_0}$ uniformly bounded.

The first step is to show that if $|\!|\!| u^\epsilon |\!|\!|_T \le M_0 = 20\, c_0\, \lambda$ (see (10.39)), the coefficients of the linear equation for $\Lambda^{2m} u^\epsilon = (I - \Delta)^m u^\epsilon$, $2m \le s$, and $x_l^2\, \Lambda^{2m} u$ with $2m \le s_1$ (assuming s, s_1 even integers) can be written so that the constants in $(\mathrm{H}_l 1)$–$(\mathrm{H}_l 5)$ are uniform for all these equations in a time interval $[0, \widetilde{T}]$ independent of ϵ.

The equations for $\Lambda^{2m} u^\epsilon$ are obtained by applying the operator Λ^{2m} to the equation (10.52), which can be written as

$$\begin{aligned} \partial_t \Lambda^{2m} u = &- \epsilon\, \Delta^2 \Lambda^{2m} u + i\mathcal{L}_{2m}(u)\Lambda^{2m} u \\ &+ f_{2m}(x, t, (\partial^\beta u)_{|\beta|\le 2m-1}, (\partial^\beta \bar{u})_{|\beta|\le 2m-1}) + \Lambda^{2m} f(x,t), \end{aligned} \tag{10.53}$$

where

$$\begin{aligned} \mathcal{L}_{2m}(u)v = &\, i a_{jk}(x, t, u, \bar{u}, \nabla u, \nabla \bar{u}) \partial^2_{x_j x_k} v \\ &+ b_{2m,2,j}(x, t, (\partial^\beta u)_{|\beta|\le 2m-1}, (\partial^\beta \bar{u})_{|\beta|\le 2m-1})\, R_j \partial_{x_j} v \\ &+ b_{2m,2,j}(x, t, (\partial^\beta u)_{|\beta|\le 2m-1}, (\partial^\beta \bar{u})_{|\beta|\le 2m-1})\, \widetilde{R}_j \partial_{x_j} \bar{v} \\ &+ c_{1,2m}(x, t, (\partial^\beta u)_{|\beta|\le 2m-1}, (\partial^\beta \bar{u})_{|\beta|\le 2m-1})\, R_{2m,1}\, v \\ &+ c_{2,2m}(x, t, (\partial^\beta u)_{|\beta|\le 2m-1}, (\partial^\beta \bar{u})_{|\beta|\le 2m-1})\, R_{2m,2}\, \bar{v}, \end{aligned} \tag{10.54}$$

where R_j, $\widetilde{R}_j$, $R_{2m,1}$, $R_{2m,2}$ are ψ.d.o. of order zero.

We observe that the principal part of $\mathcal{L}_{2m}(u)$ is independent of m. Moreover, the first-order coefficients $b_{2m,1,j}$, $b_{2m,2,j}$ depend on $2m$ as a multiplicative constant, and on the original coefficients a_{jk}, $\overrightarrow{b_1}$, $\overrightarrow{b_2}$ and their first derivatives and they verify the asymptotic flatness assumptions (H$_l$2). The term $f_{2m}(\cdot)$ involves derivatives that have been previously estimated in $L^\infty_T L^2_x$, and so putting it on the right-hand side in the $L^1_T L^2_x$-norm they appear with a factor T in front.

Similar remarks hold for the equation for $x_l^2 \Lambda^{2m} u$ after using some simple commutator identities.

Collecting this information, we can also show that there exists a $Q(\cdot)$ increasing function such that, for any $\omega \in X_{T,M_0}$ with $T > 0$ solution of the IVP (10.52),

$$\sup_{[0,T]} \sum_{|\alpha| \le s_1 - 4} \| \langle x \rangle^2 \partial_x^\alpha \partial_t \omega \|_2 \le Q(M_0) \tag{10.55}$$

holds.

All these facts will allow us to apply Lemma 10.1 to get the a priori estimate

$$|||u^\epsilon||| \le c_0(\lambda + \widetilde{T} R(M_0)) \le M_0/4 \tag{10.56}$$

for $\widetilde{T}$ small, but uniform in ϵ, where $R(\cdot)$ is a fixed increasing function. Thus, we can reapply the local existence theorem (originally on $[0, T_\epsilon]$) to extend the local solution u^ϵ to the time interval $[0, \widetilde{T}]$, with

$$|||u^\epsilon||| \le M_0 = 20 c_0 \lambda. \tag{10.57}$$

Once (10.57) has been established (as in the viscosity method for the BO explained in Chapter 9), we consider the equation for the difference $u^\epsilon - u^{\epsilon'}$, $\epsilon > \epsilon' > 0$, and reapply the argument to obtain the existence as $\epsilon \to 0$ and the uniqueness of the solution. The continuous dependence is based on Bona–Smith regularization argument [BS] (see Step 5 in the proof of Theorem 9.2).

Now we turn our attention to the proof of Lemma 10.1. One of the main ingredients in the proof is the following lemma due to Doi [Do1].

Lemma 10.2 [Do1]. *Assume that h in (10.38) verifies the assumptions (H$_l$5) (nontrapping), (H$_l$4) (regularity) and (H$_l$2) (asymptotic flatness). Then, there exists a real-valued zeroth-order classical symbol $p \in S^0$ (see (3.18)) whose seminorm is bounded in terms of the "nontrapping character" of h, the ellipticity constant γ in (H$_l$1), and the bound for the smoothness norm at $t = 0$, c_1, and a constant $\beta \in (0,1)$ (with the same dependence) such that*

$$H_h\, p = \{h,\, p\} \ge \beta \frac{|\xi|}{\langle x \rangle^2} - \frac{1}{\beta} \tag{10.58}$$

for all $(x, \xi) \in \mathbb{R}^n \times \mathbb{R}$.

Remark 10.2. The seminorm bounds for the symbol p and the constant β above are the quantitative way in which the "nontrapping" character of h enters into the proof.

We recall that

$$H_h\, p = \{h,\ p\} = \partial_{\xi_j} h\, \partial_{x_j} p - \partial_{x_j} h\, \partial_{\xi_j} p. \tag{10.59}$$

Observe that, if $\tilde{h}$ is only "approximately nontrapping" and we use the p corresponding to H_a, for $H_{\tilde{h}}$, we get a lower bound of $H_{\tilde{h}}\, p$ by $\beta|\xi|/2\langle x\rangle^2 - 2/\beta$.

To apply Doi's lemma, we need the sharp Garding inequality (see [Ho2]).

Lemma 10.3 (Sharp Garding's inequality). *Let $q \in S^1$ be a classical symbol of order 1 such that $\mathcal{R}e\, q(x,\xi) \geq 0$ for $|\xi| \geq R$, then there exist $j_0 = j_0(n)$ and $c = c(n,R)$ such that*

$$\mathcal{R}e\, \langle \Psi_q f, f\rangle \geq -c\, \|q\|_{S^1}^{(j_0)}\, \|f\|, \tag{10.60}$$

where Ψ_q denotes the ψ.d.o. with symbol q, i.e.,

$$\Psi_q\, f(x) = \int e^{ix\xi} q(x,\xi)\widehat{f}(\xi)\, d\xi. \tag{10.61}$$

Assuming Lemmas 10.2 and 10.3 we shall divide the proof of Lemma 10.1 into several steps.

Step 1. Write equation (10.40) as a system. Using

$$\overrightarrow{w} = \begin{pmatrix} u \\ \bar{u} \end{pmatrix}, \qquad \overrightarrow{f} = \begin{pmatrix} f \\ \bar{f} \end{pmatrix}, \qquad \overrightarrow{w}_0 = \begin{pmatrix} u_0 \\ \bar{u}_0 \end{pmatrix},$$

one has the system

$$\begin{cases} \partial_t \overrightarrow{w} = -\epsilon \Delta^2 I \overrightarrow{w} + (iH + B + C)\overrightarrow{w} + \overrightarrow{f}, \\ \overrightarrow{w}(x,0) = \overrightarrow{w}_0(x), \end{cases}$$

where

$$H = \begin{pmatrix} \mathcal{L} & 0 \\ 0 & -\mathcal{L} \end{pmatrix}, \qquad C = \begin{pmatrix} c_1 & c_2 \\ \bar{c}_2 & \bar{c}_1 \end{pmatrix}, \tag{10.62}$$

$$B = \begin{pmatrix} \overrightarrow{b_1} \cdot \nabla & \overrightarrow{b_2} \cdot \nabla \\ \overline{\overrightarrow{b_2}} \cdot \nabla & \overline{\overrightarrow{b_1}} \cdot \nabla \end{pmatrix} = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}, \tag{10.63}$$

and $\mathcal{L} = i a_{jk}(x,t)\partial^2_{x_j x_k}$.

Step 2. Diagonalization of the first-order terms. (To simplify the exposition take $\epsilon = 0$).

Notice that $\mathcal{L}$ is elliptic, with ellipticity constant $\gamma/2$ for $t \in [0,T]$ for T sufficiently small since

$$a_{jk}(x,t)\xi_j\xi_k = a_{jk}(x,0)\xi_j\xi_k + [a_{jk}(x,t) - a_{jk}(x,0)]\xi_j\xi_k \geq \gamma|\xi|^2 - c\,T|\xi|^2 \tag{10.64}$$

(by using the bound of $\partial_t a_{jk}(x,t)$ in ($\mathrm{H}_l 2$)).

This type of argument shall be used repeatedly.

Next we write

$$B = B_{diag} + B_{anti} = \begin{pmatrix} B_{11} & 0 \\ 0 & B_{22} \end{pmatrix} + \begin{pmatrix} 0 & B_{12} \\ B_{21} & 0 \end{pmatrix}. \tag{10.65}$$

Our goal is to eliminate B_{anti}. To do this we set

$$\Lambda = I - S, \quad \text{with} \quad S = \begin{pmatrix} 0 & S_{12} \\ S_{21} & 0 \end{pmatrix}$$

where S_{12}, S_{21} are ψ.d.o. of order -1 to be determined.

We want to write the system in the new variable

$$\overrightarrow{z} = \Lambda \overrightarrow{w} \tag{10.66}$$

for an appropriate choice of S, so that B_{anti} is eliminated.

We will use that S is a matrix of ψ.d.o. of order -1, to have that Λ is invertible in L^2 and so the estimates on $\overrightarrow{z}$ are equivalent to the estimates on $\overrightarrow{w}$.

We calculate

$$\begin{aligned} &\begin{pmatrix} \mathcal{L} & 0 \\ 0 & -\mathcal{L} \end{pmatrix} \Lambda - \Lambda \begin{pmatrix} \mathcal{L} & 0 \\ 0 & -\mathcal{L} \end{pmatrix} \\ &\qquad = - \begin{pmatrix} \mathcal{L} & 0 \\ 0 & -\mathcal{L} \end{pmatrix} \begin{pmatrix} 0 & S_{12} \\ S_{21} & 0 \end{pmatrix} + \begin{pmatrix} 0 & S_{12} \\ S_{21} & 0 \end{pmatrix} \begin{pmatrix} \mathcal{L} & 0 \\ 0 & -\mathcal{L} \end{pmatrix} \\ &\qquad = \begin{pmatrix} 0 & -\mathcal{L}S_{12} - S_{12}\mathcal{L} \\ \mathcal{L}S_{21} + S_{21}\mathcal{L} & 0 \end{pmatrix}. \end{aligned}$$

Since

$$|h(x,\xi)| = |a_{jk}(x,t)\xi_j\xi_k| \geq \gamma|\xi|^2 \quad \text{for} \quad |\xi| \geq R \tag{10.67}$$

uniformly in t, choosing $\varphi \in C_0^\infty(\mathbb{R}^n)$ with $\varphi(y) = 1$ if $|y| \leq 1$ and $\varphi(y) = 0$ if $|y| \geq 2$ we define

$$\widetilde{h}(x,t,\xi) = (h(x,\xi))^{-1}(1 - \varphi(\xi/R)) \tag{10.68}$$

and $\widetilde{\mathcal{L}} = \Psi_{\tilde{h}}$, i.e., the ψ.d.o. of order -2 with symbol $\widetilde{h}$. Notice that

$$\widetilde{\mathcal{L}}\,\mathcal{L} = I + \Psi_{\ell_1} \tag{10.69}$$

with $\ell_1 \in S^{-1}$ (uniformly in t). Define

$$S_{12} = \frac{1}{2}\, i\, B_{12}\, \widetilde{\mathcal{L}}, \qquad S_{21} = -\frac{1}{2} i\, B_{21}\, \widetilde{\mathcal{L}} \tag{10.70}$$

and

$$S = \begin{pmatrix} 0 & S_{12} \\ S_{21} & 0 \end{pmatrix}. \tag{10.71}$$

Notice that the entries of S are ψ.d.o of order -1, whose S^0 seminorms tend to zero as $R \uparrow \infty$ (see (10.68)). Thus, for R large enough Λ is invertible in $H^s(\langle x\rangle^2\, dx)$, $H^s(\langle x\rangle^{-2}\, dx)$, and $H^s(\mathbb{R}^n)$ with operator norm in the interval $(1/2, 2)$. Also if Λ^{-1} denotes the inverse of Λ, the entries of Λ^{-1} are ψ.d.o. of order zero.

Finally, from our construction

$$\begin{cases} -\mathcal{L}\, S_{12} - S_{12}\, \mathcal{L} = -B_{12} + \text{ order } 0, \\ \;\;\mathcal{L}\, S_{21} - S_{21}\, \mathcal{L} = -B_{21} + \text{ order } 0. \end{cases} \tag{10.72}$$

We then observe that

$$\begin{aligned} \Lambda B_{diag} &= (I - S)\, B_{diag} = B_{diag} - S\, B_{diag} \\ &= B_{diag}\Lambda + B_{diag}\, S - S\, B_{diag} \\ &= B_{diag}\Lambda + [(B_{diag}\, S - S\, B_{diag})\Lambda^{-1}]\, \Lambda, \end{aligned} \tag{10.73}$$

(notice that $[\,\cdot\,]$ is an operator of order zero).

Similarly,

$$\Lambda\, B_{anti} = B_{anti}\, \Lambda + C\, \Lambda, \tag{10.74}$$

by (10.70), (10.71), (10.72) with C a matrix of ψ.d.o. of order zero.

Thus, our system in $\vec{z} = \Lambda \vec{w}$ becomes

$$\begin{cases} \partial_t \vec{z} = i H \vec{z} + B_{diag}\, \vec{z} + C\, \vec{z} + \vec{g}, \\ \vec{z}(x, 0) = \vec{z_0}(x), \end{cases} \tag{10.75}$$

where $\vec{g} = \Lambda \vec{f}$, $\vec{z}_0 = \Lambda \vec{w}_0$, H, B_{diag} as before with $B_{11} = \vec{b_1} \cdot \nabla$, $B_{22} = \overline{\vec{b_1}} \cdot \nabla$ and C is a matrix of ψ.d.o. of order zero whose symbols have seminorm estimates controlled by c (not c_1).

Step 3. Energy estimates for a "gauged" system. We recall that $\mathcal{L} = a_{jk}(x,t)\,\partial^2_{x_j x_k}$ has symbol $a_{jk}(x,t)\,\xi_j\xi_k$, and our "nontrapping" assumption is on

$$h(x,\xi) = a_{jk}(x,0)\xi_j\xi_k. \tag{10.76}$$

Let $p \in S^0$ be the symbol associated to h as in Lemma 10.2 so that

$$H_h\, p = \{h,p\} \geq \beta\frac{|\xi|}{\langle x\rangle^2} - \frac{1}{\beta}.$$

Let

$$h_1(x,t,\xi) = a_{jk}(x,t)\xi_j\xi_k. \tag{10.77}$$

So

$$\begin{aligned} H_{h_1}\, p &= \frac{\partial h_1}{\partial \xi_l}\,\frac{\partial p}{\partial x_l} - \frac{\partial h_1}{\partial x_l}\,\frac{\partial p}{\partial \xi_l} \\ &= \frac{\partial h}{\partial \xi_l}\,\frac{\partial p}{\partial x_l} - \frac{\partial h}{\partial x_l}\,\frac{\partial p}{\partial \xi_l} + (a_{jk}(x,t) - a_{jk}(x,0))\frac{\partial}{\partial \xi_l}(\xi_j\xi_k)\,\frac{\partial p}{\partial x_l} \\ &\quad - \Big(\frac{\partial}{\partial x_l}\big(a_{jk}(x,t) - a_{jk}(x,0)\big)\Big)\xi_j\xi_k\,\frac{\partial p}{\partial \xi_l}. \end{aligned}$$

Thus, by "asymptotic flatness," assumption (H_l2), we see that for small T

$$H_{h_1}\, p \geq \frac{\beta}{2}\frac{|\xi|}{\langle x\rangle^2} - \frac{2}{\beta}, \tag{10.78}$$

for the same p. We now consider the ψ.d.o. of order 0, Ψ_{r_1}, whose symbol is $e^{Mp(x,t)}$ for an M large to be determined depending only on c_1, and the "nontrapping character." Notice that the seminorms of r_1 depend only on c_1, and the nontrapping character: it is elliptic. The same holds for $\Psi_{r_1}^{-1}$ (modulo order -2 errors). Also

$$H_{h_1}\, r_1 = M(H_{h_1}\, p)\, r_1 \geq \left\{M\,\frac{\beta}{2}\,\frac{|\xi|}{\langle x\rangle^2} - \frac{2M}{\beta}\right\} r_1 \tag{10.79}$$

and that modulo 0th-order operators the symbol of $i\{\mathcal{L}\Psi_{r_1} - \Psi_{r_1}\mathcal{L}\} = H_{h_1}\, r_1$, for each t.

We also consider Ψ_{r_2}, whose symbol is $e^{-Mp(x,\xi)}$ so that symbol-wise we have

$$i\{\mathcal{L}\Psi_{r_2} - \Psi_{r_2}\mathcal{L}\} = H_{h_1} r_2 = -M(H_{h_1} p) r_2 \leq -\left\{M\frac{\beta}{2}\,\frac{|\xi|}{\langle x\rangle^2} - \frac{2M}{\beta}\right\} r_2.$$

Now we define

$$\vec{\alpha} = \begin{pmatrix} \Psi_{r_1} & 0 \\ 0 & \Psi_{r_2} \end{pmatrix} \vec{z},$$

and obtain a new system for $\overrightarrow{\alpha}$, in which for M chosen appropriately we will be able to perform energy estimates. For simplicity, let

$$\Psi = \begin{pmatrix} \Psi_{r_1} & 0 \\ 0 & \Psi_{r_2} \end{pmatrix}.$$

We want to obtain the system for $\overrightarrow{\alpha}$

$$\begin{aligned} i\{\Psi H - H\Psi\} &= i \begin{pmatrix} \Psi_{r_1}\mathcal{L} - \mathcal{L}\,\Psi_{r_1} & 0 \\ 0 & \mathcal{L}\,\Psi_{r_2} - \Psi_{r_2}\,\mathcal{L} \end{pmatrix} \\ &= \begin{pmatrix} \Psi_{H_{h_1} r_1} & 0 \\ 0 & \Psi_{H_{h_1} r_2} \end{pmatrix} + \text{ 0th order.} \end{aligned}$$

Recalling that $-H_{h_1} r_1 = -M(H_{h_1} p) r_1$, $H_{h_1}\, r_2 = -M(H_{h_1}\; p)\, r_2$, and using that

$$\begin{pmatrix} \Psi_{H_{h_1} r_1} & 0 \\ 0 & \Psi_{H_{h_1} r_2} \end{pmatrix} = \begin{pmatrix} -M\Psi_{H_{h_1} p} & 0 \\ 0 & -M\Psi_{H_{h_1} p} \end{pmatrix} \begin{pmatrix} \Psi_{r_1} & 0 \\ 0 & \Psi_{r_2} \end{pmatrix} + \text{ 0th order}$$

we get

$$i\{\Psi H - H\Psi\} = \begin{pmatrix} -M\Psi_{H_{h_1} p} & 0 \\ 0 & -M\Psi_{H_{h_1} p} \end{pmatrix} \Psi \; + \text{ 0th order.}$$

Next,

$$\Psi\; B_{diag} = \begin{pmatrix} B_{11} & 0 \\ 0 & B_{22} \end{pmatrix} \Psi + \text{ 0th order}$$

and

$$\Psi\, C = (\Psi\, C \Psi^{-1})\Psi.$$

Thus, the system for $\overrightarrow{\alpha}$ is:

$$\begin{cases} \partial_t \overrightarrow{\alpha} = iH\overrightarrow{\alpha} + B_{diag}\,\overrightarrow{\alpha} - M \begin{pmatrix} \Psi_{H_{h_1} p} & 0 \\ 0 & \Psi_{H_{h_1} p} \end{pmatrix} \overrightarrow{\alpha} + C\overrightarrow{\alpha} + \overrightarrow{F}, \\ \overrightarrow{\alpha}(x,0) = \overrightarrow{\alpha_0}(x), \end{cases}$$

where $\overrightarrow{\alpha_0} = \Psi \overrightarrow{z_0}$ and $\overrightarrow{F} = \Psi \overrightarrow{g}$. It suffices to find M (see definition of Ψ_{r_1}) depending only on c_1 and the nontrapping character so that we can estimate $\overrightarrow{\alpha}$. To do this we consider

$$\langle \overrightarrow{\alpha}, \overrightarrow{\beta} \rangle = \int \alpha_1 \bar{\beta}_1 + \alpha_2 \bar{\beta}_2$$

and calculate

$$\begin{aligned}
\frac{d}{dt} \langle \overrightarrow{\alpha}, \overrightarrow{\alpha} \rangle =& \, i\big[\langle H \overrightarrow{\alpha}, \overrightarrow{\alpha} \rangle - \langle \overrightarrow{\alpha}, H \overrightarrow{\alpha} \rangle\big] + \langle B_{diag} \overrightarrow{\alpha}, \overrightarrow{\alpha} \rangle + \langle \overrightarrow{\alpha}, B_{diag} \overrightarrow{\alpha} \rangle \\
& - M\Big[\Big\langle \begin{pmatrix} \Psi_{H_{h_1} p} & 0 \\ 0 & \Psi_{H_{h_1} p} \end{pmatrix} \overrightarrow{\alpha}, \overrightarrow{\alpha} \Big\rangle + \Big\langle \overrightarrow{\alpha}, \begin{pmatrix} \Psi_{H_{h_1} p} & 0 \\ 0 & \Psi_{H_{h_1} p} \end{pmatrix} \overrightarrow{\alpha} \Big\rangle\Big] \\
& + \langle C \overrightarrow{\alpha}, \overrightarrow{\alpha} \rangle + \langle \overrightarrow{\alpha}, C \overrightarrow{\alpha} \rangle + \langle \overrightarrow{F}, \overrightarrow{\alpha} \rangle + \langle \overrightarrow{\alpha}, \overrightarrow{F} \rangle \\
=& \, i\big[\langle H \overrightarrow{\alpha}, \overrightarrow{\alpha} \rangle - \langle \overrightarrow{\alpha}, H \overrightarrow{\alpha} \rangle\big] + 2\mathcal{R}e \, \langle B_{diag} \overrightarrow{\alpha}, \overrightarrow{\alpha} \rangle \\
& - 2M\mathcal{R}e \Big\langle \begin{pmatrix} \Psi_{H_{h_1} p} & 0 \\ 0 & \Psi_{H_{h_1} p} \end{pmatrix} \overrightarrow{\alpha}, \overrightarrow{\alpha} \Big\rangle + 2\mathcal{R}e \, \langle C \overrightarrow{\alpha}, \overrightarrow{\alpha} \rangle \\
& + 2\mathcal{R}e \, \langle \overrightarrow{F}, \overrightarrow{\alpha} \rangle.
\end{aligned}$$

We recall that

$$H = \begin{pmatrix} \mathcal{L} & 0 \\ 0 & -\mathcal{L} \end{pmatrix},$$

with

$$\mathcal{L} = a_{jk}(x,t)\partial^2_{x_j x_k} = \partial_{x_j}(a_{jk}(x,t)\partial_{x_k}) - \partial_{x_j} a(x,t)\partial_{x_k} = \mathcal{L}_0 - \overrightarrow{b_3}(x,t) \cdot \nabla.$$

So

$$\begin{aligned}
i\big[\langle H \overrightarrow{\alpha}, \overrightarrow{\alpha} \rangle - \langle \overrightarrow{\alpha}, H \overrightarrow{\alpha} \rangle\big] =& \, i[\langle H_0 \overrightarrow{\alpha}, \overrightarrow{\alpha} \rangle - \langle \overrightarrow{\alpha}, H_0 \overrightarrow{\alpha} \rangle] \\
& + \langle i(\overrightarrow{b_3}(x,t) \cdot \nabla) \overrightarrow{\alpha}, \overrightarrow{\alpha} \rangle + \langle \overrightarrow{\alpha}, -i(\overrightarrow{b_3}(x,t) \cdot \nabla) \overrightarrow{\alpha} \rangle \\
=& \, [\langle H_0 \overrightarrow{\alpha}, \overrightarrow{\alpha} \rangle - \langle \overrightarrow{\alpha}, H_0 \overrightarrow{\alpha} \rangle\big] + 2\mathcal{R}e \, \langle B^1_{diag} \overrightarrow{\alpha}, \overrightarrow{\alpha} \rangle,
\end{aligned}$$

where

$$H_0 = \begin{pmatrix} \mathcal{L}_0 & 0 \\ 0 & -\mathcal{L}_0 \end{pmatrix}, \qquad B^1_{diag} = \begin{pmatrix} i\,\overrightarrow{b_3}(x,t) \cdot \nabla & 0 \\ 0 & -i\,\overrightarrow{b_3}(x,t) \cdot \nabla \end{pmatrix}.$$

Note that our asymptotic flatness assumption implies that

$$\widetilde{B}_{diag} = B_{diag} + B^1_{diag} = \begin{pmatrix} \widetilde{B}_{11} & 0 \\ 0 & \widetilde{B}_{11}, \end{pmatrix},$$

where the symbols of $\widetilde{B}_{jj}, l = 1, 2$ satisfy

$$|\partial_t \widetilde{B}_{jj}(x, t, \xi)| \leq c \frac{|\xi|}{\langle x \rangle^2},$$

and

$$|\widetilde{B}_{jj}(x, 0, \xi)| \leq c_1 \frac{|\xi|}{\langle x \rangle^2}.$$

As a consequence, for t small (depending on c) we have that

$$|\widetilde{B}_{jj}(x, t, \xi)| \leq 2c_1 \frac{|\xi|}{\langle x \rangle^2},$$

and

$$\begin{aligned} \frac{d}{dt} \langle \overrightarrow{\alpha}, \overrightarrow{\alpha} \rangle &= i[\langle H_0 \overrightarrow{\alpha}, \overrightarrow{\alpha} \rangle - \langle \overrightarrow{\alpha}, H_0 \overrightarrow{\alpha} \rangle] \\ &\quad + \mathcal{R}e\ \langle \widetilde{B}_{diag} \overrightarrow{\alpha}, \overrightarrow{\alpha} \rangle - 2M\, \mathcal{R}e\ \left\langle \begin{pmatrix} \Psi_{H_{h_1} p} & 0 \\ 0 & \Psi_{H_{h_1} p} \end{pmatrix} \overrightarrow{\alpha}, \overrightarrow{\alpha} \right\rangle \\ &\quad + 2\mathcal{R}e\ \langle C \overrightarrow{\alpha}, \overrightarrow{\alpha} \rangle + 2\mathcal{R}e\ \langle \overrightarrow{F}, \overrightarrow{\alpha} \rangle. \end{aligned} \tag{10.80}$$

Now it is easy to see that

$$\langle H_0 \overrightarrow{\alpha}, \overrightarrow{\alpha} \rangle - \langle \overrightarrow{\alpha}, H_0 \overrightarrow{\alpha} \rangle \equiv 0.$$

For the next two terms in (10.80), we get

$$\mathcal{R}e\ \langle \widetilde{B}_{11} \alpha_1, \alpha_1 \rangle - M\, \langle \Psi_{H_{h_1} p}\, \alpha_1, \alpha_2 \rangle \quad + \quad \text{similar terms in } \alpha_2.$$

We recall that

$$|\widetilde{B}_{11}(x, t, \xi)| \leq c_1 \frac{|\xi|}{\langle x \rangle^2}$$

and

$$H_{h_1}\, p \geq \frac{\beta}{2} \frac{|\xi|}{\langle x \rangle^2} - \frac{2}{\beta}.$$

We now choose M so large such that

$$M\, H_{h_1}\, p \pm \widetilde{B}_{11}(x, t, \xi) \leq \beta - \tilde{\beta}\, \frac{\langle \xi \rangle}{\langle x \rangle^2},$$

then the sharp Garding inequality (Lemma 10.3) gives

$$\mathcal{R}e\ \langle \widetilde{B}_{11} \alpha_1, \alpha_1 \rangle - M\, \mathcal{R}e\ \langle \Psi_{H_{h_1} p}\, \alpha_1, \alpha_1 \rangle \leq c \|\alpha_1\|_2^2 - \langle \Psi_{\tilde{\beta}\, \langle \xi \rangle / \langle x \rangle^2} \alpha_1, \alpha_1 \rangle.$$

Using that

$$\Psi_{\tilde{c}\,\langle\xi\rangle/\langle x\rangle^2} = \frac{1}{\langle x\rangle^2}\,\Psi_{\tilde{c}\,\langle\xi\rangle} + \Psi_{e_0},$$

with e_0 of order 0,

$$\Psi_{\tilde{c}\,\langle\xi\rangle} = \tilde{c}\Lambda^{1/2}\Lambda^{1/2},$$

and

$$\frac{1}{\langle x\rangle^2}\Lambda^{1/2}\Lambda^{1/2} = \Lambda^{1/2}\frac{1}{\langle x\rangle^2}\Lambda^{1/2} + \Psi_{e_0^1},$$

with e_0^1 of order 0, we see that

$$\langle \Psi_{\tilde{\beta}\,\langle\xi\rangle/\langle x\rangle^2}\alpha_1, \alpha_1\rangle = \tilde{\beta}\int \frac{|\Lambda^{1/2}\alpha_1|^2}{\langle x\rangle^2}(x,t)dx + O(\|\alpha_1\|_2).$$

So we pick $t_0 \in [0,T]$ such that

$$\|\overrightarrow{\alpha}(t_0)\|_2^2 \geq \frac{1}{2}\sup_{[0,T]}\|\overrightarrow{\alpha}(t)\|_2^2,$$

to get that

$$\begin{aligned}\sup_{[0,T]}\|\overrightarrow{\alpha}(t)\|_2^2 + \tilde{\beta}\int_0^T\int\frac{|\Lambda^{1/2}\alpha_1|^2}{\langle x\rangle^2}(x,t)dx\,dt &\leq 2\int_0^{t_0}\frac{d}{dt}\langle\overrightarrow{\alpha},\overrightarrow{\alpha}\rangle\,dt + 2\|\overrightarrow{\alpha_0}\|_2^2\\ &\leq c\int_0^{t_0}\|\overrightarrow{\alpha}\|_2^2\,dt + 2\int_0^{t_0}\|\overrightarrow{F}\|_2\|\overrightarrow{\alpha}\|_2\,dt + 2\|\overrightarrow{\alpha_0}\|_2^2\\ &\leq CT\sup_{[0,T]}\|\overrightarrow{\alpha}(t)\|_2^2 + 2\sup_{[0,T]}\|\overrightarrow{\alpha}(t)\|_2^2\int_0^T\|\overrightarrow{F}\|\,dt + 2\|\overrightarrow{\alpha_0}\|_2^2,\end{aligned}$$

which, upon choosing $CT < 1/2$ yields the desired estimate (10.46).

10.2 Comments

The main result in this chapter, Theorem 10.1, was obtained in [KPV10]. As it was seen, the proof is based on the artificial viscosity method (it cannot be established by a solely Picard argument; see [KT]) and uses only classical pseudodifferential operators. In this regard, the ellipticity assumption is crucial.

It was displayed in Chapter 9 that dispersive models of the form

$$\partial_t u = i(\partial_{x_1}^2 + \cdots + \partial_{x_k}^2 - \partial_{x_{k+1}}^2 - \cdots - \partial_{x_n}^2)u + f(u,\bar{u},\nabla_x u,\nabla_x\bar{u}) \qquad (10.81)$$

arise in the physical (for instance, wave propagation) and in the mathematical context, for example, related to higher-order models which can be solved by inverse scattering method.

In [KPV14], local well-posedness of the IVP associated to the equation (10.81) was obtained. The method of proof, among other arguments, employs pseudodifferential operators in the Calderón–Vaillancourt class. This approach does not seem to apply to the variable coefficient class

$$\begin{aligned}\partial_t u = i\partial_{x_k}(a_{jk}(x)\partial_{x_j}u) + (\overrightarrow{b_1}(x)\cdot\nabla)u + (\overrightarrow{b_2}(x)\cdot\nabla)\bar{u}\\ +c_1(x)u + c_2(x)u + f(u,\bar{u},\nabla_x u,\nabla_x\bar{u}),\end{aligned} \tag{10.82}$$

where $(a_{jk}(x))$ is a symmetric nondegenerate (invertible) matrix.

The local well-posedness of the IVP associated to equation (10.82) was studied in the massive work [KPRV1]. For that purpose, a new class of pseudodifferential operators was introduced, which takes into consideration the "geometry" of the nonelliptic operator. Under asymptotic flatness hypothesis of the coefficient $a_{jk}(x)$, $b_{1j}, b_{2j},\ k,j=1,\dots,n$, and nontrapping assumptions of the bicharacteristic flow associated to the symbol $a_{jk}(x)\xi_k\xi_j$ it was proved in [KPV14] that the IVP for (10.82) is locally well-posed in weighted Sobolev spaces $\mathcal{F}^s_{2k} = H^s(\mathbb{R}^n)\cap L^2(|x|^{2k}\,dx)$ for large enough values of $s,k\in\mathbb{Z}^+$ $(s>k)$. The results in [KPRV1] were extended in [KPRV2] to the case where the coefficients $a_{jk}(\,\cdot\,)$, $b_{1j}(\,\cdot\,)$, $b_{2j}(\,\cdot\,)$ depend on $(x,t,u,\bar{u},\nabla_x u,\nabla_x\bar{u})$, $k,j=1,\dots,n$, and $c_1(\cdot)$, $c_2(\cdot)$ on $(x,t,u,\bar{u})$.

In [MMTa2] and [MMTa3], the problem of finding the minimal regularity assumptions required to guarantee local well-posedness was considered. In these works, the setting was restricted to the small data regime. In [MMTa2], for the quadratic case, a translation invariant space was used instead of the weighted one. One should remark that in the small data case the crucial step in the proof presented here, the use of the integrating factor, is not necessary. Similarly, in the small data case the nontrapping condition is not relevant and the proof follows by applying the contraction mapping principle which cannot be the case for data of arbitrary size (see [KT]).

10.3 Exercises

10.1 Fill out the details of the results discussed in (i)–(v) regarding the IVP (10.7).

10.2 Prove that $f = f(u,\bar{u},\nabla\bar{u})$, $n\geq 1$, and $f=\partial_x(|u|^2u)$, $n=1$, satisfy the inequality (10.19).

10.3 Assuming that $f(u,\bar{u},\nabla u,\nabla\bar{u})$ satisfies the inequality (10.19), sketch a local existence proof for the IVP:

$$\begin{cases}\partial_t u = i\epsilon\,\Delta u + f(u,\bar{u},\nabla u,\nabla\bar{u}),\\ u(x,0)=u_0(x)\in H^s(\mathbb{R}^n), \qquad s>n/2+1,\end{cases} \tag{10.83}$$

for $\epsilon \geq 0$. This shows that under the hypothesis (10.19) the dispersion is not needed for a local theory.

10.4 Consider the IVP:

$$\begin{cases} \partial_t u = i\Delta u + P(u, \bar{u}, \nabla u, \nabla \bar{u}), \\ u(x, 0) = u_0(x), \end{cases} \tag{10.84}$$

$x \in \mathbb{R}^n, \ t \in \mathbb{R}$, where $P : \mathbb{C}^n \mapsto \mathbb{C}$ is a polynomial such that

$$P(z) = \sum_{|\alpha|=3}^{N} a_\alpha \, z^\alpha.$$

Prove that there exists $\delta > 0$ (small) and $s \gg 1$ such that for any $u_0 \in H^s(\mathbb{R})$ with $\delta_0 \equiv \|u_0\|_{s,2} \leq \delta$ the IVP (10.84) has a unique solution:

$$u \in C([0, T] : \ H^s(\mathbb{R}^n)), \quad D^{s+1/2}u \in L^2_{\text{loc}}(\mathbb{R}^n \times [0, T]) \tag{10.85}$$

(with $T = T(\delta_0) \uparrow \infty$ as $\delta_0 \downarrow 0$) which can be obtained by a fixed-point argument. Hint: Consider the equivalent integral equation form of the IVP (10.84) and prove that the operator

$$\Phi_{u_0}(u)(t) = e^{it\Delta}u_0 + \int_0^t e^{i(t-t')\Delta} P(u, \bar{u}, \nabla u, \nabla \bar{u})(\cdot, t')\, dt'$$

has a unique fixed point in an appropriate space

$$X_T^s \hookrightarrow C([0, T] : \ H^s(\mathbb{R}^n))$$

by using the estimates in Theorems 4.2 and 4.3.

10.5 (i) Prove that the symbol in (10.28) for $|\xi| \geq 1$ satisfies the estimate (10.29).

(ii) Prove that in addition, for $|\xi| \geq 1$ one has

$$|(x_j \partial_{x_j})^\gamma \, \partial_x^\alpha \partial_\xi^\beta \, \mu(x, \xi)| \leq c_{\alpha\beta\gamma} \, \langle x \rangle^{|\alpha|} \, |\xi|^{-|\beta|}, \tag{10.86}$$

$\gamma \in \mathbb{Z}^+, \alpha, \beta \in (\mathbb{Z}^+)^n$, where $\langle x \rangle = (1 + |x|^2)^{1/2}$.

Appendix A
Proof of Theorem 2.8

Definition A.1. For $k \in \mathbb{Z}$ let $\mathcal{Q}_k$ be the collection of cubes in $\mathbb{R}^n$ which are congruent to $[0, 2^{-k})^n$ and whose vertices lie on the lattice $(2^{-k}\mathbb{Z})^n$.

The cubes in

$$\mathcal{Q}^* = \bigcup_{k\in\mathbb{Z}} \mathcal{Q}_k \tag{A.1}$$

are called the *dyadic* cubes.

A.4 Proof of Theorem 2.8

As it was mentioned after the statement of Theorem 2.8 it suffices to show that the operator T_m is of weak type (1,1), that is, there exists $c_1 > 0$ such that for every $f \in L^1(\mathbb{R}^n)$

$$\sup_{\alpha>0} \alpha\, |\{x \in \mathbb{R}^n \;:\; |T_m f(x)| > \alpha\}| \leq c_1\, \|f\|_1. \tag{A.2}$$

To establish (A.2) we need the Calderón–Zygmund decomposition of L^1-functions.

Lemma A.1 (Calderón–Zygmund lemma). *Let $f \in L^1(\mathbb{R}^n)$. For any $\alpha > 0$, f can be decomposed as*

$$f = g + b = g + \sum_{j=1}^{\infty} b_j \tag{A.3}$$

such that

$$|g(x)| \leq 2^n \alpha \quad a.e \quad x \in \mathbb{R}^n, \tag{A.4}$$

$$b_j \text{ supported in } \overline{Q}_j,\; Q_j \text{ a dyadic cube with } \int_{Q_j} b_j\, dx = 0, \tag{A.5}$$

F. Linares, G. Ponce, *Introduction to Nonlinear Dispersive Equations,* Universitext, DOI 10.1007/978-1-4939-2181-2

$$\textit{the } Q_j' s \textit{ are disjoint,} \quad \sum_{j=1}^{\infty} |Q_j| \leq \alpha^{-1} \, \|f\|_1, \tag{A.6}$$

and

$$\|g\|_1 + \sum_{j=1}^{\infty} \|b_j\|_1 \leq 6 \, \|f\|_1. \tag{A.7}$$

Proof. Assume $f \geq 0$ (otherwise $f = f^+ - f^-$ and decompose each part). Since $f \in L^1(\mathbb{R}^n)$ there exists l such that $|Q|^{-1} \int_Q f \, dy < \alpha$ for any cube of side length l.

Fix $k_0 \in \mathbb{Z}$ such that

$$2^{k_0 n} \, \|f\|_1 < \alpha.$$

We start with the family of cubes in $\mathcal{Q}_{k_0}$. Let Q^0 be one of them. Divide each side of Q^0 in two to get 2^n new dyadic cubes of side length $2^{-(k_0+1)}$. Let Q^1 be such a cube; there are two possibilities:

$$\text{(a)} \ \frac{1}{|Q^1|} \int_{Q^1} f \, dy < \alpha \quad \text{or} \quad \text{(b)} \ \frac{1}{|Q^1|} \int_{Q^1} f \, dy \geq \alpha.$$

In case (b) one stops the subdivision, noticing that

$$\alpha \leq \frac{1}{|Q^1|} \int_{Q^1} f \, dy \leq \frac{2^n}{|Q^0|} \int_{Q^0} f \, dy \leq 2^n \alpha, \tag{A.8}$$

and collecting it in a sequence Q_j.

In case (a) the subdivision process continues. Thus, if $x \notin \bigcup_j Q_j$ it follows from the Lebesgue differentiation theorem (Exercise 2.6 (ii)) that

$$f(x) \leq \alpha \text{ a.e. } x \in \mathbb{R}^n \setminus \cup_j Q_j. \tag{A.9}$$

Finally, we define

$$g(x) = \begin{cases} \dfrac{1}{|Q_j|} \displaystyle\int_{Q_j} f \, dy \ \text{ if } x \in Q_j, \\ \\ f(x) \ \text{ if } x \notin Q_j, \end{cases} \tag{A.10}$$

and

$$b_j(x) = (f(x) - g(x)) \chi_{Q_j}(x), \quad j \in \mathbb{Z}^+, \tag{A.11}$$

which yields the result. □

We shall denote by Q_j^* the cube having the same center as Q_j and twice its side length as

$$\Omega = \bigcup_j Q_j \text{ and } \Omega^* = \bigcup_j Q_j^* \tag{A.12}$$

with

$$|\Omega^*| \leq \sum_j |Q_j^*| = 2^n \sum_j |Q_j|. \tag{A.13}$$

Proof of inequality (A.2). First we notice that using Calderón–Zygmund lemma

$$\begin{aligned} &|\{x \in \mathbb{R}^n \ : \ |T_m f(x)| > \alpha\}| \\ &\leq |\{x \in \mathbb{R}^n \ : \ |T_m g(x)| > \alpha/2\}| + |\{x \in \mathbb{R}^n \ : \ |T_m b(x)| > \alpha/2\}| \qquad \text{(A.14)} \\ &\leq |\{x \in \mathbb{R}^n \ : \ |T_m g(x)| > \alpha/2\}| + |\{x \notin \Omega^* \ : \ |T_m b(x)| > \alpha\}| + |\Omega^*| \\ &= E_1 + E_2 + E_3. \end{aligned}$$

From (A.13) and (A.6) in Calderón–Zygmung lemma we have that

$$E_3 = |\Omega^*| \leq 2^n \sum_j |Q_j| \leq 2^n \alpha^{-1} \|f\|_1. \tag{A.15}$$

Tchebychev's inequality and (A.4) in the Calderón–Zygmund lemma yield

$$\begin{aligned} E_1 &= |\{x \in \mathbb{R}^n \ : \ |T_m g(x)| > \alpha/2\}| \leq c \left(\frac{\|T_m g\|_2}{\alpha/2} \right)^2 \leq c \, \frac{\|g\|_2^2}{\alpha^2} \\ &\leq \frac{c}{\alpha^2} \|g\|_1 \, \|g\|_\infty \leq \frac{c}{\alpha} \|g\|_1 \leq \frac{c}{\alpha} \|f\|_1. \end{aligned} \tag{A.16}$$

Hence, it remains to prove that

$$E_2 = |\{x \notin \Omega^* \ : \ |T_m b(x)| > \alpha/2\}| \leq c\alpha^{-1} \, \|f\|_1. \tag{A.17}$$

It will suffice to show that

$$\int\limits_{x \notin Q_j^*} |T_m b_j(x)| \, dx \leq c\|b_j\|_1, \qquad j \in Z^+. \tag{A.18}$$

To establish (A.18) we follow the argument in [BeL].

Let $\varphi \in C_0^\infty(\{\xi \ : \ |\xi| < 2\})$, such that $\varphi(\xi) = 1$ for $|\xi| \leq 1$. Let $\beta(\xi) = \varphi(\xi) - \varphi(2\xi)$. Thus

$$\sum_{l=-\infty}^{\infty} \beta(2^{-l}\xi) = 1 \quad \text{for } \xi \neq 0. \tag{A.19}$$

If $m_l(\xi) = \beta(\xi)\, m(2^l \xi)$, then by hypothesis (2.18)

$$\int |(1-\Delta)^{s/2} m_l(\xi)|^2 \, d\xi < c. \tag{A.20}$$

Thus, by Plancherel's identity using the notation $K_l(x) = \widehat{m}_l(x)$, one gets that

$$\int (1+|x|^2)^s |K_l(x)|^2 \, dx < c, \tag{A.21}$$

which, combined with the Cauchy–Schwarz inequality yields the estimate

$$\int\limits_{\{x:\, \max_m |x_m| > R\}} |K_l(x)| \, dx < c\, R^{n/2-s}, \tag{A.22}$$

which is a good estimate for $R >> 1$.

Reapplying the estimates (A.20) and (A.21) for $\xi_k\, m_l(\xi)$ instead of $m_l(\xi)$ one finds that

$$\int |\nabla K_l(x)| \, dx < c. \tag{A.23}$$

Consequently, it follows that

$$\int |K_l(x+y) - K_l(x)| \, dx < c|y|. \tag{A.24}$$

We observe that as a temperate distribution,

$$K(x) = \sum_{l=-\infty}^{\infty} 2^{nl} K_l(2^l x) = \sum_{l=-\infty}^{\infty} \widehat{m}_l(2^{-l} x). \tag{A.25}$$

Assume that Q_j is a cube of side R centered at the origin. From (A.22) one has that

$$\begin{aligned} \int\limits_{x \notin Q_j^*} |2^{nl} K_l(2^l \cdot) * b_j| \, dx &\le \int\limits_{Q_j} \int\limits_{x \notin Q_j^*} |2^{nl} K_l(2^l (x-y))||b_j(y)| \, dx dy \\ &\le \|b_j\|_1 \int\limits_{\{x:\, \max_m |x_m| \ge 2^l R\}} |K_l(x)| \, dx \\ &\le c\, (2^l R)^{n/2-s}\, \|b_j\|_1. \end{aligned} \tag{A.26}$$

Now using that $\int_{Q_j} b_j dy = 0$ it follows that

$$\int\limits_{x \notin Q_j^*} 2^{nl} \int\limits_{y \in Q_j} K_l(2^l (x-y)) b_j(y) \, dy dx \tag{A.27}$$

$$= \int_{x \notin Q_j^*} 2^{nl} \int_{y \in Q_j} \big(K_l(2^l(x-y)) - K_l(2^l x) \big) \, b_j(y) \, dy dx.$$

Therefore, (A.24) yields

$$\int_{x \notin Q_j^*} |2^{nl} K_l(2^{nl} \cdot) * b_j| \, dx$$

$$\leq \int_{y \in Q_j} \int_{x \notin Q_j^*} 2^{nl} |K_l(2^l(x-y)) - K_l(2^l x)||b_j(y)| \, dx dy \tag{A.28}$$

$$\leq c \, (2^l R) \|b_j\|_1.$$

Adding in l in (A.26) for $2^l R > 1$ and in (A.28) for $2^l R \leq 1$ one gets that

$$\int_{x \notin Q_j^*} |T_m \, b_j(x)| \, dx \leq c \, \|b_j\|_1, \tag{A.29}$$

which completes the proof. □

Appendix B
Proof of Lemma 4.2

B.1 Proof of Lemma 4.2

Let

$$\Omega = \{(x,y) \in [0,1] \times [0,1] \mid x < y\} = \bigcup_j Q_j, \tag{B.1}$$

and

$$\mathcal{A} \equiv \{Q_j\}_j \tag{B.2}$$

where the Q_j's are disjoint dyadic cubes (see Definition A.1) such that if $\bar{Q}_j = \bar{I}_j \times \bar{J}_j$, ($I_j, J_j$ intervals), then

(i) $\#\,(\bar{Q}_j \cap \{(x,x) \mid x \in [0,1]\}) = 1.$
(ii) $\#\,\{Q_j \subseteq \Omega \mid \text{length side of } Q_j = 2^{-k}\} = 2^{k-1},\ k \in \mathbb{Z}^+.$

Without loss of generality assume $\|f\|_r = 1$ and define

$$F(t) = \int_{-\infty}^{t} |f(s)|^r\, ds, \tag{B.3}$$

so $F : \mathbb{R} \to [0,1]$ is a nondecreasing continuous function.

Notice that if $s < t$, then either

$$F(s) < F(t)$$

or

$$f \equiv 0, \quad \text{a.e. in}[s,t].$$

For $I = [a,b] \subseteq [0,1]$ one has that

$$F^{-1}([a,b]) = [A,B] \quad \text{with} \quad F(A) = a \quad \text{and} \quad F(B) = b. \tag{B.4}$$

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Hence

$$\int_A^B |f(s)|^r\, ds = F(B) - F(A) = b - a, \tag{B.5}$$

and

$$\|f\|_{L^r(F^{-1}(I))} = |I|^{1/r}. \tag{B.6}$$

Defining

$$B(f,g) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} K(t,s) f(s) g(t)\, ds dt \tag{B.7}$$

and

$$\tilde{B}(f,g) = \int\int_{s<t} K(t,s) f(s) g(t)\, ds dt \tag{B.8}$$

it will suffices to see that there exists $c > 0$ such that

$$|\tilde{B}(f,g)| \le c\, \|f\|_r \|g\|_{l'}, \qquad \frac{1}{l} + \frac{1}{l'} = 1. \tag{B.9}$$

We take $\|f\|_r = \|g\|_{l'} = 1$, thus

$$\begin{aligned}
|\tilde{B}(f,g)| &= |\int\int_{s<t} K(t,s) f(s) g(t)\, ds dt| \\
&= \Big| \sum_{\substack{Q_j = I_j \times J_j \\ Q_j \in \mathcal{A}}} B(\chi_{F^{-1}(I_j)} f, \chi_{F^{-1}(J_j)} g) \Big| \\
&\le \sum_{Q_j \in \mathcal{A}} c\, \|f\|_{L^r(F^{-1}(I_j))} \|g\|_{L^{l'}(F^{-1}(J_j))} \qquad \text{(B.10)} \\
&\le c \sum_{k \in \mathbb{Z}^+} (2^{-k})^{1/r} \sum_{|J_j| = 2^{-k}} \|g\|_{L^{l'}(F^{-1}(J_j))} \\
&\le c \sum_{k \in \mathbb{Z}^+} (2^{-k})^{1/r} \|g\|_{l'} \Big(\sum_{|J_j| = 2^{-k}} 1 \Big)^{1/l} \\
&\le c \sum_{k \in \mathbb{Z}^+} (2^{-k})^{1/r} (2^{k-1})^{1/l}.
\end{aligned}$$

Since by hypotheses $-\frac{1}{r} + \frac{1}{l} < 0$, this finishes the proof. □

References

[AA1] H. Added and S. Added. Equations of Langmuir turbulence and nonlinear Schrödinger equation: smoothness and approximation, J. Funct. Anal. **79** (1988), 183–210.

[AA2] H. Added and S. Added. Existence globale de solutions fortes pour les équations de la turbulence de Langmuir en dimension 2, C. R. Acad. Sci. Paris Sér. I Math. **299** (1984), 551–554.

[ABLS] J. Angulo, J.L. Bona, F. Linares, and M. Scialom. Scaling, stability and singularities for nonlinear, dispersive wave equations: the critical case, Nonlinearity **15** (2002), 759–786.

[AC] M.J. Ablowitz and P.A. Clarkson. *Solitons, Nonlinear Evolution Equations and Inverse Scattering*, London Mathematical Society Lecture Note Series, **149**. Cambridge University Press, Cambridge, 1991.

[AF] M.J. Ablowitz and A.S. Fokas. The inverse scattering transform for the Benjamin-Ono equation-a pivot to multidimensional problems, Stud. Appl. Math. **68** (1983), 1–10.

[AH] M.J. Ablowitz and R. Haberman. Nonlinear evolution equations in two and three dimensions, Phys. Rev. Lett. **35** (1975), 1185–1188.

[AL] J. Albert and F. Linares. Stability and symmetry of solitary-wave solutions to systems modeling interactions of long waves, J. Math. Pures Appl. **79** (2000), 195–226.

[AlCa] T. Alazard and R. Carles. Loss of regularity for the super-critical Schrödinger equations, Math. Ann. **343** (2009), 397–420.

[AlMn1] M. Alejo and C. Muñoz. Nonlinear stability of the mKdV breather, Comm. Math. Phys. **324** (2013), 233–262.

[AlMn2] M. Alejo and C. Muñoz. Dynamics of complex-valued modified KdV solitons with applications to the stability of breathers, arXiv:1308.0998

[AlMnVe] M. Alejo, C. Muñoz and L. Vega. The Gardner equation and the L^2-stability of the N-soliton solution of the Korteweg-de Vries equation, Trans. A.M.S. **365** (2013), 195–212.

[AnF] D. Anker and N.C. Freeman. On the soliton solutions of the Davey-Stewartson equation for long waves, Proc. R. Soc. London, Ser. A **360** (1978), 529–540.

[AS] N. Aronszajn and K.T. Smith. Theory of Bessel potentials I, Ann. Inst. Fourier (Grenoble) 11 (1961), 385–475.

[AS1] M.J. Ablowitz and H. Segur. On the evolution of packets of water waves, J. Fluid Mech. **92** (1979), 691–715.

[AS2] M.J. Ablowitz and H. Segur. Solitons and the Inverse Scattering Transform, Studies in Applied Math. Philadelphia: SIAM, 1981.

[AvHe] J.E. Avron and I.W. Herbst. Spectral and scattering theory of Schrödinger operators related to Stark effect, Comm. Math. Phys. **52** (1977) 239–254.

[B] W. Beckner. Inequalities in Fourier Analysis, Ann. of Math. **102** (1975), 159–182.

F. Linares, G. Ponce, *Introduction to Nonlinear Dispersive Equations*,
Universitext, DOI 10.1007/978-1-4939-2181-2

[BBCH] J. Bennet, N. Bez, A. Carbery, and D. Hundertmark. Heat-flow monotonicity of Strichartz norms, Anal. PDE **2** (2009), 147–158.

[BBM] T.B. Benjamin, J.L. Bona, and J.J. Mahony. Model equation for the long waves in nonlinear dispersive systems, Phil. Trans. Royal Soc. London **272** (1972), 47–78.

[BC1] R. Beals and R.R. Coifman. The spectral problem for the Davey-Stewartson and Ishimori hierarchies, Proc. Conf. on Nonlinear Evolution Equations: Integrability and Spectral Methods, Machester, U. K., 1988.

[BC2] H. Berestycki and T. Cazenave. Instabilité des états stationnaires dans les équations de Schrödinger et de Klein-Gordon non linéaires, C.R. Acad. Sci. Paris Sér. I Math. **293** (1981), 489–492.

[BeDS] I. Bejenaru and D. Da Silva. Low regularity solutions for a 2D quadratic Schrödinger equation, Trans. A.M.S. **360** (2008), 5805–5830.

[Be1] T.B. Benjamin. The stability of solitary waves, Proc. Roy. Soc. London, Ser. A **328** (1972), 153–183.

[Be2] T.B. Benjamin. Internal waves of permanent form in fluids of great depth, J. Fluid Mech. **29** (1967), 559–592.

[BeL] J. Bergh and J. Löfström. *Interpolation Spaces. An Introduction*, Grundhlehren 223, Springer-Verlag, New York, 1976.

[Bc] M. Beceanu. A centre-stable manifold for the focussing cubic NLS in $\mathbb{R}^{1+3}$, Comm. Math. Phys. **280** (2008), 145–205.

[BGa] H. Brezis and T. Gallouet. Nonlinear Schrödinger evolution equations, Nonlinear Analysis, TMS **4** (1980), 667–681.

[BGK] H. Berestycki, T. Gallouët and O. Kavian. Équations de champs scalaires Euclidiens non linéaires dans le plan, C.R. Acad. Sci. Paris Sér. I Math. **297** (1983), 307–310.

[BGT1] N. Burq, P. Gerard and N. Tzvetkov. Two singular dynamics of the nonlinear Schrödinger equation on a plane domain, Geom. Funct. Anal. **13** (2003), 1–19.

[BGT2] N. Burq, P. Gerard and N. Tzvetkov. An instability property of the nonlinear Schrödinger equation on $\mathbb{S}^d$, Math. Res. Lett. **9** (2002), 323–335.

[BGT3] N. Burq, P. Gerard and N. Tzvetkov. Strichartz inequalities and the nonlinear Schrödinger equation on compact manifolds, Amer. J. Math. **126** (2004), 569–605.

[BHHT] I. Bejenaru, S. Herr, J. Holmer and D. Tataru. On the 2D Zakharov system with L^2-Schrödinger data, Nonlinearity **22** (2009), 1063–1089.

[BHS] A. de Bouard, N. Hayashi and J-C. Saut. Global existence of small solutions to a relativistic nonlinear Schrödinger equation, Comm. Math. Phys. **189** (1997), 73–105.

[BIK] I. Bejenaru, A.D. Ionescu and C.E. Kenig. Global existence and uniqueness of Schrödinger maps in dimension $d \geq 4$, Advances in Math. **215** (2007), 263–291.

[BiL] H.A. Biagioni and F. Linares. Ill-posedness for the derivative Schrödinger and generalized Benjamin-Ono equations, Trans. A.M.S. **353** (2001), 3649–3659.

[BKl] M. Ben-Artzi and S. Klainerman. Decay and regularity for the Schrödinger equation, J. Anal. Math. **58** (1992), 25–37.

[BKPSV] B. Birnir, C.E. Kenig, G. Ponce, N. Svanstedt, and L. Vega. On the ill-posedness of the IVP for the generalized Korteweg-de Vries and nonlinear Schrödinger equations, J. London Math. Soc. **53** (1996), 551–559.

[BLi] H. Berestycki and P.-L. Lions. Nonlinear scalar field equations, Arch. Rat. Mech. Anal. **82** (1983), 313–375.

[BLiP] H. Berestycki, P.-L. Lions and L.A. Peletier. An ODE approach to the existence of positive solutions for semilinear problems in $\mathbb{R}^N$, Indiana Univ. Math. J. **30** (1981), 141–157.

[BM] I. Bialynicki-Birula and J. Mycielski. *Nonlinear wave mechanics*, Ann. Phys. **100** (1976), 62–93.

[Bn1] D.J. Benney. A general theory for interactions between short and long waves, Stud. Appl. Math. **56** (1977), 81–94.

[Bn2] J. Bona. On the stability theory of solitary waves, Proc. Roy. Soc. London, Ser. A **344** (1975), 363–374.

[BnR] D.J. Benney and G.J. Roskes. Wave instabilities, Stud. Appl. Math. **48**, (1969), 377–385.

[Bo1] J. Bourgain. Fourier transform restriction phenomena for certain lattice subsets and applications to nonlinear evolution equation, Geom. Funct. Anal. **3** (1993) 107–156, 209–262.

[Bo2] J. Bourgain. *Global Solutions of Nonlinear Schrödinger Equations*, American Mathematical Society Colloquium Publications **46**, American Mathematical Society, Providence, RI, (1999).

[Bo3] J. Bourgain. A remark on Schrödinger operators, Israel J. Math. **77** (1992), 1–16.

[Bo4] J. Bourgain. Scattering in the energy space and below for the 3D NLS, J. Anal. Math. **75** (1998), 267–297.

[Bo5] J. Bourgain. Refinements of Strichartz' inequalities and applications to 2D-NLS with critical nonlinearity, Int. Math. Res. Notices **5** (1998), 253–283.

[Bo6] J. Bourgain. On growth in time of Sobolev norms of smooth solutions of nonlinear Schrödinger equations in $\mathbb{R}^d$, J. Analyse Math. **72** (1997), 299–310.

[Bo7] J. Bourgain. Global well-posedness of defocusing critical nonlinear Schrödinger equation in the radial case. Journal A.M.S. **12** (1999), 145–171.

[Bo8] J. Bourgain. Periodic Korteweg-de Vries equation with measures as initial data, Selecta Math. **3** (1997), 115–159.

[Bo9] J. Bourgain. On the Cauchy and invariant measure problem for the periodic Zakharov system, Duke Math. J. **76** (1994), 175–202.

[Bo10] J. Bourgain. On the Cauchy problem for the Kadomtsev-Petviashvili equation, Geom. Funct. Anal. **3** (1993), 315–341.

[Bo11] J. Bourgain. On the Schrödinger maximal function in higher dimensions, arXiv:1201.3342.

[Bo12] J. Bourgain. On the Cauchy problem for the periodic KdV type equations, J. Fourier Anal. Appl. Special issue (1995), 17–86.

[BoC] J. Bourgain and J. Colliander. On well-posedness of the Zakharov system, Inter. Math. Res. Notices **11** (1996), 515–546.

[BoLi] J. Bourgain and D. Li. On an endpoint Kato-Ponce inequality, to appear in Diff. and Int. Eqs. **27** (2014), 1037–1072.

[BoW] J. Bourgain and W. Wang. Construction of blow up solutions for the nonlinear Schrödinger equations with critical nonlinearity, Ann. Scuola Norm. Sup. Pisa Cl. Sci. **25** (1997), 197–215.

[Br] A.R. Brodsky. On the asymptotic behavior of solutions of the wave equations, Proc. A.M.S. **18** (1967), 207–208.

[BRV] J.A. Barcelo, A. Ruiz and L. Vega. Some dispersive estimates for Schrödinger equations with repulsive potentials, J. Funct. Anal. **236** (2006) 1–24.

[Bs] M. Beals. Self-spreading and strength of singularities for the solutions to semilinear wave equations, Ann. of Math. **118** (1983), 187–214.

[BS] J.L. Bona and R. Smith. The initial value problem for the Korteweg-de Vries equation, Roy. Soc. London **Ser A 278** (1978), 555–601.

[BSa1] J.L. Bona and J-C. Saut. Dispersive blowup of solutions of generalized Korteweg-de Vries equations. J. Diff. Eqs. **103** (1993), 3–57.

[BSa2] J.L. Bona and J-C. Saut. Dispersive blow-up II. Schrödinger-type equations, optical and oceanic rogue waves, Chin. Ann. Math. Ser. B **31** (2010), 793–818.

[BSc] J.L. Bona and R. Scott. Solutions of the Korteweg-de Vries equation in fractional order Sobolev spaces, Duke Math. J. **43** (1976), 87–99.

[BSS] J.L. Bona, P.E. Souganidis, and W. Strauss. Stability and instability of solitary waves of Korteweg-de Vries type, Proc. Roy. Soc. London, Ser. A **A411** (1987), 395–412.

[BTo] I. Bejenaru and T. Tao. Sharp well-posedness and ill-posedness for a quadratic nonlinear Schrödinger equation, J. Funct. Anal. **233** (2006), 228–259.

[BTz] J.L. Bona and N. Tzvetkov. Sharp well-posedness result for the BBM equation, Discrete Cont. Dyn. Syst. **23** (2009), 1241–1252.

[BuKo] T. Buckmaster and H. Koch. The Korteweg-de Vries equation at H^{-1} regularity, arxiv 1112.4657

[BuPl] N. Burq and F. Planchon. On well-possedness of the Benjamin-Ono equation, Math. Ann. **340** (2008), 497–542.

[C] L. Carleson. *Some Analytical Problems Related to Statistical Mechanics*, Lecture Notes in Math. Springer-Verlag **779** (1979), 9–45.

[Ca] K. M. Case. Benjamin-Ono related equations and their solutions, Proc. Nat. Acad. Sci. USA **76** (1979), 1–3.

[Car] E. Carneiro. *A sharp inequality for the Strichartz norm*, Int. Math. Res. Notices **16** (2009), 3127–3145.

[Cb] V. Combet. Construction and characterization of solutions converging to solitons for supercritical gKdV equations, Diff. and Int. Eqs. **23** (2010), 513–568.

[CDKS] J.E. Colliander, J.-M. Delort, C.E. Kenig, and G. Staffilani. Bilinear estimates and applications to 2D NLS, Trans. A.M.S. **353** (2001), 3307–3325.

[CGD] K.W. Chow, R.H.J. Grimshaw, and E. Ding. Interaction of breathers and solitons in the extended Korteweg-de Vries equation, Wave Motion **43** (2005), 158–166.

[CH] R. Camassa and D.D. Holm. An integrable shallow water equation with peaked solutions, Phys. Rev. Lett. **71** (1993), 1661–1664.

[Ch1] H. Chihara. The initial value problem for the elliptic-hyperbolic Davey-Stewartson equation, J. Math. Kyoto Univ. **39** (1999), 41–66.

[Ch2] H. Chihara. Local existence for semilinear Schrödinger equations, Math. Japonica **42** (1995), 35–52.

[Ch3] H. Chihara. The initial value problem for Schrödinger equations on the torus, Int. Math. Res. Notices **15** (2002), 789–820.

[Ci] Y. Choi. *Well-posedness and scattering results of fifth order evolution equations*, Ph.D thesis, University of Chicago, 1994.

[CIKS] J. Colliander, A.D. Ionescu, C.E. Kenig, and G. Staffilani. Weighted low-regularity solutions of the KP-I initial-value problem, Discrete Contin. Dyn. Syst. **20** (2008), 219–258.

[CKS] W. Craig, T. Kappeler and W. Strauss. Microlocal dispersive smoothing for the Schrödinger equation, Comm. Pure Appl. Math. **48** (1995), 769–860.

[CKSTT1] J. Colliander, M. Keel, G. Staffilani, H. Takaoka, and T. Tao. A refined global well-posedness results for the Schrödinger equations with derivative, SIAM J. Math. Anal. **34** (2002), 68–86.

[CKSTT2] J. Colliander, M. Keel, G. Staffilani, H. Takaoka, and T. Tao. Resonant decompositions and the I-method for cubic nonlinear Schrödinger on $\mathbb{R}^2$, Disc. Cont. Dynam. Systems A 21 (2008), 665–686.

[CKSTT3] J. Colliander, M. Keel, G. Staffilani, H. Takaoka, and T. Tao. Global existence and scattering for rough solutions of a nonlinear Schrödinger equation on $\mathbb{R}^3$, Comm. Pure Appl. Math. **57** (2004), 987–1014.

[CKSTT4] J. Colliander, M. Keel, G. Staffilani, H. Takaoka, and T. Tao. Multi-linear estimates for periodic KdV equations, and applications, J. Funct. Anal. 211 (2004), 173–218.

[CKSTT5] J. Colliander, M. Keel, G. Staffilani, H. Takaoka, and T. Tao. Sharp global well-posedness results for periodic and non-periodic KdV and modified KdV on $\mathbb{R}$ and $\mathbb{T}$, Journal A.M.S. **16** (2003), 705–749.

[CKSTT6] J. Colliander, M. Keel, G. Staffilani, H. Takaoka, and T. Tao. Global well-posedness for KdV in Sobolev spaces of negative index, EJDE **26** (2001), 1–7.

[CKSTT7] J. Colliander, M. Keel, G. Staffilani, H. Takaoka, and T. Tao. Global well-posedness and scattering for the energy-critical nonlinear Schrödinger equation in $\mathbb{R}^3$, Annals of Math. **167** (2008), 767–865.

[Cl] M. Colin. On the local well-posedness of quasilinear Schrödinger equations in arbitrary space dimension, Comm. P.D.E. **27** (2002), 325–354.

[CM] T. Colin and G. Métivier. *Instabilities in Zakharov Equations for Laser Propagation in a Plasma*, Phase space analysis of partial differential equations, 63–81, Progr. Nonlinear Differential Equations Appl. **69**, Birkhäuser, Boston, MA, 2006.

[Cn] H. Cornille. Solutions of the generalized nonlinear Schrödinger equation in two spatial dimensions, J. Math. Phys. **20** (1979), 199–209.

[Co1] A. Cohen. Solutions of the Korteweg-de Vries equation from irregular data, Duke Math. J. **45** (1978), 149–181.

[CoK] A. Cohen and T. Kappeler. Solution to the Korteweg-de Vries equation with initial profile in $L^1_1(\mathbb{R}) \cap L^1_N(\mathbb{R}^+)$, SIAM J. Math. Anal. **18** (1987), 991–1025.

[Cr] M. Christ. Power series of a nonlinear Schrödinger equation, Mathematical aspects of nonlinear dispersive equations, Ann. of Math. Stud. **163**, Princeton Univ. Press, Princeton, NJ, (2007), 131–155.

[CrCT1] M. Christ, J. Colliander and T. Tao. Asymptotics, frequency modulation, and low regularity ill-posedness for canonical defocusing equations, Amer. J. Math. **125** (2003), 1235–1293.

[CrCT2] M. Christ, J. Colliander, and T. Tao. A priori bounds and weak solutions for the nonlinear Schrödinger equation in Sobolev spaces of negative order, J. Funct. Anal **254** (2008), 368–395.

[CrCT3] M. Christ, J. Colliander, and T. Tao. Ill-posedness for nonlinear Schrödinger and wave equations, to appear Annales IHP.

[CrHoT] M. Christ, J. Holmer, and D. Tataru. Low regularity a priori bounds for the modified Korteweg-de Vries equation, Lib. Math. (N.S.) **32** (2012), 51–75.

[CrK] M. Christ and A. Kiselev. Maximal functions associated to filtrations, J. Funct. Anal. **179** (2001), 406–425.

[CrW] M. Christ and M. Weinstein. Dispersion of small amplitude solutions of the generalized Korteweg-de Vries equation, J. Funct. Anal. **100** (1991), 87–109.

[CS] P. Constantin and J. C. Saut. Local smoothing properties of dispersive equations, Journal A.M.S. **1** (1989), 413–446.

[CST] J. Colliander, G. Staffilani and H. Takaoka. Global well-posedness for KdV below L^2, Math. Res. Lett. **6** (1999), 755–778.

[CSU] N-H. Chang, J. Shatah and K. Uhlenbeck. Schrödinger maps, Comm. Pure Appl. Math. **53** (2000), 590–602.

[Ct] M. Cotlar. A general interpolation theorem for linear operators, Revista Matemática Cuyana, **1** (1955), 57–84.

[CVV] T. Cazenave, L. Vega and M.C. Vilela, A note on the nonlinear Schrödinger equation in weak L^p spaces, Commun. Contemp. Math. **3** (2001), 153–162.

[Cz1] T. Cazenave. *An Introduction to Nonlinear Schrödinger Equations*, Textos de Métodos Matemáticos **22**, Universidade Federal de Rio de Janeiro, 1989.

[Cz2] T. Cazenave. *Semilinear Schrödinger Equations*, Courant Lectures Notes 10, AMS 2003.

[CzL] T. Cazenave and P.-L. Lions. Orbital stability of standing waves for some nonlinear Schrödinger equations, Comm. Math. Phys. **85** (1982), 549–561.

[CzW1] T. Cazenave and F. Weissler. Rapidly decaying solutions of the nonlinear Schrödinger equation, Comm. Math. Phys. **147** (1992), 75–100.

[CzW2] T. Cazenave and F. Weissler. The Cauchy problem for the nonlinear Schrödinger equation in H^1, Manuscripta Math. **61** (1988), 477–494.

[CzW3] T. Cazenave and F. Weissler. *Some Remarks on the Nonlinear Schrödinger Equation in the Critical Case*, Lecture Notes in Math. **1394**, Springer, Berlin, (1989), 18–29.

[CzW4] T. Cazenave and F.B. Weissler. The Cauchy problem for the critical nonlinear Schrödinger equation in H^s, Nonlinear Anal. TMA **14** (1990), 807–836.

[D1] B. Dodson. Global well-posedness and scattering for the defocusing, L^2-critical, nonlinear Schrödinger equation when $d \geq 3$, Journal A.M.S. **25** (2012), 429–463.

[D2] B. Dodson. Global well-posedness and scattering for the defocusing, L^2-critical, nonlinear Schrödinger equation when $d = 2$, preprint.

[D3] B. Dodson. Global well-posedness and scattering for the defocusing, L^2-critical, nonlinear Schrödinger equation when $d = 1$, preprint.

[DaR] K.D. Danov and M.S. Ruderman. Nonlinear waves on shallow water in the presence of a horizontal magnetic field, Fluid Dynamics, **18** (1983), 751–756.

[DH] K. Datchev and J. Holmer. Fast soliton scattering by attractive delta impurities, Comm. PDE **34** (2009), no. 7–9, 1074–1113.

[DHR] T. Duyckaerts, J. Holmer and S. Roudenko. Scattering for the non-radial 3D cubic nonlinear Schrödinger equation, Math. Res. Lett. **15** (2008), 1233–250.

[DJ] P.G. Drazin and R.S. Johnson. *Solitons: An Introduction*, New York: Cambridge University Press, 1989.

[DK] B. Dahlberg and C.E. Kenig. A note on the almost everywhere behavior of solutions to the Schrödinger equation, Lecture Notes in Math. **908**, Springer, Berlin-New York, (1982), 205–209.

[Dl] J. Dollard. Asymptotic convergence and the Coulomb interaction, J. Math. Phys. **5** (1964), 729–739.

[DM] T. Duyckaerts and F. Merle. Dynamic of threshold solutions for energy-critical NLS, Geom. Funct. Anal. **18** (2009), 1787–1840.

[Do1] S. Doi. Remarks on the Cauchy problem for Schrödinger-type equations, Comm. PDE **21** (1996), 163–178.

[Do2] S. Doi. On the Cauchy problem for Schrödinger type equations and the regularity of the solutions, J. Math. Kyoto Univ. **34** (1994), 319–328.

[DR] V.D. Djordjevic and L.G. Redekopp. On two-dimensional packets of capillary-gravity waves, J. Fluid Mech. **79** (1977), 703–714.

[DRu] T. Duyckaerts and S. Roudenko. Threshold solutions for the focusing 3D cubic Schrödinger equation, Rev. Mat. Iberoam. **26** (2010), 1–56.

[DS] A. Davey and K. Stewartson. On three dimensional packets of surface waves, Proc. Roy. Soc. London, Ser. A **338** (1974), 101–110.

[Du] J. Duoandikoetxea. *Fourier Analysis*, Graduate Studies in Mathematics, **29**. Ame. Math. Soc. Providence, RI, 2001.

[DW] W. Ding and Y. Wang. Local Schrödinger flow into Kähler manifolds, Sci. China Ser. A **44** (2001), 1446–1464.

[Dx] D. Dix. Nonuniqueness and uniqueness in the initial-value problem for Burgers' equation, SIAM J. Math. Anal. **27** (1996), 708–724.

[E] M.J. Esteban. Existence d'une infinité d'ondes solitaires pour des équations de champs non linéaires dans le plan, Ann. Fac. Sci. Toulouse Math. **2** (1980), 181–191.

[EKPV1] L. Escauriaza, C.E. Kenig, G. Ponce, and L. Vega. Convexity properties of solutions to the free Schrödinger equation with gaussian decay, Math. Res. Lett. **15** (2008), 957–971.

[EKPV2] L. Escauriaza, C.E. Kenig, G. Ponce, and L. Vega. Hardy's uncertainty principle, convexity and Schrödinger evolutions, J. Eur. Math. Soc. (JEMS) **10** (2008), 883–907.

[EKPV3] L. Escauriaza, C.E. Kenig, G. Ponce, and L. Vega. On uniqueness properties of solutions of the k-generalized KdV equations, J. Funct. Anal. **244** (2007), 504–535.

[ES] W. Eckhaus and P. Schuur. The emergence of solitons of the Korteweg-de Vries equation from arbitrary initial conditions, Math. Methods Appl. Sci. **5** (1983), 97–116.

[F] G. Folland. *Introduction to Partial Differential Equations*, Princeton Univ. Press, Princeton, N.J. 1976.

[Fa] L.G. Farah. Global rough solutions to the critical generalized KdV equation, J. Diff. Eqs. **249** (2010), 1968–1985.

[FaLP] L.G. Farah, F. Linares and A. Pastor. The supercritical generalized KdV equation: Global well-posedness in the energy space and below, Math. Res. Lett. **18** (2011), 357–377.

[FaPa] L.G. Farah and A. Pastor. On well-posedness and wave operator for the gKdV equation, Bull. Sci. Math. **137** (2013), 229–241.

[FePaUl] E. Fermi, J. Pasta, and S. Ulam. Studies of Nonlinear Problems I, Los Alamos Report LA1940 (1955). In Nonlinear Wave Motion, edited by A. C. Newell, 143–156. Providence, RI. AMS, 1974.

[Ff] C. Fefferman, Inequalities for strongly singular convolution operators, Acta Math. **124** (1970), 9–36.

[FFFP] L. Fanelli, V. Felli, M. Fontelos, and A. Primo. Time decay of scaling invariant electromagnetic Schrödinger equation on the plane, Comm. Math. Phys. **324** (2013), 1033–1067.

[FGr] Y. F. Fang and M. G. Grillakis. On the global existence of rough solutions for the cubic defocussing Schrödinger equation in $\mathbb{R}^{2+1}$, J. Hip. Diff. Eqs. **4** (2007), 233–257.

[FLP1] G. Fonseca, F. Linares, and G. Ponce. Global existence for the critical generalized KdV equation, Proc. A.M.S. **131** (2003), 1847–1855.

[FLP2] G. Fonseca, F. Linares, and G. Ponce. Global well-posedness for the modified Korteweg-de Vries equation, Comm. P.D.E. **24** (1999), 683–705.

[FLP3] G. Fonseca, F. Linares and G. Ponce. The IVP for the Benjamin-Ono equation in weighted Sobolev spaces II, J. Funct. Anal., **262** (2012), 2031–2049.

[FLP4] G. Fonseca, F. Linares and G. Ponce. *On persistence properties in fractional weighted spaces*, to appear in Proc. A.M.S.

[Fm] A. Friedman. *Partial Differential Equations*, Holt, Rinehart and Winston, New York, 1976.

[Fo] G. Fonseca. Growth of the H^s-norm for the modified KdV equation, Diff. and Int. Eqs. **13** (2000), 1081–1093.

[FoPo] G. Fonseca and G. Ponce. The IVP for the Benjamin-Ono equation in weighted Sobolev spaces, J. Funct. Anal. **260** (2011), 436–459.

[Fr] F.G. Friedlander. *Introduction to the Theory of Distributions*, Cambridge University Press, New York 1982.

[Fs D. Foschi. Maximizers for the Strichartz inequality, J. Eur. Math. Soc. (JEMS) bf 9 (2007), 739–774.

[FS1] A.S. Fokas and L.Y. Sung. On the solvability of the N-wave, Davey-Stewartson and Kadomtsev-Petviashvili equations, Inverse Problems **8** (1992), 673–708.

[FS2] A.S. Fokas and L.Y. Sung. The Cauchy problem for the Kadomtsev-Petviashvili-I equation without the zero mass constraint, Math. Proc. Cambridge Philos. Soc. **125** (1999), 113–138.

[FSa] A.S. Fokas and P.M. Santini. Dromions and a boundary value problem of the Davey-Stewartson I equation, Physica D **44** (1990) 99–130.

[FXC] D. Fang, J. Xie, and T. Cazenave. Scattering for the focusing energy-subcritical NLS, Sci. China Math. **54** (2011), 2037–2062.

[GaO] L. Grafakos and S. Oh. Kato-Ponce inequality, Comm. PDE **39** (2014), 1128–1157.

[Gb] M. Goldberg. Dispersive estimate for the three-dimensional Schrödinger equation with rough potentials, Amer. J. Math. **128** (2006) 731–750

[G1] Y. Giga. Solutions of the semilinear parabolic equations in L^p and regularity of weak solutions of the Navier-Stokes system, J. Diff. Eqs. **62** (1986), 186–212.

[G2] R.T. Glassey. On the blowing up of solutions to the Cauchy problem for nonlinear Schrödinger equations, J. Math. Phys. **18** (1977), 1794–1797.

[GGKM] C.S. Gardner, J.M. Greene, M.D. Kruskal and R.M. Miura. A method of solving the Korteweg-de Vries equation, Phys. Rev. Lett. **19** (1967), 1095–1097.

[GHW] R.H. Goodman, P.J. Holmes, and M.I. Weinstein. Strong NLS soliton-defect interactions, Phys. D **192** (2004), 215–248.

[GKM] C.S. Gardner, M.D. Kruskal and R. Miura. Korteweg-de Vries equation and generalizations II. Existence of conservation laws and constants of motions, J. Math. Phys. **9** (1968), 1204–1209.

[Gl1] M. Grillakis. On nonlinear Schrödinger equations, Comm. P.D.E. **25** (2000), 1827–1844.

[Gl2] M. Grillakis. Regularity and asymptotic behavior of the wave equation with critical nonlinearity, Ann. of Math. **132** (1990), 485–509.

[Gl3] M. Grillakis. Regularity for the wave equation with a critical nonlinearity, Comm. Pure Appl. Math. **45** (1992), 749–774.

[GM] L. Glangetas and F. Merle. Existence of self-similar blow-up solutions for Zakharov equation in dimension two. I. Concentration properties of blow-up solutions and instability results for Zakharov equation in dimension two. II Comm. Math. Phys. **160** (1994), 173–215, 349–389.

[GO] O.B. Gorbacheva and L.A. Ostrovsky. Nonlinear vector waves in a mechanical model of a molecular chain, Physica D **8** (1983), 223–228.

[GPS] A. Grünrock, M. Panthee, and J. Drumond Silva. A remark on global well-posedness below L^2 for the gKdV-3 equation, Diff. and Int. Eqs. **20** (2007), 1229–1236.

[Gr1] A. Grünrock. Some local well-posedness results for nonlinear Schrödinger equations below L^2, preprint.

[Gr2] A. Grünrock. A bilinear Airy type estimate with application to the 3-gkdv equation, Diff. and Int. Eqs. **18** (2005), no. 12, 1333–1339.

[Gr3] A. Grünrock. Bi- and trilinear Schrödinger estimates in one space dimension with applications to cubic NLS and DNLS, Int. Math. Res. Notices **41** (2005), 2525–2558.

[GrV] A. Grünrock and L. Vega. Local well-posedness for the modified KdV equation in almost critical $\widehat{H}^r_s$-spaces, Trans. A.M.S. **361** (2009), 5681–5694.

[GS] J.-M. Ghidaglia and J.-C. Saut . On the initial value problem for the Davey-Stewartson systems, Nonlinearity **3** (1990), 475–506.

[GSch] M. Goldberg and W. Schlag. Dispersive estimates for Schrödinger operators in dimensions one and three, Comm. Math. Phys. **251** (2004), 157–158.

[GSS] M. Grillakis, J. Shatah and W. Strauss. Stability theory of solitary waves in the presence of symmetry I, II, J. Funct. Anal. **74** (1987), 160–197. **94**, (1990), 308–348.

[Gq] Q. Guo, Nonscattering solutions to the L^2-supercritical NLS equations, preprint.

[Gu] Z. Guo, Global well-posedness of the Korteweg-de Vries equation in $H^{-3/4}(\mathbb{R})$, J. Math. Pures Appl. (9) **91** (2009), 583–597.

[GT] J. Ginibre and Y. Tsutsumi. Uniqueness of solutions for the generalized Korteweg-de Vries equation, SIAM J. Math. Anal. **20** (1989), 1388–1425.

[GTV] J. Ginibre, Y. Tsutsumi and G. Velo. On the Cauchy problem for the Zakharov system, J. Funct. Anal. **151** (1997), 384–436.

[GV1] J. Ginibre and G. Velo. On the class of nonlinear Schrödinger equations, J. Funct. Anal. **32** (1979), 1–32, 33–72.

[GV2] J. Ginibre and G. Velo. Scattering theory in the energy space for the class of nonlinear Schrödinger equations, J. Math. Pures Appl. **64** (1985), 363–401.

[GV3] J. Ginibre and G. Velo. Time decay of finite energy solutions of the nonlinear Klein-Gordon and Schrödinger equations, Ann. Inst. H. Poincaré Phys. Théor. **43** (1985), no. 4, 399–442.

[GVi] M. Goldberg and M. Visan. A counter-example to dispersive estimates for the Schrödinger operators in higher dimensions, Comm. Math. Phys. **266** (2006) 211–238.

[H] G.H. Hardy. A theorem concerning Fourier transform, J. London Math. Soc. **8** (1933), 227–331.

[HaHK] M. Hadac, S. Herr and H. Koch. Well-posedness and scattering for the KP-II equation in a critical space, Ann. Inst. H. Poincaré Anal. Non Linéaire **26** (2009), 917–941.

[H1] N. Hayashi. Global existence of small analytic solutions to nonlinear Schrödinger equations, Duke Math. J. **61** (1991), 575–592.

[H2] N. Hayashi. Local existence in time of small solutions to the Davey-Stewartson systems, Ann. Inst. H. Poincaré Phys. Théor. **65** (1996), 313–366.

[H3] N. Hayashi. Local existence in time of solutions to the elliptic-hyperbolic Davey-Stewartson system without smallness condition on the data, J. Anal. Math. **73** (1997), 133–164.

[H4] N. Hayashi. Local existence in time of small solutions to the Ishimori system, preprint.

[H5] N. Hayashi. Global existence of small analytic solutions to nonlinear Schrödinger equations, Duke Math. J. **60** (1990), 717–727.

[He] L. I. Hedberg. On certain convolution inequalities, Proc. A.M.S. **36** (1972), 505–510.

[HH1] N. Hayashi and H. Hirata. Global existence and asymptotic behaviour in time of small solutions to the elliptic-hyperbolic Davey-Stewartson system, Nonlinearity **9** (1996), 1387–1409.

[HH2] N. Hayashi and H. Hirata. Local existence in time of small solutions to the elliptic-hyperbolic Davey-Stewartson system in the usual Sobolev space, Proc. Edinburgh Math. Soc. **40** (1997), 563–581.

[Hi1] R. Hirota. Exact solutions to the Korteweg-de Vries equation for multiple collisions of solitons, Phys. Rev. Lett. **27** (1971), 1192–1194.

[Hi2] R. Hirota. Exact envelope soliton solutions of a nonlinear wave equation, J. Math. Phys. **14** (1973), 805–809.

[HMZ] J. Holmer, J. Marzuola, and M. Zworski. Fast soliton scattering by delta impurities, Comm. Math. Phys. **274** (2007), 187–216.

[HN1] N. Hayashi and P. Naumkin. Large time asymptotics of solutions to the generalized Korteweg-de Vries equation, J. Funct. Anal. **159** (1998), 110–136.

[HN2] N. Hayashi and P. Naumkin. On the modified Korteweg-de Vries equation, Math. Phys. Analysis and Geometry **4** (2001), 197–227.

[HN3] N. Hayashi and P. Naumkin. On the Davey-Stewartson and Ishimori systems, Math. Phys. Anal. Geom. **2** (1999), 53–81.

[HNT1] N. Hayashi, K. Nakamitsu and M. Tsutsumi. Nonlinear Schrödinger equations in weighted Sobolev spaces, Funkcial. Ekvac. **31** (1988), 363–381.

[HNT2] N. Hayashi, K. Nakamitsu and M. Tsutsumi. On solutions of the initial value problem for the nonlinear Schrödinger equations, J. Funct. Anal. **71** (1987), 218–245.

[HNT3] N. Hayashi, K. Nakamitsu and M. Tsutsumi. On solutions of the initial value problem for the nonlinear Schrödinger equations in one space dimension, Math. Z. **192** (1986), 637–650.

[HO] H. Hayashi and T. Ozawa. Remarks on nonlinear Schrödinger equations in one space dimension, Diff. and Int. Eqs. **2** (1994), 453–461.

[Ho1] L. Hörmander. Estimates for translation invariant operators in Lp spaces, Acta Math. **104** (1960), 93–104.

[Ho2] L. Hörmander. *The Analysis of Linear Partial Differential Operators III*, Springer, New York 1984.

[HPR] J. Holmer, G. Perelman, and S. Roudenko. A solution to the focussing 3d NLS that blows up on a contracting sphere, to appear in Trans. A.M.S.

[HR1] J. Holmer and S. Roudenko. A sharp condition for scattering of the radial 3D cubic nonlinear Schrödinger equation, Comm. Math. Phys. **282** (2008), 435–467.

[HR2] J. Holmer and S. Roudenko. Blow up solutions on a sphere for the 3D quintic NLS in the energy space, Anal. PDE **5** (2012), 475–512.

[HR3] J. Holmer and S. Roudenko. A class of solutions of the 3D cubic nonlinear Schrödinger equation that blow upon a circle, Appl. Math. Res. Express. AMRX (2011), 23–94.

[HS] N. Hayashi and J-C. Saut. Global existence of small solutions to the Davey-Stewartson and the Ishimori systems, Diff. and Int. Eqs. **8** (1995), 1657–1675.

[HoZ] J. Holmer and M. Zworski. Slow soliton interaction with delta impurities, J. Mod. Dyn. **1** (2007), 689–718.

[HuZ] D. Hundertmark and V. Zharnitsky. *On sharp Strichartz inequalities in low regularity*, Int. Math. Res. Notices (2006), 18 pp.

[I] W. Ichinose. The Cauchy problem for Schrödinger type equations with variable coefficients, Osaka J. Math. **24** (1987), 853–886.

[IK1] A. Ionescu and C. E. Kenig. Global well-posedness of the Benjamin-Ono equation in low regularity spaces, Journal A.M.S. **20** (2007), 753–798.

[IK2] A. Ionescu and C. E. Kenig. Low regularity Schrödinger maps: global well-posedness in dimension $d \geq 3$, Comm. Math. Phys. **271** (2007), 523–559.

[IKT] A. D. Ionescu, C. E. Kenig and D. Tataru. Global well-posedness of the KP-I initial-value problem in the energy space, Invent. Math. **173** (2008), 265–304.

[ILP1] P. Isaza, F. Linares, and G. Ponce. On decay properties of solutions of the k-generalized KdV equation, Comm. Math. Phys. **324** (2013), 129–146.

[ILP2] P. Isaza, F. Linares, and G. Ponce. *Propagation of regularity and decay of solutions to the k-generalized Korteweg-de Vries equation*, to appear in Comm. P.D.E.

[IM1] P. Isaza and J. Mejía. Local and global Cauchy problems for the Kadomtsev-Petviashvili(KP-II) equation in Sobolev spaces of negative indices, Comm. P.D.E. **26** (2001), 1027–1054.

[IM2] P. Isaza and J. Mejía. Global solution for the Kadomtsev-Petviashvili (KP-II) equation in Sobolev spaces of negative indices, Comm. P.D.E. **26** (2001), 1027–1054.

[IMS] P. Isaza, J. Mejía and V. Stallbohm. A regularity theorem for the Kadomtsev-Petviashvili equation with periodic boundary conditions, Nonlinear Anal. **23** (1994), 683–687.

[IN] R. Iório and W. Nunes. On equations of KP-type, Proc. Roy. Soc. Edinburgh, Section A **128** (1998), 725–743.

[Io1] R. Iório. On the Cauchy problem for the Benjamin-Ono equation, Comm. P.D.E. **11** (1986), 1031–1081.

[Io2] R. Iório. KdV, BO and friends in weighted Sobolev spaces, Functional-analytic methods for partial differential equations (Tokyo, 1989), 104–121, Lecture Notes in Math. **1450**, Springer, Berlin, 1990.

[Io3] R. Iório. Unique continuation principles for the Benjamin-Ono equation, Diff. and Int. Eqs. **16** (2003), 1281–1291.

[Is1] Y. Ishimori. Multivortex solutions of a two dimensional nonlinear wave equation, Progr. Theor. Phys. **72** (1984), 33–37.

[Is2] Y. Ishimori. Solitons in a one-dimensional Lennard/Mhy Jones lattice, Progr. Theoret. Phys. **68** (1982), 402–410.

[JK] C. Jones and T. Küpper. On the infinitely many solutions of a semilinear elliptic equation, SIAM J. Math. Anal. **17** (1986), 803–836.

[JN] F. John and L. Nirenberg. On functions of bounded mean oscillation, Comm. Pure Appl. Math. **14** (1961), 415–426.

[JP] G. James and D. Pelinovsky. Gaussian solitary waves and compactons in Fermi-Pasta-Ulam lattice with Herztian potentials, Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci. 470 (2014) 20130462, 20pp.

[JSS] J. Journé, A. Soffer, and C. Sogge. $L^p - L^{p'}$ estimates for the time dependent Schrödinger operators, Bull. A.M.S. **23** (1990), 519–524.

[K1] T. Kato. Wave operator and similarity for some non-self-adjoint operators, Math. Ann. **162** (1966), 258–279.

[K2] T. Kato. On the Cauchy problem for the (generalized) Korteweg-de Vries equation, Adv. in Math. Supp. Stud., Stud. in Appl. Math. **8** (1983), 93–128.

[K3] T. Kato. Nonlinear Schrödinger equations, Ann. Inst. Henri Poincaré, Phys. Théor. **46** (1987), 113–129.

[K4] T. Kato. On nonlinear Schrödinger equations, Lecture Notes for Physics. Nordic Summer School, 1988.

[K5] T. Kato. On the Korteweg-de Vries equation, Manuscripta Math **29** (1979), 89–99.

[Ka] O. Kavian. A remark on the blowing-up of solutions to the Cauchy problem for nonlinear Schrödinger equations, Trans. A.M.S. **299** (1987), 193–203.

[KdV] D.J. Korteweg and G. de Vries. On the change of form of long waves advancing in a rectangular canal, and on a new type of long stationary waves, Philos. Mag. **39** (1895), 422–443.

[Ke] C.E. Kenig. On the local and global well-posedness theory for the KP-I equation, Ann. Inst. H. Poincaré Anal. Non Linéaire, **21** (2004), 827–838.

[KeKo] C.E. Kenig and K.D. Köenig. On the local well-posedness of the Benjamin-Ono and modified Benjamin-Ono equations, Math. Res. Lett. **10** (2003), 879–895.

[KePi] C.E. Kenig and D. Pilod. Well-posedness for the fifth-order KdV equation in the energy space, to appear Trans. A.M.S.

[KF] S. Kruzhkov and A. Faminskii. A generalized solution for the Cauchy problem for the Korteweg–de Vries equation, Math. USRR, Sbornik **48** (1984), 93–138.

[Kg] H. Kumano-go. *Pseudo-differential Operators*, MIT Press, Cambridge 1981.

[Ki1] N. Kishimoto. Well-posedness of the Cauchy problem for the Korteweg-de Vries equation at the critical regularity, Comm. Pure Appl. Anal. **7** (2008), 1123–1143.

[Ki2] N. Kishimoto. Low-regularity bilinear estimate for a quadratic nonlinear Schrödinger equation, J. Diff. Eqs. **247** (2009), 1397–1439.

[Ki3] N. Kishimoto. Local well-posedness for the Cauchy problem for the quadratic Scrödinger equation with nonlinearity $\overline{u}^2$, Comm. Pure Appl. Anal. **7** (2008), 1123–1143.

[KiT] N. Kishimoto. and K. Tsugawa. Local well-posedness for quadratic nonlinear Schrödinger equations and the "good" Boussinesq equation, Diff. and Int. Eqs. **23** (2010), 463–493.

[Kl1] S. Klainerman. Uniform decay estimates and the Lorentz invariance of the classical wave equation, Comm. Pure Appl. Math **38** (1985), 321–332.

[Kl2] S. Klainerman. Global existence for nonlinear wave equations, Comm. Pure Appl. Math. **33** (1980), 43–101.

[KlM] S. Klainerman and M. Machedon. Space-time estimates for the null forms and the local existence theorem, Comm. Pure Appl. Math. **46** (1993), 1221–1268.

[KM] H. Koch and J. Marzuola. Small data scattering and soliton stability in $\dot{H}^{-\frac{1}{6}}$ for the quartic KdV equation, Anal. PDE **5** (2012), 145–198.

[KM1] C.E. Kenig and F. Merle. Scattering for $\dot{H}^{1/2}$ bounded solutions to the cubic defocusing NLS in three dimensions, Trans. A.M.S. **362** (2010) 1937–1962.

[KM2] C.E. Kenig and F. Merle. Global well posedness, scattering and blow up for the energy-critical, focusing, nonlinear Schrödinger equation in the radial case, Invent. Math. **166** (2006), 645–675.

[KMa] B.G. Konopelchenko and B. Matkarimov. Inverse spectral transform for the Ishimori equation. I. Initial value problem, J. Math. Phys. **31** (1990), 2737–2746.

[KN] D.J. Kaup and A.C. Newell. An exact solution for the derivative nonlinear Schrödinger equation, J. Math. Phys. **1** (1978), 798–801.

[KP] B.B. Kadomtsev and V.I. Petviashvili. On the stability of solitary waves in weakly dispersive media, Sov. Phys. Dokl. **15** (1970), 539–541.

[KPo] T. Kato and G. Ponce. Commutator estimates and the Euler and Navier-Stokes equations, Comm. Pure Appl. Math. **41** (1988), 891–907.

[KPRV1] C.E. Kenig, G. Ponce, C. Rolvung, and L. Vega. Variable coefficient Schrödinger flows for ultrahyperbolic operators, Adv. in Math. **196** (2005), 373–486.

[KPRV2] C.E. Kenig, G. Ponce, C. Rolvung, and L. Vega. The general quasilinear ultrahyperbolic Schrödinger equation, Adv. in Math. **206** (2006), 402–433.

[KPV1] C.E. Kenig, G. Ponce and L. Vega. On the (generalized) Korteweg-de Vries equation, Duke Math. J., **90** (1989) 585–610.

[KPV2] C.E. Kenig, G. Ponce, and L. Vega. Oscillatory integrals and regularity of dispersive equations, Indiana U. Math. J. **40** (1991), 33–69.

[KPV3] C.E. Kenig, G. Ponce, and L. Vega. Small solutions to non-linear Schrödinger equation, Ann. Inst. H. Poincaré Anal. Non Linéaire **10** (1993), 255–280.

[KPV4] C.E. Kenig, G. Ponce, and L. Vega. Well-posedness and scattering results for the generalized Korteweg-de Vries equation via contraction principle, Comm. Pure Appl. Math. **46** (1993), 527–620.

[KPV5] C.E. Kenig, G. Ponce, and L. Vega. On the ill-posedness of some canonical dispersive equations, Duke Math. J. **106** (2001), 617–633.

[KPV6] C.E. Kenig, G. Ponce and L. Vega. A bilinear estimate with applications to the KdV equation, Journal A.M.S. **9** (1996), 573–603.
[KPV7] C.E. Kenig, G. Ponce and L. Vega. The Cauchy problem for the Korteweg-de Vries equation in Sobolev spaces of negative indices, Duke Math. J. **71** (1993), 1–21.
[KPV8] C.E. Kenig, G. Ponce, and L. Vega. On the Zakharov and Zakharov-Schulman systems, J. Funct. Anal. **127** (1995), 204–234.
[KPV9] C.E. Kenig, G. Ponce, and L. Vega. On the initial value problem for the Ishimori system, Ann. Henri Poincaré **1** (2000), 341–384.
[KPV10] C.E. Kenig, G. Ponce, and L. Vega. The Cauchy problem for quasi-linear Schrödinger equations, Invent. Math. **158** (2004), no. 2, 343–388.
[KPV11] C.E. Kenig, G. Ponce and L. Vega. On the generalized Benjamin-Ono equation, Trans. A.M.S. **342** (1994), 155–172.
[KPV12] C.E. Kenig, G. Ponce and L. Vega. Quadratic forms for the 1-D semilinear Schrodinger equation, Trans. A.M.S. **346** (1996), 3323–3353.
[KPV13] C.E. Kenig, G. Ponce and L. Vega. Higher-order nonlinear dispersive equations, Proc. A.M.S. **122** (1994), 157–166.
[KPV14] C.E. Kenig, G. Ponce and L. Vega. Smoothing effects and local existence theory for the generalized nonlinear Schrödinger equation, Invent. Math. **134** (1998), 489–545.
[KPV15] C.E. Kenig, G. Ponce and L. Vega. A theorem of Paley-Wiener type for the Schrödinger evolutions, Ann. Scient. Éc. Norm. Sup. **47** (2014), 539–557.
[KpTo] T. Kappeler and P. Topalov. Global wellposedness of KdV in $H^{-1}(\mathbb{T}, \mathbb{R})$, Duke Math. J. **135** (2006), 327–360.
[KR] C.E. Kenig and A. Ruiz. A strong type (2, 2) estimate for a maximal operator associated to the Schrödinger equation, Trans. A.M.S. **280** (1983), 239–246.
[KrS] J. Krieger and W. Schlag. Stable manifolds for all monic supercritical focusing nonlinear Schrödinger equations in one dimension Journal A.M.S. **19** (2006), no. 4, 815–920.
[KSC] C.F.F. Karney, A. Sen and F.Y.F. Chu. Nonlinear evolution of lowe hybrid waves, Phys. Fluids **22** (1979), 940–952.
[KST] J. Krieger, W. Schlag, and D. Tataru. Slow blow-up solutions for the H1(R3) critical focusing semilinear wave equation, Duke Math. J. **147** (2009), 1–53.
[KT] C.E. Kenig and H. Takaoka. Global well-posedness of the modified Benjamin-Ono equation with initial data in $H^{1/2}$, Int. Math. Res. Notices (2006) Art. ID 95702.
[KT1] M. Keel and T. Tao. Endpoint Strichartz estimates, Amer. J. Math. **120** (1998), 955–980.
[KT2] M. Keel, T. Roy, and T. Tao. Global well-posedness of the Maxwell-Klein-Gordon equation below the energy norm, Discrete Contin. Dyn. Syst. **30** (2011), 573–621.
[KTa1] H. Koch and D. Tataru. Personal communication.
[KTa2] H. Koch and D. Tataru. A priori bounds for the 1D cubic NLS in negative Sobolev spaces, Int. Math. Res. Notices **16** (2007), Art. ID rnm053.
[KTa3] H. Koch and D. Tataru. Energy and local energy bounds for the 1-D cubic NLS equation in $H^{-1/4}$, Ann. Inst. H. Poincaré Anal. Non Linéaire **29** (2012), 955–988.
[KTV] R. Killip, T. Tao, and M. Visan. The cubic nonlinear Schrödinger equation in two dimensions with radial data, J. Eur. Math. Soc. (JEMS) **11** (2009), 1203–1258.
[KTz] H. Koch and N. Tzvetkov. On the local well-posedness of the Benjamin-Ono equation $H^s(\mathbb{R})$, Int. Math. Res. Notices **26** (2003) 1449–1464.
[KV1] R. Killip and M. Visan. The focusing energy-critical nonlinear Schrödinger equation in dimensions five and higher, Amer. J. Math. **132** (2010), 361–424.
[KV2] R. Killip and M. Visan. Energy supercritical NLS: critical $\dot{H}^s$-bounds imply scattering, Comm. PDE **35** (2010) 945–987.
[KVZ] R. Killip, M. Visan and X. Zhang. Energy-critical NLS with quadratic potentials, Comm. P.D.E. **34** (2009), 1531–1565.

[Kw1] K.M. Kwong. Uniqueness of positive solutions of $\Delta u - u + u^p = 0$ in $\mathbb{R}^n$, Arch. Rat. Mech. Anal. **105** (1989), 243–266.

[Kw2] S. Kwon. On the fifth order KdV equation: local well posedness and lack of uniform continuity of the solution map, Electron. J. Differential Equations (2008), 15 pp.

[KY] T. Kato and K. Yajima. Some examples of smooth operators and the associated smoothing effect, Reviews in Math Physics **1** (1989), 481–496.

[Kz] M. Kunze. On the existence of a maximizer for the Strichartz inequality, Comm. Math. Phys. **243** (2003), 137–164.

[Lb] G. Lamb. *Elements of Soliton Theory*, Pure and Applied Mathematics, A Wiley-Interscience Publication, John Wiley & Sons Inc., New York, 1980.

[Le] S. Lee. *Pointwise convergence of solutions to Schrödinger equations in* $\mathbb{R}^2$, Int. Math. Res. Notices (2006), 1–21.

[LiPo] F. Linares and G. Ponce. On the Davey–Stewartson systems, Ann. Inst. H. Poincaré Anal. Non Linéaire **10** (1993), 523–548.

[LiS] F. Linares and M. Scialom. On the smoothing properties of solutions to the modified Korteweg-de Vries equation, J. Diff. Eqs. **106** (1993), 141–154.

[LmPo] W.-K. Lim and G. Ponce. On the initial value problem for the one dimensional quasilinear Schrödinger equation, SIAM J. Math. Anal. **34** (2002), 435–459.

[LP] P.-L. Lions and B. Perthame. Lemmes de moments, de moyenne et de dispersion, C. R. Acad. Sci. Paris Sér. I Math. **314** (1992), 801–806.

[Lp] W. Littman. The wave operator and L^p-norms, J. Math. & Mech. **12** (1963), 55–68.

[LPS] F. Linares, G. Ponce and J.-C. Saut. On a degenerate Zakharov system, Bull. Braz. Math. Soc. (N.S.) **36** (2005), 1–23.

[LPSS] M.J. Landman, G. Papanicolaou, C. Sulem, and P.-L. Sulem. Rates of blow up for solutions of the nonlinear Schrödinger equation at the critical dimension, Phys. Rev. A (3) **38** (1988), 3837–3842.

[LS] J.E. Lin and W.A. Strauss. Decay and scattering of solutions of a nonlinear Schrödinger equation, J. Funct. Anal. **30** (1978), 245–263.

[Lx1] P. Lax. Almost periodic solutions of the KdV equation, SIAM Rev. **18** (1976), 351–375.

[Lx2] P. Lax. Integrals of nonlinear equations of evolution and solitary waves, Comm. Pure Appl. Math. **21** (1968), 467–490.

[M] B. Marshal. Mixed norm estimates for the Klein-Gordon equation, Proc. Conf. in honor of A. Zygmund, Wadsworth Int. Math ser. (1981), 638–639.

[Ma] Y. Martel. Asymptotic N-soliton-like solutions of the subcritical and critical Korteweg-de Vries equations, Amer. J. Math. **127** (2005), 1103–1140.

[McS] H. McKean and J. Shatah. The nonlinear Schrödinger equations and the nonlinear heat equation: reduction to linear form, Comm. Pure Appl. Math. **44** (1991), 1067–1080.

[Me1] F. Merle. Limit of the solution of a nonlinear Schrödigner equation at blow-up time, J. Funct. Anal. **84** (1989), 201–214.

[Me2] F. Merle. Construction of solutions with exactly k blow-up points for the Schrödinger equation with critical nonlinearity, Comm. Math. Phys. **129** (1990), 223–240.

[Me3] F. Merle. Determination of blow-up solutions with minimal mass for nonlinear Schrödinger with critical power, Duke Math. J. **69** (1993), 427–453.

[Me4] F. Merle. Existence of blow-up solutions in the energy space for the critical generalized KdV equation, Journal A.M.S. **14** (2001), 555–578.

[Me5] F. Merle. Lower bounds for the blowup rate of solutions of the Zakharov equation in dimension two, Comm. Pure Appl. Math. **49** (1996), 765–794.

[Me6] F. Merle. Blow-up results of virial type for Zakharov equations, Comm. Math. Phys. **175** (1996), 433–455.

[MeRa1] F. Merle and P. Raphael. Sharp upper bound on the blow up rate for the critical nonlinear Schrödinger equation, Geom. Funct. Anal. **13** (2003), 591–642.

[MeRa2] F. Merle and P. Raphael. The blow up dynamic and upper bound on the blow up rate for the critical nonlinear Schrödinger equation, Ann. of Math. (2) **161** (2005), 157–222.

[MG] H. Mc Gahagan. An approximation scheme for Schrödinger maps, Comm. PDE **32** (2007), 375–400.

[MGK] R. Miura, C. Gardner, and M. Kruskal. Korteweg-de Vries equation and generalizations. II. Existence of conservation laws and constants of motion, J. Math. Phys. **9** (1968), 1204–1209.

[Mi] T. Mizumachi. Large time asymptotics of solutions around solitary waves to the generalized Korteweg-de Vries equations, SIAM J. Math. Anal. **32** (2001), 1050–1080.

[MiT] T. Mizumachi and N. Tzvetkov. L^2-stability of solitary waves for the KdV equation via Pego and Weinstein's method, arXiv:1403.5321.

[MK] Y. Matsuno and D. J. Kaup. Initial value problem of the linearized Benjamin-Ono equation and its applications, J. Math. Phys. **38** (1997), 5198–5224.

[Mk] B. Muckenhoup. *Weighted norm inequalities for the Fourier transform*, Trans. A.M.S. **276** (1985), 729–742.

[MM1] Y. Martel and F. Merle. Instability of solitons for the critical generalized Korteweg-de Vries equation, Geom. Funct. Anal. **11** (2001), 74–123.

[MM2] Y. Martel and F. Merle. Asymptotic stability of solitons for subcritical generalized KdV equations, Arch. Rat. Mech. Anal. **157** (2001), 219–254.

[MM3] Y. Martel and F. Merle. A Liouville theorem for the critical generalized Korteweg-de Vries equation, J. Math. Pures Appl. **79** (2000), 339–425.

[MM4] Y. Martel and F. Merle. Blow up in finite time and dynamics of blow up solutions for the L^2-critical generalized KdV equation, Journal A.M.S. **15** (2002), 617–664.

[MM5] Y. Martel and F. Merle. Stability of blow-up profile and lower bounds for blow-up rate for the critical generalized KdV equation, Ann. of Math. **155** (2002), 235–280.

[MM6] Y. Martel and F. Merle. Nonexistence of blow-up solution with minimal L^2-mass for the critical gKdV equation, Duke Math. J. **115** (2002), 385–408.

[MM7] Y. Martel and F. Merle. Multisolitary waves for nonlinear Schrödinger equations, Ann. Inst. H. Poincaré Anal. Non Linéaire **23** (2006), 849–864.

[MM8] Y. Martel and F. Merle. Description of two soliton collision for the quartic gKdV equation, Ann. of Math. (2) **174** (2011), 757– 857.

[MMR1] Y. Martel, F. Merle, and P. Raphael. Blow up for the critical gKdV equation I: dynamics near the soliton, Acta Math. **212** (2014), 59–140.

[MMR2] Y. Martel, F. Merle, and P. Raphael. Blow up for the critical gKdV equation II: minimal mass dynamics, arXiv 1204. 4624.

[MMR3] Y. Martel, F. Merle, and P. Raphael. Blow up for the critical gKdV equation III: exotic regimes, to appear in Ann. Sc. Norm. Super. Pisa Cl. Sci.

[MMT] Y. Martel, F. Merle and T-P, Tsai. Stability and asymptotic stability in the energy space of the sum of N solitons for subcritical gKdV equations, Comm. Math. Phys. **231** (2002), 347–373.

[MMTa1] J. Marzuola, J. Metcalfe and D. Tataru. Strichartz estimates and local smoothing estimates for asypmtotically flat Schrödinger equations, J. Funct. Anal. **255** (2008), 1497–1553.

[MMTa2] J. Marzuola, J. Metcalfe and D. Tataru. Quasilinear Schrödinger equations I : small data and quadratic interactions, Adv. Math. **231** (2012), 1151–1172.

[MMTa3] J. Marzuola, J. Metcalfe and D. Tataru. Quasilinear Schrödinger equations II : small data and cubic interactions, Kyoto Journal of Mathematics, **54** (2014), 529–546

[Mn] C. Muñoz. On the inelastic two-soliton collision for gKdV equations with general nonlinearity, Int. Math. Res. Notices (2010) no. 9, 1624–1719.

[Mo1] L. Molinet. Global well-posedness in L^2 for the periodic Benjamin-Ono equation, Amer. J. Math. **130** (2008), 635–683.

[Mo2] L. Molinet. A note on ill-posedness for the KdV equation, Diff. and Int. Eqs. **24** (2011), 759–765.

[MoPi] L. Molinet and D. Pilod. The Cauchy problem for the Benjamin-Ono equation in L^2 revisited. Anal. PDE **5** (2012), 365–395.

[MPSS] D. McLaughlin, G. Papanicolaou, C. Sulem, and P.L. Sulem. The focusing singularity of the cubic Schrödinger equation, Phys. Rev. A **34** (1986), 1200–1210.

[MPTT] C. Muscalu, J.Pipher, T. Tao, and C. Thiele, Bi-parameter paraproducts. Acta Math. **193** (2004), 269–296.

[MR1] L. Molinet and F. Ribaud. Well-posedness results for the generalized Benjamin-Ono equation with small initial data, J. Math. Pures Appl. **83** (2004), 277–311.

[MR2] L. Molinet and F. Ribaud. Well-posedness results for the generalized Benjamin-Ono equation with arbitrary large initial data, Int. Math. Res. Notices, **70** (2004), 3757–3795.

[MRS] F. Merle, P. Raphael, and J. Szeftel. On collapsing ring blow up solutions to the mass supercritical NLS, to appear in Duke Math. J. **163** (2014), 369–431.

[MSa] J. Maddocks and R. Sachs. On the stability of KdV multi-solitons, Comm. Pure Appl. Math. **46** (1993), 867–901.

[MSm] S.J. Montgomery-Smith. Time decay for the bounded mean oscillation of solutions of the Schrödinger and wave equations, Duke Math. J. **91** (1998), 393–408.

[MST1] L. Molinet, J.-C. Saut and N. Tzvetkov. Well-posedness and ill-posedness results for the Kadomtsev-Petviashvili-I equation, Duke Math. J. **115** (2002), 353–384.

[MST2] L. Molinet, J.-C. Saut and N. Tzvetkov. Global well-posedness for the KP-I equation, Math. Ann. **324** (2002), 255–275.

[MST3] L. Molinet, J.-C. Saut and N. Tzvetkov. Ill-posedness issues for the Benjamin-Ono and related equations, SIAM J. Math. Anal. **33** (2001), 982–988.

[MSWX] C. Miao, S. Shao, Y. Wu, and G. Xu. The low regularity global solutions for the critical generalized KdV equation, Dyn. Partial Differ. Equ. **7** (2010), 265–288.

[MT] F. Merle and Y. Tsutsumi. L^2–concentration of blow-up solutions for the nonlinear Schrödinger equation with critical power nonlinearity, J. Diff. Eqs. **84** (1990), 205–214.

[Mu] R. Miura. The Korteweg-de Vries equation: a survey of results, SIAM Rev. **18** (1976), 412–459.

[Mu1] R. Miura. Korteweg-de Vries equation and generalizations. I. A remarkable explicit nonlinear transformation, J. Math. Phys. **9** (1968), 1202–1204.

[MV] F. Merle and L. Vega. L^2 stability of solitons for KdV equation, Int. Math. Res. Notices **13** (2003), 735–753.

[MVV1] A. Moyua, A. Vargas, and L. Vega. Restriction theorems and maximal operators related to oscillatory integrals in $\mathbb{R}^3$, Duke Math. J. **96** (1999), 547–574.

[MVV2] A. Moyua, A. Vargas, and L. Vega. Schrödinger maximal function and restriction properties of the Fourier transform, Internat. Math. Res. Notices, **16** (1996), 793–815.

[Mz] S. Mizohata. *On the Cauchy problem*, Notes and Reports in Mathematics in Science and Engineering **3**, Academic Press Inc., Orlando, FL, 1985.

[N] A.C. Newell. *Solitons in Mathematics and Physics*, Regional Conference series in Applied Math. **48**, SIAM, 1985.

[Na] K. Nakanishi. Energy scattering for nonlinear Klein-Gordon and Schrödinger equations in spatial dimensions 1 and 2, J. Funct. Anal. **169** (1999), 201–225.

[Nh] J. Nahas. A decay property of solutions to the k-generalized KdV equation, Adv. Diff. Eqs. **17** (2012), 833–858.

[NhPo1] J. Nahas and G. Ponce. On the persistence properties of semilinear Schrödinger equations, Comm. P.D.E. **34** (2009), 1–20.

[NhPo2] J. Nahas and G. Ponce. On the persistence properties of solutions of nonlinear dispersive equations in weighted Sobolev spaces, RIMS Kokyuroku Besstatsu (RIMS Proc.) (2011), 23–36

[NSc] K. Nakanishi and W. Schlag. Global dynamics above the ground state energy for the cubic NLS equation in 3D, Calc. Var. Partial Differential Equations **44** (2012), no. 1–2, 1–45.

[NSU] A. Nahmod, A. Stefanov and K. Uhlenbeck. On Schrödinger maps, Comm. Pure Appl. Math. **56** (2003), 114–151.

[NSVZ] A. Nahmod, J. Shatah, L. Vega, and C. Zeng. Schrödinger maps and their associated frame systems, Int. Math. Res. Notices **21** (2007) Art, ID rnm088, 29 pp.

[NT] H. Nawa and M. Tsutsumi. On blow up for the pseudo conformal invariant nonlinear Schrödinger equations, Funk. Ekv. **32** (1989), 417–428.

[NTT1] K. Nakanishi, H. Takaoka and Y. Tsutsumi. Counterexamples to bilinear estimates related with the KdV equation and the nonlinear Schrödinger equation, Methods Appl. Anal. **8** (2001), 569–578.

[NTT2] K. Nakanishi, H. Takaoka and Y. Tsutsumi. Local well-posedness in low regularity of the mKdV equation with periodic boundary condition, Discrete Contin. Dyn. Syst. **28** (2010), 1635–1654.

[OgT] T. Ogawa and Y. Tsutsumi. Blow-up of H^1 solutions for the one-dimensional nonlinear Schrödinger equation with critical power nonlinearity, Proc. A.M.S. **111** (1991), 487–496.

[Ol] P.J. Olver. *Hamiltonian and non-Hamiltonian Models for Water Waves*, Lecture Notes in Phys. **195**, 273–290, Springer, Berlin, 1984.

[On] H. Ono. Algebraic solitary waves in stratified fluids, J. Phys. Soc. Japan **39** (1975), 1082–1091.

[OT1] T. Ozawa and Y. Tsutsumi. Space-time estimates for null gauge forms and nonlinear Schrödinger equations, Diff. and Int. Eqs **11** (1998), 201–222.

[OT2] T. Ozawa and Y. Tsutsumi. Existence and smoothing effect of solutions for the Zakharov equations, Publ. Res. Inst. Math. Sci. **28** (1992), 329–361.

[OT3] T. Ozawa and Y. Tsutsumi. The nonlinear Schrödinger limit and the initial layer of the Zakharov equations, Diff. and Int. Eqs. **5** (1992), 721–745.

[Oz] T. Ozawa. Exact blow-up solutions to the Cauchy problem for the Davey-Stewartson systems, Proc. Roy. Soc. London, Ser. A **436** (1992), 345–349.

[P1] H. Pecher. Nonlinear small data scattering for the wave and Klein-Gordon equation, Math Z **185** (1985), 261–270.

[P2] H. Pecher. Global well-posedness below energy space for the 1-dimensional Zakharov system, Internat. Math. Res. Notices **19** (2001), 1027–1056.

[Pd] D. Pilod. On the Cauchy problem for higher-order nonlinear dispersive equations, J. Diff. Eqs. **245** (2008), 2055–2077.

[Pe1] G. Perelman. On the formation of singularities in solutions of the critical nonlinear Schrödinger equation, Ann. Henri Poincaré **2** (2001), 605–673.

[Pe2] G. Perelman. Asymptotic stability of multisoliton solutions for nonlinear Schrödinger equation. Comm PDE **29** (2004), 1051–1095.

[Pi] H.R. Pitt. Theorems on Fourier series and power series, Duke Math. J. **3** (1937), 747–755.

[Pl] F. Planchon. On the Cauchy problem in the Besov spaces for a nonlinear Schrödinger equation, Commun. Contemp. Math. **2** (2000), 243–254.

[Po] G. Ponce. On the global well-posedness of the Benjamin-Ono equation, Diff. and Int. Eqs. **4** (1991), 527–542.

[Pp] M. Poppenberg. On the local wellposedness for quasilinear Schrödinger equations in arbitrary space dimension, J. Diff. Eqs. **172** (2001), 83–115.

[PSS] A. Patera, C. Sulem, and P.L. Sulem. Numerical simulation of singular solutions to the two dimensional cubic Schrödinger equation, Comm. Pure Appl. Math. **37** (1984), 755–778.

[PW] R. Pego and M. Weistein. Asymptotic stability of solitary waves, Comm. Math. Phys. **164** (1994), 305–349.

[Ra1] P. Raphael. Stability of the log-log bound for blow up solutions to the critical non linear Schrödinger equation Math. Ann. **331** (2005), 577–609.

[Ra2] P. Raphael. Existence and stability of a solution blowing up on a sphere for the L^2 supercritical nonlinear Schrödinger equation, Duke Math. J. **134** (2006), 199–258.

[Rd] W. Rudin. *Real and Complex Analyis*, McGraw-Hill, New York (1986).

[RH] P. Rosenau, and J.M. Hyman. Compactons: Solitons with Finite Wavelength, Phys. Rev. Letters, **70** (1993) 564–567.

[RS] I. Rodnianski and W. Schlag. Time decay for solutions of the Schrödinger equations with rough and time-dependent potentials, Invent. Math. **155** (2004), 451–513.

[Ru] J. Rauch. *Partial Differential Equations*, Graduate Texts in Mathematics **128**, Springer-Verlag, New York, 1991.

[RuR] J. Rauch and M. Reed. Nonlinear microlocal analysis of semilinear hyperbolic systems in one space dimension. Duke Math. J. **49** (1982), 397–475.

[RV] A. Ruiz and L. Vega. Local regularity of solutions to wave equations with time-dependent potentials, Duke Math. J. **76**, (1994) 913–940.

[RVi] E. Ryckman and M. Visan. Global well-posedness and scattering for the defocusing energy-critical nonlinear Schrödinger equation in $\mathbb{R}^{1+4}$, Amer. J. Math. **129** (2007), 1–60.

[RZ] L. Robbiano and C. Zuily. Strichartz estimates for the Schrödinger equation with variable coefficients, Mém. Soc. Math. Fr. (N.S.) No. 101-102 (2005), vi+208 pp.

[S1] E.M. Stein. The characterization of functions arising as potentials, Bull. A.M.S. **67**, (1961), 102–104.

[S2] E.M. Stein. *Singular Integrals and Differentiability Properties of Functions*, Princeton Univ. Press, Princeton, N.J. 1970.

[S3] E.M. Stein. *Oscillatory Integrals in Fourier Analysis*, Beijing Lectures in Harmonic Analysis. Princeton University Press (1986), 307–355.

[Sa] C. Sadosky. *Interpolation of Operators and Singular Integrals, an introduction to Harmonic Analysis*, Lecture Notes Pure and Appl. Math **14**, Marcel Dekker 1979.

[SaYa] J. Satsuma and N. Yajima. Initial value problems for one dimensional self-modulation of nonlinear waves in dispersive media, Suppl. Prig. Theor. Phys. **55** (1974), 284–306.

[Sc] P. Schuur. Asymptotic analysis of soliton problems, Lecture Notes in Mathematics **1232** Springer-Verlag, Berlin, 1986.

[Sch] M. Schwarz Jr. The initial value problem for the sequence of generalized Korteweg-de Vries equations, Adv. in Math. **54** (1984), no. 1, 22–56.

[Scl1] W. Schlag. Dispersive estimates for Schrödinger operators in dimension 2, Comm. Math. Phys. **257** (2005), 87–117.

[Scl2] W. Schlag. Stable manifolds for an orbitally unstable nonlinear Schrödinger equation, Ann. of Math. (2) **169** (2009), no. 1, 139–227.

[SCMc] A. Scott, F. Chu, and D. McLaughlin. The soliton: a new concept in applied science, Proc. IEEE **97** (1973), 1143–1183.

[Se] I.E. Segal. Space time decay for solutions of wave equations, Advance in Math **22** (1976), 305–311.

[SG] C. Shen and B. Guo. Almost conservation law and global rough solutions to a nonlinear Davey-Stewartson, J. Math. Anal. Appl. **318** (206), 365–379.

[ShS] J. Shatah and M. Struwe. Regularity results for nonlinear wave equations, Ann. of Math. **138** (1993), 503–518.

[SiT] T. Simon and E. Taflin. Wave operators and analytic solutions for systems of nonlinear Klein-Gordon and nonlinear Schrödinger equations, Comm. Math. Phys. **99** (1985), 541–562.

[Sj] P. Sjölin. Regularity of solutions to the Schrödinger equation, Duke Math. J. **55** (1987), 699–715.

[Sl] D. Salort. Dispersion and Strichartz inequalities for the one-dimensional Schrödinger equation with variable coefficients, Int. Math. Res. Notice **11** (2005), 687–700.

[Sr1] W.A. Strauss. *Nonlinear Equations*, Regional conference Series in Math. **73**, A.M.S. (1989).

[Sr2] W. Strauss. Existence of solitary waves in higher dimensions, Comm. Math. Phys. **55** (1977), 149–162.

[SS] E.M. Stein and R. Shakarchi. *Complex Analysis*, Princeton Lectures in Analysis, Princeton University Press, (2003).

[SS1] C. Sulem and P.-L. Sulem. Quelques résultats de régularité pour les équations de la turbulence de Langmuir, C. R. Acad. Sci. Paris Sér. A-B **289** (1979), A173–A176.

[SS2] C. Sulem and P.-L.Sulem. *The Nonlinear Schrödinger Equation*, Applied Mathematical Sciences **139**, Springer-Verlag, New York, 1999.

[SSB] P. -L. Sulem, C. Sulem and C. Bardos, *On the continuous limit for a system of classical spins*, Comm. Math. Phys. **107** (1986), 431–454.

[SSS] A. Sidi, C. Sulem, and P.-L. Sulem. On the long time behaviour of a generalized KdV equation, Acta Appl. Math. **7** (1986), 35–47.

[ST] J.-C. Saut and R. Temam. Remarks on the Korteweg-de Vries equation, Israel J. Math. **24** (1976), 78–87.

[St1] J.-C. Saut. Remarks on the generalized Kadomtsev-Petviashvili equations, Indiana Univ. Math. J. **42** (1993), 1011–1026.

[St2] J.-C. Saut. Quelques généralisations de l'équation de Korteweg-de Vries. II, J. Diff. Eqs. **33** (1979), 320–335.

[Sta1] G. Staffilani. On the growth of high Sobolev norms of solutions for KdV and Schrödinger equations, Duke Math. J. **86** (1997), 109–142.

[Sta2] G. Staffilani. On solutions for periodic generalized KdV equations, Internat. Math. Res. Notices **18** (1997), 899–917.

[Sta3] G. Staffilani. KdV and Almost Conservation Laws, Harmonic analysis at Mount Holyoke (South Hadley, MA, 2001), Contemp. Math., 320, Amer. Math. Soc., Providence, RI, (2003), 367–381.

[Str1] R.S. Strichartz. Multipliers on fractional Sobolev spaces, J. Math. Mech. **16** (1967), 1031–1060.

[Str2] R.S. Strichartz. A priori estimates for the wave equations and some applications, J. Funct. Anal. **5** (1970), 218–235.

[Str3] R.S. Strichartz. Restriction of Fourier Transform to quadratic surfaces and decay of solutions of wave equations, Duke Math J. **44** (1977), 705–714.

[StTa] G. Staffilani and D. Tataru. Strichartz estimates for the Schrödinger operator with nonsmooth coefficients, Comm. PDE **27** (2001), 1337–1372.

[Stw] M. Struwe. Globally regular solutions to the u^5 Klein-Gordon equation, Ann. Scuola Norm. Sup. Pisa Cl. Sci. **15** (1989), 495–513.

[Su1] L.Y. Sung. An inverse scattering transform for the Davey-Stewartson II equations. III, J. Math. Anal. Appl. **183** (1994), 477–494.

[Su2] L.Y. Sung. The Cauchy problem for the Ishimori equation, J. Funct. Anal. **139** (1996), 29–67.

[Sy] A. Soyeur. The Cauchy problem for the Ishimori equations, J. Funct. Anal. **105** (1992), 233–255.

[SW] E.M. Stein and G. Weiss. *Introduction to Fourier Analysis on Euclidean Spaces*, Princeton Univ. Press, Princeton N.J. 1971.

[SWe] S. Schochet and M. Weinstein. The nonlinear Schrödinger limit of the Zakharov equations governing the Langmuir turbulence, Comm. Math. Phys. **106** (1986), 569–580.

[T] M. Taylor. *Pseudo Differential Operators*, Princeton Univ. Press, Princeton, N.J. 1981.

[T1] Y. Tsutsumi. L^2 solutions for nonlinear Schrödinger equations and nonlinear groups, Funk. Ekva. **30** (1987), 115–125.

[T2] Y. Tsutsumi. Global strong solutions for nonlinear Schrödinger equations, Nonlinear Anal. **11** (1987), 1143–1154.

[T3] Y. Tsutsumi. Rates of L^2-concentration of blow-up solutions for the nonlinear Schrödinger equation with critical power, Nonlinear Anal. TMA **15** (1990), 545–565.
[Ta1] J. Takeuchi. A necessary condition for the well-posedness of the Cauchy problem for a certain class of evolution equations, Proc. Japan Acad. **50** (1974), 133–137.
[Ta2] S. Tarama. Analyticity of solutions of the Korteweg-de Vries equation, J. Math. Kyoto Univ. **44** (2004), no. 1, 1–32.
[Td] D. Tataru. Parametrices and dispersive estimates for Schrödinger operators with variable coefficients, to appear in Amer. J. Math.
[TF1] M. Tsutsumi and I. Fukuda. On solutions of the derivative nonlinear Schrödinger equation. Existence and uniqueness theorem, Funk. Ekva. **23** (1980), 259–277.
[TF2] M. Tsutsumi and I. Fukuda. On solutions of the derivative nonlinear Schrödinger equation II, Funk. Ekva. **24** (1981), 85–94.
[Th] L. Thomann. Low regularity for a quadratic Schrödinger equation on $\mathbb{T}$, Diff. and Int. Eqs. **24** (2011), 1073–1092.
[Tk1] H. Takaoka. Well-posedness for the Zakharov system with the periodic boundary condition, Diff. and Int. Eqs. **12** (1999), 789–810.
[Tk2] H. Takaoka. Well-posedness for the Kadomtsev-Petviashvili II equation, Adv. Diff. Eqs. **5** (2000), 1421–1443.
[Tm] P. Tomas. A restriction theorem for the Fourier transform, Bull. A.M.S. **81** (1975), 477–478.
[To1] T. Tao. Spherically averaged endpoint Strichartz estimates for the two-dimensional Schrödinger equation, Comm. PDE **25** (2000), 1471–1485.
[To2] T. Tao. A sharp bilinear restrictions estimate for paraboloids, Geom. Funct. Anal. **13** (2003), 1359–1384.
[To3] T. Tao. Multilinear weighted convolution of L^2-functions, and applications to nonlinear dispersive equations, Amer. J. Math. **123** (2001), 839–908.
[To4] T. Tao. Global well-posedness of the Benjamin-Ono equation in $H^1(\mathbb{R})$, J. Hyperbolic Differ. Equ. **1** (2004), 27–49.
[To5] T. Tao. Global well-posedness and scattering for the higher-dimensional energy-critical non-linear Schrödinger equation for radial data, New York J. Math. **11** (2005), 57–80
[To6] T. Tao. Scattering for the quartic generalised Korteweg-de Vries equation, J. Diff. Eqs. **232** (2007), 623–651.
[To7] T. Tao. Nonlinear Dispersive Equations. Local and Global Analysis, CBMS Regional Conference Series in concentration Mathematics, **106**, AMS (2006).
[To8] T. Tao. A (concentration) compactness attractor for high dimension non-linear Schrödinger equations, preprint.
[Ts] M. Tsutsumi. Nonexistence and instability of solutions of nonlinear Schrödinger equations, unpublished.
[TT] H. Takaoka and N. Tzvetkov. On the local regularity of the Kadomtsev-Petviashvili-II equation, Int. Math. Res. Notices **2** (2001), 77–114.
[TTs] H. Takaoka and Y. Tsutsumi. Well-posedness of the Cauchy problem for the modified KdV equation with periodic boundary condition, Int. Math. Res. Notices **56** (2004), 3009–3040.
[TV] T. Tao and A. Vargas. *A bilinear approach to cone multipliers II, Applications*, Geom. Funct. Anal. **10** (2000), 216–258.
[TVZ] T. Tao, M. Visan and X. Zhang. Global well-posedness and scattering for the mass-critical nonlinear Schrödinger equation for radial data in higher dimensions, Duke Math. J. **140** (2007), 165–202.
[Tz1] N. Tzvetkov. On the Cauchy problem for Kadomtsev-Petviashvili equation, Comm. PDE **24** (1999), 1367–1397.
[Tz2] N. Tzvetkov. Global low-regularity solutions for Kadomtsev-Petviashvili equation, Diff. and Int. Eqs. **13** (2000), 1289–1320.

[V] L. Vega. Schrödinger equations: pointwise convergence to the initial data, Proc. A.M.S. **102** (1988), 874–878.

[Ve] S. Vento, Well-posedness for the generalized Benjamin-Ono equations with arbitrary large initial data in the critical space, Int. Math. Res. Notices (2010), 297–319.

[Vi1] M.C. Vilela. Regularity of solutions to the free Schrödinger equation with radial initial data, Illinois J. Math. **45** (2001), 361–370.

[Vi2] M.C. Vilela. Las estimaciones de Strichartz bilineales en el contexto de la ecuación de Schrödinger, Ph.D thesis, Universidad del Pais Vasco, Spain 2003.

[Vs] M. Visan. The defocusing energy-critical nonlinear Schrödinger equation in higher dimensions, Duke Math. J. **138** (2007), 281–374.

[VV] A. Vargas and L. Vega. Global well-posedness for 1D non-linear Schrödinger equation for data with an infinite L^2 norm. J. Math. Pures Appl. **80** (2001), 1029–1044.

[W] M. Weinstein. On the structure and formation of singularities of solutions to nonlinear dispersive evolution equations, Comm. PDE. **11** (1986), 545–565.

[W1] M. Weinstein. *On the Solitary traveling wave of the generalized Korteweg-de Vries equation*, Lectures in Appl. Math. **23** (1986), 23–30.

[W2] M. Weinstein. Modulational stability of ground states of nonlinear Schrödinger equations, SIAM J. Math. Anal., **16** (1985), 472–491.

[W3] M. Weinstein. Nonlinear Schrödinger equations and sharp interpolation estimates, Comm. Math. Phys. **87** (1983), 567–576.

[W4] M. Weinstein. Lyapunov stability of ground states of nonlinear dispersive evolution equations, Comm. Pure Appl. Math. **39** (1986), 51–67.

[Wa] M. Wadati. The modified Korteweg-de Vries equation, J. Phys. Soc. Japan, **47** (1972), 1681.

[Wd] R. Weder. $L^p - L^{p'}$ estimates for the Schrödinger equation on the line and inverse scattering for the Schrödinger equation with potential, J. Funct. Anal. **170** (2000), 37–68.

[X] S.L. Xiong. An analytic solution of Burgers-Korteweg-de Vries equation, Chin. Sci. Bull. **34** (1989), 1158–1162.

[Y] K. Yajima. Existence of solutions for the Schrödinger evolution equations, Comm. Math. Phys. **110** (1987), 415–426.

[Yo] K. Yosida. Functional Analysis, (6th edition), Springer-Verlag 1980.

[Z] A. Zygmund. On Fourier coefficients and transforms of functions of two variables, Studia Math. **50** (1974), 189–201.

[Za] N. J. Zabusky. A Synergetic Approach to Problems of Nonlinear Dispersive Wave Propagation and Interaction, Nonlinear Partial Diff. Eqs. Ed. W.F. Ames N.Y. Academic Press (1967), 223–258.

[ZaKr] N. J. Zabusky and M. D. Kruskal. Interaction of solitons in a collisionless plasma and recurrence of initial states, Physical Review Letters **15** (1965), 240–243.

[Zk] V.E. Zakharov. Collapse of Langmuir waves, Sov. Phys. JEPT **35** (1972), 908–914.

[ZK] V.E. Zakharov and E.A. Kuznetsov. Multi-scale expansions in the theory of systems integrable by the inverse scattering method, Physica D **18** (1986), 455–463.

[ZS] V.E. Zakharov and A.B. Shabat. Exact theory of two dimensional self modulation of waves in nonlinear media, Sov. Phys. J.E.T.P. **34** (1972), 62–69.

[ZSh] V. E. Zakharov and E. I. Shulman. Degenerated dispersive laws, motion invariant and kinetic equations, Physica D (1980), 185–250.

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F. Linares, G. Ponce, *Introduction to Nonlinear Dispersive Equations,*
Universitext, DOI 10.1007/978-1-4939-2181-2