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Christian Constanda

Mathematical Methods for Elastic Plates

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For Lia

Preface

Approximate theories of bending of thin elastic plates have been around since the middle of the nineteenth century. The reason for their existence is twofold: on the one hand, they reduce the full three-dimensional model to a simpler one in only two independent variables; on the other hand, they give prominence to the main characteristics of bending, neglecting other effects that are of lesser interest in the study of this physical process.

In spite of their good agreement with experiments and their wide use by engineers in practical applications, such theories never acquire true legitimacy until they have been validated by rigorous mathematical analysis. The study of the classical (Kirchhoff) model (Kirchhoff 1850) is almost complete (see, for example Ciarlet and Destuynder 1979; Gilbert and Hsiao 1983). In this book, we turn our attention to plates with transverse shear deformation, which include the Reissner (1944, 1945, 1947, 1976, 1985) and Mindlin (1951) models, discussing the existence, uniqueness, and approximation of their regular solutions by means of the boundary integral equation and stress function methods in the equilibrium (static) case.

With the exception of a few results of functional analysis, which are quoted from other sources, the presentation is self-contained and includes all the necessary details, from basic notation to the full-blown proofs of the lemmas and theorems.

Chapter 1 concentrates on the geometric/analytic groundwork for the investigation of the behavior of functions expressed by means of integrals with singular kernels, in the neighborhood of the boundary of the domain where they are defined.

In Chap. 2, we introduce potential-type functions and determine their mapping properties in terms of both real and complex variables, and discuss the solvability of singular integral equations.

Next, in Chap. 3, we describe the two-dimensional model of bending of elastic plates with transverse shear deformation, derive a matrix of fundamental solutions for the governing system, state the main boundary value problems, and comment on the uniqueness of their regular solutions.

All the references cited here can be found at the end of the book.

The layer and Newtonian plate potentials are introduced, respectively, in Chaps. 4 and 5, where we investigate their Hölder continuity and differentiability.

In Chap. 6, we prove the existence of regular solutions for the interior and exterior displacement, traction, and Robin boundary value problems by means of single-layer and double-layer potentials, and discuss the smoothness of the integrable solutions of these problems.

Chapter 7 is devoted to the construction of the complete integral of the system of equilibrium equations in terms of complex analytic potentials, and the clarification of the physical meaning of certain analytic constraints imposed earlier on the asymptotic behavior of the solutions.

In Chap. 8, we explain how the method of generalized Fourier series can be adapted to provide approximate solutions for the Dirichlet and Neumann problems.

Some of the results incorporated in this book have been published in Constanda (1985, 1986a, b, 1987, 1988a, b, 1989a, b, 1990a, b, 1991, 1994, 1996a, b, 1997a, b; Schiavone 1996; Thomson and Constanda 1998, 2008); additionally, Constanda (1990) is an earlier—incomplete—version compiled as research notes. Chapter 5 is based on material included in Thomson and Constanda (2011a). The technique developed in Chaps. 2–4 and 6 was later extended to the case of bending of micropolar plates in Constanda (1974), Schiavone and Constanda (1989), and Constanda (1989).

A comprehensive view and comparison of direct and indirect boundary integral equation methods for elliptic two-dimensional problems in Cartesian coordinates and Hölder spaces can be found in Constanda (1999).

Potential methods go hand in hand with variational techniques when the data functions lack smoothness. The distributional solutions of equilibrium problems with a variety of boundary conditions have been constructed by this combination of analytic procedures in Chudinovich and Constanda (1997, 1998, 1999a, b, 2000a, b, c, d, e, 2001a, b). The harmonic oscillations of plates with transverse shear deformation form the object of study in Constanda (1998), Schiavone and Constanda (1993, 1994), Thomson and Constanda (1998, 1999, 2009a, b, c, 2010, 2011a, b, 2012a, b, c, 2013), and the case that includes thermal effects has been developed in Chudinovich and Constanda (2005a, b, 2006, 2008a, b, c, 2009, 2010a, b, c, 2007).

Finally, a number of problems that impinge on the solution of this mathematical model are discussed in Chudinovich and Constanda (2000f, 2006), Constanda (1978a, b), Constanda et al. (1995), Mitric and Constanda (2005), and Constanda (2006).

Before going over to the business of mathematical analysis, I would like to thank my Springer UK editor, Lynn Brandon, for her support and guidance, and her assistant, Catherine Waite, for providing feedback from the production team in matters of formatting and style.

But above all, I am grateful to my wife for her gracious acceptance of the truth that a mathematician's work is never done.

Tulsa, January 2014 Christian Constanda

Contents

Chapter 1
Singular Kernels

1.1 Introduction

Throughout the book we make use of a number of well-established symbols and conventions. Thus, Greek and Latin subscripts take the values 1, 2 and 1, 2, 3, respectively, summation over repeated indices is understood, $x = (x_1, x_2)$ and $x = (x_1, x_2, x_3)$ are generic points referred to orthogonal Cartesian coordinates in $\mathbb{R}^2$ and $\mathbb{R}^3$, a superscript T indicates matrix transposition, $(\ldots)_{,\alpha} = \partial(\ldots)/\partial x_\alpha$, Δ is the Laplacian, and δ_{ij} is the Kronecker delta. Other notation will be defined as it occurs in the text.

The elastostatic behavior of a three-dimensional homogeneous and isotropic body is described by the equilibrium equations

$$t_{ij,j} + f_i = 0 \tag{1.1}$$

and the constitutive relations

$$t_{ij} = \lambda u_{k,k}\delta_{ij} + \mu(u_{i,j} + u_{j,i}) \tag{1.2}$$

(see, for example, Green and Zerna 1963). Here $t_{ij} = t_{ji}$ are the internal stresses, u_i the displacements, f_i the body forces, and λ and μ the Lamé constants of the material.

The components of the resultant stress vector t in a direction $n = (n_1,\ n_2,\ n_3)^{\mathrm{T}}$ are

$$t_i = t_{ij}n_j, \tag{1.3}$$

and the internal energy per unit volume (internal energy density) is

$$\mathscr{E} = \tfrac{1}{4}\,t_{ij}(u_{i,j} + u_{j,i}) = \tfrac{1}{2}\,t_{ij}u_{i,j}. \tag{1.4}$$

C. Constanda, *Mathematical Methods for Elastic Plates*,
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A *thin plate* is an elastic body that occupies a region $\bar{S} \times [-h_0/2,\ h_0/2]$ in $\mathbb{R}^3$, where S is a domain in $\mathbb{R}^2$ and $0 < h_0 = \text{const} \ll \text{diam}\, S$ is the thickness. The special form of such a body suggests that in the study of its small deformations certain simplifying assumptions may be introduced, which lead to two-dimensional theories that are easier to handle but still describe adequately the salient features of the deformation state. In what follows we are concerned exclusively with the process of bending.

The first truly systematic theory of bending of thin elastic plates was proposed by Kirchhoff (1850). Under his assumptions the displacement field becomes

$$\begin{aligned} u_\alpha &= -x_3 u_{3,\alpha}, \\ u_3 &= u_3(x_\gamma), \end{aligned} \tag{1.5}$$

and from (1.1) and (1.2) it follows that

$$\Delta\Delta u_3 = \frac{p}{D},$$

where p is the resultant load on the faces $x_3 = \pm h_0/2$ of the plate and

$$D = h_0^3 \mu \frac{\lambda + \mu}{3(\lambda + 2\mu)}$$

is the rigidity modulus. This theory, though producing good approximations in many practical cases, neglects completely the effects of the transverse shear forces since (1.2) and (1.5) yield $t_{3\alpha} = 0$ throughout the plate. It also gives rise to a few mathematical discrepancies: certain stress components are neglected in some equations but not in others. In addition, the unknown deflection u_3 can satisfy only two boundary conditions instead of the physically expected three.

Reissner (see Reissner 1944; 1976) takes transverse shear into account by assuming that

$$\begin{aligned} t_{\alpha\beta} &= \frac{h_0^2}{12}\, x_3 M_{\alpha\beta}(x_\gamma), \\ t_{\alpha 3} &= \frac{3}{2h_0}\left[1 - \left(\frac{2}{h_0}\right)^2 x_3^2\right] Q_\alpha(x_\gamma), \end{aligned}$$

and uses the principle of least work to derive a sixth order theory that accommodates three boundary conditions. While this is a more complete model than Kirchhoff's, it does not deliver the expression of the displacements but only that of their averages.

Hencky (1947), Bollé (1947), Uflyand (1948), and Mindlin (1951) introduce the effects of transverse shear deformation in a somewhat different manner. More precisely, they start with the displacement assumption

$$\begin{aligned} u_\alpha &= x_3 v_\alpha(x_\gamma), \\ u_3 &= v_3(x_\gamma) \end{aligned} \tag{1.6}$$

and arrive at the equations of an approximate sixth order theory by averaging (1.1) and (1.2) over the thickness of the plate. As in the case of Reissner's, these equations allow three conditions to be prescribed on the boundary. Unfortunately, they suffer from the same lack of rigor, due to the fact that t_{33} is neglected in the constitutive relations, which also contain so-called correction factors.

The above theories have subsequently been refined in various ways, but all their versions pursue the same ultimate goal: to offer as much valid information as possible on the characteristics of bending, while at the same time reducing the problem to a simpler one in two dimensions (see Reissner (1985) for a concise survey of this topic).

Here we are not concerned with the advantages of one theory over another from a physical standpoint, but with their mathematical treatment. As the model of our analysis we choose an approximation based solely on the kinematic assumption (1.6), thus avoiding inconsistencies that might otherwise be introduced through over-simplification. However, our technique is equally applicable—with very little modification regarding the coefficients—to all existing sixth order theories where the system of equilibrium equations is elliptic.

1.2 Geometry of the Boundary Curve

For simplicity, we use the same symbol to indicate both a point and its position vector in $\mathbb{R}^2$. Also, vector functions are not distinguished from scalar ones by any special marks, their nature being obvious from the context.

Let the boundary ∂S of S be a simple closed curve of length l, whose natural parametrization (that is, in terms of its arc length measured from some point on ∂S) is a bijection of the form

$$x = x(s), \quad s \in [0, l], \quad x(0) = x(l),$$

with inverse

$$s = s(x), \quad x \in \partial S.$$

Throughout what follows, ∂S is a C^2-curve; in other words, x is twice continuously differentiable on $[0, l]$ and

$$\frac{dx}{ds}(0+) = \frac{dx}{ds}(l-),$$
$$\frac{d^2x}{ds^2}(0+) = \frac{d^2x}{ds^2}(l-).$$

As is well known,

$$\frac{dx}{ds} = \tau(s) = \tau(x)$$

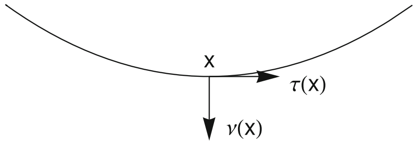

Fig. 1.1 Orientation of the local frame axes

is the unit tangent vector at $x \in \partial S$, pointing in the direction in which s increases. If we denote by $\nu(x)$ the unit *outward* (with respect to S) normal to ∂S at x, then the direction of $\tau(x)$ is chosen so that the local frame $\{\tau(x),\ \nu(x)\}$ is left-handed. In this case,

$$\tau_\alpha = \varepsilon_{\beta\alpha}\nu_\beta, \tag{1.7}$$

where $\varepsilon_{\alpha\beta}$ is the two-dimensional Ricci tensor (alternating symbol).

Figure 1.1 shows the orientation of the local frame axes.

The Frenet–Serret formulas

$$\begin{aligned} \frac{d}{ds}\tau(x) &= -\kappa(x)\nu(x), \\ \frac{d}{ds}\nu(x) &= \kappa(x)\tau(x) \end{aligned} \tag{1.8}$$

connect $\tau(x)$, $\nu(x)$, and the algebraic value $\kappa(x)$ of the curvature of ∂S at x.

1.1 Remarks. (i) The choice we made for the direction of the normal vector ensures that the formulation of the analytic arguments involving ν later on follows the well-established patterns in the literature.

(ii) If S is a domain with holes, then the above convention regarding the orientation of τ and ν applies to the boundary of each hole, as well as to the outer boundary (if there is one).

(iii) Since ∂S is a C^2-curve, we can define

$$\kappa_0 = \sup_{x \in \partial S} |\kappa(x)|. \tag{1.9}$$

It is obvious that $\kappa_0 > 0$, for $\kappa_0 = 0$ would imply that ∂S were a straight line and, therefore, not a closed curve.

Let

$$\begin{aligned} \langle x, y\rangle &= x_1y_1 + x_2y_2, \\ |x|^2 &= x_1^2 + x_2^2 \end{aligned}$$

be, respectively, the standard inner product and the Euclidean norm on $\mathbb{R}^2$.

Some of the estimates established below are not optimal. Tighter ones can be obtained, but since these are only auxiliary results, we select admissible numerical coefficients that make the inequalities easier to manipulate.

1.2 Lemma. *For all* $x,\ y \in \partial S$,

$$|\langle \nu(x),\ x-y\rangle| \le 2\kappa_0 |x-y|^2, \tag{1.10}$$
$$|\nu(x)-\nu(y)| \le 4\kappa_0 |x-y|. \tag{1.11}$$

Proof. Let s and t be the arc length coordinates of x and y. We have

$$\begin{aligned}\frac{\partial}{\partial s}|x-y|^2 &= \langle (\mathrm{grad}(x))|x-y|^2, \tau(x)\rangle = 2|x-y|\frac{x_\alpha - y_\alpha}{|x-y|}\tau_\alpha(x)\\ &= 2(x_\alpha - y_\alpha)\tau_\alpha(x) = 2\langle \tau(x),\ x-y\rangle\end{aligned}$$

and, by (1.8),

$$\begin{aligned}\frac{\partial^2}{\partial s^2}|x-y|^2 &= 2\big[\tau_\alpha(x)\tau_\alpha(x) - \kappa(x)(x_\alpha - y_\alpha)\nu_\alpha(x)\big]\\ &= 2\big[1-\kappa(x)\langle \nu(x),\ x-y\rangle\big].\end{aligned}$$

The Taylor series expansion now yields

$$\begin{aligned}|x-y|^2 &= \big[|x-y|^2\big]_{s=t} + \left[\frac{\partial}{\partial s}|x-y|^2\right]_{s=t}(s-t) + \frac{1}{2}\left[\frac{\partial^2}{\partial s^2}|x-y|^2\right]_{s=s'}(s-t)^2\\ &= \big[1-\kappa(x')\langle \nu(x'), x'-y\rangle\big](s-t)^2,\end{aligned}$$

where s' is the value of the arc length coordinate of a point x' lying between x and y on ∂S.

Suppose that $|x-y| \le 1/(2\kappa_0)$. Then

$$\begin{aligned}|1-\kappa(x')\langle \nu(x'), x'-y\rangle| &\ge 1 - |\kappa(x')\langle \nu(x'), x'-y\rangle|\\ &\ge 1 - |\kappa(x')|\,|\nu(x')|\,|x'-y| \ge 1-\kappa_0|x-y|\\ &\ge 1-\kappa_0\cdot\frac{1}{2\kappa_0} = \frac{1}{2},\end{aligned}$$

so

$$|x-y|^2 \ge \tfrac{1}{2}(s-t)^2. \tag{1.12}$$

Following the same procedure, we have

$$\begin{aligned}\frac{\partial}{\partial s}\langle \nu(y), x-y\rangle &= \nu_\alpha(y)\tau_\alpha(x) = \langle \nu(y), \tau(x)\rangle,\\ \frac{\partial^2}{\partial s^2}\langle \nu(y), x-y\rangle &= -\kappa(x)\nu_\alpha(y)\nu_\alpha(x) = -\kappa(x)\langle \nu(y), \nu(x)\rangle,\end{aligned}$$

and

$$\langle \nu(y), x-y\rangle = \big[\langle \nu(y), x-y\rangle\big]_{s=t} + \left[\frac{\partial}{\partial s}\langle \nu(y), x-y\rangle\right]_{s=t}(s-t)$$
$$+\frac{1}{2}\left[\frac{\partial^2}{\partial s^2}\langle \nu(y), x-y\rangle\right]_{s=s''}(s-t)^2,$$

where s'' is the arc length coordinate of a point x'' lying between x and y on ∂S; hence, by (1.12),

$$|\langle \nu(y), x-y\rangle| \le \tfrac{1}{2}\,|\kappa(x'')|\,|\nu(y)|\,|\nu(x'')|(s-t)^2 \le \kappa_0|x-y|^2.$$

On the other hand, if $|x-y| > 1/(2\kappa_0)$ (or, what is the same, $2\kappa_0|x-y| > 1$), then

$$\begin{aligned}|\langle \nu(y), x-y\rangle| &\le |\nu(y)|\,|x-y|\\ &\le |x-y|\cdot 2\kappa_0|x-y| = 2\kappa_0|x-y|^2.\end{aligned}$$

Combining the two cases, we conclude that for any x and y on ∂S,

$$|\langle \nu(y), x-y\rangle| \le \max\big\{\kappa_0,\ 2\kappa_0\big\}|x-y|^2 = 2\kappa_0|x-y|^2,$$

which is (1.10).

Similarly, by (1.8),

$$\begin{aligned}\nu(x)-\nu(y) &= \big[\nu(x)-\nu(y)\big]_{s=t} + \frac{\partial}{\partial s}\big[\nu(x)-\nu(y)\big]_{s=s'''}(s-t)\\ &= \kappa(x''')\tau(x''')(s-t),\end{aligned}$$

where s''' is the arc length coordinate of a point x''' lying between x and y on ∂S. Hence, in view of (1.12), for $|x-y| \le 1/(2\kappa_0)$ we have

$$|\nu(x)-\nu(y)| \le \kappa_0|s-t| \le \sqrt{2}\,\kappa_0|x-y|.$$

At the same time, for $|x-y| > 1/(2\kappa_0)$,

$$|\nu(x)-\nu(y)| \le |\nu(x)|+|\nu(y)| = 2 < 4\kappa_0|x-y|,$$

so for any pair of points x and y on ∂S,

$$|\nu(x)-\nu(y)| \le \max\big\{\sqrt{2}\,\kappa_0, 4\kappa_0\big\}|x-y| = 4\kappa_0|x-y|,$$

which is (1.11). □

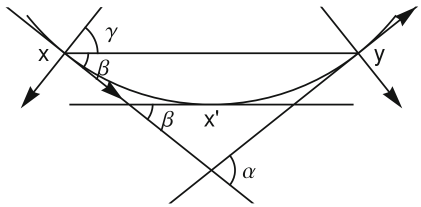

Fig. 1.2 The shorter arc joining x and y

To keep things simple, the proofs of the rest of the lemmas in this section and the next are constructed for one local boundary configuration only, but they remain valid for any other possible configuration. Also, to ensure clarity, the accompanying diagrams are not drawn to scale.

1.3 Lemma. *Let $x,\ y \in \partial S$, and let α be the angle between $\nu(x)$ and $\nu(y)$ and γ the angle between $\nu(x)$ and $x - y$. If r is a number such that*

$$0 < r \leq \frac{1}{8\kappa_0}, \tag{1.13}$$

then for all x and y satisfying $|x - y| \leq r$,

$$\tfrac{1}{2} \leq \cos\alpha \leq 1, \tag{1.14}$$
$$\tfrac{1}{2} \leq \sin\gamma \leq 1. \tag{1.15}$$

Proof. Consider the shorter arc of ∂S joining x and y (see Fig. 1.2).

By (1.11),

$$\begin{aligned}\cos\alpha = \langle \nu(x),\ \nu(y)\rangle &= 1 - \langle \nu(x), \nu(x) - \nu(y)\rangle \\ &\geq 1 - |\langle \nu(x), \nu(x) - \nu(y)|\\ &\geq 1 - |\nu(x)|\,|\nu(x) - \nu(y)| \\ &\geq 1 - 4\kappa_0 |x - y| \\ &\geq 1 - 4\kappa_0 \cdot \frac{1}{8\kappa_0} = \frac{1}{2},\end{aligned}$$

which proves (1.14).

Next, by the mean value theorem, there is a point $x' \in \partial S$ between x and y such that the support lines of $\tau(x')$ and $x - y$ are parallel. The acute angle β between the support lines of $\tau(x)$ and $x - y$ (see Fig. 1.2) is the same as the angle between $\tau(x)$ and $\tau(x')$, therefore, the same as the angle between $\nu(x)$ and $\nu(x')$. By (1.14), we have

$$\frac{1}{2} \leq \cos\beta \leq 1,$$

and (1.15) now follows from the fact that $\sin\gamma = \cos\beta$. □

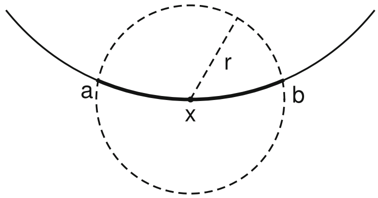

Fig. 1.3 The arc $\Sigma_{x,r}$

1.4 Lemma. *If*

$$\Sigma_{x,r} = \{y \in \partial S : |x - y| \le r\}, \quad x \in \partial S, \tag{1.16}$$

with r satisfying (1.13), *then for every* $x \in \partial S$ *and all* $y \in \Sigma_{x,r}$,

$$\tfrac{1}{2}|s - t| \le |x - y| \le |s - t|, \tag{1.17}$$

where s and t are the arc length coordinates of x and y.

Proof. Let a and b be the end-points of $\Sigma_{x,r}$ (the heavier arc in Fig. 1.3).
Direct computation shows that

$$\begin{aligned} \frac{d}{dt}\,|x - y\,| &= \frac{d}{dt}\big[(x_1 - y_1)^2 + (x_2 - y_2)^2\big]^{1/2} \\ &= \frac{1}{|x - y\,|}\left[(y_1 - x_1)\frac{dy_1}{dt} + (y_2 - x_2)\frac{dy_2}{dt}\right] \\ &= \frac{\langle \tau(y), y - x\rangle}{|x - y\,|} = \cos\beta(y), \end{aligned}$$

where $\beta(y)$ is the angle between $\tau(y)$ and $y - x$; hence, according to the mean value theorem, there is $y' \in \partial S$ between x and y such that

$$|x - y\,| = \int_s^t \cos\beta(\sigma)\,d\sigma = (t - s)\cos\beta(y'). \tag{1.18}$$

If y lies on ∂S between x and b (see Fig. 1.4), then both $\beta(y)$ and $\beta(y')$ are acute angles and $\beta(y') = \pi/2 - \gamma(y')$, where $\gamma(y')$ is the angle between $\nu(y')$ and $y' - x$; so, by (1.18),

$$|x - y\,| = (t - s)\sin\gamma(y'). \tag{1.19}$$

If, on the other hand, y lies on ∂S between a and x (see Fig. 1.5), then $\beta(y)$ and $\beta(y')$ are obtuse angles but $\gamma(y')$ is still acute and $\beta(y') = \pi/2 + \gamma(y')$; therefore, by (1.18),

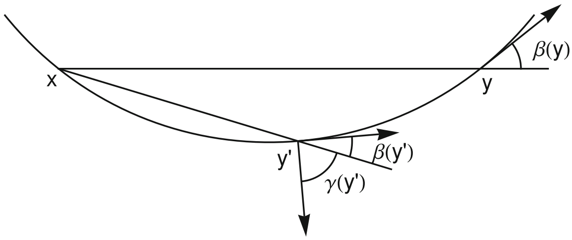

Fig. 1.4 Arc of ∂S with y between x and b

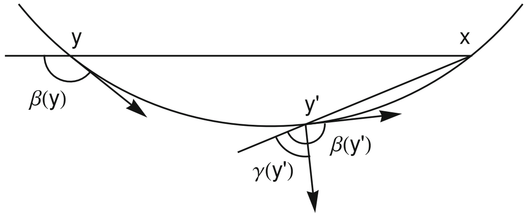

Fig. 1.5 Arc of ∂S with y between a and x

$$|x - y| = (t - s)\big(-\sin\gamma(y')\big) = (s - t)\sin\gamma(y'). \tag{1.20}$$

Equalities (1.19) and (1.20) can be written together as

$$|x - y| = |s - t|\sin\gamma(y'),$$

and (1.17) now follows from (1.15). □

1.5 Remark. From the proof of Lemma 1.4 it is clear that for any x fixed on ∂S, $|x - y|$ is a monotonic function of t on each of the intervals

$$I_1 = \{t : y(t) \in \Sigma_{x,r},\ t \le s(x)\},$$
$$I_2 = \{t : y(t) \in \Sigma_{x,r},\ t \ge s(x)\},$$

decreasing on the former and increasing on the latter. This implies that

$$|x - y'| \ne |x - y''|$$

for all $y'(t'),\ y''(t'') \in \Sigma_{x,r}$ such that $t' \ne t''$, with $t',\ t'' \in I_1$ or $t',\ t'' \in I_2$, and that there is a bijective correspondence between the points of $\Sigma_{x,r}$ and those of its projection on the tangent to ∂S at x.

1.6 Remark. A slightly modified pair of inequalities (1.17) holds for all $x,\ y \in \partial S$ if by $|s - t|$ we understand the length of the shorter arc of ∂S joining x and y. Since for $|x - y| > r$,

$$|s - t| \le l \le \frac{l}{r}|x - y|,$$

we conclude that for all $x,\ y \in \partial S$,

$$c|s-t| \le |x-y| \le |s-t|,$$

where $c = \min\{1/2,\ r/l\}$.

1.3 Properties of the Boundary Strip

Many of the results in this book are proved by considering the behavior of certain two-point functions in the neighborhood of the boundary. To help the fluency of such proofs, here we make a preliminary examination of some frequently used properties.

1.7 Lemma. *The normal displacements of ∂S defined by*

$$\partial S_\sigma = \{x \in \mathbb{R}^2 : x = \xi + \sigma\nu(\xi),\ \xi \in \partial S\},$$
$$\sigma = \text{const}, \quad 0 < |\sigma| < \frac{1}{\kappa_0},$$

where κ_0 is given by (1.9), *are well-defined C^2-curves.*

Proof. Let s and t be the arc length parameters on ∂S and ∂S_σ, respectively. Since the map

$$x = \xi + \sigma\nu(\xi) = \xi(s) + \sigma\nu(\xi(s)), \quad x \in \partial S,$$

is a C^2-parametrization of ∂S_σ in terms of s, it follows that ∂S_σ is a C^2-curve, and we may use its natural parametrization (that is, in terms of t) to discuss its differential properties.

All we need to show now is that for any distinct points $\xi,\ \xi' \in \partial S$, the support lines of $\nu(\xi)$ and $\nu(\xi')$ do not intersect at a point situated at a distance less than $1/\kappa_0$ from ∂S.

According to the assumption on σ, for any $\xi \in \partial S$,

$$1 + \sigma\kappa(\xi) \ge 1 - |\sigma|\,|\kappa(\xi)| \ge 1 - |\sigma|\kappa_0 > 0;$$

hence,

$$\frac{dx}{ds} = \frac{d\xi}{ds} + \sigma\frac{d\nu(\xi)}{ds} = \tau(\xi) + \sigma\kappa(\xi)\tau(\xi) = \big[1 + \sigma\kappa(\xi)\big]\tau(\xi). \qquad (1.21)$$

Since, in terms of the arc parameter t on ∂S_σ,

$$dx = \frac{dx}{dt}\,dt = \tau(x)\,dt,$$

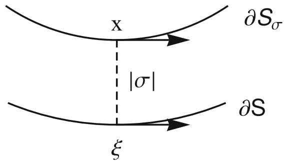

Fig. 1.6 Arcs of ∂S and ∂S_σ

it follows that, by (1.21),

$$\begin{aligned} dt &= |dx| = \left|\frac{dx}{ds}\,ds\right| = \left|\frac{dx}{ds}\right| ds \\ &= \big[1+\sigma\kappa(\xi)\big]\frac{ds}{dt}\,dt, \end{aligned}$$

so

$$\frac{ds}{dt} = \big[1+\sigma\kappa(\xi)\big]^{-1};$$

therefore,

$$\begin{aligned} \tau(x) &= \frac{dx}{dt} = \frac{dx}{ds}\,\frac{ds}{dt} \\ &= \big[1+\sigma\kappa(\xi)\big]^{-1}\big[1+\sigma\kappa(\xi)\big]\tau(\xi) = \tau(\xi). \end{aligned} \tag{1.22}$$

Suppose that there are $\xi,\ \xi' \in \partial S, \xi \neq \xi'$, such that the support lines of $\nu(\xi)$ and $\nu(\xi')$ intersect at some point x located at a distance less than $1/\kappa_0$ from ∂S; that is,

$$x = \xi + \sigma\nu(\xi) = \xi' + \sigma'\nu(\xi'), \quad |\sigma|,\ |\sigma'| < \frac{1}{\kappa_0}.$$

Then $x \in \partial S_\sigma \cap \partial S_{\sigma'}$, so, by (1.22),

$$\tau(x) = \tau(\xi) = \tau(\xi'),$$

which implies that $\nu(\xi) = \nu(\xi')$. Since this contradicts our assumption, we conclude that ∂S_σ is well defined.

Figure 1.6 illustrates an arc of ∂S and the arc of a typical curve ∂S_σ. □

1.8 Definition. Let σ_0 be a fixed number such that $0 < \sigma_0 < 1/\kappa_0$. The region

$$S_{\sigma_0} = \big\{x \in \mathbb{R}^2 : x = \xi + \sigma\nu(\xi),\ \xi \in \partial S,\ |\sigma| \le \sigma_0\big\}$$

is called the σ_0-*strip* along the boundary ∂S.

1.9 Lemma. *Let $x,\ x' \in S_{r/4}$, where r satisfies* (1.13), *be such that*

$$|x - x'| < \tfrac{1}{4}\,r.$$

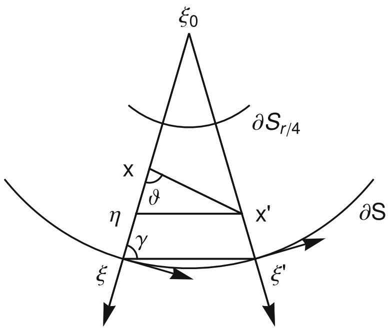

Fig. 1.7 Points ξ and ξ' on ∂S with $\nu(\xi)$ not parallel to $\nu(\xi')$

If

$$x = \xi + \sigma\nu(\xi), \quad x' = \xi' + \sigma'\nu(\xi'), \quad \xi,\, \xi' \in \partial S, \tag{1.23}$$

then

$$|\xi - \xi'| < 4|x - x'|. \tag{1.24}$$

Proof. Without loss of generality, we may assume that

$$|x - \xi| \geq |x' - \xi'|.$$

First, suppose that $\nu(\xi)$ and $\nu(\xi')$ are not parallel, and let ξ_0 be the point of intersection of their support lines. Also, let η be the point on the line through ξ and ξ_0 such that $\eta - x'$ is parallel to $\xi - \xi'$ (see Fig. 1.7).

According to the argument in the proof of Lemma 1.7, we must have

$$|\xi_0 - \xi| \geq \frac{1}{\kappa_0};$$

consequently, since $\eta \in S_{r/4}$,

$$\begin{aligned}\frac{|\eta - x'|}{|\xi - \xi'|} = \frac{|\xi_0 - \eta|}{|\xi_0 - \xi|} &= \frac{|\xi_0 - \xi| - |\eta - \xi|}{|\xi_0 - \xi|}\\ &= 1 - \frac{|\eta - \xi|}{|\xi_0 - \xi|} > 1 - \frac{r/4}{1/\kappa_0}\\ &= 1 - \tfrac{1}{4}r\kappa_0 > 1 - \tfrac{1}{32} > \tfrac{1}{2}.\end{aligned} \tag{1.25}$$

Let ϑ be the angle between $\xi - \xi_0$ and $x' - x$, and let γ be the angle between $\nu(\xi)$ and $\xi - \xi'$. By (1.25) and as seen from Fig. 1.7,

$$|\xi - \xi'| < 2|\eta - x'| \leq 2(|x - x'| + |\eta - x|) < 2\big(\tfrac{1}{4}r + \tfrac{1}{4}r\big) = r;$$

hence, by (1.15) and (1.25),

$$|x - x'| = \frac{\sin\gamma}{\sin\vartheta}\,|\eta - x'|$$
$$\geq \tfrac{1}{2}\,|\eta - x'| > \tfrac{1}{4}\,|\xi - \xi'|,$$

as required.

If $\nu(\xi)$ and $\nu(\xi')$ are parallel, then $\gamma = \pi/2$ and

$$|x - x'| = \frac{1}{\sin\vartheta}\,|\eta - x'| \geq \tfrac{1}{2}\,|\xi - \xi'|,$$

so (1.24) holds. □

1.10 Lemma. *Let* $x,\ x' \in S_{r/4}$, *where* r *satisfies* (1.13) *and* x *and* x' *are given by* (1.23), *and suppose that*

$$|x - x'| < \tfrac{1}{8}\,r.$$

Also, let $\Sigma_{\xi,r}$ *be defined by* (1.16), *and let* $s,\ s'$, *and* t *be the arc length coordinates of* $\xi,\ \xi'$, *and* y, *respectively. Then* $\xi' \in \Sigma_{\xi,r/2}$ *and*

(i) *for all* $y \in \Sigma_{\xi,r}$,

$$|x - y\,| \geq \tfrac{1}{2}\,|\xi - y\,|, \tag{1.26}$$
$$|x - y\,| \geq \tfrac{1}{2}\,|x - \xi|; \tag{1.27}$$

(ii) *for all* $y \in \Sigma_{\xi,r/2}$, *we have* $y \in \Sigma_{\xi',r}$ *and*

$$|x' - y\,| \geq \tfrac{1}{2}\,|\xi' - y\,| \geq \tfrac{1}{4}\,|s' - t|. \tag{1.28}$$

Proof. The geometric configuration (with the heavier arc representing a portion of $\Sigma_{\xi,r/2}$) is shown in Fig. 1.8.

By (1.24),

$$|\xi - \xi'| < 4|x - x'| < 4 \cdot \tfrac{1}{8}\,r = \tfrac{1}{2}\,r,$$

which means that $\xi' \in \Sigma_{\xi,r/2}$.

(i) We denote by γ the angle between $\nu(\xi)$ and $\xi - y$ and by ϑ the angle between $\nu(\xi)$ and $y - x$. If $y \in \Sigma_{\xi,r}$, then $|\xi - y\,| \leq r$, so from the sine theorem and (1.15) it follows that

$$|x - y\,| = \frac{\sin\gamma}{\sin\vartheta}\,|\xi - y\,| \geq \tfrac{1}{2}\,|\xi - y\,|,$$

which is (1.26).

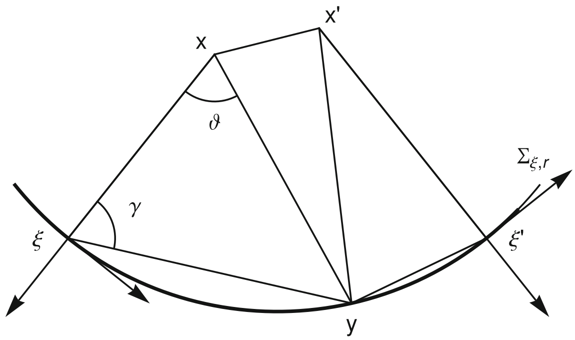

Fig. 1.8 Portions of $\Sigma_{\xi,r}$ and $\Sigma_{\xi,r/2}$ (heavier arc)

Inequality (1.27) is establish as above; that is,

$$\begin{aligned}|x-y| &= \frac{\sin\gamma}{\sin(\gamma+\vartheta)}\,|x-\xi| \\ &\geq \tfrac{1}{2}\,|x-\xi|.\end{aligned}$$

(ii) Since $y \in \Sigma_{\xi,r/2}$ and, as already shown, $\xi' \in \Sigma_{\xi,r/2}$,

$$\begin{aligned}|\xi'-y| &\leq |\xi'-\xi| + |\xi-y| \\ &< \tfrac{1}{2}\,r + \tfrac{1}{2}\,r = r.\end{aligned}$$

This means that $y \in \Sigma_{\xi',r}$, so (1.26) remains valid for x', ξ', and y, yielding

$$|x'-y| \geq \tfrac{1}{2}\,|\xi'-y|.$$

Finally, by (1.17),

$$|\xi'-y| \geq \tfrac{1}{2}\,|s'-t|,$$

which completes the proof of (1.28). □

1.11 Lemma. *Let $x,\ x' \in S_{r/4}$, where r satisfies* (1.13), *x and x' are given by* (1.23), *and*

$$|x-x'| < \tfrac{1}{8}\,r.$$

If, with the notation in Lemma 1.10,

$$\begin{aligned}\Sigma_1 &= \{y \in \Sigma_{\xi,r} : |s-t| \leq 8|x-x'|\}, \\ \Sigma_2 &= \Sigma_{\xi,r} \setminus \Sigma_1 = \{y \in \Sigma_{\xi,r} : |s-t| > 8|x-x'|\},\end{aligned} \tag{1.29}$$

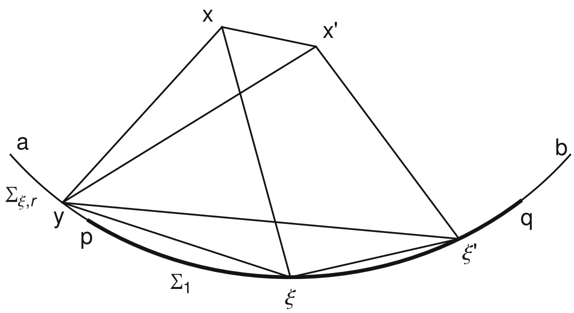

Fig. 1.9 A portion of $\Sigma_{\xi,r}$ and Σ_1 (heavier arc)

then Σ_1 *lies strictly within* $\Sigma_{\xi,r}$, $\xi' \in \Sigma_1$, *and for all* $y \in \Sigma_2$,

$$|x' - y| \geq \tfrac{1}{4}|\xi - y|, \tag{1.30}$$

$$|x - x'| < \tfrac{1}{2}|x - y|, \tag{1.31}$$

$$|\xi' - y| < 3|\xi - y|. \tag{1.32}$$

Proof. Consider the diagram in Fig. 1.9, where the heavier arc represents Σ_1. Let a, b and p, q be the boundary points of $\Sigma_{\xi,r}$ and Σ_1, respectively, with a, p and b, q on opposite sides of ξ, and let t_p and t_q, $t_p < s < t_q$, be the arc length coordinates of p and q. Then $|\xi - a| = |\xi - b| = r$, so, by (1.17) and (1.29),

$$\begin{aligned} |\xi - p| &\leq |s - t_p| = 8|x - x'| \\ &< 8 \cdot \tfrac{1}{8} r = r = |\xi - a|, \end{aligned}$$

with a similar inequality for $|\xi - q|$. This means that Σ_1 lies strictly within $\Sigma_{\xi,r}$.

Combining (1.17) and (1.24), we see that

$$|s - s'| \leq 2|\xi - \xi'| < 8|x - x'|;$$

therefore, by (1.29), $\xi' \in \Sigma_1$.

For $y \in \Sigma_2$, we use (1.29), (1.26), and (1.17) to find that

$$\begin{aligned} |x' - y| &\geq |x - y| - |x - x'| \\ &\geq |x - y| - \tfrac{1}{8}|s - t| \\ &\geq \tfrac{1}{2}|\xi - y| - \tfrac{1}{4}|\xi - y| = \tfrac{1}{4}|\xi - y|. \end{aligned}$$

Next, by (1.24), $|\xi - \xi'| < 4|x - x'| < \frac{1}{2} r < r$; hence, by (1.29), (1.17), and (1.26),

$$\begin{aligned} |x - x'| &< \tfrac{1}{8}|s - t| \leq \tfrac{1}{4}|\xi - y| \\ &\leq \tfrac{1}{4} \cdot 2|x - y| = \tfrac{1}{2}|x - y|. \end{aligned}$$

Finally, since $y \in \Sigma_2$ and, as shown above, $\xi' \in \Sigma_1$, from (1.17) and (1.29) it follows that

$$\begin{aligned}|\xi' - y| &\le |\xi - y| + |\xi - \xi'| \le |\xi - y| + |s - s'|\\ &\le |\xi - y| + 8|x - x'| < |\xi - y| + |s - t|,\end{aligned}$$

so, by (1.17),

$$|\xi' - y| \le |\xi - y| + 2|\xi - y| = 3|\xi - y|. \qquad \square$$

1.12 Lemma. *Let $x,\ x' \in S_{r/4}$, where r satisfies* (1.13),

$$|x - x'| < \tfrac{1}{4}\, r,$$

and $\Sigma_{\xi,r}$ is defined by (1.16). *Then for all $y \in \partial S \setminus \Sigma_{\xi,r}$,*

$$|x - x'| < \tfrac{1}{3}\, |x - y|, \tag{1.33}$$

$$|x - y| > \tfrac{3}{4}\, |\xi - y|, \tag{1.34}$$

$$|x' - y| > \tfrac{1}{2}\, |\xi - y|, \tag{1.35}$$

$$|\xi' - y| < 2|\xi - y|. \tag{1.36}$$

Proof. Given that $y \notin \Sigma_{\xi,r}$, we have

$$|\xi - y| > r. \tag{1.37}$$

Also, since $x \in S_{r/4}$,

$$|x - y| \ge |\xi - y| - |x - \xi| > r - \tfrac{1}{4}\, r = \tfrac{3}{4}\, r > 3|x - x'|.$$

Similarly, by (1.37) and the fact that $x \in S_{r/4}$,

$$\begin{aligned}|x - y| &\ge |\xi - y| - |x - \xi| \ge |\xi - y| - \tfrac{1}{4}\, r\\ &> |\xi - y| - \tfrac{1}{4}\, |\xi - y| = \tfrac{3}{4}\, |\xi - y|.\end{aligned}$$

Next, using (1.34) and (1.37), we deduce that

$$\begin{aligned}|x' - y| &\ge |x - y| - |x - x'| > |x - y| - \tfrac{1}{4}\, r\\ &> \tfrac{3}{4}\, |\xi - y| - \tfrac{1}{4}\, |\xi - y| = \tfrac{1}{2}\, |\xi - y|.\end{aligned}$$

Finally, by (1.24) and (1.37), we have $|\xi - \xi'| < 4|x - x'|$, so

$$\begin{aligned}|\xi' - y| &\le |\xi - y| + |\xi - \xi'| < |\xi - y| + 4|x - x'|\\ &< |\xi - y| + r < |\xi - y| + |\xi - y| = 2|\xi - y|. \qquad \square\end{aligned}$$

1.13 Lemma. *With the notation in Lemmas* 1.10 *and* 1.11, *if* $x,\ x' \in S_{r/4}$ *satisfy*

$$0 < |x - x'| < \frac{1}{16}\, r,$$
$$r \le \min\left\{\frac{1}{2},\ \frac{1}{8\kappa_0}\right\},$$

then there are constants $c_1,\ c_2,\ c_3,\ c_4 > 0$ *such that*

$$\int\limits_{\Sigma_1} \frac{1}{|x - y|^{\gamma}}\, ds(y) \le c_1 |x - x'|^{1-\gamma} \quad \forall \gamma \in [0, 1), \tag{1.38}$$

$$\int\limits_{\Sigma_1} \frac{1}{|x' - y|^{\gamma}}\, ds(y) \le c_2 |x - x'|^{1-\gamma} \quad \forall \gamma \in [0, 1), \tag{1.39}$$

$$\int\limits_{\Sigma_2} \frac{1}{|x - y|^{\gamma+1}}\, ds(y) \le \frac{c_3}{|x - x'|^{\gamma}} \quad \forall \gamma \in (0, 1), \tag{1.40}$$

$$\int\limits_{\Sigma_2} \frac{1}{|x - y|}\, ds(y) < c_4 \big|\ln |x - x'|\big|, \tag{1.41}$$

where c_1, c_2, and c_3 *depend only on* γ.

Proof. Let $\delta = |x - x'|$, let x and x' be given by (1.23), and let $s,\ s'$, and t be, as before, the arc length coordinates of $\xi,\ \xi'$, and y, respectively. We make the notation

$$\begin{aligned} \Gamma_1 &= \{t : y(t) \in \Sigma_1\} = \{t : |s - t| \le 8\delta\}, \\ \Gamma_2 &= \{t : y(t) \in \Sigma_2\} = \{t : y \in \Sigma_{\xi,r},\ |s - t| > 8\delta\}. \end{aligned} \tag{1.42}$$

For $y \in \Sigma_1 \subset \Sigma_{\xi,r}$, by (1.26) and (1.17),

$$|x - y| \ge \tfrac{1}{2}\, |\xi - y| \ge \tfrac{1}{4}\, |s - t|,$$

so, taking into account the fact that $s - 8\delta < t < s + 8\delta$ for $t \in \Gamma_1$, we find that

$$\begin{aligned} \int\limits_{\Sigma_1} \frac{1}{|x - y|^{\gamma}}\, ds(y) &\le 4^{\gamma} \int\limits_{\Gamma_1} \frac{1}{|s - t|^{\gamma}}\, dt = 4^{\gamma} \int\limits_{s-8\delta}^{s+8\delta} \frac{1}{|s - t|^{\gamma}}\, dt \\ &= 4^{\gamma} \left\{ \int\limits_{s-8\delta}^{s} \frac{1}{(s - t)^{\gamma}}\, dt + \int\limits_{s}^{s+8\delta} \frac{1}{(t - s)^{\gamma}}\, dt \right\} \\ &= \frac{4^{\gamma}}{1 - \gamma} \cdot 2(8\delta)^{1-\gamma} = \frac{2^{4-\gamma}}{1 - \gamma}\, |x - x'|^{1-\gamma}. \end{aligned}$$

Once again, for $y \in \Sigma_1 \subset \Sigma_{\xi,r}$, by (1.17),

$$|\xi - y| \le |s - t| \le 8|x - x'| \\ < 8 \cdot \tfrac{1}{16} r = \tfrac{1}{2} r,$$

so $y \in \Sigma_{\xi,r/2}$, which, according to Lemma 1.10(ii), implies that $y \in \Sigma_{\xi',r}$ and, therefore, by (1.28), $|x' - y| \ge \frac{1}{4}|s' - t|$. Also, by Lemma 1.11, $\xi' \in \Sigma_1$, which, by (1.29), implies that $|s - s'| \le 8|x - x'| = 8\delta$. As above, we then have

$$\int_{\Sigma_1} \frac{1}{|x' - y|^\gamma}\, ds(y) \le 4^\gamma \int_{\Gamma_1} \frac{1}{|s' - t|^\gamma}\, dt = 4^\gamma \Bigg\{ \int_{s-8\delta}^{s'} \frac{1}{(s' - t)^\gamma}\, dt + \int_{s'}^{s+8\delta} \frac{1}{(t - s')^\gamma}\, dt \Bigg\} \\ = \frac{4^\gamma}{1-\gamma} \cdot 2(s' - s + 8\delta)^{1-\gamma} \le \frac{2^{5-2\gamma}}{1-\gamma} |x - x'|^{1-\gamma}.$$

Let s_a and s_b, $s_a < s < s_b$, be the arc length coordinates of the end-points a and b of $\Sigma_{\xi,r}$. For $y \in \Sigma_2 \subset \Sigma_{\xi,r}$, by (1.26) and (1.17), we have

$$|x - y| \ge \tfrac{1}{2}|\xi - y| \ge \tfrac{1}{4}|s - t|;$$

hence, proceeding as above, we find that

$$\int_{\Sigma_2} \frac{1}{|x - y|^{1+\gamma}}\, ds(y) \le 4^{1+\gamma} \int_{\Gamma_2} \frac{1}{|s - t|^{1+\gamma}}\, dt \\ = 4^{1+\gamma} \Bigg\{ \int_{s_a}^{s-8\delta} \frac{1}{(s - t)^{1+\gamma}}\, dt + \int_{s+8\delta}^{s_b} \frac{1}{(t - s)^{1+\gamma}}\, dt \Bigg\} \\ = \frac{4^{1+\gamma}}{\gamma} \left\{ \frac{2}{(8\delta)^\gamma} - \frac{1}{(s - s_a)^\gamma} - \frac{1}{(s_b - s)^\gamma} \right\} \le \frac{2^{3-\gamma}}{\gamma} \frac{1}{|x - x'|^\gamma}.$$

The last integral is evaluated in the same way and yields

$$\int_{\Sigma_2} \frac{1}{|x - y|}\, ds(y) = 4\Big\{ -2\ln\delta - 2\ln 8 + \ln(s - s_a) + \ln(s - s_b) \Big\}.$$

Since $r \le 1/2$, it follows that $\ln|x - x'| \le \ln(r/16) \le \ln(1/32) < 0$. Also, by (1.17), $s - s_a \le 2|\xi - a| = 2r < 1$, so $\ln(s - s_a) < 0$, with a similar inequality for $s_b - s$; therefore,

$$\int_{\Sigma_2} \frac{1}{|x - y|^\gamma}\, ds(y) < -8\ln|x - x'| = 8\big|\ln|x - x'|\big|. \qquad \square$$

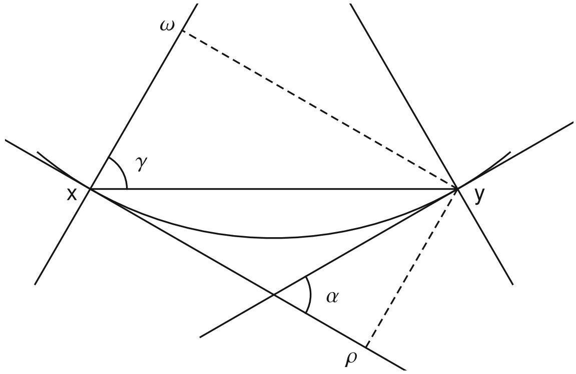

Fig. 1.10 The local frame of coordinates with the center at x

1.14 Remark. It is obvious that all the conditions in Lemmas 1.3, 1.4, 1.7, and 1.9–1.13 are satisfied if, for example, we choose $x,\ x' \in S_{r/4}$ such that

$$0 < |x - x'| < \frac{1}{16}\, r, \quad 0 < r \le \min\left\{\frac{1}{2},\ \frac{1}{8\kappa_0}\right\}. \tag{1.43}$$

From now on, we work under this assumption and use the notation

$$\begin{aligned} S_0 &= S_{r/4}, \\ S_0^+ &= \left\{x \in S_0 : x = \xi + \sigma\nu(\xi),\ \xi \in \partial S,\ -\tfrac{1}{4}\, r \le \sigma < 0\right\}, \\ S_0^- &= S_0 \setminus \bar{S}_0^+. \end{aligned} \tag{1.44}$$

1.15 Remark. For $x \in \partial S$, we introduce local coordinates $(\rho,\ \omega)$ along the positive tangent and inward normal to ∂S at x, respectively. Since ∂S is a simple C^2-curve, in accordance with Remark 1.5 there is a function f twice continuously differentiable on some closed interval $[\rho_l, \rho_r] \subseteq [-r, r]$ satisfying $f(0) = f'(0) = 0$ and such that the equation of the arc $\Sigma_{x,r}$ can be written in the form $\omega = f(\rho)$. In this local frame, x is the origin and a point $y \in \Sigma_{x,r}$ has coordinates $(\rho, \omega) = (\rho, f(\rho))$ (see Fig. 1.10).

From Lemma 1.3, the formula for the curvature of a plane curve, and (1.9), we then readily deduce that for all $y \in \Sigma_{x,r}$,

$$\begin{aligned} |f(\rho)| &= |x - y| \cos\gamma \le \frac{\sqrt{3}}{2}\, r, \\ |f'(\rho)| &= \tan\alpha \le \sqrt{3}, \\ |f''(\rho)| &\le \kappa_0\big[1 + f'^2(\rho)\big]^{3/2} \le 8\kappa_0. \end{aligned}$$

Given that $|\rho| \le r$ and in view of (1.43), this allows us to conclude that there is $c = \text{const} > 0$ depending only on ∂S and such that for all $x,\ y \in \partial S$,

$$|\rho| \le c, \quad |f(\rho)| \le c, \quad |f'(\rho)| \le c, \quad |f''(\rho)| \le c.$$

1.16 Theorem. *If $x,\ y \in \partial S$ and s and t are the arc length coordinates of x and y, then*

$$\frac{|x-y|}{|s-t|} \to 1 \quad \text{as } y \to x, \text{ uniformly on } \partial S.$$

Proof. Without loss of generality, we may assume that $y \in \Sigma_{x,r}$. Referring to the local coordinates (ρ, ω) and function f introduced in Remark 1.15, we expand f and f' in Taylor series with remainder and find that

$$\begin{aligned} f(\rho) &= f(0) + \rho f'(0) + \tfrac{1}{2}\rho^2 f''(\rho_1) = \tfrac{1}{2}\rho^2 f''(\rho_1),\\ f'(\rho) &= f'(0) + \rho f''(\rho_2) = \rho f''(\rho_2), \end{aligned} \tag{1.45}$$

with ρ_1 and ρ_2 between 0 and ρ.

For $\rho \in [\rho_l, \rho_r]$, consider the functions

$$\begin{aligned} f_1(\rho) &= |x-y| = \left[\rho^2 + f^2(\rho)\right]^{1/2},\\ f_2(\rho) &= t - s = \int_0^\rho \left[1 + f'^2(\theta)\right]^{1/2} d\theta. \end{aligned} \tag{1.46}$$

By (1.45),

$$\frac{f(\rho)}{\rho} = \tfrac{1}{2}\rho f''(\rho_1);$$

hence, the left-hand side is well defined at $\rho = 0$, where it takes the value 0. Then from (1.46),

$$\begin{aligned} f_1'(\rho) &= \frac{\rho + f(\rho)f'(\rho)}{[\rho^2 + f^2(\rho)]^{1/2}} = \frac{1 + \dfrac{f(\rho)}{\rho} f'(\rho)}{\left[1 + \left(\dfrac{f(\rho)}{\rho}\right)^2\right]^{1/2}},\\ f_2'(\rho) &= [1 + f'^2(\rho)]^{1/2}, \end{aligned} \tag{1.47}$$

so

$$f_1'(0) = 1, \quad f_2'(0) = 1$$

and we have

$$\begin{aligned} f_1(\rho) &= f_1(0) + \rho f_1'(0) + \tfrac{1}{2}\rho^2 f_1''(\rho') = \rho + \tfrac{1}{2}\rho^2 f_1''(\rho'),\\ f_2(\rho) &= f_2(0) + \rho f_2'(0) + \tfrac{1}{2}\rho^2 f_2''(\rho') = \rho + \tfrac{1}{2}\rho^2 f_2''(\rho''), \end{aligned} \tag{1.48}$$

with ρ' and ρ'' between 0 and ρ.

Differentiating the functions in (1.47) again and then replacing f and f' by their expressions (1.45), after a long but straightforward computation we arrive at

$$\begin{aligned} f_1''(\rho) &= \frac{\rho\big[\frac{1}{4}\, f''^2(\rho_1) + f''^2(\rho_2) + \frac{1}{2}\, f''(\rho_1) f''(\rho)\big]}{\big[1 + \frac{1}{4}\,\rho^2 f''^2(\rho_1)\big]^{3/2}} \\ &\quad + \frac{\rho\big[-f''(\rho_1) f''(\rho_2) + \frac{1}{8}\,\rho^2 f''^3(\rho_1) f''(\rho)\big]}{\big[1 + \frac{1}{4}\,\rho^2 f''^2(\rho_1)\big]^{3/2}}, \\ f_2''(\rho) &= \frac{f'(\rho) f''(\rho)}{\big[1 + f'^2(\rho)\big]^{1/2}}; \end{aligned} \tag{1.49}$$

therefore, by Remark 1.15, there is $c = \text{const} > 0$ depending only on ∂S and such that

$$|\rho| \le c, \quad |f_1''(\rho)| \le c, \quad |f_2''(\rho)| \le c$$

for all $\rho \in [\rho_l, \rho_r]$ (that is, for all $y \in \Sigma_{x,r}$) and all $x \in \partial S$. This implies that

$$|\rho f_\alpha''(\rho)| \le c|\rho| \to 0 \quad \text{as } \rho \to 0, \quad \alpha = 1, 2. \tag{1.50}$$

Consequently, by (1.46), (1.48), and (1.50),

$$\begin{aligned} \frac{|x-y|}{|s-t|} = \frac{f_1(\rho)}{|f_2(\rho)|} &= \frac{\rho + \frac{1}{2}\,\rho^2 f_1{}''(\rho')}{|\rho + \frac{1}{2}\,\rho^2 f_2{}''(\rho'')|} \\ &= \frac{|1 + \frac{1}{2}\,\rho f_1{}''(\rho')|}{|1 + \frac{1}{2}\,\rho f_2{}''(\rho'')|} \to 1 \quad \text{as } \rho \to 0. \end{aligned}$$ □

1.17 Theorem. *With the notation in Theorem* 1.16,

$$\frac{|x-y|}{\rho} \to 1 \quad \textit{as } y \to x, \quad \textit{uniformly on } \partial S.$$

Proof. As above, we find that

$$\frac{|x-y|}{\rho} = \frac{f_1(\rho)}{\rho} = 1 + \tfrac{1}{2}\,\rho f_1{}''(\rho') \to 1 \quad \text{as } \rho \to 0.$$ □

1.18 Remark. If f is continuously differentiable in S_0, then we can write

$$\operatorname{grad} f(x) = \left[\tau(x)\frac{\partial}{\partial s(x)} + \nu(x)\frac{\partial}{\partial \nu(x)}\right] f(x), \quad x \in \partial S, \tag{1.51}$$

where the notation $\partial/\partial s$ is preferred to $\partial/\partial \tau$.

1.19 Remark. According to the definition of S_0, the mapping from the set

$$\big\{\{\xi,\sigma\} : \xi \in \partial S,\ |\sigma| \le \tfrac{1}{4}r\big\}$$

to S_0 defined by

$$\{\xi,\sigma\} \mapsto x = \xi + \sigma\nu(\xi)$$

is a bijection.

Let g be a continuously differentiable function on S_0. Since at any point $x \in S_0$ the associated pair $\{\xi,\sigma\}$ is uniquely determined, we can extend the definition of the normal derivative $\partial g/\partial\nu(x)$ at $x \in \partial S$ to any point $x \in S_0$ by writing

$$\frac{\partial}{\partial\nu}g(x) = \langle(\operatorname{grad} g)(x), \nu(\xi)\rangle, \quad x \in S_0.$$

1.4 Integrals with Singular Kernels

Let $C(S)$ and $C^1(S)$, respectively, be the spaces of (real) continuous and continuously differentiable functions in S. We consider the set of all functions in $C(S)$ ($C^1(S)$) that are continuously extendable (continuously extendable together with their first-order derivatives) to $\bar{S} = S \cup \partial S$, and denote by $C(\bar{S})$ ($C^1(\bar{S})$) the space of the corresponding extensions. The following assertion shows that this notation is justified.

1.20 Theorem. *Let $f \in C^1(S)$, and suppose that*

$$f(x) \to l(\xi), \quad \operatorname{grad} f(x) \to \lambda(\xi) \quad \text{as } S \ni x \to \xi \in \partial S,$$

where l and λ are continuous on ∂S. Then the function

$$\tilde{f}(x) = \begin{cases} f(x), & x \in S, \\ l(x), & x \in \partial S, \end{cases}$$

has one-sided derivatives at all $x \in \partial S$ and

$$\operatorname{grad}\tilde{f}(x) = \begin{cases} \operatorname{grad} f(x), & x \in S, \\ \lambda(x), & x \in \partial S; \end{cases}$$

(that is, the operations of differentiation and extension to $\bar{S}$ commute for f).

Proof. Let $\lambda(x) = \big(\lambda_1(x), \lambda_2(x)\big)$. It is clear that $\tilde{f} \in C(\bar{S}) \cap C^1(S)$. Consequently, for $\xi = (\xi_1, \xi_2) \in \partial S$ and $x = (x_1, \xi_2) \in S$, $x_1 \ne \xi_1$, in a sufficiently small neighborhood of ξ we have

$$\left| \frac{\tilde{f}(x) - \tilde{f}(\xi)}{x_1 - \xi_1} - \lambda_1(\xi) \right| = \left| \frac{\partial}{\partial x_1} \tilde{f}(\eta) - \lambda_1(\xi) \right|$$
$$= \left| \frac{\partial}{\partial x_1} f(\eta) - \lambda_1(\xi) \right|,$$

where $\eta = (\eta_1, \xi_2)$ with η_1 between x_1 and ξ_1. The result for $\partial\tilde{f}/\partial x_1$ now follows from the fact that the right-hand side tends to zero as $x \to \xi$. The argument for $\partial\tilde{f}/\partial x_2$ is similar. □

1.21 Remark. The above spaces are also introduced for functions defined on ∂S. Let $f(x)$ be such a function, and let s be the arc length coordinate of x. Then for simplicity we also write $f(s) \equiv f\big(x(s)\big)$. In this case, the derivative of f is defined by

$$f'(s) = \lim_{t \to s} \frac{f(t) - f(s)}{t - s},$$

where $s, t \in [0, l]$, provided that the limit exists. We specify that in what follows the notation f' for the derivative does not extend to position vectors. Thus, x' will denote a point on ∂S and not dx/ds.

Clearly, if f is defined and differentiable on a domain that includes ∂S, then the derivative along ∂S of the restriction of f to ∂S coincides with $\langle \operatorname{grad} f(x), \tau(x) \rangle$.

1.22 Definition. A function f defined on $\bar{S}$ is said to be *Hölder continuous* (with index $\alpha \in (0, 1]$) on $\bar{S}$ if

$$|f(x) - f(y)| \le c|x - y|^\alpha \quad \text{for all} \quad x, y \in \bar{S}, \tag{1.52}$$

where $c = \text{const} > 0$ is independent of x and y. If S is unbounded, then the above definition must hold on every bounded subdomain of S.

We denote by $C^{0,\alpha}(\bar{S})$ the vector space of (real) Hölder continuous (with index $\alpha \in (0, 1]$) functions on $\bar{S}$, and by $C^{1,\alpha}(\bar{S})$ the subspace of $C^1(\bar{S})$ of functions whose first-order derivatives belong to $C^{0,\alpha}(\bar{S})$.

1.23 Lemma. *If* $0 < \beta < \alpha \le 1$, *then*

(i) $C^{0,\alpha}(\bar{S}) \subset C^{0,\beta}(\bar{S})$;
(ii) $fg \in C^{0,\beta}(\bar{S})$ *for all* $f \in C^{0,\alpha}(\bar{S})$ *and* $g \in C^{0,\beta}(\bar{S})$.

The proof consists in the verification of (1.52).

The spaces $C^{0,\alpha}(\partial S)$ and $C^{1,\alpha}(\partial S)$ are introduced similarly, with (1.52) required to hold for all $x, y \in \partial S$. In view of Lemma 1.4, we will not distinguish between $C^{0,\alpha}(\partial S)$ and $C^{0,\alpha}[0, l]$, which is defined by means of the inequality

$$|f(s) - f(t)| \le c|s - t|^\alpha \quad \text{for all} \quad s, t \in [0, l].$$

Obviously, Lemma 1.23 also holds for functions on ∂S.

1.24 Remark. If f is bounded in $\bar{S}$, that is, $|f(x)| \le M = \text{const}$ for all $x \in \bar{S}$, and (1.52) holds for all $x, y \in \bar{S}$ such that $|x - y| \le \delta$, where $\delta = \text{const} > 0$, then it holds (possibly with a different c) for all $x, y \in \bar{S}$. This is easily shown, since for $|x - y| > \delta$ we can write

$$|f(x) - f(y)| \le 2M < \frac{2M}{\delta^{\alpha}}|x - y|^{\alpha}.$$

1.25 Remark. If $\varphi \in C^{0,\alpha}(\partial S)$ as a function of x and $x = x(s)$, then, by Lemma 1.4, $\varphi \in C^{0,\alpha}(\partial S)$ also as a function of s, and vice versa.

1.26 Definition. A two-point function $k(x, y)$ defined and continuous for all $x \in S_0$ ($x \in \partial S$) and $y \in \partial S$, $x \ne y$, is called a *γ-singular kernel in S_0 (on ∂S)*, $\gamma \in [0, 1]$, if there is $p = \text{const} > 0$, which may depend on ∂S, such that for all $x \in S_0$ ($x \in \partial S$) and $y \in \partial S$, $x \ne y$,

$$|k(x, y)| \le \frac{p}{|x - y|^{\gamma}}.$$

If the above inequality holds and, in addition, for all $x, x' \in S_0$ ($x, x' \in \partial S$) and $y \in \partial S$ satisfying $0 < |x - x'| < \frac{1}{2}|x - y|$ we have

$$|k(x, y) - k(x', y)| \le p\,\frac{|x - x'|}{|x - y|^{\gamma+1}},$$

then $k(x, y)$ is called a *proper γ-singular kernel in S_0 (on ∂S)*.

We extend this definition to two-point matrix functions by requiring each component to satisfy the necessary properties.

1.27 Remark. A kernel may have a lower 'singularity index' γ when it is considered on ∂S rather than in S_0. For example, the function $k(x, y) = \partial\lambda/\partial\nu(y)$ is a proper 1-singular kernel in S_0, but, by Lemma 1.2, a proper 0-singular kernel on ∂S.

1.28 Lemma. *If $k(x, y)$ is γ-singular in S_0, $\gamma \in [0, 1]$, and continuously differentiable with respect to x_α for all $x \in S_0$ and $y \in \partial S$, $x \ne y$, and if the kernels $|x - y|[\partial k(x, y)/\partial x_\alpha]$ are γ-singular in S_0, then $k(x, y)$ is a proper γ-singular kernel in S_0.*

Proof. Let $x, x' \in S_0$ and $y \in \partial S$ be such that $0 < |x - x'| < \frac{1}{2}|x - y|$. For any x'' on the line between x and x' we have

$$\begin{aligned} |x'' - y| &\ge |x - y| - |x - x''| \\ &> |x - y| - \tfrac{1}{2}|x - y| \\ &= \tfrac{1}{2}|x - y|; \end{aligned}$$

consequently,

$$|k(x,\, y) - k(x',\, y)| \le |x_\alpha - x'_\alpha| \left| \frac{\partial}{\partial x_\alpha} k(x'',\, y) \right| \le p'|x - x'|\, |x - y\,|^{-\gamma-1},$$

where $p' = \text{const}$ depends only on γ. □

1.29 Remark. If $k(x,\, y)$ is a γ-singular kernel on ∂S, $\gamma \in [0, 1]$, and continuously differentiable with respect to the arc length coordinate $s(x)$ of x at all $x,\, y \in \partial S$, $x \ne y$, and if $|x - y\,|[\partial k(x,\, y)/\partial s(x)]$ is γ-singular on ∂S, then $k(x,\, y)$ is a proper γ-singular kernel on ∂S. The proof of this statement is similar to that of Lemma 1.28, use also being made of Remark 1.6.

The following assertion is proved by direct verification of the required properties.

1.30 Lemma. (i) *If* $k_1(x,\, y)$ *is* 0*-singular and* $k_2(x,\, y)$ *is* γ*-singular,* $\gamma \in [0, 1]$*, then* $k_1(x,\, y)k_2(x,\, y)$ *is* γ*-singular.*

(ii) *If* $k_1(x,\, y)$ is γ_1*-singular and* $k_2(x,\, y)$ *is* γ_2*-singular,* $0 \le \gamma_1 \le \gamma_2 \le 1$*, then* $k_1(x,\, y) + k_2(x,\, y)$ *is* γ_2*-singular.*

1.31 Remark. Lemma 1.30 also holds with ‘singular’ replaced by ‘proper singular’ in its statement.

1.32 Theorem. *If* $k(x,\, y)$ *is a* γ*-singular kernel on* ∂S*,* $\gamma \in [0, 1)$*, then the function*

$$f(x) = \int\limits_{\partial S} k(x,\, y)\, ds(y) \tag{1.53}$$

is continuous on ∂S*.*

Proof. Let $x,\, a,\, b,\, y \in \partial S$ have arc length coordinates $s,\, s - \varepsilon_1,\, s + \varepsilon_2,\, t$, respectively, with $\varepsilon_1,\, \varepsilon_2 > 0$ arbitrarily small, and let

$$I_\varepsilon(s) = \int\limits_b^a \frac{1}{|s - t|^\gamma}\, dt,$$

$$I(s) = \int\limits_{\partial S} \frac{1}{|s - t|^\gamma}\, dt.$$

Clearly,

$$|I(s) - I_\varepsilon(s)| = \frac{1}{1-\gamma}\, (\varepsilon_1^{1-\gamma} + \varepsilon_2^{1-\gamma}),$$

so $I_\varepsilon(s) \to I(s)$ uniformly with respect to s as $\varepsilon_1,\, \varepsilon_2 \to 0$. Since, by Definition 1.26 and Lemma 1.4,

$$
\begin{aligned}
|k(x,\ y)| &\le \frac{c}{|x-y\,|^{\gamma}}\\
&\le \frac{c}{|s-t|^{\gamma}}
\end{aligned}
$$

for all $x,\ y \in \partial S$, $x \ne y$, the improper integral (1.53) converges uniformly with respect to $x \in \partial S$, and the assertion follows from a well-known theorem of analysis (see, for example, Smirnov 1964). □

1.33 Theorem. *If $k(x,\ y)$ is a proper γ-singular kernel in S_0 (on ∂S), $\gamma \in [0, 1)$, and $\varphi \in C(\partial S)$, then the function*

$$
K(x) = \int\limits_{\partial S} k(x,\ y)\varphi(y)\,ds(y), \quad x \in S_0 \ \ (x \in \partial S\,),
$$

belongs to $C^{0,\beta}(S_0)$, with $\beta = 1-\gamma$ for $\gamma \in (0, 1)$ and any $\beta \in (0, 1)$ for $\gamma = 0$. In addition,

$$
\sup_{\substack{x,x' \in S_0\ (\partial S)\\ x \ne x'}} \frac{|K(x) - K(x')|}{|x-x'|^{\beta}} \le c \sup_{x\in\partial S} |\varphi(x)|,
$$

where $c = \mathrm{const} > 0$ depends only on γ.

Proof. It is obvious that $K(x)$ is an improper integral for $x \in \partial S$.

Let $\Sigma_{x,r}$, Σ_1, and Σ_2 be defined by (1.16) and (1.29). In view of Remark 1.24, we may consider $x,\ x' \in S_0$ satisfying (1.43).

Setting, as before,

$$
x = \xi + \sigma\nu(\xi), \quad x' = \xi' + \sigma'\nu(\xi'), \quad \xi,\ \xi' \in \partial S,
$$

we can write

$$
K(x) - K(x') = I_1 + I_2 + I_3,
$$

where, by Definition 1.22, Remark 1.14, and Lemmas 1.10–1.13,

$$
\begin{aligned}
|I_1| &= \left| \int\limits_{\Sigma_1} \big[k(x,\ y) - k(x',\ y)\big]\varphi(y)\,ds(y)\right|\\
&\le c_1 \sup_{x\in\partial S} |\varphi(x)| \int\limits_{\Sigma_1} \left(\frac{1}{|x-y\,|^{\gamma}} + \frac{1}{|x'-y\,|^{\gamma}}\right) ds(y)\\
&\le c_2 |x-x'|^{1-\gamma} \sup_{x\in\partial S} |\varphi(x)|,
\end{aligned}
$$

$$
\begin{aligned}
|I_2| &= \left| \int\limits_{\Sigma_2} \big[k(x,\ y) - k(x',\ y)\big]\varphi(y)\, ds(y) \right| \\
&\le c_3|x - x'| \sup_{x\in\partial S} |\varphi(x)| \int\limits_{\Sigma_2} \frac{1}{|x - y\,|^{\gamma+1}}\, ds(y) \\
&\le c_4|x - x'|^{1-\gamma} \sup_{x\in\partial S} |\varphi(x)| \quad \text{if } \gamma \in (0,\ 1), \\
|I_2| &\le c_5|x - x'|\big|\ln|x - x'|\big| \sup_{x\in\partial S} |\varphi(x)| \quad \text{if } \gamma = 0, \\
|I_3| &= \left| \int\limits_{\partial S\setminus\Sigma_{\xi,r}} \big[k(x,\ y) - k(x',\ y)\big]\varphi(y)\, ds(y) \right| \\
&\le c_6|x - x'| \sup_{x\in\partial S} |\varphi(x)| \int\limits_{\partial S\setminus\Sigma_{\xi,r}} \frac{1}{|x - y\,|^{\gamma+1}}\, ds(y) \\
&\le c_7 r^{-\gamma-1} l\, |x - x'| \sup_{x\in\partial S} |\varphi(x)| \\
&= c_8|x - x'| \sup_{x\in\partial S} |\varphi(x)|,
\end{aligned}
$$

where l is the length of the boundary curve ∂S. The assertion now follows from the fact that the constants $c_1, \dots, c_8 > 0$ are independent of x and x' (although they depend on γ).

The result is established for $x,\ x' \in \partial S$ as a particular case of the above, by setting $x = \xi$ and $x' = \xi'$. □

1.34 Remark. It is obvious that Theorem 1.33 holds if the kernel $k(x,\ y)$ is continuous on $S_0 \times \partial S$ $(\partial S \times \partial S)$.

1.35 Theorem. *If $k(x,\ y)$ is a proper 1-singular kernel in S_0 (on ∂S), $\varphi \in C^{0,\alpha}(\partial S)$, $\alpha \in (0,\ 1]$, and*

$$
\Phi(x) = \int\limits_{\partial S} k(x,\ y)\big[\varphi(y) - \varphi(\xi)\big]\, ds(y), \tag{1.54}
$$

where $x = \xi + \sigma\nu(\xi) \in S_0$ $(x = \xi \in \partial S)$, then $\Phi \in C^{0,\beta}(S_0)$ $(\Phi \in C^{0,\beta}(\partial S))$ for any $\beta \in (0,\ \alpha)$. If, in addition, $\alpha \in (0,\ 1)$ and

$$
\left| \int\limits_{\partial S\setminus\Sigma_{\xi,\delta}} k(x,\ y)\, ds(y) \right| \le c = \text{const} > 0 \tag{1.55}
$$

for all $x \in S_0$ $(x \in \partial S)$ and all $0 < \delta < r$, then $\Phi \in C^{0,\alpha}(S_0)$ $(\Phi \in C^{0,\alpha}(\partial S))$.

Proof. Clearly, Φ exists as an improper integral if $x \in \partial S$, and, by Theorem 1.32, is continuous on ∂S.

As in the proof of Theorem 1.33, let $x,\ x' \in S_0$ be chosen so that (1.43) holds. Writing

$$
\begin{aligned}
\Phi(x) - \Phi(x')& \\
&= \int_{\Sigma_1} \bigl\{k(x,\ y)\bigl[\varphi(y) - \varphi(\xi)\bigr] - k(x',\ y)\bigl[\varphi(y) - \varphi(\xi')\bigr]\bigr\}\, ds(y) \\
&\quad + \int_{\Sigma_2} \bigl\{\bigl[k(x,\ y) - k(x',\ y)\bigr]\bigl[\varphi(y) - \varphi(\xi')\bigr] \\
&\qquad\qquad - k(x,\ y)\bigl[\varphi(\xi) - \varphi(\xi')\bigr]\bigr\}\, ds(y) \\
&\quad + \int_{\partial S\setminus\Sigma_{\xi,r}} \bigl\{\bigl[k(x,\ y) - k(x',\ y)\bigr]\bigl[\varphi(y) - \varphi(\xi')\bigr] \\
&\qquad\qquad - k(x,\ y)\bigl[\varphi(\xi) - \varphi(\xi')\bigr]\bigr\}\, ds(y) \\
&= I_1 + I_2 + I_3,
\end{aligned}
$$

from Definition 1.22, Remark 1.14, and Lemma 1.13 we now find that

$$
\begin{aligned}
|I_1| &\le c_1 \int_{\Sigma_1} \left(\frac{1}{|\xi - y\,|^{1-\alpha}} + \frac{1}{|\xi' - y\,|^{1-\alpha}}\right) ds(y) \\
&\le c_2|x - x'|^{\alpha}, \\
|I_2| &\le c_3|x - x'| \int_{\Sigma_2} \frac{1}{|\xi - y\,|^{2-\alpha}}\, ds(y) + c_4|\xi - \xi'|^{\alpha} \int_{\Sigma_2} \frac{1}{|\xi - y\,|}\, ds(y) \\
&\le c_5|x - x'|^{\alpha} + c_6|x - x'|^{\alpha}\bigl|\ln|x - x'|\bigr| \quad \text{for } \alpha \in (0, 1), \\
|I_2| &\le c_3|x - x'| \int_{\Sigma_2} \frac{1}{|\xi - y\,|}\, ds(y) + c_4|\xi - \xi'| \int_{\Sigma_2} \frac{1}{|\xi - y\,|}\, ds(y) \\
&\le c_7|x - x'|\bigl|\ln|x - x'|\bigr| \quad \text{for } \alpha = 1, \\
|I_3| &\le c_8|x - x'| \int_{\partial S\setminus\Sigma_{\xi,r}} \frac{1}{|\xi - y\,|^{2-\alpha}}\, ds(y) + c_9|\xi - \xi'|^{\alpha} \int_{\partial S\setminus\Sigma_{\xi,r}} \frac{1}{|\xi - y\,|}\, ds(y) \\
&\le c_8 \frac{l}{r^{2-\alpha}}\, |x - x'| + c_{10}\frac{l}{r}\, |x - x'|^{\alpha} \le c_{11}|x - x'|^{\alpha},
\end{aligned}
$$

where $c_1, \dots,\ c_{11}$ are positive constants independent of x and x'.

This proves the first part of the assertion. For the second part, we combine the last terms in I_2 and I_3 and use the fact that

$$\int\limits_{\partial S\setminus\Sigma_1} k(x,\ y)\, ds(y)$$

is bounded for all $x,\ x' \in S_0$ satisfying the conditions of the theorem. (See Remark 1.41 below for a full explanation of this detail.)

The result for $x,\ x' \in \partial S$ is again obtained by setting $x = \xi$ and $x' = \xi'$. □

1.36 Remark. By Theorem 1.32, estimate (1.55) holds on ∂S if $k(x,\ y)$ is a γ-singular kernel on ∂S, $\gamma \in [0, 1)$.

1.37 Theorem. *Let $k(x,\ y)$ be a β-singular kernel on ∂S, $\beta \in [0, 1)$, such that*

$$g(x) = \frac{\partial}{\partial s}\int\limits_{\partial S} k(x,\ y)\, ds(y)$$

exists for all $x \in \partial S$ and $g \in C(\partial S)$, and

$$\frac{k(x',\ y) - k(x,\ y)}{s' - s} = k_0(x,\ y) + O\left(\frac{|s' - s|}{|x - y|^{\gamma+2}}\right) \tag{1.56}$$

for all $x,\ x',\ y \in \partial S$ satisfying

$$0 < |x - x'| < \frac{1}{2}|x - y|,$$

where s and s' are the arc length coordinates of x and x', and $|x - y|k_0(x,\ y)$ is a γ-singular kernel on ∂S, $\gamma \in [0, 1)$.

If $\varphi \in C^{0,\alpha}(\partial S)$, $\alpha \in (\beta, 1]$, $\alpha > \gamma$, then the function

$$F(x) = \int\limits_{\partial S} k(x,\ y)\varphi(y)\, ds(y), \quad x \in \partial S,$$

belongs to $C^1(\partial S)$ and

$$\frac{\partial}{\partial s} F(x) = \int\limits_{\partial S} k_0(x,\ y)\big[\varphi(y) - \varphi(x)\big]\, ds(y) + \varphi(x)g(x). \tag{1.57}$$

Proof. Let $G(x)$ be the function on the right-hand side in (1.57). By Theorem 1.32, $F(x)$ and the first term in $G(x)$ exist as improper integrals and are continuous on ∂S; the second term in $G(x)$ is continuous by assumption.

Let $x,\ x' \in \partial S$ be such that $0 < |x - x'| < r/8$, with r satisfying (1.43). We have

$$
\begin{aligned}
&\frac{F(x') - F(x)}{s' - s} - G(x) \\
&\quad = \frac{1}{s' - s} \int\limits_{\Sigma_1} \big\{k(x',\ y)\big[\varphi(y) - \varphi(x')\big] - k(x,\ y)\big[\varphi(y) - \varphi(x)\big]\big\}\, ds(y) \\
&\qquad + \frac{\varphi(x') - \varphi(x)}{s' - s} \int\limits_{\Sigma_1} k(x',\ y)\, ds(y) \\
&\qquad - \int\limits_{\Sigma_1} k_0(x,\ y)\big[\varphi(y) - \varphi(x)\big]\, ds(y) \\
&\qquad + \int\limits_{\Sigma_2} \left\{\frac{k(x',\ y) - k(x,\ y)}{s' - s} - k_0(x,\ y)\right\}\big[\varphi(y) - \varphi(x)\big]\, ds(y) \\
&\qquad + \int\limits_{\partial S \backslash \Sigma_{x,r}} \left\{\frac{k(x',\ y) - k(x,\ y)}{s' - s} - k_0(x,\ y)\right\}\big[\varphi(y) - \varphi(x)\big]\, ds(y) \\
&\qquad + \varphi(x)\left\{\frac{1}{s' - s}\left[\int\limits_{\partial S} k(x',\ y)\, ds(y) - \int\limits_{\partial S} k(x,\ y)\, ds(y)\right] - g(x)\right\} \\
&\quad = I_1 + I_2 + I_3 + I_4 + I_5 + I_6.
\end{aligned}
$$

By Definition 1.22, Remark 1.14, and Lemmas 1.10 and 1.13,

$$
\begin{aligned}
|I_1| &\le \frac{c_1}{|s' - s|} \int\limits_{\Sigma_1} (|x' - y|^{\alpha-\beta} + |x - y|^{\alpha-\beta})\, ds(y) \le c_2|s' - s|^{\alpha-\beta}, \\
|I_2| &\le \frac{c_3}{|s' - s|^{1-\alpha}} \int\limits_{\Sigma_1} \frac{1}{|x' - y|^{\beta}}\, ds(y) \le c_4|s' - s|^{\alpha-\beta}, \\
|I_3| &\le c_5 \int\limits_{\Sigma_1} \frac{1}{|x - y|^{1+\gamma-\alpha}}\, ds(y) \le c_6|s' - s|^{\alpha-\gamma}.
\end{aligned}
$$

By Lemma 1.11, $y \in \Sigma_2$ implies that

$$
|x - x'| < \frac{1}{2}|x - y|;
$$

hence,

$$
|I_4| \le c_7|s' - s| \int\limits_{\Sigma_2} \frac{1}{|x - y|^{2+\gamma-\alpha}}\, ds(y) \le c_8|s' - s|^{\alpha-\gamma}.
$$

Finally, by Lemma 1.12,

$$|I_5| \le c_9|s' - s| \int\limits_{\partial S \setminus \Sigma_{x,r}} \frac{1}{|x - y|^{2+\gamma-\alpha}}\, ds(y) \le c_{10}|s' - s|.$$

Since all the constants $c_1, \dots, c_{10} > 0$ are independent of x and x', we find that $I_j \to 0$ as $s' - s \to 0$, $j = 1, \dots, 5$.

In addition, by our assumption (i), $I_6 \to 0$ as $s' - s \to 0$, which proves that $F'(x)$ exists for all $x \in \partial S$ and is given by (1.57), whose right-hand side is obviously a continuous function on ∂S. □

1.38 Remark. Under the conditions in Theorem 1.37, if $g \in C^{0,\alpha}(\partial S)$ and $k_0(x, y)$ is a proper 1-singular kernel on ∂S, then, by Theorem 1.35, $F \in C^{1,\beta}(\partial S)$ for any $\beta \in (0, \alpha)$. If, furthermore, $\alpha \in (0, 1)$ and $k_0(x, y)$ satisfies estimate (1.55), then $F \in C^{1,\alpha}(\partial S)$.

1.39 Remark. In practice it is helpful to have some easily checked condition in place of assumption (1.56) in Theorem 1.37. Suppose that $k(x, y)$ is continuously differentiable with respect to the arc length coordinate s of x at all points $x, y \in \partial S$, $x \ne y$, and that $|x - y|\big[\partial k(x, y)/\partial s(x)\big]$ is a proper γ-singular kernel on ∂S, $\gamma \in [0, 1)$. Then for $x, x', y \in \partial S$ such that $0 < |x - x'| < \frac{1}{2}|x - y|$,

$$\frac{k(x', y) - k(x, y)}{s' - s} = \frac{\partial}{\partial s(x)} k(x, y) + \left[\frac{\partial}{\partial s(x)} k(x'', y) - \frac{\partial}{\partial s(x)} k(x, y)\right],$$

where s' is the arc length coordinate of x' and $x'' \in \partial S$ lies between x and x'. Since

$$\left|\frac{\partial}{\partial s(x)} k(x'', y) - \frac{\partial}{\partial s(x)} k(x, y)\right| \le c\frac{|s - s'|}{|x - y|^{\gamma+2}},$$

it follows that under the above conditions, (1.56) holds with

$$k_0(x, y) = \frac{\partial}{\partial s(x)} k(x, y).$$

1.40 Definition. Let $k(x, y)$ be defined and continuous at all points $x, y \in \partial S$, $x \ne y$. We say that $\int\limits_{\partial S} k(x, y)\, ds(y)$ exists as *principal value* if

$$\lim_{\delta \to 0} \int\limits_{\partial S \setminus \Sigma_{x,\delta}} k(x, y)\, ds(y) \tag{1.58}$$

exists for all $x \in \partial S$.

Obviously, an ordinary (even improper) integral exists as principal value, but the converse is not true in general.

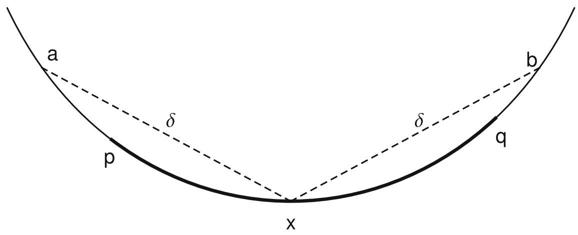

Fig. 1.11 The sets $\Sigma_{x,\delta}$ and $\Gamma_{x,\delta}$ (heavier arc)

In what follows, the principal value of an integral (if it exists) is denoted by the same symbol as an ordinary integral, the difference in meaning being either explicitly stated or understood from the context as the only possible alternative.

1.41 Remark. Let $k(x, y)$ be a 1-singular kernel on ∂S, and, with an earlier notation, let s_a, s_b and $s_p, s_q, s_a \le s_p < s_q \le s_b$, be the arc length coordinates of the end-points a, b and p, q of the sets $\Sigma_{x,\delta}$ and

$$\Gamma_{x,\delta} = \{y \in \partial S : |t - s| \le \delta\}, \tag{1.59}$$

respectively, where s and t are the arc length coordinates of x and y (see Fig. 1.11).

Since

$$\begin{aligned}
&\left| \int\limits_{\partial S \setminus \Gamma_{x,\delta}} k(x, y)\, ds(y) - \int\limits_{\partial S \setminus \Sigma_{x,\delta}} k(x, y)\, ds(y) \right| \\
&= \left| \int\limits_{\Sigma_{x,\delta} \setminus \Gamma_{x,\delta}} k(x, y)\, ds(y) \right| \le c \Bigg(\int\limits_{s_a}^{s_p} \frac{1}{|s - t|}\, dt + \int\limits_{s_q}^{s_b} \frac{1}{|s - t|}\, dt \Bigg) \\
&= c \ln \left(\frac{s - s_a}{s - s_p} \cdot \frac{s_b - s}{s_q - s} \right) = c \ln \left(\frac{s - s_a}{\delta} \cdot \frac{s_b - s}{\delta} \right),
\end{aligned}$$

Theorem 1.16 implies that if $\int\limits_{\partial S} k(x, y)\, ds(y)$ exists in the sense of principal value, then its definition can equivalently be given as

$$\lim_{\delta \to 0} \int\limits_{\partial S \setminus \Gamma_{x,\delta}} k(x, y)\, ds(y).$$

Moreover, if the limit (1.58) exists uniformly for all $x \in \partial S$, then so does the above one, and vice versa.

1.42 Remark. Let ρ be the local coordinate of $y \in \Sigma_{x,r}$ measured from x along the support line of $\tau(x)$ (see Remark 1.15), and consider the set

$$\Lambda_{x,\delta} = \{y \in \Sigma_{x,r} : |\rho| \le \delta\}, \quad \delta < \tfrac{1}{2} r.$$

Since $\delta < r/2$, all points in the neighborhood of x such that $|\rho| \le \delta$ belong to $\Sigma_{x,r}$. Denoting by $-a$ and b, $a,\ b > 0$, the ρ-coordinates of the end-points of $\Sigma_{x,\delta}$, we find that for a 1-singular kernel $k(x,\ y)$ on ∂S,

$$\begin{aligned}
&\Bigg| \int\limits_{\partial S\setminus\Sigma_{x,\delta}} k(x,\ y)\,ds(y) - \int\limits_{\partial S\setminus\Lambda_{x,\delta}} k(x,\ y)\,ds(y)\Bigg| \\
&\quad = \Bigg| \int\limits_{\Lambda_{x,\delta}\setminus\Sigma_{x,\delta}} k(x,\ y)\,ds(y)\Bigg| \le c_1 \int\limits_{\Lambda_{x,\delta}\setminus\Sigma_{x,\delta}} \frac{1}{|x-y|}\,ds(y) \\
&\quad \le c_2\Bigg(\int\limits_{-\delta}^{-a} -\frac{1}{\rho}\,d\rho + \int\limits_{b}^{\delta} \frac{1}{\rho}\,d\rho \Bigg) = c_2 \ln\Big(\frac{\delta}{a}\cdot\frac{\delta}{b}\Big),
\end{aligned}$$

where c_2 does not depend on x. Consequently, by Theorem 1.17, if

$$\int\limits_{\partial S} k(x,\ y)\,ds(y)$$

exists in the sense of principal value, then it can also be defined as

$$\lim_{\delta\to 0} \int\limits_{\partial S\setminus\Lambda_{x,\delta}} k(x,\ y)\,ds(y).$$

Furthermore, from Theorem 1.17 it follows that if either of these two equivalent limits exists uniformly with respect to $x \in \partial S$, then the other one has the same property.

1.43 Theorem. *Suppose that $k(x,\ y)$ is a proper 1-singular kernel in S_0 and, at the same time, a γ-singular kernel on ∂S, $\gamma \in [0,1)$, and let*

$$\begin{aligned}
f(x) &= \int\limits_{\partial S} k(x,\ y)\,ds(y), \quad x \in S_0 \setminus \partial S, \\
f_0(x) &= \int\limits_{\partial S} k(x,\ y)\,ds(y), \quad x \in \partial S,
\end{aligned} \tag{1.60}$$

and

$$\begin{aligned}
F(x) &= \int\limits_{\partial S} k(x,\ y)\varphi(y)\,ds(y), \quad x \in S_0 \setminus \partial S, \\
F_0(x) &= \int\limits_{\partial S} k(x,\ y)\varphi(y)\,ds(y), \quad x \in \partial S,
\end{aligned} \tag{1.61}$$

where $\varphi \in C^{0,\alpha}(\partial S)$, $\alpha \in (0, 1]$. *Also, consider the functions*

$$f^+(x) = \begin{cases} f(x), & \xi \in S_0^+, \\ l(x) + f_0(x), & x \in \partial S, \end{cases}$$
$$f^-(x) = \begin{cases} f(x), & x \in S_0^-, \\ -l(x) + f_0(x), & x \in \partial S, \end{cases} \tag{1.62}$$

and

$$F^+(x) = \begin{cases} F(x), & x \in S_0^+, \\ l(x)\varphi(x) + F_0(x), & x \in \partial S, \end{cases}$$
$$F^-(x) = \begin{cases} F(x), & x \in S_0^-, \\ -l(x)\varphi(x) + F_0(x), & x \in \partial S, \end{cases} \tag{1.63}$$

where $l \in C^{0,\alpha}(\partial S)$. *If* $f^+ \in C^{0,\alpha}(\bar{S}_0^+)$ *and* $f^- \in C^{0,\alpha}(\bar{S}_0^-)$, *then* $F^+ \in C^{0,\beta}(\bar{S}_0^+)$ *and* $F^- \in C^{0,\beta}(\bar{S}_0^-)$, *with* $\beta = \alpha$ *for* $\alpha \in (0, 1)$ *and any* $\beta \in (0, 1)$ *for* $\alpha = 1$.

Proof. From the properties of $k(x, y)$ it is clear that f_0 and F_0 are improper integrals. To prove the statement for F^+, it suffices to consider $x, x' \in \bar{S}_0^+$ satisfying (1.18). Let

$$x = \xi + \sigma\nu(\xi) \in S_0^+, \quad \xi \in \partial S, \quad x' = \xi' \in \partial S.$$

Then

$$\begin{aligned}
&\int_{\partial S} ds(y) k(x, y)\varphi(y) - l(x')\varphi(x') - \int_{\partial S} k(x', y)\varphi(y)\, ds(y) \\
&= \int_{\partial S} k(x, y)\big[\varphi(y) - \varphi(\xi)\big]\, ds(y) - \int_{\partial S} k(x', y)\big[\varphi(y) - \varphi(x')\big]\, ds(y) \\
&\qquad + \big[\varphi(\xi) - \varphi(x')\big] \int_{\partial S} k(x, y)\, ds(y) \\
&\qquad + \varphi(x')\bigg[\int_{\partial S} k(x, y)\, ds(y) - l(x') - \int_{\partial S} k(x', y)\, ds(y)\bigg];
\end{aligned} \tag{1.64}$$

that is,

$$\begin{aligned}
F^+(x) - F^+(x') = \varPhi(x) - \varPhi(x') + \big[\varphi(\xi) - \varphi(\xi')\big] f^+(x) \\
+ \big[f^+(x) - f^+(x')\big]\varphi(\xi'),
\end{aligned} \tag{1.65}$$

where $\varPhi$ is given by (1.54). Equality (1.65) is similarly obtained for $x, x' \in S_0^+$, $x, x' \in \partial S$, or $x \in \partial S$, $x' \in S_0^+$. Since, by our assumption, both f_0 and f

are bounded, (1.60) shows that $k(x, y)$ satisfies estimate (1.55). The assertion now follows from (1.65) and Theorem 1.35.

F^- is treated analogously. □

This theorem can be generalized to certain 1-singular kernels on ∂S.

1.44 Definition. A 1-singular kernel $k(x, y)$ on ∂S is called *integrable* if

$$\int_{\partial S} k(x, y)\, ds(y)$$

exists as principal value for all $x \in \partial S$, and *uniformly integrable* if the integral in (1.58) converges uniformly with respect to $x \in \partial S$.

For convenience, we extend this concept to γ-singular kernels, $\gamma \in (0, 1)$, and note that all such kernels are uniformly integrable.

1.45 Remark. If $k(x, y)$ is uniformly integrable, then $\int\limits_{\partial S} k(x, y)\, ds(y)$ is continuous on ∂S. This is shown by writing the principal value of the integral as the sum of a uniformly convergent infinite series. Evidently, any uniformly integrable kernel satisfies (1.55) on ∂S.

1.46 Theorem. *If the kernel $k(x, y)$ is 1-singular on ∂S and integrable, and if $\varphi \in C^{0,\alpha}(\partial S)$, $\alpha \in (0, 1]$, then the integral*

$$\int_{\partial S} k(x, y)\varphi(y)\, ds(y)$$

exists in the sense of principal value for all $x \in \partial S$. If $k(x, y)$ is uniformly integrable, then the above principal value exists uniformly with respect to $x \in \partial S$.

Proof. We write

$$\int_{\partial S\setminus\Sigma_{x,\delta}} k(x, y)\varphi(y)\, ds(y) = \int_{\partial S\setminus\Sigma_{x,\delta}} k(x, y)\big[\varphi(y) - \varphi(x)\big]\, ds(y) + \varphi(x) \int_{\partial S\setminus\Sigma_{x,\delta}} k(x, y)\, ds(y).$$

The result follows from the fact that, as $\delta \to 0$, the first term on the right-hand side converges uniformly since its integrand is $O(|x - y|^{\alpha-1})$. □

1.47 Theorem. *Suppose that*

(i) *$k(x, y)$ is a proper 1-singular kernel in S_0 which is integrable on ∂S;*
(ii) *f^+ and f^- defined by (1.62), where $l \in C^{0,\alpha}(\partial S)$, $\alpha \in (0, 1]$, and f_0 is understood as principal value, belong, respectively, to $C^{0,\alpha}(\bar{S}_0^+)$ and $C^{0,\alpha}(\bar{S}_0^-)$.*

Then the functions F^+ and F^- defined by (1.63), *where* $\varphi \in C^{0,\alpha}(\partial S)$ *and* F_0 *is understood as principal value*, *belong* , *respectively*, *to* $C^{0,\beta}(\bar{S}_0^+)$ and $C^{0,\beta}(\bar{S}_0^-)$ *with* $\beta = \alpha$ for $\alpha \in (0, 1)$ *and any* $\beta \in (0, 1)$ *for* $\alpha = 1$.

Proof. By Theorem 1.46, F_0 exists in the sense of principal value for all $x \in \partial S$.

As in the proof of Theorem 1.43, let $x,\ x' \in \bar{S}_0^+$, $x \neq x'$. If $x,\ x' \in S_0^+$, equality (1.65) is established immediately. If $x \in S_0^+$, $x' \in \partial S$ (or $x \in \partial S$, $x' \in S_0^+$), we write (1.64) with the integrals extended over $\partial S \setminus \Sigma_{x',\delta}$ ($\partial S \setminus \Sigma_{x,\delta}$) in the first instance, then let $\delta \to 0$. Noting that the limit of the second term on the right-hand side coincides with the improper integral $\varPhi(x')$ $\big(\varPhi(x)\big)$, we again arrive at (1.65). Finally, we see that this is also true if both $x,\ x' \in \partial S$ when the integrals in (1.64) are initially extended over $\partial S \setminus (\Sigma_{x,\delta} \cup \Sigma_{x',\delta})$. Hence, (1.65) holds for all $x,\ x' \in \bar{S}_0^+$, $x \neq x'$, and the result follows from the assumptions (i) and (ii) and Theorem 1.35.

The reasoning is similar in the case of $\bar{S}_0^-$. □

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Chapter 2
Potentials and Boundary Integral Equations

2.1 The Harmonic Potentials

In what follows, we examine the Hölder continuity and Hölder continuous differentiability on $\bar{S}^+$ and $\bar{S}^-$ of functions that are analytic in S^+ and S^-. Hence, it suffices to consider the behavior of such functions in the boundary strip S_0.

We begin by giving a brief account of the main properties of the harmonic potentials, which will be required at a later stage in the proceedings.

The *harmonic single-layer potential* is defined by

$$(v\varphi)(x) = -\int_{\partial S} (\ln|x-y|)\varphi(y)\,ds(y), \tag{2.1}$$

and the *harmonic double-layer potential* by

$$(w\varphi)(x) = -\int_{\partial S} \left[\frac{\partial}{\partial\nu(y)} \ln|x-y|\right]\varphi(y)\,ds(y), \tag{2.2}$$

where the function φ is called the *density*.

We denote by S^+ the finite domain bounded by ∂S and set

$$S^- = \mathbb{R}^2 \setminus \bar{S}^+.$$

2.1 Theorem. *If* $\varphi \in C(\partial S)$, *then* $v\varphi \in C^{0,\alpha}(\mathbb{R}^2)$ *for any index* $\alpha \in (0, 1)$.

Proof. The assertion follows from Theorem 1.33 in view of the fact that, as can easily be verified by means of Lemma 1.28, the kernel

$$k(x,\ y) = -\ln|x-y|$$

of v is a proper γ-singular kernel in S_0 for any $\gamma \in (0, 1)$. □

C. Constanda, *Mathematical Methods for Elastic Plates*,
Springer Monographs in Mathematics, DOI: 10.1007/978-1-4471-6434-0_2,

2.2 Theorem. *If* $\varphi \in C^{0,\alpha}(\partial S)$, $\alpha \in (0, 1]$, *then the restrictions of* $w\varphi$ to S^+ *and* S^- *have* $C^{0,\beta}$*-extensions to* $\bar{S}^+$ *and* $\bar{S}^-$, *respectively, with* $\beta = \alpha$ *for* $\alpha \in (0, 1)$ *and any* $\beta \in (0, 1)$ *for* $\alpha = 1$. *These extensions are given by*

$$(w\varphi)^+(x) = \begin{cases} (w\varphi)(x), & x \in S^+, \\ -\pi\varphi(x) + (w_0\varphi)(x), & x \in \partial S, \end{cases}$$
$$(w\varphi)^-(x) = \begin{cases} (w\varphi)(x), & x \in S^-, \\ \pi\varphi(x) + (w_0\varphi)(x), & x \in \partial S, \end{cases} \tag{2.3}$$

where

$$(w_0\varphi)(x) = -\int_{\partial S} \left[\frac{\partial}{\partial\nu(y)} \ln|x - y|\right]\varphi(y)\, ds(y), \quad x \in \partial S. \tag{2.4}$$

Proof. Applying Lemmas 1.28 and 1.2, we readily convince ourselves that

$$k(x,\ y) = -\frac{\partial}{\partial\nu(y)} \ln|x - y| = \frac{\langle\nu(y),\ x - y\rangle}{|x - y|^2}$$

is a proper 1-singular kernel in S_0 and 0-singular on ∂S. Consequently, $w_0\varphi$ is an improper integral.

Let $x \in S^+$, and consider a disk $\sigma_{x,\delta} \subset S^+$ with the center at x and radius δ sufficiently small. Using the divergence theorem in $S^+ \setminus \sigma_{x,\delta}$ and the fact that λ is a solution of the Laplace equation for $x \neq y$, we find that

$$\begin{aligned} 0 &= \int_{S^+\setminus\sigma_{x,\delta}} \Delta(y) \ln|x - y|\, da(y) \\ &= \int_{\partial S} \frac{\partial}{\partial\nu(y)} \ln|x - y|\, ds(y) - \int_{\partial\sigma_{x,\delta}} \frac{\partial}{\partial\nu(y)} \ln|x - y|\, ds(y) \\ &= \int_{\partial S} \frac{\partial}{\partial\nu(y)} \ln|x - y|\, ds(y) - 2\pi, \end{aligned}$$

where $\partial\sigma_{x,\delta}$ is the circular boundary of $\sigma_{x,\delta}$; hence,

$$\int_{\partial S} \frac{\partial}{\partial\nu(y)} \ln|x - y|\, ds(y) = 2\pi, \quad x \in S^+. \tag{2.5}$$

The procedure is similar for $x \in \partial S$, except that in this case $\sigma_{x,\delta}$ is replaced by $\sigma_{x,\delta} \cap S^+$ and $\partial\sigma_{x,\delta}$ by its part lying in S^+. It is not difficult to show that for a small value of δ, the length of this part is equal to $\pi\delta + O(\delta^2)$, which leads to

$$\int\limits_{\partial S} \frac{\partial}{\partial \nu(y)} \ln|x - y|\, ds(y) = \pi, \quad x \in \partial S. \tag{2.6}$$

Finally, the direct application of the divergence theorem yields

$$\int\limits_{\partial S} \frac{\partial}{\partial \nu(y)} \ln|x - y|\, ds(y) = 0, \quad x \in S^-. \tag{2.7}$$

In view of these integrals and the expression of $k(x, y)$, we now see that

$$f(x) = \int\limits_{\partial S} k(x, y)\, ds(y) = \begin{cases} -2\pi, & x \in S^+, \\ 0, & x \in S^-, \end{cases}$$

$$f_0(x) = \int\limits_{\partial S} k(x, y)\, ds(y) = -\pi, \qquad x \in \partial S. \tag{2.8}$$

From (1.62) with $l(x) = -\pi$, $x \in \partial S$, we obtain

$$f^+(x) = -2\pi, \quad x \in \bar{S}_0^+,$$
$$f^-(x) = 0, \qquad x \in \bar{S}_0^-.$$

Since

$$f^+ \in C^{0,\alpha}(\bar{S}_0^+), \quad f^- \in C^{0,\alpha}(\bar{S}_0^-),$$

the desired result follows from Theorem 1.43. □

2.3 Remark. Theorem 2.2 implies that if $\varphi \in C^{0,\alpha}(\partial S)$, then, as $S^\pm \ni x' \to x \in \partial S$, $w\varphi$ has finite limits given by

$$(w\varphi)^\pm(x) = \mp\pi\varphi(x) - \int\limits_{\partial S} \left[\frac{\partial}{\partial \nu(y)} \ln|x - y|\right]\varphi(y)\, ds(y), \quad x \in \partial S, \tag{2.9}$$

where the last term is an improper integral. It can be shown (Colton and Kress 1983) that $w\varphi$ can also be extended by continuity to $\bar{S}^+$ and $\bar{S}^-$ if $\varphi \in C(\partial S)$, but then the two extensions w^+ and w^- are merely continuous.

2.4 Theorem. *If* $\varphi \in C^{0,\alpha}(\partial S)$, $\alpha \in (0, 1]$, *then the first-order derivatives of* $v\varphi$ in S^+ *and* S^- *have* $C^{0,\beta}$*-extensions to* $\bar{S}^+$ *and* $\bar{S}^-$, *respectively, with* $\beta = \alpha$ *for* $\alpha \in (0, 1)$ *and any* $\beta \in (0, 1)$ *for* $\alpha = 1$. *These extensions are given by*

$$(\operatorname{grad} v\varphi)^+(x) = \begin{cases} (\operatorname{grad} v\varphi)(x), & x \in S^+, \\ \pi\nu(x)\varphi(x) + (\operatorname{grad} v\varphi)_0(x), & x \in \partial S, \end{cases}$$

$$(\operatorname{grad} v\varphi)^-(x) = \begin{cases} (\operatorname{grad} v\varphi)(x), & x \in S^-, \\ -\pi\nu(x)\varphi(x) + (\operatorname{grad} v\varphi)_0(x), & x \in \partial S, \end{cases}$$

where

$$(\operatorname{grad} v\varphi)_0(x) = -\int\limits_{\partial S} \big[\operatorname{grad}(x) \ln|x-y|\big]\varphi(y)\,ds(y), \quad x \in \partial S,$$

the integral being understood as principal value.

Proof. By checking the properties required in Lemma 1.28, we verify that

$$k(x,\ y) = -\operatorname{grad}(x) \ln|x-y|$$

is a proper 1-singular kernel in S_0 and on ∂S.

From (1.51) and the fact that

$$\big[\operatorname{grad}(x) + \operatorname{grad}(y)\big] \ln|x-y| = 0, \quad x \neq y,$$

it follows that for $x,\ y \in \partial S$, $x \neq y$,

$$k(x,\ y) = \left[\frac{\partial}{\partial s(y)} \ln|x-y|\right]\tau(y) + \left[\frac{\partial}{\partial \nu(y)} \ln|x-y|\right]\nu(y).$$

Consequently, using integration by parts and denoting by a and b the end-points of $\Sigma_{x,\delta}$, for $x \in \partial S$ we can write

$$\begin{aligned} \int\limits_{\partial S\setminus\Sigma_{x,\delta}} k(x,\ y)\,ds(y) &= \int\limits_{\partial S\setminus\Sigma_{x,\delta}} \left[\frac{\partial}{\partial s(y)} \ln|x-y|\right]\tau(y)\,ds(y) \\ &\quad + \int\limits_{\partial S\setminus\Sigma_{x,\delta}} \left[\frac{\partial}{\partial \nu(y)} \ln|x-y|\right]\nu(y)\,ds(y) \\ &= \big[\tau(a) - \tau(b)\big] \ln\delta - \int\limits_{\partial S\setminus\Sigma_{x,\delta}} (\ln|x-y|)\kappa(y)\nu(y)\,ds(y) \\ &\quad + \int\limits_{\partial S\setminus\Sigma_{x,\delta}} \left[\frac{\partial}{\partial \nu(y)} \ln|x-y|\right]\nu(y)\,ds(y). \end{aligned}$$

Since ∂S is a C^2-curve, the first term on the right-hand side tends to zero as $\delta \to 0$, while the other two tend to $\big(v(\kappa\nu)\big)(x)$ and $-(w_0\nu)(x)$, respectively. Therefore,

$k(x, y)$ is integrable on ∂S and

$$f_0(x) = \int\limits_{\partial S} k(x, y)\, ds(y) = \big(v(\kappa\nu)\big)(x) - (w_0\nu)(x), \quad x \in \partial S,$$

where f_0 is understood as principal value.

On the other hand, if $x \in S_0 \setminus \partial S$, then, again integrating by parts and taking (1.8) into account, we find that

$$f(x) = \int\limits_{\partial S} k(x, y)\, ds(y) = \big(v(\kappa\nu)\big)(x) - (w\nu)(x), \quad x \in S_0 \setminus \partial S.$$

By Theorems 2.1 and 2.2, the function f is $C^{0,\alpha}$-extendable to $\bar{S}_0^+$ and $\bar{S}_0^-$ and the values of the corresponding extensions on ∂S are given by the formula

$$f^{\pm}(x) = \big(v(\kappa\nu)\big)(x) \pm \pi\nu(x) + (w_0\nu)(x) = \pm\pi\nu(x) + f_0(x), \quad x \in \partial S;$$

in other words, the expressions (1.62) with $l = \pi\nu \in C^{0,1}(\partial S)$. As stated earlier, $f^+ \in C^{0,\alpha}(\bar{S}_0^+)$ and $f^- \in C^{0,\alpha}(\bar{S}_0^-)$. The assertion now follows from Theorem 1.47 with F and F_0 in (1.61) defined by

$$F(x) = -\int\limits_{\partial S} \big[\mathrm{grad}\,(x) \ln|x-y|\big]\varphi(y)\, ds(y) = (\mathrm{grad}\, v)(x), \qquad x \in S_0 \setminus \partial S,$$

$$F_0(x) = -\int\limits_{\partial S} \big[\mathrm{grad}\,(x) \ln|x-y|\big]\varphi(y)\, ds(y) = (\mathrm{grad}\, v)_0(x), \quad x \in \partial S,$$

the latter understood as principal value. □

2.5 Remark. Theorem 2.4 implies that if $\varphi \in C^{0,\alpha}(\partial S)$, then, as $S^{\pm} \ni x' \to x \in \partial S$, $(\mathrm{grad}\, v\varphi)(x')$ tends to finite limits given by

$$(\mathrm{grad}\, v\varphi)^{\pm}(x) = \pm\pi\nu(x)\varphi(x) - \int\limits_{\partial S} \big[\mathrm{grad}\,(x) \ln|x-y|\big]\varphi(y)\, ds(y), \quad x \in \partial S, \tag{2.10}$$

where the second term on the right-hand side is understood as principal value.

2.6 Remark. Theorems 2.4 and 1.20 also imply that if $\varphi \in C^{0,\alpha}(\partial S)$, $\alpha \in (0, 1]$, then the restrictions of $v\varphi$ to $\bar{S}^+$ and $\bar{S}^-$ belong, respectively, to $C^{1,\beta}(\bar{S}^+)$ and $C^{1,\beta}(\bar{S}^-)$, with $\beta = \alpha$ for $\alpha \in (0, 1)$ and any $\beta \in (0, 1)$ for $\alpha = 1$. We denote these restrictions by $(v\varphi)^+$ and $(v\varphi)^-$; hence,

$$(\mathrm{grad}\,(v\varphi)^+)(x) = (\mathrm{grad}\, v\varphi)^+(x), \quad x \in \bar{S}^+,$$
$$(\mathrm{grad}\,(v\varphi)^-)(x) = (\mathrm{grad}\, v\varphi)^-(x), \quad x \in \bar{S}^-.$$

2.7 Theorem. *If* $\varphi \in C^{1,\alpha}(\partial S)$, $\alpha \in (0,1]$, *then the restrictions of* $w\varphi$ *to* S^+ *and* S^- *have* $C^{1,\beta}$*-extensions* $(w\varphi)^+$ *and* $(w\varphi)^-$ to $\bar{S}^+$ *and* $\bar{S}^-$, *respectively, with* $\beta = \alpha$ *for* $\alpha \in (0,1)$ *and any* $\beta \in (0,1)$ *for* $\alpha = 1$. *These extensions are given by* (2.3) *and satisfy the equality* $\partial(w\varphi)^+/\partial\nu = \partial(w\varphi)^-/\partial\nu$ *on* ∂S.

Proof. Let $x \neq y$. Since

$$\Delta(y)\ln|x-y| = 0,$$

$$\big[\operatorname{grad}(x) + \operatorname{grad}(y)\big]\ln|x-y| = 0,$$

for $x \in S_0 \setminus \partial S$ and any $y \in \partial S$ we can write

$$\begin{aligned}
&\frac{\partial}{\partial x_\gamma}\left[\frac{\partial}{\partial\nu(y)}\ln|x-y|\right]\\
&\quad= \nu_\beta(y)\frac{\partial}{\partial y_\beta}\left(\frac{\partial}{\partial x_\gamma}\ln|x-y|\right) + \nu_\gamma(y)\Delta(y)\ln|x-y|\\
&\quad= \nu_\beta(y)\frac{\partial}{\partial y_\beta}\left(-\frac{\partial}{\partial y_\gamma}\ln|x-y|\right) + \nu_\gamma(y)\frac{\partial}{\partial y_\beta}\left(\frac{\partial}{\partial y_\beta}\ln|x-y|\right)\\
&\quad= \left[\nu_\beta(y)\frac{\partial}{\partial y_\gamma} - \nu_\gamma(y)\frac{\partial}{\partial y_\beta}\right]\left(\frac{\partial}{\partial x_\beta}\ln|x-y|\right)\\
&\quad= \varepsilon_{\beta\gamma}\frac{\partial}{\partial s(y)}\left(\frac{\partial}{\partial x_\beta}\ln|x-y|\right).
\end{aligned}$$

Consequently, using integration by parts, we find that for $x \in S_0 \setminus \partial S$,

$$\begin{aligned}
\frac{\partial}{\partial x_\gamma}(w\varphi)(x) &= -\int_{\partial S}\frac{\partial}{\partial x_\gamma}\left[\frac{\partial}{\partial\nu(y)}\ln|x-y|\right]\varphi(y)\,ds(y)\\
&= \varepsilon_{\beta\gamma}\frac{\partial}{\partial x_\beta}\int_{\partial S}(\ln|x-y|)\varphi'(y)\,ds(y)\\
&= \varepsilon_{\gamma\beta}\frac{\partial}{\partial x_\beta}(v\varphi')(x). \qquad (2.11)
\end{aligned}$$

From this, Theorem 2.4, and the fact that $\varphi' \in C^{0,\alpha}(\partial S)$ we deduce that $\operatorname{grad} w\varphi$ has $C^{0,\beta}$-extensions $(\operatorname{grad} w\varphi)^+$ and $(\operatorname{grad} w\varphi)^-$ to $\bar{S}^+$ and $\bar{S}^-$. By Theorem 2.2, the extensions $(w\varphi)^+$ and $(w\varphi)^-$, given by (2.3), of $w\varphi$ are Hölder continuous on $\bar{S}^+$ and $\bar{S}^-$, respectively. Since, as is obvious,

$$\operatorname{grad}(w\varphi)^+(x) = (\operatorname{grad} w\varphi)^+(x), \quad x \in S^+,$$

$$\operatorname{grad}(w\varphi)^-(x) = (\operatorname{grad} w\varphi)^-(x), \quad x \in S^-,$$

the first part of the assertion now follows from Theorem 1.20.

To complete the proof, we remark that, in view of (2.11) and (2.10), for $x \in \partial S$ Theorem 1.20 yields

$$\begin{aligned}\frac{\partial}{\partial \nu}(w\varphi)^{\pm}(x) &= \langle \operatorname{grad}(w\varphi)^{\pm}(x),\ \nu(x)\rangle = \langle (\operatorname{grad} w\varphi)^{\pm}(x),\ \nu(x)\rangle \\ &= \varepsilon_{\gamma\beta}\bigg[\frac{\partial}{\partial x_\beta}\,\nu\varphi'\bigg]^{\pm}(x)\nu_\gamma(x) \\ &= \varepsilon_{\beta\gamma}\nu_\gamma(x)\int\limits_{\partial S}\bigg(\frac{\partial}{\partial x_\beta}\ \ln|x-y|\bigg)\varphi'(y)\,ds(y),\end{aligned}$$

where the integral is understood as principal value. □

2.8 Theorem. *The function* $w_0\varphi$ *defined by* (2.4) *as the direct value on* ∂S *of the double-layer potential with density* $\varphi \in C^{0,\alpha}(\partial S)$, $\alpha \in (0, 1]$, *belongs to* $C^{1,\beta}(\partial S)$, *with* $\beta = \alpha$ *for* $\alpha \in (0, 1)$ *and any* $\beta \in (0, 1)$ *for* $\alpha = 1$.

Proof. As noted in the proof of Theorem 2.2, the kernel

$$k(x,\ y) = -\frac{\partial}{\partial \nu(y)}\ \ln|x-y|$$

is 0-singular on ∂S; consequently, $w_0(x)$ is an improper integral for all $x \in \partial S$. Clearly,

$$\begin{aligned}k_0(x,\ y) &= \frac{\partial}{\partial s(x)}k(x,\ y) \\ &= \frac{\langle \nu(y),\ \tau(x)\rangle}{|x-y|^2} - 2\frac{\langle \nu(y),\ x-y\rangle\langle \tau(x),\ x-y\rangle}{|x-y|^4}\end{aligned} \qquad (2.12)$$

is 1-singular on ∂S. Verifying the conditions of Lemma 1.28, we deduce that $k_0(x,\ y)$ is a proper 1-singular kernel on ∂S.

Next, by writing $\langle\cdot\,,\cdot\rangle$ in terms of the cosine of the angle between the vectors, we find that

$$\langle \nu(y),\ \tau(x)\rangle + \langle \nu(x),\ \tau(y)\rangle = 0, \quad x,\ y \in \partial S. \qquad (2.13)$$

Using the same technique, (2.12), and (2.13), for $x,\ y \in \partial S$, $x \neq y$, we now obtain

$$\begin{aligned}&\bigg[\frac{\partial}{\partial s(x)} + \frac{\partial}{\partial s(y)}\bigg]k(x,\ y) \\ &\quad = 2|x-y|^{-4}\big\{\langle \nu(x),\ x-y\rangle\langle \tau(y),\ x-y\rangle \\ &\qquad\qquad - \langle \nu(y),\ x-y\rangle\langle \tau(x),\ x-y\rangle + \langle \nu(y),\ \tau(x)\rangle\big\} = 0.\end{aligned} \qquad (2.14)$$

From this and (2.12) we conclude that $k_0(x,\ y)$ satisfies (1.55).

The assertion now follows from Theorem 1.37 with $\beta = \gamma = 0$ and $g(x) = -\pi$, $x \in \partial S$ (according to (2.8)), and Remarks 1.39 and 1.38. □

2.2 Other Potential-Type Functions

In this section we consider the Hölder continuity and continuous differentiability of some other useful integrals with γ-singular kernels.

2.9 Theorem. *Suppose that $k(x,\ y)$ is a continuous kernel in $S_0 \times \partial S$ and such that* grad $(x)k(x,\ y)$ *is a proper γ-singular kernel in S_0, $\gamma \in [0, 1)$, and let*

$$(v^a\varphi)(x) = \int\limits_{\partial S} k(x,\ y)\varphi(y)\,ds(y), \quad x \in S_0.$$

If $\varphi \in C(\partial S)$, then $v^a\varphi \in C^{1,\beta}(S_0)$, with $\beta = 1 - \gamma$ for $\gamma \in (0, 1)$ *and any $\beta \in (0, 1)$ for $\gamma = 0$.*

Proof. Clearly, $v^a\varphi \in C(S_0) \cap C^1(S_0^+) \cap C^1(S_0^-)$. The statement follows from the fact that for $x \in S_0 \setminus \partial S$,

$$(\text{grad}\, v^a\varphi)(x) = \int\limits_{\partial S} \text{grad}\,(x)k(x,\ y)\varphi(y)\,ds(y),$$

which, by Theorem 1.33, belongs to $C^{0,\beta}(S_0)$. □

2.10 Theorem. *Suppose that $k(x,\ y)$ is a continuous kernel on $\partial S \times \partial S$ and such that $\partial k(x,\ y)/\partial s(x)$ is a proper γ-singular kernel on ∂S, $\gamma \in [0, 1)$.* If $\varphi \in C(\partial S)$, *then the function*

$$(v_0^a\varphi)(x) = \int\limits_{\partial S} k(x,\ y)\varphi(y)\,ds(y), \quad x \in \partial S,$$

belongs to $C^{1,\beta}(\partial S)$, with $\beta = 1 - \gamma$ for $\gamma \in (0, 1)$ and any $\beta \in (0, 1)$ for $\gamma = 0$.

Proof. Consider the function

$$(v_{0\delta}^a\varphi)(x) = \int\limits_{\partial S \setminus \Sigma_{x,\delta}} k(x,\ y)\varphi(y)\,ds(y), \quad \delta > 0.$$

It is obvious that $(v_{0\delta}^a\varphi)(x) \to (v_0^a\varphi)(x)$ as $\delta \to 0$, for all $x \in \partial S$. On the other hand,

$$\frac{\partial}{\partial s}(v_{0\delta}^a\varphi)(x) = \int\limits_{\partial S \setminus \Sigma_{x,\delta}} \frac{\partial}{\partial s(x)}k(x,\ y)\varphi(y)\,ds(y),$$

which converges uniformly to $\int_{\partial S}\big[\partial k(x,\ y)/\partial s(x)\big]\varphi(y)\,ds(y)$ as $\delta\to 0$ (see the proof of Theorem 1.32). By a well-known theorem of analysis, $v_0^a\varphi$ is differentiable at all $x\in\partial S$ and

$$\frac{\partial}{\partial s}(v_0^a\varphi)(x)=\int_{\partial S}\frac{\partial}{\partial s(x)}k(x,\ y)\varphi(y)\,ds(y).$$

We complete the proof by applying Theorem 1.33 to the above integral to deduce that $\partial(v_0^a\varphi)/\partial s\in C^{0,\beta}(\partial S)$. □

2.11 Theorem. *If* $\varphi\in C(\partial S)$, *then the functions*

$$(v_{\gamma\delta}^b\varphi)(x)=\int_{\partial S}\frac{(x_\gamma-y_\gamma)(x_\delta-y_\delta)}{|x-y|^2}\,\varphi(y)\,ds(y),\qquad x\in\mathbb{R}^2,\tag{2.15}$$

$$(v_\gamma^c\varphi)(x)=\int_{\partial S}\bigg[\frac{\partial}{\partial s(y)}\big((x_\gamma-y_\gamma)\ln|x-y\,|\big)\bigg]\varphi(y)\,ds(y),\ \ x\in\mathbb{R}^2,\tag{2.16}$$

$$(v_\gamma^d\varphi)(x)=\int_{\partial S}\bigg[\frac{\partial}{\partial \nu(y)}\big((x_\gamma-y_\gamma)\ln|x-y\,|\big)\bigg]\varphi(y)\,ds(y),\ \ x\in\mathbb{R}^2,\tag{2.17}$$

belong to $C^{0,\alpha}(\mathbb{R}^2)$ *for any* $\alpha\in(0,1)$.

Proof. As mentioned earlier, it suffices to verify the Hölder continuity of these functions in S_0.

By direct verification or by means of Lemma 1.28, we easily convince ourselves that $(x_\gamma-y_\gamma)(x_\delta-y_\delta)|x-y|^{-2}$ is a proper 0-singular kernel in S_0. Similarly,

$$\frac{\partial}{\partial s(y)}\big[(x_\gamma-y_\gamma)\ln|x-y\,|\big]=-\tau_\gamma\ln|x-y\,|-\frac{(x_\gamma-y_\gamma)\langle\tau(y),\ x-y\rangle}{|x-y|^2}$$

and

$$\frac{\partial}{\partial \nu(y)}\big[(x_\gamma-y_\gamma)\ln|x-y\,|\big]=-\nu_\gamma(y)\ln|x-y\,|-\frac{(x_\gamma-y_\gamma)\langle\nu(y),\ x-y\rangle}{|x-y|^2}$$

are proper σ-singular kernels in S_0 for any $\sigma\in(0,1)$. The result now follows from Theorem 1.33. □

2.12 Theorem. *If* $\varphi\in C^{0,\alpha}(\partial S)$, $\alpha\in(0,1]$, *then the function*

$$(v_{\gamma\delta}^e\varphi)(x)=\int_{\partial S}\bigg[\frac{\partial}{\partial s(y)}\frac{(x_\gamma-y_\gamma)(x_\delta-y_\delta)}{|x-y|^2}\bigg]\varphi(y)\,ds(y),\quad x\in\mathbb{R}^2,\tag{2.18}$$

belongs to $C^{0,\beta}(\mathbb{R}^2)$, *with* $\beta=\alpha$ *for* $\alpha\in(0,1)$ *and any* $\beta\in(0,1)$ *for* $\alpha=1$.

Proof. Direct verification of the properties in Definition 1.26 shows that the kernel $k(x,\ y)$ of $v^{e}_{\gamma\delta}\varphi$ is a proper 1-singular kernel in S_0. Also, for $x,\ y \in \partial S$, $x \neq y$,

$$\begin{aligned}\frac{\partial}{\partial s(y)}\,&\frac{(x_\gamma - y_\gamma)(x_\delta - y_\delta)}{|x-y|^2}\\&= c_{\gamma\delta\rho\sigma}\frac{(x_\rho - y_\rho)(x_\sigma - y_\sigma)}{|x-y|^2}\,\frac{\partial}{\partial \nu(y)}\ln|x-y|,\end{aligned} \tag{2.19}$$

where

$$\begin{aligned}&c_{\gamma\gamma\gamma\delta} = c_{\gamma\gamma\delta\gamma} = -c_{\gamma\delta\gamma\gamma} = -c_{\delta\gamma\gamma\gamma} = \varepsilon_{\delta\gamma},\\&c_{\gamma\delta\gamma\delta} = c_{\gamma\delta\delta\gamma} = c_{\gamma\gamma\delta\delta} = 0 \quad (\gamma,\ \delta \text{ not summed}),\end{aligned} \tag{2.20}$$

which means that $k(x,\ y)$ is 0-singular on ∂S. Consequently, $v^{e}_{\gamma\delta}\varphi$ is an improper integral for $x \in \partial S$.

Since for $x,\ y \in \partial S$, $x \neq y$,

$$\lim_{y\to x}\frac{(x_\gamma - y_\gamma)(x_\delta - y_\delta)}{|x-y|^2} = \tau_\gamma(x)\tau_\delta(x),$$

we find that f and f_0 defined by (1.60) are identically zero. Hence, f^+ and f^- defined by (1.62) with $l(x) = 0$, $x \in \partial S$, belong to $C^{0,\alpha}(\partial S)$. The result now follows from Theorem 1.43. □

2.13 Theorem. *If* $\varphi \in C^{0,\alpha}(\partial S)$, $\alpha \in (0, 1]$, *then*

$$(v^{f}_{0}\varphi)(x) = \int\limits_{\partial S}\left[\frac{\partial}{\partial s(y)}\ln|x-y|\right]\varphi(y)\,ds(y), \quad x \in \partial S, \tag{2.21}$$

exists as principal value uniformly for all $x \in \partial S$. *Furthermore,* $v^{f}_{0}\varphi \in C^{0,\beta}(\partial S)$, *with* $\beta = \alpha$ *for* $\alpha \in (0, 1)$ *and any* $\beta \in (0, 1)$ *for* $\alpha = 1$.

Proof. For $x,\ y \in \partial S$, $x \neq y$, we have

$$\begin{aligned}&\left|\frac{\partial}{\partial s(y)}\ln|x-y|\right| = \frac{|\langle\tau(y),\ x-y\rangle|}{|x-y|^2} \le c_1|x-y|^{-1},\\&|x-y|\left|\frac{\partial}{\partial x_\gamma}\left[\frac{\partial}{\partial s(y)}\ln|x-y|\right]\right|\\&\qquad = \left|\frac{\tau_\gamma(y)}{|x-y|} - 2\frac{\langle\tau(y),\ x-y\rangle(x_\gamma - y_\gamma)}{|x-y|^3}\right| \le c_2|x-y|^{-1},\end{aligned}$$

where c_1 and c_2 are positive constants. Therefore, by Lemma 1.28, $\partial\ln|x-y|/\partial s(y)$ is a proper 1-singular kernel on ∂S. This kernel is also uniformly integrable since if

a and b are the end-points of $\Sigma_{x,\delta}$, then

$$\int\limits_{\partial S\setminus\Sigma_{x,\delta}} \frac{\partial}{\partial s(y)} \ln|x-y|\,ds(y) = \ln\frac{|x-a|}{|x-b|} = 0 \tag{2.22}$$

for all $0 < \delta \le r$ and all $x \in \partial S$. We can now write

$$\begin{aligned}\int\limits_{\partial S\setminus\Sigma_{x,\delta}} &\left[\frac{\partial}{\partial s(y)} \ln|x-y|\right]\varphi(y)\,ds(y)\\ &= \int\limits_{\partial S\setminus\Sigma_{x,\delta}} \left[\frac{\partial}{\partial s(y)} \ln|x-y|\right]\big[\varphi(y)-\varphi(x)\big]\,ds(y),\end{aligned}$$

and the first part of the assertion follows from Definition 1.40 and the uniform convergence, as $\delta \to 0$, of the right-hand side, whose integrand is $O(|x-y|^{\alpha-1})$; consequently,

$$(v_0^f\varphi)(x) = \int\limits_{\partial S} \left[\frac{\partial}{\partial s(y)} \ln|x-y|\right]\big[\varphi(y)-\varphi(x)\big]\,ds(y), \quad x \in \partial S, \tag{2.23}$$

in the sense of principal value.

To complete the proof, we apply Theorem 1.32 with $\xi = x$ and make use of the last part of Remark 1.45. □

2.14 Theorem. *If* $\varphi \in C^{0,\alpha}(\partial S)$, $\alpha \in (0, 1]$, *then the function*

$$(v^f\varphi)(x) = \int\limits_{\partial S} \left[\frac{\partial}{\partial s(y)} \ln|x-y|\right]\varphi(y)\,ds(y), \quad x \in S_0 \setminus \partial S, \tag{2.24}$$

is $C^{0,\beta}$*-extendable to* $\mathbb{R}^2$, *with* $\beta = \alpha$ *for* $\alpha \in (0, 1)$ *and any* $\beta \in (0, 1)$ *for* $\alpha = 1$.

Proof. In the proof of Theorem 2.13 it was shown that

$$k(x, y) = \frac{\partial \ln|x-y|}{\partial s(y)}$$

is an integrable, proper 1-singular kernel on ∂S. The same reasoning indicates that $k(x, y)$ is also a proper 1-singular kernel in S_0. In view of (2.22), formulas (1.60) yield

$$\begin{aligned} f(x) &= 0, \qquad x \in S_0 \setminus \partial S,\\ f_0(x) &= 0, \qquad x \in \partial S, \end{aligned} \tag{2.25}$$

the latter understood as principal value. From (2.25) and (1.62) with $l(x) = 0$, $x \in \partial S$, it follows that $f^+ \in C^{0,\alpha}(\bar{S}_0^+)$ and $f^- \in C^{0,\alpha}(\bar{S}_0^-)$ (both these functions are identically zero). The application of Theorem 1.47 now completes the proof. □

2.15 Remark. Since $l = 0$, (2.24) also represents the extension of $v^f\varphi$ to $\mathbb{R}^2$, that is, it holds for $x \in \mathbb{R}^2$, but for $x \in \partial S$ the integral on the right-hand side (denoted by $v_0^f\varphi$ in (2.21)) must be understood as principal value.

Alternatively, since

$$\int_{\partial S} \frac{\partial}{\partial s(y)} \ln|x-y|\,ds(y) = 0, \quad x \in \mathbb{R}^2 \setminus \partial S,$$

we see that the extension of $v^f\varphi$ to $\mathbb{R}^2$ is also given by the right-hand side of (2.23) with $x \in \mathbb{R}^2$.

2.16 Theorem. *If $\varphi \in C^{1,\alpha}(\partial S)$, $\alpha \in (0,1]$, then the function $v_0^f\varphi$ defined by* (2.21) *belongs to $C^{1,\beta}(\partial S)$, with $\beta = \alpha$* for *$\alpha \in (0,1)$ and any $\beta \in (0,1)$ for $\alpha = 1$.*

Proof. By Theorem 2.13, $v_0^f\varphi$ is Hölder continuous on ∂S.

Let $x = \psi(s) \in \partial S$ be arbitrary but fixed, and let $a = \psi(s-\delta)$ and $b = \psi(s+\delta)$ be the end-points of the arc $\Gamma_{x,\delta}$ defined by (1.59). Integrating by parts, we find that

$$\int_{\partial S\setminus\Gamma_{x,\delta}} (\ln|x-y|)\varphi'(y)\,ds(y) = \varphi(a)\ln|x-a| - \varphi(b)\ln|x-b| - \int_{\partial S\setminus\Gamma_{x,\delta}} \left[\frac{\partial}{\partial s(y)}\ln|x-y|\right]\varphi(y)\,ds(y). \quad (2.26)$$

The first term on the right-hand side can be written in the form

$$\varphi(x)(\ln|x-a| - \ln|x-b|) + \big[\varphi(a)-\varphi(x)\big]\ln|x-a| - \big[\varphi(b)-\varphi(x)\big]\ln|x-b|.$$

Since

$$\ln|x-a| - \ln|x-b| = \ln\left(\frac{|x-a|}{\delta}\cdot\frac{\delta}{|x-b|}\right)$$

and φ is differentiable on ∂S, by Theorem 1.16, this expression tends to zero as $\delta \to 0$.

In the proof of Theorem 2.13 it was shown that $\partial \ln|x-y|/\partial s(y)$ is an integrable, proper 1-singular kernel on ∂S. Setting

$$F(x) = \int_{\partial S} (\ln|x-y|)\varphi'(y)\,ds(y),$$

$$F_\delta(x) = \int_{\partial S\setminus\Gamma_{x,\delta}} (\ln|x-y|)\varphi'(y)\,ds(y),$$

and letting $\delta \to 0$ in (2.26), we see that, by Theorem 2.13 and Remark 1.41,

$$F(x) = \lim_{\delta \to 0} F_\delta(x) = -(v_0^f \varphi)(x). \tag{2.27}$$

On the other hand, by Leibniz's rule for differentiating an integral whose limits depend on the parameter,

$$F'_\delta(x) = \int\limits_{\partial S \setminus \Gamma_{x,\delta}} \left[\frac{\partial}{\partial s(x)} \ln|x-y| \right] \varphi'(y)\, ds(y) \\ + \varphi'(a) \ln|x-a| - \varphi'(b) \ln|x-b|.$$

Since $\varphi' \in C^{0,\alpha}(\partial S)$, we deduce as above that the sum of the last two terms tends to zero uniformly as $\delta \to 0$. Hence,

$$\lim_{\delta \to 0} F'_\delta(x) = \int\limits_{\partial S} \left[\frac{\partial}{\partial s(x)} \ln|x-y| \right] \varphi'(y)\, ds(y), \tag{2.28}$$

where the integral is understood as principal value and, by Theorem 2.13, the convergence is uniform with respect to x. A well-known result of analysis now implies that $F(x)$ is differentiable and that $F'(x)$ is equal to the right-hand side of (2.28). Taking (2.27) into account, we conclude that $\partial(v_0^f \varphi)/\partial s$ exists and

$$\frac{\partial}{\partial s}(v_0^f \varphi)(x) = -\int\limits_{\partial S} \left[\frac{\partial}{\partial s(x)} \ln|x-y| \right] \varphi'(y)\, ds(y), \quad x \in \partial S.$$

By Theorem 2.13, $\partial(v_0^f \varphi)/\partial s \in C^{0,\alpha}(\partial S)$, as required. □

2.17 Theorem. *If* $\varphi \in C^{0,\alpha}(\partial S)$, $\alpha \in (0,1]$, *then the functions*

$$(v^c_{\gamma 0}\varphi)(x) = \int\limits_{\partial S} \left\{ \frac{\partial}{\partial s(y)} \big[(x_\gamma - y_\gamma) \ln|x-y| \big] \right\} \varphi(y)\, ds(y), \ x \in \partial S, \tag{2.29}$$

$$(v^d_{\gamma 0}\varphi)(x) = \int\limits_{\partial S} \left\{ \frac{\partial}{\partial \nu(y)} \big[(x_\gamma - y_\gamma) \ln|x-y| \big] \right\} \varphi(y)\, ds(y), \ x \in \partial S, \tag{2.30}$$

belong to $C^{1,\beta}(\partial S)$, *with* $\beta = \alpha$ *for* $\alpha \in (0,1)$ *and any* $\beta \in (0,1)$ *for* $\alpha = 1$.

Proof. The kernel

$$
\begin{aligned}
k(x,\ y) &= \frac{\partial}{\partial s(y)}\big[(x_\gamma - y_\gamma)\ln|x-y\,|\big] \\
&= -\tau_\gamma(y)\ln|x-y\,| - \frac{(x_\gamma - y_\gamma)\langle\tau(y),\ x-y\rangle}{|x-y|^2}
\end{aligned}
$$

is δ-singular on ∂S, where $\delta \in (0, 1)$ is arbitrary, so $(v^c_{\gamma 0}\varphi)(x)$ is an improper integral for all $x \in \partial S$. Also,

$$
\begin{aligned}
k_0(x,\ y) &= \frac{\partial}{\partial s(x)}k(x,\ y) \\
&= -\tau_\gamma(y)\frac{\partial}{\partial s(x)}\ln|x-y\,| - \frac{\partial}{\partial s(x)}\frac{(x_\gamma - y_\gamma)\langle\tau(y),\ x-y\rangle}{|x-y|^2} \qquad (2.31)
\end{aligned}
$$

is 1-singular on ∂S. Using Lemma 1.28, we find that $k_0(x,\ y)$ is a proper 1-singular kernel on ∂S.

Since

$$
\hat{k}(x,\ y) = \left[\frac{\partial}{\partial s(x)} + \frac{\partial}{\partial s(y)}\right]\ln|x-y\,| = \frac{\langle\tau(x)-\tau(y),\ x-y\rangle}{|x-y|^2}
$$

is 0-singular on ∂S, the first term on the right-hand side in (2.31) can be written in the form

$$
-\hat{k}(x,\ y)\tau_\gamma(y) + \frac{\partial}{\partial s(y)}\big[\tau_\gamma(y)\ln|x-y\,|\big] + \kappa(y)\nu_\gamma(y)\ln|x-y\,|.
$$

Similarly, since

$$
\begin{aligned}
\tilde{k}_{\gamma\delta}(x,\ y) &= \left[\frac{\partial}{\partial s(x)} + \frac{\partial}{\partial s(y)}\right]\frac{(x_\gamma - y_\gamma)(x_\delta - y_\delta)}{|x-y|^2} \\
&= \frac{\big[\tau_\gamma(x)-\tau_\gamma(y)\big](x_\delta - y_\delta)}{|x-y|^2} + \frac{\big[\tau_\delta(x)-\tau_\delta(y)\big](x_\gamma - y_\gamma)}{|x-y|^2} \\
&\quad - 2\frac{(x_\gamma - y_\gamma)(x_\delta - y_\delta)\langle\tau(x)-\tau(y),\ x-y\rangle}{|x-y|^4}
\end{aligned}
$$

is 0-singular on ∂S, the second term on the right-hand side in (2.31) becomes

$$
-\tilde{k}_{\gamma\delta}(x,\ y)\tau_\delta(y) + \frac{\partial}{\partial s(y)}\frac{(x_\gamma - y_\gamma)\langle\tau(y),\ x-y\rangle}{|x-y|^2} + \frac{\kappa(y)(x_\gamma - y_\gamma)\langle\nu(y),\ x-y\rangle}{|x-y|^2}.
$$

Denoting by a and b the end-points of $\Sigma_{x,\delta}$, we find that

$$\begin{aligned}\int\limits_{\partial S\setminus\Sigma_{x,\delta}} &\frac{\partial}{\partial s(y)}\big[\tau_\gamma(y)\ln|x-y|\big]\,ds(y)\\ &= \tau_\gamma(a)\ln|x-a| - \tau_\gamma(b)\ln|x-b|\\ &= \tau_\gamma(a)\ln\frac{|x-a|}{|x-b|} + \big[\tau_\gamma(a)-\tau_\gamma(b)\big]\ln|x-b|\\ &= \big[\tau_\gamma(a)-\tau_\gamma(b)\big]\ln|x-b| \to 0,\end{aligned}$$

uniformly as $\delta \to 0$, and that

$$\begin{aligned}\int\limits_{\partial S\setminus\Sigma_{x,\delta}} &\frac{\partial}{\partial s(y)}\,\frac{(x_\gamma-y_\gamma)\langle\tau(y),\,x-y\rangle}{|x-y|^2}\,ds(y)\\ &= \frac{(x_\gamma-a_\gamma)\langle\tau(a),\,x-a\rangle}{|x-a|^2} - \frac{(x_\gamma-b_\gamma)\langle\tau(b),\,x-b\rangle}{|x-b|^2} \to 0,\end{aligned}$$

uniformly as $\delta \to 0$. Consequently, the kernel $k_0(x,\,y)$ satisfies estimate (1.55).

The result now follows from Theorem 1.37 with any $\beta \in (0,\,\alpha)$, $\gamma = 0$, and $g(x) = 0$, $x \in \partial S$, and Remarks 1.39 and 1.38.

The function $v^d_{\gamma 0}\varphi$ is treated similarly. □

2.18 Theorem. *If* $\varphi \in C^{0,\alpha}(\partial S)$, $\alpha \in (0,\,1]$, *then the function defined by*

$$(v^e_{\gamma\delta 0}\varphi)(x) = \int\limits_{\partial S}\bigg[\frac{\partial}{\partial s(y)}\,\frac{(x_\gamma-y_\gamma)(x_\delta-y_\delta)}{|x-y|^2}\bigg]\varphi(y)\,ds(y), \quad x \in \partial S, \tag{2.32}$$

belongs to $C^{1,\beta}(\partial S)$, *with* $\beta = \alpha$ *for* $\alpha \in (0,\,1)$ *and any* $\beta \in (0,\,1)$ *for* $\alpha = 1$.

Proof. From formula (2.19) and the estimates in Lemma 1.2 we see that the kernel

$$k(x,\,y) = \frac{\partial}{\partial s(y)}\,\frac{(x_\gamma-y_\gamma)(x_\delta-y_\delta)}{|x-y|^2}$$

is 0-singular on ∂S, hence $(v^e_{\gamma\delta 0}\varphi)(x)$ is an improper integral for all $x \in \partial S$. A simple calculation shows that

$$\begin{aligned}k_0(x,\,y) &= \frac{\partial}{\partial s(x)}\,k(x,\,y)\\ &= c_{\gamma\delta\lambda\mu}c_{\lambda\mu\rho\sigma}\,\frac{(x_\rho-y_\rho)(x_\sigma-y_\sigma)}{|x-y|^2}\bigg[\frac{\partial}{\partial\nu(x)}\ln|x-y|\bigg]\bigg[\frac{\partial}{\partial\nu(y)}\ln|x-y|\bigg]\\ &\quad + c_{\gamma\delta\rho\sigma}\,\frac{(x_\rho-y_\rho)(x_\sigma-y_\sigma)}{|x-y|^2}\,\frac{\partial}{\partial s(x)}\bigg[\frac{\partial}{\partial\nu(y)}\ln|x-y|\bigg],\end{aligned} \tag{2.33}$$

where the $c_{\gamma\delta\rho\sigma}$ are given by (2.20), is a 1-singular kernel on ∂S. Moreover, using Lemma 1.28, we easily convince ourselves that $k_0(x,\ y)$ is a proper 1-singular kernel on ∂S.

The first term on the right-hand side of (2.33) is 0-singular on ∂S. By (2.14), the second term can be written in the form

$$c_{\gamma\delta\rho\sigma}\left\{-\frac{\partial}{\partial s(y)}\left[\frac{(x_\rho-y_\rho)(x_\sigma-y_\sigma)}{|x-y|^2}\frac{\partial}{\partial\nu(y)}\ln|x-y|\right]\right.$$
$$\left.+\left[\frac{\partial}{\partial\nu(y)}\ln|x-y|\right]\frac{\partial}{\partial s(y)}\frac{(x_\rho-y_\rho)(x_\sigma-y_\sigma)}{|x-y|^2}\right\},$$

from which, in view of what was said above about $k(x,\ y)$, we immediately deduce by direct verification that $k_0(x,\ y)$ satisfies estimate (1.55).

The assertion now follows from Theorem 1.37 with $\beta=\gamma=0$ and $g(x)=0$, $x\in\partial S$, and Remarks 1.39 and 1.38. □

2.3 Complex Singular Kernels

In the analysis of two-dimensional problems it is often convenient to express certain properties of functions in terms of complex variables. Extending an earlier convention, for a function f given on ∂S we write $f(z)\equiv f(x)$, where $z=x_1+ix_2$, and identify z with the geometric point x.

Suppose now that $C(\partial S)$ and $C^1(\partial S)$ are complex vector spaces, and construct the complex spaces $C^{0,\alpha}(\partial S)$ and $C^{1,\alpha}(\partial S)$ by defining Hölder continuity in terms of the inequality

$$|f(z)-f(\zeta)|\le c|z-\zeta|^\alpha \quad \text{for all} \quad z,\ \zeta\in\partial S,$$

and the derivative as

$$f'(z)=\frac{d}{dz}f(z)=\lim_{\zeta\to z}\frac{f(\zeta)-f(z)}{\zeta-z},\quad z,\ \zeta\in\partial S,$$

if this limit exists.

Since $|z-\zeta|=|x-y|$, where $\zeta=y_1+iy_2$, it is obvious that Hölder continuity with respect to z and Hölder continuity with respect to x (or s, according to the discussion in Sect. 1.2), are equivalent. The same can also be said about Hölder continuous differentiability on ∂S. We can see this from the equality

$$f'(s)=\vartheta(z)f'(z),$$

where

$$\vartheta(z) = \frac{dz}{ds} = \tau_1(z) + i\tau_2(z), \tag{2.34}$$

which means that $f' \in C^{0,\alpha}(\partial S)$ in terms of z if and only if $f' \in C^{0,\alpha}(\partial S)$ in terms of s, as implied by the statement of Lemma 1.20 and the fact that both $\vartheta(z)$ and $\big[\vartheta(z)\big]^{-1} = \bar{\vartheta}(z) = \tau_1(z) - i\tau_2(z)$ belong to $C^1(\partial S)$. This shows that our somewhat loose use of the same symbol for a function on ∂S whether it is expressed in terms of z or x is justified in relation to Hölder spaces.

In the light of these arguments, and because for a kernel $k(x,\ y)$ and a density φ on ∂S

$$\int_{\partial S} k(x,\ y)\varphi(y)\, ds(y) = \int_{\partial S} k(z,\ \zeta)\varphi(\zeta)\bar{\vartheta}(\zeta)\, d\zeta,$$

we conclude that the definition of γ-singular and proper γ-singular kernels on ∂S and all the associated results established in Sect. 1.4 on the behavior on ∂S of integrals with such kernels can be understood in terms of either real or complex variables.

2.19 Theorem. *If* $\varphi \in C^{0,\alpha}(\partial S)$, $\alpha \in (0, 1]$, *then*

$$(\Psi\varphi)(z) = \int_{\partial S} \frac{\varphi(\zeta)}{\zeta - z}\, d\zeta, \quad z \in \partial S, \tag{2.35}$$

exists in the sense of principal value, uniformly for all $z \in \partial S$, *and belongs to* $C^{0,\beta}(\partial S)$, *with* $\beta = \alpha$ for $\alpha \in (0, 1)$ *and any* $\beta \in (0, 1)$ *for* $\alpha = 1$.

Proof. Let $z = x_1 + ix_2$ and $\zeta = y_1 + iy_2$. Differentiating with respect to $s(y)$ the equality

$$\log(\zeta - z) = \ln|\zeta - z| + i\theta = \ln|x - y| + i\theta,$$

where $\theta = \arg(\zeta - z)$, and using the Cauchy–Riemann relation

$$\frac{\partial}{\partial s(y)}\theta(x,\ y) = \frac{\partial}{\partial \nu(y)} \ln|x - y|,$$

we obtain

$$\frac{d\zeta}{\zeta - z} = \frac{\partial}{\partial s(y)} \ln|x - y|\, ds(y) + i\frac{\partial}{\partial \nu(y)} \ln|x - y|\, ds(y). \tag{2.36}$$

Hence, we can write

$$\int_{\partial S\setminus\Sigma_{x,\delta}} \frac{\varphi(\zeta)}{\zeta - z}\, d\zeta = \int_{\partial S\setminus\Sigma_{x,\delta}} \left[\frac{\partial}{\partial s(y)} \ln|x - y|\right]\varphi(y)\, ds(y)$$
$$+ i \int_{\partial S\setminus\Sigma_{x,\delta}} \left[\frac{\partial}{\partial \nu(y)} \ln|x - y|\right]\varphi(y)\, ds(y),$$

and the result is obtained from Theorems 2.13 and 2.2 by letting $\delta \to 0$. □

2.20 Remark. The function $\Psi\varphi$ defined by (2.35) can be expressed in terms of an improper integral. Writing

$$\int_{\partial S\setminus\Sigma_{x,\delta}} \frac{\varphi(\zeta)}{\zeta - z}\, d\zeta = \int_{\partial S\setminus\Sigma x,\delta} \frac{\varphi(\zeta) - \varphi(z)}{\zeta - z}\, d\zeta + \varphi(z) \int_{\partial S\setminus\Sigma_{x,\delta}} \frac{d\zeta}{\zeta - z},$$

replacing $(\zeta - z)^{-1} d\zeta$ by its expression in (2.36), letting $\delta \to 0$, and using formulas (2.22) and (2.8), we find that, in the sense of principal value,

$$\int_{\partial S} \frac{\varphi(\zeta)}{\zeta - z}\, d\zeta = \pi i \varphi(z) + \int_{\partial S} \frac{\varphi(\zeta) - \varphi(z)}{\zeta - z}\, d\zeta, \quad z \in \partial S, \tag{2.37}$$

where the integrand of the last term is $O(|z - \zeta|^{\alpha-1})$ if $\varphi \in C^{0,\alpha}(\partial S)$, $\alpha \in (0, 1]$.

2.21 Theorem. *If* $\varphi \in C^{1,\alpha}(\partial S)$, $\alpha \in (0, 1]$, *then* $\Psi\varphi$ *defined by* (2.35) *belongs to* $C^{1,\beta}(\partial S)$, *with* $\beta = \alpha$ *for* $\alpha \in (0, 1)$ *and any* $\beta \in (0, 1)$ *for* $\alpha = 1$.

Proof. By (2.36), (2.21), and (2.4),

$$(\Psi\varphi)(z) = v_0^f(x) - i w_0(x),$$

and the assertion follows from Theorems 2.16 and 2.8. □

2.22 Theorem. *If* $K^s : C^{0,\alpha}(\partial S) \to C^{0,\alpha}(\partial S)$, $\alpha \in (0, 1)$, *is the operator defined by*

$$(K^s\varphi)(z) = \int_{\partial S} \frac{\varphi(\zeta)}{\zeta - z}\, d\zeta, \quad z \in \partial S, \tag{2.38}$$

then $(K^s)^2 = -\pi^2 I$, *where* I *is the identity operator*.

Proof. From Theorem 2.19 it is clear that the operator composition $(K^s)^2$ is meaningful.

In Muskhelishvili (1946) it is shown that a function $f(z, \zeta)$ which is Hölder continuous with respect to both its variables z and ζ satisfies the Poincaré–Bertrand formula

$$\begin{aligned}\int\limits_{\partial S} \frac{1}{\zeta - z}\Bigg[\int\limits_{\partial S} \frac{f(\zeta, \eta)}{\eta - \zeta}\, d\eta\Bigg] d\zeta \\ = -\pi^2 f(z, z) + \int\limits_{\partial S}\Bigg[\int\limits_{\partial S} \frac{f(\zeta, \eta)}{(\zeta - z)(\eta - \zeta)}\, d\zeta\Bigg] d\eta. \end{aligned}\tag{2.39}$$

Using (2.39) and the fact that, by (2.37) with $\varphi = 1$,

$$\int\limits_{\partial S} \frac{d\zeta}{\zeta - z} = \pi i, \quad z \in \partial S, \tag{2.40}$$

in the sense of principal value, we find that for any $\varphi \in C^{0,\alpha}(\partial S)$ and $z \in \partial S$

$$\begin{aligned}\big((K^s)^2\varphi\big)(z) &= \int\limits_{\partial S} \frac{1}{\zeta - z}\Bigg[\int\limits_{\partial S} \frac{\varphi(\eta)}{\eta - \zeta}\, d\eta\Bigg] d\zeta \\ &= -\pi^2\varphi(z) + \int\limits_{\partial S}\Bigg[\int\limits_{\partial S} \frac{\varphi(\eta)}{(\zeta - z)(\eta - \zeta)}\, d\zeta\Bigg] d\eta \\ &= -\pi^2\varphi(z) + \int\limits_{\partial S}\Bigg[\frac{1}{\eta - z}\Bigg(\int\limits_{\partial S} \frac{d\zeta}{\zeta - z} - \int\limits_{\partial S} \frac{d\zeta}{\zeta - \eta}\Bigg)\varphi(\eta)\Bigg] d\eta \\ &= -\pi^2\varphi(z),\end{aligned}$$

as required. □

2.23 Theorem. *Let $f(z, \zeta)$ be a function defined on $\partial S \times \partial S$, which belongs to $C^{0,\alpha}(\partial S)$, $\alpha \in (0, 1]$, with respect to each of its variables, uniformly relative to the other one, and satisfies the inequality*

$$|f(z, \zeta) - f(z', \zeta)| < c|z - z'|\,|z - \zeta|^{\alpha - 1}, \quad c = \text{const} > 0,$$

for all $z, z', \zeta \in \partial S$ such that $0 < |z - z'| < \frac{1}{2}|z - \zeta|$. *Then the function*

$$(\Lambda f)(z) = \int\limits_{\partial S} \frac{f(z, \zeta)}{\zeta - z}\, d\zeta, \quad z \in \partial S,$$

where the integral is understood as principal value, belongs to $C^{0,\beta}(\partial S)$, with $\beta = \alpha$ for $\alpha \in (0, 1)$ and any $\beta \in (0, 1)$ for $\alpha = 1$.

Proof. Let $z = x_1 + ix_2$. Writing

$$\int_{\partial S\setminus\Sigma_{x,r}} \frac{f(z,\,\zeta)}{\zeta - z}\,d\zeta = \int_{\partial S\setminus\Sigma_{x,r}} \frac{f(z,\,\zeta) - f(z,\,z)}{\zeta - z}\,d\zeta + f(z,\,z)\int_{\partial S\setminus\Sigma_{x,r}} \frac{d\zeta}{\zeta - z},$$

from Theorem 2.19 and the fact that the integrand of the first term on the right-hand side is $O(|z-\zeta|^{\alpha-1})$ we conclude that $(\Lambda f)(z)$ exists in the sense of principal value for all $z \in \partial S$.

To establish the Hölder continuity of Λf, for $z,\, z',\, \zeta \in \partial S$ we use the decomposition

$$\begin{aligned}
2\big[(\Lambda f)(z) - (\Lambda f)(z')\big] &= \int_{\partial S}\bigg[\frac{f(z,\,\zeta) - f(z,\,z)}{\zeta - z} - \frac{f(z,\,\zeta) - f(z,\,z')}{\zeta - z'}\bigg]d\zeta\\
&\quad + \int_{\partial S}\bigg[\frac{f(z',\,\zeta) - f(z',\,z)}{\zeta - z} - \frac{f(z',\,\zeta) - f(z',\,z')}{\zeta - z'}\bigg]d\zeta\\
&\quad + \int_{\partial S}\frac{f(z,\,\zeta) - f(z',\,\zeta)}{\zeta - z}\,d\zeta + \int_{\partial S}\frac{f(z,\,\zeta) - f(z',\,\zeta)}{\zeta - z'}\,d\zeta\\
&\quad + f(z,\,z)\int_{\partial S}\frac{d\zeta}{\zeta - z} - f(z,\,z')\int_{\partial S}\frac{d\zeta}{\zeta - z'}\\
&\quad + f(z',\,z)\int_{\partial S}\frac{d\zeta}{\zeta - z} - f(z',\,z')\int_{\partial S}\frac{d\zeta}{\zeta - z'}\\
&= I_1 + I_2 + I_3 + I_4 + I_5 + I_6.
\end{aligned}$$

Let $z' = x_1' + ix_2'$ and $\zeta = y_1 + iy_2$, and let $\Sigma_{x,r}$, Σ_1, and Σ_2 be the sets defined by (1.16) and (1.29) with x and x' satisfying (1.43). By Lemmas 1.10–1.13, (2.40), and Remark 1.41,

$$\begin{aligned}
|I_{11}| &= \bigg|\int_{\Sigma_1}\bigg[\frac{f(z,\,\zeta) - f(z,\,z)}{\zeta - z} - \frac{f(z,\,\zeta) - f(z,\,z')}{\zeta - z'}\bigg]d\zeta\bigg|\\
&\le c_1\int_{\Sigma_1}(|x - y|^{\alpha-1} + |x' - y|^{\alpha-1})\,ds(y) \le c_2|z - z'|^{\alpha},\\
|I_{12}| &= \bigg|\int_{\Sigma_2}\bigg(\frac{1}{\zeta - z} - \frac{1}{\zeta - z'}\bigg)\big[f(z,\,\zeta) - f(z,\,z')\big]\,d\zeta\bigg|\\
&\le c_3|z - z'|\int_{\Sigma_2}|x - y|^{\alpha-2}\,ds(y) \le c_4|z - z'|^{\alpha} \quad \text{if } \alpha \in (0,\,1),
\end{aligned}$$

$$|I_{12}| \le c_5|z-z'|\big|\ln|z-z'|\big| \quad \text{if } \alpha = 1,$$

$$|I_{13}| = \left| \int\limits_{\partial S \setminus \Sigma_{x,r}} \left(\frac{1}{\zeta - z} - \frac{1}{\zeta - z'} \right) \big[f(z,\, \zeta) - f(z,\, z') \big]\, d\zeta \right|$$

$$\le c_6|z-z'| \int\limits_{\partial S \setminus \Sigma_{x,r}} |x-y|^{\alpha-2}\, ds(y) \le c_7|z-z'|,$$

$$|I_{14}| = \left| \big[f(z,\, z') - f(z,\, z) \big] \int\limits_{\partial S \setminus \Sigma_1} \frac{d\zeta}{\zeta - z} \right| \le c_8|z-z'|^\alpha;$$

consequently,

$$|I_1| = |I_{11} + I_{12} + I_{13} + I_{14}| \le c_9|z-z'|^\beta,$$

where the constants $c_1, \dots, c_9 > 0$ may depend on α.

Similarly,

$$|I_2| \le c_{10}|z-z'|^\beta, \quad c_{10} = \text{const} > 0.$$

Next, we find that

$$|I_{31}| = \left| \int\limits_{\Sigma_1} \big\{ \big[f(z,\, \zeta) - f(z,\, z) \big] - \big[f(z',\zeta) - f(z',z) \big] \big\} \frac{d\zeta}{\zeta - z} \right|$$

$$\le c_{11} \int\limits_{\Sigma_1} |x-y|^{\alpha-1}\, ds(y) \le c_{12}|z-z'|^\alpha,$$

$$|I_{32}| = \left| \int\limits_{\Sigma_2} \frac{f(z,\, \zeta) - f(z',\, \zeta)}{\zeta - z}\, d\zeta \right|$$

$$\le c_{13}|z-z'| \int\limits_{\Sigma_2} |x-y|^{\alpha-2}\, ds(y) \le c_{14}|z-z'|^\alpha \quad \text{if } \alpha \in (0,1),$$

$$|I_{32}| \le c_{15}|z-z'|\,\big|\ln|z-z'|\big| \quad \text{if } \alpha = 1,$$

$$|I_{33}| = \left| \int\limits_{\partial S \setminus \Sigma_{x,r}} \frac{f(z,\, \zeta) - f(z',\, \zeta)}{\zeta - z}\, d\zeta \right|$$

$$\le c_{16}|z-z'| \int\limits_{\partial S \setminus \Sigma_{x,r}} |x-y|^{\alpha-2}\, ds(y) \le c_{17}|z-z'|,$$

$$|I_{34}| = \left| \big[f(z,\, z) - f(z',\, z) \big] \left(\int\limits_{\partial S} \frac{d\zeta}{\zeta - z} - \int\limits_{\partial S \setminus \Sigma_1} \frac{d\zeta}{\zeta - z} \right) \right|$$

$$\le c_{18}|z-z'|^\alpha;$$

therefore,

$$|I_3| = |I_{31} + I_{32} + I_{33} + I_{34}| \le c_{19}|z - z'|^\beta,$$

where the constants $c_{11}, \dots, c_{19} > 0$ may depend on α. In exactly the same way, but using $\Sigma_{x',r}$ instead of $\Sigma_{x,r}$, we find that

$$|I_4| \le c_{20}|z - z'|^\beta, \quad c_{20} = \text{const} > 0.$$

Finally,

$$|I_5| = \big|\pi i\big[f(z, z) - f(z, z')\big]\big| \le c_{21}|z - z'|^\alpha,$$
$$|I_6| = \big|\pi i\big[f(z', z) - f(z', z')\big]\big| \le c_{22}|z - z'|^\alpha,$$

where c_{21} and c_{22} are positive constants.

Combining the above inequalities, we now obtain

$$|(\Lambda f)(z) - (\Lambda f)(z')| \le c_{23}|z - z'|^\beta, \quad c_{23} = \text{const} > 0,$$

as required. □

2.4 Singular Integral Equations

We discuss briefly a few concepts of functional analysis, which will enable us to find the solutions of the boundary value problems to be stated later in Sect. 3.4. The presentation is made in terms of complex variables in order to simplify the technicalities involved. Any difference between the complex and real cases will be indicated explicitly.

2.24 Theorem. *$C^{0,\alpha}(\partial S)$ is a Banach space with norm*

$$\|\varphi\|_\alpha = \|\varphi\|_\infty + |\varphi|_\alpha, \tag{2.41}$$

where

$$\|\varphi\|_\infty = \sup_{z\in\partial S} |\varphi(z)|, \quad |\varphi|_\alpha = \sup_{\substack{z,\zeta\in\partial S\\ z\ne\zeta}} \frac{|\varphi(z) - \varphi(\zeta)|}{|z - \zeta|^\alpha}.$$

Proof. As can easily be verified, (2.41) satisfies satisfies the norm axioms.

Let $\{\varphi_n\}_{n=1}^{\infty}$ be a Cauchy sequence in $C^{0,\alpha}(\partial S)$; that is, for any $\varepsilon > 0$ arbitrarily small there is a positive integer $n_0(\varepsilon)$ such that

$$\|\varphi_n - \varphi_m\|_\alpha < \varepsilon \quad \text{for all } n,\ m > n_0(\varepsilon).$$

By (2.41),

$$\|\varphi_n - \varphi_m\|_\infty < \varepsilon \quad \text{for all } n,\ m > n_0(\varepsilon),$$

which means that $\{\varphi_n\}_{n=1}^{\infty}$ is also a Cauchy sequence in $C(\partial S)$. Since $C(\partial S)$ is a complete space, there is a $\varphi \in C(\partial S)$ such that

$$\|\varphi_n - \varphi\|_\infty \to 0 \quad \text{as } n \to \infty. \tag{2.42}$$

From (2.41) we also deduce that

$$|\varphi_n - \varphi_m|_\alpha < \varepsilon \quad \text{for all } n,\ m > n_0(\varepsilon).$$

Letting $m \to \infty$ and using the uniform convergence of $\{\varphi_n\}_{n=1}^{\infty}$ on ∂S, we now obtain

$$|\varphi_n - \varphi|_\alpha < \varepsilon \quad \text{for all } n > n_0(\varepsilon). \tag{2.43}$$

Hence, there is $c = \text{const} > 0$ such that for all $z,\ \zeta \in \partial S$, $z \neq \zeta$,

$$\frac{|\varphi(z) - \varphi(\zeta)|}{|z - \zeta|^\alpha} \leq |\varphi|_\alpha \leq c;$$

in other words, $\varphi \in C^{0,\alpha}(\partial S)$. Also, from (2.42) and (2.43) it follows that

$$\|\varphi_n - \varphi\|_\alpha \to 0 \quad \text{as } n \to \infty;$$

that is, $\{\varphi_n\}_{n=1}^{\infty}$ converges in the norm (2.41), which means that $C^{0,\alpha}(\partial S)$ is complete. □

2.25 Definition. Let X and Y be normed spaces. A linear operator $K : X \to Y$ is called *compact* if it maps any bounded set in X into a relatively compact set in Y (that is, a set in which every sequence contains a convergent subsequence).

2.26 Theorem. *If $k(z,\ \zeta)$ is a proper γ-singular kernel on ∂S, $\gamma \in [0, 1)$, then the operator K defined by*

$$(K\varphi)(z) = \int\limits_{\partial S} k(z,\ \zeta)\varphi(\zeta)\, d\zeta, \quad z \in \partial S, \tag{2.44}$$

is a compact operator from $C^{0,\alpha}(\partial S)$ to $C^{0,\alpha}(\partial S)$, *with* $\alpha = 1-\gamma$ *for* $\gamma \in (0,1)$ *and any* $\alpha \in (0,1)$ *for* $\gamma = 0$.

Proof. According to Theorem 1.33 and the fact that $C^{0,\alpha}(\partial S) \subset C(\partial S)$, the operator $K : C^{0,\alpha}(\partial S) \to C^{0,\alpha}(\partial S)$ is well defined.

Let $M_1 \subset C^{0,\alpha}(\partial S)$ be a bounded set; that is,

$$\|\varphi\|_\alpha \le c = \text{const} > 0 \quad \text{for all } \varphi \in M_1. \tag{2.45}$$

Also, let $\{\theta_n\}_{n=1}^\infty \subset M_2 = K(M_1)$. We denote by $\{\varphi_n\}_{n=1}^\infty$ a sequence in M_1 such that $\theta_n = K\varphi_n$, $n = 1, 2, \dots$.

In view of (2.41), inequality (2.45) implies that

$$\sup_{z\in\partial S} |\varphi_n(z)| \le c,$$
$$|\varphi_n(z) - \varphi_n(z')| \le c|z - z'|^\alpha$$

for all $n = 1, 2, \dots$ and all $z,\ z' \in \partial S$; in other words, $\{\varphi_n\}_{n=1}^\infty$ is uniformly bounded and equicontinuous in $C(\partial S)$. By the Arzelà–Ascoli theorem (Colton and Kress 1983), it contains a uniformly convergent subsequence. For simplicity, we denote this subsequence again by $\{\varphi_n\}_{n=1}^\infty$. Hence, there is a $\varphi \in C(\partial S)$ such that

$$\|\varphi_n - \varphi\|_\infty \to 0 \quad \text{as } n \to \infty. \tag{2.46}$$

Let $\theta = K\varphi$. By Theorem 1.33, $\theta \in C^{0,\alpha}(\partial S)$. For $z \in \partial S$ we have

$$\begin{aligned}|\theta_n(z) - \theta(z)| &\le \int\limits_{\partial S} |k(z,\ \zeta)|\,|\varphi_n(\zeta) - \varphi(\zeta)|\,d\zeta \\ &\le c_1 \sup_{x\in\partial S} |\varphi_n(z) - \varphi(z)| \int\limits_{\partial S} |z-\zeta|^{-\gamma}\,d\zeta;\end{aligned}$$

consequently, by Theorem 1.32,

$$\|\theta_n - \theta\|_\infty \le c_2\|\varphi_n - \varphi\|_\infty, \quad n = 1, 2, \dots,$$

where c_1 and c_2 are positive constants. On the other hand, by Theorem 1.33,

$$|\theta_n - \theta|_\alpha \le c_3\|\varphi_n - \varphi\|_\alpha, \quad n = 1, 2, \dots.$$

The last two inequalities, (2.41), and (2.46) yield

$$\|\theta_n - \theta\|_\alpha \to 0 \quad \text{as} \quad n \to \infty,$$

which proves that K is compact on $C^{0,\alpha}(\partial S)$. □

2.27 Definition. Let X and Y be vector spaces over $\mathbb{C}$. A mapping $(\cdot\,,\cdot) : X \times Y \to \mathbb{C}$ is called a *non-degenerate bilinear form* if

(i) for any $\varphi \in X$, $\varphi \neq 0$, there is a $\psi \in Y$ such that $(\varphi,\ \psi) \neq 0$, and for any $\psi \in Y$, $\psi \neq 0$, there is a $\varphi \in X$ such that $(\varphi,\ \psi) \neq 0$;

(ii) for any $\varphi_1,\ \varphi_2,\ \varphi \in X$, $\psi_1,\ \psi_2,\ \psi \in Y$, and $\alpha_1,\ \alpha_2 \in \mathbb{C}$,

$$(\alpha_1\varphi_1 + \alpha_2\varphi_2,\ \psi) = \alpha_1(\varphi_1,\ \psi) + \alpha_2(\varphi_2,\ \psi),$$
$$(\varphi,\ \alpha_1\psi_1 + \alpha_2\psi_2) = \alpha_1(\varphi,\ \psi_1) + \alpha_2(\varphi,\ \psi_2).$$

2.28 Definition. By a *dual system* $(X,\ Y)$ we understand a pair of normed spaces X and Y together with a non-degenerate bilinear form $(\cdot\,,\cdot) : X \times Y \to \mathbb{C}$.

2.29 Definition. Let $(X,\ Y)$ be a dual system with bilinear form $(\cdot\,,\cdot)$. Two operators $K : X \to X$ and $K^* : Y \to Y$ are called *adjoint* if

$$(K\varphi,\ \psi) = (\varphi,\ K^*\psi) \tag{2.47}$$

for all $\varphi \in X$ and $\psi \in Y$.

2.30 Remark. It can be shown without difficulty (Colton and Kress 1983) that if an operator $K : X \to X$ has an adjoint $K^* : Y \to Y$ in a dual system $(X,\ Y)$, then K^* is unique, and both K and K^* are linear.

2.31 Definition. Let $(X,\ Y)$ be a dual system with bilinear form $(\cdot\,,\cdot)$, $K : X \to X$ an operator that has a (unique) adjoint $K^* : Y \to Y$, I the identity operator (which, for simplicity, is denoted by the same symbol regardless of the space where it acts), and $\omega \in \mathbb{C}$, $\omega \neq 0$, and consider the equations

$$(K - \omega I)\varphi = f, \quad f \in X, \tag{K}$$
$$(K^* - \omega I)\psi = g, \quad g \in Y, \tag{K*}$$

together with their homogeneous versions (K_0) and (K_0^*). We say that *the Fredholm Alternative holds for K in $(X,\ Y)$* if either

(i) (K_0) has only the zero solution, in which case so does (K_0^*), and (K) and (K*) have unique solutions for any $f \in X$ and $g \in Y$, respectively, or

(ii) (K_0) and (K_0^*) have the same finite number of linearly independent solutions $\{\varphi_1, \dots, \varphi_n\}$ and $\{\psi_1, \dots, \psi_n\}$, and (K) and (K*) are solvable, respectively, if and only if

$$(f, \psi_i) = 0, \quad (\varphi_i, g) = 0, \quad i = 1, \dots, n.$$

2.32 Theorem. *Let $(X,\ Y)$ be a dual system and $K : X \to X$ a compact linear operator that has a (unique) compact adjoint $K^* : Y \to Y$. Then the Fredholm Alternative holds for K* in $(X,\ Y)$.

A full, detailed proof of this assertion can be found, for example, in the monograph (Colton and Kress 1983).

2.33 Remark. Let K be the operator defined by (2.44), and consider the dual system $\big(C^{0,\alpha}(\partial S),\ C^{0,\alpha}(\partial S)\big)$, $\alpha \in (0, 1)$, with the bilinear form

$$(\varphi,\ \psi) = \int\limits_{\partial S} \varphi(\zeta)\psi(\zeta)\,d\zeta, \quad \varphi,\ \psi \in C^{0,\alpha}(\partial S), \tag{2.48}$$

which is easily seen to satisfy the conditions in Definition 2.2.7. From (2.47) and (2.48) we have

$$\begin{aligned}(K\varphi,\ \psi) &= \int\limits_{\partial S} \Big[\int\limits_{\partial S} k(z,\ \zeta)\varphi(\zeta)\,d\zeta\Big]\psi(z)\,dz\\ &= \int\limits_{\partial S} \varphi(\zeta)\Big[\int\limits_{\partial S} k(z,\ \zeta)\psi(z)\,dz\Big]d\zeta\\ &= (\varphi,\ K^*\psi),\end{aligned}$$

where

$$(K^*\varphi)(z) = \int\limits_{\partial S} k^*(z,\ \zeta)\varphi(\zeta)\,d\zeta,$$

with $k^*(z,\ \zeta) = k(\zeta,\ z)$.

This means that if $k(z,\ \zeta)$ is a proper $(1-\alpha)$-singular kernel on ∂S with respect to both z and ζ, then, by Theorems 2.26 and 2.32, the Fredholm Alternative holds for K.

The Fredholm Alterative does not hold in general for operators with 1-singular kernels. However, there is a class of such operators for which the assertion remains true.

Theorem 2.23 enables us to introduce the following concept.

2.34 Definition. An operator $K : C^{0,\alpha}(\partial S) \to C^{0,\alpha}(\partial S)$, $\alpha \in (0, 1)$, is called *α-regular singular* if it is defined by an expression of the form

$$(K\varphi)(z) = \int\limits_{\partial S} \frac{\hat{k}(z,\ \zeta)}{\zeta - z}\,\varphi(\zeta)\,d\zeta, \quad z \in \partial S, \tag{2.49}$$

where $\hat{k}(z,\ \zeta)$ belongs to $C^{0,\alpha}(\partial S)$ with respect to each variable, uniformly relative to the other one, and satisfies the inequality

$$|\hat{k}(z,\ \zeta) - \hat{k}(z',\ \zeta)| \le c|z - z'|\,|z - \zeta|^{\alpha-1}, \quad c = \text{const} > 0,$$

for all $z, z', \zeta \in \partial S$ such that $0 < |z - z'| < \frac{1}{2}|z - \zeta|$. (The value of $\hat{k}(z, \zeta)$ at $z = \zeta$ may also be understood in the sense of continuous extension.)

2.35 Theorem. *If $k(x, y)$ is a proper γ-singular kernel on ∂S, $\gamma \in [0, 1)$, with respect to both x and y, then the operator K on $C^{0,1-\gamma}(\partial S)$ defined by*

$$(K\varphi)(x) = \int_{\partial S} k(x, y)\varphi(y)\, ds(y), \quad x \in \partial S,$$

is $(1-\gamma)$-regular singular, and $\hat{k}(z, \zeta)$ in (2.49) satisfies

$$\hat{k}(z, z) = 0, \quad z \in \partial S.$$

Proof. Clearly, $(K\varphi)(x)$ is an improper integral for all $x \in \partial S$.

In accordance with our notational convention, we write

$$\int_{\partial S} k(x, y)\varphi(y)\, ds(y) = \int_{\partial S} k(z, \zeta)\varphi(\zeta)\bar{\vartheta}(\zeta)\, d\zeta = \int_{\partial S} \frac{\hat{k}(z, \zeta)}{\zeta - z}\, d\zeta,$$

where $\vartheta(z)$ is defined by (2.34) and

$$\hat{k}(z, \zeta) = (\zeta - z)k(z, \zeta)\bar{\vartheta}(\zeta).$$

By Definition 1.26, $\hat{k}(z, z) = 0$ in the sense of continuous extension.

Let $z, z', \zeta \in \partial S$ be such that $0 < |z - z'| < \frac{1}{2}|z - \zeta|$. In this case

$$|z' - \zeta| \geq |z - \zeta| - |z - z'| > |z - \zeta| - \tfrac{1}{2}|z - \zeta| = \tfrac{1}{2}|z - \zeta|. \tag{2.50}$$

Since $|\vartheta(\zeta)| = 1$, $\zeta \in \partial S$, we have

$$\begin{aligned}|\hat{k}(z, \zeta) - \hat{k}(z', \zeta)| &= |(\zeta - z)k(z, \zeta) - (\zeta - z')k(z', \zeta)| \\ &\leq |z - \zeta|\,|k(z, \zeta) - k(z', \zeta)| + |z - z'|\,|k(z', \zeta)| \\ &\leq c_1|z - z'|(|z - \zeta|^{-\gamma} + |z' - \zeta|^{-\gamma}),\end{aligned}$$

which, on the basis of (2.50), shows that

$$|\hat{k}(z, \zeta) - \hat{k}(z', \zeta)| \leq c_2|z - z'|\,|z - \zeta|^{-\gamma}$$

and that

$$|\hat{k}(z, \zeta) - \hat{k}(z', \zeta)| \leq c_3|z - z'|^{1-\gamma},$$

where the constants $c_1,\ c_2,\ c_3 > 0$ are independent of z, z', and ζ.

If $|z - z'| \geq \frac{1}{2}|z - \zeta|$, then

$$|z' - \zeta| \leq |z - z'| + |z - \zeta| \leq 3|z - z'|;$$

consequently,

$$\begin{aligned}|\hat{k}(z,\ \zeta) - \hat{k}(z',\ \zeta)| &\leq |z - \zeta|\,|k(z,\ \zeta)| + |z' - \zeta|\,|k(z',\ \zeta)| \\ &\leq c_1(|z - \zeta|^{1-\gamma} + |z' - \zeta|^{1-\gamma}) \leq c_4|z - z'|^{1-\gamma},\end{aligned}$$

where c_4 is independent of z, z', and ζ.

The Hölder continuity of $\hat{k}(z,\ \zeta)$ with respect to ζ is proved similarly by writing

$$\begin{aligned}|\hat{k}(z,\ \zeta) - \hat{k}(z,\ \zeta')| &\leq |\zeta - z|\,|k(z,\ \zeta)|\,|\vartheta(\zeta) - \vartheta(\zeta')| \\ &\quad + |\zeta - z|\,|k(z,\ \zeta) - k(z,\ \zeta')| + |\zeta - \zeta'|\,|k(z,\ \zeta')|\end{aligned}$$

and using the fact that $\vartheta \in C^{0,\alpha}(\partial S)$. □

2.36 Definition. Consider an equation of the form

$$(K - \omega I)\varphi = f \quad \text{on } \partial S, \tag{2.51}$$

where K is an α-regular singular operator, φ and f are 3×1 matrices in $C^{0,\alpha}(\partial S)$, $\alpha \in (0, 1)$, $\omega \in \mathbb{C}$, $\omega \neq 0$, and $\det\big[-\omega E_3 \pm \pi i\hat{k}(z,\ z)\big]$ (see Definition 2.34), where E_3 is the identity 3×3 matrix, do not vanish on ∂S. The number (Muskhelishvili 1946)

$$\rho = \frac{1}{2\pi}\left[\arg\frac{\det\big(-\omega E_3 - \pi i\hat{k}(z,\ z)\big)}{\det\big(-\omega E_3 + \pi i\hat{k}(z,\ z)\big)}\right]_{\partial S}, \tag{2.52}$$

where $\big[\theta(z)\big]_{\partial S}$ denotes the change in $\theta(z)$ as z traverses ∂S once anticlockwise, is called the *index* of equation (2.51).

When (2.51) is a scalar equation, the symbols det and E_3 are dropped in (2.52).

2.37 Remark. Let K be an α-regular singular operator, $\alpha \in (0, 1)$, and consider the dual system $\big(C^{0,\alpha}(\partial S),\ C^{0,\alpha}(\partial S)\big)$ with the bilinear form (2.48). Then from (2.47) and (2.49) we conclude that the kernel of the adjoint K^* of K is $-\hat{k}(\zeta,\ z)/(\zeta - z)$, which implies that, by (2.52), the index of the equation

$$(K^* - \omega I)\psi = g \text{ on } \partial S, \quad g \in C^{0,\alpha}(\partial S),$$

is equal to $-\rho$.

2.38 Theorem. *If K is an α-regular singular operator*, $\alpha \in (0, 1)$, *such that the index of the equation* (K) *is zero*, *then the Fredholm Alternative holds for K in the dual system* $\big(C^{0,\alpha}(\partial S),\ C^{0,\alpha}(\partial S)\big)$ *with the bilinear form* (2.48).

A comprehensive discussion of this assertion can be found in Muskhelishvili (1946) and Kupradze et al. (1979). Its proof consists of two stages. First, it is shown that we can always find an α-regular singular operator L and a $\vartheta \in \mathbb{C}$, $\vartheta \neq 0$, such that the equation

$$(L - \vartheta I)(K - \omega I)\varphi = (L - \vartheta I) f$$

is of the form

$$(\tilde{K} - \tilde{\omega} I)\tilde{\varphi} = \tilde{f}, \qquad (\tilde{K})$$

where $\tilde{\omega} \in \mathbb{C}$, $\tilde{\omega} \neq 0$, $\tilde{f} \in C^{0,\alpha}(\partial S)$, $\tilde{K}$ is an integral operator defined by

$$(\tilde{K}\tilde{\varphi})(z) = \int_{\partial S} \tilde{k}(z,\ \zeta)\tilde{\varphi}(\zeta)\, d\zeta, \quad z \in \partial S,$$

and $\tilde{k}(z,\ \zeta)$ is a proper $(1-\alpha)$-singular kernel on ∂S with respect to both z and ζ. By Remark 2.33, the Fredholm Alternative holds for the operator K in the dual system $\big(C^{0,\alpha}(\partial S),\ C^{0,\alpha}(\partial S)\big)$ with the bilinear form (2.48). The second part of the proof consists in showing that, since the indices of both (K) and (K*) are zero (the latter according to Remark 2.37), (K), ($\tilde{\text{K}}$) and (K*), ($\tilde{\text{K}}$*) have, respectively, the same solutions.

2.39 Remark. Let $K : C^{0,\alpha}(\partial S) \to C^{0,\alpha}(\partial S)$, $\alpha \in (0, 1)$, be an operator of the form

$$(K\varphi)(z) = \int_{\partial S} k(z,\ \zeta)\varphi(\zeta)\, d\zeta, \quad z \in \partial S,$$

and consider the corresponding equation (K)

$$-\omega\varphi(z) + \int_{\partial S} k(z,\ \zeta)\varphi(\zeta)\, d\zeta = f(z), \quad z \in \partial S.$$

Now suppose that, for f and ω real, when we change from z and ζ to x and y the transformed equation

$$-\omega\varphi(x) + \int_{\partial S} k(x,\ y)\vartheta(y)\varphi(y)\, ds(y) = f(x), \quad x \in \partial S,$$

where ϑ is defined by (2.34), is real. By Remark 2.33, the kernel of the adjoint operator K_c^* in the complex dual system $\big(C^{0,\alpha}(\partial S),\ C^{0,\alpha}(\partial S)\big)$ with the bilinear form (2.48) is

$$k_c^*(z,\ \zeta) = k(\zeta,\ z) = k(y,\ x)\vartheta(y).$$

On the other hand, it is easy to see that the kernel of the adjoint K_r^* in the real dual system $\big(C^{0,\alpha}(\partial S),\ C^{0,\alpha}(\partial S)\big)$ with the bilinear form

$$(\varphi,\ \psi) = \int\limits_{\partial S} \varphi(y)\psi(y)\,ds(y) \tag{2.53}$$

is

$$k_r^*(x,\ y) = k(y,\ x)\vartheta(x),$$

which is different from $k_c^*(z,\ \zeta)$. In Muskhelishvili (1946) it is shown that if the Fredholm Alternative holds for the operator K in the complex system, then it also holds for it in the real system, provided that we confine ourselves to real solutions of (K) and (K*) with $K^* = K_r^*$.

2.40 Remark. If $C^{0,\alpha}(\partial S)$ is understood as a space of 3×1 matrix functions, then φ is replaced by φ^{T} in (2.48) and (2.53).

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Chapter 3
Bending of Elastic Plates

3.1 The Two-Dimensional Plate Model

We consider the averaging operators $\mathscr{I}_{\alpha-1}$ and $\mathscr{J}_{\alpha-1}$ defined by

$$(\mathscr{I}_{\alpha-1}s)(x_\gamma) = \frac{1}{h_0}\left[x_3^{\alpha-1}s(x_i)\right]_{x_3=-h_0/2}^{x_3=h_0/2},$$
$$(\mathscr{J}_{\alpha-1}s)(x_\gamma) = \frac{1}{h_0}\int_{-h_0/2}^{h_0/2} x_3^{\alpha-1}s(x_i)\,dx_3. \tag{3.1}$$

Setting

$$\begin{aligned} N_{\alpha\beta} &= \mathscr{J}_1 t_{\alpha\beta},\\ N_{3\alpha} &= \mathscr{J}_0 t_{3\alpha},\\ g_\alpha &= \mathscr{J}_1 f_\alpha + \mathscr{I}_1 t_{3\alpha},\\ g_3 &= \mathscr{J}_0 f_3 + \mathscr{I}_0 t_{33}, \end{aligned} \tag{3.2}$$

from (1.1) and (3.1) we obtain the equilibrium equations

$$\begin{aligned} N_{\alpha\beta,\beta} - N_{3\alpha} + g_\alpha &= 0,\\ N_{3\beta,\beta} + g_3 &= 0. \end{aligned} \tag{3.3}$$

By analogy with (1.3), for a direction $n = (n_1, n_2)^{\mathrm{T}}$ in the middle plane we write

$$\begin{aligned} N_\alpha &= N_{\alpha\beta}n_\beta,\\ N_3 &= N_{3\beta}n_\beta. \end{aligned} \tag{3.4}$$

3.1 Remarks. It is easy to verify that $N_{3\alpha}$, $N_{\alpha\alpha}$ (α not summed), and $N_{12} = N_{21}$ are, respectively, the averaged transverse shear forces and averaged bending and

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twisting moments with respect to the middle plane, acting on the face of a vertical cross-section element of the plate perpendicular to the x_α-axis. Similarly, $\mathscr{J}_0 f_3$ and $\mathscr{J}_1 f_\alpha$ are the averaged body forces and moments, while $\mathscr{I}_0 t_{33}$ and $\mathscr{I}_1 t_{3\alpha}$ are the resultant averaged force and moments acting on the faces $x_3 = \pm h_0/2$. For simplicity, from now on we will omit the word 'averaged' when referring to forces and moments in the plate.

It can also be seen that the components of the moment with respect to the middle plane and the transverse shear force in a direction n are $\varepsilon_{\beta\alpha} N_\beta$ and N_3. If the moment is computed with respect to the origin of coordinates, then its components are $\varepsilon_{\beta\alpha}(N_\beta - x_\beta N_3)$. Clearly, knowing the N_i at a point is equivalent to knowing the $\varepsilon_{\beta\alpha}(N_\beta - x_\beta N_3)$ and N_3.

To avoid a clash of notation with the harmonic single-layer potential, in what follows we write u_i in place of v_i on the right-hand side of (1.6). Thus, from (1.6), (1.2), and (3.1) we obtain the constitutive relations

$$\begin{aligned} N_{\alpha\beta} &= h^2\big[\lambda u_{\gamma,\gamma}\delta_{\alpha\beta} + \mu(u_{\alpha,\beta} + u_{\beta,\alpha})\big],\\ N_{3\alpha} &= \mu(u_\alpha + u_{3,\alpha}), \end{aligned} \tag{3.5}$$

where $h^2 = h_0^2/12$. The same formulas and (1.3) show that

$$N_\alpha = \mathscr{J}_1 t_\alpha, \quad N_3 = \mathscr{J}_0 t_3.$$

To establish the compatibility conditions for the $N_{i\alpha}$, first we use (3.5) to deduce that

$$\begin{aligned} u_{1,1} &= \frac{1}{2h^2\mu}\big[(1-\sigma)N_{11} - \sigma N_{22}\big],\\ u_{2,2} &= \frac{1}{2h^2\mu}\big[(1-\sigma)N_{22} - \sigma N_{11}\big],\\ u_{1,2} + u_{2,1} &= \frac{1}{h^2\mu}N_{12},\\ u_{1,2} - u_{2,1} &= \frac{1}{\mu}(N_{31,2} - N_{32,1}), \end{aligned} \tag{3.6}$$

where

$$\sigma = \frac{\lambda}{2(\lambda+\mu)}$$

is Poisson's ratio. The compatibility conditions are then derived by equating $u_{\alpha,12}$ with $u_{\alpha,21}$ after computing these derivatives from (3.6). As a result, we obtain

$$\begin{aligned} &h^2(N_{31,12} - N_{32,11}) + N_{12,1} - (1-\sigma)N_{11,2} + \sigma N_{22,2} = 0,\\ &h^2(N_{32,12} - N_{31,22}) + N_{12,2} + \sigma N_{11,1} - (1-\sigma)N_{22,1} = 0. \end{aligned} \tag{3.7}$$

From (3.3) and (3.5) we find that the equilibrium equations in terms of the displacements are

$$A(\partial_x)u(x) + g(x) = 0, \tag{3.8}$$

where $A(\partial_x) = A(\partial/\partial x_\gamma)$ and $A(\xi) = A(\xi_\gamma)$ is the matrix

$$\begin{pmatrix} h^2\mu\Delta + h^2(\lambda+\mu)\xi_1^2 - \mu & h^2(\lambda+\mu)\xi_1\xi_2 & -\mu\xi_1 \\ h^2(\lambda+\mu)\xi_1\xi_2 & h^2\mu\Delta + h^2(\lambda+\mu)\xi_2^2 - \mu & -\mu\xi_2 \\ \mu\xi_1 & \mu\xi_2 & \mu\Delta \end{pmatrix}, \tag{3.9}$$

$u = (u_1, u_2, u_3)^{\mathrm{T}}$, $g = (g_1, g_2, g_3)^{\mathrm{T}}$, and $\Delta = \xi_\alpha\xi_\alpha$. Then the vector N of components

$$N_i = N_{i\beta}n_\beta$$

can be written as

$$N(x) = T(\partial_x;\, n)u(x), \tag{3.10}$$

where $T(\partial_x;\, n) = T(\partial/\partial x_\gamma;\, n)$ and $T(\xi;\, n) = T(\xi_\gamma;\, n_\delta)$ is the matrix

$$\begin{pmatrix} h^2(\lambda+2\mu)n_1\xi_1 + h^2\mu n_2\xi_2 & h^2\mu n_2\xi_1 + h^2\lambda n_1\xi_2 & 0 \\ h^2\lambda n_2\xi_1 + h^2\mu n_1\xi_2 & h^2\mu n_1\xi_1 + h^2(\lambda+2\mu)n_2\xi_2 & 0 \\ \mu n_1 & \mu n_2 & \mu n_\alpha\xi_\alpha \end{pmatrix}. \tag{3.11}$$

For brevity, we also write $T(\partial_x;\, \nu) \equiv T(\partial_x) \equiv T$.

From (1.4), (1.6), (3.1), (3.2), and (3.5) we see that the internal energy density per unit area of the middle plane is

$$\begin{aligned} E(u,\, u) = \mathscr{J}_0\mathscr{E} &= \tfrac{1}{4}\, N_{\alpha\beta}(u_{\alpha,\beta} + u_{\beta,\alpha}) + \tfrac{1}{2}\, N_{3\alpha}(u_\alpha + u_{3,\alpha}) \\ &= \tfrac{1}{2}\, \big\{h^2\big[\lambda u_{\alpha,\alpha}u_{\beta,\beta} + \mu u_{\alpha,\beta}(u_{\alpha,\beta} + u_{\beta,\alpha})\big] \\ &\qquad + \mu(u_\alpha + u_{3,\alpha})(u_\alpha + u_{3,\alpha})\big\}. \end{aligned} \tag{3.12}$$

Throughout what follows we assume that the Lamé constants satisfy the conditions

$$\lambda + \mu > 0, \quad \mu > 0. \tag{3.13}$$

3.2 Theorem. *$E(u,\, u)$ is a positive quadratic form and (3.8) an elliptic system.*

Proof. From (3.12) it follows that

$$\begin{aligned} E(u,\, u) = \tfrac{1}{2}\big\{&h^2\big[E_0(u,\, u) + \mu(u_{1,2} + u_{2,1})^2\big] \\ &+ \mu\big[(u_1 + u_{3,1})^2 + (u_2 + u_{3,2})^2\big]\big\}, \end{aligned} \tag{3.14}$$

where

$$E_0(u, u) = (\lambda + 2\mu)u_{1,1}^2 + 2\lambda u_{1,1}u_{2,2} + (\lambda + 2\mu)u_{2,2}^2. \tag{3.15}$$

We now easily verify that (3.13) are necessary and sufficient conditions for $E_0(u, u)$ to be a positive quadratic form.

The second part of the assertion is obtained from the fact that, by (3.9), the matrix $A_0(\xi)$ corresponding to the second-order derivatives in system (3.8) is invertible for all $\xi \neq 0$ since

$$\det A_0(\xi) = a_1(\xi_1^2 + \xi_2^2)^3,$$

where

$$a_1 = h^4\mu^2(\lambda+2\mu) > 0. \qquad \square$$

3.3 Theorem. $E(u, u) = 0$ *if and only if*

$$u(x) = (c_1, c_2, c_0 - c_1x_1 - c_2x_2)^{\mathrm{T}}, \tag{3.16}$$

where $c_0, c_\alpha =$ const.

Proof. Replacing (3.16) in (3.12), we see immediately that $E(u, u) = 0$.

Conversely, suppose that $E(u, u) = 0$. From (3.14) and (3.15) we get

$$\begin{aligned} u_{1,1} = u_{2,2} &= 0,\\ u_{1,2} + u_{2,1} &= 0,\\ u_{3,\alpha} + u_\alpha &= 0. \end{aligned}$$

The first two equations yield the equalities

$$\begin{aligned} u_1 &= \chi_1(x_2),\\ u_2 &= \chi_2(x_1), \end{aligned}$$

which, replaced in the second relation, lead to

$$\begin{aligned} \chi_1(x_2) &= kx_2 + c_1,\\ \chi_2(x_1) &= -kx_1 + c_2, \end{aligned}$$

where k, c_1, and c_2 are arbitrary constants. Using the compatibility condition for u_3, that is, $u_{3,12} = u_{3,21}$, from the last two equations we find that $k = 0$. Hence, $u_\alpha = c_\alpha$, so that

$$u_{3,\alpha} = -c_\alpha.$$

Integrating the equation for $\alpha = 1$ and substituting the result into that for $\alpha = 2$, we obtain

$$u_3 = c_0 - c_\alpha x_\alpha,$$

where c_0 is an arbitrary constant. □

3.4 Remarks. (i) Since the three-dimensional displacement field we are investigating is of the form

$$\big(x_3 u_1(x_1, x_2),\ x_3 u_2(x_1, x_2),\ u_3(x_1, x_2)\big)^{\mathrm{T}},$$

the most general admissible translation and rotation vectors are, respectively, of the form

$$(0,\ 0,\ a)^{\mathrm{T}},$$
$$(-x_3 b_2,\ x_3 b_1,\ x_1 b_2 - x_2 b_1)^{\mathrm{T}},$$

where a, b_1, and b_2 are arbitrary constants. Therefore, setting $a = c_0$, $b_1 = c_2$, and $b_2 = -c_1$, we conclude that (3.16) represents an arbitrary rigid displacement.

(ii) It is obvious that the columns of the matrix

$$F = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -x_1 & -x_2 & 1 \end{pmatrix} \tag{3.17}$$

form a basis for the space of rigid displacements, and that a generic rigid displacement of the form (3.16) can be written as

$$u = Fc,$$

where $c = (c_1,\ c_2,\ c_0)^{\mathrm{T}}$.

3.5 Theorem. *If* $u \in C^2(S^+) \cap C^1(\bar{S}^+)$, *then*

$$\int_{S^+} (A_{\alpha i} - x_\alpha A_{3i}) u_i \, da = \int_{\partial S} (T_{\alpha i} - x_\alpha T_{3i}) u_i \, ds,$$
$$\int_{S^+} A_{3i} u_i \, da = \int_{\partial S} T_{3i} u_i \, ds.$$

Proof. Since (3.3) and (3.8) are the same system and (3.4) and (3.10) the same 3×1 matrix, by the divergence theorem,

$$\begin{aligned}\int\limits_{S^+}(A_{\alpha i}-x_\alpha A_{3i})u_i\,da&=\int\limits_{S^+}(N_{\alpha\beta,\beta}-N_{3\alpha}-x_\alpha N_{3\beta,\beta})\,da\\&=\int\limits_{\partial S}(N_{\alpha\beta}-x_\alpha N_{3\beta})\nu_\beta\,ds=\int\limits_{\partial S}(T_{\alpha i}-x_\alpha T_{3i})u_i\,ds,\\\int\limits_{S^+}A_{3i}u_i\,da&=\int\limits_{S^+}N_{3\beta,\beta}\,da\\&=\int\limits_{\partial S}N_{3\beta}\nu_\beta\,ds=\int\limits_{\partial S}T_{3i}u_i\,ds.\end{aligned}$$

3.6 Theorem. (Betti formula) *If* $u\in C^2(S^+)\cap C^1(\bar S^+)$ *is a solution of the homogeneous system* (3.8), *then*

$$2\int\limits_{S^+}E(u,\,u)\,da=\int\limits_{\partial S}u^{\mathrm T}Tu\,ds. \tag{3.18}$$

Proof. Using (3.8), (3.3), (3.10), (3.4), the divergence theorem, and (3.12), we find that for any $u\in C^2(S^+)\cap C^1(\bar S^+)$,

$$\begin{aligned}0&=\int\limits_{S^+}u^{\mathrm T}Au\,da\\&=\int\limits_{S^+}\big[(N_{\alpha\beta,\beta}-N_{3\alpha})u_\alpha+N_{3\alpha,\alpha}u_3\big]\,da\\&=-\int\limits_{S^+}\big[N_{\alpha\beta}u_{\alpha,\beta}+N_{3\alpha}(u_\alpha+u_{3,\alpha})\big]\,da+\int\limits_{\partial S}N_iu_i\,ds\\&=-2\int\limits_{S^+}E(u,\,u)\,da+\int\limits_{\partial S}u^{\mathrm T}Tu\,ds,\end{aligned}$$

which yields the result. □

3.7 Theorem. (Reciprocity relation) *If* $u,\ \tilde u\in C^2(S^+)\cap C^1(\bar S^+)$, *then*

$$\int\limits_{S^+}(\tilde u^{\mathrm T}Au-u^{\mathrm T}A\tilde u)\,da=\int\limits_{\partial S}(\tilde u^{\mathrm T}Tu-u^{\mathrm T}T\tilde u)\,ds.$$

Proof. Let $N_{i\beta}$ and $\tilde N_{i\beta}$ be the moments and transverse shear forces generated by the displacements u and $\tilde u$, respectively. Using the equivalence of (3.3) and (3.8), together with (3.4), (3.10), and the divergence theorem, we find that

$$
\begin{aligned}
&\int\limits_{S^+} (\tilde{u}^{\mathrm{T}} A u - u^{\mathrm{T}} A \tilde{u})\, da \\
&\qquad = \int\limits_{S^+} \big[(N_{\alpha\beta,\beta} - N_{3\alpha})\tilde{u}_\alpha + N_{3\alpha,\alpha}\tilde{u}_3 - (\tilde{N}_{\alpha\beta,\beta} - \tilde{N}_{3\alpha})u_\alpha - \tilde{N}_{3\alpha,\alpha}u_3\big]\, da \\
&\qquad = \int\limits_{\partial S} (N_i \tilde{u}_i - \tilde{N}_i u_i)\, ds \\
&\qquad\quad - \int\limits_{S^+} \big[N_{\alpha\beta}\tilde{u}_{\alpha,\beta} + N_{3\alpha}(\tilde{u}_\alpha + \tilde{u}_{3,\alpha}) - \tilde{N}_{\alpha\beta}u_{\alpha,\beta} - \tilde{N}_{3\alpha}(u_\alpha + u_{3,\alpha})\big]\, da \\
&\qquad = \int\limits_{\partial S} (\tilde{u}^{\mathrm{T}} T u - u^{\mathrm{T}} T \tilde{u})\, ds,
\end{aligned}
$$

since, by (3.5), the integrand of the second integral vanishes in S^+. □

3.2 Singular Solutions

We seek a Galerkin representation for the solution of (3.8). Following the method described in Constanda (1978), if $A^*(\xi)$ is the adjoint of $A(\xi)$, then

$$u(x) = A^*(\partial_x)B(x), \tag{3.19}$$

where B is the solution of

$$(\det A)(\partial_x)B(x) + g(x) = 0. \tag{3.20}$$

More explicitly, from (3.9) we find that

$$
\begin{aligned}
A^*_{\alpha\beta}(\xi) &= h^2\mu(\lambda+2\mu)\delta_{\alpha\beta}\Delta\Delta - h^2\mu(\lambda+\mu)\Delta\xi_\alpha\xi_\beta - \mu^2\xi_\alpha\xi_\beta, \\
A^*_{33}(\xi) &= h^4\mu(\lambda+2\mu)\Delta\Delta - h^2\mu(\lambda+3\mu)\Delta + \mu^2, \\
A^*_{\alpha 3}(\xi) &= -A^*_{3\alpha}(\xi) = \mu^2\xi_\alpha(h^2\Delta - 1),
\end{aligned}
\tag{3.21}
$$

and

$$\det A(\xi) = a_1\Delta\Delta\left(\Delta - \frac{1}{h^2}\right). \tag{3.22}$$

Taking in turn each component of g equal to $\delta(|x-y|)$, where δ is the Dirac delta, and setting the other two equal to zero, from (3.19) and (3.20) we obtain the matrix of fundamental solutions

$$D(x,\ y) = A^*(\partial_x)t(x,\ y) \tag{3.23}$$

for the operator $-A$, where, by (3.20) and (3.22), $t(x,\ y)$ is a solution of

$$\Delta\Delta\left(\Delta - \frac{1}{h^2}\right)t(x,\ y) = -\frac{1}{a_1}\,\delta(|x-y|). \tag{3.24}$$

We seek t of the form

$$t(x,\ y) = b_1 \ln|x-y| + b_2|x-y|^2 \ln|x-y| + b_3 K_0\left(\frac{|x-y|}{h}\right),$$

where K_0 is the modified Bessel function of the second kind and order zero. Replacing this in (3.24) and taking into account the fact that, with respect to x,

$$\begin{aligned} \Delta(\ln|x-y|) &= 2\pi\,\delta(|x-y|),\\ \Delta\Delta(|x-y|^2\ln|x-y|) &= 8\pi\,\delta(|x-y|),\\ \left(\Delta - \frac{1}{h^2}\right)K_0\left(\frac{|x-y|}{h}\right) &= -2\pi\,\delta(|x-y|), \end{aligned}$$

we deduce that

$$\begin{aligned} t(x,\ y) &= t(|x-y|)\\ &= a_2\left[(4h^2 + |x-y|^2)\ln|x-y| + 4h^2 K_0\left(\frac{|x-y|}{h}\right)\right], \end{aligned} \tag{3.25}$$

where

$$a_2 = \frac{1}{8\pi h^2\mu^2(\lambda+2\mu)}. \tag{3.26}$$

In view of (3.21) and (3.23)–(3.25),

$$D(x,\ y) = \big(D(y,\ x)\big)^{\mathrm{T}}. \tag{3.27}$$

Along with $D(x,\ y)$, we consider the matrix of singular solutions

$$P(x,\ y;\ n) = \big(T(\partial_y;\ n)D(y,\ x)\big)^{\mathrm{T}}, \tag{3.28}$$

writing, for simplicity, $P\big(x,\ y;\ \nu(y)\big) \equiv P(x,\ y)$.

To determine the behavior of $D(x,\ y)$ and $P(x,\ y)$ in the neighborhood of $x = y$, we note (see Abramowitz and Stegun (1964), formulas (9.6.12) and (9.6.13) on p. 375) that, as $\xi \to 0$,

$$K_0(\xi) = -\big(1 + \tfrac{1}{4}\xi^2 + \tfrac{1}{64}\xi^4 + \cdots\big)\ln\xi,$$

so from (3.25) we deduce that for $|x-y|$ small,

$$t(x,\ y) = a_3|x-y|^4 \ln|x-y| + \tilde{t}(x,\ y), \tag{3.29}$$

where $\tilde{t} \in C^5(\mathbb{R}^2)$ and

$$a_3 = -\frac{1}{128\pi h^4 \mu^2(\lambda+2\mu)}.$$

We denote by $\{E_{ij}\}$ the standard ordered basis for the vector space of 3×3 matrices. From (3.23), (3.21), (3.25), (3.28), (3.11), and (3.29), we now find that for $y \in \partial S$ and x close to y,

$$\begin{aligned} D(x,\ y) = &-\frac{1}{2\pi}(\ln|x-y|)\left(a_4 E_{\gamma\gamma} + \frac{1}{\mu}E_{33}\right) \\ &+ 2a_2\mu(\lambda+\mu)\frac{(x_\alpha-y_\alpha)(x_\beta-y_\beta)}{|x-y|^2}E_{\alpha\beta} + \tilde{D}(x,\ y), \end{aligned} \tag{3.30}$$

$$\begin{aligned} P(x,\ y) = &-\frac{1}{2\pi}\left\{\mu'\varepsilon_{\alpha\beta}\left[\frac{\partial}{\partial s(y)}\ln|x-y|\right]E_{\alpha\beta}\right. \\ &+\left[\frac{\partial}{\partial \nu(y)}\ln|x-y|\right]E_3 \\ &-(\lambda'+\mu')\varepsilon_{\alpha\gamma}\left[\frac{\partial}{\partial s(y)}\frac{(x_\alpha-y_\alpha)(x_\beta-y_\beta)}{|x-y|^2}\right]E_{\gamma\beta} \\ &+\varepsilon_{\alpha\beta}\left[\frac{\partial}{\partial s(y)}\big((x_\alpha-y_\alpha)\ln|x-y|\big)\right]\left(\lambda' E_{3\beta} + \frac{1}{h^2}E_{\beta 3}\right) \\ &\left.-\mu'\left[\frac{\partial}{\partial \nu(y)}\big((x_\alpha-y_\alpha)\ln|x-y|\big)\right]E_{3\alpha}\right\} + \tilde{P}(x,\ y), \end{aligned} \tag{3.31}$$

where E_3 is the identity 3×3 matrix, $\tilde{D}(x,\ y)$ and $\tilde{P}(x,\ y)$ satisfy the conditions of Theorem 2.9 (or Theorem 2.10) with any $\gamma \in (0, 1)$,

$$a_4 = \frac{\lambda+3\mu}{2h^2\mu(\lambda+2\mu)},$$

and

$$\begin{aligned} \lambda' &= \frac{\lambda}{\lambda+2\mu}, \\ \mu' &= \frac{\mu}{\lambda+2\mu}. \end{aligned} \tag{3.32}$$

3.8 Theorem. *The columns of* $D(x, y)$ *and* $P(x, y; n)$ *are solutions of the homogeneous system* (3.8) *at all* $x \in \mathbb{R}^2$, $x \neq y$, *and for any direction* n *independent of* x.

Proof. Since $A(\xi)A^*(\xi) = \big(\det A(\xi)\big)E_3$, from (3.23), (3.22), and (3.24) we see that for $x \neq y$,

$$A(\partial_x)D(x, y) = A(\partial_x)A^*(\partial_x)t(x, y) = (\det A)(\partial_x)t(x, y) = 0.$$

Also, using (3.28) and expliciting the individual components, we easily convince ourselves that

$$A(\partial_x)P(x, y; n) = \Big(T(\partial_y; n)\big(A(\partial_x)D(x, y)\big)^{\mathrm{T}}\Big)^{\mathrm{T}} = 0. \qquad \square$$

3.9 Theorem. (Somigliana representation formula) *If the* 3×1 *matrix function* $u \in C^2(S^+) \cap C^1(\bar{S}^+)$ *is a solution of the homogeneous system* (3.8), *then*

$$\phi(x)u(x) = \int\limits_{\partial S} \big[D(x, y)T(\partial_y)u(y) - P(x, y)u(y)\big]\, ds(y), \tag{3.33}$$

where

$$\phi(x) = \begin{cases} 1, & x \in S^+, \\ \frac{1}{2}, & x \in \partial S, \\ 0, & x \in S^-. \end{cases} \tag{3.34}$$

Proof. Let $x \in S^+$, and let $\sigma_{x,\omega} \subset S^+$ be a disk with center at x and radius ω sufficiently small. Applying Theorem 3.7 in $S^+ \setminus \sigma_{x,\omega}$, with $\tilde{u}$ replaced in turn by each column of D, and making use of Theorem 3.8, we find that

$$\begin{aligned} &\int\limits_{\partial S} \big[D(x, y)T(\partial_y)u(y) - P(x, y)u(y)\big]\, ds(y) \\ &\qquad = \int\limits_{\partial\sigma_{x,\omega}} \big[D(x, y)T(\partial_y)u(y) - P(x, y)u(y)\big]\, ds(y), \end{aligned} \tag{3.35}$$

where $\partial\sigma_{x,\omega}$ is the boundary of $\sigma_{x,\omega}$.

By (3.30),

$$\int\limits_{\partial\sigma_{x,\omega}} D(x, y)T(\partial_y)u(y)\, ds(y) = O(\omega \ln \omega).$$

From (3.31) we see that for $y \in \partial\sigma_{x,\omega}$,

$$P(x,\ y)=O\left(\frac{1}{\omega}\right),$$

$$\int\limits_{\partial\sigma_{x,\omega}} P(x,\ y)\,ds(y)=-E_3+O(\omega\ln\omega);$$

consequently, since $u(x)-u(y)=O(\omega)$,

$$\begin{aligned}\int\limits_{\partial\sigma_{x,\omega}} & P(x,\ y)u(y)\,ds(y)\\ &=\int\limits_{\partial\sigma_{x,\omega}} P(x,\ y)\big[u(y)-u(x)\big]\,ds(y)\\ &\quad+\Bigg[\int\limits_{\partial\sigma_{x,\omega}} P(x,\ y)\,ds(y)\Bigg]u(x)\\ &=-u(x)+O(\omega).\end{aligned}$$

The first part of the assertion now follows from (3.35) if we let $\omega\to 0$.

The case when $x\in\partial S$ is handled in a similar way, with $\sigma_{x,\omega}$ replaced by $\sigma_{x,\omega}\cap S^+$ and $\partial\sigma_{x,\omega}$ by its part lying in S^+. As was remarked in the proof of Theorem 2.2, for ω small the length of the latter is $\pi\omega+O(\omega^2)$, which yields the required formula.

The result when $x\in S^-$ is obtained directly from (3.35). □

3.3 Case of the Exterior Domain

For y fixed and $|x|\to\infty$, we have

$$\begin{aligned}\frac{1}{|x-y|^2}&=\frac{1}{|x|^2}+2\frac{\langle x,\ y\rangle}{|x|^4}-\frac{|y|^2}{|x|^4}+4\frac{\langle x,\ y\rangle^2}{|x|^6}+O(|x|^{-5}),\\ \ln|x-y|&=\ln|x|-\frac{\langle x,\ y\rangle}{|x|^2}+\frac{1}{2}\frac{|y|^2}{|x|^2}-\frac{\langle x,\ y\rangle^2}{|x|^4}\\ &\quad+\frac{\langle x,\ y\rangle|y|^2}{|x|^4}-\frac{4}{3}\frac{\langle x,\ y\rangle^3}{|x|^6}+O(|x|^{-4}),\end{aligned}\tag{3.36}$$

and (see Abramowitz and Stegun 1964)

$$K_0\left(\frac{|x-y|}{h}\right)=O(|x|^{-1/2}e^{-|x|}).\tag{3.37}$$

Then from (3.21), (3.23), (3.25), (3.28), (3.36), and (3.37) we obtain the asymptotic relations

$$\begin{aligned} &D_{11},\ D_{22} = O(\ln|x|), \quad D_{12},\ D_{21} = O(1),\\ &D_{\alpha 3},\ D_{3\alpha} = O(|x|\ln|x|), \quad D_{33} = O(|x|^2\ln|x|),\\ &P_{\alpha\beta},\ P_{33} = O(|x|^{-1}), \quad P_{3\alpha} = O(\ln|x|), \quad P_{\alpha 3} = O(|x|^{-2}). \end{aligned} \tag{3.38}$$

This means that we cannot obtain analogs of Theorems 3.6 and 3.9 in S^- without restrictions on the behavior of u at infinity.

Let $\mathscr{A}$ be the set of 3×1 matrix functions u that, in terms of polar coordinates r, θ admit, as $r \to \infty$, an asymptotic expansion of the form

$$\begin{aligned} u_1(r,\theta) = {} & r^{-1}\big[m_0\sin\theta + 2m_1\cos\theta - m_0\sin(3\theta) + (m_2 - m_1)\cos(3\theta)\big]\\ & + r^{-2}\big[(2m_3 + m_4)\sin(2\theta) + m_5\cos(2\theta)\\ & \quad - 2m_3\sin(4\theta) + 2m_6\cos(4\theta)\big]\\ & + r^{-3}\big[2m_7\sin(3\theta) + 2m_8\cos(3\theta) + 3(m_9 - m_7)\sin(5\theta)\\ & \quad + 3(m_{10} - m_8)\cos(5\theta)\big] + O(r^{-4}),\\ u_2(r,\theta) = {} & r^{-1}\big[2m_2\sin\theta + m_0\cos\theta + (m_2 - m_1)\sin(3\theta) + m_0\cos(3\theta)\big]\\ & + r^{-2}\big[(2m_6 + m_5)\sin(2\theta) - m_4\cos(2\theta)\\ & \quad + 2m_6\sin(4\theta) + 2m_3\cos(4\theta)\big]\\ & + r^{-3}\big[2m_{10}\sin(3\theta) - 2m_9\cos(3\theta) + 3(m_{10} - m_8)\sin(5\theta)\\ & \quad + 3(m_7 - m_9)\cos(5\theta)\big] + O(r^{-4}),\\ u_3(r,\theta) = {} & -(m_1 + m_2)\ln r - \big[m_1 + m_2 + m_0\sin(2\theta)\\ & + (m_1 - m_2)\cos(2\theta)\big] + r^{-1}\big[(m_3 + m_4)\sin\theta + (m_5 + m_6)\cos\theta\\ & \quad - m_3\sin(3\theta) + m_6\cos(3\theta)\big]\\ & + r^{-2}\big[m_{11}\sin(2\theta) + m_{12}\cos(2\theta) + (m_9 - m_7)\sin(4\theta)\\ & \quad + (m_{10} - m_8)\cos(4\theta)\big] + O(r^{-3}), \end{aligned} \tag{3.39}$$

where $m_1, \dots, m_{12}$ are arbitrary constants. We also introduce the set

$$\mathscr{A}^* = \big\{u^*\colon u^* = u + u_0,\ u \in \mathscr{A},\ u_0 \text{ is of the form (3.16)}\big\}.$$

3.10 Remarks. In view of (3.12), $\mathscr{A}$ and $\mathscr{A}^*$ are classes of finite energy functions.

3.11 Remarks. For simplicity, throughout what follows we consider only the homogeneous system (3.8); that is,

$$Au = 0. \tag{3.40}$$

This is done without loss of generality since, as shown in Sect. 5.4, if g is sufficiently smooth, then (3.8) can be reduced to (3.40) by means of a particular solution constructed in the form of a Newtonian potential.

3.12 Theorem. (Somigliana representation formula) *If the* 3×1 *matrix function* $u \in C^2(S^-) \cap C^1(\bar{S}^-) \cap \mathscr{A}$ *is a solution of* (3.8), *then*

$$[1 - \phi(x)]u(x) = -\int_{\partial S} [D(x,\ y)T(\partial_y)u(y) - P(x,\ y)u(y)]\, ds(y),$$

where ϕ *is defined by* (3.34).

Proof. Consider a circle Γ_R with the center at x and radius R sufficiently large so that ∂S lies inside Γ_R. With the origin of polar coordinates at x, from (3.11), (3.21), (3.23), (3.25), (3.28), and (3.36)–(3.39) we find that for $y = (R, \theta) \in \partial\Gamma_R$,

$$\begin{aligned} T_{3i}u_i = R^{-3}\big[&(m_7 + m_9 - 2m_{11})\sin(2\theta) \\ &+ (m_8 + m_{10} - 2m_{12})\cos(2\theta)\big] + O(R^{-4}), \end{aligned}$$

$$\begin{aligned} (D_{3\alpha}T_{\alpha i} - P_{3i})u_i = \;& \frac{(4\lambda \ln R + 3\lambda + 2\mu)}{4(\lambda + 2\mu)R}\big[m_0 \sin(2\theta) + 2(m_2 - m_1)\cos(2\theta)\big] \\ &+ O(R^{-2}\ln R), \end{aligned}$$

$$(D_{\alpha i}T_{ij} - P_{\alpha j})u_j = O(R^{-2}\ln R);$$

consequently,

$$\int_{\partial\Gamma_R} [D(x,\ y)Tu(y) - P(x,\ y)u(y)]\, ds(y) = O(R^{-1}\ln R),$$

and the desired result is obtained by applying Theorem 3.9 in $S^- \cap \Gamma_R$ and letting $R \to \infty$. □

3.13 Theorem. (Betti formula) *If* $u \in C^2(S^-) \cap C^1(\bar{S}^-) \cap \mathscr{A}^*$ *is a solution of* (3.40), *then*

$$2\int_{S^-} E(u,\ u)\, da = -\int_{\partial S} u^{\mathrm{T}}Tu\, ds. \tag{3.41}$$

Proof. The required formula is obtained via the procedure used in the proof of Theorem 3.12—this time in conjunction with Theorem 3.5—after noting that for R large,

$$T_{\alpha i}u_i,\ T_{3i}u_i = O(R^{-2}), \tag{3.42}$$

which means that $u^{\mathrm{T}}Tu = O(R^{-2})$ for $u \in \mathscr{A}^*$. □

3.4 Uniqueness of Regular Solutions

Let $\mathscr{K}$, $\mathscr{L}$, $\mathscr{P}$, $\mathscr{Q}$, $\mathscr{R}$, and $\mathscr{S}$ be continuous 3×1 matrix functions prescribed on ∂S, and let σ be a continuous, positive definite 3×3 matrix function on ∂S. We consider the following interior and exterior Dirichlet, Neumann, and Robin boundary value problems:

(D$^+$) *Find* $u \in C^2(S^+) \cap C^1(\bar{S}^+)$ *satisfying*

$$\begin{aligned} Au(x) &= 0, \quad && x \in S^+, \\ u(x) &= \mathscr{P}(x), \quad && x \in \partial S. \end{aligned} \tag{3.43}$$

(N$^+$) *Find* $u \in C^2(S^+) \cap C^1(\bar{S}^+)$ *satisfying*

$$\begin{aligned} Au(x) &= 0, \quad && x \in S^+, \\ Tu(x) &= \mathscr{Q}(x), \quad && x \in \partial S. \end{aligned} \tag{3.44}$$

(R$^+$) *Find* $u \in C^2(S^+) \cap C^1(\bar{S}^+)$ *satisfying*

$$\begin{aligned} Au(x) &= 0, \quad && x \in S^+, \\ Tu(x) + \sigma(x)u(x) &= \mathscr{K}(x), \quad && x \in \partial S. \end{aligned} \tag{3.45}$$

(D$^-$) *Find* $u \in C^2(S^-) \cap C^1(\bar{S}^-) \cap \mathscr{A}^*$ *satisfying*

$$\begin{aligned} Au(x) &= 0, \quad && x \in S^-, \\ u(x) &= \mathscr{R}(x), \quad && x \in \partial S. \end{aligned} \tag{3.46}$$

(N$^-$) *Find* $u \in C^2(S^-) \cap C^1(\bar{S}^-) \cap \mathscr{A}$ *satisfying*

$$\begin{aligned} Au(x) &= 0, \quad && x \in S^-, \\ Tu(x) &= \mathscr{S}(x), \quad && x \in \partial S. \end{aligned} \tag{3.47}$$

(R$^-$) *Find* $u \in C^2(S^-) \cap C^1(\bar{S}^-) \cap \mathscr{A}$ *satisfying*

$$\begin{aligned} Au(x) &= 0, \quad && x \in S^-, \\ Tu(x) - \sigma(x)u(x) &= \mathscr{L}(x), \quad && x \in \partial S. \end{aligned} \tag{3.48}$$

3.14 Definition. A solution as stated above is called a *regular solution* of the corresponding problem.

3.15 Remark. The condition that $u \in C^1(\bar{S}^+)$ or $u \in C^1(\bar{S}^-)$ is necessary even in the case of the Dirichlet problems, to ensure the applicability of the Betti formula.

3.16 Theorem. (i) (D^+), (D^-), (N^-), (R^+), *and* (R^-) *have at most one regular solution.*

(ii) *Any two regular solutions of* (N^+) *differ by a* 3×1 *matrix of the form* (3.16).

Proof. (i) The difference u of two regular solutions of (D^+) satisfies (3.43) with $\mathscr{P} = 0$; therefore, by Theorem 3.6 and the fact that $E(u, u)$ is a positive quadratic form,

$$E(u,\, u) = 0 \quad \text{in} \;\; S^+.$$

From Theorem 3.3 it now follows that u is of the form (3.16) in $\bar{S}^+$. Since $u = 0$ on ∂S, we deduce that $u(x) = 0$, $x \in \bar{S}^+$.

The same argument, but based on Theorem 3.13 instead of Theorem 3.6, is used to prove the result for (D^-).

If u is the difference of two regular solutions of (N^-), then, as above, we conclude that u is of the form (3.16) in S^-. However, since $u \in \mathscr{A}$, we see from (3.39) that $u = 0$.

For (R^+) we write

$$(Tu)(x) = -(\sigma u)(x), \quad x \in \partial S,$$

and (3.18) leads to

$$2 \int\limits_{S^+} E(u, u)\, da + \int\limits_{\partial S} u^{\mathrm{T}}(\sigma u)\, ds = 0.$$

Since

$$u^{\mathrm{T}} \sigma u = u_i (\sigma u)_i = u_i (\sigma_{ij} u_j) = \sigma_{ij} u_i u_j = \sigma (u^{\mathrm{T}} u)$$

and σ is positive definite, we deduce that u is a rigid displacement in S^+ which vanishes on ∂S, so $u = 0$.

The argument is analogous for (R^-), with

$$(Tu)(x) = (\sigma u)(x), \quad x \in \partial S,$$

replaced in (3.41).

(ii) As in the case of (D^+), we find that the difference of two regular solutions of (N^+) is of the form (3.16). □

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Chapter 4
The Layer Potentials

4.1 Layer Potentials with Smooth Densities

Let $q = (q_{ij})$, $i = 1, \dots, n$, $j = 1, \dots, m$, be a matrix, X a space of scalar functions, and L a scalar operator on X. In what follows we write $q \in X$ if $q_{ij} \in X$, $i = 1, \dots, n$, $j = 1, \dots, m$; we also write $Lq = (Lq_{ij})$.

We introduce the *single-layer plate potential*

$$(V\varphi)(x) = \int\limits_{\partial S} D(x,\ y)\varphi(y)\, ds(y) \tag{4.1}$$

and the *double-layer plate potential*

$$(W\varphi)(x) = \int\limits_{\partial S} P(x,\ y)\varphi(y)\, ds(y), \tag{4.2}$$

where φ is a density 3×1 matrix.

If f is any of the potential-type functions defined in Sects. 2.1 and 2.2 for scalar densities, here we denote by $f\varphi$ the 3×1 vector function of components $f\varphi_i$.

4.1 Theorem. *If* $\varphi \in C(\partial S)$, *then* $V\varphi$ *and* $W\varphi$ *are analytic and satisfy* (3.40) *in* $S^+ \cup S^-$.

Proof. Clearly, $V\varphi$ and $W\varphi$ are twice continuously differentiable at any $x \notin \partial S$ and, by Theorem 3.8, are solutions of (3.40). Their analyticity follows in the usual way (see, for example, Miranda 1970). □

Let p be the vector-valued functional on $C(\partial S)$ defined by

$$pf = \int\limits_{\partial S} F^{\mathrm{T}} f\, ds, \tag{4.3}$$

where F is the matrix (3.17).

C. Constanda, *Mathematical Methods for Elastic Plates*,
Springer Monographs in Mathematics, DOI: 10.1007/978-1-4471-6434-0_4,

4.2 Theorem. *If* $\varphi \in C(\partial S)$, *then*

(i) $W\varphi \in \mathscr{A}$;
(ii) $V\varphi \in \mathscr{A}$ *if and only if* $p\varphi = 0$; *that is, by* (4.3), *if and only if*

$$(p\varphi)_\alpha = \int_{\partial S} (\varphi_\alpha - x_\alpha \varphi_3)\, ds = 0,$$
$$(p\varphi)_3 = \int_{\partial S} \varphi_3\, ds = 0.$$

Proof. (i) Expansion (3.39) for $W\varphi$ is obtained from (4.2), (3.11), (3.21), (3.23), (3.25), (3.28), (3.36), and (3.37).
(ii) Using the same formulas as above, from (4.1) we find that, as $r = |x| \to \infty$,

$$\begin{aligned}
&(V\varphi)_1(r,\theta)\\
&\quad = -a_2\mu^2\big[(p\varphi)_3 r(2\ln r + 1)\cos\theta + (p\varphi)_1(2\ln r + 2 + \cos(2\theta)) + (p\varphi)_2 \sin(2\theta)\big]\\
&\qquad + (V\varphi)_1^{\mathscr{A}}(r,\theta),\\
&(V\varphi)_2(r,\theta)\\
&\quad = -a_2\mu^2\big[(p\varphi)_3 r(2\ln r + 1)\sin\theta + (p\varphi)_2(2\ln r + 2 - \cos(2\theta)) + (p\varphi)_1 \sin(2\theta)\big]\\
&\qquad + (V\varphi)_2^{\mathscr{A}}(r,\theta),\\
&(V\varphi)_3(r,\theta)\\
&\quad = a_2\mu(p\varphi)_3\big[\mu r^2\ln r - 4h^2(\lambda + 2\mu)\ln r - 4h^2(\lambda + 3\mu)\big]\\
&\qquad + a_2\mu\big[(p\varphi)_1\cos\theta + (p\varphi)_2\sin\theta\big]\big[\mu r(2\ln r + 1) - 4h^2(\lambda + 2\mu)r^{-1}\big]\\
&\qquad + (V\varphi)_3^{\mathscr{A}}(r,\theta),
\end{aligned}$$

where $(V\varphi)^{\mathscr{A}} \in \mathscr{A}$. This can be written more compactly in the form

$$(V\varphi)(r,\theta) = M^\infty(r,\theta)(p\varphi) + (V\varphi)^{\mathscr{A}}(r,\theta), \tag{4.4}$$

where $M^\infty(r,\theta)$ is the 3×3 matrix function with columns

$$\begin{aligned}
M^{\infty(1)}(r,\theta) &= -a_2\mu\big(\mu(2\ln r + 2 + \cos(2\theta)),\ \mu\sin(2\theta),\\
&\qquad\qquad - (\mu r(2\ln r + 1) - 4h^2(\lambda + 2\mu)r^{-1})\cos\theta\big)^{\mathrm{T}},\\
M^{\infty(2)}(r,\theta) &= -a_2\mu\big(\sin(2\theta),\ \mu(2\ln r + 2 - \cos(2\theta)),\\
&\qquad\qquad - (\mu r(2\ln r + 1) - 4h^2(\lambda + 2\mu)r^{-1})\sin\theta\big)^{\mathrm{T}},\\
M^{\infty(3)}(r,\theta) &= -a_2\mu\big(\mu r(2\ln r + 1)\cos\theta,\ \mu r(2\ln r + 1)\sin\theta,\\
&\qquad\qquad - \mu r^2\ln r + 4h^2(\lambda + 2\mu)\ln r + 4h^2(\lambda + 3\mu)\big)^{\mathrm{T}}
\end{aligned}$$

and a_2 is the constant (3.26). The assertion now follows immediately from (4.4). □

4.3 Remark. The requirement that the solutions of the exterior boundary value problems belong to $\mathscr{A}$ or $\mathscr{A}^*$ is justified by the fact that such solutions will be sought in the form of single-layer or double-layer plate potentials.

In view of Theorem 4.1, to investigate the continuity and differentiability of $V\varphi$ and $W\varphi$ in $\bar{S}^+$ and $\bar{S}^-$ it suffices to consider their behavior in $\bar{S}_0^+$ and $\bar{S}_0^-$.

4.4 Theorem. *If $\varphi \in C(\partial S)$, then $V\varphi \in C^{0,\alpha}(\mathbb{R}^2)$ for any $\alpha \in (0, 1)$.*

Proof. From (4.1) and (3.30) we see that for $x \in S_0 \setminus \partial S$,

$$V\varphi = \frac{1}{2\pi}\left(a_4 E_{\gamma\gamma} + \frac{1}{\mu} E_{33}\right)(v\varphi) + 2a_2\mu(\lambda + \mu)E_{\alpha\beta}(v^b_{\alpha\beta}\varphi) + (V\varphi)^{\mathscr{A}},$$

where $v\varphi$ and $v^b_{\alpha\beta}\varphi$ are defined by (2.1) and (2.15), respectively, and

$$(V\varphi)^{\mathscr{A}}(x) = \int\limits_{\partial S} \tilde{D}(x,\ y)\varphi(y)\, ds(y).$$

The assertion now follows from Theorems 2.1, 2.11, and 2.9. □

4.5 Theorem. *If $\varphi \in C^{0,\alpha}(\partial S)$, $\alpha \in (0, 1]$, then $W\varphi$ has $C^{0,\beta}$-extensions $(W\varphi)^+$ and $(W\varphi)^-$ to $\bar{S}^+$ and $\bar{S}^-$, respectively, with $\beta = \alpha$ for $\alpha \in (0, 1)$ and any $\beta \in (0, 1)$ for $\alpha = 1$. These extensions are given by*

$$\begin{aligned} (W\varphi)^+(x) &= \begin{cases} (W\varphi)(x), & x \in S^+, \\ -\tfrac{1}{2}\,\varphi(x) + (W_0\varphi)(x), & x \in \partial S, \end{cases} \\ (W\varphi)^-(x) &= \begin{cases} (W\varphi)(x), & x \in S^-, \\ \tfrac{1}{2}\,\varphi(x) + (W_0\varphi)(x), & x \in \partial S, \end{cases} \end{aligned} \tag{4.5}$$

where

$$(W_0\varphi)(x) = \int\limits_{\partial S} P(x,\ y)\varphi(y)\, ds(y), \quad x \in \partial S, \tag{4.6}$$

the integral being understood as principal value.

Proof. As mentioned at the beginning of Sect. 2.1, it suffices to perform the analysis in the boundary strip S_0 (see (1.44)). Thus, following the argument in the proof of Theorem 4.4, from (4.2) and (3.31) we find that for $x \in S_0 \setminus \partial S$,

$$\begin{aligned} W\varphi = -\frac{1}{2\pi}\bigg[&\mu'\varepsilon_{\alpha\beta}E_{\alpha\beta}(v^f\varphi) - E_3(w\varphi) - (\lambda' + \mu')\varepsilon_{\alpha\gamma}E_{\gamma\beta}(v^e_{\alpha\beta}\varphi) \\ &+ \frac{1}{2}\,\varepsilon_{\alpha\beta}\left(\lambda' E_{3\beta} + \frac{1}{h^2}E_{\beta 3}\right)(v^c_\alpha\varphi) - \frac{1}{2}E_{3\alpha}(v^d_\alpha\varphi)\bigg] + \tilde{W}\varphi, \end{aligned} \tag{4.7}$$

where $v_\alpha^c\varphi$, $v_\alpha^d\varphi$, $v_{\alpha\beta}^e\varphi$, $v^f\varphi$, and $w\varphi$ are defined by (2.2), (2.16)–(2.18), and (2.24), respectively,

$$(\tilde{W}\varphi)(x) = \int\limits_{\partial S} \tilde{P}(x,\ y)\varphi(y)\,ds(y),$$

and the kernel $\tilde{P}(x,\ y)$ satisfies the conditions of Theorem 2.9 with any $\gamma \in (0, 1)$.

The assertion now follows from (4.7), Theorems 2.11, 2.12, 2.14, 2.9, and 2.2, and Remarks 2.15 and 2.3. □

4.6 Theorem. *If* $\varphi \in C^{0,\alpha}(\partial S)$, $\alpha \in (0, 1]$, *then the first-order derivatives of* $V\varphi$ *in* S^+ *and* S^- *have* $C^{0,\beta}$*-extensions to* $\bar{S}^+$ *and* $\bar{S}^-$, *respectively, with* $\beta = \alpha$ *for* $\alpha \in (0, 1)$ *and any* $\beta \in (0, 1)$ *for* $\alpha = 1$. *These extensions are given by*

$$((V\varphi)_{,\gamma})^+(x) = \begin{cases} (V\varphi)_{,\gamma}\,(x), & x \in S^+, \\ f^\gamma(x) + ((V\varphi)_{,\gamma})_0(x), & x \in \partial S, \end{cases}$$

$$((V\varphi)_{,\gamma})^-(x) = \begin{cases} (V\varphi)_{,\gamma}\,(x), & x \in S^-, \\ -f^\gamma(x) + ((V\varphi)_{,\gamma})_0(x), & x \in \partial S, \end{cases}$$

where the components of the vector function f^γ *are*

$$\begin{aligned} f_\alpha^\gamma &= 4\pi a_2\mu\big[(\lambda + 2\mu)\delta_{\alpha\beta} - (\lambda + \mu)\nu_\alpha\nu_\beta\big]\nu_\gamma\varphi_\beta, \\ f_3^\gamma &= \tfrac{1}{2\mu}\,\nu_\gamma\varphi_3 \end{aligned} \tag{4.8}$$

and

$$((V\varphi)_{,\gamma})_0(x) = \int\limits_{\partial S} \frac{\partial}{\partial x_\gamma}\,D(x,\ y)\varphi(y)\,ds(y), \quad x \in \partial S,$$

the integral being understood as principal value.

Proof. Once again, we restrict our attention to the boundary strip S_0. For $x \in S_0\backslash\partial S$ and $y \in \partial S$ we write

$$\begin{aligned} D_{\alpha\beta} &= D_{\alpha\beta}^* + \tilde{D}_{\alpha\beta}, \\ D_{33} &= D_{33}^* + \tilde{D}_{33}, \end{aligned} \tag{4.9}$$

where, by (3.30),

$$\begin{aligned} D_{\alpha\beta}^*(x,\ y) &= 2a_2\mu\bigg[-(\lambda + 3\mu)\delta_{\alpha\beta}\ln|x - y| \\ &\qquad + (\lambda + \mu)\frac{(x_\alpha - y_\alpha)(x_\beta - y_\beta)}{|x - y|^2}\bigg], \\ D_{33}^*(x,\ y) &= -\frac{1}{2\pi\mu}\ln|x - y|, \end{aligned} \tag{4.10}$$

and $\tilde{D}_{\alpha\beta}$ and $\tilde{D}_{33}$ satisfy the conditions of Theorem 2.9 with any $\gamma \in (0, 1)$. From (3.21), (3.23), and (3.25) it follows that

$$D_{\alpha 3} = -D_{3\alpha} = -a_2\mu^2(x_\alpha - y_\alpha)(2\ln|x - y| + 1), \tag{4.11}$$

which also satisfies the conditions of Theorem 2.9 with any $\gamma \in (0, 1)$.

Using (1.7) and the equality

$$\varepsilon_{\beta\sigma}\varepsilon_{\mu\rho} = \delta_{\beta\mu}\delta_{\sigma\rho} - \delta_{\beta\rho}\delta_{\sigma\mu}, \tag{4.12}$$

we find that

$$\begin{aligned}
&\varepsilon_{\beta\sigma}\frac{\partial}{\partial s(y)}\,\frac{(x_\alpha - y_\alpha)(x_\sigma - y_\sigma)}{|x-y|^2}\\
&\quad= \left[\nu_\beta(y)\frac{\partial}{\partial y_\sigma} - \nu_\sigma(y)\frac{\partial}{\partial y_\beta}\right]\frac{(x_\alpha - y_\alpha)(x_\sigma - y_\sigma)}{|x-y|^2}\\
&\quad= -\delta_{\alpha\beta}\frac{\partial}{\partial\nu(y)}\,\ln|x-y|\\
&\qquad -2\nu_\sigma(y)\frac{(x_\alpha - y_\alpha)(x_\beta - y_\beta)(x_\sigma - y_\sigma)}{|x-y|^4}.
\end{aligned} \tag{4.13}$$

Next, (4.13) and (1.51) yield

$$\begin{aligned}
&\frac{\partial}{\partial\nu(y)}\,\frac{(x_\alpha - y_\alpha)(x_\beta - y_\beta)}{|x-y|^2}\\
&\quad= -\nu_\alpha(y)\frac{x_\beta - y_\beta}{|x-y|^2} - \nu_\beta(y)\frac{x_\alpha - y_\alpha}{|x-y|^2}\\
&\qquad + 2\nu_\sigma(y)\frac{(x_\alpha - y_\alpha)(x_\beta - y_\beta)(x_\sigma - y_\sigma)}{|x-y|^4}\\
&\quad= \left[\nu_\alpha(y)\frac{\partial}{\partial y_\beta} + \nu_\beta(y)\frac{\partial}{\partial y_\alpha}\right]\ln|x-y|\\
&\qquad - \varepsilon_{\beta\sigma}\frac{\partial}{\partial s(y)}\,\frac{(x_\alpha - y_\alpha)(x_\sigma - y_\sigma)}{|x-y|^2}\\
&\qquad - \delta_{\alpha\beta}\frac{\partial}{\partial\nu(y)}\,\ln|x-y|\\
&\quad= \big[\nu_\alpha(y)\tau_\beta(y) + \nu_\beta(y)\tau_\alpha(y)\big]\frac{\partial}{\partial s(y)}\,\ln|x-y|\\
&\qquad + \big[2\nu_\alpha(y)\nu_\beta(y) - \delta_{\alpha\beta}\big]\frac{\partial}{\partial\nu(y)}\,\ln|x-y|\\
&\qquad - \varepsilon_{\beta\sigma}\frac{\partial}{\partial s(y)}\,\frac{(x_\alpha - y_\alpha)(x_\sigma - y_\sigma)}{|x-y|^2}.
\end{aligned} \tag{4.14}$$

Finally, from (4.10), (4.14), (1.7), and (1.51) we obtain

$$\begin{aligned}
\frac{\partial}{\partial x_\gamma} D^*_{\alpha\beta}(x,\ y) &= -\frac{\partial}{\partial y_\gamma} D^*_{\alpha\beta}(y, x)\\
&= 2a_2\mu\bigg\{(\lambda+3\mu)\delta_{\alpha\beta}\bigg[\varepsilon_{\sigma\gamma}\nu_\sigma(y)\frac{\partial}{\partial s(y)} + \nu_\gamma(y)\frac{\partial}{\partial \nu(y)}\bigg]\ln|x-y|\\
&\qquad + (\lambda+\mu)\bigg[\varepsilon_{\gamma\sigma}\nu_\sigma(y)\frac{\partial}{\partial s(y)} - \nu_\gamma(y)\frac{\partial}{\partial \nu(y)}\bigg]\frac{(x_\alpha-y_\alpha)(x_\beta-y_\beta)}{|x-y|^2}\bigg\}\\
&= 2a_2\mu\bigg\{2\nu_\gamma(y)\big[(\lambda+2\mu)\delta_{\alpha\beta} - (\lambda+\mu)\nu_\alpha(y)\nu_\beta(y)\big]\frac{\partial}{\partial \nu(y)}\ \ln|x-y|\\
&\qquad + \big[(\lambda+3\mu)\delta_{\alpha\beta}\varepsilon_{\sigma\gamma}\nu_\sigma(y)\\
&\qquad\quad - (\lambda+\mu)\big(\varepsilon_{\sigma\beta}\nu_\alpha(y) + \varepsilon_{\sigma\alpha}\nu_\beta(y)\big)\nu_\sigma(y)\nu_\gamma(y)\big]\frac{\partial}{\partial s(y)}\ \ln|x-y|\\
&\qquad - (\lambda+\mu)\bigg[\varepsilon_{\sigma\gamma}\nu_\sigma(y)\frac{\partial}{\partial s(y)}\ \frac{(x_\alpha-y_\alpha)(x_\beta-y_\beta)}{|x-y|^2}\\
&\qquad\qquad + \varepsilon_{\sigma\beta}\nu_\gamma(y)\frac{\partial}{\partial s(y)}\ \frac{(x_\alpha-y_\alpha)(x_\sigma-y_\sigma)}{|x-y|^2}\bigg]\bigg\}.
\end{aligned}$$

Similarly,

$$\frac{\partial}{\partial x_\gamma} D^*_{33}(x,\ y) = \frac{1}{2\pi\mu}\bigg[\nu_\gamma(y)\frac{\partial}{\partial \nu(y)} + \varepsilon_{\sigma\gamma}\nu_\sigma(y)\frac{\partial}{\partial s(y)}\bigg]\ln|x-y|.$$

In view of (4.1) and the above calculation, for $x \in S_0 \setminus \partial S$ and $y \in \partial S$ we can now write

$$\begin{aligned}
(V\varphi)_{,\gamma} &= -\pi^{-1} w f^\gamma + v g^\gamma + v^b_{\alpha\gamma} p^\alpha\\
&\quad + v^e_{\alpha\beta} q^{\alpha\beta\gamma} + v^f r^\gamma + \mathscr{V}^\gamma\varphi.
\end{aligned} \tag{4.15}$$

Here, $w\varphi$, $v\varphi$, $v^b_{\alpha\gamma}\varphi$, $v^e_{\alpha\beta}\varphi$, and $v^f\varphi$ are defined by (2.1), (2.2), (2.15), (2.18), and (2.24), respectively, the densities f^γ (given by (4.8)), g^γ, p^α, $q^{\alpha\beta\gamma}$, and r^γ are 3×1 vector functions of class $C^{0,\alpha}(\partial S)$, and

$$(\mathscr{V}^\gamma\varphi)(x) = \int\limits_{\partial S} \mathscr{F}(x,\ y) t^\gamma(y)\, ds(y),$$

where t^γ is another 3×1 vector function of class $C^{0,\alpha}(\partial S)$ and $\mathscr{F}(x,\ y)$ is a proper ε-singular kernel in S_0 for any $\varepsilon \in (0, 1)$. The assertion now follows from Theorems 2.1, 2.2, 2.11, 2.12, 2.14, and 1.33. □

4.7 Remark. From Theorems 4.4 and 4.6 we conclude that if $\varphi \in C^{0,\alpha}(\partial S)$, $\alpha \in (0, 1]$, then the restrictions of $V\varphi$ to $\bar{S}^+$ and $\bar{S}^-$ belong, respectively, to $C^{1,\beta}(\bar{S}^+)$ and $C^{1,\beta}(\bar{S}^-)$, with $\beta = \alpha$ for $\alpha \in (0, 1)$ and any $\beta \in (0, 1)$ for $\alpha = 1$. Denoting these two functions by $(V\varphi)^+$ and $(V\varphi)^-$, we can write

$$\begin{aligned} ((V\varphi)^+)_{,\gamma}(x) &= ((V\varphi)_{,\gamma})^+(x), \quad x \in \bar{S}^+, \\ ((V\varphi)^-)_{,\gamma}(x) &= ((V\varphi)_{,\gamma})^-(x), \quad x \in \bar{S}^-. \end{aligned}$$

In view of Theorem 4.4, we make the notation

$$(V\varphi)^\pm(x) = (V\varphi)^\pm_0(x) = (V_0\varphi)(x), \quad x \in \partial S. \tag{4.16}$$

Consider the function $T(V\varphi)$ defined on $S_0 \backslash \partial S$ (see Remark 1.19).

4.8 Corollary. *If $\varphi \in C^{0,\alpha}(\partial S)$, $\alpha \in (0, 1]$, then the restrictions of $T(V\varphi)$ to S_0^+ and S_0^- have $C^{0,\beta}$-extensions to $\bar{S}_0^+$ and $\bar{S}_0^-$, respectively, with $\beta = \alpha$ for $\alpha \in (0, 1)$ and any $\beta \in (0, 1)$ for $\alpha = 1$. These extensions are given by*

$$\begin{aligned} (T(V\varphi))^+(x) &= \begin{cases} T(V\varphi)(x), & x \in S_0^+, \\ \frac{1}{2}\,\varphi(x) + (T(V\varphi))_0(x), & x \in \partial S, \end{cases} \\ (T(V\varphi))^-(x) &= \begin{cases} T(V\varphi)(x), & x \in S_0^-, \\ -\frac{1}{2}\,\varphi(x) + (T(V\varphi))_0(x), & x \in \partial S, \end{cases} \end{aligned} \tag{4.17}$$

where

$$(T(V\varphi))_0(x) = \int\limits_{\partial S} T(\partial_x)D(x,\, y)\varphi(y)\,ds(y),$$

the integral being understood as principal value.

Proof. By (3.11), for $x \in S_0 \backslash \partial S$ we have

$$\begin{aligned} (T(V\varphi))_\alpha &= h^2\big\{\lambda\nu_\alpha(V\varphi)_{\beta,\beta} + \mu\nu_\beta\big[(V\varphi)_{\beta,\alpha} + (V\varphi)_{\alpha,\beta}\big]\big\}, \\ (T(V\varphi))_3 &= \mu\nu_\alpha\big[(V\varphi)_\alpha + (V\varphi)_{3,\alpha}\big], \end{aligned} \tag{4.18}$$

and the $C^{0,\beta}$-extendability of $T(V\varphi)$ to $\bar{S}_0^+$ and $\bar{S}_0^-$ follows from Theorem 4.6. For $x \in \partial S$, the same theorem and (4.18) yield

$$\begin{aligned} (T(V\varphi))^\pm_\alpha &= \pm h^2\big[\lambda\nu_\alpha f^\beta_\beta + \mu\nu_\beta(f^\beta_\alpha + f^\alpha_\beta)\big] + (T(V\varphi))_{0\alpha} \\ &= \pm\tfrac{1}{2}\,\varphi_\alpha + (T(V\varphi))_{0\alpha}, \\ (T(V\varphi))^\pm_3 &= \mu\nu_\alpha f^\alpha_3 + (T(V\varphi))_{03} = \pm\tfrac{1}{2}\,\varphi_3 + (T(V\varphi))_{03}, \end{aligned}$$

which completes the proof. □

4.9 Remark. In view of Remark 4.7, we can write

$$\begin{aligned}(T(V\varphi)^+)(x) &= (T(V\varphi))^+(x), \quad x \in \bar{S}_0^+,\\ (T(V\varphi)^-)(x) &= (T(V\varphi))^-(x), \quad x \in \bar{S}_0^-.\end{aligned} \tag{4.19}$$

4.10 Theorem. *If* $\varphi \in C^{1,\alpha}(\partial S)$, $\alpha \in (0,1]$, *then the restrictions of* $W\varphi$ *to* S^+ *and* S^- *have* $C^{1,\beta}$*-extensions* $(W\varphi)^+$ *and* $(W\varphi)^-$ *to* $\bar{S}^+$ *and* $\bar{S}^-$, *respectively, with* $\beta = \alpha$ *for* $\alpha \in (0,1)$ *and any* $\beta \in (0,1)$ *for* $\alpha = 1$. *These extensions are given by* (4.5) *and satisfy*

$$T(W\varphi)^+ = T(W\varphi)^- \quad \text{on } \partial S.$$

Proof. Let u be a continuously differentiable 3×1 matrix function on ∂S. From (3.11), (1.7), and the easily verified equality

$$\nu_\beta u_{\beta,\gamma} - \nu_\gamma u_{\beta,\beta} = \varepsilon_{\beta\gamma} \frac{\partial}{\partial s} u_\beta,$$

for $y \in \partial S$ we obtain

$$\begin{aligned}T_{\gamma\beta} u_\beta &= h^2\big[\lambda \nu_\gamma u_{\beta,\beta} + \mu \nu_\beta (u_{\beta,\gamma} + u_{\gamma,\beta})\big]\\ &= h^2\Big\{2\mu\varepsilon_{\beta\gamma}\frac{\partial}{\partial s} u_\beta + \Big[(\lambda+\mu)\nu_\gamma \frac{\partial}{\partial y_\beta} - \mu\varepsilon_{\beta\gamma}\frac{\partial}{\partial s} + \mu\delta_{\beta\gamma}\frac{\partial}{\partial \nu}\Big]u_\beta\Big\}.\end{aligned} \tag{4.20}$$

Using (4.10)–(4.12), after a lengthy but straightforward calculation we find that for $x \in S_0 \backslash \partial S$ and $y \in \partial S$,

$$\begin{aligned}&\Big[(\lambda+\mu)\nu_\gamma(y)\frac{\partial}{\partial y_\beta} - \mu\varepsilon_{\beta\gamma}\frac{\partial}{\partial s(y)} + \mu\delta_{\beta\gamma}\frac{\partial}{\partial \nu(y)}\Big]D^*_{\alpha\beta}(x, y)\\ &\quad = \frac{1}{2\pi}\Big[-\delta_{\alpha\gamma}\frac{\partial}{\partial \nu(y)} \ln|x-y| + \varepsilon_{\alpha\gamma}\frac{\partial}{\partial s(y)}\ln|x-y|\Big].\end{aligned}$$

Consequently, using (4.20), (3.27) and integration by parts, we see that

$$\begin{aligned}&\int_{\partial S} \big[T_{\gamma\beta}(\partial_y) D^*_{\beta\alpha}(y, x)\big]\varphi_\gamma(y)\, ds(y)\\ &\quad = h^2\Big\{2\mu\varepsilon_{\gamma\beta}\int_{\partial S} D^*_{\alpha\beta}(x, y)\varphi'_\gamma(y)\, ds(y)\\ &\qquad - \frac{1}{2\pi}\int_{\partial S}\Big[\frac{\partial}{\partial \nu(y)} \ln|x-y|\Big]\varphi_\alpha(y)\, ds(y)\\ &\qquad - \frac{1}{2\pi}\varepsilon_{\alpha\gamma}\int_{\partial S}(\ln|x-y|)\varphi'_\gamma(y)\, ds(y)\Big\}.\end{aligned}$$

Similarly,

$$\int_{\partial S} \big[T_{3\beta}(\partial_y)D_{\beta\alpha}(y,\ x)\big]\varphi_3(y)\,ds(y) = \mu \int_{\partial S} D_{\alpha\beta}(x,\ y)\nu_\beta(y)\varphi_3(y)\,ds(y)$$

and, by (4.11),

$$\begin{aligned}\int_{\partial S} \big[T_{33}(\partial_y)D_{3\alpha}(y, x)\big]\varphi_3(y)\,ds(y) \\ = 2a_2\mu^3\bigg\{\int_{\partial S} (\ln|x-y|)\nu_\alpha(y)\varphi_3(y)\,ds(y) \\ - \int_{\partial S}\bigg[\frac{\partial}{\partial\nu(y)}\ln|x-y|\bigg](x_\alpha - y_\alpha)\varphi_3(y)\,ds(y)\bigg\} + c_\alpha,\end{aligned}$$

where the c_α are combinations of λ and μ.

By means of the same procedure and (3.27), we arrive at

$$\begin{aligned}\int_{\partial S} \big[T_{\gamma\beta}(\partial_y)D_{\beta 3}(y,\ x)\big]\varphi_\gamma(y)\,ds(y) \\ = 2h^2\mu\bigg\{\varepsilon_{\gamma\beta}\int_{\partial S} D_{3\beta}(x,\ y)\varphi'_\gamma(y)\,ds(y) \\ - a_2\mu(2\lambda+3\mu)\int_{\partial S}(\ln|x-y|)\nu_\gamma(y)\varphi_\gamma(y)\,ds(y) \\ + a_2\mu^2\int_{\partial S}\bigg[\frac{\partial}{\partial\nu(y)}\ln|x-y|\bigg](x_\gamma - y_\gamma)\varphi_\gamma(y)\,ds(y) \\ - a_2\mu^2\varepsilon_{\gamma\beta}\int_{\partial S}(\ln|x-y|)(x_\beta - y_\beta)\varphi'_\gamma(y)\,ds(y)\bigg\} + c'_\gamma x_\gamma + c',\end{aligned}$$

$$\int_{\partial S} \big[T_{3\beta}(\partial_y)D_{\beta 3}(y,\ x)\big]\varphi_3(y)\,ds(y) = \mu\int_{\partial S} D_{3\beta}(x,\ y)\nu_\beta(y)\varphi_3(y)\,ds(y),$$

$$\begin{aligned}\int_{\partial S} \big[T_{33}(\partial_y)D^*_{33}(y,\ x)\big]\varphi_3(y)\,ds(y) \\ = -\frac{1}{2\pi}\int_{\partial S}\bigg[\frac{\partial}{\partial\nu(y)}\ln|x-y|\bigg]\varphi_3(y)\,ds(y),\end{aligned}$$

where the c'_γ and c' are combinations of λ and μ.

These relations, (4.1), (4.2), (4.9), and (3.28) yield

$$\begin{aligned} W\varphi = \nu\biggl(2h^2a_2\mu^2\vartheta - \frac{h^2}{2\pi}\,\sigma' - 2a_2\mu^3\varpi\biggr)\\ +w\biggl(\frac{h^2}{2\pi}\,\varphi - 2a_2\mu^3\rho\biggr)\\ + V(2h^2\mu\sigma' + \mu\varpi) + 2a_2\mu^3(\tilde{W}\varphi) + \mathscr{W}\varphi, \end{aligned} \tag{4.21}$$

where, as functions of x, the specified densities and $\tilde{W}\varphi$ are

$$\begin{aligned} \vartheta &= \bigl(0,\ 0,\ (2\lambda + 3\mu)\nu_\gamma\varphi_\gamma + \mu\varepsilon_{\gamma\beta}x_\gamma\varphi'_\beta\bigr)^{\mathrm{T}},\\ \sigma &= (-\varphi_2,\ \varphi_1,\ 0)^{\mathrm{T}},\\ \varpi &= (\nu_1\varphi_3,\ \nu_2\varphi_3,\ 0)^{\mathrm{T}},\\ \rho &= (x_1\varphi_3,\ x_2\varphi_3,\ -h^2x_\gamma\varphi_\gamma)^{\mathrm{T}},\\ \tilde{W}\varphi &= \bigl(x_1(w\varphi_3),\ x_2(w\varphi_3),\ h^2x_\gamma\bigl(v(\varepsilon_{\beta\gamma}\varphi'_\beta) - w\varphi_\gamma\bigr)\bigr)^{\mathrm{T}}, \end{aligned} \tag{4.22}$$

and $\mathscr{W}\varphi$ is a potential-type function whose kernel satisfies the conditions of Theorem 2.9 with any $\gamma \in (0, 1)$. Since $\varphi \in C^{1,\alpha}(\partial S)$ and $\nu \in C^1(\partial S)$, we see immediately that

$$\vartheta \in C^{0,\alpha}(\partial S), \quad \varpi \in C^1(\partial S), \quad \sigma,\ \rho \in C^{1,\alpha}(\partial S).$$

By Theorem 4.5, the restrictions of $W\varphi$ to S^+ and S^- have $C^{0,\beta}$-extensions $(W\varphi)^+$ and $(W\varphi)^-$ to $\bar{S}^+$ and $\bar{S}^-$, respectively. On the other hand, by Theorems 2.4, 2.7, 2.9, and 4.6, the restrictions of $(W\varphi)_{,\gamma}$ to S^+ and S^- have $C^{0,\beta}$-extensions $((W\varphi)_{,\gamma})^+$ and $((W\varphi)_{,\gamma})^-$, given by (2.45), to $\bar{S}^+$ and $\bar{S}^-$. Since

$$\begin{aligned} ((W\varphi)^+)_{,\gamma}(x) = ((W\varphi)_{,\gamma})^+(x), \ \ x \in S^+,\\ ((W\varphi)^-)_{,\gamma}(x) = ((W\varphi)_{,\gamma})^-(x), \ \ x \in S^-, \end{aligned}$$

the first part of the assertion follows from Theorem 1.20.

For the second part, first we deduce from (2.11) and (2.10) that

$$\begin{aligned} &\biggl(\frac{\partial}{\partial x_\gamma}\,(w\varphi)(x)\biggr)^{\pm}\\ &\quad= \varepsilon_{\gamma\beta}\biggl\{\pm\pi\nu_\beta(x)\varphi'(x)\\ &\qquad\qquad + \int\limits_{\partial S}\biggl(\frac{\partial}{\partial x_\beta}\,\ln|x-y|\biggr)\varphi'(y)\,ds(y)\biggr\}, \end{aligned} \tag{4.23}$$

where the integral is understood as principal value. Next, we convince ourselves by direct verification that for $x \in \partial S$,

$$\varepsilon_{\gamma\beta}\nu_\gamma(x_\beta\varphi_3)' = \varphi_3 + \varepsilon_{\gamma\beta}x_\beta\nu_\gamma\varphi_3',$$

$$\varepsilon_{\gamma\alpha}\nu_\gamma(x_\beta\varphi_3)' = -\nu_\alpha\nu_\beta\varphi_3 + \varphi_3\delta_{\alpha\beta} + \varepsilon_{\gamma\alpha}x_\beta\nu_\gamma\varphi_3',$$

$$\varepsilon_{\gamma\beta}\nu_\gamma(x_\alpha\varphi_3)' = -\nu_\alpha\nu_\beta\varphi_3 + \varphi_3\delta_{\alpha\beta} + \varepsilon_{\gamma\beta}x_\alpha\nu_\gamma\varphi_3',$$

$$\varepsilon_{\beta\alpha}\nu_\beta(x_\gamma\varphi_\gamma)' = \varphi_\alpha - \nu_\alpha\nu_\gamma\varphi_\gamma + \varepsilon_{\beta\alpha}x_\gamma\nu_\beta\varphi_\gamma',$$

$$\varepsilon_{\gamma\alpha}\nu_\beta\nu_\gamma\varphi_\beta' - \varepsilon_{\gamma\beta}\nu_\alpha\nu_\beta\varphi_\gamma' = \varepsilon_{\gamma\alpha}\varphi_\gamma'.$$

Finally, rewriting (4.21) in the form

$$W\varphi = V(2h^2\mu\sigma' + \mu\varpi) + \hat{W}\varphi, \tag{4.24}$$

starting from (3.11), and making use of (2.9), (2.10), (4.20), (4.22), (4.23), and the above relations, after another simple but rather lengthy computation we get

$$\begin{aligned}(T(\hat{W}\varphi)^\pm)_\alpha &= ((T(\hat{W}\varphi))_\alpha)^\pm = (T_{\alpha\beta}(\hat{W}\varphi)_\beta)^\pm\\ &= \mp\tfrac{1}{2}\,\mu\big[\nu_\alpha\varphi_3 + h^2(\varepsilon_{\gamma\beta}\nu_\alpha\nu_\beta\varphi_\gamma' + \varepsilon_{\gamma\alpha}\nu_\gamma\nu_\beta\varphi_\beta' + \varepsilon_{\gamma\alpha}\varphi_\gamma')\big] + (\hat{\mathscr{W}}\varphi)_\alpha\\ &= \mp\mu\big(\tfrac{1}{2}\,\nu_\alpha\varphi_3 + h^2\varepsilon_{\gamma\alpha}\varphi_\gamma'\big) + (\hat{\mathscr{W}}\varphi)_\alpha,\end{aligned}$$

$$(T(\hat{W}\varphi)^\pm)_3 = ((T(\hat{W}\varphi))_3)^\pm = (\hat{\mathscr{W}}\varphi)_3,$$

where $\hat{\mathscr{W}}\varphi$ has the same properties as $\mathscr{W}\varphi$. On the other hand, by Remark 4.9, Corollary 4.8, (4.24), and (4.22),

$$\begin{aligned}&(T(V(2h^2\mu\sigma' + \mu\varpi))^\pm)_\alpha\\ &\quad= ((T(V(2h^2\mu\sigma' + \mu\varpi)))_\alpha)^\pm\\ &\quad= \pm\mu\big(\tfrac{1}{2}\,\nu_\alpha\varphi_3 + h^2\varepsilon_{\gamma\alpha}\varphi_\gamma'\big) + (T(V(2h^2\mu\sigma' + \mu\varpi)))_{0\alpha},\end{aligned}$$

$$\begin{aligned}&(T(V(2h^2\mu\sigma' + \mu\varpi))^\pm)_3\\ &\quad= ((T(V(2h^2\mu\sigma' + \mu\varpi)))_\alpha)^\pm\\ &\quad= (T(V(2h^2\mu\sigma' + \mu\varpi)))_{03};\end{aligned}$$

consequently,

$$(T(W\varphi)^+)(x) = (T(W\varphi)^-)(x) = (\hat{\mathscr{W}}\varphi)(x) + (T(V\varphi))_0(x), \quad x \in \partial S,$$

which completes the proof. □

4.2 Layer Potentials with Integrable Densities

We denote by $L^p(\partial S)$, $p \geq 1$, the space of functions f on ∂S that are measurable and such that $|f|^p$ is Lebesgue integrable over ∂S. As is well known (see, for example, Weir 1973), the space $L^p(\partial S)$ is complete with respect to the norm

$$\|f\|_p = \left\{ \int_{\partial S} |f(y)|^p \, ds(y) \right\}^{1/p}.$$

Also, every function $f \in L^1(\partial S)$ can be written in the form

$$f = f_1 - f_2, \tag{4.25}$$

where f_1 and f_2 are the limits almost everywhere of increasing sequences of step functions $\{\varphi_n^{(1)}\}_{n=1}^{\infty}$ and $\{\varphi_n^{(2)}\}_{n=1}^{\infty}$, respectively, for which the corresponding sequences of integrals $\Big\{ \int\limits_{\partial S} \varphi_n \Big\}_{n=1}^{\infty}$ and $\Big\{ \int\limits_{\partial S} \psi_n \Big\}_{n=1}^{\infty}$ are bounded.

4.11 Definition. Let $f \in L^1(\partial S)$. A point $x \in \partial S$ such that

$$\{\varphi_n^{(\alpha)}(x)\}_{n=1}^{\infty} \to f_\alpha(x),$$

where the f_α are as in (4.25), is called a *Lebesgue point* for f.

4.12 Lemma. *If $f \in L^1(\partial S)$ and x is a Lebesgue point for f, then*

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_s^{s+\varepsilon} f(t) \, dt = f(s).$$

The proof of this assertion is based on (4.25) and Definition 4.11.

4.13 Theorem. *Let $k(x,\ y)$ be a proper γ-singular kernel in S_0, $\gamma \in [0, 1]$, let*

$$x = \xi \mp \delta\nu(\xi) \in S_0^{\pm}, \quad \xi \in \partial S, \quad \delta > 0,$$

and suppose that

$$\lim_{\delta \to 0} \int_{\partial S} k(x,\ y) \, ds(y) = l^{\pm}(\xi) \tag{4.26}$$

and

$$\lim_{\delta \to 0} \int_{\partial S \setminus \Sigma_{\xi,\delta}} k(\xi,\ y) \, ds(y) = l(\xi), \tag{4.27}$$

where $\Sigma_{\xi,\delta}$ *is defined by* (1.16). *If* $\varphi \in L^1(\partial S)$, *then*

$$\lim_{\delta\to 0}\Bigg[\int\limits_{\partial S} k(x,\ y)\varphi(y)\,ds(y) - \int\limits_{\partial S\setminus\Gamma_{\xi,8\delta}} k(\xi,\ y)\varphi(y)\,ds(y)\Bigg] = \big[l^{\pm}(\xi) - l(\xi)\big]\varphi(\xi)$$

for almost all $\xi \in \partial S$, *where* $\Gamma_{\xi,8\delta}$ *is defined by* (1.59).

Proof. Let ξ be a Lebesgue point for φ, let

$$|x-\xi| = \delta < r/8,$$

with r chosen as in (1.43), and let Σ_1 and Σ_2 be the sets (1.29) constructed with x and x' replaced, respectively, by ξ and x. Then

$$\Gamma_{\xi,8\delta} = \Sigma_1,$$

and since here, obviously,

$$\partial S = \Sigma_1 \cup \Sigma_2 \cup (\partial S\setminus\Sigma_{\xi,r}) = \Gamma_{\xi,8\delta} \cup \Sigma_2 \cup (\partial S\setminus\Sigma_{\xi,r}),$$

we can write

$$\begin{aligned}
I &= \int\limits_{\partial S} k(x,\ y)\varphi(y)\,ds(y) - \int\limits_{\partial S\setminus\Gamma_{\xi,8\delta}} k(\xi,\ y)\varphi(y)\,ds(y)\\
&= \int\limits_{\partial S} k(x,y)[\varphi(y)-\varphi(\xi)]\,ds(y) - \int\limits_{\partial S\setminus\Gamma_{\xi,8\delta}} k(\xi,y)[\varphi(y)-\varphi(\xi)]\,ds(y)\\
&\quad + \varphi(\xi)\Bigg[\int\limits_{\partial S} k(x,y)\,ds(y) - \int\limits_{\partial S\setminus\Gamma_{\xi,8\delta}} k(\xi,y)\,ds(y)\Bigg]\\
&= \int\limits_{\Sigma_1} k(x,y)[\varphi(y)-\varphi(\xi)]\,ds(y) + \int\limits_{\Sigma_2} k(x,y)[\varphi(y)-\varphi(\xi)]\,ds(y)\\
&\quad + \int\limits_{\partial S\setminus\Sigma_{\xi,r}} k(x,y)[\varphi(y)-\varphi(\xi)]\,ds(y)\\
&\quad - \int\limits_{\Sigma_2} k(\xi,y)[\varphi(y)-\varphi(\xi)]\,ds(y) - \int\limits_{\partial S\setminus\Sigma_{\xi,r}} k(\xi,y)[\varphi(y)-\varphi(\xi)]\,ds(y)\\
&\quad + \varphi(\xi)\Bigg[\int\limits_{\partial S} k(x,y)\,ds(y) - \int\limits_{\partial S\setminus\Gamma_{\xi,8\delta}} k(\xi,y)\,ds(y)\Bigg]\\
&= I_1 + I_2 + I_3 + I_4,
\end{aligned}$$

where

$$I_1 = \int_{\Sigma_1} k(x,\ y)\big[\varphi(y) - \varphi(\xi)\big]\, ds(y),$$

$$I_2 = \int_{\Sigma_2} \big[k(x,\ y) - k(\xi,\ y)\big]\big[\varphi(y) - \varphi(\xi)\big]\, ds(y),$$

$$I_3 = \int_{\partial S \setminus \Sigma_{\xi,r}} \big[k(x,\ y) - k(\xi,\ y)\big][\varphi(y) - \varphi(\xi)\big]\, ds(y),$$

$$I_4 = \varphi(\xi)\Bigg[\int_{\partial S} k(x,\ y)\, ds(y) - \int_{\partial S \setminus \Gamma_{\xi,8\delta}} k(\xi,\ y)\, ds(y)\Bigg].$$

Let $f(y) = \varphi(y) - \varphi(\xi)$, and let Γ_1 and Γ_2 be defined by (1.42). By Lemmas 1.10 and 4.12, as $\delta \to 0$,

$$\begin{aligned} |I_1| &\le c_1 \int_{\Sigma_1} |x - y|^{-\gamma} |f(y)|\, ds(y) \\ &\le c_2 \int_{\Sigma_1} |x - \xi|^{-\gamma} |f(y)|\, ds(y) \\ &\le c_3 \delta^{-\gamma} \int_{\Gamma_1} |f(t)|\, dt \to c|f(0)| = 0, \end{aligned}$$

where the positive constants $c_1,\ c_2,\ c_3$, and c do not depend on x.

By Lemma 1.11,

$$\begin{aligned} |I_2| &\le c_1 \int_{\Sigma_2} |x - \xi|\, |\xi - y|^{-\gamma-1} |f(y)|\, ds(y) \\ &\le c_4 \delta \int_{\Gamma_2} |s - t|^{-\gamma-1} |f(t)|\, dt, \end{aligned}$$

where $c_4 = \mathrm{const} > 0$ does not depend on x. Setting

$$g(t) = \int_s^t |f(\sigma)|\, d\sigma,$$

we see that

$$\frac{1}{t - s}\, g(t) \to |f(s)| = 0 \quad \text{as } t \to s,$$

g is absolutely continuous (Thomson and Constanda 1999), and $g' = |f|$ almost everywhere in Γ_2. Let a and b, $a < s < b$, be the arc coordinates of the end-points of $\Sigma_{\xi,r}$. Integrating by parts, we obtain

$$\begin{aligned}
|I_2| &\le c_4\delta\Bigg[\int\limits_a^{s-8\delta}(s-t)^{-\gamma-1}g'(t)\,dt + \int\limits_{s+8\delta}^{b}(t-s)^{-\gamma-1}g'(t)\,dt\Bigg]\\
&\le c_5\Bigg\{\delta^{-\gamma}\big[g(s-8\delta)-g(s+8\delta)\big]\\
&\quad - \delta\big[(s-a)^{-\gamma-1}g(a)-(b-s)^{-\gamma-1}g(b)\big]\\
&\quad - (\gamma+1)\delta\Bigg[\int\limits_a^{s-8\delta}(s-t)^{-\gamma-2}g(t)\,dt - \int\limits_{s+8\delta}^{b}(t-s)^{-\gamma-2}g(t)\,dt\Bigg]\Bigg\},
\end{aligned}$$

where $c_5 = \text{const} > 0$ does not depend on x. According to Lemma 4.12, we have

$$\delta^{-\gamma}g(s\pm 8\delta)\to 0 \quad \text{as} \quad \delta\to 0,$$

which also yields

$$\lim_{\delta\to 0}\delta\int\limits_a^{s-8\delta}(s-t)^{-\gamma-2}g(t)\,dt = \lim_{\delta\to 0}\delta\int\limits_{s+8\delta}^{b}(t-s)g(t)\,dt = 0.$$

Hence, $|I_2|\to 0$ as $\delta\to 0$.

From Lemma 1.12 it follows that

$$\begin{aligned}
|I_3| &\le c_1\int\limits_{\partial S\setminus\Sigma_{\xi,r}}|x-\xi|\,|\xi-y|^{-\gamma-1}|f(y)|\,ds(y)\\
&\le c_6 r^{-2}\delta\int\limits_{\partial S}|f(y)|\,ds(y)\to 0 \quad \text{as } \delta\to 0,
\end{aligned}$$

where $c_6 = \text{const} > 0$. Finally, as noted in Remark 1.41,

$$\int\limits_{\Sigma_{\xi,8\delta}\setminus\Gamma_{\xi,8\delta}} k(\xi,\ y)\,ds(y)\to 0 \quad \text{as } \delta\to 0.$$

Consequently, from our assumption on $k(x,\ y)$ we deduce that

$$I_4\to\big[l^{\pm}(\xi)-l(\xi)\big]\varphi(\xi) \quad \text{as } \delta\to 0,$$

which completes the proof. □

4.14 Theorem. *If* $k(x,\ y)$ *is a* γ*-singular kernel on* ∂S, $\gamma \in [0, 1)$, *and* $\varphi \in L^1(\partial S)$, *then*

$$\int_{\partial S} k(x,\ y)\varphi(y)\,ds(y)$$

exists for almost all $x \in \partial S$.

Proof. By Theorem 1.32, the function $\int_{\partial S} |k(x,\ y)|\,ds(x)$ is continuous on ∂S, therefore,

$$|\varphi(y)| \int_{\partial S} |k(x,\ y)|\,ds(x)$$

belongs to $L^1(\partial S)$. By Tonelli's Theorem (see, for example, Weir 1973), $|k(x,\ y)\varphi(y)|$ belongs to $L^1(\partial S \times \partial S)$, and the assertion now follows from Fubini's Theorem (Weir 1973). □

4.15 Remark. Using Lebesgue's dominated convergence theorem, it is easy to show that if $\Lambda_{x,\delta}$ is any small neighborhood of x on ∂S of arc length $\delta > 0$ and $k(x,\ y)$ and φ are as in Theorem 4.14, then

$$\lim_{\delta \to 0} \int_{\partial S \setminus \Lambda_{x,\delta}} k(x,\ y)\varphi(y)\,ds(y) = \int_{\partial S} k(x,\ y)\varphi(y)\,ds(y)$$

for almost all $x \in \partial S$.

4.16 Theorem. *If* $\varphi \in L^1(\partial S)$, *then*

$$\int_{\partial S} \left[\frac{\partial}{\partial s(y)} \ln|x - y|\right]\varphi(y)\,ds(y)$$

exists in the sense of principal value for almost all $x \in \partial S$.

Proof. Since $\partial \ln|x - y|/\partial\nu(y)$ is 0-singular on ∂S, from Theorem 4.14 it follows that

$$\int_{\partial S} \left[\frac{\partial}{\partial \nu(y)} \ln|x - y|\right]\varphi(y)\,ds(y)$$

exists for almost all $x \in \partial S$. Also, in Prössdorf (1978) it is shown that the function

$$\int_{\partial S} \frac{\varphi(\zeta)}{\zeta - z}\,d\zeta$$

exists in the sense of principal value for almost all $x \in \partial S$. The result is now obtained by means of (2.36). □

4.17 Theorem. *Suppose that $k(x, y)$ is a proper γ-singular kernel in S_0, $\gamma \in [0, 1)$,*

$$x = \xi \mp \delta\nu(\xi) \in S_0^{\pm}, \xi \in \partial S, \delta > 0,$$

and $\varphi \in L^1(\partial S)$; then

$$\lim_{\delta \to 0} \int_{\partial S} k(x, y)\varphi(y)\, ds(y) = \int_{\partial S} k(\xi, y)\varphi(y)\, ds(y) \tag{4.28}$$

for almost all $\xi \in \partial S$.

Proof. By Theorem 1.33, $l^{\pm}(\xi)$ and $l(\xi)$ defined by (4.26) and (4.27) are all equal to $\int_{\partial S} k(\xi, y)\, ds(y)$, and (4.28) now follows from Theorem 4.13 and Remark 4.15. □

Having completed this preparatory work, we can turn to the study of the behavior of the layer plate potentials with L^2-densities.

4.18 Theorem. *If $\varphi \in L^2(\partial S)$, then*

(i) *$V\varphi$ and $W\varphi$ are analytic in $\mathbb{R}^2 \setminus \partial S$;*
(ii) *$A(V\varphi) = A(W\varphi) = 0$ in $\mathbb{R}^2 \setminus \partial S$;*
(iii) *Theorem 4.2 holds for $V\varphi$ and $W\varphi$.*

The proof of this assertion is based on classical results concerning the analyticity of solutions of systems of partial differential equations (see, for example, Miranda 1970), the definition of $V\varphi$ and $W\varphi$, and their asymptotic expansions for $|x|$ large.

4.19 Theorem. *If*

$$x = \xi \mp \delta\nu(\xi) \in S_0^{\pm}, \xi \in \partial S, \delta > 0,$$

and $\varphi \in L^2(\partial S)$, then

$$\lim_{\delta \to 0} (V\varphi)(x) = (V\varphi)(\xi)$$

for almost all $\xi \in \partial S$.

Proof. From (4.1) and (3.30) we see that the kernel of $V\varphi$ is a proper γ-singular kernel in S_0, with any $\gamma \in (0, 1)$. Also, since ∂S is a set of finite measure, we have $\varphi \in L^1(\partial S)$, and the assertion now follows from Theorem 4.17. □

4.20 Theorem. *If*

$$x = \xi \mp \delta\nu(\xi) \in S_0^{\pm}, \xi \in \partial S, \delta > 0,$$

and $\varphi \in L^2(\partial S)$, then

$$\lim_{\delta \to 0} (W\varphi)(x) = \mp\tfrac{1}{2}\,\varphi(\xi) + \int_{\partial S} P(\xi, y)\varphi(y)\, ds(y) = \mp\tfrac{1}{2}\,\varphi(\xi) + (W_0\varphi)(\xi)$$

for almost all $\xi \in \partial S$, where the integral is understood as principal value.

Proof. We examine the terms of the kernel $P(x, y)$ of $W\varphi$ one by one, using the expression (3.31). As above, $\varphi \in L^1(\partial S)$.

(i) First, $\partial \ln|x-y|/\partial s(y)$ satisfies the conditions of Theorem 4.13 with $\gamma = 1$ and $l^{\pm} = l = 0$ (see Theorem 2.14). By Remark 4.15 and Theorems 4.16 and 4.13,

$$\lim_{\delta\to 0} \int\limits_{\partial S} \left[\frac{\partial}{\partial s(y)} \ln|x-y| \right] \varphi(y)\,ds(y) = \int\limits_{\partial S} \left[\frac{\partial}{\partial s(y)} \ln|\xi - y| \right] \varphi(y)\,ds(y)$$

for almost all $\xi \in \partial S$, where the right-hand side is understood as principal value.

(ii) Next, $\partial \ln|x-y|/\partial \nu(y)$ satisfies the conditions of Theorem 4.3 with $\gamma = 1$, $l^{+} = 2\pi$, $l^{-} = 0$, and $l = \pi$ (see (2.5)–(2.7)). Also, this is a 0-singular kernel on ∂S. Consequently, by Remark 4.15 and Theorem 4.13,

$$\lim_{\delta\to 0} \int\limits_{\partial S} \left[\frac{\partial}{\partial \nu(y)} \ln|x-y| \right] \varphi(y)\,ds(y) = \pm\pi\varphi(\xi) + \int\limits_{\partial S} \left[\frac{\partial}{\partial \nu(y)} \ln|\xi - y| \right] \varphi(y)\,ds(y)$$

for almost all $\xi \in \partial S$.

(iii) The kernel $\partial\big[(x_\alpha - y_\alpha)(x_\beta - y_\beta)|x-y|^{-2}\big]/\partial s(y)$ satisfies the conditions of Theorem 4.13 with $\gamma = 1$ and $l^{\pm} = l = 0$, and is 0-singular on ∂S (see Theorem 2.14). Hence, by Remark 4.15 and Theorem 4.13,

$$\lim_{\delta\to 0} \int\limits_{\partial S} \left[\frac{\partial}{\partial s(y)} \frac{(x_\alpha - y_\alpha)(x_\beta - y_\beta)}{|x-y|^2} \right] \varphi(y)\,ds(y) = \int\limits_{\partial S} \left[\frac{\partial}{\partial s(y)} \frac{(\xi_\alpha - y_\alpha)(\xi_\beta - y_\beta)}{|\xi-y|^2} \right] \varphi(y)\,ds(y)$$

for almost all $\xi \in \partial S$.

(iv) Finally, the remaining terms are proper γ-singular kernels in S_0, with any $\gamma \in (0, 1)$; therefore, by Theorem 4.17, they satisfy (4.28).

The assertion now follows from (i)–(iv), (3.31), and (4.2). □

4.21 Theorem. *If*

$$x = \xi \mp \delta\nu(\xi) \in S_0^{\pm},\ \xi \in \partial S,\ \delta > 0,$$

and $\varphi \in L^2(\partial S)$, *then*

$$\lim_{\delta\to 0} (T(V\varphi))(x) = \pm\tfrac{1}{2}\,\varphi(\xi) + \int\limits_{\partial S} T(\partial_\xi) D(\xi,\ y)\varphi(y)\,ds(y)$$

for almost all $\xi \in \partial S$, *where the integral is understood as principal value.*

Proof. From the expressions derived in the proof of Theorem 4.6 it is clear that the kernel of $\partial(V\varphi)_i/\partial x_\alpha$ consists of the same type of terms as the kernel of $W\varphi$. To complete the proof, we use (4.15) and (4.18). □

4.22 Remark. It is not difficult to show that the same results hold for $V\varphi$, $W\varphi$, and $T(V\varphi)$ as $S_0^\pm \ni x \to \xi \in \partial S$ on any direction different from that of $\tau(\xi)$. However, for our purposes it suffices to have the corresponding limiting formulas established as $x \to \xi$ along the normal at ξ.

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Chapter 5
The Newtonian Potential

5.1 Definition

The nonhomogeneous system (3.8)—that is,

$$A(\partial_x)u(x) + g(x) = 0, \quad x \in S^+ \text{ or } x \in S^-, \tag{5.1}$$

where $A(\partial_x)$ is defined by (3.9)—can be reduced to its homogeneous version if we know a particular solution of it. Our aim is to construct such a solution and to determine conditions under which this solution has all the required smoothness properties.

5.1 Definition. The integral

$$U(x) = \int_S D(x, y)g(y)\,da(y), \tag{5.2}$$

where S denotes either S^+ or S^- and $D(x, y)$ is the matrix (3.23) of fundamental solutions for $A(\partial_x)$, is called the *Newtonian plate potential* of density g.

Here, we consider only the case of the interior domain S^+; the case of S^- is treated in exactly the same way, with additional restrictions on $g(x)$ as $|x| \to \infty$, to ensure that $U(x)$ exists as an improper integral.

Refining expression (3.30) and writing it more compactly, we see that for x close to y,

$$\begin{aligned} D(x, y) = (\ln|x-y|)\Big(d_1 E_{\gamma\gamma} - \frac{1}{2\pi\mu} E_{33}\Big) \\ + d_2 \frac{(x_\alpha - y_\alpha)(x_\beta - y_\beta)}{|x-y|^2} E_{\alpha\beta} \\ + C + O(|x-y|\ln|x-y|), \end{aligned} \tag{5.3}$$

C. Constanda, *Mathematical Methods for Elastic Plates*,
Springer Monographs in Mathematics, DOI: 10.1007/978-1-4471-6434-0_5,

where C is a constant 3×3 matrix and

$$d_1 = -\frac{\lambda + 3\mu}{4\pi h^2 \mu(\lambda + 2\mu)},$$
$$d_2 = \frac{\lambda + \mu}{4\pi h^2 \mu(\lambda + 2\mu)}. \tag{5.4}$$

The properties of the series expansions for the modified Bessel functions allow us to differentiate (5.3) as necessary.

Let $\sigma(a, \rho)$ be the disk with the center at a, radius ρ, and circular boundary $\partial\sigma(a, \rho)$. All subsequent integration over circularly symmetric domains is performed in terms of polar coordinates with the pole at the center of symmetry.

We make the notation

$$I_{\alpha\beta} = \int\limits_{\partial\sigma(x,1)} (x_\alpha - y_\alpha)(x_\beta - y_\beta)\, ds(y) = \pi\delta_{\alpha\beta}, \tag{5.5}$$

$$I_{\alpha\beta\rho\eta} = \int\limits_{\partial\sigma(x,1)} (x_\alpha - y_\alpha)(x_\beta - y_\beta)(x_\rho - y_\rho)(x_\eta - y_\eta)\, ds(y)$$
$$= \frac{\pi}{4}(\delta_{\alpha\beta}\delta_{\rho\eta} + \delta_{\alpha\rho}\delta_{\beta\eta} + \delta_{\alpha\eta}\delta_{\beta\rho}), \tag{5.6}$$

where the integrals have been computed as indicated above.

5.2 The First-Order Derivatives

5.2 Theorem. *If* $g \in L^\infty(S^+)$, then $\partial U(x)/\partial x_\alpha$ *exists at each point* $x \in S^+$ $(x \in \partial S)$ *and*

$$\frac{\partial}{\partial x_\alpha} U(x) = \int\limits_{S^+} \frac{\partial}{\partial x_\alpha} D(x, y) g(y)\, da(y).$$

Proof. Consider the function

$$U^\omega(x) = \int\limits_{S^+} k^\omega(x, y) g(y)\, da(y),$$

where

$$k^\omega(x, y) = \begin{cases} 0, & 0 \le |x - y| \le \omega, \\ \dfrac{1}{2}\left\{\sin\left[\pi\left(\dfrac{|x - y|}{\omega} - \dfrac{3}{2}\right)\right] + 1\right\} D(x, y), & \omega < |x - y| < 2\omega, \\ D(x, y), & 2\omega \le |x - y| < \infty. \end{cases}$$

Obviously, $k^{\omega}(x, y)$ has first-order derivatives with respect to the coordinates of $x \in S^+$ ($x \in \partial S$); hence, $\partial U^{\omega}(x)/\partial x_\alpha$ exists at each point $x \in S^+$ ($x \in \partial S$) and

$$\frac{\partial}{\partial x_\alpha} U^{\omega}(x) = \int_{S^+} \frac{\partial}{\partial x_\alpha} k^{\omega}(x, y) g(y)\, da(y),$$

where the kernel of the integral satisfies

$$\begin{aligned}
&\frac{\partial}{\partial x_\alpha} k^{\omega}(x, y) = 0 \quad \text{if } 0 \le |x - y| < \omega,\\
&\frac{\partial}{\partial x_\alpha} k^{\omega}(x, y) = \left\{ \frac{\pi}{2\omega} \frac{x_\alpha - y_\alpha}{|x - y|} \cos\left[\pi\left(\frac{|x - y|}{\omega} - \frac{3}{2}\right)\right]\right\} D(x, y)\\
&\qquad\qquad + \frac{1}{2}\left\{ \sin\left[\pi\left(\frac{|x - y|}{\omega} - \frac{3}{2}\right)\right] + 1\right\} \frac{\partial}{\partial x_\alpha} D(x, y)\\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{if } \omega < |x - y| < 2\omega,\\
&\frac{\partial}{\partial x_\alpha} k^{\omega}(x, y) = \frac{\partial}{\partial x_\alpha} D(x, y) \quad \text{if } 2\omega < |x - y| < \infty.
\end{aligned}$$

Next,

$$\begin{aligned}
&\left|U^{\omega}(x) - U(x)\right|\\
&\quad = \left| \int_{S^+} \left[k^{\omega}(x, y) - D(x, y)\right] g(y)\, da(y) \right|\\
&\quad = \left| - \int_{S^+ \cap \sigma(x,\omega)} D(x, y) g(y)\, da(y) \right.\\
&\qquad \left. + \frac{1}{2} \int_{S^+ \cap \sigma(x,2\omega) \setminus S^+ \cap \sigma(x,\omega)} \left\{ \sin\left[\pi\left(\frac{|x - y|}{\omega} - \frac{3}{2}\right)\right] - 1\right\} D(x, y) g(y)\, da(y) \right|\\
&\quad \le \|g\|_\infty \int_{\sigma(x,\omega)} |D(x, y)|\, da(y) + \|g\|_\infty \int_{\sigma(x,2\omega)\setminus\sigma(x,\omega)} |D(x, y)|\, da(y) \to 0,
\end{aligned}$$

uniformly as $\omega \to 0$, since $D(x, y)$ has only a logarithmic singularity at $x = y$, which means that the behavior of the right-hand side is similar to

$$\int_0^{\omega} \rho \ln \rho \, d\rho = \tfrac{1}{2}\, \omega^2 \ln \omega - \tfrac{1}{4}\, \omega^2 \to 0 \quad \text{as } \omega \to 0.$$

Here, $\|g\|_\infty$ is the norm on the space $L^\infty(S^+)$.

Also,

$$
\begin{aligned}
&\left|\int\limits_{S^+} \frac{\partial}{\partial x_\alpha} k^\omega(x,y) g(y)\, da(y) - \int\limits_{S^+} \frac{\partial}{\partial x_\alpha} D(x,y) g(y)\, da(y)\right| \\
=&\left| - \int\limits_{S^+\cap\sigma(x,\omega)} \frac{\partial}{\partial x_\alpha} D(x,y) g(y)\, da(y) \right.\\
&+ \int\limits_{S^+\cap\sigma(x,2\omega)\setminus S^+\cap\sigma(x,\omega)} \left\{\left[\frac{\pi}{2\omega}\frac{x_\alpha - y_\alpha}{|x-y|}\cos\left(\pi\left(\frac{|x-y|}{\omega} - \frac{3}{2}\right)\right)\right] D(x,y)\right.\\
&\qquad \left.\left. + \frac{1}{2}\left[\sin\left(\pi\left(\frac{|x-y|}{\omega}-\frac{3}{2}\right)\right) - 1\right]\frac{\partial}{\partial x_\alpha} D(x,y)\right\} g(y)\, da(y)\right|,
\end{aligned}
$$

which leads to

$$
\begin{aligned}
&\left|\int\limits_{S^+} \frac{\partial}{\partial x_\alpha} k^\omega(x,y) g(y)\, da(y) - \int\limits_{S^+} \frac{\partial}{\partial x_\alpha} D(x,y) g(y)\, da(y)\right| \\
&\le \|g\|_\infty \int\limits_{\sigma(x,\omega)} \left|\frac{\partial}{\partial x_\alpha} D(x,y)\right| da(y) \\
&\quad + \frac{\pi}{2\omega}\|g\|_\infty \int\limits_{\sigma(x,2\omega)\setminus\sigma(x,\omega)} \left|\frac{x_\alpha - y_\alpha}{|x-y|} D(x,y)\right| da(y) \\
&\quad + \|g\|_\infty \int\limits_{\sigma(x,2\omega)\setminus\sigma(x,\omega)} \left|\frac{\partial}{\partial x_\alpha} D(x,y)\right| da(y) \\
&\le c\|g\|_\infty \int\limits_{\sigma(x,\omega)} \frac{1}{|x-y|}\, da(y) + \frac{\pi}{2\omega}\|g\|_\infty \int\limits_{\sigma(x,2\omega)\setminus\sigma(x,\omega)} \left|\frac{x_\alpha - y_\alpha}{|x-y|} D(x,y)\right| da(y) \\
&\quad + c\|g\|_\infty \int\limits_{\sigma(x,2\omega)\setminus\sigma(x,\omega)} \frac{1}{|x-y|}\, da(y) \to 0,
\end{aligned}
$$

uniformly as $\omega \to 0$. The second integral on the right-hand side tends to zero as $\omega \to 0$ because its behavior is similar to

$$
\begin{aligned}
\frac{1}{\omega}\int\limits_{\omega}^{2\omega} \rho \ln\rho\, d\rho &= \frac{1}{\omega}\left[\tfrac{1}{2}\rho^2\ln\rho - \tfrac{1}{4}\rho^2\right]_\omega^{2\omega} \\
&= \tfrac{1}{4}\,\omega(6\ln\omega + 8\ln 2 - 3) \to 0 \quad \text{as } \omega \to 0.
\end{aligned}
$$

The assertion now follows from a well-known theorem of real analysis. □

5.3 Lemma. *If* $\beta \in (0, 1]$, *then there is a constant* c *such that*

$$|x' - x''|^\beta \ln \frac{1}{|x' - x''|} \le c|x' - x''|^\alpha, \quad 0 < \alpha < \beta \le 1, \quad x',\, x'' \in \mathbb{R}^2.$$

Proof. Consider the function h defined by

$$h(t) = \frac{1}{1-\gamma}\, t^\gamma + t \ln t, \tag{5.7}$$

where $\gamma \in (0, 1)$ and $t \in [0, \infty)$. We can write h as

$$h(t) = t^\gamma \chi(t), \tag{5.8}$$

with

$$\chi(t) = \frac{1}{1-\gamma} + t^{1-\gamma} \ln t.$$

Obviously, $h(0) = 0$. We claim that $\chi(t) > 0$ on $0 \le t < \infty$. The function χ has a turning point if and only if

$$0 = \chi'(t) = (1-\gamma)t^{-\gamma} \ln t + t^{-\gamma},$$

which occurs at

$$t = e^{1/(\gamma-1)}.$$

Differentiating χ a second time yields

$$\chi''(t) = -\gamma(1-\gamma)t^{-\gamma-1} \ln t + (1-\gamma)t^{-\gamma-1} - \gamma t^{-\gamma-1};$$

so, since $\gamma < 1$,

$$\begin{aligned}\chi''\big(e^{1/(\gamma-1)}\big) &= -\gamma(1-\gamma)e^{(\gamma+1)/(1-\gamma)} \frac{1}{\gamma-1} \\ &\quad + (1-\gamma)e^{(\gamma+1)/(1-\gamma)} - \gamma e^{(\gamma+1)/(1-\gamma)} \\ &= (1-\gamma)e^{(\gamma+1)/(1-\gamma)} > 0.\end{aligned}$$

Also,

$$\begin{aligned}\chi\big(e^{1/(\gamma-1)}\big) &= \frac{1}{1-\gamma} + e^{-1} \frac{1}{\gamma-1} \\ &= \frac{1}{1-\gamma}\left(1 - \frac{1}{e}\right) > 0.\end{aligned}$$

Hence, χ has a minimum turning point at $t = e^{1/(\gamma-1)}$, and $\chi(e^{1/(\gamma-1)}) > 0$. Since this is the only turning point of the function, $\chi(t) > 0$ on $0 \le t < \infty$. Therefore, $h(t)$ given by (5.8) has exactly one root, which occurs at $t = 0$. Now, since

$$h\big(e^{1/(\gamma-1)}\big) = \frac{1}{1-\gamma}\, e^{\gamma/(\gamma-1)}\left(1 - \frac{1}{e}\right) > 0,$$

we find that $h(t) \ge 0$ on $0 \le t < \infty$. Replacing t by $|x' - x''|$ in (5.7) yields

$$\frac{1}{1-\gamma}\,|x' - x''|^{\gamma} + |x' - x''| \ln |x' - x''| \ge 0$$

for any $\gamma \in (0, 1)$.

With $c = \dfrac{1}{1-\gamma}$, this is rewritten as

$$|x' - x''| \ln \frac{1}{|x' - x''|} \le c|x' - x''|^{\gamma},$$

so

$$|x' - x''|^{\beta} \ln \frac{1}{|x' - x''|} \le c|x' - x''|^{\alpha}, \quad 0 < \alpha < \beta \le 1,$$

which proves the assertion. □

5.4 Theorem. *If $g \in L^{\infty}(S^+)$, then $U \in C^{1,\alpha}(\partial S)$, $\alpha \in (0, 1)$.*

Proof. By Theorem 5.2,

$$Q(x) = \frac{\partial}{\partial x_\eta}\, U(x) = \int_{S^+} q(x, y) g(y)\, da(y) \quad \text{on } \partial S,$$

where

$$q(x, y) = \frac{\partial}{\partial x_\eta}\, D(x, y).$$

By (5.3),

$$q(x, y) = O(|x - y|^{-1}) \text{as} |x - y| \to 0.$$

Let $x', x'' \in \partial S$ and $\xi = |x' - x''|$. Then

$$Q(x') - Q(x'') = J_1(x', x'') + J_2(x', x'') + J_3(x', x''), \tag{5.9}$$

where

$$J_1(x', x'') = \int\limits_{S^+ \cap \sigma(x', 2\xi)} q(x', y) g(y)\, da(y),$$

$$J_2(x', x'') = -\int\limits_{S^+ \cap \sigma(x', 2\xi)} q(x'', y) g(y)\, da(y),$$

$$J_3(x', x'') = \int\limits_{S^+ \setminus \sigma(x', 2\xi)} \big[q(x', y) - q(x'', y)\big]\, g(y)\, da(y).$$

Taking into account the singularity of $q(x, y)$, we find that

$$\begin{aligned} |J_1| &\le \int\limits_{S^+ \cap \sigma(x', 2\xi)} |q(x', y)||g(y)|\, da(y) \\ &\le c_1 \|g\|_\infty \int\limits_{\sigma(x', 2\xi)} \frac{1}{|x' - y|}\, da(y) \\ &= 2\pi c_1 \|g\|_\infty \cdot 2\xi, \end{aligned}$$

where, as mentioned earlier, the integration was performed in terms of polar coordinates with the pole at x'; hence,

$$|J_1| \le c_2 |x' - x''|. \tag{5.10}$$

Similarly, using polar coordinates with the pole at x'', we get

$$\begin{aligned} |J_2| &\le \int\limits_{S^+ \cap \sigma(x', 2\xi)} |q(x'', y)||g(y)|\, da(y) \\ &\le c_1 \|g\|_\infty \int\limits_{\sigma(x'', 3\xi)} \frac{1}{|x'' - y|}\, da(y) \\ &= 2\pi c_1 \|g\|_\infty \cdot 3\xi, \end{aligned}$$

so

$$|J_2| \le c_3 |x' - x''|. \tag{5.11}$$

Estimating J_3 is less straightforward. First, we have

$$|J_3| \le \int\limits_{S^+ \setminus \sigma(x', 2\xi)} |q(x', y) - q(x'', y)|\, |g(y)|\, da(y).$$

By the mean value theorem,

$$\begin{aligned}|q(x', y) - q(x'', y)| &\le |x'_\beta - x''_\beta| \left| \frac{\partial}{\partial x_\beta} q(x''', y) \right| \\ &\le c_4 |x' - x''|\,|x''' - y|^{-2},\end{aligned}$$

where x''' lies between x' and x''. Also, for $y \in S^+ \setminus \sigma(x', 2\xi)$,

$$|x''' - y| \ge |x' - y| - |x' - x'''| > |x' - y| - \tfrac{1}{2}\,|x' - y| = \tfrac{1}{2}\,|x' - y|;$$

therefore, when $y \in S^+ \setminus \sigma(x', 2\xi)$,

$$|q(x', y) - q(x'', y)| \le c_5 |x' - x''|\,|x' - y|^{-2},$$

which implies that

$$|J_3| \le c_5 \|g\|_\infty |x' - x''| \int\limits_{S^+ \setminus \sigma(x', 2\xi)} \frac{1}{|x' - y|^2}\, da(y);$$

so, by extending the domain of integration, we see that

$$|J_3| \le c_5 \|g\|_\infty |x' - x''| \int\limits_{\sigma(x', M) \setminus \sigma(x', 2\xi)} \frac{1}{|x' - y|^2}\, da(y),$$

where M is the largest distance between any two points in $\bar{S}^+$; consequently,

$$\begin{aligned}|J_3| &\le 2\pi c_5 \|g\|_\infty |x' - x''| \left(\ln \frac{M}{2} - \ln |x' - x''| \right) \\ &= c_6 |x' - x''| + c_7 |x' - x''| \ln \frac{1}{|x' - x''|},\end{aligned}$$

and from Lemma 5.3 we now conclude that

$$|J_3| \le c_6 |x' - x''| + c_8 |x' - x''|^\alpha, \quad \alpha \in (0, 1);$$

in other words,

$$|J_3| \le c_9 |x' - x''|^\alpha. \tag{5.12}$$

Combining (5.9)–(5.12), we arrive at the required inequality

$$|Q(x') - Q(x'')| \le c|x' - x''|^\alpha, \quad \alpha \in (0, 1),$$

where c is a constant independent of x' and x''. □

5.3 The Second-Order Derivatives

5.5 Theorem. *If* $g \in C^{0,\alpha}(S^+)$, $\alpha \in (0, 1]$, *then the integral*

$$M(x) = \int_{S^+} m(x, y)g(y)\, da(y),$$

where

$$m(x, y) = \frac{\partial^2}{\partial x_\alpha \partial x_\beta} D(x, y),$$

exists in the sense of principal value at every point $x \in S^+$ *and is given by the formula*

$$M(x) = \int_{S^+} m(x, y)[g(y) - g(x)]\, da(y) + \Bigg[\int_{S^+ \setminus \sigma(x,\omega)} m(x, y)\, da(y) + \lim_{\varepsilon \to 0} \int_{\sigma(x,\omega) \setminus \sigma(x,\varepsilon)} m(x, y)\, da(y) \Bigg] g(x), \tag{5.13}$$

where $\omega > \varepsilon$ *and* $\sigma(x, \omega) \subseteq S^+$.

Proof. The principal value of the integral is computed as

$$\begin{aligned} \lim_{\varepsilon \to 0} \int_{S^+ \setminus \sigma(x,\varepsilon)} m(x, y)g(y)\, da(y) &= \lim_{\varepsilon \to 0} \Bigg\{ \int_{S^+ \setminus \sigma(x,\varepsilon)} m(x, y)[g(y) - g(x)]\, da(y) \\ &\quad + \Bigg[\int_{S^+ \setminus \sigma(x,\omega)} m(x, y)\, da(y) \\ &\quad + \int_{\sigma(x,\omega) \setminus \sigma(x,\varepsilon)} m(x, y)\, da(y) \Bigg] g(x) \Bigg\}, \end{aligned}$$

where ω is chosen so that $\omega > \varepsilon$ and $\sigma(x, \omega) \subseteq S^+$.

The first integral on the right-hand side of this expression converges since the integrand is $O(|x - y|^{\alpha-2})$, $\alpha \in (0, 1]$; hence,

$$\lim_{\varepsilon \to 0} \int\limits_{S^+ \setminus \sigma(x,\varepsilon)} m(x, y) g(y)\, da(y)$$
$$= \int\limits_{S^+} m(x, y)[g(y) - g(x)]\, da(y)$$
$$+ \Bigg[\int\limits_{S^+ \setminus \sigma(x,\omega)} m(x, y)\, da(y) + \lim_{\varepsilon \to 0} \int\limits_{\sigma(x,\omega) \setminus \sigma(x,\varepsilon)} m(x, y)\, da(y) \Bigg] g(x). \quad (5.14)$$

We claim that the third integral has a limit as $\varepsilon \to 0$. By (5.3),

$$\frac{\partial}{\partial x_\beta} D_{\rho\eta}(x, y) = d_1 \delta_{\rho\eta} \frac{x_\beta - y_\beta}{|x - y|^2}$$
$$+ d_2 \frac{\delta_{\rho\beta}(x_\eta - y_\eta) + \delta_{\eta\beta}(x_\rho - y_\rho)}{|x - y|^2}$$
$$- 2 \frac{(x_\rho - y_\rho)(x_\eta - y_\eta)(x_\beta - y_\beta)}{|x - y|^4} + O(\ln |x - y|)$$

and

$$\frac{\partial^2}{\partial x_\alpha \partial x_\beta} D_{\rho\eta}(x, y)$$
$$= d_1 \delta_{\rho\eta} \Bigg[\frac{\delta_{\alpha\beta}}{|x - y|^2} - 2 \frac{(x_\alpha - y_\alpha)(x_\beta - y_\beta)}{|x - y|^4} \Bigg]$$
$$+ d_2 \Bigg[\frac{\delta_{\rho\beta}\delta_{\alpha\eta} + \delta_{\eta\beta}\delta_{\alpha\rho}}{|x - y|^2} - 2 \frac{\delta_{\rho\beta}(x_\alpha - y_\alpha)(x_\eta - y_\eta) + \delta_{\eta\beta}(x_\alpha - y_\alpha)(x_\rho - y_\rho)}{|x - y|^4} \Bigg]$$
$$- 2 d_2 \Bigg[\frac{\delta_{\alpha\rho}(x_\eta - y_\eta)(x_\beta - y_\beta) + \delta_{\alpha\eta}(x_\rho - y_\rho)(x_\beta - y_\beta) + \delta_{\alpha\beta}(x_\rho - y_\rho)(x_\eta - y_\eta)}{|x - y|^4}$$
$$- 4 \frac{(x_\alpha - y_\alpha)(x_\beta - y_\beta)(x_\rho - y_\rho)(x_\eta - y_\eta)}{|x - y|^6} \Bigg] + O(|x - y|^{-1}).$$

From (5.3) we also deduce that

$$\frac{\partial^2}{\partial x_\alpha \partial x_\beta} D_{\rho 3}(x, y) = O(|x - y|^{-1})$$

and

$$\frac{\partial}{\partial x_\beta} D_{33}(x, y) = -\frac{1}{2\pi\mu} \frac{x_\beta - y_\beta}{|x - y|^2} + O(|x - y| \ln |x - y|),$$

which lead to

$$\frac{\partial^2}{\partial x_\alpha \partial x_\beta} D_{33}(x, y) = -\frac{1}{2\pi\mu}\left[\frac{\delta_{\alpha\beta}}{|x-y|^2} - 2\frac{(x_\alpha - y_\alpha)(x_\beta - y_\beta)}{|x-y|^4}\right] + O(\ln|x-y|);$$

therefore,

$$\begin{aligned}
&\int\limits_{\sigma(x,\omega)\setminus\sigma(x,\varepsilon)} m_{\rho\eta}(x, y)\, da(y)\\
&= \ln\frac{\omega}{\varepsilon}\big[d_1(2\pi\delta_{\rho\eta}\delta_{\alpha\beta} - 2\delta_{\rho\eta}I_{\alpha\beta}) + 2\pi d_2(\delta_{\rho\beta}\delta_{\alpha\eta} + \delta_{\eta\beta}\delta_{\alpha\rho})\\
&\qquad - 2d_2(\delta_{\rho\beta}I_{\alpha\eta} + \delta_{\eta\beta}I_{\alpha\rho} + \delta_{\alpha\rho}I_{\eta\beta} + \delta_{\alpha\eta}I_{\rho\beta} + \delta_{\alpha\beta}I_{\rho\eta}) + 8d_2 I_{\alpha\beta\rho\eta}\big]\\
&\quad + \int\limits_{\sigma(x,\omega)\setminus\sigma(x,\varepsilon)} O(|x-y|^{-1})\, da(y).
\end{aligned}$$

Taking (5.5) and (5.6) into account, we find that

$$\begin{aligned}
&\int\limits_{\sigma(x,\omega)\setminus\sigma(x,\varepsilon)} m_{\rho\eta}(x, y)\, da(y)\\
&= \ln\frac{\omega}{\varepsilon}\big[2\pi d_1(\delta_{\rho\eta}\delta_{\alpha\beta} - \delta_{\rho\eta}\delta_{\alpha\beta}) + 2\pi d_2(\delta_{\rho\beta}\delta_{\alpha\eta} + \delta_{\eta\beta}\delta_{\alpha\rho})\\
&\qquad - 2\pi d_2(2\delta_{\rho\beta}\delta_{\alpha\eta} + 2\delta_{\eta\beta}\delta_{\alpha\rho} + \delta_{\alpha\beta}\delta_{\rho\eta})\\
&\qquad + 2\pi d_2(\delta_{\alpha\beta}\delta_{\rho\eta} + \delta_{\alpha\rho}\delta_{\beta\eta} + \delta_{\alpha\eta}\delta_{\beta\rho})\big]\\
&\quad + \int\limits_{\sigma(x,\omega)\setminus\sigma(x,\varepsilon)} O(|x-y|^{-1})\, da(y);
\end{aligned}$$

hence,

$$\int\limits_{\sigma(x,\omega)\setminus\sigma(x,\varepsilon)} m_{\rho\eta}(x, y)\, da(y) = \int\limits_{\sigma(x,\omega)\setminus\sigma(x,\varepsilon)} O(|x-y|^{-1})\, da(y),$$

from which we readily infer that

$$\lim_{\varepsilon\to 0} \int\limits_{\sigma(x,\omega)\setminus\sigma(x,\varepsilon)} m_{\rho\eta}(x, y)\, da(y) \ \text{ exists.}$$

Similarly,

$$\int\limits_{\sigma(x,\omega)\setminus\sigma(x,\varepsilon)} m_{\rho 3}(x, y)\, da(y) = \int\limits_{\sigma(x,\omega)\setminus\sigma(x,\varepsilon)} \frac{\partial^2}{\partial x_\alpha \partial x_\beta} D_{\rho 3}(x, y)\, da(y)$$
$$= \int\limits_{\sigma(x,\omega)\setminus\sigma(x,\varepsilon)} O\left(\frac{1}{|x-y|}\right) da(y),$$

so

$$\lim_{\varepsilon\to 0} \int\limits_{\sigma(x,\omega)\setminus\sigma(x,\varepsilon)} m_{\rho 3}(x, y)\, da(y) \;\text{ exists.}$$

Finally,

$$\int\limits_{\sigma(x,\omega)\setminus\sigma(x,\varepsilon)} m_{33}(x, y)\, da(y)$$
$$= \int\limits_{\sigma(x,\omega)\setminus\sigma(x,\varepsilon)} \frac{\partial^2}{\partial x_\alpha \partial x_\beta} D_{33}(x, y)\, da(y)$$
$$= \ln\frac{\omega}{\varepsilon}\left(-\frac{1}{\mu}\,\delta_{\alpha\beta} + \frac{1}{\pi\mu}\, I_{\alpha\beta}\right)$$
$$+ \int\limits_{\sigma(x,\omega)\setminus\sigma(x,\varepsilon)} O(\ln|x-y|)\, da(y).$$

By (5.5),

$$\int\limits_{\sigma(x,\omega)\setminus\sigma(x,\varepsilon)} m_{33}(x, y)\, da(y) = \int\limits_{\sigma(x,\omega)\setminus\sigma(x,\varepsilon)} O(\ln|x-y|)\, da(y),$$

which implies that

$$\lim_{\varepsilon\to 0} \int\limits_{\sigma(x,\omega)\setminus\sigma(x,\varepsilon)} m_{33}(x, y)\, da(y) \;\text{ exists.}$$

Therefore, we have shown that

$$\lim_{\varepsilon\to 0} \int\limits_{\sigma(x,\omega)\setminus\sigma(x,\varepsilon)} m(x, y)\, da(y) \;\text{ exists,}$$

and from (5.14) we conclude that

$$\lim_{\varepsilon\to 0} \int\limits_{S^+\setminus\sigma(x,\varepsilon)} m(x,y)\,da(y) \ \text{ exists,}$$

thus proving the assertion. □

5.6 Remark. Since both

$$\int\limits_{S^+} D(x,y)g(y)\,da(y) \quad \text{and} \quad \int\limits_{S^+} \frac{\partial}{\partial x_\alpha} D(x,y)g(y)\,da(y)$$

exist as improper integrals (provided that g is bounded), their principal values obviously exist and coincide with the values of the improper integrals themselves, so we may write

$$\begin{aligned}
&\int\limits_{S^+} D(x,y)g(y)\,da(y) \\
&\quad = \int\limits_{S^+} D(x,y)\big[g(y)-g(x)\big]\,da(y) \\
&\qquad + \Bigg[\int\limits_{S^+\setminus\sigma(x,\omega)} D(x,y)\,da(y) \\
&\qquad + \lim_{\varepsilon\to 0} \int\limits_{\sigma(x,\omega)\setminus\sigma(x,\varepsilon)} D(x,y)\,da(y)\Bigg]g(x)
\end{aligned}$$

and

$$\begin{aligned}
&\int\limits_{S^+} \frac{\partial}{\partial x_\alpha} D(x,y)g(y)\,da(y) \\
&\quad = \int\limits_{S^+} \frac{\partial}{\partial x_\alpha} D(x,y)\big[g(y)-g(x)\big]\,da(y) \\
&\qquad + \Bigg[\int\limits_{S^+\setminus\sigma(x,\omega)} \frac{\partial}{\partial x_\alpha} D(x,y)\,da(y) \\
&\qquad + \lim_{\varepsilon\to 0} \int\limits_{\sigma(x,\omega)\setminus\sigma(x,\varepsilon)} \frac{\partial}{\partial x_\alpha} D(x,y)\,da(y)\Bigg]g(x).
\end{aligned}$$

In the next assertion, S^* denotes a bounded domain in $\mathbb{R}^2$ such that $S^* \subset S^+$.

5.7 Theorem. *If* $g \in C^{0,\beta}(S^+)$, $\beta \in (0, 1)$, *then* $M \in C^{0,\alpha}(S^*)$, *where* $0 < \alpha < \beta < 1$ *and* $M(x)$ *defined by* (5.13) *is understood in the sense of principal value.*

Proof. Let $\rho > 0$ be the minimum distance between ∂S and the boundary of S^*. By Theorem 5.5, $M(x)$ exists for $x \in S^*$ in the sense that

$$M(x) = \int_{S^+} m(x, y)\big[g(y) - g(x)\big]\, da(y) + \Bigg[\int_{S^+ \setminus \sigma(x,\rho)} m(x, y)\, da(y)\Bigg] g(x) + \Bigg[\lim_{\varepsilon \to 0} \int_{\sigma(x,\rho)\setminus\sigma(x,\varepsilon)} m(x, y)\, da(y)\Bigg] g(x).$$

Since $x \in S^*$, the disk $\sigma(x, \rho)$ is contained entirely within S^+.

We have already shown in the proof of Theorem 5.5 that the limit of the third integral on the right-hand side exists. Given that m is, in fact, a function of $x - y$ and we are integrating over an annular region with the center at x, this integral is a constant matrix depending on ρ; that is, it is independent of x.

For simplicity, from now on we consider m and f to be scalar functions instead of a matrix-valued function and a vector-valued function, respectively.

The function M can be written in the form

$$M(x) = M_1(x) + M_2(x) + M_3(x), \tag{5.15}$$

where

$$M_1(x) = \int_{S^+} m(x, y)\big[g(y) - g(x)\big]\, da(y),$$

$$M_2(x) = g(x) \int_{S^+ \setminus \sigma(x,\rho)} m(x, y)\, da(y),$$

$$M_3(x) = \Bigg[\lim_{\varepsilon \to 0} \int_{\sigma(x,\rho)\setminus\sigma(x,\varepsilon)} m(x, y)\, da(y)\Bigg] g(x) = c_\rho g(x).$$

Let $x', x'' \in S^*$ be such that

$$\xi = |x' - x''| < \tfrac{1}{2}\rho.$$

First, we have

$$M_1(x') - M_1(x'') = J_1(x', x'') + J_2(x', x'') + J_3(x', x'') + J_4(x', x''), \tag{5.16}$$

where

$$J_1(x', x'') = \int\limits_{S^+\setminus\sigma(x',\rho)} \big[m(x', y) - m(x'', y)\big]\big[g(y) - g(x'')\big]\, da(y)$$
$$+ \big[g(x'') - g(x')\big] \int\limits_{S^+\setminus\sigma(x',\rho)} m(x', y)\, da(y),$$

$$J_2(x', x'') = \int\limits_{\sigma(x',2\xi)} m(x', y)\big[g(y) - g(x')\big]\, da(y)$$
$$- \int\limits_{\sigma(x',2\xi)} m(x'', y)\big[g(y) - g(x'')\big]\, da(y),$$

$$J_3(x', x'') = \big[g(x'') - g(x')\big] \int\limits_{\sigma(x',\rho)\setminus\sigma(x',2\xi)} m(x', y)\, da(y),$$

$$J_4(x', x'') = \int\limits_{\sigma(x',\rho)\setminus\sigma(x',2\xi)} \big[m(x', y) - m(x'', y)\big]\big[g(y) - g(x'')\big]\, da(y).$$

We estimate the above integrals. First,

$$|J_1(x', x'')| \le \int\limits_{S^+\setminus\sigma(x',\rho)} |m(x', y) - m(x'', y)|\, |g(y) - g(x'')|\, da(y)$$
$$+ |g(x'') - g(x')| \int\limits_{S^+\setminus\sigma(x',\rho)} |m(x', y)|\, da(y).$$

By the mean value theorem,

$$|m(x', y) - m(x'', y)| \le |x'_\alpha - x''_\alpha| \left| \frac{\partial}{\partial x_\alpha} m(x''', y) \right| \le c_1 \frac{|x' - x''|}{|x''' - y|^3},$$

where x''' lies between x' and x''. Also, since

$$\begin{aligned} |x' - x'''| &< |x' - x''| \\ &< \tfrac{1}{2}\, |x' - y|, \end{aligned}$$

for $y \in S^+ \setminus \sigma(x', \rho)$ we have

$$\begin{aligned} |x''' - y| &\ge |x' - y| - |x' - x'''| \\ &> |x' - y| - \tfrac{1}{2}\, |x' - y| = \tfrac{1}{2}\, |x' - y|. \end{aligned}$$

Therefore, when $y \in S^+ \setminus \sigma(x', \rho)$,

$$|m(x', y) - m(x'', y)| \le c_2 \frac{|x' - x''|}{|x' - y|^3}, \tag{5.17}$$

from which

$$\begin{aligned} |J_1(x', x'')| \le c_3 |x' - x''| \int\limits_{S^+ \setminus \sigma(x', \rho)} \frac{|x'' - y|^\beta}{|x' - y|^3}\, da(y) \\ + c_4 |x' - x''|^\beta \int\limits_{S^+ \setminus \sigma(x', \rho)} \frac{1}{|x' - y|^2}\, da(y). \end{aligned}$$

The integrals on the right-hand side do not pose a problem since x' lies outside the domain of integration and $|x'' - y|$ is bounded in $S^+ \setminus \sigma(x', \rho)$; hence,

$$|J_1(x', x'')| \le c_5 |x' - x''|^\beta, \tag{5.18}$$

where c_5 is a constant depending on ρ.

Estimating J_2, we arrive at

$$\begin{aligned} |J_2(x', x'')| \le & \int\limits_{\sigma(x', 2\xi)} |m(x', y)|\, |g(y) - g(x')|\, da(y) \\ & + \int\limits_{\sigma(x', 2\xi)} |m(x'', y)|\, |g(y) - g(x'')|\, da(y) \\ & \le c_6 \int\limits_{\sigma(x', 2\xi)} |x' - y|^{\beta - 2}\, da(y) + c_6 \int\limits_{\sigma(x', 2\xi)} |x'' - y|^{\beta - 2}\, da(y). \end{aligned}$$

From Fig. 5.1 we see that $\sigma(x', 2\xi) \subset \sigma(x'', 3\xi)$; so, using polar coordinates with the pole at x' and x'', respectively, to evaluate the integrals, we obtain

$$\begin{aligned} |J_2(x', x'')| & \le c_6 \int\limits_{\sigma(x', 2\xi)} |x' - y|^{\beta - 2}\, da(y) + c_6 \int\limits_{\sigma(x'', 3\xi)} |x'' - y|^{\beta - 2}\, da(y) \\ & = \frac{2\pi}{\beta} c_6 \left\{ 2^\beta |x' - x''|^\beta + 3^\beta |x' - x''|^\beta \right\}, \end{aligned}$$

which means that

$$|J_2(x', x'')| \le c_7 |x' - x''|^\beta. \tag{5.19}$$

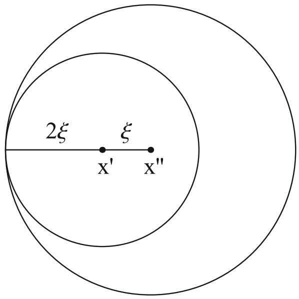

Fig. 5.1 The disks $\sigma(x', 2\xi)$ and $\sigma(x'', 3\xi)$

Now, by Lemma 5.3,

$$
\begin{aligned}
|J_3(x', x'')| &\le |g(x'') - g(x')| \int\limits_{\sigma(x',\rho)\setminus\sigma(x',2\xi)} |m(x', y)|\, da(y) \\
&\le c_8|x' - x''|^\beta \int\limits_{\sigma(x',\rho)\setminus\sigma(x',2\xi)} \frac{1}{|x' - y|^2}\, da(y) \\
&= 2\pi c_8 |x' - x''|^\beta \left(\ln\frac{\rho}{2} + \ln\frac{1}{|x' - x''|} \right) \\
&\le c_9|x' - x''|^\beta + c_{10}|x' - x''|^\alpha \le c_{11}|x' - x''|^\alpha,
\end{aligned}
\tag{5.20}
$$

where $0 < \alpha < \beta < 1$ and c_{11} depends on ρ.

Regarding J_4, from the assumption on g it follows that

$$
\begin{aligned}
|J_4(x', x'')| &\le \int\limits_{\sigma(x',\rho)\setminus\sigma(x',2\xi)} |m(x', y) - m(x'', y)|\, |g(y) - g(x'')|\, da(y) \\
&\le c_{12} \int\limits_{\sigma(x',\rho)\setminus\sigma(x',2\xi)} |m(x', y) - m(x'', y)|\, |x'' - y|^\beta\, da(y).
\end{aligned}
$$

It can be shown that (5.17) holds for $y \in \sigma(x', \rho) \setminus \sigma(x', 2\xi)$, so

$$
|J_4(x', x'')| \le c_{13}|x' - x''| \int\limits_{\sigma(x',\rho)\setminus\sigma(x',2\xi)} \frac{|x'' - y|^\beta}{|x' - y|^3}\, da(y).
$$

For $y \in \sigma(x', \rho) \setminus \sigma(x', 2\xi)$,

$$
\begin{aligned}
|x'' - y| &\le |x' - x''| + |x' - y| \\
&\le \tfrac{1}{2}|x' - y| + |x' - y| = \tfrac{3}{2}|x' - y|;
\end{aligned}
$$

therefore, since $\beta < 1$, we use polar coordinates again and arrive at

$$|J_4(x', x'')| \leq c_{14}|x' - x''| \int\limits_{\sigma(x',\rho)\setminus\sigma(x',2\xi)} |x' - y|^{\beta-3}\, da(y)$$
$$= \frac{2\pi}{1-\beta} c_{14}|x' - x''|\left\{2^{\beta-1}|x' - x''|^{\beta-1} - \rho^{\beta-1}\right\},$$

which yields

$$|J_4(x', x'')| \leq c_{15}|x' - x''|^{\beta}, \tag{5.21}$$

where c_{15} depends on ρ.

Estimates (5.18)–(5.21) together with (5.16) now lead to the conclusion that for $|x' - x''| < \rho/2$,

$$|M_1(x') - M_1(x'')| \leq c_{16}|x' - x''|^{\alpha}, \tag{5.22}$$

where $0 < \alpha < \beta < 1$ and c_{16} depends on ρ.

Next,

$$M_2(x') - M_2(x'') = J_5(x', x'') + J_6(x', x'') + J_7(x', x'') + J_8(x', x''), \tag{5.23}$$

where

$$J_5(x', x'') = [g(x') - g(x'')] \int\limits_{S^+\setminus\sigma(x',\rho)} m(x', y)\, da(y),$$

$$J_6(x', x'') = g(x'') \int\limits_{S^+\setminus\sigma(x',\rho)} \left[m(x', y) - m(x'', y)\right] da(y),$$

$$J_7(x', x'') = g(x'') \int\limits_{\sigma(x'',\rho)\setminus\sigma(x',\rho)} m(x'', y)\, da(y),$$

$$J_8(x', x'') = -g(x'') \int\limits_{\sigma(x',\rho)\setminus\sigma(x'',\rho)} m(x'', y)\, da(y).$$

The estimation of J_5 starts from the inequality

$$|J_5(x', x'')| \leq |g(x') - g(x'')| \int\limits_{S^+\setminus\sigma(x',\rho)} |m(x', y)|\, da(y)$$
$$\leq c_{17}|x' - x''|^{\beta} \int\limits_{S^+\setminus\sigma(x',\rho)} \frac{1}{|x' - y|^2}\, da(y).$$

Since x' is outside the domain of integration, this yields

$$|J_5(x',x'')| \le c_{18}|x'-x''|^\beta, \tag{5.24}$$

where c_{18} depends on ρ.

Recalling (5.17), we see that

$$\begin{aligned}|J_6(x',x'')| &\le |f(x'')| \int\limits_{S^+\setminus\sigma(x',\rho)} |m(x',y)-m(x'',y)|\,da(y)\\ &\le c_2\Big[\sup_{x\in S^*}|f(x)|\Big]\,|x'-x''| \int\limits_{S^+\setminus\sigma(x',\rho)} \frac{1}{|x'-y|^3}\,da(y);\end{aligned}$$

hence,

$$|J_6(x',x'')| \le c_{19}|x'-x''|, \tag{5.25}$$

where c_{19} depends on ρ.

For J_7, we have

$$\begin{aligned}|J_7(x',x'')| &\le |g(x'')| \int\limits_{\sigma(x'',\rho)\setminus\sigma(x',\rho)} |m(x'',y)|\,da(y)\\ &\le c_{20} \int\limits_{\sigma(x'',\rho)\setminus\sigma(x',\rho)} \frac{1}{|x''-y|^2}\,da(y).\end{aligned}$$

Extending the domain of integration and changing to polar coordinates with the pole at x'' leads to

$$\begin{aligned}|J_7(x',x'')| &\le c_{20} \int\limits_{\sigma(x'',\rho+|x'-x''|)\setminus\sigma(x'',\rho-|x'-x''|)} \frac{1}{|x''-y|^2}\,da(y)\\ &\le 2\pi c_{20}\left[\rho+|x'-x''|-\big(\rho-|x'-x''|\big)\right]\frac{1}{\rho-|x'-x''|}\\ &= 4\pi c_{20}\frac{|x'-x''|}{\rho-|x'-x''|}.\end{aligned}$$

Since

$$\rho-|x'-x''| > \rho-\tfrac{1}{2}\rho = \tfrac{1}{2}\rho,$$

we arrive at

$$|J_7(x',x'')| \le \frac{8\pi}{\rho}c_{20}|x'-x''|;$$

that is,

$$|J_7(x', x'')| \leq c_{21}|x' - x''|, \tag{5.26}$$

where c_{21} depends on ρ.

Similarly, noting that

$$|x'' - y| \geq |x' - y|$$

for $y \in \sigma(x', \rho) \setminus \sigma(x'', \rho)$, we find that

$$\begin{aligned}
|J_8(x', x'')| &\leq |f(x'')| \int\limits_{\sigma(x',\rho)\setminus\sigma(x'',\rho)} |m(x'', y)|\, da(y) \\
&\leq c_{22} \int\limits_{\sigma(x',\rho)\setminus\sigma(x'',\rho)} \frac{1}{|x'' - y|^2}\, da(y) \\
&\leq c_{22} \int\limits_{\sigma(x',\rho)\setminus\sigma(x'',\rho)} \frac{1}{|x' - y|^2}\, da(y) \\
&\leq c_{22} \int\limits_{\sigma(x',\rho+|x'-x''|)\setminus\sigma(x',\rho-|x'-x''|)} \frac{1}{|x' - y|^2}\, da(y).
\end{aligned}$$

Just as in the case of J_7, the above inequality yields

$$|J_8(x', x'')| \leq c_{23}|x' - x''|, \tag{5.27}$$

where c_{23} depends on ρ.

Combining (5.24)–(5.27) and also taking (5.23) into account, we see that for $|x' - x''| < \rho/2$,

$$|M_2(x') - M_2(x'')| \leq c_{24}|x' - x''|^{\beta}, \tag{5.28}$$

where c_{24} depends on ρ.

Finally, it is obvious that

$$|M_3(x') - M_3(x'')| \leq |c_\rho||x' - x''|^{\beta}. \tag{5.29}$$

Thus, from (5.15), (5.22), (5.28), and (5.29) we conclude that for $|x'-x''| < \rho/2$,

$$|M(x') - M(x'')| \leq c_{25}|x' - x''|^{\alpha},$$

where $0 < \alpha < \beta < 1$ and c_{25} depends on ρ.

We can easily show that M is bounded:

$$\begin{aligned}|M(x)| &\le \int\limits_{S^+} |m(x,y)|\,|g(y)-g(x)|\,da(y)\\ &\quad + |g(x)| \int\limits_{S^+\setminus\sigma(x,\rho)} |m(x,y)|\,da(y) + |c_\rho||g(x)|\\ &\le c_{26}\int\limits_{S^+} |x-y|^{\beta-2}\,da(y) + c_{27}\int\limits_{S^+\setminus\sigma(x,\rho)} \frac{1}{|x-y|^2}\,da(y) + |c_\rho|\,|g(x)| \le N.\end{aligned}$$

Hence, for $|x'-x''| \ge \rho/2$,

$$|M(x') - M(x'')| \le 2N = 2N\,\frac{|x'-x''|^\alpha}{|x'-x''|^\alpha} \le 2N\left(\frac{2}{\rho}\right)^\alpha |x'-x''|^\alpha,$$

and we conclude that for all $x', x'' \in S^*$,

$$|M(x') - M(x'')| \le c|x'-x''|^\alpha,$$

where $0 < \alpha < \beta < 1$ and c depends on ρ. □

5.8 Theorem. *If* $g \in C^{0,\beta}(S^+)$, $\beta \in (0,1)$, *then* $U \in C^{2,\alpha}(\Omega)$, *where* Ω *is an arbitrary domain in* $\mathbb{R}^2$ *whose closure lies in* S^+, *and* $0 < \alpha < \beta < 1$. *Additionally,*

$$\frac{\partial^2}{\partial x_\alpha \partial x_\beta}\,U(x) = \gamma(\alpha,\beta)g(x) + M(x), \tag{5.30}$$

where M *is defined by* (5.13) *and* $\gamma(\alpha,\beta)$ *is a constant symmetric* 3×3 *matrix with entries*

$$\gamma_{\rho\eta}(\alpha,\beta) = \pi d_1 \delta_{\alpha\beta}\delta_{\rho\eta} + \tfrac{1}{2}\pi d_2(\delta_{\alpha\rho}\delta_{\beta\eta} + \delta_{\alpha\eta}\delta_{\beta\rho} - \delta_{\alpha\beta}\delta_{\rho\eta}), \tag{5.31}$$

$$\gamma_{\rho 3}(\alpha,\beta) = \gamma_{3\rho}(\alpha,\beta) = 0, \tag{5.32}$$

$$\gamma_{33}(\alpha,\beta) = -\frac{1}{2\mu}\,\delta_{\alpha\beta}. \tag{5.33}$$

Proof. By Theorem 5.2 with $D(x,y) = D(x-y)$ (which reflects the structure of D more accurately),

$$\begin{aligned}\frac{\partial^2}{\partial x_\alpha \partial x_\beta}\,U(x) &= \frac{\partial^2}{\partial x_\alpha \partial x_\beta}\int\limits_{S^+} D(x-y)g(y)\,da(y)\\ &= \frac{\partial}{\partial x_\alpha}\int\limits_{S^+} \frac{\partial}{\partial x_\beta}\,D(x-y)g(y)\,da(y).\end{aligned}$$

According to Theorem 2.8 in Kupradze et al. (1979, p. 184),

$$\frac{\partial}{\partial x_\alpha}\int\limits_{S^+\setminus\sigma(x,\omega)}\frac{\partial}{\partial x_\beta}D(x-y)g(y)\,da(y)$$
$$=\int\limits_{S^+\setminus\sigma(x,\omega)}\frac{\partial^2}{\partial x_\alpha\partial x_\beta}D(x-y)g(y)\,da(y)$$
$$-\int\limits_{\partial\sigma(x,\omega)}\frac{y_\alpha-x_\alpha}{|x-y|}\frac{\partial}{\partial x_\beta}D(x-y)g(y)\,ds(y). \tag{5.34}$$

We rewrite the second integral on the right-hand side of (5.34) as

$$-\int\limits_{\partial\sigma(x,\omega)}\frac{y_\alpha-x_\alpha}{|x-y|}\frac{\partial}{\partial x_\beta}D(x-y)g(y)\,ds(y)$$
$$=\omega\int\limits_{\partial\sigma(x,1)}(x_\alpha-y_\alpha)\frac{\partial}{\partial x_\beta}D(\omega(x-y))g(x+\omega(y-x))\,ds(y).$$

The first two components of the integral around $\partial\sigma(x,1)$ are

$$\omega\int\limits_{\partial\sigma(x,1)}(x_\alpha-y_\alpha)\frac{\partial}{\partial x_\beta}D_{\rho\eta}(\omega(x-y))g_\eta(x+\omega(y-x))\,ds(y)$$
$$+\,\omega\int\limits_{\partial\sigma(x,1)}(x_\alpha-y_\alpha)\frac{\partial}{\partial x_\beta}D_{\rho 3}(\omega(x-y))g_3(x+\omega(y-x))\,ds(y)$$
$$=\int\limits_{\partial\sigma(x,1)}(x_\alpha-y_\alpha)\big\{d_1\delta_{\rho\eta}(x_\beta-y_\beta)+d_2\big[\delta_{\rho\beta}(x_\eta-y_\eta)+\delta_{\eta\beta}(x_\rho-y_\rho)$$
$$-\,2(x_\rho-y_\rho)(x_\eta-y_\eta)(x_\beta-y_\beta)\big]+O(\omega\ln\omega)\big\}g_\eta(x+\omega(y-x))\,ds(y)$$
$$+\int\limits_{\partial\sigma(x,1)}(x_\alpha-y_\alpha)(O(\omega\ln\omega))g_3(x+\omega(y-x))\,ds(y).$$

As $\omega\to 0$ in the above expression, from (5.5), (5.6), and (5.31)–(5.33) it follows that the first two components are

$$g_\eta(x)\big[d_1\delta_{\rho\eta}I_{\alpha\beta}+d_2(\delta_{\beta\eta}I_{\alpha\rho}+\delta_{\beta\rho}I_{\alpha\eta}-2I_{\alpha\beta\rho\eta})\big]$$
$$=g_\eta(x)\big\{\pi d_1\delta_{\rho\eta}\delta_{\alpha\beta}$$
$$+\,\pi d_2\big[\delta_{\beta\eta}\delta_{\alpha\rho}+\delta_{\beta\rho}\delta_{\alpha\eta}-\tfrac{1}{2}(\delta_{\alpha\beta}\delta_{\rho\eta}+\delta_{\alpha\rho}\delta_{\beta\eta}+\delta_{\alpha\eta}\delta_{\beta\rho})\big]\big\}$$
$$=\gamma_{\rho\eta}(\alpha,\beta)g_\eta(x)+\gamma_{\rho 3}(\alpha,\beta)g_3(x).$$

The third component of the integral around $\partial\sigma(x,1)$ is found analogously. First, we see that

$$\omega \int\limits_{\partial\sigma(x,1)} (x_\alpha - y_\alpha)\frac{\partial}{\partial x_\beta} D_{3\mu}(\omega(x-y))g_\mu(x+\omega(y-x))\,ds(y)$$
$$+\,\omega \int\limits_{\partial\sigma(x,1)} (x_\alpha - y_\alpha)\frac{\partial}{\partial x_\beta} D_{33}(\omega(x-y))g_3(x+\omega(y-x))\,ds(y)$$
$$= \int\limits_{\partial\sigma(x,1)} (x_\alpha - y_\alpha)(O(\omega\ln\omega))g_\mu(x+\omega(y-x))\,ds(y)$$
$$+ \int\limits_{\partial\sigma(x,1)} (x_\alpha - y_\alpha)\Big[-\frac{1}{2\pi\mu}(x_\beta - y_\beta) + O(\omega^2\ln\omega)\Big]g_3(x+\omega(y-x))\,ds(y).$$

As $\omega \to 0$, from (5.5) we deduce that the third component is

$$-\frac{1}{2\pi\mu} I_{\alpha\beta} g_3(x) = -\frac{1}{2\mu}\delta_{\alpha\beta} g_3(x) = \gamma_{3\mu}(\alpha,\beta)g_\mu(x) + \gamma_{33}(\alpha,\beta)g_3(x).$$

Equality (5.30) is now obtained by letting $\omega \to 0$ in (5.34).

By Theorem 5.5, $M(x)$ exists in the sense of principal value and, by Theorem 5.7, $M \in C^{0,\alpha}(\Omega)$, which completes the proof. □

5.4 A Particular Solution of the Nonhomogeneous System

5.9 Theorem. *If* $g \in C^{0,\beta}(S^+)$, $\beta \in (0,1)$, *then* $U(x)$ *defined by* (5.2) *is a regular solution in* S^+ *of system* (5.1).

Proof. The regularity of U has been shown in the proofs of Theorems 5.4 and 5.8.

By (3.9),

$$\big[A(\partial_x)U(x)\big]_\alpha = h^2\mu \sum_{\beta=1}^{2} \frac{\partial^2}{\partial x_\beta^2} U_\alpha(x)$$
$$+\, h^2(\lambda+\mu)\sum_{\rho=1}^{2} \frac{\partial^2}{\partial x_\alpha \partial x_\rho} U_\rho(x)$$
$$-\,\mu U_\alpha(x) - \mu\frac{\partial}{\partial x_\alpha} U_3(x),$$

and by Theorem 5.8,

$$\begin{aligned}
\frac{\partial^2}{\partial x_\beta^2}\, U_\alpha(x) &= \int_{S^+} \left[\frac{\partial^2}{\partial x_\beta^2}\, D(x,y)\big(g(y)-g(x)\big) \right]_\alpha da(y) \\
&\quad + \left[\int_{S^+\setminus\sigma(x,\omega)} \frac{\partial^2}{\partial x_\beta^2}\, D_{\alpha k}(x,y)\, da(y) \right] g_k(x) \\
&\quad + \lim_{\varepsilon\to 0} \left[\int_{\sigma(x,\omega)\setminus\sigma(x,\varepsilon)} \frac{\partial^2}{\partial x_\beta^2}\, D_{\alpha k}(x,y)\, da(y) \right] g_k(x) \\
&\quad + \big[\gamma(\beta,\beta) g(x)\big]_\alpha, \\
\frac{\partial^2}{\partial x_\alpha \partial x_\rho}\, U_\rho(x) &= \int_{S^+} \left[\frac{\partial^2}{\partial x_\alpha \partial x_\rho}\, D(x,y)\big(g(y)-g(x)\big) \right]_\rho da(y) \\
&\quad + \left[\int_{S^+\setminus\sigma(x,\omega)} \frac{\partial^2}{\partial x_\alpha \partial x_\rho}\, D_{\rho k}(x,y)\, da(y) \right] g_k(x) \\
&\quad + \lim_{\varepsilon\to 0} \left[\int_{\sigma(x,\omega)\setminus\sigma(x,\varepsilon)} \frac{\partial^2}{\partial x_\alpha \partial x_\rho}\, D_{\rho k}(x,y)\, da(y) \right] g_k(x) \\
&\quad + \big[\gamma(\alpha,\rho) g(x)\big]_\rho.
\end{aligned}$$

Also, Remark 5.6 and Theorem 5.2 imply that

$$\begin{aligned}
U_\alpha(x) &= \int_{S^+} \big[D(x,y)\big(g(y)-g(x)\big)\big]_\alpha\, da(y) \\
&\quad + \left[\int_{S^+\setminus\sigma(x,\omega)} D_{\alpha k}(x,y)\, da(y) \right] g_k(x) \\
&\quad + \lim_{\varepsilon\to 0} \left[\int_{\sigma(x,\omega)\setminus\sigma(x,\varepsilon)} D_{\alpha k}(x,y)\, da(y) \right] g_k(x), \\
\frac{\partial}{\partial x_\alpha}\, U_3(x) &= \int_{S^+} \left[\frac{\partial}{\partial x_\alpha}\, D(x,y)\big(g(y)-g(x)\big) \right]_3 da(y) \\
&\quad + \left[\int_{S^+\setminus\sigma(x,\omega)} \frac{\partial}{\partial x_\alpha}\, D_{3k}(x,y)\, da(y) \right] g_k(x) \\
&\quad + \lim_{\varepsilon\to 0} \left[\int_{\sigma(x,\omega)\setminus\sigma(x,\varepsilon)} \frac{\partial}{\partial x_\alpha}\, D_{3k}(x,y)\, da(y) \right] g_k(x);
\end{aligned}$$

consequently,

$$\begin{aligned}\big[A(\partial_x)U(x)\big]_\alpha &= h^2\mu\sum_{\beta=1}^{2}\big[\gamma(\beta,\beta)g(x)\big]_\alpha\\ &\quad + h^2(\lambda+\mu)\sum_{\rho=1}^{2}\big[\gamma(\alpha,\rho)g(x)\big]_\rho\\ &\quad + \int\limits_{S^+}\big[A(\partial_x)D(x,y)\big(g(y)-g(x)\big)\big]_\alpha\,da(y)\\ &\quad + \Bigg[\int\limits_{S^+\setminus\sigma(x,\omega)}\big[A(\partial_x)D(x,y)\big]_{\alpha k}\,da(y)\Bigg]g_k(x)\\ &\quad + \lim_{\varepsilon\to 0}\Bigg[\int\limits_{\sigma(x,\omega)\setminus\sigma(x,\varepsilon)}\big[A(\partial_x)D(x,y)\big]_{\alpha k}\,da(y)\Bigg]g_k(x).\end{aligned}$$

Next,

$$\begin{aligned}&\int\limits_{S^+}\big[A(\partial_x)D(x,y)\big(g(y)-g(x)\big)\big]_\alpha\,da(y)\\ &\quad = \int\limits_{S^+}\big[-\delta(|x-y|)E_3\big(g(y)-g(x)\big)\big]_\alpha\,da(y)\\ &\quad = -\int\limits_{S^+}\delta(|x-y|)g_\alpha(y)\,da(y)\\ &\qquad + \Bigg[\int\limits_{S^+}\delta(|x-y|)\,da(y)\Bigg]g_\alpha(x)\\ &\quad = -g_\alpha(x)+g_\alpha(x)=0.\end{aligned}$$

By Theorem 3.8, the integrals over $S^+\setminus\sigma(x,\omega)$ and $\sigma(x,\omega)\setminus\sigma(x,\varepsilon)$ are also zero. Therefore, using (5.31) and (5.32), we obtain

$$\begin{aligned}\big[A(\partial_x)U(x)\big]_\alpha &= \sum_{\beta=1}^{2}\big\{h^2\mu\gamma_{\alpha\eta}(\beta,\beta)g_\eta(x)+h^2\mu\gamma_{\alpha 3}(\beta,\beta)g_3(x)\\ &\qquad + h^2(\lambda+\mu)\gamma_{\beta\mu}(\alpha,\beta)g_\mu(x)+h^2(\lambda+\mu)\gamma_{\beta 3}(\alpha,\beta)g_3(x)\big\}\\ &= \sum_{\beta=1}^{2}\big\{h^2\mu\big[\pi d_1\delta_{\alpha\eta}+\tfrac12\pi d_2(2\delta_{\alpha\beta}\delta_{\eta\beta}-\delta_{\alpha\eta})\big]g_\eta(x)\\ &\qquad + h^2(\lambda+\mu)\big[\pi d_1\delta_{\alpha\beta}\delta_{\beta\mu}+\tfrac12\pi d_2\delta_{\alpha\mu}\big]g_\mu(x)\big\};\end{aligned}$$

so, by (5.4),

$$
\begin{aligned}
\big[A(\partial_x)U(x)\big]_\alpha &= \big[h^2\mu(2\pi d_1 - \pi d_2) + h^2(\lambda+\mu)\pi d_2 \\
&\quad + h^2\mu\pi d_2 + h^2(\lambda+\mu)\pi d_1\big]g_\alpha(x) \\
&= h^2\pi\big[(\lambda+3\mu)d_1 + (\lambda+\mu)d_2\big]g_\alpha(x) \\
&= \frac{1}{4\mu(\lambda+2\mu)}\big[(\lambda+\mu)^2 - (\lambda+3\mu)^2\big]g_\alpha(x) = -g_\alpha(x),
\end{aligned}
$$

which means that

$$\big[A(\partial_x)U(x)\big]_\alpha + g_\alpha(x) = 0.$$

Similarly,

$$\big[A(\partial_x)U(x)\big]_3 = \mu\sum_{\beta=1}^{2}\bigg\{\frac{\partial}{\partial x_\beta}\,U_\beta(x) + \frac{\partial^2}{\partial x_\beta^2}\,U_3(x)\bigg\},$$

which leads to

$$
\begin{aligned}
&\big[A(\partial_x)U(x)\big]_3 \\
&\quad = \mu\sum_{\beta=1}^{2}\big[\gamma(\beta,\beta)g(x)\big]_3 + \int\limits_{S^+}\big[A(\partial_x)D(x,y)\big(g(y)-g(x)\big)\big]_3\,da(y) \\
&\qquad + \bigg[\int\limits_{S^+\backslash\sigma(x,\omega)}\big[A(\partial_x)D(x,y)\big]_{3k}\,da(y)\bigg]g_k(x) \\
&\qquad + \lim_{\varepsilon\to 0}\bigg[\int\limits_{\sigma(x,\omega)\backslash\sigma(x,\varepsilon)}\big[A(\partial_x)D(x,y)\big]_{3k}\,da(y)\bigg]g_k(x).
\end{aligned}
$$

As before, the integrals on the right-hand side vanish, and (5.32) and (5.33) yield

$$
\begin{aligned}
\big[A(\partial_x)U(x)\big]_3 &= \mu\sum_{\beta=1}^{2}\big[\gamma_{3\alpha}(\beta,\beta)g_\alpha(x) + \gamma_{33}(\beta,\beta)g_3(x)\big] \\
&= \mu\sum_{\beta=1}^{2}\Big(-\frac{1}{2\mu}\Big)g_3(x);
\end{aligned}
$$

hence,

$$\big[A(\partial_x)U(x)\big]_3 + g_3(x) = 0,$$

as required. □

5.10 Remark. Consider the generic boundary value problem

$$\begin{aligned} A(\partial_x)u(x) + g(x) &= 0, \qquad x \in S^+, \\ Bu(x) &= \mathscr{U}(x), \quad x \in \partial S \end{aligned} \tag{5.35}$$

for system (5.1), where $g \in L^\infty(S^+)$, B is any of the boundary operators generated by the Dirichlet, Neumann, or Robin conditions, and $\mathscr{U}$ is a continuous 3×1 matrix function prescribed on ∂S (see Sect. 3.4). By Theorem 5.9 in conjunction with the smoothness results established in Sects. 5.2 and 5.3, the substitution

$$u = v + U \tag{5.36}$$

transforms (5.35) into the boundary value problem

$$\begin{aligned} A(\partial_x)v(x) &= 0, \qquad x \in S^+, \\ Bv(x) &= \mathscr{U}(x) - BU(x), \quad x \in \partial S \end{aligned} \tag{5.37}$$

for the homogeneous system (5.1). According to Theorem 5.2, the boundary data function $\mathscr{U} - BU$ is continuous on ∂S, and if (5.37) has a regular solution v, then (5.35) also has a regular solution u given by (5.36). This justifies the statement made in Remark 3.11.

Reference

Kupradze, V.D., Gegelia, T.G., Basheleishvili, M.O., Burchuladze, T.V.: Three-Dimensional Problems of the Mathematical Theory of Elasticity and Thermoelasticity. North-Holland, Amsterdam (1979)

Chapter 6
Existence of Regular Solutions

6.1 The Dirichlet and Neumann Problems

In view of Theorems 4.2 and 4.10 and Remarks 4.7 and 4.9, we may seek the solutions of problems (N^+) and (N^-) in the form of $(V\varphi)^+$ and $(V\varphi)^-$ with $\varphi \in C^{0,\alpha}(\partial S)$, that of ($\mathrm{D}^+$) in the form of $(W\varphi)^+$ with $\varphi \in C^{1,\alpha}(\partial S)$, and that of ($\mathrm{D}^-$) as the sum of $(W\varphi)^-$ with $\varphi \in C^{1,\alpha}(\partial S)$ and some 3×1 matrix u_0 of the form (3.16). By Theorem 4.5, Corollary 4.8, and Remark 4.9, the boundary value problems (3.43), (3.44) and (3.46), (3.47) are reduced, respectively, to the (systems of) singular integral equations

$$-\tfrac{1}{2}\,\varphi(x) + \int\limits_{\partial S} P(x,\ y)\varphi(y)\,ds(y) = \mathscr{P}(x), \tag{$\mathscr{D}^+$}$$

$$\tfrac{1}{2}\,\varphi(x) + \int\limits_{\partial S} T(\partial_x)D(x,\ y)\varphi(y)\,ds(y) = \mathscr{Q}(x), \tag{$\mathscr{N}^+$}$$

$$\tfrac{1}{2}\,\varphi(x) + \int\limits_{\partial S} P(x,\ y)\varphi(y)\,ds(y) = \mathscr{R}(x) - u_0(x), \tag{$\mathscr{D}^-$}$$

$$-\tfrac{1}{2}\,\varphi(x) + \int\limits_{\partial S} T(\partial_x)D(x,\ y)\varphi(y)\,ds(y) = \mathscr{S}(x), \tag{$\mathscr{N}^-$}$$

where $x \in \partial S$ and φ is an unknown density.

Let ($\mathscr{D}_0^+$), ($\mathscr{N}_0^+$), ($\mathscr{D}_0^-$), and ($\mathscr{N}_0^-$) be the associated homogeneous equations.

6.1 Theorem. *If $\mathscr{P} \in C^{1,\alpha}(\partial S)$, $\alpha \in (0, 1)$, then any solution $\varphi \in C^{0,\alpha}(\partial S)$ of equation ($\mathscr{D}^+$) belongs to $C^{1,\alpha}(\partial S)$. A similar statement holds for ($\mathscr{D}^-$) if $\mathscr{R} \in C^{1,\alpha}(\partial S)$.*

C. Constanda, *Mathematical Methods for Elastic Plates*,
Springer Monographs in Mathematics, DOI: 10.1007/978-1-4471-6434-0_6,

Proof. By (4.6), we can write $(\mathscr{D}^+)$ in the form

$$\big(W_0 - \tfrac{1}{2}\, I\big)\varphi = \mathscr{P}, \tag{6.1}$$

where, in view of (3.31), (2.36), (2.38), (2.2), (2.32), (2.29), and (2.30), $W_0\varphi$ admits the decomposition

$$W_0\varphi = -\frac{1}{2\pi}\,(\mu'\varepsilon_{\alpha\beta}E_{\alpha\beta}K^s\varphi + K^w\varphi); \tag{6.2}$$

here, K^s is defined by (2.38),

$$\begin{aligned} K^w\varphi = &- (E_3 - i\mu'\varepsilon_{\alpha\beta}E_{\alpha\beta})(w_0\varphi) \\ &- (\lambda' + \mu')\varepsilon_{\alpha\gamma}E_{\gamma\beta}(v^e_{\alpha\beta\,0}\varphi) \\ &+ \tfrac{1}{2}\,\varepsilon_{\alpha\beta}(\lambda' E_{3\beta} + h^{-2}E_{\beta 3})(v^c_{\alpha\,0}\varphi) \\ &- \tfrac{1}{2}\,E_{3\alpha}(v^d_{\alpha\,0}\varphi) - 2\pi(\tilde{W}_0\varphi), \end{aligned} \tag{6.3}$$

$$(\tilde{W}_0\varphi)(x) = \int\limits_{\partial S} \tilde{P}(x,\, y)\varphi(y)\,ds(y), \quad x \in \partial S,$$

and $\tilde{P}(x,\, y)$ satisfies the conditions of Theorem 2.10 with any $\gamma \in (0, 1)$. Applying the operator

$$-2\pi(\mu'\varepsilon_{\alpha\beta}E_{\alpha\beta}K^s - \pi E_3 I)$$

to both sides of (6.1) and making use of Theorem 2.22, we obtain

$$\begin{aligned} &\big[\pi^2(E_3 - \mu'^2 E_{\gamma\gamma})I - \mu'\varepsilon_{\alpha\beta}E_{\alpha\beta}K^sK^w + \pi E_3K^w\big]\varphi \\ &\quad = 2\pi(\mu'\varepsilon_{\alpha\beta}E_{\alpha\beta}K^s - \pi E_3 I)\mathscr{P}. \end{aligned} \tag{6.4}$$

Clearly, any solution of (6.1) is also a solution of (6.4). By Theorem 2.21, $C^{1,\alpha}(\partial S)$ is invariant under K^s; consequently, the right-hand side of (6.4) belongs to $C^{1,\alpha}(\partial S)$. By Theorems 2.8, 2.18, 2.17, and 2.10, K^w maps $C^{0,\alpha}(\partial S)$ into $C^{1,\alpha}(\partial S)$. A further application of Theorem 2.21 now shows that every $C^{0,\alpha}$-solution of (6.4) belongs to $C^{1,\alpha}(\partial S)$, which proves the assertion.

The case of $(\mathscr{D}^-)$ is treated in the same way. □

6.2 Theorem. *The Fredholm Alternative holds for the pairs of integral equations* $(\mathscr{D}^+)$, $(\mathscr{N}^-)$ *and* $(\mathscr{N}^+)$, $(\mathscr{D}^-)$ *in the (real) dual system* $\big(C^{0,\alpha}(\partial S),\, C^{0,\alpha}(\partial S)\big)$, $\alpha \in (0, 1)$, *equipped with the bilinear form*

$$(\varphi,\, \psi) = \int\limits_{\partial S} \varphi^{\mathrm{T}}(y)\psi(y)\,ds(y). \tag{6.5}$$

Proof. Denoting by $\mathscr{D}$ and $\mathscr{N}$ the integral operators occurring in $(\mathscr{D}^{\pm})$ and $(\mathscr{N}^{\pm})$, respectively, we see that, by (3.28), for any $\varphi,\ \psi \in C^{0,\alpha}(\partial S)$

$$\begin{aligned}(\mathscr{D}\varphi,\ \psi) &= \int\limits_{\partial S}\Big[\int\limits_{\partial S} P(x,\ y)\varphi(y)\,ds(y)\Big]^{\mathrm{T}}\psi(x)\,ds(x)\\ &= \int\limits_{\partial S}\Big[\int\limits_{\partial S}\big(T(\partial_y)D(y,x)\big)^{\mathrm{T}}\varphi(y)\,ds(y)\Big]^{\mathrm{T}}\psi(x)\,ds(x),\\ &= \int\limits_{\partial S}\varphi^{\mathrm{T}}(y)\Big[\int\limits_{\partial S} T(\partial_y)D(y,x)\psi(x)\,ds(x)\Big]ds(y)\\ &= (\varphi,\ \mathscr{N}\psi).\end{aligned}$$

Owing to the symmetry of the bilinear form (6.5), we also have

$$(\mathscr{N}\varphi,\ \psi) = (\varphi,\ \mathscr{D}\psi),$$

which means that $\mathscr{D}$ and $\mathscr{N}$ are mutually adjoint in the given dual system. Since $\mathscr{D} = W_0$, it is then natural to write $\mathscr{N} = W_0^*$, the adjoint of W_0. Therefore,

$$(W_0^*\varphi)(x) = \int\limits_{\partial S} T(\partial_x)D(x,\ y)\varphi(y)\,ds(y),$$

and the integral equations $(\mathscr{D}^+)$, $(\mathscr{N}^+)$, $(\mathscr{D}^-)$, and $(\mathscr{N}^-)$ can be written in the alternative form

$$\big(W_0 - \tfrac{1}{2}\,I\big)\varphi = \mathscr{P}, \tag{$\mathscr{D}^+$}$$

$$\big(W_0^* + \tfrac{1}{2}\,I\big)\varphi = \mathscr{Q}, \tag{$\mathscr{N}^+$}$$

$$\big(W_0 + \tfrac{1}{2}\,I\big)\varphi = \mathscr{R} - u_0, \tag{$\mathscr{D}^-$}$$

$$\big(W_0^* - \tfrac{1}{2}\,I\big)\varphi = \mathscr{S}. \tag{$\mathscr{N}^-$}$$

From (6.2), (6.3), Theorem 2.19, and the fact (pointed out in the proof of Theorem 6.1) that K^w maps $C^{0,\alpha}(\partial S)$ into $C^{1,\alpha}(\partial S)$, it is clear that $C^{0,\alpha}(\partial S)$ is invariant under the operator W_0.

The kernel k^w of K^w is a proper γ-singular kernel on ∂S with respect to both x and y, for any $\gamma \in (0, 1)$. Hence, by Theorem 2.35, K^w is α-regular singular and its complex kernel $\hat{k}^w$ satisfies

$$\hat{k}^w(z,\ z) = 0, \quad z \in \partial S.$$

Also, (2.38) shows that the same can be said about K^s, except that in this case

$$\hat{k}^s(z,\ z) = E_3, \quad z \in \partial S.$$

Then, by (6.2), $\mathscr{D}$ itself is α-regular singular and

$$\hat{k}(z,\, z) = -\frac{1}{2\pi}\big[\mu'\varepsilon_{\alpha\beta}E_{\alpha\beta}\hat{k}^s(z,\, z) + \hat{k}^w(z,\, z)\big]$$
$$= -\frac{1}{2\pi}\mu'\varepsilon_{\alpha\beta}E_{\alpha\beta}, \quad z \in \partial S.$$

Consequently, and in view of (3.32) and (3.13),

$$\det\big[-\tfrac{1}{2}E_3 \pm \pi i\hat{k}(z,\, z)\big] = -\tfrac{1}{8}(1-\mu'^2) < 0,$$

from which we immediately deduce that the index ρ of the complex version of $(\mathscr{D}^+)$, defined by (2.52), is zero. According to Theorem 2.38, this means that the Fredholm Alternative holds for the pair $(\mathscr{D}^+)$, $(\mathscr{N}^-)$ in the (complex) dual system $\big(C^{0,\alpha}(\partial S),\, C^{0,\alpha}(\partial S)\big)$ with the bilinear form

$$(\varphi,\, \psi) = \int\limits_{\partial S} \varphi^{\mathrm{T}}(\zeta)\psi(\zeta)\, d\zeta;$$

therefore, by Remark 2.39, it also holds for $(\mathscr{D}^+)$, $(\mathscr{N}^-)$ in the (real) dual system $\big(C^{0,\alpha}(\partial S),\, C^{0,\alpha}(\partial S)\big)$ with the bilinear form (6.5).

The argument is similar for the pair $(\mathscr{D}^-)$, $(\mathscr{N}^+)$. □

6.3 Theorem. $(\mathscr{D}_0^-)$ *has exactly three linearly independent* $C^{0,\alpha}$*-solutions.*

Proof. In view of Theorem 6.1, it suffices to prove the assertion in $C^{1,\alpha}(\partial S)$, $\alpha \in (0,\, 1)$.

It is clear that a 3×1 matrix u_0 of the form (3.16) is a solution of the homogeneous interior Neumann problem (N^+). Since $Tu_0 = 0$, replacing u by u_0 in (3.33), we obtain

$$\tfrac{1}{2}u_0(x) + \int\limits_{\partial S} P(x,\, y)u_0(y)\, ds(y) = 0, \quad x \in \partial S;$$

that is, u_0 is a solution of $(\mathscr{D}_0^-)$; hence, $f^{(1)}$, $f^{(2)}$, and $f^{(3)}$, where

$$\begin{aligned} f^{(1)}(x) &= (1,\, 0,\, -x_1)^{\mathrm{T}},\\ f^{(2)}(x) &= (0,\, 1,\, -x_2)^{\mathrm{T}},\\ f^{(3)}(x) &= (0,\, 0,\, 1)^{\mathrm{T}} \end{aligned} \tag{6.6}$$

are the columns of the matrix F defined in (3.17), are three linearly independent solutions of $(\mathscr{D}_0^-)$.

Let $f^{(0)}$ be an arbitrary $C^{1,\alpha}$-solution of $(\mathscr{D}_0^-)$. Then

$$f = f^{(0)} - c_i f^{(i)} \tag{6.7}$$

is also a $C^{1,\alpha}$-solution of $(\mathscr{D}_0^-)$, for any constants c_i. This means that

$$(Wf)^- = 0 \quad \text{on } \partial S;$$

consequently, by Theorems 4.1 and 4.2(i), Wf is a regular solution of the homogeneous exterior Dirichlet problem (D$^-$). By Theorem 3.16(i),

$$(Wf)^- = 0 \quad \text{in } \bar{S}^-.$$

This yields

$$T(Wf)^- = 0 \quad \text{on } \partial S,$$

which, in turn, by Theorem 4.10, implies that

$$T(Wf)^+ = 0 \quad \text{on } \partial S,$$

and we deduce that $(Wf)^+$ is a regular solution of the homogeneous interior Neumann problem (N$^+$). Hence, by Theorem 3.16(ii),

$$(Wf)^+ = \left(Wf^{(0)}\right)^+ - c_i\left(Wf^{(i)}\right)^+ = \tilde{u} \quad \text{in } \bar{S}^+, \tag{6.8}$$

where $\tilde{u}$ is of the form (3.16).

Without loss of generality, suppose that the origin of coordinates lies in S^+. We choose the c_i so that $\tilde{u} = 0$, for example, by asking that

$$(Wf)^+(0) = 0.$$

This is equivalent to the system of linear equations

$$c_i\left(Wf^{(i)}\right)^+(0) = \left(Wf^{(0)}\right)^+(0). \tag{6.9}$$

Let $\{c_1^*, c_2^*, c_3^*\}$ be a solution of the homogeneous system (6.9). Then, setting $f^* = c_i^* f^{(i)}$, we obtain

$$\left(Wf^*\right)^+(0) = 0. \tag{6.10}$$

Taking $f^{(0)} = 0$ and $c_i = c_i^*$ in (6.7), we see that, as above, $(Wf^*)^+$ is a regular solution of the homogeneous problem (N$^+$); therefore, by Theorem 3.16(ii), $(Wf^*)^+$ is of the form (3.16). In view of (6.10), we conclude that

$$(Wf^*)^+ = 0 \quad \text{in } \bar{S}^+,$$

so

$$T(Wf^*)^+ = 0 \quad \text{on } \partial S.$$

By Theorem 4.10,

$$T(Wf^*)^- = 0 \quad \text{on } \partial S.$$

Thus, $(Wf^*)^-$ is a regular solution of the homogeneous exterior Neumann problem (N^-). Hence, by Theorem 3.16(i),

$$(Wf^*)^- = 0 \quad \text{in } \bar{S}^-.$$

Since

$$(Wf^*)^+ = 0 \quad \text{in } \bar{S}^+,$$

from (4.5) it follows that

$$f^* = (Wf^*)^- - (Wf^*)^+ = 0 \quad \text{on } \partial S.$$

This means that the homogeneous system (6.9) has only the trivial solution; therefore, (6.9) has a unique solution $\{c_1,\ c_2,\ c_3\}$, for which, by (6.8),

$$(Wf)^+ = 0 \quad \text{in } \bar{S}^+.$$

But, as was established earlier, we also have

$$(Wf)^- = 0 \quad \text{in } \bar{S}^-.$$

Using (4.5) again, we now obtain

$$f = (Wf)^- - (Wf)^+ = 0 \quad \text{on } \partial S.$$

Hence, according to (6.7), any $C^{1,\alpha}$-solution of $(\mathscr{D}_0^-)$ can be expressed uniquely as a linear combination of the $f^{(i)}$. □

6.4 Lemma. *If* $\varphi \in C^{0,\alpha}(\partial S)$, $\alpha \in (0, 1)$, *is a regular solution of equation* $(\mathscr{N}^-)$, *then*

$$p\varphi = -p\mathscr{S};$$

that is,

$$\int\limits_{\partial S} (\varphi_\alpha - x_\alpha \varphi_3)\, ds = -\int\limits_{\partial S} (\mathscr{S}_\alpha - x_\alpha \mathscr{S}_3)\, ds,$$

$$\int\limits_{\partial S} \varphi_3\, ds = -\int\limits_{\partial S} \mathscr{S}_3\, ds.$$

Proof. Setting

$$u(y) = (c_1,\ c_2,\ c_0 - c_1 y_1 - c_2 y_2)^{\mathrm{T}}$$

in Theorem 3.9 and taking into account the fact that $Tu = 0$ for such a choice, we find that for $x \in \partial S$,

$$\int\limits_{\partial S} \big[P_{j\alpha}(x,\ y) - y_\alpha P_{j3}(x,\ y)\big]\, ds(y) = -\tfrac{1}{2}\,(\delta_{j\alpha} - x_\alpha \delta_{j3}),$$

$$\int\limits_{\partial S} P_{j3}(x,\ y)\, ds(y) = -\tfrac{1}{2}\,\delta_{j3},$$

or, in view of (3.28),

$$\int\limits_{\partial S} \big[T_{\alpha k}(\partial_y) - y_\alpha T_{3k}(\partial_y)\big] D_{kj}(y,\ x)\, ds(y) = -\tfrac{1}{2}\,(\delta_{j\alpha} - x_\alpha \delta_{j3}),$$

$$\int\limits_{\partial S} T_{3k}(\partial_y) D_{kj}(y,\ x)\, ds(y) = -\tfrac{1}{2}\,\delta_{j3}. \tag{6.11}$$

Multiplying $(\mathscr{N}^-)_3$ and the combinations $(\mathscr{N}^-)_\alpha - x_\alpha \times (\mathscr{N}^-)_3$ by $ds(x)$ and integrating the resulting expressions over ∂S, we obtain the equalities

$$-\frac{1}{2}\int\limits_{\partial S} \varphi_3(x)\, ds(x) + \int\limits_{\partial S}\bigg[\int\limits_{\partial S} T_{3k}(\partial_x) D_{kj}(x,\ y)\, ds(x)\bigg]\varphi_j(y)\, ds(y)$$
$$= \int\limits_{\partial S} \mathscr{S}_3(x)\, ds(x)$$

and

$$-\frac{1}{2}\int\limits_{\partial S} \big[\varphi_\alpha(x) - x_\alpha \varphi_3(x)\big]\, ds(x)$$
$$+ \int\limits_{\partial S}\bigg\{\int\limits_{\partial S} \big[T_{\alpha k}(\partial_x) D_{kj}(x,\ y) - x_\alpha T_{3k}(\partial_x) D_{kj}(x,\ y)\big]\, ds(x)\bigg\}\varphi_j(y)\, ds(y)$$
$$= \int\limits_{\partial S} \big[\mathscr{S}_\alpha(x) - x_\alpha \mathscr{S}_3(x)\big]\, ds(x),$$

and the desired formulas follow from (6.11). □

6.5 Theorem. (i) *The interior Dirichlet problem* (D^+) *has a unique regular solution for any* $\mathscr{P} \in C^{1,\alpha}(\partial S), \alpha \in (0, 1)$. *This solution can be represented as the extension*

$(W\varphi)^+$ to $\bar{S}^+$ of the restriction to S^+ of a double-layer potential $W\varphi$ with density $\varphi \in C^{1,\alpha}(\partial S)$.

(ii) *The exterior Neumann problem* (N^-) *has a unique regular solution for $\mathscr{S} \in C^{0,\alpha}(\partial S)$, $\alpha \in (0,1)$, if and only if*

$$p\mathscr{S} = 0;$$

that is,

$$\int_{\partial S} (\mathscr{S}_\alpha - x_\alpha \mathscr{S}_3)\, ds = 0, \qquad (6.12)$$
$$\int_{\partial S} \mathscr{S}_3\, ds = 0.$$

This solution can be represented as the restriction $(V\varphi)^-$ to $\bar{S}^-$ of a single-layer potential $V\varphi$ with density $\varphi \in C^{0,\alpha}(\partial S)$.

Proof. By Theorem 6.2, the Fredholm Alternative holds for the pairs ($\mathscr{D}^+$), ($\mathscr{N}^-$) and ($\mathscr{D}^-$), ($\mathscr{N}^+$) in the (real) dual system $\big(C^{0,\alpha}(\partial S),\ C^{0,\alpha}(\partial S)\big)$ with the bilinear form (6.5).

Let u be a regular solution of (N^-), and consider a disk Γ_R of sufficiently large radius R so that $\bar{S}^+ \subset \Gamma_R$. Applying Theorem 3.5 in $S^- \cap \Gamma_R$, we find that

$$\int_{\partial S} (\mathscr{S}_\alpha - x_\alpha \mathscr{S}_3)\, ds - \int_{\partial \Gamma_R} (T_{\alpha i} - x_\alpha T_{3i}) u_i\, ds = 0,$$
$$\int_{\partial S} \mathscr{S}_3\, ds - \int_{\partial \Gamma_R} T_{3i} u_i\, ds = 0,$$

from which (6.12) are obtained by letting $R \to \infty$ and taking (3.42) into account.

Suppose now that (6.12) hold, and let $\varphi^{(0)}$ be a solution of ($\mathscr{N}_0^-$). By (4.17) and (4.19), this is equivalent to

$$T\left(V\varphi^{(0)}\right)^- = 0 \quad \text{on } \partial S.$$

Since

$$A\left(V\varphi^{(0)}\right)^- = 0 \quad \text{in } S^-$$

and, by Lemma 6.4 and Theorem 4.2(ii),

$$(V\varphi^{(0)})^- \in \mathscr{A},$$

it follows that $\left(V\varphi^{(0)}\right)^-$ is a solution of the homogeneous exterior Neumann problem (N^-). By Theorem 3.16(i),

$$\left(V\varphi^{(0)}\right)^{-} = 0 \quad \text{in } \bar{S}^{-};$$

hence, by Theorem 4.4,

$$\left(V\varphi^{(0)}\right)^{-} = 0 = \left(V\varphi^{(0)}\right)^{+} \quad \text{on } \partial S.$$

Next, $\left(V\varphi^{(0)}\right)^{+}$ is a solution of the homogeneous interior Dirichlet problem (D^{+}); consequently, by Theorem 3.16(i),

$$\left(V\varphi^{(0)}\right)^{+} = 0 \quad \text{in } \bar{S}^{+}.$$

Then

$$T\left(V\varphi^{(0)}\right)^{+} = 0 \quad \text{on } \partial S,$$

and (4.17) and (4.19) yield

$$\varphi^{(0)} = T\left(V\varphi^{(0)}\right)^{+} - T\left(V\varphi^{(0)}\right)^{-} = 0 \quad \text{on } \partial S,$$

from which we conclude that ($\mathscr{N}_0^{-}$) has only the zero solution. According to the Fredholm Alternative, so does ($\mathscr{D}_0^{+}$); therefore, ($\mathscr{D}^{+}$) and ($\mathscr{N}^{-}$) have unique solutions $\varphi \in C^{0,\alpha}(\partial S)$.

To complete the proof, we remark that in the case of (N^{-}), from Lemma 6.4, (6.12), and Theorem 4.2(ii) it follows that $(V\varphi)^{-} \in \mathscr{A}$; in other words, $(V\varphi)^{-}$ is a regular solution of (N^{-}). At the same time, in the case of (D^{+}), Theorem 6.1 yields $\varphi \in C^{1,\alpha}(\partial S)$; hence, by Theorem 4.10, $(W\varphi)^{+}$ is a regular solution of the problem.

The uniqueness of these solutions was established in Theorem 3.16(i). □

6.6 Theorem. *The interior Neumann problem* (N^{+}) *is solvable for* $\mathscr{Q} \in C^{0,\alpha}(\partial S)$, $\alpha \in (0,1)$, *if and only if*

$$p\mathscr{Q} = 0;$$

that is,

$$\int_{\partial S} (\mathscr{Q}_\alpha - x_\alpha \mathscr{Q}_3)\, ds = 0,$$

$$\int_{\partial S} \mathscr{Q}_3\, ds = 0. \tag{6.13}$$

The regular solution is unique up to a 3×1 *matrix of the form* (3.16) *and can be represented as the restriction* $(V\varphi)^{+}$ to $\bar{S}^{+}$ *of a single-layer potential* $V\varphi$ *with density* $\varphi \in C^{0,\alpha}(\partial S)$.

Proof. By Theorem 6.2 and the Fredholm Alternative, $(\mathscr{N}^+)$ is solvable if and only if

$$(f^{(i)}, \mathscr{Q}) = \int\limits_{\partial S} (f^{(i)})^{\mathrm{T}} \mathscr{Q}\, ds = 0,$$

where the $f^{(i)}$ are defined by (6.6). Writing these conditions explicitly, we see that they coincide with (6.13). Consequently, if the equalities (6.13) hold, then there is a density $\varphi \in C^{0,\alpha}(\partial S)$ for which $(V\varphi)^+$ is a regular solution of (N$^+$). The uniqueness of this solution is discussed in Theorem 3.16(ii). □

6.7 Theorem. *The exterior Dirichlet problem* (D$^-$) *has a unique regular solution for any* $\mathscr{R} \in C^{1,\alpha}(\partial S)$. *This solution can be represented as the sum of the extension* $(W\varphi)^-$ to $\bar{S}^-$ *of the restriction to* S^- *of a double-layer potential* $W\varphi$ *with density* $\varphi \in C^{1,\alpha}(\partial S)$, *and a particular* 3×1 matrix u_0 *of the form* (3.16).

Proof. According to Theorem 6.2 and the Fredholm Alternative, $(\mathscr{N}_0^+)$ has exactly three linearly independent $C^{0,\alpha}$-solutions $g^{(i)}$. Without loss of generality, suppose that the sets $\{f^{(i)}\}$ and $\{g^{(i)}\}$ have been biorthonormalized Kupradze et al. (1979); that is, we have

$$\left(f^{(i)},\, g^{(j)}\right) = \delta_{ij}.$$

Taking $u_0 = c_i f^{(i)}$, where

$$c_i = \int\limits_{\partial S} \left(g^{(i)}\right)^{\mathrm{T}} \mathscr{R}\, ds,$$

we see that

$$(g^{(j)},\, \mathscr{R} - c_i f^{(i)}) = \int\limits_{\partial S} \left(g^{(j)}\right)^{\mathrm{T}} (\mathscr{R} - c_i f^{(i)})\, ds = 0.$$

Consequently, by the Fredholm Alternative, $(\mathscr{D}^-)$ has a solution $\varphi \in C^{0,\alpha}(\partial S)$. By Theorem 6.1, $\varphi \in C^{1,\alpha}(\partial S)$. Since, by Theorem 4.2(i),

$$(W\varphi)^- + u_0 \in \mathscr{A}^*,$$

it follows that $(W\varphi)^- + u_0$ is a regular solution of (D$^-$). The uniqueness of this solution is guaranteed by Theorem 3.16(i). □

6.8 Remark. Restrictions (6.13) and (6.12), which are necessary and sufficient for the solvability of (N$^+$) and (N$^-$), respectively, have a direct physical meaning. By Remark 3.1, they represent the condition that the transverse shear force and the bending and twisting moments acting on ∂S be zero.

The regular solutions to all our boundary value problems have been found in closed form. But one question still remains unanswered: what is the mechanical significance of the class $\mathscr{A}$ that intervenes so essentially in the proceedings? Is its introduction really necessary? Could there be regular solutions outside this class as well? The boundary integral equation method, while elegant and precise, offers no answer. To settle this outstanding matter, in Chap. 7 we change over to a different technique of investigation, equally powerful, which allows us to obtain the complete integral of system (3.40).

6.2 The Robin Problems

Let Φ be the matrix whose columns $g^{(i)}$, $i = 1, 2, 3$, are three linearly independent solutions of $(\mathscr{N}^+)$ (see the proof of Theorem 6.7), chosen so that

$$p\Phi = E_3.$$

For the Robin problems (R^+) and (R^-) we seek solutions of the form

$$u = (V(\varphi - \Phi(p\varphi)))^+ + F(p\varphi), \tag{6.14}$$

$$u = (V(\varphi - \Phi(p\varphi)))^- + F(p\varphi). \tag{6.15}$$

Then, in view of (4.16), the boundary conditions in (3.45) and (3.48)—that is,

$$Tu + \sigma u = \mathscr{K} \quad \text{and} \quad Tu - \sigma u = \mathscr{L} \quad \text{on } \partial S,$$

give rise, respectively, to the boundary integral equations

$$\left(W_0^* + \tfrac{1}{2} I\right)(\varphi - \Phi(p\varphi)) + \sigma V_0(\varphi - \Phi(p\varphi)) + \sigma F(p\varphi) = \mathscr{K}, \tag{$\mathscr{R}^+$}$$

$$\left(W_0^* - \tfrac{1}{2} I\right)(\varphi - \Phi(p\varphi)) - \sigma V_0(\varphi - \Phi(p\varphi)) - \sigma F(p\varphi) = \mathscr{L}. \tag{$\mathscr{R}^-$}$$

Since the dominant terms in the kernels of $(\mathscr{R}^+)$ and $(\mathscr{R}^-)$ are the same as in those of $(\mathscr{N}^+)$ and $(\mathscr{N}^-)$, the index of each of these equations is zero, so the Fredholm Alternative can be applied to them.

6.9 Theorem. *Let* $\sigma \in C^{0,\alpha}(\partial S)$, $\alpha \in (0, 1)$.

(i) *The interior Robin problem* $(\mathscr{R}^+)$ *has a unique solution* $\varphi \in C^{0,\alpha}(\partial S)$ *for any* $\mathscr{K} \in C^{0,\alpha}(\partial S)$. *Then the* (*unique*) *solution of* (R^+) *is given by* (6.14).

(ii) *The exterior Robin problem* $(\mathscr{R}^-)$ *has a unique solution* $\varphi \in C^{0,\alpha}(\partial S)$ *for any* $\mathscr{L} \in C^{0,\alpha}(\partial S)$. *Then the* (*unique*) *solution of* (R^-) *is given by* (6.15).

Proof. Since $\sigma \in C^{0,\alpha}(\partial S)$, the operators occurring in $(\mathscr{R}^+)$ and $(\mathscr{R}^-)$ map $C^{0,\alpha}(\partial S)$ to $C^{0,\alpha}(\partial S)$.

(i) Consider a solution $\bar{\varphi}$ of the homogeneous equation $(\mathscr{R}_0^+)$; in other words, a function $\bar{\varphi} \in C^{0,\alpha}(\partial S)$ such that

$$(W_0^* + \tfrac{1}{2} I)(\bar{\varphi} - \Phi(p\bar{\varphi})) + \sigma V_0(\bar{\varphi} - \Phi(p\bar{\varphi})) + \sigma F(p\bar{\varphi}) = 0. \tag{6.16}$$

This means that $(V(\bar{\varphi} - \Phi(p\bar{\varphi})))^+ + F(p\bar{\varphi})$ is the (unique) solution of the homogeneous problem (R$^+$); therefore,

$$(V(\bar{\varphi} - \Phi(p\bar{\varphi})))^+ + F(p\bar{\varphi}) = 0. \tag{6.17}$$

Since

$$p(\bar{\varphi} - \Phi(p\bar{\varphi})) = 0,$$

from (4.16), (6.16), and (6.17) we deduce that the function

$$\mathscr{U}^- = (V(\bar{\varphi} - \Phi(p\bar{\varphi})))^- + F(p\bar{\varphi})$$

satisfies

$$\begin{aligned}
A\mathscr{U}^- &= 0 \quad \text{in } S^-,\\
\mathscr{U}^- &= (V(\bar{\varphi} - \Phi(p\bar{\varphi})))_0^- = V_0(\bar{\varphi} - \Phi(p\bar{\varphi})) + F(p\bar{\varphi})\\
&= (V(\bar{\varphi} - \Phi(p\bar{\varphi})))_0^+ + F(p\bar{\varphi}) = 0 \quad \text{on } \partial S,\\
\mathscr{U}^-(x) &= \big[M^\infty p(\bar{\varphi} - \Phi(p\bar{\varphi})) + \mathscr{U}^{\mathscr{A}} + F(p\bar{\varphi})\big](x)\\
&= (\mathscr{U}^{\mathscr{A}} + F(p\bar{\varphi}))(x) \quad \text{as } |x| \to \infty.
\end{aligned}$$

By Theorem 3.16,

$$\begin{aligned}
F(p\bar{\varphi}) &= 0,\\
(V(\bar{\varphi} - \Phi(p\bar{\varphi})))^- + F(p\bar{\varphi}) = (V(\bar{\varphi} - \Phi(p\bar{\varphi})))^- &= 0.
\end{aligned}$$

Since the columns of F are linearly independent, this implies that

$$p\bar{\varphi} = 0, \quad (V\bar{\varphi})^- = 0.$$

Also, (6.17) yields

$$(V\bar{\varphi})^+ = 0;$$

hence, by Remark 4.9 and (4.17) , $\bar{\varphi} = 0$.

Since the homogeneous equation ($\mathscr{R}_0^+$) has only the zero solution, the Fredholm Alternative states that ($\mathscr{R}^+$) has a unique solution $\varphi \in C^{0,\alpha}(\partial S)$. By Remark 4.7 and Theorem 4.1, u given by (6.14) belongs to $C^{1,\alpha}(\bar{S}^+)$ and satisfies $Au = 0$ (in S^+), so it is the (unique) solution of (R$^+$).

(ii) If $\bar{\varphi}$ is a solution of the homogeneous equation ($\mathscr{R}_0^-$), that is,

$$(W_0^* - \tfrac{1}{2} I)(\bar{\varphi} - \Phi(p\bar{\varphi})) - \sigma V_0(\bar{\varphi} - \Phi(p\bar{\varphi})) - \sigma F(p\bar{\varphi}) = 0,$$

then, by (4.4), the function

$$\mathscr{U}^- = (V(\bar{\varphi} - \Phi(p\bar{\varphi})))^- + F(p\bar{\varphi})$$

satisfies

$$\begin{aligned} A\mathscr{U}^- &= 0 \quad \text{in } S^-, \\ T\mathscr{U}^- - \sigma\mathscr{U}^- &= 0 \quad \text{on } \partial S, \\ \mathscr{U}^-(x) &= \big[M^\infty p(\bar{\varphi} - \Phi(p\bar{\varphi})) + \mathscr{U}^{\mathscr{A}} + F(p\bar{\varphi})\big](x) \\ &= \big(\mathscr{U}^{\mathscr{A}} + F(p\bar{\varphi})\big)(x) \quad \text{as } |x| \to \infty, \end{aligned}$$

which is the homogeneous problem (R^-). Applying Theorem 3.16 once more, we conclude that

$$F(p\bar{\varphi}) = 0, \quad (V(\bar{\varphi} - \Phi(p\bar{\varphi})))^- + F(p\bar{\varphi}) = (V(\bar{\varphi} - \Phi(p\bar{\varphi})))^- = 0;$$

hence, as above, $p\bar{\varphi} = 0$ and

$$(V\bar{\varphi})_0^- = 0 = V_0\bar{\varphi} = (V\bar{\varphi})_0^+,$$

so $(V\bar{\varphi})^+ = 0$ is the unique solution of the homogeneous problem (D^+). Remark 4.9 and (4.17) now imply that $\bar{\varphi} = 0$. Consequently, by the Fredholm Alternative, ($\mathscr{R}^-$) has a unique solution $\varphi \in C^{0,\alpha}(\partial S)$.

The function u given by (6.15) belongs to $C^{1,\alpha}(\bar{S}^-)$ and satisfies $Au = 0$ (in S^-). Also, $u \in \mathscr{A}^*$ since $p\big(\varphi - \Phi(p\varphi)\big) = 0$, which means that u is the (unique) solution of (R^-). □

6.10 Remark. The sole purpose of the term $\Phi(p\varphi)$ in the density of $V^\pm$ is to ensure that p applied to the density yields zero. This term can be replaced by any other that has the same effect. For example, in Schiavone (1996) the correction term in the density is $F(pF)^{-1}(p\varphi)$.

6.3 Smoothness of the Integrable Solutions

We conclude this chapter by taking a closer look at the regularity properties of the L^2-solutions of the singular integral equations corresponding to the interior and exterior Dirichlet and Neumann boundary value problems.

6.11 Theorem. *Suppose that*

$$\lambda\varphi(x) + \int\limits_{\partial S} k(x,\ y)\varphi(y)\, ds(y) = f(x) \tag{6.18}$$

for almost all $x \in \partial S$, *where* $k(x,\ y)$ *is a proper* γ*-singular kernel on* ∂S, $\gamma \in [0, 1)$, $\lambda \in \mathbb{R}$, $\lambda \neq 0$, *and* $f \in C^{0,\alpha}(\partial S)$, $\alpha \in (0, 1]$. *If* $\varphi \in L^p(\partial S)$ *is a solution of* (6.18), *then* $\varphi \in C^{0,\beta}(\partial S)$, *with* $\beta = \min\{\alpha, 1-\gamma\}$ *if* $\gamma \in (0,1)$, $\beta = \alpha$ *if* $\alpha \in (0,1)$ *and* $\gamma = 0$, *and any* $\beta \in (0,1)$ *if* $\alpha = 1$ *and* $\gamma = 0$.

Proof. Let

$$K(x) = \int\limits_{\partial S} k(x,\ y)\varphi(y)\, ds(y),$$

which, by Theorem 4.14, exists for almost all $x \in \partial S$. We have

$$|K(x)| \le \int\limits_{\partial S} \big[|k(x,\ y)|^{2-\gamma}|\varphi(y)|^p\big]^{1/[p(2-\gamma)]} \times |\varphi(y)|^{(1-\gamma)/(2-\gamma)}|k(x,\ y)|^{(p-1)/p}\, ds(y).$$

Setting

$$p_1 = p(2-\gamma), \quad p_2 = \frac{p(2-\gamma)}{1-\gamma}, \quad p_3 = \frac{p}{p-1}$$

and noting that

$$\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = 1$$

and that the three factors of the integrand on the right-hand side above belong to $L^{p_1}(\partial S)$, $L^{p_2}(\partial S)$, and $L^{p_3}(\partial S)$, respectively, we apply the generalized Hölder inequality and Theorem 1.32 to obtain

$$\begin{aligned}|K(x)| &\le \Bigg[\int\limits_{\partial S} |k(x,\ y)|^{2-\gamma}|\varphi(y)|^p\, ds(y)\Bigg]^{1/[p(2-\gamma)]}\\ &\quad\times\Bigg[\int\limits_{\partial S} |\varphi(y)|^p\, ds(y)\Bigg]^{(1-\gamma)[p(2-\gamma)]}\Bigg[\int\limits_{\partial S} |k(x,\ y)|\, ds(y)\Bigg]^{(p-1)/p}\\ &= c_1\|\varphi\|_p^{(1-\gamma)/(2-\gamma)}\Bigg[\int\limits_{\partial S} |k(x,\ y)|^{2-\gamma}|\varphi(y)|^p\, ds(y)\Bigg]^{1/[p(2-\gamma)]},\end{aligned}$$

where $c_1 = \text{const} > 0$. Then, by Fubini's Theorem,

$$\begin{aligned}&\int\limits_{\partial S} |K(x)|^{p(2-\gamma)}\, ds(x)\\ &\quad\le c_1\|\varphi\|_p^{p(1-\gamma)} \int\limits_{\partial S}\Bigg[\int\limits_{\partial S} |k(x,\ y)|^{2-\gamma}|\varphi(y)|^p\, ds(y)\Bigg] ds(x)\\ &\quad= c_1\|\varphi\|_p^{p(1-\gamma)} \int\limits_{\partial S}\Bigg[\int\limits_{\partial S} |k(x,\ y)|^{2-\gamma}\, ds(x)\Bigg]|\varphi(y)|^p\, ds(y).\end{aligned}$$

Since

$$|k(x,\ y)|^{2-\gamma} \leq c_2|x-y|^{-\gamma(2-\gamma)}, \quad c_2 = \text{const} > 0,$$

and $0 \leq \gamma(2-\gamma) < 1$, from Theorem 1.32 it follows that

$$\int\limits_{\partial S} |K(x)|^{p(2-\gamma)}\, ds(y) \leq c_3\|\varphi\|_p^{p(1-\gamma)} \int\limits_{\partial S} |\varphi(y)|^p\, ds(y) = c_3\|\varphi\|_p^{p(2-\gamma)},$$

where $c_3 = \text{const} > 0$. This means that $K \in L^{p(2-\gamma)}(\partial S)$. Then (6.18) yields $\varphi \in L^{p(2-\gamma)}(\partial S)$. Applying the argument successively n times, we deduce that $\varphi \in L^{p(2-\gamma)^n}(\partial S)$ for any positive integer n; hence, $\varphi \in L^\infty(\partial S)$.

If we now repeat the proof of Theorem 1.33 with the integrals understood in the sense of Lebesgue, we conclude that $K \in C^{0,\delta}(\partial S)$, with $\delta = 1-\gamma$ for $\gamma \in (0,1)$ and any $\delta \in (0,1)$ for $\gamma = 0$. The result now follows from (6.18). □

6.12 Theorem. *Suppose that equations* ($\mathscr{D}^\pm$) *and* ($\mathscr{N}^\pm$) *hold almost everywhere on* ∂S, *and that* $\mathscr{P}$, $\mathscr{Q}$, $\mathscr{R}$, $\mathscr{S} \in C^{0,\alpha}(\partial S)$, $\alpha \in (0,1)$. *If* $\varphi \in L^2(\partial S)$ *is a solution of any of the above equations, then* $\varphi \in C^{0,\alpha}(\partial S)$.

Proof. ($\mathscr{D}^\pm$) and ($\mathscr{N}^\pm$) are of the form

$$(K - \omega I)\varphi = g, \tag{6.19}$$

where, as seen in the proof of Theorem 6.2, K is α-regular singular and $\omega \in \mathbb{R}$, $\omega \neq 0$. In Muskhelishvili (1946) it is shown that we can always find an α-regular singular operator L that maps $L^2(\partial S)$ to $L^2(\partial S)$, and a $\vartheta \in \mathbb{R}$, $\vartheta \neq 0$, such that the equation

$$(L - \vartheta I)(K - \omega I)\varphi = (L - \vartheta I)g \tag{6.20}$$

is of the form (6.18), where $\lambda \in \mathbb{R}$, $\lambda \neq 0$, $f \in C^{0,\alpha}(\partial S)$, and $k(x,\ y)$ is a proper $(1-\alpha)$-singular kernel on ∂S. Since every solution of (6.19) is also a solution of (6.20), the assertion follows from Theorem 6.11 with $p = 2$. □

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Chapter 7
Complex Variable Treatment

7.1 Complex Representation of the Stresses

We revert to the original notation, where S is a bounded simply connected domain in $\mathbb{R}^2$, whose boundary ∂S is a simple closed contour.

In agreement with (3.40), we consider the homogeneous system (3.3); that is,

$$\begin{aligned} N_{\alpha\beta,\beta} - N_{3\alpha} &= 0, \\ N_{3\beta,\beta} &= 0, \end{aligned} \tag{7.1}$$

and investigate its analytic solutions in S.

From the second equation (7.1) we deduce that there is a function $\mathscr{G}(x_\gamma)$ such that

$$\begin{aligned} N_{31} &= \mathscr{G}_{,2}\,, \\ N_{32} &= -\mathscr{G}_{,1}\,. \end{aligned} \tag{7.2}$$

This and the first equation (7.1) yield

$$\begin{aligned} N_{11,1} + (N_{12} - \mathscr{G})_{,2} &= 0, \\ (N_{12} + \mathscr{G})_{,1} + N_{22,2} &= 0. \end{aligned}$$

Hence, there are functions $\mathscr{H}_\alpha(x_\gamma)$ such that

$$\begin{aligned} N_{11} &= \mathscr{H}_{1,2}\,, \qquad & N_{12} - \mathscr{G} &= -\mathscr{H}_{1,1}, \\ N_{22} &= -\mathscr{H}_{2,1}\,, \qquad & N_{12} + \mathscr{G} &= \mathscr{H}_{2,2}. \end{aligned} \tag{7.3}$$

Obviously, we must have

$$\mathscr{G} - \mathscr{H}_{1,1} = -\mathscr{G} + \mathscr{H}_{2,2}. \tag{7.4}$$

C. Constanda, *Mathematical Methods for Elastic Plates*,
Springer Monographs in Mathematics, DOI: 10.1007/978-1-4471-6434-0_7,

Let $\mathscr{B}(x_\gamma)$ be such that

$$\mathscr{G} = \mathscr{B}_{,12}\,. \tag{7.5}$$

Then from (7.4) it follows that there is a function $\mathscr{C}(x_\gamma)$ satisfying

$$\begin{aligned}\mathscr{B}_{,2} - \mathscr{H}_1 &= -\mathscr{C}_{,2}\,,\\ \mathscr{B}_{,1} - \mathscr{H}_2 &= \mathscr{C}_{,1}\,,\end{aligned}$$

in which case (7.2), (7.3), and (7.5) imply that

$$\begin{aligned}N_{11} &= (\mathscr{C} + \mathscr{B})_{,22}\,,\\ N_{22} &= (\mathscr{C} - \mathscr{B})_{,11}\,,\\ N_{12} &= -\mathscr{C}_{,12}\,,\\ N_{31} &= \mathscr{B}_{,122}\,,\\ N_{32} &= -\mathscr{B}_{,112}\,.\end{aligned} \tag{7.6}$$

The stress functions $\mathscr{B}$ and $\mathscr{C}$ generate a deformation state if and only if $N_{\alpha\beta}$ and $N_{3\alpha}$ given by (7.6) satisfy the compatibility relations (3.7). Replacing (7.6) in (3.7), we obtain the Cauchy–Riemann system

$$\begin{aligned}(h^2\Delta\mathscr{B}_{,12} - \mathscr{B}_{,12})_{,1} &= (1-\sigma)(\Delta\mathscr{C} + \mathscr{B}_{,22} - \mathscr{B}_{,11})_{,2}\,,\\ (h^2\Delta\mathscr{B}_{,12} - \mathscr{B}_{,12})_{,2} &= -(1-\sigma)(\Delta\mathscr{C} + \mathscr{B}_{,22} - \mathscr{B}_{,11})_{,1}\,.\end{aligned}$$

Hence,

$$\Delta\mathscr{B}_{,12} - \frac{1}{h^2}\mathscr{B}_{,12} = \frac{2}{h^2}\,\mathrm{Re}\Omega_0, \tag{7.7}$$

$$\Delta\mathscr{C} + \mathscr{B}_{,22} - \mathscr{B}_{,11} = \frac{2}{1-\sigma}\,\mathrm{Im}\Omega_0, \tag{7.8}$$

where Ω_0 is an arbitrary analytic function of $z = x_1 + ix_2$ in S.

Let $\eta(z,\,\bar{z})$ be an arbitrary real solution in S of the equation

$$\Delta\eta - \frac{1}{h^2}\eta = 0. \tag{7.9}$$

Then from (7.7) we find that

$$\mathscr{B}_{,12} = \mathrm{Re}\big[\eta(z,\,\bar{z}) - 2\Omega_0(z)\big]. \tag{7.10}$$

For simplicity, in what follows we omit the explicit mention of z and $\bar{z}$ in the symbols of functions.

Applying the operator $(\Delta - h^{-2})\partial^2/\partial x_1 \partial x_2$ to (7.8) and using (7.10), we obtain

$$\Delta\left(\Delta\mathscr{C},_{12} - \frac{1}{h^2}\mathscr{C},_{12}\right) = \frac{8}{h^2}\sigma_1 \mathrm{Re}\Omega_0'',$$

where $\sigma_1 = \frac{1}{4}(1-2\sigma)(1-\sigma)^{-1}$. Therefore,

$$\mathscr{C},_{12} = \mathrm{Re}(\theta - 2\sigma_1 \bar{z}\,\Omega_0' + 2\omega_0), \tag{7.11}$$

where θ is another arbitrary real solution of (7.9) in S and ω_0 is an arbitrary analytic function in S. From (7.10) and (7.11) we deduce that

$$(\mathscr{C} + \mathscr{B}),_{12} = \mathrm{Re}(\eta + \theta - 2\Omega_0 + 2\omega_0 - 2\sigma_1 \bar{z}\,\Omega_0'), \tag{7.12}$$
$$(\mathscr{C} - \mathscr{B}),_{12} = \mathrm{Re}(-\eta + \theta + 2\Omega_0 + 2\omega_0 - 2\sigma_1 \bar{z}\,\Omega_0'). \tag{7.13}$$

Differentiating (7.12) with respect to x_2 and replacing the result in (7.8) differentiated with respect to x_1, and dealing similarly with (7.13) and (7.8), we find $(\mathscr{C} - \mathscr{B}),_{111}$ and $(\mathscr{C} + \mathscr{B}),_{222}$, which we then combine with $(\mathscr{C} - \mathscr{B}),_{122}$ and $(\mathscr{C} + \mathscr{B}),_{112}$ obtained directly from (7.12) and (7.13). Thus, we arrive at

$$\Delta\big[(\mathscr{C} + \mathscr{B}),_2 + i(\mathscr{C} - \mathscr{B}),_1\big] = 4(\eta,_z - 2\sigma_1\Omega_0'),$$

where $(\ldots),_z = \partial(\ldots)/\partial z$. This implies that

$$\begin{aligned}(\mathscr{C} + \mathscr{B}),_2 &= 2\mathrm{Re}(2h^2\eta,_z - \sigma_1 \bar{z}\,\Omega_0 + \Omega_1),\\ (\mathscr{C} - \mathscr{B}),_1 &= 2\mathrm{Im}(2h^2\eta,_z - \sigma_1 \bar{z}\,\Omega_0 - \Omega_2),\end{aligned} \tag{7.14}$$

where Ω_α are arbitrary analytic functions in S.

Since this representation has been obtained by differentiating the exact formula (7.8), it may contain too much arbitrariness. Replacing (7.14) in (7.12), (7.13), and (7.8), we see that

$$\mathrm{Re}(4h^2\eta,_{zz} - \theta + \Omega_1' - \Omega_2' - 2\omega_0) = 0, \tag{7.15}$$
$$2(1-\sigma)(\Omega_1' + \Omega_2') + (3 - 2\sigma)\Omega_0 = 0. \tag{7.16}$$

Now (7.16) yields

$$\Omega_0 = -\frac{2(1-\sigma)}{3-2\sigma}(\Omega_1' + \Omega_2'). \tag{7.17}$$

Since η and θ are solutions of (7.9), from (7.15) we find that

$$\begin{aligned}2\mathrm{Re}\,\omega_0 &= \mathrm{Re}(\Omega_1' - \Omega_2'),\\ \theta &= 4h^2\mathrm{Re}\,\eta,_{zz}\,.\end{aligned} \tag{7.18}$$

Substituting (7.17) and (7.18) in (7.10)–(7.12) and (7.14) and setting

$$\kappa = \frac{1-2\sigma}{3-2\sigma} = \frac{\mu}{2\lambda+3\mu}, \tag{7.19}$$

we obtain

$$\begin{aligned}(\mathscr{C}+\mathscr{B})_{,2} &= \text{Re}\big[4h^2\eta_{,z} + \kappa\bar{z}(\Omega_1' + \Omega_2') + 2\Omega_1\big],\\ (\mathscr{C}-\mathscr{B})_{,1} &= \text{Im}\big[4h^2\eta_{,z} + \kappa\bar{z}(\Omega_1' + \Omega_2') - 2\Omega_2\big],\\ \mathscr{C}_{,12} &= \text{Re}\big[4h^2\eta_{,zz} + \kappa\bar{z}(\Omega_1'' + \Omega_2'') + \Omega_1' - \Omega_2'\big],\\ \mathscr{B}_{,12} &= \text{Re}\big[\eta + (1+\kappa)(\Omega_1' + \Omega_2')\big].\end{aligned} \tag{7.20}$$

Finally, from the above relations and formulas (7.6) we conclude that

$$\begin{aligned}N_{11} &= -\text{Im}\big[4h^2\eta_{,zz} + \kappa\bar{z}(\Omega_1'' + \Omega_2'') - \kappa\Omega_2' + (2-\kappa)\Omega_1'\big],\\ N_{22} &= \text{Im}\big[4h^2\eta_{,zz} + \kappa\bar{z}(\Omega_1'' + \Omega_2'') + \kappa\Omega_1' - (2-\kappa)\Omega_2'\big],\\ N_{12} &= -\text{Re}\big[4h^2\eta_{,zz} + \kappa\bar{z}(\Omega_1'' + \Omega_2'') + \Omega_1' - \Omega_2'\big],\\ N_{31} &= -\text{Im}\big[2\eta_{,z} + (1+\kappa)(\Omega_1'' + \Omega_2'')\big],\\ N_{32} &= -\text{Re}\big[2\eta_{,z} + (1+\kappa)(\Omega_1'' + \Omega_2'')\big].\end{aligned} \tag{7.21}$$

Since an arbitrary solution of (7.9) can be expressed in terms of an arbitrary analytic function in S (Miranda 1970), the bending and twisting moments and the transverse shear forces are represented in terms of three arbitrary analytic functions of z in S. Functions of this type are known in the literature as complex potentials.

7.2 The Traction Boundary Value Problem

We consider the Neumann boundary conditions

$$N_i = N_{i\alpha}\nu_\alpha = \tilde{N}_i \quad \text{on } \partial S. \tag{7.22}$$

According to Remark 3.1, the resultant force and complex moment acting on an arc t_0t of ∂S are

$$\begin{aligned}\big[\mathscr{N}\big]_{t_0}^{t} &= \int_{t_0}^{t} N_3\,ds = \tilde{\mathscr{N}},\\ \big[\mathscr{M}\big]_{t_0}^{t} &= \int_{t_0}^{t} \big[-N_2 + x_2N_3 + i(N_1 - x_1N_3)\big]\,ds = \tilde{\mathscr{M}}.\end{aligned} \tag{7.23}$$

From (7.6) and (7.22) we obtain

$$\begin{aligned} N_3 &= \frac{d}{ds}\mathscr{B},_{12}\,, \\ -N_2 + x_2 N_3 &+ i(N_1 - x_1 N_3) \\ &= \frac{d}{ds}\big[(\mathscr{C}-\mathscr{B}),_1 \; + i(\mathscr{C}+\mathscr{B}),_2 \; - iz\mathscr{B},_{12}\big]. \end{aligned} \tag{7.24}$$

Using this in (7.23), we find that on ∂S,

$$\begin{aligned} \mathscr{B},_{12} &= \tilde{\mathscr{N}} + \beta_1, \quad \beta_1 \in \mathbb{R}, \\ (\mathscr{C}-\mathscr{B}),_1 \; + i(\mathscr{C}+\mathscr{B}),_2 \; - iz\mathscr{B},_{12} &= \tilde{\mathscr{M}} + \beta_2, \quad \beta_2 \in \mathbb{C}. \end{aligned} \tag{7.25}$$

Setting $\Omega_1 + \Omega_2 = \rho$ and $\Omega_1 - \Omega_2 = \vartheta$, from (7.20) and (7.25) we now deduce that

$$\begin{aligned} 4h^2\eta,_{\bar{z}} \; + \kappa z\bar{\rho}' + \rho + \bar{\vartheta} &= z\tilde{\mathscr{N}} - i\tilde{\mathscr{M}} + \beta_1 z - i\beta_2, \\ \eta + \tfrac{1}{2}(1+\kappa)(\rho' + \bar{\rho}') &= \tilde{\mathscr{N}} + \beta_1 \quad \text{on } \partial S. \end{aligned} \tag{7.26}$$

Hence, the traction boundary value problem reduces to finding η, ρ, and ϑ satisfying (7.9) in S and (7.26) on ∂S.

7.3 The Displacement Boundary Value Problem

Consider the Dirichlet boundary conditions

$$u_i = \tilde{u}_i \quad \text{on } \partial S. \tag{7.27}$$

We introduce the complex displacements, moments, and force by

$$\begin{aligned} \Gamma &= u_1 + iu_2\,, \\ \Theta &= u_3\,, \\ \Phi &= N_{11} - N_{22} + 2iN_{12}\,, \\ \Psi &= N_{11} + N_{22}, \\ \Lambda &= N_{31} + iN_{32}. \end{aligned} \tag{7.28}$$

Then the constitutive relations (3.5) become

$$\begin{aligned} \Phi &= 4h^2\mu\Gamma,_{\bar{z}}\,, \\ \Psi &= 4h^2(\lambda+\mu)\mathrm{Re}\Gamma,_z\,, \\ \Lambda &= \mu(\Gamma + 2\Theta,_{\bar{z}}), \end{aligned} \tag{7.29}$$

and from (7.21), the third equation (7.28), and the first equation (7.29) we find that

$$\Gamma = -\frac{i}{2h^2\mu}\big[4h^2\eta,_{\bar z} + \kappa z\,(\bar\Omega_1' + \bar\Omega_2') + \bar\Omega_1 - \bar\Omega_2 + \theta_1\big], \tag{7.30}$$

where θ_1 is an analytic function of z in S. This, (7.21), and (7.29) yield

$$\operatorname{Im}\theta'_1 = -\kappa\operatorname{Im}(\Omega'_1 + \Omega_2');$$

therefore,

$$\theta_1 = -\kappa(\Omega_1 + \Omega_2) - c_1 z - c_2, \tag{7.31}$$

where $c_1 \in \mathbb{R}$ and $c_2 \in \mathbb{C}$.

Setting $\Omega_\alpha = \omega'_\alpha$, from (7.30) and (7.31) we get

$$\Gamma = -\frac{i}{2h^2\mu}\big[4h^2\eta,_{\bar z} + \kappa z\,(\bar\omega_1'' + \bar\omega_2'') - \kappa(\omega_1' + \omega_2') + \bar\omega_1' - \bar\omega_2' - c_1 z - c_2\big]. \tag{7.32}$$

From this, (7.21), and the third equation (7.29) we obtain

$$\begin{aligned}\Theta = \frac{i}{4h^2\mu}\big[&\kappa z\,(\bar\omega_1' + \bar\omega_2') - \kappa\bar z\,(\omega_1' + \omega_2') + \bar\omega_1 - \bar\omega_2\\ &- c_1 z\bar z - c_2\bar z - 2h^2(1+\kappa)(\bar\omega_1'' + \bar\omega_2'') + \theta_2\big],\end{aligned} \tag{7.33}$$

where θ_2 is an analytic function of z in S. Since Θ is real, we must have

$$\begin{aligned}&c_1 = 0,\\ &\theta_2 = -(\omega_1 - \omega_2) + 2h^2(1+\kappa)(\omega_1'' + \omega_2'') + \bar c_2 z - ic_3,\end{aligned} \tag{7.34}$$

where $c_3 \in \mathbb{R}$. We set

$$\begin{aligned}\psi &= -\frac{2}{\mu}\eta,\\ \Omega &= \frac{i\kappa}{2h^2\mu}(\omega_1' + \omega_2'),\\ \omega &= \frac{i}{2h^2\mu}(\omega_1 - \omega_2),\\ l &= l_1 + il_2 = \frac{ic_2}{2h^2\mu},\\ m &= \frac{c_3}{4h^2\mu}.\end{aligned} \tag{7.35}$$

From this and (7.32)–(7.34) we then conclude that

$$\begin{aligned}\varGamma &= i\psi_{,\bar z} + z\bar{\varOmega}' + \varOmega + \bar{\omega}' + l,\\ \varTheta &= \mathrm{Re}\left(4h^2\frac{\lambda+2\mu}{\mu}\varOmega' - \bar z\varOmega - \omega - l\bar z + m\right).\end{aligned} \tag{7.36}$$

7.1 Remarks. The matrix u_0 defined by the terms containing l and m is of the form (3.16); consequently, it represents a rigid displacement. These terms are unessential and in what follows we assume them to be incorporated in $\varOmega$ and ω, respectively.

Let $z \in \partial S$. Writing $\tilde u = \tilde u_1 + i\tilde u_2$, from (7.27) we find that

$$\begin{aligned} i\psi_{,\bar z} + z\bar{\varOmega}' + \varOmega + \bar{\omega}' &= \tilde u,\\ \mathrm{Re}\left(4h^2\frac{\lambda+2\mu}{\mu}\varOmega' - \bar z\varOmega - \omega\right) &= \tilde u_3.\end{aligned} \tag{7.37}$$

Thus, the displacement boundary value problem reduces to finding a solution ψ in S of (7.9) and arbitrary analytic functions $\varOmega$ and ω in S satisfying (7.37) on ∂S.

From (7.29), (7.35), and (7.36) we see that

$$\begin{aligned}\varPhi &= 4h^2\mu(i\psi_{,\bar z\bar z} + z\bar{\varOmega}'' + \bar{\omega}''),\\ \varPsi &= 4h^2(\lambda+\mu)(\varOmega' + \bar{\varOmega}'),\\ \varLambda &= i\mu\psi_{,\bar z} + 4h^2(\lambda+2\mu)\bar{\varOmega}''.\end{aligned} \tag{7.38}$$

Comparing the definitions of η, ρ, ϑ and ψ, $\varOmega$, ω, we can rewrite the traction boundary conditions (7.26) as

$$\begin{aligned} i\psi_{,\bar z} + z\bar{\varOmega}' - \frac{2\lambda+3\mu}{\mu}\varOmega + \bar{\omega}' &= -\frac{1}{2h^2\mu}(\tilde{\mathscr M} + iz\tilde{\mathscr N} + i\beta_1 z + \beta_2),\\ i\psi - 4h^2\frac{\lambda+2\mu}{\mu}(\varOmega' - \bar{\varOmega}') &= -\frac{2i}{\mu}(\tilde{\mathscr N} + \beta_1);\end{aligned} \tag{7.39}$$

similarly, using (7.23), (7.24), and (7.20), we rewrite the resultant force and complex moment acting on the arc t_0t of ∂S as

$$\begin{aligned}\big[\mathscr N\big]_{t_0}^{t} &= \big[-\tfrac12\mu\psi - 2h^2(\lambda+2\mu)(\varOmega' - \bar{\varOmega}')\big]_{t_0}^{t},\\ \big[\mathscr M\big]_{t_0}^{t} &= \big[\tfrac12 i\mu z\psi - 2h^2(\lambda+2\mu)z(\varOmega - \bar{\varOmega}')\\ &\quad - 2h^2\mu(i\psi_{,\bar z} + z\bar{\varOmega}' + \bar{\omega}') + 2h^2(2\lambda+3\mu)\varOmega\big]_{t_0}^{t}.\end{aligned} \tag{7.40}$$

7.2 Remarks. A representation similar to (7.38) has been derived in the case of Reissner's theory (Green and Zerna 1963). In our notation, this is

$$\begin{aligned}
\Phi &= 4h^2\mu\left[i\varpi,_{\bar{z}\bar{z}} + \frac{\lambda+2\mu}{4(\lambda+\mu)}(z\bar{\Omega}'' + \bar{\omega}'')\right],\\
\Psi &= \frac{h^2\mu(3\lambda+2\mu)}{\lambda+\mu}(\Omega' + \bar{\Omega}'),\\
\Lambda &= \mu(\tfrac{5}{6}i\varpi,_{\bar{z}} + 4h^2\bar{\Omega}''),
\end{aligned}$$

where ϖ is an arbitrary analytic solution in S of the equation

$$\Delta\varpi - \frac{5}{6h^2}\varpi = 0.$$

7.3 Remarks. Θ given by the second equation (7.36) satisfies $\Delta\Delta\Theta = 0$. From (7.36) and (7.19) we obtain

$$\Gamma = -2\Theta,_{\bar{z}} + i\psi,_{\bar{z}} + 4h^2\mu^{-1}(\lambda+2\mu)\bar{\Omega}''.$$

Hence, in this theory, as in Kirchhoff's, $\Theta = u_3$ remains a biharmonic function. In addition, Kirchhoff's theory also leads to the second equation (1.5); that is,

$$\Gamma = -2\Theta,_{\bar{z}}.$$

Here, Γ contains two correction terms, of which one is a solution of (7.9) and the other is harmonic.

7.4 Remarks. If instead of (3.5) we adopt Mindlin's constitutive relations (Mindlin 1951), then, ignoring rigid displacements, we obtain

$$\begin{aligned}
\Gamma &= i\chi,_{\bar{z}} - 2k^2\left(z\bar{\Omega}' + \Omega + \bar{\omega}' + \frac{8h^2}{1-\sigma}\bar{\Omega}''\right),\\
\Theta &= 2\mathrm{Re}(\bar{z}\,\Omega + \omega),\\
\Phi &= \frac{2Eh^2}{1+\sigma}\left[i\chi,_{\bar{z}\bar{z}} - 2k^2\left(z\bar{\Omega}'' + \bar{\omega}'' + \frac{8h^2}{1-\sigma}\bar{\Omega}'''\right)\right],\\
\Psi &= -8k^2\frac{Eh^2}{1-\sigma}\mathrm{Re}\Omega',\\
\Lambda &= k^2\frac{E}{2(1-\sigma)}\left[i\chi,_{\bar{z}} + (k^2-1)(z\bar{\Omega}' - \Omega + \bar{\omega}') + \frac{8h^2}{1-\sigma}\bar{\Omega}''\right],
\end{aligned}$$

where E is Young's modulus, k^2 a correction coefficient introduced by Mindlin, and χ an arbitrary real solution of the equation

$$\Delta\chi - \frac{k^2}{h^2}\chi = 0.$$

7.4 Arbitrariness in the Complex Potentials

Suppose that the functions ψ_*, Ω_*, and ω_* generate the same stress state as ψ, Ω, and ω. Then from the second equation (7.38) we find that

$$\mathrm{Re}\Omega'_* = \mathrm{Re}\Omega';$$

therefore,

$$\Omega_* = \Omega + i d_1 z + d_2, d_1 \in \mathbb{R}, d_2 \in \mathbb{C}. \tag{7.41}$$

Using the third equation (7.38) and (7.41), we obtain

$$\psi_{*,\bar{z}} = \psi_{,\bar{z}}\,,$$

which yields

$$\psi_* = \psi + d_3 z + d_4, d_3,\ d_4 \in \mathbb{C}.$$

Since ψ and ψ_* are real functions, it follows that $d_3 = 0$ and $d_4 \in \mathbb{R}$, and the fact that both ψ and ψ_* are solutions of (7.9) leads to

$$\psi_* = \psi. \tag{7.42}$$

From the first equation (7.38), (7.41), and (7.42) we deduce that

$$\omega_* = \omega + d_5 z + i d_6, \quad d_5,\ d_6 \in \mathbb{C}. \tag{7.43}$$

Choosing d_1, d_2, and d_5 so that

$$d_1 = \frac{\kappa}{2h^2\mu(1+\kappa)}\beta_1, \quad d_2 - \kappa\bar{d}_5 = \frac{\kappa}{2h^2\mu}\beta_2,$$

we make the terms β_1 and $i\beta_1 z + \beta_2$ vanish in (7.39). To reduce the arbitrariness of Ω and ω we may impose, for example, the additional conditions

$$\Omega(0) = 0, \quad \omega(0) = 0.$$

If we also want the displacements to remain unchanged, then, according to (7.36) and (7.41)–(7.43), we must require that

$$d_5 = -\bar{d}_2, \quad d_6 \in \mathbb{R}.$$

Thus, the functions Ω and ω are completely determined if we ask, say, that they satisfy

$$\Omega(0) = 0, \quad \mathrm{Im}\Omega'(0) = 0, \quad \omega(0) = 0.$$

7.5 Bounded Multiply Connected Domain

Let S be multiply connected. For simplicity, we introduce the notation

$$\theta(z,\ \bar{z}) \in \mathscr{U} \quad \Leftrightarrow \quad \theta(z,\ \bar{z}) \text{ is single-valued in } S.$$

From (7.36) and (7.38) we see that the moments, forces, and displacements are single-valued if

$$\begin{aligned} i\psi_{,\bar{z}} \ + \ z\bar{\Omega}' + \Omega + \bar{\omega}' &\in \mathscr{U}, \\ \mathrm{Re}\left(4h^2\frac{\lambda+2\mu}{\mu}\Omega' - \bar{z}\,\Omega - \omega\right) &\in \mathscr{U}, \\ i\psi_{,\bar{z}\bar{z}} \ + \ z\bar{\Omega}'' + \bar{\omega}'' &\in \mathscr{U}, \\ \mathrm{Re}\Omega' &\in \mathscr{U}, \\ i\psi_{,\bar{z}} \ + \ 4h^2\frac{\lambda+2\mu}{\mu}\bar{\Omega}'' &\in \mathscr{U}. \end{aligned} \tag{7.44}$$

Clearly, from the first equation (7.44) it follows that

$$\Omega'' \in \mathscr{U}, \tag{7.45}$$

which, in view of (7.44), yields $\psi_{,\bar{z}} \in \mathscr{U}$. Then

$$\psi_{,z\bar{z}} \in \mathscr{U}$$

also, consequently,

$$\psi \in \mathscr{U}. \tag{7.46}$$

From the third equation (7.44)–(7.46) we deduce that

$$\omega'' \in \mathscr{U}, \tag{7.47}$$

and from the second and fourth equations (7.44) we get

$$\mathrm{Re}(\bar{z}\,\Omega + \omega) \in \mathscr{U}. \tag{7.48}$$

Also, the first equation (7.44) and (7.46) lead to

$$z\bar{\Omega}' + \Omega + \bar{\omega}' \in \mathscr{U}. \tag{7.49}$$

This means that the necessary single-valuedness conditions are the fourth equation (7.44) and (7.45)–(7.49).

Next, suppose that the boundary of S consists of $n+1$ disjoint simple closed curves of which one, ∂S, encloses all the remaining ones, ∂S_k, $k=1,\ldots,n$. According to a well-known argument in three-dimensional elasticity (Muskhelishvili 1949), we can choose arbitrary points z_k inside the contours ∂S_k and write

$$\begin{aligned}\Omega &= \frac{1}{2\pi i}\sum_{k=1}^{n}(c_k z + d_k)\log(z - z_k) + \tilde{\Omega},\\ \omega &= \frac{1}{2\pi i}\sum_{k=1}^{n}(p_k z + q_k)\log(z - z_k) + \tilde{\omega},\end{aligned} \tag{7.50}$$

where $c_k,\ d_k,\ p_k,\ q_k \in \mathbb{C}$, $k=1,\ldots,n$, and $\tilde{\Omega}$ and $\tilde{\omega}$ are analytic functions in S. From the fourth equation (7.44) and (7.48)–(7.50) we find that the coefficients must satisfy

$$\operatorname{Re} c_k = 0, \quad d_k + \bar{p}_k = 0, \quad \operatorname{Re} q_k = 0. \tag{7.51}$$

Traversing ∂S_k once anticlockwise, from (7.40), (7.46), and (7.50) we obtain the resultant force and moment on ∂S_k in the form

$$\begin{aligned}\mathscr{N}_k &= -[\mathscr{N}]_{\partial S_k} = -4h^2(\lambda + 2\mu)\operatorname{Im} c_k,\\ \mathscr{M}_k &= -[\mathscr{M}]_{\partial S_k} = 2h^2\big[-(2\lambda + 3\mu)d_k + \mu\bar{p}_k\big].\end{aligned}$$

Combining these relations with (7.51), we deduce that

$$\begin{aligned}c_k &= -2\pi i c\mathscr{N}_k, \quad & d_k &= -2\pi c\mathscr{M}_k,\\ p_k &= 2\pi c\bar{\mathscr{M}}_k, \quad & q_k &= -2\pi i c s_k,\end{aligned}$$

where $c = \big[8\pi h^2(\lambda + 2\mu)\big]^{-1}$ and $s_k \in \mathbb{R}$. From this and (7.50) we now conclude that

$$\begin{aligned}\Omega &= -c\sum_{k=1}^{n}(z\mathscr{N}_k - i\mathscr{M}_k)\log(z - z_k) + \tilde{\Omega},\\ \omega &= -c\sum_{k=1}^{n}(iz\bar{\mathscr{M}}_k + s_k)\log(z - z_k) + \tilde{\omega}.\end{aligned} \tag{7.52}$$

7.5 Remarks. The terms $s_k \log(z - z_k)$, although many-valued, do not alter the single-valuedness of the force, moments, and displacements. These terms occur only in the expression of Θ, in the form

$$\operatorname{Re}\big[s_k \log(z - z_k)\big] = s_k \log|z - z_k| \in \mathscr{U}.$$

7.6 Unbounded Multiply Connected Domain

Suppose that the curve ∂S has expanded to infinity. Introducing the notation

$$\mathsf{N} = \sum_{k=1}^{n} \mathscr{N}_k,$$

$$\mathsf{M} = \sum_{k=1}^{n} \mathscr{M}_k,$$

$$s = \sum_{k=1}^{n} s_k$$

and proceeding as in Muskhelishvili (1949), from (7.52) we find that the complex potentials admit the expansions

$$\Omega = -c(\mathsf{N}z - i\,\mathsf{M}) \log z + \sum_{n=-\infty}^{\infty} a_n z^n,$$

$$\omega = -c(i\,\bar{\mathsf{M}}z + s) \log z + \sum_{n=-\infty}^{\infty} b_n z^n, \tag{7.53}$$

where $a_n, b_n \in \mathbb{C}$. Then (7.38) and (7.53) yield the complex moments and force in the form

$$\Phi = 4h^2\mu\Big[i\psi,_{\bar{z}\bar{z}} - c\,\mathsf{N}z\bar{z}^{-1} + ic\,\bar{\mathsf{M}}z\bar{z}^{-2} + ic\mathsf{M}\bar{z}^{-1} + cs\bar{z}^{-2} + \sum_{n=-\infty}^{\infty} n(n-1)(\bar{a}_n z + \bar{b}_n)\bar{z}^{n-2}\Big],$$

$$\Psi = 4h^2(\lambda+\mu)\Big[-2c\,\mathsf{N}\ln|z| - 2c\,\mathsf{N} + ic(\mathsf{M}z^{-1} - \bar{\mathsf{M}}\bar{z}^{-1}) + \sum_{n=-\infty}^{\infty} n(a_n z^{n-1} + \bar{a}_n\bar{z}^{n-1})\Big], \tag{7.54}$$

$$\Lambda = i\mu\psi,_{\bar{z}} + 4h^2(\lambda+2\mu)\Big[-c\,\mathsf{N}\bar{z}^{-1} + ic\,\bar{\mathsf{M}}\bar{z}^{-2} + \sum_{n=-\infty}^{\infty} n(n-1)\bar{a}_n\bar{z}^{n-2}\Big].$$

To investigate the behavior of Φ, Ψ, and Λ as $|z| \to \infty$, we need to know the asymptotics of ψ. Since (Abramowitz and Stegun 1964)

$$\frac{d}{d\xi} K_0(\xi) = -K_1(\xi),$$
$$\frac{d}{d\xi} K_1(\xi) = -K_0(\xi) - \frac{1}{\xi} K_1(\xi), \tag{7.55}$$

and, as $|\xi| \to \infty$,

$$K_0(\xi) = \left(\frac{\pi}{2\xi}\right)^{1/2} e^{-\xi} + \cdots,$$
$$K_1(\xi) = \left(\frac{\pi}{2\xi}\right)^{1/2} e^{-\xi} + \cdots, \tag{7.56}$$

and since $(2\pi)^{-1}K_0(h^{-1}|x - y|)$ is a fundamental solution of (7.9), just as in harmonic potential theory we deduce that ψ admits the representation

$$\psi(x) = \int\limits_{\cup\partial S_k} \Big[\psi(y)\partial\nu(y)K_0(h^{-1}|x - y|) - K_0(h^{-1}|x - y|)\partial\nu(y)\psi(y)\Big]\, ds(y).$$

From this, (7.55), and (7.56) we see that ψ and its derivatives vanish as $|x| \to \infty$. Therefore, (7.54) shows that Φ, Ψ, and Λ are bounded at infinity if and only if

$$\mathsf{N} = 0, \quad a_n = 0 \quad (n \geq 2), \quad b_n = 0 \quad (n \geq 3). \tag{7.57}$$

Next, using (7.53) and (7.57) in (7.36), we obtain

$$\begin{aligned}\Gamma &= (a_1 + \bar{a}_1)z + 2\bar{b}_2\bar{z} + 2ic\,\mathsf{M}\ln|z| - ic\bar{\mathsf{M}}z\bar{z}^{-1} + a_0 + \bar{b}_1 + ic\,\mathsf{M}\\ &\quad + a_{-1}z^{-1} - cs\bar{z}^{-1} - \bar{a}_{-1}z\bar{z}^{-2} + O(|z|^{-2}),\\ \Theta &= \operatorname{Re}\Big[ic(\bar{\mathsf{M}}z - \mathsf{M}\bar{z})\log z + cs\ln|z| - b_2z^2 - a_1z\bar{z}\\ &\quad - (\bar{a}_0 + b_1)z + 4h^2\frac{\lambda + 2\mu}{\mu}a_1 - a_{-1}z^{-1}\bar{z} - b_0\Big] + O(|z|^{-1}).\end{aligned} \tag{7.58}$$

Since $u_3 = \Theta$ occurs in the internal energy density (3.14) only in terms of its derivatives, we conclude that for a finite energy solution—that is, $\Gamma = O(1)$ and $\Theta = O(\ln|z|)$ as $|z| \to \infty$—we must have

$$\mathsf{M} = 0, \quad a_1 = i\alpha \quad (\alpha \in \mathbb{R}), \quad b_2 = 0, \quad b_1 = -\bar{a}_0. \tag{7.59}$$

In view of (7.41), we may discard the a_0-term and the a_1-term. Setting

$$-cs = a \in \mathbb{R}$$

and

$$\mathrm{Re} b_0 = b,$$

from (7.53), (7.57), and (7.59) we conclude that, as $|z| \to \infty$,

$$\Omega = \sum_{n=-\infty}^{-1} a_n z^n,$$
$$\omega = a \log z + b + \sum_{n=-\infty}^{-1} b_n z^n. \tag{7.60}$$

These formulas and (7.36) (with an arbitrary ψ satisfying (7.9)) yield the general analytic finite energy solution of (3.40) in S.

7.6 Remark. It is interesting to note that, although ψ is responsible for the sixth order character of this bending theory, it plays no active role in the far-field pattern of the solution, which depends exclusively on the structure of Ω and ω.

7.7 Remark. A straightforward calculation shows that for $a = b$ in (7.60), the expansion of u coincides with (3.39), which characterizes the class $\mathscr{A}$. Also, from (7.36), (7.56), and (7.60) we obtain the asymptotic relations

$$\Phi = O(|z|^{-2}), \quad \Psi = O(|z|^{-2}),$$
$$\Gamma = O(|z|^{-1}), \quad \Theta = O(\ln|z|), \quad \Lambda = O(|z|^{-3}).$$

These imply that the Betti formula in the exterior domain, proved in Theorem 3.13, holds for all solutions satisfying (7.57) and (7.59). Consequently, the condition that $u \in \mathscr{A}$, which was shown to be sufficient for the solvability of the exterior Neumann problem, turns out to be also necessary if we want a unique solution. Removing the restriction $a = b$ in (7.60) means that the regular solution of this problem is unique up to an arbitrary vertical translation.

7.7 Example

Consider an infinite plate with a circular hole of radius ρ, acted upon at the hole by a normal force cx_3, $c = \text{const} > 0$, parallel to the middle plane of the plate. Choosing the origin at the center of the hole and following the averaging procedure set out in Sect. 3.1, we arrive at the boundary and far-field conditions

$$N_{rr} = h^2 c, \quad N_{r\theta} = N_{3r} = 0 \quad \text{if } |z| = \rho,$$
$$N_{\alpha\beta} = N_{3\alpha} = 0 \quad \text{as } |z| \to \infty, \tag{7.61}$$

where N_{rr}, $N_{r\theta}$, and N_{3r} are the physical polar components of the $N_{i\alpha}$, defined by

$$\begin{aligned}N_{rr} &= \tfrac{1}{2}\mathrm{Re}(e^{-2i\theta}\Phi + \Psi),\\ N_{r\theta} &= \tfrac{1}{2}\mathrm{Im}(e^{-2i\theta}\Phi - \Psi),\\ N_{3r} &= \mathrm{Re}(e^{-i\theta}\Lambda).\end{aligned}$$

To solve the problem, we use a semi-inverse method, setting

$$\begin{aligned}a_{-1} &= a_{-2} = \ldots = 0,\\ b_{-1} &= b_{-2} = \ldots = 0,\\ \psi &= 0\end{aligned}$$

in (7.60), the last value being justified by the arbitrariness in ω as shown by (7.43). Then from (7.60) and (7.38) we see that (7.61) are satisfied if

$$a = -c\rho^2(2\mu)^{-1},$$

in which case (7.36) and (7.60) yield

$$\Gamma = -\frac{c\rho^2}{2\mu}\bar{z}^{-1}, \quad \Theta = \frac{c\rho^2}{2\mu}\ln|z| - b.$$

It is clear that this exterior Neumann problem has a unique solution in $\mathscr{A}$, corresponding to $b = -c\rho^2(2\mu)^{-1}$. If this restriction is removed, then the solution is determined up to an arbitrary vertical translation, as noted in Remark 7.7.

7.8 Physical Significance of the Restrictions

From (7.54) and (7.57) we find that, as $|z| \to \infty$, the limiting values of Φ, Ψ, and Λ are

$$\begin{aligned}\Phi_\infty &= 8h^2\mu\bar{b}_2,\\ \Psi_\infty &= 8h^2(\lambda + \mu)\mathrm{Re}a_1,\\ \Lambda_\infty &= 0;\end{aligned}$$

that is, the bending and twisting moments are uniformly distributed at infinity, whereas the transverse shear force vanishes. We can see that the second and third equations (7.59) are equivalent to $\Phi_\infty = \Psi_\infty = 0$.

In view of (1.6), the rotations in the vertical coordinate planes in $\mathbb{R}^3$ are given by

$$\varepsilon_\alpha = \tfrac{1}{2}(u_\alpha - u_{3,\alpha}).$$

From this, (7.28), (7.58), and the second and third equations (7.59) we deduce that, as $|z| \to \infty$, the complex vertical rotation is

$$\begin{aligned}\varepsilon = \varepsilon_1 + i\varepsilon_2 &= \tfrac{1}{2}(\Gamma - 2\Theta,_{\bar{z}})\\ &= 2ic\,\mathsf{M}\ln|z| - ic\,\bar{\mathsf{M}}z\bar{z}^{-1} + a_0 + \bar{b}_1 + ic\,\mathsf{M} + O(|z|^{-1}).\end{aligned}$$

Hence, the first and fourth equations (7.59) are equivalent to $\varepsilon_\infty = 0$. In this case, $\Theta = O(\ln|z|)$.

7.8 Remark. In view of the above arguments, we conclude that an analytic solution of (3.40) is of finite energy if and only if the corresponding bending and twisting moments, transverse shear force, and rotation in the vertical coordinate plane vanish at infinity. Then, by Remark 7.6, $\mathscr{A}$ is the class of all finite energy solutions of (3.40) that contain no vertical translation.

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Chapter 8
Generalized Fourier Series

8.1 The Interior Dirichlet Problem

In this chapter we suspend the convention of summation over repeated indices, as well as that regarding the values taken by Latin subscripts. Greek subscripts and superscripts continue to take the values $1,\ 2$.

8.1 Definition. Let X be a normed space. A subset $\mathscr{X} \subset X$ is called a *fundamental set in* X if span $\mathscr{X}$ is dense in X.

The following assertion is a well-known result of functional analysis.

8.2 Theorem. *If X is a Hilbert space, then $\mathscr{X} \subset X$ is a fundamental set in X if and only if the orthogonal complement of $\mathscr{X}$ in X consists of the zero vector alone.*

Let ∂S_* be a simple closed C^2-curve such that ∂S lies strictly in the domain S_*^+ enclosed by ∂S_*, and let $\{x^{(k)} \in \partial S_*,\ k = 1,\ 2,\ \ldots\}$ be a countable set of points densely distributed on ∂S_*. We set $S_*^- = \mathbb{R}^2 \setminus \bar{S}_*^+$, and denote by $D^{(i)}$ the columns of the matrix D.

8.3 Theorem. *The set*

$$\{f^{(i)},\ \theta^{(jk)},\ i,\ j = 1,\ 2,\ 3,\ k = 1,\ 2,\ \ldots\}, \tag{8.1}$$

where the $f^{(i)}$ are defined by (6.6) *and*

$$\theta^{(jk)}(x) = D^{(j)}(x,\ x^{(k)}), \tag{8.2}$$

is linearly independent on ∂S and fundamental in $L^2(\partial S)$.

Proof. Suppose that there are a positive integer N and real numbers c_i and c_{jk}, $i,\ j = 1,\ 2,\ 3, k = 1,\ 2,\ \ldots,\ N$, not all zero, such that

$$\sum_{i=1}^{3} c_i f^{(i)}(x) + \sum_{j=1}^{3} \sum_{k=1}^{N} c_{jk} \theta^{(jk)}(x) = 0, \quad x \in \partial S. \tag{8.3}$$

C. Constanda, *Mathematical Methods for Elastic Plates*,
Springer Monographs in Mathematics, DOI: 10.1007/978-1-4471-6434-0_8,

Setting

$$\varpi(x) = \sum_{i=1}^{3} c_i f^{(i)}(x) + \sum_{j=1}^{3}\sum_{k=1}^{N} c_{jk}\theta^{(jk)}(x), \tag{8.4}$$

from (8.2), (8.3), and Theorem 3.8 we see that

$$\begin{aligned} A\varpi &= 0 \quad \text{in } S^+, \\ \varpi &= 0 \quad \text{on } \partial S; \end{aligned}$$

that is, ϖ is a regular solution of the homogeneous interior Dirichlet problem. By Theorem 3.16(i), $\varpi = 0$ in $\bar{S}^+$. Then, using analyticity arguments, we deduce that

$$\varpi = 0 \quad \text{in } S_*^+. \tag{8.5}$$

Let $x^{(p)}$ be any of the (finitely many) points $x^{(1)}, \ldots, x^{(N)}$. In view of (8.4) and (8.2), we write

$$\varpi_l(x) = \sum_{i=1}^{3} c_i f_l^{(i)}(x) + \sum_{j=1}^{3}\sum_{k=1}^{N} c_{jk} D_{jl}(x, x^{(k)}), \quad l = 1, 2, 3,$$

and remark that, according to (3.30), as $x \to x^{(p)}$, all the terms on the right-hand side remain bounded except $c_{lp} D_{ll}(x, x^{(p)})$, which is of order $O(\ln|x - x^{(p)}|)$. This clearly contradicts the equality (8.5), and we conclude that all the c_{jk} in (8.3) must be zero. Since the $f^{(i)}$ are linearly independent, we deduce that the c_i are also zero. Hence, the set (8.1) is linearly independent on ∂S.

Now let $\varphi \in L^2(\partial S)$ be such that for all $i, j = 1, 2, 3$ and $k = 1, 2, \ldots,$

$$\int_{\partial S} (f^{(i)})^{\mathrm{T}} \varphi \, ds = \int_{\partial S} (\theta^{(jk)})^{\mathrm{T}} \varphi \, ds = 0. \tag{8.6}$$

By (8.2) and (3.27), this is equivalent to

$$\int_{\partial S} D(x^{(k)}, y)\varphi(y)\, ds(y) = 0, \quad k = 1, 2, \ldots, \tag{8.7}$$

$$\int_{\partial S} [\varphi_\alpha(y) - y_\alpha \varphi_3(y)]\, ds(y) = \int_{\partial S} \varphi_3(y)\, ds(y) = 0. \tag{8.8}$$

Consider the single-layer plate potential

$$(V\varphi)(x) = \int_{\partial S} D(x, y)\varphi(y)\, ds(y).$$

Since, by Theorem 4.18(i), $V\varphi$ is continuous on ∂S_* and the points $x^{(k)}$, $k = 1, 2, \dots,$ are densely distributed on ∂S_*, from (8.7) it follows that $V\varphi = 0$ on ∂S_*. In view of Theorem 4.18(ii), (iii), we have

$$\begin{aligned} A(V\varphi) &= 0 \quad \text{in } S_*^-, \\ V\varphi &= 0 \quad \text{on } \partial S_*, \\ V\varphi &\in \mathscr{A}. \end{aligned}$$

This means that $V\varphi$ is a regular solution in $\bar{S}_*^-$ of the homogeneous exterior Dirichlet problem (D$^-$); consequently, by Theorem 3.16(i), $V\varphi = 0$ in $\bar{S}_*^-$. The analyticity of the single-layer plate potential V in $\mathbb{R}^2 \backslash \partial S$ now implies that

$$V\varphi = 0 \quad \text{in } S^-. \tag{8.9}$$

In turn, this yields $(T(V\varphi))^- = 0$ in S^-. Letting $S^- \ni x' \to x \in \partial S$ along the support line of $\nu(x)$, from Theorem 4.21 we find that

$$-\tfrac{1}{2}\,\varphi(x) + \int\limits_{\partial S} T(\partial_x)D(x,\ y)\varphi(y)\,ds(y) = 0$$

for almost all $x \in \partial S$, where the integral is understood as principal value. By Theorem 6.12, $\varphi \in C^{\,0,\alpha}(\partial S)$ with any $\alpha \in (0, 1)$. Then $V\varphi$ is continuous in $\mathbb{R}^2$ and

$$\begin{aligned} A(V\varphi) &= 0 \quad \text{in } S^+, \\ V\varphi &= 0 \quad \text{on } \partial S; \end{aligned}$$

that is, $V\varphi$ is a regular solution in $\bar{S}^+$ of the homogeneous problem (D$^+$). Consequently, by Theorem 3.16(i), $V\varphi = 0$ in $\bar{S}^+$. From this and (8.9) we deduce that

$$(T(V\varphi))^+ = (T(V\varphi))^- = 0 \quad \text{on } \partial S,$$

and (4.17) yields $\varphi = 0$.

Since $L^2(\partial S)$ is a Hilbert space, we now apply Theorem 8.2 to conclude that (8.1) is a fundamental set in $L^2(\partial S)$. □

Let u be the (unique) regular solution of (D$^+$). By Theorem 3.9 and (3.43),

$$u(x) = \int\limits_{\partial S} D(x,\ y)\psi(y)\,ds(y) - H(x), \quad x \in S^+, \tag{8.10}$$

$$H(x) = \int\limits_{\partial S} D(x,\ y)\psi(y)\,ds(y), \quad x \in S^-, \tag{8.11}$$

where we have used the notation

$$H(x) = \int\limits_{\partial S} P(x,\ y)\mathscr{P}(y)\, ds(y), \quad x \in \mathbb{R}^2\backslash\partial S, \tag{8.12}$$

$$\psi(y) = (Tu)(y), \qquad y \in \partial S. \tag{8.13}$$

Formula (8.11) yields

$$\int\limits_{\partial S} D(x^{(k)},\ y)\psi(y)\, ds(y) = H(x^{(k)}), \quad k = 1,\ 2,\ \dots,$$

which, by (3.27) and (8.2), is equivalent to

$$\int\limits_{\partial S} (\theta^{(jk)})^{\mathrm{T}}\psi\, ds = H_j(x^{(k)}), \quad j = 1,\ 2,\ 3,\ k = 1,\ 2,\ \dots. \tag{8.14}$$

We arrange the elements of (8.1) in the order

$$f^{(1)},\ f^{(2)},\ f^{(3)},\ \theta^{(11)},\ \theta^{(21)},\ \theta^{(31)},\ \dots,\ \theta^{(1k)},\ \theta^{(2k)},\ \theta^{(3k)},\ \dots,$$

and denote the new sequence by $\{\theta^{(m)}\}_{m=1}^{\infty}$. Let $\{\omega^{(n)}\}_{n=1}^{\infty}$ be the orthonormalized fundamental sequence constructed from the set $\{\theta^{(m)}\}_{m=1}^{\infty}$ in $L^2(\partial S)$ by means of the Gram–Schmidt process. Then

$$\omega^{(n)} = \sum_{m=1}^{n} k_{nm}\theta^{(m)}, \quad n = 1,\ 2,\ \dots,$$

where k_{nm} are well-determined numbers. Writing

$$\psi^{(n)} = \sum_{r=1}^{n} p_r\omega^{(r)}, \quad n = 1,\ 2,\ \dots, \tag{8.15}$$

with the coefficients on the right-hand side given by

$$p_r = \int\limits_{\partial S} (\omega^{(r)})^{\mathrm{T}}\psi\, ds = \sum_{m=1}^{r} k_{rm} \int\limits_{\partial S} (\theta^{(m)})^{\mathrm{T}}\psi\, ds, \quad r = 1,\ 2,\ \dots, \tag{8.16}$$

and setting

$$u^{(n)}(x) = \int\limits_{\partial S} D(x,\ y)\psi^{(n)}(y)\, ds(y) - H(x), \quad x \in S^+, \tag{8.17}$$

from (8.10) we see that for $x \in S^+$,

$$\begin{aligned}
\left|u(x) - u^{(n)}(x)\right| &\le \sum_{i=1}^{3} \left|u_i(x) - u_i^{(n)}(x)\right| \\
&\le \sum_{i=1}^{3} \int_{\partial S} \left|\left(D^{(i)}(y,\,x)\right)^{\mathrm{T}}\left[\psi(y) - \psi^{(n)}(y)\right]\right| ds(y) \\
&\le \sum_{i=1}^{3} \|D^{(i)}(x,\cdot)\|_2 \|\psi - \psi^{(n)}\|_2.
\end{aligned}$$

Since the $\|D^{(i)}(x,\cdot)\|_2$ are uniformly bounded on any closed subdomain $S' \subset S^+$ and $\|\psi - \psi^{(n)}\|_2 \to 0$ as $n \to \infty$, we conclude that $u^{(n)} \to u$, uniformly on S'.

Clearly, each $u^{(n)}$ is a solution of the equation $Au = 0$ in S^+.

8.4 Remark. According to (8.13), u is a regular solution of the interior Neumann problem

$$\begin{aligned}
Au &= 0 \quad \text{in } S^+, \\
Tu &= \psi \quad \text{on } \partial S;
\end{aligned}$$

therefore, by Theorem 6.6,

$$\int_{\partial S} (f^{(i)})^{\mathrm{T}} \psi \, ds = 0, \quad i = 1,\, 2,\, 3, \tag{8.18}$$

which is equivalent to (6.13). Since $\theta^{(i)} = f^{(i)}$, $i = 1,\, 2,\, 3$, from (8.16) and (8.18) it now follows that

$$\begin{aligned}
&p_1 = p_2 = p_3 = 0, \\
&p_r = \sum_{m=4}^{r} k_{rm} \int_{\partial S} (\theta^{(m)})^{\mathrm{T}} \psi \, ds, \quad r = 4,\, 5,\, \dots .
\end{aligned} \tag{8.19}$$

Hence, the approximate solution $u^{(n)}$ is given by (8.17), where H and $\psi^{(n)}$ are given by (8.12) and (8.15), respectively, with the p_r as in (8.19) and fully determined for $r = 4,\, 5,\, \dots$ by (8.12) and (8.14).

8.2 The Interior Neumann Problem

With the notation introduced in the preceding section, we can prove the following assertion.

8.5 Theorem. *The set*

$$\{f^{(i)},\ \vartheta^{(jk)},\ i,\ j=1,\ 2,\ 3,\ k=1,\ 2,\ \ldots\},\tag{8.20}$$

where the $f^{(i)}$ *are defined by* (6.6) *and*

$$\vartheta^{(jk)}(x)=T(\partial_x)D^{(j)}(x,\ x^{(k)}),\tag{8.21}$$

is linearly independent on ∂S *and fundamental in* $L^2(\partial S)$.

Proof. As in the proof of Theorem 8.3, suppose that there are a positive integer N and real numbers c_i and c_{jk}, $i,\ j=1,\ 2,\ 3,\ k=1,\ 2,\ \ldots,\ N$, not all zero, such that

$$\sum_{i=1}^{3}c_i f^{(i)}(x)+\sum_{j=1}^{3}\sum_{k=1}^{N}c_{jk}\vartheta^{(jk)}(x)=0,\quad x\in\partial S.\tag{8.22}$$

Then, taking (8.21), (8.22), and Theorem 3.8 into consideration, we find that the 3×1 matrix

$$\varpi(x)=\sum_{j=1}^{3}\sum_{k=1}^{N}c_{jk}D^{(j)}(x,\ x^{(k)})\tag{8.23}$$

is a regular solution of the interior Neumann problem

$$A\varpi=0\quad\text{in } S^+,$$
$$T\varpi=-\sum_{i=1}^{3}c_i f^{(i)}\quad\text{on } \partial S;$$

consequently, by (6.13),

$$\int_{\partial S}(f^{(l)})^{\mathrm{T}}\Big[-\sum_{i=1}^{3}c_i f^{(i)}\Big]\,ds=0,\quad l=1,\ 2,\ 3,$$

which implies that the coefficients c_1, c_2, and c_3 are all equal to zero. This yields

$$T\varpi=0\quad\text{on } \partial S,$$

so, by Theorem 3.16(ii),

$$\varpi=\sum_{i=1}^{3}\beta_i f^{(i)}\quad\text{in } S^+$$

for some constants β_i, $i = 1, 2, 3$. From this and (8.23) it follows that

$$\tilde{\varpi}(x) = \sum_{j=1}^{3} \sum_{k=1}^{N} c_{jk} D^{(j)}(x, x^{(k)}) - \sum_{i=1}^{3} \beta_i f^{(i)}(x) = 0, \quad x \in \bar{S}^{+}.$$

By analyticity, $\tilde{\varpi} = 0$ in $\bar{S}_*^{+}$, and the linear independence of the set (8.20) on ∂S is established by means of the argument used in the proof of Theorem 8.3.

Suppose now that for all $i, j = 1, 2, 3$ and $k = 1, 2, \dots$ the function $\varphi \in L^2(\partial S)$ satisfies

$$\int_{\partial S} (f^{(i)})^{\mathrm{T}} \varphi \, ds = \int_{\partial S} (\vartheta^{(jk)})^{\mathrm{T}} \varphi \, ds = 0.$$

According to (8.21) and (3.28), this means that

$$\int_{\partial S} \left[\varphi_\alpha(y) - y_\alpha \varphi_3(y)\right] ds(y) = \int_{\partial S} \varphi_3(y) \, ds(y) = 0, \tag{8.24}$$

$$\int_{\partial S} P(x^{(k)}, y)\varphi(y) \, ds(y) = 0, \quad k = 1, 2, \dots. \tag{8.25}$$

By Theorem (4.18)(i), the double-layer plate potential

$$(W\varphi)(x) = \int_{\partial S} P(x, y)\varphi(y) \, ds(y)$$

is continuous on ∂S_*. Since the $x^{(k)}$ are densely distributed on ∂S_*, from (8.25) we deduce that $W\varphi = 0$ on ∂S_*. Then, by Theorem 4.18(ii), (iii), $W\varphi$ is a regular solution of the exterior Dirichlet problem

$$\begin{aligned} A(W\varphi) &= 0 \quad \text{in } S_*^{-}, \\ W\varphi &= 0 \quad \text{on } \partial S_*, \\ W\varphi &\in \mathscr{A}; \end{aligned}$$

hence, by Theorem 3.16(i), $W\varphi = 0$ in $\bar{S}_*^{-}$. The analyticity of $W\varphi$ in $\mathbb{R}^2 \backslash \partial S$ now yields $W\varphi = 0$ in S^{-}. Letting $S^{-} \ni x' \to x \in \partial S$ along the support line of $\nu(x)$, from Theorem 4.20 we find that

$$\tfrac{1}{2}\varphi(x) + \int_{\partial S} P(x, y)\varphi(y) \, ds(y) = 0$$

for almost all $x \in \partial S$, where the integral is understood as principal value. By Theorem 6.12, $\varphi \in C^{0,\alpha}(\partial S)$ for any $\alpha \in (0, 1)$, which, in view of Theorem 6.3, implies that

$$\varphi = \sum_{i=1}^{3} \gamma_i f^{(i)} \quad \text{on } \partial S, \quad \gamma_i = \text{const} > 0.$$

Then, by (8.24),

$$\int_{\partial S} \left[(f^{(l)})^{\mathrm{T}} \sum_{i=1}^{3} \gamma_i f^{(i)} \right] ds = 0, \quad l = 1, 2, 3,$$

and we conclude that all the γ_i are zero; that is, $\varphi = 0$. The desired result now follows from Theorem 8.2. □

Let u be a regular solution of (N^+). By Theorem 3.9 and (3.44),

$$u(x) = -\int_{\partial S} P(x, y)\rho(y)\, ds(y) + L(x), \quad x \in S^+, \tag{8.26}$$

$$L(x) = \int_{\partial S} P(x, y)\rho(y)\, ds(y), \quad x \in S^-, \tag{8.27}$$

where

$$L(x) = \int_{\partial S} D(x, y)\mathscr{Q}(y)\, ds(y), \quad x \in \mathbb{R}^2 \backslash \partial S, \tag{8.28}$$

$$\rho(x) = u(x), \quad x \in \partial S.$$

We rearrange the elements of the subset $\{\vartheta^{(jk)},\ j = 1, 2, 3,\ k = 1, 2, \dots\}$ of (8.20) in the order

$$\vartheta^{(11)}, \vartheta^{(21)}, \vartheta^{(31)}, \dots, \vartheta^{(1k)}, \vartheta^{(2k)}, \vartheta^{(3k)}, \dots,$$

denote the new sequence by $\{\vartheta^{(m)}\}_{m=1}^{\infty}$, and use the Gram–Schmidt process to construct the orthonormal sequence $\{\eta^{(n)}\}_{n=1}^{\infty}$ in $L^2(\partial S)$; thus,

$$\eta^{(n)} = \sum_{m=1}^{n} \kappa_{nm} \vartheta^{(m)}, \quad n = 1, 2, \dots, \tag{8.29}$$

where κ_{nm} are well-determined numerical coefficients. Also, let $\{\tilde{f}^{(i)}\}_{i=1}^{3}$ be the orthonormalized set obtained from $\{f^{(i)}\}_{i=1}^{3}$.

We claim that $\{\tilde{f}^{(i)},\ \eta^{(n)},\ i = 1, 2, 3,\ n = 1, 2, \ldots\}$ is a fundamental orthonormal set in $L^2(\partial S)$. To convince ourselves of this, we need to verify only that

$$\int_{\partial S} (\tilde{f}^{(i)})^{\mathrm{T}} \eta^{(n)}\, ds = 0, \quad i = 1, 2, 3,\ n = 1, 2, \ldots.$$

But this is obviously true, since the $\tilde{f}^{(i)}$ and the $\eta^{(n)}$ are finite linear combinations of the $f^{(i)}$ and $\vartheta^{(jk)}$, respectively, and, by (6.6), (8.21), and Theorems 3.5 and 3.8,

$$\begin{aligned}
\int_{\partial S} (f^{(\alpha)})^{\mathrm{T}} \vartheta^{(jk)}\, ds &= \int_{\partial S} (T_{\alpha l} - x_\alpha T_{3l}) D_l^{(j)}(x,\ x^{(k)})\, ds \\
&= \int_{S^+} (A_{\alpha l} - x_\alpha A_{3l}) D_l^{(j)}(x,\ x(k))\, da = 0, \\
\int_{\partial S} (f^{(3)})^{\mathrm{T}} \vartheta^{(jk)}\, ds &= \int_{\partial S} T_{3l} D_l^{(j)}(x,\ x^{(k)}) \\
&= \int_{S^+} A_{3l} D_l^{(j)}(x,\ x^{(k)})\, da = 0.
\end{aligned}$$

Without loss of generality, suppose that $n > 3$, and let

$$\rho^{(n)} = \sum_{i=1}^{3} \tilde{q}_i \tilde{f}^{(i)} + \sum_{r=1}^{n-3} q_r \eta^{(r)}, \tag{8.30}$$

where

$$\begin{aligned}
\tilde{q}_i &= \int_{\partial S} (\tilde{f}^{(i)})^{\mathrm{T}} \rho\, ds, \quad i = 1, 2, 3, \\
q_r &= \int_{\partial S} (\eta^{(r)})^{\mathrm{T}} \rho\, ds = \sum_{m=1}^{r} \kappa_{rm} \int_{\partial S} (\vartheta^{(m)})^{\mathrm{T}} \rho\, ds, \quad r = 1, 2, \ldots.
\end{aligned} \tag{8.31}$$

Setting

$$u^{(n)}(x) = -\int_{\partial S} P(x,\ y) \rho^{(n)}(y)\, ds(y) + L(x), \quad x \in S^+, \tag{8.32}$$

and using (8.26), just as in Sect. 8.1 we find that $u^{(n)} \to u$ as $n \to \infty$, uniformly on any closed subdomain $S' \subset S^+$.

From (8.27) it follows that

$$\int\limits_{\partial S} P(x^{(k)},\ y)\rho(y)\,ds(y) = L(x^{(k)}), \quad k = 1,\ 2,\ \dots .$$

By (3.28) and (8.21), this is the same as

$$\int\limits_{\partial S} (\vartheta^{(jk)})^{\mathrm{T}}\rho\,ds = L_j(x^{(k)}), \quad j = 1,\ 2,\ 3,\ k = 1,\ 2,\ \dots . \tag{8.33}$$

Applying Theorem 3.9 to $\tilde{f}^{(i)}$ in S^+, from (8.32) and (8.30) we now obtain the approximate solution in the form

$$u^{(n)}(x) = \sum_{i=1}^{3} \tilde{q}_i \tilde{f}^{(i)}(x) - \sum_{r=1}^{n-3} q_r \int\limits_{\partial S} P(x,\ y)\eta^{(r)}(y)\,ds(y) + L(x), \quad x \in S^+,$$

where the first term on the right-hand side is a rigid displacement independent of n, $L(x)$ is given by (8.28), $\eta^{(r)}$ by (8.29), and the q_r are computed by means of (8.31), (8.33), and (8.28). Since the coefficients $\tilde{q}_i$ cannot be found in terms of the boundary data of the problem, we conclude that, in agreement with Theorem 6.6, the exact solution is determined in the limit up to an arbitrary rigid displacement.

8.3 The Exterior Dirichlet Problem

The construction of a fundamental sequence in the space of the solution for exterior problems meets with the usual difficulties that arise from the behavior of the matrices $D(x,\ y)$ and $P(x,\ y)$ for $y \in \partial S$ and $|x|$ large. To overcome these obstacles, we need to establish some auxiliary results.

Let S be a finite domain in $\mathbb{R}^2$, and let $\mathscr{G}$ be a linear functional on $C(\partial S)$.

8.6 Theorem. *Consider the* 3×1 *vector function*

$$\chi(x) = \mathscr{G}_y(D(x,\ y)\varphi(y)), \quad \varphi \in X,\ x \in S^-, \tag{8.34}$$

where the subscript y *indicates that* $\mathscr{G}$ *is applied to its argument regarded as a function of the point* y. *Then* $\chi \in \mathscr{A}$ *if*

$$\begin{aligned} \tau_\alpha &= \mathscr{G}_y(\varphi_\alpha(y) - y_\alpha\varphi_3(y)) = 0,\\ \tau_3 &= \mathscr{G}_y(\varphi_3(y)) = 0. \end{aligned} \tag{8.35}$$

Proof. We define

$$
\begin{aligned}
D_{11}^{\infty}(x,\ y) &= -a_2\mu^2(2\ln r + 2 + \cos 2\theta),\\
D_{22}^{\infty}(x,\ y) &= -a_2\mu^2(2\ln r + 2 - \cos 2\theta),\\
D_{33}^{\infty}(x,\ y) &= a_2\mu\big[\mu r^2\ln r - 4h^2(\lambda+2\mu)\ln r - 4h^2(\lambda+3\mu)\big]\\
&\quad - a_2\mu\big\{y_1\big[\mu r(2\ln r+1) - 4h^2(\lambda+2\mu)r^{-1}\big]\cos\theta\\
&\qquad + y_2\big[\mu r(2\ln r+1) - 4h^2(\lambda+2\mu)r^{-1}\big]\sin\theta\big\},\\
D_{12}^{\infty}(x,\ y) &= D_{21}^{\infty}(x,\ y) = -a_2\mu^2\sin 2\theta,\\
D_{13}^{\infty}(x,\ y) &= -a_2\mu^2\big[r(2\ln r+1)\cos\theta\\
&\qquad - y_1(2\ln r + 2 + \cos 2\theta) - y_2\sin 2\theta\big],\\
D_{23}^{\infty}(x,\ y) &= -a_2\mu^2\big[r(2\ln r+1)\sin\theta\\
&\qquad - y_2(2\ln r + 2 - \cos 2\theta) - y_1\sin 2\theta\big],\\
D_{31}^{\infty}(x,\ y) &= a_2\mu\big[\mu r(2\ln r+1) - 4h^2(\lambda+2\mu)r^{-1}\big]\cos\theta,\\
D_{32}^{\infty}(x,\ y) &= a_2\mu\big[\mu r(2\ln r+1) - 4h^2(\lambda+2\mu)r^{-1}\big]\sin\theta,
\end{aligned}
\tag{8.36}
$$

where $(r,\ \theta)$ are the polar coordinates of x, and, for $|x|$ large, write (8.34) in the form

$$
\begin{aligned}
\chi(x) &= \mathscr{G}_y(D^{\infty}(x,\ y)\varphi(y)) + \mathscr{G}_y((D(x,\ y) - D^{\infty}(x,\ y))\varphi(y))\\
&= \chi^{\infty}(x) + \tilde{\chi}(x).
\end{aligned}
$$

Using (3.21), (3.23), (3.25), (3.36), (3.37), and (8.36), we find by direct calculation that $\tilde{\chi} \in \mathscr{A}$ and that

$$
\begin{aligned}
\chi_1^{\infty}(x) &= -a_2\mu^2\big[\tau_3 r(2\ln r+1)\cos\theta\\
&\qquad + \tau_1(2\ln r + 2 + \cos 2\theta) + \tau_2\sin 2\theta\big],\\
\chi_2^{\infty}(x) &= -a_2\mu^2\big[\tau_3 r(2\ln r+1)\sin\theta\\
&\qquad + \tau_2(2\ln r + 2 - \cos\theta) + \tau_1\sin 2\theta\big],\\
\chi_3^{\infty}(x) &= a_2\mu\tau_3\big[\mu r^2\ln r - 4h^2(\lambda+3\mu)\big]\\
&\quad + a_2\mu(\tau_1\cos\theta + \tau_2\sin\theta)\big[\mu r(2\ln r+1) - 4h^2(\lambda+2\mu)r^{-1}\big],
\end{aligned}
$$

which means that $\chi^{\infty} = 0$ if (8.35) hold. □

8.7 Remark. Obviously, Theorem 4.2(ii) is a particular case of Theorem 8.6 with $\mathscr{G}$ defined on $C(\partial S)$ by

$$\mathscr{G}\sigma = \int_{\partial S} \sigma\, ds.$$

8.8 Theorem. *For any (fixed)* $y \in \partial S$,
(i) $A(\partial_x)D^\infty(x, y) = 0$, $x \in S^-$;
(ii) *the columns of* $D - D^\infty$ *belong to* $\mathscr{A}$.

Proof. (i) We can easily convince ourselves that the columns of D^∞ are generated by (7.28) and (7.36) with $l = m = 0$, $\psi = 0$, and Ω and ω given, respectively, by

$$\begin{aligned}
&\Omega(z) = -a_2\mu^2(\log z + 1), \quad \omega(z) = -a_2\mu^2 z \log z,\\
&\Omega(z) = -ia_2\mu^2(\log z + 1), \quad \omega(z) = ia_2\mu^2 z \log z,\\
&\Omega(z) = -a_2\mu^2\big[(z - \zeta)\log z - \zeta\big], \quad \omega(z) = a_2\mu^2(\bar{\zeta} z \log z + 4h^2),
\end{aligned}$$

where $z = x_1 + ix_2$ and $\zeta = y_1 + iy_2$.

(ii) This assertion is proved by computing the entries of the matrix $D - D^\infty$ explicitly and verifying that its columns exhibit the far-field pattern (3.39) stipulated in the definition of the class $\mathscr{A}$. □

Let the curve ∂S_* now be chosen so that it lies strictly inside the domain S^+.

8.9 Theorem. The set (8.1), constructed as in Theorem 8.3, is linearly independent on ∂S and fundamental in $L^2(\partial S)$.

Proof. Suppose that there are a positive integer N and real numbers c_i and c_{jk}, $i, j = 1, 2, 3, k = 1, 2, \dots, N$, not all zero, such that (8.3) holds, and let ϖ again be defined by (8.4). Then grad $\varpi_i = 0$ on ∂S, $i = 1, 2, 3$. Since $\varpi \in C^1(S_*^-)$, from expression (3.11) we immediately see that

$$T\varpi = 0 \quad \text{on } \partial S. \tag{8.37}$$

Using the representation

$$\varpi = \varpi^\infty + \tilde{\varpi} + \sum_{i=1}^{3} c_i f^{(i)}, \tag{8.38}$$

where the functions ϖ^∞ and $\tilde{\varpi}$ are defined by means of the columns of the matrices D^∞ and $D - D^\infty$, respectively, from Theorem 8.4 and (8.37) we deduce that $\tilde{\varpi}$ is a regular solution of the exterior Neumann problem

$$\begin{aligned}
A\tilde{\varpi} &= 0 \quad \text{in } S^-,\\
T\tilde{\varpi} &= -T\varpi^\infty \quad \text{on } \partial S,\\
\tilde{\varpi} &\in \mathscr{A}.
\end{aligned}$$

According to Theorem 6.5(ii),

$$\int_{\partial S} (f^{(i)})^{\mathrm{T}} T\varpi^\infty \, ds = 0, \quad i = 1, 2, 3.$$

Consider a circle Γ_R with the center at the origin and radius R sufficiently large so that $\bar{S}^+ \subset \Gamma_R$ strictly. By Theorem 3.7 applied to $f^{(i)}$ and ϖ^∞ in $\Gamma_R \backslash \bar{S}^+$, the above equality yields

$$\int\limits_{\partial \Gamma_R} (f^{(i)})^{\mathrm{T}} T \varpi^\infty \, ds = 0, \quad i = 1, 2, 3.$$

Direct calculation now shows that these relations are equivalent to

$$\begin{aligned} \sum_{k=1}^{N} (c_{\alpha k} - x_\alpha^{(k)} c_{3k}) &= 0, \\ \sum_{k=1}^{N} c_{3k} &= 0. \end{aligned} \tag{8.39}$$

Let $\mathscr{G}$ be the linear functional defined on $C(\partial S)$ by

$$\mathscr{G}\sigma = \sum_{k=1}^{N} \sigma(x^{(k)}), \quad \sigma \in C(\partial S),$$

and let $\varphi_c \in C(\partial S)$ be such that

$$\varphi_c(x^{(k)}) = (c_{1k}, c_{2k}, c_{3k})^{\mathrm{T}}, \quad k = 1, 2, \ldots, N.$$

Then

$$\begin{aligned} \sum_{j=1}^{3} \sum_{k=1}^{N} c_{jk} \theta^{(jk)}(x) &= \sum_{j=1}^{3} \sum_{k=1}^{N} c_{jk} D^{(j)}(x, x^{(k)}) \\ &= \sum_{k=1}^{N} D(x, x^{(k)}) \varphi_c(x^{(k)}) \\ &= \mathscr{G}_y(D(x, y)\varphi_c(y)). \end{aligned}$$

In view of this and the definition of $\mathscr{G}$, (8.39) are equivalent to (8.35); therefore, by Theorem 8.6 and (8.38), $\varpi \in \mathscr{A}^*$. From (8.3) and (8.4) we then see that ϖ is the regular solution in S^- of the homogeneous Dirichlet problem

$$\begin{aligned} A\varpi &= 0 \quad \text{in } S^-, \\ \varpi &= 0 \quad \text{on } \partial S, \\ \varpi &\in \mathscr{A}^*. \end{aligned}$$

By Theorem 3.16(i), $\varpi = 0$ in $\bar{S}^-$. Due to the analyticity of ϖ, we have $\varpi = 0$ in S_*^-, and the linear independence of the set (8.1) on ∂S is established by the argument used in the proof of Theorem 8.3.

Suppose now that equalities (8.6) hold for some $\varphi \in L^2(\partial S)$. Since the points $x^{(k)}$ are densely distributed on ∂S_*, we deduce from Theorem 4.18(i), (ii) that the single-layer potential V of density φ is a regular solution of the homogeneous interior Dirichlet problem

$$\begin{aligned} A(V\varphi) &= 0 \quad \text{in } S_*^+,\\ V\varphi &= 0 \quad \text{on } \partial S_*; \end{aligned}$$

hence, by Theorem 3.16(i), $V\varphi = 0$ in $\bar{S}_*^+$. Due to the analyticity of $V\varphi$ in $\mathbb{R}^2\backslash\partial S$, we conclude that $V\varphi = 0$ in S^+, and so, $(T(V\varphi))^+ = 0$ in S^+. Letting $x' \in S^+$ tend to $x \in \partial S$ along the support line of $\nu(x)$, we apply Theorem 4.21 to obtain the equation

$$\tfrac{1}{2}\varphi(x) + \int\limits_{\partial S} T(\partial_x)D(x,\ y)\varphi(y)\,ds(y) = 0$$

for almost all $x \in \partial S$, the integral being understood as principal value. Theorem 6.12 now indicates that $\varphi \in C^{0,\alpha}(\partial S)$, with any $\alpha \in (0, 1)$. Hence, $V\varphi \in C(\mathbb{R}^2)$, which means that $V\varphi = 0$ on ∂S.

The first three equalities in (8.6) are equivalent to $p\varphi = 0$, where p is the functional defined in (4.3). Then, by Theorem 4.2(ii), $V\varphi \in \mathscr{A}$, so $V\varphi$ is a regular solution of the homogeneous Dirichlet problem

$$\begin{aligned} A(V\varphi) &= 0 \quad \text{in } S^-,\\ V\varphi &= 0 \quad \text{on } \partial S,\\ V &\in \mathscr{A}; \end{aligned}$$

hence, by Theorem 3.16(i), $V\varphi = 0$ in S^-. This implies that $(T(V\varphi))^- = 0$, and, by (4.17), $\varphi = 0$. As in the proof of Theorem 8.3, we finally deduce that (8.1) is a fundamental set in $L^2(\partial S)$. □

8.10 Remark. In classical three-dimensional elasticity (Kupradze et al. 1979) there is no need for the $f^{(i)}$ to be included in the set (8.1).

Let u be the (unique) regular solution of (D^-). According to Theorem 6.7, we can write

$$u = \tilde{u} + \sum_{i=1}^{3} c_i f^{(i)}, \tag{8.40}$$

where $\tilde{u} \in \mathscr{A}$ and, as shown in the proof of that theorem,

$$c_i = \int\limits_{\partial S} (g^{(i)})^{\mathrm{T}} \mathscr{R}\, ds.$$

By (3.45) and Theorem 3.12 applied to $\tilde{u}$,

$$\tilde{u}(x) = -\int_{\partial S} D(x, y)\psi(y)\, ds(y) + H(x), \quad x \in S^-,$$

$$H(x) = \int_{\partial S} D(x, y)\psi(y)\, ds(y), \quad x \in S^+,$$

where

$$H(x) = \int_{\partial S} P(x, y)\Big[\mathscr{R}(y) - \sum_{i=1}^{3} c_i f^{(i)}(y)\Big]\, ds(y), \quad x \in \mathbb{R}^2 \backslash \partial S, \tag{8.41}$$

$$\psi(y) = (T\tilde{u})(y), \quad y \in \partial S.$$

Since, by (8.40) and (8.41), $\tilde{u}$ is a regular solution of the exterior Neumann problem

$$A\tilde{u} = 0 \quad \text{in } S^-,$$

$$T\tilde{u} = \psi \quad \text{on } \partial S,$$

$$\tilde{u} \in \mathscr{A},$$

from Theorem 6.5(ii) it follows that

$$\int_{\partial S} (f^{(i)})^{\mathrm{T}} \psi\, ds = 0, \quad i = 1, 2, 3,$$

which is equivalent to (6.12). This fact allows us now to proceed as in Sect. 8.1 and construct a similar scheme for the approximation of $\tilde{u}$.

8.4 The Exterior Neumann Problem

Let the curve ∂S_* and the points $x^{(k)}$ be as described in Sect. 8.3.

8.11 Theorem. *The set*

$$\{\vartheta^{(jk)}, \ j = 1, 2, 3, \ k = 1, 2, \dots\}, \tag{8.42}$$

where the $\vartheta^{(jk)}$ are defined by (8.21), is linearly independent on ∂S and fundamental in $L^2(\partial S)$.

Proof. Suppose that there are a positive integer N and real numbers c_{jk}, $j = 1, 2, 3$, $k = 1, 2, \dots, N$, not all zero, such that

$$\sum_{j=1}^{3}\sum_{k=1}^{N} c_{jk}\vartheta^{(jk)}(x) = 0, \quad x \in \partial S.$$

Representing ϖ defined by (8.23) in the form $\varpi = \varpi^{\infty} + \tilde{\varpi}$, where ϖ^{∞} and $\tilde{\varpi}$ are constructed in terms of D^{∞} and $D - D^{\infty}$, respectively, just as in the proof of Theorem 8.9 (this time with $\varpi \in \mathscr{A}$) we deduce that the set (8.42) is linearly independent on ∂S.

An argument similar to that used in the proof of Theorem 8.5 now shows that if

$$\int\limits_{\partial S} (\vartheta^{(jk)})^{\mathrm{T}}\varphi \, ds = 0, \quad j = 1, 2, 3, \; k = 1, 2, \dots,$$

for some $\varphi \in L^2(\partial S)$, then the double-layer potential W of density φ satisfies $W = 0$ in S^+. Hence, as $S^+ \ni x' \to x \in \partial S$ along the support line of $\nu(x)$, Theorem 4.20 yields

$$-\tfrac{1}{2}\varphi(x) + \int\limits_{\partial S} P(x, y)\varphi(y)\, ds(y) = 0$$

for almost all $x \in \partial S$, where the integral is understood in the sense of principal value. By Theorems 6.12 and 6.3, $\varphi(x) = 0$, $x \in \partial S$.

The fact that (8.42) is a fundamental set in $L^2(\partial S)$ now follows directly from Theorem 8.2. □

The generalized Fourier series approximation $u^{(n)}$ of the (unique) regular solution u of (N$^-$) is constructed just as in Sect. 8.2, the procedure being simplified here by the absence of the rigid displacements $f^{(i)}$ in (8.42).

8.5 Numerical Example

This illustration is based on the solution in S^+ of the homogeneous system (3.8), generated by (7.36) with $\psi = 0$, $\Omega = z^2$, $\omega = z + 1$, and $l = m = 0$. Below, we use the procedure described in Sect. 8.1 to reconstruct the solution from its values on the boundary ∂S.

Consider the interior Dirichlet problem for a disk with $h = 0.5$ and $\lambda = \mu = 1$, where ∂S is the unit circle centered at the origin and the boundary conditions are

$$\mathscr{P}_1(x) = 2(x_1^2 + 1), \quad \mathscr{P}_2(x) = 2x_1x_2, \quad \mathscr{P}_3(x) = 4x_1 - 1, \quad x \in \partial S.$$

Let ∂S_* be the circle concentric with ∂S and of radius $r_o > 1$. We introduce polar coordinates with the pole at the origin and choose the points $x^{(k)}$, $k = 1, 2, \ldots,$ on ∂S_* to be those corresponding to the polar angles

$$0,\ \pi,\ \tfrac{1}{2}\pi,\ \tfrac{3}{2}\pi,\ \tfrac{1}{4}\pi,\ \tfrac{3}{4}\pi,\ \tfrac{5}{4}\pi,\ \tfrac{7}{4}\pi,\ \ldots,$$

in this order. Obviously, the set $\{x^{(k)}\}_{k=1}^{\infty}$ is densely distributed on ∂S_*.

The approximation scheme described in Sect. 8.1 has been implemented on a PC with the *Mathematica*® software. Since for this type of problem the default numerical integration technique in *Mathematica*® takes inordinately long to execute and, additionally, does not cope well with zero intermediate results, the integrals over ∂S have been evaluated by means of Simpson's rule with $n_s = 36$ equal 'strips'. Consequently, all the functions defined on ∂S have been discretized at the Simpson nodes, and the approximate values of the solution have been computed at a few specified individual points $x^{(i)}$ in S^+.

8.12 Remark. In the Gram–Schmidt method applied to the subset $\{\theta^{(m)}\}_{m=1}^{n_v}$ consisting of the first n_v elements of the complete set $\{\theta^{(m)}\}_{m=1}^{\infty}$, we begin by constructing the orthogonal vector functions

$$\begin{aligned} \Omega^{(1)} &= \theta^{(1)}, \\ \Omega^{(m)} &= \theta^{(m)} - \sum_{q=1}^{m-1} \frac{\langle \theta^{(m)},\ \Omega^{(q)} \rangle_2}{\|\Omega^{(q)}\|_2^2}\, \Omega^{(q)}, \quad m = 2, \ldots, n_v, \end{aligned} \tag{8.43}$$

and then the orthonormalized vector functions

$$\omega^{(m)} = \frac{1}{\|\Omega^{(m)}\|_2}\, \Omega^{(m)}, \quad m = 1, \ldots, n_v, \tag{8.44}$$

where $\langle \cdot\,, \cdot \rangle_2$ and $\|\cdot\|_2$ are the inner product and norm on $L^2(\partial S)$. Combining (8.43) and (8.44), we can also write

$$\Omega^{(m)} = \theta^{(m)} - \sum_{q=1}^{m-1} \langle \theta^{(m)},\ \omega^{(q)} \rangle_2 \omega^{(q)}, \quad m = 2, \ldots, n_v,$$

and

$$\omega^{(m)} = \frac{1}{\|\Omega^{(m)}\|_2} \left\{ \theta^{(m)} + \sum_{q=1}^{m-1} l_{mq} \omega^{(q)} \right\}, \quad m = 2, \ldots, n_v, \tag{8.45}$$

where

$$l_{mq} = -\langle \theta^{(m)},\ \omega^{(q)} \rangle_2.$$

Expressing the Gram–Schmidt transformation as

$$\omega^{(m)} = \sum_{q=1}^{m} k_{mq}\theta^{(q)}, \quad m = 1, \dots, n_{\mathrm{v}}, \tag{8.46}$$

from (8.45) and (8.46) we see that

$$k_{mm} = \frac{1}{\|\Omega^{(m)}\|_2}, \quad m = 1, \dots, n_{\mathrm{v}},$$

and

$$\begin{aligned}\omega^{(m)} &= k_{mm}\sum_{q=1}^{m-1} l_{mq}\omega^{(q)} + k_{mm}\theta^{(m)}\\ &= \sum_{q=1}^{m-1} k_{mm}l_{mq}\Bigl(\sum_{s=1}^{q} k_{qs}\theta^{(s)}\Bigr) + k_{mm}\theta^{(m)}, \quad m = 2, \dots, n_{\mathrm{v}}.\end{aligned}$$

These equalities permit us to compute all the coefficients k_{mq}, $q = 1, \dots, m$, $m = 1, \dots, n_{\mathrm{v}}$. Obviously, $k_{mq} = 0$ for $q = m+1, \dots, n_{\mathrm{v}}$.

For the sake of symmetry, in our approximating procedure we work with the first $n_{\mathrm{o}} = 32 = 2^5$ points $x^{(k)}$ on ∂S_*, so the circle is fully traversed five times. Since the first three elements in the sequence (8.1) are the rigid displacements, the number of vector functions $\theta^{(m)}$ (hence, also $\omega^{(m)}$) available after each of these passages is $9,\ 15,\ 27,\ 51$, and 99.

The data compiled in Tables 8.1–8.17 are the values of the exact solution u and the approximate solution $u^{(m)}$ at selected points $x^{(\mathrm{i})} \in S^+$ designated by their polar coordinates, for $m = 9,\ 15,\ 27,\ 51,\ 99$ and various radii r_{o} of ∂S_*. The exact errors $|u - u^{(m)}|$ at these points are also computed, where

$$|u - u^{(m)}| = \bigl(|u_1 - u_1^{(m)}|,\ |u_2 - u_2^{(m)}|,\ |u_3 - u_3^{(m)}|\bigr)^{\mathrm{T}}.$$

8.13 Remark. Tables 8.1, 8.2, 8.3, and 8.4 show the results at four points $x^{(\mathrm{i})}$ located on different radial directions at an increasing distance from the center of the disk toward its boundary, for $r_{\mathrm{o}} = 2$ and $r_{\mathrm{o}} = 4$, respectively (see Remark 8.14(ii)). The numbers in these tables lead to several conclusions.

(i) The approximation worsens as we get closer to the boundary. This is explained by the fact that the matrix functions D and P are singular on ∂S.

(ii) The smallest errors are yielded by $u^{(51)}$ and $u^{(99)}$, with the former having a slight 'edge' over the latter for both $r_{\mathrm{o}} = 2$ and $r_{\mathrm{o}} = 4$.

(iii) The approximation is better for $r_{\mathrm{o}} = 4$ at the points closer to the center, and for $r_{\mathrm{o}} = 2$ at the points farther away from from it.

Table 8.1 The values of $u^{(m)}$ and u at selected points $x^{(i)}$ in S^+ for $r_o = 2$

$x^{(i)}$	$(0.05,\ 0)$	$(0.3,\ \pi/4)$	$(0.6,\ \pi/2)$	$(0.9,\ 3\pi/4)$
$u^{(9)}$	0.909099	1.093230	1.416320	2.332110
	0.000000	0.065845	0.000034	−0.361680
	−0.756856	0.011257	−1.000300	−3.434730
$u^{(15)}$	1.020470	1.190750	1.369730	2.311760
	0.000000	0.098177	0.000037	−0.642597
	−0.750471	0.039260	−1.000040	−3.525950
$u^{(27)}$	1.007360	1.179880	1.359420	2.325780
	0.000000	0.089996	0.000030	−0.606286
	−0.750132	0.041542	−1.000040	−3.559670
$u^{(51)}$	1.007420	1.179940	1.359310	2.331510
	0.000000	0.090023	0.000030	−0.601356
	−0.750129	0.041555	−1.000040	−3.559750
$u^{(99)}$	1.007580	1.179940	1.360440	2.063980
	0.000022	0.089946	−0.000932	−0.538252
	−0.750123	0.041559	−1.000070	−3.539530
u	1.007500	1.180000	1.360000	2.620000
	0.000000	0.090000	0.000000	−0.810000
	−0.750125	0.041568	−1.000000	−3.666500

8.14 Remark. Tables 8.5, 8.6, 8.7, and 8.8 contain, respectively, the values of $u^{(51)}$ and $u^{(99)}$ and the corresponding errors at the same points as in the first four tables, for several values of r_o.

(i) It is clear that $u^{(99)}$ is a better approximation than $u^{(51)}$ away from the boundary (more precisely, at the first three points) for $r_o = 1.1$. The situation is reversed for $r_o = 1.5$, and there is not much to choose between the two for $r_o = 10$.

(ii) The best approximations $u^{(51)}$ and $u^{(99)}$ in the computed set are for $r_o = 2$, followed closely by those for $r_o = 4$. The errors increase when r_o is significantly smaller than 2 or larger than 4. This is explained by the singularity of $D(x, y)$ and $P(x, y)$ on ∂S and the asymptotics (3.38) of these functions as $y \in \partial S$ and $|x| \to \infty$.

(iii) The anomalous error numbers generated by $u^{(99)}$ at $(0.9,\ 3\pi/4)$ for $r_o = 1.1$ have the same explanation. Other points with a polar radius of 0.9 were tried in this case, with similar results.

8.15 Remark. Tables 8.9 and 8.10 widen the scope of the investigation into how the approximation errors for $u^{(51)}$ and $u^{(99)}$ vary as the point $x^{(i)}$ moves from the center of the disk toward ∂S in polar radius increments of 0.1, for $r_o = 2$ and $r_o = 4$.

(i) For $r_o = 2$, both errors are zero up to 4 decimal places until the polar radius of the point reaches 0.3, and to 3 decimal places until 0.6, after which it starts deteriorating fast. For $r_o = 4$, the thresholds are 0.5 and 0.6, respectively.

Table 8.2 The computational errors $|u - u^{(m)}|$ at the points in Table 8.1 for $r_o = 2$

$x^{(i)}$	$(0.05,\ 0)$	$(0.3,\ \pi/4)$	$(0.6,\ \pi/2)$	$(0.9, 3\pi/4)$
$\|u - u^{(9)}\|$	0.098401	0.086770	0.056320	0.287890
	0.000000	0.024155	0.000034	0.448320
	0.006731	0.030311	0.000030	0.231770
$\|u - u^{(15)}\|$	0.012970	0.010750	0.009730	0.308240
	0.000000	0.008177	0.000037	0.167403
	0.000346	0.002308	0.000040	0.140550
$\|u - u^{(27)}\|$	0.000140	0.000120	0.000580	0.294220
	0.000000	0.000004	0.000030	0.203714
	0.000007	0.000026	0.000040	0.106830
$\|u - u^{(51)}\|$	0.000080	0.000060	0.000690	0.288490
	0.000000	0.000023	0.000030	0.208644
	0.000004	0.000013	0.000040	0.106750
$\|u - u^{(99)}\|$	0.000080	0.000060	0.000440	0.556020
	0.000022	0.000054	0.000932	0.271748
	0.000002	0.000009	0.000070	0.126970

Table 8.3 The values of $u^{(m)}$ and u at the points $x^{(i)}$ in Table 8.1 for $r_o = 4$

$x^{(i)}$	$(0.05,\ 0)$	$(0.3,\ \pi/4)$	$(0.6,\ \pi/2)$	$(0.9,\ 3\pi/4)$
$u^{(9)}$	0.891649	1.079830	1.426260	2.395640
	0.000000	0.065017	0.000040	−0.336053
	−0.755540	0.018704	−1.000030	−3.498200
$u^{(15)}$	0.894734	1.083160	1.418590	2.421150
	0.000000	0.065388	0.000040	−0.343568
	−0.754609	0.024425	−1.000030	−3.552410
$u^{(27)}$	1.007450	1.179960	1.359160	2.324730
	0.000000	0.089989	0.000040	−0.607881
	−0.750127	0.041560	−1.000030	−3.558930
$u^{(51)}$	1.007500	1.180000	1.359120	2.324910
	0.000000	0.090000	0.000055	−0.607857
	−0.750125	0.041568	−1.000030	−3.559470
$u^{(99)}$	1.007500	1.180000	1.359050	1.459230
	0.000000	0.090002	0.000079	0.329970
	−0.750125	0.041568	−1.000030	−0.050139
u	1.007500	1.180000	1.360000	2.620000
	0.000000	0.090000	0.000000	−0.810000
	−0.750125	0.041568	−1.000000	−3.666500

Table 8.4 The computational errors $|u - u^{(m)}|$ at the points in Table 8.3 for $r_o = 4$

$x^{(i)}$	$(0.05,\ 0)$	$(0.3,\ \pi/4)$	$(0.6,\ \pi/2)$	$(0.9, 3\pi/4)$
$\lvert u - u^{(9)} \rvert$	0.115851	0.100170	0.066260	0.224360
	0.000000	0.024983	0.000040	0.473947
	0.005415	0.022864	0.000030	0.315500
$\lvert u - u^{(15)} \rvert$	0.112766	0.096840	0.058590	0.198850
	0.000000	0.024612	0.000040	0.466432
	0.004484	0.017143	0.000030	0.114090
$\lvert u - u^{(27)} \rvert$	0.000050	0.000040	0.000840	0.295270
	0.000000	0.000011	0.000040	0.202119
	0.000002	0.000008	0.000030	0.107570
$\lvert u - u^{(51)} \rvert$	0.000000	0.000000	0.000880	0.295090
	0.000000	0.000000	0.000055	0.202143
	0.000000	0.000000	0.000030	0.107030
$\lvert u - u^{(99)} \rvert$	0.000000	0.000000	0.000950	1.160770
	0.000000	0.000002	0.000079	1.139970
	0.000000	0.000000	0.000030	3.616361

Table 8.5 The values of $u^{(51)}$ and u at the points $x^{(i)}$ in Table 8.1 for various radii r_o

$x^{(i)}$	$(0.05,\ 0)$	$(0.3,\ \pi/4)$	$(0.6,\ \pi/2)$	$(0.9,\ 3\pi/4)$
$r_o = 1.1$	0.984393	1.148190	1.468830	2.221520
	0.000000	0.097241	0.001287	−0.415976
	−0.759148	0.003493	−1.032630	−3.419870
$r_o = 1.5$	1.005800	1.178670	1.363540	2.340940
	0.000000	0.091092	−0.000231	−0.599018
	−0.750251	0.041138	−1.000250	−3.560160
$r_o = 2$	1.007420	1.179940	1.359310	2.331510
	0.000000	0.090023	0.000030	−0.601356
	−0.750129	0.041555	−1.000040	−3.559750
$r_o = 4$	1.007500	1.180000	1.359120	2.324910
	0.000000	0.090000	−0.000055	−0.607857
	−0.750125	0.041568	−1.000030	−3.559470
$r_o = 10$	0.890012	1.078890	1.423330	2.412190
	0.000000	0.064548	0.000122	−0.344890
	−0.754919	0.023032	−1.000420	−3.545090
u	1.007500	1.180000	1.360000	2.620000
	0.000000	0.090000	0.000000	−0.810000
	−0.750125	0.041568	−1.000000	−3.666500

Table 8.6 The computational errors $|u - u^{(51)}|$ at the points in Table 8.5

$x^{(i)}$	(0.05, 0)	(0.3, $\pi/4$)	(0.6, $\pi/2$)	(0.9, $3\pi/4$)
$r_o = 1.1$	0.023107	0.031810	0.108830	0.398480
	0.000000	0.007241	0.001287	0.394024
	0.009023	0.038075	0.032630	0.246630
$r_o = 1.5$	0.001700	0.001330	0.003540	0.279060
	0.000000	0.001092	0.000230	0.210982
	0.000126	0.000430	0.000250	0.106340
$r_o = 2$	0.000080	0.000060	0.000690	0.288490
	0.000000	0.000023	0.000030	0.208644
	0.000004	0.000013	0.000040	0.106750
$r_o = 4$	0.000000	0.000000	0.000880	0.295090
	0.000000	0.000000	0.000055	0.202143
	0.000000	0.000000	0.000030	0.107030
$r_o = 10$	0.117488	0.101110	0.063330	0.207810
	0.000000	0.025452	0.000122	0.465110
	0.004794	0.022648	0.000420	0.121410

Table 8.7 The values of $u^{(99)}$ and u at the points $x^{(i)}$ in Table 8.1 for various radii r_o

$x^{(i)}$	(0.05, 0)	(0.3, $\pi/4$)	(0.6, $\pi/2$)	(0.9, $3\pi/4$)
$r_o = 1.1$	1.006880	1.150640	0.873030	160.400000
	0.000000	0.047808	−0.043748	−120.054000
	−0.750165	0.038164	−1.009930	−6.325140
$r_o = 1.5$	1.007030	1.175570	1.345790	2.103650
	−0.002429	0.089554	−0.022097	−0.505978
	−0.750366	0.040829	−1.000750	−3.532410
$r_o = 2$	1.007580	1.179940	1.360440	2.063980
	0.000022	0.089946	−0.000932	−0.538252
	−0.750123	0.041559	−1.000070	−3.539530
$r_o = 4$	1.007500	1.180000	1.359050	1.459230
	0.000000	0.090002	0.000079	0.329970
	−0.750125	0.041568	−1.000030	−0.050139
$r_o = 10$	0.890082	1.078900	1.422220	2.408500
	0.000017	0.064634	0.000412	−0.360479
	−0.754916	0.023071	−1.001010	−3.541330
u	1.007500	1.180000	1.360000	2.620000
	0.000000	0.090000	0.000000	−0.810000
	−0.750125	0.041568	−1.000000	−3.666500

Table 8.8 The computational errors $|u - u^{(99)}|$ at the points in Table 8.7

$x^{(i)}$	(0.05, 0)	(0.3, $\pi/4$)	(0.6, $\pi/2$)	(0.9, $3\pi/4$)
$r_o = 1.1$	0.000620	0.029360	0.486970	157.780000
	0.000000	0.042192	0.043748	119.244000
	0.000040	0.003404	0.009930	2.658640
$r_o = 1.5$	0.000470	0.004430	0.014210	0.516350
	0.002429	0.000446	0.022097	0.304022
	0.000241	0.000739	0.000750	0.134090
$r_o = 2$	0.000080	0.000060	0.000440	0.556020
	0.000022	0.000054	0.000932	0.271748
	0.000002	0.000009	0.000070	0.126970
$r_o = 4$	0.000000	0.000000	0.000950	1.160770
	0.000000	0.000002	0.000079	1.139970
	0.000000	0.000000	0.000030	3.616361
$r_o = 10$	0.117418	0.101100	0.062220	0.211500
	0.000017	0.025366	0.000412	0.449521
	0.004791	0.018497	0.001010	0.125170

(ii) The errors for $u^{(51)}$ and $u^{(99)}$ remain comparable at 0.7 and 0.8; the latter becomes worse at 0.9, especially for $r_o = 4$.

8.16 Remark. Tables 8.11, 8.12 and 8.13, 8.14 show the approximate and exact values of the solution, and the corresponding errors, at three points very close to ∂S (specifically, with polar radius 0.99), for $r_o = 2$ and $r_o = 4$. As expected, the approximations are unsatisfactory, with the 'badness' worst at (0.99, 0), which, of the three points, is closest to a Simpson node and, therefore, most affected by the singularities of D and P.

8.17 Remark. Tables 8.15, 8.16, and 8.17 contain the values of the approximations $u^{(51)}$ and $u^{(99)}$ and the error of the former at the same close-to-the-boundary points used in Tables 8.11, 8.12, 8.13, and 8.14, for the set of incremental radii r_o of ∂S_* considered earlier. The conclusions regarding the error size for each of these radii are similar to those in Remark 8.16.

We make a few general comments.

8.18 Remarks. (i) The size of the errors did not diminish when we doubled the number of points $x^{(k)}$ on ∂S_*, to construct $u^{(195)}$. We suspect that the main reason for this is the numerical instability of the classical Gram–Schmidt process.

(ii) Simpson's rule of numerical integration over $[0, 2\pi]$ was also tried with 48 and 96 equal strips, but the changes in the final figures were not significant.

(iii) The values of $u^{(m)}$ were computed for more radii r_o, from 0.01 to 100, and the conclusion was that $r_o = 2$ and $r_o = 4$ fall within the optimal range for the error. Placing ∂S_* too close to ∂S or too far away from it led to badly distorted results.

Table 8.9 The values of $u^{(51)}$, $u^{(99)}$, and u at selected points $x^{(i)}$ in S^+ for $r_o = 2$ and $r_o = 4$

$x^{(i)}$	$u^{(51)}$		$u^{(99)}$		u
	$r_o = 2$	$r_o = 4$	$r_o = 2$	$r_o = 4$	
(0.05, 0)	1.007420	1.007500	1.007580	1.007500	1.007500
	0.000000	0.000000	0.000022	0.000000	0.000000
	−0.750129	−0.750125	−0.750123	−0.750125	−0.750125
(0.1, $\pi/3$)	1.014930	1.015000	1.015070	1.015000	1.015000
	0.008662	0.008660	0.008635	0.008659	0.008660
	−0.750504	−0.750500	−0.750500	−0.750500	−0.750500
(0.2, $\pi/6$)	1.099910	1.100000	1.100030	1.100000	1.100000
	0.034650	0.034641	0.034481	0.034638	0.034641
	−0.140916	−0.140903	−0.140905	−0.140903	−0.140903
(0.3, $\pi/4$)	1.179940	1.180000	1.179940	1.180000	1.180000
	0.090023	0.090000	0.089946	0.090002	0.090000
	0.041555	0.041568	0.041559	0.041568	0.041568
(0.4, $\pi/9$)	1.442430	1.442570	1.442350	1.442530	1.442570
	0.102863	0.102849	0.102984	0.102867	0.102846
	0.819218	0.819245	0.819216	0.819243	0.819245
(0.5, $4\pi/9$)	1.265210	1.265120	1.265170	1.265080	1.265080
	0.085503	0.085478	0.085823	0.085449	0.085505
	−0.587587	−0.587586	−0.587581	−0.587587	−0.587586
(0.6, $\pi/2$)	1.359310	1.359120	1.360440	1.359050	1.360000
	0.000030	−0.000055	−0.000932	0.000079	0.000000
	−1.000040	−1.000030	−1.000070	−1.000030	−1.000000
(0.7, $5\pi/4$)	1.970290	1.970030	1.820640	1.969680	1.980000
	0.490459	0.490576	0.829947	0.489950	0.490000
	−3.231610	−3.231600	−3.242060	−3.231590	−3.232340
(0.8, $10\pi/9$)	2.696250	2.698700	2.895320	2.699320	2.770270
	0.372738	0.374538	0.413757	0.374755	0.411384
	−4.247290	−4.247350	−4.269450	−4.247350	−4.277650
(0.9, $3\pi/4$)	2.331510	2.324910	2.063980	1.459230	2.620000
	−0.601356	−0.607857	−0.538252	0.329970	−0.810000
	−3.559750	−3.559470	−3.539530	−0.050139	−3.666500

(iv) The convergence of $u^{(m)}$ in the vicinity of the boundary is, as expected, much slower than near the center. At points close to ∂S, a considerably higher value of m needs to be considered to bring the error to a reasonable size.

(v) The accuracy of the approximation can be improved if use is made of a more powerful computer and a more sophisticated numerical algorithm. For example, the stability of the Gram–Schmidt scheme can be enhanced by means of a modified version where the construction of the vector function $\Omega^{(m)}$ involves its orthogonalization against any errors introduced in the computation of its predecessors. Also, a

Table 8.10 The computational errors $|u - u^{(51)}|$ and $|u - u^{(99)}|$ at the points in Table 8.9

$x^{(i)}$	$\|u - u^{(51)}\|$		$\|u - u^{(99)}\|$	
	$r_o = 2$	$r_o = 4$	$r_o = 2$	$r_o = 4$
(0.05, 0)	0.000080	0.000000	0.000080	0.000000
	0.000000	0.000000	0.000022	0.000000
	0.000004	0.000000	0.000002	0.000000
(0.1, $\pi/3$)	0.000070	0.000000	0.000070	0.000000
	0.000002	0.000000	0.000025	0.000001
	0.000004	0.000000	0.000000	0.000000
(0.2, $\pi/6$)	0.000090	0.000000	0.000030	0.000000
	0.000009	0.000000	0.000160	0.000003
	0.000013	0.000000	0.000002	0.000000
(0.3, $\pi/4$)	0.000060	0.000000	0.000060	0.000000
	0.000023	0.000000	0.000054	0.000002
	0.000013	0.000000	0.000009	0.000000
(0.4, $\pi/9$)	0.000140	0.000000	0.000220	0.000040
	0.000017	0.000003	0.000138	0.000021
	0.000027	0.000000	0.000029	0.000002
(0.5, $4\pi/9$)	0.000130	0.000040	0.000090	0.000000
	0.000002	0.000027	0.000318	0.000056
	0.000001	0.000000	0.000005	0.000001
(0.6, $\pi/2$)	0.000690	0.000880	0.000440	0.000950
	0.000030	0.000055	0.000932	0.000021
	0.000040	0.000030	0.000070	0.000030
(0.7, $5\pi/4$)	0.009710	0.009970	0.159360	0.010320
	0.000459	0.000576	0.339947	0.000050
	0.000730	0.000740	0.009720	0.000750
(0.8, $10\pi/9$)	0.074020	0.071570	0.125050	0.070950
	0.038646	0.036846	0.002373	0.036629
	0.030360	0.030300	0.008200	0.030300
	0.288490	0.295090	0.556020	1.160770
	0.208644	0.202143	0.271748	1.139970
	0.106750	0.107030	0.126970	3.616361

faster and more efficient numerical integration procedure could be adopted, and the computer algebra capabilities of *Mathematica*® could be exploited if the machine has sufficient available memory. But such refinements go beyond the scope of this book, which is concerned chiefly with the analytic handling of the problem.

8.19 Remark. For the interested reader, below are the contents of the *Mathematica*® notebook used to generate the numerical results in this section. The commands in each cell are preceded by brief explanatory comments.

Table 8.11 The values of $u^{(m)}$ and u at selected points $x^{(i)}$ close to ∂S for $r_o = 2$

$x^{(i)}$	$(0.99,\ 0)$	$(0.99,\ \pi/16)$	$(0.99,\ \pi/36)$
$u^{(9)}$	14.318600	3.714870	1.883640
	0.000000	0.453832	−0.187059
	7.291790	3.604700	1.773780
$u^{(15)}$	14.405100	3.815040	1.958140
	0.000000	0.570787	−0.136224
	7.208620	3.528960	1.712860
$u^{(27)}$	14.394200	3.806260	1.952360
	0.000000	0.581351	−0.132348
	7.168420	3.497190	1.683650
$u^{(51)}$	14.260300	3.866190	1.947120
	0.000000	0.613252	−0.111941
	7.167530	3.496620	1.683580
$u^{(99)}$	14.364800	3.973990	1.998120
	0.424528	0.361206	−0.086232
	7.019800	3.605200	1.682940
u	3.940300	3.865690	3.925410
	0.000000	0.375068	0.170193
	2.979700	2.903230	2.964560

Table 8.12 The computational errors $|u - u^{(m)}|$ at the points in Table 8.11 for $r_o = 2$

$x^{(i)}$	$(0.99,\ 0)$	$(0.99,\ \pi/16)$	$(0.99,\ \pi/36)$
$\lvert u - u^{(9)}\rvert$	10.378300	0.150820	2.041770
	0.000000	0.078764	0.357252
	4.312090	0.701470	1.190780
$\lvert u - u^{(15)}\rvert$	10.464800	0.050650	1.967270
	0.000000	0.195719	0.306417
	4.228920	0.625730	1.251700
$\lvert u - u^{(27)}\rvert$	10.453900	0.059430	1.973050
	0.000000	0.206283	0.302541
	4.188720	0.593960	1.280910
$\lvert u - u^{(51)}\rvert$	10.320000	0.000050	1.978290
	0.000000	0.238184	0.282134
	4.187830	0.593390	1.280980
$\lvert u - u^{(99)}\rvert$	10.424500	0.108300	1.927290
	0.424528	0.013862	0.256425
	4.040100	0.701970	1.281620

Table 8.13 The values of $u^{(m)}$ and u at the points $x^{(i)}$ in Table 8.11 for $r_o = 4$

$x^{(i)}$	$(0.99,\ 0)$	$(0.99,\ \pi/16)$	$(0.99,\ \pi/36)$
$u^{(9)}$	14.200100	3.625110	1.808630
	0.000000	0.460078	−0.186517
	7.238340	3.562120	1.735350
$u^{(15)}$	14.170700	3.604000	1.791320
	0.000000	0.467818	−0.183938
	7.164520	3.496130	1.676040
$u^{(27)}$	14.392900	3.806970	1.952520
	0.000000	0.583507	−0.131415
	7.166160	3.496450	1.682600
$u^{(51)}$	14.393000	3.807050	1.952580
	0.000122	0.584242	−0.131048
	7.167680	3.497100	1.683410
$u^{(99)}$	14.393300	3.806120	1.952220
	0.000147	0.583194	−0.131500
	7.168040	3.497150	1.683520
u	3.940300	3.865690	3.925410
	0.000000	0.375068	0.170193
	2.979700	2.903230	2.964560

Table 8.14 The computational errors $|u - u^{(m)}|$ at the points in Table 8.13 for $r_o = 4$

$x^{(i)}$	$(0.99,\ 0)$	$(0.99,\ \pi/16)$	$(0.99,\ \pi/36)$
$\lvert u - u^{(9)}\rvert$	10.259800	0.240580	2.116780
	0.000000	0.085010	0.356710
	4.258640	0.658890	1.229210
$\lvert u - u^{(15)}\rvert$	10.230400	0.261690	2.134090
	0.000000	0.092750	0.354131
	4.184820	0.592900	1.288520
$\lvert u - u^{(27)}\rvert$	10.452600	0.058720	1.972890
	0.000000	0.208439	0.301608
	4.186460	0.593220	1.281960
$\lvert u - u^{(51)}\rvert$	10.452700	0.058640	1.972830
	0.000122	0.209174	0.301241
	4.187980	0.593870	1.281150
$\lvert u - u^{(99)}\rvert$	10.453000	0.059570	1.973190
	0.000147	0.208126	0.301693
	4.188340	0.593920	1.281040

Table 8.15 The values of $u^{(51)}$ and u at the points $x^{(i)}$ in Table 8.11 for various radii r_o

$x^{(i)}$	$(0.99,\ 0)$	$(0.99,\ \pi/16)$	$(0.99,\ \pi/36)$
$r_o = 1.1$	14.158700	4.011060	1.980900
	0.000000	0.738428	−0.041818
	7.128330	3.379650	1.598850
$r_o = 1.5$	14.151600	3.897410	1.932960
	0.000000	0.618801	−0.104884
	7.167320	3.493720	1.681880
$r_o = 2$	14.260300	3.866190	1.947120
	0.000000	0.613252	−0.111941
	7.167530	3.496620	1.683580
$r_o = 4$	14.393000	3.807050	1.952580
	0.000122	0.584242	−0.131048
	7.167680	3.497100	1.683410
$r_o = 10$	14.184800	3.616360	1.799440
	0.005259	0.445375	−0.190697
	7.163660	3.497540	1.675870
u	3.940300	3.865690	3.925410
	0.000000	0.375068	0.170193
	2.979700	2.903230	2.964560

Table 8.16 The values of $u^{(99)}$ and u at the points in Table 8.11 for various radii r_o

$x^{(i)}$	$(0.99,\ 0)$	$(0.99,\ \pi/16)$	$(0.99,\ \pi/36)$
$r_o = 1.1$	615.922000	−619.646000	−70.536600
	0.032329	675.962000	250.783000
	−33.759500	22.981500	10.222800
$r_o = 1.5$	6.729380	8.452130	1.337370
	4.033270	−5.983060	−1.401630
	8.731610	3.165590	1.844720
$r_o = 2$	14.364800	3.973990	1.998120
	0.424528	0.361206	−0.086232
	7.019800	3.605200	1.682940
$r_o = 4$	14.393300	3.806120	1.952220
	0.000147	0.583194	−0.131500
	7.168040	3.497150	1.683520
$r_o = 10$	14.199000	3.630510	1.808370
	0.012549	0.432949	−0.192421
	7.157130	3.494560	1.671720
u	3.940300	3.865690	3.925410
	0.000000	0.375068	0.170193
	2.979700	2.903230	2.964560

Table 8.17 The computational error $|u - u^{(51)}|$ at the points in Table 8.15

$x^{(i)}$	(0.99, 0)	(0.99, $\pi/16$)	(0.99, $\pi/36$)
$r_o = 1.1$	10.218400	0.145370	1.944510
	0.000000	0.363360	0.212011
	4.148630	0.476420	1.365710
$r_o = 1.5$	10.211300	0.031720	1.992450
	0.000000	0.243733	0.275077
	4.187620	0.590490	1.282680
$r_o = 2$	10.320000	0.000500	1.978290
	0.000000	0.238184	0.282131
	4.187830	0.593390	1.280980
$r_o = 4$	10.452700	0.058640	0.020250
	0.000122	0.209172	0.301241
	4.187980	0.593870	1.281150
$r_o = 10$	10.244500	0.249330	1.799440
	0.005259	0.070307	0.360890
	4.183960	0.594310	1.288690

The physical parameters:

```
lam = 1;
mu = 1;
h = 1/2;
```

Some auxiliary expressions:

```
xy[x1_, x2_, y1_, y2_] := ((x1 - y1)^2 +
        (x2 - y2)^2)^(1/2);
lxy[x1_, x2_, y1_, y2_] := Log[xy[x1, x2, y1, y2]];
txy[x1_, x2_, y1_, y2_] := (1/(8* Pi*h^2*mu^2*
        (lam + 2*mu)))*((4*h^2 +
        xy[x1, x2, y1, y2]^2)*lxy[x1, x2, y1, y2] +
        4*h^2*BesselK[0, xy[x1, x2, y1, y2]/h]);
```

The matrix $D(x, y)$ of fundamental solutions:

```
D11[x1_, x2_, y1_, y2_] := ((h^2*mu*(lam + 2*mu)*
        (D[txy[x1, x2, y1, y2], {x1, 4}] +
        2*D[txy[x1, x2, y1, y2], x1, x1, x2, x2] +
        D[txy[x1, x2, y1, y2], {x2, 4}]) -
        h^2*mu*(lam + mu)*
        (D[txy[x1, x2, y1, y2], {x1, 4}] +
        D[txy[x1, x2, y1, y2], x1, x1, x2, x2]) -
        mu^2*D[txy[x1, x2, y1, y2], {x1, 2}]));
```

```
D12[x1_, x2_, y1_, y2_] := ((-h^2*mu*(lam + mu)*
        (D[txy[x1, x2, y1, y2], x1, x1, x1, x2] +
        D[txy[x1, x2, y1, y2], x1, x2, x2, x2]) -
        mu^2*D[txy[x1, x2, y1, y2], x1, x2]));
D13[x1_, x2_, y1_, y2_] := ((mu^2*
        (h^2*(D[txy[x1, x2, y1, y2], {x1, 3}] +
        D[txy[x1, x2, y1, y2], x1, x2, x2]) -
        D[txy[x1, x2, y1, y2], x1])));
D21[x1_, x2_, y1_, y2_] := D12[y1, y2, x1, x2];
D22[x1_, x2_, y1_, y2_] := ((h^2*mu*(lam + 2*mu)*
        (D[txy[x1, x2, y1, y2], x1, 4] +
        2*D[txy[x1, x2, y1, y2], x1, x1, x2, x2] +
        D[txy[x1, x2, y1, y2], {x2, 4}]) -
        h^2*mu*(lam + mu)*
        (D[txy[x1, x2, y1, y2], x1, x1, x2, x2] +
        D[txy[x1, x2, y1, y2], {x2, 4}]) -
        mu^2*D[txy[x1, x2, y1, y2], {x2, 2}]));
D23[x1_, x2_, y1_, y2_] := ((mu^2*
        (h^2*(D[txy[x1, x2, y1, y2], x1, x1, x2] +
        D[txy[x1, x2, y1, y2], {x2, 3}]) -
        D[txy[x1, x2, y1, y2], x2])));
D31[x1_, x2_, y1_, y2_] := D13[y1, y2, x1, x2];
D32[x1_, x2_, y1_, y2_] := D23[y1, y2, x1, x2];
D33[x1_, x2_, y1_, y2_] := ((h^4*mu*(lam + 2*mu)*
        (D[txy[x1, x2, y1, y2], {x1, 4}] +
        2*D[txy[x1, x2, y1, y2], x1, x1, x2, x2] +
        D[txy[x1, x2, y1, y2], {x2, 4}]) -
        h^2*mu*(lam + 3*mu)*
        (D[txy[x1, x2, y1, y2], {x1, 2}] +
        D[txy[x1, x2, y1, y2], {x2, 2}]) +
        mu^2*txy[x1, x2, y1, y2]))
MatrixD[x1_, x2_, y1_, y2_] := {{D11[x1, x2, y1, y2],
        D12[x1, x2, y1, y2], D13[x1, x2, y1, y2]},
        {D21[x1, x2, y1, y2], D22[x1, x2, y1, y2],
        D23[x1, x2, y1, y2]}, {D31[x1, x2, y1, y2],
        D32[x1, x2, y1, y2], D33[x1, x2, y1, y2]}};
```

The matrix $P(x, y)$ *of singular solutions:*

```
P11[x1_, x2_, y1_, y2_] := (h^2*(lam + 2*mu)*nu1y*
        D[D11[y1, y2, x1, x2], y1] + h^2*mu*nu2y*
        D[D11[y1, y2, x1, x2], y2] + h^2*mu*nu2y*
        D[D21[y1, y2, x1, x2], y1] + h^2*lam*nu1y*
        D[D21[y1, y2, x1, x2], y2]);
P12[x1_, x2_, y1_, y2_] := (h^2*lam*nu2y*
```

```
        D[D11[y1, y2, x1, x2], y1] + h^2*mu*nu1y*
        D[D11[y1, y2, x1, x2], y2] + h^2*mu*nu1y*
        D[D21[y1, y2, x1, x2], y1] + h^2*(lam + 2*mu)*
        nu2y*D[D21[y1, y2, x1, x2], y2]);
P13[x1_, x2_, y1_, y2_] := (mu*nu1y*
        D11[y1, y2, x1, x2] + mu*nu2y*
        D21[y1, y2, x1, x2] + mu*(nu1y*
        D[D31[y1, y2, x1, x2], y1] + nu2y*
        D[D31[y1, y2, x1, x2], y2]));
P21[x1_, x2_, y1_, y2_] := (h^2*(lam + 2*mu)*nu1y*
        D[D12[y1, y2, x1, x2], y1] + h^2*mu*nu2y*
        D[D12[y1, y2, x1, x2], y2] + h^2*mu*nu2y*
        D[D22[y1, y2, x1, x2], y1] + h^2*lam*nu1y*
        D[D22[y1, y2, x1, x2], y2]);
P22[x1_, x2_, y1_, y2_] := (h^2*lam*nu2y*
        D[D12[y1, y2, x1, x2], y1] + h^2*mu*nu1y*
        D[D12[y1, y2, x1, x2], y2] + h^2*mu*nu1y*
        D[D22[y1, y2, x1, x2], y1] + h^2*(lam + 2*mu)*
        nu2y*D[D22[y1, y2, x1, x2], y2]);
P23[x1_, x2_, y1_, y2_] := (mu*nu1y*
        D12[y1, y2, x1, x2] + mu*nu2y*
        D22[y1, y2, x1, x2] + mu*(nu1y*
        D[D32[y1, y2, x1, x2], y1] + nu2y*
        D[D32[y1, y2, x1, x2], y2]));
P31[x1_, x2_, y1_, y2_] := (h^2*(lam + 2*mu)*nu1y*
        D[D13[y1, y2, x1, x2], y1] + h^2*mu*nu2y*
        D[D13[y1, y2, x1, x2], y2] + h^2*mu*nu2y*
        D[D23[y1, y2, x1, x2], y1] + h^2*lam*nu1y*
        D[D23[y1, y2, x1, x2], y2]);
P32[x1_, x2_, y1_, y2_] := (h^2*lam*nu2y*
        D[D13[y1, y2, x1, x2], y1] + h^2*mu*nu1y*
        D[D13[y1, y2, x1, x2], y2] + h^2*mu*nu1y*
        D[D23[y1, y2, x1, x2], y1] + h^2*(lam + 2*mu)*
        nu2y*D[D23[y1, y2, x1, x2], y2]);
P33[x1_, x2_, y1_, y2_] := (mu*nu1y*
        D13[y1, y2, x1, x2] + mu*nu2y*
        D23[y1, y2, x1, x2] + mu*(nu1y*
        D[D33[y1, y2, x1, x2], y1] + nu2y*
        D[D33[y1, y2, x1, x2], y2]));
MatrixP[x1_, x2_, y1_, y2_] := {{P11[x1, x2, y1, y2],
        P12[x1, x2, y1, y2], P13[x1, x2, y1, y2]},
        {P21[x1, x2, y1, y2], P22[x1, x2, y1, y2],
        P23[x1, x2, y1, y2]}, {P31[x1, x2, y1, y2],
        P32[x1, x2, y1, y2], P33[x1, x2, y1, y2]}};
```

The radii of the circles ∂S *and* ∂S_*:

```
rInner = 1;
rOuter = 2;
```

The number n_o *of points on* ∂S_**, chosen, for symmetry, to be a power of* 2:

```
nO = 32;
```

The (even) number n_s *of equal subintervals for numerical integration by Simpson's rule from 0 to* 2π:

```
nS = 36;
```

The polar angles of the points $x^{(m)}$*, which are placed around the circle in equal increments of* $\pi/2$, then $\pi/4$, *then* $\pi/8$, *then* $\pi/16$, *etc., without replicating the values generated by earlier passages:*

```
AngleList = {0, Pi};
For[n = 3, n <= nO, n++, {quot = n - 1, den = 1;
        While[(quot = Floor[quot/2])! = 0, den = den*2;
        phi = (2*(n - den) - 1)* Pi/den]};
        AppendTo[AngleList, phi]; Print[AngleList]];
```

The Cartesian coordinates of the points $x^{(m)}$ *on* ∂S_**, computed from their polar angles:*

```
Do[{xm1[m] = rOuter*Cos[AngleList[[m]]], xm2[m] =
        rOuter*Sin[AngleList[[m]]]}, {m, 1, nO}];
```

The indexed entries of $D(x, y)$:

```
MatrixD[1, 1] = D11[x1, x2, y1, y2];
MatrixD[1, 2] = D12[x1, x2, y1, y2];
MatrixD[1, 3] = D13[x1, x2, y1, y2];
MatrixD[2, 1] = D21[x1, x2, y1, y2];
MatrixD[2, 2] = D22[x1, x2, y1, y2];
MatrixD[2, 3] = D23[x1, x2, y1, y2];
MatrixD[3, 1] = D31[x1, x2, y1, y2];
MatrixD[3, 2] = D32[x1, x2, y1, y2];
MatrixD[3, 3] = D33[x1, x2, y1, y2];
For[i = 1, i <= 3, i++, For[j = 1, j <= 3, j++,
        MatrixD[i, j]]];
```

The columns of $D(x, y)$:

```
Do[DCol[i] = {MatrixD[1, i], MatrixD[2, i],
        MatrixD[3, i]}, {i, 1, 3}];
```

The first $3n_{\rm o}+3$ *terms of the complete set* $\theta^{(i,m)}$ *of vector functions, in double index notation:*

```
theta[1, 0] = {1, 0, -x1};
theta[2, 0] = {0, 1, -x2};
theta[3, 0] = {0, 0, 1};
Do[{Do[theta[i, m] = DCol[i] /. {y1 -> xm1[m],
        y2 -> xm2[m]}, {i, 1, 3}], Print[m]},
        {m, 1, nO}];
```

The same set in single index notation:

```
theta[1] = theta[1, 0];
theta[2] = theta[2, 0];
theta[3] = theta[3, 0];
Do[{theta[m] = If[(m - 1)/3 \[Element] Integers,
        theta[1, (m - 1)/3], If[(m - 2)/3
        \[Element] Integers, theta[2, (m - 2)/3],
        theta[3, (m - 3)/3]]], Print[m]},
        {m, 4, 3*nO + 3}];
```

The $3n_{\rm o}+3$ *vector functions* $\theta^{(m)}$ *in terms of the polar angle* α on ∂S:

```
Do[{thetaPolar[m] = theta[m] /. {x1 -> rInner*
         Cos[\[Alpha]], x2 -> rInner* Sin[\[Alpha]]},
        Print[m]}, {m, 1, 3*nO + 3}];
```

Discretization of a function as the set of its $n_{\rm s}+1$ *values at the Simpson nodes of rank from 0 to* $n_{\rm s}$ on $[0,\ 2\pi]$:

```
Discretize[f_, \[Alpha]_] := Module[{g, step},
step = (2*Pi)/nS; g[i_] := N[f /. \[Alpha] -> step*i];
        Do[g[i], {i, 0, nS}]; Table[g[i], {i, 0, nS}]];
```

Discretization of the set of all $3n_{\rm o}+3$ *vector functions* $\theta^{(m)}$:

```
Do[{thetaDiscr[m] = Discretize[thetaPolar[m],
        \[Alpha]], Print[m]}, {m, 1, 3*nO + 3}];
```

Simpson's rule of integration from 0 to 2π with n_s equal subintervals, for a function already discretized at the Simpson nodes:

```
Simpson[f_] := Module[odd, even, odd = 0; even = 0;
        Do[If[Mod[i, 2] === 0, odd = odd + f[[i]],
        even = even + f[[i]]], i, 2, nS]; N[(1/3)*
        (2* Pi/nS)*(f[[1]] + 2*even + 4*odd +
        f[[nS + 1]])]];
```

The L^2-inner product of two discretized vector functions on $[0, 2\pi]$:

```
IP[f_, g_] := Module[{fg}, fg = Table[f[[i]].g[[i]],
        {i, 1, nS + 1}]; Simpson[fg]];
```

Orthonormalization of the discretized vector functions $\theta^{(m)}$ as discretized vector functions $\omega^{(m)}$ on ∂S, via the Gram–Schmidt process:

```
Do[{OmegaDiscr[m] = thetaDiscr[m] -
        Sum[(IP[thetaDiscr[m], OmegaDiscr[q]]/
        IP[OmegaDiscr[q], OmegaDiscr[q]])*
        OmegaDiscr[q], {q, 1, m - 1}], omegaDiscr[m] =
        (1/IP[OmegaDiscr[m], OmegaDiscr[m]])^(1/2)*
        OmegaDiscr[m], Print[m]}, {m, 1, 3*nO + 3}];
```

The coefficients $k_{m,n}$ of the Gram–Schmidt transformation (*the $\omega^{(m)}$ expressed as linear combinations of the $\theta^{(n)}$*):

```
For[m = 1, m <= 3*nO + 3, m++, k[m, m] =
        (1/IP[OmegaDiscr[m], OmegaDiscr[m]])^(1/2);
        For[q = 1, q <= m - 1, q++, k[m, q] = 0;
        For[s = q, s <= m - 1, s++, k[m, q] = k[m, q] -
        IP[thetaDiscr[m], omegaDiscr[s]]*k[s, q]];
        k[m, q] = k[m, m]*k[m, q]]; Print[m]];
```

The zero coefficients above the leading diagonal of the transformation matrix:

```
Do[Do[If[q <= m, k[m, q] = k[m, q], k[m, q] = 0],
        {q, 1, 3*nO + 3}], {m, 1, 3*nO + 3}];
```

The vector function $\mathscr{P}$ prescribed on ∂S:

```
ScriptP[x1_, x2_] := {2*(x1^2 + 1), 2*x1*x2,
        4*x1 - 1};
```

$\mathscr{P}$ in polar coordinates:

```
ScriptPPolar[\[Alpha]_] := ScriptP[x1, x2] /.
        {x1 -> rInner* Cos[\[Alpha]], x2 -> rInner*
         Sin[\[Alpha]]}; ScriptPPolar[\[Alpha]];
```

$\mathscr{P}$ discretized:

```
ScriptPPolarDiscr = Discretize[ScriptPPolar[\[Alpha]],
        \[Alpha]];
```

$P(x, y)$ with $y \in \partial S$ in polar coordinates:

```
MatrixPPolar[x1_, x2_, \[Alpha]_] :=
        MatrixP[x1, x2, y1, y2] /.
        {nu1y -> Cos[\[Alpha]], nu2y -> Sin[\[Alpha]],
        y1 -> rInner*Cos[\[Alpha]],
        y2 -> rInner*Sin[\[Alpha]]}
```

The set of n_o matrices generated by $P(x, y)$ with $y \in \partial S$ in polar coordinates and x replaced in turn by every one of the points $x^{(m)}$ on ∂S_*:

```
Do[{MatrixPPolarOuter[m] = MatrixPPolar[x1, x2,
        \[Alpha]] /. {x1 -> xm1[m], x2 -> xm2[m]},
        Print[m]}, {m, 1, nO}];
```

The same set, discretized at the Simpson nodes:

```
Do[{MatrixPPolarOuterDiscr[n] = Discretize[
        MatrixPPolarOuter[n], \[Alpha]], Print[n]},
        {n, 1, nO}];
```

The number n_v of vector functions $\theta^{(m)}$ (so also $\omega^{(m)}$) chosen from the full $3n_\mathrm{o} + 3$-set for the calculation of the approximate solution (*because of the way the θ-set is constructed, these vectors will always be the first n_v ones in the full set*):

```
nV = 51;
```

The set of n_o vectors generated by the vector function H at every one of the points $x^{(m)}$ on ∂S_* (*not needed if* $n_\mathrm{v} = 3$):

```
Do[{Hxm[m] = IP[MatrixPPolarOuterDiscr[m],
        ScriptPPolarDiscr], Print[m]}, {m, 1, nO}];
```

The L^2-inner product of $\theta^{(m)}$ and ψ on $[0, 2\pi]$ for $m \geq 4$:

```
Do[{IntThetaPsi[m] = If[(m - 1)/3 \[Element] Integers,
        Hxm[(m - 1)/3][[1]], If[(m - 2)/3
        \[Element] Integers, Hxm[(m - 2)/3][[2]],
        Hxm[(m - 3)/3][[3]]]], Print[m]}, {m, 4, nV}];
```

The first three coefficients p_r (as remarked, they are zero):

```
p[1] = 0;
p[2] = 0;
p[3] = 0;
```

The rest of the p_r (for $r \geq 4$):

```
Do[{p[r] = Sum[k[r, m]*IntThetaPsi[m], {m, 4, r}],
        Print[r]}, {r, 4, nV}];
```

The vector function ψ discretized at the Simpson nodes:

```
psiDiscr = p[1]*omegaDiscr[1]; Do[lbr psiDiscr =
        psiDiscr + p[r]*omegaDiscr[r], Print[r]},
        {r, 2, nV}];
```

A specific point $x^{(\mathrm{i})}$ in S^+ where the solution is computed (its Cartesian coordinates are constructed from its polar coordinates to allow easier radial and angular variations, as desired):

```
PolarRadius = 0.05;
PolarAngle = 0;
xI1 = PolarRadius*Cos[PolarAngle]; xI2 =
        PolarRadius*Sin[PolarAngle];
```

$D(x, y)$ with $y \in \partial S$ in polar coordinates:

```
MatrixDPolar[x1_, x2_, \[Alpha]_] :=
        MatrixD[x1, x2, y1, y2] /.
        {y1 -> rInner* Cos[\[Alpha]], y2 -> rInner*
         Sin[\[Alpha]]};
```

$D(x, y)$ as above and $x = x^{(\mathrm{i})}$:

```
MatrixDPolarInner[\[Alpha]_] := MatrixDPolar[x1, x2,
        \[Alpha]] /. {x1 -> xI1, x2 -> xI2};
```

$D(x, y)$ now discretized at the Simpson nodes:

```
MatrixDPolarInnerDiscr =
        Discretize[MatrixDPolarInner[\[Alpha]],
        \[Alpha]];
```

The integral of $D(x, y)\psi(y)$ with respect to the polar angle α over $[0, 2\pi]$:

```
IntMatrixDpsi = IP[MatrixDPolarInnerDiscr, psiDiscr];
```

$P(x, y)$ with $y \in \partial S$ in polar coordinates and $x = x^{(\mathrm{i})}$:

```
MatrixPPolarInner[\[Alpha]_] := MatrixPPolar[x1, x2,
        \[Alpha]] /. {x1 -> xI1, x2 -> xI2};
```

$P(x, y)$ as above, discretized at the Simpson nodes:

```
MatrixPPolarInnerDiscr =
        Discretize[MatrixPPolarInner[\[Alpha]],
        \[Alpha]];
```

The vector $H(x^{(\mathrm{i})})$:

```
HInner = IP[MatrixPPolarInnerDiscr,
        ScriptPPolarDiscr];
```

The approximate solution u at $(x^{(\mathrm{i})})$:

```
uxi = IntMatrixDpsi - HInner
```

8.20 Remark. The exact solution, computed—as mentioned at the beginning of this section—from (7.36) with $\psi = 0$, $\Omega = z^2$, $\omega = z + 1$, and $l = m = 0$, is

$$u(x_1,\ x_2) = (3x_1^2 + x_2^2 + 1,\ 2x_1x_2,\ -x_1^3 - x_1x_2^2 + 5x_1 - 1)^{\mathrm{T}}.$$

Figures 8.1, 8.2, and 8.3 show the graphs of the three components u_1, u_2, and u_3 of this solution defined in S^+. The discs on the top and bottom of the coordinate boxes enclosing the graphs are horizontal cross sections of the cylinder $x_1^2 + x_2^2 \le 1$. The boundary curves (shown in thicker lines) of the surfaces representing the u_i are the graphs of the components of u prescribed on ∂S; that is,

$$\mathscr{P}(x_1, x_2) = \big(2(x_1^2 + 1),\ 2x_1x_2,\ 4x_1 - 1\big)^{\mathrm{T}}, \quad x \in \partial S.$$

Fig. 8.1 Graph of u_1

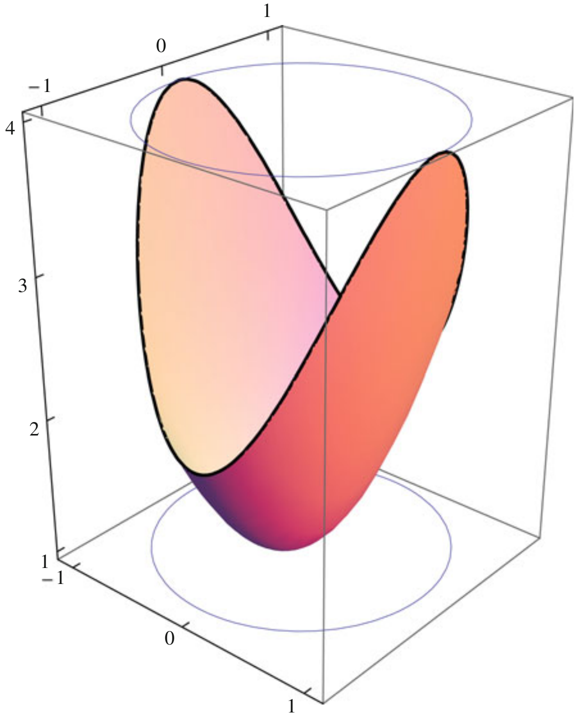


Fig. 8.2 Graph of u_2

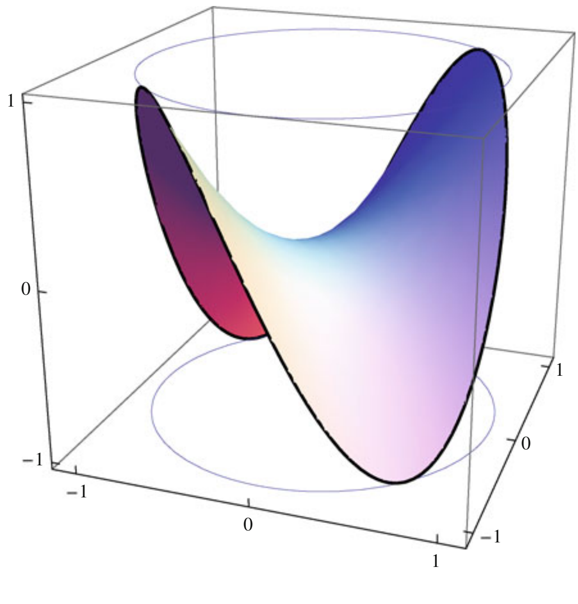
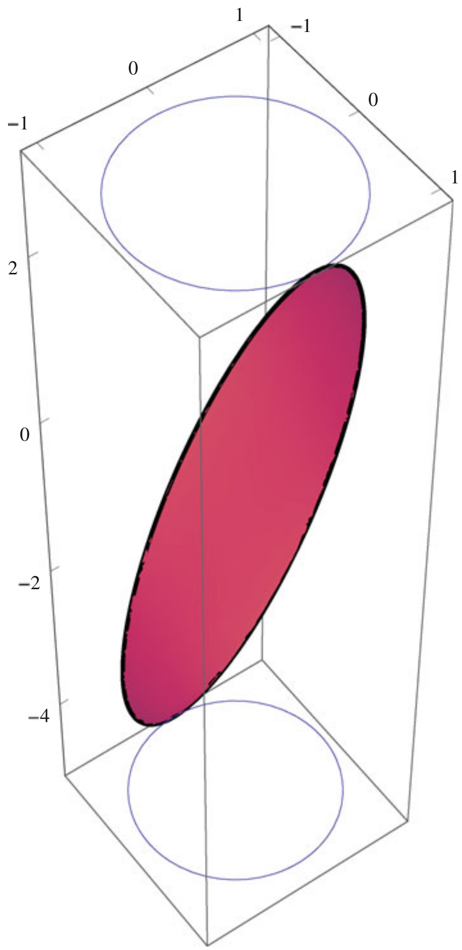


Fig. 8.3 Graph of u_3



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