Solutions Manual for

Fluid Mechanics: Fundamentals and Applications

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CHAPTER 10

APPROXIMATE SOLUTIONS OF THE NAVIER-STOKES EQUATION

General and Introductory Problems, Modified Pressure, Fluid Statics

10-1C

Solution We are to discuss the difference between an "exact" solution and an approximate solution of the Navier-Stokes equation.

Analysis **In an "exact" solution, we begin with the full Navier-Stokes equation**. As we solve the problem, some terms may drop out due to the specified geometry or other simplifying assumptions in the problem. **In an approximate solution, we eliminate some terms in the Navier-Stokes equation right from the start**. In other words, we *begin* with a reduced or simplified *approximate* form of the equation.

Discussion The approximations are based on the class of flow problem and/or the *region* in which such approximations are appropriate (e.g. irrotational, boundary layer, etc.).

10-2C

Solution We are to label regions in a flow field where certain approximations are likely to be appropriate.

Assumptions **1** The flow is incompressible. **2** The flow is steady in the mean (we ignore the unsteady flow field close to the rotating blades).

Analysis A boundary layer grows along the floor, both upstream and downstream of the fan. The flow upstream of the fan is largely irrotational except very close to the floor. The air is nearly static far upstream and far above the fan. Downstream of the fan, the flow is most likely swirling and turbulent, and none of the approximations are expected to be appropriate there. In other words, the full Navier-Stokes equation must be solved in that region. **We sketch all these regions in Fig. 1**.

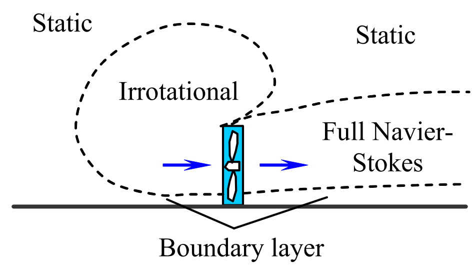


FIGURE 1
Regions of appropriate approximations for the flow produced by a box fan sitting on the floor of a large room.

Discussion The regions sketched in Fig. 1 are not well defined, nor are they necessarily to scale.

10-3C

Solution We are to discuss the role of nondimensionalization of the Navier-Stokes equations.

Analysis When we properly nondimensionalize the Navier-Stokes equation, **all the terms are re-written in the form of some nondimensional parameter times a quantity of order unity**. Thus, we can simply compare the orders of magnitude of the nondimensional parameters to see which terms (if any) can be ignored because they are very small compared to other terms. For example, if the Strouhal number is much smaller than the Euler number, we can ignore the term that contains the Strouhal number, but must retain the term that contains the Euler number.

Discussion This method works only if the characteristic scales of the problem (length, speed, frequency, etc.) are chosen properly.

10-4C

Solution We are to discuss the most significant danger that arises with an approximate solution, and we are to come up with an example.

Analysis The danger of an approximate solution of the Navier-Stokes equation is this: **If the approximation is not appropriate to begin with, our solution will be incorrect** – even if we perform all the mathematics correctly. There are many examples. For instance, we may assume that a boundary layer exists in a region of flow. However, if the Reynolds number is not large enough, the boundary layer is too thick and the boundary layer approximations break down. Another example is that we may assume a fluid statics region, when in reality there are swirling eddies in that region. The unsteady motion of the eddies makes the problem unsteady and dynamic – the approximation of fluid statics would be inappropriate.

Discussion When you make an approximation and solve the problem, it is best to go back and verify that the approximation is appropriate.

10-5C

Solution We are to discuss the criteria used to determine whether an approximation of the Navier-Stokes equation is appropriate or not

Analysis We determine if an approximation is appropriate **by comparing the orders of magnitude of the various terms in the equations of motion**. If the neglected terms are negligibly small compared to other terms, then the approximation is appropriate. If not, then it is not appropriate to neglect those terms.

Discussion It is important that the proper scales be used for the nondimensionalization of the equation. Otherwise, the order of magnitude analysis may be incorrect.

10-6C

Solution We are to discuss the physical significance of the four nondimensional parameters in the nondimensionalized incompressible Navier-Stokes equation.

Analysis The four parameters are discussed individually below:

- **Strouhal number**: St is the ratio of some characteristic flow time to some period of oscillation. If St << 1, the oscillation period is very large compared to the characteristic flow time, and the problem is quasi-steady; the unsteady term in the Navier-Stokes equation may be ignored. If St >> 1, the oscillation period is very short compared to the characteristic flow time, and the unsteadiness dominates the problem; the unsteady term must remain.
- **Euler number**: Eu is the ratio of a characteristic pressure difference to a characteristic pressure due to fluid inertia. If Eu << 1, pressure gradients are very small compared to inertial pressure, and the pressure term can be neglected in the Navier-Stokes equation. If Eu >> 1, the pressure term is very large compared to the inertial term, and must remain in the equation.
- **Froude number**: Fr is the ratio of inertial forces to gravitational forces. Note that Fr appears in the *denominator* of the nondimensionalized Navier-Stokes equation. If Fr << 1, gravitational forces are very large compared to inertial forces, and the gravity term must remain in the Navier-Stokes equation. If Fr >> 1, gravitational forces are negligible compared to inertial forces, and the gravity term in the Navier-Stokes equation can be ignored.
- **Reynolds number**: Re is the ratio of inertial forces to viscous forces. Note that Re appears in the *denominator* of the nondimensionalized Navier-Stokes equation. If Re << 1, viscous forces are very large compared to inertial forces, and the viscous term must remain. (In fact, it may dominate the other terms, as in creeping flow). If Re >> 1, viscous forces are negligible compared to inertial forces, and the viscous term in the Navier-Stokes equation can be ignored. Note that this applies only to regions outside of boundary layers, because the characteristic length scale for a boundary layer is generally much smaller than that for the overall flow.

Discussion You must keep in mind that the approximations discussed here are appropriate only in certain *regions* of the flow field. In other regions of the same flow field, different approximations may apply.

10-7C

Solution We are to discuss the criterion for using modified pressure.

Analysis **Modified pressure can be used only when there are *no free surface effects* in the problem**.

Discussion Modified pressure is simply a combination of thermodynamic pressure and hydrostatic pressure. It turns out that if there are **no free surface effects**, the hydrostatic pressure component is independent of the flow pressure component, and these two can be separated.

10-8C

Solution We are to discuss which nondimensional parameter is eliminated by use of the modified pressure.

Analysis Modified pressure effectively combines the effects of actual pressure and gravity. In the nondimensionalized Navier-Stokes equation in terms of modified pressure, **the Froude number disappears**. The reason Froude number is eliminated is because **the gravity term is eliminated from the equation.**

Discussion Keep in mind that we can employ modified pressure only for flows without free surface effects.

10-9

Solution We are to plug the given scales for this flow problem into the nondimensionalized Navier-Stokes equation to show that only two terms remain in the region consisting of most of the tank.

Assumptions **1** The flow is incompressible. **2** $d << D$. **3** D is of the same order of magnitude as H.

Analysis The characteristic frequency is taken as the inverse of the characteristic time, $f = 1/t_{\text{drain}}$. The Strouhal number is thus

Strouhal number:
$$\text{St} = \frac{fL}{V} = \frac{H}{t_{\text{drain}}V} \sim 1 \qquad (1)$$

St is of order of magnitude 1 since the order of magnitude of t_{drain} is H/V. The Euler number is

Euler number:
$$\text{Eu} = \frac{P_0 - P_\infty}{\rho V^2} = \frac{\rho g H}{\rho V^2} \sim \frac{V_{\text{jet}}^{\ 2}}{V^2} \sim \frac{D^4}{d^4} \qquad (2)$$

where we have used the order of magnitude estimate that $V_{\text{jet}} \sim \sqrt{gH}$. We have also used conservation of mass, namely $V_{\text{jet}}d^2 = V_{\text{tank}}D^2$. Similarly, the Froude number is

Froude number:
$$\text{Fr} = \frac{V}{\sqrt{gH}} \sim \frac{V}{V_{\text{jet}}} \sim \frac{d^2}{D^2} \qquad (3)$$

Finally, the Reynolds number is

Reynolds number:
$$\text{Re} = \frac{\rho V H}{\mu} \sim \frac{\rho V D}{\mu} = \frac{\rho V_{\text{jet}} D}{\mu}\frac{V}{V_{\text{jet}}} \sim \text{Re}_{\text{jet}}\frac{d^2}{D^2} \qquad (4)$$

We plug Eqs. 1 through 4 into the nondimensionalized incompressible Navier-Stokes equation and compare orders of magnitude of each term,

Nondimensionalized incompressible Navier-Stokes equation:
$$\underbrace{[\text{St}]\frac{\partial \vec{V}^*}{\partial t^*}}_{\sim 1} + \underbrace{\left(\vec{V}^*\cdot\vec{\nabla}^*\right)\vec{V}^*}_{\sim 1} = \underbrace{-[\text{Eu}]\vec{\nabla}^* P^*}_{\sim \frac{D^2}{d^2}} + \underbrace{\left[\frac{1}{\text{Fr}^2}\right]\vec{g}^*}_{\sim \frac{D^2}{d^2}} + \underbrace{\left[\frac{1}{\text{Re}}\right]\vec{\nabla}^{*2}\vec{V}^*}_{\sim \text{Re}_{\text{jet}}\frac{d^2}{D^2}} \qquad (5)$$

Clearly, the first two terms (the unsteady and inertial terms) in Eq. 5 are negligible compared to the second two terms (the pressure and gravity terms) since $D >> d$. The last term (the viscous term) is a little trickier. We know that if the flow remains laminar, the order of magnitude of Re_{jet} is at most 10^3. Thus, in order for the viscous term to be of the same order of magnitude as the inertial term, d^2/D^2 must be of order of magnitude 10^{-3}. Thus, provided that these criteria are met, the only two remaining terms in the Navier-Stokes equation are the pressure and gravity terms. The final dimensional form of the equation is the same as that of fluid statics,

Incompressible Navier-Stokes equation for fluid statics:
$$\vec{\nabla}P = \rho\vec{g} \qquad (6)$$

The criteria for Carrie's approximation to be appropriate depends on the desired precision. For 1% error, D must be at least 10 times greater than d to ignore the unsteady term and the inertial term. The viscous term, however, depends on the value of Re_{jet}. To be safe, Carrie should assume the highest possible value of Re_{jet}, for which we know from the above order of magnitude estimates that D must be at least $10^{3/2}$ times greater than d.

Discussion We cannot use the modified pressure in this problem since there is a free surface.

10-10

Solution We are to sketch the profile of modified pressure and shade in the region representing hydrostatic pressure.

Assumptions **1** The flow is incompressible. **2** The flow is fully developed. **3** Gravity acts vertically downward. **4** There are no free surface effects in this flow field.

Analysis By definition, modified pressure $P' = P + \rho gz$. So we add hydrostatic pressure component ρgz to the given profile for P to obtain the profile for P'. Recall from Example 9-16, that for the case in which gravity does not act in the x-z plane, the pressure would be constant along any slice $x = x_1$. Thus we infer that here (with gravity), the linear increase in P as we move down vertically in the channel is due to hydrostatic pressure. Therefore, when we add ρgz to P to obtain the modified pressure, it turns out that P' is *constant* at this horizontal location.

We show two solutions in Fig. 1: (a) datum plane $z = 0$ located at the bottom wall, and (b) datum plane $z = 0$ located at the top wall. The shaded region in Fig. 1b represents the hydrostatic pressure component. P' is constant along the slice $x = x_1$ for either case, and the datum plane can be drawn at any arbitrary elevation.

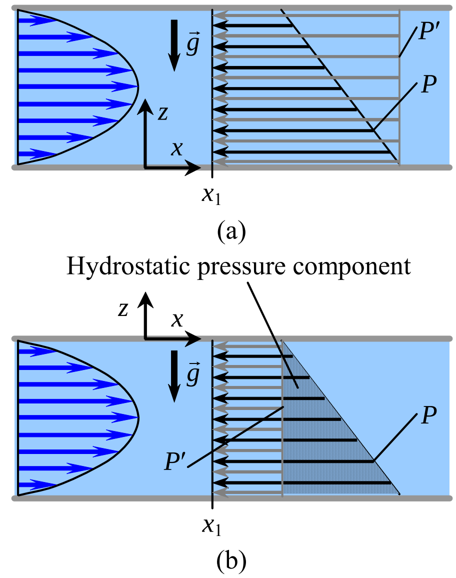


FIGURE 1
Actual pressure P (black arrows) and modified pressure P' (gray arrows) for fully developed planar Poiseuille flow. (a) Datum plane at bottom wall and (b) datum plane at top wall. The hydrostatic pressure component ρgz is the shaded area in (b).

Discussion It should be apparent why it is advantageous to use modified pressure; namely, the gravity term is eliminated from the Navier-Stokes equation, and P' is in general simpler than P.

10-11

Solution We are to discuss how modified pressure varies with downstream distance in planar Poiseuille flow.

Assumptions **1** The flow is incompressible. **2** The flow is fully developed. **3** Gravity acts vertically downward. **4** There are no free surface effects in this flow field.

Analysis For fully developed planar Poiseuille flow between two parallel plates, we know that pressure P decreases linearly with x, the distance down the channel. Modified pressure is defined as $P' = P + \rho gz$. However, since the flow is horizontal, elevation z does not change as we move axially down the channel. Thus we conclude that **modified pressure *P′* decreases linearly with *x***. We sketch both P and P' in Fig. 1 at two axial locations, $x = x_1$ and $x = x_2$. The shaded region in Fig. 1 represents the hydrostatic pressure component ρgz. Since channel height is constant, the hydrostatic component does not change with x. P' is constant along any vertical slice, but its magnitude decreases linearly with x as sketched.

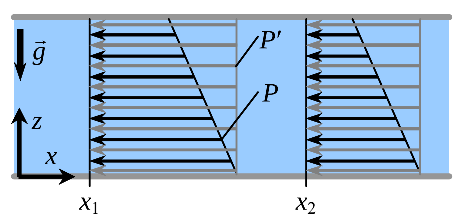


FIGURE 1
Actual pressure P (black arrows) and modified pressure P' (gray arrows) at two axial locations for fully developed planar Poiseuille flow.

Discussion The pressure gradient dP'/dx in terms of modified pressure is the same as the pressure gradient $\partial P/\partial x$ in terms of actual pressure.

10-12

Solution We are to generate an "exact" solution of the Navier-Stokes equation for fully developed Couette flow, using modified pressure. We are to compare to the solution of Chap. 9 that does *not* use modified pressure.

Assumptions We number and list the assumptions for clarity:

1 The plates are infinite in x and z (z is out of the page in the figure associated with this problem).
2 The flow is steady.
3 This is a parallel flow (we assume the y component of velocity, v, is zero).
4 The fluid is incompressible and Newtonian, and the flow is laminar.
5 Pressure P = constant with respect to x. In other words, there is no applied pressure gradient pushing the flow in the x direction; the flow establishes itself due to viscous stresses caused by the moving upper wall. In terms of modified pressure, P' is also constant with respect to x.
6 The velocity field is purely two-dimensional, which implies that $w = 0$ and $\frac{\partial}{\partial z}\left(\text{any velocity component}\right) = 0$.
7 Gravity acts in the negative z direction.

Analysis To obtain the velocity and pressure fields, we follow the step-by-step procedure outlined in Chap. 9.

Step 1 Set up the problem and the geometry. See the figure associated with this problem.
Step 2 List assumptions and boundary conditions. We have already listed seven assumptions. The boundary conditions come from imposing the no slip condition: (1) At the bottom plate ($y = 0$), $u = v = w = 0$. (2) At the top plate ($y = h$), $u = V$, $v = 0$, and $w = 0$. (3) At $z = 0$, $P = P_0$, and thus $P' = P + \rho gz = P_0$.
Step 3 Write out and simplify the differential equations. We start with the continuity equation in Cartesian coordinates,

Continuity:
$$\frac{\partial u}{\partial x} + \underbrace{\cancel{\frac{\partial v}{\partial y}}}_{\text{Assumption 3}} + \underbrace{\cancel{\frac{\partial w}{\partial z}}}_{\text{Assumption 6}} = 0 \qquad \text{or} \qquad \frac{\partial u}{\partial x} = 0 \tag{1}$$

Equation 1 tells us that u is not a function of x. In other words, it doesn't matter where we place our origin – the flow is the same at any x location. I.e., the flow is fully developed. Furthermore, since u is not a function of time (Assumption 2) or z (Assumption 6), we conclude that u is at most a function of y,

Result of continuity:
$$u = u\left(y\right) \text{ only} \tag{2}$$

We now simplify the x momentum equation as far as possible:

$$\rho\left(\underbrace{\cancel{\frac{\partial u}{\partial t}}}_{\text{Assumption 2}} + \underbrace{u\cancel{\frac{\partial u}{\partial x}}}_{\text{Continuity}} + \underbrace{v\cancel{\frac{\partial u}{\partial y}}}_{\text{Assumption 3}} + \underbrace{w\cancel{\frac{\partial u}{\partial z}}}_{\text{Assumption 6}}\right) = \underbrace{-\cancel{\frac{\partial P'}{\partial x}}}_{\text{Assumption 5}} + \mu\left(\underbrace{\cancel{\frac{\partial^2 u}{\partial x^2}}}_{\text{Continuity}} + \frac{\partial^2 u}{\partial y^2} + \underbrace{\cancel{\frac{\partial^2 u}{\partial z^2}}}_{\text{Assumption 6}}\right) \quad \rightarrow \quad \frac{d^2 u}{dy^2} = 0 \tag{3}$$

All other terms in Eq. 3 have disappeared except for a lone viscous term, which must then itself equal zero. Notice that we have changed from a partial derivative ($\partial/\partial y$) to a total derivative (d/dy) in Eq. 3 as a direct result of Eq. 2. We do not show the details here, but you can show in similar fashion that every term except the pressure term in the y momentum equation goes to zero, forcing that lone term to also be zero,

y momentum:
$$\frac{\partial P'}{\partial y} = 0 \tag{4}$$

The same thing happens to the z momentum equation; the result is

z momentum:
$$\frac{\partial P'}{\partial z} = 0 \tag{5}$$

In other words, P' is not a function of y or z. Since P' is also not a function of time (Assumption 2) or x (Assumption 5), P' is a constant,

Result of y and z momentum:
$$P' = \text{ constant} = C_3 \tag{6}$$

Step 4 Solve the differential equations. Continuity, y momentum, and z momentum have already been "solved", resulting in Eqs. 2 and 6. Equation 3 (x momentum) is integrated twice to get

Integration of x momentum: $$u = C_1 y + C_2 \qquad (7)$$

where C_1 and C_2 are constants of integration.

Step 5 We apply boundary condition (3), $P' = P_0$ at $z = 0$. Eq. 6 yields $C_3 = P_0$, and

Final solution for pressure field: $$\boxed{P' = P_0 \quad \rightarrow \quad P = P_0 - \rho g z} \qquad (9)$$

We next apply boundary conditions (1) and (2) to obtain constants C_1 and C_2.

Boundary condition (1): $$u = C_1(0) + C_2 = 0 \qquad \text{or} \qquad C_2 = 0$$

and

Boundary condition (2): $$u = C_1(h) + 0 = V \qquad \text{or} \qquad C_1 = \frac{V}{h}$$

Finally, Eq. 7 becomes

Final result for velocity field: $$\boxed{u = V\frac{y}{h}} \qquad (10)$$

The velocity field reveals a simple linear velocity profile from $u = 0$ at the bottom plate to $u = V$ at the top plate.

Step 6 Verify the results. You can plug in the velocity and pressure fields to verify that all the differential equations and boundary conditions are satisfied.

We verify that **the results are identical to those of Example 9-15**. Thus, we get the same result using modified pressure throughout the calculation as we do using the regular (thermodynamic) pressure throughout the calculation.

Discussion Since there are no free surfaces in this problem, the gravity term in the Navier-Stokes equation is absorbed into the modified pressure, and the pressure and gravity terms are combined into one term. This is possible since flow pressure and hydrostatic pressure are uncoupled.

10-13

Solution We are to write all three components of the Navier-Stokes equation in terms of modified pressure, and show that they are equivalent to the equations with regular pressure. We are also to discuss the advantage of using modified pressure.

Analysis In terms of modified pressure, the Navier-Stokes equation is written in Cartesian components as

x component:
$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right)=-\frac{\partial P'}{\partial x}+\mu\nabla^2 u \qquad (1)$$

and

y component:
$$\rho\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}\right)=-\frac{\partial P'}{\partial y}+\mu\nabla^2 v \qquad (2)$$

and

z component:
$$\rho\left(\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\right)=-\frac{\partial P'}{\partial z}+\mu\nabla^2 w \qquad (3)$$

The definition of modified pressure is

Modified pressure:
$$P' = P + \rho g z \qquad (4)$$

When Eq. 4 is plugged into Eqs. 1 and 2, the gravity term disappears since z is independent of x and y. The result is

x component:
$$\boxed{\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right)=-\frac{\partial P}{\partial x}+\mu\nabla^2 u} \qquad (5)$$

and

y component:
$$\boxed{\rho\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}\right)=-\frac{\partial P}{\partial y}+\mu\nabla^2 v} \qquad (6)$$

However, when Eq. 4 is plugged into Eq. 3, the result is

z component:
$$\boxed{\rho\left(\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\right)=-\frac{\partial P}{\partial z}-\rho g+\mu\nabla^2 w} \qquad (7)$$

Equations 5 through 7 are the appropriate components of the Navier-Stokes equation in terms of regular pressure, so long as gravity acts downward (in the $-z$ direction).

The advantage of using modified pressure is that the gravity term disappears from the Navier-Stokes equation.

Discussion Modified pressure can be used only when there are no free surfaces.

10-14

Solution We are to sketch the profile of actual pressure and shade in the region representing hydrostatic pressure.

Assumptions **1** The flow is incompressible. **2** Gravity acts vertically downward. **3** There are no free surface effects in this flow field.

Analysis By definition, modified pressure $P' = P + \rho gz$. Thus, to obtain actual pressure P, we subtract the hydrostatic component ρgz from the given profile of P'. Using the given value of P at the mid-way point as a guide, **we sketch the actual pressure in Fig. 1** such that the difference between P' and P increases linearly. In other words, we subtract the hydrostatic pressure component ρgz from the modified pressure P' to obtain the profile for actual pressure P.

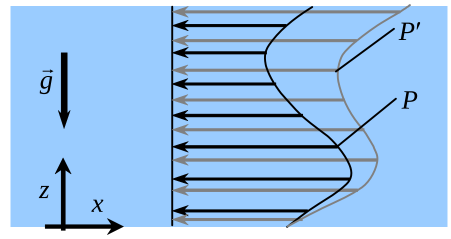


FIGURE 1
Actual pressure P (black arrows) and modified pressure P' (gray arrows) for the given pressure field.

Discussion We assume that there are no free surface effects in the problem; otherwise modified pressure should not be used. The datum plane is set in the problem statement, but any arbitrary elevation could be used instead. If the datum plane were set at the top of the domain, P' would be less than P everywhere because of the negative values of z in the transformation from P to P'.

10-15

Solution We are to solve the Navier-Stokes equation in terms of modified pressure for the case of steady, fully developed, laminar flow in a round pipe. We are to obtain expressions for the pressure and velocity fields, and compare the actual pressure at the top of the pipe to that at the bottom of the pipe.

Assumptions We make the same assumptions as in Example 9-18, except we use modified pressure P' in place of actual pressure P.

Analysis The Navier-Stokes equation with gravity, written in terms of modified pressure P', is identical to the Navier-Stokes equation with no gravity, written in terms of actual pressure P. In other words, all of the algebra of Example 9-18 remains the same, except we use modified pressure P' in place of actual pressure P. The velocity field does not change, and the result is

Axial velocity field:
$$\boxed{u = \frac{1}{4\mu}\frac{dP}{dx}\left(r^2 - R^2\right)} \qquad (1)$$

The modified pressure field is

Modified pressure field:
$$P' = P'(x) = P_1' + \frac{dP'}{dx}dx \qquad (2)$$

where P_1' is the modified pressure at location $x = x_1$. In Example 9-18, the actual pressure varies only with x. In fact it decreases linearly with x (note that the pressure gradient is negative for flow from left to right). Here, Eq. 2 shows that modified pressure behaves in the same way, namely P' varies only with x, and in fact decreases linearly with x.

We simply subtract the hydrostatic pressure component ρgz from modified pressure P' (Eq. 2) to obtain the final expression for actual pressure P,

Actual pressure field:
$$\boxed{P = P' - \rho gz \quad \rightarrow \quad P = P_1' + \frac{dP'}{dx}dx - \rho gz} \qquad (3)$$

Since the pipe is horizontal, the bottom of the pipe is lower than the top of the pipe. Thus, z_{top} is greater than z_{bottom}, and therefore by Eq. 3 P_{top} *is less than* P_{bottom}. This agrees with our experience that pressure increases downward.

Discussion Since there are no free surfaces in this flow, the gravity term does not directly influence the velocity field, and a hydrostatic component is added to the pressure field. You can see the advantage of using modified pressure.

Creeping Flow

10-16C

Solution We are to name each term in the Navier-Stokes equation, and then discuss which terms remain when the creeping flow approximation is made.

Analysis The terms in the equation are identified as follows:

- **I** ***Unsteady*** **term**
- **II** ***Inertial*** **term**
- **III** ***Pressure*** **term**
- **IV** ***Gravity*** **term**
- **V** ***Viscous*** **term**

When the creeping flow approximation is made, **only terms III (pressure) and V (viscous) remain**. The other three terms are very small compared to these two and can be ignored. The significance is that all unsteady and inertial effects (terms I and II) have disappeared, as has gravity. We are left with a flow in which pressure forces and viscous forces must balance. Another significant result is that density has disappeared from the creeping flow equation, as discussed in the text.

Discussion There are other acceptable one-word descriptions of some of the terms in the equation. For example, the inertial term can also be called the convective term, the advective term, or the acceleration term.

10-17

Solution We are to estimate the maximum speed of honey through a hole such that the Reynolds number remains below 0.1, at two different temperatures.

Analysis The density of honey is equal to its specific gravity times the density of water,

Density of honey:
$$\rho_{\text{honey}} = SG_{\text{honey}}\rho_{\text{water}} = 1.42\left(998.0 \text{ kg/m}^3\right) = 1420 \text{ kg/m}^3 \qquad (1)$$

We convert the viscosity of honey from poise to standard SI units,

Viscosity of honey at 20^o:
$$\mu_{\text{honey}} = 190 \text{ poise}\left(\frac{\text{g}}{\text{cm}\cdot\text{s}\cdot\text{poise}}\right)\left(\frac{\text{kg}}{1000 \text{ g}}\right)\left(\frac{100 \text{ cm}}{\text{m}}\right) = 19.0\frac{\text{kg}}{\text{m}\cdot\text{s}} \qquad (2)$$

Finally, we plug Eqs. 1 and 2 into the definition of Reynolds number, and set Re = 0.1 to solve for the maximum speed to ensure creeping flow,

Maximum speed for creeping flow at 20^o:
$$V_{\max} = \frac{\text{Re}_{\max}\mu_{\text{honey}}}{\rho_{\text{honey}}D} = \frac{(0.1)(19.0 \text{ kg/m}\cdot\text{s})}{\left(1420 \text{ kg/m}^3\right)(0.0040 \text{ m})} = \mathbf{0.33\ m/s} \qquad (3)$$

At the higher temperature of 40^{o}C, the calculations yield $V_{\max}$ = **0.035 m/s**. Thus, it is much easier to achieve creeping flow with honey at lower temperatures since the viscosity of honey increases rapidly as the temperature drops.

Discussion We used Re < 0.1 as the maximum Reynolds number for creeping flow, but experiments reveal that in many flows, the creeping flow approximation is acceptable at Reynolds numbers as high as nearly 1.0.

10-18

Solution For each case we are to calculate the Reynolds number and determine if the creeping flow approximation is appropriate.

Assumptions **1** The values given are characteristic scales of the motion.

Properties For water at $T = 20°C$, $\rho = 998.0$ kg/m^3 and $\mu = 1.002 \times 10^{-3}$ kg/m·s. For unused engine oil at $T = 140°C$, $\rho = 816.8$ kg/m^3 and $\mu = 6.558 \times 10^{-3}$ kg/m·s. For air at $T = 30°C$, $\rho = 1.164$ kg/m^3 and $\mu = 1.872 \times 10^{-5}$ kg/m·s.

Analysis (*a*) The Reynolds number of the microorganism is

$$\text{Re} = \frac{\rho DV}{\mu} = \frac{\left(998.0\text{ kg/m}^3\right)\left(5.0\times10^{-6}\text{ m}\right)\left(0.2\text{ mm/s}\right)}{1.002\times10^{-3}\text{ kg/m}\cdot\text{s}}\left(\frac{\text{m}}{1000\text{ mm}}\right) = \mathbf{9.96\times10^{-4}}$$

Since Re << 1, the creeping flow approximation is certainly **appropriate**.

(*b*) The Reynolds number of the oil in the gap is

$$\text{Re} = \frac{\rho DV}{\mu} = \frac{\left(816.8\text{ kg/m}^3\right)\left(0.0012\text{ mm}\right)\left(20.0\text{ m/s}\right)}{6.558\times10^{-3}\text{ kg/m}\cdot\text{s}}\left(\frac{\text{m}}{1000\text{ mm}}\right) = \mathbf{2.99}$$

Since Re > 1, the creeping flow approximation is ***not*** **appropriate**.

(*c*) The Reynolds number of the fog droplet is

$$\text{Re} = \frac{\rho DV}{\mu} = \frac{\left(1.164\text{ kg/m}^3\right)\left(10\times10^{-6}\text{ m}\right)\left(3.0\text{ mm/s}\right)}{1.872\times10^{-5}\text{ kg/m}\cdot\text{s}}\left(\frac{\text{m}}{1000\text{ mm}}\right) = \mathbf{1.87\times10^{-3}}$$

Since Re << 1, the creeping flow approximation is certainly **appropriate**.

Discussion At room temperature, the oil viscosity increases by a factor of more than a hundred, and the Reynolds number of the bearing of Part (b) would be of order 10^{-2}, which *is* in the creeping flow range.

10-19

Solution We are to estimate the speed and Reynolds number from a multiple-image photograph.

Assumptions **1** The characteristic speed is taken as the average over 10 images.

Properties For water at $T = 20^\circ\text{C}$, $\rho = 998.0\ \text{kg/m}^3$ and $\mu = 1.002 \times 10^{-3}$ kg/m·s.

Analysis By measurement with a ruler, we estimate the sperm's diameter as 2.4 μm, and it moves about 7.7 μm in 10 frames. This represents a time of

Time for 10 frames:
$$T = \frac{10\ \text{frames}}{200\ \text{frames/s}} = 0.050\ \text{s}$$

Thus the sperm's speed is

Approximate speed:
$$V = \frac{x}{T} = \frac{7.7\ \mu\text{m}}{0.050\ \text{s}}\left(\frac{\text{m}}{10^6\ \mu\text{m}}\right) = \mathbf{1.5 \times 10^{-4}\ m/s}$$

and its Reynolds number is

Reynolds number:
$$\text{Re} = \frac{\rho D V}{\mu} = \frac{\left(998.0\ \text{kg/m}^3\right)\left(2.4\times10^{-6}\ \text{m}\right)\left(1.5\times10^{-4}\ \text{m/s}\right)}{1.002\times10^{-3}\ \text{kg/m}\cdot\text{s}} = 3.59\times10^{-4} \cong \mathbf{3.6\times10^{-4}}$$

Since Re << 1, the creeping flow approximation is certainly appropriate.

Note: Students' answers may differ widely since the measurements from the photograph are not very accurate. We report the final answer to only two significant digits because of the inherent error in measuring distances from the photograph.

Discussion If you use the cell's length rather than its diameter as the characteristic length scale, Re increases by a factor of about two, but the flow is still well within the creeping flow regime.

10-20

Solution We are to compare the number of body lengths per second of a swimming human and a swimming sperm.

Analysis We let *BLPS* denote "body lengths per second". For the human swimmer,

Human:
$$BLPS_{\text{human}} = \frac{100\ \text{m/min}}{1.8\ \text{m/body length}}\left(\frac{\text{min}}{60\ \text{s}}\right) = \mathbf{0.93\ body\ length/s}$$

For the sperm, we use the speed calculated in Problem 10-19. The total body length of the sperm (head and tail) is about 40 μm, as measured from the figure.

Sperm:
$$BLPS_{\text{sperm}} = \frac{1.5\times10^{-4}\ \text{m/s}}{40\ \mu\text{m/body length}}\left(\frac{10^6\ \mu\text{m}}{\text{m}}\right) = \mathbf{3.8\ body\ length/s}$$

So, on an equal basis of comparison, the sperm swims faster than the human! This result is perhaps surprising since the human benefits from inertia, while the sperm feels no inertial effects. However, we must keep in mind that the sperm's body is designed to swim, while the human body is designed for multiple uses – it is not optimized for swimming.

Note: Students' answers may differ widely since the measurements from the photograph are not very accurate.

Discussion Perhaps a more fair comparison would be between a *fish* and a sperm.

10-21

Solution We are to calculate how fast air must move vertically to keep a water drop suspended in the air.

Assumptions **1** The drop is spherical. **2** The creeping flow approximation is appropriate.

Properties For air at T = 25°C, ρ = 1.184 kg/m³ and μ = 1.849 × 10⁻⁵ kg/m·s. The density of the water at T = 25°C is 997.0 kg/m³.

Analysis Since the drop is sitting still, its downward force must exactly balance its upward force when the vertical air speed V is "just right". The downward force is the weight of the particle:

Downward force on the particle:
$$F_{\text{down}} = \pi \frac{D^3}{6} \rho_{\text{particle}} g \qquad (1)$$

The upward force is the aerodynamic drag force acting on the particle plus the buoyancy force on the particle. The aerodynamic drag force is obtained from the creeping flow drag on a sphere,

Upward force on the particle:
$$F_{\text{up}} = 3\pi\mu VD + \pi \frac{D^3}{6} \rho_{\text{air}} g \qquad (2)$$

We equate Eqs. 1 and 2, i.e., $F_{\text{down}} = F_{\text{up}}$,

Balance:
$$\pi \frac{D^3}{6} \left(\rho_{\text{particle}} - \rho_{\text{air}}\right) g = 3\pi\mu VD$$

and solve for the required air speed V,

$$V = \frac{D^2}{18\mu}\left(\rho_{\text{particle}} - \rho_{\text{air}}\right) g = \frac{\left(30\times10^{-6}\text{ m}\right)^2}{18\left(1.849\times10^{-5}\text{ kg/m}\cdot\text{s}\right)}\left[\left(998.0-1.184\right)\text{ kg/m}^3\right]\left(9.81\text{ m/s}^2\right) = \mathbf{0.0264\ m/s}$$

Finally, we must verify that the Reynolds number is small enough that the creeping flow approximation is appropriate.

Check of Reynolds number:
$$\text{Re} = \frac{\rho_{\text{air}} VD}{\mu} = \frac{\left(1.184\text{ kg/m}^3\right)\left(0.0264\text{ m/s}\right)\left(30\times10^{-6}\text{ m}\right)}{1.849\times10^{-5}\text{ kg/m}\cdot\text{s}} = \mathbf{0.0507}$$

Since Re << 1, The creeping flow approximation is appropriate.

Discussion Notice that although air density does appear in the calculation of V, it is very small compared to the density of water. (If we ignore ρ_{air} in that calculation, we get V = 0.0265 m/s, an error of less than 0.4%. However, ρ_{air} *is* required in the calculation of Reynolds number – to verify that the creeping flow approximation is appropriate.

10-22

Solution We are to discuss why density is not a factor in aerodynamic drag on a particle in creeping flow.

Analysis It turns out that fluid density drops out of the creeping flow equations, since **the terms that contain ρ in the Navier-Stokes equation are negligibly small compared to the pressure and viscous terms (which do not contain ρ)**. Another way to think about this is: In creeping flow, there is no fluid inertia, and since inertia is associated with fluid mass (density), density cannot contribute to the aerodynamic drag on a particle moving in creeping flow. In creeping flow, there is a balance between pressure forces and viscous forces, neither of which depend on fluid density.

Discussion Density does have an *indirect* influence on creeping flow drag. Namely, ρ is needed in the Reynolds number calculation, and Re determines whether the flow is in the creeping flow regime or not.

10-23

Solution We are to generate a characteristic pressure scale for flow through a slipper-pad bearing.

Assumptions **1** The flow is steady and incompressible. **2** The flow is two-dimensional in the *x*-*y* plane. **3** The creeping flow approximation is appropriate.

Analysis The *x* component of the creeping flow momentum equation is

x momentum:

$$\frac{\partial P}{\partial x} \approx \mu \nabla^2 u \quad \rightarrow \quad \frac{\partial P}{\partial x} \approx \mu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \underbrace{\cancel{\frac{\partial^2 u}{\partial z^2}}}_{2-D} \right)$$

We plug in the characteristic scales to get

Orders of magnitude:

$$\underbrace{\frac{\partial P}{\partial x}}_{\frac{\Delta P}{L}} \approx \mu \underbrace{\frac{\partial^2 u}{\partial x^2}}_{\frac{V}{L^2}} + \mu \underbrace{\frac{\partial^2 u}{\partial y^2}}_{\frac{V}{h_0^2}} \qquad (1)$$

The first term on the right of Eq. 1 is clearly much smaller than the second term on the right since $h_0 << L$. Equating the orders of magnitude of the two remaining terms,

Characteristic pressure scale:

$$\frac{\Delta P}{L} \sim \mu \frac{V}{h_0^2} \rightarrow \boxed{\Delta P \sim \frac{\mu V L}{h_0^2}} \qquad (2)$$

Discussion The characteristic pressure scale differs from that in the text because there are two length scales in this problem rather than just one.

10-24

Solution We are to find a characteristic velocity scale for *v*, compare the inertial terms of the *x* momentum equation to the pressure and viscous terms, and discuss how the creeping flow equations can still be used even if Re is not small.

Assumptions **1** The flow is steady and incompressible. **2** The flow is two-dimensional in the *x*-*y* plane. **3** Gravity forces are negligible.

Analysis (*a*) We use the continuity equation to obtain the characteristic velocity scale for *v*,

Continuity:

$$\underbrace{\frac{\partial u}{\partial x}}_{\frac{V}{L}} + \underbrace{\frac{\partial v}{\partial y}}_{\frac{v}{h_0}} = 0 \rightarrow \boxed{v \sim \frac{V h_0}{L}} \qquad (1)$$

(*b*) We analyze the orders of magnitude of each term in the steady, 2-D, incompressible *x* momentum equation without gravity,

x momentum:

$$\underbrace{\rho u \frac{\partial u}{\partial x}}_{\frac{\rho V^2}{L}} + \underbrace{\rho v \frac{\partial u}{\partial y}}_{\rho \frac{V h_0}{L}\frac{V}{h_0} = \frac{\rho V^2}{L}} = -\underbrace{\frac{\partial P}{\partial x}}_{\frac{\mu V}{h_0^2}} + \underbrace{\mu \frac{\partial^2 u}{\partial x^2}}_{\frac{\mu V}{L^2}} + \underbrace{\mu \frac{\partial^2 u}{\partial y^2}}_{\frac{\mu V}{h_0^2}} \qquad (2)$$

where we have also used the result of Problem 10-23. The first viscous term of Eq. 2 is clearly much smaller than the second viscous term since $h_0 << L$. We multiply the order of magnitude of all the remaining terms by $L/(\rho V^2)$ to compare terms,

Comparison of orders of magnitude:

$$\underbrace{\rho u \frac{\partial u}{\partial x}}_{1} + \underbrace{\rho v \frac{\partial u}{\partial y}}_{1} = -\underbrace{\frac{\partial P}{\partial x}}_{\frac{\mu}{\rho V h_0}\frac{L}{h_0}} + \underbrace{\mu \frac{\partial^2 u}{\partial y^2}}_{\frac{\mu}{\rho V h_0}\frac{L}{h_0}} \qquad (3)$$

We recognize the Reynolds number based on gap height, $\text{Re} = \rho V h_0/\mu$. Since the pressure and viscous terms contain the product of 1/Re, which is large for creeping flow, and L/h_0, which is also large, it is clear that **the inertial terms (left side of Eq. 3) are negligibly small compared to the pressure and viscous terms**.

(*c*) Since the pressure and viscous terms contain the product of 1/Re and L/h_0, when $h_0 << L$, **the creeping flow equations can still be appropriate even if Reynolds number is not less than one**. For example, if $L/h_0 \sim 10{,}000$ and Re ~ 10, the pressure and viscous terms are still three orders of magnitude larger than the inertial terms.

Discussion In the limit as $L/h_0 \rightarrow \infty$, the inertial terms disappear regardless of the Reynolds number. This limiting case is the Couette flow problem of Chap. 9.

10-25

Solution We are to analyze the *y* momentum equation by order of magnitude analysis, and we are to comment about the pressure gradient $\partial P/\partial y$.

Assumptions **1** The flow is steady and incompressible. **2** The flow is two-dimensional in the *x*-*y* plane. **3** Gravity forces are negligible.

Analysis We analyze the orders of magnitude of each term in the steady, 2-D, incompressible *y* momentum equation without gravity,

y momentum:

$$\underbrace{\rho u\frac{\partial v}{\partial x}}_{\rho V\frac{Vh_0}{L}\frac{1}{L}=\frac{\rho V^2 h_0}{L^2}} + \underbrace{\rho v\frac{\partial v}{\partial y}}_{\rho\frac{V^2 h_0^2}{L^2}\frac{1}{h_0}=\frac{\rho V^2 h_0}{L^2}} = -\underbrace{\frac{\partial P}{\partial y}}_{\frac{\mu V L}{h_0^3}} + \underbrace{\mu\frac{\partial^2 v}{\partial x^2}}_{\mu\frac{Vh_0}{L}\frac{1}{L^2}=\frac{\mu V h_0}{L^3}} + \underbrace{\mu\frac{\partial^2 v}{\partial y^2}}_{\mu\frac{Vh_0}{L}\frac{1}{h_0^2}=\frac{\mu V}{L h_0}} \qquad (1)$$

The first viscous term of Eq. 1 is clearly much smaller than the second viscous term, since $h_0 << L$. We multiply the order of magnitude of all the remaining terms by $L^2/(\rho V^2 h_0)$ to compare terms,

Comparison of orders of magnitude:

$$\underbrace{\rho u\frac{\partial v}{\partial x}}_{1} + \underbrace{\rho v\frac{\partial v}{\partial y}}_{1} = -\underbrace{\frac{\partial P}{\partial y}}_{\frac{\mu}{\rho V h_0}\left(\frac{L}{h_0}\right)^3} + \underbrace{\mu\frac{\partial^2 v}{\partial y^2}}_{\frac{\mu}{\rho V h_0}\left(\frac{L}{h_0}\right)} \qquad (2)$$

We recognize the Reynolds number based on gap height, $\text{Re} = \rho V h_0/\mu$. Since the pressure and viscous terms contain the product of 1/Re, which is large for creeping flow, and L/h_0, which is also large, it is clear that the inertial terms (left side of Eq. 2) are negligibly small compared to the pressure and viscous terms. This is expected, of course, for creeping flow. Now we compare the pressure and viscous terms. Both contain 1/Re, but the pressure term has and additional factor of $(L/h_0)^2$, which is very large. Thus the pressure term is the only remaining term in Eq. 2. How can this be? Since there are no terms that can balance the pressure term, the pressure term itself must be very small. In other words, the *y* momentum equation reduces to

Final form of y momentum:

$$\boxed{\frac{\partial P}{\partial y} \approx 0} \qquad (3)$$

In other words, **pressure is a function of *x*, but a very weak, negligible function of *y***.

Discussion The result here is very similar to that for boundary layers, where we also find that $\partial P/\partial y \approx 0$ through the boundary layer.

10-26

Solution We are to list boundary conditions and solve the *x* momentum equation for *u*. Then we are to nondimensionalize our result.

Assumptions **1** The flow is steady and incompressible. **2** Gravity forces are negligible. **3** The flow is two-dimensional in the *x*-*y* plane. **4** *P* is not a function of *y*.

Analysis (*a*) From the figure associated with this problem, we write two boundary conditions on *u*,

Boundary condition (1): $$\boxed{u = V \text{ at } y = 0 \text{ for all } x} \qquad (1)$$

and

Boundary condition (2): $$\boxed{u = 0 \text{ at } y = h \text{ for all } x} \qquad (2)$$

We note that *h* is not a constant, but rather a function of *x*.

(*b*) We write the creeping flow *x* momentum equation, and integrate once with respect to *y*, noting that *P* is not a function of *y*. This is a *partial* integration.

Integration of x momentum: $$\frac{\partial^2 u}{\partial y^2} = \frac{1}{\mu}\frac{dP}{dx} \quad \rightarrow \quad \frac{\partial u}{\partial y} = \frac{1}{\mu}\frac{dP}{dx} y + f_1(x)$$

We integrate again to obtain

Second integration: $$u = \frac{1}{2\mu}\frac{dP}{dx} y^2 + y f_1(x) + f_2(x) \qquad (3)$$

We apply boundary conditions to find the two unknown functions of *x*. From Eq. 1,

Result of boundary condition (1): $$f_2(x) = V$$

and from Eq. 2,

Result of boundary condition (2): $$f_1(x) = \frac{-V - \frac{1}{2\mu}\frac{dP}{dx} h^2}{h}$$

From these, the final expression for *u* is obtained,

Final expression for u, dimensional: $$\boxed{u(x, y) = V\left(1 - \frac{y}{h}\right) + \frac{h^2}{2\mu}\frac{dP}{dx}\frac{y}{h}\left(\frac{y}{h} - 1\right)} \qquad (4)$$

We recognize two distinct components of the velocity profile in Eq. 4, namely *a Couette flow component and a Poiseuille flow component*. Thus, the axial velocity is a superposition of Couette flow due to the moving bottom wall and Poiseuille flow due to the pressure gradient.

(*c*) We nondimensionalize Eq. 4 by applying the nondimensional variables given in the problem statement. After some algebra,

Nondimensional expression for u: $$\boxed{u^* = (1 - y^*) + \frac{h^{*2}}{2}\frac{dP^*}{dx^*} y^*(y^* - 1)} \qquad (5)$$

Discussion Although we have a final expression for *u*, it is in terms of the pressure gradient *dP*/*dx*, which is not known. Pressure boundary conditions and further algebra are required to solve for the pressure field.

10-27

Solution We are to generate an expression for axial velocity for a slipper-pad bearing with arbitrary gap shape.

Assumptions **1** The flow is steady and incompressible. **2** Gravity forces are negligible. **3** The flow is two-dimensional in the *x*-*y* plane. **4** *P* is not a function of *y*.

Analysis In the solution of Problem 10-26, we never used the fact that *h*(*x*) was linear. In fact, our solution is in terms of *h*(*x*), the specific form of which was never specified. Thus, the solution of Problem 10-26 is still appropriate, and no further work needs to be done here. The result is

Expression for u for arbitrary h(x):
$$\boxed{u(x,y)=V\left(1-\frac{y}{h}\right)+\frac{h^2}{2\mu}\frac{dP}{dx}\frac{y}{h}\left(\frac{y}{h}-1\right)} \qquad (1)$$

Discussion As gap height *h*(*x*) changes, so does the pressure distribution.

10-28

Solution We are to prove the given equation for the slipper-pad bearing.

Assumptions **1** The flow is steady and incompressible. **2** The flow is two-dimensional in the *x*-*y* plane.

Analysis We solve the 2-D continuity equation for *v* by integration,

Continuity:
$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \quad \rightarrow \quad \int_0^h \frac{\partial v}{\partial y}dy=-\int_0^h \frac{\partial u}{\partial x}dy \quad \rightarrow \quad v(h)-v(0)=-\int_0^h \frac{\partial u}{\partial x}dy \qquad (1)$$

But the no-slip condition tells us that *v* = 0 at both the bottom (*y* = 0) and top (*y* = *h*) plates. Thus Eq. 1 reduces to

Result of continuity:
$$\int_0^h \frac{\partial u}{\partial x}dy=0 \qquad (2)$$

The 1-D Leibnitz theorem is discussed in Chap. 4 and is repeated here:

1-D Leibnitz theorem:
$$\frac{d}{dx}\int_{a(x)}^{b(x)} G(x,y)\,dy=\int_a^b \frac{\partial G}{\partial x}dy+\frac{db}{dx}G(x,b)-\frac{da}{dx}G(x,a) \qquad (3)$$

In our case (comparing Eqs. 2 and 3), *a* = 0, *b* = *h*(*x*), and *G* = *u*. Thus,

$$\frac{d}{dx}\int_0^h u\,dy=\int_0^h \frac{\partial u}{\partial x}dy+\frac{dh}{dx}u(x,h) \qquad (4)$$

But *u*(*h*) = 0 for all values of *x* (no-slip condition). Finally then, we combine Eqs. 2 and 4 to yield the desired result,

Final result:
$$\boxed{\frac{d}{dx}\int_0^h u\,dy=0}$$

Discussion This result could also be obtained by control volume conservation of mass. Now we finally have the means of calculating the pressure distribution in the slipper-pad bearing.

10-29

Solution We are to prove the given equation for flow through a 2-D slipper-pad bearing.

Assumptions **1** The flow is steady and incompressible. **2** Gravity forces are negligible. **3** The flow is two-dimensional in the *x*-*y* plane. **4** *P* is not a function of *y*.

Analysis We substitute the expression for *u* from Problem 10-26 into the equation of Problem 10-28,

$$\frac{d}{dx}\int_0^h u dy = 0 \quad \rightarrow \quad \frac{d}{dx}\int_0^h \left[V\left(1-\frac{y}{h}\right)+\frac{h^2}{2\mu}\frac{dP}{dx}\frac{y}{h}\left(\frac{y}{h}-1\right)\right] dy = 0 \qquad (1)$$

The integral in Eq. 1 is easily evaluated since both *h* and *dP*/*dx* are functions of *x* only. After some algebra,

$$\frac{d}{dx}\left[V\frac{h}{2}-\frac{h^3}{12\mu}\frac{dP}{dx}\right] = 0 \qquad (2)$$

Finally, we take the *x* derivative, recognizing that *h* and *dP*/*dx* are functions of *x*,

Steady, 2-D Reynolds equation for lubrication:

$$\boxed{\frac{d}{dx}\left(h^3\frac{dP}{dx}\right) = 6\mu V\frac{dh}{dx}} \qquad (3)$$

Discussion For a given geometry (*h* as a known function of *x*), we can integrate Eq. 3 to obtain the pressure distribution along the slipper-pad bearing.

10-30

Solution We are to find the pressure distribution for flow through a 2-D slipper-pad bearing with linearly decreasing gap height and atmospheric pressure at both ends of the slipper-pad.

Assumptions **1** The flow is steady and incompressible. **2** Gravity forces are negligible. **3** The flow is two-dimensional in the *x*-*y* plane. **4** *P* is not a function of *y*.

Analysis We integrate the Reynolds equation of Problem 10-29, and rearrange:

First integration:

$$h^3\frac{dP}{dx} = 6\mu Vh + C_1 \quad \rightarrow \quad \frac{dP}{dx} = 6\mu Vh^{-2} + C_1h^{-3} \qquad (1)$$

where C_1 is a constant of integration. Next we substitute the given equation for *h*,

$$\frac{dP}{dx} = 6\mu V\left(h_0+\alpha x\right)^{-2} + C_1\left(h_0+\alpha x\right)^{-3} \qquad (2)$$

Equation 2 is in the desired form, i.e., *dP*/*dx* as a function of *x*. We integrate Eq. 2,

Second integration:

$$P = -\frac{6\mu V}{\alpha}\left(h_0+\alpha x\right)^{-1} - \frac{C_1}{2\alpha}\left(h_0+\alpha x\right)^{-2} + C_2 \qquad (3)$$

where C_2 is a second constant of integration. We plug in the two boundary conditions on *P* to find constants C_1 and C_2, namely $P = P_{atm}$ at $x = 0$ and $P = P_{atm}$ at $x = L$. After some algebra, the results are

Constants:

$$C_1 = -\frac{12\mu Vh_0h_L}{h_0+h_L} \quad \text{and} \quad C_2 = P_{atm} + \frac{6\mu V}{\alpha\left(h_0+h_L\right)} \qquad (4)$$

with which we generate our final expression for *P* from Eq. 3. After some algebra,

Pressure distribution:

$$\boxed{P = P_{atm} + 6\mu Vx\left[\frac{h_0-h_L+\alpha x}{\left(h_0+h_L\right)\left(h_0+\alpha x\right)^2}\right]} \qquad (5)$$

Discussion There are other equivalent ways to write the expression for *P*, but Eq. 5 is about as compact as we can get.

10-31E *[Also solved using EES on enclosed DVD]*

Solution We are to calculate α, we are to calculate P_{gage} at a given x location, and we are to plot nondimensional gage pressure as a function of nondimensional axial distance for the case of a slipper-pad bearing with linearly decreasing gap height. Finally, we are to estimate the total force that this slipper-pad bearing can support.

Assumptions **1** The flow is steady and incompressible. **2** Gravity forces are negligible in the oil flow. **3** The flow is two-dimensional in the *x-y* plane. **4** P is not a function of y.

Properties Unused engine oil at T = 40°C: ρ = 876.0 kg/m³, μ = 0.2177 kg/m·s.

Analysis (*a*) The convergence is calculated by its definition (see Problem 10-30), and its tangent is also calculated,

$$\alpha = \frac{h_L - h_0}{L} = \frac{(0.0005 - 0.001)\text{ inch}}{1.0\text{ inch}} = \mathbf{-0.0005}$$

$$\rightarrow \qquad \tan\alpha = \mathbf{-0.0005}$$

Note that we must set α to radians when taking the tangent.

(*b*) At x = 0.5 inches (0.0127 m), we calculate $P_{gage} = P - P_{atm}$ using the result of Problem 10-30; the gage pressure at the mid-way point is

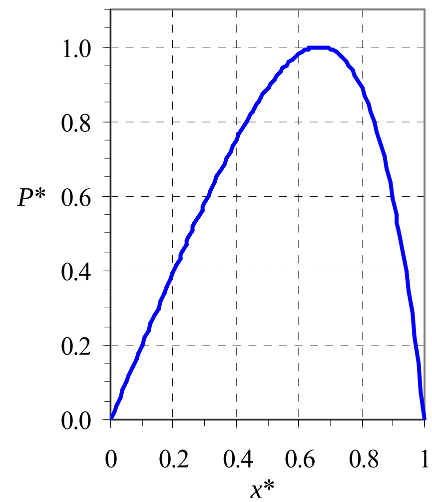


FIGURE 1
Nondimensional gage pressure in a slipper-pad bearing as a function of nondimensional axial distance along the slipper-pad.

$$P_{gage} = P - P_{atm} = 6\mu Vx\left[\frac{h_0 - h_L + \alpha x}{(h_0 + h_L)(h_0 + \alpha x)^2}\right]$$

$$= 6\left(0.2177\frac{\text{kg}}{\text{m}\cdot\text{s}}\right)\left(3.048\frac{\text{m}}{\text{s}}\right)(0.0127\text{ m})\left(\frac{\text{N s}^2}{\text{kg m}}\right)\left(\frac{\text{Pa m}^2}{\text{N}}\right)\left[\frac{\left[(2.54-1.27)\times10^{-5}\text{ m}\right] - 0.0005(0.0127\text{ m})}{\left[(2.54+1.27)\times10^{-5}\text{ m}\right]\left[2.54\times10^{-5}\text{ m} - 0.0005(0.0127\text{ m})\right]^2}\right]$$

$$= 2.32\times10^{7}\text{ Pa} = 3370\text{ psig} = \mathbf{229\ atm}$$

The gage pressure in the middle of the slipper-pad is more than 200 atmospheres. This is quite large, and illustrates how a small slipper-pad bearing can support a large amount of force.

(*c*) We repeat the calculations of Part (b) for values of x between 0 and L. We nondimensionalize both x and P_{gage} using $x^* = x/L$ and $P^* = (P - P_{atm})h_0^2/\mu VL$. A plot of P^* versus x^* is shown in Fig. 1. The gage pressure is constrained to be zero at both ends of the pad, but reaches a peak near the middle, but more towards the end. For these conditions the maximum value of P^* is 1.0.

(*d*) To calculate the total weight that the slipper-pad bearing can support, we integrate pressure over the surface area of the plate. We used the trapezoidal rule to integrate numerically in a spreadsheet. The result is

Total vertical force (load): $F_{load} = \int_{x=0}^{x=L} P_{gage}\, b\, dx = 62{,}600\text{ N} = \mathbf{14{,}100\ lbf}$

You can also obtain a reasonable estimate by simply taking the average pressure in the gap times the area – this yields F_{load} = 62,000 N = 13,900 lbf.

Discussion This slipper-pad bearing can hold an enormous amount of weight (7 tons!) due to the extremely high pressures encountered in the oil passage.

10-32

Solution We are to discuss what happens to the load when the oil temperature increases.

Analysis Oil viscosity appears only once in the equation for gap pressure. Thus, pressure and load increase linearly as oil viscosity increases. However, as the oil heats up, its viscosity goes down rapidly. For example, at $T = 40^{o}C$, $\mu = 0.2177$ kg/m·s, but at $T = 80^{o}C$, μ drops to 0.03232 kg/m·s. This is more than a factor of six decrease in viscosity for only a $20^{o}C$ increase in temperature. So, **the load would decrease rapidly as oil temperature rises**.

Discussion This problem illustrates why engineers need to look at extreme operating conditions when designing products – just in case.

10-33

Solution We are to see if the Reynolds number is low enough that the flow can be approximated as creeping flow.

Assumptions **1** The flow is steady. **2** The flow is two-dimensional in the *x*-*y* plane.

Properties Unused engine oil at $T = 40^{o}C$: $\rho = 876.0$ kg/m^3, $\mu = 0.2177$ kg/m·s.

Analysis We base Re on the largest gap height, h_0,

Reynolds number:

$$\text{Re} = \frac{\rho h_0 V}{\mu} = \frac{\left(876.0 \text{ kg/m}^3\right)\left(2.54\times 10^{-5} \text{ m}\right)\left(3.048 \text{ m/s}\right)}{0.2177 \text{ kg/m}\cdot\text{s}} = 0.312$$

We see that the Reynolds number is less than one, but we cannot say that Re << 1. So, the flow is not really in the creeping flow regime. However, the creeping flow approximation is generally reasonable up to Reynolds numbers near one. Also, as discussed in Problem 10-24, **the creeping flow approximation is still reasonable** in this case since L/h_0 is so large.

Discussion The error introduced by making the creeping flow approximation is probably less than the error associated with measurement of gap height.

10-34

Solution We are to calculate how much the gap compresses when the load on the bearing is doubled.

Analysis There are several ways to approach this problem: You can try to integrate the pressure distribution analytically to calculate the total load, or you can integrate numerically on a spreadsheet or math program. This is an "inverse" problem in that we can calculate the load for a given value of h_0, but we cannot do the reverse calculation directly – we must do it implicitly. One way to do this is graphically – plot load as a function of h_0, and pick off the value of h_0 where the load has doubled. Another way is by trial and error, or by a convergence technique like Newton's method. It turns out that **the load is doubled when h_0 = 0.0008535 inches (2.168×10^{-5} m). This represents a decrease in initial gap height of about 14.7%.**

Discussion The relationship between gap height and load is clearly nonlinear. When the load doubles, the gap height decreases by less than 15%.

10-35

Solution We are to estimate the speed at which a human being swimming in water would be in the creeping flow regime.

Properties For water at $T = 20^{o}C$, $\rho = 998.0$ kg/m^3 and $\mu = 1.002 \times 10^{-3}$ kg/m·s.

Analysis The characteristic length scale of a human body is of order 1 m. To be in the creeping flow regime, the Reynolds number of the body should be below 1. Thus,

$$\text{Re} = \frac{\rho L V}{\mu} \quad \rightarrow \quad V = \frac{\mu \text{Re}}{\rho \text{L}} = \frac{\left(1.002\times10^{-3}\ \text{kg/m}\cdot\text{s}\right)(1)}{\left(998\ \text{kg/m}^3\right)(1\ \text{m})} \sim \mathbf{1\times10^{-6}\ m/s}$$

So, we would have to move at about one-millionth of a meter per second, or less. This speed is so slow that it is not measurable. Natural currents in the water, even in a "stagnant" pool of water, would be much greater than this. Hence, **we could never experience creeping flow in water**.

Discussion If we were to use a Reynolds number of 0.1 instead of 1, the result would be even slower.

Inviscid Flow

10-36C

Solution We are to discuss the approximation associated with the Euler equation.

Analysis The Euler equation is simply the Navier-Stokes equation with *the viscous term neglected*; **it is therefore an inviscid approximation of the Navier-Stokes equation**. **The Euler equation is appropriate in high Reynolds number regions of the flow where net viscous forces are negligible, far away from walls and wakes.**

Discussion The Euler equation is not appropriate very close to solid walls, since frictional forces are always present there. Note that the same Euler equation is appropriate in an *irrotational* region of flow as well.

10-37C

Solution We are to discuss the main difference between the steady, incompressible Bernoulli equation when applied to irrotational regions of flow vs. rotational but inviscid regions of flow.

Analysis The Bernoulli equation itself is identical in these two cases, but the "constant" for the case of rotational but inviscid regions of flow is constant only along streamlines of the flow, not everywhere. **For irrotational regions of flow, the same Bernoulli constant holds everywhere**.

Discussion A simple example is that of solid body rotation, which is rotational but inviscid. In this flow, as discussed in the text, the Bernoulli "constant" changes from one streamline to another.

10-38

Solution We are to show that the given vector identity is satisfied in Cartesian coordinates.

Analysis We expand each term in the vector identity carefully. The first term is

$$\left(\vec{V}\cdot\vec{\nabla}\right)\vec{V}=\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right)\vec{i}+\left(u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}\right)\vec{j}+\left(u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\right)\vec{k} \qquad (1)$$

The second term is

$$\vec{\nabla}\left(\frac{V^2}{2}\right)=\frac{1}{2}\left[\left(\frac{\partial u^2}{\partial x}+\frac{\partial v^2}{\partial x}+\frac{\partial w^2}{\partial x}\right)\vec{i}+\left(\frac{\partial u^2}{\partial y}+\frac{\partial v^2}{\partial y}+\frac{\partial w^2}{\partial y}\right)\vec{j}+\left(\frac{\partial u^2}{\partial z}+\frac{\partial v^2}{\partial z}+\frac{\partial w^2}{\partial z}\right)\vec{k}\right]$$

which reduces to

$$\vec{\nabla}\left(\frac{V^2}{2}\right)=\left(u\frac{\partial u}{\partial x}+v\frac{\partial v}{\partial x}+w\frac{\partial w}{\partial x}\right)\vec{i}+\left(u\frac{\partial u}{\partial y}+v\frac{\partial v}{\partial y}+w\frac{\partial w}{\partial y}\right)\vec{j}+\left(u\frac{\partial u}{\partial z}+v\frac{\partial v}{\partial z}+w\frac{\partial w}{\partial z}\right)\vec{k} \qquad (2)$$

The third term is

$$\vec{V}\times\left(\vec{\nabla}\times\vec{V}\right)=\left[v\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)-w\left(\frac{\partial u}{\partial z}-\frac{\partial w}{\partial x}\right)\right]\vec{i}+\left[w\left(\frac{\partial w}{\partial y}-\frac{\partial v}{\partial z}\right)-u\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)\right]\vec{j}+\left[u\left(\frac{\partial u}{\partial z}-\frac{\partial w}{\partial x}\right)-v\left(\frac{\partial w}{\partial y}-\frac{\partial v}{\partial z}\right)\right]\vec{k} \qquad (3)$$

When we substitute Eqs. 1 through 3 into the given equation, we see that all the terms disappear, and **the equation is satisfied**. We show this for the *x* direction only (all terms with unit vector $\vec{i}$):

$$\cancel{u\frac{\partial u}{\partial x}}+\cancel{v\frac{\partial u}{\partial y}}+\cancel{w\frac{\partial u}{\partial z}}=\cancel{u\frac{\partial u}{\partial x}}+\cancel{v\frac{\partial v}{\partial x}}+\cancel{w\frac{\partial w}{\partial x}}-\cancel{v\frac{\partial v}{\partial x}}+\cancel{v\frac{\partial u}{\partial y}}+\cancel{w\frac{\partial u}{\partial z}}-\cancel{w\frac{\partial w}{\partial x}} \qquad (4)$$

The algebra is similar for the $\vec{j}$ and $\vec{k}$ terms, and the vector identity is shown to be true for Cartesian coordinates.

Discussion Since we have a vector identity, it must be true regardless of our choice of coordinate system.

10-39

Solution We are to use an alternative method to show that the Euler equation given in the problem statement reduces to the Bernoulli equation for regions of inviscid flow.

Analysis We take the dot product of both sides of the equation with $\vec{V}$. The Euler equation dotted with velocity becomes

$$\vec{\nabla}\left(\frac{P}{\rho}+\frac{V^2}{2}+gz\right)\cdot\vec{V}=\left(\vec{V}\times\vec{\zeta}\right)\cdot\vec{V} \qquad (1)$$

The cross product on the right side of Eq. 1 is a vector that is perpendicular to $\vec{V}$. However, the dot product of two perpendicular vectors is zero by definition of the dot product. Thus, the right hand side of Eq. 1 is identically zero,

$$\vec{\nabla}\left(\frac{P}{\rho}+\frac{V^2}{2}+gz\right)\cdot\vec{V}=0 \qquad (2)$$

Now we use the same argument on the left hand side of Eq. 2, but in reverse. Namely, there are three ways for the dot product of the two vectors in Eq. 2 to be identically zero: (a) the first vector is zero,

Option (a): $$\vec{\nabla}\left(\frac{P}{\rho}+\frac{V^2}{2}+gz\right)=0 \qquad (3)$$

(b) the second vector is zero,

Option (b): $$\vec{V}=0 \qquad (4)$$

or (c) the two vectors are everywhere perpendicular to each other,

Option (c): $$\vec{\nabla}\left(\frac{P}{\rho}+\frac{V^2}{2}+gz\right)\perp\vec{V} \qquad (5)$$

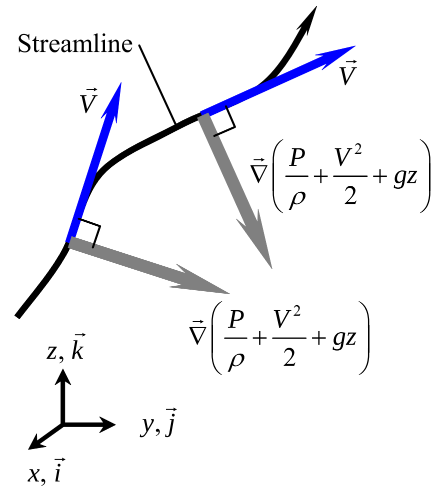


FIGURE 1

Along a streamline, $\vec{\nabla}\left(\frac{P}{\rho}+\frac{V^2}{2}+gz\right)$ is a vector everywhere perpendicular to the streamline; hence $\frac{P}{\rho}+\frac{V^2}{2}+gz$ is constant along the streamline.

Option (a) represents the *restricted* case in which the quantity in parentheses in Eq. 3 is constant everywhere. Option (b) is the *trivial* case in which there is no flow (fluid statics). Option (c) is the most *general* option, and we work with Eq. 5. Since $\vec{V}$ is everywhere parallel to streamlines of the flow, $\vec{\nabla}\left(\frac{P}{\rho}+\frac{V^2}{2}+gz\right)$ must therefore be everywhere *perpendicular* to streamlines (Fig. 1). Finally, we argue that the gradient of a scalar is a vector that points perpendicular to an imaginary surface on which the scalar is constant. Thus, we argue that the scalar $\frac{P}{\rho}+\frac{V^2}{2}+gz$ must be *constant along a streamline*. Our final result is the steady incompressible Bernoulli equation for inviscid regions of flow,

$$\boxed{\frac{P}{\rho}+\frac{V^2}{2}+gz=\text{ constant along streamlines}} \qquad (6)$$

Discussion Since we have a vector identity, it must be true regardless of our choice of coordinate system.

10-40

Solution We are to expand the Euler equation into Cartesian coordinates.

Analysis We begin with the vector form of the Euler equation,

Euler equation:

$$\rho\left(\frac{\partial \vec{V}}{\partial t}+\left(\vec{V}\cdot\vec{\nabla}\right)\vec{V}\right)=-\vec{\nabla}P+\rho\vec{g} \quad (1)$$

The *x* component of Eq. 1 is

x component:

$$\boxed{\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right)=-\frac{\partial P}{\partial x}+\rho g_x} \quad (2)$$

The *y* component of Eq. 1 is

y component:

$$\boxed{\rho\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}\right)=-\frac{\partial P}{\partial y}+\rho g_y} \quad (3)$$

The *z* component of Eq. 1 is

z component:

$$\boxed{\rho\left(\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\right)=-\frac{\partial P}{\partial z}+\rho g_z} \quad (4)$$

Discussion The expansion of the Euler equation into components is identical to that of the Navier-Stokes equation, except that the viscous terms are gone.

10-41

Solution We are to expand the Euler equation into cylindrical coordinates.

Analysis We begin with the vector form of the Euler equation,

Euler equation:

$$\rho\left(\frac{\partial \vec{V}}{\partial t}+\left(\vec{V}\cdot\vec{\nabla}\right)\vec{V}\right)=-\vec{\nabla}P+\rho\vec{g} \quad (1)$$

We must be careful to include the "extra" terms in the convective acceleration. The *r* component of Eq. 1 is

r component:

$$\boxed{\rho\left(\frac{\partial u_r}{\partial t}+u_r\frac{\partial u_r}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_r}{\partial \theta}-\frac{u_\theta{}^2}{r}+u_z\frac{\partial u_r}{\partial z}\right)=-\frac{\partial P}{\partial r}+\rho g_r} \quad (2)$$

The *y* component of Eq. 1 is

θ component:

$$\boxed{\rho\left(\frac{\partial u_\theta}{\partial t}+u_r\frac{\partial u_\theta}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_\theta}{\partial \theta}+\frac{u_r u_\theta}{r}+u_z\frac{\partial u_\theta}{\partial z}\right)=-\frac{1}{r}\frac{\partial P}{\partial \theta}+\rho g_\theta} \quad (3)$$

The *z* component of Eq. 1 is

z component:

$$\boxed{\rho\left(\frac{\partial u_z}{\partial t}+u_r\frac{\partial u_z}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_z}{\partial \theta}+u_z\frac{\partial u_z}{\partial z}\right)=-\frac{\partial P}{\partial z}+\rho g_z} \quad (4)$$

Discussion The expansion of the Euler equation into components is identical to that of the Navier-Stokes equation, except that the viscous terms are gone.

10-42

Solution We are to calculate the pressure field and the shape of the free surface for solid body rotation of water in a container.

Assumptions **1** The flow is steady and incompressible. **2** The flow is rotationally symmetric, meaning that all derivatives with respect to θ are zero. **3** Gravity acts in the negative z direction.

Properties For water at $T = 20^{o}$C, $\rho = 998.0$ kg/m^3 and $\mu = 1.002 \times 10^{-3}$ kg/m·s.

Analysis We reduce the components of the Euler equation in cylindrical coordinates (Problem 10-41) as far as possible, noting that $u_r = u_z = 0$ and $u_\theta = \omega r$. The θ component disappears. The r component reduces to

r component of Euler equation:
$$-\rho \frac{u_\theta^{\,2}}{r} = -\frac{\partial P}{\partial r} \quad \rightarrow \quad \frac{\partial P}{\partial r} = \rho \omega^2 r \qquad (1)$$

and the z component reduces to

z component of Euler equation:
$$0 = -\frac{\partial P}{\partial z} - \rho g \quad \rightarrow \quad \frac{\partial P}{\partial z} = -\rho g \qquad (2)$$

We find $P(r,z)$ by cross integration. First we integrate Eq. 1 with respect to r,

$$P = \frac{\rho \omega^2 r^2}{2} + f(z) \qquad (3)$$

Note that we add a function of z instead of a constant of integration since this is a *partial* integration. We take the z derivative of Eq. 3, equate to Eq. 2, and integrate,

$$\frac{\partial P}{\partial z} = f'(z) = -\rho g \quad \rightarrow \quad f(z) = -\rho g z + C_1 \qquad (4)$$

Plugging Eq. 4 into Eq. 3 yields our expression for $P(r,z)$,

$$P = \frac{\rho \omega^2 r^2}{2} - \rho g z + C_1 \qquad (5)$$

Now we apply the boundary condition at the origin to find the value of constant C_1,

Boundary condition: At $r = 0$ and $z = 0$, $P = P_{\text{atm}} = C_1 \quad \rightarrow \quad C_1 = P_{\text{atm}}$

Finally, Eq. 5 becomes

Pressure field:
$$\boxed{P = \frac{\rho \omega^2 r^2}{2} - \rho g z + P_{\text{atm}}} \qquad (6)$$

At the free surface, we know that $P = P_{\text{atm}}$, and Eq. 6 yields the equation for the shape of the free surface,

Free surface shape:
$$\boxed{z_{\text{surface}} = \frac{\omega^2 r^2}{2g}} \qquad (7)$$

Discussion Since we know the velocity field from the start, the Euler equation is not needed for obtaining the velocity field. Instead, it is used only to calculate the pressure field. Similarly, the continuity equation is identically satisfied and is not needed here.

10-43

Solution We are to calculate the pressure field and the shape of the free surface for solid body rotation of engine oil in a container.

Assumptions **1** The flow is steady and incompressible. **2** The flow is rotationally symmetric, meaning that all derivatives with respect to θ are zero. **3** Gravity acts in the negative *z* direction.

Analysis In Problem 10-42, water density appears only as a constant in the pressure equation. Thus, nothing is different here except the value of density, and **the results are identical to those of Problem 10-42**.

Discussion In solid body rotation, the density of the fluid does not affect the shape of the free surface. For oil (less dense than water), pressure increases with depth at a slower rate compared to water.

10-44

Solution We are to calculate the Bernoulli constant for solid body rotation of water in a container.

Assumptions **1** The flow is steady and incompressible. **2** The flow is rotationally symmetric, meaning that all derivatives with respect to θ are zero. **3** Gravity acts in the negative *z* direction.

Analysis From Problem 10-42, we have the pressure field,

Pressure field:
$$P = \frac{\rho\omega^2 r^2}{2} - \rho g z + P_{\text{atm}} \tag{1}$$

The Bernoulli equation for steady, incompressible, inviscid regions of flow is

$$\frac{P}{\rho} + \frac{V^2}{2} + gz = C_r = \text{constant along streamlines} \tag{2}$$

The velocity field is $u_r = u_z = 0$ and $u_\theta = \omega r$, $V^2 = \omega^2 r^2$, and Eq. 2 becomes

$$C_r = \frac{P}{\rho} + \frac{\omega^2 r^2}{2} + gz \tag{3}$$

Substitution of Eq. 1 into Eq. 3 yields the final expression for C_r,

Bernoulli "constant":
$$C_r = \frac{\omega^2 r^2}{2} - gz + \frac{P_{\text{atm}}}{\rho} + \frac{\omega^2 r^2}{2} + gz \rightarrow \boxed{C_r = \frac{P_{\text{atm}}}{\rho} + \omega^2 r^2} \tag{4}$$

Discussion Streamlines in this flow field are circles about the *z* axis (lines of constant *r*). The Bernoulli "constant" C_r is constant along any given streamline, but changes from streamline to streamline. This is typical of rotating flow fields.

10-45

Solution For a given volume flow rate, we are to generate an expression for u_r assuming inviscid flow, and then discuss the velocity profile shape for a real (viscous) flow.

Assumptions **1** The flow remains radial at all times (no u_θ component). **2** The flow is steady, two-dimensional, and incompressible.

Analysis If the flow were inviscid, we could not enforce the no-slip condition at the walls of the duct. At any r location, the volume flow rate must be the same,

Volume flow rate at any r location: $$\dot{V} = u_r r b \Delta\theta \qquad (1)$$

where $\Delta\theta$ is the angle over which the contraction is bound (see Fig. 1). Thus,

$$u_r = \frac{\dot{V}}{rb\Delta\theta} \qquad (2)$$

At radius $r = R$, Eq. 2 becomes

Radial velocity at r = R: $$u_r(R) = \frac{\dot{V}}{Rb\Delta\theta} \qquad (3)$$

Upon substitution of Eq. 3 into Eq. 2, we get

Radial velocity at any r location: $$\boxed{u_r = \frac{R}{r}u_r(R)} \qquad (4)$$

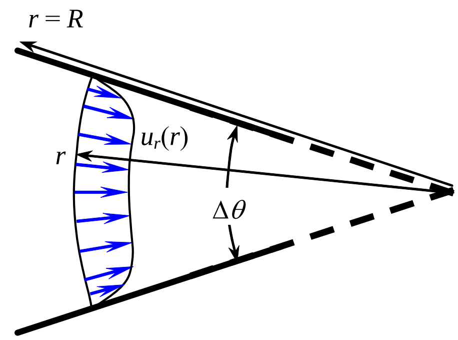


FIGURE 1
Possible shape of the velocity profile for a real (viscous) flow.

In other words, the radial velocity component increases as the reciprocal of r as r approaches zero (the origin).

In a real flow (with viscous effects), we would expect that the velocity near the center of the duct is somewhat larger, while that near the walls is somewhat smaller. Right at the walls, of course, the velocity is zero by the no-slip condition. In Fig. 1 is a sketch of what the velocity profile might look like in a real flow.

Discussion In either case, the radial velocity is infinite at the origin. This is actually a portion of a line sink, as discussed in this chapter.

10-46

Solution We are to show that the region of flow given by this velocity field is inviscid.

Assumptions **1** The flow is steady. **2** The flow is incompressible. **3** The flow is two-dimensional in the x-y plane.

Analysis We consider the viscous terms of the x and y momentum equations:

x momentum viscous terms: $$\mu\left(\underbrace{\frac{\partial^2 u}{\partial x^2}}_{0} + \underbrace{\frac{\partial^2 u}{\partial y^2}}_{0} + \underbrace{\frac{\partial^2 u}{\partial z^2}}_{0\text{ (2-D)}}\right) = 0 \qquad (1)$$

y momentum viscous terms: $$\mu\left(\underbrace{\frac{\partial^2 v}{\partial x^2}}_{0} + \underbrace{\frac{\partial^2 v}{\partial y^2}}_{0} + \underbrace{\frac{\partial^2 v}{\partial z^2}}_{0\text{ (2-D)}}\right) = 0 \qquad (2)$$

Since the viscous terms are identically zero in both components of the Navier-Stokes equation, **this region of flow can indeed be considered inviscid**.

Discussion With the viscous terms removed, the Navier-Stokes equation is reduced to the Euler equation.

Irrotational (Potential) Flow

10-47C

Solution We are to discuss the flow property that determines whether a region of flow is rotational or irrotational.

Analysis The **vorticity** determines whether a region of flow is rotational or irrotational. Specifically, if the vorticity is zero (or negligibly small), the flow is approximated as irrotational, but if the vorticity is not negligibly small, the flow is rotational.

Discussion Another acceptable answer is the rate of rotation vector or the angular velocity vector of a fluid particle.

10-48

Solution We are to show that the vorticity components are zero in an irrotational region of flow.

Analysis The first component of vorticity becomes

r-component of vorticity vector:
$$\zeta_r = \frac{1}{r}\frac{\partial u_z}{\partial \theta} - \frac{\partial u_\theta}{\partial z} = \frac{1}{r}\frac{\partial^2 \phi}{\partial \theta \partial z} - \frac{1}{r}\frac{\partial^2 \phi}{\partial z \partial \theta} = 0$$

which is valid as long as ϕ is a smooth function of θ and z. Similarly, the second component of vorticity becomes

θ-component of vorticity vector:
$$\zeta_\theta = \frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r} = \frac{\partial^2 \phi}{\partial z \partial r} - \frac{\partial^2 \phi}{\partial r \partial z} = 0$$

which is valid as long as ϕ is a smooth function of r and z. Finally, the third component of vorticity becomes

z-component of vorticity vector:
$$\zeta_z = \frac{1}{r}\frac{\partial}{\partial r}(r u_\theta) - \frac{1}{r}\frac{\partial u_r}{\partial \theta} = \frac{1}{r}\frac{\partial^2 \phi}{\partial r \partial \theta} - \frac{1}{r}\frac{\partial^2 \phi}{\partial \theta \partial r} = 0$$

which is valid as long as ϕ is a smooth function of r and z. Thus **all three components of vorticity are zero**.

Discussion By mathematical identity, the velocity potential function is definable only when the vorticity vector is zero; therefore the results are not surprising. Note that in a three-dimensional flow, ϕ must be a smooth function of r, θ, and z.

10-49

Solution We are to verify that the Laplace equation holds in an irrotational flow field in cylindrical coordinates.

Analysis We plug in the components of velocity from Problem 10-48 into the left hand side of the Laplace equation in cylindrical coordinates,

Laplace equation in cylindrical coordinates:
$$\nabla^2 \phi = \frac{1}{r}\frac{\partial}{\partial r}(r u_r) + \frac{1}{r^2}\frac{\partial}{\partial \theta}(r u_\theta) + \frac{\partial u_z}{\partial z} \quad (1)$$

But since r is not a function of θ, we simplify Eq. 1 to

$$\nabla^2 \phi = \frac{1}{r}\frac{\partial}{\partial r}(r u_r) + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z} \quad (2)$$

We recognize the terms on the right side of Eq. 2 as those of the incompressible form of the continuity equation in cylindrical coordinates; Eq. 2 is thus equal to zero, and the Laplace equation holds,

$$\boxed{\nabla^2 \phi = 0} \quad (3)$$

Discussion The Laplace equation is valid for any incompressible irrotational region of flow, regardless of whether the flow is two- or three-dimensional.

10-50

Solution We are to identify regions in the flow field that are irrotational, and regions that are rotational.

Assumptions **1** The air in the room would be calm if not for the presence of the hair dryer.

Analysis **Flow in the air far away from the hair dryer and its jet is certainly irrotational. As the air approaches the inlet, it is irrotational except very close to the surface of the hair dryer. Flow in the jet is rotational, but flow outside of the jet can be approximated as irrotational**.

Discussion Flow near solid walls is nearly always rotational because of the viscous rotational boundary layer that grows there. There are sharp velocity gradients in a jet, so the vorticity cannot be zero in that region, and the flow must be rotational in the jet as well.

10-51

Solution We are to compare the Bernoulli equation and its restrictions for inviscid, rotational regions of flow and viscous, irrotational regions of flow.

Assumptions **1** The flow is incompressible and steady.

Analysis The Bernoulli equation is the same in both cases, namely

Steady incompressible Bernoulli equation:

$$\boxed{\frac{P}{\rho}+\frac{V^2}{2}+gz=C} \quad (1)$$

However, in an inviscid, rotational region of flow, Eq. 1 is applicable only along a streamline. The Bernoulli "constant" C is constant along any particular streamline, but may change from streamline to streamline. In a viscous, irrotational region of flow, however, the Bernoulli constant is constant everywhere, even across streamlines. Thus, **the inviscid, rotational region of flow has more restrictions on the use of the Bernoulli equation**.

Discussion In either case, the viscous terms in the Navier-Stokes equation disappear, but for different reasons.

10-52

Solution For a given set of streamlines, we are to sketch the corresponding set of equipotential curves and explain how we obtain them.

Assumptions **1** The flow is steady and incompressible. **2** The flow is two-dimensional in the plane of the figure associated with this problem. **3** The flow in the region shown in the figure is irrotational.

Analysis **Some possible equipotential lines are sketched in Fig. 1**. We draw these based on the fact that the streamlines and equipotential curves must intersect at 90° angles. To find the "correct" shape, it helps to sketch a few extra streamlines in between the given ones to guide in construction of the equipotential curves. These "interpolated" streamlines are shown in Fig. 1 as thin, dotted blue lines.

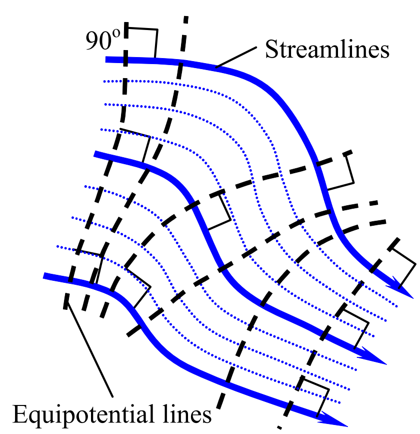


FIGURE 1
Possible equipotential curves (dashed black lines) and intermediate streamlines (dotted blue lines).

Discussion The exact shape of the equipotential curves is not known, and individuals may sketch curves of other shapes that are equally valid. The important thing to emphasize is that the curves of constant ϕ are everywhere *perpendicular* to the streamlines.

10-53

Solution We are to discuss the role of the momentum equation in an irrotational region of flow.

Assumptions **1** The flow is steady and incompressible. **2** The region of interest in the flow field is irrotational.

Analysis Although it is true that the momentum equation is not required in order to solve for the velocity field, **it *is* required in order to solve for the *pressure* field**. In particular, the Navier-Stokes equation reduces to the Bernoulli equation in an irrotational region of flow.

Discussion Mathematically, it turns out that in an irrotational flow field the continuity equation is *uncoupled* from the momentum equation, meaning that we can solve continuity for ϕ by itself, without need of the momentum equation. However, the momentum equation cannot be solved by itself.

10-54

Solution For a given velocity field, we are to assess whether the flow field is irrotational. If so, we are to generate an expression for the velocity potential function.

Assumptions **1** The flow is steady. **2** The flow is incompressible. **3** The flow is two-dimensional in the *x*-*y* plane.

Analysis For the flow to be irrotational, the vorticity must be zero. Since the flow is planar in the *x*-*y* plane, the only non-zero component of vorticity is in the *z* direction,

z-component of vorticity:
$$\zeta_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0 - 0 = 0 \qquad (1)$$

Since the vorticity is zero, **this flow field can be considered irrotational**, and we should be able to generate a velocity potential function that describes the flow. In two dimensions we have

Velocity components in terms of potential function:
$$u = \frac{\partial \phi}{\partial x} \qquad v = \frac{\partial \phi}{\partial y} \qquad (2)$$

We pick one of these (the first one) and integrate to obtain an expression for ϕ,

Velocity potential function:
$$\frac{\partial \phi}{\partial x} = u = ax + b \qquad \phi = a\frac{x^2}{2} + bx + f(y) \qquad (3)$$

Note that we have added a function of *y* rather than a constant of integration since we have performed a partial integration with respect to *x*. Using Eq. 2, we differentiate Eq. 3 with respect to *y* and equate the result to the *v* component of velocity,

$$\frac{\partial \phi}{\partial y} = f'(y) = v = -ay + c \qquad (4)$$

Equation 4 is integrated with respect to *y* to find function *f*(*y*),

$$f(y) = -a\frac{y^2}{2} + cy + C_1 \qquad (5)$$

This time, a constant of integration (C_1) is added since this is a total integration. Finally, we plug Eq. 5 into Eq. 3 to obtain our final expression for the velocity potential function,

Result, velocity potential function:
$$\boxed{\phi = a\frac{\left(x^2 - y^2\right)}{2} + bx + cy + C_1} \qquad (6)$$

Discussion You should plug Eq. 6 into Eq. 2 to verify that it is correct.

10-55

Solution We are to discuss similarities and differences between two approximations: inviscid regions of flow and irrotational regions of flow.

Assumptions **1** The flow is incompressible and steady.

Analysis The two approximations are similar in that in both cases, the viscous terms in the Navier-Stokes equation drop out, leaving the Euler equation. Also, in both cases the Bernoulli equation results form integration of the Euler equation. However, these two approximations differ significantly from each other. **When making the inviscid flow approximation, we assume that the viscous terms are negligibly small**. A good example, as discussed in this chapter, is solid body rotation. In this case, although the fluid itself is viscous, all effects of viscosity are gone, and the flow field can be considered "inviscid" (although it is rotational). On the other hand, **the irrotational approximation is made when the vorticity (a measure of rotationality of fluid particles) is negligibly small**. In this case, viscosity still acts on fluid particles – it shears them and distorts them, yet the net rate of rotation of fluid particles is zero. In other words, in an irrotational region of flow, the net viscous force on a fluid particle is zero, but viscous stresses on the fluid particle are certainly *not* zero. Examples of irrotational, but viscous flows include any irrotational flow field with curved streamlines, such as a line vortex, a doublet, irrotational flow over a circular cylinder, etc. Freestream flow is both inviscid and irrotational since fluid particles do not shear or distort or rotate, and viscosity does not enter into the picture.

Discussion In either case, the viscous terms in the Navier-Stokes equation disappear, but for different reasons. In the inviscid flow approximation, the viscous terms disappear because we neglect viscosity. In the irrotational flow approximation, the viscous terms disappear because they cancel each other out due to the fact that the vorticity (hence the rate of rotation) of fluid particles is negligibly small.

10-56

Solution We are to calculate the velocity components from a given potential function, verify that the velocity field is irrotational, and generate an expression for ψ.

Assumptions **1** The flow is steady and incompressible. **2** The flow is two-dimensional in the *x*-*y* plane. **3** The flow is irrotational in the region in which Eq. 1 applies.

Analysis (*a*) The velocity components are found by taking the *x* and *y* partial derivatives of ϕ,

Velocity components:
$$u = \frac{\partial \phi}{\partial x} = 10x + 2 \qquad v = \frac{\partial \phi}{\partial y} = -10y - 4 \tag{1}$$

(*b*) We plug in *u* and *v* from Eq. 1 into the *z* component of vorticity to get

z-component of vorticity:
$$\zeta_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0 - 0 = 0 \tag{2}$$

Since $\zeta_z = 0$, and the only component of vorticity in a 2-D flow in the *x*-*y* plane is in the *z* direction, the vorticity is zero, and **the flow is irrotational in the region of interest**.

(*c*) The stream function is found by integration of the velocity components. We begin by integrating the *x* component, $\partial\psi/\partial y = u$, and then taking the *x* derivative to compare with the known value of *v*,

$$\psi = 10xy + 2y + f(x) \quad \rightarrow \quad v = -\frac{\partial \psi}{\partial x} = -10y - f'(x) = -10y - 4 \tag{3}$$

From which we see that $f'(x) = 4$. Integrating with respect to *x*,

$$f(x) = 4x + \text{constant} \tag{4}$$

The constant is arbitrary since velocity components are always derivatives of ψ. Thus,

Stream function:
$$\psi = 10xy + 2y + 4x + \text{constant} \tag{5}$$

Discussion You can verify that the partial derivatives of Eq. 5 yield the same velocity components as those of Eq. 1.

10-57

Solution We are to show that the stream function for a planar irrotational region of flow satisfies the Laplace equation in cylindrical coordinates.

Assumptions **1** This region of flow is planar in the r-θ plane. **2** The flow is incompressible. **3** This region of flow is irrotational.

Analysis We defined stream function ψ as

Planar flow stream function in cylindrical coordinates: $$u_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta} \qquad u_\theta = -\frac{\partial \psi}{\partial r} \tag{1}$$

We also know that for irrotational flow the vorticity must be zero. Since the only non-zero component of vorticity is in the z direction,

z-component of vorticity: $$\zeta_z = \frac{1}{r}\left(\frac{\partial(ru_\theta)}{\partial r} - \frac{\partial u_r}{\partial \theta}\right) = \frac{1}{r}\left(\frac{\partial}{\partial r}\left(-r\frac{\partial \psi}{\partial r}\right) - \frac{\partial}{\partial \theta}\left(\frac{1}{r}\frac{\partial \psi}{\partial \theta}\right)\right) = 0 \tag{2}$$

Since r is not a function of θ, it can come outside the derivative operator in the last term. Also, the negative sign in both terms can be disposed of. Thus,

Result of irrotationality condition: $$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \psi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \psi}{\partial \theta^2} = 0 \tag{3}$$

Since Eq. 3 is the Laplace equation for 2-D planar flow in the r-θ plane, we have shown that **the stream function indeed satisfies the Laplace equation**.

Discussion Since the Laplace equation for stream function is satisfied in Cartesian coordinates for the case of 2-D planar flow in the x-y plane, it must also be satisfied for the same flow in cylindrical coordinates. All we have done is use a different coordinate system to describe the *same* flow.

10-58

Solution We are to write the Laplace equation in two dimensions (r and θ) in spherical polar coordinates.

Assumptions **1** The flow is independent of angle ϕ (about the x axis). **2** The flow is irrotational.

Analysis We look up the Laplace equation in spherical polar coordinates in any vector analysis book. Ignoring derivatives with respect to ϕ, we get

Laplace equation, axisymmetric flow, (r,θ): $$\boxed{\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial \phi}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial \phi}{\partial \theta}\right) = 0}$$

Discussion Even though ϕ satisfies the Laplace equation in an irrotational region of flow, ψ does *not* for the present case of axisymmetric flow.

10-59

Solution We are to prove that the given stream function exactly satisfies the continuity equation for the case of axisymmetric flow in spherical polar coordinates.

Assumptions **1** The flow is axisymmetric, implying that there is no variation rotationally around the axis of symmetry. **2** The flow is incompressible.

Analysis We plug the stream function into the continuity equation, and perform the algebra,

$$\frac{1}{r}\frac{\partial}{\partial r}\left(-\frac{1}{\sin\theta}\frac{\partial\psi}{\partial\theta}\right)+\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\frac{1}{r}\frac{\partial\psi}{\partial r}\right)=0 \qquad (1)$$

since θ is not a function of r and vice-versa, Eq. 1 can be rearranged as

$$-\frac{1}{r\sin\theta}\frac{\partial^2\psi}{\partial r\partial\theta}+\frac{1}{r\sin\theta}\frac{\partial^2\psi}{\partial\theta\partial r}=0 \qquad -\frac{\partial^2\psi}{\partial r\partial\theta}+\frac{\partial^2\psi}{\partial\theta\partial r}=0 \qquad (2)$$

Equation 2 is identically satisfied as long as ψ is a smooth function of r and θ.

Discussion If ψ were *not* smooth, the order of differentiation (r then θ versus θ then r) would be important and Eq. 2 would not necessarily be zero. In the definition of stream function, it is somewhat arbitrary whether we put the negative sign on u_r or u_θ, and you may find the opposite sign convention in other textbooks.

10-60

Solution We are to generate expressions for velocity potential function and stream function for the case of a uniform stream of magnitude V inclined at angle α.

Assumptions **1** The flow is planar, incompressible, and irrotational. **2** The flow is uniform everywhere in the flow field, with magnitude V and inclination angle α.

Analysis For planar flow in Cartesian coordinates, we write

$$u = \frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y} = V\cos\alpha \qquad v = \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x} = V\sin\alpha \qquad (1)$$

By integrating the first of these with respect to x, and then differentiating the result with respect to y, we generate an expression for the velocity potential function for a uniform stream,

$$\phi = Vx\cos\alpha + f(y) \qquad v = \frac{\partial \phi}{\partial y} = f'(y) = V\sin\alpha \qquad (2)$$

Integrating with respect to y,

$$f(y) = Vy\sin\alpha + \text{constant} \qquad (3)$$

The constant is arbitrary since velocity components are always derivatives of ϕ. We set the constant to zero, knowing that we can always add an arbitrary constant later on if desired. Thus,

Velocity potential function:
$$\boxed{\phi = Vx\cos\alpha + Vy\sin\alpha} \qquad (4)$$

We do a similar analysis for the stream function, beginning again with Eq. 1.

$$\psi = Vy\cos\alpha + g(x) \qquad v = -\frac{\partial \psi}{\partial x} = -g'(x) = V\sin\alpha \qquad (5)$$

Integrating with respect to x,

$$g(x) = -Vx\sin\alpha + \text{constant} \qquad (6)$$

The constant is arbitrary since velocity components are always derivatives of ψ. We set the constant to zero, knowing that we can always add an arbitrary constant later on if desired. Thus,

Stream function:
$$\boxed{\psi = Vy\cos\alpha - Vx\sin\alpha} \qquad (7)$$

Discussion You should be able to obtain the same answers by starting with the *opposite* equations in Eq. 1 (i.e., integrate first with respect to y to obtain ϕ and with respect to x to obtain ψ).

10-61

Solution We are to generate expressions for the stream function and the velocity potential function for a line source, beginning with the first equation above.

Assumptions **1** The flow is steady and incompressible. **2** The flow is irrotational in the region of interest. **3** The flow is two-dimensional in the *x*-*y* or *r*-θ plane.

Analysis To find the stream function, we integrate the first equation with respect to θ, and then differentiate with respect to the other variable *r*,

$$\frac{\partial\psi}{\partial\theta} = \frac{\dot{V}/L}{2\pi} \quad\rightarrow\quad \psi = \frac{\dot{V}/L}{2\pi}\theta + f(r) \quad\rightarrow\quad \frac{\partial\psi}{\partial r} = f'(r) = -u_\theta = 0 \qquad (1)$$

We integrate Eq. 1 to obtain

$$f(r) = \text{constant} \qquad (2)$$

We set the arbitrary constant of integration to zero since we can add back a constant as desired at any time without changing the flow. Thus,

Line source at the origin:
$$\boxed{\psi = \frac{\dot{V}/L}{2\pi}\theta} \qquad (3)$$

We perform a similar analysis for ϕ by beginning with the first equation:

$$\frac{\partial\phi}{\partial r} = \frac{\dot{V}/L}{2\pi r} \quad\rightarrow\quad \phi = \frac{\dot{V}/L}{2\pi}\ln r + f(\theta) \quad\rightarrow\quad \frac{\partial\phi}{\partial\theta} = f'(\theta) = ru_\theta = 0 \qquad (4)$$

We integrate Eq. 4 to obtain

$$f(\theta) = \text{constant}$$

We set the arbitrary constant of integration to zero since we can add back a constant as desired at any time without changing the flow. Thus,

Line source at the origin:
$$\boxed{\phi = \frac{\dot{V}/L}{2\pi}\ln r} \qquad (5)$$

Discussion You can easily verify by differentiation that Eqs. 3 and 5 yield the correct velocity components. Also note that if $\dot{V}/L$ is negative, the flow field is that of a line sink rather than a line source.

10-62

Solution We are to generate expressions for the stream function and the velocity potential function for a line vortex.

Assumptions **1** The flow is steady and incompressible. **2** The flow is irrotational in the region of interest. **3** The flow is two-dimensional in the *x*-*y* or r-θ plane.

Analysis To find the stream function we integrate the first equation with respect to θ, and then differentiate with respect to the other variable *r*,

$$\frac{\partial \psi}{\partial \theta} = 0 \quad \rightarrow \quad \psi = f(r) \quad \rightarrow \quad \frac{\partial \psi}{\partial r} = f'(r) = -u_\theta = -\frac{\Gamma}{2\pi r} \qquad (1)$$

We integrate Eq. 1 to obtain

$$f(r) = -\frac{\Gamma}{2\pi}\ln r + \text{constant} \qquad (2)$$

We set the arbitrary constant of integration to zero since we can add back a constant as desired at any time without changing the flow. Thus,

Line vortex at the origin:

$$\boxed{\psi = -\frac{\Gamma}{2\pi}\ln r} \qquad (3)$$

We perform a similar analysis for ϕ. Beginning with the first equation:

$$\frac{\partial \phi}{\partial r} = 0 \quad \rightarrow \quad \phi = f(\theta) \quad \rightarrow \quad \frac{\partial \phi}{\partial \theta} = f'(\theta) = ru_\theta = \frac{\Gamma}{2\pi} \qquad (4)$$

We integrate Eq. 4 to obtain

$$f(\theta) = \frac{\Gamma}{2\pi}\theta + \text{constant}$$

We set the arbitrary constant of integration to zero since we can add back a constant as desired at any time without changing the flow. Thus,

Line vortex at the origin:

$$\boxed{\phi = \frac{\Gamma}{2\pi}\theta} \qquad (5)$$

Discussion You can easily verify by differentiation that Eqs. 3 and 5 yield the correct velocity components. Also note that if Γ is positive, the vortex is counterclockwise, and if Γ is negative, the vortex is clockwise.

10-63

Solution For a given stream function, we are to calculate the velocity potential function

Assumptions **1** The flow is steady and incompressible. **2** The flow is two-dimensional in the r-θ plane. **3** The flow is approximated as irrotational.

Analysis There are two ways to approach this problem: (1) Calculate the velocity components from the stream function, and then integrate to obtain ϕ. (2) Superpose a freestream and a doublet to generate ϕ directly. We show both methods here.

Method (1): We calculate the velocity components everywhere in the flow field by differentiating the stream function,

Velocity components:

$$u_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta} = V_\infty \cos\theta\left(1-\frac{a^2}{r^2}\right) \qquad u_\theta = -\frac{\partial \psi}{\partial r} = -V_\infty \sin\theta\left(1+\frac{a^2}{r^2}\right) \tag{1}$$

Now we integrate to obtain the velocity potential function. We begin by integrating the expression for u_r in Eq. 1,

$$u_r = \frac{\partial \phi}{\partial r} = V_\infty \cos\theta\left(1-\frac{a^2}{r^2}\right) \quad \rightarrow \quad \phi = V_\infty \cos\theta\left(r+\frac{a^2}{r}\right)+f(\theta) \tag{2}$$

We differentiate Eq. 2 with respect to θ and divide by r to get

$$u_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta} = -V_\infty \sin\theta\left(1+\frac{a^2}{r^2}\right)+\frac{f'(\theta)}{r} = -V_\infty \sin\theta\left(1+\frac{a^2}{r^2}\right) \tag{3}$$

Equation 3 reduces to $f'(\theta) = 0$, or $f(\theta)$ = constant. The constant is arbitrary, and we set it to zero for convenience. Hence, Eq. 2 reduces to

Velocity potential, flow over a cylinder:

$$\boxed{\phi = V_\infty \cos\theta\left(r+\frac{a^2}{r}\right)} \tag{4}$$

Method (2): The velocity potential functions for a freestream and a doublet are superposed (added) to yield

Superposition:

$$\phi = V_\infty r\cos\theta + K\frac{\cos\theta}{r} \tag{5}$$

To find the doublet strength (K), we set the radial velocity component u_r to zero at the cylinder surface ($r = a$),

$$u_r = \frac{\partial \phi}{\partial r} = V_\infty \cos\theta - K\frac{\cos\theta}{r^2} \quad \rightarrow \quad 0 = V_\infty \cos\theta - K\frac{\cos\theta}{a^2} \tag{6}$$

Equation 6 reduces to $K = a^2 V_\infty$. Hence, Eq. 5 becomes

Velocity potential, flow over a cylinder:

$$\boxed{\phi = V_\infty \cos\theta\left(r+\frac{a^2}{r}\right)} \tag{7}$$

Discussion Both methods yield the same answer, as they must.

10-64

Solution We are to discuss D'Alembert's paradox.

Analysis D'Alembert's paradox states that **with the irrotational flow approximation, the aerodynamic drag force on any non-lifting body of any shape immersed in a uniform stream is zero**. It is a paradox because we know from experience that bodies in a flow field have non-zero aerodynamic drag.

Discussion Irrotational flow over a non-lifting immersed body has neither pressure drag nor viscous drag. In a real flow, both of these drag components are present.

Boundary Layers

10-65C

Solution We are to explain why the boundary layer approximation bridges the gap between the Euler equation and the Navier-Stokes equation.

Analysis The Euler equation neglects the viscous terms compared to the inertial terms. For external flow around a body, this is a reasonable approximation over the majority of the flow field, except very close to the body, where viscous effects dominate. The Navier-Stokes equation, on the other hand, includes both viscous and inertial terms, but is much more difficult to solve. The boundary layer equations bridge the gap between these two: **we solve the simpler Euler equation away from walls, and then fit in a thin boundary layer to account for the no-slip condition at walls**.

Discussion Students' discussions should be in their own words.

10-66C

Solution

(*a*) ***False***: If the Reynolds number at a given *x* location were to increase, all else being equal, viscous forces would decrease in magnitude relative to inertial forces, rendering the boundary layer thinner.

(*b*) ***False***: Actually, as *V* increases, so does Re, and the boundary layer thickness decreases with increasing Reynolds number.

(*c*) ***True***: Since μ appears in the denominator of the Reynolds number, Re decreases as μ increases, causing the boundary layer thickness to increase.

(d) ***False***: Since ρ appears in the numerator of the Reynolds number, Re increases as ρ increases, causing the boundary layer thickness to decrease.

10-67C

Solution We are to name three flows (other than flow along a wall) where the boundary layer approximation is appropriate, and we are to explain why.

Analysis The boundary layer approximation is appropriate for the three basic types of shear layers: **wakes**, **jets**, and **mixing layers**. These flows have a predominant flow direction, and for high Reynolds numbers, the shear layer is very *thin*, causing the viscous terms to be much smaller than the inertial terms, just as in the case of a boundary layer along a wall.

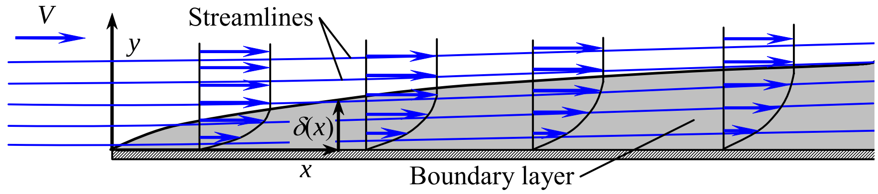
Discussion For the wake and the mixing layer, there is an irrotational outer flow in the streamwise direction, but for the jet, the flow outside the jet is nearly stagnant.

10-68C

Solution We are to sketch several streamlines and discuss whether the curve representing $\delta(x)$ is a streamline or not.

Analysis **Five streamlines are sketched in Fig. 1**. In order to satisfy conservation of mass, the streamlines must cross the curve $\delta(x)$. Thus, $\delta(x)$ cannot itself be a streamline of the flow.

FIGURE 1
Several streamlines and the curve representing δ as a function of x for a flat plate boundary layer. Since streamlines cross the curve $\delta(x)$, $\delta(x)$ cannot itself be a streamline of the flow.



Discussion As the boundary layer grows in thickness, streamlines diverge slowly away from the wall (and become farther apart from each other) in order to conserve mass. However, the upward displacement of the streamlines is not as fast as the growth of $\delta(x)$.

10-69C

Solution We are to define trip wire and explain its purpose.

Analysis A trip wire is **a rod or wire stretched normal to the streamwise direction along the wall**. Its purpose is to create a large disturbance in the laminar boundary layer that causes the boundary layer to "trip" to turbulence much more quickly than it would otherwise.

Discussion Dimples on a golf ball serve the same purpose.

10-70

Solution We are to calculate the location of transition and turbulence along a flat plate boundary layer.

Assumptions **1** The flow is incompressible and steady in the mean. **2** Freestream disturbances are small. **3** The surface of the plate is very smooth.

Properties The density and viscosity of air at $T = 30^{\circ}$C are 1.164 kg/m^3 and 1.872×10^{-5} kg/m·s respectively.

Analysis Transition begins at the critical Reynolds number, which is approximately 100,000 for "clean" flow along a smooth flat plate. Thus,

Beginning of transition:

$$\mathrm{Re}_{x,\mathrm{critical}} = \frac{\rho V x_{\mathrm{critical}}}{\mu} = 100,000 \qquad (1)$$

Solving for x yields

$$x_{\mathrm{critical}} = \frac{100,000\mu}{\rho V} = \frac{100,000\left(1.872\times10^{-5}\ \mathrm{kg/m\cdot s}\right)}{\left(1.164\ \mathrm{kg/m^3}\right)\left(25.0\ \mathrm{m/s}\right)} = 6.43\times10^{-2}\ \mathrm{m} \qquad (2)$$

That is, transition begins at $\boldsymbol{x \approx}$ **6 or 7 cm**. Fully turbulent flow in the boundary layer occurs at approximately 30 times x_{critical}, at $\mathrm{Re}_{x,\mathrm{transition}} \approx 3 \times 10^6$. So, the boundary layer is expected to be fully turbulent by $\boldsymbol{x \approx}$ **2 m**.

Discussion Final results are given to only one significant digit, since the locations of transition and turbulence are only approximations. The actual locations are influenced by many things, such as noise, roughness, vibrations, freestream disturbances, etc.

10-71E

Solution We are to assess whether the boundary layer on the surface of a fin is laminar or turbulent or transitional.

Assumptions **1** The flow is steady and incompressible. **2** The fin surface is smooth.

Properties The density and viscosity of water at T = 40°F are 62.42 lbm/ft^3 and 1.038 × 10^{-3} lbm/ft·s, respectively. The kinematic viscosity is thus ν = 1.663 × 10^{-5} ft^2/s.

Analysis Although the fin is not a flat plate, the flat plate boundary layer values are useful as a reasonable approximation to determine whether the boundary layer is laminar or turbulent. We calculate the Reynolds number at the trailing edge of the fin, using c as the approximate streamwise distance along the flat plate,

$$\mathrm{Re}_x = \frac{Vc}{\nu} = \frac{(6.0\ \text{mi/hr})(1.6\ \text{ft})}{1.663\times10^{-5}\ \text{ft}^2/\text{s}}\left(\frac{5280\ \text{ft}}{\text{mi}}\right)\left(\frac{\text{hr}}{3600\ \text{s}}\right) = 8.47\times10^5$$

The critical Reynolds number for transition to turbulence is 1 × 10^5 for the case of a smooth flat plate with very clean, low-noise freestream conditions. Our Reynolds number is higher than this. The engineering value of critical Reynolds number for real engineering flows is $\mathrm{Re}_{x,cr}$ = 5 × 10^5. Since Re_x is greater than $\mathrm{Re}_{x,cr}$, but less than $\mathrm{Re}_{x,transition}$ (30 × 10^5), **the boundary layer is most likely transitional, but may be fully turbulent by the trailing edge of the fin**.

Discussion In a real-life situation, the freestream flow is not very "clean" – there are eddies and other disturbances, the fin surface is not perfectly smooth, and the vehicle may be vibrating. Thus, transition and turbulence are likely to occur much earlier than predicted for a smooth flat plate, and the boundary layer may be fully turbulent (or nearly so) by the trailing edge of the fin.

10-72

Solution We are to assess whether the boundary layer on the surface of a sign is laminar or turbulent or transitional.

Assumptions **1** The flow is steady and incompressible. **2** The sign surface is smooth.

Properties The density and viscosity of air at T = 25°C are 1.184 kg/m^3 and 1.849 × 10^{-5} kg/m·s respectively. The kinematic viscosity is thus ν = 1.562 × 10^{-5} m^2/s.

Analysis Since the air flow is parallel to the sign, this flow is that of a flat plate boundary layer. We calculate the Reynolds number at the downstream edge of the sign, using W as the streamwise distance along the flat plate,

$$\mathrm{Re}_x = \frac{VW}{\nu} = \frac{(5.0\ \text{m/s})(0.45\ \text{m})}{1.562\times10^{-5}\ \text{m}^2/\text{s}} = 1.44\times10^5 \qquad (1)$$

The critical Reynolds number for transition to turbulence is 1 × 10^5 for the case of a smooth flat plate with very clean, low-noise freestream conditions. Our Reynolds number is higher than this, but just barely so. The engineering value of critical Reynolds number for transition to a turbulent boundary layer in real engineering flows is $\mathrm{Re}_{x,cr}$ = 5 × 10^5; our value of Re_x is less than $\mathrm{Re}_{x,cr}$. Since Re_x is a bit greater than $\mathrm{Re}_{x,critical}$, but less than $\mathrm{Re}_{x,cr}$ (5 × 10^5), and much less than $\mathrm{Re}_{x,transition}$ (30 × 10^5), **the boundary layer is laminar for a while, and then becomes transitional by the trailing edge of the fin**.

Discussion The flow over the sign is not very "clean" – there are eddies from the passing vehicles, and other atmospheric disturbances. In addition, the sign surface is not perfectly smooth, and most signs tend to oscillate somewhat in the wind. Thus, transition and turbulence are likely to occur much earlier than predicted for a smooth flat plate. The boundary layer on this sign is definitely transitional, but probably not turbulent, by the downstream edge of the sign.

10-73E

Solution We are to assess whether the boundary layer on the wall of a wind tunnel is laminar or turbulent or transitional.

Assumptions **1** The flow is steady and incompressible. **2** The surface of the wind tunnel is smooth. **3** There are minimal disturbances in the freestream flow.

Properties The density and viscosity of air at T = 80°F are 0.07350 lbm/ft^3 and 1.247 × 10^{-5} lbm/ft·s respectively. The kinematic viscosity is thus ν = 1.697 × 10^{-4} ft^2/s.

Analysis We calculate the Reynolds number at the downstream end of the wall, using L = 1.5 ft as the streamwise distance along the flat plate,

$$\mathrm{Re}_x = \frac{VL}{\nu} = \frac{(7.5\ \mathrm{ft/s})(1.5\ \mathrm{ft})}{1.697\times10^{-4}\ \mathrm{ft^2/s}} = 6.63\times10^4 \qquad (1)$$

The engineering value of critical Reynolds number for transition to a turbulent boundary layer in real engineering flows is $\mathrm{Re}_{x,cr}$ = 5 × 10^5; our value of Re_x is much less than $\mathrm{Re}_{x,cr}$. In fact, our Reynolds number is even lower than the critical Reynolds number for transition to turbulence (1 × 10^5) for the case of a smooth flat plate with very clean, low-noise freestream conditions. Since the flow is clean and Re_x is less than $\mathrm{Re}_{x,critical}$, **the boundary layer is definitely laminar**.

Discussion There is typically a contraction just upstream of the test section of a wind tunnel. Upstream of that are typically some screens and/or honeycombs to make the flow clean and uniform. Thus, the disturbances are likely to be quite small, and the boundary layer is most likely laminar.

10-74

Solution We are to generate an expression for the outer flow velocity at point 2 in the boundary layer.

Assumptions **1** The flow is steady, incompressible, and laminar. **2** The boundary layer approximation is appropriate.

Analysis The boundary layer approximation tells us that P is constant *normal* to the boundary layer, but not necessarily *along* the boundary layer. Therefore, at any streamwise location along the boundary layer, the pressure in the outer flow region just above the boundary layer is the same as that at the wall. In the outer flow region, the Bernoulli equation reduces to

Outer flow:
$$U\frac{dU}{dx} = -\frac{1}{\rho}\frac{dP}{dx} \quad \rightarrow \quad \frac{dU}{dx} = -\frac{1}{\rho U}\frac{dP}{dx} \qquad (1)$$

For small values of Δx, we can approximate $U_2 \approx U_1 + (dU/dx)\Delta x$, and $P_2 \approx P_1 + (dP/dx)\Delta x$. Substitution of these approximations into Eq. 1 yields

$$U_2 \approx U_1 - \frac{1}{\rho U_1}\frac{dP}{dx}\Delta x = U_1 - \frac{1}{\rho U_1}\frac{P_2 - P_1}{\Delta x}\Delta x \quad \rightarrow \quad \boxed{U_2 \approx U_1 - \frac{P_2 - P_1}{\rho U_1}} \qquad (2)$$

Discussion It turns out that U_2 does not depend on Δx or μ, but only on P_1, P_2, U_1, and ρ.

10-75

Solution We are to estimate U_2, and explain whether it is less than, equal to, or greater than U_1.

Assumptions **1** The flow is steady, incompressible, and laminar. **2** The boundary layer approximation is appropriate.

Properties The density and viscosity of air at $T = 25^{\circ}$C are 1.184 kg/m^3 and 1.849 × 10^{-5} kg/m·s respectively.

Analysis The Bernoulli equation is valid in the outer flow region. Thus, we know that as P increases, U decreases, and vice-versa. In this case, P increases, and thus **we expect U_2 to be less than U_1**. From the results of Problem 10-74,

$$U_2 \approx U_1 - \frac{P_2 - P_1}{\rho U_1} = 10.3\ \frac{\text{m}}{\text{s}} - \frac{2.44\ \text{N/m}^2}{\left(1.186\ \text{kg/m}^3\right)\left(10.3\ \text{m/s}\right)}\left(\frac{\text{kg m}}{\text{N s}^2}\right) = \mathbf{10.1\ m/s}$$

Thus, the outer flow velocity indeed decreases by a small amount.

Discussion The approximation is first order, and thus appropriate only if the distance between x_1 and x_2 is small.

10-76

Solution We are to list the five steps of the boundary layer procedure.

Analysis **We list the five steps below**, with a description of each:

Step 1 Solve for the outer flow, ignoring the boundary layer (assuming that the region of flow outside the boundary layer is approximately inviscid and/or irrotational). Transform coordinates as necessary to obtain $U(x)$.

Step 2 Assume a thin boundary layer – so thin in fact that it does not affect the outer flow solution of Step 1.

Step 3 Solve the boundary layer equations. For this step we use the no-slip boundary condition at the wall, $u = v = 0$ at $y = 0$, the known outer flow condition at the edge of the boundary layer, $u \rightarrow U(x)$ as $y \rightarrow \infty$, and some known starting profile, $u = u_{\text{starting}}(y)$ at $x = x_{\text{starting}}$.

Step 4 Calculate quantities of interest in the flow field. For example, once the boundary layer equations have been solved (Step 3), we can calculate $\delta(x)$, shear stress along the wall, total skin friction drag, etc.

Step 5 Verify that the boundary layer approximations are appropriate. In other words, verify that the boundary layer is indeed *thin* – otherwise the approximation is not justified.

Discussion Students' discussions should be in their own words.

10-77

Solution We are to list at least three "red flags" to look for when performing boundary layer calculations.

Analysis **We list four below**. (Students are asked to list at least three.)

- The boundary layer approximation breaks down if Reynolds number is not large enough. For example, $\delta/L \sim 0.01$ (1%) for $\text{Re}_L = 10{,}000$.
- The assumption of zero pressure gradient in the y direction breaks down if wall curvature is of similar magnitude as δ. In such cases, centripetal acceleration effects due to streamline curvature cannot be ignored. Physically, the boundary layer is not "thin" enough for the approximation to be appropriate when δ is not << R.
- When Reynolds number is too *high*, the boundary layer does not remain laminar. The boundary layer approximation itself may still be appropriate, but the laminar boundary layer equations are *not* valid if the flow is transitional or fully turbulent. The laminar boundary layer on a smooth flat plate under clean flow conditions begins to transition towards turbulence at $\text{Re}_x \approx 1 \times 10^5$. In practical engineering applications, walls may not be smooth and there may be vibrations, noise, and fluctuations in the freestream flow above the wall, all of which contribute to an even earlier start of the transition process.
- If flow separation occurs, the boundary layer approximation is no longer appropriate in the separated flow region. The main reason for this is that a separated flow region contains *reverse flow*, and the parabolic nature of the boundary layer equations is lost.

Discussion Students' discussions should be in their own words.

10-78

Solution We are to prove that $\tau_w = 0.332\dfrac{\rho U^2}{\sqrt{\mathrm{Re}_x}}$ for a flat plate boundary layer.

Assumptions **1** The flow is steady and incompressible. **2** The Reynolds number is in the range where the Blasius solution is appropriate.

Analysis Equation 4 of Example 10-10 gives the definition of similarity variable η, which we re-write in terms of y as a function of η,

y as a function of η:
$$\eta = y\sqrt{\frac{U}{\nu x}} \quad \rightarrow \quad y = \eta\sqrt{\frac{\nu x}{U}} \tag{1}$$

From the chain rule and Eq. 1, we obtain an expression for $d/d\eta$,

Derivative with respect to similarity variable η:
$$\frac{d}{d\eta} = \frac{d}{dy}\frac{dy}{d\eta} = \frac{d}{dy}\sqrt{\frac{\nu x}{U}} \tag{2}$$

We apply Eq. 2 above to Eq. 8 of Example 10-10,

$$\left.\frac{d\left(u/U\right)}{d\eta}\right)_{\eta=0} = \left.\frac{du}{dy}\right)_{y=0}\sqrt{\frac{\nu x}{U^3}} = 0.332 \tag{3}$$

But by definition, $\tau_w = \mu\left.\dfrac{\partial u}{\partial y}\right)_{y=0}$, and Eq. 3 yields

Shear stress at the wall:
$$\tau_w = \mu\left.\frac{du}{dy}\right)_{y=0}\sqrt{\frac{\nu x}{U^3}} = 0.332\rho\nu\sqrt{\frac{U^3}{\nu x}} = 0.332\rho U^2\sqrt{\frac{\nu}{Ux}} = 0.332\frac{\rho U^2}{\sqrt{\mathrm{Re}_x}} \tag{4}$$

which is the desired expression for the shear stress at the wall in physical variables.

Discussion The chain rule algebra is valid here since U and x are functions of x only – they are not functions of y.

10-79E

Solution We are to calculate δ, δ^*, and θ at the end of the wind tunnel test section.

Assumptions **1** The flow is steady and incompressible. **2** The surface of the wind tunnel is smooth. **3** The boundary layer remains laminar all the way to the end of the test section.

Properties The density and viscosity of air at $T = 80^{\circ}$F are 0.07350 lbm/ft^3 and 1.247×10^{-5} lbm/ft·s respectively.

Analysis In Problem 10-73E, we calculated the Reynolds number at the downstream end of the wall, $\text{Re}_x = 6.63 \times 10^4$ (keeping an extra digit for the calculations). All of the desired quantities are functions of Re_x:

Boundary layer thickness:

$$\delta = \frac{4.91x}{\sqrt{\text{Re}_x}} = \frac{4.91(1.5\text{ ft})}{\sqrt{6.63\times10^4}} = 0.0286\text{ ft} \approx 0.34\text{ in} \qquad (1)$$

and

Displacement thickness:

$$\delta^* = \frac{1.72x}{\sqrt{\text{Re}_x}} = \frac{1.72(1.5\text{ ft})}{\sqrt{6.63\times10^4}} = 0.0100\text{ ft} \approx 0.12\text{ in} \qquad (2)$$

and

Momentum thickness:

$$\theta = \frac{0.664x}{\sqrt{\text{Re}_x}} = \frac{0.664(1.5\text{ ft})}{\sqrt{6.63\times10^4}} = 0.00387\text{ ft} \approx 0.046\text{ in} \qquad (3)$$

Thus, $\boldsymbol{\delta}$ **= 0.34 inches**, $\boldsymbol{\delta^*}$ **= 0.12 inches**, and $\boldsymbol{\theta}$ **= 0.046 inches** at the end of the wind tunnel test section. As expected, $\delta > \delta^* > \theta$.

Discussion All answers are given to two significant digits.

10-80

Solution We are to determine which orientation of a rectangular flat plate produces the higher drag.

Assumptions **1** The flow is steady and incompressible. **2** The Reynolds number is high enough for a laminar boundary layer to form on the plate, but not high enough for the boundary layer to become turbulent.

Analysis Reynolds number appears in the denominator of the equation for shear stress along the wall of a laminar boundary layer. Thus, wall shear stress decreases with increasing *x*, the distance down the plate. Hence, the average wall shear stress is higher for the case with the plate oriented with its short dimension aligned with the wind (case (b) of Fig. P10-80). Since the surface area of the plate is the same regardless of orientation, the plate with the higher average value of τ_w has the higher overall drag. **Case (b) has the higher drag**.

Discussion Another way to think about this problem is that since the boundary layer is thinner near the leading edge, the shear stress is higher there, and the front portion of the plate contributes to more of the total drag than does the rear portion of the plate.

10-81

Solution We are to define displacement thickness and discuss whether it is larger or smaller than boundary layer thickness.

Assumptions **1** The flow is steady and incompressible. **2** The boundary layer growing on the flat plate is laminar.

Analysis The two definitions of displacement thickness are:

- Displacement thickness is the distance that a streamline just outside of the boundary layer is deflected away from the wall due to the effect of the boundary layer.
- Displacement thickness is the imaginary increase in thickness of the wall, as seen by the outer flow, due to the effect of the growing boundary layer.

For a laminar boundary layer, **δ is larger than δ^***. δ is defined by the overall thickness of the boundary layer, whereas δ^* is an integrated thickness across the boundary layer that averages the mass deficit across the boundary layer. Therefore, it is not surprising that δ^* is less than δ.

Discussion The definitions given by students should be in their own words.

10-82

Solution The acceleration of air through the round test section of a wind tunnel is to be calculated.

Assumptions **1** The flow is steady and incompressible. **2** The walls are smooth, and disturbances and vibrations are kept to a minimum. **3** The boundary layers is laminar.

Properties The kinematic viscosity of air at 20°C is $\nu = 1.516 \times 10^{-5}$ m^2/s.

Analysis (*a*) As the boundary layer grows along the wall of the wind tunnel test section, air in the region of irrotational flow in the central portion of the test section accelerates in order to satisfy conservation of mass. The Reynolds number at the end of the test section is

$$\text{Re}_x = \frac{Vx}{\nu} = \frac{(2.0 \text{ m/s})(0.60 \text{ m})}{1.516\times10^{-5} \text{ m}^2/\text{s}} = 7.92\times10^4$$

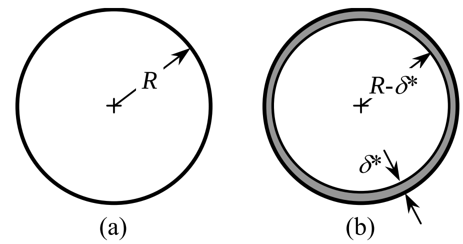


FIGURE 1
Cross-sectional views of the test section of the wind tunnel: (a) beginning of test section, and (b) end of test section.

The engineering value of critical Reynolds number for transition to a turbulent boundary layer in real engineering flows is $\text{Re}_{x,cr} = 5 \times 10^5$; our value of Re_x is less than $\text{Re}_{x,cr}$. In fact, Re_x is lower than the critical Reynolds number, $\text{Re}_{x,\text{critical}} \approx 1 \times 10^5$, for a smooth flat plate with a clean free stream. Since the walls are smooth and the flow is clean, we may assume that the boundary layer on the wall remains laminar throughout the length of the test section. We estimate the displacement thickness at the end of the test section,

$$\delta^* \approx \frac{1.72x}{\sqrt{\text{Re}_x}} = \frac{1.72(0.60 \text{ m})}{\sqrt{7.92\times10^4}} = 3.67\times10^{-3} \text{ m} = 3.67 \text{ mm} \qquad (1)$$

Two cross-sectional views of the test section are sketched in Fig. 1, one at the beginning and one at the end of the test section. The effective radius at the end of the test section is reduced by δ^* as calculated by Eq. 1. We apply conservation of mass to calculate the air speed at the end of the test section,

$$V_{\text{end}} A_{\text{end}} = V_{\text{beginning}} A_{\text{beginning}} \quad \rightarrow \quad V_{\text{end}} = V_{\text{beginning}} \frac{\pi R^2}{\pi (R-\delta^*)^2} \qquad (2)$$

We plug in the numerical values to obtain

Result:

$$V_{\text{end}} = (2.0 \text{ m/s}) \frac{(0.20 \text{ m})^2}{(0.20 \text{ m} - 3.67\times10^{-3} \text{ m})^2} = \mathbf{2.08 \text{ m/s}} \qquad (3)$$

Thus the air speed increases by approximately 4% through the test section, due to the effect of displacement thickness.

Discussion The same displacement thickness technique may be applied to turbulent boundary layers; however, a different equation for $\delta^*(x)$ is required.

10-83

Solution The acceleration of air through the square test section of a wind tunnel is to be calculated and compared to that through a round wind tunnel test section.

Assumptions **1** The flow is steady and incompressible. **2** The walls are smooth, and disturbances and vibrations are kept to a minimum. **3** The boundary layers is laminar.

Properties The kinematic viscosity of air at 20°C is $\nu = 1.516 \times 10^{-5}$ m^2/s.

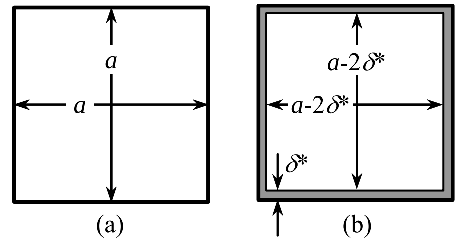


FIGURE 1
Cross-sectional views of the test section of the wind tunnel: (a) beginning of test section, and (b) end of test section.

Analysis (*a*) As the boundary layer grows along the walls of the wind tunnel test section, air in the region of irrotational flow in the central portion of the test section accelerates in order to satisfy conservation of mass. The Reynolds number at the end of the test section is

$$\mathrm{Re}_x = \frac{Vx}{\nu} = \frac{(2.0 \text{ m/s})(0.60 \text{ m})}{1.516\times10^{-5} \text{m}^2/\text{s}} = 7.92\times10^4$$

The engineering value of critical Reynolds number for transition to a turbulent boundary layer in real engineering flows is $\mathrm{Re}_{x,cr} = 5 \times 10^5$; our value of Re_x is less than $\mathrm{Re}_{x,cr}$. In fact, Re_x is lower than the critical Reynolds number, $\mathrm{Re}_{x,critical} \approx 1 \times 10^5$, for a smooth flat plate with a clean free stream. Since the walls are smooth and the flow is clean, we may assume that the boundary layer on the wall remains laminar throughout the length of the test section. We estimate the displacement thickness at the end of the test section,

$$\delta^* \approx \frac{1.72x}{\sqrt{\mathrm{Re}_x}} = \frac{1.72(0.60 \text{ m})}{\sqrt{7.92\times10^4}} = 3.67\times10^{-3} \text{ m} = 3.67 \text{ mm} \quad (1)$$

Two cross-sectional views of the test section are sketched in Fig. 1, one at the beginning and one at the end of the test section. The effective dimensions at the end of the test section are reduced by $2\delta^*$. We apply conservation of mass to calculate the air speed at the end of the test section,

$$V_{\text{end}} A_{\text{end}} = V_{\text{beginning}} A_{\text{beginning}} \quad \rightarrow \quad V_{\text{end}} = V_{\text{beginning}} \frac{a^2}{(a-2\delta^*)^2} \quad (2)$$

We plug in the numerical values to obtain

Result:

$$V_{\text{end}} = (2.0 \text{ m/s}) \frac{(0.40 \text{ m})^2}{\left[0.40 \text{ m} - 2\left(3.67\times10^{-3} \text{ m}\right)\right]^2} = \mathbf{2.08 \text{ m/s}} \quad (3)$$

Thus the air speed increases by approximately 4% through the test section, due to the effect of displacement thickness.

The result for the square test section is identical to that of the round test section. We might have expected the square test section to do better since its cross-sectional area is larger than that of the round test section. However, the square test section also has more wall surface area than does the round test section, and thus, the acceleration due to displacement thickness on the walls is the same in both cases.

Discussion The same displacement thickness technique may be applied to turbulent boundary layers; however, a different equation for $\delta^*(x)$ is required.

10-84

Solution The height of a boundary layer scoop in a wind tunnel test section is to be calculated.

Assumptions **1** The flow is steady and incompressible. **2** The walls are smooth. **3** The boundary layers starts growing at x = 0.

Properties The kinematic viscosity of air at 20°C is $\nu = 1.516 \times 10^{-5}$ m²/s.

Analysis (*a*) As the boundary layer grows along the wall of the wind tunnel test section, the Reynolds number increases. The Reynolds number at location x is

Re$_x$:
$$\text{Re}_x = \frac{Vx}{\nu} = \frac{(65.0 \text{ m/s})(1.45 \text{ m})}{1.516\times10^{-5} \text{ m}^2\text{/s}} = 6.22\times10^6$$

Since Re_x is greater than the transition Reynolds number, $\text{Re}_{x,\text{transition}} \approx 3 \times 10^6$, we assume that the boundary layer is turbulent throughout the length of the test section. We estimate the boundary layer thickness at the location of the scoop,

Table 10-4a:
$$\delta \approx \frac{0.16x}{(\text{Re}_x)^{1/7}} = \frac{0.16(1.45 \text{ m})}{(6.22\times10^6)^{1/7}} = 2.48\times10^{-2} \text{ m} = 24.8 \text{ mm} \qquad (1)$$

or,

Table 10-4b:
$$\delta \approx \frac{0.38x}{(\text{Re}_x)^{1/5}} = \frac{0.38(1.45 \text{ m})}{(6.22\times10^6)^{1/5}} = 2.41\times10^{-2} \text{ m} = 24.1 \text{ mm} \qquad (1)$$

We design the scoop height to be greater than or equal to the boundary layer thickness at the location of the scoop. Thus, we set $\boldsymbol{h \approx \delta \approx}$ **25. mm, or about an inch**.

Discussion The suction pressure of the scoop must be adjusted carefully so as not to suck too much or too little – otherwise it would disturb the flow. Since the early portion of the boundary layer is laminar, the actual boundary layer thickness will be somewhat lower than that calculated here. Thus, our calculation represents an upper limit. However, some of the large turbulent eddies in the boundary layer may actually exceed height δ, so our calculated h may actually not be sufficient to remove the complete boundary layer.

10-85E

Solution The acceleration of air through the round test section of a wind tunnel is to be calculated.

Assumptions **1** The flow is steady and incompressible. **2** The walls are smooth, and disturbances and vibrations are kept to a minimum. **3** The boundary layers is laminar.

Properties The kinematic viscosity of air at 70°F is $\nu = 1.643 \times 10^{-4}$ ft²/s.

Analysis (*a*) As the boundary layer grows along the wall of the wind tunnel test section, air in the region of irrotational flow in the central portion of the test section accelerates in order to satisfy conservation of mass. The Reynolds number at the end of the test section is

$$\mathrm{Re}_x = \frac{Vx}{\nu} = \frac{(5.0\text{ ft/s})(10.0\text{ in})}{1.643\times10^{-4}\text{ ft}^2\text{/s}}\left(\frac{1\text{ft}}{12\text{ in}}\right) = 2.54\times10^4$$

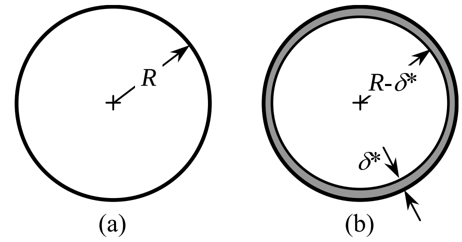


FIGURE 1
Cross-sectional views of the test section of the wind tunnel: (a) beginning of test section, and (b) end of test section.

The engineering value of critical Reynolds number for transition to a turbulent boundary layer in real engineering flows is $\mathrm{Re}_{x,cr} = 5 \times 10^5$; our value of Re_x is less than $\mathrm{Re}_{x,cr}$. In fact, Re_x is lower than the critical Reynolds number, $\mathrm{Re}_{x,\text{critical}} \approx 1 \times 10^5$, for a smooth flat plate with a clean free stream. Since the walls are smooth and the flow is clean, we may assume that the boundary layer on the wall remains laminar throughout the length of the test section. We estimate the displacement thickness at the end of the test section,

$$\delta^* \approx \frac{1.72x}{\sqrt{\mathrm{Re}_x}} = \frac{1.72(10.0\text{ in})}{\sqrt{2.54\times10^4}} = 0.108\text{ in} \qquad (1)$$

Two cross-sectional views of the test section are sketched in Fig. 1, one at the beginning and one at the end of the test section. The effective radius at the end of the test section is reduced by δ^* as calculated by Eq. 1. We apply conservation of mass to calculate the air speed at the end of the test section,

$$V_{\text{end}}A_{\text{end}} = V_{\text{beginning}}A_{\text{beginning}} \quad \rightarrow \quad V_{\text{end}} = V_{\text{beginning}}\frac{\pi R^2}{\pi\left(R-\delta^*\right)^2} \qquad (2)$$

We plug in the numerical values to obtain

Result:

$$V_{\text{end}} = (5.0\text{ ft/s})\frac{(6.0\text{ in})^2}{(6.0\text{ in} - 0.108\text{ in})^2} = \mathbf{5.18\text{ ft/s}} \qquad (3)$$

Thus the air speed increases by approximately 4% through the test section, due to the effect of displacement thickness.

To eliminate this acceleration, the engineers can either diverge the test section walls, or add some suction along the sides to remove some air.

Discussion The same displacement thickness technique may be applied to turbulent boundary layers; however, a different equation for $\delta^*(x)$ is required.

10-86E

Solution We are to determine if a boundary layer is laminar, turbulent, or transitional, and then compare the laminar and turbulent boundary layer thicknesses.

Properties The kinematic viscosity of air at 70°F is $\nu = 1.643 \times 10^{-4}$ ft²/s.

Analysis First, we calculate the Reynolds number at the end of the plate,

Re_x, end of plate:
$$\mathrm{Re}_x = \frac{Vx}{\nu} = \frac{(15.5 \text{ ft/s})(10.6 \text{ ft})}{1.643\times10^{-4}\ \text{ft}^2\text{/s}} = 1.00\times10^6$$

The engineering value of critical Reynolds number for transition to a turbulent boundary layer in real engineering flows is $\mathrm{Re}_{x,cr} = 5 \times 10^5$; our value of Re_x is greater than $\mathrm{Re}_{x,cr}$, leading us to suspect that the boundary layer is turbulent. However, Re_x is lower than the transition Reynolds number, $\mathrm{Re}_{x,\text{transition}} \approx 3 \times 10^6$, for a smooth flat plate with a clean free stream. Thus, we suspect that **this boundary layer is laminar at the front of the plate, and then transitional farther downstream**. If the plate is vibrating and/or the freestream is noisy, the boundary layer may possibly be fully turbulent by the end of the plate.

If the boundary layer were to remain laminar to the end of the plate, its thickness would be

Laminar:
$$\delta = \frac{4.91x}{\sqrt{\mathrm{Re}_x}} = \frac{4.91(10.6 \text{ ft})}{\sqrt{1.00\times10^6}} = 0.0520 \text{ ft} = \mathbf{0.625\ in} \qquad (1)$$

If the boundary layer at the end of the plate were fully turbulent (and turbulent from the beginning of the plate), its thickness would be

Turbulent, Table 10-4a:
$$\delta \approx \frac{0.16x}{(\mathrm{Re}_x)^{1/7}} = \frac{0.16(10.6 \text{ ft})}{(1.00\times10^6)^{1/7}} = 0.236 \text{ ft} = \mathbf{2.83\ in} \qquad (1)$$

or,

Turbulent, Table 10-4b:
$$\delta \approx \frac{0.38x}{(\mathrm{Re}_x)^{1/5}} = \frac{0.38(10.6 \text{ ft})}{(1.00\times10^6)^{1/5}} = 0.254 \text{ ft} = \mathbf{3.05\ in} \qquad (1)$$

Thus, the turbulent boundary layer thickness is about 4.5 to 4.9 times thicker than the corresponding laminar boundary layer thickness at the same Reynolds number. We expect the actual boundary layer thickness to lie somewhere between these two extremes.

Discussion The difference between the two turbulent boundary layer equations for δ is about 7 or 8 percent. This is larger than we might hope, but keep in mind that at $\mathrm{Re}_x = 1.00 \times 10^6$, the boundary layer is not yet fully turbulent, and the equations for δ are not accurate at such low Reynolds numbers.

10-87

Solution The apparent thickness of a flat plate is to be calculated.

Assumptions **1** The flow is steady and incompressible. **2** The walls are smooth. **3** The boundary layers starts growing at x = 0.

Properties The kinematic viscosity of air at 20°C is $\nu = 1.516 \times 10^{-5}$ m²/s.

Analysis (*a*) As the boundary layer grows along the plate, the Reynolds number increases. The Reynolds number at location x is

Re_x:
$$\text{Re}_x = \frac{Vx}{\nu} = \frac{(5.0\text{ m/s})(0.25\text{ m})}{1.516\times10^{-5}\,\text{m}^2/\text{s}} = 8.2454\times10^4$$

The engineering value of critical Reynolds number for transition to a turbulent boundary layer in real engineering flows is $\text{Re}_{x,cr} = 5 \times 10^5$; our value of Re_x is less than $\text{Re}_{x,cr}$. In fact, Re_x is lower than the critical Reynolds number, $\text{Re}_{x,\text{critical}} \approx 1 \times 10^5$, for a smooth flat plate with a clean free stream. Since Re_x is lower than the critical Reynolds number, and since the walls are smooth and the flow is clean, we may assume that the boundary layer on the wall remains laminar, at least to location x. We estimate the displacement thickness at x = 25 cm,

$$\delta^* \approx \frac{1.72x}{\sqrt{\text{Re}_x}} = \frac{1.72(0.25\text{ m})}{\sqrt{8.2454\times10^4}} = 1.4975\times10^{-3}\text{ m} = 0.14975\text{ cm} \qquad (1)$$

This "extra" thickness is seen by the outer flow. Since the plate is 0.75 cm thick, and since a similar boundary layer forms on the bottom as on the top, the total apparent thickness of the plate is

Apparent thickness:
$$h_{\text{apparent}} = h + 2\delta^* = 0.75\text{ cm} + 2(0.14975\text{ cm}) = \mathbf{1.05\,cm}$$

Discussion We have kept 5 digits of precision in intermediate steps, but report our final answer to three significant digits. The Reynolds number is pretty close to critical. If the freestream air flow were noisy and/or the plate were rough or vibrating, we might expect the boundary layer to be transitional, and then the apparent thickness would be greater.

10-88 [*Also solved using EES on enclosed DVD*]

Solution We are to plot the mean boundary layer profile $u(y)$ at the end of a flat plate using three different approximations.

Assumptions **1** The plate is smooth. **2** The boundary layer is turbulent from the beginning of the plate. **3** The flow is steady in the mean. **4** The plate is infinitesimally thin and is aligned parallel to the freestream.

Properties The kinematic viscosity of air at 20°C is $\nu = 1.516 \times 10^{-5}$ m²/s.

Analysis First we calculate the Reynolds number at $x = L$,

$$\mathrm{Re}_x = \frac{Vx}{\nu} = \frac{(80.0\text{ m/s})(17.5\text{ m})}{1.516\times10^{-5}\text{ m}^2/\text{s}} = 9.23\times10^7$$

This value of Re_x is well above the transitional Reynolds number for a flat plate boundary layer, so the assumption of turbulent flow from the beginning of the plate is reasonable.

Using the column (a) values of Table 10-4, we estimate the boundary layer thickness and the local skin friction coefficient at the end of the plate,

$$\delta \approx \frac{0.16x}{(\mathrm{Re}_x)^{1/7}} = 0.204\text{ m} \qquad C_{f,x} \approx \frac{0.027}{(\mathrm{Re}_x)^{1/7}} = 1.97\times10^{-3} \qquad (1)$$

We calculate the friction velocity by using the definition of $C_{f,x}$,

$$u_* = \sqrt{\tau_w/\rho} = U\sqrt{C_{f,x}/2} = (80.0\text{ m/s})\sqrt{(1.97\times10^{-3})/2} = 2.51\text{ m/s} \qquad (2)$$

where $U(x) = V$ = constant for a flat plate. It is trivial to generate a plot of the one-seventh-power law. We follow Example 10-13 to plot the log law, namely,

$$y = \frac{\nu}{u_*}e^{\kappa\left(\frac{u}{u_*}-B\right)} \qquad (3)$$

Since we know that u varies from 0 at the wall to U at the boundary layer edge, we are also able to plot the log law velocity profile in physical variables. Finally, Spalding's law of the wall is also written in terms of y as a function of u. **We plot all three profiles on the same plot for comparison (Fig. 1)**. All three are close, and we cannot distinguish the log law from Spalding's law on this scale.

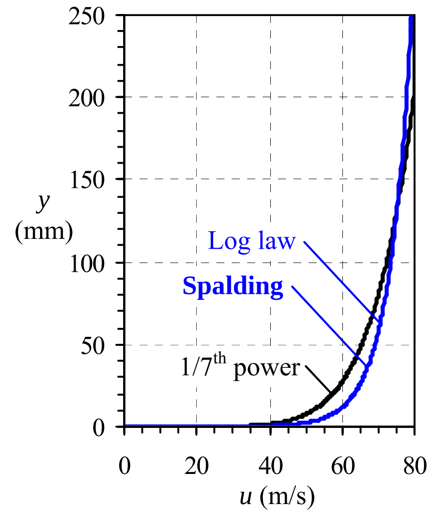


FIGURE 1
Comparison of turbulent flat plate boundary layer profile expressions in physical variables at $\mathrm{Re}_x = 9.23 \times 10^7$: one-seventh-power approximation, log law, and Spalding's law of the wall.

Discussion Neither the one-seventh-power law nor the log law are valid real close to the wall, but Spalding's law is valid all the way to the wall. However, on the scale shown in Fig. 1, we cannot see the differences between the log law and the Spalding law very close to the wall.

10-89

Solution We are to discuss the difference between a favorable and an adverse pressure gradient.

Analysis **When the pressure decreases downstream, the boundary layer is said to experience to a favorable pressure gradient. When the pressure increases downstream, the boundary layer is subjected to an adverse pressure gradient.** The term "favorable" is used because the boundary layer is unlikely to separate off the wall. On the other hand, "adverse" or "unfavorable" indicates that the boundary layer is more likely to separate off the wall.

Discussion A favorable pressure gradient occurs typically at the front of a body, whereas an adverse pressure gradient occurs typically at the back portion of a body.

10-90

Solution We are to discuss the role of an inflection point in a boundary layer profile.

Analysis As sketched in Fig. 10-124, **the existence of an inflection point in the boundary layer profile indicates an adverse or unfavorable pressure gradient**. The reason for this is due to the fact that the second derivative of the velocity profile $u(y)$ at the wall is directly proportional to the pressure gradient (Eq. 10-86). In an adverse pressure gradient field, dP/dx is *positive*, and thus, $\partial^2 u/\partial y^2)_{y=0}$ is also *positive*. However, since $\partial^2 u/\partial y^2$ must be *negative* as u approaches $U(x)$ at the edge of the boundary layer, there has to be an *inflection point* ($\partial^2 u/\partial y^2 = 0$) somewhere in the boundary layer.

Discussion If the adverse pressure gradient is large enough, the boundary layer separates off the wall, leading to reverse flow near the wall.

10-91

Solution We are to compare laminar and turbulent boundary layer separation, and explain why golf balls have dimples.

Analysis **Turbulent boundary layers are more "full" than are laminar boundary layers**. Because of this, **a turbulent boundary layer is much less likely to separate compared to a laminar boundary layer under the same adverse pressure gradient**. A smooth golf ball, for example, would maintain a laminar boundary layer on its surface, and the boundary layer would separate fairly easily, leading to large aerodynamic drag. **Golf balls have dimples (a type of surface roughness) in order to create an early transition to a turbulent boundary layer**. Flow still separates from the golf ball surface, but much farther downstream in the boundary layer, resulting in significantly reduced aerodynamic drag.

Discussion Turbulent boundary layers have more skin friction drag than do laminar boundary layers, but this effect is less significant than the pressure drag caused by flow separation. Thus, a rough golf ball (at appropriate Reynolds numbers) ends up with less overall drag, compared to a smooth golf ball at the same conditions.

10-92

Solution We are to generate expressions for δ^* and θ, and compare to Blasius.

Analysis First, we set $U(x) = V$ = constant for a flat plate. We integrate using the definition of δ^*,

$$\delta^* = \int_0^\infty \left(1-\frac{u}{U}\right)dy = \int_0^\delta \left[1-\frac{y}{\delta}\right]dy = \left[y-\frac{y^2}{2\delta}\right]_{y=0}^{y=\delta}$$

We integrate only to $y = \delta$, since beyond that, the integrand is identically zero. After substituting the limits of integration, we obtain δ^* as a function of δ,

$$\delta^* = [\delta-\delta/2]-[0-0] = \delta/2 \quad \rightarrow \quad \boldsymbol{\delta^* = \delta/2}$$

Similarly,

$$\theta = \int_0^\infty \frac{u}{U}\left(1-\frac{u}{U}\right)dy = \int_0^\delta \left[\frac{y}{\delta}-\frac{y^2}{\delta^2}\right]dy = \left[\frac{y^2}{2\delta}-\frac{y^3}{3\delta^2}\right]_{y=0}^{y=\delta}$$

After substituting the limits of integration, we obtain θ as a function of δ,

$$\theta = [\delta/2-\delta/3]-[0-0] = \delta/6 \quad \rightarrow \quad \boldsymbol{\theta = \delta/6}$$

The ratios are $\boldsymbol{\delta^*/\delta}$ **= 1/2 = 0.500**, and $\boldsymbol{\theta/\delta}$ **= 1/6 = 0.167**, to three significant digits. We compare these approximate results to those obtained from the Blasius solution, i.e., $\delta^*/\delta = 1.72/4.91 = 0.350$, and $\theta/\delta = 0.664/4.91 = 0.135$. Thus, **our approximate velocity profile yields** $\boldsymbol{\delta^*/\delta}$ **to about 43% error, and** $\boldsymbol{\theta/\delta}$ **to about 23% error**.

Discussion The linear approximation is not very accurate.

10-93

Solution We are to generate an expression for δ/x, and compare to Blasius.

Analysis By definition of local skin friction coefficient $C_{f,x}$,

$$C_{f,x} = \frac{2\tau_w}{\rho U^2} = \frac{2}{\rho U^2}\left(\mu \frac{du}{dy}\right)_{y=0} = \frac{2}{\rho U^2}\left[\mu \frac{U}{\delta}\right]_{y=0} = \frac{2\mu}{\rho U \delta} \qquad (1)$$

For a flat plate, the Kármán integral equation reduces to

$$C_{f,x} = 2\frac{d\theta}{dx} = 2\left(\frac{1}{6}\right)\frac{d\delta}{dx} \qquad (2)$$

where we have used the expression for θ as a function of δ from Problem 10-92. Substitution of Eq. 1 into Eq. 2 gives

$$\frac{d\delta}{dx} = 3C_{f,x} = \frac{6\mu}{\rho U \delta}$$

We separate variables and integrate,

$$\delta d\delta = \frac{6\mu}{\rho U}dx \quad \rightarrow \quad \frac{\delta^2}{2} = \frac{6\mu}{\rho U}x \quad \rightarrow \quad \frac{\delta}{x} = \sqrt{12\frac{\mu}{\rho U x}}$$

or, collecting terms, rounding to three digits, and setting $\rho Ux/\mu = \mathrm{Re}_x$,

$$\boxed{\frac{\delta}{x} = \frac{3.46}{\sqrt{\mathrm{Re}_x}}}$$

Compared to the Blasius result, $\delta/x = 4.91/\sqrt{\mathrm{Re}_x}$, our approximation based on the sine function velocity profile yields less than 30% error.

Discussion The Kármán integral equation is useful for obtaining approximate relations, and is "forgiving" because of the integration. Even so, the linear approximation is not very good. Nevertheless, a 30% error is sometimes reasonable for "back of the envelope" calculations. The sine wave approximation does much better, as in the next problem.

10-94

Solution We are to compare the sine wave approximation to the Blasius velocity profile.

Analysis We plot both profiles in Fig. 1. There is not much difference between the two, and thus, **the sine wave profile is a very good approximation of the Blasius profile**.

Discussion The slope of the two profiles at the wall is indistinguishable on the plot (Fig. 1); thus, the sine wave approximation should yield reasonable results for skin friction (shear stress) along the wall as well.

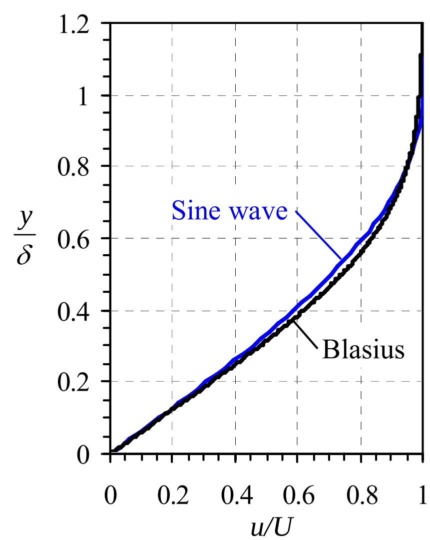


FIGURE 1
Comparison of Blasius and sine wave velocity profiles.

10-95

Solution We are to generate expressions for δ^* and θ, and compare to Blasius.

Analysis First, we set $U(x) = V$ = constant for a flat plate. We integrate using the definition of δ^*,

$$\delta^* = \int_0^\infty \left(1 - \frac{u}{U}\right) dy = \int_0^\delta \left[1 - \sin\left(\frac{\pi}{2}\frac{y}{\delta}\right)\right] dy = \left[y + \cos\left(\frac{\pi}{2}\frac{y}{\delta}\right)\frac{2\delta}{\pi}\right]_{y=0}^{y=\delta}$$

We integrate only to $y = \delta$, since beyond that, the integrand is identically zero. After substituting the limits of integration, we obtain δ^* as a function of δ,

$$\delta^* = [\delta + 0] - \left[0 + \frac{2\delta}{\pi}\right] = \delta - \frac{2}{\pi}\delta \quad \rightarrow \quad \boldsymbol{\delta^* = 0.3634\delta}$$

Similarly,

$$\theta = \int_0^\infty \frac{u}{U}\left(1 - \frac{u}{U}\right) dy = \int_0^\delta \left[\sin\left(\frac{\pi}{2}\frac{y}{\delta}\right) - \sin^2\left(\frac{\pi}{2}\frac{y}{\delta}\right)\right] dy$$

$$= \left\{-\cos\left(\frac{\pi}{2}\frac{y}{\delta}\right)\frac{2\delta}{\pi} - \left[\frac{y}{2} - \frac{\delta}{2\pi}\sin\left(\pi\frac{y}{\delta}\right)\right]\right\}_{y=0}^{y=\delta}$$

where we obtained the integral for $\sin^2$ from integration tables. After substituting the limits of integration, we obtain θ as a function of δ,

$$\theta = \left\{0 - \left[\frac{\delta}{2} - 0\right]\right\} - \left\{-\frac{2\delta}{\pi} - [0 - 0]\right\} = -\frac{\delta}{2} + \frac{2\delta}{\pi} \quad \rightarrow \quad \boldsymbol{\theta = 0.1366\delta}$$

The ratios are $\boldsymbol{\delta^*/\delta = 0.363}$, and $\boldsymbol{\theta/\delta = 0.137}$, to three significant digits. We compare these approximate results to those obtained from the Blasius solution, i.e., δ^*/δ = 1.72/4.91 = 0.350, and θ/δ = 0.664/4.91 = 0.135. Thus, **our approximate velocity profile yields $\boldsymbol{\delta^*/\delta}$ to less than 4% error, and $\boldsymbol{\theta/\delta}$ to about 1% error**.

Discussion Integration is "forgiving", and reasonable results can be obtained from integration, even when the velocity profile shape is not exact.

10-96

Solution We are to generate an expression for δ/x, and compare to Blasius.

Analysis By definition of local skin friction coefficient $C_{f,x}$,

$$C_{f,x} = \frac{2\tau_w}{\rho U^2} = \frac{2}{\rho U^2}\left(\mu \frac{du}{dy}\right)_{y=0} = \frac{2}{\rho U^2}\left[\mu U \cos\left(\frac{\pi}{2}\frac{y}{\delta}\right)\frac{\pi}{2\delta}\right]_{y=0} = \frac{2\mu}{\rho U}\left[\frac{\pi}{2\delta}\right] = \frac{\mu\pi}{\rho U \delta} \qquad (1)$$

For a flat plate, the Kármán integral equation reduces to

$$C_{f,x} = 2\frac{d\theta}{dx} = 2\left(0.1366\right)\frac{d\delta}{dx} \qquad (2)$$

where we have used the expression for θ as a function of δ from Problem 10-95. Substitution of Eq. 1 into Eq. 2 gives

$$\frac{d\delta}{dx} = \frac{C_{f,x}}{2\left(0.1366\right)} = \frac{\mu\pi}{0.2732\rho U \delta}$$

We separate variables and integrate,

$$\delta d\delta = \frac{\mu\pi}{0.2732\rho U}dx \quad \rightarrow \quad \frac{\delta^2}{2} = \frac{\mu\pi}{0.2732\rho U}x \quad \rightarrow \quad \frac{\delta}{x} = \sqrt{\frac{2\pi}{0.2732}\frac{\mu}{\rho U x}}$$

Collecting terms, rounding to three digits, and setting $\rho Ux/\mu = \mathrm{Re}_x$,

$$\boxed{\frac{\delta}{x} \approx \frac{4.80}{\sqrt{\mathrm{Re}_x}}}$$

Compared to the Blasius result, $\delta/x = 4.91/\sqrt{\mathrm{Re}_x}$, **the approximation yields less than 3% error**.

Discussion The Kármán integral equation is useful for approximations, and is "forgiving" because of the integration.

10-97

Solution We are to compare H for laminar vs. turbulent boundary layers, and discuss its significance.

Analysis Shape factor H is defined as the ratio of displacement thickness to momentum thickness. Thus,

Shape factor:
$$H = \frac{\delta^*}{\theta} = \frac{\delta^*/x}{\theta/x} \qquad (1)$$

For the laminar boundary layer on a flat plate, Eq. 1 becomes

Laminar:
$$H = \frac{\delta^*/x}{\theta/x} = \frac{1.72/\sqrt{\mathrm{Re}_x}}{0.664/\sqrt{\mathrm{Re}_x}} = \mathbf{2.59}$$

For the turbulent boundary layer, using both columns for comparison, Eq. 1 yields

Table 10-4a:
$$H = \frac{\delta^*/x}{\theta/x} = \frac{0.020/\left(\mathrm{Re}_x\right)^{1/7}}{0.016/\left(\mathrm{Re}_x\right)^{1/7}} = \mathbf{1.25}$$

Table 10-4b:
$$H = \frac{\delta^*/x}{\theta/x} = \frac{0.048/\left(\mathrm{Re}_x\right)^{1/5}}{0.037/\left(\mathrm{Re}_x\right)^{1/5}} = \mathbf{1.30}$$

Thus, **the shape factor for a laminar boundary layer is about twice that of turbulent boundary layer**. This implies that the smaller the value of H, the more full is the boundary layer. We may also infer that the smaller the value of H, the less likely is the boundary layer to separate. ***H*** **depends on the shape of the velocity profile – hence its name, shape factor**.

Discussion In fact, at the separation point of a laminar boundary layer, $H \approx 3.5$.

10-98

Solution We are to calculate H for an infinitesimally thin boundary layer.

Analysis By definition,

Shape factor:
$$H = \frac{\delta^*}{\theta} = \frac{\int_0^\infty \left(1 - \frac{u}{U}\right) dy}{\int_0^\infty \frac{u}{U}\left(1 - \frac{u}{U}\right) dy} \tag{1}$$

But for the limiting case under consideration, $u/U = 1$ through the entire boundary layer, yielding $\delta^* = 0$ and $\theta = 0$. To calculate the ratio in Eq. 1, we use l'Hopital's rule, where the variable u approaches U in the limit,

$$H = \lim_{u \to U} \frac{\frac{d}{du}\int_0^\infty \left(1 - \frac{u}{U}\right) dy}{\frac{d}{du}\int_0^\infty \frac{u}{U}\left(1 - \frac{u}{U}\right) dy} = \lim_{u \to U} \frac{-\int_0^\infty \frac{1}{U} dy}{\int_0^\infty \left(\frac{1}{U} - 2\frac{u}{U}\frac{1}{U}\right) dy} = \frac{\int_0^\infty \frac{1}{U} dy}{\int_0^\infty \frac{1}{U} dy} = \mathbf{1}$$

In other words, H is always greater than unity for any real boundary layer.

Discussion Since turbulent boundary layers are fuller than laminar boundary layers, it is no surprise that $H_{turbulent}$ is closer to unity than is $H_{laminar}$.

10-99

Solution We are to integrate an expression for δ.

Analysis We start with Eq. 5 of Example 10-14,

$$\frac{d\delta}{dx} = \frac{72}{14} 0.027 \left(\mathrm{Re}_x\right)^{-1/7} = 0.139 \left(\frac{Ux}{\nu}\right)^{-1/7}$$

Integration with respect to x yields

$$\delta = \frac{7}{6}(0.139)\left(\frac{Ux}{\nu}\right)^{6/7} \frac{\nu}{U} \quad \rightarrow \quad \frac{\delta}{x} = 0.162 \left(\frac{Ux}{\nu}\right)^{6/7} \frac{\nu}{Ux}$$

or, collecting terms, rounding to two digits, and setting $Ux/\nu = \mathrm{Re}_x$,

$$\boxed{\frac{\delta}{x} \approx \frac{0.16}{\left(\mathrm{Re}_x\right)^{1/7}}}$$

Discussion This approximate result is based on the 1/7th power law.

10-100

Solution We are to generate an expression for δ/x.

Analysis For a flat plate, the Kármán integral equation reduces to

$$C_{f,x} = 2\frac{d\theta}{dx} = 2(0.097)\frac{d\delta}{dx}$$

where we have used the given expression for θ as a function of δ. Substitution of the given expression for $C_{f,x}$ gives

$$\frac{d\delta}{dx} = \frac{C_{f,x}}{2(0.097)} = \frac{0.059(\mathrm{Re}_x)^{-1/5}}{0.194} = 0.304\left(\frac{Ux}{\nu}\right)^{-1/5}$$

Integration with respect to x yields

$$\delta = \frac{5}{4}(0.304)\left(\frac{Ux}{\nu}\right)^{4/5}\frac{\nu}{U} \quad \rightarrow \quad \frac{\delta}{x} = 0.380\left(\frac{Ux}{\nu}\right)^{4/5}\frac{\nu}{Ux}$$

or, collecting terms, rounding to two digits, and setting $Ux/\nu = \mathrm{Re}_x$,

$$\boxed{\frac{\delta}{x} \approx \frac{0.38}{(\mathrm{Re}_x)^{1/5}}}$$

The result is identical to that of Table 10-4, column (b).

Discussion The Kármán integral equation is useful for obtaining approximate relations like those of Table 10-4.

Review Problems

10-101C

Solution

(*a*) ***True***: We do not have to make the 2-D approximation in order to define the velocity potential function – ϕ can be defined for any flow if the vorticity is zero.

(*b*) ***False***: The stream function is definable for any two-dimensional flow field, regardless of the value of vorticity.

(*c*) ***True***: The velocity potential function is valid only for irrotational flow regions where the vorticity is zero.

(d) ***True***: The stream function is defined from the continuity equation, and is valid only for two-dimensional flows. Note that some researchers have defined three-dimensional forms of the stream function, but these are beyond the scope of the present introductory text book.

10-102

Solution We are to compute the viscous term for the given velocity field, and show that the flow is rotational.

Assumptions **1** The flow is steady. **2** The flow is two-dimensional in the r-θ plane.

Analysis We consider the viscous terms of the θ component of the Navier-Stokes equation,

Viscous terms:

$$\mu\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_\theta}{\partial r}\right)-\frac{u_\theta}{r^2}+\frac{1}{r^2}\frac{\partial^2 u_\theta}{\partial \theta^2}-\frac{2}{r^2}\frac{\partial u_r}{\partial \theta}+\frac{\partial^2 u_\theta}{\partial z^2}\right)$$

$$=\mu\left(\frac{\omega}{r}\quad-\frac{\omega}{r}\quad+0\quad-0\quad+0\right)=0$$

The viscous terms are zero, implying that **there are no net viscous forces acting on fluid elements**. This does not necessarily mean that the flow is inviscid – it could also mean that the flow is irrotational, since the viscous terms disappear in both inviscid and irrotational flows. We obtain the z component of vorticity in cylindrical coordinates from Chap. 4,

z component of vorticity: $$\zeta_z=\frac{1}{r}\left(\frac{\partial(ru_\theta)}{\partial r}-\frac{\partial u_r}{\partial \theta}\right)=\frac{1}{r}\left(\frac{\partial(\omega r^2)}{\partial r}-0\right)=2\omega$$

Thus, since the vorticity is non-zero, **this flow field is rotational**. Finally, since the flow is not irrotational, the only other way that the net viscous force can be zero is if the flow is inviscid. We conclude, then, that **this flow field is also inviscid**.

Discussion The vorticity is twice the angular velocity, as discussed in Chap. 4.

10-103

Solution We are to calculate the viscous stress tensor for a given velocity field.

Assumptions **1** The flow is steady. **2** The flow is two-dimensional in the r-θ plane.

Analysis The viscous stress tensor is given in Chap. 9 as

Viscous stress tensor in cylindrical coordinates:

$$\tau_{ij}=\begin{pmatrix}\tau_{rr} & \tau_{r\theta} & \tau_{rz}\\ \tau_{\theta r} & \tau_{\theta\theta} & \tau_{\theta z}\\ \tau_{zr} & \tau_{z\theta} & \tau_{zz}\end{pmatrix}=\begin{pmatrix}2\mu\frac{\partial u_r}{\partial r} & \mu\left(r\frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right)+\frac{1}{r}\frac{\partial u_r}{\partial \theta}\right) & \mu\left(\frac{\partial u_r}{\partial z}+\frac{\partial u_z}{\partial r}\right)\\ \mu\left(r\frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right)+\frac{1}{r}\frac{\partial u_r}{\partial \theta}\right) & 2\mu\left(\frac{1}{r}\frac{\partial u_\theta}{\partial \theta}+\frac{u_r}{r}\right) & \mu\left(\frac{\partial u_\theta}{\partial z}+\frac{1}{r}\frac{\partial u_z}{\partial \theta}\right)\\ \mu\left(\frac{\partial u_r}{\partial z}+\frac{\partial u_z}{\partial r}\right) & \mu\left(\frac{\partial u_\theta}{\partial z}+\frac{1}{r}\frac{\partial u_z}{\partial \theta}\right) & 2\mu\frac{\partial u_z}{\partial z}\end{pmatrix} \quad (1)$$

We plug in the velocity field from Problem 10-102 into Eq. 1, and we get

$$\tau_{ij}=\begin{pmatrix}\tau_{rr} & \tau_{r\theta} & \tau_{rz}\\ \tau_{\theta r} & \tau_{\theta\theta} & \tau_{\theta z}\\ \tau_{zr} & \tau_{z\theta} & \tau_{zz}\end{pmatrix}=\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix} \quad (2)$$

Thus, we conclude that there are no viscous stresses in this flow field (solid body rotation). Thus, **this flow can be considered inviscid**.

Discussion Since the fluid moves as a solid body, no fluid particles move relative to any other fluid particles; hence we expect no viscous stresses.

10-104

Solution We are to compute the viscous term for the velocity field and show that the flow is irrotational.

Assumptions **1** The flow is steady. **2** The flow is two-dimensional in the *r*-θ plane.

Analysis First, we write out and simplify the viscous terms of the θ component of the Navier-Stokes equation,

$$\mu\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_\theta}{\partial r}\right)-\frac{u_\theta}{r^2}+\frac{1}{r^2}\frac{\partial^2 u_\theta}{\partial \theta^2}-\frac{2}{r^2}\frac{\partial u_r}{\partial \theta}+\frac{\partial^2 u_\theta}{\partial z^2}\right)=\mu\left(\frac{\Gamma}{2\pi r^3}-\frac{\Gamma}{2\pi r^3}\quad+0\quad-0\quad+0\right)=0$$

The viscous terms are zero, implying that **there are no net viscous forces acting on fluid elements**. This does not necessarily mean that the flow is inviscid – it could also mean that the flow is irrotational, since the viscous terms disappear in both inviscid and irrotational flows.

We obtain the *z* component of vorticity in cylindrical coordinates from Chap. 4,

z component of vorticity:
$$\zeta_z=\frac{1}{r}\left(\frac{\partial\left(ru_\theta\right)}{\partial r}-\frac{\partial u_r}{\partial \theta}\right)=\frac{1}{r}\left(0-0\right)=0$$

Thus, since the vorticity is zero, **this flow field is irrotational**.

Discussion We cannot say for sure whether the flow is inviscid unless we calculate the viscous shear stresses, as in the following problem.

10-105

Solution We are to calculate the viscous stress tensor for a given velocity field.

Assumptions **1** The flow is steady. **2** The flow is two-dimensional in the *r*-θ plane.

Analysis The viscous stress tensor in cylindrical coordinates is given in Chap. 9 as

$$\tau_{ij}=\begin{pmatrix}\tau_{rr} & \tau_{r\theta} & \tau_{rz}\\ \tau_{\theta r} & \tau_{\theta\theta} & \tau_{\theta z}\\ \tau_{zr} & \tau_{z\theta} & \tau_{zz}\end{pmatrix}=\begin{pmatrix} 2\mu\frac{\partial u_r}{\partial r} & \mu\left(r\frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right)+\frac{1}{r}\frac{\partial u_r}{\partial \theta}\right) & \mu\left(\frac{\partial u_r}{\partial z}+\frac{\partial u_z}{\partial r}\right)\\ \mu\left(r\frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right)+\frac{1}{r}\frac{\partial u_r}{\partial \theta}\right) & 2\mu\left(\frac{1}{r}\frac{\partial u_\theta}{\partial \theta}+\frac{u_r}{r}\right) & \mu\left(\frac{\partial u_\theta}{\partial z}+\frac{1}{r}\frac{\partial u_z}{\partial \theta}\right)\\ \mu\left(\frac{\partial u_r}{\partial z}+\frac{\partial u_z}{\partial r}\right) & \mu\left(\frac{\partial u_\theta}{\partial z}+\frac{1}{r}\frac{\partial u_z}{\partial \theta}\right) & 2\mu\frac{\partial u_z}{\partial z}\end{pmatrix} \quad (1)$$

We plug in the velocity field from Problem 10-104 into Eq. 1, and we get

$$\tau_{ij}=\begin{pmatrix}\tau_{rr} & \tau_{r\theta} & \tau_{rz}\\ \tau_{\theta r} & \tau_{\theta\theta} & \tau_{\theta z}\\ \tau_{zr} & \tau_{z\theta} & \tau_{zz}\end{pmatrix}=\begin{pmatrix} 0 & \mu\frac{\Gamma}{\pi r^2} & 0\\ \mu\frac{\Gamma}{\pi r^2} & 0 & 0\\ 0 & 0 & 0\end{pmatrix} \quad (2)$$

Thus, we conclude that there are indeed some non-zero viscous stresses in this flow field. Hence, **this flow is not inviscid, even though it is irrotational**.

Discussion The fluid particles move relative to each other, generating viscous shear stresses. However, the *net* viscous force on a fluid particle is zero since the flow is irrotational.

10-106

Solution We are to calculate modified pressure P' and sketch profiles of P' at two vertical locations in the pipe.

Assumptions **1** The flow is incompressible. **2** Gravity acts vertically downward. **3** There are no free surface effects in this flow field.

Analysis By definition, modified pressure $P' = P + \rho gz$. So we add hydrostatic pressure component ρgz to the given profile $P = P_{atm}$ to obtain the profile for P'. At any z location in the pipe,

Modified pressure: $P' = P + \rho gz \quad \rightarrow \quad P' = P_{atm} + \rho gz$

We see that P' is uniform at any vertical location (P' does not vary radially), but P' varies with elevation z. At $z = z_1$,

Modified pressure at z_1: $\boxed{P_1' = P_{atm} + \rho g z_1}$ (1)

and at $z = z_2$,

Modified pressure at z_2: $\boxed{P_2' = P_{atm} + \rho g z_2}$ (2)

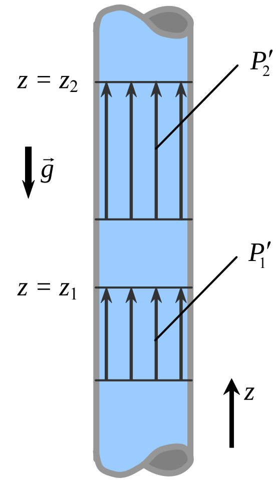


FIGURE 1
Modified pressure profiles at two vertical locations in the pipe.

Comparing Eqs. 1 and 2, the modified pressure is higher at location z_2 since z_2 is higher in elevation than z_1. In this problem there is no forced pressure gradient in terms of actual pressure. However, in terms of *modified* pressure, there is a linearly decreasing modified pressure along the axis of the pipe. In other words, there *is* a forced pressure gradient in terms of modified pressure.

Discussion Since modified pressure eliminates the gravity term from the Navier-Stokes equation, we have replaced the effect of gravity by a gradient of modified pressure P'.

10-107

Solution We are to calculate the required pressure drop between two axial locations of a horizontal pipe that would yield the same volume flow rate as that of the vertical pipe of Problem 10-106.

Assumptions **1** The flow is incompressible. **2** Gravity acts vertically downward. **3** There are no free surface effects in this flow field.

Analysis For the vertical case of Problem 10-106, we know that at $z = z_1$,

Modified pressure at z_1: $P_1' = P_{atm} + \rho g z_1$ (1)

and at $z = z_2$,

Modified pressure at z_2: $P_2' = P_{atm} + \rho g z_2$ (2)

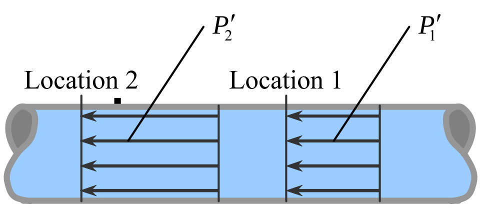


FIGURE 1
Modified pressure profiles at two horizontal locations in the pipe.

Since modified pressure effectively eliminates gravity from the problem, we expect that at the same volume flow rate, *the difference in modified pressure from z_2 to z_1 does not change with changes in the orientation of the pipe*. From Eqs. 1 and 2,

Change in modified pressure from z_2 to z_1: $P_2' - P_1' = \rho g \left(z_2 - z_1\right)$ (3)

The modified pressure profiles at two axial locations in the horizontal pipe are sketched in Fig. 1. We convert the modified pressures in Eq. 3 to actual pressures using the definition of modified pressure, $P = P' - \rho gz$. We note however, that for the horizontal pipe, z does not change along the pipe. Thus we conclude that the required difference in actual pressure is

Change in pressure from location 2 to location 1: $\boxed{P_2 - P_1 = \rho g \left(z_2 - z_1\right)}$ (4)

where z_2 and z_1 are the elevations of the vertical pipe case.

Discussion In order to achieve the same flow rate, the forced pressure gradient in the horizontal case must be the same as the hydrostatic pressure difference supplied by gravity in the vertical case.

Design and Essay Problem

10-108

Solution We are to discuss the velocity overshoot in Fig. 10-136.

Analysis **The velocity overshoot is a direct result of the displacement effect and the effect of inertia**. At very *low* values of Re_L (less than about 10^1), where the displacement effect is most prominent, the velocity overshoot is almost non-existent. This can be explained by the lack of inertia at these low Reynolds numbers. Without inertia, there is no mechanism to accelerate the flow around the plate; rather, viscosity *retards* the flow everywhere in the vicinity of the plate, and the influence of the plate extends tens of plate lengths beyond the plate in all directions. At *moderate* values of Reynolds number (Re_L between about 10^1 and 10^4), the displacement effect is significant, and inertial terms are no longer negligible. Hence, fluid is able to accelerate around the plate and the velocity overshoot is significant. At very *high* values of Reynolds number ($Re_L > 10^4$), inertial terms dominate viscous terms, and the boundary layer is so thin that the displacement effect is almost negligible – the small displacement effect leads to very small velocity overshoot at high Reynolds numbers.

Discussion We can imagine that the flat plate appears thicker from the point of view of the outer flow, and therefore, the flow must accelerate around this "fat" plate.