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A Berry–Esseen Theorem and a Law of the Iterated Logarithm for Asymptotically Negatively Associated Sequences

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Abstract Negatively associated sequences have been studied extensively in recent years. Asymptotically negative association is a generalization of negative association. In this paper a Berry–Esseen theorem and a law of the iterated logarithm are obtained for asymptotically negatively associated sequences.

Keywords Berry–Esseen theorem, law of the iterated logarithm, negative association, asymptotically negative association

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1 Introduction and Results

A finite family of random variables $\{X_i; 1 \le i \le n\}$ is said to be negatively associated (NA) if, for every pair of disjoint subsets A and B of $\{1, 2, \dots, n\}$,

$$\mathrm{Cov}(f(X_i; i \in A), g(X_j; j \in B)) \le 0 \tag{1.1}$$

whenever f and g are coordinate-wise nondecreasing and the above covariance exists. An infinite family is negatively associated if every finite subfamily is negatively associated. The concept of the negative association was introduced by Alam and Saxena [1] and Joag-Dev and Proschan [2]. As proved by Joag-Dev and Proschan [2], some well-known multivariate distributions have the NA property. The NA property has found important and wide applications in areas such as multivariate statistics and reliability theory. In the past few decades, a lot of effort was dedicated to proving the limit theorems of NA random variables. We refer to Joag-Dev and Proschan [2] for fundamental properties, Newman [3] for the central limit theorem, Su, et al. [4] for the moment inequality and functional central limit theorem, Pan and Lu [5] for the uniform convergence rate in the central limit theorem and Shao and Su [6] for the law of the iterated logarithm.

We first define notations and introduce an asymptotically negatively association assumption that will be used throughout the paper. Let $\Phi(x)$ denote the standard normal distribution function and $\log x = \ln(x \vee e)$. For any real number x, let $[x]$ denote the integer part of x, $x^+ = \max(0, x)$ and $x^- = \max(0, -x)$ (except for the definition of $\rho^-(\cdot)$). Let $a \wedge b = \min(a, b)$, $a \vee b = \max(a, b)$ and C denote a positive constant, which may take different values in different expressions.

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Definition 1.1 *A sequence of random variables $\{X_n; n \geq 1\}$ is said to be asymptotically negatively associated (ANA), if*

$$\rho^-(r) := \sup\{\rho^-(S,T) : S,T \in N, \mathrm{dist}(S,T) \geq r\} \to 0 \ \ (r \to \infty),$$

where

$$\rho^-(S,T) := 0 \vee \sup\left\{\frac{\mathrm{Cov}(f(X_i; i \in S), g(X_j; j \in T))}{(\mathrm{Var} f(X_i; i \in S))^{1/2} \cdot (\mathrm{Var}\ g(X_j; j \in T))^{1/2}} : f, g \in \mathscr{F}\right\},$$

$\mathscr{F} = \{f = f(x_1, \ldots, x_p) : f$ *is coordinatewise increasing;* $p \geq 1\}$.

The above definition was introduced by Zhang [7, 8]. An NA sequence is obviously an ANA sequence with $\rho^-(1) = 0$. Compared to NA, ANA defines a strictly larger class of random variables (for detail examples, see Zhang [7]). Consequently, the study of the limit theorems for ANA sequences is of much interest. Zhang [8] established the central limit theorem. Wang and Lu [9] established the moment inequality and the functional central limit theorem. The main purpose of this paper is to establish a uniform error bound of the Berry–Esseen type in normal approximation and a law of the iterated logarithm for ANA random variables. Let $\{X_n; n \geq 1\}$ be a sequence of random variables; denote $S_n = \sum_{j=1}^n X_j$, $\sigma_n^2 = \mathrm{Var}(S_n)$.

Theorem 1.1 *Let $\theta > 2$ and $\{X_n; n \geq 1\}$ be an ANA sequence of random variables with $EX_n = 0$, and $\sup_{j \in N} E|X_j|^{2+\delta} < \infty$ for some $0 < \delta \leq 1$. Assume*

(C1) $u(r) := \sup_{j \in N} \sum_{k:|k-j| \geq r} |\mathrm{Cov}(X_j, X_k)| = O(r^{-\theta_1})$, $\theta_1 > \max(1, \frac{\delta}{1+\delta}(\theta - 1))$;

(C2) $\rho^-(r) = O(r^{-\theta_2})$, $\theta_2 > \theta - 1$;

and,

(C3) $\inf_{n \in N} \sigma_n^2 / n > 0$.

Then there exists a positive constant C such that

$$\sup_x \left| P\left(\frac{S_n}{\sigma_n} < x\right) - \Phi(x)\right| \leq C(n^{-\beta_1} + n^{-\beta_2} + n^{-\beta_3}), \tag{1.2}$$

where $\beta_1 = \frac{1}{\theta}(\theta_1 - \frac{\delta}{1+\delta}(\theta - 1))$, $\beta_2 = \frac{1}{\theta}(\theta_2 - (\theta - 1))$ and $\beta_3 = \frac{\delta(1-2/\theta)}{2(1+\delta)}$.

Remark 1.1 We provide a uniform convergence rate in the central limit theorem under a power decay of the covariance of an ANA sequence. Let $\delta = 1$ and θ be sufficiently large. Then θ_1, θ_2 are sufficiently large and the maximum convergence rate of (1.2) is close to $O(n^{-1/4})$. Pan and Lu [5] [resp. Birkel [10]] obtained a convergence rate $O(n^{-1/2} \log n)$ for NA [resp. PA] sequences if $u(r)$ decreases exponentially to 0. However, it seems that the method of Birkel [10], and Pan and Lu [5] cannot work for ANA sequences.

Remark 1.2 If conditions (C1) and (C2) are replaced by weaker conditions:

(C1)$'$ $u(n) = O((\log n)^{-3(1+\tau)})$ for some $\tau > 0$;

and,

(C2)$'$ $\rho^-(n) = O((\log n)^{-(3+2/\delta)(1+\tau)})$,

then

$$\sup_x \left| P\left(\frac{S_n}{\sigma_n} < x\right) - \Phi(x)\right| \leq O((\log n)^{-1-\tau}).$$

The proof is similar to the one of Theorem 1.1.

Theorem 1.2 *Let $\{X_n; n \geq 1\}$ be a weakly stationary ANA sequence of random variables with $EX_n = 0$, and $\sup_{j \in N} E|X_j|^{2+\delta} < \infty$ for some $\delta > 0$. Denote $\sigma^2 = \mathrm{Var}(X_1) + 2\sum_{j=2}^\infty \mathrm{Cov}(X_1, X_j)$. Assume (C1)$'$, (C2)$'$ and*

(C3)$'$ $\quad \mathrm{Var} X_1 - 2\sum_{j=2}^\infty \mathrm{Cov}(X_1, X_j)^- > 0.$

Then

$$\limsup_{n \to \infty} \frac{S_n}{(2\sigma^2 n \log\log n)^{1/2}} = 1 \quad a.s. \tag{1.3}$$

Remark 1.3 It is easy to see that conditions (C1)′ and (C3)′ imply $0 < \sigma^2 < \infty$. In the stationary case, condition (C3)′ implies (C3). In fact,

$$\begin{aligned}\frac{\sigma_n^2}{n} &= \mathrm{Var}(X_1) + 2n^{-1}\sum_{j=2}^{n}(n+1-j)\mathrm{Cov}(X_1, X_j)\\ &\geq \mathrm{Var}(X_1) - 2n^{-1}\sum_{j=2}^{n}(n+1-j)\mathrm{Cov}(X_1, X_j)^-\\ &\geq \mathrm{Var}(X_1) - 2\sum_{j=2}^{n}\mathrm{Cov}(X_1, X_j)^-.\end{aligned}$$

Thus

$$\inf_{n\in N}\sigma_n^2/n \geq \mathrm{Var}(X_1) - 2\sum_{j=2}^{\infty}\mathrm{Cov}(X_1, X_j)^- > 0.$$

2 Proof of Theorem 1.1

In order to prove Theorem 1.1, we need the following lemmas.

Lemma 2.1 (Lin [11]) *Suppose $f_1(t)$ and $f_2(t)$ are the characteristic functions corresponding to distribution functions $F_1(x)$ and $F_2(x)$, respectively. Then $\forall\ T > 0, b > \frac{1}{2\pi}$,*

$$\begin{aligned}\sup_x|F_1(x) - F_2(x)| \leq b\max_{k=1,2}\sup_x&\left|\int_{-T}^{T}\frac{f_1(t)-f_2(t)}{it}h_k(t)e^{-itx}dt\right|\\ &+ bT\sup_x\int_{|y|\leq C(b)/T}|F_2(x+y) - F_2(x)|dy,\end{aligned}\tag{2.1}$$

where

$$h_1(t) = \begin{cases}(1-\frac{|t|}{T})e^{ita/T}, & |t| < T,\\ 0, & |t| \geq T,\end{cases} \qquad h_2(t) = \begin{cases}(1-\frac{|t|}{T})e^{-ita/T}, & |t| < T,\\ 0, & |t| \geq T.\end{cases}$$

Here the constants $C(b)$ and a depend only on b. Furthermore, we have

$$\sup_x\left|\int_{-T}^{T}\frac{f_1(t)-f_2(t)}{it}h_k(t)e^{-itx}dt\right| \leq \sup_x|F_1(x) - F_2(x)| \quad (k = 1, 2).\tag{2.2}$$

Lemma 2.2 (Zhang [7]) *Suppose that $\{X_n; n \geq 1\}$ is an ANA sequence of random variables with finite variance. Then, for any real $\lambda_1, \ldots, \lambda_n$,*

$$\begin{aligned}&\left|E\exp\left(i\sum_{k=1}^{n}\lambda_kX_k\right) - \prod_{k=1}^{n}E\exp(i\lambda_kX_k)\right|\\ &\leq 4\sum_{1\leq j\neq k\leq n}|\lambda_k||\lambda_j|\{-\mathrm{Cov}(X_k, X_j) + 8\rho^-(1)||X_j||_{2,1}||X_k||_{2,1}\},\end{aligned}$$

where $||X||_{2,1} = \int_0^\infty P^{1/2}(|X| \geq x)dx$.

Lemma 2.3 (Wang and Lu [9]) *Let $\{X_n; n \geq 1\}$ be an ANA sequence of random variables with $EX_n = 0$, and $E|X_n|^p < \infty$ for some $p \geq 2$. Assume that $\rho^-(N) \leq r$ for $N \geq 1$, $0 < r < (\frac{1}{6p})^{p/2}$. Then there exists a positive constant D such that*

$$E\max_{1\leq k\leq n}\left|\sum_{i=1}^{k}X_i\right|^p \leq D\left\{\sum_{i=1}^{n}E|X_i|^p + \left(\sum_{i=1}^{n}EX_i^2\right)^{p/2}\right\}.$$

Proof of Theorem 1.1 The basic approach of this proof is based on Wood [12] but the details are quite different. Without loss of generality, we can assume that $n \geq 8$. For given $\theta > 2$ and $0 < \delta \leq 1$, let $\{p = p(n)\}$ and $\{q = q(n)\}$ be two arrays of positive integers such that

$p = [n^{(\delta+1/\theta)/(1+\delta)}]$ and $q = [n^{1/\theta}]$. Define $\{k = k(n)\}$ such that $k = [\frac{n}{p+q}]$. It is not difficult to observe that $k \geq 1$ if n is large enough. Define

$$A(j) = \{i \in N : (j-1)(p+q) < i \leq j(p+q) - q\}, \quad j = 1, \ldots, k,$$
$$B(j) = \{i \in N : j(p+q) - q < i \leq j(p+q)\}, \quad j = 1, \ldots, k-1,$$
$$B(k) = \{i \in N : k(p+q) - q < i \leq n\},$$
$$W_j = \sum_{i \in A(j)} X_i, \qquad V_j = \sum_{i \in B(j)} X_i, \qquad j = 1, \ldots, k,$$
$$S_{n1} = \sum_{j=1}^{k} W_j \qquad \text{and} \qquad S_{n2} = \sum_{j=1}^{k} V_j.$$

For $a_n > 0$, using the fact that $S_n = S_{n1} + S_{n2}$ and that $|\Phi(x + a_n) - \Phi(x)| \leq (2\pi)^{-1/2} a_n$, we have

$$\sup_x \left| P\left(\frac{S_n}{\sigma_n} < x\right) - \Phi(x)\right| \leq \sup_x \left| P\left(\frac{S_{n1}}{\sigma_n} < x\right) - \Phi(x)\right| + (2\pi)^{-1/2} a_n + P(|S_{n2}| \geq a_n \sigma_n). \tag{2.3}$$

Let $\bar{X}_i = a_n\sigma_n \wedge (X_i \vee (-a_n\sigma_n))$, $\bar{S_{n2}} = \sum_{j=1}^{k} \sum_{i \in B(j)} \bar{X}_i$. By the Markov inequality and $\sup_j E|X_j|^{2+\delta} < \infty$, we have

$$\begin{aligned} P(|S_{n2}| \geq a_n\sigma_n) &\leq P(|\bar{S_{n2}}| \geq a_n\sigma_n) + \sum_{j=1}^{k} \sum_{i \in B(j)} P(|X_i| > a_n\sigma_n) \\ &\leq P(|\bar{S_{n2}}| \geq a_n\sigma_n) + Ckq/(a_n\sigma_n)^{2+\delta}. \end{aligned} \tag{2.4}$$

Note that $\bar{X}_1, \bar{X}_2, \ldots,$ are also ANA random variables. Using the Markov inequality, Lemma 2.3 and $\sup_j E|X_j|^{2+\delta} < \infty$, for $\gamma \geq 2 + \delta$, we have

$$\begin{aligned} P(|\bar{S_{n2}}| \geq a_n\sigma_n) &\leq C(a_n\sigma_n)^{-\gamma} \Bigg\{ \sum_{j=1}^{k} \sum_{i \in B(j)} E|\bar{X}_i|^{\gamma} + \Bigg(\sum_{j=1}^{k} \sum_{i \in B(j)} E\bar{X}_i^{\,2} \Bigg)^{\gamma/2} \\ &\quad + \Bigg(\sum_{j=1}^{k} \sum_{i \in B(j)} E|X_i|^{2+\delta}/(a_n\sigma_n)^{1+\delta} \Bigg)^{\gamma} \Bigg\} \\ &\leq C\{kq/(a_n\sigma_n)^{2+\delta} + (kq/(a_n\sigma_n)^2)^{\gamma/2}\}. \end{aligned} \tag{2.5}$$

Combining (2.4) and (2.5), by condition (C3) we have

$$P(|S_{n2}| \geq a_n\sigma_n) \leq C\{kq/(a_n^{2+\delta} n^{1+\delta/2}) + (kq/(a_n^2 n))^{\gamma/2}\}. \tag{2.6}$$

Let $\{W_j^{'}; 1 \leq j \leq k\}$ be a sequence of independent random variables such that $W_j^{'}$ and W_j have the same distribution for each $j = 1, \ldots, k$. Then applying the triangle inequality to Lemma 2.1 (2.1) we get

$$\begin{aligned} &\sup_x \left| P\left(\frac{S_{n1}}{\sigma_n} < x\right) - \Phi(x)\right| \\ &\leq b \max_{\nu=1,2} \sup_x \left| \int_{-T}^{T} \frac{E\exp(it\sum_{j=1}^{k} W_j/\sigma_n) - E\exp(it\sum_{j=1}^{k} W_j^{'}/\sigma_n)}{it} h_\nu(t) e^{-itx} dt \right| \\ &\quad + b \max_{\nu=1,2} \sup_x \left| \int_{-T}^{T} \frac{E\exp(it\sum_{j=1}^{k} W_j^{'}/\sigma_n) - E\exp(it\sum_{j=1}^{k} W_j^{'}/\sigma_{n1})}{it} h_\nu(t) e^{-itx} dt \right| \\ &\quad + b \max_{\nu=1,2} \sup_x \left| \int_{-T}^{T} \frac{E\exp(it\sum_{j=1}^{k} W_j^{'}/\sigma_{n1}) - \exp(-t^2/2)}{it} h_\nu(t) e^{-itx} dt \right| + \frac{bC^2(b)}{\sqrt{2\pi}T} \\ &:= I_1 + I_2 + I_3 + \frac{bC^2(b)}{\sqrt{2\pi}T}, \end{aligned} \tag{2.7}$$

where $\sigma_{n1}^2 = \sum_{j=1}^{k} \mathrm{Var}(W_j)$.

From Lemma 2.2, it follows that

$$
\begin{aligned}
I_1 &\le \int_{-T}^{T} \left| \frac{E\exp(it\sum_{j=1}^{k} W_j/\sigma_n) - \prod_{j=1}^{k} E\exp(itW_j/\sigma_n)}{t} \right| dt \\
&\le \frac{4T^2}{\sigma_n^2} \sum_{1\le i\ne j\le k} \{-\mathrm{Cov}(W_i, W_j) + 8\rho^-(q)||W_i||_{2,1}||W_j||_{2,1}\} \\
&\le \frac{4T^2}{\sigma_n^2} \Big\{ \sum_{1\le i\ne j\le k} \{-\mathrm{Cov}(W_i, W_j)\} + C\rho^-(q) \sum_{1\le i\ne j\le k} (E|W_i|^{2+\delta} E|W_j|^{2+\delta})^{1/(2+\delta)} \Big\} \\
&:= \frac{4T^2}{\sigma_n^2} \{I_{11} + I_{12}\}, \qquad (2.8)
\end{aligned}
$$

where the inequality $||X||_{2,1} \le (\frac{2+\delta}{\delta})(E|X|^{2+\delta})^{1/(2+\delta)}$ (cf. Ledoux and Talagrand [13, p. 251]) is used. By (C1), it follows that

$$
I_{11} \le \sum_{i,j\le n: |i-j|\ge q+1} |\mathrm{Cov}(X_i, X_j)| \le Cnu(q+1) \le Cnq^{-\theta_1}. \qquad (2.9)
$$

By Theorem 3.1 of Zhang [7] and (C2), it follows that

$$
I_{12} \le C\rho^-(q)k^2 p(\sup_j E|X_j|^{2+\delta})^{2/(2+\delta)} \le Cnkq^{-\theta_2}. \qquad (2.10)
$$

Combining (2.8)–(2.10), by (C3) we get

$$
I_1 \le C\{q^{-\theta_1} + kq^{-\theta_2}\}T^2. \qquad (2.11)
$$

By Taylor's theorem and the independence of $\{W_j'\}$ we get

$$
\begin{aligned}
I_2 &\le 2TE \left| \frac{\sum_{j=1}^{k} W_j'}{\sigma_n} - \frac{\sum_{j=1}^{k} W_j'}{\sigma_{n1}} \right| \\
&\le 2T \left| \frac{\sigma_{n1}}{\sigma_n} - 1 \right| \left[E\Big(\sum_{j=1}^{k} W_j'/\sigma_{n1} \Big)^2 \right]^{1/2} \\
&\le 2T\sigma_n^{-2}|\sigma_{n1}^2 - \sigma_n^2|. \qquad (2.12)
\end{aligned}
$$

Using $\sigma_n^2 = \mathrm{Var}(\sum_{j=1}^{k}(W_j + V_j))$ and Theorem 3.1 of Zhang [7] we get

$$
\begin{aligned}
&|\sigma_{n1}^2 - \sigma_n^2| \\
&\le \left| \sum_{j=1}^{k} \mathrm{Var}(V_j) \right| + 2\left| \sum_{j=1}^{k} \mathrm{Cov}(W_j, V_j) \right| + \left| \sum_{j=1}^{k} \mathrm{Var}(W_j + V_j) - \mathrm{Var}\Big(\sum_{j=1}^{k} (W_j + V_j) \Big) \right| \\
&\le Ckq \sup_j EX_j^2 + 2\left| \sum_{j=1}^{k} \mathrm{Cov}(W_j, V_j) \right| + \left| \sum_{1\le i\ne j\le k} \mathrm{Cov}(W_j + V_j, W_i + V_i) \right|. \qquad (2.13)
\end{aligned}
$$

By (C1),

$$
\left| \sum_{j=1}^{k} \mathrm{Cov}(W_j, V_j) \right| \le k \sum_{i=1}^{\min(p,q)} u(i) \le Ck, \qquad (2.14)
$$

and

$$
\begin{aligned}
\left| \sum_{1\le i\ne j\le k} \mathrm{Cov}(W_j + V_j, W_i + V_i) \right| &= 2\left| \sum_{j=1}^{k-1} \mathrm{Cov}\Big(W_j + V_j, \sum_{i=j+1}^{k} (W_i + V_i) \Big) \right| \\
&\le 2(k-1) \sum_{i=1}^{p+q} u(i) \le Ck. \qquad (2.15)
\end{aligned}
$$

Thus, by (2.12)–(2.15) and (C3) we get

$$
I_2 \le C(kq + k)n^{-1}T \le C\frac{q}{p}\, T. \qquad (2.16)
$$

Now we estimate I$_3$. Using Lemma 2.1 (2.2) we have

$$I_3 \le b \sup_x \left| P\left(\frac{\bar{S_{n1}}}{\sigma_{n1}} < x \right) - \Phi(x) \right|,$$

where $\bar{S_{n1}} = \sum_{j=1}^k W_j^{'}$. Note that $W_1^{'}, \dots, W_k^{'}$ are independent random variables, $EW_j^{'} = 0$, $j = 1, \dots, k$. Thus applying the Berry–Esseen inequality for independent random variables (see, Petrov [14]), we have

$$I_3 \le \frac{C}{\sigma_{n1}^{2+\delta}} \sum_{j=1}^k E|W_j^{'}|^{2+\delta}.$$

By Theorem 3.1 of Zhang [7] and condition (C3) we get

$$E|W_j^{'}|^{2+\delta} = E|W_j|^{2+\delta} \le Cp^{(2+\delta)/2} \sup_j E|X_j|^{2+\delta} \le Cp^{(2+\delta)/2}, \qquad j = 1, \dots, k,$$

and

$$\sigma_{n1} \ge C\sqrt{kp}.$$

Thus

$$I_3 \le Ck^{-\delta/2}. \tag{2.17}$$

Now, associated with (2.3), (2.6), (2.7), (2.11), (2.16) and (2.17), we have

$$\sup_x \left| P\left(\frac{S_n}{\sigma_n} < x \right) - \Phi(x) \right| \le C\left\{ q^{-\theta_1}T^2 + kq^{-\theta_2}T^2 + \frac{q}{p}T + \frac{1}{T} + k^{-\delta/2} + a_n + kq/(a_n^{2+\delta}n^{1+\delta/2}) + (kq/(a_n^2 n))^{\gamma/2} \right\}. \tag{2.18}$$

Finally, we choose

$$T = (q/p)^{-1/2} \qquad \text{and} \qquad a_n = n^{-\frac{\delta(1-2/\theta)}{2(1+\delta)}}.$$

Substitute p, q, k, T and a_n into (2.18). Let $\gamma \ge \max(\theta - 2, 2 + \delta)$. We immediately obtain the result.

3 Proof of Theorem 1.2

In order to prove Theorem 1.2, the following lemmas are required.

Lemma 3.1 *Under the assumption of Theorem* 1.2*, if* $\{n_k; k \ge 1\}$ *is a nondecreasing sequence of positive integers such that* $\sum_{k=1}^\infty (\log n_k)^{-1-\tau} < \infty$*, then*

$$\sum_{k=1}^\infty \sup_x \left| P\left(\frac{S_{n_k}}{n_k^{1/2}\sigma} < x \right) - \Phi(x) \right| < \infty.$$

Proof

$$\begin{aligned} &\sum_{k=1}^\infty \sup_x \left| P\left(\frac{S_{n_k}}{n_k^{1/2}\sigma} < x \right) - \Phi(x) \right| \\ &\le \sum_{k=1}^\infty \sup_x \left| P\left(\frac{S_{n_k}}{\sigma_{n_k}} < \frac{n_k^{1/2}\sigma}{\sigma_{n_k}} x \right) - \Phi\left(\frac{n_k^{1/2}\sigma}{\sigma_{n_k}} x \right) \right| + \sum_{k=1}^\infty \sup_x \left| \Phi\left(\frac{n_k^{1/2}\sigma}{\sigma_{n_k}} x \right) - \Phi(x) \right| \\ &:= K_1 + K_2. \end{aligned} \tag{3.1}$$

Then, by Remark 1.2, we get

$$K_1 \le C \sum_{k=1}^\infty (\log n_k)^{-1-\tau} < \infty. \tag{3.2}$$

By Theorem 3.1 of Zhang [7] and Remark 1.3, we have

$$\frac{n_k^{1/2}\sigma}{\sigma_{n_k}} \ge \frac{n_k^{1/2}\sigma}{Cn_k^{1/2}} > C, \qquad \text{for } k \in N.$$

Thus, by Petrov [14, p. 114], Remark 1.3, (C1)′ and the property of being stationary, we have

$$\begin{aligned}K_2 &\le (2\pi e)^{-1/2}\sum_{k=1}^{\infty}\left|\frac{n_k^{1/2}\sigma}{\sigma_{n_k}}-1\right| \le C\sum_{k=1}^{\infty}\sigma_{n_k}^{-2}|n_k\sigma^2-\sigma_{n_k}^2|\\ &\le C\sum_{k=1}^{\infty}\left|2\sum_{j=n_k+1}^{\infty}\mathrm{Cov}(X_1,X_j)+\frac{2}{n_k}\sum_{j=2}^{n_k}(j-1)\mathrm{Cov}(X_1,X_j)\right|\\ &\le C\sum_{k=1}^{\infty}u(n_k)+C\sum_{k=1}^{\infty}n_k^{-1}\sum_{j=1}^{n_k}u(j)\\ &\le C\sum_{k=1}^{\infty}(\log n_k)^{-3-3\tau}<\infty. \end{aligned} \tag{3.3}$$

Combining (3.1)–(3.3), we immediately obtain the result.

Lemma 3.2 *Let $\{X_n\}$ and $\{n_k\}$ be sequences satisfying the conditions of Lemma* 3.1, *and $\{g(n)\}$ be a nondecreasing sequence of positive numbers. Then the following statements are equivalent*:

(A) $\sum_{k=1}^{\infty}P(|S_{n_k}|>g(n_k)n_k^{1/2}\sigma)<\infty$;

and,

(B) $\sum_{k=1}^{\infty}g^{-1}(n_k)\exp(-\frac{1}{2}g^2(n_k))<\infty$.

Proof By Lemma 3.1, the proof follows easily from the proof of Lemma 9 of Petrov [14, p. 311].

Proof of Theorem 1.2 Without loss of generality we may suppose that $\sigma^2=1$. It suffices to show that $\forall\,\varepsilon>0$

$$\limsup_{n\to\infty}\frac{|S_n|}{h(n)}\le 1+\varepsilon \quad \text{a.s.} \tag{3.4}$$

and

$$\limsup_{n\to\infty}\frac{S_n}{h(n)}\ge 1-\varepsilon \quad \text{a.s.,} \tag{3.5}$$

where $h(n)=(2n\log\log n)^{1/2}$.

To prove (3.4), let $n_k=[e^{k^{\alpha}}]$, $0<\alpha<1$. We write $g(n)=(2\log\log n)^{1/2}$, then

$$g^{-1}(n_k)\exp\left(-\frac{1}{2}(1+\varepsilon)^2g^2(n_k)\right)\le Ck^{-\alpha(1+\varepsilon)^2},$$

for sufficiently large k. Fixing $\varepsilon>0$, we have $0<\alpha<1$ such that $\alpha(1+\varepsilon)^2>1$, then

$$\sum_{k=1}^{\infty}g^{-1}(n_k)\exp\left(-\frac{1}{2}(1+\varepsilon)^2g^2(n_k)\right)<\infty.$$

By Lemma 3.2 and the Borel–Cantelli lemma we get

$$\limsup_{k\to\infty}\frac{|S_{n_k}|}{h(n_k)}\le 1+\varepsilon \quad \text{a.s.}$$

Let

$$M_k=\max_{n_k\le n<n_{k+1}}|S_n-S_{n_k}|/h(n_k),$$

for $k\ge 1$. For each $k\ge 1$,

$$|S_n|/h(n)\le|S_{n_k}|/h(n_k)+M_k,$$

for $n_k\le n<n_{k+1}$. Thus it suffices to show that

$$M_k\to 0 \qquad \text{a.s.} \tag{3.6}$$

Define

$$\bar{X}_i=i^{1/2}I(X_i>i^{1/2})+X_iI(|X_i|\le i^{1/2})-i^{1/2}I(X_i<-i^{1/2}), \quad \text{for } i\in N,$$

and

$$\bar{S_n} = \sum_{i=1}^{n} \bar{X}_i, \quad \text{for } n \in N.$$

We have

$$\begin{aligned}
\sum_{k=1}^{\infty} P(M_k > \varepsilon) &= \sum_{k=1}^{\infty} P(\max_{n_k \le n < n_{k+1}} |S_n - S_{n_k}| > \varepsilon(2n_k \log\log n_k)^{1/2}) \\
&\le \sum_{k=1}^{\infty} P\Big(\bigcup_{n_k \le i < n_{k+1}} \{|X_i| > i^{1/2}\}\Big) \\
&\quad + \sum_{k=1}^{\infty} P(\max_{n_k \le n < n_{k+1}} |\bar{S_n} - \bar{S_{n_k}}| > \varepsilon(2n_k \log\log n_k)^{1/2}) \\
&:= H_1 + H_2. \qquad (3.7)
\end{aligned}$$

First, for some $0 < \delta \le 1$, by condition $\sup_j E|X_j|^{2+\delta} < \infty$, we have

$$H_1 \le \sum_{k=1}^{\infty} \sum_{i=n_k}^{n_{k+1}-1} P(|X_i| > i^{1/2}) \le \sum_{i=1}^{\infty} i^{-(2+\delta)/2} \sup_j E|X_j|^{2+\delta} < \infty. \qquad (3.8)$$

By conditions $EX_n = 0$ and $\sup_j E|X_j|^{2+\delta} < \infty$, it is easy to show that, $\forall\, \varepsilon > 0$,

$$\max_{n_k \le n < n_{k+1}} |E(\bar{S_n} - \bar{S_{n_k}})| \le \frac{\varepsilon}{2}(2n_k \log\log n_k)^{1/2}, \qquad k \to \infty.$$

Thus

$$H_2 \le \sum_{k=1}^{\infty} P\Big(\max_{n_k \le n < n_{k+1}} |\bar{S_n} - \bar{S_{n_k}} - E(\bar{S_n} - \bar{S_{n_k}})| > \frac{\varepsilon}{2}(2n_k \log\log n_k)^{1/2}\Big).$$

By definition of ANA, $\bar{X}_{n_k} - E\bar{X}_{n_k}, \ldots, \bar{X}_{n_{k+1}-1} - E\bar{X}_{n_{k+1}-1}$ are also ANA random variables with $E(\bar{X}_i - E\bar{X}_i) = 0$, $i = n_k, \ldots, n_{k+1}-1$. Thus, by the Markov inequality, Lemma 2.3 and $\sup_j E|X_j|^{2+\delta} < \infty$, we have, for some $p > \max(2+\delta, \frac{2}{1-\alpha})$,

$$\begin{aligned}
H_2 &\le C\sum_{k=1}^{\infty}(n_k \log\log n_k)^{-p/2} E \max_{n_k \le n < n_{k+1}} |\bar{S_n} - \bar{S_{n_k}} - E(\bar{S_n} - \bar{S_{n_k}})|^p \\
&\le C\sum_{k=1}^{\infty}(n_k \log\log n_k)^{-p/2}\Big\{\sum_{i=n_k}^{n_{k+1}-1} E|\bar{X}_i|^p + \Big(\sum_{i=n_k}^{n_{k+1}-1} E\bar{X}_i^{\,2}\Big)^{p/2}\Big\} \\
&\le C\sum_{k=1}^{\infty}(n_k \log\log n_k)^{-p/2} \sum_{i=n_k}^{n_{k+1}-1} i^{p/2} P(|X_i| > i^{1/2}) \\
&\quad + C\sum_{k=1}^{\infty}(n_k \log\log n_k)^{-p/2} \sum_{i=n_k}^{n_{k+1}-1} E|X_i|^p I(|X_i| \le i^{1/2}) + C\sum_{k=1}^{\infty}\Big(\frac{n_{k+1} - n_k}{n_k \log\log n_k}\Big)^{p/2} \\
&:= H_{21} + H_{22} + H_{23}, \qquad (3.9)
\end{aligned}$$

where

$$\begin{aligned}
H_{21} &\le C\sum_{k=1}^{\infty}(n_k \log\log n_k)^{-p/2} \sum_{i=n_k}^{n_{k+1}-1} i^{(p-2-\delta)/2} \sup_j E|X_j|^{2+\delta} \\
&\le C\sum_{k=1}^{\infty} e^{-\frac{\delta}{2}(k+1)^{\alpha}} < \infty, \qquad (3.10)
\end{aligned}$$

$$H_{22} \le C\sum_{k=1}^{\infty}(n_k \log\log n_k)^{-p/2}\sum_{i=n_k}^{n_{k+1}-1} i^{(p-2-\delta)/2}\sup_j E|X_j|^{2+\delta} < \infty, \tag{3.11}$$

$$H_{23} \le C\sum_{k=1}^{\infty} k^{-\frac{(1-\alpha)p}{2}}(\log k)^{-p/2} < \infty. \tag{3.12}$$

From (3.7)–(3.12), we have $\sum_{k=1}^{\infty} P(M_k > \varepsilon) < \infty$, $\forall\, \varepsilon > 0$. Hence $M_k \to 0$ a.s. as desired.

We proceed to prove (3.5). Fix $N > 9$ and $0 < r < 1$, let

$$C_k = \{S_{N^k} - S_{N^{k-1}+N^{k/2}} > (1-r)h(N^k - N^{k-1} - N^{k/2})\}, \quad k \in N.$$

The first thing we need to show is that

$$\sum_{k=1}^{\infty} P(C_k) = \infty. \tag{3.13}$$

We will use the inequality

$$\begin{aligned} P(C_k) \ge\ & P(S_{N^k} > (1-r/2)h(N^k - N^{k-1} - N^{k/2})) \\ & - P\Big(S_{N^{k-1}+N^{k/2}} \ge \frac{r}{2}h(N^k - N^{k-1} - N^{k/2})\Big). \end{aligned} \tag{3.14}$$

Note that, for sufficiently large k,

$$\begin{aligned} &\left(\frac{N-1}{2}\right)^{1/2}(N^{k-1}+N^{k/2})^{1/2}g(N^{k-1}+N^{k/2}) \\ &\quad \le h(N^k - N^{k-1} - N^{k/2}) \le (N^k)^{1/2}g(N^k), \end{aligned} \tag{3.15}$$

$$g^{-1}(N^{k-1}+N^{k/2})\exp\left(-\frac{1}{2}\left(\frac{r}{2}\right)^2\left(\frac{N-1}{2}\right)g^2(N^{k-1}+N^{k/2})\right) \le Ck^{-r^2(N-1)/8} \tag{3.16}$$

and

$$g^{-1}(N^k)\exp\left(-\frac{1}{2}\left(1-\frac{r}{2}\right)^2 g^2(N^k)\right) \ge C(\log k)^{-1/2}k^{-(1-\frac{r}{2})^2}. \tag{3.17}$$

Choose positive constants N, r, such that $r^2(N-1)/8 > 1$. Then, by (3.15)–(3.17) and Lemma 3.2, we get

$$\sum_{k=1}^{\infty} P\Big(S_{N^{k-1}+N^{k/2}} \ge \frac{r}{2}h(N^k - N^{k-1} - N^{k/2})\Big) < \infty, \tag{3.18}$$

$$\sum_{k=1}^{\infty} P(S_{N^k} > (1-r/2)h(N^k - N^{k-1} - N^{k/2})) = \infty. \tag{3.19}$$

Hence (3.13) follows immediately from (3.14), (3.18) and (3.19).

Let ξ_k be the indicator function of C_k. Then

$$\begin{aligned} P\Big(\sum_{k=1}^{\infty}\xi_k \le \frac{1}{2}\sum_{k=1}^{n} P(C_k)\Big) &\le P\Big(\sum_{k=1}^{n}\xi_k \le \frac{1}{2}\sum_{k=1}^{n} P(C_k)\Big) \\ &\le P\Big(\Big|\sum_{k=1}^{n}\xi_k - \sum_{k=1}^{n}P(C_k)\Big| \ge \frac{1}{2}\sum_{k=1}^{n}P(C_k)\Big) \\ &\le \frac{4\mathrm{Var}(\sum_{k=1}^{n}\xi_k)}{(\sum_{k=1}^{n}P(C_k))^2}. \end{aligned} \tag{3.20}$$

By (C2)$'$, we have

$$\mathrm{Var}\Big(\sum_{k=1}^{n}\xi_k\Big) = \sum_{k=1}^{n}\mathrm{Var}(\xi_k) + 2\sum_{k=1}^{n-1}\sum_{j=1}^{n-k}\mathrm{Cov}(\xi_k, \xi_{k+j})$$

$$\leq \sum_{k=1}^{n} P(C_k) + 2\sum_{k=1}^{\infty}\sum_{j=1}^{\infty} \rho^-(N^{k+j-1} + N^{(k+j)/2} - N^k)$$
$$\leq \sum_{k=1}^{n} P(C_k) + C\sum_{k=1}^{\infty}\sum_{j=1}^{\infty} (\log N^{(k+j)/2})^{-(3+2/\delta)(1+\tau)}$$
$$= \sum_{k=1}^{n} P(C_k) + C. \tag{3.21}$$

Since $\sum_{k=1}^{\infty} P(C_k) = \infty$, letting $n \to \infty$ in (3.20) and (3.21) gives $P(\sum_{k=1}^{\infty} \xi_k < \infty) = 0$. Hence

$$P(C_k \quad \text{i.o.}) = 1. \tag{3.22}$$

Let

$$B_k = \{S_{N^{k-1}+N^{k/2}} > -2h(N^{k-1} + N^{k/2})\}, \quad k \in N.$$

Using the conclusion of the first half of the proof, we have $P(B_k \bigcap C_k \text{ i.o.}) = 1$. It is straightforward to notice that choosing N sufficiently large implies, for arbitrary $\varepsilon > r > 0$, that

$$P(S_{N^k} > (1-\varepsilon)h(N^k) \text{ i.o.})$$
$$\geq P(S_{N^k} > (1-r)h(N^k - N^{k-1} - N^{k/2}) - 2h(N^{k-1} + N^{k/2}) \text{ i.o.})$$
$$\geq P(B_k \cap C_k \text{ i.o.}) = 1. \tag{3.23}$$

Therefore (3.5) is proved by (3.23).

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