ANALYSIS OF ENGINEERING STRUCTURES

B. Bedenik and Colin Besant

Horwood Series in Engineering Science

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SICILLLUM MAIORAT CIVITATIS CICESTRIE
Mediaeval Seals of Mayors of Chichester, 1502 & 1530,
the design *motif* for the Horwood Publishing colophon

> It is a massy wheel
> Fixed on the summit of the highest mount,
> To whose huge spokes ten thousand lesser things
> are mortis'd and adjoin'd: which when it falls,
> Each small annexement, petty consequence,
> Attends the boist'rous ruin.
> Shakespeare: *Hamlet* II, iii

Branko Bedenik obtained a diploma in civil engineering in 1973 at the University of Ljubljana. He worked as a structural engineer with a civil engineering company in Maribor and subsequently became an assistant professor at the University. He went to Imperial College to undertake research into triaxial stresses in concrete and obtained a PhD degree in this subject in 1983. He returned to the University of Maribor and was promoted to associate professor in 1989. Professor Bedenik has lectured extensively in the field of structures to diploma students as well as postgraduates. He has also led a research group in the field of structures that has been sponsored by the civil engineering industry as well as the research council of Slovenia. He is well known for the design of important buildings in Slovenia including the University Library and University Sports Centre in Maribor and has also contributed to the design of many bridges and buildings in Slovenia and elsewhere.

Colin Besant obtained a BSc(Eng) degree in mechanical engineering with first class honours from the University of London in 1959. He subsequently went to Imperial College to undertake research in the field of nuclear engineering and obtained a PhD degree in 1962. He worked for Rolls-Royce and Associates on nuclear submarine systems and was also a senior scientific officer with the United Kingdom Authority. He returned to Imperial College in 1964 as a lecturer in the Department of Mechanical Engineering. He became Reader in Mechanical Engineering in 1975 and Professor of Computer Aided Manufacture in 1988. He was also made a Fellow of the Fellowship of Engineering in 1983.

Professor Besant has been closely involved with the teaching of stress analysis to undergraduates whilst at Imperial College. He is now Head of the Computer Aided Systems Engineering Section in the Department of Mechanical Engineering.

Analysis of Engineering Structures

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Table of contents

SYNOPSIS

This book provides a fundamental and thorough coverage of statics for students of structural engineering. The methods for structural analyses are explained in detail, based on basic static, kinematic and energy methods.

A whole chapter deals with calculations relating to the deformation of structures, as the authors believe that it is a basis for achieving a good understanding of structural behaviour.

Much attention is paid to the practical aspects of the subject and each piece of theoretical analysis is followed by worked examples and in conclusion a simple bridge is analysed by the various methods that have been presented.

The finite element method, as an extension of the displacement method, is covered only to provide an understanding of computer applications presented using the structural analysis program OCEAN, which can be downloaded from the internet (*kamen.uni-mb.si/lak/ocean*).

An innovative approach that enables influence lines to be calculated (using ψ-functions developed by Bedenik) in much simpler manner than any previously known method is also described.

Basic matrix algebra is given in appendices to provide readers with the necessary tools to understand the text.

PREFACE

Structural analysis is a field of engineering that has undergone the greater changes in the last few decades. The matrix formulation of structural analysis has provided a bookkeeping scheme which makes it possible to deal with structures of a size and complexity which were previously too complex to contemplate.

The use of computers, whose developments were intimately connected with that of matrix methods, finally enabled the structural engineer to perform such analyses and apply them to the design process to create much more functional, economic and aesthetic structures and buildings.

Matrix algebra has also enabled a structural engineer to model any structure with a finite number of degrees of freedom and arrive at a physical model, which comes near to the real structure in geometry and in its behaviour.

The "danger" of all computer analyses lies in the fact that a structural engineer can loose the "physical feeling" of the structure. The longhand methods presented in this book should help an engineer to distinguish between the critical or less important results of a computer analysis.

A major focus of the book is the numerous worked examples that are related to practical applications. These examples will not only provide guidance to students but also provide a reference for practicing engineers.

Acknowledgements

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Finally, Professor Bedenik is indebted to his wife, Marjetka and his daughter Tina for their patience and support during the unending hours spent on the preparation of this book.

1

Introduction

Structural analysis is a science, which ensures that structures are safe and fulfill the functions for which they were built. Safety requirements must be met so that a structure is able to serve its purpose with the minimum of costs. Structural concepts arise from the work of engineers from different fields with a common aim that the structure is functional, aesthetic and economic.

Detailed planning of the structure usually comes from several studies made by town planners, investors, users, architects and other engineers. In general an architect is responsible to the investor and a structural engineer works in collaboration with the architect as an equal partner in a project. In some structures such as industrial halls, bridges and sports halls, a structural engineer has the main influence on the overall structural design and an architect is involved in aesthetic details.

After the preliminary design of the structure, an approximate analysis of loads and stresses in all elements must be carried out including the determination of deformation in individual elements as well as in structure as a whole. This preliminary analysis is a check to show where and how the structure can be improved and reduced in costs. It is possible that the initial design proves to be uneconomic and the structure has to be changed in individual elements or as a whole.

The process of analysis has then to be repeated until the structure as a whole is optimal from all points of view, followed by final analysis and dimensioning. The whole process can be divided into:

- initial design
- preliminary dimensioning
- optimisation (when necessary, change of individual elements of the structure or change of the structure entirely must be made)
- final analysis and dimensioning.

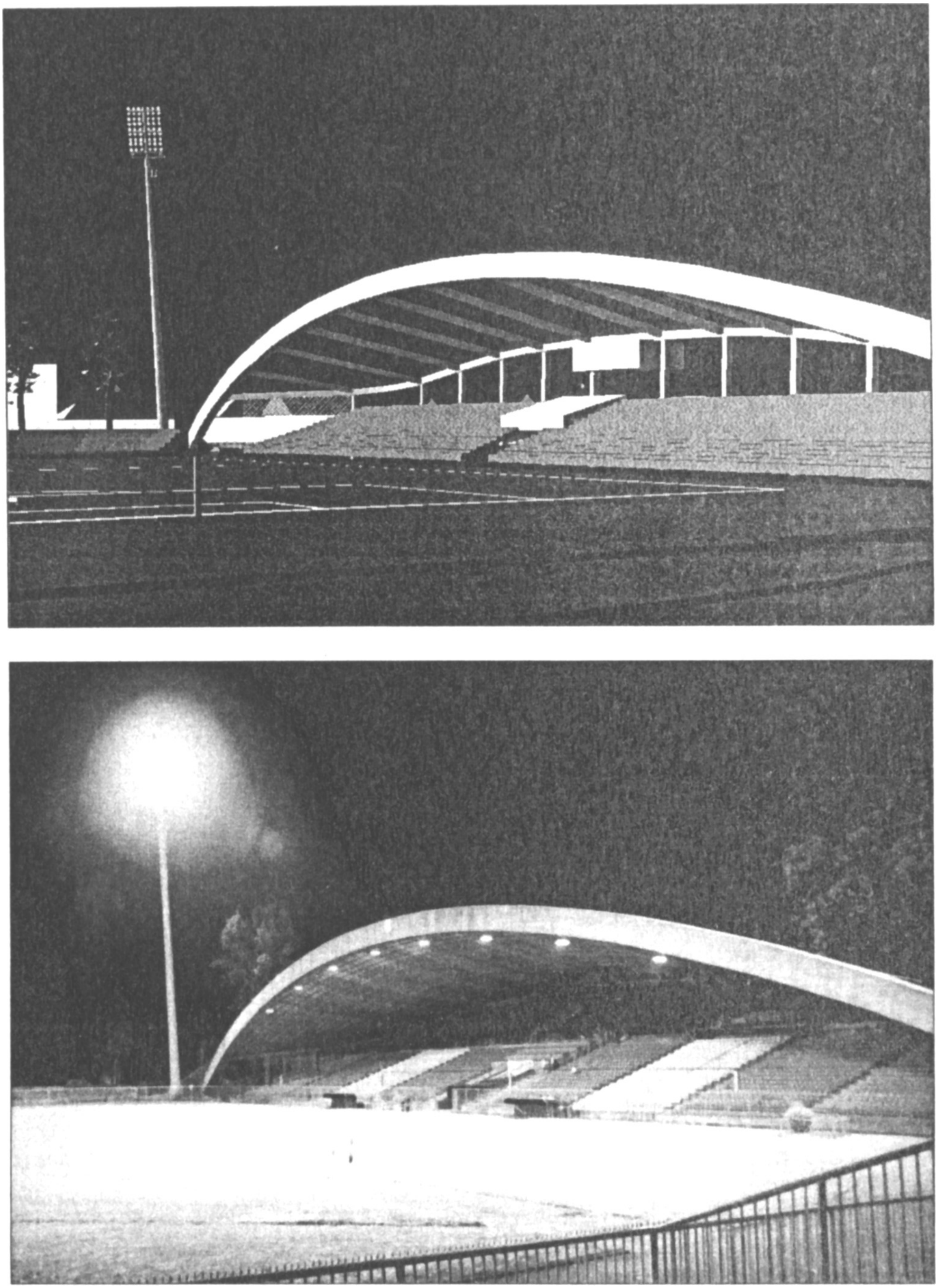


Figure 1.1: An example of CAD design (top picture) and photograph (bottom picture) of the sports stadium in the city of Maribor

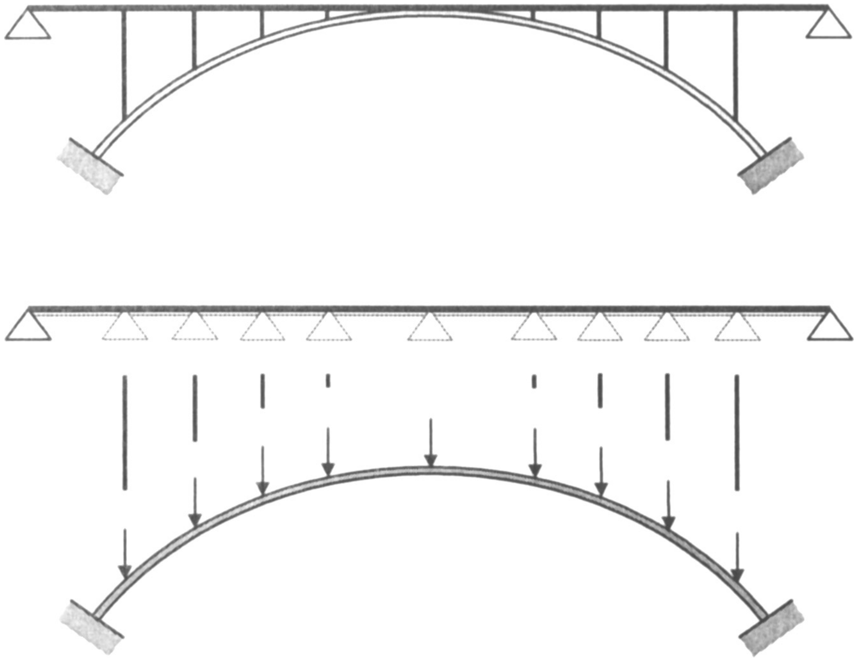


Figure 1.2: Idealisation of an arched bridge

It is obvious that the processes of structural analysis and design are closely related, since each change in element dimensions influences the optimal structural shape, weight and stiffness. These quantities are known only after the elements have been designed. Thus, analysis and design are mutually interacting and the process is called structural analysis.

1.1 Types of Structures

Structural analysis deals with a number of different structures:

- *Buildings (residential, industrial)*
- *Bridges*
- *Underground structures, tunnels*
- *Industrial structures, power stations, reactor containers*
- *Planes, missiles*
- *Vehicles (automobiles, railcars, ships)*
- *Machines, cranes, elevators, aerials, electricity pylons*

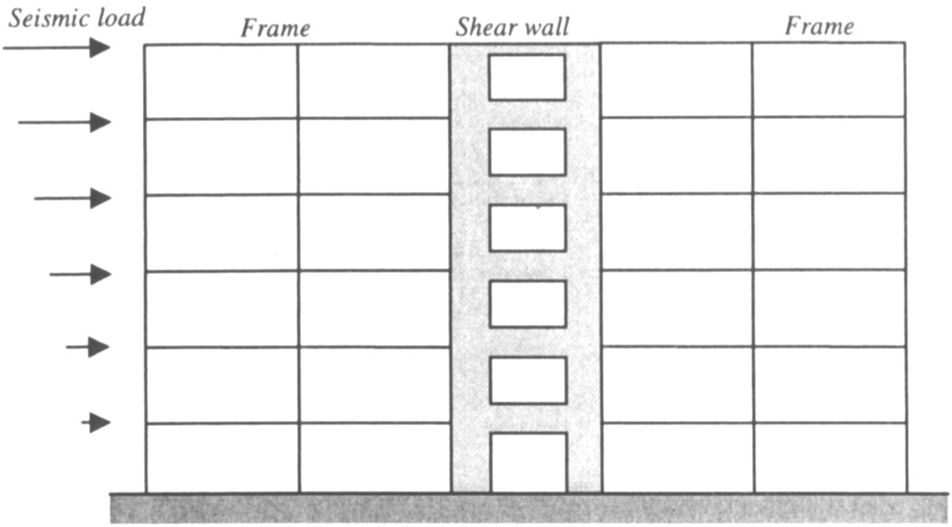


Figure 1.3: Modelling of a building

Structures can be divided according to the nature of their components into three main classes:

- *Linear or uniaxial members*: truss elements, beams, columns, arches and their combinations. Elements of this type are simple to analyse and are therefore suitable for elementary presentation of structural theory. It is possible to idealise even complex structures as assemblies of such members.

- *Two-dimensional elements* such as plates, shells and walls. Although the analysis of such elements has been considered as a branch of the theory of elasticity, modern computational methods facilitates analysis to any degree of accuracy.

- *Three-dimensional elements* such as machine parts, pressure vessels, soil and rock foundations. Some structural joints must also be included as such elements in a detailed stress analysis using the theory of elasticity or plasticity.

Although there are several computer programs available today, in practice it is common to analyse structures using very simple models consisting of linear elements by the elementary methods presented in this book.

1.2 Loads

The nature and magnitude of loads must be determined before a structure can be analysed though these are only crude approximations in the initial design. The most important loads are:

- *Dead load (D)*, which can be exactly determined only after the structure has been designed. It is obvious, the smaller the ratio of the dead load to the other loads, the more efficient is the structure. Some structures, such as long span bridges, can carry dead loads many times higher than live loads. For such structures a shape optimisation has to be performed to gain an optimal and efficient structure.

- *Live load (L)* is the useful load carried by the structure. If it is caused by human activities it should be determined by the use of probability theory. Building Codes (i.e. Eurocode 1) determine the most unfavourable cases that can occur in a lifetime of the structure. In bridges, the live load is moving, and an analysis has to determine the most unfavourable position of vehicles using *influence lines* (covered in Ch. 9).

- *Wind, earthquake and aerodynamic forces.* Effects of these forces must be calculated including dynamic effects as they act in cycles and cause inertia forces in the structure. The field of structural dynamics, which is not included in this text, is rapidly developing and full dynamic analysis is possible using appropriate computer programs. In building frames equivalent static forces are taken into consideration although it is known that interaction between a forced vibration and properties of structures exist. It is known that a stiffer and heavier structure carries higher dynamic forces than a slim and light structure. This has been proved in recent earthquakes, where slim and economically reinforced concrete structures underwent only slight damage, and oversized and therefore minimally reinforced structures were heavily damaged or collapsed.

- *Earth pressure, gas and liquid pressures.* Earth pressure varies between the extreme active and passive cases and is dependant on soil-structure interaction. Gas and liquid pressures are well known, controlled and act hydrostatically on a surface.

- *Self-strains* due to supports settlements, pre-stressing, creep, shrinkage of concrete, welding and temperature gradient.

Beside active loads, a change of length or misfit of structural elements can take place causing huge stresses in a structure as a whole or in an individual element. Specification of loads is usually included in building codes, but it is the structural engineer who has to find the most unfavourable combination of loads, which can also be time dependant as with creep or relaxation in pre-stressing steel.

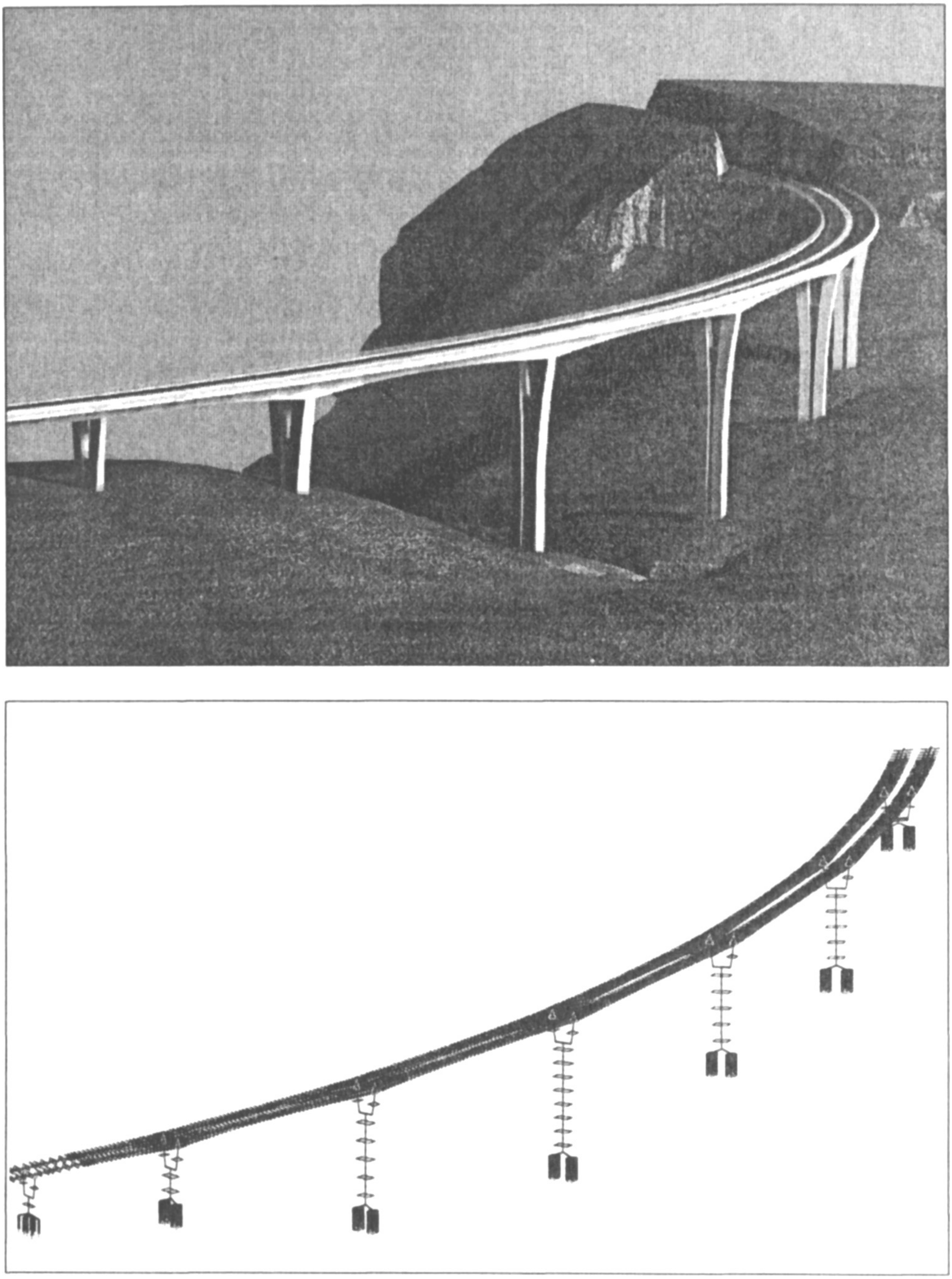
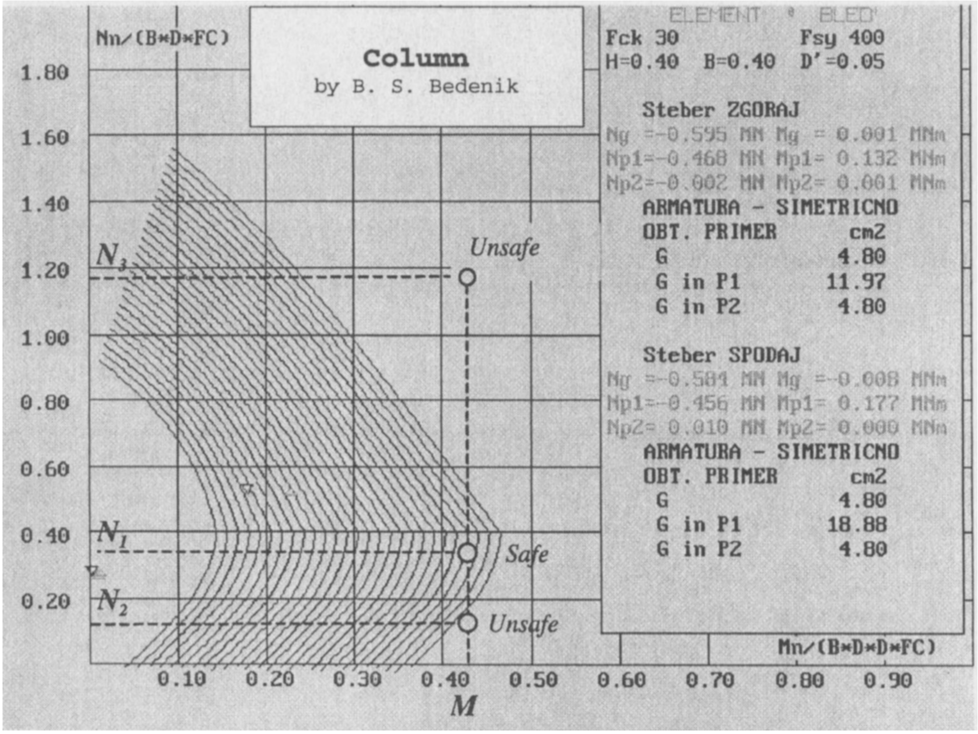


Figure 1.4: Viaduct 'Crni Kal' – total length 1067 m
CAD simulation and Finite element model (Courtesy of Ponting Ltd.)

The probability of maximum loads due to several causes ocurring at the same time will decrease with the number of loads considered.

In fact, a loading case of maximum normal force and maximum bending moment acting simultaneously is not always critical. The concrete column in Fig. 1.5 is carrying both a compressive normal force and a bending moment. It can be observed from the interaction curve, that at the same reinforcement ratio at constant bending moment *M*, by increasing the normal force from N_1 to N_3 the element goes from the unsafe through to the safe and again unsafe condition.



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      ELEMENT  ®  BLED'
Fck 30            Fsy 400
H=0.40  B=0.40    D'=0.05

   Steber ZGORAJ
Ng =-0.595 MN Mg = 0.001 MNm
Np1=-0.468 MN Mp1= 0.132 MNm
Np2=-0.002 MN Mp2= 0.001 MNm
   ARMATURA - SIMETRICNO
   OBT. PRIMER          cm2
    G                  4.80
    G in P1           11.97
    G in P2            4.80

   Steber SPODAJ
Ng =-0.584 MN Mg =-0.008 MNm
Np1=-0.456 MN Mp1= 0.177 MNm
Np2= 0.010 MN Mp2= 0.000 MNm
   ARMATURA - SIMETRICNO
   OBT. PRIMER          cm2
    G                  4.80
    G in P1           18.88
    G in P2            4.80
```

Figure 1.5: Interaction curve for a concrete column

The determination of loads acting on a structure is a complex and difficult task and is readily underestimated in practice.

Loads determined by building codes are approximate only, usually on the safe side, and are in general inside *±10 %* accuracy.

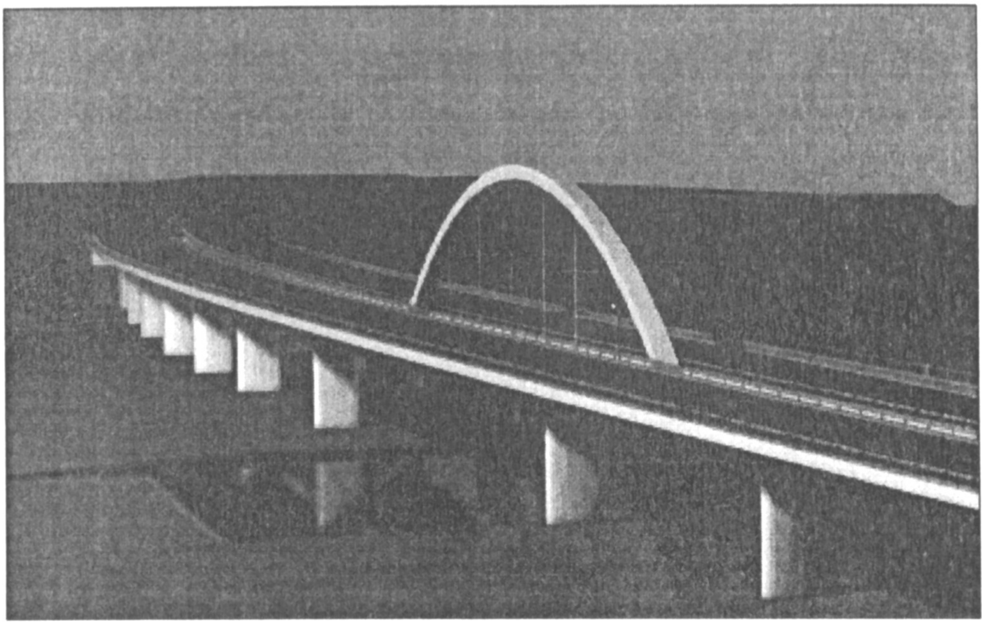


Figure 1.6: Mura bridge - CAD simulation (Courtesy of Ponting Ltd. Maribor)

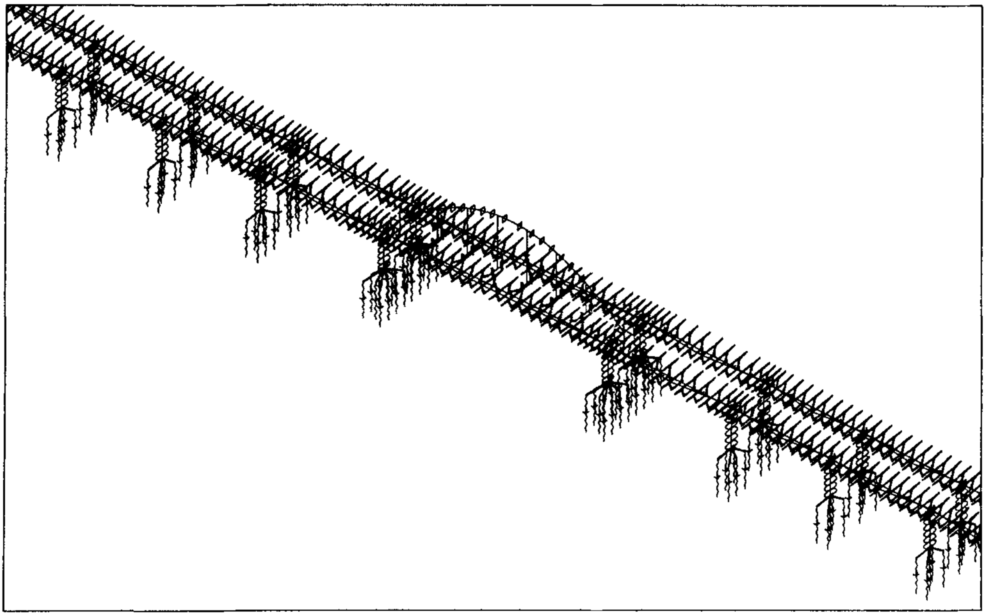


Figure 1.7: Mura bridge - Finite element model (Courtesy of Ponting Ltd.)

1.3 Idealisation and modelling of structures

Real structures are too complex for an exact analysis, often they have to be modelled by a simple models, which are then appropriate for analysis.

This modelling is a very important task for a structural engineer and requires experience and judgement such that the resulting model satisfies the compromise between reality and simplicity. All of the examples in this book are only models of real structures. The modelling can be viewed from three aspects:

1.3.1 Geometry and interconnection of structure elements

As an example we take a three-dimensional space frame. Structural analysis is carried out on individual elements of plates, beams and columns, which are in reality connected into a three-dimensional space grid. As *3D* analysis is complex, we introduce an additional simplification and treat individual elements in single planes as plane frames.

The concrete bridge from Fig. 1.2 is assembled from arch, columns and a deck. The modelling for the analysis is done such that elements act individually and independent of each other. Such an analysis could be sufficient for the preliminary design and dimensional determination. The final design must be carried out on the whole structure and this task should not present a significant problem.

In tall buildings sufficient horizontal stiffness must be assured either by shear walls in both directions or by a stiff core (in which elevators and stairs are situated), as shown in Fig. 1.3. Such shear walls are usually treated as bending elements under horizontal loading, whether this assumption is justified depends on their proportions.

The consequence of such an idealisation is, that all interconnecting joints are taken as points, which of course is not true and it is impossible to calculate stresses at these joints.

On the other hand, when considering plane frames it is assumed, that elements are fully clamped to each other, which is not justified when dealing with concrete members if insufficient reinforcement is present.

A similar situation arises in tall buildings in outer spans of continuous plates of thickness $h_p = 14 - 20\ cm$ and walls of thickness $d_s = 14 - 18\ cm$ where it is not justified to consider plates as fully clamped.

Similar cases occur at the foundation of columns and walls as they interact with the footing itself, which rotates under internal forces and is stabilised after a certain time. Rotation of the foundation is dependent on the footing and soil stiffness, matters that are often unknown. To avoid the problem, we assume that the footing is infinitely stiff and deforms as a rigid body, the consequence is a linear soil pressure distribution, which can easily be calculated but leads to a conservative design resulting in high bending moments.

Figure 1.8: The bridge at Kozina (Courtesy of Ponting Ltd.)

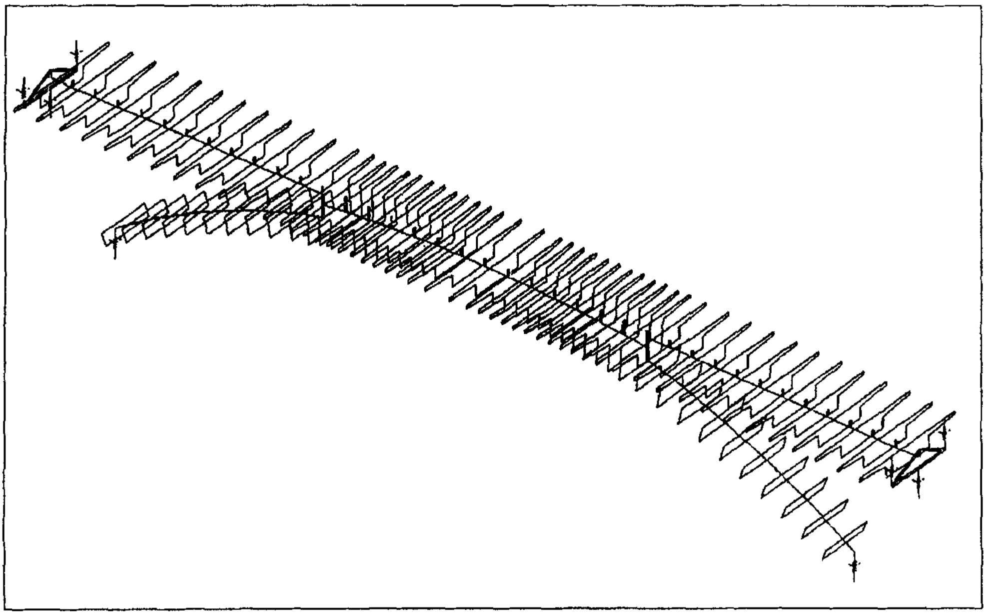


Figure 1.9: The finite element model of the bridge at Kozina
(Courtesy of Ponting Ltd.)

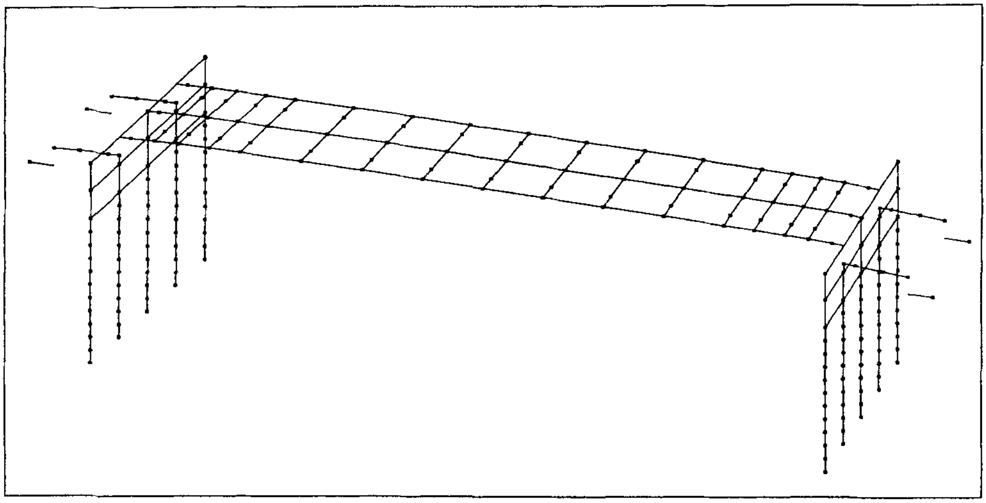


Figure 1.10: The bridge on East Highway, Ljubljana (finite element model)
(Courtesy of Gradis BP Ltd. Maribor)

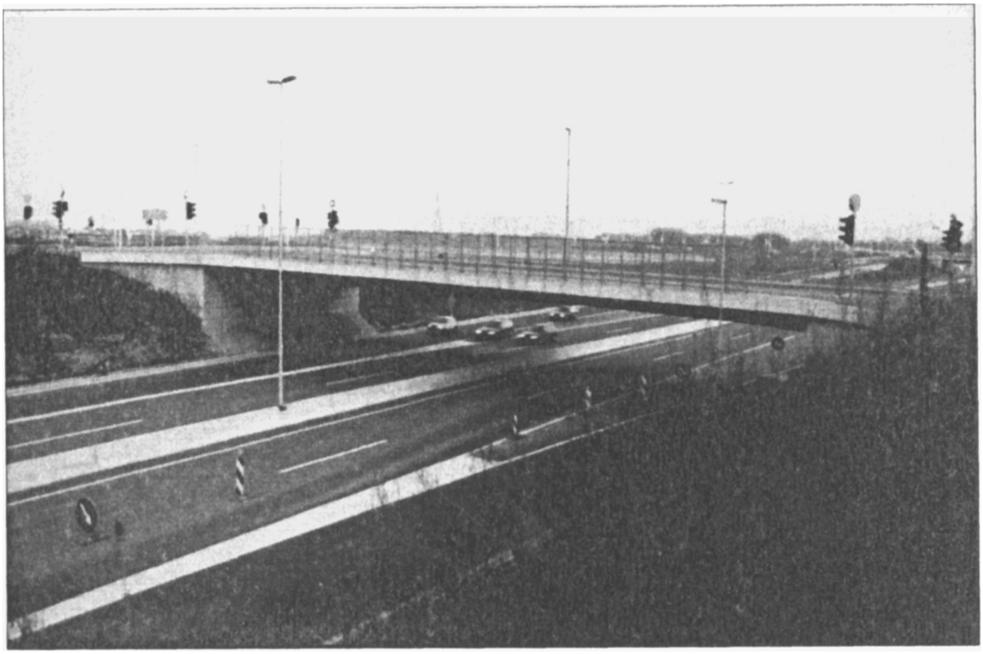


Figure 1.11: The bridge on East Highway, Ljubljana
(Courtesy of Gradis BP Ltd.)

1.3.2 Element connections and support conditions

Idealising the joints and support conditions of a structure makes further simplifications. A typical example is in truss type structures with the usual simplification of frictionless pins connecting members, allowing full rotation of individual bars even when joints are welded or elements are continuous over joints. Standard beam connections of steel beams are usually considered as simple supports, even though they are capable of resisting considerable bending moments.

Continuous foundation plates must be calculated as elements on an elastic foundation. The pressure in reality under the plate could vary from idealised values by up to $\pm 100\ \%$.

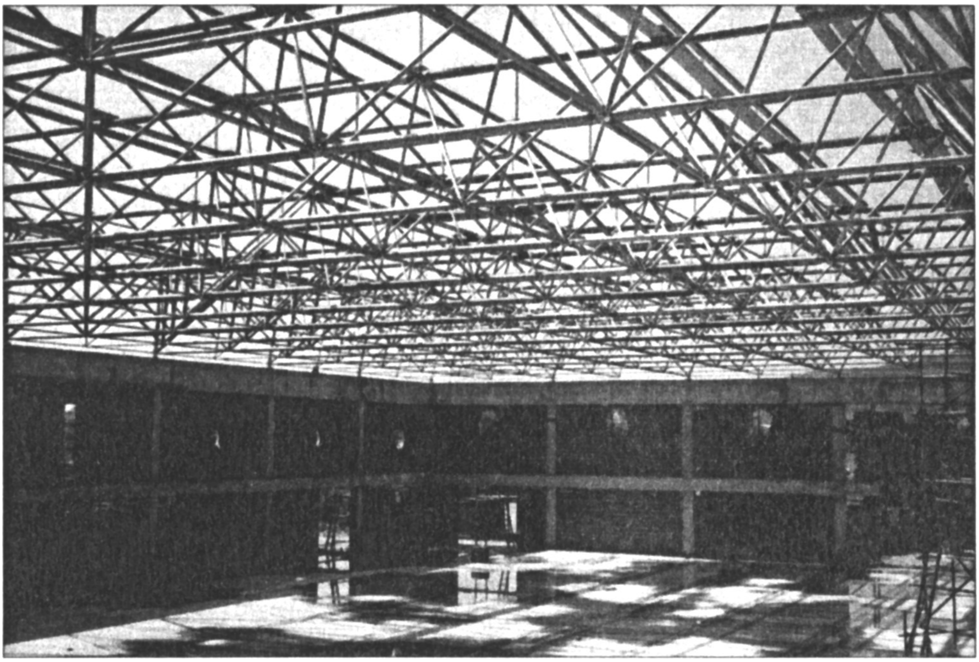


Figure 1.12: University sports centre Maribor under construction (Design by B. S. Bedenik)

1.3.3 Material behaviour and stability

Most engineering materials used in structures possess a linearly elastic range, for which Hooke's law holds, but only over a limited range of stresses or specific strains. The allowable stresses under service loads are controlled by factors of safety in codes and are sufficiently low so that elastic action prevails and linear behaviour is valid.

Figure 1.14a shows specific deformations for concrete of different qualities. It can be seen, that concrete behaves linearly up to $\varepsilon \cong 0.1\,\%$, at higher specific strains and related stresses the rise is non-linear until the ultimate load is reached at around $\varepsilon \cong 0.3\,\%$.

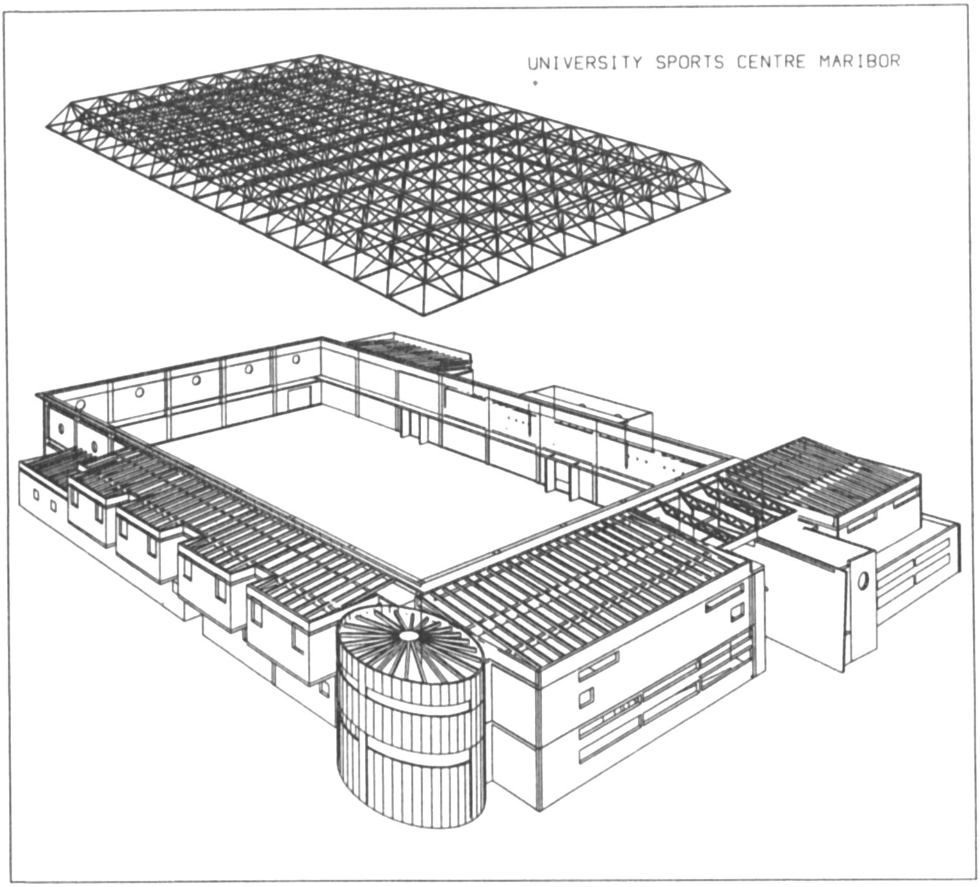
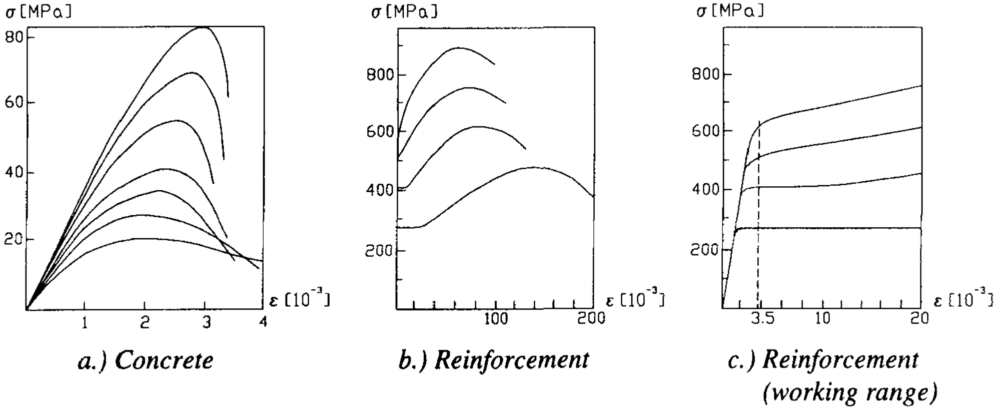


Figure 1.13: University sports centre Maribor
(Design and CAD model by B. S. Bedenik 1992)

In ductile materials, such as mild reinforcement steel (lower curve in Fig 1.14b), continued deformation under constant yield stress would reach strains up to $\varepsilon \cong 10-15\,\%$. This is the reason why such structures are capable of absorbing much more energy when overloaded such as under earthquake conditions. On the other hand high strength reinforcement exhibits up to 3 times lower specific deformations than mild reinforcement steel.



a.) Concrete

b.) Reinforcement

c.) Reinforcement (working range)

Figure 1.14: Nonlinear material behaviour

Analysis of simple structures does not require *stability analysis*, though a concrete column having slenderness over *λ>50* requires simplified second order analysis according to empirical relationship equations which are usually stated in national codes.

Dealing with more complicated structures will necessitate the use of stability theory♣ and an Euler buckling load determination, which will be dependent on boundary conditions on individual elements of the structure or on the structure as a whole. Nonlinear material behaviour and stability of structures is beyond the scope of this book.

♣ ***Petersen**: Statik und Stabilitet der Baukonstruktionen, Vieweg, Wiesbaden 1982*

2

Definitions and basic concepts

2.1 Sign conventions

Throughout this book we will use the Cartesian co-ordinate system of three mutually perpendicular axes. Only right-handed systems will be used, which means the following:

- *If two positive axes are chosen, i.e. x and y, the positive direction for the z-axis will be that in which a right-handed screw would advance when turned if the x-axis is rotated into the y-axis by the shortest way.*

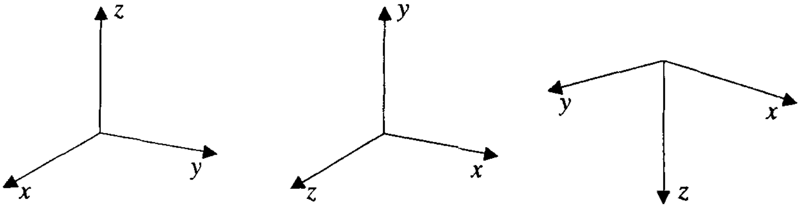


Figure 2.1: Right-handed co-ordinate systems

2.2 Forces and moments

Consider a force F (of magnitude F) in the direction defined by the angles α, β *and* γ, which is enclosed within the x-, y- and z-axes of a Cartesian co-ordinate system.

Figure 2.2 shows force F, represented in magnitude and direction by a vector, which is the main diagonal of a rectangular prism of sides F_x, F_y and F_z. F_x, F_y and F_z are projections of F on x-, y- and z-axes and we usually say that F_x, F_y and F_z are *components of force F* in the three co-ordinate directions. From Fig. 2.2:

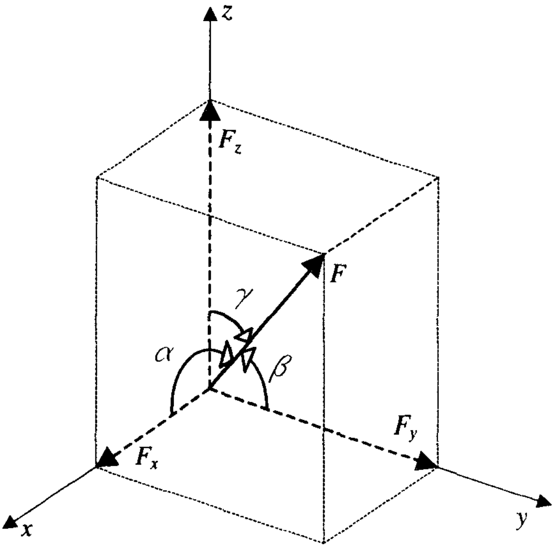


Figure 2.2: Force *F* and its components

$$F_x = F \cdot cos\,\alpha$$
$$F_y = F \cdot cos\,\beta \tag{2.1}$$
$$F_z = F \cdot cos\,\gamma$$

where $cos\,\alpha$, $cos\,\beta$ and $cos\,\gamma$ are direction cosines of force *F* and can be denoted by *l, m* and *n* respectively:

$$l = cos\,\alpha$$
$$m = cos\,\beta \tag{2.2}$$
$$n = cos\,\gamma$$

The components F_x, F_y and F_z completely define the magnitude and direction of force *F*, and can be written in matrix form as:

$$\mathrm{F} = \begin{Bmatrix} F_x \\ F_y \\ F_z \end{Bmatrix} = \{F_x \ \ F_y \ \ F_z\}^T \tag{2.3}$$

In Eqn. (2.3) force *F* is expressed in matrix form of *3 x 1* order (a three component *column matrix*, which can be further written in *transposed* form using symbol ^{T}), whose elements are the components of force F in the three co-ordinate directions. This matrix is called a *force vector.*

- ❖ *A force vector has magnitude and direction and can be represented by a single straight line in space.*

The components of force F are dependent on magnitude and direction only and are independent of the point of action. From Fig. 2.2, if the force acts at any other point than at the co-ordinate origin, the components would still be F_x, F_y and F_z and given by Eqn. (2.1). Dealing with several forces F_1, F_2, F_3 ... F_n as shown in Fig. 2.3, the components of forces are:

$$F_1 = \begin{Bmatrix} F_{x1} \\ F_{y1} \\ F_{z1} \end{Bmatrix} \quad F_2 = \begin{Bmatrix} F_{x2} \\ F_{y2} \\ F_{z2} \end{Bmatrix} \quad \ldots \quad F_n = \begin{Bmatrix} F_{xn} \\ F_{yn} \\ F_{zn} \end{Bmatrix} \tag{2.4}$$

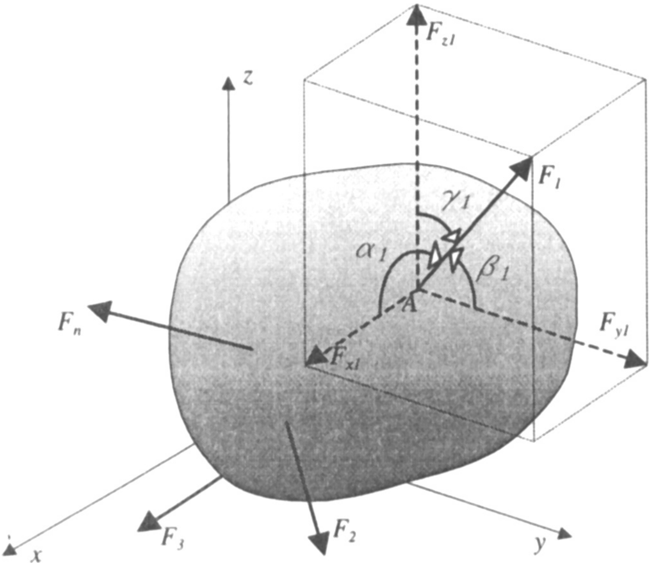


Figure 2.3: Force vectors

with $F_{x1} = F_1 \cdot \cos\alpha_1$, $F_{x2} = F_2 \cdot \cos\alpha_2$ etc. The resultant of all n forces is a vector sum:

$$F_1 + F_2 + \ldots + F_n = \begin{Bmatrix} F_{x1} \\ F_{y1} \\ F_{z1} \end{Bmatrix} + \begin{Bmatrix} F_{x2} \\ F_{y2} \\ F_{z2} \end{Bmatrix} + \ldots + \begin{Bmatrix} F_{xn} \\ F_{yn} \\ F_{zn} \end{Bmatrix} =$$

$$= \begin{Bmatrix} F_{x1} + F_{x2} + \ldots + F_{xn} \\ F_{y1} + F_{y2} + \ldots + F_{yn} \\ F_{z1} + F_{z2} + \ldots + F_{zn} \end{Bmatrix} = \begin{Bmatrix} \Sigma F_{xi} \\ \Sigma F_{yi} \\ \Sigma F_{zi} \end{Bmatrix} \tag{2.5}$$

The resultant force is zero only, when *all three components* are zero, which leads to the following equation:

$$\begin{Bmatrix} \Sigma F_{xi} \\ \Sigma F_{yi} \\ \Sigma F_{zi} \end{Bmatrix} = 0 \tag{2.6}$$

Eqn. (2.6) by itself does not ensure that there is no resultant moment. Consider the moments produced by force F_1 as from Fig. 2.3.

The moment of force F_1 about the x-axis is equal to the sum of the moments of its components F_y and F_z, and the moment of component F_x equals zero, as F_x is parallel to the x-axis. If the force is applied at point A, then:

$$M_x = F_{z1} \cdot y_1 - F_{y1} \cdot z_1$$

Similarly we can write moments about y- in z-axis:

$$\begin{aligned} M_y &= F_{x1} \cdot z_1 - F_{z1} \cdot x_1 \\ M_z &= F_{y1} \cdot x_1 - F_{x1} \cdot y_1 \end{aligned} \tag{2.7}$$

If no moment exists for forces F_1, F_2 ... F_n about any of the co-ordinate axis, then the following equation must be fulfilled:

$$\begin{Bmatrix} \Sigma(F_{zi} \cdot y_i - F_{yi} \cdot z_i) \\ \Sigma(F_{xi} \cdot z_i - F_{zi} \cdot x_i) \\ \Sigma(F_{zi} \cdot x_i - F_{xi} \cdot y_i) \end{Bmatrix} = 0 \tag{2.8}$$

Eqn. (2.8) is the summation of the vector products

$$\Sigma\, r_i \times F_i ,$$

where the force vector is

$$F_i = (F_{xi} \;\; F_{yi} \;\; F_{zi})^T$$

The position vector written for the co-ordinate origin is

$$r_i = \{(0 - x_i)\ (0 - y_i)\ (0 - z_i)\}^T = \{-x_i\ -y_i\ -z_i\}^T$$

which can be written in a determinant form:

$$\sum \begin{vmatrix} i & j & k \\ r_{xi} & r_{yi} & r_{zi} \\ F_{xi} & F_{yi} & F_{zi} \end{vmatrix} = 0 , \tag{2.8a}$$

where ***i***, ***j*** and ***k*** are unit vectors in the *x*-, *y*- and *z*-axis respectively.

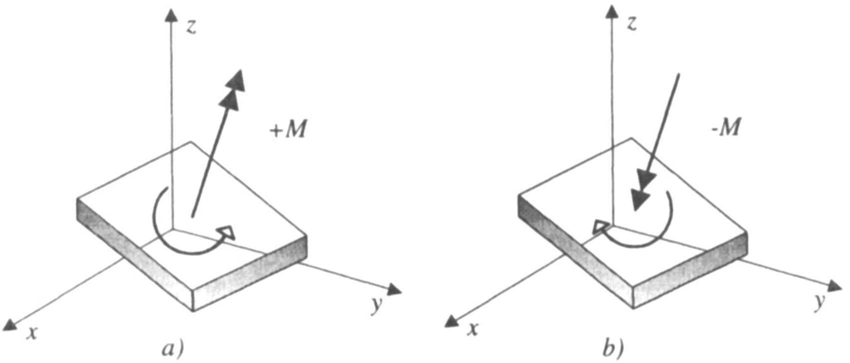
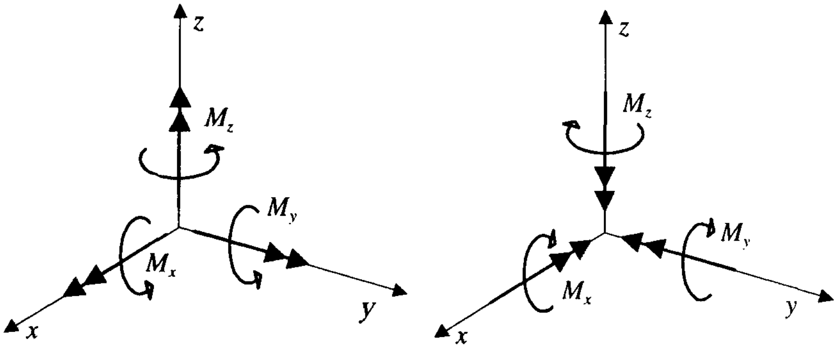


Figure 2.4: Definition of moments

Let us suppose, that moment *M* acts on the plane as shown in Fig. 2.4a); the moment is shown as a vector perpendicular to the plane with a double arrow. The length of the vector represents the magnitude of the moment, the direction is always perpendicular to the plane on which the moment is acting and the arrow shows the advance of a right-handed screw if turned by the action of the moment. Thus if moment *M* is reversed, the arrow points in the opposite direction as in Fig. 2.4b).

A moment acting about any co-ordinate axis is positive if the arrow of the vector points in the positive direction of that axis. All moments in Fig. 2.5a) are *positive* and those in Fig. 2.5b) are *negative.*



a) positive moments *b) negative moments*

Figure 2.5: Vector representation of moments

In case of several moments M_1, M_2 ... M_n, shown in Fig. 2.6 as vectors, each moment can be represented by its components on a co-ordinate axis as with forces in Fig. 2.2 using Eqn. (2.5):

$$M_1 = \begin{Bmatrix} M_{x1} \\ M_{y1} \\ M_{z1} \end{Bmatrix} \quad M_2 = \begin{Bmatrix} M_{x2} \\ M_{y2} \\ M_{z2} \end{Bmatrix} \quad M_R = \begin{Bmatrix} M_{xR} \\ M_{yR} \\ M_{zR} \end{Bmatrix} \tag{2.9}$$

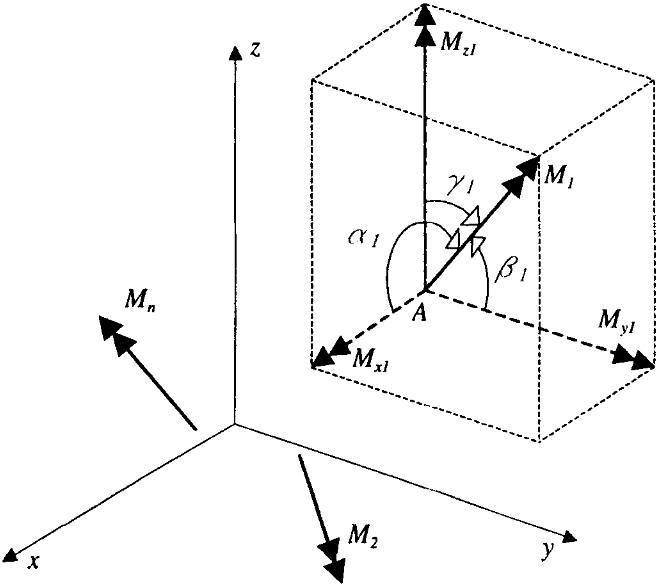


Figure 2.6: System of moments

The resultant moment of n moments is a vector sum:

$$M_1 + M_2 + \ldots + M_n = \begin{Bmatrix} M_{x1} \\ M_{y1} \\ M_{z1} \end{Bmatrix} + \begin{Bmatrix} M_{x2} \\ M_{y2} \\ M_{z2} \end{Bmatrix} + \ldots + \begin{Bmatrix} M_{xn} \\ M_{yn} \\ M_{zn} \end{Bmatrix} =$$

$$= \begin{Bmatrix} M_{x1} + M_{x2} + \ldots + M_{xn} \\ M_{y1} + M_{y2} + \ldots + M_{yn} \\ M_{z1} + M_{z2} + \ldots + M_{zn} \end{Bmatrix} = \begin{Bmatrix} \Sigma M_{xi} \\ \Sigma M_{yi} \\ \Sigma M_{zi} \end{Bmatrix} \tag{2.10}$$

If the resultant moment is zero, then:

$$\begin{Bmatrix} \Sigma M_{xi} \\ \Sigma M_{yi} \\ \Sigma M_{zi} \end{Bmatrix} = 0 \tag{2.11}$$

or using the summation principle:

$$M_i = 0 \quad i = x, y, z$$

Using the principle, where a column matrix in Eqn. (2.5) was called a force vector, the column matrices for moment components

$$\begin{Bmatrix} M_{x1} \\ M_{y1} \\ M_{z1} \end{Bmatrix} , \quad \begin{Bmatrix} M_{x2} \\ M_{y2} \\ M_{z2} \end{Bmatrix} , \quad \begin{Bmatrix} M_{xR} \\ M_{yR} \\ M_{zR} \end{Bmatrix} ,$$

are called *moment vectors*. From Figs. 2.1 and 2.4 one can see, that *vectors of forces and moments* can be represented as a *straight line* in space.

- ❖ *Vectors, which can be represented as straight lines, are physical quantities, having magnitude and direction.*

Some other physical vectors are velocity and acceleration vectors, which are of course well known to the reader.

Generalised force vector

Forces F_1, F_2 ... F_n and moments M_1, M_2 ... M_n, acting on a body (Fig. 2.7), can be represented (substituted) by a single force F and a single moment M; F and M being resultants of all forces and moments. From Eqn. (2.6) and (2.11) follows:

$$F_R = \begin{Bmatrix} \Sigma F_x \\ \Sigma F_y \\ \Sigma F_z \end{Bmatrix} \quad \text{and} \quad M_R = \begin{Bmatrix} \Sigma M_x \\ \Sigma M_y \\ \Sigma M_z \end{Bmatrix} \tag{2.12}$$

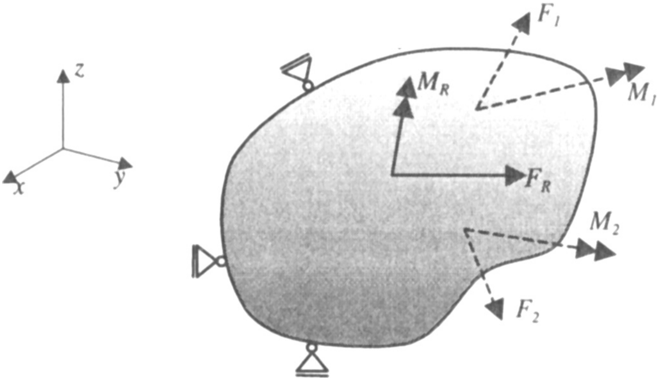


Figure 2.7: Generalised force

Both F_R and M_R can be denoted by a single letter F:

$$F = \begin{Bmatrix} F_R \\ M_R \end{Bmatrix} = \begin{Bmatrix} \Sigma F_x \\ \Sigma F_y \\ \Sigma F_z \\ \Sigma M_x \\ \Sigma M_y \\ \Sigma M_z \end{Bmatrix}, \tag{2.13}$$

and when only one force acts the equation reduces to:

$$F = \begin{Bmatrix} F_x \\ F_y \\ F_z \\ M_x \\ M_y \\ M_z \end{Bmatrix} \tag{2.13a}$$

The quantity F in equation (2.13a) is called the generalised force vector or ***generalised force***, which is always composed of two vectors: ***a force and a moment vector*** and ***can not be represented by a single straight line in space***!
In one single plane (i.e. x-y plane) the generalised force reduces to:

$$F = \begin{Bmatrix} F_x \\ F_y \\ M_z \end{Bmatrix} = \begin{Bmatrix} F_R \\ M_R \end{Bmatrix} \qquad (2.13b)$$

As moment M_z is caused by forces F_x and F_y, the ***generalised force F*** always refers to a certain point and ***is not independent of the position*** any more, as is each force or moment vector.

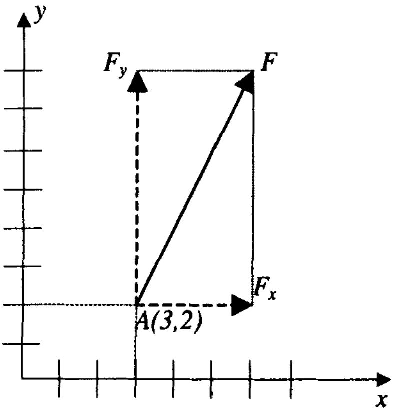


Figure 2.8: A generalised force in a plane

Consider the case in Fig. 2.8. The generalised force on origin (*0*, *0*) equals:

$$F_0 = \begin{Bmatrix} F\cos\alpha \\ F\sin\alpha \\ 3F\sin\alpha - 2F\cos\alpha \end{Bmatrix} = F\begin{Bmatrix} \cos\alpha \\ \sin\alpha \\ 3\sin\alpha - 2\cos\alpha \end{Bmatrix} = F\begin{Bmatrix} 0.447 \\ 0.894 \\ 1.789 \end{Bmatrix}$$

but at point A produces no moment, therefore:

$$F_A = \begin{Bmatrix} F\cos\alpha \\ F\sin\alpha \\ 0 \end{Bmatrix}$$

The procedure as described above is called ***reduction of a force to the chosen point.***

2.3 Equilibrium of a body

Figure 2.9 shows a body under action of *n* forces and *m* moments. From Newton's second law the following must be true if the body is to be at rest:

- *The resultant force must be zero*
- *The resultant moment (caused by n forces and m moments) must be zero*

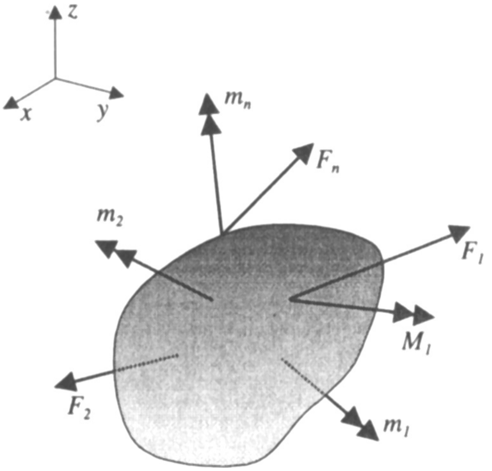


Figure 2.9: A body under a system of forces and moments[1]

From equations (2.7), (2.9) and (2.12), the necessary and sufficient condition for equilibrium is:

$$\begin{Bmatrix} \Sigma F_x \\ \Sigma F_y \\ \Sigma F_z \end{Bmatrix} = 0 \qquad (2.14a)$$

and

$$\begin{Bmatrix} \Sigma\left(F_{zl} \cdot y_l - F_{yl} \cdot z_l\right) \\ \Sigma\left(F_{xl} \cdot z_l - F_{zl} \cdot x_l\right) \\ \Sigma\left(F_{yl} \cdot x_l - F_{xl} \cdot y_l\right) \end{Bmatrix} + \begin{Bmatrix} \Sigma m_{xj} \\ \Sigma m_{yj} \\ \Sigma m_{zj} \end{Bmatrix} \qquad (2.14b)$$

---

[1] *m denotes moments due to its own mass, M denotes moments of a generalised force*

which is usually written simply as:

$$\Sigma M_x = 0 \quad \Sigma M_y = 0 \quad \text{and} \quad \Sigma M_z = 0 \tag{2.15}$$

It should be noted that equations (2.15) are independent of the choice of the coordinate system.

❖ *If a body is in equilibrium, equations (2.14) and (2.15) are satisfied for any co-ordinate system and at any origin.*

In two dimensional plane systems that lie for example in the *x*-*y* plane, if forces and moments act on this plane, the equations reduce to

$$\Sigma F_x = 0 \quad \Sigma F_y = 0 \quad and \quad \Sigma M_z = 0 \tag{2.16}$$

and the expression $\Sigma M_z = 0$ is simply written as $\Sigma M = 0$.

2.4 Displacements (and rotations)

Figure 2.10 shows a body in the *x*-*y* plane, loaded by force *F* in the same plane acting at point *B*. Force *F* will in general cause a displacement at all points of the body, except at the points where displacements are suppressed by supports (points *1*, *2*, *3*).

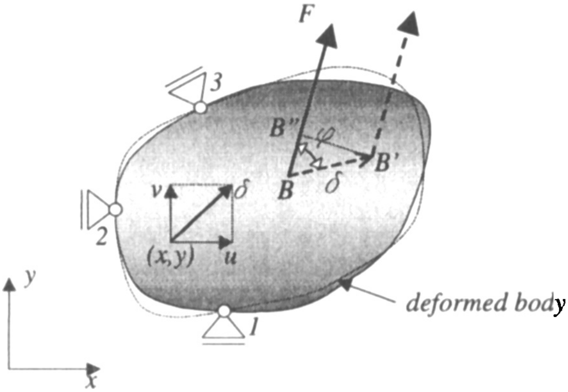


Figure 2.10: Deformation of a body

The *movement* of an arbitrary point (*x*, *y*) is denoted by δ, its components in *x* and *y* directions are denoted by *u* and *v*; we say therefore that the point *moves by* δ, exhibiting two *displacements u* and *v*.

$$\delta = \begin{Bmatrix} u \\ v \end{Bmatrix} = \{u \ \ v\}^T \tag{2.17}$$

The column matrix δ is a vector of displacements or *displacement vector* at point (x, y). It has to be noted that the displacement vector δ_B at point B in general does not coincide with the direction of force F. The component of the displacement vector δ_B in the direction of force F is the *corresponding displacement* δ_F (a deflection under gravitational loads).

$$\delta_F = BB'' = \delta_B \cdot cos\varphi \tag{2.18}$$

In addition to the displacement δ at any point (x, y) a body will in general rotate at all points through a rotation φ ; in plane structures it is the rotation about the z axis, but in a three dimensional body we have all three rotations:

$$\varphi = \left\{\varphi_x \;\; \varphi_y \;\; \varphi_z\right\}^T \tag{2.19}$$

Column matrix (2.19) is a *rotation vector* (vector of rotations). Similarly a displacement vector will have three components in space:

$$\delta = \left\{u \;\; v \;\; w\right\}^T, \tag{2.20}$$

If we combine both displacements and rotations in a single vector δ

$$\delta = \left\{u \;\; v \;\; w \;\; \varphi_x \;\; \varphi_y \;\; \varphi_z\right\}^T \tag{2.21}$$

then it is called the *generalised displacement.*

- *A generalised displacement consists of two vectors and can not be represented by a single straight line in space!*

Previous discussion on *corresponding displacement* holds also for three-dimensional space, therefore at the generalised force

$$\left\{F_x \;\; F_y \;\; F_z \;\; M_x \;\; M_y \;\; M_z\right\}^T \tag{2.22}$$

a generalised displacement is produced that is as follows:

$$\left\{u_x \;\; v_y \;\; w_z \;\; \varphi_x \;\; \varphi_y \;\; \varphi_z\right\}^T \tag{2.23}$$

The components of the displacement vector are the corresponding displacements to the respective forces. In general displacements $(u \; v \; w)$ will not be in the same directions as forces $(F_x \; F_y \; F_z)$ and rotations $(\varphi_x \; \varphi_y \; \varphi_z)$ will not be in the same directions as moments $(M_x \; M_y \; M_z)$.

2.5 Stresses

Figure 2.11 shows a prismatic element of constant cross sectional area, loaded by an axial force *F*. Let us imagine that the element is cut in section *1-1* perpendicular to the longitudinal axis, such that the lower part of the element is isolated as a *free body*.

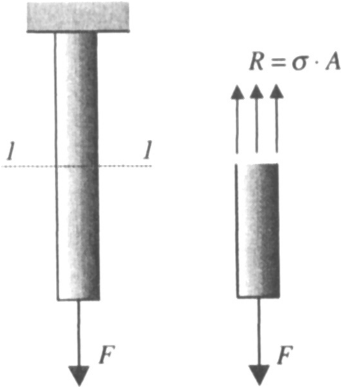


Figure 2.11: Element under tension and a *free body*

As the free body has to be in equilibrium, it is obvious that in section *1-1* a force σ per unit area acts, such that the resultant of all these forces σ ($R = \sigma \cdot A$) is in equilibrium with force *F*. Hence:

$$\sigma = \frac{F}{A} \tag{2.24}$$

The quantity σ is the force per unit area and is called the ***stress*** in the element, *R* is the *internal force*, sometimes called the ***stress resultant*** of σ .

Let us consider a three-dimensional body in equilibrium, which is loaded by arbitrary forces and moments. Imagine the body is cut by an imaginary plane that divides it into two free bodies.

As both free bodies have to be in equilibrium, an internal force *R* must exist on the imaginary plane, having *magnitude, direction* and *point of action* to satisfy equilibrium equations.

In general force *R* acts on an imaginary plane in an arbitrary direction. Its component on the normal to the surface is the *normal* or *axial force* R_n and the component tangential to the surface is the *tangential* or *shear force* R_s.

A distribution of the force *R* is usually not uniformly distributed across the surface. Let us suppose that on a small area *dA* a force of magnitude *dR* acts. The ratio *dR/dA* is the *stress at the point*, which is at the centre of gravity of area *dA*. Stress *dR/dA* has an arbitrary direction with regard to the plane.

The component of *dR/dA* at the normal is the *normal stress*, usually denoted by σ, the component of dR/dA at the tangent is the *tangential* or *shear stress*, denoted by τ.

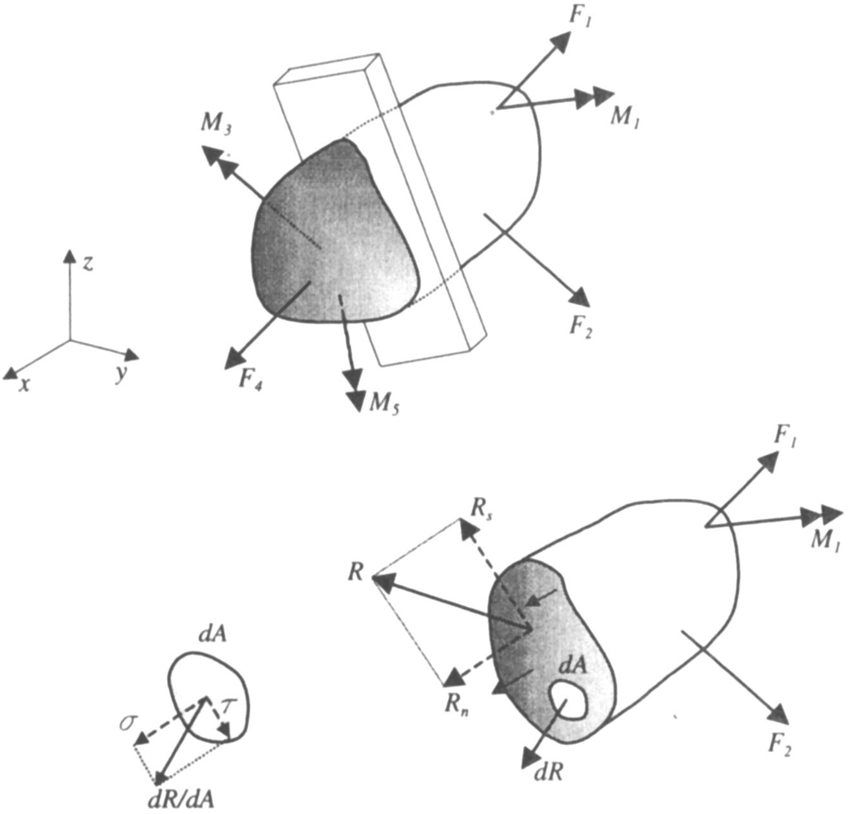


Figure 2.12: Free body in space

The stress notation is as follows:

- Normal stresses are denoted by σ. Subscript at σ denotes co-ordinate axis, which is a normal to the plane, on which the stress acts. Thus, σ_x acts on a plane, perpendicular to the *x*-axis, etc.

- Tangential stresses are denoted by τ with two subscripts. The first subscript denotes the plane, on which the stress τ acts, the second subscript gives the direction of the tangential stress.

The sign convention is as follows:

- *Normal stress is positive if it is tensile*; it means that it is directed *away* from the surface (along normal)
- The positive direction of the shear stress is dependent on the positive direction of the normal stress at that section. The *rule is*: ***if a positive normal stress acts in the direction of the positive axis then a shear stress is positive if acting in the positive direction of the corresponding axis.***

The sign conventions can be briefly written as:

- ***If the normal to the plane is in the positive co-ordinate axis direction, then the positive normal and shear stresses act in the direction of the corresponding positive co-ordinate axis and vice versa.***

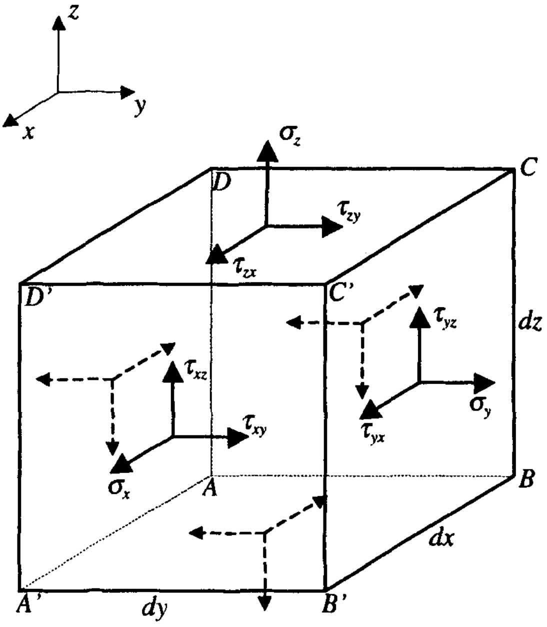


Figure 2.13: Stresses in a parallelepiped

Consider the equilibrium of the parallelepiped from figure 2.13. Moments about the *z*-axis are caused by the shear forces on planes *A'B'C'D'* and *ABCD* of magnitude ($\tau_{xy}\, dy\, dz$) and by shear forces on planes *BB'C'C* and *AA'D'D* of magnitude ($\tau_{yx}\, dx\, dz$).

If the body is in equilibrium all rotations have to be zero and therefore the sum of all moments about the z-axis must be zero:

$$\left(\tau_{xy}\cdot dy\cdot dz\right)\cdot dx=\left(\tau_{yx}\cdot dx\cdot dz\right)\cdot dy$$

(*force*) × *lever arm* = (*force*) × *lever arm*

From the above equation it is clear that:

$$\tau_{xy}=\tau_{yx} \tag{2.25}$$

From rotations about the x- and y-axes in similar manner as above:

$$\tau_{yz}=\tau_{zy} \quad\text{and}\quad \tau_{xz}=\tau_{zx} \tag{2.25a}$$

The state of stress on the faces of the parallelepiped, defined by nine components of *stress tensor 3 x 3*, is actually *defined by six components only* because of the symmetry as shown in equations (2.25) and (2.25a).

$$\sigma=\left\{\sigma_x\ \sigma_y\ \sigma_z\ \tau_{xy}\ \tau_{yz}\ \tau_{zx}\right\}^T \tag{2.26}$$

The column matrix in equation (2.26) is called the *stress vector* at the point (x, y, z). Shear stresses τ_{xy} and τ_{yx} are equal in magnitude and are called *complementary shear stresses*; the same stands for stresses τ_{yz} and τ_{zy} and for stresses τ_{xz} and τ_{zx}.

2.6 Specific deformations (strains)

Displacements that arise in engineering structures *are small under normal loads* in comparison with the dimensions of the structure; therefore, all the definitions discussed in this chapter, *are applicable only* if the deformations *are small.*

2.6.1 Axial deformation (normal strain)

Let us consider a *prismatic bar* in figure 2.14. Suppose that for some reason (*axial force at the end of the bar, increase in temperature*) the bar extends from the initial length L to L'. The distance between $A'B'$ after deformation is:

$$x+u+\left(\frac{\partial u}{\partial v}\right)\cdot dx-\left(x+u\right)=\partial x\cdot\left(\frac{\partial u}{\partial v}\right)\cdot dx$$

Axial specific deformation or *direct specific deformation* at point A is defined as:

$$\varepsilon = \frac{Elongation\,of\ \overline{AB}}{Initial\,length\,of\ \overline{AB}}$$

and is hence dimensionless. From figure 2.14:

$$\varepsilon = \frac{\overline{A'B'} - \overline{AB}}{\overline{AB}} = \frac{\left[dx + \frac{\partial u}{\partial x} \cdot dx\right] - dx}{dx} = \frac{du}{dx}, \tag{2.27}$$

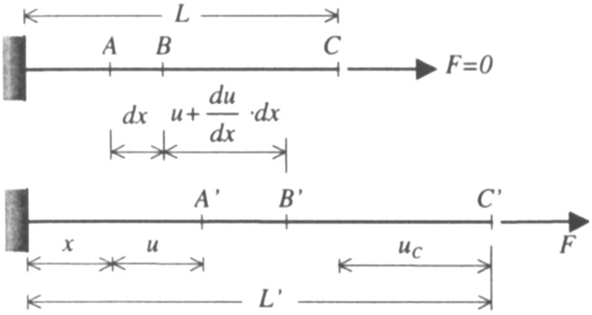


Figure 2.14: Axial deformation

and further, if increase (du/dx) *is constant over the whole length of the bar*:

$$\varepsilon = \frac{u_{free\ end} - u_{at\ sup\,port}}{L} = \frac{(L' - L) - 0}{L} = \frac{L' - L}{L}$$

$$\varepsilon = \frac{L' - L}{L} = \frac{\Delta L}{L} \tag{2.28}$$

Many structures are built from straight elements of constant cross section (trusses, frames, etc.). Axial forces only usually load these elements and the quantity (du/dx) is constant over the entire length. Equation (2.28) in such cases represents *axial specific deformation* or *axial strain*, which is positive under a tensile axial force and negative under a compressive axial force.

Consider now a two-dimensional case as in figure 2.15. The axial strain at a point (x,y) of the element depends on the element direction, as ε_x is defined as the ratio between the elongation and initial length. An increase in length depends on the direction, as it can be different in the x and y direction.

That is the reason why in two-dimensional (plane) cases we have to use two indexes or subscripts. Thus ε_x is the axial strain of the element initially in x-direction and ε_y is the axial strain of the element initially in y-direction.

After deformation from the initial position $m_1(x, y)$ and $m_2(x+dx, y)$ the points move to m_1' and m_2' (Figure 2.15).

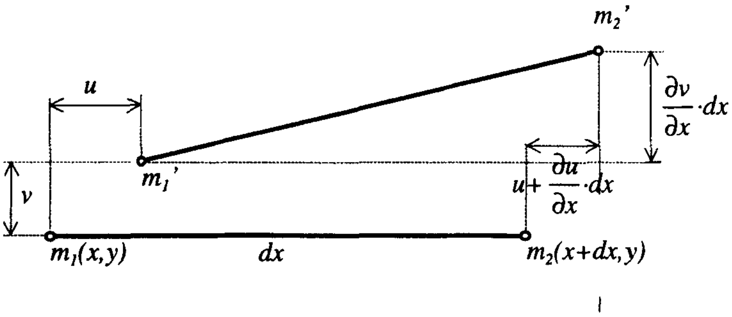


Figure 2.15: Displacement of a bar in a plane

According to the definition of strain we have:

$$\varepsilon_x = \frac{\overline{m_1'm_2'} - \overline{m_1m_2}}{\overline{m_1m_2}}$$

It is obvious that the element of length dx not only displaces in x direction by u but may rotate about the z axis, which is perpendicular to the paper. A rotation does not have influence on the axial strain, which is by definition the ratio between *change in length/ length*. The element length after the deformation is:

$$m_1'm_2' = \sqrt{\left[dx + \frac{\partial u}{\partial x}\cdot dx\right]^2 + \left[\frac{\partial v}{\partial x}\cdot dx\right]^2}$$

$$= \sqrt{(dx)^2 + 2\cdot\frac{\partial u}{\partial x}\cdot(dx)^2 + \left(\frac{\partial u}{\partial x}\right)^2\cdot(dx)^2 + \left(\frac{\partial v}{\partial x}\right)^2\cdot(dx)^2}$$

or if we neglect fourth order quantities:

$$m_1'm_2' = \sqrt{(dx)^2 + 2\cdot\frac{\partial u}{\partial x}(dx)^2} = dx\sqrt{\left(1 + 2\cdot\frac{\partial u}{\partial x}\right)}$$

If the deformations are small, the above equation can be rearranged by adding a small square term as follows:

$$m_1'm_2' = dx\sqrt{\left(1+2\cdot\frac{\partial u}{\partial x}\right)+\left(\frac{\partial u}{\partial x}\right)^2} = dx\left(1+\frac{\partial u}{\partial x}\right)$$

from which we can express the specific strain:

$$\varepsilon_x = \frac{\overline{m_1'm_2'}-\overline{m_1m_2}}{\overline{m_1m_2}} = \frac{dx\left(1+\frac{\partial u}{\partial x}\right)-dx}{dx} = \frac{du}{dx} \tag{2.29a}$$

Similarly we can show that specific strain in the *y* direction is:

$$\varepsilon_y = \frac{dv}{dy} \tag{2.29b}$$

In a three-dimensional body further deformation in the *z* direction gives:

$$\varepsilon_z = \frac{dw}{dz} \tag{2.29c}$$

2.6.2 Shear strains

Shear strains are determined from figure 2.16, which shows two elements m_1m_2 of length *dx* and m_1m_3 of length *dy*.

After deformation the points move to a new position m_1', m_2' *in* m_3' and elements *dx* and *dy* rotate through angles φ_1 and φ_2. *Shear strain* is defined as the change in value of the angle between elements before and after the deformation.

$$\gamma_{xy} = \varphi_1 + \varphi_2 \tag{2.30}$$

From figure 2.16 we can evaluate angle φ_1:

$$\varphi_1 \cong sin\varphi_1 = \frac{\frac{\partial v}{\partial x}\cdot dx}{m_1'm_2'} = \frac{\frac{\partial v}{\partial x}\cdot dx}{m_1m_2\cdot(1+\varepsilon_x)} = \frac{\frac{\partial v}{\partial x}\cdot dx}{dx\cdot(1+\varepsilon_x)},$$

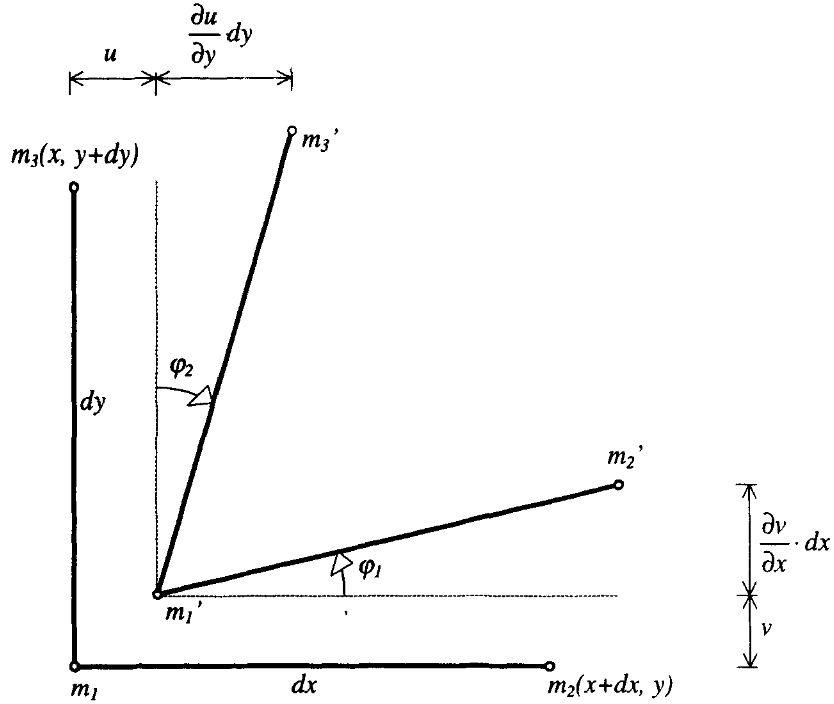


Figure 2.16: Definition of shear strains

and if we say that quantity ε_x is small in comparison with unity:

$$\varphi_1 = \frac{\partial v}{\partial x}$$

Similarly for angle φ_2 :

$$\varphi_2 = \frac{\partial u}{\partial y}$$

The total shear strain is the sum of both angles:

$$\gamma_{xy} = \varphi_1 + \varphi_2 = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \tag{2.31}$$

In a three-dimensional body all three shear strains exist and it is easy to show that:

$$\begin{aligned}\gamma_{xy} &= \frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\\ \gamma_{yz} &= \frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\\ \gamma_{zx} &= \frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\end{aligned} \tag{2.32}$$

where u, v and w are the components of the displacement at the point in directions x, y and z respectively.

The above equations show that specific strains *are all zero* if a rigid body has *constant displacements* u, v and w throughout (note that the rigid body translates only, see Ch. 4 on kinematics of a body). They also show that at any point (x, y, z) of a body only six independent strains exist, which can be written as a *strain vector*:

$$\varepsilon = \left\{\varepsilon_x \;\; \varepsilon_y \;\; \varepsilon_z \;\; \gamma_{xy} \;\; \gamma_{yz} \;\; \gamma_{zx}\right\}^T, \tag{2.33}$$

2.7 Stress-strain relations

In the previous two sections stresses and deformations were discused as being independent from each other, since deformations can be caused from factors other than stresses. From Hooke's experiments it is known that if a body is subjected to a uniform stress σ_x (Figure 2.17), it follows the law:

$$\sigma_x = E\cdot\varepsilon_x \quad \text{or} \quad \frac{\sigma_x}{\varepsilon_x}=E \quad \text{or} \quad \varepsilon_x=\frac{\sigma_x}{E} \tag{2.34}$$

where E is an experimentally determined constant called *modulus of elasticity* or *Young's modulus*. Since specific deformations or strains are defined as the ratio change in *length/initial length* and are hence dimensionless it is obvious that E has the same dimensions as stress. The proportionality of stress and strain is known as *Hooke's law*.

When dealing with materials, having the same properties in all directions (*isotropic material*), the following will be true:

$$\frac{\sigma_y}{\varepsilon_y}=E \quad \textit{and} \quad \frac{\sigma_z}{\varepsilon_z}=E \tag{2.35}$$

If the stress σ_x is removed from the block on figure 2.17 then *all deformations* would disappear. Such a material is called a *perfectly elastic material.*

Most engineering materials exhibit such a property but to certain level of stresses only, which is called the *proportional limit* of the material and the material in such a condition is called a *linear elastic material.*

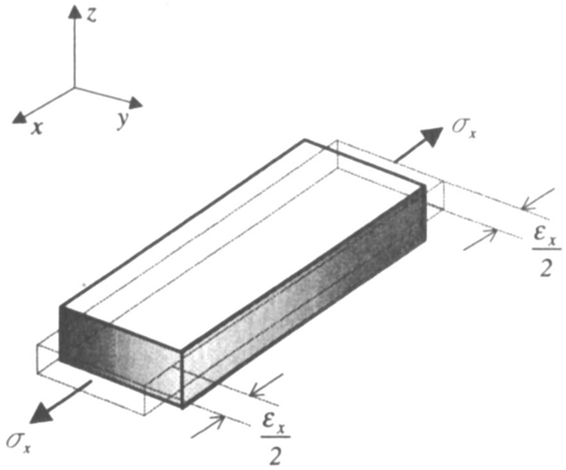


Figure 2.17: Hooke's law

Note that under stress σ_x there always exist lateral strains in addition to the axial strain ε_x, which are proportional to strain ε_x, but are of opposite sign:

$$\varepsilon_y = \varepsilon_z = -\nu \cdot \varepsilon_x = -\nu \cdot \frac{\sigma_x}{E} \tag{2.36}$$

The constant ν is called *Poisson's ratio* of the material. It is usually adequate to assume that E and ν has the ***same magnitude under tension and compression.***

Suppose the body in Fig. 2.17 is subjected to all three stresses σ_x, σ_y and σ_z simultaneously. If direction x is considered, the stress σ_x will cause an axial strain ε_x, but stresses σ_y and σ_z will also cause ***negative lateral strains.*** The total specific deformation in x direction is therefore:

$$\varepsilon_x = \frac{\sigma_x}{E} - \nu \cdot \frac{\sigma_y}{E} - \nu \cdot \frac{\sigma_z}{E} = \frac{1}{E} \cdot \left[\sigma_x - \nu \cdot \left(\sigma_y + \sigma_z\right)\right] \tag{2.37a}$$

Similarly for other directions:

$$\varepsilon_y = \frac{\sigma_y}{E} - \nu \cdot \frac{\sigma_z}{E} - \nu \cdot \frac{\sigma_x}{E} = \frac{1}{E} \cdot \left[\sigma_y - \nu \cdot \left(\sigma_z + \sigma_x\right)\right] \tag{2.37b}$$

$$\varepsilon_z = \frac{\sigma_z}{E} - \nu \cdot \frac{\sigma_x}{E} - \nu \cdot \frac{\sigma_y}{E} = \frac{1}{E} \cdot \left[\sigma_z - \nu \cdot \left(\sigma_x + \sigma_y\right)\right] \tag{2.37c}$$

The above equations completely define the deformation of the body under the normal stresses σ_x, σ_y and σ_z. It is worth remembering that *normal stresses produce only normal strains* (no shear strains). In general it should be noted that:

- *Normal strains ε_x, ε_y and ε_z are functions of stresses σ_x, σ_y and σ_z and are independent of shear strains, though they can act simultaneously.*

- *Shear strains (rotations!) are functions of shear stresses and are independent of any normal stresses.*

- *Shear strains are dependent on corresponding shear stresses only i.e. γ_{xy} is a function of τ_{xy} only and is independent of shear stresses τ_{yz} and τ_{zx}.*

Shear strains are given by equations:

$$\gamma_{xy} = \frac{\tau_{xy}}{G} \qquad \gamma_{yz} = \frac{\tau_{yz}}{G} \qquad \gamma_{zx} = \frac{\tau_{zx}}{G} \tag{2.38}$$

The constant G is the *shear modulus* of the material. Constants of the material are E, G and ν and are related by the equation:

$$G = \frac{E}{2\cdot(1+\nu)} \cong \frac{E}{3} \tag{2.39}$$

Poisson's ratio for steel is nearly constant and has the value of approximately *0.30*, for concrete it varies between *0.12* and *0.20*.

The relation between stresses and strains can be written in matrix form:

$$\left\{\begin{array}{c} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{array}\right\} = \frac{1}{E} \left[\begin{array}{ccc|ccc} 1 & -\nu & -\nu & 0 & 0 & 0 \\ \nu & 1 & -\nu & 0 & 0 & 0 \\ -\nu & -\nu & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 2(1+\nu) & 0 & 0 \\ 0 & 0 & 0 & 0 & 2(1+\nu) & 0 \\ 0 & 0 & 0 & 0 & 0 & 2(1+\nu) \end{array}\right] \left\{\begin{array}{c} \sigma_x \\ \sigma_y \\ \sigma_z \\ \hline \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{array}\right\} \tag{2.40}$$

or shorter

$$\{\varepsilon\} = [N]\{\sigma\}$$

The above equation can be inversely tranformed

$$\{\sigma\} = [N]^{-1}\{\varepsilon\} = [D]\cdot\{\varepsilon\}$$

or explicitly:

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \hline \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{Bmatrix} = \frac{E}{2(1+v)(1-2v)} \left[\begin{array}{ccc|ccc} 2(1-v) & 2v & 2v & 0 & 0 & 0 \\ 2v & 2(1-v) & 2v & 0 & 0 & 0 \\ 2v & 2v & 2(1-v) & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & (1-2v) & 0 & 0 \\ 0 & 0 & 0 & 0 & (1-2v) & 0 \\ 0 & 0 & 0 & 0 & 0 & (1-2v) \end{array} \right] \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \hline \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix} \tag{2.41}$$

Let us consider two special cases:

a) *Plane stress* is defined as

$$\sigma_z = \tau_{yz} = \tau_{zx} = 0 \, ,$$

i.e. only stresses σ_x, σ_y and τ_{xy} exist. Equation (2.40) is reduced to:

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \frac{1}{E} \cdot \left[\begin{array}{c|c|c} 1 & -v & 0 \\ \hline -v & 1 & 0 \\ \hline 0 & 0 & \dfrac{(1-v)}{2} \end{array} \right] \cdot \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} \tag{2.42}$$

If the Eqn. (2.42) is solved for stresses:

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{E}{(1-v)^2} \cdot \left[\begin{array}{c|c|c} 1 & v & 0 \\ \hline v & 1 & 0 \\ \hline 0 & 0 & \dfrac{(1-v)}{2} \end{array} \right] \cdot \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} \tag{2.43}$$

It has to be emphasised that in a *state of plane stress* ε_z *is not equal to zero*, but is given by the equation

$$\varepsilon_z = -\frac{v(\sigma_x + \sigma_y)}{E}$$

b) Plane strain is defined as

$$\varepsilon_z = \gamma_{yz} = \gamma_{zx} = 0,$$

i.e. only strains $\varepsilon_x, \varepsilon_y$ and γ_{xy} exist. Equation (2.41) then reduces to:

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{zy} \end{Bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \cdot \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \frac{(1-2\nu)}{2} \end{bmatrix} \cdot \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} \qquad (2.44)$$

or if expressed in terms of strains:

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \frac{1+\nu}{E} \cdot \begin{bmatrix} 1-\nu & -\nu & 0 \\ -\nu & 1-\nu & 0 \\ 0 & 0 & 2 \end{bmatrix} \cdot \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} \qquad (2.45)$$

Again it has to be emphasised that in a *state of plane strain σ_z is not equal to zero*, but is derived from the equation

$$0 = \frac{1}{E}\left[\sigma_z - \nu\left(\sigma_x + \sigma_y\right)\right]$$

and hence

$$\sigma_z = \nu\left(\sigma_x + \sigma_y\right)$$

2.8 Discrete element deformations and displacements

When considering structural elements as discrete elements it is of interest to find the overall element deformations and the corresponding joint displacements.

Under the action of external loads, internal actions and stresses will develop, resulting in internal discrete element deformations and in displacements of the whole structure.

Such deformations can be caused by axial forces, bending moments, torsion moments and shear forces either separately or in any combination acting simultaneously.

2.8.1 *Axial deformations*

Tensile stress in the bar at section *x* is:

$$\sigma_x = \frac{F_x}{A}$$

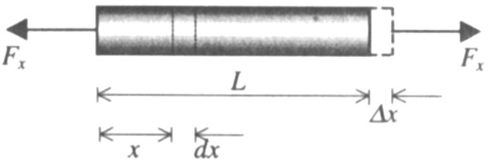


Figure 2.18: Axial deformation

According to Hooke's law the axial strain is:

$$\varepsilon_x = \frac{\sigma_x}{E} = \frac{F_x}{E \cdot A}$$

The total axial deformation is derived by integration over the whole length:

$$\Delta_x = \int_0^L \varepsilon_x \cdot dx = \int_0^L \frac{F_x}{E \cdot A} \cdot dx \qquad (2.46)$$

At a constant *axial rigidity EA* the total deformation is simply:

$$\Delta_x = \frac{F_x \cdot L}{E \cdot A} \qquad (2.47)$$

2.8.2 *Shear displacement and deformations*

Consider a member of rectangle cross section $B \cdot H$ with shear forces Q_y acting in the *x-y* plane. The shearing stresses at an arbitrary distance from neutral axis are:

$$\tau_{xy} = \frac{Q_y \cdot S_z}{I_z \cdot b},$$

where I_z is the moment of inertia and S_z is the static moment about the neutral axis of that portion of the section, lying outside the part for which the shear stress is being considered.

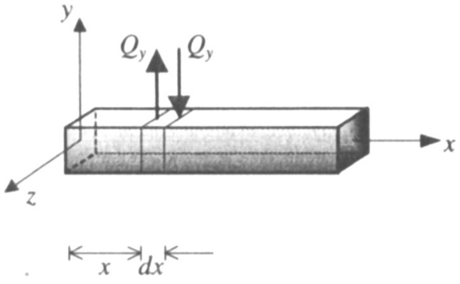


Figure 2.19: Shear force

The shear strain is given by the equation (2.38):

$$\gamma_{xy} = \frac{\tau_{xy}}{G}$$

The relative displacement in the *y* direction between two sections is given as:

$$dv = \frac{K}{G \cdot A_x} \cdot Q_y \cdot dx$$

The expression $G \cdot A_x / K$ is called *shearing rigidity*, *K* is a factor that depends on the shape of the cross section and is given in appendix B.
The total relative displacement of two end sections is

$$\Delta y = \int_0^l dv = \int_0^l \frac{K}{G \cdot A_x} \cdot Q_y \cdot dx \tag{2.48}$$

and for constant *G*, A_x and *K* is:

$$\Delta_Y = \frac{K}{G \cdot A_x} \cdot Q_y \cdot L \tag{2.49}$$

2.8.3 Bending displacements and deformations

If a member is loaded by two equal bending moments (couples) M_z about the *z*-axis, the stress in the *x* direction in a cross section at a distance *y* from the neutral axis is given by:

$$\sigma_x = -\frac{M_z \cdot y}{I_z}$$

The bending (or flexural) strain is given by:

$$\varepsilon_x = \frac{\sigma_x}{E} = -\frac{M_z \cdot y}{E \cdot I_z}$$

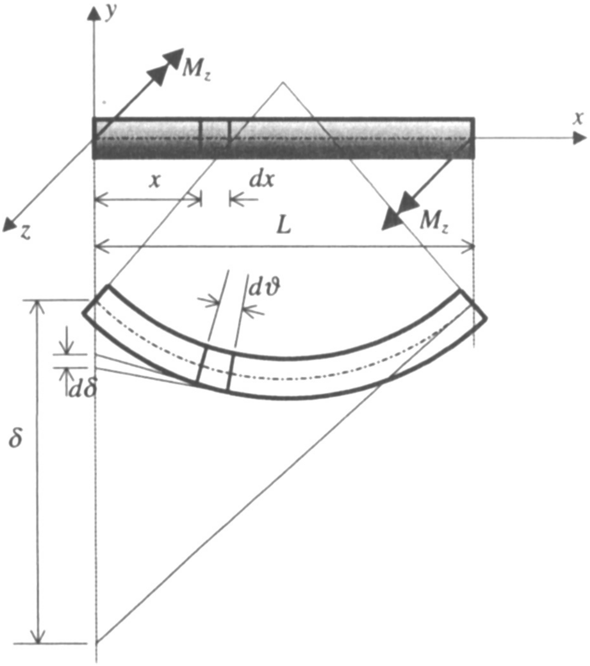


Figure 2.20: Pure bending

The quantity EI_z is called *flexural rigidity*. The relative angle of rotation $d\theta$ between two cross sections is

$$d\theta = -\frac{\varepsilon_x \cdot dx}{y} = \frac{M_z \cdot dx}{E \cdot I_z}$$

and the total rotation between bar ends:

$$\theta_z = \int_0^L d\theta = \int_0^l \frac{M_z \cdot dx}{E \cdot I_z} \tag{2.50}$$

which can be explicitly integrated when EI_z = constant:

$$\theta_z = \frac{M_z \cdot L}{E \cdot I_z} \tag{2.51}$$

The distance between end tangents from Fig. 2.20 is:

$$\delta = \int_0^L d\delta = \int_0^L x \cdot d\theta = \int_0^L \frac{M_z \cdot x \cdot dx}{E \cdot I_z} = \frac{M \cdot L^2}{2 \cdot E \cdot I_z} \tag{2.52}$$

2.8.4 *Torsion deformations*

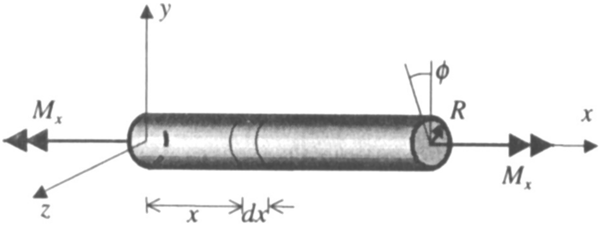


Figure 2.21: Torsion loading

The torsion stress for a circular cross section bar from Fig. 2.21 at a section at a distance r from the x axis is:

$$\tau = \frac{M_x \cdot r}{I_x},$$

where I_x is the polar moment of inertia about x axis (given for some cross sections in appendix B).

$$\tau_{max} = \frac{M_x \cdot R}{I_x}$$

The shear strain γ is given by

$$\gamma = \frac{\tau}{G} = \frac{M_x \cdot r}{G \cdot I_x} \qquad \Rightarrow \qquad \gamma_{max} = \frac{M_x \cdot R}{G \cdot I_x}$$

where G is the shear modulus of elasticity, and GI_x is called the *torsional rigidity* of the element. The relative angle of rotation between two sections is

$$d\varphi = \frac{\gamma_{max} \cdot dx}{R} = \frac{M_x \cdot dx}{G \cdot I_x}$$

The total relative angle of twist between end sections is

$$\varphi = \int_0^L d\varphi = \int_0^L \frac{M_x \cdot dx}{G \cdot I_x} \tag{2.53}$$

or simplified when GI_x is constant over the whole length:

$$\varphi = \frac{M_x \cdot L}{G \cdot I_x} \tag{2.54}$$

Note: For members with noncircular cross sections, *warping* may play a significant role.

3

Statically determinate structures

3.1 Supports and reactions

A *plane structure* is considered if it lies in one plane in space as shown in Fig. 3.1. Usually loads act in the same plane, but it is not always the case (as in floor plates carrying loads normal to its plane). On a plane a movement of a point is defined by three components of a displacement in the Cartesian co-ordinate system.

As already mentioned in Ch. 2, the displacements and forces are *conjugate quantities* which means the following: if the displacement in a direction is zero (i.e. $u=0$), then a force in the same direction must exist ($F_x \neq 0$) to prevent that displacement. These forces are called *reaction forces* or simply *reactions* and will be denoted by symbol R_i ; R will show the direction of the force and index i defines the point of a support. *Supports* are the points on a structure that do not permit rigid body movement, sometimes forces at these points will be called *constraints*.
Reactions are given by equations:

$$\begin{aligned} X_A &= k_x \cdot u \\ Y_B &= k_y \cdot v \\ M_C &= K \cdot \varphi , \end{aligned} \tag{3.1}$$

as shown in Fig. 3.2; possible displacements u, v and φ depend on the characteristics of springs. Constants k_x and k_y are *spring constants* having units of *kN/m*, which means the *force required to shorten or extend a spring for a unit of length.*

The constant K is the *rotational spring constant* of unit *kNm*, it is a moment which produces rotation of an element by an angle of *1 radian.*

Quantities k_x , k_y and K can in practice have different values such that in supports both reactions and displacements can occur; such supports are called *elastic supports.* The structural engineer must always assume realistic support conditions, though in elementary statics the values of k_x , k_y and K will either be assumed zero or infinite.

If the values of k_x, k_y and K are zero, then the displacements u, v and φ will occur, when k_x, k_y and K are infinite, the reactions will occur at zero displacements.

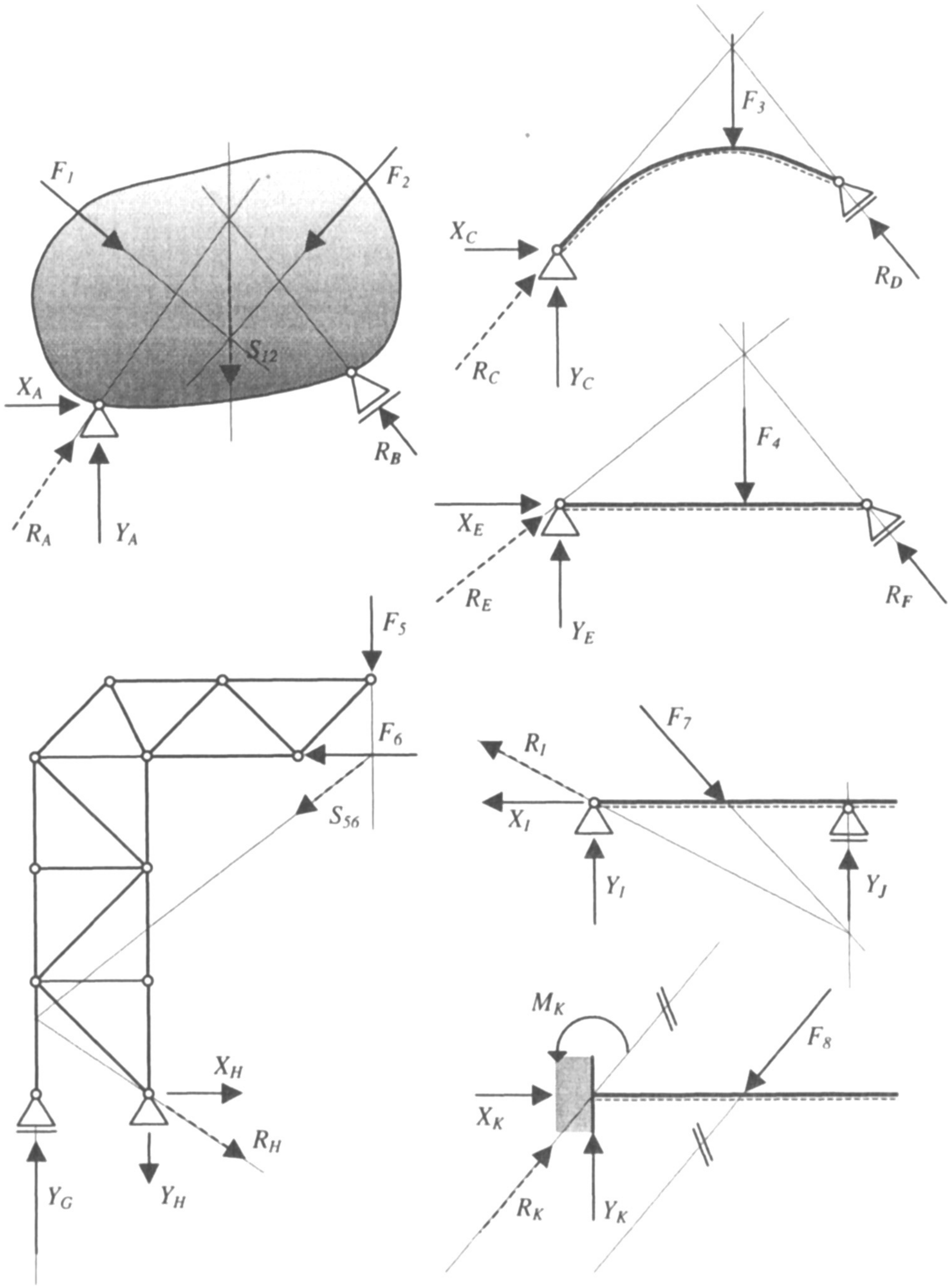


Figure 3.1: Plane structures

Equation (3.2) relates displacements and forces as conjugate quantities:

$$\begin{Bmatrix} u_x \neq 0 \\ u_y \neq 0 \\ \varphi_z \neq 0 \end{Bmatrix} \Leftrightarrow \begin{Bmatrix} R_x = 0 \\ R_y = 0 \\ M_z = 0 \end{Bmatrix} \tag{3.2}$$

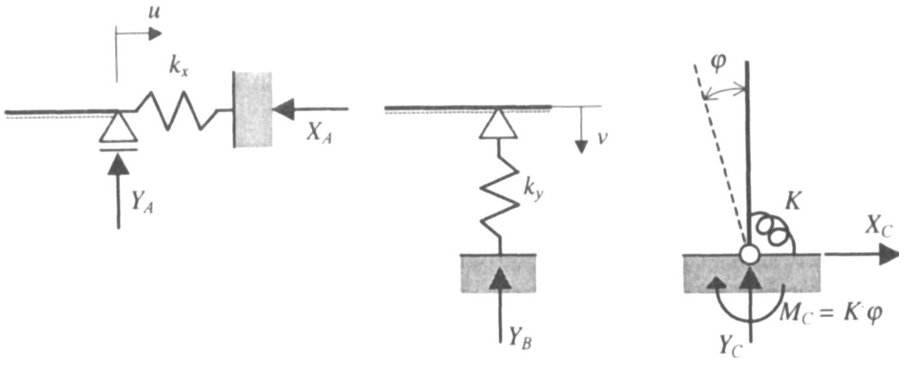
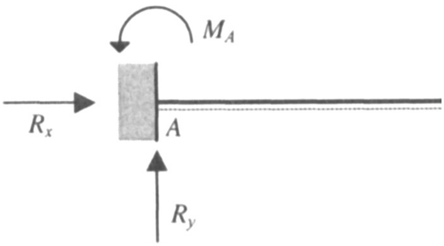


Figure 3.2: Elastic supports

If all three displacements are suppressed we get a *fixed* or *clamped support*, as shown in Fig. 3.3, where all three displacements at point A are zero (i.e. two displacements and a rotation).



$$\begin{Bmatrix} u_A = 0 \\ v_A = 0 \\ \varphi_A = 0 \end{Bmatrix} \Leftrightarrow \begin{Bmatrix} R_x \neq 0 \\ R_y \neq 0 \\ M_A \neq 0 \end{Bmatrix}$$

Figure 3.3: Fixed support

On the contrary, the displacements at all other points of the structure are possible.

- *At point A in figure 3.3 all three displacements are zero, therefore three reaction forces R_x, R_y and M_A must exist as conjugate quantities to suppress these displacements.*

3.2 Principle of a free body

Graphically the equations in the previous section can be represented as the principle of a free body. As seen from Ch. 2 there are three equilibrium equations in a plane:

$$\begin{aligned} &\Sigma X_i = 0 \\ &\Sigma Y_i = 0 \\ &\Sigma M_i = 0, \end{aligned} \tag{3.3}$$

which are sufficient to calculate reactions R_x , R_y and M at point A of the cantilever structure in Fig. 3.3. It must be stated that the magnitude of these three reactions are such that the displacements u_x, v_y and φ_z are *zero.*

If the rotation at point A is released, therefore $\varphi_z \neq 0$ (or because of the conjugate properties also $M = 0$), then at point A only two reactions remain (R_x and R_y) or in general one *generalised reaction force R* of magnitude

$$R = \sqrt{R_x^2 + R_y^2} \tag{3.4}$$

and is inclined at an angle of

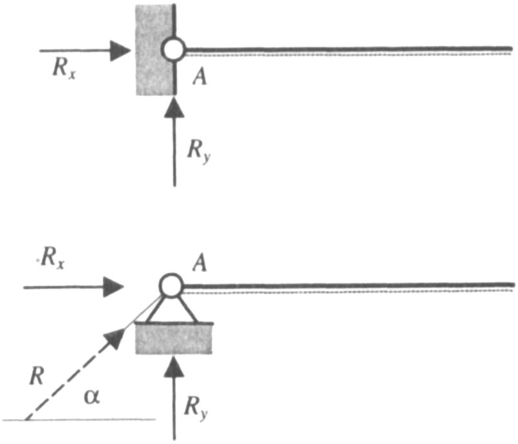
$$\alpha = tan^{-1}\left(\frac{R_y}{R_x}\right). \tag{3.5}$$

A support with two displacements suppressed (i.e. u_x and v_y) is called a *pin support* and is shown in Fig. 3.4. In both cases the displacements u_x and v_y are zero, but the rotation at point is possible, as symbolically shown by the circle (hinge). A free body can substitute the pin support as shown in Fig. 3.5.

It is obvious that element AB would at this state rotate around point A under any loading (self weight, external forces) and would not be in equilibrium any more. ($\varphi_z \neq 0$, $M = 0$).

The third equilibrium equation can not be satisfied any more and the structure becomes *unstable or a mechanism.* (Note: in this state the element AB could be additionally supported at point B by at least one reaction force to become stable again).

As there are in general three possible movements at point A (u, v and φ_z) it is possible to release any of the displacements (u or v) but the rotation φ_z has to be zero ($\varphi_z = 0$). In this way we get a *clamped support which is guided* in one of the directions (one of the displacements u or v is possible).



$$\begin{Bmatrix} u_A = 0 \\ v_A = 0 \\ \varphi_A \neq 0 \end{Bmatrix} \Leftrightarrow \begin{Bmatrix} R_x \neq 0 \\ R_y \neq 0 \\ M_A \equiv 0 \end{Bmatrix} \tag{3.6}$$

Figure 3.4: Pin (non-movable) support

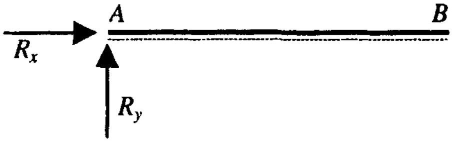
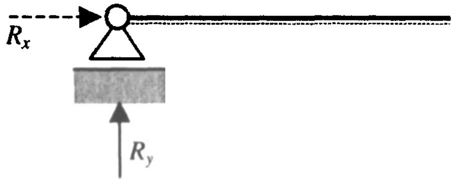


Figure 3.5: Free body of a pin support

The rotation is in both cases prevented by a spring. If a spring is *very stiff* we are discussing a *full clamped* condition, but on contrary if a *spring is weak* and its rotational stiffness becomes zero, then we get a *roller support*, symbolically shown in Fig. 3.6.

At a roller support only one reaction occurs, which is always perpendicular to the possible displacement (i.e. possible is displacement *u* and reaction *Y* exists).

It has to be stated that an effect of a movable support can be achieved by so-called *swinging supports,* i.e. bars, which through hinges connect two elements or rigid bodies. In such bars only axial forces can occur which acts as a reaction, the displacement is always perpendicular to the bar axis (see Ch. 4 on kinematics).



$$\left\{\begin{matrix} u_A \neq 0 \\ v_A = 0 \\ \varphi_A \neq 0 \end{matrix}\right\} \Leftrightarrow \left\{\begin{matrix} R_x = 0 \\ R_y \neq 0 \\ M_A \equiv 0 \end{matrix}\right\} \qquad (3.7)$$

Figure 3.6: Roller (movable) support

The release of the last possible displacement v_A enables free movement of a point A. Such a point *can not be supported with reactions* but can be a connection point between two elements in statically indeterminate structures.

Let us follow step by step the release of displacement quantities of generalised displacement and its influence on reactions in the table bellow:

Support	*Displacements*	*Reactions*
Elastic	$u \neq 0$ $v \neq 0$ $\varphi \neq 0$	$X = k_x.u$ $Y = k_y.v$ $M = K.\varphi$
Clamped	$u = 0$ $v = 0$ $\varphi = 0$	X Y M
Non-moveable *(pin support)*	$u = 0$ $v = 0$ $\varphi \neq 0$	X Y M=0
Clamped-movable, *guided* in y direction	$u = 0$ $v \neq 0$ $\varphi = 0$	X Y=0 M
Roller support *Movable* in y direction	$u = 0$ $v \neq 0$ $\varphi \neq 0$	X Y=0 M=0
Roller support *Movable* in x direction	$u \neq 0$ $v = 0$ $\varphi \neq 0$	X=0 Y M=0
Movable *Swinging support,* Rotation about absolute pole	$u = 0$ $v = 0$ $\varphi \neq 0$	N *(Axial force in* *swinging bar)*

3.3 Plane truss

3.3.1 Introduction

The truss is one of the major types of engineering structures. It provides both a practical and an economical solution to many engineering situations, especially in the design of bridges and buildings.

A truss is a structural system in which, due its construction and configuration, all members are subjected only to pure tension or compression forces. If all the members lie in one plane and the truss is loaded in that plane, it is referred to as a *plane truss*. An analysis is performed on an idealised structure, which fulfils the following conditions:

- *Member connections are made by frictionless hinges or pins*
- *Members are straight and connected at their extremities only*
- *Loads are acting at truss joints only (self weight forces are reduced to hinges)*

If all of the above conditions are met then only axial forces occur in members. In practice it is difficult to meet all of the above conditions as the elements of tension and compressive parts are usually continuous beams and are not connected by hinges and the diagonals are often connected to beams by welding or by welded or bolted steel plates.

The loading causes deformations of the structure, which has to be small if the compatibility conditions are to be fulfilled and the basic static analysis can be applied.

The elements of a truss, which are loaded directly outside joints have to be analysed locally as beams carrying shear forces and bending moments at the same time as an axial force. Bending moments should never be neglected since, in combination with a compressive axial force, induce buckling of bars.

3.3.2 Modelling of trusses

The main concern with truss modelling is the kinematic stability. The simplest possible kinematically stable truss is a triangular truss.

A triangle consists of three joints (*j*) and three elements (*m*); each additional joint requires two additional elements and can be written as

$$m - \underset{\substack{\uparrow \\ \textit{forces in the basic triangle}}}{3} = 2 \cdot j - \underset{\substack{\uparrow \\ \textit{displacements of the basic triangle}}}{6} \tag{3.8}$$

Eqn. (3.8) links the number of elements *m* to the number of joints. If the structure is to be in equilibrium, three swinging supports can be added as additional truss elements and Eqn. (3.8) transforms into:

$$m = 2 \cdot j - 3\,, \tag{3.9}$$

where j means a *number of free joints*. Eqn. (3.9) assures internal determinacy, if there are more than three reactions the structure becomes *externally indeterminate* with no influence on internal determinacy.
Kinematic stability of trusses is determined by the equation

$$f = 2 \cdot j - m - 3 \qquad (\textit{3 reactions included}), \tag{3.10a}$$

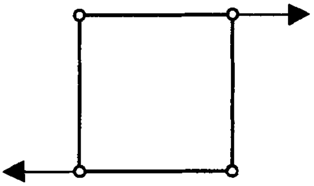
and can be used for externally determined trusses or by a general equation

$$f = 2 \cdot j - m - p \qquad (p = \textit{number of suppressed displacements}), \tag{3.10b}$$

if the condition of internal determinacy from Eqn. (3.9) is met.

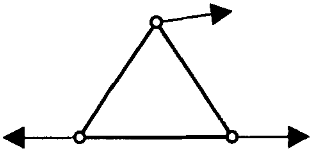
- *Trusses can be internally or externally indeterminate independent of either condition.*

Let us consider basic cases of truss modelling:

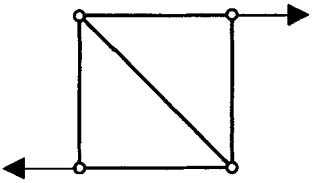


kinematically *unstable*
$f = 2j - m - 3 = 8 - 4 - 3 = 1$
(one degree of freedom)

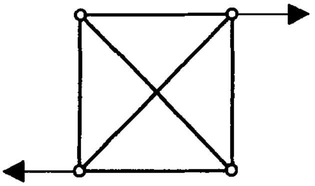
kinematically *stable, statically determinate*
$f = 2j - m - 3 = 4 - 1 - 3 = 0$



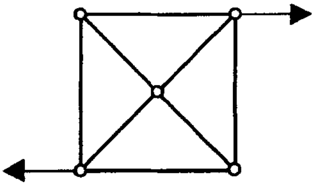
kinematically *stable, statically determinate*
$f = 2j - m - 3 = 6 - 3 - 3 = 0$



kinematically *stable, statically determinate*
$f = 2j - m - 3 = 8 - 5 - 3 = 0$



Internally indeterminate truss
$f = 2j - m - 3 = 8 - 6 - 3 = -1$



Internally indeterminate truss
f = 2j – m - 3 = 10 – 8 - 3 = -1

Reactions in trusses are calculated by a beam analogy, as the reactions are external forces and the whole truss can be represented as a rigid body (Figure 3.7).

Displacements in trusses at roller supports are considerable and much higher than in beams and therefore much attention has to be paid to support design and its execution.

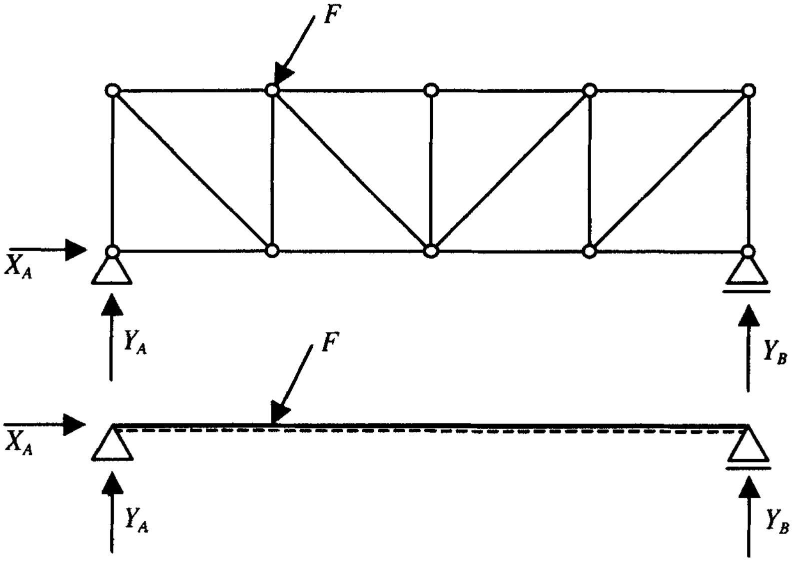


Figure 3.7: Analogy of truss and beam

Let us calculate the stability of the truss in Fig. 3.7, that is externally in equilibrium with three reactions:

f = 2j – m - 3 = 2.10 – 17 - 3 = 0,

As *f=0* the truss is *externally and internally determinate.*

3.3.3 Methods of truss analysis

The analysis of statically determinate and stable plane trusses can be accomplished by a number of relatively easy methods that can contribute to a better understanding of basic statics:

1. The *Graphical Method* will be used when a new truss is to be designed and we will be looking for all forces in the truss.

2. The *Method of Sections* (Ritter's method) is used, when forces in some bars only are desirable.

3. The *Method of Joints* (projection method) is used in orthogonal systems i.e. if two bars are perpendicular to each other at any loading.

4. The *Kinematics Method* is the simplest method though the knowledge of kinematics is necessary. It is used to calculate forces in individual bars and no reaction calculation is necessary (see example in Ch. 4).

When using the first three methods the structure has to be equilibrated prior to any other calculations; the kinematics method requires no reaction force calculation.

Example 3.1: Calculate all forces in the truss from Fig. 3.8 by graphical and analytical methods ($F = 50$ kN).

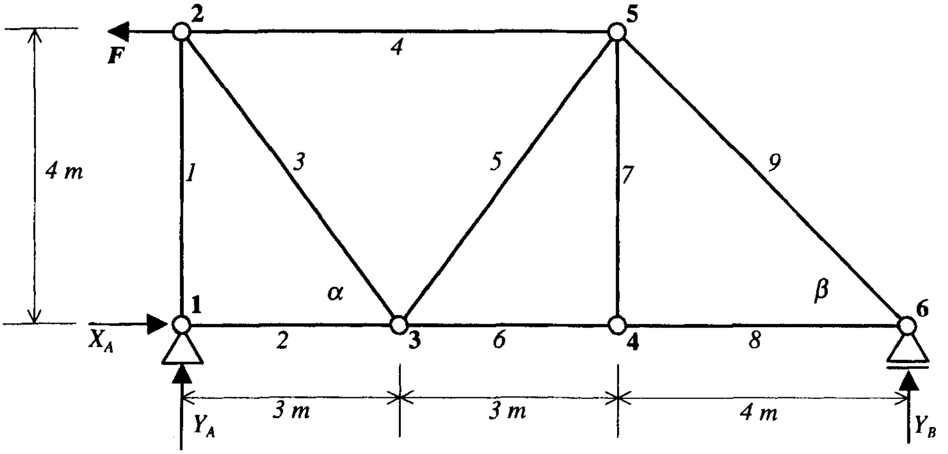


Figure 3.8: Plane truss

3.3.3.1 Graphical solution (Maxwell's diagram, Cremona's diagram)

A graphical solution is achieved using a *polygon of forces*. Reaction directions are determined from the fact that a force F can be equilibrated by two components, if all three forces meet at the same point. The direction of the reaction at support B is known as the displacement is in the x direction. Since the reaction can only be perpendicular to the displacement, it has to be in the y direction. The common point at the Y_B line then determines the direction of the reaction R, which has to act through the support at A. Let us draw the *polygon of forces* in a chosen scale:

Entire free body

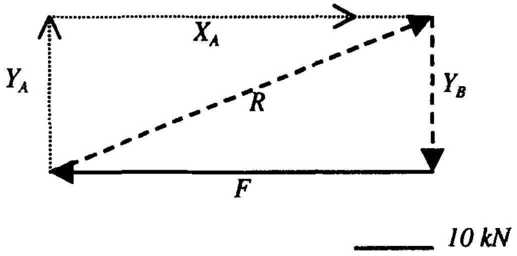


Figure 3.9: Polygon of forces at equilibrium

The reaction magnitudes are determined from the polygon of forces (*1 cm = 10 kN*) and are $X_A = F = 50\ kN$, $Y_A = 20\ kN$, $Y_B = -20\ kN$. As soon as the reactions are found we can successively equilibrate joints in which only two unknown forces occur.

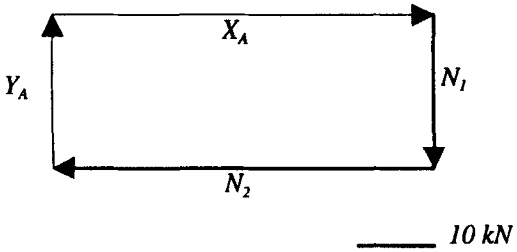


Figure 3.10: Equilibrium at joint *1*

The direction of arrows from the polygon determines the nature of the axial force. If a force is ***directed toward the joint***, the element is in ***compression*** and if it ***pulls away*** from the joint the element is ***in tension***. As each element by itself is a free body in equilibrium, the direction of the axial force has to be reversed on the opposite side of the element.

We can now proceed to joint *2*, where only two forces N_3 and N_4 are still unknown. Joint *3* at this moment can not be solved as there are still three unknown forces N_3, N_5 and N_6.

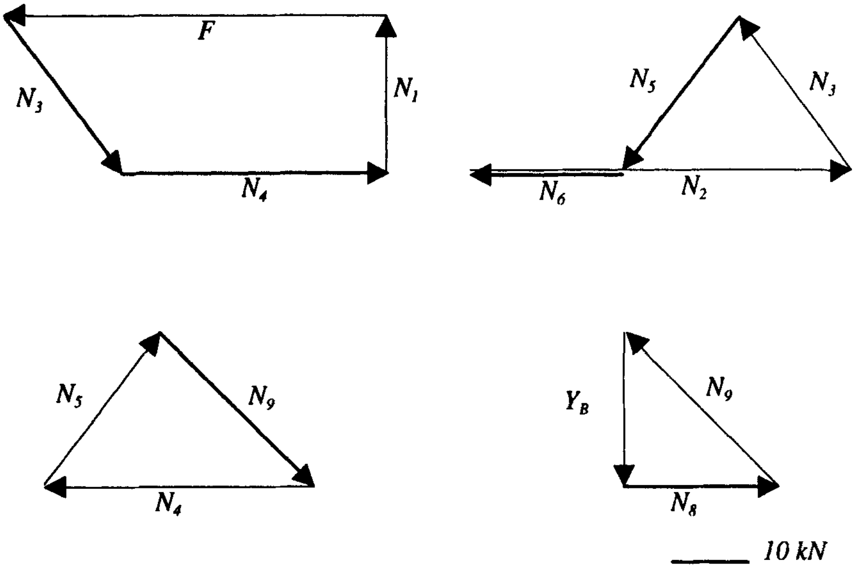


Figure 3.11: Equilibrium of joints *2*, *3*, *5* and *6*

At joint *4* forces N_6 and N_8 equilibrate each other but N_7 equals zero. In this way we can equilibrate all joints of the structure. The whole procedure can be drawn in a single diagram as in Fig. 3.12 (Maxwell's diagram), but the diagram becomes complex and hard to follow.

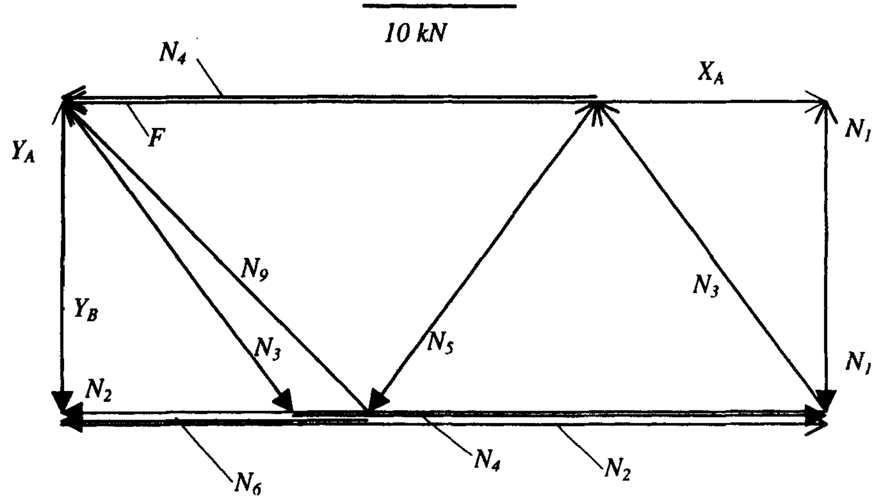


Figure 3.12: Maxwell's (Cremona's) diagram

The magnitudes of axial forces are scaled from the diagram and are as follows:

$N_1 = -20\ kN,\ N_2 = -50\ kN,\ N_3 = 25\ kN,\ N_4 = 35\ kN,$
$N_5 = -25\ kN,\ N_6 = -20\ kN,\ N_7 = 0\ kN,\ N_8 = -20\ kN$
$N_9 = 28\ kN.$

3.3.3.2 Analytical solutions

Let us at first calculate some basic geometrical relations from Fig. 3.8

$$tg\alpha = \frac{4}{3} \Rightarrow \alpha = 53.13^\circ;\ \sin\alpha = 0.800;\ \cos\alpha = 0.600$$

$$tg\beta = 1 \Rightarrow \beta = 45^\circ;\ \sin\beta = \cos\beta = 0.707$$

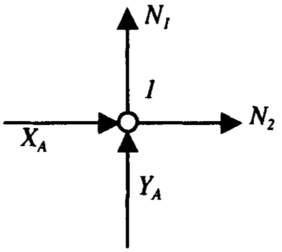
and then equilibrate the structure using the basic equations of equilibrium.

$$\Sigma M_A = 0\ :\ Y_B \cdot 10 + F \cdot 4 = 0 \ \Rightarrow\ Y_B = -0.4F = -20\,kN$$

$$\Sigma Y = 0\ :\ Y_A + Y_B = 0 \ \Rightarrow\ Y_A = +0.4F = 20\,kN$$

$$\Sigma X = 0\ :\ X_A - F = 0 \ \Rightarrow\ X_A = F = 50\,kN$$

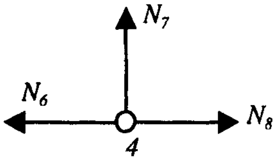
Forces in elements *1* and 2, connected at joint *1*, can be calculated using the ***projection method*** based on the principle of a free body. All forces in elements are supposed to act as positive tensile forces (i.e. away from a joint):



$$\sum X = 0:\ X_A + N_2 = 0 \Rightarrow N_2 = -F = -50\,kN\ (compression)$$

$$\sum Y = 0:\ Y_A + N_1 = 0 \Rightarrow N_1 = -0.4 \cdot F = -20\,kN\ \ (compression)$$

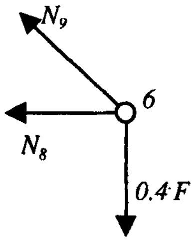
Using the projection method we can instantly conclude that force N_7 equals zero and that forces N_6 and N_8 are equal in magnitude:



$$\Sigma Y = 0:\ N_7 = 0$$

$$\Sigma X = 0: \mathrm{N}_8 - N_6 = 0 \ \Rightarrow\ N_8 = N_6 = -0.5657F = -28.285\,kN$$

Now we proceed to the last unsolved joint *6* (support *B*):



$$\Sigma X = 0:\ -N_8 - N_9 \cdot \cos\beta = 0$$

$$N_8 = -N_9 \cdot \cos\beta = -\frac{0.4 \cdot F \cdot \cos\beta}{\sin\beta} = \frac{-0.4 \cdot F}{tg\beta} = -0.4F = -20\ kN$$

$$\Sigma Y = 0:\ N_9 \sin\beta = 0.4 \cdot F \ \Rightarrow\ N_9 = 0.5657 \cdot F = 28.285\ kN$$

The method of joints is most effective when the forces in all the members of the truss are to be determined. If, however, the force in only one member or the forces in very few members are desired, another method, the *method of sections* (*Ritter's method*) is more efficient.

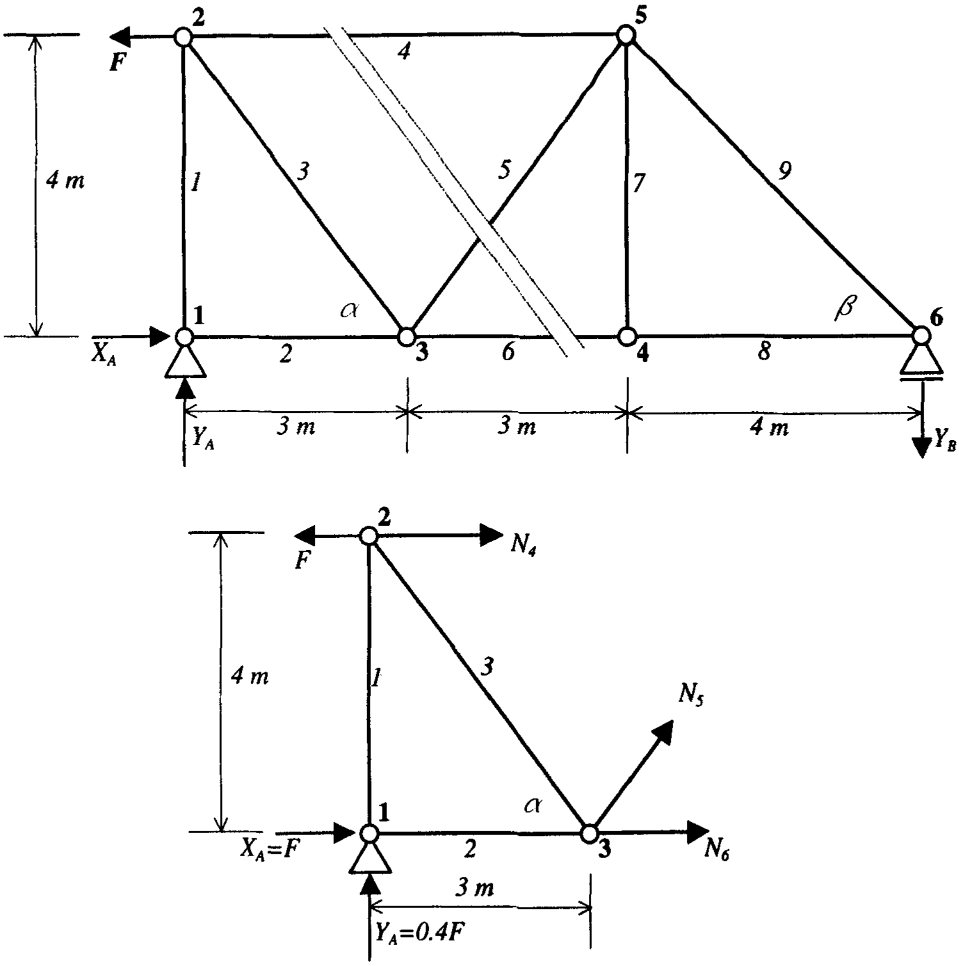


Figure 3.13: Method of sections (Ritter's method)

The method of sections is a process of analysing a truss as a series of independent but stable sections or free bodies. The structure is cut in a section such that in the section only three unknowns occur which can be determined by applying the moment equilibrium equation to the point where two unknowns meet (see Fig. 3.13).

$$\Sigma M_3 = 0$$

$$-N_4.4 + F \cdot 4 - (0.4 \cdot F) \cdot 3 = 0$$

$$N_4 = 0.7 \cdot F = 35.000 \quad kN \quad (Tension)$$

If two unknown forces are parallel then another section has to be chosen or one of the forces has to be calculated by any other of the methods.

Note that no independent check of the computation is available if only one force is to be calculated, therefore it is desirable to check calculated forces on a free body applying any other point of rotation.

If there are more than three unknown forces at the section then redundant forces have to be calculated by the kinematic method or projection method prior to the application of the method of sections.

3.3.4 Internally indeterminate trusses

An indeterminate truss is obtained if with a pure triangular truss with $f=0$ one or more elements are added. The method of solution is called Henneberg's method. All redundant elements are removed in such a way that the determinate truss remains stable.

The statically determinate truss is then solved by one of the methods explained above. At joints, where redundant elements were removed, we apply pairs of unknown forces $X_i = 1 (i = 1, n)$, n being the degree of static indeterminacy. The truss must to be solved for each of the redundant forces X_i.

The deformations of the truss are calculated by equation

$$\Delta = \int \frac{N\overline{N}}{EA} \cdot dx$$

and from equations of compatibility, unknown forces X_i are calculated. The method of superposition is applied to evaluate forces in all members (see example 6.5).

3.4 Differential relations on beams

3.4.1 Undeformed beam

Let us consider a *simply supported beam* loaded by the *uniform load* over the whole span of intensity $q = q_x$ = *constant* and a *differentially small part of length dx*, shown as free body in Fig. 3.14:

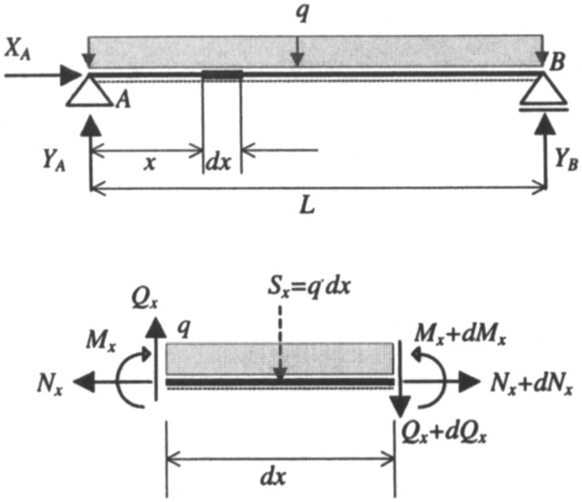


Figure 3.14: Simply supported beam

The equilibrium equation in the *x* direction is

$$\Sigma X = 0 \; : \; N_x - (N_x + dN_x) = 0 \, ,$$

from where $dN_x = 0$, since on length *dx* no additional force in the *x* direction exists. Consider now equilibrium in the *y* direction:

$$\Sigma Y = 0 \; : \; Q_x - q.dx - (Q_x + dQ_x) = 0$$

or

$$dQx = - q.dx$$

❖ *The change in shear force on the differential region equals the negative change in loading on the same region.*

The equation is divided by *dx*:

$$\frac{dQ_x}{dx} = -q \tag{3.11}$$

❖ *The differential of shear force is equal to the negative external loading.*

Consider now the sum of moments about point 2 on the differential body:

$$Q_x.dx + M_x - (q.dx)\cdot\frac{dx}{2} - (M_x + dM_x) = 0,$$

and hence

$$Q_x \cdot dx - q \cdot \frac{dx^2}{2} - dM_x = 0$$

If the *deformation is small*, quantities of second order (dx^2) can be neglected and we are considering *theory of first order*. Equation is simplified to:

$$\frac{dM_x}{dx} = Q_x \tag{3.12}$$

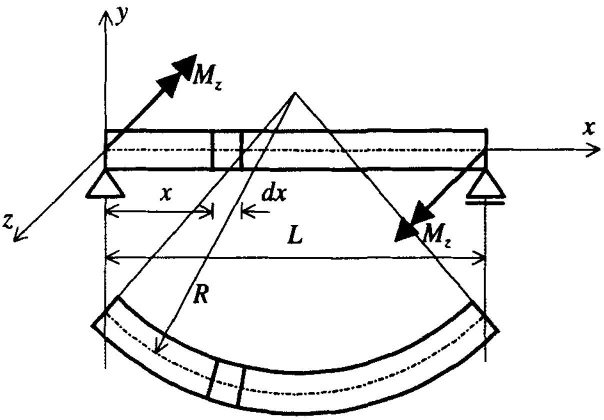
❖ *The differential of the bending moment is equal to the shear force.*

If equation (3.12) is differentiated again

$$\frac{d^2M_x}{dx^2} = \frac{dQ_x}{dx} = -q \tag{3.13}$$

❖ *The second derivative of the bending moment is equal to the negative external loading.*

3.4.2 Deformed beam



R = radius of curvature $\frac{1}{R}$ *= measure of curvature*

Figure 3.15: A beam under pure bending

The equation for the radius of curvature can be found in any mathematics text book and is given by:

$$\frac{1}{R} = \frac{y''}{\sqrt{\left(1 + y'^2\right)^3}} \tag{3.14}$$

If the second order quantities are neglected (y'^2) and the expression under the square root equal unity, the theory of first order only is considered:

$$\frac{1}{R} \cong \frac{d^2y}{dx^2} \tag{3.15}$$

Consider now the basic geometrical relations with regard to a *constant curvature 1/R, R* being measured from the centre of gravity of the cross section:

$$\frac{\Delta x}{y} = \frac{dx}{R} \quad \text{or} \quad \frac{\Delta x}{dx} = \frac{y}{R}, \tag{3.16}$$

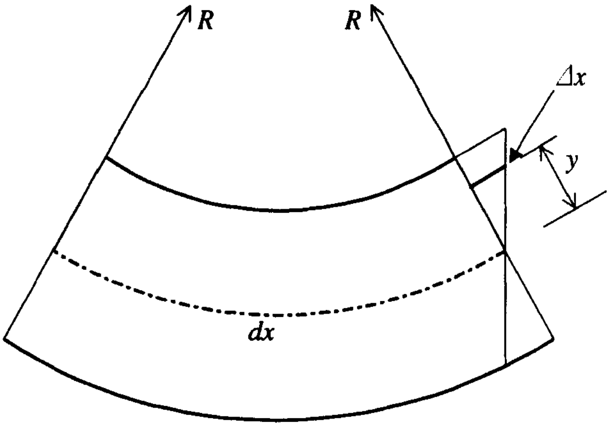


Figure 3.16: Deformed differential body

But as $\Delta L / L = \varepsilon$ it follows that

$$\frac{\Delta x}{dx} = \frac{y}{R} = \varepsilon_x$$

and after insertion of stresses due to Hook's law:

$$\sigma = \varepsilon \cdot E = \frac{y}{R} \cdot E$$

The stress is $\sigma = F / A$, which gives the differential force

$$dF = \frac{E}{R} \cdot y \cdot dA,$$

and integrated over the whole cross section gives the total force:

$$F = \frac{E}{R} \int y \cdot dA \tag{3.17}$$

From equilibrium in the x direction, the sum of all forces is zero i.e. $\Sigma X = 0$:

$$F = \frac{E}{R} \int y \cdot dA = 0 \tag{3.18}$$

From Eqn. (3.18) we can conclude, that equilibrium will take place when the resultant force in the x direction vanishes. Eqn. (3.18) also equals zero, when R is infinite, which is true when the beam is undeformed. This does not makes sense in our case.
It follows that the expression

$$\int y \cdot dA = 0$$

must be true. As the above expression is the *static moment about the x axis* which is zero for a symmetrical cross sections, it then follows that the *neutral axis must coincide with the centre of gravity of the section.* The above statement is true for a simple cross section otherwise we have to consider the centre of torsion or shear centre of a cross section.
The moment equilibrium equation is

$$dM_x = dF \cdot y = \frac{E}{R} \cdot y \cdot dA \cdot y = \frac{E}{R} \cdot y^2 \cdot dA$$

which after integration gives the relation to the bending moment

$$M_x = \frac{E}{R} \int y^2 dA \tag{3.19}$$

It is known that the expression $\int y^2 dA$ is a second moment of inertia of a cross section I, hence:

$$M_x = \frac{EI}{R} \quad \Rightarrow \quad \frac{1}{R} = \frac{M_x}{EI} = \frac{d^2 y}{dx^2} \tag{3.20}$$

and finally

$$\frac{d^2 y}{dx^2} = \frac{M_x}{EI} \tag{3.21}$$

which is a *basic differential equation of pure bending* also known as the moment-curvature relation.

- *The second derivative of the deflected shape is a bending moment, reduced by bending stiffness EI.*

It is now straightforward that further derivation gives shear forces and loads. The third derivative of a deflection curve gives the shear force

$$\mathrm{EI}\,\frac{d^3y}{dx^3}=Q$$

and the fourth derivative of a deflection curve gives external loading:

$$\mathrm{EI}\,\frac{d^4y}{dx^4}=-q$$

If the equation of pure bending is given then after first integration along the beam axis, an expression dy/dx arises which of course is a *tangent to the deflection curve* or *rotation.*

It has to be emphasised that loading, shear force and bending moments are independent of bending stiffness, but on the other hand, rotations and deflections depend on the *bending stiffness EI.*

Differential relations between loading, shear forces, bending moments, rotations and deflections are shown in the following table:

Loading	*Load function* $f(x)$	*Shear force* $Q(x)$	*Bending moment* $M(x)$	*Rotation* $EI\varphi$	*Deflection* EIy
Concentrated force	$\langle x^{-1}\rangle$	x^0	x^1	x^2	x^3
Uniform	x^0	x^1	x^2	x^3	x^4
Linear	x^1	x^2	x^3	x^4	x^5
Parabola[♣]	x^2	x^3	x^4	x^5	x^6
Periodic	*sin* *cos*	*-cos* *sin*	*-sin* *-cos*	*cos* *-sin*	*sin* *cos*

Concentrated forces and moments are the only loadings where a sudden change occurs in functions of shear forces or bending moments. All other loadings produce smooth and continuous functions.

As can be concluded from the above derivation the basic differential relations between loading, shear force and bending moment enable easy physical interpretation between these quantities. In this way from one known diagram of internal forces another two can easily be reconstructed in sense but not in magnitude. The magnitude of such quantities can be determined with the application of boundary conditions.

3.4.3 Integration of a load function

Let the load function be $q(x) = q = constant$ along the whole length of the simply supported beam as shown in Fig. 3.17. From equation (3.11):

[♣] *Parabolic loading occurs in pre-stressed beams, as the cable position is a square function of longitudinal distance. It will also be used in the deformation determination by Mohr's method.*

$$\frac{dQ_x}{dx} = -q(x)$$

$$\int dQ_x = \int -q dx$$

$$Q_x = -q \cdot x + C_1$$

The constant C_1 is determined from boundary conditions[♣] on the beam i.e. from known values for the shear force:

at $x = 0 \;\Rightarrow\; Q_x = Y_A$
at $x = L \;\Rightarrow\; Q_x = -Y_B$

Inserting values at $x = 0$:

$$Q_x = -q \cdot 0 + C_1 \;\Rightarrow\; Q_x = C_1 \;\Rightarrow\; C_1 = Y_A$$

The constant C_1 is now inserted into the shear force equation such that:

$$Q_x = -q \cdot x + Y_A = \frac{q \cdot L}{2} - q \cdot x \tag{3.22}$$

Using a direct integration of a load function with the use of known boundary conditions we derived the *equation of shear forces at any section.* It can be seen that shear forces changes linearly.
Now we integrate again:

$$M_x = \int \left(q \cdot \frac{L}{2} - q \cdot x \right) dx + C_2$$

$$M_x = \frac{q \cdot L \cdot x}{2} - \frac{q \cdot x^2}{2} + C_2$$

Boundary conditions: at $x = 0 \;\Rightarrow\; M = 0$
at $x = L \;\Rightarrow\; M = 0$

at $x = 0 \;\Rightarrow\; C_2 = 0$

[♣] *Boundary condition represents a known value at a certain point, usually we choose a point where a value of the function is zero.*

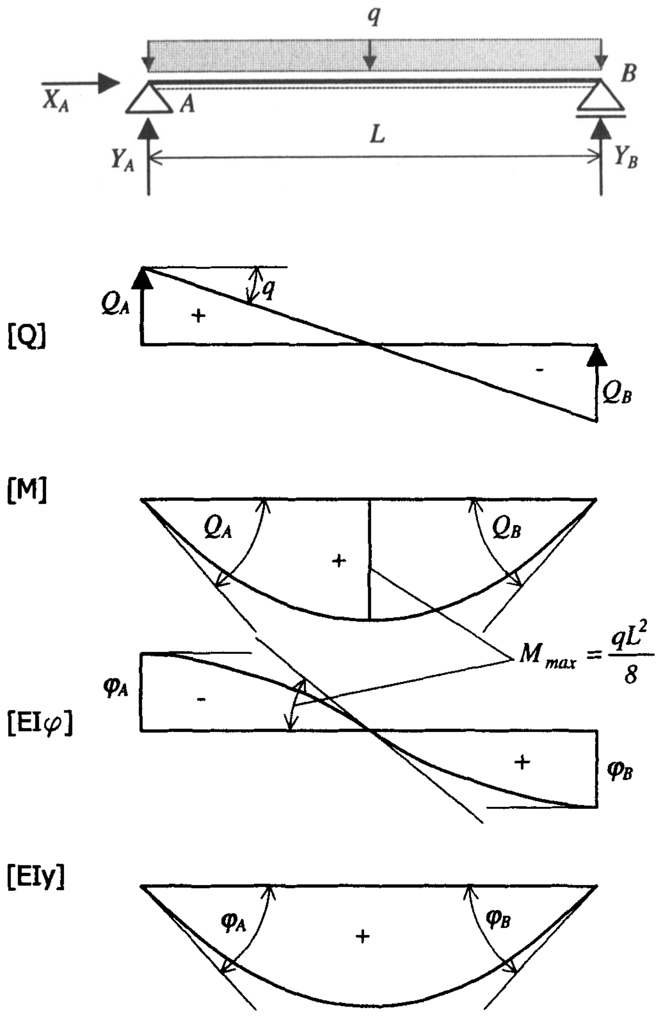


Figure 3.17: Graphical representation of integration

The equation for the bending moments is now

$$M_x = \frac{q \cdot L}{2} x - \frac{q \cdot x^2}{2}, \tag{3.23}$$

from where values of M_x can be calculated at any point:

$$\text{at } x = \frac{L}{4} \Rightarrow M_x = \frac{q \cdot L \cdot L}{2 \cdot 4} - \frac{q \cdot L^2}{2 \cdot 16} \Rightarrow M_x = \frac{3 \cdot q \cdot L^2}{32}$$

$$\text{at } x=\frac{L}{2} \Rightarrow M_x=\frac{q\cdot L\cdot L}{2\cdot 2}-\frac{q\cdot L^2}{2\cdot 4} \Rightarrow M_x=\frac{q\cdot L^2}{8}$$

$$\text{at } x=\frac{3L}{4} \Rightarrow M_x=\frac{q\cdot L\cdot 3L}{2\cdot 4}-\frac{q}{2}\cdot\frac{q\cdot L^2}{16} \Rightarrow M_x=\frac{3\cdot q\cdot L^2}{32}$$

Integrating the moment equation:

$$\frac{d\varphi}{dx}EI=-M_x$$

$$d\varphi\cdot EI=-M_x\cdot dx=\left(\frac{q\cdot x^2}{2}-\frac{q\cdot L\cdot x}{2}\right)dx$$

$$\varphi\cdot EI=\frac{q\cdot x^3}{6}-\frac{q\cdot L\cdot x^2}{4}+C_3$$

Boundary conditions: $\text{at } x=\frac{L}{2} \Rightarrow \varphi=0 \Rightarrow 0=\frac{q\cdot L^3}{6\cdot 8}-\frac{q\cdot L\cdot L^2}{4\cdot 4}+C_3$

$$C_3=\frac{q\cdot L^3}{24}$$

The constant C_3 is inserted into equation of rotation:

$$\varphi\cdot EI=\frac{q\cdot x^3}{6}-\frac{q\cdot L\cdot x^2}{4}+\frac{q\cdot L^3}{24} \tag{3.24}$$

and the values of rotation at supports A and B are evaluated:

$$\text{at } x=0 \Rightarrow \varphi_A=\frac{q\cdot L^3}{24\cdot EI}$$

$$\text{at } x=L \Rightarrow \varphi_B=-\frac{q\cdot L^3}{24\cdot EI}$$

Integrating the rotation:

$$EI\cdot\frac{dy}{dx}=\varphi$$

$$\mathrm{EI}\, y = \frac{q \cdot x^4}{6 \cdot 4} - \frac{q \cdot L \cdot x^3}{4 \cdot 3} + \frac{q \cdot L^3 \cdot x}{24} + C_4$$

Boundary conditions: at $x = 0 \Rightarrow y = 0$

at $x = L \Rightarrow y = 0 \ , \ C_4 = 0$

The equation of the deflection curve is:

$$EI \cdot y = \frac{q \cdot x^4}{24} - \frac{q \cdot L \cdot x^3}{12} + \frac{q \cdot L^3 \cdot x}{24} \tag{3.25}$$

The maximum deflection of a simply supported beam is at the point of zero rotation (as the rotation is a derivative of the deflection) that is at $x = L/2$:

$$y_{max} = \frac{5qL^4}{384EI} \tag{3.26}$$

3.5 Analysis of determinate beams

3.5.1 Beams with straight axes

Example 3.2: Simply supported beam

Consider the beam from Fig. 3.18, loaded by the uniform load of intensity q, a *chosen positive side of the beam* is marked by a dotted line. The positive side of a beam is an *arbitrary side* but usually it is a *bottom side*, as under gravitational loads a tension occurs at that side.

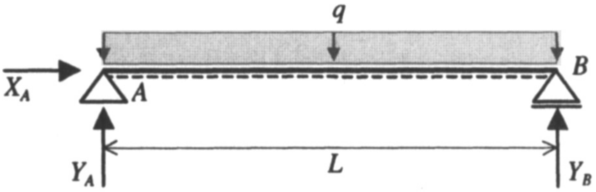


Figure 3.18: Simply supported beam

The beam is at first equilibrated. Write down the equilibrium equations on the free body at which displacements are prevented by reaction forces X_A, Y_A and Y_B, which are supposed to act in the positive directions of corresponding co-ordinate axes. From the sum of forces in the x and y directions:

$$\Sigma X = 0: \ X_A = 0$$

$$\Sigma Y = 0: \; Y_A + Y_B - q.L = 0$$

The third equilibrium equation is needed for the determination of unknowns. The moment equation can be written for any point in the structure:

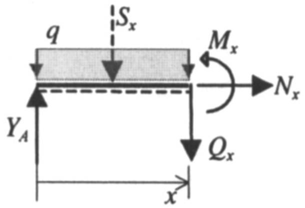
$$\Sigma M_A = 0: \qquad Y_B \cdot L - (q.L) \cdot \frac{L}{2} = 0,$$

from which:

$$Y_B = \frac{qL}{2}$$

and then from the second equation $Y_A = Y_B = \frac{qL}{2}$.

Internal forces at an arbitrary section at distance x are determined in a way that the beam is cut into two free bodies at the distance x. As originally the beam was at this section connected together by internal forces, these internal forces are now applied as external forces on the free bodies.



Internal forces Q_x and M_x are now determined from equilibrium equations:

$$Y_A - q \cdot x - Q_x = 0$$

$$Y_A \cdot x - (q \cdot x) \cdot \frac{x}{2} - M_x = 0$$

hence

$$Q_x = Y_A - q \cdot x$$

$$M_x = Y_A \cdot x - (q \cdot x) \cdot \frac{x}{2} = \frac{qL}{2} \cdot x - \frac{qx^2}{2}$$

As seen from the above equations, the shear forces change linearly while the bending moments follow a quadratic parabola. Both functions are now drawn in the diagrams of shear forces [Q] and bending moments [M].

The maximum bending moment is at the point where the shear force is zero:

$$Q_x = Y_A - q.x = 0 \qquad \Rightarrow \qquad x_{max} = \frac{Y_A}{q} \tag{3.27}$$

which is inserted into the equation of bending moments

$$M_{max} = Y_A \cdot x_{max} - \frac{q \cdot x_{max}^2}{2} = \frac{Y_A^2}{2 \cdot q} \tag{3.28}$$

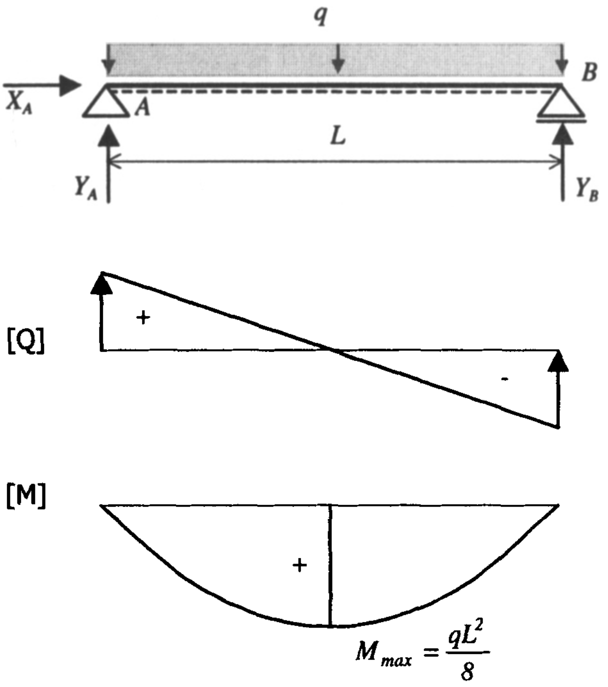


Figure 3.19: Diagrams of internal forces

Example 3.3: Cantilever beam (Fig. 3.20)

From the sum of forces in the *x* and *y* directions

$$\Sigma X = 0: \qquad X_A = 0$$

$$\Sigma Y = 0: \quad Y_A - q \cdot L = 0 \quad Y_A = q \cdot L$$

followed by the sum of moments about support *A*

$$\Sigma M_A = 0: \qquad M_A + (q.L) \cdot \frac{L}{2} = 0$$

$$M_A = -(q.L) \cdot \frac{L}{2} = -\frac{q \cdot L^2}{2}$$

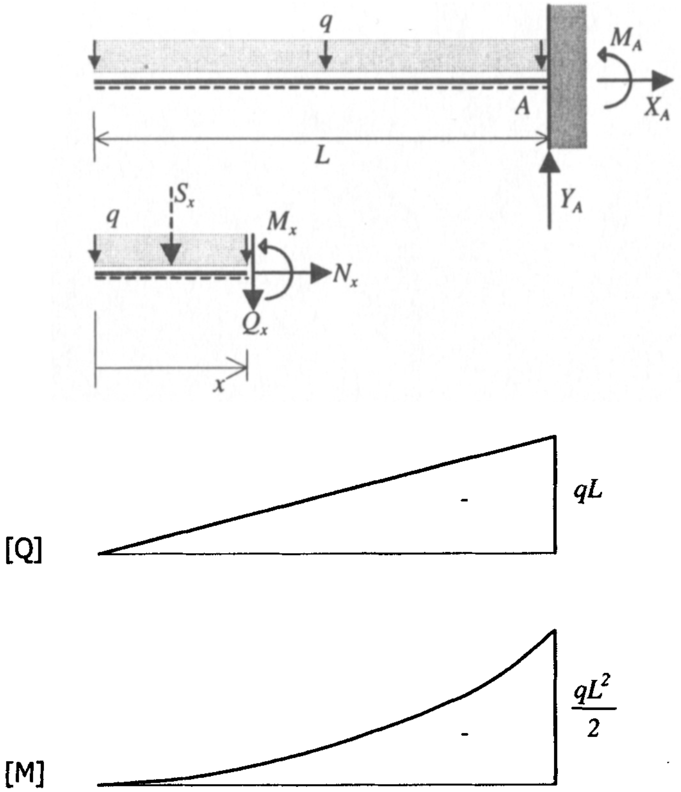


Figure 3.20: Cantilever beam and diagrams of internal forces

From the equilibrium equations related to the free body, equations of shear forces Q_x and bending moments M_x are derived

$$Q_x = -q \cdot x \qquad M_x = -\frac{q \cdot x^2}{2}$$

and then drawn as diagrams of shear forces [Q] and bending moments [M] in Fig. 3.20.

3.5.2 Beams with bent axes

Since the given structure is not a horizontal beam as discused earlier, it is necessary to introduce a *local co-ordinate systems* on the inclined parts of the structure.

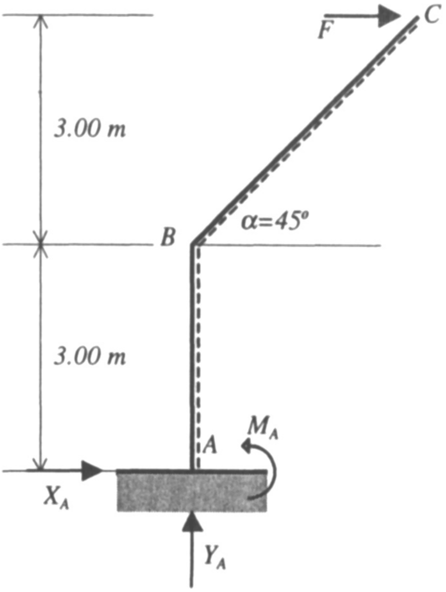


Figure 3.21: Beam with bent axis

From the equilibrium equations all reactions are calculated: $X_A = -F$, $Y_A = 0$ and $M_A = 6F$. The beam is cut and a *local co-ordinate system* is chosen in such a way that it *corresponds* to *the chosen positive side of the beam.*

$$N_x = F \cdot cos\alpha$$

$$Q_x = F \cdot sin\alpha \tag{3.29}$$

$$M_x = F \cdot sin\alpha \cdot x$$

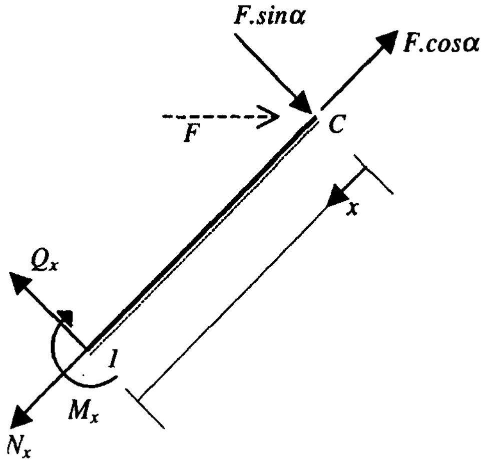


Figure 3.22: Free body – beam with bent axis

Diagrams of internal forces are drawn perpendicular to the beam axis as shown in Fig. 3.23.

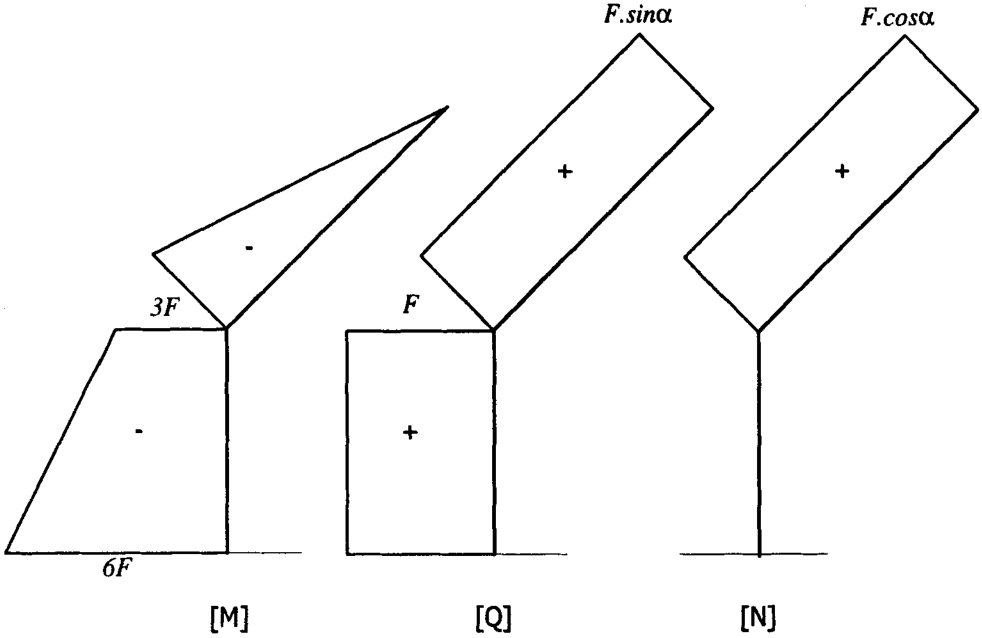


Figure 3.23: Diagrams of internal forces

3.5.3 Beams with curved axis

Example 3.3: Three-hinged semicircular arch

Three-hinged arch is a determinate structure though four reactions occur at the supports but an additional independent fourth moment equation can be written for a hinge at point *B* of the arch.

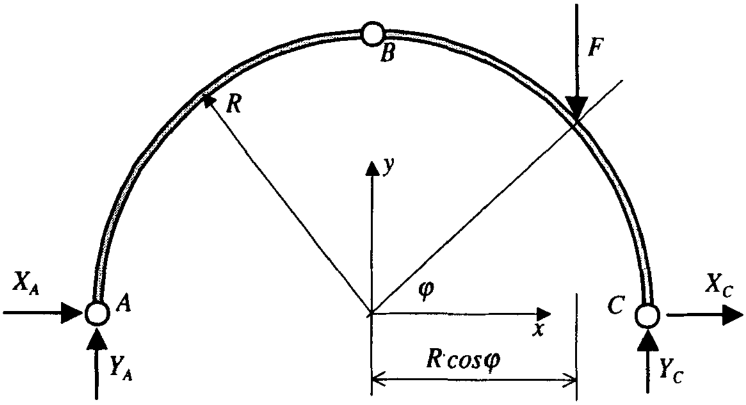


Figure 3.24: Three-hinged semicircular arch as a free body

From equilibrium of the entire free body:

$$\Sigma M_A = 0 \quad : \quad -F \cdot (R + R \cdot \cos\varphi) + Y_C \cdot 2 \cdot R = 0$$

$$Y_C = \frac{F \cdot R \cdot (1 + \cos\varphi)}{2 \cdot R} = \frac{F}{2} \cdot (1 + \cos\varphi)$$

In the case of $\varphi = \pi / 2$, that is when force *F* acts in the middle of an arch, so $\cos\varphi = 0$ and the reaction is $Y_C = F / 2$ as expected.

$$\Sigma Y = 0 \quad Y_A + Y_C = F \quad \Rightarrow \quad Y_A = F - \frac{F}{2} \cdot (1 + \cos\varphi) = \frac{F}{2} \cdot (1 - \cos\varphi)$$

The sum of moments on the free body *BC* about point *B* gives:

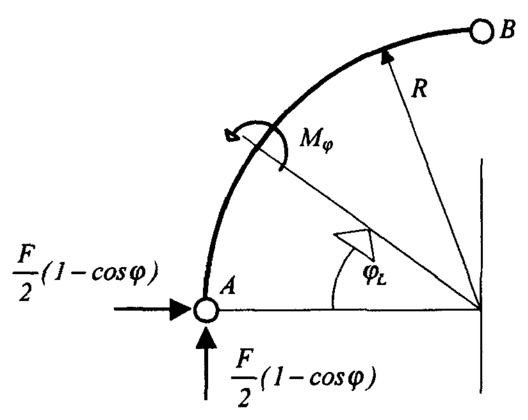
$$\Sigma M_B = 0: \quad Y_C \cdot R + X_C \cdot R - F \cdot R \cdot \cos\varphi = 0$$

$$X_C = -\frac{F}{2}\cdot\left(1-\cos\varphi\right) \qquad if \quad \varphi \le \frac{\pi}{2}$$

$$\Sigma X = 0: \quad X_A = -X_C = \frac{F}{2}\cdot\left(1-\cos\varphi\right)$$

Internal forces are calculated for the position of force F at $\varphi = 45^0$:

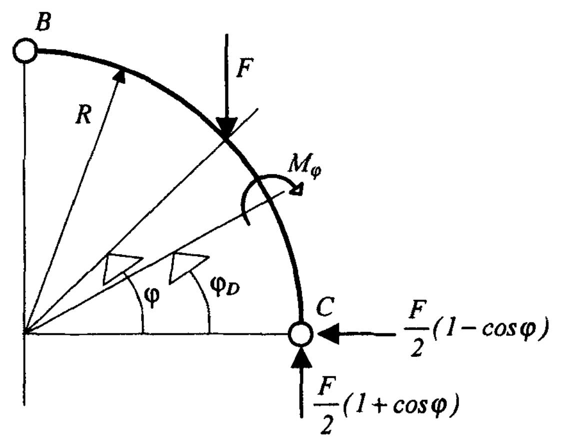
a) Free body AB (Figure 3.25a)



φ_L	M_φ
0	0
15	-0.033·FR
30	-0.054·FR
45	-0.061·FR
60	-0.054·FR
75	-0.033·FR
90	0

$$M_\varphi = \frac{F}{2}\cdot R\cdot\left(1-\cos\varphi\right)\cdot\left[1-\cos\varphi_L-\sin\varphi_L\right]$$

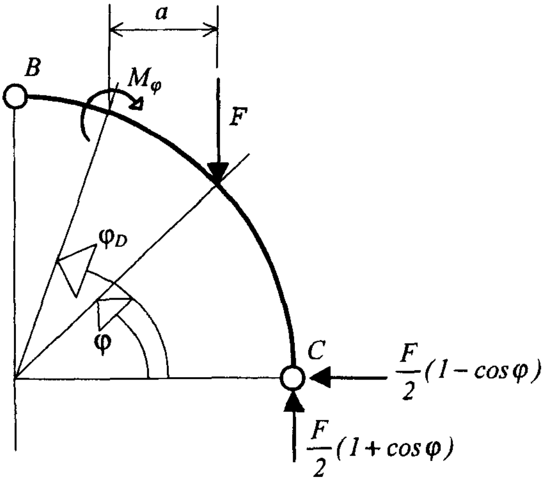
b.) Free body from point C to force F (Figure 3.25b)



$\varphi_D<\varphi=45°$	M_φ
0	0
5	-0.010·FR
10	-0.012·FR
20	-0.001·FR
30	0.041·FR
40	0.106·FR
45	0.146·FR

$$M_\varphi = \frac{F\cdot R}{2}\cdot\left[\left(1+\cos\varphi\right)\cdot\left(1-\cos\varphi\right)-\left(1-\cos\varphi\right)\cdot\sin\varphi_D\right]$$

c) Free body from force F to the point B (Figure 3.25c)



$\varphi_D > \varphi$	M_φ
45	0.146 FR
50	0.128 FR
60	0.093 FR
70	0.059 FR
80	0.028 FR
90	0

$$a = R\cdot(1-\cos\varphi_D) - R\cdot(1-\cos\varphi) = R\cdot(\cos\varphi - \cos\varphi_D)$$

$$M_\varphi = \frac{F}{2}\cdot(1+\cos\varphi) - R\cdot(1-\cos\varphi_D) - \\ -\frac{F}{2}\cdot(1-\cos\varphi)\cdot R\cdot\sin\varphi_D - F\cdot R\cdot(\cos\varphi - \cos\varphi_D)$$

The internal forces diagrams are drawn in *polar co-ordinates* i.e. in the radial direction or always *perpendicular to the axis.*

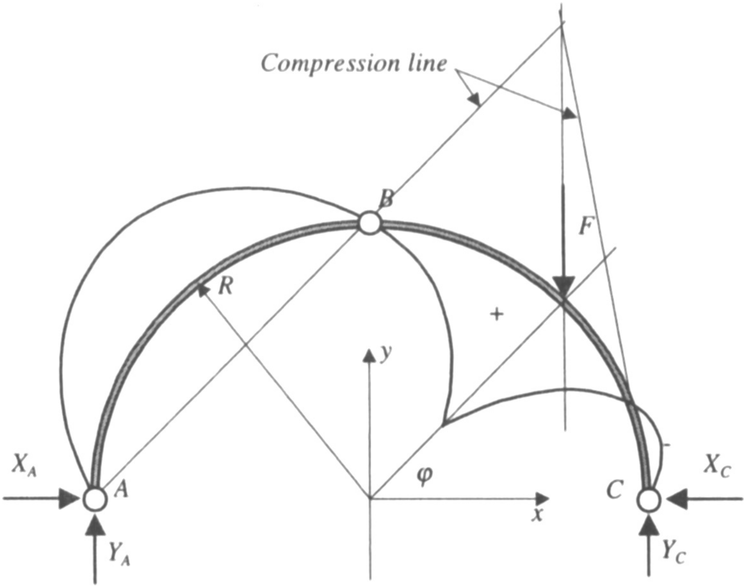


Figure 3.26: Moment diagram for a concentrated force F

Note from Fig. 3.26 that all points of the structure *inside the compression line are under positive* bending moments and *outside the compression line under negative bending moments* (*Note*: Imagine that if a structure would have a shape of a compression line, then no bending moments in a structure would occur).

Calculate reactions and internal forces for self-weight of the structure, specific weight is *200 kg* per unit length ($q = 200\ \ kg \cdot 10\, m \cdot s^{-2}\ = 2\ \ kN/m$).

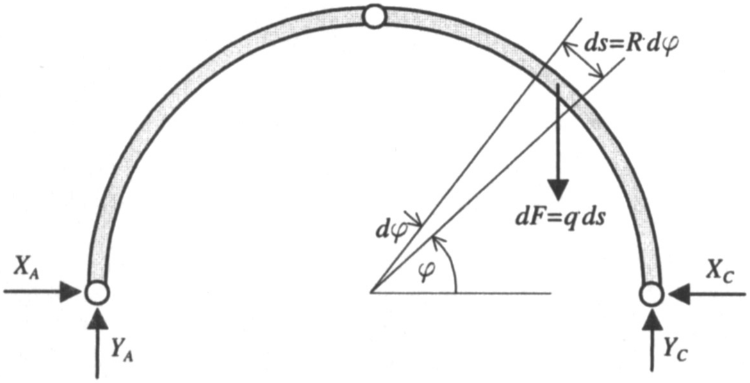


Figure 3.27: Three-hinged arch loaded by self-weight

Let us at first determine the differential quantities of length *ds* and force *dF*:

$$dF = q \cdot R \cdot d\varphi$$

$$ds = R \cdot d\varphi$$

The total force is given by integration:

$$F = \int_0^{\pi} q \cdot R \cdot d\varphi = q \cdot R \cdot \pi$$

Reactions are determined by summing reactions of *dF* over the entire body:

$$Y_C = \int_0^{\pi} \frac{q \cdot R \cdot d\varphi}{2} \cdot (1 + \cos\varphi) = \frac{q}{2} \cdot R \cdot \left(\int_0^{\pi} d\varphi + \int_0^{\pi} \cos\varphi \cdot d\varphi \right) = \frac{q \cdot R \cdot \pi}{2}$$

$$X_A = 2 \cdot \int_0^{\frac{\pi}{2}} \frac{q \cdot R \cdot d\varphi}{2} \cdot (1 - \cos\varphi)$$

$$X_A = \frac{q \cdot R \cdot \pi}{2} - \frac{q \cdot R}{2} \cdot \int_0^{\frac{\pi}{2}} \cos\varphi \cdot d\varphi = q \cdot R \cdot \left(\frac{\pi}{2} - 1 \right)$$

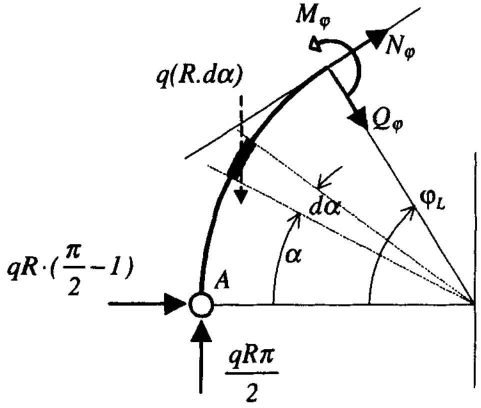


Figure 3.28: Three-hinged arch - self-weight free body

a.) Shear forces:

$$\begin{aligned}&-\left(\frac{\pi}{2}-1\right)\cdot q\cdot R\cdot cos\varphi+\frac{\pi}{2}\cdot q\cdot R\cdot sin\varphi-\int_{\alpha=0}^{\varphi} q\cdot R\cdot d\alpha\cdot sin\varphi+Q_X=0\\&Q_X=q\cdot R\cdot\left[\left(\frac{\pi}{2}-1\right)\cdot cos\varphi+\left(\varphi-\frac{\pi}{2}\right)\cdot sin\varphi\right]\end{aligned}\tag{3.30}$$

b.) Axial forces:

$$\begin{aligned}&\left(\frac{\pi}{2}-1\right)\cdot q\cdot R\cdot sin\varphi+\frac{\pi}{2}\cdot q\cdot R\cdot cos\varphi-\int_{\alpha=0}^{\varphi} q\cdot R\cdot d\varphi\, cos\varphi\cdot+N_X=0\\&N_X=q\cdot R\cdot\left[\left(\varphi-\frac{\pi}{2}\right)\cdot cos\varphi-\left(\frac{\pi}{2}-1\right)\cdot sin\varphi\right]\end{aligned}\tag{3.31}$$

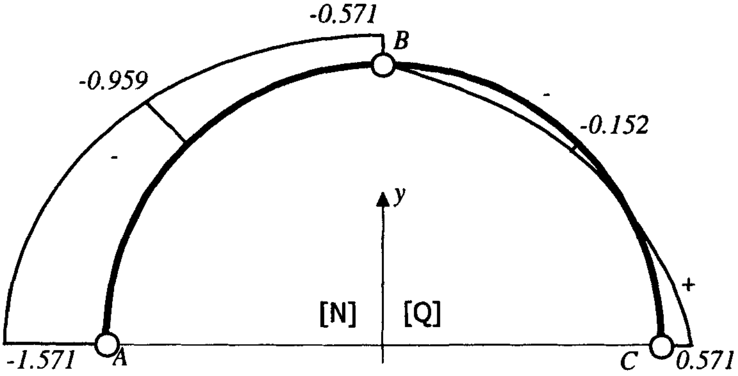


Figure 3.29: Axial and shear forces for self-weight (*qR)

c.) Bending moments:

$$\left(\frac{\pi}{2}-1\right)\cdot q\cdot R\cdot R\cdot sin\varphi-\frac{\pi}{2}q\cdot R^2\cdot(1-cos\varphi)+$$

$$+\int_{\alpha=0}^{\varphi} q\cdot R\cdot d\alpha\cdot R\cdot(cos\alpha-cos\varphi)+M_x=0$$

$$M_X = q \cdot R^2 \cdot \left[\frac{\pi}{2} \cdot (1 - \sin\varphi) + \left(\varphi - \frac{\pi}{2} \right) \cdot \cos\varphi \right] \tag{3.32}$$

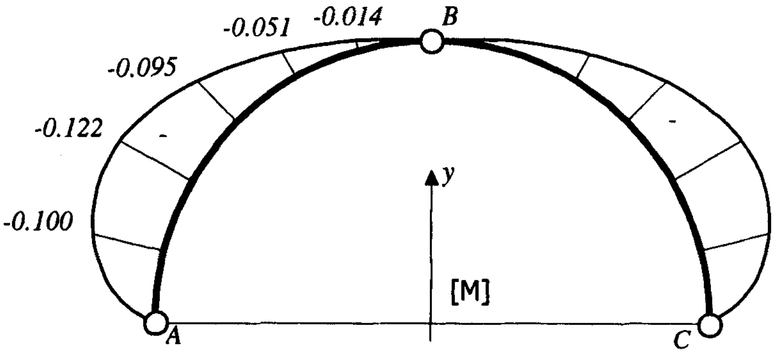


Figure 3.30: Bending moments for self-weight ($*qR^2$)

3.6 Statically determined structures in space

Space structures in general have complicated geometries and the use of vector algebra is unavoidable for the determination of internal forces and reactions. In the general case, six internal force components act at any section of the member. The calculation can be divided into two steps:

- Geometry calculation and
- Equilibrium calculation

3.6.1 Geometry calculation

At first we have to define the geometry of element axes and the principal section axes at which internal forces are required. Consider a member whose axis is defined by the parametric equations:

$$x = f_1(\varphi)$$

$$y = f_2(\varphi) \tag{3.33}$$

$$z = f_3(\varphi)$$

The unit vectors along the *x*, *y* and *z* axes are denoted by *i*, *j* and *k*. The section on which the forces are required is cut and the principal axes are defined by *N* for the axial (normal) force, *S* for the strong and *W* for the weak bending axes. The corresponding unit vectors are *n*, *s* and *w* (Figure 3.31).
The unit normal vector is determined by equation:

$$n = \frac{(dx)i + (dy)j + (dz)k}{ds} = \frac{(dx)i + (dy)j + (dz)k}{\sqrt{(dx)^2 + (dy)^2 + (dz)^2}} \tag{3.34}$$

Let us define the principal bending axes. If the strong bending axis is parallel to the *xy* plane (or horizontal, as is the case in gravitational loads), then this axis is normal to the *N* and *Z* axes, the unit vector is found from equation:

$$s = \frac{n \times k}{|n \times k|} \tag{3.35}$$

or by words: the cross product must be divided by its absolute value to obtain the unit vector. The weak bending axis is normal to the *N* and *S* and its unit vector is:

$$w = n \times s \tag{3.36}$$

Since both *n* and *s* are unit vectors their cross product is also a unit vector.

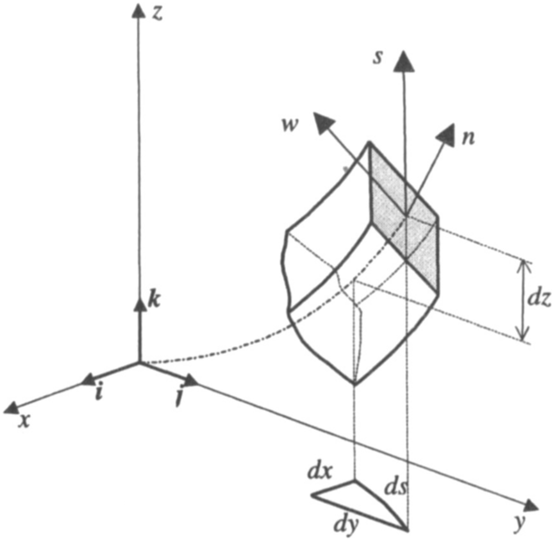


Figure 3.31: Curved beam in space

3.6.2 Equilibrium

Consider the free body of Fig. 3.32. Force *F* is applied at point *A* and produces a resultant force vector F_R and resultant moment M_R, which can be found from the vector equilibrium equations:

$$\Sigma F = 0: \quad F + F_R = 0 \quad \Rightarrow \quad F_R = F$$

$$\Sigma M_B = 0: \quad L \times F_R + M_R = 0 \quad \Rightarrow \quad M_R = -L \times F_R, \qquad (3.37)$$

L is the lever arm from the point of the force *F* applied to a chosen point *B*.

Axial and shear forces *N*, Q_S and Q_W are the components of the resultant force F_R along the *n*, *s* and *w* axes and can be found by a dot multiplication with corresponding unit vectors:

$$N = F_R \cdot n$$

$$Q_S = F_R \cdot s \tag{3.38}$$

$$Q_W = F_R \cdot w$$

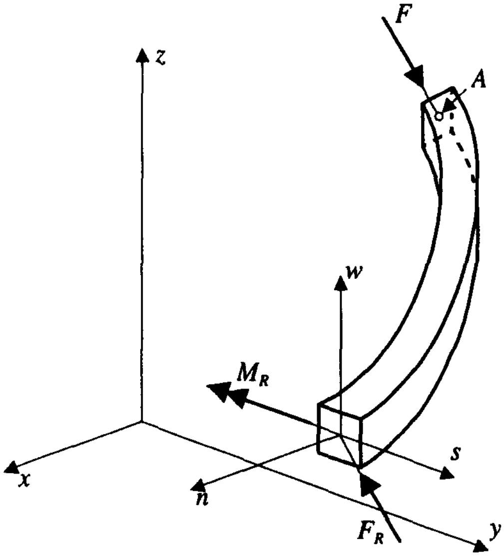


Figure 3.32: Definition of axes on a free body

The torsion and bending moments T, M_S and M_W are the components of the resultant moment M_R along the corresponding axes:

$$T = M_R \cdot n$$

$$M_S = M_R \cdot s \tag{3.39}$$

$$M_W = M_R \cdot w$$

When all internal forces are determined, deflections and stresses can be calculated.

Example 3.4: Determine all six internal forces along a helicoidal stairs beam due to the vertical force F applied at the top of the stairs at point A.

At first we determine the beam geometry. The helix geometry is given in terms of angle φ by the parametric equations:

$$x = R \cdot cos\varphi \qquad y = R \cdot sin\varphi \qquad z = \frac{H}{\pi} \cdot \varphi \tag{3.40}$$

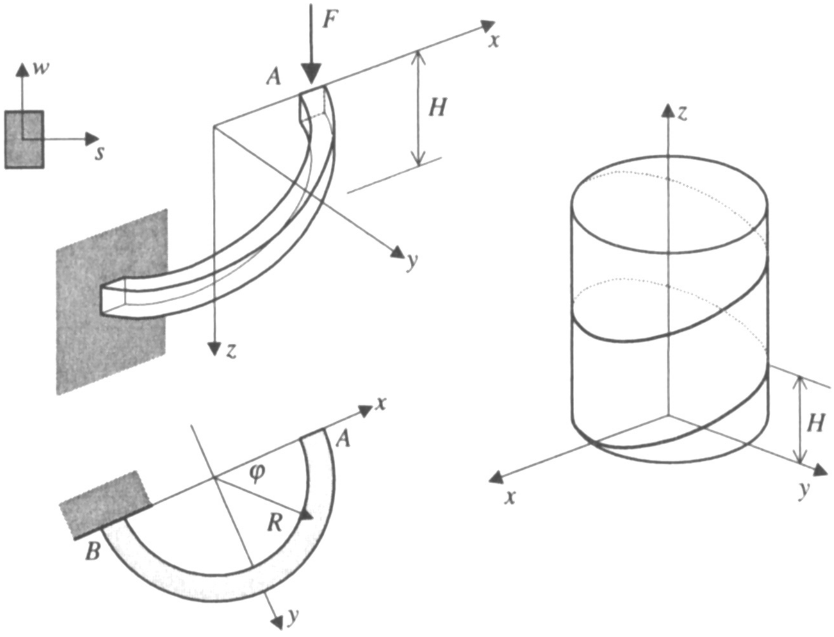


Figure 3.33: Helicoidal stairs beam

To find the normal unit vector n we need the derivatives of the parametric equations

$$\begin{aligned} dx &= -R \cdot sin\varphi \cdot d\varphi \\ dy &= R \cdot cos\varphi \cdot d\varphi \\ dz &= \frac{H}{\pi} \cdot d\varphi \end{aligned} \tag{3.41}$$

which are inserted into the equation of the normal

$$n=\frac{-i\cdot R\cdot sin\varphi\cdot d\varphi+j\cdot R\cdot cos\varphi\cdot d\varphi+\frac{H}{\pi}\cdot d\varphi}{\sqrt{R^2\cdot sin^2\varphi\cdot d^2\varphi+R^2\cdot cos^2\varphi\cdot d^2\varphi+\left(\frac{H}{\pi}\right)^2\cdot d^2\varphi}}$$

$$n=\frac{1}{\sqrt{1+\left(\frac{H}{\pi\cdot R}\right)^2}}\cdot\left[-i\cdot sin\varphi+j\cdot cos\varphi+k\cdot\frac{H}{\pi\cdot R}\right] \tag{3.42}$$

The denominator depends on helix properties only and will be a constant for a given geometry and hence denoted by K. The strong bending axis lies in a horizontal plane and can thus be found by a cross product

$$n\times k=\frac{1}{K}\cdot\begin{vmatrix} i & j & k \\ -sin\varphi & cos\varphi & \frac{H}{\pi\cdot R} \\ 0 & 0 & 1\end{vmatrix}=\frac{1}{K}\cdot(i\cdot cos\varphi+j\cdot sin\varphi) \tag{3.43}$$

The absolute value of this product is

$$|n\times k|=\frac{1}{K}\cdot(cos^2\varphi+sin^2\varphi)=\frac{1}{K},$$

hence

$$s=\frac{n\times k}{|n\times k|}=i\cdot cos\varphi+j\cdot sin\varphi \tag{3.44}$$

and finally the unit vector along the weak axis is in a similar way:

$$w=s\times n=\frac{1}{K}\cdot\begin{vmatrix} i & j & k \\ cos\varphi & sin\varphi & 0 \\ -sin\varphi & cos\varphi & \frac{H}{\pi\cdot R}\end{vmatrix}=\frac{1}{K}\cdot\left(i\cdot\frac{H}{\pi\cdot R}\cdot sin\varphi-j\cdot\frac{H}{\pi\cdot R}\cdot cos\varphi+k\right)$$

From the equilibrium equations at an arbitrary section the resultant force F_R and moment M_R can be found:

$$\Sigma F=0 \qquad \Rightarrow \qquad F_R=-k\cdot F$$

$$\Sigma M_B = 0 \qquad \Rightarrow \qquad M_R = -L \times F$$

using the lever arm:

$$L = i \cdot R \cdot (1 - \cos\varphi) - j \cdot R \cdot \sin\varphi - k \cdot \frac{H}{\pi} \cdot \varphi \tag{3.45}$$

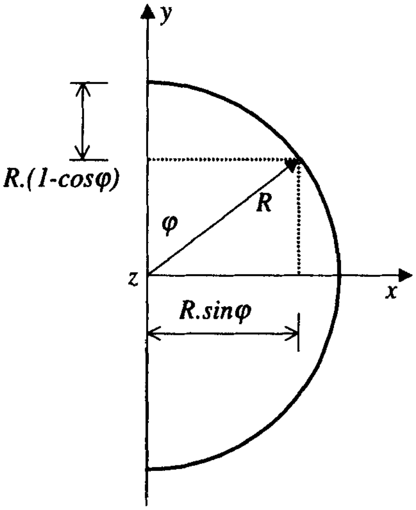


Figure 3.34: A view from *z*-axis

The resultant moment is:

$$L \times F = M_R = \begin{vmatrix} i & j & k \\ R \cdot (1 - \cos\varphi) & R \cdot \sin\varphi & \frac{H}{\pi} \cdot \varphi \\ 0 & 0 & F \end{vmatrix} =$$

$$= i \cdot R \cdot \sin\varphi \cdot F + j \cdot (R \cdot (1 - \cos\varphi)) \cdot F + 0 = \tag{3.46}$$

$$= F \cdot R \cdot [i \cdot \sin\varphi + j \cdot (1 - \cos\varphi)]$$

As the resultant forces F_R and M_R are determined; using dot products all internal forces can be evaluated:

$$N = F_R \cdot n = \frac{1}{K} \cdot \begin{pmatrix} -i \cdot sin\varphi & j \cdot cos\varphi & \frac{H}{\pi \cdot R} \end{pmatrix} \cdot \begin{Bmatrix} 0 \\ 0 \\ -R \cdot F \end{Bmatrix} = -\frac{1}{K} \cdot \frac{H \cdot F}{\pi \cdot R} \tag{3.47}$$

$$Q_S = F_R \cdot s = (cos\varphi \;\; sin\varphi \;\; 0) \cdot \begin{Bmatrix} 0 \\ 0 \\ -k \cdot F \end{Bmatrix} = 0 \tag{3.48}$$

$$Q_W = F_R \cdot w = \frac{1}{K} \cdot \begin{pmatrix} \frac{H}{\pi \cdot R} \cdot sin\varphi & -\frac{H}{\pi \cdot R} \cdot cos\varphi & 1 \end{pmatrix} \cdot \begin{Bmatrix} 0 \\ 0 \\ -k \cdot F \end{Bmatrix} = -\frac{F}{K} \tag{3.49}$$

$$T = M_R \cdot n = F \cdot R \cdot (sin\varphi \;\; (1 - cos\varphi) \;\; 0) \cdot \begin{Bmatrix} -sin\varphi \\ cos\varphi \\ \frac{H}{\pi \cdot R} \end{Bmatrix} \cdot \frac{1}{K}$$

$$T = \frac{F \cdot R}{K} \cdot \left(-sin^2\varphi + cos\varphi - cos^2\varphi\right) = -\frac{F \cdot R}{K} \cdot (1 - cos\varphi) \tag{3.50}$$

The bending moments about the principal strong axis are

$$\begin{aligned} M_S = M_R \cdot s = F_R \cdot (sin\varphi \;\; 1 - cos\varphi \;\; 0) \cdot \begin{Bmatrix} cos\varphi \\ sin\varphi \\ 0 \end{Bmatrix} = \\ = F_R \cdot (sin\varphi \cdot cos\varphi + sin\varphi - sin\varphi \cdot cos\varphi) = F_R \cdot sin\varphi \end{aligned} \tag{3.51}$$

and about the weak axis are:

$$M_{S0} = M_R \cdot w = F \cdot R \cdot \left(sin\varphi \;\; 1 - cos\varphi \;\; 0 \right) \cdot \frac{1}{K} \cdot \begin{Bmatrix} \dfrac{H}{\pi \cdot R} \cdot sin\varphi \\ -\dfrac{H}{\pi \cdot R} \cdot cos\varphi \\ 1 \\ \end{Bmatrix} =$$

$$= -\frac{F \cdot R}{K} \cdot \frac{H}{\pi \cdot R} \cdot \left(sin^2\varphi - cos\varphi + cos^2\varphi \right) = -\frac{F \cdot H}{K \cdot \pi} \cdot \left(1 - cos\varphi \right) \tag{3.52}$$

Numerical example: $R = 1.20\,m$, $H = 2.70\,m$, $F = 10\,kN$

$$K = \sqrt{1 + \left(\frac{H}{\pi \cdot R} \right)^2} = 1.230$$

Table 3.1: Internal forces in stairs beam

φ [deg]	N [kN]	Q_W [kN]	T [kNm]	M_S [kNm]	M_W [kNm]
0	-5.823	-8.130	0.000	0.000	0.000
30	-5.823	-8.130	-1.307	6.000	0.936
60	-5.823	-8.130	-4.878	10.392	3.494
90	-5.823	-8.130	-9.756	12.000	6.987
120	-5.823	-8.130	-14.634	10.392	10.481
150	-5.823	-8.130	-18.605	6.000	13.038
180	-5.823	-8.130	-19.512	0.000	13.974

4

Kinematics of structures

4.1 Connections and reactions

Members of structures are connected to each other through *rigid connections* or by *hinges*. A hinged connection enables independent rotation of all connected elements but the displacement is common to all elements at the joint. If more than two elements are connected in a hinge then we have double, triple etc. hinges.

A rigid connection permits no relative rotation between elements at a common displacement and a rigid connection of *n* elements has *(n - 1) rigid angles.*

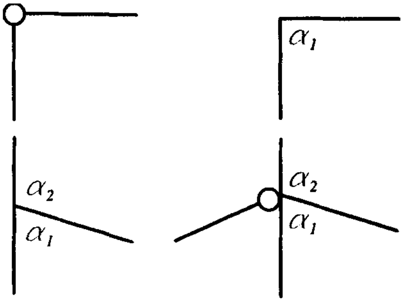
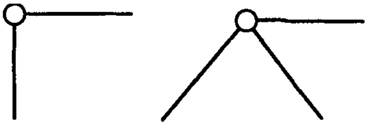


Figure 4.1: Connections of elements



Double hinge *Triple hinge*

Figure 4.2: Multiple hinges

A *support* is a member of the structure that *disables a displacement* of the structure in *at least one direction.* As the displacement or rotation is disabled it can only be done so by reactions at the supports. The supports can be *fixed* (*3* reactions), *pinned* (2 reactions), *roller* (*1* reaction), *swinging* (*1* reaction), *guided* (2 reactions) or *elastic* (*1-3* reactions).

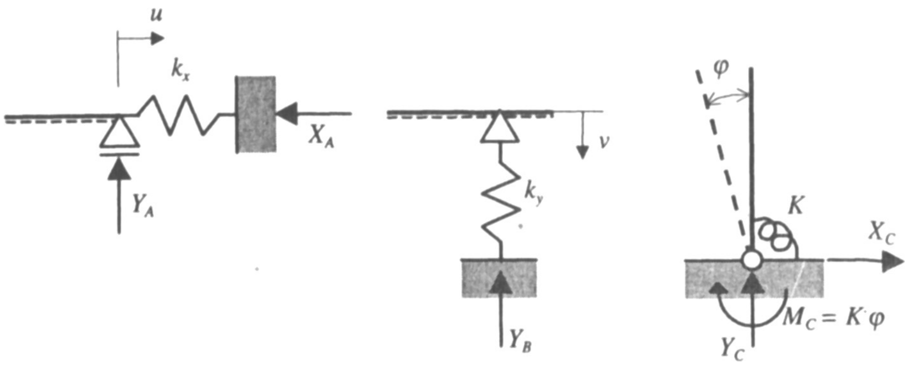


Figure 4.3: Types of supports

4.2 Stability

The shape of a structure and its supports has to be chosen in a way that structure is *stable*. If joints of a structure can displace and elements of a structure remain straight then such a structure is ***kinematically unstable or a mechanism*** and can not be used in practice.

❖ ***If elements deform during joint displacements then a structure is stable.***

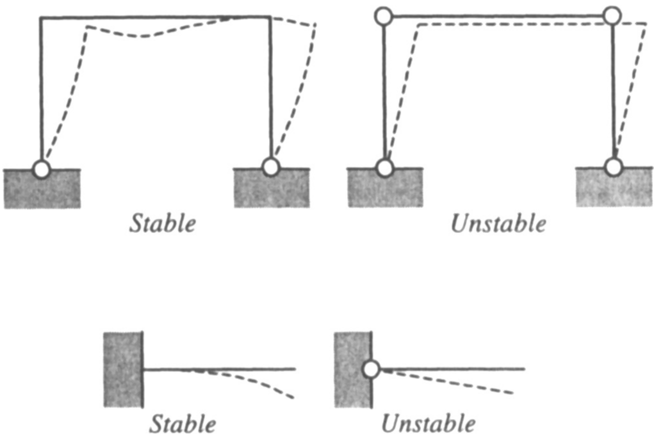


Figure 4.4: Stability of structures

4.3 Analysis of structural elements

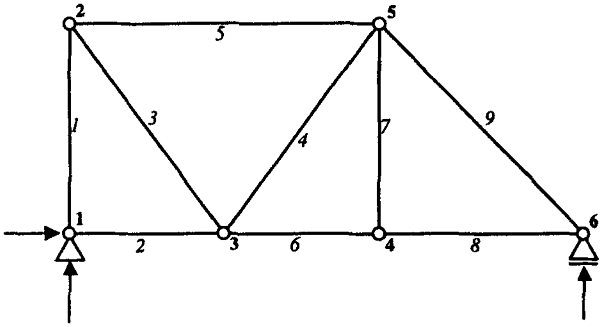
In the determination of kinematical stability we have to define the *number of structural members*, which can be defined by:

j	joints
m	elements (bars, beams, cables)
k	rigid angles
p	unknowns in supports

The *number of structural members* is:

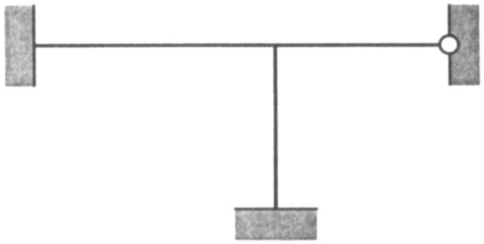
$$e = m + k + p \tag{4.1}$$

Example 4.1: Determine the number of structural members



$m = 9$
$k = 0$
$p = 3$

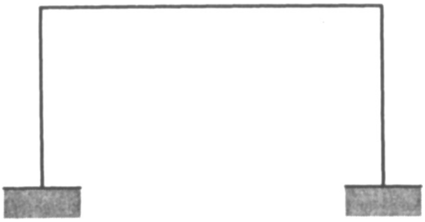
$e = 9 + 0 + 3 = 12$



$m = 3$
$k = 2$
$p = 8$

$e = 3 + 2 + 8 = 13$

The definition of structural members is not always uniform as can be seen in example *a)* in the picture below, where 3 elements are connected through two rigid angles but in example *b)* one element only is taken with no rigid angles.



a) $m = 3$
$k = 2$
$p = 6$
$e = 3 + 2 + 6 = 11$

b) $m = 1$
$k = 0$
$p = 6$
$e = 1 + 0 + 6 = 7$

It can be proved that this virtual inconsistency has no influence on kinematical stability determination.

4.4 Kinematical stability (*Geometrical conditions of kinematical stability*)

Consider element *ij* that translates into a new position *i'j'* simultaneously rotating through angle ψ_{ij}.

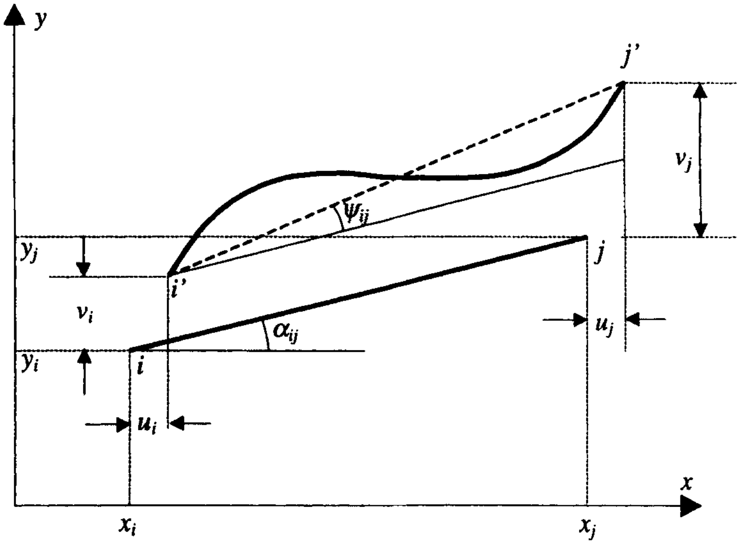


Figure 4.5: Displacement of an element

The basic geometric relations are determined from Fig. 4.5:

$$\cos\alpha_{ij} = \frac{x_j - x_i}{L_{ij}} \quad \Rightarrow \quad x_j - x_i = L_{ij} \cdot \cos\alpha_{ij} \tag{4.2}$$

$$\sin\alpha_{ij} = \frac{y_j - y_i}{L_{ij}} \quad \Rightarrow \quad y_j - y_i = L_{ij} \cdot \sin\alpha_{ij} \tag{4.3}$$

Let the change in length be δL_{ij} and the angle change (element rotation) be $\delta\alpha_{ij} = \psi_{ij}$. Variation of (4.2) and (4.3) gives:

$$\delta(x_j - x_i) = \delta(L_{ij} \cdot cos\alpha_{ij})$$

$$\delta(y_j - y_i) = \delta(L_{ij} \cdot sin\alpha_{ij}) \tag{4.4}$$

and after derivation:

$$\delta x_j - \delta x_i = \delta L_{ij} \cdot cos\,\alpha_{ij} - L_{ij} \cdot \delta\alpha_{ij} \cdot sin\,\alpha_{ij}$$

$$\delta y_j - \delta y_i = \delta L_{ij} \cdot sin\,\alpha_{ij} - L_{ij} \cdot \delta\alpha_{ij} \cdot cos\,\alpha_{ij} \tag{4.5}$$

Define the change in length and angle by joint displacements as follows:

$$\delta x_j = u_j \qquad \delta x_i = u_i$$

$$\delta y_j = v_j \qquad \delta y_i = v_i$$

$$\delta\alpha_{ij} = \psi_{ij}$$

which can be inserted in equations (4.5):

$$u_j - u_i = \delta L_{ij} \cdot cos\,\alpha_{ij} - L_{ij} \cdot \delta\psi_{ij} \cdot sin\,\alpha_{ij}$$

$$v_j - v_i = \delta L_{ij} \cdot sin\,\alpha_{ij} - L_{ij} \cdot \delta\psi_{ij} \cdot cos\,\alpha_{ij}$$

and written in matrix form :

$$\begin{Bmatrix} \Delta u \\ \Delta v \end{Bmatrix} = \begin{bmatrix} cos\,\alpha_{ij} & -L_{ij} \cdot sin\,\alpha_{ij} \\ sin\,\alpha_{ij} & L_{ij} \cdot cos\,\alpha_{ij} \end{bmatrix} \begin{Bmatrix} \delta L_{ij} \\ \psi_{ij} \end{Bmatrix} \tag{4.6}$$

The determinant is then calculated:

$$det = cos^2\,\alpha_{ij} \cdot L_{ij} + sin^2\,\alpha_{ij} \cdot L_{ij} = L_{ij}$$

and solved by unknowns:

$$\begin{Bmatrix} \delta L_{ij} \\ \psi_{ij} \end{Bmatrix} = \frac{1}{L_{ij}} \cdot \begin{bmatrix} L_{ij} \cdot cos\,\alpha_{ij} & L_{ij} \cdot sin\,\alpha_{ij} \\ -\,sin\,\alpha_{ij} & cos\,\alpha_{ij} \end{bmatrix} \begin{Bmatrix} \Delta u \\ \Delta v \end{Bmatrix} \tag{4.7}$$

or explicitly:

$$\begin{Bmatrix} \delta L_{ij} \\ \psi_{ij} \end{Bmatrix} = \begin{Bmatrix} \Delta u \cdot cos\alpha_{ij} + \Delta v \ \cdot sin\alpha_{ij} \\ \dfrac{-\Delta v \cdot sin\alpha_{ij}}{L_{ij}} + \dfrac{\Delta u \cdot cos\alpha_{ij}}{L_{ij}} \end{Bmatrix} \tag{4.8}$$

The change in length is thus

$$\delta L_{ij} = (u_j - u_i) \cdot cos\,\alpha_{ij} + (v_j - v_i) \cdot sin\,\alpha_{ij} \tag{4.9}$$

and the change of angle is

$$\delta\alpha_{ij} = \psi_{ij} = \frac{(v_j - v_i) \cdot cos\,\alpha_{ij}}{L_{ij}} - \frac{(u_j - u_i) \cdot sin\,\alpha_{ij}}{L_{ij}} \tag{4.10}$$

Equation (4.9) represents *m conditions* (*m = number of elements*) that the elements of a *structure will deform*, but equation (4.10) gives *no additional conditions*, as a rotation by ψ_{ij} not always means element deformation.

The rotation of *two rigidly connected elements* are not independent of each other. From Fig. 4.6 we can observe that the *sum of angles* before and after deformation is the same:

$$\begin{aligned} \psi_{ij} + \tau_{ij} &= \psi_{ik} + \tau_{ik} \\ \tau_{ij} - \tau_{ik} &= \psi_{ik} - \psi_{ij} \end{aligned} \tag{4.11}$$

From equation (4.10):

$$\begin{aligned} \tau_{ij} - \tau_{ik} = {} & \frac{1}{L_{ik}}(v_k - v_i)cos\,\alpha_{ik} - \frac{1}{L_{ik}}(u_k - u_i)sin\,\alpha_{ik} - \\ & - \frac{1}{L_{ik}}(v_j - v_i)cos\,\alpha_{ij} + \frac{1}{L_{ik}}(u_j - u_i)sin\,\alpha_{ij} \end{aligned} \tag{4.12}$$

Equation (4.12) equals zero only if two elements remain undeformed, which is obviously an additional condition of kinematic stability and can be written for *each rigid connection k*. The *influence of support displacements* on element deformations is shown in Figs. 3.2 and 4.3 as an arbitrary support displacement Δ_i that can be represented by displacements u and v in the following equation:

$$\Delta_i = u_i \cdot cos\,\beta + v_i \cdot sin\,\beta \tag{4.13a}$$

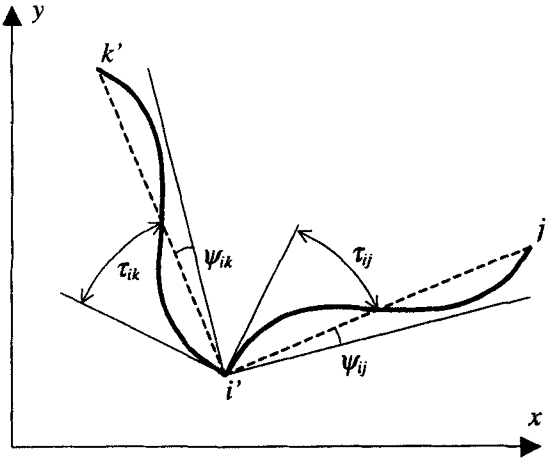


Figure 4.6: Rotation of two rigidly connected elements

If a support rotation of angle ψ is present, the total angle of rotation is:

$$\Theta_i = \psi_{ij} + \tau_{ij}$$

Substituting into (4.10):

$$\tau_{ij} = \Theta_i - \psi_{ij} = \Theta_i - \frac{(v_j - v_i)\cdot cos\alpha_{ij}}{S_{ij}} + \frac{(u_j - u_i)\cdot sin\alpha_{ij}}{S_{ij}} \tag{4.13b}$$

Equations (4.13a) and (4.13b) relate support rotations and displacements to the deformation of an element. The *deformation of an element is zero* only if the *displacements at supports are suppressed in a corresponding direction.*
The total number of conditions therefore equals:

$$2j = e = m + k + p, \tag{4.14}$$

which represents the relation between two possible displacements at joints and quantities δ_s, τ, Δ and Θ, are related to *element deformation.*

These conditions are called *geometrical conditions of joint displacements.* If the number of conditions equals the number of all possible displacements (2 displacements per joint in plane structures) then these conditions represent *sufficient and necessary conditions* to calculate all deformations δ_s, τ, Δ and Θ.

The system of equations can be solved only if

$$det A \neq 0 \tag{4.15}$$

but a body will move without any deformation, only when δ_s, τ, Δ and Θ *are all equal to zero*. This gives a homogenous system of equations (4.9), (4.12) and (4.13). Such a system can only have a non-trivial solution if $\det A = 0$, which is in contradiction with equation (4.15).

- *Structures are therefore stable (are under displacements deformed) only if conditions from equations (4.14) and (4.15) are met.*

Let us denote by a letter f the *degree of kinematic stability*:

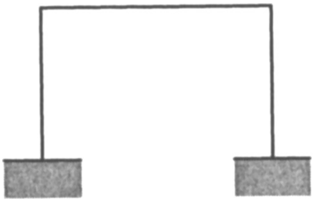
$$f = 2j - (m + k + p) \tag{4.16}$$

Clearly there are three possible conditions:

$$\begin{array}{ll} f < 0 & \textit{multiple stable system} \\ f = 0 & \textit{simply stable system} \\ f > 0 & \textit{unstable system} \end{array} \tag{4.17}$$

Example 4.2: Calculate the stability of the frame structure at the following successive release of deformational quantities

Consider at first the frame from Figure 4.7 with both supports clamped.



$j = 4$
$m = 3$
$k = 2$
$p = 6$

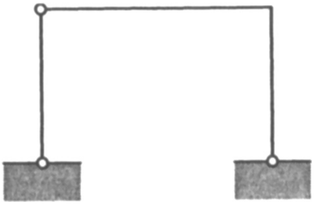
$f = 2 \cdot 4 - (3 + 2 + 6)$
$f = -3$

or $j = 2$
$m = 1$
$k = 0$
$p = 6$

$f = 2 \cdot 2 - (1 + 0 + 6)$
$f = -3$

Figure 4.7

As $f = -3$ the structure has *multiple stability*. Three quantities are redundant or in statics it is *3x statically indeterminate*. Three deformational quantities therefore can be released; for example in figure 4.7a three moments were released, one in a rigid connection and two at the supports.



$j = 4$
$m = 3$
$k = 1$
$p = 4$

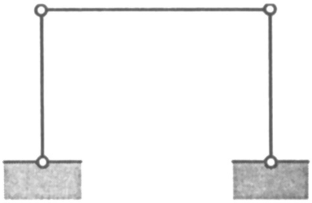
$f = 8 - 3 - 1 - 4 = 0$

or $j=3$
$m = 2$
$k = 0$
$p = 4$

$f = 6 - 2 - 4 = 0$

Figure 4.7a

The structure from Fig. 4.7a is ***simply stable or statically determinate*** as $f=0$. If further quantities are released (i.e. moment by a hinge in the upper right corner) we obtain (Figure 4.7b):



$j = 4$
$m = 3$
$k = 0$
$p = 4$

$f = 8 - 4 - 3 = 1$

Figure 4.7b

It is obvious that ***the structure is unstable*** as $f > 1$; such structures are ***kinematically unstable or statically over-determined.***

Example 4.3: Plane truss with one diagonal element missing

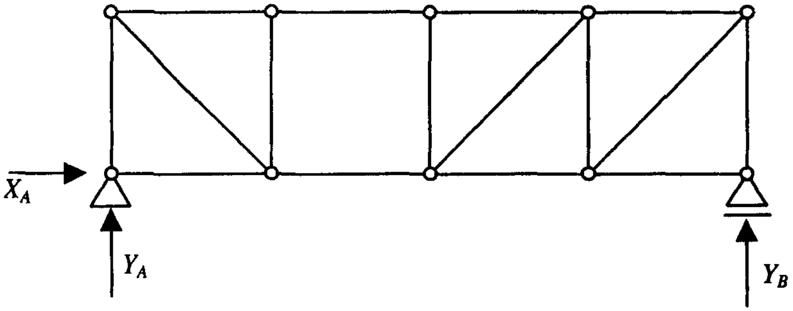


Figure 4.8: Unstable truss

$j = 10$ $m = 16$ $k = 0$ $p = 3$

$f = 2 \cdot 10 - (16 + 3) = 1$

The truss is unstable as the condition from equation 3.10a ($f = 2j - m - 3$) is not fulfilled; from the figure it is obvious that a diagonal in the second field is missing.

Example 4.4: Three-hinged frame

$j = 3 \quad m = 2 \quad k = 0 \quad p = 4$

$f = 6 - 6 = 0$

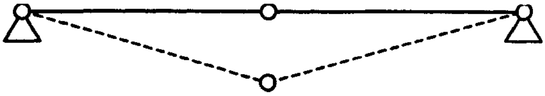


Figure 4.9: Instability of a beam

Though $f=0$ the beam is unstable because the condition $det A \neq 0$ is not met, as three joints lie on the same straight line.

Example 4.5: Mixed bridge structure

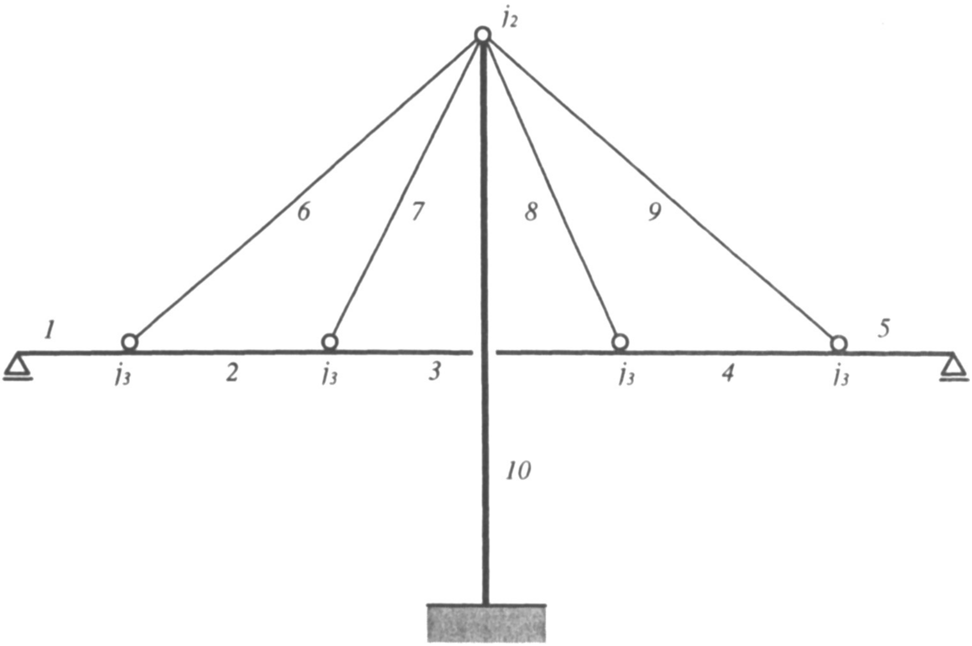


Figure 4.10: Mixed bridge structure (pile and beam are not in contact)

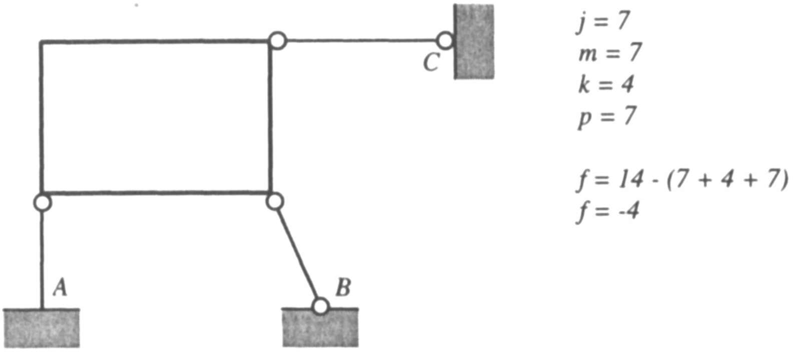
$j = 8 \qquad m = 10 \qquad k = 4 \qquad p = 5$

$f = 16 - (10 + 4 + 5) = -3$

The structure is *3x statically indeterminate* since $f = -3$, possible redundants could be axial forces in cables 6, 7 and 8.

Example 4.6: Mixed system

The system from figure 4.11 is *multiple stable or 4x statically indeterminate*, redundant forces are *M, Q* and *N* in an arbitrary section of the inner box and moment in the clamped support at *A*.



$j = 7$
$m = 7$
$k = 4$
$p = 7$

$f = 14 - (7 + 4 + 7)$
$f = -4$

Figure 4.11: Stability of mixed structure

4.5 Kinematics of rigid bodies

4.5.1 Degrees of freedom

Any rigid body lying in plane has three degrees of freedom. The movement of a rigid body is defined by three parameters.

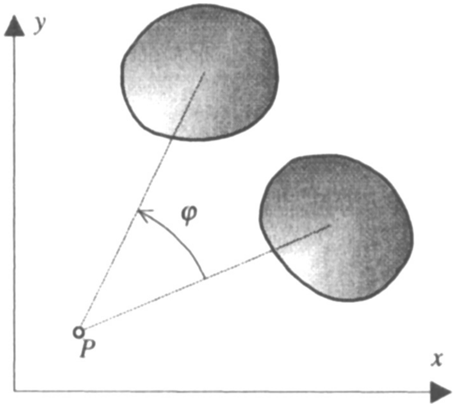


Figure 4.12: Rotation of a rigid body

Degrees of freedom are determined by the equation

$$f = 3 s - v,$$

where: s = *number of rigid bodies*
v = *number of ties between bodies* (single, double,...)

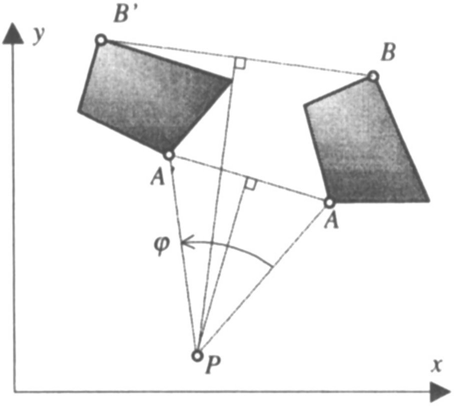


Figure 4.13: Final displacement of a body

We are interested in the starting and final position of a body only, such a displacement will be called a ***final displacement of a body.*** If the ***pole of rotation P lies at infinity*** then the only movement will be a ***translation*** of a body.

Elementary displacement of a body is performed, if the displacements are small such that a ***chord can be substituted by a tangent.***

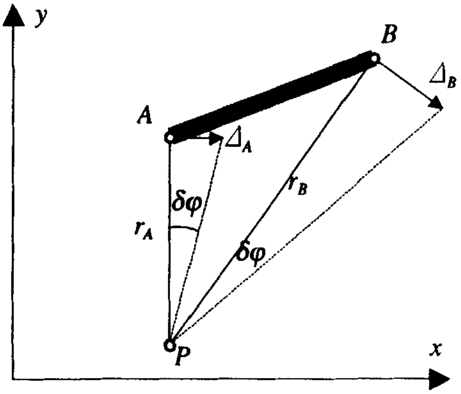


Figure 4.14: Elementary displacement of a body

Consider now small rotations such that $d\varphi$ can be substituted by φ:

$$tg\delta\varphi = \frac{\Delta_A}{r_A} \quad \Rightarrow \quad \Delta_A = r_A \cdot \varphi \quad in \quad \Delta_B = r_B \cdot \varphi \tag{4.18}$$

4.5.2 Addition of rotations

A rotation is a *vector* lying *perpendicular to the plane*, therefore $\delta\varphi_1$ and $\delta\varphi_2$ are *parallel vectors*, which can be added into a resultant vector.

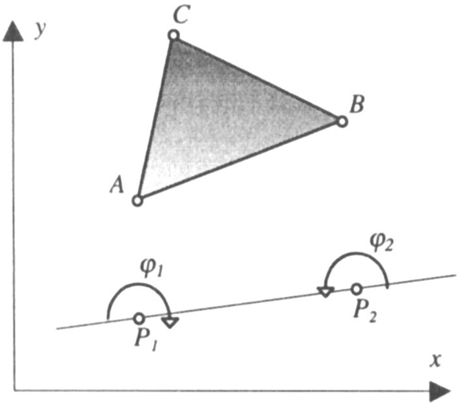


Figure 4.15: Line of common rotation

The *pole of a common rotation* lies on the same line through poles P_1 and P_2. Position of a resultant rotation can be determined from a moment equilibrium equation:

$$\varphi_1 \cdot L = \varphi \cdot x$$

$$\varphi = \varphi_1 + \varphi_2$$

$$x = \frac{\varphi_1}{\varphi_1 + \varphi_2} \cdot L \tag{4.19}$$

where L is the distance between poles P_1 and P_2.

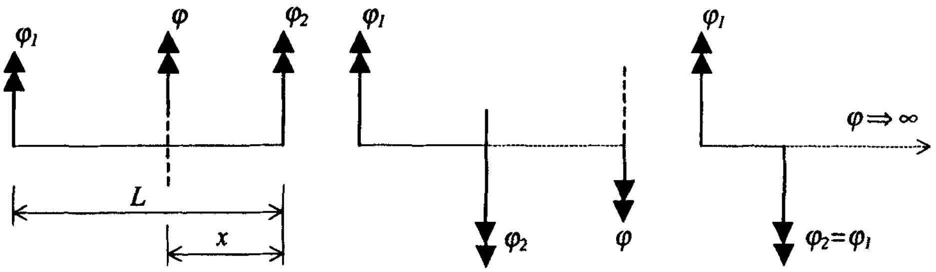
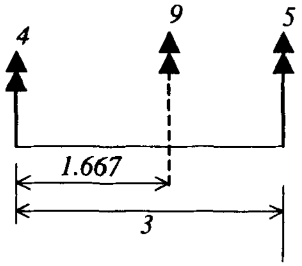


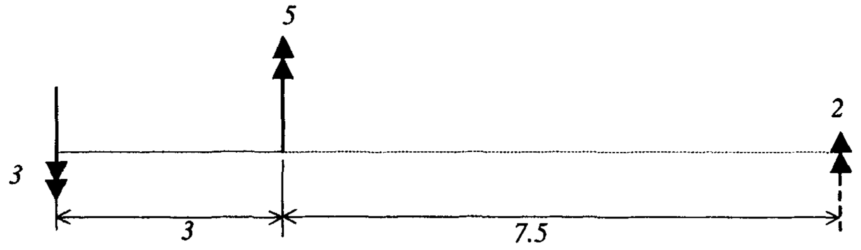
Figure 4.16: Addition of rotations

Consider a case of two rotations in the same direction:



$$x = \frac{5}{5+4} \cdot 3 = 1.667 \ m$$

- ❖ *If two rotations are in the same direction then the pole of common rotation lies on a line between both poles nearer to the greater rotation.*



$$x = \frac{5}{5-3} \cdot 3 = 7.5 \ m$$

- ❖ *If two rotations are in the opposite direction, then the pole of common rotation lies on a line through both poles outside both poles on the side of the greater rotation.*

- *If two rotations are in the opposite direction and are of the same magnitude (a couple), then the pole of common rotation lies at infinity.*

4.5.3 Relative rotation of two bodies

Let the body *I.* rotate about P_1 by φ_1 and the body *II.* about P_2 by φ_2. We are looking for a *relative* or interacting *rotation* of two bodies. Both bodies *I.* and *II.* are now rotated by φ_1 about P_1, the body *I.* is in its initial position but the body *II.* rotated by φ_2 about P_2 and by φ_1 about P_1.

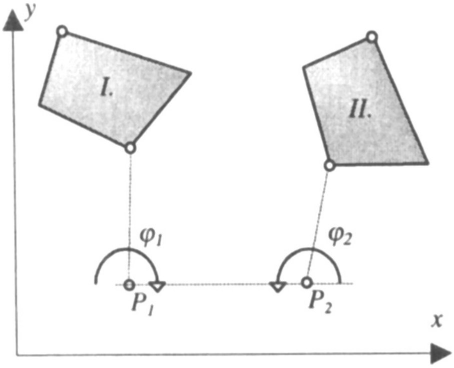


Figure 4.17: Relative rotation of two bodies

$$\varphi_{12} = \varphi_2 - \varphi_1 \tag{4.20}$$

The position of the *relative pole* P_{12} is given by the equation:

$$x = \frac{\varphi_2}{\varphi_2 - \varphi_1} \cdot L \tag{4.21}$$

Poles P_1 and P_2 are called *absolute poles* and will in further text be denoted by P_{10} and P_{20} for consistency. From equation (4.21) it follows that:

- *If rotations* φ_1 *and* φ_2 *act in the same direction, the relative pole lies outside of the absolute poles on the side of the greater rotation*
- *If rotations* φ_1 *and* φ_2 *act in the opposite direction, the relative pole lies between both absolute poles on the side of the greater rotation*

- *If rotations φ_1 and φ_2 are of the same magnitude and act in the same direction, the relative pole lies at infinity*

The relative pole is always related to two rigid bodies, therefore the number of relative poles is given by the equation:

$$p = \binom{2}{n} = \frac{n \cdot (n-1)}{1 \cdot 2}, \tag{4.22}$$

n is the number of rigid bodies.

4.6 Kinematics of three chained bodies

Figure 4.18 shows a structure, consisting of three rigid bodies interconnected as a chain by relative poles P_{12} and P_{23}. Degrees of freedom are determined by the equation:

$$f = 3 \cdot s - v = 3 \cdot 3 - 2 \cdot 2 = 5$$

The chain (of three rigid bodies) has therefore *5 degrees of freedom*. We say that a movement, which determined by *5* parameters, has to be prevented if the structure is to become stable ($f = 0$).

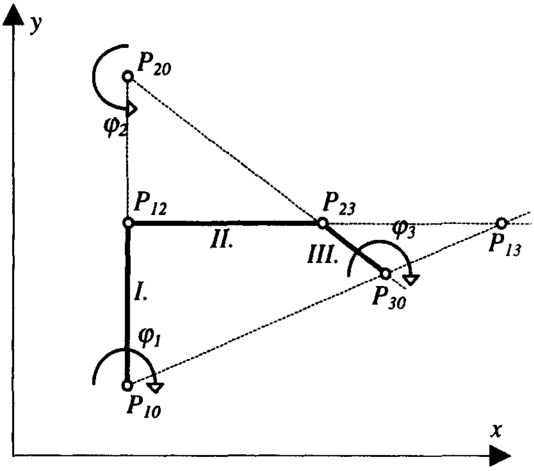


Figure 4.18: Kinematic chain of three rigid bodies

The positions of *absolute poles of supported bodies* are usually known. Bodies *III.* and *I.* are immovable at absolute poles P_{30} and P_{10} (at each support two displacements are suppressed). If, for instance, a clockwise direction of the rotation φ_1 is chosen, then all other parameters can be determined.

Let us first determine the position of absolute pole P_{20}. As absolute and relative poles lie on the same line, pole P_{20} must lie on the intersection of two lines through P_{10}-P_{12} and P_{30}-P_{23}.

For each of the related bodies a rotation can be calculated using Eqn. (4.21):

$$\varphi_2 = -\frac{x}{L-x}\cdot\varphi_1 = -\frac{a}{b}\cdot\varphi_1$$

where a is the distance P_{10}-P_{12} and b the distance of P_{12}-P_{20}. The minus sign implies that the rotations are of *opposite direction if the relative pole lies between the absolute poles.* Similarly when rotation φ_3 is calculated, the quantity c is the distance between poles P_{20}-P_{23} and d is the distance P_{23}-P_{30}.

$$\varphi_3 = -\frac{c}{d}\cdot\varphi_2 = +\frac{a}{b}\cdot\frac{c}{d}\cdot\varphi_1$$

The number of relative poles is given by (4.22)

$$p = \frac{3\cdot(3-1)}{2} = 3,$$

so it is obvious that pole P_{13} is not yet determined. It can be found on the intersection of line through P_{10} and P_{30} with the line through P_{12} and P_{23}. As we can see, relative pole P_{13} lies *outside of the absolute poles* therefore the rotations φ_1 and φ_3 are of the same direction.

- *Practical rule for a pole determination is:* two equal indexes from two poles are deleted and the remaining indexes determine the new pole. For instance if we consider poles P_{12} and P_{10}, index 1 is deleted and the line through both poles therefore determines absolute pole P_{20}.

4.7 Determination of internal forces by kinematics

The procedure for determining the internal forces (and reactions) by the kinematics method is relatively simple and will be shown in the following example.

Example 4.7: Calculate bending moment in mid-span by the kinematics method.

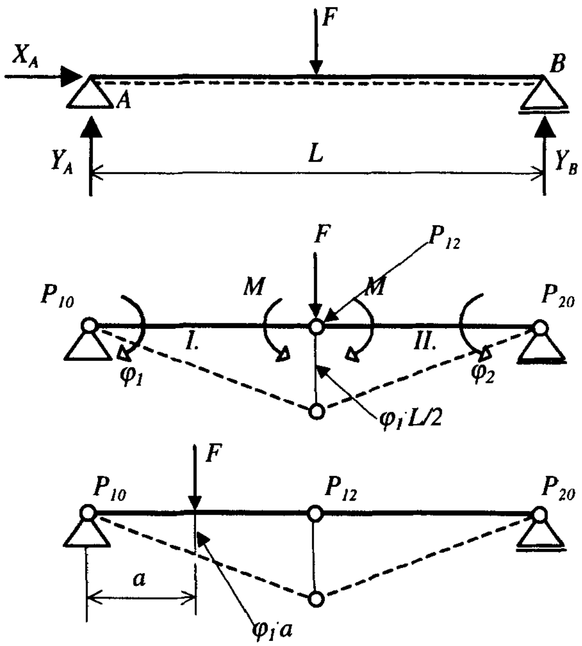


Figure 4.19: Simple beam

At the point where a *bending moment is desired a rotation is released* (a hinge is inserted) and a beam becomes unstable or a mechanism of two rigid bodies, which can be only kept in place by the application of a moment couple M (internally in equilibrium) at the point. Let the rigid body I rotate about the absolute pole P_{I0} by a clockwise angle φ_I. From the figure it can be deduced that rotations φ_I and φ_2 are equal in magnitude but in the opposite direction.

The total work done by the moments M and by the force F must be zero if no displacements are desired and hence:

$$-M\varphi_I - M\varphi_2 + F\cdot\varphi_I\cdot\frac{L}{2}=0$$

$$2 \cdot M \cdot \varphi_1 = F \cdot \varphi_1 \cdot \frac{L}{2}$$

$$M = \frac{F \cdot L}{4}$$

If the equations are set for a *general position* of the force *F* (i.e. at the distance *a* from the left support) then only right hand side of the equation changes:

$$2 \cdot M \cdot \varphi_1 = F \cdot \varphi_1 \cdot a$$

$$M = \frac{F \cdot a}{2}$$

We have noticed that no reaction calculation was necessary as in *structures in equilibrium the work done by reactions is always zero.*
Let us now calculate the same example by the static method:

$$\Sigma M_A = 0: \;\; Y_B \cdot L = a \cdot F$$

$$Y_B = \frac{a}{L} \cdot F$$

$$M = Y_B \cdot \frac{L}{2} = \frac{F \cdot a}{2}$$

or from the left

$$\Sigma M_B = 0: \;\; Y_A \cdot L = (L - a) \cdot F$$

$$Y_A = \frac{L-a}{L} \cdot F$$

$$M = \frac{L-a}{L} \cdot F \cdot \frac{L}{2} - F \cdot \left(\frac{L}{2} - a \right) = F \cdot \frac{a}{2}$$

Example 4.8: Analyse frame structure by the kinematics method

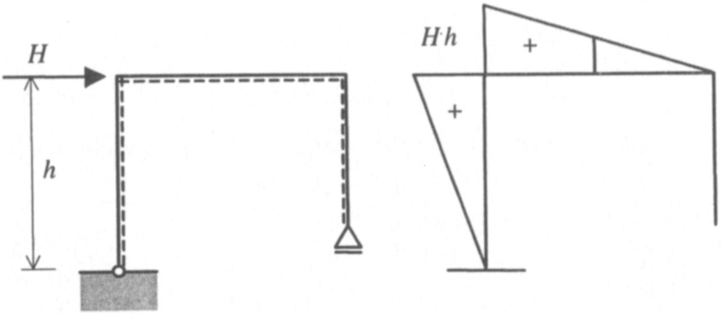
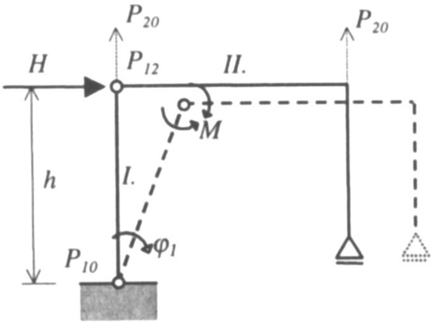


Figure 4.20: Frame structure and bending moments diagram

a) Moment at left corner

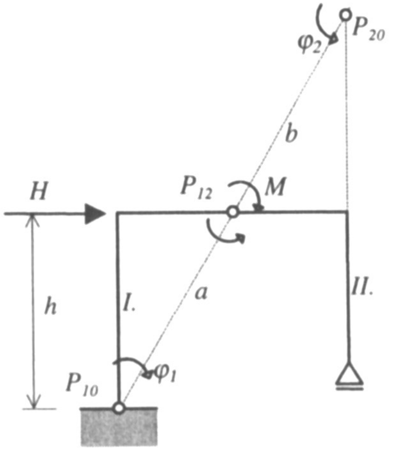


$$\varphi_I \cdot M - \varphi_I \cdot h \cdot H = 0$$

$$M = H \cdot h$$

Note: As absolute pole P_{20} lies at infinity body *II* moves translatory only.

b) Moment at mid-span



$$\varphi_I \cdot a = \varphi_2 \cdot b$$

$$\varphi_2 = \frac{a}{b} \cdot \varphi_I$$

$$H \cdot h \cdot \varphi_I - M \cdot \varphi_I - M \cdot \varphi_2 = 0$$

$$M = \frac{H \cdot h}{2}$$

In general:

$$M \cdot (\varphi_2 + \varphi_I) = H \cdot h \cdot \varphi_I$$

$$M = H \cdot h \cdot \frac{\varphi_I}{\varphi_I + \varphi_2} = H \cdot h \cdot \frac{b}{a+b}$$

Example 4.9: Analysis of arched structure from figure 4.21 by the kinematics method

The structure is determinate since

$$f = 3 \cdot s - p - v = 3 - 3 = 0$$

and after a hinge insertion it becomes unstable or a mechanism:

$$f = 3 \cdot 2 - 3 - 2 = 1$$

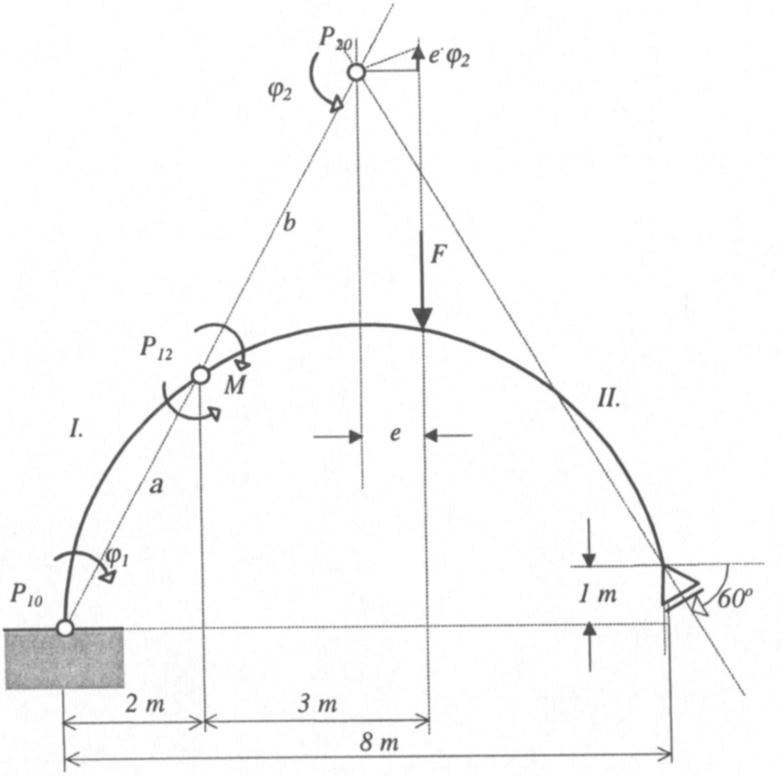


Figure 4.21: Arch structure

From basic geometry the rotation φ_2 is determined:

$$\varphi_2 = \frac{a}{b} \cdot \varphi_1 = \frac{2}{3-e} \cdot \varphi_1$$

or

$$\varphi_1 = \frac{3-e}{2} \cdot \varphi_2$$

The work done by internal bending moments must equal the work done by the external force, which moves distance ($e.\varphi_2$):

$$W_M = -M\varphi_1 - M\varphi_2$$

$$W_F = -F \cdot e \cdot \varphi_2$$

The total work done is therefore:

$$-M \cdot \varphi_1 - M \cdot \varphi_2 - F \cdot e \cdot \varphi_2 = 0$$

$$M \cdot (\varphi_1 + \varphi_2) = -F \cdot e \cdot \varphi_2$$

from where the bending moment at the point is found:

$$M = -F \cdot e \cdot \frac{2}{5-e}$$

Static method calculation

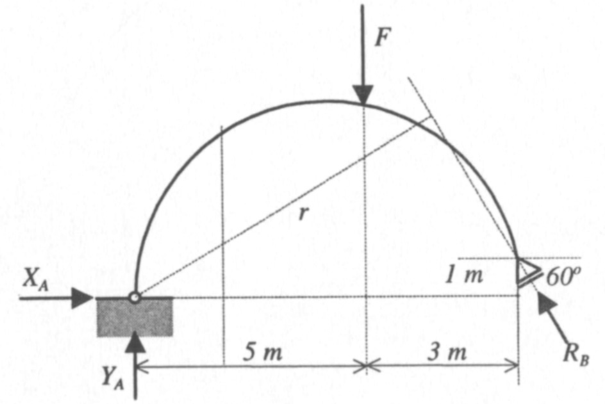


Figure 4.22: Disposition of an arched structure

From the sum of moments about point A the reaction R_B is calculated:

$$R_B \cdot r = F \cdot 5$$

$$R_B = \frac{5}{r} \cdot F = 0.673 \cdot F$$

Reaction R_B is resolved into X and Y components and reactions Y_A and X_A are calculated from equilibrium equations in both directions:

$$Y_A = F - 0.583 \cdot F = 0.417 \cdot F$$

$$X_A = 0.337 \cdot F$$

Suppose that the arch is semicircular of radius $R = 4.00\ m$, then y ordinate at $x = 2.00\ m$ equals $3.464\ m$ and the bending moment at $x = 2.00\ m$ is therefore:

$$M = 0.417 \cdot 2 - 0.337 \cdot 3.464$$

$$M = -0.333 \cdot F$$

By the kinematics method: (*e* can be *measured*)

$$e = 5 - \frac{8.577}{2} = 0.712m$$

$$M = F \cdot \frac{2 \cdot 0.712}{5 - 0.712} = 0.332 \cdot F$$

Example 4.10: Analysis of plane truss in Fig. 4.23 by the kinematics method

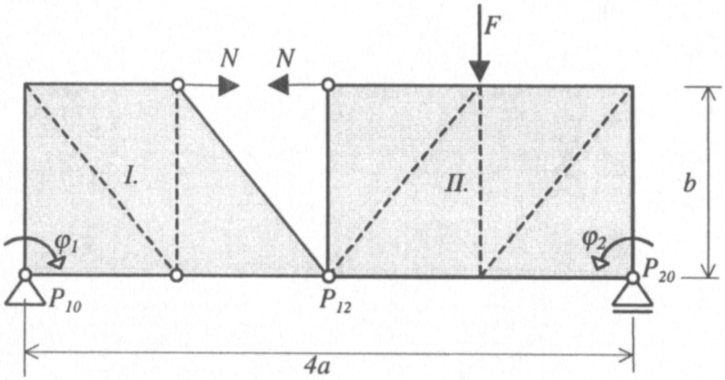


Figure 4.23: Plane truss (element *8* is cut)

To find the axial force in element *8* it is cut and the truss becomes a kinematic chain of two rigid bodies:

$$f = 3 \cdot s - p - v = 6 - 3 - 2 = 1$$

As the relative pole lies between absolute poles the rotations are equal and act in the opposite direction:

$$\varphi_1 = -\varphi_2$$

The sum of work done by external and internal forces must be zero, therefore:

$$N \cdot \varphi_1 \cdot b + N \cdot \varphi_2 \cdot b + F \cdot \varphi_2 \cdot a = 0$$

$$N = -F \cdot \frac{a}{2 \cdot b}$$

Using the static method we have to calculate reactions first and then by the method of sections, the force in the element can be found:

$$Y_A \cdot 4 \cdot a = F \cdot a \qquad \Rightarrow \qquad Y_A = \frac{F}{4}$$

$$N \cdot b + F \cdot 2 \cdot a = 0$$

$$N = -Y_A \cdot \frac{2 \cdot a}{b} = -\frac{F}{4} \cdot \frac{2 \cdot a}{b} = -\frac{F \cdot a}{2 \cdot b}$$

Let us calculate the axial force in diagonal 7. The diagonal is cut and the truss becomes a mechanism of four bodies as shown in Fig. 4.24:

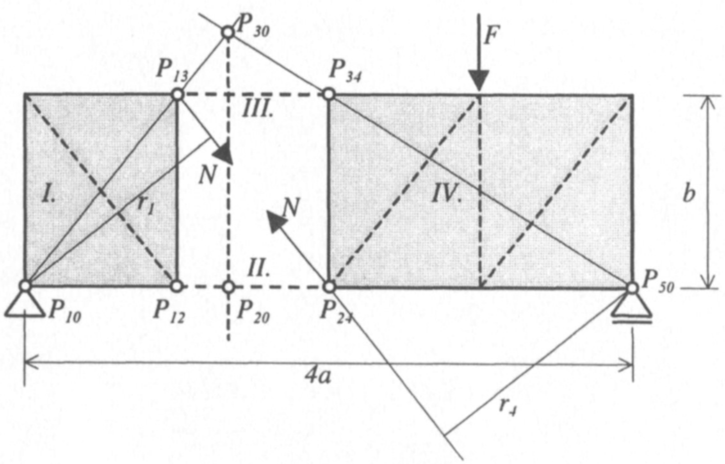


Figure 4.24: Plane truss (diagonal 7 is cut)

The number of relative poles is

$$p = \frac{n \cdot (n-1)}{2} = \frac{4 \cdot (4-1)}{2} = 6$$

therefore two relative poles P_{14} and P_{23} are still missing, but from the figure it is easy to deduce that they lie at infinity and bodies *I.* and *IV.* as well as *II.* and *III.* undergo equal rotations ($\varphi_1 = \varphi_4$ and $\varphi_2 = \varphi_3$), as shown in Fig. 4.25.

From the basic geometry

$$tg\alpha = \frac{b}{a} \qquad sin\alpha = \frac{(r_1 + r_4)}{4 \cdot a} \qquad r_1 + r_4 = 4 \cdot a \cdot sin\alpha$$

and from the sum of work done by external and internal forces we get:

$$N \cdot \varphi_1 \cdot r_1 + N \cdot \varphi_4 \cdot r_4 = F \cdot \varphi_4 \cdot a$$

$$N \cdot (r_1 + r_4) = F \cdot a$$

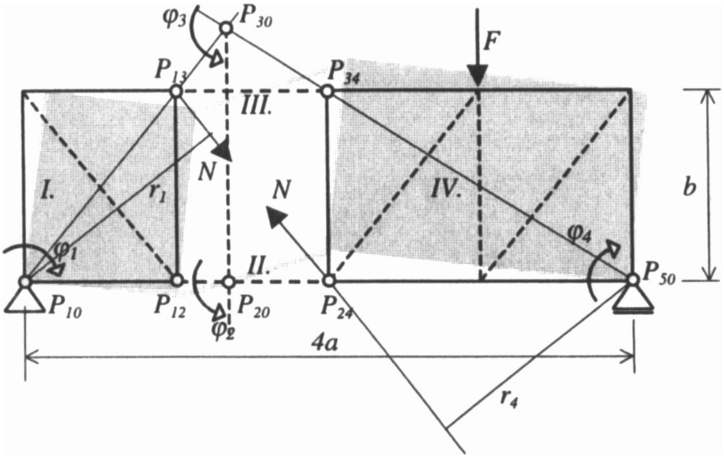


Figure 4.25: Rotation of rigid bodies (exceedingly enlarged)

The force in diagonal *7* is finally:

$$N = F \cdot \frac{a}{r_1 + r_4} = \frac{F}{4 \cdot sin\alpha}$$

5

Basic concepts of structural analysis

5.1 Superposition of actions and displacements

Usually we assume that a displacement at a point, caused by a force, is *linearly dependent on the magnitude of a force.* The assumption is based on the proportionality and reversibility of stresses and deformations at any small particle of the material, from which the structure is made. This phenomenon is called *elasticity.*

The assumption of elasticity is not always true as the deflections of a structure can significantly change the geometry and thereby change the manner in which potential energy is stored in the deflected structure.

The proportionality between loads F (applied at some point in some particular direction) and displacement (at any point in any direction) is assumed and therefore the individual effects of several forces can be summed up (*method of a superposition*).

If a force acts at a specified point in a specified direction and is increased from zero to F_n it would produce a movement of the structure Δ_m at all points (unless movement is prevented by reactions adequate to ensure equilibrium):

$$\Delta_m \propto F_n$$

Therefore a general case must be valid:

$$\Delta_m = f_{mn} \cdot F_n \tag{5.1}$$

as shown in Fig. 5.1.

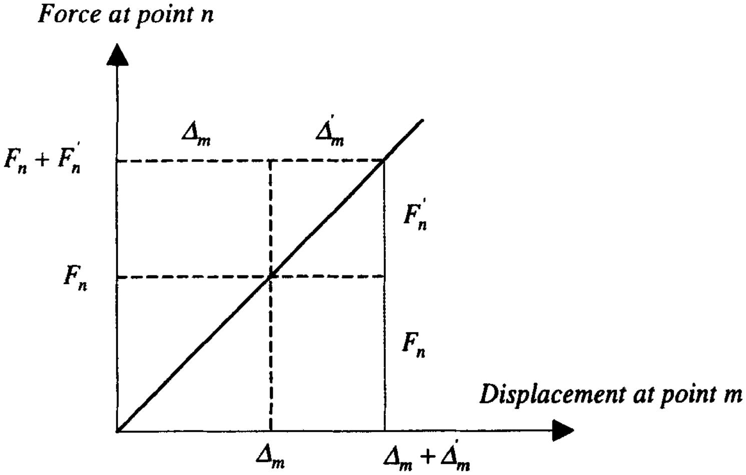


Figure 5.1: Linearity of loads and deformations

The coefficient f_{mn} in the equation (5.1) is a ***flexibility influence coefficient***, describing a relation between force F and displacement Δ. The ***first subscript*** of f_{mn} denotes the ***direction of a measured deformation*** and the ***second subscript*** denotes the ***direction of a force*** that caused the deformation. The coefficient is numerically equal but the dimension can be different as a force can cause rotation and vice versa.

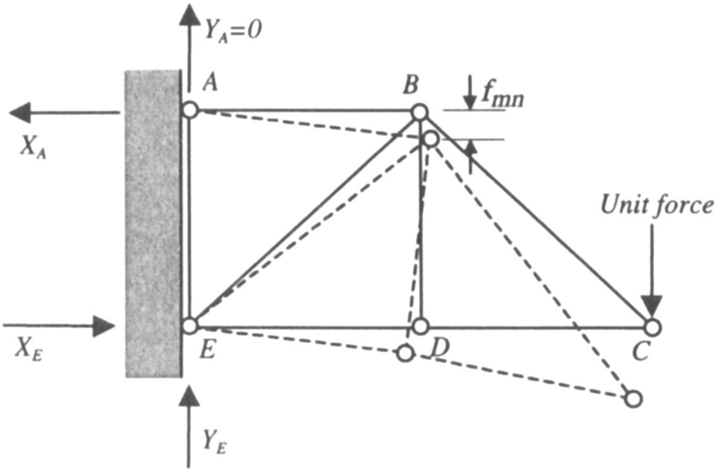


Figure 5.2: Direction of a chosen deformation

Referring to Fig. 5.2 the direction m defines the vertical displacement of point B and n defines the direction of a unit force at point C.

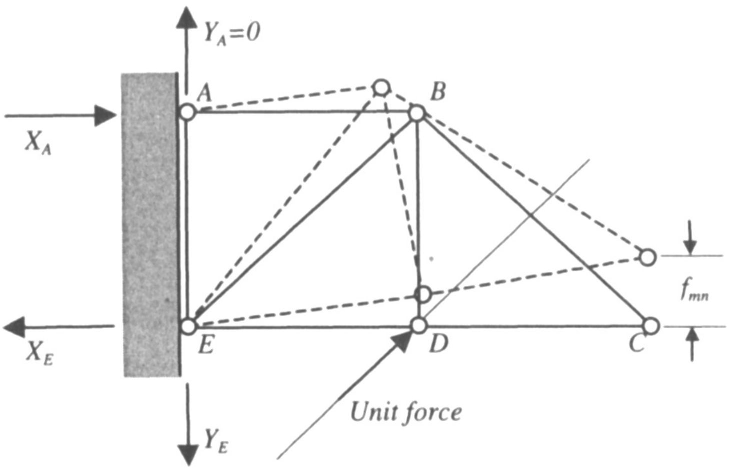


Figure 5.3: Direction of a chosen deformation

From Fig. 5.3 the direction *m* is vertical at point *C* and *n* is the direction of unit force at point *D*.

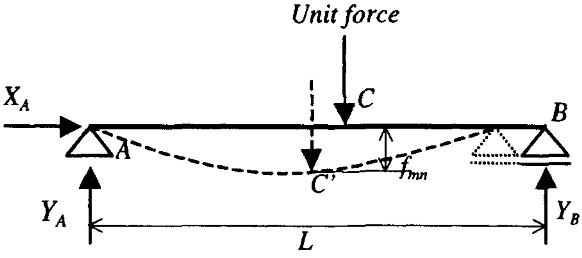


Figure 5.4: Gravitational load

In Fig. 5.4 the beam is loaded at point *C*, both directions *m* and *n* are vertical, which is common with gravitational loads.
If equation (5.1) is valid, then equally

$$\Delta'_m = f_{mn} \cdot F'_n, \tag{5.2}$$

where Δ'_m represents the deformation caused by a force F'_n. The deformations and loads may be added since the vector consistency has been maintained:

$$\Delta_m + \Delta'_m = f_{mn} \cdot F_n + f_{mn} \cdot F'_n = f_{mn} \cdot (F_n + F'_n) \tag{5.3}$$

- *Forces and displacements may be added or superimposed (method of a superposition)*

The displacement and load may be written inversely as

$$F_n = k_{nm} \cdot \Delta_m \tag{5.4}$$

where k_{nm} represents the *stiffness influence coefficient*, which is numerically equal to the *force in a specified direction n* to *produce unit displacement* at another point in a specified direction *m*, at a condition so that *all other displacements are prevented* at that point.
If this relation is valid then, using the superposition method, equation

$$F_n + F_n' = k_{nm} \cdot \Delta_m + k_{nm} \cdot \Delta_m' = k_{nm} \cdot (\Delta_m + \Delta_m') \tag{5.5}$$

must be valid. Equation (5.5) indicates that the total displacement equals the sum of individual displacements caused by forces F_n and F_n'.

It is assumed that the *structure is in equilibrium* and *the effect of reactions is taken into account* when calculating the stiffness coefficients.

It has to be emphasised that for any two points there are an infinite number of influence coefficients f_{mn} and k_{nm} depending on the specified (chosen) directions *m* and *n*. The number of coefficients may be reduced to as many as needed for the analysis of the structure. Note that at different boundary conditions stiffness coefficients change.

The relationship between the flexibility and stiffness coefficients is of the greatest importance in structural analysis, both coefficients are reciprocally related by the equation:

$$f_{mn} = \frac{1}{k_{nm}} \tag{5.6}$$

5.2 Non-linear behaviour (superposition is not valid)

The principle of superposition may be used if loads and deflections are linearly dependent, which in general relates linear dependence between stresses and strains.

In some cases, though linearity between stresses and strains is valid, the geometry of a structure changes significantly and superposition is not valid for two reasons. Either the geometry relations are violated (*Case A*) or the change alters the way the potential energy is stored in the structure (*Case B*).

Case A) Figure 5.5 shows an elastic string of zero mass and of length *2L* fastened between two points. A force of magnitude *2F* is applied to the middle of the string.

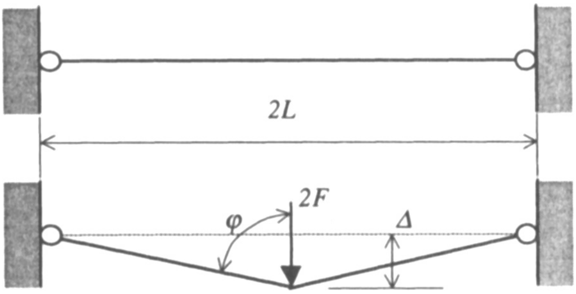


Figure 5.5: Non-linearity of a string

Because of the symmetry the deflection is vertical only in the direction of the force $2F$. From equilibrium, at the middle of the string:

$$N \cdot cos\varphi = F$$

The tension force in the string is:

$$N = \frac{F}{cos\varphi} = F \cdot \frac{\sqrt{L^2 + \Delta^2}}{\Delta}$$

If the elasticity of the string would be such that unit extension would be produced by a tensile force k, then

$$\sqrt{L^2 + \Delta^2} - L = \frac{\sqrt{L^2 + \Delta^2}}{\Delta} \cdot \frac{F}{k} = \frac{N}{k}$$

or if rearranged

$$\frac{F}{k} = \Delta \cdot \left(1 - \frac{L}{\sqrt{L^2 + \Delta^2}} \right) \tag{5.7}$$

It can be observed that a very small Δ produces a high rate of increase in Δ in respect to the force F, as shown in Fig. 5.6.

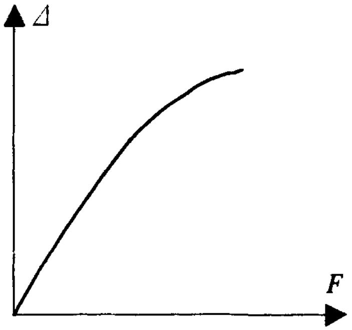


Figure 5.6: Non-linearity of a displacement with respect to a force

Case B) The second case is shown in Fig. 5.7, where a simple beam is supported in a way that movement at support *B* is possible and is acted on by a pair of forces *H* with a slight initial eccentricity of *e*. Potential energy is stored in the structure in two ways:

1. Direct *compression* of the beam due to axial force *H*
2. Deflection due to *constant bending* along the whole length

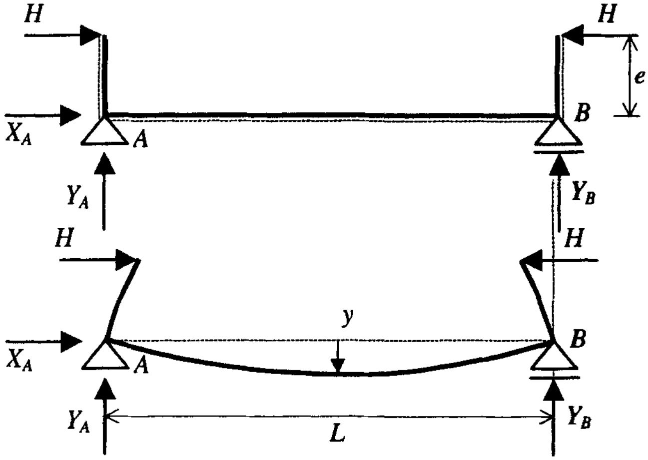


Figure 5.7: Geometrical non-linearity (buckling)

Maximal stresses in the beam along axis are:

$$\sigma_{max} = \frac{H}{A} + H \cdot \frac{(e+y)}{W} = \frac{H}{A} \cdot \left(1 + \frac{A}{W} \cdot (e+y)\right) \tag{5.8}$$

The deflection y obviously depends on the magnitude of horizontal force H, but the stresses will not vary linearly even by the linear stress-strain law.

This example of *non-linear behaviour in compression* is of great structural importance and is called *buckling*.

The principle of superposition is valid if all deformations are so small that no significant change of the structure occurs, as even small changes in geometry can have a considerable influence on structural behaviour.

5.3 Compatibility

Deformation of any structure as a result of applied loads arises from the deformation of the elements of which it is composed as can be concluded from Fig. 5.7. Deformations are a combination of two causes: shortening due to the compressive axial force and from bending due to the bending moment. The bending moment gives a compressive stress on the concave side of the beam and tensile stress on the convex side of the beam.
Consider now a simple truss in Fig. 5.8:

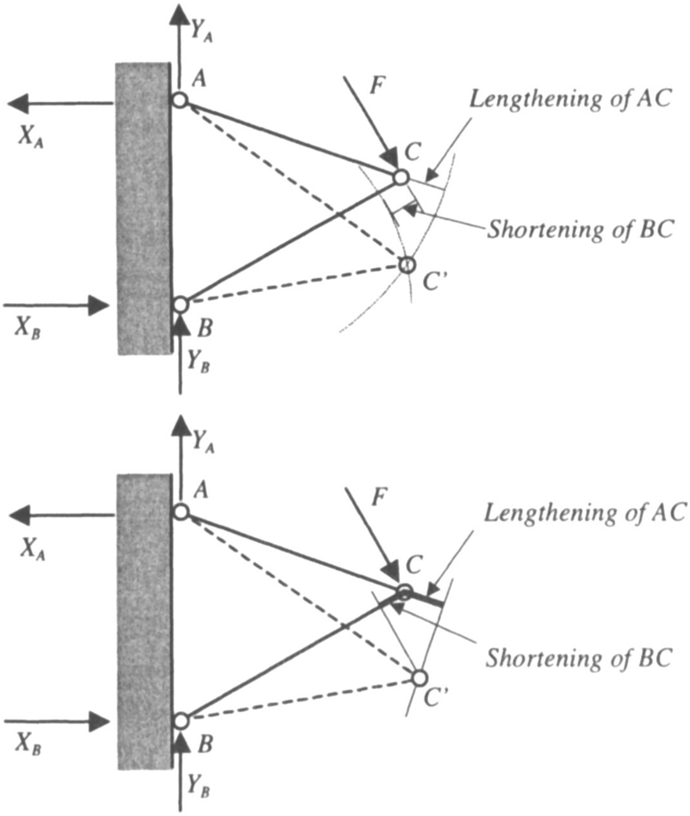


Figure 5.8: Truss compatibility (Villiot diagram)

At applied load F element AC stretches by Δ_{AC} and element BC shortens by Δ_{BC}. The new position of point C is found where two arcs with centres at A and C meet having radii equal to the lengths of the respective elements.

As the deformation of elements are assumed small, tangents can be used instead of chords to obtain a picture of deformations of satisfactory accuracy. This method is called the *Villiot diagram of displacements.*

The truss in Fig. 5.9 is internally indeterminate, as from Eqn. (3.9), $m = 2j$ so that the number of unknown forces in elements is greater than the number of equations related to the joints ($m = 7$, $j = 3$ = *free joints*). As explained earlier the number of support forces (*4* in our case) has no influence on *internal* indeterminacy.

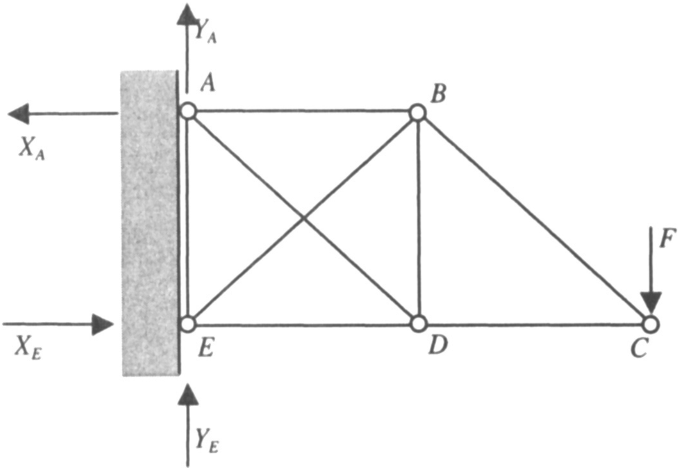


Figure 5.9: Internally indeterminate truss

The redundant element AD is removed and thus the truss becomes determinate. The displacement of point B will be to the right as element AB lengthens and downward due to the shortening of element EB. Point D moves to the left due to the shortening of element ED and downward as diagonal AD lengthens.

All these element deformations must occur in such a manner that the structure after deformation, remains a compatible whole.

Therefore the length of AD from Fig. 5.10 must equal the deformed length of element AD from the original truss and the compatibility equation can be written as equation (5.9):

$$\begin{bmatrix}\textit{Increase of distance AD} \\ \textit{from Fig. 5.10} \\ \end{bmatrix} - \begin{bmatrix}\textit{Shortening of AD due to} \\ \textit{redundant force R from} \\ \textit{Fig. 5.11}\end{bmatrix} = \begin{bmatrix}\textit{Increase of distance AD due} \\ \textit{to tensile force from} \\ \textit{Fig.5.9}\end{bmatrix} \tag{5.9}$$

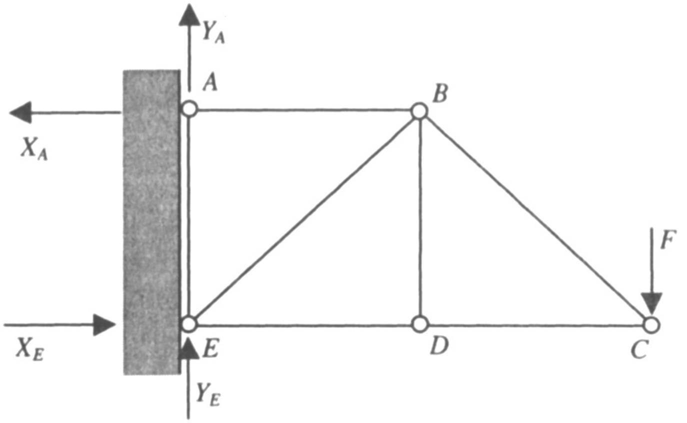


Figure 5.10: Determinate truss

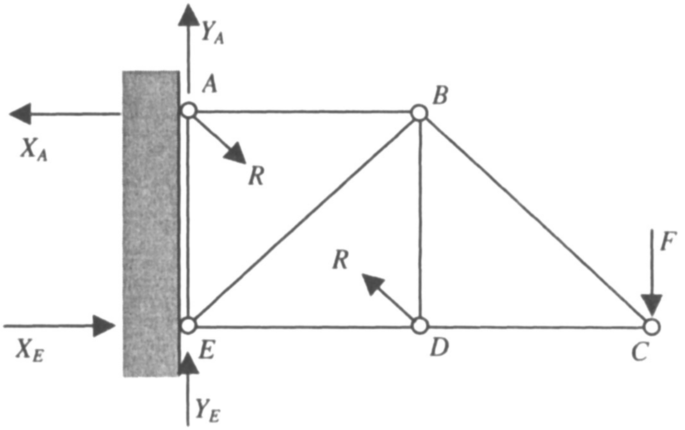


Figure 5.11: Force *R* restores compatibility

The redundant force *R*, which cannot be calculated from equilibrium equations, can thus be calculated from the compatibility equation (5.10).

The use of compatibility will be shown on the *1x* indeterminate beam from Fig. 5.12, where the right hand support is removed as a redundant quantity.

Load q will produce deflection δ_q, therefore the beam is loaded by force *F*, which produces deflection δ_F. From compatibility on the original beam the sum of both deflections must equal zero.

The deflection produced by uniform load q is (see Ch.. 6 for deformation calculation):

$$\delta_q = \frac{q \cdot L^4}{8 \cdot EI} \qquad (5.10a)$$

and due to force F

$$\delta_F = \frac{F \cdot L^3}{3 \cdot EI} \qquad (5.10b)$$

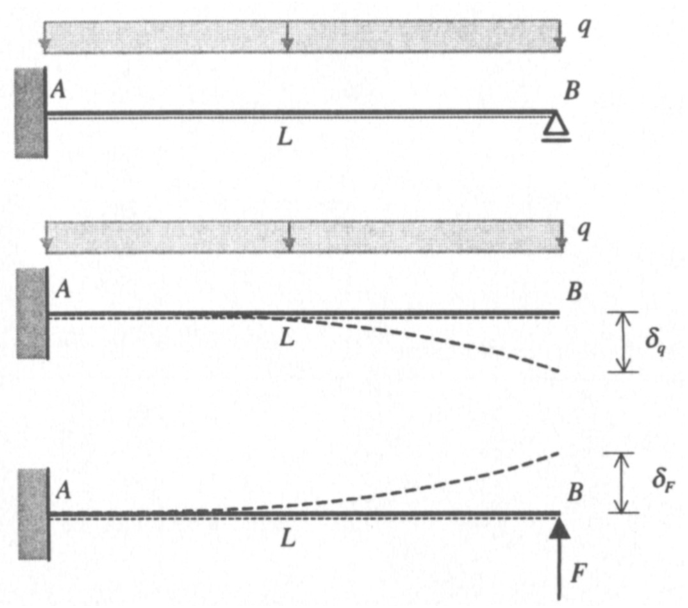


Figure 5.12: Compatibility of deflections

From the compatibility of deflections $\delta_q = \delta_F$, force F can be calculated:

$$\frac{F \cdot L^3}{3 \cdot EI} = \frac{q \cdot L^4}{8 \cdot EI} \qquad (5.10c)$$

$$F = \frac{3}{8} \cdot q \cdot L$$

Force F therefore restores the initial state (compatibility) and represents a vertical reaction at the right hand support.

5.4 Work

Consider element *AB* in Fig. 5.13a loaded by a tensile force *F* at the free end. The element will lengthen by *e* as can be shown in the force-deformation diagram Fig. 5.13b.

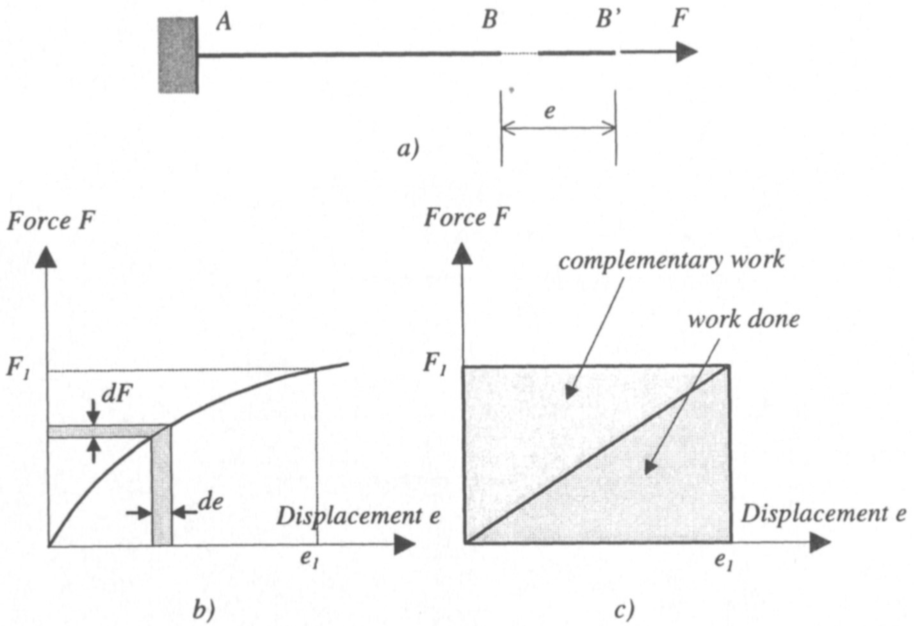


Figure 5.13: Work done by a force

The area below the curve is the *work done by the force* in moving its point of application from *B to B′*, and the *potential energy* accumulated in the element (deformational energy, strain energy) *is equal to work done*:

$$U = strain\ energy\ = work\ done = \int_0^e F \cdot de \tag{5.11}$$

Integration must be carried through the entire range of deformation. The area above the curve is the *complementary work* and is equal to

$$\int_0^{F_1} e \cdot dF\ , \tag{5.12}$$

or written in an explicit form:

$$C = complementary\ \ energy = complementary\ work\ done = \int_0^{F_1} e \cdot dF \qquad (5.13)$$

Equations (5.11) and (5.13) are very important in structural analysis though conceptual difficulties arise because no physical meaning can be defined for the complementary energy. However, if linear behaviour is assured then both energies will be equal as shown in Fig. 5.13c.

5.5 Linearity of load and deformation

If an *elastic body is loaded* by a system of forces and is in equilibrium (maintained by reaction forces) *it will deform.* Points of load application (as all other points except supports) will move in a general direction, as shown in (Figure 5.14).

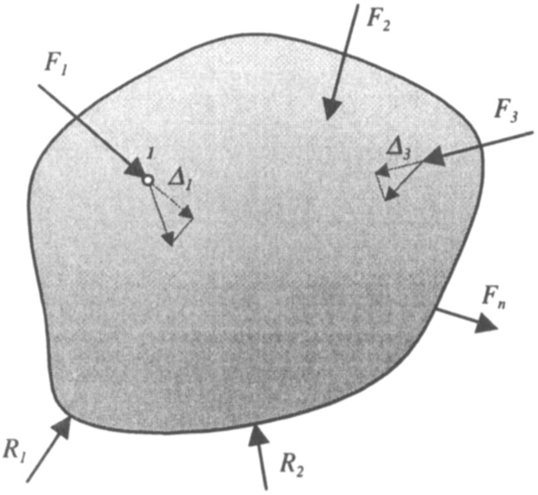


Figure 5.14: A rigid body under system of loads

If point *1* is the application point of force F_1, then at this point a corresponding displacement exists caused by all forces:

$$\Delta_1 = f_{11} \cdot F_1 + f_{12} \cdot F_2 + f_{13} \cdot F_3 + ... + f_{1n} \cdot F_n \qquad (5.14)$$

Coefficients f_{1n} are flexibility coefficients or displacements in the direction of force F_1 caused by forces F_1, F_2 ... F_n. The summation may include all independent forces but not the reactions, as they were calculated from applied forces to ensure equilibrium.
Displacements at other points are:

$$\Delta_2 = f_{21} \cdot F_1 + f_{22} \cdot F_2 + f_{23} \cdot F_3 + \ldots\ldots + f_{2n} \cdot F_n$$
$$\Delta_3 = f_{31} \cdot F_1 + f_{32} \cdot F_2 + f_{33} \cdot F_3 + \ldots\ldots + f_{3n} \cdot F_n$$
$$\vdots$$
$$\Delta_n = f_{n1} \cdot F_1 + f_{n2} \cdot F_2 + f_{n3} \cdot F_3 + \ldots\ldots + f_{nn} \cdot F_n$$

or written in a matrix form:

$$\{\Delta\} = [f] \cdot \{F\} \tag{5.15}$$

where $\{\Delta\}$ represents generalised displacement vector ($n \times 1$) caused by forces $\{F\}$ of ($n \times 1$) and matrix $[f]$ is a flexibility matrix of order ($n \times n$).
Using equation (3.1) we finally obtain:

$$\{F\} = [k] \cdot \{u\} \Rightarrow \{u\} = \frac{1}{[k]} \cdot \{F\} = [f] \cdot \{F\} \tag{5.16}$$

$$[k]^{-1} = [f] \tag{5.17}$$

The various phases of loading and structural behaviour are shown in Fig. 5.15.

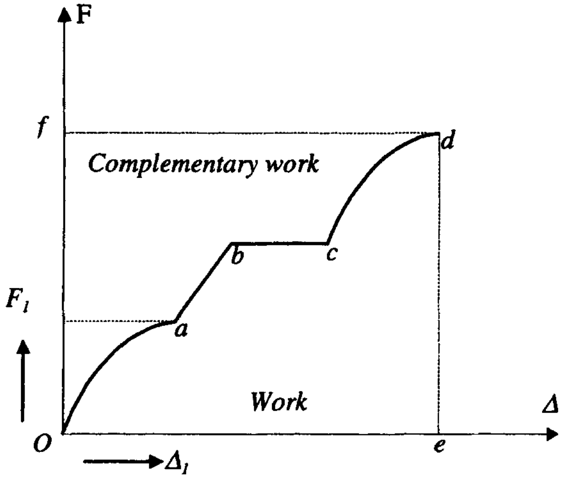


Figure 5.15: A general relation between load and deformation

Work done by force F_1 from Fig. 5.15 is

$$\int F_l \cdot d\Delta_l\,, \tag{5.18}$$

which is the area between the curve *Oabcde* and the deformation axis; complementary work done by the same force is the area between *Oabcdf* and the load axis:

$$\int \Delta_1 \cdot dF_1 \tag{5.19}$$

It has to be emphasised that on line *bc*, work increases even though force F_l remains unchanged since the area under the curve increases. The work done by force F_l in this case was done on the displacements caused by *other forces.*

In general work done by a force is

$$\int F_1 \cdot d\Delta_1 = \eta_1 \cdot F_1 \cdot \Delta_1\,, \tag{5.20}$$

Coefficient η_1 depends on the shape of load and deflection relation and is

$$0 < \eta_1 < 1$$

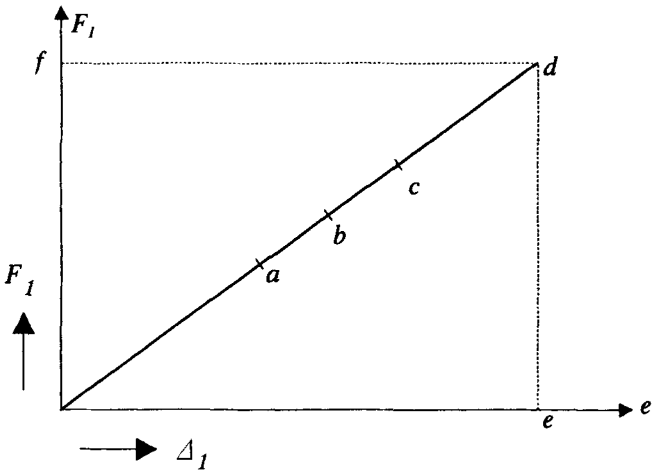


Figure 5.16: Linear increase in deformation

It is easy to see from Fig. 5.16 that

$$\frac{1}{2}Oed = \frac{1}{2} \cdot F_l \cdot \Delta_l$$

and

$$\eta_l = \frac{1}{2}$$

The linear relation could exist for any other load, therefore the work done is:

$$U = \frac{1}{2} \cdot F_1 \cdot \Delta_1 + \frac{1}{2} \cdot F_2 \cdot \Delta_2 + \ldots\ldots + \frac{1}{2} \cdot F_n \cdot \Delta_n = \frac{1}{2} \cdot \Sigma\, F \cdot \Delta \tag{5.21}$$

$$U = \sum_{1}^{n} \eta_i \cdot F_i \cdot \Delta_i$$

Later we shall see that the work done at the final position is independent of the way we reached that position, as the principle of the conservation of energy requires:

$$W = \frac{1}{2} \cdot F^T \cdot \Delta \tag{5.22}$$

The transposition sign is used, as work is a scalar quantity. In general, forces will change independently and we will write the total work by the following equation:

$$W = \eta \cdot F^T \cdot \Delta \tag{5.23}$$

A pressure p acting on a small elementary area represents distributed loads (uniform load). The stress vector is

$$p = \{p_x \quad p_y \quad p_z \quad m_x \quad m_y \quad m_z\}^T,$$

and the corresponding displacements are:

$$\delta = \{u \quad v \quad w \quad \varphi_x \quad \varphi_y \quad \varphi_z\}^T$$

In the above equation each coefficient corresponds to the appropriate term in the load vector and the work done is:

$$F = \int p^T \cdot dA \tag{5.24}$$

$$U = \frac{1}{2} \cdot \int p^T \cdot \delta \cdot dA \tag{5.25}$$

The total work on a structure is therefore:

$$U_{total} = work\ done\ by\ all\ forces = \frac{1}{2} \cdot F^T \cdot \Delta + \frac{1}{2} \cdot \int p^T \cdot \delta \cdot dA \tag{5.26}$$

5.6 Strain energy

In this book, only conservative structural systems will be considered. This implies that *all work done* by external forces on corresponding displacements will be *converted into kinetic and strain energy* and no energy loses will occur.

According to the principle of *conservation of energy*, when a structure is gradually loaded, *the kinetic energy is zero* and *the work done by external loads W is equal to the strain energy.*

The energy is related to the internal forces and deformations they cause and is stored in the structure as a potential energy due to axial forces, bending moments, shearing forces and twisting moments (torsion).

In the special case of elastic structures the potential energy is released in a way that the structure returns into its initial position. The potential energy in form of strains in the case of elastic structures is given by equation:

$$U = \frac{1}{2} \cdot \sigma^T \cdot \varepsilon \tag{5.27}$$

σ and ε are stress and strain vectors.

Normally stresses and deformations are expressed by internal forces (axial forces, bending moments, shearing forces and torque) and by the displacement vector. Energy in this way is given by

$$U = \frac{1}{2} \cdot F^T \cdot e \tag{5.28}$$

There will be equality between work done by the applied forces and the strain energy U stored in the structure $(\eta = 0.5)$:

$$U = work\ done = \eta \sum_{1}^{n} F_i \cdot \Delta_i + \eta \int p^T \cdot \delta \cdot dA = energy\ stored \tag{5.29}$$

The same can be written for complementary energy:

$$C = complementary\ work\ done = (1 - \eta) \sum_{1}^{n} F_i \cdot \Delta_i + (1 - \eta) \int p^T \cdot \delta \cdot dA \tag{5.30}$$

5.7 Superposition of strain energies

Energy is stored in a structure at each application of the load. Therefore it is thought that the energy, stored as a result of each load acting separately, could be added to give the effect of the loads acting simultaneously as the energy is scalar.

There are some exceptions as can be seen from Fig. 5.14 in the region *bc*, point *1* was displaced due to the action of other forces but F_1 remained unchanged. The result is increase in strain energy as F_1 actually moved and work was done.

❖ *If a system of forces initiate work done by some other forces then simple addition of energy is inadmissible.*

Example 5.1: An element is subjected to axial force and torque. The energy stored is from two kinds of deformation these being due to axial and shear stresses. Both effects are different therefore the energy can simply be added.

If shear forces, beside torque, acted upon the same member then energy from both sources cannot be added as they cause deformations on the same plane. In such cases addition of the energies may only be made if effects *of all loads on the displacements of others* is taken into account.

Example 5.2: The truss in Fig. 5.17 is made of elements of equal cross section and of the same material. The structure is symmetric about the vertical line as is the system of loads F_2 and F_3 and no horizontal movement of the structure occurs (no displacement u at the point of application of force F_1). Force F_1 causes the displacement, which does not correspond to forces F_2 or F_3. It follows that

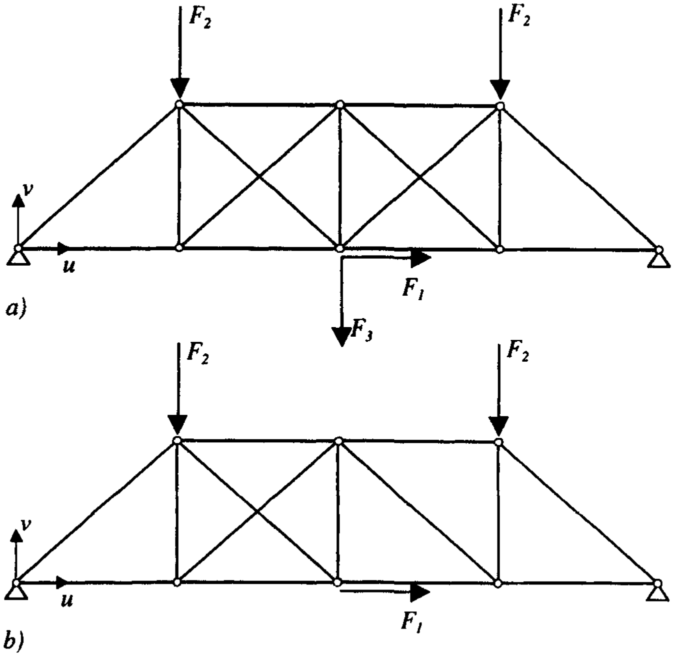


Figure 5.17: Superposition of work done by forces

$$u_{F_2+F_1} = u_{F_2} + u_{F_1}$$

$$u_{F_3+F_1} = u_{F_3} + u_{F_1}$$

$$u_{F_2+F_3} = u_{F_2} + u_{F_3}$$

The above conclusions are deducted from the symmetry of the structure but, if the structure is not symmetric (Figure 5.17b) then:

$$u_{F_2+F_1} \neq u_{F_2} + u_{F_1},$$

as the application of forces F_2 cause displacement corresponding to the direction of the force F_1.

5.8 Reciprocity of influence coefficients

It has been explained earlier that:

- f_{mn} is a *flexibility influence coefficient*, that is a displacement in the direction *m* caused by a unit force in the direction *n*
- k_{nm} is a *stiffness influence coefficient*, that is a force in the direction *n* that causes a unit displacement in the direction *m*

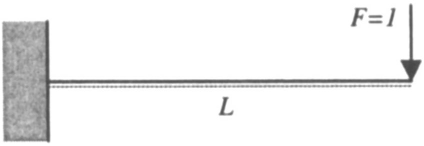


Figure 5.18: Flexibility

Flexibility: $$f_{mn} = \frac{F \cdot L^3}{3 \cdot EI} = \frac{L^3}{3 \cdot EI}$$

Stiffness: $$k_{nm} = \frac{1}{f_{mn}} = \frac{3 \cdot EI}{L^3}$$

5.8.1 Betty-Maxwell theorem

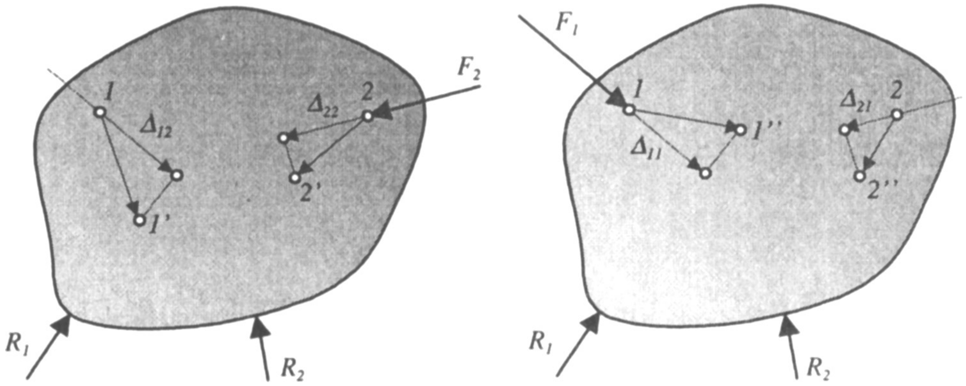


Figure 5.19: Loading of a body by two forces

Force F_2 causes the displacement of point *1* to *1'* and of point *2* to *2'* together with rotations at both points; the displacements in the directions of forces F_1 and F_2 are the corresponding displacements:

$$\Delta_{12} = f_{12} \cdot F_2 \qquad and \qquad \Delta_{22} = f_{22} \cdot F_2$$

If force F_1 is applied points move to *1"* and 2" and the corresponding displacements are:

$$\Delta_{21} = f_{21} \cdot F_1 \qquad and \qquad \Delta_{11} = f_{11} \cdot F_1$$

Let us first load a body by the force F_2, the work done is

$$\frac{1}{2} \cdot F_2 \cdot \Delta_{22} = \frac{1}{2} \cdot F_2 \cdot \left(f_{22} \cdot F_2\right),$$

and after that by the force F_1, which produces work:

$$\frac{1}{2} \cdot F_1 \cdot \Delta_{11} = \frac{1}{2} \cdot F_1 \cdot \left(f_{11} \cdot F_1\right)$$

but causes the displacement of point of application of force F_2, which produces additional work:

$$F_2 \cdot \Delta_{21} = F_2 \cdot \left(f_{21} \cdot F_1\right)$$

It is very important to realise that the coefficient ½ was omitted as the point *2* moved at constant F_2. The *total work done by the two forces* and therefore the *strain energy stored* in the body is:

$$U_1 = \frac{1}{2} \cdot f_{22} \cdot F_2^2 + \frac{1}{2} \cdot f_{11} \cdot F_1^2 + f_{21} \cdot F_1 \cdot F_2 \tag{5.31}$$

Let us load the body in a reverse order, at first by force F_1,

$$\frac{1}{2} \cdot F_1 \cdot \Delta_{11} = \frac{1}{2} \cdot F_1 \cdot (f_{11} \cdot F_1),$$

and then by force F_2

$$\frac{1}{2} \cdot F_2 \cdot \Delta_{22} = \frac{1}{2} \cdot F_2 \cdot (f_{22} \cdot F_2)$$

which will displace point *1* as well, thus

$$F_1 \cdot \Delta_{12} = F_1 \cdot (f_{12} \cdot F_2)$$

The total work is:

$$U_2 = \frac{1}{2} \cdot f_{11} \cdot F_1^2 + \frac{1}{2} \cdot f_{22} \cdot F_2^2 + f_{12} \cdot F_1 \cdot F_2 \tag{5.32}$$

According to the principle of conservation of energy, the energies obtain by the different ways of loading must be equal, therefore:

$$U_1 = U_2$$

and finally from Eqns. (5.31) and (5.32):

$$f_{12} = f_{21} \tag{5.33}$$

This is the *reciprocal theorem* or *Betty-Maxwell theorem of reciprocity* that can be graphically represented as shown in Fig. 5.20.

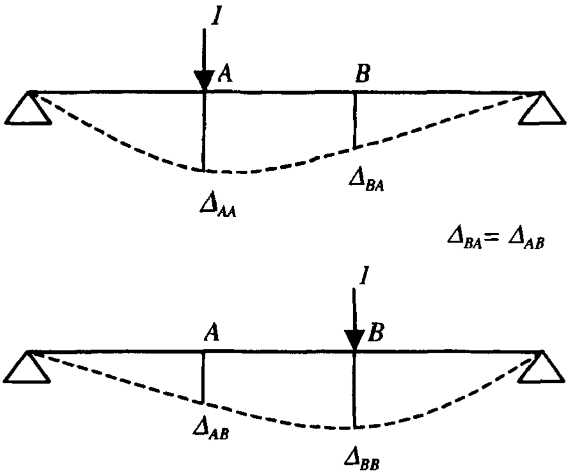


Figure 5.20: Betty-Maxwell theorem

- *Theorem: The displacement at point B caused by the force at point A is equal to the displacement at point A caused by the equal force at point B.*

- *The consequence of the theorem is that all structural matrices are symmetrical*

5.8.2 Stiffness coefficients

The structure in Fig. 5.21 is in equilibrium such that the work done by reactions is zero. Consider points *r* and *s* on the structure.

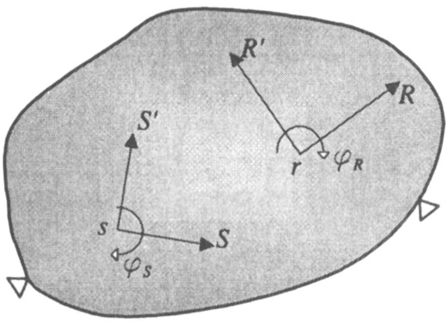


Figure 5.21: Stiffness coefficients

Let us displace point *r* by a *unit displacement* in the direction of force *R* while six forces suppress all other displacements at *r* and *s*.

$$k_{RR},\ k_{R'R},\ k_{\varphi_R R},\ k_{SR},\ k_{S'R}\ \ and\ \ k_{\varphi_S R}$$

are the influence coefficients valid only for the displacements of *r* in the direction of force *R*. If the magnitude of the displacements is q_R the forces are:

$$k_{RR} \cdot q_R,\ k_{R'R} \cdot q_R,\ k_{\varphi_R R} \cdot q_R \qquad at\ point\ \ r$$

and

$$k_{SR} \cdot q_R,\ k_{S'R} \cdot q_R,\ k_{\varphi_S R} \cdot q_R \qquad at\ point\ s$$

In this state the structure is loaded by the displacement q_S at point *s* (all other displacements are suppressed). The total work done is

$$U_1 = \frac{1}{2} \cdot k_{RR} \cdot q_R^2 + k_{SR} \cdot q_R \cdot q_S + \frac{1}{2} \cdot k_{SS} \cdot q_S^2 \tag{5.34}$$

and by reverse order of loading by displacements (first q_S and then q_R)

$$U_2 = \frac{1}{2} \cdot k_{RR} \cdot q_R^2 + k_{RS} \cdot q_R \cdot q_S + \frac{1}{2} \cdot k_{SS} \cdot q_S^2 \tag{5.35}$$

Since $U_1 = U_2$ we get the reciprocity of stiffness coefficients $k_{RS} = k_{SR}$

5.8.3 Application of Betty-Maxwell theorem

In many structures a number of different load conditions are possible. For instance, a traffic load moving across a bridge will cause different forces, depending on its location. In such cases, the structural engineer must determine that load position which will be critical on its effect in terms of the forces on the structure. An influence line (*Green's function*) is a useful tool to accomplish this task.

Consider the example in Fig. 5.22. Let *F* be at point *x* causing a vertical deflection δ_X and deflection δ_C at point *C*. Using flexibility coefficients:

$$\delta_X = f_{XX} \cdot F \tag{5.36}$$

$$\delta_C = f_{CX} \cdot F \tag{5.37}$$

Suppose that the beam is now loaded at point *C* by a unit force causing displacements at points *X* and *C* of f_{XC} and f_{CC}.

$\delta_X^C = f_{XC} \cdot 1$ and $\delta_C^C = f_{CC} \cdot 1$

From the reciprocal theorem:

$f_{XC} = f_{CX}$

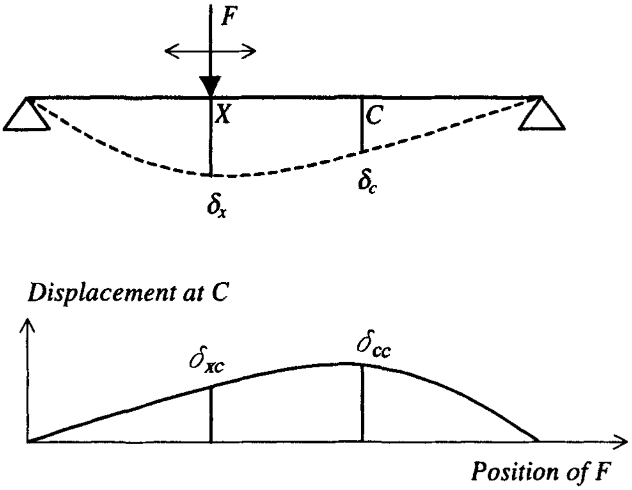


Figure 5.22: Displacement of a chosen point caused by a moving load

- ❖ ***A vertical displacement of any point of the beam caused by unit force at C equals a vertical displacement at point C due to a unit load at any other point on the beam.***

- ❖ ***Theorem: The deflected shape of a structure caused by a unit force is the influence line for deformation corresponding to that unit force.***

Example 5.3: Find the influence line for the deflection of point *A*.

At point *A* a unit force is applied and *the deformed shape of the beam is the influence line for the deflection* of point *A* (note that real numerical values will be calculated in Ch. 6).

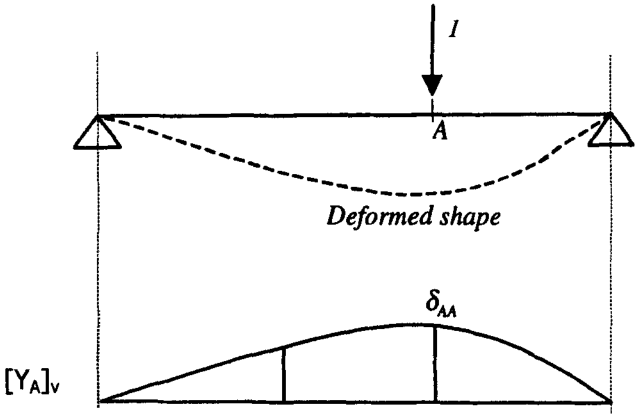


Figure 5.23: Influence line for deflection at *A*

5.9 Betti's theorem

A force system $\{F\}$ acts at arbitrary point on a structure causing displacements $\{\Delta_P\}$ at all points of the structure. The equations may be written in matrix form

$$\{\Delta_P\} = [f] \cdot \{F\} \tag{5.38}$$

where $[f]$ is a *flexibility matrix* that is symmetrical about a leading diagonal, hence

$$[f] = [f]^T$$

Some other system of forces $\{Q\}$ will cause displacements $\{\Delta_Q\}$:

$$\{\Delta_Q\} = [f] \cdot \{Q\}$$

Using a matrix algebra rules

$$\begin{aligned}\{F\}^T \cdot \{\Delta_Q\} = \{F\}^T \cdot [f] \cdot \{Q\} = \left[\{F\}^T \cdot [f] \cdot \{Q\}\right]^T = \\ = \{Q\}^T \cdot [f]^T \cdot \{F\} = \{Q\}^T \cdot [f] \cdot \{F\} = \{Q\}^T \cdot \{\Delta_P\}\end{aligned}$$

we obtain

$$\{F\}^T \cdot \{\Delta_Q\} = \{Q\}^T \cdot \{\Delta_P\} \tag{5.39}$$

5.10 Mueller-Breslau principle

5.10.1 The principle

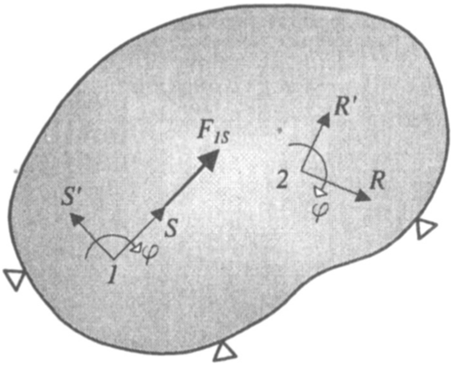


Figure 5.24: Work done by two force systems

The structure shown in Fig. 5.24 is in equilibrium so that the work done by the reactions is zero. Consider two different load systems applied to the structure in turn.

In the first system let the load act at point *1* in the direction S, the force F_{1S} causes displacements Δ. The displacements at point *2* are zero as no discontinuity can occur (except at point *1*).

First system	*At point 1*			*At point 2*		
Forces (F)	F_{1s}	*0*	*0*	F_{2R}	$F_{2R'}$	$F_{2\varphi}$
Displacements(Δ_p)	Δ_{1S}	$\Delta_{1S'}$	φ_1	*0*	*0*	*0*

Now let the second system of forces be obtained by the displacement of point *2* of magnitude δ_{2R} in the direction of R but no other displacements are permissible at point *2*. There is no load at point *1* that moves by δ in all three possible directions.

Second system	*At point 1*			*At point 2*		
Forces (Q)	*0*	*0*	*0*	Q_{2R}	$Q_{2R'}$	$Q_{2\varphi}$
Displacements (Δ_Q)	δ_{1S}	$\delta_{1S'}$	$\delta_{1\varphi}$	δ_{2R}	*0*	*0*

The reciprocal Betti's theorem is now applied:

$$\{F\}^T \cdot \{\Delta_Q\} = \{Q\}^T \cdot \{\Delta_P\}$$

$$[F_{1S} \quad 0 \quad 0 \quad F_{2R} \quad F_{2R'} \quad F_{2\varphi}]\cdot\begin{bmatrix}\delta_{1S}\\ \delta_{1S'}\\ \delta_{1\varphi}\\ \delta_{2R}\\ 0\\ 0\end{bmatrix}=[0 \quad 0 \quad 0 \quad Q_{2R} \quad Q_{2R'} \quad Q_{2\varphi}]\cdot\begin{bmatrix}\Delta_{1S}\\ \Delta_{1S'}\\ \Delta_{1\varphi}\\ 0\\ 0\\ 0\end{bmatrix}$$

The application of the corresponding matrix multiplication gives:

$$F_{1S}\cdot\delta_{1S}+0\cdot\delta_{1S}+0\cdot\delta_{1\varphi}+F_{2R}\cdot\delta_{2R}+0+0=0$$

$$F_{2R}\cdot\delta_{2R}=-F_{1S}\cdot\delta_{1S}$$

which gives the so called *transformation equation*

$$F_{2R}=-\frac{\delta_{1S}}{\delta_{2R}}\cdot F_{1S} \tag{5.40}$$

- The force F_{2R} (at point 2 in the direction of R) as a result of unit force F_{1S} (at point 1 in the direction of S) can be determined independently using the displacement δ_{1S}, caused by unit displacement of point 2, corresponding to force F_{2R}.

- *Theorem: The deflected shape of a structure due to the particular unit distortion represenst the influence line for the effect corresponding to that distortion.*

5.10.2 Application of the principle

Example 5.4: Determination of a shear force by the Mueller-Breslau principle

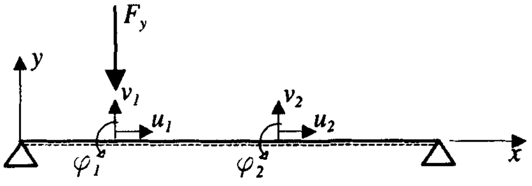


Figure 5.25: Shear force determination at point 2

First system	*At point 1*			*At point 2*		
Forces (F)	*0*	F_y	*0*	N_2	Q_2	M_2
Displacements(Δ_p)	u_1	v_1	φ_1	*0*	*0*	*0*

Second system	*At point 1*			*At point 2*		
Forces (Q)	*0*	*0*	*0*	N_2	Q_2	M_2
Displacements (Δ_Q)	u_1	v_1	φ_1	*0*	v_2	*0*

$$F_Y \cdot v_1 + Q_2 \cdot v_2 = 0$$

which gives the transformation equation:

$$Q_2 = -\frac{v_1}{v_2} \cdot F_Y$$

What does the above equation mean? It says that from *three measured quantities the fourth quantity can be calculated.* Practically it means that if a beam is loaded by a force *F* and two displacements are measured then by the use of the transformation equation, the shear force Q_2 can be calculated. Let us emphasise again that all deformations have to be small in comparison with structure dimensions.

Example 5.5: Determine the influence line for bending moment at point P

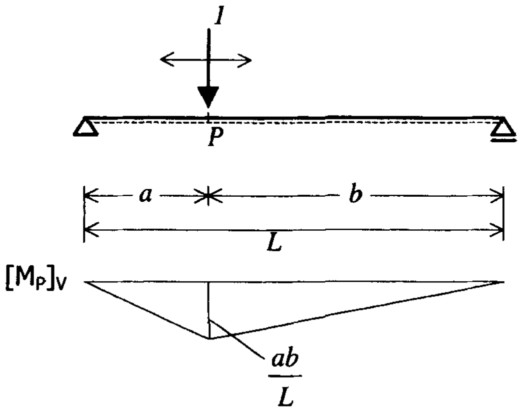


Figure 5.26: Influence line for a bending moment

Let us at first determine a bending moment by the static method. If the force is at point *P* the reactions are:

$$Y_A \cdot L = 1 \cdot b \quad \text{and} \quad Y_A = \frac{b}{L},$$

from which the bending moment at *P* is determined:

$$M_P = Y_A \cdot a = \frac{a \cdot b}{L} \tag{5.41}$$

According to the principle described, we have to determine the corresponding unit displacement (in this case a unit rotation at *P*) since we are looking for the bending moment.

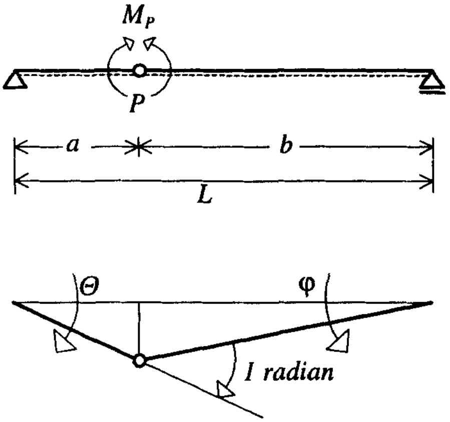


Figure 5.27: Mueler-Breslau principle

From geometry:

$$tg\theta = \frac{y_P}{a} \quad \Rightarrow \quad y_P = a \cdot \theta$$

$$tg\varphi = \frac{y_P}{b} \quad \Rightarrow \quad y_P = b \cdot \varphi$$

or from equality of y_p:

$$\varphi = \frac{a}{b} \cdot \theta \tag{5.42}$$

From the sum of both rotations at the supports:

$$\varphi + \theta = \frac{a}{b} \cdot \theta + \theta = 1 \qquad \Rightarrow \qquad 1 = \theta \cdot \left(1 + \frac{a}{b} \right)$$

or

$$\theta = \frac{1}{1 + \dfrac{a}{b}} = \frac{b}{a+b}, \tag{5.43}$$

Hence:

$$y_P = a \cdot \theta = \frac{a \cdot b}{a+b} = \frac{a \cdot b}{L}, \tag{5.44}$$

which is the same result as obtained by the static method. The same procedure is used to determine the shear force at *P*.

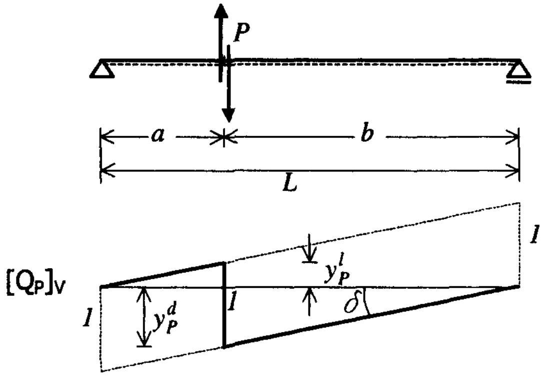


Figure 5.28: Mueler-Breslau principle to determine shear force

$$tg\delta = \frac{y_P^L}{a} \qquad \Rightarrow \qquad y_P^L = a \cdot \delta = \frac{a}{L}$$

$$\frac{y_P^D}{b} = \frac{1}{L} \qquad \Rightarrow \qquad y_P^D = \frac{b}{L}$$

The above displacements are all unit displacements and can be usefully applied in practice.

Example 5.6: Consider a continuous beam over four supports

a) When is Y_B a maximum? If the load is on spans *AB* and *BC*!

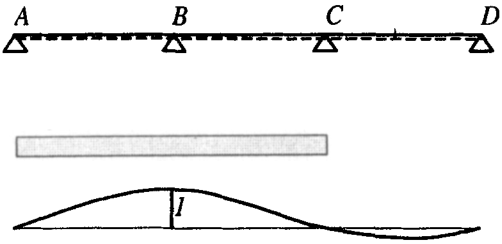


Figure 5.29: A reaction determination by the Mueler-Breslau principle

b) When is M_C a maximum?If the load is on spans *BC* and *CD*!

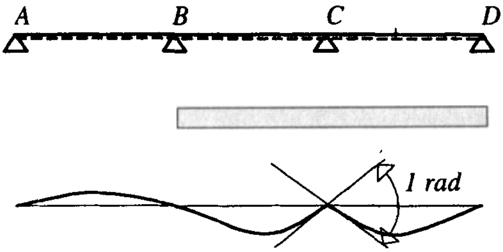


Figure 5.30: A bending moment determination by the Mueler-Breslau principle

c) When is Q_E a maximum? If the load is on spans *BC* and *ED* or on spans *AB* and *CE*!

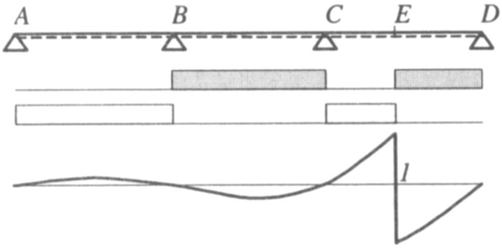


Figure 5.31: A shear force determination by the Mueler-Breslau principle

d) M_E will be positive and a maximum, if the load is on *CD* and *AB* and will be negative, if the load is on *BC*.

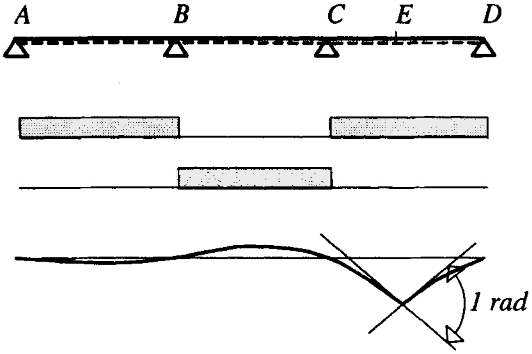


Figure 5.32: A field moment determination by the Mueler-Breslau principle

5.11 The principle of virtual work

A generalized system of forces represented by the vector *F* will produce both a reaction system of forces and an internal stress system denoted by a vector σ.

The energy stored in the structure in terms of internal stresses and strains will be different at any point of the structure and must be expressed as a function of their product for each elementary volume of material:

$$U = \int \left\{ \int_{0}^{\varepsilon} \sigma^T d\varepsilon \right\} dV \qquad (5.45)$$

For instance, the total energy in a skeletal structure composed of straight beam elements will be derived by integration over the entire cross-section (stresses vary from point to point) and subsequently by integration over the entire length of a beam (internal forces vary along length of an element).

The calculation can be simplified if it is noted that external forces produce *internal actions* F_N (axial force, shear force and bending moment in a *2D* case) at a certain section. It is then straightforward to calculate stresses from these internal actions at any section.

- A load system *F* will produce reactions and internal forces F_N, from which stresses σ can be calculated. A structure will change shape as a result of displacements, internal displacements and specific strains denoted by Δ, *e* and ε respectively.

A system of concurrent external forces $F_1, F_2 \ldots F_n$, which has a resultant force F_R, is acting upon a small particle. If a particle moves along by a displacement Δ_R in the direction of the force F_R, work will be done by all forces.

If Δ_R is small, so that the directions of forces and their magnitudes remain unchanged, then the work done through the corresponding displacements $\Delta_1, \ldots, \Delta_n$ is:

$$F_R \cdot \Delta_R = F_1 \cdot \Delta_1 + F_2 \cdot \Delta_2 + \ldots\ldots.. + F_n \cdot \Delta_n \tag{5.46}$$

Now, as the whole system is in equilibrium ($F_R = 0$)

$$F_1 \cdot \Delta_1 + F_2 \cdot \Delta_2 + \ldots\ldots.. + F_n \cdot \Delta_n = 0 \tag{5.47}$$

or

$$\sum F_i \cdot \Delta_i = 0 \tag{5.48}$$

❖ *Work done on systems in equilibrium is always zero.*

When a system is *elastic and in equilibrium* equation (5.47) can be written:

$$W_E = W_I \tag{5.49}$$

❖ *Work done by external and internal forces is equal.*

Let us imagine that points of a structure move by virtual displacements (these displacements must be possible but not necessarily produced) denoted by:

$\overline{\Delta}$	Displacements
$\overline{e}$	Deformations of elements
$\overline{\varepsilon}$	Specific strains

Work done by all forces is:

$$F^T \cdot \overline{\Delta} = F_N{}^T \cdot \overline{e} \tag{5.50}$$

If a system remains in equilibrium, energy is stored as deformational energy, given in an elemental form:

$$\int \sigma^T \cdot \overline{\varepsilon} \cdot d(vol) \qquad \text{or} \qquad \int_V \sigma^T \cdot \overline{\varepsilon},$$

leading to equation

$$F^T \cdot \bar{\Delta} = F_I{}^T \cdot \bar{e} = \int_V \sigma^T \cdot \bar{\varepsilon}^{\,*} \qquad (5.51)$$

The above equation is the form in which the principle of virtual work is used in structural analysis. The second term ($F \cdot \bar{e}$) includes *all internal forces* and their corresponding displacements (i.e. *N*, *Q* and *M* in *2D* cases).

Consider now a truss structure in which only axial forces occur:

$$F^T \cdot \bar{e} = \sum_1^n F \cdot \bar{e} \qquad n \ldots \textit{number of elements of a structure}$$

If a bending element is considered (see Ch 2.8.3), the effect of bending is

$$\Sigma \int M \cdot d\bar{\varphi}$$

and a virtual deformation $d\bar{\varphi}$ must be integrated at first over the entire length of a beam and after that a summation over all elements must be executed. The same procedure must be used considering shear forces *Q* and torque *T*.

The influence of a *uniform load* must be included in the *work done by external forces*:

$$P^T \cdot \bar{\Delta} + \int p^T \cdot \bar{\delta} \cdot dA = F^T \cdot e = \int_V \sigma^T \bar{\varepsilon} \qquad (5.52)$$

The principle of virtual work thus relates two independent systems:

❖ *system of forces in equilibrium*
F external forces
σ internal stresses (or internal forces F_I)

❖ *system of geometrically compatible deformations*
Δ displacements
ε specific strains

$$\sum F \cdot \Delta = \int_V \sigma \cdot \varepsilon \cdot dV \qquad (5.53)$$

---

[*] *Transformation for truss elements:*

$$\int_V \sigma^T \cdot \varepsilon = \int_V \frac{N}{A} \cdot \frac{\Delta L}{L} = \int N \cdot \Delta L \cdot \frac{dV}{A \cdot L} = N \cdot \Delta L \cdot \int \frac{dV}{A \cdot L} = N \cdot \Delta L = N \cdot e$$

We will remember that *one of the systems is always real* (the structure for which a solution is to be found) and that *the other system is virtual.* Therefore two possibilities exist:

- *Theorem of virtual forces*, in which a system of real displacements and deformations is related to virtual forces and stresses

  Σ (Virtual external forces)· (Real displacements) =
  = $\int$ (Virtual stresses)· (Real strains)· dV

- *Theorem of virtual displacements*, in which a system of real forces and stresses is related to virtual displacements and deformations

  Σ (Real external forces)· (Virtual displacements) =
  = $\int$ (Real stresses)· (Virtual strains)· dV

The first equation enables determination of displacements (*theorem of unit force*) and the second equation enables determination of forces (*theorem of unit displacement*).

5.12 Application of the principle of virtual work

In equation

$$P^T \cdot \overline{\Delta} = F^T \cdot \overline{e} = \int \sigma^T \cdot \overline{\varepsilon} \cdot dV$$

are, as mentioned previously, two systems (system of forces in equilibrium and system of compatible deformations).

If the deformed structure is chosen as a system of displacements then $\overline{\Delta}$ becomes real displacements corresponding to forces, $\overline{e}$ are real element deformations caused by internal forces and $\overline{\varepsilon}$ denotes specific strains at all points.

Let us choose the system of forces in such a way that a unit force acts at a point in the direction of the desired displacement. System P is in this case represented by *unit force only* as the work done by the reactive forces equals zero. Let us denote internal forces by F_I, they being the consequence of unit force and reactive forces. Let σ_I represent resultant stresses in the structure.

$$1 \cdot \Delta = F_I^T \cdot e = \int \sigma_I^T \cdot \varepsilon \cdot dV \tag{5.54}$$

Notation of $\Delta, \overline{e}, \overline{\varepsilon}$ is not necessary any more as these quantities are in this case real and finite values, Δ represents the quantity to be found. *Equation (5.54) is valid for all geometrically possible cases including temperature loading, creep, shrinkage etc.*

- *Equation (5.54) can explicitly be written for all linear elastic structures*

In such structures internal actions can be expressed by *axial force N, shear force Q, bending moment M* and *torsion moment T*, their corresponding deformations were given in Ch. 2 in Eqns. (2.47-2.54).

If all these deformations are the consequence of unit load, they can be summed up by

$$\Delta=\sum\frac{N_U\cdot N\cdot L}{E\cdot A}+\sum\int\frac{M_U\cdot M}{EI}\cdot ds+\sum K\cdot\int\frac{Q_U\cdot Q}{GA}\cdot ds \tag{5.55}$$

and if the temperature influence is taken into account, the total deformation is given in a general form

$$\Delta=\int_0^L\left[N_U\cdot\left(\frac{N}{E\cdot A}+\alpha_T\cdot T\right)+Q_U\cdot\frac{Q}{A_S\cdot G}+M_U\cdot\left(\frac{M}{EI}+\alpha_T\cdot\frac{\Delta T}{h}\right)\right]\cdot ds, \tag{5.56}$$

where Σ denotes summation from all elements and $\int ds$ the integration along the whole length of each of the elements. In many structures elements are of constant cross-section over the entire length and the equation can be simplified into:

$$\Delta=\sum\frac{N_U\cdot N\cdot L}{E\cdot A}+\sum\frac{1}{EI}\cdot\int M_U\cdot M\cdot ds+\sum\frac{K}{GA}\cdot\int Q_U\cdot Q\cdot ds \tag{5.56a}$$

In a three dimensional case there are six internal forces and in addition to the above quantities there is an influence of torsional moment given by:

$$\sum\int\frac{T_U\cdot T}{G\cdot I_P}\cdot dx \tag{5.57}$$

The notation is as follows:

❄	I_p	*polar second moment of area*
❄	N_U	*axial force caused by virtual force F=1*
❄	N	*axial force due to external loading*
❄	E	*modulus of elasticity*
❄	A	*cross-section*
❄	α_T	*temperature dilatation coefficient*
❄	T	*temperature increase*
❄	ΔT	*temperature gradient between upper on bottom fibre of an element*
❄	Q_U	*shear force caused by virtual force F=1*
❄	Q	*shear force due to external loading*
❄	A_s	*shear cross-section*
❄	G	*shear modulus*
❄	M_U	*bending moment caused by virtual force F = 1*
❄	M	*bending moment due to external loading*
❄	h	*height of a beam*

Example 5.7: Calculate the deflection at free end of the cantilever beam and show the ratio of shear and bending deformation.

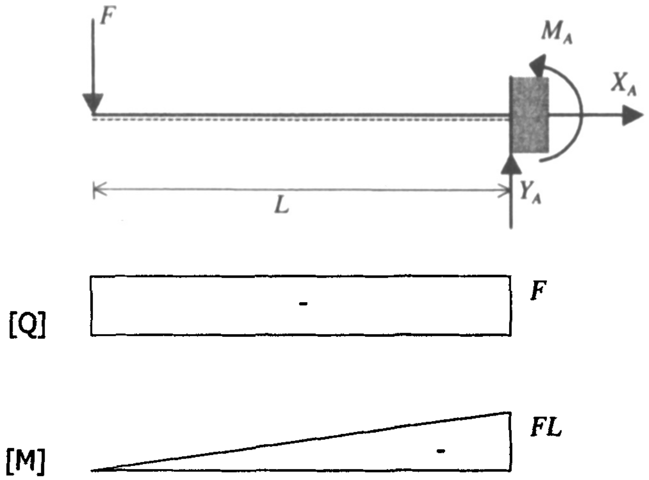


Figure 5.33: Diagrams of shear forces and bending moments

The unit force $F = 1$ is applied at the free end of the cantilever and the deflection is calculated from equation (5.55):

$$\delta_a = \int_0^L \left[\overline{Q} \cdot \frac{Q}{A_S \cdot G} + \overline{M} \cdot \frac{M}{EI} \right] \cdot ds$$

$$\delta_a = \int_0^L \left[(-1) \cdot \frac{(-F)}{A_S \cdot G} + (-x) \cdot \frac{(-F \cdot x)}{EI} \right] \cdot dx = \int_0^L \left(\frac{F}{A_S \cdot G} + \frac{F \cdot x^2}{EI} \right) \cdot dx =$$

$$= \left| \frac{F \cdot x}{A_S \cdot G} + \frac{F \cdot x^3}{3 \cdot EI} \right| = \frac{F \cdot L}{A_S \cdot G} + \frac{F \cdot L^3}{3 \cdot EI}$$

From the equation the bending part equals

$$\frac{F \cdot L^3}{3 \cdot EI}$$

and the shearing part is

$$\frac{F \cdot L}{A_S \cdot G}$$

Let us put both deflections into a ratio:

$$R = \frac{FL^3}{3EI} \cdot \frac{A_S G}{FL} = \frac{FL^3}{3E\dfrac{BH^3}{12}} \cdot \frac{\dfrac{BH}{1.2} \cdot \dfrac{E}{2(1+v)}}{FL} \cong 1{,}41 \cdot \left(\frac{L}{H} \right)^2$$

From the above equation the *shearing part in beams of L/H > 10 is less than 7%* and in *plates of L/H > 30 is less than 1%.* As we can see the shearing part of the deflection is always less than 10% and is in comparison with the total deflection negligible in practice.

- *Shearing part of the deformation is in engineering practice negligible.*

- *Note: Computer programs take into account all deformations (axial, shear, bending), therefore the results of hand calculations differ from that calculated by programs.*

Equation (5.55), because of the above reasons, is usually simplified as:

$$\Delta_i = \int \frac{\overline{M} \cdot M}{EI} \cdot dx \qquad (5.58)$$

In the above equation vector Δ includes both *displacements and rotations*. If a displacement is to be found a virtual force $\overline{F} = 1$ will be applied and a virtual moment $\overline{M} = 1$ will be applied when seeking rotations.

5.13 Castigliano's theorems

The total strain energy resulting from a system of forces is

$$U = \sum_{1}^{n} \int F d\Delta$$

In a linear elastic structure subjected to loads $F_1 ... F_n$ the corresponding deformation is given by the equation (5.14):

$$\Delta_1 = f_{11} \cdot F_1 + f_{12} \cdot F_2 + f_{13} \cdot F_3 + ... + f_{1n} \cdot F_n$$

The work done by each applied load is

$$\frac{1}{2} \Sigma \, F_i \cdot \Delta_i$$

and therefore the work done by all loads, in this case of three forces, is for the sake of simplicity and taking $f_{12} = f_{21}$:

$$U = \frac{1}{2} F_1 (f_{11} \cdot F_1 + f_{12} \cdot F_2 + f_{13} \cdot F_3) + \frac{1}{2} F_2 (f_{21} \cdot F_1 + f_{22} \cdot F_2 + f_{23} \cdot F_3)$$

$$+ \frac{1}{2} F_3 (f_{31} \cdot F_1 + f_{32} \cdot F_2 + f_{33} \cdot F_3) = \qquad (5.59)$$

$$= \frac{1}{2} f_{11} \cdot F_1^2 + \frac{1}{2} f_{22} \cdot F_2^2 + \frac{1}{2} f_{33} \cdot F_3^2 + f_{12} \cdot F_1 \cdot F_2 + f_{23} \cdot F_2 \cdot F_3 + f_{31} \cdot F_3 \cdot F_1$$

The rate at which U increases with F_1 is given by differentiating equation (5.59) with respect to F_1 and since the loads are independent variables and the reactions produce no work:

$$\frac{\partial U}{\partial F_1} = f_{11} \cdot F_1 + f_{12} \cdot F_2 + f_{13} \cdot F_3 = \Delta_1$$

which is exactly the corresponding displacement Δ_1. The differentiation with respect to forces F_2 and F_3 gives a similar result, hence a general equation can be written

$$\frac{\partial U}{\partial F_i} = \Delta_i \qquad (5.60)$$

Equation (5.60) is known as *Second Castigliano's theorem.*

- ❖ *In an elastic system, the partial derivative of the strain energy U in respect to any selected force equals the displacement associated with this force.*

Since in linear elastic structures strain energy equals complementary energy $U = C$:

$$\frac{\partial C}{\partial \Delta_i} = \frac{\partial U}{\partial \Delta_i} = F_i \qquad (5.61)$$

Equation (5.61) is known as *First Castigliano's theorem.*

- ❖ *In an elastic system, the partial derivative of the strain energy U in respect to any selected displacement equals the force associated with this displacement.*

6

Deformations

6.1 Integration of a load function

Example 6.1: An integration of a load function will be shown on a simple cantilever beam loaded by an uniform load as in Fig. 6.1.

We begin from the basic differential relation (Eqn. 3.11) between a uniform load and a shear force:

$$\frac{dQ}{dx} = -q(x)$$

$$dQ_x = -q(x) \cdot dx = -q \cdot dx$$

$$Q_x = -q \cdot x + C_1$$

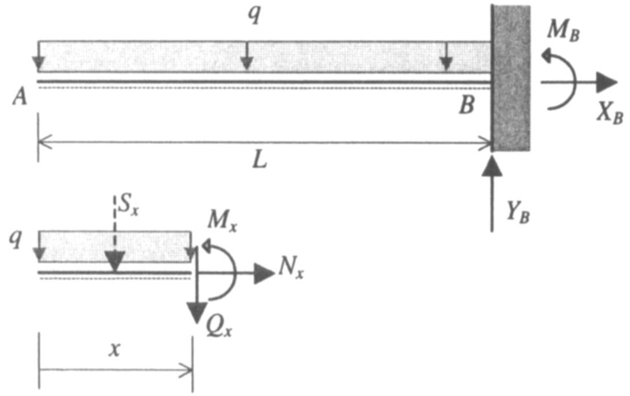


Figure 6.1: Cantilever beam and a free body

The constant C_1 is determined from boundary conditions for shear force on the beam which is zero at the free end:

$$\text{at } x = 0 \;\Rightarrow\; Q_A = 0 = Q_x$$

$$0 = -q \cdot 0 + C_1 \;\Rightarrow\; C_1 = 0$$

$$Q_x = -q \cdot x \tag{6.1}$$

Note: The same result is gained from the boundary condition at support *B*, that is at $x = L$: $-Q_A = -q \cdot L + C_1 \;\Rightarrow\; C_1 = 0$

A relation between shear force and bending moment is given by Eqn. (3.12) from Ch. 3:

$$\frac{dM_x}{dx} = Q_x$$

$$dM_x = Q_x \cdot dx = -q \cdot x \cdot dx$$

$$M_x = -q \cdot \frac{x^2}{2} + C_2$$

The boundary condition is: at $x = 0 \;\Rightarrow\; M_A = 0$

$$0 = -q \cdot \frac{0^2}{2} + C_2 \;\Rightarrow\; C_2 = 0$$

$$M_x = -q \cdot \frac{x^2}{2} \tag{6.2}$$

If this equation is integrated, we obtain the expression for rotation:

$$EI \cdot \varphi = \int M_x \cdot dx = \int \left(-\frac{q \cdot x^2}{2} \right) \cdot dx = -\frac{q \cdot x^3}{6} + C_3$$

The constant C_3 is once more determined from boundary conditions for the beam rotations. A rotation equals an angle of the tangent to the deformation line. A rotation is therefore zero at the point where the tangent is horizontal or parallel to the *x*-axis (in this example). Furthermore, by the definition of clamped support there is no rotation at that point, therefore:

$$\text{at } x = L \;\Rightarrow\; \varphi = 0 \quad \text{that is} \quad \varphi_B = 0$$

$$0 = -\frac{q \cdot L^3}{6} + C_3 \quad \Rightarrow \quad C_3 = \frac{q \cdot L^3}{6}$$

$$-EI \cdot \varphi = \frac{q \cdot L^3}{6} - \frac{q \cdot x^3}{6} \tag{6.3}$$

Equation (6.3) is a general equation for rotations of a beam that is loaded by a uniform load. The maximum rotation is at the free end at $x = 0$ and has the value of:

$$\varphi_{max} = \frac{q \cdot L^3}{6 \cdot EI} \tag{6.4}$$

Now we integrate the rotation:

$$-EI \cdot \frac{dy}{dx} = \frac{q \cdot L^3}{6} - \frac{q \cdot x^3}{6} \quad \Rightarrow \quad -EI \cdot y = \frac{q \cdot L^3 \cdot x}{6} - \frac{q \cdot x^4}{24} + C_4$$

Boundary conditions: the displacement y at support B is suppressed, hence:

at $x = L \quad \Rightarrow \quad y = 0$ therefore $y_B = 0$

$$0 = \frac{q \cdot L^3 \cdot L}{6} - \frac{q \cdot L^4}{24} + C_4 \quad \Rightarrow \quad C_4 = -\frac{q \cdot L^4}{8}$$

$$EI \cdot y = -(\frac{q \cdot L^3 \cdot x}{6} - \frac{q \cdot x^4}{24} - \frac{q \cdot L^4}{8}) \tag{6.5}$$

The above equation is a general equation of deflection of a beam that is loaded by a uniform load. The deflection has a maximum value at $x = 0$:

$$y_{max} = \frac{q \cdot L^4}{8 \cdot EI} \tag{6.6}$$

at $x = \frac{L}{2} \quad \Rightarrow \quad y = \frac{17 \cdot q \cdot L^4}{384 \cdot EI}$

Example 6.2: A cantilever beam is at the free end loaded by the concentrated force F. As the load function for a concentrated force is singular at its point of application we begin from the moment equation derived from a free body as shown in Ch. 3.

$$M_x = -F \cdot x$$

which gives, after integration

$$EI \cdot \varphi = -\frac{F \cdot x^2}{2} + C_3$$

Boundary condition: at $x = L \;\Rightarrow\; \varphi = 0 \;\Rightarrow\; C_3 = \frac{F \cdot L^2}{2}$

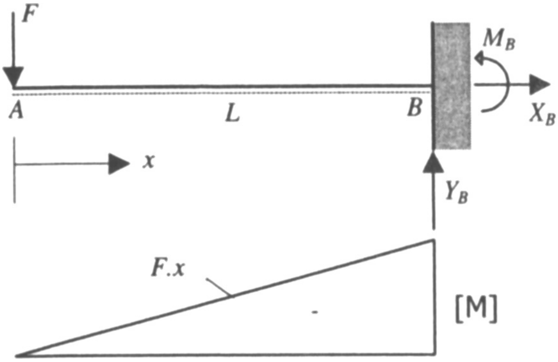


Figure 6.2: Cantilever beam loaded by the concentrated force

$$EI \cdot \varphi = -\frac{F \cdot x^2}{2} + \frac{F \cdot L^2}{2} \tag{6.7}$$

$$\varphi_{max} = \frac{F \cdot L^2}{2 \cdot EI} \tag{6.8}$$

$$EI \cdot y = \frac{F \cdot L^2 \cdot x}{2} - \frac{F \cdot x^3}{6} + C_4$$

Boundary condition: at $x = L \;\Rightarrow\; y = 0 \;\Rightarrow\; C_4 = \frac{F \cdot L^3}{3}$

$$EI \cdot y = \frac{F \cdot L^2 \cdot x}{2} - \frac{F \cdot x^3}{6} + \frac{F \cdot L^3}{3} \tag{6.9}$$

The maximum deflection for a cantilever beam that is loaded by the concentrated force at its free end is:

$$y_{max}^{F} = \frac{FL^3}{3EI} \tag{6.10}$$

Example 6.3: Simple beam loaded by the uniform load of intensity q

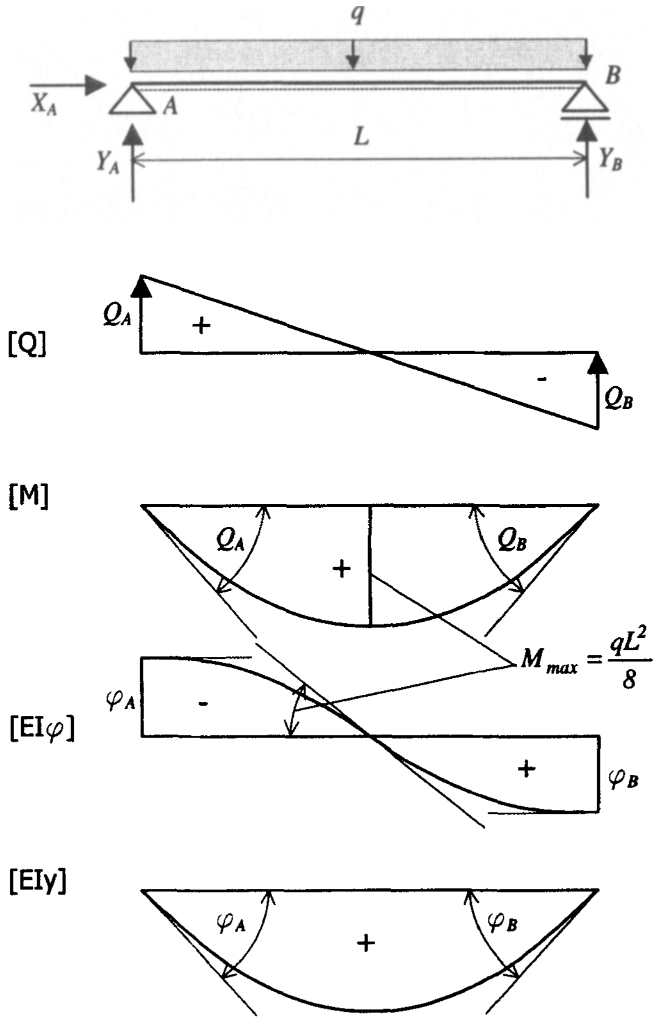


Figure 6.3: Simple beam and diagrams

The load function is $q(x) = q = constant$ is inserted into familiar equations:

$$\frac{dQ_x}{dx} = -q(x)$$

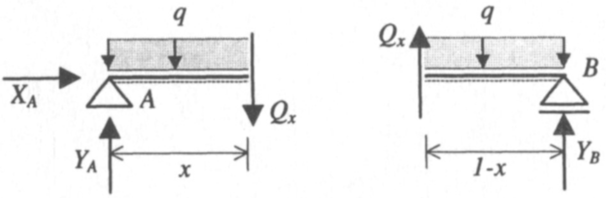
$$Q_x = -q \cdot x + C_1$$

The constant C_1 is determined from boundary conditions:

$$\text{at } x = 0 \Rightarrow Q_x = Y_A$$

$$\text{at } x = L \Rightarrow Q_x = -Y_B$$

$$Q_x = -q \cdot x + C_1$$



$$\text{at } x = 0 \Rightarrow Q_x = -q \cdot 0 + C_1$$

$$Q_x = C_1 \Rightarrow C_1 = Y_A$$

$$Q_x = -q \cdot x + Y_A = \frac{q \cdot L}{2} - q \cdot x \tag{6.11}$$

By the direct integration of the load function using boundary conditions, the equation of shear forces at any section was evaluated. Let us now check if the equation is valid at point *B*. Inserting $x = L$ we get

$$Q_{(L)} = \frac{q \cdot L}{2} - q \cdot L = -\frac{q \cdot L}{2} = Q_B = -Y_B$$

The result is indeed the reaction at point *B*. From Eqn. (6.11) it is obvious that it represents a straight line whose derivation is the inclination of the line to the *x*-axis and from Eqn. (3.11) also the uniform load intensity.

Performing the second integration a bending moment is gained:

$$M_x = \int\left(q\cdot\frac{L}{2} - q\cdot x\right)dx + C_2$$

$$M_x = \frac{q\cdot L\cdot x}{2} - \frac{q\cdot x^2}{2} + C_2$$

Boundary conditions: at $x = 0 \Rightarrow M = 0$

at $x = L \Rightarrow M = 0$

at $x = 0 \Rightarrow M_x = \frac{q\cdot L\cdot 0}{2} - \frac{q\cdot 0^2}{2} + C_2 \Rightarrow C_2 = 0$

$$M_x = \frac{q\cdot L\cdot x}{2} - \frac{q\cdot x^2}{2} \tag{6.12}$$

Now, if the bending moment is integrated

$$\frac{d\varphi}{dx}EI = -M_x$$

$$EI\cdot d\varphi = -M_x\cdot dx = \left(\frac{q\cdot x^2}{2} - \frac{q\cdot L\cdot x}{2}\right)dx$$

$$EI\cdot\varphi = \frac{q\cdot x^3}{6} - \frac{q\cdot L\cdot x^2}{4} + C_3$$

Using boundary conditions: at $x = \frac{L}{2} \Rightarrow \varphi = 0 \Rightarrow 0 = \frac{q\cdot L^3}{6\cdot 8} - \frac{q\cdot L\cdot L^2}{4\cdot 4} + C_3$

$$C_3 = \frac{q\cdot L^3}{24}$$

constant C_3 is obtained and after the insertion into the equation of rotation

$$EI\cdot\varphi = \frac{q\cdot x^3}{6} - \frac{q\cdot L\cdot x^2}{4} + \frac{q\cdot L^3}{24} \tag{6.13}$$

a general equation of rotation is evaluated.

Calculate now the maximum values:

$$\text{at } x=0 \;\Rightarrow\; \varphi_A=\frac{q\cdot L^3}{24\cdot EI} \tag{6.14}$$

$$\text{at } x=L \;\Rightarrow\; \varphi_B=-\frac{q\cdot L^3}{24\cdot EI}$$

Now we proceed with the integration of the equation of rotation:

$$EI\cdot\frac{dy}{dx}=\varphi$$

$$y=\frac{q\cdot x^4}{6\cdot 4}-\frac{q\cdot L\cdot x^3}{4\cdot 3}+\frac{q\cdot L^3\cdot x}{24}+C_4$$

Boundary conditions: at $x=0 \Rightarrow y=0$

at $x=L \Rightarrow y=0,\ C_4=0$

$$EI\cdot y=\frac{q\cdot x^4}{24}-\frac{q\cdot L\cdot x^3}{12}+\frac{q\cdot L^3\cdot x}{24} \tag{6.15}$$

Equation (6.15) is a basic equation of the deflection line (deformation). The maximum deflection is found from the condition of zero rotation (as the rotation is a derivative of deformation) at $x = L/2$:

$$y_{\max}=\frac{5qL^4}{384EI} \tag{6.16}$$

6.2 Theorem of unit load *(theorem of virtual forces)*

We proceed from the equations from Ch. 5.12, defining the use of principle of virtual work for deformation calculation. The equation of a volume integral (5.58) is used

$$\Delta_i=\int\frac{\overline{M}\cdot M}{EI}\cdot dx,$$

$\overline{M}$ is a bending moment caused by the *unit force or unit moment* applied at the point and in the direction at which the deformation is to be found, M is bending moment caused by external loading.

6.2.1 Integration of the equation

Let us at first find the deflection of the cantilever beam at the point A and in the direction of the applied force F from Fig. 6.4:

$$\Delta_i = \int \frac{\overline{M} \cdot M}{EI} \cdot dx = \frac{1}{EI} \int_0^L (-F \cdot x) \cdot (-1 \cdot x) dx = \frac{1}{EI} \cdot \left| \frac{F \cdot x^3}{3} \right|_0^L = \frac{FL^3}{3EI}$$

$$y_A = \frac{FL^3}{3EI}$$

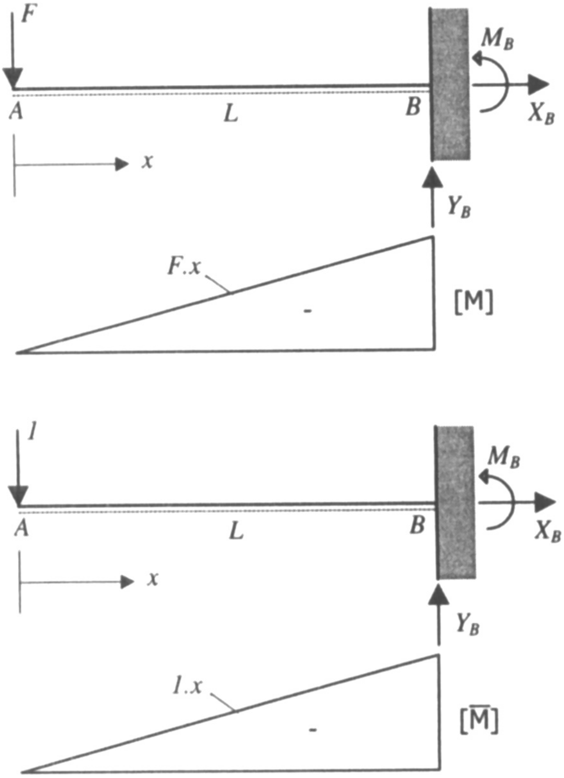


Figure 6.4: Theorem of unit forces (determination of a deflection)

The same procedure is used to calculate the rotation at point A (Fig. 6.5), but as the rotation is to be found, a unit moment at that point has to be applied:

$$\varphi_A = \int_0^L \frac{1 \cdot F \cdot x}{EI} \cdot dx = \frac{FL^2}{2EI} \tag{6.17}$$

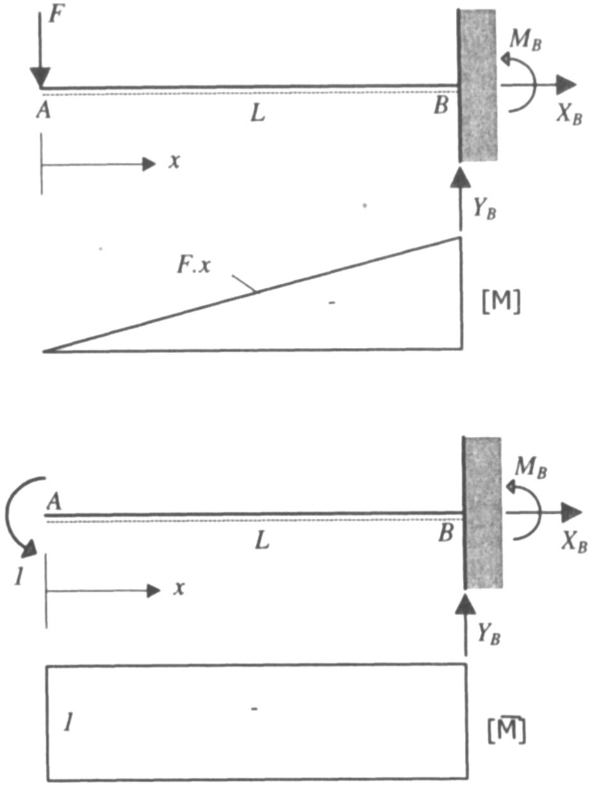


Figure 6.5: Theorem of unit forces (determination of a rotation)

6.2.2 Tables

It is useful to calculate integrals of different moment functions explicitly over the entire length and write them in the tables that are given in appendices *B* in tables *B.4* and *B.5*.

Only the maximum values of bending moments are necessary, denoted by letters *"j"* and *"k"* for real and virtual moments respectively. It is also obvious that both values can be exchanged for each other. For instance, from Fig. 6.4 the real maximum moment is $j = FL$ and the virtual moment is $k = L$.

In the table *B.4* we find two corresponding triangles, in this case in the second row and in the second column and read the value of the integral, that will have to be multiplied by the length of the integration *L*:

$$EI \cdot y_A = \frac{1}{3} \cdot j \cdot k \cdot L = \frac{1}{3} F \cdot L \cdot L \cdot L = \frac{F \cdot L^3}{3} \tag{6.18}$$

Example 6.4: Calculate the deflection of the simple beam loaded by a uniform load

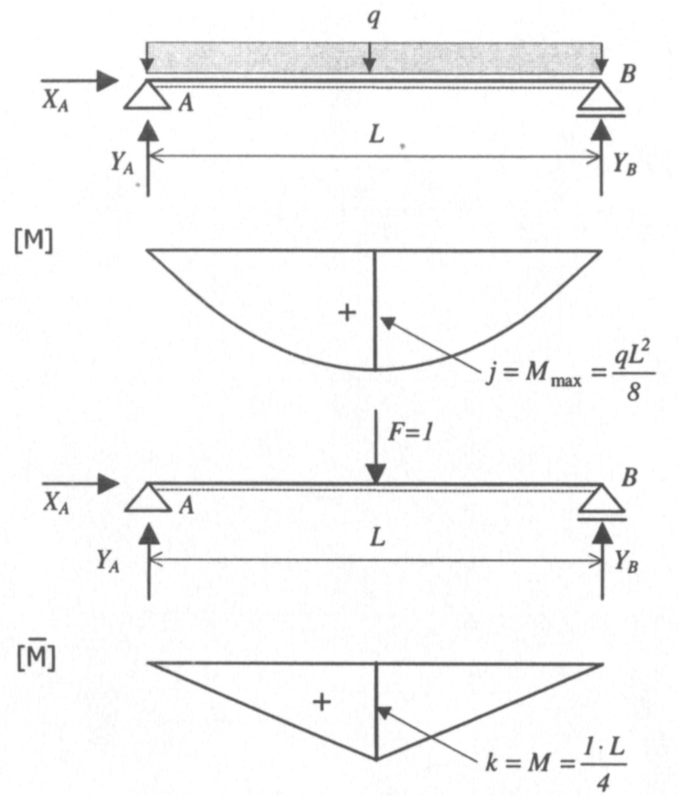


Figure 6.6: Integration by tables

From table *B.5* row *8* and column *6*:

$$EI\delta = \frac{1}{3} \cdot j \cdot k \cdot (1 + \alpha \cdot \beta) \cdot L$$

$$EI\delta = \frac{1}{3} \cdot \frac{qL^2}{8} \cdot \frac{L}{4}(1 + 0.5 \cdot 0.5) \cdot L$$

$$EI\delta = \frac{5qL^4}{384}$$

Because of symmetry only half of the beam can be observed, hence from table *B.5*, row *11*, column *2*:

$$EI\delta = (\frac{5}{12} \cdot j \cdot k \cdot \frac{L}{2}) \cdot 2 = \frac{5}{12} \cdot \frac{qL^2}{8} \cdot \frac{L}{4} \cdot L = \frac{5qL^4}{384} \qquad (6.19)$$

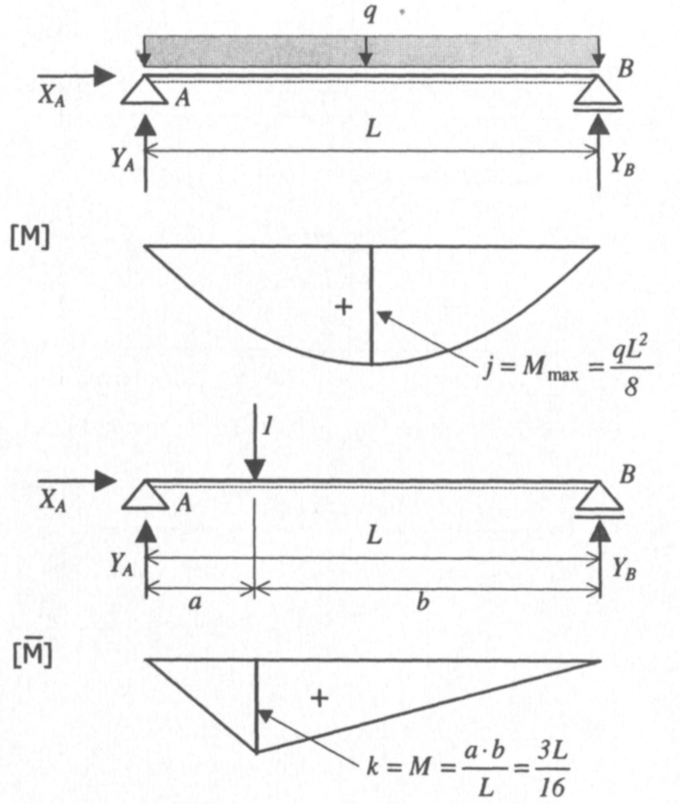


Figure 6.7: Deflection of an arbitrary point

Calculate the deflection at $x = L/4$ (Fig. 6.7). From table *B.5*, row *8*, column *6*:

$$EI\delta_a = \frac{1}{3} \cdot j \cdot k \cdot (1 + \alpha \cdot \beta) \cdot L$$

$$EI\delta_a = \frac{1}{3} \cdot \frac{qL^2}{8} \cdot \frac{3 \cdot L}{16} \cdot (1 + \frac{1}{4} \cdot \frac{3}{4}) \cdot L$$

$$EI\delta_a = \frac{57 \cdot qL^4}{256 \cdot 24} \quad \Rightarrow \quad \delta_a = \frac{19qL^4}{2048EI}$$

What is the *ratio of this deflection* against maximum deflection in the middle of the beam?

$$\frac{\delta_a}{\delta_{\max}} = \frac{19}{2048} : \frac{5}{384} = \frac{19}{2048} \cdot \frac{384}{5} = 0.71$$

6.2.3 Method of Vereshagin

The equation of virtual work is used for deflection determination:

$$EI \cdot \delta = \int M\overline{M} \cdot dx = \int_0^L (F \cdot x) \cdot (x) \cdot dx = \frac{F \cdot L^3}{3}$$

$$\delta_{max} = \frac{F \cdot L^3}{3 \cdot EI}$$

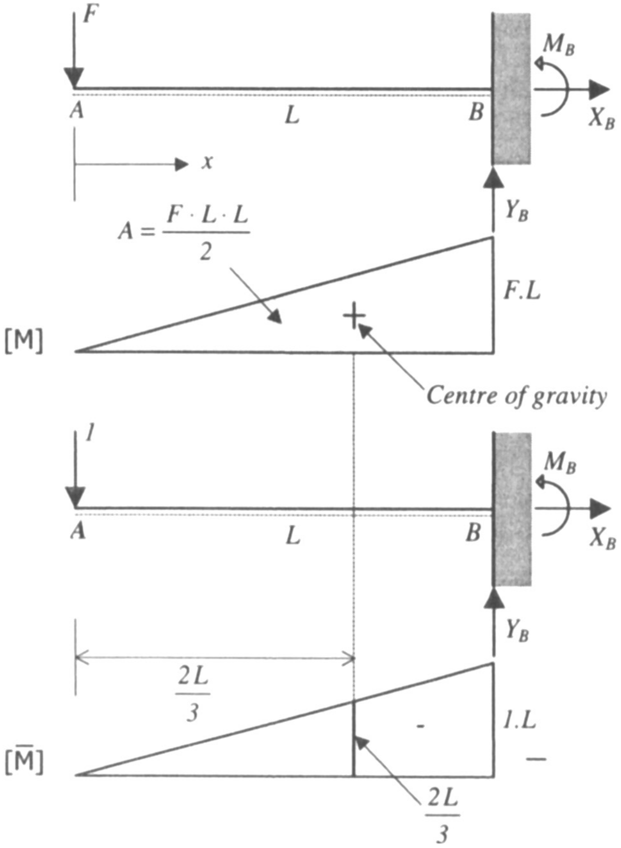
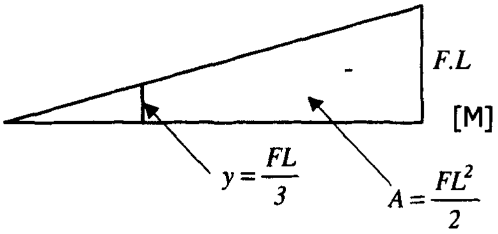
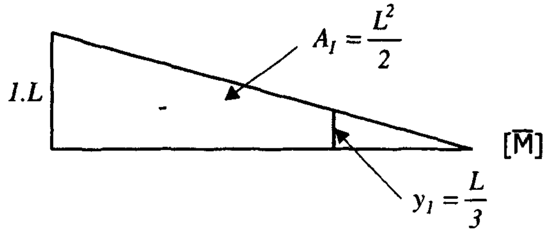


Figure 6.8: Cantilever deflection by Vereshagin

The deformation by the Vereshagin method is calculated *as the product between the area of the first diagram and the value of the second diagram* (*under the centre of gravity of the first diagram*). The diagrams can be interchanged.





$$EI\delta = A\cdot y_1 = \frac{FL^2}{2}\cdot\frac{L}{3} = \frac{FL^3}{6}$$

or if the diagrams are interchanged:

$$EI\delta = A_1\cdot y = \frac{L^2}{2}\cdot\frac{FL}{3} = \frac{FL^3}{6}$$

The method is even simpler if one of the diagrams has a constant value, as only the area of the other diagram is needed for the integral evaluation (Fig. 6.9).

$$EI\cdot\varphi_A = \frac{q\cdot L^3}{6}\cdot 1$$

$$\varphi_A = \frac{q\cdot L^3}{6EI}$$

To check the result let us directly integrate both moment functions

$$EI\cdot\varphi_A = \int_0^L M\overline{M}\cdot dx = \int_0^L \frac{q\cdot x^2}{2}\cdot 1\cdot dx = \frac{q\cdot x^3}{6}\Big|_0^L = \frac{q\cdot L^3}{6}$$

which gives exactly the same result as above calculated by the Vereshagin method.

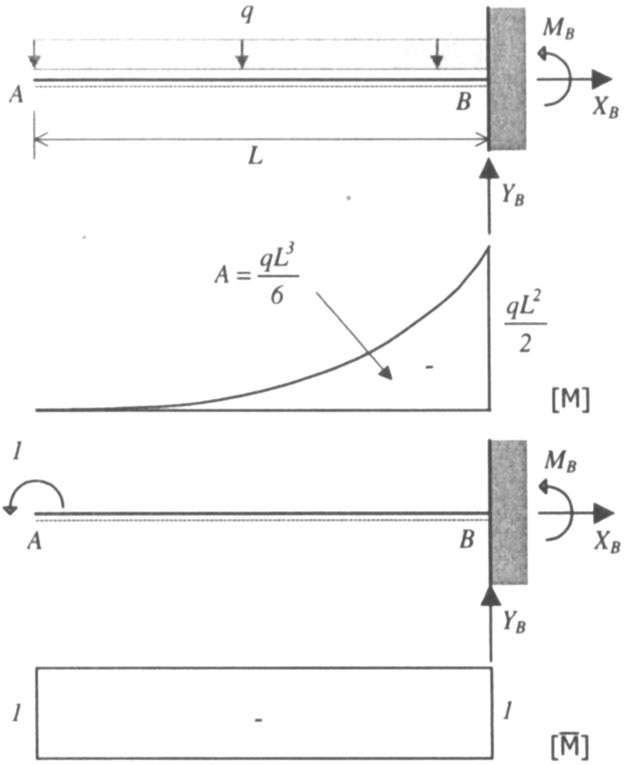


Figure 6.9: Rotation at A by the Vereshagin method

6.3 Mohr's method

The method is based on differential relations with respect to a beam since successive integration of a load function gives shear force, bending moment, rotation and deflection.

A *conjugate beam* will be loaded by a moment diagram of an *original beam*. The boundary conditions on a conjugate beam are derived from the equality of shear forces on the original beam with rotations on the conjugate beam and on the equality of bending moments on the original beam with deflections on the conjugate beam. It means that:

- ❖ *At the point where the shear force on the original beam equals zero the rotation on the conjugate beam must be also zero and vice versa.*
- ❖ *At the point where the bending moment on the original beam equals zero the deflection on the conjugate beam must be also zero and vice versa.*

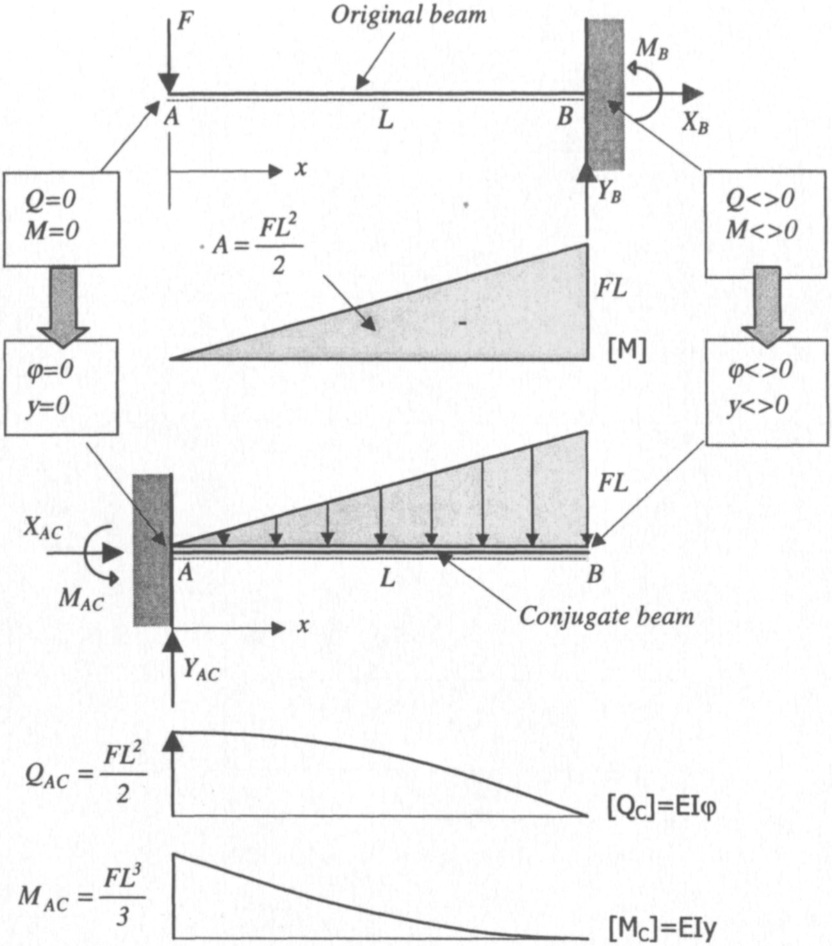


Figure 6.10: A conjugate beam or Mohr's method

Diagrams of shear forces and bending moments on the conjugate beam or better, their values at chosen points, are *EI-times* the values of rotations and displacements on the original beam. Therefore, the calculated value is reduced by *EI* to get a real value of rotation or displacement at that point.

Mohr's method will mostly be used for the calculation of maximum values of deformation as from the example above, the reaction Q_{AC} on the conjugate beam represents the maximum rotation and the clamping moment M_{AC} represents the maximum deflection.

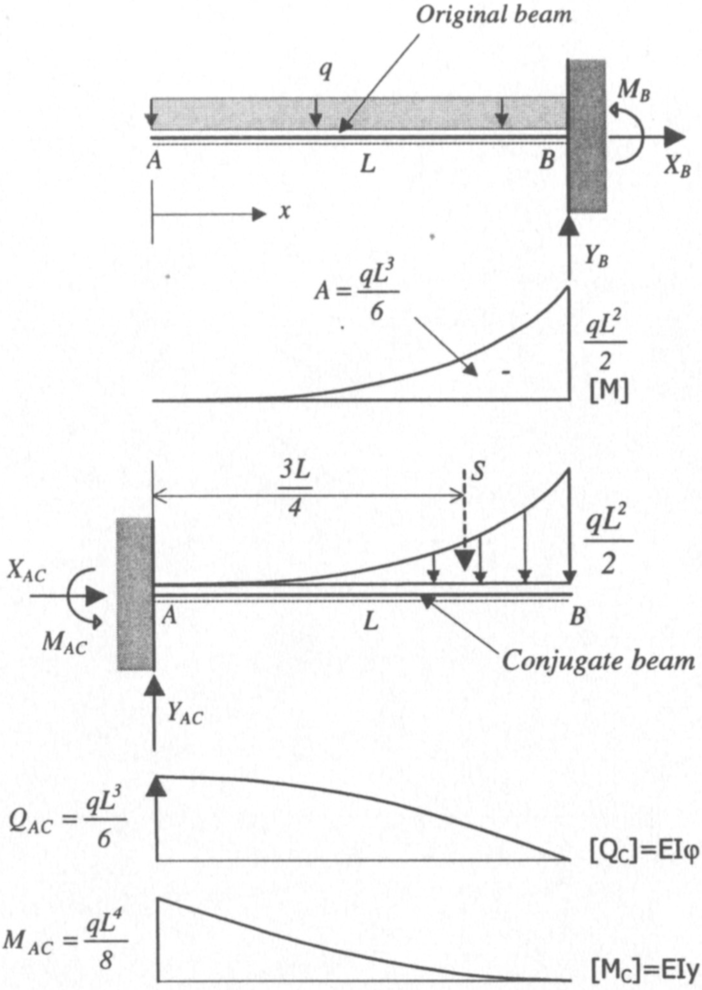


Figure 6.11: Deflection of a cantilever beam with a uniform load

$$S=\frac{1}{3}\cdot\frac{q\cdot L^2}{2}\cdot L=\frac{q\cdot L^3}{6}=EI\cdot\varphi_{max} \tag{6.20}$$

$$M_{AC}=S\cdot\frac{3}{4}\cdot L=\frac{q\cdot L^3}{6}\cdot\frac{3}{4}\cdot L=\frac{q\cdot L^4}{8}=EI\cdot y_{max} \tag{6.21}$$

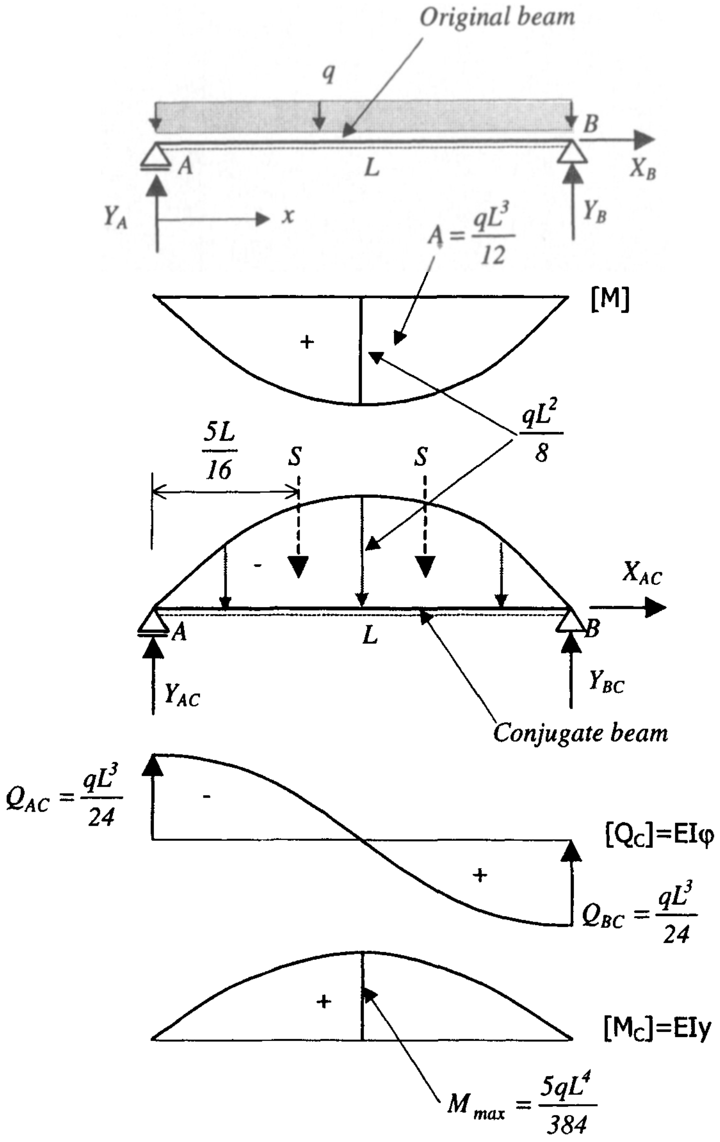


Figure 6.12: Mohr's method in a simple beam

Calculate the maximum deflection of a simple beam loaded by a uniform load (Figure 6.12). From symmetry it is clear that the maximum deflection is in the middle of the beam therefore a bending moment on the conjugate beam at that point will be calculated. The total substitution force is:

$$2 \cdot S = \frac{2}{3} \cdot \frac{q \cdot L^3}{8} = \frac{q \cdot L^3}{12}$$

therefore the reaction at the support equals

$$Y_{AC} = \frac{q \cdot L^3}{24}$$

Since the reactions are known and they equal shear forces at the supports, their values reduced by *EI*, represents the rotation at those points:

$$\varphi_A = -\frac{qL^3}{24EI} \qquad \varphi_B = \frac{qL^3}{24EI}$$

To find the deflection we need to calculate bending moments in the middle of the span:

$$M_x + \frac{q \cdot L^2}{24} \cdot \frac{L}{2} + \frac{q \cdot L^3}{24} \cdot \frac{3}{16} \cdot L = 0$$

$$M_{XC} = \frac{q \cdot L^4}{24} \cdot \left(\frac{1}{2} - \frac{3}{16}\right) = \frac{q \cdot L^4}{24} \cdot \left(\frac{8-3}{16}\right) = \frac{5qL^4}{384}$$

The maximum deflection is the value of the bending moment in the middle reduced by the bending stiffness *EI*:

$$y_{max} = \frac{M_{XC}}{EI} = \frac{5qL^4}{384EI}$$

6.4 Deformation of trusses

Trusses are structures in which only axial forces occur. Therefore for the calculation of deformations we use Equation (5.55)

$$\Delta = \Sigma \frac{N_U \cdot N \cdot L}{E \cdot A} + \Sigma \int \frac{M_U \cdot M}{EI} \cdot ds + \Sigma K \cdot \int \frac{Q_U \cdot Q}{GA} \cdot ds \qquad (5.55)$$

and take into account its first part only:

$$\Delta = \Sigma \frac{N_U \cdot N \cdot L}{E \cdot A}$$

A truss deformation at a chosen point is calculated in three steps:

1. a truss is loaded by an *external load* and forces in all elements are calculated
2. a truss is loaded by a *unit force* at a point in a chosen direction and forces in all elements are calculated
3. the *summation* is done over all elements according to the above equation

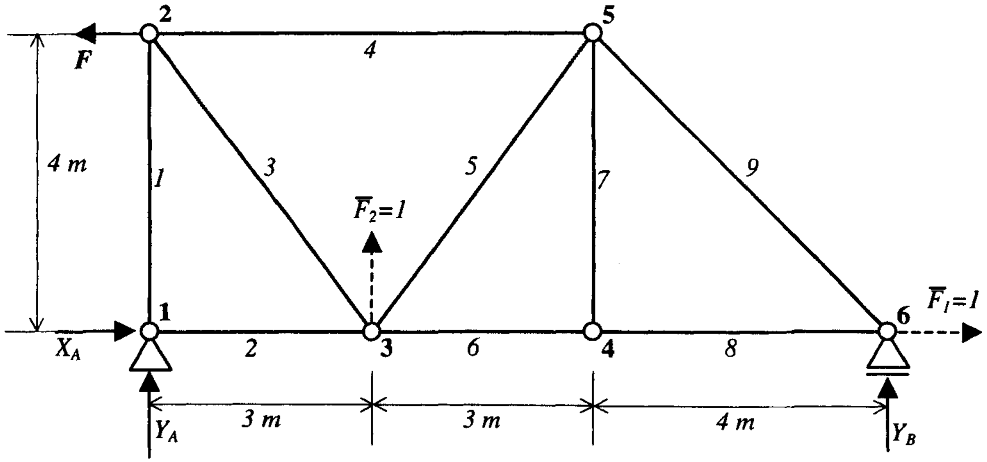


Figure 6.13: Plane truss

As trusses usually consist of several elements it is the best to execute the summation in tables as shown in the following example.

Example 6.5: Calculate the horizontal displacement of joint *6* and vertical displacement of joint *3* in the truss from example 3.1 (Figure 6.13).
The cross-section of all elements is *5* cm^2, the material is steel with modulus of elasticity $E=210\ GPa$.

Table 6.1

Element	L [m]	N_0 [kN]	N_1 [1]	N_2 [1]	N_0N_1L [kNm]	N_0N_2L [kNm]
1	*4.00*	*-20.000*	*0*	*0.700*	*0*	*-56.000*
2	*3.00*	*-50.000*	*1.000*	*0*	*-150.000*	*0*
3	*5.00*	*25.000*	*0*	*-0.875*	*0*	*-109.375*
4	*6.00*	*35.000*	*0*	*0.525*	*0*	*110.250*
5	*5.00*	*-25.000*	*0*	*-0.375*	*0*	*46.875*
6	*3.00*	*-20.000*	*1.000*	*-0..300*	*-60.000*	*18.000*
7	*4.00*	*0*	*0*	*0*	*0*	*0*
8	*4.00*	*-20.000*	*1.000*	*-0.300*	*-80.000*	*24.000*
9	*5.66*	*28.285*	*0*	*0.424*	*0*	*67.879*
Σ		-	-	-	***-290.000***	***101.629***

Real displacements have to be reduced by *EA*, hence:

$$u_6 = \frac{-290.000}{210 \cdot 10^6 \cdot 5.00 \cdot 10^{-4}} = -2.762 \cdot 10^{-3} \ m = -2.762 \ mm$$

$$v_3 = \frac{101.629}{210 \cdot 10^6 \cdot 5.00 \cdot 10^{-4}} = -0.968 \cdot 10^{-3} \ m = -0.968 \ mm$$

Note: If cross-sections are not equal for all elements, the reduction by axial stiffness *EA* has to be performed in the table for each element separately.

6.5 Beams of variable height

Consider the cantilever beam from Fig. 6.14. The height is linearly changing from H_1 at the free end to H_2 at the support.

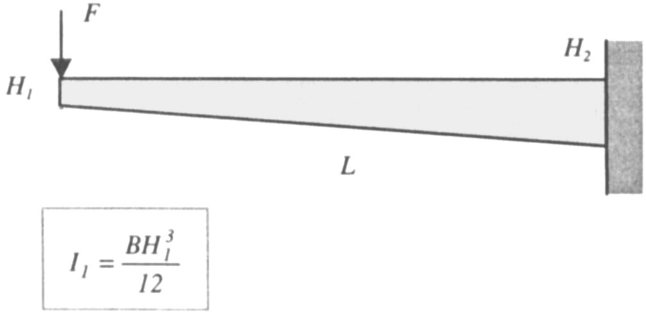


Figure 6.14: Cantilever beam of variable height

Let us at first write a function describing the change of beam height as a function of the position along the longitudinal axis

$$H_x = H_1 + \frac{\Delta H}{L} \cdot x \qquad \Delta H = H_2 - H_1 \qquad (6.22)$$

and then equations of real and virtual moments:

$$M_x = F \cdot x \qquad \overline{M} = 1 \cdot x$$

The function of second moment of area is:

$$I = \frac{B \cdot H_x^3}{12} = \frac{B}{12} \cdot \left(H_1 + \frac{\Delta H}{L} \cdot x \right)^3 = \frac{B}{12} \cdot (a \cdot x + b)^3$$

$$a = \frac{\Delta H}{L} \qquad b = H_1 \qquad X = H_1 + \frac{\Delta H}{L} \cdot x \tag{6.23}$$

Deformations are calculated by integration using the virtual work method:

$$\delta = \int_0^L \frac{M \cdot \overline{M}}{EI} \cdot dx = \int_0^L \frac{F \cdot x^2}{E \cdot \frac{B}{12} \cdot \left(H_1 + \frac{\Delta H}{L} \cdot x \right)^3} \cdot dx = \frac{12 \cdot F}{E \cdot B} \cdot \int_0^L \frac{x^2}{\left(H_1 + \frac{\Delta H}{L} \cdot x \right)^3} \cdot dx =$$

$$= \frac{12 \cdot F}{E \cdot B} \cdot \frac{1}{\left(\frac{\Delta H}{L} \right)^3} \left. \left(\ln\left(H_1 + \frac{\Delta H}{L} \cdot x \right) + \frac{2 \cdot H_1}{H_1 + \frac{\Delta H}{L} \cdot x} - \frac{H_1^2}{2 \cdot \left(H_1 + \frac{\Delta H}{L} \cdot x \right)^2} \right) \right|_0^L =$$

$$= \frac{12 \cdot F}{E \cdot B} \cdot \left(\frac{L}{\Delta H} \right)^3 \left(\ln(H_1 + \Delta H) + \frac{2 \cdot H_1}{H_1 + \Delta H} - \frac{H_1^2}{2 \cdot (H_1 + \Delta H)^2} - \ln H_1 - 2 + \frac{1}{2} \right) =$$

$$= \frac{12 \cdot F}{E \cdot B} \cdot \left(\frac{L}{\Delta H} \right)^3 \left(\ln\left(1 + \frac{\Delta H}{H_1} \right) + \frac{2 \cdot H_1}{H_1 + \Delta H} - \frac{H_1^2}{2 \cdot (H_1 + \Delta H)^2} - \frac{3}{2} \right)$$

The above equation can be rearranged so as to give a real deflection compared with the deflection at constant *EI*:

$$\Delta_1 = \frac{F \cdot L^3}{3EI} \cdot 3 \cdot \left(\frac{H_1}{\Delta H} \right)^3 \left(\ln\left(1 + \frac{\Delta H}{H_1} \right) + \frac{2 \cdot H_1}{H_1 + \Delta H} - \frac{H_1^2}{2 \cdot (H_1 + \Delta H)^2} - \frac{3}{2} \right)$$

$$\Delta_1 = \frac{F \cdot L^3}{3EI} \cdot 3 \cdot \left(\frac{H_1}{\Delta H} \right)^3 \left(\ln\left(1 + \frac{\Delta H}{H_1} \right) + \frac{H_1}{(H_1 + \Delta H)^2} \cdot \frac{3 \cdot H_1 + 4 \cdot \Delta H}{2} - \frac{3}{2} \right) \tag{6.24}$$

or can be written:

$$\Delta_l = \frac{F \cdot L^3}{3EI} \cdot K \tag{6.25}$$

$$K = 3 \cdot \left(\frac{H_l}{\Delta H}\right)^3 \left(\ln\left(1 + \frac{\Delta H}{H_l}\right) + \frac{H_l}{(H_l + \Delta H)^2} \cdot \frac{3 \cdot H_l + 4 \cdot \Delta H}{2} - \frac{3}{2}\right)^{♣} \tag{6.26}$$

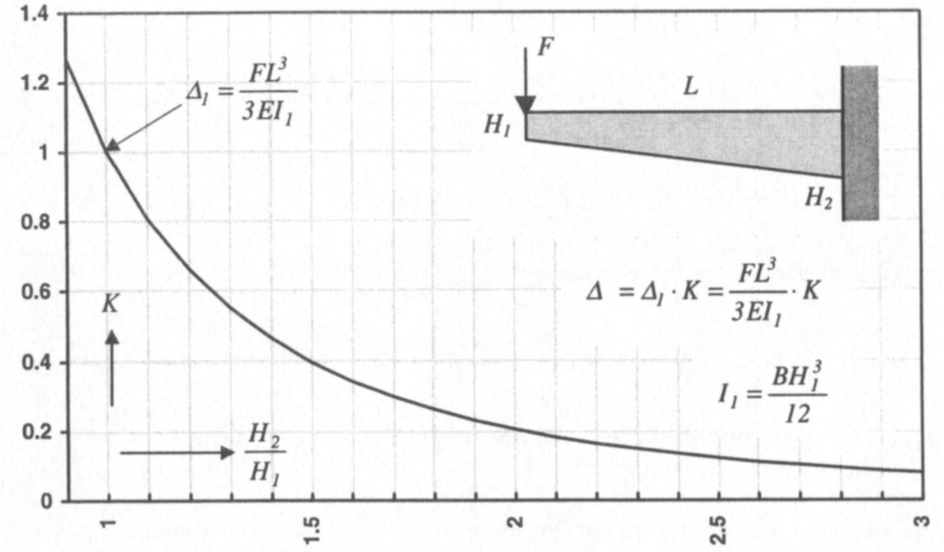


Figure 6.15: Deflection of cantilever beam of variable height

Example 6.6: Calculate the deflection of the cantilever beam of variable height using the following data:

$L = 3.00\ m,\ F = 25\ kN$

$B = 0.20\ m,\ H_l = 0.20\ m,\ H_2 = 0.50\ m,\ E = 20\ GPa$

[♣] *Note: The equation is valid for all* $\Delta H <> 0$

$$EI = E \cdot \frac{(0.2)^4}{12} = 2667.67 \ kNm^2$$

$$\Delta_1 = \frac{F \cdot L^3}{3EI} = 0.084 \ m$$

$$\Delta H = 0.50 - 0.20 = 0.30 \ m$$

$$K = 3 \cdot \left(\frac{H_1}{\Delta H} \right)^3 \left(\ln\left(1 + \frac{\Delta H}{H_1} \right) + \frac{H_1}{(H_1 + \Delta H)^2} \cdot \frac{3 \cdot H_1 + 4 \cdot \Delta H}{2} - \frac{3}{2} \right) =$$

$$K = 3 \cdot \left(\frac{0.2}{0.3} \right)^3 \left(\ln\left(1 + \frac{0.3}{0.2} \right) + \frac{0.2}{(0.2 + 0.3)^2} \cdot \frac{3 \cdot 0.2 + 4 \cdot 0.3}{2} - \frac{3}{2} \right) = 0.121$$

$$\Delta = \Delta_1 \cdot K = 0.084 \cdot 0.121 = 0.010 \ m$$

Note: The constant *K* can be also read from Fig. 6.15 at ratio $H_2 / H_1 = 2.5$.

6.6 Deformation at pre-stressing

Example 6.7: Calculate the deformation line for the cantilever beam due to the pre-stressing force N_P

The equation of eccentricity against the centre of gravity of the cross-section is:

$$e = \frac{H \cdot x}{3L} \tag{6.27}$$

The bending moment at an arbitrary distance x is

$$M = N_p \cdot \frac{H}{3 \cdot L} \cdot x \tag{6.28}$$

and is integrated with the bending moment caused by a unit force used to calculate the deflection at the free end resulting from the pre-stressing force N_p.

$$\delta_{NP} = \frac{1}{EI} \cdot \frac{1}{3} \cdot \frac{N_p \cdot H}{3} \cdot L^2 = \frac{N_p \cdot H \cdot L^2}{9 \cdot EI} \tag{6.29}$$

As the deflection for uniform load q is

$$\delta_q = \frac{q \cdot L^4}{8 \cdot EI}$$

from the condition of equal displacements $\delta_{NP} = \delta_q$ the force in the pre-stressing cable is calculated to have a zero total displacement at the free end.

$$\frac{N_p \cdot H \cdot L^2}{9 \cdot EI} = \frac{q \cdot L^4}{8 \cdot EI}$$

$$N_p = \frac{9}{8} \cdot \frac{q \cdot L^2}{H} \tag{6.30}$$

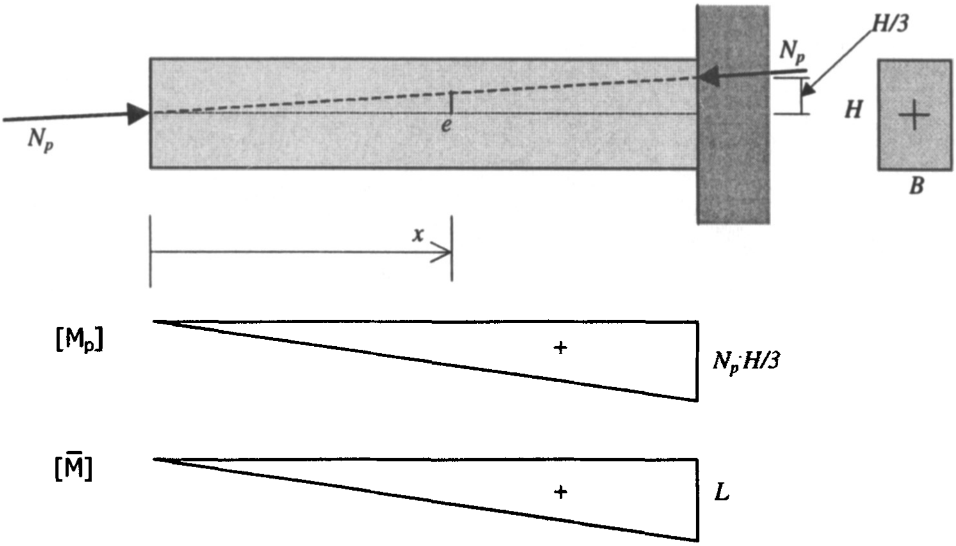


Figure 6.16: Pre-stressing cable in a straight line

The force in the cable to produce a total zero displacement is therefore:

$$N_p = \frac{9}{8} \cdot \frac{L}{H} \cdot q \cdot L \tag{6.31}$$

A general equation of the deformation line is derived by integration using boundary conditions and procedures explained earlier in this book:

$$M_x = N_p \cdot \frac{H}{3 \cdot L} \cdot x$$

$$EI \cdot \varphi = N_p \cdot \frac{H}{6 \cdot L} \cdot x^2 + C_3 \;\Rightarrow\; x = L \;\Rightarrow\; \varphi = 0 \;:\; C_3 = -N_p \cdot \frac{H \cdot L}{6} \tag{6.32}$$

$$EI \cdot \varphi = N_p \cdot \frac{H}{6 \cdot L} \cdot x^2 - N_p \cdot \frac{H \cdot L}{6}$$

$$EI \cdot y = N_p \cdot \frac{H}{18 \cdot L} \cdot x^3 - N_p \cdot \frac{H \cdot L}{6} \cdot x + C_4$$

$$C_4 = N_p \cdot \frac{H \cdot L^2}{6} - N_p \cdot \frac{H \cdot L^2}{18} = N_p \cdot \frac{H \cdot L^2}{9}$$

$$EI \cdot y = N_p \cdot H \cdot L^2 \cdot \left(\frac{x^3}{18 \cdot L^3} - \frac{x}{6 \cdot L} + \frac{1}{9} \right)$$

The maximum deflection is at $x = 0$ and is:

$$y_{max} = \frac{N_p \cdot H \cdot L^2}{9EI} \tag{6.33}$$

Example 6.8: Calculate the deformation line for the cantilever beam due to the prestressing force N_P if the cable line is parabolic along the longitudinal axis and its tangent is horizontal at the support

The basic geometrical relations are given in the equations below and in practice, constant n is limited to $0 < n <= 0.4$:

$$x^2 = \frac{L^2}{n \cdot H} \cdot y$$

$$x = L\sqrt{\frac{y}{n \cdot H}} = K_1 \cdot \sqrt{y} \qquad K_1 = L \cdot \sqrt{n \cdot H} \tag{6.34}$$

$$x = K_1 \cdot \sqrt{y}$$

The bending moment due to the cable force is:

$$M_X = N_P \cdot n \cdot H \cdot \sqrt{\frac{x}{L}} = K \cdot \sqrt{x}$$

$$(6.35)$$

$$K = \frac{N_p \cdot n \cdot H}{\sqrt{L}}$$

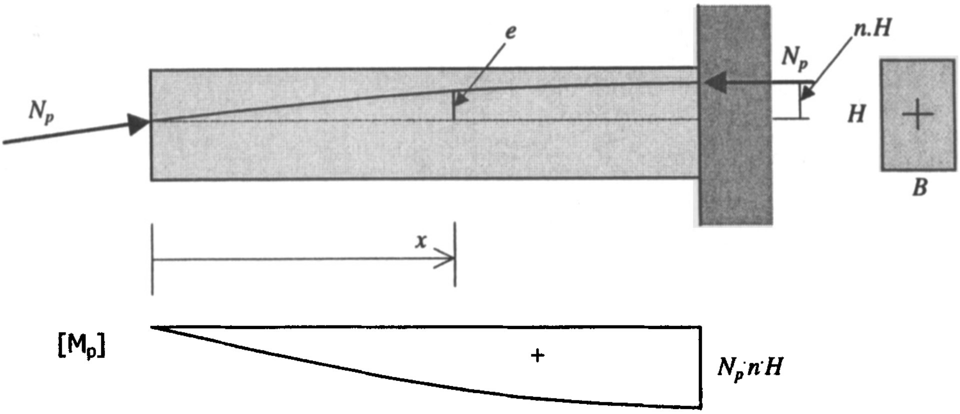


Figure 6.17: Parabolic cable line

A general equation of the deformation line is derived by integration using boundary conditions:

$$EI \cdot \varphi = K \cdot \frac{2}{3} \cdot \sqrt{x^3} + C_3 \quad \text{and hence:} \quad C_3 = -K \cdot \frac{2}{3} \cdot \sqrt{L^3}$$

$$EI \cdot \varphi = \frac{2}{3} \cdot K \cdot \left(\sqrt{x^3} - \sqrt{L^3} \right)$$

$$EI \cdot y = \frac{2}{3} \cdot K \cdot \left(\frac{2}{5} \sqrt{x^5} - x \cdot \sqrt{L^3} \right) + C_4$$

$$C_4 = \frac{2}{5} \cdot K \cdot \sqrt{L^5}$$

$$EI \cdot y = \frac{2}{3} \cdot K \cdot \left(\frac{2}{5}\sqrt{x^5} - x \cdot \sqrt{L^3} \right) + \frac{2}{5} \cdot K \cdot \sqrt{L^5}$$

$$EI \cdot y = \frac{N_p \cdot n \cdot H}{\sqrt{L}} \cdot \left(\frac{4}{15}\sqrt{x^5} - \frac{2}{3} \cdot x \cdot \sqrt{L^3} + \frac{2}{5} \cdot \sqrt{L^5} \right) \tag{6.36}$$

The maximum deflection is at the free end and is given as:

$$y_{max} = \frac{2}{5EI} \cdot N_p \cdot n \cdot H \cdot L^2 \tag{6.37}$$

A comparison between both deflections for cable in a straight line and a parabolic cable shows that at the same conditions (*n* = *1/3*) deflections differ by *20 %* and it is inadmissible to replace a parabola by a straight line.

Example 6.9: Calculate deflections due to the *self weight, pre-stressing, creep of concrete and relaxation in the cable* for the beam shown in Fig. 6.18. (*E = 24 GPa, B/H = 0.3/0.5 m, EI = 75000 kNm²*).

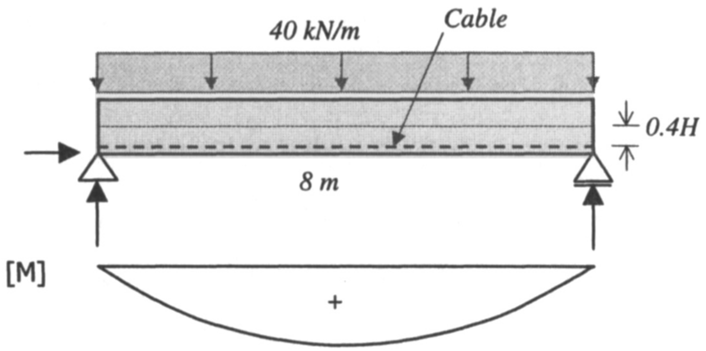


Figure 6.18: Pre-stressed beam

The deflection due to the self-weight is:

$$y_{max} = \frac{5 \cdot q \cdot L^4}{384 \cdot EI_p} = \frac{5 \cdot 40 \cdot 8^4}{384 \cdot 75000} = 0.0285 \ m,$$

which is compared with the *permissible deflection* given by a Code (usually as $L/300$):

$$y_{perm} = \frac{L}{300} = \frac{800}{300} = 2.667 \ cm$$

The stresses are calculated from the maximum bending moment:

$$M_{max} = \frac{q \cdot L^2}{8} = \frac{40 \cdot 8^2}{8} = 320 \ kNm$$

$$W = \frac{B \cdot H^2}{6} = \frac{0.3 \cdot (0.5)^2}{6} = 0.0125 \ m^3$$

$$\sigma_{max} = \pm \frac{M}{W} = \pm 25600 \ \frac{kN}{m^2} = \pm 25.6 \ MPa$$

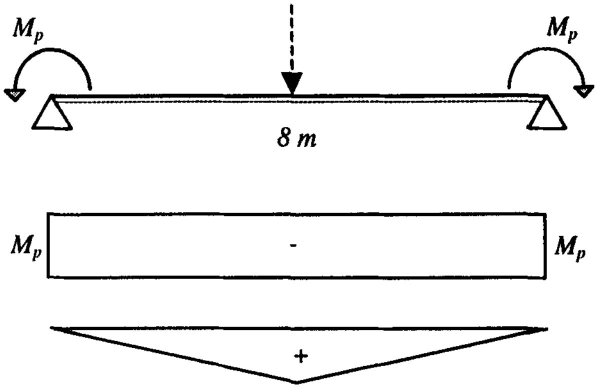


Figure 6.19: Beam under pre-stressing moment

The cable force will be calculated from the condition that the deflection at mid-span does not exceed the maximum allowable deflection. Let the position of the cable be at *0.4H* under the centre of gravity of the section, hence:

$$M_p = F_p \cdot 0.4 \cdot H = F_p \cdot 0.2$$

$$EI \cdot y_p = \frac{L}{4} \cdot L \cdot \frac{1}{2} \cdot M_p = \frac{L^2}{8} \cdot F_p \cdot 0.2 = \frac{F_p \cdot L^2}{40}$$

$$F_p = \frac{40 \cdot EI \cdot y_p}{L^2} = \frac{40 \cdot 75000 \cdot 2.667 \cdot 10^{-2}}{8^2} = 1250 \ kN$$

$$M_p = F_p \cdot 0.2 = 250 \ kNm$$

The initial deflection due to self-weight q and pre-stressing is:

$$y_{t_0} = 0.0285 - 0.02667 = 0.0018\ m$$

Suppose now that after t = *360 days* due to the relaxation the cable force falls by *20 %*, therefore after one year the deformation will be:

$$y_{t_1} = 0.0285 - (1 - 0.2) \cdot 0.02667 = 0.714\ \ cm$$

Now we take into account the creep of concrete, which causes the fall in modulus of elasticity E by say *40 %*, and the total deformation after one year will be approximately

$$y_{t_2} = \frac{0.714}{0.6} = 1.19\,cm\,,$$

which is still inside the allowable limits.

7

Stiffness and flexibility

7.1 Coefficients of stiffness for prismatic elements

A typical element whose x-axis is lying along the centroidal line of the element and is defined as positive in the direction from joint *1* to joint *2* is shown in Fig. 7.1. The y- and z-axis complete the right-handed co-ordinate system and are chosen to be ***the principal axes for the element cross-section.*** It is assumed that the shear centre of the section coincides with the centroid and therefore the force applied in any one principal plane causes displacements in that plane only.

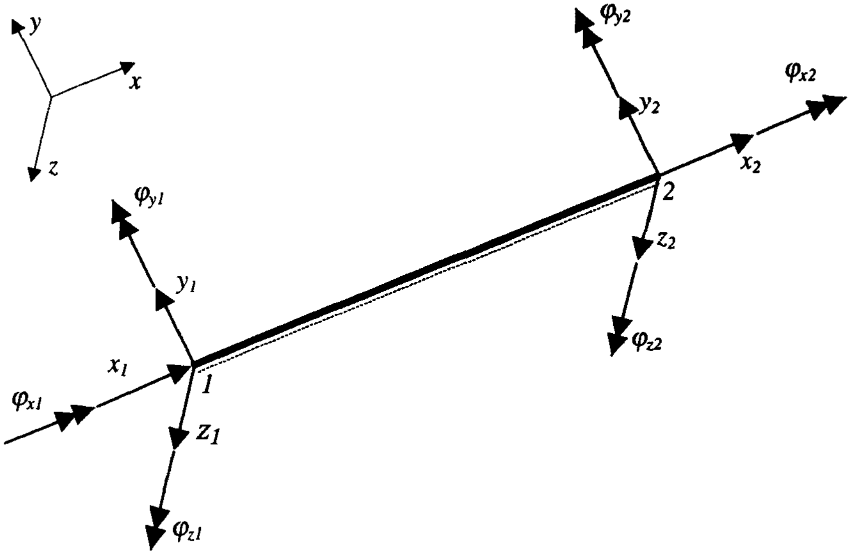


Figure 7.1: Degrees of freedom

The displacement vector is:

$$\{u_1\}=\{\ x_1\ \ y_1\ \ z_1\ \ \varphi_{x1}\ \ \varphi_{y1}\ \ \varphi_{z1}\ \}$$

$$\{u_2\}=\{\ x_2\ \ y_2\ \ z_2\ \ \varphi_{x2}\ \ \varphi_{y2}\ \ \varphi_{z2}\ \}$$

There are thus *12* possible displacement components; a element has *12 degrees of freedom*. Each of these possible displacements can be suppressed by a corresponding force as shown in Fig. 7.2 and denoted as the force vector:

$$\{F_1\}=\{N_1\ \ Q_{Y1}\ \ Q_{Z1}\ \ T_1\ \ M_{Y1}\ \ M_{Z1}\}$$

$$\{F_2\}=\{N_2\ \ Q_{Y2}\ \ Q_{Z2}\ \ T_2\ \ M_{Y2}\ \ M_{Z2}\}$$

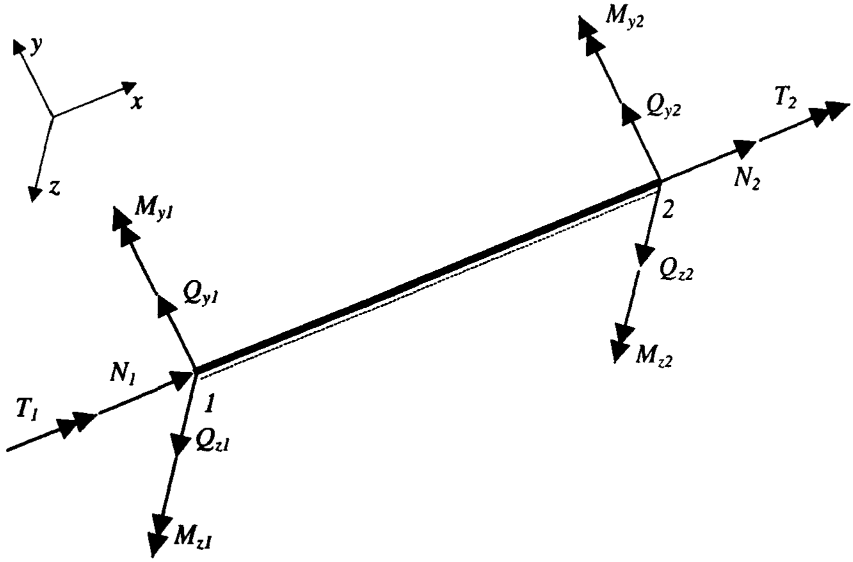


Figure 7.2: Corresponding forces

The properties of the element are designated as:

A	*Cross-section*
L	*Element length*
v	*Poisson's ratio*
E	*Modulus of elasticity*
$G=\dfrac{E}{2\cdot(1+v)}$	*Shear modulus*

The principal second moments of area (in bending) are:

I_y ... Principal second moments of area about y axis
I_z ... Principal second moments of area about z axis

The polar second moment of area should be denoted by I_x but is usually written as I_P.

7.2 Stiffness matrix of a clamped element (m_6)

The *stiffness coefficients* are the actions imposed by a neighbouring body that can be either a support or an element if *unit displacement* occurs at each joint of the element in a corresponding direction while *all other displacements are kept zero.*

The resulting forces must be in equilibrium and six equations may be drawn up:

$$N_1 + N_2 = 0$$

$$Q_{Y1} + Q_{Y2} = 0 \qquad (7.1)$$

$$Q_{Z1} + Q_{Z2} = 0$$

$$T_1 + T_2 = 0$$

$$M_{Y1} + M_{Y2} + Q_{Z1} \cdot L = 0 \qquad (7.2)$$

$$M_{Z1} + M_{Z2} - Q_{Y1} \cdot L = 0$$

Consider first, displacement x_1 from Fig. 7.3 below:

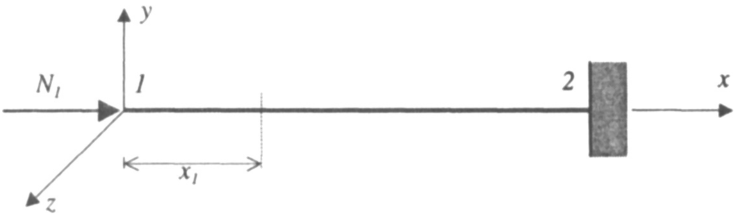


Figure 7.3: Axial force at joint 1

From Eqn. (2.47) according to Hooke's law the force that caused the displacement x_I is equal to:

$$N_I = \frac{E \cdot A}{L} \cdot x_I, \tag{7.3a}$$

hence from Eqn. (7.1):

$$N_2 = -\frac{E \cdot A}{L} \cdot x_I \tag{7.3b}$$

The displacement at the other end, in a similar manner, gives:

$$N_2 = \frac{E \cdot A}{L} \cdot x_2 \qquad \textit{and} \qquad N_I = -\frac{E \cdot A}{L} \cdot x_2$$

Displacements x_I and x_2 produce forces in the axial direction only, therefore all other forces are zero.

Figure 7.4: Axial force at joint *2*

As displacements are assumed to be small, the application of rotations φ_{XI} and φ_{X2} will produce torsion restraints only, hence:

$$\begin{aligned} T_I &= \frac{GI_x}{L} \cdot \varphi_{XI} \qquad & T_2 &= -\frac{GI_x}{L} \cdot \varphi_{XI} \\ T_2 &= \frac{GI_x}{L} \cdot \varphi_{X2} \qquad & T_I &= -\frac{GI_x}{L} \cdot \varphi_{X2} \end{aligned} \tag{7.4}$$

The stiffness coefficients involving rotations φ_Z and φ_Y as well as displacements y and z will be determined using Castigliano's theorem derived in Ch. 5.13. (The reader should verify that the use of principle of virtual work leads to the same results).

Rotation φ_Z : The element *1-2* is initially straight and as such is given an end rotation of φ_{Z2} at joint 2.

A bending moment at an arbitrary section is

$$M = -M_{Z1} + Q_{Y1} \cdot x\,, \tag{7.5}$$

From equilibrium at joint 2

$$M_{Z1} + M_{Z2} - Q_{Y1} \cdot L = 0\,,$$

hence:

$$M = -M_{Z2} + Q_{Y1} \cdot (L - x)$$

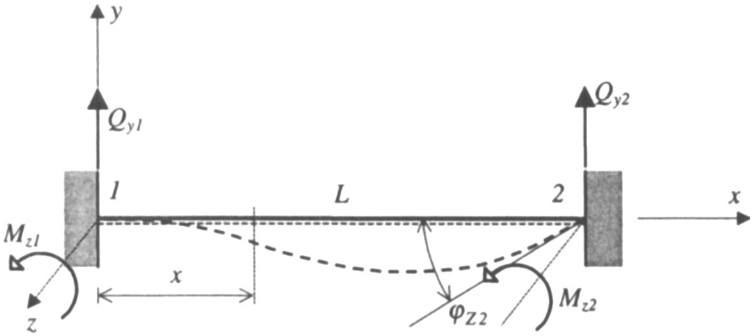


Figure 7.5: Rotation of joint 2 by φ_{Z2}

The deformational (strain) energy is

$$U = \int_0^L \frac{M^2 \cdot dx}{2EI_Z} =$$

$$= \frac{1}{2EI_Z} \cdot \int_0^L \left[M_{Z2}^2 - 2 \cdot M_{Z2} \cdot Q_{Y1} \cdot (L - x) + Q_{Y1}^2 \cdot \left(L^2 - 2 \cdot L \cdot x + x^2 \right) \right] \cdot dx =$$

$$= \frac{1}{2EI_Z} \cdot \Big| M_{Z2}^2 \cdot x - 2 \cdot M_{Z2} \cdot Q_{Y1} \cdot L \cdot x + M_{Z2} \cdot Q_{Y1}^2 \cdot x^2 + Q_{Y1}^2 \cdot L^2 \cdot x$$

$$- Q_{Y1}^2 \cdot L \cdot x^2 + Q_{Y1}^2 \cdot \frac{x^3}{3} \Big|_0^L$$

$$U = \frac{1}{2EI_Z} \cdot \left[M_{Z2}^2 \cdot L - M_{Z2} \cdot Q_{Y1} \cdot L^2 + Q_{Y1}^2 \cdot \frac{L^3}{3} \right] \tag{7.6}$$

Let us find now shear force Q_{Y1} using Castigliano's theorem:

$$\frac{\partial U}{\partial Q_{Y1}} = y_1 = 0 \tag{7.7}$$

$$0 = \frac{1}{2EI_Z} \cdot \left[- M_{Z2} \cdot L^2 + 2 \cdot Q_{Y1} \cdot \frac{L^3}{3} \right]$$

$$\boxed{Q_{Y1} = \frac{3 \cdot M_{Z2}}{2 \cdot L}} \tag{7.8}$$

$$\frac{\partial U}{\partial M_{Z2}} = \varphi_{Z2}$$

$$\frac{1}{2EI_Z} \left[2 \cdot M_{Z2} \cdot L - Q_{Y1} \cdot L^2 \right] = \varphi_{Z2}$$

Inserting Q_{Y1} from Eqn. (7.8)

$$\frac{1}{2EI_Z} \left[2 \cdot M_{Z2} \cdot L - \frac{3 \cdot M_{Z2}}{2} \cdot L \right] = \varphi_{Z2}$$

$$M_{Z2} \cdot \left[2 - \frac{3}{2} \right] = \frac{2EI_Z}{L} \cdot \varphi_{Z2}$$

$$\boxed{M_{Z2} = \frac{4EI_Z}{L} \cdot \varphi_{Z2}} \tag{7.9}$$

The shear force Q_{Y1} is then

$$Q_{Y1} = \frac{3}{2 \cdot L} \cdot M_{Z2} = \frac{3}{2 \cdot L} \cdot \frac{4EI_Z}{L} \cdot \varphi_{Z2} = \frac{6EI_Z}{L^2} \cdot \varphi_{Z2}$$

$$\boxed{Q_{Y1} = \frac{6EI_Z}{L^2} \cdot \varphi_{Z2}} \qquad (7.10)$$

From the equilibrium equation (7.2):

$$M_{Z1} = Q_{Y1} \cdot L - M_{Z2} = L \cdot \frac{6EI_Z}{L^2} \cdot \varphi_{Z2} - \frac{4EI_Z}{L} \cdot \varphi_{Z2} = \frac{2EI_Z}{L} \cdot \varphi_{Z2}$$

$$\boxed{M_{Z1} = \frac{2EI_Z}{L} \cdot \varphi_{Z2}} \qquad (7.11)$$

or

$$M_{Z1} = \frac{M_{Z2}}{2} \qquad (7.11a)$$

The free body in equilibrium is shown in Fig. 7.6:

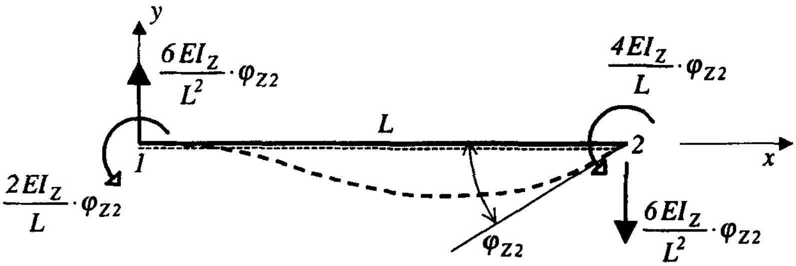


Figure 7.6: Element in equilibrium at rotation φ_{Z2}

Suppose now that the *positive side* of the element *is chosen as the lower side* of the beam and draw diagrams of bending moments [M] and shear forces [Q].

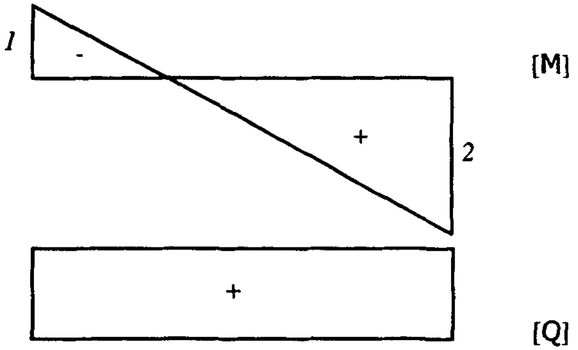


Figure 7.7: Internal forces at rotation φ_{Z2}

Similar expressions are derived for the rotation φ_{Y2} about the y-axis, the forces in equilibrium are:

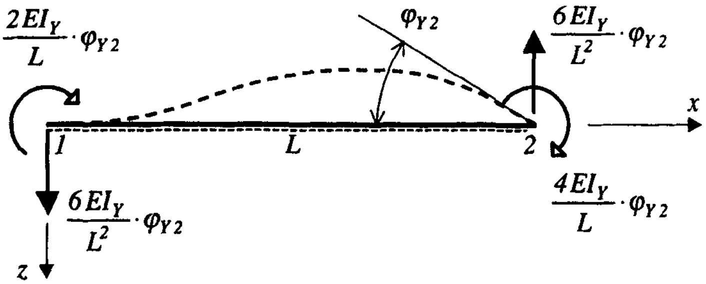


Figure 7.8: Element in equilibrium under rotation φ_{Y2}

Displacement y: The element is initially clamped and is given the displacement y_2 at joint 2 (Fig. 7.9)

A bending moment at an arbitrary section is

$$M_{Z1} + M - Q_{Y1} \cdot x = 0$$

$$M = Q_{Y1} \cdot x - M_{Z1} = -Q_{Y2} \cdot x - M_{Z1} \tag{7.12}$$

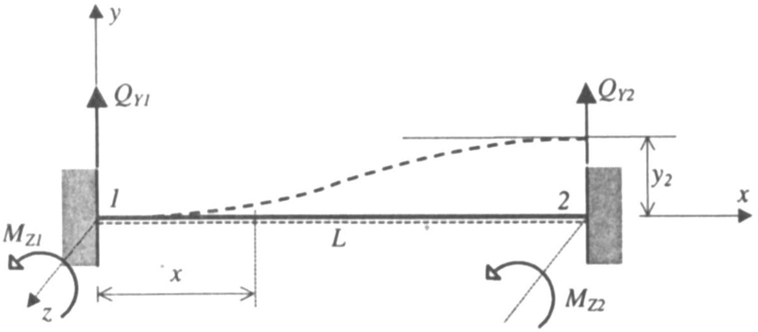


Figure 7.9: Element loaded by the displacement y_2

which gives the strain energy

$$U = \int_0^L \frac{M^2 \cdot dx}{2EI_Z} = \frac{1}{2EI_Z} \cdot \int_0^L \left[Q_{Y2}^2 \cdot x^2 + 2 \cdot Q_{Y2} \cdot M_{Z1} \cdot x + M_{Z1}^2 \right] \cdot dx$$

$$U = \frac{1}{2EI_Z} \cdot \left[Q_{Y2}^2 \cdot \frac{L^3}{3} + 2 \cdot Q_{Y2} \cdot M_{Z1} \cdot \frac{L^2}{2} + M_{Z1}^2 \cdot L^3 \right] \tag{7.13}$$

The derivative of M_{Z1} gives the rotation φ_{Z1}, which is zero due to the boundary condition at joint *1*:

$$\frac{\partial U}{\partial M_{Z1}} = 0 = 2 \cdot M_{Z1} \cdot L + Q_{Y2} \cdot L^2$$

$$Q_{Y2} = -\frac{2 \cdot M_{Z1}}{L} \tag{7.14}$$

The derivation of Q_{Y2} gives

$$\frac{\partial U}{\partial Q_{Y2}} = y_2 = \frac{1}{2EI_Z} \cdot \left(2 \cdot Q_{Y2} \frac{L^3}{3} + M_{Z1} \cdot L^2 \right)$$

$$U = \frac{1}{2EI_Z} \cdot \left[Q_{Y2}^2 \cdot \frac{L^3}{3} + 2 \cdot Q_{Y2} \cdot M_{Z1} \cdot \frac{L^2}{2} + M_{Z1}^2 \cdot L^3 \right]$$

$$y_2 = \frac{1}{2EI_Z} \cdot M_{Z1} \cdot L^2 \cdot \left(-\frac{4}{3} + 1 \right)$$

$$y_2 = -\frac{L^2}{6EI_Z} \cdot M_{Z1}$$

$$\boxed{M_{Z1} = -\frac{6EI_Z}{L^2} \cdot y_2} \qquad (7.15)$$

The bending moment at the other end is calculated from the equilibrium equation

$$M_{Z2} = -Q_{Y2} \cdot L - M_{Z1} = -\left(-\frac{2 \cdot M_{Z1}}{L} \right) \cdot L - M_{Z1} = -\frac{6EI_Z}{L^2} \cdot y_2$$

$$\boxed{M_{Z2} = -\frac{6EI_Z}{L^2} \cdot y_2} \qquad (7.16)$$

and finally the shear force Q_{Y2} from Eqn. (7.14):

$$Q_{Y2} = -\frac{2 \cdot M_{Z1}}{L} = \frac{12EI_Z}{L^3} \cdot y_2$$

$$\boxed{Q_{Y2} = \frac{12EI_Z}{L^3} \cdot y_2} \qquad (7.17)$$

As no other force acts in the y direction, the shear force Q_{Y1} equilibrates Q_{Y2}.

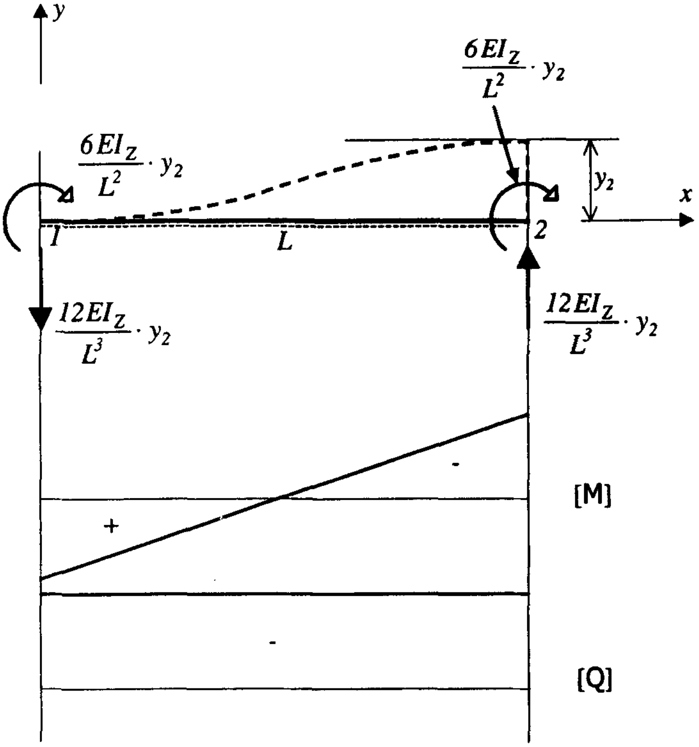


Figure 7.10: Element in equilibrium under displacement y_2

The remaining coefficients (rotations φ_Y and displacements z) can be determined in a similar manner and are written in the stiffness matrix in Fig. 7.11.

If all elements related to the z direction are crossed out from the equation in Fig. 7.11, we get the stiffness matrix which relates forces and displacements for a plane element:
The above equations may be written in a matrix form

$$\{F\}=[K]\cdot\{u\},\tag{7.18}$$

$[K]$ is a stiffness matrix of an element, $\{F\}$ is vector of forces at joints and $\{u\}$ is a vector of displacements at joints.

$$\begin{Bmatrix} N_1 \\ Q_{Y1} \\ Q_{Z1} \\ T_1 \\ M_{Y1} \\ M_{Z1} \\ \hdashline N_2 \\ Q_{Y2} \\ Q_{Z2} \\ T_2 \\ M_{Y2} \\ M_{Z2} \end{Bmatrix} = \left[\begin{array}{ccc|ccc|ccc|ccc}
\frac{EA}{L} & 0 & 0 & 0 & 0 & 0 & -\frac{EA}{L} & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{12EI_Z}{L^3} & 0 & 0 & 0 & \frac{6EI_Z}{L^2} & 0 & \frac{-12EI_Z}{L^3} & 0 & 0 & 0 & \frac{6EI_Z}{L^2} \\
0 & 0 & \frac{12EI_Y}{L^3} & 0 & \frac{-6EI_Y}{L^2} & 0 & 0 & 0 & \frac{-12EI_Y}{L^3} & 0 & \frac{-6EI_Y}{L^2} & 0 \\
\hline
0 & 0 & 0 & \frac{EJ_X}{2L(1+v)} & 0 & 0 & 0 & 0 & 0 & \frac{-EJ_X}{2L(1+v)} & 0 & 0 \\
0 & 0 & \frac{-6EI_Y}{L^2} & 0 & \frac{4EI_Y}{L} & 0 & 0 & 0 & \frac{6EI_Y}{L^2} & 0 & \frac{2EI_Y}{L} & 0 \\
0 & \frac{6EI_Z}{L^2} & 0 & 0 & 0 & \frac{4EI_Z}{L} & 0 & \frac{-6EI_Z}{L^2} & 0 & 0 & 0 & \frac{2EI_Z}{L} \\
\hline
-\frac{EA}{L} & 0 & 0 & 0 & 0 & 0 & \frac{EA}{L} & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{-12EI_Z}{L^3} & 0 & 0 & 0 & \frac{-6EI_Z}{L^2} & 0 & \frac{12EI_Z}{L^3} & 0 & 0 & 0 & \frac{-6EI_Z}{L^2} \\
0 & 0 & \frac{-12EI_Y}{L^3} & 0 & \frac{6EI_Y}{L^2} & 0 & 0 & 0 & \frac{12EI_Y}{L^3} & 0 & \frac{6EI_Y}{L^2} & 0 \\
\hline
0 & 0 & 0 & \frac{-EJ_X}{2L(1+v)} & 0 & 0 & 0 & 0 & 0 & \frac{EJ_X}{2L(1+v)} & 0 & 0 \\
0 & 0 & \frac{-6EI_Y}{L^2} & 0 & \frac{2EI_Y}{L} & 0 & 0 & 0 & \frac{6EI_Y}{L^2} & 0 & \frac{4EI_Y}{L} & 0 \\
0 & \frac{6EI_Z}{L^2} & 0 & 0 & 0 & \frac{2EI_Z}{L} & 0 & \frac{-6EI_Z}{L^2} & 0 & 0 & 0 & \frac{4EI_Z}{L}
\end{array}\right] \cdot \begin{Bmatrix} x_1 \\ y_1 \\ z_1 \\ \hdashline \varphi_{X1} \\ \varphi_{Y1} \\ \varphi_{Z1} \\ \hdashline x_2 \\ y_2 \\ z_2 \\ \hdashline \varphi_{X2} \\ \varphi_{Y2} \\ \varphi_{Z2} \end{Bmatrix}$$

Figure 7.11: Stiffness matrix of a prismatic element

$$\begin{Bmatrix} N_1 \\ Q_1 \\ M_1 \\ \hdashline N_2 \\ Q_2 \\ M_2 \end{Bmatrix} = \left[\begin{array}{ccc|ccc}
\frac{EA}{L} & 0 & 0 & -\frac{EA}{L} & 0 & 0 \\
0 & \frac{12EI}{L^3} & \frac{6EI}{L^2} & 0 & \frac{-12EI}{L^3} & \frac{6EI}{L^2} \\
0 & \frac{6EI}{L^2} & \frac{4EI}{L} & 0 & \frac{-6EI}{L^2} & \frac{2EI}{L} \\
\hline
-\frac{EA}{L} & 0 & 0 & \frac{EA}{L} & 0 & 0 \\
0 & \frac{-12EI}{L^3} & \frac{-6EI}{L^2} & 0 & \frac{12EI}{L^3} & \frac{-6EI}{L^2} \\
0 & \frac{6EI}{L^2} & \frac{2EI}{L} & 0 & \frac{-6EI}{L^2} & \frac{4EI}{L}
\end{array}\right] \begin{Bmatrix} x_1 \\ y_1 \\ \varphi_1 \\ \hdashline x_2 \\ y_2 \\ \varphi_2 \end{Bmatrix}$$

Figure 7.12: A relation between forces and displacements
for a plane rectangular element (no shear deformation is included)

Equation (7.18) can not be solved for a single element as an element can move without any distortions as shown in Fig. 7.13.

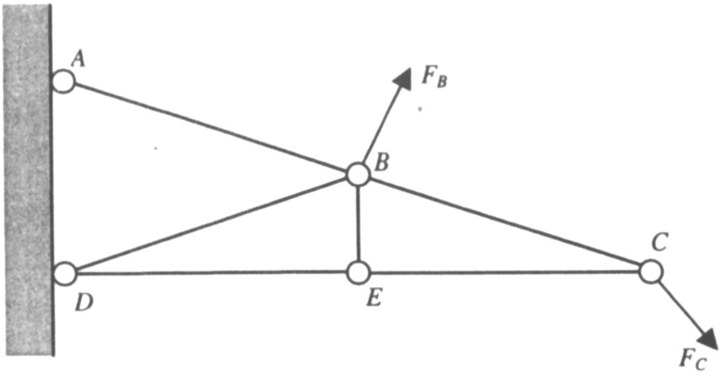


Figure 7.13: Plane truss ($N_{BE} = 0$)

Element BE is perpendicular to elements DE and EC, therefore the axial force in element BE must be zero, even though forces F_B and F_C cause deformation of the whole structure.

A similar case is on the structure from Fig. 7.14 where an additional force at joint E is applied in the direction of the element BE causing the tensile axial force in BE of the same magnitude F_E irrespective of all other applied forces.

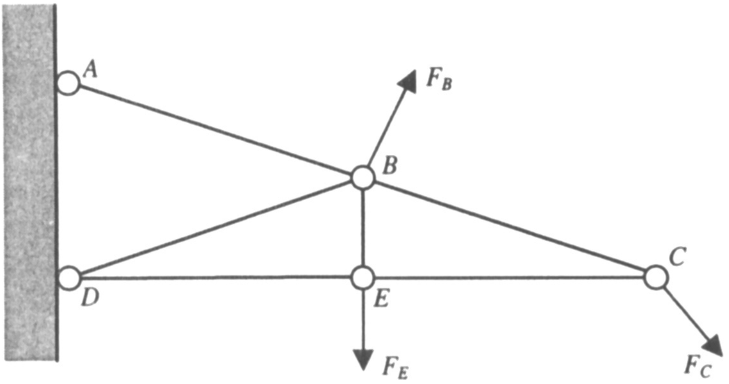


Figure 7.14: Plane truss ($N_{BE} \neq 0$)

Equation

$$\{u\}=[K]^{-1}\cdot\{F\} \tag{7.19}$$

therefore *can not be directly solved for one element* only.

An equation similar to Eqn. (7.18) can be written in terms of element distortions, i.e. the displacements at joint 2 (joint B in Fig. 7.15) are expressed *relative to the displacements of joint 1* (Joint *A* in Fig. 7.15).

A total distortion may be split into two phases:

a) A rigid body movement from *AB* to *A'B'*. The position is define by the final values of displacements at joint *A* (by x_1 and y_1)

b) A deformation of *A'B'* into the final positions *A'B''*.

As joint *A'* is already in its final position, additional displacements at *B'* are only required and will be denoted by *element distortions e*.

The vector of element distortions *e* has, in this plane case, three components (e_x, e_y, φ_z) but in a general case in space will have six components (e_x, e_y, e_z, φ_x, φ_y, φ_z). Forces on the element due to element distortion *e* are uniquely related and will in this case act at the joint as follows:

$$\varphi_Z = \varphi_{Z2} - \varphi_{Z1}$$

$$e_y = y_2 - y_1 - \varphi_{z1} \cdot L$$

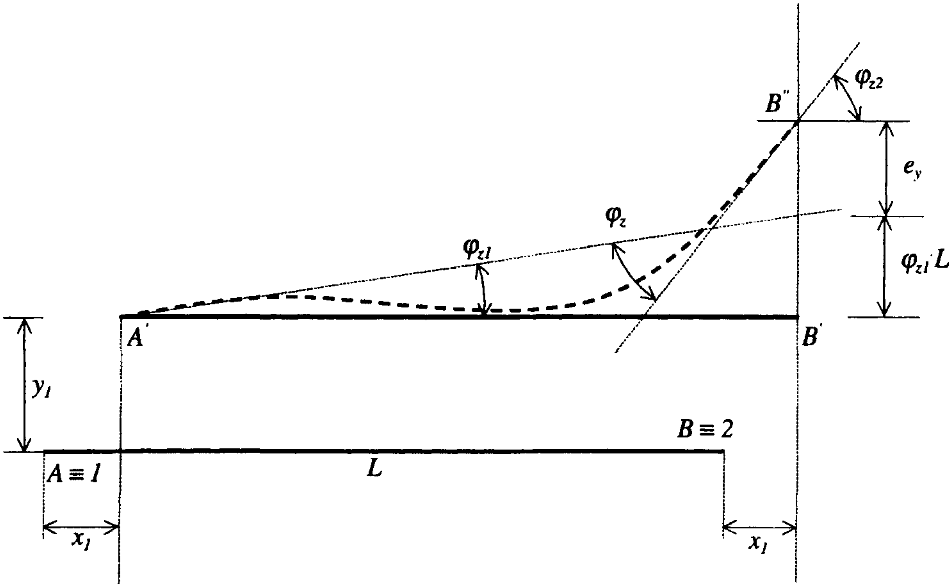


Figure 7.15: Relative joint displacements

The simplest way to deduce a new stiffness matrix of order 6×6 is to set all displacements at joint A to zero.

$$\begin{Bmatrix} N_2 \\ Q_{Y2} \\ Q_{Z2} \\ T_2 \\ M_{Y2} \\ M_{Z2} \end{Bmatrix} = \left[\begin{array}{ccc|ccc} \frac{EA}{L} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{12EI_Z}{L^3} & 0 & 0 & 0 & -\frac{6EI_Z}{L^2} \\ 0 & 0 & \frac{12EI_Y}{L^3} & 0 & \frac{6EI_Y}{L^2} & 0 \\ \hline 0 & 0 & 0 & \frac{EI_X}{2L(1+v)} & 0 & 0 \\ 0 & 0 & \frac{6EI_Y}{L^2} & 0 & \frac{4EI_Y}{L^3} & 0 \\ 0 & -\frac{6EI_Z}{L^2} & 0 & 0 & 0 & \frac{4EI_Z}{L} \end{array} \right] \cdot \begin{Bmatrix} e_X \\ e_Y \\ e_Z \\ \varphi_X \\ \varphi_Y \\ \varphi_Z \end{Bmatrix}$$

or simply

$$\{F\} = [K] \cdot \{e\} \tag{7.20}$$

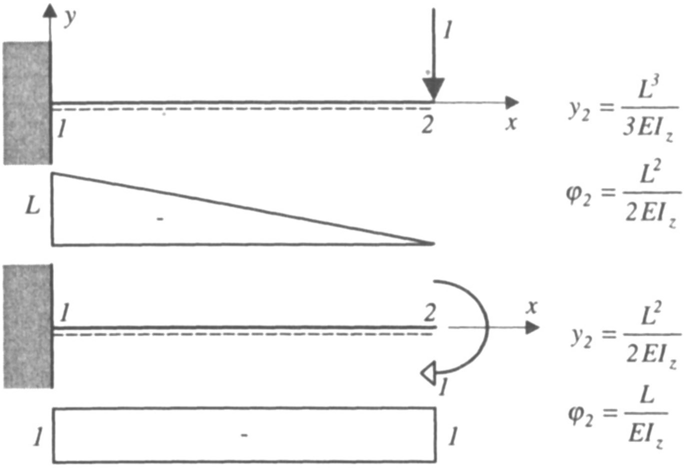
Solving for element deformations gives

$$\{e\} = [K]^{-1} \cdot \{F\} = [D] \cdot \{F\}, \tag{7.21}$$

where $[D]$ is a *flexibility matrix of an element.*

$$[D] = \left[\begin{array}{ccc|ccc} \frac{L}{EA} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{L^3}{3EI_Z} & 0 & 0 & 0 & \frac{L^2}{2EI_Z} \\ 0 & 0 & \frac{L^3}{3EI_Y} & 0 & -\frac{L^2}{2EI_Y} & 0 \\ \hline 0 & 0 & 0 & \frac{2L(1+v)}{EI_X} & 0 & 0 \\ 0 & 0 & -\frac{L^2}{2EI_Y} & 0 & \frac{L}{EI_Y} & 0 \\ 0 & \frac{L^2}{2EI_Z} & 0 & 0 & 0 & \frac{L}{EI_Z} \end{array} \right]$$

The terms of flexibility matrix $[D]$ are *flexibility influence coefficients*. They are the displacements which occur at joint 2 if the same joint is loaded by a unit force and all displacements at joint 1 are held to zero. Consider the example below:



$$y_2 = \frac{L^3}{3EI_z} \qquad \varphi_2 = \frac{L^2}{2EI_z}$$

$$y_2 = \frac{L^2}{2EI_z} \qquad \varphi_2 = \frac{L}{EI_z}$$

Figure 7.16: Displacement and rotation of an element

7.3 Stiffness matrix of a clamped-hinged element *(m_5)*

The stiffness coefficients will be determined by the theorem of unit force considering elements in the *x*-*y* plane only.

Figure 7.17: Clamped-hinged element

a) *Rotation* φ_{Z1}

$$EI \cdot \varphi_{Z1}' = \frac{1}{3} \cdot M_{Z1} \cdot 1 \cdot L = \frac{M_{Z1} \cdot L}{3}$$

$$M_{Z1} = \frac{3EI}{L} \cdot \varphi_{Z1} \tag{7.22}$$

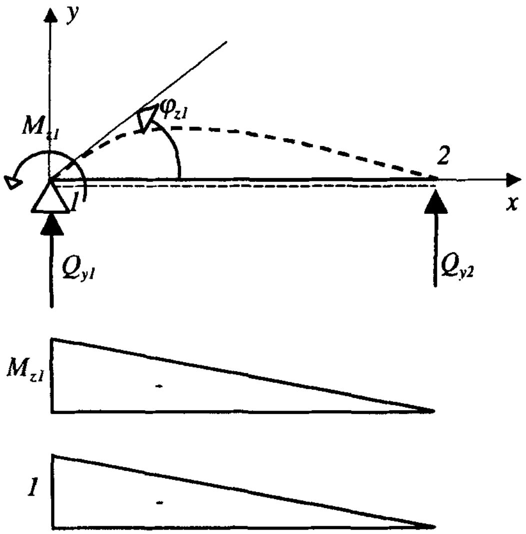


Figure 7.18: Forces induced by the rotation φ_{z1}

From the equilibrium condition $\Sigma M_1 = 0$:

$$Q_{Y2} \cdot L + M_{Z1} = 0$$

$$Q_{Y2} = -\frac{M_{Z1}}{L} = -\frac{3EI}{L^2} \cdot \varphi_{Z1} \tag{7.23}$$

$$Q_{Y1} = \frac{3EI}{L^2} \cdot \varphi_{Z1} \tag{7.24}$$

b) *Displacement* y_1

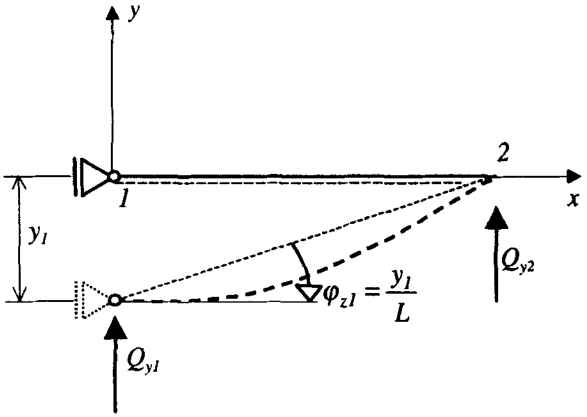


Figure 7.19: Deformation of an element under displacement y_1

The element is at first displaced by y_1 (dotted line) and then rotated by φ_{Z1} to have a horizontal tangent at joint *1* (initially clamped) in the final position.

$$M_{Z1} = \frac{3 \cdot E \cdot I}{L} \cdot \frac{y_1}{L} = \frac{3 \cdot E \cdot I}{L^2} \cdot y_1 \tag{7.25}$$

$$Q_{Y2} = -\frac{3 \cdot E \cdot I}{L^3} \cdot y_1 \tag{7.26}$$

$$Q_{Y1} = +\frac{3 \cdot E \cdot I}{L^3} \cdot y_1 \tag{7.27}$$

$$\left\{\begin{array}{c} N_1 \\ Q_{Y1} \\ M_{Z1} \\ \hdashline N_2 \\ Q_{Y2} \\ 0 \end{array}\right\} = \left[\begin{array}{ccc:ccc} \frac{EA}{L} & 0 & 0 & -\frac{EA}{L} & 0 & - \\ 0 & \frac{3EI}{L^3} & \frac{3EI}{L^2} & 0 & \frac{-3EI}{L^3} & - \\ 0 & \frac{3EI}{L^2} & \frac{3EI}{L} & 0 & \frac{-3EI}{L^2} & - \\ \hdashline -\frac{EA}{L} & 0 & 0 & \frac{EA}{L} & 0 & - \\ 0 & \frac{-3EI}{L^3} & \frac{-3EI}{L^2} & 0 & \frac{3EI}{L^3} & - \\ - & - & - & - & - & - \end{array}\right] \left\{\begin{array}{c} x_1 \\ y_1 \\ \varphi_{Z1} \\ \hdashline x_2 \\ y_2 \\ \varphi_{Z2} \end{array}\right\}$$

The same result is derived by the reduction of the whole matrix (of the clamped element on both sides) into prime degrees of freedom by partitioning of the matrix

$$[K] = \begin{bmatrix} [K_{aa}] & [K_{ab}] \\ [K_{ba}] & [K_{bb}] \end{bmatrix} \tag{7.28}$$

The reduction is done by *unessential degree of freedom* that is *the rotation at joint 2*

$$[K_{bb}] = \left[\frac{4EI_Z}{L}\right] \qquad [K_{bb}]^{-1} = \left[\frac{L}{4EI_Z}\right]$$

$$[K_c] = [K_{aa}] - [K_{ab}][K_{bb}]^{-1}[K_{ba}] \tag{7.29}$$

and after matrix multiplications the same result as previous is obtained.

7.4 Stiffness matrix for a truss element (m_4)

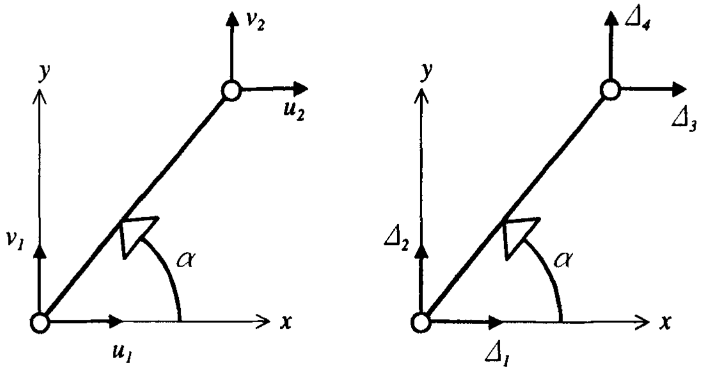


Figure 7.20: Degrees of freedom of a truss element

The stiffness matrix for a truss element consists of terms from Eqns. (7.3a) and (7.3b) only and is written in a global co-ordinate system using Eqn. (10.34):

$$[K]=\frac{EA}{L}\cdot\left[\begin{array}{cc:cc} \cos^2\alpha & \sin\alpha\cdot\cos\alpha & -\cos^2\alpha & -\sin^2\alpha \\ \sin\alpha\cdot\cos\alpha & \sin^2\alpha & -\sin\alpha\cdot\cos\alpha & -\sin^2\alpha \\ \hdashline -\cos^2\alpha & -\sin\ \ \cdot\cos & \cos^2\alpha & \sin\alpha\cdot\cos\alpha \\ -\sin\alpha\cdot\cos\alpha & -\sin^2\alpha & \sin\alpha\cdot\cos\alpha & \sin^2\alpha \end{array}\right] \qquad (7.30)$$

The reader should note that the matrix is symmetric as expected from Betty-Maxwell's theorem derived earlier.

8

The Force Method

(Method of consistent deformations)

8.1 A degree of static indeterminacy (*DSI*)

A *degree of static indeterminacy* n can be calculated from a number of equilibrium equations E and a number of unknown forces N on a structure by the equation:

$$n = N - E\,, \tag{8.1}$$

therefore structures can be classified as:

$n = 0$ *statically determinate structures*
$n > 0$ *statically indeterminate structures*
$n < 0$ *unstable structures (mechanisms)*

Each structure consists of j joints and m elements. Joints can be rigid (j_3) or pinned (j_2) and are connected by elements m_6, m_5, m_4 and m_3.

Note that the supports in Fig. 8.1 are not denoted, as the *support forces are included* in the elements connected to supports[♣].

Equilibrium equations are set for joints as free bodies so that in rigid joints there are three equations and in pinned joint only two equations:

$$E = 3 \cdot j_3 + 2 \cdot j_2 + 3 \cdot \left(m_6 + m_5 + m_4 + m_3\right)$$

[♣] If several elements are connected to a support, it has to be included into Eq.8.2 but the number of suppresed displacements p at that support has to be deduced from the equation

Unknown forces appear at element ends where a structure is connected together; the number of forces depends on the element type:

$$N = 6 \cdot m_6 + 5 \cdot m_5 + 4 \cdot m_4 + 3 \cdot m_3$$

A degree of static indeterminacy n (*DSI*) is therefore:

$$\begin{aligned} n = N - E = 6 \cdot m_6 + 5 \cdot m_5 + 4 \cdot m_4 + 3 \cdot m_3 - \\ - 3 \cdot j_3 - 2 \cdot j_2 - 3 \cdot m_6 - 3 \cdot m_5 - 3 \cdot m_4 - 3 \cdot m_3 = \\ = 3 \cdot m_6 + 2 \cdot m_5 + m_4 - 3 \cdot j_3 - 2 \cdot j_2 \end{aligned}$$

or

$$n = 3 \cdot (m_6 - j_3) + 2 \cdot (m_5 - j_2) + m_4 \qquad (8.2)$$

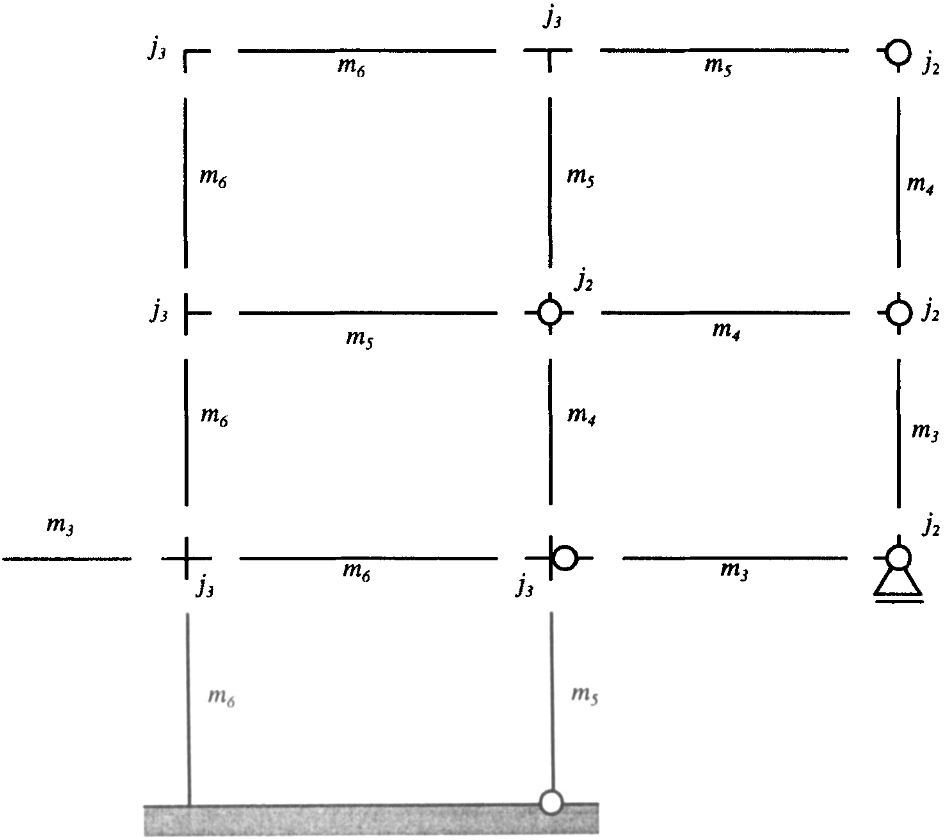


Figure 8.1: Definition of joint and element types

As can be seen from Eqn. (8.2) an element m_3 has no influence on the degree of indeterminacy as it is a basic statically determinate element (i.e. cantilever), which can be solved by three basic equilibrium equations.

Example 8.1: Determine a degree of static indeterminacy for the rigid frame in Fig. 8.2.

$$m_6 = 6,\ m_5 = 0,\ m_4 = 0,\ j_3 = 4,\ j_2 = 0$$

$$n = 3(6 - 4) + 2(0 - 0) + 0 = 6$$

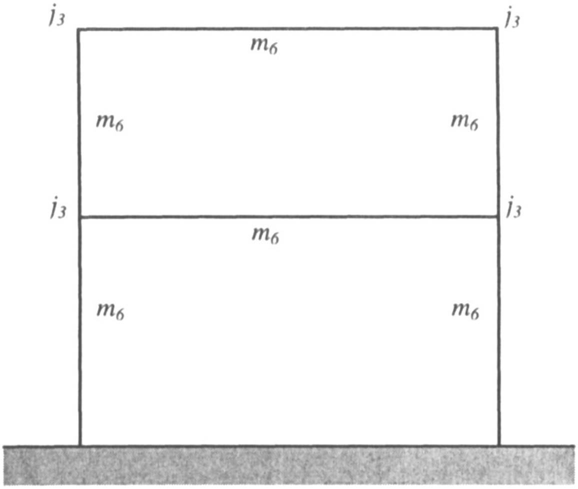


Figure 8.2: Rigid frame - definition of joints and elements

Example 8.2: Determine the degree of static indeterminacy for a cable stayed bridge shown in Fig. 8.3 (Note that the deck and pylon are not connected)

$$m_6 = 3,\ m_5 = 1,\ m_4 = 6,\ j_3 = 4,\ j_2 = 1$$

The structure is $n = 3(3 - 4) + 2(1 - 1) + 6 = 3$ *times statically indeterminate*; redundant or unknown forces could be chosen as axial forces in the three out of four cables.

8.2 Primary structure

We consider a structure of degree n of static indeterminacy; that is, the number of unknown forces exceeds the number of available equilibrium equations by n. The statics equations must be supplemented by a number of equations of geometry equal to the degree of redundancy n of the structure. In other words, we have to write n additional elasticity or *deformational equations* on the *primary structure*, which is a statically determinate structure.

A primary structure is obtained in such a way that an indeterminate structure is transformed into determinate structure *by removal of n redundant forces*, which can be reactive forces at supports or internal forces at an arbitrary section. The *primary structure obtained must be stable*, which can be proved as shown in Ch. 4.

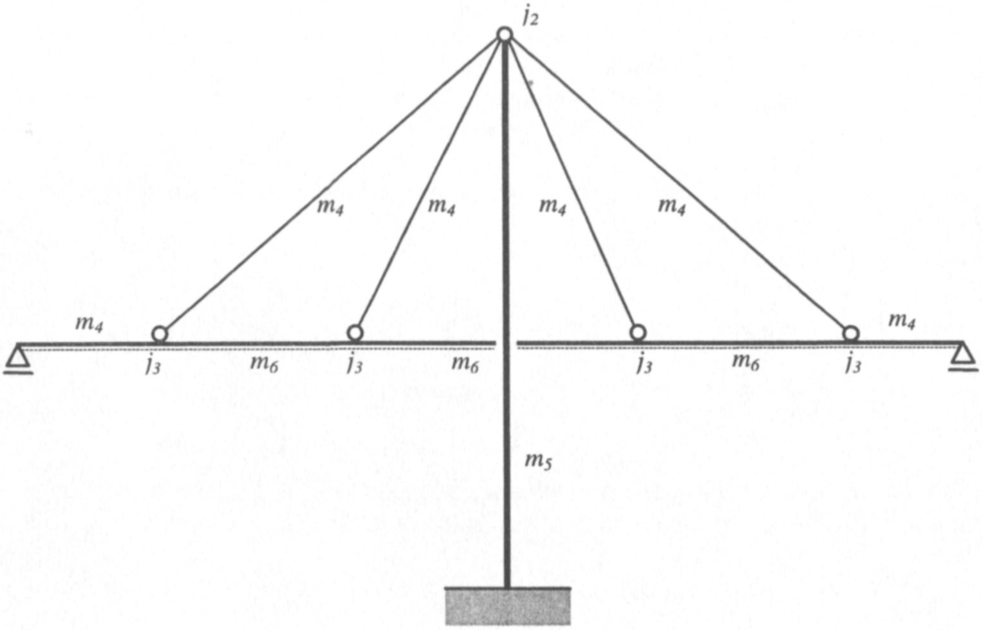


Figure 8.3: A cable stayed bridge

8.3 Deformation of primary structure

Consider the frame in Fig. 8.4a, which is reduced to the primary structure in Fig. 8.4b.

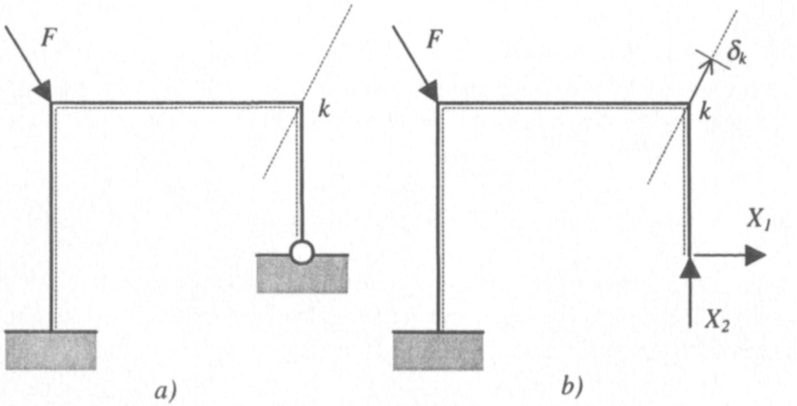


Figure 8.4: Basic and primary structure

Calculate the deformation of joint *k* in the direction as shown in Fig. 8.4b. External force F causes a deformation at joint *k*:

$$\delta_{kO} = \int \frac{M_k \cdot M_0}{EI} \cdot ds \tag{8.3}$$

where: M_0 *bending moments on primary structure caused by external load*

M_k *bending moments on primary structure caused by unit force in the direction of the desired deformation*

Furthermore, unknown forces X_1 and X_2 will cause deformations at joint *k* given by the following equations:

$$\delta_{k1} = \int \frac{M_k \cdot M_1}{EI} \cdot ds \tag{8.4}$$

$$\delta_{k2} = \int \frac{M_k \cdot M_2}{EI} \cdot ds \tag{8.5}$$

Moment M_k is as defined above and moments M_1 and M_2 are caused by unit forces $X_1 = 1$ and $X_2 = 1$. Real deformations due to unknown forces X_1 and X_2 are:

$$X_1 \cdot \delta_{k1} \quad and \quad X_2 \cdot \delta_{k2}$$

The total deformation at joint *k* is therefore:

$$\delta_k = \delta_{kO} + X_1 \cdot \delta_{k1} + X_2 \cdot \delta_{k2}$$

If there are *n* redundants, the equation can be written as:

$$\delta_k = \delta_{kO} + \sum_{i=1}^{n} X_i \cdot \delta_{ki} \tag{8.6}$$

Equation (8.6) means that a *deformation can be calculated at any point of the structure*, hence we can choose *n* points *k* and write down *n* equations. If the deformation δ_k at all these points would be known then all *n* unknowns X_i could be calculated.

8.4 Elasticity equations

Static indeterminate *internal forces* and *moments* X_i always occur as pairs being in equilibrium with each other. These pairs of forces produce no inter-related deformation on the whole structure, hence for these points the deformation $\delta_k = 0$ and the equation (8.6) reduces to:

$$\delta_{k0} + \sum_{i=1}^{n} X_i \cdot \delta_{ki} = 0 \tag{8.7}$$

Equations (8.7) are *elasticity equations of static*. The number of these equations exactly equals the number of redundants in a structure.
The coefficients are:

δ_{k0} *deformation at joint k caused by an external load*
δ_{ki} *deformation at joint k caused by a unit force applied at i*

If *bending moments only* are taken into account (see Example 5.7) then the coefficients are given by:

$$\delta_{k0} = \int \frac{M_k \cdot M_0}{EI} \cdot ds \tag{8.8}$$

$$\delta_{ki} = \int \frac{M_i \cdot M_k}{EI} \cdot ds \tag{8.9}$$

Example 8.3: The force method will be outlined step by step in terms of a one-time statically indeterminate beam:

- *Step 1*: Determine the degree of static indeterminacy using Eqn. (8.2)

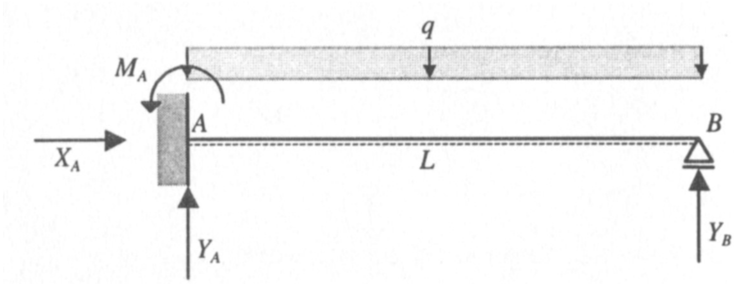


Figure 8.5a: Statically indeterminate beam

$$n = 3 \cdot (m_6 - j_3) + 2 \cdot (m_5 - j_2) + m_4 = 3(0\text{-}0) + 2(0\text{-}0) + 1 = 1$$

- *Step 2*: Evaluate a statically determined stable primary structure. In this example clamping moment M_A or reaction Y_B can be released.

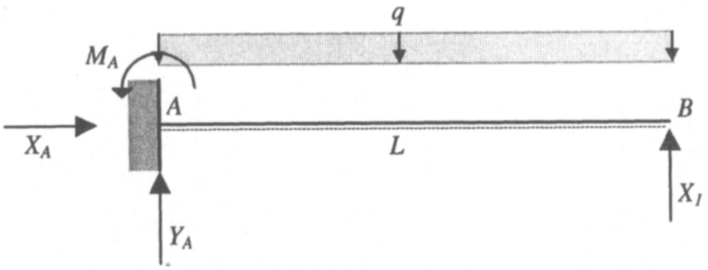


Figure 8.5b: Primary structure

❖ *Step 3*: Calculate the deformation of a primary structure caused by external loads

$$\delta_{BO} = \frac{1}{EI}\frac{1}{4}\frac{qL^2}{2}L \cdot L = \frac{qL^4}{8EI}$$

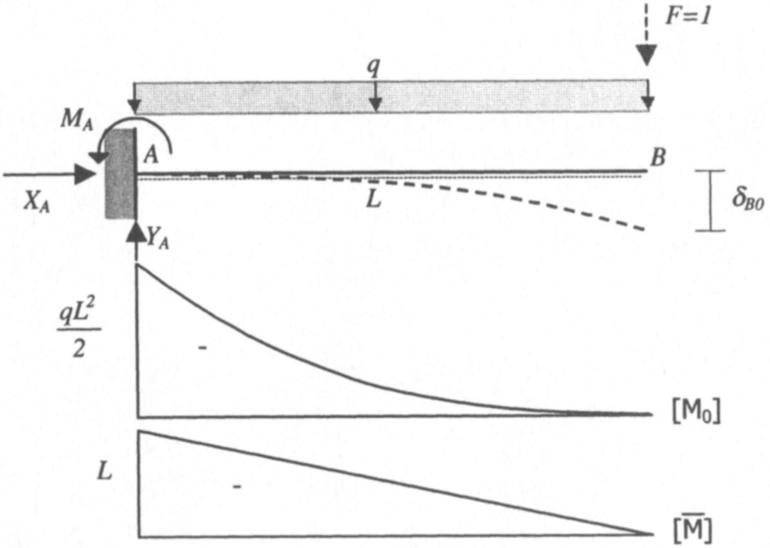


Figure 8.5c: Deformation due to an external load

❖ *Step 4*: Calculate the deformation of a primary structure caused by the redundant force $X_1 = 1$

$$\delta_{B1} = \frac{1}{EI}\frac{1}{3}L \cdot (-L) \cdot L = -\frac{L^3}{3EI}$$

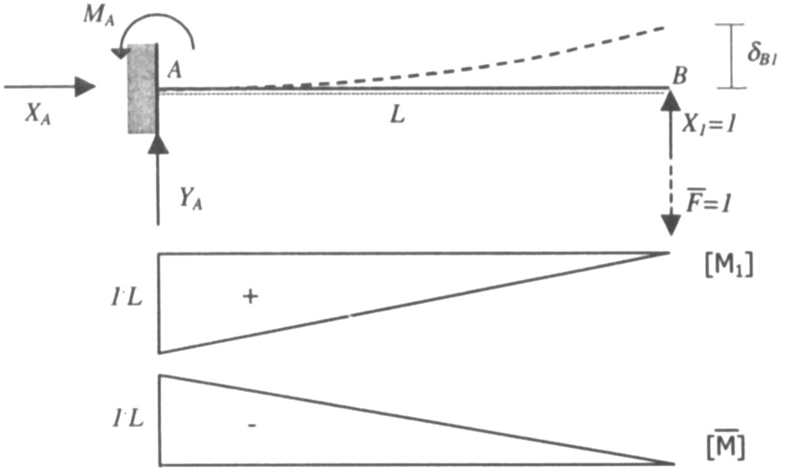


Figure 8.5d: Deformation due to a redundant force $X_I = 1$

- *Step 5*: Using elasticity equation (8.7) calculate the unknown force X_I such that the displacement at joint B is zero.

$$\delta_{BO} + X_I \cdot \delta_{BI} = 0$$

$$\frac{qL^4}{8EI} + X_I \cdot \left(-\frac{L^3}{3EI}\right) = 0$$

$$X_I = \frac{3qL}{8}$$

- *Step 6*: Force X_I becomes an ***external load*** on the primary structure, which can be solved using basic equilibrium equations.

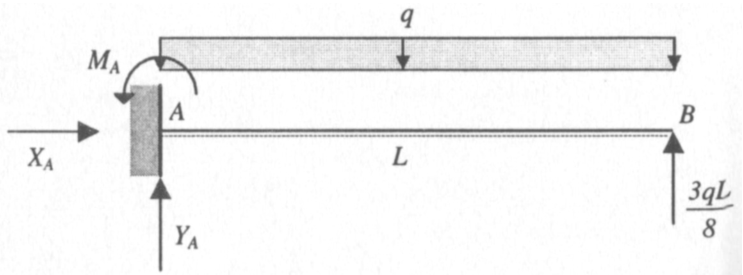


Figure 8.5e: Primary structure is loaded by all forces

$$\Sigma Y = 0 : \; Y_A - qL + \frac{3qL}{8} = 0 \quad \Rightarrow \quad Y_A = \frac{5qL}{8}$$

$$\Sigma M_A = 0 : \; M_A - qL\frac{L}{2} + \frac{3qL}{8}L = 0 \quad \Rightarrow \quad M_A = \frac{qL^2}{8}$$

Find the equilibrium for a small part of the beam of length *x* from the right hand end:

$$M_x = \frac{3qL}{8}x - \frac{qx^2}{2}$$

$$Q_x = \frac{3qL}{8} - qx$$

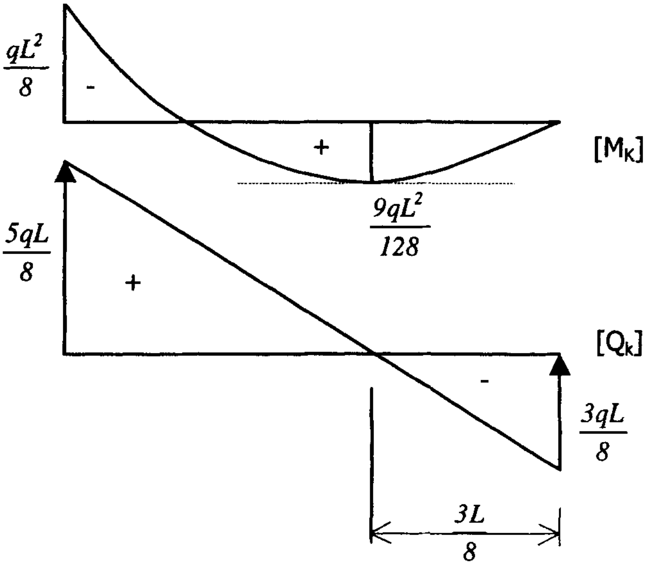


Figure 8.5f: Diagrams of bending moments and shear forces

The *maximum bending moment* is at the position where the shear force equals zero:

$$x_{max} = \frac{3L}{8} \qquad\qquad M_{max} = \frac{9qL^2}{128}$$

Final diagrams of bending moments and shear forces are shown in Fig. 8.5f.

When we need to calculate values at some critical points, we can use the method of superposition, i.e. bending moment at joint *A* is:

$$M_A = M_0 + M_1 \cdot X_1 = -\frac{qL^2}{2} + L\cdot\frac{3qL}{8} = -\frac{qL^2}{8}$$

Example 8.4: Using the force method solve the structure from Fig. 8.6a, which consists of beam *BD* and two truss elements *AD* and *CD*. The beam is clamped at joint *B*.

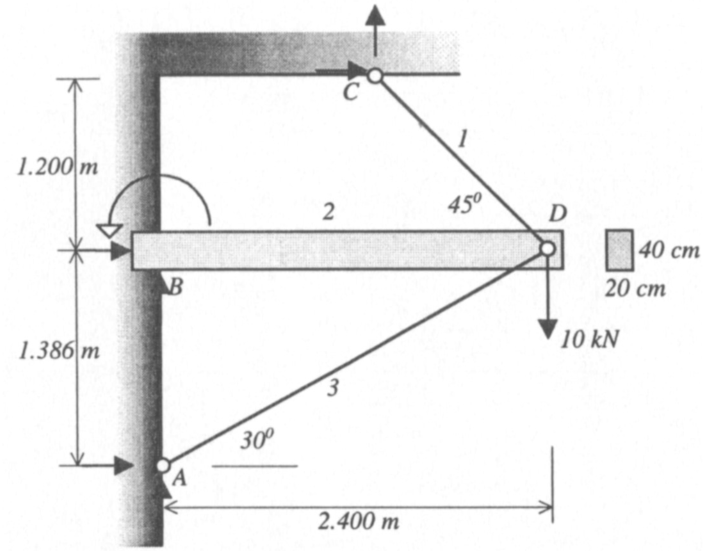


Figure 8.6a: Mixed structure

Element data:

$$A_1 = A_3 = 3\ cm^2 = 3 \cdot 10^{-4}\ m^2$$

$$E_1 = E_3 = 200\ GPa = 200 \cdot 10^6\ kPa$$

$$A_2 = 800\ cm^2,\ I_2 = 10.67 \cdot 10^{-4}\ m^4$$

$$E_2 = 25\ GPa$$

$$E_1 A_1 = 60000\ kN \qquad E_2 A_2 = 2000000\ kN \qquad E_2 I_2 = 26675\ kNm^2$$

The structure has also two *truss elements*, which can carry axial forces only, hence the work of the axial forces must be taken into account. For ease of understanding, the deformation for all three will be calculated separately.

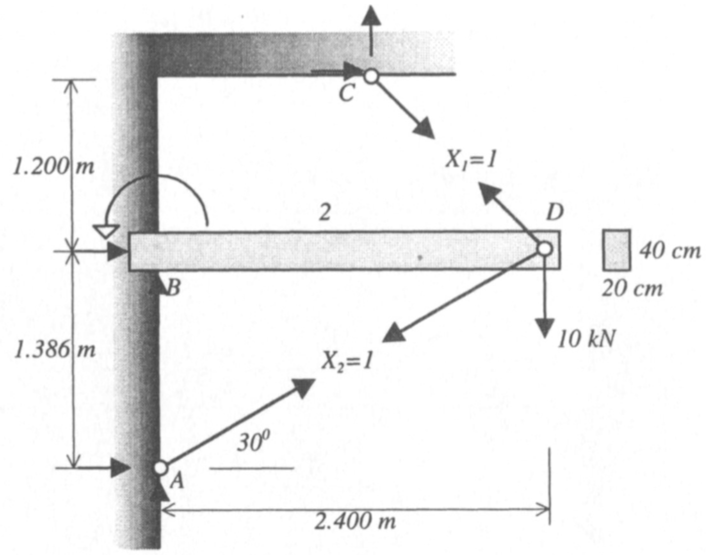


Figure 8.6b: Primary structure

Elasticity equations in the direction of redundant forces are ♥

$$a_{10}+a_{11}\cdot X_1+a_{12}\cdot X_2=-\frac{X_1L_1}{(EA)_1}$$

$$a_{20}+a_{21}\cdot X_1+a_{22}\cdot X_2=-\frac{X_2L_3}{(EA)_3},$$

where the right hand side terms represent elongation or contraction of elements *1* and *3* due to axial forces in respective elements and the left hand side terms include all three deformations. At first we calculate and draw diagrams of bending moments, shear forces and axial forces for external load and redundant forces X_1 and X_2. Integration of deformations is performed for each influence separately:

♥ *In subsequent text coefficients δ will be denoted by letter a as is usual in mathematical textbooks*

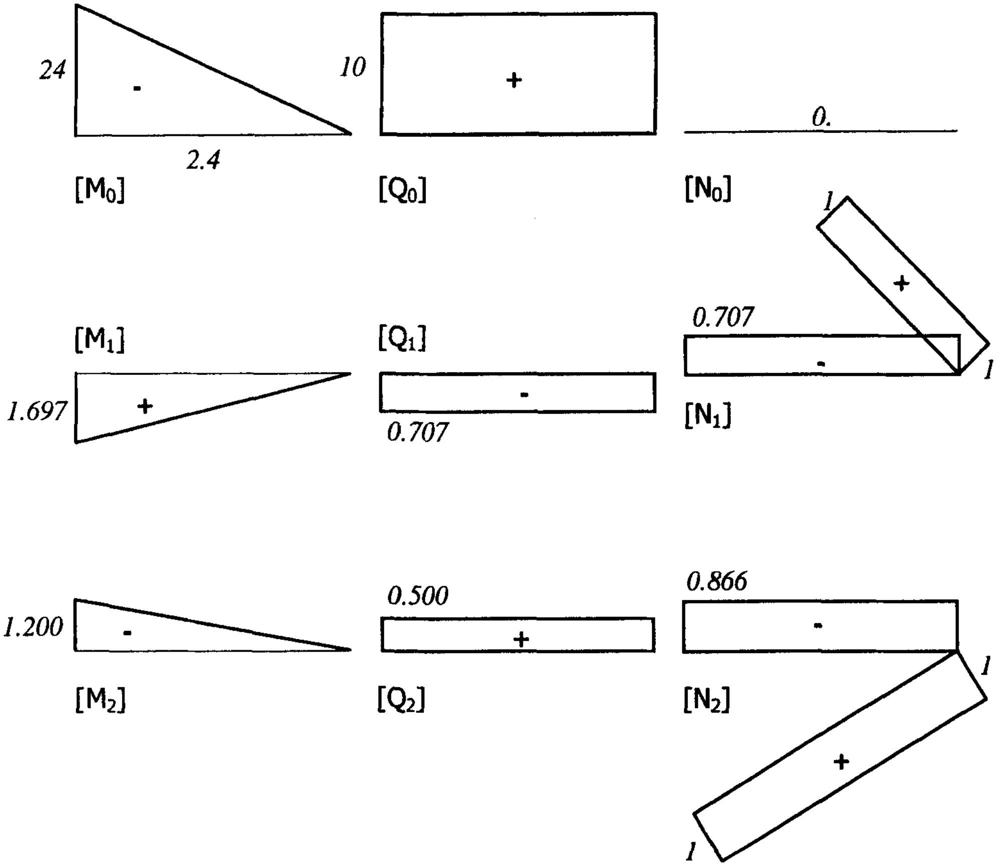


Figure 8.7: Diagrams of the primary structure

a) Contribution of bending moments

$$a_{11}^{M} = \frac{1}{3}\frac{M\overline{M}}{(EI)_2}2.4 = \frac{1.697^2}{3\cdot 26675}\cdot 2.4 = 8.638\cdot 10^{-5}$$

$$a_{22}^{M} = \frac{1.200^{\;2}}{3\cdot 26675}\cdot 2.4 = 4.319 \cdot 10^{-5}$$

$$a_{12}^{M} = \frac{1.697\cdot(-1.200)}{3\cdot 26675}\cdot 2.4 = -6.108\cdot 10^{-5}$$

$$a_{10}^{M} = \frac{(-24)\cdot 1.697}{3\cdot 26675}\cdot 2.4 = -1.221\cdot 10^{-3}$$

$$a_{20}^{M}=\frac{(-24)\cdot(-1.200)}{3\cdot 26675}\cdot 2.4=0.8637\cdot 10^{-3}$$

b) *Contribution of shear forces* (shear shape factor for rectangular cross-section $K=1.2$)

$$G_2=\frac{E}{2(1+\nu)}=\frac{25}{2(1+0.18)}=10.593\ GPa$$

$$A_S=\frac{A}{K}=\frac{A}{1.2}=\frac{800}{1.2}=666.667\cdot 10^{-4}\,m^2$$

$$(GA_S)_2=706200\ kN$$

$$a_{11}^{Q}=\frac{Q\overline{Q}}{(GA_S)_2}2.4=\frac{0.707^2}{706200}\cdot 2.4=1.699\cdot 10^{-6}$$

$$a_{22}^{Q}=\frac{0.500^2}{706200}\cdot 2.4=0.8496\cdot 10^{-6}$$

$$a_{12}^{Q}=\frac{(-0.707)\cdot 0.500}{706200}\cdot 2.4=-1.201\cdot 10^{-6}$$

$$a_{10}^{Q}=\frac{10\cdot(-0.707)}{706200}\cdot 2.4=-2.403\cdot 10^{-5}$$

$$a_{20}^{Q}=\frac{10\cdot 0.500}{706200}\cdot 2.4=1.699\cdot 10^{-5}$$

c) *Contribution of axial forces*

$$a_{11}^{N}=\sum\frac{N\overline{N}}{(EA)_i}L_i=\frac{1\cdot 1}{60000}\cdot 1.697+\frac{0.707^2}{2\cdot 10^6}\cdot 2.4$$
$$=2.828\cdot 10^{-5}+6\cdot 10^{-7}=2.888\cdot 10^{-5}$$

$$a_{22}^{N}=\frac{1\cdot 1}{60000}\cdot 2.771+\frac{0.866^2}{2\cdot 10^6}\cdot 2.4=4.619\cdot 10^{-5}+9\cdot 10^{-7}=4.709\cdot 10^{-5}$$

$$a_{12}^{N}=\frac{(-0.707)\cdot(-0.866)}{2\cdot 10^6}\cdot 2.4=7.348\cdot 10^{-7}$$

$$a_{10}^{N}=0\quad ;\quad a_{20}^{N}=0$$

Table 8.1

$all*10^{-5}$	M	Q	N	Σ
a_{11}	8.638	0.16992	2.888	11.696
a_{22}	4.318	0.08496	4.709	9.112
a_{12}	-6.108	-0.12015	0.074	-6.154
a_{10}	-122.100	-2.40300	0.000	-124.503
a_{20}	86.370	1.69900	0.000	88.069

The equations are

$$\begin{bmatrix} 11.696 & -6.155 \\ -6.155 & 9.113 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \begin{Bmatrix} 124.503 \\ -88.069 \end{Bmatrix},$$

giving the solution $X_1 = 8.624\ kN$ and $X_2 = -3.840\ kN$.

❖ If shear forces are neglected

$$\begin{bmatrix} 11.526 & -6.034 \\ -6.034 & 9.028 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \begin{Bmatrix} 122.10 \\ -86.37 \end{Bmatrix}$$

with a solution $X_1 = 8.591\ kN$ and $X_2 = -3.825\ kN$.

❖ If axial forces (***inadmissible in this example!***) are neglected

$$\begin{bmatrix} 8.638 & -6.108 \\ -6.108 & 4.318 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \begin{Bmatrix} 122.10 \\ -86.37 \end{Bmatrix}$$

The above system of equations ***has no trivial solution as the determinant is zero***. Practically it means that joint *D* does not move at all, as the flexibility of truss elements was infinite.

In practice such mixed systems are often solved in a way, that in the matrix of bending coefficients, axial flexibility of *L/EA* is added to the diagonal terms:

$$\begin{bmatrix} 8.638+2.888 & -6.108 \\ -6.108 & 4.318+4.709 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \begin{Bmatrix} 122.10 \\ -86.37 \end{Bmatrix},$$

giving the solution $X_1 = 8.611\ kN$ and $X_2 = -3.742\ kN$. The solution is not exact but acceptable being within ±2 %.
Internal forces are calculated using the method of superposition e.g.:

$$M_B = -24 + 1.697 \cdot X_1 - 1.2 \cdot X_2 =$$

$$= -24 + 1.697 \cdot 8.624 - 1.2 \cdot (-3.840) = -4.757\ kNm$$

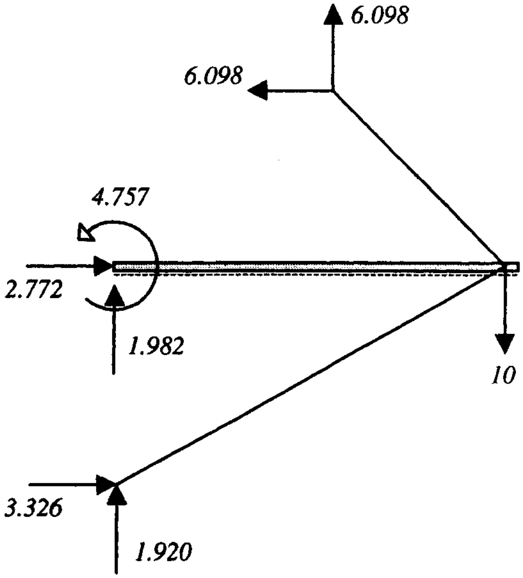


Figure 8.8: Free body in equilibrium

Example 8.5: Calculate the structure from example 8.4 using a different primary structure. In this case we cut element *CD* and release the clamping moment (insert hinge) at joint *B*.

External loading causes axial forces only and a bending moment occurs on the beam due to $X_2=1$.

The influence of shear forces will be neglected, hence:

$$a_{11} = \frac{1.414^2}{60000} \cdot 2.771 + \frac{1.1}{60000} \cdot 1.697 + \frac{1.932^2}{2 \cdot 10^6} \cdot 2.4 = 12.51 \cdot 10^{-5}$$

$$a_{22} = \frac{0.833^2}{60000} \cdot 2.771 + \frac{0.722^2}{2 \cdot 10^6} \cdot 2.4 + \frac{1}{3}\frac{1.1}{26675} \cdot 2.4 = 6.266 \cdot 10^{-5}$$

$$a_{12} = \frac{1.414 \cdot 0.833}{60000} \cdot 2.771 + \frac{1.932 \cdot 0.722}{2 \cdot 10^6} \cdot 2.4 = 5.607 \cdot 10^{-5}$$

$$a_{10} = \frac{(-20) \cdot 1.414}{60000} \cdot 2.771 + \frac{17.321 \cdot (-1.932)}{2 \cdot 10^6} \cdot 2.4 = -1.346 \cdot 10^{-3}$$

$$a_{20} = \frac{(-20) \cdot 0.833}{60000} \cdot 2.771 + \frac{17.321 \cdot (-0.722)}{2 \cdot 10^6} \cdot 2.4 = -0.784 \cdot 10^{-3}$$

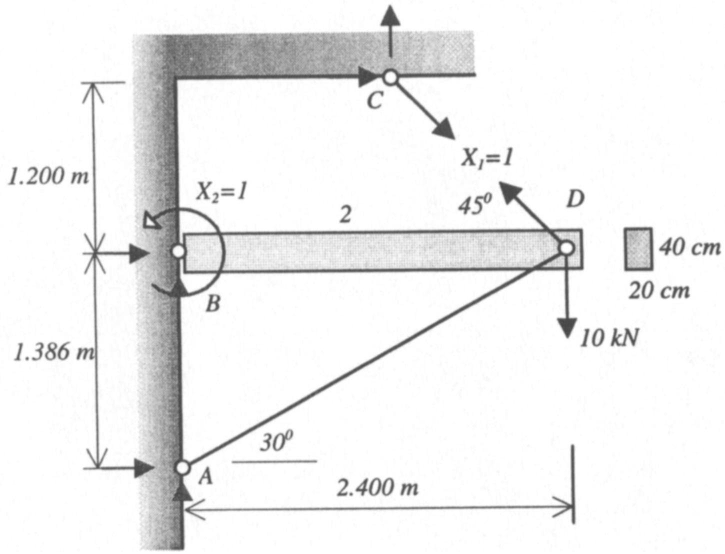


Figure 8.9: Primary structure

$$\begin{bmatrix} 12.51 & 5.607 \\ 5.607 & 6.266 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \begin{Bmatrix} 134.6 \\ 78.4 \end{Bmatrix}$$

The solution is $X_1 = 8.601\ kN$ and $X_2 = 4.815\ kNm$.

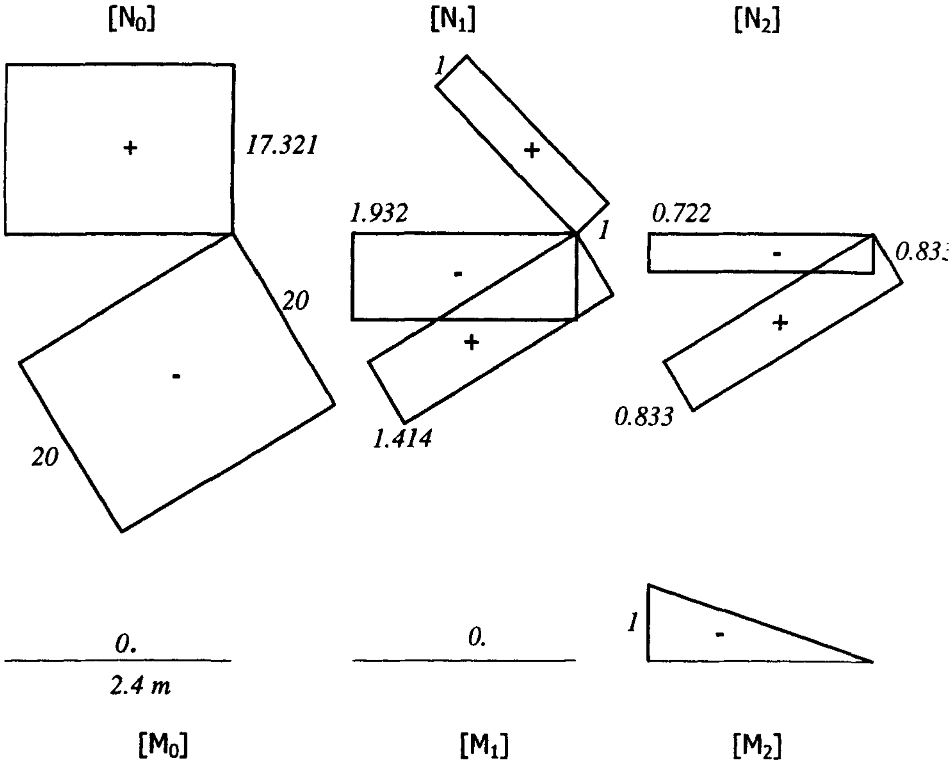


Figure 8.10: Internal forces on the primary structure

In this example redundant force X_2 represents the clamping moment at B, other forces are determined by the method of superposition:

$$N_1 = -20 + 1.414 \cdot X_1 + 0.833 \cdot X_2 =$$

$$-20 + 12.162 + 4.011 = -3.827\ kN$$

$$N_2 = 17.321 - 1.932 \cdot X_1 - 0.722 \cdot X_2 =$$

$$17.321 - 16.617 - 3.476 = -2.773\ kN$$

$$N_3 = X_1 = 8.601\ kN$$

The reactions are:

$$X_A = -N_1 \cdot cos30^0 = 3.827 \cdot 0.866 = 3.314\ kN$$

$$Y_A = -N_1 \cdot sin30^0 = 3.827 \cdot 0.5 = 1.914\ kN$$

$$X_B = -N_2 = 2.773\ kN$$

$$X_C = -N_3 \cdot cos45^0 = -6.082\ kN$$

$$Y_C = N_3 \cdot sin45^0 = 6.082\ kN$$

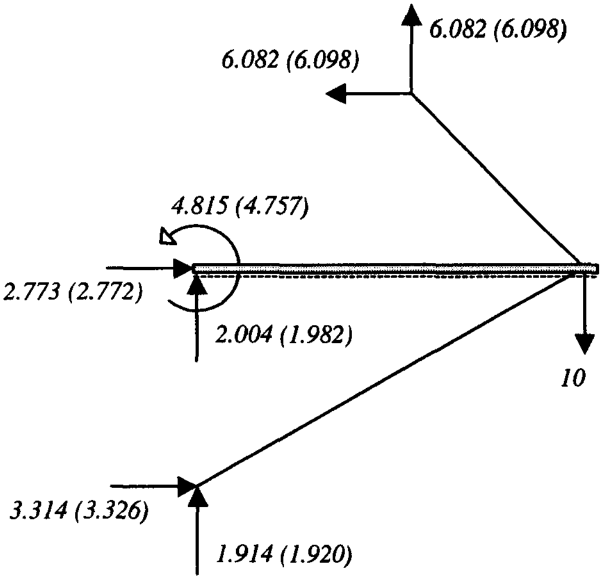


Figure 8.11: Free body at equilibrium
(In brackets are values if shear forces are not neglected)

The reaction Y_B is calculated from the sum of forces in the y-direction

$$Y_B = -Y_A - Y_C + 10 = 2.004\ kN$$

and the reaction is the shear force on the beam *BD*. It is always good practice to check the results by an independent equation, e.g. check if the sum of moments about joint *B* equals the clamping moment M_B.

$$6.082 \cdot 1.2 \cdot 2 - 10 \cdot 2.4 + 3.314 \cdot 1.386 \ ?= 4.815$$

$$4.810 \ ?= 4.815$$

As seen the error is very small which can result from rounding errors using calculators.

Example 8.6: Calculation of a simple bridge structure (see Chapter 12)

Example 8.7: Calculation of a *statically indeterminate truss*

The truss in Fig. 8.12 is two-times statically indeterminate as from the equation 3.10b

$$f = 2j - m - p = 12 - 10 - 4 = -2$$

Redundant forces are one of the reactions and one of the elements; hence the truss is externally and internally indeterminate.

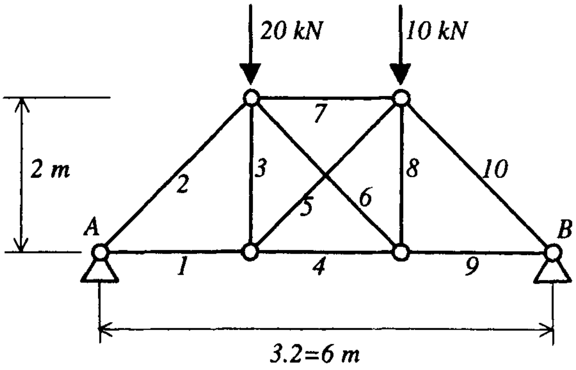


Figure 8.12: Statically indeterminate truss

A statically determinate structure is obtained if support B becomes a roller support (reaction X_B is removed) and element 6 is cut (axial force N_6 is removed).

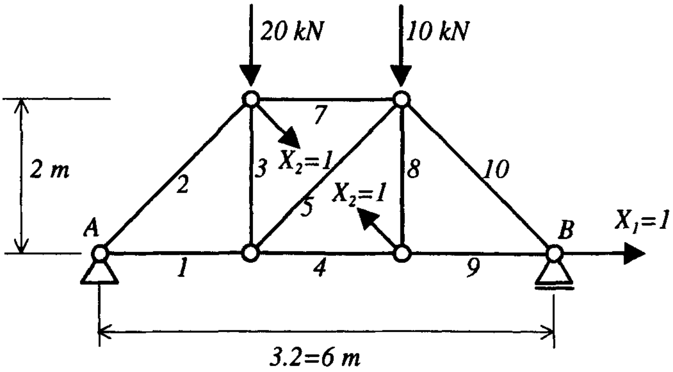


Figure 8.13: Primary structure

Now all axial forces on the primary structure due to external loads and due to unknown forces $X_1 = 1$ and $X_2 = 1$ are calculated and shown in the *table 8.2*.

Table 8.2

Element	L(m)	N_0	N_1	N_2
1	2.000	16.667	1.000	0
2	2.828	-23.570	0.000	0
3	2.000	-3.333	0.000	-0.707
4	2.000	13.333	1.000	-0.707
5	2.828	4.714	0.000	1.000
6	2.828	-	-	1.000
7	2.000	-16.667	0.000	-0.707
8	2.000	0.000	0.000	-0.707
9	2.000	13.333	1.000	0
10	2.828	-18.856	0.000	0

The coefficients of the equation of elasticity are calculated using equation

$$a_{ii} = \Sigma \frac{N\overline{N}}{(EA)_i} L_i$$

and are written in a tabular form in table 8.3, by taking the axial stiffness *EA* the same for all elements:

Table 8.3

Element	a_{11}	a_{22}	a_{12}	a_{10}	a_{20}
1	*2.000*	*0*	*0*	*33.334*	*0*
2	*0*	*0*	*0*	*0*	*0*
3	*0*	*1.000*	*0*	*0*	*4.714*
4	*2.000*	*1.000*	*-1.414*	*26.667*	*-18.852*
5	*0*	*2.828*	*0*	*0*	*13.332*
6	-	*2.828*	-	-	-
7	*0*	*1.000*	*0*	*0*	*23.567*
8	*0*	*1.000*	*0*	*0*	*0*
9	*2.000*	*0*	*0*	*26.667*	*0*
10	*0*	*0*	*0*	*0*	*0*
Sum	*6.000*	*9.657*	*-1.414*	*86.667*	*22.760*

The equations are given in matrix form:

$$\begin{bmatrix} 6.000 & -1.414 \\ -1.414 & 9.657 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \begin{Bmatrix} -86.667 \\ -22.760 \end{Bmatrix}$$

and the solution is: $X_1 = -15.536\ kN$ and $X_2 = -4.632\ kN$.

Final forces in a truss are calculated by a method of a superposition from equation

$$N_i = N_0 + X_1 \cdot N_1 + X_2 \cdot N_2$$

and it is very convenient to evaluate the above equation in tabular form:

Table 8.4

Element	N_0	$N_1 \cdot X_1$	$N_2 \cdot X_2$	N_K	N_K^*
1	16.667	-15.536	0	1.131	16.667
2	-23.570	0.000	0	-23.570	-23.570
3	-3.333	0.000	3.275	-0.058	-1.667
4	13.333	-15.536	3.275	1.072	15.000
5	4.715	0.000	-4.632	0.083	2.357
6	-	-	-4.632	-4.632	-2.357
7	-16.667	0.000	3.275	-13.392	-15.000
8	0.000	0.000	3.275	3.275	1.667
9	13.333	-15.536	0	-2.203	13.333
10	-18.856	0.000	0	-18.856	-18.856

Support B moves horizontally for some reason

Final axial forces N_K are shown in Fig. 8.14.

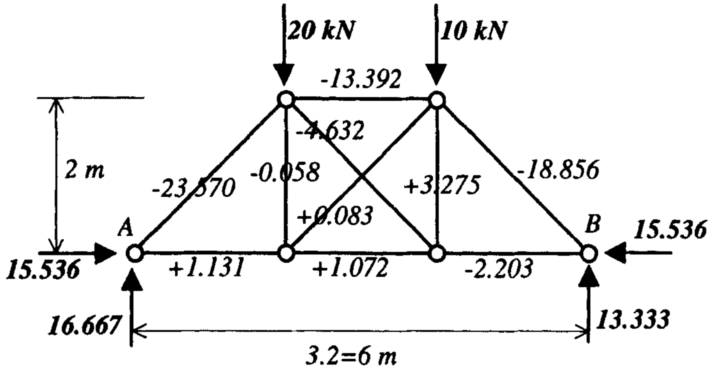


Figure 8.14: Free body and axial forces in the truss

Note: It has to be emphasised that *the horizontal reactions* $X_A = -X_B = 15.536$ *kN*, which occur at externally indeterminate trusses *can be very large*. If these forces for any reason can not be transmitted to an unmoveable terrain by suitable under-structures, then the supports will move and therefore $X_1 = 0$. Axial forces in the truss are for this reason of different values as the elasticity equation is:

$$9.657 \cdot X_2 = -22.760 \;\Rightarrow\; X_2 = -2.357 \text{ kN } (X_1 = 0!)$$

The results for this case are shown in right hand column of *Table 8.4*.

8.5 Special loads

In many cases, secondary effects such as support displacements, temperature changes and pre-strain can be neglected. However, in most cases the engineer must at least evaluate their possible magnitudes, instead of allowing for an arbitrary increase in stress, as is often done.
The three major secondary effects, beside loads, that may act on a structure are *support displacements, temperature changes and pre-strain.*

8.5.1 Support displacements

In practice structures have displacements under the foundations and foundation rotations are usually assumed to be zero. Possible displacements at supports can be u and v, but a support may rotate by an angle φ as well:

$$u = -\frac{R}{c_X} \qquad v = -\frac{R}{c_Y} \qquad \varphi = -\frac{M}{C} \tag{8.10}$$

A displacement can be dependant or independent on a reaction force. In both cases in the elasticity equation

$$\delta_{k_0} + \sum_{i=1}^{n} X_i \cdot \delta_{k_i} = 0$$

On the right hand side of the equation an actual none-zero displacement at support k should be considered.

Example 8.8: Suppose that the support A of the continuous beam from Fig. 8.15 is displaced by $v = 2\ cm$ (*The displacement is independent of the reaction force*).

There is no external loading, hence $\delta_{k0} = 0$.

$$EI \cdot a_{11} = \frac{1}{3} \cdot 5 \cdot 5 \cdot 10 = \frac{250}{3}$$

$$a_{11} \cdot X_1 + \delta_{k_0} = -\delta_k$$

$$X_1 = -\frac{\delta_k}{a_{11}} = -\frac{3 \cdot v \cdot EI}{250}$$

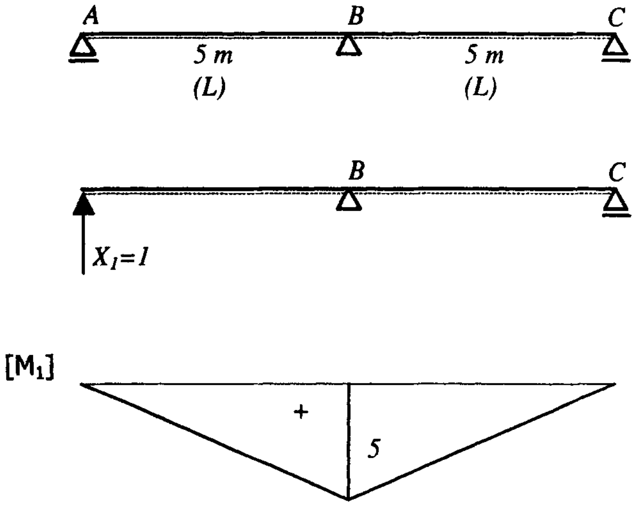


Figure 8.15: Continuous beam
(The displacement is *independent* of the reaction force)

Numerical example: The continuous beam from Fig. 8.15 is made of concrete of quality *C30/36* and has dimensions *B/H = 0.2/0.4* m. An external reason causes the displacement of *2 cm* in the *y*-direction at the support *A*.

$$E = 32\ GPa \qquad I = \frac{BH^3}{12} = 1.067 \cdot 10^{-3}\ m^4 \qquad EI = 34133\ kNm^2$$

$$X_1 = -\frac{\delta_k}{a_{11}} = -\frac{3 \cdot v \cdot EI}{250} = -\frac{3 \cdot 0.02 \cdot 34133}{250} = -8.192\ kN$$

$$M_B = 5 \cdot X_1 = -40.960\ kNm$$

The bending moment at support *B* is negative as the upper fibers of the beam are in tension. Let us calculate the stress at support *B*:

$$\sigma = \pm\frac{M}{W} = \pm\frac{40.960 \cdot 10^{-3}}{\frac{0.2 \cdot 0.4^2}{6}} = \pm 7.68\ MPa$$

The calculated stress is much higher than the concrete in tension can resist (approximately *2.5 MPa*), therefore the beam has to be reinforced on the upper side over a much longer region than would be necessary for static indeterminacy.

Example 8.9: Suppose that support *A* of the continuous beam from Fig. 8.16 is displaced as a result of external loading (*The displacement is dependent on the reaction force*).

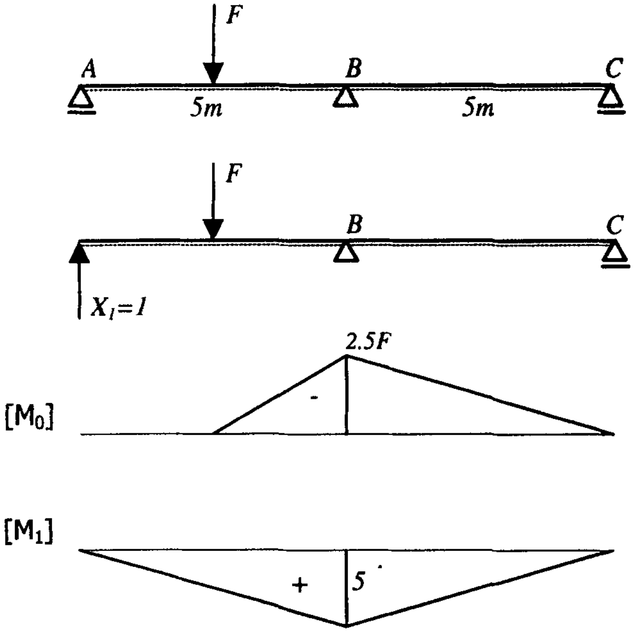


Figure 8.16: Continuous beam
(The displacement is *dependent* on the reaction force)

As an illustration let us first solve the structure when the support does not move at all:

$$a_{11} = \frac{1}{3} \cdot 5 \cdot 5 \cdot 10 = \frac{250}{3}$$

$$a_{10} = \frac{1}{3} \cdot (-2.5 \cdot F) \cdot 5 \cdot 5 + \frac{1}{6} \cdot (-2.5 \cdot F) \cdot (2.5 + 10) \cdot 2.5 =$$
$$= -20.883 \cdot F - 13.021 \cdot F = -33.854 \cdot F$$

$$X_1 = -\frac{-33.854 \cdot F}{\frac{250}{3}} = 0.4062 \cdot F$$

$$M_B = -2.5 \cdot F + 5 \cdot X_1 = -2.5 \cdot F + 5 \cdot 0.4062 \cdot F = -0.469 \cdot F$$

Now, if the displacement is *dependent on the reaction force* the equation is:

$$a_{11} \cdot X_1 + a_{10} = -\frac{X_1}{c_A} \cdot EI = -\frac{EI}{c_A} \cdot X_1$$

$$\left(a_{11} + \frac{EI}{c_A} \right) \cdot X_1 = -a_{10}$$

$$X_1 = -\frac{a_{10}}{a_{11} + \frac{EI}{c_A}} = \frac{33.854 \cdot F}{\frac{250}{3} + \frac{EI}{c_A}}$$

$$M_B = -2.5 \cdot F + 5 \cdot \frac{33.854 \cdot F}{\frac{250}{3} + \frac{34133}{c_A}}$$

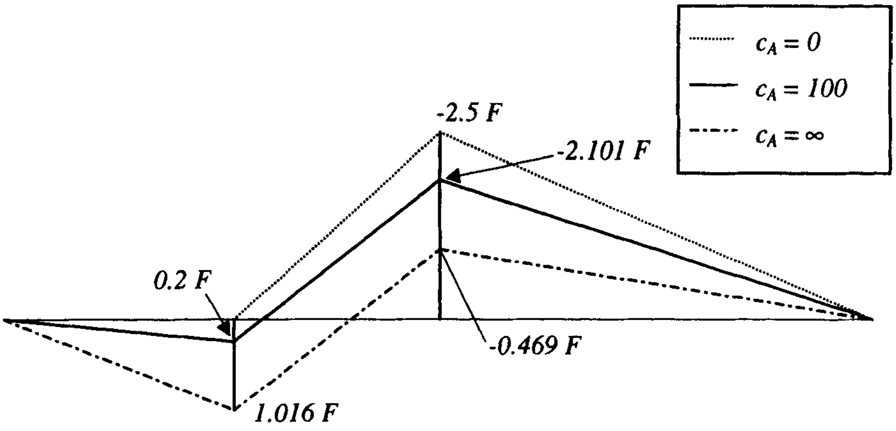


Figure 8.17: Diagram of bending moments (c_A=0- ∞)

At first we choose an infinite spring c_A (support is unmovable), therefore the bending moment at B is:

$$M_B = -2.5 \cdot F + 5 \cdot \frac{33.854 \cdot F}{\frac{250}{3} + \frac{34133}{\infty}} = -2.5 \cdot F + 5 \cdot \frac{33.854 \cdot F}{\frac{250}{3} + 0} = -0.469 \cdot F$$

then $c_A = 100 \ \frac{kN}{m}$

$$M_B = -2.5 \cdot F + 5 \cdot \frac{33.854 \cdot F}{\frac{250}{3} + \frac{34133}{100}} = -2.101 \cdot F$$

and finally $c_A = 0$, which actually means that support at A was removed.

$$M_B = -2.5 \cdot F + 5 \cdot \frac{33.854 \cdot F}{\frac{250}{3} + \frac{34133}{0}} = -2.5 \cdot F + 5 \cdot \frac{33.854 \cdot F}{\frac{250}{3} + \infty} = -2.5 \cdot F$$

The last example is a cantilever case of beam AB.

8.5.2 Temperature loading

If a structure is subjected to a temperature change the deformation can be calculated by the equation (see Ch 5.12):

$$\delta_{k_0}^{(T)} = \int N_K \cdot \alpha_t \cdot T \cdot ds + \int M_K \cdot \alpha_t \cdot \frac{\Delta T}{H} \cdot ds \tag{8.11}$$

where:

$T = \frac{T_{up} - T_{lo}}{2}$	*Average rise in temperature*	(8.12)
$\Delta T = T_{lo} - T_{up}$	*Temperature gradient*	(8.13)
H	*Depth of a beam*	
α_T	*Thermal coefficient of expansion* $\alpha_T = 10^{-5}$ */degree* (concrete, steel)	

The first part of Eqn. (8.11) arises from Hooke's law, the second part is derived as follows. A stress from Hooke's law is calculated

$$\sigma = E\varepsilon = E \cdot \alpha_T \cdot \frac{\Delta T}{2} \tag{8.14}$$

and is inserted into the equation for stresses at bending (see Ch. 2.8.3) and a bending moment is calculated from;

$$\sigma = \frac{M}{I} \cdot \frac{H}{2} = E \cdot \alpha_T \cdot \frac{\Delta T}{2}$$

$$M = EI \cdot \alpha_T \cdot \frac{\Delta T}{H} \tag{8.15}$$

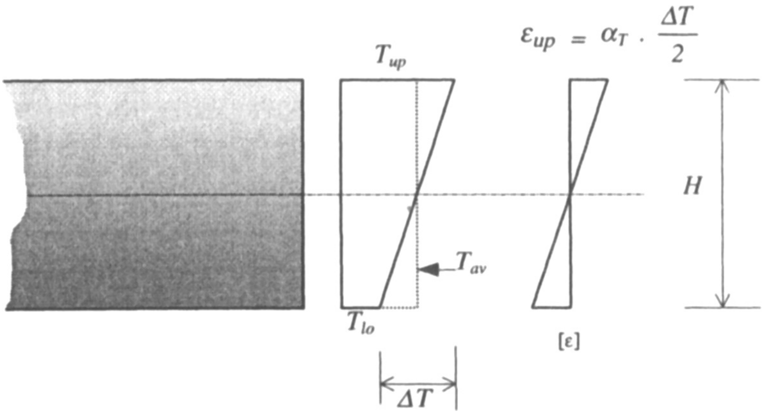


Figure 8.18: Thermal loading of a prismatic element

The term $\delta_{k0}^{(T)}$ is a part of *external loading* in the equation:

$$\delta_{k0} + \sum_{i=1}^{n} X_i \cdot \delta_{ki} = 0 \qquad i, k = 1 \dots n$$

Example 8.10: Calculate bending moments due to *unequal heating* of the continuous plate from Fig. 8.19.

The bending moments caused by temperature gradient ΔT on the primary structure are given by Eqn. (8.15) and are constant across the whole length:

$$M_0 = \alpha_t \cdot \frac{\Delta T}{H} \cdot EI$$

The coefficients are:

$$a_{11} = \frac{1}{3} \cdot \frac{L}{2} \cdot \frac{L}{2} \cdot 2 \cdot L = \frac{L^3}{6} \qquad a_{10} = \frac{1}{2} \cdot \frac{L}{2} \cdot M_0 \cdot 2 \cdot L = \frac{M_0 \cdot L^2}{2}$$

The coefficients are inserted into the elasticity equation and X_1 is calculated:

$$a_{11} \cdot X_1 + a_{10} = 0$$

$$X_1 = -\frac{a_{10}}{a_{11}} = -\frac{M_0 \cdot L^2}{2} \cdot \frac{6}{L^3} = -3 \cdot \frac{M_0}{L}$$

$$X_1 = -\frac{3}{L} \cdot \alpha_t \cdot \frac{\Delta T}{H} \cdot EI$$

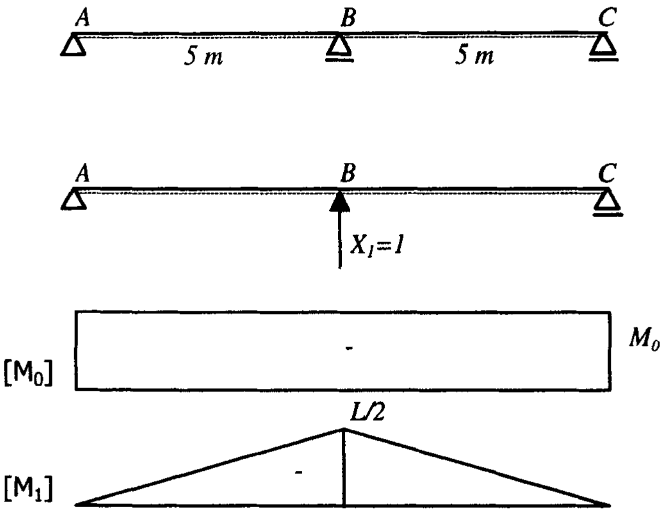


Figure 8.19: Continuous plate

Bending moments are calculated using the method of a superposition:

$$M_A = M_0 = -\alpha_t \cdot \frac{\Delta T}{H} \cdot EI$$

$$M_B = -\alpha_t \cdot \frac{\Delta T}{H} \cdot EI + \frac{L}{2} \cdot \frac{3}{L} \cdot \alpha_t \cdot \frac{\Delta T}{H} \cdot EI = \frac{1}{2} \cdot \alpha_t \cdot \frac{\Delta T}{H} \cdot EI$$

Consider now a plate thickness of 0.18 *m*, the width is a band of *1.00 m*, positioned at the uppermost floor of a building and heated by $\Delta T = 30^\circ C$.

$$EI = 32 \cdot 10^6 \cdot \frac{1.0 \cdot 0.18^3}{12} = 15552 \ kNm^2$$

$$M_A = \alpha_t \cdot \frac{\Delta T}{H} \cdot EI = 10^{-5} \cdot \frac{32}{0.18} \cdot 15552 = 27.648 \ kNm$$

$$M_B = 13.824 \, kNm$$

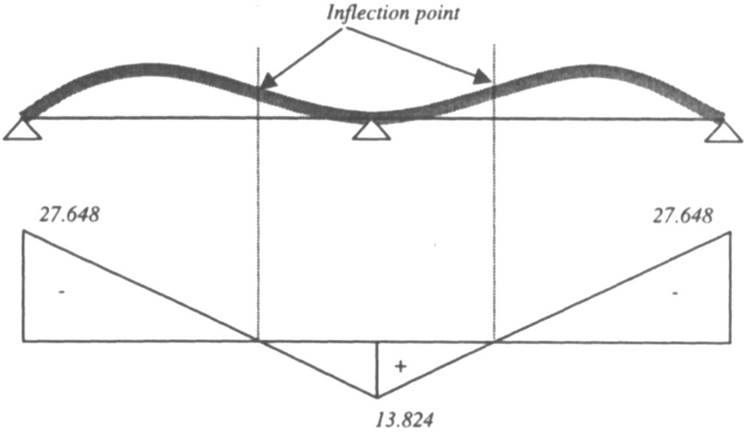


Figure 8.20: Deflection of a plate and bending moments

The stresses at joints *A* and *C* are

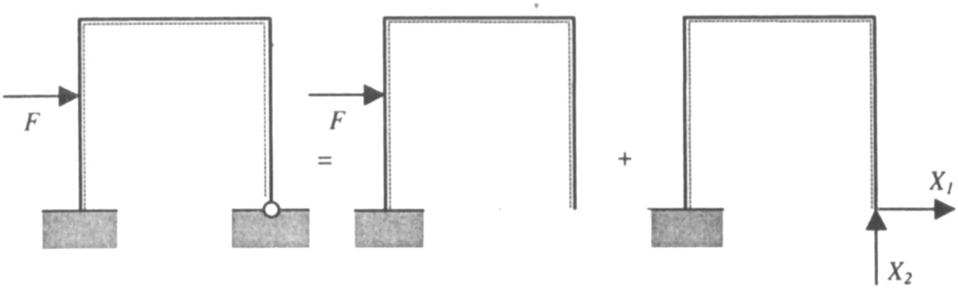
$$\sigma = \pm \frac{M}{W} = \pm \frac{M}{\frac{1 \cdot 0.18^2}{6}} = \pm 5.12 \ MPa ,$$

which is well over the allowable stresses for concrete in tension.

8.6 Deformation of statically indeterminate structures

The procedure is outlined in the following two steps:

1. *Solve an indeterminate system*



$$a_{10} + a_{11} \cdot X_1 + a_{12} \cdot X_2 = 0$$

$$a_{20} + a_{21} \cdot X_1 + a_{22} \cdot X_2 = 0$$

The solution of the above equations gives unknowns X_1 and X_2. Bending moments are determined by the method of superposition:

$$M_j = M_{j0} + \sum_{i=1}^{n} M_i \cdot X_i$$

2. *Calculate the deformation* at an arbitrary point k by the equation:

$$EI \cdot \delta_k = \int M_j \cdot \overline{M}_k \cdot ds\,,$$

where bending moments due to $F_k = 1$ are:

$$\overline{M}_k = \overline{M}_{k_0} + \sum_{i=1}^{n} \overline{M}_i \overline{X}_i$$

It is obvious that an indeterminate system has to be solved twice; first for external loads and then for load $F = 1$, applied at the point where the deflection is desired.

Example 8.11: Determine the deflection at point *C* for the beam in Fig. 8.21.

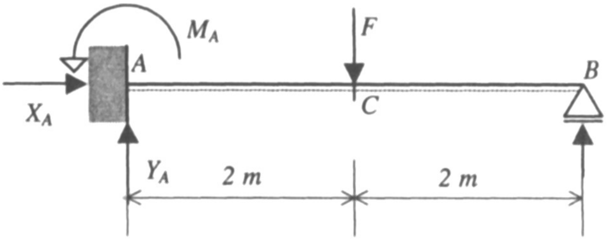


Figure 8.21: Indeterminate beam

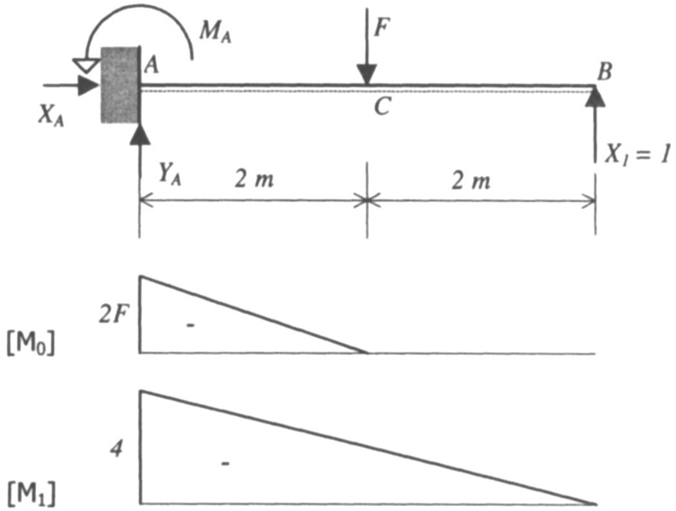


Figure 8.22: Primary structure and moments M_0 and M_1

$$a_{10} = -\frac{2 \cdot F \cdot 2}{2} \cdot \frac{5}{6} \cdot 4 = -\frac{40 \cdot F}{6}$$

$$a_{11} = \frac{1}{3} \cdot 4 \cdot 4 \cdot 4 = \frac{64}{3}$$

$$X_1 = -\frac{a_{10}}{a_{11}} = \frac{10 \cdot F}{32}$$

$$M_A = -2 \cdot F + 4 \cdot X_1 = -\frac{24 \cdot F}{32}$$

$$M_C = \frac{20 \cdot F}{32}$$

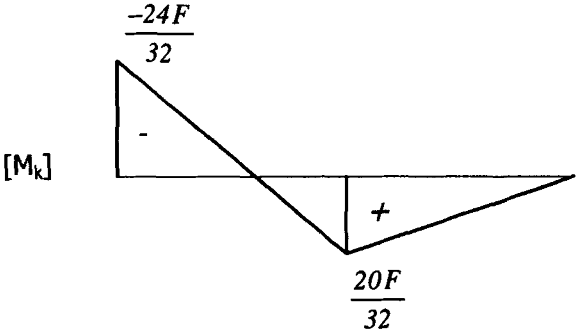


Figure 8.23: Final diagram of bending moments M_K due to force F

Now the structure is solved for virtual force $\overline{F} = 1$ at point C. The final bending moment diagram is shown on Fig. 8.24.

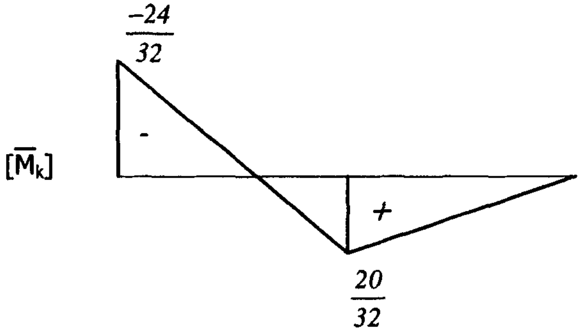


Figure 8.24: Final diagram of bending moments M_K due to force $F = 1$

Deformations are calculated using table *B.4*:

$$EI \cdot \delta_C = \int M \cdot \overline{M} \cdot ds = -\frac{1}{3} \cdot \frac{40}{32} \cdot \frac{40 \cdot F}{32} \cdot 4 + \frac{1}{3} \cdot \frac{64}{32} \cdot \frac{64 \cdot F}{32} \cdot 2 =$$

$$= -2.083 \cdot F + 2.667 \cdot F = 0.583 \cdot F$$

$$\delta_C = \frac{0.583 \cdot F}{EI}$$

Let us now see the result if the *final diagram of bending moments* is integrated with the diagram of *bending moments on the primary structure caused by unit force*, applied at the point and in the direction of the desired deflection.

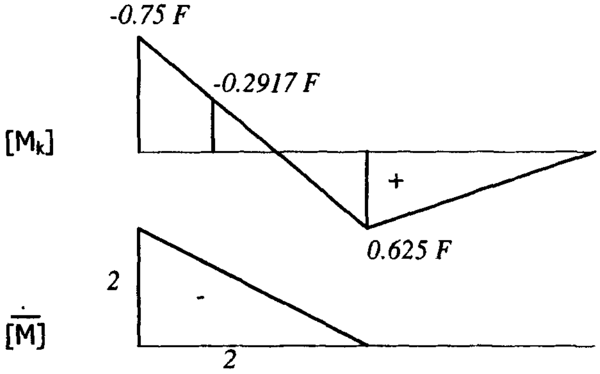


Figure 8.25: Final diagram of bending moments
and diagram of bending moments on the primary structure

Integration is performed by the Vereshagin's method

$$EI \cdot \Delta = \frac{2 \cdot 2}{2} \cdot 0.2917 \cdot F = 0.583 \cdot F\,,$$

and *gives exactly the same results as above.* This simplified procedure is called *the reduction statement* and is proved in Section 8.7 below.

8.7 The reduction statement

Using the principle of virtual displacements a deformation is given by:

$$\delta_m = \int \frac{M \cdot \overline{M}}{EI} \cdot ds$$

and is valid for statically determinate and indeterminate structures. Bending moments of an indeterminate system are calculated from an elasticity equation

$$\delta_{K_0} + \Sigma \delta_{K_i} \cdot X_i = 0$$

and by equation

$$M = M_0 + \Sigma X_i \cdot M_i$$

In a similar manner virtual bending moments are determined by:

$$\bar{\delta}_{j0} + \Sigma \bar{\delta}_{jk} \cdot X_k = 0$$

$$\overline{M} = \overline{M}_0 + \Sigma \overline{X}_k \cdot \overline{M}_k$$

If at first virtual bending moments are inserted into the former equation

$$\delta_m = \int \left(\overline{M}_0 + \Sigma \overline{X}_k \cdot \overline{M}_k\right) \cdot M \cdot \frac{ds}{EI}$$

$$\delta_m = \int \overline{M}_0 \cdot M \cdot \frac{ds}{EI} + \Sigma \overline{X}_k \int M \cdot \overline{M}_k \cdot \frac{ds}{EI}$$

and then from real moments caused by external loads we obtain:

$$\delta_m = \int \overline{M}_0 \cdot M \cdot \frac{ds}{EI} + \Sigma \overline{X}_k \cdot \int \left(M_0 + \Sigma X_i \cdot M_i\right) \cdot \overline{M}_k \cdot \frac{ds}{EI} =$$

$$= \int \overline{M}_0 \cdot M \cdot \frac{ds}{EI} + \Sigma \overline{X}_k \cdot \left\{ \int M_0 \cdot \overline{M}_k \cdot \frac{ds}{EI} + \Sigma X_i \cdot \int M_i \cdot \overline{M}_k \cdot \frac{ds}{EI} \right\}$$

hence:

$$\delta_m = \int \frac{\overline{M}_0 \cdot M \cdot ds}{EI} + \Sigma \overline{X}_k \cdot \left(\delta_{k_0} + \Sigma X_i \delta_{k_i}\right)$$

As the right hand part of the above equation is the elasticity equation, which equals zero, it follows that:

$$\delta_m = \int \frac{M \cdot \overline{M}_0 \cdot ds}{EI} \qquad (8.16)$$

❖ *The reduction statement states that deformations of an indeterminate system can be determined if bending moments of an indeterminate system are known and if virtual bending moments of the primary system are known.*

8.8 Application of the reduction statement

The method of forces is a useful tool for the analysis of simple structures and gives a student a good understanding of structural behaviour through deformation calculations. However, there *is no independent control* of the calculated unknowns, since the equilibrium of the whole structure is established using unknowns as external loads.

The reduction statement offers a simple and effective control of the calculated results. *A new primary structure is determined which has to be different from the one from which unknowns were determined.* Using the equation

$$\delta_m = \int M_0 \cdot \overline{M} \cdot \frac{ds}{EI}$$

some arbitrary but *known* (*usually zero*) *deformation* is calculated fulfilling the condition:

$$\delta_m = \int M_0 \cdot \overline{M} \cdot \frac{ds}{EI} = 0$$

Example 8.12: Solve the rigid frame from Fig. 8.26 and prove the results by the reduction statement.

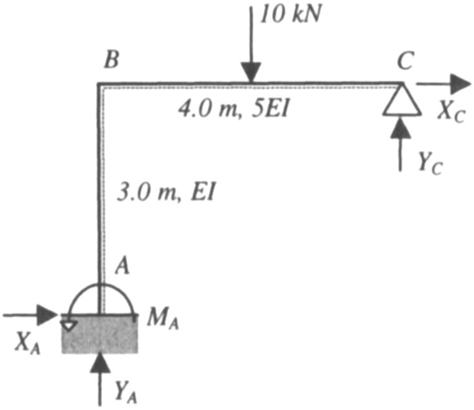


Figure 8.26: Rigid frame
(Stiffness of the beam is *5x* stiffness of the column)

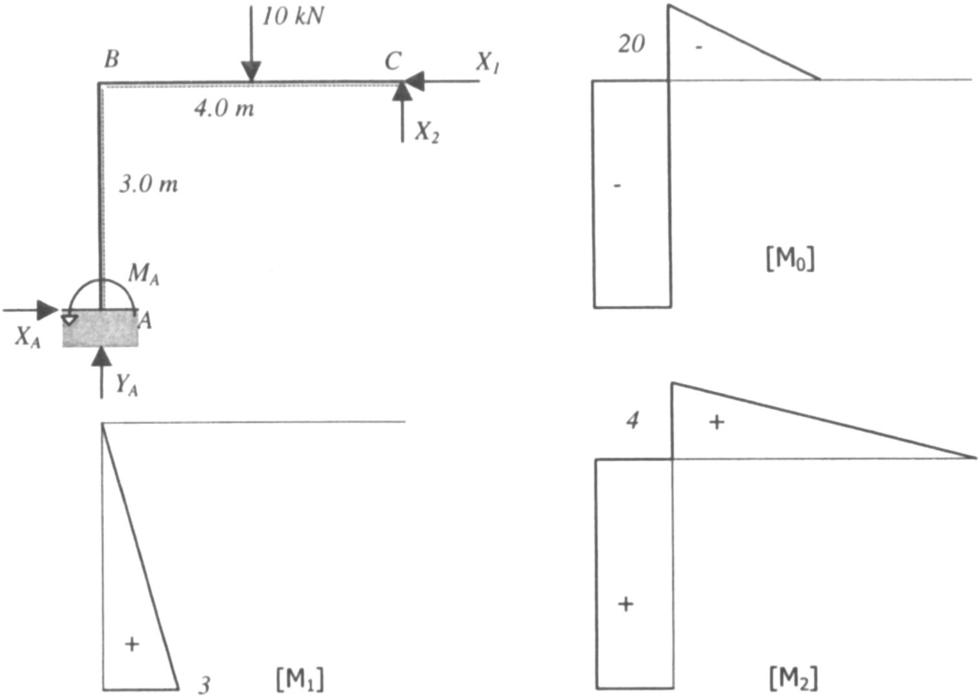


Figure 8.27: Primary structure and bending moments

$$a_{11} = \frac{3^3}{3} = 9 \qquad a_{22} = \frac{1}{5}\frac{4^3}{3} + 4^2 \cdot 3 = \frac{64}{3} + 48 = 52.267$$

$$a_{12} = \frac{1}{2} \cdot 3 \cdot 4 \cdot 3 = 18$$

$$a_{10} = \frac{1}{2} \cdot (-20) \cdot 3 \cdot 3 = -90$$

$$a_{20} = \frac{1}{5}\frac{(-20) \cdot 2}{2} \cdot 3.333 - 20 \cdot 3 \cdot 4 = -253.333$$

$$\begin{bmatrix} 9 & 18 \\ 18 & 52.267 \end{bmatrix} \cdot \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \begin{Bmatrix} 90 \\ 253.333 \end{Bmatrix}$$

$$\begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \begin{Bmatrix} 0.984 \\ 4.508 \end{Bmatrix}$$

$$M_A = -20 + 3 \cdot 0.984 + 4 \cdot 4.508 = 0.984 \ kNm$$

$$M_B = -20 + 4 \cdot 4.508 = -1.968 \ kNm$$

$$M_C = 2 \cdot 4.508 = 9.016 \ kNm$$

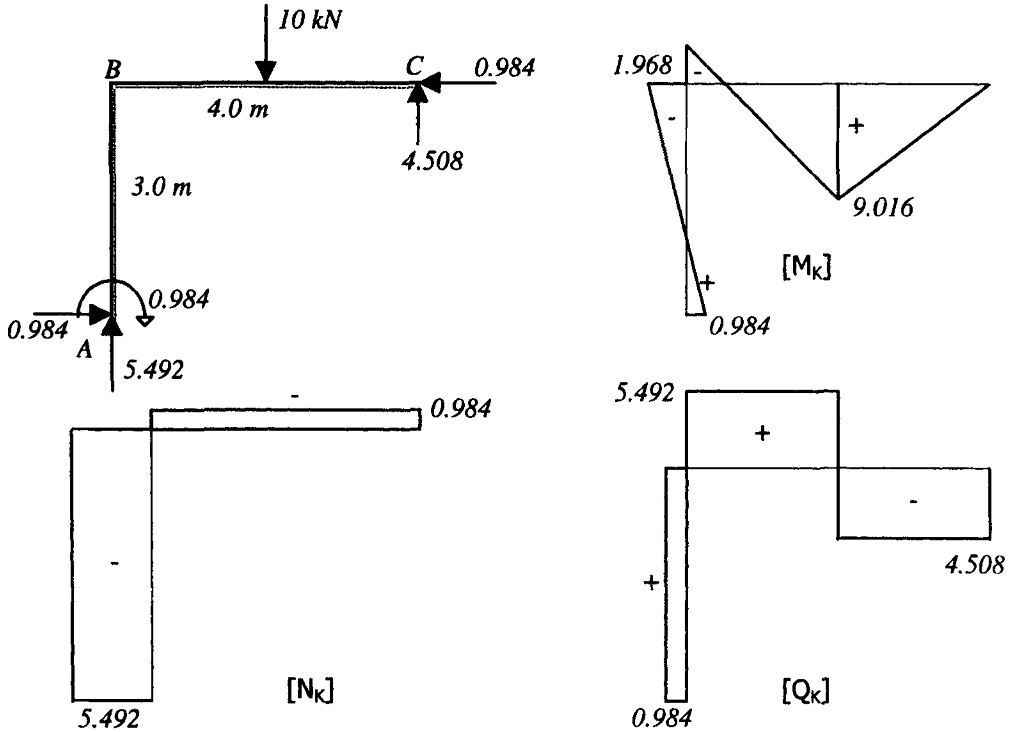
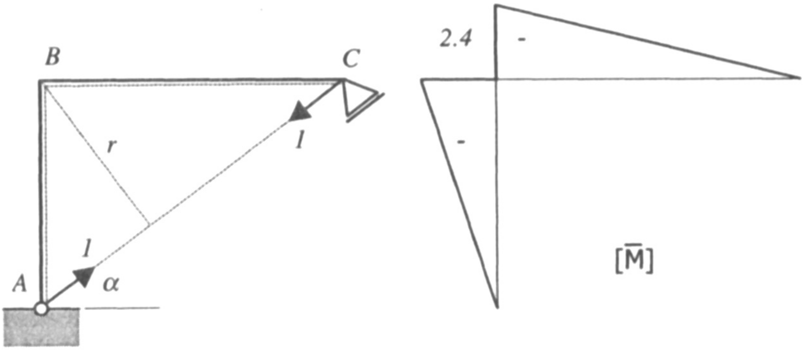


Figure 8.28: Free body diagrams and diagrams of internal forces

A check of the results using the reduction statement is done in such a way, that a new primary structure, which has to be different from the original primary structure, is chosen and then a known deformation is calculated (see Fig. 8.29).



$$tg\alpha = \frac{3}{4} \quad \Rightarrow \quad \alpha = 36.87^\circ$$

$$\cos\alpha = \frac{r}{3} \quad \Rightarrow \quad r = 3 \cdot \cos\alpha = 2.4\ m$$

Figure 8.29: Primary structure used by the reduction statement

Calculate now the displacement between joints *A* and *C* by integration of final moments from Fig. 8.28 with virtual moments from Fig. 8.29:

$$EI \cdot \delta = \frac{(-2.4)}{2} 3 \cdot (-0.984) + \frac{1}{5} \cdot \frac{1}{3} \cdot (-2.4) \cdot (-1.968) \cdot 4 +$$

$$+ \frac{1}{5} \cdot \frac{1}{6} (-2.4) \cdot 10 \cdot (1 + 0.5) \cdot 4 =$$

$$= 3.542 + 1.260 - 4.800 = 0.02 \cong 0$$

As expected the result is zero and a check by the reduction statement proved that results calculated by the force method were correct.

9

The Displacement Method

9.1 Introduction

Statically indeterminate structures are solved by the displacement method as if unknown displacements and rotations were chosen. From a system of equilibrium equations we calculate deformations from which internal forces and reactions are calculated.

The displacement method is superior to the force method when the number of unknown forces exceeds the number of unknown displacements and rotations.

The concept of stiffness matrix, introduced in Ch. 7, will be used for a structural stiffness matrix assembly and it is a foundation for the displacement and later the finite element method.

Two simplified longhand methods will be shown: a *classical deformational method* and a *moment distribution method* (Cross's method). In the classical deformational method a system of equations will be established but contrary to the force method the determination of coefficients of the equations is much simpler as they depend on single individual elements only. The moment distribution method remains the most powerful tool for an engineer without computing equipment and can help in better understanding of simple structures.

At the end of the chapter a new method for influence line determination is presented using ψ functions as a further development of well-known ω numbers.

9.2 Kinematics (deformational) indeterminacy

Displacements and rotations of joints define a deformational state of a structure; joints can be rigid or pinned.

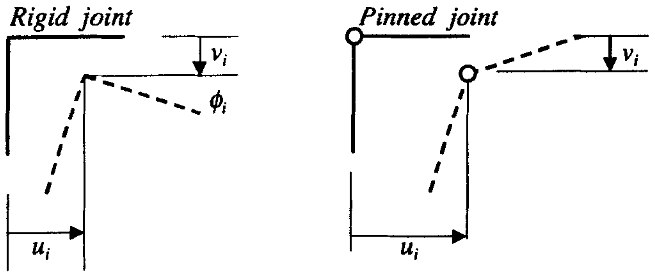


Figure 9.1: Rigid and pinned joints

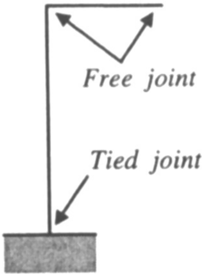


Figure 9.2: Free and tied joints

Two displacements u and v and a rotation φ are possible in a rigid joint, whilst only two displacements u and v are possible in a pinned joint. We shall distinguish between *free* and *tied* joints; tied joints are supports in which p_1 rotations φ_i and p_2 displacements u_i and v_i can occur.

Example 9.1: Determination of joints

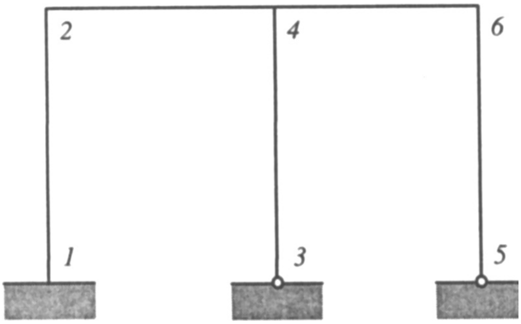


Figure 9.3: Determination of joints

Number of all joints:	*6*
Number of tied joints:	*3 (1, 3, 5)*
Number of free joints:	*3 (2, 4, 6)*

At the supports seven quantities are prescribed, they are:

$$u_1 = v_1 = \varphi_1 = u_3 = v_3 = u_5 = v_5 = 0$$

If on a structure there are k rigid joints and g pinned joints then the total of all *deformational quantities* is:

$$3k + 2g$$

from which p *prescribed* deformational quantities at supports (p_1 rotations and p_2 displacements) are deduced, hence:

$$3k + 2g - p \tag{9.1}$$

The above quantities can be divided into *unknown rotations*

$$k - p_1$$

and into unknown *displacements*

$$2(k+g) - p_2 = 2j - p_2,$$

where p_1 means the number of suppressed rotations (at supports) and p_2 means the number of suppressed displacements (at supports), j is the number of all joints.

Note that the total number of unknowns will not be $3 k + 2 g - p$, as some displacements are dependant on each other.

Joints of a structure are interconnected to each other with elements (beams and columns); therefore the *compatibility equations* (4.9) at joints must be applicable:

$$\Delta = -\frac{u_k - u_j}{y_k - y_j} = \frac{v_k - v_j}{x_k - x_j} \tag{9.2}$$

Equation (9.2) relates the deformation Δ (a rotation of an element) and displacements of joints u and v. The number of relations that exist depends on the number of elements in a structure, hence the number of deformational quantities is:

$$n = 3 \cdot k + 2 \cdot g - p - m \tag{9.3}$$

or if we distinguish between rotations and displacements:

Unknown rotations: $b = k - p_1$ (9.4)

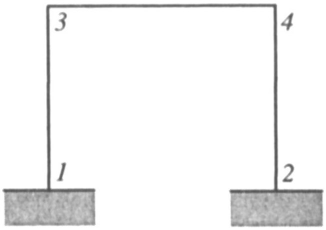
Unknown displacements: $c = 2 \cdot k + 2 \cdot g - p_2 - m$ (9.5)

or $c = 2 \cdot j - p_2 - m$

A *degree of deformational indeterminacy* (DDI) is then given by:

$$n = b + c \tag{9.6}$$

Examples 9.2: Determination of degree of deformational indeterminacy

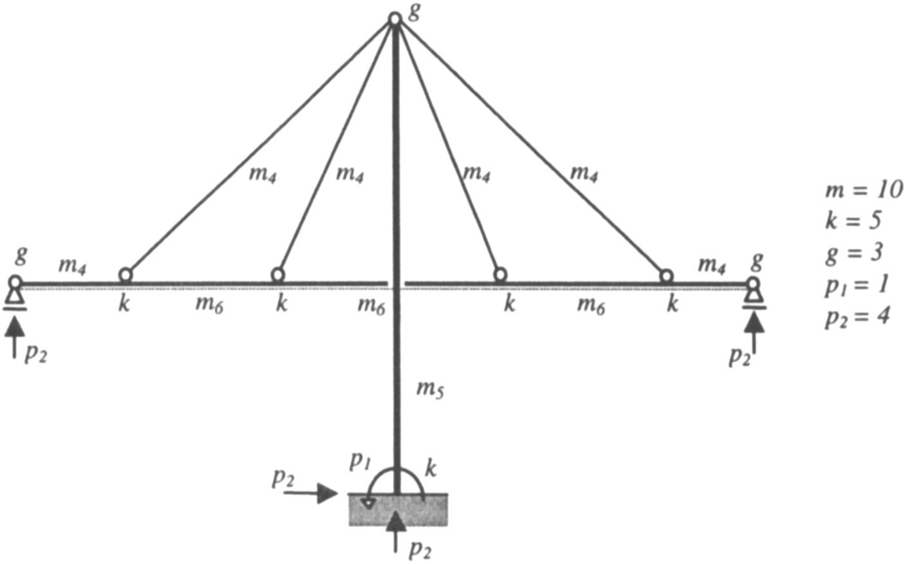


Rotations: $b = 4 - 2 = 2$

Displacements: $c = 2 \cdot 4 + 0 - 4 - 3 = 1$

$n = 2 + 1 = 3$

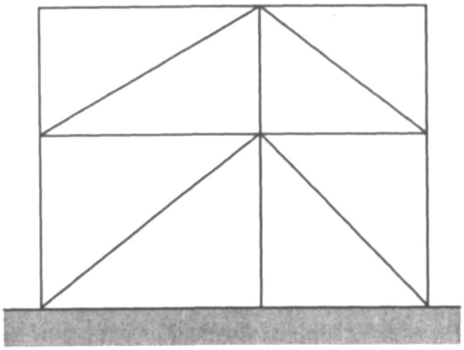
Unknown rotations are φ_3 and φ_4, displacement occurs at the top of column *1-3*, at the same time column *2-4* moves but its displacement is the same as that of column *1-3* (compatibility condition).



Rotations: $b = 5 - 1 = 4$

Displacements: $c = 2 \cdot 5 + 2 \cdot 3 - 4 - 10 = 2$

$n = 4 + 2 = 6$



$m=14$

$k=9$

$g=0$

$p_1=3$

$p_2=6$

Rotations:

$b = 9 - 3 = 6$

Displacements:

$c = 2\cdot 9 + 0 - 6 - 14 = -2$

The above examples confirmed that rotations and displacements influence the *DDI* separately as eventual over-determinacy in displacements Δ $(c \leq 0)$ *has no influence on the number of unknown joint rotations* φ.

The conclusion comes from the fact that the rotations in the structure can occur without any sway movements of the structure.

The above separation of rotations and displacements will be used in the classical longhand deformational method and in Cross's moment distribution method where structures will be classified as *non-sway* ($c <= 0$) and *sway* ($c > 0$) structures.

9.3 Structure stiffness matrix

The structure stiffness matrix $[K]$ relates the forces and displacements of a structure composed of elements. The force X_i at a joint and in the direction i is linearly related through the *corresponding displacements* Δ_i by equation:

$$X_i = k_{i1} \cdot \Delta_1 + k_{i2} \cdot \Delta_2 + ... + k_{ij} \cdot \Delta_j + ...$$

$$\{X\} = [K] \cdot \{\Delta\} \qquad (9.7)$$

Each element k_{ij} of this stiffness matrix is defined as the force that must be applied to the complete structure at node i to produce unit displacement at node j, all others are kept zero. Consider now one single joint connecting several elements *1,2...m*. Figure 9.4 shows the internal element forces $S_i^{(m)}$ of element m along node i, as well as their equal and opposite reaction forces on the joint. For instance, the *i-th* force on the *m-th* element is

$$S_i^{(m)} = k_{i1}^{(m)} \cdot \Delta_1 + k_{i2}^{(m)} \cdot \Delta_2 + ...k_{ij}^{(m)} \cdot \Delta_j + ... \qquad (9.8)$$

where the j extends over all nodes attached to the element. The equilibrium equation for the forces in the direction i at the joint is

$$\Sigma X_i = 0 : \quad -S_{i1} - S_{i2} - S_{im} + F_i = 0 ,$$

from which, inserting Eqn. (9.8), we get

$$\left(k_{i1}^{1}+k_{i1}^{2}+\ldots..+k_{i1}^{m}+\ldots..\right)\cdot\Delta_{1}+\left(k_{i2}^{1}+k_{i2}^{2}+\ldots..+k_{i2}^{m}+\ldots..\right)\cdot\Delta_{2}+\ldots..=F_{i}$$

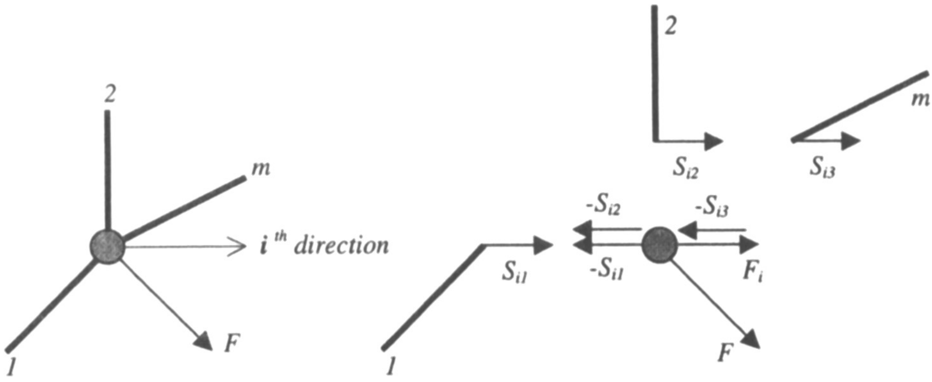


Figure 9.4: Joint and element forces

The structure stiffness coefficient K_{ij} is therefore obtained by superposition of the element stiffness $k_{ij}^{(m)}$:

$$K_{ij}=k_{ij}^{1}+k_{ij}^{2}+\ldots..+k_{ij}^{m}+\ldots=\sum_{m}k_{ij}^{(m)} \tag{9.10}$$

It is essential that each node of the structure be carefully labeled, and that the nodal numbering of each element corresponds to that of the structure. The element stiffness matrices are then written and superimposed, or *assembled.*

Example 9.3: Assemble the stiffness matrix for the structure of the Fig. 9.5

Element data:

$A_1 = A_3 = 3\ cm^2$

$A_2 = 800\ cm^2$

$I_2 = 10.67 \cdot 10^{-4}\ m^4$

$E_1 = E_3 = 200\ GPa$

$E_2 = 25\ GPa$

$E_1 \cdot A_1 = 60 \cdot 10^3\ kN$

$E_2 \cdot A_2 = 2 \cdot 10^6\ kN$

$E_2 \cdot I_2 = 26.675 \cdot 10^3\ kNm^2$

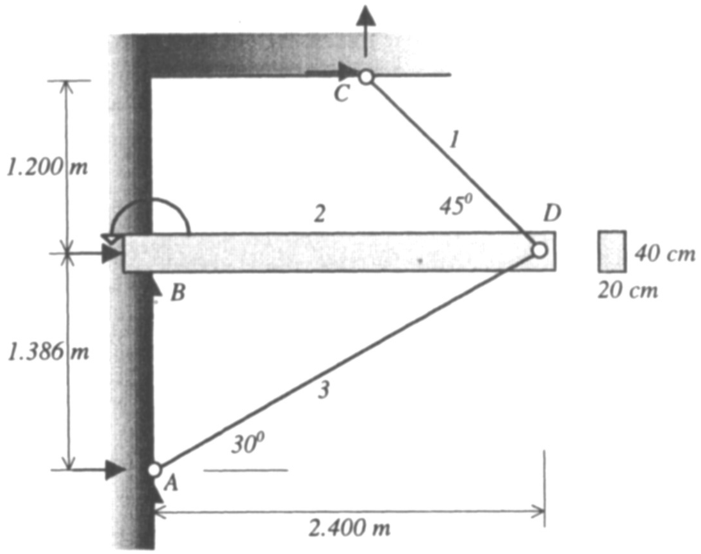


Figure 9.5: Mixed structure

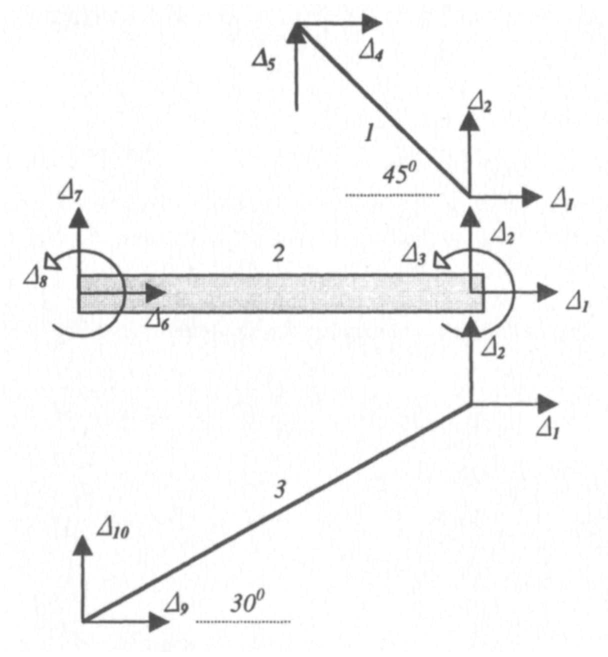


Figure 9.6: Degrees of freedom at elements

First calculate axial and bending stiffness of individual elements:

$$L_1 = 1.697\,m \quad \Rightarrow \quad \frac{E \cdot A}{L} = 35355\ \frac{kN}{m}$$

$$L_2 = 2.40\ m \quad \Rightarrow \quad \frac{E \cdot A}{L} = 833333\ \frac{kN}{m}$$

$$E \cdot I = 26.675\ kNm^2$$

$$L_3 = 2.771\ m \quad \Rightarrow \quad \frac{E \cdot A}{L} = 21653\ \frac{kN}{m}$$

Elements *1* and *3* are *truss elements* (two force elements) therefore equation (7.30) is applied, element 2 is a *beam element* and equation (7.11) is used:

$$\left[k^{(1)}\right] = \begin{array}{r|rr|rr|} & 1 & 2 & 4 & 5 \\ 1 & 17677 & -17677 & -17677 & 17677 \\ 2 & -17677 & 17677 & 17677 & -17677 \\ \hline 4 & -17677 & 17677 & 17677 & -17677 \\ 5 & 17677 & -17677 & -17677 & 17677 \end{array}$$

$$\left[k^{(2)}\right] = \begin{array}{r|rrr|rrr|} & 1 & 2 & 3 & 6 & 7 & 8 \\ 1 & 833333 & 0 & 0 & & & sim. \\ 2 & 0 & 23155 & 27786 & & & \\ 3 & 0 & 27786 & 44458 & & & \\ \hline 6 & -833333 & 0 & 0 & 833333 & & \\ 7 & 0 & -23155 & -27786 & 0 & 23155 & \\ 8 & 0 & 27786 & 22229 & 0 & 27786 & 44458 \end{array}$$

$$
\left[k^{(3)}\right]=
\begin{array}{c|cc|cc}
 & 1 & 2 & 9 & 10 \\
\hline
1 & 16240 & 9376 & -16240 & -9376 \\
2 & 9376 & 5413 & 9376 & -5413 \\
\hline
9 & -16240 & -9376 & 16240 & 9376 \\
10 & -9376 & -5413 & 9376 & 5413
\end{array}
$$

The structure stiffness matrix (*10x10*) is obtained by assembling the above element stiffness into the locations as indicated by numbering of the rows and columns and by the appropriate dotted lines inside the matrix.

$$
[K]=
\begin{array}{c|cc|c|cc|ccc|cc}
 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
\hline
1 & 867250 & & & & & & & & & \\
2 & -8301 & 46245 & & & & & & & & \\
\hline
3 & 0 & -27786 & 44458 & & & & & \text{sym.} & & \\
\hline
4 & -17677 & 17677 & 0 & 17677 & & & & & & \\
5 & 17677 & -17677 & 0 & -17677 & 17677 & & & & & \\
\hline
6 & -833333 & 0 & 0 & 0 & 0 & 833333 & & & & \\
7 & 0 & -23155 & 27786 & 0 & 0 & 0 & 23155 & & & \\
8 & 0 & -27786 & 22229 & 0 & 0 & 0 & -27786 & 44458 & & \\
\hline
9 & -16240 & -9376 & 0 & 0 & 0 & 0 & 0 & 0 & 16240 & \\
10 & -9376 & -5413 & 0 & 0 & 0 & 0 & 0 & 0 & 9376 & 5413
\end{array}
$$

The physical meaning of these numbers should be clearly understood. For instance stiffness $K_{11} = 867250$ is about *20*-times the value of $K_{22} = 46245$, indicating that the horizontal force required to stretch the structure a specified amount horizontally is *20*-times as much as a vertical force at the same point causing the same amount of vertical displacement.

9.4 Matrix formulation

The matrix displacement method is the most powerful of the various methods for structural analysis if used in conjunction with appropriate computers. In its generalisation as the *finite element method* it is capable of analysing any solid body though in this text will be restricted to the analysis of framed structures.

The displacement method relates the forces to the displacements by the structure stiffness matrix $[K]$:

$$\{X\} = [K] \cdot \{\Delta\}$$

At each node (joint) a force and its corresponding displacement exist as conjugate quantities.

For a structure having f degrees of freedom we group together nodes with unknown displacements and nodes with known displacements (usually at supports) in a partitioned form (Eqn. 9.10):

$$\begin{array}{rl} \textit{Known forces} \Rightarrow \\ \textit{Reactions} \Rightarrow \end{array} \begin{Bmatrix} X_\alpha \\ X_\beta \end{Bmatrix} = \begin{bmatrix} K_{\alpha\alpha} & K_{\alpha\beta} \\ K_{\beta\alpha} & K_{\beta\beta} \end{bmatrix} \cdot \begin{Bmatrix} \Delta_\alpha \\ \Delta_\beta \end{Bmatrix} \begin{array}{l} \Rightarrow \quad \textit{Unknown displacements} \\ \Rightarrow \textit{Known (specified) displacements} \end{array}$$

The solution is obtained in two steps:

$$\{X_\alpha\} = [K_{\alpha\alpha}] \cdot \{\Delta_\alpha\} + [K_{\alpha\beta}] \cdot \{\Delta_\beta\}, \tag{9.11}$$

from which

$$\{\Delta_\alpha\} = [K_{\alpha\alpha}]^{-1} \cdot (\{X_\alpha\} - [K_{\alpha\beta}] \cdot \{\Delta_\beta\}) \tag{9.12}$$

Note that the matrix $[K_{\alpha\alpha}]$ is a square matrix of order ($f \times f$) which has to be inverted, corresponding to a solution of f simultaneous equations.

The displacements thus found are then substituted into the remaining equations to solve for unknown forces or reactions:

$$\{X_\beta\} = [K_{\beta\alpha}] \cdot \{\Delta_\alpha\} + [K_{\beta\beta}] \cdot \{\Delta_\beta\} \tag{9.13}$$

If all $\{\Delta_\beta\}$ equal zero (support deformations are zero) the equations simplify into:

$$\{\Delta_\alpha\} = [K_{\alpha\alpha}]^{-1} \cdot \{X_\alpha\} \tag{9.14}$$

$$\{X_\beta\} = [K_{\beta\alpha}] \cdot \{\Delta_\alpha\} \tag{9.15}$$

Let us here load the structure at the free joint by the force $F_y = -10\ kN$.

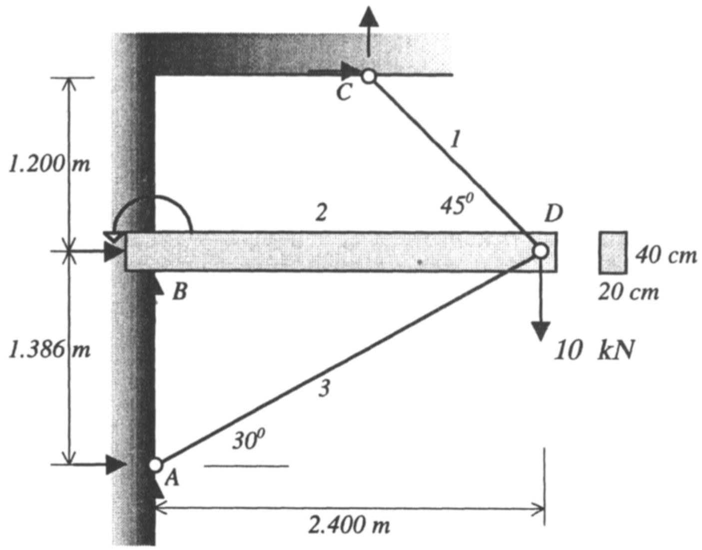


Figure 9.7: Application of the load

$$\begin{Bmatrix} 0 \\ -10 \\ 0 \\ X_4 \\ X_5 \\ X_6 \\ X_7 \\ X_8 \\ X_9 \\ X_{10} \end{Bmatrix} = \left[\begin{array}{ccc|ccccccc} 867250 & -8301 & 0 & & & & & & & \\ -8301 & 46245 & -27786 & & & & & & & \\ 0 & -27786 & 44458 & & & & & & & \\ \hline -17677 & 17677 & 0 & 17677 & & & & & & \\ 17677 & -17677 & 0 & -17677 & 17677 & & & & & \\ -833333 & 0 & 0 & 0 & 0 & 833333 & & & & \\ 0 & -23155 & 27786 & 0 & 0 & 0 & 23155 & & & \\ 0 & -27786 & 22229 & 0 & 0 & 0 & -27786 & 44458 & & \\ -16240 & -9376 & 0 & 0 & 0 & 0 & 0 & 0 & 16240 & \\ -9376 & -5413 & 0 & 0 & 0 & 0 & 0 & 0 & 9376 & 5413 \end{array}\right] \cdot \begin{Bmatrix} \Delta_1 \\ \Delta_2 \\ \Delta_3 \\ \Delta_4 \\ \Delta_5 \\ \Delta_6 \\ \Delta_7 \\ \Delta_8 \\ \Delta_9 \\ \Delta_{10} \end{Bmatrix}$$

Let us emphasise here that the first column contains displacements in the directions *1*, *2* and *7 caused by unit force in the direction 1*, the second column displacements due to unit force in the direction *2* and the third column displacements due to unit moment in the direction *3*.

Equation (9.14) can now be solved to yield unknown displacements

$$\{\Delta_\alpha\} = [K_{\alpha\alpha}]^{-1} \cdot \{X_\alpha\}$$

and after inserting the known forces we get:

$$\begin{Bmatrix} \Delta_1 \\ \Delta_2 \\ \Delta_3 \end{Bmatrix} = \begin{bmatrix} 867250 & -8301 & 0 \\ -8301 & 46245 & -27786 \\ 0 & -27786 & 44458 \end{bmatrix}^{-1} \cdot \begin{Bmatrix} 0 \\ -10 \\ 0 \end{Bmatrix}$$

$$\begin{Bmatrix} \Delta_1 \\ \Delta_2 \\ \Delta_3 \end{Bmatrix} = 10^{-6} \cdot \begin{bmatrix} 1.156 & 0.332 & 0.208 \\ 0.332 & 34.723 & 21.702 \\ 0.208 & 21.702 & 36.057 \end{bmatrix} \cdot \begin{Bmatrix} 0 \\ -10 \\ 0 \end{Bmatrix},$$

which gives the solution

$$\begin{Bmatrix} \Delta_1 \\ \Delta_2 \\ \Delta_3 \end{Bmatrix} = \begin{Bmatrix} -3.324 \cdot 10^{-6} \ m \\ -3.472 \cdot 10^{-4} \ m \\ -2.170 \cdot 10^{-4} \ rd \end{Bmatrix}$$

When the displacements are known from equation $\{X_\beta\} = \lfloor K_{\beta\alpha} \rfloor \cdot \{\Delta_\alpha\}$ reactions can be calculated

$$\begin{Bmatrix} X_4 \\ X_5 \\ X_6 \\ X_7 \\ X_8 \\ X_9 \\ X_{10} \end{Bmatrix} = \begin{Bmatrix} -6.079 \\ 6.079 \\ 2.770 \\ 2.010 \\ 4.824 \\ 3.309 \\ 1.911 \end{Bmatrix}$$

The internal element forces, that is the end forces and moments of elements *1*,*2* and *3*, can now be found by applying element stiffness matrices, which are multiplied by corresponding displacements. For instance, find internal forces in element *1*:

$$\begin{Bmatrix} N_{1x} \\ N_{2y} \\ N_{4x} \\ N_{5y} \end{Bmatrix} = \left[\begin{array}{cc|cc} 17677 & -17677 & -17677 & 17677 \\ -17677 & 17677 & 17677 & -17677 \\ \hline -17677 & 17677 & 17677 & -17677 \\ 17677 & -17677 & -17677 & 17677 \end{array}\right] \begin{Bmatrix} -3.324.10^{-6} \\ -3.472.10^{-4} \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 6.079 \\ -6.079 \\ -6.079 \\ 6.079 \end{Bmatrix}$$

These forces are given in a global co-ordinate system and have to be transformed into the local co-ordinate system; in this case it is calculated simply at joint C using equation (10.35) with $\alpha = -45^0$:

$$\begin{Bmatrix} N_C \\ Q_C \end{Bmatrix} = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} \begin{Bmatrix} 6.079 \\ -6.079 \end{Bmatrix} = \begin{bmatrix} 0.707 & -0.707 \\ 0.707 & 0.707 \end{bmatrix} \begin{Bmatrix} 6.079 \\ -6.079 \end{Bmatrix} = \begin{Bmatrix} 8.597 \\ 0 \end{Bmatrix}$$

$N_1 = 8.597\ kN.$

It is always good practice to draw a free body in equilibrium to check for global (not only of an element) equilibrium as shown in Fig. 9.9.

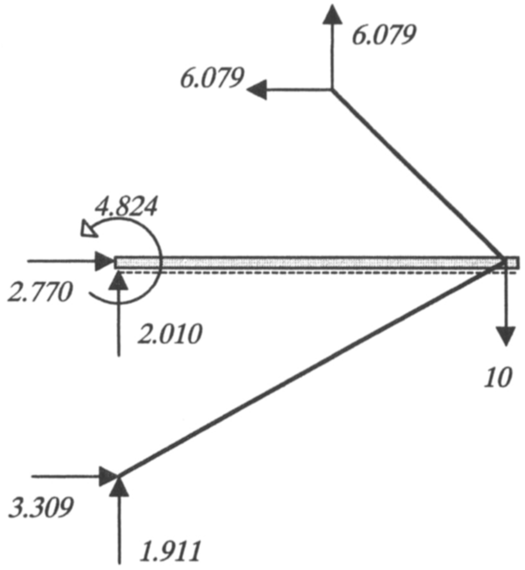


Figure 9.8: Free body in equilibrium

The above procedure can be outlined in the following six steps necessary for the formulation of the matrix displacement method:

- *Step 1*: Identify individual elements of the structure. The interconnection points between these elements are called joints.
- *Step 2*: At each joint identify and number the nodes for which forces and corresponding displacements exist. Number the nodes with unknown displacements first.
- *Step 3*: Calculate the stiffness matrices for all elements, adhering to the numbering established in step 2.
- *Step 4*: Assemble the structure stiffness matrix by superposition of the element stiffness matrices.
- *Step 5*: Write the matrix equation $\{X\}=[K]\cdot\{\Delta\}$, substitute known values of forces and displacements, partition into Eqn. (9.12) and (9.13), solve for unknown displacements $\{\Delta_\alpha\}$ and the unknown reactions $\{\Delta_\beta\}$.
- *Step 6*: Calculate internal element forces by the equation $\{S\}_m=[k_m]\cdot\{\Delta\}$ for each element separately.

9.5 Slope-Deflection method

9.5.1 Basics

The general matrix displacement method explained in Ch. 9.4 is suitable for use with computers. If axial and shear distortions are neglected a much simpler method appropriate for longhand calculations can be derived; it is applicable to planar structures only and is called the slope-deflection or classic displacement method.

The method of nodal numbering will follow the classical notation. The joints of an element of length L (Figure 9.9) are designated by letters j and k, elements are loaded by end moments M_j and M_k only (therefore end shear forces are opposite), joints can rotate by angles φ_j and φ_k and a relative translation between element ends is Δ_k.

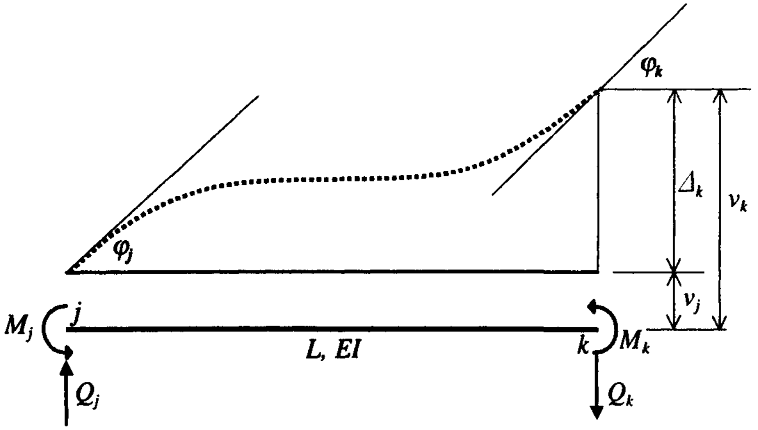


Figure 9.9: Slope-deflection method convention

From a stiffness matrix of a planar element (Equation 7.11) appropriate values are taken and written in the following equation

$$\begin{Bmatrix} M_j \\ M_k \end{Bmatrix} = \begin{Bmatrix} M_j \\ M_k \end{Bmatrix}_F + \begin{bmatrix} k_{jj} & k_{jk} & -\frac{1}{L}\cdot\left(k_{jj}+k_{jk}\right) \\ k_{kj} & k_{kk} & -\frac{1}{L}\cdot\left(k_{kj}+k_{kk}\right) \end{bmatrix} \cdot \begin{Bmatrix} \varphi_j \\ \varphi_k \\ \Delta_k \end{Bmatrix} \tag{9.16}$$

where the first column on the right hand side of the equation represents moments caused by external forces and the second column moments due to the end deformations of an element.

For the specific case of a straight prismatic element of length L the rotational end stiffness are known

$$k_{jj} = k_{kk} = \frac{4 \cdot EI}{L} \qquad k_{jk} = k_{kj} = \frac{2 \cdot EI}{L} \tag{9.17}$$

and after insertion into (9.16) we get:

$$\begin{Bmatrix} M_j \\ M_k \end{Bmatrix} = \begin{Bmatrix} M_j \\ M_k \end{Bmatrix}_F + EI \cdot \begin{bmatrix} \frac{4}{L} & \frac{2}{L} & -\frac{6}{L^2} \\ \frac{2}{L} & \frac{4}{L} & -\frac{6}{L^2} \end{bmatrix} \cdot \begin{Bmatrix} \varphi_j \\ \varphi_k \\ \Delta_k \end{Bmatrix} \tag{9.18}$$

The sign convention of the fixed-end moments from Fig. 9.10 is: *counter-clockwise moments on the element ends are positive.* Equation (9.18) can be expanded to include shear forces at the element ends as:

$$\begin{Bmatrix} M_j \\ M_k \\ Q_j \\ Q_k \end{Bmatrix} = \begin{Bmatrix} M_j \\ M_k \\ Q_j \\ Q_k \end{Bmatrix}_F + \begin{bmatrix} \frac{4EI}{L} & \frac{2EI}{L} & -\frac{6EI}{L^2} \\ \frac{2EI}{L} & \frac{4EI}{L} & -\frac{6EI}{L^2} \\ -\frac{6EI}{L^2} & -\frac{6EI}{L^2} & \frac{12EI}{L^3} \\ -\frac{6EI}{L^2} & -\frac{6EI}{L^2} & \frac{12EI}{L^3} \end{bmatrix} \begin{Bmatrix} \varphi_j \\ \varphi_k \\ \Delta_{kj} \end{Bmatrix} \tag{9.19}$$

or explicitly

$$M_j = (M_j)_F + \left(\frac{4EI}{L} \varphi_j + \frac{2EI}{L} \varphi_k - \frac{6EI}{L^2} \Delta_k \right) \tag{9.20}$$

$$M_k = (M_k)_F + \left(\frac{2EI}{L} \varphi_j + \frac{4EI}{L} \varphi_k - \frac{6EI}{L^2} \Delta_k \right) \tag{9.21}$$

$$Q_j = (Q_j)_F - \left(\frac{6EI}{L^2} \cdot \varphi_j + \frac{6EI}{L^2} \cdot \varphi_k - \frac{12EI}{L^3} \Delta_k \right) \tag{9.22}$$

$$Q_k = (Q_K)_F - \left(\frac{6EI}{L^2} \cdot \varphi_j + \frac{6EI}{L^2} \cdot \varphi_k - \frac{12EI}{L^3} \Delta_k \right) \tag{9.23}$$

Equation (9.19) can be written in a matrix form

$$\{F\} = \{F\}_0 + [K] \cdot \{u\}, \tag{9.24}$$

$\{F\}$ is a force vector (final forces at joints), $\{F\}_0$ is a vector of fixed-end moments and shear forces due to the external loads on the element, $\{u\}$ is a vector of joint displacements, $[K]$ is so called *"stiffness matrix of the element type h"*, that is if the element is clamped at both ends. If one end of the element is pinned (*element type g*) then $M_k = 0$ and therefore:

$$\begin{Bmatrix} M_j \\ M_k = 0 \\ Q_j \\ Q_k \end{Bmatrix} = \begin{bmatrix} \frac{3EI}{L} & -\frac{3EI}{L^2} \\ 0 & 0 \\ -\frac{3EI}{L^2} & \frac{3EI}{L^3} \\ -\frac{3EI}{L^2} & \frac{3EI}{L^3} \end{bmatrix} \begin{Bmatrix} \varphi_j \\ \Delta_k \end{Bmatrix} \qquad (9.25)$$

The procedure of calculation is as follows:

- Define a geometrically determinate system, that is a system where all *rotations* φ_i and *displacements of elements* Δ_i *are zero* and write stiffness relations for each element for two types of loads:

  1. *External loading* on the element
  2. Loading by *rotations* of both ends and by a *relative displacement* between element ends

- The geometrical (compatibility) relations are satisfied by writing displacements common to several elements and by inserting the given support conditions. We have to write down

  $b = k - p_1$ *Equations of joint rotations* φ_i and

  $c = 2 \cdot k + 2 \cdot g - p_2 - m$ *Equations for element displacements* Δ_i

- Equilibrium must be satisfied by the equilibrium equation for each rotation and displacement separately. Equations are then solved for these displacements.
- The displacements found in previous step are substituted into the slope-deflection equations to find the element end forces. All other element forces can then be determined by statics.

Statically the problem is solved when all rotations φ_i and displacements Δ_i are found. As the joint rotations are suppressed in step one; external loading causes fixed-end moments at the element ends. Fixed-end moments can be determined by the force method (see example below) or can be found in Tables *B.6* and *B.7* for the most common loads.

Positive signs of moments were defined earlier, that is a counter-clockwise moment is positive.

Example 9.4: Fixed-end moment for the element of type "g"

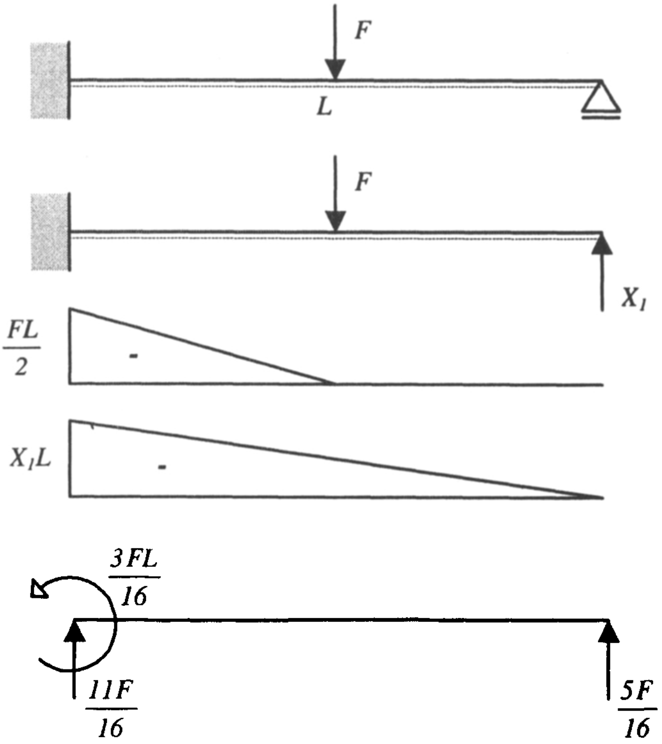


Figure 9.10: Determination of fixed-end moment

$$a_{10} = \frac{1}{6} \cdot \frac{F \cdot L}{2}\left(\frac{L}{2} + 2 \cdot L\right) \cdot \frac{L}{2} = \frac{5 \cdot F \cdot L^3}{48} \qquad a_{11} = \frac{L^3}{3}$$

$$X_1 = \frac{5 \cdot F}{16} \qquad M_A = -\frac{F \cdot L}{2} + \frac{5 \cdot F}{16} \cdot L = FL \cdot \left(-\frac{1}{2} + \frac{5}{16}\right) = -\frac{3FL}{16}$$

9.5.2 Deformational equations

The structure is $b + c = n$ times statically indeterminate, hence b equations for rotations of joints φ_i and c equations for the displacements of elements Δ_i must be written.

In practice we will distinguish between three cases:

- Joints can only rotate (*non-sway systems* $c \leq 0$). Only b deformational equations for joint rotations φ_i must be written.
- Beside joint rotations, element displacements Δ_i can occur, but they are independent of each other (*simply sway systems* $c > 0$, $n = b + c$ deformational equations)
- Displacement of elements Δ_i are dependent of each other (*general sway systems* $c > 0$, $n = b + c$ deformational equations)

9.5.3 Non-sway systems $c \leq 0$

Example 9.5: Analyse the continuous beam below

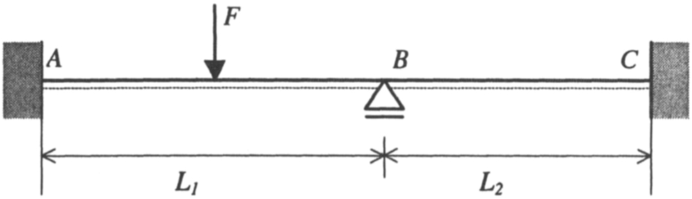


Figure 9.11: Continuous beam

Static indeterminacy:

$$n = 3 \cdot (m_6 - j_3) + 2 \cdot (m_5 - j_2) + m_4 =$$
$$= 3 \cdot (2 - 0) + 2 \cdot (0 - 1) + 0 = 4$$

Kinematics stability:

$$f = 2 \cdot v - (n + k + p) = 6 - (2 + 1 + 7) = -4$$

Deformational indeterminacy:

$$n = b + c$$

$$b = k - p_1 = 3 - 2 = 1$$

$$c = 2 \cdot (k + g) - p_2 - e = 2 \cdot (3 + 0) - 5 - 2 = -1$$

As seen, the structure is four times statically indeterminate hence using the force method four equations must be solved (i.e. for X_c, Y_c, Y_B and M_c); on the contrary the only deformational unknown is the rotation φ_B.

1. Force-deformation relation (including boundary conditions)

a.) Element AB

Fixed-end moments

$$M_{AB}^{F} = \frac{F \cdot L_1}{8} \qquad M_{BA}^{F} = -\frac{F \cdot L_1}{8}$$

Slope-deflection equations

$$M_{AB} = \frac{F \cdot L_1}{8} + EI \cdot \left(\frac{4}{L_1} \cdot \varphi_{AB} + \frac{2}{L_1} \cdot \varphi_{BA} - 6 \cdot \frac{\Delta_{BA}}{L_1^2} \right) \quad or$$

$$M_{AB} = \frac{F \cdot L_1}{8} + EI \cdot \left(0 + \frac{2}{L_1} \cdot \varphi_B - 0 \right)$$

$$M_{BA} = -\frac{F \cdot L_1}{8} + EI \cdot \left(0 + \frac{4}{L_1} \cdot \varphi_B - 0 \right)$$

b.) Element BC (no external loading)

$$M_{BC} = 0 + EI \cdot \left(\frac{4}{L_2} \cdot \varphi_B + 0 - 0 \right)$$

$$M_{CB} = 0 + EI \cdot \left(0 + \frac{2}{L_2} \cdot \varphi_B - 0 \right)$$

2. Geometry conditions:

$$\varphi_{AB} = \varphi_{CB} = \Delta_{BA} = \Delta_{BC} = 0$$

$$\varphi_{BA} = \varphi_{BC} = \varphi_C$$

3. Equilibrium at B:

$$\sum M_B = 0 \quad : \quad M_{BA} + M_{BC} = 0$$

$$\left(-\frac{F\cdot L_1}{8}+\frac{4\cdot EI}{L_1}\cdot\varphi_B\right)+\frac{4\cdot EI}{L_2}\cdot\varphi_B=0$$

$$-\frac{F\cdot L_1}{8}+4\cdot EI\cdot\varphi_B\cdot\left(\frac{1}{L_1}+\frac{1}{L_2}\right)=0$$

$$\varphi_B=\frac{F\cdot L_1}{8}\cdot\frac{1}{4\cdot EI}\cdot\frac{1}{\frac{1}{L_1}+\frac{1}{L_2}}=\frac{F\cdot L_1}{32\cdot EI}\cdot\frac{L_1\cdot L_2}{L_1+L_2}$$

4. *End moments calculation*; φ_B from above is re-substituted into the equations from step *1*.

$$M_{AB}=\frac{F\cdot L_1}{8}+EI\cdot\frac{2}{L_1}\cdot\varphi_B$$

$$M_{AB}=\frac{F\cdot L_1}{8}+EI\cdot\frac{2}{L_1}\cdot\varphi_B\cdot\frac{F\cdot L_1}{32\cdot EI}\cdot\frac{L_1\cdot L_2}{L_1+L_2}=\frac{F\cdot L_1}{8}+\frac{F}{16}\cdot\frac{L_1\cdot L_2}{L_1+L_2}$$

$$M_{BA}=-\frac{F\cdot L_1}{8}+EI\cdot\frac{4}{L_1}\cdot\varphi_B=-\frac{F\cdot L_1}{8}+\frac{F}{8}\cdot\frac{L_1\cdot L_2}{L_1+L_2}$$

$$M_{BC}=EI\cdot\frac{4}{L_2}\cdot\varphi_B=\frac{F}{8}\cdot\frac{L_1}{L_2}\cdot\frac{L_1\cdot L_2}{L_1+L_2}$$

$$M_{CB}=EI\cdot\frac{2}{L_2}\cdot\varphi_B=\frac{F}{16}\cdot\frac{L_1}{L_2}\cdot\frac{L_1\cdot L_2}{L_1+L_2}$$

If $L_1=L_2=L$:

$$M_A=\frac{5FL}{32}\qquad M_{BA}=-M_{BC}=-\frac{FL}{16}\qquad M_C=\frac{FL}{32}$$

The equilibrium is restored on free bodies of individual elements and shear forces as well as reactions are calculated, for instance:

$$Y_A=\frac{F}{2}+\frac{1}{L}\left(\frac{5\cdot FL}{32}-\frac{FL}{16}\right)=\frac{19F}{32}$$

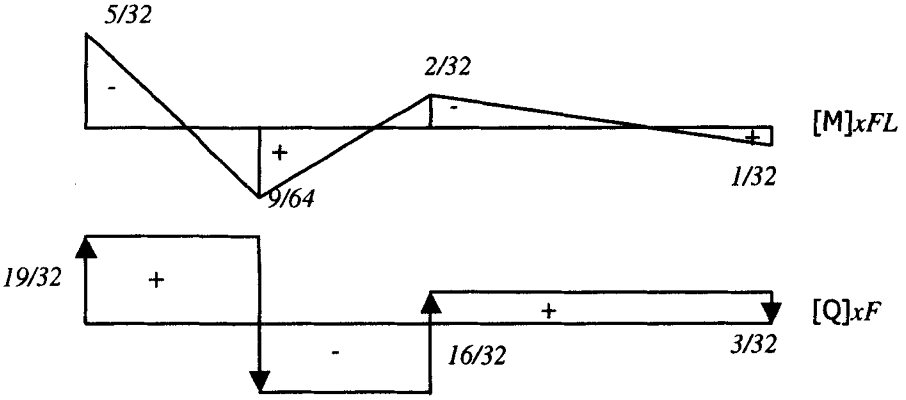


Figure 9.12: Final diagrams of bending moments and shear forces

Example 9.6: Solve the non-sway frame structure ($EI = const.$)

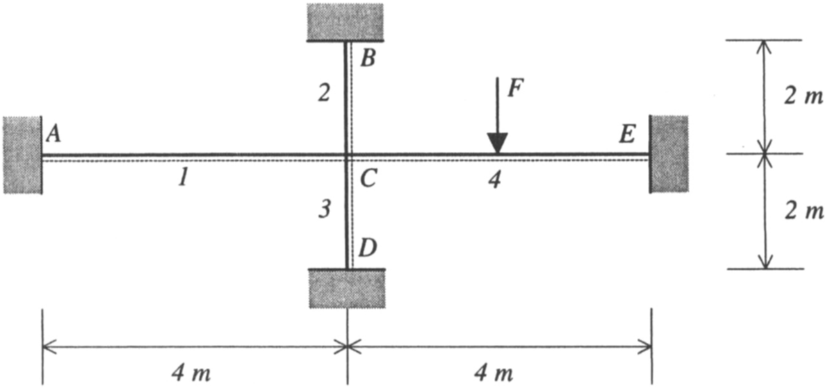


Figure 9.13: Non-sway frame structure

At first we shall determine the degree of deformational indeterminacy:

$$m=4,\quad k=5,\quad g=0,\quad p_1=4,\quad p_2=8$$

$$b = k - p_1 = 5 - 4 = 1$$

$$c = 2 \cdot k + 2 \cdot g - p_2 - m$$

$$c = 2 \cdot 5 - 8 - 4 = -2$$

$$c \leq 0$$

The structure has one unknown deformational quantity only, hence only one deformational equilibrium equation for the rotation at joint *C* has to be written (the unknown is therefore the rotation φ_C).

The rotation at *C* is calculated from the condition, that all moments have to be in equilibrium:

$$(\Sigma M_i)_C = 0$$

Moments at a particular joint can be caused by external loading, by the rotation of that particular joint and by the rotations of neighbouring joints; in the example 9.6 the only external loading is force *F* acting in the middle of element *4*.

$$a_{j0} = \frac{F \cdot L}{8} = \frac{F \cdot 4}{8} = \frac{F}{2}$$

$$a_{jj} = \sum_{1}^{4} \frac{4}{L_i} = \frac{4}{L_1} + \frac{4}{L_2} + \frac{4}{L_3} + \frac{4}{L_4} = 6$$

$$a_{jk} = \sum_{1}^{4} \frac{2}{L_i} = 0$$

The equilibrium equation is

$$a_{j0} + a_{jj} \cdot \varphi_j = 0$$

from where the rotation at joint *C* is calculated:

$$EI \cdot \varphi_C = -\frac{a_{j0}}{a_{jj}} = -\frac{F}{2 \cdot 6} = -\frac{F}{12}$$

Moments in individual elements are calculated using equations (9.20-9.21)

$$M_{AC} = \frac{2EI}{L} \cdot \varphi_C = -\frac{2}{4} \cdot \frac{F}{12} = -\frac{F}{24}$$

$$M_{CA} = \frac{4EI}{L} \cdot \varphi_C = -\frac{F}{12}$$

$$M_{CE} = -\frac{F}{12} + M_0 = -\frac{F}{12} + \frac{F}{2} = \frac{5F}{12}$$

$$M_{EC} = -\frac{F}{24} - M_0 = -\frac{7F}{12}$$

$$M_{BC} = \frac{2EI}{L} \cdot \varphi_j = -\frac{2}{2} \cdot \frac{F}{12} = -\frac{F}{12}$$

$$M_{CB} = \frac{4EI}{2} \cdot \varphi_j = -\frac{F}{6}$$

$$M_{CD} = -\frac{F}{6}$$

$$M_{DC} = -\frac{F}{12}$$

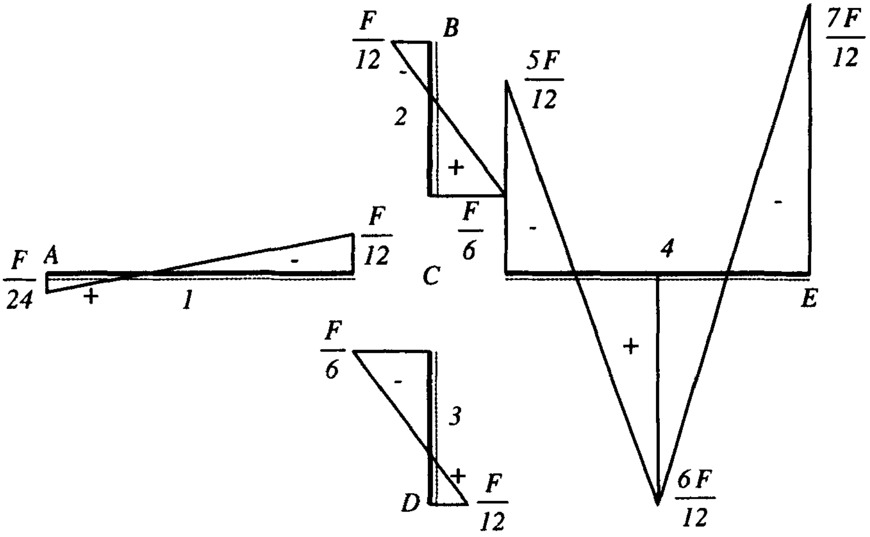


Figure 9.14: Diagrams of bending moments

Shear and axial forces are calculated from the equilibrium at each individual element, for instance for element *4*:

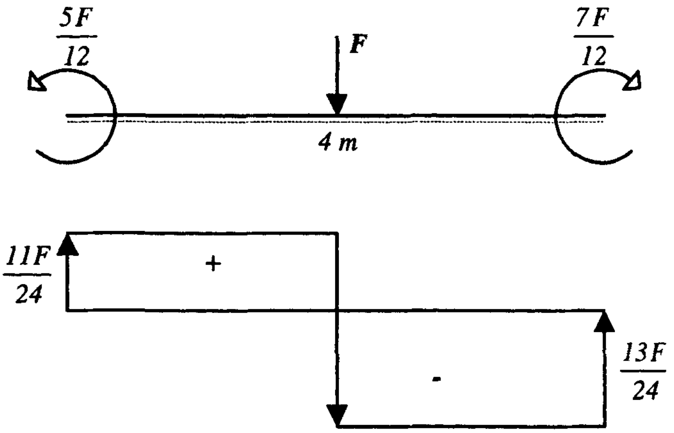


Figure 9.15: Determination of shear forces

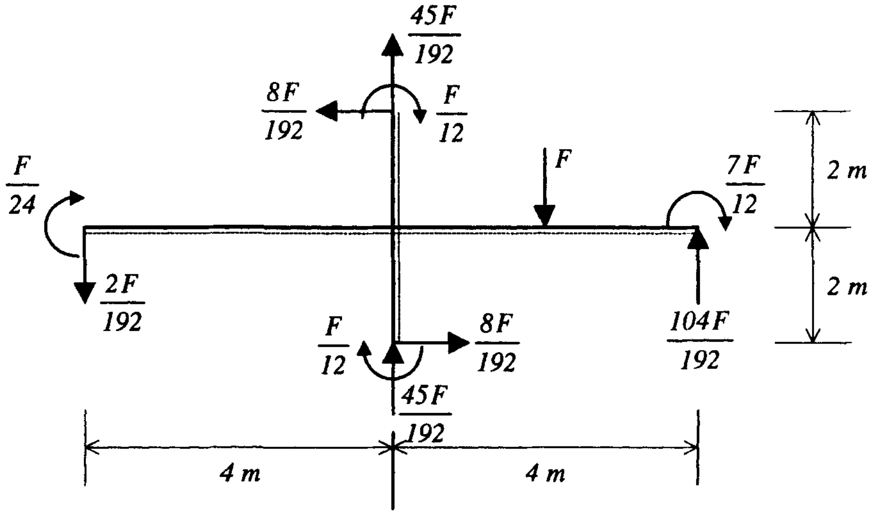


Figure 9.16: Free body in equilibrium

9.5.4 Simply sway structures

If a structure has *c* element displacements, then beside *b* equilibrium equations, *c* supplemental equations must be written from the equilibrium of all forces on a free body related to the displacement Δ_i on a particular kinematics chain.

Simply sway structures are defined as structures at which a displacement of a rigid body (i.e. column) is dependant on that particular displacement Δ_i only.

The missing *c* equilibrium equations are derived from the equilibrium of the kinematics chain by the method of virtual work. The work of all forces on a chain is calculated and an equivalent force is derived causing the same work done. (see the example 9.7).

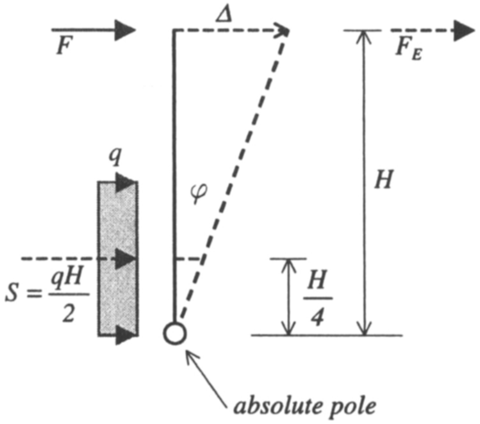


Figure 9.17: Work done on a kinematics chain

The work done at the rotation φ of the element about an absolute pole is

$$W = F\cdot(H\cdot\varphi)+S\cdot(\frac{H}{4}\varphi) = F\cdot(H\cdot\varphi)+\frac{qH}{2}\frac{H}{4}\varphi = H\varphi(F+\frac{qH}{8})$$

$$F_E = \frac{W}{\Delta} = F+\frac{qH}{8},$$

hence the diagonal terms of kinematics chain displacements (from Fig 7.11: second row, second column) are

$$\frac{12EI}{L^3}\Delta,$$

which are in fact equivalent forces on the kinematics chain. The displacement causes moments at the other end and therefore the out of diagonal terms are(from Fig 7.11: sixth row, eighth column):

$-\frac{6EI}{L^2}\Delta$ at positive *y* displacement and

$+\frac{6EI}{L^2}\Delta$ at positive *x* displacement

Note: Considering elements of type "*g*", the coefficients *12* and *6* equal *3*.

Example 9.7: Simple sway frame *(F = 10 kN, g = 10 kN/m, p = 5 kN/m)*

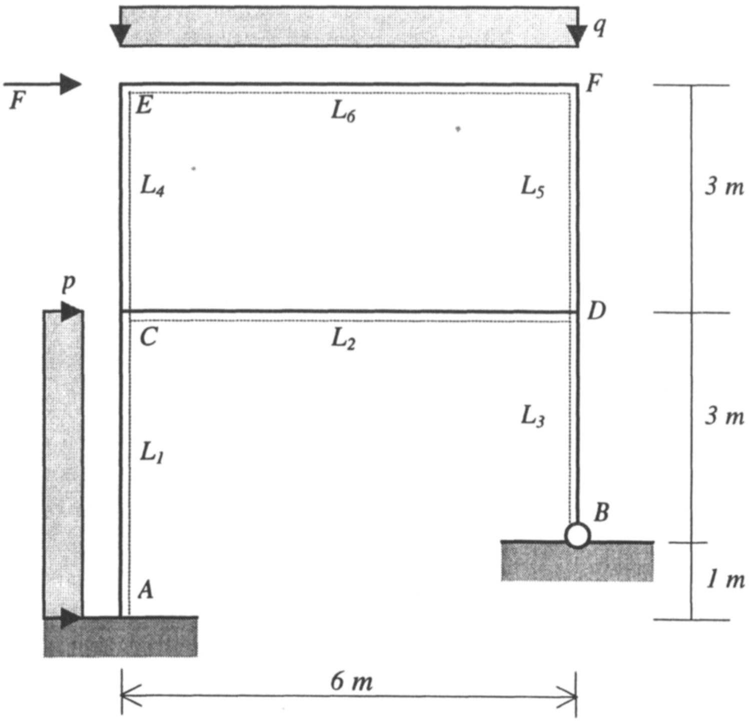


Figure 9.18: Two-story frame structure

Kinematics analysis:

$$g=1,\quad k=5,\quad m=6,\quad p_1=1,\quad p_2=4$$

$$b=5-1=4$$

$$c=2\cdot 5+2\cdot 1-4-6=2$$

The structure is hence $b+c=4+2=6$ times *kinematically indeterminate* having unknown rotations φ_C, φ_D, φ_E and φ_F and unknown story displacements Δ_1 and Δ_2.

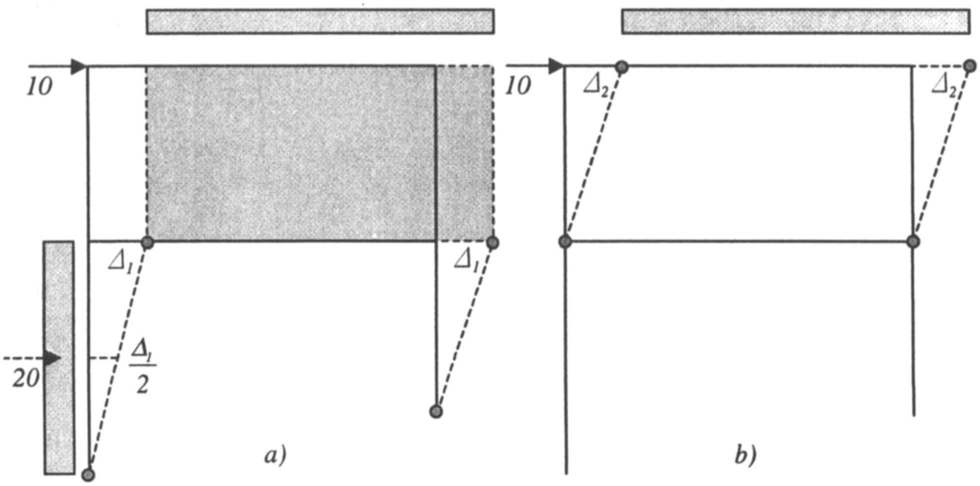


Figure 9.19: Kinematics chains

The matrix form of the equilibrium equations has the following terms:

$$\left[\begin{array}{cccc|c|c} a_{CC} & a_{CD} & a_{CE} & - & a_{C1} & a_{C2} \\ & a_{DD} & - & a_{DF} & a_{D1} & a_{D2} \\ & & a_{EE} & a_{EF} & & a_{E2} \\ & & & a_{FF} & & a_{F2} \\ \hline & & & & a_{11} & - \\ & & & & - & a_{22} \end{array}\right] \left\{\begin{array}{c} \varphi_C \\ \varphi_D \\ \varphi_E \\ \varphi_F \\ \hline \Delta_1 \\ \Delta_2 \end{array}\right\} = \left\{\begin{array}{c} a_{C0}+0 \\ 0+0 \\ a_{E0}+0 \\ a_{F0}+0 \\ 0+a_{10} \\ 0+a_{20} \end{array}\right\}$$

Calculation of equation coefficients gives:

$$a_{CC} = \frac{4}{L_1} + \frac{4}{L_2} + \frac{4}{L_4} = \frac{4}{4} + \frac{4}{6} + \frac{4}{3} = 3.000$$

$$a_{DD} = \frac{4}{3} + \frac{4}{6} + \frac{3}{3} = 3.000$$

$$a_{EE} = \frac{4}{3} + \frac{4}{6} = 2$$

$$a_{FF} = 2$$

$$a_{CD} = \frac{2}{6} = 0.333$$

$$a_{CE} = \frac{2}{3} = 0.667$$

$$a_{DF} = \frac{2}{3} = 0.667$$

$$a_{EF} = \frac{2}{6} = 0.333$$

$$a_{C1} = \frac{6}{L_1^{\,2}} = \frac{6}{4^2} = 0.375$$

$$a_{D1} = \frac{3}{L_3^{\,2}} = 0.333$$

$$a_{C2} = a_{D2} = a_{E2} = a_{F2} = \frac{6}{L_4^{\,2}} = \frac{6}{3^2} = 0.667$$

$$a_{11} = \frac{12}{L_1^{\,3}} + \frac{3}{L_3^{\,3}} = \frac{12}{4^3} + \frac{3}{3^3} = 0.299$$

$$a_{22} = \frac{12}{L_4^{\,3}} + \frac{12}{L_5^{\,3}} = 2 \cdot \frac{12}{3^3} = 0.889$$

The right hand side (load) coefficients are:

$$a_{C0} = -\frac{p \cdot L_1^2}{12} = -\frac{5 \cdot 4^2}{12} = -6.67 \qquad a_{D0} = 0$$

$$a_{E0} = \frac{g \cdot L_6^2}{12} = \frac{5 \cdot 6^2}{12} = 30 \qquad a_{F0} = -\frac{g \cdot L_6^2}{12} = -30$$

The total work done at the displacement of the first story, from Fig. 9.19a, is

$$W = p \cdot H \cdot \frac{\Delta}{2} + F \cdot \Delta$$

from which the horizontal force is derived by equation

$$X = \frac{W}{\Delta} = \frac{pH}{2} + F = 10 + 10 = 20$$

hence the load coefficient is

$$a_{10} = 20$$

The same procedure for the displacement of the second story (Figure 9.19b) gives:

$$a_{20} = F = 10$$

The complete equations are:

$$\left[\begin{array}{cccc|c|c} 3.000 & 0.333 & 0.667 & 0 & 0.375 & 0.667 \\ & 3.000 & 0 & 0.667 & 0.333 & 0.667 \\ & & 2.000 & 0.333 & 0 & 0.667 \\ & Sym. & & 2.000 & 0 & 0.667 \\ \hline & & & & 0.299 & 0 \\ & & & & 0 & 0.889 \end{array}\right] \left\{\begin{array}{c} \varphi_C \\ \varphi_D \\ \varphi_E \\ \varphi_F \\ \hline \Delta_1 \\ \Delta_2 \end{array}\right\} = \left\{\begin{array}{c} 6.667 \\ 0 \\ -30 \\ +30 \\ \hline +20 \\ +10 \end{array}\right\}$$

Equations are solved for unknown deformations, hence:

$$\left\{\begin{array}{c} \varphi_C \\ \varphi_D \\ \varphi_E \\ \varphi_F \\ \hline \Delta_1 \\ \Delta_2 \end{array}\right\} = \left\{\begin{array}{c} -15.538 \\ -26.305 \\ -31.009 \\ 9.298 \\ \hline 115.853 \\ 58.915 \end{array}\right\}$$

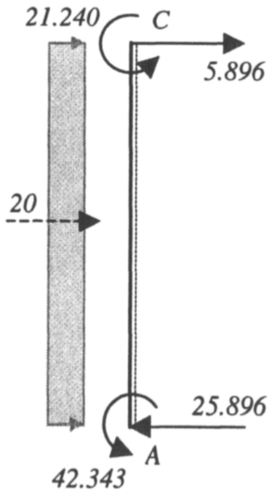
Now internal forces on idividual elements from the basic relations on elements can be calculated. Note that positive rotations and moments are by convention counterclockwise.

For element *1*, for which the free body is drawn below, the calculation is as follows.

$$\begin{Bmatrix} M_{AC} \\ M_{CA} \\ Q_{AC} \\ Q_{CA} \end{Bmatrix} = \begin{Bmatrix} M_{ACO} \\ M_{CAO} \\ Q_{ACO} \\ Q_{CAO} \end{Bmatrix} + \begin{bmatrix} +\dfrac{4}{L_1} & +\dfrac{2}{L_1} & +\dfrac{6}{L_1^2} \\ +\dfrac{2}{L_1} & +\dfrac{4}{L_1} & +\dfrac{6}{L_1^2} \\ -\dfrac{6}{L_1^2} & -\dfrac{6}{L_1^2} & -\dfrac{12}{L_1^3} \\ +\dfrac{6}{L_1^2} & +\dfrac{6}{L_1^2} & +\dfrac{12}{L_1^3} \end{bmatrix} \begin{Bmatrix} \varphi_A = 0 \\ \varphi_C \\ \Delta_1 \end{Bmatrix} =$$

$$= \begin{Bmatrix} +6.667 \\ -6.667 \\ 10 \\ -10 \end{Bmatrix} + \begin{bmatrix} 1 & 0.5 & 0.375 \\ 0.5 & 1 & 0.375 \\ -0.375 & -0.375 & -0.1875 \\ +0.375 & +0.375 & +0.1875 \end{bmatrix} \begin{Bmatrix} 0 \\ -15.538 \\ +115.853 \end{Bmatrix}$$

$$\begin{Bmatrix} M_{AC} \\ M_{CA} \\ Q_{AC} \\ Q_{CA} \end{Bmatrix} = \begin{Bmatrix} +6.667 \\ -6.667 \\ -10 \\ -10 \end{Bmatrix} + \begin{Bmatrix} +35.676 \\ 27.907 \\ -15.896 \\ +15.896 \end{Bmatrix} = \begin{Bmatrix} 42.343 \\ 21.240 \\ -25.896 \\ +5.896 \end{Bmatrix}$$



All other moments are calculated explicitly (EI=1!):

$$M_{CD}=\frac{4}{L_2}\varphi_C+\frac{2}{L_2}\varphi_D=\frac{4}{6}(-15.538)+\frac{2}{6}(-26.305)=-19.127$$

$$M_{DC}=\frac{2}{L_2}\varphi_C+\frac{4}{L_2}\varphi_D=\frac{2}{6}(-15.538)+\frac{4}{6}(-26.305)=-22.716$$

$$M_{DB}=\frac{3}{L_3}\varphi_D+\frac{3}{L_3^2}\Delta_1=\frac{3}{3}(-26.305)+\frac{3}{9}(115.853)=12.313$$

$$M_{CE}=\frac{4}{L_4}\varphi_C+\frac{2}{L_4}\varphi_E+\frac{6}{L_4^2}\Delta_2=\frac{4}{3}(-15.538)+\frac{2}{3}(-31.009)+\frac{6}{9}(58.915)=-2.113$$

$$M_{EC}=\frac{2}{L_4}\varphi_C+\frac{4}{L_4}\varphi_E+\frac{6}{L_4^2}\Delta_2=\frac{2}{3}(-15.538)+\frac{4}{3}(-31.009)+\frac{6}{9}(58.915)=-12.427$$

$$M_{DF}=\frac{4}{L_5}\varphi_D+\frac{2}{L_5}\varphi_F+\frac{6}{L_5^2}\Delta_2=\frac{4}{3}(-26.305)+\frac{2}{3}(9.298)+\frac{6}{9}(58.915)=10.402$$

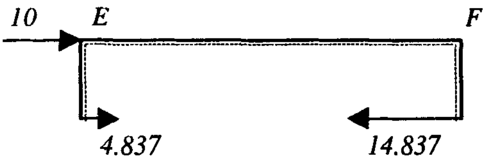
$$M_{FD}=\frac{2}{L_5}\varphi_D+\frac{4}{L_5}\varphi_F+\frac{6}{L_5^2}\Delta_2=\frac{2}{3}(-26.305)+\frac{4}{3}(9.298)+\frac{6}{9}(58.915)=34.137$$

$$M_{EF}=M_{EF0}+\frac{4}{L_6}\varphi_E+\frac{2}{L_6}\varphi_F=30+\frac{4}{6}(-31.009)+\frac{2}{6}(9.298)=12.427$$

$$M_{FE}=M_{FE0}+\frac{2}{L_6}\varphi_E+\frac{4}{L_6}\varphi_F=-30+\frac{2}{6}(-31.009)+\frac{4}{6}(9.298)=-34.138$$

A check of the calculation can be performed at an arbitrary horizontal section; the sum of all forces at that section must equal zero:

$$F\overset{?}{=}Q_{EC}+Q_{FD}$$



$$10\overset{?}{=}\frac{-12.427-2.113}{3}+\frac{34.137+10.402}{3}=-4.8467+14.8463=9.9997$$

A similar check can be done at joints, the sum of all moments must equal zero. Let us calculate the maximum positive bending moment on element *EF*. First, calculate the shear force at *E*:

$$Y_E = Q_{EF} = \frac{qL}{2} + \frac{M_{EF} + M_{FE}}{L} = 30 + \frac{12.427 - 34.137}{6} = 30 - 3.618 = 26.382\ kN$$

The position of the maximum moment is where shear force equals zero

$$x_{max} = \frac{Q_{EF}}{q} = \frac{25.382}{10} = 2.638\ m$$

hence

$$M_{max} = Q_{EF} \cdot x_{max} - \frac{q \cdot x_{max}^2}{2} - M_{EF} = 22.374\ kNm$$

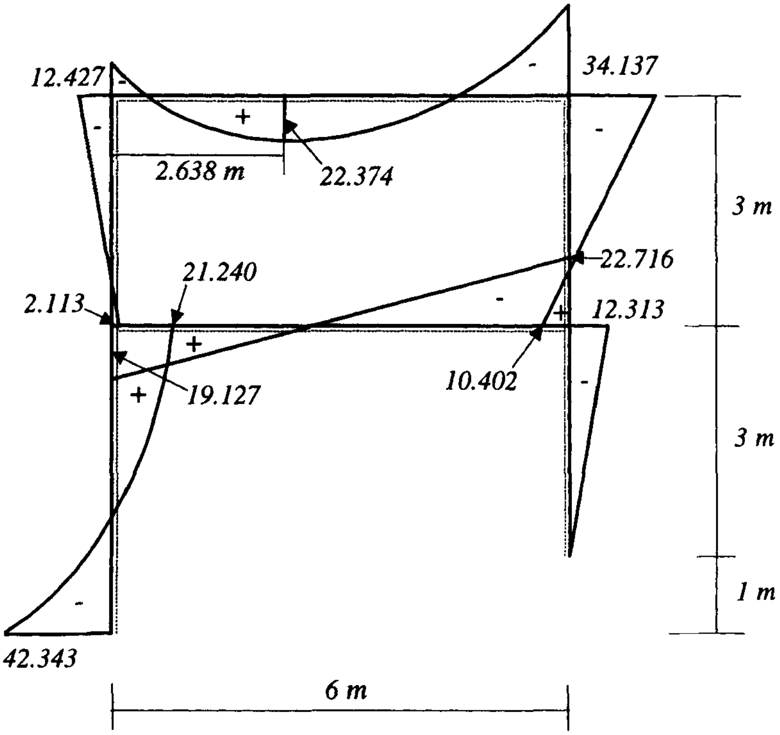


Figure 9.20: Final diagram of bending moments [kNm]

Structures with general (composed) sway movements should not be solved by this classical method as the coefficient determination is rather complicated and requires a complete kinematics analysis.

9.6 The moment distribution method (Cross's method)

The method of moment distribution is a numerical application of the displacement method in which the desired quantities are determined by a method of successive approximation that is suitable for longhand calculations.
The method is applicable to structures that satisfy the following:

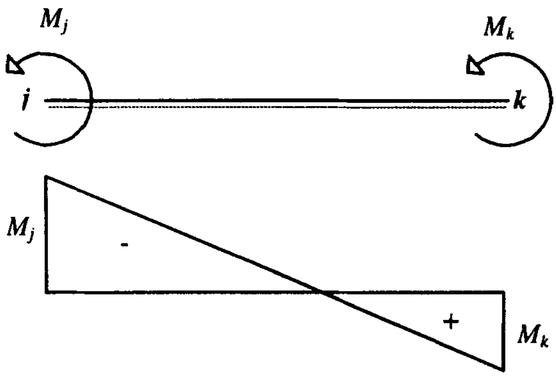
- *Plane problem*
- *Shear deformations can be neglected*
- *Deformations are small*

Equation $\{F\}=[K]\cdot\{u\}$ is written in a form for plane problems where rotations of joints *1* and 2 are only considered:

$$\begin{Bmatrix} M_{Z1} \\ M_{Z2} \end{Bmatrix} = \begin{bmatrix} K_{6,6} & K_{6,12} \\ K_{12,6} & K_{12,12} \end{bmatrix} \cdot \begin{Bmatrix} \varphi_{Z1} \\ \varphi_{Z2} \end{Bmatrix} \tag{9.26}$$

$$\begin{Bmatrix} M_{Z1} \\ M_{Z2} \end{Bmatrix} = \begin{bmatrix} \frac{4EI_Z}{L} & \frac{2EI_Z}{L} \\ \frac{2EI_Z}{L} & \frac{4EI_Z}{L} \end{bmatrix} \cdot \begin{Bmatrix} \varphi_{Z1} \\ \varphi_{Z2} \end{Bmatrix} \tag{9.27}$$

Loading by a counter-clockwise positive moments causes bending moments in a beam in accordance with the convention in the figure below:



Due to the above convention the equation is rewritten as:

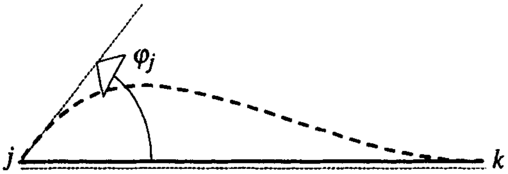
$$\begin{Bmatrix} M_{jk} \\ M_{kj} \end{Bmatrix} = \frac{2EI_Z}{L} \cdot \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \cdot \begin{Bmatrix} \varphi_j \\ \varphi_k \end{Bmatrix} \tag{9.28}$$

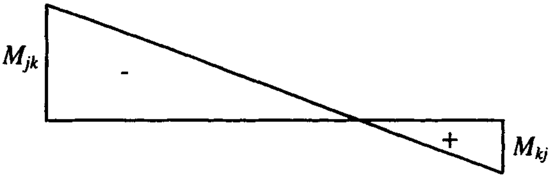
From the condition that $\varphi_k = 0$ both end moments are

$$M_{jk} = \frac{4EI}{L} \cdot \varphi_j \tag{9.29}$$

$$M_{kj} = \frac{2EI}{L} \cdot \varphi_j = \frac{M_{jk}}{2} \tag{9.30}$$

from which we can see that for prismatic members a *carry-over factor is 0.50.*





Consider now a case when joint *B* from Fig. 9.21 is loaded by a moment *M*:

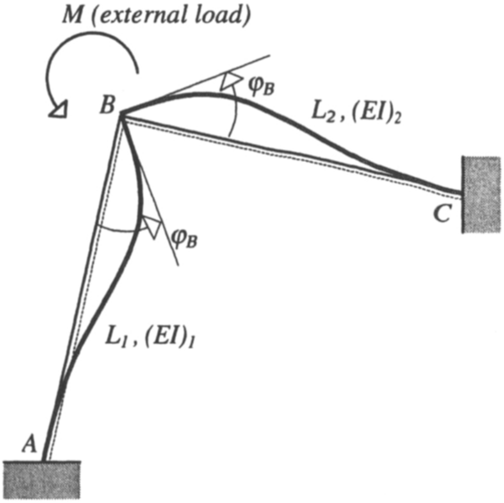


Figure 9.21: Distribution of moments

The sum of all moments at the rotation of the joint must be zero, hence:

$$M = M_{BA} + M_{BC} = \left(\frac{4 \cdot (EI)_1}{L_1} + \frac{4 \cdot (EI)_2}{L_2} \right) \cdot \varphi_B \tag{9.31}$$

$$\varphi_B = \frac{M}{\left(\frac{4 \cdot (EI)_1}{L_1} + \frac{4 \cdot (EI)_2}{L_2} \right)}$$

$$M_{AB} = \frac{2 \cdot (EI)_1}{L_1} \cdot \varphi_B = \frac{M_{BA}}{2}$$

$$M_{CB} = \frac{2 \cdot (EI)_2}{L_2} \cdot \varphi_B = \frac{M_{BC}}{2}$$

$$M_{BA} = \frac{4 \cdot (EI)_1}{L_1} \cdot \varphi_B = \frac{4 \cdot (EI)_1}{L_1} \cdot \frac{M}{\frac{4 \cdot (EI)_1}{L_1} + \frac{4 \cdot (EI)_2}{L_2}} \tag{9.32}$$

In the majority of structures the modulus of elasticity is the same for all members therefore *E* cancels out:

$$M_{BA} = M \cdot \frac{\frac{I_1}{L_1}}{\frac{I_1}{L_1} + \frac{I_2}{L_2}} = \frac{k_1}{\sum k} \cdot M = r_{BA} \cdot M \tag{9.33}$$

The coefficient *r* is the *distribution factor.*

If joint *C* would be pinned, the equations would become:

$$M = M_{BA} + M_{BC} = \frac{4 \cdot (EI)_1}{L_1} \cdot \varphi_B + \frac{3 \cdot (EI)_2}{L_2} \cdot \varphi_B$$

$$M_{AB} = \frac{2 \cdot (EI)_1}{L_1} \cdot \varphi_B = \frac{M_{BA}}{2}$$

$$M_{BC} = \frac{3 \cdot (EI)_2}{L_2} \cdot \varphi_B$$

$$M_{CB} = 0$$

and the moment at B on the element adjacent to A is:

$$M_{BA} = M \cdot \frac{\frac{4 \cdot (EI)_1}{L_1}}{\frac{4 \cdot (EI)_1}{L_1} + \frac{3 \cdot (EI)_2}{L_2}} = M \cdot \frac{\frac{I_1}{L_1}}{\frac{I_1}{L_1} + \frac{3}{4} \cdot \frac{I_2}{L_2}} \tag{9.34}$$

$$M_{BC} = M \cdot \frac{\frac{3}{4} \cdot \frac{I_2}{L_2}}{\frac{I_1}{L_1} + \frac{3}{4} \cdot \frac{I_2}{L_2}} \tag{9.35}$$

The stiffness coefficient for a *one-sided pinned element* is *0.75* of the value for a fixed element and the *carry-over factor is zero.*

The moment distribution procedure begins with the moments due to loads on a geometrically determinate structure; that is, all joints are prevented from movement by fixed-end moments.

The structure is next gradually released into its final deformed shape by allowing one joint at the time to rotate. Each time the joint is released the unbalanced moment on the joint is distributed to adjacent elements (whose opposite ends are fixed at this stage) in accordance to the distribution factors at that joint.

A fraction of these moments is carried over to the far end of the elements in accordance with the carry-over factors.

As the joints are successively released the residual unbalanced moments become smaller and smaller and finally converges to the correct solution.

The successful application of the procedure depends on an efficient tabular scheme as shown in the following example.

Example 9.8: Analyse the continuous beam of the Fig. 9.22 by the moment distribution method ($E = const.$, $I_1 = 3 \cdot I_2$)

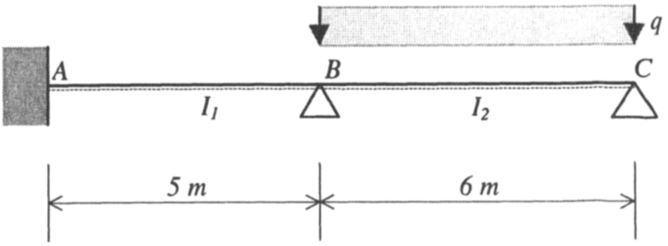


Figure 9.22: Continuous beam

1. Fixed-end moments:

$$M_{BC} = q \cdot \frac{6^2}{8} = 4.5 \cdot q$$

Free-span moment:

$$M^0_{BC} = \frac{q \cdot L^2}{8} = 4.5 \cdot q$$

2. Carry-over factors:

$$\mathit{from}\ B\ \mathit{to}\ A \ \rightarrow\ 0.5$$

$$\mathit{from}\ B\ \mathit{to}\ C \ \rightarrow\ 0$$

3. Stiffness:

$$k_{AB} = \frac{4 \cdot EI_{AB}}{L_{AB}} = \frac{4 \cdot 3}{5} = 2.4$$

$$k_{BC} = \frac{3 \cdot EI_{BC}}{L_{BC}} = \frac{3 \cdot 1}{6} = 0.5$$

Total stiffness:

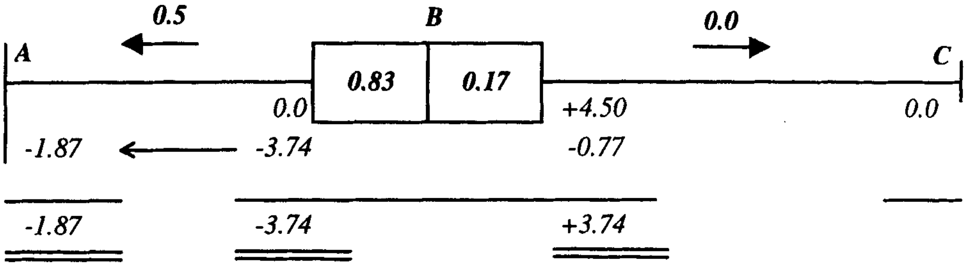
$$k = 2.4 + 0.5 = 2.9$$

4. Distribution coefficients:

$$r_{AB} = \frac{2.4}{2.9} = 0.83$$

$$r_{BC} = \frac{0.5}{2.9} = 0.17$$

Iteration scheme:



All reactions and internal forces can be obtained by statics from these end moments and the moment and shear forces diagrams are drawn as shown in figure below.

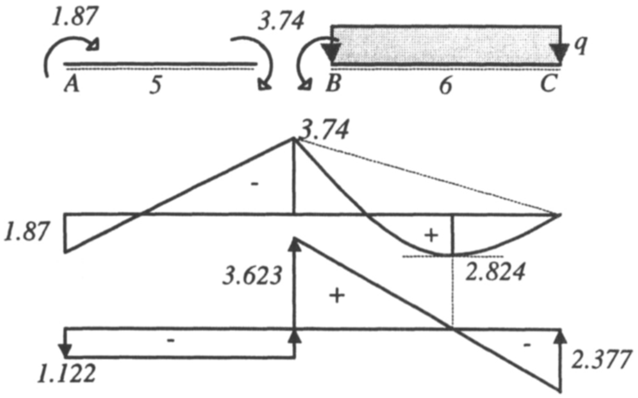


Figure 9.23: Free bodies and internal force diagrams

9.7 Influence lines by the displacement method
(Using Mueller-Breslau principle)

Let us recall the theorem by Mueller-Breslau: *The deflected shape of a structure due to the particular unit distortion represents the influence line for the effect corresponding to that distortion.*
The task here is to actually determine the deflected shape due to a *unit displacement at the point and in the direction* of the quantity to be found, which can be calculated by different methods. In this text we shall use Mohr's method as derived in Ch. 6.3.

9.7.1 Determination of ψ functions

Due to the unit virtual displacement or rotation at a point and in the direction of the quantity to be found, bending moments occur on indeterminate structures as shown in Figure 9.24, where an element of type *"g"* (fixed-pinned ends) is considered.

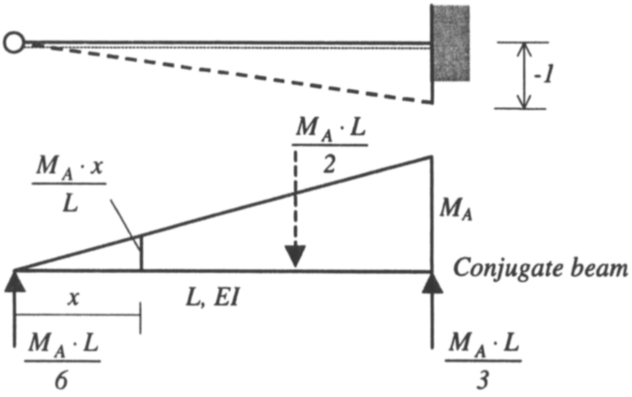


Figure 9.24: Loading of an element by unit displacement

The equation of bending moments on the conjugate beam due to the bending moment loading from the original beam is (see Ch. 6.3):

$$EI \cdot \overline{M} = \frac{M_A \cdot L}{6} \cdot x - \frac{M_A \cdot x}{L} \cdot \frac{x}{2} \cdot \frac{x}{3} = \frac{M_A \cdot L}{6} \cdot x - \frac{M_A \cdot x^3}{6 \cdot L} \tag{9.36}$$

$$y = \frac{M_A}{6EI} \cdot \left(L \cdot x - \frac{x^3}{L} \right)$$

$$(9.37)$$

$$y = \frac{M_A}{6EI} \cdot \left(L^2 \cdot \xi - \xi^3 \frac{L^3}{L} \right) = \frac{M_A \cdot L^2}{6EI} \cdot \left(\xi - \xi^3 \right)$$

The terms inside brackets are so called ω numbers (i.e. $\omega = \xi - \xi^3$) that can be found in several books tabulated for different divisions of elements. However, practically in the evaluation of these ω numbers difficulties occur as this term has to be multiplied by a coefficient outside brackets dependant on different boundary conditions.

Due to the above mentioned difficulties a new method has been derived using so called ψ functions, otherwise based on the preceding method as follows:

It should be noted that after the solution of an indeterminate structure, the support moments differ, since ω numbers relate to *fixed-end moments* that are for a unit displacement

$$M_{50} = \frac{3EI}{L^2} \cdot \Delta = \frac{3EI}{L^2} \qquad \textit{for a fixed-pinned element } m_5$$

$$M_{60} = \frac{6EI}{L^2} \cdot \Delta = \frac{6EI}{L^2} \qquad \textit{for a fixed end element } m_6, \tag{9.38}$$

equation (9.37) is rearranged as follows:

$$\psi = \frac{M_{50} \cdot L^2}{6EI} \cdot \left(\xi - \xi^3\right) = \frac{3EI \cdot L^2}{L^2 \cdot 6EI} \cdot \left(\xi - \xi^3\right) \cdot \frac{M_A}{M_{50}} = 0.5 \cdot \left(\xi - \xi^3\right) \cdot \frac{M_A}{M_{50}}$$

$$\psi_1 = 0.5 \cdot \left(\xi - \xi^3\right) \cdot \frac{M_A}{M_{50}}, \tag{9.39}$$

M_A is here the *actual moment* at the support and M_0 is the moment due to a unit displacement.

Equation (9.39) is dimensionless; therefore the *basic straight line* from the kinematics chain can be directly added:

$$\psi_3 = \xi + 0.5 \cdot \left(\xi - \xi^3\right) \cdot \frac{M_A}{M_{50}} \tag{9.40}$$

Using the same procedure ψ functions for fixed elements of type "*h*" can also be determined for all possible loads and all ψ functions are given in appendices in tables *B.1-B.3*.

Example 9.9: Determine influence lines for the continuous beam from Fig. 9.25

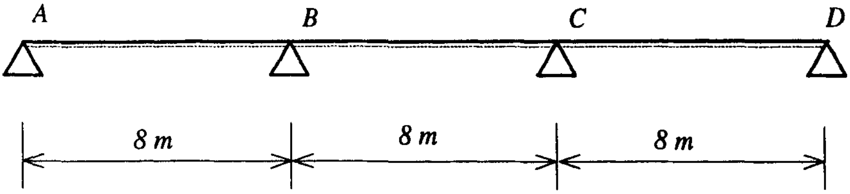


Figure 9.25: Continuous beams over three spans

Let us at first determine influence lines for reactions Y_A and Y_B (influence lines are the same for Y_C and Y_D due to symmetry).

Perform an unit displacement at A and solve two indeterminate systems for unknown rotations φ_B and φ_C:

$$a_{BB} \cdot \varphi_B + a_{BC} \cdot \varphi_C + a_{B0} = 0$$

$$a_{CB} \cdot \varphi_B + a_{CC} \cdot \varphi_C + a_{C0} = 0$$

$$a_{BB} = a_{CC} = \frac{3EI}{8} + \frac{4EI}{8} = \frac{7EI}{8}$$

$$a_{BC} = \frac{2EI}{8}$$

$$a_{B0} = -\frac{3 \cdot EI}{L^2} = -\frac{3 \cdot EI}{8^2} = -\frac{3EI}{64}$$

$$\varphi_{C0} = 0$$

$$\begin{bmatrix} 7 & 2 \\ 2 & 7 \end{bmatrix} \cdot \begin{Bmatrix} \varphi_B \\ \varphi_C \end{Bmatrix} = \begin{Bmatrix} 0.375 \\ 0 \end{Bmatrix}$$

$$\varphi_B = 0.0583 \qquad \varphi_C = -0.0167$$

From known rotations moments at B and C can be calculated; suppose that the bending stiffness $EI = 10000\ kNm^2$:

$$M_{BA} = -468.75 + \frac{3 \cdot EI}{L} \cdot \varphi_B = -468.75 + \frac{3 \cdot 10000}{8} \cdot 0.0583 = -250.0 \ kNm$$

$$M_{CD} = \frac{3 \cdot EI}{L} \cdot \varphi_C = \frac{3 \cdot 10000}{8} \cdot -0.0167 = -62.5 \ kNm$$

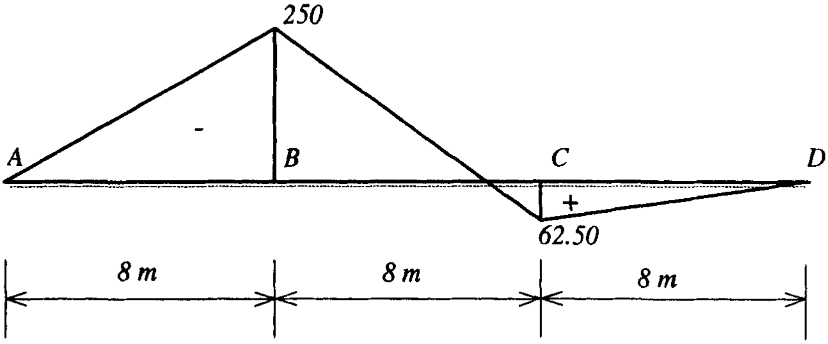


Figure 9.26: Diagram of moments due to the displacement of support *A*

In a similar way solve for the structure due to the displacement at *B*; note that only load coefficients change:

$$a_{BO} = \frac{3 \cdot EI}{L^2} - \frac{6 \cdot EI}{L^2} = -\frac{3EI}{64}$$

$$\varphi_{CO} = -\frac{6 \cdot EI}{L^2} = -\frac{6EI}{64}$$

$$\begin{bmatrix} 7 & 2 \\ 2 & 7 \end{bmatrix} \cdot \begin{Bmatrix} \varphi_B \\ \varphi_C \end{Bmatrix} = \begin{Bmatrix} 0.375 \\ 0.750 \end{Bmatrix}$$

$$\varphi_B = 0.025 \qquad \varphi_C = 0.100$$

$$M_{BA} = 468.75 + \frac{3 \cdot EI}{L} \cdot \varphi_B = 562.50 \ kNm$$

$$M_{CD} = \frac{3 \cdot EI}{L} \cdot \varphi_C = 376.50 \ kNm$$

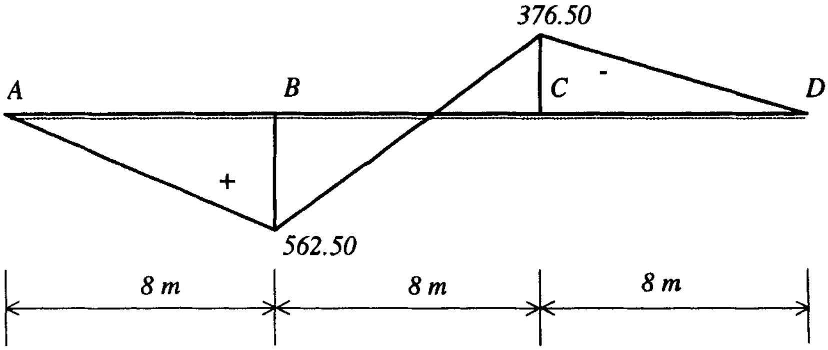


Figure 9.27: Diagram of moments due to the displacement of support *B*

Let us now determine the influence *line for the reaction at support A* using ψ functions.

1. *Field AB*:

$$M_{50} = \frac{3EI}{L^2} \cdot \Delta = \frac{3 \cdot 10000}{8^2} \cdot 1 = 468.75$$

$$K_A = \frac{M_A}{M_{50}} = \frac{250}{468.75} = 0.533$$

$$\psi_2 = 1 - \xi - 0.5(\xi - \xi^3) \cdot K_A = 1 - 1.267\xi + 0.267\xi^3$$

2. *Field BC*:

$$M_{60} = \frac{6EI}{L^2} \cdot \Delta = \frac{6 \cdot 10000}{8^2} \cdot 1 = 937.50$$

$$K_B = \frac{M_B}{M_{60}} = \frac{-250}{937.50} = -0.267$$

$$K_C = \frac{M_C}{M_{60}} = \frac{62.50}{937.50} = 0.067$$

$$\psi_{13} = \psi_7 + \psi_{10} = (2\xi - 3\xi^2 + \xi^3) \cdot K_B + (\xi - \xi^3) \cdot K_C$$

3. *Field CD*:

$$M_{50} = \frac{3EI}{L^2} \cdot \Delta = \frac{3 \cdot 10000}{8^2} \cdot 1 = 468.75$$

$$K_A = \frac{M_A}{M_{50}} = \frac{62.50}{468.75} = 0.133$$

$$\psi_4 = 0.5(2\xi - 3\xi^2 + \xi^3) \cdot K_A$$

Calculated values for each field in steps of $0.1 \cdot L$ are drawn in the diagram, which represents the *influence line for* Y_A (Figure 9.28).

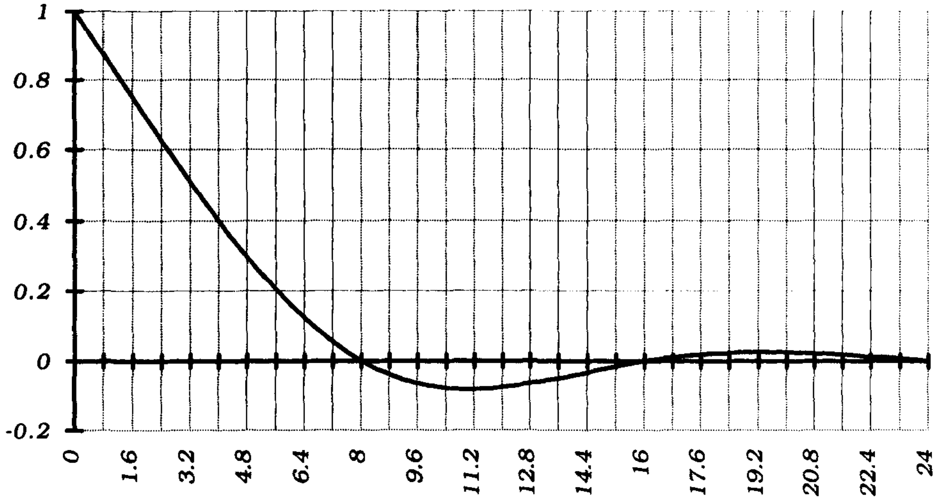


Figure 9.28: Influence lines for the reaction Y_A

Using the same procedure the influence line for the reaction at support *B* can be determined as shown in Fig. 9.29.

As soon as the influence lines for reactions are determined, influence lines for shear forces can be directly constructed from the values of influence lines for reactions. At the point of the searched shear force, the reaction influence line is shifted by unit, as shown in Fig. 9.30 for the distance of *3.2 m* from support *A* in field one and in Fig. 9.31 for a distance of *12.0 m* from support *A* in field two.

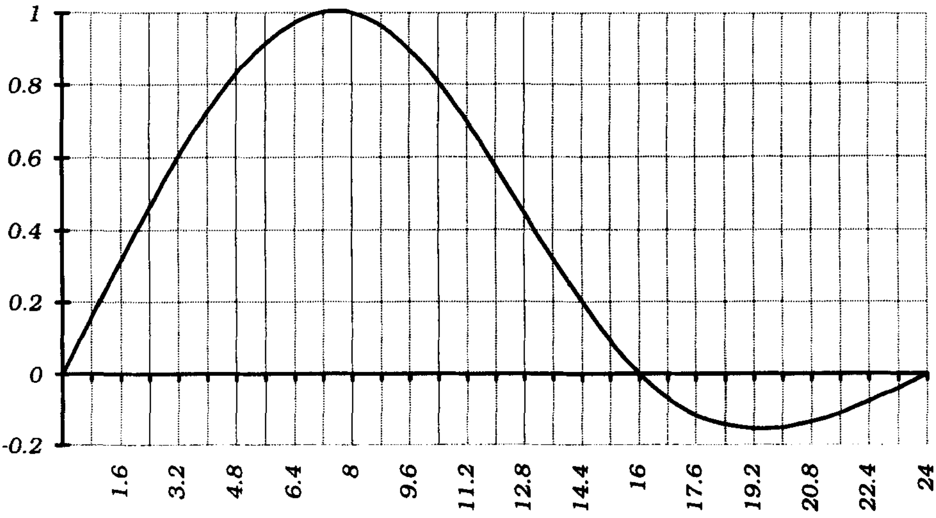


Figure 9.29: Influence line for the reaction Y_B

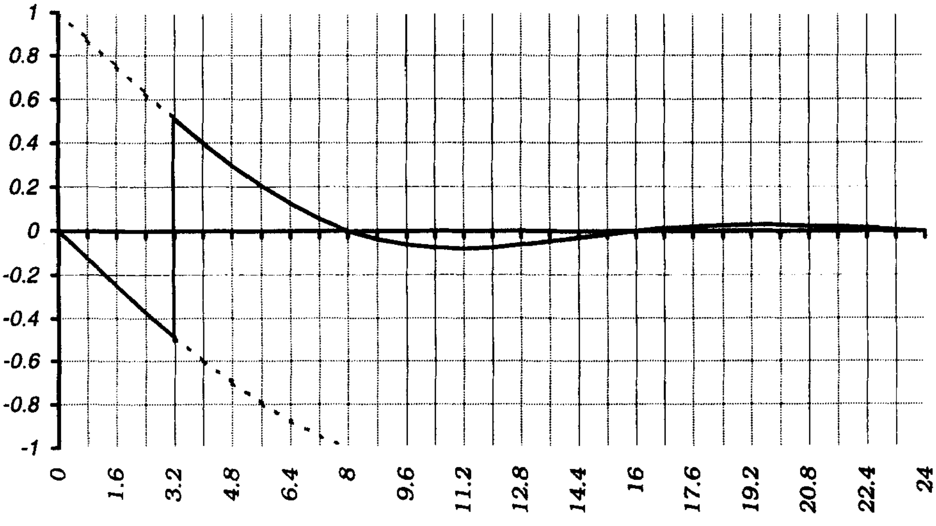


Figure 9.30: Influence line for the shear force $Q_{3.2}$

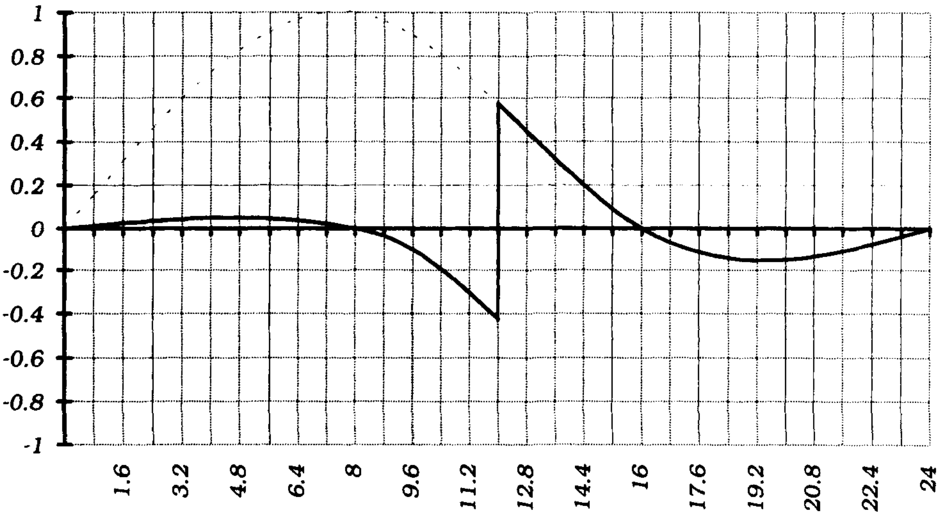


Figure 9.31: Influence line for the shear force $Q_{12.0}$

The influence lines for *moments at supports* are determined by the use of reaction influence lines that are multiplied by an appropriate span between supports, in our case by $L = 8.00\ m$.

Figure 9.32 represents the influence line for the bending moment at support *B*; first, field values of the indeterminate system were subtracted from the basic unit line on the kinematics chain *AB* and the difference was multiplied by the span between supports. In the other two fields no kinematic movement occurs, therefore reaction values are simply multiplied by the value of span between supports.

Influence lines for field *bending moments at outer fields* are again determined from reaction influence lines, a value for a chosen point is determined in the following way (Figure 9.33):

- First determine the value of the reaction *left of the chosen point*, in our example for a point *3.2 m* from the support, the value of reaction is *0.51*;
- The moment at that point is the reaction multiplied by the distance(*3.2 m)*;
- All moments *right of the chosen point* are simply *3.2* times the value of the reaction influence line Y_A;
- All moments *left of the chosen point* are reaction value *0.51* times the distance to the chosen point, from which moments of the indeterminate system has to be subtracted.

As an example calculate the value at a distance *2.4 m*:

$$(M_{3.2})_{2.4}= 0.51 \cdot 2.4 - (0.8-0.749) = 1.173,$$

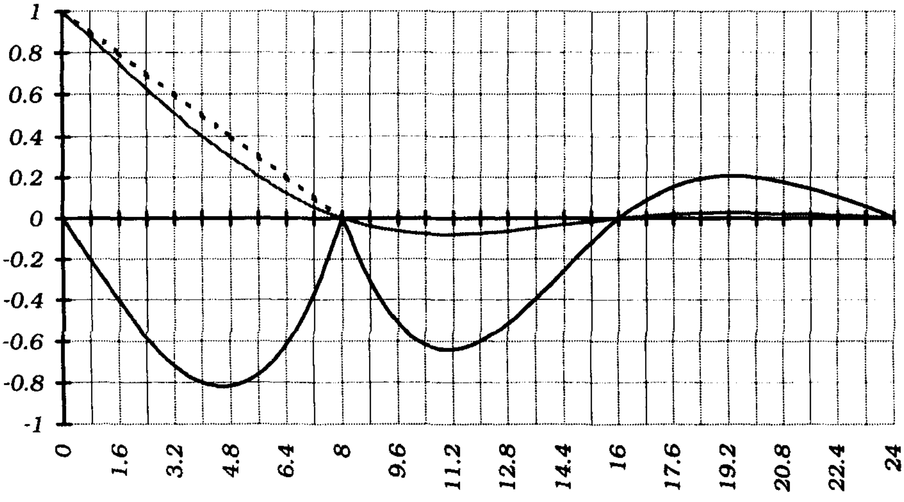


Figure 9.32: Influence line for the support moment $M_{8.0}$

where quantity *0.8* represents the value of the basic line on the kinematics chain, quantity *0.749* is the value of influence line Y_A at point *2.4 m* from the support.

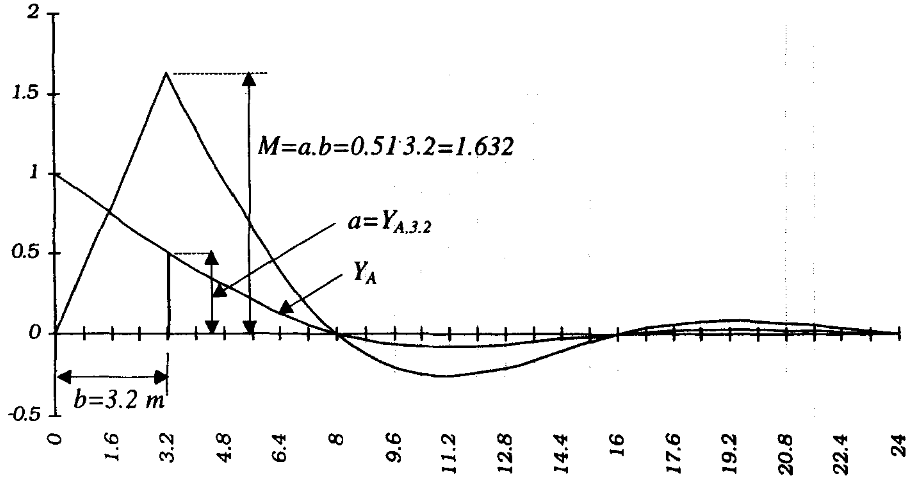


Figure 9.33: Construction of influence line for the field moment $M_{3.2}$

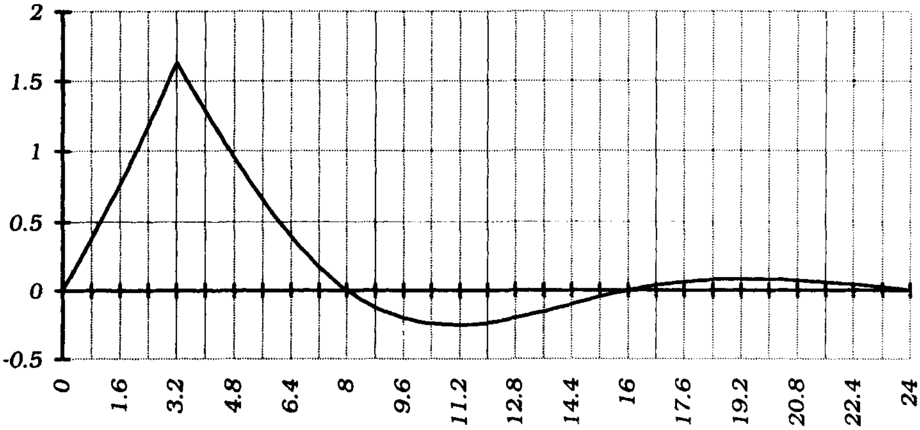


Figure 9.34: Influence line for the field moment $M_{3.2}$

Influence lines for field *bending moments at inner fields* are again determined from reaction influence lines, a value for a chosen point is determined in the following way (Figure 9.35):

- At first determine the value of *all reactions left of the chosen point*, in our example for a point at *11.2 m*, they are $Y_B = 0.695$ and $Y_A = -0.080$;
- The moment at that point is the sum of the reaction times the distance, that is

$$3.2\ 0.695 + 11.2 \cdot (-0.080) = 1.327;$$

- All moments *right of the chosen point* are simply the sum of multiplication between the reaction values left of the chosen point ($Y_B = 0.695$, $Y_A = -0.080$) and appropriate distances *3.2* and *11.2*;
- All moments *left of the chosen point* are simply the sum of multiplication between the reaction values right of the chosen point ($Y_C = 0.447$, $Y_D = -0.064$) and appropriate distances *4.8* and *12.8*;

$$M_{11.2} = 0.447 \cdot 4.8 + 12.8 \cdot (-0.064) = 1.326;$$

9.7.2 *Evaluation of influence lines*

Influence lines were determined using the Mueller-Breslau principle therefore they represent corresponding forces for unit loading.

A searched value is hence determined if a value for a unit force is multiplied by the actual value of the force at that position. If loading is continuous then the intensity of the loading is multipied by the area of an influence line under the loading.

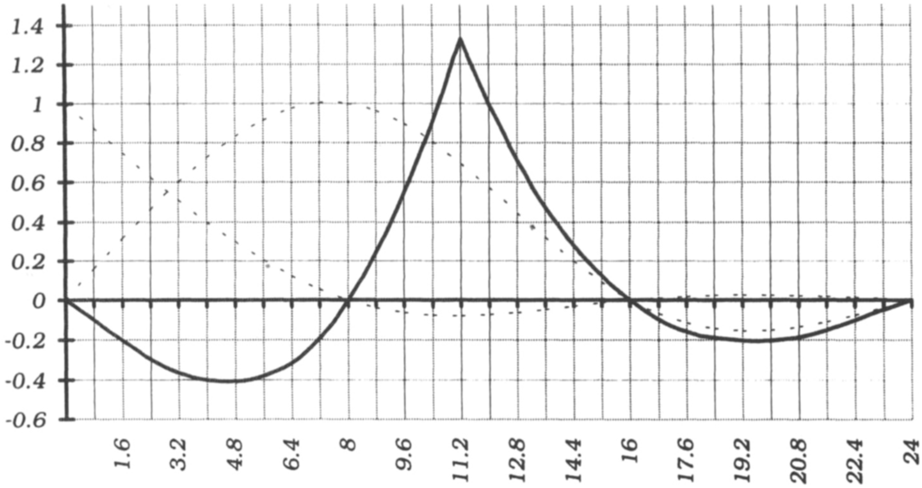


Figure 9.35: Influence line for the field moment $M_{II,2}$

Example 9.10: Evaluation of the reaction Y_A and moment M_B for the continuous beam from example 9.9

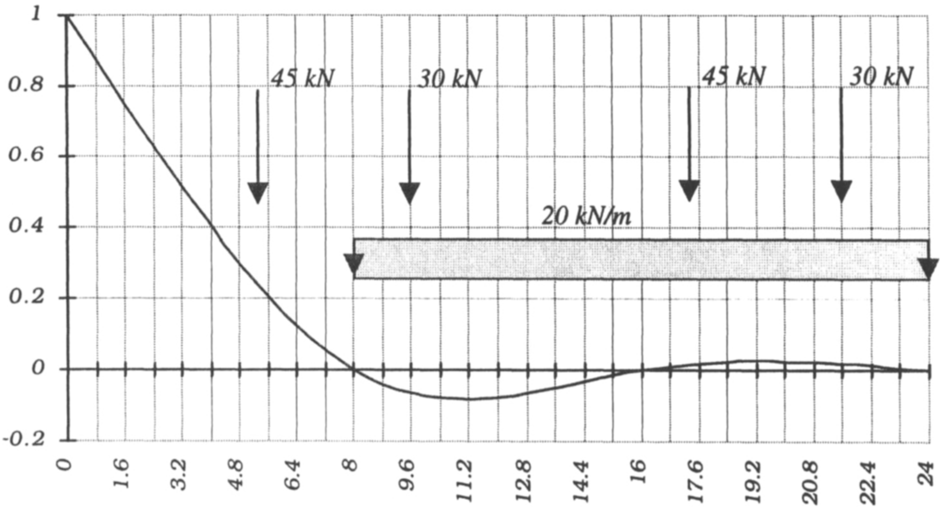


Figure 9.36: Position of loads for the evaluation of the reaction Y_A

At first we calculate the reaction for a concentrated force:

$$(Y_A)_F = 45\cdot 0.23 + 30(-0.05) + 45\cdot 0.023 + 30\cdot 0.015 = 10.335\ kN$$

For uniform loading we will calculate an *approximate value*, as the influence line will be taken as a parabola (which in general is not true):

$$(Y_A)q = 20\cdot(2/3).8\cdot(-0.08) + 20.(2/3)\cdot 8\cdot 0.026 = -5.760\ kN$$

The total reaction for all loads in Fig. 9.36 is:

$$Y_A = 10.335 - 5.760 = 4.575\ kN$$

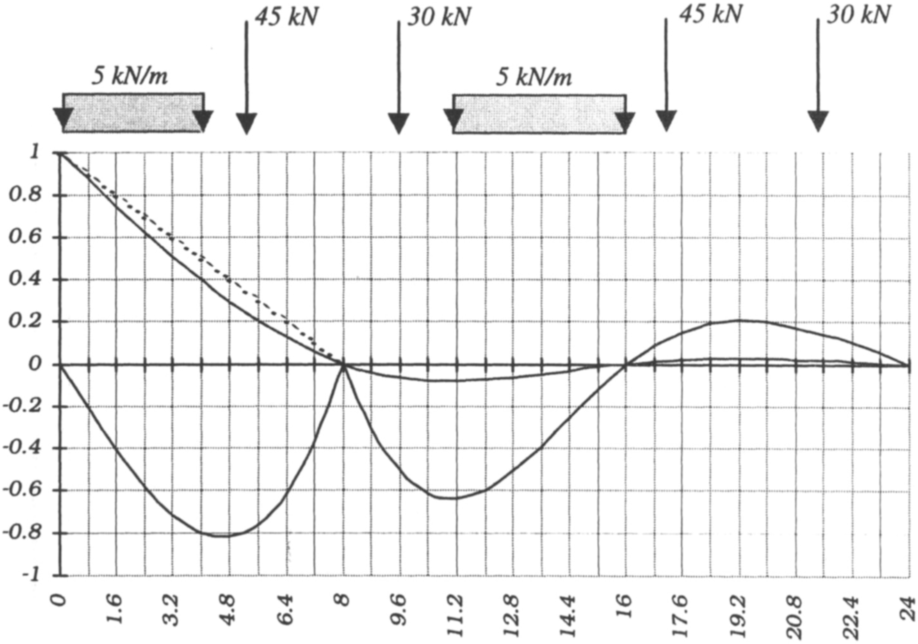


Figure 9.37: Position of loads for the evaluation of the support moment $M_{8.0}$

The exact area of influence line can be obtained by the integration of ψ functions in the region of loading. Alternatively, as ψ functions are suitable for programming, it is easy to calculate values, using a computer, in desirable small steps and to evaluate maximum values for different positions of moving loads.

Calculate now moment M_B at support B for the position of loads from Fig. 9.37. Moment for concentrated forces is (measured from the diagram):

$$(M_B)_F = 45\cdot(-0.79+0.13) + 30(-0.52+0.15) = -40.800\ kNm$$

An approximate value is calculated for uniform load:

$$(M_B)q=5\cdot(4\cdot(-0.81)/2+(2/3)\cdot 4\cdot(-0.11)) +5\cdot 4.8\cdot(-0.63)/2= -17.127\ kNm$$

The total moment is therefore for the loading in Fig. 9.37:

$$M_B = -40.050 - 17.133 = -57.927\ kNm$$

10

The Finite Element Method

10.1 Introduction

The finite element method is an extension of the matrix displacement method, which has been explained in Chapters 9.3 and 9.4 and is applicable to any solid body and not only to straight prismatic bars as in the displacement method in Ch. 9.

The generalisation depends on discretisation of the body into finite elements and on feasibility to calculate stiffness matrices for these finite elements.

10.2 Basic concept

The basic idea is shown in Fig. 10.1; the body to be analysed is modelled as an assembly of finite elements interconnected at specified joints or nodal points.

In the case of the plane stress body shown in Fig. 10.1, a division (or decomposition) into triangular elements is a good choice because of the ease with which such elements can simulate irregular boundaries even though triangles have straight edges. Elements are interconnected to each other at a joint only; the connection can be done by compatible displacements (the force method) or by forces in equilibrium (the displacement method).

A comparison between both methods in Chapters 8 and 9 show that the displacement method is superior to the force method, above all because of its convenient automatic treatment and is thus applicable to computer applications. In the further text the displacement finite element method will be only treated.

An essential requirement of a numerical method is that of convergence; that is, the solution should come closer and closer to the correct one of the prototype structure with increasingly finer subdivision into finite elements. The reason for inaccuracy is in the fact that elements are interconnected at joints only and not along the common edge. A choice of an interpolation function will therefore play a major role in the accuracy of an element.

Displacements of common joints are unknown quantities hence a numbering of joints should be done in accordance with corresponding displacements at joints.

Let us now assume that it is possible to determine structural stiffness values or better a relation between displacements and corresponding forces (see Ch. 9.2) by the equation:

$$\{X\}=[K]\cdot\{\Delta\} \tag{10.1}$$

At each joint, forces and their corresponding displacements will exist as conjugate quantities. If a structure has f degrees of freedom then the equations for unknown displacements are written first and after that the equations for known displacements (usually at supports) in a partitioned form (Eqn. 10.2).

$$\begin{array}{ll} \textit{Known forces} \Rightarrow \\ \textit{Reactions} \Rightarrow \end{array} \begin{Bmatrix} X_\alpha \\ X_\beta \end{Bmatrix} = \begin{bmatrix} K_{\alpha\alpha} & K_{\alpha\beta} \\ K_{\beta\alpha} & K_{\beta\beta} \end{bmatrix} \cdot \begin{Bmatrix} \Delta_\alpha \\ \Delta_\beta \end{Bmatrix} \begin{array}{ll} \Rightarrow \textit{Unknown displacements} \\ \Rightarrow \textit{Known(specified)displacements} \end{array}$$

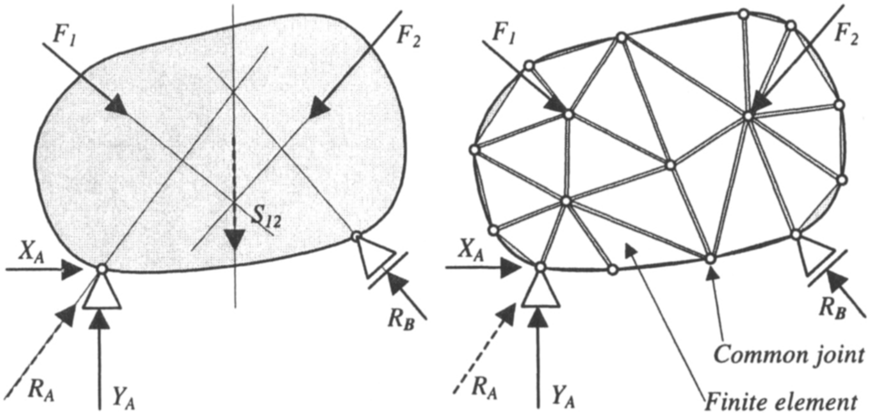


Figure 10.1: Finite element model

The solution for the unknown joint displacements and the unknown joint forces is derived in two steps:

$$\{X_\alpha\}=[K_{\alpha\alpha}]\cdot\{\Delta_\alpha\}+[K_{\alpha\beta}]\cdot\{\Delta_\beta\}, \tag{10.3}$$

from where:

$$\{\Delta_\alpha\}=[K_{\alpha\alpha}]^{-1}\cdot(\{X_\alpha\}-[K_{\alpha\beta}]\cdot\{\Delta_\beta\}) \tag{10.4}$$

The above equation is inserted into the rest of the equations and the unknown forces (reactions) are calculated:

$$\{X_\beta\} = \lfloor K_{\beta\alpha} \rfloor \cdot \{\Delta_\alpha\} + \lfloor K_{\beta\beta} \rfloor \cdot \{\Delta_\beta\} \qquad (10.5)$$

If all $\{\Delta_\beta\}$ equal zero (at unmovable supports) the equations are simplified as:

$$\{\Delta_\alpha\} = [K_{\alpha\alpha}]^{-1} \cdot \{X_\alpha\} \qquad (10.6)$$

and

$$\{X_\beta\} = \lfloor K_{\beta\alpha} \rfloor \cdot \{\Delta_\alpha\} \qquad (10.7)$$

The solution of above equations requires a matrix inversion[♥] of order of unknown displacements that can be, in realistic problems, in the order of *10000* or more. It is apparent that such problems can be only tackled with a help of adequate computing power. Once the reactions and displacements have been calculated it only remains to solve for internal stresses and strains by the use of element stiffness matrices.

10.3 A derivation of the element stiffness matrices

The principle of a minimum potential energy (π) is used, satisfying:

-Equations of elasticity $\{\sigma\} = [E] \cdot \{\varepsilon\}$ and
-Equilibrium equations

Total potential energy is a sum of work done by internal and external forces:

$$\pi = U - W, \qquad (10.8)$$

where the internal work done is due to element deformations

$$U = \int_V \sigma \cdot \varepsilon$$

and the external work due to forces displacements

$$W = \sum F_i \cdot \Delta_i$$

[♥] *Structural Analysis Package OCEAN, described in Ch. 13, uses for the solution of equations the frontal method and not matrix inversion.*

The total work is hence

$$\pi = \int_V \sigma^T \cdot \varepsilon \cdot dV - \{F\}^T \{u\} \tag{10.9}$$

Let us describe strains and element displacements through an interpolation matrix $[B]$ and stresses and strains by a matrix of material properties $[D]$

$$\{\varepsilon\} = [B] \cdot \{u_i\} \tag{10.10}$$

$$\{\sigma\} = [D] \cdot \{\varepsilon\} \tag{10.11}$$

and insert the above equations into the equation for potential energy:

$$\pi = \int \{u\}^T \cdot [B]^T \cdot [D] \cdot [B] \cdot dV \cdot \{u\} - \{F\} \cdot \{u\}^T \tag{10.12}$$

A structure is in equilibrium when the potential energy has the smallest value, a derivation to the displacement gives:

$$\frac{\partial \pi}{\partial u_i} = 0$$

$$\pi = \int [B]^T \cdot [D] \cdot [B] \cdot dV \cdot \{u_i\} - \{F\} = 0 \tag{10.13}$$

It is obvious that the above equation has the same form as

$$\{F\} = [K] \cdot \{u\}, \tag{10.14}$$

where a stiffness matrix has a different value:

$$[K] = \int_V [B]^T \cdot [D] \cdot [B] \cdot dV \tag{10.15}$$

If we succeed in obtaining the stiffness matrix $[K]$ for an element, a further procedure is exactly the same as with the displacement method.

10.4 Stiffness matrix of a prismatic element

As an example of stiffness matrix derivation we shall derive a stiffness matrix for a *straight bending prismatic element*, which was actually already derived in Ch. 7 by the use of Castiglian's theorem.
A stiffnes matrix formulation from equation (10.15) can be for this element written as:

$$[K] = \int [\varepsilon]^T \cdot [\sigma] \cdot dx\,, \tag{10.15a}$$

where $[\varepsilon]$ represents virtual strains and $[\sigma]$ real stresses.

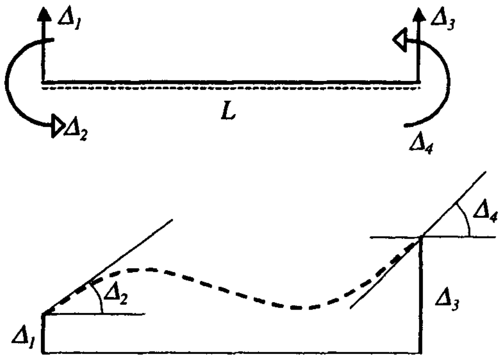


Figure 10.2: Straight bending prismatic element

- *Step 1*: To describe the deformed element shape from Fig. 10.2, we assume a four-term displacement function

$$v = \alpha_1 + \alpha_2 \cdot x + \alpha_3 \cdot x^2 + \alpha_4 \cdot x^3 \tag{10.16}$$

As the joint displacements also include end rotations we need the slope, which is obtained by differentiation of the assumed displacement function and both equations are written in a matrix form:

$$\varphi = \frac{dv}{dx} = \alpha_2 + 2 \cdot \alpha_3 \cdot x + 3 \cdot \alpha_4 \cdot x^2 \tag{10.17}$$

$$\begin{Bmatrix} v \\ \dfrac{dv}{dx} \end{Bmatrix} = \begin{bmatrix} 1 & x & x^2 & x^3 \\ 0 & 1 & 2x & 3x^2 \end{bmatrix} \cdot \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{Bmatrix} \tag{10.18}$$

or shorter

$$\{u(x)\} = [N(x)] \cdot \{\alpha\} \tag{10.19}$$

The matrix $[N]$ is a function of position along the beam and is therefore called a position or field matrix.

❖ *Step 2*: Calculate the joint displacements $\{\Delta\}$ in terms of the coefficients $\{\alpha\}$ for all four possible load cases by inserting appropriate co-ordinates of the element joints (that is the beam ends)

$$\begin{Bmatrix} \Delta_1 \\ \Delta_2 \\ \Delta_3 \\ \Delta_4 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & L & L^2 & L^3 \\ 0 & 1 & 2L & 3L^2 \end{bmatrix} \cdot \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{Bmatrix}$$

or

$$\{\Delta\} = [A] \cdot \{\alpha\}, \tag{10.20}$$

where the matrix $[A]$ depends on joint co-ordinates only.

For the determination of the complete stiffness matrix, the coefficients are dependent on all four virtual displacements, applied one at a time:

$$[\Delta] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = [I]$$

$$\{\alpha\} = [A]^{-1} \cdot \{\Delta\} = [A]^{-1} \cdot [I] = [A]^{-1} \tag{10.21}$$

A matrix inversion is possible if a matrix is a square matrix; that is, the number of coefficients $\{\alpha\}$ *must match the number of possible displacements* at joints.

$$[A]^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -\frac{3}{L^2} & \frac{2}{L} & +\frac{3}{L^2} & -\frac{1}{L} \\ +\frac{2}{L^3} & \frac{1}{L^2} & -\frac{2}{L^3} & +\frac{1}{L^2} \end{bmatrix} \tag{10.22}$$

- *Step 3*: Calculate the internal strains $\{\varepsilon\} = [B] \cdot \{u_i\}$ as a function of position. In beams under bending these distortions are conveniently represented by the curvature $[\phi]$ of a beam, which can be obtained from displacements by two successive differentiations of Eqn. (10.16):

$$[\phi(x)] = \frac{d^2v}{dx^2} = [0 \quad 0 \quad 2 \quad 6x] \cdot [\alpha]$$

or

$$\{\varepsilon(x)\} = [B(x)] \cdot \{\alpha\}, \tag{10.23}$$

where the matrix $[B(x)]$ relates the internal strains $\{\varepsilon(x)\}$ to the coefficients $[\alpha]$. Insert equation (10.21) into (10.23)

$$\{\varepsilon(x)\} = [B(x)] \cdot [A]^{-1} \tag{10.24}$$

and rewrite for further use as:

$$\{\varepsilon(x)\}^T = [[B(x)] \cdot [A]^{-1}]^T = [[A]^{-1}]^T \cdot [B(x)]^T \cdot \tag{10.25}$$

- *Step 4*: The internal forces $[\sigma(x)]$, which are also functions of position, can be represented by bending moments $[M(x)]$. The moments are linearly related to the curvature through the beam stiffness *EI* (see Eqns. 2.51 and 3.20):

$$[M] = EI \cdot [\phi]$$

or from Hook's law

$$[\sigma(x)] = [D] \cdot [\varepsilon(x)], \tag{10.26}$$

The matrix $[D]$ is in general of complicated form but in our case it is just a scalar quantity (bending stiffness)

$$[D]=EI \tag{10.27}$$

Equation (10.24) is inserted into (10.27) and a relation between moments due to the unit joint displacements is:

$$[\sigma(x)]=[D]\cdot[B(x)]\cdot[A]^{-1} \tag{10.28}$$

❖ *Step 5*: Now insert equations (10.25) and (10.28) into (10.15a). Since the matrix $[A]^{-1}$ is not a function of position it can be taken outside the integral:

$$[k]=[A^{-1}]^T\int_L[B(x)]^T[D]\cdot[B(x)]\cdot dx\cdot[A^{-1}] \tag{10.29}$$

We perform the indicated matrix operations, integrate term by term, and finally arrive at the result:

$$[k]=\frac{EI}{L^3}\cdot\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}$$

The result is the same as derived in Ch. 7 in Fig. 7.12, which is not surprising, as the result is an exact solution of the beam theory since the displacement function $\{u\}$ was chosen to match the differential beam equation.

The theory is based on the principle of minimum potential energy and the solution, using other elements, should become better as the elements become smaller.

Let us emphasise that the displacements are exact at joints only, values between joints depend on the choice of the displacement function and hence the choice of an element in finite element analysis for an analysis of different structures is of the greatest importance.

10.5 Transformations

We have already shown in Ch. 9 that the displacement method is suitable for computer applications but rather difficult for longhand calculations, as the assemblage of individual element stiffness into the structure stiffness matrix is rather involved.

A generality of the method is obtained if a *local co-ordinate system* of individual elements is introduced and before assembling the structural stiffness matrix all quantities are transformed into the *global co-ordinate system.*

We shall consider here a plane system, but the procedure can easily be applied to a general space system.

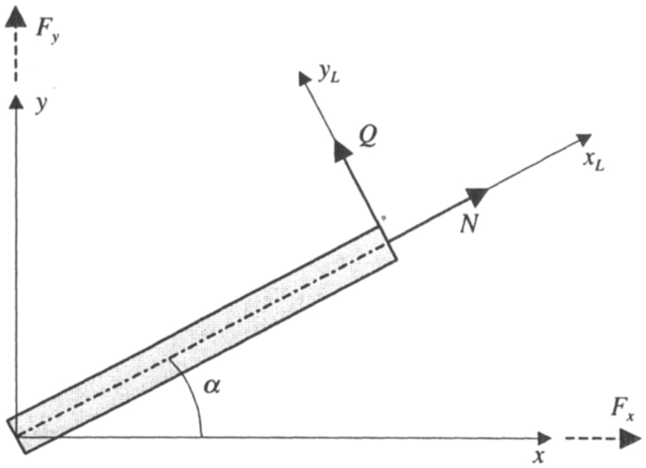


Figure 10.3: Local co-ordinate system

The projection of internal forces on global co-ordinate axis gives:

$$\begin{aligned} F_X &= N \cdot cos\alpha - Q \cdot sin\alpha \\ F_Y &= N \cdot sin\alpha + Q \cdot cos\alpha \end{aligned} \tag{10.31}$$

or written in a matrix form:

$$\begin{Bmatrix} F_X \\ F_Y \end{Bmatrix} = \begin{bmatrix} cos\alpha & -sin\alpha \\ +sin\alpha & cos\alpha \end{bmatrix} \cdot \begin{Bmatrix} N \\ Q \end{Bmatrix} \tag{10.32}$$

Now we transform forces from global into the local co-ordinate system:

$$\begin{Bmatrix} N \\ Q \end{Bmatrix} = \begin{bmatrix} cos\alpha & +sin\alpha \\ -sin\alpha & cos\alpha \end{bmatrix} \cdot \begin{Bmatrix} F_X \\ F_Y \end{Bmatrix} \tag{10.33}$$

$$\begin{Bmatrix} N \\ Q \end{Bmatrix} = [T] \cdot \begin{Bmatrix} F_X \\ F_Y \end{Bmatrix}$$

or

$$\begin{gathered}\{\bar{F}\}=[T]\cdot\{F\}\\ \{F\}=[T]^{-1}\cdot\{\bar{F}\}=[T]^{T}\cdot\{\bar{F}\}\end{gathered}\tag{10.34}$$

The matrix $[T]$ is a *transformation matrix*, with a crossbar denoting forces given in a local co-ordinate system. In the same way local displacements can be transformed into global displacements using the following equation:

$$\begin{Bmatrix} X_L \\ Y_L \end{Bmatrix}=[T]\cdot\begin{Bmatrix} X \\ Y \end{Bmatrix} \qquad \{\bar{\Delta}\}=[T]\cdot\{\Delta\}\tag{10.35}$$

Example 10.1: For a truss element (pinned at both ends) from Fig. 10.4 calculate the stiffness matrix $[\bar{k}]$ in the local co-ordinate system, transformation matrix $[T]$ and stiffness matrix $[K]$ of the element in the global co-ordinate system.

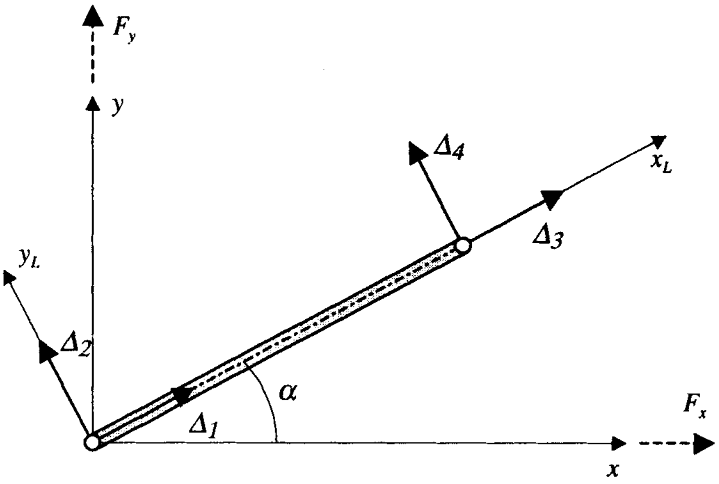


Figure 10.4: A truss element

The stiffness matrix of a truss element is:

$$
\left[\bar{k}\right]=\begin{array}{c}\\ 1\\ 2\\ 3\\ 4\end{array}\!\!\begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4\end{array}\\ \left[\begin{array}{cc|cc} \frac{EA}{L} & 0 & -\frac{EA}{L} & 0\\ 0 & 0 & 0 & 0\\ \hline -\frac{EA}{L} & 0 & \frac{EA}{L} & 0\\ 0 & 0 & 0 & 0 \end{array}\right]\end{array}=\frac{EA}{L}\cdot\left[\begin{array}{cc|cc} 1 & 0 & -1 & 0\\ 0 & 0 & 0 & 0\\ \hline -1 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 \end{array}\right]
$$

The transformation matrix needs to transform two vectors at each end of the element:

$$
\left[T\right]=\begin{bmatrix}\lambda & 0\\ 0 & \lambda\end{bmatrix}=\left[\begin{array}{cc|cc} \cos\alpha & \sin\alpha & 0 & 0\\ -\sin\alpha & \cos\alpha & 0 & 0\\ \hline 0 & 0 & \cos\alpha & \sin\alpha\\ 0 & 0 & -\sin\alpha & \cos\alpha \end{array}\right]
$$

The stiffness matrix is then obtained by the following procedure:

$$
\left\{\bar{\Delta}\right\}=\left[T\right]\cdot\left\{\Delta\right\}
$$

$$
\left\{\bar{F}\right\}=\left[k\right]\cdot\left\{\bar{\Delta}\right\}
$$

$$
\left\{F\right\}=\left[T\right]^T\cdot\left\{\bar{F}\right\}=\left[T\right]^T\cdot\left[k\right]\cdot\left\{\bar{\Delta}\right\}=\left[T\right]^T\cdot\left[k\right]\cdot\left[T\right]\cdot\left\{\Delta\right\}
$$

$$
\left[k\right]=\left[T\right]^T\cdot\left[\bar{k}\right]\cdot\left[T\right]=\frac{EA}{L}\cdot\left[\begin{array}{cc|cc} \cos^2\alpha & & & \\ \sin\alpha\cdot\cos\alpha & \sin^2\alpha & & \\ \hline -\cos^2\alpha & -\sin\alpha\cdot\cos\alpha & \cos^2\alpha & \\ -\sin\alpha\cdot\cos\alpha & -\sin^2\alpha & \sin\alpha\cdot\cos\alpha & \sin^2\alpha \end{array}\right]
$$

10.6 Structure stiffness matrix

For the sake of completeness let us repeat the procedure described in Ch. 9.3. The structure stiffness matrix $\left[K\right]$relates the forces and displacements of a structure composed of elements. The force X_i at the joint and in the direction i is linearly related through the *corresponding displacements* Δ_i by equation:

$$
X_i=k_{i1}\cdot\Delta_1+k_{i2}\cdot\Delta_2+.....+k_{ij}\cdot\Delta_j+.....
$$

$$
\left\{X\right\}=\left[K\right]\cdot\left\{\Delta\right\} \qquad (9.7)
$$

Each element k_{ij} of this stiffness matrix is defined as the force that must be applied to the complete structure at node i to produce unit displacement at node j, all others are kept zero. Consider now one single joint connecting several elements *1,2...m*. Figure 9.4 shows the internal element forces $S_i^{(m)}$ of element *m* along node *i*, as well as their equal and opposite reaction forces on the joint. For instance, the *i-th* force on the *m-th* element is

$$S_i^{(m)} = k_{i1}^{(m)} \cdot \Delta_1 + k_{i2}^{(m)} \cdot \Delta_2 + ...k_{ij}^{(m)} \cdot \Delta_j + ... \tag{9.8}$$

where the j extends over all nodes attached to the element. The equilibrium equation for the forces in the direction i at the joint is

$$\Sigma X_i = 0: \quad -S_{i1} - S_{i2} - S_{im} + F_i = 0\,,$$

from which, inserting Eqn. (9.8), we get

$$\left(k_{i1}^1 + k_{i1}^2 + ... + k_{i1}^m + ...\right) \cdot \Delta_1 + \left(k_{i2}^1 + k_{i2}^2 + ... + k_{i2}^m + ...\right) \cdot \Delta_2 + ... = F_i$$

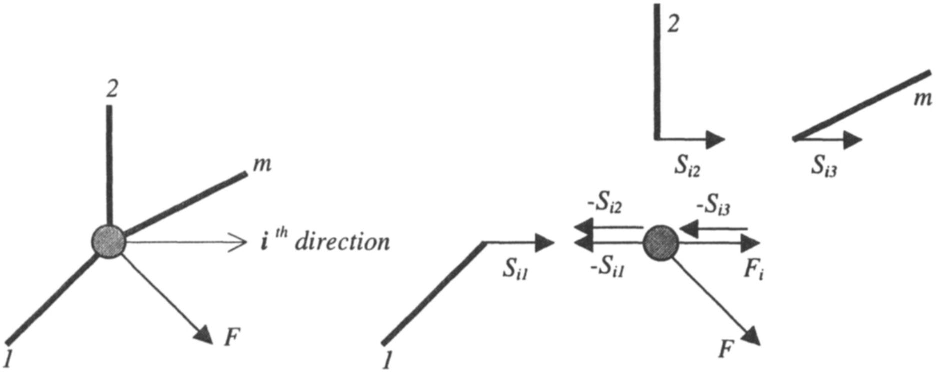


Figure 9.4: Joint and element forces

The structure stiffness coefficient K_{ij} is therefore obtained by the superposition of the element stiffness $k_{ij}^{(m)}$:

$$K_{ij} = k_{ij}^1 + k_{ij}^2 + ... + k_{ij}^m + ... = \sum_m k_{ij}^{(m)} \tag{9.9}$$

It is essential that each node of the structure be carefully labeled, and that the nodal numbering of each element corresponds to that of the structure. The element stiffness matrices are then written and superimposed, or *assembled.*

$$K_{ij} = k_{ij}^1 + k_{ij}^2 + ... + k_{ij}^m + ... = \sum_m k_{ij}^{(m)} \tag{9.10}$$

10.7 Calculation of strains and stresses

After the joint displacements $\{\Delta_\alpha\}$ and the reactions $\{X_\beta\}$ have been determined by use of equations (10.2-10.5), it remains to compute the internal strains and stresses using previously derived equations:

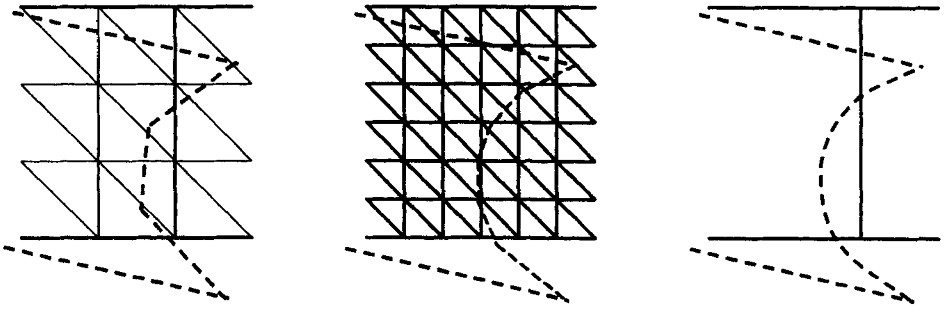
$$[\varepsilon(x)] = [B(x)] \cdot [A]^{-1} \cdot [\Delta]$$

$$[\sigma(x)] = [D] \cdot [B(x)] \cdot [A]^{-1} \cdot [\Delta]$$

Internal forces can now be calculated from stresses and strains using equations derived in Ch. 2.8 for truss and beam elements or in a general case, using the theory of elasticity. Some applications of the finite element method are presented in chapter 12.

10.8 Convergence of results

We have already referred to the fact that the finite element method is a process of establishing equilibrium by minimising potential energy of a geometrically compatible displacement system.



a) Course mesh b) Fine mesh c) Exact solution

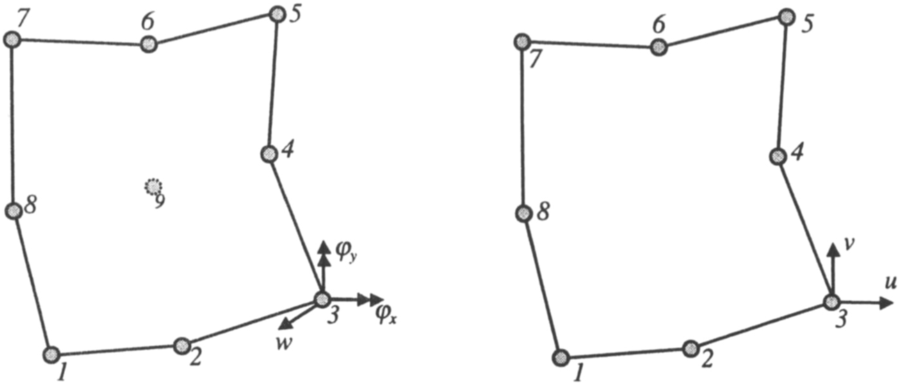
Figure 10.5: Convergence of a solution

For such a system it can be proved that any approximate solution leads to an underestimation of the displacements, practically it means that stiffness of individual elements are too large.

This conclusion is true for all elements (Figure 10.5), where the deformational state *between joints* was not described by a function, which is an exact solution to the differential equation (plane stress, plane strain, 3D elements).

The use of area co-ordinates, higher order polynomials and isoparametric functions improves element qualities substantially. For instance, a computer program for plate bending PLATE uses 9-noded isoparametric elements having 27 degrees of freedom and a program for shear walls calculation SWALL 8-noded isoparametric elements having 16 degrees of freedom (Figure 10.6).

Details of these elements are out of scope of this text and the readers interested in this subject should find details in specialised books on finite elements.



a) 9-noded plate element
27 degrees of freedom

b) 8-noded wall element
16 degrees of freedom

Figure 10.6: Isoparametric elements

In the application of the finite element method to frame and truss structures (OCEAN, modules FRAME, P-TRUSS and S-TRUSS) the interpolation function is an exact solution of the differential equation hence the results are exact (at joints) and any further division into more elements does not improve results. The subdivision of elements is required in influence line calculations or when deformations are required between master joints of the structure.

11

Inelastic material behaviour in structures

It is important to be able to predict the initiation of the inelastic response of materials that are subjected to various stress states. The term inelastic is used to define the material response in relation to the stress-strain diagram that is non-linear and that retains a permanent strain or returns to an unstrained state on complete unloading. The term plastic or plasticity is used to describe the inelastic behaviour of a material that retains a permanent set on complete unloading.

The condition for the initiation of yield in ductile metals, such as structural steels, is discussed in this chapter. The use of uniaxial stress-strain data and their limitations are discussed together with a general description of non-linear material behaviour.

11.1 The use of uniaxial stress-strain data

Material properties are usually obtained under uniaxial conditions based on tension or compression tests. These tests are normally performed at room temperature in a testing machine that has a head speed that is usually below *20* mm/min. Material properties determined from such tests are used in the design of structures although, in practice, structures may be subjected to temperatures much higher or lower than room temperature and at loading rates much higher than that provided by most testing machines. In addition, the shape of a structural member may be such that biaxial or triaxial stresses arise which can be very different from the uniaxial stress experienced by a test specimen.

The stress-strain relationship may be greatly affected by the rate at which a load is applied. If a normal ductile material is considered, then the stress-strain relationship has an elastic range followed by a non-linear inelastic or plastic range. If the loading rate is very high, then the magnitude of the inelastic strain that precedes fracture can be reduced compared to that from normal load rates that are experienced under test conditions. Furthermore, high load rates result in an apparent increase in yield stress and modulus of elasticity. The material response is also less ductile under such conditions and, in the case of extremely high load rates, the response resembles brittleness.

Temperature also has a considerable influence on material behaviour especially when combined with high load rates. For example, if structural metals are subjected to very low temperatures and very high load rates, then brittle fracture might occur. Thus, the selection of materials for applications involving low temperatures and high load rates is important if failures are to be avoided.

If metals are subjected to elevated temperatures, then under constant load, the strain may increase until fracture occurs and this condition is known as creep.

11.2 Non-linear material response

If a specimen of a particular material is tested under tensile conditions, the shape of the stress-strain curve will be dependent on the material. However, if the load is applied and slowly removed, certain features of the stress-strain curve are similar for all materials.

For example, for small loads, the relation between stress and strain is linear. If loading is increased to a sufficiently large value, then the relationship between stress and strain becomes non-linear. The material response may then be classified as elastic or plastic depending on its response to the loading condition as shown in Figs. 11.1(a) and 11.1(b) respectively.

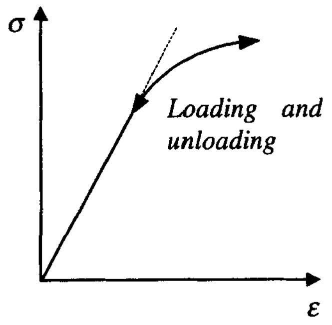


Fig 11.1a: Non-linear elastic response

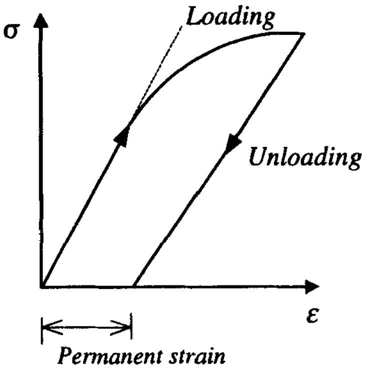
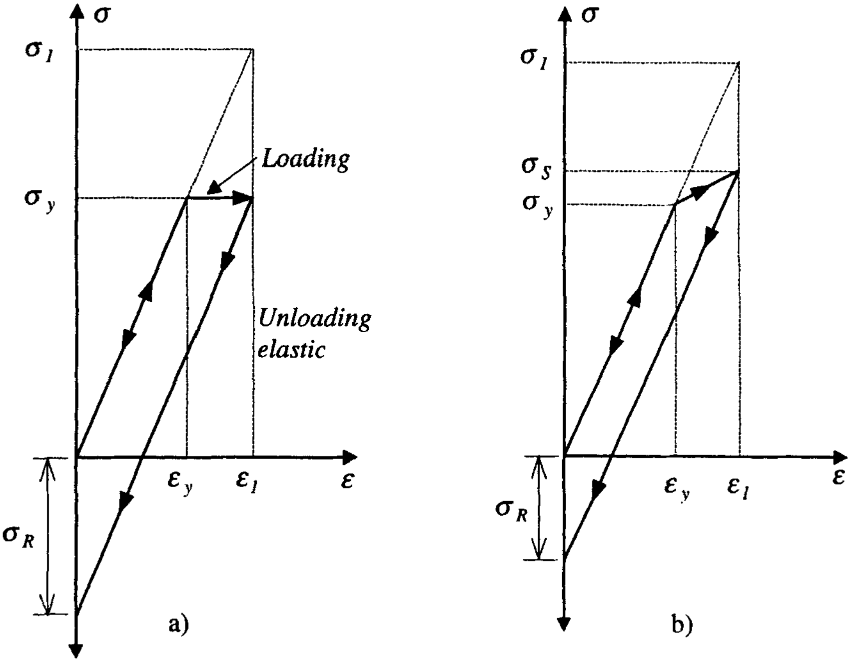


Fig. 11.1b: Non-linear plastic response

If the unloading path coincides with the loading path, the process is reversible and the material is said to be elastic as in Fig. 11.1a. If, however, the unloading path does not follow the loading path, then the behaviour of the material is said to be inelastic as shown in Fig. 11.1b. A material that behaves in a plastic manner does not return to an unstrained state when the load is released.

With some materials, the transition from linear elastic response to inelastic response may be abrupt or gradual. For an abrupt transition, the change occurs at the yield stress, σ_y, as shown in Fig. 11.2a. In the case of a gradual transition, the yield stress is arbitrarily defined as the stress that corresponds to a given permanent strain, ε_s, where usually $\varepsilon_s = 0.002$.

The material stress-strain relationships are idealised for ease of use in calculations. Two idealised models are shown in Fig. 11.2.



Note : $\sigma_y \langle \sigma_1 \langle 2\sigma_y$

Fig. 11.2: Idealised stress-strain curves
(a) Elastic – perfectly plastic response (b) Elastic – strained-hardening response

In the stress-strain relationship given in Fig. 11.2, repeated cycling of the load will lead to the yield stress being increased to σ_l, due to a residual stress, σ_R, being built-up in the material. Thus, $\sigma_1 = \sigma_y + \sigma_R$, and provided σ_R remains less than $-\sigma_y$, the material will behave elastically as cycles are repeated.

If the nominal stress, σ_l, exceeds $2\sigma_y$, then on repeated cycles the material will no longer "shake down" to elastic action and permanent plastic deformation will occur with each cyclic loading. This is shown in Fig. 11.3. Therefore, $2\sigma_y$ is the threshold beyond which some plastic action occurs.

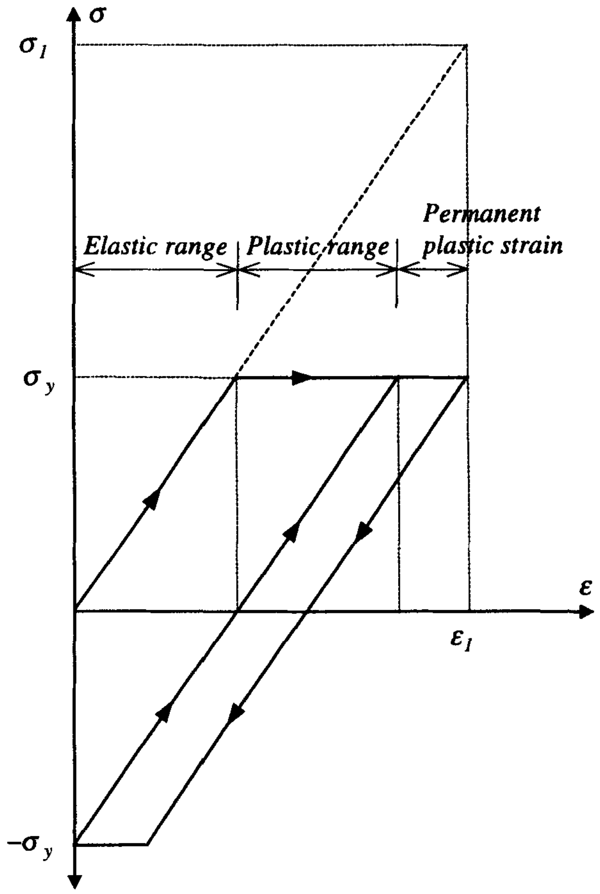


Fig. 11.3: Idealised stress-strain curve for $\sigma_l \rangle 2\sigma_y$

In the design and analysis of structures, a fundamental issue is the determination of the strength of a structure. Of considerable importance are the deformations and stresses that occur within the structure. The performance of a structure will depend on its shape and size, the properties of the material and the nature of the applied loads. It is also important to decide on how the strength of a structure should be defined.

In order to illustrate some of these issues, a structure may be considered consisting of a number of elements. On loading the structure, one of the elements may reach its yield stress whilst at the same time other elements may have only reached half of the yield stress. A further increase in the load will, therefore, cause yielding in one element, but the structure may not fail due to more modest loading on the remaining elements that make up the structure, since these elements may not yet have reached their yield point. Once all the elements of the structure have reached their yield point, then the structure will not be able to cope with any further increase in loading.

This can be verified by an analysis of the simple structure shown in Fig. 11.4 that is loaded at point *A* with a load *W*.

The elements in the structure are assumed to be connected to the top support and each other with perfect pin joints that permit free rotations.

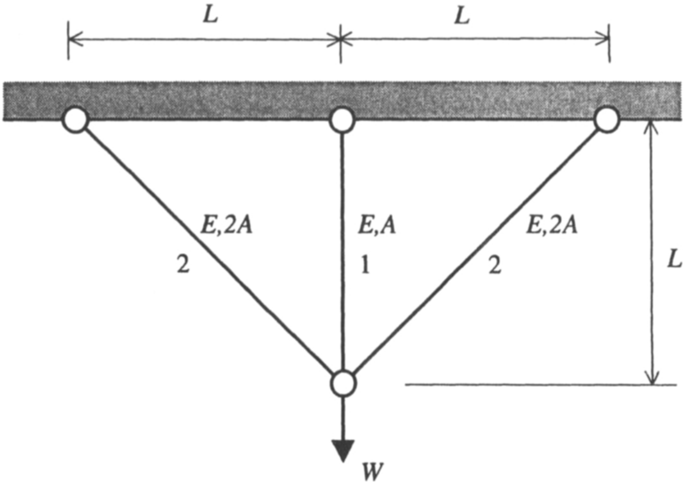


Fig. 11.4: Simple structure

Assuming that all the elements remain elastic, then the forces in the elements are as follows:

$$N_1 = \frac{2W}{2+\sqrt{2}} \quad \text{and} \quad N_2 = \frac{N_1}{2}$$

and the extension in the elements are:

$$\Delta_1 = \frac{N_1 L}{EA} \quad \text{and} \quad \Delta_2 = \frac{\Delta_1}{4}\sqrt{2}$$

where A = cross-sectional area of the elements.

This simple analysis is based on linear elastic theory and provides basic data on structural deformation and stresses in members. It can easily be seen that the load W on the structure can be increased to a level such that the stress in element 1 will go into the plastic region but the structure will not fail since the stress in elements 2 will remain in the elastic region.

11.3 Definition of plastic moments

Plastic analysis is based on idealised perfect elastoplastic behaviour, as shown in Fig.11.3. The yield point and the limit of stress-strain proportionality are assumed to occur at the same point. In the plastic range the stress-strain diagram is assumed to be a horizontal line. For most structural materials the strain at the end of the plastic range is approximately 12 times the strain at the initiation of yielding.

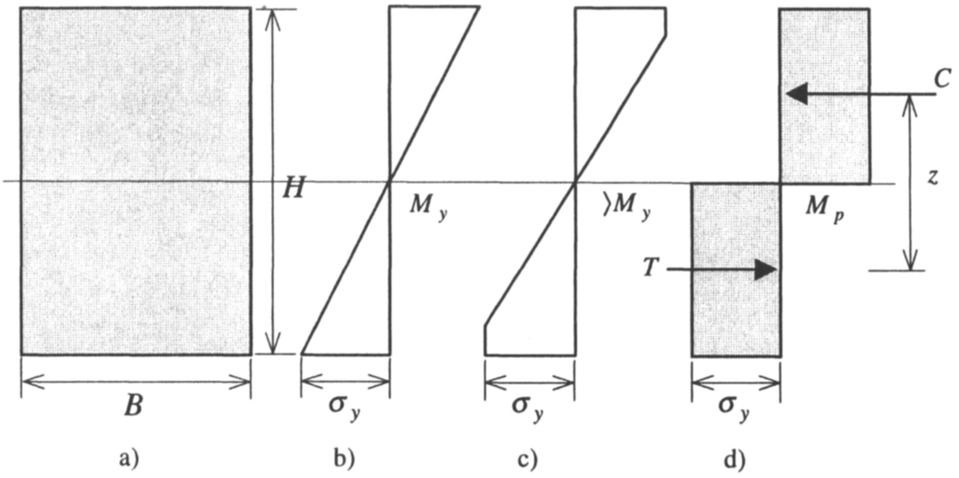


Fig.11.5: Bending stresses under increasing moments

Consider a beam that is bent as a result of loading, then the maximum stress will occur at the outer fibre (Fig. 11.5) and as the bending moment is increased a point will be reached when yielding takes place in the region as shown in Fig. 11.5c.The moment associated with first yield is called the *yield moment*, M_y. As the moment is further increased, yielding will progress towards the interior of the beam until the section is fully yielded (Fig.11.5d). At this point the forces in the section are:

$$C=\sigma_y A_c \quad and \quad T=\sigma_y A_t \tag{11.1}$$

and since C is equal to T,

$$M_p=C\cdot z=T\cdot z, \tag{11.2}$$

where A_c and A_t are the compressive and tensile areas, respectively and C and T the resulting compressive and tensile forces, respectively, z is the lever arm between both forces. The moment M_p associated with yielding over the full depth of the beam is called the *plastic moment*.

Example 11.1: Determine the plastic moment of resistance of the I beam section with wide flanges HE-M 160 (EuroCode 53-62) shown in Fig 11.6 if the yield stress of the material is 340 MPa in tension and in compression.

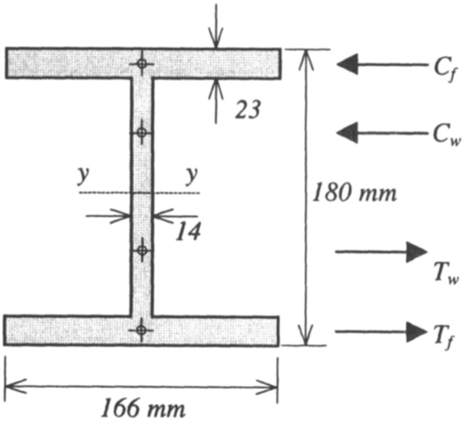


Fig. 11.6: I section with wide flanges

The plastic moment is determined separatelly from the contribution of flanges and the web as follows:

$$M_p = 340 \cdot 10^3 \cdot (0.166 \cdot 0.023 \cdot 0.157 + 0.014 \cdot 0.067 \cdot 0.067) = 225.172\ kNm$$

The elastic moment is using the moment of resistance $W = 566$ cm^3 for I section:

$$M_e = \sigma_y \cdot W = 340 \cdot 10^3 \cdot 566 \cdot 10^{-6} = 192.340\ kNm$$

from which the *shape factor* α can be calculated:

$$\alpha_{pe} = \frac{M_p}{M_e} = \frac{225.172}{192.340} = 1.170 \qquad (11.3)$$

Shape factors can be found in relevant books and vary between *1.12-1.18* for *I* sections, equals *1.27* for a hollow circular section(pipe), *1.5* for a rectangular section and is *1.7* for a solid circular section.

11.4 Formation of plastic hinges

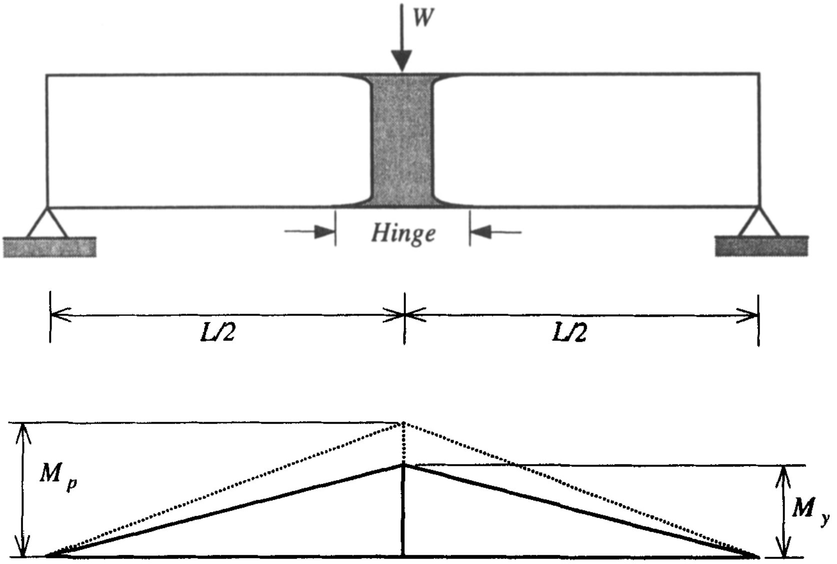


Fig. 11.7: Formation of plastic hinge

Consider the simply supported beam shown in Fig. 11.7, which is loaded by a concentrated load at the midspan. The moment diagram shows that as the load increases beyond the load that first causes yielding, the yielding proceeds towards the centre of the beam and also moves outwards.When the fully plastic moment is reached, the load is at its ultimate value, which is:

$$W_u = \frac{4M_p}{L} \tag{11.4}$$

When a section of a beam experiences a fully plastic stress, as shown in Fig. 11.7, then we say a *plastic hinge* is formed. Such a plastic hinge has no additional moment resistence. In static-determinate structures, the formation of a plastic hinge changes the restraint characteristics of the the structure. This results in a redistribution of internal forces causing other sections to reach their full strength and develop plastic hinges until the structure forms a *collapse mechanism*. The loading associated with the onset of collapse is called the *ultimate load*.

For design purposes we define the ratio of the ultimate load or stress to the working load or stress as *f*, thus

$$f = \frac{ultimate\ load}{working\ load} \tag{11.5}$$

The *plastic* design method may result in considerable savings in material and also provides a better evaluation of the safety of a structure.

11.5 Analysis of structures using plastic moments

There are two methods of analysing beams when using plastic analysis. These are the *equilibrium (statical)* method and the *virtual-work (mechanism)* method. Plastic analysis should satisfy three conditions which are:

1. *Mechanism condition.* The ultimate load is reached when a collapse mechanism forms.
2. *Eqilibrium condition.* There is static equilibrium until collapse.
3. *Plastic-moment condition.* The moment must not exceed M_p.

The equilibrium method satisfies the first two conditions and is applicable to solving most practical problems in structural engineering. It will therefore be described here along with some worked examples. The procedure consists of the following steps:

1. Draw a composite moment diagram such that a mechanism is formed.
2. Compute the ultimate stress using static equilibrium equations.
3. Check to see the M_{max} is less or equal to M_p.

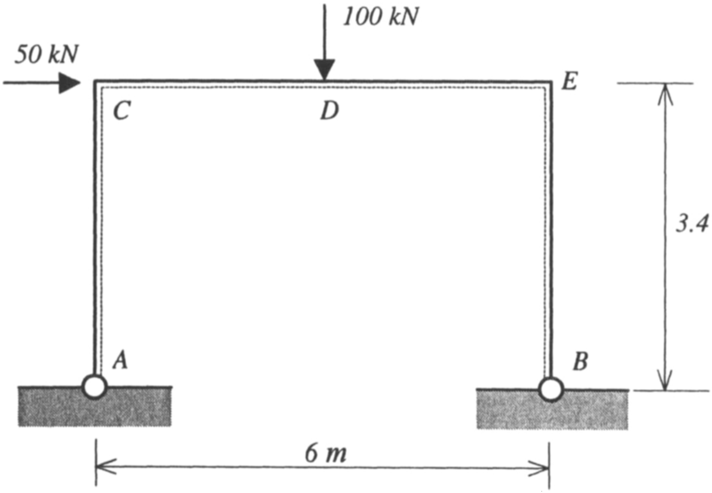


Fig. 11.8: Portal frame

To illustrate the method consider the portal frame shown in Fig. 11.8. The frame is supported at locations *A* and *B* by fixed supports. A central load $F_y = 100\ kN$ is applied at *D*, the centre of the beam *CE* and a load $F_x = 50\ kN$ at point C. the bending moment moment diagram for this structure is shown in Fig. 11.9 where it can be seen that the maximum bending moment of *139.253 kNm* occurs at E.

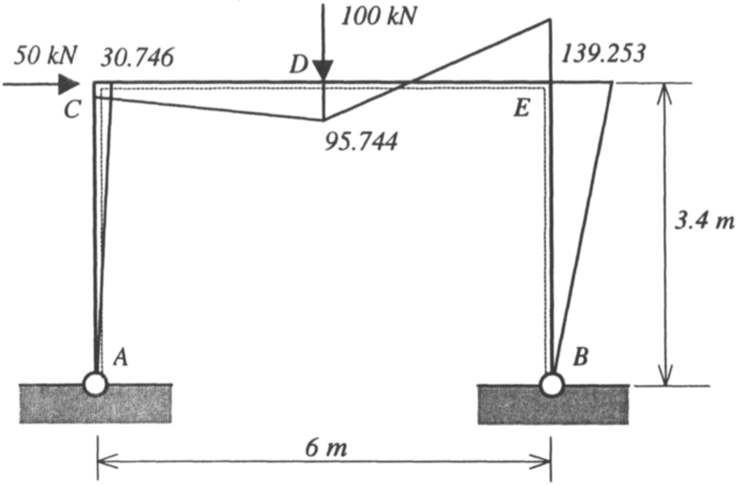


Fig. 11.9: Bending moments at initial forces [kNm]

If $M_p = 225.172\ kNm$ (from Example 11.1) then:

$$f = \frac{M_p}{M_{max}} = \frac{225.172}{139.253} = 1.617 \qquad (11.6)$$

From this simple analysis it can be seen that the structure is safe.

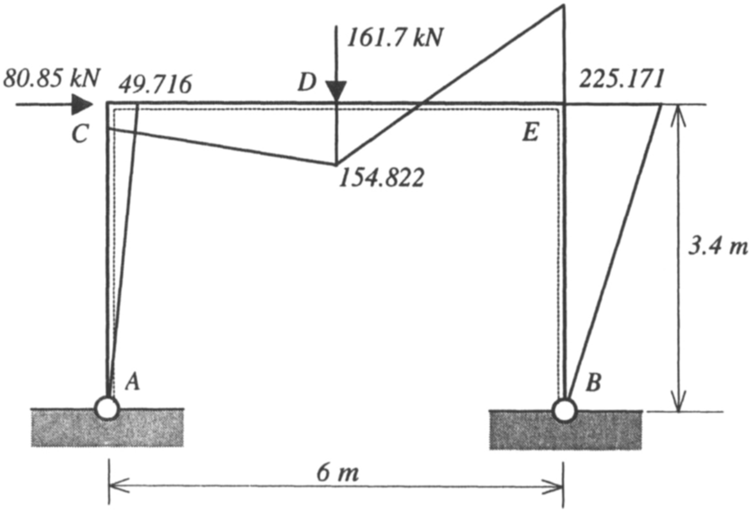


Fig. 11.10: Bending moments at initial forces times *f* [kNm]

If now the loads are increased by factor *f* to $F_y = 161.7\ kN$ and $F_x = 80.85\ kN$ then the bending moment diagram changes significantly as shown in Fig. 11.10. The maximum bending moment now becomes *225 kNm* at point *E* and therefore *f =1*. This means that a plastic hinge forms at point *E*.

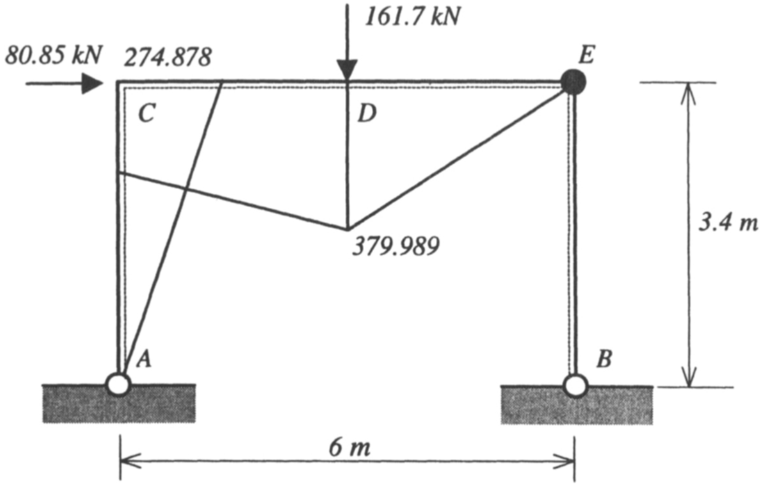


Fig. 11.11 Plastic hinge formed at E

The analysis of the structure must now be repeated with a new constraint at point *E*, this restraint being one of a zero restraining moment. The results of this new analysis is shown in Fig. 11.11. It can now be seen that M_{max} occurs at point *D* and that *f* exceeds unity. This means that a hinge forms at the midspan of beam *CE* and a mechanism is formed. The structure is now therefore unstable and will collapse as a three chained body (see Ch.4) as indicated in Fig. 11.12.

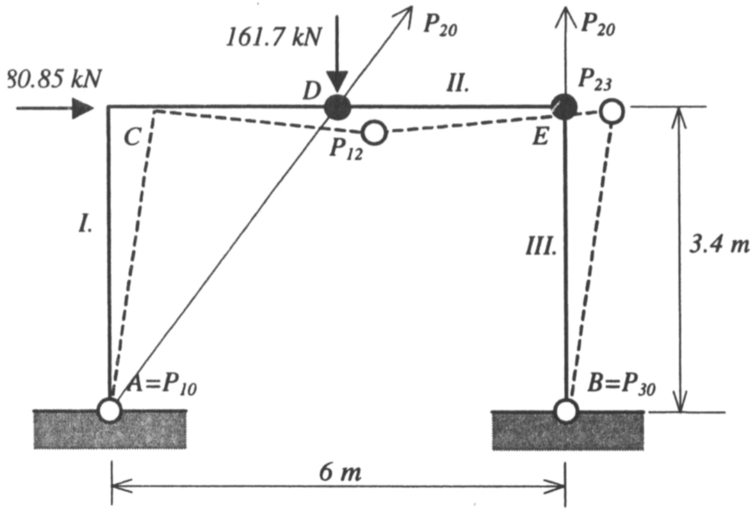


Fig. 11.12 Plastic hinges at D and E - mechanism

The whole procedure is repeated for the same portal frame except that the supports at *A* and *B* are *clamped*. The resulting bending moments are shown in Figs 11.13, 11.14, 11.15 and 11.16.

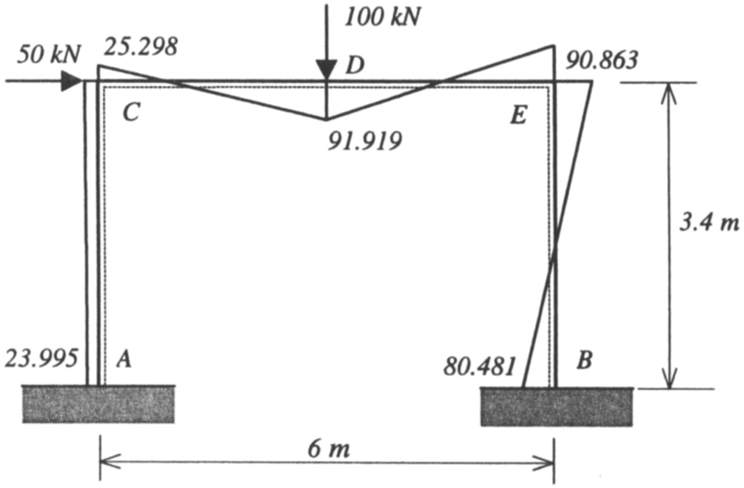


Fig. 11.13: Bending moments at initial forces [kNm]

$$f = \frac{M_p}{M_{max}} = \frac{225.172}{91.919} = 2.45$$

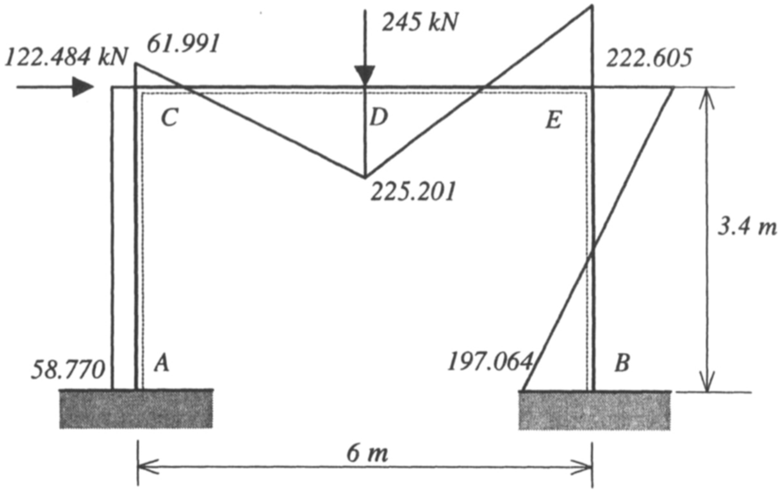


Fig. 11.14 Bending moments at initial forces times *f* [kNm]

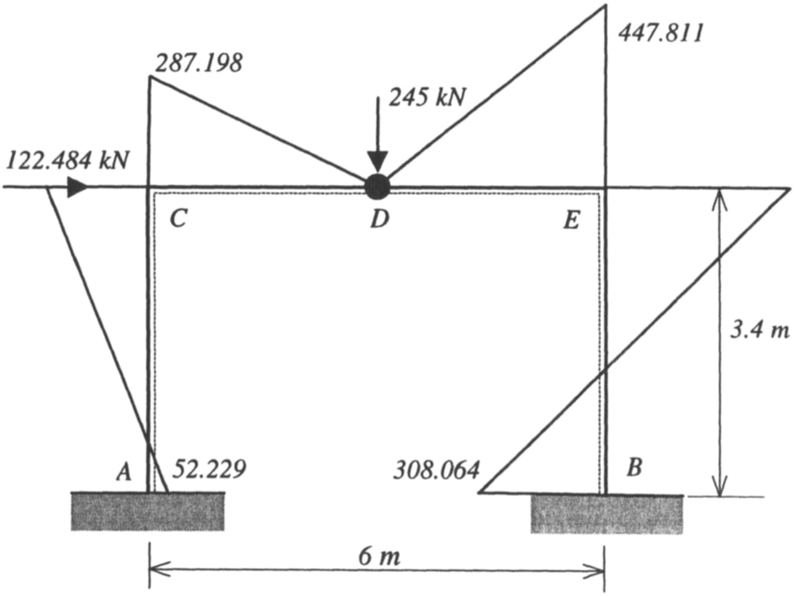


Fig. 11.15 Bending moments after first hinge at D [kNm]

From Fig. 11.15 it is obvious that after first hinge formed at D further hinges will immediatelly form at E, B and then at C which will cause the collapse of the structure.

12

A simple bridge analysis

12.1 Disposition of the structure

A bridge on a secondary road is bridging a deep valley and is founded in the riverbed on a rock; both outside supports are resting on deep piles such that no displacements at the supports are possible (see a simulation of an elastic support under column in Ch. 13, Fig.13.6).

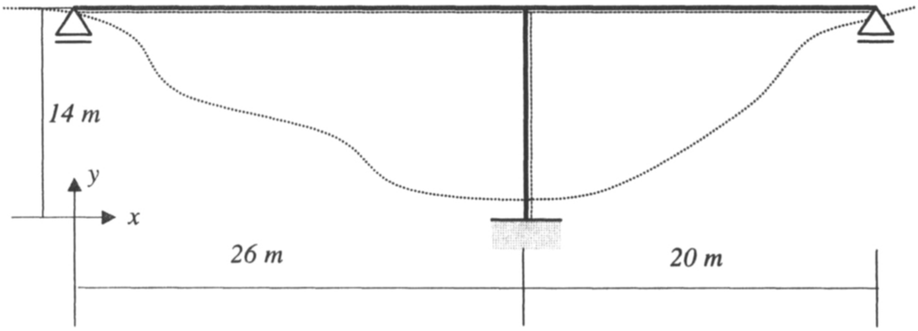


Figure 12.1: Static system of the bridge

12.2 Geometrical properties

At first we calculate cross-section areas, centres of gravity, second moments of area (moments of inertia) for the bridge deck (beam) and for the column:

Cross-section: $A = 4.80 \cdot 1.60 - 4.20 \cdot 1.00 + 2 \cdot 1.20 \cdot 0.3 = 4.20\ m^2$

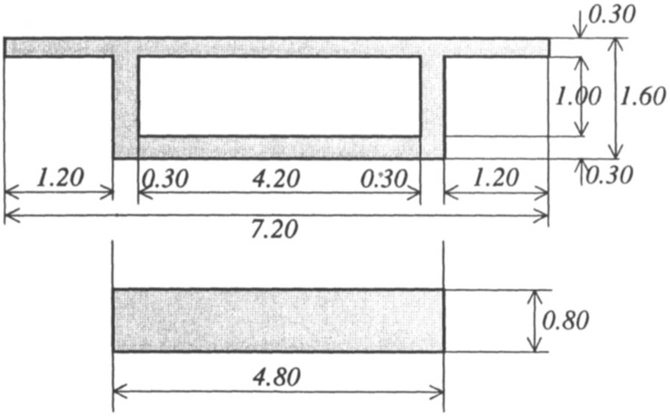


Figure 12.2: Cross-section of the beam and column

The centre of gravity in the *z*-direction is due to the symmetry at $z_T = 3.60\ m$, the centre of gravity in the *y*-direction is calculated from the static moment about the *z*-axis, which is for convenience taken at the bottom of the beam section:

$$y_T = \frac{S_z}{A}$$

$$A_1 = 7.20 \cdot 0.30 = 2.16\ m^2$$

$$A_2 = 1.30 \cdot 0.30 = 0.39\ m^2$$

$$A_3 = 4.20 \cdot 0.30 = 1.26\ m^2$$

$$A = A_1 + 2 \cdot A_2 + A_3 = 4.20\ m^2$$

The static moment is the sum of partial static moments:

$$S_z = A_1 \cdot 1.45 + 2 \cdot A_2 \cdot 0.65 + A_3 \cdot 0.15$$

$$S_z = 3.132 + 0.507 + 0.189 = 3.828\ m^3$$

$$y_T = \frac{S_z}{A} = \frac{3.828}{4.20} = 0.91\ m$$

The moment of inertia is calculated using Steiner's statement:

$$I_Z = \frac{7.2 \cdot (0.3)^3}{12} + 2 \cdot \frac{0.3 \cdot (1.3)^3}{12} + \frac{4.2 \cdot (0.3)^3}{12} +$$

$$+ A_1 \cdot (0.54)^2 + 2 \cdot A_2 \cdot (0.26)^2 + A_3 \cdot (0.76)^2 =$$

$$= 0.0162 + 0.10985 + 0.00945 + 0.629 + 0.0527 + 0.727 =$$

$$= 0.1255 + 1.41 = 1.546 \;\; m^4$$

$$I_Z = 1.546 \;\; m^4$$

The self weight of the beam is:

$$q = A \cdot \rho \cdot g = 4.2 \cdot 25 = 105.0 \; \frac{kN}{m}$$

For the column only the moment of inertia about the weak axis is needed

$$I_C = \frac{4.8 \cdot (0.8)^3}{12} = 0.204 \;\; m^4 \qquad A = 4.2 \cdot 0.8 = 3.36 \; m^2 ,$$

and for later purposes let us calculate the ratio between both beam and column stiffnesses:

$$K = \frac{I_Z}{I_C} = 7.578$$

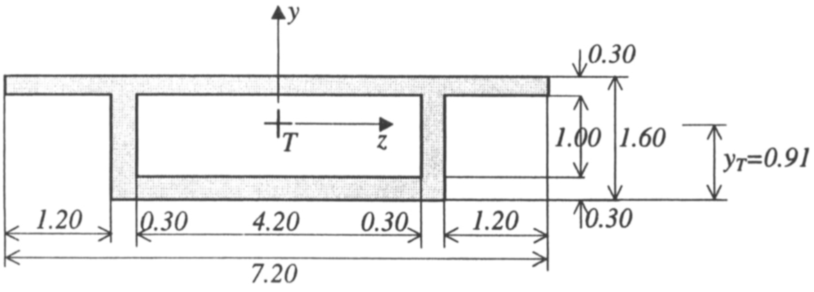


Figure 12.3: Cross-section and centre of gravity

12.3 Analysis by the force method

A degree of static indeterminacy *n* (DSI) is determined by equation (8.2):

$$n = 3 \cdot (m_6 - j_3) + 2 \cdot (m_5 - j_2) + m_4$$

$$n = 3 \cdot (1 - 1) + 2 \cdot (0 - 0) + 2 = 2$$

To get the primary structure we remove supports at *A* and *C*. The loading on the structure is self weight only (Fig. 12.4).

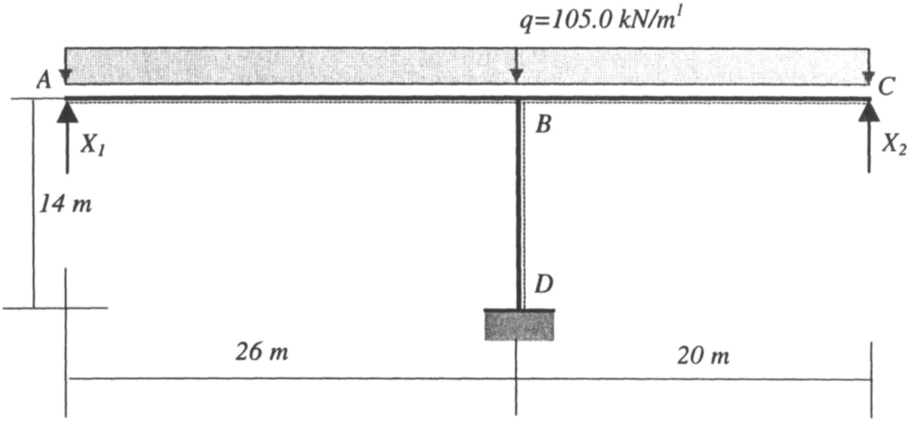


Figure 12.4: Primary structure

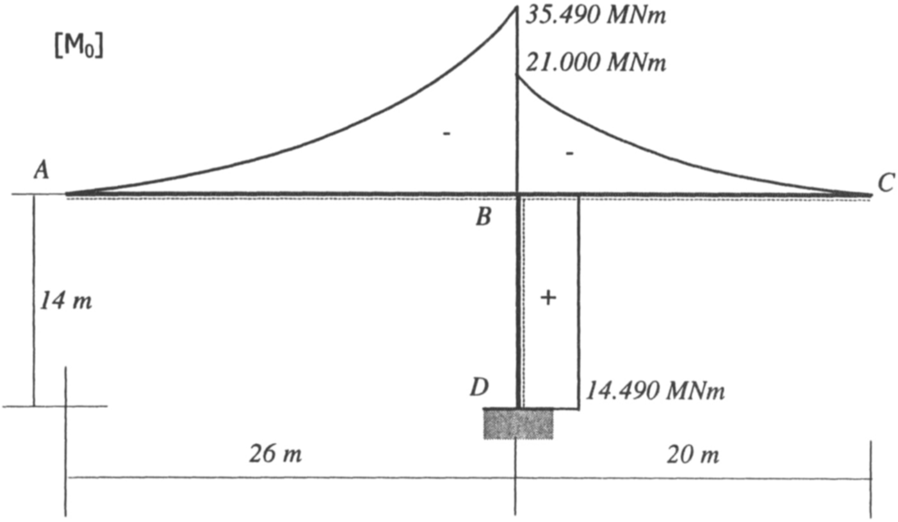


Figure 12.5a: Bending moments for external loading

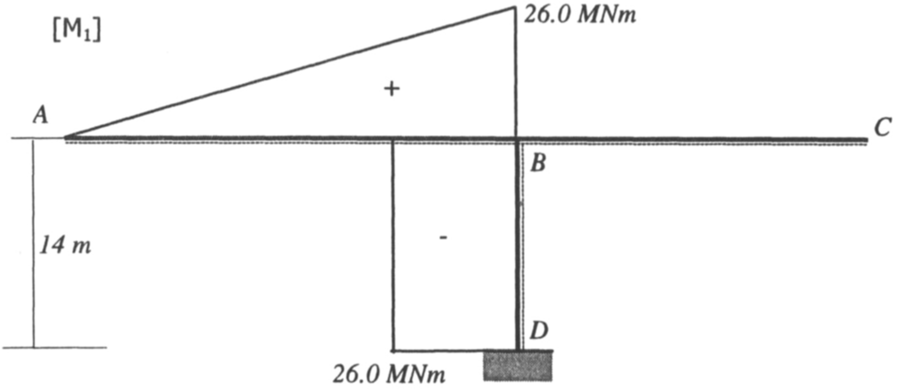


Figure 12.5b: Bending moments for unknown force X_1

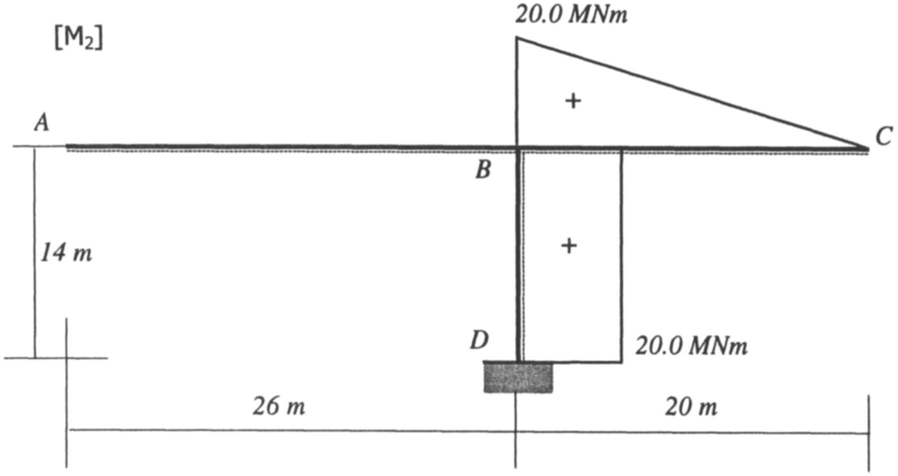


Figure 12.5c: Bending moments for unknown force X_2

The equations of compatibility are:

$$a_{11} \cdot X_1 + a_{12} \cdot X_2 + a_{10} = 0$$

$$a_{21} \cdot X_1 + a_{22} \cdot X_2 + a_{20} = 0$$

For comparison we will at first calculate the case of equal moments of inertia for beam and column, and we will further simplify the calculation with $E = 1$:

$$a_{11} = \frac{1}{3} \cdot (26)^3 + 26 \cdot 26 \cdot 14 = 15323$$

$$a_{22} = \frac{1}{3} \cdot (20)^3 + 20 \cdot 20 \cdot 14 = 8267$$

$$a_{11} = -26 \cdot 20 \cdot 14 = -7280$$

$$a_{10} = \int_0^{26} \frac{0.105 \cdot x^2}{2} \cdot 1 \cdot x \cdot dx + 14.49 \cdot (-26) \cdot 14 =$$

$$= \frac{0.105 \cdot (26^4)}{2 \cdot 4} - 5274 = -5997.01 - 5274 = -11272.0$$

$$a_{20} = -\frac{1}{4} \cdot 21 \cdot 20 \cdot 20 + 14.49 \cdot 20 \cdot 14 = 1957$$

$$\begin{bmatrix} 15323 & -7280 \\ -7280 & 8267 \end{bmatrix} \cdot \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \begin{Bmatrix} 11272 \\ -1957 \end{Bmatrix}$$

$$X_1 = 1.071 \ MN$$

$$X_2 = 0.707 \ MN$$

Now real values for moments of inertia are taken into the calculation:

$$\delta = \int \frac{\overline{M} \cdot M}{EI} \cdot ds$$

$$a_{11} = 9836 \qquad a_{22} = 5769 \qquad a_{12} = -7280$$

$$a_{10} = 5656 \qquad a_{20} = 3923$$

$$\begin{bmatrix} 10237 & -7280 \\ -7280 & 5952 \end{bmatrix} \cdot \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \begin{Bmatrix} 6066 \\ -3780 \end{Bmatrix}$$

$$X_1 = 1.082 \ MN$$

$$X_2 = 0.689 \ MN$$

Bending moments are calculated by the method of superposition (note values in brackets, which stand for equal moments of inertia):

$$M_A = 0$$

$$M_{BA} = -35.49 + 26 \cdot 1.082 = -7.358 \ (-7.644) \ MNm$$

$$M_{BC} = -21 + 20 \cdot 0.689 = -7.220 \ (-6.860) \ MNm$$

$$M_{BD} = 14.49 - 26 \cdot 1.082 + 20 \cdot (0.689) = 0.138 \ (0.784) \ MNm$$

Shear forces are calculated from equilibrium on each beam separately:

$$Q_{AB} = \frac{0.105 \cdot 26}{2} - \frac{7.358}{26} = 1.365 - 0.283 = 1.082 \ MN$$

$$Q_{BA} = \frac{0.105 \cdot 26}{2} + \frac{7.358}{26} = 1.365 + 0.283 = 1.648 \ MN$$

$$Q_{BC} = \frac{0.105 \cdot 20}{2} + \frac{7.220}{20} = 1.050 + 0.361 = 1.411 \ MN$$

$$Q_{CB} = \frac{0.105 \cdot 20}{2} + \frac{7.220}{20} = 1.050 - 0.361 = 0.689 \ MN$$

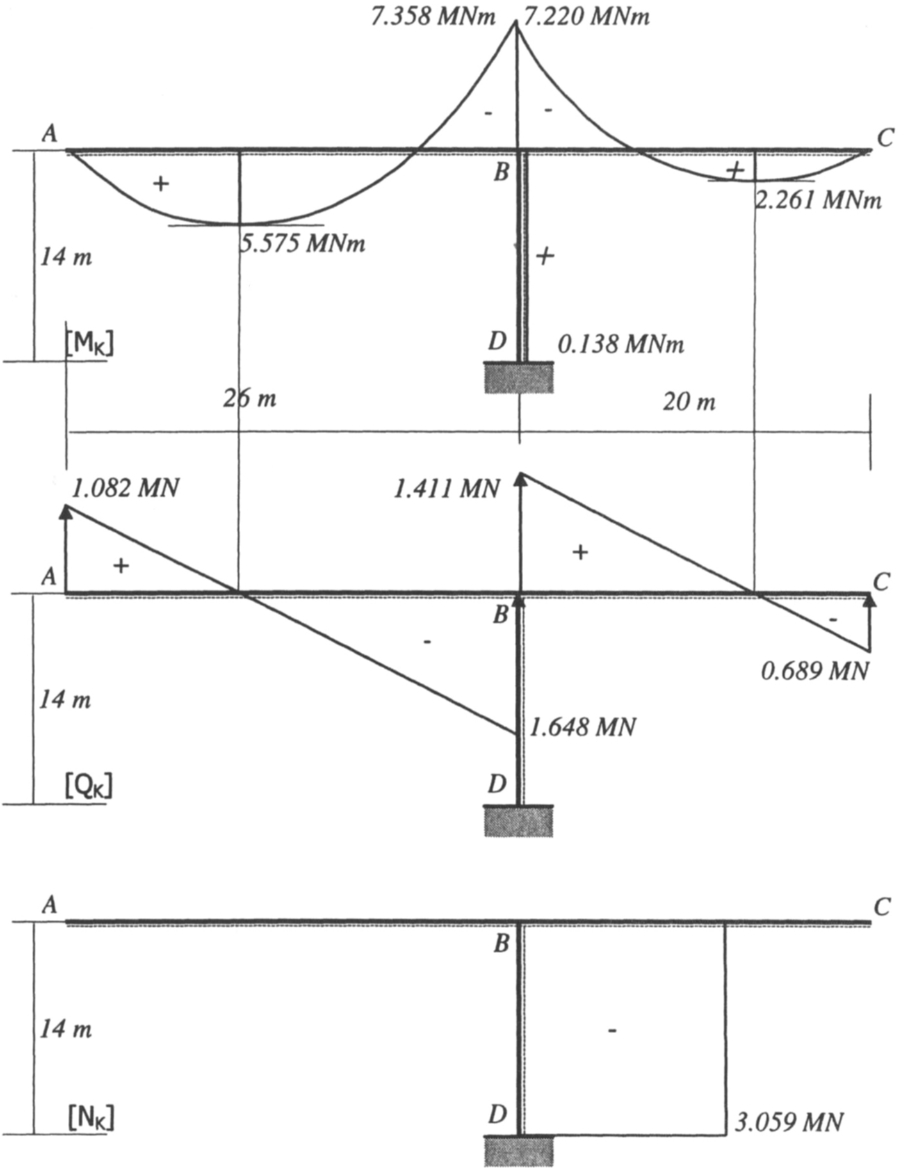


Figure 12.6: Final diagrams of internal forces (The force method)

There are no axial forces on the beams as both outside supports are roller supports and that is the reason for zero shear forces on the column (*X* reaction at *D* equals zero).

12.4 Analysis by the displacement method

Equations (9.4) and (9.5) give a degree of deformational indeterminacy (*DDI*):

Unknown rotations: $b = k - p_1 = 2 - 1 = 1$

Unknown displacements: $c = 2 \cdot k + 2 \cdot g - p_2 - m = 2 \cdot 2 - 2 \cdot 2 - 4 - 3 = 1$

or $c = 2 \cdot j - p_2 - m = 2 \cdot 4 - 4 - 3 = 1$

The system is hence a sway system as the structure can displace at the deck level.

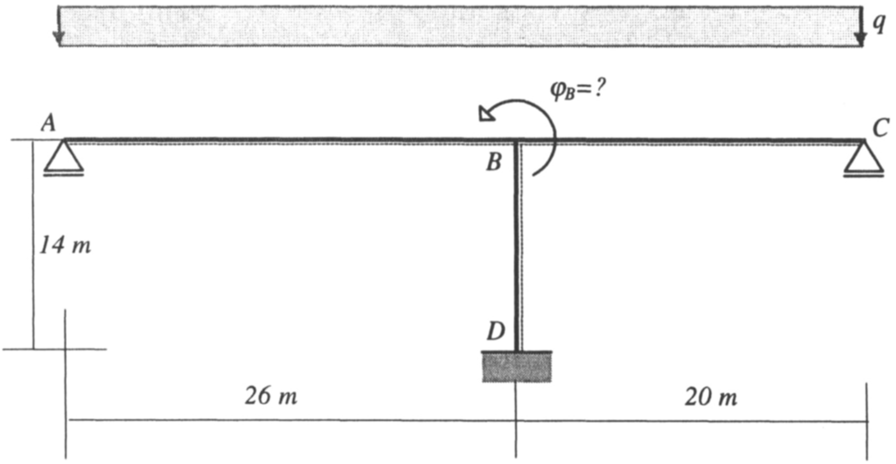


Figure 12.7: Unknown rotation

Let us at first analyse the non-sway structure:

$$a_{11} \cdot \varphi_B + a_{10} = 0$$

$$a_{11} = \frac{3 \cdot 7.578}{26} + \frac{3 \cdot 7.578}{20} + \frac{4}{14} = 2.297$$

$$a_{10} = -\frac{q \cdot L_1^2}{8} + \frac{q \cdot L_2^2}{8} = -\frac{0.105}{8} \cdot 26^2 + \frac{0.105}{8} \cdot 20^2$$

$$a_{10} = -8.8725 + 5.250 = 3.6225 \ \ MNm$$

$$\varphi_B = -\frac{-3.6228}{2.297} = 1.577$$

Now we calculate bending moments due to rotation φ_B :

$$M_{BA}^{\varphi}=\frac{3\cdot EI}{L_1}\cdot\varphi_B=\frac{3\cdot 7.578}{26}\cdot 1.577=1.379\ \ MNm$$

$$M_{BC}^{\varphi}=\frac{3\cdot EI}{L_2}\cdot\varphi_B=\frac{3\cdot 7.578}{20}\cdot 1.577=1.793\ \ MNm$$

$$M_{BD}^{\varphi}=\frac{4\cdot EI}{14}\cdot\varphi_B=\frac{4}{14}\cdot 1.577=0.451\ \ MNm$$

$$M_{BA}=-8.873+1.379=-7.494\ \ MNm$$

$$M_{BC}=5.25+1.916=7.166\ \ MNm$$

$$M_{BD}=0.451\ \ MNm$$

$$M_{DB}=\frac{M_{BD}}{2}=0.226\ \ MNm$$

The real structural system is a sway system as $c=1$.

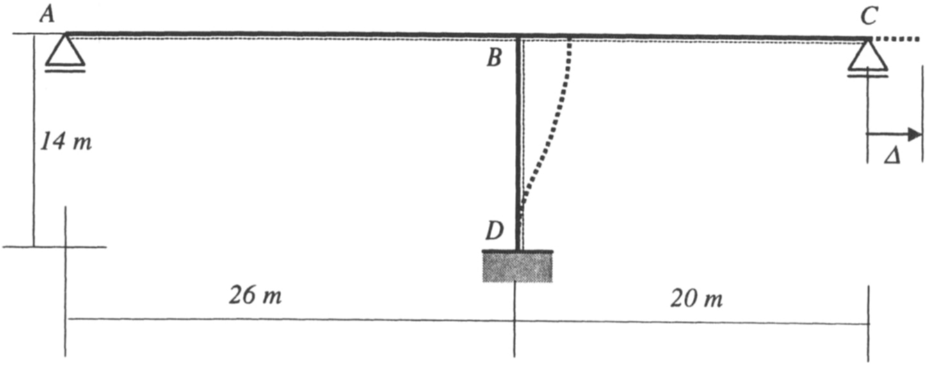


Figure 12.8: Displacement at the deck level

The equations for a sway systems are (see example 9.7):

$$a_{BB}\cdot\varphi_B+a_{B1}\cdot\Delta=a_{B0}$$

$$a_{1B}\cdot\varphi_B+a_{11}\cdot\Delta=a_{10}$$

The kinematic chain can move in *x*-direction only hence the work done by the external loading is zero and $a_{10} = 0$.

$$a_{11} = \frac{12}{H^3} = \frac{12}{14^3} = 0.004373$$

$$a_{B1} = -\frac{6}{H^2} = -\frac{6}{14^2} = -0.030612$$

$$\begin{bmatrix} 2.297 & -0.030612 \\ -0.030612 & 0.004373 \end{bmatrix} \cdot \begin{Bmatrix} \varphi_B \\ \Delta_B \end{Bmatrix} = \begin{Bmatrix} 3.6225 \\ 0 \end{Bmatrix}$$

$$\varphi_B = 1.7393$$

$$\Delta = 12.1756$$

$$M_{BA} = -8.873 + \frac{3 \cdot EI}{L_1} \cdot \varphi_B = -8.873 + \frac{3 \cdot 7.578}{26} \cdot 1.7393 =$$

$$M_{BA} = -8.873 + 1.520 = -7.352 \ MNm$$

$$M_{BC} = 5.250 + \frac{3 \cdot EI}{L_2} \cdot \varphi_B = 5.250 + \frac{3 \cdot 7.578}{20} \cdot 1.7393 =$$

$$M_{BC} = 5.250 + 1.977 = 7.227 \ MNm$$

$$M_{BD} = \frac{4 \cdot EI}{14} \cdot \varphi_B - \frac{6EI}{14^2} \cdot \Delta = \frac{4}{14} \cdot 1.7393 - \frac{6}{14^2} \cdot (12.1756) =$$

$$M_{BD} = 0.497 - 0.372 = 0.124 \ MNm$$

$$M_{DB} = \frac{2 \cdot EI}{14} \cdot \varphi_B - \frac{6EI}{14^2} \cdot \Delta = \frac{2}{14} \cdot 1.7393 - \frac{6}{14^2} \cdot (12.1756) =$$

$$M_{DE} = 0.247 - 0.372 = -0.124 \ MNm$$

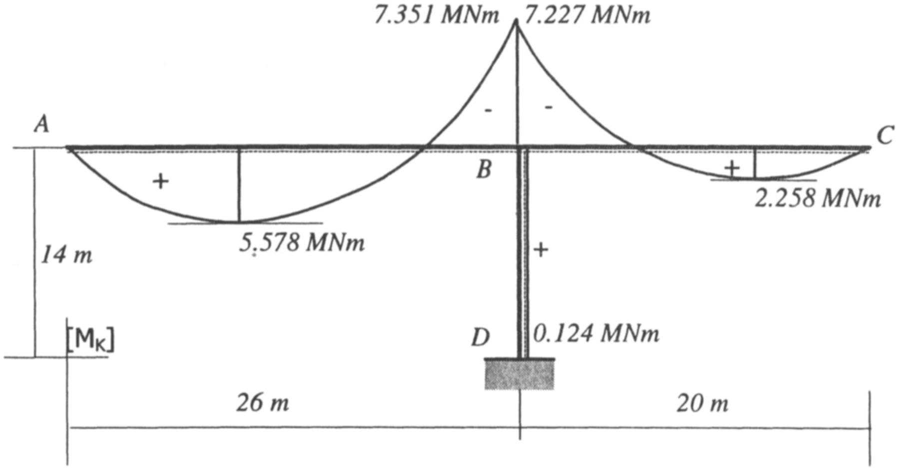


Figure 12.9: Final diagram of bending moments (the displacement method)

A comparison of the results to the force method gives small differences, which is due to the inaccuracy of the calculations using a pocket calculator.

12.5 Analysis by the moment distibution method

Using this method fix-end and free-span moments are first calculated:

$$M_{AB}^{0} = \frac{q \cdot L^2}{8} = \frac{0.105 \cdot (26)^2}{8} = 8.873 \ MNm$$

$$M_{BC}^{0} = \frac{q \cdot L^2}{8} = \frac{0.105 \cdot (20)^2}{8} = 5.25 \ MNm$$

$$M_{BA} = -\frac{q \cdot L^2}{8} = -8.873 \ MNm$$

$$M_{BC} = \frac{q \cdot L^2}{8} = 5.25 \ MNm$$

followed by the stiffness and distribution coefficients calculations:

$$k_{BA} = \frac{3 \cdot 7.578}{26} = 0.874$$

$$k_{BC} = \frac{3 \cdot 7.578}{20} = 1.137$$

$$k_{BD} = \frac{4}{14} = 0.286$$

$$\sum k = 0.874 + 1.137 + 0.286 = 2.297$$

$$r_{BA} = \frac{0.874}{2.297} = 0.381$$

$$r_{BC} = \frac{1.137}{2.296} = 0.495$$

$$r_{BD} = \frac{0.286}{2.296} = 0.125$$

The only free joint to rotate is at *B*, the moment is distributed according to the distribution coefficients and half of its part on column at *B* is transferred to joint *D*.

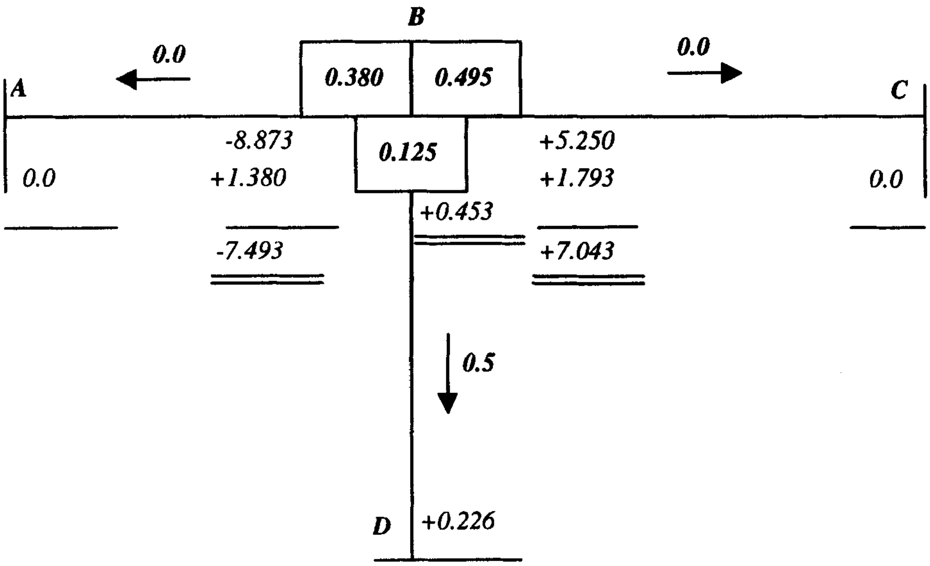


Figure 12.10: Iteration scheme (non-sway system)

The iteration was performed on the non-sway system, which at this moment is not in the equilibrium as column bending moments cause a shear force

$$\Delta H = \frac{M_{BD} + M_{DB}}{H} = \frac{0.453 + 0.226}{14} = 0.049 \ MN,$$

that can only be equalised by additional moments on both ends of the column (as no horizontal reactions occur) of magnitude:

$$\Delta M = \frac{M_{BD} + M_{DB}}{2} = 0.340 \ MNm.$$

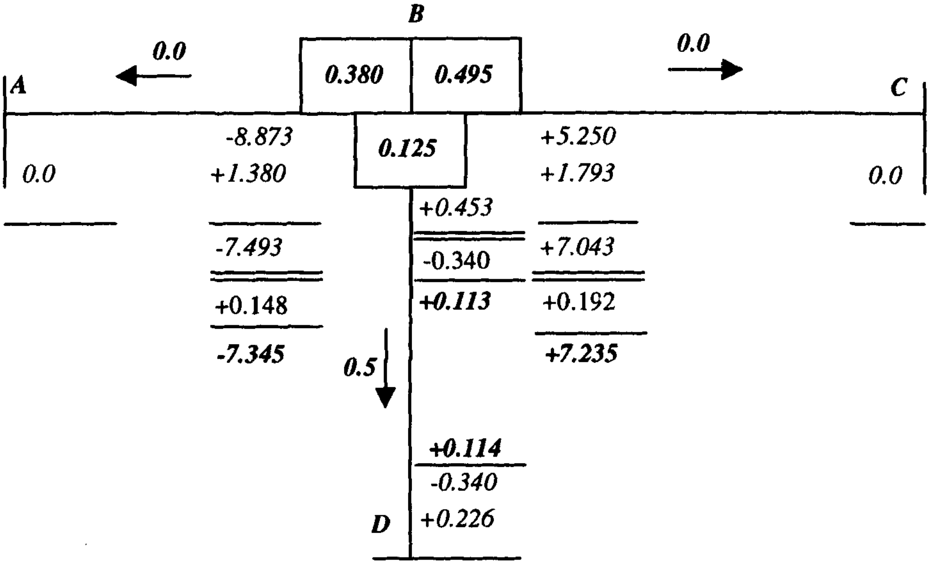


Figure 12.11: Iteration scheme after first iteration (sway system)

The iterative procedure can be repeated until desirably small differences in the horizontal force at the deck level ΔH occur.

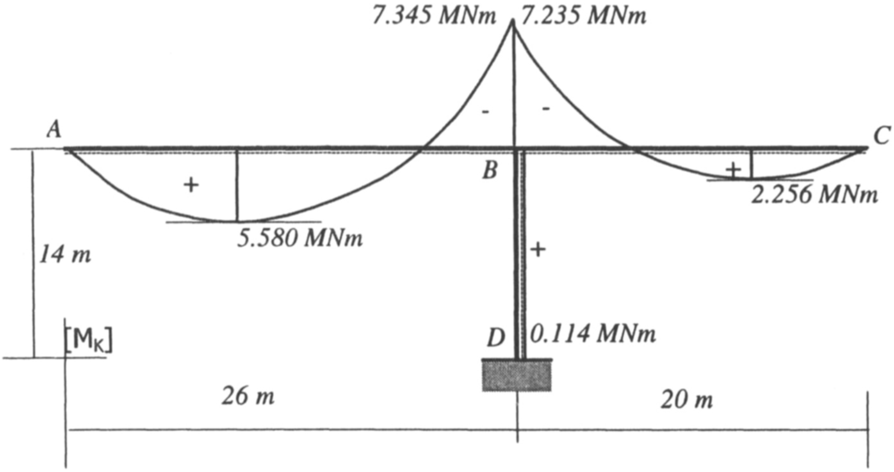


Figure 12.12: Final diagram of bending moments (Cross's method)

13

Computer applications

13.1 Introduction

In this chapter we shall consider the finite element computer program OCEAN that was used to carry out analyses for examples presented in previous chapters. Some basic but important applications from everyday practice are calculated by the use of the finite element method.

13.2 Structural analysis package OCEAN

13.2.1 Program description

OCEAN is a computer program for structural analyses running in a unique user-friendly graphic environment. It is written in PowerBASIC language consisting of approximately *46000* statements.

The structural analysis package OCEAN implements the following subprograms: Plate bending (module PLATE), Frame analysis (module FRAME), Space and plane truss analysis (modules P-TRUSS and S-TRUSS), Seismic design of shear walled structures (module WALL), Plates on columns (module PL-COL) and Shear wall analysis (module SWALL). Some of these sub-modules perform instantaneous reinforcement dimensioning in accordance with *EuroCode 2*. The total loads for specified loading cases are combined using Code factors by default or by any user defined load factors in the post-processor during reinforcement calculations.

The methods of analysis are completely transparent to the program user as the input is entered through the mouse driven pre-processor menu. There is no numerical input; the mesh generation is automatic in local co-ordinate systems using a mouse for the generation of substructures on the basis of the *DXF* format databases, created in architectural *3D* design.

The pre- and post-processor modules are mouse driven, using *pop-up menus*, enabling the input data to be given simply by pointing to the specific attributes using geometry displayed on the screen.

OCEAN uses mouse input for data preparation and colour graphics to display quantities or their derivatives on isoparametric surfaces inside the microcomputer environment OCEAN, which was written in a general form. This enables utilisation of different FEM procedures using the same input-output algorithms and utilises some unique and innovative features such as:

- Complete graphic input and output
- Special frontal method for a solution of simultaneous equations enabling analysis of more than one structure at the same time
- Special algorithm for the solution of non-linear equations in column design
- Utilisation of optimisation algorithms inside the analysis process
- Instantaneous reinforcement calculations of concrete structures
- Ease of use with no need of manuals
- *Some AI implemention at certain levels of an analysis*

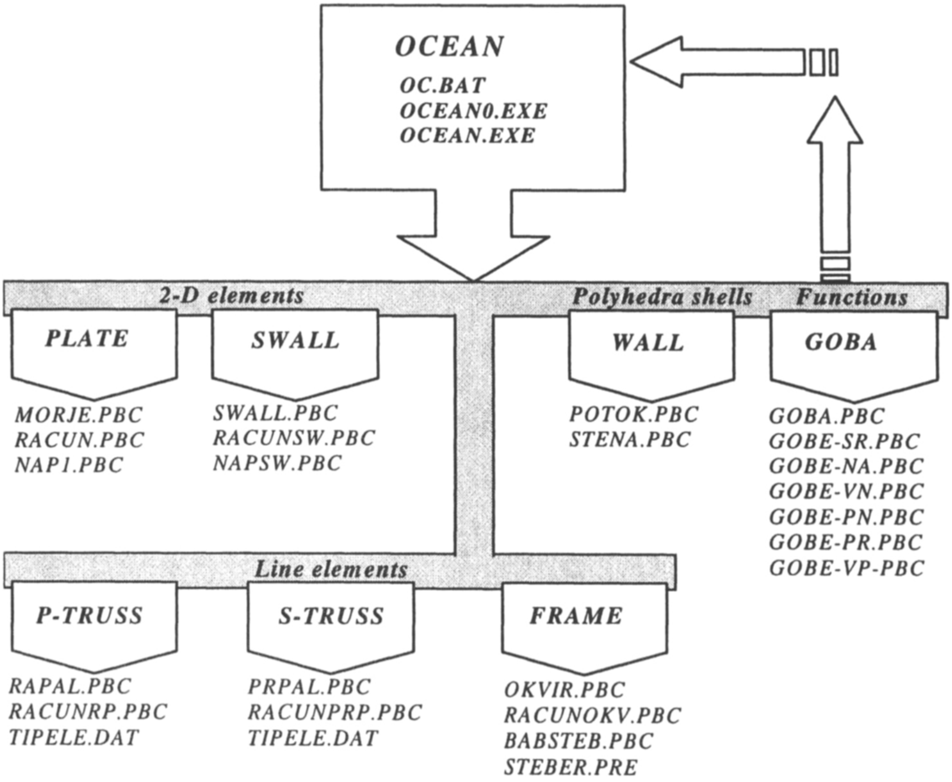


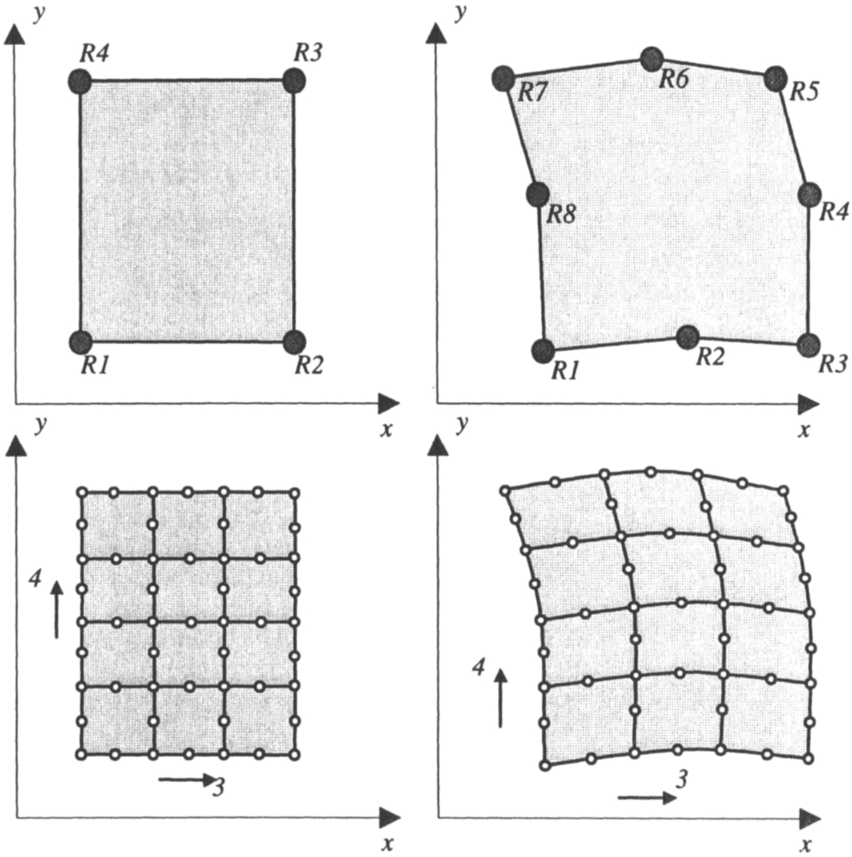
Figure 13.1: Structure of the program and element types

As seen from Fig. 13.1 the package includes only one execution module (OCEAN.EXE), all other modules are chained files of type *PBC* (PowerBasic Chain Module), and can be called from within any other module of the same chain.

13.2.2 Structure generation

A structure is generated by regions, which define part or the whole structure. Regions are divided either into finite elements (modules PLATE and SWALL) or into fictitious elements; their nodes are connected to form finite elements (modules FRAME, TRUSS and WALL). There are two choices of regions (Fig.13.2):

- An *arbitrary quadrilateral*
- A *region of an arbitrary shape*, defined by 8 joints



a) Arbitrary quadrilateral *b) 8-nodded arbitrary shape*

Figure 13.2: Regional joints and division

As an example of the generation procedure, generation of the structure from example 3.1 will be shown. Regional joints are: *R1(0, 0)*, *R2(10, 0)*, *R3(10, 4)* and *R4(0, 4)*.

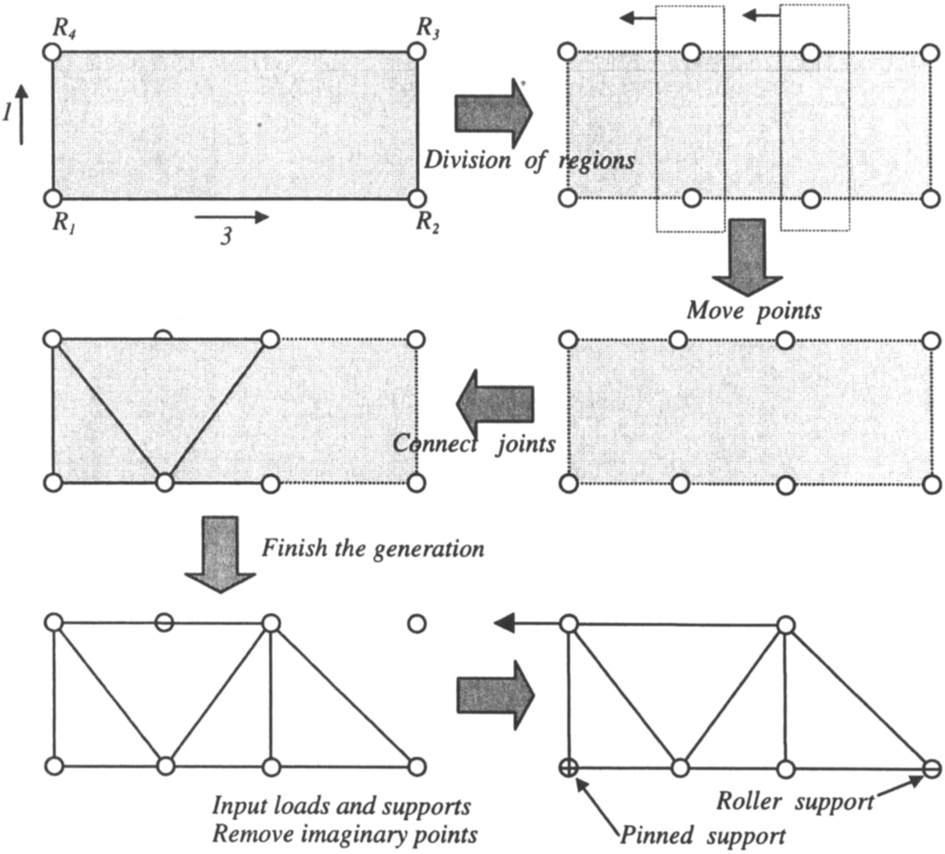


Figure 13.3: Generation procedure

The generation procedure is in principle the same for all types of structures with the exception of modules PLATE and SWALL, where a division in a direction results in finite element generation directly.

Results of the finite element analysis are the same as in longhand calculations in truss structures only, frame analysis by finite elements includes work done by moments, shear forces and axial forces and are a little different from longhand calculations as shown in Fig. 13.4 below.

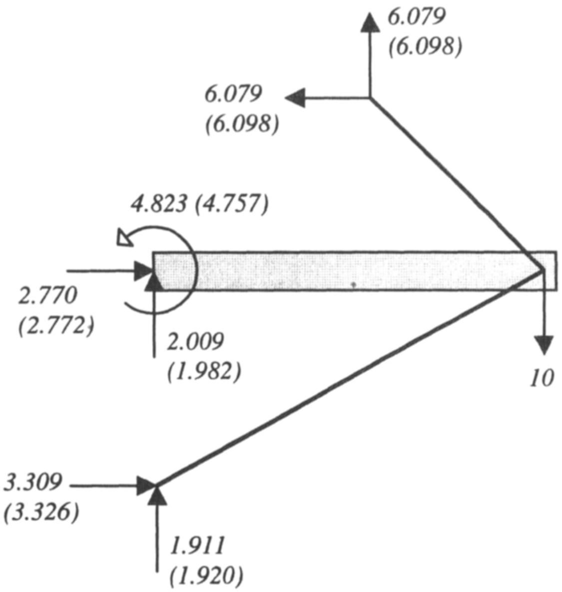


Figure 13.4: Free body in equilibrium
(In brackets are the longhand calculation results)

Let us calculate the example shown in Fig. 13.5 using module FRAME. Note the imaginary points *1* and *3* in figure 13.5 below. These two points were generated during region definition but do not take part in the analysis at all and could well be removed.

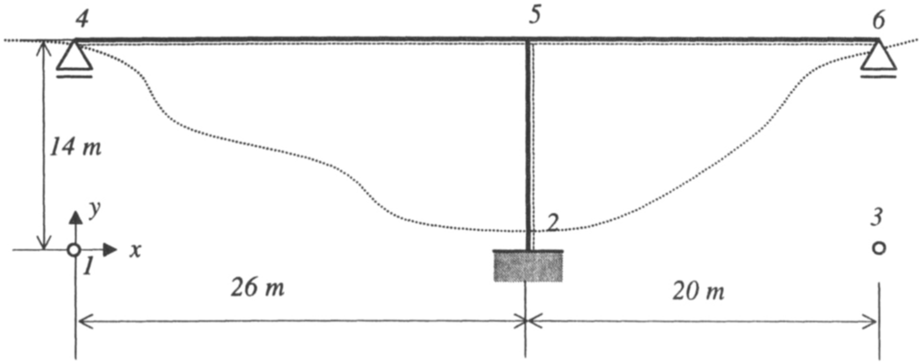


Figure 13.5: Nodes of the computer model

The geometry and loads were taken from Ch. 11; here we present only a part of computer listing which should be used by the reader to compare the results from computer and longhand analysis.

```
P R O G R A M   F R A M E                                     Page  1
by  B. S. BEDENIK 1999                          13:04:02      02-04-1999

STRUCTURE : Example11

MATERIAL   1
Cross-section             4.200000 m2
Moment of Inertia         1.546000 m4
Modulus of elasticity     30.000    GPa
Specific weight           0.000    kN/m3

MATERIAL   2
Cross-section             3.360000 m2
Moment of Inertia         0.204000 m4
Modulus of elasticity    30.000    GPa
Specific weight           0.000    kN/m3
.
.
.
.
Displacements        Direction
 Joint        X          Y          Rotation

  4      -0.002023  0.000000  0.000998
  5      -0.002023 -0.000423 -0.000289
  6      -0.002023  0.000000 -0.000265
.
.
.
.
 Internal forces       LOAD CASE  1

 Elem. i  A [m2]      I [m4]       N [kN]       Q [kN]         M [kNm]

   1   4  4.2000    1.546000        0.000     1086.573          0.000
       5                            0.000    -1643.427      -7239.101
   2   2  3.3600    0.204000    -3049.067        0.000       -126.305
       5                        -3049.067        0.000        126.305
   3   5  4.2000    1.546000       -0.000     1405.640       7112.796
       6                           -0.000     -694.360          0.000
```

If a support under columns can be displaced under loading for any reason, a computer model of an elastic support can be simulated as given in Figure 13.6. The *horizontal elements are very stiff* and hence have the same rotation as the column itself, but the simulation columns have stiffness of the soil under the foundation, varying A, L and modulus E.

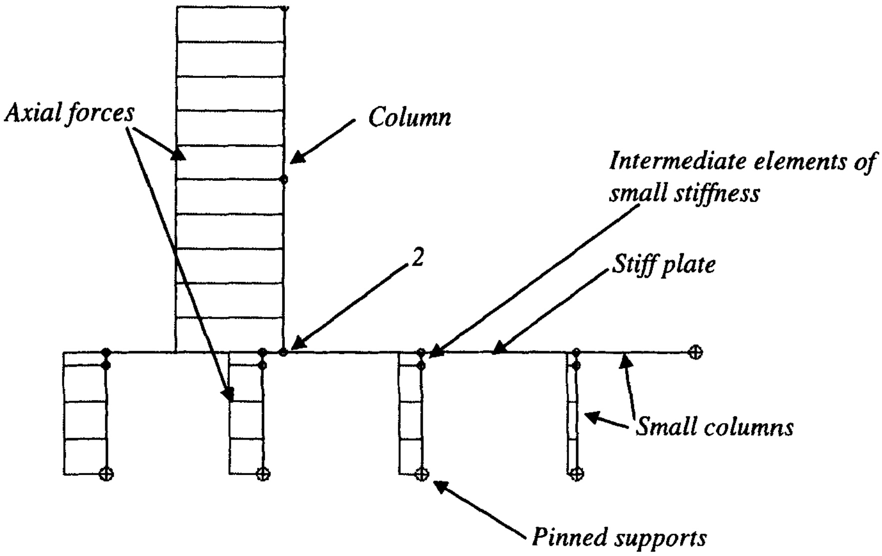


Figure 13.6: A simulation of an elastic support under a column

13.3 Practical examples of 2D structures

As mentioned previously, the plate bending module PLATE uses *9-nodded isoparametric element with 27 degrees of freedom* and shear walls module SWALL uses *8-nodded isoparametric element with 16 degrees of freedom* (Figure 10.6). Some practical examples of analyses using these 2D elements are shown in Figures 13.7-17.

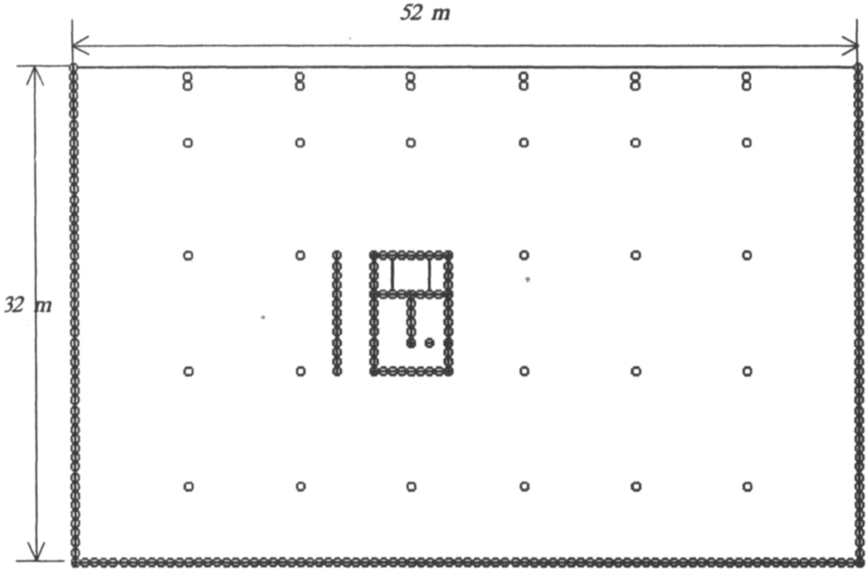


Figure 13.7: A plan of a house(PLATE)

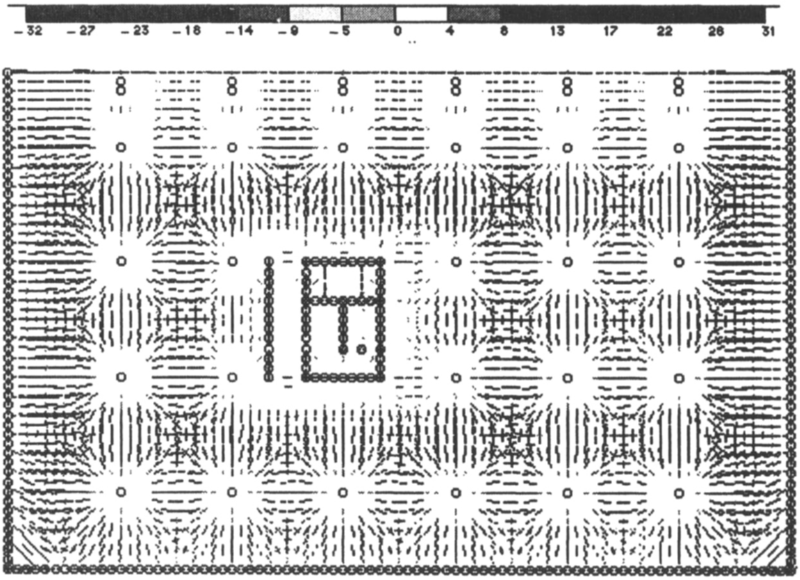


Figure 13.8: House – principle positive bending moments

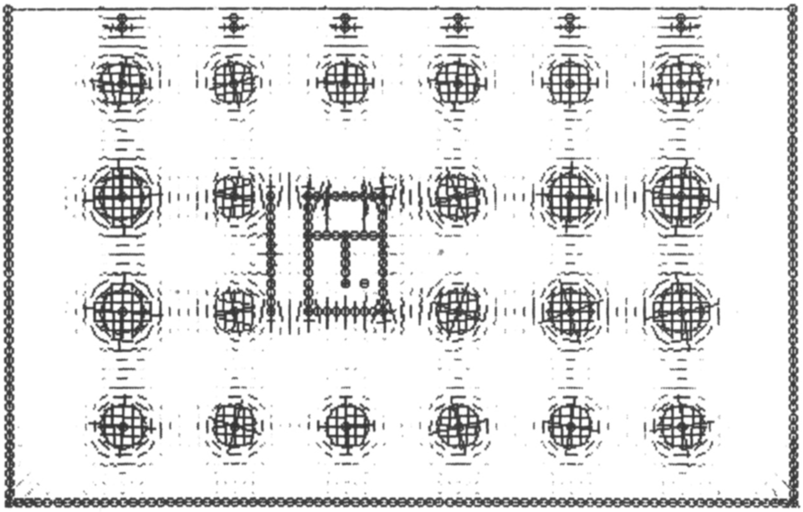


Figure 13.9: House – principle negative bending moments

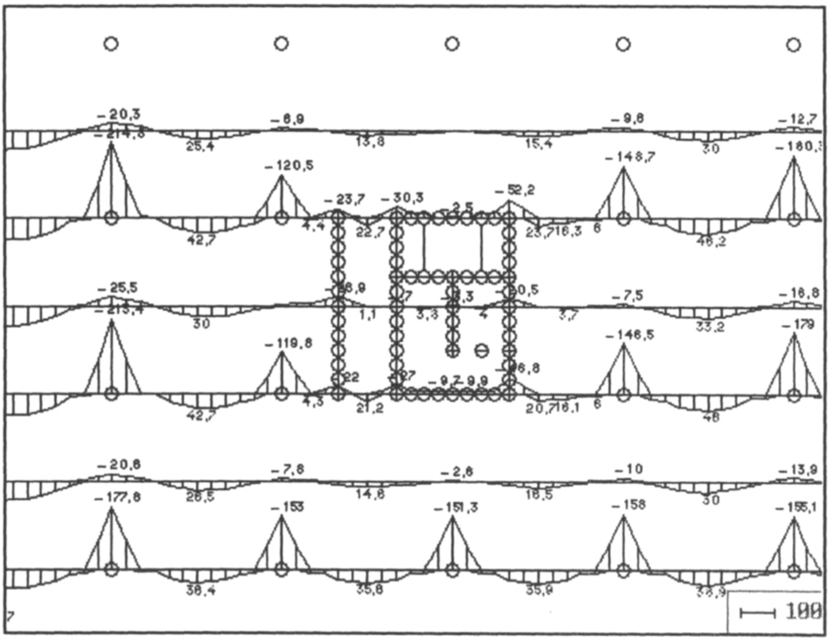


Figure 13.10: House– envelopes of bending moments in X-direction (kNm)

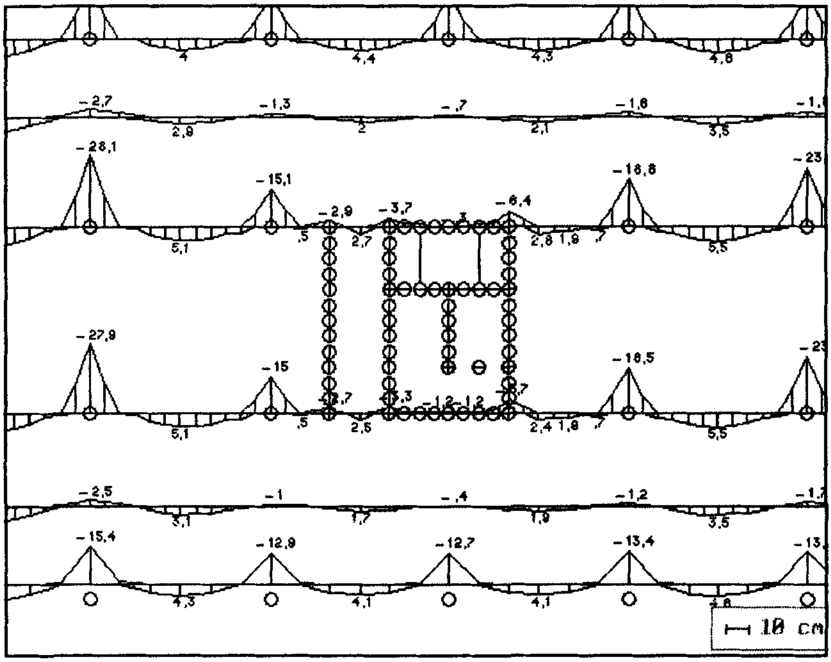


Figure 13.11: House – envelopes of reinforcement in X-direction (cm^2)

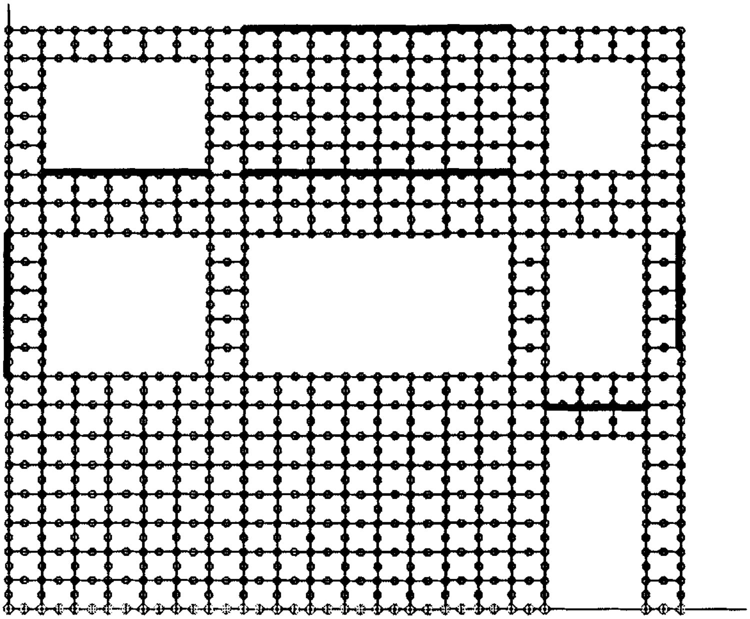


Figure 13.12: Concrete frame (SWALL)

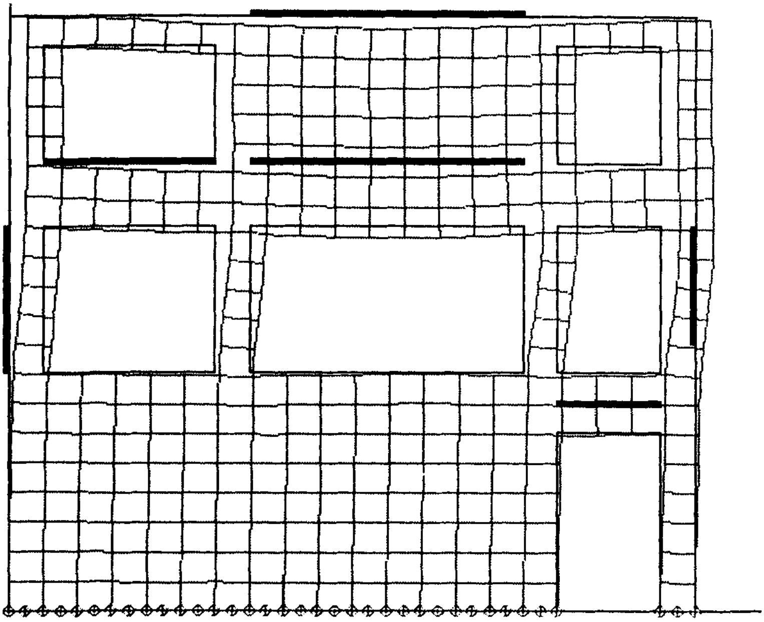


Figure 13.13: Concrete frame – displacements at earthquake (SWALL)

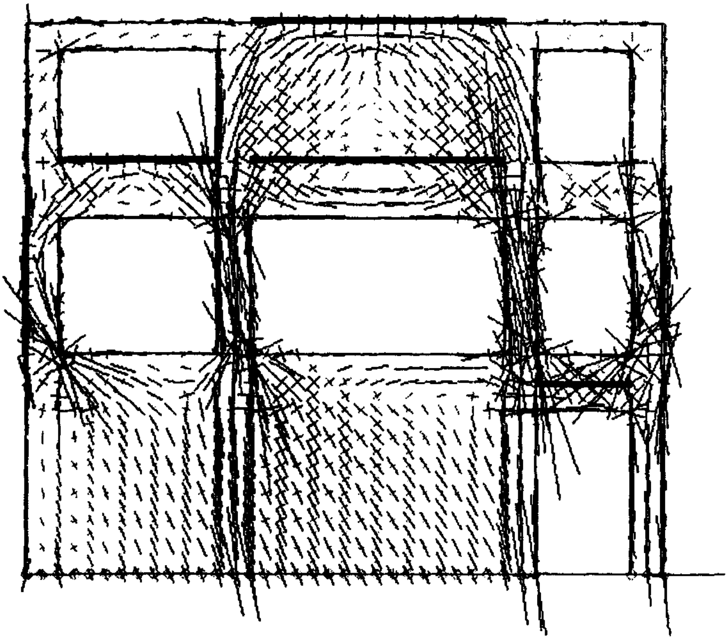


Figure 13.14: Concrete frame – principle stresses at earthquake (SWALL)

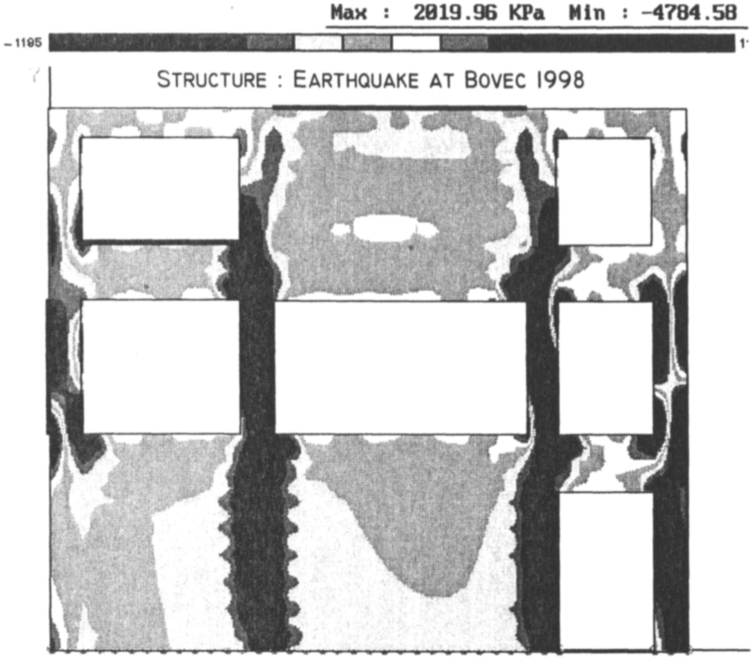


Figure 13.15: Concrete frame – stresses σ_y at earthquake (SWALL)

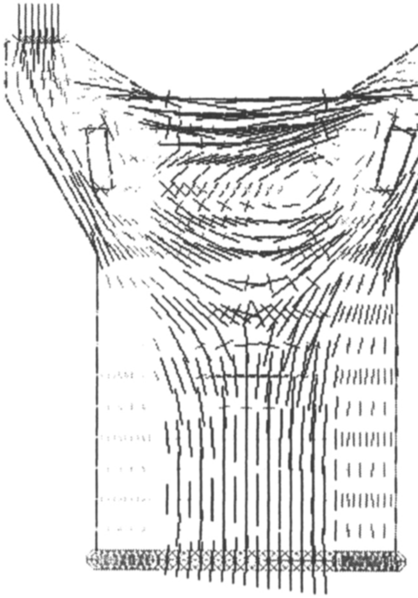


Figure 13.16: A bridge column – principle stresses at earthquake (SWALL)

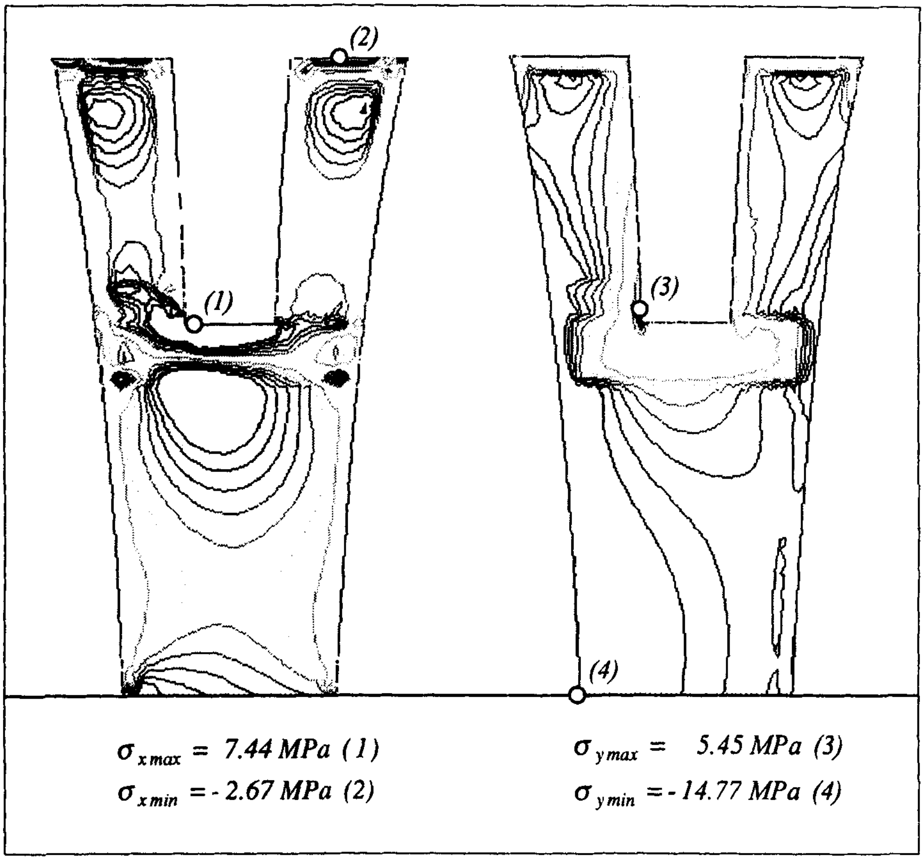


Figure 13.17: A bridge column – stresses σ_x and σ_y at earthquake (SWALL)

Appendix A

Basics of matrix algebra

A.1 Introduction

The fundamental concepts of matrix algebra, definitions and matrix manipulations, are presented in a concise manner to provide adequate background for understanding the text in this book.

A.2 Definitions

A matrix is an array of elements which can be written as:

$$[A]=\begin{bmatrix} A_{11} & A_{12} & \dots & A_{1n} \\ A_{21} & A_{22} & \dots & A_{2n} \\ \vdots & \vdots & . & \vdots \\ A_{m1} & A_{m2} & \dots & A_{mn} \end{bmatrix} \quad \begin{matrix} i=1,2,\dots,m \\ j=1,2,\dots,n \end{matrix}, \qquad (A.1)$$

where i and j indicate the number of rows and columns in the array, the above matrix is thus of order $i \times j$. The coefficients A are the elements which constitute the matrix $[A]$. Their locations are determined by their subscripts; for instance, the element A_{i1} is in the i^{th} row and in the first column, A_{2j} is in the second row and in the j^{th} column.

If i and j are not equal ($i \neq j$), the matrix is a *rectangular matrix*, if $i = j$ the matrix is a *square matrix*, i.e.:

$$[A]=\begin{bmatrix} A_{11} & A_{12} & \dots & A_{1n} \\ A_{21} & A_{22} & \dots & A_{2n} \\ : & : & . & : \\ A_{n1} & A_{n2} & \dots & A_{nn} \end{bmatrix} \tag{A.2}$$

Elements A_{11}, A_{22} ... A_{nn} form the *main diagonal* of a square matrix. For $i = 1$ and $j > 1$, the matrix is said to be a *row matrix*, i.e.,

$$[A]=[A_1 \;\; A_2 \;\; \dots \;\; A_n] \qquad j = 1,2,\dots,n \tag{A.3}$$

and a *column matrix* for $i > 1$ and $j = 1$:

$$\{A\}\equiv[A]=\begin{bmatrix} A_1 \\ A_2 \\ : \\ A_m \end{bmatrix}\equiv\begin{Bmatrix} A_1 \\ A_2 \\ . \\ A_m \end{Bmatrix} \qquad i = 1,2,\dots,m \tag{A.4}$$

The *diagonal matrix* is a square matrix where all off-diagonal elements are zero. A special case of a diagonal matrix is *unit matrix*, symbolised by letter I, where all diagonal elements equal unity.

$$[I]=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{A.5}$$

A matrix is a *symmetric matrix* if

$$A_{ij} = A_{ji},$$

hence all off-diagonal elements are *symmetric about the main diagonal*, i.e.:

$$[A]=\begin{bmatrix} 3 & 5 & 2 & 1 \\ 5 & 7 & 3 & 2 \\ 2 & 3 & 8 & 4 \\ 1 & 2 & 4 & 4 \end{bmatrix}$$

The *transpose* of a matrix is found by writing A_{ij} elements of $[A]$ as A_{ji} elements of $[A]^T$ as shown in the example below:

If

$$[A]=\begin{bmatrix} 1 & 7 & 9 & 4 \\ 5 & 5 & 2 & 1 \\ 3 & 7 & 8 & 2 \end{bmatrix}$$

then

$$[A]^T=\begin{bmatrix} 1 & 5 & 3 \\ 7 & 5 & 7 \\ 9 & 2 & 8 \\ 4 & 1 & 2 \end{bmatrix}$$

The transpose of a square symmetric matrix is the matrix itself:

$$[A]=[A]^T$$

Determinants are defined only for square matrices. Thus, for a given square matrix $[A]$, the determinant is defined as the number that results on performing the following arithmetic operation on $\lfloor A \rfloor$:

$$|A|=\sum_{i=1}^{n} A_{ij}C_{ij} \quad (j=1,2,...,n) \tag{A.7}$$

or

$$|A|=\sum_{j=1}^{n} A_{ij}C_{ij} \quad (i=1,2,...,n) \tag{A.8}$$

The elements A_{ij} in Eqns. (A.7) and (A.8) are the i^{th} row and j^{th} column elements of the matrix $\lfloor A \rfloor$, C_{ij} are the *cofactors* corresponding to the i^{th} row and j^{th} column and are defined as

$$C_{ij}=(-1)^{i+j}M_{ij}, \tag{A.9}$$

where M_{ij} are the *minors* of matrix $\lfloor A \rfloor$, defined as the determinant of the matrix, which results after the i^{th} row and the j^{th} column elements are deleted from the matrix $\lfloor A \rfloor$. The sign of a cofactor is determined by the odd-even judgement: if the sum of the row and the column is even, the cofactor sign is positive and vice versa. Equations (A.7), (A.8) and (A.9) are considered by the following example.

$$[A]=\begin{bmatrix} +5 & -1 & +8 \\ -1 & +7 & -5 \\ +8 & -5 & +4 \end{bmatrix} \tag{A.10}$$

Let us first apply Eqn. (A.7) and arbitrarily select $j = 2$:

$$|A| = \sum_{i=1}^{3} A_{i2} \cdot C_{i2} \quad = A_{12}C_{12} + A_{22}C_{22} + A_{32}C_{32}$$

From Eqn. (A.10) column 2 of matrix $[A]$is:

$$A_{12} = -1 \qquad A_{22} = 7 \qquad A_{32} = -5\,,$$

For $j = 2$ equation (A.9) becomes:

$$C_{ij} = (-1)^{i+2} M_{ij} \quad (i = 1,2,3)\,,$$

therefore:

$$C_{12} = -M_{12}$$

$$C_{22} = +M_{22}$$

$$C_{32} = -M_{32}$$

By definition, the minor M_{12} of matrix $[A]$ is the determinant of that matrix after deleting the first row and the second column, or:

$$[M_{12}] = \begin{bmatrix} -1 & -5 \\ +8 & +4 \end{bmatrix} = (-1) \cdot 4 - (-5) \cdot 8 = 36$$

likewise

$$[M_{22}] = \begin{bmatrix} 5 & 8 \\ 8 & 4 \end{bmatrix} = 5 \cdot 4 - 8 \cdot 8 = -44$$

$$[M_{32}] = \begin{bmatrix} +5 & +8 \\ -1 & -5 \end{bmatrix} = 5 \cdot (-5) - (-1) \cdot 8 = -17\,,$$

hence

$$C_{12} = -36 \qquad C_{22} = -44 \qquad C_{32} = 17$$

and the determinant is:

$$|A| = (-1) \cdot (-36) + 7 \cdot (-44) + (-5) \cdot 17 = -357$$

Let us present some practical rules:

a.) If the elements of any row or column have a common factor, b, then the following holds true:

$$\begin{bmatrix} A_{11}\cdot b & A_{12}\cdot b \\ A_{21} & A_{22} \end{bmatrix} = b\cdot\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

$$\begin{bmatrix} A_{11}\cdot b & A_{12} \\ A_{21}\cdot b & A_{22} \end{bmatrix} = b\cdot\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

$$\begin{bmatrix} A_{11}\cdot b & A_{12}\cdot b \\ A_{21}\cdot b & A_{22}\cdot b \end{bmatrix} = b^2\cdot\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

b.) When rows and corresponding columns are interchanged the value of the determinant *is not altered*:

$$\begin{bmatrix} 2 & 4 \\ 5 & 6 \end{bmatrix} = \begin{bmatrix} 2 & 5 \\ 4 & 6 \end{bmatrix} = -8$$

c.) When two rows (or columns) are interchanged *the sign* of the determinant is changed:

$$\begin{bmatrix} 2 & 4 \\ 5 & 6 \end{bmatrix} = -\begin{bmatrix} 4 & 2 \\ 6 & 5 \end{bmatrix} = 8 = \begin{bmatrix} 5 & 6 \\ 2 & 4 \end{bmatrix}$$

d.) If two rows (or columns) are *identical* then the value of the determinant is zero.

e.) If a matrix $[A]$ has any dependant rows or columns, its determinant equals zero. In the example below the third column is the sum of the first and second column:

$$\left|[A]\right| = \left|\begin{bmatrix} 2 & 4 & 6 \\ 5 & 6 & 11 \\ 7 & 2 & 9 \end{bmatrix}\right| = 0$$

or explicitly choosing $j = 1$ in Eqn. (A.7):

$$|A| = 2\cdot\left|\begin{bmatrix} 6 & 11 \\ 2 & 9 \end{bmatrix}\right| - 4\cdot\left|\begin{bmatrix} 5 & 11 \\ 7 & 9 \end{bmatrix}\right| + 6\cdot\left|\begin{bmatrix} 5 & 6 \\ 7 & 2 \end{bmatrix}\right| =$$

$$= 2\cdot 32 - 4\cdot(-32) + 6\cdot(-32) =$$

$$= 64 + 128 - 192 = 0$$

If the value of the *determinant is zero* the matrix is said to be a *singular matrix.* In structural analysis it always means that *the structure is unstable.*

A.3 Matrix algebra manipulation

A.3.1 Matrix addition and subtraction

Two matrices $[A]_{ixj}$ and $[B]_{ixj}$ can be added (subtracted) by adding (subtracting) each element A_{ij} of matrix $[A]$ to the corresponding element B_{ij} of matrix $[B]$.

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \pm \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} = \begin{bmatrix} (A_{11} \pm B_{11}) & (A_{12} \pm B_{12}) \\ (A_{21} \pm B_{21}) & (A_{22} \pm B_{22}) \end{bmatrix} \qquad \text{(A.11)}$$

or

$$[A] \pm [B] = [C], \qquad \text{(A.12)}$$

where C_{ij} in general is given by

$$C_{ij} = A_{ij} \pm B_{ij} \qquad \text{(A.13)}$$

A.3.2 Matrix multiplication

Multiplication of two matrices $[A]_{i \times j}$ and $[B]_{j \times k}$ is performed by multiplying each element of row m of $[A]_{i \times j}$ by its corresponding elements in column n of $[B]_{j \times k}$ and the products summed. Mathematically this can be written as follows:

$$[A] \cdot [B] = [C], \qquad \text{(A.14)}$$

where an element C_{mn} of $[C]_{i \times k}$ is given by:

$$C_{mn} = \sum_{k=1}^{j} A_{mk} \cdot B_{kn} \qquad \text{(A.15)}$$

and j is the number of columns in $[A]$ or rows in $[B]$.

Numerical example: Given are two matrices $[A]$ and $[B]$.

$$[A] = \begin{bmatrix} 1 & 2 \\ 5 & 4 \end{bmatrix} \qquad [B] = \begin{bmatrix} 2 & 5 & 6 \\ 1 & 2 & 3 \end{bmatrix}$$

Required is product:

$$[A]\cdot[B]=[C]$$

Solution:

$$\begin{bmatrix} 1 & 2 \\ 5 & 4 \end{bmatrix}\cdot\begin{bmatrix} 2 & 5 & 6 \\ 1 & 2 & 3 \end{bmatrix}=\begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \end{bmatrix}$$

From Eqn. (A.15) we get elements C_{ij}:

$$C_{11}=A_{11}\cdot B_{11}+A_{12}\cdot B_{21}=1\cdot 2+2\cdot 1=4$$

$$C_{12}=A_{11}\cdot B_{12}+A_{12}\cdot B_{22}=1\cdot 5+2\cdot 2=9$$

$$C_{13}=A_{11}\cdot B_{13}+A_{12}\cdot B_{23}=1\cdot 6+2\cdot 3=12$$

$$C_{21}=A_{21}\cdot B_{11}+A_{22}\cdot B_{21}=5\cdot 2+4\cdot 1=14$$

$$C_{22}=A_{21}\cdot B_{12}+A_{22}\cdot B_{22}=5\cdot 5+4\cdot 2=33$$

$$C_{23}=A_{21}\cdot B_{13}+A_{22}\cdot B_{23}=5\cdot 6+4\cdot 3=42$$

Matrix $[C]$ is:

$$[C]=\begin{bmatrix} 4 & 9 & 12 \\ 14 & 33 & 42 \end{bmatrix}$$

Note the following rule:

- ❖ *Two matrices can be multiplied only if the number of rows in the first matrix equals the number of columns in the second matrix (j = r).*

$$[A]_{ixj}\cdot[B]_{rxs}=[C]_{ixs} \tag{A.16}$$

A.3.3 Properties of matrix multiplication

Associative law

$$[D]=([A]\cdot[B])\cdot[C]=[A]\cdot([B]\cdot[C]) \tag{A.17}$$

Any element of $[D]$ is given by:

$$D_{ij}=\sum_r\sum_s A_{ir}\cdot B_{rs}\cdot C_{si} \tag{A.18}$$

Distributive law

$$[D]=[A]\cdot([B]+[C])=[A]\cdot[B]+[A]\cdot[C] \tag{A.19}$$

Commutative law

$$[A]\cdot[B]\neq[B]\cdot[A] \tag{A.20}$$

- ❖ *If both $[A]$ and $[B]$ are symmetric matrices their product in general does not yield a symmetric matrix*
- ❖ *If a matrix is multiplied or divided by a scalar quantity, a, it is equivalent to multiplying or dividing each element of a matrix by a.*

$$2\cdot\begin{bmatrix}5 & 4\\ 7 & 2\end{bmatrix}=\begin{bmatrix}2\cdot 5 & 2\cdot 4\\ 2\cdot 7 & 2\cdot 2\end{bmatrix}=\begin{bmatrix}10 & 8\\ 14 & 4\end{bmatrix}$$

A.4 Matrix inversion

Consider first a linear equation

$$y=a\cdot x$$

and solving for unknown x we get:

$$x=\frac{y}{a}=a^{-1}\cdot y \tag{A.21}$$

Similarly, if x and y represent vectors, the solution is found by matrix inversion $[A]^{-1}$ defined as:

$$[A]\cdot[A]^{-1}=[I], \tag{A.22}$$

which can be accomplished for square matrices only.

The inversion procedure is as follows:

Step 1: Form a new matrix of co-factors C_{ij} using Eqn. (A.9), referred to as $adj[A]$.

$$C_{ij}=(-1)^{i+j}\cdot M_{ij}$$

Step 2: Transpose $adj[A]$ and call the result $adj[A]^T$ (transposed matrix of co-factors).

Step 3: Calculate the determinant of matrix $[A]$

$$|A| = \sum_{i=1}^{n} A_{ij} C_{ij} \quad (j = 1,2,\ldots,n)$$

Step 4: Divide $adj[A]^T$ by the determinant to obtain the inverted matrix:

$$[A]^{-1} = \frac{adj[A]^T}{|A|} \tag{A.23}$$

Numerical example

$$[A] = \begin{bmatrix} +5 & -1 & +8 \\ -1 & +7 & -5 \\ +8 & -5 & +4 \end{bmatrix}$$

Step 1:

$$C_{11} = M_{11} \qquad C_{12} = -M_{12} \qquad C_{13} = M_{13}$$

$$C_{21} = -M_{21} \qquad C_{22} = M_{22} \qquad C_{23} = -M_{23}$$

$$C_{31} = M_{31} \qquad C_{32} = -M_{32} \qquad C_{33} = M_{33},$$

where:

$$M_{11} = \begin{vmatrix} +7 & -5 \\ -5 & +4 \end{vmatrix} = 3 \qquad M_{12} = \begin{vmatrix} -1 & -5 \\ +8 & +4 \end{vmatrix} = 36 \qquad M_{13} = \begin{vmatrix} -1 & +7 \\ +8 & -5 \end{vmatrix} = -51$$

$$M_{21} = \begin{vmatrix} -1 & +8 \\ -5 & +4 \end{vmatrix} = 36 \qquad M_{22} = \begin{vmatrix} +5 & +8 \\ +8 & +4 \end{vmatrix} = -44 \qquad M_{23} = \begin{vmatrix} +5 & -1 \\ +8 & -5 \end{vmatrix} = -17$$

$$M_{31} = \begin{vmatrix} -1 & +8 \\ +7 & -5 \end{vmatrix} = -51 \qquad M_{32} = \begin{vmatrix} +5 & +8 \\ -1 & -5 \end{vmatrix} = -17 \qquad M_{33} = \begin{vmatrix} +5 & -1 \\ -1 & +7 \end{vmatrix} = 34$$

Hence:

$$adj[A] = \begin{bmatrix} +3 & -36 & -51 \\ -36 & -44 & +17 \\ -51 & +17 & +34 \end{bmatrix}$$

Step 2: Transpose matrix $adj\left[A\right]$:

$$adj\left[A\right]^T = \begin{bmatrix} +3 & -36 & -51 \\ -36 & -44 & +17 \\ -51 & +17 & +34 \end{bmatrix}$$

Step 3: Calculate determinant $\left|A\right| = -357$

Step 4: Divide adj $\left[A\right]^T$ by the determinant of matrix $\left[A\right]$:

$$\left[A\right]^{-1} = \frac{adj\left[A\right]^T}{\left|A\right|} = \frac{1}{-357} \cdot \begin{bmatrix} +3 & -36 & -51 \\ -36 & -44 & +17 \\ -51 & +17 & +34 \end{bmatrix}$$

$$\left[A\right]^{-1} = \begin{bmatrix} -0.00840 & +0.10084 & +0.14286 \\ +0.10084 & +0.12325 & -0.04762 \\ +0.14286 & -0.04762 & -0.09524 \end{bmatrix}$$

Check

$$\left[A\right] \cdot \left[A\right]^{-1} = \left[I\right],$$

hence

$$\begin{bmatrix} +5 & -1 & +8 \\ -1 & +7 & -5 \\ +8 & -5 & +4 \end{bmatrix} \cdot \begin{bmatrix} +3 & -36 & -51 \\ -36 & -44 & +17 \\ -51 & +17 & +34 \end{bmatrix} \cdot \frac{1}{-357} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Since the identity matrix is obtained, the result of inversion is correct.

A.5 A solution of simultaneous equations by matrix inversion

The use of matrix algebra is of great importance in structural analysis. As the inversion of a matrix is done once only and the coefficients are the properties of structure it is easy to calculate multiple load cases by changing the right hand side vector only. Suppose that equilibrium equations, as shown in Ch. 9, are explicitly written as:

$$\begin{aligned} +5x \quad -y \quad +8z &= A \\ -x \quad +7y \quad -5z &= B \\ +8x \quad -5y \quad +4z &= C \end{aligned} \tag{A.24}$$

or shorter in matrix form

$$[A]\cdot\{X\}=\{B\} \tag{A.25}$$

then the unknowns are calculated using the equation:

$$\{X\}=[A]^{-1}\cdot\{B\}, \tag{A.26}$$

$[A]^{-1}$ is the inverted matrix of coefficients, dependent on structural properties only. A solution for a general loading is then

$$\begin{Bmatrix} x \\ y \\ z \end{Bmatrix}=\begin{bmatrix} -0.00840 & +0.10084 & +0.14286 \\ +0.10084 & +0.12325 & -0.04762 \\ +0.14286 & -0.04762 & -0.09524 \end{bmatrix}\cdot\begin{Bmatrix} A \\ B \\ C \end{Bmatrix} \tag{A.27}$$

and for an arbitrary load vector $\{B\}$ a direct multiplication gives unknowns $\{X\}$. It is important in structural analysis to understand that the first column represents a solution for unit force A, the second column for unit force B and the third column for unit force C.

A.6 Matrix partitioning

In structural analysis problems it is often convenient to partition a matrix into smaller submatrices (see Ch. 9.4) in which we group together known values of forces or displacements and on the other hand unknown values of displacements and reactions. Drawing vertical and horizontal dotted lines, as shown below, represents this descripition.

$$[K]=\left[\begin{array}{cc|cc} K_{11} & K_{12} & K_{13} & K_{14} \\ K_{21} & K_{22} & K_{23} & K_{24} \\ \hline K_{31} & K_{32} & K_{33} & K_{34} \\ K_{41} & K_{42} & K_{43} & K_{44} \end{array}\right]$$

or in shorthand matrix notation:

$$[K]=\begin{bmatrix} K_{\alpha\alpha} & K_{\alpha\beta} \\ K_{\beta\alpha} & K_{\beta\beta} \end{bmatrix} \tag{A.28}$$

Properties of partitioned matrices

a.) Two partitioned matrices $[A]_{ixj}$ *and* $[B]_{ixj}$ can be added or subtracted in terms of their submatrices if and only if both matrices are of the same order and partitioned in the same manner.

b.) Two partitioned matrices $[A]_{m\times n}$ and $[B]_{n\times q}$ can be multiplied in terms of their submatrices if and only if the submatrices of $[A]$ and the corresponding submatrices of $[B]$ obey the law of matrix multiplication, hence the multiplication of submatrices $[A]_{r\times s}$ and $[B]_{p\times q}$ is possible only if $p = s$.

Example:

$$[A]=\left[\begin{array}{cc|c} 1 & 2 & 3 \\ 4 & 5 & 6 \\ \hline 7 & 8 & 9 \end{array}\right] \qquad [B]=\left[\begin{array}{c|c} 3 & 5 \\ 1 & 3 \\ \hline 4 & 2 \end{array}\right]$$

$$[A]\cdot[B]=\left[\begin{array}{cc|c} 1 & 2 & 3 \\ 4 & 5 & 6 \\ \hline 7 & 8 & 9 \end{array}\right]\cdot\left[\begin{array}{c|c} 3 & 5 \\ 1 & 3 \\ \hline 4 & 2 \end{array}\right]=$$

$$\left[\begin{array}{c|c} \begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix}\cdot\begin{bmatrix} 3 \\ 1 \end{bmatrix}+\begin{bmatrix} 3 \\ 6 \end{bmatrix}\cdot[4] & \begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix}\cdot\begin{bmatrix} 5 \\ 3 \end{bmatrix}+\begin{bmatrix} 3 \\ 6 \end{bmatrix}\cdot[2] \\ \hline [7\;\;8]\cdot\begin{bmatrix} 3 \\ 1 \end{bmatrix}+[9]\cdot[4] & [7\;\;8]\cdot\begin{bmatrix} 5 \\ 3 \end{bmatrix}+[9]\cdot[2] \end{array}\right]$$

The evaluation of all submatrices multiplication gives:

$$[A]\cdot[B]=\begin{bmatrix} 17 & 17 \\ 41 & 47 \\ 65 & 77 \end{bmatrix}$$

Inversion of a partitioned matrix

The inverse of a partitioned (*stiffness*) matrix can be obtained in terms of its submatrices as a new matrix $[D]$, referred to as a *flexibility matrix* in structural analysis:

$$[D]=[K]^{-1} \tag{A.29}$$

or given in a partitioned form

$$\begin{bmatrix} D_{\alpha\alpha} & D_{\alpha\beta} \\ D_{\beta\alpha} & D_{\beta\beta} \end{bmatrix}=\begin{bmatrix} K_{\alpha\alpha} & K_{\alpha\beta} \\ K_{\beta\alpha} & K_{\beta\beta} \end{bmatrix}^{-1}, \tag{A.30}$$

where the submatrices of $[D]$ are given by:

$$[D_{\alpha\alpha}]=\left[[K_{\alpha\alpha}]-[K_{\alpha\beta}]\cdot[K_{\beta\beta}]^{-1}\cdot[K_{\beta\alpha}]\right]^{-1} \tag{A.31}$$

$$\left[D_{\beta\beta}\right]=\left[\left[K_{\beta\beta}\right]-\left[K_{\beta\alpha}\right]\cdot\left[K_{\alpha\alpha}\right]^{-1}\cdot\left[K_{\alpha\beta}\right]\right]^{-1} \tag{A.32}$$

$$\left[D_{\alpha\beta}\right]=-\left[K_{\alpha\alpha}\right]^{-1}\cdot\left[K_{\alpha\beta}\right]\cdot\left[D_{\beta\beta}\right] \tag{A.33}$$

$$\left[D_{\beta\alpha}\right]=-\left[K_{\beta\beta}\right]^{-1}\cdot\left[K_{\beta\alpha}\right]\cdot\left[D_{\alpha\alpha}\right] \tag{A.34}$$

The above procedure is used in the displacement method in Ch. 9, where unknown displacements were determined in a simplified manner as the submatrix $\left[K_{\beta\beta}\right]$ equals zero due to the zero displacements at the supports.

Appendix B

Tables

***Table B.1**: Functions ψ for a fixed-pinned element*

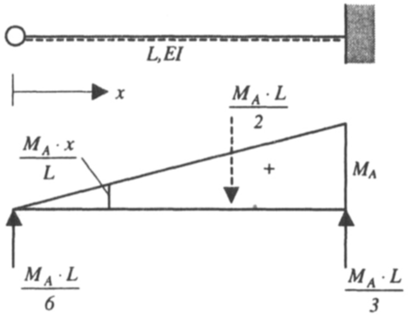
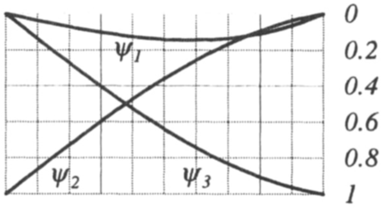
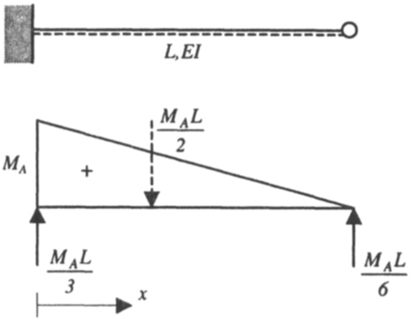
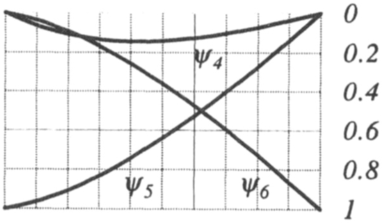
 	$M_{50} = \frac{3EI}{L^2}$ $\xi = \frac{x}{L}$ $K_A = \frac{M_A}{M_{50}}$ $\psi_1 = 0.5(\xi - \xi^3) \cdot K_A$ $\psi_2 = 1 - \xi - 0.5(\xi - \xi^3) \cdot K_A$ $\psi_3 = \xi + 0.5(\xi - \xi^3) \cdot K_A$
 	$M_{50} = \frac{3EI}{L^2}$ $\xi = \frac{x}{L}$ $K_A = \frac{M_A}{M_{50}}$ $\psi_4 = 0.5(2\xi - 3\xi^2 + \xi^3) \cdot K_A$ $\psi_5 = 1 - \xi + 0.5(2\xi - 3\xi^2 + \xi^3) \cdot K_A$ $\psi_6 = \xi - 0.5(2\xi - 3\xi^2 + \xi^3) \cdot K_A$

Table B.2: Functions ψ for a fixed element

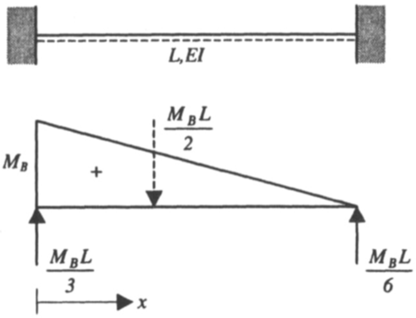
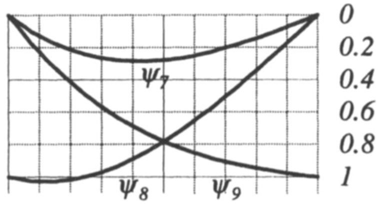
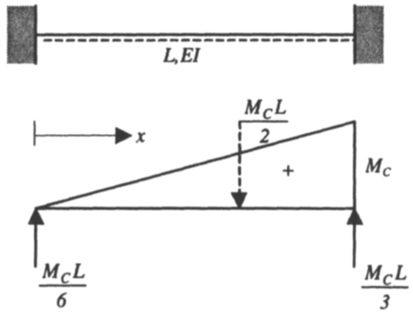
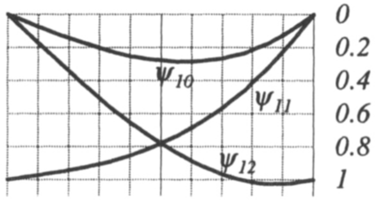
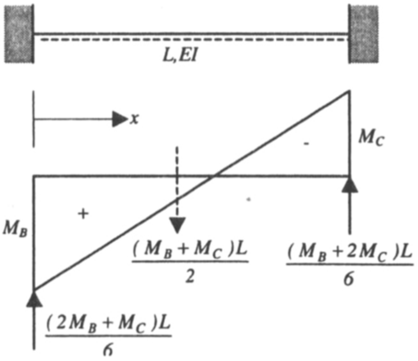
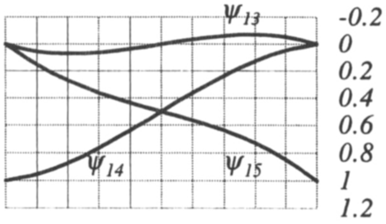
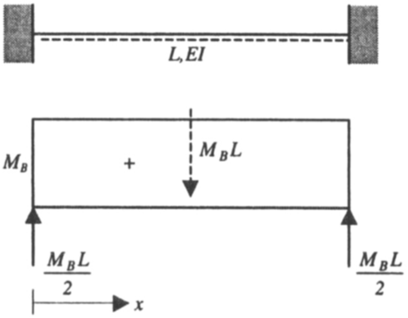
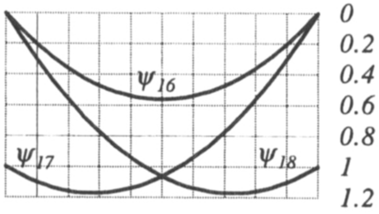
 	$M_{60} = \frac{6EI}{L^2}$ $\xi = \frac{x}{L}$ $K_B = \frac{M_B}{M_{60}}$ $\psi_7 = \left(2\xi - 3\xi^2 + \xi^3\right) \cdot K_B$ $\psi_8 = 1 - \xi + \left(2\xi - 3\xi^2 + \xi^3\right) \cdot K_B$ $\psi_9 = \xi + \left(2\xi - 3\xi^2 + \xi^3\right) \cdot K_B$
 	$M_{60} = \frac{6EI}{L^2}$ $\xi = \frac{x}{L}$ $K_C = \frac{M_C}{M_{60}}$ $\psi_{10} = \left(\xi - \xi^3\right) \cdot K_C$ $\psi_{11} = 1 - \xi + \left(\xi - \xi^3\right) \cdot K_C$ $\psi_{12} = \xi + \left(\xi - \xi^3\right) \cdot K_C$

Table B.3: Functions ψ for a fixed element

 	$M_{60}=\frac{6EI}{L^2}$ $\xi=\frac{x}{L}$ $K_B=\frac{M_B}{M_{60}}$ $K_C=\frac{M_C}{M_{60}}$ $\psi_{13}=\psi_7-\psi_{10}$ $\psi_{14}=1-\xi+\psi_8-\psi_{11}$ $\psi_{15}=\xi+\psi_9-\psi_{12}$
 	$M_{60}=\frac{6EI}{L^2}$ $\xi=\frac{x}{L}$ $K_B=\frac{M_B}{M_{60}}$ $\psi_{16}=3\left(\xi-\xi^2\right)\cdot K_B$ $\psi_{17}=1-\xi+3\left(\xi-\xi^2\right)\cdot K_B$ $\psi_{18}=\xi+3\left(\xi-\xi^2\right)\cdot K_B$

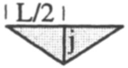
		1	2	3	4	5	6	7
								$\int j^2 ds$
1		jk	$\frac{1}{2}jk$	$\frac{1}{2}j(k_1+k_2)$	0	$\frac{1}{4}jk$	$\frac{1}{2}jk$	j^2
2		$\frac{1}{2}jk$	$\frac{1}{3}jk$	$\frac{1}{6}j(k_1+2k_2)$	$-\frac{1}{6}jk$	0	$\frac{1}{6}jk(1+\alpha)$	$\frac{1}{3}j^2$
3		$\frac{1}{2}jk$	$\frac{1}{6}jk$	$\frac{1}{6}j(2k_1+k_2)$	$\frac{1}{6}jk$	$\frac{1}{4}jk$	$\frac{1}{6}jk(1+\beta)$	$\frac{1}{3}j^2$
4		$\frac{1}{2}jk$	$\frac{1}{4}jk$	$\frac{1}{4}j(k_1+k_2)$	0	$\frac{1}{8}jk$	$\frac{jk}{12\beta}\left(3-4\alpha^2\right)$	$\frac{1}{3}j^2$
5		$\frac{1}{2}jk$	$\frac{1}{6}jk(1+\gamma)$	$\frac{1}{6}j\left[k_1(1+\delta) + k_2(1+\gamma)\right]$	$\frac{1}{6}jk(1-2\gamma)$	$\frac{1}{12}jk(1+2\delta-\gamma)$	$\frac{jk}{6\beta\gamma}\left(2\gamma-\gamma^2-\alpha^2\right)$ $\gamma \ge \alpha$	$\frac{1}{3}j^2$
6		$\frac{1}{2}k(j_1+j_2)$	$\frac{1}{6}k(j_1+2j_2)$	$\frac{1}{6}\left[j_1(2k_1+k_2) + j_2(k_1+2k_2)\right]$	$\frac{1}{6}k(j_1-j_2)$	$\frac{1}{4}j_1k$	$\frac{1}{6}k\begin{bmatrix} j_1(1+\beta)+ \\ +j_2(1+\alpha) \end{bmatrix}$	$\frac{1}{3}\begin{bmatrix} j_1j_1+j_1j_2 \\ +j_2j_2 \end{bmatrix}$
7		0	$-\frac{1}{6}jk$	$\frac{1}{6}j(k_1-k_2)$	$\frac{1}{3}jk$	$\frac{1}{4}jk$	$\frac{1}{6}jk(1-2\alpha)$	$\frac{1}{3}j^2$

Table B.4**: Values of volume integrals* $\int \frac{M_j M_k}{EI} ds$; $EI \cdot \Delta = L \cdot$ ***table value

		1	2	3	4	5	6	7
								$\int j^2 ds$
8	 quadratic parabola	$\frac{2}{3} jk$	$\frac{1}{3} jk$	$\frac{1}{3} j(k_1 + k_2)$	0	$\frac{1}{6} jk$	$\frac{1}{3} jk(1+\alpha\beta)$	$\frac{8}{15} j^2$
9	 quadratic parabola	$\frac{1}{3} jk$	$\frac{1}{6} jk$	$\frac{1}{6} j(k_1 + k_2)$	0	$\frac{1}{12} jk$	$\frac{1}{6} jk(1-2\alpha\beta)$	$\frac{1}{5} j^2$
10	 quadratic parabola	$\frac{2}{3} jk$	$\frac{1}{4} jk$	$\frac{1}{12} j(5k_1 + 3k_2)$	$\frac{1}{6} jk$	$\frac{7}{24} jk$	$\frac{1}{12} jk\left(5-\alpha-\alpha^2\right)$	$\frac{8}{15} j^2$
11	 quadratic parabola	$\frac{2}{3} jk$	$\frac{5}{12} jk$	$\frac{1}{12} j(3k_1 + 5k_2)$	$-\frac{1}{6} jk$	$\frac{1}{24} jk$	$\frac{1}{12} jk\left(5-\beta-\beta^2\right)$	$\frac{8}{15} j^2$
12	 quadratic parabola	$\frac{1}{3} jk$	$\frac{1}{4} jk$	$\frac{1}{12} j(k_1 + 3k_2)$	$-\frac{1}{6} jk$	$-\frac{1}{24} jk$	$\frac{1}{12} jk\left(1+\alpha+\alpha^2\right)$	$\frac{1}{5} j^2$
13	 quadratic parabola	$\frac{1}{3} jk$	$\frac{1}{12} jk$	$\frac{1}{12} j(3k_1 + k_2)$	$\frac{1}{6} jk$	$\frac{5}{24} jk$	$\frac{1}{12} jk\left(1+\beta+\beta^2\right)$	$\frac{1}{5} j^2$
14	 cubic parabola	$\frac{1}{4} jk$	$\frac{1}{5} jk$	$\frac{1}{20} j(k_1 + 4k_2)$	$-\frac{3}{20} jk$	$-\frac{1}{20} jk$	$\frac{1}{20} jk(1+\alpha)\left(1+\alpha^2\right)$	$\frac{1}{7} j^2$

Table B.5**: Values of volume integrals* $\int \frac{M_j M_k}{EI} ds$; $EI \cdot \Delta = L \cdot$ ***table value

***Table B6*:**
Fixed end moments for a fixed element

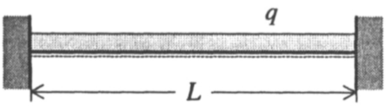
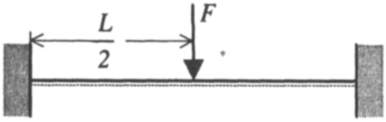
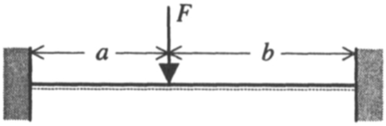
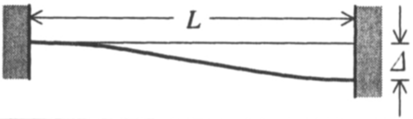
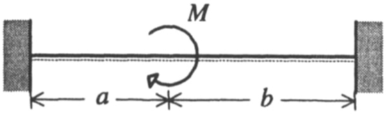
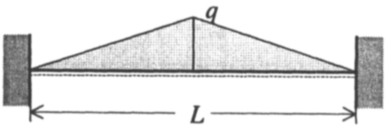
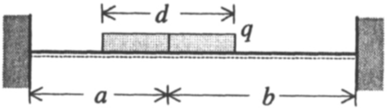
$\frac{qL^2}{12}$	↺		↻	$\frac{qL^2}{12}$
$\frac{FL}{8}$	↺		↻	$\frac{FL}{8}$
$\frac{Fab^2}{L^2}$	↺		↻	$\frac{Fa^2b}{L^2}$
$\frac{6EI}{L^2}\Delta$	↺		↻	$-\frac{6EI}{L^2}\Delta$
$\frac{Mb}{L^2}(2a-b)$	↺		↻	$\frac{Ma}{L^2}(2b-a)$
$\frac{qL^2}{30}$	↺		↻	$\frac{qL^2}{20}$
$\frac{5qL^2}{96}$	↺		↻	$\frac{5qL^2}{96}$
	↺		↻	
$\frac{qd}{L^2}\left(ab^2+\frac{(a-2b)d^2}{12}\right)$				$\frac{qd}{L^2}\left(a^2b+\frac{(b-2a)d^2}{12}\right)$

***Table B7*:**
Fixed end moments for a pinned-fixed element

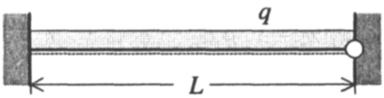
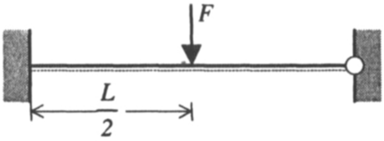
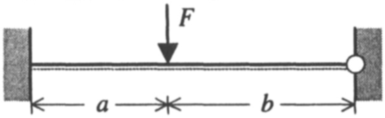
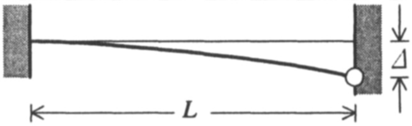
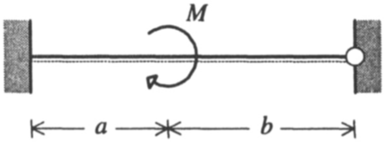
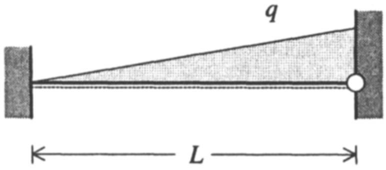
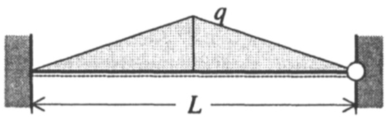
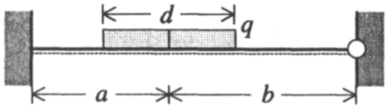
$\frac{qL^2}{8}$	↺		
$\frac{3FL}{16}$	↺		
$\frac{Fab}{2L^2}(L+b)$	↺		
$\frac{3EI}{L^2}\Delta$	↺		
$\frac{M}{2L^2}\left(L^2-3b^2\right)$	↺		
$\frac{7}{120}qL^2$	↺		
$\frac{5}{64}qL^2$	↺		
$\frac{qbd}{8L^2}\left(4a(L+b)-d^2\right)$	↺		

***Table B8**: Polar moments of inertia I_X*

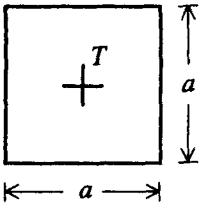
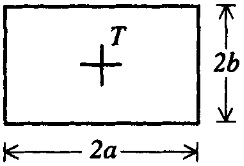
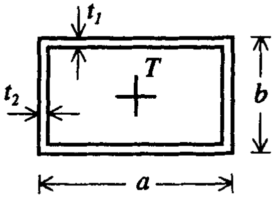
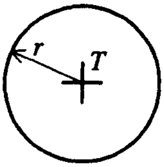
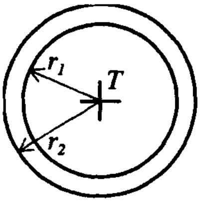
	$J_x = 0.1406a^4$
	$J_x = ab^3\left[\frac{16}{3} - 3.36\frac{b}{a}\left(1 - \frac{b^4}{12a^4}\right)\right]$
	$J_x = \frac{2t_1t_2(a-t_2)^2(b-t_1)}{at_2 + bt_1 - t_2^2 - t_1^2}$
	$J_x = \frac{\pi r^4}{2}$
	$J_x = \frac{\pi\left(r_2^4 - r_1^4\right)}{2}$

***Table B8**: Polar moments of inertia* I_X (continued)

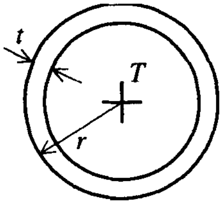
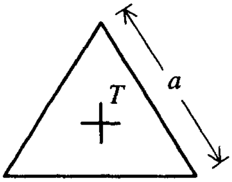
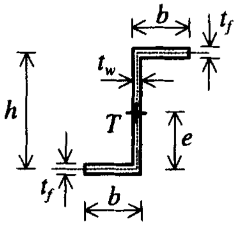
	$J_x = 8\pi r t^3$
	$J_x = \dfrac{a^4\sqrt{3}}{80}$
	$J_x = \dfrac{2bt_f^3 + ht_w^3}{3}$

***Table B9**: Shear shape factors*

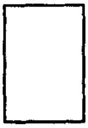
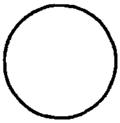
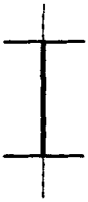
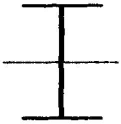
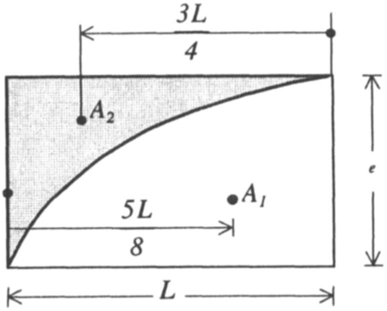
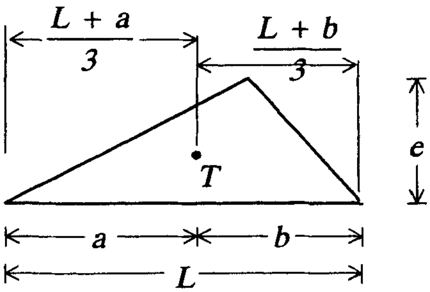
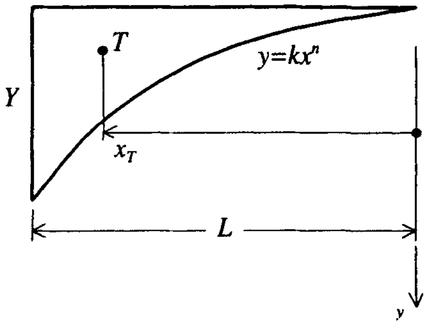
	$K = 1.2$
	$K = 1.11$
	$K \approx \frac{1.2A}{A_f}$ A_f... *flange area*
	$K \approx \frac{A}{A_w}$ A_w... *web area*

Table B10: Areas and centres of gravity

	$A_1 = \frac{2eL}{3}$ $A_2 = \frac{eL}{3}$
	$A = \frac{eL}{3}$
	$A = \frac{YL}{n+1}$ $x_T = \frac{n+1}{n+2}L$

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Index