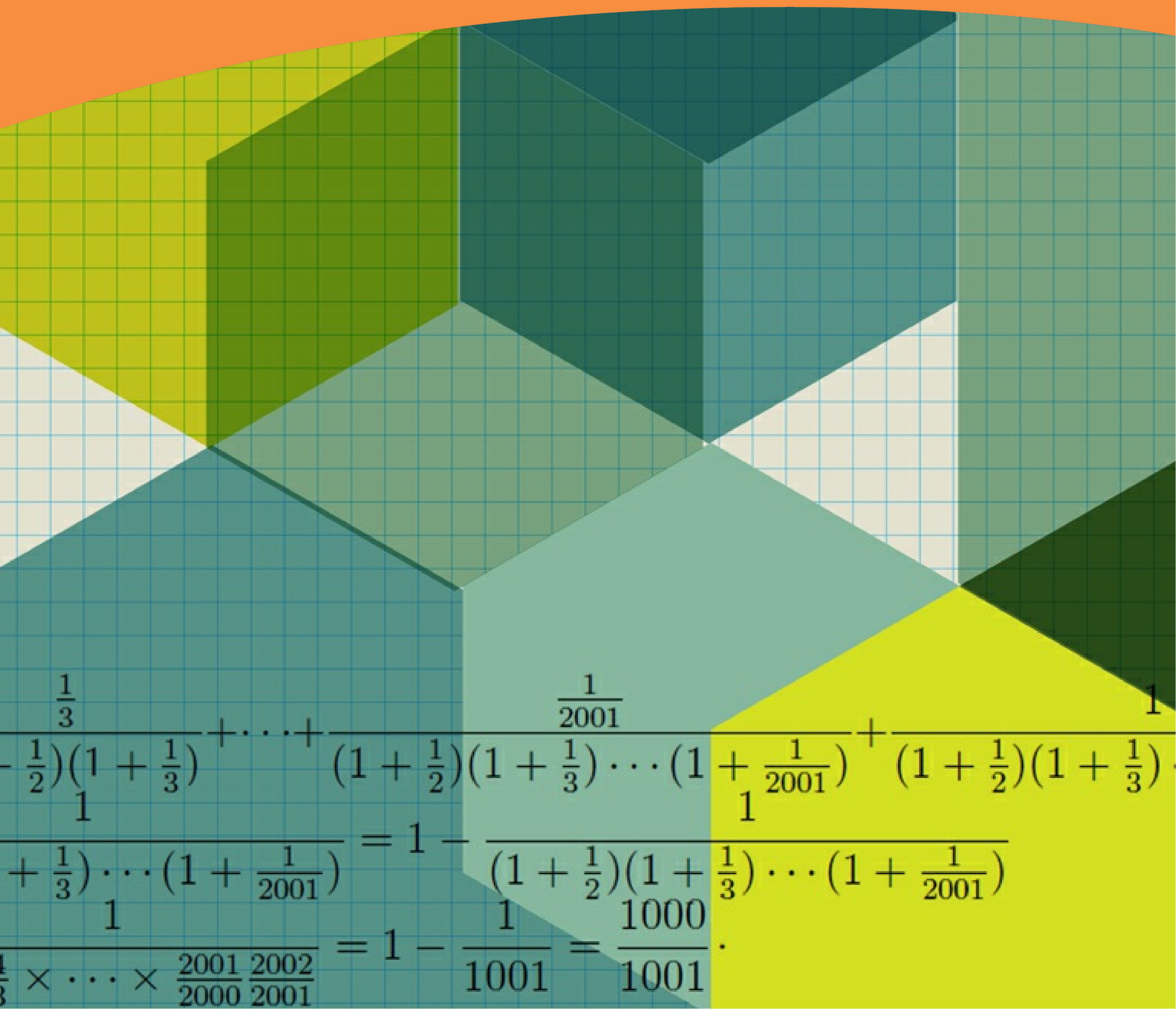
Elementary Algebra Exercise Book II

Hao Wang; Wenlong Wang



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WENLONG WANG AND HAO WANG

ELEMENTARY ALGEBRA EXERCISE BOOK II

Elementary Algebra Exercise Book II
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CONTENTS

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PREFACE

The series of elementary algebra exercise books is designed for undergraduate students with any background and senior high school students who like challenging problems. This series should be useful for non-math college students to prepare for GRE general test - quantitative reasoning and GRE subject test - mathematics. All the books in this series are independent and helpful for learning elementary algebra knowledge.

The number of stars represents the difficulty of the problem: the least difficult problem has zero star and the most difficult problem has five stars. With this difficulty indicator, each reader can easily pick suitable problems according to his/her own level and goal.

Many thanks to Lina Zhang for translating and typing the our handwriting notes into Latex.

4 TRIGONOMETRIC FUNCTIONS

4.1 Given $-\frac{\pi}{2} < x < 0$ and $\sin x + \cos x = \frac{1}{5}$, (1) find the value of $\sin x - \cos x$; (2) find the value of $\dfrac{3\sin^2\frac{x}{2} - 2\sin\frac{x}{2}\cos\frac{x}{2} + \cos^2\frac{x}{2}}{\tan x + \cot x}$.

Solution: (1) The equation $\sin x + \cos x = \frac{1}{5}$ implies that $\sin x = \frac{1}{5} - \cos x$. Substitute it into $\sin^2 x + \cos^2 x = 1$ to obtain $25\cos^2 x - 5\cos x - 12 = 0$. Then, $\cos x = -\frac{3}{5}$ or $\cos x = \frac{4}{5}$. Since $-\frac{\pi}{2} < x < 0$, we have $\cos x = \frac{4}{5}$ and $\sin x = -\frac{3}{5}$. Hence, $\sin x - \cos x = -\frac{7}{5}$.

(2) The expression is equal to $\dfrac{2\sin^2\frac{x}{2} - \sin x + 1}{\frac{\sin x}{\cos x} + \frac{\cos x}{\sin x}} = (2 - \cos x - \sin x) \cdot \sin x \cdot \cos x = (2 - \frac{4}{5} + \frac{3}{5}) \cdot \frac{-3}{5} \cdot \frac{4}{5} = -\frac{108}{125}$.

4.2 Find the range of $y = \dfrac{\sin x + 1}{\cos x + 2}$.

Solution: The equation implies that $y\cos x + 2y = \sin x + 1$, then $\sin x - y\cos x = 2y - 1 \Rightarrow \sqrt{1+y^2}\sin(x-\phi) = 2y - 1$, where $\sin\phi = \dfrac{y}{\sqrt{1+y^2}}$. Since $\sin(x-\phi) \leqslant 1$, that is $\sqrt{1+y^2} \geqslant |2y-1|$. Squaring both sides of the equation, we can obtain $3y^2 - 4y \leqslant 0$. Therefore, $0 \leqslant y \leqslant \frac{4}{3}$, that is, the range of y is $[0, \frac{4}{3}]$.

4.3 Given $f(\theta) = -\frac{1}{2} + \dfrac{\sin\frac{5}{2}\theta}{2\sin\frac{\theta}{2}}$ $(0 < \theta < \pi)$, (1) express $f(\theta)$ as a polynomial of $\cos\theta$. (2) If $a \in R$, find the range of a where there is at least one intersection of the curve $y = a\cos\theta + a$ with the curve $y = f(\theta)$.

Solution: (1) $f(\theta) = \dfrac{\sin\frac{5}{2}\theta - \sin\frac{1}{2}\theta}{2\sin\frac{1}{2}\theta} = \dfrac{\cos\frac{3}{2}\theta\sin\theta}{\sin\frac{1}{2}\theta} = 2\cos\frac{3}{2}\theta\cos\frac{\theta}{2} = \cos 2\theta + \cos\theta = 2\cos^2\theta + \cos\theta - 1$.
(2) According to the given condition and (1), we have

$$\begin{cases} y = 2\cos^2\theta + \cos\theta - 1 \\ y = a\cos\theta + a \end{cases}$$

It is easy to figure out that $(\cos\theta + 1)(2\cos\theta - 1) = a(\cos\theta + 1)$. Since $0 < \theta < \pi$ and $\cos\theta + 1 \neq 0$, we have $2\cos\theta - 1 = a$. On the other hand, $-1 < \cos\theta < 1$, thus

$-1 < \dfrac{a+1}{2} < 1$, which is equivalent to $-3 < a < 1$.

4.4 Given $\tan\alpha = \dfrac{\sin\beta - \cos\beta}{\sin\beta + \cos\beta}$, show $\sin\beta - \cos\beta = \pm\sqrt{2}\sin\alpha$.

Solution: The equation is equivalent to $\cot\alpha = \dfrac{\sin\beta + \cos\beta}{\sin\beta - \cos\beta}$, then $\cot^2\alpha + 1 = (\dfrac{\sin\beta + \cos\beta}{\sin\beta - \cos\beta})^2 + 1 = \dfrac{2}{1 - 2\sin\beta\cos\beta}$, hence $\dfrac{1}{\sin^2\alpha} = \dfrac{2}{(\sin\beta - \cos\beta)^2}$, that is $(\sin\beta - \cos\beta)^2 = 2\sin^2\alpha$. Thus, $\sin\beta - \cos\beta = \pm\sqrt{2}\sin\alpha$.

4.5 Given $e^x - e^{-x} = 2\tan\theta, e^x + e^{-x} = 2\sec\theta, 0 < \theta < \dfrac{\pi}{2}$, solve for x.

Proof: Adding the given equations, we obtain $e^x = \tan\theta + \sec\theta = \dfrac{1 + \sin\theta}{\cos\theta} = \dfrac{1 - \cos(\frac{\pi}{2} + \theta)}{\sin(\frac{\pi}{2} + \theta)} = \tan(\dfrac{\pi}{4} + \dfrac{\theta}{2})$. Since $e^x > 0, 0 < \theta < \dfrac{\pi}{2}$, then $\dfrac{\pi}{4} < \dfrac{\pi}{4} + \dfrac{\theta}{2} < \dfrac{\pi}{2}, \tan(\dfrac{\pi}{2} + \dfrac{\theta}{2}) > 0$, then $x = \ln\tan(\dfrac{\pi}{4} + \dfrac{\theta}{2})$.

4.6 If acute angles x and y satisfy $\sin y \csc x = \cos(x + y), x + y \neq \dfrac{\pi}{2}$, evaluate the maximum value of $\tan y$.

Solution: Since $\sin y \csc x = \cos x\cos y - \sin x \sin y \Rightarrow \sin y(\sin x + \csc x) = \cos x \cos y \Rightarrow \tan y = \dfrac{\cos x}{\sin x + \csc x} = \dfrac{\sin x\cos x}{1 + \sin^2 x} = \dfrac{\sin x \cos x}{2\sin^2 x + \cos^2 x} = \dfrac{\tan x}{1 + 2\tan^2 x} \leqslant \dfrac{\tan x}{2\sqrt{2}\tan x} = \dfrac{\sqrt{2}}{4}$, the equality holds if and only if $\tan x = \dfrac{\sqrt{2}}{2}$. Therefore, the maximum value is $\dfrac{\sqrt{2}}{4}$.

4.7 Given $\sin\alpha + \sin\beta = \dfrac{\sqrt{2}}{2}$, evaluate the range of $\cos\alpha + \cos\beta$.

Solution: Let $t = \cos\alpha + \cos\beta \cdots ①$ and $\dfrac{\sqrt{2}}{2} = \sin\alpha + \sin\beta \cdots ②$.
$①^2 + ②^2 \Rightarrow t^2 + \dfrac{1}{2} = 2 + 2\cos(\alpha - \beta)$, that is $2\cos(\alpha - \beta) = t^2 - \dfrac{3}{2}$. Since $-1 \leqslant \cos(\alpha - \beta) \leqslant 1 \Rightarrow -2 \leq t^2 - \dfrac{3}{2} \leq 2$, then $t^2 \leqslant \dfrac{7}{2}, t \in [-\dfrac{\sqrt{14}}{2}, \dfrac{\sqrt{14}}{2}]$, hence $(\cos\alpha + \cos\beta) \in [-\dfrac{\sqrt{14}}{2}, \dfrac{\sqrt{14}}{2}]$.

4.8 Solve the inequality $\arcsin \frac{x-3}{2x-1} > \frac{\pi}{6}$.

Solution: $\arcsin \frac{1}{2} = \frac{\pi}{6} \Rightarrow \arcsin \frac{x-3}{2x-1} > \arcsin \frac{1}{2} \Rightarrow \frac{x-3}{2x-1} > \frac{1}{2}$.
On the other hand, by the domain of arcsine function, we can obtain $-1 \leqslant \frac{x-3}{2x-1} \leqslant 1$. Subsequently, we have $x \leqslant -2$ as the solution.

4.9 Show $\arctan \frac{4}{3} + \operatorname{arccot} \frac{5}{12} + \arctan \frac{56}{33} = \pi$.

Proof: Since $\tan(\arctan \frac{4}{3} + \arctan \frac{56}{33}) = \frac{\frac{4}{3} + \frac{56}{33}}{1 - \frac{4}{3}\frac{56}{33}} = -\frac{12}{5}$, and $\arctan \frac{4}{3} \in (\frac{\pi}{4}, \frac{\pi}{2})$, $\arctan \frac{56}{33} \in (\frac{\pi}{4}, \frac{\pi}{2})$, then $(\arctan \frac{4}{3} + \arctan \frac{56}{33}) \in (\frac{\pi}{2}, \pi)$. In addition $\operatorname{arccot} \frac{5}{12} \in (0, \frac{\pi}{2})$, it implies $(\pi - \operatorname{arccot} \frac{5}{12}) \in (\frac{\pi}{2}, \pi) \Rightarrow \arctan \frac{4}{3} + \arctan \frac{56}{33} = \pi - \operatorname{arccot} \frac{5}{12}$. Therefore, $\arctan \frac{4}{3} + \operatorname{arccot} \frac{5}{12} + \arctan \frac{56}{33} = \pi$.

4.10 ★ If $f(n) = cos \frac{n\pi}{5}, (n \in N^*)$, evaluate the value of $f(1) + f(2) + \cdots + f(2000)$.

Solution: Assume the period of the function is T, then $T = \frac{2\pi}{\frac{\pi}{5}} = 10$. Thus $f(1) + f(2) + f(3) + f(4) + f(5) + f(6) + f(7) + f(8) + f(9) + f(10) = \cos \frac{\pi}{5} + \cos \frac{2\pi}{5} + \cos \frac{3\pi}{5} + \cos \frac{4\pi}{5} + \cos \frac{5\pi}{5} + cos \frac{6\pi}{5} + \cos \frac{7\pi}{5} + \cos \frac{8\pi}{5} + \cos \frac{9\pi}{5} + cos \frac{10\pi}{5}$. Since $\cos \frac{\pi}{5} = \cos \frac{9\pi}{5} = -\cos \frac{4\pi}{5} = -\cos \frac{6\pi}{5}$, $\cos \frac{2\pi}{5} = \cos \frac{8\pi}{5} = -\cos \frac{3\pi}{5} = -\cos \frac{7\pi}{5}$, $\cos \frac{5\pi}{5} = -\cos \frac{10\pi}{5}$, then $f(1) + f(2) + \cdots + f(10) = 0$. As a conclusion, $f(1) + f(2) + \cdots + f(2000)$=0.

4.11 ★ Find the monotony interval of the function $y = \cos^2 x + \sin x$.

Solution: Since $y = \cos^2 x + \sin x = -\sin^2 x + \sin x + 1$, let $t = \sin x$, then $y = -t^2 + t + 1 = -(t - \frac{1}{2})^2 + \frac{5}{4}$. It is monotonically increasing when $x \in (-\infty, \frac{1}{2}]$, and it is monotonically decreasing when $x \in [\frac{1}{2}, \infty)$. Since $t = \sin x \leqslant \frac{1}{2} \Rightarrow 2k\pi + \frac{5\pi}{6} \leqslant x \leqslant$

$2k\pi+\frac{13\pi}{6}, (k \in Z)$, $t=\sin x \geqslant \frac{1}{2} \Rightarrow 2k\pi+\frac{\pi}{6} \leqslant x \leqslant 2k\pi+\frac{5\pi}{6}, (k \in Z)$. Additionally since the function $t=\sin x$ is increasing in the interval $[2k\pi-\frac{\pi}{2}, 2k\pi+\frac{\pi}{2}]$, $(k \in Z)$, and it is decreasing in the interval $[2k\pi+\frac{\pi}{2}, 2k\pi+\frac{3\pi}{2}]$, $(k \in Z)$. As a conclusion, the increasing interval $y=\cos^2 x+\sin x$ is $[2k\pi-\frac{\pi}{2}, 2k\pi+\frac{\pi}{6}]\bigcup[2k\pi+\frac{\pi}{2}, 2k\pi+\frac{5\pi}{6}] \quad (k \in Z)$, the decreasing interval is $[2k\pi+\frac{\pi}{6}, 2k\pi+\frac{\pi}{2}]\bigcup[2k\pi+\frac{5\pi}{6}, 2k\pi+\frac{3\pi}{2}] \quad (k \in Z)$.

4.12 ★★ Find the domain and range of the function $y=\sqrt{\arccos(x^2+x+1)}$.

Solution: According to the domain of square root, we can obtain $0 \leqslant x^2+x+1 \leqslant 1$. Since $x^2+x+1 \leqslant 1$, then $-1 \leqslant x \leqslant 0$. Hence, the domain of the function is $[-1,0]$. Since $x^2+x+1=(x+\frac{1}{2})^2+\frac{3}{4} \geqslant \frac{3}{4} \Rightarrow \frac{3}{4} \leqslant x^2+x+1 \leqslant 1 \Rightarrow 0 \leqslant \arccos(x^2+x+1) \leqslant \arccos\frac{3}{4} \Rightarrow 0 \leqslant \sqrt{\arccos(x^2+x+1)} \leqslant \sqrt{\arccos\frac{3}{4}}$. Therefore, the range of the function is $[0, \sqrt{\arccos\frac{3}{4}}]$.

4.13 For arbitrary real number x and integer n, the equation $f(\sin x)=\sin(4n+1)x$ always holds. Evaluate $f(\cos x)$.

Solution: Since $f(\sin x)=\sin(4n+1)x$ and $\cos x=\sin(\frac{\pi}{2}-x)$, then $f(\cos x)=f[\sin(\frac{\pi}{2}-x)]=\sin[(4n+1)(\frac{\pi}{2}-x)]=\sin[2n\pi+\frac{\pi}{2}-(4n+1)x]=\sin[\frac{\pi}{2}-(4n+1)x]=\cos(4n+1)x$.

4.14 ★★ Given $x, y \in [-\frac{\pi}{4}, \frac{\pi}{4}], a \in R$, and x, y are the roots of the equation system,

$$\begin{cases} x^3+\sin x-2a=0 \\ 4y^3+\sin y\cos y+a=0 \end{cases}$$

find the value of $\cos(x+2y)$.

Solution: The second equation implies $4y^3+\sin y\cos y=-a$. Multiply the equation by -2 to obtain $(-2y)^3+\sin(-2y)=2a$. The first equation implies $x^3+\sin x=2a$, then $f(x)=f(-2y)$. Let $f(t)=t^3+\sin t$. Since the function $f(t)$ is increasing where $t \in [-\frac{\pi}{2}, \frac{\pi}{2}] \Rightarrow x=-2y$, therefore $x+2y=0$. As a conclusion, $\cos(x+2y)=1$.

4.15 Let α, β, γ form a geometric sequence with the common ratio 2, $\alpha \in [0, 2\pi]$, and $\sin\alpha, \sin\beta, \sin\gamma$ also form a geometrical sequence, find the values of α, β, γ.

Solution: Let $\beta = 2\alpha, \gamma = 4\alpha$, and $\frac{sin\beta}{\sin\alpha} = \frac{\sin\gamma}{\sin\beta} \Rightarrow \frac{\sin 2\alpha}{\sin\alpha} = \frac{\sin 4\alpha}{\sin 2\alpha} \Rightarrow \cos\alpha = 2\cos^2\alpha - 1 \Rightarrow 2\cos^2\alpha - \cos\alpha - 1 = 0$. The roots of $2\cos^2\alpha - cos\alpha - 1 = 0$ are $\cos\alpha = 1$ and $\cos\alpha = -\frac{1}{2}$. When $\cos\alpha = 1$, then $\sin\alpha = 0$, it does not satisfy the condition that the first term is nonzero. Therefore, $\cos\alpha \neq 1$. When $\cos\alpha = -\frac{1}{2}$, since $\alpha \in [0, 2\pi]$, then $\alpha = \frac{2\pi}{3}$ or $\alpha = \frac{4\pi}{3}$. When $\alpha = \frac{2\pi}{3}$, then $\beta = \frac{4\pi}{3}, \gamma = \frac{8\pi}{3}$. When $\alpha = \frac{4\pi}{3}$, then $\beta = \frac{8\pi}{3}, \gamma = \frac{16\pi}{3}$.

4.16 Compare $\arcsin\frac{1}{3}, \arctan\sqrt{2}, \arccos\frac{3}{4}$.

Solution: Let $\arcsin\frac{1}{3} = \alpha$, then $\sin\alpha = \frac{1}{3} < \frac{1}{2}$, thus $0 < \alpha < \frac{\pi}{6}$.
Let $\arctan\sqrt{2} = \beta$, then $\tan\beta = \sqrt{2}$. Since $1 < \sqrt{2} < \sqrt{3}$, then $\frac{\pi}{4} < \beta < \frac{\pi}{3}$.
Let $\arccos\frac{3}{4} = \gamma$, then $\cos\gamma = \frac{3}{4}$. Since $\frac{\sqrt{2}}{2} < \frac{3}{4} < \frac{\sqrt{3}}{2}$, then $\frac{\pi}{6} < \gamma < \frac{\pi}{4}$.
As a conclusion, $\alpha < \gamma < \beta$, therefore,$\arcsin\frac{1}{3} < \arccos\frac{3}{4} < \arctan\sqrt{2}$.

4.17 ★ Given $\vec{a} = (\cos\frac{3}{2}x, \sin\frac{3}{2}x), \vec{b} = (\cos\frac{x}{2}, -\sin\frac{x}{2}), x \in [0, \frac{\pi}{2}]$. (1) Solve $\vec{a}\cdot\vec{b}$ and $|\vec{a}+\vec{b}|$. (2) If the minimum value of $f(x) = \vec{a}\cdot\vec{b} - 2\lambda|\vec{a}+\vec{b}|$ is $-\frac{3}{2}$, compute λ.

Solution: (1) $\vec{a}\cdot\vec{b} = \cos\frac{3}{2}x\cos\frac{x}{2} - \sin\frac{3}{2}x\sin\frac{x}{2} = \cos 2x, x \in [0, \frac{\pi}{2}]$.
$|\vec{a}+\vec{b}| = \sqrt{(\cos\frac{3}{2}x + \cos\frac{x}{2})^2 + (\sin\frac{3}{2}x - \sin\frac{x}{2})^2} = 2\sqrt{\cos^2 x} = 2\cos x, x \in [0, \frac{\pi}{2}]$.
(2) $f(x) = \cos 2x - 4\lambda\cos x = 2\cos^2 x - 4\lambda\cos x - 1 = 2(\cos x - \lambda)^2 - 1 - 2\lambda^2$. Since $x \in [0, \frac{\pi}{2}]$, then $\cos x \in [0, 1]$.
When $\lambda < 0, \cos x = 0$, then $f(x)_{min} = -1$. It does not satisfy the given condition.
When $0 \leqslant \lambda \leqslant 1, \cos x = \lambda$, then $f(x)_{min} = -1 - 2\lambda^2 = -\frac{3}{2}$, then $\lambda = \frac{1}{2}$.
When $\lambda \geqslant 1, \cos x = 1$, then $f(x)_{min} = 1 - 4\lambda = -\frac{3}{2}$, then $\lambda = \frac{5}{8} < 1$. It does not satisfy the condition $\lambda \geqslant 1$. After all, $\lambda = \frac{1}{2}$.

4.18 Let $\triangle ABC, \sin A + \cos A = \frac{\sqrt{2}}{2}, AC = 2, AB = 3$, find the value of $\tan A$ and the area of $\triangle ABC$.

Solution: Since $\sin A + \cos A = \sqrt{2}\cos(A - 45^0) = \frac{\sqrt{2}}{2} \Rightarrow \cos(A - 45^0) = \frac{1}{2}$. Additionally, $0 < A < 180^0$, then $A - 45^0 = 60^0 \Rightarrow A = 105^0 \Rightarrow \tan A = \tan(45^0 + 60^0) = \frac{1+\sqrt{3}}{1-\sqrt{3}} = -2 - \sqrt{3}$. Since $\sin A = \sin(45^0 + 60^0) = \sin 45^0 \cos 60^0 + \cos 45^0 \sin 60^0 = \frac{\sqrt{2}+\sqrt{6}}{4}$, we have $S_{\triangle ABC} = \frac{1}{2}AC\cdot AB\cdot\sin A = \frac{1}{2}\times 2\times 3\times\frac{\sqrt{2}+\sqrt{6}}{4} = \frac{3}{4}(\sqrt{2}+\sqrt{6})$.

4.19 Find the symmetric center, the symmetric axis equation of the function $y = 3 - 2\cos(2x - \frac{\pi}{3})$ and the value of x when y has the maximum and the minimum.

Solution : Since the symmetric center of $y = \cos x$ is $(k\pi + \frac{\pi}{2}, 0) \quad (k \in Z)$, and the symmetric axis equation is $k\pi \quad (k \in Z)$. Thus, $2x - \frac{\pi}{3} = k\pi + \frac{\pi}{2} \Rightarrow x = \frac{k\pi}{2} + \frac{5\pi}{12} \quad (k \in Z)$. Since $2x - \frac{\pi}{3} = k\pi$, then $x = \frac{k\pi}{2} + \frac{\pi}{6} \quad (k \in Z)$, therefore the the symmetric center of the function $y = 3 - 2\cos(2x - \frac{\pi}{3})$ is $(\frac{k\pi}{2} + \frac{5\pi}{12}, 3) \quad (k \in Z)$, and the symmetric axis equation is $x = \frac{k\pi}{2} + \frac{\pi}{6} \quad (k \in Z)$. When $2x - \frac{\pi}{3} = 2k\pi \Rightarrow x = k\pi + \frac{\pi}{6} \quad (k \in Z)$, the minimum of $y = 3 - 2\cos(2x - \frac{\pi}{3})$ is 1. When $2x - \frac{\pi}{3} = (2k+1)\pi \Rightarrow x = k\pi + \frac{2\pi}{3} \quad (k \in Z)$, the maximum of $y = 3 - 2\cos(2x - \frac{\pi}{3})$ is 5.

4.20 ★ If the equation $(2\cos\theta - 1)x^2 - 4x + 4\cos\theta + 2 = 0$ has two distinct positive roots, and θ is an acute angle. Find the range of θ.

Solution: Assume the two roots are $x_1, x_2 > 0$. Since $\Delta = (-4)^2 - 4(2\cos\theta - 1)(4\cos\theta + 2) > 0$, then $-\frac{\sqrt{3}}{2} < \cos\theta < \frac{\sqrt{3}}{2}$ ①. Since $x_1 + x_2 = \frac{4}{2\cos\theta - 1} > 0$, then $\cos\theta > \frac{1}{2}$ ②. Since $x_1 x_2 = \frac{4\cos\theta + 2}{2\cos\theta - 1} > 0$, then $\cos\theta < -\frac{1}{2}$ or $\cos\theta > \frac{1}{2} \cdots$ ③. According to ①, ②, ③, we can obtain $\frac{1}{2} < \cos\theta < \frac{\sqrt{3}}{2}$. Since θ is an acute angle, then $30^0 < \theta < 60^0$.

4.21 Let the function $f(x) = -a\cos 2x - 2\sqrt{3}a\sin x\cos x + 2a + b$, its domain is $[0, \frac{\pi}{2}]$, the range is $[-5, 1]$. Evaluate a and b.

Solution: $f(x) = -a\cos 2x - \sqrt{3}a\sin 2x + 2a + b = -2a\cos(2x - \frac{\pi}{3}) + 2a + b$. Since $x \in [0, \frac{\pi}{2}] \Rightarrow -\frac{\pi}{3} \leqslant 2x - \frac{\pi}{3} \leqslant \frac{2\pi}{3}$, then $-\frac{1}{2} \leqslant \cos(2x - \frac{\pi}{3}) \leqslant 1$.
When $a > 0$, then $b \leqslant f(x) \leqslant 3a + b \Rightarrow$

$$\begin{cases} 3a + b = 1 \\ b = -5 \end{cases}$$

we can obtain $a = 2, b = -5$.
When $a < 0$, then $3a + b \leqslant f(x) \leqslant b \Rightarrow$

$$\begin{cases} 3a + b = -5 \\ b = 1 \end{cases}$$

we can obtain $a = -2, b = 1$.

4.22 ★ Given $\tan(\cos^{-1}\sqrt{x}) = \sin(\cot^{-1}\frac{1}{2})$, find the value of x.

Solution: Let $\cos^{-1}\sqrt{x} = \theta$, then $cos\theta = \sqrt{x}, \tan\theta = \frac{\sin\theta}{cos\theta} = \frac{\sqrt{1-\cos^2\theta}}{\cos\theta} = \frac{\sqrt{1-x}}{\sqrt{x}}$. Let $\cot^{-1}\frac{1}{2} = \phi$, then $\cot\phi = \frac{1}{2}, \sin\phi = \frac{1}{\csc\phi} = \frac{1}{\sqrt{1+cot^2\phi}} = \frac{2}{\sqrt{5}}$. The equation is equal to $\frac{\sqrt{1-x}}{\sqrt{x}} = \frac{2}{\sqrt{5}} \Rightarrow \frac{1-x}{x} = \frac{4}{5}$, therefore $x = \frac{5}{9}$.

4.23 ★ Let $0 < \theta < \pi$, find the maximum value of $\sin\frac{\theta}{2}(1+\cos\theta)$.

Solution: Since $0 < \theta < \pi$, then $sin\frac{\theta}{2}(1+\cos\theta) = 2sin\frac{\theta}{2}\cos^2\frac{\theta}{2} = \sqrt{2}\sqrt{2\sin^2\frac{\theta}{2}\cos^4\frac{\theta}{2}} \leqslant \sqrt{2}\sqrt{(\frac{2\sin^2\frac{\theta}{2}+\cos^2\frac{\theta}{2}+\cos^2\frac{\theta}{2}}{3})^3} = \sqrt{2}\sqrt{(\frac{2}{3})^3} = \sqrt{2}\times\frac{2}{3}\times\sqrt{\frac{2}{3}} = \frac{4\sqrt{3}}{9}$. Hence the maximum value of $\sin\frac{\theta}{2}(1+\cos\theta)$ is $\frac{4\sqrt{3}}{9}$.

4.24 ★ Find the value of $\sin^4\frac{\pi}{16}+\sin^4\frac{3\pi}{16}+\sin^4\frac{5\pi}{16}+\sin^4\frac{7\pi}{16}$.

Solution: The quantity is equal to $\sin^4\frac{\pi}{16}+\sin^4\frac{3\pi}{16}+\sin^4(\frac{\pi}{2}-\frac{3\pi}{16})+sin^4(\frac{\pi}{2}-\frac{\pi}{16}) = \sin^4\frac{\pi}{16}+\sin^4\frac{3\pi}{16}+\cos^4\frac{3\pi}{16}+\cos^4\frac{\pi}{16} = (\sin^2\frac{\pi}{16}+\cos^2\frac{\pi}{16})^2-2\sin^2\frac{\pi}{16}\cos^2\frac{\pi}{16}+(\sin^2\frac{3\pi}{16}+\cos^2\frac{3\pi}{16})^2-2\sin^2\frac{3\pi}{16}\cos^2\frac{3\pi}{16} = 2-\frac{1}{2}(\sin^2\frac{\pi}{8}+\sin^2\frac{3\pi}{8}) = 2-\frac{1}{2}[\sin^2\frac{\pi}{8}+\sin^2(\frac{\pi}{2}-\frac{\pi}{8})]$
$= 2-\frac{1}{2}(\sin^2\frac{\pi}{8}+\cos^2\frac{\pi}{8}) = \frac{3}{2}$.

4.25 ★ Given vector $\vec{m} = (\cos\theta, \sin\theta), \vec{n} = (\sqrt{2} - \sin\theta, \cos\theta), \theta \in (\pi, 2\pi)$, and $|\vec{m} + \vec{n}| = \frac{8\sqrt{2}}{5}$, find the value of $\cos(\frac{\theta}{2} + \frac{\pi}{8})$.

Solution: From the given condition, we have $\vec{m}+\vec{n} = (\cos\theta - \sin\theta + \sqrt{2}, \sin\theta + \cos\theta)$, then $|\vec{m}+\vec{n}| = \sqrt{(\cos\theta - \sin\theta + \sqrt{2})^2 + (\cos\theta + \sin\theta)^2} = \sqrt{4 + 2\sqrt{2}(\cos\theta - \sin\theta)} = \sqrt{4 + 4\cos(\theta + \frac{\pi}{4})} = 2\sqrt{1 + \cos(\theta + \frac{\pi}{4})}$. Since $|\vec{m}+\vec{n}| = \frac{8\sqrt{2}}{5} \Rightarrow 2\sqrt{1 + \cos(\theta + \frac{\pi}{4})} = \frac{8\sqrt{2}}{5} \Rightarrow \cos(\theta + \frac{\pi}{4}) = \frac{7}{25}$. Sine $\cos(\theta + \frac{\pi}{4}) = 2\cos^2(\frac{\theta}{2} + \frac{\pi}{8}) - 1 \Rightarrow \cos^2(\frac{\theta}{2} + \frac{\pi}{8}) = \frac{16}{25}$. Since $\pi < \theta < 2\pi \Rightarrow \frac{5\pi}{8} < \frac{\theta}{2} + \frac{\pi}{8} < \frac{9\pi}{8} \Rightarrow cos(\frac{\theta}{2} + \frac{\pi}{8}) < 0$. Thus, $\cos(\frac{\theta}{2} + \frac{\pi}{8}) = -\frac{4}{5}$.

4.26 ★★ Given $\alpha, \beta \in (0, \frac{\pi}{4}), 3\sin\beta = \sin(2\alpha + \beta), 4\tan\frac{\alpha}{2} = 1 - \tan^2\frac{\alpha}{2}$, evaluate $\alpha + \beta$.

Solution: Since $4\tan\frac{\alpha}{2} = 1 - \tan^2\frac{\alpha}{2} \Rightarrow \frac{4\tan\frac{\alpha}{2}}{1 - tan^2\frac{\alpha}{2}} = 1 \Rightarrow 2\tan\alpha = 1 \Rightarrow \tan\alpha = \frac{1}{2}$.
Since $3\sin\beta = \sin(2\alpha + \beta) = \sin(\alpha + \beta)\cos\alpha + \cos(\alpha + \beta)\sin\alpha$ ①,
$3\sin\beta = 3\sin(\alpha + \beta - \alpha) = 3\sin(\alpha + \beta)\cos\alpha - 3\cos(\alpha + \beta)\sin\alpha$ ②.
② − ① $\Rightarrow \sin(\alpha + \beta)\cos\alpha = 2\cos(\alpha + \beta)\sin\alpha \Rightarrow \tan(\alpha + \beta) = 2\tan\alpha = 1$. For $\alpha, \beta \in (0, \frac{\pi}{4})$, thus $\alpha + \beta = \frac{\pi}{4}$.

4.27 ★ Find the domain and range of the function $y = \frac{\pi}{4} - \frac{1}{2}\arctan\sqrt{2\cos x - 1}$.

Solution: the function is defined if and only if $2\cos x - 1 \geqslant 0$, that is $\cos x \geqslant \frac{1}{2}$.
Thus the domain of y is $2n\pi - \frac{\pi}{3} \leqslant x \leqslant 2n\pi + \frac{\pi}{3} \quad (n \in Z)$.
Since $0 \leqslant \sqrt{2\cos x - 1} \leqslant 1$, then $0 \leqslant \arctan\sqrt{2\cos x - 1} \leqslant \frac{\pi}{4}$, thus $\frac{\pi}{8} \leqslant y \leqslant \frac{\pi}{4}$.

Therefore, the domain of the function is $x \in [2n\pi - \frac{\pi}{3}, 2n\pi + \frac{\pi}{3}] \quad (n \in Z)$, and the range is $y \in [\frac{\pi}{8}, \frac{\pi}{4}]$.

4.28 ★ Given $\triangle ABC$, $\frac{a^3 + b^3 + c^3}{a + b + c} = c^2$, and $\sin A\sin B = \frac{3}{4}$. Judge the shape of $\triangle ABC$.

Solution: $\frac{a^3 + b^3 + c^3}{a + b + c} = c^2 \Rightarrow a^3 + b^3 = ac^2 + bc^2 \Rightarrow (a+b)(a^2 - ab + b^2) = c^2(a+b)$. Since $a+b \neq 0$, then $a^2 - ab + b^2 = c^2$ ①. The cosine theorem is $c^2 = a^2 + b^2 - 2ab\cos C$ ②. According to ①, ②, we get $\cos C = \frac{1}{2} \Rightarrow C = 60^0 \Rightarrow A + B = 120^0$. Since $\sin A\sin B = \frac{3}{4} \Rightarrow \frac{1}{2}[\cos(A - B) - \cos(A + B)] = \frac{3}{4}$, then $\cos(A - B) = \frac{3}{2} - \frac{1}{2} = 1$, thus $A - B = 0 \Rightarrow A = B$. Therefore $\triangle ABC$ is a right triangle.

4.29 ★★ Let $-\frac{\pi}{2} \leqslant x \leqslant \frac{\pi}{2}$, $f(x)$ satisfies $2f(-\sin x) + 3f(\sin x) = 4\sin x\cos x$.
(1) Show $f(x)$ is an odd function. (2) Find the analytic expression of $f(x)$.

(1) Proof: Since $2f(-\sin x)+3f(\sin x) = 4\sin x\cos x$ ①, substitute $-\sin x$ into ① to obtain $2f(\sin x)+3f(-\sin x) = -4\sin x\cos x$ ②. ①+②$\Rightarrow 5f(\sin x)+5f(-\sin x) = 0 \Rightarrow f(\sin x) = -f(-\sin x)$, therefore $f(x)$ is an odd function.
(2) Solution: ① − ② $\Rightarrow$ $f(\sin x) - f(-\sin x) = 8\sin x\cos x$. Since $f(\sin x) = -f(-\sin x) \Rightarrow 2f(\sin x) = 8\sin x\cos x$, then $f(\sin x) = 4\sin x\sqrt{1-\sin^2 x}, (-1 \leqslant x \leqslant 1)$. Therefore, $f(x) = 4x\sqrt{1-x^2}, (-1 \leqslant x \leqslant 1)$.

4.30 ★ Solve the equation $\sin x + \cos x + \sin x\cos x = 1$.

Solution 1: Multiply both sides of the equation by 2 and adding 1, we obtain $2(\sin x + \cos x) + 2\sin x\cos x + 1 = 3 \Rightarrow (\sin x + \cos x)^2 + 2(\sin x + \cos x) - 3 = 0 \Rightarrow [(\sin x + \cos x) - 1][(\sin x + \cos x) + 3] = 0 \Rightarrow \sin x + \cos x = 1$ or $\sin x + \cos x = -3$. Since $-1 \leqslant \sin x \leqslant 1, -1 \leqslant cosx \leqslant 1$, we have $\sin x + \cos x \neq -3$.
Since $\sin x + \cos x = 1 \Rightarrow \sqrt{2}\sin(x+\frac{\pi}{4}) = 1 \Rightarrow sin(x+\frac{\pi}{4}) = \frac{\sqrt{2}}{2} \Rightarrow x+\frac{\pi}{4} = 2k\pi+\frac{\pi}{4}$, or $x+\frac{\pi}{4} = (2k+1)\pi-\frac{\pi}{4} \Rightarrow x = 2k\pi$, or $x = 2k\pi+\frac{\pi}{2}, (k \in Z)$. Therefore, the solution of the equation is $\{x|x = 2k\pi, k \in Z\}\bigcup\{x|x = 2k\pi + \frac{\pi}{2}, k \in Z\}$.
Solution 2: Assume $\sin x + \cos x = u$, then $\sin x\cos x = \frac{u^2-1}{2}$. Substitute it into the equation $\sin x + \cos x + \sin x\cos x = 1$: $u + \frac{u^2-1}{2} = 1 \Rightarrow u^2 + 2u - 3 = 0 \Rightarrow u = 1$ or $u = -3$. Since $-1 \leqslant \sin x \leqslant 1, -1 \leqslant \cos x \leqslant 1$, then $u = \sin x + \cos x \neq -3$. Hence, $\sin x + \cos x = 1 \Rightarrow \sqrt{2}\sin(x + \frac{\pi}{4}) = 1 \Rightarrow \sin(x + \frac{\pi}{4}) = \frac{\sqrt{2}}{2} \Rightarrow x + \frac{\pi}{4} = 2k\pi + \frac{\pi}{4}$, or $x + \frac{\pi}{4} = (2k+1)\pi - \frac{\pi}{4} \Rightarrow x = 2k\pi$, or $x = 2k\pi + \frac{\pi}{2}, (k \in Z)$. Therefore, the solution of the equation is $\{x|x = 2k\pi, k \in Z\}\bigcup\{x|x = 2k\pi + \frac{\pi}{2}, k \in Z\}$.

4.31 ★ If $0 < x < 45^0$, and $\lg\tan x - \lg\sin x = \lg\cos x - \lg\cot x + \lg 9 - \lg\sqrt{8}$, find the value of $\cos x - \sin x$.

Solution: The given equation is equal to $\lg(\sin x\cos x) - \lg(\tan x\cot x) = \lg\sqrt{8} - \lg 9$, then $\sin x\cos x = \frac{2\sqrt{2}}{9}$. Since $(\cos x - \sin x)^2 = 1 - 2\sin x\cos x = 1 - \frac{4\sqrt{2}}{9} = \frac{9-4\sqrt{2}}{9}, 0 < x < 45^0, \cos x > \sin x$. Thus $\cos x - \sin x = \frac{1}{3}\sqrt{9-4\sqrt{2}} = \frac{1}{3}(2\sqrt{2}-1)$.

4.32 ★ Find all positive integer solutions which satisfy the equation $\tan^{-1} x + \cot^{-1} y = \tan^{-1} 3$.

Solution: $\tan^{-1} x + \cot^{-1} y = \tan^{-1} 3 \Rightarrow \tan^{-1} x + \tan^{-1} \frac{1}{y} = \tan^{-1} 3 \Rightarrow \tan(\tan^{-1} x + \tan^{-1} \frac{1}{y}) = \tan(\tan^{-1} 3) \Rightarrow \frac{x + \frac{1}{y}}{1 - \frac{x}{y}} = 3 \Rightarrow x = \frac{3y-1}{y+3} = 3 - \frac{10}{y+3}$. Since x, y are positive integers, then $y+3$ is the divisor of 10. Thus $y = 2$ or $y = 7$, $x = 1$ or $x = 2$. As a conclusion, the positive integer solutions are

$$\begin{cases} x = 1 \\ y = 2 \end{cases}$$

or

$$\begin{cases} x = 2 \\ y = 7 \end{cases}$$

4.33 ★★ Check the sign of the formula $\frac{\sin(\cos\theta)}{\cos(\sin 2\theta)}$ when θ is in the second quadrant. If $\pi < \alpha + \beta < \frac{4\pi}{3}, -\pi < \alpha - \beta < -\frac{\pi}{3}$, find the range of $2\alpha - \beta$.

Solution: (1) $2k\pi + \frac{\pi}{2} < \theta < 2k\pi + \pi.(k \in Z) \Rightarrow -1 < \cos\theta < 0$. The condition $4k\pi + \pi < 2\theta < 4k\pi + 2\pi$ gives $-1 < \sin 2\theta < 0$. Thus $\sin(\cos\theta) < 0, \cos(\sin 2\theta) > 0$. Therefore $\frac{\sin(\cos\theta)}{\cos(\sin 2\theta)} < 0$.

(2) Assume $x = \alpha+\beta, y = \alpha-\beta, 2\alpha-\beta = mx+ny$. Then $2\alpha-\beta = m\alpha+m\beta+n\alpha-n\beta = (m+n)\alpha + (m-n)\beta$. Comparing the coefficients, we have

$$\begin{cases} m+n=2 \\ m-n=-1 \end{cases}$$

Hence $m = \frac{1}{2}, n = \frac{3}{2} \Rightarrow 2\alpha - \beta = \frac{1}{2}x + \frac{3}{2}y$. Since $\pi < x < \frac{4\pi}{3}, -\pi < y < -\frac{\pi}{3}$, then $-\pi < \frac{1}{2}x + \frac{3}{2}y < \frac{\pi}{6}$. As a conclusion, the range of $2\alpha - \beta$ is $(-\pi, \frac{\pi}{6})$.

4.34 ★★ Let $\tan\alpha + \sin\alpha = m, \tan\alpha - \sin\alpha = n$. Show $m^2 - n^2 = 4\sqrt{mn}$.

Solution: Multiplying the two equations together to obtaion $mn = \tan^2\alpha - \sin^2\alpha = \tan^2\alpha(1-\cos^2\alpha) = \tan^2\alpha\sin^2\alpha$ ①.
Adding the two equations: $2\tan\alpha = m+n \Rightarrow \tan\alpha = \frac{m+n}{2}$ ②.
Using subtraction for the two equations: $2\sin\alpha = m-n \Rightarrow \sin\alpha = \frac{m-n}{2}$ ③.
Substituting ② and ③ into ① $\Rightarrow mn = (\frac{m+n}{2})^2(\frac{m-n}{2})^2 \Rightarrow m^2 - n^2 = 4\sqrt{mn}$.

4.35 ★ Given $\frac{a}{\cos\alpha} = \frac{b}{\cos 2\alpha} = \frac{c}{\cos 3\alpha} \neq 0$, show $\sin^2\frac{\alpha}{2} = \frac{2b-a-c}{4b}$.

Proof: Applying the equal radio theorem, we have $\frac{b}{\cos 2\alpha} = \frac{a+c}{\cos\alpha + 3\cos\alpha} \neq 0$. Since $\cos\alpha \neq 0, \cos 2\alpha \neq 0 \Rightarrow \frac{b}{\cos 2\alpha} = \frac{a+c}{2\cos\alpha\cos 2\alpha} \neq 0$. In particular, $b \neq 0, \cos\alpha = \frac{a+c}{2b}$. Therefore $\sin^2\frac{\alpha}{2} = \frac{1-\cos\alpha}{2} = \frac{1}{2}(1 - \frac{a+c}{2b}) = \frac{2b-a-c}{4b}$.

4.36 ★★ Let $\sin^2(n+1)\theta = \sin^2 n\theta + \sin^2(n-1)\theta$, and $(n+1)\theta, n\theta, (n-1)\theta$ are the three interior angles of a triangle, find the integer value of n.

Solution: $\sin^2(n+1)\theta = \sin^2 n\theta + \sin^2(n-1)\theta \Rightarrow \sin^2(n+1)\theta - \sin^2(n-1)\theta = \sin^2 n\theta \Rightarrow [\sin(n+1)\theta - \sin(n-1)\theta][\sin(n+1)\theta + \sin(n-1)\theta] = \sin^2 n\theta \Rightarrow 2\sin\theta\cos n\theta 2\sin n\theta\cos\theta = \sin^2 n\theta \Rightarrow \sin 2n\theta\sin 2\theta = \sin^2 n\theta$ $(*)$. Since $(n+1)\theta + n\theta + (n-1)\theta = \pi$, then $n\theta = \frac{\pi}{3}$. Substituting it into $(*)$, then $\sin 2\theta = \frac{1}{2}\tan\frac{\pi}{3} = \frac{\sqrt{3}}{2} \Rightarrow 2\theta = \frac{\pi}{3} \Rightarrow \theta = \frac{\pi}{6}$. Since $n\theta = \frac{\pi}{3}$, we have $n = 2$.

4.37 ★★ Given $\sin\alpha + 3\cos\alpha = 2$, compute $\dfrac{\sin\alpha - \cos\alpha}{\sin\alpha + \cos\alpha}$.

Solution: Let $\dfrac{\sin\alpha - \cos\alpha}{\sin\alpha + \cos\alpha} = k \Rightarrow (1-k)\sin\alpha = (1+k)\cos\alpha$ ①. Denote $\sin\alpha = 2 - 3\cos\alpha$ ②. Applying ② ÷ ①, we have $\cos\alpha = \dfrac{1-k}{2-k}$. Substituting it into ①, we have $\sin\alpha = \dfrac{1+k}{2-k}$ $(k \neq 2)$. Since $\sin^2\alpha + \cos^2\alpha = 1$, then $(\dfrac{1+k}{2-k})^2 + (\dfrac{1-k}{2-k})^2 = 1$. It can be written as $k^2 + 4k - 2 = 0$. Solving the equation, we have $k = -2 \pm \sqrt{6}$. As a conclusion, $\dfrac{\sin\alpha - \cos\alpha}{\sin\alpha + \cos\alpha} = -2 \pm \sqrt{6}$.

4.38 ★★ Let a, b, c are the three side lengths of $\triangle ABC$, and they form a geometric sequence, and $\cos B = \dfrac{3}{4}$. (1) Find the value of $\cot A + \cot C$. (2)Let $\overrightarrow{BA}\overrightarrow{BC} = \dfrac{3}{2}$, compute $a + c$.

Solution: (1) $\cos B = \dfrac{3}{4} \Rightarrow 0 < B < \dfrac{\pi}{2} \Rightarrow \sin B = \sqrt{1 - (\dfrac{3}{4})^2} = \dfrac{\sqrt{7}}{4}$. Since a, b, c form a geometric sequence, applying the sine theorem, we have $\sin^2 B = \sin A \sin C$. Therefore $\cot A + \cot C = \dfrac{\cos A}{\sin A} + \dfrac{\cos C}{\sin C} = \dfrac{\sin(A+C)}{\sin A \sin C} = \dfrac{\sin B}{\sin^2 B} = \dfrac{1}{\sin B} = \dfrac{4\sqrt{7}}{7}$.
(2) $\overrightarrow{BA}\overrightarrow{BC} = \dfrac{3}{2} \Rightarrow ca\cos B = \dfrac{3}{2}$. Additionally $\cos B = \dfrac{3}{4}$, thus $ca = 2$. Since $b^2 = ac = 2$, applying the cosine theorem $b^2 = a^2 + c^2 - 2ac\cos B$, we have $a^2 + c^2 = b^2 + 2ac\cos B = 5 \Rightarrow (a+c)^2 = a^2 + c^2 + 2ac = 5 + 4 = 9$. Therefore, $a + c = 3$.

4.39 ★★★ If $\log_{\tan\theta}\cos\theta = \dfrac{2}{3}, (\theta \in (0, \dfrac{\pi}{2}))$, find the value of $\log_{\csc^2\theta}(\dfrac{\sin 2\theta}{2})$.

Solution: Changing the base number of the given equation, we have $\dfrac{\lg\cos\theta}{\lg\sin\theta - \lg\cos\theta} = \dfrac{2}{3} \Rightarrow \dfrac{\lg\cos\theta}{\lg\sin\theta} = \dfrac{2}{5} \Rightarrow \log_{\sin\theta}\cos\theta = \dfrac{2}{5}$. Hence, $\log_{\csc^2\theta}(\dfrac{\sin 2\theta}{2}) = -\log_{\sin^2\theta}(\sin\theta\cos\theta) = -\log_{\sin\theta}(\sin\theta\cos\theta)^{\frac{1}{2}} = -\dfrac{1}{2}\log_{\sin\theta}(\sin\theta\cos\theta) = -\dfrac{1 + \log_{\sin\theta}\cos\theta}{2} = -\dfrac{1 + \frac{2}{5}}{2} = -\dfrac{7}{10}$.

4.40 ★ Given $f(x) = 2acos^2 x + b\sin x\cos x, f(0) = 2, f(\dfrac{\pi}{3}) = \dfrac{1}{2} + \dfrac{\sqrt{3}}{2}$, find the set of x values that satisfy the formula $f(x) > 2$.

Solution: $f(0) = 2a = 2 \Rightarrow a = 1$. Since $f(\frac{\pi}{3}) = \frac{1}{2}a + \frac{\sqrt{3}}{4}b = \frac{1}{2} + \frac{\sqrt{3}}{2}$, substituting $a = 1$ into this formula, we have $b = 2$. Thus $f(x) = 2\cos^2 x + 2\sin x \cos x = \sin 2x + \cos 2x + 1$. Since$f(x) > 2$, then $\sin 2x + \cos 2x + 1 > 2 \Rightarrow \sin(2x + \frac{\pi}{4}) > \frac{\sqrt{2}}{2} \Rightarrow 2k\pi + \frac{\pi}{4} < (2x + \frac{\pi}{4}) < 2k\pi + \frac{3\pi}{4}.(k \in Z)$. Therefore the set of x values that satisfy the formula $f(x) > 2$ is $\{x | k\pi < x < k\pi + \frac{\pi}{4}, k \in Z\}$.

4.41 ★★ Let a, b, c are the three side lengths of $\triangle ABC$, and they form a geometric sequence. $\sin B + \cos B = m^2$. Find the range of m.

Solution: Since a, b, c form a geometric sequence, then $b^2 = ac$. Applying the sine theorem, we have $\sin^2 B = \sin A \sin C$. Then $1 - \cos^2 B = -\frac{1}{2}[\cos(A+C) - \cos(A-C)] \Rightarrow 2\cos^2 B + \cos B - 1 = 1 - \cos(A - C) \Rightarrow 2cos^2 B + \cos B - 1 \geqslant 0 \Rightarrow \cos B \geqslant \frac{1}{2}$, or $\cos B \leqslant -1$ (truncated). Hence $0 < B \leqslant \frac{\pi}{3}$. Additionally since $\sin B + \cos B = \sqrt{2}\sin(B + \frac{\pi}{4}) \Rightarrow 1 \leqslant m^2 \leqslant \sqrt{2} \Rightarrow -\sqrt[4]{2} \leqslant m \leqslant -1$, or $1 \leqslant m \leqslant \sqrt[4]{2}$.

4.42 ★ Let α, β are the two real roots of the equation $x^2+2(\sin\theta+1)x+\sin^2\theta=0$, and $|\alpha-\beta|\leqslant 2\sqrt{2}$. Find the range of θ.

Solution: Since the equation has real roots, then $\Delta = 4(\sin\theta+1)^2-4\sin^2\theta = 8\sin\theta+4\geqslant 0 \Rightarrow \sin\theta \geqslant -\frac{1}{2}$. Applying the Vieta theorem, we have

$$\begin{cases} \alpha+\beta=-2(\sin\theta+1) \\ 2\alpha\beta=\sin^2\theta \end{cases}$$

Hence $(\alpha-\beta)^2=(\alpha+\beta)^2-4\alpha\beta=8|\sin\theta|+4$ ①. Since $|\alpha-\beta|\leqslant 2\sqrt{2}\Rightarrow(\alpha-\beta)^2\leqslant 8$ ②. According to ① and ②, we have $8|\sin\theta|+4\leqslant 8\Rightarrow|\sin\theta|\leqslant\frac{1}{2}\Rightarrow-\frac{1}{2}\leqslant\sin\theta\leqslant\frac{1}{2}$. Therefore $k\pi-\frac{\pi}{6}\leqslant\theta\leqslant k\pi+\frac{\pi}{6}\quad(k\in Z)$.

4.43 ★★ Let A, B, C are the three interior angles of triangle $\triangle ABC$, show that $\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\leqslant\frac{1}{8}$.

Proof 1: $\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}=-\frac{1}{2}(\cos\frac{A+B}{2}-\cos\frac{A-B}{2})\sin\frac{C}{2}=-\frac{1}{2}(\sin^2\frac{C}{2}-\sin\frac{C}{2}cos\frac{A-B}{2})=-\frac{1}{2}(\sin^2\frac{C}{2}-\sin\frac{C}{2}\cos\frac{A-B}{2}+\frac{1}{4}\cos^2\frac{A-B}{4}-\frac{1}{4}\cos^2\frac{A-B}{4})=-\frac{1}{2}[(\sin\frac{C}{2}-\frac{1}{2}\cos\frac{A-B}{2})^2-\frac{1}{4}\cos^2\frac{A-B}{2}]=\frac{1}{8}\cos^2\frac{A-B}{2}-\frac{1}{2}(\sin\frac{C}{2}-\frac{1}{2}\cos\frac{A-B}{2})^2\leq\frac{1}{8}$.

Proof 2: Since $\sin^2\frac{A}{2}=\frac{1-\cos A}{2}=\frac{1}{2}(1-\frac{b^2+c^2-a^2}{2bc})=\frac{a^2-(b-c)^2}{4bc}\leqslant\frac{a^2}{4bc}$, and $\sin^2\frac{A}{2}\geqslant 0\Rightarrow\sin\frac{A}{2}\leqslant\frac{a}{2\sqrt{bc}}$. Similarly ,we have $\sin\frac{B}{2}\leqslant\frac{b}{2\sqrt{ac}}$, $\sin\frac{C}{2}\leqslant\frac{c}{2\sqrt{ab}}$. Hence $\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\leqslant\frac{a}{2\sqrt{bc}}\frac{b}{2\sqrt{ac}}\frac{c}{2\sqrt{ab}}=\frac{1}{8}$.

4.44 ★ Given vectors $\vec{a}=(2\cos\frac{x}{2},\tan(\frac{x}{2}+\frac{\pi}{4}))$, $\vec{b}=(\sqrt{2}\sin(\frac{x}{2}+\frac{\pi}{4}),\tan(\frac{x}{2}-\frac{\pi}{4}))$. Let $f(x)=\vec{a}\cdot\vec{b}$. Find the value of x when $f(x)=\sqrt{2}$, $(0<x<\frac{\pi}{2})$.

Solution: $f(x)=\vec{a}\cdot\vec{b}=2\sqrt{2}\cos\frac{x}{2}\sin(\frac{x}{2}+\frac{\pi}{4})+\tan(\frac{x}{2}+\frac{\pi}{4})\tan(\frac{x}{2}-\frac{\pi}{4})=2\sqrt{2}\cos\frac{x}{2}(\frac{\sqrt{2}}{2}\sin\frac{x}{2}+\frac{\sqrt{2}}{2}\cos\frac{x}{2})+\frac{1+\tan\frac{x}{2}}{1-\tan\frac{x}{2}}\frac{\tan\frac{x}{2}-1}{1+\tan\frac{x}{2}}=2\sin\frac{x}{2}\cos\frac{x}{2}+2\cos^2\frac{x}{2}-1=\sin x+\cos x=\sqrt{2}\sin(x+\frac{\pi}{4})=\sqrt{2}$. Thus $\sin(x+\frac{\pi}{4})=1$. On the other hand, $0<x<\frac{\pi}{2}$, thus $x=\frac{\pi}{4}$.

4.45 ★ Let $f(x)=\sin(\omega x+\phi)(\omega>0, 0\leqslant\phi\leqslant\pi)$ is an even function defined in $\mathbb{R}$. Its graph is symmetric about the point $M(\frac{3\pi}{4},0)$. It is monotone on the interval $[0,\frac{\pi}{2}]$. Find the values of ω and ϕ.

Solution: Since $f(-x) = f(x) \Rightarrow \sin(-\omega x + \phi) = \sin(\omega x + \phi)$, then $2\cos\phi\sin\omega x = 0$. Since $x \in R, \omega > 0$, then $\cos\phi = 0$. In other words, since $0 \leqslant \phi \leqslant \pi$, we have $\phi = \frac{\pi}{2}$. Since its graph is symmetric about the point $M(\frac{3\pi}{4}, 0)$, then $f(\frac{3\pi}{4} - x) = -f(\frac{3\pi}{4} + x)$, then $f(\frac{3\pi}{4}) = -f(\frac{3\pi}{4})$ when $x = 0$. Since $f(\frac{3\pi}{4}, 0)$ is a point of the graph, then $f(\frac{3\pi}{4}) = -f(\frac{3\omega\pi}{4} + \frac{\pi}{2}) = \cos\frac{3\omega\pi}{4}$, that is $\cos\frac{3\omega\pi}{4} = 0$. Since $\omega > 0$, then $\frac{3\omega\pi}{4} = \frac{\pi}{2} + k\pi, k = 0, 1, 2, \cdots \Rightarrow \omega = \frac{2}{3}(2k+1)$. When $k = 0, \omega = \frac{2}{3}$, $f(x) = \sin(\frac{2}{3}x + \frac{\pi}{2})$ is decreasing on the interval $[0, \frac{\pi}{2})$. When $k = 1, \omega = 2$, $]f(x) = \sin(2x + \frac{\pi}{2})$ is decreasing on the interval $[0, \frac{\pi}{2}]$. When $k \geqslant 2, \omega \geqslant \frac{10}{3}$, $f(x) = \sin(\omega x + \frac{\pi}{2})$ is not a monotone function on the interval $[0, \frac{\pi}{2})$. After all, $\omega = \frac{2}{3}$ or $\omega = 2$.

4.46 ★★ Let sides a, b, c correspond to angle A, B, C in $\triangle ABC$, show $\frac{a\cos B - b\cos A}{\sin(A-B)} = \frac{c}{\sin C}$.

Proof: Since $a\cos B - b\cos A = \frac{1}{c}(ac\cos B - bc\cos A) = \frac{a^2+c^2-b^2}{2c} - \frac{b^2+c^2-a^2}{2c} = \frac{a^2-b^2}{c}$, we have $\frac{a\cos B - b\cos A}{\sin(A-B)} = \frac{a^2-b^2}{c\sin(A-B)} = \frac{(\frac{c}{\sin C}\sin A)^2 - (\frac{c}{\sin C}\sin B)^2}{c\sin(A-B)} = \frac{c^2\sin^2 A - c^2\sin^2 B}{c\sin^2 C\sin(A-B)} = \frac{c(\sin A - \sin B)(\sin A + \sin B)}{\sin^2 C\sin(A-B)} = \frac{c2\sin\frac{A-B}{2}\cos\frac{A+B}{2}2\sin\frac{A+B}{2}\cos\frac{A-B}{2}}{\sin^2 C\sin(A-B)} = \frac{c\sin(A-B)\sin(A+B)}{\sin^2 C\sin(A-B)} = \frac{c\sin(A+B)}{\sin^2 C} = \frac{c\sin C}{\sin^2 C} = \frac{c}{\sin C}$.

4.47 ★★ If $\theta \in (0, \frac{\pi}{6})$, compare $\tan(\sin\theta), \tan(\tan\theta), \tan(\cos\theta)$.

Solution: Since $\theta \in (0, \frac{\pi}{6})$, then $0 < \sin\theta < \cos\theta < 1$. Since $\tan\theta = \frac{\sin\theta}{\cos\theta}$, then $\sin\theta = \tan\theta\cos\theta$. On the other hand, $0 < \cos\theta < 1$, we have $\sin\theta < \tan\theta$. Since $0 < \tan\theta < \tan\frac{\pi}{6} = \frac{\sqrt{3}}{3}, 1 > \cos\theta > \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$, then $1 > \cos\theta > \tan\theta > 0$. Hence $0 < \sin\theta < \tan\theta < \cos\theta < 1$. Since $y = \tan x$ is increasing in the interval $(0, 1)$, then $\tan(\sin\theta) < \tan(\tan\theta) < \tan(\cos\theta)$.

4.48 ★★ Find the value of m which satisfies the inequality $\cos^2\alpha + 2m\sin\alpha - 2m - 2 < 0$.

Solution: $\cos^2\alpha + 2m\sin\alpha - 2m - 2 < 0 \Rightarrow \sin^2\alpha - 2m\sin\alpha + 2m + 1 > 0$. Let $\sin\alpha = t$, then $-1 \leqslant t \leqslant 1$. Assume $f(t) = t^2 - 2mt + 2m + 1 = (t-m)^2 - m^2 + 2m + 1 > 0, t \in [-1, 1]$. (1) If $m < -1$, then $f(t)_{min} = 2 + 4m$ at $t = -1$. Let $2 + 4m > 0$, then $m > -\frac{1}{2}$. It is in contradiction with $m < -1$. Therefore $m > -\frac{1}{2}$ should be rejected. (2) If $-1 \leqslant m \leqslant 1$, then $f(t)_{min} = -m^2 + 2m + 1$ at $t = m$. Let $-m^2 + 2m + 1 > 0$, that is $m^2 - 2m - 1 < 0$, then $1 - \sqrt{2} < m \leqslant 1$. (3) If $m > 1$, then $f(t)_{min} = 2 > 0$ at $t = 1$. After all, $m > 1 - \sqrt{2}$.

4.49 ★★ Let the angles A, B, C of $\triangle ABC$ form an arithmetic progression, a, b, c are the side lengths corresponding to angles A, B, C, and $c - a$ is equal to the altitude h on the side AC. Find the value of $\sin\dfrac{C-A}{2}$.

Solution: From the given condition, we have $h = c - a = \dfrac{h}{\sin A} - \dfrac{h}{\sin C}$. The equation is equivalent to $\sin C - \sin A = \sin A \sin C$. Thus $2\sin\dfrac{C-A}{2}\cos\dfrac{C+A}{2} = \dfrac{1}{2}[\cos(C-A) - \cos(C+A)] \cdots (*)$. Since $A + C = 2B$ and $A + B + C = 180^0$, then $A + C = 120^0$. Substituting it into $(*)$, we have $\sin\dfrac{C-A}{2} = \dfrac{1}{2}[\cos(C-A) + \dfrac{1}{2}] \Rightarrow \sin\dfrac{C-A}{2} = -\dfrac{1}{2}[-\cos(C-A) + 1 - \dfrac{3}{2}] \Rightarrow \sin\dfrac{C-A}{2} = -\dfrac{1-\cos(C-A)}{2} + \dfrac{3}{4}] \Rightarrow (\sin\dfrac{C-A}{2})^2 + \sin\dfrac{C-A}{2} - \dfrac{3}{4} = 0$. Hence, $\sin\dfrac{C-A}{2} = \dfrac{1}{2}$ or $\sin\dfrac{C-A}{2} = -\dfrac{3}{2}$ (rejected). Therefore, $\sin\dfrac{C-A}{2} = \dfrac{1}{2}$.

4.50 ★ Given the function $f(x) = a\sin + b\cos x$. (1) If $f(\frac{\pi}{4}) = \sqrt{2}$ and the maximum value of $f(x)$ is $\sqrt{10}$, find the value of a, b. (2) If $f(\frac{\pi}{3}) = 1$ and the minimum value of $f(x)$ is k, find the range of k.

Solution: (1) It is easy to figure out that $a\sin\frac{\pi}{4} + b\cos\frac{\pi}{4} = \sqrt{2}$. Thus $\frac{\sqrt{2}}{2}(a+b) = \sqrt{2} \Rightarrow a + b = 2$. On the other hand, $f(x) = a\sin + b\cos x = \sqrt{a^2+b^2}\sin(x+\theta)$. Since the maximum value of $f(x)$ is $\sqrt{10}$ when $\sin(x+\theta) = 1$, then $\sqrt{a^2+b^2} = \sqrt{10}$, that is $a^2 + b^2 = 10$. Since

$$\begin{cases} a + b = 2 \\ a^2 + b^2 = 10 \end{cases}$$

we have

$$\begin{cases} a = -1 \\ b = 3 \end{cases}$$

or

$$\begin{cases} a = 3 \\ b = -1 \end{cases}$$

(2) From the given condition, we have $\frac{\sqrt{3}}{2}a + \frac{1}{2}b = 1$, that is $b = 2 - \sqrt{3}a$. On the other hand, $f(x) = a\sin + b\cos x = \sqrt{a^2+b^2}\sin(x+\theta)$. The condition $\sin(x+\theta) = -1$ can lead to the minimum value of $f(x)$. Hence $-\sqrt{a^2+b^2} = k, (k < 0)$. For the equation system

$$\begin{cases} b = 2 - \sqrt{3}a \\ a^2 + b^2 = k^2 \end{cases}$$

Eliminating b, we obtain $4a^2 - 4\sqrt{3}a + 4 - k^2 = 0$. Since $a \in R$, then $\Delta = 48 - 64 + 16k^2 \geqslant 0 \Rightarrow k^2 \geqslant 1$. Since $k < 0$, we have $k \leqslant -1$.

4.51 ★ Evaluate the equation $\sqrt{1+x^2} + \frac{\sqrt{1+x^2}}{x} = 2\sqrt{2}$ by applying the trigonometric functions.

Solution: Let $x = \tan\theta, \theta \in (-\frac{\pi}{2}, \frac{\pi}{2}), \theta \neq 0$. The equation is equivalent to $\frac{1}{\cos\theta} + \frac{1}{\sin\theta} = 2\sqrt{2} \Rightarrow \sin\theta + \cos\theta = 2\sqrt{2}\sin\theta\cos\theta \Rightarrow \sqrt{2}\sin(\theta + \frac{\pi}{4}) = \sqrt{2}\sin 2\theta \Rightarrow \sin(\theta + \frac{\pi}{4}) = \sin 2\theta$. Hence $2\theta = 2k\pi + \theta + \frac{\pi}{4}$ or $2\theta = (2k+1)\pi - \theta - \frac{\pi}{4}, (k \in Z)$. Thus $\theta = 2k\pi + \frac{\pi}{4}$ or $\theta = \frac{2k\pi}{3} + \frac{\pi}{4}, (k \in Z)$. Since $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$, then $\theta = \frac{\pi}{4}$ or $\theta = -\frac{5\pi}{12}$. Therefore $x = 1$ or $x = -2 - \sqrt{3}$.

4.52 ★★ If $\sin(A + B) = \tan\frac{A+B}{2}$ in $\triangle ABC$, and the three side lengths a, b, c form an arithmetic sequence, evaluate the radius of its circumcircle and the radius of its incircle.

Solution: Since $\sin(A + B) = \tan\frac{A+B}{2} \Rightarrow 2\sin\frac{A+B}{2}\cos\frac{A+B}{2} = \frac{\sin\frac{A+B}{2}}{\cos\frac{A+B}{2}} \Rightarrow 2\cos^2\frac{A+B}{2} = 1 \Rightarrow \cos(A + B) = 0 \Rightarrow A + B = \frac{\pi}{2}$, we have $b = c\cos A, a = c\sin A$. Since a, b, c form an arithmetic sequence, that is $2b = a + c$, thus $2c\cos A = c\sin A + c$, hence $2\cos A = \sin A + 1 \cdots$(i). On the other hand $\sin^2 A + \cos^2 A = 1 \cdots$(ii). According to (i) and (ii), we have $\sin A = \frac{3}{5}, \cos A = \frac{4}{5}$. Assume the radius of incircle is r and the radius of circumcircle is R. Since $\triangle ABC$ is a right triangle, then $r = \frac{a+b-c}{2} = \frac{c\sin A + c\cos A - c}{2}$. Applying the since theorem $\frac{c}{\sin 90^0} = 2R$, we have $R = \frac{c}{2}$. Therefore, $\frac{r}{R} = \frac{c\sin A + c\cos A - c}{c} = \sin A + \cos A - 1 = \frac{2}{5}$.

4.53 ★★ Let a, b, c are real numbers, find the sufficient and necessary condition for that $a\sin x + b\cos x + c > 0$ always holds for any real number x.

Solution: (1) When a, b are not zero at the same time, we have $a\sin x + b\cos x + c > 0 \Leftrightarrow \sqrt{a^2 + b^2}\sin(x + \phi) + c > 0 \Leftrightarrow \sin(x + \phi) > -\frac{c}{\sqrt{a^2+b^2}}$. The sufficient and necessary condition for that the formula always holds is $-\frac{c}{\sqrt{a^2+b^2}} < -1$. That is $\sqrt{a^2 + b^2} < c$.
(2) When a, b are both zero at the same time, then $c > 0$.
As a conclusion, for any real number x, the sufficient and necessary condition for that $a\sin x + b\cos x + c > 0$ always holds is $\sqrt{a^2 + b^2} < c$.

4.54 ★★ Let the function $f(x) = \frac{3}{2}\sin\omega x + \frac{3\sqrt{3}}{2}\cos\omega x + 1, (\omega > 0)$, and its period is π. If α, β are the two roots of the equation $f(x) = 0$, and $\alpha \neq k\pi + \beta, (k \in Z)$, compute $\tan(\alpha + \beta)$.

Since the period $T = \dfrac{2\pi}{\omega} = \pi$, then $\omega = 2$. Since α, β are the two roots of the equation $f(x) = 0$, we have

$$\begin{cases} 3\sin(2\alpha + \frac{\pi}{3}) + 1 = 0, \\ 3\sin(2\beta + \frac{\pi}{3}) + 1 = 0. \end{cases}$$

Simplifying the equation system, we have $\sin(2\alpha + \dfrac{\pi}{3}) - \sin(2\beta + \dfrac{\pi}{3}) = 0$. That is $2\cos(\alpha+\beta+\dfrac{\pi}{3})\sin(\alpha-\beta) = 0$. Since $\alpha - \beta \neq k\pi, (k \in Z)$, then $\sin(\alpha - \beta) \neq 0$. Thus $\cos(\alpha+\beta+\dfrac{\pi}{3}) = 0$. Hence $\alpha+\beta+\dfrac{\pi}{3} = k\pi+\dfrac{\pi}{2} \quad (k \in Z) \Rightarrow \alpha+\beta = k\pi+\dfrac{\pi}{6} \quad (k \in Z)$. Therefore, $\tan(\alpha+\beta) = \dfrac{\sqrt{3}}{3}$.

4.55 ★★ Given $\sin\theta = \sqrt{|\sin t|}$, $\cos\theta = \sqrt{|\cos t|}$, and $0 \leqslant \theta \leqslant \dfrac{\pi}{2}$. Find the value of t such that θ is in the interval $[0, \dfrac{\pi}{4}]$.

Solution: Since $0 \leqslant \theta \leqslant \dfrac{\pi}{2}$, then $0 \leqslant \theta \leqslant \dfrac{\pi}{4} \Leftrightarrow 0 \leqslant \tan\theta \leqslant 1$. Since $\tan\theta = \dfrac{\sin\theta}{\cos\theta} = \dfrac{\sqrt{|\sin t|}}{\sqrt{|\cos t|}} = \sqrt{|\tan t|}$, then $0 \leqslant \tan\theta \leqslant 1 \Leftrightarrow 0 \leqslant \sqrt{|\tan t|} \leqslant 1 \Leftrightarrow 0 \leqslant |\tan t| \leqslant 1 \Leftrightarrow -1 \leqslant \tan t \leqslant 1$. Since $y = \tan t$ is increasing on the interval $(-\dfrac{\pi}{2}, \dfrac{\pi}{2})$ and the period $T = \pi$, then the solution of inequality $-1 \leqslant \tan t \leqslant 1$ is $k\pi - \dfrac{\pi}{4} \leqslant t \leqslant k\pi + \dfrac{\pi}{4} \quad (k \in Z)$.

4.56 ★★★ The side lengths a, b, c correspond to the angles A, B, C in $\triangle ABC$. If $a = (\sqrt{3}-1)c$, and $\dfrac{\cot B}{\cot C} = \dfrac{c}{2a-c}$, value A, B, C.

Solution: Applying the given equation and the since theorem, we have $\dfrac{\cos B}{\sin B}\dfrac{\sin C}{\cos C} = \dfrac{\sin C}{2\sin A - \sin C} \Rightarrow (2\sin A - \sin C)\cos B = \sin B\cos C \Rightarrow 2\sin A\cos B = \sin(B+C)$. On the other hand, $\sin(B+C) = \sin A$, then $\cos B = \dfrac{1}{2}, B = \dfrac{\pi}{3}, A + C = \dfrac{2\pi}{3}$ ①.

From the given condition $\dfrac{a}{c} = \sqrt{3}-1 \Rightarrow \dfrac{\sin A}{\sin C} + 1 = \sqrt{3} \Rightarrow \dfrac{2\sin\frac{A+C}{2}\cos\frac{A-C}{2}}{\sin C} = \sqrt{3} \Rightarrow \dfrac{2\frac{\sqrt{3}}{2}\cos\frac{A-C}{2}}{\sin C} = \sqrt{3} \Rightarrow \cos\dfrac{A-C}{2} = \sin C = \cos(\dfrac{\pi}{2} - C)$. Since A, B, C are three interior angles of a triangle, thus $\dfrac{C-A}{2} = \dfrac{\pi}{2} - C$. That is $3C - A = \pi$ ②. According to ① and ②, we have $C = \dfrac{5}{12}\pi, A = \dfrac{\pi}{4}, B = \dfrac{\pi}{3}$.

4.57 ★★ If the positive numbers a, b, c form an arithmetic sequence, and $a + b = c$, $\arctan\dfrac{1}{a} + \arctan\dfrac{1}{b} + \arctan\dfrac{1}{c} = \dfrac{\pi}{2}$. Find the values of a, b, c.

Solution: Since a, b, c form an arithmetic sequence, then $a + c = 2b$. On the other hand $a + b = c$, solving the above two equations, we have $a = \dfrac{b}{2}, c = \dfrac{3}{2}b$. Let $\arctan\dfrac{1}{a} = \alpha, \arctan\dfrac{1}{b} = \beta, \arctan\dfrac{1}{c} = \gamma$. Since a, b, c are positive numbers, then α, β, γ are acute angles. Hence $\tan\alpha = \dfrac{1}{a}, \tan\beta = \dfrac{1}{b}, \tan\gamma = \dfrac{1}{c}$. Since $\arctan\dfrac{1}{a} + \arctan\dfrac{1}{b} = \dfrac{\pi}{2} - \arctan\dfrac{1}{c} \Rightarrow \tan(\alpha+\beta) = \tan(\dfrac{\pi}{2} - \gamma) \Rightarrow \dfrac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta} = \cot\gamma \Rightarrow \dfrac{\frac{1}{a} + \frac{1}{b}}{1 - \frac{1}{a}\frac{1}{b}} = c \Rightarrow a + b = abc - c \Rightarrow \dfrac{b}{2} + b = \dfrac{b}{2}\cdot b\cdot\dfrac{3}{2}b - \dfrac{3}{2}b \Rightarrow b^3 - 4b = 0 \Rightarrow b = 2$, $b = 0$ (rejected), $b = -2$ (rejected). Therefore, $a = 1, b = 2, c = 3$.

4.58 ★★★ If $x \in [-1, 1]$, show $\arcsin x + \arccos x = \dfrac{\pi}{2}$.

Proof: The function $\arcsin x$ and $\arccos x$ are defined for $x \in [-1, 1]$. Applying the induction formula and the definition of inverse cosine function, we obtain $\sin(\dfrac{\pi}{2} - \arccos x) = \cos(\arccos x) = x$. Since $0 \leqslant \arccos x \leqslant \pi$, thus $-\pi \leqslant -\arccos x \leqslant 0$, then $-\dfrac{\pi}{2} \leqslant \dfrac{\pi}{2} - \arccos x \leqslant \dfrac{\pi}{2}$, that is $\dfrac{\pi}{2} - \arccos x \in [-\dfrac{\pi}{2}, \dfrac{\pi}{2}]$. Applying the definition of inverse sine function, we have $\arcsin x = \dfrac{\pi}{2} - \arccos x$. After all, $\arcsin x + \arccos x = \dfrac{\pi}{2}$.

4.59 ★★★ Given $\sin x + \cos x = \sqrt{2}\sin(x + \frac{\pi}{4})$, $x \in [-\frac{\pi}{6}, \frac{\pi}{2}]$. Find the maximum and minimum values of the function $y = (\sin x + 1)(\cos x + 1)$.

Solution: Let $\sin x + \cos x = u$, then $\sin x \cos x = \frac{u^2 - 1}{2}$, and $u = \sqrt{2}\sin(x + \frac{\pi}{4})$. When $x = \frac{\pi}{4}$, $u_{\max} = \sqrt{2}$. When $x = -\frac{\pi}{6}$, u reaches the minimum value. Since $\sin(-\frac{\pi}{6})\cos\frac{\pi}{6} = \frac{u^2-1}{2} \Rightarrow -\frac{\sqrt{3}}{4} = \frac{u^2-1}{2} \Rightarrow u^2 = \frac{2-\sqrt{3}}{2}$. Since $u > 0$, then $u_{\min} = \sqrt{\frac{2-\sqrt{3}}{2}} = \frac{\sqrt{3}-1}{2}$, then $u \in [\frac{\sqrt{3}-1}{2}, \sqrt{2}]$. Since $y = (\sin x + 1)(\cos x + 1) = \sin x + \cos x + \sin x \cos x + 1 = \frac{1}{2}(u+1)^2$, and y is increasing on the interval $[\frac{\sqrt{3}-1}{2}, \sqrt{2}]$, therefore, $y_{\min} = \frac{2+\sqrt{3}}{4}$, $y_{\max} = \frac{3+2\sqrt{2}}{2}$.

4.60 ★★ Given $\frac{\sin\theta}{x} = \frac{\cos\theta}{y}$, and $\frac{\sin^2\theta}{y^2} + \frac{\cos^2\theta}{x^2} = \frac{6}{x^2+y^2}$. Find the value of θ.

Solution: Since $\tan\theta = \frac{x}{y}$, we have $\sin^2\theta = \frac{1}{\csc^2\theta} = \frac{1}{1+\cot^2\theta} = \frac{\tan^2\theta}{\tan^2\theta+1} = \frac{x^2}{x^2+y^2}$, and $\cos^2\theta = \frac{1}{\sec^2\theta} = \frac{1}{1+\tan^2\theta} = \frac{y^2}{x^2+y^2}$. Substituting $\sin^2\theta$ and $\cos^2\theta$ into the given second equation, we have $\frac{x^2}{(x^2+y^2)y^2} + \frac{y^2}{(x^2+y^2)x^2} = \frac{6}{x^2+y^2} \Rightarrow \frac{x^2}{y^2} + \frac{y^2}{x^2} = 6 \Rightarrow (\frac{x^2}{y^2})^2 - 6(\frac{x^2}{y^2}) + 1 = 0$. Thus $\frac{x^2}{y^2} = 3 \pm 2\sqrt{2} = (\sqrt{2} \pm 1)^2$. Hence $\tan\theta = \pm(\sqrt{2}+1)$ or $\tan\theta = \pm(\sqrt{2}-1)$. Applying $\tan\theta = \pm(\sqrt{2}+1)$, we have $\theta = n\pi \pm \frac{3\pi}{8}, (n \in Z)$. Applying $\tan\theta = \pm(\sqrt{2}-1)$, we have $\theta = n\pi \pm \frac{\pi}{8}, (n \in Z)$.

4.61 ★★★ Given in $\triangle ABC$, $\frac{\sin A}{\sin B} = \frac{m}{n}$, $\frac{\cos A}{\cos B} = -\frac{p}{q}$. Show $\cos C = \frac{mp-nq}{np-mq}$.

Solution : From the given condition, we have $\frac{\sin(B+C)}{\sin B} = \frac{m}{n} \Rightarrow \frac{\sin B\cos C + \cos B\sin C}{\sin B} = \frac{m}{n} \Rightarrow \cos C + \cot B\sin C = \frac{m}{n} \Rightarrow \cot B\sin C = \frac{m}{n} - \cos C$ ①. Since $\frac{\cos(B+C)}{\cos B} = -\frac{p}{q} \Rightarrow \frac{\cos B\cos C - \sin B\sin C}{\cos B} = -\frac{p}{q} \Rightarrow \tan B\sin C = \frac{p}{q} + \cos C$ ②. Applying ① × ②, we obtain $\sin^2 C = (\frac{m}{n} - \cos C)(\frac{p}{q} + \cos C) \Rightarrow 1 = \frac{mp}{nq} + \cos C(\frac{m}{n} - \frac{p}{q})$. Therefore $\cos C = \frac{mp-nq}{np-mq}$.

4.62 ★★★ Given $\cos\theta + \cos\phi = a$, $\sin\theta + \sin\phi = b$. Compute $\cos(\theta+\phi)$ and $\sin 2\theta + \sin 2\phi$.

Solution: Since $\frac{\sin\theta+\sin\phi}{\cos\theta+\cos\phi} = \frac{b}{a}$, on the other hand, $\frac{\sin\theta+\sin\phi}{\cos\theta+\cos\phi} = \frac{2\sin\frac{\theta+\phi}{2}\cos\frac{\theta-\phi}{2}}{2\cos\frac{\theta+\phi}{2}\cos\frac{\theta-\phi}{2}} = \tan\frac{\theta+\phi}{2}$, then $\tan\frac{\theta+\phi}{2} = \frac{b}{a}$. Assume $\tan\frac{\theta+\phi}{2} = t$, then $\cos(\theta+\phi) = \frac{1-t^2}{1+t^2} = \frac{a^2-b^2}{a^2+b^2}$, $\sin(\theta+\phi) = \frac{2t}{1+t^2} = \frac{2ab}{a^2+b^2}$ (Applying trigonometric function formulas). Since $2(\cos\theta+\cos\phi)(\sin\theta+\sin\phi) = 2ab \Rightarrow \sin 2\theta + \sin 2\phi + 2\sin(\theta+\phi) = 2ab$, we have $\sin 2\theta + \sin 2\phi = 2ab - \frac{4ab}{a^2+b^2} = 2ab(1 - \frac{2}{a^2+b^2})$.

4.63 ★★★ If $3\tan^{-1}\dfrac{1}{2+\sqrt{3}} - \tan^{-1}\dfrac{1}{x} = \tan^{-1}\dfrac{1}{3}$, evaluate the value of x.

Solution: Let $\tan^{-1}\dfrac{1}{2+\sqrt{3}} = \alpha$, then $\tan\alpha = \dfrac{1}{2+\sqrt{3}}$, $\tan 3\alpha = \dfrac{3\tan\alpha - \tan^3\alpha}{1-3\tan^2\alpha} = \dfrac{\frac{3}{2+\sqrt{3}} - \frac{1}{(2+\sqrt{3})^3}}{1-\frac{3}{(2+\sqrt{3})^2}} = \dfrac{3(2+\sqrt{3})^2-1}{(2+\sqrt{3})^3-3(2+\sqrt{3})} = \dfrac{20+12\sqrt{3}}{20+12\sqrt{3}} = 1$. Hence, $3\alpha = \tan^{-1}1$. Therefore, the equation $3\tan^{-1}\dfrac{1}{2+\sqrt{3}} - \tan^{-1}\dfrac{1}{x} = \tan^{-1}\dfrac{1}{3}$ is equivalent to the equation $\tan^{-1}1 - \tan^{-1}\dfrac{1}{3} = \tan^{-1}\dfrac{1}{x}$. Since $\tan(\tan^{-1}1 - \tan^{-1}\dfrac{1}{3}) = \tan(\tan^{-1}\dfrac{1}{x}) \Rightarrow \dfrac{1-\frac{1}{3}}{1+\frac{1}{3}} = \dfrac{1}{x}$. After all, $x = 2$.

4.64 ★★★ Let $\sin\alpha = p\sin\beta, \cos\alpha = q\cos\beta, \sin\alpha + \cos\alpha = r(\sin\beta + \cos\beta)$, show $(p-r)^2(1-q^2) + (q-r)^2(1-p^2) = 0$.

Solution: From the given conditions, we have $p^2\sin^2\beta + q^2\cos^2\beta = \sin^2\alpha + \cos^2\alpha = \sin^2\beta + \cos^2\beta$. Dividing both sides of the equation by $\cos^2\beta$, we have $p^2\tan^2\beta + q^2 = \tan^2\beta + 1$, that is, $\tan^2\beta = \dfrac{q^2-1}{1-p^2}$ ①. Since $p\sin\beta + q\cos\beta = r(\sin\beta + \cos\beta) \Rightarrow (p-r)\sin\beta = (r-q)\cos\beta$, then $\tan\beta = \dfrac{r-q}{p-r}$ ②. Applying ① and ②, we have $\dfrac{q^2-1}{1-p^2} = \dfrac{(r-q)^2}{(p-r)^2}$. Simplifying the formula, we obtain $(p-r)^2(1-q^2+(q-r)^2(1-p^2 = 0$.

4.65 ★★★ Let a, b, c are the side lengths of triangle ABC corresponding to angles A, B, C, $(\sin B + \sin C + \sin A)(\sin B + \sin C - \sin A) = 3\sin B\sin C$. b, c are the two roots of equation $x^2 - 3x + 4\cos A = 0$, and $b > c$. The radius of circumcircle of $\triangle ABC$ is 1. Find the value of $\angle A$, a, b, c.

Solution: From the given conditions, we have $b+c = 3$, $bc = 4\cos A$. Applying the sine law, $b = 2R\sin B = 2\sin B$, $c = 2R\sin C = 2\sin C$. Adding the two equations together, we obtain $\sin B + \sin C = \dfrac{b+c}{2} = \dfrac{3}{2}$ ①. Multiplying the two equations, we obtain $\sin B\sin C = \dfrac{bc}{4} = \cos A$ ②. Simplifying the equation $(\sin B + \sin C + \sin A)(\sin B + \sin C - \sin A) = 3\sin B\sin C$, we get $(\sin B + \sin C)^2 - \sin A^2 = 3\sin B\sin C$ ③. Submit ① and ② into ③, then $\dfrac{9}{4} - \sin^2 A = 3\cos A \Rightarrow 4\cos^2 A - 12\cos A + 5 = 0 \Rightarrow \cos A = \dfrac{1}{2}$ or $\cos A = \dfrac{5}{2}$(rejected). Hence $\angle A = 60^0$. According to the equations system

$$\begin{cases} b+c=3 \\ bc=2 \end{cases}$$

and $b > c$, we have $b = 2, c = 1, a = 2R\sin A = \sqrt{3}$.

4.66 ★★★ Show $\tan x + \frac{1}{2}\tan\frac{x}{2} + \frac{1}{2^2}\tan\frac{x}{2^2} + \cdots + \frac{1}{2^n}\tan\frac{x}{2^n} = \frac{1}{2^n}\cot\frac{x}{2^n} - 2\cot 2x$.

Proof: Since $\cot x - \tan x = \frac{1-\tan^2 x}{\tan x} = 2\frac{1-\tan^2 x}{2\tan x} = 2\frac{1}{\tan 2x} = 2\cot 2x$, then $\tan x = \cot x - 2\cot 2x$. Similarly $\frac{1}{2}\tan\frac{x}{2} = \frac{1}{2}\cot\frac{x}{2} - \cot x$, $\frac{1}{2^2}\tan\frac{x}{2^2} = \frac{1}{2^2}\cot\frac{x}{2^2} - \frac{1}{2}\cot\frac{x}{2}, \cdots, \frac{1}{2^n}\tan\frac{x}{2^n} = \frac{1}{2^n}\cot\frac{x}{2^n} - \frac{1}{2^{n-1}}\cot\frac{x}{2^{n-1}}$. Adding the above equations, we have $\tan x + \frac{1}{2}\tan\frac{x}{2} + \frac{1}{2^2}\tan\frac{x}{2^2} + \cdots + \frac{1}{2^n}\tan\frac{x}{2^n} = \frac{1}{2^n}\cot\frac{x}{2^n} - 2\cot 2x$.

4.67 ★★★★ If $0 \leqslant x \leqslant 1$, show $\arcsin x - \arcsin\frac{x-\sqrt{1-x^2}}{\sqrt{2}} = \frac{\pi}{4}$.

Proof: Let $\arcsin x = \alpha, \arcsin\frac{x-\sqrt{1-x^2}}{\sqrt{2}} = \beta$. Since $\sin\alpha = x, 0 \leqslant x \leqslant 1$, then $0 \leqslant \alpha \leqslant \frac{\pi}{2}$. Hence $\cos\alpha = \sqrt{1-\sin^2\alpha} = \sqrt{1-x^2}$. Since $0 \leqslant x^2 \leqslant 1 \Rightarrow -1 \leqslant -x^2 \leqslant 0 \Rightarrow 0 \leqslant 1-x^2 \leqslant 1 \Rightarrow 0 \leqslant \sqrt{1-x^2} \leqslant 1 \Rightarrow -1 \leqslant -\sqrt{1-x^2} \leqslant 0 \Rightarrow -1 \leqslant x - \sqrt{1-x^2} \leqslant 1$. That is $-\frac{1}{\sqrt{2}} \leqslant \frac{x-\sqrt{1-x^2}}{\sqrt{2}} \leqslant \frac{1}{\sqrt{2}}$. Since $\sin\beta = \frac{x-\sqrt{1-x^2}}{\sqrt{2}}$, then $-\frac{\pi}{4} \leqslant \beta \leqslant \frac{\pi}{4}$. Since $\sin(\alpha - \frac{\pi}{4}) = \sin\alpha\cos\frac{\pi}{4} - \cos\alpha\sin\frac{\pi}{4} = x\frac{1}{\sqrt{2}} - \sqrt{1-x^2}\frac{1}{\sqrt{2}} = \frac{x-\sqrt{1-x^2}}{\sqrt{2}} = \sin\beta$. Since $0 \leqslant \alpha \leqslant \frac{\pi}{2}$, then $-\frac{\pi}{4} \leqslant \alpha - \frac{\pi}{4} \leqslant \frac{\pi}{4}$. Hence $\alpha - \frac{\pi}{4} = \beta$, that is $\alpha - \beta = \frac{\pi}{4}$.

Therefore, $\arcsin x - \arcsin\frac{x-\sqrt{1-x^2}}{\sqrt{2}} = \frac{\pi}{4}$.

4.68 ★★★ Solve the equation system

$$\begin{cases} \arcsin x \arcsin y = \frac{\pi^2}{12} \cdots ① \\ \arccos x \arccos y = \frac{\pi^2}{24} \cdots ② \end{cases}$$

Solution: Applying $\arccos x = \frac{\pi}{2} - \arcsin x, \arccos y = \frac{\pi}{2} - \arcsin y$ to rewrite the given equation ② as the formula $(\frac{\pi}{2} - \arcsin x)(\frac{\pi}{2} - \arcsin y) = \frac{\pi^2}{24}$. Let $\alpha = \arcsin x, \beta = \arcsin y$, then the given equation system is equivalent to

$$\begin{cases} \alpha\beta = \frac{\pi^2}{12} \\ \alpha\beta - (\alpha+\beta)\frac{\pi}{2} = -\frac{5\pi^2}{24} \end{cases}$$

That is,

$$\begin{cases} \alpha\beta = \dfrac{\pi^2}{12} \\ \alpha + \beta = \dfrac{7\pi}{12} \end{cases}$$

Assume α, β are the roots of equation $12z^2 - 7\pi z + \pi^2 = 0$, then $\alpha = \frac{\pi}{3}, \beta = \frac{\pi}{4}$, or $\alpha = \frac{\pi}{4}, \beta = \frac{\pi}{3}$. Those are $\arcsin x = \frac{\pi}{3}, \arcsin y = \frac{\pi}{4}$, or $\arcsin x = \frac{\pi}{4}, \arcsin y = \frac{\pi}{3}$. The solutions are $x_1 = \frac{\sqrt{3}}{2}, y_1 = \frac{\sqrt{2}}{2}$ or $x_2 = \frac{\sqrt{2}}{2}, y_2 = \frac{\sqrt{3}}{2}$. We can verify that $x_1 = \frac{\sqrt{3}}{2}, y_1 = \frac{\sqrt{2}}{2}$ and $x_2 = \frac{\sqrt{2}}{2}, y_2 = \frac{\sqrt{3}}{2}$ are both the roots of the system.

4.69 ★★★ Let A, B, C are the three angles of $\triangle ABC$ corresponding to the side lengths a, b, c, and they form a geometric sequence, and $b^2 - a^2 = ac$. Find the value of $\angle B$.

Solution: Since A, B, C form a geometric sequence, we assume $A = \frac{1}{q}B, C = qB$.

Since $A+B+C=\pi$, then $\frac{1}{q}B+B+qB=\pi$, that is $B=\frac{q\pi}{q^2+q+1}$. Since $b^2-a^2=ac$, according to the cosine law $b^2=a^2+c^2-2ac\cos B$, we have $ac=c^2-2ac\cos B$. Since $c\neq 0$, thus $a=c-2a\cos B$. Applying the sine law, we have $\sin A=\sin C-2\sin A\cos B \Rightarrow \sin A=\sin C-[\sin(A+B)+\sin(A-B)] \Rightarrow \sin A=\sin C-\sin C-\sin(A-B) \Rightarrow \sin A=\sin(B-A) \Rightarrow A=B-A \Rightarrow A=\frac{1}{2}B$. Hence, $\frac{1}{2}B=\frac{1}{q}B$. After all, $q=2, B=\frac{2\pi}{2^2+2+1}=\frac{2\pi}{7}$.

4.70 ★★★ If $0\leqslant x\leqslant\frac{\pi}{2}$, show $\cot\frac{x}{2^n}-\cot x\geqslant n,(n\in N)$.

Proof: (1) When $n=1$, $\cot\frac{x}{2}-\cot x=\frac{1+\cos x}{\sin x}-\frac{\cos x}{\sin x}=\frac{1}{\sin x}$. Since $0<x\leqslant\frac{\pi}{2}, 0<\sin x\leqslant 1$, then $\frac{1}{\sin x}\geqslant 1$, that is $\cot\frac{x}{2}-\cot x\geqslant 1$. The equation holds when $x=\frac{\pi}{2}$.

(2) Assume the inequation holds when $n=k,(k\in N)$, that is $\cot\frac{x}{2^k}-\cot x\geqslant k$, then $\cot\frac{x}{2^{k+1}}-\cot x=\frac{1+\cos\frac{x}{2^k}}{\sin\frac{x}{2^k}}-\cot x=\frac{1}{\sin\frac{x}{2^k}}+\cot\frac{x}{2^k}-\cot x$ when $n=k+1$. Since $\frac{1}{\sin\frac{x}{2^k}}>1,(0<x\leqslant\frac{\pi}{2}), k\in N$, then $\cot\frac{x}{2^{k+1}}-\cot x>k+1$. Therefore, for all $n\in N$, $\cot\frac{x}{2^n}-\cot x\geqslant n$ holds.

4.71 ★★★ Find the sum of the formula $\tan^{-1}\frac{1}{1+1\cdot 2}+\tan^{-1}\frac{1}{1+2\cdot 3}+\tan^{-1}\frac{1}{1+3\cdot 4}+\cdots$.

Solution: The nth term is $\tan^{-1}\frac{1}{1+n\cdot(n+1)}=\tan^{-1}\frac{(n+1)-n}{1+n\cdot(n+1)}=\tan^{-1}(n+1)-\tan^{-1}n$. Substituting $n=1,2,3,\cdots$, into the above equation and adding these equations, we have $\tan^{-1}\frac{1}{1+1\cdot 2}+\tan^{-1}\frac{1}{1+2\cdot 3}+\tan^{-1}\frac{1}{1+3\cdot 4}+\cdots=(\tan^{-1}2-\tan^{-1}1)+(\tan^{-1}3-\tan^{-1}2)+(\tan^{-1}4-\tan^{-1}3)+\cdots=-\tan^{-1}1+\tan^{-1}\infty=-\frac{\pi}{4}+\frac{\pi}{2}=\frac{\pi}{4}$.

4.72 ★★★ Let $A+B+C=\pi$, and $\sin(A+\frac{C}{2})=n\sin\frac{C}{2}$. Show $\tan\frac{A}{2}\tan\frac{B}{2}=\frac{n-1}{n+1}$.

Proof: $\sin(A+\frac{C}{2})=\sin(A+\frac{180^0-A-B}{2})=\sin(90^0-\frac{B-A}{2})=\cos\frac{B-A}{2}\cdots①$. Since $\sin(A+\frac{C}{2})=n\sin\frac{C}{2}$, then $\sin\frac{C}{2}=\cos\frac{A+B}{2}$. Hence $\sin(A+\frac{C}{2})=n\cos\frac{A+B}{2}\cdots②$.

Applying ① and ②, we have $\cos\dfrac{B-A}{2}=n\cos\dfrac{A+B}{2}$, that is $\cos\dfrac{A}{2}\cos\dfrac{B}{2}+\sin\dfrac{A}{2}\sin\dfrac{B}{2}=n(\cos\dfrac{A}{2}\cos\dfrac{B}{2}-\sin\dfrac{A}{2}\sin\dfrac{B}{2})\Rightarrow(n+1)\sin\dfrac{A}{2}\sin\dfrac{B}{2}=(n-1)\cos\dfrac{A}{2}\cos\dfrac{B}{2}$. Therefore $\tan\dfrac{A}{2}\tan\dfrac{B}{2}=\dfrac{n-1}{n+1}$.

4.73 ★★★★ If $\tan\alpha,\tan\beta$ are the two roots of the equation $x^2+px+q=0$, express $\sin^2(\alpha+\beta)+p\sin(\alpha+\beta)\cos(\alpha+\beta)+q\cos^2(\alpha+\beta)$ by p and q.

Solution: According to the relation between roots and coefficients, we have $\tan\alpha+\tan\beta=-p$, that is $p=-(\tan\alpha+\tan\beta),q=\tan\alpha\tan\beta$. Hence the quantity is equal to $\sin^2(\alpha+\beta)-(\tan\alpha+\tan\beta)\sin(\alpha+\beta)\cos(\alpha+\beta)+\tan\alpha\tan\beta\cos^2(\alpha+\beta)=\sin^2(\alpha+\beta)-(\tan\alpha+\tan\beta)\sin(\alpha+\beta)\cos(\alpha+\beta)+\tan\alpha\tan\beta[1-\sin^2(\alpha+\beta)]=\sin^2(\alpha+\beta)(1-\tan\alpha\tan\beta)-(\tan\alpha+\tan\beta)\sin(\alpha+\beta)\cos(\alpha+\beta)+\tan\alpha\tan\beta=\sin(\alpha+\beta)\cos(\alpha+\beta)\{\dfrac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)}(1-\tan\alpha\tan\beta)-(\tan\alpha+\tan\beta)\}+\tan\alpha\tan\beta=\sin(\alpha+\beta)\cos(\alpha+\beta)\{\dfrac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}(1-\tan\alpha\tan\beta)-(\tan\alpha+\tan\beta)\}+\tan\alpha\tan\beta=\tan\alpha\tan\beta=q$.

4.74 ★★ If $(1-\tan\theta)(1+\sin 2\theta)=1+\tan\theta$, evaluate the value of θ.

Solution: The given equation is equivalent to $(1-\dfrac{\sin\theta}{\cos\theta})(\sin\theta+\cos\theta)^2=1+\dfrac{\sin\theta}{\cos\theta}\Rightarrow(\cos\theta-\sin\theta)(\cos\theta+\sin\theta)^2=\cos\theta+\sin\theta\Rightarrow(\cos\theta+\sin\theta)[(\cos\theta-\sin\theta)(\cos\theta+\sin\theta)-1]=0$. If $\cos\theta+\sin\theta=0$, then $\tan\theta=-1$, thus $\theta=n\pi+\dfrac{3\pi}{4}\quad(n\in Z)$. If $(\cos\theta-\sin\theta)(\cos\theta+\sin\theta)-1=0$, then $\cos^2\theta-\sin^2\theta=1$, thus $\cos 2\theta=1$, hence $2\theta=2n\pi$, that is $\theta=n\pi\quad(n\in Z)$. Therefore, $\theta=n\pi+\dfrac{3\pi}{4}$ or $\theta=n\pi\quad(n\in Z)$.

4.75 ★★ Let A,B,C are the interior angles of triangle ABC, and $\cot\dfrac{A}{2},\cot\dfrac{B}{2},\cot\dfrac{C}{2}$ form an arithmetic sequence. Show $\cot\dfrac{A}{2}\cot\dfrac{C}{2}=3$.

Proof: From the given conditions, we have $\cot\dfrac{A}{2}+\cot\dfrac{C}{2}=2\cot\dfrac{B}{2}\Rightarrow\dfrac{\cos\frac{A}{2}}{\sin\frac{A}{2}}+\dfrac{\cos\frac{C}{2}}{\sin\frac{C}{2}}=2\dfrac{\cos\frac{B}{2}}{\sin\frac{B}{2}}=2\dfrac{\sin\frac{A+C}{2}}{\cos\frac{A+C}{2}}\Rightarrow\dfrac{\sin\frac{A+C}{2}}{\sin\frac{A}{2}\sin\frac{C}{2}}=\dfrac{2\sin\frac{A+C}{2}}{\cos\frac{A+C}{2}}$. Since A,B,C are the interior angles of triangle ABC, then $\sin\dfrac{A+C}{2},\cos\dfrac{A+C}{2},\sin\dfrac{A}{2},\sin\dfrac{C}{2}$ are all nonzero. Hence $\cos\dfrac{A+C}{2}=2\sin\dfrac{A}{2}\sin\dfrac{C}{2}\Rightarrow\cos\dfrac{A}{2}\cos\dfrac{C}{2}-\sin\dfrac{A}{2}\sin\dfrac{C}{2}=2\sin\dfrac{A}{2}\sin\dfrac{C}{2}\Rightarrow\cos\dfrac{A}{2}\cos\dfrac{C}{2}=3\sin\dfrac{A}{2}\sin\dfrac{C}{2}$. Therefore, $\cot\dfrac{A}{2}\cot\dfrac{C}{2}=3$.

4.76 ★★★★ If $\alpha \in (0, \frac{\pi}{2})$, $\beta \in (0, \frac{\pi}{2})$, and $\alpha + \beta = \theta$ is a constant. Find the minimum value of $\csc\alpha + \csc\beta$.

Solution: $\csc\alpha + \csc\beta = \frac{1}{\sin\alpha} + \frac{1}{\sin\beta} = \frac{\sin\alpha + \sin\beta}{\sin\alpha\sin\beta} = \frac{2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}}{\sin\alpha\sin\beta} = \frac{4\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}}{2\sin\alpha\sin\beta}$ $= \frac{4\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}}{\cos(\alpha-\beta) - \cos(\alpha+\beta)} = \frac{2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}}{\frac{1+\cos(\alpha-\beta)}{2} - \frac{1+\cos(\alpha+\beta)}{2}} = \frac{2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}}{\cos^2\frac{\alpha-\beta}{2} - \cos^2\frac{\alpha+\beta}{2}} = \sin\frac{\alpha+\beta}{2}$ $[\frac{1}{\cos\frac{\alpha-\beta}{2} - \cos\frac{\alpha+\beta}{2}} + \frac{1}{\cos\frac{\alpha-\beta}{2} + \cos\frac{\alpha+\beta}{2}}]$. Since $\alpha + \beta = \theta$ is a constant, then the above quantity is equal to $\sin\frac{\theta}{2}[\frac{1}{\cos\frac{\alpha-\beta}{2} - \cos\frac{\theta}{2}} + \frac{1}{\cos\frac{\alpha-\beta}{2} + \cos\frac{\theta}{2}}]$. This function reaches the minimum value when $\cos\frac{\alpha-\beta}{2} = 1$, i.e. $\alpha = \beta$. The minimum value of $\csc\alpha + \csc\beta$ is $\sin\frac{\theta}{2}[\frac{1}{1-\cos\frac{\theta}{2}} + \frac{1}{1+\cos\frac{\theta}{2}}] = \sin\frac{\theta}{2}\frac{2}{2\sin^2\frac{\theta}{2}} = \frac{2}{\sin\frac{\theta}{2}}$. As a conclusion, $(\csc\alpha + \csc\beta)_{min} = \frac{2}{\sin\frac{\theta}{2}}$ $(0 < \theta < \pi)$.

4.77 ★★★★ Let c be the hypotenuse length of $\triangle ABC$, $\angle C = 90^0$, the area is S. Find the values of $a, b, \angle A, \angle B$.

Solution: From the given conditions, we have $c^2 = a^2 + b^2$, $S = \frac{1}{2}ab$. Hence $a^2 + 2ab + b^2 = c^2 + 4S$, that is $a + b = \sqrt{c^2 + 4S} \cdots ①$. Similarly we have $a - b = \sqrt{c^2 - 4S} \cdots ②$. According to ① and ②, $a = \frac{1}{2}(\sqrt{c^2 + 4S} + \sqrt{c^2 - 4S})$, $b = \frac{1}{2}(\sqrt{c^2 + 4S} - \sqrt{c^2 - 4S})$. We also can obtain $\tan A = \frac{a}{b} = \frac{\frac{1}{2}(\sqrt{c^2 + 4S} + \sqrt{c^2 - 4S})}{\frac{1}{2}(\sqrt{c^2 + 4S} - \sqrt{c^2 - 4S})} = \frac{(\sqrt{c^2 + 4S} + \sqrt{c^2 - 4S})^2}{c^2 + 4S - c^2 + 4S} = \frac{c^2 + 4S + 2\sqrt{c^4 - 16S^2} + c^2 - 4S}{8S} = \frac{c^2 + \sqrt{c^4 - 16S^2}}{4S} \Rightarrow A = \arctan \frac{c^2 + \sqrt{c^4 - 16S^2}}{4S}$. Similarly $\tan B = \frac{b}{a} = \frac{\frac{1}{2}(\sqrt{c^2 + 4S} - \sqrt{c^2 - 4S})}{\frac{1}{2}(\sqrt{c^2 + 4S} + \sqrt{c^2 - 4S})} = \frac{c^2 + 4S - 2\sqrt{c^4 - 16S^2} + c^2 - 4S}{c^2 + 4S - c^2 + 4S} = \frac{2c^2 - 2\sqrt{c^4 - 16S^2}}{8S} = \frac{c^2 - \sqrt{c^4 - 16S^2}}{4S} \Rightarrow B = \arctan \frac{c^2 - \sqrt{c^4 - 16S^2}}{4S}$.

4.78 ★★ If $\tan\alpha, \tan\beta$ are the two roots of the equation $x^2 - 2(\log_3 12 + \log_4 12)x - \log_3 12 \log_4 12 = 0$, show $\sin(\alpha + \beta) + 2\sin\alpha\sin\beta = 0$.

Proof: The Vieta's formulas lead to $\tan\alpha + \tan\beta = 2(\log_3 12 + \log_4 12)$, $\tan\alpha\tan\beta = -\log_3 12 \log_4 12$. Then $\frac{\sin\alpha}{\cos\alpha} + \frac{\sin\beta}{\cos\beta} = \frac{\sin(\alpha+\beta)}{\cos\alpha\cos\beta} = 2(\log_3 4 + \log_4 3 + 2) \cdots ①$, $\frac{\sin\alpha\sin\beta}{\cos\alpha\cos\beta} = -(\log_3 4 + 1)(\log_4 3 + 1) = -(\log_3 4 + \log_4 3 + 2) \cdots ②$. divided ① by ②, then $\frac{\sin(\alpha+\beta)}{\sin\alpha\sin\beta} = -2 \Rightarrow \sin(\alpha + \beta) + 2\sin\alpha\sin\beta = 0$.

4.79 ★★★★ Let $\frac{\cos\theta\cos\frac{\phi}{2}}{\cos(\theta - \frac{\phi}{2})} + \frac{\cos\phi\cos\frac{\theta}{2}}{\cos(\phi - \frac{\theta}{2})} = 1$, show $\cos\theta + \cos\phi = 1$.

Proof: $\frac{\cos\theta\cos\frac{\phi}{2}}{\cos(\theta - \frac{\phi}{2})} + \frac{\cos\phi\cos\frac{\theta}{2}}{\cos(\phi - \frac{\theta}{2})} = 1 \Rightarrow \frac{\cos(\theta + \frac{\phi}{2}) + \cos(\theta - \frac{\phi}{2})}{2\cos(\theta - \frac{\phi}{2})} + \frac{\cos(\phi + \frac{\theta}{2}) + \cos(\phi - \frac{\theta}{2})}{2\cos(\phi - \frac{\theta}{2})} = 1 \Rightarrow \frac{\cos(\theta + \frac{\phi}{2}) + \cos(\theta - \frac{\phi}{2})}{2\cos(\theta - \frac{\phi}{2})} - \frac{1}{2} + \frac{\cos(\phi + \frac{\theta}{2}) + \cos(\phi - \frac{\theta}{2})}{2\cos(\phi - \frac{\theta}{2})} - \frac{1}{2} = 0 \Rightarrow \frac{\cos(\theta + \frac{\phi}{2})}{\cos(\theta - \frac{\phi}{2})} = -\frac{\cos(\phi + \frac{\theta}{2})}{\cos(\phi - \frac{\theta}{2})} \Rightarrow \frac{\cos\theta\cos\frac{\phi}{2} - \sin\theta\sin\frac{\phi}{2}}{\cos\theta\cos\frac{\phi}{2} + \sin\theta\sin\frac{\phi}{2}} = -\frac{\cos\phi\cos\frac{\theta}{2} - \sin\phi\sin\frac{\theta}{2}}{\cos\phi\cos\frac{\theta}{2} + \sin\phi\sin\frac{\theta}{2}}$. Cancellate the denominator to rewrite the equation as $\frac{\cos\theta\cos\frac{\phi}{2}}{\sin\theta\sin\frac{\phi}{2}} = \frac{\sin\phi\sin\frac{\theta}{2}}{\cos\phi\cos\frac{\theta}{2}} \Rightarrow \frac{\cos\theta\cos\frac{\phi}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\sin\frac{\phi}{2}} = \frac{2\sin\frac{\phi}{2}\cos\frac{\phi}{2}\sin\frac{\theta}{2}}{\cos\phi\cos\frac{\theta}{2}} \Rightarrow \frac{\cos\theta}{2\sin\frac{\theta}{2}\sin\frac{\phi}{2}} = \frac{2\sin\frac{\phi}{2}\sin\frac{\theta}{2}}{\cos\phi} \Rightarrow \cos\theta\cos\phi = 4\sin^2\frac{\theta}{2}\sin^2\frac{\phi}{2} = (1 - \cos\theta)(1 - \cos\phi) \Rightarrow \cos\theta\cos\phi = 1 - \cos\theta - \cos\phi + \cos\theta\cos\phi$. Hence $\cos\theta + \cos\phi = 1$.

4.80 ★★★★ Given $\sin\{2\cos^{-1}(\cot 2\tan^{-1} x)\} = 0$, evaluate the value of x.

Solution: Let $\tan^{-1} x = \theta$, then $\tan\theta = x, \cot 2\theta = \dfrac{1}{\tan 2\theta} = \dfrac{1-\tan^2\theta}{2\tan\theta} = \dfrac{1-x^2}{2x}$. Thus the given equation yields that $\sin\{2\cos^{-1}\dfrac{1-x^2}{2x}\} = 0$. Hence $2\cos^{-1}\dfrac{1-x^2}{2x} = n\pi, (n \in N)$. That is $\cos^{-1}\frac{1-x^2}{2x} = \frac{n\pi}{2}$. Dividing the equation by the cosine function, we have $\dfrac{1-x^2}{2x} = \cos\dfrac{n\pi}{2}, (n \in N)$. Since n is an arbitrary real number, then $\cos\dfrac{n\pi}{2} = 0$, or 1, or -1. If $\dfrac{1-x^2}{2x} = 0$, we have $x = \pm 1$. If $\dfrac{1-x^2}{2x} = 1$, then $x = -1 \pm \sqrt{2}$.

If $\dfrac{1-x^2}{2x} = -1$, then $x = 1 \pm \sqrt{2}$. As a conclusion, the solutions are $x = \pm 1$, or $x = -1 \pm \sqrt{2}$, or $x = 1 \pm \sqrt{2}$.

4.81 ★★★ Given $\tan^{-1} x + \dfrac{1}{2}\sec^{-1} 5x = \dfrac{\pi}{4}$, find the value of x.

Solution: Let $\tan^{-1} x = \theta, \dfrac{1}{2}\sec^{-1} 5x = \phi$. Then $x = \tan\theta, 5x = \sec 2\phi$. Since $\theta + \phi = \dfrac{\pi}{4}$, we have $2\phi = \dfrac{\pi}{2} - 2\theta$. Hence $5x = \sec(\dfrac{\pi}{2} - 2\theta) = \csc 2\theta = \dfrac{1}{2\sin\theta\cos\theta} = \dfrac{\sin^2\theta + \cos^2\theta}{2\sin\theta\cos\theta} = \dfrac{1}{2}(\tan\theta + \dfrac{1}{\tan\theta})$. That is $10x = x + \dfrac{1}{x} \Rightarrow 9x^2 = 1 \Rightarrow x = \pm\dfrac{1}{3}$.

4.82 ★★★ If a, b, c are the side lengths of triangle ABC, show $\dfrac{a\sin(B-C)}{b^2-c^2} = \dfrac{b\sin(C-A)}{c^2-a^2} = \dfrac{c\sin(A-B)}{a^2-b^2}$.

Proof: Let $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C} = k$. We obtain that $\dfrac{a\sin(B-C)}{b^2-c^2} = \dfrac{k\sin A\sin(B-C)}{b^2-c^2} = \dfrac{k\sin(B+C)\sin(B-C)}{b^2-c^2} = \dfrac{k[(\sin B\cos C)^2 - (\cos B\sin C)^2]}{b^2-c^2}$
$= \dfrac{k[\sin^2 B(1-\sin^2 C) - (1-\sin^2 B)\sin^2 C]}{b^2-c^2} = \dfrac{k(\sin^2 B - \sin^2 C)}{b^2-c^2}$. Since $\dfrac{\sin B}{b} = \dfrac{\sin C}{c} = \dfrac{1}{k}$, applying the geometric theorem, we have $\dfrac{\sin^2 B - \sin^2 C}{b^2-c^2} = \dfrac{1}{k^2}$, then $\dfrac{a\sin(B-C)}{b^2-c^2} = \dfrac{1}{k}$. Similarly, $\dfrac{b\sin(C-A)}{c^2-a^2} = \dfrac{1}{k}$, $\dfrac{c\sin(A-B)}{a^2-b^2} = \dfrac{1}{k}$. Therefore $\dfrac{a\sin(B-C)}{b^2-c^2} = \dfrac{b\sin(C-A)}{c^2-a^2} = \dfrac{c\sin(A-B)}{a^2-b^2}$.

4.83 ★★★ For a triangle ABC, if $\tan A \tan B > 1$, show the triangle is an acute triangle.

Proof 1: Since $\tan A \tan B > 1 \Rightarrow \tan A \tan B - 1 > 0 \Rightarrow \dfrac{\sin A \sin B - \cos A \cos B}{\cos A \cos B} > 0 \Rightarrow -\dfrac{\cos(A+B)}{\cos A \cos B} > 0 \Rightarrow \dfrac{\cos C}{\cos A \cos B} > 0$. Hence $\cos A \cos B \cos C > 0$. Therefore $\cos A > 0$. Otherwise, if $\cos A < 0$ which means A is an obtuse angle, applying $\cos A \cos B \cos C > 0$, we have $\cos B \cos C < 0$ which means one of B and C is an obtuse angle. Hence $A+B+C > 180^0$. The conclusion is contradicting to $A+B+C = 180^0$. Therefore $\cos A > 0$. Similarly, we obtain $\cos B > 0, \cos C > 0$. As a conclusion, $\triangle ABC$ is an acute triangle.

Proof 2: Since $\tan A \tan B > 1$, then $\tan A$ and $\tan B$ are the same sign. If $\tan A$ and $\tan B$ are both negative, we obtain A and B are both obtuse angles which is contradictory with the given condition. If $\tan A$ and $\tan B$ are both positive, we obtain A and B are both acute angles. Since $0 > 1 - \tan A \tan B = \dfrac{\tan A + \tan B}{\tan(A+B)} = -\dfrac{\tan A + \tan B}{\tan C}$. On the other hand, since $\tan A + \tan B > 0$, then $\tan C > 0$, hence C is also an acute angle. Consequently, $\triangle ABC$ is an acute triangle.

4.84 ★ Given $|A| < 1, \sin\alpha = A\sin(\alpha+\beta)$. Show $\tan(\alpha+\beta) = \dfrac{\sin\beta}{\cos\beta - A}$.

Solution: $\sin\alpha = A\sin(\alpha+\beta) \Rightarrow \sin[(\alpha+\beta)-\beta] = A\sin(\alpha+\beta) \Rightarrow \sin(\alpha+\beta)\cos\beta - \cos(\alpha+\beta)\sin\beta = A\sin(\alpha+\beta) \Rightarrow \sin(\alpha+\beta)(\cos\beta - A) = \cos(\alpha+\beta)\sin\beta \Rightarrow \tan(\alpha+\beta) = \dfrac{\sin\beta}{\cos\beta - A}$.

4.85 ★★ Given $0 < x < \pi$, find the minimum value of function $f(x) = \sin x + \dfrac{4}{\sin x}$.

Solution: Since $0 < \sin x \leqslant 1$ for $0 < x < \pi$, then the minimum value of $f(x)$ is equal to the minimum value of $f(x)$ for $0 < x \leqslant \dfrac{\pi}{2}$. Assume $0 < x_2 < x_1 \leqslant \dfrac{\pi}{2}$, then $f(x_1) - f(x_2) = (\sin x_1 + \dfrac{4}{\sin x_1}) - (\sin x_2 + \dfrac{4}{\sin x_2}) = \dfrac{-(\sin x_1 - \sin x_2)(4 - \sin x_1 \sin x_2)}{\sin x_1 \sin x_2}$. Since $0 < \sin x_2 < \sin x_1 \leqslant 1, 4 - \sin x_1 \sin x_2 > 0$, we have $f(x_1) - f(x_2) < 0$, that is $f(x_1) < f(x_2)$. Therefore $f(x)$ is decreasing on the interval $(0, \dfrac{\pi}{2}]$. The minimum value of $f(x)$ is 5 at $x = \dfrac{\pi}{2}$. Consequently, the minimum value of $f(x)$ is 5 for $0 < x < \pi$.

4.86 ★★★ If α and β are two acute angles that satisfy the equation $\sin^2\alpha + \sin^2\beta = \sin(\alpha+\beta)$. Show $\alpha + \beta = \dfrac{\pi}{2}$.

Proof: $\sin^2\alpha + \sin^2\beta = \sin(\alpha+\beta) \Rightarrow \sin^2\alpha + \sin^2\beta = \sin\alpha\cos\beta + \cos\alpha\sin\beta \Rightarrow \sin\alpha(\sin\alpha - \cos\beta) = \sin\beta(\cos\alpha - \sin\beta)$. Since $0 < \alpha, \beta < \dfrac{\pi}{2}$, we have $\sin\alpha > 0, \sin\beta > 0$. Hence $\sin\alpha - \cos\beta$ and $\cos\alpha - \sin\beta$ are the same signs, or they are both zero at the same time.

(1) If

$$\begin{cases} \sin\alpha - \cos\beta > 0 \\ \cos\alpha - \sin\beta > 0 \end{cases}$$

then

$$\begin{cases} \sin\alpha > \cos\beta > 0 \\ \cos\alpha > \sin\beta > 0 \end{cases}$$

$\Rightarrow \sin^2\alpha + \cos^2\alpha > \sin^2\beta + \cos^2\beta$, which means $1 > 1$. It does not hold.

(2) If

$$\begin{cases} \sin\alpha - \cos\beta < 0 \\ \cos\alpha - \sin\beta < 0 \end{cases}$$

then

$$\begin{cases} \cos\beta > \sin\alpha > 0 \\ \sin\beta > \cos\alpha > 0 \end{cases}$$

$\Rightarrow \sin^2\beta + \cos^2\beta > \sin^2\alpha + \cos^2\alpha$, which means $1 > 1$. It does not hold. The above two cases are both false. Therefore we have

$$\begin{cases} \cos\beta - \sin\alpha = 0 \cdots ① \\ \sin\beta - \cos\alpha = 0 \cdots ② \end{cases}$$

Checking $①^2 + ②^2$, we obtain $\sin\alpha\cos\beta + \cos\alpha\sin\beta = 1$ which implies that $\sin(\alpha+\beta) = 1$. Since α and β are acute angles, then $\alpha + \beta = \frac{\pi}{2}$.

4.87 ★★★ Given a, b, c in the interval $(0, \frac{\pi}{2})$, and $a = \cos a, b = \sin(\cos b), c = \cos(\sin c)$, compare their values.

Solution: Their order is $b < a < c$.
Otherwise, assume $b \geqslant a$. Since cosine function is decreasing on the interval $(0, \frac{\pi}{2})$, then $0 < \cos b \leqslant \cos a = a < \frac{\pi}{2}$. Applying the relation that is $\sin x < x$ when $x \in (0, \frac{\pi}{2})$, we have $0 < \sin(\cos b) < \cos b \leqslant \cos a = a$ which means $b < a$. It contradicts to the assumption. Therefore $b < a$. Next, assume $c \leqslant a$. Since cosine function is decreasing in the interval $(0, \frac{\pi}{2})$, thus $0 < \sin c < c \leqslant a < \frac{\pi}{2}$, hence $\cos(\sin c) > \cos c \geqslant \cos a = a$ which means $c > a$. It contradicts to the assumption. Therefore $c > a$. As a conclusion, $b < a < c$.

4.88 ★★★ Given the three side lengths a, b, c corresponding to angles A, B, C of an obtuse triangle ABC, $\sin C = \frac{k}{\sqrt{2}}$, $k \in Z$, and equation $x^2 - 2kx + 3k^2 - 7k + 3 = 0$ has real roots. The formula $(c-b)\sin^2 A + b\sin^2 B = c\sin^2 C$ holds. Find the values of A, B, C.

Solution: Since the equation has real roots, then $\Delta = 4k^2 - 4(3k^2 - 7k + 3) \geqslant 0$, that is $2k^2 - 7k + 3 \leqslant 0 \Rightarrow \frac{1}{2} \leqslant k \leqslant 3$. Since k is an integer, then $k = 1$ or 2 or 3. Since $k = \sqrt{2}\sin C$, and $0 < \sin C < 1$ in the obtuse triangle ABC, we have $k = 1$, $\sin C = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$, $\angle C = 45^0$ or $\angle C = 135^0$. Since $(c-b)\sin^2 A + b\sin^2 B = c\sin^2 C$, we apply the sine law $a = 2R\sin A, b = 2R\sin B, c = 2R\sin C$ to obtain that $(c-b)a^2 + b^3 - c^3 = 0$. By solving the equation $(b-c)(b^2 + c^2 - a^2 + bc) = 0$, we have $b = c$ or $b^2 + c^2 - a^2 + bc = 0$. When $b = c$, $B = 45^0$ or $B = 135^0$. $\angle B = \angle C = 45^0$ and $\angle B = \angle C = 135^0$ do not hold, since they conflict with the given condition that $\triangle ABC$ is an obtuse triangle and $\angle A + \angle B + \angle C = 180^0$. When $b^2 + c^2 - a^2 + bc = 0$, we can apply the cosine law to obtain that $\cos A = \frac{b^2 + c^2 - a^2}{2bc} = \frac{-bc}{2bc} = -\frac{1}{2}$. Therefore, $\angle A = 120^0, \angle B = 15^0, \angle C = 45^0$.

4.89 ★★★★ If $a_1, a_2, a_3, \cdots, a_n$ are positive numbers which are all less than 1, show $\sum_{k=1}^{n} \arctan a_k = \frac{n\pi}{4} - \sum_{k=1}^{n} \arctan \frac{1-a_k}{1+a_k}$.

Proof: Let $\alpha = \arctan a_k, \beta = \arctan \frac{1-a_k}{1+a_k}, (k = 1, 2, 3, \cdots, n)$. Since $0 < a_k < 1$, then $0 < \alpha < \frac{\pi}{4}, 0 < \beta < \frac{\pi}{2}$, then $0 < \alpha + \beta < \pi$. Hence $\tan(\alpha+\beta) = \frac{a_k + \frac{1-a_k}{1+a_k}}{1 - \frac{a_k(1-a_k)}{1+a_k}} = \frac{a_k^2+1}{1+a_k^2} = 1 \Rightarrow \alpha + \beta = \frac{\pi}{4}$. Therefore $\arctan a_k + \arctan \frac{1-a_k}{1+a_k} = \frac{\pi}{4}$. Substituting separately $k = 1, 2, 3, \cdots, n$ into the equation and adding these equations, we have $\sum_{k=1}^{n} (\arctan a_k + \arctan \frac{1-a_k}{1+a_k}) = \frac{n\pi}{4}$. It can be shown that $\sum_{k=1}^{n} \arctan a_k = \frac{n\pi}{4} - \sum_{k=1}^{n} \arctan \frac{1-a_k}{1+a_k}$.

4.90 ★★★★ Given complex number $z = \frac{\sqrt{5}}{2} \sin \frac{A+B}{2} + i \cos \frac{A-B}{2}$, where A, B, C are the interior angles of $\triangle ABC$, and $|z| = \frac{3\sqrt{2}}{4}$. (1) Compute $\tan A \tan B$. (2) If $|AB| = 6$, calculate the area of $\triangle ABC$ when $\angle C$ reaches its maximum value.

Solution: (1) By the given condition, we have $|z|^2 = [\frac{\sqrt{5}}{2}\sin\frac{A+B}{2}]^2 + [\cos\frac{A-B}{2}]^2 = [\frac{3\sqrt{2}}{4}]^2 \Rightarrow \frac{5}{4}\frac{1-\cos(A+B)}{2} + \frac{1+\cos(A-B)}{2} = \frac{9}{8} \Rightarrow 4\cos(A-B) = 5\cos(A+B) \Rightarrow 9\sin A\sin B = \cos A\cos B$. Hence $\tan A\tan B = \frac{1}{9}$.

(2) $\tan C = -\tan(A+B) = -\frac{\tan A+\tan B}{1-\tan A\tan B} = -\frac{\tan A+\tan B}{1-\frac{1}{9}} = -\frac{9}{8}(\tan A + \tan B) \leqslant -\frac{9}{4}\sqrt{\tan A\tan B} = -\frac{3}{4}$, and $\tan C$ gets the maximum value if and only if $\tan A = \tan B = \frac{1}{3}$. It means $\triangle ABC$ is an isosceles triangle when $\angle C$ reaches its maximum value. The value of altitude h on the side AB is $h = \frac{|AB|}{2}\tan A = 1$. Therefore $S_{\triangle ABC} = \frac{1}{2}|AB|h = 3$.

4.91 ★★★ Given a, b, c are three side lengths of $\triangle ABC$, $a+b = 10$, $(a+b+c)(a+b-c) = 3ab$, compute the maximal area and the minimal perimeter of $\triangle ABC$.

Solution: From the given conditions and the cosine theorem, we have

$$\begin{cases} (a+b+c)(a+b-c) = 3ab \\ c^2 = a^2+b^2-2ab\cos C \end{cases}$$

$\Rightarrow$

$$\begin{cases} (a+b)^2 - c^2 = 3ab \\ a^2+b^2-c^2 = 2ab\cos C \end{cases}$$

$\Rightarrow$

$$\begin{cases} a^2+b^2-c^2 = ab \\ a^2+b^2-c^2 = 2ab\cos C \end{cases}$$

Thus $\cos C = \frac{1}{2} \Rightarrow C = \frac{\pi}{3}$. Let the area is S. Since $b = 10-a$, we have $S = \frac{1}{2}ab\sin C = \frac{1}{2}a(10-a)\frac{\sqrt{3}}{2} = -\frac{\sqrt{3}}{4}(a-5)^2 + \frac{25\sqrt{3}}{4}$. $S_{\max} = \frac{25\sqrt{3}}{4}$ when $a = b = 5$. Let the perimeter of $\triangle ABC$ is p, then $p = a+b+c = 10+\sqrt{a^2+b^2-2ab\cos C} = 10+\sqrt{a^2+(10-a)^2-2a(10-a)\frac{1}{2}} = 10+\sqrt{3(a-5)^2+25}$. When $a = b = 5$, $p_{min} = 15$.

4.92 ★★★★ Show $\frac{1}{\sin 2\alpha}+\frac{1}{\sin 4\alpha}+\cdots+\frac{1}{\sin 2^n\alpha}=\cot\alpha-\cot 2^n\alpha$.

Proof: $\frac{1}{\sin 2\alpha}=\frac{\sin\alpha}{\sin\alpha\sin 2\alpha}=\frac{\sin(2\alpha-\alpha)}{\sin\alpha\sin 2\alpha}=\frac{\sin 2\alpha\cos\alpha-\cos 2\alpha\sin\alpha}{\sin\alpha\sin 2\alpha}=\cot\alpha-\cot 2\alpha$. $\frac{1}{\sin 4\alpha}=\frac{\sin 2\alpha}{\sin 2\alpha\sin 4\alpha}=\frac{\sin(4\alpha-2\alpha)}{\sin 2\alpha\sin 4\alpha}=\frac{\sin 4\alpha\cos 2\alpha-\cos 4\alpha\sin 2\alpha}{\sin 2\alpha\sin 4\alpha}=\cot 2\alpha-\cot 2^2\alpha$. Applying the recurrence relation, we have $\frac{1}{\sin 8\alpha}=\cot 2^2\alpha-\cot 2^3\alpha,\cdots,\frac{1}{\sin 2^n\alpha}=\cot 2^{n-1}\alpha-\cot 2^n\alpha$. Adding all above equations , we obtain $\frac{1}{\sin 2\alpha}+\frac{1}{\sin 4\alpha}+\cdots+\frac{1}{\sin 2^n\alpha}=\cot\alpha-\cot 2^n\alpha$.

4.93 ★★★ Let $y\cos\alpha-x\sin\alpha=a\cos 2\alpha, y\sin\alpha+x\cos\alpha=2a\sin 2\alpha$, show $(x+y)^{\frac{2}{3}}+(x-y)^{\frac{2}{3}}=2a^{\frac{2}{3}}$.

Proof: From the two given conditions, we obtain $x=\frac{\begin{vmatrix}\cos\alpha & a\cos 2\alpha\\ \sin\alpha & 2a\sin 2\alpha\end{vmatrix}}{\begin{vmatrix}\cos\alpha & -\sin\alpha\\ \sin\alpha & \cos\alpha\end{vmatrix}}=2a\sin 2\alpha\cos\alpha-a\sin\alpha\cos 2\alpha=2a2\sin\alpha\cos^2\alpha-a\sin\alpha(\cos^2\alpha-\sin^2\alpha)=a(3\sin\alpha\cos^2\alpha+\sin^3\alpha)$.
$y=\frac{\begin{vmatrix}a\cos 2\alpha & -\sin\alpha\\ 2a\sin 2\alpha & \cos\alpha\end{vmatrix}}{\begin{vmatrix}\cos\alpha & -\sin\alpha\\ \sin\alpha & \cos\alpha\end{vmatrix}}=a\cos\alpha\cos 2\alpha+2a\sin 2\alpha\sin\alpha=a[\cos\alpha(\cos^2\alpha-\sin^2\alpha)+4\sin^2\alpha\cos\alpha]=a(\cos^3\alpha-\sin^2\alpha\cos\alpha+4\sin^2\alpha\cos\alpha)=a(\cos^3\alpha+3\sin^2\alpha\cos\alpha)$. Thus $x+y=a(\sin^3\alpha+3\sin^2\alpha\cos\alpha+3\sin\alpha\cos^2\alpha+\cos^3\alpha)=a(\sin\alpha+\cos\alpha)^3$. $x-y=a(\sin^3\alpha-3\sin^2\alpha\cos\alpha+3\sin\alpha\cos^2\alpha-\cos^3\alpha)=a(\sin\alpha-\cos\alpha)^3$. Hence $(x+y)^{\frac{2}{3}}+(x-y)^{\frac{2}{3}}=a^{\frac{2}{3}}[(\sin\alpha+\cos\alpha)^2+(\sin\alpha-\cos\alpha)^2]=a^{\frac{2}{3}}[2(\sin^2\alpha+\cos^2\alpha)]=2a^{\frac{2}{3}}$.

4.94 ★★★★★ Let the incircle radius of triangle ABC is r, the circumcircle radius of triangle ABC is R, show $r\leqslant\frac{1}{2}R$.

Proof: Let a,b,c are the side lengths of $\triangle ABC$, $p=\frac{1}{2}(a+b+c)$, the area of $\triangle ABC$ is S. Then $\frac{r}{R}=\frac{S}{p}\div\frac{abc}{4S}=\frac{4S^2}{pabc}=\frac{4p(p-a)(p-b)(p-c)}{pabc}=4\sqrt{\frac{(p-a)(p-b)}{ab}}\sqrt{\frac{(p-b)(p-c)}{bc}}\sqrt{\frac{(p-c)(p-a)}{ca}}=4\sqrt{\frac{c^2-a^2+2ab-b^2}{4ab}}\sqrt{\frac{a^2-b^2+2bc-c^2}{4bc}}\sqrt{\frac{b^2-a^2+2ac-c^2}{4ca}}=4\sqrt{\frac{2ab(1-\cos C)}{4ab}}\sqrt{\frac{2bc(1-\cos A)}{4bc}}\sqrt{\frac{2ac(1-\cos B)}{4ca}}=4\sqrt{\frac{(1-\cos C)}{2}}\sqrt{\frac{(1-\cos A)}{2}}\sqrt{\frac{(1-\cos B)}{2}}=4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$.

And $\sin\frac{A}{2}\sin\frac{B}{2} = \sin[(\frac{A}{4}+\frac{B}{4})+(\frac{A}{4}-\frac{B}{4})]\sin[(\frac{A}{4}+\frac{B}{4})-(\frac{A}{4}-\frac{B}{4})] = [\sin(\frac{A}{4}+\frac{B}{4})\cos(\frac{A}{4}-\frac{B}{4})+\cos(\frac{A}{4}+\frac{B}{4})\sin(\frac{A}{4}-\frac{B}{4})][\sin(\frac{A}{4}+\frac{B}{4})\cos(\frac{A}{4}-\frac{B}{4})-\cos(\frac{A}{4}+\frac{B}{4})\sin(\frac{A}{4}-\frac{B}{4})] = \sin^2(\frac{A}{4}+\frac{B}{4})\cos^2(\frac{A}{4}-\frac{B}{4})-\cos^2(\frac{A}{4}+\frac{B}{4})\sin^2(\frac{A}{4}-\frac{B}{4}) = \sin^2(\frac{A}{4}+\frac{B}{4})\cos^2(\frac{A}{4}-\frac{B}{4})-[1-\sin^2(\frac{A}{4}+\frac{B}{4})]\sin^2(\frac{A}{4}-\frac{B}{4}) = \sin^2(\frac{A}{4}+\frac{B}{4})[\cos^2(\frac{A}{4}-\frac{B}{4})+\sin^2(\frac{A}{4}-\frac{B}{4})]-\sin^2(\frac{A}{4}-\frac{B}{4}) = \sin^2(\frac{A}{4}+\frac{B}{4})-\sin^2(\frac{A}{4}-\frac{B}{4})$. $\sin\frac{A}{2}\sin\frac{B}{2}$ reaches its maximum value when $A=B$. Following the same logic, we can prove that $\sin\frac{A}{2}\sin\frac{C}{2} = \sin^2(\frac{A}{4}+\frac{C}{4})-\sin^2(\frac{A}{4}-\frac{C}{4})$. $\sin\frac{A}{2}\sin\frac{C}{2}$ reaches its maximum value when $A=C$. $\sin\frac{B}{2}\sin\frac{C}{2} = \sin^2(\frac{B}{4}+\frac{C}{4})-\sin^2(\frac{B}{4}-\frac{C}{4})$. $\sin\frac{B}{2}\sin\frac{C}{2}$ reaches its maximum value when $B=C$. Hence $\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$ reaches its maximum value when $A=B=C=\frac{\pi}{3}$. Therefore $\frac{r}{R} = 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} \leqslant 4\sin^3\frac{\pi}{6} = \frac{1}{2}$. As a summary, $r \leqslant \frac{1}{2}R$.

4.95 ★★★★★ Solve the equation $\cos^2\theta-\cos^2\phi = 2\cos^3\theta(\cos\theta-\cos\phi)-2\sin^3\theta(\sin\theta-\sin\phi)$.

Solution: $\cos^2\theta-\cos^2\phi = 2\cos^3\theta(\cos\theta-\cos\phi)-2\sin^3\theta(\sin\theta-\sin\phi) \Rightarrow \cos^2\theta-\cos^2\phi = \frac{\cos 3\theta+3\cos\theta}{2}(\cos\theta-\cos\phi)-\frac{3\sin\theta-\sin 3\theta}{2}(\sin\theta-\sin\phi) \Rightarrow 2(\cos^2\theta-\cos^2\phi) = \cos 3\theta\cos\theta+\sin 3\theta\sin\theta-\cos 3\theta\cos\phi-\sin 3\theta\sin\phi+3\cos^2\theta-3\sin^2\theta-3\cos\theta\cos\phi+3\sin\theta\sin\phi \Rightarrow \cos 2\theta-\cos(3\theta-\phi)-3\cos(\theta+\phi) = 3\sin^2\theta-\cos^2\theta-2\cos^2\phi \Rightarrow \cos 2\theta-\cos(3\theta-\phi)-3\cos(\theta+\phi) = 3-4\cos^2\theta-2\cos^2\phi = 3-2(1+\cos 2\theta)-(1+\cos 2\phi) = -2\cos 2\theta-\cos 2\phi \Rightarrow 3\cos 2\theta-3\cos(\theta+\phi)-\cos(3\theta-\phi)+\cos 2\phi = 0 \Rightarrow -6\sin\frac{3\theta+\phi}{2}\sin\frac{\theta-\phi}{2}+2\sin\frac{3\theta+\phi}{2}\sin\frac{3(\theta-\phi)}{2} = 0 \Rightarrow \sin\frac{3\theta+\phi}{2}(\sin\frac{3(\theta-\phi)}{2}-3\sin\frac{\theta-\phi}{2}) = 0 \Rightarrow \sin\frac{3\theta+\phi}{2}(-4\sin^3\frac{\theta-\phi}{2}) = 0$. Thus $\sin\frac{3\theta+\phi}{2} = 0$ or $\sin\frac{\theta-\phi}{2} = 0$. Therefore, we have $\frac{3\theta+\phi}{2} = n\pi \quad (n\in N)$ or $\frac{\theta-\phi}{2} = n\pi \quad (n\in N)$. As a conclusion, $\theta = n\pi, \phi = -n\pi \quad (n\in N)$.

4.96 ★★★★★ The interior angles A, B, C satisfy $\sin A\cos B-\sin B = \sin C-\sin A\cos C$. If the perimeter of $\triangle ABC$ is 12, find the maximal area.

Solution: The given equation can be written as $\sin A(\cos B+\cos C) = \sin B+\sin C$ where $\cos B+\cos C \neq 0$. Otherwise, if $\cos B+\cos C = 0$, we have $\cos B = \cos(\pi-C)$. Since $0<B<\pi, 0<\pi-C<\pi$, then $B = \pi-C$, that is $B+C = \pi$. It contradicts to the condition $A+B+C = \pi$. Hence $\cos B+\cos C \neq 0$. Therefore $\sin A = \frac{\sin B+\sin C}{\cos B+\cos C} = \frac{2\sin\frac{B+C}{2}\cos\frac{B-C}{2}}{2\cos\frac{B+C}{2}\cos\frac{B-C}{2}} = \tan\frac{B+C}{2}$. Since $\frac{B+C}{2} = \frac{\pi}{2}-\frac{A}{2}$, then $\tan\frac{B+C}{2} = \tan(\frac{\pi}{2}-\frac{A}{2}) = \cot\frac{A}{2}$. Then $\sin A = \cot\frac{A}{2} \Rightarrow 2\sin\frac{A}{2}\cos\frac{A}{2} = \frac{\cos\frac{A}{2}}{\sin\frac{A}{2}}$.

Since $0 < \frac{A}{2} < \frac{\pi}{2}$, then $\cos\frac{A}{2} \neq 0$. Hence $\sin^2\frac{A}{2} = \frac{1}{2} \Rightarrow \sin\frac{A}{2} = \frac{\sqrt{2}}{2}$. Therefore $A = \frac{\pi}{2}$. After all, $\triangle ABC$ is a right triangle. Let a, b, c are the side lengths corresponding to the angles A, B, C, and a is the hypotenuse. We have $b + c + \sqrt{b^2 + c^2} = 12$. Since $b > 0, c > 0$, then $b + c \geqslant 2\sqrt{bc}$, $\sqrt{b^2+c^2} \geqslant \sqrt{2bc}$. Hence $2\sqrt{bc} + \sqrt{2}\sqrt{bc} \leqslant 12$. That is $\sqrt{bc} \leqslant \frac{12}{2+\sqrt{2}} = 6(2-\sqrt{2})$. $S_{\triangle ABC} = \frac{1}{2}bc \leqslant \frac{1}{2}36(2-\sqrt{2})^2 = 36(3-2\sqrt{2})$. When $b = c$, we have the the maximal area $(S_{\triangle ABC})_{max} = 36(3-2\sqrt{2})$.

4.97 ★★★★★ Given $a\sin x + b\cos x = 0$, $A\sin 2x + B\cos 2x = C$ where a and b are not zero at the same time, and $0 \leqslant x \leqslant 180^0$, show $2abA + (b^2 - a^2)B + (a^2 + b^2)C = 0$.

Solution: (1) When $a = 0, b \neq 0$, we have $b\cos x = 0$, then $\cos x = 0, 0 \leqslant x \leqslant 180^0$. Hence $x = 90^0$. That is $A\sin 180^0 + B\cos 180^0 = C$. Solving the equation, we have $-B = C$. That is $2abA + (b^2 - a^2)B + (a^2 + b^2)C = b^2(B + C) = 0$.
(2) When $b = 0, a \neq 0$, we have $a\sin x = 0$, then $\sin x = 0, 0 \leqslant x \leqslant 180^0$. Hence $x = 0^0$ or $x = 180^0$. That is $A\sin 0 + B\cos 0 = C$ or $A\sin 360^0 + B\cos 360^0 = C$. Solving the equation, we have $B = C$. That is $2abA + (b^2 - a^2)B + (a^2 + b^2)C = a^2(C - B) = 0$.
(3) When $a \neq 0, b \neq 0$, the equation system is

$$\begin{cases} a\sin x + b\cos x = 0, & ① \\ A\sin 2x + B\cos 2x = C. & ② \end{cases}$$

Since $\cos x \neq 0$, equation ① results in $\tan x = -\frac{b}{a}$. We have $\sin 2x = 2\sin x \cos x = 2\tan x \cos^2 x = \frac{2\tan x}{1+\tan^2 x} = \frac{2(-\frac{b}{a})}{1+(-\frac{b}{a})^2} = -\frac{2ab}{a^2+b^2}$ ③ and $\cos 2x = 2\cos^2 x - 1 == \frac{2}{1+\tan^2 x} - 1 = \frac{2}{1+(-\frac{b}{a})^2} - 1 = \frac{a^2-b^2}{a^2+b^2}$ ④. Substituting ③ and ④ into ②, we have $A(-\frac{2ab}{a^2+b^2}) + B(\frac{a^2-b^2}{a^2+b^2}) = C \Rightarrow -\frac{2abA}{a^2+b^2} + \frac{(a^2-b^2)B}{a^2+b^2} = C \Rightarrow -2abA + (a^2-b^2)B = (a^2+b^2)C \Rightarrow 2abA + (b^2-a^2)B + (a^2+b^2)C = 0$.

4.98 ★★★★ Let a, b, c be the side lengths of $\triangle ABC$ corresponding to the angles A, B, C, show $(\cot\frac{A}{4} - \csc\frac{A}{2}) : (\cot\frac{B}{2} + \cot\frac{C}{2}) = (b+c-a) : 2a$.

Proof: We have $\frac{b+c-a}{2a} = \frac{\sin B + \sin C - \sin A}{2\sin A} = \frac{2\sin\frac{B+C}{2}\cos\frac{B-C}{2} - 2\sin\frac{A}{2}\cos\frac{A}{2}}{4\sin\frac{A}{2}\cos\frac{A}{2}} = \frac{\cos\frac{B-C}{2} - \cos\frac{B+C}{2}}{2\sin\frac{A}{2}} = \frac{\sin\frac{B}{2}\sin\frac{C}{2}}{\sin\frac{A}{2}}$ ①. Since $\cot\frac{A}{4} - \csc\frac{A}{2} = \frac{\cos\frac{A}{4}}{\sin\frac{A}{4}} - \frac{1}{\sin\frac{A}{2}} = \frac{\cos\frac{A}{4}}{\sin\frac{A}{4}} - \frac{1}{2\sin\frac{A}{4}\cos\frac{A}{4}} = \frac{2\cos^2\frac{A}{4} - 1}{2\sin\frac{A}{4}\cos\frac{A}{4}} = \frac{\cos\frac{A}{2}}{\sin\frac{A}{2}}$ and $\cot\frac{B}{2} + \cot\frac{C}{2} = \frac{\cos\frac{B}{2}}{\sin\frac{B}{2}} + \frac{\cos\frac{C}{2}}{\sin\frac{C}{2}} = \frac{\sin\frac{B+C}{2}}{\sin\frac{B}{2}\sin\frac{C}{2}} = \frac{\cos\frac{A}{2}}{\sin\frac{B}{2}\sin\frac{C}{2}}$. Then $\frac{\cot\frac{A}{4} - \csc\frac{A}{2}}{\cot\frac{B}{2} + \cot\frac{C}{2}} = \frac{\cos\frac{A}{2}}{\sin\frac{A}{2}}\frac{\sin\frac{B}{2}\sin\frac{C}{2}}{\cos\frac{A}{2}} = \frac{\sin\frac{B}{2}\sin\frac{C}{2}}{\sin\frac{A}{2}}$ ②. From ① and ②, we have that $(\cot\frac{A}{4} - \csc\frac{A}{2}) : (\cot\frac{B}{2} + \cot\frac{C}{2}) = (b+c-a) : 2a$.

4.99 ★★★★★ If the equation $a\sin x + b\cos x + c = 0$ has two distinct solutions α, β which are in the interval $[0, 2\pi]$, show that $\tan\frac{\alpha+\beta}{2} = \frac{a}{b}$ when $b \neq 0$.

Proof: The equation implies that $a\sin x + b\cos x = -c$. Then $\frac{a}{\sqrt{a^2+b^2}}\sin x + \frac{b}{\sqrt{a^2+b^2}}\cos x = -\frac{c}{\sqrt{a^2+b^2}}$. Let $\cos\varphi = \frac{a}{\sqrt{a^2+b^2}}, \sin\varphi = \frac{b}{\sqrt{a^2+b^2}}$. Then $\sin(x+\varphi) = -\frac{c}{\sqrt{a^2+b^2}}$. Since α, β are two distinct solutions which are between 0 and 2π, then $\sin(\alpha+\varphi) = -\frac{c}{\sqrt{a^2+b^2}}$ ①, $\sin(\beta+\varphi) = -\frac{c}{\sqrt{a^2+b^2}}$ ②. Checking ① − ②, we have $\sin(\alpha+\varphi) - \sin(\beta+\varphi) = 0$. That is $2\cos(\frac{\alpha+\beta}{2}+\varphi)\sin\frac{\alpha-\beta}{2} = 0$. Since $\alpha \neq \beta$ and $\alpha, \beta \in [0, \pi]$, then $\frac{\alpha-\beta}{2} \neq 0$. Hence $\cos(\frac{\alpha+\beta}{2}+\varphi) = 0$. That is $\cos\frac{\alpha+\beta}{2}\cos\varphi - \sin\frac{\alpha+\beta}{2}\sin\varphi = 0 \Rightarrow \cos\frac{\alpha+\beta}{2}\frac{a}{\sqrt{a^2+b^2}} - \sin\frac{\alpha+\beta}{2}\frac{b}{\sqrt{a^2+b^2}} = 0$. Since $b \neq 0$, then $\cos\frac{\alpha+\beta}{2} \neq 0$. Dividing the equation by $\cos\frac{\alpha+\beta}{2}$ and multiplying it by $\sqrt{a^2+b^2}$, we have $a - b\tan\frac{\alpha+\beta}{2} = 0$. Therefore, $\tan\frac{\alpha+\beta}{2} = \frac{a}{b}$.

4.100 ★★★★★ Show $\frac{a^3}{2}\csc^2(\frac{1}{2}\tan^{-1}\frac{a}{b})+\frac{b^3}{2}\sec^2(\frac{1}{2}\tan^{-1}\frac{b}{a})=(a+b)(a^2+b^2)$.

Proof: Let $\tan^{-1}\frac{a}{b}=\theta$, then $\tan\theta=\frac{a}{b}$, $\csc^2\frac{\theta}{2}=\frac{1}{\sin^2\frac{\theta}{2}}=\frac{2}{1-\cos\theta}=\frac{2}{1-\frac{b}{\sqrt{a^2+b^2}}}=\frac{2\sqrt{a^2+b^2}}{\sqrt{a^2+b^2}-b}=\frac{2\sqrt{a^2+b^2}}{\sqrt{a^2+b^2}-b}\frac{\sqrt{a^2+b^2}+b}{\sqrt{a^2+b^2}+b}=\frac{2(a^2+b^2)+2b\sqrt{a^2+b^2}}{a^2}$. We have the equation $\frac{a^3}{2}\csc^2(\frac{1}{2}\tan^{-1}\frac{a}{b})=\frac{a^3}{2}\csc^2\frac{\theta}{2}=a(a^2+b^2)+ab\sqrt{a^2+b^2}$.

Let $\tan^{-1}\frac{b}{a}=\phi$, then $\tan\phi=\frac{b}{a}$, $\sec^2\frac{\phi}{2}=\frac{1}{\cos^2\frac{\phi}{2}}=\frac{2}{1+\cos\phi}=\frac{2}{1+\frac{a}{\sqrt{a^2+b^2}}}=\frac{2\sqrt{a^2+b^2}}{\sqrt{a^2+b^2}+a}=\frac{2\sqrt{a^2+b^2}}{\sqrt{a^2+b^2}+a}\frac{\sqrt{a^2+b^2}-a}{\sqrt{a^2+b^2}-a}=\frac{2(a^2+b^2)-2a\sqrt{a^2+b^2}}{b^2}$. We have the equation $\frac{b^3}{2}\sec^2(\frac{1}{2}\tan^{-1}\frac{b}{a})=\frac{b^3}{2}\sec^2\frac{\phi}{2}=b(a^2+b^2)-ab\sqrt{a^2+b^2}$.

Consequently, $\frac{a^3}{2}\csc^2(\frac{1}{2}\tan^{-1}\frac{a}{b})+\frac{b^3}{2}\sec^2(\frac{1}{2}\tan^{-1}\frac{b}{a})=a(a^2+b^2)+ab\sqrt{a^2+b^2}+b(a^2+b^2)-ab\sqrt{a^2+b^2}=(a+b)(a^2+b^2)$.

4.101 ★★★★★ The side lengths a,b,c of $\triangle ABC$ form a harmonic series, show that $\cos\frac{B}{2}=\sqrt{\frac{\sin C\sin A}{\cos C+\cos A}}$.

Proof: From the given condition that side lengths a, b, c form a harmonic series, then $\frac{1}{a}+\frac{1}{c}=\frac{2}{b}$. Hence $b=\frac{2ac}{a+c}$. Let the area of $\triangle ABC$ is S, and the half of its perimeter is p. We have $\cos\frac{B}{2}=\sqrt{\frac{1+\cos B}{2}}=\sqrt{\frac{a^2+c^2-b^2+2ac}{4ac}}=\sqrt{\frac{(a+c+b)(a+c-b)}{4ac}}=\sqrt{\frac{p(p-b)}{ac}}=\sqrt{\frac{S^2}{ac(p-a)(p-c)}}=\sqrt{\frac{bc\sin Aab\sin C}{4ac(p-a)(p-c)}}=\sqrt{\frac{2ab^2c\sin A\sin C}{b(a+c)[b^2-(c-a)^2]}}=\sqrt{\frac{2abc\sin A\sin C}{ab^2+b^2c+ac^2+a^2c-c^3-a^3}}=\sqrt{\frac{2abc\sin A\sin C}{a(b^2+c^2-a^2)+c(a^2+b^2-c^2)}}=\sqrt{\frac{\sin A\sin C}{\frac{b^2+c^2-a^2}{2bc}+\frac{a^2+b^2-c^2}{2ab}}}=\sqrt{\frac{\sin A\sin C}{\cos A+\cos C}}$.

4.102 ★★★★★ If $y=\sin^{10}x+10\sin^2x\cos^2x+\cos^{10}x, -\frac{\pi}{2}<x<\frac{\pi}{2}$. Find the maximum value and minimum value of y.

Solution1: Applying the double angle formula and half angle formula, we obtain $y=\sin^{10}x+10\sin^2x\cos^2x+\cos^{10}x=(\sin^2x)^5+\frac{5}{2}(2\sin x\cos x)^2+(\cos^2x)^5=(\frac{1-\cos 2x}{2})^5+\frac{5}{2}\sin 2x+(\frac{1+\cos 2x}{2})^5=\frac{1}{32}[(1-\cos 2x)^5+(1+\cos 2x)^5]+\frac{5}{2}\sin^2 2x=\frac{2+20\cos^2 2x+10\cos^4 2x}{32}+\frac{5}{2}(1-\cos^2 2x)=\frac{1+10\cos^2 2x+5\cos^4 2x+40(1-\cos^2 2x)}{16}=\frac{5\cos^4 2x-30\cos^2 2x+41}{16}=\frac{5}{16}(\cos^4 2x-6\cos^2 2x+\frac{41}{5})=\frac{5}{16}(\cos^2 2x-3)^2-\frac{1}{4}$.

When $\cos^2 2x=0$ (i.e. $x=\pm\frac{\pi}{4}$), y has the maximum value $y_{max}=\frac{5}{16}\times 9-\frac{1}{4}=2\frac{9}{16}$.

When $\cos^2 2x=1$ (i.e. $x=0$), y has the minimum value $y_{min}=\frac{5}{16}\times 4-\frac{1}{4}=1$.

Solution2: We can check the derivative of y, $\frac{dy}{dx}=10\sin^9x\cos x+10(2\sin x\cos^3x-2\sin^3x\cos x)-10\cos^9x\sin x=10\sin x\cos x[\sin^8x+2(\cos^2x-\sin^2x)-\cos^8x]=5\sin 2x[(\sin^4x+\cos^4x)(\sin^4x-\cos^4x)+2\cos 2x]=5\sin 2x[(1-\frac{1}{2}\sin^2 2x)(\sin^2x-\cos^2x)+2\cos 2x]=5\sin 2x[(1-\frac{1}{2}\sin^2 2x)(-\cos 2x)+2\cos 2x]=5\sin 2x\cos 2x(1+\frac{1}{2}\sin^2 2x)$. Let $\frac{dy}{dx}=0$, we obtain the stationary point:

(1) $x_1=0$ when $\sin 2x=0$. We find that the sign of $\frac{dy}{dx}$ changes from negative to positive, then $y_{min}(0)=1$ is a local minimum.

(2) $x_{2,3}=\pm\frac{\pi}{4}$ when $\cos 2x=0$. We find that the sign of $\frac{dy}{dx}$ changes from positive to negative, then $y_{max}(\pm\frac{\pi}{4})=2\frac{9}{16}$ are local maxima.

(3)If $1+\frac{1}{2}\sin^2 2x=0$, then $\sin^2 2x=-2$. The equation has no solution.

4.103 ★★★★★ If n is an arbitrary natural number, show that $\sin\alpha + \sin 2\alpha + \sin 3\alpha + \cdots + \sin n\alpha = \dfrac{\sin\frac{n\alpha}{2}\sin\frac{(n+1)\alpha}{2}}{\sin\frac{\alpha}{2}}$.

Proof: prove the conclusion by the method of mathematical induction.
(1) When $n = 1$, the left side of the equation is $\sin\alpha$, the right side of the equation is $\dfrac{\sin\frac{\alpha}{2}\sin\alpha}{\sin\frac{\alpha}{2}}$ which equals $\sin\alpha$. The left side equals the right side. The equation holds.
(2) Assume the equation holds when $n = k$. Then $\sin\alpha+\sin 2\alpha+\sin 3\alpha+\cdots+\sin k\alpha = \dfrac{\sin\frac{k\alpha}{2}\sin\frac{(k+1)\alpha}{2}}{\sin\frac{\alpha}{2}}$. We add $\sin(k+1)\alpha$ to both sides, then $\sin\alpha + \sin 2\alpha + \cdots + \sin k\alpha + \sin(k+1)\alpha = \dfrac{\sin\frac{k\alpha}{2}\sin\frac{(k+1)\alpha}{2} + \sin\frac{\alpha}{2}\sin(k+1)\alpha}{\sin\frac{\alpha}{2}} = \dfrac{\sin\frac{(k+1)\alpha}{2}[\sin\frac{k\alpha}{2} + 2\sin\frac{\alpha}{2}\cos\frac{(k+1)\alpha}{2}]}{\sin\frac{\alpha}{2}} = \dfrac{\sin\frac{(k+1)\alpha}{2}[\sin\frac{k\alpha}{2} + \sin\frac{(k+2)\alpha}{2} - \sin\frac{k\alpha}{2}]}{\sin\frac{\alpha}{2}} = \dfrac{\sin\frac{(k+1)\alpha}{2}\sin\frac{[(k+1)+1]\alpha}{2}}{\sin\frac{\alpha}{2}}$. Hence the equation holds when $n = k + 1$.
According to (1) and (2), the equation holds for all natural numbers n.

4.104 ★★★★★ If the maximum value of $F(x) = |\cos^2 x + 2\sin x\cos x - \sin^2 x + Ax + B|$, denoted M, is considered with parameters A and B for $0 \leqslant x \leqslant \dfrac{3}{2}\pi$, find the values of A and B such that M has the minimum value.

Solution: $F(x) = |\cos 2x + \sin 2x + Ax + B| = |\sqrt{2}\sin(2x + \dfrac{\pi}{4}) + Ax + B|$.
Let $f_1(x) = \sqrt{2}\sin(2x + \dfrac{\pi}{4}), (0 \leqslant x \leqslant \dfrac{3}{2}\pi)$. Then $f_1(x)$ has the maximum value $\sqrt{2}$ at $x = \dfrac{\pi}{8}$ or $x = \dfrac{9\pi}{8}$. $f_1(x)$ has the minimum value $-\sqrt{2}$ at $x = \dfrac{5\pi}{8}$.
Let $f_2(x) = Ax + B$ which is a monotone function. If A and B are not both zero, then the sign of $f_2(x)$ can not change twice. Hence when $x = \dfrac{\pi}{8}$, $x = \dfrac{5\pi}{8}$, or $x = \dfrac{9\pi}{8}$, there is at least point of $f_2(x)$ which sign is same as the sign of $f_1(x)$, and their sum is large than $\sqrt{2}$. Otherwise, for a pair of A and B with at least one of them nonzero, the maximum value of $F(x)$ is less than $\sqrt{2}$. Then

$$\begin{cases} F(\frac{\pi}{8}) = |\sqrt{2} + \frac{\pi A}{8} + B| < \sqrt{2} \\ F(\frac{5\pi}{8}) = |-\sqrt{2} + \frac{5\pi A}{8} + B| < \sqrt{2} \\ F(\frac{9\pi}{8}) = |\sqrt{2} + \frac{9\pi A}{8} + B| < \sqrt{2} \end{cases}$$

$\Rightarrow$

$$\begin{cases} \frac{\pi A}{8} + B < 0, & ① \\ \frac{5\pi A}{8} + B > 0, & ② \\ \frac{9\pi A}{8} + B < 0, & ③ \end{cases}$$

According to ① and ②, we have $A > 0$. According to ② and ③, we have $A < 0$. It is a contradiction. Therefore, M has the minimum value $\sqrt{2}$ occurring at $A = B = 0$.

5 SEQUENCES

5.1 Given $a_{n+1} = \dfrac{2a_n}{a_n + 2}$, $a_1 = 2$. (1) Show sequence $\frac{1}{a_n}$ is an arithmetic sequence. (2) Find the explicit formula for a_n.

Solution: (1) $a_{n+1} = \dfrac{2a_n}{a_n+2} \Rightarrow \dfrac{2}{a_{n+1}} = \dfrac{a_n+2}{a_n} = 1 + \dfrac{2}{a_n} \Rightarrow \dfrac{1}{a_{n+1}} - \dfrac{1}{a_n} = \dfrac{1}{2}$. Hence sequence $\{\dfrac{1}{a_n}\}$ is an arithmetic sequence and its common difference is $\dfrac{1}{2}$.

(2)From the conclusion of (1), we have that $\dfrac{1}{a_2} - \dfrac{1}{a_1} = \dfrac{1}{2}, \dfrac{1}{a_3} - \dfrac{1}{a_2} = \dfrac{1}{2}, \cdots, \dfrac{1}{a_n} - \dfrac{1}{a_{n-1}} = \dfrac{1}{2}$. We sum up these equations to obtain $\dfrac{1}{a_n} - \dfrac{1}{a_1} = \dfrac{1}{2}(n-1)$. Since $a_1 = 2$, then $a_n = \dfrac{2}{n} \quad (n \in N^*)$.

5.2 Let a_n be the number of the integer roots of the equation $f(x) = x^2 + x + \dfrac{1}{2}, x \in [n, n+1]$, where $n \in N^*$. (1) Find the general term of $\{a_n\}$. (2) Let $b_n = \dfrac{1}{a_n a_{n+1}}$, compute the nth partial sum of $\{b_n\}$ which is denoted by S.

Solution: (1) Since $f(x)$ is increasing on $[n, n+1]$, then the range is $[f(n), f(n+1)]$. Hence $a_n = f(n+1) - f(n) = [(n+1)^2 + (n+1) + \frac{1}{2}] - (n^2 + n + \frac{1}{2}) = 2n+2, (n \in N^*)$.
(2) $b_n = \frac{1}{a_n a_{n+1}} = \frac{1}{(2n+2)[2(n+1)+2]} = \frac{1}{4}\frac{1}{(n+1)(n+2)} = \frac{1}{4}(\frac{1}{n+1} - \frac{1}{n+2})$.
Hence, $S = \frac{1}{4}[(\frac{1}{2} - \frac{1}{3}) + (\frac{1}{3} - \frac{1}{4}) + \cdots + (\frac{1}{n+1} - \frac{1}{n+2})] = \frac{1}{4}(\frac{1}{2} - \frac{1}{n+2}) = \frac{n}{8n+16} \quad (n \in N^*)$.

5.3 Let sequence $\{a_n\}$ satisfy $a_1 = a_2 = 1$ and $a_n + a_{n-1} + a_{n-2} = n^2$ for $n \geqslant 3$, compute a_{1996}.

Solution: From the given condition, we have $a_1 = a_2 = 1$ and $a_1 + a_2 + a_3 = 9$. Hence $a_3 = 7$. Since $a_n + a_{n-1} + a_{n-2} = n^2$ and $a_{n-1} + a_{n-2} + a_{n-3} = (n-1)^2$, adding them to obtain $a_n - a_{n-3} = 2n - 1$, which implies that that $a_{n-3} - a_{n-6} = 2(n-3) - 1, \cdots$. Thus $a_{1996} - a_{1993} = 2 \times 1996 - 1$, $a_{1993} - a_{1990} = 2 \times 1993 - 1, \cdots, a_4 - a_1 = 2 \times 4 - 1$. Adding the above equations to obtain $a_{1996} - a_1 = \frac{2(1996+4) \times 665}{2} - 665 = 2000 \times 665 - 665 = 1329335$. Therefore, $a_{1996} = 1329335 + 1 = 1329336$.

5.4 If sequence $\{a_n\}$ satisfies $a_1 = 3$, and $a_{n+1} = 2a_n + 1 \quad (n \in N^*)$. Find the general term a_n of the sequence.

Solution: Adding 1 to both sides of the equation $a_{n+1} = 2a_n + 1$ to obtain $a_{n+1} + 1 = 2(a_n + 1)$. We apply the recurrent relation to obtain $a_n + 1 = 2(a_{n-1} + 1), a_{n-1} + 1 = 2(a_{n-2} + 1), \cdots, a_2 + 1 = 2(a_1 + 1)$. Multiplying the above equations and applying $a_1 = 3$ to generate $a_n = 2^{n+1} - 1 \quad (n \in N^*)$.

5.5 Let the function $f(x) = \log_2 x - \log_x 2 \quad (0 < x < 1)$, and the sequence $\{a_n\}$ satisfies $f(2^{a_n}) = 2n$. Find a_n.

Solution: Since $f(x) = \log_2 x - \log_x 2 = \log_2 x - \frac{1}{\log_2 x}$, then $f(2^{a_n}) = \log_2 2^{a_n} - \frac{1}{\log_2 2^{a_n}} = a_n - \frac{1}{a_n} = 2n$. It leads to $a_n^2 - 2na_n - 1 = 0$. Hence $a_n = n \pm \sqrt{n^2+1}$. Since $0 < x < 1$, then $0 < 2^{a_n} < 1$ which means $a_n < 0$. Therefore $a_n = n - \sqrt{n^2+1} \quad (n \in N^*)$.

5.6 Given the general term of sequence $\{a_n\}$ as $a_n = 2n^2 - n$, do there exist nonzero constants p, q such that the sequence $\{\frac{a_n}{pn+q}\}$ is an arithmetic sequence?

Solution: Assume that there exist nonzero constants p, q such that the sequence $\{\frac{a_n}{pn+q}\}$ is an arithmetic sequence. Then $\frac{a_1}{p+q}, \frac{a_2}{2p+q}, \frac{a_3}{3p+q}$ form an arithmetic sequence. From $a_1 = 1, a_2 = 6, a_3 = 15$, we have $(\frac{6}{2p+q}) \times 2 = \frac{1}{p+q} + \frac{15}{3p+q}$.

Then $pq + 2q^2 = 0$. Since $q \neq 0$, then $p = -2q$, then $\frac{a_n}{pn+q} = \frac{2n^2 - n}{-2qn + q} = -\frac{n}{q}$ when $p = -2q$. We show that $\{\frac{a_n}{pn+q}\}$ is an arithmetic sequence and the common difference is $-\frac{1}{q}$.

5.7 Given the sequence $\{a_n\}$, $a_1 = 1$, $a_{n+1} = a_n + 3n \quad (n \in N^*)$, find a_{10}.

Solution: $a_{n+1} = a_n + 3n \Rightarrow a_{n+1} - a_n = 3n$. Then $a_n - a_{n-1} = 3(n-1)$, $a_{n-1} - a_{n-2} = 3(n-2)$, $a_{n-2} - a_{n-3} = 3(n-3)$, $\cdots$, $a_3 - a_2 = 3 \times 2$, $a_2 - a_1 = 3 \times 1$. Adding the above equations to obtain $a_n - a_1 = 3[1+2+3+\cdots+(n-1)] = 3 \times \frac{n(n-1)}{2}$. Hence $a_n = a_1 + \frac{3n(n-1)}{2} = \frac{3}{2}n^2 - \frac{3}{2}n + 1$. Therefore, $a_{10} = 150 - 15 + 1 = 136$.

5.8 Given two arithmetic sequences $\{a_m\} : 1, 5, 9, \cdots$ and $\{b_n\} : 3, 10, 17, \cdots$. Consider their first 200th terms and find out the number of the terms with same values.

Solution: Since $a_m = 4m - 3, (m = 1, 2, \cdots)$, $b_n = 7n + 3, (0 = 0, 1, 2, \cdots)$, $a_m < b_n$, then $n = \frac{4m-6}{7} \quad (m, n \in N^*)$ when $4m - 3 = 7n + 3$. Thus $m_1 = 5$, $m_2 = 12$, $\cdots$. Assume $m_k = 7k - 2 \leqslant 200, \quad (k \in N^*)$. Hence $k = 1, 2, 3, \cdots, 28$. Hence, there are 28 terms with same values.

5.9 Assume the four roots of the equation $x^2-x+a=0$ and the equation $x^2-x+b=0$ form an arithmetic sequence with the first term $\frac{1}{4}$. If $a<b$, determine the values of a, b.

Solution: Let the four roots of the two equations are $\frac{1}{4}$, $\frac{1}{4}+d$, $\frac{1}{4}+2d$, $\frac{1}{4}+3d$. Applying the relation between roots and coefficients, we have $\frac{1}{4}+\frac{1}{4}+3d=\frac{1}{4}+d+\frac{1}{4}+2d=1$. Then $d=\frac{1}{6}$. Thus the two roots of the equation $x^2-x+a=0$ are $\frac{1}{4}$ and $\frac{3}{4}$. Hence $a=\frac{3}{16}$. And the two roots of $x^2-x+b=0$ are $\frac{1}{4}+\frac{1}{6}=\frac{5}{12}$ and $\frac{1}{4}+2\times\frac{1}{6}=\frac{7}{12}$. Hence, $b=\frac{5}{12}\times\frac{7}{12}=\frac{35}{144}$.

5.10 ★ Let $f(x)=(\sqrt{x}+\sqrt{2})^2 \quad (x\geqslant 0)$ and for $\{a_n\}$, $a_1=2$, $n\geqslant 2$, $a_n>0$, $S_n=f(S_{n-1})$. Find the general term of $\{a_n\}$.

Solution: According to $f(x)=(\sqrt{x}+\sqrt{2})^2$ and $S_n=f(S_{n-1})$, we have $S_n=(\sqrt{S_{n-1}}+\sqrt{2})^2$. It means $\sqrt{S_n}-\sqrt{S_{n-1}}=\sqrt{2}$. Thus $\sqrt{S_n}-\sqrt{S_{n-1}}=\sqrt{2}$, $\sqrt{S_{n-1}}-\sqrt{S_{n-2}}=\sqrt{2}, \cdots$, $\sqrt{S_3}-\sqrt{S_2}=\sqrt{2}$, $\sqrt{S_2}-\sqrt{S_1}=\sqrt{2}$. Adding the above equations to obtain $\sqrt{S_n}-\sqrt{S_1}=(n-1)\sqrt{2}$. Since $S_1=a_1=2$, then $\sqrt{S_n}=\sqrt{2}+(n-1)\sqrt{2}=n\sqrt{2}$. Hence $S_n=2n^2, \quad (n\in N^*)$. $a_n=S_n-S_{n-1}=2n^2-2(n-1)^2=4n-2$ when $n\geqslant 2$. And $a_1=2$ when $n=1$. Therefore, $a_n=4n-2 \quad (n\in N^*)$.

5.11 ★ Let each term of the sequence $\{a_n\}$ is nonzero, show that $\frac{1}{a_1a_2}+\frac{1}{a_2a_3}+\cdots+\frac{1}{a_na_{n+1}}=\frac{n}{a_1a_{n+1}}$.

Solution: Let the common difference of the sequence $\{a_n\}$ is d.
$\frac{1}{a_n}-\frac{1}{a_{n+1}}=\frac{a_{n+1}-a_n}{a_na_{n+1}}=\frac{d}{a_na_{n+1}}\Rightarrow\frac{1}{a_na_{n+1}}=\frac{1}{d}(\frac{1}{a_n}-\frac{1}{a_{n+1}})$. Thus $\frac{1}{a_1a_2}+\frac{1}{a_2a_3}+\cdots+\frac{1}{a_na_{n+1}}=\frac{1}{d}(\frac{1}{a_1}-\frac{1}{a_2}+\frac{1}{a_2}-\frac{1}{a_3}+\cdots+\frac{1}{a_n}-\frac{1}{a_{n+1}})=\frac{1}{d}(\frac{1}{a_1}-\frac{1}{a_{n+1}})=\frac{1}{d}(\frac{a_{n+1}-a_1}{a_1a_{n+1}})=\frac{1}{d}\frac{nd}{a_1a_{n+1}}=\frac{n}{a_1a_{n+1}}$.

5.12 ★ Compute the sum of the sequence.
(1) Given $\frac{1}{2}$, $2\frac{3}{4}$, $4\frac{7}{8}$, $6\frac{15}{16}$, $\cdots$, compute the nth partial sum S_n.
(2) $S=a^n+a^{n-1}b+a^{n-2}b^2+\cdots+a^{n-r}b^r+\cdots+ab^{n-1}+b^n$, where $a\neq 0, b\neq 0, n\in N^*$, evaluate S.
(3) Given 1, $1+2$, $1+2+2^2$, $\cdots$, $1+2+2^2+\cdots+2^{n-1}$, compute the nth partial sum S_n of the sequence.

Solution: (1) Let $M = \frac{1}{2}+\frac{3}{4}+\frac{7}{8}+\frac{15}{16}+\cdots+\frac{2^n-1}{2^n} = (1-\frac{1}{2})+(1-\frac{1}{4})+(1-\frac{1}{8})+\cdots+(1-\frac{1}{2^n}) = n-(\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots+\frac{1}{2^n}) = n-\frac{\frac{1}{2}[1-(\frac{1}{2})^n]}{1-\frac{1}{2}} = n-(1-\frac{1}{2^n}) = \frac{1}{2^n}+n-1 \quad (n \in N^*)$.
Let $N = 2+4+6+\cdots+2(n-1) = (n-1)2+\frac{(n-1)(n-2)}{2}2 = n^2-n \quad (n \in N^*)$.
Thus $S_n = M+N = \frac{1}{2^n}+n-1+n^2-n = \frac{1}{2^n}+n^2-1 \quad (n \in N^*)$.
(2) Multiplying the equation by a or by b to obtain

$$\begin{cases} aS = a^{n+1}+a^nb+a^{n-1}b^2+\cdots+a^{n-r+1}b^r+\cdots+a^2b^{n-1}+ab^n, \\ bS = a^nb+a^{n-1}b^2+a^{n-2}b^3+\cdots+a^{n-r}b^{r+1}+\cdots+ab^n+b^{n+1}. \end{cases}$$

The first equation minus the second equation, we have $(a-b)S = a^{n+1}-b^{n+1}$. Hence,

$$S = \begin{cases} \frac{a^{n+1}-b^{n+1}}{a-b}, a \neq b \\ (n+1)a^n, a = b \end{cases}$$

(3) Since $a_n = (1+2+2^2+\cdots+2^{n-1})(2-1) = 2^n-1$, then $S_n = \sum_{k=1}^{n}(2^k-1) = \sum_{k=1}^{n}2^k - \sum_{k=1}^{n}1 = \frac{2(1-2^n)}{1-2} - n = 2^{n+1}-n-2 \quad (n \in N^*)$.

5.13 ★★ Given sequence $\{a_n\}$, $a_1 = 1$, and sequence $\{b_n\}$, $b_1 = 0$, with the relationships $a_n = \frac{1}{3}(2a_{n-1}+b_{n-1})$ and $b_n = \frac{1}{3}(a_{n-1}+2b_{n-1})$ for $n \geqslant 2$. Find a_n, b_n.

Solution: $a_n+b_n = \frac{1}{3}(2a_{n-1}+b_{n-1})+\frac{1}{3}(a_{n-1}+2b_{n-1}) = a_{n-1}+b_{n-1} = a_{n-2}+b_{n-2} = \cdots = a_1+b_1 = 1 \quad ①$.

And $a_n-b_n = \frac{1}{3}(2a_{n-1}+b_{n-1})-\frac{1}{3}(a_{n-1}+2b_{n-1}) = \frac{1}{3}(a_{n-1}-b_{n-1}) = (\frac{1}{3})^2(a_{n-2}+b_{n-2}) = \cdots = (\frac{1}{3})^{n-1}(a_1-b_1) = (\frac{1}{3})^{n-1} \quad ②$.
According to ① and ②, we have $a_n = \frac{1}{2}(1+\frac{1}{3^{n-1}})$, $b_n = \frac{1}{2}(1-\frac{1}{3^{n-1}})$.

5.14 If the nth partial sum of the arithmetic sequence is 30 and the $2n$th partial sum is 100, compute the $3n$th partial sum.

Solution 1: Let the first term be a_1 and the common difference be d. From the given conditions, we have

$$\begin{cases} na_1+\frac{n(n-1)}{2}d = 30, \\ 2na_1+\frac{2n(2n-1)}{2}d = 100. \end{cases}$$

Solving this equation system to obtain $d = \frac{40}{n^2}$, $a_1 = \frac{10}{n} + \frac{20}{n^2}$. Thus $S_{3n} = 3na_1 + \frac{3n(3n-1)}{2}d = 3n\frac{10(n+2)}{n^2} + \frac{3n(3n-1)}{2}\frac{40}{n^2} = 210$.

Solution 2: According to the properties of an arithmetic sequence, S_n, $S_{2n} - S_n$, $S_{3n} - S_{2n}$ form an arithmetic sequence. Hence $2(S_{2n} - S_n) = S_n + (S_{3n} - S_{2n})$. Therefore $S_{3n} = 3(S_{2n} - S_n) = 3(100 - 30) = 210$.

Solution 3: The formula of the nth partial sum of an arithmetic sequence implies that S_n is a quadratic function. We have

$$\begin{cases} An^2 + Bn = 30, \\ A(2n)^2 + B2n = 100, \end{cases}$$

where A, B are constants. Solving the equation system to obtain $A = \frac{20}{n^2}$, $B = \frac{10}{n}$. Therefore, $S_{3n} = A(3n)^2 + B3n = \frac{20}{n^2}(3n)^2 + \frac{10}{n}3n = 210$.

5.15 ★★ The vertex coordinates of the quadratic function $f(x) = ax^2 + bx + c$ is $(\frac{3}{2}, -\frac{1}{4})$, and $f(3) = 2$. For an arbitrary real number x, the sequences $\{a_n\}$ and $\{b_n\}$ satisfy $f(x)g(x) + a_n x + b_n = x^{n+1} \quad (n \in N^*)$, where $g(x)$ is defined on the set of real numbers $\mathbb{R}$. Find the general terms of the sequences $\{a_n\}$ and $\{b_n\}$.

Solution: From the given condition, we have $f(x) = a(x - \frac{3}{2})^2 - \frac{1}{4} \quad (a \neq 0)$. Since $f(3) = 2$, then $a(3 - \frac{3}{2})^2 - \frac{1}{4} = 2$. Solving the equation to obtain $a = 1$. Hence $f(x) = x^2 - 3x + 2$, $x \in R$, $f(1) = 0$, $f(2) = 0$.
Applying $f(1)g(1) + a_n + b_n = 1$ to obtain $a_n + b_n = 1 \quad ①$. On the other hand, $f(2)g(2) + 2a_n + b_n = 2^{n+1}$, then $2a_n + b_n = 2^{n+1} \quad ②$. According to ① and ②, we have $a_n = 2^{n+1} - 1$, $b_n = 2 - 2^{n+1}$, $n \in N^*$.

5.16 ★★ Given the arithmetic sequence $\{a_n\}$, let $b_n = (\frac{1}{2})^{a_n}$, and $b_1 + b_2 + b_3 = \frac{21}{8}$, $b_1b_2b_3 = \frac{1}{8}$. Find the general term of the sequence $\{a_n\}$.

Solution: From the given condition, we have $b_1b_2b_3 = (\frac{1}{2})^{a_1}(\frac{1}{2})^{a_2}(\frac{1}{2})^{a_3} = (\frac{1}{2})^{a_1+a_2+a_3} = \frac{1}{8} = (\frac{1}{2})^3$. Hence $a_1+a_2+a_3 = 3$. Since $\{a_n\}$ is an arithmetic sequence, we assume $a_1 = a_2 - d$ and $a_3 = a_2 + d$, where d is the common difference. Thus $a_2 - d + a_2 + a_2 + d = 3$, then $a_2 = 1$. $b_1 + b_2 + b_3 = (\frac{1}{2})^{a_1} + (\frac{1}{2})^{a_2} + (\frac{1}{2})^{a_3} = (\frac{1}{2})^{1-d} + \frac{1}{2} + (\frac{1}{2})^{1+d} = \frac{21}{8}$. Solving the equation, we have $2^d + 2^{-d} = \frac{17}{4}$. That means $d = 2$ or $d = -2$.
When $d = 2$,$a_1 = 1 - d = -1, a_n = -1 + 2(n - 1) = 2n - 3 \quad (n \in N^*)$.
When $d = -2$,$a_1 = 1 - d = 3, a_n = 3 - 2(n - 1) = -2n + 5 \quad (n \in N^*)$.

5.17 ★ Given $\{a_n\}$ as a sequence with positive terms, S_n denotes the nth partial sum, and $2\sqrt{S_n} = a_n + 1 \quad (n \in N^*)$. Find the general term of the sequence $\{a_n\}$.

Solution: Since $2\sqrt{S_n} = a_n + 1$, then $2\sqrt{a_1} = a_1 + 1$ for $n = 1$. It means that $(\sqrt{a_1} - 1)^2 = 0$. Thus $a_1 = 1$. We have $4S_n = (a_n + 1)^2, 4S_{n-1} = (a_{n-1} + 1)^2$ for $n \geqslant 2$. Subtracting the second equation from the first equation to obtain $4a_n = (a_n+1)^2 - (a_{n-1}+1)^2 \Rightarrow (a_n - 1)^2 - (a_{n-1}+1)^2 = 0 \Rightarrow (a_n + a_{n-1})(a_n - a_{n-1} - 2) = 0$. Since $a_n + a_{n+1} > 0$, then $a_n - a_{n-1} - 2 = 0$ which means $a_n - a_{n-1} = 2$. Hence $\{a_n\}$ is an arithmetic sequence with the first term 1 and the common difference 2, $a_n = 1 + (n - 1) \times 2 = 2n - 1 \quad (n \in N^*)$.

5.18 ★ Let the nth partial sum of $\{a_n\}$ be S_n, $a_1 = 1$, $a_{n+1} = \frac{n+2}{n}S_n \quad (n = 1, 2, 3, \cdots)$. Show that (1) the sequence $\{\frac{S_n}{n}\}$ is a geometric sequence; (2) $S_{n+1} = 4a_n$.

Proof: (1) Since $a_{n+1} = \frac{n+2}{n}S_n$ and $a_{n+1} = S_{n+1} - S_n \quad (n = 1, 2, 3, \cdots)$, then $(n + 2)S_n = n(S_{n+1} - S_n) \Rightarrow nS_{n+1} = 2(n + 1)S_n \Rightarrow \frac{S_{n+1}}{n+1} = 2\frac{S_n}{n}$. Therefore the sequence $\{\frac{S_n}{n}\}$ is a geometric sequence.

(2) We apply (1) to obtain $\frac{S_{n+1}}{n+1} = 4\frac{S_{n-1}}{n-1}, (n \geqslant 2)$. Hence $S_{n+1} = 4(n+1)\frac{S_{n-1}}{n-1} = 4a_n, (n \geqslant 2)$. Since $a_2 = 3S_1 = 3$, then $S_2 = a_1 + a_2 = 4 = 4a_1$. Therefore $S_{n+1} = 4a_n$ holds for an arbitrary positive integer $n \geqslant 1$.

5.19 ★ Let the common difference of arithmetic sequence $\{a_n\}$ and the common ratio of geometric sequence $\{b_n\}$ be both d (where $d \neq 1$ and $d \neq 0$), $a_1 = b_1$, $a_4 = b_4$, $a_{10} = b_{10}$. (1) Find the values of a_1 and d. (2) Is b_{16} a term of $\{a_n\}$? If it is a term of $\{a_n\}$, which term is it? If it is not a term of $\{a_n\}$, explain the reason.

Solution: (1) Since $a_n = a_1 + (n-1)d$, $b_n = b_1 d^{n-1} = a_1 d^{n-1}$, and $a_4 = b_4$, $a_{10} = b_{10}$, we have

$$\begin{cases} a_1 + 3d = a_1 d^3 \\ a_1 + 9d = a_1 d^9 \end{cases}$$

$\Rightarrow$

$$\begin{cases} 3d = a_1(d^3 - 1) \\ 9d = a_1(d^9 - 1) \end{cases}$$

Dividing the first equation by the second equation leads to $d^6 + d^3 - 2 = 0$ which means that $d^3 = 1$ or $d^3 = -2$. Since $d \neq 1$, then $d^3 = -2$. Hence $d = -\sqrt[3]{2}$. Substituting it into the equation, we have $a_1 = \sqrt[3]{2}$. Therefore $a_1 = \sqrt[3]{2}$, $d = -\sqrt[3]{2}$.
(2) We apply (1) to obtain that the general terms of $\{a_n\}$ and $\{b_n\}$ are $a_n = (2-n)\sqrt[3]{2}$, $b_n = \sqrt[3]{2}(-\sqrt[3]{2})^{n-1} = -(-\sqrt[3]{2})^n$. Hence $b_{16} = -32\sqrt[3]{2}$. Since $(2-n)\sqrt[3]{2} = -32\sqrt[3]{2}$, then $n = 34$. Therefore, b_{16} is the 34th term of $\{a_n\}$.

5.20 ★ Given the sequence $\{a_n\}$, $a_1 = 1$, $a_{n+1} = S_n + (n+1)$, $(n \in N^*)$.
(1) Show the sequence $\{a_n + 1\}$ is a geometric sequence. (2) Find the general term a_n and the nth partial sum S_n.

(1) Proof: We apply $a_{n+1} = S_n + (n+1)$ to obtain $S_n = a_{n+1} - (n+1)$ and $S_{n-1} = a_n - n$. Then $a_n = S_n - S_{n-1} = [a_{n+1} - (n+1)] - (a_n - n)$. Thus $a_{n+1} = 2a_n + 1 \Leftrightarrow a_{n+1} + 1 = 2(a_n + 1) \Leftrightarrow \frac{a_{n+1}+1}{a_n+1} = 2$. Therefore $\{a_n + 1\}$ is a geometric sequence with common ratio 2.
(2) Solution: Since $a_1 + 1 = 2$, then $a_n + 1 = 2 \cdot 2^{n-1} = 2^n$ which is $a_n = 2^n - 1$. $S_n = (2-1) + (2^2 - 1) + \cdots + (2^n - 1) = (2 + 2^2 + \cdots + 2^n) - n = \frac{2(1-2^n)}{1-2} - n = 2^{n+1} - n - 2 \quad (n \in N^*)$.

5.21 ★★ Let $P_1(x_1, y_1)$, $P_2(x_2, y_2)$,$\cdots$,$P_n(x_n, y_n)$ $(n \geqslant 3)$ be the points on quadratic curve C, and $a_1 = |OP_1|^2$, $a_2 = |OP_2|^2$,$\cdots$,$a_n = |OP_n|^2$ form an arithmetic sequence with common difference $d \quad (d \neq 0)$, and $S_n = a_1 + a_2 + \cdots + a_n$.
(1) If the curve C is $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1 \quad (a = 10, b = 5)$, the point $P_1(10, 0)$, and $S_3 = 255$. Determine the point P_3.
(2) If the curve C is $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1 \quad (a > b > 0)$, the point $P_1(a, 0)$, Find the minimum value of S_n as d varies.

Solution: (1) Applying the point $P_1(10, 0)$, we have $a_1 = |OP_1|^2 = 10^2 = 100$, $S_3 = a_1 + \dfrac{a_1 + a_3}{2} + a_3 = \dfrac{3}{2}(a_1 + a_3) = 255$. Then $a_3 = |OP_3|^2 = 70$. Thus $x_3^2 + y_3^2 = 70 \quad ①$, $\dfrac{x_3^2}{100} + \dfrac{y_3^2}{25} = 1 \quad ②$. Applying ① and ② to obtain $x_3 = \pm 2\sqrt{15}$, $y_3 = \pm\sqrt{10}$. Therefore the coordinates of the point P_3 are $(2\sqrt{15}, \sqrt{10})$, $(2\sqrt{15}, -\sqrt{10})$, $(-2\sqrt{15}, \sqrt{10})$, $(-2\sqrt{15}, -\sqrt{10})$.
(2) Since $a_1 = |OP_1|^2 = a^2$, then $d < 0$ and $a_n = |OP_n|^2 = a^2 + (n-1)d \geqslant b^2$. That means $\dfrac{b^2 - a^2}{n-1} \leqslant d < 0$. Since $n \geqslant 3$, then $S_n = na^2 + \dfrac{n(n-1)}{2}d$ is increasing in $[\dfrac{b^2 - a^2}{n-1}, 0)$. Therefore $(S_n)_{min} = na^2 + \dfrac{n(n-1)}{2}\dfrac{b^2 - a^2}{n-1} = \dfrac{n(a^2 + b^2)}{2}$.

5.22 Let an arithmetic sequence has twelve terms where $S_{even} : S_{odd} = 32 : 27$ and the sum of sequence is 354. Find the common difference d.

Solution1: From the given condition, we have $S_{even} - S_{odd} = \dfrac{1}{2}nd = 6d$. Since $S_{even} : S_{odd} = 32 : 27$, let $S_{even} = 32t$, $S_{odd} = 27t$, then $32t + 27t = 354$. Thus $t = 6$. Hence $S_{even} = 32 \times 6 = 192$, $S_{odd} = 27 \times 6 = 162$. Therefore $S_{even} - S_{odd} = 30 = 6d$, then $d = 5$.

Solution2: $\dfrac{S_{even}}{S_{odd}} = \dfrac{32}{27} \Rightarrow \dfrac{S_{even}}{S_{even} + S_{odd}} = \dfrac{32}{32+27} \Rightarrow S_{even} = \dfrac{32}{59} \times 354 = 192, S_{odd} = 354 - 192 = 162$. Since $S_{even} - S_{odd} = \dfrac{1}{2}nd = 6d$, then $6d = 192 - 162 = 30$. Therefore $d = 5$.

5.23 ★ Let $\{a_n\}$, $a_1 = 1$, $na_{n+1} = (n+1)a_n + 1 \quad (n \geqslant 2)$. Compute the the nth partial sum S_n.

Solution: $na_{n+1} = (n+1)a_n + 1 \quad (n \geqslant 2) \Rightarrow n(a_{n+1} + 1) = (n+1)(a_n + 1) \quad (n \geqslant 2)$.

Let $b_n = a_n + 1$, then $b_{n+1} = \dfrac{n+1}{n}b_n$. Thus $b_1 = 2$, $b_2 = 2 \times 2$, $b_3 = 3 \times 2$, $b_4 = 4 \times 2$,$\cdots$, $b_n = n \times 2$. Therefore $S_n = a_1 + a_2 + \cdots + a_n = b_1 + b_2 + \cdots + b_n - n = 2(1 + 2 + \cdots + n) - n = n^2 \quad (n \geqslant 2)$.

5.24 ★★ Given the quadratic function $f(x) = n(n+1)x^2 - (2n+1)x + 1$, and n is chosen as all natural numbers, compute the sum of lengths of all line segments on the x-axis intercepted by the graph.

Solution: $n(n+1)x^2-(2n+1)x+1 = 0$when $f(x) = 0$. Thus $(nx-1)[(n+1)x-1] = 0$. Then $x_1 = \frac{1}{n}$, $x_2 = \frac{1}{n+1}$. Let the parabola intersects x-axis at the point A_n and the point B_n, then the sum of lengths of the intercepted line segments $S_n = |A_1B_1| + |A_2B_2| + \cdots + |A_nB_n| = (1-\frac{1}{2}) + (\frac{1}{2}-\frac{1}{3}) + \cdots + (\frac{1}{n} - \frac{1}{n+1}) = 1 - \frac{1}{n+1}$. $\lim\limits_{n\to\infty} S_n = \lim\limits_{n\to\infty}(1-\frac{1}{n+1}) = 1$.

5.25 ★★ (1) Given an arithmetic sequence $\{a_n\}$ that satisfies $a_1 = -60$, $a_{17} = -12$. Let $b_n = |a_n|$. Evaluate the 30th partial sum of $\{b_n\}$.
(2) If the general term of the arithmetic sequence $\{a_n\}$ is $a_n = 10-3n$, compute $|a_1| + |a_2| + \cdots + |a_n|$.

Solution: (1) Let the common difference of the arithmetic sequence $\{a_n\}$ is d. Since $a_1 = -60$, $a_{17} = -12$, then $-60+16d = -12$. Thus $d = 3$. Hence $a_n = -60+(n-1)3$. It means $a_n = 3n-63$. $3n-63 = 0$ if $a_n = 0$. Then $n = 21$. $a_{21} = 0$, $a_{22} = 3$.
Method 1: $S_{21} = \frac{-60+0}{2}\times 20 = -600$. $S_{30}-S_{21} = (30-21+1)\times 3 + \frac{(30-21+1)(30-21)}{2} \times 3 = 165$. Therefore the 30th partial sum of $\{b_n\}$ is $S_{30} = |S_{21}| + |S_{30} - S_{21}| = 765$.
Method 2: Since $a_n = 3n-63$, $S_{30} = a_1 + a_2 + \cdots + a_{30} - 2(a_1 + a_2 + \cdots + a_{20}) = \frac{-60+27}{2} \times 30 - 2\frac{-60-3}{2} \times 20 = 765$.

(2) Since $a_n = 10-3n$, then $a_1 > 0, a_2 > 0, a_3 > 0$, $a_4, a_5, \cdots, a_n < 0$. Hence

$$|a_1| + |a_2| + \cdots + |a_n| = \begin{cases} a_1 + a_2 + \cdots + a_n, (n \leqslant 3) \\ a_1 + a_2 + a_3 - a_4 - \cdots - a_n, (n \geqslant 4) \end{cases}$$

$$= \begin{cases} \frac{a_1+a_n}{2}n, (n \leqslant 3) \\ 2(a_1+a_2+a_3) - (a_1+a_2+\cdots+a_n), (n \geqslant 4) \end{cases}$$

$$= \begin{cases} \frac{-3n^2+17n}{2}, (n \leqslant 3) \\ 24 - \frac{-3n^2+17n}{2}, (n \geqslant 4) \end{cases}$$

$$= \begin{cases} \frac{-3n^2+17n}{2}, (n \leqslant 3) \\ \frac{3n^2-17n+48}{2}, (n \geqslant 4) \end{cases}$$

5.26 ★ Given $f(x)$ is a linear function, and $f(8) = 15$. $f(2), f(5), f(4)$ form a geometric sequence. Denote $S_n = f(1) + f(2) + \cdots + f(n)$. Compute $\lim_{n\to\infty}(\frac{S_n}{n^2})$.

Solution: Let $f(x) = kx + b$. From the given condition, we have

$$\begin{cases} 8k + b = 15 \\ (5k+b)^2 = (2k+b)(4k+b) \end{cases}$$

$\Rightarrow$

$$\begin{cases} k = 4 \\ b = -17 \end{cases}$$

Then $f(x) = 4x - 17$. Consist the sequence $-13, -9, -5, \cdots, (4n-17)$ when $x = 1, 2, \cdots, n$. $S_n = \dfrac{(-13 + 4n - 17)n}{2} = 2n^2 - 15n$. $\lim_{n\to\infty}(\dfrac{S_n}{n^2}) = \lim_{n\to\infty}\dfrac{2n^2 - 15n}{n^2} = 2 - \lim_{n\to\infty}\dfrac{15}{n} = 2$.

5.27 ★ Let all terms of the arithmetic sequence $\{a_n\}$ are positive. Show $\dfrac{1}{\sqrt{a_1}+\sqrt{a_2}} + \dfrac{1}{\sqrt{a_2}+\sqrt{a_3}} + \cdots + \dfrac{1}{\sqrt{a_{n-1}}+\sqrt{a_n}} = \dfrac{n-1}{\sqrt{a_1}+\sqrt{a_n}}$.

Proof: Let $M = \dfrac{1}{\sqrt{a_1}+\sqrt{a_2}} + \dfrac{1}{\sqrt{a_2}+\sqrt{a_3}} + \cdots + \dfrac{1}{\sqrt{a_{n-1}}+\sqrt{a_n}}$, the common difference is d. We have $\dfrac{1}{\sqrt{a_{n-1}}+\sqrt{a_n}} = \dfrac{\sqrt{a_{n-1}}-\sqrt{a_n}}{a_{n-1}-a_n} = -\dfrac{1}{d}(\sqrt{a_{n-1}}-\sqrt{a_n}) \Rightarrow M = -\dfrac{1}{d}(\sqrt{a_1}-\sqrt{a_2}+\sqrt{a_2}-\sqrt{a_3}+\cdots+\sqrt{a_{n-1}}-\sqrt{a_n}) = -\dfrac{1}{d}(\sqrt{a_1}-\sqrt{a_n}) = -\dfrac{1}{d}\dfrac{a_1-a_n}{\sqrt{a_1}+\sqrt{a_n}} = -\dfrac{1}{d}\dfrac{a_1-[a_1+(n-1)d]}{\sqrt{a_1}+\sqrt{a_n}} = -\dfrac{1}{d}\dfrac{-(n-1)d}{\sqrt{a_1}+\sqrt{a_n}} = \dfrac{n-1}{\sqrt{a_1}+\sqrt{a_n}}$.

5.28 ★ Solve the nth partial sum of the sequence $1, 3a, 5a^2, 7a^3, \cdots, (2n-1)a^{n-1}$.

Solution: When $a=1$, the sequence is $1, 3, 5, 7, \cdots, (2n-1)$. $S_n = \dfrac{[1+(2n-1)]n}{2} = n^2 \quad (n \in N^*)$.
When $a \neq 1$, $S_n = 1+3a+5a^2+7a^3+\cdots+(2n-1)a^{n-1}$ ①. Multiplying the equation ① by a to obtain $aS_n = a+3a^2+5a^3+7a^4+\cdots+(2n-1)a^n$ ②. Using ① − ②, we have $(1-a)S_n = 1+2a+2a^2+2a^3+\cdots+2a^{n-1}-(2n-1)a^n = 1-(2n-1)a^n+2(a+a^2+a^3+\cdots+a^{n-1}) = 1-(2n-1)a^n+2\dfrac{a(1-a^{n-1})}{1-a} = 1-(2n-1)a^n+\dfrac{2(a-a^n)}{1-a}$.
While $1-a \neq 0$, then $S_n = \dfrac{1-(2n-1)a^n}{1-a} + \dfrac{2(a-a^n)}{(1-a)^2} \quad (n \in N^*)$.

5.29 ★★ Let the nth partial sum of the sequence $\{a_n\}$ is $S_n = 2n^2$, $\{b_n\}$ is a geometric sequence, and $a_1 = b_1$, $b_2(a_2-a_1) = b_1$. (1) Find the general term of $\{a_n\}$ and $\{b_n\}$. (2) Let $c_n = \dfrac{a_n}{b_n}$, evaluate the nth partial sum T_n of the sequence $\{c_n\}$.

Solution: (1) $S_1 = a_1 = 2$ when $n=1$. $a_n = S_n - S_{n-1} = 2n^2-2(n-1)^2 = 4n-2$ when $n \geqslant 2$. $4n-2 = 2 = a_1$ when $n=1$. Hence the general term of $\{a_n\}$ is $a_n = 4n-2 = 2+4(n-1)$. Therefore $\{a_n\}$ is an arithmetic sequence with the fist term 2 and the common difference 4.
Let common ratio of $\{b_n\}$ is q. Since $b_2(a_2-a_1) = b_1$ and $b_2 = b_1q$, then $b_1qd = b_1$. Thus $q = \dfrac{1}{d} = \dfrac{1}{4}$. Otherwise, $b_1 = a_1$, then $b_n = b_1q^{n-1} = \dfrac{2}{4^{n-1}}$.
(2) $c_n = \dfrac{a_n}{b_n} = (2n-1)4^{n-1}$. $T_n = c_1+c_2+\cdots+c_n = 1+3\times4+5\times4^2+\cdots+(2n-1)4^{n-1}$ ①. Multiplying the equation ① by 4 to obtain $4T_n = 1\times4+3\times4^2+5\times4^3+\cdots+(2n-1)4^n$ ②. Using ①−②, we have $3T_n = -1-2\times(4+4^2+\cdots+4^{n-1})+(2n-1)4^n = -1-2\dfrac{4(1-4^{n-1})}{1-4}+(2n-1)4^n = \dfrac{5}{3}+\dfrac{1}{3}(6n-5)4^n = \dfrac{1}{3}[(6n-5)4^n+5]$. Therefore $T_n = \dfrac{1}{9}[(6n-5)4^n+5] \quad (n \in N^*)$.

5.30 ★ The sequence $\{a_n\}$ is a geometric sequence, $a_1 = 8$, $b_n = \log_2 a_n$. If the first 7th partial sum S_7 of $\{b_n\}$ is the maximum value, and $S_7 \neq S_8$. Find the range of the common ratio q of the sequence $\{a_n\}$.

Solution: $b_{n+1} - b_n = \log_2 a_{n+1} - \log_2 a_n = \log_2 \frac{a_{n+1}}{a_n} = \log_2 q$. Then $\{b_n\}$ is an arithmetic sequence, and its first term is $b_1 = \log_2 a_1 = 3$, its common difference is $\log_2 q$. From the given condition, we have

$$\begin{cases} b_7 \geqslant 0 \\ b_8 < 0 \end{cases}$$

Then

$$\begin{cases} 3 + 6\log_2 q \geqslant 0 \\ 3 + 7\log_2 q < 0 \end{cases}$$

Thus $-\frac{1}{2} \leqslant \log_2 q < -\frac{3}{7}$. Hence $q \in [\frac{\sqrt{2}}{2}, 2^{-\frac{3}{7}})$.

5.31 ★★ For a positive number, its decimal part, integer part and itself form a geometric sequence. Find this number.

Solution: Let this number is x, its integer part is $[x]$, its decimal part is $x - [x]$. From the given condition, we have $x(x - [x]) = [x]^2$ which means $x^2 - [x]x - [x]^2 = 0$ where $[x] > 0$, $0 < x - [x] < 1$. Solve the equation, then $x = \frac{1+\sqrt{5}}{2}[x]$. Since $0 < x - [x] < 1$, then $0 < \frac{1+\sqrt{5}}{2}[x] - [x] < 1$. Thus $0 < \frac{\sqrt{5}-1}{2}[x] < 1$. Hence $0 < [x] < \frac{1+\sqrt{5}}{2} < 2$. Therefore $[x] = 1 \Rightarrow x = \frac{1+\sqrt{5}}{2}$.

5.32 ★★ The sequence $\{a_n\}$ has k terms (k is a fixed number). Its nth partial sum $S_n = 2n^2 + n \quad (n \leqslant k, n \in N^*)$. If we remove one term (neither the first term nor the last term) from the k terms, the average value of the remaining $(k-1)$ terms is 79.
(1) Find the general term for $\{a_n\}$. (2) Determine k and which term is removed.

Solution: (1) From the given condition, we have $S_1 = a_1 = 3$, $a_n = S_n - S_{n-1} = (2n^2+n) - [2(n-1)^2 + (n-1)] = 4n - 1, (n \geqslant 2)$. Since a_1 satisfies the above formula, then $a_n = 4n - 1, \quad (n \leqslant k, n \in N^*)$.
(2) Let the removed term be the tth term, then $1 < t < k$. From the given condition, we have $S_k - a_t = 79(k-1)$. Thus $2k^2 + k - (4t - 1) = 79k - 79 \Rightarrow 4t = 2k^2 - 78k + 80 \Rightarrow 4 < 2k^2 - 78k + 80 < 4k \Rightarrow 38 < k < 40$. Since $k \in N^*$, then $k = 39$. Hence $t = \frac{k^2 - 39k + 40}{2} = 20$. Therefore the removed term is the 20th term.

5.33 ★★ Given $f(x) = x^2 - (2n+1)x + n^2 + 5n - 7$.
(1) If the y-ordinate of the vertex of the graph of $f(x)$ form a sequence $\{a_n\}$, show $\{a_n\}$ is an arithmetic sequence.
(2) If the distance from the vertex of the graph of $f(x)$ to x-axis form a sequence $\{b_n\}$, evaluate the nth partial sum of $\{b_n\}$.

Solution: (1) $f(x) = x^2 - (2n+1)x + n^2 + 5n - 7 = [x-(n+1)]^2 + 3n - 8$, then $a_n = 3n-8$. Since $a_{n+1} - a_n = [3(n+1)-8] - (3n-8) = 3$, then $\{a_n\}$ is an arithmetic sequence which common difference is 3.
(2) Applying (1), we have $b_n = |3n-8|$. When $1 \leqslant n \leqslant 2$, then $b_n = 8-3n$, $b_1 = 5$, $S_n = \dfrac{(5+8-3n)n}{2} = \dfrac{13n-3n^2}{2}$. When $n \geqslant 3$, then $b_n = 3n-8$, $S_n = 5+2+1+4+7+\cdots+(3n-8) = 7 + \dfrac{(1+3n-8)(n-2)}{2} = \dfrac{3n^2-13n+28}{2}$.
Thus

$$S_n = \begin{cases} \dfrac{13n-3n^2}{2}, (1 \leqslant n \leqslant 2) \\ \dfrac{3n^2-13n+28}{2}, (n \geqslant 3) \end{cases}$$

5.34 ★★★ If we insert a number a between two positive numbers, the three numbers form an arithmetic sequence. If we insert two numbers b and c, the four numbers form a geometric sequence. Show (1) $2a > b+c$ (2) $(a+1)^2 \geqslant (b+1)(c+1)$.

Proof: (1) Let the two positive numbers be m and n $(m, n > 0)$. Then $m+n = 2a$ ①, $mc = b^2$ ②, $nb = c^2$ ③. Applying ① to obtain $a > 0$. Applying ② and ③ to obtain $b, c > 0$. Thus $\dfrac{b^2}{c} + \dfrac{c^2}{b} = m+n = 2a$. Then $2abc = b^3 + c^3 = (b+c)(b^2+c^2-bc) \geqslant (b+c)(2bc-bc) = (b+c)bc$. Hence $2a > b+c$.

(2) Applying (1) to obtain $a = \dfrac{m+n}{2} \geqslant \sqrt{mn} = \sqrt{bc}$. Thus $a^2 \geqslant bc$. Applying (1) again, we have $2a > b + c$. Thus

$$\begin{cases} a^2 \geqslant bc \\ 2a \geqslant b+c \end{cases}$$

$\Rightarrow a^2 + 2a \geqslant bc + b + c \Rightarrow (a+1)^2 \geqslant bc + b + c + 1 = (b+1)(c+1)$.

5.35 ★ For an arbitrary real number x, $[x]$ denotes the integer part of x. It means $[x]$ is the biggest integer number which satisfies $[x] \leqslant x$.
(1) evaluate $[\log_2 1] + [\log_2 2] + \cdots + [\log_2 1024]$.
(2) Deduce the formula of $[\log_2 1] + [\log_2 2] + \cdots + [\log_2(2^n - 1)]$.

Solution: (1) $[\log_2 1] = 0$, $[\log_2 2] = [\log_2 3] = 1$, $[\log_2 4] = [\log_2 5] = [\log_2 6] = [\log_2 7] = 2, \cdots$, $[\log_2 512] = [\log_2 513] = \cdots = [\log_2 1023] = 9$, $[\log_2 1024] = 10$. Thus $[\log_2 1] + [\log_2 2] + \cdots + [\log_2 1024] = 2 + 2 \times 2^2 + 3 \times 2^3 + \cdots + 9 \times 2^9 + 10 = 8204$.

(2) Let $[\log_2 1] + [\log_2 2] + \cdots + [\log_2(2^n - 1)] = S_n$. Applying (1) to obtain $S_n = 2 + 2 \times 2^2 + 3 \times 2^3 + \cdots + (n-1) \times 2^{n-1}$, $2S_n = 2^2 + 2 \times 2^3 + 3 \times 2^4 + \cdots + (n-1) \times 2^n$. The second equation minus the first equation, we have $S_n = (n-1) \times 2^n - (2 + 2^2 + 2^3 + \cdots + 2^{n-1}) = (n-1)2^n - (2^n - 1) = n2^n - 2^{n+1} + 1 \quad (n \in N^*)$.

5.36 ★★ How many terms are same in the first 100the terms of the arithmetic sequence $5, 8, 11, \cdots$ and the arithmetic sequence $3, 7, 11, \cdots$? Evaluate the sum of these same terms.

Solution: The general term of the arithmetic sequence $5, 8, 11, \cdots$ is $a_n = 3n + 2$, the general term of the geometric sequence $3, 7, 11, \cdots$ is $b_n = 4m - 1$, $(m, n \in N^*)$. Let $3n + 2 = 4m - 1$, then $n = \dfrac{4}{3}m - 1$. Let $m = 3k, (k \in N^*)$, then $n = 4k - 1$. Hence the general term of the same terms of the two sequences is $c_k = 12k - 1$. Let $5 \leqslant 12k - 1 \leqslant 302$, then $\dfrac{1}{2} \leqslant k \leqslant 25\dfrac{1}{4}$. Thus $k = 1, 2, \cdots, 25$. Therefore there are 25 terms which are same in the first 100th terms of the two sequences.
Thus $\{c_k\}$ is an arithmetic sequence whose first term is $c_1 = 11$ and common difference is $d = c_2 - c_1 = 12$. Hence $S_{25} = \dfrac{[11 + 11 + (25-1) \times 12] \times 25}{2} = 3875$.

5.37 ★★★ (1) Consider a geometric sequence $\{a_n\}$, $a_1 = 1$, and it has even number of terms. The sum of all odd terms is 85. The sum of all even terms is 170. Evaluate the common ratio q and the number of terms n.

(2) All terms of the geometric sequence $\{a_n\}$ are positive, and it has even number of terms. The sum of all terms is four times of the sum of all even terms. The product of the 2th term and 4th term is nine times of the sum of the 3th term and 4th term. Compute a_1, the common ratio q, and the term number n when the nth partial sum of sequence $\{\lg a_n\}$ reaches the maximum value.

(3) The nth partial sum of the geometric sequence $\{a_n\}$ whose terms are all positive is 80. The largest term is 54. The $2n$th partial sum is 6560. Find the common ratio q.

Solution: (1) Since the term number is even, then $\frac{S_{even}}{S_{odd}} = q = \frac{170}{85} = 2$, $S_{even} = \frac{1 \times [1-(q^2)^{\frac{n}{2}}]}{1-q^2} = 85$. Thus $2^n = 256 = 2^8 \Rightarrow n = 8$.

(2) $S_n = 4S_{even} \Rightarrow S_{even} + S_{odd} = 4S_{even} \Rightarrow S_{odd} = 3S_{even} \Rightarrow \frac{S_{even}}{S_{odd}} = \frac{1}{3} = q$. From the condition $a_2a_4 = 9(a_3+a_4)$, we have $a_1^2q^4 = 9(a_1q^2+a_1q^3)$. Thus $a_1^2 - 108a_1 = 0$. Then $a_1 = 108$. Such that the nth partial sum of sequence $\{\lg a_n\}$ reaches the maximum value, we have $\lg a_n = \lg(a_1q^{n-1}) > 0$, then $108(\frac{1}{3})^{n-1} > 1$. Thus $(\frac{1}{3})^{n-1} > \frac{1}{108}$. $n-1 < \log_{\frac{1}{3}}\frac{1}{108} = \log_{\frac{1}{3}}(\frac{1}{3^3}\frac{1}{4}) = 3 + \log_3 4 \Rightarrow n < 4 + \log_3 4 \leqslant 4 + 1 = 5$. Therefore the nth partial sum of sequence $\{\lg a_n\}$ reaches the maximum value when $n = 5$.

(3) From the given condition, we know that the last nth partial sum of the positive sequence $\{\lg a_n\}$ is larger than the first nth partial sum, then $q > 0$, $a_n = 54$, $a_n = a_1q^{n-1} = 54$ ①, $S_n = \frac{a_1(1-q^n)}{1-q} = 80$ ②, $q^n = \frac{S_{2n}-S_n}{S_n} = \frac{6560-80}{80} = 81$. Applying ① to obtain $\frac{a_1}{q} = \frac{54}{81} = \frac{2}{3}$. Thus $a_1 = \frac{2}{3}q$. Substituting it into ②, we have $\frac{\frac{2}{3}q(1-81)}{1-q} = 80$. Thus $q = 3$.

5.38 ★★★ The function is defined on $(-1,1)$, $f(\frac{1}{2}) = -1$ and satisfied that $f(x)+f(y) = f(\frac{x+y}{1+xy})$ for $x, y \in (-1,1)$

(1) If the sequence $\{f(x_n)\}$ satisfies that $x_1 = \frac{1}{2}$ and $x_{n+1} = \frac{2x_n}{1+x_n^2}$. Compute $f(x_n)$.

(2) Show $\frac{1}{f(x_1)} + \frac{1}{f(x_2)} + \cdots + \frac{1}{f(x_n)} > -\frac{2n+5}{n+2}$.

(1) Solution: $f(x_1) = f(\frac{1}{2}) = -1$. $f(x_{n+1}) = f(\frac{2x_n}{1+x_n^2}) = f(\frac{x_n+x_n}{1+x_nx_n}) = f(x_n) + f(x_n) = 2f(x_n) \Rightarrow \frac{f(x_{n+1})}{f(x_n)} = 2$. Thus $\{f(x_n)\}$ is a geometric sequence and its first term is -1, the common ratio is 2. Therefore $f(x_n) = -2^{n-1}, \quad (n \in N^*)$.

(2) Proof: $\frac{1}{f(x_1)}+\frac{1}{f(x_2)}+\cdots+\frac{1}{f(x_n)}=-(1+\frac{1}{2}+\frac{1}{2^2}+\cdots+\frac{1}{2^{n-1}})=-\frac{1-\frac{1}{2^n}}{1-\frac{1}{2}}=-(2-\frac{1}{2^{n-1}})=-2+\frac{1}{2^{n-1}}>-2$. On the other hand, $-\frac{2n+5}{n+2}=-\frac{2(n+2)+1}{n+2}=-(2+\frac{1}{n+2})=-2-\frac{1}{n+2}<-2$. Thus $\frac{1}{f(x_1)}+\frac{1}{f(x_2)}+\cdots+\frac{1}{f(x_n)}>-\frac{2n+5}{n+2}$.

5.39 ★★★ If the sequence $\{a_n\}$ and $\{b_n\}$ satisfy $b_n=\frac{a_1+2a_2+3a_3+\cdots+na_n}{1+2+3+\cdots+n}$, and $\{b_n\}$ is a geometric sequence. Show $\{a_n\}$ is also a geometric sequence.

Proof: Since $(1+2+3+\cdots+n)b_n=a_1+2a_2+3a_3+\cdots+na_n$, then $\frac{n(n+1)}{2}b_n=a_1+2a_2+3a_3+\cdots+na_n$ ①. We also have $\frac{(n-1)n}{2}b_{n-1}=a_1+2a_2+3a_3+\cdots+(n-1)a_{n-1}$ ②. Checking ① − ② to obtain $\frac{n(n+1)}{2}b_n-\frac{(n-1)n}{2}b_{n-1}=na_n, \quad (n=2,3,\cdots)\Rightarrow a_n=\frac{1}{2}[(n+1)b_n-(n-1)b_{n-1}], \quad (n=2,3,4,\cdots)$ $(*)$. Since $\{b_n\}$ is a geometric sequence, let the common difference be d. Substituting $b_n=b_1+(n-1)d$ into $(*)$, we obtain $a_n=\frac{1}{2}\{(n+1)[b_1+(n-1)d]-(n-1)[b_1+(n-2)d]\}=b_1+\frac{3}{2}(n-1)d,(n=2,3,\cdots)$. For $b_n=\frac{a_1+2a_2+3a_3+\cdots+na_n}{1+2+3+\cdots+n}$, let $n=1$, then $b_1=a_1$. Thus $a_n=a_1+\frac{3}{2}(n-1)d,(n=2,3,\cdots)$. Hence $a_n-a_{n-1}=[a_1+\frac{3}{2}(n-1)d]-[a_1+\frac{3}{2}(n-2)d]=\frac{3}{2}d$ (constant). Therefore $\{a_n\}$ is also a geometric sequence.

3.40 ★★★ Let the first nth partial sum S_n of sequence $\{a_n\}$ satisfy $S_n=1-\frac{2}{3}a_n \quad (n\in N^*)$.
(1) Calculate S_n and a_n. (2) If we let T_n denote the first nth partial sum of sequence $\{a_nS_n\}$, compute $\lim_{n\to\infty}T_n$.

Solution: (1) $S_n=1-\frac{2}{3}a_n\Rightarrow S_n=1-\frac{2}{3}(S_n-S_{n-1})\Rightarrow\frac{5}{3}S_n=1+\frac{2}{3}S_{n-1}\Rightarrow\frac{5}{3}(S_n-1)=\frac{2}{3}(S_{n-1}-1)\Rightarrow\frac{S_n-1}{S_{n-1}-1}=\frac{2}{5}$. Since $S_1=1-\frac{2}{3}S_1$, then $S_1=\frac{3}{5}$, $S_1-1=\frac{3}{5}-1=-\frac{2}{5}$. Then $\{S_n-1\}$ is a geometric sequence and its first term is $-\frac{2}{5}$, the common ratio is $\frac{2}{5}$. $S_n-1=(-\frac{2}{5})(\frac{2}{5})^{n-1}=-(\frac{2}{5})^n$. Therefore $S_n=1-(\frac{2}{5})^n$, $a_n=S_n-S_{n-1}=[1-(\frac{2}{5})^n]-[1-(\frac{2}{5})^{n-1}]=(\frac{2}{5})^{n-1}-(\frac{2}{5})^n=(\frac{2}{5})^n(\frac{5}{2}-1)=\frac{3}{2}(\frac{2}{5})^n \quad (n\in N^*)$.

(2) Since $a_n = \frac{3}{2}(\frac{2}{5})^n = \frac{3}{2}\frac{2}{5}(\frac{2}{5})^{n-1} = \frac{3}{5}(\frac{2}{5})^{n-1}$, then $\{a_n\}$ is a geometric sequence and its first term is $\frac{3}{5}$, the common ratio is $\frac{2}{5}$. $a_nS_n = \frac{3}{5}(\frac{2}{5})^{n-1}[1-(\frac{2}{5})^n] = \frac{3}{5}(\frac{2}{5})^{n-1} - \frac{6}{25}[(\frac{2}{5})^2]^{n-1}$. Thus $\lim\limits_{n\to\infty} T_n = \lim\limits_{n\to\infty} a_nS_n = \frac{\frac{3}{5}}{1-\frac{2}{5}} - \frac{\frac{6}{25}}{1-(\frac{2}{5})^2} = 1 - \frac{2}{7} = \frac{5}{7}$.

5.41 ★ Let the common ratio of the geometric sequence $\{a_n\}$ is $q > 1$. The square of the 17th term is equal to the 24th term. Compute the range of the integer number n which satisfies $a_1 + a_2 + a_3 + \cdots + a_n > \frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \cdots + \frac{1}{a_n}$.

Solution: $a_{17}^2 = a_{24} \Rightarrow (a_1q^{16})^2 = a_1q^{23}$. Since $q > 1$ and $a_1 \neq 0$, then $a_1 = q^{-9}$.

Since $a_1 + a_2 + a_3 + \cdots + a_n > \frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \cdots + \frac{1}{a_n}$, then $\frac{a_1(1-q^n)}{1-q} > \frac{\frac{1}{a_1}(1-\frac{1}{q^n})}{1-\frac{1}{q}}$, which means $a_1 > \frac{1}{a_1q^{n-1}}$ ①. Substituting $a_1 = q^{-9}$ into ①, we have $q^{-18} > q^{1-n}$. Since $q > 1$, then $-18 > 1 - n$. Thus $n > 19$. On the other hand, $(n \in N^*)$, then $n \geqslant 20$. Hence the range of the integer number n is $[20, +\infty)$.

5.42 ★★ Given the arithmetic sequence $\{a_n\}$ and the x-dependent equations $a_i x^2 + 2a_{i+1}x + a_{i+2} = 0, (i = 1, 2, \cdots, n)$, and a_1 and the common difference d are both nonzero real numbers. (1) Show these equations have same solutions. (2) If another solution is β_i, then $\dfrac{1}{\beta_1+1}, \dfrac{1}{\beta_2+1}, \cdots, \dfrac{1}{\beta_n+1}$ form a geometric sequence.

Proof: (1) Since $a_i x^2 + 2a_{i+1}x + a_{i+2} = 0$ and $\{a_n\}$ is a geometric sequence which means $2a_{i+1} = a_i + a_{i+2}$, then $a_i x^2 + (a_i + a_{i+2})x + a_{i+2} = 0$. Then $(x^2+x)a_i + (x+1)a_{i+2} = 0$. Since a_1 and the common difference d are both nonzero real numbers, then $a_i \neq 0$ and $a_{i+2} \neq 0$. Thus $x^2 + x = 0$ and $x + 1 = 0$. Hence $x = -1$. Therefore these equations have same solutions $x = -1$.
(2) Applying the relation of roots and coefficient to obtain $\beta_i + (-1) = -\dfrac{2a_{i+1}}{a_i} = -\dfrac{2(a_i+d)}{a_i} = -2 - \dfrac{2d}{a_i} \Rightarrow \beta_i = -1 - \dfrac{2d}{a_i} \Rightarrow \dfrac{1}{\beta_{i+1}+1} - \dfrac{1}{\beta_i+1} = \dfrac{1}{-1-\frac{2d}{a_{i+1}}+1} - \dfrac{1}{-1-\frac{2d}{a_i}+1} = \dfrac{a_i - a_{i+1}}{2d} = \dfrac{-d}{2d} = -\dfrac{1}{2}$ (constant). Then $\frac{1}{\beta_1+1}, \frac{1}{\beta_2+1}, \cdots, \frac{1}{\beta_n+1}$ form a geometric sequence.

5.43 ★★★ Given $f(x) = \sqrt{x^2-4}\quad (x \leqslant -2)$. (1) Find the inverse function $f^{-1}(x)$. (2) Let $a_1 = 1, a_n = -f^{-1}(a_{n-1})$, evaluate a_n. (3) If $b_1 = \dfrac{1}{a_1+a_2}, b_2 = \dfrac{1}{a_2+a_3}, \cdots, b_n = \dfrac{1}{a_n+a_{n+1}}, \cdots$, compute the first nth partial sum of $\{b_n\}$.

Solution: (1) Since $y = f(x) = \sqrt{x^2-4}\quad (x \leqslant -2)$, then $x = -\sqrt{y^2+4}$, which means $f^{-1}(x) = -\sqrt{x^2+4}, (x \geqslant 0)$.
(2) From the given condition and (1), we have $a_n = \sqrt{a_{n-1}^2+4}$. Squaring both sides of the equation, then $a_n^2 = a_{n-1}^2 + 4$, that is $a_n^2 - a_{n-1}^2 = 4$. Hence $a_2^2 - a_1^2 = 4$, $a_3^2 - a_2^2 = 4, \cdots, a_n^2 - a_{n-1}^2 = 4$. Adding the above equations and applying $a_1 = 1$ to obtain $a_n^2 = 4n - 3$. Thus $a_n = \sqrt{4n-3}, \quad (n \in N^*)$.
(3) $S_n = b_1 + b_2 + \cdots + b_n = \dfrac{1}{a_1+a_2} + \dfrac{1}{a_2+a_3} + \cdots + \dfrac{1}{a_n+a_{n+1}} = \dfrac{a_2-a_1}{4} + \dfrac{a_3-a_2}{4} + + \dfrac{a_{n+1}-a_n}{4} = \dfrac{a_{n+1}-a_1}{4} = \dfrac{\sqrt{4n+1}-1}{4}, \quad (n \in N^*)$.

5.44 ★★★ For the arithmetic sequence $\{a_n\}$, $a_1 = 1$, the common difference is d, the first nth partial sum is A_n. For the geometric sequence $\{b_n\}$, $b_1 = 1$, the common ratio is q ($|q| < 1$), the first nth partial sum is B_n. Let $S_n = B_1 + B_2 + \cdots + B_n$. If $\lim\limits_{n\to\infty}(\dfrac{A_n}{n} - S_n)$=1, evaluate d and q.

Solution: From the given condition, we have $A_n = n + \frac{n(n-1)}{2}d$, $B_n = \frac{1(1-q^n)}{1-q}$, $S_n = \frac{(1-q)+(1-q^2)+\cdots+(1-q^n)}{1-q} = \frac{n - \frac{q(1-q^n)}{1-q}}{1-q} = \frac{(1-q)n - q(1-q^n)}{(1-q)^2}$. Since $\lim\limits_{n\to\infty}(\frac{A_n}{n} - S_n) = 1$, then $\lim\limits_{n\to\infty}[1 + \frac{n-1}{2}d - \frac{(1-q)n - q(1-q^n)}{(1-q)^2}] = 1 \overset{\lim\limits_{n\to\infty} q^n=0}{\Rightarrow}$ $\lim\limits_{n\to\infty}[\frac{n-1}{2}d - \frac{(1-q)n - q}{(1-q)^2}]] = 0 \Rightarrow \lim\limits_{n\to\infty}[(\frac{d}{2} - \frac{1}{1-q})n - \frac{d}{2} + \frac{q}{(1-q)^2}] = 0 \Rightarrow$

$$\begin{cases} \frac{d}{2} - \frac{1}{1-q} = 0 \\ -\frac{d}{2} + \frac{q}{(1-q)^2} = 0 \end{cases}$$

$\Rightarrow$

$$\begin{cases} d = 4 \\ q = \frac{1}{2} \end{cases}$$

5.45 ★★ If the product of the first 3th terms of an increasing geometric sequence $\{a_n\}$ is 512, and subtracting 1,3,9 from these three terms respectively form an arithmetic sequence. Show $\frac{1}{a_1} + \frac{2}{a_2} + \frac{3}{a_3} + \cdots + \frac{n}{a_n} < 1$.

Proof: Let the first 3th terms of the increasing geometric sequence be $\frac{a}{q}, a, aq$. From the given condition, we have $a^3 = 512$, then $a = 8$. Similarly from the given condition, we have $(\frac{a}{q} - 1) + (aq - 9) = 2(a - 3)$, then $(\frac{8}{q} - 1) + (8q - 9) = 10$. Solving the equation, we have $q = \frac{1}{2}$ or $q = 2$. Since $\{a_n\}$ is an increasing sequence, then $q = 2$, $a_1 = \frac{8}{2} = 4$, $a_n = 4 \times 2^{n-1} = 2^{n+1}$. Let $S = \frac{1}{a_1} + \frac{2}{a_2} + \frac{3}{a_3} + \cdots + \frac{n}{a_n} = \frac{1}{4} + \frac{2}{8} + \frac{3}{16} + \cdots + \frac{n}{2^{n+1}}$, ①, then $\frac{1}{2}S = \frac{1}{8} + \frac{2}{16} + \frac{3}{32} + \cdots + \frac{n}{2^{n+2}}$, ②. (① − ②) × 2 leads to $S = \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^n} - \frac{n}{2^{n+1}} = 1 - \frac{1}{2^n} - \frac{n}{2^{n+1}} < 1$.

5.46 ★★ Let the common difference of arithmetic sequence $\{a_n\}$ is nonzero, $\{b_n\}$ is a geometric sequence. If $a_1 = 3$, $b_1 = 1$, $a_2 = b_2$, $3a_5 = b_3$. For an arbitrary positive number n, there are constants α and β such that $a_n = \log_\alpha b_n + \beta$ always holds. Evaluate $\alpha + \beta$.

Solution: Let the common difference of $\{a_n\}$ is d, the common ratio of $\{b_n\}$ is q. Since $a_1 = 3$, $b_1 = 1$, $a_2 = b_2$, $3a_5 = b_3$, then

$$\begin{cases} 3 + d = q \\ 3(3 + 4d) = q^2 \end{cases}$$

Thus

$$\begin{cases} d = 6 \\ q = 9 \end{cases}$$

Hence $a_n = 3 + (n - 1)6 = 6n - 3$, $b_n = 9^{n-1}$. Then $6n - 3 = \log_\alpha 9^{n-1} + \beta = n \log_\alpha 9 - \log_\alpha 9 + \beta$ always holds for an arbitrary positive number n. Thus $\log_\alpha 9 = 6$, $\beta - \log_\alpha 9 = -3 \Rightarrow \alpha^6 = 9 = 3^2$. Since $\alpha > 0$, then $\alpha = 3^{\frac{2}{6}} = 3^{\frac{1}{3}} = \sqrt[3]{3}$, $\beta = -3 + \frac{\log_3 9}{\log_3 \alpha} = -3 + \frac{2}{\frac{1}{3}} = 3$. Therefore $\alpha + \beta = \sqrt[3]{3} + 3$.

5.47 ★★ The third-order arithmetic sequence $\{a_n\}$ is 1,2,8,22,47,$\cdots$, find the value of a_{10}.

Solution: Since $\{a_n\}$ is a third-order arithmetic sequence, then let $a_n = An^3 + Bn^2 + Cn + D$ where A, B, C, D are undetermined coefficients. From the given condition, we have the following relationships: When $n = 1$, $A + B + C + D = 1$ ①; When $n = 2$, $8A + 4B + 2C + D = 2$ ②; When $n = 3$, $27A + 9B + 3C + D = 8$ ③; When $n = 4$, $64A + 16B + 4C + D = 22$ ④. According to the ①, ②, ③ and ④, we have $A = \frac{1}{2}$, $B = -\frac{1}{2}$, $C = -1$, $D = 2$. Thus the general term of $\{a_n\}$ is $a_n = \frac{1}{2}n^3 - \frac{1}{2}n^2 - n + 2$. Therefore $a_{10} = \frac{1}{2} \times 10^3 - \frac{1}{2} \times 10^2 - 10 + 2 = 442$.

5.48 ★ For the sequence $\{a_n\}$, $f_n(x) = a_1x + a_2x^2 + \cdots + a_nx^n$, and $a_1 = 3$, $f_n(1) = p(1 - \frac{1}{2^n})$. (1) Evaluate p. (2) Find the general term of $\{a_n\}$. (3) Find the minimum value of positive integer n such that $3f_n(2) \geqslant 2005f_n(1)$.

Solution: (1) From the given condition, we have $f_n(1) = a_1 + a_2 + \cdots + a_n = p(1 - \frac{1}{2^n})$. Since $a_1 = 3$, then $\frac{p}{2} = 3$. Thus $p = 6$.
(2) when $n \geqslant 2$, $a_n = 6(1 - \frac{1}{2^n}) - 6(1 - \frac{1}{2^{n-1}}) = 3(\frac{1}{2})^{n-1}$, $\quad (n \in N^*)$.
(3) Since $f_n(2) = 2a_1 + 4a_2 + \cdots + 2^na_n = 2 \times 3 + 4 \times \frac{3}{2} + \cdots + 2^n\frac{3}{2^{n-1}} = 6n$ and $3f_n(2) \geqslant 2005f_n(1)$, then $3 \times 6n \geqslant 2005 \times 6(1 - \frac{1}{2^n})$. Then $3n \geqslant 2005(1 - \frac{1}{2^n})$. When $n \leqslant 10$, it does not hold. When$n > 10$, $\frac{2005}{2^n} < 1$. Then $3n > 2004$ which means $n > 668$. Since $n \in N^*$, then the minimum value of n is 669.

5.49 ★★★ Given $a_0 = a_1 = 1$, and $a_0a_n + a_1a_{n-1} + \cdots + a_na_0 = 2^na_n$. Show $a_n = \frac{1}{n!}$ holds for all $n \in N^*$.

Proof: (1) When $n = 0$ or 1, $a_0 = a_1 = \frac{1}{1!} = 1$. $p(0)$ and $p(1)$ hold. (2) Assume when $n = k$, $p(k)$ holds which means $a_k = \frac{1}{k!}$. Applying the recurrence relation, we have $a_{k+1} + \frac{1}{1!k!} + \frac{1}{2!(k-1)!} + \cdots + \frac{1}{k!1!} + a_{k+1} = 2^{k+1}a_{k+1}$ when $n = k + 1$. Then $(2^{k+1} - 2)a_{k+1} = \frac{1}{(k+1)!}(C_{k+1}^1 + C_{k+1}^2 + \cdots + C_{k+1}^k) = \frac{1}{(k+1)!}(2^{k+1} - 2)$. Thus $a_{k+1} = \frac{1}{(k+1)!}$. When $n = k+1$, $p(k+1)$ also holds. Therefore for all integers $n \geqslant 0$, $a_n = \frac{1}{n!}$ holds.

5.50 ★★ Let A, B, C be the three interior angles of $\triangle ABC$. $\lg A, \lg B, \lg C$ form an arithmetic sequence. Find the range of B.

Solution: From the given condition, we have $\lg A + \lg C = 2\lg B$, then $B^2 = AC$. Thus $C > B > A$ and $B < \frac{\pi}{2}$. Hence $[\pi - (A + C)]^2 = AC \leqslant (\frac{A+C}{2})^2$. Since $\frac{A+C}{2} \leqslant B < \frac{\pi}{2} \Rightarrow \pi - (A + C) \geqslant \frac{A+C}{2} \Rightarrow A + C \leqslant \frac{2\pi}{3} \Rightarrow B \geqslant \pi - \frac{2\pi}{3} = \frac{\pi}{3}$. Therefore $\frac{\pi}{3} \leqslant B < \frac{\pi}{2}$.

5.51 ★★★ For $\{a_n\}$, $a_1 = 1$, $8a_{n+1}a_n - 16a_{n+1} + 2a_n + 5 = 0$, $(n \in N^*)$. And $b_n = \frac{1}{a_n - \frac{1}{2}}$, $(n \in N^*)$.
(1) Find the value of b_1, b_2, b_3, b_4. (2) Find the general term of $\{b_n\}$, and the nth partial sum of $\{a_n b_n\}$, denoted as $\{S_n\}$.

Solution: (1) Since $b_n = \frac{1}{a_n - \frac{1}{2}}$, then $a_n = \frac{1}{b_n} + \frac{1}{2}$. Substituting it into the equation $8a_{n+1}a_n - 16a_{n+1} + 2a_n + 5 = 0$, we have $\frac{4}{b_{n+1}b_n} - \frac{6}{b_{n+1}} + \frac{3}{b_n} = 0$. It means $b_{n+1} = 2b_n - \frac{4}{3}$. Since $a_1 = 1$, then $1 = \frac{1}{b_1} + \frac{1}{2}$. Thus $b_1 = 2$. Hence $b_2 = 2b_1 - \frac{4}{3} = \frac{8}{3}$, $b_3 = 2b_2 - \frac{4}{3} = 4$, $b_4 = 2b_3 - \frac{4}{3} = \frac{20}{3}$.
(2) From (1), we have $b_{n+1} = 2b_n - \frac{4}{3}$. Then $b_{n+1} - \frac{4}{3} = 2(b_n - \frac{4}{3})$. Since $b_1 - \frac{4}{3} = \frac{2}{3} \neq 0$, then $\{b_n - \frac{4}{3}\}$ is a geometric sequence and its first term is $\frac{2}{3}$, the common ratio q is 2. Hence $b_n - \frac{4}{3} = \frac{2}{3}2^{n-1} = \frac{1}{3}2^n$. Thus $b_n = \frac{1}{3}2^n + \frac{4}{3}$, $(n \in N^*)$. Since $b_n = \frac{1}{a_n - \frac{1}{2}}$, then $a_n b_n = \frac{1}{2}b_n + 1$. Therefore $S_n = a_1b_1 + a_2b_2 + \cdots + a_nb_n = \frac{1}{2}(b_1 + b_2 + +b_n) + n = \frac{1}{2}[(b_1 - \frac{4}{3}) + (b_2 - \frac{4}{3}) + \cdots + (b_n - \frac{4}{3})] + \frac{5}{3}n = \frac{\frac{2}{3}(1 - 2^n)}{2(1-2)} + \frac{5}{3}n = \frac{1}{3}(2^n + 5n - 1)$, $(n \in N^*)$.

5.52 ★★★ If the increasing sequence $\{a_n\}$ satisfies $a_{n+2} = a_{n+1} + a_n$ when $n \geqslant 1$, $a_7 = 120$. Find the value of a_8.

Solution: Let $a_1 = x$, $a_2 = y$, $x, y \in N^*$. From the given condition, we have $x < y$, and $a_3 = x + y$, $a_4 = x + 2y$, $a_5 = 2x + 3y$, $a_6 = 3x + 5y$, $a_7 = 5x + 8y$, $a_8 = 8x + 13y$. Since $a_7 = 120$, then $5x + 8y = 120$. We have

$$\begin{cases} x = 8t \\ y = 15 - 5t \end{cases}$$

where t is a integer number. Since $y > x > 0$, then $15t - 5t > 8t > 0$. Thus $0 < t < \frac{15}{13}$. Then $t = 1$. Hence $x = 8, y = 10$. Therefore $a_8 = 8 \times 8 + 13 \times 10 = 194$.

5.53 ★★ If the nth partial sum of an arithmetic sequence $\{a_n\}$ with positive common difference is S_n, $a_3a_4 = 117$, $a_2 + a_5 = 22$. (1) Find the general term a_n. (2) If the arithmetic sequence $\{b_n\}$ satisfies $b_n = \frac{S_n}{n+c}$, evaluate the nonzero constant c. (3) Calculate the maximum value of $f(n) = \frac{b_n}{(n+36)b_{n+1}} \quad (n \in N^*)$.

Solution: (1) Since $\{a_n\}$ is an arithmetic sequence, then $a_3 + a_4 = a_2 + a_5 = 22$. On the other hand, $a_3a_4 = 117$, then a_3, a_4 are the two roots of the equation $x^2 - 22x + 117 = 0$. Since the common difference $d > 0$, then $a_3 < a_4$. Solving the equation to obtain $a_3 = 9, a_4 = 13$. Then

$$\begin{cases} a_1 + 2d = 9 \\ a_1 + 3d = 13 \end{cases}$$

$\Rightarrow$

$$\begin{cases} a_1 = 1 \\ d = 4 \end{cases}$$

Thus $a_n = 1 + (n-1)4 = 4n - 3 \quad (n \in N^*)$.
(2) From (1), we have $S_n = n \times 1 + \frac{n(n-1)}{2}4 = 2n^2 - n$. Then $b_n = \frac{S_n}{n+c} = \frac{2n^2 - n}{n+c} \quad (n \in N^*) \Rightarrow b_1 = \frac{1}{1+c}, b_2 = \frac{6}{2+c}, b_3 = \frac{15}{3+c}$. Since $\{b_n\}$ is an arithmetic sequence, then $2b_2 = b_1 + b_3$. Thus $\frac{12}{2+c} = \frac{1}{1+c} + \frac{15}{3+c} \Rightarrow 2c^2 + c = 0$. Therefore $c = -\frac{1}{2}$ or $c = 0$. Since c is nonzero, then $c = -\frac{1}{2}$.
(3) From (2), we have $b_n = \frac{2n^2 - n}{n - \frac{1}{2}} = 2n$, then $f(n) = \frac{2n}{2(n+1)(n+36)} = \frac{n}{n^2 + 37n + 36} = \frac{1}{n + \frac{36}{n} + 37} \leqslant \frac{1}{2\sqrt{n\frac{36}{n}} + 37} = \frac{1}{49}$. $f(n)_{max} = \frac{1}{49}$ when $n = 6$. Thus the maximum value of $f(n)$ is $\frac{1}{49}$.

5.54 ★★★ Given the function $f(x) = \dfrac{1}{\sqrt{x^2-4}}$, $(x < -2)$.

(1) Let $a_1 = 1$, $\dfrac{1}{a_{n+1}} = -f^{-1}(a_n)$, $(n \in N^*)$. Evaluate a_n.

(2) Let $S_n = a_1^2 + a_2^2 + \cdots + a_n^2$, $b_n = S_{n+1} - S_n$. Determine whether there exists the minimum value of positive integer m such that $b_n < \dfrac{m}{25}$ holds for $n \in N^*$. If yes, find the value of m. Otherwise, explain the reason.

Solution: (1) Since $y = f(x) = \dfrac{1}{\sqrt{x^2-4}}, (x < -2)$, then $x = -\sqrt{4+\dfrac{1}{y^2}}$. Thus $f^{-1}(x) = -\sqrt{4+\dfrac{1}{x^2}}, (x > 0)$. Since $\dfrac{1}{a_{n+1}} = -f^{-1}(a_n) = \sqrt{4+\dfrac{1}{a_n^2}} \Rightarrow \dfrac{1}{a_{n+1}^2} - \dfrac{1}{a_n^2} = 4$. Hence $\{\dfrac{1}{a_n^2}\}$ is an arithmetic sequence and $a_1 = 1$, the common difference d is 4. Then $\dfrac{1}{a_n^2} = \dfrac{1}{a_1^2} + 4(n-1) = 4n-3$. Since $a_n > 0$, then $a_n = \dfrac{1}{\sqrt{4n-3}}, (n \in N^*)$.

(2) From the given condition and (1), we have $b_n = S_{n+1} - S_n = a_{n+1}^2 = \dfrac{1}{4(n+1)-3} = \dfrac{1}{4n+1}$. Since $b_n < \dfrac{m}{25}$, then $\dfrac{1}{4n+1} < \dfrac{m}{25}$. Thus $m > \dfrac{25}{4n+1}$ holds for $n \in N^*$. $\dfrac{25}{4n+1} \leqslant 5 \Rightarrow m > 5$. Therefore there exists the minimum value of positive integer $m = 6$ such that $b_n < \dfrac{m}{25}$ holds for $n \in N^*$.

5.55 ★★★ The nth partial sum of a geometric sequence $\{a_n\}$ is S, the product is p, the sum of the reciprocal of every term is T. Show $p^2 = (\dfrac{S}{T})^n$, $(n \in N^*)$.

Proof: (1) When the common ratio $q = 1$, then $S = na_1, T = \dfrac{n}{a_1}, p = a_1^n, p^2 = a_1^{2n} \Rightarrow (\dfrac{S}{T})^n = (a_1^2)^n = a_1^{2n}$. That means $p^2 = (\dfrac{S}{T})^n$ holds.

(2) when the common ratio $q \neq 1$, then $S = \dfrac{a_1(1-q^n)}{1-q}, T = \dfrac{\frac{1}{a_1}(1-\frac{1}{q^n})}{1-\frac{1}{q}} = \dfrac{q^n-1}{a_1q^{n-1}(q-1)}, p = a_1^n q^{1+2+\cdots+(n-1)} = a_1^n q^{\frac{n(n-1)}{2}}, p^2 = a_1^{2n} q^{n(n-1)}$, then $(\dfrac{S}{T})^n = [\dfrac{a_1(1-q^n)}{1-q} \dfrac{a_1q^{n-1}(q-1)}{q^n-1}]^n = a_1^{2n} q^{n(n-1)} = p^2$. As a conclusion, $p^2 = (\dfrac{S}{T})^n, (n \in N^*)$ holds.

5.56 ★★★★ There are n numbers which form a sequence. Their numerators form an arithmetic sequence, the first term is a, the common difference is d. The denominators form a geometric sequence, the first term is b, the common ratio is q. Show the first nth sum S_n satisfies $S_n = \dfrac{(a-aq-d)(1-q^n)+nd(1-q)}{bq^{n-1}(1-q)^2}$.

Proof: From the given condition, we have $S_n = \frac{a}{b} + \frac{a+d}{bq} + \frac{a+2d}{bq^2} + \cdots + \frac{a+(n-1)d}{bq^{n-1}} = \frac{1}{bq^{n-1}}\{(1+q+q^2+\cdots+q^{n-1})a + [q^{n-2}+2q^{n-3}+3q^{n-4}+\cdots+(n-3)q^2+(n-2)q+(n-1)]d\} \cdots (*)$. For the above equation, we find that the coefficient for a is $\frac{1-q^n}{1-q}$ ①.
The coefficient for d can be solved by the following method:
Let $A = q^{n-2} + 2q^{n-3} + 3q^{n-4} + \cdots + (n-3)q^2 + (n-2)q + (n-1)$. Dividing the both sides by q to obtain $\frac{A}{q} = q^{n-3} + 2q^{n-4} + \cdots + (n-3)q + (n-2) + \frac{(n-1)}{q}$. Subtracting the above two equations, we have $A - \frac{A}{q} = q^{n-2} + q^{n-3} + q^{n-4} + \cdots + q + 1 - \frac{n-1}{q}$.

Then $A(1-\frac{1}{q}) = \frac{1-q^{n-1}}{1-q} - \frac{n-1}{q} \Rightarrow A = \frac{q}{q-1}\frac{(1-q^{n-1})q-(n-1)(1-q)}{(1-q)q} = \frac{n(1-q)-(1-q^n)}{(1-q)^2}$ ②. Substituting ① and ② into $(*)$, we have $S_n = \frac{1}{bq^{n-1}}[\frac{1-q^n}{1-q}a + \frac{n(1-q)-(1-q^n)}{(1-q)^2}d] = \frac{(1-q^n)(1-q)a + [n(1-q)-(1-q^n)]d}{bq^{n-1}(1-q)^2}$
$= \frac{(a-aq-d)(1-q^n)+nd(1-q)}{bq^{n-1}(1-q)^2}$.

5.57 ★★★★ We put $n^2 (n \geqslant 4)$ positive numbers as n rows and n columns:

$$\begin{array}{ccccc} a_{11} & a_{12} & a_{13} & a_{14} & \cdots a_{1n} \\ a_{21} & a_{22} & a_{23} & a_{24} & \cdots a_{2n} \\ a_{31} & a_{32} & a_{33} & a_{34} & \cdots a_{3n} \\ a_{41} & a_{42} & a_{43} & a_{44} & \cdots a_{4n} \\ & & & \cdots & \cdots \\ a_{n1} & a_{n2} & a_{n3} & a_{n4} & \cdots a_{nn} \end{array}$$

where the numbers in each row form an arithmetic sequence and the numbers in each column form a geometric sequence which common ratios are all equal. Given $a_{24} = 1$, $a_{42} = \frac{1}{8}$, $a_{43} = \frac{3}{16}$. Evaluate $a_{11} + a_{22} + a_{33} + a_{44} + \cdots + a_{nn}$.

Solution: Let the common difference of the sequence from the first row is d, the common ratio of the sequences from columns is q. then the common difference of the 4th term is dq^3. Then

$$\begin{cases} a_{24} = (a_{11} + 3d)q = 1 \\ a_{42} = (a_{11} + d)q^3 = \dfrac{1}{8} \\ a_{43} = \dfrac{1}{8} + dq^3 = \dfrac{3}{16} \end{cases}$$

Solving the equations system, we have $a_{11} = d = q = \pm\frac{1}{2}$. Since these n^2 numbers are all positive, then $a_{11} = d = q = \frac{1}{2}$. For any $1 \leqslant k \leqslant n$, $a_{kk} = a_{1k}q^{k-1} = [a_{11} + (k-1)d]q^{k-1} = k\frac{1}{2^k}$, $S = a_{11} + a_{22} + a_{33} + \cdots + a_{nn} = \frac{1}{2} + 2\frac{1}{2^2} + 3\frac{1}{2^3} + \cdots + n\frac{1}{2^n}$ Since $\frac{1}{2}S = \frac{1}{2^2} + 2 \cdot \frac{1}{2^3} + 3 \cdot \frac{1}{2^4} + \cdots + n \cdot \frac{1}{2^{n+1}}$. Subtracting these two equations, we have $\frac{1}{2}S = \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \cdots + \frac{1}{2^n} - n\frac{1}{2^{n+1}} = \frac{\frac{1}{2}(1 - \frac{1}{2^n})}{1 - \frac{1}{2}} - n\frac{1}{2^{n+1}}$. Therefore $S = 2 - \frac{1}{2^{n-1}} - \frac{n}{2^n}, (n \geqslant 4)$.

5.58 ★★★★ Let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ satisfy $b_n = a_n - a_{n+2}$, $c_n = a_n + 2a_{n+1} + 3a_{n+2}$, $(n \in N^*)$. Show that $\{a_n\}$ is an arithmetic sequence if and only if $\{c_n\}$ is an arithmetic sequence and $b_n \leqslant b_{n+1} \quad (n \in N^*)$.
Proof: "$\Rightarrow$": Let the common difference of the arithmetic sequence $\{a_n\}$ be d_1, then $b_{n+1} - b_n = (a_{n+1} - a_{n+3}) - (a_n - a_{n+2}) = (a_{n+1} - a_n) - (a_{n+3} - a_{n+2}) = d_1 - d_1 = 0$. Thus $b_n \leqslant b_{n+1}, (n \in N^*)$. On the other hand, $c_{n+1} - c_n = (a_{n+1} - a_n) + 2(a_{n+2} - a_{n+1}) + 3(a_{n+3} - a_{n+2}) = d_1 + 2d_1 + 3d_1 = 6d_1$ (constant), then $\{c_n\}$ is an arithmetic sequence.

"$\Leftarrow$": Let the common difference of the arithmetic sequence $\{c_n\}$ is d_2, and $b_n \leqslant b_{n+1}$, $(n \in N^*)$. Since $c_n = a_n + 2a_{n+1} + 3a_{n+2}$ ①, then $c_{n+2} = a_{n+2} + 2a_{n+3} + 3a_{n+4}$ ②. Using ① − ②, we have $c_n - c_{n+2} = (a_n - a_{n+2}) + 2(a_{n+1} - a_{n+3}) + 3(a_{n+2} - a_{n+4}) = b_n + 2b_{n+1} + 3b_{n+2}$. On the other hand, $c_n - c_{n+2} = (c_n - c_{n+1}) + (c_{n+1} - c_{n+2}) = -2d_2$, then $b_n + 2b_{n+1} + 3b_{n+2} = -2d_2$ ③, and $b_{n+1} + 2b_{n+2} + 3b_{n+3} = -2d_2$ ④. Using ④−③, we have $(b_{n+1} - b_n) + 2(b_{n+2} - b_{n+1}) + 3(b_{n+3} - b_{n+2}) = 0$ ⑤. Since $b_{n+1} - b_n \geqslant 0$, $b_{n+2} - b_{n+1} \geqslant 0$, $b_{n+3} - b_{n+2} \geqslant 0$, we have $b_{n+1} - b_n = 0, (n \in N^*)$ by ⑤. Assume $b_n = d_3$, then $a_n - a_{n+2} = d_3$ (constant). Then $c_n = a_n + 2a_{n+1} + 3a_{n+2} = 4a_n + 2a_{n+1} - 3d_3$, $c_{n+1} = 4a_{n+1} + 2a_{n+2} - 3d_3 = 4a_{n+1} + 2(a_n - d_3) - 3d_3 = 4a_{n+1} + 2a_n - 5d_3$. Subtracting the above two equations, we have $c_{n+1} - c_n = 2(a_{n+1} - a_n) - 2d_3$ which means $a_{n+1} - a_n = \frac{1}{2}(c_{n+1} - c_n) + d_3 = \frac{1}{2}d_2 + d_3$ (constant) $\quad (n \in N^*)$.
Therefore $\{a_n\}$ is an arithmetic sequence.

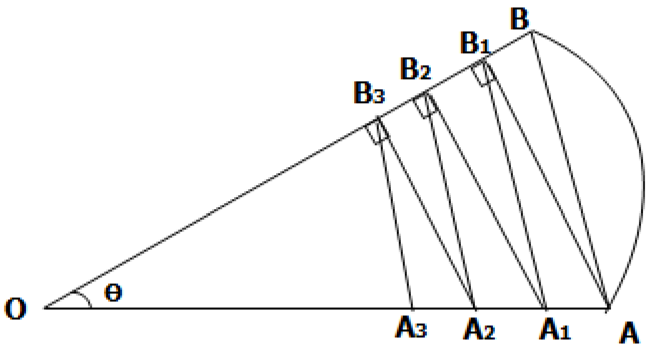


Figure 1

5.59 ★★★★ See Figure 1, let the radius of sector AOB be R, $\angle AOB = \theta(0 < \theta < \frac{\pi}{2})$. AB_1 is perpendicular to OB, B_1A_1 is parallel to AB, A_1B_2 is perpendicular to OB, B_2A_2 is parallel to AB, and keep going, then we obtain two sequences of points $\{A_n\}$ and $\{B_n\}$ on OA an OB. Let the areas of $\triangle ABB_1$, $\triangle A_1B_1B_2$,$\cdots$,$\triangle A_nB_nB_{n+1}\cdots$ be S_1, S_2,$\cdots$,S_{n+1},$\cdots$. Evaluate the sum S of all these areas.

Solution: From the given condition, we have $\angle ABO = \frac{\pi - \theta}{2} = \frac{\pi}{2} - \frac{\theta}{2}$. Considering the line perpendicular to AB through the point O, we have $AB = 2R\cos\angle ABO = 2R\cos(\frac{\pi}{2} - \frac{\theta}{2}) = 2R\sin\frac{\theta}{2}$.
In right triangle $\triangle AB_1B$, $BB_1 = AB\cos\angle ABO = 2R\sin\frac{\theta}{2}\cos(\frac{\pi}{2} - \frac{\theta}{2}) = 2R\sin^2\frac{\theta}{2} = (1 - \cos\theta)R$, $AB_1 = 2R\sin\frac{\theta}{2}\sin(\frac{\pi}{2} - \frac{\theta}{2}) = R\sin\theta$, $OB_1 = R - BB_1 = R\cos\theta$, $OB_2 = R\cos^2\theta$, $OB_3 = R\cos^3\theta$,$\cdots$. Thus $S_{\triangle ABB_1} = \frac{1}{2}BB_1 \cdot AB_1 = \frac{1}{2}(1 - \cos\theta)R \cdot R\sin\theta = \frac{1}{2}(1 - \cos\theta)R^2\sin\theta = S_1$. Since $\triangle ABB_1 \backsim \triangle A_1B_1B_2 \backsim \triangle A_2B_2B_3 \cdots \backsim \triangle A_nB_nB_{n+1}$.

On the other hand, $AB||A_1B_1||A_2B_2$, we have $q = \dfrac{S_2}{S_1} = \cdots = \dfrac{A_1B_1^2}{AB^2} = \dfrac{OB_1^2}{OB^2} = \dfrac{R^2\cos^2\theta}{R^2} = \cos^2\theta$. Hence $S = \lim\limits_{n\to\infty}\sum\limits_{k=1}^{n} S_k = \dfrac{\frac{1}{2}(1-\cos\theta)R^2\sin\theta}{1-\cos^2\theta} = \dfrac{1}{2}R^2\dfrac{\sin\theta}{1+\cos\theta} = \dfrac{1}{2}R^2\cdot\dfrac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}} = \dfrac{1}{2}R^2\tan\dfrac{\theta}{2}$.

5.60 ★★★★ Given $a > 0$ and $a \neq 1$. The sequence $\{a_n\}$ is a geometric sequence. The first term is a, the common ratio is also a. If $b_n = a_n \lg a_n \quad (n \in N^*)$. Does there exist a such that every term of $\{b_n\}$ is less than its next term? If yes, find the range of a. If no, please explain the reason.

Solution: Assume there exists a real number a such that $b_n < b_{n+1}$ for all $n \in N^*$. From the given condition, we have $a_n = a\cdot a^{n-1} = a^n$, then $b_n = a_n \lg a_n = a^n \lg a^n = na^n \lg a$. Thus $na^n \lg a < (n+1)a^{n+1}\lg a$ for all $n \in N^*$.

(1) When $a > 1$, since $\lg a > 0$, then $n < (n+1)a$ for all $n \in N^*$ which means $a > \dfrac{n}{n+1}$ for all $n \in N^*$. Thus $\dfrac{n}{n+1} < 1 < a$ always holds. Therefore $b_n < b_{n+1}$ always holds for $a > 1$ and $n \in N^*$.

(2) When $0 < a < 1$, since $\lg a < 0$, then $n > (n+1)a$ for all $n \in N^*$ which means $a < \frac{n}{n+1}$ for all $n \in N^*$. Since $\frac{n}{n+1} = 1 - \frac{1}{n+1}$ is increasing when n is increasing. Thus the minimum value of $\frac{n}{n+1}$ is $\frac{1}{2}$ when $n = 1$. That means that $\frac{n}{n+1} \geqslant \frac{1}{2}$ always holds. Thus $a < \frac{n}{n+1}$ always holds when $0 < a < \frac{1}{2}$. Therefore $b_n < b_{n+1}$ always holds when $0 < a < \frac{1}{2}$ and $n \in N^*$.

According to (1) and (2), the range of a is $0 < a < \frac{1}{2}$ or $a > 1$.

5.61 ★★★★ Given a sequence $\{a_n\}$ with $a_1 = 0$, $a_n = \frac{1}{4}(a_{n-1}+3) \quad (n = 2, 3, \cdots)$. Find the general term a_n.

Solution: From the given condition $a_n = \frac{1}{4}(a_{n-1}+3)$, we have $a_{n-1} = \frac{1}{4}(a_{n-2}+3)$. Then $a_n - a_{n-1} = \frac{1}{4}(a_{n-1} - a_{n-2})$, $a_{n-1} - a_{n-2} = \frac{1}{4}(a_{n-2} - a_{n-3})$,$\cdots$,$a_3 - a_2 = \frac{1}{4}(a_2 - a_1)$. Multiplying all the above equations together to obtain $a_n - a_{n-1} = (\frac{1}{4})^{n-2}(a_2 - a_1) = \frac{3}{4}(\frac{1}{4})^{n-2} \quad (n \geqslant 2)$. Then $a_n - a_{n-1} = \frac{3}{4}(\frac{1}{4})^{n-2}$, $a_{n-1} - a_{n-2} = \frac{3}{4}(\frac{1}{4})^{n-3}$,$\cdots$,$a_2 - a_1 = \frac{3}{4}$. Adding all the above equations to obtain $a_n - a_1 = \frac{3}{4}[1 + \frac{1}{4} + \cdots + (\frac{1}{4})^{n-3} + (\frac{1}{4})^{n-2}]$. Since $a_1 = 0$, then $a_n = \frac{3}{4} \cdot \frac{1-(\frac{1}{4})^{n-1}}{1-\frac{1}{4}} = 1 - (\frac{1}{4})^{n-1} \quad (n \in N^*)$.

5.62 ★★★ Consider a sequence $\{a_n\}$ with $a_1 = a_2 = 1$, $a_3 = 2$, and $a_n a_{n+1} a_{n+2} \neq 1$ an for arbitrary natural number n, and $a_n a_{n+1} a_{n+2} a_{n+3} = a_n + a_{n+1} + a_{n+2} + a_{n+3}$. Evaluate $a_1 + a_2 + \cdots + a_{100}$.

Solution: Since $a_n a_{n+1} a_{n+2} a_{n+3} = a_n + a_{n+1} + a_{n+2} + a_{n+3}$, then $a_1 a_2 a_3 a_4 = a_1 + a_2 + a_3 + a_4$, and we have $a_1 = a_2 = 1$, $a_3 = 2$, thus $a_4 = 4$. From the given condition, we have $a_n a_{n+1} a_{n+2} a_{n+3} = a_n + a_{n+1} + a_{n+2} + a_{n+3}$, $a_{n+1} a_{n+2} a_{n+3} a_{n+4} = a_{n+1} + a_{n+2} + a_{n+3} + a_{n+4}$. Subtracting the second equation from the first equation to obtain $a_{n+1} a_{n+2} a_{n+3}(a_n - a_{n+4}) = a_n - a_{n+4}$ which means $(a_n - a_{n+4})(a_{n+1} a_{n+2} a_{n+3} - 1) = 0$. Since $a_n a_{n+1} a_{n+2} \neq 1$, then $a_{n+4} = a_n$. Therefore $\sum_{i=1}^{100} a_i = \frac{100}{4}(a_1 + a_2 + a_3 + a_4) = 200$.

5.63 ★★★★ If two sequences $\{a_n\}$ and $\{b_n\}$ satisfy $a_1 = 1$, $a_2 = r \quad (r > 0)$, $b_n = a_n a_{n+1}$, and $\{b_n\}$ is a geometric sequence with common ratio $q \quad (q > 0)$. Let $c_n = a_{2n-1} + a_{2n} \quad (n \in N^*)$. (1) Find the general term of $\{c_n\}$. (2) If $d_n = \frac{\lg c_{n+1}}{\lg c_n}$, $r = 2^{19.2} - 1$, $q = \frac{1}{2}$, find the maximum and minimum terms.

Solution: (1) Since $\{b_n\}$ is a geometric sequence with common ratio q, then $\frac{b_{n+1}}{b_n} = q$. Since $b_n = a_n a_{n+1}$, then $\frac{a_{n+1}a_{n+2}}{a_n a_{n+1}} = q$. It means $\frac{a_{n+2}}{a_n} = q, (n \in N^*)$. Thus the sequence a_1, a_3, $a_5, \cdots, a_{2n-1}, \cdots$ and the sequence a_2, a_4, $a_6, \cdots, a_{2n}, \cdots$ are both geometric sequences with common ratio q. Then $a_{2n-1} = a_1 q^{n-1} = q^{n-1}$, $a_{2n} = a_2 q^{n-1} = r \cdot q^{n-1}, (n \in N^*)$. Therefore $c_n = a_{2n-1} + a_{2n} = q^{n-1} + rq^{n-1} = (1+r)q^{n-1}, (n \in N^*)$.

(2) Since $c_n = (1+r)q^{n-1} = (1 + 2^{19.2} - 1)(\frac{1}{2})^{n-1} = 2^{20.2-n}$, then $d_n = \frac{\lg c_{n+1}}{\lg c_n} = \frac{\lg 2^{20.2-(n+1)}}{\lg 2^{20.2-n}} = 1 + \frac{1}{n - 20.2} \quad (n \in N^*)$.
When $n - 20.2 > 0$ which means $n \geqslant 21 \quad (n \in N^*)$, then d_n is decreasing as n is increasing. Thus $1 < d_n \leqslant d_{21} = 1 + \frac{1}{21 - 20.2} = 2.25$ ①.
When $n - 20.2 < 0$ which means $n \leqslant 20 \quad (n \in N^*)$, then d_n is decreasing as n is increasing. Thus $1 > d_n \geqslant d_{20} = 1 + \frac{1}{20 - 20.2} = -4$ ②.
By applying ① and ②, we have $d_{20} \leqslant d_n \leqslant d_{21}, (n \in N^*)$. Therefore the maximum term of $\{d_n\}$ is $d_{21} = 2.25$, and minimum term is $d_{20} = -4$.

5.64 ★★★ Given $c_1 = 1$, $c_2 = 1$, $c_{n+2} = c_{n+1} + c_n$, find the general term c_n.

Solution: From the given condition, we have that the roots of the characteristic equation $x^2 - x - 1 = 0$ are $\frac{1+\sqrt{5}}{2}$ and $\frac{1-\sqrt{5}}{2}$. Assume $c_n = A(\frac{1+\sqrt{5}}{2})^n + B(\frac{1-\sqrt{5}}{2})^n$.
When $n = 1$, then $A\frac{1+\sqrt{5}}{2} + B\frac{1-\sqrt{5}}{2} = 1$.
When $n = 2$, then $A(\frac{1+\sqrt{5}}{2})^2 + B(\frac{1-\sqrt{5}}{2})^2 = 1$.
By solving the above equations, we have $A = \frac{1}{\sqrt{5}}$, $B = -\frac{1}{\sqrt{5}}$. Thus $c_n = \frac{1}{\sqrt{5}}[(\frac{1+\sqrt{5}}{2})^n - (\frac{1-\sqrt{5}}{2})^n]$.

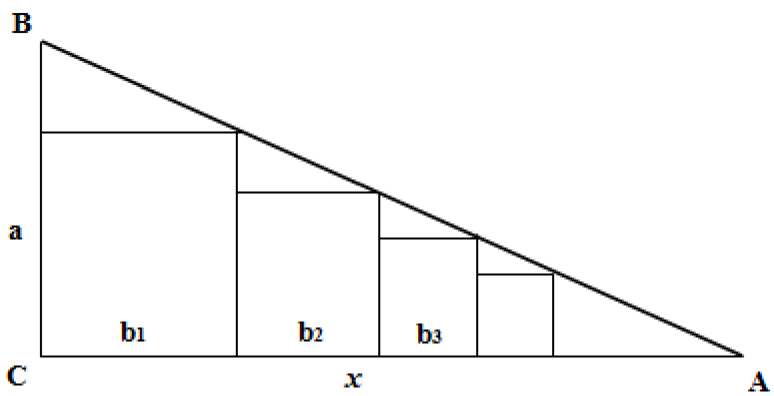


Figure 2

5.65 ★★★★ See Figure 2, in $Rt\triangle ABC$ ($Rt\triangle$ represents right triangle), there are infinitely many squares S_1, S_2, S_3, $S_4, \cdots$, and the leg $BC = a$. The sum of areas of all these squares is half of the area of $Rt\triangle ABC$. Compute the length of the other leg AC.

Solution: Let the side length of the first square is b_1, from left to right, the others are b_2, $b_3, \cdots, b_n, \cdots$. Let $AC = x$.
By applying the similarity of triangles, we have $\frac{a-b_1}{a} = \frac{b_1}{x} \Rightarrow b_1 = \frac{ax}{a+x} = \frac{a}{\frac{a}{x}+1} = \frac{a}{\cot B+1} = \frac{\tan B}{1+\tan B}a$. Similarly, we have $b_2 = \frac{\tan B}{1+\tan B}b_1 = (\frac{\tan B}{1+\tan B})^2 a$, $b_3 = (\frac{\tan B}{1+\tan B})^3 a, \cdots, b_n = (\frac{\tan B}{1+\tan B})^n a$. Then $b_1^2 = (\frac{\tan B}{1+\tan B}a)^2$, $b_2^2 = [(\frac{\tan B}{1+\tan B})^2 a]^2$, $b_3^2 = [(\frac{\tan B}{1+\tan B})^3 a]^2, \cdots$, $b_n^2 = [(\frac{\tan B}{1+\tan B})^n a]^2$. Thus $\{b_n^2\}$ is a geometric sequence with the first term $(\frac{\tan B}{1+\tan B}a)^2$, the common ratio is $(\frac{\tan B}{1+\tan B})^2$. From the given condition, we have $\lim_{n\to\infty}\{b_n^2\} = \frac{1}{4}ax$, which means $\frac{(\frac{\tan B}{1+\tan B})^2 a^2}{1-(\frac{\tan B}{1+\tan B})^2} = \frac{1}{4}ax$. Since $x = a\tan B \Rightarrow \frac{a^2\cdot\tan^2 B}{1+2\tan B} = \frac{1}{4}a^2\tan B$. Then $\tan B = \frac{1}{2}$. Therefore $AC = a\cdot\tan B = \frac{1}{2}a$.

5.66 ★★★★ If the sequence $\{a_n\}$ satisfies $a_1 = 1$, $a_2 = r \quad (r > 0)$. $\{a_n a_{n+1}\}$ is a geometric sequence with the common ratio q $(q > 0)$. Let $b_n = a_{2n-1} + a_{2n} \quad (n \in N^*)$.
(1) Find the range of q such that $a_n a_{n+1} + a_{n+1}a_{n+2} > a_{n+2}a_{n+3}$.
(2) Find b_n and $\lim\limits_{n\to\infty} \frac{1}{S_n}$ where $S_n = b_1 + b_2 + \cdots + b_n$.

Solution: (1) $a_1 a_2 = r$ when $n = 1$. Since $\{a_n a_{n+1}\}$ is a geometric sequence with the common ratio q. Then $a_n a_{n+1} = r \cdot q^{n-1} \quad (n \in N^*)$. $a_2 a_3 = rq$ when $n = 2$. Thus $a_3 = q$. $a_3 a_4 = rq^2$ when $n = 3$. Thus $a_4 = rq$. $a_4 a_5 = rq^3$ when $n = 4$. Thus $a_5 = q^2 \cdots$. Hence the sequence $\{a_n\}$ is $1, r, q, rq, q^2, rq^2, \cdots, q^{n-1}, rq^{n-1}, \cdots$. Since $a_n a_{n+1} + a_{n+1}a_{n+2} > a_{n+2}a_{n+3}$, then $rq^{n-1} + rq^n > rq^{n+1}$. Therefore $0 < q < \frac{1+\sqrt{5}}{2}$.
(2) By applying $b_n = a_{2n-1} + a_{2n} \quad (n \in N^*)$ and (1), we have $b_1 = 1 + r$, $b_2 = (1 + r)q$, $b_3 = (1 + r)q^2$,$\cdots$,$b_n = (1 + r)q^{n-1}$. Thus $S_n = b_1 + b_2 + \cdots + b_n = (1 + r)(1 + q + q^2 + \cdots + q^{n-1}) = (1 + r)\frac{1-q^n}{1-q}$.

$$\lim_{n\to\infty} \frac{1}{S_n} = \lim_{n\to\infty} \frac{1-q}{(1+r)(1-q^n)} = \begin{cases} (1+r)n, & q = 1, n \in N^*; \\ \frac{1-q}{1+r}, & 0 < q < 1; \\ 0, & q > 1. \end{cases}$$

5.67 ★★★ Given the sequence $\{a_n\}$ with $a_1 = 2$, $a_{n+1} = \frac{a_n^2 + 3}{2a_n}$. The sequence $\{b_n\}$ satisfies $b_n = 3 - a_n^2, (n \in N)$. Show (1) $b_n < 0$. (2) $|\frac{b_{n+1}}{b_n}| < \frac{1}{2}$. (3) $|b_n| < (\frac{1}{2})^{n-1}, (n \geqslant 2)$.

Proof: (1) From the given condition, we know that $\{a_n\}$ is a positive sequence, and $a_n = \frac{a_{n-1}^2 + 3}{2a_{n-1}} = \frac{a_{n-1}}{2} + \frac{3}{2a_{n-1}} > 2\sqrt{\frac{a_{n-1}}{2} \cdot \frac{3}{2a_{n-1}}} = \sqrt{3}$. Thus $b_n = 3 - a_n^2 < 3 - (\sqrt{3})^2 = 0$. This means $b_n < 0$.

(2) $|\frac{b_{n+1}}{b_n}| - \frac{1}{2} = |\frac{3 - a_{n+1}^2}{3 - a_n^2}| - \frac{1}{2} = \frac{3 - (\frac{a_n^2+3}{2a_n})^2}{3 - a_n^2} - \frac{1}{2} = -\frac{a_n^2 + 3}{4a_n^2} < 0$. Thus $|\frac{b_{n+1}}{b_n}| < \frac{1}{2}$.
(3) $|b_n| = |b_1| \cdot |\frac{b_2}{b_1}| \cdot |\frac{b_3}{b_2}| \cdots |\frac{b_n}{b_{n-1}}| < |-1| \cdot \frac{1}{2} \cdot \frac{1}{2} \cdots \frac{1}{2} = (\frac{1}{2})^{n-1}, (n \geqslant 2)$.

5.68 ★★★★ Let the first nth partial sum of $\{a_n\}$ be S_n, and $a_1 = 1$, $S_{n+1} = 4a_n + 2, (n \in N^*)$.
(1) Let $b_n = a_{n+1} - 2a_n$, show $\{b_n\}$ is a geometric sequence.
(2) Let $c_n = \frac{a_n}{2^n}$, show $\{c_n\}$ is a geometric sequence.
(3) Calculate $S_n = a_1 + a_2 + \cdots + a_n$.

(1) Proof: Since $a_{n+1} = S_{n+1} - S_n = 4a_n - 4a_{n-1}$, then $a_{n+1} - 2a_n = 2(a_n - 2a_{n-1})$. Since $b_n = a_{n+1} - 2a_n$, then $b_n = 2b_{n-1}, (n \geqslant 2)$. On the other hand, $b_1 = a_2 - 2a_1 = S_2 - 3a_1 = (4a_1 + 2) - 3a_1 = a_1 + 2 = 3$, $q = \frac{b_n}{b_{n-1}} = 2$. Thus $\{b_n\}$ is a geometric sequence with the first term 3 and the common ratio 2.
(2) Proof: By Applying (1), we have $b_n = 3 \cdot 2^{n-1}$, then $a_{n+1} - 2a_n = 3 \cdot 2^{n-1}$. Since $c_n = \frac{a_n}{2^n}$, then $c_{n+1} - c_n = \frac{1}{2^{n+1}}(a_{n+1} - 2a_n) = \frac{1}{2^{n+1}} \cdot 3 \cdot 2^{n-1} = \frac{3}{4}$, and $c_1 = \frac{a_1}{2} = \frac{1}{2}$. Thus $\{c_n\}$ is a geometric sequence with the first term $\frac{1}{2}$, the common ratio is $\frac{3}{4}$.
(3) Solution: By Applying (2), we have $c_n = \frac{1}{2} + (n-1) \cdot \frac{3}{4} = \frac{3}{4}n - \frac{1}{4}$. Then $a_n = 2^n \cdot c_n = 2^{n-2}(3n-1)$. When $n \geqslant 2$, then $S_n = 4a_{n-1} + 2 = 4 \times 2^{n-3}(3n-4) + 2 = 2^{n-1}(3n-4) + 2$.
When $n = 1$, then $S_1 = a_1 = 1$ which also satisfies the above equation. Therefore $S_n = 2^{n-1}(3n-4) + 2$ always holds for all $n \in N^*$.

5.69 ★★★★★ The adjacent terms of $\{a_n\}$, a_n and a_{n+1}, are the roots of the equation $x^2 - c_n x + (\frac{1}{3})^n = 0$, and $a_1 = 2$. Find the sum of infinite sequence $c_1, c_2, \cdots, c_n, \cdots$.

Solution: By applying the Vieta's theorem, we have $c_n = a_n + a_{n+1}$, $a_n \cdot a_{n+1} = (\frac{1}{3})^n$ ①. Then $a_{n+1} \cdot a_{n+2} = (\frac{1}{3})^{n+1}$ ②. Dividing the equation ② by the equation ①, we have $\frac{a_{n+2}}{a_n} = \frac{1}{3} \Rightarrow \sum_{n=1}^{\infty} c_n = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} a_{n+1} = \sum_{k=0}^{\infty} a_{2k+1} + 2\sum_{k=1}^{\infty} a_{2k} + \sum_{k=1}^{\infty} a_{2k+1} = 2\sum_{k=0}^{\infty} a_{2k+1} + 2\sum_{k=1}^{\infty} a_{2k} - a_1$.
Since $a_1 = 2$, $a_1 a_2 = \frac{1}{3}$, then $a_2 = \frac{1}{6}$. By applying $\frac{a_{n+2}}{a_n} = \frac{1}{3}$ and the formula for the sum of infinite decreasing geometric sequence, we have $\sum_{n=1}^{\infty} c_n = 2 \cdot \frac{2}{1-\frac{1}{3}} + 2 \cdot \frac{\frac{1}{6}}{1-\frac{1}{3}} - 2 = \frac{9}{2}$.

5.70 ★★★★ Given $\{a_n\}$ is an arithmetic sequence, $\{b_n\}$ is a geometric sequence, and $a_1 = b_2$, $a_2 = b_2$, $a_1 \neq a_2$, $a_n > 0$, $n = 1, 2, \cdots$. Show $a_n < b_n$ when $n > 2$.

Proof: Let the common difference be d and the common ratio be q. Since $a_2 = a_1 + d$, $b_2 = b_1 q = a_1 q$, $a_2 = b_2$, we have $a_1 + d = a_1 q$. Then $q = 1 + \frac{d}{a_1}$.
On the other hand, $a_n > 0, n = 1, 2, \cdots$, then $d > 0$. Thus $q > 1$. Hence $a_n - b_n = a_1 + (n-1)d - a_1 q^{n-1} = a_1(1-q^{n-1}) + (n-1)d = a_1(1-q)(1+q+\cdots+q^{n-2}) + (n-1)d < a_1(1-q)(n-1) + (n-1)d = (n-1)[a_1(1-q)+d] = (n-1)(a_2 - b_2) = 0$.
Therefore $a_n < b_n$ when $n > 2$.

5.71 ★★★★ For an arbitrary real number sequence $A = (a_1, a_2, \cdots)$, we define ΔA as the sequence $A = (a_2 - a_1, a_3 - a_2, \cdots)$. Its nth term is $a_{n+1} - a_n$. Assume the differences between the adjacent two terms are all 1, and $a_{19} = a_{92} = 0$. Evaluate a_1.

Solution 1: Let $a_{n+1} - a_n = b_n$, then $b_n - b_{n-1} = 1$. Thus $\{b_n\}$ is an arithmetic sequence and its common difference is 1. Hence $b_n = b_1 + n - 1$. This means $a_{n+1} - a_n = b_1 + n - 1$. We obtain $a_n = a_1 + \sum_{k=1}^{n-1}(b_1 + k - 1) = a_1 + (n-1)b_1 + \frac{(n-1)(n-2)}{2}$. Since $b_1 = a_2 - a_1$, then $a_n = (n-1)a_2 - (n-2)a_1 + \frac{(n-1)(n-2)}{2}$.
$a_{19} = 0$ when $n = 19$. Then $18a_2 - 17a_1 = -\frac{18 \times 17}{2}$ ①.
$a_{92} = 0$ when $n = 92$. Then $91a_2 - 90a_1 = -\frac{91 \times 90}{2}$ ②.
Solving the equation ① and the equation ② to generate $a_1 = 819$.
Solution 2: Let the first term of the sequence ΔA be d. Then the sequence ΔA is $(d, d+1, d+2, \cdots)$. Its nth term is $d + (n-1)$. Thus the nth term of sequence A is $a_n = a_1 + \sum_{k=2}^{n-1}(a_{k+1} - a_k) = a_1 + d + (d+1) + \cdots + (d+n-2) = a_1 + (n-1)d + \frac{1}{2}(n-1)(n-2)$.
This shows that a_n is a quadratic polynomial with respect to n, and the coefficient of its first term is $\frac{1}{2}$. Since $a_{19} = a_{92} = 0$, then $a_n = \frac{1}{2}(n-19)(n-92)$. Therefore $a_1 = \frac{1}{2}(1-19)(1-92) = 819$.
Solution 3: From the given condition, we obtain that $\{a_n\}$ is a second order arithmetic sequence. Thus its general term is a quadratic polynomial whit respect to n. Since $a_{19} = a_{92} = 0$, we let $a_n = A(n-19)(n-92)$ where A is an undetermined coefficient. Since $a_3 - 2a_2 + a_1 = 1$, then $A[(3-19)(3-92) - 2(2-19)(2-92) + (1-19)(1-92)] = 1$. By solving the equation, we have $A = \frac{1}{2}$. Therefore $a_1 = \frac{1}{2}(1-19)(1-92) = 819$.

5.72 ★★★ Given sequence $\{a_n\}$ with $a_1 = 1$, $a_1 + a_2 + \cdots + a_n = n^2 a_n, (n \in N^*)$. Find the general term of the sequence $\{a_n\}$ and S_{100}.

Solution: From the given condition $a_1 + a_2 + \cdots + a_n = n^2 a_n$, we have $a_1 + a_2 + \cdots + a_{n-1} = (n-1)^2 a_{n-1}$ by substituting n for $n-1$. Subtracting the second equation from the first one, we have $a_n + (n-1)^2 a_{n-1} = n^2 a_n$. This means $a_n = \frac{n-1}{n+1} a_{n-1} = \frac{n-1}{n+1}\frac{n-2}{n} a_{n-2} = \frac{n-1}{n+1}\frac{n-2}{n}\frac{n-3}{n-1} a_{n-3} = \cdots = \frac{n-1}{n+1}\frac{n-2}{n}\frac{n-3}{n-1}\cdots\frac{1}{3}a_1 = \frac{2}{n(n+1)}$, $(n \in N^*)$.
$S_{100} = a_1 + a_2 + \cdots + a_{100} = 100^2 \frac{2}{100(100+1)} = \frac{200}{101}$.

5.73 ★★★ Given the nth partial sum of sequence $\{a_n\}$ as $S_n = 1 + ka_n$, k is a constant and $k \neq 1$. Find a_n and S_n.

Solution: Since $S_n = 1 + ka_n$, we have $S_1 = a_1 = 1 + ka_1$, then $a_1 = \frac{1}{1-k}$, $a_2 = S_2 - S_1 = 1 + ka_2 - a_1$. This means $(1-k)a_2 = 1 - \frac{1}{1-k} = -\frac{k}{1-k}$. Thus $a_2 = -\frac{k}{(1-k)^2}$. Similarly, $a_3 = \frac{k^2}{(1-k)^3}, \cdots, a_{n-1} = (-1)^{n-2}\frac{k^{n-2}}{(1-k)^{n-1}}$, $a_n = (-1)^{n-1}\frac{k^{n-1}}{(1-k)^n}$. Thus $\frac{a_2}{a_1} = -\frac{k}{(1-k)^2}/\frac{1}{1-k} = -\frac{k}{1-k}$, $\frac{a_3}{a_2} = \frac{k^2}{(1-k)^3}/-\frac{k}{(1-k)^2} = -\frac{k}{1-k}, \cdots, \frac{a_n}{a_{n-1}} = -\frac{k}{1-k}$. Hence $\{a_n\}$ is a geometric sequence with $a_1 = \frac{1}{1-k}$ and $q = -\frac{k}{1-k}$. Therefore $a_n = (-1)^{n-1}\frac{k^{n-1}}{(1-k)^n}, (n \in N^*)$, $S_n = 1 + ka_n = 1 + k(-1)^{n-1}\frac{k^{n-1}}{(1-k)^n} = 1 + (-1)^{n-1}(\frac{k}{1-k})^n, (k \neq 1, n \in N^*)$.

5.74 ★★★★ The three sides of $\triangle ABC$ form an arithmetic sequence, and the difference between the largest angle and the smallest angle is 90^0. Show the ratio of the three sides is $(\sqrt{7}+1) : \sqrt{7} : (\sqrt{7}-1)$.

Proof: Let the side lengths of $\triangle ABC$ be $a-d, a, a+d$, the smallest angle is α, the largest angle is $90^0+\alpha$. From the given condition, we have $\frac{a-d}{\sin\alpha} = \frac{a+d}{\sin(90^0+\alpha)} = \frac{a}{\sin(90^0-2\alpha)} \Rightarrow \frac{a-d+a+d}{\sin\alpha+\sin(90^0+\alpha)} = \frac{a}{\sin(90^0-2\alpha)} \Rightarrow \frac{2a}{\sin\alpha+\cos\alpha} = \frac{a}{\cos 2\alpha} \Rightarrow$ $2\cos 2\alpha = \sin\alpha+\cos\alpha \Rightarrow 2(\cos^2\alpha-\sin^2\alpha) = \sin\alpha+\cos\alpha \Rightarrow \cos\alpha-\sin\alpha = \frac{1}{2} \Rightarrow$ $\sqrt{2}\sin(45^0-\alpha) = \frac{1}{2} \Rightarrow \sin(45^0-\alpha) = \frac{\sqrt{2}}{4}$. Then $\cos(45^0-\alpha) = \sqrt{1-(\frac{\sqrt{2}}{4})^2} = \frac{\sqrt{14}}{4}$.
Thus $\sin\alpha = \sin[45^0-(45^0-\alpha)] = \sin 45^0\cos(45^0-\alpha)-\cos 45^0\sin(45^0-\alpha) = \frac{\sqrt{2}}{2}\frac{\sqrt{14}}{4} - \frac{\sqrt{2}}{2}\frac{\sqrt{2}}{4} = \frac{\sqrt{7}-1}{4}$, $\sin(90^0+\alpha) = \cos\alpha = \cos[45^0-(45^0-\alpha)] = \cos 45^0\cos(45^0-\alpha)+\sin 45^0\sin(45^0-\alpha) = \frac{\sqrt{2}}{2}\frac{\sqrt{14}}{4}+\frac{\sqrt{2}}{2}\frac{\sqrt{2}}{4} = \frac{\sqrt{7}+1}{4}$. $\sin(90^0-2\alpha) = \cos 2\alpha = \cos^2\alpha-\sin^2\alpha = (\frac{\sqrt{7}+1}{4})^2-(\frac{\sqrt{7}-1}{4})^2 = \frac{\sqrt{7}}{4}$.
As a conclusion, the ratio of the three sides is $(\sqrt{7}+1) : \sqrt{7} : (\sqrt{7}-1)$.

5.75 ★★★★★ Given the sequence $\{a_n\}$, $a_1 = 5$, $a_{n+1} = \frac{5a_n+6}{a_n+4}$. Find the general term a_n.

Solution: Let $x = \frac{5x+6}{x+4}$ which means $x^2-x-6=0$, then the two fixed points of $f(x) = \frac{5x+6}{x+4}$ are $x=3$, $x=-2$. Hence $a_{n+1}-3 = \frac{5a_n+6}{a_n+4}-3 = \frac{2(a_n-3)}{a_n+4}$ ①, $a_{n+1}+2 = \frac{5a_n+6}{a_n+4}+2 = \frac{7(a_n+2)}{a_n+4}$ ②. By ① ÷ ②, then $\frac{a_{n+1}-3}{a_{n+1}+2} = \frac{2}{7}(\frac{a_n-3}{a_n+2})$. Hence $\{\frac{a_n-3}{a_n+2}\}$ is a geometric sequence with the first term $\frac{a_1-3}{a_1+2} = \frac{2}{7}$ and the common ratio $\frac{2}{7} \Rightarrow \frac{a_n-3}{a_n+2} = \frac{2}{7}(\frac{2}{7})^{n-1} = (\frac{2}{7})^n$. Therefore $a_n = \frac{3\cdot 7^n+2^{n+1}}{7^n-2^n}$.

5.76 ★★★★ Given sequence $\{a_n\}$, $a_1 = 2$, $a_{n+1} = \frac{a_n}{2}+\frac{1}{a_n}$ $(n \in N^*)$. Show $\sqrt{2} < a_n < \sqrt{2}+\frac{1}{n}$.

Proof:(1) $\sqrt{2} < a_1 = 2 < \sqrt{2}+\frac{1}{1} = \sqrt{2}+1$ when $n=1$. Thus $p(1)$ holds.
(2) Assume $p(k)$ holds when $n=k$. This means $\sqrt{2} < a_k < \sqrt{2}+\frac{1}{k}$.

When $n = k+1$, then $a_{k+1} = \frac{a_k}{2} + \frac{1}{a_k} \geqslant 2\sqrt{\frac{a_k}{2}\frac{1}{a_k}} = \sqrt{2}$, and the equation holds if and only if $\frac{a_k}{2} = \frac{1}{a_k}$ which means $a_k = \sqrt{2}$. Since $a_k > \sqrt{2}$, then equal sign cannot hold for the above formula. Then $a_{k+1} > \sqrt{2}$. Since $a_{k+1} = \frac{a_k}{2} + \frac{1}{a_k} < \frac{\sqrt{2}+\frac{1}{k}}{2} + \frac{1}{\sqrt{2}} = \sqrt{2} + \frac{1}{2k} \leqslant \sqrt{2} + \frac{1}{k+1}$. Hence $p(k+1)$ holds when $n = k+1$.
$\sqrt{2} < a_n < \sqrt{2} + \frac{1}{n}$ always holds for all $n \in N^*$.

5.77 ★★★★ Given sequence $\{a_n\}$, $3a_{n+1} + a_n = 4 \quad (n \geqslant 1)$, $a_1 = 9$, the nth partial sum is S_n and $|S_n - n - 6| < \frac{1}{125}$. Find the smallest positive integer number n.

Solution: By applying the recurrence relation, we have $3(a_{n+1} - 1) = -(a_n - 1)$ where $n = 1$ is a root of equation $3n^2 + n = 4$. Let $b_n = a_n - 1$, $b_{n+1} = a_{n+1} - 1$, then $b_{n+1} = -\frac{1}{3}b_n$, $b_1 = a_1 - 1 = 8$. Hence $\{b_n\}$ is a geometric sequence with the first term 8 and the common ratio $-\frac{1}{3}$. Therefore $b_n = 8(-\frac{1}{3})^{n-1}$. $S_n - n = (a_1 - 1) + (a_2 - 1) + \cdots + (a_n - 1) = b_1 + b_2 + \cdots + b_n = \frac{8 \cdot [1 - (-\frac{1}{3})^n]}{1 - (-\frac{1}{3})} = 6 - 6 \cdot (-\frac{1}{3})^n$. Since $|S_n - n - 6| < \frac{1}{125} \Rightarrow 2 \cdot 3^{1-n} < \frac{1}{125} \Rightarrow 3^{n-1} > 250$. Therefore the smallest positive integer number such that the inequality holds is $n = 7$.

5.78 ★★★★★ Given sequence $\{a_n\}$, $a_1 = 1$, $a_2 = 2$, $a_{n+2} = 4a_{n+1} - 3a_n + 2$, find the general term of $\{a_n\}$.

Solution: By applying the recurrence relation, we have $a_{n+2} - a_{n+1} = 3(a_{n+1} - a_n) + 2$. Let $a_{n+1} - a_n = c_n$, then $c_1 = 1$, $c_{n+1} = 3c_n + 2$, $\frac{c_{n+1}}{3^n} = \frac{c_n}{3^{n-1}} + 2(\frac{1}{3})^n = \frac{c_n}{3^{n-1}} + 2 \times \frac{1}{3} \times (\frac{1}{3})^{n-1}$. Thus $\frac{c_n}{3^{n-1}} = c_1 + \sum_{k=1}^{n-1} 2 \times (\frac{1}{3})^k = 1 + 2 \times (\frac{1}{3})\frac{1 - (\frac{1}{3})^{n-1}}{1 - \frac{1}{3}} = 2 - (\frac{1}{3})^{n-1}$. Then $c_n = 2 \times 3^{n-1} - 1$, $a_{n+1} = a_n + 2 \cdot 3^{n-1} - 1$. Therefore $a_n = a_1 + \sum_{k=1}^{n-1}(2 \cdot 3^{k-1} - 1) = 1 + 2 \times \frac{3^{n-1} - 1}{2} - (n-1) = 3^{n-1} - n + 1 \quad (n \in N^*)$.

5.79 ★★★★ Let the sequence $\{a_n\}$ satisfy $(2-a_n)\cdot a_{n+1}=1, n\geqslant 1$. Show $\lim\limits_{n\to\infty} a_n=1$.

Proof: From the given condition, we have $a_{n+1}=\dfrac{1}{2-a_n}$, then $a_n=\dfrac{1}{2-a_{n-1}}$. Subtracting both sides by 1 to obtain $a_n-1=\dfrac{a_{n-1}-1}{2-a_{n-1}}$. Thus $\dfrac{1}{a_n-1}=-1+\dfrac{1}{a_{n-1}-1}=-2+\dfrac{1}{a_{n-2}-1}=\cdots=-(n-1)+\dfrac{1}{a_1-1}=\dfrac{1-(n-1)(a_1-1)}{a_1-1}$. Hence $a_n=\dfrac{a_1-1}{1-(n-1)(a_1-1)}+1$. Therefore $\lim\limits_{n\to\infty} a_n=\lim\limits_{n\to\infty}[\dfrac{a_1-1}{1-(n-1)(a_1-1)}+1]=1$.

5.80 ★★★★★ If $\{a_n\}$ and $\{b_n\}$ are both positive infinite sequences, $a_1=a$, $b_1=b$, and a_n, b_n, a_{n+1} are arithmetic, b_n, a_{n+1}, b_{n+1} are geometric. (1) Show $\{\sqrt{b_n}\}$ is an arithmetic sequence. (2) Find the general term of $\{b_n\}$. (3) Compare a_n and b_n.

(1) Proof: From the given condition, we have $a_{n+1}^2=b_nb_{n+1}$, $a_n^2=b_{n-1}b_n$. Thus $a_n+a_{n+1}=\sqrt{b_n\cdot b_{n-1}}+\sqrt{b_n\cdot b_{n+1}}=2b_n$. Divide the both sides by $\sqrt{b_n}$, then $\sqrt{b_{n-1}}+\sqrt{b_{n+1}}=2\sqrt{b_n}$. Hence $\{\sqrt{b_n}\}$ is an arithmetic sequence.

(2) Solution: Since $\sqrt{b_1} = \sqrt{b}$, $a_2^2 = b_1 b_2$ and $a_1 + a_2 = 2b_1$, we have $b_2 = \frac{(2b-a)^2}{b}$. On the other hand, since $\{b_n\}$ is a positive sequence and $a_n < b_n$, then $\sqrt{b_2} = \frac{2b-a}{\sqrt{b}}$. Thus $d = \sqrt{b_2} - \sqrt{b_1} = \frac{2b-a}{\sqrt{b}} - \sqrt{b} = \frac{b-a}{\sqrt{b}} > 0$, $\sqrt{b_3} = \sqrt{b_1} + (3-1)\frac{b-a}{\sqrt{b}} = \frac{3b-2a}{\sqrt{b}}$(all terms are positive)$\cdots$, $\sqrt{b_n} = \frac{nb-(n-1)a}{\sqrt{b}}$. Thus $b_n = \frac{[nb-(n-1)a]^2}{b}$.
(3) Solution: Since $\{\sqrt{b_n}\}$ is an increasing sequence and $a_n^2 = b_{n-1}b_n$, then $b_{n-1} < a_n < b_n$ which means $a_n < b_n$.

5.81 ★★★★★ Given sequence $\{a_n\}$, $a_1 > 0$, $a_{n+1} = \sqrt{\frac{3+a_n}{2}}$. (1)Find the range of a_1 such that $a_{n+1} > a_n$ for any positive number n. (2) If $a_1 = 4$, $b_n = |a_{n+1} - a_n| \quad (n \in N^*)$, and let the nth partial sum of $\{b_n\}$ be S_n, show $S_n < \frac{5}{2}$.

(1) Solution: $a_{n+1} - a_n = \sqrt{\frac{3+a_n}{2}} - \sqrt{\frac{3+a_{n-1}}{2}} = \frac{a_n - a_{n-1}}{2(\sqrt{\frac{3+a_n}{2}} + \sqrt{\frac{3+a_{n-1}}{2}})}$ when $n \geqslant 2$. Since the denominator is positive, then $a_{n+1} - a_n > 0 \Leftrightarrow a_n - a_{n-1} > 0 \Leftrightarrow \cdots \Leftrightarrow a_2 - a_1 = \sqrt{\frac{3+a_1}{2}} - a_1 > 0$, and $a_1 > 0$. Thus $0 < a_1 < \frac{3}{2}$. Hence the range of a_1 is $a_1 \in (0, \frac{3}{2})$.
(2) Proof: By applying the method of (1), we obtain that $a_{n+1} - a_n < 0$ holds for any positive n when $a_1 > \frac{3}{2}$. Since $a_1 = 4$, $a_{n+1} - a_n < 0$, then $b_n = |a_{n+1} - a_n| = a_n - a_{n+1}$. Hence $S_n = b_1 + b_2 + \cdots + b_n = (a_1 - a_2) + (a_2 - a_3) + \cdots + (a_n - a_{n+1}) = 4 - a_{n+1}$. Additionally, since $a_{n+2} < a_{n+1}$ which means $\sqrt{\frac{3+a_{n+1}}{2}} < a_{n+1}$, then $a_{n+1} > \frac{3}{2}$. Therefore $S_n < 4 - \frac{3}{2} = \frac{5}{2}$.

5.82 ★★★★★ Let the positive sequence $a_0, a_1, a_2, \cdots, a_n, \cdots$ satisfy $\sqrt{a_n a_{n-2}} - \sqrt{a_{n-1}a_{n-2}} = 2a_{n-1} \quad (n \geqslant 2)$, and $a_0 = a_1 = 1$. Find the general term of $\{a_n\}$.

Solution: The given equation leads to $\sqrt{a_n a_{n-2}} - 2a_{n-1} = \sqrt{a_{n-1}a_{n-2}}$. Dividing both sides by $\sqrt{a_{n-1}a_{n-2}}$ to obtain $\sqrt{\frac{a_n}{a_{n-1}}} - 2\sqrt{\frac{a_{n-1}}{a_{n-2}}} = 1$ ①. Thus $\sqrt{\frac{a_{n-1}}{a_{n-2}}} - 2\sqrt{\frac{a_{n-2}}{a_{n-3}}} = 1$ ②, $\cdots$, $\sqrt{\frac{a_2}{a_1}} - 2\sqrt{\frac{a_1}{a_0}} = 1$ n-1. By applying ①$\times 1 +$ ②$\times 2 +$ ③$\times 2^2 + \cdots +$ n-1$\times 2^{n-2}$, we have $\sqrt{\frac{a_n}{a_{n-1}}} - 2^{n-1}\sqrt{\frac{a_1}{a_0}} = 1 + 2 + 2^2 + \cdots + 2^{n-2}$. This means $\sqrt{\frac{a_n}{a_{n-1}}} = 1 + 2 + 2^2 + \cdots + 2^{n-2} + 2^{n-1} = 2^n - 1$. Hence $a_n = (2^n-1)^2 a_{n-1} = (2^n-1)^2(2^{n-1}-1)^2 a_{n-2} = \cdots = \prod_{k=1}^{n}(2^k-1)^2$.

As a conclusion,

$$a_n = \begin{cases} 1, n = 0; \\ \prod_{k=1}^{n} (2^k - 1)^2, n \in N^*. \end{cases}$$

5.83 ★★★★★ Given sequence $\{a_n\}$, $a_1 = 2$, the nth partial sum is S_n. a_n is the arithmetic mean of $3S_n - 4$ and $2 - \frac{5}{2}S_{n-1}$ for any $n \in N^*$. (1) Show $\{a_n\}$ is a geometric sequence, and find the general term a_n. (2) Show $\frac{1}{2}(\log_2 S_n + \log_2 S_{n+2}) < \log_2 S_{n+1}$. (3) If $b_n = \frac{4}{a_n} - 1$, $c_n = \log_2(\frac{4}{a_n})^2$. Let T_n is the nth partial sum of $\{b_n\}$, and R_n is the nth partial sum of $\{c_n\}$. Does there exist a positive integer n such that $T_n > R_n$. If yes, find its range. If no, please explain the reason.

(1) Proof: $2a_n = 3S_n - 4 + 2 - \frac{5}{2}S_{n-1}$ when $n \geqslant 2$ which leads to $2(S_n - S_{n-1}) = 3S_n - 2 - \frac{5}{2}S_{n-1}$. Then $S_n = \frac{1}{2}S_{n-1} + 2$, $S_{n+1} = \frac{1}{2}S_n + 2$. Since $a_1 = 2$, then $2 + a_2 = \frac{1}{2} \times 2 + 2$. Thus $a_2 = 1$. Hence $\frac{a_{n+1}}{a_n} = \frac{S_{n+1} - S_n}{S_n - S_{n-1}} = \frac{(\frac{1}{2}S_n + 2) - (\frac{1}{2}S_{n-1} + 2)}{S_n - S_{n-1}} = \frac{1}{2}$. Therefore $\frac{a_2}{a_1} = \frac{1}{2}$. After all, $\{a_n\}$ is a geometric sequence with the common ratio $\frac{1}{2}$.

(2) Proof: From (1), we have $S_n = \dfrac{2(1-\frac{1}{2^n})}{1-\frac{1}{2}} = 4-(\dfrac{1}{2})^{n-2}$. To show $\dfrac{1}{2}(\log_2 S_n + \log_2 S_{n+2}) < \log_2 S_{n+1}$, we only need to show $S_nS_{n+2} < S_{n+1}^2$. Since $S_nS_{n+2} = [4-(\dfrac{1}{2})^{n-2}][4-(\dfrac{1}{2})^n] = 16-5(\dfrac{1}{2})^{n-2}+(\dfrac{1}{2})^{2n-2}$, $S_{n+1}^2 = [4-(\dfrac{1}{2})^{n-1}]^2 = 16-4(\dfrac{1}{2})^{n-2}+(\dfrac{1}{2})^{2n-2}$, then $S_nS_{n+2} < S_{n+1}^2$. Therefore $\dfrac{1}{2}(\log_2 S_n + \log_2 S_{n+2}) < \log_2 S_{n+1}$.
(3) Proof: From the given condition and (1), we have $b_n = \dfrac{4}{a_n} - 1 = \dfrac{4}{\frac{1}{2^{n-2}}} - 1 = 2^n - 1$, $c_n = \log_2(\dfrac{4}{a_n})^2 = \log_2(2^n)^2 = 2n$. $T_n = 2(1+2+2^2+\cdots+2^{n-1}) - n = 2(2^n-1) - n = 2^{n+1} - n - 2$. $R_n = 2(1+2+3+\cdots+n) = 2 \times \dfrac{n(n+1)}{2} = n^2 + n$. $T_n < R_n$ when $n = 1, 2, 3$. $T_n > R_n$ when $n = 4, 5$ which means $2^{n+1} > n^2 + 2n + 2$. When $n \geqslant 6$, then $2^{n+1} = (1+1)^{n+1} = c_{n+1}^0 + c_{n+1}^1 + c_{n+1}^2 + \cdots + c_{n+1}^{n+1} > 2(c_{n+1}^0 + c_{n+1}^1 + c_{n+1}^2) = n^2 + 3n + 4 > n^2 + 2n + 2$. Thus $T_n > R_n$ when $n \geqslant 4$.

5.84 ★★★★ Given sequence $\{a_n\}$, $a_k > 0 \quad (k = 1, 2, \cdots, n)$, and $S_n = \dfrac{1}{2}(a_n + \dfrac{1}{a_n})$. Find a_n and S_n.

Solution: Since $a_1 = S_1 = \dfrac{1}{2}(a_1 + \dfrac{1}{a_1})$, then $a_1 = 1$. Since $a_2 = S_2 - S_1 = \dfrac{1}{2}(a_2 + \dfrac{1}{a_2}) - 1$, then $a_2 = -1 + \sqrt{2}$, $S_2 = a_2 + S_1 = \sqrt{2}$. Since $a_3 = S_3 - S_2 = \dfrac{1}{2}(a_3 + \dfrac{1}{a_3}) - \sqrt{2}$, then $a_3 = -\sqrt{2} + \sqrt{3}$, $S_3 = a_3 + S_2 = \sqrt{3}, \cdots$. We have the conjecture: $a_n = -\sqrt{n-1} + \sqrt{n}$, $S_n = \sqrt{n}, (n \in N^*)$.
Now we show the conjecture using mathematical induction.
Proof: (1) When $n = 1$, then $a_1 = \sqrt{1} = S_1$. $p(1)$ is correct.
(2) Suppose $p(k)$ is correct when $n = k$. This means $a_k = -\sqrt{k-1} + \sqrt{k}$ and $S_k = \sqrt{k}$ both hold. Then when $n = k+1$, we have $a_{k+1} = S_{k+1} - S_k = \dfrac{1}{2}(a_{k+1} + \dfrac{1}{a_{k+1}}) - \sqrt{k} \Rightarrow 2a_{k+1} = a_{k+1} + \dfrac{1}{a_{k+1}} - 2\sqrt{k} \Rightarrow a_{k+1}^2 + 2\sqrt{k}a_{k+1} - 1 = 0 \Rightarrow (a_{k+1} + \sqrt{k})^2 = k+1$. Since $a_{k+1} > 0$, then $a_{k+1} = -\sqrt{k} + \sqrt{k+1}$. Thus $S_{k+1} = a_{k+1} + S_k = (-\sqrt{k} + \sqrt{k+1}) + \sqrt{k} = \sqrt{k+1}$. Hence $p(k+1)$ is correct when $n = k+1$.
Therefore $a_n = -\sqrt{n-1} + \sqrt{n}$ and $S_n = \sqrt{n}$ both hold for any $n \in N^*$.

5.85 ★★★★★ Given sequence $\{a_n\}$, $a_1 = 0$, $a_{n+1} = 2a_n + n^2 \quad (n \in N^*)$. Find the nth partial sum S_n of $\{a_n\}$.

Solution: $a_{n+1} = 2a_n + n^2$ implies $a_{n+1} - 2a_n = n^2$.
Since $S_n = a_1 + a_2 + \cdots + a_n$, then $2S_n = 2a_1 + 2a_2 + \cdots + 2a_n$. Subtracting the second equation from the first equation, we have $-S_n = a_1 + (a_2 - 2a_1) + (a_3 - 2a_2) + \cdots + (a_n - 2a_{n-1}) - 2a_n$. This means $-S_n = 0 + 1^2 + 2^2 + \cdots + (n-1)^2 - 2a_n$.

Thus $S_n = 2a_n - \frac{n(n-1)(2n-1)}{6}$ $(n \in N^*)(*)$. Let $a_n = an^2 + bn + c$, then $a(n+1)^2 + b(n+1) + c = 2(an^2 + bn + c) + n^2$. Simplifying the equation to generate $(a+1)n^2 + (b-2a)n - [(a+b) - c] = 0$. Thus $a = -1, b = -2, c = -3$. Hence $a_{n+1} + [(n+1)^2 + 2(n+1) + 3] = 2(a_n + n^2 + 2n + 3) \Rightarrow \frac{a_{n+1} + [(n+1)^2 + 2(n+1) + 3]}{a_n + n^2 + 2n + 3} = 2$. $a_1 + 1^2 + 2 \times 1 + 3 = 6$ when $n = 1$. Therefore $\{a_n + n^2 + 2n + 3\}$ is a geometric sequence with the first term 6 and the common ratio 2. Then $a_n + n^2 + 2n + 3 = 6 \times 2^{n-1}$, $2a_n = 6 \times 2^n - 2 \times n^2 - 4n - 6$. By substituting it into $(*)$, we have $S_n = 6 \times 2^n - 2n^2 - 4n - 6 - \frac{n(n-1)(2n-1)}{6}$ $(n \in N^*)$.

5.86 ★★★★★ Let the function $f_1(x) = \frac{2}{1+x}$. Define $f_{n+1}(x) = f_1[f_n(x)]$, and $a_n = \frac{f_n(0) - 1}{f_n(0) + 2}$, $n \in N^*$. (1) Find the general term of $\{a_n\}$. (2) If $T_{2n} = a_1 + 2a_2 + \cdots + 2na_{2n}$, $Q_n = \frac{4n^2+n}{4n^2+4n+1}$, $n \in N^*$. Compare $9T_{2n}$ and Q_n.

Solution: (1) From the given condition, we have $f_1(0) = 2$, $a_1 = \frac{2-1}{2+2} = \frac{1}{4}$, $f_{n+1}(0) = f_1[f_n(0)] = \frac{2}{1+f_n(0)}$. Thus $a_{n+1} = \frac{f_{n+1}(0) - 1}{f_{n+1}(0) + 2} = \frac{\frac{2}{1+f_n(0)} - 1}{\frac{2}{1+f_n(0)} + 2} = \frac{1 - f_n(0)}{4 + 2f_n(0)} = -\frac{1}{2}\frac{f_n(0) - 1}{f_n(0) + 2} = -\frac{1}{2}a_n$. This means $\frac{a_{n+1}}{a_n} = -\frac{1}{2}$. Thus $\{a_n\}$ is a geometric sequence with the first term $\frac{1}{4}$ and the common ratio $-\frac{1}{2}$, and $a_n = \frac{1}{4}(-\frac{1}{2})^{n-1}, n \in N^*$.
(2) $T_{2n} = a_1 + 2a_2 + \cdots + (2n-1)a_{2n-1} + 2na_{2n}$ ①. Subtracting both sides of ① by $-\frac{1}{2}$ to obtain $-\frac{1}{2}T_{2n} = (-\frac{1}{2})a_1 + (-\frac{1}{2})2a_2 + \cdots + (-\frac{1}{2})(2n-1)a_{2n-1} + (-\frac{1}{2})2na_{2n} = a_2 + 2a_3 + \cdots + (2n-1)a_{2n} - na_{2n}$ ②. Using ① − ②, we have $\frac{3}{2}T_{2n} = a_1 + a_2 + a_3 + \cdots + a_{2n} + na_{2n} = \frac{\frac{1}{4}[1 - (-\frac{1}{2})^{2n}]}{1 + \frac{1}{2}} + n\frac{1}{4}(-\frac{1}{2})^{2n-1} = \frac{1}{6} - \frac{1}{6}(-\frac{1}{2})^{2n} + \frac{n}{4}(-\frac{1}{2})^{2n-1}$. Thus $T_{2n} = \frac{1}{9} - \frac{1}{9}(-\frac{1}{2})^{2n} + \frac{n}{6}(-\frac{1}{2})^{2n-1} = \frac{1}{9} - \frac{1}{9}\frac{1}{2^{2n}} - \frac{n}{6} \cdot 2 \cdot \frac{1}{2^{2n}} = \frac{1}{9}(1 - \frac{3n+1}{2^{2n}})$. This means $9T_{2n} = 1 - \frac{3n+1}{2^{2n}}$. $Q_n = \frac{4n^2+n}{4n^2+4n+1} = 1 - \frac{3n+1}{(2n+1)^2}$.
When $n = 1$, then $2^{2n} = 4$, $(2n+1)^2 = 9$. Thus $9T_{2n} < Q_n$.
When $n = 2$, then $2^{2n} = 16$, $(2n+1)^2 = 25$. Thus $9T_{2n} < Q_n$.
When $n \geqslant 3$, then $2^{2n} = [(1+1)^n]^2 = (c_n^0 + c_n^1 + \cdots + c_n^n)^2 > (2n+1)^2$. Thus $9T_{2n} > Q_n$.

5.87 ★★★★★ Let x_1 and x_2 be the two real roots of equation $x^2 - 6x + 1 = 0$. Show $x_1^n + x_2^n$ is always an integer but not a multiple of 5, for any natural number n.

Proof: According to the relation of roots and coefficients, we have $x_1+x_2 = 6$, $x_1x_2 = 1$. Let $a_n = x_1^n + x_2^n$, then $a_1 = x_1 + x_2 = 6$, $a_2 = x_1^2 + x_2^2 = (x_1+x_2)^2 - 2x_1x_2 = 34$. Since $x_1^n + x_2^n = (x_1+x_2)(x_1^{n-1} + x_2^{n-1}) - x_1x_2(x_1^{n-2} + x_2^{n-2})$, then $a_n = 6a_{n-1} - a_{n-2} \quad (n \geqslant 3)$. Let b_n be the remainder of a_n divided by 5. By applying the above recursive formula, we have $b_n = b_{n-1} - b_{n-2}$, $b_{n+2} = b_{n+1} - b_n$, $b_{n+3} = b_{n+2} - b_{n+1} = (b_{n+1} - b_n) - b_{n+1} = -b_n$. Then $b_{n+6} = -b_{n+3} = b_n$. Thus $\{b_n\}$ is a sequence whose period is 6.
Since $a_1 = 6$, then $b_1 = 1$. Since $a_2 = 34$, then $b_2 = 4$. Since $a_3 = x_1^3 + x_2^3 = (x_1 + x_2)[(x_1+x_2)^2 - 3x_1x_2] = 198$. Thus $b_3 = 3$. Since $a_4 = x_1^4 + x_2^4 = [(x_1+x_2)^2 - 2x_1x_2]^2 - 2(x_1x_2)2 = 1154$, then $b_4 = -1$. Since $a_5 = x_1^5 + x_2^5 = (x_1^3 + x_2^3)(x_1^2 + x_2^2) - x_1^3x_2^2 - x_1^2x_2^3 = (x_1+x_2)[(x_1+x_2)^2 - 3x_1x_2][(x_1+x_2)^2 - 2x_1x_2] - (x_1x_2)^2(x_1+x_2) = 6726$, then $b_5 = -4$. Since $a_6 = x_1^6 + x_2^6 = (x_1^2)^3 + (x_2^2)^3 = (x_1^2 + x_2^2)(x_1^4 - x_1^2x_2^2 + x_2^4) = [(x_1+x_2)^2 - 2x_1x_2]\{[(x_1+x_2)^2 - 2x_1x_2]^2 - 3(x_1x_2)^2\} = 39202$, then $b_6 = -3$. Therefore $b_n \neq 0$ for any natural number n and an is not a multiple of 5.

5.88 ★★★★★ Let sequence $\{a_n\}$ and sequence $\{b_n\}$ satisfy $a_0 = 1$, $b_0 = 0$, and

$$\begin{cases} a_{n+1} = 7a_n + 6b_n - 3 & \text{①} \\ b_{n+1} = 8a_n + 7b_n - 4 & \text{②} \end{cases} \quad (n = 0, 1, 2, \cdots)$$

Show $a_n \quad (n = 0, 1, 2, \cdots)$ are complete squares.

Proof: By applying the equation ①, we get $b_n = \frac{1}{6}(a_{n+1} - 7a_n + 3)$. Substituting it into ②, we get $b_{n+1} = \frac{1}{6}(7a_{n+1} - a_n - 3)$ ③. From the equation ①, we get $b_{n+1} = \frac{1}{6}(a_{n+2} - 7a_{n+1} + 3)$ ④. From the equation ③ and equation ④, we get $a_{n+2} = 14a_{n+1} - a_n - 6$. This means $a_{n+2} - \frac{1}{2} = 14(a_{n+1} - \frac{1}{2}) - (a_n - \frac{1}{2})$ where $\frac{1}{2}$ is the root of the equation $x = 14x - x - 6$.
Let $d_n = a_n - \frac{1}{2}$, then $d_0 = 1 - \frac{1}{2} = \frac{1}{2}, d_1 = a_1 - \frac{1}{2} = 7a_0 - 6b_0 - 3 - \frac{1}{2} = \frac{7}{2}, d_{n+2} = 14d_{n+1} - d_n$. The characteristic equation is $x^2 = 14x - 1$, and the characteristic roots are $x_{1,2} = 7 \pm 4\sqrt{3}$. Then $d_n = c_1(7 + 4\sqrt{3})^n + c_2(7 - 4\sqrt{3})^n$. Since $d_0 = \frac{1}{2}$, $d_1 = \frac{7}{2}$, then

$$\begin{cases} c_1 + c_2 = \frac{1}{2} \\ 7(c_1 + c_2) + 4\sqrt{3}(c_1 - c_2) = \frac{7}{2} \end{cases}$$

Solving the equations to obtain $c_1 = c_2 = \frac{1}{4}$. Thus $d_n = \frac{1}{4}[(7 + 4\sqrt{3})^n + (7 - 4\sqrt{3})^n]$. $a_n = d_n + \frac{1}{2} = \frac{1}{4}[(2+\sqrt{3})^{2n} + 2(2+\sqrt{3})^n(2-\sqrt{3})^n + (2-\sqrt{3})^{2n}] = \frac{1}{4}[(2+\sqrt{3})^n + (2-\sqrt{3})^n]^2$. Let $(2+\sqrt{3})^n = A_n + B_n\sqrt{3}$ $(A_n, B_n$ are all positive integer number). Then $(2-\sqrt{3})^n = A_n - B_n\sqrt{3}$. Hence $a_n = \frac{1}{4}(A_n + B_n\sqrt{3} + A_n - B_n\sqrt{3})^2 = \frac{1}{4}(2A_n)^2 = A_n^2$. Therefore a_n $(n = 0, 1, 2, \cdots)$ are complete squares.

5.89 ★★★★★ Given sequences $\{a_n\}$ and $\{b_n\}$, $a_1 = 1$, $a_2 = -1$, $b_1 = 2$, $b_2 = -3$, and $a_{n+1} = 3a_n - 2b_n$, $b_{n+1} = 5a_n - 4b_n$. Find the general terms of sequence $\{a_n\}$ and $\{b_n\}$.

Solution: From the given condition, we have $a_{n+2} = 3a_{n+1} - 2b_{n+1} = 3a_{n+1} - 2(5a_n - 4b_n) = 3a_{n+1} - 2(5a_n + 2a_{n+1} - 6a_n) = -a_{n+1} + 2a_n$. let $a_{n+2} - r_1a_{n+1} = r_2(a_{n+1} - r_1a_n)$. Comparing the coefficients to obtain $r_1 + r_2 = -1$, $r_1r_2 = -2$. Then r_1, r_2 are the two roots of the characteristic equation $x^2 + x - 2 = 0$. Thus $r_1 = 1, r_2 = -2$. This means $a_{n+2} - a_{n+1} = -2(a_{n+1} - a_n) \Rightarrow a_n - a_{n-1}{=}{-2}(a_{n-1} - a_{n-2}) \Rightarrow \frac{a_n - a_{n-1}}{a_{n-1} - a_{n-2}} = -2, \frac{a_{n-1} - a_{n-2}}{a_{n-2} - a_{n-3}} = -2, \cdots, \frac{a_3 - a_2}{a_2 - a_1} = -2$. Multiplying the above equations to obtain $\frac{a_n - a_{n-1}}{a_2 - a_1} = (-2)^{n-2}$. Thus $a_n - a_{n-1} = (-2)^{n-1}$, $a_{n-1} - a_{n-2} = (-2)^{n-2}, \cdots$, $a_2 - a_1 = (-2)$. Adding the above equations to obtain $a_n - a_1 = -2 + (-2)^2 + \cdots + (-2)^{n-1}$. Hence $a_n = 1 + \frac{-2[1 - (-2)^{n-1}]}{1 - (-2)} = \frac{1 - (-2)^n}{3}$ $(n \in N^*)$. similarly, $b_n = \frac{1 + 5(-2)^{n-1}}{3}$ $(n \in N^*)$.

6 FUNCTIONS

6.1 Let the function $f(x) = \dfrac{1-2x}{1+x}$, and the graphs of $g(x)$ and $y = f^{-1}(x+1)$ are symmetric about $y = x$. Evaluate $g(2)$.

Solution 1: Since $y = f(x) = \dfrac{1-2x}{1+x}$, then $x = \dfrac{1-y}{y+2}$. Thus $f^{-1}(x) = \dfrac{1-x}{x+2}$, $f^{-1}(x+1) = \dfrac{-x}{x+3}$. Hence $g(x)$ and $f(x) = \dfrac{-x}{x+3}$ are inverse functions for each other. Since $2 = -\dfrac{x}{x+3}$, then $g(2) = -2$.
Solution 2: Since $y = f^{-1}(x+1)$, then $x = f(y) - 1$. Thus $g(x) = f(x) - 1$. Then $g(2) = f(2) - 1 = -2$.

6.2 Compute the range of $x = \dfrac{1}{\log_{\frac{1}{2}} \frac{1}{3}} + \dfrac{1}{\log_{\frac{1}{5}} \frac{1}{3}}$.

Solution: $x = \dfrac{1}{\log_{\frac{1}{2}} \frac{1}{3}} + \dfrac{1}{\log_{\frac{1}{5}} \frac{1}{3}} = \log_{\frac{1}{3}} \dfrac{1}{2} + \log_{\frac{1}{3}} \dfrac{1}{5} = \log_{\frac{1}{3}} \dfrac{1}{10} = \log_3 10$, and $2 = \log_3 9 < \log_3 10 < \log_3 27 = 3$. Hence $x \in (2, 3)$.

6.3 Let x_1 and x_2 be the two real roots of the equation $x^2 - (k-2)x + (k^2+3k+5) = 0 \quad (k \in R)$. Find the maximum value of $x_1^2 + x_2^2$.

Solution: According to the Vieta's theorem, we have $x_1+x_2 = k-2$, $x_1x_2 = k^2+3k+5$. Then $x_1^2 + x_2^2 = (x_1 + x_2)^2 - 2x_1x_2 = (k-2)^2 - 2(k^2 + 3k + 5) = -(k+5)^2 + 19$. Since the equation has two real roots, then $\Delta = (k-2)^2 - 4(k^2 + 3k + 5) \geqslant 0$. This means $3k^2 + 16k + 16 \leqslant 0$. The range is $-4 \leqslant k \leqslant -\dfrac{4}{3}$. To find the maximum value of $x_1^2+x_2^2$, we need to find the maximum value of $y = -(k+5)^2+19$ when $-4 \leqslant k \leqslant -\dfrac{4}{3}$. Since the symmetric axis $k = -5$ is not in $[-4, -\frac{4}{3}]$, then the function is decreasing in $[-4, -\frac{4}{3}]$. Therefore the the maximum value is $y_{max} = -(-4+5)^2 + 19 = 18$.

6.4 If the function $f(x)$ is defined for all real numbers R, and $f(10+x) = f(10-x)$, $f(20-x) = -f(20+x)$. Is $f(x)$ a periodic function? And determine $f(x)$ is odd or even.
Solution: From the first given equation, we have $f[10 + (10-x]) = f[10 - (10-x)]$. Thus $f(x) = f(20-x) \quad ①$. Combining the given second equation, we have $f(x) = -f(20+x) \quad ②$.

Then $f(40+x) = f[20 + (20+x)] = -f(20+x) = f(x)$. Hence $f(x)$ is a periodic function. By applying ① and ②, we have $f(-x) = f(20+x) = -f(x)$. Therefore $f(x)$ is an odd function.

6.5 The function $F(x)$ is an odd function, and $a > 0, a \neq 1$. Determine the function $G(x) = F(x)(\frac{1}{a^x - 1} + \frac{1}{2})$ is odd or even.

Proof: Since $F(x)$ is an odd function, then $F(-x) = -F(x)$. Let $g(x) = \frac{1}{a^x - 1} + \frac{1}{2} = \frac{a^x + 1}{2(a^x - 1)}$, then $g(-x) = \frac{a^{-x} + 1}{2(a^{-x} - 1)} = \frac{\frac{1}{a^x} + 1}{2(\frac{1}{a^x} - 1)} = \frac{a^x + 1}{2(1 - a^x)} = -g(x)$. This means $g(x)$ is also an odd function. $G(x) = F(x)g(x)$ holds in R when $a > 0$ and $a \neq 1$. Since $G(-x) = F(-x)g(-x) = F(x)g(x)$, then $G(x)$ is an even function.

6.6 Given the set $M = \{x, xy, \lg(xy)\}$, the set $N = \{0, |x|, y\}$, and $M = N$, evaluate $(x + \frac{1}{y}) + (x^2 + \frac{1}{y^2}) + (x^3 + \frac{1}{y^3}) + \cdots + (x^{2011} + \frac{1}{y^{2011}})$.

Proof: Since $M = N$, we get that at least one element of M is zero. From the definition of logarithmic function, we have $xy \neq 0$. This means x and y are both nonzero. Thus $\lg(xy) = 0$. Then $xy = 1$. Hence $M = \{x, 1, 0\}$, $N = \{0, |x|, \frac{1}{x}\}$. Additionally, by applying the equal of sets, we have

$$\begin{cases} x = |x| \\ 1 = \dfrac{1}{x} \end{cases}$$

or

$$\begin{cases} x = \frac{1}{x} \\ 1 = |x| \end{cases}$$

But it is contradicting to the element distinction in a set when $x = 1$. Thus $x = -1$, $y = -1$. Then $x^{2k+1} + \dfrac{1}{y^{2k+1}} = -2$, $x^{2k} + \dfrac{1}{y^{2k}} = 2, (k = 0, 1, 2, \cdots)$.
Therefore $(x + \dfrac{1}{y}) + (x^2 + \dfrac{1}{y^2}) + (x^3 + \dfrac{1}{y^3}) + \cdots + (x^{2011} + \dfrac{1}{y^{2011}}) = -2$.

6.7 Given $f(x) = \dfrac{1}{\sqrt[3]{x^2 + 2x + 1} + \sqrt[3]{x^2 - 1} + \sqrt[3]{x^2 - 2x + 1}}$, solve $f(1) + f(3) + f(5) + \cdots + f(2011)$.

Solution: $f(x) = \dfrac{1}{\sqrt[3]{x^2 + 2x + 1} + \sqrt[3]{x^2 - 1} + \sqrt[3]{x^2 - 2x + 1}}$

$= \frac{\sqrt[3]{x+1} - \sqrt[3]{x-1}}{(x-1) - \sqrt[3]{(x+1)^2(x-1)} + \sqrt[3]{(x^2-1)(x+1)} - \sqrt[3]{(x^2-1)(x-1)} + \sqrt[3]{(x-1)^2(x+1)} - (x-1)}$
$= \frac{\sqrt[3]{x+1} - \sqrt[3]{x-1}}{2 - \sqrt[3]{(x+1)(x^2-1)} + \sqrt[3]{(x^2-1)(x+1)} - \sqrt[3]{(x^2-1)(x-1)} + \sqrt[3]{(x^2-1)(x-1)}}$
$= \dfrac{1}{2}(\sqrt[3]{x+1} - \sqrt[3]{x-1})$.
Thus $f(1) + f(3) + f(5) + \cdots + f(2011) = \dfrac{1}{2}(\sqrt[3]{2} - 0 + \sqrt[3]{4} - \sqrt[3]{2} + \sqrt[3]{6} - \sqrt[3]{4} + \cdots + \sqrt[3]{2010} - \sqrt[3]{2008} + \sqrt[3]{2012} - \sqrt[3]{2010}) = \dfrac{\sqrt[3]{2012}}{2}$.

6.8 Given $f(x) = a \sin x + b\sqrt[3]{x} + 4, (a, b \in R)$, and $f(\lg \log_3 10) = 5$. Find the value of $f(\lg \lg 3)$.

Solution: Since $f(x) - 4 = a \sin x + b\sqrt[3]{x}$, then $f(x) - 4$ is an odd function. Thus $f(-x) - 4 = -(f(x) - 4)$. This means $f(-x) = -f(x) + 8$. Additionally, since $\lg \lg 3 = -\lg \log_3 10$, then $f(\lg \lg 3) = f(-\lg \log_3 10) = -f(\lg \log_3 10) + 8 = -5 + 8 = 3$.

6.9 Given $f(x)$ is an odd function, $g(x)$ is an even function, and $f(x) - g(x) = x^2 - x$. Find $f(x)$ and $g(x)$.

Solution: Since $f(x)$ is an odd function , then $f(-x) = -f(x)$. Since $g(x)$ is an even function , then $g(-x) = g(x)$. Thus $f(x) - g(x) = x^2 - x \Rightarrow f(-x) - g(-x) = x^2 + x \Rightarrow -f(x) - g(x) = x^2 + x \Rightarrow f(x) + g(x) = -x^2 - x$. Then

$$\begin{cases} f(x) - g(x) = x^2 - x \\ f(x) + g(x) = -x^2 - x \end{cases}$$

$\Rightarrow f(x) = -x, g(x) = -x^2$.

6.10 ★ If the domain of the function $y = f(x^2)$ is $[-\frac{1}{4}, 1]$, find the domain of $g(x) = f(x+a) + f(x-a)$.

Solution: Since the domain of the function $y = f(x^2)$ is $[-\frac{1}{4}, 1]$, then $-\frac{1}{4} \leqslant x \leqslant 1$. Thus $0 \leqslant x^2 \leqslant 1$. Hence the domain of the function $y = f(x)$ is $\{x|0 \leqslant x \leqslant 1\}$. Then the domain of $g(x)$ is the solution set of the following system:

$$\begin{cases} 0 \leqslant x+a \leqslant 1 \\ 0 \leqslant x-a \leqslant 1 \end{cases}$$

Then

$$\begin{cases} -a \leqslant x \leqslant 1-a \\ a \leqslant x \leqslant 1+a \end{cases}$$

The domain of $g(x) = f(x+a) + f(x-a)$ is $\{x| -a \leqslant x \leqslant 1+a\}$ when $-\frac{1}{2} < a < 0$. The domain of $g(x) = f(x+a) + f(x-a)$ is $\{x|a \leqslant x \leqslant 1-a\}$ when $0 \leqslant a \leqslant \frac{1}{2}$. Note that $x \in \emptyset$ when $a < -\frac{1}{2}$ or $a > \frac{1}{2}$.

6.11 ★ Given the range of $y = \frac{ax^2 + 8x + b}{x^2 + 1}$ as $\{y|1 \leqslant y \leqslant 9\}$, find the value of a and b.

Solution: Since $y = \frac{ax^2 + 8x + b}{x^2 + 1}$, then $(y-a)x^2 - 8x + y - b = 0$. Since $x \in R$, then $\Delta = 64 - 4(y-a)(y-b) \geqslant 0$ when $y \neq a$. Simplifying the formula to generate $y^2 - (a+b)y + ab - 16 \leqslant 0$. Since the range of y is $\{y|1 \leqslant y \leqslant 9\}$, then 1 and 9 are the two roots of the equation with respect to y. According to the relationship between roots and coefficients, we have

$$\begin{cases} a+b = 10 \\ ab - 16 = 9 \end{cases}$$

Then $a = b = 5$.
$x = \frac{a-b}{8} \in R$ when $y = a$. Therefore $a = b = 5$.

6.12 ★ For arbitrary $x, y \in R$, the function $y = f(x)$ always satisfies $f(x+y) = f(x) + f(y) - 1$. And $f(x) > 1$ when $x > 0$ and $f(3) = 4$. (1) Show $y = f(x)$ is an increasing function. (2) Find the maximum value and the minimum value of $f(x)$ in $[1, 2]$.

(1) Proof: Let $x_1, x_2 \in R$, and $x_1 < x_2$, then $f(x_2) = f(x_2 - x_1 + x_1) = f(x_2 - x_1) + f(x_1) - 1$. Thus $f(x_2) - f(x_1) = f(x_2 - x_1) - 1$. Since $x_1 < x_2$, $x_2 - x_1 > 0$, we have $f(x_2 - x_1) > 1$ which means $f(x_2 - x_1) - 1 > 0$. Hence $f(x_2) - f(x_1) > 0$. Then $f(x_2) > f(x_1)$. Therefore $y = f(x)$ is an increasing function.

(2) Solution: From (1), we know that $f(x)$ is an increasing function in R. Then $f(x)$ is an increasing function in $[1,2]$. The minimum value of $f(x)$ is $f(1)=2$ when $x=1$. The maximum value of $f(x)$ is $f(2)=2f(1)-1=3$ when $x=2$. Therefore the maximum value of $f(x)$ is 3 and the minimum value of $f(x)$ is 2 when $x \in [1,2]$.

6.13 For all ordered pairs of positive integers (x,y), $f(x,1)=1$ holds. $f(x,y)=0$ and $f(x+1,y)=y[f(x,y)+f(x,y-1)]$ both hold when $y>x$. Evaluate $f(5,5)$.

Solution: Since $f(x,1)=1$, then $f(1,1)=1$, $f(2,2)=f(1+1,2)=2[f(1,2)+f(1,1)]=2[0+f(1,1)]=2f(1,1)=2=2\times 1$. $f(3,3)=f(2+1,3)=3[f(2,3)+f(2,2)]=3f(2,2)=3\times 2\times 1,\cdots$. Thus $f(5,5)=5\times 4\times 3\times 2\times 1=120$.

6.14 ★ If the real numbers x and θ satisfy $\log_3(x+7)+2\cos(\theta+2012)=4$. Compute $|x-2|+|x-722|$.

Solution: $\log_3(x+7)+2\cos(\theta+2012)=4 \Rightarrow 2 \leqslant \log_3(x+7)=4-2\cos(\theta+2012) \leqslant 6 \Rightarrow 9 \leqslant x+7 \leqslant 729 \Rightarrow 2 \leqslant x \leqslant 722$. Thus $|x-2|+|x-722|=x-2+722-x=720$.

6.15 ★ Let $A = [1, b] \quad (b > 1)$. The function $f(x) = \frac{1}{2}x^2 - x + \frac{3}{2}$. The range of $f(x)$ is A when $x \in A$. Find the value of b.

Solution: $f(x) = \frac{1}{2}x^2 - x + \frac{3}{2} = \frac{1}{2}(x-1)^2 + 1$ is a parabolic curve, and its symmetric axis is $x = 1$, the vertex is $(1, 1)$. Thus $f(x)$ is an increasing function when $x \in [1, b] \quad (b > 1)$. Then $f(x)$ reaches the maximum value $f(b)$ when $x = b$. Since $f(b) \in [1, b]$, then $f(b) = b$. This means $\frac{1}{2}(b-1)^2 + 1 = b$. Then $b^2 - 4b + 3 = 0$. Hence $b = 1$ or $b = 3$. Since $b > 1$, then $b = 3$.

6.16 Given the function $f(x) = ax^2 + bx + 1$ where a, b are real numbers, $x \in R$. $F(x) = \begin{cases} f(x), (x > 0) \\ -f(x), (x < 0) \end{cases}$
(1) If $f(-1) = 0$, and the range of $f(x)$ is $[0, +\infty)$, Find the analytic formula of $F(x)$. (2) Under the condition of (1), $g(x) = f(x) - kx$ is a monotone function when $x \in [-2, 2]$. Find the range of k.

Solution: (1) From the given condition, we have $\begin{cases} a - b + 1 = 0 \\ -\dfrac{b}{2a} = -1 \end{cases}$. By solving the equation system, we have $\begin{cases} a = 1 \\ b = 2 \end{cases}$. Thus $F(x) = \begin{cases} x^2 + 2x + 1, (x > 0) \\ -x^2 - 2x - 1, (x < 0) \end{cases}$.
(2) $g(x) = x^2 + (2-k)x + 1$ is a monotone function when $x \in [-2, 2]$ if and only if $-\frac{2-k}{2} \geqslant 2$ or $-\frac{2-k}{2} \leqslant -2$. Then $k \geqslant 6$ or $k \leqslant -2$. Thus the range of k is $(-\infty, -2] \cup [6, +\infty)$.

6.17 ★ The graph of $f(x) = kx + b$ intersects x axis at A and intersects y axis at B. $\overrightarrow{AB} = 2i + 2j$, ($i$ is the unit vector of positive x axis, j is the unit vector of positive y axis), and $g(x) = x^2 - x - 6$.

(1) Evaluate k and b. (2) Find the minimum value of $\frac{g(x)+1}{f(x)}$ when $f(x) > g(x)$.

Solution: (1) From the given condition, we have $A(-\frac{b}{k}, 0)$, $B(0, b)$. Then $\overrightarrow{AB} = \{\frac{b}{k}, b\}$. Thus $\frac{b}{k} = 2, b = 2, k = 1$.
(2) Since $f(x) > g(x)$, then $x + 2 > x^2 - x - 6$. thus $-2 < x < 4$. $\frac{g(x)+1}{f(x)} = \frac{x^2 - x - 5}{x+2} = x + 2 + \frac{1}{x+2} - 5$. Since $x + 2 > 0$, then $\frac{g(x)+1}{f(x)} \geqslant 2\sqrt{(x+2)\frac{1}{x+2}} - 5 = -3$, and the equation holds if and only if $x + 2 = 1$ i.e. $x = -1$. Hence the minimum value of $\frac{g(x)+1}{f(x)}$ is -3.

6.18 If the equation $(2-2^{-|x-3|})^2 = 3+a$ with respect to x has real roots, find the range of real number a.

Solution: We simply the given equation to get $a = (2-2^{-|x-3|})^2 - 3$. Let $t = 2^{-|x-3|}$, then $0 < t \leqslant 1, a = f(t) = (t-2)^2 - 3$. Since $a = f(t)$ is decreasing on $(0,1]$, then $f(1) \leqslant f(t) < f(0)$. This means $-2 \leqslant f(t) < 1$. Thus the range of real number a is $a \in [-2, 1)$.

6.19 If the maximum value of the function $f(x) = -3x^2 - 3x + 4b^2 + \frac{9}{4} \quad (b > 0)$ on $[-b, 1-b]$ is 25, find the value of b.

Solution: From the given condition, we have $f(x) = -3(x+\frac{1}{2})^2 + 4b^2 + 3$.
(1) The maximum value of $f(x)$ is $4b^2+3 = 25$ when $-b \leqslant -\frac{1}{2} \leqslant 1-b$ i.e. $\frac{1}{2} \leqslant b \leqslant \frac{3}{2}$. Then $b^2 = \frac{11}{2}$. It is contradicting to $\frac{1}{2} \leqslant b \leqslant \frac{3}{2}$.
(2) $f(x)$ is decreasing on the interval $[-b, 1-b]$ when $-\frac{1}{2} \leqslant -b$ i.e. $0 < b < \frac{1}{2}$. Then $f(-b) = (b+\frac{3}{2})^3 < 25$.
(3) $f(x)$ is increasing on the interval $[-b, 1-b]$ when $-\frac{1}{2} > 1-b$ i.e. $b > \frac{3}{2}$. Hence $f(1-b) = b^2 + 9b - \frac{15}{4} = 25$. Then $b = \frac{5}{2}$.

6.20 ★ If for all $x, y \in R$, $f(x+y) = f(x) + f(y)$ holds. (1) Show $f(x)$ is an odd function. (2) if $f(-3) = a$, express $f(12)$ as a function a.

(1) Proof: Obviously, the domain of $f(x)$ is R. Let $y = -x$ where $f(x+y) = f(x)+f(y)$, then $f(0) = f(x) + f(-x)$. Let $x = y = 0$ where $f(0) = f(0) + f(0)$, then $f(0) = 0$. Thus $f(x) + f(-x) = 0$ which means $f(x) = -f(-x)$. Hence $f(x$ is an odd function.
(2) Solution: Since $f(-3) = a$, $f(x+y) = f(x) + f(y)$, and $f(x)$ is an odd function, we have $f(12) = 2f(6) = 4f(3) = -4f(-3) = -4a$.

6.21 ★ If M is a set of functions that satisfy the following conditions: (1) the domain of $f(x)$ is $[-1, 1]$. (2) If $x_1, x_2 \in [-1, 1]$, then $|f(x_1) - f(x_2)| \leqslant 4|x_1 - x_2|$. Determine whether the function $g(x) = x^2 + 2x - 1$ defined on the interval $[-1, 1]$ belongs to the set M.

Proof: From the given condition, we know that $g(x)$ satisfies the condition (1) obviously. Let $x_1, x_2 \in [-1, 1]$. Then $|x_1| \leqslant 1$, $|x_2| \leqslant 1$. Since $|g(x_1) - g(x_2)| = |(x_1^2 + 2x_1 - 1) - (x_2^2 + 2x_2 - 1)| = |(x_1 - x_2)(x_1 + x_2 + 2)| \leqslant |x_1 - x_2||x_1 + x_2 + 2| \leqslant (|x_1| + |x_2| + 2)|x_1 - x_2| \leqslant 4|x_1 - x_2|$. Thus $g(x)$ satisfies the condition (2). Hence $g(x) \in M$.

6.22 ★ Given the set $A = \{x|(x-2)[x-(3a+1)] < 0\}$, $B = \{x|\dfrac{(x-2a)}{x-(a^2+1)} < 0\}$. (1) Find $A \cap B$ when $a = 2$. (2) Find the range of the real number a such that $B \subseteq A$.
Solution: (1) $A = \{x|(x-2)(x-7) < 0\} = (2,7)$, $B = \{x|\dfrac{x-4}{x-5} < 0\} = (4,5)$. Thus $A \cap B = (4,5)$.
(2) Since $B = (2a, a^2+1)$, then $A = (3a+1, 2)$ when $a < \dfrac{1}{3}$. In order to have $B \subseteq A$, we must have $\begin{cases} 2a \geqslant 3a+1 \\ a^2+1 \leqslant 2 \end{cases}$ for which $a = -1$. $A = \phi$ When $a = \dfrac{1}{3}$. There is no a such that $B \subseteq A$. Then $A = (2, 3a+1)$ when $a > \dfrac{1}{3}$. In order to have $B \subseteq A$, we must have $\begin{cases} 2a \geqslant 2 \\ a^2+1 \leqslant 3a+1 \end{cases}$. Thus $1 \leqslant a \leqslant 3$. As a conclusion, the range of the real number a such that $B \subseteq A$ is $[1,3] \cup \{-1\}$.

6.23 ★ The statement p is that the equation $a^2x^2 + ax - 2 = 0$ has solutions on the interval $[-1,1]$, and the statement q is that there is only one real number x such that the inequality $x^2 + 2ax + 2a \leqslant 0$ holds. If the statement "p or q" is a false statement, find the range of a.

Solution: $a^2x^2 + ax - 2 = 0 \Rightarrow (ax+2)(ax-1) = 0$. Obviously, $a \neq 0$, then $x = -\frac{2}{a}$ or $x = \frac{1}{a}$. Since $x \in [-1,1]$, then $|\frac{2}{a}| \leqslant 1$ or $|\frac{1}{a}| \leqslant 1$. Thus $|a| \geqslant 1$. Since there is only one real number x such that the inequality $x^2 + 2ax + 2a \leqslant 0$ holds, then there is only one intersection of the parabolic curve $y = x^2 + 2ax + 2a$ and x-axis. Thus $\Delta = 4a^2 - 8a = 0$. Hence $a = 0$ or $a = 2$. Then $|a| \geqslant 1$ or $a = 0$ when the statement "p or q" is a true statement. Therefore the range of a is $\{a | -1 < a < 1 \text{ or } 0 < a < 1\}$ when the statement "p or q" is a false statement.

6.24 ★ The function $f(x)$ is defined on the interval $[0,1]$, and $f(0) = f(1)$. If for arbitrary distinct $x_1, x_2 \in [0,1]$, $|f(x_2) - f(x_1)| < |x_2 - x_1|$ holds. Show $|f(x_2) - f(x_1)| < \frac{1}{2}$.

Proof: Let $0 \leqslant x_1 \leqslant x_2 \leqslant 1$. (1) If $x_2 - x_1 \leqslant \frac{1}{2}$, then $|f(x_2) - f(x_1)| < |x_2 - x_1| \leqslant \frac{1}{2}$. (2) If $x_2 - x_1 > \frac{1}{2}$, since $f(0) = f(1)$, then $|f(x_2) - f(x_1)| = |f(x_2) - f(1) + f(0) - f(x_1)| \leqslant |f(x_2) - f(1)| + |f(0) - f(x_1)| < (1 - x_2) + (x_1 - 0) = 1 - (x_2 - x_1) < \frac{1}{2}$.
As a conclusion, $|f(x_2) - f(x_1)| < \frac{1}{2}$.

6.25 ★ Given $f(x)$ is an increasing function on the interval $(0, +\infty)$, and $f(1) = 0$, $f(x) + f(y) = f(x, y)$, show $|f(x)| > |f(y)|$ when $0 < x < y < 1$.

Proof: Since $f(x)$ is an increasing function on the interval $(0, +\infty)$, and $0 < x < y$, we have that $f(x) < f(y)$. This means $f(x) - f(y) < 0$ ①. Since $0 < x < y < 1$, then $f(x) + f(y) = f(xy) < f(1) = 0$. This means $f(x) + f(y) < 0$ ②. By applying ① × ②, we have $[f(x)]^2 - [f(y)]^2 > 0$. Thus $|f(x)| > |f(y)|$.

6.26 Let $f(x)$ is a function defined on $R \to R$. Show $f(x)$ can be expressed as the sum of an odd function and an even function.
Proof: Assume $f((x) = g(x) + h(x)$ where $g(x)$ is an even function and $h(x)$ is odd function. Then $f(-x) = g(-x) + h(-x) = g(x) - h(x)$. Thus $\begin{cases} f(x) = g(x) + h(x) \\ f(-x) = g(x) - h(x) \end{cases}$.
Solving the equation system, we have $g(x) = \frac{1}{2}[f(x) + f(-x)], h(x) = \frac{1}{2}[f(x) - f(-x)]$.
Conversely, since $g(-x) = \frac{1}{2}[f(-x) + f(x)] = g(x)$, then $g(x)$ is an even function. Since $h(-x) = \frac{1}{2}[f(-x) - f(x)] = -\frac{1}{2}[f(x) - f(-x)] = -h(x)$, then $h(x)$ is an odd function. Therefore $f((x)$ can be expressed as the sum of an odd function and an even function.

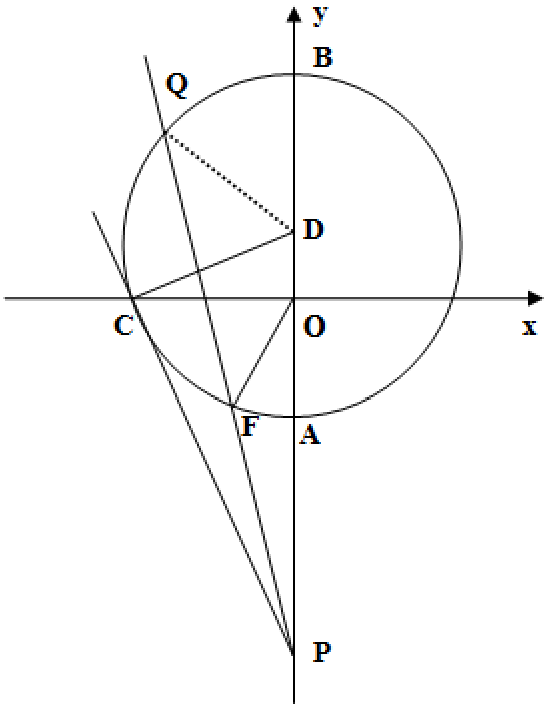


Figure 3

6.27 As shown in Figure 3, the circle $\odot D$ intersects y-axis at the points A and B, and intersects x-axis at the point C on the left. The straight line $y = -2\sqrt{2}x - 8$ intersects y-axis at the point P. The coordinates of the center D is $(0, 1)$. (1) Show PC is a tangent line of the circle $\odot D$. (2) Determine whether there exists a point E on the straight line CP such that $S_{\triangle EOP} = 4S_{\triangle COD}$. If yes, find the coordinates of E. If no, please explain the reason. (3) When the straight line CP turns around the point P, it intersects the inferior arc $\overset{\frown}{AC}$ at the point F (here F does not coincide with A or C). We connect OF. Let $PF = m$, $OF = n$, find the relation between m and n, and determine the range of the variable n.

(1) Proof: The straight line $y = -2\sqrt{2}x - 8$ passing through C intersects x-axis at $C(-2\sqrt{2}, 0)$ and y-axis at $P(0, -8)$. Then $\cot\angle OCD = \dfrac{|CO|}{|OD|} = 2\sqrt{2}$, $\cot\angle OPC = \dfrac{|OP|}{|OC|} = 2\sqrt{2}$. Since $\angle OPC + \angle PCO = 90^0$, then $\angle OCD + \angle PCO = 90^0$. Hence, PC is a tangent line of the circle $\odot D$.
(2) Let the point $E(x, y)$ on the straight line CP such that $S_{\triangle EOP} = 4S_{\triangle COD}$. Then $\dfrac{1}{2} \times 8 \times |x| = 4 \times \dfrac{1}{2} \times 1 \times 2\sqrt{2}$. Thus $x = \pm\sqrt{2}$. Since $y = -2\sqrt{2}x - 8$, then $y = -12$ when $x = \sqrt{2}$ and $y = -4$ when $x = -\sqrt{2}$. Thus there exists a point $E(\sqrt{2}, -12)$ or $E(-\sqrt{2}, -4)$ on the straight line CP such that $S_{\triangle EOP} = 4S_{\triangle COD}$.

(3) Let the straight line PF intersects the arc $\widehat{AC}$ at the point F, and intersects the arc $\widehat{BC}$ at the point Q. We connect DQ. By applying the cutting theorem, we have $PC^2 = PF \cdot PQ$ ①. In $\triangle CPD$ and $\triangle OPC$, $\angle PCD = \angle POC = 90^0$, $\angle CPD = \angle OPC$. Thus $\triangle CPD \backsim \triangle OPC$, $\frac{PC}{PO} = \frac{PD}{PC}$ which means $PC^2 = PO \cdot PD$ ②. According to ① and ②, we have $PO \cdot PD = PF \cdot PQ$. Additionally, since $\angle FPO = \angle DPQ$, then $\triangle FPO \backsim \triangle DPQ$. Hence $\frac{PF}{FO} = \frac{PD}{DQ} = \frac{m}{n}$. Since $PD = 9$, $DQ = CD = \sqrt{(2\sqrt{2})^2 + 1^2} = 3$. Thus $\frac{m}{n} = \frac{9}{3} = 3$, $OA = 3 - 1 = 2$. Therefore $m = 3n \quad (2 < n < 2\sqrt{2})$.

6.28 ★ Given the function $f(x) = \log_2(x+1)$, and the point (x, y) moves on the graph of $f(x)$, the point $(\frac{x}{3}, \frac{y}{2})$ moves on the graph of $y = g(x)$. Find the maximum value of the function $p(x) = g(x) - f(x)$.

Solution: From the given condition, we have $g(x) = \frac{1}{2}\log_2(3x+1)$. Then $P(x) = \frac{1}{2}\log_2(3x+1) - \log_2(x+1) = \log_2\sqrt{\frac{3x+1}{(x+1)^2}}$. Let $u = \frac{3x+1}{(x+1)^2}$, then $ux^2 + (2u-3)x + u - 1 = 0$. Since u has meaning, then $\Delta = (2u-3)^2 - 4u(u-1) = -8u + 9 \geqslant 0$. This means $u \leqslant \frac{9}{8}$. Thus $p_{max}(x) = \log_2\sqrt{\frac{9}{8}} = \log_2 3 - \frac{3}{2}$.

6.29 ★ Suppose the function $f(x) = \sqrt{2 - \frac{x+3}{x+1}}$ has the domain A, and the function $g(x) = \lg[(x-a-1)(2a-x)](a < 1)$ has the domain B. (1) Find A and B. (2) If $B \subseteq A$, find the range of the real number a.

Solution: (1) From the given condition, we have $f(x) = \sqrt{\frac{2x+2-x-3}{x+1}} = \sqrt{\frac{x-1}{x+1}}$. Since $\frac{x-1}{x+1} \geqslant 0$, then $x \geqslant 1$ or $x \leqslant -1$. Thus $A = (-\infty, -1] \cup [1, +\infty)$. Since $\begin{cases} (x-a-1)(2a-x) > 0 \\ a < 1 \end{cases} \Rightarrow \begin{cases} [x-(a+1)](x-2a) < 0 \\ a < 1 \end{cases}$, then $2a < x < a+1$.

Thus $B = (2a, a+1)$.
(2) Since $B \subseteq A$, then $2a \geqslant 1$ or $a+1 \leqslant -1$. This means $a \geqslant \frac{1}{2}$, or $a \leqslant -2$. Since $a < 1$, then $\frac{1}{2} \leqslant a < 1$, or $a \leqslant -2$. For $B \subseteq A$, the range of real number a is $(-\infty, -2] \cup [\frac{1}{2}, 1)$.

6.30 ★ The graph of the linear function $f(x) = ax + b$ passes through the point $(10, 13)$, and its x-intercept is $(p, 0)$ and its y-intercept is $(0, q)$, where p is a prime number and q is a positive integer number. Find all linear functions that satisfy the above conditions.

Solution: Since the x- and y-intercepts of the linear function $f(x) = ax + b$ are $(p, 0)$ and $(0, q)$ respectively, we have $\begin{cases} ap + b = 0 \\ b = q \end{cases}$. Solving these equations to obtain $a = -\frac{p}{q}, b = q$. Thus $y = -\frac{q}{p}x + q$. This means $\frac{x}{p} + \frac{y}{q} = 1$. Since the linear function passes through $(10, 13)$, we have $10q + 13p = pq$. Then $(p-10)(q-13) = 130$. Since p is a prime number, then p is only 11 or 23. Hence, $q = 143$ when $p = 11$; $q = 23$ when $p = 23$. The linear functions which satisfy the conditions are $y = -13x + 143$; $y = -x + 23$.

6.31 Given a quadratic function $f(x) = ax^2 + bx + c \quad (a > 0)$. The two roots of the equation $f(x) - x = 0$ are x_1,x_2, which satisfy $0 < x_1 < x_2 < \frac{1}{a}$. In addition, the graph of $f(x)$ is symmetric about the straight line $x = x_0$. Show $x_0 < \frac{x_1}{2}$.

Proof: From the given condition, we have $f(x) - x = ax^2 + (b-1)x + c$. Since the two roots of the equation $f(x) - x = 0$ are x_1,x_2, which satisfy $0 < x_1 < x_2 < \frac{1}{a}$, we have $0 < x_1 < \frac{b-1}{-2a} < x_2 < \frac{1}{a}$, and $\frac{b-1}{-2a} - x_1 = x_2 - \frac{b-1}{-2a} < \frac{1}{a} - \frac{b-1}{-2a}$. This means $-\frac{b}{a} < x_1$. Thus $x_0 = -\frac{b}{2a} < \frac{x_1}{2}$.

6.32 ★★ Given $f(x) = x^4 + ax^3 + bx^2 + cx + d$ (a,b are constants), and $f(1) = 2009f(2) = 4018f(3) = 6027$. Evaluate $\frac{1}{4}[f(11) + f((-7)]$.

Solution: Let $n = 2009$, $F(x) = f(x) - nx$, then $F(1) = F(2) = F(3) = 0$. Thus $F(x) = (x-1)(x-2)(x-3)(x-r)$.

$\frac{1}{4}[f(11)+f(-7)] = \frac{1}{4}[F(11)+F((-7)+11n-7n] = \frac{1}{4}[F(11)+F(-7)]+n = \frac{1}{4}[10\times 9\times 8\times(11-r)+(-8)\cdot(-9)\cdot(-10)\cdot(-7-r)]+2009 = \frac{1}{4}[10\times 9\times 8\times(11-r+7+r)]+2009 = \frac{1}{4}\times 10\times 9\times 8\times 18+2009 = 5249$.

6.33 ★★ If $a > b > c$, show that the equation $3x^2-2(a+b+c)x+ab+bc+ca = 0$ has two real roots with one located in the interval (c, b) and the other one in the interval (b, a).

Proof: Let $f(x) = 3x^2-2(a+b+c)x+ab+bc+ca$. Since $\Delta = [-2(a+b+c)]^2-4\times 3\cdot(ab+bc+ca) = 4(a+b+c)^2-12(ab+bc+ca) = 2(2a^2+2b^2+2c^2-2ab-2bc-2ca) = 2[(a-b)^2+(b-c)^2+(c-a)^2]$. Since $a > b > c$, then $\Delta > 0$. Thus the equation has two distinct real roots.
$f(a) = 3a^2-2(a+b+c)a+ab+bc+ca = a^2-ca+bc-ab = (a-c)(a-b) > 0$.
$f(b) = 3b^2-2(a+b+c)b+ab+bc+ca = b^2-ab-bc+ca = (b-a)(b-c) < 0$.
$f(c) = 3c^2-2(a+b+c)c+ab+bc+ca = c^2-ca-bc+ab = (c-a)(c-b) > 0$.
The two x-intercepts of the graph are in the interval (c, b) and (b, a). Thus the equation has two real roots with one located in the interval (c, b) and the other one in the interval (b, a).

6.34 ★ Let p be a real number, and the graph of the quadratic function $y = x^2-2px-p$ has two distinct x-intercepts $A(x_1, 0)$, $B(x_2, 0)$. (1) Show $2px_1+x_2^2+3p > 0$. (2) If the distance between the two points A and B is not larger than $|2p-3|$. Find the maximum value of p.

(1) Proof: According to the relationship between roots and coefficients, we have $x_1+x_2 = 2p$, $x_1x_2 = -p$. From the above equations, we have $x_2^2 = 2px_2+p$. Since $\Delta = (-2p)^2-4(-p) = 4p^2+4p > 0$, then $2px_1+x_2^2+3p = 2px_1+(2px_2+p)+3p = 2p(x_1+x_2)+4p = 4p^2+4p > 0$.
(2) Solution: $AB = |x_2-x_1| = \sqrt{(x_2+x_1)^2-4x_2x_1} = \sqrt{4p^2+4p} \leqslant |2p-3|$. Squaring both sides to generate $4p^2+4p \leqslant 4p^2-12p+9$. Solving the inequality to obtain $p \leqslant \frac{9}{16}$. Thus the maximum value of p is $\frac{9}{16}$.

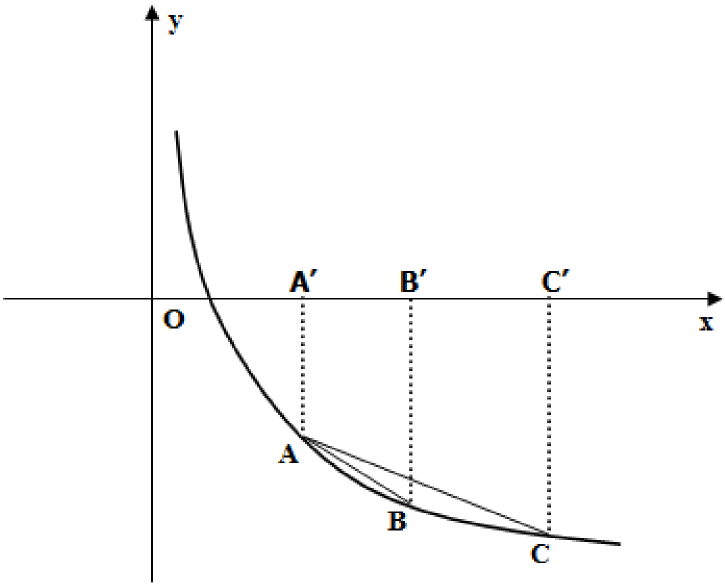


Figure 4

6.35 ★★ As shown in Figure 4, A, B, C are three points on the graph of the function $y = \log_{\frac{1}{3}} x$. Their x-coordinates are t, $t+2$, $t+4$ $\quad (t \geqslant 1)$, respectively. (1) Let the area $S_{\triangle ABC}$ be S, find $S = f(t)$. (2) Determine the monotonicity of $S = f(t)$. (3) Find the maximum value of $S = f(t)$.

Solution: (1) Starting from the points A, B, C, we draw lines AA', BB',CC' perpendicular to x-axis with the foot points A', B', C'. Then $S = S_{\text{trapezoid}}AA'B'B + S_{\text{trapezoid}}BB'C'C - S_{\text{trapezoid}}AA'C'C = -[\log_{\frac{1}{3}} t + \log_{\frac{1}{3}}(t+2)](t+2-t) \times \frac{1}{2} - [\log_{\frac{1}{3}}(t+2) + \log_{\frac{1}{3}}(t+4)] \cdot [(t+4)-(t+2)]\frac{1}{2} + [\log_{\frac{1}{3}} t + \log_{\frac{1}{3}}(t+4)](t+4-t) \times \frac{1}{2} = \log_{\frac{1}{3}} \frac{t^2+4t}{(t+2)^2} = \log_3(1 + \frac{4}{t^2+4t}), (t \geqslant 1)$.
(2) Let $u = t^2 + 4t$ be an increasing function on the interval $[1, +\infty)$, and $u \geqslant 5$. Let $v = 1 + \frac{4}{u}$ be a decreasing function on the interval $[5, +\infty)$, and $1 < v \leqslant \frac{9}{5}$, which means $S = \log_3 v$ is an increasing function on the interval $(1, \frac{9}{5}]$. Hence, the composite function $S = f(t) = \log_3(1 + \frac{4}{t^2+4t})$ is a decreasing function on the interval $[1, +\infty)$.

(3) According to (2), S reaches the maximum value when $t = 1$, and its maximum value is $S_{max} = f(1) = \log_3 \frac{9}{5} = 2 - \log_3 5$.

6.36 ★★ Given $f(2x-1) = x^2 \quad (x \in R)$, find the range of the function $f[f(x)]$.

Solution 1 : Let $A(2x-1)^2 + B(2x-1) + C \equiv x^2$. Comparing the coefficients to obtain $A = \frac{1}{4}, B = \frac{1}{2}, C = \frac{1}{4}$. Thus $f(x) = \frac{1}{4}x^2 + \frac{1}{2}x + \frac{1}{4} = \frac{1}{4}(x+1)^2$. Hence $f[f(x)] = \frac{1}{4}[\frac{1}{4}(x+1)^2 + 1]^2 = \frac{1}{64}[(x+1)^2 + 4]^2$. Then $f[f(x)] \geqslant \frac{1}{4}$ (The equation holds when $x = -1$). This means the minimum value of $f[f(x)]$ is $\frac{1}{4}$. The range of the function $f[f(x)]$ is $[1/4, +\infty)$.
Solution 2 : Let $2x - 1 = t$, then $x = \frac{t+1}{2}$. From the given condition, we have $f(t) = (\frac{t+1}{2})^2$. Notice that $t \in R$ and $x \in R$, then $f(x) = (\frac{x+1}{2})^2 = \frac{1}{4}(x+1)^2$, $f[f(x)] = \frac{1}{4}[\frac{1}{4}(x+1)^2+1]^2 = \frac{1}{64}[(x+1)^2+4]^2$. Thus $f[f(x)] \geqslant \frac{1}{4}$ (The equation holds when $x = -1$). The range of the function $f[f(x)]$ is $[1/4, +\infty)$.

6.37 ★★ Let $A = [-1, 0] \cup (1, 2]$ and $B = [0, 2]$. The map $f : A \to B$ maps x to $y = |x|$. Show that $f : A \to B$ is a one-to-one map and find its inverse map.

Proof: Let $x_1, x_2 \in A$, and $x_1 \neq x_2$, with $f(x_1) = |x_1|$, $f(x_2) = |x_2|$. If $|x_1| = |x_2|$, according the given condition $x_1 \neq x_2$, then $x_1 = -x_2$. This means x_1, x_2 are opposite numbers. Without loss of generality, let $x_1 > 0, x_2 < 0$, according to the given A, we have $x_1 \in (1, 2], x_2 \in [-1, 0)$. Obviously, $|x_1| > 1$, $|x_2| \leqslant 1$ for this case. Thus $|x_1| \neq |x_2|$, which is contradicting to the given condition.

On the other hand, let $y_1 \in B$, then $0 \leqslant y_1 \leqslant 2$. Notice $B = [0,1] \cup (1,2]$. Then $y_1 \in [0,1]$, or $y_1 \in (1,2]$. When $y_1 \in [0,1]$, then there is a unique element in A, $x_1 = -y_1$, corresponding to it. When $y_1 \in (1,2]$, then there is a unique element in A, $x_2 = y_1$, corresponding to it. According to the property of a one-to-one map, we know that $f : A \to B$ is a one-to-one map, and its inverse map is $x = \begin{cases} y, (y \in (1,2]) \\ -y, (y \in [0,1]) \end{cases}$.

6.38 ★★ Given the set $A = \{(x,y)|x^2+mx-y+2 = 0, x \in R\}$, $B = \{(x,y)|x-y+1 = 0, 0 \leqslant x \leqslant 2\}$. If $A \cap B \neq \phi$, find the range of the real number m.

Solution: The problem is equivalent to the following: the equation system $\begin{cases} y = x^2 + mx + 2 \\ y = x + 1 \end{cases}$ has solutions on the interval $[0,2]$. This means $x^2 + (m-1)x + 1 = 0$ has solutions on the interval $[0,2]$. Let $f(x) = x^2 + (m-1)x + 1$, since $f(0) = 1$, then the parabolic curve $y = f(x)$ passes through the point $(0,1)$. Thus the parabolic curve $y = f(x)$ has an x-intercept in the interval $[0,2]$. It is equivalent to $f(2) = 2^2 + 2(m-1) + 1 \leqslant 0$ ①

or $\begin{cases} \Delta = (m-1)^2 - 4 \geqslant 0 \\ 0 < \dfrac{1-m}{2} < 2 \\ f(2) = 2^2 + 2(m-1) + 1 > 0 \end{cases}$ ②.

From ①, we have $m \leqslant -\frac{3}{2}$. From ②, we have $-\frac{3}{2} < m \leqslant -1$. Thus the range of m is $(-\infty, -1]$.

6.39 ★★ If the functions $f(x)$ and $g(x)$ defined for the real numbers satisfy $f(0) = 0$, and for arbitrary $x, y \in R$, $g(x-y) = f(x)f(y) + g(x)g(y)$ holds. Show $[f(x)]^{2012} + [g(x)]^{2012} \leqslant 1$.

Proof : Let $x = y$, then $[f(x)]^2 + [g(x)]^2 = g(0)(*)$. In $(*)$, let $x = 0$, then $[f(0)]^2 + [g(0)]^2 = g(0)$. Since $f(0) = 0$, then $[g(0)]^2 = g(0)$. Thus $g(0) = 0$ or $g(0) = 1$.
(1) If we substitute $g(0) = 0$ into $(*)$, then $[f(x)]^2 + [g(x)]^2 = g(0)$. Thus $f(x) = g(x) = 0$. Hence $[f(x)]^{2012} + [g(x)]^{2012} \leqslant 1$.
(2) If we substitute $g(0) = 1$ into $(*)$, then $[f(x)]^2 + [g(x)]^2 = 1$. Thus $|f(x)| \leqslant 1$, $|g(x)| \leqslant 1$. Hence $[f(x)]^{2012} \leqslant [f(x)]^2, [g(x)]^{2012} \leqslant [g(x)]^2$. Therefore $[f(x)]^{2012} + [g(x)]^{2012} \leqslant [f(x)]^2 + [g(x)]^2 \leqslant 1$.

6.40 ★★ Let $f(x) = x^2 + px + q$, $A = \{x|x = f(x)\}$, $B = \{x|f[f(x)] = x\}$. (1) Show $A \subseteq B$. (2) If $A = \{-1, 3\}$, find B.

(1) Proof: Let x_0 is an arbitrary element in the set A which means $x_0 \in A$. Since $A = \{x|x = f(x)\}$, then $x_0 = f(x_0)$. Thus $f[f(x_0)] = f(x_0) = x_0$. Thus $x_0 \in B$. Therefore $A \subseteq B$.
(2) Solution: Since $A = \{-1, 3\} = \{x|x^2 + px + q = x\}$, then the equation $x^2 + (p-1)x + q = 0$ has two roots -1 and 3. By applying the Vieta's theorem, we have $\begin{cases} -1 + 3 = -(p-1) \\ (-1) \times 3 = q \end{cases} \Rightarrow \begin{cases} p = -1 \\ q = -3 \end{cases}$. Then $f(x) = x^2 - x - 3$. Thus the elements in the set B is the roots of the equation $f[f(x)] = x$. This means $(x^2 - x - 3)^2 - (x^2 - x - 3) - 3 = x$. Simplifying the equation to generate $(x^2 - x - 3)^2 - x^2 = 0 \Rightarrow (x^2 - x - 3 + x)(x^2 - x - 3 - x) = 0 \Rightarrow (x^2 - 3)(x^2 - 2x - 3) = 0$. Thus $x_1 = -\sqrt{3}$, $x_2 = \sqrt{3}$, $x_3 = -1$, $x_4 = 3$. Therefore $B = \{-\sqrt{3}, -1, \sqrt{3}, 3\}$.

6.41 ★★ Let $f(x) = \dfrac{4^x}{4^x + 2}$, compute $f(\frac{1}{1001}) + f(\frac{2}{1001}) + \cdots + f(\frac{1000}{1001})$.

Solution: Since $f(1-x) = \dfrac{4^{1-x}}{4^{1-x} + 2} = \dfrac{4}{4 + 2 \times 4^x} = \dfrac{2}{4^x + 2}$, then $f(x) + f(1-x) = \dfrac{4^x}{4^x + 2} + \dfrac{2}{4^x + 2} = 1$. Thus $f(\frac{1}{1001}) + f(\frac{2}{1001}) + \cdots + f(\frac{1000}{1001}) = f(\frac{1}{1001}) + f(\frac{1000}{1001}) + f(\frac{2}{1001}) + f(\frac{999}{1001}) + \cdots + f(\frac{500}{1001}) + f(\frac{501}{1001}) = \underbrace{1 + 1 + \cdots + 1}_{500} = 500$.

6.42 ★★★ Given vectors $\vec{a} = (\cos\frac{3}{2}x, \sin\frac{3}{2}x)$, $\vec{b} = (\cos\frac{x}{2}, -\sin\frac{x}{2})$, and $x \in [0, \frac{\pi}{2}]$. If the minimum value of $f(x) = ab - 2\lambda|a + b|$ is $-\frac{3}{2}$. Find the value of λ.

Solution: $ab = (\cos\frac{3}{2}x, \sin\frac{3}{2}x)(\cos\frac{x}{2}, -\sin\frac{x}{2}) = \cos\frac{3x}{2}\cos\frac{x}{2} - \sin\frac{3x}{2}\sin\frac{x}{2} = \cos 2x$, $|a+b| = \sqrt{(\cos\frac{3x}{2} + \cos\frac{x}{2})^2 + (\sin\frac{3x}{2} - \sin\frac{x}{2})^2} = \sqrt{2(1+\cos 2x)} = 2|\cos x|$. Since $x \in [0, \frac{\pi}{2}]$, then $\cos x > 0$. Thus $|a+b| = 2\cos x$. Hence $f(x) = ab - 2\lambda|a+b| = \cos 2x - 4\lambda\cos x + 2\lambda^2 - 2\lambda^2 = 2(\cos x - \lambda)^2 - 1 - 2\lambda^2$. Since $x \in [0, \frac{\pi}{2}]$, then $0 \leqslant \cos x \leqslant 1$.
(1) If $\lambda < 0$, $f(x)$ reaches the minimum value -1 if and only if $\cos x = 0$. It is contradicting to the given condition.
(2) If $0 \leqslant < \lambda \leqslant 1$, $f(x)$ reaches the minimum value $-1 - 2\lambda^2$ if and only if $\cos x = \lambda$. Since $-1 - 2\lambda^2 = -\frac{3}{2}$, then $\lambda = \frac{1}{2}$.
(3) If $\lambda > 1$, $f(x)$ reaches the minimum value $1 - 4\lambda$ if and only if $\cos x = 1$. Since $1 - 4\lambda = -\frac{3}{2}$, then $\lambda = \frac{5}{8}$. It is contradicting to $\lambda > 1$.
As a conclusion, $\lambda = \frac{1}{2}$.

6.43 ★★ Given $f(x) = x^2 + (\lg a + 2)x + \lg b$, and $f(-1) = -2$. $f(x) \geqslant 2x$ holds for all $x \in R$. Evaluate $a + b$.

Solution: Since $f(-1) = -2 \Rightarrow 1 - (\lg a + 2) + \lg b = -2 \Rightarrow \lg a = \lg 10b \Rightarrow a = 10b$. Since $f(x) \geqslant 2x$ holds for all $x \in R$, then $x^2 + (\lg a + 2)x + \lg b \geqslant 2x \Rightarrow x^2 + (\lg a)x + \lg b \geqslant 0$. Since the efficient of x^2 is $1 > 0$, then $\Delta = \lg^2 a - 4\lg b \leqslant 0 \Rightarrow \lg^2 a - 4(\lg a - 1) \leqslant 0 \Rightarrow (\lg a - 2)^2 \leqslant 0$. Since $(\lg a - 2)^2 \geqslant 0$, then $(\lg a - 2)^2 = 0$. Thus $a = 100$, $b = 10$. Therefore $a + b = 110$.

6.44 ★★ Let the equation of the curve C is $y = x^3 - x$. The graph of C shifts $t \quad (t \neq 0)$ units to the positive x direction and then shifts s units to the positive y direction. Then we obtain the curve C_1. (1) Write the equation of the curve C_1. (2) Show the curves C and C_1 are symmetric about the point $A(\frac{t}{2}, \frac{s}{2})$. (3) If there is only one intersection between the curves C and C_1, show $s = \frac{t^3}{4} - t$.

(1) Solution: The equation of the curve C_1 is $y = (x-t)^3 - (x-t) + s \quad (t \neq 0)$.
(2) Proof: We choose an arbitrary point $B_1(x_1, y_1)$ on the curve C. Assume $B_2(x_2, y_2)$ is the symmetric point of B_1 about $A(\frac{t}{2}, \frac{s}{2})$, then $\frac{x_1 + x_2}{2} = \frac{t}{2}$, $\frac{y_1 + y_2}{2} = \frac{s}{2}$. Thus $x_1 = t - x_2$, $y_1 = s - y_2$. By substituting them into the equation of C, we have $s - y_2 = (t - x_2)^3 - (t - x_2)$. This means $y_2 = (x_2 - t)^3 - (x_2 - t) + s$. Then we show that the point $B_2(x_2, y_2)$ is on the curve C_1. Similarly, we can show that the symmetric point of C_1 about $A(\frac{t}{2}, \frac{s}{2})$ is on the curve C. Therefore the curves C and C_1 are symmetric about the point $A(\frac{t}{2}, \frac{s}{2})$.

(3) Proof: Since the curves C and C_1 have a unique intersection point, then the equation system $\begin{cases} y = x^3 - x \\ y = (x-t)^3 - (x-t) + s \end{cases}$ has only one solution. By eliminating y, we obtain $3tx^2 - 3t^2x + (t^3 - t - s) = 0$. Then $\Delta = 9t^4 - 12t(t^3 - t - s) = 0$. This means $t(t^3 - 4t - 4s) = 0$. Since $t \neq 0$, then $t^3 - 4t - 4s = 0$. Thus $s = \dfrac{t^3}{4} - t \quad (t \neq 0)$.

6.45 ★★ Given $f(x) = \lg(x + \sqrt{x^2+1})$, show $f(x)$ and $f^{-1}(x)$ are both odd functions.

Proof: Let $y = f(x) = \lg(x + \sqrt{x^2+1}) \Rightarrow x + \sqrt{x^2+1} = 10^y \Rightarrow x^2 + 1 = x^2 + 10^{2y} - 2x \cdot 10^y \Rightarrow x = \dfrac{10^y - 10^{-y}}{2}$. Thus $f^{-1}(x) = \dfrac{10^x - 10^{-x}}{2}, (x \in R)$. Since $f(-x) = \lg(-x + \sqrt{x^2+1}) = \lg \dfrac{1}{\sqrt{x^2+1}+x} = -\lg(x + \sqrt{x^2+1}) = -f(x)$, then $f(x)$ is an odd function.
Since $f^{-1}(-x) = \dfrac{10^{-x} - 10^x}{2} = -\dfrac{10^x - 10^{-x}}{2} = -f(x)$, then $f^{-1}(x)$ is an odd function on R.

6.46 ★★★ Given $f(x) = \dfrac{x+1-u}{u-x} \quad (u \in R)$. (1) Is the grapy of the function $y = f(x)$ centrally symmetric? If it is centrally symmetric, please point out its symmetric center. (2) Find the range of $f(x)$ for $x \in [u+1, u+2]$.

Solution: (1) From the given condition, we have $f(x) = \frac{x+1-u}{u-x} = -1 - \frac{1}{x-u}$, then the graph of $y = f(x)$ can be obtained by shifting the graph of $g(x) = -\frac{1}{x}$ in the horizontal direction. Since the graph of $g(x) = -\frac{1}{x}$ is centrally symmetric about the original point, then the grapy of the function $y = f(x)$ is centrally symmetric about $(u, -1)$, the symmetric center is $(u, -1)$.
(2) Since $f(x)+2 = \frac{x+1-u}{u-x}+2 = \frac{u+1-x}{u-x}$, $f(x)+\frac{3}{2} = \frac{x+1-u}{u-x}+\frac{3}{2} = \frac{u+2-x}{2(u-x)}$. Thus $[f(x)+2][f(x)+\frac{3}{2}] = \frac{u+1-x}{u-x} \cdot \frac{u+2-x}{2(u-x)} = \frac{(x-u-1)(x-u-2)}{2(u-x)^2}$. On the other hand, since $x \in [u+1, u+2]$, $(u-x)^2 > 0$, then $(x-u-1) \geqslant 0, (x-u-2) \leqslant 0$. Thus $[f(x)+2][f(x)+\frac{3}{2}] \leqslant 0$. Hence $-2 \leqslant f(x) \leqslant -\frac{3}{2}$.

6.47 ★★ If $M = \{z | z = \frac{t}{1+t} + i\frac{1+t}{t}, t \in R, t \neq -1, t \neq 0\}$, $N = \{z | z = \sqrt{2}[\cos(\arcsin t) + i \cdot \cos(\arccos t)], t \in R, |t| \leqslant 1\}$. Determine how many elements in $M \cap N$.

Solution: The points of the set M are on the curve M: $\begin{cases} x = \frac{t}{1+t} \\ y = \frac{1+t}{t} \end{cases}$ $(t \in R, t \neq -1, t \neq 0)$.
The points of the set N are on the curve N: $\begin{cases} x = \sqrt{2(1-t^2)} \\ y = \sqrt{2}t \end{cases}$ $(t \in R, |t| \leqslant 1)$.
The general equations of the curves M and N are $xy = 1 (x \neq 0, 1), x^2 + y^2 = 2, (0 \leqslant x \leqslant \sqrt{2})$, respectively. Thus the x-coordinates of the intersection points of M and N satisfy $x^2 + \frac{1}{x^2} = 2$. This means $x = \pm 1$. Obviously, $M \cap N = \phi$. Hence, $M \cap N$ has zero elements.

6.48 ★★★ Given $f(x) = \begin{cases} 0, (x < a) \\ (\frac{x-a}{a-b})^2, (a \leqslant x \leqslant b) \\ 1, (x > b) \end{cases}$. (1) Show $f(x) \geqslant \frac{1}{4}$ always holds for arbitrary $x \geqslant \frac{a+b}{2}$. (2) Is there a real number c such that $f(c) \geqslant \frac{a+b}{2}$? If c exists, find its range. If c does not exist, please explain the reason.

(1) Proof: When $x \geqslant \frac{a+b}{2}$, ① if $\frac{a+b}{2} \leqslant x \leqslant b$, then $f(x) = \frac{1}{(a-b)^2}(x-a)^2$ is an increasing function. Thus $f(x) \geqslant \frac{1}{(a-b)^2}(\frac{a+b}{2} - a)^2 = \frac{1}{4}$. ② if $x > b$, then $f(x) = 1 > \frac{1}{4}$. Thus $f(x) \geqslant \frac{1}{4}$ holds when $x \geqslant \frac{a+b}{2}$.

(2) Solution: When $a+b \leqslant 0$, since $f(x) \geqslant 0$, then $f(c) \geqslant \frac{a+b}{2}$ always holds for arbitrary real number c. When $\frac{a+b}{2} > 1$, since $f(x) \leqslant 1$, then c does not exist. When $0 < \frac{a+b}{2} \leqslant 1$, if $c > b$, then $f(x) = 1$. If $a < c \leqslant b$, then $f(c) = \frac{(c-a)^2}{(a-b)^2} \geqslant \frac{a+b}{2}$. We get that $(b-a)\sqrt{\frac{a+b}{2}} + a \leqslant c \leqslant b$. Hence $f(c) \geqslant \frac{a+b}{2}$ when $c \geqslant (b-a)\sqrt{\frac{a+b}{2}} + a$. After all, when $a+b > 2$, the real number c, which satisfies $f(c) \geqslant \frac{a+b}{2}$, does not exist. When $a+b \leqslant 0$, the real number c, which satisfies $f(c) \geqslant \frac{a+b}{2}$, exists. When $0 < a+b \leqslant 2$, $c \in [(b-a)\sqrt{\frac{a+b}{2}} + a, +\infty]$ makes $f(c) \geqslant \frac{a+b}{2}$ hold.

6.49 ★★ Determine how many elements in the set $\{(x,y) | \lg(x^3 + \frac{1}{3}y^3 + \frac{1}{9}) = \lg x + \lg y\}$.

Solution: From the properties of the logarithmic function, we get $x > 0, y > 0$. Then $\lg(x^3+\frac{1}{3}y^3+\frac{1}{9}) = \lg x+\lg y \Rightarrow x^3+\frac{1}{3}y^3+\frac{1}{9} = xy \Rightarrow x^3+\frac{1}{3}y^3+\frac{1}{9} \geqslant 3\sqrt[3]{x^3(\frac{1}{3}y^3)\frac{1}{9}} = xy$, the equation holds if and only if $\begin{cases} x^3 = \frac{1}{9} \\ \frac{1}{3}y^3 = \frac{1}{9} \end{cases}$. Thus $x = \sqrt[3]{\frac{1}{9}}, y = \sqrt[3]{\frac{1}{3}}$. Hence there is a unique point $(\sqrt[3]{\frac{1}{9}}, \sqrt[3]{\frac{1}{3}})$ in the set. Therefore the set has one element.

6.50 ★★ Given the functions $f(x) = 3^x - 1$ and $g(x) = \log_9(3x+1)$. (1) If $f^{-1}(x) \leqslant g(x)$, find the range D of x. (2) Let the function $H(x) = g(x) - \frac{1}{2}f^{-1}(x)$. What is the range of $H(x)$ when $x \in D$?

Solution: (1) Since $f(x) = 3^x - 1$, then $f^{-1}(x) = \log_3(x+1)$. Since $f^{-1}(x) \leqslant g(x) \Rightarrow \log_3(x+1) \leqslant \log_9(3x+1) \Rightarrow \log_9(x+1)^2 \leqslant \log_9(3x+1) \Rightarrow \begin{cases} (x+1)^2 \leqslant 3x+1 \\ x+1 > 0 \end{cases}$.

Then $0 \leqslant x \leqslant 1$. Thus $x \in D = [0, 1]$.
(2) $H(x) = g(x) - \frac{1}{2} f^{-1}(x) = \log_9(3x+1) - \frac{1}{2}\log_3(x+1) = \log_9(3x+1) - \log_9(x+1) = \log_9 \frac{3x+1}{x+1}, x \in [0, 1]$. Let $t = \frac{3x+1}{x+1} = 3 - \frac{2}{x+1}$. Obviously, t is increasing on the interval $[0, 1]$, then $1 \leqslant t \leqslant 2$. Hence $0 \leqslant H(x) \leqslant \log_9 2$. Therefore the range of $H(x)$ is $\{y | 0 \leqslant y \leqslant \log_9 2\}$.

6.51 ★★★ The function $f(x) = \log_2(x + m)$, and the numbers $f(0)$, $f(2)$, $f(6)$ form an arithmetic sequence. (1) Find the value of the real number m. (2) If a, b, c are distinct positive numbers, and they form a geometric sequence, determine the order of $f(a) + f(c)$ and $2f(b)$.

Solution: (1) Since $f(0)$, $f(2)$, $f(6)$ form an arithmetic sequence, then $2\log_2(2+m) = \log_2 m + \log_2(m+6) \Rightarrow (m+2)^2 = m(m+6)$, and $m > 0$. Thus $m = 2$.
(2) Since $f(x) = \log_2(x+2)$, then $2f(b) = 2\log_2(b+2) = \log_2(b+2)^2$, $f(a) + f(c) = \log_2(a+2) + \log_2(c+2) = \log_2[(a+2)(c+2)]$. Thus $(a+2)(c+2) = ac + 2(a+c) + 4 > ac + 4\sqrt{ac} + 4 = b^2 + 4b + 4 = (b+2)^2$. Hence $\log_2[(a+2)(c+2) > \log_2(b+2)^2$. This means $\log_2(a+2) + \log_2(c+2) > 2\log_2(b+2)$. Therefore $f(a) + f(c) > 2f(b)$.

6.52 ★★ Given the function $f(x) = x^{-k^2+k+2}$ $(k \in Z)$ and $f(2) < f(3)$. (1) Evaluate k. (2) Is there a positive number p such that the range of the function $g(x) = 1 - pf(x) + (2p-1)x$ is $[-4, \frac{17}{8}]$ when $x \in [-1, 2]$. If p exists, find its range. If p does not exist, please explain the reason.

Solution: (1) Since $f(2) < f(3)$, then $-k^2 + k + 2 > 0$. Thus $k^2 - k - 2 < 0$. Since $k \in Z$, then $k = 0$ or $k = 1$.
(2) From (1), we have $f(x) = x^2, g(x) = 1 - px^2 + (2p-1)x = -p(x - \frac{2p-1}{2p})^2 + \frac{4p^2+1}{4p}$. When $\dfrac{2p-1}{2p} \in [-1, 2]$ which means $p \in [\dfrac{1}{4}, +\infty)$, then $\dfrac{4p^2+1}{4p} = \dfrac{17}{8}$. We have $p = 2$ or $p = \dfrac{1}{8}$. Since $p \in [\dfrac{1}{4}, +\infty)$, then $p = 2$. Thus $g(-1) = -4$, $g(2) = -1$. When $\dfrac{2p-1}{2p} \in (2, +\infty)$ Since $p > 0$, then such p does not exist. When $\dfrac{2p-1}{2p} \in (-\infty, -1)$ which means $p \in (0, \dfrac{1}{4})$, then $g(-1) = \dfrac{17}{8}$, $g(2) = -4$. Thus such p does not exist. After all, $p = 2$.

6.53 ★★★ Suppose $f(x)$ is an increasing function defined on $(0, +\infty)$, and $f(\dfrac{x}{y}) = f(x) - f(y)$, $f(2) = 1$. Solve the inequality $f(x) - f(\dfrac{1}{x-3}) \leqslant 2$.

Proof: Since $1 = f(2)$, then $2 = f(2)+f(2)$. And $f(\dfrac{x}{y}) = f(x)-f(y) \Rightarrow f(y)+f(\dfrac{x}{y}) = f(x)$. Let $y = 2$, $\dfrac{x}{y} = 2$, we have $x = 2y = 4$. Then $f(2)+f(2) = f(4)$. Thus $f(4) = 2$. Hence $f(x) - f(\dfrac{1}{x-3}) \leqslant 2 \Rightarrow f[x(x-3)] \leqslant f(4)$. Since $f(x)$ is an increasing function on $(0, +\infty)$, we have $\begin{cases} x(x-3) \leqslant 4 \\ x > 0 \\ x - 3 > 0 \end{cases}$. Solving the equation system to obtain $3 < x \leqslant 4$. Therefore the solution set of the inequality is $\{x|3 < x \leqslant 4\}$.

6.54 ★★ Let $f(x) = x^2 + ax + b\cos x$, $\{x|f(x) = 0, x \in R\} = \{x|f(f(x)) = 0, x \in R\} \neq \phi$. Find all values of a and b which satisfy the conditions.

Solution: Let $x_0 \in \{x|f(x) = 0, x \in R\}$, then $b = f(0) = f(f(x_0)) = 0$. Thus $f(x) = x(x+a)$, $f(f(x)) = f(x)(f(x)+a) = x(x+a)(x^2+ax+a)$. Obviously, $a = 0$ satisfies the problem. If $a \neq 0$, since the roots of $x^2+ax+a = 0$ are neither 0 nor $-a$. Hence $x^2 + ax + a = 0$ does not have real roots. Therefore $\Delta = a^2 - 4a < 0$. We have $0 < a < 4$.
After all, all a and b which satisfy the conditions are $0 \leqslant a < 4$, $b = 0$.

6.55 ★★★ Suppose $a, b, c \in R$, and their absolute values are no more than 1. Show $ab + bc + ca + 1 \geqslant 0$.

Proof: We introduce the function $f(a) = ab + bc + ca + 1$. Since $a \in [-1, 1]$, then we only need to show $f(-1) \geqslant 0$, $f(1) \geqslant 0$. Then we show $f(a) \geqslant 0$.
When $a = -1$, then $f(-1) = -b + bc - c + 1 = (b-1)(c-1)$. Since $b, c \in [-1, 1]$, then $(b-1)(c-1) \geqslant 0$. Thus $f(-1) \geqslant 0$.
When $a = 1$, then $f(1) = b + bc + c + 1 = (b+1)(c+1)$. Since $b, c \in [-1, 1]$, then $(b+1)(c+1) \geqslant 0$. Thus $f(1) \geqslant 0$. Therefore $ab + bc + ca + 1 \geqslant 0$.

6.56 ★★ The function $f(x) = \dfrac{a \cdot 2^x - 1}{2^x + 1}(a \in R)$ is an odd function. (1) Evaluate a. (2) Find the inverse function of $f(x)$. (3) For arbitrary $k \in (0, +\infty)$, solve the inequality of $f^{-1}(x) > \log_2 \dfrac{1+x}{k}$.

Solution: (1) Since the $f(x)$ is an odd function, we have $f(0) = 0$. Then $\dfrac{a-1}{2} = 0$.

Thus $a = 1$. While $f(x) + f(-x) = \dfrac{2^x - 1}{2^x + 1} + \dfrac{2^{-x} - 1}{2^{-x} + 1} = \dfrac{2^x - 1}{2^x + 1} + \dfrac{1 - 2^x}{1 + 2^x} = 0$ satisfies that $f(x)$ is an odd function.
(2) Since $y = f(x) = \dfrac{2^x - 1}{2^x + 1} = 1 - \dfrac{2}{2^x + 1}$, then $2^x = \dfrac{1+y}{1-y}, (-1 < y < 1)$. Thus $f^{-1}(x) = \log_2 \dfrac{1+x}{1-x}, (-1 < x < 1)$.
(3) Since $f^{-1}(x) = \log_2 \dfrac{1+x}{1-x} > \log_2 \dfrac{1+x}{k} \Rightarrow \Rightarrow \begin{cases} \dfrac{1+x}{1-x} > \dfrac{1+x}{k} \\ -1 < x < 1 \end{cases} \Rightarrow \begin{cases} x > 1 - k \\ -1 < x < 1 \end{cases}$.
When $0 < k < 2$, the solution set of the inequality is $\{x | 1 - k < x < 1\}$. When $k \geqslant 2$, the solution set of the inequality is $\{x | -1 < x < 1\}$.

6.57 ★★ The function $f(x)$ is defined for real numbers, and $f(x+2)f(1 - f(x)) = 1 + f(x)$. (1) Show that $f(x)$ is a periodic function. (2) If $f(1) = 2 + \sqrt{3}$, find the value of $f(2013)$.

(1) Proof: Since $f(x+2)(1 - f(x)) = 1 + f(x)$ holds on R, then $f(1) \neq 1$. Thus $f(x+2) = \dfrac{1 + f(x)}{1 - f(x)}$. Hence $f(x+4) = f[(x+2)+2] = \dfrac{1 + f(x+2)}{1 - f(x+2)} = \dfrac{1 + \frac{1+f(x)}{1-f(x)}}{1 - \frac{1+f(x)}{1-f(x)}} = \dfrac{2}{-2f(x)} = -\dfrac{1}{f(x)}$. On the other hand, $f(x+8) = f[(x+4)+4] = -\dfrac{1}{f(x+4)} = f(x)$, then $f(x)$ is a periodic function with the period 8.
(2)Solution: Since $f(1) = 2 + \sqrt{3}$ and $f(x)$ is a periodic function with the period 8, we have $f(8k+1) = 2 + \sqrt{3}, (k \in Z)$. Thus $f(2009) = f(251 \times 8 + 1) = f(1) = 2 + \sqrt{3}$, $f(2013) = f(2009 + 4) = -\dfrac{1}{f(2009)} = -\dfrac{1}{2 + \sqrt{3}} = \sqrt{3} - 2$.

6.58 ★★★ Find the range of the function $y = x + \sqrt{x^2 - 3x + 2}$.

Solution: From the given condition, we have $\sqrt{x^2 - 3x + 2} = y - x \geqslant 0$. Then we square both sides of the equation to obtain $x^2 - 3x + 2 = y^2 - 2xy + x^2$. This means $(2y - 3)x = y^2 - 2$. From the above equation, we have $y \neq \frac{3}{2}$ and $x = \frac{y^2 - 2}{2y - 3}$. Additionally, since $y \geqslant x$, then $y \geqslant \frac{y^2 - 2}{2y - 3} \Rightarrow \frac{y^2 - 3y + 2}{2y - 3} \geqslant 0 \Rightarrow \frac{(y-1)(y-2)}{y - \frac{3}{2}} \geqslant 0$.

Then $1 \leqslant y < \frac{3}{2}$ or $y \geqslant 2$.

Now we have $y_0 \in [2, +\infty)$ arbitrarily. Let $x_0 = \frac{y_0^2 - 2}{2y_0 - 3}$, then $x_0 - 2 = \frac{y_0^2 - 2}{2y_0 - 3} - 2 = \frac{(y_0 - 2)^2}{2y_0 - 3} \geqslant 0$. Thus $x_0 \geqslant 2$. Hence $x_0^2 - 3x_0 + 2 \geqslant 0$, and $y_0 = x_0 + \sqrt{x_0^2 - 3x_0 + 2}$.

We have $y \in [1, \frac{3}{2})$ arbitrarily. Let $x_0 = \frac{y_0^2 - 2}{2y_0 - 3}$, then $x_0 - 1 = \frac{y_0^2 - 2}{2y_0 - 3} - 1 = \frac{(y_0 - 1)^2}{2y_0 - 3} \leqslant 0$. Thus $x_0 \leqslant 1$. Hence $x_0^2 - 3x_0 + 2 \geqslant 0$, and $y_0 = x_0 + \sqrt{x_0^2 - 3x_0 + 2}$.

As a conclusion, the range of the function $y = x + \sqrt{x^2 - 3x + 2}$ is $[1, \frac{3}{2}) \cup [2, +\infty)$.

6.59 ★★ Given the coefficient of the quadratic term of the quadratic function $f(x)$ is a, and the solution of the inequality $f(x) > -2x$ is $(1,3)$. (1) If the function $f(x)+6a=0$ has two equal roots, find the analytic form of $f(x)$. (2) If the maximum value of $f(x)$ is a positive number, find the range of a.

Solution: (1) Since the solution of the inequality $f(x) > -2x$ is $(1,3)$, we let $f(x)+2x = a(x-1)(x-3)$ and $a<0$. Then $f(x) = a(x-1)(x-3)-2x = ax^2-(2+4a)x+3a$ ①. By substituting the equation $f(x)+6a=0$ into ①, we have $ax^2-(2+4a)x+9a=0$ ②. Since the equation ② has two equal roots, then $\Delta = [-(2+4a)]^2-36a^2=0$. Thus $a=1$ or $a=-\frac{1}{5}$. Since $a<0$, then $a=-\frac{1}{5}$. By substituting $a=-\frac{1}{5}$ into ①, we have $f(x) = -\frac{1}{5}x^2-\frac{6}{5}x-\frac{3}{5}$.
(2) Since $f(x) = ax^2-2(1+2a)x+3a = a(x-\frac{1+2a}{a})^2-\frac{a^2+4a+1}{a}$ and $a<0$, then the maximum value of $f(x)$ is $-\frac{a^2+4a+1}{a}$. Since $\begin{cases} -\frac{a^2+4a+1}{a} > 0 \\ a<0 \end{cases}$, then $a<-2-\sqrt{3}$ or $-2+\sqrt{3}<a<0$. Therefore the range of a is $(-\infty,-2-\sqrt{3})$ or $(-2+\sqrt{3},0)$ when the maximum value of $f(x)$ is positive.

6.60 ★★ Given $f(x)=(x^n+c)^m$, $g(x)=(ax^m+1)$, $h(x)=(bx^n+1)$, and $f(x)\equiv g(x)h(x)$ where m,n are both positive integers. Compute $|a+b+c|$.

Solution: Since $f(x)\equiv g(x)h(x)$, then $(x^n+c)^m \equiv (ax^m+1)(bx^n+1)$. Comparing the leading terms, we have $mn=m+n \Rightarrow (m-1)(n-1)=1$. Since m,n are both positive integers, then $m=2$, $n=2$. Thus $(x^2+c)^2 \equiv (ax^2+1)(bx^2+1) \Rightarrow x^4+2cx^2+c^2 \equiv abx^4+(a+b)x^2+1 \Rightarrow \begin{cases} ab=1 \\ a+b=2c \\ c^2=1 \end{cases}$. Then $a=1,b=1,c=1$ or $a=-1,b=-1,c=-1$. Hence $|a+b+c|=3$.

6.61 ★★★ Given the function $f(x)=ax^2+4x+b$ $(a<0, a,b\in R)$, the two real roots of the equation $f(x)=0$ with respect to x are x_1,x_2, and the two real roots of the equation $f(x)=x$ with respect to x are α,β.
(1) If $|\alpha-\beta|=1$, Find the relation formula between a and b. (2) If a and b are both negative integers, and $|\alpha-\beta|=1$, find the analytic expression of $f(x)$. (3) If $\alpha<1<\beta<2$, show $(x_1+1)(x_2+1)<7$.

Solution: (1) Since $f(x)=ax^2+4x+b$ and $f(x)=x$, then $ax^2+3x+b=0$. From the given condition, we have $\begin{cases} \alpha+\beta=-\frac{3}{a} \\ \alpha\beta=\frac{b}{a} \\ |\alpha-\beta|=1 \end{cases}$. Then $a^2+4ab=9$.

(2) Since a, b are both negative integers, then $a+4b$ is also a negative integer, and $a+4b \leqslant -5$. Since $a^2+4ab=9$, then $a(a+4b)=9$. Thus $a=-1, a+4b=-9$. Then $b=-2$. Hence $f(x)=-x^2+4x-2$.
(3) Let $g(x)=ax^2+3x+b$, then the sufficient and necessary condition of $\alpha<1<\beta<2$ is $\begin{cases} g(1)>0 \\ g(2)<0 \end{cases}$, that is $\begin{cases} a+b+3>0 \\ 4a+b+6<0 \end{cases}$. Since $x_1+x_2=-\frac{4}{a}$, $x_1x_2=\frac{b}{a}$, then $(x_1+1)(x_2+1)-7=x_1x_2+(x_1+x_2)-6=\frac{b}{a}-\frac{4}{a}-6=\frac{-6a+b-4}{a}=\frac{\frac{10}{3}g(1)-\frac{7}{3}g(2)}{a}$. Since $g(1)>0, g(2)<0, a<0$, then $(x_1+1)(x_2+1)-7<0$. Thus we show that $(x_1+1)(x_2+1)<7$.

6.62 ★★ Let $f(x)=ax^2+bx+c \quad (a>b>c)$, $f(1)=0$, $g(x)=ax+b$.
(1) Show the graphs of $y=f(x)$ and $y=g(x)$ have two intersection points.
(2) Let the two intersection points of the graphs of $y=f(x)$ and $y=g(x)$ are A, B, their projections on x-axis are A_1, B_1. Find the range of $|A_1B_1|$.

(1) Proof: From the given condition, we have $\begin{cases} a+b+c=0 \\ a>b>c \end{cases} \Rightarrow a>0, c<0$. Let $ax^2+bx+c=ax+b$, then $ax^2+(b-a)x+c-b=0$, $\Delta=(b-a)^2-4a(c-b)=(b+a)^2-4ac$. Since $a>0, c<0$, then $\Delta>0$. Thus the graphs of $y=f(x)$ and $y=g(x)$ have two intersection points.
(2) Solution: Let the two roots of $ax^2+(b-a)x+c-b=0$ are x_1, x_2, then $x_1+x_2=\frac{a-b}{a}$, $x_1x_2=\frac{c-b}{a}$. Thus $|A_1B_1|=|x_1-x_2|=\sqrt{(x_1+x_2)^2-4x_1x_2}=\frac{\sqrt{(b+a)^2-4ac}}{a}$. Since $b=-(a+c)$, then $|A_1B_1|=\frac{\sqrt{c^2-4ac}}{a}=\sqrt{(\frac{c}{a})^2-4(\frac{c}{a})}=\sqrt{(\frac{c}{a}-2)^2-4}$. Since $b=-(a+c)$ and $a>b>c, a>0, c<0$, then $\begin{cases} a>-(a+c) \\ -(a+c)>c \end{cases}$.

Solve the equation system, then $-2<\frac{c}{a}<-\frac{1}{2}$. Thus $\frac{3}{2}<|A_1B_1|<2\sqrt{3}$. Hence the range of $|A_1B_1|$ is $(\frac{3}{2}, 2\sqrt{3})$.

6.63 ★★ The function $f(x)$ is defined for real numbers, and for arbitrary $x, y \in R$, $f(x)+f(y)=f(x+y)-xy-1$. Since $f(1)=1$, Find the integer number n such that $f(n)=n$.

Solution: For the given equation, let $y=1$, then $f(x+1)=f(x)+x+2$. Thus $f(x+1)>x+1$ when $x\in Z$. For the given equation, let $x=y=0$, then $2f0)=f(0)-1$. Thus $f(0)=-1$. Let $x=-1, y=1$, then $f(-1)+f(1)=f(0)$. Thus $f(-1)=f(0)-f(1)=-2$. Let $x=-2, y=1$, then $f(-2)+f(1)=f(-1)+1$. Thus $f(-2)=f(-1)-f(1)+1=-2$. When $x\in Z, x\leqslant -3$, then $f(x)>x$. Therefore the integer number n such that $f(n)=n$ is $n=1$ or $n=-2$.

6.64 ★★ Given the odd function $f(x)$ defined for real numbers, and $f(x) > 0$ when $x \geqslant 0$. Does there exist a real number λ such that $f(\cos 2\theta - 3) + f(4\lambda - 2\lambda\cos\theta) > f(0)$ hold for all $\theta \in [0, \frac{\pi}{2}]$? If yes, find its range. If no, please explain the reason.

Solution: Since $f(x)$ is an odd function and is an increasing function on $[0, +\infty)$, then $f(x)$ is an increasing function on R, and $f(0) = 0$. Since $f(\cos 2\theta - 3) + f(4\lambda - 2\lambda\cos\theta) > f(0) = 0 \Rightarrow f(\cos 2\theta - 3) > -f(4\lambda - 2\lambda\cos\theta) = f(2\lambda\cos\theta - 4\lambda) \Rightarrow \cos 2\theta - 3 > 2\lambda\cos\theta - 4\lambda \Rightarrow \cos^2\theta - \lambda\cos\theta + 2\lambda - 2 > 0$. If the inequality hold for arbitrary $\theta \in [0, \frac{\pi}{2}]$, we should have λ is lager than the maximum value of $y = \dfrac{2 - \cos^2\theta}{2 - \cos\theta}$. Let $t = \cos\theta \in [0, 1]$, then $y = \dfrac{2 - t^2}{2 - t} \Rightarrow t^2 - yt + 2y - 2 = 0$, $\Delta = y^2 - 8y + 8 \geqslant 0$. Thus $y_{max} = 4 - 2\sqrt{2}$. Hence $\lambda \in (4 - 2\sqrt{2}, +\infty)$.

6.65 ★★★ If the quadratic function $f(x) = ax^2 + bx + 1$ $(a, b \in R, a > 0)$, and the function $f(x) = x$ has two real roots x_1, x_2. (1) If $x_1 < 2 < x_2 < 4$, let the symmetric axis of $f(x)$ is $x = x_0$. Show $x_0 > -1$. (2) If $|x_1| < 2, |x_2 - x_1| = 2$, find the range of b.

(1) Proof: Let $g(x) = f(x) - x = ax^2 + (b-1)x + 1$, then the two roots of $g(x) = 0$ are x_1, x_2. Since $a > 0$ and $x_1 < 2 < x_2 < 4$, we have $\begin{cases} g(2) < 0 \\ g(4) > 0 \end{cases} \Rightarrow \begin{cases} 4a + 2b - 1 < 0 \\ 16a + 4b - 3 > 0 \end{cases}$

$\Rightarrow \begin{cases} 3+3\cdot\dfrac{b}{2a}-\dfrac{3}{4a}<0 & ① \\ -4-2\cdot\dfrac{b}{2a}+\dfrac{3}{4a}<0 & ② \end{cases}$ By ① + ②, we have $\dfrac{b}{2a}<1$. Thus $x_0=-\dfrac{b}{2a}>-1$.

(2) Solution: $(x_1-x_2)^2=(x_1+x_2)^2-4x_1x_2=(\dfrac{b-1}{a})^2-\dfrac{4}{a}=4 \Rightarrow 2a+1=\sqrt{(b-1)^2+1}$. Since $x_1x_2=\dfrac{1}{a}>0$, then the signs of x_1,x_2 are same. Thus $|x_1|<2, |x_2-x_1|=2$ are equivalent to $\begin{cases} 0<x_1<2<x_2 \\ 2a+1=\sqrt{(b-1)^2+1} \end{cases}$ or $\begin{cases} x_2<-2<x_1<0 \\ 2a+1=\sqrt{(b-1)^2+1} \end{cases}$ $\Rightarrow \begin{cases} g(2)<0 \\ g(0)>0 \\ 2a+1=\sqrt{(b-1)^2+1} \end{cases}$ or $\begin{cases} g(-2)<0 \\ g(0)>0 \\ 2a+1=\sqrt{(b-1)^2+1} \end{cases}$. Then we have $b<\dfrac{1}{4}$ or $b>\dfrac{7}{4}$.

6.66 ★★★ Given a linear function $f(x)=kx+h \quad (k\neq 0)$. $f(m)>0$ and $f(n)>0$ when $m<n$.
(1) Show $f(x)>0$ holds for arbitrary $x\in(m,n)$. (2) By applying the condition of (1), show if $a,b,c\in R$ and $|a|<1,|b|<1,|c|<1$, then $ab+bc+ca>-1$.

Proof: (1) When $k>0$, then the function $f(x)=kx+h$ is an increasing function on $x\in R$, $m<x<n$, $f(x)>f(m)>0$. When $k<0$, then the function $f(x)=kx+h$ is a decreasing function on $x\in R$, $m<x<n$, $f(x)>f(n)>0$. Thus $f(x)>0$ holds for arbitrary $x\in(m,n)$.
(2) Rewrite $ab+bc+ca+1$ as $(b+c)a+bc+1$. Introduce the function $f(x)=(b+c)x+bc+1$. Then $f(a)=(b+c)a+bc+1$. $f(a)=bc+1=-c^2+1$ when $b+c=0$ which means $b=-c$. Since $|c|<1$, then $f(a)=-c^2+1>0$. When $b+c\neq 0$, then $f(x)=(b+c)x+bc+1$ is a linear function. Since $|b|<1,|c|<1$, then $f(1)=b+c+bc+1=(1+b)(1+c)>0$, $f(-1)=-b-c+bc+1=(1-b)(1-c)>0$. From (1), we obtain all $f(a)>0$ for $|a|<1$. Thus $(b+c)a+bc+1=ab+bc+ca+1>0$. Hence we show that $ab+bc+ca>-1$.

6.67 ★★ Let $P(x+a,y_1)$, $Q(x,y_2)$, $R(2+a,y_3)$ be the three distinct points on the graph of the inverse function of the function $f(x)=2^x+a$. If there is one but the only one real number x such that y_1,y_2,y_3 form an arithmetic sequence, find the range of a. And Compute the area of $\triangle PQR$ when the distance from the origin to the point R is smallest.

Solution: From the given condition, $P(x+a,y_1)$, $Q(x,y_2)$, $R(2+a,y_3)$ are on the graph of the function $f^{-1}(x)=\log_2(x-a)$. Then $y_1=\log_2 x$, $y_2=\log_2(x-a)$, $y_3=1$. Since $y_1+y_3=2y_2$, then $\log_2 x+1=2\log_2(x-a) \Leftrightarrow \begin{cases} x>a \\ (x-a)^2=2x \end{cases}$.

From the given condition, the equation $x^2-(2a+2)x+a^2=0$ has an unique root. Then $\Delta=(2a+2)^2-4a^2=0$. Thus $a=-\frac{1}{2}$ or $a\geqslant 0$. Additionally, $|OR|^2=(2+a)^2+1$. When $a=-\frac{1}{2}$, then $|OR|$ reaches the minimum value, for which $x=\frac{1}{2}$, $P(0,-1)$, $Q(\frac{1}{2},0)$, $R(\frac{3}{2},1)$. Since $|PQ|=\sqrt{(\frac{1}{2}-0)^2+(0+1)^2}=\frac{\sqrt{5}}{2}$, the distance from the point R to the straight line $PQ: 2x-y-1=0$ is $d=\frac{1}{\sqrt{5}}$. Hence $S_{\triangle PQR}=\frac{1}{2}|PQ|\cdot d=\frac{1}{4}$.

6.68 ★★ Given the function $f(x)=2^x-\frac{a}{2^x}$. (1) The graph of $y=g(x)$ is the graph of $y=f(x)$ shifted to the right by 2 units. Find the analytic expression of $g(x)$. (2) The graph of $y=h(x)$ and the graph of $y=g(x)$ are symmetric about the line $y=1$. Find the analytic expression of $h(x)$. (3) Let $F(x)=\frac{1}{a}f(x)+h(x)$, the minimum value of $F(x)$ is m, and $m>2+\sqrt{7}$, find the range of the real number a.

Solution: (1) $g(x)=f(x-2)=2^{x-2}-\frac{a}{2^{x-2}}$
(2) An arbitrary point on $y=h(x)$ is $P(x,y)$. Then the symmetric point of P about the linear line $y=1$ is $Q(x,2-y)$ and the point Q is on the graph $y=g(x)$. Thus $h(x)=2-2^{x-2}+\frac{a}{2^{x-2}}$.
(3) $F(x)=\frac{1}{a}(2^x-\frac{a}{2^x})+2-2^{x-2}+\frac{a}{2^{x-2}}=(\frac{1}{a}-\frac{1}{4})2^x+(4a-1)\frac{1}{2^x}+2$.
① When $a<0$, $\frac{1}{a}-\frac{1}{4}<0$, $4a-1<0$, then $F(x)<2$. It is contradictory with $m>2+\sqrt{7}$.
② When $0<a\leqslant\frac{1}{4}$, $\frac{1}{a}-\frac{1}{4}>0$, $4a-1\leqslant 0$, then $F(x)$ is an increasing function on R. Thus $F(x)$ has no minimum value.
③ When $\frac{1}{4}<a<4$, $\frac{1}{a}-\frac{1}{4}>0$, $4a-1>0$, then $F(x)\geqslant 2\sqrt{\frac{(4-a)(4a-1)}{4a}}+2=m$.
Since $m>2+\sqrt{7}$, then $\begin{cases}\frac{1}{4}<a<4\\ \frac{(4-a)(4a-1)}{a}>7\end{cases}$. Then $\frac{1}{2}<a<2$.

6.69 ★★ If an odd function $f(x)$ is defined on R, and $f(x)=2x-x^2$ when $x\geqslant 0$. Let the range of the function $y=f(x), x\in[a,b]$ be $[\frac{1}{b},\frac{1}{a}]$, $(a\neq b)$. Find the values of a,b.

Solution: Since $y=f(x)$ is an odd function, then $f(x)=x^2+2x$ when $x<0$. Thus $f(x)=\begin{cases}2x-x^2,(x\geqslant 0)\\ x^2+2x,(x<0)\end{cases}$. Since $[a,b]$ and $[\frac{1}{b},\frac{1}{a}]$ exist at the same time, and $a\neq b$, then $a<b$, $\frac{1}{b}<\frac{1}{a}$. Thus the signs of a,b are same.

① When $1 \leqslant a < b$, since $\begin{cases} 2a - a^2 = \dfrac{1}{a} \\ 2b - b^2 = \dfrac{1}{b} \end{cases} \Rightarrow \begin{cases} (a-1)(a^2-a-1) = 0 \\ (b-1)(b^2-b-1) = 0 \end{cases}$ We have $a = 1, b = \dfrac{1+\sqrt{5}}{2}$.

② When $-\infty < a < b \leqslant -1$, since $\begin{cases} 2a + a^2 = \dfrac{1}{a} \\ 2b + b^2 = \dfrac{1}{b} \end{cases} \Rightarrow \begin{cases} (a+1)(a^2+a-1) = 0 \\ (b+1)(b^2+b-1) = 0 \end{cases}$ We have $a = \dfrac{-1-\sqrt{5}}{2}, b = -1$.

③ When $0 < a < b < 1$, since $\begin{cases} 2a - a^2 = \dfrac{1}{b} \\ 2b - b^2 = \dfrac{1}{a} \end{cases}$. The equation system has no solution.

④ When $-1 < a < b < 0$, since $\begin{cases} 2a + a^2 = \dfrac{1}{b} \\ 2b + b^2 = \dfrac{1}{a} \end{cases}$. The equation system has no solution.

⑤ When $0 < a < 1 < b < +\infty$, since $\dfrac{1}{a} = 1$, then $a = 1$. It is contradicting to $a < 1$.

⑥ When $-\infty < a < -1 < b < 0$, since $\dfrac{1}{b} = -1$, then $b = -1$. It is contradicting to $b > -1$.

As a conclusion, we have $\begin{cases} a = 1 \\ b = \dfrac{1+\sqrt{5}}{2} \end{cases}$ or $\begin{cases} a = \dfrac{-1-\sqrt{5}}{2} \\ b = -1 \end{cases}$.

6.70 ★★ Given a function $f(x)$, and $f(x+y) - f(y) = (x+2y+1)x$ holds for all x, y, and $f(1) = 0$.
(1) Evaluate $f(0)$. (2) If $f(x_1)+2 < \log_a x_2$ holds for arbitrary $x_1 \in (0, \frac{1}{2})$, $x_2 \in (0, \frac{1}{2})$, find the range of a.

Solution: (1) Let $x = 1$ and $y = 0$ in the formula $f(x+y) - f(y) = (x+2y+1)x$, then $f(1) - f(0) = 2$. Since $f(1) = 0$, then $f(0) = -2$.

(2) Let $y = 0$ in the formula $f(x+y) - f(y) = (x+2y+1)x$, then $f(x) - f(0) = (x+1)x$. From (1), we have $f(0) = -2$. Then $f(x) + 2 = x^2 + x$. Since $x_1 \in (0, \frac{1}{2})$, then $f(x_1) + 2 = x_1^2 + x_1 = (x_1 + \frac{1}{2})^2 - \frac{1}{4}$ is increasing when $x_1 \in (0, \frac{1}{2})$. Thus $f(x_1) + 2 \in (0, \frac{3}{4})$. To make $f(x_1) + 2 < \log_a x_2$ hold for arbitrary $x_1 \in (0, \frac{1}{2})$, $x_2 \in (0, \frac{1}{2})$, when $a > 1$, $\log_a x_2 < \log_a \frac{1}{2}$, obviously the inequality does not hold. when $0 < a < 1$, $\log_a x_2 > \log_a \frac{1}{2}$, then $\begin{cases} 0 < a < 1 \\ \log_a \frac{1}{2} \geqslant \frac{3}{4} \end{cases}$. Solving the inequality system to generate $\frac{\sqrt[3]{4}}{4} \leqslant a < 1$.

6.71 ★★★ If on the interval $[1, 2]$, the function $f(x) = x^2 + px + q \quad (p \in [-4, -2])$ and $g(x) = x + \frac{1}{x^2}$ reach the same minimum value on the same point. Find the maximum value of $f(x)$ on the interval.

Solution: When $x \in [1, 2]$, $g(x) = x + \frac{1}{x^2} = \frac{x}{2} + \frac{x}{2} + \frac{1}{x^2} \geqslant 3 \cdot \sqrt[3]{\frac{x}{2} \cdot \frac{x}{2} \cdot \frac{1}{x^2}} = \frac{3}{2}\sqrt[3]{2}$, and the equation holds if and only if $\frac{x}{2} = \frac{1}{x^2}$ i.e. $x = \sqrt[3]{2} \in [1, 2]$. Thus $g(x)_{\min} = \frac{3}{2}\sqrt[3]{2}$.
On the other hand, $f(x) = x^2 + px + q = (x + \frac{p}{2})^2 + \frac{4q - p^2}{4}$. Since $f(x)$ and $g(x)$ reach the same minimum value on the same point, then $-\frac{p}{2} = \sqrt[3]{2}$, $\frac{4q - p^2}{4} = \frac{3}{2}\sqrt[3]{2}$.
Thus $p = -2\sqrt[3]{2}, q = \frac{3}{2}\sqrt[3]{2} + \sqrt[3]{4}$.
Since the symmetric axis of $f(x) = x^2 + px + q$ is $x = \sqrt[3]{2}$ and the coefficient of the quadratic term is positive, then the function $f(x)$ is decreasing on the interval $[1, \sqrt[3]{2}]$. Thus $f(1) = 1 - \frac{\sqrt[3]{2}}{2} + \sqrt[3]{4}$ when $x = 1$. The function $f(x)$ is increasing on the interval $[\sqrt[3]{2}, 2]$. Thus $f(2) = 4 - \frac{5}{2}\sqrt[3]{2} + \sqrt[3]{4}$ when $x = 2$. Since $f(2) > f(1)$. Therefore the maximum value of $f(x)$ on the interval $[1, 2]$ is $f_{\max}(x) = f(2) = 4 - \frac{5}{2}\sqrt[3]{2} + \sqrt[3]{4}$.

6.72 ★★★ Suppose the function $y = f(x)$ is a periodic function defined on R with the period 5. The function $y = f(x)$ is an odd function on the interval $[-1, 1]$. $y = f(x)$ is a linear function on the $[0, 1]$, and it is a quadratic function on the interval $[1, 4]$. The function reaches its minimum value -5 occurring at $x = 2$.
(1) Show $f(1)+f(4) = 0$. (2) Find the analytic expression of $y = f(x)$ when $x \in [1, 4]$.

(3) Find the analytic expression of $y = f(x)$ when $x \in [4, 9]$.

(1) Proof: Since $f(x)$ is a periodic function with the period 5, then $f(4) = f(4-5) = f(-1)$. Since $y = f(x)$ is an odd function on the interval $[-1, 1]$, then $f(1) = -f(-1) = -f(4)$. Thus $f(1) + f(4) = 0$.
(2) Solution: According to the given condition, we assume $f(x) = a(x-2)^2 - 5 (a > 0)$ when $x \in [1, 4]$. From (1), then $f(1)+f(4) = 0$, that is $a(1-2)^2 - 5 + a(4-2)^2 - 5 = 0$. Thus $a = 2$. Hence $f(x) = 2(x-2)^2 - 5, (1 \leqslant x \leqslant 4)$.
(3) Solution: Since $y = f(x)$ is an odd function on the interval $[-1, 1]$, then $f(0) = 0$. Since $y = f(x)$ is a linear function on the $[0, 1]$, we choose $f(x) = kx \quad (0 \leqslant x \leqslant 1)$. Since $f(1) = 2(1-2)^2 - 5 = -3$, then $k = -3$. Thus, when $0 \leqslant x \leqslant 1$, then $f(x) = -3x$. When $-1 \leqslant x < 0$, then $f(x) = -f(-x) = -3x$. when $-1 \leqslant x \leqslant 1$, then $f(x) = -3x$. Hence, when $4 \leqslant x \leqslant 6$, which is equivalent to $-1 \leqslant x - 5 \leqslant 1$, then $f(x) = f(x-5) = -3(x-5) = -3x + 15$. when $6 < x \leqslant 9$, which is equivalent to $1 < x - 5 \leqslant 4$, then $f(x) = f(x-5) = 2[(x-5) - 2]^2 - 5 = 2(x-7)^2 - 5$.
Thus $f(x) = \begin{cases} -3x + 15, (4 \leqslant x \leqslant 6) \\ 2(x-7)^2 - 5, (6 < x \leqslant 9) \end{cases}$.

6.73 ★★★ Given the function $f(x) = |x - a|$, $g(x) = x^2 + 2ax + 1$ (a is a positive constant). The y-intercepts of the graphs of $f(x)$ and $g(x)$ are equal.
(1) Evaluate a. (2) What is the monotone increasing interval of $f(x) + g(x)$. (3) If $n \in N^*$, show $10^{f(n)} \cdot (\frac{4}{5})^{g(n)} < 4$.

(1) Solution: From the given condition, we have $f(0) = g(0)$. Then $|a| = 1$. Since $a > 0$, then $a = 1$.
(2) Solution: $f(x) + g(x) = |x-1| + x^2 + 2x + 1$. When $x \geqslant 1$, then $f(x) + g(x) = x^2 + 3x$. It is monotonous increasing on the interval $[1 + \infty)$. When $x < 1$, then $f(x) + g(x) = x^2 + x + 2$. It is monotonous decreasing on the interval $[-\frac{1}{2}, 1)$.
(3) Proof: Let $T_n = 10^{f(n)} \cdot (\frac{4}{5})^{g(n)}$. Solving the inequality $\frac{T_{n+1}}{T_n} < 1$ with $T_n > 0$ to obtain $10(\frac{4}{5})^{2n+3} < 1$. By solving the inequality, we have $n > \frac{1}{2\lg 0.8} - \frac{3}{2} \approx 3.7$. Since $n \in N^*$, then $n \geqslant 4$. Thus $T_1 \leqslant T_2 \leqslant T_3 \leqslant T_4$, while $T_4 > T_5 > T_6 > \cdots$.
Therefore $10^{f(n)} \cdot (\frac{4}{5})^{g(n)} \leqslant 10^{f(4)} \cdot (\frac{4}{5})^{g(4)} = 10^3 \cdot (\frac{4}{5})^{25} < 4$.

6.74 ★★ If the symmetric axis of the quadratic function $f(x) = x^2 + bx + c$ is on the right of y-axis. Its y-intercept is $P(0, -3)$, and its x-intercepts are A, B. Its vertex is Q. If the area of $\triangle QAB$ is 8, find the analytic expression of the quadratic function.

Solution: Let $A(x_1,0), B(x_2,0)$, then x_1, x_2 are the two roots of the equation $x^2+bx+c=0$. Since $x_1+x_2=-b$, $x_1\cdot x_2=c$, then $AB=|x_1-x_2|=\sqrt{(x_1+x_2)^2-4x_1x_2}=\sqrt{b^2-4c}$. The vertex of the parabolic curve is $\frac{4c-b^2}{4}$. Thus $S_{\triangle QAB}=\frac{1}{2}\sqrt{b^2-4c}\cdot|\frac{4c-b^2}{4}|$ which is equivalent to $8=\frac{1}{8}\sqrt{b^2-4c}\cdot(b^2-4c)$. Then $\sqrt{b^2-4c}=4$ ①. Since the intersection of the parabolic curve and y-axis is $(0,-3)$, then substituting $c=-3$ into ① to generate $b=\pm 2$. When $b=2$, then the symmetric axis is $x=-\frac{2}{2}=-1$ which is on the left of y-axis. It is contradicting to the given conditions. Thus $b=-2$. When $b=-2$, the analytic expression of the quadratic function is $y=x^2-2x-3$.

6.75 ★★★ If $k\in R$, $f(x)=\frac{x^4+kx^2+1}{x^4+x^2+1}$. $f(a), f(b), f(c)$ form three sides of a triangle for arbitrary real numbers a,b,c. Find the range of k.

Solution: To have $f(x)>0$, we only need $x^4+kx^2+1>0$.
$x^4+kx^2+1>0$ holds when $k\geqslant 0$. We need $\Delta=k^2-4<0$ when $k<0$. This means $-2<k<0$. Thus $f(x)>0$ when $k>-2$.
① When $k=1$, then $f(x)=1$ which satisfies the given conditions.
② When $k>1$, then $f(x)=1+\frac{(k-1)x^2}{x^4+x^2+1}\geqslant 1$ and the equation holds when $x=0$.

Additionally, $f(x) = 1 + \dfrac{(k-1)x^2}{x^4+x^2+1} \leqslant 1 + \dfrac{(k-1)x^2}{3x^2} = \dfrac{k+2}{3}$ and the equation holds when $x = 1$. Thus $f_{\min}(x) = 1$, $f_{\max}(x) = \dfrac{k+2}{3}$.
According to the property that the sum of two sides is larger than the third side, we have $2 \times 1 > \dfrac{k+2}{3}$. Then $k < 4$. Thus $1 < k < 4$ satisfies the given condition.
③ When $-2 < k < 1$, similarly with ②, we have $f_{\max}(x) = 1, f_{\min}(x) = \dfrac{k+2}{3}$. Since $2 \times \dfrac{k+2}{3} > 1$, then $k > -\dfrac{1}{2}$. Therefore $-\dfrac{1}{2} < k < 1$.
As a conclusion, the range of k is $-\dfrac{1}{2} < k < 4$.

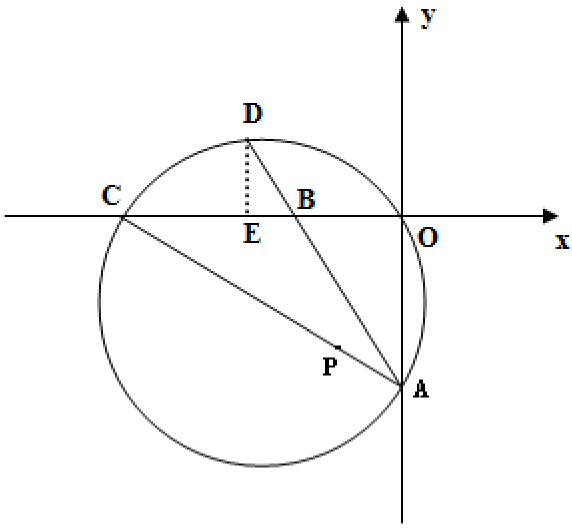


Figure 5

6.76 ★★★ As shown in Figure 5, in the Cartesian coordinates, the points B, C are on the negative x-axis, the points A is on the negative y-axis. The circle with diameter AC intersects the extended line of AB at the point D. $\overset{\frown}{CD} = \overset{\frown}{AO}$. If $AB = 10$, and the lengths of AO and BO are the two roots of the quadratic function $x^2+kx+48 = 0$, $AO > BO > 0$.
(1) Find the coordinates of D. (2) If the point P is on the straight line AC, and $AP = \dfrac{1}{4}AC$. Is the point $(-2, -10)$ on the straight line DP? Please explain the reason.

Solution: (1) Since the lengths of AO and BO are the two roots of the quadratic function $x^2 + kx + 48 = 0$, then $AO \cdot BO = 48$ ①. In $Rt\triangle ABO$, $AB^2 = AO^2 + BO^2$. Since $AO > BO > 0$, $AB = 10$, then $AO^2 + BO^2 = 100$ ②. From ① and ②, we have $AO = 8$, $BO = 6$.

Since $\widehat{CD} = \widehat{AO}$, then $\angle BAC = \angle BCA$ which means $\widehat{CDO} = \widehat{AOD}$. Thus $AB = BC = 10$, $AD = CO = CB + BO = 16$. Then $DB = AD - AB = 6$. We draw $DE \perp BC$ at the point E through the point D as shown in Figure 5, then $Rt\triangle DEB \backsim Rt\triangle AOB$. Thus $\dfrac{DE}{AO} = \dfrac{DB}{AB} = \dfrac{EB}{OB}$. This means $DE = \dfrac{AO \cdot DB}{AB} = \dfrac{8 \times 6}{10} = \dfrac{24}{5}$, $EB = \dfrac{DB \cdot OB}{AB} = \dfrac{6 \times 6}{10} = \dfrac{18}{5}$. Since $EO = EB + BO = \dfrac{18}{5} + 6 = \dfrac{48}{5}$, then D is $(-\dfrac{48}{5}, \dfrac{24}{5})$.

(2) Since the point P is on AC, $AP = \dfrac{1}{4}AC$, $A(0, -8)$, $C(-16, 0)$, $P(-4, -6)$. Let the straight line passing through the points D and P is $y = kx + b$. Substituting the coordinates of D and P into the linear function to generate $\begin{cases} -\dfrac{48}{5}k + b = \dfrac{24}{5} \\ -4k + b = -6 \end{cases}$. Then $k = -\dfrac{27}{14}, b = -\dfrac{96}{7}$. Hence the straight line passing through the points D and P is $y = -\dfrac{27}{14}x - \dfrac{96}{7}$. By substituting -2, the x-coordinate of the point $(-2, -10)$, into the equation of the straight line DP, we have $y = -\dfrac{69}{7} \neq -10$. Thus the point $(-2, -10)$ is not on the straight line DP.

6.77 ★★★ Let $M = \{(x, y) || xy| = 1, x > 0\}$, $N = \{(x, y) | \arctan x + \operatorname{arccot} y = \pi\}$. Show $M \cup N = M$.

Proof: In the set M, $|xy| = 1$. This means $xy = 1$ or $xy = -1$. Since $x > 0$, then the graph of the reciprocal function is in the first and fourth quadrants.
In the set N, $\arctan x + \operatorname{arccot} y = \pi$, then $\arctan x = \pi - \operatorname{arccot} y$. Thus $x = \tan(\pi - \operatorname{arccot} y) = -\tan(\operatorname{arccot} y) = -\dfrac{1}{\cot(\pi - \operatorname{arccot} y)} = -\dfrac{1}{y}$. Then $xy = -1$. Thus the reciprocal function is in the second and fourth quadrants. Since $-\dfrac{\pi}{2} < \arctan x < \dfrac{\pi}{2}, 0 < \operatorname{arccot} y < \pi$, if $x < 0$, then $-\dfrac{\pi}{2} < \arctan x < 0$, $y > 0$, $0 < \operatorname{arccot} y < \dfrac{\pi}{2}$. At this time $-\dfrac{\pi}{2} < \arctan x + \operatorname{arccot} y < \dfrac{\pi}{2}$. It is contradicting to $\arctan x + \operatorname{arccot} y = \pi$. Thus $x \geqslant 0$. Then $N = \{(x, y) | xy = -1, x > 0\}$ is the reciprocal function $xy = -1$ in the fourth quadrant. Hence $N \subset M$. therefore $M \cup N = M$.

6.78 ★★★ The real numbers a, b, c satisfy $\frac{a}{m+2}+\frac{b}{m+1}+\frac{c}{m}=0$ where m is a positive integer. For $f(x)=ax^2+bx+c$, (1) if $a\neq 0$, show $af(\frac{m}{m+1})<0$. (2) If $a\neq 0$, show the equation $f(x)=0$ has real roots on the interval $(0,1)$.

Proof: (1) From the given condition, we have $af(\frac{m}{m+1})=a[a(\frac{m}{m+1})^2+b\frac{m}{m+1}+c]$ ①. Since $\frac{a}{m+2}+\frac{b}{m+1}+\frac{c}{m}=0$, then $c=-(\frac{am}{m+2}+\frac{bm}{m+1})$. By substituting it into ①, we have $af(\frac{m}{m+1})=a^2[(\frac{m}{m+1})^2-\frac{m}{m+2}]=a^2m^2[\frac{1}{(m+1)^2}-\frac{1}{m^2+2m}]$. Since $(m+1)^2>m^2+2m>0$, then $\frac{1}{(m+1)^2}-\frac{1}{m^2+2m}<0$. Thus $af(\frac{m}{m+1})<0$.

(2) If $a>0$, from the conclusion of (1), we have $f(\frac{m}{m+1})<0$. We only need to prove one of $f(0)$ and $f(1)$ is larger than 0. Since $f(0)=c$, $f(1)=a+b+c$, then $f(0)>0$ holds if $c>0$. We prove the conclusion. If $c\leqslant 0$, we only need to prove $f(1)=a+b+c>0$. By applying $\frac{a}{m+2}+\frac{b}{m+1}+\frac{c}{m}=0$, we have $b=-[\frac{a(m+1)}{m+2}+\frac{c(m+1)}{m}]$. Thus $f(1)=a-\frac{a(m+1)}{m+2}-\frac{c(m+1)}{m}+c=\frac{a}{m+2}-\frac{c}{m}$. Since $a>0, m>0, c\leqslant 0$, then $f(1)>0$. We prove the conclusion. Hence when $a>0$, the equation $f(x)=0$ has real roots on the interval $(0,1)$. Similarly, when $a<0$, the equation $f(x)=0$ also has real roots on the interval $(0,1)$. Therefore, the equation $f(x)=0$ has real roots on the interval $(0,1)$ when $a\neq 0$.

6.79 ★★★ If the function $f(x)$ is the inverse function of $y = \dfrac{2}{10^x+1} - 1 \quad (x \in R)$, the graph of $g(x)$ and the graph of $y = -\dfrac{1}{x+2}$ are symmetric about the straight line $x = -2$. Let $F(x) = f(x) + g(x)$.
(1) Find the analytic expression and the domain of $F(x)$. (2) Determine whether there exist two distinct points A, B on the graph of the function $F(x)$ such that the straight line AB is perpendicular to y-axis. If yes, find the coordinates of A, B. If no, please explain the reason.

Solution: (1) Since $y = \dfrac{2}{10^x+1} - 1$, then its inverse function is $f(x) = \lg\dfrac{1-x}{1+x}$. Additionally, the graph of $g(x)$ and the graph of $y = -\dfrac{1}{x+2}$ are symmetric about the straight line $x = -2$, then $g(x) = \dfrac{1}{x+2}$. Thus $F(x) = \lg\dfrac{1-x}{1+x} + \dfrac{1}{x+2}, x \in (-1, 1)$.
(2) Let the two distinct points A, B on the graph of the function $F(x)$ be $A(x_1, y_1)$, $B(x_2, y_2)$, $-1 < x_1 < x_2 < 1$, then $y_1 - y_2 = F(x_1) - F(x_2) = \lg\dfrac{1-x_1}{1+x_1} + \dfrac{1}{x_1+2} - \lg\dfrac{1-x_2}{1+x_2} - \dfrac{1}{x_2+2} = \lg(\dfrac{1-x_1}{1+x_1}\dfrac{1+x_2}{1-x_2}) + (\dfrac{1}{x_1+2} - \dfrac{1}{x_2+2}) = \lg(\dfrac{1-x_1}{1+x_1}\dfrac{1+x_2}{1-x_2}) + \dfrac{x_2-x_1}{(x_1+2)(x_2+2)}$. Since $-1 < x_1 < x_2 < 1$, then $\dfrac{1+x_2}{1+x_1} > 1, \dfrac{1-x_1}{1-x_2} > 1, x_2 - x_1 > 0, (x_1+2)(x_2+2) > 0$. Thus $\lg(\dfrac{1-x_1}{1+x_1}\dfrac{1+x_2}{1-x_2}) > 0$, $\dfrac{x_2-x_1}{(x_1+2)(x_2+2)} > 0$. Hence $y_1 > y_2$, which means $F(x)$ is monotone decreasing on $(-1, 1)$. Therefore there do not exist two distinct points A, B on the graph of the function $F(x)$ such that the straight line AB is perpendicular to y-axis.

6.80 ★★★ Given the function $f(x) = ax^2 + (b+1)x + b - 2 \quad (a \neq 0)$. If there is a real number x_0 such that $f(x_0) = x_0$ holds, then x_0 is called the fixed point of $f(x)$. (1) Find the fixed point of $f(x)$ when $a = 2, b = -2$. (2) If for an arbitrary real number b, the function $f(x)$ has two distinct fixed points. Find the range of the real number a. (3) On the condition of (2), if the x-coordinates of the points A, B on the curve of $f(x)$ are the fixed points of $f(x)$, and the straight line $y = kx + \dfrac{1}{2a^2+1}$ is the perpendicular bisector of the line segment AB, find the range of the real number b.

Solution: (1) Since $f(x) = ax^2 + (b+1)x + b - 2, (a \neq 0)$, $a = 2, b = -2$, then $f(x) = 2x^2 - x - 4$. Assume x is a fixed point, then $2x^2 - x - 4 = x$. Thus $x_1 = -1$, $x_2 = 2$. Hence the fixed points of $f(x)$ are $2x^2 - x - 4 = x$ are $x_1 = -1$ and $x_2 = 2$.
(2) Since $f(x) = x$, then $ax^2 + bx + b - 2 = 0$. Since the function $f(x)$ has two distinct fixed points, then $\Delta > 0$. Thus $b^2 - 4a(b-2) > 0 \Rightarrow b^2 - 4ab + 8a > 0$ always holds for arbitrary $b \in R$. This means $\Delta_b < 0$, $16a^2 - 32a < 0$. Hence $0 < a < 2$.

(3) Let $A(x_1, x_1)$, $B(x_2, x_2)$. Since the straight line $y = kx + \dfrac{1}{2a^2+1}$ is the perpendicular bisector of the line segment AB, then $k = -1$. Let the midpoint of AB is $M(x_0, x_0)$. From (2), we have $x_0 = -\dfrac{b}{2a}$. Since M is on the straight line $y = kx + \dfrac{1}{2a^2+1}$, then $-\dfrac{b}{2a} = \dfrac{b}{2a} + \dfrac{1}{2a^2+1}$. By simplifying the equation, we have $b = -\dfrac{a}{2a^2+1} = -\dfrac{1}{2a+\frac{1}{a}} \geqslant -\dfrac{1}{2\sqrt{2a \cdot \frac{1}{a}}} = -\dfrac{\sqrt{2}}{4}$, and the equation holds if and only if $a = \dfrac{\sqrt{2}}{2}$. Hence $b \in [-\dfrac{\sqrt{2}}{4}, +\infty)$.

6.81 ★★★ The curve of the quadratic function $y = x^2 - (2m+4)x + m^2 - 4$ passes through y-axis, and its y-intercept is below the origin. The curve of the quadratic function y passes through x-axis, and its two x-intercepts are A, B. A is on the left of B. The distances from A, B to the origin are $|AO|, |OB|$. $|AO|$ and $|BO|$ satisfy $3(|OB| - |AO|) = 2|AO| \cdot |OB|$. The intersection of the straight line $y = kx + k$ and the curve of the quadratic function is P. the tangent function of acute angle $\angle POB$ is 4.
(1) Find the analytic expression of the quadratic function. (2) Find the analytic expression of the linear function $y = kx + k$. (3) Find the coordinates of P.

Solution: (1) Let the coordinates of A, B be $A(x_1, 0)$, $B(x_2, 0)$, and $x_1 < x_2$, then x_1, x_2 are the two real roots of the equation $x^2 - (2m+4)x + m^2 - 4 = 0$. $\Delta = [-(2m+4)]^2 - 4(m^2-4) > 0$, then $m > -2$. On the other hand, since the quadratic function has y-intercept below the origin, and $x_1 < x_2$, then $x_1 < 0, x_2 > 0$. Since $3(|OB| - |AO|) = 2|AO||OB|$, then $3[x_2 - (-x_1)] = 2(-x_1)x_2$. $3(2m+4) = -2(m^2-4) \Rightarrow m^2 + 3m + 2 = 0$. Thus $m_1 = -1$, $m_2 = -2$. Since $m > -2$, $m_1 = -1$. Hence the analytic expression of the quadratic function is $y = x^2 - 2x - 3$.
(2) Since $y = x^2 - 2x - 3$, then $A(-1, 0)$, $B(3, 0)$. Since the straight line $y = kx + k$ intersects the quadratic curve, then $\begin{cases} y = x^2 - 2x - 3 \\ y = kx + k \end{cases}$. Thus $\begin{cases} x_1 = -1 \\ y_1 = 0 \end{cases}$ or $\begin{cases} x_2 = k+3 \\ y_2 = k^2 + 4k \end{cases}$. Since $\angle POB$ is an acute angle, then the point is on the right of y-axis. Thus, P is $(k+3, k^2+4k)$, and $k+3 > 0$. Since $\tan \angle POB = 4$, then $\dfrac{|k^2+4k|}{k+3} = 4$. When $\dfrac{|k^2+4k|}{k+3} = 4$, we solve the equation to get $k_1 = 2\sqrt{3}$, $k_2 = -2\sqrt{3}$. We can validate that $k_1 = 2\sqrt{3}$ and $k_2 = -2\sqrt{3}$ are both the solutions of the equation. Additionally, $k_2 + 3 < 0$, then $k_1 = 2\sqrt{3}$. When $\dfrac{k^2+4k}{k+3} = -4$, we solve the equation to get $k_3 = -2$, $k_4 = -6$. We can validate that $k_3 = -2$ and $k_4 = -6$ are both the solutions of the equation. Additionally, $k_4 + 3 < 0$, then $k_3 = -2$. Thus the analytic expression of the linear function is $y = 2\sqrt{3}x + 2\sqrt{3}$ or $y = -2x - 2$.
(3) From (2), the analytic expression of the linear function is $y = 2\sqrt{3}x + 2\sqrt{3}$ or $y = -2x - 2$.

When $y = 2\sqrt{3}x + 2\sqrt{3}$, then $\dfrac{2\sqrt{3}x + 2\sqrt{3}}{x} = 4$. Thus $x = 2\sqrt{3} + 3$, $y = 12 + 8\sqrt{3}$. Hence, P is $(2\sqrt{3} + 3, 12 + 8\sqrt{3})$.
When $y = -2x - 2$, then $\dfrac{-2x - 2}{-x} = 4$. Thus $x = 1$, $y = -4$. Hence, P is $(1, -4)$.

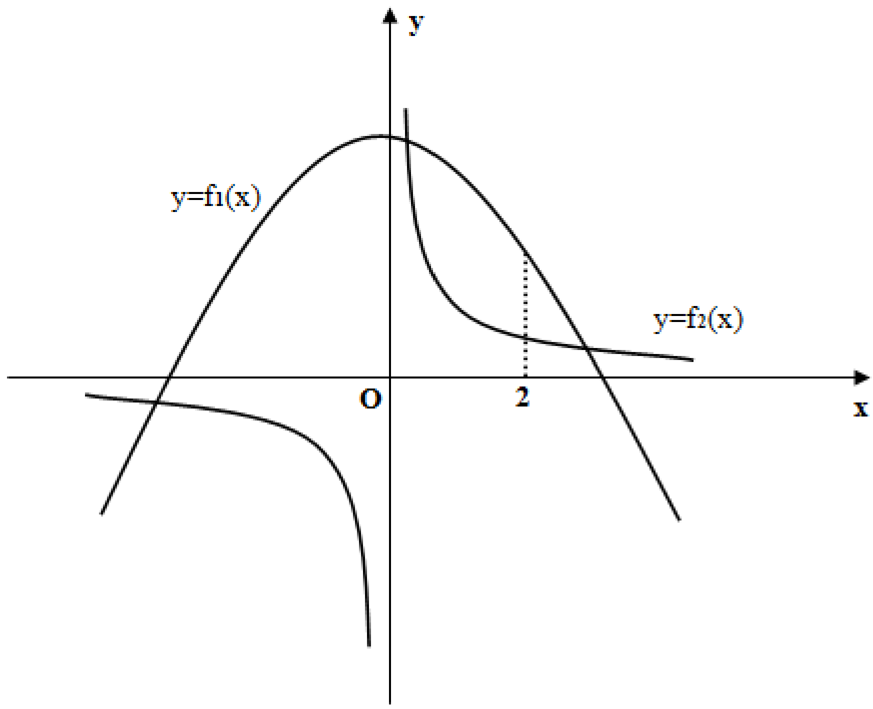


Figure 6

6.82 ★★★ If the quadratic curve $y = f_1(x)$ has the origin as its vertex and passes through the point $(1, 1)$. The distance between the two intersection points of the reciprocal curve $y = f_2(x)$ and the diagonal line $y = x$ is 8. Let $f(x) = f_1(x) + f_2(x)$. (1) Find the analytic expression of $f(x)$. (2) Show the function $f(x) = f(a)$ with respect to x has three real roots when $a > 3$.

(1) Solution: From the given condition, we let $f_1(x) = ax^2$. Since $f(1) = 1$, then $a = 1$. Thus $f_1(x) = x^2$. Let $f_2(x) = \dfrac{k}{x}$ $(k > 0)$. The intersection points of its graph and the diagonal line $y = x$ are $A(\sqrt{k}, \sqrt{k})$ and $B(-\sqrt{k}, -\sqrt{k})$. Since $|AB| = 8$, then $\sqrt{(\sqrt{k} + \sqrt{k})^2 + (\sqrt{k} + \sqrt{k})^2} = 8$. Thus $k = 8$. Then $f_2(x) = \dfrac{8}{x}$. Hence $f(x) = x^2 + \dfrac{8}{x}$.
(2) Proof: Since $f(x) = f(a)$, then $x^2 + \dfrac{8}{x} = a^2 + \dfrac{8}{a}$ which is equivalent to $\dfrac{8}{x} = -x^2 + a^2 + \dfrac{8}{a}$. The curve $f_2(x) = \dfrac{8}{x}$ and the curve $f_3(x) = -x^2 + a^2 + \dfrac{8}{a}$ are sketched in Figure 6. From the figure, we observe that $f_2(x)$ and $f_3(x)$ have one intersection point in the third quadrant. Then $f(x) = f(a)$ has a negative solution.

Since $f_2(2) = 4$, $f_3(2) = -4 + a^2 + \dfrac{8}{a}$. When $a > 3$, the point $(2, f(2))$ on the curve $f_3(x)$ in the first quadrant is above the curve $f_2(x)$. Thus the curves $f_2(x)$ and $f_3(x)$ have two intersection points in the first quadrant. Hence $f(x) = f(a)$ has two positive solutions. Therefore $f(x) = f(a)$ has three real solutions.

6.83 ★★★★ If the domain of the function $f(x)$ is symmetric about the origin but does not include zero. For an arbitrary real number x in the domain, there exist x_1, x_2 in the domain such that $x = x_1 - x_2$, $f(x_1) \neq f(x_2)$, and the following conditions hold: (A) If $0 < |x_1 - x_2| < 2a$, then $f(x_1 - x_2) = \frac{f(x_1)f(x_2)+1}{f(x_2)-f(x_1)}$. (B) $f(a) = 1$ (a is a positive constant). (C) $f(x) > 0$ when $0 < x < 2a$. Show (1) $f(x)$ is an odd function. (2) $f(x)$ is a periodic function. And evaluate the period. (3) $f(x)$ is a decreasing function on $(0, 4a)$.

Proof: (1) From the given condition, we have $f(x) = f(x_1 - x_2) = \frac{f(x_1)f(x_2)+1}{f(x_2)-f(x_1)} = -f(x_2 - x_1) = -f(-x)$. Thus $f(x)$ is an odd function.
(2) Since $f(a) = 1$, then $f(-a) = -f(a) = -1$. Thus $f(-2a) = f(-a - a) = \dfrac{f(-a)f(a) + 1}{f(a) - f(-a)} = 0$. If $f(x) \neq 0$, then $f(x + 2a) = f[x - (-2a)] = \dfrac{f(x)f(-2a) + 1}{f(-2a) - f(x)} = -\dfrac{1}{f(x)}$, $f(x + 4a) = f[(x + 2a) + 2a] = -\dfrac{1}{f(x + 2a)} = f(x)$.

If $f(x) = 0$, then $f(x+a) = f[x-(-a)] = \dfrac{f(x)f(-a)+1}{f(-a)-f(x)} = -1$, $f(x+3a) = f[(x+a)+2a] = -\dfrac{1}{f(x+a)} = 1$. $f(x+4a) = f[(x+3a)-(-a)] = \dfrac{f(x+3a)f(-a)+1}{f(-a)-f(x+3a)} = 0$.
Thus $f(x+4a) = f(x)$.
Hence $f(x)$ is a periodic function, and the period is $4a$.
(3) When $0 < x_1 < x_2 \leqslant 2a$, then $0 < x_2 - x_1 < 2a$. Thus $f(x_1) > 0$, $f(x_2) \geqslant 0$ ($f(x_2) = -f(-2a) = 0$ when $x_2 = 2a$). and $\dfrac{f(x_2)f(x_1)+1}{f(x_1)-f(x_2)} = f(x_2-x_1) > 0$. Hence $f(x_1) > f(x_2)$.
When $2a < x_1 < x_2 < 4a$, then $0 < x_1 - 2a < x_2 - 2a < 2a$. Thus $f(x_1 - 2a) > f(x_2 - 2a) > 0$. Hence $f(x) = f[(x-2a)+2a] = -\dfrac{1}{f(x-2a)}$. Then $f(x_1) - f(x_2) = -\dfrac{1}{f(x_1-2a)} + \dfrac{1}{f(x_2-2a)} > 0$. Hence $f(x)$ is a decreasing function on $(2a, 4a)$.
Therefore, $f(x)$ is a decreasing function on $(0, 4a)$.

6.84 ★★★★ If the inverse function of the function $f(x) = \log_a(x + \sqrt{x^2-2}) \quad (a > 0, a \neq 1)$ is $f^{-1}(x)$ and let $g(n) = \dfrac{\sqrt{2}}{2} f^{-1}(n + \log_a \sqrt{2})$. If $g(n) < \dfrac{3^n + 3^{-n}}{2} \quad (n \in N^*)$. Find the range of a.

Solution: Since $x + \sqrt{x^2-2} > 0$, then the domain of $f(x)$ is $[\sqrt{2}, +\infty)$. Thus $x + \sqrt{x^2-2} \geqslant \sqrt{2}$.
When $a > 1$, the domain of $f^{-1}(x)$ is $[\log_a \sqrt{2}, +\infty)$. When $0 < a < 1$, the domain of $f^{-1}(x)$ is $(-\infty, \log_a \sqrt{2}]$. Since $y = \log_a(x+\sqrt{x^2-2})$, we have $x + \sqrt{x^2-2} = a^y$ ①. Rationalizing the numerator of ① to obtain $x - \sqrt{x^2-2} = 2a^{-y}$ ②. From ① and ②, we have $x = \dfrac{a^y + 2a^{-y}}{2}$. Since $n + \log_a \sqrt{2} \in [\log_a \sqrt{2}, +\infty)$, then $a > 1$, and then $f^{-1}(x) = \dfrac{a^x + 2a^{-x}}{2} (x \geqslant \log_a \sqrt{2})$. Hence $g(n) = \dfrac{\sqrt{2}}{2} f^{-1}(n + \log_a \sqrt{2}) = \dfrac{\sqrt{2}}{2}\dfrac{1}{2}[a^{n+\log_a \sqrt{2}} + 2a^{-(n+\log_a \sqrt{2})}] = \dfrac{\sqrt{2}}{4}[\sqrt{2}a^n + 2a^{-n}\dfrac{\sqrt{2}}{2}] = \dfrac{a^n + a^{-n}}{2}$. Since $g(n) < \dfrac{3^n + 3^{-n}}{2}$, then $a^n + a^{-n} < 3^n + 3^{-n} \Rightarrow 3^n a^{2n} + 3^n < 3^{2n} a^n + a^n \Rightarrow (3^n a^n - 1)(a^n - 3^n) < 0 \Rightarrow \dfrac{1}{3} < a < 3$. Since $a > 1$, then $1 < a < 3$.

6.85 ★★★★ The straight line l with dip angle 45^0 passes through the point $A(1-2)$ and the point B where B is in the first quadrant and $|AB| = 3\sqrt{2}$.
(1) Fine the coordinates of the point B. (2) If the straight line l passes through the hyperbolic curve $C : \frac{x^2}{a^2} - y^2 = 1(a > 0)$ at the two points E and F, and the middle point of the line segment EF is $(4, 1)$. Evaluate a. (3) For an arbitrary point P in the plane, when Q is moving on the line segment AB, we denote the minimum value of $|PQ|$ as the distance from the point P to the line segment AB. If the point P moves on the x-axis, find the function for the distance $h(t)$ from the point $P(t, 0)$ to the line segment AB.

Solution: (1) Let the equation of the straight line l is $y = \tan 45^0 x + b = x + b$. Since the straight line passes through the point $A(1 - 2)$, then $-2 = 1 + b$. Thus $b = -3$. And $y = x - 3$. Let the point $B = (x, y)$. Since $\begin{cases} y = x - 3 \\ (x-1)^2 + (y+2)^2 = (3\sqrt{2})^2 \end{cases}$ and $x > 0, y > 0$, then $x = 4, y = 1$. Hence, the coordinate of B is $(4, 1)$.
(2) Since $\begin{cases} y = x - 3 \\ \frac{x^2}{a^2} - y^2 = 1 \end{cases}$, then $(\frac{1}{a^2} - 1)x^2 + 6x - 10 = 0$. Let $E(x_1, y_1)$, $F(x_2, y_2)$.
Since the middle point of EF is $(4, 1)$, then $x_1 + x_2 = -\frac{6a^2}{1-a^2} = 8$. Thus $a = 2$.
(3) Let the coordinates of an arbitrary point Q on the line segment AB is $(x, x - 3)$.
$|PQ| = \sqrt{(t-x)^2 + (x-3)^2}$.

Denote $f(x) = \sqrt{(t-x)^2+(x-3)^2} = \sqrt{2(x-\frac{t+3}{2})^2+\frac{(t-3)^2}{2}}(1 \leqslant t \leqslant 4)$. When $1 \leqslant \frac{t+3}{2} \leqslant 4$, i.e. $-1 \leqslant t \leqslant 5$, then $|PQ|_{\min} = f(\frac{t+3}{2}) = \frac{|t-3|}{\sqrt{2}}$. When $\frac{t+3}{2} > 4$, i.e. $t > 5$, then $f(x)$ is monotonous decreasing on $[1,4]$. Thus $|PQ|_{\min} = f(4) = \sqrt{(t-4)^2+1}$. When $\frac{t+3}{2} < 1$, i.e. $t < -1$, then $f(x)$ is increasing on $[1,4)$. Thus $|PQ|_{\min} = f(4) = \sqrt{(t-1)^2+4}$.

As a conclusion, $h(t) = \begin{cases} \sqrt{(t-1)^2+4} & (t < -1) \\ \frac{|t-3|}{\sqrt{2}} & (-1 \leqslant t \leqslant 5) \\ \sqrt{(t-4)^2+1} & (t > 5) \end{cases}$

6.86 ★★★ Let the function $f(x) = a^x + \frac{x-2}{x+1}$ $(a > 1)$. Show (1) The function $f(x)$ is increasing on $(-1,+\infty)$. (2) The equation $f(x) = 0$ has no negative roots.

Proof: (1) Let $-1 < x_1 < x_2$, then $f(x_1) - f(x_2) = a^{x_1} + \frac{x_1-2}{x_1+1} - a^{x_2} - \frac{x_2-2}{x_2+1} = a^{x_1} - a^{x_2} + \frac{3(x_1-x_2)}{(x_1+1)(x_2+1)}$. Since $-1 < x_1 < x_2$, then $x_1+1 > 0, x_2+1 > 0, x_1-x_2 < 0$, then $\frac{3(x_1-x_2)}{(x_1+1)(x_2+1)} < 0$.
Since $-1 < x_1 < x_2$ and $a > 1$, then $a^{x_1} < a^{x_2}$, $a^{x_1} - a^{x_2} < 0$. Thus $f(x_1) - f(x_2) < 0$ which is equivalent to $f(x_1) < f(x_2)$. Hence $f(x)$ is increasing on $(-1,+\infty)$.
(2) Assume x_0 is a negative root of the equation $f(x) = 0$, and $x_0 \neq -1$, then $a^{x_0} + \frac{x_0-2}{x_0+1} = 0 \Rightarrow a^{x_0} = \frac{2-x_0}{x_0+1} = \frac{3-(x_0+1)}{x_0+1} = \frac{3}{x_0+1} - 1$ ①. When $-1 < x_0 < 0$, i.e. $0 < x_0+1 < 1$, then $\frac{3}{x_0+1} - 1 > 2$. Since $a > 1$, then $a^{x_0} < 1$, then the formula ① does not hold. When $x_0 < -1$, i.e. $x_0+1 < 0$, then $\frac{3}{x_0+1} - 1 < -1$. Since $a^{x_0} > 0$, then $a^{x_0} < 1$, then the formula ① does not hold.
As a conclusion, the equation $f(x) = 0$ has no negative roots.

6.87 ★★★★ Let $f(x) = ax^2+bx+c$ $(a \neq 0)$. If $|f(0)| \leqslant 1, |f(1)| \leqslant 1, |f(-1)| \leqslant 1$. Show $|f(x)| \leqslant \frac{5}{4}$ holds for any $x \in [-1,1]$.

Proof: Since $f(-1) = a-b+c$, $f(1) = a+b+c$, $f(0) = c$, then $a = \frac{1}{2}(f(1)+f(-1)-2f(0))$, $b = \frac{1}{2}(f(1)-f(-1))$, $c = f(0)$. Substituting a, b, c into $f(x) = ax^2+bx+c$ and simplifying the equation to obtain $f(x) = f(1)(\frac{x^2+x}{2})+f(-1)(\frac{x^2-x}{2})+f(0)(1-x^2)$.

When $-1 \leqslant x < 0$, then $|f(x)| \leqslant |f(1)||\frac{x^2+x}{2}| + |f(-1)||\frac{x^2-x}{2}| + |f(0)||1-x^2| \leqslant |\frac{x^2+x}{2}| + |\frac{x^2-x}{2}| + |1-x^2| = -(\frac{x^2+x}{2}) + (\frac{x^2-x}{2}) + (1-x^2) = -x^2-x+1 = -(x+\frac{1}{2})^2 + \frac{5}{4} \leqslant \frac{5}{4}$.
When $0 \leqslant x \leqslant 1$, then $|f(x)| \leqslant |f(1)||\frac{x^2+x}{2}| + |f(-1)||\frac{x^2-x}{2}| + |f(0)||1-x^2| \leqslant |\frac{x^2+x}{2}|+|\frac{x^2-x}{2}|+|1-x^2| = \frac{x^2+x}{2}+\frac{-x^2+x}{2}+1-x^2 = -x^2+x+1 = -(x-\frac{1}{2})^2+\frac{5}{4} \leqslant \frac{5}{4}$.
As a conclusion, $|f(x)| \leqslant \frac{5}{4}$ holds for any $x \in [-1, 1]$.

6.88 ★★★★★ (1) If x is an arbitrary positive integer, and the values of the quadratic function $f(x) = ax^2 + bx + c$ are all integers. Show $2a$, $a - b$, c are all integers.
(2) Write the inverse statement of the above statement. Judge the inverse statement is true or false and provide your reason.

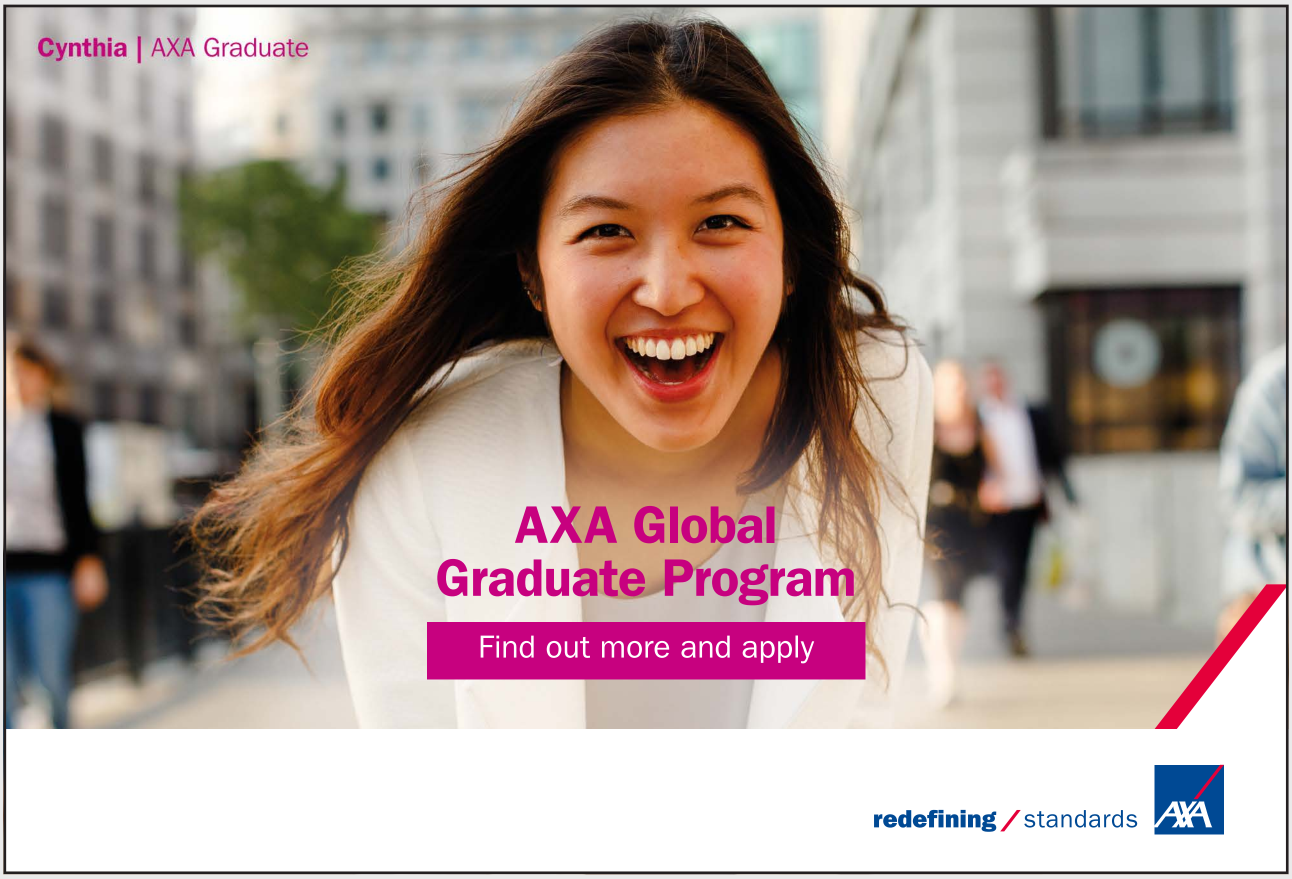
Proof (1): From the given condition, the values of the quadratic function $f(x) = ax^2 + bx + c$ are all integers when x is an arbitrary positive integer, we have $f(0) = c$ is an integer when $x = 0$. Similarly, when $x = -1$, then $f(-1) = a - b + c$ is an integer. Thus $a - b = f(-1) - c$ is an integer. When $x = -2$, then $f(-2) = (-2)^2a + (-2)b + c$ is an integer. Thus $2a = f(-2) - 2f(-1) + c$ is an integer. Hence $2a$, $a - b$, c are all integers.
(2) The inverse statement is that if $2a$, $a - b$, c are all integers, then the values of the quadratic function $f(x) = ax^2 + bx + c$ are all integers when x is an arbitrary positive integer. This inverse statement is true and the proof is provided below.
If $2a$, $a - b$, c are all integers, then $f(x) = ax^2 + bx + c = ax^2 + ax - ax + bx + c = ax(x+1) - (a-b)x + c$. When x is an integer, then $x(x+1)$ is an even function. Thus $\frac{1}{2}x(x+1)$ is an integer. Additionally, $2a$ is an integer, then $2a \cdot \frac{1}{2}x(x+1)$ is an integer. Since $a - b$, c are integers, then $-(a - b)x + c$ is an integer. Hence the values of the quadratic function $f(x) = ax^2 + bx + c$ are all integers when x is an arbitrary positive integer.
An alternative proof: If $2a$, $a - b$, c are all integers, then when x is an even number (let $x = 2k$), we have $f(2k) = a(2k)^2 + b(2k) + c = 2a \cdot 2k^2 + [2a - 2(a - b)]k + c$ is an integer.
When x is an odd number, let $x = 2k-1$, we have $f(2k-1) = a(2k-1)^2+b(2k-1)+c = (4k^2 - 4k)a + +a + 2kb - b + c = 2a(2k^2 - 2k) + [2a - 2(a - b)] + (a - b) + c$ is an integer. Therefore, the inverse statement is true.

6.89 ★★★★★ Let the function $f_n(x)(n \in N^*)$ satisfy $f_1(x) = 2$, $f_{n+1}(x) = xf_n(x)+1$. Find the analytic expression of $f_n(x)$ and prove the conclusion.

Solution: Since $f_1(x) = 2$, $f_{n+1}(x) = xf_n(x)+1$, then $f_2(x) = xf_1(x)+1 = 2x+1$, $f_3(x) = xf_2(x)+1 = 2x^2+x+1$, $f_4(x) = xf_3(x)+1 = 2x^3+x^2+x+1, \cdots$. Thus we have $f_n(x) = 2x^{n-1}+x^{n-2}+\cdots+x+1$ and the proof is provided below.
(1) When $n = 1$, then $f_1(x) = 2x = 2$. Thus $p(1)$ holds.
(2) Assume $p(k)$ holds when $n = k$, i.e. $f_k(x) = 2x^{k-1}+x^{k-2}+\cdots+x+1$.
When $n = k+1$, we have $f_{k+1}(x) = xf_k(x)+1 = x(2x^{k-1}+x^{k-2}+\cdots+x+1)+1 = 2x^k+x^{k-1}+\cdots+x+1$. Thus $f_{k+1}(x) = 2x^k+x^{k-1}+\cdots+x+1$. $p(k+1)$ holds.
$f_n(x) = 2x^{n-1}+x^{n-2}+\cdots+x+1$ holds for all $n \in N^*$.

6.90 ★★★★★ If the function $f(x)$ is defined on R, $f(0) = 2008$, and for any $x \in R$, $f(x+2)-f(x) \leqslant 3\cdot 2^x$ and $f(x+6)-f(x) \geqslant 63\cdot 2^x$ both hold. Compute $f(2008)$.

Solution: From the given condition, we have $f(x+2)-f(x) = -[f(x+4)-f(x+2)]-[f(x+6)-f(x+4)]+[f(x+6)-f(x)] \geqslant -3\cdot 2^{x+2}-3\cdot 2^{x+4}+63\cdot 2^x = -12\cdot 2^x-48\cdot 2^x+63\cdot 2^x = 3\cdot 2^x$. Since $f(x+2)-f(x) \leqslant 3\cdot 2^x$, then $f(x+2)-f(x) = 3\cdot 2^x$. Thus $f(2008) = f(2008)-f(2006)+f(2006)-f(2004)+\cdots+f(2)-f(0)+f(0) = 3(2^{2006}+2^{2004}+\cdots+2^2+1)+f(0) = 3\dfrac{4^{1004}-1}{4-1}+2008 = 2^{2008}+2007$.



6.91 ★★★★★ If the function $f(x)$ is defined on $(0,+\infty)$ and satisfies $f(x)+f(y)=f(xy)$, and $f(x)<0$ when $x>1$. If the inequality $f(\sqrt{x^2+y^2})\leqslant f(\sqrt{xy})+f(a)$ always holds for any $x,y\in(0,+\infty)$, find the range of the real number a.

Solution: Let $x_1,x_2\in(0,+\infty)$, and $x_1<x_2$, then $\dfrac{x_2}{x_1}>1$. We get $f(x_1)-f(x_2)=f(x_1)-f(x_1\dfrac{x_2}{x_1})=f(x_1)-[f(x_1)+f(\dfrac{x_2}{x_1})]=-f(\dfrac{x_2}{x_1})>0$. Since $x>1$, then $f(x)<0$. Thus $f(x_1)-f(x_2)>0$. Hence $f(x_1)>f(x_2)$. We obtain that the function $f(x)$ is decreasing on $(0,+\infty)$. Then $f(\sqrt{x^2+y^2})\leqslant f(\sqrt{xy})+f(a)\Rightarrow f(\sqrt{x^2+y^2})\leqslant f(a\sqrt{xy})\Rightarrow\sqrt{x^2+y^2}\geqslant a\sqrt{xy}$. Thus $a\leqslant\dfrac{\sqrt{x^2+y^2)}}{\sqrt{xy}}$. Since $\sqrt{xy}\leqslant\sqrt{\dfrac{x^2+y^2}{2}}$, then $\dfrac{\sqrt{x^2+y^2}}{\sqrt{xy}}\geqslant\sqrt{2}$. Hence $a\leqslant\sqrt{2}$. Additionally, since $a>0$, we have $0<a\leqslant\sqrt{2}$.

After all, the range of the real number a is $(0,\sqrt{2}]$.

6.92 ★★★★★ Given $f(x)=\dfrac{x}{x+1}(x\neq-1)$.

(1) Find the intervals on which $f(x)$ is monotone. (2) If $a>b>0$, $c=\dfrac{1}{(a-b)b}$, show $f(a)+f(c)>\dfrac{3}{4}$.

(1) Solution: Since $f(x)=\dfrac{x}{x+1}=1-\dfrac{1}{x+1}$, then $f(x)$ is an monotone function on the interval $(-\infty,-1)\cup(-1,+\infty)$. Let $-\infty<x_1<x_2<-1\cup-1<x_1<x_2<+\infty$, we have $f(x_2)-f(x_1)=1-\dfrac{1}{x_2+1}-1+\dfrac{1}{x_1+1}=\dfrac{x_2-x_1}{(x_1+1)(x_2+1)}>0$. Thus $f(x_2)>f(x_1)$. Hence $f(x)$ is a monotone increasing function on the intervals $(-\infty,-1)\cup(-1,+\infty)$.
(2) If $x>y>0$, since $f(x)+f(y)=\dfrac{x}{x+1}+\dfrac{y}{y+1}=\dfrac{xy+xy+x+y}{xy+x+y+1}>\dfrac{xy+x+y}{xy+x+y+1}=f(xy+x+y)$. And $xy+x+y>x+y$. From (1), we have $f(xy+x+y)>f(x+y)$. Thus $f(x)+f(y)>f(x+y)$. On the other hand, $c=\dfrac{1}{(a-b)b}\geqslant\dfrac{1}{(\frac{a-b+b}{2})^2}=\dfrac{4}{a^2}>0$, then $a+c\geqslant a+\dfrac{4}{a^2}=\dfrac{a}{2}+\dfrac{a}{2}+\dfrac{4}{a^2}\geqslant3\sqrt[3]{\dfrac{a}{2}\cdot\dfrac{a}{2}\cdot\dfrac{4}{a^2}}=3$. Therefore $f(a)+f(c)>f(a+c)\geqslant f(3)=\dfrac{3}{4}$.

6.93 ★★★★★ If the monotone function $f(x)$ defined on R satisfies $f(3)=\log_2 3$, and for any $x,y\in R$, $f(x+y)=f(x)+f(y)$. (1) Determine $f(x)$ is odd or even. (2) If $f(k3^x)+f(3^x-9^x-2)<0$ holds for any $x\in R$, find the range of the real number k.

Solution: (1) Since $f(x+y)=f(x)+f(y)\quad(x,y\in R)$. Let $x=y=0$, then $f(0)=f(0)+f(0)$. Thus $f(0)=0$. Let $y=-x$, then $f(0)=f(x)+f(-x)$, i.e. $f(x)+f(-x)=0$. Hence $f(-x)=-f(x)$ holds for any $x,y\in R$. Therefore $f(x)$ is an odd function.
(2) Since $f(3)=\log_2 3>0$, $f(3)>f(0)$. Since $f(x)$ is a monotone function, then $f(x)$ is an increasing function on R. And Since $f(x)$ is an odd function according to (1), we get $f(k3^x)<-f(3^x-9^x-2)=f(-3^x+9^x+2)$. Hence $k3^x<-3^x+9^x+2\Rightarrow 3^{2x}-(k+1)3^x+2>0$ holds for any $x\in R$. Let $t=3^x$, the question is equivalent to the following: $t^2-(k+1)t+2>0$ holds for any $t>0$. Let $f(t)=t^2-(k+1)t+2$, the symmetric axis is $x=\dfrac{k+1}{2}$. When $\dfrac{k+1}{2}<0$, i.e. $k<-1$, then $f(0)=2>0$ satisfies the given problem. When $\dfrac{k+1}{2}\geqslant 0$, i.e. $k\geqslant -1$, for any $t>0$, $f(t)>0$ always holds $\Leftrightarrow \begin{cases}\dfrac{k+1}{2}\geqslant 0\\ \Delta=(k+1)^2-8<0\end{cases}\Leftrightarrow -1\leqslant k\leqslant -1+2\sqrt{2}$.
As a conclusion, when $k<-1+2\sqrt{2}$, $f(k3^x)+f(3^x-9^x-2)<0$ holds for any $x\in R$. Therefore the range of the real number k is $(-\infty,-1+2\sqrt{2})$.

6.94 ★★★★★ Let the function $f(x)=\begin{cases}0 & (x=0)\\ -\dfrac{1}{2}x & (4^{k-1}\leqslant |x|<2\cdot 4^{k-1}, k\in Z)\\ 2x & (2\cdot 4^{k-1}\leqslant |x|\leqslant 4^k, k\in Z)\end{cases}$
(1) What is the domain of $f(x)$? (2) We rotate the curve of $y=f(x)$ around the origin by $\dfrac{\pi}{2}$ to obtain the curve of $y=g(x)$. Find the analytic expression of $g(x)$. (3) For the function $f(x)$ defined on R, if we rotate the curve of $y=f(x)$ around the origin by $\dfrac{\pi}{2}$ to obtain the same curve, show that the function $f(x)=x$ has a unique solution.

(1) Solution: Let the domain of the function $f(x)$ is D. For any $x\in R$, $x\in D$ when $x=0$; when $x\neq 0$, then $|x|>0$. There exists an integer k such that $4^{k-1}\leqslant |x|\leqslant 4^k$, then $x\in D$, which means $R\subseteq D$. Hence $D=R$. Therefore the domain of $f(x)$ is $x\in R$.
(2) Solution: We rotate an arbitrary point (x_0,y_0) on $y=f(x)$ around the origin by $\dfrac{\pi}{2}$, then the coordinates of the new point is $(-y_0,x_0)$. $f(0)=0$ when $x_0=0$, then $g(0)=0$. When $4^{k-1}\leqslant |x|<2\cdot 4^{k-1}$, then $f(x_0)=-\dfrac{1}{2}x_0$. Thus $g(\dfrac{1}{2}x_0)=x_0$. Let $\dfrac{1}{2}x_0=x_1$, then $g(x_1)=2x_1$. Thus $2\times 4^{k-2}\leqslant |x_1|<4^{k-1}$. When $2\times 4^{k-1}\leqslant |x_0|\leqslant 4^k$, $f(x_0)=2x_0$. Thus $g(-2x_0)=x_0$. Let $-2x_0=x_1$, then $g(x_1)=-\dfrac{1}{2}x_1$. Thus $4^k\leqslant |x_1|\leqslant 2\times 4^k$.

As a conclusion, $g(x) = \begin{cases} 0 & (x=0) \\ 2x & (2\cdot 4^{k-2} \leqslant |x| < 4^{k-1}, k \in Z) \\ -\dfrac{1}{2}x & (4^k \leqslant |x| \leqslant 2\cdot 4^k, k \in Z) \end{cases}$

(3) Proof: Let $f(0) = y_0$, then $(0, y_0)$ is on the curve of the function $f(x)$. We rotate the point twice (by $\frac{\pi}{2}$ each time) in the same direction around the origin to obtain the point $(0, -y_0)$ which is still on the curve of $y = f(x)$. Since $y_0 = f(0) = -y_0$, then $y_0 = 0, f(0) = 0$. Hence $x = 0$ is a solution of the equation $f(x) = x$.
Assume $f(x_0) = x_0$, then the point (x_0, x_0) is on the curve of $y = f(x)$. If it rotates three $\frac{\pi}{2}$ around the origin to generate the point $(x_0, -x_0)$. And the point is also on the curve of $y = f(x)$. Hence $x_0 = f(x_0) = -x_0$. Then $x_0 = 0$.
After all, the function $f(x) = x$ has a unique solution $x = 0$.

6.95 ★★★★★ Let N be the set of natural numbers, and $k \in N$. If the function $f : N \to N$ is strictly increasing, and for every $n \in N$, $f(f(n)) = kn$. Show for an arbitrary $n \in N$, $\frac{2k}{k+1}n \leqslant f(n) \leqslant \frac{k+1}{2}n$.

Proof: Let $a, b \in N$, and $a < b$. Since $f : N \to N$ is a strictly increasing, we have $f(a+1) - f(a) > 0$. Thus $f(a+1) - f(a) \geqslant 1$. Then $f(b) - f(a) = [f(b) - f(b-1)] + [f(b-1) - f(b-2)] + \cdots + [f(a+1) - f(a)] \geqslant 1 + 1 + \cdots + 1 = b - a$.

From the above conclusion, we have $f(f(f(n)))-f(f(n)) \geqslant f(f(n))-f(n) \geqslant f(n)-n$, which is equivalent to $kf(n)-kn \geqslant kn-f(n) \geqslant f(n)-n$. Since $kf(n)-kn \geqslant kn-f(n)$, then $f(n) \geqslant \frac{2k}{k+1}n$. Since $kn-f(n) \geqslant f(n)-n$, then $f(n) \leqslant \frac{k+1}{2}n$.

Therefore for any $n \in N$, $\frac{2k}{k+1}n \leqslant f(n) \leqslant \frac{k+1}{2}n$ holds.

6.96 ★★★★★ Given the function $f(x) = a - \frac{b}{|x|}$ $(x \neq 0)$. (1) If the function $f(x)$ is increasing on $(0,+\infty)$, find the range of b. (2) When $b = 2$, if $f(x) < x$ holds on $(1,+\infty)$, find the range of a. (3) If the range of $f(x)$ is also $[m,n]$ when $x \in [m,n]$, we call $f(x)$ as the closed function on $[m,n]$. Find the conditions that a, b should satisfy.

Solution:(1) When $x \in (0,+\infty)$, then $f(x) = a - \frac{b}{|x|} = a - \frac{b}{x}$ is an increasing function. Let $0 < x_1 < x_2$, then $f(x_1) < f(x_2)$. Thus $f(x_2)-f(x_1) = -\frac{b}{x_2}+\frac{b}{x_1} = \frac{b(x_2-x_1)}{x_1x_2} > 0$. Since $0 < x_1 < x_2$, then $x_2 - x_1 > 0$, $x_2x_1 > 0$. Thus $b > 0$ which is equivalent to $b \in (0,+\infty)$.

(2) When $b = 2$, $f(x) = a - \frac{b}{|x|} < x$ holds on $(1,+\infty)$, i.e. $a < x + \frac{2}{x}$. Since $x + \frac{2}{x} \geqslant 2\sqrt{x\frac{2}{x}} = 2\sqrt{2}$ and the equation holds if and only if $x = \frac{2}{x}$ which is equivalent to $x = \sqrt{2}$ and $\sqrt{2} \in (1,+\infty)$. Thus the minimum value of $x + \frac{2}{x}$ is $2\sqrt{2}$ when $x \in (1,+\infty)$. Hence $a \leqslant 2\sqrt{2}$. Therefore the range of a is $(-\infty, 2\sqrt{2}]$.

(3) From the given condition, we know that the domain of $f(x) = a - \frac{b}{|x|}$ is $\{x|x \neq 0\}$. Let $f(x)$ be a closed function on $[m,n]$, then $mn > 0$, and $b \neq 0$.

(i) If $0 < m < n$, when $b > 0$, then $f(x) = a - \frac{b}{|x|}$ is an increasing function on $(0,+\infty)$. We have $\begin{cases} f(m) = m \\ f(n) = n \end{cases}$. Thus the equation $a - \frac{b}{x} = x$ has two distinct roots on $(0,+\infty)$. This means $x^2 - ax + b = 0$ has two distinct roots on $(0,+\infty)$. Hence $\Delta = a^2 - 4b > 0$, $x_1 + x_2 = a > 0$, $x_1x_2 = b > 0$. Then $a > 0, b > 0$ and $a^2 - 4b > 0$. When $b < 0$, then $f(x) = a - \frac{b}{|x|} = a + \frac{-b}{x}$ is an decreasing function on $(0,+\infty)$. We have $\begin{cases} f(m) = n \\ f(n) = m \end{cases} \Rightarrow \begin{cases} a - \frac{b}{m} = n \\ a - \frac{b}{n} = m \end{cases} \Rightarrow \begin{cases} a = 0 \\ mn = -b \end{cases}$. Thus $a = 0, b < 0$.

(ii) If $m < n < 0$, when $b > 0$, then $f(x) = a - \frac{b}{|x|} = a + \frac{b}{x}$ is a decreasing function on

$(-\infty, 0)$. We have $\begin{cases} f(m) = n \\ f(n) = m \end{cases} \Rightarrow \begin{cases} a + \dfrac{b}{m} = n \\ a + \dfrac{b}{n} = m \end{cases} \Rightarrow \begin{cases} a = 0 \\ mn = b \end{cases}$. Thus $a = 0, b > 0$.
When $b < 0$, then $f(x) = a - \dfrac{b}{|x|} = a + \dfrac{b}{x}$ is an increasing function on $(-\infty, 0)$. We have $\begin{cases} f(m) = m \\ f(n) = n \end{cases}$. Thus the equation $a + \dfrac{b}{x} = x$ has two distinct roots on $(-\infty, 0)$. This means $x^2 - ax - b = 0$ has two distinct roots on $(-\infty, 0)$. Hence $\Delta = a^2 + 4b > 0$, $x_1 + x_2 = a < 0$, $x_1 x_2 = -b > 0$. Then $a < 0, b < 0$ and $a^2 + 4b > 0$.
After all, $a = 0, b \neq 0$ or $a < 0, b < 0$ and $a^2 + 4b > 0$ or $a > 0, b > 0$ and $a^2 - 4b > 0$. Thus a, b should satisfy the conditions: $a = 0, b \neq 0$ or $ab > 0$ and $a^2 - 4|b| > 0$.

6.97 ★★★★★ The function $f(t)$ satisfies $f(x + y) = f(x) + f(y) + xy + 1$ and $f(-2) = -2$. (1) Evaluate $f(1)$. (2) Show $f(t) > t$ always holds for any positive integer t larger than 1. (3) Compute the number of integers which satisfy $f(t) = t$, and explain the reason.

(1) Solution: Let $x = y = 0$, then $f(0) = -1$. Let $x = y = -1$, since $f(-2) = -2$, then $f(-2) = 2f(-1) + 2$. Thus $f(-1) = -2$. Let $x = 1, y = -1$, then $f(0) = f(1) + f(-1)$. Thus $f(1) = f(0) - f(-1) = 1$.
(2) Solution: Let $x = 1$, then $f(y+1) = f(y) + y + 2$. Thus $f(y+1) - f(y) = y + 2 \quad (*)$. When $y \in N$, then $f(y + 1) - f(y) > 0$. Since $f(y + 1) > f(y)$ and $f(1) = 1$, then $f(y) > 0$ holds for any integer y. Thus when $y \in N$, $f(y+1) = f(y) + 1 + y + 1 > y + 1$. Then $f(t) > t$ always holds for any positive integer t larger than 1.
(3) From $(*)$ and (1), we have $f(-3) = -1$, $f(-4) = 1$. Now we can show that $f(t) > t$ when $t \leqslant -4$.
Since $t \leqslant -4$, then $-(t+2) \geqslant 2 > 0$. From $(*)$, we have $f(t) - f(t+1) = -(t+2) > 0$ which is equivalent to $f(-5) - f(-4) > 0$, $f(-6) - f(-5) > 0, \cdots$, $f(t+1) - f(t+2) > 0$, $f(t) - f(t+1) > 0$. Adding the above inequalities to generate $f(t) - f(-4) > 0$. Thus $f(t) > f(-4) = 1$. Hence $t \leqslant -4$.
Therefore, the number of integers t which satisfy $f(t) = t$ is two, and $t = 1$ or $t = -2$.

6.98 ★★★★★ The function $f(x)$ is defined on $(-1, 1)$, and $f(\frac{1}{2}) = 1$. $f(x) - f(y) = f(\frac{x-y}{1-xy})$ for $x, y \in (-1, 1)$. The sequence $\{x_n\}$, $x_1 = \dfrac{1}{2}$, $x_{n+1} = \dfrac{2x_n}{1 + x_n^2}$. (1) Show $f(x)$ is an odd function on $(-1, 1)$. (2) Find the analytic expression of $f(x_n)$. (3) Is there a natural number m such that $\dfrac{1}{f(x_1)} + \dfrac{1}{f(x_2)} + \cdots + \dfrac{1}{f(x_n)} < \dfrac{m-8}{4}$ for any $n \in N^*$. If m exists, find its minimum value. If m does not exist, please explain the reason.

(1) Proof: Let $x = y = 0$, then $f(0) = 0$. Let $x = 0$, then $f(0) - f(y) = f(-y)$. Thus $f(-y) + f(y) = 0$. Hence for arbitrary $x \in (-1, 1)$, $f(-x) + f(x) = 0$ holds. Therefore $f(x)$ is an odd function on $(-1, 1)$.

(2) Solution: Since the sequence $\{x_n\}$, $x_1 = \frac{1}{2}$, $x_{n+1} = \frac{2x_n}{1+x_n^2}$, then $0 < x_n < 1$. Since $f(x_n) - f(-x_n) = f[\frac{x_n - (-x_n)}{1 - x_n(-x_n)}] = f(\frac{2x_n}{1+x_n^2})$. Additionally, since $f(x)$ is an odd function on $(-1, 1)$, then $f(x_{n+1}) = 2f(x_n)$. Thus $\frac{f(x_{n+1})}{f(x_n)} = 2$. Since $f(\frac{1}{2}) = 1$, $x_1 = \frac{1}{2}$, then $f(x_1) = 1$. Hence the sequence $\{f(x_n)\}$ is a geometric sequence with the first term 1 and the common ratio 2. Thus $f(x_n) = 2^{n-1}$.

(3) Solution: $\frac{1}{f(x_1)} + \frac{1}{f(x_2)} + \cdots + \frac{1}{f(x_n)} = 1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{n-1}} = \frac{1 - \frac{1}{2^n}}{1 - \frac{1}{2}} = 2 - \frac{1}{2^{n-1}} (n \in N^*)$.

Assume there is a natural number m such that $\frac{1}{f(x_1)} + \frac{1}{f(x_2)} + \cdots + \frac{1}{f(x_n)} < \frac{m-8}{4}$ for arbitrary $n \in N^*$. Then $2 - \frac{1}{2^{n-1}} < \frac{m-8}{4}$ holds. Thus $\frac{m-8}{4} \geqslant 2$. We have $m \geqslant 16$. Therefore there is a natural number m, and $\frac{1}{f(x_1)} + \frac{1}{f(x_2)} + \cdots + \frac{1}{f(x_n)} < \frac{m-8}{4}$ holds when $m \geqslant 16$ for arbitrary $n \in N^*$. The minimum value of m is 16.

6.99 ★★★★★ The domain of the function $f(x)$ is R^+, for arbitrary $x, y \in R^+$, $f(xy) = f(x) + f(y)$ holds. (1) Show $f(\frac{1}{x}) = -f(x)$ when $x \in R^+$. (2) If $f(x) < 0$ holds when $x > 1$, show $f(x)$ has an inverse function. (3) Let $f^{-1}(x)$ is the inverse function of $f(x)$. Show that in the domain of $f^{-1}(x)$, $f^{-1}(x_1+x_2) = f^{-1}(x_1) \cdot f^{-1}(x_2)$.

(1) Proof: Let $y = \frac{1}{x}$ in the given equation, then $f(x) + f(\frac{1}{x}) = f(x \cdot \frac{1}{x}) = f(1)$. Let $x = y = 1$, then $f(1) = f(1) + f(1)$. Thus $f(1) = 0$. Hence $f(x) + f(\frac{1}{x}) = 0$. Therefore $f(\frac{1}{x}) = -f(x)$ when $x \in R^+$.
(2) Proof: Let $x_1, x_2 \in R^+$, and $x_1 < x_2$, then $\frac{x_2}{x_1} > 1$. Thus $f(x_2) - f(x_1) = f(x_2) + f(\frac{1}{x_1}) = f(\frac{x_2}{x_1}) < 0$. Hence the function $f(x)$ is decreasing in R^+. Therefore $f(x)$ has an inverse function.
(3) Proof: Since $x_1, x_2, x_1 + x_2$ are in the domain of $f^{-1}(x)$, then $f^{-1}(x_1)$, $f^{-1}(x_2)$, $f^{-1}(x_1 + x_2) \in R^+$. Thus $f[f^{-1}(x_1) \cdot f^{-1}(x_2)] = f[f^{-1}(x_1)] + f[f^{-1}(x_2)] = x_1 + x_2 = f[f^{-1}(x_1 + x_2)]$. Hence $f^{-1}(x_1 + x_2) = f^{-1}(x_1)f^{-1}(x_2)$.

6.100 ★★★★★ The function $f(x) = 2x^3 + (m-x)^3 \quad (m \in N^*)$.
(1) If $x_1, x_2 \in (0, m)$, show $f(x_1) + f(x_2) \geqslant 2f(\frac{x_1+x_2}{2})$. (2) If $a_n = f(n) \quad (n = 1, 2, \cdots, m-1)$, show $a_1 + a_{m-1} \geqslant a_2 + a_{m-2}$. (3) For arbitrary $a, b, c \in [\frac{m}{2}, \frac{2}{3}m]$, can the values of $f(a)$, $f(b)$, $f(c)$ form the three side lengths of a triangle? Please explain the reason.

(1) Proof: From the given condition, we have $x_1, x_2 \in (0, m)$, $f(x_1) = 2x_1^3 + (m-x_1)^3$, $f(x_2) = 2x_2^3 + (m-x_2)^3$. Since $x_1^3 + x_2^3 - 2(\frac{x_1+x_2}{2})^3 = \frac{3}{4}(x_1+x_2)(x_1-x_2)^2$, $x_1, x_2 \in (0, m)$, then $\frac{3}{4}(x_1+x_2)(x_1-x_2)^2 \geqslant 0$. Thus $x_1^3 + x_2^3 \geqslant 2(\frac{x_1+x_2}{2})^3$ which is equivalent to $2x_1^3 + 2x_2^3 \geqslant 2 \times 2(\frac{x_1+x_2}{2})^3$. Similarly, $(m-x_1)^3 + (m-x_2)^3 \geqslant 2(\frac{m-x_1+m-x_2}{2})^3 = 2(m - \frac{x_1+x_2}{2})^3$. Therefore $f(x_1) + f(x_2) \geqslant 2f(\frac{x_1+x_2}{2})$.
(2) Proof: From (1), we have $a_1 + a_3 \geqslant 2a_2$, $a_2 + a_4 \geqslant 2a_3$, $a_3 + a_5 \geqslant 2a_4, \cdots$, $a_{m-3} + a_{m-1} \geqslant 2a_{m-2}$. Adding the above $(m-3)$ inequalities to generate $a_1 + a_{m-1} \geqslant a_2 + a_{m-2}$.
(3) Solution: Since $f(x) = 2x^3 + (m-x)^3$, then $f'(x) = 6x^2 - 3(m-x)^2 = 3x^2 + 6mx - 3m^2$. Obviously, $f'(x) > 0$ when $x \in [\frac{m}{2}, \frac{2}{3}m]$. This means $f(x)$ is an increasing function on $[\frac{m}{2}, \frac{2}{3}m]$. The minimum value of $f(x)$ is $f(x)_{min} = 2 \times \frac{m^3}{8} + \frac{m^3}{8} = \frac{3}{8}m^3$ when $x = \frac{m}{2}$. The maximum value of $f(x)$ is $f(x)_{max} = 2 \times \frac{8}{27}m^3 + \frac{1}{27}m^3 = \frac{17}{27}m^3$ when $x = \frac{2}{3}m$.

We let $a \leqslant b \leqslant c$, then $\frac{3}{8}m^3 \leqslant f(a) \leqslant f(b) \leqslant f(c) \leqslant \frac{17}{27}m^3$. Thus $f(a) + f(b) \geqslant \frac{3}{8}m^3 \cdot 2 = \frac{3}{4}m^3 > \frac{17}{27}m^3 \geqslant f(c)$.
Therefore $f(a)$, $f(b)$, $f(c)$ can be the three side lengths of a triangle.

6.101 ★★★★★ Given $f(x) = \frac{x}{\sqrt{1-x^2}}$, and $f_n(x) = \underbrace{f(f\cdots(f(x)))}_{nf}, n \in N^*$. Find the analytic expression of $f_n(x)$ and prove it.

Solution: From the given condition, we have $f_1(x) = f(x) = \frac{x}{\sqrt{1-x^2}}$, $f_2(x) = f(f(x)) = \frac{f(x)}{\sqrt{1-[f(x)]^2}} = \cdots = \frac{x}{\sqrt{1-2x^2}}$, $f_3(x) = f(f(f(x))) = f(f_2(x)) = \cdots = \frac{x}{\sqrt{1-3x^2}}, \cdots$. Then we generalize $f_n(x) = \frac{x}{\sqrt{1-nx^2}} \quad (n \in N^*)$.
Now we prove the conclusion by mathematical induction.
(1) When $n = 1$, $f_1(x) = \frac{x}{\sqrt{1-x^2}} = f(x)$. $p(1)$ holds.
(2) Assume $p(k)$ holds when $n = k$. This means that $f_k(x) = \frac{x}{\sqrt{1-kx^2}}$ holds.
When $n = k+1$, $f_{k+1}(x) = \underbrace{f(f\cdots(f(x)))}_{(k+1)f} = f(f_k(x)) = \frac{f_k(x)}{\sqrt{1-f_k^2(x)}} = \frac{\frac{x}{\sqrt{1-kx^2}}}{\sqrt{1-\frac{x^2}{1-kx^2}}} = \frac{x}{\sqrt{1-(k+1)x^2}}$. Then $f_{k+1}(x) = \frac{x}{\sqrt{1-(k+1)x^2}}$. Hence $p(k+1)$ also holds.
Therefore, for all $n \in N^*$, $f_n(x) = f(x) = \frac{x}{\sqrt{1-nx^2}}$ always holds.

6.102 ★★★★★ Let the function $f_n(\theta) = \sin^n\theta + (-1)^n\cos^n\theta, \quad 0 \leqslant \theta \leqslant \frac{\pi}{4}$, where n is a positive integer.
(1) Determine the monotonicity of $f_1(\theta)$ and $f_3(\theta)$. Prove your conclusions.
(2) Show $2f_6(\theta) - f_4(\theta) = (\cos^4\theta - \sin^4\theta)(\cos^2\theta - \sin^2\theta)$.
(3) For an arbitrary given positive integer n, find the maximum value and minimum value of the function $f_n(\theta)$.

(1) Solution: We can show that $f_1(\theta)$ and $f_3(\theta)$ are both increasing functions on $[0, \frac{\pi}{4}]$.
Now we provide the proof for the monotonicity of $f_1(\theta)$.
Since $f_1(\theta) = \sin\theta - \cos\theta$, let $\theta_1, \theta_2 \in [0, \frac{\pi}{4}]$, and $\theta_1 < \theta_2$, then $f_1(\theta_1) - f_1(\theta_2) = (\sin\theta_1 - \cos\theta_1) - (\sin\theta_2 - \cos\theta_2) = (\sin\theta_1 - \sin\theta_2) + (\cos\theta_2 - \cos\theta_1)$. Since $\sin\theta_1 < \sin\theta_2$), $\cos\theta_2 < \cos\theta_1$), then $f_1(\theta_1) - f_1(\theta_2) < 0$. Thus $f_1(\theta_1) < f_1(\theta_2)$. Hence $f_1(\theta)$ is increasing on $[0, \frac{\pi}{4}]$.

Similarly, $f_3(\theta)$ is increasing on $[0, \frac{\pi}{4}]$

(2) Proof: The left-hand side of the equation $2f_6(\theta) - f_4(\theta) = 2(\sin^6\theta + \cos^6\theta) - (\sin^4\theta + \cos^4\theta) = 2(\sin^2\theta + \cos^2\theta)(\sin^4\theta - \sin^2\theta\cos^2\theta + \cos^4\theta) - (\sin^4\theta + \cos^4\theta) = \sin^4\theta - 2\sin^2\theta\cos^2\theta + \cos^4\theta = (\sin^2\theta + \cos^2\theta)^2 - 4\sin^2\theta\cos^2\theta = 1 - \sin^2 2\theta = \cos^2 2\theta$.
The right-hand side of the equation $= (\cos^2\theta + \sin^2\theta)(\cos^2\theta - \sin^2\theta)^2 = \cos^2 2\theta$.
Thus, The left-hand side of the equation equals the right-hand side.

(3) When $n = 1$, the function $f_1(\theta)$ is increasing on $[0, \frac{\pi}{4}]$, then $f_1(\theta)_{\max} = f_1(\frac{\pi}{4}) = 0$, $f_1(\theta)_{\min} = f_1(0) = -1$. When $n = 2$, $f_2(\theta)_{\max} = f_2(\theta)_{\min} = 1$. When $n = 3$, the function $f_3(\theta)$ is increasing on $[0, \frac{\pi}{4}]$, then $f_3(\theta)_{\max} = f_3(\frac{\pi}{4}) = 0$, $f_3(\theta)_{\min} = f_3(0) = -1$. When $n = 4$, the function $f_4(\theta) = 1 - \frac{1}{2}\sin^2 2\theta$ is decreasing on $[0, \frac{\pi}{4}]$, then $f_4(\theta)_{\max} = f_4(0) = 1$, $f_4(\theta)_{\min} = f_4(\frac{\pi}{4}) = \frac{1}{2}$. Now we discuss the case $n \geqslant 5$.

When n is an odd number, for arbitrary $\theta_1, \theta_2 \in [0, \frac{\pi}{4}]$, and $\theta_1 < \theta_2$, since $f_n(\theta_1) - f_n(\theta_2) = (\sin^n\theta_1 - \sin^n\theta_2) + (\cos^n\theta_2 - \cos^n\theta_1)$, and $0 \leqslant \sin\theta_1 < \sin\theta_2 < 1$, $0 < \cos\theta_2 < \cos\theta_1 \leqslant 1$. Thus $\sin^n\theta_1 < \sin^n\theta_2$, $\cos^n\theta_2 < \cos^n\theta_1$. Hence $f_n(\theta_1) < f_n(\theta_2)$. Then $f_n(\theta)$ is increasing on $[0, \frac{\pi}{4}]$. We have $f_n(\theta)_{\max} = f_n(\frac{\pi}{4}) = 0$, $f_n(\theta)_{\min} = f_n(0) = -1$.

When n is an even number, on one hand, $f_n(\theta) = \sin^n\theta + \cos^n\theta \leqslant \sin^2\theta + \cos^2\theta \leqslant 1 = f_n(0)$, and on the other hand, for an arbitrary positive integer $l \geqslant 2$, we have $2f_{2l}(\theta) - f_{2l-2}(\theta) = (\cos^{2l-2}\theta - \sin^{2l-2}\theta)(\cos^2\theta - \sin^2\theta) \geqslant 0$, then $f_n(\theta) \geqslant \frac{1}{2}f_{n-2}(\theta) \geqslant \cdots \geqslant \frac{1}{2^{\frac{n}{2}-1}}f_2(\theta) = \frac{1}{2^{\frac{n}{2}-1}} = f_n(\frac{\pi}{4})$. Thus $f_n(\theta)_{\max} = f_n(0) = 1$, $f_n(\theta)_{\min} = f(\frac{\pi}{4}) = 2\sqrt{(\frac{1}{2})^n}$.

As a conclusion, when n is an odd number, the maximum value of $f_n(\theta)$ is 0, the minimum value of $f_n(\theta)$ is -1. When n is an even number, the maximum value of $f_n(\theta)$ is 1, the minimum value of $f_n(\theta)$ is $2\sqrt{(\frac{1}{2})^n}$.