Mathematics Fundamentals

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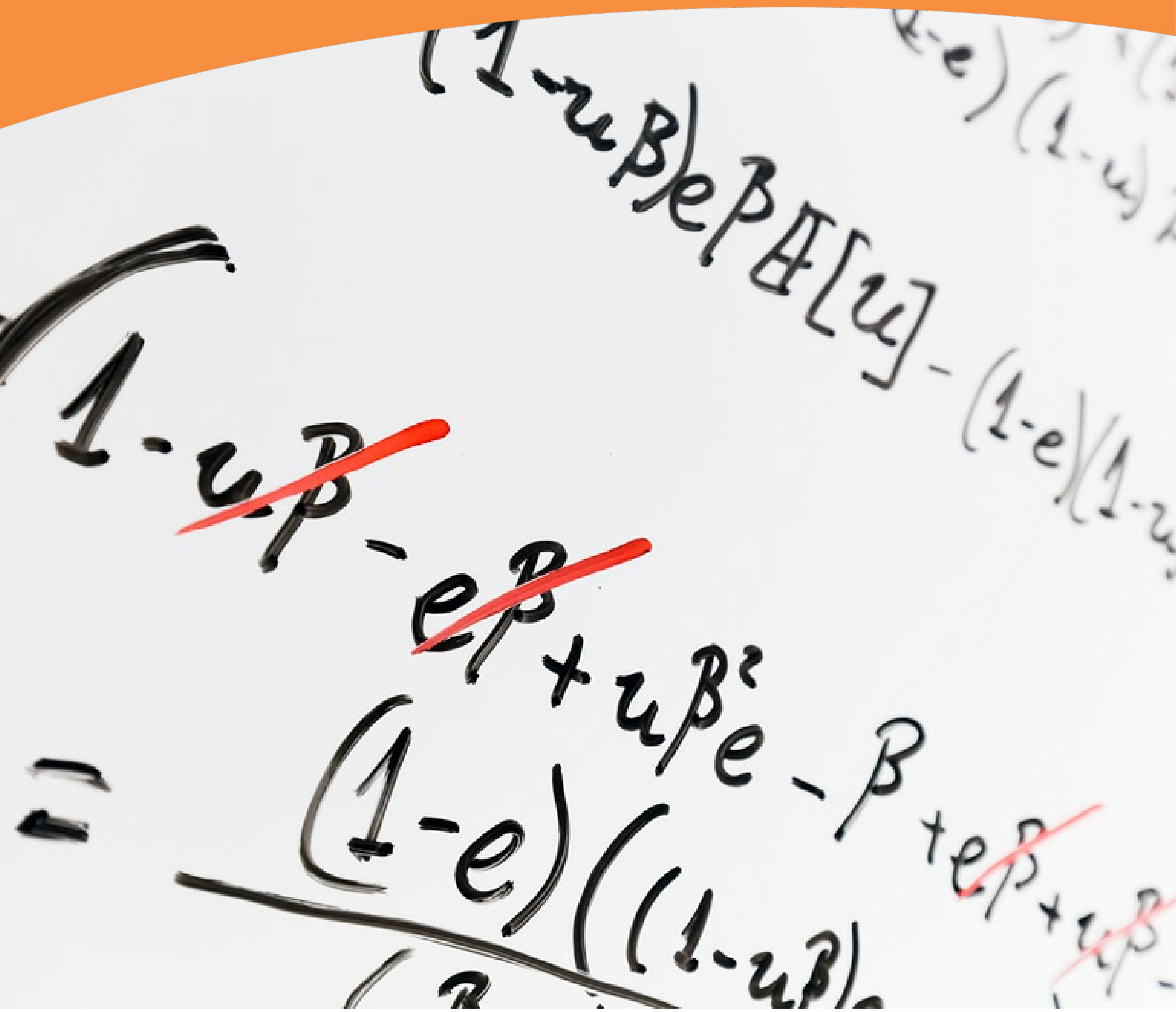

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1st edition

ISBN 978-87-403-1864-7
Peer review by Dr Miccal Matthews, The University of Western Australia

CONTENTS

1. *Numbers and operations*

Types of Numbers

The **natural numbers** are the usual numbers we use for counting:

$$1,\ 2,\ 3,\ 4,\ \cdots$$

We also have the very special number zero

$$0$$

and the negative numbers

$$\cdots -4,\ -3,\ -2,\ -1$$

which are used to indicate deficits (that is, debts) or to indicate opposite directions. These numbers collectively form the **integers**

$$\cdots -4,\ -3,\ -2,\ -1,\ 0,\ 1,\ 2,\ 3,\ 4,\ \cdots$$

The basic operations of arithmetic

Hopefully, the everyday operations $+$, $-$, $\times$ and $\div$ are familiar:

(i) $3+4=7$

(ii) $11-6=5$

(iii) $4\times 9=36$

(iv) $24\div 6=4$

Arithmetic involving negative numbers

When negative numbers are involved we need to be careful, and must use brackets if necessary.

Multiplication examples:

(i) $4\times(-7)=-28$

(ii) $(-5)\times 6=-30$

(iii) $(-3)\times(-6)=18$

(iv) $3\times 5=15$

positive × negative = negative
negative × positive = negative
negative × negative = positive
positive × positive = positive

Other examples:

(i) $13+(-8)=5$

(ii) $21-(-3)=24$

(iii) $-16+13=-3$

(iv) $-16-13=-29$

(v) $(-12)\div 4=-3$

(vi) $\dfrac{-12}{-4}=3$

BIDMAS. *The order of operations*

Consider the simple question:

What is $2 + 3 \times 4$?

Which operation ($+$ or $\times$) do we perform first?

Mathematical expressions are *not* simply calculated from left to right. They are performed in the following order:

B: Expressions within Brackets $(\ldots)$

I: Indices (powers, square roots, etc)

DM: Divisions ($\div$) and Multiplications ($\times$)

AS: Additions ($+$) and Subtractions ($-$)

We will consider Indices in detail later

A couple of things to note:

- Division and multiplication have the same precedence.
- Addition and subtraction have the same precedence.
- Operations with the same precedence are performed left to right.

Some BIDMAS examples

(i) $2 + 3 \times 4 = 2 + 12 = 14$

(ii) $(4 + 6) \times (12 - 4) = 10 \times 8 = 80$

(iii) $4 + 7 \times (6 - 2) = 4 + 7 \times 4 = 4 + 28 = 32$

(iv) $3 \times (4 + 8) - 2 \times (10 - 6) = 3 \times 12 - 2 \times 4 = 36 - 8 = 28$

When brackets are involved we can omit the multiplication sign. So instead of $3 \times (4 + 8) - 2 \times (10 - 6)$ we could write $3(4 + 8) - 2(10 - 6)$

More examples

$$\begin{aligned} &4 \div 2 \times 3 \div 3 \\ &= 2 \times 3 \div 3 \\ &= 6 \div 3 \\ &= 2 \end{aligned}$$

$$\begin{aligned} &-[2 - 3(4 - 6 \div 2 - (-3))] \\ &= -[2 - 3(4 - 3 + 3)] \\ &= -[2 - 3 \times (1 + 3)] \\ &= -(2 - 3 \times 4) \\ &= -(2 - 12) \\ &= -(-10) \\ &= 10 \end{aligned}$$

When brackets are **nested** we usually use different sorts, e.g. $(\cdots)$ $[\cdots]$ $\{\cdots\}$

Word problems

We are sometimes asked quesions in written or spoken conversation that we will be able to use mathematics to help us solve.

Our first task is to turn the 'word problem' into a mathematics problem.

Example: What is three times the difference between 12 and 4?

Answer:

$$
\begin{aligned}
&3 \times (12 - 4) \\
&= 3 \times 8 \\
&= 24
\end{aligned}
$$

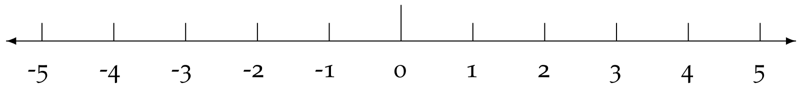
Another example: Joy takes the product of the sum of two and six, and the difference between nineteen and nine. She then result the answer by four. What number does she arrive at?

Answer:

$$
\begin{gathered}
\{(2+6) \times (19-9)\} \div 4 \\
= \{8 \times 10\} \div 4 \\
= 80 \div 4 \\
= 20
\end{gathered}
$$

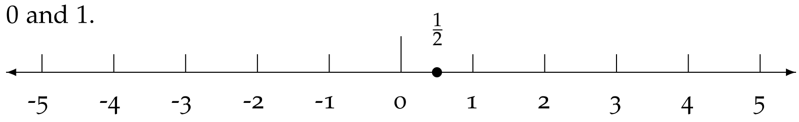
The number line

The **integers** sit nicely on a **number line**:



Fractions

These are numbers that sit between the integers. For example, the number "one-half", which we write "$\frac{1}{2}$", is exactly halfway between 0 and 1.



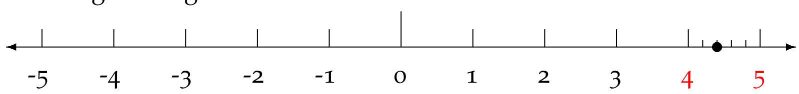
Fractions are *ratios* of integers, that is,

$$\frac{p}{q}$$

Note that p or q or both can be *negative*

where the **numerator** p and **denominator** q are integers. When the numerator (top line) is smaller than the denominator (bottom line) we call this a **proper fraction**.

The number "four and two-fifths" is written "$4\frac{2}{5}$" and is between 4 and 5. To pinpoint this number, imagine splitting up the interval between 4 and 5 into fifths (five equal pieces). Now, starting at 4, move right along two of these fifths.

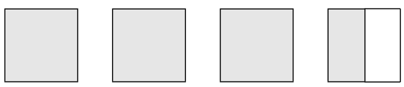


We call $4\frac{2}{5}$ a **mixed numeral**, as it is the **sum** of an integer and a proper fraction.

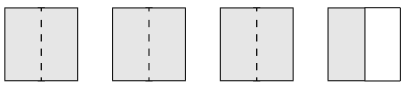
Converting mixed numerals to improper fractions

An **improper fraction** is one in which the numerator is larger than the denominator.

Example: We wish to convert the mixed numeral $3\frac{1}{2}$ into an **improper fraction**. We have three wholes and one half:



or three lots of two halves (making six) plus one half is seven halves:



$$\text{so} \quad 3\frac{1}{2} = \frac{7}{2}$$

Arithmetically, we write

$$3\frac{1}{2} = \frac{3 \times 2 + 1}{2} = \frac{6+1}{2} = \frac{7}{2}$$

Some examples

$$4\frac{1}{3} = \frac{4 \times 3 + 1}{3} = \frac{12+1}{3} = \frac{13}{3}$$

$$2\frac{1}{4} = \frac{2 \times 4 + 1}{4} = \frac{8+1}{4} = \frac{9}{4}$$

$$1\frac{5}{6} = \frac{1 \times 6 + 5}{6} = \frac{6+5}{6} = \frac{11}{6}$$

$$8\frac{3}{5} = \frac{8 \times 5 + 3}{5} = \frac{40+3}{5} = \frac{43}{5}$$

Note: Improper fractions turn out to be used more often in maths, mainly because mixed numbers are confusing in algebra.

Multiplication of fractions

To multiply two fractions simply multiply their numerators and denominators separately. For example,

$$\frac{1}{2} \times \frac{5}{6} = \frac{1 \times 5}{2 \times 6} = \frac{5}{12}$$

Think of this as finding "one-half of five-sixths"

Another example is

$$\frac{3}{5} \times \frac{2}{7} = \frac{3 \times 2}{5 \times 7} = \frac{6}{35}$$

Division of fractions

The notation $2 \div 3$ means the same as $\frac{2}{3}$, that is "two-thirds". Now think of 2 and 3 as fractions, that is

$$2 \div 3 = \frac{2}{1} \div \frac{3}{1}$$

We can also use fraction multiplication to say that

$$\frac{2}{3} = \frac{2}{1} \times \frac{1}{3}$$

Comparing these last two we have

$$\frac{2}{1} \div \frac{3}{1} = \frac{2}{1} \times \frac{1}{3}$$

That is, dividing by a fraction is equivalent to multiplying by its reciprocal, that is, the fraction having been *inverted*. For example,

$$\frac{2}{7} \div \frac{5}{9} = \frac{2}{7} \times \frac{9}{5} = \frac{18}{35}$$

More examples

$$\frac{1}{3}\times\frac{2}{7}\div\frac{5}{6} = \frac{2}{21}\div\frac{5}{6} = \frac{2}{21}\times\frac{6}{5} = \frac{12}{105}$$

$$\frac{\frac{1}{4}\times\frac{7}{8}}{\frac{3}{5}} = \frac{\frac{7}{32}}{\frac{3}{5}} = \frac{7}{32}\div\frac{3}{5} = \frac{7}{32}\times\frac{5}{3} = \frac{35}{96}$$

$$2\frac{5}{4}\div\frac{2}{3}\times 3\frac{1}{7} = \frac{13}{4}\div\frac{2}{3}\times\frac{22}{7} = \frac{13}{4}\times\frac{3}{2}\times\frac{22}{7} = \frac{858}{56}$$

Factorization of numbers

This is the process of expressing a given number as a product of two (or more) smaller numbers. For example,

$$28 = 4\times 7$$

$$\begin{aligned}108 &= 2\times 54\\ &= 2\times 6\times 9\end{aligned}$$

$$\begin{aligned}64 &= 2\times 32\\ &= 2\times 2\times 16\\ &\cdots\\ &= 2\times 2\times 2\times 2\times 2\times 2\end{aligned}$$

Later we'll write this as $64 = 2^6$

Cancellation of numbers

The process of simplifying fractions by looking for factors which are common to both the numerator and the denominator and then eliminating (cancelling) them. For example,

$$\frac{18}{20} = \frac{\cancel{2}\times 9}{\cancel{2}\times 10} = \frac{9}{10}$$

$$\frac{100}{250} = \frac{10\times\cancel{10}}{25\times\cancel{10}} = \frac{10}{25} = \frac{\cancel{5}\times 2}{\cancel{5}\times 5} = \frac{2}{5}$$

$$\frac{24}{108} = \frac{\cancel{4}\times 6}{\cancel{4}\times 27} = \frac{6}{27} = \frac{\cancel{3}\times 2}{\cancel{3}\times 9} = \frac{2}{9}$$

Adding and subtracting fractions

Consider the sum

$$\frac{1}{3}+\frac{2}{5}$$

We can't add these fractions yet because thirds are different to fifths. However, we can make use of **reverse cancellation** to make the denominators the same:

$$\begin{aligned}\frac{1}{3}+\frac{2}{5} &= \frac{1\times 5}{3\times 5} + \frac{2\times 3}{5\times 3}\\ &= \frac{5}{15}+\frac{6}{15}\end{aligned}$$

Now we just have a total of $5+6=11$ fifteenths, that is

$$\begin{aligned}\frac{1}{3}+\frac{2}{5} &= \frac{5+6}{15}\\ &= \frac{11}{15}\end{aligned}$$

The same applies to subtraction. For example, $\frac{9}{20}-\frac{2}{5}$. Following the above pattern the denominator would be $20\times 5=100$, but we can simplify the numbers a little as follows. Since 5 divides into 20 we could just do this instead:

$$\begin{aligned}\frac{9}{20}-\frac{2}{5} &= \frac{9}{20}-\frac{2\times 4}{5\times 4}\\ &= \frac{9}{20}-\frac{8}{20}\\ &= \frac{9-8}{20}\\ &= \frac{1}{20}\end{aligned}$$

Two examples

$$\frac{1}{4}+\frac{4}{7} = \frac{1\times 7}{4\times 7}+\frac{4\times 4}{7\times 4} = \frac{7}{28}+\frac{16}{28} = \frac{23}{28}$$

$$\frac{2}{9}+\frac{1}{6}-\frac{5}{54} = \frac{12}{54}+\frac{9}{54}-\frac{5}{54} = \frac{21}{54}-\frac{5}{54} = \frac{16}{54}$$

which if required can be simplified:
$= \frac{\cancel{2}\times 8}{\cancel{2}\times 27} = \frac{8}{27}$

Decimals

Decimals are a convenient and useful way of writing fractions with denominators 10, 100, 1000, etc.

$\frac{2}{10}$ is written as 0.2 $\frac{4}{100}$ is written as 0.04

$\frac{35}{100}$ is written as 0.35 $\frac{612}{1000}$ is written as 0.612

The number 237.46 is shorthand for

$$2 \times 100 + 3 \times 10 + 7 \times 1 + \frac{4}{10} + \frac{6}{100}$$

The decimal point separates the whole numbers from the fractions. To the right of the decimal point, we read the names of the digits individually. For example, 237.46 is read as 'two hundred and thirty-seven point four six'.

The places after the decimal point are called the decimal places. We say that 25.617 has 3 decimal places.

Some fractions expressed as decimals

We can write

$\frac{3}{10}$	as	0.3	$-\frac{4}{10}$	as	-0.4
$\frac{47}{100}$	as	0.47	$\frac{12}{10} = 1 + \frac{2}{10}$	as	1.2
$-\frac{23}{100}$	as	-0.23	$\frac{327}{100} = 3 + \frac{27}{100}$	as	3.27
$\frac{931}{1000}$	as	0.931	$\frac{28}{1000}$	as	0.028
$\frac{6}{100}$	as	0.06	$\frac{5}{1000}$	as	0.005

Common fractions expressed as decimals

All fractions can be written as decimals. However, the decimal sequence may go on forever.

Examples

$\frac{1}{10} = 0.1$ $\frac{1}{100} = 0.01$ $\frac{1}{1000} = 0.001$

$\frac{1}{2} = 0.5$ $\frac{1}{4} = 0.25$ $\frac{1}{8} = 0.125$

$\frac{3}{4} = 0.75$ $\frac{1}{5} = 0.2$ $\frac{2}{5} = 0.4$

$\frac{1}{3} = 0.3333\cdots$ $\frac{2}{3} = 0.6666\cdots$ $-\frac{2}{5} = -0.4$

$\frac{1}{7} = 0.142857142857\cdots$ $\frac{22}{7} = 3 + \frac{1}{7} = 3.1428\cdots$

The $\cdots$ indicate that the sequence goes on forever

Fixing the number of decimal places

When working with decimals we usually limit ourselves to a certain number of decimal places.

Examples To three decimal places

$$\frac{1}{3} = 0.333 \qquad \frac{1}{7} = 0.143$$

$$\frac{1}{8} = 0.125 \qquad \frac{2}{5} = 0.400$$

Note that we have 'rounded up' the 2 to 3 because the 8 after the 2 in $\frac{1}{7} = 0.14285714\cdots$ is larger than 5 (out of 10)

Note that we can *'pad out'* with zeros to the left in the decimal for $\frac{2}{5}$

Multiplying and dividing decimal numbers by 10

To multiply a decimal number by 10 simply move the decimal point to the left, and to divide a decimal number by 10 move the decimal point to the right

Examples

$$0.47 \times 10 = 4.7 \qquad 0.06 \times 10 = 0.6$$

$$-0.12 \times 10 = -1.2 \qquad 0.000234 \times 10 = 0.00234$$

$$0.000234 \times 100 = 0.0234 \qquad 0.56 \times 100 = 56.0$$

$$0.378 \div 10 = 0.0378 \qquad -0.04 \div 10 = -0.004$$

$$2.571 \div 10 = 0.2571 \qquad 4.63 \div 100 = 0.0463$$

Exercises 1

1.1 Evaluate the following

(i) $3(4+2\times 6)+12\div 4$ (ii) $6+2[25-3(2+5)]$

1.2 What is the difference between the product of 6 and 7, and the sum of 3 and 8?

1.3 Convert to improper fractions

$(a)\quad 1\frac{1}{4}$ $(b)\quad 2\frac{3}{8}$

1.4 Evaluate (that is, calculate)

$(a)\quad \frac{3}{4}\times\frac{1}{5}$ $(b)\quad \frac{1}{3}\div\frac{2}{5}$

1.5 What fraction is two-thirds of four-fifths?

1.6 Evaluate

(i) $\frac{3}{4}+\frac{2}{5}$ (ii) $\frac{2}{3}-\frac{2+4}{5-1}$

1.7 Reduce to simplest form by using cancellation:

$$\frac{78}{102}$$

1.8 What is one-quarter of the sum of one-third and two-sevenths?

Problem set 1

1.1 Evaluate the following.

(a) $4+4\times 4\div 4-4$
(b) $4+4\times(4\div 4-4)$
(c) $2\times 4-(8-1)$
(d) $-14\div 2-(-20)\div(-5)$
(e) $(-4\times 12)-[32-(-4)]$
(f) $[4\times(3+4)-21]\div[2\times 14\div 7+3]$
(g) $3\times 8-[-16-(-4)]$
(h) $[-3\times(3-4)-19]\div[8\div 4\times 7-6]$
(i) $(8-4)\div(4+4)-8$
(j) $4+4\times 4\div(4-4)$

1.2 Compute the following, leaving your answer in the simplest fraction form.

(a) $\frac{1}{2}+\frac{3}{4}+\frac{5}{12}$ (b) $\frac{3}{4}-\frac{4}{5}+\frac{3}{10}$

(c) $\frac{4}{24}+1\frac{2}{3}$ (d) $1\frac{2}{3}-2\frac{3}{5}+1\frac{4}{5}$

(e) $\frac{4}{5}\times\frac{3}{7}$ (f) $2\frac{2}{3}\div 4\frac{2}{9}$

(g) $\frac{7}{9}\times 1\frac{1}{8}\div\frac{5}{7}$ (h) $2\frac{3}{7}\div\frac{34}{14}$

1.3 What is half of the difference between a third of 60 and a quarter of 80?

1.4 What is the sum of half of 45 and a third of the product of two thirds and $1\frac{1}{8}$?

1.5 James and his wife Sweet Li have a 12 year old daughter called Lyn. James is 50 year old and Lyn is 12. Half of his age added to five thirds of Lyn's age gives the age of Sweet Li. How old is Sweet Li?

1.6 Swee Khum put a third of her savings in the bank, a third in bonds, a quarter of the remainder in stocks and the rest in fixed deposit. If her total amount is $600,000 how much did she put in fixed deposit?

2. *Algebra*

Algebra is a powerful tool for expressing fundamental relationships and solving real-world problems.

An example: In the expression $E = mc^2$, the **variables** are the energy (E) of an object and its mass (m), c^2 means $c \times c$ and c is a constant representing the inconveniently large value of the speed of light in a vacuum, i.e. 29,792,458 metres per second.

Such constants appearing in relationships are called **parameters**

The famous equation above summarises the relationship between the variables E and m

An **equation** is something with an equals sign in it

We can also use algebra to express our rules for multiplying and dividing fractions:

$$\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd} \qquad \frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c} = \frac{ad}{bc}$$

An example of algebra in action

A simplified version of the formula for the length of time it takes a child to say "are we nearly there yet?" on a long car journey is

$$T = \frac{1 + A}{C^2}$$

This formula was proposed by Prof. Dwight Barkley, Warwick University

where

T = time to say "are we nearly there yet?" (in hours),
A = number of activities provided and
C =number of children.

So, 2 children with 3 activities will take

$$\begin{aligned} T &= \frac{1+3}{2^2} \\ &= \frac{4}{4} \\ &= 1 \quad \text{hour to get restless} \end{aligned}$$

Solving equations

In many situations we are told the numerical values of all variables in an equation except one and asked to determine the value of that variable. That is, we are asked to **solve** for the unknown variable.

Example. In the restlessness problem above we might know that there were 2 children in the back and it took them an hour and a half to complain. How many activities were they given? We have

$$1\frac{1}{2} = \frac{1+A}{2^2} \qquad \text{that is} \qquad \frac{3}{2} = \frac{1+A}{2^2}$$

To answer questions like this we need to know how to re-arrange equations.

Simple Equations

Here are some typical equations with an unknown variable:

$$x + 4 = 7$$

$$2x + 7 = \frac{3x}{2} + 8$$

$$\frac{2}{x} + 4 = \frac{3}{x}$$

Our aim is to find out what value of the unknown variable x makes the equation work. That is, we wish to "solve for x".

Operations on equations

Let's look at

$$x + 4 = 7$$

To get x on its own, we can "remove" the other operations (in this case the "+ 4") by performing operations that "undo" them.

To maintain equality, operations must be applied to **the entirety of both sides**. Operations include:

(1) Multiply (or divide) both sides of an equation by a non-zero number

(2) Add (or subtract) the same number to (from) both sides of an equation

Using these operations effectively is just a matter of technique and practice.

Example: To solve

$$x + 4 = 7$$

subtract 4 from both sides:

$$x + \cancel{4} - \cancel{4} = 7 - 4$$
$$\Rightarrow \quad x = 3$$

The symbol $\Rightarrow$ means 'implies that', and the diagonal bars indicate cancellation.

It's always a good idea to check that the value we have obtained is in fact correct. Clearly $3 + 4 = 7$ so we are correct.

Example: To solve $3x = 12$ divide each side by 3,

$$\Rightarrow \quad \frac{\cancel{3}x}{\cancel{3}} = \frac{12}{3}$$
$$\Rightarrow \quad x = 4$$

Check: $3 \times 4 = 12$ ✓

Example: To solve $y - 4 = 2y + 10$ you need to get the two y terms on the same side,

$$\Rightarrow \quad y - 4 - y = 2y + 10 - y$$
$$\Rightarrow \quad -4 = y + 10 \quad \text{(since "two } y\text{'s" minus "one } y\text{" is "one } y\text{".)}$$
$$\Rightarrow \quad -4 - 10 = y + 10 - 10$$
$$\Rightarrow \quad -14 = y$$

Check: $-14 - 4 = -18$ and $2 \times (-14) + 10 = -18$ hence ✓

Example: This one presents a new problem: $3x + 2 = 2(x + 7)$.

Normally, we perform the operations in the brackets first but there is nothing we can do with $x + 7$. We need to **expand the brackets**. We do this with the aid of

The Distributive Law: $a(b + c) = ab + ac$
where a, b and c all represent terms.

Basically, the term out the front, a is multiplied by (distributed amongst) each term inside the brackets, that is, b and c.

Let's first test this law. Say $a = 9$, $b = 4$, $c = 3$. Then the left-hand side **(LHS)** of the equation is

$$9(4 + 3) = 9 \times 7 = 63$$

and the right-hand side (**RHS**) of the equation is

$$9 \times 4 + 9 \times 3 = 36 + 27 = 63$$

Some examples

$$4(3+x) = 12+4x$$
$$3(4-2x) = 12-6x$$
$$-x(y-5z) = -xy+5xz$$

You need to be careful when negative signs are involved

and *really* careful when more negative signs are involved

Watch out when brackets are near other operations:

$$4x - 4(3+5x) = 4x - 12 - 20x = -16x - 12$$

The $4x$ part is *not* involved in the expansion of the brackets

We can solve equations involving brackets by making use of the Distributive Law.

Example:

$$3x+2 = 2(x+7)$$
$$\Rightarrow \quad 3x+2 = 2x+14$$
$$\Rightarrow \quad x+2 = 14$$
$$\Rightarrow \quad x = 12$$

expand the bracket

then subtract $2x$ from both sides

and finally, subtract 2 from both sides

and recall that the symbol $\Rightarrow$ means 'implies that'.

Another example: To solve

$$2t-1 = 3-2\left[2t-3\left(1-t\right)\right]$$

first expand the brackets (from the inside out):

$$\Rightarrow \quad 2t-1 = 3-2\left[2t-3+3t\right]$$
$$\Rightarrow \quad 2t-1 = 3-2\left[5t-3\right]$$
$$\Rightarrow \quad 2t-1 = 3-10t+6$$
$$\Rightarrow \quad 2t-1 = 9-10t$$
$$\Rightarrow \quad 12t-1 = 9$$
$$\Rightarrow \quad 12t = 10$$

and hence

$$t = \frac{10}{12}$$

which simplifies to $t = \frac{5}{6}$

Equations with fractions

Fractions create extra challenges and it's a good idea to deal with them early on. If we have an equation of the form

$$\frac{A}{B} = \frac{C}{D}$$

(where A, B, C, D are mathematical expressions) we can multiply both sides by the denominators,

$$\frac{A}{B} = \frac{C}{D}$$

$$\Rightarrow \quad \cancel{B} \times D \times \frac{A}{\cancel{B}} = \frac{C}{\cancel{D}} \times B \times \cancel{D}$$

$$\Rightarrow \quad AD = BC$$

An example

$$\frac{2}{x-1} = \frac{3}{x-2}$$
$$\Rightarrow \quad 2(x-2) = 3(x-1)$$
$$\Rightarrow \quad 2x-4 = 3x-3$$
$$\Rightarrow \quad 2x-1 = 3x$$
$$\Rightarrow \quad -1 = x$$

IMPORTANT: When the denominators are 'moved to the top line' they must be placed in brackets. We can then expand the brackets if desired.

Check: If $x = -1$ then the LHS (left-hand side) is

$$\frac{2}{x-1} = \frac{2}{-1-1} = \frac{2}{-2} = -\frac{2}{2} = -1$$

and similarly, the RHS (right-hand side) of the equation is -1

Example

$$\frac{x}{2} + \frac{x}{3} = 5$$

We can't cross multiply here but we can still get rid of the fractions by multiplying through by their denominators:

$$\Rightarrow \quad 2 \times 3 \left(\frac{x}{2} + \frac{x}{3}\right) = 2 \times 3 \times 5$$

and expand the bracket:

$$\Rightarrow \quad \not{2} \times 3 \times \frac{x}{\not{2}} + 2 \times \not{3} \times \frac{x}{\not{3}} = 6 \times 5$$
$$\Rightarrow \quad 3x + 2x = 30$$
$$\Rightarrow \quad 5x = 30$$
$$\Rightarrow \quad x = \frac{30}{5}$$
$$\Rightarrow \quad x = 6$$

Check: $\frac{6}{2} + \frac{6}{3} = 3 + 2 = 5$ ✓

Simplifying expressions

As you can see, when re-arranging equations we may need to "tidy up", or simplify the other side of an equation (or expression).

Example

$$6z - [z + 4(z-5) - 2(3-4z)] = \cdots$$
$$\Rightarrow \quad 6z - [z + 4z - 20 - 6 + 8z] = \cdots$$
$$\Rightarrow \quad 6z - z - 4z + 20 + 6 - 8z = \cdots$$
$$\Rightarrow \quad z + 26 - 8z = \cdots$$
$$\Rightarrow \quad -7z + 26 = \cdots$$

Mathematical modelling

We can use algebra to solve real-world problems. Given a problem which is stated in words (a 'word problem') we can convert it into an algebraic equation which we can hopefully solve to find the solution to the real-world problem.

Example: A man is 12 times as old as his son. In 9 years time, the man will be 3 times as old as his son. How old are each now?

Start by identifying the unknowns and assigning a letter to them. Let s be the son's current age. We could call the father's current age "f" but we already know what it is in terms of s (ie. $12 \times s$).

In 9 years time the son's age will be $s + 9$ and the father's $12s + 9$.

Now, we are told that

$$12s + 9 = 3\,(s + 9)$$

$$\Rightarrow 12s + 9 = 3s + 27 \quad \Rightarrow \quad 9s = 18 \quad \Rightarrow \quad s = 2$$

Hence, the son is currently 2 years old, and his father 24. In 9 years time, the son will be 11 and his father 33, which is correct!

Exercises 2

2.1 Solve the following equations

(a) $4y - 3 = 3y + 7$

(b) $9x + 8 = 8x - 4$

2.2 In the formula

$$x = \frac{5+t}{b-t}$$

if $x = 2$ and $t = 1$, then what is the value of b?

2.3 Solve the following equations

(a) $\frac{1}{3x+1} = \frac{2}{x-1}$

(b) $-2y + 1 + 3(2y - 4) = 2y - 1$

2.4 Sally's age is 12 years plus half her age. How old is she?

Problem set 2

2.1 Solve the following equations for the unknown.

(a) $x + 3 = 5$

(b) $4y - 5 = y + 10$

(c) $8x - 4 = 16$

(d) $\dfrac{2z}{z+2} = 4$

(e) $3 - 2x = 4$

(f) $7x + 7 = 2(x + 1)$

(g) $x - \dfrac{1}{2} = 2$

(h) $\dfrac{2y-3}{4} = \dfrac{6y+7}{3}$

(i) $t = 2 - 2[2t - 3(1 - t)]$

(j) $\dfrac{3}{2}(4a - 3) = 2[a - (4a - 3)]$

(k) $\dfrac{x+3}{x} = \dfrac{2}{5}$

(l) $\dfrac{1}{p-1} = \dfrac{2}{p-2}$

(m) $\dfrac{x}{2} - \dfrac{x}{5} = 1$

(n) $\dfrac{y-4}{2} + \dfrac{y}{3} = 4$

(o) $\dfrac{2}{y} + 4 = \dfrac{3}{y}$

(p) $\dfrac{2-y}{3} - \dfrac{1-y}{2} = y$

(q) $y - \dfrac{1-y}{2} + 4 = 7$

(r) $\dfrac{y}{3} - (3 - y) = 4 - y$

(s) $\dfrac{d-1}{2} - \dfrac{d}{3} = 0$

(t) $\dfrac{4-p}{3} = -8$

(s) $\dfrac{d-1}{2} - \dfrac{d}{3} = 0$

(t) $\dfrac{4-p}{3} = -8$

2.2 If $2\frac{1}{4}$ cakes are served in slices of $\frac{1}{8}$ of a cake, how many slices are served?

2.3 Lian earns $2,000 a fortnight and spends $1200 of it. What proportion of his salary does he save?

2.4 Nadia buys a packet of 60 samosas for a tea party. She takes out half for her family. She takes the remaining to the party. On the way she meets a friend who takes a sixth of the samosas. A little further on she sees a hungry cat and gives it a fifth of the samosas she has. Then she suddenly feels hungry and eats a quarter of the remaining samosas. How many arrive at the tea party?

2.5 I have a naan recipe that calls for $4\frac{2}{3}$ cups of flour for 14 servings of naan. I want to make one serving only. How much flour should I use?

2.6 Suppose Mike travels a certain distance on the first day and twice the distance on the next day. If the total distance he travelled is 60 km, how far does he travel on the first day?

2.7 Joy, Pam, Sandra and Lilin each make a donation to the Guide Dogs Association. Sandra gives twice as much as Lilin, Pam gives three times as much as Sandra and Joy gives four times as much as Pam. If their total gift is $132, find the amount of Lilin's donation, and hence the amount donated by each.

3. Simultaneous equations

All of the equations that we have solved so far are called "linear equations in one unknown". Now we will look at situations where we have two linear equations in two unknowns.

Example

David buys movie tickets for seven adults and one child for $92. Grace buys tickets for ten adults and three children for $144. What are the costs of the individual tickets?

Let a be the cost of an adult ticket, and c be the cost of a child's ticket. Then the equations representing the information given are:

$$\begin{array}{rcrcrl} 7a & + & c & = & 92 & (1) \\ 10a & + & 3c & = & 144 & (2) \end{array}$$

Note: The (1) and (2) are labels

These are called *simultaneous equations*. (Many real-life problems contain many equations in many unknowns and require computer methods to solve them.)

A method of solving simultaneous equations

We manipulate the two equations to produce an equation in only one unknown to solve.

Once we have solved this equation we substitute this value into the remaining equation, and then solve for the second unknown.

Here is how the method performs the first step:

$$\begin{array}{rcrcrl} 7a & + & c & = & 92 & (1) \\ 10a & + & 3c & = & 144 & (2) \end{array}$$

Start by multiplying both sides of equation (1) by 3 to produce "$3c$" in both equations:

$$\begin{array}{rcrcrl} 21a & + & 3c & = & 276 & (3) \\ 10a & + & 3c & = & 144 & (2) \end{array}$$

Note that we have given this equation a new label

Plan is to subtract equation (2) from equation (3) to get rid of c.

If we subtract equation (2) from equation (3) we get

$$21a + \cancel{3c} - 10a - \cancel{3c} = 276 - 144$$

In *tabular form* we can write this as

$$\begin{array}{rrcrcrl} & 21a & + & 3c & = & 276 & (3) \\ (-) & 10a & + & 3c & = & 144 & (2) \\ \hline & 11a & & & = & 132 & \end{array}$$

Note that the $(-)$ is simply an indicator of our intention

Since the terms are lined up we can perform the subtraction (or addition) term by term.

We now have an easy equation to solve:

$$11a = 132 \quad \Rightarrow \quad a = \frac{132}{11} = 12$$

That is, the cost of an adult ticket is $12

Because we *eliminated* a variable (in this case, c) this method is usually called the elimination method.

We still need to determine how much a child's ticket costs.

All we have to do is substitute the value for a into *any* of the labelled equations. For example, if we choose equation (1) we get

$$\begin{aligned} 7a + c &= 92 \\ \Rightarrow\ 7 \times 12 + c &= 92 \\ \Rightarrow\ 84 + c &= 92 \\ \Rightarrow\ c &= 92 - 84 \\ \Rightarrow\ c &= 8 \end{aligned}$$

That is, the cost of an child's ticket is $8

We can check our calculations by substituting both values into the *other* equation (equation (2)):

$$10a + 3c = 10 \times 12 + 3 \times 8 = 120 + 24 = 144 \quad \checkmark$$

Note: The biggest source of errors is getting subtractions wrong

Example

Solve the pair of simulataneous equations

$$\begin{array}{rcrcrl} 3x & - & 2y & = & 10 & (1) \\ 2x & + & 3y & = & 11 & (2) \end{array}$$

Notice that if we multiply equation (1) by 2 and equation (2) by 3 then each equation will have the term $6x$ and we will be able to subtract one from the other. That is,

$$\begin{array}{rrcrcrl} (-) & 6x & - & 4y & = & 20 & (3) \\ & 6x & + & 9y & = & 33 & (4) \\ \hline & & & 13y & = & 13 & \end{array}$$

Recall that the $(-)$ is simply an indicator of our intention

So $y = 1$, and substitution of this into say equation (1) gives

$$3x - 2 \times 1 = 10 \quad \Rightarrow \quad 3x - 2 = 10 \quad \Rightarrow \quad 3x = 12 \quad \Rightarrow \quad x = 4$$

Finally, we should check our result by substituting both values into equation (2):

$$2 \times 4 + 3 \times 1 = 8 + 3 = 11 \quad \checkmark$$

Another example

Solve the pair of simulataneous equations

$$\begin{array}{rcrcrl} 4x & - & 2y & = & 24 & (1) \\ -3x & + & 5y & = & -25 & (2) \end{array}$$

Notice that if we multiply equation (1) by 3 and equation (2) by 4 then each equation will have the term 12x and we will be able to add the equations together. That is,

$$\begin{array}{rrcrcrl} (+) & 12x & - & 6y & = & 72 & (3) \\ & -12x & + & 20y & = & -100 & (4) \\ \hline & & & 14y & = & -28 & \end{array}$$

The (+ is simply an indicator of our intention

So $y = -2$, and substitution of this into say equation (1) gives

$$4x - 2 \times (-2) = 24 \quad \Rightarrow \quad 4x + 4 = 24 \quad \Rightarrow \quad 4x = 20 \quad \Rightarrow \quad x = 5$$

We should check our result by substituting both values into equation (2): $-3 \times 5 + 5 \times (-2) = -15 - 10 = -25$ ✓

The father and son problem revisited

A man is 12 times as old as his son. In 9 years time, he will be only 3 times as old as his son. How old are each now?

Solution: Let s be the son's current age and m be the man's current age. We are told that $m = 12s$ and $m + 9 = 3(s + 9)$

We need to re-arrange these equations slightly:

$$m = 12s \quad \Rightarrow \quad m - 12s = 0 \quad (1)$$

$$m + 9 = 3(s + 9) \quad \Rightarrow \quad m + 9 = 3s + 27 \quad \Rightarrow \quad m - 3s = 18 \quad (2)$$

If we subtract equation (1) from equation (2) we get

$$-3s - (-12s) = 18 - 0 \Rightarrow -3s + 12s = 18 \Rightarrow 9s = 18 \Rightarrow s = 2$$

Substituting this into the equation $m = 12s$ gives $m = 12 \times 2 = 24$

Hence the son is currently 2 and his father is currently 24 years old

An example of modelling with simultaneous equations

Bruce is a building contractor. If he hires 8 bricklayers and 2 roofers, his daily payroll is \$960, while 10 bricklayers and 5 roofers require a daily payroll of \$1500. What is the daily wage of a bricklayer and the daily wage of a roofer?

Solution. Let B be a bricklayer's daily wage and R a roofer's daily wage. We are told that

$$\begin{array}{rcrcrl} 8B & + & 2R & = & 960 & (1) \\ 10B & + & 5R & = & 1500 & (2) \end{array}$$

Note that we drop the \$ signs when doing algebra

We can keep the numbers smaller by replacing eq. (2) by a new equation, eq. (3), which equals eq. (2) minus eq. (1).That is,

$$\begin{array}{rcrcrl} 8B & + & 2R & = & 960 & (1) \\ 2B & + & 3R & = & 540 & (3) \end{array}$$

Multiply equation (3) by 4 to get

$$\begin{array}{rrcrcrl} (-) & 8B & + & 2R & = & 960 & (1) \\ & 8B & + & 12R & = & 2160 & (4) \end{array}$$

Now it is simply a matter of subtracting equation (1) from equation (4) to get

$$10R = 1200 \quad \Rightarrow \quad R = \frac{1200}{10} = \frac{120 \times \cancel{10}}{1 \times \cancel{10}} = 120$$

That is, a roofer's daily wage is \$120.

Substitution of $R = 120$ into, say equation (1) gives

$$8B + 2R = 960 \quad \Rightarrow \quad 8B + 2 \times 120 = 960 \quad \Rightarrow \quad 8B + 240 = 960$$

and hence

$$8B = 960 - 240 \quad \Rightarrow \quad 8B = 720 \quad \Rightarrow \quad B = \frac{720}{8} = \frac{\cancel{8} \times 90}{\cancel{8} \times 1} = 90$$

We can verify these by substitution into eq. (2) to get $10B + 5R = 1500$, so $10 \times 90 + 5 \times 120 = 1500$ and hence $900 + 600 = 1500$ ✓

That is, a bricklayer's daily wage is \$90.

Exercises 3

3.1 Solve the pairs of simultaneous equations

(a)
$$\begin{aligned} 2x + 3y &= 10 \quad (1) \\ x + 2y &= 6 \quad (2) \end{aligned}$$

(b)
$$\begin{aligned} 2x + 3y &= 19 \quad (1) \\ 5x - 6y &= -20 \quad (2) \end{aligned}$$

(c)
$$\begin{aligned} 2x + 4y &= 4 \quad (1) \\ 5x - 6y &= 2 \quad (2) \end{aligned}$$

3.2 The sum of two numbers is 28 and their difference is 12. What are the numbers?

Problem set 3

3.1 Solve the following sets of simultaneous equations.

(a) $2x + y = 8$, $3x - y = 7$

(b) $2x + 3y = 19$, $4x - y = 3$

(c) $x + 2y = 10$, $x - y = 1$

(d) $4x - y = 0$, $2x - y = -2$

(e) $2x + y = 4$, $5x + 2y = 9$

(f) $x + 3y = 3$, $-2x + y = 8$

(g) $2x - 3y = 5$, $5x + 2y = -16$

(h) $x + y = 5$, $2x + y = 7$

(i) $4x + y = 10$, $2x + 3y = 10$

(j)* $2x + 3y = 8$, $3x - 2y = -7$

* This one is too difficult for a Test question

3.2 Tickets for an ice-skating display are sold at $5 for adults and $2 for children. If 101 tickets were sold altogether for a take of $394, find the number of adults and children who attended.

3.3 An island contains foxes and rabbits. An ecologist counts both species to study their interaction and how their populations change over time. Last year she found the total number of foxes and rabbits was 7,290 and that the fox population was one-eighth of the rabbit population. How many of each species was present?

3.4 The Roman emperor Augustus was fond of gold and silver sovereigns. Every gold sovereign was to weigh 50 grams and every silver one to weigh 40 grams. One year Augustus suspected that the jeweller was cheating him and delivering sub-weight sovereigns. A consignment contained 30 gold and 20 silver sovereigns weighed 2,250 grams, and a consignment of 15 gold and 25 silver sovereigns weighed 1,550 grams? Was the jeweller cheating Augustus?

4. *Quadratics*

In Chapter 2 we expanded single brackets. For example,

$$2(2x+1) = 4x+2$$

$$-3(x-2) = -3x+6$$

$$x(2y+3z) = 2xy+3xz$$

Double bracket expansion
To expand $(a+b)(c+d)$ we use the *distributive law*:

$$(a+b)(c+d) = (a+b)c+(a+b)d = ac+bc+ad+bd$$

Example

$$(x+1)(2x+3) = 2x^2+3x+2x+3 = 2x^2+5x+3$$

Note that each term in the first set of brackets gets multiplied by each term in the second set of brackets, or
FOIL - First (ac), Outside (ad), Inside (bc), Last (bd)

As always, watch out for negative signs

Examples

$$(3x+1)(x+4) = 3x^2+12x+x+4 = 3x^2+13x+4$$

$$(x-4)(2x-3) = 2x^2-3x-8x+12 = 2x^2-11x+12$$

x^2 means x times x

More than one variable
Expressions can contain any number of variables. For example,

$$3x^2+4xy+3y-z^2$$

In this expression xy means x times y

The expanding brackets idea is the same as for one variable.
Examples

(i) $x(3y+2) = 3xy+2x$

(ii) $(2x+4)(3y-5) = 6xy-10x+12y-20$

(iii) $3(2x+1)-4(z-5) = 6x+3-4z+20 = 6x-4z+23$

Note that we don't write expressions like $x3y$. We always move the number to the left of a term, and write $3xy$

Quadratic expressions

Double bracket expressions with one variable like this:

$$(x+1)(3x+2)$$

are an important type of expression called a quadratic.

When expanded out and terms collected they are always of the form

$$ax^2 + bx + c$$

Note that if $a = 0$ it isn't a quadratic (but b or c or both can be zero)

where a, b and c are numbers.

The numbers a, b and c are called the coefficients of the quadratic.

We say that a is the "coefficient of x^2", b is the "coefficient of x" and c is the "*constant* coefficient".

An exercise

Two of the following are not quadratic expressions. Which two?

(a) $x^2 + 3x + 4$	(f) $2x^2 + 5x$
(b) $x^2 - 12$	(g) $3x - 8$
(c) $-3x^2 + 4x + 12$	(h) $3 - 5x - 2x^2$
(d) $4 - x^2$	(i) $x^3 + 2x + 5$
(e) $x^2 + \frac{1}{4}x + \frac{3}{5}$	(j) $\frac{2}{7}x^2 - \frac{1}{4}x + \frac{3}{5}$

Answer: (g) and (i)

Factorization of quadratic expressions

Factorization is the reverse of expanding brackets. In other words, we start with a quadratic expression, say:

$$x^2 + 5x + 4$$

and return it to double bracket form (that is, two factors). In this case the answer turns out to be:

$$(x+1)(x+4)$$

To see how this is done, we'll work with quadratic expressions where $a = 1$ and note that

$$(x+A)(x+B) = x^2 + (A+B)x + AB$$

So, to find A and B we can see that:

- the constant coefficient is the *product* of A and B and
- the coefficient of x is the *sum* of A and B.

Example: To factorize

$$x^2 + 5x + 6 = (x + A)(x + B)$$

we need to find numbers A and B such that

- their product is 6
- their sum is 5

To do this, look at the factorizations of 6:

2×3 or 1×6 or even $(-2) \times (-3)$ or $(-1) \times (-6)$

The corresponding sums are

$2 + 3$ or $1 + 6$ or even $(-2) + (-3)$ or $(-1) + (-6)$

Only 2 and 3 sum to 5, so

$$x^2 + 5x + 6 = (x + 2)(x + 3)$$

Note that the order of brackets doesn't matter, we can write

$$x^2 + 5x + 6 = (x + 3)(x + 2)$$

Example: To factorize

$$x^2 - 7x + 12$$

We need the factorizations of 12. These are

$$1 \times 12 \qquad 2 \times 6 \qquad 3 \times 4$$

and

$$(-1) \times (-12) \qquad (-2) \times (-6) \qquad (-3) \times (-4)$$

The corresponding sums are

$$13 \qquad 8 \qquad 7 \qquad -13 \qquad -8 \qquad -7$$

We want the sum of the two numbers to be -7 so we must choose -3 and -4. That is,

$$x^2 - 7x + 12 = (x-3)(x-4)$$

or

$$x^2 - 7x + 12 = (x-4)(x-3)$$

Example. To factorize $x^2 - 3x - 4$, observe that the factorizations of -4 are

$$1 \times (-4) \qquad 2 \times (-2) \qquad (-1) \times 4$$

The corresponding sums are -3, 0 and 3 and we want the sum of the two numbers to be -3 so we must choose 1 and -4. That is,

$$x^2 - 3x - 4 = (x+1)(x-4)$$

Note: We can always check the answer by expanding the brackets:
$(x+1)(x-4) = x^2 - 4x + x - 4$
$= x^2 - 3x - 4$ ✓

Example. To factorize $x^2 - 22x + 21$ we need to investigate the factors of 21:

$$1 \times 21 \qquad 3 \times 7 \qquad (-1) \times (-21) \qquad (-3) \times (-7)$$

The factors must add up to -22 so clearly we want -1 and -21 so

$$x^2 - 22x + 21 = (x-1)(x-21)$$

Special cases

Notice the following:

$$(x+4)(x-4) \;=\; x^2 - \cancel{4x} + \cancel{4x} - 16 \;=\; x^2 - 16$$

and

$$(x-3)(x+3) \;=\; x^2 + \cancel{3x} - \cancel{3x} - 9 \;=\; x^2 - 9$$

In general, we have the difference of two squares

$$(x+A)(x-A) \;=\; x^2 - A^2$$

Notice also that

$$(x+3)^2 \;=\; (x+3)(x+3) \;=\; x^2 + 6x + 9$$

and

$$(x-3)^2 = (x-3)(x-3) = x^2-6x+9$$

In general, we have the perfect squares

$$(x+A)^2 = x^2+2Ax+A^2$$

and

$$(x-A)^2 = x^2-2Ax+A^2$$

A simple case

When b and c are both positive, we only have to consider factors which are positive. For example, to factorize

$$x^2+13x+40$$

we only need to investigate the positive factorizations of 40:

$$1\times 40 \qquad 2\times 20 \qquad 4\times 10 \qquad 5\times 8$$

The factors must add up to 13 so clearly we want 5 and 8 and so

$$x^2+13x+40 = (x+5)(x+8)$$

Another example. To factorize $x^2+8x+16$ we investigate the positive factorizations of 16:

$$1\times 16 \qquad 2\times 8 \qquad 4\times 4$$

The factors must add up to 8 so clearly we want 4 and 4 and so

$$x^2+8x+16 = (x+4)(x+4)$$

Factorizing general quadratics

The most general form for a quadratic is

$$ax^2+bx+c \qquad a\neq 0$$

When $a\neq 1$ the factorizing becomes more difficult.

In general, when we see an expression of the form

$$ax^2+b+c$$

we want to factorize it by writing it in the form

$$(Ax+B)(Cx+D)$$

Note that

- A and C multiply to produce the coefficient of x^2, i.e. a
- B and D multiply to produce the constant coefficient, i.e. c

We'll then need to play around with factorizations of a and c to get the right value of b.

Example

To factorize $7x^2 + 15x + 2$ we need to write it in the form

$$(Ax + B)(Cx + D)$$

The only factorisation of 7 is 1×7 so

$$7x^2 + 15x + 2 = (7x + B)(x + D)$$

The numbers B and D must multiply to 2, so one of them must be 1 and the other must be 2. The question is, which one is which? There are two possibilities:

$$(7x + 2)(x + 1) \quad \text{or} \quad (7x + 1)(x + 2)$$

Expand each of these to find out which:

$$\begin{aligned} (7x + 2)(x + 1) &= 7x^2 + 7x + 2x + 2 = 7x^2 + 9x + 2 \quad \text{X} \\ (7x + 1)(x + 2) &= 7x^2 + 14x + x + 2 = 7x^2 + 15x + 2 \quad \checkmark \end{aligned}$$

A way to factorize when $a \neq 1$

We associate with a given quadratic $ax^2 + bx + c$ the associated quadratic

$$x^2 + bx + ac$$

and factorize it instead. If the answer is

$$x^2 + bx + ac = (x + A)(x + B)$$

then the factorization for the original quadratic is

$$ax^2 + bx + c = (ax + A)(x + \frac{B}{a})$$

Example. To factorize $6x^2 + 25x + 4$ instead factorize

$$x^2 + 25x + 24 = (x + 1)(x + 24)$$

(after a bit of work) and the required factorization is

$$6x^2 + 25x + 4 = (6x + 1)(x + 4)$$

Example. To factorize $2x^2 - 10x + 8$ instead factorize

$$x^2 - 10x + 16 = (x - 2)(x - 8)$$

(after a bit of work) and the required factorization is

$$2x^2 - 10x + 8 = (2x - 2)(x - 4)$$

Check:

$$(2x - 2)(x - 4) = 2x^2 - 8x - 2x + 8 = 2x^2 - 10x + 8 \quad \checkmark$$

Our first example To factorize $6x^2 - 4x - 10$ instead factorize

$$x^2 - 4x - 60 = (x - 10)(x + 6)$$

(after quite a bit of work) and the required factorization is

$$6x^2 - 4x - 10 = (6x - 10)(x + 1)$$

Check:

$$(6x - 10)(x + 1) = 6x^2 + 6x - 10x - 10 = 6x^2 - 4x - 10 \quad \checkmark$$

Algebraic fractions

Algebraic fractions (or *rational expressions*) are fractions which involve algebraic variables. For example,

$$\frac{x+3}{x+1} \qquad \frac{x}{x-3} \qquad \frac{x+y}{x^2-y-4}$$

$$\frac{1}{x+1} \qquad \frac{x^2+1}{x^2-4} \qquad \frac{x^2+y^2+z^2}{1+z}$$

Note that we can never divide by 0, so in for example

$$\frac{x}{x-3}$$

we aren't allowed to let x be 3, that is, we must have $x \neq 3$.

Simplifying rational expressions

Recall that we can cancel out common factors in numerical fractions (if there are any): For example

$$\frac{56}{24} = \frac{\cancel{8} \times 7}{\cancel{8} \times 3} = \frac{7}{3}$$

We can also cancel out common factors in algebraic fractions (if there are any). To do this, we need to factorize the numerator and denominator and see if there are any factors common to both.

Example

To simplify $\frac{x^2-x-6}{x^2+x-12}$ factorise both quadratics:

$$\begin{aligned} \frac{x^2 - x - 6}{x^2 + x - 12} &= \frac{(x-3)(x+2)}{(x+4)(x-3)} \\ &= \frac{\cancel{(x-3)}(x+2)}{(x+4)\cancel{(x-3)}} \\ &= \frac{x+2}{x+4} \end{aligned}$$

In other words, the original expression produces the same number as (the simpler) expression

$$\frac{x+2}{x+4}$$

whatever x value we choose to use (except those that would give 0 in a denominator).

For example, if $x = 5$ the original expression produces the value

$$\begin{aligned} \frac{5^2 - 5 - 6}{5^2 + 5 - 12} &= \frac{25 - 5 - 6}{25 + 5 - 12} \\ &= \frac{14}{18} = \frac{7 \times \cancel{2}}{9 \times \cancel{2}} = \frac{7}{9} \end{aligned}$$

The simplified expression also produces this value

$$\frac{5+2}{5+4} = \frac{7}{9}$$

Let's try another number. If $x = 3$ then the original expression produces the value

$$\frac{3^2 - 3 - 6}{3^2 + 3 - 12} = \frac{0}{0} \qquad \text{(huh?)}$$

but the simplified expression produces the value

$$\frac{3+2}{3+4} = \frac{5}{7}$$

What's gone wrong?

One of the few downsides of the number 0 is that it doesn't always behave like other numbers. In particular,

Division by 0is not defined

When we have algebraic expressions in the denominator, values of variables which make the denominator 0 are not allowed. So

$$\frac{x^2 - x - 6}{x^2 + x - 12} = \frac{\cancel{(x-3)}(x+2)}{(x+4)\cancel{(x-3)}} = \frac{x+2}{x+4} \qquad x \neq 3, -4$$

Adding and subtracting algebraic fractions

The rules for adding and subtracting algebraic fractions are the same as for numerical fractions. For example, to calculate

$$\frac{x+1}{x-3} \quad + \quad \frac{2x+7}{x-5}$$

we still need a common denominator:

$$\begin{aligned} & \frac{(x+1)(x-5)}{(x-3)(x-5)} + \frac{(2x+7)(x-3)}{(x-5)(x-3)} \\ = \quad & \frac{(x+1)(x-5)+(x-3)(2x+7)}{(x-3)(x-5)} \\ = \quad & \frac{x^2-4x-5+2x^2+x-21}{x^2-8x+15} \text{ (expand brackets)} \\ = \quad & \frac{3x^2-3x-26}{x^2-8x+15} \text{ (group like terms)} \end{aligned}$$

Multiplying and dividing algebraic fractions

The rules for multiplying and dividing algebraic fractions are the same as for numerical fractions. For example, multiplication:

$$\frac{x+1}{x-3} \times \frac{2x+7}{x-5} = \frac{(x+1)(2x+7)}{(x-3)(x-5)}$$

and

$$\frac{2x+1}{x+3} \div \frac{4x+5}{x-6} = \frac{2x+1}{x+3} \times \frac{x-6}{4x+5}$$

$$= \frac{(2x+1)(x-6)}{(x+3)(4x+5)} = \frac{2x^2-11x-6}{4x^2+17x+15}$$

Recall that dividing by a fraction is equivalent to multiplying by its reciprocal

Exercises 4

4.1 Expand brackets and collect terms in the following

(i) $(x+2)(x-4)$

(ii) $(-x+5)(-3x+4)$

(iii) $(2-t)(3t+4)$

(iv) $\left(y+\frac{1}{2}\right)\left(y+\frac{1}{4}\right)$

4.2 Factorise the following quadratic expressions.

(i) x^2-3x+2

(ii) x^2-x-12

(iii) $x^2+14x+40$

(iv) $3x^2+8x+4$

(v) $4x^2-5x-6$

4.3 Simplify the following rational expressions.

$(i)\quad \dfrac{x^2-x-2}{x^2+2x+1}$ $\qquad (ii)\quad \dfrac{x^2-1}{x^2-2x+1}$

Problem set 4

4.1 Expand brackets and collect terms in the following

(a) $(x+2)(x+4)$

(b) $(x-1)(x-2)$

(c) $(x-2)(x+2)$

(d) $(x-1)(1-x)$

(e) $(2x+3)(x+2)$

(f) $(x+1)^2$

(g) $(2x-1)^2$

(h) $(x+2)^2$

(i) $x(x-1)(x+1)$

(j) $(x-2y)(x+2y)$

(k) $(2x-3y)^2$

(l) $(x^2-1)(x^2+1)$

(m) $(x^2+x)(x^2-x)$

(n) $(x+3y)(x-3y)$

(o) $(x+1)(x+2)(x+3)$

(p) $(x-1)(x+1)(x+2)$

4.2 Factorise the following quadratic expressions.

(a) x^2+5x+6

(b) x^2-5x+6

(c) x^2-x-6

(d) x^2+x-6

(e) $x^2-2x-15$

(f) x^2-100

4.3 Simplify the following rational expressions.

$(i)\quad \dfrac{x^2-9}{x+3}$ $\qquad (ii)\quad \dfrac{x^2+5x-14}{x^2+10x+21}$

5. Indices

Indices (sometimes called powers) are notation used to shorthand expressions involving numbers multiplied by themselves a number of times.

Some of the powers of 2 are

$$\begin{aligned} 2^2 &= 2 \times 2 = 4 \\ 2^3 &= 2 \times 2 \times 2 = 8 \\ 2^4 &= 2 \times 2 \times 2 \times 2 = 16 \\ 2^5 &= \underbrace{2 \times 2 \times \cdots \times 2}_{5 \text{ terms}} = 32 \end{aligned}$$

For any positive integer (counting number) n,

$$2^n = \underbrace{2 \times 2 \times \cdots \times 2}_{n \text{ terms}}$$

Similarly,

$$3^n = \underbrace{3 \times 3 \times \cdots \times 3}_{n \text{ terms}}$$

For example,

$$3^4 = 3 \times 3 \times 3 \times 3 = 81$$

In general For any number k and positive integer n,

$$k^n = \underbrace{k \times k \times \cdots \times k}_{n \text{ terms}}$$

We say 'k to the n^{th} power' or 'k to the n'. We can also take powers of negative numbers, e.g.

$$\begin{aligned} (-2)^4 &= \underbrace{(-2) \times (-2)} \times \underbrace{(-2) \times (-2)} \\ &= \quad 4 \quad \times \quad 4 \\ &= \quad 16 \end{aligned}$$

Note: "-2^4" is read as "$-(2)^4$" so $-2^4 = -(2 \times 2 \times 2 \times 2) = -16$.

We can alos take powers of fractions. For example,

$$\left(\frac{1}{2}\right)^3 = \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}$$

$$\left(-\frac{3}{7}\right)^2 = \left(-\frac{3}{7}\right) \times \left(-\frac{3}{7}\right) = \frac{9}{49}$$

Properties of indicies (powers)

There are some shortcuts (or laws of indicies) we can develop.
Observe that

$$\begin{aligned} 2^3 \times 2^5 &= \underbrace{2 \times 2 \times 2}_{3 \text{ terms}} \times \underbrace{2 \times 2 \times 2 \times 2 \times 2}_{5 \text{ terms}} \\ &= \underbrace{2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2}_{3+5=8 \text{ terms}} \\ &= 2^8 \end{aligned}$$

We say that 2 cubed times 2 to the 5^{th} is 2 to the 8^{th}.

In general, the rule is clearly

$$\begin{aligned} k^m \times k^n &= \underbrace{k \times k \times \cdots \times k}_{m \text{ terms}} \times \underbrace{k \times k \times \cdots \times k}_{n \text{ terms}} \\ &= \underbrace{k \times k \times k \times \cdots \times k \times k}_{m+n \text{ terms}} \end{aligned}$$

That is,

$$k^m \times k^n = k^{m+n}$$

Here are another two rules. Observe that

$$\begin{aligned} \frac{7^5}{7^3} &= \frac{7 \times 7 \times 7 \times 7 \times 7}{7 \times 7 \times 7} \\ &= \frac{7 \times 7 \times \cancel{7} \times \cancel{7} \times \cancel{7}}{\cancel{7} \times \cancel{7} \times \cancel{7}} = 7 \times 7 = 7^2 \end{aligned}$$

We say that 7 to the 5^{th} divided by 7 cubed is 7 squared.

In general, the rule is

$$\frac{k^m}{k^n} = k^{m-n}$$

Observe that

$$\begin{aligned}(3\times 5)^3 &= (3\times 5)\times(3\times 5)\times(3\times 5)\\ &= 3\times 5\times 3\times 5\times 3\times 5\\ &= \underbrace{3\times 3\times 3}\times\underbrace{5\times 5\times 5}\\ &= \quad 3^3 \quad\times\quad 5^3\end{aligned}$$

In general, the rule is

$$(jk)^n = j^n k^n$$

A rule for fractions

$$\left(\frac{2}{7}\right)^3 = \left(\frac{2}{7}\right)\times\left(\frac{2}{7}\right)\times\left(\frac{2}{7}\right)$$

$$= \frac{2\times 2\times 2}{7\times 7\times 7} = \frac{2^3}{7^3}$$

In general, the rule is

$$\left(\frac{j}{k}\right)^n = \frac{j^n}{k^n}$$

A rule for powers of powers

$$(5^4)^3 = 5^4\times 5^4\times 5^4$$

$$= \underbrace{5\times 5\times 5\times 5}_{4\text{ terms}}\times\underbrace{5\times 5\times 5\times 5}_{4\text{ terms}}\times\underbrace{5\times 5\times 5\times 5}_{4\text{ terms}}$$

$$= \underbrace{5\times 5\times 5\times 5\times 5\times 5\times 5\times 5\times 5\times 5\times 5\times 5}_{3\times 4=12\text{ terms}} = 5^{12}$$

In general, the rule is

$$(k^m)^n = k^{m\times n}$$

The laws of indicies so far

We so far have five Laws of indicies. They are

$$k^m\times k^n = k^{n+m} \qquad \frac{k^m}{k^n} = k^{m-n}$$

$$(jk)^n = j^n\times k^n \qquad \left(\frac{j}{k}\right)^n = \frac{j^n}{k^n} \qquad (k^m)^n = k^{mn}$$

Some commonly occuring powers

$2^2 = 4$ $2^3 = 8$ $2^4 = 16$ $2^5 = 32$ $2^6 = 64$

and

$3^2 = 9$ $3^3 = 27$ $3^4 = 81$

Some commonly occuring squares (that is, to the power of 2)

$2^2 = 4$ $3^2 = 9$ $4^2 = 16$ $5^2 = 25$ $6^2 = 36$

$7^2 = 49$ $8^2 = 64$ $9^2 = 81$ $10^2 = 100$

The powers of 10

$10^2 = 100$ $10^3 = 1,000$ $10^4 = 10,000$

$10^5 = 100,000$ $10^6 = 1,000,000$ etc.

The laws of indicies used in algebraic calculations

The laws can be very useful when working with calculations and expressions. For example,

$$\frac{16^9}{2^{33}} = \frac{(2^4)^9}{2^{33}} = \frac{2^{36}}{2^{33}} = 2^{36-33} = 2^3 = 2 \times 2 \times 2 = 8$$

$$\frac{y^7}{y^3} = y^{7-3} = y^4$$

$$(z^4)^2 = z^{4\times 2} = z^8$$

$$(5x^3)^2 = 5^2(x^3)^2 = 5^2x^{3\times 2} = 25x^6$$

Simplifying algebraic expressions

$$(3y^2)(4y^3) = 3\times y^2\times 4\times y^3 = 12y^2y^3 = 12y^{2+3} = 12y^5$$

$$\frac{12z^5}{3z^2} = \frac{4\times \cancel{3}z^5}{\cancel{3}z^2} = 4\times\frac{z^5}{z^2} = 4z^{5-2} = 4z^3$$

$$\frac{x^5}{x^2x^2} = \frac{x^5}{x^{2+2}} = \frac{x^5}{x^4} = x^{5-4} = x^1$$

The power of 1

What does x^1 represent? We can see the answer by cancelling common factors:

$$\frac{x^5}{x^4} = \frac{xxxxx}{xxxx} = \frac{x\cancel{x}\cancel{x}\cancel{x}\cancel{x}}{\cancel{x}\cancel{x}\cancel{x}\cancel{x}} = x$$

So, $x^1 = x$ and we have another index law:

$$k^1 = k$$

Examples:

$$5^1 = 5 \qquad (-3)^1 = -3$$

$$\left(\frac{1}{2}\right)^1 = \frac{1}{2} \qquad \left(-\frac{2}{7}\right)^1 = -\frac{2}{7}$$

The power of 0

Similarly, we can extend indices to powers *other than counting numbers*. For example,

$$\frac{3^2}{3^2} = \frac{9}{9} = 1$$

The index rule $\frac{k^m}{k^n} = k^{m-n}$ gives us

$$\frac{3^2}{3^2} = 3^{2-2} = 3^0$$

Hence $3^0 = 1$. In fact:

$k^0 = 1$ for any value of k (except 0)

Examples:

$$5^0 = 1 \qquad \left(-\frac{2}{7}\right)^0 = 1$$

$$140,291,090^0 = 1 \qquad \left(\frac{987,654,321}{123,456,789}\right)^0 = 1$$

Negative powers

What about this situation:

$$\frac{3^2}{3^4} = \frac{\not{3} \times \not{3}}{\not{3} \times 3 \times 3 \times \not{3}} = \frac{1 \times 1}{1 \times 3 \times 3 \times 1} = \frac{1}{3 \times 3} = \frac{1}{3^2}$$

The index rule $\frac{k^m}{k^n} = k^{m-n}$ gives us

$$\frac{3^2}{3^4} = 3^{2-4} = 3^{-2}$$

Hence $3^{-2} = \frac{1}{3^2}$ and in fact

$$k^{-n} = \frac{1}{k^n} \text{ for any values of } k, n$$

Examples

$$5^{-1} = \frac{1}{5} \qquad 10^{-3} = \frac{1}{10^3} = \frac{1}{1000}$$

$$(-4)^{-2} = \frac{1}{(-4)^2} = \frac{1}{(-4) \times (-4)} = \frac{1}{16}$$

Note that

$$\left(\frac{a}{b}\right)^{-1} = \frac{1}{\left(\frac{a}{b}\right)^1} = 1 \div \frac{a}{b} = 1 \times \frac{b}{a} = \frac{b}{a}$$

This is another example of a reciprocal

In all, we have nine Laws of Indices

$$k^m \times k^n = k^{n+m} \qquad \frac{k^m}{k^n} = k^{m-n}$$

$$(jk)^n = j^n \times k^n \qquad \left(\frac{j}{k}\right)^n = \frac{j^n}{k^n} \qquad (k^m)^n = k^{mn}$$

$$k^{-n} = \frac{1}{k^n} \qquad \left(\frac{j}{k}\right)^{-1} = \frac{k}{j} \qquad k^1 = k \qquad k^0 = 1$$

How do we interpret powers that are fractions, eg. $4^{\frac{1}{2}}$? Using the index law $k^m \times k^n = k^{n+m}$ we can say

$$\begin{aligned} 4^{\frac{1}{2}} \times 4^{\frac{1}{2}} &= 4^{\frac{1}{2}+\frac{1}{2}} \\ &= 4^1 \\ &= 4 \end{aligned}$$

That is, "$4^{\frac{1}{2}}$" is a number which, when multiplied by itself, gives 4.

Clearly, an answer is 2 because $2 \times 2 = 4$. We refer to "$4^{\frac{1}{2}}$" as the square root of 4 and write

$$\sqrt{4}$$

Note that $(-2) \times (-2) = 4$ as well so we will define the square root (that is, power of $\frac{1}{2}$) to represent only the *positive* answer. That is,

$$4^{\frac{1}{2}} \;=\; \sqrt{4} \;=\; 2$$

Some other examples

$$9^{\frac{1}{2}} = 3 \qquad \text{because } 3 \times 3 = 9$$

$$\sqrt{16} = 4 \qquad \text{because } 4 \times 4 = 16$$

$$25^{\frac{1}{2}} = 5 \qquad \text{because } 5 \times 5 = 25$$

$$\sqrt{36} = 6 \qquad \text{because } 6 \times 6 = 36$$

Note that

$$\sqrt{0} = 0 \qquad \text{and} \qquad \sqrt{1} = 1$$

Unfortunately, most square roots aren't whole numbers. For example,

$$\sqrt{2} = 1.414\cdots \qquad \sqrt{5} = 2.236\cdots$$

Mathematicians leave such answers in square-root form. Other examples are

$$\sqrt{11} \qquad \sqrt{17} \qquad \sqrt{24} \qquad \sqrt{50}$$

The index laws apply to fractional powers as well. For example,

$$\left(\frac{4}{9}\right)^{\frac{1}{2}} = \frac{4^{\frac{1}{2}}}{9^{\frac{1}{2}}} = \frac{2}{3} \qquad \text{and} \qquad \sqrt{\frac{1}{36}} = \frac{\sqrt{1}}{\sqrt{36}} = \frac{1}{6}$$

What about "$8^{\frac{1}{3}}$"? Notice that

$$8^{\frac{1}{3}} \times 8^{\frac{1}{3}} \times 8^{\frac{1}{3}} = 8^{\frac{1}{3}+\frac{1}{3}+\frac{1}{3}} = 8^1 = 8$$

That is, "$8^{\frac{1}{3}}$" is a number which, when multiplied by itself three times, gives 8.

Clearly, an answer is 2 and "$8^{\frac{1}{3}}$" is an alternative way of expressing the cube root of 8:

$$8^{\frac{1}{3}} = \sqrt[3]{8}.$$

In general,

$$k^{\frac{1}{n}} = \sqrt[n]{k} \quad \text{(the "}n^{\text{th}}\text{ root of } k\text{")}.$$

Example

$$625^{\frac{1}{4}} = \sqrt[4]{625} = 5 \qquad \text{because} \quad 5^4 = 625$$

That is, the 4^{th} root of 625 is 5

Note that this time, $(-2) \times (-2) \times (-2) = -8$ so there is only one answer for $8^{\frac{1}{3}}$ (that is, 2). In fact, we have

$$(-8)^{\frac{1}{3}} = \sqrt[3]{-8} = -2$$

Note: On the other hand, $(-4)^{\frac{1}{2}} = \sqrt{-4}$ does not exist.

That is, there is no such number

Even and odd numbers

We refer to $2, 4, 6, 8, \cdots$ as the even numbers.

We refer to $1, 3, 5, 7, \cdots$ as the odd numbers.

The situation is that if n is odd, then we can take the n^{th} root of a negative number, but if n is an even, then we cannot.

So you can't take the square root of a negative number

Mathematicians had to invent a new type of number, called the *imaginary number* $i = \sqrt{-1}$ to handle this situation in practical problems.

What if the numerator isn't 1. For example, what is "$8^{\frac{2}{3}}$"?

The laws of Indices tell us. Recall that $8 = 2^3$. Now

$$8^{\frac{2}{3}} = \left(2^3\right)^{\frac{2}{3}} = 2^{\cancel{3} \times \frac{2}{\cancel{3}}} \quad \text{using } (k^m)^n = k^{mn} = 2^2 = 4$$

Examples

$$16^{\frac{3}{2}} = \left(4^2\right)^{\frac{3}{2}} = 4^{\cancel{2} \times \frac{3}{\cancel{2}}} = 4^3 = 4 \times 4 \times 4 = 64$$

$$27^{-\frac{4}{3}} = \left(3^3\right)^{-\frac{4}{3}} = 3^{\cancel{3} \times \left(-\frac{4}{\cancel{3}}\right)}$$

$$= 3^{-4} = \frac{1}{3^4} = \frac{1}{3 \times 3 \times 3 \times 3} = \frac{1}{81}$$

Surds

Recall that instead of $\sqrt{9}$ we can write 3, and $\sqrt{16}$ we can write 4. Square roots, cube roots, etc. that can't be written more simply, for example $\sqrt{2}$ and $\sqrt{5}$ are called surds Thanks to fractional powers and index laws, we have two very useful formulae for simplifying expressions involving surds:

$$\sqrt{a}\sqrt{a} = a \qquad \sqrt{a}\sqrt{b} = \sqrt{ab}$$

from

$$k^n \times k^m = k^{n+m} \qquad j^n \times k^n = (jk)^n$$

Examples:

$$\sqrt{7}\sqrt{7} = 7 \qquad \sqrt{3}\sqrt{5} = \sqrt{15}$$

Note: Similar expressions exist for other roots. For example

$$(\sqrt[3]{a})^3 = a \qquad \sqrt[3]{a}\sqrt[3]{b} = \sqrt[3]{ab}$$

Simplifying surd expressions

Expressions involving surds can be simplified using the usual laws of algebra, and the rules just mentioned. Recall that

$$a(b+c) = ab + ac \qquad (a+b)(c+d) = ac + ad + bc + bd$$

and

$$\sqrt{a}\sqrt{a} = a \qquad \sqrt{a}\sqrt{b} = \sqrt{ab}$$

Examples

$$(1+\sqrt{7})(1-\sqrt{7}) \;=\; 1-\cancel{\sqrt{7}}+\cancel{\sqrt{7}}-7 \;=\; 1-7 \;=\; -6$$

$$(\sqrt{3}+\sqrt{7})(\sqrt{3}+\sqrt{7}) \;=\; 3+\sqrt{3}\sqrt{7}+\sqrt{7}\sqrt{3}+7$$
$$=\; 3+\sqrt{21}+\sqrt{21}+7 \;=\; 10+2\sqrt{21}$$

We can sometimes simplify a surd itself. Especially if the number under the root sign can be factorized. For example,

$$\sqrt{20} \;=\; \sqrt{4\times 5} \;=\; \sqrt{4}\sqrt{5} \;=\; 2\sqrt{5}$$

$$\sqrt{75} \;=\; \sqrt{3\times 25} \;=\; \sqrt{3}\sqrt{25} \;=\; \sqrt{3}\times 5 \;=\; 5\sqrt{3}$$

Exercises 5

5.1 Simplify

(i) $2^4 \times 2^9$

(ii) $\frac{3^{11}}{3^5}$

(iii) $(5\times 7)^3$

5.2 Remove the brackets in

(i) $\left(\frac{7}{3}\right)^2$

(ii) $(5^4)^7$

5.3 Use the Laws of Indices to simplify the following

(i) $x^{-3}x^5$

(ii) $(5x^3)^{-2}x^6$

(iii) $(5\times 7)^3$

(iv) $\left(3x^{-2}\right)^{-2}x^{-3}y^{-3}$

(v) $\frac{5y^{-2}z}{(2y^3z^{-5})^2}$

5.4 Simplify the following surds

(i) $\sqrt{48}$

(ii) $\sqrt{3}(5+\sqrt{2})$

(iii) $(3+\sqrt{5})(2-\sqrt{5})$

(iv) $(2\sqrt{5}-1)(2\sqrt{5}+1)$

Problem set 5

5.1 Use index laws to simplify each of the following expressions

(a) $x^3\times x^4\times x$

(b) x^2y^3x

(c) $(xy)^2$

(d) $\dfrac{a^2b^5}{ab}$

(e) $\dfrac{2x^4y^2}{6x^2y^2}$

(f) $\left(4x^2\right)^3$

(g) $\dfrac{(3x^2z)^3}{(2x)^2}$

(h) $\left(\dfrac{9x^4}{(3x)^2}\right)^5$

5.2 Express the following numbers without indices

(a) $\dfrac{2^4}{4^2}$
(b) $\dfrac{5^{12}}{25^5}$
(c) 4^{-1}
(d) 10^{-2}
(e) $\dfrac{3^5}{27^2}$
(f) $(-2)^{-4}$
(g) $(-3)^{-3}$
(h) 0^{-6}
(i) $(-6)^0$

5.3 Use index laws to simplify each of the following expressions

(a) $y^{-2}y^{-3}$
(b) $x^{-3} \times x^3$
(c) $(a^{-3}b^{-4})(a^{-1}b)$
(d) $\dfrac{4g^{-3}h^{-5}}{12g^{-5}h^3}$
(e) $(-9x^{-1}y^2)^{-2}$
(f) $\left(\dfrac{3ab^2}{c}\right)^{-1}$
(g) $\dfrac{6a^3b^{-2}}{a^2b^{-1}} \div \dfrac{8a^3b}{b}$
(h) $\dfrac{(-2ab)^3}{(2a^{-1}b)^2}$
(i) $\left(\dfrac{3x^2y^{-1}}{2}\right)^0$
(j) $\dfrac{x^6}{x^3} \times \left(\dfrac{4x^3}{12x}\right)^2$

5.4 Use index laws to simplify each of the following expressions

(a) $3^x \times 2^x \times 3^{1-x}$
(b) $\dfrac{2^2}{2^x}$
(c) $\dfrac{x^y y^x}{x^{-y}}$
(d) $\dfrac{(9^a)^b}{(3^{2a})^b}$

5.5 Evaluate each of the following expressions.

(a) $8^{\frac{1}{3}}$
(b) $32^{\frac{1}{5}}$
(c) $81^{\frac{1}{2}}$
(d) $125^{\frac{1}{3}}$
(e) $81^{\frac{1}{4}}$
(f) $64^{\frac{1}{3}}$
(g) $10000^{-\frac{1}{4}}$
(h) $144^{-\frac{1}{2}}$
(i) $32^{\frac{2}{5}}$
(j) $36^{\frac{3}{2}}$
(k) $49^{-\frac{3}{2}}$
(l) $64^{\frac{5}{6}}$
(m) $64^{-\frac{1}{6}}$
(n) $81^{\frac{3}{4}}$
(o) $(-27)^{\frac{2}{3}}$

5.6 Evaluate each of the following expressions.

(a) $27^{\frac{2}{3}}$
(b) $27^{-\frac{2}{3}}$
(c) $4^{\frac{7}{2}}$
(d) $16^{-\frac{1}{2}}$
(e) $100^{\frac{3}{2}}$
(f) $16^{-\frac{3}{4}}$
(g) $169^{-\frac{1}{2}}$
(h) $32^{\frac{4}{5}}$
(i) $27^{\frac{4}{3}}$
(j) $64^{-\frac{5}{6}}$
(k) $\left(\dfrac{4}{9}\right)^{\frac{3}{2}}$
(l) $\left(\dfrac{4}{9}\right)^{-\frac{3}{2}}$
(m) $81^{-\frac{1}{2}}$

5.7 Simplify each of the following expressions

(a) $3x^{\frac{1}{2}} \times 4x^{\frac{2}{3}}$

(b) $5x^{\frac{2}{5}} \times 6x^{-\frac{1}{10}}$

(c) $(4x^{\frac{1}{2}})^3 \div (9x^{\frac{1}{3}})^{\frac{3}{2}}$

(d) $(64x^2)^{-\frac{1}{6}} \div (32x^{\frac{5}{2}})^{-\frac{2}{5}}$

(e) $\dfrac{x^{-\frac{2}{3}}y^{\frac{5}{2}}}{x^{-\frac{5}{3}}y^{-1}}$

(f) $\dfrac{x^{\frac{3}{7}}y^{-\frac{4}{5}}}{x^{-\frac{2}{7}}y^{\frac{3}{5}}}$

5.8 Simplify each of the following expressions

(a) $3x^{\frac{1}{2}} \times 2x^{-\frac{1}{2}}$

(b) $(8x^3y^5)^{-\frac{2}{3}}$

(c) $(x^{\sqrt{5}})^{3\sqrt{5}}$

(d) $\dfrac{3x^{\sqrt{27}}}{x^{\sqrt{12}}}$

(e) $3x^{-\frac{1}{2}} \times 4x^{\frac{2}{3}}$

(f) $(x^{2\sqrt{7}})^{\sqrt{7}}$

(g) $\dfrac{x^{-3}y^5}{x^2y^{-7}}$

(h) $7x^{\frac{1}{4}} \times 2x^{-\frac{1}{2}}$

(i) $(9x)^{\frac{1}{2}} \times (27x)^{\frac{1}{3}}$

(j) $(9x)^{\frac{1}{2}} \times (27x)^{-\frac{1}{3}}$

(k) $3x^{-\frac{1}{3}} \div 2x^{\frac{3}{4}}$

5.9 Simplify each of the following expressions

(a) $\sqrt{12}$

(b) $\sqrt{72}$

(c) $\sqrt{150}$

(d) $\sqrt[3]{96}$

(e) $\sqrt{2}(3-\sqrt{2})$

(f) $\sqrt{5}(2\sqrt{3}+\sqrt{2})$

(g) $(1-\sqrt{2})\sqrt{3}$

(h) $(7+3\sqrt{5})\sqrt{3}$

(i) $(6-2\sqrt{5})(6+2\sqrt{5})$

(j) $(3+\sqrt{7})^2$

(k) $(4\sqrt{3}-1)(2\sqrt{5}+1)$

(l) $(\sqrt{5}-\sqrt{3})(2\sqrt{5}+4\sqrt{3})$

6. *Re-arranging formulae*

In Chapter 2, we saw how to solve equations with a single variable. If there are several variables in the equation we can still isolate one of them by the same methods.

Example. Solve $3P = 5v + 12$ for v. We get

$$3P - 12 = 5v \quad \Rightarrow \quad \frac{3P - 12}{5} = v$$

We call this process isolating v

Example. Solve $\dfrac{H+4}{t} = \dfrac{3}{b+2}$ for H. Cross-multiply to get

$$(H+4)(b+2) = 3t \quad \Rightarrow \quad H(b+2) + 4(b+2) = 3t$$

$$\Rightarrow \quad H(b+2) = 3t - 4(b+2)$$

and so

$$H = \frac{3t - 4(b+2)}{b+2}$$

We have isolated H

Equations involving indices

We can use index laws to isolate variables in more sophisticated equations.

Example. To solve $A = b^5$ for b, we need to "undo" the operation "raise to the power of 5".

This can be done by raising both sides to the power of $\frac{1}{5}$:

$$\begin{aligned} & (A)^{\frac{1}{5}} = \left(b^5\right)^{\frac{1}{5}} \\ \Rightarrow \quad & A^{\frac{1}{5}} = b^{5 \times \frac{1}{5}} \quad \text{by index law } (k^m)^n = k^{mn} \\ \Rightarrow \quad & A^{\frac{1}{5}} = b^1 \\ \Rightarrow \quad & A^{\frac{1}{5}} = b \end{aligned}$$

More examples

$$\begin{aligned} Z = u^3 \quad &\Rightarrow \quad u = Z^{\frac{1}{3}} \\ K = p^{\frac{1}{4}} \quad &\Rightarrow \quad p = K^4 \\ h = w^{\frac{5}{2}} \quad &\Rightarrow \quad w = h^{\frac{2}{5}} \\ R = g^{-\frac{1}{7}} \quad &\Rightarrow \quad g = R^{-7} \end{aligned}$$

To solve $c = ax^2 + b$ for x, we proceed as follows:

$$c = ax^2 + b \quad \Rightarrow \quad c - b = ax^2$$

$$\Rightarrow \quad \frac{c-b}{a} = x^2 \quad \Rightarrow \quad \left(\frac{c-b}{a}\right)^{\frac{1}{2}} = x \quad \text{or} \quad \sqrt{\frac{c-b}{a}} = x$$

To solve $P = \frac{4r^3c}{b^2}$ for r, we proceed as follows:

$$P = \frac{4r^3c}{b^2} \quad \Rightarrow \quad Pb^2 = 4r^3c$$

$$\Rightarrow \quad \frac{Pb^2}{4c} = r^3 \quad \Rightarrow \quad \left(\frac{Pb^2}{4c}\right)^{\frac{1}{3}} = r$$

Note that we could also solve for b. We would get

$$b = \sqrt{\frac{4r^3c}{P}}$$

Exercises 6

6.1

(i) Solve $z^3p^{\frac{1}{4}} = a^4w^{\frac{2}{3}}$ for w

(ii) Solve $y = \sqrt{x^3 - c}$ for x

(iii) Solve $m = \dfrac{12s^4y^3}{7p^2}$ for s

(iv) Solve $f^2h^{\frac{1}{3}} = vm^{\frac{1}{4}}$ for h

Problem set 6

6.1 In each of the following scientific formulae, isolate the variable indicated

(a) Temperature scales

$$\text{Solve} \quad C = \frac{5}{9}(F - 32) \quad \text{for } F$$

(b) Volume of a sphere

$$\text{Solve} \quad V = \frac{4}{3}\pi r^3 \quad \text{for } r$$

(c) Ideal gas law

$$\text{Solve} \quad PV = nRT \quad \text{for } T$$

(d) Newton's law of gravity

$$\text{Solve} \quad F = \frac{GMm}{r^2} \quad \text{for } r$$

(e) Brightness of a star

$$\text{Solve} \quad b = \frac{L}{4\pi d^2} \quad \text{for } d$$

(f) Buckling of a column

$$\text{Solve} \quad L = kd^{\frac{2}{3}}\left(\frac{E}{p}\right)^{\frac{1}{3}} \quad \text{for } p$$

(g) Wavelength of gas particles

$$\text{Solve} \quad L = \sqrt{\frac{h^2}{2\pi mkT}} \quad \text{for } m$$

6.2 In each of the following business formulae, isolate the variable indicated

(a) Linear demand equation

$$\text{Solve} \quad Q = a - bP \quad \text{for } P$$

(b) Compound interest

$$\text{Solve} \quad A = P(1+r)^n \quad \text{for } r$$

(c) Money supply

$$\text{Solve} \quad MV = PY \quad \text{for } Y$$

(d) Relationship between marginal revenue and price elasticity

$$\text{Solve} \quad R = P\left(1 - \frac{1}{e}\right) \quad \text{for } e$$

(e) Gravity model of trade

$$\text{Solve} \quad F = K \times \frac{SD}{d^2} \quad \text{for } S$$

7. Exponentials and logarithms

Exponential equations

Question: 2 to which power gives us 8?. The answer is 3 because we know that $2 \times 2 \times 2 = 2^3 = 8$. We have just solved the equation

$$2^y = 8$$

This is a new sort of equation because the variable to be isolated is a power or exponent. . We call it an exponential equation.

Anything that grows (or shrinks) *as a percentage of its size* follows an exponential equation and they arise in many areas of Science (eg. population growth, radioactive decay) and Economics (eg. compound interest, economic growth measurement).

Logarithms

To help in solving exponential equations, we can rewrite them as logarithmic equations:

$$2^3 = 8 \quad \text{is the same as} \quad 3 = \log_2 8$$

The notation "$\log_2 8$" is read *"logarithm to the base 2 of* 8*"* and poses the question *"what power of the base (* 2 *) gives us* 8*?"*.

In general, we say:

$$b^y = x \quad \text{is the same as} \quad y = \log_b x$$

The notation "$\log_b x$" is read *"logarithm to the base b of x"* and poses the question *"what power of b gives us x?"*

Notice that the exponent (y) becomes an ordinary variable when the equation is changed to logarithmic form. This will help us to solve exponential equations.

Some examples of logarithms

$$\log_9 81 = 2 \quad \text{because} \quad 9^2 = 81$$
$$\log_3 27 = 3 \quad \text{because} \quad 3^3 = 27$$
$$\log_{10} 1000 = 3 \quad \text{because} \quad 10^3 = 1000$$
$$\log_2 32 = 5 \quad \text{because} \quad 2^5 = 32$$
$$\log_{10} 1,000,000 = 6 \quad \text{because} \quad 10^6 = 1,000,000$$
$$\log_3 81 = 4 \quad \text{because} \quad 3^4 = 81$$

A special case: $\log_b b$

$$\log_3 3 = 1 \quad \text{because} \quad 3^1 = 3$$
$$\log_7 7 = 1 \quad \text{because} \quad 7^1 = 7$$

Clearly, this is true for any base, so $\log_b b = 1$

Another special case: $\log_b 1$

$$\log_3 1 = 0 \quad \text{because} \quad 3^0 = 1$$
$$\log_9 1 = 0 \quad \text{because} \quad 9^0 = 1$$

Clearly, this is also true for any base, so $\log_b 1 = 0$

Negative numbers

We only consider positive bases for logarithms. We also don't define logarithms with a base of 1.

Negative bases create serious mathematical complications

Note that we can't take the log of a negative number. For example, $\log_2(-4)$ makes no sense because 2 raised to any power will always give a *positive* number. However, the log of a positive number may be negative. Recall the index law

$$k^{-n} = \frac{1}{k^n}$$

We have

$$\log_2\left(\frac{1}{8}\right) = -3 \quad \text{because} \quad 2^{-3} = \frac{1}{2^3} = \frac{1}{8}$$

Examples of applications of logarithms

Example: *Binary Search* in Computer Science is used to track down specific values in an ordered list. Think of it this way:

I have chosen a number between 1 and 16. When you guess I will say 'higher', 'lower' or 'correct'.

The maximum number of guesses required to identify the number is given by

$$\log_2 16 = 4$$

In general, the maximum number of guesses is

$$\log_2 N$$

where N is the number of items in the list.

Example: pH in Chemistry measures the acidity or alkalinity of an aqueous (water based) solution. It is related logarithmically to the amount of activity of the hydrogen ion:

$$pH = -\log_{10}(H^+)$$

A pH of 7 is neutral (eg. pure water), lower values are acidic and higher values are alkaline.

Example: In many real-life situations (eg. newspaper articles, balance sheets in accounting) numbers starting with 1 occur more often than numbers starting with 2, etc.

Benford's Law gives the frequency of each starting digit:

$$P = \log_{10}\left(\frac{d+1}{d}\right)$$

where P is the frequency and d is the starting digit $(1, 2, 3 \ldots, 9)$. Around 30% of such numbers start with a '1', 18% with a '2', down to just 5% starting with a '9'.

There are many other applications, and in a wide diversity of fields, for example,

Earthquakes (the Richter scale)
Signal processing (signal-noise ratio)
Psychology (Hick's law)
Physics (entropy)
Music (intervals)

The Laws of Logarithms

Logarithms are obviously related to indices, and there are closely related laws of logarithms. For all suitable values of x, y and b we have

$$\log_b b = 1 \quad \text{(related to } k^1 = k\text{)}$$

$$\log_b 1 = 0 \quad \text{(related to } k^0 = 1\text{)}$$

$$\log_b(xy) = \log_b x + \log_b y \quad \text{(related to } k^m \times k^n = k^{n+m}\text{)}$$

$$\log_b\left(\frac{x}{y}\right) = \log_b x - \log_b y \quad \text{(related to } \frac{k^m}{k^n} = k^{m-n}\text{)}$$

$$\log_b\left(x^r\right) = r\ \log_b x \quad \text{(related to } (k^m)^n = k^{mn}\text{)}$$

In any given investigation we are usually using the same base b (for example, if we are discussing Binary Searches) and we can omit it from the laws of logarithms and write

$$\log b = 1$$

$$\log 1 = 0$$

$$\log(xy) = \log x + \log y$$

$$\log\left(\frac{x}{y}\right) = \log x - \log y$$

$$\log\left(x^r\right) = r\ \log x$$

Recently, we have used the laws of indices to simplify expressions involving indices (powers).

Similarly, we can use the laws of logarithms to simplify expressions involving logarithms.

Examples of simplifying equations involving logarithms

$$\log_2 4 + \log_2 16 = \log_2(4 \times 16) = \log_2 64$$

$$\log_3 55 - \log_3 11 = \log_3 \frac{55}{11} = \log_3 5$$

$$2\log_7 6 = \log_7 6^2 = \log_7 36$$

$$\log_2\left(x^2\right) + \log_2\left(x^3\right) = \log_2\left(x^2x^3\right) = \log_2 x^5$$

$$3\log_7\left(y^{\frac{1}{2}}\right) = \log_7 y^{3\times\frac{1}{2}} = \log_7 y^{\frac{3}{2}}$$

We can use algebra and log laws to express logarithms as algebraic expressions based on simpler logarithms.

Example: Let $\log 2 = x$ and $\log 3 = y$. Express each of the following in terms of x and y:

$$\log 12 \qquad \log 18 \qquad \log\left(\frac{9}{2}\right) \qquad \log\left(\frac{1}{6}\right)$$

To do this we need to write the numbers in terms of 2 and 3.

Firstly, we have $12 = 3 \times 4 = 3 \times 2^2$ so

$$\log 12 = \log(3 \times 2^2) = \log 3 + \log 2^2 = \log 3 + 2\log 2 = y + 2x$$

Similarly, we have $18 = 2 \times 9 = 2 \times 3^2$ and so

$$\log 18 = \log(2 \times 3^2) = \log 2 + \log 3^2$$
$$= \log 2 + 2\log 3 = x + 2y$$

Also, $\frac{9}{2} = \frac{3^2}{2}$ and so

$$\log\left(\frac{9}{2}\right) = \log\left(\frac{3^2}{2}\right) = \log 3^2 - \log 2$$
$$= 2\log 3 - \log 2 = 2y - x$$

Also, $\frac{1}{6} = \frac{1}{3\times 2}$ and so

$$\log\left(\frac{1}{6}\right) = \log\left(\frac{1}{3 \times 2}\right) = \log 1 - \log(3 \times 2)$$
$$= \log 1 - (\log 3 + \log 2) = \log 1 - \log 3 - \log 2$$
$$= -y - x \qquad \text{since } \log 1 = 0$$

Note that these are true no matter which base b we use!

Exercises 7

7.1 Convert the following exponential equations into logarithmic form

(i) $2^6 = 64$ (ii) $121 = 11^2$

(iii) $27^{\frac{4}{3}} = 81$ (iv) $12 = \sqrt{144}$

7.2 Convert the following logarithmic equations into exponential form

(i) $\log_5 125 = 3$ (ii) $\log_4 64 = 3$

(iii) $\log_7 x = 3$ (iv) $\log_4 8 = x$

7.3 Use the laws of logarithms to simplify the following

(i) $\log_2 9 + \log_2 3$

(ii) $\log_3 7 - \log_3 28$

(iii) $\log_8 16 - 1$

7.4 Let $\log_4 3 = s$ and $\log_4 5 = t$. Express in terms of s and t:

(i) $\log_4 75$ (ii) $\log_4\left(\frac{5}{9}\right)$

Problem set 7

7.1 Write the following logarithmic equations in exponential form.

(a) $\log_{10} 100 = 2$
(b) $\log_{10} 1000 = 3$
(c) $\log_{10} 1 = 0$
(d) $\log_3 9 = 2$
(e) $\log_3 1 = 0$
(f) $\log_3 \left(\frac{1}{9}\right) = -2$
(g) $\log_n 256 = 2$
(h) $\log_n 14 = 3$
(i) $\log_n 62 = 4$
(j) $\log_2 \left(\frac{1}{8}\right) = -3$
(k) $\log_2 1 = 0$
(l) $\log_2 n = 1024$
(m) $\log_2 64 = 6$
(n) $\log_9 81 = 2$
(o) $\log_9 \left(\frac{1}{3}\right) = -\frac{1}{2}$
(p) $\log_9 3 = \frac{1}{2}$
(q) $\log_n 2 = 2$
(r) $\log_{10} x = \frac{1}{2}$
(s) $\log_5 100 = x$
(t) $\log_x 12 = 2$

7.2 Write the following exponential equations in logarithmic form.

(a) $2^5 = 32$
(b) $2^3 = 8$
(c) $2^0 = 1$
(d) $8^2 = 64$
(e) $8^{-1} = \frac{1}{8}$
(f) $8^{\frac{1}{3}} = 2$
(g) $16^1 = 16$
(h) $81^{-\frac{1}{2}} = \frac{1}{9}$
(i) $3^{-2} = \frac{1}{9}$
(j) $3^{-3} = \frac{1}{27}$
(k) $3^5 = 243$
(l) $25^{\frac{1}{4}} = \sqrt{5}$
(m) $10^4 = 10000$
(n) $3^x = 81$
(o) $25^x = 5$
(p) $x^5 = 7$
(q) $3^{x^2} = 81$
(r) $20^{3x+1} = 3$

7.3 Simplify the following using logarithm laws.

(a) $\log 5 + \log 2$

(b) $\log 8 - \log 2$

(c) $3 \log 2 - 2 \log 3$

(d) $\log 4 - \log 20$

(e) $\log 12 - \log 3$

(f) $\log 16 - \log 4$

(g) $\log_2 12 - \log_2 3$

(h) $\log_{10} 25 + \log_{10} 4$

(i) $\log_{10} 25 - \log_{10} 250$

(j) $\log_5 20 + \log_5 4$

(k) $\log\left(\frac{x^2}{\sqrt{y}}\right) + \log\left(x^{-2}y^3\right)$

(l) $2\log\left(x^{\frac{1}{2}}y^2\right) - \log\left(y^2x^{-1}\right)$

(m) $\frac{\log_{10} 27}{\log_{10} 9}$

(n) $\log_2 8 + \log_2 18 - \log_2 3$

(o) $\log_2 10 - \log_2 5$

(p) $\log \sqrt{xy} + \frac{1}{2}\log(xy) - \log(xy)$

7.4 Let $x = \log 3$ and $y = \log 4$. Express the following in terms of x and y.

(a) $\log 12$

(b) $\log 36$

(c) $\log\left(\frac{3}{4}\right)$

(d) $\log 6$

(e) $\log\left(\frac{3}{2}\right)$

(f) $\log\left(\frac{3}{8}\right)$

(g) $\log 144$

7.5 Let $\log_5 2 = x$ and $\log_5 3 = y$. Express the following in terms of x and y.

(a) $\log_5 6$

(b) $\log_5 12$

(c) $\log_5\left(\frac{3}{10}\right)$

(d) $\log_5 60$

(e) $\log_5\left(\frac{3}{2}\right)$

(f) $\log_5\left(\frac{3}{8}\right)$

(g) $\log_5 30$

8. Equations involving logarithms and/or exponentials

In Chapter 2, we discussed how to solve equations which contain a single variable, and in Chapter 6 we extended these ideas to cover multiple variables and variables raised to powers.

Now we look at equations with variables as powers (these are called exponential equations), e.g.

$$5(2^x) = 20$$

and equations involving logarithms, e.g.

$$\log_5 x - \log_5 8 = 2$$

To solve these we need to be familiar with:

- The laws of logarithms and the laws of indices/exponentials
- Switching between exponential form and logarithm form:

$$\underbrace{b^y = x}_{\text{exponential form}} \qquad \underbrace{\log_b x = y}_{\text{logarithm form}}$$

Equations involving logarithms

To solve these we can collect log terms and then switch to the exponential form.

A simple example: Solve $\log_2(3x) = 5$. Switch to exponential form:

$$3x = 2^5 \qquad \Rightarrow \qquad 3x = 32 \qquad \Rightarrow \qquad x = \frac{32}{3}$$

A not so simple example: Solve $\log_4 x - \log_4 (x-1) = 2$. Use the laws of logs to combine the terms on the left-hand side:

$$\log_4 \left(\frac{x}{x-1} \right) = 2$$

Now switch to exponential form:

$$\frac{x}{x-1} = 4^2 \quad \Rightarrow \quad \frac{x}{x-1} = 16$$

Now cross-multiply to get

$$x = 16(x-1) \quad \Rightarrow \quad x = 16x - 16 \quad \Rightarrow \quad 16 = 15x \quad \Rightarrow \quad x = \frac{16}{15}$$

Equations involving exponentials

To solve, make the base on each side of the equation the same.

A simple example: Solve $4^x = 16$. Recall that $16 = 4^2$.Then

$$4^x = 4^2$$

and since the base is the same on both sides of the equation then the indices must also be same. Hence $x = 2$.

A not so simple example: Solve $2^x 2^{3x-1} = 8$. Use the laws of indices to combine the terms on the left-hand side:

$$\begin{aligned} 2^{x+3x-1} &= 8 \\ \Rightarrow \quad 2^{4x-1} &= 2^3 \quad \text{(make the bases the same)} \\ \Rightarrow \quad 4x-1 &= 3 \quad \text{(equate the powers)} \\ \Rightarrow \quad 4x &= 4 \quad \Rightarrow \quad x = 1 \end{aligned}$$

A harder example: Solve $\frac{5^{3x}}{5^{2x-1}} = 10$ Use the laws of indices to simplify the left-hand side:

$$5^{3x-(2x-1)} = 10 \quad \Rightarrow \quad 5^{x+1} = 10$$

However, 10 can't be expressed as an easy power of 5 so switch to logarithm form:

$$x + 1 = \log_5 10 \quad \Rightarrow \quad x = \log_5 10 - 1$$

This is a mathematically *exact* answer but hard to visualise. We know that $\log_5 5 = 1$ and $\log_5 25 = 2$ so $\log_5 10$ lies somewhere on the numberline between 1 and 2. This means that the numerical answer is between 0 and 1. A calculator tells us that it is

$$x \approx 0.43$$

The symbol $\approx$ means "is approximately equal to"

Exponential equations These are equations of the form

$$y = a(b^x)$$

for some value a and some *positive* base value b. We usually omit the brackets and write $y = ab^x$.

Example: The simplest example is

$$y = 2^x$$

Exponentials grow very quickly. Lets look at $y = 2^x$:

x	1	2	3	4	5	6	7	8	9	10	11	12
y	2	4	8	16	32	64	128	256	515	1,024	2,048	4,096

Exponentials with larger bases grow even faster. Example, $y = 5^x$

x	1	2	3	4	5	6
y	5	25	125	625	3,125	15,725

Exponential models

In general, exponentials describe situations where every one unit increase in x changes y by a proportion of its current value. Applications of exponentials are many and varied:

- **animal populations:** the more animals there are in a population, the faster the population size increases.
- **viral infections:** the more people who are infected, the greater the spread of the infection will be.
- **compound interest:** the more money you have in an account, the greater the growth of your capital.

Since we are usually interested in quantities which change exponentially over time, we'll replace x with t (for time) and y with a suitable variable, e.g. P for population size at time t:

$$P = ab^t$$

These are all examples of **exponential growth**.

Initial population size

The initial population size is the size of the population when we started our investigation. We call this time $t = 0$. We usually label the initial population size as P_0. If we put $t = 0$ and $P = P_0$ in the equation $P = ab^t$ we get

$$P_0 = a \times b^0 \quad \Rightarrow \quad P_0 = a \times b^0 \quad \Rightarrow \quad P_0 = a \times 1 \quad \Rightarrow \quad P_0 = a$$

In other words, a is just the initial population size. For this reason we usually replace a with P_0 and write

$$P = P_0 b^t$$

The bases b tells us how quickly the population grows. The larger the value of b, the faster the population grows.

Example: Bacterial growth. If we start with $N_0 = 8$ bacteria and they double each generation, then after n generations we'll have

$$N = 8(2^n)$$

Example: Compound interest. If we invest $A_0 = \$1,000$ at a 5% interest rate compounded annually, then after t years we'll have

$$A = 1000\left(1 + \frac{5}{100}\right)^t$$

A more involved example: A bacterial colony numbered 3 million at the start of a study (time $t = 0$), and after two weeks numbers 12 million. Assuming exponential growth, predict when the bacteria population will exceed 150 million.

Using the units *millions of bacteria* for the colony size P and *weeks* for time t, we have the exponential model

$$P = 3b^t$$

We know that at $t = 2$ weeks we have 12 million bacteria, so

$$12 = 3b^2$$

and from this we can find the growth rate of the bacteria:

$$12 = 3b^2 \quad \Rightarrow \quad \frac{12}{3} = b^2 \quad \Rightarrow \quad b^2 = 4 \quad \Rightarrow \quad b = 2$$

Note that $b = -2$ is also a mathematical solution but it makes no biological sense, so we ignore it

Hence the growth model is

$$P = 3\left(2^t\right)$$

Note that we need the brackets here because without them it would be 32 to the power of t. To find the time t when the population reaches 150 million bacteria, we have to solve the exponential equation

$$\begin{aligned} 150 &= 3\left(2^t\right) \\ \Rightarrow \quad 50 &= 2^t \\ \Rightarrow \quad \log_2 50 &= t \end{aligned}$$

A calculator gives us the value $t = 5.64$ weeks, that is, about 5 weeks and '4 or 5' days, after the beginning of the study.

Exponential decay

We call exponential functions where the population gets smaller over time exponential decay. For every one unit increase in t, the population reduces by a proportion of the current value of P.

In such situations the base b is a fraction between 0 and 1. Applications of exponential decay models are also varied:

- **population decay (extinction):** as the population gets smaller, fewer animals die.
- **radioactive decay:** the "half life" of a radioactive substance is the time it takes the radioactivity to halve. It takes another half-life to halve again (ie. reduce to one quarter of the original level).
- **depreciation of assets:** a new car reduces in value rapidly just after being bought but doesn't change much after a few years.

Example: A pigeon population initially has 1000 members, but two years later it has declined to 250 members. Assuming the exponential model $P = P_0 b^t$ with t measured in years, after how many more years will the pigeon population reach 100?

Since the initial population was 1000, then $P_0 = 1000$ and so

$$P = 1000b^t$$

After two years it has declined to $P = 250$, that is,

$$250 = 1000b^2 \quad \Rightarrow \quad b^2 = \frac{250}{1000} = \frac{1}{4} \quad \Rightarrow \quad b = \sqrt{\frac{1}{4}} = \frac{\sqrt{1}}{\sqrt{4}} = \frac{1}{2}$$

We wish to know when $P = 100$. That is,

$$100 = 1000\left(\frac{1}{2}\right)^t \quad \Rightarrow \quad \left(\frac{1}{2}\right)^t = \frac{100}{1000} = \frac{1}{10} \quad \Rightarrow \quad t = \log_{\frac{1}{2}} \frac{1}{10}$$

A calculator gives us the value $t = 3.32$ years, that is, about 3 years and 4 months, after the beginning of the study.

A very special growth rate

To get around using logarithms with different bases for every different growth rate, we tend to use an agreed common base (and adjust the power to get different growth rates instead).

For many good mathematical reasons, this common base number is 2.718281828459045... It arises naturally in many varied branches of Mathematics. This number is so widely used (and annoying to write!) that it gets its own letter:

$$e = 2.718281828459045\ldots$$

where the letter e is in honour of Swiss mathematician Leonhard Euler (1707-1783).

The natural logarithm

We use logarithms with base e so much that they get their own special notation:

$$\text{“}\log_e\text{” is equivalent to “}\ln\text{”}$$

The “l” stands for “logarithm” and the “n” stands for “natural” (since the base e occurs naturally in so many settings). An expression like

$$\frac{\log_e 250}{\log_e 15} \quad \text{is equivalent to} \quad \frac{\ln 250}{\ln 15}$$

Some examples of switching between exp and log forms:

$$e^x = 4 \quad \Rightarrow \quad x = \ln 4$$

$$\ln y = 7 \quad \Rightarrow \quad y = e^7$$

The index and log laws for e and $\ln$
The laws of indices and laws of logarithms still hold for e and ln:

$$e^s\, e^t = e^{s+t} \qquad \frac{e^s}{e^t} = e^{s-t}$$

$$(e^s)^t = e^{st} \qquad e^{-s} = \frac{1}{e^s}$$

$$e^0 = 1 \qquad e^1 = e$$

$$\ln(st) = \ln s + \ln t \qquad \ln\left(\frac{s}{t}\right) = \ln s - \ln t$$

$$\ln(s^r) = r\ \ln s$$

$$\ln 1 = 0 \qquad \ln e = 1$$

Solving equations The method of solving exponential and logarithmic equations is the same, except that now we take logarithms with the base e (that is, natural logarithms).
Examples

$$(i) \quad e^{2x} = 1 \quad \Rightarrow \quad 2x = \ln 1 \quad \Rightarrow \quad 2x = 0 \quad \Rightarrow \quad x = \frac{0}{2} = 0$$

$$(ii) \quad \ln x + 2\ln 2 = \ln 16 \quad \Rightarrow \quad \ln x + \ln 2^2 = \ln 16$$

$$\Rightarrow \ \ln x + \ln 4 = \ln 16 \ \Rightarrow \ \ln(4x) = \ln 16 \ \Rightarrow \ 4x = 16 \ \Rightarrow \ x = 4$$

$$(iii) \quad \frac{e^{3x}}{e^4} = e^2 \quad \Rightarrow \quad e^{3x-4} = e^2 \quad \Rightarrow \quad 3x - 4 = 2$$

$$\Rightarrow \quad 3x = 6 \quad \Rightarrow \quad x = 2$$

Exponential models with base e

In this new format, our growth model is

$$P = P_0 e^{kt}$$

where k is used to set the required growth rate.
The bacterial colony example again: The model is now

$$P = 3e^{kt}$$

After two weeks its size is 12 million so

$$12 = 3e^{2k} \quad \Rightarrow \quad 4 = e^{2k} \quad \Rightarrow \quad \ln 4 = 2k \quad \Rightarrow \quad k = \frac{\ln 4}{2}$$

In this case we can simplify this slightly:

$$k \ = \ \frac{\ln(2^2)}{2} \ = \ \frac{\cancel{2}\ln 2}{\cancel{2}} \quad \Rightarrow \quad k = \ln 2$$

A tip: don't replace k with this expression in the growth model until it is needed

To find the time t when the population reaches 150 million bacteria we have to solve the exponential equation

$$150 = 3e^{kt}$$
$$\Rightarrow \quad e^{kt} = \frac{150}{3} = 50$$
$$\Rightarrow \quad kt = \ln 50$$
$$\Rightarrow \quad t = \frac{\ln 50}{k}$$

and since $k = \ln 2$ then

$$t = \frac{\ln 50}{\ln 2}$$

Recall that our previous answer was

$$t = \log_2 50$$

A calculator says that they are numerically the same answer:

$$t = 5.64$$

Which model to use?

If we wish to describe exponential growth we use

$$P = P_0 b^t \qquad b > 1$$

or

$$P = P_0 e^{kt} \qquad k > 0$$

If we wish to describe exponential decay we use

$$P = P_0 b^t \qquad b \text{ fraction between 0 and 1}$$

or

$$P = P_0 e^{-kt} \qquad k > 0$$

Note that k is always positive. It's the sign in the exponent that tells us if it's growth or decay

Another pigeon problem: A population of pigeons was measured to be 500 in 2012, but had reduced to 300 by 2015. Assuming the population law $P = P_0 e^{-kt}$, when will the population reach 100?

Let t be time in years, starting at $t = 0$ in 2012. Now $P_0 = 500$ and so

$$P = 500e^{-kt}$$

Also, 2015 represents $t = 3$, so we have $P = 300$ and hence

$$300 = 500e^{-3k} \quad \Rightarrow \quad e^{-3k} = \frac{300}{500} \quad \Rightarrow \quad e^{-3k} = \frac{3}{5}$$

Now 'take logs of both sides' (that is, convert to log form):

$$\Rightarrow \quad -3k = \ln\left(\tfrac{3}{5}\right) \quad \Rightarrow \quad k = -\frac{ln\left(\frac{3}{5}\right)}{3}$$

A calculator tells us that this is about $k = 0.17$ (that is, the population is decining at about 17% per year),

but it's best to leave it in mathematical form

To find the time t when the population reaches 100 we solve

$$100 = 500e^{-kt} \qquad \Rightarrow \qquad \frac{1}{5} = e^{-kt}$$

and hence

$$-kt = \ln\left(\frac{1}{5}\right) \quad \Rightarrow \quad t = \ln\left(\frac{1}{5}\right) \div (-k)$$

From before we have that $-k = \frac{ln\left(\frac{3}{5}\right)}{3}$ and hence

$$t = \frac{\ln\left(\frac{1}{5}\right)}{1} \div \frac{\ln\left(\frac{3}{5}\right)}{3} = \frac{\ln\left(\frac{1}{5}\right)}{1} \times \frac{3}{\ln\left(\frac{3}{5}\right)}$$

That is,

$$t = \frac{3\ln\left(\frac{1}{5}\right)}{\ln\left(\frac{3}{5}\right)}$$

A calculator tells us that this is about 9.5 years, so roughly the year 2021. It depends what month the measurements were done.

Exercises 8

8.1 Solve the following equations

(i) $\log_5 x + \log_5 8 = 2$ (ii) $2^{2x+1} = \dfrac{1}{32}$

(iii) $\left(3^{2x-1}\right)^2 = 27$

8.2 Simplify the following

(i) $\ln 27 - \ln 3 + \ln 4$ (ii) $\dfrac{16e^5}{2e^3}$

8.3 Solve the following equations

(i) $\ln(2x) + \ln 6 = \ln(4x)$ (ii) $\dfrac{e^{5x+1}}{e^{3x-2}} = e^4$

8.4 A fish population numbered 20 at the start of a study (start time $t = 0$) and after five years numbered 480. Assuming that the population law $P = P_0 e^{kt}$ predict when the population will exceed $1,000$.

8.5 An industrial machine which cost $7000 in 2010 is valued at $1000 in 2015. Assuming the decay law $P = P_0 e^{-kt}$, when will the machine be worth $700?

Problem set 8

8.1 Solve the following exponential equations for the unknown.

(a) $2^x = 64$
(b) $4^2x = 128$
(c) $9^x = 27$
(d) $5^{-x} = 125$
(e) $8^{2x} = 32$
(f) $10^{x+1} = \frac{1}{100}$
(g) $3^{x^3} = 81$
(h) $4^{\frac{x}{2}-1} = 128$
(i) $4^x = 7$
(j) $3^{x+1} = 10$
(k) $9^{2x+1} = 50$
(l) $7^{x-1} = 4$
(m) $2^{\frac{x}{2}+1} = 64$
(n) $5^{2x-1} = 20$

8.2 Solve the following logarithmic equations for the unknown.

(a) $\log_5 n = 4$
(b) $\log_8 4 = n$
(c) $\log_2 n = \frac{1}{3}$
(d) $\log_n \left(\frac{1}{64}\right) = -2$
(e) $\log_{\frac{1}{3}} n = 3$
(f) $\log_6 6 = n$
(g) $\log_n 8 = \frac{3}{4}$
(h) $\log_4 3 + \log_4(x+2) = 2$
(i) $x = \log_4 72 - \log_4 9$
(j) $x = \log_7 98 - \log_7 2$
(k) $\log_5 x - \log_5 7 = \log_5 6$
(l) $\log_3 63 - \log_3(7x) = \log_3 2$
(m) $\log_5 4 + \log_5(2x-3) = 2$
(n) $\log_2 2 + \log_2(x+2) - \log_2(3x-5) = 3$
(o) $\log_2 4 + \log_2(x-1) - \log_2(3x-4) = 2$

8.3 In each of the scientific formulae below, isolate the indicated variable.

(a) pH. Using p as the symbol for pH and H for $[H^+]$

$$p = -\log_{10} H \qquad \text{find } H$$

(b) Arrhenius equation

$$K = Ae^{\frac{E}{RT}} \qquad \text{find } T$$

(c) Relationship of rate constants at two different temperatures

$$\log_e\left(\frac{k}{K}\right) = \frac{E}{R}\left(\frac{t-T}{tT}\right) \qquad \text{find } K$$

(d) Benford's Law

$$P = \log_{10}\left(\frac{d+1}{d}\right) \qquad \text{find } d$$

8.4 A population starts with 1000 individuals and doubles in size every decade.

(a) Use the growth model $N = N_0 b^t$ to find how long will it take for the population to reach $500,000$.
(b) Repeat part (a) using the growth model $N = N_0 e^{kt}$.

8.5 An endangered species numbered $7,200$ at the start of the year 2003 and at the start of 2014 numbered 800. Assuming the population follows the decay law $N = N_0e^{-kt}$ predict in what year the population will fall below 120.

8.6 According to Wikipedia the number of articles on their site grew exponentially between October 2002 and mid 2006. There were $80,000$ articles in October 2002 and $166,000$ in October 2003. When does the model predict that Wikipedia reached $1,500,000$ articles? Use the growth model $N = N_0e^{kt}$.)

8.7 DDT is a pesticide that was very effective at controlling marlaria-carrying mosquitos until its toxic effects on animals (including cancer in humans) were established. Also, it remains active for many years in the environment because it decays exponentially with a half-life of 15 years. How long does it take for 100 grams of DDT to decay down to 1 gram? Use model $A = A_0e^{-kt}$.

9. *Quadratic equations*

In Chapter 4 we looked at *quadratic expressions*:

$$ax^2 + bx + c \qquad\qquad \text{with} \quad a \neq 0$$

For example: $2x^2 + x - 6 \qquad\qquad x^2 - 3x + 4$

Quadratic equations are equations of the kind

$$x^2 - 4x + 3 = 0 \qquad\qquad 2x^2 + 7x - 4 = 0$$

$$5x^2 - x - 2 = 0 \qquad\qquad -x^2 - 20x + 40 = 0$$

In other words, a quadratic equation is a quadratic expression equal to 0. The format $ax^2 + bx + c = 0$ is called the standard form of a quadratic equation.

Solving quadratic equations

If the left hand side can be factorized then the solution is easy:

$$x^2 - 4x + 3 = 0$$

We can factorize the left-hand side:

$$(x - 1)(x - 3) = 0$$

We know that

$$0 \times \text{anything} = 0$$

so, if one of the factors above is 0, the entire left-hand side is 0 and we have a solution to the equation. Hence

$$x - 1 = 0 \qquad \text{or} \qquad x - 3 = 0$$

and so

$$x = 1 \qquad \text{or} \qquad x = 3$$

As we can see, there can be more than one solution to a quadratic equation. We'll see later there can be zero, one or two solutions.

Some examples

Example. $x^2 - x - 6 = 0$. Factorize to get

$$(x - 3)(x + 2) = 0 \quad \Rightarrow \quad x = 3 \quad \text{or} \quad x = -2$$

Example. $x^2 - 9 = 0$. Factorize to get

$$(x - 3)(x + 3) = 0 \quad \Rightarrow \quad x = 3 \quad \text{or} \quad x = -3$$

Example. $t(t+2)-1=3-t$. Write this in standard form:

$$t(t+2)-1=3-t \quad \Rightarrow \quad t^2+2t-1=3-t \quad \Rightarrow \quad t^2+3t-4=0$$

Now factorize to get

$$(t+4)(t-1)=0 \quad \Rightarrow \quad t=-4 \quad \text{or} \quad t=1$$

Example. $x^2+4x+4=0$. Factorize to get

$$x^2-4x+4=0 \quad \Rightarrow \quad (x-2)(x-2)=0 \quad \Rightarrow \quad x=2$$

We see that there is only one solution in this case.

Modelling using quadratic equations

Example: The product of two consecutive positive even numbers is 48. Find the numbers.

As with earlier problems, we follow some general steps:

Step 1: Define an appropriate variable: Let x be the smaller of the two. Then the other one is $x+2$.

Step 2: Form a quadratic equation: The product of the two is then $x(x+2)$ and this must be 48 so

$$x(x+2)=48 \quad \Rightarrow \quad x^2+2x-48=0$$

Step 3: Solve the quadratic equation:

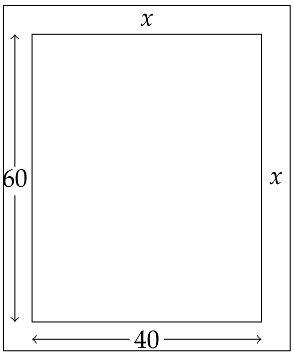
$$x^2+2x-48=(x+8)(x-6)=0 \quad \Rightarrow \quad x=-8 \quad \text{or} \quad x=6$$

Step 4: Interpret the solution(s): The numbers must be positive so $x=6$ and hence the numbers are 6 and $6+2=8$.

Check: $6\times 8=48$ ✓

Example: A framer at a photo gallery wants to frame a print with a *border of uniform width* all around the print. To make it pleasing to the eye, the area of the border should equal the area of the print. If the print measures 60 cm by 40 cm, how wide should the border be?

It's always a good idea to draw a sketch



where $x=$ is the width of the pathway. Note that the area of the picture is

$$A_{\text{picture}}=60\text{ cm}\times 40\text{ cm}=2400\text{ cm}^2$$

Observe is that the lengths of the sides of the frame are

$$40+2x\text{ cm} \qquad \text{and} \qquad 60+2x\text{ cm}$$

and hence the area of the frame is

$$A_{\text{frame}}=(40+2x)(60+2x)\text{ cm}^2$$

We are told that $A_{\text{frame}} = 2A_{\text{picture}}$. That is,

$$
\begin{aligned}
(40+2x)(60+2x) &= 4800 \\
\Rightarrow \quad 2400 + 80x + 120x + 4x^2 &= 4800 \\
\Rightarrow \quad 4x^2 + 200x - 2400 &= 0 \\
\Rightarrow \quad x^2 + 50x - 600 &= 0 \\
\Rightarrow \quad (x+60)(x-10) &= 0 \\
\Rightarrow \quad x = -60 \quad \text{or} \quad x &= 10
\end{aligned}
$$

The answer must be positive so $x = 10$ and hence the frame should be 60 cm by 80 cm.

Check: $60 \times 80 = 4800 =$ twice 2400 ✓

Factorization of quadratics revisited

In many cases it's very difficult to determine the factorization of a quadratic expression. This is because it can involve square roots!

Some examples

(a) $x^2 - 7 = (x - A)(x - B)$ where

$$A = \sqrt{7} \qquad \text{and} \qquad B = -\sqrt{7}$$

(b) $x^2 - 2x - 1 = (x - A)(x - B)$ where

$$A = 1 + \sqrt{2} \qquad \text{and} \qquad B = 1 - \sqrt{2}$$

(c) $x^2 + 5x + 3 = (x - A)(x - B)$ where

$$A = \frac{-5 + \sqrt{13}}{2} \qquad \text{and} \qquad B = \frac{-5 - \sqrt{13}}{2}$$

The Quadratic Formula

There is a general formula that will give us the factorization. If

$$ax^2 + bx + c = a(x - A)(x - B) \qquad a \neq 0$$

then

$$A = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad B = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

Since A and B have very similar structure, we write

$$A, B = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

(The symbol "$\pm$" means "plus or minus".) The plus expression corresponds to A and the minus expression corresponds to B.

The values A and B are called the roots of the quadratic equation, even if they turn out not to involve square roots.

Let's check that the formula works on a simple example:

$$x^2 - 3x + 2 = (x - 2)(x - 1)$$

To use the quadratic formula, first identify the coefficients:

$$a = 1 \qquad b = -3 \qquad c = 2$$

The quadratic formula is $A, B = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Hence $x^2 - 3x + 2 = (x - A)(x - B)$ where

$$A, B = \frac{3 \pm \sqrt{9 - 8}}{2} = \frac{3 \pm \sqrt{1}}{2} = \frac{3 \pm 1}{2} = \frac{4}{2}, \frac{2}{2} = 2,\ 1$$

Hence $A = 2$ and $B = 1$ are the roots of the quadratic equation, so

$$x^2 - 3x + 2 = (x - 2)(x - 1)$$

Example: Solve $x^2 - 2x - 1 = 0$. We have

$$a = 1 \qquad b = -2 \qquad c = -1$$

First calculate

$$b^2 - 4ac = (-2)^2 - 4 \times 1 \times (-1) = 4 + 4 = 8$$

and then

$$A, B = \frac{2 \pm \sqrt{8}}{2} = \frac{2 \pm 2\sqrt{2}}{2} = \frac{\cancel{2}}{\cancel{2}} \pm \frac{\cancel{2}\sqrt{2}}{\cancel{2}} = 1 \pm \sqrt{2}$$

where we've used $\sqrt{8} = \sqrt{4 \times 2} = \sqrt{4}\sqrt{2} = 2\sqrt{2}$. That is, the roots of the quadratic equation $x^2 - 2x - 1 = 0$ are

$$A = 1 + \sqrt{2} \quad \text{and} \quad B = 1 - \sqrt{2}$$

Example: Solve $3x^2 + 6x - 4 = 0$. We have

$$a = 3 \qquad b = 6 \qquad c = -4$$

First calculate

$$b^2 - 4ac \;=\; 6^2 - 4 \times 3 \times (-4) \;=\; 36 + 48 \;=\; 84$$

and then

$$A, B \;=\; \frac{-6 \pm \sqrt{84}}{6} \;=\; -\frac{6}{6} \pm \frac{2\sqrt{21}}{6} \;=\; -1 \pm \frac{\sqrt{21}}{3}$$

where we've used $\sqrt{84} = \sqrt{4 \times 21} = \sqrt{4}\sqrt{21} = 2\sqrt{21}$. That is, the roots of the quadratic equation $3x^2 + 6x - 4 = 0$ are

$$A = -1 + \frac{\sqrt{21}}{3} \quad \text{and} \quad B = -1 - \frac{\sqrt{21}}{3}$$

The number of solutions
This depends on the value of

$$b^2 - 4ac$$

which is called the discriminant of the quadratic equation.

- If the discriminant is positive, we have two solutions, one for the positive component and one for the negative.
- If the discriminant is zero, then the solutions are

  $$A, B = \frac{-b \pm 0}{2a} \qquad \Rightarrow \qquad A \text{ and } B \text{ both } = \frac{-b}{2a}$$

  We have one solution.
- If the discriminant is negative, then we can't take its square root and hence A and B are undefined. In other words, we have no solutions to the equation.

The *discriminant* of the quadratic equation $ax^2 + bx + c = 0$ is given its own symbol:

$$\Delta = b^2 - 4ac$$

Summary:

- If $\Delta > 0$ then there are two distinct solutions
- If $\Delta = 0$ then there is only one solution
- If $\Delta < 0$ then there are no solutions

Example: The discriminant of the quadratic equation

$$2x^2 + 5x + 4 = 0$$

is

$$\Delta = b^2 - 4ac \;=\; 5^2 - 4 \times 2 \times 4 \;=\; 25 - 32 \;=\; -7$$

and hence there is no solution to the quadratic equation.

An alternative form for quadratics

Instead of the standard form $ax^2 + bx + c$, there is an alternative form:

$$a(x + \alpha)^2 + \beta$$

where the numbers α and β (the Greek letters alpha and beta) are related to a, b and c. To convert from the alternative form to the standard form is easy. We just expand the bracket and tidy up.

Example:

$$\begin{aligned} & 2(x+3)^2 + 5 \\ =\ & 2(x+3)(x+3) + 5 \\ =\ & 2(x^2 + 6x + 9) + 5 \\ =\ & 2x^2 + 12x + 23 \end{aligned}$$

Completing the square

Converting from the standard to the alternative form requires the following process. It is called completing the square and the aim is to re-arrange a quadratic in such a way as to achive our objective.

If the quadratic is of the form $x^2 + bx + c$ there are three steps:

(1) halve b. Call it α.

(2) square α and both add and subtract it from the quadratic

(3) write this quadratic in the form $(x + \alpha)^2 + \beta$

If the quadratic is of the form $ax^2 + bx + c$ there is a preliminary step of factoring out a:

$$ax^2 + bx + c \quad = \quad a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right)$$

Examples

(a) $x^2 + 6x + 14$. Half of 6 is 3 and the square of this is 9 so write

$$x^2 + 6x + 14 = (x^2 + 6x + 9) + (14 - 9)$$

and observe that the first three terms are the expansion of $(x + 3)^2$. So

$$x^2 + 6x + 14 = (x + 3)^2 + 5$$

(b) $x^2 - 10x + 12$. Half of -10 is -5 and the square of this is 25 so write

$$x^2 - 10x + 12 = (x^2 - 10x + 25) + (12 - 25)$$

and observe that the first three terms are the expansion of $(x - 5)^2$. So

$$x^2 - 10x + 12 = (x - 5)^2 - 13$$

Exercises 9

9.1 Solve the following quadratic equations

(i) $(y - 2)^2 = -9y$

(ii) $2(x - 3)^2 = 6 - 5x + x^2$

(iii) $(2x + 2)^2 - (x - 5)^2 = 2x(x + 7)$

Hint: You'll first have to write them in standard form

9.2 Solve the following quadratic equations

(i) $x^2 + 2x - 4 = 0$

(ii) $11x^2 - 10x - 1 = 0$

(iii) $(x + 2)(x + 3) = -x^2 - 2x + 1$

9.3 How many solutions do each of the following quadratic equations have?

(i) $x^2 - 4x + 2 = 0$

(ii) $x^2 + 2x + 7 = 0$

(iii) $4x^2 + 4x + 1 = 0$

9.4 Complete the square for the following quadratics

(i) $x^2 + 6x - 4$

(ii) $x^2 - 8x + 20$

(iii) $x^2 + 18x - 12$

(iv) $x^2 - 10x$

Problem set 9

9.1 Solve the following equations by factorising.

(a) $x^2 - 3x = 0$
(b) $x^2 - 5x + 4 = 0$
(c) $x^2 - 5x - 14 = 0$
(d) $x^2 - x - 42 = 0$
(e) $3x^2 - 6x = -3$
(f) $(x-2)(x+1) = 4$
(g) $x + \dfrac{15}{x} = 8$
(h) $x = \dfrac{5}{x-4}$

9.2 Solve the following equations by using the quadratic formula.

(a) $x^2 - 5x + 4 = 0$
(b) $3x^2 + 7x + 2 = 0$
(c) $x^2 + 4x + 1 = 0$
(d) $-2x^2 + 3x - 1 = 0$
(e) $x^2 + 3x + 4 = 0$
(f) $\frac{3}{2}x^2 + 4x + 1 = 0$
(g) $x^2 + x - 1 = 0$
(h) $2x^2 - 2x - 3 = 0$
(i) $2 - 3x^2 = 0$

9.3 Determine the number of solutions for the following equations.

(a) $x^2 + 3x + 1 = 0$
(b) $3x^2 + 4x + 2 = 0$
(c) $2x^2 + 3x + 5 = 0$
(d) $2x^2 + 4x + 2 = 0$
(e) $\frac{3}{2}x^2 + 4x + 6 = 0$
(f) $x^2 + 5x - 1 = 0$
(g) $4x^2 - 5x + 4 = 0$

9.4 Complete the square for the following quadratic expressions

(a) $x^2 + 2x + 3$
(b) $x^2 - 4x - 8$
(c) $x^2 + 10x + 7$
(d) $x^2 - 6x$
(e) $x^2 + 12x + 36$
(f) $x^2 - 20x - 1$

9.5 The product of two consecutive odd numbers is 99. Find the two numbers.

9.6 One positive number exceeds three times another positive number by 5. The product of the numbers is 68. Find the numbers.

Note that $\sqrt{25 + 12 \times 68} = 29$

9.7 A group of zoologists was studying the effect on the body weight of rats of varying the amount of yeast in their diet. By changing the percentage P of yeast in the diet, the average weight gain, G (in grams), over time was estimated to be

$$G = -200P^2 + 200P + 20$$

What percentage of yeast would you expect to give an average weight gain of 70 grams?

9.8 You wish to make a square-bottomed box with a height of three cm and a volume of 75 cm^3. You will take a piece of square cardboard, cut a three cm square from each corner and fold up the sides. What sized piece of cardboard do you need?

Hint: it might help to draw a picture.

9.9 A rectangular park has dimensions 40 metres by 50 metres. A pathway is to be added all around the park which will increase the total area to 3000 m^2. How wide is the pathway going to be?

Hint: it might help to draw a picture.

10. Functions and graphs

So far we have mostly examined ways to find *specific* values of variables from equations. Now we will explore mathematical **relationships** *between* variables. For example,

- The *cost of a taxi ride* is **related to** *how far you travel*.
- The *amount of interest earned on a deposit* is **related to** *how long you leave it in the bank*.
- The *distance travelled by a ball* is **related to** *how hard you throw it*.

An example: a (simplified) taxi fare costs $2 per kilometre (km) plus a $4 "flagfall".

If you travel 5 km, the taxi ride costs $\$4 + \$2 \times 5 = \$14$

If you travel 10 km, the taxi ride costs $\$4 + \$2 \times 10 = \$24$

If you travel x km, the taxi ride costs $\$4 + 2x$

If we define y to be the cost of the taxi ride, we can just write

$$y = 2x + 4$$

We say that y depends on x or "y is a function of x":

$$y = f(x) \qquad \text{where in this case} \qquad f(x) = 2x + 4$$

For example, when $x = 7$, we have $y = f(7) = 2 \times 7 + 4 = 18$. In other words, a 7 km ride costs $18.

Using function notation $f(x)$ to describe a variable like y is a useful way to make clear that the value of y is determined by the value another variable x via an algebraic expression.

Note that "$f(x)$" does *not* stand for "f multiplied by x"

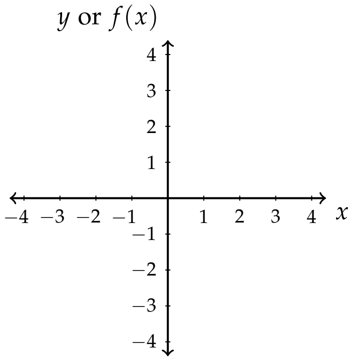
Some examples:

$$\begin{aligned} y = f(x) &= x^2 + 1 \\ f(2) &= 2^2 + 1 = 4 + 1 = 5 \\ f(0) &= 0^2 + 1 = 0 + 1 = 1 \\ f(-1) &= (-1)^2 + 1 = 1 + 1 = 2 \end{aligned}$$

$$\begin{aligned} y = g(x) &= 2^x \\ g(1) &= 2^1 = 2 \\ g(2) &= 2^2 = 4 \\ g\left(\tfrac{1}{2}\right) &= 2^{\frac{1}{2}} = \sqrt{2} \end{aligned}$$

Graphical representation of functions

It's useful to get a *visual* sense of how a function (y or $f(x)$) changes as x changes. Since two variables are involved, draw two number lines at right angles (through the 0 point on each line):



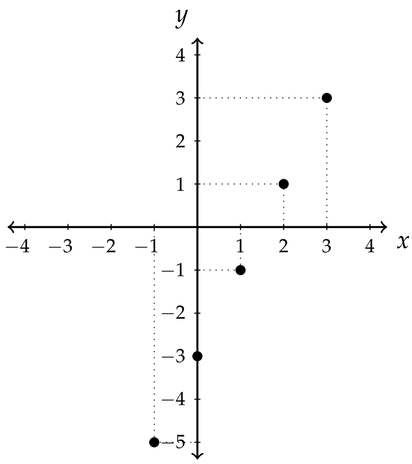
These number lines are called the "*y-axis*" (vertical) and the "*x-axis*" (horizontal).
To visually examine the behaviour of a function, say

$$y = 2x - 3$$

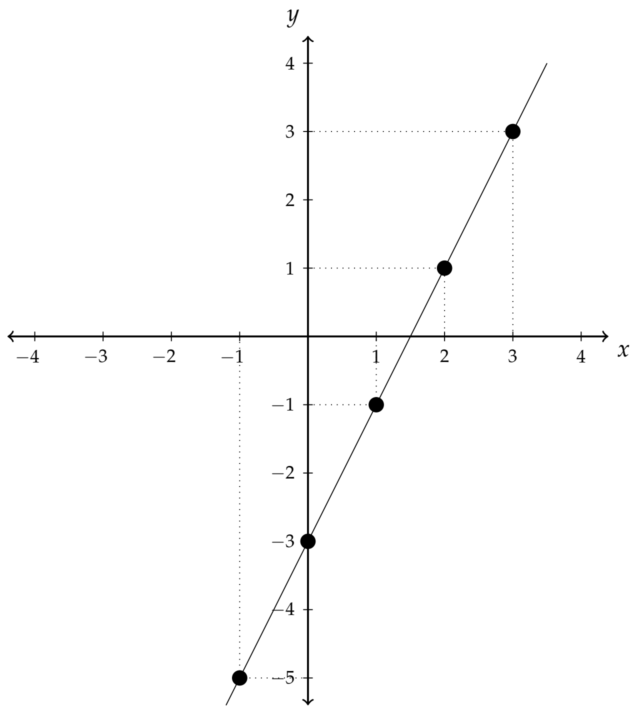
start by finding out the value of y for different values of x.

$$x = 2 \quad \text{then} \quad y = 2(2) - 3 = 1$$
$$x = 1 \quad \text{then} \quad y = 2(1) - 3 = -1$$
$$x = 3 \quad \text{then} \quad y = 2(3) - 3 = 3$$
$$x = 4 \quad \text{then} \quad y = 2(4) - 3 = 5$$
$$x = 0 \quad \text{then} \quad y = 2(0) - 3 = -3$$
$$x = -1 \quad \text{then} \quad y = 2(-1) - 3 = -5$$

Now represent them on the picture by cross-referencing points on the x-axis with their corresponding points on the y-axis:



If we tried x-values between the whole numbers, the plotted points would follow the same pattern so we can join them up into a smooth line:



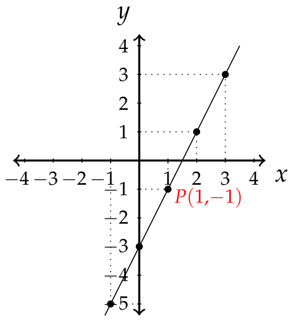
We have now plotted a graph of the function $y = 2x - 3$.

Co-ordinates

Rather than writing: when $x = 1,\ y = -1$ we use co-ordinates:

$$(1, -1)$$

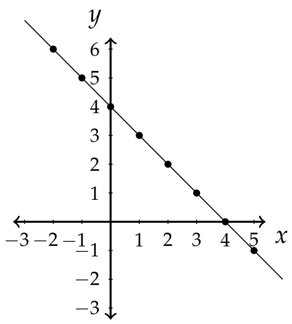
These are like a map reference and the x-value is always given first.
To signify a point of interest we can give it a label:



Example
To plot the graph of $y = 4 - x$ by proceeding at follows:

- Calculate some co-ordinates, that is, find some *points on the graph*
- Plot the points on a set of axes
- Draw a continuous line which includes all of the points

Here is the graph of the function $y = 4 - x$



Linear functions

Functions $y = 2x - 3$ and $y = 4 - x$ are called linear functions because their graphs are straight lines. Such functions are common and easy to interpret. They have the general form

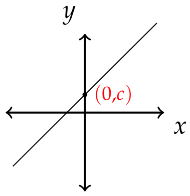
$$y = mx + c$$

where m and c are numbers.

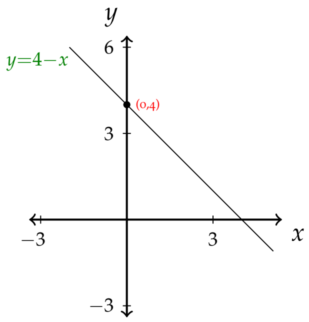
If we understand how m and c relate to the straight line graph they produce then we can draw the graph without plotting lots of individual points. For example,

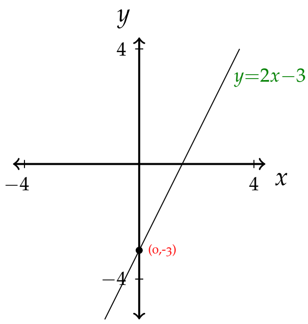
$$x = 0 \quad \Rightarrow \quad y = m0 + c \quad \Rightarrow \quad y = c$$

Note that the co-ordinate $(0, c)$ is on the y-axis. This is the place where the line cuts the y-axis, and for this reason it is called the y-intercept.



The y-intercepts of $y = 4 - x$ and $y = 2x - 3$ are:





Note that we usually don't list all of the numbers on the numberline. Just enough for us to get an idea of the scale of things

The gradient

This is defined as the change in the y-values between two points on the line divided by the change in their x-values.

Example: $y = 2x - 3$

$$\text{If} \quad x = 2 \quad \text{then we have} \quad y = 1$$

$$\text{If} \quad x = 3 \quad \text{then we have} \quad y = 3$$

So y increases by 2 while x increases by 1, so the gradient is

$$\frac{\text{change in } y\text{-value}}{\text{change in } x\text{-value}} = \frac{2}{1} = 2$$

The gradient is the same if you use different points:

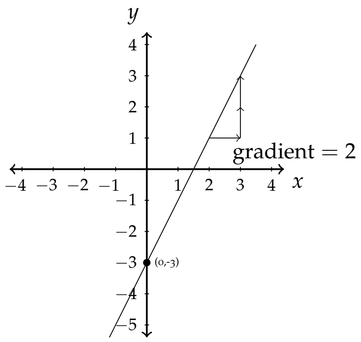
$$\text{If} \quad x = -1 \quad \text{then we have} \quad y = -5$$

$$\text{If} \quad x = 4 \quad \text{then we have} \quad y = 5$$

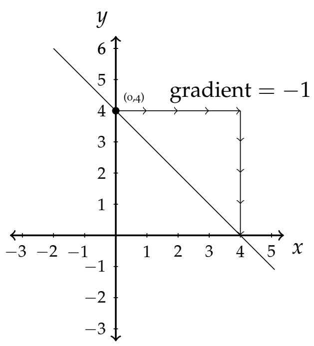
So y increases by 10 while x increases by 5, so the gradient is

$$\frac{\text{change in } y\text{-value}}{\text{change in } x\text{-value}} = \frac{10}{5} = 2$$

Here is the graph of $y = 2x - 3$ showing its y-intercept and indicating its gradient.



Here is the graph of $y = 4 - x$ showing its y-intercept and indicating its gradient.



We observe that the gradient is the value of m in $y = mx + c$. A useful way to understand the gradient is to think of it as the change in y for every one unit increase in x. This is useful for describing linear relationships in context.

Example: The taxi ride (from last time). The equation that described the taxi fare was

$$y = 2x + 4$$

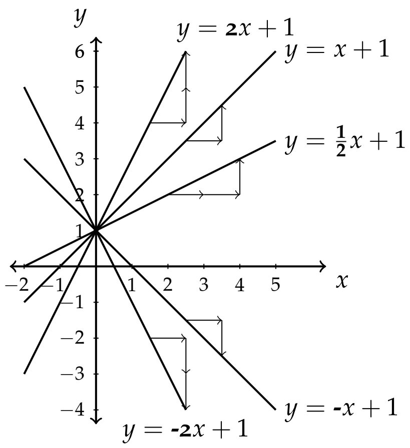
where

$$x = \text{distance travelled (km)}$$
$$y = \text{cost(\$)}$$

Recall that c is the y-intercept

The gradient of 2 represents the change in cost for every 1 km travelled. In other words, \$2 per km.

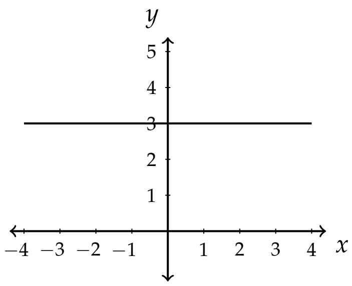
Physical meaning of the gradient



The larger the value of m, the steeper the line. A positive gradient means going uphill left-to-right, and a negative gradient means going downhill left-to-right.

Horizontal lines

What is the equation of this *horizontal* line?



Hint: What is the gradient, and what is the y-intercept?

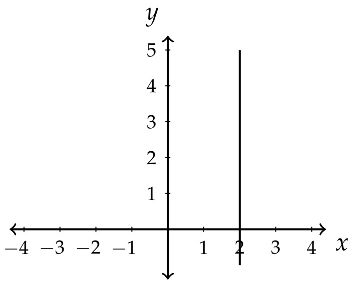
The gradient is zero, and the y-intercept is 3, so the equation is simply $y = 3$. In general, the horizontal line passing through $(0, c)$ has equation

$$y = c$$

A special case is the x-axis. It has equation $y = 0$.

Vertical lines

The only lines which don't fit the $y = mx + c$ format are *vertical*:



The equation of this line is $x = 2$. In general, the vertical line passing through $(a, 0)$ has equation

$$x = a$$

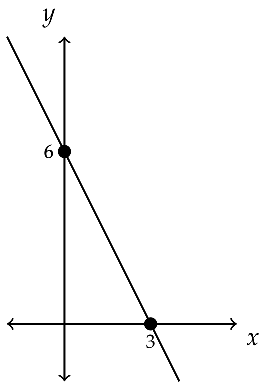
A special case is the y-axis. It has equation $x = 0$.

The x-intercept

This is where the line crosses the x-axis. That is, the place on the line where $y = 0$.

Example: $y = -2x + 6$. We immediately observe that the gradient is $m = -2$ and the y-intercept is $c = 6$. The x-intercept is where $y = 0$ so we solve

$$0 = -2x + 6 \quad \Rightarrow \quad 2x = 6 \quad \Rightarrow \quad x = 3$$



In general, the quick way to plot an accurate graph of $y = mx + c$ is to find both intercepts, mark them on the axes and draw the straight line which contains them. We call this a sketch of the graph.

Finding the equation of a line

We can work out the equation of a line given sufficient information.

Example: Find the equation of the line with gradient 2 which contains the point $(4, 7)$. We know that we are working with a line $y = mx + c$ and we are told that $m = 2$, so

$$y = 2x + c$$

We also know that when $x = 4$ the equation gives $y = 7$. Substitute these x and y values into the equation and solve for c:

$$\begin{aligned} 7 &= 2(4) + c \\ \Rightarrow \quad 7 &= 8 + c \\ \Rightarrow \quad 7 - 8 &= c \\ \Rightarrow \quad c &= -1 \end{aligned}$$

Hence, the equation of the straight line is $y = 2x - 1$

Example: The equation of the line containing $(2, 5)$ and $(5, 11)$.

Method 1: Substituting the two co-ordinates into the equation of a line $y = mx + c$ gives us a pair of simultaneous equations to be solved for m and c:

$$\begin{array}{ccccccc} (5,11) & \Rightarrow & 11 & = & m \times 5 & + \; c & (1) \\ (2,5) & \Rightarrow & 5 & = & m \times 2 & + \; c & (2) \\ \hline & & 6 & = & 3m & & \end{array}$$

$$\text{So} \quad m = \frac{6}{3} = 2$$

Substituting $m = 2$ into say equation (2) gives

$$\begin{aligned} 5 &= 2(2) + c \\ \Rightarrow \quad 5 &= 4 + c \\ \Rightarrow \quad c &= 1 \end{aligned}$$

Hence, the equation is $y = 2x + 1$.

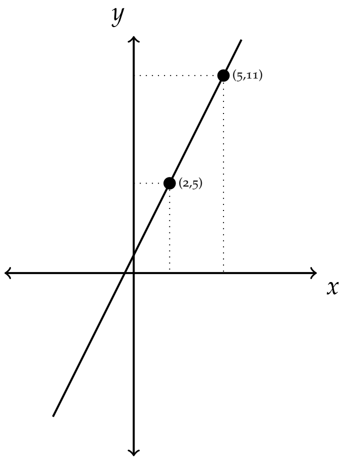
Method 2: Use the definition of gradient to find m:

$$m = \frac{\text{change in } y \text{ value}}{\text{change in } x \text{ value}} = \frac{11-5}{5-2} = \frac{6}{3} = 2$$

So the equation (so far) is $y = 2x + c$. To find c, substitute one of the points, say $(2,5)$ into this equation and solve for c:

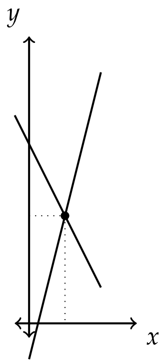
$$5 = 2(2) + c \quad \Rightarrow \quad 5 = 4 + c \quad \Rightarrow \quad 5 - 4 = c \quad \Rightarrow \quad c = 1$$

Hence, the equation of the straight line is $y = 2x + 1$



Intersecting lines

We often wish to find the co-ordinates of the point where two lines meet or intersect.



Example: Where do $y = 4x - 1$ and $y = -2x + 5$ intersect?

The point of intersection has the same y-value on both lines, so we can just equate the right-hand sides:

$$4x - 1 = -2x + 5 \quad \Rightarrow \quad 6x = 6 \quad \Rightarrow \quad x = 1$$

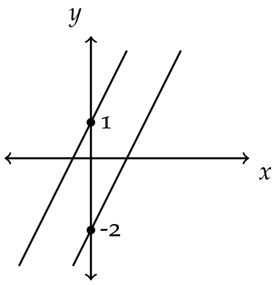
Substituting back into either of the line equations gives

$$y = 4x - 1 = 4(1) - 1 = 4 - 1 = 3$$

Hence, the coordinates of the point of intersection are $(1,3)$.

Parallel lines Parallel lines have the same gradient. For example,

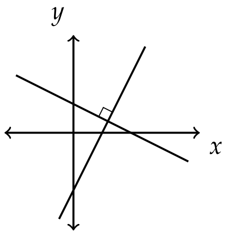
$$y = 2x + 1 \qquad \text{and} \qquad y = 2x - 2$$



Note that parallel lines never intersect

Perpendicular lines Perpendicular lines (that is, lines at right-angles to each other) have gradients which multiply to give -1.For example,

$$y = 2x - 2 \text{ and } y = -\tfrac{1}{2}x + 1 \qquad \left(2 \times -\tfrac{1}{2} = -1\right)$$



If a given line has gradient m then any line perpendicular to it must have gradient $-\frac{1}{m}$.

Graphs of exponential and logarithm functions

Exponential growth and decay

When we discussed exponentials and logarithms we observed that exponenial growth functions grow very quickly. For example, the function $y = 2^x$

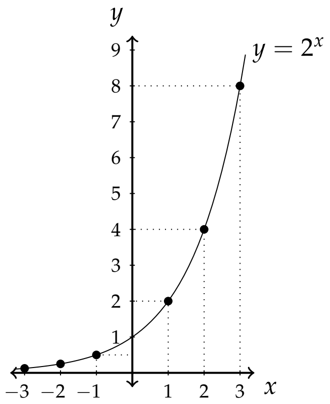
x	1	2	3	4	5	6	7	8	9	10	11	12
y	2	4	8	16	32	64	128	256	515	1,024	2,048	4,096

The graph of 2^x

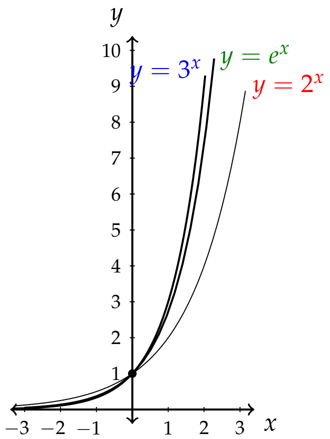
To plot the graph of the function $y = 2^x$ we'll need to determine some points on the graph, plot them and then draw a smooth line between them. In all, we have:

x	-4	-3	-2	-1	0	1	2	3	4
y	$\frac{1}{16}$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8	16

and the graph is



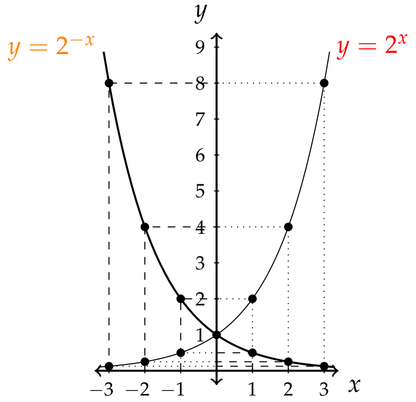
More graphs of exponentials
Here is a plot of the functions 2^x, 3^x, and between them e^x.



Note that the graphs **never** go below the x-axes, and that **all** of the graphs have y-intercept 1

These graphs of exponential growth functions give us a graphical interpretation of things like population growth and compound interest.

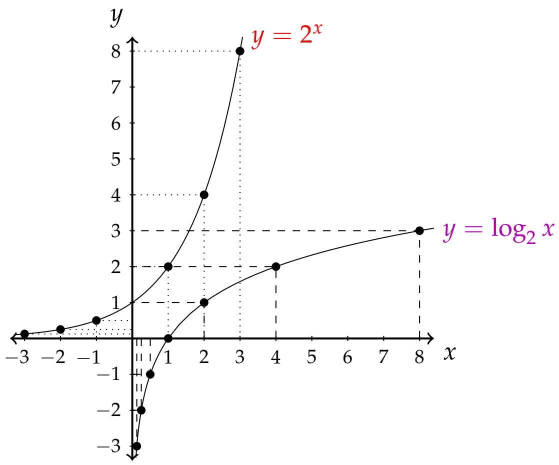
The graph of 2^{-x}
Here is a graph of 2^{-x}, along with 2^x.



Note that the graphs are **'left-right-symmetric'**

These graphs of exponential decay functions give us a graphical interpretation of things like radioactive decay and depreciation.

The graph of $\log_2 x$
Here is a graph of $\log_2 x$, along with 2^x.



Note that there is a diagonal symmetry, and observe that the graph of $\log_2 x$ **never** crosses the y-axis

Exercises 10

10.1 Sketch the graphs of the following linear functions

(i) $y = 4x - 8$

(ii) $y = 9 - 3x$

(iii) $y = 2x + 1$

10.2 Find the equation of the horizontal line which contains the point $(4, -5)$.

10.3 Find the equation of the line which is perpendicular to $y = 2x + 5$ and has y-intercept 6.

10.4 Find the equation of the line containing $(-2, -4)$ and $(3, 1)$.

Problem set 10

10.1 Find the indicated values of the following functions.

$$f(x) = -3x^2 + 2x + 4 \qquad g(x) = \log_2 x$$

$$h(u) = \sqrt{11-u} \qquad F(v) = 3^{2v-4}$$

$$G(t) = 5 - 3t \qquad H(s) = \frac{s-3}{s+1}$$

(a) $f(2)$
(b) $f(-1)$
(c) $g(8)$
(d) $g\left(\frac{1}{4}\right)$
(e) $h(7)$
(f) $h(11)$
(g) $F(3)$
(h) $F(2)$
(i) $G\left(\frac{1}{4}\right)$
(j) $G\left(-\frac{2}{5}\right)$
(k) $H(2)$
(l) $H(-3)$

10.2 Sketch the graphs of the following functions

(a) $y = 3x + 6$
(b) $y = -4x - 4$
(c) $y = -2t + 6$
(d) $z = 5x - 4$

10.3 Find the equations of the following lines.

(a) Gradient 2 and contains the point $(0, 4)$
(b) Gradient -3 and contains the point $(-1, 3)$
(c) Contains the points $(3, 4)$ and $(6, -2)$
(d) Contains the point $(3, 1)$ and is parallel to the line $y = 2x - 1$
(e) Perpendicular $y = 4x - 1$ and contains the point $(4, 2)$

10.4 The line L_1 has equation $y = 3x + 1$. Find the equations of the following five lines.

(a) L_2 which is parallel to L_1 and contains the point $(-1, 4)$
(b) L_3 which is perpendicular to L_1 and has x-intercept 3
(c) L_4 perpendicular to L_1 and intersects it at the point $(2, -3)$
(d) L_5 which is horizontal and has the same y-intercept as L_1
(e) L_6 which contains $(3, 7)$ and has the same y-intercept as L_1

10.5 Sketch the following pairs of lines from the previous question on the same axes.

(a) L_1 and L_3
(b) L_2 and L_6
(c) L_4 and L_5

10.6 Find the intersection point of each pair of lines in the previous question.

10.7 An application. This is an example of a problem in the area of *Operations Research* (or *Management Systems* as it is called in Business). Consider the lines

$$L_1: \ x + y = 4$$

$$L_2: \ x + 2y = 6$$

$$L_3: \ 2x + y = 7$$

(a) Sketch the graphs of all three on the same set of axes.

(b) Find the points of intersection between:

(i) L_1 and L_2

(ii) L_1 and L_3

(iii) L_2 and the y-axis

(iv) L_3 and the x-axis

(c) At each of those intersection points find the value of

$$z = 3x + 5y$$

(d) At which intersection point does z attain its greatest value?

11. *Quadratic functions and Parabolae*

Recall that quadratics are expressions of the form

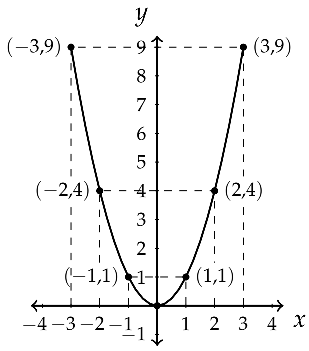
$$ax^2 + bx + c$$

where a, b and c are given numbers.

Quadratic functions are functions of the form

$$y = f(x) = ax^2 + bx + c$$

Here is the graph of $y = x^2$. We call this graph a parabola.

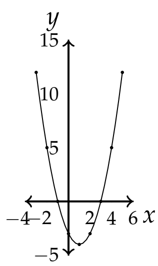


Quadratic functions

Different values of a, b and c produce different parabolic graphs. We can obtain the graph of any function by choosing a series of x-values and for each, calculating the corresponding y-value and plotting that co-ordinate. Finally, draw a smooth curve which contains all of the points.

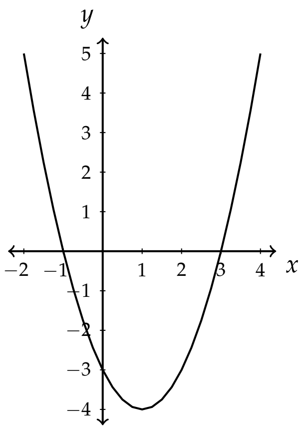
Example: The quadratics function $y = x^2 - 2x - 3$. We have

x	-3	-2	-1	0	1	2	3	4	5
y	12	5	0	-3	-4	-3	0	5	12



Obviously we would like to avoid plotting lots of points in order to sketch a parabola. There are four main properties of parabolae:

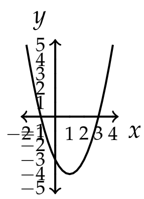
- the convexity
- the $y-$intercept
- the $x-$intercept(s) (if they exist)
- the vertex (or turning point)



Each of these properties can be determined from the coefficients a, b and c in

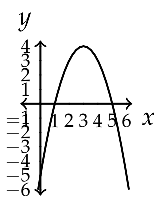
$$y = ax^2 + bx + c$$

Convexity



$$y = x^2 - 2x - 3$$

Concave up (like a smile)



$$y = -x^2 + 6x - 5$$

Concave down (like a frown)

If the coefficient of x^2 (that is, the value of a) is:

- positive, the parabola is concave up (smile)
- negative, the parabola is concave down (frown)

The y-intercept

This is where the parabola passes through the $y-$axis. In other words, the $y-$intercept is the value of y when $x = 0$. So, for $y = ax^2 + bx + c$, the $y-$intercept is:

$$y = a(0)^2 + b(0) + c \qquad \Rightarrow \qquad y = c$$

The co-ordinate of the $y-$intercept is hence $(0, c)$.

Example: The parabola $y = 2x^2 - 4x + 6$ has y-intercept 6 and is concave up

Example: The parabola $y = -5x^2 - 3$ has y-intercept -3 and is concave down

Example: The parabola $y = x - 4x^2$ has y-intercept 0 and is concave down

The x-intercept(s)

These are where the parabola passes through the $x-$axis. In other words, the $x-$intercepts are the values of x when $y = 0$. So, for $y = ax^2 + bx + c$, we need to solve

$$ax^2 + bx + c = 0$$

That is, we have to find *the roots of the quadratic equation*. We do this either by factorising or using the quadratic formula.

Recall that there may be 0, 1 or 2 solutions

Example: The parabola $y = x^2 - 2x - 3$. We must solve

$$x^2 - 2x - 3 = 0$$

Luckily, we can factorize:

$$(x-3)(x+1) = 0 \qquad \Rightarrow \qquad x = 3 \text{ or } x = -1$$

Hence there are two x-intercepts and they are $x = 3$ and $x = -1$.

The vertex (turning point)

The symmetry of parabolae tells us that the vertex has an $x-$value half way between the $x-$intercepts.

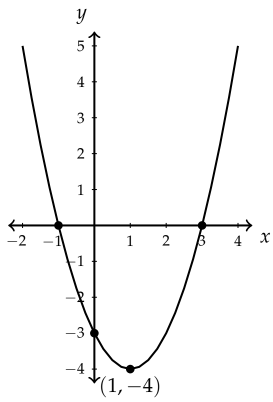
Example: The parabola $y = x^2 - 2x - 3$. In this case, the $x-$value of the vertex is $x = 1$. To find the corresponding $y-$value we substitute this value into the function:

$$y = 1^2 - 2(1) - 3 = -4$$

The vertex is hence at $(1, -4)$.

For the function $y = x^2 - 2x - 3$ we now have the four main properties:

- concave up
- $y-$intercept: $(0, -3)$
- $x-$intercepts: $(-1, 0)$ and $(3, 0)$
- vertex: $(1, -4)$



and so we can draw the Here is a sketch of the parabola:

Note that we write in the coordinates of any points of interest (for example, the vertex) which are not on one of the axes.

A formula for the vertex *x*-coordinate

What if the $x-$intercepts weren't so easy to work with (or perhaps didn't even exist)?

The quadratic formula provides the clue to a general method:

To solve $ax^2 + bx + c = a(x - A)(x - B) = 0$ then

$$A, B = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

If we remove the term added on or taken off, we are left with an $x-$value halfway between the roots:

$$x = \frac{-b}{2a}$$

This is the x-coordinate of the vertex.

Example: The parabola $y = x^2 - 2x - 3$ has $a = 1$ and $b = -2$, so

$$x = \frac{-b}{2a} = \frac{2}{2} = 1$$

and substitution of this into $y = x^2 - 2x - 3$ gives us the

y-coordinate of the vertex: $y = 1^2 - 2(1) - 3 = -4$.

An example A sketch of the graph of $y = x^2 - 6x + 8$. We have:

- The graph is concave up since $a = 1$ is positive.
- The $y-$intercept is $c = 8$.
- To find the $x-$intercepts we solve $x^2 - 6x + 8 = 0$. We can do this using the quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad \Rightarrow \quad x = \frac{6 \pm \sqrt{36 - 4 \times 1 \times 8}}{2}$$

That is,

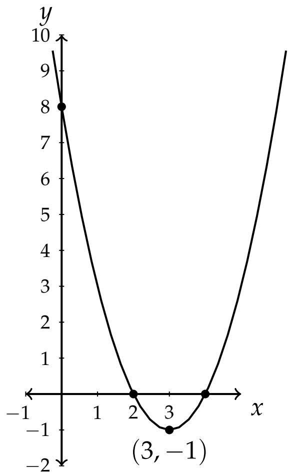
$$x = \frac{6 \pm \sqrt{36-32}}{2} = \frac{6 \pm \sqrt{4}}{2} = \frac{6 \pm 2}{2} = \frac{8}{2}, \frac{4}{2} = 4,\ 2.$$

- To find the vertex:

$$x = \frac{-b}{2a} = \frac{6}{2} = 3 \quad \Rightarrow \quad y = 3^2 - 6 \times 3 + 8 = 9 - 18 + 8 = -1$$

So the vertex is at $(3, -1)$.

The sketch of $y = x^2 - 6x + 8$ is:



Another example
A sketch of the graph of $y = -x^2 - 2x - 2$. We have that

- The graph is concave down since $a = -1$ is negative.
- The $y-$intercept is $c = -2$.
- To find the $x-$intercepts we solve $-x^2 - 2x - 2 = 0$. We can do this using the quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad \Rightarrow \quad x = \frac{2 \pm \sqrt{4 - 4 \times (-1) \times (-2)}}{2 \times (-1)}$$

$$\Rightarrow \quad x = \frac{2 \pm \sqrt{4-8}}{-2} = \frac{2 \pm \sqrt{-4}}{-2}$$

but we can't take the square root of a negative number and hence there are no x-intercepts.

- To find the vertex:

$$x = \frac{-b}{2a} = \frac{2}{-2} = -1$$

$$\Rightarrow \quad y = -(-1)^2 - 2 \times (-1) - 2 = -1 + 2 - 2 = -1$$

So the vertex is at $(-1, -1)$.

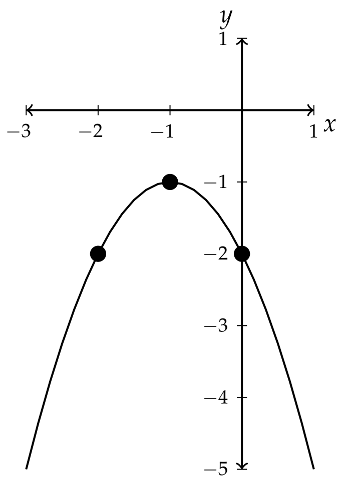
We don't have much to go on, just the vertex and the y-intercept.

We'd like to get a *point symmetrically across* from the y-intercept. It's x-coordinate will be twice that of the vertex:

$$x = -2 \quad \Rightarrow \quad y \; = \; -(-2)^2 - 2(-2) - 2 \; = \; -4 + 4 - 2 \; = = -2$$

and from this we have another point on the graph: $(-2, -2)$.

The graph of $y = -x^2 - 2x - 2$ is



Exercises 11

11.1 Sketch the graph of the quadratic function $y = x^2 - 2x - 8$ indicating

- the convexity (concave up or down)
- the $y-$intercept
- the vertex co-ordinates
- whatever $x-$intercepts exist

Problem set 11

11.1 Sketch the graphs of the following quadratic functions, in each case indicating

- the convexity (concave up or down)
- the $y-$intercept
- the vertex co-ordinates
- whatever $x-$intercepts exist

(a) $y = 2x^2 + 8x + 6$

(b) $y = x^2 - 2x - 8$

(c) $y = x^2 - 4x + 4$

(d) $y = -x^2 + 4x - 3$

(e) $y = x^2 - 4x + 12$

(f) $y = x^2 - 6x$

(g) $y = x^2 - 4$

(h) $y = x^2 - 7x + 10$

11.2 Complete the square for the following quadratic functions and use the result to verify the co-ordinates of the vertex of the parabola they repesent.

(a) $y = x^2 + 8x + 3$

(b) $y = x^2 - 14x$

12. *Calculus*

Rate of change

In many situations we are interested in how quickly (or slowly) something is changing. For example,

- how quickly a car is accelerating
- how quickly a virus is spreading
- how quickly a company is growing

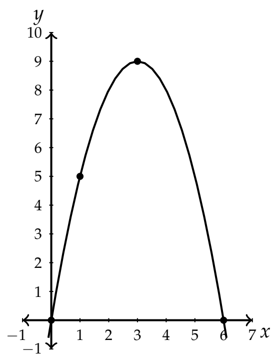
We refer to how quickly something of interest is changing as its rate of change. In Mathematics we use functions to describe things of interest. How can we calculate the rate of change of a *function*?

The basic Idea

Consider climbing a hill. Initially it may be quite steep, but as we approach the top it starts to level off. The usually refer to the steepness as its slope or gradient.

It's a measure of the rate of change of height. The simplest situation is a straight line, for example $y = 2x + 1$. Its slope/gradient is $m = 2$ *anywhere* on it.

What about a parabola? For example, $y = 6x - x^2$. Its graph is:



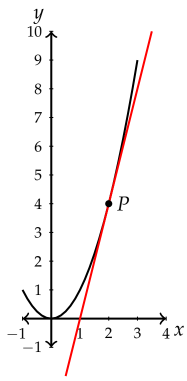
What is its slope at

- $x = 0$?
- $x = 1$?
- $x = 3$?

Notice that the slope/gradient depends on the value of x. How do we find the gradient at any point x? Can we obtain a "formula" or expression for the gradient of the graph in terms of x?

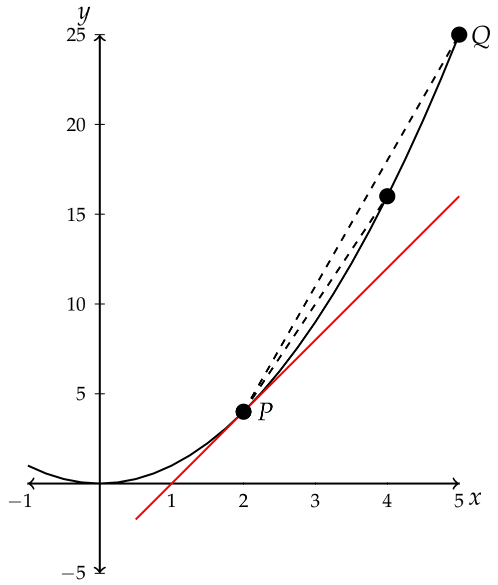
The simplest case

Let's look at the simplest case, $y = x^2$. We wish to know how steep it is, that is, what is its *slope/gradient* when say, $x = 2$. The gradient we are after is the gradient of the line that just touches the parabola at the point $P(2, 4)$. This line is called the tangent to the parabola at P.



Chords

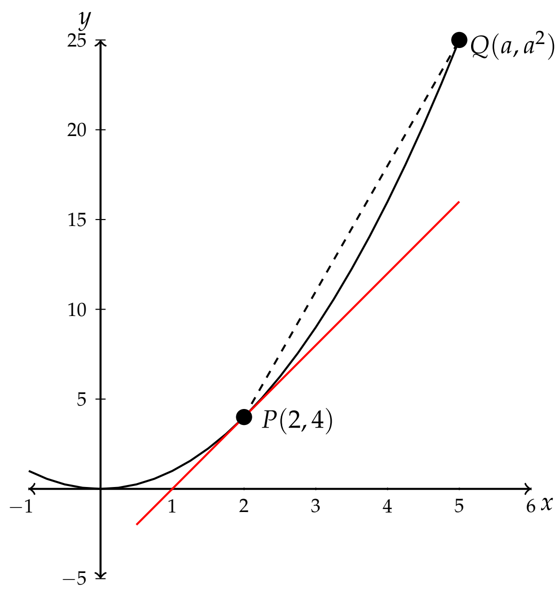
We approximate the tangent at the point to the graph of $y = x^2$ at the point $x = 2$ by a sequence of chords. A chord is a straight lines joining two points on a curve.



Each chord joins our point of interest P to another point, Q on the parabola. As Q moves in towards P, the chord's gradient approaches that of the tangent. Let's take an arbitrary point Q on the parabola. Its co-ordinates will be $Q(a, a^2)$ for some vlaue of a. Then the gradient of PQ is

$$\frac{a^2 - 4}{a - 2}$$

Graphically, we have



Limits

What value does function $\frac{a^2-4}{a-2}$ approaches as a approaches 2? Mathematicians refer to that value as the limit of the function as a approaches 2, and write

$$\lim_{a\to 2} \frac{a^2-4}{a-2}$$

We can't substitute $a = 2$ in the expression because we'll get $\frac{0}{0}$. We first factorise the numerator: $a^2 - 4 = (a+2)(a-2)$, so

$$\lim_{a\to 2} \frac{a^2-4}{a-2} = \lim_{a\to 2} \frac{(a+2)\cancel{(a-2)}}{\cancel{(a-2)}} = \lim_{a\to 2}(a+2) = 2+2 = 4$$

So the gradient of the tangent line to the function $y = x^2$ at $x = 2$ is $m = 4$. We can if needed calculate that the y-intercept is $c = -4$, and hence the equation of the tangent line is $y = 4x - 4$.

Examples

$$\lim_{a\to 5} \frac{a^2-11a+30}{a-5} = \lim_{a\to 5} \frac{\cancel{(a-5)}(a-6)}{\cancel{a-5}} = \lim_{a\to 5}(a-6) = 5-6 = -1$$

Our task is always to factorize the numerator. Observe that the bottom line immediately tells us one of the factors of the top line. Our job is to find the other.

Example

$$\lim_{a\to 3} \frac{a^2+3a+2}{a-3} = \lim_{a\to 3} \frac{(a+1)(a+2)}{a-3}$$

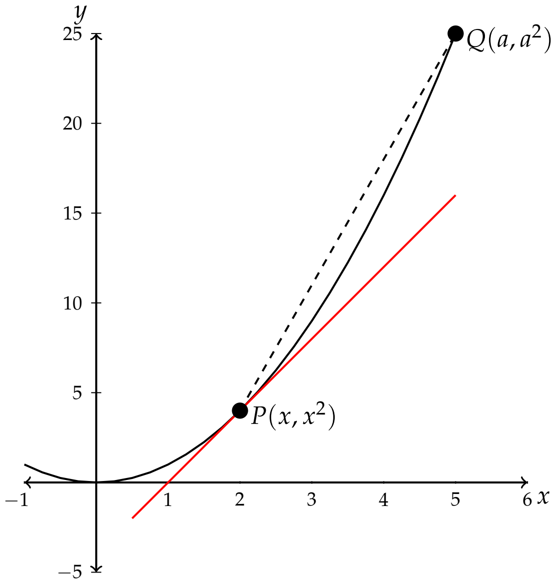
but nothing cancels, and we will end up with $\frac{20}{0}$, but we can't divide by zero! We say that the limit does not exist

Tangents

Recall from last lecture our investigation of the tangent to the parabola $y = x^2$ at $x = 2$. Its gradient is that of the chord PQ as a approaches 2. That is,

$$m = \lim_{a \to 2} \frac{a^2 - 4}{a - 2} = \lim_{a \to 2} \frac{(a+2)\cancel{(a-2)}}{\cancel{(a-2)}} = \lim_{a \to 2}(a+2) = 2+2 = 4$$

We can do the same calculation for any point $P(x, x^2)$.



The gradient of the tangent line is that of the chord PQ in the limit as a approaches x. That is,

$$\lim_{a \to x} \frac{a^2 - x}{a - x} = \lim_{a \to x} \frac{(a+x)\cancel{(a-x)}}{\cancel{(a-x)}} = \lim_{a \to x}(a+x) = x+x = 2x$$

Example
The gradient of the tangent line at the point $(3, 9)$ will be

$$m = 2 \times 3 = 6$$

The derivative

We have an expression for the gradient of the tangent line to the function $f(x) = x^2$. It is $2x$. Note that this is another function. We write it as

$$f'(x) = 2x$$

and call this function the derivative of $f(x) = x^2$.

We could do the same sort of calculation for *any* function $f(x)$. For example, $f(x) = x^n$ or $f(x) = e^x$ or $f(x) = \log x$.

If we are writing our function in the $y = \cdots$ form then we write the derivative in the fractional form $\frac{dy}{dx} = \cdots$, reminiscent of the fraction in the limit calculation.

Example

$$\text{if} \qquad y = x^2 \qquad \text{then} \qquad \frac{dy}{dx} = 2x$$

The operation of finding the derivative is called differentiation.

Some derivatives

By performing a process similar to what we did previously, we can show that for any number n,

$$\text{if} \qquad f(x) = x^n \qquad \text{then} \qquad f'(x) = nx^{n-1}$$

Another way of writing this is

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

Examples

$$\frac{d}{dx}\left(x^3\right) = 3x^2$$

$$\frac{d}{dx}\left(x^{\frac{1}{2}}\right) = \frac{1}{2}x^{-\frac{1}{2}} \qquad \text{or equivalently} \qquad \frac{d}{dx}\left(\sqrt{x}\right) = \frac{1}{2\sqrt{x}}$$

$$\frac{d}{dx}\left(x^{-1}\right) = -x^{-2} \qquad \text{or equivalently} \qquad \frac{d}{dx}\left(\frac{1}{x}\right) = -\frac{1}{x^2}$$

Two special ones are

$$\frac{d}{dx}(x) = \frac{d}{dx}\left(x^1\right) = 1x^0 = 1$$

and

$$\frac{d}{dx}(1) = \frac{d}{dx}\left(x^0\right) = 0x^{-1} = 0$$

Derivatives of sums and multiples

If a function is the sum/difference of two others then its derivative is the sum/difference of the respective derivatives.

Examples

$$g(x) = x^2 + x^3 \qquad \Rightarrow \qquad g'(x) = 2x + 3x^2$$

$$h(x) = x^6 - x^4 \qquad \Rightarrow \qquad h'(x) = 6x^5 - 4x^3$$

If a function is a multiple of a another function then its derivative is a multiple of its derivative.

Examples

$$j(x) = 4x^2 \qquad \Rightarrow \qquad j'(x) = 4(2x) = 8x$$

$$k(x) = -2x^3 \qquad \Rightarrow \qquad k'(x) = -2(3x^2) = -6x^2$$

We can use both rules together, for example,

$$p(x) = 4x^3 - 7x^{-2} + 5 \qquad \Rightarrow \qquad p'(x) = 12x^2 + 14x^{-3}$$

Note that for any constant, C we have

$$\frac{d}{dx}(C) = \frac{d}{dx}(C \times 1) = C \times \frac{d}{dx}(1) = C \times 0 = 0$$

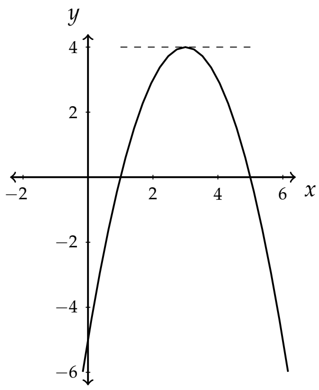
An application of derivatives

Derivatives can be used to determine the maximum (or minimum) values of functions.

Example: What is the maximum value attained by the function

$$y = -x^2 + 6x - 5$$

The solution to this problem is to realise that at its point of maximum value the tangent to the function will be horizontal:



That is, the gradient of the tangent line will be zero, and hence the derivative of the function will be zero.

We can easily find the derivative of $y = f(x) = -x^2 + 6x - 5$. It's

$$f'(x) = -2x + 6$$

and this is zero where

$$-2x + 6 = 0 \qquad \Rightarrow \qquad 2x = 6 \qquad \Rightarrow \qquad x = 3$$

Hence the maximum value of the function occurs when $x = 3$. To find the actual value of the function we simply calculaute $y(3)$. That is,

$$y_{\max} = -3^2 + 6(3) - 5 = -9 + 18 - 5 = 9 - 5 = 4$$

Derivatives of exp and log functions

By performing a process similar to what we did for power functions (like $f(x) = x^2$) we can show that

$$\text{if} \qquad f(x) = e^x \qquad \text{then} \qquad f'(x) = e^x$$

That is, e^x is its own derivative.

We can also show that

$$\text{if} \qquad f(x) = \ln x \qquad \text{then} \qquad f'(x) = \frac{1}{x} \qquad \text{and} \qquad x \neq 0$$

When constants (k or a or b) are involved we have the following:

$$\frac{d}{dx}\left(e^{kx}\right) = ke^{kx} \qquad \text{and} \qquad \frac{d}{dx}\left(e^{-kx}\right) = -\,ke^{-kx}$$

$$\frac{d}{dx}\Big(\ln(ax)\Big) = \frac{1}{x} \qquad \text{and} \qquad \frac{d}{dx}\left(\log_b x\right) = \frac{1}{\ln(b)\ x}$$

Antiderivatives

Not surprisingly, antiderivatives are the opposites of derivatives. Since $3x^2$ is the derivative of x^3, then x^3 is an antiderivative of $3x^2$.

However, x^3 plus *any* constant c is also an antiderivative because

$$\frac{d}{dx}\left(x^3 + c\right) = \frac{d}{dx}(x^3) + \frac{d}{dx}(c) = 3x^2 + 0 = 3x^2$$

We write this symbolically as follows:

$$\int 3x^2\, dx = x^3 + c$$

and say that *the antiderivative of* $3x^2$ is $x^3 + c$ where c can be *any* constant. Similarly,

$$\int x^2\, dx = \frac{x^3}{3} + C \qquad \qquad \text{where } C = \frac{c}{3}$$

The $\int$ symbol is actually a very tall, very thin letter S (for sum)

Some antiderivatives

By considering the derivatives we've seen so far we can deduce the following antiderivatives.

$$\int x^n\, dx = \frac{x^{n+1}}{n+1} + C \qquad \qquad n \neq -1$$

$$\int \frac{1}{x}\,dx = \ln x + C \qquad \int e^x\,dx = e^x + C$$

and more complicated things if constants like k and a are involved.

We can also calculate sums and multiples for antiderivatives. For example,

$$\int \left(5x^3 + 7x^{-2}\right)dx = 5\int x^3 dx + 7\int x^{-2} dx$$
$$= 5\left(\frac{x^4}{4}\right) + 7\left(\frac{x^{-1}}{-1}\right) + C = \frac{5x^4}{4} - \frac{7}{x} + C$$

Note the generic $+C$. It must always be there

Integrals

Another name for antiderivative is integral. We say for instance, that the integral of $3x^2$ is $x^3 + C$.

Actually, it's called an indefinite integral. In general, we write

$$F(x) = \int f(x)\,dx$$

and say that $F(x)$ is the integral of $f(x)$. In addition, we call $f(x)$ the integrand.

A definite integral is one which is to be *evaluated between two points*. We write

$$\int_a^b f(x)\,dx = \Big[F(x)\Big]_a^b \qquad \text{which means} \qquad F(b) - F(a)$$

Example

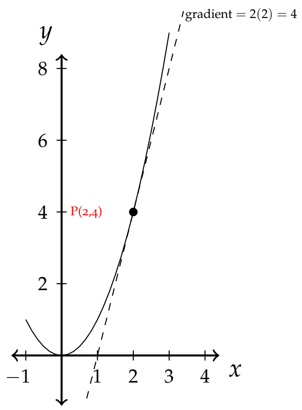
$$\int_2^5 3x^2\,dx = \Big[x^3 + C\Big]_2^5 = 5^3 + \not{C} - \left(2^3 + \not{C}\right) = 125 - 8 = 117$$

Note that C will always cancel out

Geometric meaning of the derivative

Recall that the derivative has the geometric interpretation of being the gradient (slope) of a curve.

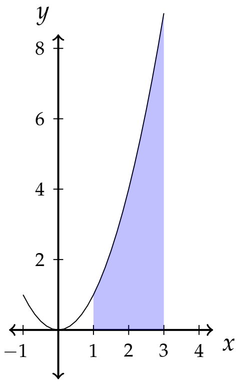
Example: Recall that $\frac{d}{dx}(x^2) = 2x$. Then when say, $x = 2$ we have



The gradient of the tangent line to the parabola $y = x^2$ is $m = 2x$, where x is the x-value of the point of interest.

Geometric meaning of the integral

It turns out that integrals tell us the areas under curves. For example, if the curve is the graph of $y = x^2$ and the units are cm, then what is the area of the shaded region below?

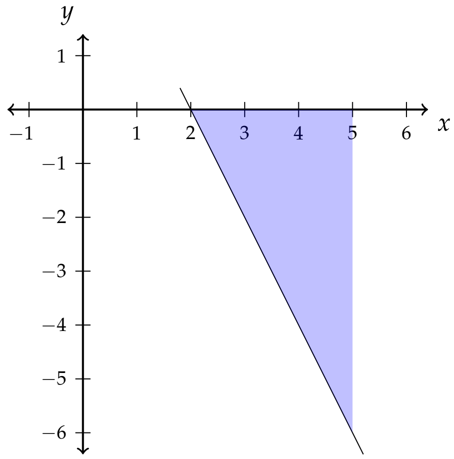


The answer is

$$\text{Area} = \int_1^3 x^2\,dx = \left[\frac{x^3}{3}\right]_1^3 = \frac{3^3}{3} - \frac{1^3}{3} = \frac{27}{3} - \frac{1}{3} = \frac{26}{3} \approx 8.67 \text{ cm}^2$$

There's one thing we have to be careful of. If the area is below the x-axis then the integral will be negative, and so the area must be the negative of the integral because area must be positive.

Example: Line is the graph of $y = 4 - 2x$ and the units are metres. What is the area of the shaded region below?



The answer is

$$\text{Area} = -\int_2^5 (4-2x)dx = -\left[4x - x^2\right]_2^5 = -\left(4(5) - 5^2\right) + \left(4(2) - 2^2\right)$$

$$= -(20-25) + (8-4) = -(-5) + 4 = 5 + 4 = 9 \text{ m}^2$$

which we can in fact check because the region is a triangle:

$$\text{Area} = \frac{1}{2}\text{ base} \times \text{height} = \frac{3 \times 6}{2} = \frac{18}{2} = 9 \text{ m}^2$$

Exercises 12

12.1 Find if possible the following limits

(i) $\lim_{a \to 2} \frac{a^2 - a - 2}{a - 2}$

(ii) $\lim_{s \to 4} \frac{s^2 - 2s - 8}{s - 4}$

(iii) $\lim_{t \to 6} \frac{t^2 - 5t + 4}{t - 6}$

(iv) $\lim_{z \to 3} \frac{4z^2 - 11z - 3}{z - 3}$

12.2 Differentiate the following functions

(i) $f(x) = 2x^3 + 4x^2 + 3x + 5$

(ii) $g(t) = 2t(t^2 + t + 1)$

(iii) $z = \frac{3x^4}{4} + \frac{1}{x} + \sqrt{x} + 4x$

12.3 Integrate the following functions

(i) $f(x) = 5x^2 + 3x + 4$

(ii) $g(x) = \frac{3x^4}{4} + \frac{1}{x^2} + \sqrt{x}$

12.4 Find the area bounded by the graph of the function $y = 12 - 3x^2$ and the x-axis

Problem set 12

12.1 Evaluate the following limits if they exist.

$$(a) \qquad \lim_{x\to 4} \frac{x^2-16}{x-4}$$

$$(b) \qquad \lim_{x\to 2} \frac{x^2+2x-8}{x-2}$$

$$(c) \qquad \lim_{x\to 4} \frac{x^2-x-12}{x-1}$$

$$(d) \qquad \lim_{x\to 3} \frac{x^2-6x+9}{x-3}$$

$$(e) \qquad \lim_{x\to 2} \frac{4x^2-16}{x-2}$$

$$(f) \qquad \lim_{x\to -7} \frac{x^2+15x+56}{x+7}$$

12.2 Differentiate each of the following functions.

$$(a) \qquad y = x^3 + x^2 + x + 6$$

$$(b) \qquad y = 3x^4 - x^3 + x^2 + x - 2$$

$$(c) \qquad y = x^2 + \sqrt{x} + \frac{1}{x} - 3$$

12.3 Differentiate

$$y = \left(\sqrt{x} + \frac{1}{x}\right)\left(x^2 + \frac{1}{x^2}\right)$$

12.4 Perform the following integrations

$$(a) \qquad \int \left(x^2 + x + 6\right) dx$$

$$(b) \qquad \int \left(3x^4 - x^3 - 2\right) dx$$

$$(c) \qquad \int \left(x^2 + \sqrt{x} + \frac{4}{x^2}\right) dx$$

12.5 Evaluate the following definite integrals.

$$(a) \qquad \int_0^1 \left(x^3 + x^2\right) dx$$

$$(b) \qquad \int_2^3 \left(5x^4 + 5x - 2\right) dx$$

$$(c) \qquad \int_{-2}^2 \left(x^2 + 4x - 3\right) dx$$

12.6 Find the AREA bounded by the graph of $f(x)$ and the x-axis between the points indicated for the functions below. [HINT: Sketch the graph first!]

$$(a) \qquad f(x) = x^2 + 1, \qquad x = -1 \text{ to } x = 1$$

$$(b) \qquad f(x) = x^2 - x, \qquad x = 0 \text{ to } x = 2$$

Appendix 1: Answers to exercises

Exercises in Chapter 1

In various places colours are used to focus attention

1.1

(*i*)

$$\begin{aligned} &3(4+2\times 6)+12\div 4\\ &=3\times(4+12)+3\\ &=3\times 16+3\\ &=48+3\\ &=51 \end{aligned}$$

(*ii*)

$$\begin{aligned} &6+2[25-3(2+5)]\\ &=6+2[25-3\times 7]\\ &=6+2[25-21]\\ &=6+2\times 4\\ &=6+8\\ &=14 \end{aligned}$$

1.2 Difference of the product of 6 and 7 and the sum of 3 and 8 is the difference of 42 and 11, that is, $42-11$, which is 31. In symbols we have

$$(6\times 7)-(3+8) \;=\; 42-11 \;=\; 31$$

1.3 (*a*) $1\frac{1}{4}=\frac{4\times1+1}{4}=\frac{4+1}{4}=\frac{5}{4}$ (*b*) $2\frac{3}{8}=\frac{2\times8+3}{8}=\frac{16+3}{8}=\frac{19}{8}$

1.4 (*a*) $\frac{3}{4}\times\frac{1}{5}=\frac{3\times1}{4\times5}=\frac{3}{20}$ (*b*) $\frac{1}{3}\div\frac{2}{5}=\frac{1}{3}\times\frac{5}{2}=\frac{1\times5}{3\times2}=\frac{5}{6}$

1.5 Two-thirds of four-fifths is

$$\frac{2}{3}\times\frac{4}{5} \;=\; \frac{2\times4}{3\times5} \;=\; \frac{8}{15}$$

1.6

(*ii*) $$\frac{3}{4}+\frac{2}{5}=\frac{3\times5}{4\times5}+\frac{2\times4}{5\times4}=\frac{15}{20}+\frac{8}{20}=\frac{15+8}{20}=\frac{23}{20}$$

(*ii*) $$\frac{2}{3}-\frac{2+4}{5-1}=\frac{2}{3}-\frac{6}{4}=\frac{2\times4}{3\times4}-\frac{6\times3}{4\times3}=\frac{8}{12}-\frac{18}{12}=\frac{8-18}{12}=\frac{-10}{12}$$

which is the same as $-\frac{10}{12}$

1.7 In simplest form:

$$\frac{78}{102}=\frac{\not{2}\times39}{\not{2}\times51}=\frac{39}{51}=\frac{\not{3}\times13}{\not{3}\times17}=\frac{13}{17}$$

1.8 One-quarter of the sum of one-third and two-sevenths is

$$\begin{aligned} &\frac{1}{4}\times\left(\frac{1}{3}+\frac{2}{7}\right)\\ &=\frac{1}{4}\times\frac{13}{21}\\ &=\frac{13}{84} \end{aligned}$$

Exercises in Chapter 2

2.1

(i) $$4y - 3 = 3y + 7$$

$$\Rightarrow \quad 4y - \cancel{3} + \cancel{3} = 3y + 7 + 3 \quad \Rightarrow \quad 4y = 3y + 10$$

$$\Rightarrow \quad 4y - 3y = \cancel{3y} + 10 - \cancel{3y} \quad \Rightarrow \quad y = 10$$

Check:

$4 \times 10 - 3 = 40 - 3 = 37$ and $3 \times 10 + 7 = 30 + 7 = 37$ hence ✓

(ii) $9x + 8 = 8x - 4 \quad \Rightarrow \quad 9x = 8x - 12 \quad \Rightarrow \quad x = -12$

Check:

$$\begin{aligned} 9 \times (-12) + 8 &= -108 + 8 = -100 \\ 8 \times (-12) - 4 &= -96 - 4 = -100 \end{aligned} \quad ✓$$

2.2 In the formula $x = \frac{5+t}{b-t}$, if $x = 2$ and $t = 1$, then

$$2 = \frac{5+1}{b-1} \quad \Rightarrow \quad \frac{2}{1} = \frac{6}{b-1}$$

$$\Rightarrow \quad 2(b-1) = 6 \quad \Rightarrow \quad \frac{\cancel{2}(b-1)}{\cancel{2}} = \frac{6}{2}$$

$$\Rightarrow \quad b - 1 = 3 \quad \Rightarrow \quad b = 4$$

Check: $$\frac{5+t}{b-t} = \frac{5+1}{4-1} = \frac{6}{3} = 2 \quad ✓$$

2.3 (i) Cross-multiply:

$$\frac{1}{3x+1} = \frac{2}{x-1} \quad \Rightarrow \quad x - 1 = 2(3x+1)$$

$$\Rightarrow \quad x - 1 = 6x + 2 \quad \Rightarrow \quad -3 = 5x \quad \Rightarrow \quad x = -\frac{3}{5}$$

(ii) Expand the bracket:

$$-2y + 1 + 3(2y - 4) = 2y - 1 \quad \Rightarrow \quad -2y + 1 + 6y - 12 = 2y - 1$$

$$\Rightarrow \quad 4y - 11 = 2y - 1 \quad \Rightarrow \quad 2y = 10 \quad \Rightarrow \quad y = 5$$

2.4 Let a be Sally's age. If it's 12 years plus half her age then

$$a = 12 + \frac{a}{2} \quad \Rightarrow \quad 2a = 24 + a \quad \Rightarrow \quad a = 24$$

Exercises in Chapter 3

3.1

(a)
$$\begin{array}{rcrcrl} 2x & + & 3y & = & 10 & (1) \\ x & + & 2y & = & 6 & (2) \end{array}$$

Multiply equation (2) by 2 to get

$$\begin{array}{crcrcrl} (-) & 2x & + & 3y & = & 10 & (1) \\ & 2x & + & 4y & = & 12 & (3) \\ \hline & & & y & = & 2 & \end{array}$$

Substituting this value into, say equation (2) gives

$$x + 2y = 6 \quad \Rightarrow \quad x + 4 = 6 \quad \Rightarrow \quad x = 2$$

Finally, check by substituting both values into equation (1):

$$2x + 3y = 10 \quad \Rightarrow \quad 4 + 6 = 10 \quad \checkmark$$

(b)
$$\begin{array}{rcrcrl} 2x & + & 3y & = & 19 & (1) \\ 5x & - & 6y & = & -20 & (2) \end{array}$$

Multiply equation (1) by 2 to get

$$\begin{array}{crcrcrl} & 4x & + & 6y & = & 38 & (3) \\ (+) & 5x & - & 6y & = & -20 & (2) \\ \hline & 9x & & & = & 18 & \end{array}$$

Hence $x = 2$ and substituting this into, say equation (1) gives

$$2x + 3y = 19 \quad \Rightarrow \quad 4 + 3y = 19 \quad \Rightarrow \quad 3y = 15 \quad \Rightarrow \quad y = 5$$

Finally, check by substituting both values into equation (2):

$$5x - 6y = -20 \quad \Rightarrow \quad 10 - 30 = -20 \quad \checkmark$$

(c)
$$\begin{array}{rcrcrl} 2x & + & 4y & = & 4 & (1) \\ 5x & - & 6y & = & 2 & (2) \end{array}$$

multiply equation (1) by 5 and equation (2) by 2 to get

$$\begin{array}{crcrcrl} & 10x & + & 20y & = & 20 & (3) \\ (-) & 10x & - & 12y & = & 4 & (4) \\ \hline & & & 32y & = & 16 & \end{array}$$

and hence

$$y = \frac{16}{32} = \frac{\not{16} \times 1}{\not{16} \times 2} \quad \Rightarrow \quad y = \frac{1}{2}$$

Substituting this value into, say equation (1) gives

$$2x + 4y = 4 \quad \Rightarrow \quad 2x + 2 = 4 \quad \Rightarrow \quad 2x = 2 \quad \Rightarrow \quad x = 1$$

Finally, check by substituting both values into equation (2):

$$5x - 6y = 2 \quad \Rightarrow \quad 5 - 3 = 2 \quad \checkmark$$

3.2 Let the numbers be m and n, with m larger than n. We are told that

$$\begin{aligned} m + n &= 28 \qquad (1) \\ m - n &= 12 \qquad (2) \end{aligned}$$

Adding these equations together gives

$$m + m + \cancel{n} - \cancel{n} = 28 + 12 \quad \Rightarrow \quad 2m = 40 \quad \Rightarrow \quad m = \frac{40}{2} = 20$$

Substitute this back into say equations (1) to get

$$20 + n = 28 \quad \Rightarrow \quad n = 28 - 20 \quad \Rightarrow \quad n = 8$$

That is, the required numbers are 20 and 8. Check

$$20 + 8 = 28 \ \checkmark \qquad \text{and} \qquad 20 - 8 = 12 \ \checkmark$$

Exercises in Chapter 4

4.1 Expanding brackets and collecting terms gives:

(i) $$(x+2)(x-4) = x^2 - 4x + 2x - 8 = x^2 - 2x - 8$$

(ii) $$(-x+5)(-3x+4) = 3x^2 - 4x - 15x + 20 = 3x^2 - 19x + 20$$

(iii) $$(2-t)(3t+4) = 6t + 8 - 3t^2 - 4t = -3t^2 + 2t + 8$$

(iv) $$\left(y+\frac{1}{2}\right)\left(y+\frac{1}{4}\right) = y^2 + \frac{1}{2}y + \frac{1}{4}y + \frac{1}{2}\times\frac{1}{4} = y^2 + \frac{3}{4}y + \frac{1}{8}$$

4.2 To factorize

(i) $x^2 - 3x + 2$. The factors of 2 are

$$1\times 2 \qquad (-1)\times(-2)$$

and the sum must be -3 so we have

$$x^2 - 3x + 2 = (x-1)(x-2)$$

(ii) $x^2 - x - 12$. The factors of -12 are

$$1\times(-12), \quad 2\times(-6), \quad 3\times(-4), \quad 4\times(-3), \quad 6\times(-2), \quad 12\times(-1)$$

and the sum must be -1 so we have

$$x^2 - x - 12 = (x+3)(x-4)$$

(iii) $x^2 + 14x + 40$. The positive factors of 40 are

$$1\times 40 \qquad 2\times 20 \qquad 4\times 10 \qquad 5\times 8$$

and the sum must be 14 so

$$x^2 + 14x + 40 = (x+4)(x+10)$$

(iv) The original quadratic is $3x^2 + 8x + 4$, so the associated quadratic is $x^2 + 8x + 12$. After a little bit of work we find that

$$x^2 + 8x + 12 = (x+2)(x+6)$$

and hence the required factorization is

$$3x^2 + 8x + 4 = (3x+2)\left(x+\frac{6}{3}\right)$$

That is,

$$3x^2 + 8x + 4 = (3x+2)(x+2)$$

(v) The original quadratic is $4x^2 - 5x - 6$, so the associated quadratic is $x^2 - 5x - 24$. After a little bit of work we find that

$$x^2 - 5x - 24 = (x+4)(x-8)$$

and hence the required factorization is

$$4x^2 - x - 5 = (4x + 4)\left(x - \frac{8}{4}\right)$$

That is,

$$4x^2 - 5x - 6 = (4x + 3)(x - 2)$$

4.3 Factorizing and cancelling gives:

(*i*)

$$\frac{x^2 - x - 2}{x^2 + 2x + 1} = \frac{\cancel{(x+1)}(x-2)}{\cancel{(x+1)}(x+1)} = \frac{x-2}{x+1}$$

(*ii*)

$$\frac{x^2 + 5x - 14}{x^2 + 10x + 21} = \frac{\cancel{(x+7)}(x-2)}{\cancel{(x+7)}(x+3)} = \frac{x-2}{x+3}$$

Exercises in Chapter 5

5.1

(*i*) $$2^4 \times 2^9 = 2^{4+9} = 2^{13}$$

(*ii*) $$\frac{3^{11}}{3^5} = 3^{11-5} = 3^6$$

(*iii*) $$(5 \times 7)^3 = 5^3 \times 7^3$$

5.2

(*i*) $$\left(\frac{7}{3}\right)^2 = \frac{7^2}{3^2}$$

(*ii*) $$(5^4)^7 = 5^{4\times 7} = 5^{28}$$

5.3 Using the Laws of Indices we have

(*i*) $$x^{-3}x^5 = x^{-3+5} = x^2$$

(*ii*) $$\frac{(y^{-1})^7}{y^{-3}} = \frac{y^{-1\times 7}}{y^{-3}} = \frac{y^{-7}}{y^{-3}}$$

$$= y^{-7-(-3)} = y^{-7+3} = y^{-4}$$

(*iii*) $$(5x^3)^{-2}x^6 = 5^{-2}(x^3)^{-2}x^6 = 5^{-2}x^{-6}x^6 = 5^{-2}x^{-6+6}$$

$$= 5^{-2}x^0 = 5^{-2}\times 1 = 5^{-2} = \frac{1}{5^2} = \frac{1}{25}$$

(*iv*)
$$\left(3x^{-2}\right)^{-2} x^{-3} = 3^{-2}\left(x^{-2}\right)^{-2} x^{-3}$$
$$= \frac{1}{3^2}\, x^4 x^{-3} = \frac{1}{9}\, x^{4-3} = \frac{x}{9}$$

(*v*)
$$\frac{5y^{-2}z}{(2y^3z^{-5})^2} = \frac{5y^{-2}z}{2^2(y^3)^2(z^{-5})^2} = \frac{5y^{-2}z}{4y^6z^{-10}}$$
$$= \frac{5}{4}\, y^{-2-6} z^{1-(-10)} = \frac{5}{4}\, y^{-8} z^{1+10} = \frac{5}{4}\, y^{-8} z^{11}$$

5.4 Simplifying gives

(*i*)
$$\sqrt{48} = \sqrt{16\times 3} = \sqrt{16}\sqrt{3} = 4\sqrt{3}$$

(*ii*)
$$\sqrt{3}(5+\sqrt{2}) = \sqrt{3}\times 5 + \sqrt{3}\sqrt{2} = 5\sqrt{3}+\sqrt{6}$$

(*iii*)
$$(3+\sqrt{5})(2-\sqrt{5}) = 6-3\sqrt{5}+2\sqrt{5}-\sqrt{5}\sqrt{5} = 6+(2-3)\sqrt{5}-5$$
$$= (6-5)+(-1)\sqrt{5} = 1-\sqrt{5}$$

(*iv*)
$$(2\sqrt{5}-1)(2\sqrt{5}+1) = 4\sqrt{5}\sqrt{5}+\cancel{2\sqrt{5}}-\cancel{2\sqrt{5}}-1$$
$$= 4\times 5-1 = 20-1 = 19$$

Exercises in Chapter 6

6.1 (i)

$$z^3 p^{\frac{1}{4}} = a^4 w^{\frac{2}{3}} \quad \Rightarrow \quad \frac{z^3 p^{\frac{1}{4}}}{a^4} = w^{\frac{2}{3}} \quad \Rightarrow \quad \left(\frac{z^3 p^{\frac{1}{4}}}{a^4}\right)^{\frac{3}{2}} = w$$

We usually write the isolated variable on the left-hand side:

$$w = \left(\frac{z^3 p^{\frac{1}{4}}}{a^4}\right)^{\frac{3}{2}}$$

(ii)

$$y = \sqrt{x^3 - c} \quad \Rightarrow \quad y^2 = x^3 - c$$
$$\Rightarrow \quad y^2 + c = x^3 \quad \Rightarrow \quad x = \sqrt[3]{y^2 + c}$$

(iii)

$$m = \frac{12s^4y^3}{7p^2} \quad \Rightarrow \quad 7p^2m = 12s^4y^3$$
$$\Rightarrow \quad \frac{7p^2m}{12y^3} = s^4 \quad \Rightarrow \quad s = \sqrt[4]{\frac{7p^2m}{12y^3}}$$

(iv)

$$f^2 h^{\frac{1}{3}} = vm^{\frac{1}{4}} \quad \Rightarrow \quad h^{\frac{1}{3}} = \frac{vm^{\frac{1}{4}}}{f^2} \quad \Rightarrow \quad h = \left(\frac{vm^{\frac{1}{4}}}{f^2}\right)^3$$

Exercises in Chapter 7

Recall that the Laws of Logarithms are

$$\log_b b = 1 \qquad \log_b 1 = 0 \qquad \log_b (x^r) = r\ \log_b x$$

$$\log_b(xy) = \log_b x + \log_b y \qquad \log_b \left(\frac{x}{y}\right) = \log_b x - \log_b y$$

7.1

(i) $2^6 = 64$ so $6 = \log_2 64$

(ii) $121 = 11^2$ so $\log_{11} 121 = 2$

(iii) $27^{\frac{4}{3}} = 81$ so $\frac{4}{3} = \log_{27} 81$

(iv) $12 = \sqrt{144}$ so $\log_{144} 12 = \frac{1}{2}$

7.2

(i) $\log_5 125 = 3$ so $125 = 5^3$

(ii) $\log_4 64 = 3$ so $64 = 4^3$

(iii) $\log_7 x = 3$ so $x = 7^3$

(iv) $\log_4 8 = x$ so $8 = 4^x$

7.3

(*i*) $\log_2 9 + \log_2 3 = \log_2 27$ since $9 \times 3 = 27$

(*ii*)

$\log_3 7 - \log_3 28 = \log_3 \left(\frac{1}{4}\right)$ since $\frac{7}{28} = \frac{1 \times \cancel{7}}{4 \times \cancel{7}} = \frac{1}{4}$

(*iii*) In $\log_8 16 - 1$ write 1 as $\log_8 8$ to get

$$\log_8 16 - 1 = \log_8 16 - \log_8 8 = \log_8 \left(\frac{16}{8}\right) = \log_8 2$$

Note that log takes precedence over subtraction. If we'd wanted the subtraction to be done first, we'd write $\log_8(16-1)$

7.4 We have $\log_4 3 = s$ and $\log_4 5 = t$. Now

(*i*) $$\log_4 75 = log_4(3 \times 25) = log_4(3 \times 5^2) = log_4 3 + log_4(5^2)$$
$$= log_4 3 + 2 log_4 5 = s + 2t$$

(*ii*) $$\log_4 \left(\frac{5}{9}\right) = \log_4 \left(\frac{5}{3^2}\right) = \log_4 5 - \log_4(3^2)$$
$$= \log_4 5 - 2\log_4 3 = t - 2s$$

Exercises in Chapter 8

8.1

(*i*) $$\log_5 x + \log_5 8 = 2 \Rightarrow \log_5(8x) = 2 \Rightarrow 8x = 5^2$$
$$\Rightarrow 8x = 25 \Rightarrow x = \frac{25}{8}$$

(*ii*) $$2^{2x+1} = \frac{1}{32} \Rightarrow 2^{2x+1} = 2^{-5} \Rightarrow 2x + 1 = -5$$
$$\Rightarrow 2x = -6 \Rightarrow x = -\frac{6}{2} \Rightarrow x = -3$$

(*iii*) $$\left(3^{2x-1}\right)^2 = 27 \Rightarrow 3^{2(2x-1)} = 3^3 \Rightarrow 3^{4x-2} = 3^3$$
$$\Rightarrow 4x - 2 = 3 \Rightarrow 4x = 5 \Rightarrow x = \frac{5}{4}$$

8.2

(*i*) $$\ln 27 - \ln 3 + \ln 4 = \ln\left(\tfrac{27}{3}\right) + \ln 4 = \ln 9 + \ln 4 = \ln 36$$

(*ii*) $$(\frac{16e^5}{2e^3} = \frac{16}{2} e^{5-3} = 8e^2$$

8.3

(*i*) $$\ln(2x) + \ln 6 = \ln(4x+2) \Rightarrow \ln(12x) = \ln(4x+2)$$
$$\Rightarrow 12x = 4x + 2 \Rightarrow 8x = 2 \Rightarrow x = \frac{2}{8} = \frac{1}{4}$$

(*ii*) $$\frac{e^{5x+1}}{e^{3x-2}} = e^4 \Rightarrow e^{5x+1-(3x-2)} = e^4$$
$$\Rightarrow 5x + 1 - 3x + 2 = 4 \Rightarrow 2x + 3 = 4 \Rightarrow 2x = 1 \Rightarrow x = \frac{1}{2}$$

8.4 The data gives immediately that $P_0 = 20$ so $P = 20e^{kt}$. We are told that after 5 years there were 480 fish, so

$$480 = 20e^{5k} \;\Rightarrow\; e^{5k} = \frac{480}{20} = 24 \;\Rightarrow\; 5k = \ln 24 \;\Rightarrow\; k = \frac{\ln 24}{5}$$

This is about 0.64, that is, the growth rate is about 64%. To find out when the population exceeds 1000 we solve

$$P(t) = 1000 \quad\Rightarrow\quad 20e^{kt} = 1000 \quad\Rightarrow\quad e^{kt} = \frac{1000}{20} = \frac{100}{2} = 50$$

$$\Rightarrow \quad kt = \ln 50 \quad\Rightarrow\quad t = \frac{\ln 50}{k}$$

That is,

$$t = \ln 50 \,\div\, k = \frac{\ln 50}{1} \div \frac{\ln 24}{5} = \frac{\ln 50}{1} \times \frac{5}{\ln 24} = \frac{5\ln 50}{\ln 24}$$

A calculator would tell us that $t \approx 6.15$ years, that is, about 6 years and 2 months after the beginning of the study.

8.5 The machine cost \$7000 in 2010 and is valued at \$1000 in 2015. Assuming the decay law $P = P_0e^{-kt}$, when will the machine be worth \$700? Let t be time in years starting at 0 in 2010. Then $P_0 = 7000$, so

$$P = 7000e^{-kt}$$

Now, 2015 represents $t = 5$, so we have $P = 1000$ and hence

$$1000 = 7000e^{-5t} \quad \Rightarrow \quad e^{-5k} = \frac{1000}{7000} \quad \Rightarrow \quad e^{-5k} = \frac{1}{7}$$

Now 'take logs of both sides' (that is, switch to log form):

$$\Rightarrow \quad -5k = \ln\left(\frac{1}{7}\right) \quad \Rightarrow \quad k = -\frac{\ln\left(\frac{1}{7}\right)}{5}$$

A calculator would tell us that this is about 0.39 (that is, decining at 39% per year), but it's best to leave it in mathematical form. To find the time t when the machine will be worth \$700 we solve

$$700 = 7000e^{-kt} \qquad \Rightarrow \qquad \frac{1}{10} = e^{-kt}$$

and hence

$$-kt = \ln\left(\frac{1}{10}\right) \quad \Rightarrow \quad t = \ln\left(\frac{1}{10}\right) \div (-k)$$

From before we have that $-k = \frac{\ln\left(\frac{1}{7}\right)}{5}$ and hence

$$t = \frac{\ln\left(\frac{1}{10}\right)}{1} \div \frac{\ln\left(\frac{1}{7}\right)}{5} = \frac{\ln\left(\frac{1}{10}\right)}{1} \times \frac{5}{\ln\left(\frac{1}{7}\right)}$$

That is,

$$t = \frac{5\ln\left(\frac{1}{10}\right)}{\ln\left(\frac{1}{7}\right)}$$

A calculator will tell us that this is about 6 years, so roughly the year 2016.

Exercises in Chapter 9

9.1 We need to write them in standard form and then factorize:

(i)

$$(y-2)^2 = -9y \quad \Rightarrow \quad y^2 - 4y + 4 = -9y$$
$$\Rightarrow \quad y^2 + 5y + 4 = 0$$
$$\Rightarrow \quad (y+1)(y+4) = 0 \quad \Rightarrow \quad y = -1 \text{ or } -4$$

(ii)

$$2(x-3)^2 = 6 - 5x + x^2$$
$$\Rightarrow \quad 2x^2 - 12x + 18 = 6 - 5x + x^2$$
$$\Rightarrow \quad x^2 - 7x + 12 = 0 \quad \Rightarrow \quad (x-3)(x-4) \quad \Rightarrow \quad x = 3 \text{ or } 4$$

(iii)

$$(2x+2)^2 - (x-5)^2 = 2x^2 + 19$$
$$\Rightarrow \quad 4x^2 + 8x + 4 - (x^2 - 10x + 25) = 2x^2 + 19$$
$$\Rightarrow \quad x^2 + 18x - 40 = 0 \quad \Rightarrow \quad (x-2)(x+20) \quad \Rightarrow \quad x = 2 \text{ or } -20$$

9.2 The quadratic formula is

$$ax^2 + bx + c = a(x-A)(x-B) \;\Rightarrow\; A, B = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

(i) $x^2 + 2x - 4 = 0$. We have

$$a = 1,\ b = 2,\ c = -4 \;\Rightarrow\; b^2 - 4ac = 2^2 - 4 \times 1 \times (-4) = 4 + 16 = 20$$

and hence

$$A, B = \frac{-2 \pm \sqrt{20}}{2} = \frac{-2 + 2\sqrt{5}}{2} = \frac{-2}{2} \pm \frac{2\sqrt{5}}{2} = -1 \pm \sqrt{5}$$

(ii) $11x^2 - 10x - 1 = 0$. We have

$$a = 11,\ b = 10,\ c = -1 \;\Rightarrow\; b^2 - 4ac = 10^2 - 4 \times 11 \times (-1) = 144$$

and hence

$$A, B = \frac{10 \pm \sqrt{144}}{22} = \frac{10 \pm 12}{22} = \frac{22}{22}, \frac{-2}{22} = 1, -\frac{1}{11}$$

(iii) $(x+2)(x+3) = -x^2 - 2x + 1$. We first need to put the equation into standard form. We have

$$x^2 + 5x + 6 = -x^2 - 2x + 1 \qquad \Rightarrow \qquad 2x^2 + 7x + 5 = 0$$

We have

$$a = 2,\ b = 7,\ c = 5 \;\Rightarrow\; b^2 - 4ac = 49 - 4 \times 2 \times 5 = 49 - 40 = 9$$

and hence

$$A, B = \frac{-7 \pm \sqrt{9}}{4} = \frac{-7 \pm 3}{4} = \frac{-7+3}{4}, \frac{-7-3}{4} = -1, -\frac{10}{4}$$

9.3 The discriminant is

$$\Delta = b^2 - 4ac$$

(i) $x^2 - 4x + 2 = 0$. We have

$\Delta = b^2 - 4ac = (-4)^2 - 4 \cdot 1 \cdot 2 = 16 - 8 = 8 > 0 \Rightarrow$ two solutions

(ii) $x^2 + 2x + 7 = 0$. We have

$\Delta = b^2 - 4ac = 2^2 - 4 \cdot 1 \cdot 7 = 4 - 28 = -24 < 0 \Rightarrow$ no solutions

(iii) $4x^2 + 4x + 1 = 0$. We have

$$\Delta = b^2 - 4ac = 4^2 - 4 \cdot 4 \cdot 1 = 16 - 16 = 0 \Rightarrow \text{one solution}$$

9.4 Completing the square gives

(i) $x^2 + 6x - 4$. Half of 6 is 3 and the square of this is 9 so write

$$x^2 + 6x - 4 = (x^2 + 6x + 9) + (-4 - 9) = (x + 3)^2 - 13$$

(ii) $x^2 - 8x + 20$. Half of -8 is -4 and the square of this is 16 so write

$$x^2 - 8x + 20 = (x^2 - 8x + 16) + (20 - 16) = (x - 4)^2 + 4$$

(iii) $x^2 + 18x - 12$. Half of 18 is 9 and the square of this is 81 so write

$$x^2 + 18x - 12 = (x^2 + 18x + 81) + (-12 - 81) = (x + 9)^2 - 93$$

(iv) $x^2 - 10x$. Half of -10 is -5 and the square of this is 25 so write

$$x^2 - 10x = (x^2 - 10x + 25) + (-25) = (x - 5)^2 - 25$$

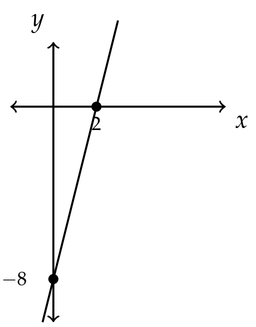
Exercises in Chapter 10

10.1 We need to find the y-intercept and the x-intercept.

(i) $y = 4x - 8$. The y-intercept is $c = -8$ and to get the x-intercept solve

$$0 = 4x - 8 \quad \Rightarrow \quad 4x = 8 \quad \Rightarrow \quad x = 2$$

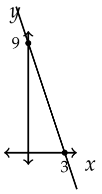
and hence the graph is



(*ii*) $y = 9 - 3x$. The y-intercept is $c = 9$ and to get the x-intercept solve

$$0 = 9 - 3x \quad \Rightarrow \quad 3x = 9 \quad \Rightarrow \quad x = 3$$

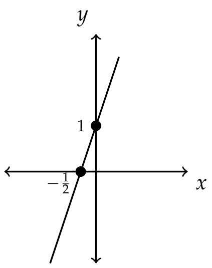
and hence the graph is



(*iii*) $y = 2x + 1$. The y-intercept is $c = 1$ and to get the x-intercept solve

$$0 = 2x + 1 \quad \Rightarrow \quad 2x = -1 \quad \Rightarrow \quad x = -\frac{1}{2}$$

and hence the graph is



10.2 Horizontal lines have equation $y = c$ (that is, $m = 0$). Now, the y-coordinate of the point $(4, -5)$ is $c = -5$ so the equation of the line must be

$$y = -5$$

10.3 We are told that the y-intercept is 6 and hence $c = 6$. Observe that the given line ($y = 2x + 5$) has gradient 2. As the desired line is perpendicular to this line then it must have gradient

$$m = -\frac{1}{2}$$

The general equation of a line is $y = mx + c$

and hence the equation of the desired line is

$$y = -\frac{1}{2}x + 6$$

10.4 To find the equation of the line containing the points $(-2, -4)$ and $(3, 1)$ we can use either of Method 1 or Method 2.

Method 1: Substituting the two co-ordinates into the equation of a line $y = mx + c$ gives us a pair of simultaneous equations to be solved for m and c:

$$\begin{array}{rcrcrcrl} (-2,-4) & \Rightarrow & -4 & = & m \times (-2) & + & c & (1) \\ (3,1) & \Rightarrow & 1 & = & m \times 3 & + & c & (2) \\ \hline & & -5 & = & -5m & & & \end{array}$$

$$\text{So} \quad m = \frac{-5}{-5} = 1$$

Substituting $m = 1$ into say equation (2) gives

$$1 = 3 + c$$
$$\Rightarrow \quad c = -2$$

Hence the equation is

$$y = x - 2$$

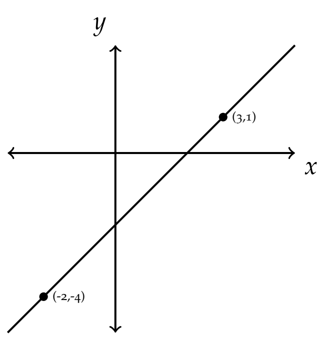
Method 2: The points are $(-2, -4)$ and $(3, 1)$. Use the definition of gradient to find m:

$$m = \frac{\text{change in } y \text{ value}}{\text{change in } x \text{ value}} = \frac{1 - (-4)}{3 - (-2)} = \frac{1+4}{3+2} = \frac{5}{5} = 1$$

So the equation (so far) is $y = x + c$. To find c, substitute one of the points, say $(3, 1)$ into this equation and solve for c:

$$1 = 3 + c \quad \Rightarrow \quad c = -2$$

Hence, the equation of the straight line is $y = x - 2$



Exercises in Chapter 11

11.1 To sketch the graph of $y = x^2 - 2x - 8$ we need

- concave up
- the $y-$intercept is at $(0, -8)$
- to find the $x-$intercepts:

$$\begin{aligned} x^2 - 2x - 8 &= 0 \\ \Rightarrow \quad (x+2)(x-4) &= 0 \end{aligned}$$

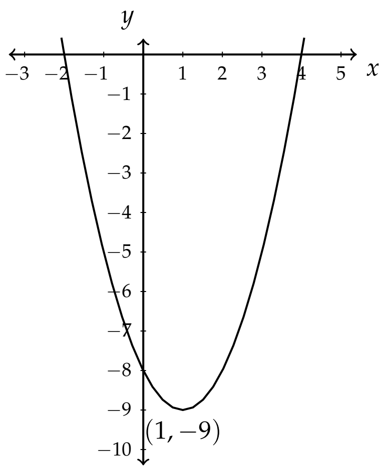
  So, the $x-$intercepts are $(-2, 0)$ and $(4, 0)$.
- the vertex occurs halfway between the $x-$intercepts, at $x = 1$ (or use

$$x = \frac{-b}{2a} = \frac{-(-2)}{2(1)} = 1)$$

  The $y-$value at $x = 1$ is then

$$\begin{aligned} y &= (1)^2 - 2\,(1) - 8 \\ &= 1 - 2 - 8 \\ &= -9 \end{aligned}$$

  The vertex is $(1, -9)$.



Exercises in Chapter 12

12.1 In each case, we need to factorise the numerators

$$(i) \quad \lim_{a\to 2} \frac{a^2 - a - 2}{a - 2} = \lim_{a\to 2} \frac{\cancel{(a-2)}(a+1)}{\cancel{a-2}} = \lim_{a\to 2}(a+1) = 3$$

$$(ii) \quad \lim_{s\to 4} \frac{s^2 - 2s - 8}{s - 4} = \lim_{s\to 4} \frac{\cancel{(s-4)}(s+2)}{\cancel{s-4}} = \lim_{s\to 4}(s+2) = 6$$

$$(iii) \quad \lim_{t\to 6} \frac{t^2 - 5t + 4}{t - 6} = \lim_{t\to 6} \frac{(t-1)(t-4)}{t - 6} \quad \text{does not exist}$$

$$(iv) \quad \lim_{z\to 3} \frac{4z^2 - 11z - 3}{z - 3} = \lim_{z\to 3} \frac{\cancel{(z-3)}(4z+1)}{\cancel{z-3}} = \lim_{z\to 3}(4z+1) = 13$$

12.2 Recall that

$$\frac{d}{dx}(x^n) = nx^{n-1} \qquad \frac{d}{dx}(x) = 1 \qquad \frac{d}{dx}(1) = 0$$

(i) $f(x) = 2x^3 + 4x^2 + 3x + 5$. Then

$$f'(x) = 2(3x^2) + 4(2x^1) + 3(1) + 5(0) = 6x^2 + 8x + 3$$

(ii) $g(t) = 2t(t^2 + t + 1) = 2t^3 + 2t^2 + 2t$. Then

$$g'(t) = 6t^2 + 4t + 2$$

(iii) $z = \frac{3x^4}{4} + \frac{1}{x} + \sqrt{x} + 4x$. We have

$$z = \frac{3}{4}x^4 + x^{-1} + x^{\frac{1}{2}} + 4x$$

$$\Rightarrow \frac{dz}{dx} = \frac{3}{\cancel{4}}(\cancel{4}x^3) - x^{-2} + \frac{1}{2}x^{-\frac{1}{2}} + 4$$

$$= 3x^3 - \frac{1}{x^2} + \frac{1}{2\sqrt{x}} + 4$$

12.3 Recall that Recall that

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \qquad n \neq -1$$

(i) If $f(x) = 5x^2 + 3x + 4$ then

$$\int f(x)dx = \frac{5x^3}{3} + \frac{3x^2}{2} + 4x + C$$

(ii) If $g(x) = \frac{3x^4}{4} + \frac{1}{x^2} + \sqrt{x}$ then we have

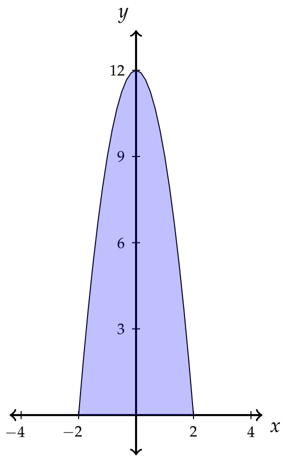
$$g(x) = \frac{3}{4}x^4 + x^{-2} + x^{\frac{1}{2}}$$

and hence

$$\int g(x)dx = \frac{3}{4}\left(\frac{x^5}{5}\right) + \frac{x^{-1}}{-1} + \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + C = \frac{3x^5}{20} - \frac{1}{x} + \frac{2}{3}x^{\frac{3}{2}} + C$$

12.4 We have to integrate $y = f(x) = 12 - 3x^2$, but from where to where? We need to know its x-intercepts. That is,

$$12 - 3x^2 = 0 \quad \Rightarrow \quad 3x^2 = 12 \quad \Rightarrow \quad x^2 = 4 \quad \Rightarrow \quad x = \pm 2$$



We have

$$F(x) = \int f(x)dx = 12x - x^3 + C$$

Hence the required area is

$$\text{Area} = \Big[F(x)\Big]_{-2}^{2} = \left(12(2) - 2^3 + \cancel{C}\right) - \left(12(-2) - (-2)^3 + \cancel{C}\right)$$

$$= (24 - 8) - (-24 + 8) = 16 - (-16) = 16 + 16 = 32$$

Appendix 2: Solutions to problem sets

Problem Set 1

1.1 Evaluate the following.

(a)
$$\begin{aligned} & 4+4\times 4\div 4-4 \\ = \; & 4+16\div 4-4 \\ = \; & 4+4-4 \\ = \; & 4 \end{aligned}$$

(b)
$$\begin{aligned} & 4+4\times(4\div 4-4) \\ = \; & 4+4\times(1-4) \\ = \; & 4+4\times(-3) \\ = \; & 4-12 \\ = \; & -8 \end{aligned}$$

(c)
$$\begin{aligned} & 2\times 4-(8-1) \\ = \; & 2\times 4-7 \\ = \; & 8-7 \\ = \; & 1 \end{aligned}$$

(d)
$$\begin{aligned} & -14\div 2-(-20)\div(-5) \\ = \; & -7-(4) \\ = \; & -11 \end{aligned}$$

(e)
$$\begin{aligned} & (-4\times 12)-[32-(-4)] \\ = \; & (-48)-[36] \\ = \; & -84 \end{aligned}$$

(f)
$$\begin{aligned} & [4\times(3+4)-21]\div[2\times 14\div 7+3] \\ = \; & [4\times 7-21]\div[28\div 7+3] \\ = \; & [28-21]\div[4+3] \\ = \; & 7\div 7 \\ = \; & 1 \end{aligned}$$

(g)
$$\begin{aligned} & 3\times 8-[-16-(-4)] \\ = \; & 24-[-16+4] \\ = \; & 24-[-12] \\ = \; & 24+12 \\ = \; & 36 \end{aligned}$$

(h)
$$\begin{aligned} & [-3\times(3-4)-19]\div[8\div 4\times 7-6] \\ = \; & [(-3)\times(-1)-19]\div[2\times 7-6] \\ = \; & [3-19]\div[14-6] \\ = \; & -16\div 8 \\ = \; & -2 \end{aligned}$$

(i)
$$\begin{aligned} & (8-4)\div(4+4)-8 \\ & 4\div 8-8 \\ = \; & \frac{1}{2}-8 \\ = \; & \frac{1}{2}-\frac{16}{2} \\ = \; & -\frac{15}{2} \end{aligned}$$

(j)
$$\begin{aligned} & 4+4\times 4\div(4-4) \\ & 4+4\times 4\div 0 \end{aligned}$$

but dividion by zero is not allowed

1.2 Compute the following, leaving your answer in the simplest fraction form.

(a) $\frac{1}{2}+\frac{3}{4}+\frac{5}{12}$ (2 and 4 divide into 12)

$$\begin{aligned} &= \frac{1\times 6}{2\times 6}+\frac{3\times 3}{4\times 3}+\frac{5}{12} \\ &= \frac{6}{12}+\frac{9}{12}+\frac{5}{12} \\ &= \frac{6+9+5}{12} \\ &= \frac{20}{12} \\ &= \frac{5\times \not{4}}{3\times \not{4}} \\ &= \frac{5}{3} \end{aligned}$$

(b) $\frac{3}{4}-\frac{4}{5}+\frac{3}{10}$ (4, 5 and 10 divide into 20)

$$\begin{aligned} &= \frac{3\times 5}{4\times 5}-\frac{4\times 4}{5\times 4}+\frac{3\times 2}{10\times 2} \\ &= \frac{15-16+6}{20} \\ &= \frac{5}{20} \\ &= \frac{\not{5}\times 1}{\not{5}\times 4} \\ &= \frac{1}{4} \end{aligned}$$

(c) $\frac{4}{24}+1\frac{2}{3}$

$$\begin{aligned} &= \frac{1\times \not{4}}{6\times \not{4}}+\frac{5}{3} \\ &= \frac{1}{6}+\frac{5\times 2}{3\times 2} \\ &= \frac{1+10}{6} \\ &= \frac{11}{6} \end{aligned}$$

(d) $1\frac{2}{3}-2\frac{3}{5}+1\frac{4}{5}$

$$\begin{aligned} &= \frac{5}{3}-\frac{13}{5}+\frac{9}{5} \\ &= \frac{25-39+27}{15} \\ &= \frac{13}{15} \end{aligned}$$

(e) $\frac{4}{5}\times\frac{3}{7}$

$$\begin{aligned} &= \frac{4\times 3}{5\times 7} \\ &= \frac{12}{35} \end{aligned}$$

(f) $2\frac{2}{3}\div 4\frac{2}{9}$

$$\begin{aligned} &= \frac{8}{3}\div\frac{38}{9} \\ &= \frac{8}{3}\times\frac{9}{38} \\ &= \frac{4\times \not{2}}{1\times \not{3}}\times\frac{\not{3}\times 3}{\not{2}\times 19} \\ &= \frac{4\times 3}{1\times 19} \\ &= \frac{12}{19} \end{aligned}$$

(g) $\frac{7}{9}\times 1\frac{1}{8}\div\frac{5}{7}$

$$\begin{aligned} &= \frac{7}{\not{9}}\times\frac{\not{9}}{8}\times\frac{7}{5} \\ &= \frac{7\times 1\times 7}{1\times 8\times 5} \\ &= \frac{49}{40} \end{aligned}$$

(h) $2\frac{3}{7}\div\frac{34}{14}$

$$\begin{aligned} &= \frac{17}{7}\times\frac{14}{34} \\ &= \frac{\not{17}}{\not{7}}\times\frac{\not{7}\times 2}{\not{17}\times 2} \\ &= \frac{2}{2} \\ &= 1 \end{aligned}$$

1.3 What is half of the difference between a third of 60 and a quarter of 80?

$$\frac{1}{2} \times \left(\frac{1}{3} \times 60 - \frac{1}{4} \times 80 \right) = \frac{1}{2}(20 - 20) = 0$$

1.4 What is the sum of half of 45 and a third of the product of two thirds and $1\frac{1}{8}$?

$$\begin{aligned} & \frac{1}{2} \times 45 + \frac{1}{3} \times \frac{2}{3} \times 1\frac{1}{8} \\ = \; & \frac{1}{2} \times \frac{45}{1} + \frac{1}{3} \times \frac{2}{3} \times \frac{9}{8} \\ = \; & \frac{45}{2} + \frac{1}{4} = \frac{91}{4} \end{aligned}$$

1.5 James and his wife Sweet Li have a 12 year old daughter called Lyn. James is 50 year old and Lyn is 12. Half of his age added to five thirds of Lyn's age gives the age of Sweet Li. How old is Sweet Li?

$$\frac{1}{2} \times 50 + \frac{5}{3} \times 12 = 25 + 20 = 45, \text{ so Sweet Li is } 45$$

1.6 Swee Khum put a third of her savings in the bank, a third in bonds, a quarter of the remainder in stocks and the rest in fixed deposit. If her total amount is \$600,000 how much did she put in fixed deposit?

After Swee Khum put a third in the bank and a third in bonds, she is left with one third of her savings, which is \$200,000. She then puts a quarter in stocks, so she is left with $\frac{3}{4} \times \$200,000 = \$150,000$, *so she puts \$150,000 in fixed deposit.*

Problem Set 2

2.1 Solve the following equations for the unknown.

(a) $x + 3 = 5$

$$\begin{aligned} x + 3 &= 5 \\ \Rightarrow \quad x + 3 - 3 &= 5 - 3 \\ \Rightarrow \quad x &= 2 \end{aligned}$$

(b) $4y - 5 = y + 10$

$$\begin{aligned} 4y - 5 &= y + 10 \\ \Rightarrow \quad 4y - 5 + 5 &= y + 10 + 5 \\ \Rightarrow \quad 4y &= y + 15 \\ \Rightarrow \quad 4y - y &= y + 15 - y \\ \Rightarrow \quad 3y &= 15 \\ \Rightarrow \quad \frac{3y}{3} &= \frac{15}{3} \\ \Rightarrow \quad y &= 5 \end{aligned}$$

(c) $8x - 4 = 16$

$$\begin{aligned} 8x - 4 &= 16 \\ \Rightarrow \quad 8x - 4 + 4 &= 16 + 4 \\ \Rightarrow \quad \frac{8x}{8} &= \frac{20}{8} \\ \Rightarrow \quad x &= \frac{20}{8} \\ \Rightarrow \quad &= \frac{5}{2} \end{aligned}$$

(d) $\dfrac{2z}{z+2} = 4$

First we get z out of the bottom line by multiplying both sides by $z + 2$.

$$\begin{aligned} \Rightarrow \quad \frac{z+2}{1}\frac{2z}{z+2} &= 4(z+2) \\ \Rightarrow \quad 2z &= 4z + 8 \\ \Rightarrow \quad 2z - 2z &= 4z + 8 - 2z \\ \Rightarrow \quad 0 &= 2z + 8 \\ \Rightarrow \quad 0 - 8 &= 2z + 8 - 8 \\ \Rightarrow \quad -8 &= 2z \\ \Rightarrow \quad -4 &= z \end{aligned}$$

(e) $3-2x=4$

$$\begin{aligned} 3-2x &= 4 \\ \Rightarrow \quad -3+3-2x &= 4-3 \\ \Rightarrow \quad -2x &= 1 \\ \Rightarrow \quad \frac{-2x}{-2} &= \frac{1}{-2} \\ \Rightarrow \quad x &= -\frac{1}{2} \end{aligned}$$

(f) $7x+7=2(x+1)$

$$\begin{aligned} 7x+7 &= 2(x+1) \\ \Rightarrow \quad 7x+7 &= 2x+2 \\ \Rightarrow \quad 7x+7-7-2x &= 2x+2-7-2x \\ \Rightarrow \quad 5x &= -5 \\ \Rightarrow \quad x &= -1 \end{aligned}$$

(g) $x-\frac{1}{2}=2$

$$\begin{aligned} x-\frac{1}{2} &= 2 \\ \Rightarrow \quad x &= 2\frac{1}{2} \\ &= \frac{5}{2} \end{aligned}$$

(h) $\frac{2y-3}{4}=\frac{6y+7}{3}$

First multiply both sides by 3 and by 4 to remove the fractions.

$$\begin{aligned} \Rightarrow \quad 3\times \not{4}\times\frac{2y-3}{\not{4}} &= \not{3}\times 4\times\frac{6y+7}{\not{3}} \\ \Rightarrow \quad 3\,(2y-3) &= 4\,(6y+7) \\ \Rightarrow \quad 6y-9 &= 24y+28 \\ \Rightarrow \quad 6y-9+9-24y &= 24y+28+9-24y \\ \Rightarrow \quad -18y &= 37 \\ \Rightarrow \quad y &= -\frac{37}{18} \end{aligned}$$

(i) $t=2-2[2t-3(1-t)]$

First we simplify the bracket term.

$$\begin{aligned} t &= 2-2[2t-3+3t] \\ &= 2-2[5t-3] \\ &= 2-10t+6 \\ &= 8-10t \\ \Rightarrow \quad t+10t &= 8-10t+10t \\ \Rightarrow \quad 11t &= 8 \\ \Rightarrow \quad t &= \frac{8}{11} \end{aligned}$$

(j) $\frac{3}{2}(4a-3) = 2[a-(4a-3)]$

First we simplify the bracket term on the right-hand side, and them multiply both sides by 2 to remove the denominator on the left-hand side.

$$
\begin{aligned}
\Rightarrow \quad \frac{3}{2}(4a-3) &= 2[a-(4a-3)] \\
&= 2[a-4a+3] \\
&= 2[3-3a] \\
&= 6-6a \\
\Rightarrow \quad \cancel{2} \times \frac{3}{\cancel{2}}(4a-3) &= 2 \times (6-6a) \\
\Rightarrow \quad 3(4a-3) &= 12-12a \\
\Rightarrow \quad 12a-9 &= 12-12a \\
\Rightarrow \quad 12a-9+9+12a &= 12-12a+9+12a \\
\Rightarrow \quad 24a &= 21 \\
\Rightarrow \quad a &= \frac{21}{24} \\
&= \frac{7}{8}
\end{aligned}
$$

(k) $\frac{x+3}{x} = \frac{2}{5}$

Cross-multiply to get

$$
\begin{aligned}
\Rightarrow \quad 5(x+3) &= 2x \\
\Rightarrow \quad 5x+15 &= 2x \\
\Rightarrow \quad 3x &= -15 \\
\Rightarrow \quad x &= -5.
\end{aligned}
$$

(l) $\dfrac{1}{p-1} = \dfrac{2}{p-2}$
Cross-multiply to get

$$\begin{aligned} \Rightarrow \quad p-2 &= 2(p-1) \\ \Rightarrow \quad p-2 &= 2p-2 \\ \Rightarrow \quad p-2+2-p &= 2p-2+2-p \\ \Rightarrow \quad 0 &= p \end{aligned}$$

(m) $\dfrac{x}{2} - \dfrac{x}{5} = 1$
Multiply throught by 2 *and by* 5 *to get*

$$\begin{aligned} \frac{x}{2} - \frac{x}{5} &= 1 \\ \Rightarrow \quad \not{2} \times 5 \times \frac{x}{\not{2}} - 2 \times \not{5} \times \frac{x}{\not{5}} &= 2 \times 5 \times 1 \\ \Rightarrow \quad 5x - 2x &= 10 \\ \Rightarrow \quad 3x &= 10 \\ \Rightarrow \quad x &= \frac{10}{3} \end{aligned}$$

(n) $\dfrac{y-4}{2} + \dfrac{y}{3} = 4$
Multiply through by 2 *and by* 3 *to get*

$$\begin{aligned} \Rightarrow \quad \not{2} \times 3 \times \frac{y-4}{\not{2}} + 2 \times \not{3} \times \frac{y}{\not{3}} &= 2 \times 3 \times 4 \\ \Rightarrow \quad 3(y-4) + 2y &= 24 \\ \Rightarrow \quad 3y - 12 + 2y &= 24 \\ \Rightarrow \quad 5y &= 36 \\ \Rightarrow \quad y &= \frac{36}{5} \end{aligned}$$

(o) $\dfrac{2}{y} + 4 = \dfrac{3}{y}$
Multiply both sides by y *to get*

$$\begin{aligned} \Rightarrow \quad y\left(\frac{2}{y} + 4\right) &= \not{y} \times \frac{3}{\not{y}} \\ \Rightarrow \quad \not{y} \times \frac{2}{\not{y}} + y \times 4 &= 3 \\ \Rightarrow \quad 2 + 4y &= 3 \\ \Rightarrow \quad 4y &= 1 \\ \Rightarrow \quad y &= \frac{1}{4} \end{aligned}$$

(p) $\dfrac{2-y}{3} - \dfrac{1-y}{2} = y$
Multiply through by 2 *and by* 3 *to get*

$$\begin{aligned}
\Rightarrow \quad 2 \times \not{3} \times \frac{2-y}{\not{3}} - \not{2} \times 3 \times \frac{1-y}{\not{2}} &= 2 \times 3 \times y \\
\Rightarrow \quad 2(2-y) - 3(1-y) &= 6y \\
\Rightarrow \quad 4 - 2y - 3 + 3y &= 6y \\
\Rightarrow \quad 1 + y &= 6y \\
\Rightarrow \quad 1 &= 5y \\
\text{so } \; y &= \frac{1}{5}
\end{aligned}$$

(q) $y - \frac{1-y}{2} + 4 = 7$

First transfer all the numbers to the right-hand side, and then multiply both sides by 2.

$$\begin{aligned}
\Rightarrow \quad y - \frac{1-y}{2} &= 3 \\
\Rightarrow \quad 2y - (1-y) &= 6 \quad \text{(don't forget the brackets here!)} \\
\Rightarrow \quad 2y - 1 + y &= 6 \\
\Rightarrow \quad 3y &= 7 \\
\Rightarrow \quad y &= \frac{7}{3}
\end{aligned}$$

(r) $\frac{y}{3} - (3-y) = 4 - y$

Multiplying both sides by 3 gives

$$\begin{aligned}
y - 3(3-y) &= 3(4-y) \\
\Rightarrow \quad y - 9 + 3y &= 12 - 3y \\
\Rightarrow \quad 4y &= 21 - 3y \\
\Rightarrow \quad 7y &= 21 \\
\Rightarrow \quad y &= 3
\end{aligned}$$

(s) $\frac{d-1}{2} - \frac{d}{3} = 0$

Multiply through by 2 and by 3 to get

$$\begin{aligned}
\not{2} \times 3 \times \frac{d-1}{\not{2}} - 2 \times \not{3} \times \frac{d}{\not{3}} &= 2 \times 3 \times 0 \\
\Rightarrow \quad 3(d-1) - 2d &= 0 \\
\Rightarrow \quad 3d - 3 - 2d &= 0 \\
\Rightarrow \quad d &= 3
\end{aligned}$$

(t) $\dfrac{4-p}{3} = -8$

Multiply both sides by 3 to get

$$\begin{aligned} 4 - p &= -24 \\ \Rightarrow \quad -p &= -28 \\ \Rightarrow \quad \frac{-p}{-1} &= \frac{-28}{-1} \\ \Rightarrow \quad p &= 28 \end{aligned}$$

(u) $\dfrac{d-1}{2} - \dfrac{d}{3} = 0$

Multiply through by 2 and by 3 to get

$$\begin{aligned} \not{2} \times 3 \times \frac{d-1}{\not{2}} - 2 \times \not{3} \times \frac{d}{\not{3}} &= 2 \times 3 \times 0 \\ \Rightarrow \quad 3(d-1) - 2d &= 0 \\ \Rightarrow \quad 3d - 3 - 2d &= 0 \\ \Rightarrow \quad d &= 3 \end{aligned}$$

(v) $\dfrac{4-p}{3} = -8$

Multiply both sides by 3 to get

$$\begin{aligned} 4 - p &= -24 \\ \Rightarrow \quad -p &= -28 \\ \Rightarrow \quad \frac{-p}{-1} &= \frac{-28}{-1} \\ \Rightarrow \quad p &= 28 \end{aligned}$$

2.2 If $2\frac{1}{4}$ cakes are served in slices of $\frac{1}{8}$ of a cake, how many slices are served?

$$\begin{aligned} & 2\frac{1}{4} \div \frac{1}{8} \\ = & \frac{9}{4} \times \frac{8}{1} \\ = & \frac{9}{\not{4}} \times \frac{\not{4} \times 2}{1} \\ = & 18 \text{ slices} \end{aligned}$$

2.3 Lian earns \$2,000 a fortnight and spends \$1200 of that. What proportion of his salary does he save?

Lian saves \$2,000 - \$1,200 = \$800, which is

$$\begin{aligned} & \frac{800}{2000} \\ = & \frac{8}{20} \\ = & \frac{2}{5} \text{ which is } \tfrac{2}{5} \text{ of his salary.} \end{aligned}$$

2.4 Nadia buys a packet of 60 samosas for a tea party. She takes out half for her family. She takes the remaining to the party. On the way she meets a friend who takes a sixth of the samosas. A little further on she sees a hungry cat and gives it a fifth of the samosas she has. The she suddenly feels hungry and eats a quarter of the remaining samosas. How many does she arrive with at the tea party?

Nadia starts with half the samosas for her party, so she takes $\frac{1}{2} \times 60 = 30$ *samosas. Her friend takes* $\frac{1}{6}$ *of these, so she is left with* $\frac{5}{6} \times 30 = 25$ *samosas. She gives a cat* $\frac{1}{5}$ *of these, so she is left with* $\frac{4}{5} \times 25 = 20$ *samosas. She eats* $\frac{1}{4}$ *of these, so she is left with* $\frac{3}{4} \times 20 = 15$ *samosas. She arrives at the party with 15 samosas.*

2.5 I have a naan recipe that calls for $4\frac{2}{3}$ cups of flour for 14 servings of naan. I want to make one serving only. How much flour should I use?

$$\begin{aligned} & 4\frac{2}{3} \div 14 \\ = & \frac{14}{3} \times \frac{1}{14} \\ = & \frac{1}{3} \end{aligned}$$

so I need $\frac{1}{3}$ *cup of flour.*

2.6 Suppose Mike travels a certain distance on the first day and twice the distance on the next day. If the total distance he travelled is 60 km, how far does he travel on the first day?

Let x denote the distance travelled on the first day. The distance travelled on the second day is $2x$*. Since the total distance travelled is* 60km*, we must have*

$$x + 2x = 60 \quad \Rightarrow \quad 3x = 60 \quad \Rightarrow \quad x = 20$$

so Mike travels 20 *km on the first day.*

2.7 Joy, Pam, Sandra and Lilin each make a donation to the Guide Dogs Association. Sandra gives twice as much as Lilin, Pam gives three times as much as Sandra and Joy gives four times as much as Pam. If their total gift is $132, find the amount of Lilin's donation, and hence the amount donated by each.

Let the amount given by Lilin be x. Then Sandra gives $2x$*, Pam gives* $3 \times 2x = 6x$ *and Joy gives* $4 \times 6x = 24x$*. Total donation was $132, so*

$$\begin{aligned} x + 2x + 6x + 24x &= 132 \\ \Rightarrow \quad 33x &= 132 \\ \Rightarrow \quad x &= \frac{132}{33} \\ \Rightarrow \quad x &= \frac{\cancel{11} \times 12}{\cancel{11} \times 3} \quad \textit{(via 11 times table)} \\ \Rightarrow \quad x &= \frac{12}{3} \\ &= \frac{\cancel{3} \times 4}{\cancel{3} \times 1} = 4 \end{aligned}$$

Thus Lilin gave $4, Sandra gave $8, Pam gave $24 and Joy gave $96. The total of these amounts is $132 as required.

Problem Set 3

Please note that these solutions present only one option for solving these problems. You may have taken different steps. The important thing is that you get the right answers at the end.

3.1 Solve the following sets of simultaneous equations.

(a)

$$\begin{array}{rrcrcr} & 2x & + & y & = & 8 & \quad (1)\\ (+) & 3x & - & y & = & 7 & \quad (2)\\ \hline & 5x & & & = & 15 & \end{array}$$

$$\Rightarrow \quad x = \frac{15}{5} = 3$$

Substitute this back into either of the equations, for example, Eq. (1) to get

$$2 \times 3 + y = 8$$
$$\Rightarrow \quad 6 + y = 8$$
$$\Rightarrow \quad y = 8 - 6 = 2$$

We can check this solution by substituting these values of x and y into the LHS of the other equation (2):

$$3 \times 3 - 2 = 7 = \text{ RHS}$$

so the solution is correct: $x = 3, y = 2$.

(b)

$$\begin{array}{rcrcl} 2x & + & 3y & = & 19 \quad (1)\\ 4x & - & y & = & 3 \quad (2) \end{array}$$

Multiply Eq. (1) by 2 to get

$$\begin{array}{rrcrcr} & 4x & + & 6y & = & 38 & \quad (3)\\ (-) & 4x & - & y & = & 3 & \quad (2)\\ \hline & & & 7y^{\star} & = & 35 & \end{array}$$

$$\Rightarrow \quad y = \frac{35}{7} = 5$$

⋆ Note that $6y - (-y) = 7y$.

Substitute this back into either of the original equations, for example, Eq. (1) to get

$$2x + 3 \times 5 = 19$$
$$\Rightarrow \quad 2x + 15 = 19$$
$$\Rightarrow \quad 2x = 19 - 15$$
$$\Rightarrow \quad 2x = 4 \quad \Rightarrow \quad x = 2$$

We can check this solution by substituting these values of x and y into the LHS of the other equation (2):

$$4 \times 2 - 5 = 3 = \text{ RHS}$$

so the solution is correct: $x = 2, y = 5$

(c)

$$\begin{array}{rrcrcr} & x & + & 2y & = & 10 \\ (-) & x & - & y & = & 1 \\ \hline & & & 3y^\star & = & 9 \end{array} \qquad \begin{array}{c}(1)\\(2)\\ \\ \end{array}$$

$$\Rightarrow \quad y = 3$$

* Note that $2y - (-y) = 3y$.

Substitute this back into either of the original equations, for example, Eq. (2) to get

$$x - 3 = 1 \quad \Rightarrow \quad x = 4$$

We can check this solution by substituting these values of x and y into the LHS of the other equation (1):

$$4 + 2 \times 3 = 10 = \text{ RHS}$$

so the solution is correct: $x = 4, y = 3$

(d)

$$\begin{array}{rrcrcr} & 4x & - & y & = & 0 \\ (-) & 2x & - & y & = & -2 \\ \hline & 2x & & & = & 2^\star \end{array} \qquad \begin{array}{c}(1)\\(2)\\ \\ \end{array}$$

$$\Rightarrow \quad x = 1$$

* Note that $-y - (-y) = 0$ and $0 - (-2) = 2$.

Substitute this back into either of the original equations, for example, Eq. (1) to get

$$4 \times 1 - y = 0 \quad \Rightarrow \quad 4 = y$$

We can check this solution by substituting these values of x and y into the LHS of the other equation (2):

$$2 \times 1 - 4 = -2 = \text{ RHS}$$

so the solution is correct: $x = 1, y = 4$

(e)

$$\begin{array}{rcrcr} 2x & + & y & = & 4 \\ 5x & + & 2y & = & 9 \end{array} \qquad \begin{array}{c}(1)\\(2)\end{array}$$

Multiply Eq. (1) by 2 to get

$$\begin{array}{rrcrcr} & 4x & + & 2y & = & 8 \\ (-) & 5x & + & 2y & = & 9 \\ \hline & -x & & & = & -1 \end{array} \qquad \begin{array}{c}(3)\\(2)\\ \\ \end{array}$$

$$\Rightarrow \quad x = 1$$

Substitute this back into either of the original equations, for example, Eq. (1) to get

$$2 \times 1 + y = 4 \quad \Rightarrow \quad y = 2$$

We can check this solution by substituting these values of x and y into the LHS of the other equation (2):

$$5 \times 1 + 2 \times 2 = 9 = \text{ RHS}$$

so the solution is correct: $x = 1, y = 2$

(f)

$$\begin{array}{rcrcr} x & + & 3y & = & 3 \\ -2x & + & y & = & 8 \end{array} \qquad \begin{array}{c}(1)\\(2)\end{array}$$

Multiply Eq. (1) by 2 to get

$$\begin{array}{rrcrcr} & 2x & + & 6y & = & 6 \\ (+) & -2x & + & y & = & 8 \\ \hline & & & 7y & = & 14 \end{array} \qquad \begin{array}{c}(3)\\(2)\\ \\ \end{array}$$

$$\Rightarrow \quad y = 2$$

Substitute this back into either of the original equations, for example, Eq. (1) to get

$$x + 3 \times 2 = 3 \quad \Rightarrow \quad x = 3 - 6 \quad \Rightarrow \quad x = -3$$

(g)

$$\begin{array}{rcrcrr} 2x & - & 3y & = & 5 & (1) \\ 5x & + & 2y & = & -16 & (2) \end{array}$$

Do 5× Eq. (1) and do 2× Eq. (2) to get

$$\begin{array}{lrcrcrr} & 10x & - & 15y & = & 25 & (3) \\ (-) & 10x & + & 4y & = & -32 & (4) \\ \hline & & & -19y & = & 57 & \\ & & \Rightarrow & y & = & \frac{57}{-19} & \\ & & \Rightarrow & y & = & -\frac{\cancel{19} \times 3}{\cancel{19} \times 1} = -3 & \end{array}$$

Substitute this back into either of the original equations, for example, Eq. (1) to get

$$2x - 3(-3) = 5$$
$$\Rightarrow \quad 2x + 9 = 5$$
$$\Rightarrow \quad 2x = 5 - 9$$
$$\Rightarrow \quad x = \frac{-4}{2} = -2$$

We can check this solution by substituting these values of x and y into the LHS of the other equation (2):

$$5(-2) + 2(-3) = -10 - 6 = -16 = \text{ RHS}$$

so the solution is correct: $x = -2, y = -3$

(h)

$$\begin{array}{lrcrcrr} & x & + & y & = & 5 & (1) \\ (-) & 2x & + & y & = & 7 & (2) \\ \hline & -x & & & = & -2 & \\ & & \Rightarrow & x & = & 2 & \end{array}$$

Substitute this back into either of the original equations, for example, Eq. (1) to get

$$2 + y = 5 \quad \Rightarrow \quad y = 3$$

We can check this solution by substituting these values of x and y into the LHS of the other equation (2):

$$2 \times 2 + 3 = 7 = \text{ RHS}$$

so the solution is correct: $x = 2, y = 3$

(i)

$$\begin{array}{rcrcl} 4x & + & y & = & 10 \quad (1) \\ 2x & + & 3y & = & 10 \quad (2) \end{array}$$

Do $2\times$ Eq. (2) to get

$$\begin{array}{rrcrcr} & 4x & + & y & = & 10 \quad (1) \\ (-) & 4x & + & 6y & = & 20 \quad (3) \\ \hline & & & -5y & = & -10 \\ & & \Rightarrow & y & = & \frac{-10}{-5} \\ & & & & = & 2 \end{array}$$

Substitute this back into either of the original equations, for example, Eq. (1) to get

$$4x + 2 = 10$$
$$\Rightarrow \quad 4x = 10 - 2$$
$$\Rightarrow \quad x = \frac{8}{4}$$
$$= 2$$

We can check this solution by substituting these values of x and y into the LHS of the other equation (2):

$$2 \times 2 + 3 \times 2 = 10 = \text{ RHS}$$

so the solution is correct: $x = 2, y = 2$

(j)* *This one is too difficult for a Test question but if you like fraction arithmetic ....*

$$\begin{array}{rcrcr} 2x & + & 3y & = & 8 \quad (1) \\ 3x & - & 2y & = & -7 \quad (2) \end{array}$$

Multiply Eq. (1) by 3 and Eq. (2) by 2 to get

$$\begin{array}{rrcrcr} & 6x & + & 9y & = & 24 \quad (3) \\ (-) & 6x & - & 4y & = & -14 \quad (4) \\ \hline & & & 13y & = & 38 \\ & & \Rightarrow & y & = & \frac{38}{13} \end{array}$$

Substitute this back into either of the original equations, for example, Eq. (1) to get

$$2x + 3 \times \frac{38}{13} = 8$$
$$\Rightarrow \quad 26x + 3 \times 38 = 13 \times 8$$
$$\Rightarrow \quad 26x + 114 = 104$$
$$\Rightarrow \quad 26x = -10$$
$$\Rightarrow \quad x = \frac{-10}{26}$$
$$= -\frac{\cancel{2} \times 5}{\cancel{2} \times 13}$$
$$= -\frac{5}{13}$$

We can check this solution by substituting these values of x and y into the LHS of the other equation (2):

$$3 \times \left(-\frac{5}{13}\right) - 2 \times \left(\frac{38}{13}\right) = -\frac{15}{13} - \frac{76}{13}$$
$$= -\frac{91}{13}$$
$$= -\frac{\cancel{13} \times 7}{\cancel{13} \times 1}$$
$$= -7 = \text{ RHS}$$

so the solution is correct.

3.2 Tickets for an ice-skating display are sold at \$5 for adults and \$2 for children. If 101 tickets were sold altogether for a take of \$394, find the number of adults and children who attended.

Let a denote the number of adults and c the number of children. We are told that 101 tickets were sold so we must have

$$a + c = 101 \qquad (1)$$

We are also told that

$$5a + 2c = 394 \qquad (2)$$

To solve this set of equations multiply Eq. (1) by 5 to get

$$\begin{array}{ccccc} 5a & + & 5c & = & 505 \qquad (3) \\ 5a & + & 2c & = & 394 \qquad (2) \end{array}$$

Do Eq. (3) - Eq. (2) to get

$$\cancel{5a} - \cancel{5a} + 5c - 2c = 505 - 394 \quad \Rightarrow \quad 3c = 111 \quad \Rightarrow \quad c = \frac{111}{3}$$

It turns out that $111 = 3 \times 37$ so

$$c = \frac{111}{3} = \frac{37 \times \cancel{3}}{1 \times \cancel{3}} \quad \Rightarrow \quad c = \frac{37}{1} \quad \Rightarrow \quad y = 37$$

Substitute this back into either of the original equations, for example, Eq. (1) to get

$$a + c = 101 \quad \Rightarrow \quad a + 37 = 101 \quad \Rightarrow \quad a = 101 - 37 \quad \Rightarrow \quad a = 64$$

That is, 64 adults and 37 children attended the display.

3.3 An island contains foxes and rabbits. An ecologist counts both species to study their interaction and how their populations change over time. Last year she found the total number of foxes and rabbits was 7,290 and the foxes were only 1 eighth of the rabbit population. How many of each species was present?

Let f be the number of foxes and r be the number of rabbits. We are told that $r + f = 7290$. We are also told that the foxes were only 1 eighth of the rabbit population, that is, the rabbit population is eight times as large as the fox population. That is $r = 8f$ which we can write as $r - 8f = 0$. Hence the two simultaneous equations are

$$\begin{array}{ccccc} r & + & f & = & 7290 \qquad (1) \\ r & - & 8f & = & 0 \qquad (2) \end{array}$$

Do Eq. (1) - Eq. (2) to get

$$\cancel{r} - \cancel{r} + f - (-8f) = 7290 - 0 \quad \Rightarrow \quad f + 8f = 7290 \quad \Rightarrow \quad 9f = 7290 \quad \Rightarrow \quad f = \frac{7290}{9} = 810$$

by using a calculator. The rabbit population is eight times this much so

$$r = 8 \times 810 = 6480$$

3.4 The ancient Roman emperor Augustus was fond of gold and silver sovereigns. By royal decree every gold sovereign was to weigh 50 grams and every silver one to weigh 40 grams. One year a new jeweller was hired to prepare the sovereigns. Augustus suspected that the jeweller was cheating him and delivering sub-weight sovereigns. A consignment contained 30 gold and 20 silver sovereigns weighed $2,250$ grams, and a consignment of 15 gold and 25 silver sovereigns weighed $1,550$ grams? Was the jeweller cheating Augustus?

Let g be the weight of a gold sovereign and s the weight of a silver sovereign. Then

$$\begin{array}{rcrcl} 30g & + & 20s & = & 2250 \qquad (1) \\ 15g & + & 25s & = & 1550 \qquad (2) \end{array}$$

Do $2\times$ Eq. (2) to get

$$\begin{array}{rcrcl} 30g & + & 20s & = & 2250 \qquad (1) \\ 30g & + & 50s & = & 3100 \qquad (3) \end{array}$$

Now do Eq. (3) - Eq. (1) to get

$$\cancel{30g} - \cancel{30g} + 50s - 20s = 3100 - 2250 \quad \Rightarrow \quad 30s = 850$$

$$\Rightarrow \quad s = \frac{850}{30} = \frac{85}{3} = 28.333 \text{ grams approx.}$$

but a silver sovereign should weigh 40 grams, so it looks like the jeweller was cheating Augustus. However, we should first check what the situation is with the gold soverigns. Substituting $s = \frac{85}{3}$ into, say Eq. (1) gives

$$30g + 20 \times \frac{85}{3} = 2250 \quad \Rightarrow \quad 3 \times 30g + 20 \times 85 = 3 \times 2250$$

$$\Rightarrow \quad 90g + 1700 = 6750$$

$$\Rightarrow \quad 90g = 5050 \quad \Rightarrow \quad g = \frac{5050}{90} = \frac{505}{9} = 56.111 \text{ grams approx.}$$

so the gold soverigns are actually heavier than required!

Problem Set 4

4.1 Expand the following.

(a) $(x+2)(x+4) = x^2+6x+8$

(b) $(x-1)(x-2) = x^2-3x+2$

(c) $(x-2)(x+2) = x^2-4$

(d) $(x-1)(1-x) = -x^2+2x-1$

(e) $(2x+3)(x+2) = 2x^2+7x+6$

(f) $(x+1)^2 = x^2+2x+1$

(g) $(2x-1)^2 = 4x^2-4x+1$

(h) $(x+2)^2 = x^2+4x+4$

(i) $x(x-1)(x+1) = x^3-x$

(j) $(x-2y)(x+2y) = x^2-4y^2$

(k) $(2x-3y)^2 = 4x^2-12xy+9y^2$

(l) $(x^2-1)(x^2+1) = x^4-1$

(m) $(x^2+x)(x^2-x) = x^4-x^2$

(n) $(x+3y)(x-3y) = x^2-9y^2$

(o) $(x+1)(x+2)(x+3) = x^3+6x^2+11x+6$

(p) $(x-1)(x+1)(x+2) = x^3+2x^2-x-2$

4.2 Factorise the following quadratic expressions.

(a) $x^2+5x+6 = (x+2)(x+3)$

(b) $x^2-5x+6 = (x-2)(x-3)$

(c) $x^2-x-6 = (x+2)(x-3)$

(d) $x^2+x-6 = (x-2)(x+3)$

(e) $x^2-2x-15 = (x-5)(x+3)$

(f) $x^2-100 = (x+10)(x-10)$

4.3 Simplify the following rational expressions.

(a)
$$\begin{aligned}\frac{x^2-9}{x+3} &= \frac{(x-3)\cancel{(x+3)}}{\cancel{x+3}}\\ &= x-3\end{aligned}$$

(b)
$$\begin{aligned}\frac{x^2+5x-14}{x^2+10x+21} &= \frac{\cancel{(x+7)}(x-2)}{\cancel{(x+7)}(x+3)}\\ &= \frac{x-2}{x+3}\end{aligned}$$

Problem Set 5

There are often several different (but equally efficient) ways to solve index problems. Only one version is presented here so don't worry if you get the same answer by a different method.

5.1 *Use index laws to simplify each of the following expressions*

(a) $x^3 \times x^4 \times x = x^{3+4+1} = x^8$

(b) $x^2y^3x = x^{2+1}y^3 = x^3y^3$

(c) $(xy)^2 = x^2y^2$

(d) $\dfrac{a^2b^5}{ab} = \dfrac{a^2}{a} \times \dfrac{b^5}{b} = a^{2-1} \times b^{5-1} = ab^4$

(e) $\dfrac{2x^4y^2}{6x^2y^2} = \dfrac{1 \times \cancel{2}}{3 \times \cancel{2}} \times \dfrac{x^4}{x^2} \times \dfrac{y^2}{y^2} = \dfrac{1}{3} \times x^2 \times y^0 = \dfrac{1}{3}x^2$ *or* $\dfrac{x^2}{3}$

(f) $\left(4x^2\right)^3 = 4^3x^{2\times 3} = 64x^6$ *(note that* $4 \times 4 \times 4 = 16 \times 4 = 8 \times 8$*)*

(g) $\dfrac{(3x^2z)^3}{(2x)^2} = \dfrac{3^3x^6z^3}{2^2x^2} = \dfrac{27}{4} \times x^{6-2} \times z^3 = \dfrac{27}{4}x^4z^3$ *or* $\dfrac{27x^4z^3}{4}$

(h) $\left(\dfrac{9x^4}{(3x)^2}\right)^5 = \left(\dfrac{9x^4}{9x^2}\right)^5 = (x^{4-2})^5 = (x^2)^5 = x^{10}$

5.2 *Express the following numbers without indices*

(a) $\dfrac{2^4}{4^2} = \dfrac{16}{16} = 1$ *or by index laws* $\dfrac{2^4}{(2^2)^2} = \dfrac{2^4}{2^4} = 1$

(b) $\dfrac{5^{12}}{25^5} = \dfrac{5^{12}}{(5^2)^5} = \dfrac{5^{12}}{5^{10}} = 5^{12-10} = 5^2 = 25$

(c) $4^{-1} = \dfrac{1}{4}$

(d) $10^{-2} = \dfrac{1}{10^2} = \dfrac{1}{100}$

(e) $\dfrac{3^5}{27^2} = \dfrac{3^5}{(3^3)^2} = \dfrac{3^5}{3^6} = 3^{5-6} = 3^{-1} = \dfrac{1}{3}$

(f) $(-2)^{-4} = \dfrac{1}{(-2)^4} = \dfrac{1}{16}$

(g) $(-3)^{-3} = \dfrac{1}{(-3)^3} = -\dfrac{1}{27}$

(h) $0^{-6} = \frac{1}{0^6} = \frac{1}{0\times0\times0\times0\times0\times0} = \frac{1}{0}$ *but this is not defined.*

(i) $(-6)^0 = 1$

5.3 *Use index laws to simplify each of the following expressions*

(a) $y^{-2}y^{-3} = y^{-2+(-3)} = y^{-5}$

(b) $x^{-3} \times x^3 = x^{-3+3} = x^0 = 1$

(c) $(a^{-3}b^{-4})(a^{-1}b) = a^{-3}b^{-4}a^{-1}b = a^{-3+(-1)}b^{-4+1} = a^{-4}b^{-3}$

(d) $\dfrac{4g^{-3}h^{-5}}{12g^{-5}h^3} = \dfrac{4}{12} \times \dfrac{g^{-3}}{g^{-5}} \times \dfrac{h^{-5}}{h^3} = \dfrac{1}{3} \times g^{-3-(-5)} \times h^{-5-3} = \dfrac{1}{3}g^2h^{-8}$

(e) $(-9x^{-1}y^2)^{-2} = (-9)^{-2}x^{-1\times(-2)}y^{2\times(-2)} = \dfrac{1}{(-9)^2}x^2y^{-4} = \dfrac{1}{81}x^2y^{-4}$

(f) $\left(\dfrac{3ab^2}{c}\right)^{-1} = \dfrac{c}{3ab^2}$ *(recognising that a power of -1 represents reciprocal)*

(g) $\dfrac{6a^3b^{-2}}{a^2b^{-1}} \div \dfrac{8a^3b}{b} = \dfrac{6a^3b^{-2}}{a^2b^{-1}} \times \dfrac{b}{8a^3b} = \dfrac{{}^3\cancel{6}a^3b^{-2+1}}{{}_4\cancel{8}a^{2+3}b^{-1+1}} = \dfrac{3a^3b^{-1}}{4a^5b^0} = \dfrac{3}{4}a^{-2}b^{-1}$ *or* $\dfrac{3}{4a^2b}$

(h) $\dfrac{(-2ab)^3}{(2a^{-1}b)^2} = \dfrac{(-2)^3a^3b^3}{2^2a^{-1\times2}b^2} = \dfrac{-8}{4}a^{3-(-2)}b^{3-2} = -2a^5b$

(i) $\left(\dfrac{3x^2y^{-1}}{2}\right)^0 = 1$ *(always!)*

(j) $\dfrac{x^6}{x^3} \times \left(\dfrac{4x^3}{12x}\right)^2 = x^{6-3} \times \left(\dfrac{1\times\cancel{4}}{3\times\cancel{4}}x^{3-1}\right)^2 = x^3 \times \left(\dfrac{1}{3}x^2\right)^2 = x^3 \times \dfrac{1}{9}x^4 = \dfrac{1}{9}x^7$ *or* $\dfrac{x^7}{9}$

5.4 *index laws to simplify each of the following expressions*

(a) $3^x \times 2^x \times 3^{1-x} = 3^{x+1-x}2^x = 3 \times 2^x$ *(note that the "$\times$" is needed here)*

(b) $\dfrac{2^2}{2^x} = 2^{2-x}$

(c) $\dfrac{x^yy^x}{x^{-y}} = x^{y-(-y)}y^x = x^{2y}y^x$

(d) $\dfrac{(9^a)^b}{(3^{2a})^b} = \dfrac{9^{ab}}{3^{2ab}} = \dfrac{(3^2)^{ab}}{3^{2ab}} = \dfrac{3^{2ab}}{3^{2ab}} = 1$

5.5 *Evaluate each of the following expressions.*

(a) $8^{\frac{1}{3}} = (2^3)^{\frac{1}{3}} = 2^{3\times\frac{1}{3}} = 2^1 = 2$

(b) $32^{\frac{1}{5}} = (2^5)^{\frac{1}{5}} = 2^{5\times\frac{1}{5}} = 2^1 = 2$

(c) $81^{\frac{1}{2}} = (9^2)^{\frac{1}{2}} = 9^{2\times\frac{1}{2}} = 9^1 = 9$

(d) $125^{\frac{1}{3}} = (5^3)^{\frac{1}{3}} = 5^{3\times\frac{1}{3}} = 5^1 = 5$

(e) $81^{\frac{1}{4}} = (3^4)^{\frac{1}{4}} = 3^{4\times\frac{1}{4}} = 3^1 = 3$

(f) $64^{\frac{1}{3}} = (4^3)^{\frac{1}{3}} = 4^{3\times\frac{1}{3}} = 4^1 = 4$

(g) $10000^{-\frac{1}{4}} = (10^4)^{-\frac{1}{4}} = 10^{-1} = \dfrac{1}{10}$

(h) $144^{-\frac{1}{2}} = (12^2)^{-\frac{1}{2}} = 12^{-1} = \dfrac{1}{12}$

(i) $32^{\frac{2}{5}} = (2^5)^{\frac{2}{5}} = 2^{5\times\frac{2}{5}} = 2^2 = 4$

(j) $36^{\frac{3}{2}} = (6^2)^{\frac{3}{2}} = 6^{2\times\frac{3}{2}} = 6^3 = 216$

(k) $49^{-\frac{3}{2}} = (7^2)^{-\frac{3}{2}} = 7^{2\times(-\frac{3}{2})} = 7^{-3} = \dfrac{1}{343}$

(l) $64^{\frac{5}{6}} = (2^6)^{\frac{5}{6}} = 2^5 = 32$

(m) $64^{-\frac{1}{6}} = (2^6)^{-\frac{1}{6}} = 2^{-1} = \dfrac{1}{2}$

(n) $81^{\frac{3}{4}} = (3^4)^{\frac{3}{4}} = 3^3 = 27$

(o) $(-27)^{\frac{2}{3}} = [(-3)^3]^{\frac{2}{3}} = (-3)^2 = 9$

5.6 *Evaluate each of the following expressions.*

(a) $27^{\frac{2}{3}} = (3^3)^{\frac{2}{3}} = 3^2 = 9$

(b) $27^{-\frac{2}{3}} = (3^3)^{-\frac{2}{3}} = 3^{-2} = \dfrac{1}{9}$

(c) $4^{\frac{7}{2}} = (2^2)^{\frac{7}{2}} = 2^7 = 128$

(d) $16^{-\frac{1}{2}} = (4^2)^{-\frac{1}{2}} = 4^{-1} = \dfrac{1}{4}$

(e) $100^{\frac{3}{2}} = (10^2)^{\frac{3}{2}} = 10^3 = 1000$

(f) $16^{-\frac{3}{4}} = (2^4)^{-\frac{3}{4}} = 2^{-3} = \dfrac{1}{8}$

(g) $169^{-\frac{1}{2}} = (13^2)^{-\frac{1}{2}} = 13^{-1} = \dfrac{1}{13}$

(h) $32^{\frac{4}{5}} = (2^5)^{\frac{4}{5}} = 2^4 = 16$

(i) $27^{\frac{4}{3}} = (3^3)^{\frac{4}{3}} = 3^4 = 81$

(j) $64^{-\frac{5}{6}} = (2^6)^{-\frac{5}{6}} = 2^{-5} = \dfrac{1}{32}$

(k) $\left(\dfrac{4}{9}\right)^{\frac{3}{2}} = \left[\left(\dfrac{2}{3}\right)^2\right]^{\frac{3}{2}} = \left(\dfrac{2}{3}\right)^3 = \dfrac{8}{27}$

(l) $\left(\dfrac{4}{9}\right)^{-\frac{3}{2}} = \left[\left(\dfrac{2}{3}\right)^2\right]^{-\frac{3}{2}} = \left(\dfrac{2}{3}\right)^{-3} = \dfrac{27}{8}$

(m) $81^{-\frac{1}{2}} = (9^2)^{-\frac{1}{2}} = 9^{-1} = \dfrac{1}{9}$

5.7 *Simplify each of the following expressions*

(a) $3x^{\frac{1}{2}} \times 4x^{\frac{2}{3}} = 12x^{\frac{1}{2}+\frac{2}{3}} = 12x^{\frac{7}{6}}$

(b) $5x^{\frac{2}{5}} \times 6x^{-\frac{1}{10}} = 30x^{\frac{2}{5}+(-\frac{1}{10})} = 30x^{\frac{3}{10}}$

(c) $(4x^{\frac{1}{2}})^3 \div (9x^{\frac{1}{3}})^{\frac{3}{2}} = \dfrac{64x^{\frac{3}{2}}}{27x^{\frac{1}{2}}} = \dfrac{64x}{27}$

(d) $(64x^2)^{-\frac{1}{6}} \div (32x^{\frac{5}{2}})^{-\frac{2}{5}} = 64^{-\frac{1}{6}}x^{-\frac{1}{3}} \times \dfrac{1}{(32x^{\frac{5}{2}})^{-\frac{2}{5}}}$

$= \dfrac{1}{64^{\frac{1}{6}}x^{\frac{1}{3}}} \times (32x^{\frac{5}{2}})^{\frac{2}{5}} = \dfrac{32^{\frac{2}{5}}x}{2x^{\frac{1}{3}}} = \dfrac{4x}{2x^{\frac{1}{3}}} = 2x^{\frac{2}{3}}$

(e) $\dfrac{x^{-\frac{2}{3}}y^{\frac{5}{2}}}{x^{-\frac{5}{3}}y^{-1}} = \dfrac{x^{\frac{5}{3}}y^{\frac{5}{2}}y}{x^{\frac{2}{3}}} = xy^{\frac{7}{2}}$

(f) $\dfrac{x^{\frac{3}{7}}y^{-\frac{4}{5}}}{x^{-\frac{2}{7}}y^{\frac{3}{5}}} = \dfrac{x^{\frac{3}{7}}x^{\frac{2}{7}}}{y^{\frac{3}{5}}y^{\frac{4}{5}}} = \dfrac{x^{\frac{5}{7}}}{y^{\frac{7}{5}}}$

5.8 *Simplify each of the following expressions*

(a) $3x^{\frac{1}{2}} \times 2x^{-\frac{1}{2}} = 6x^{\frac{1}{2}-\frac{1}{2}} = 6x^0 = 6$

(b) $(8x^3y^5)^{-\frac{2}{3}} = 8^{-\frac{2}{3}}x^{-2}y^{-\frac{10}{3}} = \dfrac{1}{(2^3)^{\frac{2}{3}}x^2y^{\frac{10}{3}}} = \dfrac{1}{4x^2y^{\frac{10}{3}}}$

(c) $(x^{\sqrt{5}})^{3\sqrt{5}} = x^{\sqrt{5}\times 3\sqrt{5}} = x^{15}$

(d) $\dfrac{3x^{\sqrt{27}}}{x^{\sqrt{12}}} = \dfrac{3x^{\sqrt{9\times 3}}}{x^{\sqrt{4\times 3}}} = \dfrac{3x^{3\sqrt{3}}}{x^{2\sqrt{3}}} = 3x^{\sqrt{3}}$

(e) $3x^{-\frac{1}{2}} \times 4x^{\frac{2}{3}} = 12x^{-\frac{1}{2}+\frac{2}{3}} = 12x^{\frac{1}{6}}$

(f) $(x^{2\sqrt{7}})^{\sqrt{7}} = x^{2\times 7} = x^{14}$

(g) $\dfrac{x^{-3}y^5}{x^2y^{-7}} = \dfrac{y^5y^7}{x^2x^3} = \dfrac{y^{12}}{x^5}$

(h) $7x^{\frac{1}{4}} \times 2x^{-\frac{1}{2}} = 14x^{\frac{1}{4}+(-\frac{1}{2})} = 14x^{-\frac{1}{4}}$

(i) $(9x)^{\frac{1}{2}} \times (27x)^{\frac{1}{3}} = 3x^{\frac{1}{2}} \times 3x^{\frac{1}{3}} = 9x^{\frac{1}{2}+\frac{1}{3}} = 9x^{\frac{5}{6}}$

(j) $(9x)^{\frac{1}{2}} \times (27x)^{-\frac{1}{3}} = \dfrac{3x^{\frac{1}{2}}}{(27x)^{\frac{1}{3}}} = \dfrac{3x^{\frac{1}{2}}}{3x^{\frac{1}{3}}} = x^{\frac{1}{2}-\frac{1}{3}} = x^{\frac{1}{6}}$

(k) $3x^{-\frac{1}{3}} \div 2x^{\frac{3}{4}} = \dfrac{3x^{-\frac{1}{3}}}{2x^{\frac{3}{4}}} = \dfrac{3}{2x^{\frac{3}{4}} \times x^{\frac{1}{3}}} = \dfrac{3}{2x^{\frac{3}{4}+\frac{1}{3}}} = \dfrac{3}{2x^{\frac{13}{12}}}$

5.9 *Simplify each of the following expressions*

(a) $\sqrt{12} = \sqrt{4\times 3} = \sqrt{4}\sqrt{3} = 2\sqrt{3}$

(b) $\sqrt{72} = \sqrt{36\times 2} = \sqrt{36}\sqrt{2} = 6\sqrt{2}$

(c) $\sqrt{150} = \sqrt{25\times 6} = \sqrt{25}\sqrt{6} = 5\sqrt{6}$

(d) $\sqrt[3]{96} = \sqrt[3]{8\times 12} = \sqrt[3]{8}\sqrt[3]{12} = 2\sqrt[3]{12}$

(e) $\sqrt{2}(3-\sqrt{2}) = 3\sqrt{2} - \sqrt{2}\sqrt{2} = 3\sqrt{2} - 2$

(f) $\sqrt{5}(2\sqrt{3}+\sqrt{2}) = 2\sqrt{15} + \sqrt{10}$

(g) $(1-\sqrt{2})\sqrt{3} = \sqrt{3} - \sqrt{6}$

(h) $(7+3\sqrt{5})\sqrt{3} = 7\sqrt{3} + 3\sqrt{15}$

(i) $(6-2\sqrt{5})(6+2\sqrt{5}) = 6^2 - (2\sqrt{5})^2 = 36 - 2^2\times 5 = 16$

(j) $(3+\sqrt{7})^2 = (3+\sqrt{7})(3+\sqrt{7}) = 9 + 3\sqrt{7} + 3\sqrt{7} + 7 = 16 + 6\sqrt{7}$

(k) $(4\sqrt{3}-1)(2\sqrt{5}+1) = 8\sqrt{15} + 4\sqrt{3} - 2\sqrt{5} - 1$

(l) $(\sqrt{5}-\sqrt{3})(2\sqrt{5}+4\sqrt{3}) = 2\times 5 + 4\sqrt{15} - 2\sqrt{15} - 4\times 3 = 10 + 2\sqrt{15} - 12 = 2\sqrt{15} - 2$

Problem Set 6

6.1 *For each of the following formulae, isolate the indicated variable*

In each case, think of the operation that has been applied to both sides. These are listed for the first two parts to get you started.

$$\begin{aligned} \textit{(a)}\quad \textit{Solve}\quad C &= \frac{5}{9}(F-32) \quad \textit{for } F \\ \Rightarrow \frac{9}{5}C &= F-32 \quad \textit{[multiply both sides by } \tfrac{9}{5}\textit{]} \\ \Rightarrow 32+\frac{9}{5}C &= F \quad \textit{[add 32 to both sides]} \end{aligned}$$

$$\begin{aligned} \textit{(b)}\quad \textit{Solve}\quad V &= \frac{4}{3}\pi r^3 \quad \textit{for } r \\ \Rightarrow \frac{V}{\pi} &= \frac{4}{3}r^3 \quad \textit{[divide both sides by } \pi\textit{]} \\ \Rightarrow \frac{3V}{4\pi} &= r^3 \quad \textit{[multiply both sides by } \tfrac{3}{4}\textit{]} \\ \Rightarrow r &= \left(\frac{3V}{4\pi}\right)^{\frac{1}{3}} \quad \textit{or} \quad \sqrt[3]{\frac{3V}{4\pi}} \quad \textit{[raise both sides to the power of } \tfrac{1}{3}\textit{]} \end{aligned}$$

$$\begin{aligned} \textit{(c)}\quad \textit{Solve}\quad PV &= nRT \quad \textit{for } T \\ \Rightarrow \frac{PV}{nR} &= T \end{aligned}$$

$$\begin{aligned} \textit{(d)}\quad \textit{Solve}\quad F &= \frac{GMm}{r^2} \quad \textit{for } r \\ \Rightarrow r^2F &= GMm \\ \Rightarrow r^2 &= \frac{GMm}{F} \\ \Rightarrow r &= \sqrt{\frac{GMm}{F}} \end{aligned}$$

$$\begin{aligned} \textit{(e)}\quad \textit{Solve}\quad b &= \frac{L}{4\pi d^2} \quad \textit{for } d \\ \Rightarrow 4\pi d^2 b &= L \\ \Rightarrow d^2 &= \frac{L}{4\pi b} \\ \Rightarrow d &= \sqrt{\frac{L}{4\pi b}} \end{aligned}$$

$$
\begin{aligned}
\text{(f)}\quad \textit{Solve}\quad L &= kd^{\frac{2}{3}}\left(\frac{E}{p}\right)^{\frac{1}{3}} \quad \textit{for } p\\
\Rightarrow\quad L^3 &= k^3 d^2 \frac{E}{p}\\
\Rightarrow\quad L^3 p &= k^3 d^2 E\\
\Rightarrow\quad p &= \frac{k^3 d^2 E}{L^3}
\end{aligned}
$$

$$
\begin{aligned}
\text{(g)}\quad \textit{Solve}\quad L &= \sqrt{\frac{h^2}{2\pi mkT}} \quad \textit{for } m\\
\Rightarrow\quad L^2 &= \frac{h^2}{2\pi mkT}\\
\Rightarrow\quad mL^2 &= \frac{h^2}{2\pi kT}\\
\Rightarrow\quad m &= \frac{h^2}{2\pi kTL^2}
\end{aligned}
$$

6.2 *For each of the following business formulae, isolate the indicated variable*

$$\begin{aligned} \textbf{(a)} \quad \text{Solve} \quad Q &= a - bP \quad \text{for } P \\ \Rightarrow \quad bP &= a - Q \\ \Rightarrow \quad P &= \frac{a-Q}{b} \end{aligned}$$

$$\begin{aligned} \textbf{(b)} \quad \text{Solve} \quad A &= P(1+r)^n \quad \text{for } r \\ \Rightarrow \quad \frac{A}{P} &= (1+r)^n \\ \Rightarrow \quad \left(\frac{A}{P}\right)^{\frac{1}{n}} &= 1+r \\ \Rightarrow \quad \left(\frac{A}{P}\right)^{\frac{1}{n}} - 1 &= r \end{aligned}$$

$$\begin{aligned} \textbf{(c)} \quad \text{Solve} \quad MV &= PY \quad \text{for } Y \\ \Rightarrow \quad \frac{MV}{P} &= Y \end{aligned}$$

$$\begin{aligned} \textbf{(d)} \quad \text{Solve} \quad R &= P\left(1 - \frac{1}{e}\right) \quad \text{for } e \\ \Rightarrow \quad \frac{R}{P} &= 1 - \frac{1}{e} \\ \Rightarrow \quad \frac{1}{e} &= 1 - \frac{R}{P} \\ &= \frac{P}{P} - \frac{R}{P} \\ &= \frac{P-R}{P} \\ \Rightarrow \quad e &= \frac{P}{P-R} \end{aligned}$$

$$\begin{aligned} \textbf{(e)} \quad \text{Solve} \quad F &= K \times \frac{SD}{d^2} \quad \text{for } S \\ \Rightarrow \quad d^2F &= KSD \\ \Rightarrow \quad \frac{d^2F}{KD} &= S \end{aligned}$$

Problem Set 7

7.1 *The required exponential forms are*

(a) $10^2 = 100$
(b) $10^3 = 1000$
(c) $10^0 = 1$
(d) $3^2 = 9$
(e) $3^0 = 1$
(f) $3^{-2} = \frac{1}{9}$
(g) $n^2 = 256$
(h) $n^3 = 14$
(i) $n^4 = 62$
(j) $2^{-3} = \frac{1}{8}$
(k) $2^0 = 1$
(l) $2^{1024} = n$
(m) $2^6 = 64$
(n) $9^2 = 81$
(o) $9^{-\frac{1}{2}} = \frac{1}{3}$
(p) $9^{\frac{1}{2}} = 3$
(q) $n^2 = 2$
(r) $10^{\frac{1}{2}} = x$
(s) $5^x = 100$
(t) $x^2 = 12$

7.2 *The required logarithmic forms are*

(a) $5 = \log_2 32$
(b) $3 = \log_2 8$
(c) $0 = \log_2 1$
(d) $2 = \log_8 64$
(e) $-1 = \log_8 \left(\frac{1}{8}\right)$
(f) $\frac{1}{3} = \log_8 2$
(g) $1 = \log_{16} 16$
(h) $-\frac{1}{2} = \log_{81} \left(\frac{1}{9}\right)$
(i) $-2 = \log_3 \left(\frac{1}{9}\right)$
(j) $-3 = \log_3 \left(\frac{1}{27}\right)$
(k) $5 = \log_3 243$
(l) $\frac{1}{4} = \log_{25} \sqrt{5}$
(m) $4 = \log_{10} 10000$
(n) $x = \log_3 81$
(o) $x = \log_{25} 5$
(p) $5 = \log_x 7$
(q) $x^2 = \log_3 81$
(r) $3x + 1 = \log_{20} 3$

7.3 (a) $\log 5 + \log 2 = \log(5 \times 2) = \log 10$

(b) $\log 8 - \log 2 = \log\left(\frac{8}{2}\right) = \log 4$

(c) $3\log 2 - 2\log 3 = \log\left(2^3\right) - \log\left(3^2\right) = \log\left(\frac{8}{9}\right)$

(d) $\log 4 - \log 20 = \log\left(\frac{4}{20}\right) = \log\left(\frac{1}{5}\right)$

(e) $\log 12 - \log 3 = \log\left(\frac{12}{3}\right) = \log 4$

(f) $\log 16 - \log 4 = \log\left(\frac{16}{4}\right) = \log 4$

(g) $\log_2 12 - \log_2 3 = \log_2\left(\frac{12}{3}\right) = \log_2 4 = 2$

(h) $\log_{10} 25 + \log_{10} 4 = \log_{10}(25 \times 4) = \log_{10} 100 = 2$

(i) $\log_{10} 25 - \log_{10} 250 = \log_{10}\left(\frac{25}{250}\right) = \log_{10}\left(\frac{1}{10}\right) = -1$

(j) $\log_5 20 + \log_5 4 = \log_5(20 \times 4) = \log_5 80$

(k) $\log\left(\frac{x^2}{\sqrt{y}}\right) + \log(x^{-2}y^3) = \log\left(x^2y^{-\frac{1}{2}}\right) + \log(x^{-2}y^3) = \log\left(x^2y^{-\frac{1}{2}} \times x^{-2}y^3\right)$

$$= \log\left(x^0y^{\frac{5}{2}}\right) = \tfrac{5}{2}\log y$$

(l) $2\log\left(x^{\frac{1}{2}}y^2\right) - \log\left(y^2x^{-1}\right) = \log\left(\left(x^{\frac{1}{2}}y^2\right)^2\right) - \log\left(y^2x^{-1}\right) = \log\left(xy^4\right) - \log\left(y^2x^{-1}\right)$

$$= \log\left(\frac{xy^4}{x^{-1}y^2}\right) = \log\left(x^2y^2\right) = \log\left((xy)^2\right) = 2\log(xy)$$

(m) $\frac{\log_{10} 27}{\log_{10} 9} = \frac{\log_{10}\left(3^3\right)}{\log_{10}\left(3^2\right)} = \frac{3\cancel{\log_{10} 3}}{2\cancel{\log_{10} 3}} = \frac{3}{2}$

(n) $\log_2 8 + \log_2 18 - \log_2 3 = \log_2\left(\frac{8 \times 18}{3}\right) = \log_2(8 \times 6) = \log_2 48$

(o) $\log_2 10 - \log_2 5 = \log_2 \left(\frac{10}{5}\right) = \log_2 2 = 1$

(p) $\log \sqrt{xy} + \frac{1}{2}\log(xy) - \log(xy) = \log\left((xy)^{\frac{1}{2}}\right) + \frac{1}{2}\log(xy) - \log(xy)$
$= \frac{1}{2}\log(xy) + \frac{1}{2}\log(xy) - \log(xy) = \left(\frac{1}{2} + \frac{1}{2} - 1\right)\log(xy) = 0$

7.4 *In this question, we are given that* $x = \log 3$ *and* $y = \log 4$.

(a) $\log 12 = \log(3 \times 4) = \log 3 + \log 4 = x + y$

(b) $\log 36 = \log 9 + \log 4 = \log\left(3^2\right) + \log 4 = 2\log 3 + \log 4 = 2x + y$

(c) $\log\left(\frac{3}{4}\right) = \log 3 - \log 4 = x - y$

(d) $\log 6 = \log 2 + \log 3 = \log\left(4^{\frac{1}{2}}\right) + \log 3 = \frac{1}{2}\log 4 + \log 3 = \frac{1}{2}y + x$

(e) $\log\left(\frac{3}{2}\right) = \log 3 - \log 2 = \log 3 - \log\left(4^{\frac{1}{2}}\right) = \log 3 - \frac{1}{2}\log 4 = x - \frac{1}{2}y$

(f) $\log\left(\frac{3}{8}\right) = \log 3 - \log 8 = \log 3 - (\log 4 + \log 2) = \log 3 - \left(\log 4 + \log\left(4^{\frac{1}{2}}\right)\right)$
$= \log 3 - \left(\log 4 + \frac{1}{2}\log 4\right) = x - \left(y + \frac{1}{2}y\right) = x - \frac{3}{2}y$

(g) $\log 144 = \log\left(12^2\right) = 2\log 12 = 2(\log 3 + \log 4) = 2(x + y) = 2x + 2y$

7.5 *In the question, we are given* $x = \log_5 2$ *and* $y = \log_5 3$.

(a) $\log_5 6 = \log_5 3 + \log_5 2 = y + x$

(b) $\log_5 12 = \log_5 4 + \log_5 3 = \log_5\left(2^2\right) + \log_5 3$
$= 2\log_5 2 + \log_5 3 = 2x + y$

(c) $\log_5\left(\frac{3}{10}\right) = \log_5 3 - \log_5 10 = \log_5 3 - (\log_5 2 + \log_5 5) = y - (x + 1) = y - x - 1$

(d) $\log_5 60 = \log_5 6 + \log_5 10 = \log_5 2 + \log_5 3 + \log_5 2 + \log_5 5 = 2x + y + 1$

(e) $\log_5\left(\frac{3}{2}\right) = \log_5 3 - \log_5 2 = y - x$

(f) $\log_5\left(\frac{3}{8}\right) = \log_5 3 - \log_5 8 = \log_5 3 - \log_5\left(2^3\right) = \log_5 3 - 3\log_5 2 = y - 3x$

(g) $\log_5 30 = \log_5 3 + \log_5 2 + \log_5 5 = x + y + 1$

Problem Set 8

8.1 *(a)*
$$\begin{aligned} 2^x &= 64 \\ 2^x &= 2^6 \\ x &= 6 \end{aligned}$$

(b)
$$\begin{aligned} 4^2x &= 128 \\ 16x &= 128 \\ x &= 8 \end{aligned}$$

(c)
$$\begin{aligned} 9^x &= 27 \\ \left(3^2\right)^x &= 3^3 \\ 3^{2x} &= 3^3 \\ 2x &= 3 \\ x &= \frac{3}{2} \end{aligned}$$

(d)
$$\begin{aligned} 5^{-x} &= 125 \\ 5^{-x} &= 5^3 \\ -x &= 3 \\ x &= -3 \end{aligned}$$

(e)
$$\begin{aligned} 8^{2x} &= 32 \\ (2^3)^{2x} &= 2^5 \\ 6x &= 5 \\ x &= \frac{5}{6} \end{aligned}$$

(f)
$$\begin{aligned} 10^{x+1} &= \frac{1}{100} \\ 10^{x+1} &= 10^{-2} \\ x+1 &= -2 \\ x &= -3 \end{aligned}$$

(g)
$$\begin{aligned} 3^{x^3} &= 81 \\ 3^{x^3} &= 3^4 \\ x^3 &= 4 \\ x &= \sqrt[3]{4} \text{ or } 4^{\frac{1}{3}} \approx 1.59 \end{aligned}$$

(h)
$$\begin{aligned} 4^{\frac{x}{2}-1} &= 128 \\ (2^2)^{\frac{x}{2}-1} &= 2^7 \\ 2(\frac{x}{2}-1) &= 7 \\ \frac{2x}{2} - 2 &= 7 \\ x - 2 &= 7 \\ x &= 9 \end{aligned}$$

(i)
$$\begin{aligned} 4^x &= 7 \\ x &= \log_4 7 \approx 1.43 \end{aligned}$$

(j)
$$\begin{aligned} 3^{x+1} &= 10 \\ x+1 &= \log_3 10 \\ x &= \log_3 10 - 1 \approx 1.10 \end{aligned}$$

(k)
$$\begin{aligned} 9^{2x+1} &= 50 \\ 2x+1 &= \log_9 50 \\ 2x &= \log_9 50 - 1 \\ x &= \frac{\log_9 50 - 1}{2} \approx 0.39 \end{aligned}$$

(l)
$$\begin{aligned} 7^{x-1} &= 4 \\ x-1 &= \log_7 4 \\ x &= \log_7 4 + 1 \approx 1.71 \end{aligned}$$

(m)
$$\begin{aligned} 2^{\frac{x}{2}+1} &= 64 \\ 2^{\frac{x}{2}+1} &= 2^6 \\ \frac{x}{2} + 1 &= 6 \\ \frac{x}{2} &= 5 \\ x &= 10 \end{aligned}$$

(n)
$$\begin{aligned} 5^{2x-1} &= 20 \\ 2x-1 &= \log_5 20 \\ 2x &= \log_5 20 + 1 \\ x &= \frac{\log_5 20 + 1}{2} \\ &= \frac{\log_5 4 + \log_5 5 + 1}{2} \\ &= \frac{\log_5 4 + 2}{2} \approx 1.43 \end{aligned}$$

8.2 *(a)*
$$\begin{aligned} \log_5 n &= 4 \\ n &= 5^4 \\ n &= 625 \end{aligned}$$

(b)
$$\begin{aligned} \log_8 4 &= n \\ 8^n &= 4 \\ 2^{3n} &= 2^2 \\ 3n &= 2 \\ n &= \frac{2}{3} \end{aligned}$$

(c)
$$\begin{aligned} \log_2 n &= \frac{1}{3} \\ n &= 2^{\frac{1}{3}} \text{ or } \sqrt[3]{2} \approx 1.26 \end{aligned}$$

(d)
$$\begin{aligned} \log_n \left(\frac{1}{64}\right) &= -2 \\ n^{-2} &= \frac{1}{64} \\ n^{-2} &= \frac{1}{8^2} \\ n^{-2} &= 8^{-2} \\ n &= 8 \end{aligned}$$

(e)
$$\begin{aligned} \log_{\frac{1}{3}} n &= 3 \\ \left(\frac{1}{3}\right)^3 &= n \\ \frac{1}{27} &= n \end{aligned}$$

(f)
$$\begin{aligned} \log_6 6 &= n \\ 6^n &= 6 \\ n &= 1 \end{aligned}$$

(g)
$$\begin{aligned} \log_n 8 &= \frac{3}{4} \\ n^{\frac{3}{4}} &= 8 \\ n^{\frac{3}{4}} &= 2^3 \\ \left(n^{\frac{1}{4}}\right)^3 &= 2^3 \\ n^{\frac{1}{4}} &= 2 \\ n &= 2^4 \\ &= 16 \end{aligned}$$

(h)
$$\begin{aligned}\log_4 3+\log_4(x+2)&=2\\ \log_4[3(x+2)]&=2\\ \log_4(3x+6)&=2\\ 3x+6&=4^2\\ 3x+6&=16\\ 3x&=10\\ x&=\frac{10}{3}\end{aligned}$$

(i)
$$\begin{aligned}x&=\log_4 72-\log_4 9\\ x&=\log_4\left(\frac{72}{9}\right)\\ x&=\log_4 8\\ 4^x&=8\\ (2^2)^x&=2^3\\ 2x&=3\\ x&=\frac{3}{2}\end{aligned}$$

(j)
$$\begin{aligned}x&=\log_7 98-\log_7 2\\ x&=\log_7\left(\frac{98}{2}\right)\\ x&=\log_7 49\\ 7^x&=49\\ 7^x&=7^2\\ x&=2\end{aligned}$$

(k)
$$\begin{aligned}\log_5 x-\log_5 7&=\log_5 6\\ \log_5\left(\frac{x}{7}\right)-\log_5 6&=0\\ \log_5\left(\frac{x}{7}\div 6\right)&=0\\ \log_5\left(\frac{x}{42}\right)&=0\\ \frac{x}{42}&=5^0\\ \frac{x}{42}&=1\\ x&=42\end{aligned}$$

(l)
$$\begin{aligned}\log_3 63-\log_3(7x)&=\log_3 2\\ \log_3\left(\frac{63}{7x}\right)-\log_3 2&=0\\ \log_3\left(\frac{9}{x}\div 2\right)&=0\\ \log_3\left(\frac{9}{2x}\right)&=0\\ \frac{9}{2x}&=3^0\\ \frac{9}{2x}&=1\\ 9&=2x\\ \frac{9}{2}&=x\end{aligned}$$

(m)
$$\begin{aligned}\log_5 4+\log_5(2x-3)&=2\\ \log_5[4(2x-3)]&=2\\ \log_5(8x-12)&=2\\ 8x-12&=5^2\\ 8x&=25+12\\ x&=\frac{37}{8}\end{aligned}$$

(n)
$$\begin{aligned}\log_2 2+\log_2(x+2)-\log_2(3x-5)&=3\\ \log_2\left[\frac{2(x+2)}{3x-5}\right]&=3\\ \log_2\left[\frac{2x+4}{3x-5}\right]&=3\\ \frac{2x+4}{3x-5}&=2^3\\ \frac{2x+4}{3x-5}&=8\\ 2x+4&=8(3x-5)\\ 2x+4&=24x-40\\ 44&=22x\\ 2&=x\end{aligned}$$

(o)
$$\begin{aligned}\log_2 4+\log_2(x-1)-\log_2(3x-4)&=2\\ \log_2\left[\frac{2(x-1)}{3x-4}\right]&=2\\ \frac{4x-4}{3x-4}&=2^2\\ \frac{4x-4}{3x-4}&=4\\ 4x-4&=4(3x-4)\\ 4x-4&=12x-16\\ 12&=8x\\ \frac{3}{2}&=x\end{aligned}$$

8.3 *(a) pH. Using p as the symbol for pH and H as the symbol for* $[H^+]$

$$p = -\log_{10} H \quad \Rightarrow \quad -p = \log_{10} H \quad \Rightarrow \quad H = 10^{-p}$$

(b) Arrhenius equation

$$K = Ae^{\frac{E}{RT}} \quad \Rightarrow \quad \frac{K}{A} = e^{\frac{E}{RT}} \quad \Rightarrow \quad \log_e\left(\frac{K}{A}\right) = \frac{E}{RT}$$

$$\Rightarrow \quad RT\log_e\left(\frac{K}{A}\right) = E \quad \Rightarrow \quad T = \frac{E}{R\log_e\left(\frac{K}{A}\right)}$$

(c) Relationship of rate constants at two different temperatures

$$\log_e\left(\frac{k}{K}\right) = \frac{E}{R}\left(\frac{t-T}{tT}\right)$$

$$\Rightarrow \quad \frac{k}{K} = e^{\frac{E}{R}\left(\frac{t-T}{tT}\right)}$$

$$\Rightarrow \quad \frac{K}{k} = \frac{1}{e^{\frac{E}{R}\left(\frac{t-T}{tT}\right)}}$$

$$\Rightarrow \quad K = \frac{k}{e^{\frac{E}{R}\left(\frac{t-T}{tT}\right)}} \text{ or } ke^{-\frac{E}{R}\left(\frac{t-T}{tT}\right)} \text{ or even } ke^{\frac{E}{R}\left(\frac{T-t}{tT}\right)}$$

(d) Benford's Law

$$P = \log_{10}\left(\frac{d+1}{d}\right) \quad \Rightarrow \quad \frac{d+1}{d} = 10^P \quad \Rightarrow \quad d+1 = 10^P d$$

$$\Rightarrow \quad 1 = 10^P d - d \quad \Rightarrow \quad 1 = d\left(10^P - 1\right) \quad \Rightarrow \quad \frac{1}{10^P - 1} = d$$

8.4 *A population starts with 1000 individuals and doubles in size every decade.*

(a) *Use the growth model $N = N_0 b^t$ to find how long will it take for the population to reach 500,000.*
The initial population size is $N_0 = 1000$ and "doubling every decade" means that measuring t in decades is a good idea because we can then use $b = 2$. That is, $N = 1000\left(2^t\right)$. When $N = 500,000$ we have

$$\begin{aligned} 500,000 &= 1000\left(2^t\right) \\ \Rightarrow \quad \frac{500,000}{1,000} &= 2^t \\ \Rightarrow \quad \frac{500 \times \cancel{1000}}{1 \times \cancel{1000}} &= 2^t \\ \Rightarrow \quad 500 &= 2^t \\ \Rightarrow \quad t &= \log_2 500 \;\approx\; 8.97 \text{ decades} = \; 89.7 \text{ years} \end{aligned}$$

(b) *Repeat part (a) using the growth model $N = N_0 e^{kt}$.*
Firstly, we need to get an expression for k. Since the population is doubling every decade, we know that there will be 2000 individuals when $t = 1$ decades:

$$2000 = 1000e^{k(1)} \quad \Rightarrow \quad 2 = e^k \quad \Rightarrow \quad k = \ln 2 \;\approx\; 0.69$$

When $N = 500,000$ we have

$$\begin{aligned} 500,000 &= 1000e^{kt} \quad \textit{so we can solve for } t \\ \Rightarrow \quad 500 &= e^{kt} \\ \Rightarrow \quad kt &= \ln 500 \\ \Rightarrow \quad t &= \ln 500 \div k \\ &= \ln 500 \div \ln 2 \quad \textit{from above} \\ &= \frac{\ln 500}{\ln 2} \;\approx\; 8.97 \end{aligned}$$

8.5 *An endangered species numbered 7,200 at the start of 2003 and at the start of 2014 numbered 800. Assuming the population follows the decay law $N = N_0 e^{-kt}$ predict in what year the population will fall below 120.*
If we use 2003 as our starting time (that is, $t = 0$) then 2014 is $t = 11$. This makes $N_0 = 7200$ and $N = 800$ when $t = 11$:

$$\begin{aligned} N = 800 &= 7200e^{-k(11)} \quad \textit{and solve for } k \\ \Rightarrow \quad \frac{800}{7200} &= e^{-11k} \\ \Rightarrow \quad \frac{1 \times \cancel{800}}{9 \times \cancel{800}} &= e^{-11k} \\ \Rightarrow \quad \frac{1}{9} &= e^{-11k} \\ \Rightarrow \quad -11k &= \ln\left(\frac{1}{9}\right) = \ln\left(9^{-1}\right) \\ &= -\ln 9 \quad \textit{using log laws} \\ \Rightarrow \quad k &= \frac{-\ln 9}{-11} \\ &= \frac{\ln 9}{11} \;\approx\; 0.20 \end{aligned}$$

To find out when the population falls below 120 *start with* $N = 120$*:*

$$\begin{aligned}
120 &= 7200e^{-kt} \quad \textit{so we can solve for t} \\
\Rightarrow \quad \frac{120}{7200} &= e^{-11k} \\
\Rightarrow \quad \frac{1 \times \cancel{120}}{60 \times \cancel{120}} &= e^{-11k} \\
\Rightarrow \quad \frac{1}{60} &= e^{-kt} \\
\Rightarrow \quad -kt &= \ln\left(\frac{1}{60}\right) \\
&= -\ln 60 \quad \textit{(see above)} \\
\Rightarrow \quad t &= -\ln 60 \div -k \\
&= \ln 60 \div \frac{\ln 9}{11} \quad \textit{from above} \\
&= \frac{\ln 60}{1} \times \frac{11}{\ln 9} = \frac{11 \ln 60}{\ln 9} \approx 20.50
\end{aligned}$$

That is, July 2023 *is when the population will fall below* 120*.*

8.6 *According to Wikipedia (en.wikipedia.org/wiki/Wikipedia:Modelling_Wikipedia%27s_growth), the number of articles on their site grew exponentially between October 2002 and mid 2006. There were 80,000 articles in October 2002 and 166,000 in October 2003. When does the model predict that Wikipedia reached 1,500,000 articles? (Use the growth model* $N = N_0 e^{kt}$*.)*

Firstly, we need to get an expression for k. It makes sense to use thousands of articles as the units of N (although it doesn't matter if you use just articles instead) and set October 2002 as $t = 0$ *years. This makes* $N_0 = 80$ *and there were 166 thousand articles at* $t = 1$*:*

$$166 = 80e^{k(1)} \quad \Rightarrow \quad \frac{166}{80} = e^{-11k} \quad \Rightarrow \quad \frac{2 \times \cancel{83}}{2 \times \cancel{40}} = e^{-11k} \quad \Rightarrow \quad k = \ln\left(\frac{83}{40}\right) \approx 0.73$$

When $N = 1500$ *we have*

$$\begin{aligned}
1500 &= 80e^{kt} \quad \textit{so we can solve for t} \\
\Rightarrow \quad \frac{1500}{80} &= e^{-11k} \\
\Rightarrow \quad \frac{75 \times \cancel{20}}{4 \times \cancel{20}} &= e^{-11k} \\
\Rightarrow \quad kt &= \ln\left(\frac{75}{4}\right) \\
\Rightarrow \quad t &= \ln\left(\frac{75}{4}\right) \div k \\
&= \ln\left(\frac{75}{4}\right) \div \ln\left(\frac{83}{40}\right) \quad \textit{from above} \\
&= \frac{\ln\left(\frac{75}{4}\right)}{\ln\left(\frac{83}{40}\right)} \approx 4.02
\end{aligned}$$

Hence Wikipedia would have reached 1,500,000 articles around October 2006. As it happens, growth of new articles had started to slow down by then and this exponential model no longer worked.

8.7 *DDT is a pesticide that was very effective at controlling marlaria-carrying mosquitos until its toxic effects on animals (including cancer in humans) were established. Furthermore, it remains active for many years in the environment because it decays exponentially with a half-life of 15 years.*

How long does it take for 100 grams of DDT to decay down to 1 gram? (Use the decay model $A = A_0 e^{-kt}$.)

Firstly, we need to get an expression for k. Since the half-life is $t = 15$ years, we know that there will be 50 grams left at this time:

$$50 = 100e^{-k(15)} \quad \Rightarrow \quad \frac{1}{2} = e^{-15k} \quad \Rightarrow \quad -15k = \ln\left(\frac{1}{2}\right) = \ln\left(2^{-1}\right) = -\ln 2 \quad \Rightarrow \quad k = \frac{-\ln 2}{-15} = \frac{\ln 2}{15}$$

When $A = 1$ gram we have

$$\begin{aligned}
1 &= 100e^{-kt} \quad \textit{so we can solve for } t \\
\Rightarrow \quad \frac{1}{100} &= e^{-kt} \\
\Rightarrow \quad -kt &= \ln\left(\frac{1}{100}\right) \\
&= \ln\left(100^{-1}\right) \\
&= -\ln 100 \\
\Rightarrow \quad t &= -\ln 100 \div -k \\
&= \ln 100 \div \frac{\ln 2}{15} \quad \textit{from above} \\
&= \frac{\ln 100}{1} \times \frac{15}{\ln 2} \\
&= \frac{15 \ln 100}{\ln 2} \approx 99.66
\end{aligned}$$

That is, t is between 99 and 100 years. That's a long time for something to remain toxic!

Problem Set 9

9.1 *Solve the following equations by factorising.*

(a)

$$\begin{aligned} x^2 - 3x &= 0 \\ \Rightarrow \quad x(x-3) &= 0 \\ \Rightarrow \quad x = 0 &\text{ or } x - 3 = 0 \\ \Rightarrow \quad x = 0 &\text{ or } x = 3 \end{aligned}$$

(b)

$$\begin{aligned} x^2 - 5x + 4 &= 0 \\ \Rightarrow \quad (x-4)(x-1) &= 0 \\ \Rightarrow \quad x - 4 = 0 &\text{ or } x - 1 = 0 \\ \Rightarrow \quad x = 4 &\text{ or } x = 1 \end{aligned}$$

(c)

$$\begin{aligned} x^2 - 5x - 14 &= 0 \\ \Rightarrow \quad (x-7)(x+2) &= 0 \\ \Rightarrow \quad x - 7 = 0 &\text{ or } x + 2 = 0 \\ \Rightarrow \quad x = 7 &\text{ or } x = -2 \end{aligned}$$

(d)

$$\begin{aligned} x^2 - x - 42 &= 0 \\ \Rightarrow \quad (x-7)(x+6) &= 0 \\ \Rightarrow \quad x - 7 = 0 &\text{ or } x + 6 = 0 \\ \Rightarrow \quad x = 7 &\text{ or } x = -6 \end{aligned}$$

(e)

$$\begin{aligned} 3x^2 - 6x &= -3 \\ \Rightarrow \quad 3x^2 - 6x + 3 &= 0 \end{aligned}$$

Divide through by 3:

$$\begin{aligned} \Rightarrow \quad x^2 - 2x + 1 &= 0 \\ \Rightarrow \quad (x-1)^2 &= 0 \\ \Rightarrow \quad x - 1 &= 0 \\ \Rightarrow \quad x &= 1 \end{aligned}$$

(f)

$$(x-2)(x+1) = 4$$

Expand and convert to standard form:

$$\begin{aligned} \Rightarrow \quad x^2 - x - 2 &= 4 \\ \Rightarrow \quad x^2 - x - 6 &= 0 \\ \Rightarrow \quad (x+2)(x-3) &= 0 \\ \Rightarrow \quad x + 2 = 0 &\text{ or } x - 3 = 0 \\ \Rightarrow \quad x = -2 &\text{ or } x = 3 \end{aligned}$$

(g)

$$x + \frac{15}{x} = 8$$

Multiply through by x:

$$\begin{aligned} \Rightarrow \quad x^2 + 15 &= 8x \\ \Rightarrow \quad x^2 - 8x + 15 &= 0 \\ \Rightarrow \quad (x-5)(x-3) &= 0 \\ \Rightarrow \quad x - 5 = 0 &\text{ or } x - 3 = 0 \\ \Rightarrow \quad x = 5 &\text{ or } x = 3 \end{aligned}$$

(h)

$$x = \frac{5}{x-4}$$

Cross multiply:

$$\begin{aligned} \Rightarrow \quad \frac{x}{1} &= \frac{5}{x-4} \\ \Rightarrow \quad x(x-4) &= 1 \times 5 \\ \Rightarrow \quad x^2 - 4x - 5 &= 0 \\ \Rightarrow \quad (x-5)(x+1) &= 0 \\ \Rightarrow \quad x - 5 = 0 &\text{ or } x + 1 = 0 \\ \Rightarrow \quad x = 5 &\text{ or } x = -1 \end{aligned}$$

9.2 *Solve the following equations by using the quadratic formula.*

(a) $x^2 - 5x + 4 = 0$, *so* $a = 1$, $b = -5$, $c = 4$:

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ \Rightarrow \quad x &= \frac{-(-5) \pm \sqrt{(-5)^2 - 4 \times 1 \times 4}}{2 \times 1} \\ &= \frac{5 \pm \sqrt{25 - 16}}{2} \\ &= \frac{5 \pm \sqrt{9}}{2} \\ &= \frac{5 \pm 3}{2} \\ \Rightarrow \quad x &= \frac{5+3}{2} \text{ or } x = \frac{5-3}{2} \\ \Rightarrow \quad x &= \frac{8}{2} \text{ or } x = \frac{2}{2} \\ \Rightarrow \quad x &= 4 \text{ or } x = 1 \end{aligned}$$

Note: *we could have factorised this one:*

$$(x-4)(x-1) = 0$$

(b) $3x^2 + 7x + 2 = 0$, *so* $a = 3$, $b = 7$, $c = 2$:

$$\begin{aligned} x &= \frac{-7 \pm \sqrt{7^2 - 4 \times 3 \times 2}}{2 \times 3} \\ &= \frac{-7 \pm \sqrt{49 - 24}}{6} \\ &= \frac{-7 \pm \sqrt{25}}{6} \\ &= \frac{-7 \pm 5}{6} \\ \Rightarrow \quad x &= \frac{-7+5}{6} \text{ or } x = \frac{-7-5}{6} \\ \Rightarrow \quad x &= -\frac{1}{3} \text{ or } x = -2 \end{aligned}$$

(c) $x^2 + 4x + 1 = 0$, *so* $a = 1$, $b = 4$, $c = 1$:

$$\begin{aligned} x &= \frac{-4 \pm \sqrt{4^2 - 4 \times 1 \times 1}}{2 \times 1} \\ &= \frac{-4 \pm \sqrt{16 - 4}}{2} \\ &= \frac{-4 \pm \sqrt{12}}{2} \end{aligned}$$

Recall that $\sqrt{12} = 2\sqrt{3}$, *so*

$$\begin{aligned} &= \frac{-4 \pm 2\sqrt{3}}{2} \\ &= \frac{2(-2 \pm \sqrt{3})}{2} \\ &= \frac{{}^{1}\not{2}(-2 \pm \sqrt{3})}{\not{2}_{1}} \\ &= -2 \pm \sqrt{3} \\ \Rightarrow \quad x &= -2 + \sqrt{3} \text{ or } x = -2 - \sqrt{3} \end{aligned}$$

(d) $-2x^2 + 3x - 1 = 0$, *so* $a = -2$, $b = 3$, $c = -1$:

$$\begin{aligned} x &= \frac{-3 \pm \sqrt{3^2 - 4 \times (-2) \times (-1)}}{2 \times (-2)} \\ &= \frac{-3 \pm \sqrt{9 - 8}}{-4} \\ &= \frac{-3 \pm \sqrt{1}}{-4} \\ &= \frac{-3 \pm 1}{-4} \\ \Rightarrow \quad x &= \frac{-3+1}{4} \text{ or } x = \frac{-3-1}{4} \\ \Rightarrow \quad x &= \frac{1}{2} \text{ or } x = 1 \end{aligned}$$

(e) $x^2 + 3x + 4 = 0$, *so* $a = 1$, $b = 3$, $c = 4$:

$$\begin{aligned} x &= \frac{-3 \pm \sqrt{3^2 - 4 \times 1 \times 4}}{2 \times 1} \\ &= \frac{-3 \pm \sqrt{9 - 16}}{2} \\ &= \frac{-3 \pm \sqrt{-7}}{2} \end{aligned}$$

We can't take the square root of a negative number so there are no solutions.

(f) $\frac{3}{2}x^2 + 4x + 1 = 0$

Multiply through by 2 to remove fraction:

$3x^2 + 8x + 2 = 0$, *so* $a = 3$, $b = 8$, $c = 2$:

$$x = \frac{-8 \pm \sqrt{8^2 - 4 \times 3 \times 2}}{2 \times 3} = \frac{-8 \pm \sqrt{40}}{6}$$

Recall that $\sqrt{40} = 2\sqrt{10}$, *so*

$$= \frac{-8 \pm 2\sqrt{10}}{6} = \frac{2(-4 \pm \sqrt{10})}{6} = \frac{{}^1\not{2}(-4 \pm \sqrt{10})}{\not{6}_3} = \frac{-4 \pm \sqrt{10}}{3}$$

$$\Rightarrow \quad x = \frac{-4 + \sqrt{10}}{3} \text{ or } x = \frac{-4 - \sqrt{10}}{3}$$

(g) $x^2 + x - 1 = 0$, *so* $a = 1$, $b = 1$, $c = -1$:

$$x = \frac{-1 \pm \sqrt{1^2 - 4 \times 1 \times (-1)}}{2 \times 1} = \frac{-1 \pm \sqrt{1+4}}{2} = \frac{-1 \pm \sqrt{5}}{2}$$

$$\Rightarrow \quad x = \frac{-1 + \sqrt{5}}{2} \text{ or } x = \frac{-1 - \sqrt{5}}{2}$$

(h) $2x^2 - 2x - 3 = 0$, *so* $a = 2$, $b = -2$, $c = -3$:

$$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4 \times 2 \times (-3)}}{2 \times 2} = \frac{2 \pm \sqrt{4 + 24}}{4} = \frac{2 \pm \sqrt{28}}{4} = \frac{2 \pm 2\sqrt{7}}{4} = \frac{2(1 \pm \sqrt{7})}{4} = \frac{{}^1\not{2}(1 \pm \sqrt{7})}{\not{4}_2} = \frac{1 \pm \sqrt{7}}{2}$$

$$\Rightarrow \quad x = \frac{1 + \sqrt{7}}{2} \text{ or } x = \frac{1 - \sqrt{7}}{2}$$

(i) $2 - 3x^2 = 0$

No x term is the same as $0x$:

$-3x^2 + 0x + 2 = 0$, *so* $a = -3$, $b = 0$, $c = 2$:

$$x = \frac{-0 \pm \sqrt{0^2 - 4 \times (-3) \times 2}}{2 \times (-3)} = \frac{\pm\sqrt{24}}{-6} = \frac{\pm 2\sqrt{6}}{-6} = \frac{\pm {}^1\not{2}\sqrt{6}}{-\not{6}_3} = \frac{\pm\sqrt{6}}{-3}$$

$$\Rightarrow \quad x = -\frac{\sqrt{6}}{3} \text{ or } x = \frac{\sqrt{6}}{3}$$

9.3 *Determine the number of solutions for the following equations.*
All we need to do in each case is check the discriminant $b^2 - 4ac$*.*

(a) $x^2 + 3x + 1 = 0$ *In this case we have* $a = 1$, $b = 3$ *and* $c = 1$ *so*

$$b^2 - 4ac = 3^2 - 4 \times 1 \times 1 = 9 - 4 = 5 > 0 \quad \Rightarrow \quad \text{two solutions}$$

(b) $3x^2 + 4x + 2 = 0$

$$b^2 - 4ac = 4^2 - 4 \times 3 \times 2 = 16 - 24 = -8 < 0 \quad \Rightarrow \quad \text{no solutions}$$

(c) $2x^2 + 3x + 5 = 0$

$$b^2 - 4ac = 3^2 - 4 \times 2 \times 5 = 9 - 40 = -31 < 0 \quad \Rightarrow \quad \text{no solutions}$$

(d) $2x^2 + 4x + 2 = 0$

$$b^2 - 4ac = 4^2 - 4 \times 2 \times 2 = 16 - 16 = 0 \quad \Rightarrow \quad \text{one solution}$$

Note: *you get the same conclusion if you divide through by 2 first:* $x^2 + 2x + 1 = 0$

$$b^2 - 4ac = 2^2 - 4 \times 1 \times 1 = 4 - 4 = 0 \quad \Rightarrow \quad \text{one solution}$$

(e) $\frac{3}{2}x^2 + 4x + 6 = 0$

$$b^2 - 4ac = 4^2 - 4 \times \frac{3}{2} \times 6 = 16 - 36 = -24 < 0 \quad \Rightarrow \quad \text{no solutions}$$

Note: *you get the same conclusion if you multiply through by 2 first:* $3x^2 + 8x + 12 = 0$

$$b^2 - 4ac = 8^2 - 4 \times 3 \times 12 = 64 - 144 < 0 \quad \Rightarrow \quad \text{no solutions}$$

(f) $x^2 + 5x - 1 = 0$

$$b^2 - 4ac = 5^2 - 4 \times 1 \times (-1) = 25 + 4 = 29 > 0 \quad \Rightarrow \quad \text{two solutions}$$

(g) $4x^2 - 5x + 4 = 0$

$$b^2 - 4ac = (-5)^2 - 4 \times 4 \times 4 = 25 - 64 = -39 < 0 \quad \Rightarrow \quad \text{no solutions}$$

9.4 *Complete the square for the following quadratic expressions*

$$(a)\quad x^2+2x+3 = x^2+2x+\left(\frac{2}{2}\right)^2-\left(\frac{2}{2}\right)^2+3$$
$$= x^2+2x+1-1+3$$
$$= (x+1)^2-1+3$$
$$= (x+1)^2+2$$

$$(b)\quad x^2-4x-8 = x^2-4x+\left(\frac{-4}{2}\right)^2-\left(\frac{-4}{2}\right)^2-8$$
$$= x^2-4x+4-4-8$$
$$= (x-2)^2-4-8$$
$$= (x-2)^2-12$$

$$(c)\quad x^2+10x+7 = x^2+10x+\left(\frac{10}{2}\right)^2-\left(\frac{10}{2}\right)^2+7$$
$$= x^2+10x+25-25+7$$
$$= (x+5)^2-25+7$$
$$= (x+5)^2-18$$

$$(d)\quad x^2-6x = x^2-6x+\left(\frac{-6}{2}\right)^2-\left(\frac{-6}{2}\right)^2$$
$$= x^2-6x+9-9$$
$$= (x-3)^2-9$$

$$(e)\quad x^2+12x+36 = x^2+12x+\left(\frac{12}{2}\right)^2-\left(\frac{12}{2}\right)^2+36$$
$$= x^2+12x+36-36+36$$
$$= (x+6)^2-36+36$$
$$= (x+6)^2$$

(In other words, $x^2+12x+36$ is already a perfect square.)

$$(f)\quad x^2-20x-1 = x^2-20x+\left(\frac{-20}{2}\right)^2-\left(\frac{-20}{2}\right)^2-1$$
$$= x^2-20x+100-100-1$$
$$= (x-10)^2-100-1$$
$$= (x-10)^2-101$$

9.5 *The product of two consecutive odd numbers is 99. Find the two numbers.*

Let n be the smaller number. The next odd number is then $n+2$. Since the product of the two numbers is 99 we have

$$n(n+2) = 99 \quad \Rightarrow \quad n^2 + 2n - 99 = 0$$

You might recognise that the factorisation is

$$(n-9)(n+11) = 0$$

in which case $n - 9 = 0$ or $n + 11 = 0$. Hence $n = 9$ or -11.

Answer: *the numbers are either 9 and 11 or -11 and -9.*

If you don't recognise the factorisation, the quadratic formula gives us that

$$\begin{aligned} n &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-2 \pm \sqrt{2^2 - 4 \times 1 \times (-99)}}{2 \times 1} \\ &= \frac{-2 \pm \sqrt{400}}{2} \\ &= \frac{-2 \pm 20}{2} \\ \Rightarrow \quad x &= \frac{-2+20}{2} \text{ or } x = \frac{-2-20}{2} \\ \Rightarrow \quad x &= 9 \text{ or } x = -11 \end{aligned}$$

9.6 *One positive number exceeds three times another positive number by 5. The product of the numbers is 68. Find the numbers. Note that $\sqrt{25 + 12 \times 68} = 29$.*

Let n be the smaller number. Then the larger one is $3n + 5$. Since the product of the two numbers is 68 we have

$$\begin{aligned} n(3n+5) &= 68 \\ \Rightarrow \quad 3n^2 + 5n &= 68 \\ \Rightarrow \quad 3n^2 + 5n - 68 &= 0 \end{aligned}$$

The quadratic formula then gives us that

$$\begin{aligned} n &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-5 \pm \sqrt{5^2 - 4 \times 3 \times (-68)}}{2 \times 3} \\ &= \frac{-5 \pm \sqrt{25 + 12 \times 68}}{6} \\ &= \frac{-5 \pm 29}{6} \\ \Rightarrow \quad x &= \frac{-5+29}{6} \text{ or } x = \frac{-5-29}{6} \\ &= 4 \text{ or } x = -\frac{34}{6} \end{aligned}$$

Since we are told the number is positive, we have

Answer: *the numbers are are 4 and $3 \times 4 + 5 = 17$.*

9.7 *A group of zoologists was studying the effect on the body weight of rats of varying the amount of yeast in their diet. By changing the percentage P of yeast in the diet, the average weight gain, G (in grams), over time was estimated to be* $G = -200P^2 + 200P + 20$. *What percentage of yeast would you expect to give an average weight gain of 70 grams?*

We have to solve $-200P^2 + 200P + 20 = 70$. *That is*

$$-200P^2 + 200P - 50 = 0$$

Note that $200 = 4 \times 50$ *so we can divide through by 50 to get*

$$-4P^2 + 4P - 1 = 0$$

$$a = -4, b = 4, c = -1$$

$$\begin{aligned} P &= \frac{-4 \pm \sqrt{4^2 - 4 \times (-4) \times (-1)}}{2 \times (-4)} \\ &= \frac{-4 \pm \sqrt{16 - 16}}{-8} \\ &= \frac{-4 \pm 0}{-8} \\ &= \frac{1}{2} \end{aligned}$$

Answer: *The percentage of yeast required is* $\frac{1}{2}\%$

Note: *you can reduce the number of negative signs by dividing the quadratic equation through by* -1 *as well:*

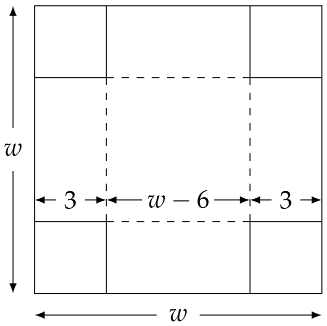
$$4P^2 - 4P + 1 = 0$$

$$a = 4, b = -4, c = 1$$

This version leads to the same answer.

9.8 *You wish to make a square-bottomed box with a height of three cm and a volume of 75 cm*^{3}*. You will take a piece of square cardboard, cut a three cm square from each corner and fold up the sides. What sized piece of cardboard do you need?* **Hint:** *it might help to draw a picture.*

Let w = *side length of of the piece of square cardboard:*



The base area of the box is $(w-6)^2$ *and its volume equals 75 cm*^{3}*, so we have*

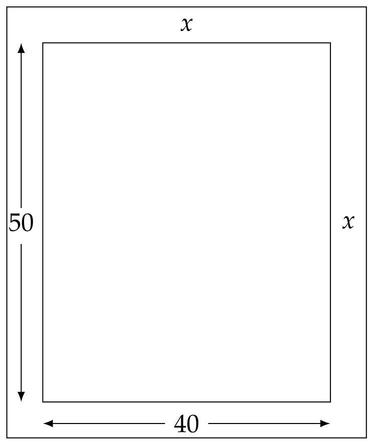
$$\begin{aligned} \text{base} \times \text{height} &= (w-6)^2 \times 3 = 75 \\ \Rightarrow \quad (w-6)^2 = 25 \quad &\Rightarrow \quad w^2 - 12w + 36 = 25 \\ \Rightarrow \quad w^2 - 12w + 11 = 0 \quad &\Rightarrow \quad (w-11)(w-1) = 0 \quad \textit{(or use the quadratic formula)} \\ &\Rightarrow \quad w = 11 \text{ or } 1 \end{aligned}$$

$w = 1$ *isn't a plausible answer since we can't fold up 3 cm sides if the width is only 1 cm.*

Answer: *we need a square piece of cardboard of width 11 cm.*

9.9 *A rectangular park has dimensions 40 metres by 50 metres. A pathway is to be added all around the park which will increase the total area to 3000 m*^{2}*. How wide is the pathway going to be?* **Hint:** *Draw a picture.*

Let x = *width of the pathway:*



The side lengths of the total area (park and pathway) are $40 + 2x$ *and* $50 + 2x$ *and the total area equals 75 cm*^{3}*, so we have*

$$\begin{aligned} (40+2x)(50+2x) &= 3000 \\ \Rightarrow \quad 2000 + 180x + 4x^2 = 3000 \quad &\Rightarrow \quad 4x^2 + 180x - 1000 = 0 \\ \Rightarrow \quad x^2 + 45x - 250 &= 0 \\ \textit{quad (dividing through by 4)} \Rightarrow \quad (w-5)(w+50) &= 0 \quad \textit{(or use the quadratic formula)} \\ \Rightarrow \quad w &= 5 \text{ or } -50 \end{aligned}$$

$w = -50$ *is negative so it isn't a plausible path width.*

Problem Set 10

10.1 *Find the indicated values of the following functions.*

$$f(x) = -3x^2 + 2x + 4 \qquad g(x) = \log_2 x \qquad h(u) = \sqrt{11-u}$$

$$F(v) = 3^{2v-4} \qquad G(t) = 5 - 3t \qquad H(s) = \frac{s-3}{s+1}$$

(a) $$\begin{aligned} f(2) &= -3(2^2) + 2(2) + 4 \\ &= -12 + 4 + 4 \\ &= -4 \end{aligned}$$

(b) $$\begin{aligned} f(-1) &= -3(-1)^2 + 2(-1) + 4 \\ &= -3 - 2 + 4 \\ &= -1 \end{aligned}$$

(c) $$\begin{aligned} g(8) &= \log_2 8 \\ &= 3 \end{aligned}$$

(d) $$\begin{aligned} g\left(\frac{1}{4}\right) &= \log_2\left(\frac{1}{4}\right) \\ &= \log_2\left(2^{-2}\right) \\ &= -2 \end{aligned}$$

(e) $$\begin{aligned} h(7) &= \sqrt{11-7} \\ &= \sqrt{4} \\ &= 2 \end{aligned}$$

(f) $$\begin{aligned} h(11) &= \sqrt{11-11} \\ &= \sqrt{0} \\ &= 0 \end{aligned}$$

(g) $$\begin{aligned} F(3) &= 3^{2(3)-4} \\ &= 3^{6-4} \\ &= 3^2 \\ &= 9 \end{aligned}$$

(h) $$\begin{aligned} F(2) &= 3^{2(2)-4} \\ &= 3^{4-4} \\ &= 3^0 \\ &= 1 \end{aligned}$$

(i) $$\begin{aligned} G\left(\frac{1}{4}\right) &= 5 - 3\left(\frac{1}{4}\right) \\ &= \frac{20}{4} - \frac{3}{4} \\ &= \frac{17}{4} \end{aligned}$$

(j) $$\begin{aligned} G\left(-\frac{2}{5}\right) &= 5 - 3\left(-\frac{2}{5}\right) \\ &= \frac{25}{5} + \frac{6}{5} \\ &= \frac{31}{5} \end{aligned}$$

(k) $$\begin{aligned} H(2) &= \frac{2-3}{2+1} \\ &= \frac{-1}{3} \\ &= -\frac{1}{3} \end{aligned}$$

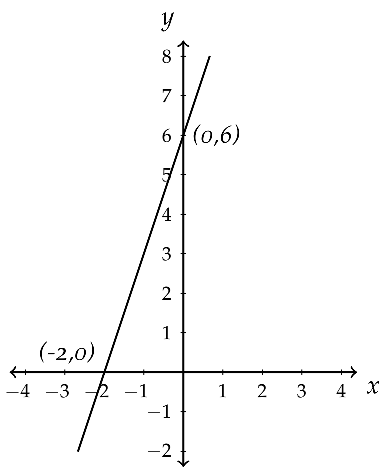
(l) $$\begin{aligned} H(-3) &= \frac{-3-3}{-3+1} \\ &= \frac{-6}{-2} \\ &= 3 \end{aligned}$$

10.2 *Sketch the graphs of the following functions*

(a) $y = 3x + 6$

To find the x−intercept:

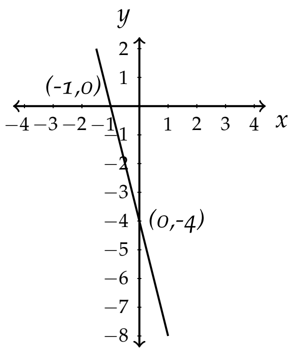
$$\begin{aligned} 0 &= 3x + 6 \\ \Rightarrow \quad -6 &= 3x \\ \Rightarrow \quad -2 &= x \end{aligned}$$



(b) $y = -4x - 4$

To find the x−intercept:

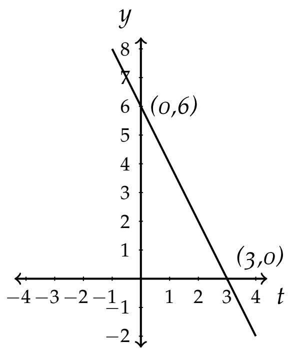
$$\begin{aligned} 0 &= -4x - 4 \\ \Rightarrow \quad 4x &= -4 \\ \Rightarrow \quad x &= -1 \end{aligned}$$



(c) $y = -2t + 6$

To find the t−intercept:

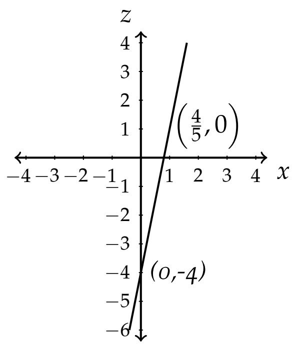
$$\begin{aligned} 0 &= -2t + 6 \\ \Rightarrow \quad 2t &= 6 \\ \Rightarrow \quad t &= 3 \end{aligned}$$



(d) $z = 5x - 4$

To find the x−intercept:

$$\begin{aligned} 0 &= 5x - 4 \\ \Rightarrow \quad 4 &= 5x \\ \Rightarrow \quad \frac{4}{5} &= x \end{aligned}$$



10.3 *Find the equations of the following lines.*

(a) *Gradient* 2 *and contains the point* $(0,4)$
Equation is $y = 2x + c$ *and point* $(0,4)$ *tells us that the y-intercept is* 4 *so the equation is* $y = 2x + 4$.

(b) *Gradient* -3 *and contains the point* $(-1,3)$
The equation is $y = -3x + c$ *and the point* $(-1,3)$ *is on the line so*

$$\begin{aligned} 3 &= -3(-1) + c \\ \Rightarrow \quad 3 &= 3 + c \quad \Rightarrow \quad 0 = c \end{aligned}$$

The equation is $y = -3x$.

(c) *Contains the points* $(3,4)$ *and* $(6,-2)$
***Method* 1 :** *The equation is* $y = mx + c$ *and the points tell us that*

$$\begin{array}{ccrcrcrcl} (3,4) & \Rightarrow & & 4 & = & 3m & + & c & (1) \\ (6,-2) & \Rightarrow & (-) & -2 & = & 6m & + & c & (2) \\ \hline & & & 6 & = & -3m & & & \\ & & \Rightarrow & -2 & = & m & & & \end{array}$$

Substitute one of the points, say $(3,4)$, *into* $y = -2x + c$:

$$\begin{aligned} 4 &= -2(3) + c \\ \Rightarrow \quad 4 = -6 + c \quad &\Rightarrow \quad 10 = c \end{aligned}$$

The equation is $y = -2x + 10$.
***Method* 2 :** *The gradient of the line is*

$$\begin{aligned} m &= \frac{\text{change in } y \text{ value}}{\text{change in } x \text{ value}} \\ &= \frac{-2-4}{6-3}\star \\ &= \frac{-6}{3} = -2 \end{aligned}$$

We now find c in the same way as for Method 1.

(d) *Contains the point* $(3,1)$ *and is parallel to the line with equation* $y = 2x - 1$
The gradient is $m = 2$ *and the point* $(3,1)$ *is on the line so*

$$\begin{aligned} 1 &= 2(3) + c \\ \Rightarrow \quad 1 &= 6 + c \\ \Rightarrow \quad -5 &= c \end{aligned}$$

The equation is $y = 2x - 5$.

(e) *Perpendicular to the line* $y = 4x - 1$ *and contains the point* $(4,2)$
The gradient is $m = -\frac{1}{4}$ *and the point* $(4,2)$ *is on the line so*

$$\begin{aligned} 2 &= -\frac{1}{4}(4) + c \\ \Rightarrow \quad 2 &= -1 + c \\ \Rightarrow \quad 3 &= c \end{aligned}$$

The equation is $y = -\frac{1}{4}x + 3$.

10.4 *The line L_1 has equation $y = 3x + 1$. Find the equations of the following six lines.*

(a) The line L_2 which is parallel to L_1 and contains the point $(-1, 4)$

(b) The line L_3 which is perpendicular to L_1 and has x-intercept 3

(c) The line L_4 which is perpendicular to L_1 and intersects it at the point $(2, -3)$

(d) The line L_5 which is horizontal and has the same y-intercept as L_1

(e) The line L_6 which contains the point $(3, 7)$ and has the same y-intercept as L_1

Note that the line $L_1: \; y = 3x + 1$ has gradient 3 and y-intercept 1

(a) L_2 *is parallel to* L_1 *so* $y = 3x + c$ *and it contains the point* $(-1, 4)$ *so*

$$\begin{aligned} 4 &= 3(-1) + c \\ \Rightarrow \quad 4 &= -3 + c \\ \Rightarrow \quad 7 &= c \end{aligned}$$

The equation is $y = 3x + 7$.

(b) L_3 *is perpendicular to* L_1 *so* $y = -\frac{1}{3}x + c$ *and has x-intercept 3. Hence,* $(3, 0)$ *is on* L_3 *so*

$$\begin{aligned} 0 &= -\tfrac{1}{3}(3) + c \\ \Rightarrow \quad 0 &= -1 + c \\ \Rightarrow \quad c &= 1 \end{aligned}$$

The equation is $y = -\frac{1}{3}x + 1$.

(c) L_4 *is perpendicular to* L_1 *so* $y = -\frac{1}{3}x + c$ *and it intersects* L_1 *at* $(2, -3)$ *so the point* $(2, -3)$ *must be on* L_4 *as well and hence*

$$\begin{aligned} -3 &= -\tfrac{1}{3}(2) + c \\ \Rightarrow \quad -\tfrac{9}{3} &= -\tfrac{2}{3} + c \\ \Rightarrow \quad c &= -\tfrac{7}{3} \end{aligned}$$

The equation is $y = -\frac{1}{3}x - \frac{7}{3}$.

(d) L_5 *is horizontal so it has the equation* $y = c$. *The line has the same y-intercept as* L_1, *that is* $c = 1$ *and hence the equation of* L_5 *is*

$$y = 1$$

(e) L_6 *has the same y-intercept as* L_1 *so* $y = mx + 1$ *and it contains the point* $(3, 7)$ *so*

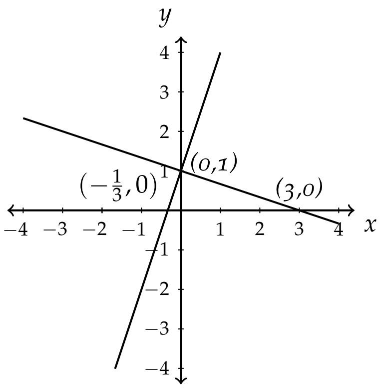
$$\begin{aligned} 7 &= 3m + 1 \\ \Rightarrow \quad 7 - 1 &= -3m \\ \Rightarrow \quad 6 &= 3m \\ \Rightarrow \quad 2 &= m \end{aligned}$$

The equation is $y = 2x + 1$.

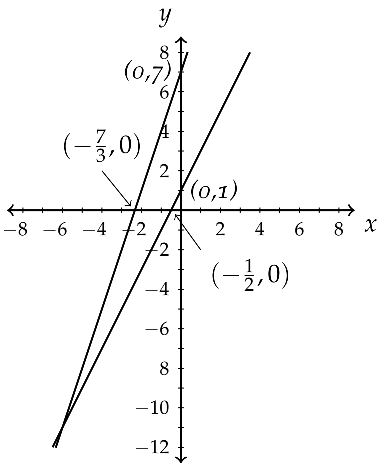
10.5 *Sketch the following pairs of lines from the previous question on the same axes.*

(All x−intercepts worked out via the same method as Question 2.)

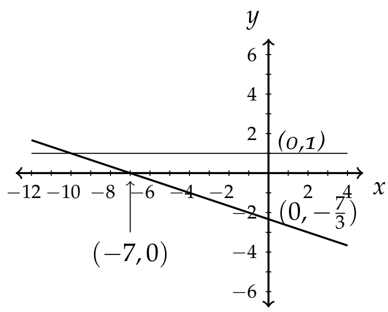
(a) $L_1: \ y = 3x + 1$ *and* $L_3: \ y = -\frac{1}{3}x + 1$



(b) $L_2: \ y = 3x + 7$ *and* $L_6: \ y = 2x + 1$



(c) $L_4: \ y = -\frac{1}{3}x - \frac{7}{3}$ *and* $L_5: \ y = 1$



10.6 *Find the intersection point of each pair of lines in the previous question.*

(a) $L_1: \ y = 3x + 1$ *and* $L_3: \ y = -\frac{1}{3}x + 1$ *intersect where*

$$3x + 1 = -\frac{1}{3}x + 1 \quad \Rightarrow \quad 9x + 3 = -x + 3 \quad \Rightarrow \quad 8x = 0 \quad \Rightarrow \quad x = 0$$

Use, say, L_1 *to find the* $y-$*value at the intersection:* $y = 3(0) + 1 = 1$. *Hence the intersection point is* $(0, 1)$. *(Check the graph to see that this must be the correct answer.)*

(b) $L_2: \ y = 3x + 7$ *and* $L_6: \ y = 2x + 1$ *intersect where*

$$3x + 7 = 2x + 1 \quad \Rightarrow \quad x = -6$$

Use, say, L_6 *to find the* $y-$*value at the intersection:*

$$y = 2(-6) = -12 + 1 = -11$$

so the intersection point is $(-6, -11)$. *(Check the graph to see that this answer is in the right area.)*

(c) $L_4: \ y = -\frac{1}{3}x - \frac{7}{3}$ *and* $L_5: \ y = 1$ *intersect where*

$$-\frac{1}{3}x - \frac{7}{3} = 1 \quad \Rightarrow \quad -x - 7 = 3 \quad \Rightarrow \quad -x = 10 \quad \Rightarrow \quad x = -10$$

Since L_5 *is horizontal the intersection point is* $(-10, 1)$. *(Check the graph to see that this answer is in the right area.)*

10.7 **Application:** *this is an example of a problem in the area of Operations Research (or Management Systems as it is called in Business). Consider the lines*

$$L_1: \ x + y = 4 \qquad L_2: \ x + 2y = 6 \qquad L_3: \ 2x + y = 7$$

(a) *Sketch the graphs of all three functions on the same set of axes.*
Re-arrange the equations into $y = mx + c$ *form:*

$$L_1: \ y = -x + 4 \qquad L_2: \ y = -\frac{1}{2}x + 3 \qquad L_3: \ y = -2x + 7$$

See below for the graphs.

(b) *Find the points of intersection between:*

(i) L_1 *and* L_2

$$-x + 4 = -\frac{1}{2}x + 3 \quad \Rightarrow \quad -2x + 8 = -x + 6 \quad \Rightarrow \quad 2 = x$$

Use, say, L_1 *to find the* $y-$*value at the intersection:*

$$y = -(2) + 4 = 2.$$

Hence, the intersection point is $(2, 2)$.

(ii) L_1 *and* L_3

$$-x + 4 = -2x + 7 \quad \Rightarrow \quad x = 3$$

Use, say, L_1 *to find the* $y-$*value at the intersection:*

$$y = -(3) + 4 = 1.$$

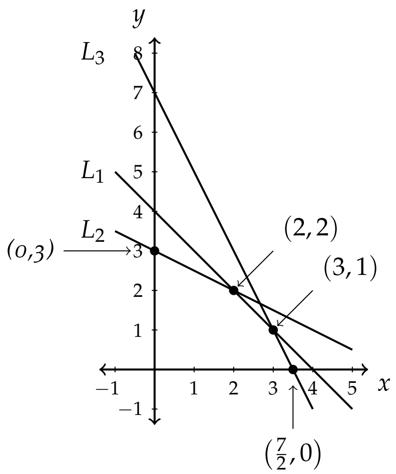
Hence, the intersection point is $(3, 1)$.

(iii) L_2 *and the* $y-$*axis This is just the* $y-$*intercept of* L_2*:* $(0, 3)$.

(iv) L_3 *and the* $x-$*axis This is the* $x-$*intercept of* L_3*:*

$$0 = -2x + 7 \quad \Rightarrow \quad 2x = 7 \quad \Rightarrow \quad x = \frac{7}{2}$$

Hence, the intersection point is $\left(\frac{7}{2}, 0\right)$.



(c) At each of the intersection points in (b) find the value of $z = 3x + 5y$

(i) $(2,2)$*:* $z = 3(2) + 5(2) = 16$*.*

(ii) $(3,1)$*:* $z = 3(3) + 5(1) = 14$

(iii) $(0,3)$*:* $z = 3(0) + 5(3) = 15$

(iv) $\left(\frac{7}{2}, 0\right)$*:* $z = 3\left(\frac{7}{2}\right) + 5(0) = \frac{21}{2}$ *or* $10\frac{1}{2}$

(d) At which intersection point does $z = 3x + 5y$ *attain its greatest value?*
At $(2,2)$ *(greatest value* 16*).*

Problem Set 11

11.1 *Sketch the graphs of the following quadratic functions, in each case indicating*

- *the convexity (concave up or down),*
- *the* $y-$*intercept,*
- *the vertex co-ordinates and*
- *whatever* $x-$*intercepts exist.*

(a) $y = 2x^2 + 8x + 6$

- *concave up*
- *the* $y-$*intercept is at* $(0,6)$
- *to find the* $x-$*intercepts:*

$$\begin{aligned} 2x^2 + 8x + 6 &= 0 \\ \Rightarrow \quad x^2 + 4x + 3 &= 0 \\ \Rightarrow \quad (x+1)(x+3) &= 0 \end{aligned}$$

So, the $x-$*intercepts are* $(-1,0)$ *and* $(-3,0)$*. (You can also use the quadratic formula to find the roots but trying to factorise first is an excellent shortcut.)*

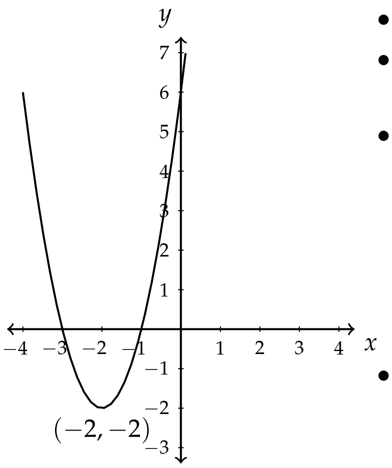
- *the vertex occurs halfway between the* $x-$*intercepts, at* $x = -2$ *(or use*

$$x = \frac{-b}{2a} = \frac{-(4)}{2(1)} = -2)$$

The $y-$*value at* $x = -2$ *is then*

$$\begin{aligned} y &= 2(-2)^2 - 8(-2) + 6 \\ &= 8 - 16 + 6 \\ &= -2 \end{aligned}$$

The vertex is $(-2,-2)$.



(b) $y = x^2 - 2x - 8$

- *concave up*
- *the* $y-$*intercept is at* $(0,-8)$
- *to find the* $x-$*intercepts:*

$$\begin{aligned} x^2 - 2x - 8 &= 0 \\ \Rightarrow \quad (x+2)(x-4) &= 0 \end{aligned}$$

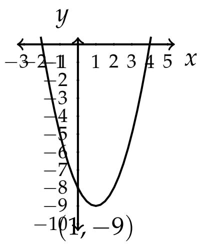
  So, the $x-$*intercepts are* $(-2,0)$ *and* $(4,0)$.
- *the vertex occurs halfway between the* $x-$*intercepts, at* $x = 1$ *(or use*

$$x = \frac{-b}{2a} = \frac{-(-2)}{2(1)} = 1)$$

  The $y-$*value at* $x = 1$ *is then*

$$\begin{aligned} y &= (1)^2 - 2(1) - 8 \\ &= 1 - 2 - 8 \\ &= -9 \end{aligned}$$

  The vertex is $(1,-9)$.



(c) $y = x^2 - 4x + 4$

- *concave up*
- *the* $y-$*intercept is at* $(0, 4)$
- *to find the* $x-$*intercepts:*

$$\begin{aligned} x^2 - 4x + 4 &= 0 \\ \Rightarrow \quad (x-2)(x-2) &= 0 \end{aligned}$$

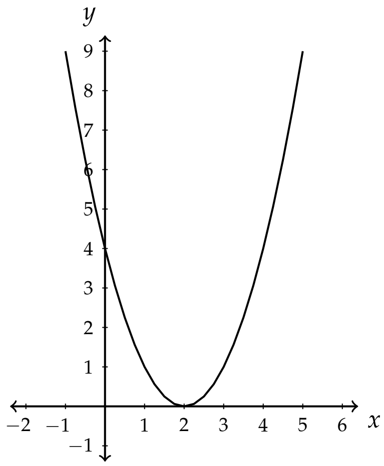
So, the $x-$*intercept is* $(2, 0)$.

- *in this case the vertex occurs halfway is also at the* $x-$*intercept (or use*

$$x = \frac{-b}{2a} = \frac{-(-4)}{2(1)} = 2)$$

The $y-$*value at* $x = 2$ *is clearly 0 so the vertex is* $(2, 0)$.

Note: *this is a feature of perfect squares.*



(d) $y = -x^2 + 4x - 3$

- *concave down*
- *the* $y-$*intercept is at* $(0, -3)$
- *to find the* $x-$*intercepts:*

$$\begin{aligned} -x^2 + 4x - 3 &= 0 \\ \Rightarrow \quad x^2 - 4x + 3 &= 0 \\ \Rightarrow \quad (x-1)(x-3) &= 0 \end{aligned}$$

So, the $x-$*intercepts are* $(1, 0)$ *and* $(3, 0)$.

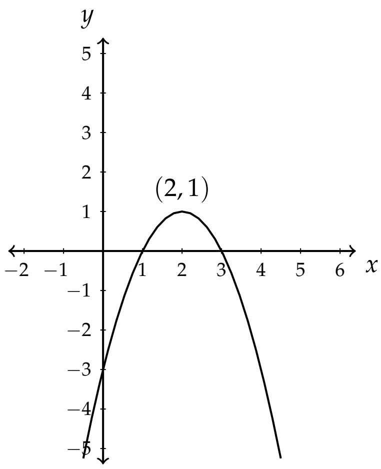
- *the vertex occurs halfway between the* $x-$*intercepts, at* $x = 2$ *(or use*

$$x = \frac{-b}{2a} = \frac{-(4)}{2(-1)} = 2)$$

The $y-$*value at* $x = 2$ *is then*

$$\begin{aligned} y &= -(2)^2 + 4(2) - 3 \\ &= -4 + 8 - 3 \\ &= 1 \end{aligned}$$

The vertex is $(2, 1)$.



(e) $y = x^2 - 4x + 12$

- *concave up*
- *the* $y-$*intercept is at* $(0, 12)$
- *the* $x-$*intercepts look difficult so use the discriminant:*

$$\begin{aligned} b^2 - 4ac &= (-4)^2 - 4(1)(12) \\ &= 16 - 48 \\ &< 0 \end{aligned}$$

So, there are no $x-$*intercepts.*

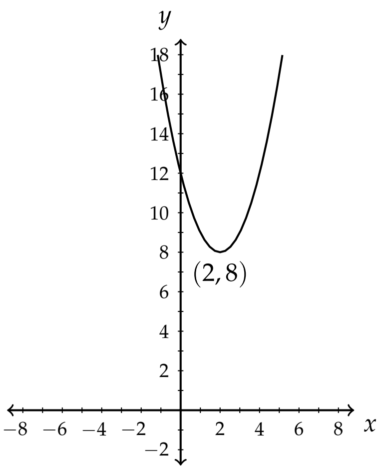
- *the* $x-$*coordinate of the vertex is =*

$$x = \frac{-b}{2a} = \frac{-(-4)}{2(-1)} = 2$$

The $y-$*value at* $x = 2$ *is then*

$$\begin{aligned} y &= (2)^2 - 4\,(2) + 12 \\ &= 4 - 8 + 12 \\ &= 8 \end{aligned}$$

The vertex is $(2, 8)$.



(f) $y = x^2 - 6x$

- *concave up*
- *in this case* $c = 0$ *so the* $y-$*intercept is at* $(0, 0)$.
- *we already know that one of the* $x-$*intercepts is* $(0, 0)$. *Factorizing confirms this and locates the other one:*

$$\begin{aligned} x^2 - 6x &= 0 \\ \Rightarrow \quad x(x-6) &= 0 \end{aligned}$$

So, the other $x-$*intercept is* $(6, 0)$.

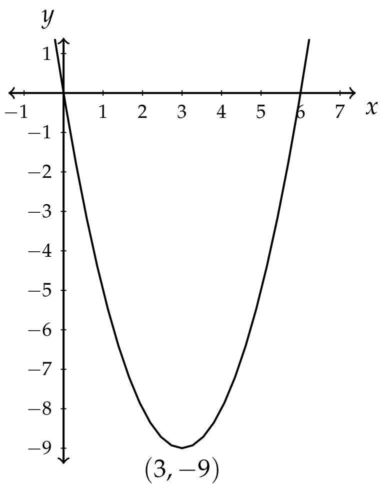
- *the vertex occurs halfway between the* $x-$*intercepts, at* $x = 3$ *(or use*

$$x = \frac{-b}{2a} = \frac{-(-6)}{2(1)} = 3)$$

The $y-$*value at* $x = 3$ *is then*

$$\begin{aligned} y &= -\,(3)^2 - 6\,(3) \\ &= 9 - 18 \\ &= -9 \end{aligned}$$

The vertex is $(3, -9)$.



(g) $y = x^2 - 4$

- *concave up*
- *the y−intercept is at* $(0, -4)$
- *to find the x−intercepts:*

$$\begin{aligned} x^2 - 4 &= 0 \\ \Rightarrow \quad (x-2)(x+2) &= 0 \end{aligned}$$

So, the x−intercepts are $(-2, 0)$ *and* $(2, 0)$.

- *the vertex occurs halfway between the x−intercepts, at* $x = 0$ *(or use*

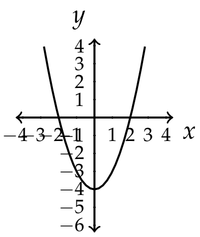
$$x = \frac{-b}{2a} = \frac{-(0)}{2(1)} = 0)$$

The y−value at $x = 0$ *is then*

$$\begin{aligned} y &= -(0)^2 - 4 \\ &= -4 \end{aligned}$$

(or realise that it is also the y−intercept!). The vertex is $(0, -4)$.

Note: *this is a feature of the difference of two squares.*



(h) $y = x^2 - 7x + 10$

- *concave up*
- *the y−intercept is at* $(0, 10)$
- *to find the x−intercepts:*

$$\begin{aligned} x^2 - 7x + 10 &= 0 \\ \Rightarrow \quad (x-2)(x-5) &= 0 \end{aligned}$$

So, the x−intercepts are $(2, 0)$ *and* $(5, 0)$.

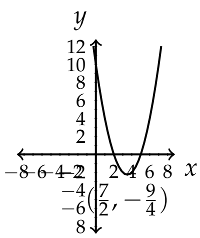
- *the vertex occurs halfway between the x−intercepts, at* $x = \frac{7}{2}$ *(or use*

$$x = \frac{-b}{2a} = \frac{-(-7)}{2(1)} = \frac{7}{2})$$

The y−value at $x = \frac{7}{2}$ *is then*

$$\begin{aligned} y &= \left(\frac{7}{2}\right)^2 - 7\left(\frac{7}{2}\right) + 10 \\ &= \frac{49}{4} - \frac{49}{2} + 10 \\ &= -\frac{9}{4} \end{aligned}$$

The vertex is $(\frac{7}{2}, -\frac{9}{4})$.



11.2 *Complete the square for the following quadratic functions and use the result to verify the co-ordinates of the vertex of the parabola they repesent.*

$$
\begin{aligned}
(a)\quad y &= x^2+8x+3\\
&= x^2+8x+\left(\frac{8}{2}\right)^2-\left(\frac{8}{2}\right)^2+3\\
&= x^2+8x+16-16+3\\
&= (x+4)^2-16+3\\
y &= (x+4)^2-13
\end{aligned}
$$

$(x+4)^2$ can't be negative so the smallest value of y occurs when

$$(x+4)^2=0$$

ie. $x=-4$. The $y-$value at $x=-4$ is then -13, so the coordinates of the vertex are $(-4,-13)$.
The $x-$value of the vertex is also given by

$$x=\frac{-b}{2a}=\frac{-(8)}{2(1)}=-4$$

At $x=-4$:

$$
\begin{aligned}
y &= (-4)^2+8(-4)+3\\
&= 16-32+3\\
&= -13
\end{aligned}
$$

The answers are the same.

$$
\begin{aligned}
(b)\quad y &= x^2-14x\\
&= x^2-14x+\left(\frac{-14}{2}\right)^2-\left(\frac{-14}{2}\right)^2\\
&= x^2-14x+49-49\\
&= (x-7)^2-49
\end{aligned}
$$

$(x-7)^2$ can't be negative so the smallest value of y occurs when

$$(x-7)^2=0$$

ie. $x=7$. The $y-$value at $x=7$ is then -49, so the coordinates of the vertex are $(7,-49)$.
The $x-$value of the vertex is also given by

$$x=\frac{-b}{2a}=\frac{-(-14)}{2(1)}=7$$

At $x=7$:

$$
\begin{aligned}
y &= (7)^2-14(7)\\
&= 49-98\\
&= -49
\end{aligned}
$$

The answers are the same.

Problem Set 12

12.1 *Evaluate the following limits if they exist.*

$$(a)\quad \lim_{x\to 4}\frac{x^2-16}{x-4} = \lim_{x\to 4}\frac{\cancel{(x-4)}(x+4)}{\cancel{x-4}} = \lim_{x\to 4}(x+4) = 4+4 = 8$$

$$(b)\quad \lim_{x\to 2}\frac{x^2+2x-8}{x-2} = \lim_{x\to 2}\frac{\cancel{(x-2)}(x+4)}{\cancel{x-2}} = \lim_{x\to 2}(x+4) = 2+4 = 6$$

$$(c)\quad \lim_{x\to 4}\frac{x^2-x-12}{x-1} = \lim_{x\to 4}\frac{(x-4)(x+3)}{x-1} = \frac{(4-4)(4+3)}{4-1} = \frac{0\times 7}{3} = \frac{0}{3} = 0$$

$$(d)\quad \lim_{x\to 3}\frac{x^2-6x+9}{x-3} = \lim_{x\to 3}\frac{\cancel{(x-3)}(x-3)}{\cancel{x-3}} = \lim_{x\to 3}(x-3) = 3-3 = 0$$

$$(e)\quad \lim_{x\to 2}\frac{4x^2-16}{x-2} = \lim_{x\to 2}\frac{4(x^2-4)}{x-2} = \lim_{x\to 2}\frac{4\cancel{(x-2)}(x+2)}{\cancel{x-2}} = \lim_{x\to 2}(4(x+2)) = 4(2+2) = 4\times 4 = 16$$

$$(f) \quad \lim_{x\to -7} \frac{x^2+15x+56}{x+7} = \lim_{x\to -7} \frac{\cancel{(x+7)}(x+8)}{\cancel{x+7}} = \lim_{x\to -7}(x+8) = -7+8 = 1$$

12.2 *Differentiate each of the following functions.*

$$(a) \quad y = x^3+x^2+x+6 \quad \Rightarrow \quad \frac{dy}{dx} = 3x^2+2x+1$$

$$(b) \quad y = 3x^4-x^3+x^2+x-2 \quad \Rightarrow \quad \frac{dy}{dx} = 12x^3-3x^2+2x+1$$

$$(c) \quad y = x^2+\sqrt{x}+\frac{1}{x}-3 \quad \Rightarrow \quad \frac{dy}{dx} = 2x+\frac{1}{2\sqrt{x}}-\frac{1}{x^2}$$

12.3 *To differentiate*

$$y = \left(\sqrt{x}+\frac{1}{x}\right)\left(x^2+\frac{1}{x^2}\right)$$

we need to expand the brackets. We get

$$y = x^2\sqrt{x}+\frac{\sqrt{x}}{x^2}+\frac{x^2}{x}+\frac{1}{x^3} = x^2x^{\frac{1}{2}}+x^{\frac{1}{2}}x^{-2}+x+x^{-3} = x^{\frac{5}{2}}+x^{-\frac{3}{2}}+x+x^{-3}$$

and because $\frac{d}{dx}(x^n) = nx^{n-1}$ *we get*

$$\frac{dy}{dx} = \frac{5}{2}x^{\frac{3}{2}}-\frac{3}{2}x^{-\frac{5}{2}}+1-3x^{-4}$$

12.4 *Perform the following integrations*

$$(a) \quad \int \left(x^2+x+6\right)dx = \frac{x^3}{3}+\frac{x^2}{2}+6x+C$$

$$(b) \quad \int \left(3x^4-x^3-2\right)dx = \frac{3x^5}{5}-\frac{x^4}{4}-2x+C$$

$$(c) \quad \int \left(x^2+\sqrt{x}+\frac{4}{x^2}\right)dx = \frac{x^3}{3}+\frac{2x^{\frac{3}{2}}}{3}-\frac{4}{x}+C$$

12.5 *Evaluate the following definite integrals.*

$$(a) \quad \int_0^1 \left(x^3+x^2\right)dx = \left[\frac{x^4}{4}+\frac{x^3}{3}\right]_0^1 = \left(\frac{1^4}{4}+\frac{1^3}{3}\right)-\left(\frac{0^4}{4}+\frac{0^3}{3}\right) = \frac{1}{4}+\frac{1}{3} = \frac{7}{12}$$

$$(b) \quad \int_2^3 \left(5x^4+5x-2\right)dx = \left[x^5+\frac{5x^2}{2}-2x\right]_2^3 = \left(3^5+\frac{5(3^2)}{2}-2(3)\right)-\left(2^5+\frac{5(2^2)}{2}-2(2)\right)$$

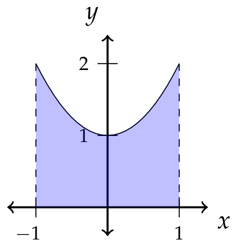
$$= \left(243+\frac{45}{2}-6\right)-\left(32+\frac{20}{2}-4\right) = 237+\frac{45}{2}-28-\frac{20}{2} = 209+\frac{25}{2} = \frac{418+25}{2} = \frac{443}{2}$$

$$(c) \quad \int_{-2}^2 \left(x^2+4x-3\right)dx = \left[\frac{x^3}{3}+2x^2-3x\right]_{-2}^2 = \left(\frac{2^3}{3}+2(2^2)-3(2)\right)-\left(\frac{(-2)^3}{3}+2((-2)^2)-3(-2)\right)$$

$$= \left(\frac{8}{3}+8-6\right)-\left(-\frac{8}{3}+8+6\right) = \frac{8}{3}+2+\frac{8}{3}-14 = \frac{16}{3}-12 = \frac{16-36}{3} = -\frac{20}{3}$$

12.6 *Find the AREA bounded by the graph of f(x) and the x-axis between the points indicated for the functions below. The graphs are shown below.*

Hint: Sketch the graph first.



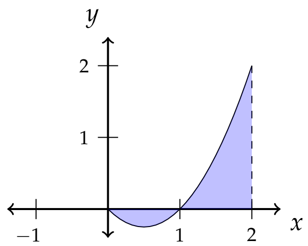
$(a) \quad f(x) = x^2 + 1, \qquad x = -1 \text{ to } x = 1$

The region is above the x-axis so

$$\text{Area} = \int_{-1}^{1} \left(x^2+1\right) dx = \left[\frac{x^3}{3} + x\right]_{-1}^{1}$$

$$= \left(\frac{1}{3}+1\right) - \left(\frac{-1}{3} - 1\right)$$

$$= \frac{1}{3} + 1 + \frac{1}{3} + 1 = \frac{2}{3} + 2 = \frac{8}{3}$$



$(b) \quad f(x) = x^2 - x, \qquad x = 0 \text{ to } x = 2$

Some of the region is above the x-axis and some below so

$$\text{Area} = -\int_{0}^{1} \left(x^2 - x\right) dx + \int_{1}^{2} \left(x^2 - x\right) dx$$

$$= -\left[\frac{x^3}{3} - \frac{x^2}{2}\right]_0^1 + \left[\frac{x^3}{3} - \frac{x^2}{2}\right]_1^2$$

$$= -\left(\frac{1}{3} - \frac{1}{2}\right) + 0 + \left(\frac{8}{3} - \frac{4}{2}\right) - \left(\frac{1}{3} - \frac{1}{2}\right)$$

$$= -\frac{1}{3} + \frac{1}{2} + \frac{8}{3} - 2 - \frac{1}{3} + \frac{1}{2} = \frac{6}{3} - 1 = 1$$

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