# Real Functions in Several Variables: Volume II

Continuous Functions in Several Variables

Leif Mejlbro

bookboon
The eBook company

Leif Mejlbro

# Real Functions in Several Variables

## Volume-II Continuous Functions in Several Variables

Real Functions in Several Variables: Volume-II Continuous Functions in Several Variables
2nd edition

ISBN 978-87-403-0908-9

# Contents

# Preface

The topic of this series of books on *"Real Functions in Several Variables"* is very important in the description in e.g. *Mechanics* of the real 3-dimensional world that we live in. Therefore, we start from the very beginning, modelling this world by using the coordinates of $\mathbb{R}^3$ to describe e.g. a motion in space. There is, however, absolutely no reason to restrict ourselves to $\mathbb{R}^3$ alone. Some motions may be rectilinear, so only $\mathbb{R}$ is needed to describe their movements on a line segment. This opens up for also dealing with $\mathbb{R}^2$, when we consider plane motions. In more elaborate problems we need higher dimensional spaces. This may be the case in *Probability Theory* and *Statistics*. Therefore, we shall in general use $\mathbb{R}^n$ as our abstract model, and then restrict ourselves in examples mainly to $\mathbb{R}^2$ and $\mathbb{R}^3$.

For rectilinear motions the familiar *rectangular coordinate system* is the most convenient one to apply. However, as known from e.g. *Mechanics*, circular motions are also very important in the applications in engineering. It becomes natural alternatively to apply in $\mathbb{R}^2$ the so-called *polar coordinates* in the plane. They are convenient to describe a circle, where the rectangular coordinates usually give some nasty square roots, which are difficult to handle in practice.

Rectangular coordinates and polar coordinates are designed to model each their problems. They supplement each other, so difficult computations in one of these coordinate systems may be easy, and even trivial, in the other one. It is therefore important always in advance carefully to analyze the geometry of e.g. a domain, so we ask the question: Is this domain best described in rectangular or in polar coordinates?

Sometimes one may split a problem into two subproblems, where we apply rectangular coordinates in one of them and polar coordinates in the other one.

It should be mentioned that in *real life* (though not in these books) one cannot always split a problem into two subproblems as above. Then one is really in trouble, and more advanced mathematical methods should be applied instead. This is, however, outside the scope of the present series of books.

The idea of polar coordinates can be extended in two ways to $\mathbb{R}^3$. Either to *semi-polar* or *cylindric coordinates*, which are designed to describe a cylinder, or to *spherical coordinates*, which are excellent for describing spheres, where rectangular coordinates usually are doomed to fail. We use them already in daily life, when we specify a place on Earth by its longitude and latitude! It would be very awkward in this case to use rectangular coordinates instead, even if it is possible.

Concerning the contents, we begin this investigation by modelling point sets in an $n$-dimensional Euclidean space $E^n$ by $\mathbb{R}^n$. There is a subtle difference between $E^n$ and $\mathbb{R}^n$, although we often identify these two spaces. In $E^n$ we use *geometrical methods* without a coordinate system, so the objects are independent of such a choice. In the coordinate space $\mathbb{R}^n$ we can use ordinary calculus, which in principle is not possible in $E^n$. In order to stress this point, we call $E^n$ the "abstract space" (in the sense of calculus; not in the sense of geometry) as a warning to the reader. Also, whenever necessary, we use the colour black in the "abstract space", in order to stress that this expression is theoretical, while variables given in a chosen coordinate system and their related concepts are given the colours blue, red and green.

We also include the most basic of what mathematicians call *Topology*, which will be necessary in the following. We describe what we need by a function.

Then we proceed with limits and continuity of functions and define continuous curves and surfaces, with parameters from subsets of $\mathbb{R}$ and $\mathbb{R}^2$, resp..

Continue with (partial) differentiable functions, curves and surfaces, the chain rule and Taylor's formula for functions in several variables.

We deal with maxima and minima and extrema of functions in several variables over a domain in $\mathbb{R}^n$. This is a very important subject, so there are given many worked examples to illustrate the theory.

Then we turn to the problems of integration, where we specify four different types with increasing complexity, *plane integral, space integral, curve (or line) integral and surface integral*.

Finally, we consider *vector analysis*, where we deal with vector fields, Gauß's theorem and Stokes's theorem. All these subjects are very important in theoretical Physics.

The structure of this series of books is that each subject is *usually* (but not always) described by three successive chapters. In the first chapter a brief theoretical theory is given. The next chapter gives some practical guidelines of how to solve problems connected with the subject under consideration. Finally, some worked out examples are given, in many cases in several variants, because the standard solution method is seldom the only way, and it may even be clumsy compared with other possibilities.

I have as far as possible structured the examples according to the following scheme:

**A** *Awareness*, i.e. a short description of what is the problem.

**D** *Decision*, i.e. a reflection over what should be done with the problem.

**I** *Implementation*, i.e. where all the calculations are made.

**C** *Control*, i.e. a test of the result.

This is an ideal form of a general procedure of solution. It can be used in any situation and it is not linked to Mathematics alone. I learned it many years ago in the Theory of Telecommunication in a situation which did not contain Mathematics at all. The student is recommended to use it also in other disciplines.

From high school one is used to immediately to proceed to **I**. *Implementation*. However, examples and problems at university level, let alone situations in real life, are often so complicated that it in general will be a good investment also to spend some time on the first two points above in order to be absolutely certain of what to do in a particular case. Note that the first three points, **ADI**, can always be executed.

This is unfortunately not the case with **C** *Control*, because it from now on may be difficult, if possible, to check one's solution. It is only an extra securing whenever it is possible, but we cannot include it always in our solution form above.

I shall on purpose not use the logical signs. These should in general be avoided in Calculus as a shorthand, because they are often (too often, I would say) misused. Instead of $\wedge$ I shall either write "and", or a comma, and instead of $\vee$ I shall write "or". The arrows $\Rightarrow$ and $\Leftrightarrow$ are in particular misunderstood by the students, so they should be totally avoided. They are not telegram short hands, and from a logical point of view they usually do not make sense at all! Instead, write in a plain language what you mean or want to do. This is difficult in the beginning, but after some practice it becomes routine, and it will give more precise information.

When we deal with multiple integrals, one of the possible pedagogical ways of solving problems has been to colour variables, integrals and upper and lower bounds in blue, red and green, so the reader by the colour code can see in each integral what is the variable, and what are the parameters, which do not enter the integration under consideration. We shall of course build up a hierarchy of these colours, so the order of integration will always be defined. As already mentioned above we reserve the colour black for the theoretical expressions, where we cannot use ordinary calculus, because the symbols are only shorthand for a concept.

The author has been very grateful to his old friend and colleague, the late Per Wennerberg Karlsson, for many discussions of how to present these difficult topics on real functions in several variables, and for his permission to use his textbook as a template of this present series. Nevertheless, the author has felt it necessary to make quite a few changes compared with the old textbook, because we did not always agree, and some of the topics could also be explained in another way, and then of course the results of our discussions have here been put in writing for the first time.

The author also adds some calculations in MAPLE, which interact nicely with the theoretic text. Note, however, that when one applies MAPLE, one is forced first to make a geometrical analysis of the domain of integration, i.e. apply some of the techniques developed in the present books.

The theory and methods of these volumes on "Real Functions in Several Variables" are applied constantly in higher Mathematics, Mechanics and Engineering Sciences. It is of paramount importance for the calculations in *Probability Theory*, where one constantly integrate over some point set in space.

It is my hope that this text, these guidelines and these examples, of which many are treated in more ways to show that the solutions procedures are not unique, may be of some inspiration for the students who have just started their studies at the universities.

Finally, even if I have tried to write as careful as possible, I doubt that all errors have been removed. I hope that the reader will forgive me the unavoidable errors.

Leif Mejlbro
March 21, 2015

# Introduction to volume II,<br>Continuous Functions in Several Variables

This is the second volume in the series of books on *Real Functions in Several Variables*. We start in Chapter 5 with the necessary theoretical background. Here we assume that the theory of volume I is known by the reader.

We introduce maps and functions, including vector functions, and we give some guidelines on how to visualize such functions. This is not always an easy task, because we easily are forced to consider graphs lying in spaces of dimension $\geq 4$, where very few human beings have a geometrical understanding of what is going on.

Then we introduce the *continuous functions*, starting with defining the basic concept of what we understand by taking a limit. We must apparently have some sense of "distance" in order to say that two points are close to each other. We therefore make use of the topological notions of norm and distance already introduced in volume I.

Continuous functions are then defined as functions, for which "the image points are lying close together, whenever the points themselves are close to each other". We of course make this more precise in the text.

The first application of continuous functions is to introduce *continuous curves*. The safest description of such curves, though it is not always necessary, is to use a *parametric description* of them. This is also done in MAPLE, and at the same time we get a sense of direction of a motion along the curve from an initial point to a final point.

Then we use the continuous curves to define (curve) connected sets, which are the only connected sets we shall consider here. (There exist sets which are connected, but not curve connected; but they will not be of interest to us.) A set $A$ is (curve) connected, if any two points $\mathbf{x}$ and $\mathbf{y} \in A$ can always be connected with a continuous curve, which lies entirely in $A$. If $A \in \mathbb{R}^n$ is *open*, then any two points can always be connected by a continuous curve of a very special and convenient structure. The curve consists of concatenated line segments, where each of them is parallel to one of the axes in $\mathbb{R}^n$. This property will be very useful in the theory of integration later on.

If furthermore, two curves connecting any two given points $\mathbf{x}$ and $\mathbf{y} \in A$ can be transformed continuously into each other without leaving $A$ during this transformation process, then $A$ in some sense "does not contain holes", and $A$ is called *simply connected*. As one would expect, simply connected sets have better properties than sets, which are only connected.

Once we have introduced continuous curves, using a parametric description, where the parameter set $I$ of course is a one-dimensional interval, it is formally straightforward to replace this one-dimensional parameter interval $I$ for a one-dimensional curve by a two-dimensional interval to get a two-dimensional surface. Then we discover that it is not essential that the parameter set indeed is an interval. A two-dimensional connected set will suffice.

The vague definition above of a surface is of course not precise, so we must first get rid of all pathological cases, but in general a continuous function $\mathbf{r} : E \to \mathbb{R}^n$, where $E$ is a two-dimensional connected set, defines a two-dimensional surface $\mathcal{F}$ in $\mathbb{R}^n$. If $n = 3$, we can visualize the process of the function $\mathbf{r}$ as taking a two-dimensional plate of shape $E$ and then bend, compress and stretch this plate, such that we in the end obtain the surface $\mathcal{F}$ of the wanted shape in e.g. $\mathbb{R}^3$.

The above gives the general idea, although matters are not always that easy.

A parameter set $E \subseteq \mathbb{R}^2$ may have a non-empty boundary $\partial E$. We would expect that it is mapped by $\mathbf{r}$ into the "boundary" $\delta\mathcal{F}$ of the surface $\mathcal{F}$. Since topologically $\mathcal{F} = \partial\mathcal{F}$ is equal to its own boundary, we must describe, what is meant by the "boundary" of the different notation $\delta\mathcal{F}$ in $\mathcal{F}$. Usually, $\delta\mathcal{F} = \mathbf{r}(\partial E)$, but is easy to construct examples, where $\delta\mathcal{F}\,(\subseteq \mathbf{r}(\partial E))$ is not equal to $\mathbf{r}(\partial E)$.

Finally we recall (without proofs) the three main theorems for continuous functions, and we show some of their simplest implications, which will be used over and over again in the following volumes.

Chapter 6 on practical guidelines is very short in this volume.

Then follows a fairly long Chapter 7 with examples, following more or less the same structure as the theoretical Chapter 5, so the reader may consult both chapter, when reading this book.

Chapter 8 on Formulæ is identical with Chapter 4 in volume I. It is convenient to have these formulæ at the end of the books as reference, although many people alternatively may use MAPLE or MATHEMATICA instead.

The index is the same in all volumes, and it covers the whole text.

# 5 Continuous maps and functions in several variables

## 5.1 Maps in general

We shall restrict ourselves to the concept of a map from a subset of $\mathbb{R}^n$ into $\mathbb{R}^m$, i.e. a map is here defined on a set $D \subseteq \mathbb{R}^n$ in a coordinate space,

$$\mathbf{f} : D \to \mathbb{R}^m, \qquad \mathbf{x} \mapsto \mathbf{f}(\mathbf{x}), \qquad \text{where } D \subseteq \mathbb{R}^n.$$

This is the precise notation, but it is in general too complicated, so we shall allow ourselves to use a shorthand like

$$\mathbf{f} : D \to \mathbb{R}^m, \qquad \text{where } D \subseteq \mathbb{R}^n.$$

If $\mathbb{R}^m$ and $\mathbb{R}^n$ already are given, we shall just write

$$\mathbf{f}(\mathbf{x}) \quad \text{for } \mathbf{x} \in D, \qquad \text{or just } \mathbf{f}(\mathbf{x}) \text{ or } \mathbf{f}.$$

The notation

$$D \xrightarrow{\mathbf{f}} \mathbb{R}^m$$

may be useful, when we put several maps together into the same schematic structure in order to get a feeling of what is going on, when we e.g. form some compositions of maps.

The map $\mathbf{f} : D \to \mathbb{R}^m$ has its *domain* $D \subseteq \mathbb{R}^n$, and we call $\mathbf{f}(D)$ $[\subseteq \mathbb{R}^m]$ its *range*. The map is said to be *surjective* $\mathbf{f} : D \to \mathbf{f}(D)$, i.e. every point of $\mathbf{f}(D)$ is the image of at least one point of $D$. If every point of $\mathbf{f}(D)$ is the image of precisely one point $\mathbf{x} \in D$, then $\mathbf{f}$ is called *injective.*. If $\mathbf{f} : D \to \mathbb{R}^m$ is injective, then as seen above, it is both an injective and surjective map of $D$ onto the range $\mathbf{f}(D)$, and we call in this case $\mathbf{f}$ a *bijective map* or a *1-1 map*.

We shall use a little of our previously introduced *Topology*. We say that a map $\mathbf{f} : D \to \mathbb{R}^m$ is *bounded*, if there exists a ball $B$ of finite radius in $\mathbb{R}^m$, such that $\mathbf{f}(D) \subseteq B$. The terminology agrees with what one would expect. A ball of finite radius must be bounded, and so is every subset of this ball.

It must be emphasized that a map $\mathbf{f} : D \to \mathbb{R}^m$ is specified by the operations defined by $\mathbf{f}$ itself, *as well of its specified domain D!* If we for some reason extend the domain $D$ to some other $D_1$, in which the operations given by $\mathbf{f}$ still make sense, or we let $D_1 \subset D$ be a real subset of $D$, so $\mathbf{f}$ is defined by restriction to $D_1$, *then* $\mathbf{f}_1 : D_1 \to \mathbb{R}^m$ *is not considered as the same map as* $\mathbf{f} \to \mathbb{R}^m$, although they are strongly related. We note the following important special cases: Given a map $\mathbf{f} : D \to \mathbb{R}^m$.

1) If $\mathbf{f}_1 : D_1 \to \mathbb{R}^m$ satisfies

$$D_1 \subset D \qquad \text{and} \qquad \mathbf{f}_1(\mathbf{x}) = \mathbf{f}(\mathbf{x}) \text{ for all } \mathbf{x} \in D_1,$$

then $(\mathbf{f}_1, D_1)$ is called a *restriction* of $(\mathbf{f}, D)$.

2) If $\mathbf{f}_1 : D_1 \to \mathbb{R}^m$ satisfies

$$D \subset D_1 \qquad \text{and} \qquad \mathbf{f}_1(\mathbf{x}) = \mathbf{f}(\mathbf{x}) \text{ for all } \mathbf{x} \in D,$$

then $(\mathbf{f}_1, D_1)$ is called an *extension* of $(\mathbf{f}, D)$.

There are of course other possibilities, but they are not as important as the two cases described above.

In practice we shall want to specify the map $\mathbf{f}$ by its coordinates in $D \subseteq \mathbb{R}^n$. This may be written in the following way, or similarly,

$$\mathbf{f}(\mathbf{x}) = \cdots , \qquad \text{where } \mathbf{x} \in \cdots ,$$

where we for $\mathbf{x} \in \cdots$ write a specification of $D$ using equations or inequalities between expressions in its coordinates.

One problem often occurs in practice. We may by some theoretical analysis have derived the structure of the map $\mathbf{f}$, but somehow we have not specified its domain $D$. Then the normal procedure is to analyze $\mathbf{f}$ in order to find the *maximal domain*, in which $\mathbf{f}$ can be defined. Some guidelines are given in Section 5.2 and Chapter 6. This maximal domain is defined by Mathematics alone. We may therefore later for physical reasons be forced to restrict this (mathematical) maximal domain, when we interpret the model in the real world. One example is that we may get a relation (a map) in which the temperature in Kelvin occurs. The maximal domain of the map may in a mathematical sense allow the temperature to be negative, which of course is not possible in Physics.

## 5.2 Functions in several variables

Assume that the map $\mathbf{f} : D \to \mathbb{R}$ maps into the real line $\mathbb{R}$, i.e. $m = 1$. In this case, when the range is one-dimensional it is customary to call $\mathbf{f}$ a function, and we change the notation to $f : D \to \mathbb{R}$.

Let $f : D \to \mathbb{R}$ be a function, where the domain $D \subseteq \mathbb{R}^n$ is of dimension $\geq 2$. Then $f$ is called a *function in several (real) variables*. In the present case we have $n$ variables. Using the well-known theory of real functions in one real variable it is possible to derive simple properties of $f$ by restricting $f$ to one-dimensional subsets of $D$.

We shall in the following illustrate the question of maximal domain of a given function. This was introduced in Section 5.1 in general for maps.

1) Given $f_1(x,y) = \exp\left(x^2 + 2y^2\right)$ in $\mathbb{R}^2$. Since exp is defined for all $z \in \mathbb{R}$, and $z = x^2 + 2y^2 \in \mathbb{R}$ for all $(x,y) \in \mathbb{R}^2$, the maximal domain is $\mathbb{R}^2$.

2) Given

$$f_2(x,y) = \sqrt{x} + \sqrt{y} + \frac{1}{xy}$$

in $\mathbb{R}^2$. The square root $\sqrt{z}$ is only defined in the real for $z \geq 0$, so we must require that both $x \geq 0$ and $y \geq 0$. However, a denominator must never be zero, so we also require that $xy \neq 0$, and we conclude that the maximal domain is the open first quadrant $\mathbb{R}_+^2$.

3) Given $f_3(x,y) = \ln(x-1) + \sqrt{2-y}$ in $\mathbb{R}^2$. The logarithm is only defined, if $z = x - 1 > 0$, i.e. $z > 1$, and the square root is only defined for $z = 2 - y \geq 0$, i.e. for $y \leq 2$. We conclude that the maximal domain of $f_3$ is $D_3 = ]1, +\infty[ \times ]-\infty, 2]$, where we usually would prefer just to write $x > 1$ and $y \leq 2$ instead.

4) The function

$$f_4(x,y) = \frac{1}{x^2+2y^2-2x+1}$$

in $\mathbb{R}^2$ is defined, when the denominator is $\neq 0$, i.e. when

$$0 \neq x^2+2y^2-2x+1 = (x-1)^2+2y^2.$$

The only requirement is that $(x,y) \neq (1,0)$, so the maximal domain of $f_4$ is $\mathbb{R}^2 \setminus \{(1,0)\}$.

5) Given in $\mathbb{R}^2$ the function

$$f_5(x,y) = \sqrt{4-x^2-y^2} + \sqrt{y}.$$

The requirements of the domain are $y \geq 0$ and $4-x^2-y^2 \geq 0$, i.e. $x^2+y^2 \leq 4 = 2^2$, so the maximal domain $D$ is the closed half-disc on Figure 5.1.

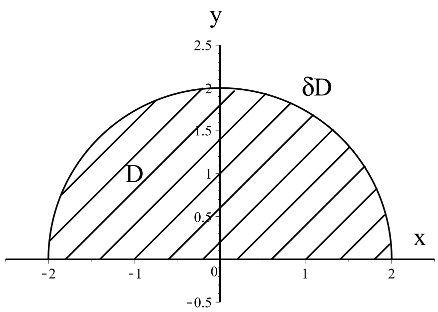



Figure 5.1: The maximal domain of $f_5$ is a closed half-disc.

Its boundary $\partial D$ is composed of the line segment $[-2,2]$ on the $x$-axis, where $y=0$, and the half-circle $x^2+y^2=2^2=4$, $y \geq 0$, in the upper half-plane, i.e. $y = +\sqrt{4-x^2}$. The restriction of $f_5$ to $\partial D$ is given by

$$\begin{cases} F_{5,1}(x) = f_5(x,0) = \sqrt{4-x^2}, & \text{for } x \in [-2,2], \\ F_{5,2}(x) = f_5\left(x, \sqrt{4-x^2}\right) & \text{for } x \in [-2,2]. \end{cases}$$

It is a coincidence that $F_{5,1}$ and $F_{5,2}$ look the same. The reader should note the construction above, because such restrictions to the boundary will be very important in the following chapters, when we shall find the maximum and minimum of a function.

6) A commonly used restriction is the restriction of a function to a line. We may in $\mathbb{R}^2$ use the following parametric description,

$$\varphi(t) := (x_0 + \alpha t, y_0 + \beta t), \qquad t \in \mathbb{R},$$

where $(\alpha, \beta) \neq (0,0)$. If $\alpha = 0$ (and $\beta \neq 0$), we get the vertical line (parametric description)

$$\varphi(y) = (x_0, \beta y), \qquad y \in \mathbb{R},$$

where we clearly cannot use $x$ as a parameter. If $\alpha \neq 0$, we may for convenience choose $\alpha = 1$, so by some reformulation we get

$$\varphi(x) = (x, y_0 + \beta x), \qquad x \in \mathbb{R}.$$

The parametric description i $t$ above is the safest to apply. It is also used in MAPLE. If we use the other possibilities, there is an unexplainable tendency of forgetting the possibility of a vertical line.

7) Consider in $\mathbb{R}^2$ the function

$$f_7(x,y) = \frac{x-y}{x}.$$

Its maximal domain in mathematical sense is given by $x \neq 0$, i.e. the maximal domain consists of all points in $\mathbb{R}^2$, except for the points on the $y$-axis.

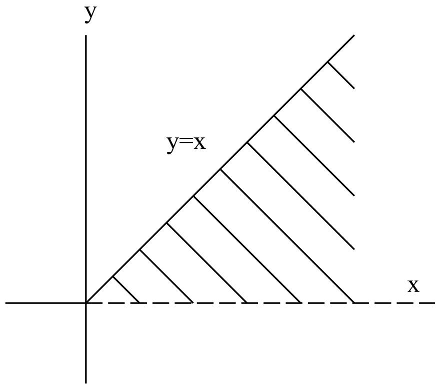



Figure 5.2: The thermodynamical domain of the function $f_7$. This is clearly not equal to the maximal domain of $f_7$ in the mathematical sense.

We may interpret $f_7(x,y)$ in *Thermodynamics* as the theoretical efficiency of a given engine, which interacts with two heat reservoirs, a cold one of temperature $y$, and a warmer one of temperature $x$. Then we must require of thermodynamical reasons that

$$x > 0, \quad y > 0, \quad \text{and} \quad x \geq y,$$

because temperatures measured in Kelvin are always positive. This means that the *thermodynamical domain* is the restriction given in Figure 5.2.

## 5.3 Vector functions

Consider the map $\mathbf{f} : D \to \mathbb{R}^m$, $D \subseteq \mathbb{R}^n$, where $m > 1$. Then we call $\mathbf{f}$ a *vector function*. It is written in the following way,

$$\mathbf{f} = (f_1, \dots, f_m), \qquad \mathbf{f}(\mathbf{x}) = (f_1(\mathbf{x}), \dots, f_m(\mathbf{x})).$$

The functions $f_1, \dots, f_m$ are called the *coordinate functions*. Using the ordinary orthonormal basis in $\mathbb{R}^m$ and the inner (dot) product, the *projections* of $\mathbf{f}(\mathbf{x})$ onto the lines defined by the basis vectors are given by

$$f_1(\mathbf{x}) = \mathbf{e}_1 \cdot \mathbf{f}(\mathbf{x}), \; \cdots, \; f_m(\mathbf{x}) = \mathbf{e}_m \cdot \mathbf{f}(\mathbf{x}).$$

The *maximal domain* of a vector function $\mathbf{f} = (f_1, \dots, f_m)$ is defined as the intersection of all the maximal domains of its coordinate functions $f_1, \dots, f_m$.

If $n = m > 1$, i.e. domain and range are of the same dimension $> 1$, then the vector function $\mathbf{f} : D \to \mathbb{R}^m$ is called a *vector field*.

If $n = 1$, and all coordinate functions are differentiable in the variable $t \in D \subseteq \mathbb{R}$, then we define

$$\frac{d\mathbf{f}}{dt} := \left(\frac{df_1}{dt}, \dots, \frac{df_m}{dt}\right).$$

Similarly, if they are all integrable for $t \in [a, b]$,

$$\int_a^b \mathbf{f}(t)\,\mathrm{d}t = \left(\int_a^b f_1(t)\,\mathrm{d}t, \dots, \int_a^b f_m(t)\,\mathrm{d}t\right).$$

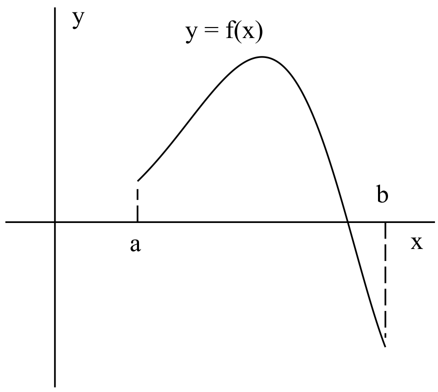



Figure 5.3: The graph of a function $f$ defined in the interval $I = [a, b]$.

## 5.4 Visualization of functions

Nothing can be more instructive than an illustrative figure. In the case of describing a map we e.g. sketch its *graph*.

Let us first consider an ordinary function in one variable

$$f : I \to \mathbb{R}, \qquad \text{where } I \subseteq \mathbb{R}.$$

Then its *graph* is defined as the set

$$\left\{(x, y) \in \mathbb{R}^2 \mid y = f(x),\, x \in I\right\} \subset \mathbb{R}^2.$$

In the given case, the graph is a curve in the plane $\mathbb{R}^2$, cf. Figure 5.3.

A function $f : D \to \mathbb{R}$ in several variables has similarly given a graph. If e.g. $D \subseteq \mathbb{R}^2$, and $f : D \to \mathbb{R}$, then the *graph* of $f$ is given by

$$\left\{(x, y, z) \in \mathbb{R}^3 \mid z = f(x, y),\, (x, y) \in D\right\}.$$

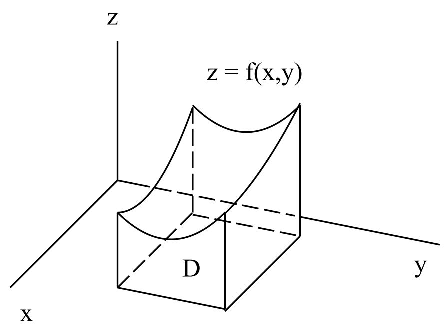



Figure 5.4: The graph of a function $f$ defined in the interval $I = [a, b]$.

In this case the graph becomes a *surface* in $\mathbb{R}^3$, cf. Figure 5.4
However, it is often difficult – even in MAPLE – to sketch the graph of a function in two variables, so instead one may introduce *level curves* of $f$. These are defined by fixing $z = \alpha$, where the constant $\alpha$ is a value of the range of $f$. Cf. Figure 5.5.

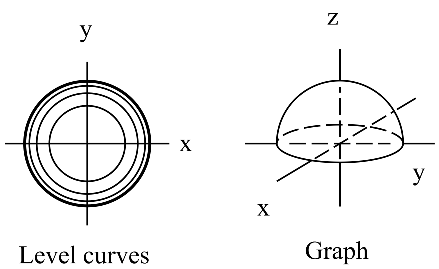



Figure 5.5: To the left we depict the level curves of the function $z = f(x, y) = 1 - x^2 - y^2$ for $\alpha = 0$, 0.2, 0.4, 0.6 and 0.8. The level curves are not equally spaced. To the right we have for comparison sketched the graph of $z = 1 - x^2 - y^2$. The level curves are in the $xy$-plane, while the graph lies in the $xyz$-space. We note that when the level curves are close to each others, the graph is very steep.

If the domain $D$ is of dimension 3 (or higher), the graph description of the function $f : D \to \mathbb{R}$ becomes impossible, because the graph is then at least a curved 3-dimensional space in the 4-dimensional $\mathbb{R}^4$. The author has only met one person, who actually could argue geometrically in $E^4$, namely his late professor in Geometry back in the 1970s. He told us young people that he could "see" some "vague shadows" in $E^4$. Not many people have this gift, so we must instead use the idea of level curves. We define in analogy with the above a *level surface* in the following way for a function $f : D \to \mathbb{R}$, where $D \subseteq \mathbb{R}^3$,

$$\left\{(x,y,z) \in \mathbb{R}^3 \mid f(x,y,z) = \alpha, (x,y,z) \in D\right\}, \qquad \alpha \in f(D) \text{ fixed.}$$

In general, the level surfaces may be complicated to sketch. However, the idea is not quite impossible in all cases.

Obviously, vector functions are far more difficult to visualize, unless one restricts oneself to only considering each coordinate function separately. Another possibility is to sketch the so-called *field lines*, which are curves which in each point take the value (a vector) of the vector function as its tangent.

## 5.5 Implicit given functions

We quite often end up – in particular in the applications in Physics – with an equation in some variables, which clearly are dependent of each other, but where it is not obvious which variable should be chosen as a function of the others, and where the function expression may be quite complicated. In order to explain this problem, let us for simplicity consider the case of three variables, which satisfy a relation like e.g.

(5.1) $F(x,y,z)=0,$

where $F : D \to \mathbb{R}$, $D \subseteq \mathbb{R}^3$, is a function in three variables. If $F$ is continuous, then (5.1) describes a surface in $\mathbb{R}^3$, cf. Section 5.4.

This surface is far from always a graph of a function. If e.g. $F(x,y,z) = x^2+y^2+z^2-1$, then (5.1) describes the unit sphere. When we solve the equation (5.1) with respect to e.g. $z$, we get two possible values,

$$x = \pm\sqrt{1-x^2-y^2} \qquad \text{for } x^2+y^2 \le 1,$$

defined in the closed unit disc, and the "function" is not unique. But *locally* we can in the *open* unit disc choose one of the two possible signs and obtain a graph of a continuous function, e.g.

(5.2) $z = Z(x,y) = +\sqrt{1-x^2-y^2}, \qquad \text{for } x^2+y^2 < 1,$

the graph of which is the open upper half of the unit sphere. (We may of course extend this function by continuity to the closed unit disc by adding $z = Z(x,y) = 0$ for $x^2+y^2 = 1$ to the definition, but this is not the point here.)

The example of the unit sphere above illustrates the primitive and yet efficient way of isolating one of the variables as a function of the others. We fix a point $(x,y)$ in the projection of the domain $D \subset \mathbb{R}^3$ onto $\mathbb{R}^2$ and then solve with respect to the remaining variable $z$. If there is just one solution, then we have found $z = Z(x,y)$ at this particular point $(x,y)$. If there are several possible values of $z$, then we must choose one of these. It is usually done, such that

(5.3) $z = Z(x,y)$

is *locally continuous* in the neighbourhood of some given point $(x_0,y_0)$. In this case we say that $z$ is *implicitly* given by (5.1), i.e. an expression of the type

$$F(x,y,z)=0,$$

while (5.3), i.e.

$$z = Z(x,y) \qquad \text{in a neighbourhood of } (x_0,y_0)$$

*explicitly* expresses $z$ *locally* as a function in a neighbourhood of the given point $(x_0,y_0)$. In the *explicit case* $z = Z(x,y)$ is just an ordinary function in two variables.

Note in 5.3) the difference between $z$, which is a *variable*, and $Z$, which is a *function*, here in two variables. Strictly speaking, the two symbols $z$ and $Z$ must not be confused. They are related, but not identical. However, it is nevertheless customary to let $z$ alone denote both the variable $z$ and the function $Z$ in order to avoid too many symbols.

## 5.6 Limits and continuity

The definition of a limit of a function in one variable is easy to generalize to limits of functions in several variables, when the *absolute value* $|\cdot|$ in $\mathbb{R}$ is replaced by the previously introduced norm $\|\cdot\|$ in $\mathbb{R}^n$. We recall that $\|\cdot\|$ is here defined as the *Euclidean norm*, i.e.

$$\|\mathbf{x}\| = \sqrt{x_1^2 + \cdots + x_n^2} \qquad \text{for } \mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n.$$

Let $\mathbf{x} \in \mathbb{R}^m$ be a fixed vector. By the symbol

$$\mathbf{x} \to \mathbf{x}_0$$

we shall understand that whenever we are given an $\varepsilon > 0$, then we restrict $\mathbf{x}$ to the open ball $B(\mathbf{x}, \varepsilon)$, where

$$\|\mathbf{x} - \mathbf{x}_0\| < \varepsilon \qquad \text{for all } \mathbf{x} \in B(\mathbf{x}, \varepsilon).$$

More generally, given a set $A \subseteq \mathbb{R}^m$, let $\mathbf{x}_0 \in \overline{A}$, i.e. the closure of $A$, where we assume that $\mathbf{x}_0$ is not an isolated point of $\overline{A}$. This means that

$$A \cap B(\mathbf{x}_0, r) \neq \emptyset \qquad \text{for all radii } r > 0.$$

Then we say that

$$\mathbf{x} \to \mathbf{x}_0 \qquad \text{in } A,$$

if

$$\|\mathbf{x} - \mathbf{x}_0\| \to 0 \qquad \text{and } \mathbf{x} \in A \setminus \{\mathbf{x}_0\},$$

or, more explicitly, if for every given $\varepsilon > 0$, the point $\mathbf{x}$ is restricted to the set

$$(A \cap B(\mathbf{x}_0, \varepsilon)) \setminus \{\mathbf{x}_0\}, \qquad \text{on which } \|\mathbf{x} - \mathbf{x}_0\| < \varepsilon.$$

We assumed above that $\mathbf{x}_0 \in A$ was bounded, so we could apply balls of centre $\mathbf{x}_0$ and then shrink them by letting the radius $r \to 0+$. If $A$ is unbounded, we also have to define, what is meant by $\mathbf{x} \to \infty$ on $A$, when $k \geq 2$. We define

$$\mathbf{x} \to \infty \quad \text{in } A, \qquad \text{if } \|\mathbf{x}\| \to +\infty \text{ and } \mathbf{x} \in A.$$

Note the difference in notation between the symbol $\infty$ for the *unspecified infinity* and the *signed infinities* $+\infty$ and $-\infty$. The latter two are linked to the two direction of the real line $\mathbb{R} = ]-\infty, +\infty[$. The unspecified infinity $\infty$ "lies far away in all possible directions at the same time". A natural sequence of "neighbourhoods" of $\infty$ is given by e.g. $\mathbb{R}^m \setminus B[\mathbf{0}, n]$, $n \in \mathbb{N}$, where we let $n \to +\infty$, or similarly. When $n$ increases, then clearly $\mathbb{R}^m \setminus B[\mathbf{0}, n]$ decreases, and points in $\mathbb{R}^m \setminus B[\mathbf{0}, n]$ satisfy $\|\mathbf{x}\| > n$.

Once these concepts have been specified we can build them together and e.g. define

$$\lim_{\mathbf{x} \to \mathbf{x}_0,\, \mathbf{x} \in A} \mathbf{f}(\mathbf{x}) = \mathbf{a}, \quad \text{also written} \quad \mathbf{f}(\mathbf{x}) \to \mathbf{a} \text{ for } \mathbf{x} \to \mathbf{x}_0 \text{ in } A.$$

This means that for every $\varepsilon > 0$ there exists a $\delta > 0$, such that

$$\|\mathbf{f}(\mathbf{x}) - \mathbf{a}\| < \varepsilon, \qquad \text{whenever } \|\mathbf{x} - \mathbf{x}_0\| < \delta \text{ and } \mathbf{x} \in A.$$

Similarly, for an unbounded set $A$,

$$\lim_{\mathbf{x}\to\infty,\,\mathbf{x}\in A} \mathbf{f}(\mathbf{x}) = \mathbf{a}, \quad \text{also written} \quad \mathbf{f}(\mathbf{x}) \to \mathbf{a} \text{ for } \mathbf{x} \to \infty \text{ in } A,$$

means that for every $\varepsilon > 0$ there is an $R > 0$, such that

$$\|\mathbf{f}(\mathbf{x}) - \mathbf{a}\| < \varepsilon, \qquad \text{whenever } \|\mathbf{x}\| > R \text{ and } \mathbf{x} \in A.$$

The rules of omputation known from the 1-dimensional case, i.e. sum, difference, and if $m = 1$, product and quotient (provided that the denominator is always $\neq 0$) are easily extended to limits in several variables.

We also obtain some new rules of computation like e.g.: If (for images in the same $\mathbb{R}^m$)

$$\lim_{\mathbf{x}\to\mathbf{x}_0,\,\mathbf{x}\in A} \mathbf{f}(\mathbf{x}) = \mathbf{a} \in \mathbb{R}^m \qquad \text{and} \qquad \lim_{\mathbf{x}\to\mathbf{x}_0,\,\mathbf{x}\in A} \mathbf{g}(\mathbf{x}) = \mathbf{b} \in \mathbb{R}^m,$$

then

$$\lim_{\mathbf{x}\to\mathbf{x}_0,\,\mathbf{x}\in A} \{\mathbf{f}(\mathbf{x}) \cdot \mathbf{g}(\mathbf{x})\} = \mathbf{a} \cdot \mathbf{b},$$

where "·" is the *inner* (or dot) *product.*.

When we restrict ourselves to $\mathbb{R}^3$, i.e. choose $m = 3$, we get a similar result for the *vector* (or cross) *product.*

Another important result is that

$$\lim_{\mathbf{x}\to\mathbf{x}_0,\,\mathbf{x}\in A} \mathbf{f}(\mathbf{x}) = \mathbf{a} = (a_1, \dots, a_m)\,,$$

if and only if for all coordinate functions,

$$\lim_{\mathbf{x}\to\mathbf{x}_0,\,\mathbf{x}\in A} f_1(\mathbf{x}) = a_1, \;\cdots\;, \lim_{\mathbf{x}\to\mathbf{x}_0,\,\mathbf{x}\in A} f_m(\mathbf{x}) = a_m.$$

We shall briefly sketch some methods, which may show us, if a function $f(\mathbf{x})$ has a limit for $\mathbf{x} \to \mathbf{x}_0$, or if this is not the case. We shall illustrate the methods in $RR^2$, where we for simplicity choose $\mathbf{x}_0 = \mathbf{0}$.

1) *A direct proof of convergence for* $\mathbf{x} \to \mathbf{0}$ *by comparing the magnitudes of the numerator and the denominator.* As an illustrative example we consider the function

$$f_1(x,y) = \frac{xy^2}{x^2 + y^2} \qquad \text{for } (x,y) \neq (0,0).$$

The numerator is a homogeneous monomial in $(x,y)$ of degree $1 + 2 = 3$, while the denominator is a homogeneous polynomial in $(x,y)$ of degree 2. Thus, if $\varrho$ denotes the radius in polar coordinates, then we have roughly $\varrho^3$ in the numerator and $\varrho^2$ in the denominator, so $f_1(x,y) \sim \varrho$, which tends towards 0 for $\varrho \to 0+$.

More precisely, in polar coordinates,

$$x = \varrho \cos\varphi \qquad \text{and} \qquad y = \varrho \sin\varphi,$$

so

$$f_1(x,y) = \frac{xy^2}{x^2+y^2} = \frac{\varrho\cos\varphi \cdot \varrho^2 \sin^2\varphi}{\varrho^2} = \varrho\cos\varphi\sin^2\varphi \qquad \text{for } \varrho > 0 \text{ and } \varphi \in \mathbb{R}.$$

To prove that $f_1(x,y) \to 0$ for $(x,y) \to (0,0)$, i.e. for $\varrho \to 0+$, we simply use the definition and estimate,

$$|f_1(x,y) - 0| = \left|\varrho\cos\varphi\sin^2\varphi - 0\right| \le \varrho \to 0 \qquad \text{for } \varrho \to 0+,$$

from which we conclude that $f_1(x,y) \to 0$ for $(x,y) \to (0,0)$.

2) *A proof of divergence for* $\mathbf{x} \to \mathbf{0}$ *by comparing the magnitudes of the numerator and the denominator.* If we change $f_1$ above to

$$f_2(x,y) = \frac{xy^2}{x^4+y^4} \qquad \text{for } (x,y) \neq (0,0),$$

then the numerator is a monomial of degree 3, and the denominator is a homogeneous polynomial of degree 4. In this case we get $f_2(x,y) \sim 1/\varrho$, so we would expect divergence for $\varrho \to 0+$. To prove this we again apply polar coordinates, so

$$f_2(x,y) = \frac{xy^2}{x^4+y^4} = \frac{\varrho^3 \cos\varphi \sin^2\varphi}{\varrho^4\left(\cos^4\varphi + \sin^4\varphi\right)} = \frac{1}{\varrho} \cdot \frac{\cos\varphi\sin^2\varphi}{\cos^4\varphi+\sin^4\varphi}.$$

If $\varphi = n\pi/2$, $n \in \mathbb{Z}$, i.e. if $(x,y)$ lies on either the $x$-axis or the $y$-axis, then clearly $f_2(x,y) = 0$, and in the limit $\varrho \to 0+$ we also get 0. If instead $\varphi \neq n\pi/2$, $n \in \mathbb{N}$, is kept fixed, then clearly $|f_2(x,y)| \to +\infty$ for $\varrho \to 0+$, so $f(x,y)$ is divergent for $(x,y) \to (0,0)$. The argument above shows also that $f_2(x,y)$ does not diverge towards $\infty$ either.

3) *Proof of divergence by restricting ourselves to straight lines.* Consider again

$$f_2(x,y) = \frac{xy^2}{x^4+y^4} \qquad \text{for } (x,y) \neq (0,0),$$

above. We have seen already that $f(0,y) = f(x,0) = 0$, so along the axes we get the limit 0 at $(0,0)$. A straight line through $(0,0)$ is either given by the vertical $y$-axis, or it is described by the equation $y = \alpha\, x$ for some constant $\alpha \in \mathbb{R}$. Then by insertion for $(x,y) = (x,\alpha y)$ on this line,

$$f_2(x,\alpha x) = \frac{x^3\alpha^2}{x^4\left(1+\alpha^4\right)} = \frac{1}{x} \cdot \frac{\alpha^2}{1+\alpha^4}.$$

Choose any $\alpha \neq 0$, and the $\alpha$-factor is a constant $\neq 0$, while $|1/x| \to +\infty$ for $x \to 0$, and $f_2(x,y)$ diverges for $(x,y) \to (0,0)$.

Another illustrative example is the following, where both the numerator and the denominator are homogeneous polynomials of the same degree 2. We consider the function

$$f_3(x,y) = \frac{xy}{x^2+y^2} \qquad \text{for } (x,y) \neq (0,0).$$

Clearly, $f_3(x,0) = f_3(0,y) = 0$, so *if* the function converges, then the limit must necessarily be 0. This is not the case, for if we restrict ourselves to the straight line $y = \alpha x$ and exclude $(0,0)$, then we get

$$f_3(x,\alpha x) = \frac{\alpha}{1+\alpha^2},$$

which for $\alpha \neq 0$ is a constant $\neq 0$ along this straight line, so this must also be the limit along this line. But then we have found a different candidate of the limit, contradicting that the limit is unique. Hence, $f_3(x,y)$ is divergent for $(x,y) \to (0,0)$.

A variant is of course to use polar coordinates, in which case

$$f_3(x,y) = \cos\varphi\sin\varphi = \frac{1}{2}\sin 2\varphi,$$

independent of $\varrho$, so along a straight half-line of angle $\varphi$ the value of $f_3(x,y)$ is given by $(\sin 2\varphi)/2$, which is a nonconstant function in the angle $\varphi$, and we conclude again that $f_3(x,y)$ is divergent for $(x,y) \to (0,0)$.

4) *Analysis of level curves.* In this case consider the function

$$f_4(x,y) = \frac{x}{x^2+y^2} \qquad \text{for } (x,y) \neq (0,0).$$

Let us first try the already known methods. The numerator is homogeneous of degree 1, and the denominator is homogeneous of degree 2, so according to 2) we would expect divergence. Using polar coordinates we get

$$f_4(x,y) = \frac{x}{x^2+y^2} = \frac{\varrho\cos\varphi}{\varrho^2} = \frac{1}{\varrho}\cos\varphi.$$

Fix $\varphi \neq n\pi + \pi/2$, $n \in \mathbb{Z}$, so $\cos\varphi$ is a constant $\neq 0$. Then clearly

$$|f_4(x,y)| = \frac{1}{\varrho}\,|\cos\varphi| \to +\infty \qquad \text{for } \varrho \to 0+,$$

so $f_4(x,y)$ is divergent for $(x,y) \to (0,0)$, and the only possible limit is the unspecified $\infty$. But since $f_4(0,y) = 0$ for all $y \neq 0$, this is not tending towards $\infty$ for $y \to 0$, so $f_4(x,y)$ is just divergent.

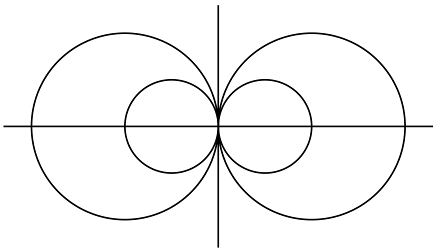


Figure 5.6: Some level curves of $f_4(x,y)$.

*Alternatively* we may analyze the level curves $f_4(x,y) = c$. If $c = 0$, then $x = 0$, so the level curves of $f_4$ corresponding to the value 0 are the positive and the negative $y$-axes.

If instead $c \neq 0$, and $(x,y) \neq (0,0)$, then

$$f_4(x,y) = \frac{x}{x^2+y^2} = c, \quad \text{if and only if} \quad x^2+y^2 = \frac{1}{c}\,x,$$

which we rewrite as

$$\left(x - \frac{1}{2c}\right)^2 + y^2 = \frac{1}{4c^2}.$$

The level curve corresponding to the value $c \neq 0$ is therefore, with the exception of the point $(0,0)$, the circle of centre $\left(\frac{1}{2c}, 0\right)$ and radius $\frac{1}{2|c|} > 0$. Cf. Figure 5.6. When we approach $(0,0)$ along the level curve (a circle or the $y$-axis) of constant $c$, we get the limit $c$ at $(0,0)$. Since $c \in \mathbb{R}$ is arbitrary, no unique limit exists, and $f_4(x,y)$ diverges for $(x,y) \to (0,0)$.

5) *The possibility of restriction to other curves than straight lines.* The method above in 3), where we approach the point $\mathbf{x}_0$ along straight lines, is only applicable to prove that we have *divergence.* We shall below see that *even if the limit is the same on the restriction of all straight lines, this does not imply that the limit exists!* So the same limit on all straight lines is only a *necessary* and not a sufficient condition for that the limit exists.

Consider the function

$$f_5(x,y) = \frac{x^2 y}{x^4 + y^2} \qquad \text{for } (x,y) \neq (0,0).$$

If $x = 0$, i.e. we restrict ourselves to the $y$-axis, then

$$f_5(0,y) = 0 \to 0 \qquad \text{for } y \to 0.$$

Then we restrict ourselves to the straight line of equation $y = \alpha x$, $\alpha \in \mathbb{R}$. Then

$$f_5(x,\alpha x) = \frac{x^2 \cdot \alpha x}{x^4 + \alpha^2 x^2} = \frac{\alpha x}{x^2 + \alpha^2}.$$

If $\alpha = 0$, then clearly

$$f_5(x,0) = 0 \to 0 \qquad \text{for } x \to 0.$$

If $\alpha \neq 0$, then

$$|f_5(x,\alpha x) - 0| = \left| \frac{\alpha x}{x^2 + \alpha^2} \right| \leq \left| \frac{\alpha x}{\alpha^2} \right| = \frac{1}{|\alpha|} \cdot |x| \to 0 \qquad \text{for } x \to 0.$$

Thus we have proved that the limit of $f_5(x,y)$ exists on the restriction to every straight line through $(0,0)$, when $(x,y) \to (0,0)$, and the common value of these limits is 0, and the *necessary* condition is fulfilled.

It is *not sufficient!* To prove this we take a closer look on the denominator $x^4 + y^2$, which is not a homogeneous polynomial in $(x,y)$. The idea is to choose curves, on which $x^4$ and $y^2$ are comparable through the limit process. If we choose the curves $y = \alpha x^2$, $\alpha \in \mathbb{R} \setminus \{0\}$, i.e. a family of parabolas, then $x^4 + y^2 = x^4 \left\{1 + \alpha^2\right\}$, which is $x^4$ times a constant depending on $\alpha$. Then we get by insertion for fixed $\alpha$ that

$$f_5\left(x, \alpha x^2\right) = \frac{x^2 \cdot \alpha x^2}{x^4 + \alpha^2 x^4} = \frac{\alpha}{1 + \alpha^2} \to \frac{\alpha}{1 + \alpha^2} \qquad \text{for } x \to 0.$$

Hence, the limits exist for $(x,y) \to (0,0)$ along these parabolas, but the values are different for different $\alpha$, so we get lot of different candidates for the limit. This is not possible, because the limit – if it exists – is unique. Hence, the limit of $f_5(x,y)$ does not exist for $(x,y) \to (0.0)$.

We emphasize that the methods described in 2)–5) can only be applied to prove *divergence.* To prove *convergence* we either use a direct proof using some estimate like

$$|f(\mathbf{x}) - a| \leq g(\mathbf{x}),$$

where we know – or prove – that $g(\mathbf{x}) \to 0$ for $\mathbf{x} \to \mathbf{x}_0$, or we prove that $f(\mathbf{x})$ is a (local) *contraction.* This means that there exists a constant $\alpha \in [0,1[$, such that

$$|f(\mathbf{x}) - f(\mathbf{y})| < \alpha \|\mathbf{x} - \mathbf{y}\| \qquad \text{for } \mathbf{x},\ \mathbf{y} \text{ lying close to each other.}$$

## 5.7 Continuous functions

As in the one-dimensional case we use the concept of a limit, introduced in Section 5.6 to define continuity of a function in several variables.

**Definition 5.1** *Consider a (vector) function* $\mathbf{f} : A \to \mathbb{R}^m$*, where* $A \subseteq \mathbb{R}^n$*, and let* $\mathbf{x}_0 \in A$ *be a given point. We say that* $\mathbf{f}$ *is continuous at* $\mathbf{x}_0$*, if*

$$\mathbf{f}(\mathbf{x}) \to \mathbf{f}(\mathbf{x}_0) \qquad \text{for } \mathbf{x} \to \mathbf{x}_0 \text{ in } A.$$

*We say that* $\mathbf{f}$ *is continuous in a subset* $B \subseteq A$*, if* $\mathbf{f}$ *is continuous at all points of* $B$*.*

The traditional way of stating that $\mathbf{f}$ is continuous at $\mathbf{x}_0 \in A$ is the following:

To every given $\varepsilon > 0$ we can find $\delta > 0$, such that

$$\|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{x}_0)\| < \varepsilon, \qquad \text{whenever } \mathbf{x} \in A \text{ and } \|\mathbf{x} - \mathbf{x}_0\| < \delta.$$

The usual rules of computation, known from real functions in one real variable, are easily carried over to our present case:

Given two (vector) functions $\mathbf{f}$, $\mathbf{g} : A \to \mathbb{R}^m$, and assume that they are both continuous at a given point $\mathbf{x}_0 \in A$. Then the *sum* and *difference* and *inner* (dot) *product* of $\mathbf{f}$ and $\mathbf{g}$ are all continuous, i.e.

$$\mathbf{f} + \mathbf{g}, \quad \mathbf{f} - \mathbf{g} \quad \text{and} \quad \mathbf{f} \cdot \mathbf{g} \qquad \text{are all continuous.}$$

If $m = 3$, then the *vector* (cross) *product*

$$\mathbf{f} \times \mathbf{g} \qquad \text{is continuous,} \qquad \left(\text{in } \mathbb{R}^3\right).$$

If $m = 1$, then the *scalar product* (note, no notation of the scalar product)

$$fg \qquad \text{is continuous,} \qquad (\text{in } \mathbb{R}),$$

and also the *scalar quotient*

$$\frac{f}{g} \qquad \text{is continuous at } \mathbf{x}_0\text{, provided that } g(\mathbf{x}) \neq 0 \text{ in a neighbourhood of } \mathbf{x}_0.$$

Assume that $\mathbf{f} : A \to \mathbb{R}^m$, where $A \subseteq \mathbb{R}^n$, and $\mathbf{g} : B \to \mathbb{R}^n$, $B \subseteq \mathbb{R}^k$, are continuous in their respective domains. If furthermore, $\mathbf{g}(B) \subseteq A$, then the *composition*composition

$$\mathbf{f} \circ \mathbf{g} : B \to \mathbb{R}^m$$

exists and is continuous in $B \subseteq \mathbb{R}^k$.It is not hard to prove that *a vector function* $\mathbf{f}$ *is continuous, if and only if all its coordinate functions are continuous.*

We defined in Section 5.6 the limit of a function $\mathbf{f}(\mathbf{x})$ for $\mathbf{x} \to \mathbf{x}_0$ in $A$, where we only required that $\mathbf{x}_0 \in \overline{A}$ is not an isolated point of the closure $\overline{A}$ of $A$. Assume that $\mathbf{x}_0 \in \overline{A} \setminus A$ and that $\lim_{\mathbf{x} \to \mathbf{x}_0,\, \mathbf{x} \in A} \mathbf{f}(\mathbf{x}) = \mathbf{a}$ exists. Then we can extend the domain of $\mathbf{f}$ to also including $\mathbf{x}_0$, where the extension is defined by

$$\tilde{\mathbf{f}}(\mathbf{x}) = \begin{cases} \mathbf{f}(\mathbf{x}) & \text{for } \mathbf{x} \in A, \\ \lim_{\mathbf{x} \to \mathbf{x}_0,\, \mathbf{x} \in A} \mathbf{f}(\mathbf{x}) = \mathbf{a} & \text{for } \mathbf{x} = \mathbf{x}_0. \end{cases}$$

It follows immediately from this construction that if the extension is defined in $\mathbf{x}_0 \in \overline{A} \setminus A$, then the extension is automatically continuous at this point $\mathbf{x}_0$.

We have already met an example of this type in Section 5.6, where we proved that

$$f_1(x, y) = \frac{xy^2}{x^2 + y^2} \qquad \text{for } (x, y) \neq (0, 0),$$

has the limit

$$\lim_{(x,y) \to (0,0)} f_1(x, y) = 0.$$

Hence, the continuous extension of $f_1$, defined in all of $\mathbb{R}^2$, is given by

$$\tilde{f}_1(x, y) = \begin{cases} \dfrac{x^2 y}{x^2 + y^2} & \text{for } (x, y) \neq (0, 0), \\ 0 & \text{for } (x, y) = (0, 0). \end{cases}$$

Sometimes one may be able to *factorize* the function under consideration and then cancel the common factor, which becomes zero in the limit in both the numerator and the denominator. One of the simplest examples is

$$f_6(x, y) = \frac{x^2 - y^2}{x - y} \qquad \text{for } y \neq x.$$

In fact,

$$f_6(x,y) = \frac{x^2-y^2}{x-y} = \frac{(x-y)(x+y)}{x-y} = x+y \qquad \text{for } y \neq x.$$

Only the factor $x-y$ in both the numerator and the denominator is 0 at the exception set, and we cancel them by division in the set where $y \neq x$. Since the quotient $x+y$ also makes sense for $y = x$ (formally we take the limit to this set), the continuous extension of $f_6$ is defined by

$$\tilde{f}_6(x,y) = x+y \qquad \text{for } (x,y) \in \mathbb{R}^2.$$

A more sophisticated example using the same idea is given by

$$f_7(x,y) = \frac{\sin(x+y)}{x+y} \qquad \text{for } y \neq -x.$$

A common trick in mathematics is to give an "unpleasant expression" a new name. In this case we put $t := x+y$, and the restriction is then $t \neq 0$, in which case

$$f_7(x,y) = \frac{\sin t}{t}, \qquad t = x+y \neq 0.$$

It is well-known from the theory of real functions in one real variable that

$$\lim_{t \to 0} \frac{\sin t}{t} = 1,$$

which means that $f_7(x,y)$ has the continuous extension to all of $\mathbb{R}^2$,

$$\tilde{f}_7(x,y) = \begin{cases} \dfrac{\sin(x+y)}{x+y} & \text{for } x+y \neq 0, \\[2ex] 1 & \text{for } x+y = 0. \end{cases}$$

## 5.8 Continuous curves

### 5.8.1 Parametric description

Intuitively, a continuous curve in $\mathbb{R}^m$ is a path, along which e.g. a particle moves from an initial point to a final point, i.e. we have a sense of which direction the particle moves along the path. We coin these ideas in the following definition.

**Definition 5.2** *A* continuous curve *in* $\mathbb{R}^m$ *is a continuous map* $\mathbf{r} : I \to \mathbb{R}^m$ *of a real interval* $I \subseteq \mathbb{R}$. *If* $I$ *has the left end point* $a$ *(including the possibility of* $-\infty$*) and the right end point* $b$ *(including the possibility of* $+\infty$*), we call* $\mathbf{r}(a)$ *the* initial point *of the curve, and* $\mathbf{r}(b)$ *the* final point *of the curve.*

The curve inherits the orientation of the interval $I$, so roughly speaking, "we are just taking the interval $I$, and then bend and stretch it" as described by the map $\mathbf{r} : I \to \mathbb{R}^m$.

Given a continuous curve $\mathbf{r} : I \to \mathbb{R}^m$. Its image is given by

$$\mathcal{K} = \{\mathbf{x} \in \mathbb{R}^m \mid \mathbf{x} = \mathbf{r}(t),\, t \in I\} = \{\mathbf{r}(t) \mid t \in I\}.$$

This is often a better way to describe the curve than the formal definition above. Note, however, that it is always safe to use Definition 5.2 in the applications, and this is also the most common construction in MAPLE, where we e.g. in $\mathbb{R}^2$ write

$$[r_1(t), r_2(t), t = a..b]\,,$$

where $(x, y) = \mathbf{r}(t)$, $t \in [a, b]$.

We call $\mathbf{x} = \mathbf{r}(t)$, $t \in I$, a *parametric description* of the curve $\mathcal{K}$, and $t$ is the parameter, and $I$ the parameter interval.

Given a continuous curve $\mathbf{r} : I \to \mathbb{R}^m$. Assume that $n$ different parameters $t_1, \dots, t_n$, where $n \geq 2$, all are mapped into the same point on the curve,

$$\mathbf{r}(t_1) = \mathbf{r}(t_2) = \cdots = \mathbf{r}(t_n) = \mathbf{u} \in \mathbb{R}^m.$$

Then we call the common point $\mathbf{u} \in \mathbb{R}^m$ a *multiple point* (of the curve). If $n = 2$, we may call it a *double point* instead.

**Remark 5.1** Even if Definition 5.2 looks very straightforward, it is *not*. It was a shock for the mathematicians, when the Italian mathematician appr. 1900 constructed a continuous curve, which passed through all points in e.g. the unit square. And even worse a couple of years later, when Osgood modified Peano's construction obtaining a continuous curve without multiple points, which Peano's curve had, and of *positive area!* In particular, the unit one-dimensional interval $[0, 1]$, clearly of no area, was mapped continuously and bijectively onto a set of positive area. However, although such *space filling curves* are of interest in their own right, we shall not consider them further in this series of books. We shall be more interested in *differential curves*, for which such phenomena do not occur. $\Diamond$

Sometimes we may want to consider continuous curves, which are composed of axiparallel line segments. We therefore give such curves a name, namely *step lines*, because we step from one coordinate to the next one, when we run through the curve, only changing one coordinate at a time, which therefore locally can be used as a parameter. Cf. Figure 5.7.

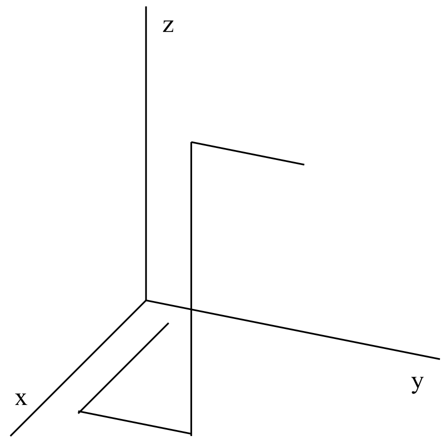



Figure 5.7: An example of a step line.

We list the most commonly used parametric descriptions of curves.

1) A *plane curve* of the equation

$$y = Y(x), \qquad x \in I,$$

is already given by its parametric description, when we use $t = x \in I$ as its parameter. Its graph is

$$\mathcal{K} = \{(x, Y(x)) \mid x \in I\}.$$

2) A *straight line (segment)* in $\mathbb{R}^m$ is given by the parametric description

$$\mathbf{x} = \mathbf{a} + \mathbf{v}t, \qquad t \in I,$$

where $\mathbf{a}$ and $\mathbf{v} \in \mathbb{R}^m$ are constant vectors, and $\mathbf{v} \neq \mathbf{0}$, and $I \subseteq \mathbb{R}$ is some given interval.

If $I = \mathbb{R}$, we get an *oriented line* in $\mathbb{R}^m$. If $I = [a, +\infty[$, $]a, +\infty[$, $]-\infty, b[$ or $]-\infty, b]$, we get an *oriented half line* in $\mathbb{R}^m$. Finally, if $I$ is bounded, we get an *oriented line segment.* The orientation is inherited from the usual orientation of $I \subseteq \mathbb{R}$ with respect to the order relation $\leq$. The vector $\mathbf{v} \neq \mathbf{0}$ is called the *direction vector* of the line. This is quite often chosen as a unit vector,

3) A *circle* of radius $a > 0$ and centre $(0, 0) \in \mathbb{R}^2$ of equation

$$x^2 + y^2 = a^2,$$

is considered as a curve with the parametric description (in polar coordinates)

$$x = a\,\cos\varphi, \quad y = a\,\sin\varphi, \qquad \varphi \in [0, 2\pi[,$$

or in MAPLE-notation,

$$[a \cdot \cos(t), b \cdot \sin(t), t = 0..2\pi].$$

The circle inherits its orientation from the interval $[0, 2\pi[$. In the present case it is also *positively oriented* in the plane $\mathbb{R}^2$. This means that the curve moves counterclockwise around the centre $(0, 0)$.

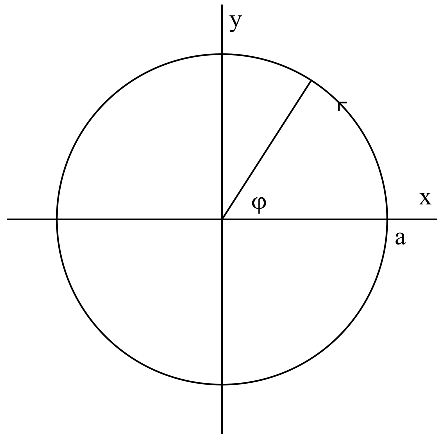



Figure 5.8: A circle of radius $a$ as a curve of positive orientation in the plane $\mathbb{R}^2$.

If we want a parameter description in the negative sense of the plane, just replace $t$ by $-t$, so we get instead

$$x = a \cos\varphi, \quad y = -a \sin\varphi, \qquad \varphi \in [0, 2\pi[.$$

Tbe parameter descriptions above describe the circle run through just once (and without double points). Other choices of $I$ are possible, like e.g. $]-\pi, \pi]$, where the initial point is $(-1, 0)$ on the negative $x$-axis. If $I = \mathbb{R}$, then the circle is run through infinitely many times, just to mention a few of the many possibilities.

The parameter $\varphi \in [0, 2\pi[$ can be interpreted as the angle of the radius vector from $(0, 0)$ to the point $(x, y) \neq (0, 0)$ under consideration.

4) A modification of the description of the circle above gives us the parametric description of an *ellipse* of the equation

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1.$$

In this case we just multiply the $y$-coordinate of the circle by the *affinity factor* $b/a$, so the basic parametric description of the ellipse becomes

$$x = a \cos\varphi, \quad y = b \sin\varphi, \qquad \varphi \in [0, 2\pi[,$$

where $a$, $b > 0$ are the two half axes of the ellipse. Cf. Figure 5.9.

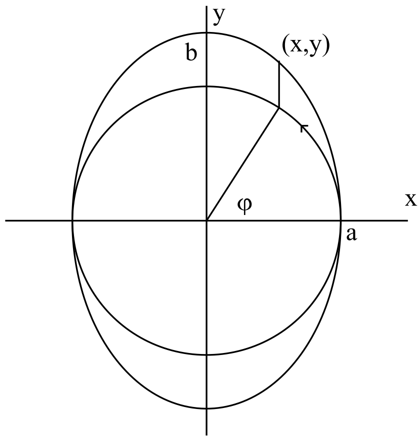



Figure 5.9: An ellipse of half axes $a$ and $b$ in the plane $\mathbb{R}^2$ is obtained from a circle by an application of an affinity.

### 5.8.2 Change of parameter of a curve

Given a continuous curve of parametric description

$$\mathbf{x} = \mathbf{r}(t), \qquad t \in I.$$

As mentioned earlier, we may interpret the curve as the path of a particle. If we change the speed of this particle, we get another curve,

$$\mathbf{x} = \mathbf{r}_1(u), \qquad u \in J.$$

The path itself is of course the same in the two cases, but the parameters do not match, so that is why we say that we have a different curve.

The change from $\mathbf{r}$ to $\mathbf{r}_1$ is given by a uniquely determined function

$$\Phi : I \to J, \qquad u = \Phi(t),$$

such that

$$\mathbf{r}_1(u) = \mathbf{r}(t) = \mathbf{r}(\Phi(u)) = (\mathbf{r} \circ \Phi)(u).$$

In fact, every point of $I$ must by the monotony of the map correspond to precisely one point of $J$, and *vice versa*, and this gives us a bijective function $\Phi : I \to J$.

We call $\Phi : I \to J$ a *change of parameter*.

## 5.9 Connectedness

Using continuous curves we can introduce a new important topological concept, which will be used in the sequel. Given a set $A$, it is important that we can move from one point $\mathbf{x} \in A$ to another point $\mathbf{y} \in A$ along a continuous curve without leaving $A$ during this motion. We coin this property in the following definition.

**Definition 5.3** *A set $A \subseteq \mathbb{R}^m$ is called* connected*, if any two points $\mathbf{x}, \mathbf{y} \in A$ can be connected with a continuous curve lying in $A$, i.e. we can find a continuous function $\mathbf{r} : [a, b] \to \mathbb{R}^m$, such that*

$$\mathbf{x} = \mathbf{r}(\mathbf{a}), \qquad \mathbf{y} = \mathbf{r}(\mathbf{b}), \qquad \{\mathbf{r}(t) \mid t \in [a, b]\} \subseteq A.$$

In particular *starshaped sets* $A$ are connected, because there exists a point $\mathbf{x}_0 \in A$, which can be reached from any other point $\mathbf{x} \in A$ by a straight line segment in $A$. So when we construct a path from $\mathbf{x} \in A$ to $\mathbf{y} \in A$, we just take the detour via $\mathbf{x}_0$.

In particular, a *convex set* $A$ is connected, because the straight line segment between two points $\mathbf{x}$, $\mathbf{y} \in A$ also lies totally in $A$.

One can prove that if a subset $I \subseteq \mathbb{R}$ of the real line is connected, then it is an interval. This may seem obvious, and we have already tacitly used this property, when we described the process of changing parameters.

It will also be convenient to consider any set $A = \{\mathbf{x}_0\}$ consisting of just one point as connected.

If $A$ is an open and connected set, we call it an *open domain.* If we add some of its boundary points to $A$, we just call the result a *domain.* And if we add all of its boundary points, then we call it a *closed domain.*

We mention without proof the following theorem, which will be useful in the next volumes of this series. In particular in connection with line integrals.

**Theorem 5.1** *Assume that $A$ is an open domain. Then any two points $\mathbf{x}$, $\mathbf{y} \in A$ from $A$ can be connected by a step line, i.e. a continuous curve consisting og only axiparallel line segments.*

Consider the two connected sets of Figure 5.10, i.e. a disc and an annulus, They clearly do not have the same topological shape, because the annulus contains a hole, which the disc does not. We therefore introduce the following:

Let $A$ be a connected set. Let $\mathbf{x}$, $\mathbf{y} \in A$ be two points, connected with two continuous curves entirely in $A$,

$$\mathbf{r}_0 : [0,1] \to A, \quad \mathbf{r}_1 : [0,1] \to A, \quad \text{where} \quad \mathbf{r}_0(0) = \mathbf{r}_1(0) = \mathbf{x} \text{ and } \mathbf{r}_0(1) = \mathbf{r}_1(1) = \mathbf{y}.$$

Assume that we can change $\mathbf{r}_0$ continuously, so that we in the end get to $\mathbf{r}_1$, i.e. we can deform the path of $\mathbf{r}_0$ continuously until we reach the path of $\mathbf{r}_1$.

More precisely, we can find a family of maps

$$\mathbf{r}(t,\alpha) : [0,1] \times [0,1] \to \mathbb{R}^m,$$

such that $\mathbf{r}(t,\alpha)$ is *continuous* in the variables $(t,\alpha) \in [0,1] \times [0,1]$ satisfying the conditions

$$\mathbf{r}(t,0) = \mathbf{r}_0(t), \qquad \mathbf{r}(t,1) = \mathbf{r}_1(t), \qquad \text{for all } t \in [0,1].$$

We say that $A$ is *simply connected*, if all curves $\mathbf{r}(\cdot,\alpha)$, $\alpha \in [0,1]$ lie entirely in $A$. In some sense the set $A$ does not have "holes".

In $\mathbb{R}^2$ it is easy to understand, what a hole is. However, the reader must be careful in higher dimensions. If e.g. we just remove the centre of a solid ball in $\mathbb{R}^3$, then the remaining set is still simply connected, even if one would believe that the removed point was a "hole". Cf. Figure 5.11. However, if we remove all points of the $z$-axis, or even a tube as on Figure 5.11, then the remaining set is no longer simply connected. Consider e.g. two circles in this set, one circling around the $z$-axis, while the other one does not. Then one cannot change one of them continuously to the other one without cutting the $z$-axis, so we get outside $A$ by this continuous transformation.

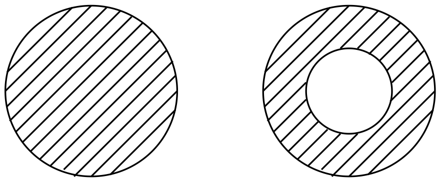



Figure 5.10: The disc to the left is simply connected, while the annulus to the right is not, though it of course is connected.

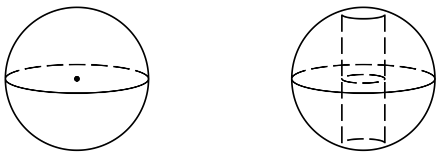



Figure 5.11: A simply connected and a not simply connected set in $\mathbb{R}^3$.

## 5.10 Continuous surfaces in $\mathbb{R}^3$

Surfaces are like curves also important in the applications. We shall here for convenience restrict ourselves to surfaces in the 3-dimensional space $\mathbb{R}^3$. The primitive idea is described in the following way: Take a plane plate and hammer it into a wanted shape. The hammering is then described by some continuous function.

### 5.10.1 Parametric description and continuity

We shall of course generalize the definition of a (1-dimensional) curve to a 2-dimensional surface. So instead of a 1-dimensional parameter interval $I \subseteq \mathbb{R}$ one is tempted to replace it by a 2-dimensional interval like $I \times J \subseteq \mathbb{R}^2$, where $I,\ J \subseteq \mathbb{R}$ are intervals. This actually is sufficient in many cases.

However, a closer look shows that we may allow more general 2-dimensional parameter sets $E \subseteq \mathbb{R}^2$. In fact, it suffices that $E$ is connected, i.e. a domain in $\mathbb{R}^2$. This is in agreement with our primitive idea in the introduction above, namely that $E$ is some connected 2-dimensional plate, which should be bent and stretched or compressed to give the wanted surface in $\mathbb{R}^3$.

Glancing at the previous definition of a curve we see that a surface should have the structure

$$\mathcal{F} = \left\{ \mathbf{x} \in \mathbb{R}^3 \mid \mathbf{x} = \mathbf{r}(u,v),\, (u,v) \in E \right\}, \qquad \text{where } E \subseteq \mathbb{R}^2.$$

Here, $\mathbf{r} : E \to \mathbb{R}^3$ is a continuous vector function in two variables.

The above illustrates the general idea of a parametric description of a surface, which we illustrate on Figure 5.12.

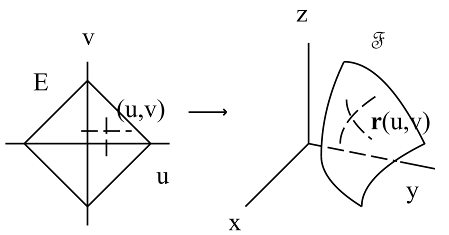



Figure 5.12: The parametric description $\mathbf{r} : E \to \mathbb{R}^3$ of a surface $\mathcal{F}$.

We call $\mathbf{x} = \mathbf{r}(u,v)$, $(u,v) \in E$ a *parametric description* of the surface $\mathcal{F}$. Let $(u,v) \in E$ be a point in the parameter domain. The vertical line segment in $E$ through $(u,v)$ is 1-dimensional. It is therefore mapped into a continuous curve on the surface $\mathcal{F}$. This curve is called the *parameter curve* on $\mathcal{F}$ through $(u,v)$. Similarly, when we consider a horizontal line segment through $(u,v) \in E$.

The sloppy definition above of a surface includes some pathological cases, which we should avoid in practice. If e.g. $\mathbf{r}(u,v) = \mathbf{R}(u)$ is independent of $v$, then the "surface" generates to a curve, which one would not consider as a surface. Furthermore, since already curves can be space filling, the same is true for surfaces even for continuous parametric descriptions. Since we do not have the concept of a "null set" at hand, it is here not easy to give a precise definition of a surface, so we allow ourselves only to sketch the main points.

1) The *parametric map* $\mathbf{r} : E \to \mathbb{R}^3$ should not only be continuous. It should also be differentiable "almost everywhere". (Differentiable functions are the subject of Volume III.) This only means that we allow some – though not too many – exceptional points, in which we do not have differentiability.

2) The *parametric curves* should at "almost every point $\mathbf{r}(u,v) \in \mathcal{F}$ have two parameter curves, which have linearly independent tangent vectors with respect to the parameters $(u,v) \in E$.

The simplest surfaces in $\mathbb{R}^3$ are probably the following:

1) *A plane in* $\mathbb{R}^3$. Given two linearly independent vectors $\mathbf{b}$, $\mathbf{c}$ in $\mathbb{R}^3$, and let $\mathbf{a}$ just be a point in $\mathbb{R}^3$. Then

$$\mathbf{x} = \mathbf{r}(u, v) = \mathbf{a} + \mathbf{b}u + \mathbf{c}v, \qquad (u, v) \in \mathbb{R}^2,$$

is a parametric description of a plane through the point $\mathbf{a} \in \mathbb{R}^3$.

2) *A graph of a function.* Assume that the surface $\mathcal{F}$ is the graph of a function in two variables,

$$z = Z(x, y), \qquad \text{for } (x, y) \in E.$$

Then this is clearly a parametric description. In fact, replace $(x, y)$ with $(u, v) \in E$.

3) *A sphere of radius* $a > 0$ *and centre at* $\mathbf{0}$. I this case the most commonly used parametric description is

$$\mathbf{x} = (x, y, z) = \mathbf{r}(\theta, \varphi) = (a\,\sin\theta\cos\varphi, a\,\sin\theta\sin\varphi, a\,\cos\theta), \qquad \theta \in [0, \pi], \quad \varphi \in [0, 2\pi].$$

The construction is the following: Write the rectangular coordinates $(x, y, z)$ as functions in the spherical coordinates $(r, \theta, \varphi)$ introduced in Chapter 1 (volume I), and then keep $r = a > 0$ fixed.

4) *An ellipsoidal surface.* In rectangular coordinates an ellipsoidal surface is given in its canonical form by

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 = 1.$$

When we modify the parametric description of the sphere above we get the following parametric description of the surface

$$(x, y, z) = (a \sin\theta\cos\varphi, b \sin\theta\sin\varphi, c \cos\theta), \qquad \theta \in [0, \pi], \quad \varphi \in [0, 2\pi].$$

In the next two sections we introduce other commonly occurring surfaces, which are also easily described.

### 5.10.2 Cylindric surfaces

A *cylindric surface* is the union of all straight lines, the *generators*, in a space, which are parallel and which all intersect a given curve. We shall here for convenience confine ourselves to the case, where the given curve lies in a plane, and the generators are all perpendicular to this plane, supplied with the extra assumption that the cylindric surface may consist of only *line segments* of the generators.

If the given curve $\mathcal{L}$ lies in the $xy$-plane, the cylindric surface above $\mathcal{L}$ is illustrated by taking a sheet of paper and fold it along the curve $\mathcal{L}$.

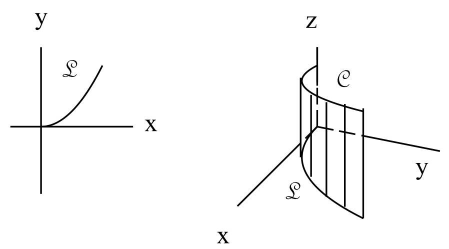



Figure 5.13: The given plane curve $\mathcal{L}$ in the $xy$-plane and the corresponding perpendicular cylindric surface $\mathcal{C}$ in the $xyz$-space.

The parametric description of a cylindric surface is constructed in the following way:

First assume that in the $xy$-plane the given curve $\mathcal{L}$ has been given a parametric description of the form

$$\mathcal{L} = \left\{(x, y) \in \mathbb{R}^2 \mid x = X(t),, y = Y(t), t \in I\right\}.$$

Then the cylindric surface $\mathcal{C}$ is described by adding $J(t)$ as a $z$-interval above the point of the curve $(X(t), Y(t), 0)$ of the same parameter $t \in I$, hence

$$\mathcal{C} = \left\{(x, y, z) \in \mathbb{R}^3 \mid x = X(t),\ y = Y(t),\ z \in J(t),\ t \in I\right\}.$$

### 5.10.3 Surfaces of revolution

A *surface of revolution* is constructed in the following way:

Given an *axis of revolution* – usually chosen as the $z$-axis – and a so-called *meridian curve* $\mathfrak{M}$ in the *meridian half-plane* $\{(\varrho, z) \mid \varrho \geq 0,\ z \in \mathbb{R}\}$.

Assume that the meridian curve has the following parametric description,

$$\mathfrak{M} = \{(\varrho, z) \mid \varrho = P(t) \geq 0,\ z = Z(t),\ t \in I\}.$$

When we rotate $\mathfrak{M}$ in $\mathbb{R}^3$ around the $z$-axis, the surface of revolution $\mathcal{O}$ is described in *semi-polar*, or *cylindric, coordinates* (cf. Chapter 1 in Volume I) by

$$\mathcal{O}: \qquad \varrho = P(t) \geq 0 \quad \text{and} \quad z = Z(t), \qquad \text{for } t \in I \text{ and } \varphi \in [0, 2\pi].$$

If we use rectangular coordinates, we of course get

$$\mathcal{O}: \qquad x = P(t)\cos\varphi,\ y = P(t)\sin\varphi,\ z = Z(t), \qquad \text{for } t \in I \text{ and } \varphi \in [0, 2\pi[,$$

cf. Figure 5.14.

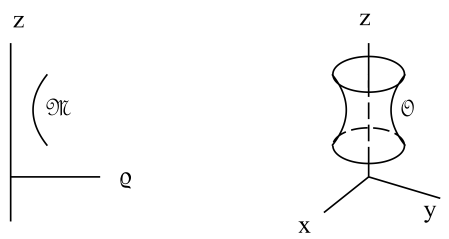



Figure 5.14: The meridian curve $\mathfrak{M}$ in the meridian half-plane, and the corresponding surface of revolution $\mathcal{O}$ in the space $\mathbb{R}^3$.

If in particular $\mathfrak{M}$ is a half-circle of radius $a > 0$ and centre at $\mathbf{0}$, then the surface of revolution becomes a sphere of centre $\mathbf{0}$ and radius $a$. A parametric description of $\mathfrak{M}$ is

$$\mathfrak{M}: \qquad \varrho = a\,\sin\theta, \quad z = a\,\cos\theta, \qquad \theta \in [0, \pi],$$

so the parametric description of the sphere is the well-known description in spherical coordinates with $r = a$ fixed,

$$\mathcal{O}: \qquad x = a\,\sin\theta\cos\varphi, \quad y = a\,\sin\theta\sin\varphi, \quad z = a\,\cos\theta, \qquad \theta \in [0, \pi], \quad \varphi \in [0, 2\pi[,$$

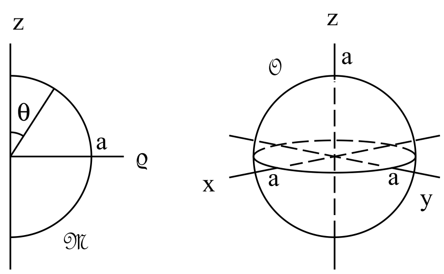



Figure 5.15: The meridian curve $\mathfrak{M}$ is a half-circle, and the surface of revolution $\mathcal{O}$ is a sphere in the $xyz$-space.

so this is a way to derive the spherical coordinates from the parametric descripton of $\mathfrak{M}$. Note that $\theta$ here is measured positively from the vertical $z$-axis towards the horizontal $\varrho$-axis, i.e. apparently in the negative orientation of the meridian half-plane. Cf. also Figure 5.15.

If instead the meridian curve is a circle lying in the *open* meridian half-plan, so it does not touch the axis of rotation, then its parametric description may be given by

$$\varrho = a + b\,\cos t, \quad z = b\,\sin t, \qquad \text{for } t \in [0, 2\pi[, \quad \text{where } 0 < b < a,$$

cf. Figure 5.16.

The surface of revolution is a *torus* of parametric description in *semi-polar* or *cylindric*, *coordinates*

$$\mathcal{O}; \qquad \varrho = a + b\,\cos t, \quad z = b\,\sin t, \qquad \text{for } t \in [0, 2\pi[ \text{ and } \varphi \in [0, 2\pi[.$$

Clearly, $(\varrho - a)^2 + z^2 = b^2$, which is an *equation* of the torus surface in semi-polar coordinates. The equation in *rectangular coordinates*, is

$$\left(\sqrt{x^2+y^2} - a\right)^2 + z^2 = b^2, \qquad \text{where } 0 < b < a,$$

because $x = \varrho\,\cos\varphi$ and $y = \varrho\,\sin\varphi$, so $\varrho = \sqrt{x^2+y^2}$.

### 5.10.4 Boundary curves, closed surfaces and orientation of surfaces

When we consider a curve, then it is obvious that its initial point and final point – if they exist – are the points, where the curve stops in some sense. We note that a curve does not necessarily have initial and final points. One example is the unit circle, where we can continue moving along it without ever reaching one of its end points, because they do not exist.

Similarly, a surface $\mathcal{F}$ may have a *boundary curve* $\delta\mathcal{F}$, where the surface $\mathcal{F}$ in some sense stops. Also here we may expect cases, where such a boundary curve does not exist. We shall return to this later.

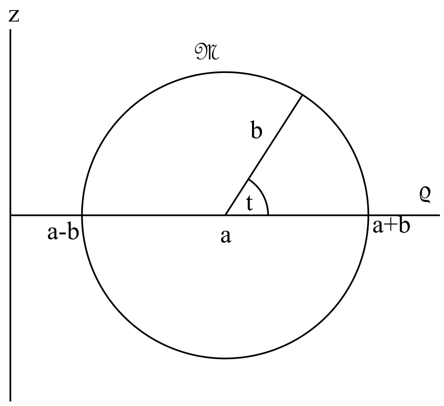



Figure 5.16: The meridian curve $\mathfrak{M}$ is a circle in the open meridian half-plane. The surface of revolution $\mathcal{O}$ is a *torus* in the $xyz$-space. A figure of the torus is given in MAPLE. However, for some obscure reason it has not been possible for the author to put it here.

We exclude here all space filling surfaces, so every surface $\mathcal{F}$ under consideration will not have interior points in $\mathbb{R}^3$, thus $\mathcal{F}$ is equal to its *topological* boundary in $\mathbb{R}^3$, i.e. $\mathcal{F} = \partial\mathcal{F}$. The boundary curves of surfaces we are considering here are intrinsic boundary curves with respect to the surface $\mathcal{F}$ itself and they have nothing to do with the boundaries of sets in $\mathbb{R}^3$. It is for this reason that we use the notation $\partial\mathcal{F}$ for such boundary curves.

We get a hint of what is the meaning of $\delta\mathcal{F}$, when we consider the parametric domain $E \subseteq \mathbb{R}^2$ of $\mathcal{F} \subset \mathbb{R}^3$. Clearly, $E$ has a usual *topological* boundary $\partial E$ in $\mathbb{R}^2$, and when we use the picture that $E$ is hammered into the shape of $\mathcal{F}$ in $\mathbb{R}^3$ by the application of the map $\mathbf{r} : E \to \mathbb{R}^3$, we would expect that $\delta\mathcal{F} = \mathbf{r}(\partial E)$. This is very often the case, though not always, which we shall show in the following.

Consider the unit sphere in spherical coordinates. Then the parameter domain is

$$E = [0, \pi] \times [0, 2\pi[,$$

which has the topological boundary

$$\partial E = \{0\} \times [0, 2\pi] \cup \{\pi\} \times [0, 2\pi] \cup [0, \pi] \times \{0\} \cup [0, \pi] \times \{2\pi\},$$

while the sphere $\mathcal{F}$ does not have a boundary curve, $\delta\mathcal{F} = \emptyset$. By the "hammering" with the function $\mathbf{r}$ we identify the two horizontal sides, $[0, \pi] \times \{0\}$ and $[0, \pi] \times \{2\pi\}$, leaving us with a cylinder. And then all points of $\{0\} \times [0, 2\pi]$ are identified, i.e. hammered and glued together to get the North Pole, and all points of $\{\pi\} \times [0, 2\pi]$ are identified i.e. hammered and glued together to get the South Pole. So all points of a possible boundary curve simply disappear by this proces.

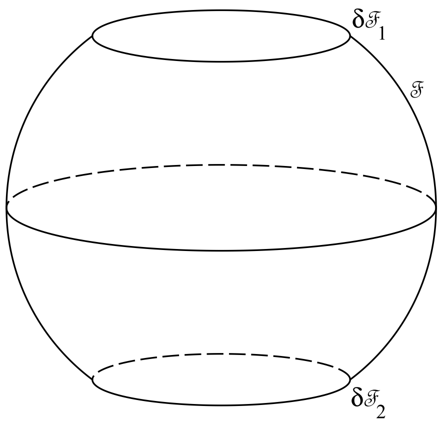



Figure 5.17: The boundary curve $\delta\mathcal{F} = \delta\mathcal{F}_1 \cup \delta\mathcal{F}_2$ is not connected. Its branches are two circles lying in parallel planes at different latitudes.

A boundary curve of a surface is not necessarily connected. If we cut the sphere with two parallel planes and let the surface $\mathcal{F}$ be the part of the sphere, which lies between the two planes. Then the boundary curve consists of two parallel circles at differet latitudes, cf. Figure 5.17.

Each connected component of $\delta\mathcal{F}$ is called a *branch* of $\delta\mathcal{F}$, and each branch is a continuous curve in space.

If a surface $\mathcal{F}$ does not have a boundary curve in this sense, $\delta\mathcal{F} = \emptyset$, then we call $\mathcal{F}$ a *closed surface*. We have already seen some closed surfaces; the *sphere*, the *ellipsoidal surface*, and the *torus*, all considered in the previous Section 5.10.3.

Assume that $\mathcal{F}$ is a closed surface. If e.g. $\mathcal{F}$ is the sphere, then it is obvious that we can talk of the inside and the outside of the sphere, so we can talk of a direction out of the ball, which has the sphere as its boundary. This is the general idea of the new concept *orientation*.

Not all surfaces have an inside and an outside, so we must find a means to decide when this is the case. First note that if the surface is divided into sufficiently small pieces, which overlap each others, then we *locally* can talk of two sides of the surface. We paint one of them red, and the other one blue, and then go to the neighbouring piece of the surface (with an overlap). Paint this neighbouring piece of the surface according to the colours in their overlap. Proceed in this way, until either all local pieces of surface have been painted, in which case we can define e.g. blue the inside and red the outside, and we have obtained an *orientation*. Or, we come to a piece of the surface, which according to this procedure should have each both sides painted both red and blue, which is not possible. In this case we say that the surface cannot be oriented.

The simplest example of a surface, which cannot be oriented, is the so-called *Möbius's strip*. Take a strip of paper and twist it once before gluing the ends of the strip together, cf. Figure 5.18.

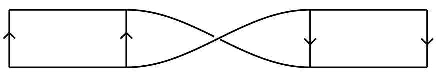


Figure 5.18: *Möbius's strip*. When the strip of paper is twisted once, we switch the local orientation, denoted bu the arrows. When we glue the two ends together, we end up with a strip, which *globally* has only one surface!

we shall in this series of books on *Real Functions in Several Real Variables* only consider surfaces, which can be oriented.

## 5.11 Main theorems for continuous functions

We shall in this section quote (without proofs) the three main teorems for continuous functions, here restricted to the spaces $\mathbb{R}^n$. They will be very important in the applications in the sequel.

**1^st^ main theorem for continuous functions**. *Let* $A \subseteq \mathbb{R}^n$ *be a* connected *set, and let* $\mathbf{f} : A \to \mathbb{R}^k$ *be* continuous. *Then the range,* $\mathbf{f}(A)$*, is also connected.*

It should be noted that even if $\mathbf{f} : A \to \mathbb{R}^k$ is continuous and $\mathbf{f}(A)$ is connected, we cannot conclude that the domain $A$ itself is connected. Consider $f = \sin : A \to \mathbb{R}$, where $A \subseteq \mathbb{R}$. Then $f$ is continuous and $\sin(A) = ]-1,1[$ is connected, while

$$A := \bigcup_{n \in \mathbb{Z}} \left] -\frac{\pi}{2} + 2n\pi, \frac{\pi}{2} + 2n\pi \right[ \qquad \text{is not connected.}$$

An important case is when $k = 1$, in which case $f(A)$ is a connected set in $\mathbb{R}$, i.e. an interval. We shall later use this observation over and over again.

Let $A \subset \mathbb{R}^n$. If $A$ is bounded and closed, we call it *compact*. Compact sets are very important in Mathematics, and the next two main theorems are dealing with them.

**$2^{\text{nd}}$ main theorem for continuous functions**. *Let $A \subset \mathbb{R}^n$ be a compact set. If $\mathbf{f} : A \to \mathbb{R}^k$ is continuous, then its range $\mathbf{f}(A)$ is also compact.*

Again we consider the special case, when $k = 1$. If $f : A \to \mathbb{R}$ is continuous, and $A$ is compact, then $f(A) \subset \mathbb{R}$ is also compact. In particular, $f(A)$ contains its upper and lower bounds. We therefore conclude that there must exist points $\mathbf{a}$, $\mathbf{b} \in A$, such that

$$f(\mathbf{a}) \le f(\mathbf{x}) \le f(\mathbf{b}) \qquad \text{for all } \mathbf{x} \in A.$$

Clearly $f(\mathbf{a})$ is the *minimum*, and $f(\mathbf{b})$ is the *maximum* of $f$ on $A$, so we can in principle find points in $A$, in which these extrema are attained. Unfortunately, the $2^{\text{nd}}$ main theorem does not give any hint of how to find these points in $A$. We shall later give some results in this direction.

Finally, we turn to the $3^{\text{rd}}$ main theorem for continuous functions. For some reason this is in general the most difficult one to understand for the reader. Let us start with the strict definition of *continuity* as it was given half a century ago,

$$(5.4) \qquad \forall\, \varepsilon > 0 \,\forall\, \mathbf{x} \in A \,\exists\, \delta > 0 \,\forall\, \mathbf{y} \in A : \|\mathbf{x} - \mathbf{y}\| < \delta \;\Rightarrow\; \|\mathbf{f}(\mathbf{x}) - \mathbf{y}\| < \varepsilon.$$

Here, $\forall$ is read "forall", and $\exists$ is read "there exists".

Today one would use a lesser formal language like the following: First define the growth of the function by

$$\Delta\mathbf{f}(\mathbf{x}, \mathbf{h}) := \mathbf{f}(\mathbf{x} + \mathbf{h}) - \mathbf{f}(\mathbf{x}).$$

Then *continuity* at the *fixed point* $\mathbf{x} \in A$ means that $\Delta\mathbf{f}(\mathbf{x}, \mathbf{x}) \to \mathbf{0}$, when $\mathbf{h} \to \mathbf{0}$, which more explicitly means that *to every $\varepsilon > 0$ we can find $\delta = \delta(\varepsilon, \mathbf{x}) > 0$, depending on both $\varepsilon$ and $\mathbf{x}$, such that*

$$\|\mathbf{h}\| < \delta \qquad \text{implies that} \qquad \|\Delta\mathbf{f}(\mathbf{x}, \mathbf{h})\| < \varepsilon.$$

It is not hard to show that this is the same as the more stringent definition (5.4).

In the applications we often need a stronger property of $f$ than just continuity. It is important for many proofs that we can choose $\delta = \delta(\varepsilon) > 0$ above, *independently of the point* $\mathbf{x} \in A$. This means that $\delta$ is chosen after $\varepsilon > 0$, but before $\mathbf{x} \in A$. When a function $f$ has this property, we call it *uniformly continous*. The same pair $(\varepsilon, \delta)$ can be used everywhere in $A$, so the formal mathematical definition becomes

$$(5.5) \qquad \forall\, \varepsilon > 0 \,\exists\, \delta > 0 \,\forall\, \mathbf{x} \in A \,\forall\, \mathbf{y} \in A : \|\mathbf{x} - \mathbf{y}\| < \delta \;\Rightarrow\; \|\mathbf{f}(\mathbf{x}) - \mathbf{y}\| < \varepsilon.$$

When we compare (5.4) and (5.5), we see that the difference is, at we in (5.4) write

$$\forall\, \varepsilon > 0 \,\forall\, \mathbf{x} \in A \,\exists\, \delta = \delta(\varepsilon, \mathbf{x}) > 0 \,\cdots,$$

i.e. the choice of $\delta$ depends on both $\varepsilon$ and $\mathbf{x}$, while we in (5.5) have interchanged two groups of quantors, so

$$\forall\, \varepsilon > 0 \,\exists\, \delta = \delta(\varepsilon) > 0 \,\forall\, \mathbf{x} \in A \,\cdots,$$

i.e. $\forall \mathbf{x} \in A$ follows *after* the specification of $\delta$.

This makes a very big difference, and *uniform continuity* is clearly a stronger concept than just *continuity*.

**3rd main theorem for continuous functions.** *Let $A \subset \mathbb{R}^n$ be compact, and let $\mathbf{f} \to \mathbb{R}^k$ be continuous. Then $\mathbf{f}$ is uniformly continuous.*

The latter two main theorems show that the compact sets (i.e. closed and bounded sets) in $\mathbb{R}^n$ are very important. It is for that reason that they have been given a special name.

To show its importance we give a simple and useful consequence of the 3rd main theorem below.

**Theorem 5.2** *Given a continuous function $f : I \times [a,b] \to \mathbb{R}$, where $[a,b]$ is a compact interval, while $I \neq \emptyset$ is just an interval. We define a function $F : I \to \mathbb{R}$ by*

$$F(x) := \int_a^b f(x,t)\,dt, \qquad \text{for } x \in I.$$

*Then $F$ is continuous on $I$.*

PROOF. Choose any fixed $\mathbf{x} \in I^\circ$ (the interior of $I$), and then a compact interval

$$J = [x-c, x+c] \subset I^\circ,$$

which is possible, because $I^\circ \neq \emptyset$ is open. Then we have

$$\Delta := F(x+h) - F(x) = \int_a^b \{f(x+h,t) - f(x,t)\}\,\mathrm{d}t \qquad \text{for } |h| < c.$$

The restriction of the continuous function $f$ to the compact set $J \times [a,b]$ is according to the $3^{\text{rd}}$ main theorem *uniformly continuous.* Hence, to every given $\varepsilon > 0$ there is a $\delta = \delta(\varepsilon) > 0$ depending only on $\varepsilon$, such that

$$|f(x,t) - f(y,u)| < \frac{\varepsilon}{b-a}, \qquad \text{if } (x,t),\ (y,u) \in J \times [a,b] \text{ and } \|(x,t) - (y,u)\| < \delta.$$

Note that we only for technical reasons have divided $\varepsilon$ by the length $b-a$ of the interval $[a,b]$.

The above is in particular true, if $u = t \in [a,b]$ and $y = x+h$, where $|h| < \min\{c,\delta\}$. When $h$ is chosen in this way, then we get the estimates

$$|\Delta| \le \int_a^b |f(x+h,t) - f(x,t)|\,\mathrm{d}t \le \int_a^b \frac{\varepsilon}{b-a}\,\mathrm{d}t = \frac{\varepsilon}{b-a} \cdot (b-a) = \varepsilon,$$

and we have proved that $F$ is continuous at every point in the interior of $I$. If $\mathbf{x} \in I$ is an end point, then just modify the $J$ interval above. However, if the end point $\mathbf{x}$ of $I$ does not belong to $I$, we cannot conclude anything. However, Theorem 5.2 does not claim anything in this case. □

# 6 A useful procedure

## 6.1 The domain of a function

**Problem 6.1** *Let the structure of a function $f(x, y, \dots)$ be given. Find the maximum domain of this function, based on this structure.*

**Procedure**.

1) Divide the function into subfunctions according to the signs $+$ and $-$, i.e. write $f = f_+ + f_-$, where $f_+ \geq 0$ and $f_- \leq 0$ (and $f_+ \cdot f_- = 0$).
2) Find the domain for each of the subfunctions (if possible, sketch a figure).
3) Then the domain of $f$ is the intersection of the domains of subfunctions. (Sketch a figure).

If $\mathbf{f}(x, y, \dots)$ is a *vector function*, then apply the above separately for each coordinate function. The domain is the intersection of all the domains of the coordinate functions.

One should in particular be aware of the following rules:

1) **Never** divide by 0.

   Analyze the set of zeros for the denominator, if it exists.

2) In real analysis, **never** take the square root of a negative number.

   Find the set of zeros of the radicand of the square root. Check the sign in the domains which are bounded by this set of zeros.

3) In real analysis **never** take the logarithm of a negative number or of 0.

   Find the set of zeros of the expression which we are going to take the logarithm of. Check the sign in the domains which are bounded by this set of zeros.

**Remark 6.1** Experience tells that the square root is in particular difficult to handle. A professor once told me that "if one can handle the square root, then one can handle anything in mathematics!". Notice that pocket calculators does not like square roots either. $\Diamond$

# 7 Examples of continuous functions in several variables

## 7.1 Maximal domain of a function

**Example 7.1** *Find and sketch in each of the following cases the domain of the given function.*

*1)* $f(x,y) = \ln|1-x^2-y^2|$.

*2)* $f(x,y) = \sqrt{-x^2-y^2}$.

*3)* $f(x,y) = \ln(1-x^2-y^2) + \sqrt{(x-\frac{1}{2})(x^2+y^2)}$.

*4)* $f(x,y) = \ln[y(x^2+y^2+2y)]$.

*5)* $f(x,y) = y\sqrt{2-x^2} + \text{ Arctan } \dfrac{y}{x}$.

*6)* $f(x,y) = \sqrt{3-x^2-y^2} + 2\,\text{Arcsin}(x^2-y^2)$.

*7)* $f(x,y) = \text{ Arcsin}(2-x^2-y)$.

*8)* $f(x,y) = \sqrt{xy-1}$.

*9)* $f(x,y) = \sqrt{y+\sin x} + \sqrt{-y+\sin x}$.

*10)* $f(x,y) = x^y$.

*11)* $f(x,y) = \ln y + \ln(x^2+y^2+2y)$.

**A** Domain of a function.

**D** Analyze the domain and the sketch the set.

**I** 1) The function $\ln|1-x^2-y^2|$ is defined for $|1-x^2-y^2|>0$, i.e. for $x^2+y^2 \neq 1$. The domain is $\mathbb{R}^2$ with the exception of the unit circle:

$$\mathbb{R}^2 \setminus \{(x,y) \mid x^2+y^2=1\}.$$

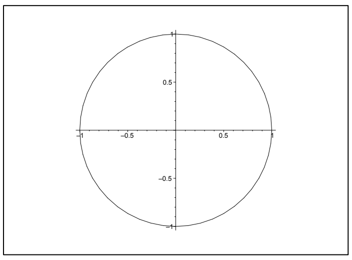



Figure 7.1: The domain of $f(x,y) = \ln|1-x^2-y^2|$

2) The requirement of the function $\sqrt{-x^2-y^2}$ is that $-x^2-y^2 \geq 0$, i.e. the domain is only the point $\{(0,0)\}$.

3) The function $\ln(1-x^2-y^2)+\sqrt{(x-\frac{1}{2})(x^2+y^2)}$ is defined for

$$1-x^2-y^2>0 \qquad \text{and} \qquad \left(x-\frac{1}{2})(x^2+y^2\right) \geq 0.$$

We first conclude that $x^2+y^2<1$, so the domain must be contained in the open unit disc.

Then note that both requirements are fulfilled for $(x,y)=(0,0)$, thus $(0,0)$ belongs to the domain.

Finally, when $0<x^2+y^2<1$ we also have the requirement $x \geq \frac{1}{2}$.

Summarizing the domain is

$$\{(0,0)\} \cup \{(x,y) \mid x \geq \frac{1}{2},\, x^2+y^2<1\}.$$

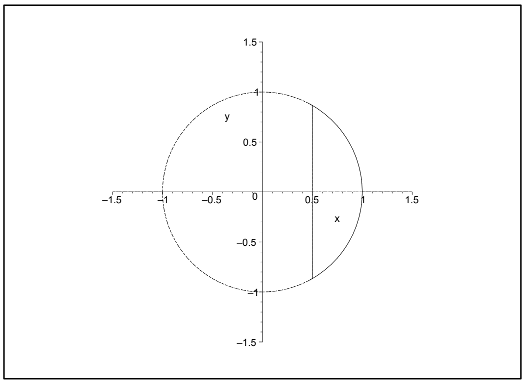



Figure 7.2: The domain of $f(x,y)=\ln(1-x^2-y^2)+\sqrt{(x-\frac{1}{2})(x^2+y^2)}$

4) The function $\ln(y(x^2+y^2+2y))$ is defined for

$$y(x^2+y^2+2y)=y\{x^2+(y+1)^2-1\}>0.$$

Here we get two possibilities:

a) When both $y>0$ and $x^2+(y+1)^2>1$, we see that we can reduce to $y>0$, because then also $(y+1)^2>1$.
b) The second possibility is that $y<0$ and $x^2+(y+1)^2<1$. In this case we reduce to $x^2+(y+1)^2<1$, because this inequality determines an open disc in the lower half plane of centre $(0,-1)$ and radius 1, and $y<0$ is automatically satisfied.

Summarizing we obtain the domain

$$\{(x,y) \mid y>0\} \cup \{(x,y) \mid x^2+(y+1)^1<1\},$$

i.e. the union of the upper half plane and the afore mentioned circle in the lower half plane.

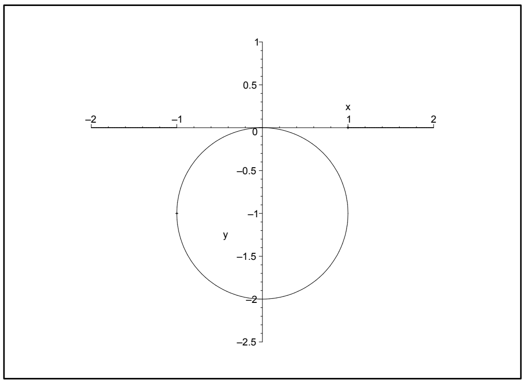



Figure 7.3: The domain of $f(x,y) = \ln[y(x^2+y^2+2y)]$.

5) The function $y\sqrt{2-x^2} + \text{ Arctan } \dfrac{y}{x}$ is defined for

$$2 - x^2 \geq 0 \qquad \text{and} \qquad x \neq 0,$$

i.e. the domain is the union of two vertical strips, which are neither open nor closed,

$$\{(x,y) \mid -\sqrt{2} \leq x < 0\} \cup \{(x,y) \mid 0 < x \leq \sqrt{2}\}.$$

This can also be written

$$[-\sqrt{2}, \sqrt{2}] \times \mathbb{R} \setminus \{(0)\} \times \mathbb{R}.$$

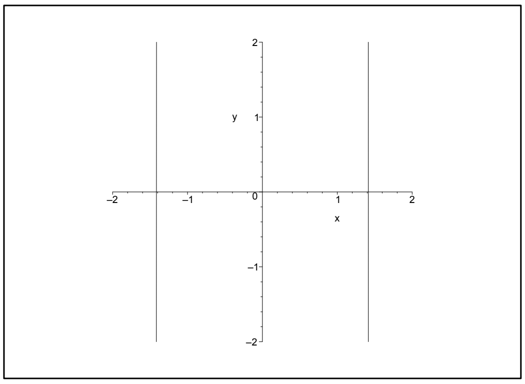



Figure 7.4: The domain of $f(x,y) = y\sqrt{2-x^2} + \text{Arctan } \dfrac{y}{x}$.

6) The function $\sqrt{3-x^2-y^2} + 2\,\text{Arcsin}(x^2-y^2)$ is defined for

$$x^2 + y^2 \leq 3 \qquad \text{and} \qquad -1 \leq x^2 - y^2 \leq 1,$$

i.e. for

$$\sqrt{x^2+y^2} \leq \sqrt{3}, \qquad x^2 - y^2 \leq 1, \qquad y^2 - x^2 \leq 1.$$

The domain is that component of the intersection with the disc which also contains the point $(0,0)$.

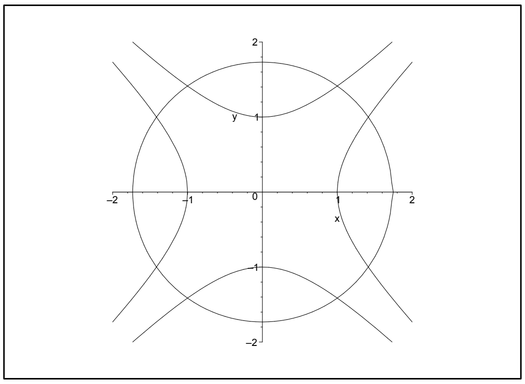



Figure 7.5: The domain of $f(x,y) = \sqrt{3 - x^2 - y^2} + 2\,\mathrm{Arcsin}(x^2 - y^2)$.

7) The function $\mathrm{Arcsin}(2 - x^2 - y)$ is defined for

$$-1 \le 2 - x^2 - y \le 1,$$

i.e. when the following two conditions are fulfilled:

$$y \le 3 - x^2 \qquad \text{and} \qquad y \ge 1 - x^2.$$

Summarizing the domain becomes

$$\{(x,y) \mid 1-x^2 \le y \le 3-x^2\},$$

which is the closed set which lies between the two arcs of parabolas.

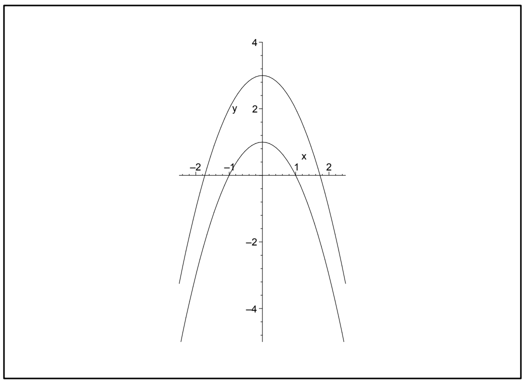


Figure 7.6: The domain of $f(x,y) = \text{Arcsin}(2-x^2-y)$.

8) The function $\sqrt{xy-1}$ is defined for $xy \ge 1$ i.e. the sets in the first and third quadrant, which are bounded by the hyperbola $y = \dfrac{1}{x}$ and which is not close to any of the axes:

$$\{(x,y) \mid x>0,\, y>0,\, xy \ge 1\} \cup \{(x,y) \mid x<0,\, y<0,\, xy \ge 1\}.$$

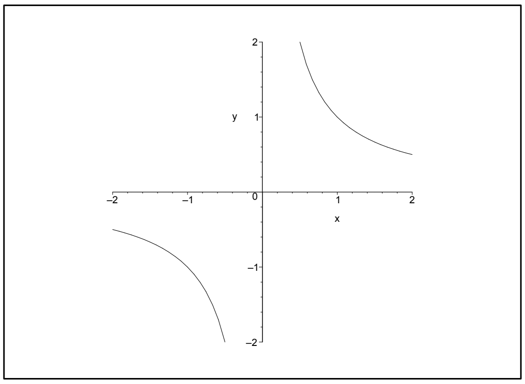


Figure 7.7: The domain of $f(x,y) = \sqrt{xy-1}$.

9) The function $\sqrt{y+\sin x} + \sqrt{-y+\sin x}$ is defined when both

$$y + \sin x \ge 0 \qquad \text{and} \qquad -y+\sin x \ge 0,$$

i.e. when

$$-\sin x \le y \le \sin x.$$

Hence the condition $\sin x \geq 0$, i.e. $x \in [2p\pi, \pi + 2p\pi]$, $p \in \mathbb{Z}$, and the domain is

$$\bigcup_{p \in \mathbb{Z}} \{(x,y) \mid 2p\pi \leq x \leq 2p\pi + \pi,\ |y| \leq \sin x\}.$$

On the figure the domain is the union of every second of the connected subsets.

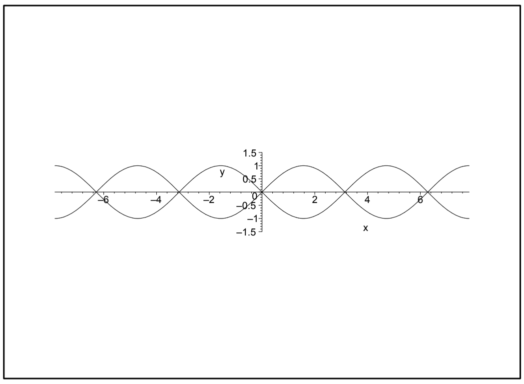



Figure 7.8: The domain of $f(x,y) = \sqrt{y + \sin x} + \sqrt{-y + \sin x}$.

10) This is a very difficult example. First notice that the function $f(x,y) = x^y$ is at least defined when $x,\ y > 0$.

When $x = 0$ the function is defined for every $y > 0$.

When $x < 0$, the function is defined for every $y = \dfrac{p}{2q+1}$, where $p \in \mathbb{Z}$ and $q \in \mathbb{N}_0$.

When $y < 0$ is not a rational number of odd denominator, we must necessarily require that $x > 0$.

When $y = -\dfrac{p}{2q+1}$, $p \in \mathbb{N}$, $q \in \mathbb{N}_0$, then $x^y$ is also defined for $x < 0$, though not for $x = 0$.

Remark. It is a matter of definition whether one can put $x^0 = 1$ for $x < 0$. This may be practical in some cases, though not in everyone. $\Diamond$.

This domain is fairly complicated:

$$\{(x,y) \mid x > 0\} \cup \{(0,y) \mid y > 0\} \cup \bigcup_{p,\, q \in \mathbb{N}_0} \{(x,y) \mid x < 0,\ y = -p/(2q+1)\},$$

where one may discuss whether the point $(0,0)$ should be included or not.

11) When the function $f(x,y) = \ln y + \ln(x^2 + y^2 + 2y)$ is defined, we must at least require that $y > 0$, because $\ln y$ in particular should be defined.

If on the other hand $y > 0$, then clearly also $x^2 + y^2 + 2y > 0$, no matter the choices of $x$ and $y > 0$, thus $f(x,y)$ is defined for $y > 0$, i.e. in the upper half plane.

**Example 7.2** *Describe in each of the following cases the domain of the given function.*

*1)* $f(x,y,z) = \sqrt{1-|x|-|y|-|z|}$.

*2)* $f(x,y,z) = \ln(\sqrt{1-|x|-|y|-|z|})$.

*3)* $f(x,y,z) = \text{Arcsin}(x^2+y^2-4)$.

*4)* $f(x,y,z) = \sqrt[4]{x^2+4y^2+9z^2-1}$.

*5)* $f(x,y,z) = \text{Arctan}\,\dfrac{x+z}{y}$.

*6)* $f(x,y,z) = \exp(3x+2y+5z)$.

**A** Domain of functions in three variables.

**D** Analyze in each case the function. There will here be given no sketches.

**I** 1) The function $\sqrt{1-|x|-|y|-|z|}$ is defined for $|x|+|y|+|z| \le 1$,

$$\{(x,y,z) \mid |x|+|y|+|z| \le 1\}.$$

This set is a closed tetrahedron in the space.

2) The function $\ln(\sqrt{1-|x|-|y|-|z|})$ is defined in the corresponding *open* tetrahedron in space,

$$\{(x,y,z) \mid |x|+|y|+|z| < 1\}.$$

3) The function $\text{Arcsin}(x^2+y^2+z^2-4)$ is defined when

$$-1 \le x^2+y^2+z^2-4 \le 1,$$

i.e. in the shell

$$\left\{(x,y,z) \mid (\sqrt{3})^2 \le x^2+y^2+z^2 \le (\sqrt{5})^2\right\},$$

of centre $(0,0,0)$, inner radius $\sqrt{3}$ and outer radius $\sqrt{5}$.

4) The function $\sqrt[4]{x^2+4y^2+9z^2-1}$ is defined outside an ellipsoid,

$$\left\{(x,y,z) \;\middle|\; x^2+\left(\frac{y}{\frac{1}{2}}\right)^2+\left(\frac{z}{\frac{1}{3}}\right)^2 \ge 1\right\},$$

where the half axes are $1$, $\dfrac{1}{2}$ and $\dfrac{1}{3}$.

5) The function $\text{Arctan}\,\dfrac{x+z}{y}$ is defined for $y \ne 0$.

6) The function $\exp(3x+2y+5z)$ is of course defined in the whole space $\mathbb{R}^2$.

## 7.2 Level curves and level surfaces

**Example 7.3** *Let*

$$f(x,y) = \ln(2-2x^2-3y^2)+2-4x^2-6y^2, \qquad (x,y) \in A.$$

*1) Sketch the domain A.*

*2) Describe the level curves of the function. It is convenient to introduce a new variable u, such that $f(x,y) = F(u(x,y))$.*

*3) Sketch the level curve corresponding to $f(x,y) = 0$.*

*4) Find the range $f(A)$.*

**A** Domain and level curves.

**D** Describe the set given by $2-2x^2-3y^2>0$, where $f(x,y)$ is defined. Then change the parameter to $u$.

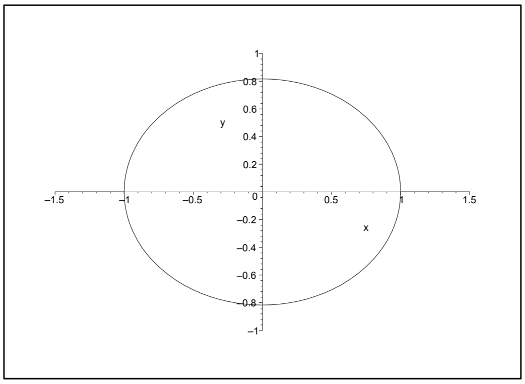
**I** 1) The function is defined, if and only if

$$u = u(x,y) = 2-2x^2-3y^2 > 0,$$

i.e. for

$$\left(\frac{x}{1}\right)^2 + \left(\frac{y}{\sqrt{\frac{2}{3}}}\right)^2 < 1,$$

which describes an open ellipsoidal disc of centrum $(0,0)$ and half axes 1 and $\sqrt{\frac{2}{3}}$.




2) If we define

$$u = u(x,y) = 2 - 2x^2 - 3y^2 > 0,$$

i.e. $u \in ]0,2]$, then

$$\begin{aligned} f(x,y) &= \ln(2-2x^2-3y^2) + 2 - 4x^2 - 6y^2 \\ &= \ln(2-2x^2-3y^2) + 2(2-2x^2-3y^2) - 2 \\ &= \ln u + 2u - u. \end{aligned}$$

This is clearly an increasing function in $u \in ]0,2]$. Every level curve

$$f(x,y) = \ln u + 2u - 2 = c$$

corresponds to

$$u = 2 - 2x^2 - 3y^2 = k \in ]0,2],$$

where $k$ is unique according to the above.

Then by a rearrangement,

$$2x^2 + 3y^2 = 2 - k, \qquad k \in ]0,2].$$

If $k = 2$, then the level "curve" degenerates to the point $(0,0)$.

If $0 < k < 2$, then the level curve is an ellipse

$$\left(\frac{x}{\sqrt{\frac{2-k}{2}}}\right)^2 + \left(\frac{y}{\sqrt{\frac{2-k}{3}}}\right)^2 = 1$$

with the half axes $\sqrt{\dfrac{2-k}{2}}$ and $\sqrt{\dfrac{2-k}{3}}$.

3) When

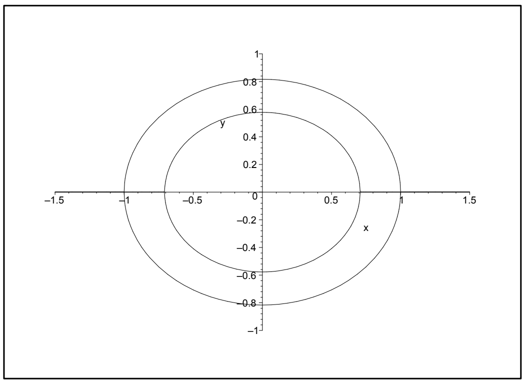
$$f(x,y) = \ln u + 2u - 2 = 0,$$

it follows that $u = 1$ is a solution. Since the function of $u$ is strictly increasing, it follows that $u = 1$ is the only solution, so $k = 1$.

According to 2) the level curve $f(x,y) = 0$ is the ellipse

$$\left(\frac{x}{\sqrt{\frac{1}{2}}}\right)^2 + \left(\frac{y}{\sqrt{\frac{1}{3}}}\right)^2 = 1$$

of centre $(0,0)$ and half axes $\sqrt{\dfrac{1}{2}}$ and $\sqrt{\dfrac{1}{3}}$.




4) We obtain the range by changing the variable to $u$,

$$f(x,y) = F(u) = \ln u + 2u - 2, \qquad u \in ]0,2],$$

because the value $u$ is attained precisely on one level curve.

Since $F'(u) = \dfrac{1}{u} + 2$, we see that $F(u)$ is increasing.

When $n \to 0+$, we get $F(u) \to -\infty$. When $u = 2$, we get

$$F(u) = \ln 2 + 4 - 2 = 2 + \ln 2.$$

Since $F(u)$ is continuous, the connected interval $]0,2]$ is mapped into the connected interval $]-\infty, 2+\ln 2]$. Here we apply the third main theorem of continuous functions.

The range is $f(A) = ]-\infty, 2+\ln 2]$.

**Example 7.4** *Sketch for each for the functions $f : \mathbb{R}^2 \to \mathbb{R}$ below the level curves given by $f(x,y) = C$ for the given values of the constant $C$.*

*1)* $f(x,y) = x^2 + y^2, \qquad C \in \{1,2,3,4,5\},$

*2)* $f(x,y) = x^2 - 4x + y^2, \qquad C \in \{-3,-2,-1,0,1\},$

*3)* $f(x,y) = x^2 - 2y, \qquad C \in \{-2,-1,0,1,2\},$

*4)* $f(x,y) = \max\{|x|,|y|\}, \qquad C \in \{1,2,3\},$

*5)* $f(x,y) = |x| + |y|, \qquad C \in \{1,2,3\},$

*6)* $f(x,y) = (x^2+y^2+1)^2 - 4x^2, \qquad C \in \{\frac{1}{2},1,3\},$

*7)* $f(x,y) = x^2 + y^2(1+x)^3, \qquad C \in \{-4,0,\frac{1}{4},1,4\}.$

**A** Level curves.

**D** Whenever it is necessary, start by analyzing the given function.

**I** 1) The level curves are circles of centrum $(0,0)$ and radii $\sqrt{C}$, i.e. $1$, $\sqrt{2}$, $\sqrt{3}$, $2$, $\sqrt{5}$.

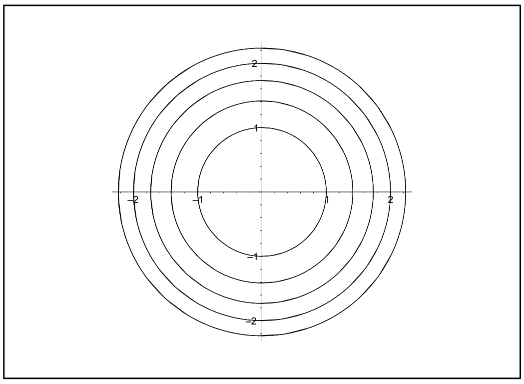



Figure 7.9: The level curves $x^2+y^2=C$, $C=1$, $\sqrt{2}$, $\sqrt{3}$, $2$, $\sqrt{5}$.

2) Since

$$f(x,y) = x^2 - 4x + y^2 = (x-2)^2 + y^2 - y,$$

we can also write the equation $f(x,y) = C$ of the level curves in the form

$$(x-2)^2 + y^2 = 4 + C.$$

The level curves are circles of centre $(2,0)$ and radius $\sqrt{4+C}$, i.e. $1$, $\sqrt{2}$, $\sqrt{3}$, $2$, $\sqrt{5}$.

It follows that we obtain the same system as in 1), only translated to the centre $(2,0)$.

3) The equation of the level curves $f(x,y) = C$ can also be written

$$y = \frac{1}{2}\,x^2 - \frac{C}{2}, \qquad C \in \{-2,-1,0,1,2\}.$$

These are parabolas of top points at $\left(0, -\dfrac{C}{2}\right)$.

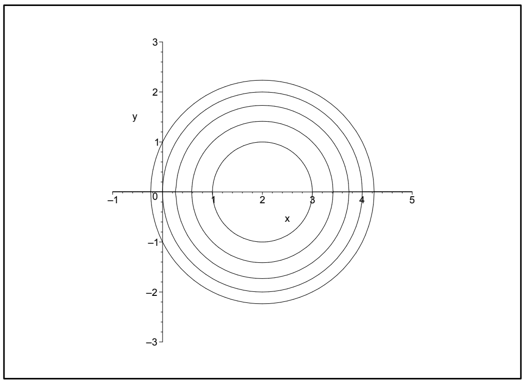



Figure 7.10: The level curves $x^2 - 4x + y^2 = C$, $C = -3, -2, -1, 0, 1$.

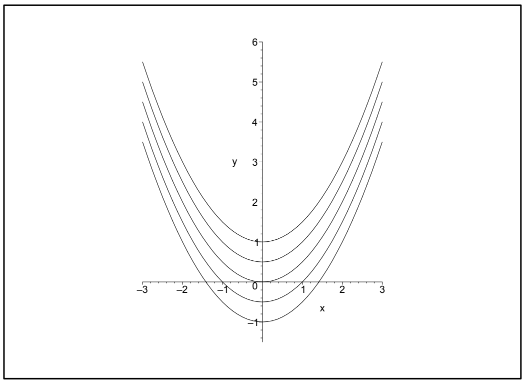



Figure 7.11: The level curves $x^2 - 2y = C$, $C = -2, -1, 0, 1, 2$.

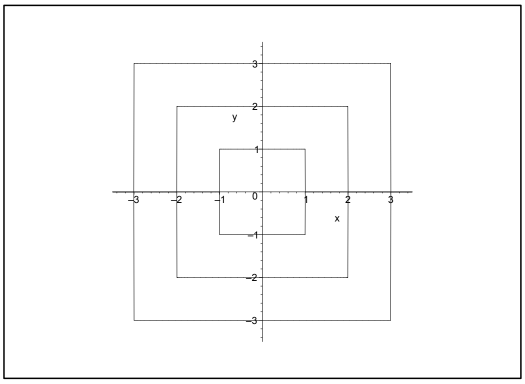



Figure 7.12: The level curves $\max\{|x|, |y|\} = C$, $C = 1, 2, 3$.

4) The level curves are the boundary of the squares of centre $(0,0)$ and edge length $2C$.
5) The level curves are the boundaries of the squares of centre $(0,0)$ and the corners $(\pm C, 0)$ and $(0, \pm C)$.

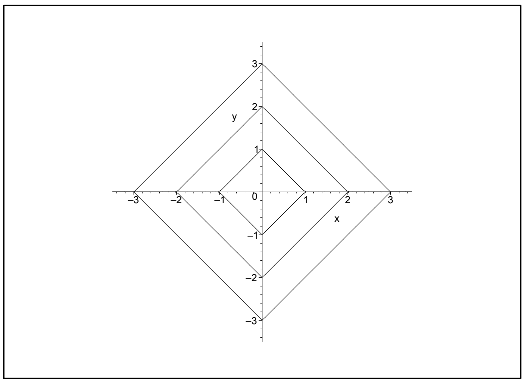



Figure 7.13: The level curves $|x| + |y| = C$, $C = 1, 2, 3$.

6) First note that

$$\begin{aligned} f(x,y) &= (x^2+y^2+1)^2 - 4x^2 \\ &= (x^2+y^2+1-2x)(x^2+y^2+1+2x) \\ &= \{(x-1)^2+y^2\}\{(x+1)^2+y^2\}. \end{aligned}$$

The level curves $f(x,y) = C$ can then be interpreted as the curves composed of the points $(x,y)$, for which the *product* of the distances to $(1,0)$ and $(-1,0)$ is equal to $\sqrt{C}$.

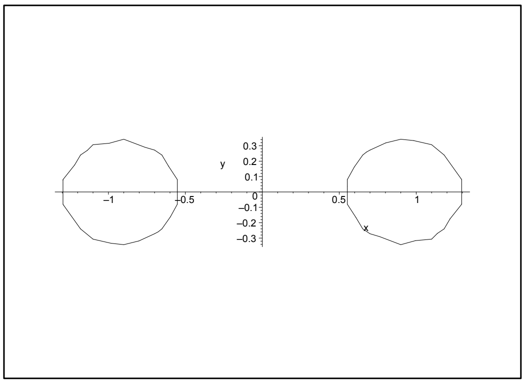



Figure 7.14: The level curve $(x^2+y^2+1)^2 - 4x^2 = \dfrac{1}{2}$.

7) First note that when $x = -1$, then $f(-1,0) = 1$. This means that we shall be particular careful in the case of $C = 1$.

Here we get five cases which are treated successively.

a) When $C = -4$, it follows from our first remark that $x \neq -1$. Clearly, $y \neq 0$, because $x^2 = -4$ does not have any real solution. The level curves are given by

$$y^2 = -\frac{4+x^2}{(1+x)^3} > 0.$$

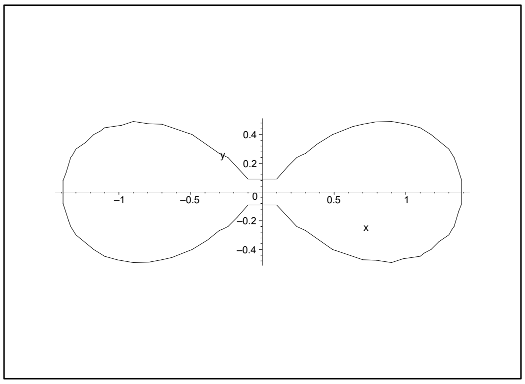



Figure 7.15: The level curve $(x^2+y^2+1)^2-4x^2=1$. Though it cannot be seen (due to some error in the programme of sketching) the curves continue through $(0,0)$.

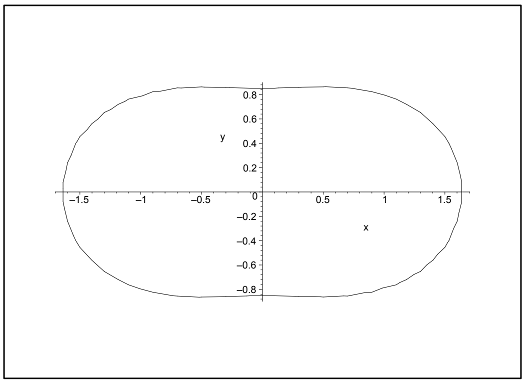



Figure 7.16: The level curve $(x^2+y^2+1)^2-4x^2=3$.

Accordingly, $x<-1$, and

$$y^2=-\frac{1}{(1+x)^2}\left(x-1+\frac{5}{1-x}\right),$$

i.e.

$$y=\pm\frac{1}{|1+x|}\sqrt{1-x-\frac{5}{1+x}}=\pm\frac{1}{|1+x|}\sqrt{2+|1+x|+\frac{5}{|1+x|}},$$

for $x<-1$.

We get two level curves, which lie symmetrically to each other with respect to the $X$ axis where the line $x=-1$ and the $X$ axis are the asymptotes.

b) When $C=0$, we again find that $x\neq -1$. Note that if $y=0$, then $x=0$ is a solution, hence the point $(0,0)$ belongs to the solutions. When $y\neq 0$, we get

$$y=\pm\left|\frac{x}{1+x}\right|\cdot\frac{1}{\sqrt{|1+x|}},\qquad x<-1.$$

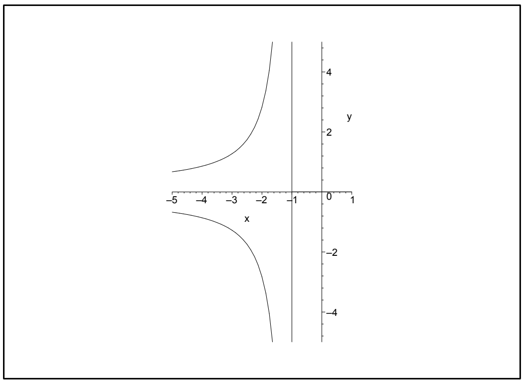



Figure 7.17: The level curves $x^2 + y^2(1+x)^3 = -4$.

The level "curves" are the point $(0,0)$ and two symmetric curves with respect to the $X$ axis. These are closer the asymptotes than the level curves for $C = -4$.

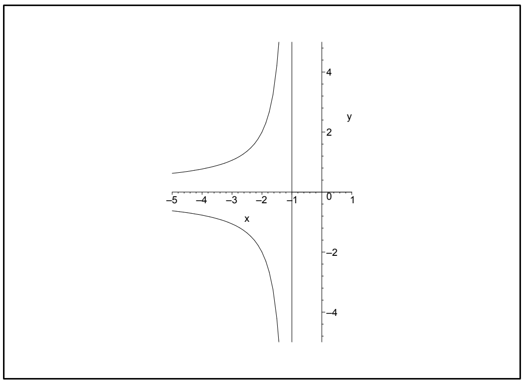



Figure 7.18: The level curves $x^2 + y^2(1+x)^3 = 0$, where the point $(0,0)$ should be added.

c) If $C = \dfrac{1}{4}$, then $x \neq -1$, and

$$y^2 = \frac{\frac{1}{4} - x^2}{(1+x)^3} = -\frac{(x-\frac{1}{2})(x+\frac{1}{2})}{(x+1)^2} \geq 0.$$

We note that $y = 0$, if and only if $x = \pm\dfrac{1}{2}$.

Then the right hand side is positive, when either $|x| < \dfrac{1}{2}$ or $x < -1$.

The level curves are two symmetric curves for $x < -1$ with respect to the $X$ axis, where the $X$ axis and the line $x = -1$ are the asymptotes, supplied with a closed curve for $x \in \left[-\dfrac{1}{2}, \dfrac{1}{2}\right]$.

d) For $C = 1$ we are in the exceptional case mentioned above where $x = -1$ is a level curve.

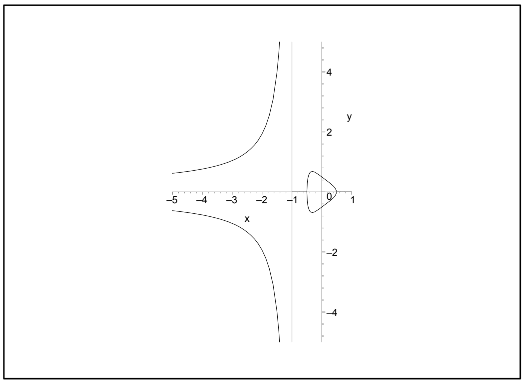



Figure 7.19: The level curves $x^2 + y^2(1+x)^3 = \frac{1}{4}$.

When $x \neq -1$, we get

$$y^2 = \frac{1-x^2}{(1+x)^3} = \frac{1-x}{(1+x)^2} \geq 0,$$

thus $x \leq 1$. When $x = 1$, we only get the solution $y = 0$, i.e. we get the point $(1, 0)$.

The level curves are the line $x = -1$, two symmetric curves with respect to the $X$ axis for $x < -1$, and a curve with the $X$ axis as an axis of symmetry for $x \in ]-1, 1]$ and the line $x = -1$ as an asymptote.

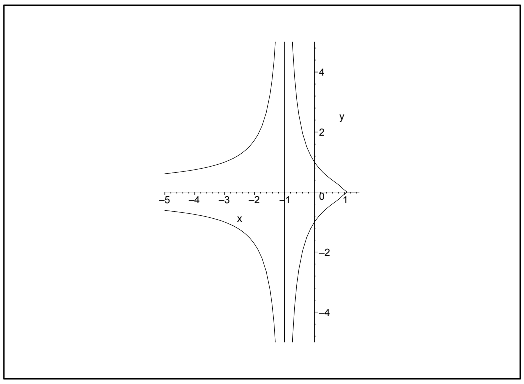



Figure 7.20: The level curves $x^2 + y^2(1+x)^3 = 1$.

e) When $C = 4$, we get

$$y^2 = \frac{4-x^2}{(1+x)^3} \geq 0.$$

It follows that $(\pm 2, 0)$ are solutions and that we only get solutions for either $x \leq -2$ or $-1 < x \leq 2$.

We obtain two curves, each symmetric with respect to the $X$ axis. Furthermore, one of these curves has $x = -1$ as an asymptote.

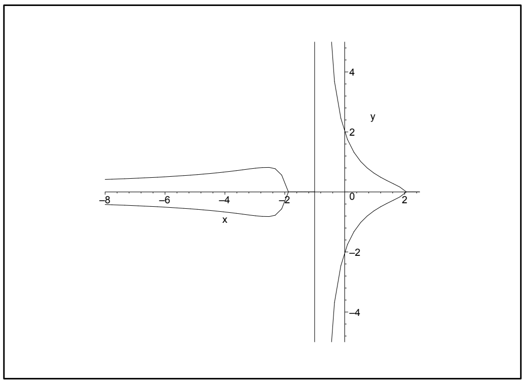



Figure 7.21: The level curves $x^2 + y^2(1+x)^3 = 4$.

**Example 7.5** *Describe the level surfaces for the following functions:*

*1)* $f(x,y,z) = x$ *for* $(x,y,z) \in \mathbb{R}^3$,

*2)* $f(x,y,z) = \max\{|x|,|y|,|z|\}$ *for* $(x,y,z) \in \mathbb{R}^3$,

*3)* $f(x,y,z) = \sqrt{\max\{|x|,|y|,|z|\}}$ *for* $(x,y,z) \in \mathbb{R}^3$,

*4)* $f(x,y,z) = z - x^2 - y^2$ *for* $(x,y,z) \in \mathbb{R}^3$,

*5)* $f(x,y,z) = \dfrac{x^2+y^2+z^2-a^2}{z}$ *for* $z \neq 0$.

**A** Level surfaces in space.

**D** Analyze the function. The sketches are left to the reader, because there are difficulties here with the MAPLE programs. (I am not clever enough to get the right drawings.)

**I** 1) Obviously, the level surfaces

$$f(x,y,z) = x = c$$

are all planes parallel to the $YZ$ plane, where $c \in \mathbb{R}$.

2) The level surfaces are the boundaries of all cubes of centrum $(0,0,0)$ and edge length $2c$ for $c > 0$, supplied with the point $(0,0,0)$ when $c = 0$.

Only $c \geq 0$ is possible.

3) The level surfaces are the same as in 2), only the edge length is here $2c^2$ for $c > 0$. When $c = 0$ we obtain as before the point $(0,0,0)$.

4) Since $f(x,y,z) = z - x^2 - y^2 = c$ can also be written

$$z - c = x^2 + y^2,$$

we obtain all paraboloids of revolution with top point at $(0,0,c)$, through the unit circle in the plane $z = 1 + c$ and with the $Z$ axis as the axis of revolution.

5) First we rewrite

$$f(x,y,z) = \frac{x^2+y^2+z^2-a^2}{z} = c, \qquad z \neq 0,$$

to

$$x^2+y^2+z^2-a^2 = cz, \qquad z \neq 0,$$

i.e.

$$x^2+y^2+\left(z-\frac{C}{2}\right)^2 = a^2+\frac{c^2}{4}, \qquad z \neq 0+.$$

The level surfaces are spheres of centrum $\left(0,0,\frac{c}{2}\right)$ and radius $\sqrt{a^2+\frac{c^2}{4}}$, with the exception of the points in the $XY$ plane, i.e. with the exception of the circle

$$x^2+y^2=a^2, \qquad z=0.$$

**Example 7.6** *Consider the function* $f(\mathbf{x}) = \mathbf{x} \cdot \mathbf{e}$, $\mathbf{x} \in \mathbb{R}^k$, *where* $\mathbf{e}$ *is a constant unit vector.*

*1) Sketch the level curves of the function in the case of* $k = 2$.

*2) Describe the level surfaces of the function in the case of* $k = 3$.

**A** Level curves and level surfaces.

**D** Sketch if possible a figure and analyze.

**I** 1) The level curves are all the straight lines $\ell$, which are perpendicular to the line generated by the vector $\mathbf{e}$.

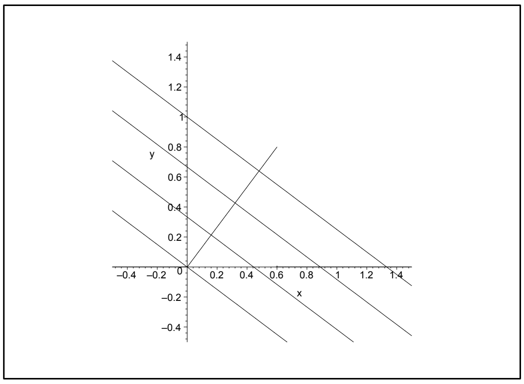



Figure 7.22: Some level curves when $\mathbf{e} = \left(\frac{3}{5}, \frac{4}{5}\right)$.

2) Analogously the level surfaces for $k = 3$ are all planes $\pi$, which are perpendicular to the line generated by the vector $\mathbf{e}$.

**Example 7.7** *Let* $a$ *be a positive constant. Find the domain of the function*

$$f(x, y, z) = \ln\left(a^2 - 3x^2 - y^2 - 2z^2\right).$$

*The describe the level surfaces for* $f$, *and find the range of the function.*

**A** Domain, level surfaces, range.

**D** Just follow the text.

**I** The function is defined for

$$3x^2 + y^2 + 2z^2 < a^2,$$

which describes the open ellipsoid with the half axes

$$\frac{a}{\sqrt{3}}, \quad a, \quad \frac{a}{\sqrt{2}}.$$

The level surfaces are all the ellipsoidal surfaces

$$3x^2 + y^2 + 2 <^2 = b^2, \qquad 0 < b < a,$$

with the half axes

$$\frac{b}{\sqrt{3}}, \quad b, \quad \frac{b}{\sqrt{2}}.$$

The value of the function on such a level surface is $\ln\left(a^2 - b^2\right)$.

The range of $f$ is the same as the range of the function

$$g(t) = \ln\left(a^2 - t^2\right), \qquad t \in [0, a[,$$

so the range is $]-\infty, 2\ln a]$.

**Example 7.8** *Sketch the domain $A$ of the function*

$$f(x,y) = \ln\left(225 - 25x^2 - 9y^2\right).$$

*Indicate the boundary $\partial A$ of $A$, and sketch the level curve of $f$, which contains the point*

$$(x,y) = \left(\frac{3}{2}, \frac{5}{2}\right).$$

**A** Domain and level curve.

**D** Since ln is only defined on $\mathbb{R}_+$, the domain is given by the requirement that the expression inside the ln is positive.

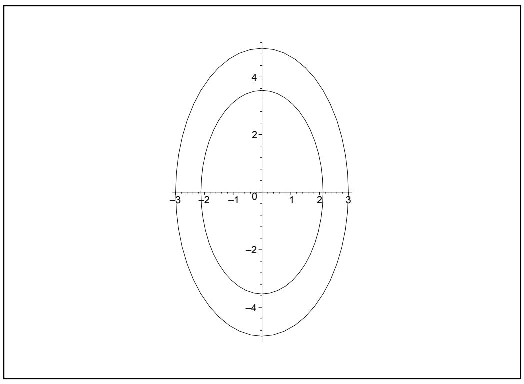



Figure 7.23: The domain $A$ and the level curve through $\left(\frac{3}{2}, \frac{5}{2}\right)$.

**I** The function is defined for

$$225 - 25x^2 - 9y^2 > 0, \quad \text{i.e. for} \quad (5x)^2 + (3y)^2 < 15^2,$$

hence

$$\left(\frac{x}{3}\right)^2 + \left(\frac{y}{5}\right)^2 < 1.$$

The domain is an open ellipsoidal disc of centrum $(0,0)$ and half axes 3 and 5.

the level curve is given by

$$\ln\left(225 - 25x^2 - 9y^2\right) = f\left(\frac{3}{2},\frac{5}{2}\right) = \ln\left\{225 - \left(5\cdot\frac{3}{2}\right)^2 - \left(3\cdot\frac{5}{2}\right)^2\right\},$$

i.e. by

$$225 - 25x^2 - 9y^2 = 225\left(1 - \frac{1}{4} - \frac{1}{4}\right) = \frac{225}{2},$$

hence by a rearrangement,

$$(5x)^2 + (3y)^2 = \left(\frac{15}{\sqrt{2}}\right)^2.$$

This can also be written

$$\left(\frac{x}{\frac{3}{2}\sqrt{2}}\right)^2 + \left(\frac{y}{\frac{5}{2}\sqrt{2}}\right)^2 = 1.$$

Thus the level curve is an ellipse of centrum $(0,0)$ and half axes $\frac{3}{2}\sqrt{2} = \frac{3}{\sqrt{2}}$ and $\frac{5}{2}\sqrt{2} = \frac{5}{\sqrt{2}}$.

## 7.3 Continuous functions

**Example 7.9** *The range of each of the following functions in two variables is not the whole plane but $\mathbb{R}^2 \setminus M$, where $M \neq \emptyset$. Find the point set $M$ in each case and explain why $f : \mathbb{R}^2 \setminus \to \mathbb{R}$ is continuous. Finally, check whether the function has a continuous extension to either $\mathbb{R}^2$ or to $\mathbb{R}^2 \setminus L$, where $L \subset M$.*

*1)* $f(x,y) = \dfrac{x^2 - y^2}{x^2 + y^2}$,

*2)* $f(x,y) = \dfrac{x^3 + y^3}{x^2 + y^2}$,

*3)* $f(x,y) = \dfrac{x^2 y}{\sqrt{x^2 + y^2}}$,

*4)* $f(x,y) = \dfrac{xy}{\sqrt{x^2 + y^2}}$,

*5)* $f(x,y) = \dfrac{3x - 2y}{2x - 3y}$,

*6)* $f(x,y) = \dfrac{x^2 - y^2}{\operatorname{Arctan}(x - y)}$,

*7)* $f(x,y) = \dfrac{x^3 - y^3}{x - y}$,

*8)* $f(x,y) = \dfrac{1 - e^{xy}}{xy}$.

**A** Examination of functions, continuous extension.

**D** Find the set of exceptional points. Since the numerator and the denominator are continuous in $\mathbb{R}^2$ in all cases, it is only a matter of determining the zero set of the denominator. A possible continuous extension can only take place at points in which both the numerator and the denominator are zero, so this set should be examined too.

**I** 1) The denominator is clearly only zero at $(0,0)$, so $M = \{(0,0)\}$.

If we use polar coordinates, we get for $\varrho > 0$,

$$f(x,y) = \frac{\varrho^2\cos^2\varphi - \varrho^2\sin^2\varphi}{\varrho^2} = \cos^2\varphi - \sin^2\varphi = \cos 2\varphi,$$

and it is obvious that we cannot have a continuous extension to $(0,0)$, because there is no restriction on $\varphi$.

2) Here also $M = \{(0,0)\}$. By using polar coordinates we get

$$f(x,y) = \frac{\varrho^3\cos^3\varphi + \varrho^3\sin^3\varphi}{\varrho^2} = \varrho\{\cos^3\varphi + \sin^3\varphi),$$

which tends to 0 for $\varrho \to 0$. Hence, the function has a continuous extension to $(0,0)$ given by $f(0,0) = 0$.

3) Again $M = \{(0,0)\}$. By using polar coordinates we get

$$f(x,y) = \frac{\varrho^3\cos^2\varphi\sin\varphi}{\varrho} = \varrho^2\cos^2\varphi\sin\varphi,$$

which tends to 0 for $\varrho \to 0$. Hence the function has a continuous extension given by $f(0,0) = 0$.

4) Also here $M = \{(0,0)\}$. Again by polar coordinates,

$$f(x,y) = \frac{\varrho^2\sin\varphi\cos\varphi}{\varrho} = \varrho\sin\varphi\cos\varphi \to 0 \qquad \text{for } \varrho \to 0.$$

By continuous extension we get $f(0,0) = 0$.

5) Here

$$M = \{(x,y) \mid 2x = 3y\} = \left\{(x,y) \;\middle|\; y = \frac{2}{3}\,x\right\}.$$

The only possibility of a continuous extension must take place on that subset where the numerator is also zero, i.e. on $\{(0,0)\}$. Using polar coordinates we get

$$f(x,y) = \frac{3\cos\varphi - 2\sin\varphi}{2\cos\varphi - 3\sin\varphi},$$

which clearly does not have a limit, when $\varrho \to 0$, and $\varphi \in [0, 2\pi[$. In this case we do not have a continuous extension.

6) Here $M = \{(x,y) \mid y = x\}$. Since

$$f(x,y) = \frac{x+y}{\dfrac{\operatorname{Arctan}(x-y)}{x-y}}, \qquad (x,y) \notin M,$$

where

$$\frac{\operatorname{Arctan} t}{t} \to 1 \qquad \text{for } t \to 0,$$

it is possible to extend the function to all of $M$ by

$$f(x,x) = 2x, \qquad (x,x) \in M.$$

7) Here we also have $M = \{(x,y) \mid y = x\}$. We get by a division

$$f(x,y) = \frac{x^3 - y^3}{x - y} = x^2 + xy + y^2, \qquad (x,y) \notin M.$$

Clearly, the latter expression can be continuously extended to all of $\mathbb{R}^2$. On $M$ we get

$$f(x,x) = 3x^2, \qquad (x,x) \in M.$$

8) Here $M = \{(x,y) \mid x = 0 \text{ or } y = 0\}$, i.e. the union of the coordinate axes.

Since

$$\frac{1 - e^t}{t} = -\frac{e^t - e^0}{t - 0} \to -1 \qquad \text{for } t \to 0,$$

it follows from an application of the substitution $t = xy$ that $f$ can be extended to the axes by

$$f(0,y) = f(x,0) = -1.$$

**Example 7.10** *In each of the following cases one shall find the domain $D$ of the given function $f$, and explain why $f$ is continuous. Then show that $f$ has a continuous extension to a point set $B$, where $B \supset D$.*

*1)* $f(x,y) = \dfrac{x+y-1}{\sqrt{x}-\sqrt{1-y}}$,

*2)* $f(x,y) = (x+y)\ln\sinh(x+y)$,

*3)* $f(x,y) = \dfrac{\text{Arcsin}(xy-2)}{\text{Arctan}(3xy-6)}$,

*4)* $f(x,y) = \exp\left(-\dfrac{1}{(x-y)^2}\right)$.

**A** Examination of functions and continuous extensions.

**D** Find the point set where the numerator and the denominator are defined and continuous.

Then check a possible extension to the set where both the numerator and the denominator are zero.

**I** 1) The numerator is defined in $\mathbb{R}^2$. The numerator is defined and continuous when $x \geq 0$ and $1-y \geq 0$, i.e. for $y \leq 1$.

The denominator is zero, when $\sqrt{x} = \sqrt{1-y}$ for $x \geq 0$ and $y \leq 1$. A squaring shows that the denominator is zero when

$$x+y=1, \qquad x \geq 0, \qquad y \leq 1,$$

and we see that the numerator is zero on the same set. We see that the domain is

$$D = \{(x,y) \mid x \geq 0,\, y \leq 1,\, x+y \neq 1\} = D_1 \cup D_2.$$

In the two subdomains $D_1$ (the "lower triangular domain") and $D_2$ (the "upper triangular domain") both the numerator and the denominator are continuous, and the denominator is not zero in these two sets, so the function us continuous on $D$.
It has already above been given a hint that there is a possible continuous extension to the line $x+y=1$ for $x \geq 0$ and $y \leq 1$, because both the numerator and the denominator are here 0. We get by a simple rearrangement for $(x,y) \in D$, i.e. in particular for $x+y \neq 1$, that

$$f(x,y) = \frac{x-(1-y)}{\sqrt{x}-\sqrt{1-y}} = \frac{(\sqrt{x})^2-(\sqrt{1-y})^2}{\sqrt{x}-\sqrt{1-y}} = \sqrt{x}+\sqrt{1-y}.$$

This expression is continuous on the set

$$\{(x,y) \mid x \geq 0,\, y \leq 1\},$$

and we have found our continuous extension of the original function.

2) Here $f(x,y)$ is defined and continuous for $\sinh(x+y) > 0$, i.e. when $x+y>0$, and the domain is

$$D = \{(x,y) \mid x+y > 0\}.$$

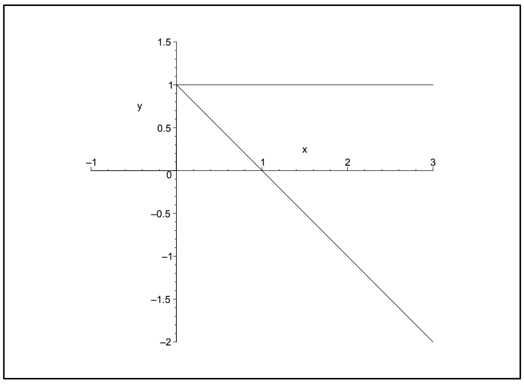


Figure 7.24: The domain of $f(x,y) = \dfrac{x+y-1}{\sqrt{x}-\sqrt{1-y}}$.

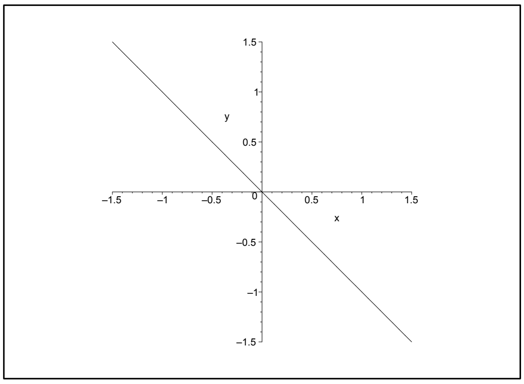


Figure 7.25: The domain of $f(x,y) = (x+y)\ln\sinh(x+y)$ lies above the oblique line.

By putting $t = x+y > 0$ we exploit that $f(x,y)$ actually is a function in $x+y$. Then

$$f(x,y) = g(t) = t\ln\sinh t = \frac{t}{\sinh t}\{\sinh t \cdot \ln\sinh t\}.$$

Here, $\dfrac{t}{\sinh t} \to 1$ for $t \to 0+$, and $\sinh t \cdot \ln\sinh t \to 0$ for $\sinh t \to 0+$, i.e. for $t \to 0+$. We therefore conclude for $z = \sinh t$ that

$$\lim_{t \to 0+} t\,\ln\sinh t = 0.$$

Then by the substitution $t = x+y$,

$$(x+y)\ln\sinh(x+y) \to 0 \qquad \text{for } x+y \to 0+.$$

Hence, the function can be extended continuously to the set

$$\overline{D} = \{(x,y) \mid x+y \geq 0\},$$

where we for $x+y=0$ put

$$\overline{f}(x,-x)=0, \qquad x\in\mathbb{R}.$$

3) The numerator $\operatorname{Arcsin}(xy-2)$ is defined and continuous, when $-1\le xy-2\le 1$, i.e. when $1\le xy\le 3$.

The denominator $\operatorname{Arctan}(3xy-6)$ is defined and continuous for every $(x,y)\in\mathbb{R}^2$.

The denominator is zero for $xy=2$, and we see that the numerator is zero on the same set.

Thus the domain is

$$D=\{(x,y)\mid 1\le xy<2 \text{ or } 2<xy\le 3\}.$$

We see that the domain has four connected components.

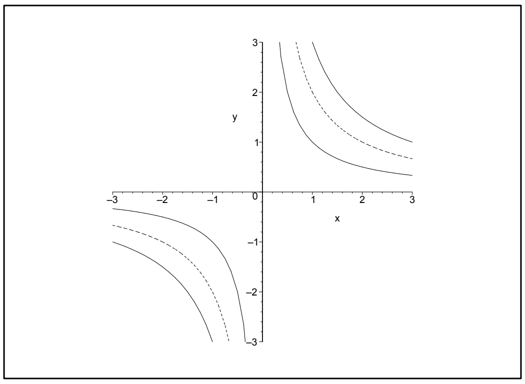



Figure 7.26: The domain of $f(x,y)=\dfrac{\operatorname{Arcsin}(xy-2)}{\operatorname{Arctan}(3xy-6)}$ is the union of the sets which lie between the hyperbolas in the first and third quadrant, with the exception of the dotted hyperbola in the "middle" of each set.

Since both the numerator and the denominator are zero on the exceptional hyperbola of the equation $xy=2$, there is a possibility of a continuous extension to this hyperbola. We shall now examine this possibility.

First note that

$$\frac{\operatorname{Arcsin} t}{\operatorname{Arctan} 3t}=\frac{1}{3}\cdot\frac{\operatorname{Arcsin} t}{t}\cdot\frac{3t}{\operatorname{Arctan} 3t}\to\frac{1}{3} \qquad \text{for } t\to 0.$$

Then by the substitution $t=xy-2$,

$$f(x,y)=\frac{\operatorname{Arcsin}(xy-2)}{\operatorname{Arctan}(3xy-6)}\to\frac{1}{3} \qquad \text{for } xy\to 2.$$

Hence, we can extend $f$ continuously to the set

$$B=\{(x,y)\mid\le xy\le 3\}$$

by putting

$$\overline{f}(x,y) = \begin{cases} \dfrac{\operatorname{Arcsin}(xy-2)}{\operatorname{Arctan}(3xy-6)} & \text{for } xy \in [1,3] \setminus \{2\}, \\ \dfrac{1}{3} & \text{for } xy = 2. \end{cases}$$

4) The function is defined and continuous for $y \neq x$, so the domain is given by

$$D = \{(x,y) \mid y \neq x\}.$$

Since

$$\lim_{t \to 0} \exp\left(-\frac{1}{t^2}\right) = 0,$$

it follows by the substitution $t = x - y$ that $f(x,y)$ can be extended to all of $\mathbb{R}^2$ by the continuous extension

$$\overline{f}(x,y) = \begin{cases} \exp\left(-\dfrac{1}{(x-y)^2}\right) & \text{for } y \neq x, \\ 0 & \text{for } y = x. \end{cases}$$

**Example 7.11** *Sketch in each of the cases below the domain of the given function or vector function. Then examine whether the (vector) function has a limit for $(x, y) \to (0, 0)$, and find this limit when it exists.*

*1)* $f(x, y) = \dfrac{\sin(xy)}{x}$,

*2)* $f(x, y) = \dfrac{1}{x} \sin y$,

*3)* $f(x, y) = x \sin \dfrac{1}{y}$,

*4)* $\mathbf{f}(x, y) = \left( \dfrac{\ln(1 + x^2 + y^2)}{\sqrt{x^2 + y^2}}, \dfrac{\ln x + \ln y}{\ln(xy)} \right)$,

*5)* $\mathbf{f}(x, y) = \left( \dfrac{x \sin y}{\sqrt{x^2 + y^2}}, \dfrac{x^2y^2 + x^2 + y^2}{x^2 + 3y^2} \right)$,

*6)* $f(x, y) = \left( \dfrac{x}{x + y}, \sqrt{x + y} \right)$.

**A** Domains; limits.

**D** Analyze the function; take the limit.

**I** 1) The function is defined for $x \neq 0$, i.e. everywhere except on the $Y$ axis,

$$D = \{(x, y) \mid x \neq 0\}.$$

There is of course no need to sketch the domain in this case.

By using polar coordinates we get from $x = \varrho \cos \varphi \neq 0$ in $D$ that $\varrho > 0$ and $\cos \varphi \neq 0$. This shows that in $D$,

$$|f(x, y)| = \left| \frac{\sin(\varrho^2 \cos \varphi \sin \varphi)}{\varrho \cos \varphi} \right| \leq \frac{\varrho^2 |\cos \varphi|\,|\sin \varphi|}{\varrho |\cos \varphi|} = \varrho |\sin \varrho|,$$

which tends to 0 for $\varrho \to 0+$, hence

$$\lim_{(x,y) \to (0,0)} f(x, y) = 0.$$

Alternatively one can use directly that

$$|f(x, y) - 0| = \left| \frac{\sin(xy)}{x} \right| \leq \frac{|xy|}{|x|} = |y| \to 0$$

for $|y| \leq \sqrt{x^2 + y^2} \to 0$.

2) The domain is the same as in 1).
The limit does not exist, because e.g.

$$f(x, x) = \frac{\sin x}{x} \to 1 \qquad \text{for } x \to 0,$$

$$f(x, -x) = -\frac{\sin x}{x} \to -1 \qquad \text{for } x \to 0.$$

3) The function is defined for $y \neq 0$, i.e. at the points outside the $X$ axis. There is no need either to sketch this set.

   The limit is 0, because

$$|f(x,y) - 0| = |x| \cdot \left|\sin \frac{1}{y}\right| \leq |x| \to 0 \qquad \text{for } (x,y) \to (0,0).$$

4) The vector function is defined (and continuous), when
   a) $1 + x^2 + y^2 > 0$ (always fulfilled),
   b) $x^2 + y^2 > 0$ (i.e. $(x,y) \neq (0,0)$),
   c) $x > 0$,
   d) $y > 0$,
   e) $xy > 0$,
   f) $xy \neq 1$.

   Summarizing we see that the domain is the open first quadrant, with the exception of a branch of a hyperbola,

$$D = \{(x,y) \mid x > 0,\ y > 0\} \setminus \{(x,y) \mid xy = 1\}.$$

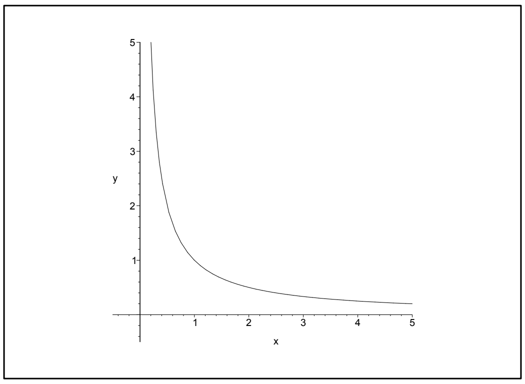



Figure 7.27: The vector function is defined in the first quadrant with the exception of the branch of the hyperbola.

Since

$$\begin{aligned}\frac{\ln(1+x^2+y^2)}{\sqrt{x^2+y^2}} &= \frac{1}{\sqrt{x^2+y^2}}\left\{(x^2+y^2) + (x^2+y^2)\varepsilon(x^2+y^2)\right\} \\ &= \sqrt{x^2+y^2}\{1+\varepsilon(x^2+y^2)\} \to 0\end{aligned}$$

for $(x,y) \to (0,0)$, and

$$\frac{\ln x + \ln y}{\ln(xy)} = \frac{\ln(xy)}{\ln(xy)} = 1 \qquad \text{for } (x,y) \in D,$$

we conclude that

$$\lim_{\substack{(x,y)\to(0,0)\\(x,y)\in D}} f(x,y) = (0,1).$$

5) The vector function is defined for $(x, y) \neq (0, 0)$.

Let us estimate the first coordinate function,

$$\left|\frac{x\sin t}{\sqrt{x^2+y^2}}\right| = \frac{|x|}{\sqrt{x^2+y^2}}\,|\sin y| \leq 1\cdot|\sin y| \to 0$$

for $(x, y) \to (0, 0)$. We see that the first coordinate function converges towards 0 by the limit.

In the examination of the second coordinate function we use polar coordinates $0 < \varphi < \dfrac{\pi}{2}$, $\varrho > 0$. We get by insertion

$$\frac{x^2y^2+x^2+y^2}{x^2+3y^2} = \frac{\varrho^4\cos^2\varphi\cdot\sin^2\varphi+\varrho^2}{\varrho^2(1+2\sin^2\varphi)} = \frac{1}{1+2\sin^2\varphi} + \varrho^2\cdot\frac{\sin^2\varphi\cos^2\varphi}{1+2\sin^2\varphi}.$$

The latter term converges towards 0 for $\varrho \to 0$; but the first term depends on $\varphi$ and not on $\varrho$.

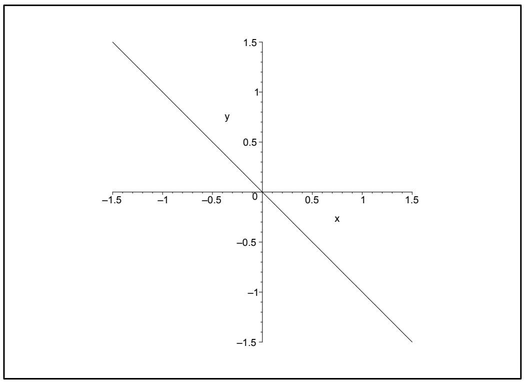



Figure 7.28: **Example 7.11.5**. The domain is the half plane which lies above the line.

Since the second coordinate function cannot be extended continuously to $(0,0)$, neither can the vector function itself be extended continuously to $(0,0)$.

6) The vector function

$$\mathbf{f}(x,y) = \left( \frac{x}{x+y}, \sqrt{x+y} \right)$$

is defined for $x+y \neq 0$ and $x+y \geq 0$, so the domain is

$$\{(x,y) \mid x+y > 0\}.$$

The first coordinate function does not have a limit for $(x,y) \to (0,0)$ in the domain. In fact if we in particular restrict ourselves to the positive $X$ axis where $y=0$, then

$$\lim_{x\to 0+} f_1(x,0) = \lim_{x\to 0+} \frac{x}{x+0} = 1.$$

If we instead restrict ourselves to the positive $Y$ axis we get

$$\lim_{y\to 0+} f_1(0,y) = \lim_{y\to 0+} \frac{0}{0+y} = 0.$$

Since $1 \neq 0$, the limit does not exist.

**Example 7.12** *Let* $f : \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R}$ *be given by*

$$f(x,y) = \frac{x^2y^2}{x^2y^2 + (x-y)^2}.$$

*Show that*

$$\lim_{x\to 0}\left(\lim_{y\to 0} f(x,y)\right) = \lim_{y\to 0}\left(\lim_{x\to 0} f(x,y)\right) = 0,$$

*and that* $f$ *nevertheless* does not *have a limit for* $(x,y) \to (0,0)$.

**A** Limits.

**D** Calculate the successive limits and finally the limit along the line $y = x$.

**I** If $x \neq 0$, then

$$x^2y^2 \to 0 \text{ and } x^2y^2 + (x-y)^2 \to x^2 \neq 0 \qquad \text{for } y \to 0,$$

hence

$$\lim_{y\to 0} f(x,y) = 0 \qquad \text{for } x \neq 0.$$

Note also that

$$\lim_{y\to 0} f(0,y) = \lim_{y\to 0} \frac{0}{y^2} = 0.$$

Since $f(x,y) = f(y,x)$, it follows immediately that

$$\lim_{x\to 0}\left(\lim_{y\to 0} f(x,y)\right) = \lim_{y\to 0}\left(\lim_{x\to 0} f(x,y)\right) = 0.$$

Then consider the limit $(x,y) \to (0,0)$ along the line $y = x$. This is given by

$$\lim_{x\to 0} f(x,x) = \lim_{x\to 0} \frac{x^4}{x^4 + 0^2} = 1 \neq 0.$$

We conclude that $f$ does not have a limit for $(x,y) \to (0,0)$.

**Example 7.13** *Let* $f : \mathbb{R}^2 \to \mathbb{R}$ *be given by*

$$f(x,y) = \begin{cases} \sin \dfrac{1}{x} \sin y, & x \neq 0, \\ 0, & x = 0. \end{cases}$$

*Prove that* $f(x,y) \to 0$ *for* $(x,y) \to (0,0)$*; and that we nevertheless do not have*

$$\lim_{x\to 0}\left(\lim_{y\to 0} f(x,y)\right) = \lim_{y\to 0}\left(\lim_{x\to 0} f(x,y)\right).$$

**A** Limits.

**D** Use the definition of a limit in 1), and the rules of calculations in 2).

**I** If $x \neq 0$, then

$$|f(x,y) - f(0,0)| = \left|\sin \frac{1}{x}\right| \cdot |\sin y| \leq |\sin y| \to 0 \quad \text{for } (x,y) \to (0,0),$$

and it follows trivially for $x = 0$ that

$$|f(0,y) - f(0,0)| = 0 \to 0 \qquad \text{for } (x,y) \to (0,0).$$

We conclude that

$$\lim_{(x,y)\to(0,0)} f(x,y) = 0.$$

Then it follows immediately that

$$\lim_{y\to 0} f(x,y) = \begin{cases} \lim_{y\to 0} \sin \dfrac{1}{x} \cdot \sin y = 0, & \text{for } x \neq 0, \\ \lim_{y\to 0} 0 = 0, & \text{for } x = 0, \end{cases}$$

thus

$$\lim_{x\to 0}\left(\lim_{y\to 0} f(x,y)\right) = 0.$$

On the other hand, $\sin \dfrac{1}{x} \cdot \sin y$ for $y \neq p\pi$, $p \in \mathbb{Z}$, does not have a limit for $x \to 0$, so

$$\lim_{y\to 0}\left(\lim_{x\to 0} f(x,y)\right)$$

is not defined.

**Example 7.14** *Find the domain $A$ of*

$$f(x,y) = \frac{xy}{x+y}.$$

*Show that $f$ cannot be continuously extended to a point set $B \supset A$.*
*Then let*

$$D = \{(x,y) \mid 0 \le x,\, 0 \le y,\, x^2 + y^2 > 0\},$$

*and consider the function $g : D \to \mathbb{R}$ given by*

$$g(x,y) = \frac{xy}{x+y}.$$

*Sketch $D$, and prove that $g$ has a continuous extension to the point set $D \cup \{(0,0)\}$. Compare with the formula of thenresulting resistance of a connection in parallel of two resistances.*

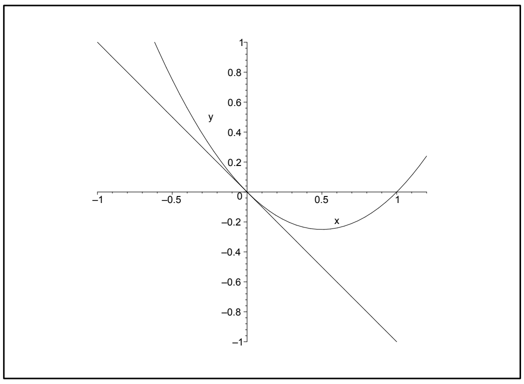
**A** Domain; continuous extension; limit.

**D** Find the point set, in which the denominator is 0, and then indicate $A$. Examine the limit in $D$.

**I** Clearly,

$$A = \{(x,y) \mid y \ne -x\}.$$

Furthermore, $(0,0)$ is the only point in which both the numerator and the denominator are zero, so there is only a possibility of a continuous extension to the set $A \cup \{(0,0)\}$.




When we restrict ourselves to the curve $y = -x + x^2$, $x \neq 0$, we get

$$\lim_{x \to 0} f(x, -x + x^2) = \lim_{x \to 0} \frac{-x^2 + x^3}{x^2} = -1.$$

On the other hand, it is obvious that $f(x,0) = 0 \to 0$ for $x \to 0$, so we get two different limits by approaching $(0,0)$ along two different curves. Hence, the limit does not exist, and $f$ cannot be extended continuously.

The set $D$ is the closed first quadrant with the exception of the point $(0,0)$. Since $x \geq 0$ and $y \geq 0$ in $D$, we have the estimate

$$0 < \max\{x, y\} \leq x + y \qquad \text{for every } (x,y) \in D,$$

and hence

$$|g(x,y) - 0| = \left| \frac{x}{x+y} \right| \cdot |y| \leq |y| \to 0 \qquad \text{for } (x,y) \to (0,0) \text{ in } D.$$

This shows that $g$ can be extended continuously to $(0,0)$, when we define $g(0,0) = 0$.

By the rearrangement

$$\frac{1}{g(x,y)} = \frac{x+y}{xy} = \frac{1}{x} + \frac{1}{y}, \qquad x > 0, \quad y > 0,$$

we get the connection to the formula of the resulting resistance for a connection in parallel. From the above follows that

$$g(x,y) = \frac{xy}{x+y}$$

in $D^\circ$ can be extended to $D^\circ \cup \{(0,0)\}$.

## 7.4 Description of curves

**Example 7.15** *In the following there are given some curves. In each case one shall find an equation of the curve by eliminating the parameter $t$. Indicate the name of the curve.*

*1)* $\mathbf{r}(t) = \left(a\,\dfrac{1-t^2}{1+t^2}, b\,\dfrac{2t}{1+t^2}\right)$, *for* $t \in \mathbb{R}$.

*2)* $\mathbf{r}(t) = \left(\dfrac{a}{\cos t}, b\,\tan t\right)$, *for* $t \in \left]-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right[ \cup \left]\dfrac{\pi}{2}, \dfrac{3\pi}{2}\right[$.

*3)* $\mathbf{r}(t) = \left(at^2, 2at\right)$, *for* $t \in \mathbb{R}$.

*4)* $\mathbf{r}(t) = (a\sin t, a\cos 2t)$, *for* $t \in [-\pi, \pi]$.

**A** Description of curves.

**D** Eliminate the parameter.

**I** 1) It follows from $x = a\,\dfrac{1-t^2}{1+t^2}$ and $y = b\,\dfrac{2t}{1+t^2}$ that

$$\frac{x}{a} = \frac{1-t^2}{1+t^2} \qquad \text{and} \qquad \frac{y}{b} = \frac{2t}{1+t^2},$$

where the idea is that the two right hand sides are independent of the arbitrary constants $a$ and $b$.
We get by squaring and adding

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = \left(\frac{1-t^2}{1+t^2}\right)^2 + \left(\frac{2t}{1+t^2}\right)^2 = \frac{(1-2t^2+t^4)+4t^2}{(1+t^2)^2} = \frac{1+2t^2+t^t}{1+2t^2+t^4} = 1.$$

Thus the curve is a subset of an ellipse of centre $(0,0)$ and the half axes $a$ and $b$.

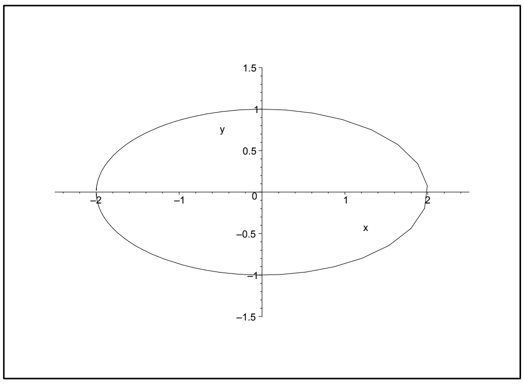



Figure 7.29: the curve for $a = 2$ and $b = 1$.

Then

$$\frac{1-t^2}{1+t^2} = -1 + \frac{2}{1+t^2} \leq 1$$

with equality for $t = 0$, so $\dfrac{1-t^2}{1+t^2}$ runs through the interval $]-1,1]$ (twice), when $t$ runs through $\mathbb{R}$. Since $\dfrac{2t}{1+t^2}$ changes its sign for $t = 0$, we conclude that the arc of the curve is the ellipse with the exception of the point $(-a, 0)$.

2) It follows from $x = \dfrac{a}{\cos t}$ and $y = b \tan t$ that

$$\frac{x}{a} = \frac{1}{\cos t} \qquad \text{and} \qquad \frac{y}{b} = \frac{\sin t}{\cos t},$$

so the parameter $t$ is eliminated by

$$\left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2 = \frac{1-\sin^2 t}{\cos^2 t} = 1.$$

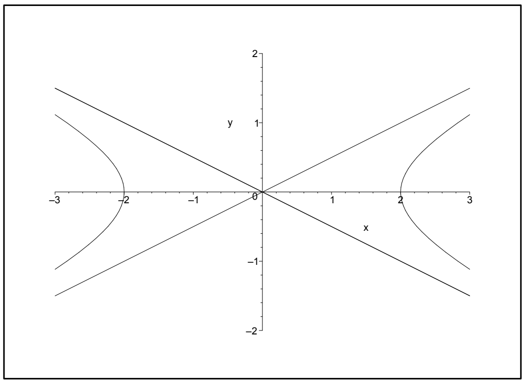



Figure 7.30: The curves for $a = 2$ and $b = 1$.

This describes an hyperbola of the half axes $a$ and $b$ and of centre $(0, 0)$. The two intervals $\left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$ and $\left]\frac{\pi}{2}, \frac{3\pi}{2}\right[$ corresponds to the two branches.

3) Here, $x = at^2$ and $y = 2at$, so $t = \dfrac{y}{2a}$. Then by insertion,

$$x = at^2 = \frac{1}{4a}\, y^2,$$

which is the equation of a parabola with top point $(0, 0)$ and the $X$ axis as its axis.

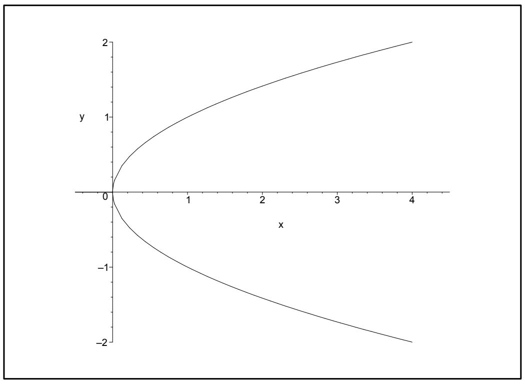



Figure 7.31: The curve for $a = \dfrac{1}{4}$.

4) When

$$(x, y) = \mathbf{r}(t) = (a \sin t, a \cos 2t), \qquad t \in [-\pi, \pi],$$

and $a > 0$, it follows that

$$y = a \cos 2t = a\left(1 - 2\sin^2 t\right) = 1 - \frac{2}{a}(a \sin t)^2 = a - \frac{2}{a}\, x^2,$$

i.e.

$$y = a - \frac{2}{a}\,x^2, \qquad x \in [-\pi, \pi],$$

which is a part of a parabolic arc. Note that we use that

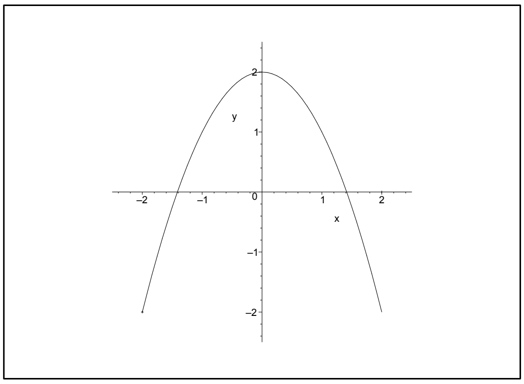
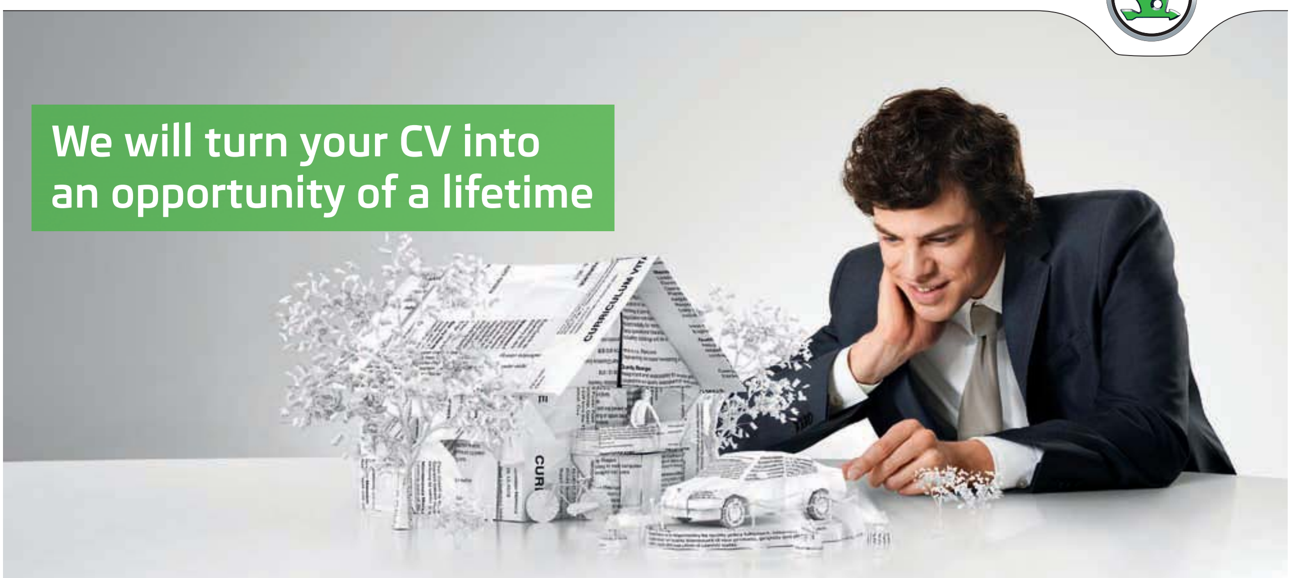



Figure 7.32: The curve for $a = 2$.

$$|x| = |a \sin t| \le a,$$

when we find the domain $[-a, a]$, where we can have both $x = -a$ and $x = a$.





**Example 7.16** *Prove that the curve given by*

$$\mathbf{r}(t) = \left(3r + t^2, t - t^2, 3 - 5t + t^2\right), \qquad t \in \mathbb{R},$$

*lies in a plane, and find an equation of this plane.*

**A** Space curve lying in a plane.

**D** Put the coordinate functions of the curve into the general equation of a plane and find the coefficients.

**I** In general the equation of a plane is given by

$$ax + by + cz = k.$$

Then by insertion of

$$(x, y, z) = (3t + t^2, t - t^2, 3 - 5t + t^2),$$

we get

$$\begin{aligned} k &= a(3t + t^2) + b(t - t^2) + c(3 - 5t + t^2) \\ &= t^2(a - b + c) + t(3a + b - 5c) + 3c, \qquad t \in \mathbb{R}. \end{aligned}$$

This should hold for every $t$, so we must necessarily have

$$\left\{ \begin{aligned} a - b + c &= 0, \\ 3a + b - 5c &= 0, \\ 3c &= k. \end{aligned} \right.$$

It follows that if $k = 0$, then we only get $(a, b, c) = (0, 0, 0)$ as a solution.

By choosing $k \neq 0$, e.g. $k = 3$, we get $c = 1$, and then by insertion

$$\left\{ \begin{aligned} a - b &= -c = -1, \\ 3a + b &= 5c = 5. \end{aligned} \right.$$

An addition shows that $4a = 4$, i.e. $a = 1$, and it follows that $b = 2$.

Hence an equation of the plane is

$$x + 2y + z = 3,$$

and we have at the same time proved that the curve lies in this plane.

**Example 7.17** *Prove that the curve given by*

$$\mathbf{r}(t) = \left(2t\sqrt{1-t}, 2(1-t)\sqrt{t}, 1-2t\right), \qquad t \in [0,1],$$

*lies on a sphere of centre* $(0,0,0)$.

**A** A space curve lying on a sphere.

**D** Put the coordinate functions into the equation of the sphere and find its radius $r$.

**I** The general equation of a sphere of centrum $(0,0,0)$ is

$$x^2 + y^2 + z^2 = r^2.$$

By putting

$$x = 2r\sqrt{1-t}, \qquad y = 2(1-t)\sqrt{t}, \qquad z = 1-2t,$$

we get

$$\begin{aligned} x^2+y^2+z^2 &= 4t^2(1-t) + 4(1-t)^2t + (1-2t)^2 \\ &= 4t(1-t)\{t+(1-t)\} + (1-2t)^2 \\ &= (4t-4t^2) + (1-4t+4t^2) = 1, \end{aligned}$$

and we conclude that the curve lies on the unit sphere.

**Example 7.18** *Prove that the curve given by*

$$\mathbf{r}(t) = \left(a(1-\sin t)\cos t, b(\sin t + \cos^2 t), c\cos t\right), \qquad t \in [-\pi, \pi],$$

*lies on an hyperboloid.*

**A** A space curve lying on an hyperboloid.

**D** Calculate $\left(\frac{x}{a}\right)^2$, $\left(\frac{y}{b}\right)^2$ and $\left(\frac{z}{c}\right)^2$, which are three expressions which are independent of the constants $a$, $b$ and $c$. Then compare.

**I** We calculate

$$\begin{aligned} \left(\frac{x}{a}\right)^2 &= (1-\sin t)^2\cos^2 t &&= (1-2\sin t+\sin^2 t)\cos^2 t \\ & &&= \cos^2 t - 2\sin t\cos^2 t + \sin^2 t\cos^2 t, \\ \left(\frac{y}{b}\right)^2 &= (\sin t+\cos^2 t)^2 &&= \sin^2 t + 2\sin t\cos^2 t + \cos^4 t \\ \left(\frac{z}{c}\right)^2 &= &&= \cos^2 t. \end{aligned}$$

Hence

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1 + \cos^2 t = 1 + \left(\frac{z}{c}\right)^2,$$

and by a rearrangement

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^1 - \left(\frac{z}{c}\right)^2,$$

and we conclude that the curve lies on an hyperboloid with one sheet.

**Example 7.19** *Sketch the so-called* cycloid *given by*

$$\mathbf{r}(t) = (a(t - \sin t), a(1 - cost)), \qquad t \in \mathbb{R}.$$

**A** Sketch of a curve.

**D** If one does not have MAPLE at hand, start by finding some points of the curve. One may exploit the geometrical meaning of

$$\mathbf{r}(t) = a(t, 1) - a(\sin t, \cos t), \qquad t \in \mathbb{R},$$

where the former term on the right hand side is a rectilinear and even motion, while the latter term is a circular motion. Thus the curve describes the motion of a point on a wheel, which is rolling along the $X$ axis.

**I** Clearly, $\mathbf{r}(t)$ is periodical of period $2\pi$, so it suffices to sketch one period and a little bit of the neighbouring periods.

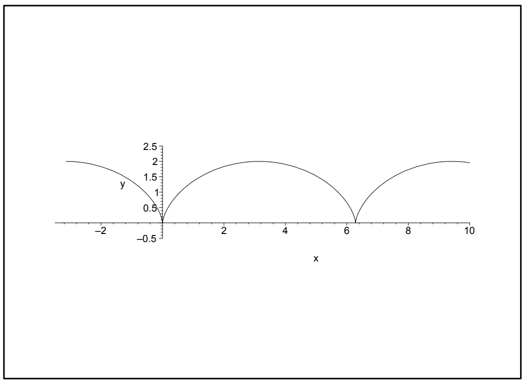



Figure 7.33: The cycloid for $a = 1$.

**Example 7.20** *Find in each of the cases below an equation of the given curve by eliminating the parameter $t$, and then sketch the curve.*

*1)* $\mathbf{r}(t) = \left( \dfrac{3t}{1+t^3}, \dfrac{3t^2}{1+t^3} \right)$, *for* $t \in \mathbb{R} \setminus \{-1\}$.

*2)* $\mathbf{r}(t) = (\cos t, \sin t \cos t)$, *for* $t \in \mathbb{R}$.

*3)* $\mathbf{r}(t) = \left(a \cos^3 t, a \sin^3 t\right)$, *for* $t \in [-\pi, \pi]$.

*4)* $\mathbf{r}(t) = \left(a(1-3t^2), at(3-t^2)\right)$, *for* $t \in \mathbb{R}$.

**A** Description of curves.

**D** Eliminate the parameter.

**I** 1) When $t \neq -1$, we get

$$x = \frac{3t}{1+t^3} \qquad \text{and} \qquad y = \frac{3t^2}{1+t^3}.$$

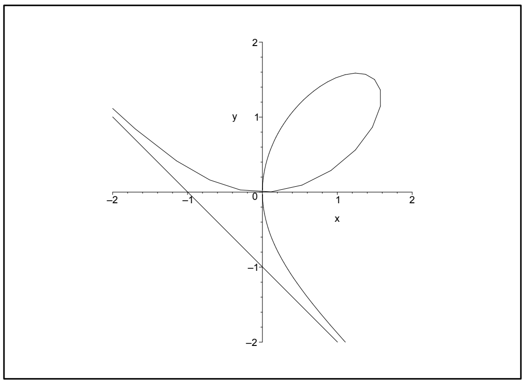
For $t = 0$ we get the point $(x, y) = (0, 0)$.
For $t \neq 0$ and $t \neq -1$ we get $t = \dfrac{y}{x}$, where $x \neq 0$ and $y \neq 0$, so by insertion

$$x = \frac{3t}{1+t^3} = \frac{3y/x}{1+(y/x)^3} = \frac{3x^2 y}{x^3+y^3}.$$

When $x \neq 0$ this is reduced to

$$x^3 + y^3 = 3xy.$$

Finally, we see that $(x, y) = (0, 0)$, which corresponds to $t = 0$, also satisfies this equation, so we can remove the restriction.
Note that the line $y = x$ is an axis of symmetry.

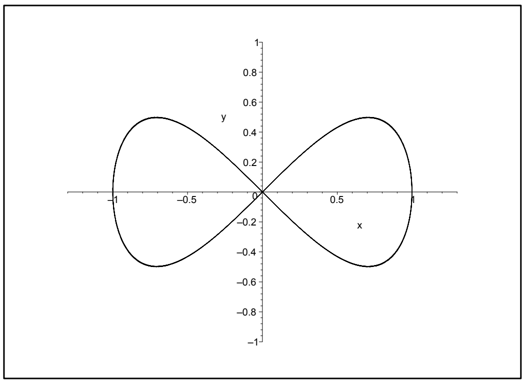



2) Here, $x = \cos t$ and $y = \sin t \cos t$, hence

$$y^2 = \sin^2 t \cos^2 = (1 - \cos^2 t) \cos^2 = (1 - x)x^2,$$

or written more conveniently,

$$y^2 = (1 - x^2)x^2, \qquad \text{hence } y = \pm |x| \sqrt{1 - x^2}, \quad x \in [-1, 1].$$




3) From

$$x = a \cos^3 t, \qquad y = a \sin^3 t,$$

we get by elimination

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}} \left\{ \cos^2 t + \sin^2 t \right\} = a^{\frac{2}{3}}.$$

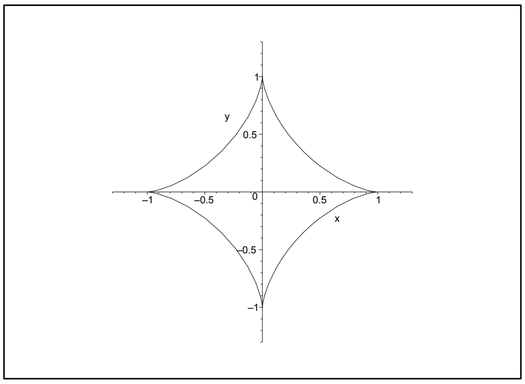



Figure 7.34: The curve for $a = 1$.

Note that

$$\mathbf{r}'(t) = 3a\,\sin t \cdot \cos(-\cos t, \sin t)$$

is $\mathbf{0}$ for $t = p \cdot \dfrac{-\pi}{2}$, $p = -2, -1, 0, 1, 2$, corresponding to the cusps on the curve.

4) First note that $\mathbf{r}'(t) = a(-6t, 3 - 3t^2)$, so $y'(t) = 0$ for $t = \pm 1$. It follows from

$$x(t) = a(1 - 3t^2) \qquad \text{and} \qquad y(t) = at(3 - t^2)$$

that $x(t)$ is largest for $t = 0$, corresponding to $x(t) \le x(0) = a$. For this value the point on the curve is $\mathbf{r}(0) = (a, 0)$.

Furthermore, we see that the $X$ axis is an axis of symmetry.

Note

a) that $x(t) = 0$ for $t = \pm \dfrac{1}{\sqrt{3}}$, corresponding to

$$(x, y) = \left(0, \pm \frac{8a}{3\sqrt{3}}\right),$$

b) that the curve has a horizontal tangent for $y'(t) = 0$, i.e. for $t = \pm 1$, corresponding to

$$(x, y) = (-2a, \pm 2a),$$

c) and that $y(t) = 0$ for $t = 0$ and $t = \pm\sqrt{3}$, corresponding to

$$(0, 0) \qquad \text{and} \qquad (-8a, 0).$$

d) that $y$ and $t$ have the same sign for $0 < |t| < \sqrt{3}$, and opposite sign for $|t| > \sqrt{3}$. The latter means that we are allowed to square by the elimination of $t$.

It follows from

$$\frac{x}{a} = 1 - 3t^2 \qquad \text{and} \qquad \frac{y}{a} = t(3 - t^2)$$

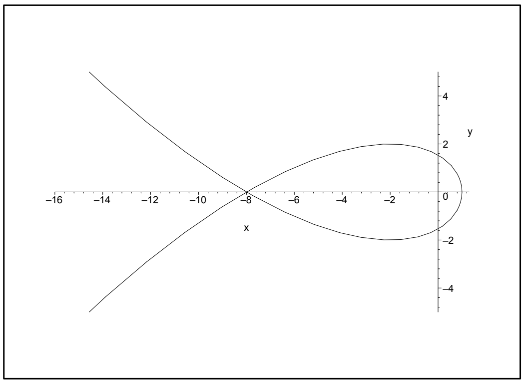



Figure 7.35: The curve for $a = 1$.

that

$$t^2 = \frac{1}{3}\left\{1 - \frac{x}{a}\right\},$$

so we finally get by a squaring,

$$\frac{y^2}{a^2} = t^2\left(3 - t^2\right)^2 = \frac{1}{3}\left\{1 - \frac{x}{a}\right\}\left(3 - \frac{1}{3}\left\{1 - \frac{x}{a}\right\}\right)^2 = \frac{1}{27}\left(1 - \frac{x}{a}\right)\left(8 + \frac{x}{a}\right)^2,$$

thus

$$y^2 = \frac{1}{27a}(a - x)(8a + x)^2.$$

Note that $|y|$ tends faster towards $+\infty$ than $|x|$ for $|t| \to +\infty$.

**Example 7.21** *Sketch the point set $A$ in the first quadrant of the plane, which is bounded by the three curves given by*

$$\mathbf{r}(t) = (\cos t, 1 + \sin t), \quad t \in \left[0, \frac{\pi}{2}\right],$$

$$\mathbf{r}(t) = (1 + \cos t, sint), \quad t \in \left[0, \frac{\pi}{2}\right],$$

$$\mathbf{r}(t) = (2\cos t, 2\sin t), \quad t \in \left[0, \frac{\pi}{2}\right].$$

**A** A set bounded by given curves.

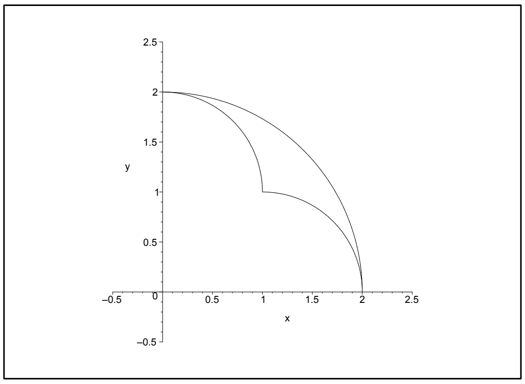
**D** Identify the curves and sketch the set.

**I** All three curves are quarter circles, which follows from

$$\mathbf{r}_1(t) = (0, 1) + (\cos t, \sin t), \quad t \in \left[0, \frac{\pi}{2}\right],$$

$$\mathbf{r}_2(t) = (1, 0) + (\cos t, \sin t), \quad t \in \left[0, \frac{\pi}{2}\right],$$

$$\mathbf{r}_3(t) = (0, 0) + 2(\cos t, \sin t), \quad t \in \left[0, \frac{\pi}{2}\right].$$


2.5
2
1.5
y
1
0.5
–0.5
0
0.5
1
1.5
2
2.5
x
–0.5

**Example 7.22** *Let $\alpha$ be a non-negative constant, and let the curve $\mathcal{K}$ be given by the equation*

$$\varrho = \frac{c}{1 + a\,\cos\varphi}, \qquad \varphi \in I,$$

*where $I$ is a symmetric interval around the point 0, which is as big as possible. Prove that $\mathcal{K}$ is (a part of) a conical section.*

**A** Conical section in polar coordinates.

**D** Multiply by the denominator and reduce to rectangular coordinates, where $c$ as usual denotes some positive constant.

**I** Since $\varrho \geq 0$ and $c > 0$, we must have $1 + \alpha \cos\varphi > 0$. Therefore, in order to find $I$ we must find possible zeros of the denominator, i.e. we shall examine the equation

$$1 + \alpha\cos\varphi = 0, \qquad \text{i.e. } \cos\varphi = -\frac{1}{\alpha}.$$

Since $\alpha \geq 0$, we have to distinguish between the cases

$$\alpha = 0, \qquad 0 < \alpha < 1, \qquad \alpha = 1, \qquad \alpha > 1.$$

1) If $\alpha = 0$, then $\varrho = c$, which is the polar equation of a circle of radius $c > 0$. The circle is clearly a conical section, and $I = \mathbb{R}$.
2) If $0 < \alpha < 1$, then $1 + \alpha\cos\varphi \geq 1 - \alpha > 0$ for every $\varphi$, and the denominator is always positive, and we get $I = \mathbb{R}$. When we multiply by the denominator we get by using rectangular coordinates,

$$c = \varrho + \alpha\varrho\cos\varphi = \alpha x,$$

hence by a rearrangement,

$$\sqrt{x^2 + y^2} = c - \alpha x \geq 0.$$

We get in particular the condition $x < \dfrac{c}{\alpha}$, which should be checked at the very end of this example.
When this restriction is satisfied we can square, obtaining

$$x^2 + y^2 = c^2 - 2\alpha c x + \alpha^2 x^2.$$

Then by a rearrangement,

$$(1 - \alpha^2)x^2 + 2\alpha c x + y^2 = c^2, \qquad 0 < \alpha < 1,$$

i.e.

$$(1-\alpha^2)\left\{x^2 + \frac{2\alpha c}{1-\alpha^2}\,x + \left(\frac{\alpha c}{1-\alpha^2}\right)^2\right\} + y^2 = c^2 + \frac{\alpha^2 c^2}{1-\alpha^2} = \frac{c^2}{1-\alpha^2}.$$

This can be written in the following canonical way

$$\left\{\frac{x + \dfrac{\alpha c}{1-\alpha^2}}{\dfrac{c}{1-\alpha^2}}\right\}^2 + \left\{\frac{y}{\dfrac{c}{\sqrt{1-\alpha^2}}}\right\}^2 = 1.$$

This is the equation of an ellipse, hence a conical section of

$$\text{centre: } \left(-\frac{\alpha c}{1-\alpha^2},0\right) \qquad \text{and half axes: } \frac{c}{1-\alpha^2} \quad \text{and} \quad \frac{c}{\sqrt{1-\alpha^2}}.$$

Note that

$$-\frac{\alpha c}{1-\alpha^2}+\frac{c}{1-\alpha^2}=\frac{c}{1+\alpha}<\frac{c}{\alpha} \quad \text{for } 0<\alpha<1,$$

and we conclude that the earlier restriction for the squaring is automatically fulfilled.

3) If $\alpha = 1$, the denominator is $1+\cos\varphi = 0$ for $\varphi =$ an odd multiple of $\pi$, and $>0$ otherwise. The searched for interval is $I = ]-\pi,\pi[$.

When we multiply with the denominator we get

$$c = \varrho+\varrho\cos\varphi = \sqrt{x^2+y^2}+x,$$

hence

$$\sqrt{x^2+y^2}=c-x\ge 0, \qquad \text{i.e. } x \le c.$$

Assuming this we get by squaring,

$$x^2+y^2=c^2-2cx+x^2,$$

so after some reduction we obtain the equation of the parabola

$$x=-\frac{1}{2c}\,y^2+\frac{c}{2}.$$

Clearly, this expression is $\le c$, so $\mathcal{K}$ is the whole of the parabola, and a parabola is also a conical section.

4) If $\alpha > 1$, then $1+\alpha\cos\varphi = 0$ for

$$\cos\varphi = -\frac{1}{\alpha} \in ]-1,0[,$$

i.e. the largest possible symmetric domain interval $I$ is

$$I = \left]-\text{Arccos}\left(-\frac{1}{\alpha}\right),\text{Arccos}\left(-\frac{1}{\alpha}\right)\right[.$$

In this interval we get as in 2) that

$$\sqrt{x^2+y^2}=c-\alpha x\ge 0, \qquad \text{i.e. } x\le\frac{c}{\alpha},$$

and the calculations are then continued in the usual way under this assumption by a squaring,

$$x^2+y^2=c^2+\alpha^2x^2-2\alpha cx \qquad \text{for } x\le\frac{c}{\alpha}.$$

Then by a rearrangement,

$$(1-\alpha^2)\left\{x^2+\frac{2\alpha c}{1-\alpha^2}\,x+\left(\frac{\alpha c}{1-\alpha^2}\right)^2\right\}+y^2=\frac{c^2}{1-\alpha^2}<0,$$

hence by norming

$$\left\{\frac{x-\dfrac{\alpha c}{\alpha^2-1}}{\dfrac{c}{\alpha^2-1}}\right\}^2-\left\{\frac{y}{\dfrac{c}{\sqrt{\alpha^2-1}}}\right\}^2=1.$$

Thus, for $\varphi \in I$ we get an arc of an hyperbola, which again is a conical section.

## 7.5 Connected sets

**Example 7.23** *Examine if the point set*

$$A=\{(x,y)\mid (x^2+y^2+2x)(y^2-x)<0\}$$

*is connected.*

**A** Connected set.

**D** First find the boundary curves of $A$. Sketch a figure.

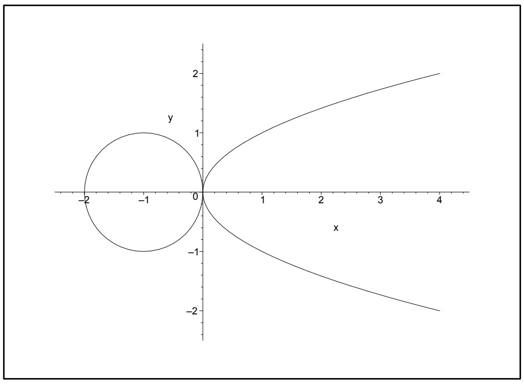



Figure 7.36: The set $A$ consists of the points which either lies inside the circle or inside the parabola.

**I** Since $(x^2+y^2+2x)(y^2-x)$ is continuous in $\mathbb{R}^2$, the boundary $\partial A$ is given by

$$0=(x^2+y^2+2x)(y^2-x)=\{(x+1)^2+y^2-1\}(y^2-x),$$

i.e. the boundary is composed of the circle of equation

$$(x+1)^2+y^2=1$$

of centre $(-1,0)$ and radius 1, and the parabola of equation $x=y^2$. The plane $\mathbb{R}^2$ is in this way divided into three subregions in which $f(x,y)$ due to the continuity must have a fixed sign in each of these.

The set $A$ is characterized by the condition $f(x,y)<0$.

Inserting the centre $(-1,0)$ of the circle we get

$$f(-1,0)=-1\cdot 1=-1<0,$$

so by the continuity it follows that the open disc is contained in $A$.

The point $(1,0)$ lies inside the parabola, and the value is

$$f(1,0) = 3 \cdot (-1) = -3 < 0,$$

so the interior of the parabola is also a subset of $A$.

This is sufficient to declare that the set is not connected, because it is impossible to connect $(-1,0) \in A$ with $(1,0) \in A$ by any continuous curve without intersecting at least one of the zero curves, which do *not* lie in $A$. We therefore conclude that $A$ is not connected.

REMARK. Since $(0,1)$ is a point in the latter component, and

$$f(0,1) = 1 \cdot 1 = 1 > 0,$$

the third component of $\mathbb{R}^2$ does not contain any point from $A$, and $A$ consists of precisely the union of the open disc and the open interior of the parabola. However, one was never asked this question. $\Diamond$

**Example 7.24** *Give an example of a point set which fulfils the following condition: $A$ is not connected, but its closure $\overline{A}$ is connected.*

**A** Connected sets.

**D** Analyze the concept of connected sets and give examples.

**I** According to **Example 7.27** below an extreme example is $A = \mathbb{Q}$, which is not connected in $\mathbb{R}$, while $\overline{A} = \mathbb{R}$ is connected.

A simpler example is $A = \mathbb{R} \setminus \{0\}$ where $\overline{A} = \mathbb{R}$.

Another example is given by **Example 7.23**, because one by the closure also include the point $(0, 0)$, which can be reached by a continuous curve from both components.

**Example 7.25** *Show by an example that two connected point sets $A$ and $B$ do not necessarily have a connected intersection.*

**A** Connected sets.

**D** Sketch an "amoebe" in the plane.

**I** Sketch two "half moons" which only intersect in their tips, we see that the intersection has got two components, and the intersection is not connected. Clearly, each "half moon" is connected.

The sketches are left to the reader.

**Example 7.26** *Examine if the domain of the function*

$$f(x, y) = \text{Arcsin}(x^2 + y^2 - 3)$$

*is simply connected.*

**A** Simply connected sets.

**D** Find the domain and analyze.

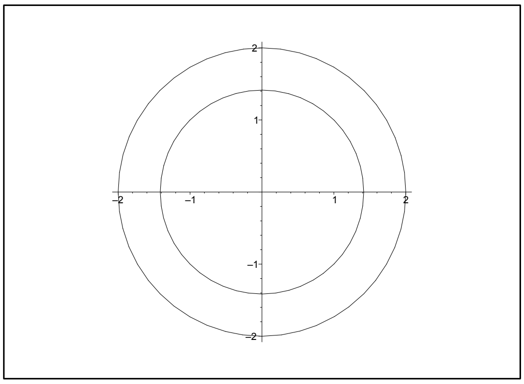
**I** The function $f(x, y)$ is defined for

$$-1 \leq x^2 + y^2 - 3 \leq 1,$$

i.e. for

$$2 \leq x^2 + y^2 \leq 4.$$

This set is an annulus (a set containing a "hole") of inner radius $\sqrt{2}$ and outer radius 2.
This set is clearly not simply connected.



**Example 7.27** *Prove that the set of rational numbers is not connected. Formulate a similar result for a set in the plane.*

**A** Connected sets.

**D** Analyze the definition of connected sets.

**I** Let $x$, $y \in \mathbb{Q}$, where e.g. $x < y$. Every continuous curve in $\mathbb{R}$, which connects $x$ and $y$, will then contain the interval $[x, y]$, which also contains irrational numbers, i.e. points outside $\mathbb{Q}$. We conclude that $\mathbb{Q}$ is not connected.

The set $\{(x, y) \mid x \in \mathbb{Q},\, y \in \mathbb{Q}\}$ is not connected in the plane.

**Example 7.28** *Check in each of the cases below if the domain of the given function is connected.*

*1)* $f(x, y, z) = \ln|1 - x^2 - y^2 - z^2|$.

*2)* $f(x, y, z) = \ln(1 - x^2 - y^2 - z^2)$.

*3)* $f(x, y, z) = \sqrt{y^2 - x^2} + \sqrt{z^2 - 1}$.

*4)* $f(x, y, z) = \sqrt{y - x} + \sqrt{z - 1}$.

*5)* $f(x, y, z) = \ln(1 - y^2) + \sqrt{x^2 - 4} + \sqrt{9 - x^2}$.

**A** Connected domains.

**D** First find the domain. Then analyze.

1) The function is defined for $x^2 + y^2 + z^2 \neq 1$, i.e. everywhere with the exception of the unit sphere. The set can obviously be divided into two connected components, so it is not connected.

2) In this case the domain is the open unit ball, which is connected.

3) It suffices to realize that the domain has one part lying in the half space $z \geq 1$ and another part in the half space $z \leq -1$ and no point in between. Hence the set is not connected.

4) The domain is given by $y \geq x$ and $z \geq 1$, i.e. the union of two *half spaces* (convex sets) and thus connected.

5) The function is independent of $z$, and defined for

$$1 - y^2 > 0, \qquad x^2 - 4 \geq 0, \qquad 9 - x^2 \geq 0,$$

so the domain is

$$[-3,-2] \times \,]-1,1[\, \times \mathbb{R} \cup [2,3] \times \,]-1,1[\, \times \mathbb{R}.$$

This set contains two connected components, hence it is not itself connected.

## 7.6 Description of surfaces

**Example 7.29** *In the following there are given some surfaces in the form* $\mathbf{x} = \mathbf{r}(u,v)$, $(u,v) \in \mathbb{R}^2$. *Find in each of these cases an equation of the surface by eliminating the parameters* $(u,v)$, *and then describe the type of the surface.*

*1)* $\mathbf{r}(u,v) = (u, u+2v, v-u)$.

*2)* $\mathbf{r}(u,v) = (u, \sin v, 3\cos v)$.

*3)* $\mathbf{r}(u,v) = (u\,\cos v, u\,\sin v, u^2 \sin 2v)$.

*4)* $\mathbf{r}(u,v) = (a(\cos v - u\,\sin v), b(\sin v + u\,\cos v), cu)$.

*5)* $\mathbf{r}(u,v) = (u\,\cos v, 2u\,\sin v, u^2)$.

*6)* $\mathbf{r}(u,v) = (u+v, u-v, 4v^2)$.

*7)* $\mathbf{r}(u,v) = (u+v, u^2+v^2, u^3+v^3)$.

**A** Description of surfaces.

**D** Eliminate $(u,v)$ to obtain some known relationship between $x$, $y$, $z$.

**I** 1) Here

$$x = u, \qquad y = u+2v, \qquad z = v-u,$$

hence

$$y - 2z = u + 2v - 2v + 2u = 3u = 3x,$$

or

$$3x - y + 2z = 0.$$

This is the equation of a plane through $(0,0,0)$ with the normal vector $(3,-1,2)$.

2) Here

$$x = u, \qquad y = \sin v, \qquad z = 3\cos v,$$

i.e.

$$y^2 + \left(\frac{z}{3}\right)^2 = 1, \qquad x = u, \quad u \in \mathbb{R}.$$

This is a cylindric surface with the $X$ axis as its axis and the ellipse of centrum $(0,0)$ and half axes 1 and 3 in the $YZ$ plane as the generating curve.

3) It follows from

$$x = u\,\cos v, \qquad y = u\,\sin v, \qquad z = u^2 \sin 2v$$

that

$$2xy = 2u^2 \cos v \cdot \sin v = u^2 \sin 2v = z,$$

i.e.

$$z = 2xy,$$

which describes an hyperbolic paraboloid.

4) Here

$$\frac{x}{a} = \cos v - u\,\sin v, \quad \frac{y}{b} = \sin v + u\,\cos v, \quad \frac{z}{c} = u,$$

hence

$$\left(\frac{x}{a}\right)^2 = \cos^2 v - 2u\,\sin v \cdot \cos v + u^2 \sin^2 v,$$

$$\left(\frac{y}{b}\right)^2 = \sin^2 v + 2u\,\sin v \cdot \cos v + u^2 \cos^2 v,$$

and accordingly

$$\left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^2=1+u^2=1+\left(\frac{z}{c}\right)^2.$$

This is the equation of an hyperboloid with one sheet.

5) It follows from

$$x=u\,\cos v,\qquad \frac{y}{2}=u\,\sin v,\qquad z=u^2$$

that

$$x^2+\left(\frac{y}{2}\right)^2=u^2=z,$$

which is the equation of an elliptic paraboloid.

6) It follows from

$$x=u+v,\qquad y=u-v,\qquad z=4v^2$$

that $2v=x-y$, i.e.

$$z=4v^2=(x-y)^2.$$

This is the equation of a cylindric surface with the line $y=x$ as its axis and a parabola as its generating curve.

7) It follows from

$$x=u+v,\qquad y=u^2+v^2,\qquad z=u^3+v^3$$

that

$$2z=2(u^3+v^3)=(u+v)(2u^2-2uv+2v^2)=x(2y-2uv),$$

where

$$2uv=(u+v)^2-(u^2+v^2)=x^2-y.$$

Then by insertion,

$$2z=x(3y-x^2).$$

This equation contains terms of first, second and third order.

**Example 7.30** *Sketch the following cylindric surfaces.*

*1)* $x = \cos\varphi$, $y = \sin\varphi$, $\varphi \in \left[\frac{\pi}{6}, \frac{\pi}{2}\right]$, $z \in [1, 2\varphi]$.

*2)* $xy = 1$, $y \in \left[\frac{1}{2}, 2\right]$, $z \in [0, x]$.

*3)* $y = e^{-x}$, $z \in [y, 1]$.

*4)* $x = y^2$, $z \in [x, y]$.

**A** Cylindric surfaces.

**D** First sketch the projection onto the $XY$ plane.

**I** 1) Here we get a circular arc in the $XY$- plane.

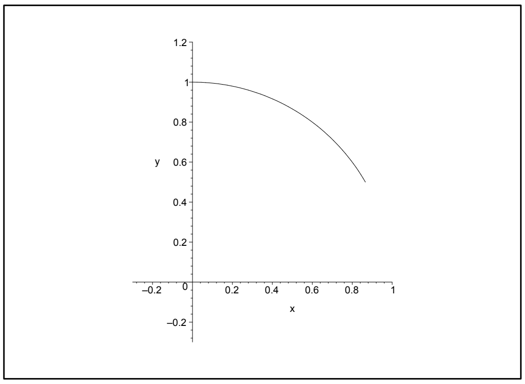



Figure 7.37: The projection onto the $XY$ plane.

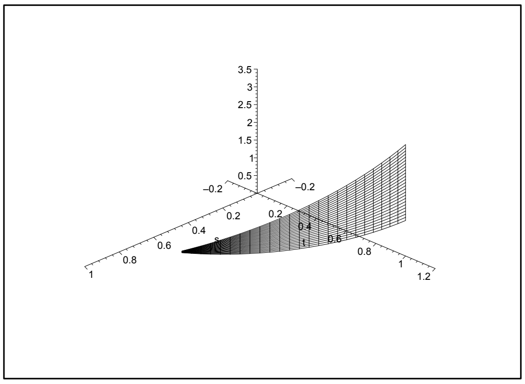


Figure 7.38: The cylindric surface of 1).

2) The projection onto the $XY$ plane is an arc of an hyperbola, lying in the first quadrant. Note that $x \in [\frac{1}{2}, 2]$.

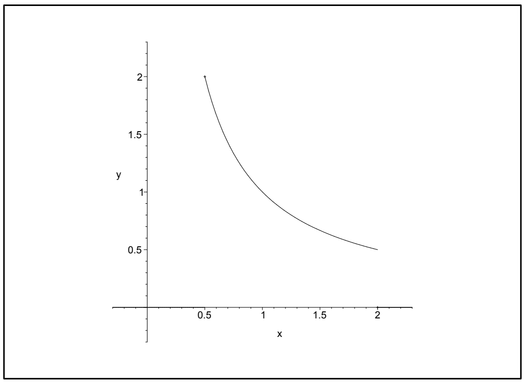



Figure 7.39: The projection onto the $XY$ plane.

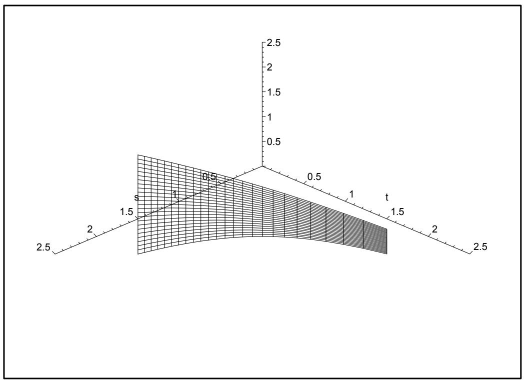



Figure 7.40: The cylindric surface of 2).

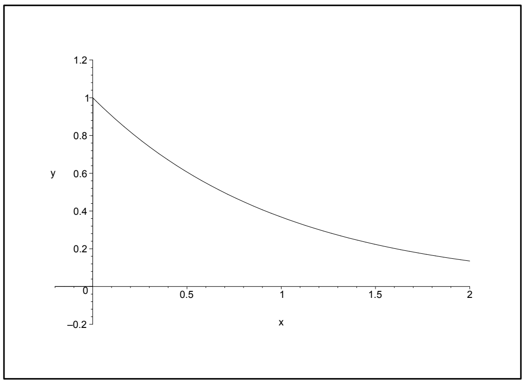



Figure 7.41: The projection onto the $XY$ plane.

3) Since $y \leq 1$, we must have $x \geq 0$.
4) From $x = y^2 \leq z \leq y$ we get the condition $0 \leq y \leq 1$. On the figure the surface looks wrong. There may here be an error in the MAPLE programme, though I am not sure.

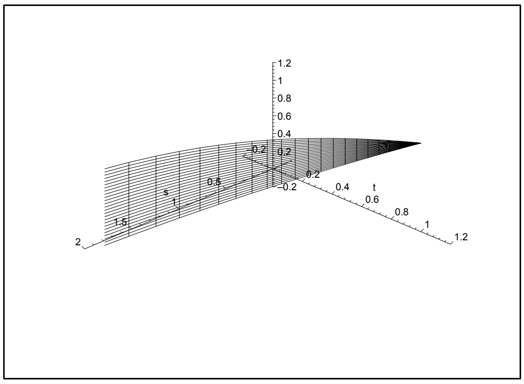



Figure 7.42: The cylindric surface of 3).

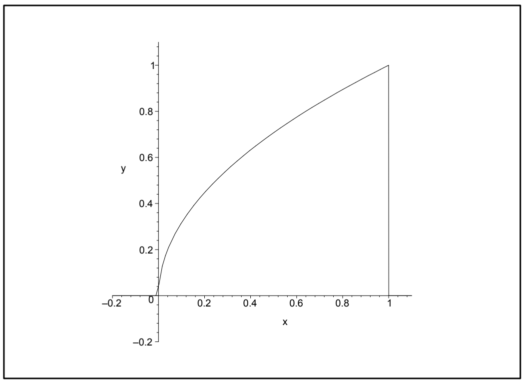



Figure 7.43: The projection onto the $XY$ plane.

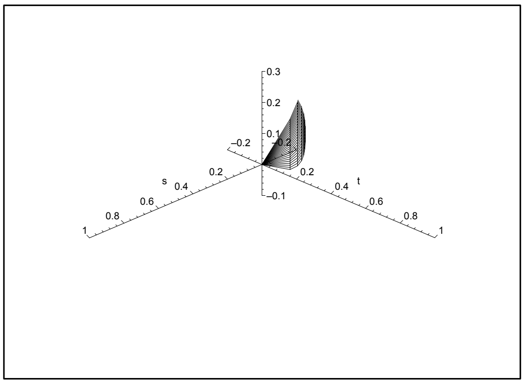



Figure 7.44: The cylindric surface of 4).

**Example 7.31** *In the following there are given some equations of meridian curves. Set up in each case an equation of the corresponding surface of revolution $\mathcal{O}$ and find the name of $\mathcal{O}$.*

*1)* $z = \varrho$.

*2)* $\varrho = |z|$.

*3)* $\varrho = a$.

*4)* $z^2 + 2\varrho^2 = 2az$.

*5)* $z^2 - \varrho^2 = a^2$.

*6)* $\varrho^2 - z^2 = a^2$.

**A** Surfaces of revolution with a given meridian curve.

**D** First sketch the meridian curve in the $PZ$ half plane.

**I** 1) This is a cone of vertex $(0,0,0)$.

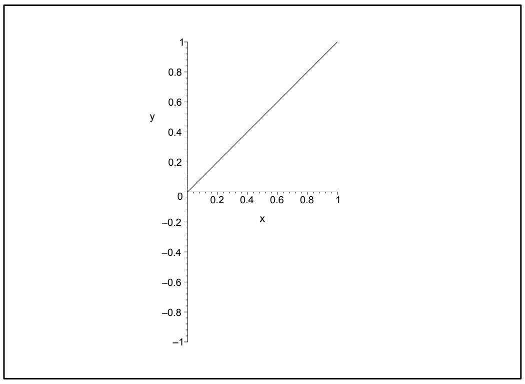



Figure 7.45: The meridian curve of 1).

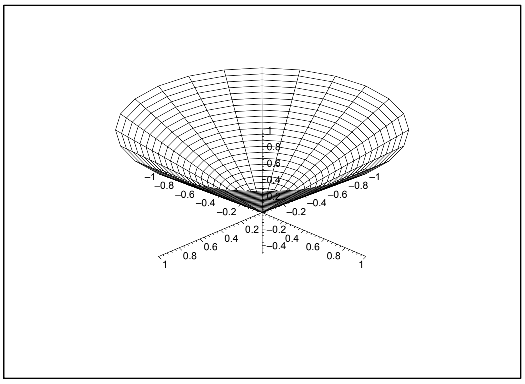


Figure 7.46: The surface of 1).

2) This is a double cone of vertex $(0, 0, 0)$.

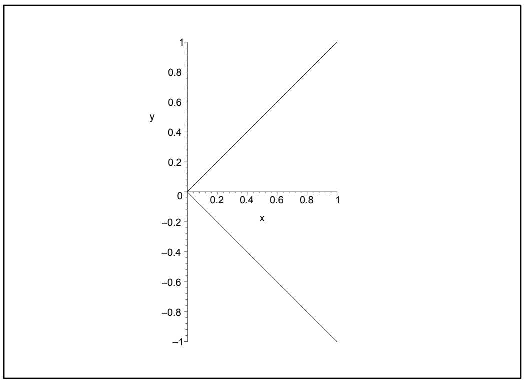



Figure 7.47: The meridian curve of 2).

3) This is clearly a cylinder.

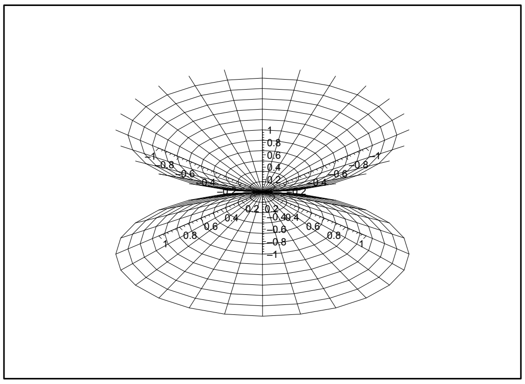


Figure 7.48: The surface of 2).

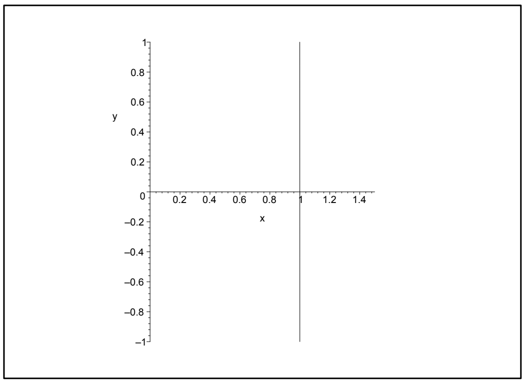



Figure 7.49: The meridian curve of 3).

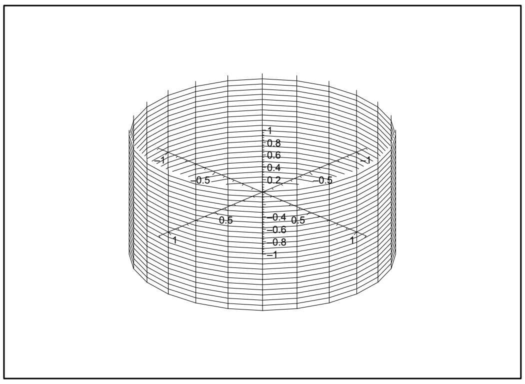


Figure 7.50: The surface of 3).

4) It follows by a small rearrangement that the equation is equivalent to

$$(z-a)^2 + 2\varrho^2 = a^2,$$

i.e. in the canonical form

$$\left(\frac{\varrho}{\frac{a}{\sqrt{2}}}\right)^2 + \left(\frac{z-a}{a}\right)^2 = 1, \qquad \varrho \geq 0.$$

The meridian curve is an half ellipse in the $PZ$ half plane of centre $(0, a)$ and half axes $\frac{a}{\sqrt{2}}$ and $a$.

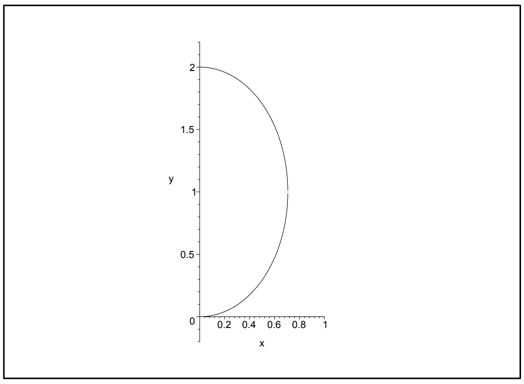



Figure 7.51: The meridian curve of 4).

The surface of revolution is the surface of an ellipsoid of centre $(0, 0, a)$ and half axes $\frac{a}{\sqrt{2}}$, $\frac{a}{\sqrt{2}}$ and $a$. Notice that one of the top points lies at $(0, 0, 0)$. Also note that the scales are different on the axes on the figure.

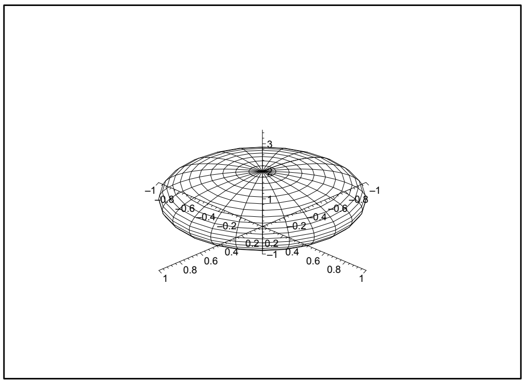


Figure 7.52: The surface of 4).

5) In this case the meridian curves consist of two halves of branches of an hyperbola.

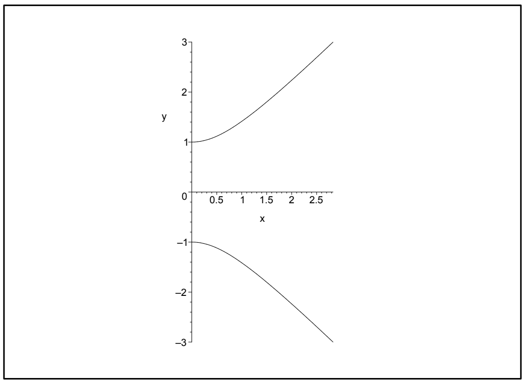



Figure 7.53: The meridian curves of 5).

By the revolution we get an hyperboloid with two sheets. Only the upper sheet is sketched on the figure (and we use different scales on the axes). There is a similar surface in the lower half space.

6) The curve $\varrho^2 - z^2 = a^2$, $\varrho \geq 0$, is a branch of an hyperbola with its top point at $(a, 0)$ and its half axes $a$ and $a$. The surface of revolution is an hyperboloid with one sheet and of centre $(0, 0, 0)$ and with the $Z$ axis as its axes of revolution and with the half axes $a$, $a$, $a$.

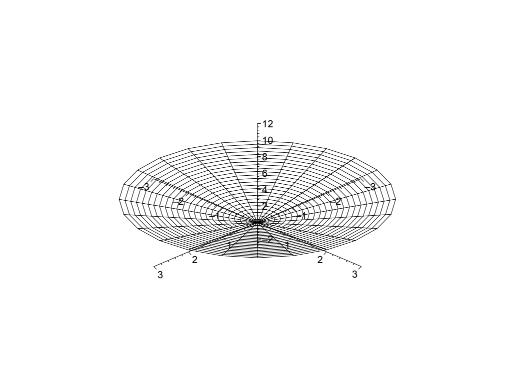


Figure 7.54: The upper surface of 5).

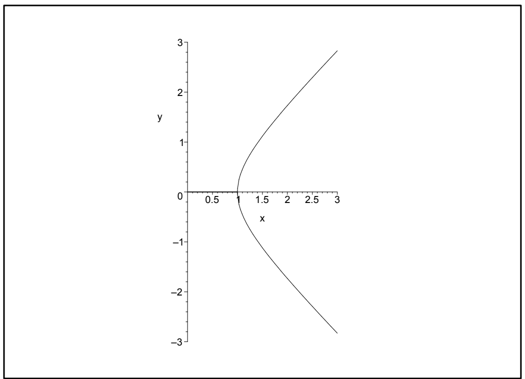



Figure 7.55: The meridian curve of 6).

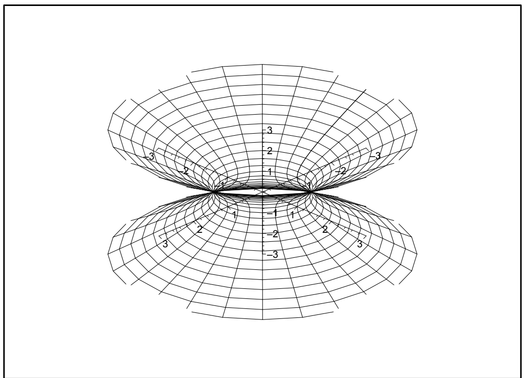


Figure 7.56: The surface of 6).

# 8 Formulæ

Some of the following formulæ can be assumed to be known from high school. It is highly recommended that one *learns most of these formulæ in this appendix by heart.*

## 8.1 Squares etc.

The following simple formulæ occur very frequently in the most different situations.

$$
\begin{aligned}
&(a+b)^2 = a^2+b^2+2ab, \qquad && a^2+b^2+2ab = (a+b)^2,\\
&(a-b)^2 = a^2+b^2-2ab, && a^2+b^2-2ab = (a-b)^2,\\
&(a+b)(a-b) = a^2-b^2, && a^2-b^2 = (a+b)(a-b),\\
&(a+b)^2 = (a-b)^2+4ab, && (a-b)^2 = (a+b)^2-4ab.
\end{aligned}
$$

## 8.2 Powers etc.

**Logarithm:**

$$
\begin{aligned}
\ln|xy| &= \ln|x|+\ln|y|, && x,y\neq 0,\\
\ln\left|\frac{x}{y}\right| &= \ln|x|-\ln|y|, && x,y\neq 0,\\
\ln|x^r| &= r\ln|x|, && x\neq 0.
\end{aligned}
$$

**Power function, fixed exponent:**

$$(xy)^r = x^r\cdot y^r,\, x,y>0 \quad \text{(extensions for some } r\text{)},$$

$$\left(\frac{x}{y}\right)^r = \frac{x^r}{y^r},\, x,y>0 \quad \text{(extensions for some } r\text{)}.$$

**Exponential, fixed base:**

$$
\begin{aligned}
&a^x\cdot a^y = a^{x+y}, \quad a>0 && \text{(extensions for some } x,\, y\text{)},\\
&(a^x)^y = a^{xy},\, a>0 && \text{(extensions for some } x,\, y\text{)},
\end{aligned}
$$

$$a^{-x} = \frac{1}{a^x},\, a>0, \qquad \text{(extensions for some } x\text{)},$$

$$\sqrt[n]{a} = a^{1/n},\, a\geq 0, \qquad n\in\mathbb{N}.$$

**Square root:**

$$\sqrt{x^2} = |x|, \qquad x\in\mathbb{R}.$$

**Remark 8.1** It happens quite frequently that students make errors when they try to apply these rules. They must be mastered! In particular, as one of my friends once put it: "If you can master the square root, you can master everything in mathematics!" Notice that this innocent looking square root is one of the most difficult operations in Calculus. Do not forget the *absolute value!* $\Diamond$

## 8.3 Differentiation

Here are given the well-known rules of differentiation together with some rearrangements which sometimes may be easier to use:

$$\{f(x) \pm g(x)\}' = f'(x) \pm g'(x),$$

$$\{f(x)g(x)\}' = f'(x)g(x) + f(x)g'(x) = f(x)g(x)\left\{\frac{f'(x)}{f(x)} + \frac{g'(x)}{g(x)}\right\},$$

where the latter rearrangement presupposes that $f(x) \neq 0$ and $g(x) \neq 0$.
If $g(x) \neq 0$, we get the usual formula known from high school

$$\left\{\frac{f(x)}{g(x)}\right\}' = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}.$$

It is often more convenient to compute this expression in the following way:

$$\left\{\frac{f(x)}{g(x)}\right\} = \frac{d}{dx}\left\{f(x)\cdot\frac{1}{g(x)}\right\} = \frac{f'(x)}{g(x)} - \frac{f(x)g'(x)}{g(x)^2} = \frac{f(x)}{g(x)}\left\{\frac{f'(x)}{f(x)} - \frac{g'(x)}{g(x)}\right\},$$

where the former expression often is *much easier* to use in practice than the usual formula from high school, and where the latter expression again presupposes that $f(x) \neq 0$ and $g(x) \neq 0$. Under these assumptions we see that the formulæ above can be written

$$\frac{\{f(x)g(x)\}'}{f(x)g(x)} = \frac{f'(x)}{f(x)} + \frac{g'(x)}{g(x)},$$

$$\frac{\{f(x)/g(x)\}'}{f(x)/g(x)} = \frac{f'(x)}{f(x)} - \frac{g'(x)}{g(x)}.$$

Since

$$\frac{d}{dx}\ln|f(x)| = \frac{f'(x)}{f(x)}, \qquad f(x) \neq 0,$$

we also name these the *logarithmic derivatives.*

Finally, we mention the rule of **differentiation of a composite function**

$$\{f(\varphi(x))\}' = f'(\varphi(x)) \cdot \varphi'(x).$$

We first differentiate the function itself; then the insides. This rule is a 1-dimensional version of the so-called *Chain rule.*

## 8.4 Special derivatives.

**Power like:**

$$\frac{d}{dx}(x^\alpha) = \alpha \cdot x^{\alpha-1}, \qquad \text{for } x > 0, \quad \text{(extensions for some } \alpha\text{)}.$$

$$\frac{d}{dx}\ln|x| = \frac{1}{x}, \qquad \text{for } x \neq 0.$$

**Exponential like:**

$$\frac{d}{dx}\exp x = \exp x, \qquad \text{for } x \in \mathbb{R},$$

$$\frac{d}{dx}\left(a^x\right) = \ln a \cdot a^x, \qquad \text{for } x \in \mathbb{R} \text{ and } a > 0.$$

**Trigonometric:**

$$\frac{d}{dx}\sin x = \cos x, \qquad \text{for } x \in \mathbb{R},$$

$$\frac{d}{dx}\cos x = -\sin x, \qquad \text{for } x \in \mathbb{R},$$

$$\frac{d}{dx}\tan x = 1 + \tan^2 x = \frac{1}{\cos^2 x}, \qquad \text{for } x \neq \frac{\pi}{2} + p\pi, p \in \mathbb{Z},$$

$$\frac{d}{dx}\cot x = -\left(1 + \cot^2 x\right) = -\frac{1}{\sin^2 x}, \qquad \text{for } x \neq p\pi, p \in \mathbb{Z}.$$

**Hyperbolic:**

$$\frac{d}{dx}\sinh x = \cosh x, \qquad \text{for } x \in \mathbb{R},$$

$$\frac{d}{dx}\cosh x = \sinh x, \qquad \text{for } x \in \mathbb{R},$$

$$\frac{d}{dx}\tanh x = 1 - \tanh^2 x = \frac{1}{\cosh^2 x}, \qquad \text{for } x \in \mathbb{R},$$

$$\frac{d}{dx}\coth x = 1 - \coth^2 x = -\frac{1}{\sinh^2 x}, \qquad \text{for } x \neq 0.$$

**Inverse trigonometric:**

$$\frac{d}{dx}\text{ Arcsin } x = \frac{1}{\sqrt{1-x^2}}, \qquad \text{for } x \in \,]-1,1\,[,$$

$$\frac{d}{dx}\text{ Arccos } x = -\frac{1}{\sqrt{1-x^2}}, \qquad \text{for } x \in \,]-1,1\,[,$$

$$\frac{d}{dx}\text{ Arctan } x = \frac{1}{1+x^2}, \qquad \text{for } x \in \mathbb{R},$$

$$\frac{d}{dx}\text{ Arccot } x = \frac{1}{1+x^2}, \qquad \text{for } x \in \mathbb{R}.$$

**Inverse hyperbolic:**

$$\frac{d}{dx}\text{ Arsinh } x = \frac{1}{\sqrt{x^2+1}}, \qquad \text{for } x \in \mathbb{R},$$

$$\frac{d}{dx}\text{ Arcosh } x = \frac{1}{\sqrt{x^2-1}}, \qquad \text{for } x \in \,]1,+\infty\,[,$$

$$\frac{d}{dx}\text{ Artanh } x = \frac{1}{1-x^2}, \qquad \text{for } |x| < 1,$$

$$\frac{d}{dx}\text{ Arcoth } x = \frac{1}{1-x^2}, \qquad \text{for } |x| > 1.$$

**Remark 8.2** The derivative of the trigonometric and the hyperbolic functions are to some extent exponential like. The derivatives of the inverse trigonometric and inverse hyperbolic functions are power like, because we include the logarithm in this class. $\Diamond$

## 8.5 Integration

The most obvious rules are dealing with linearity

$$\int \{f(x) + \lambda g(x)\}\, dx = \int f(x)\, dx + \lambda \int g(x)\, dx, \qquad \text{where } \lambda \in \mathbb{R} \text{ is a constant,}$$

and with the fact that differentiation and integration are "inverses to each other", i.e. modulo some arbitrary constant $c \in \mathbb{R}$, which often tacitly is missing,

$$\int f'(x)\, dx = f(x).$$

If we in the latter formula replace $f(x)$ by the product $f(x)g(x)$, we get by reading from the right to the left and then differentiating the product,

$$f(x)g(x) = \int \{f(x)g(x)\}'\, dx = \int f'(x)g(x)\, dx + \int f(x)g'(x)\, dx.$$

Hence, by a rearrangement

**The rule of partial integration:**

$$\int f'(x)g(x)\, dx = f(x)g(x) - \int f(x)g'(x)\, dx.$$

The differentiation is moved from one factor of the integrand to the other one by changing the sign and adding the term $f(x)g(x)$.

**Remark 8.3** This technique was earlier used a lot, but is almost forgotten these days. It must be revived, because MAPLE and pocket calculators apparently do not know it. It is possible to construct examples where these devices cannot give the exact solution, unless you first perform a partial integration yourself. $\diamond$

**Remark 8.4** This method can also be used when we estimate integrals which cannot be directly calculated, because the antiderivative is not contained in e.g. the catalogue of MAPLE. The idea is by a succession of partial integrations to make the new integrand smaller. $\diamond$

**Integration by substitution:**

If the integrand has the special structure $f(\varphi(x)) \cdot \varphi'(x)$, then one can change the variable to $y = \varphi(x)$:

$$\int f(\varphi(x)) \cdot \varphi'(x)\, dx = \text{“}\int f(\varphi(x))\, d\varphi(x)\text{”} = \int_{y=\varphi(x)} f(y)\, dy.$$

**Integration by a monotonous substitution**:

If $\varphi(y)$ is a *monotonous* function, which maps the $y$-interval *one-to-one* onto the $x$-interval, then

$$\int f(x)\,dx = \int_{y=\varphi^{-1}(x)} f(\varphi(y))\varphi'(y)\,dy.$$

**Remark 8.5** This rule is usually used when we have some "ugly" term in the integrand $f(x)$. The idea is to put this ugly term equal to $y = \varphi^{-1}(x)$. When e.g. $x$ occurs in $f(x)$ in the form $\sqrt{x}$, we put $y = \varphi^{-1}(x) = \sqrt{x}$, hence $x = \varphi(y) = y^2$ and $\varphi'(y) = 2y$. $\Diamond$

## 8.6 Special antiderivatives

**Power like:**

$$\int \frac{1}{x}\,dx = \ln|x|, \qquad \text{for } x \neq 0. \text{ (Do not forget the numerical value!)}$$

$$\int x^\alpha\,dx = \frac{1}{\alpha+1}x^{\alpha+1}, \qquad \text{for } \alpha \neq -1,$$

$$\int \frac{1}{1+x^2}\,dx = \text{Arctan}\, x, \qquad \text{for } x \in \mathbb{R},$$

$$\int \frac{1}{1-x^2}\,dx = \frac{1}{2}\ln\left|\frac{1+x}{1-x}\right|, \qquad \text{for } x \neq \pm 1,$$

$$\int \frac{1}{1-x^2}\,dx = \text{Artanh}\, x, \qquad \text{for } |x| < 1,$$

$$\int \frac{1}{1-x^2}\,dx = \text{Arcoth}\, x, \qquad \text{for } |x| > 1,$$

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \text{Arcsin}\, x, \qquad \text{for } |x| < 1,$$

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = -\text{Arccos}\, x, \qquad \text{for } |x| < 1,$$

$$\int \frac{1}{\sqrt{x^2+1}}\,dx = \text{Arsinh}\, x, \qquad \text{for } x \in \mathbb{R},$$

$$\int \frac{1}{\sqrt{x^2+1}}\,dx = \ln(x+\sqrt{x^2+1}), \qquad \text{for } x \in \mathbb{R},$$

$$\int \frac{x}{\sqrt{x^2-1}}\,dx = \sqrt{x^2-1}, \qquad \text{for } x \in \mathbb{R},$$

$$\int \frac{1}{\sqrt{x^2-1}}\,dx = \text{Arcosh}\, x, \qquad \text{for } x > 1,$$

$$\int \frac{1}{\sqrt{x^2-1}}\,dx = \ln|x+\sqrt{x^2-1}|, \qquad \text{for } x > 1 \text{ eller } x < -1.$$

There is an error in the programs of the pocket calculators TI-92 and TI-89. The *numerical signs* are missing. It is obvious that $\sqrt{x^2-1} < |x|$ so if $x < -1$, then $x + \sqrt{x^2-1} < 0$. Since you cannot take the logarithm of a negative number, these pocket calculators will give an error message.

**Exponential like:**

$$\int \exp x \, dx = \exp x, \qquad \text{for } x \in \mathbb{R},$$

$$\int a^x \, dx = \frac{1}{\ln a} \cdot a^x, \qquad \text{for } x \in \mathbb{R}, \text{ and } a > 0, a \neq 1.$$

**Trigonometric:**

$$\int \sin x \, dx = -\cos x, \qquad \text{for } x \in \mathbb{R},$$

$$\int \cos x \, dx = \sin x, \qquad \text{for } x \in \mathbb{R},$$

$$\int \tan x \, dx = -\ln|\cos x|, \qquad \text{for } x \neq \frac{\pi}{2} + p\pi, \quad p \in \mathbb{Z},$$

$$\int \cot x \, dx = \ln|\sin x|, \qquad \text{for } x \neq p\pi, \quad p \in \mathbb{Z},$$

$$\int \frac{1}{\cos x} \, dx = \frac{1}{2} \ln\left(\frac{1+\sin x}{1-\sin x}\right), \qquad \text{for } x \neq \frac{\pi}{2} + p\pi, \quad p \in \mathbb{Z},$$

$$\int \frac{1}{\sin x} \, dx = \frac{1}{2} \ln\left(\frac{1-\cos x}{1+\cos x}\right), \qquad \text{for } x \neq p\pi, \quad p \in \mathbb{Z},$$

$$\int \frac{1}{\cos^2 x} \, dx = \tan x, \qquad \text{for } x \neq \frac{\pi}{2} + p\pi, \quad p \in \mathbb{Z},$$

$$\int \frac{1}{\sin^2 x} \, dx = -\cot x, \qquad \text{for } x \neq p\pi, \quad p \in \mathbb{Z}.$$

**Hyperbolic:**

$$\int \sinh x \, dx = \cosh x, \qquad \text{for } x \in \mathbb{R},$$

$$\int \cosh x \, dx = \sinh x, \qquad \text{for } x \in \mathbb{R},$$

$$\int \tanh x \, dx = \ln \cosh x, \qquad \text{for } x \in \mathbb{R},$$

$$\int \coth x \, dx = \ln|\sinh x|, \qquad \text{for } x \neq 0,$$

$$\int \frac{1}{\cosh x} \, dx = \operatorname{Arctan}(\sinh x), \qquad \text{for } x \in \mathbb{R},$$

$$\int \frac{1}{\cosh x} \, dx = 2 \operatorname{Arctan}(e^x), \qquad \text{for } x \in \mathbb{R},$$

$$\int \frac{1}{\sinh x} \, dx = \frac{1}{2} \ln\left(\frac{\cosh x - 1}{\cosh x + 1}\right), \qquad \text{for } x \neq 0,$$

$$\int \frac{1}{\sinh x}\,dx = \ln\left|\frac{e^x-1}{e^x+1}\right|, \qquad \text{for } x \neq 0,$$

$$\int \frac{1}{\cosh^2 x}\,dx = \tanh x, \qquad \text{for } x \in \mathbb{R},$$

$$\int \frac{1}{\sinh^2 x}\,dx = -\coth x, \qquad \text{for } x \neq 0.$$

## 8.7 Trigonometric formulæ

The trigonometric formulæ are closely connected with circular movements. Thus $(\cos u, \sin u)$ are the coordinates of a point $P$ on the unit circle corresponding to the angle $u$, cf. figure A.1. This geometrical interpretation is used from time to time.

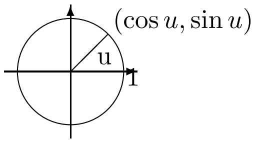



Figure 8.1: The unit circle and the trigonometric functions.

**The fundamental trigonometric relation**:

$$\cos^2 u + \sin^2 u = 1, \qquad \text{for } u \in \mathbb{R}.$$

Using the previous geometric interpretation this means according to *Pythagoras's theorem*, that the point $P$ with the coordinates $(\cos u, \sin u)$ always has distance 1 from the origo $(0,0)$, i.e. it is lying on the boundary of the circle of centre $(0,0)$ and radius $\sqrt{1} = 1$.

**Connection to the complex exponential function**:

The *complex exponential* is for imaginary arguments defined by

$$\exp(\mathrm{i}\,u) := \cos u + \mathrm{i}\ \sin u.$$

It can be checked that the usual functional equation for exp is still valid for complex arguments. In other word: The definition above is extremely conveniently chosen.

By using the definition for $\exp(\mathrm{i}\,u)$ and $\exp(-\mathrm{i}\,u)$ it is easily seen that

$$\cos u = \frac{1}{2}(\exp(\mathrm{i}\,u) + \exp(-\mathrm{i}\,u)),$$

$$\sin u = \frac{1}{2i}(\exp(\mathrm{i}\,u) - \exp(-\mathrm{i}\,u)),$$

**Moivre's formula**: We get by expressing $\exp(\mathrm{i}nu)$ in two different ways:

$$\exp(\mathrm{i}nu) = \cos nu + \mathrm{i}\,\sin nu = (\cos u + \mathrm{i}\,\sin u)^n.$$

**Example 8.1** If we e.g. put $n = 3$ into Moivre's formula, we obtain the following typical application,

$$\begin{aligned}\cos(3u) + \mathrm{i}\,\sin(3u) &= (\cos u + \mathrm{i}\,\sin u)^3\\ &= \cos^3 u + 3\mathrm{i}\,\cos^2 u\cdot\sin u + 3\mathrm{i}^2\cos u\cdot\sin^2 u + \mathrm{i}^3\sin^3 u\\ &= \{\cos^3 u - 3\cos u\cdot\sin^2 u\} + \mathrm{i}\{3\cos^2 u\cdot\sin u - \sin^3 u\}\\ &= \{4\cos^3 u - 3\cos u\} + \mathrm{i}\{3\sin u - 4\sin^3 u\}\end{aligned}$$

When this is split into the real- and imaginary parts we obtain

$$\cos 3u = 4\cos^3 u - 3\cos u, \qquad \sin 3u = 3\sin u - 4\sin^3 u. \quad \diamond$$

**Addition formulæ**:

$$\sin(u+v) = \sin u\,\cos v + \cos u\,\sin v,$$

$$\sin(u-v) = \sin u\,\cos v - \cos u\,\sin v,$$

$$\cos(u+v) = \cos u\,\cos v - \sin u\,\sin v,$$

$$\cos(u-v) = \cos u\,\cos v + \sin u\,\sin v.$$

**Products of trigonometric functions to a sum**:

$$\sin u\,\cos v = \frac{1}{2}\sin(u+v) + \frac{1}{2}\sin(u-v),$$

$$\cos u\,\sin v = \frac{1}{2}\sin(u+v) - \frac{1}{2}\sin(u-v),$$

$$\sin u\,\sin v = \frac{1}{2}\cos(u-v) - \frac{1}{2}\cos(u+v),$$

$$\cos u\,\cos v = \frac{1}{2}\cos(u-v) + \frac{1}{2}\cos(u+v).$$

**Sums of trigonometric functions to a product**:

$$\sin u + \sin v = 2\sin\left(\frac{u+v}{2}\right)\cos\left(\frac{u-v}{2}\right),$$

$$\sin u - \sin v = 2\cos\left(\frac{u+v}{2}\right)\sin\left(\frac{u-v}{2}\right),$$

$$\cos u + \cos v = 2\cos\left(\frac{u+v}{2}\right)\cos\left(\frac{u-v}{2}\right),$$

$$\cos u - \cos v = -2\sin\left(\frac{u+v}{2}\right)\sin\left(\frac{u-v}{2}\right).$$

**Formulæ of halving and doubling the angle:**

$$\sin 2u = 2\sin u\,\cos u,$$

$$\cos 2u = \cos^2 u - \sin^2 u = 2\,\cos^2 u - 1 = 1 - 2\,\sin^2 u,$$

$$\sin\frac{u}{2} = \pm\sqrt{\frac{1-\cos u}{2}} \qquad \text{followed by a discussion of the sign,}$$

$$\cos\frac{u}{2} = \pm\sqrt{\frac{1+\cos u}{2}} \qquad \text{followed by a discussion of the sign,}$$

## 8.8 Hyperbolic formulæ

These are very much like the trigonometric formulæ, and if one knows a little of Complex Function Theory it is realized that they are actually identical. The structure of this section is therefore the same as for the trigonometric formulæ. The reader should compare the two sections concerning similarities and differences.

**The fundamental relation**:

$$\cosh^2 x - \sinh^2 x = 1.$$

**Definitions**:

$$\cosh x = \frac{1}{2}\left(\exp(x) + \exp(-x)\right), \qquad \sinh x = \frac{1}{2}\left(\exp(x) - \exp(-x)\right).$$

**"Moivre's formula"**:

$$\exp(x) = \cosh x + \sinh x.$$

This is trivial and only rarely used. It has been included to show the analogy.

**Addition formulæ**:

$$\sinh(x+y) = \sinh(x)\cosh(y) + \cosh(x)\sinh(y),$$

$$\sinh(x-y) = \sinh(x)\cosh(y) - \cosh(x)\sinh(y),$$

$$\cosh(x+y) = \cosh(x)\cosh(y) + \sinh(x)\sinh(y),$$

$$\cosh(x-y) = \cosh(x)\cosh(y) - \sinh(x)\sinh(y).$$

**Formulæ of halving and doubling the argument**:

$$\sinh(2x) = 2\sinh(x)\cosh(x),$$

$$\cosh(2x) = \cosh^2(x) + \sinh^2(x) = 2\cosh^2(x) - 1 = 2\sinh^2(x) + 1,$$

$$\sinh\left(\frac{x}{2}\right) = \pm\sqrt{\frac{\cosh(x)-1}{2}} \qquad \text{followed by a discussion of the sign,}$$

$$\cosh\left(\frac{x}{2}\right) = \sqrt{\frac{\cosh(x)+1}{2}}.$$

**Inverse hyperbolic functions**:

$$\operatorname{Arsinh}(x) = \ln\left(x + \sqrt{x^2+1}\right), \qquad x \in \mathbb{R},$$

$$\operatorname{Arcosh}(x) = \ln\left(x + \sqrt{x^2-1}\right), \qquad x \geq 1,$$

$$\operatorname{Artanh}(x) = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right), \qquad |x| < 1,$$

$$\operatorname{Arcoth}(x) = \frac{1}{2}\ln\left(\frac{x+1}{x-1}\right), \qquad |x| > 1.$$

## 8.9 Complex transformation formulæ

$$\cos(\mathrm{i}x) = \cosh(x), \qquad \cosh(\mathrm{i}x) = \cos(x),$$
$$\sin(\mathrm{i}x) = \ \mathrm{i}\sinh(x), \qquad \sinh(\mathrm{i}x) = \ \mathrm{i}\sin x.$$

## 8.10 Taylor expansions

The generalized binomial coefficients are defined by

$$\binom{\alpha}{n} := \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{1\cdot 2\cdots n},$$

with $n$ factors in the numerator and the denominator, supplied with

$$\binom{\alpha}{0} := 1.$$

The Taylor expansions for *standard functions* are divided into *power like* (the radius of convergency is finite, i.e. $= 1$ for the standard series) and*exponential like* (the radius of convergency is infinite).
**Power like**:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \qquad |x| < 1,$$

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n, \qquad |x| < 1,$$

$$(1+x)^n = \sum_{j=0}^{n} \binom{n}{j} x^j, \qquad n \in \mathbb{N}, x \in \mathbb{R},$$

$$(1+x)^\alpha = \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n, \qquad \alpha \in \mathbb{R}\setminus\mathbb{N}, |x| < 1,$$

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1}\frac{x^n}{n}, \qquad |x| < 1,$$

$$\mathrm{Arctan}(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}, \qquad |x| < 1.$$

**Exponential like**:

$$\exp(x) = \sum_{n=0}^{\infty} \frac{1}{n!} x^n, \qquad x \in \mathbb{R}$$

$$\exp(-x) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{n!} x^n, \qquad x \in \mathbb{R}$$

$$\sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+1)!} x^{2n+1}, \qquad x \in \mathbb{R},$$

$$\sinh(x) = \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} x^{2n+1}, \qquad x \in \mathbb{R},$$

$$\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n)!} x^{2n}, \qquad x \in \mathbb{R},$$

$$\cosh(x) = \sum_{n=0}^{\infty} \frac{1}{(2n)!} x^{2n}, \qquad x \in \mathbb{R}.$$

## 8.11 Magnitudes of functions

We often have to compare functions for $x \to 0+$, or for $x \to \infty$. The simplest type of functions are therefore arranged in an hierarchy:

1) logarithms,

2) power functions,

3) exponential functions,

4) faculty functions.

When $x \to \infty$, a function from a higher class will always dominate a function form a lower class. More precisely:

**A)** A *power function* dominates a *logarithm* for $x \to \infty$:

$$\frac{(\ln x)^\beta}{x^\alpha} \to 0 \qquad \text{for } x \to \infty, \quad \alpha,\, \beta > 0.$$

**B)** An *exponential* dominates a *power function* for $x \to \infty$:

$$\frac{x^\alpha}{a^x} \to 0 \qquad \text{for } x \to \infty, \quad \alpha,\, a > 1.$$

**C)** The *faculty function* dominates an *exponential* for $n \to \infty$:

$$\frac{a^n}{n!} \to 0, \qquad n \to \infty, \quad n \in \mathbb{N}, \quad a > 0.$$

**D)** When $x \to 0+$ we also have that a *power function* dominates the *logarithm:*

$$x^\alpha \ln x \to 0-, \qquad \text{for } x \to 0+, \quad \alpha > 0.$$

# Index