Real Functions in Several Variables: Volume I

Point sets in Rn

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Contents

Preface

The topic of this series of books on *"Real Functions in Several Variables"* is very important in the description in e.g. *Mechanics* of the real 3-dimensional world that we live in. Therefore, we start from the very beginning, modelling this world by using the coordinates of $\mathbb{R}^3$ to describe e.g. a motion in space. There is, however, absolutely no reason to restrict ourselves to $\mathbb{R}^3$ alone. Some motions may be rectilinear, so only $\mathbb{R}$ is needed to describe their movements on a line segment. This opens up for also dealing with $\mathbb{R}^2$, when we consider plane motions. In more elaborate problems we need higher dimensional spaces. This may be the case in *Probability Theory* and *Statistics*. Therefore, we shall in general use $\mathbb{R}^n$ as our abstract model, and then restrict ourselves in examples mainly to $\mathbb{R}^2$ and $\mathbb{R}^3$.

For rectilinear motions the familiar *rectangular coordinate system* is the most convenient one to apply. However, as known from e.g. *Mechanics*, circular motions are also very important in the applications in engineering. It becomes natural alternatively to apply in $\mathbb{R}^2$ the so-called *polar coordinates* in the plane. They are convenient to describe a circle, where the rectangular coordinates usually give some nasty square roots, which are difficult to handle in practice.

Rectangular coordinates and polar coordinates are designed to model each their problems. They supplement each other, so difficult computations in one of these coordinate systems may be easy, and even trivial, in the other one. It is therefore important always in advance carefully to analyze the geometry of e.g. a domain, so we ask the question: Is this domain best described in rectangular or in polar coordinates?

Sometimes one may split a problem into two subproblems, where we apply rectangular coordinates in one of them and polar coordinates in the other one.

It should be mentioned that in *real life* (though not in these books) one cannot always split a problem into two subproblems as above. Then one is really in trouble, and more advanced mathematical methods should be applied instead. This is, however, outside the scope of the present series of books.

The idea of polar coordinates can be extended in two ways to $\mathbb{R}^3$. Either to *semi-polar* or *cylindric coordinates*, which are designed to describe a cylinder, or to *spherical coordinates*, which are excellent for describing spheres, where rectangular coordinates usually are doomed to fail. We use them already in daily life, when we specify a place on Earth by its longitude and latitude! It would be very awkward in this case to use rectangular coordinates instead, even if it is possible.

Concerning the contents, we begin this investigation by modelling point sets in an n-dimensional Euclidean space E^n by $\mathbb{R}^n$. There is a subtle difference between E^n and $\mathbb{R}^n$, although we often identify these two spaces. In E^n we use *geometrical methods* without a coordinate system, so the objects are independent of such a choice. In the coordinate space $\mathbb{R}^n$ we can use ordinary calculus, which in principle is not possible in E^n. In order to stress this point, we call E^n the "abstract space" (in the sense of calculus; not in the sense of geometry) as a warning to the reader. Also, whenever necessary, we use the colour black in the "abstract space", in order to stress that this expression is theoretical, while variables given in a chosen coordinate system and their related concepts are given the colours blue, red and green.

We also include the most basic of what mathematicians call *Topology*, which will be necessary in the following. We describe what we need by a function.

Then we proceed with limits and continuity of functions and define continuous curves and surfaces, with parameters from subsets of $\mathbb{R}$ and $\mathbb{R}^2$, resp..

Continue with (partial) differentiable functions, curves and surfaces, the chain rule and Taylor's formula for functions in several variables.

We deal with maxima and minima and extrema of functions in several variables over a domain in $\mathbb{R}^n$. This is a very important subject, so there are given many worked examples to illustrate the theory.

Then we turn to the problems of integration, where we specify four different types with increasing complexity, *plane integral, space integral, curve (or line) integral and surface integral*.

Finally, we consider *vector analysis*, where we deal with vector fields, Gauß's theorem and Stokes's theorem. All these subjects are very important in theoretical Physics.

The structure of this series of books is that each subject is *usually* (but not always) described by three successive chapters. In the first chapter a brief theoretical theory is given. The next chapter gives some practical guidelines of how to solve problems connected with the subject under consideration. Finally, some worked out examples are given, in many cases in several variants, because the standard solution method is seldom the only way, and it may even be clumsy compared with other possibilities.

I have as far as possible structured the examples according to the following scheme:

A *Awareness*, i.e. a short description of what is the problem.

D *Decision*, i.e. a reflection over what should be done with the problem.

I *Implementation*, i.e. where all the calculations are made.

C *Control*, i.e. a test of the result.

This is an ideal form of a general procedure of solution. It can be used in any situation and it is not linked to Mathematics alone. I learned it many years ago in the Theory of Telecommunication in a situation which did not contain Mathematics at all. The student is recommended to use it also in other disciplines.

From high school one is used to immediately to proceed to **I**. *Implementation*. However, examples and problems at university level, let alone situations in real life, are often so complicated that it in general will be a good investment also to spend some time on the first two points above in order to be absolutely certain of what to do in a particular case. Note that the first three points, **ADI**, can always be executed.

This is unfortunately not the case with **C** *Control*, because it from now on may be difficult, if possible, to check one's solution. It is only an extra securing whenever it is possible, but we cannot include it always in our solution form above.

I shall on purpose not use the logical signs. These should in general be avoided in Calculus as a shorthand, because they are often (too often, I would say) misused. Instead of $\wedge$ I shall either write "and", or a comma, and instead of $\vee$ I shall write "or". The arrows $\Rightarrow$ and $\Leftrightarrow$ are in particular misunderstood by the students, so they should be totally avoided. They are not telegram short hands, and from a logical point of view they usually do not make sense at all! Instead, write in a plain language what you mean or want to do. This is difficult in the beginning, but after some practice it becomes routine, and it will give more precise information.

When we deal with multiple integrals, one of the possible pedagogical ways of solving problems has been to colour variables, integrals and upper and lower bounds in blue, red and green, so the reader by the colour code can see in each integral what is the variable, and what are the parameters, which do not enter the integration under consideration. We shall of course build up a hierarchy of these colours, so the order of integration will always be defined. As already mentioned above we reserve the colour black for the theoretical expressions, where we cannot use ordinary calculus, because the symbols are only shorthand for a concept.

The author has been very grateful to his old friend and colleague, the late Per Wennerberg Karlsson, for many discussions of how to present these difficult topics on real functions in several variables, and for his permission to use his textbook as a template of this present series. Nevertheless, the author has felt it necessary to make quite a few changes compared with the old textbook, because we did not always agree, and some of the topics could also be explained in another way, and then of course the results of our discussions have here been put in writing for the first time.

The author also adds some calculations in MAPLE, which interact nicely with the theoretic text. Note, however, that when one applies MAPLE, one is forced first to make a geometrical analysis of the domain of integration, i.e. apply some of the techniques developed in the present books.

The theory and methods of these volumes on “Real Functions in Several Variables” are applied constantly in higher Mathematics, Mechanics and Engineering Sciences. It is of paramount importance for the calculations in *Probability Theory*, where one constantly integrate over some point set in space.

It is my hope that this text, these guidelines and these examples, of which many are treated in more ways to show that the solutions procedures are not unique, may be of some inspiration for the students who have just started their studies at the universities.

Finally, even if I have tried to write as careful as possible, I doubt that all errors have been removed. I hope that the reader will forgive me the unavoidable errors.

Leif Mejlbro
March 21, 2015

Introduction to volume I,
Point sets in $\mathbb{R}^n$. The maximal domain of a function

In this first volume of the series of books on *Real Functions in Several Variables* we start in Chapter 1 by giving a small theoretical introduction to what is needed in order to get started on the main subject. We shall work in *Euclidean space* E^n, which in rectangular coordinates is similar to the vector space $\mathbb{R}^n$, also called the *coordinate space*. The difference may at the first glance seem very small, and yet this difference is quite important. If we ever prove something in E^n, then this is done *geometrically* without any coordinate axes. This may be very strange to most younger readers, who have never learned Geometry in school using only ruler and compasses. For that reason I have in lack of better words called objects in E^n for "abstract" or "theoretical", though they are neither "abstract" nor "purely theoretical".

Once we have chosen a rectangular coordinate system in E^n, i.e. defined the n orthonormal basic vectors, then we have also defined the *rectangular coordinates* $(x_1, \dots, x_n) \in \mathbb{R}^n$ of an element $\mathbf{x} \in E^n$. The reason for this transformation from the Euclidean space E^n to its corresponding coordinate space $\mathbb{R}^n$ is of course that it is often easier to compute things in $\mathbb{R}^n$ than to argue geometrically in E^n.

Obviously, $E^2 \sim \mathbb{R}^2$ and $E^3 \sim \mathbb{R}^3$ are very important examples of $E^n \sim \mathbb{R}^n$, so the main emphasis is put on these two cases, though we cannot totally rule out higher dimensional spaces.

We introduce the dot product in all $\mathbb{R}^n$ and use it to define the *norm* (or length) and *angle*.

In $E^3 \sim \mathbb{R}^3$ (and only in this space) we also introduce the important *cross product* or *vector product*, which is applied in particular in Physics.

Even if rectangular coordinates may seem natural in the beginning, they are not well suited for all our problems. When we consider Mechanics in the plane E^2, there are clearly two very important motions, which we should be able to describe in a reasonable way, namely the *rectilinear motion*, where rectangular coordinates clearly are most appropriate, and the *circular motion*, where we in a rectangular description almost always end up with some nasty square roots. To ease matters we instead introduce the *polar coordinates* in the plane. In this case E^2 and the corresponding polar coordinate space $\subset \mathbb{R}^2$ are clearly not of the same geometrical shape. The circular motion is usually easy to describe in polar coordinates, when the coordinate system is put properly.

Once we have started introducing another coordinate system like the polar coordinates instead of the usual rectangular coordinate system, we may of course proceed by introducing other useful coordinate systems, like *semi-polar coordinates* in $\mathbb{R}^3$, which are designed to describe bodies of revolution with the z-axis as the axis of revolution, and the *spherical coordinates* in in $\mathbb{R}^3$, which are convenient, when we are dealing with spheres and balls in E^3.

All these new coordinate systems are only defined in Chapter 1. However, their applications will be demonstrated over and over again in the following volumes.

We continue with introducing the most basic of what is called *Topology*. We define the interior, exterior, boundary and closure of (abstract) sets. We shall also need all these abstract concepts in the following.

We give some examples of typical sets, which will be used frequently in the following. For the same reason we also include a section on the classical cones and conical sections from Geometry, because we cannot assume that all readers have seen them before.

The short Chapter 2 describes some guidelines of how to solve some typical problems in this book.

Chapter 3 contains a lot of examples describing the theoretical text from Chapter 1.

A short list of useful formulæ is given in Chapter 4.

The table of contents and the index cover all volumes, which are organized with succeeding page numbers. Unfortunately, it has not been possible to organize the index such that the number of the volume is also given.

1 Basic concepts

1.1 Introduction

We shall start by defining the *model number spaces* $\mathbb{R}^n$, so they are at hand, when we in the next section consider the corresponding *Euclidean spaces* E^n. There is a bijective correspondence between E^n and its coordinate space $\mathbb{R}^n$, when we use the obvious orthonormal basis. The subtle difference is that we argue in E^n in an "abstract way" on the geometry of the set, while we set up some rules of computation in the coordinate space $\mathbb{R}^n$. In other words, E^n contains the abstract geometrical objects, which then are described analytically in the coordinate space $\mathbb{R}^n$. In rectangular coordinates a point set $A \subseteq E^n$ has the same geometry as its set of coordinates $\tilde{A} \subseteq \mathbb{R}^n$, so one may hardly see the difference. However, whenever it is convenient to use another coordinate system, which is not rectangular, e.g. polar or spherical coordinates, then the set of coordinates $\tilde{A} \subseteq \mathbb{R}^n$ has apparently a different geometry from that of the original set $A \subseteq E^n$.

Whenever there is a need to distinguish between the "abstract space" of $A \subseteq E^n$ and its coordinate set $\tilde{A} \subseteq \mathbb{R}^n$, we shall use the following colour code: black in the "abstract" space E^n, and blue, red, green, etc. in the coordinate space $\mathbb{R}^n$. This is, however, not needed in the first volumes, and it only becomes convenient, when we are describing plane or space integrals, etc., where we calculate analytically the value of these integrals.

So first we *define* the *model number spaces* $\mathbb{R}^n$, and then discuss $\mathbb{R}^n$ as a real *vector space*, followed by introducing the most commonly used coordinate systems, i.e. *rectangular coordinates* (in $\mathbb{R}^n$ in general), *polar coordinates* (only in $\mathbb{R}^2$), *semi-polar coordinates*, also called *cylindric coordinates* (only in $\mathbb{R}^3$), and finally the *spherical coordinates*. These are here only defined in $\mathbb{R}^3$, but it is not hard to prove that *generalized spherical coordinates* can be defined in any number space $\mathbb{R}^n$, where $n \geq 3$.

In the following sections we turn to *point sets* in the *Euclidean space* E^n. To ease matters for the reader we shall, as already mentioned above, whenever it is felt convenient, identify a point set $A \subseteq E^n$ with its coordinate set $\tilde{A} \subseteq \mathbb{R}^n$ in rectangular coordinates. Note, however, that in principle A and $\tilde{A}$ are *not* the same set, although they may look alike!

We introduce some necessary abstract topological concepts like *open* and *closed sets*, *boundary sets*, *convex* and *starshaped sets*, etc.. These may seem very strange for the unexperienced reader, but they are needed, when we later shall describe limits and continuity of functions.

In the last section of this chapter we describe *functions in several variables*, and extend them to *vector functions*. We also describe how to visualize functions in several variables. Finally, we mention the problem of *implicit given functions*. It is not possible here to give a correct proof of the Theorem of implicit given function, though it clearly is very important.

1.2 The real linear space $\mathbb{R}^n$

The real number space $\mathbb{R}^n$ is considered as a real *vector space* $(\mathbb{R}^n, +, \cdot, \mathbb{R})$, also called a *linear space*. The elements of $\mathbb{R}^n$ are ordered sets of n real numbers, which are called the *coordinates* of the point. Hence, an element of $\mathbb{R}^n$ is written

$$\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}^n, \qquad \text{where } x_1, \dots, x_n \in \mathbb{R}.$$

Although we have not proved it yet, we mention that this is a description of $\mathbf{x}$ in *rectangular coordinates*, so when $\mathbf{x} \in \mathbb{R}^n$ is identified with the corresponding element in the *Euclidean space* E^n, which is also denoted by $\mathbf{x}$, then $\mathbf{x}$ is interpreted, depending on the actual situation, either as a point $\mathbf{x} \in E^n$, or as a *vector* $\overrightarrow{x} \in E^n$ pointing from $\mathbf{0} = (0, \dots, 0)$, or $\overrightarrow{0} = (0, \dots, 0)$ to the end point $\mathbf{x}$.

The *addition* in the vector space $\mathbb{R}^n$ is defined by

$$\mathbf{x} + \mathbf{y} = (x_1, \dots, x_n) + (y_1, \dots, y_n) = (x_1 + y_1, \dots, x_n + y_n)\,,$$

so we add the coordinates at place j, $j = 1, \dots, n$.

The addition is clearly commutative,

$$\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}, \qquad \text{for } \mathbf{x}, \mathbf{y} \in \mathbb{R}^n.$$

The *neutral element* is the *zero point* (or *zero vector* $\mathbf{0}$, because

$$\mathbf{x} + \mathbf{0} = (x_1, \dots, x_n) + (0, \dots, 0) = (x_1, \dots, x_n) = \mathbf{x}.$$

The *scalar multiplication* by $\lambda \in \mathbb{R}$ of $\mathbf{x} \in \mathbb{R}^n$ is defined by

$$\lambda\mathbf{x} = \lambda\,(x_1, \dots, x_n) = (\lambda x_1, \dots, \lambda x_n)\,,$$

so each coordinate is multiplied by the same scalar λ. This can be interpreted as a stretching. We note that we have no notation for the scalar product. In fact, there is no way to misunderstand the concatenation $\lambda\mathbf{x}$, and we shall later use the most obvious notation "$\cdot$" for another important product in $\mathbb{R}^n$.

A natural *basis* of $\mathbb{R}^n$ is given by the vectors of the coordinates

$$\mathbf{e}_1 = (1, 0, \cdots, 0), \cdots, \mathbf{e}_n = (0, \dots, 0, 1),$$

where e.g. $\mathbf{e}_j$ has 1 on its j-th coordinate, while all other coordinates are 0. In fact, it is obvious that we have

$$\mathbf{x} = (x_1, \dots, x_n) = x_1\mathbf{e}_1 + \cdots + x_n\mathbf{x}_n,$$

and if

$$x_1\mathbf{e}_1 + \cdots + x_n\mathbf{e}_n = \mathbf{0} = (0, \dots, 0),$$

then necessarily all $x_j = 0$, so the description of $\mathbf{x}$ is unique, and $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ is indeed a basis of $\mathbb{R}^n$.

One usually adds a so-called *inner product* in $\mathbb{R}^n$. This is a function denoted by $\cdot : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$. It is in order to avoid confusion that we do not introduce a notation for the scalar product of a scalar and a vector.

The *inner product* of two elements $\mathbf{x}$, $\mathbf{y} \in \mathbb{R}^n$ is defined in the following way:

$$\mathbf{x} \cdot \mathbf{y} := (x_1, \dots, x_n) \cdot (y_1, \dots, y_n) = x_1 y_1 + \dots + x_n y_n = \sum_{j=1}^{n} x_j y_j .$$

This is actually a geometrical concept, which shall be demonstrated in the following. Note in particular that

$$\mathbf{e}_i \cdot \mathbf{e}_j := \delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ \\ 0 & \text{if } i \neq j. \end{cases}$$

The symbol δ_{ij} defined above is called the *Kronecker symbol.* Due to this relation one may say that the vectors $\mathbf{e}_1, \dots, \mathbf{e}_n$ are perpendicular to each other.

Since $\mathbf{e}_j \cdot \mathbf{e}_j = 1$, we call the $\mathbf{e}_j$ *unit vectors.* They form an *orthonormal system.*

We call

$$x_j = \mathbf{x} \cdot \mathbf{e}_j$$

the *projection* of $\mathbf{x}$ onto the line defined by the unit vector $\mathbf{e}_j$. It is interpreted as the (signed) length of the orthogonal projection of $\mathbf{x}$ onto the line defined by the unit vector $\mathbf{e}_j$.

Using *Pythagoras's theorem* repeatedly $n-1$ times we easily derive that

$$\|\mathbf{x}\| := \sqrt{\mathbf{x} \cdot \mathbf{x}} = \sqrt{x_1^2 + \dots + x_n^2} \qquad \text{for } \mathbf{x} \in \mathbb{R}^n,$$

is the *length* (also called the *norm*) of the vector $\overrightarrow{x} \sim \mathbf{x}$. Hence, whenever we are given an inner product – in general satisfying some conditions, which are not given here – then we can talk about the length of a vector, and even of the angle between two vectors. We shall see below, how this is done.

We mention the following properties of the norm $\|\mathbf{x}\|$ defined above for $\mathbf{x} \in \mathbb{R}^n$. Let $\mathbf{x}$, $\mathbf{y} \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$ be given. Then

1) $\|\mathbf{x}\| > 0 \quad \text{for } \mathbf{x} \neq \mathbf{0} \quad (\text{and } \|\mathbf{0}\| = 0)$

2) $\|\lambda \mathbf{x}\| = |\lambda| \|\mathbf{x}\|$

3) $\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$ (triangle inequality)

4) $|\mathbf{x} \cdot \mathbf{y}| \leq \|\mathbf{x}\| \, \|\mathbf{y}\|$ (Cauchy-Schwarz's inequality)

The proofs of the first two claims are straightforward (left to the reader) by using the coordinate description.

Cauchy-Schwarz's inequality is proved in the following way: Let $\mathbf{x}$, $\mathbf{y} \in \mathbb{R}^n$ be given points, and let $\lambda \in \mathbb{R}$ be a scalar. Then

$$\begin{aligned} 0 &\leq \|\lambda \mathbf{x} + \mathbf{y}\|^2 = \sum_{j=1}^{n} (\lambda x_j + y_j)^2 = \lambda^2 \sum_{j=1}^{n} x_j^2 + 2\lambda \sum_{j=1}^{n} x_j \, y_j + \sum_{j=1}^{n} y_j^2 \\ &= \lambda^2 \|\mathbf{x}\|^2 + 2\lambda (\mathbf{x} \cdot \mathbf{y}) + \|\mathbf{y}\|^2, \end{aligned}$$

which holds for all $\lambda \in \mathbb{R}$. This is a real polynomial in λ of second degree, and it is nonnegative for all $\lambda \in \mathbb{R}$. Hence, its discriminant is not positive,

$$4(\mathbf{x} \cdot \mathbf{y})^2 - 4\|\mathbf{x}\|^2 \, \|\mathbf{y}\|^2 \le 0,$$

so by a rearrangement,

$$|\mathbf{x} \cdot \mathbf{y}| \le \|\mathbf{x}\| \, \|\mathbf{y}\|,$$

and the claim is proved.

We prove below, after we have defined the angle between two vectors, that the equality sign holds if and only if $\mathbf{x}$ and $\mathbf{y}$ are proportional, i.e. there exists a $\lambda \in \mathbb{R}$, such that either $\mathbf{x} = \lambda \mathbf{y}$ or $\mathbf{y} = \lambda \mathbf{x}$. (We cannot rule out the possibilities of either $\mathbf{x} = \mathbf{0}$ or $\mathbf{y} = \mathbf{0}$.)

Once we have proved *Cauchy-Schwarz's inequality*, we get the *triangle inequality* in the following way:

$$\begin{aligned} \|\mathbf{x}+\mathbf{y}\|^2 &= (\mathbf{x}+\mathbf{y}) \cdot (\mathbf{x}+\mathbf{y}) = \mathbf{x} \cdot \mathbf{x} + 2\mathbf{x} \cdot \mathbf{y} + \mathbf{y} \cdot \mathbf{y}, \\ &\le \|\mathbf{x}\|^2 + 2\|\mathbf{x}\| \, \|\mathbf{y}\| + \|\mathbf{y}\|^2 = (\|\mathbf{x}\| + \|\mathbf{y}\|)^2, \end{aligned}$$

hence, by taking the square root,

$$\|\mathbf{x}+\mathbf{y}\| \le \|\mathbf{x}\| + \|\mathbf{y}\|.$$

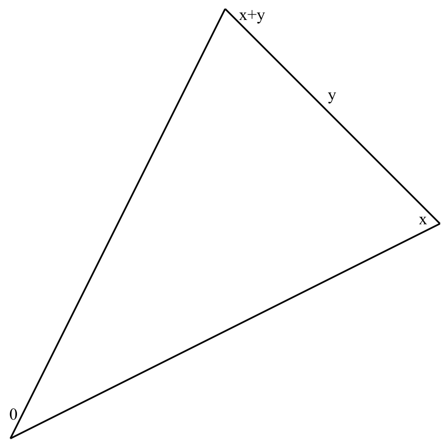


Figure 1.1: The triangle inequality

Remark 1.1 The vectors $\overrightarrow{x}$ and $\overrightarrow{x+y}$ form a triangle, if we add the vector $\mathbf{y}$ from $\mathbf{x}$, cf. Figure 1.1. The triangle inequality says that the length from $\mathbf{0}$ to $\mathbf{x}+\mathbf{y}$ is at most equal to the length of the broken path from $\mathbf{0}$ via $\mathbf{x}$ to $\mathbf{x}+\mathbf{y}$. $\diamondsuit$

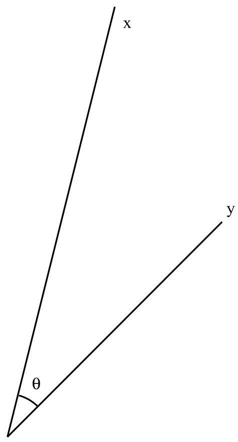


Figure 1.2: The angle between two vectors

Let $\mathbf{x}$, $\mathbf{y} \neq \mathbf{0}$ be two non-zero vectors from $\mathbb{R}^n$ (or E^n). Then they span an ordinary plane, so we can use the usual geometrical argument of trigonometry in this plane. In fact, we only use Pythagoras's theorem and the high school definition of cosine. In particular, the angle $\theta \in [0, \pi]$ between $\mathbf{x}$ and $\mathbf{y}$ is uniquely determined by the relation

$$\mathbf{x} \cdot \mathbf{y} = \|\mathbf{x}\| \, \|\mathbf{y}\| \, \cos\theta,$$

thus

$$\cos\theta = \frac{\mathbf{x} \cdot \mathbf{y}}{\|\mathbf{x}\| \, \|\mathbf{y}\|} \qquad \text{for } \mathbf{x},\, \mathbf{y} \neq \mathbf{0},$$

which defines θ uniquely in the interval $[0, \pi]$.

Note in particular that if $\mathbf{x}$, $\mathbf{y} \neq \mathbf{0}$, and we have equality in *Cauchy-Schwarz's inequality*, then $\cos\theta = \pm 1$, so we have either $\theta = 0$ or $\theta = \pi$. In either cases $\mathbf{x}$ and $\mathbf{y}$ are proportional. When $\mathbf{x} = \mathbf{0}$ or $\mathbf{y} = \mathbf{0}$ this statement is of course trivial.

1.3 The vector product

The three-dimensional case $\mathbb{R}^3$ has through centuries been thoroughly studied, because it models the daily space which we live in. It was very early realized by physicists and mathematicians that it would be quite convenient to introduce yet another product, denoted $\times : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$. It is *in rectangular coordinates* defined by

$$\begin{aligned} \mathbf{x} \times \mathbf{y} &= (x_1, x_2, x_3) \times (y_1, y_2, y_3) \\ &= (x_2y_3 - x_3y_2, x_3y_1 - x_1y_3, x_1y_2 - y_2x_1) \end{aligned} \tag{1.1}$$

and it works only in $\mathbb{R}^3$!

If the reader is familiar with how to calculate (3×3)-determinants, then (1.1) can also formally be written in the following way,

$$\mathbf{x} \times \mathbf{y} = \begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \end{vmatrix}, \tag{1.2}$$

where $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ form an orthonormal basis, and (x_1, x_2, x_3), $(y_1, y_2, y_3) \in \mathbb{R}^3$ are the coordinates of $\mathbf{x}$, $\mathbf{y}$, resp., expanded with respect to this basis.

It is easy to remember the structure of this determinant. We put the coordinates of the first factor in the first row, the coordinates of the second factor in the second row, and the three basis vectors in the third row.

By using *Linear Algebra* we immediately get the following results:

1) When $\mathbf{x}$ and $\mathbf{y}$ are interchanged, then the first two rows in the determinant are interchanged, so the determinant changes its sign, and we obtain that

   $$\mathbf{y} \times \mathbf{x} = -\mathbf{x} \times \mathbf{y}.$$

   This means that the vector product is *anticommutative*.

2) It is easy to see that

   $$(\mathbf{x} \times \mathbf{y}) \cdot \mathbf{z} = \begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix}. \tag{1.3}$$

Since the value of the determinant does not change, when we change the rows cyclically, we immediately get the following result,

$$(\mathbf{x}\times\mathbf{y})\cdot\mathbf{z} = \begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix} = \begin{vmatrix} y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \\ x_1 & x_2 & x_3 \end{vmatrix} = (\mathbf{y}\times\mathbf{z})\cdot\mathbf{x} = \mathbf{x}\cdot(\mathbf{y}\times\mathbf{z}),$$

which shows that we can interchange the two products if we only keep the order of the vectors $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$. Hence,

$$(\mathbf{x}\times\mathbf{y})\cdot\mathbf{z} = \mathbf{x}\cdot(\mathbf{y}\times\mathbf{z}).$$

3) By choosing $\mathbf{z} = \mathbf{x}$, or $\mathbf{z} = \mathbf{y}$ it also follows from (1.3) that

$$(\mathbf{x}\times\mathbf{y})\cdot\mathbf{x} = 0 \qquad \text{and} \qquad (\mathbf{x}\times\mathbf{y})\cdot\mathbf{y} = 0.$$

This means that $(\mathbf{x}\times\mathbf{y})$ is perpendicular to both $\mathbf{x}$ and $\mathbf{y}$, and since all vectors lie in $\mathbb{R}^3$, the vector $(\mathbf{x}\times\mathbf{y})$ is either $\mathbf{0}$ or normal to the plane spanned by $\mathbf{x}$ and $\mathbf{y}$.

4) The products $\cdot$ and $\times$ are actually geometrically connected with the "abstract" *Euclidean space* E^3, which means that they are independent of our specific choice of orthonormal basis. This means that we can choose the basis, such that $\mathbf{x}$ and $\mathbf{y}$ lie in the plane spanned by $\mathbf{e}_1$ and $\mathbf{e}_2$, which means that

$$\mathbf{x} = (x_1, x_2, 0) \qquad \text{and} \qquad \mathbf{y} = (y_1, y_2, 0)\,.$$

Then we get from (1.1) that

$$(\mathbf{x}\times\mathbf{y}) = (0, 0, x_1y_2 - y_1x_2) = \left(0, 0, \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix}\right),$$

and it is well-known that the absolute value of the third coordinate, $|x_1y_2 - y_1x_2|$ is the area of the parallelogram defined by the vectors $\mathbf{x}$ and $\mathbf{y}$.

When we look closer at the sign of $x_1y_2 - y_1x_2$, it follows that when $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{x}\times\mathbf{y}$ are all $\neq \mathbf{0}$, then $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{x}\times\mathbf{y}$ in this order defines a right hand system of vectors. This means that if $\mathbf{x}$ is directed along your right thumb, and $\mathbf{y}$ along your right forefinger, then $\mathbf{x}$, $\mathbf{y}$ must point along your right middle finger. This is also a way to find out the direction, in which $\mathbf{x}\times\mathbf{y}$ is pointing.

The length of $\mathbf{x}\times\mathbf{y}$ is as noted above equal to the *area* of the parallelogram, which is spanned by $\mathbf{x}$ and $\mathbf{y}$.

5) When we combine 3) and 4) above it follows that $|(\mathbf{x}\times\mathbf{y})\cdot\mathbf{z}|$ is the volume of the parallelepipedum spanned by the three vectors $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$.

6) Finally, we shall consider the *double vector product* $\mathbf{x}\times(\mathbf{y}\times\mathbf{z})$, which by 3) must be orthogonal to both $\mathbf{x}$ and $\mathbf{y}\times\mathbf{z}$. It must therefore in particular lie in the plane spanned by $\mathbf{y}$ and $\mathbf{z}$, so there are real constants α and β, such that

$$\mathbf{x}\times(\mathbf{y}\times\mathbf{z}) = \alpha\mathbf{y} + \beta\mathbf{z}.$$

This is orthogonal to $\mathbf{x}$, so

$$0 = \mathbf{x} \cdot (\alpha \mathbf{y} + \beta \mathbf{z}) = \alpha(\mathbf{x} \cdot \mathbf{y}) + \beta(\mathbf{x} \cdot \mathbf{z}).$$

This is only possible, if there exists a real constant λ, such that

$$\alpha = \lambda(\mathbf{x} \cdot \mathbf{z}) \qquad \text{and} \qquad \beta = -\lambda(\mathbf{x} \cdot \mathbf{y}).$$

Finally, by insertion,

$$\mathbf{x} \times (\mathbf{y} \times \mathbf{z}) = \lambda\{(\mathbf{x} \cdot z)\mathbf{y} - (\mathbf{x} \cdot \mathbf{y})\mathbf{z}\}.$$

This shows that $\mathbf{x} \times (\mathbf{y} \times \mathbf{z})$ and $(\mathbf{x} \cdot z)\mathbf{y} - (\mathbf{x} \cdot \mathbf{y})\mathbf{z}$ are proportional. Then use the coordinates of $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$ with respect to the orthonormal basis $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ and prove that $\lambda = 1$. (Left to the reader.) It follows that

$$\mathbf{x} \times (\mathbf{y} \times \mathbf{z}) = (\mathbf{x} \cdot z)\mathbf{y} - (\mathbf{x} \cdot \mathbf{y})\mathbf{z}.$$

These results on the vector product in $\mathbb{R}^3$ will later be important in our treatment of e.g. integration in $\mathbb{R}^3$.

1.4 The most commonly used coordinate systems

When we are given the Euclidean space E^n and want to describe it by coordinates in $\mathbb{R}^n$, it is obvious that the coordinate system can be chosen in many ways. We shall always try to choose the coordinate system in such a way that the calculations become as easy as possible. This is of course a very vague statement, which does not help the reader, so we here list the most commonly used coordinate systems. Concerning the choice of which one, the reader should be guided by e.g. the geometry of the domain, or in case of integration, of the structure of the integrand.

1) The *rectangular coordinate system* in $\mathbb{R}^n$, $n \in \mathbb{N}$ arbitrarily chosen. This is the most obvious coordinate system to start with. As already mentioned previously, its basis is given by the vectors

$$\mathbf{e}_1 = (1,0,\dots,0), \; \dots \;, \mathbf{e}_n = (0,\dots,0,1),$$

in general,

$$\mathbf{e}_j = (\delta_{1j}, \delta_{2j}, \dots, \delta_{nj}),$$

where δ_{ij} is the *Kronecker symbol*, defined by

$$\delta_{ij} = \begin{cases} 1 & \text{for } i = j, \\ 0 & \text{for } i \neq j. \end{cases}$$

The domain in the Euclidean space E^n is congruent with the corresponding coordinate domain in $\mathbb{R}^n$, and one hardly notices the difference.

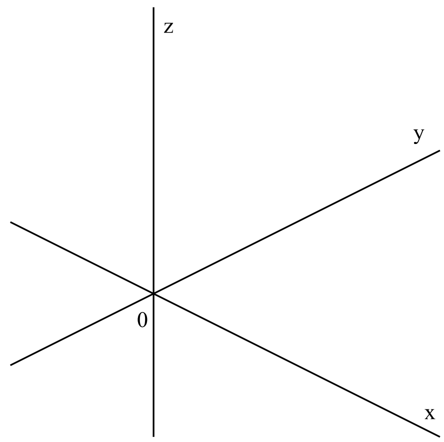


Figure 1.3: The usual way to draw the rectangular coordinate system in $\mathbb{R}^3$.

If $\mathbf{x} = (x_1, \dots, x_n)$ and $\mathbf{y} = (y_1, \dots, y_n)$ with respect to the basis above, then the *inner product* of $\mathbf{x}$ and $\mathbf{y}$ is defined by

$$\mathbf{x} \cdot \mathbf{y} = x_1 y_1 + \cdots + x_n y_n = \sum_{j=1}^{n} x_j y_j .$$

In the important special case of $\mathbb{R}^3$ we also define the *vector product* by

$$\begin{aligned}\mathbf{x} \times \mathbf{y} &= (x_1, x_2, x_3) \times (y_1, y_2, y_3) \\ &= (x_2y_3 - x_3y_2, x_3y_1 - x_1y_3, x_1y_2 - y_2x_1)\,.\end{aligned}$$

The rectangular system is well designed for linear problems, e.g. rectilinear motions. In the case of integration, the domain of integration should be limited by straight lines. If this condition is not satisfied, one may by the following reductions end up with almost incalculable integrals.

2) *Polar coordinates* in the plane. These can only be used in dimension 2.

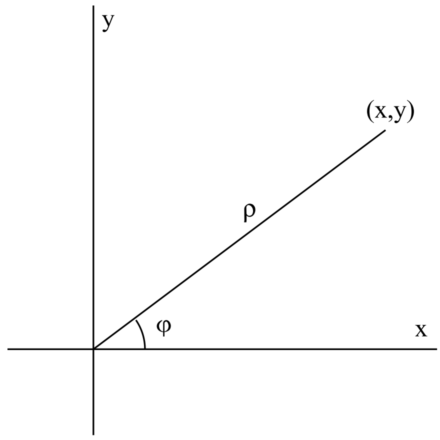


Figure 1.4: The coordinate system in polar coordinates

Assume that the point P in the Euclidean space E^2 has the rectangular coordinates (x, y), cf. Figure 1.4. The distance ϱ from origo $O\colon (0,0)$ to $P\colon (x,y)$ is by *Pythagoras's theorem* given by

$$\varrho = \sqrt{x^2 + y^2}.$$

It then follows by high school trigonometry that

$$x = \varrho \cos\varphi \qquad \text{and} \qquad y = \varrho \sin\varphi,$$

where φ is the angle measured from the X-axis in the positive sense of the plane.

If $\varrho = 0$, i.e. $P = O$, so we are at origo, then the angle φ is undetermined. Every $\varphi \in \mathbb{R}$ will do in this case.

If $x \neq 0$, then

$$\tan\varphi = \frac{y}{x},$$

so we may choose

$$\varphi = \begin{cases} \operatorname{Arctan}\left(\dfrac{y}{x}\right) & \text{for } x > 0, \\ \operatorname{Arctan}\left(\dfrac{y}{x}\right) + \pi & \text{for } x < 0. \end{cases}$$

If instead $y \neq 0$, then

$$\cot \varphi = \frac{x}{y},$$

so we may choose

$$\varphi = \begin{cases} \text{Arccot}\left(\dfrac{x}{y}\right) & \text{for } y > 0, \\[2ex] \text{Arccot}\left(\dfrac{x}{y}\right) + \pi & \text{for } y < 0. \end{cases}$$

Note, however, that when $\varrho > 0$, the angle is only specified modulo 2π, so we can always add a multiple of 2π to the angle φ without changing x and y.

Summing up, we get the following correspondence between rectangular coordinates $(x, y) \in \mathbb{R}^2 \setminus \{(0,0)\}$ and polar coordinates (ϱ, φ), where $\varrho > 0$, and φ belongs to some half open interval of length 2π,

$$(1.4) \qquad \begin{cases} x = \varrho \cos \varphi, \qquad y = \varrho \sin \varphi, \\[1ex] \varrho = \sqrt{x^2 + y^2} \\[1ex] \tan \varphi = \dfrac{y}{x} \text{ for } x \neq 0, \quad \text{and} \quad \cot \varphi = \dfrac{x}{y} \text{ for } y \neq 0. \end{cases}$$

Experience shows that students are not too happy with the polar coordinates, when they first meet them. This is probably due to the fact that the angle φ is not uniquely determined, in general only modulo 2π. Nevertheless, they are very useful, and when circular motions are considered, they are better than rectangular coordinates, so they are very important in e.g. Mechanics. We shall here illustrate this by the simplest possible example. The *unit circle* is explicitly described in polar coordinates by the simple equation

$$\varrho = 1.$$

This unit circle is implicitly described in rectangular coordinates by

$$\sqrt{x^2+y^2} = 1, \qquad \text{or} \qquad x^2+y^2=1,$$

so by solving this equation with respect to y we get the more messy explicit expression,

$$y = \begin{cases} \sqrt{1-x^2} & \text{for } x \in [-1,1] \text{ ¡nd } y \geq 0, \\ -\sqrt{1-x^2} & \text{for } x \in [-1,1] \text{ and } y \leq 0. \end{cases}$$

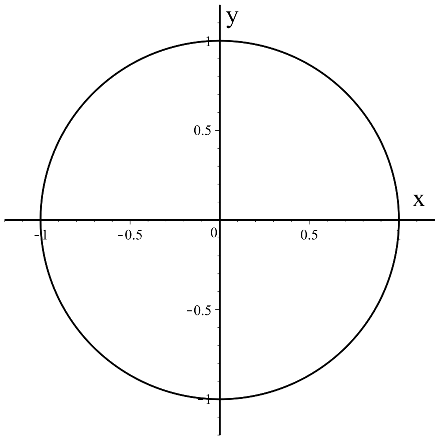


Figure 1.5: The unit circle in E^2.

When we compare Figure 1.5 and Figure 1.6 it is obvious that although the two sets are in correspondence, they do not look like each other. This means that in polar coordinates the geometry is quite different in the Euclidean plane E^2 and the coordinate plane $\mathbb{R}^2$. Therefore, they must not be confused!

The polar coordinates are used, whenever we are dealing with circular motion or domains, which are discs. Also, when the integrand contains expressions which are functions in $\sqrt{x^2+y^2}$ in the rectangular coordinates, one should rewrite the problem in polar coordinates, because then we may get rid of at least some of these square roots. The drawback is of course that the angle φ in (1.4) is only specified modulo 2π, so we must choose an half-open φ-interval of length 2π, e.g. $]-\pi,\pi]]$, or $]\,0,2\pi]$, or more general, $]\,\alpha,\alpha+2\pi]$ for some constant α, depending on the geometry of the domain under consideration.

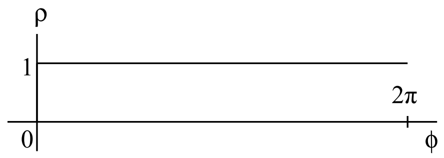


Figure 1.6: The parameter set in polar coordinates of the unit circle in $\mathbb{R}^2$.

We note that the description of the *inner product* in polar coordinates is not an easy job, and we shall not derive it.

3) *Semi-polar coordinates* in E^3. These can only be applied in the Euclidean space E^3. Also in this case, the corresponding domain in the coordinate space $\mathbb{R}^3$ is distorted compared with the original set in E^3.

Given the usual rectangular basis $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ in E^n, the idea is to apply the polar coordinates in the plane spanned by $\mathbf{e}_1$ and $\mathbf{e}_2$, and keep the rectangular coordinate along the $\mathbf{e}_3$-axis.

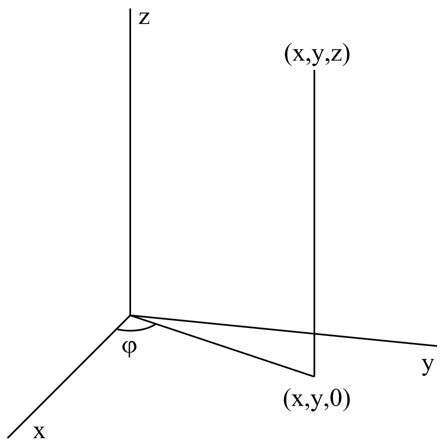


Figure 1.7: The geometry of the definition of the semi-polar coordinates in $\mathbb{R}^3$.

It follows from the above that

$$\begin{cases} x = \varrho\cos\varphi, \quad y = \varrho\sin\varphi, \quad z = z, \\ \varrho = \sqrt{x^2+y^2}, \\ \tan\varphi = \dfrac{y}{x} \text{ for } x \neq 0 \qquad \text{and} \qquad \cot\varphi = \dfrac{x}{y} \text{ for } y \neq 0. \end{cases}$$

If $(x,y) \neq (0,0)$, then φ is determined modulo 2π. On the z-axis, where $(x,y) = (0,0)$, the angle φ is undetermined, and any $\varphi \in \mathbb{R}$ can be used.

When the angle φ is kept fixed, while $\varrho \geq 0$ and $z \in \mathbb{R}$ vary, we describe a half plane, which we call the *meridian half plane*. In such a meridian half plane (ϱ, z) are ordinary rectangular coordinates.

If instead $\varrho > 0$ is kept fixed, while φ and z vary, we describe a *cylindric surface* with the z-axis as its axis of rotation. For that reason the semi-polar coordinates are also called *cylindric coordinates.*

The semi-polar coordinates are typically used, when we are dealing with rotational bodies in E^3, or, if a rectangular coordinate system in $\mathbb{R}^3$ e.g. the variables (x,y) only appear in the combined form $\sqrt{x^2+y^2}$.

4) *Spherical coordinates in* $\mathbb{R}^3$. It was noted above in 3), semi-polar coordinates, that for fixed φ we describe the *meridian half plane* in the rectangular coordinates (ϱ, z), $\varrho \geq 0$ and $z \in \mathbb{R}$.

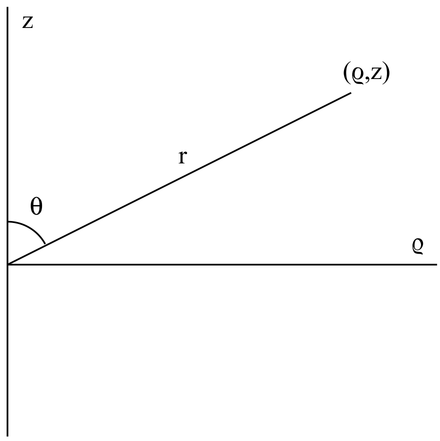


Figure 1.8: The meridian half plane for fixed ϱ.

Let $r = \sqrt{\varrho^2 + z^2}$ denote the Euclidean distance between $(0,0)$ and (ϱ, z), and let $\vartheta \in [0,\pi]$ denote the angle *positive from the z-axis towards the vector of coordinates* (ϱ, z), cf. Figure 1.8. Then clearly,

$$z = r\cos\theta \quad \text{and} \quad \varrho = r\sin\theta, \qquad \text{for } \theta \in [0,\pi] \text{ and } r = \sqrt{z^2+\varrho^2}.$$

Since we already have

$$x = \varrho \cos\varphi \quad \text{and} \quad y = \varrho \sin\varphi$$

for $\varphi \in I$, where I is some interval of length 2π, where we for convenience here put $I = [0, 2\pi]$, we get by insertion

$$(1.5) \qquad \begin{cases} x = r \sin\theta \cos\varphi, \\ y = r \sin\theta \sin\varphi, \qquad \theta \in [0,\pi], \quad \varphi \in 0, 2\pi, \quad r \in [0, +\infty[. \\ z = r \cos\theta, \end{cases}$$

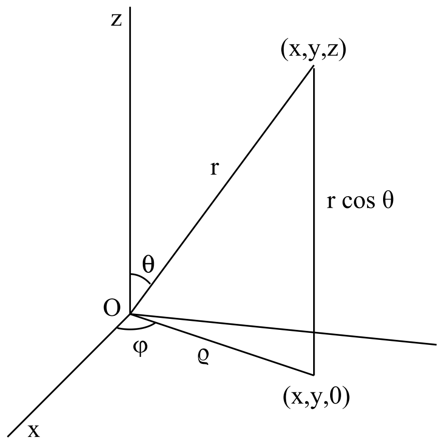


Figure 1.9: The geometry of the definition of the spherical coordinates in $\mathbb{R}^3$.

We call (r, θ, φ) the *spherical coordinates* in $\mathbb{R}^3$. If $r > 0$ is kept fixed, then (1.5) describes a *sphere* of radius r.

If we let $r =$ the radius of the Earth and specify $\varphi \in [-\pi, \pi] \sim [-180°, 180°]$, and define $\vartheta := \frac{\pi}{2} - \theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \sim [-90°, 90°]$, then φ is the degree of longitude, and ϑ is the degree of latitude. It is well-known that these two spherical coordinates with success have been applied for centuries in Geography and Astronomy.

Spherical coordinates are in particular applied, when we are dealing with a sphere, or when the rectangular coordinates (x, y, z) also appear in the form $\sqrt{x^2 + y^2 + z^2}$.

If instead $\theta \in]0, \pi[$ is kept fixed, then (1.5) describes a *cone*,, and – as already seen above – when φ is a constant, then (1.5) describes a *meridian half plane*.

5) It is possible to extend this construction of spherical coordinates to $\mathbb{R}^n$ for $n > 3$. In fact, if (x, y, z, t) are the rectangular coordinates in $\mathbb{R}^4$, then we can start by using the spherical coordinates above in the variables (x, y, z). When φ and θ are kept fixed, we again obtain a *meridian half plane*. This time the rectangular coordinates are (r, t). Let (r, t) be a vector in this half plane, and

define $R = \sqrt{r^2 + t^2}$ and $\vartheta \in [0, \pi]$ as the angle between the t-axis and the vector (r, t), *measured from the* t*-axis*. Then,

$$t = R \cos\vartheta \quad \text{and} \quad r = R \sin\vartheta, \qquad \vartheta \in [0, \pi] \text{ and } R = \sqrt{r^2 + t^2} = \sqrt{x^2 + y^2 + z^2 + t^2},$$

and we obtain by insertion the rectangular coordinates $(x, y, z, t) \in \mathbb{R}^4$ expressed in the hyper-spherical coordinates $(R, \varphi, \theta, \vartheta)$.

Continue this construction to higher dimensions, whenever needed. Note, however, that this construction will not be used in this series of books.

Remark 1.2 The author has actually used this construction in an analysis of solid balls in E^n. These have an unexpected geometry, when $n > 3$, and one cannot just conclude that "they behave as the solid balls in the usual Euclidean space E^3". One example is the following: Choose any small ε, $\delta \in]0, 1[$, and let B_n denote the unit ball in $\mathbb{R}^n$ of n-dimensional volume $|B_n|$. Let A_n denote the subset of B_n, which is obtained by restricting e.g. the x_1-coordinate, so

$$A_n := \left\{ (x_1, \dots, x_n) \mid x_1^2 + \cdots + x_n^2 \leq 1 \text{ and } -\varepsilon \leq x_1 \leq \varepsilon \right\}$$

with its n-dimensional volume denoted by $|A_n|$

Then there exists an $N \in \mathbb{N}$, such that for all $n \geq N$ most of the volume of B_n lies the slab A_n, or more precisely,

$$|A_n| \geq (1 - \delta) \, |B_n| \, . \qquad \diamond$$

1.5 Point sets in space

We shall in this section introduce the most necessary of what mathematicians call *Topology.* We shall use the Euclidean space E^n as our model space, and whenever necessary we shall choose a rectangular coordinate system and use the equivalent coordinate space $\mathbb{R}^n$. This means that at least in E^2 and E^3 it should be possible to visualize the sets. In particular, the sets are easily drawn in the Euclidean plane E^2.

The formal definition of a set A in the Euclidean space E^n is given by

$$A = \{\mathbf{x} \in E^n \mid p(\mathbf{x})\},$$

where p denotes a property, which is satisfied for all $\mathbf{x} \in A$. In plain words this is expressed as "A is the set of $\mathbf{x} \in \mathbb{R}^n$, for which property $p(\mathbf{x})$ is true".

If $A \subseteq E^n$ allows some symmetry, it is convenient to introduce the axes, such that these are in harmony with this symmetry. Such a choice will usually have the effect that the corresponding coordinate set $\tilde{A} \subset \mathbb{R}^n$ becomes simple.

In the Euclidean plane $E^2 \sim \mathbb{R}^2$ it is easy to draw the most important sets for the applications. This does not mean that all plane sets can be reasonably drawn. For instance, we have problems in *drawing* the set

$$\{(x, y) \mid x \in [0, 1] \cap \mathbb{Q},\, y \in [0, 1] \cap \mathbb{Q}\},$$

which is the set of all points in the square $[0, 1]^2$ of rational coordinates. However, we shall in the following mostly avoid such pathological sets, so in general they are not at problem.

We shall, whenever necessary or convenient, use the following conventions on drawings in $E^2 \sim \mathbb{R}^2$: What is included in a set is marked by

1) a hatching (2-dimensional),

2) a full-drawn line (1-dimensional),

3) a small circle or just a point (0-dimensional).

In particular, a dot-and-dash line is only limiting a hatched set, and the points on such a line do not belong to the set. Cf. Figure 1.10 to the left.

Note that if a closed curve without double points surrounds a set which together with the curve is totally included in the set, we do not hatch the set inside the closed curve. Cf. Figure 1.10 to the right.

1.5.1 Interior, exterior and boundary of a set

Given a Euclidean space E^n with its usual Euclidean distance, which in rectangular coordinates is given by

$$\text{dist}_n(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\| = \sqrt{\sum_{j=1}^{n} (x_j - y_j)^2}.$$

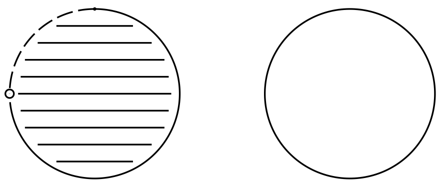


Figure 1.10: Visualization of two discs. On the left disc part of the boundary is not included, so we are forced to hatch the interior. To the right, the full boundary is included, so there is no need to hatch the interior.

where $(x_1, \dots, x_n)$ and $(y_1, \dots, y_n) \in \mathbb{R}^n$ are the coordinates of $\mathbf{x}$ and $\mathbf{y}$, resp.. Then it is possible to introduce solid balls in $E^n \sim \mathbb{R}^n$ as the points of distance smaller than (or equal to) a given radius from a given centre $\mathbf{x}_0$.

The *open ball* $B(\mathbf{x}_0, r)$ of radius $r > 0$ and centre $\mathbf{x}_0 \in E_n \sim \mathbb{R}^n$ is given by

$$B(\mathbf{x}_0, r) := \{\mathbf{x} \in E^n \mid \text{dist}_n(\mathbf{x}, \mathbf{x}_0) < r\} = \{\mathbf{x} \in \mathbb{R}^n \mid \|\mathbf{x} - \mathbf{x}_0 < r\}.$$

The *closed ball* $B[\mathbf{x}_0, r]$ of radius $r > 0$ and centre $\mathbf{x}_0 \in E_n \sim \mathbb{R}^n$ is given by

$$B[\mathbf{x}_0, r] := \{\mathbf{x} \in E^n \mid \text{dist}_n(\mathbf{x}, \mathbf{x}_0) \le r\} = \{\mathbf{x} \in \mathbb{R}^n \mid \|\mathbf{x} - \mathbf{x}_0 \le r\}.$$

In the latter case we may allow $r = 0$, in which case the closed ball of centre $\mathbf{x}_0$ and radius 0 is just the centre, $B[\mathbf{x}_0, 0] = \{\mathbf{x}_0\}$. These balls are fundamental in describing more general objects.

1) If $\mathbf{x}_1 \in A$, and there exists an $r > 0$, such that $B(\mathbf{x}_1, r) \subseteq A$, then we call $\mathbf{x}_1$ an *interior point* of A. The set of all interior points of A is called the *interior* of A, and it is denoted by A°.

2) If $\mathbf{x}_2 \notin A$, and there exists an $r > 0$, such that $B(\mathbf{x}_2, r) \cap A = \emptyset$, then we call $\mathbf{x}_2$ an *exterior point* of A. The set of all exterior points of A is called the *exterior* of A. If $\complement A := E^2 \setminus A$ denotes the complementary set of A, then the exterior of A is the interior of the complement of A, i.e. the set $(\complement A)^\circ$. The point $\mathbf{x}_2$ on Figure 1.11 is exterior.

3) The remaining part $E^n \setminus \left\{A^\circ \cup (\complement A)^\circ\right\}$ is called the *boundary* of A. It is denoted by ∂A. Due to the "symmetry" it follows that A and $\complement A$ have the same boundary, so

$$\partial A = \partial(\complement A) = E^n \setminus \left\{A^\circ \cup (\complement A)^\circ\right\}.$$

On Figure 1.11 the points $\mathbf{x}_3 \in A$ and $\mathbf{x}_4 \notin A$ are both boundary points.

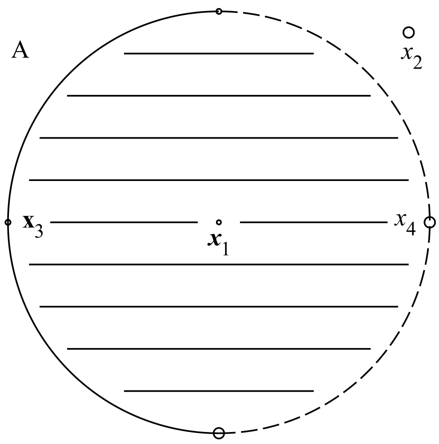


Figure 1.11: A set $A \subseteq E^2$ divides E^2 into three sets, 1) the *interior* S° of A, 2) the *exterior* $(\complement A)^\circ$ of A, and 3) the *boundary* ∂A of A, which is the remaining set $E^2 \setminus \left\{A^\circ \cup (\complement A)^\circ\right\}$.

A boundary point $\mathbf{x} \in \partial A$ is characterized in the following way: For every $r > 0$, the open ball $B(\mathbf{x}, r)$ contains points from both the interior A° and the exterior $(\complement A)^\circ$, i.e.

$$B(\mathbf{x}, \mathbf{r}) \cap A^\circ \neq \emptyset \qquad \text{and} \qquad B(\mathbf{x}, \mathbf{r}) \cap (\complement A)^\circ \neq \emptyset.$$

Note that the boundary point $\mathbf{x} \in \partial A$ may or may not be a point in A.

The *union* of the *interior* and the *boundary* is called the *closure* of A. It is denoted by $\overline{A}$, hence

$$\overline{A} = A^\circ \cup \partial A = A \cup \partial A.$$

A set A is called *open*, if it does not contain any boundary point, i.e. if

$$A \cap \partial A = \emptyset, \qquad \text{or equivalently,} \qquad A = A^\circ.$$

Summing up we see that

A is open, if and only if $A \cap \partial A = \emptyset$,

and

A is closed, if and only if $\partial A \subseteq A$.

A set A is called a *neighbourhood* of $\mathbf{x} \in A$, if there exists an $r > 0$, such that $B(\mathbf{x}, r) \subseteq A$. In particular, when $A = A^\circ$ is open, then A is a neighbourhood of all its points.

A boundary point P of A is called an *isolated point*, if there exists an $r > 0$, such that $B(P, r) \cap A = \{P\}$, i.e. if P is the only point from A in a neighbourhood of P.

In the rectangular coordinate space $\mathbb{R}^n$ we have already used the distance

$$\text{dist}_n(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}_n\|_n := \sqrt{\sum_{j=1}^{n} (x_j - y_j)^2}.$$

The open/closed balls are written

$$B(\mathbf{x},r)=\left\{\mathbf{y}\in\mathbb{R}^n \mid \sum_{j=1}^{n}\left(x_j-y_j\right)^2<r^2\right\} \text{ and } B[\mathbf{x},r]=\left\{\mathbf{y}\in\mathbb{R}^n \mid \sum_{j=1}^{n}\left(x_j-y_j\right)^2\le r^2\right\}.$$

The importance of these new topological concepts will be demonstrated in connection with limits and continuity in the next volume of this series.

1.5.2 Starshaped and convex sets

Concerning the shapes of the sets under consideration the situation is very simple in the 1-dimensional case of E^1, where it usually suffices only to consider intervals. However, even in the two-dimensional case of E^2 concerning the shapes of sets, the situation becomes far more complicated, and it is not always obvious which type of sets we should look at.

Clearly, the n-dimensional intervals

$$I_1\times I_2\times\cdots\times I_n:=\left\{(x_1,x_2,\dots,x_n)\mid x_1\in I_1,\,x_2\in I_2,\,\dots,\,x_n\in I_n\right\}$$

are obvious candidates, where each I_j is of one of the following four types,

$$I_j=\,]a_j,b_j[\,,\;]a_j,b_j)\,,\;[a_j,b_j[\,,\;[a_j,b_j]\,.$$

The balls defined previously are also often used sets.

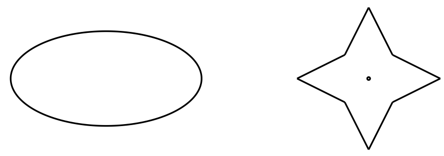


Figure 1.12: A convex and a starshaped set.

We may, however, also be interested in sets having some weaker geometrical properties.

A set $A \subseteq E^n$ is called *starshaped* with respect to a point $\mathbf{x}_0 \in A$, if for every $\mathbf{x} \in A$, the straight *line segment* $[\mathbf{x}_0, \mathbf{x}]$ from $\mathbf{x}_0$ to $\mathbf{x}$ lies totally in A. The set to the right of Figure 1.12 illustrates why the set is called *starshaped*. Every line segment from the centre to any other point in A lies in A. However, if we choose two points from adjacent arms of the star, it is obvious that the line segment between them is not totally contained in A, so we cannot in general choose the point $\mathbf{x}_0$ arbitrarily.

If the line segment between any two points of A also lies in A, then we say that this (clearly) starshaped set is *convex*. The set to the left of Figure 1.12 is convex.

Finally, we say that a set $A \subset E^n$ is *bounded*, if there exists an $R > 0$, such that $A \subseteq B(\mathbf{0}, R)$, i.e. A is contained in a ball of finite radius. Any centre $\mathbf{x}_0$ may of course be used here instead.

1.5.3 Catalogue of frequently used point sets in the plane and the space

We shall in this section give a summary of frequently used point sets in $E^2 \sim \mathbb{R}^2$ and $E^3 \sim \mathbb{R}^3$.

1) If I, $J \subset \mathbb{R}$ are ordinary one-dimensional intervals, we define their *product set* by

$$I \times J := \left\{(x, y) \in \mathbb{R}^2 \mid x \in I,\, y \in J\right\}.$$

If $J = I$, we often write I^2 instead of $I \times I$.

If I and J are bounded, then $I \times J$ is a *rectangle*. In particular, I^2 is a *square*, if I is a bounded interval.

Let I be a bounded interval. Then $I \times \mathbb{R}$ is called a *strip*, and $I \times [a, +\infty[$ and $I \times]-\infty, a]$ are called *half-strips*, cf. Figure 1.13.

The set $\mathbb{R}_+ \times \mathbb{R}_+ = \mathbb{R}_+^2$ is the open first *quadrant*, and $\mathbb{R} \times \mathbb{R}_+$ is the upper *half-plane*, cf. Figure 1.14.

We mention the possibilities of the open right half-plane $\mathbb{R}_+ \times \mathbb{R}$, the open left half-plane $\mathbb{R}_- \times \mathbb{R}$ and the open lower half-plane $\mathbb{R} \times \mathbb{R}_-$ and variants of these.

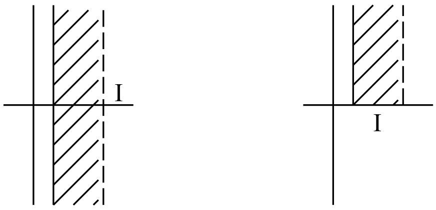


Figure 1.13: A strip and a half-strip.

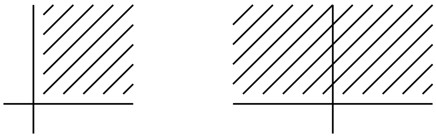


Figure 1.14: The first quadrant and the upper half-plane.

2) Let $A \subset \mathbb{R}^2$ be a bounded plane set, and let $I \subseteq \mathbb{R}$ be an interval. We define a *cylinder* in $\mathbb{R}^3$ by

$$A \times I := \left\{(x, y, z) \in \mathbb{R}^3 \mid (x, y) \in A,\ z \in J\right\},$$

cf. Figure 1.15.

When the interval I is bounded, then the length $|I|$ of I is called the *height* of the cylinder.

If A is a polygon, we also call the cylinder a *prism*. Special cases are a *parallelepipedum*, where A is a rectangle, and a *cube*,, where A is a square.

3) Assume that the coordinate system has been chosen, such that the coordinate description of the set A only contains the first two coordinates (x, y) in the form $x^2 + y^2$. Then A is rotational symmetric with respect to the z-axis, and the three-dimensional set A can be fully described by

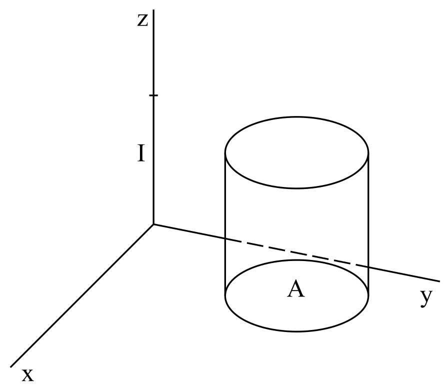


Figure 1.15: A (bounded) cylinder.

one (two-dimensional) *meridian half-plane*, in which we can use either the rectangular coordinates (ϱ, z) or the polar coordinates (r, θ), as described earlier. Then the point set A can be described as a *body of revolution*, which is obtained by revolving the so-called *meridian section*, cf. Figure 1.16

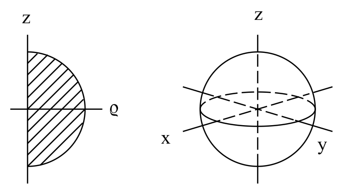


Figure 1.16: The meridian section to the left is a half disc in the right (ϱ, z)-half-plane. The body of revolution is a solid ball

4) A *torus* is the body of revolution, which is obtained by revolving a disc with respect to a line, which does not meet the disc. If the coordinate system is placed conveniently with the z-axis as the axis of revolution, then the disc in the meridian half-plane (i.e. the meridian section) is described by the inequality

$$z^2 + (\varrho - a)^2 \le b^2, \qquad \text{where } 0 < b < a,$$

cf. Figure 1.21.

Since $\varrho = \sqrt{x^2+y^2}$, the *torus* T is then described in $\mathbb{R}^3$ by

$$T = \left\{ (x,y,z) \in \mathbb{R}^3 \mid \left(\sqrt{x^2+y^2} - a \right)^2 + z^2 \le b^2 \right\},$$

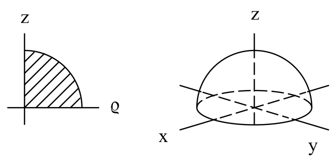


Figure 1.17: The meridian section to the left is a quarter of a disc in the right (ϱ, z)-half-plane. The body of revolution is a solid half ball

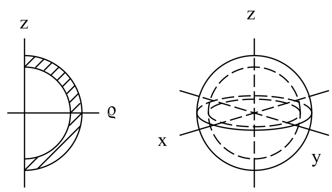


Figure 1.18: The meridian section to the left is the half of a solid ring in the right (ϱ, z)-half-plane. The body of revolution is a solid shell of a ball

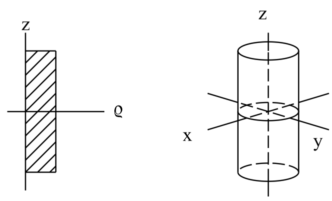


Figure 1.19: The meridian section to the left is a rectangle with one of its sides on the z-axis. The body of revolution is the cylinder to the right.

where $0 < b < a$.

5) Consider a (solid) cone of revolution K of height $h > 0$ and radius a of its basis. If the coordinate

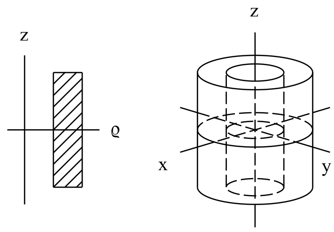


Figure 1.20: The meridian section to the left is a rectangle without one of its sides on the z-axis. The body of revolution to the right is the shell of a cylinder.

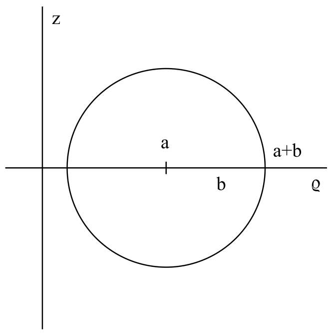


Figure 1.21: The meridian section of a torus.

axes are put as on Figure 1.22, then the cone is described in cylinder coordinates by

$$\frac{z}{h}+\frac{\varrho}{a}\le 1 \qquad \text{and} \qquad z\ge 0.$$

Using that

$$0<z<h\left(1-\frac{\varrho}{a}\right) \qquad \text{and} \qquad \varrho=\sqrt{x^2+y^2}\le a,$$

we obtain the following rectangular coordinate description of the cone K,

$$K=\left\{(x,y,z)\mid x^2+y^2\le a^2,\, 0\le z\le h\left(1-\frac{\sqrt{x^2+y^2}}{a^2}\right)\right\}.$$

If instead we choose the triangle as in Figure 1.23, then the hypothenuse of the triangle has the equation

$$z=\frac{\varrho h}{a}.$$

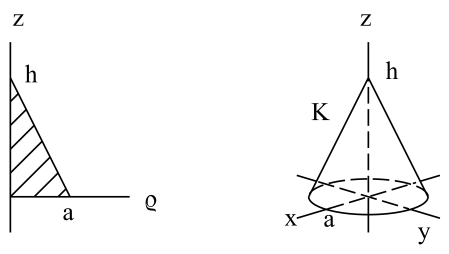


Figure 1.22: A triangle in the meridian half-plane, and the cone K of height h and radius a of its basis, which is the body of revolution of the triangle in the meridian half-plane.

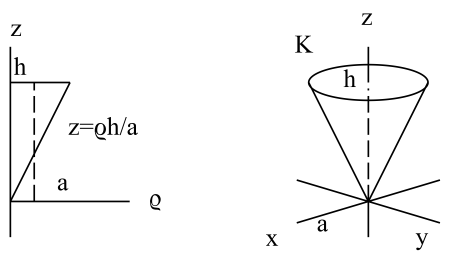


Figure 1.23: A triangle in the meridian half-plane, and the cone K of height h and radius a of its basis, which is the body of revolution of the triangle in the meridian half-plane.

We therefore conclude that the triangle in the meridian half-plane is described by

$$\left\{(\varrho, z) \mid \varrho \geq 0,\, \frac{\varrho h}{a} \leq z \leq h\right\}.$$

Since $\varrho = \sqrt{x^2+y^2} \geq 0$, it follows that the cone K in this case is described in rectangular coordinates by

$$\left\{(x, y, z) \mid x^2+y^2 \leq a^2,\, \frac{h}{a}\sqrt{x^2+y^2} \leq z \leq h\right\}.$$

In particular we see, that the rectangular description contains the ugly looking square root, $\sqrt{x^2+y^2}$, which may obscure the reader's feeling of what is going on.

Note on Figure 1.23 that we fix ϱ (the vertical dashed line) to find the corresponding z-interval. This technique will be used over and over again in this series of books on *Real Functions in Several Variables*.

1.6 Quadratic equations in two or three variables; short theoretical review

1.6.1 Quadratic equations in two variables. Conic sections

The general quadratic equation in two variables is given by

$$(1.6)\quad Ax^2 + By^2 + 2Cxy + 2Dx + 2Ey + F = 0,$$

where A, B, C, D, E, $F \in \mathbb{R}$, and $(A, B, C) \neq (0, 0, 0)$.
If $C \neq 0$, then this equation can also be written in the following way,

$$(x \;\; y)\begin{pmatrix} A & C \\ C & B \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + 2(D \;\; E)\begin{pmatrix} x \\ y \end{pmatrix} + F = 0.$$

If we apply some orthogonal substitution of the form

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} q_{11} & -q_{21} \\ q_{21} & q_{11} \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \qquad q_{11}^2 + q_{21}^2 = 1,$$

then we may obtain by a suitable choice of q_{11} and q_{21} above that this equation is reduced to

$$\lambda_1 \, x_1^2 + \lambda_2 \, y_2^2 + 2(D \;\; E) \begin{pmatrix} q_{11} & -q_{21} \\ q_{21} & q_{11} \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + F = 0,$$

hence,

$$\lambda_1 \, x_1^2 + \lambda_2 \, y_1^2 + 2D_1 \, x_1 + 2E_1 \, y_1 + F = 0,$$

where the term $2C_1 \, x_1 \, y_1$ has disappeared, because we have obtained that $C_1 = 0$ for some suitable choice of (q_{11}, q_{21}), which defines an orthogonal substitution.

We have proved that if we choose a specific orthogonal substitution, then the general quadratic equation (1.6) is reduced to

(1.7) $A \, x^2 + B \, y^2 + 2Dx + 2Ey + F = 0, \qquad$ where $(A, B) \neq (0, 0)$,

and where we for convenience write (x, y) instead of (x_1, y_1).

I. Both coefficients are $\neq 0$

When both $A \neq 0$ and $B \neq 0$, then the reduced equation (1.7) can be written

$$A \left(x + \frac{D}{A} \right)^2 + B \left(y + \frac{E}{B} \right)^2 = \frac{D^2}{A} + \frac{E^2}{B} - F.$$

This equation is simplified, when we introduce the new variables

$$x_1 = x + \frac{D}{A}, \qquad t_1 = y + \frac{E}{B}, \qquad \text{and the constant } K = \frac{D^2}{A} + \frac{E^2}{B} - F.$$

Then the reduced equation becomes

$$A \, x_1^2 + B \, y_1^2 = K.$$

We have here two possibilities: Either $K \neq 0$ or $K = 0$. If $K \neq 0$, then we “norm” the equation by dividing it by K to get

$$\frac{x_1^2}{K/A} + \frac{y_1^2}{K/B} = 1.$$

It is customary to introduce new constants by

$$a = \sqrt{\left| \frac{K}{A} \right|} \qquad \text{and} \qquad b = \sqrt{\left| \frac{K}{B} \right|}.$$

Depending on the signs of A, B and K we then get three possibilities,

Ellipse $\quad \dfrac{x_1^2}{a^2} + \dfrac{y_1^2}{b^2} = 1$

Hyperbola $\quad \dfrac{x_1^2}{a^2} - \dfrac{y_1^2}{b^2} = 1 \qquad \left(\text{and also } -\dfrac{x_1^2}{a^2} + \dfrac{y_1^2}{b^2} = 1 \right)$

Empty set $\quad -\dfrac{x_1^2}{a^2} - \dfrac{y_1^2}{b^2} = 1.$

If instead $K = 0$, then we put

$$a = \sqrt{\frac{1}{|A|}} \qquad \text{and} \qquad b = \sqrt{\frac{1}{|B|}}.$$

Then, depending on the signs of A and B we get the following two possibilities:

A point $\qquad \dfrac{x_1^2}{a^2} + \dfrac{y_1^2}{b^2} = 0,$

Two straight lines $\qquad \dfrac{x_1^2}{a^2} - \dfrac{y_1^2}{b^2} = 0.$

We shall in the following briefly discuss these possibilities. For simplicity we again write (x, y) instead of (x_1, y_1).

The ellipse. The normed equation of the ellipse is given by

$$(1.8) \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \qquad \text{where } a,\, b > 0.$$

In the special case where $a = b$, formula (1.8) describes a *circle* of centre $(0, 0)$ and radius $r = a =$.

In general, (1.8) has the two coordinate axes as *axes of symmetry.* The ellipse cuts the x-axis at the points $A_+ : (a, 0)$ and $A_- : (-a, 0)$, and the y-axis at the points $B_+ : (0.b)$ and $B_- : (0, -b)$. These four points are called the *top point* of the ellipse. The numbers a and b (or more correctly the line segments from $O : (0, 0)$ to $A_+ : (a, 0)$, and from $O : (0, 0)$ to $B_+ : (0, b)$) are called the *semi-axes* of the ellipse. The larger of a and b is called the *major semi-axis*, and the smaller of them is called the *minor semi-axis* of the ellipse. Let us assume in the following that $a > b$. Then we define the *eccentricity* e of the ellipse by

$$e := \sqrt{1 - \frac{b^2}{a^2}}, \qquad 0 < e < 1,$$

where we formally may add $e = 0$ in the limiting case $b = a$, when the ellipse becomes a circle.

The *foci* of the ellipse (in singular: *focus*) are when $a > b$ the points

$$F_+ : (ea, 0) \qquad \text{and} \qquad F_- : (-ea, 0).$$

If $P : (x, y)$ lies on the ellipse, then a small computation shows that

$$\left|\overrightarrow{F_+P}\right| = a - ex \qquad \text{and} \qquad \left|\overrightarrow{F_-P}\right| = a + ex,$$

hence by addition,

$$(1.9) \quad \left|\overrightarrow{F_+P}\right| + \left|\overrightarrow{F_-P}\right| = 2a,$$

i.e. equal twice the major semi-axis. It is possible to prove that a relation like (1.9) only holds for an ellipse of foci F_+ and F_- and the major semi-axis a, where we of course must require that $\left|\overrightarrow{F_-F_+}\right| < 2a$.

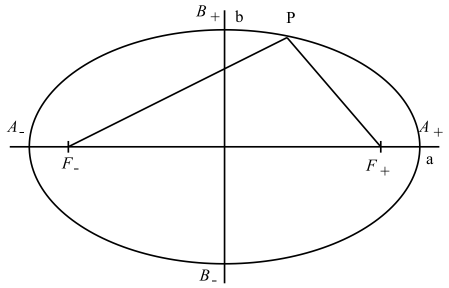


Figure 1.24: An ellipse.

The hyperbola. The normed equation of the hyperbola is for a convenient choice of the variables of the form

$$(1.10) \quad \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

The coordinate axes are the *axes of symmetry*. The hyperbola (1.10) intersects the x-axis at the two *top points* $A_+ : (a, 0)$ and $A_- : (-a, 0)$, and it has no point in common with the y-axis. The positive numbers a and b are called the *semi-axes* of the hyperbola.

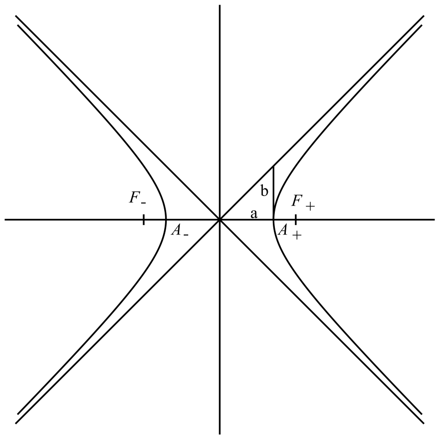


Figure 1.25: An hyperbola.

The lines $y = \pm \frac{b}{a}\, x$ are the *asymptotes* of the hyperbola. They are found by replacing 1 on the right hand side of (1.10) by 0 and then solving the equation. It is obvious that b is the length of the line segment perpendicular to the x-axis from A to the asymptote in the first quadrant.

The *eccentricity* e of the hyperbola is defined by

$$e := \sqrt{1 + \frac{b^2}{a^2}}, \qquad e > 1.$$

The *foci* are defined by their coordinates, i.e.

$$F_+ : (ea, 0) \qquad \text{and} \qquad F_- : (-ea, 0).$$

If $P : (x, y)$ is a point on the hyperbola in the right half plane (i.e. closest to the focus F_+), then one likewise proves that

$$\left|\overrightarrow{F_+P}\right| = ex - a \qquad \text{and} \qquad \left|\overrightarrow{F_-P}\right| = ex + a,$$

hence by subtraction,

$$\left|\overrightarrow{F_-P}\right| - \left|\overrightarrow{F_+P}\right| = 2a,$$

so in general for $P : (x, y)$ just a point on the hyperbola,

$$(1.11) \quad \left|\left|\overrightarrow{F_+P}\right| - \left|\overrightarrow{F_-P}\right|\right| = 2a.$$

It is possible to prove that if F_+ and F_- are two fixed points in the plane, then all points P, which satisfy (1.11), describe an hyperbola of foci F_+ and F_- and a one of the semi-axes. The other one, b, is then obtained from the equation

$$\left|\overrightarrow{F_-F_+}\right| = 2ae = 2a\sqrt{1 + \frac{b^2}{a^2}} = 2\sqrt{a^2 + b^2},$$

hence

$$b=\frac{1}{2}\sqrt{\left|\overrightarrow{F_-F_+}\right|^2-4a^2}.$$

A point. The general equation is here

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=0,$$

where $O:\ (0,0)$ is the only solution.

Two lines. The general solution is

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=0,$$

which is rewritten as

$$\left(\frac{x}{a}-\frac{y}{b}\right)\left(\frac{x}{a}+\frac{y}{b}\right)=0.$$

The solutions are the two lines

$$bz+ay=0 \qquad \text{and} \qquad bx-ay=0.,$$

which describe two lines through $(0,0)$.

II. Precisely one of the constants A and B is 0.
We may assume that $A\neq 0$ and $B=0$. Then (1.7) is written

(1.12) $A\,x^2+2D\,x+2E\,t+F=0.$

If also $E\neq 0$, then this equation is rewritten as

$$A\left(x+\frac{D}{A}\right)^2=-2E\left(y-\frac{1}{2E}\left\{\frac{D^2}{A}-F\right\}\right),$$

so if we put

$$x_1=x+\frac{D}{A},\qquad y_1=y-\frac{1}{2E}\left\{\frac{D^2}{A}-F\right\}\quad \text{and}\quad a=-\frac{A}{2E},$$

then we get the structure,

$$y_1=ax_1^2 \qquad \text{(a parabola).}$$

If instead $E=0$, then (1.12) becomes

$$A\,x^2+2D\,x+F=0,$$

which is written as

$$x_1^2=k, \qquad \text{(no solution, one line, or, two parallel lines),}$$

where

$$x_1 = x + \frac{D}{A} \qquad \text{and} \qquad k = \frac{1}{A}\left\{\frac{D^2}{A} - F\right\}.$$

As usual we write in the following for convenience (x, y) instead of (x_1, y_1). Then the analysis above shows that we have two cases.

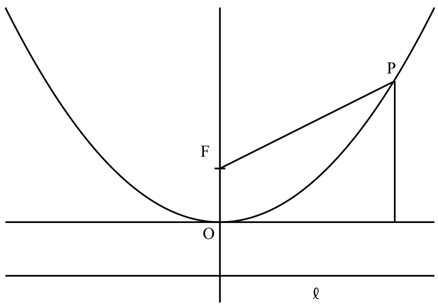


Figure 1.26: A parabola.

The parabola. The normed equation is here

(1.13) $y = a\,x^2, \qquad a \neq 0.$

It intersects the coordinate axes only at the origo, $O : (0,0)$, which is called the *top point* of the parabola, and the y-axis is the only *axis of symmetry*.

One usually instead put $p = \frac{1}{a}$, and then (1.13) is written

(1.14) $x^2 = py,$

where p is called the *parameter of the parabola*. The *focus* of the parabola is $F : \left(0, \frac{p}{4}\right)$, and the line ℓ of the equation $y = -\frac{p}{4}$ is called the *directrix* of the parabola. Its geometric meaning is that if P is any point on the parabola, then

$$\left|\overrightarrow{FP}\right| = \operatorname{dist}(P, \ell),$$

i.e. the distance from P to the focus is equal to the distance from P to the directrix ℓ.

The empty set, one line, or two parallel lines. In the case the equation is

$$x^2 = k.$$

If $k < 0$, then we have no solution.
If $k = 0$, then the line $x = 0$, $y \in \mathbb{R}$, is the only solution.
If $k > 0$, then the two parallel lines $x = \pm\sqrt{k}$, $y \in \mathbb{R}$, are the solutions.

We call the ellipses, the hyperbolas and the parabolas the (non-degenerated) *conic sections*, because they can be obtained as the intersection of a cone with a plane. The other cases mentioned above are then called the degenerated conic sections.

1.6.2 Quadratic equations in three variables. Conic sectional surfaces.

The *general quadratic equation* in three variables has the form

$$(1.15)\quad A\,x^2 + B\,y^2 + C\,z^2 + 2D\,xy + 2E\,xz + 2F\,yz + 2g\,x + 2H\,y + 2I\,z + J = 0,$$

where A, B, $\dots$, $J \in \mathbb{R}$ are real constants, and where $(A, B, C, D, E, F) \neq (0, 0, 0, 0, 0, 0)$.

As usual, the product terms $2D\,xy + 2E\,xz + 2F\,yz$ are a nuisance, when $(D, E, F) \neq (0, 0, 0)$, so the first task is to transform (1.15) into some new variables x_1, y_1, z_1, such that the new coefficients are all zero, $(D_1, E_1, F_1) = (0, 0, 0)$.

We note that (1.15) can be written

$$(1.16)\quad (x\ \ y\ \ z)\mathfrak{A}\begin{pmatrix} x \\ y \\ z \end{pmatrix} + 2(G\ \ H\ \ I)\begin{pmatrix} x \\ y \\ z \end{pmatrix} + J = 0, \qquad \text{where} \quad \mathfrak{A} := \begin{pmatrix} A & D & E \\ D & B & F \\ E & F & C \end{pmatrix}.$$

We find by methods described previously a coordinate transformation

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \mathbf{Q} \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}, \qquad \text{where } \mathbf{Q} = (\mathbf{q}_1 \;\; \mathbf{q}_2 \;\; \mathbf{q}_3)\,, \qquad \text{with } \mathbf{q}_3 = \mathbf{q}_1 \times \mathbf{q}_2,$$

such that

$$(x_1 \;\; y_1 \;\; z_1)\, \mathbf{Q}^T \mathfrak{A} \mathbf{Q} \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} = \lambda_1\, x_1^2 + \lambda_2\, y_1^2 + \lambda_3\, z_1^2.$$

Just find the eigenvalues and the corresponding eigenvectors of $\mathfrak{A}$. (Here MAPLE may be used to ease the computations.)

When we use this coordinate transformation, then (1.16) is reduced to

$$\lambda_1\, x_1^2 + \lambda_2\, y_1^2 + \lambda_3\, z_1^2 + 2(G \;\; H \;\; I)\mathbf{Q} \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + J = 0.$$

Then introduce

$$G_1 = (G \;\; H \;\; I)\mathbf{q}_1, \qquad H_1 = (G \;\; H \;\; I)\mathbf{q}_2, \qquad I_1 = (G \;\; H \;\; I)\mathbf{q}_3,$$

and the equation is reduced to

(1.17) $\lambda_1\, x_1^2 + \lambda_2\, y_1^2 + \lambda_3\, z_1^2 + 2G_1\, x_1 + 2H_1\, y_1 + 2I_1\, z_1 + J = 0.$

It follows from the analysis above that it suffices to consider the simpler equation

$$A\, x^2 + B\, y^2 + C\, z^2 + 2G\, x + 2H\, y + 2I\, z + J = 0, \qquad (A, B, C) \neq (0, 0, 0),$$

where we again have simplified the notation.

We shall split the investigation into three cases.

I. First case, $A \neq 0$, $B \neq 0$ and $C \neq 0$.

In this case, (1.17) can be rewritten as

$$A\left(x + \frac{G}{A}\right)^2 + B\left(y + \frac{H}{B}\right)^2 + C\left(z + \frac{I}{C}\right)^2 = \frac{G^2}{A} + \frac{H^2}{B} + \frac{I^2}{C} - J.$$

If we put

$$x_1 = x + \frac{G}{A}, \qquad y_1 = y + \frac{H}{B}, \qquad z_1 = z + \frac{I}{C}, \qquad K = \frac{G^2}{A} + \frac{H^2}{B} + \frac{I^2}{C} - J,$$

then the equation (1.17) is reduced to the simpler form

$$A\, x_1^2 + B\, y_1^2 + C\, z_1^2 = K.$$

Let us first assume that $K \neq 0$. Then it is customary to norm the equation by dividing it by K,

$$\frac{x_1^2}{K/A} + \frac{y_1^2}{K/B} + \frac{z_1^2}{K/C} = 1.$$

We write for short,

$$a := \sqrt{\left|\frac{K}{A}\right|}, \qquad b := \sqrt{\left|\frac{K}{B}\right|}, \qquad c := \sqrt{\left|\frac{K}{C}\right|}.$$

Then we obtain the canonical form

$$\pm\left(\frac{x_1}{a}\right)^2 \pm\left(\frac{y_1}{b}\right)^2 \pm\left(\frac{z_1}{c}\right)^2 = 1,$$

with all possible choices of the signs, i.e. in principle eight subcases in total, which, however, by some trivial argument of symmetry (where we rename the variables) can be reduced to four. These are

ellipsoid	$\left(\frac{x_1}{a}\right)^2 + \left(\frac{y_1}{b}\right)^2 + \left(\frac{z_1}{c}\right)^2 = 1,$
hyperboloid of one sheet	$\left(\frac{x_1}{a}\right)^2 + \left(\frac{y_1}{b}\right)^2 - \left(\frac{z_1}{c}\right)^2 = 1,$
hyperboloid of two sheets	$\left(\frac{x_1}{a}\right)^2 - \left(\frac{y_1}{b}\right)^2 - \left(\frac{z_1}{c}\right)^2 = 1,$
empty set	$-\left(\frac{x_1}{a}\right)^2 - \left(\frac{y_1}{b}\right)^2 - \left(\frac{z_1}{c}\right)^2 = 1.$

If instead $K = 0$, then we put

$$a := \sqrt{\frac{1}{|A|}}, \qquad b := \sqrt{\frac{1}{|B|}}, \qquad c := \sqrt{\frac{1}{|C|}},$$

from which we get the two possibilities,

a point	$\left(\frac{x_1}{a}\right)^2 + \left(\frac{y_1}{b}\right)^2 + \left(\frac{z_1}{c}\right)^2 = 0,$
conic sectional conic surface	$\left(\frac{x_1}{a}\right)^2 + \left(\frac{y_1}{b}\right)^2 - \left(\frac{z_1}{c}\right)^2 = 0.$

We shall briefly describe these possibilities in the following, where we again for short write (x, y, z) instead of (x_1, y_1, z_1).

1. The ellipsoid has the canonical equation

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 = 1.$$

The semi-axes are clearly a, b and c.

2. The hyperboloid with one sheet. The normed equation is

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 - \left(\frac{z}{c}\right)^2 = 1,$$

with one minus sign on the left hand side of the equation. The corresponding surface is connected, i.e. it consists of one surface. (This is only indicated on the figure, because the author has not been clever enough to make the right figure.)

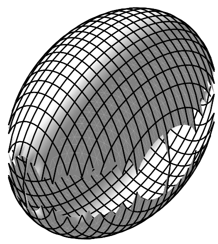


Figure 1.27: An ellipsoid.

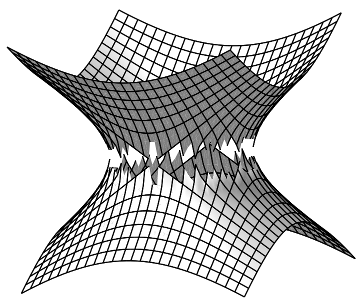


Figure 1.28: An hyperboloid with one sheet.

An important special case is obtained, when $a = b$, in which case

$$\frac{x^2 + y^2}{a^2} - \frac{z^2}{c^2} = 1.$$

This *hyperboloid of one sheet* is obtained by revolving the (two dimensional) hyperbola

$$\left(\frac{x}{a}\right)^2 - \left(\frac{z}{c}\right)^2 = 1, \qquad y = 0,$$

in the XZ-plane around the z-axis.

It is possible to prove the following

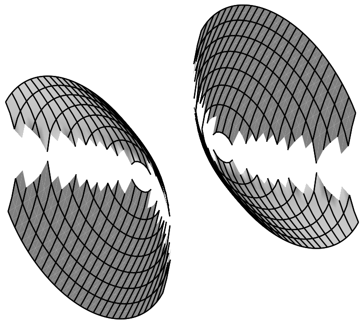


Figure 1.29: An hyperboloid with two sheets.

Theorem 1.1 *An* hyperboloid of one sheet *contains two systems of straight lines. Two different lines from the same system are always oblique. Two lines, one from each system, always lie in the same plane.*

3. The hyperboloid with two sheets. The normed equation is

$$\left(\frac{x}{a}\right)^2-\left(\frac{y}{b}\right)^2-\left(\frac{z}{c}\right)^2=1.$$

It is characterized by having *two* minus signs on the left hand side of the normed equation. The corresponding surface is split into *two* connected components.

4. The conic sectional conic surface has the canonical equation

$$\left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^2-\left(\frac{z}{c}\right)^2=0.$$

It is clearly a cone with $O:(0,0,0)$ as its centrum.

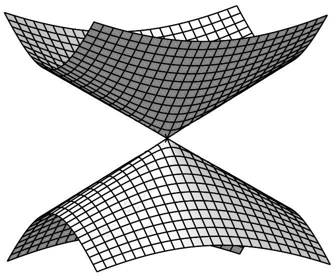


Figure 1.30: A conic sectional conic surface.

5. A point. The equation is

$$\left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^2+\left(\frac{z}{c}\right)^2=0,$$

which is only satisfied for $O:(0,0,0)$.

II. Second case. Here we assume that $A\neq 0$, $B\neq 0$ and $C=0$. Then (1.17) is reduced to

(1.18) $A\,x^2+B\,y^2+2G\,x+2H\,y+2I\,z+J=0.$

First assume that also $I\neq 0$. Then (1.18) can be reformulated as

$$A\left(x+\frac{G}{A}\right)^2+B\left(y+\frac{H}{B}\right)^2=-2I\left(z-\frac{1}{2I}\left\{\frac{G^2}{A}+\frac{H^2}{B}-J\right\}\right).$$

If we put

$$x_1=x+\frac{G}{A},\qquad y_1=y+\frac{H}{B},\qquad z_1=z-\frac{1}{2I}\left\{\frac{G^2}{A}+\frac{H^2}{B}-J\right\},\qquad L=-2I,$$

then (1.18) is reduced to

$$A\,x_1^2 + B\,y_1^2 = L\,z_1.$$

By assumption, $L = -2I \neq 0$, so when we divide by L, we get

$$\frac{x_1^2}{L/A} + \frac{y_1^2}{L/B} = z_1.$$

Then write for short,

$$a := \sqrt{\left|\frac{L}{A}\right|} \qquad \text{and} \qquad b := \sqrt{\left|\frac{L}{B}\right|},$$

and we get the two possibilities,

elliptic paraboloid $\left(\frac{x_1}{a}\right)^2 + \left(\frac{y_1}{b}\right)^2 = z_1,$

hyperbolic paraboloid $\left(\frac{x_1}{a}\right)^2 - \left(\frac{y_1}{b}\right)^2 = z_1.$

If instead $I = 0$, then (1.18) is written

$$A\left(x + \frac{G}{A}\right)^2 + B\left(y + \frac{H}{B}\right)^2 = \frac{G^2}{A} + \frac{H^2}{B} - J.$$

We simplify by writing

$$x_1 = x + \frac{G}{A}, \qquad y_1 = y + \frac{H}{B}, \qquad K = \frac{G^2}{A} + \frac{H^2}{B} - J,$$

because then (1.18) takes the simpler form

(1.19) $A\,x_1^2 + B\,y_1^2 = K.$

Again we must split into the two cases, $K \neq 0$ and $K = 0$. If $K \neq 0$, then we write for short

$$a := \sqrt{\left|\frac{K}{A}\right|}, \qquad b := \sqrt{\left|\frac{K}{B}\right|}.$$

We obtain the following three possibilities,

elliptic cylindric surface $\left(\frac{x_1}{a}\right)^2 + \left(\frac{y_1}{b}\right)^2 = 1,$

hyperbolic cylindric surface $\left(\frac{x_1}{a}\right)^2 - \left(\frac{y_1}{b}\right)^2 = 1,$

empty set $-\left(\frac{x_1}{a}\right)^2 - \left(\frac{y_1}{b}\right)^2 = 1.$

If instead $K = 0$, we put

$$a := \sqrt{\frac{1}{|A|}} \qquad \text{and} \qquad b := \sqrt{\frac{1}{|B|}}.$$

Then we get the two possibilities,

the z_1-axis	$\left(\frac{x_1}{a}\right)^2 + \left(\frac{y_1}{b}\right)^2 = 0,$
two planes through the z_1-axis	$\left(\frac{x_1}{a}\right)^2 - \left(\frac{y_1}{b}\right)^2 = 0.$

We shall in the following briefly sketch the possibilities above. Again we write for short (x, y, z) instead of (x_1, y_1, z_1).

1. **The elliptic paraboloid.** The canonical equation is

$$\left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^2=z.$$

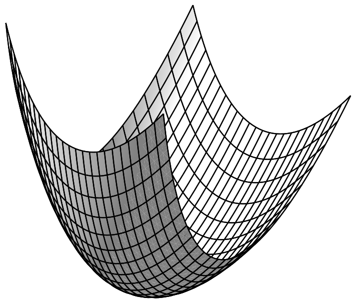


Figure 1.31: An elliptic paraboloid.

2. **The hyperbolic paraboloid.** The canonical equation is

$$\left(\frac{x}{a}\right)^2-\left(\frac{y}{b}\right)^2=z.$$

It is possible to prove the following theorem

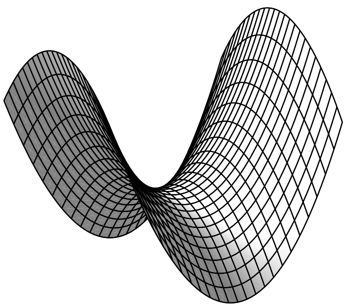


Figure 1.32: An hyperbolic paraboloid.

Theorem 1.2 *An* hyperbolic paraboloid *contains two systems of straight lines. Two different lines from the same system are always oblique with respect to each other. Two lines from different systems will always intersect each other.*

3. The elliptic cylindric surface. The canonical equation is

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1.$$

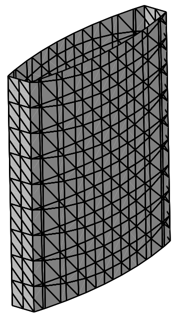


Figure 1.33: An elliptic cylindric surface.

4. The hyperbolic cylindric surface. The canonical equation is

$$\left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2 = 1.$$

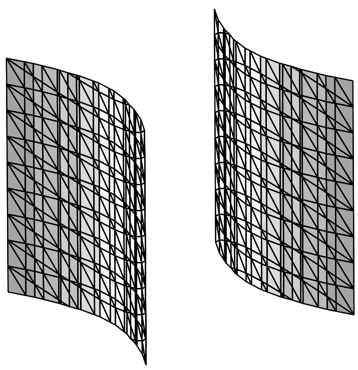


Figure 1.34: An hyperbolic cylindric surface.

III. The third case. Here we assume that $A \neq 0$, while $B = C = 0$. Then (1.17) is reduced to

(1.20) $A\,x^2 + 2G\,x + 2H\,y + 2I\,z + J = 0.$

If $(H, I) \neq (0, 0)$, e.g. $I \neq 0$, then (1.20) is reformulated as

$$A\left(x + \frac{G}{A}\right)^2 + 2H\,y + 2I\left(z + \frac{1}{2I}\left\{J - \frac{G^2}{A}\right\}\right) = 0.$$

We put

$$x_1 = x + \frac{G}{A}, \qquad y_1 = y \qquad \text{and} \qquad z_1 = z + \frac{1}{2I}\left\{J - \frac{G^2}{A}\right\},$$

from which

$$A\,x_1^2 + 2H\,y_1 + 2I\,z_1 = 0.$$

Then apply the orthogonal substitution

$$\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & I/\sqrt{H^2+I^2} & H/\sqrt{H^2+I^2} \\ 0 & -H/\sqrt{H^2+I^2} & I/\sqrt{H^2+i^2} \end{pmatrix} \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}$$

to reduce the equation above to

$$A\,x_2^2 + 2\sqrt{H^2+I^2}\,z_2 = 0.$$

This structure invites to put $p := -2\sqrt{H^2+I^2}/A$, so we get

parabolic cylindric surface $\qquad x_2^2 = p\,z_2,$

which is the canonical equation.

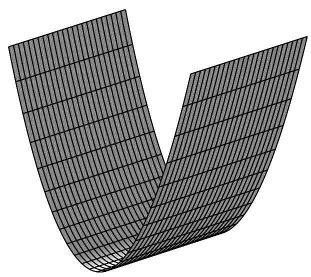


Figure 1.35: A parabolic cylindric surface.

If instead $(H, I) = (0, 0)$, then (1.20) reduces to

$$A\left(x + \frac{G}{A}\right)^2 = \frac{G^2}{A} - J.$$

Writing

$$x_1 = x + \frac{G}{A} \qquad \text{and} \qquad k = \frac{1}{A}\left\{\frac{G^2}{A}.J\right\},$$

we see that this case can be written in the form

$$x_1^2 = k^2,$$

i.e. the *empty set*, *one plane*, or *two (parallel) planes*.

1.6.3 Summary of the canonical cases in three variables

Equation	Name	$(0,0,0)$	Generators
$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$	Ellipsoid	Centrum	None
$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$	Hyperboloid of one sheet	Centrum	Two systems of lines
$\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$	Hyperboloid of two sheets	Centrum	None
$-\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$	Empty set	—	—
$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=0$	Point $(0,0,0)$	—	—
$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=0$	Conic sectional conic surface	Centrum	Lines through the centrum
$\frac{x^2}{a^2}+\frac{y^2}{b^2}=z$	Elliptic paraboloid	Toppoint	None
$\frac{x^2}{a^2}-\frac{y^2}{b^2}=z$	Hyperbolic paraboloid	Toppoint	Two systems of lines
$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$	Elliptic cylindric surface	Centrum	Lines parallel with the z-axis
$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$	Hyperbolic cylindric surface	Centrum	Lines parallel with the z-axis
$-\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$	Empty set	—	—
$\frac{x^2}{a^2}+\frac{y^2}{b^2}=0$	$z-$ axis	—	—
$\frac{x^2}{a^2}-\frac{y^2}{b^2}=0$	Two planes through the z-axis	—	—
$x^2=p\,z$	Parabolic cylindrical surface	Toppoint	Lines parallel with the y-axis
$x^2=k>0$	Two planes parallel with the YZ-plane	—	—
$x^2=0$	YZ-plane	—	—
$x^2=k<0$	Empty set	—	—

2 Some useful procedures

2.1 Introduction

In this chapter we collect some simple and useful practical procedures, like integration of trigonometric polynomials, the technique of partial fractions, when MAPLE is not at hand, integration of a quotient of two polynomials, and how to find the domain of a given function. All these will be important in the following chapter.

2.2 Integration of trigonometric polynomials

Problem 2.1 *Calculate the integral*

$$\int \sin^m x \cos^n x \, dx \qquad \textit{for } m,\, n \in \mathbb{N}_0.$$

Notation: By the *degree* of the product $\sin^m x \cos^n x$ we shall understand the sum $m+n$ of the exponents.

Split the problem into a simpler one: There are two main cases, *odd* and *even* degree. Each of these is again split into two *subcases:*

1) $m+n$ *odd.*
   a) m even and n odd, i.e. $m = 2p$ and $n = 2q+1$, $p,\, q \in \mathbb{N}_0$,
   b) m odd and n even, i.e. $m = 2p+1$ and $n = 2q$, $p,\, q \in \mathbb{N}_0$.
2) $m+n$ *even.*
   a) m and n are both odd, i.e. $m = 2p+1$ and $n = 2q+1$, $p,\, q \in \mathbb{N}_0$,
   b) m and n are both even, i.e. $m = 2p$ and $n = 2q$, $p,\, q \in \mathbb{N}_0$.

   The most difficult case occurs in 2b), where both m and n are even.

Method of solution:

1) a) $m = 2p$ and $n = 2p+1$.
   Apply the substitution $u = \sin x$ corresponding to $m = 2p$ *even*:

$$\begin{aligned}\int \sin^{2p} x \cos^{2q+1} x \,\mathrm{d}x &= \int \sin^{2p} x \cdot \cos^{2q} x \cdot \cos x \,\mathrm{d}x \\ &= \int \sin^{2p} x \cdot (1-\sin^2 x)^q \,\mathrm{d}\sin x \\ &= \int_{u=\sin x} u^{2p}(1-u^2)^q \,\mathrm{d}u,\end{aligned}$$

   where the integral is a usual polynomial in u of degree $2p+2q$.

b) $m = 2p+1$ and $n = 2q$.
Apply the substitution $u = \cos x$ corresponding to $n = 2q$ *even*:

$$\begin{aligned}\int \sin^{2p+1} x \cdot \cos^{2q} x \, \mathrm{d}x &= \int \sin^{2p} x \cdot \cos^{2q} x \cdot \sin x \, \mathrm{d}x \\ &= \int (1-\cos^2 x)^p \cos^{2q} x \cdot (-1) \, \mathrm{d}\cos x \\ &= -\int_{u=\cos x} (1-u^2)^p u^{2q} \, \mathrm{d}u,\end{aligned}$$

where the integral is a usual polynomial in u og degree $2p+2q$.

2) When the degree $m+n$ is *even*, the trick is to change the problem to a similar one by doubling the angle, thereby halving the degree. Therefore, we use the formulæ

$$\cos^2 x = \frac{1}{2}(1+\cos 2x), \quad \sin^2 x = \frac{1}{2}(1-\cos 2x), \quad \sin x \cos x = \frac{1}{2}\sin 2x.$$

a) $m = 2p+1$ and $n = 2q+1$ are both *odd*.
The integrand is transformed in the following way:

$$\begin{aligned}\sin^{2p+1} x \cdot \cos^{2q+1} x &= (\sin^2 x)^p \cdot (\cos^2 x)^q \cdot \sin x \cos x \\ &= \left\{\frac{1}{2}(1-\cos 2x)\right\}^p \left\{\frac{1}{2}(1+\cos 2x)\right\}^q \cdot \frac{1}{2}\sin 2x.\end{aligned}$$

Hence we are in a special case of 1b), so by the substitution $u = \cos 2x$ we get

$$\begin{aligned}\int \sin^{2p+1} x \cos^{2q+1} x \, \mathrm{d}x &= \frac{1}{2^{p+q+1}} \int (1-\cos 2x)^p (1+\cos 2x)^q \sin 2x \, \mathrm{d}x \\ &= \frac{1}{2^{p+q+1}} \int (1-\cos 2x)^p (1+\cos 2x)^q \cdot \left(-\frac{1}{2}\right) \mathrm{d}\cos 2x \\ &= -\frac{1}{2^{p+q+2}} \int_{u=\cos 2x} (1-u)^p (1+u)^q \, \mathrm{d}u.\end{aligned}$$

b) $m = 2p$ and $n = 2q$ are both *even*.
In this case there is no final formula, but there is a procedure by which we can reduce the problem to a sum of several problems of the types 1a) and 2b) of lower degree. The result is obtained after a finite number of steps.

The integrand is rewritten in the following way:

$$\sin^{2p} x \cos^{2q} x = \left\{\frac{1}{2}(1-\cos 2x)\right\}^p \left\{\frac{1}{2}(1+\cos 2x)\right\}^q.$$

The left hand side is a trigonometrical polynomial of degree $2p+2q$ in the angle x. The right hand side is a trigonometric polynomial of degree $p+q$ in the doubled angle $2x$. Each term of this polynomial must be handled separately, depending on whether the degree $j(\leq p+q)$ is *odd* (case 1a) or 1b)) or *even* (case 2b)).

Remark 2.1 It is of course in principle possible to create a specific solution formula, but it will be more confusing than the description of the procedure given above. $\Diamond$

MAPLE. When $m, n \in \mathbb{N}_0$ are explicitly given as numbers, an application of MAPLE is of course the easiest method. When either m or $n \in \mathbb{N}_0$ is not specified, one applies the method above.

2.3 Complex decomposition of a fraction of two polynomials

Problem 2.2 *Write the quotient* $\dfrac{P(x)}{Q(x)}$ *of two polynomials as a sum of elementary fractions.*

Remark 2.2 This problem occurs typically in connection with integration, and in courses on series also in telescopic summation. If the denominator has complex conjugated roots of at least order 2, a complex decomposition is usually the easiest method. If the order is 1, then real decomposition may be applied instead. We shall here show the method of *complex decomposition.* $\Diamond$

Procedure.

1) If the degree of the numerator is $\geq$ the degree of the denominator, we first perform a *division by the denominator*,

$$\frac{P(x)}{Q(x)} = P_1(x) + \frac{R(x)}{Q(x)},$$

where the residual polynomial $R(x)$ (the new numerator) has a degree smaller than the degree of $Q(x)$. We save the resulting polynomial $P_1(x)$ for the last step.

2) The *denominator* $Q(x)$ is then factorized into polynomials of degree one (with *complex* roots):

$$Q(x) = c \cdot (x - a_1)^{p_1} \cdots (x - a_k)^{p_k}.$$

Check that the sum $p_1 + \cdots + p_k$ of all exponents is equal to the degree of $Q(x)$. If $Q(x)$ is a real polynomial, check that the complex conjugated roots occur of the *same degree.*

3) To ease matters, choose the simplest one of the two polynomials $P(x)$ and $R(x)$. The following method gives the same result, whether $P(x)$ or $R(x)$ is used. Since it is here theoretically most correct to use $R(x)$, we shall use $R(x)$ in the rest of this description, and it is left to the reader to write $P(x)$ instead of $R(x)$, whenever this will give us simpler calculations.

4) The fraction is rewritten in the following way

$$\frac{R(x)}{Q(x)} = \frac{1}{c} \cdot \frac{R(x)}{(x-a_1)^{p_1} \cdots (x-a_k)^{p_k}}.$$

We get the coefficient of the special simple fraction

$$\frac{1}{(x-a_1)^{p_1}}$$

by "covering by one's hand" the factor $(x-a_1)^{p_1}$ in the denominator and then putting $x = a_1$ into the remainder part:

$$b_{1,p_1} = \frac{1}{c} \cdot \left. \frac{R(x)}{(x-a_2)^{p_2} \cdots (x-a_k)^{p_k}} \right|_{x=a_1}.$$

Save the result

$$\frac{b_{1,p_1}}{(x-a_1)^{p_1}}$$

for the last step in this procedure.

5) Repeat 4) on any other of the factors

$$(x-a_2)^{p_2}, \quad \cdots \quad (x-a_k)^{p_k},$$

in the denominator and save all the found special fractions.

6) Subtract all the found special fractions from $\dfrac{R(x)}{Q(x)}$ and reduce:

$$\frac{1}{c} \cdot \frac{R(x)}{(x-a_1)^{p_1} \cdots (x-a_k)^{p_k}} - \frac{b_{1,p_1}}{(x-a_1)^{p_1}} - \cdots - \frac{b_{k,p_k}}{(x-a_k)^{p_k}} = \frac{1}{d} \cdot \frac{R_1(x)}{(x-a_1)^{q_1}} \cdots (x-a_k)^{q_k}.$$

If the calculations are made without errors, then

$$q_1 < p_1, \quad \cdots \quad q_k < p_k.$$

Check this! (A weak test.)

7) Repeat 4), 5) and 6) on the reduced fraction

$$\frac{1}{d} \cdot \frac{R_1(x)}{(x-a_1)^{q_1} \cdots (x-a_k)^{q_k}}.$$

Remember in each step to write down the elementary fractions which have been found. The process must necessarily stop after a finite number of steps, because the degree of the denominator is becoming smaller by each iteration.

8) Finally, collect all the found elementary fractions together with the polynomial from 1).

The description above is the standard procedure. My experience has shown me that one often can find shortcuts, which are impossible to systemize here. I shall therefore here only give one example of many possibilities.

Example 2.1 Let us here try to decompose the fractional function

$$\frac{1}{x^4-1}.$$

1) *The standard procedure* as described above. The denominator has the simple roots 1, i, -1, $-i$, hence

$$\begin{aligned}
\frac{1}{x^4-1} &= \frac{1}{(x-1)(x-i)(x+1)(x+i)} \\
&= \frac{1}{(1-i)(1+1)(1+i)}\cdot\frac{1}{x-1}+\frac{1}{(i-1)(i+1)(i+i)}\cdot\frac{1}{x-i} \\
&\quad +\frac{1}{(-1-1)(-1-i)(-1+i)}\cdot\frac{1}{x+1}+\frac{1}{(-i-1)(-i-i)(-i+1)}\cdot\frac{1}{x+i} \\
&= \frac{1}{4}\cdot\frac{1}{x-1}-\frac{1}{4i}\cdot\frac{1}{x-i}-\frac{1}{4}\cdot\frac{1}{x+1}+\frac{1}{4i}\cdot\frac{1}{x+i} \\
&= \frac{1}{4}\cdot\frac{1}{x-1}-\frac{1}{4}\cdot\frac{1}{x+1}-\frac{1}{4i}\left\{\frac{1}{x-i}-\frac{1}{x+i}\right\} \\
&= \frac{1}{4}\cdot\frac{1}{x-1}-\frac{1}{4}\cdot\frac{1}{x+1}-\frac{1}{2}\cdot\frac{1}{x^2+1}.
\end{aligned}$$

This is of course fairly tiresome, though it works.

2) *Alternatively* it is seen that

$$x^4-1=\left(x^2\right)^2-1=(x^2+1)(x^2-1),$$

so if we write $u=x^2$, and first decompose with respect to u followed by a decomposition with respect to x, we easily get in two simpler steps that

$$\begin{aligned}
\frac{1}{x^4-1} &= \frac{1}{u^2-1}=\frac{1}{(u-1)(u+1)}=\frac{1}{2}\,\frac{1}{u-1}-\frac{1}{2}\,\frac{1}{u+1} \\
&= \frac{1}{2}\,\frac{1}{x^2-1}-\frac{1}{2}\,\frac{1}{x^2+1}=\frac{1}{2}\,\frac{1}{(x-1)(x+1)}-\frac{1}{2}\,\frac{1}{x^2+1} \\
&= \frac{1}{4}\,\frac{1}{x-1}-\frac{1}{4}\,\frac{1}{x+1}-\frac{1}{2}\,\frac{1}{x^2+1}. \qquad \Diamond
\end{aligned}$$

3) **MAPLE**. This is easy here, because the rational function does not contain extra parameters:

$$\text{convert}\left(\frac{1}{x^4-1},\text{parfrac},x\right)$$

$$-\frac{1}{2\,(x^2+1)}-\frac{1}{4(x+1)}+\frac{1}{4(x-1)}$$

Here, “parfrac” is of course a shorthand for “partial fraction”.

In the latter two cases we should of course continue with a *complex decomposition* of $\frac{1}{x^2+1}$. The simple details are left to the reader.

2.4 Integration of a fraction of two polynomials

Problem 2.3 *Calculate* $\int \frac{P(x)}{Q(x)}\,dx$, *where* $P(x)$ *and* $Q(x)$ *are (real) polynomials.*

Procedure.

1) Decompose $\frac{P(x)}{Q(x)}$ as described in the previous chapter on *complex decomposition.*

   Then $\frac{P(x)}{Q(x)}$ is written as a sum of a polynomial $P_1(x)$ and some elementary fractions of the type $\frac{c}{(x-a)^p}$, i.e. we perform a partial fraction construction.

2) The polynomial $P_1(x)$ is integrated in the usual way.

3) The elementary fractions where $p > 1$ are also integrated in the usual way

$$\int \frac{c}{(x-a)^p}\,\mathrm{d}x = -\frac{c}{p-1}\cdot\frac{1}{(x-a)^{p-1}},$$

   no matter whether a is real or complex. If $P(x)$ and $Q(x)$ are real, then any complex fraction of the type $\frac{c}{(x-a)^p}$ will be accompanied by its complex conjugated fraction $\frac{\overline{c}}{(x-\overline{a})^p}$. This means that the integration of such a pair of complex conjugated fractions can be reduced to

$$\begin{aligned}\int\left\{\frac{c}{(x-a)^p}+\frac{\overline{c}}{(x-\overline{a})^p}\right\}\mathrm{d}x &= -\frac{1}{p-1}\left\{\frac{c}{(x-a)^{p-1}}+\frac{\overline{c}}{(x-\overline{a})^{p-1}}\right\}\\ &= -\frac{2}{p-1}\,\mathrm{Re}\left\{\frac{c}{(x-a)^{p-1}}\cdot\frac{(x-\overline{a})^{p-1}}{(x-\overline{a})^{p-1}}\right\}\\ &= -\frac{2}{p-1}\cdot\frac{\mathrm{Re}\{c\cdot(x-\overline{a})^{p-1}\}}{\{x^2-2\,\mathrm{Re}\,a\cdot x+|a|^2\}^{p-1}}\end{aligned}$$

4) If $p = 1$, and a is real, then of course

$$\int \frac{x}{x-a}\,dx = c\cdot\ln|x-a|.$$

5) If $p = 1$, and a is complex, then both $\frac{c}{x-a}$ and $\frac{\overline{c}}{x-\overline{a}}$ occur in the decomposition. A direct integration is not possible, unless one is familiar with the theory of *Complex Functions.* Instead we add the two elementary fractions *before* the integration. (Note that when $p > 1$, this is done *after* the integration, cf. 3)).

More precisely we put $a = \alpha + i\,\beta$ and $c = r + i\,s$. Then

$$\begin{aligned}
\frac{c}{x-a} + \frac{\overline{c}}{x-\overline{a}} &= \frac{r+i\,s}{x-\alpha-i\,\beta} + \frac{r-i\,s}{x-\alpha+i\,\beta} \\
&= \frac{(r+i\,s)(x-\alpha-i\,\beta)+(r-i\,s)(x-\alpha-i\,\beta)}{(x-\alpha)^2+\beta^2} \\
&= \frac{2r(x-\alpha)}{(x-\alpha)^2+\beta^2} - \frac{2s\beta}{(x-\alpha)^2+\beta^2},
\end{aligned}$$

whence

$$\begin{aligned}
\int \left\{ \frac{c}{x-a} + \frac{\overline{c}}{x-\overline{a}} \right\} \mathrm{d}x &= r \int \frac{2(x-\alpha)}{(x-\alpha)^2+\beta^2}\,\mathrm{d}x - 2s \int \frac{1}{1+\left(\dfrac{x-\alpha}{\beta}\right)^2}\,\frac{1}{\beta}\,\mathrm{d}x \\
&= r \cdot \ln\left\{(x-\alpha)^2+\beta^2\right\} - 2s\,\mathrm{Arctan}\left(\frac{x-\alpha}{\beta}\right).
\end{aligned}$$

6) The final result is obtained by gathering all the results from 2), 3), 4) and 5).

Example 2.2 In Example 2.1 we found the decomposition

$$\frac{1}{x^4-1} = \frac{1}{4}\,\frac{1}{x-1} - \frac{1}{4}\,\frac{1}{x+1} - \frac{1}{2}\,\frac{1}{x^2+1},$$

from which we immediately get

$$\int \frac{1}{x^4-1}\,dx = \frac{1}{4}\,\ln\left|\frac{x-1}{x+1}\right| - \frac{1}{2}\,\text{Arctan}\,x, \qquad x \neq \pm 1. \qquad \Diamond$$

ALTERNATIVELY it is straightforward here to apply MAPLE instead. The details are left to the reader.

3 Examples of point sets, conics and conical sections

3.1 Point Sets

Example 3.1 *Sketch the point set A, the interior A°, the boundary ∂A and the closure $\overline{A}$ in each of the cases below.*
Furthermore, examine whether A is open, closed or nothing of that kind.
Finally, check whether A is bounded or unbounded.

1) $\{(x, y) \mid xy \neq 0\}$.

2) $\{(x, y) \mid 0 < x < 1, \, 1 \leq y \leq 3\}$.

3) $\{(x, y) \mid y \geq x^2, \, |x| < 2\}$.

4) $\{(x, y) \mid x^2 + y^2 - 2x + 6y \leq 15\}$.

A Examination of point sets in the plane.

D Each set is analyzed on a figure.

I 1) The set $A = \{(x, y) \mid xy \neq 0\}$ is the whole plane with the exception of the X and the Y axes. It is obvious that it is *open*,

$$A = A^\circ.$$

The boundary ∂A is the union of the X and the Y axes.
The closure is $\overline{A} = A^\circ \cup \partial A = \mathbb{R}^2$, i.e. the whole plane.
Finally, A is clearly *not bounded.*

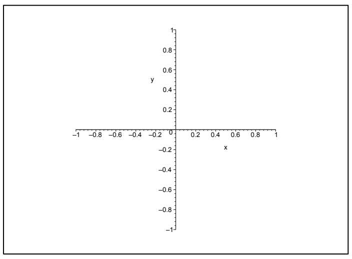


Figure 3.1: The set of Example 3.1.1

2) It is easy to sketch the rectangle $A =]0, 1[\times [1, 3]$. We see that

$$A^\circ =]0, 1[\times]1, 3[.$$

The boundary of the rectangle is rather complicated to describe formally:

$$\begin{aligned} \partial A &= \{(x, y) \mid 0 \leq x \leq 1, \, y = 1\} \cup \{(x, y) \mid 0 \leq x \leq 1, \, y = 3\} \\ &\quad \cup \{(x, y) \mid x = 0, \, 1 \leq y \leq 3\} \cup \{(x, y) \mid x = 1, \, 1 \leq y \leq 3\}. \end{aligned}$$

This example shows why one shall often prefer a figure instead of a formally correct mathematical description.

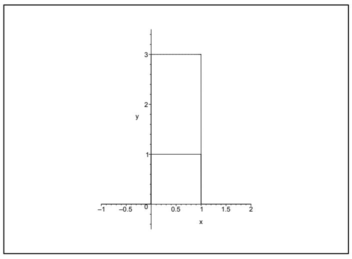


Figure 3.2: The set of Example 3.1.2

The closure is

$$\overline{A} = [0,1] \times [1,3].$$

The set A is neither open nor closed.
Obviously, the set is bounded (it is e.g. contained in the disc of centre $(0,0)$ and radius 4).

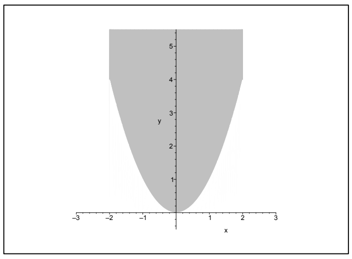


Figure 3.3: The set of Example 3.1.3

3) The set

$$A = \{(x,y) \mid y > x^2,\ |x| < 2\},$$

is also easily sketched. Here

$$A^\circ = \{(x,y) \mid y > x^2,\ |x| < 2\}$$

and

$$\partial A = \{(x,y) \mid x = -2,\ y \geq 4\} \cup \{(x,y) \mid |x| \leq 2,\ y = x^2\} \cup \{(x,y) \mid x = 2,\ y \geq 4\},$$

and

$$\overline{A} = \{(x,y) \mid y \geq x^2,\ |x| \leq 2\}.$$

The set A is neither open nor closed.
The set is clearly not bounded.

4) Since

$$x^2 + y^2 - 2x + 6y \leq 15$$

can be rewritten as

$$x^2 - 2x + 1 + y^2 + 6y + 9 \leq 9 \leq 15 + 1 + 9 = 25 = 5^2,$$

i.e. put into the form

$$(x-1)^2 + (y+3)^2 \leq 25 = 5^2,$$

it follows that

$$A = \{(x,y) \mid (x-1)^2 + (y+3)^2 \leq 5^2\} = \overline{K}((1,3);5).$$

This describes a closed disc of centre $(1,-3)$ and radius 5, thus $A = \overline{A}$.

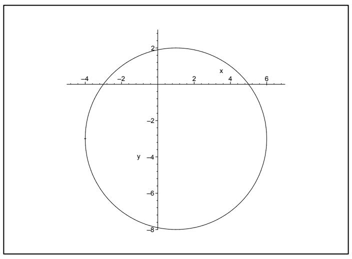


Figure 3.4: The set of Example 3.1.4

Then

$$A^\circ = K((1,-3);5) = \{(x,y) \mid (x-1)^2 + (y+3)^2 < 5^2\}$$

and

$$\partial A = \{(x,y) \mid (x-1)^2 + (y+3)^2 = 5^2\},$$

and $A = \overline{A}$ is closed and bounded.

Remark. Note that whenever a set like the one under consideration is described by an inequality between simple algebraic expressions, one will *usually* obtain the open set A° by only using the inequality signs $<$ or $>$ without equality sign, obtain the closed set by using $\leq$ or $\geq$ everywhere, and finally get the boundary by only using equality sign $=$. This is unfortunately only a rule of thumb, and one must be aware of that there are exceptions from this rule. $\Diamond$

Example 3.2 *Sketch in each of the following cases the point set A. Examine whether A is open or closed or none of the kind.*

1) $\{(x, y) \mid 3x^2 + 2y^2 < 6\}$.

2) $\{(x, y) \mid x^2 + y^2 \leq 1,\ y > 0\}$.

3) $\{(x, y) \mid x^2(1 - x^2 - y^2) > 0\}$.

4) $\{(x, y) \mid 0 < x - y \leq 1,\ y > 4\}$.

5) $\left\{(x, y) \mid x^2 + y^2 \geq \sqrt{x^2 + y^2}\right\}$.

6) $\{(x, y) \mid \max\{|x|, |y|\} \leq 1\}$.

7) $\{(x, y) \mid |x| + |y| < 1\}$.

8) $\{(x, y) \mid x \leq y \leq 4 - x^2\}$.

9) $\{(x, y) \mid (x - 1)(x^2 + y^2) \geq 0\}$.

10) $\{(x, y) \mid (y^2 - 1)(y - 3) > 0\}$.

A Examination of point sets in the plane.

D Analyze each set on a figure, e.g. by first examining the function. (Neither LaTeX nor MAPLE er may be well fit for the sketches in every one of the cases).

I 1) It follows from the rearrangement

$$A = \{(x, y) \mid 3x^2 + 2y^2 < 6\} = \left\{ (x, y) \;\middle|\; \left(\frac{x}{\sqrt{2}}\right)^2 + \left(\frac{y}{\sqrt{3}}\right)^2 < 1 \right\}$$

that the set is an open ellipsoidal disc of centre $(0, 0)$ and length of the half axes $\sqrt{2}$ and $\sqrt{3}$. The set is open.

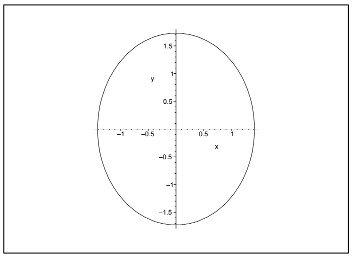


Figure 3.5: The set of Example 3.2.1

2) The set

$$A = \{(x, y) \mid x^2 + y^2 \leq 1,\ y > 0\}$$

is the intersection of the closed unit disc and the open upper half plan. The set is neither open nor closed.

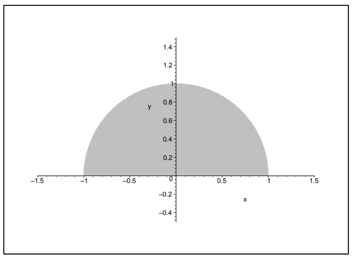


Figure 3.6: The set of Example 3.2.2

3) The set

$$A = \{(x, y) \mid x^2(1 - x^2 - y^2) > 0\} = \{(x, y) \mid x \neq 0,\ x^2 + y^2 < 1\}$$

is the open unit disc wich the exception of the points on the Y axis (where $x = 0$).
The set is open.

4) The set $A = \{(x, y) \mid 0 < x - y \leq 1,\ y > 4\}$ is the intersection of the three half planes

$$\{(x, y) \mid x > y\}, \qquad \{(x, y) \mid y \geq x - 1\}, \qquad \{(x, y) \mid y > 4\}.$$

This set is neither open nor closed.

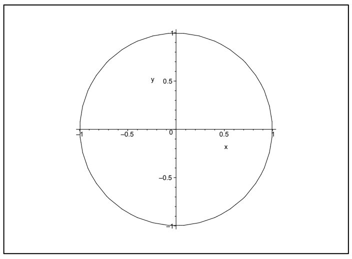


Figure 3.7: The set of Example 3.2.3

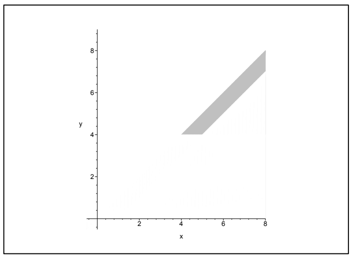


Figure 3.8: The set of Example 3.2.4

5) The set

$$\begin{aligned} A &= \{(x,y) \mid x^2+y^2 \ge \sqrt{x^2+y^2} \ge 1\} \\ &= \{(0,0)\} \cup \{(x,y) \mid \sqrt{x^2+y^2} \ge 1\} \\ &= \{(0,0)\} \cup \{(x,y) \mid x^2+y^2 \ge 1\} \end{aligned}$$

is the complementary set of a disc (centre $(0,0)$ and radius 1), supplied with the point $(0,0)$. The set is closed.

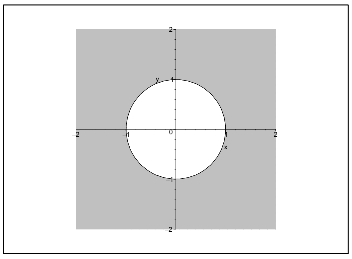


Figure 3.9: The set of Example 3.2.5

6) The set

$$A = \{(x,y) \mid \max\{|x|,|y|\} \le 1\} = [-1,1] \times [-1,1]$$

is a closed square.

7) The set $A = \{(x,y) \mid |x| + |y| < 1\}$ is the open square bounded by the lines

$$x+y=1, \quad -x+y=1, \quad x-y=1, \quad -x-y=1.$$

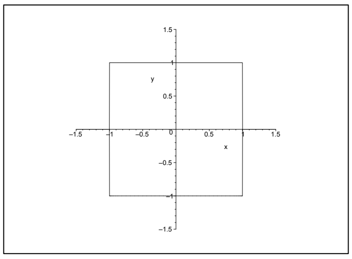


Figure 3.10: The set of Example 3.2.6

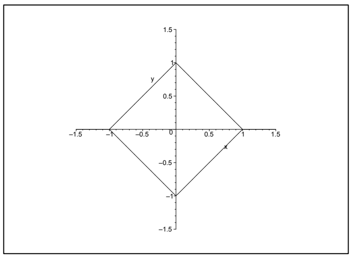


Figure 3.11: The set of Example 3.2.7

8) The set $A = \{(x,y) \mid x \le y \le 4 - x^2\}$ lies above the line $y = x$ and below the parabola $y = 4 - x^2$. These curves cut each other when $x^2 + x = 4$, i.e. when $x = -\dfrac{1}{2} \pm \dfrac{1}{2}\sqrt{17}$.

9) Since we always have $x^2 + y^2 \ge 0$ and $x^2 + y^2 = 0$ only for $(x,y) = (0,0)$, we get that

$$A = \{(x,y) \mid (x-1)(x^2+y^2) \ge 0\} = \{(0,0)\} \cup \{(x,y) \mid x \ge 1\}$$

is the union of a point $(0,0)$ and a closed half plane $x \ge 1$. It follows that A is closed.

10) The set

$$\begin{aligned} A &= \{(x,y) \mid (y^2-1)(y-3) > 0\} = \{(x,y) \mid (y+1)(y-1)(y-3) > 0\} \\ &= \{(x,y) \mid -1 < y < 1\} \cup \{(x,y) \mid 3 < y\} \end{aligned}$$

is open.

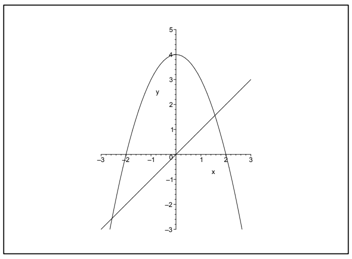


Figure 3.12: The set of Example 3.2.8.

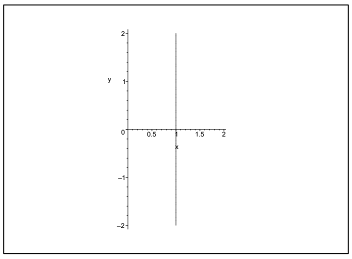


Figure 3.13: The set of Example 3.2.9.

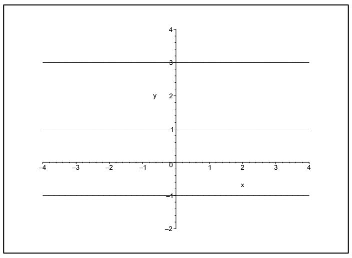


Figure 3.14: The set of Example 3.2.10.

Example 3.3 *Examine in each of the following cases, possibly by means of a sketch of a figure, the given point set. Do these sets have names?*

1) $A = \{(x,y,z) \mid \max\{|x|,|y|,|z| \le 1\}$.

2) $A = \{(x,y,z) \mid |x|+|y|+|z| \le 1\}$.

3) $A = \{(x,y,z) \mid x>0,\, y>0,\, z>0\}$.

4) $A = \{(x,y,z) \mid 0<x<y\}$.

5) $A = \{(x,y,z) \mid 0<y\}$.

6) $A = \{(x,y,z) \mid x^2+2y^2 \le 8\}$.

Remark. It is difficult in all cases to let LaTeX or MAPLE sketch the three dimensional figures. The readers are kindly asked to sketch them themselves. ◊

A Point sets in the three dimensional space $\mathbb{R}^3$.

D Analyze each set, possibly on a figure.

I 1) The set

$$A = \{(x,y,z) \mid \max\{|x|,|y|,|z|\} \le 1\} = [-1,1]^3$$

is a closed cube of centre $(0,0,0)$ and edge length 2.

2) The set

$$A = \{(x,y,z) \mid |x| + |y| + |z| \leq 1\}$$

is a closed dodecahedron. It is obtained by taking the intersection of the eight half spaces

$$\begin{array}{ll} x+y+z \leq 1, & x+y+z \geq -1, \\ x+y-z \leq 1, & x+y-z \geq -1, \\ x-y+z \leq 1, & x-y+z \geq -1, \\ x-y-z \leq 1, & x-y-z \geq -1. \end{array}$$

3) The set

$$A = \{(x,y,z) \mid x > 0,\ y > 0,\ z > 0\}$$

is the open first octant.

4) The set $A = \{(x,y,z) \mid 0 < x < y\}$ is the intersection of two open half spaces, hence itself open. The axis of the set is the Z axis, and the projection onto the XY plane in the direction of the Z axis is the angular set which lies between the line $y = x$ and the Y axis in the first quadrant.
5) The set $A = \{(x,y,z) \mid 0 < y\}$ is the open half space which is given by the inequality $y > 0$, i.e. bounded by the XZ plane where $y = 0$.
6) The set

$$A = \{(x,y,z) \mid x^2 + 2y^2 \leq 8\} = \left\{(x,y,z) \;\middle|\; \left(\frac{x}{2\sqrt{2}}\right)^2 + \left(\frac{y}{2}\right)^2 \leq 1\right\}$$

is the closed cylinder over the ellipse in the XY plane with centre $(0,0)$ and half axes $2\sqrt{2}$ and 2. The figure shows the projection of the set onto the XY plane in the direction of the Z axis, hence a cross section.

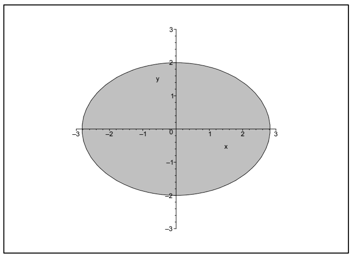


Figure 3.15: The projection onto the XY plane of the set of Example 3.3.6.

Example 3.4 *In each of the following cases a plane point set A is given in polar coordinates. Sketch the point set and find a name of it.*

1) $0 \le \varphi \le \dfrac{\pi}{2}, \qquad 0 \le \varrho \le a\cos\varphi.$

2) $0 \le \varphi \le \dfrac{\pi}{4}, \qquad 0 \le \varrho \le a\cos\varphi + a\sin\varphi.$

3) $-\pi < \varphi \le \pi, \qquad (\varrho - a)^2 \ge |\varrho - a|a.$

4) $\begin{cases} 0 \le \varphi \le \ \text{Arctan}\,\dfrac{b}{a}, & 0 \le \varrho \le \dfrac{a}{\cos\varphi}, \\ \text{Arctan}\,\dfrac{b}{a} \le \varphi \le \dfrac{\pi}{2}, & 0 \le \varrho \le \dfrac{b}{\sin\varphi}. \end{cases}$.

A Point sets in the plane given in polar coordinates.

D Analyze the point sets and sketch them.

I 1) When $0 \le \varrho \le a\cos\varphi$ a multiplication by $\varrho \ge 0$ gives

$$0 \le \varrho^2 \le a\varrho\cos\varphi,$$

i.e.

$$x^2 + y^2 \le ax,$$

and then by a rearrangement

$$\left(x - \frac{a}{2}\right)^2 + y^2 \le \left(\frac{a}{2}\right)^2.$$

Since $0 \le \varphi \le \dfrac{\pi}{2}$, we get a closed half disc in the first quadrant of centre $\left(\dfrac{a}{2}, 0\right)$ and radius $\dfrac{a}{2}$.

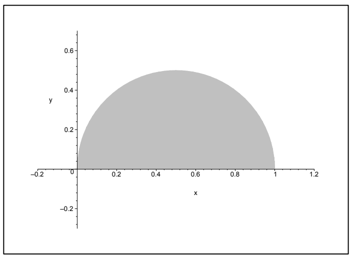


Figure 3.16: The set of Example 3.4.1.

2) By a multiplication by ϱ we get

$$\varrho^2 \le a\varrho\cos\varphi + a\varrho\sin\varphi,$$

thus in rectangular coordinates

$$x^2 + y^2 \le ax + ay,$$

which is reduced to

$$\left(x-\frac{a}{2}\right)^2+\left(y-\frac{a}{2}\right)^2\le\frac{a^2}{2}=\left(\frac{a}{\sqrt{2}}\right)^2.$$

This expression describes a closed disc of centre $\left(\frac{a}{2},\frac{a}{2}\right)$ and radius $\frac{a}{\sqrt{2}}$. From the condition $0\le\varphi\le\frac{\pi}{4}$ follows that the set A is that part of the disc, which lies in in this angular set (a circumference angle).

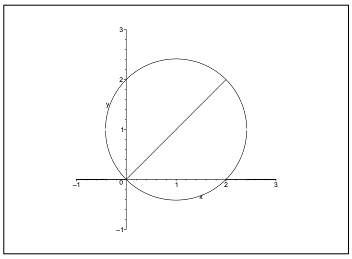


Figure 3.17: The set of Example 3.4.2.

3) It follows from $(\varrho-a)^2\ge|\varrho-a|a$ that either $\varrho=a$ or $|\varrho-a|\ge a$, hence

$$\varrho-a\ge a \qquad \text{or} \qquad \varrho-a\le -a.$$

Summarizing we get

$$\varrho=a \quad \text{or} \quad \varrho\ge 2a \quad \text{or} \quad \varrho=0,$$

since $\varrho<0$ is not possible.

The point set is the union of a point $\{(0,0)\}$, a circumference $\varrho=a$ and the closed complementary set of a disc $\varrho\ge 2a$, since we have no restrictions on the angle $-\pi<\varrho\le\pi$.

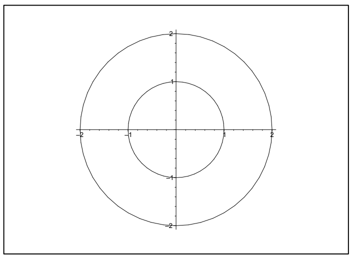


Figure 3.18: The set of Example 3.4.3.

4) Since $\cos\varphi>0$ for $0\le\varphi\le$ Arctan $\frac{b}{a}$, the condition $0\le\varrho\le\frac{a}{\cos\varphi}$ is equivalent to

$$0\le\varrho\cos\varphi=x\le a,\qquad 0\le\varphi\le \text{ Arctan }\frac{b}{a}.$$

Analogously, $\sin\varphi > 0$ for $\operatorname{Arctan}\dfrac{b}{a} \le \varphi \le \dfrac{\pi}{2}$, thus $0 \le \varrho \le \dfrac{b}{\sin\varphi}$ is equivalent to

$$0 \le \varrho\sin\varphi = y \le b, \qquad \operatorname{Arctan}\frac{b}{a} \le \varphi \le \frac{\pi}{2}.$$

The two cases are described by each their triangle, and the conclusion is that the set in rectangular coordinates is just the rectangle $A = [0,a] \times [0,b]$.

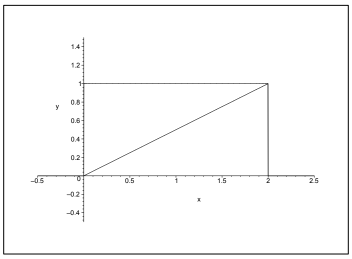


Figure 3.19: The set of Example 3.4.4.

Example 3.5 *Sketch and describe in polar coordinates the set A, where A is given below in rectangular coordinates.*

1) $A = \left\{(x,y) \mid x \geq 0,\ \left(x^2+y^2\right)^2 \geq x^2+y^2\right\}.$

2) $A = \left\{(x,y) \;\middle|\; x>0,\ \frac{1}{2}+y^2 \leq x^2 \leq 1-y^2\right\}.$

A Point sets in the plane, given in rectangular coordinates should be described in polar coordinates instead.

D Sketch the sets and use that $x = \varrho\cos\varphi$ and $y = \varrho\sin\varphi$.

I 1) The point set A is the intersection of a closed complementary set of a disc and the closed right half plane supplied by the point $(0,0)$.
In polar coordinates this is described by

$$-\frac{\pi}{2} \leq \varphi \leq \frac{\pi}{2} \qquad \text{and} \qquad \varrho^2 \geq \varrho.$$

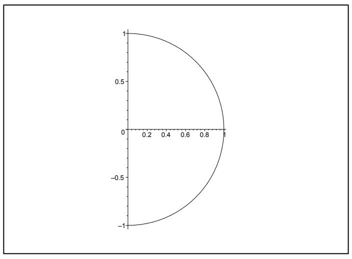


Figure 3.20: The set of Example 3.5.1.

2) Since $x > 0$, the point set lies in the open right half plane. It follows from $x^2 \leq 1-y^2$ that $x^2+y^2 \leq 1$, so the point set lies in the unit disc.
Finally, $\frac{1}{2}+y^2 \leq x^2$ describes the interior of a branch of a hyperbola. The two limiting curves

$$x^2+y^2 = 1 \qquad \text{and} \qquad x^2-y^2 = \frac{1}{2}$$

cut each other at the points $\left(\frac{\sqrt{3}}{2}, \pm\frac{1}{2}\right)$, so A lies in the angular set $-\frac{\pi}{6} \leq \varphi \leq \frac{\pi}{6}$.

In polar coordinates the upper is described by $\varrho \leq 1$, and the lower bound is given by

$$\frac{1}{2} + \varrho^2\sin^2\varrho \leq \varrho^2\cos^2\varphi,$$

hence by a rearrangement,

$$\frac{1}{2} \leq \varrho^2\left\{\cos^2\varphi - \sin^2\varphi\right\} = \varrho^2\cos 2\varphi.$$

Summarizing we get the following polar description

$$-\frac{\pi}{6} \le \varphi \le \frac{\pi}{6} \qquad \text{and} \qquad \frac{1}{\sqrt{2\cos 2\varphi}} \le \varrho \le 1.$$

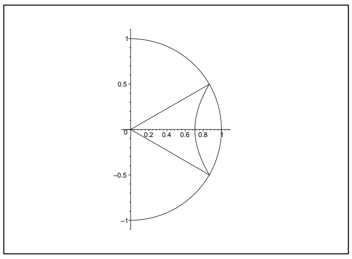


Figure 3.21: The set of Example 3.5.2.

Example 3.6 *Sketch the following subsets of* $\mathbb{R}^2$*, and if any of them is star shaped.*

1) $\{(x,y) \mid y > 3x^2\}$.

2) $\{(x,y) \mid x^2 + y^2 > 1\}$.

3) $\{(x,y) \mid y > -x^2\}$.

4) $\{(x,y) \mid x > 0,\, y > -x^2\}$.

A Analysis of sets concerning if they are star shaped.

D Start by sketching the sets. In this case I have had problems with the sketching programs, so the sets are only given by their boundaries and not by the more desirable hatching.

I 1) Here, A in the interior of a parabola. Obviously, this set is star shaped (and even convex).

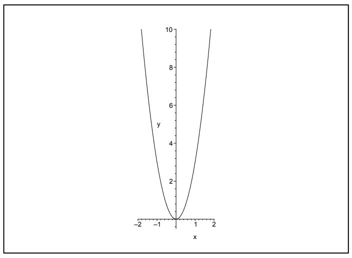


Figure 3.22: The set of Example 3.6.1.

2) This set is the complementary of a disc, so it cannot be star shaped. For any point from the set the unit disc shades for some other points.

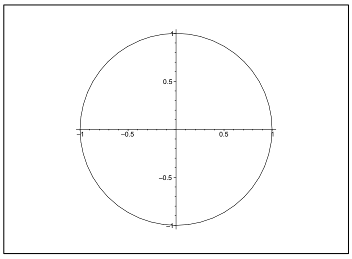


Figure 3.23: The set of Example 3.6.2.

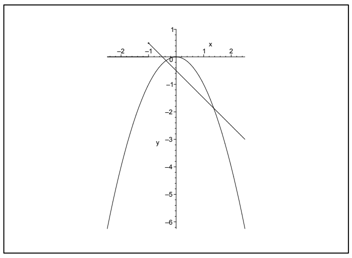


Figure 3.24: The set of Example 3.6.3.

3) The set is the exterior of a parabola. If $(x_0, y_0) \in A$ is any point we can always find a straight line through (x_0, y_0), which cuts the parabola in two different points. The points on the line beyond the most distant intersection point cannot be connected with (x_0, y_0) by a straight line inside A, so A is not star shaped seen from any point.
4) This set A is a part of the set in Example 3.6.3, hence it lies in the right half plane.
First note that

$$y + \lambda^2 = -2\lambda(x - \lambda)$$

is a tangent of the parabola for every $\lambda > 0$. This can also be written

$$y + 2\lambda x = \lambda^2, \qquad \lambda > 0.$$

Indirect proof. Assume that A indeed was star shaped from a point (x, y). Then

$$y + 2\lambda x \geq \lambda^2 \qquad \text{for all } \lambda > 0,$$

which can also be written

$$y \geq \lambda(\lambda - 2x) \qquad \text{for all } \lambda > 0.$$

This is of course not possible for any $(x, y) \in A$. In fact, the right hand side of this inequality tends to $+\infty$ for $\lambda \to +\infty$, while y remains constant, and the inequality is violated.

We conclude from this contradiction that A is not star shaped.

Example 3.7 . *Sketch the point sets given below, and indicate which ones are convex.*

1) $\{(x, y) \mid -5 < y < -3x^2\}$.

2) $\{(x, y) \mid x^2 + 3y^2 > 2\}$.

3) $\{(x, y) \mid y > -x^2\}$.

4) $\{/x, y) \mid x \geq 0,\ y \leq 0\}$.

A Examination of convexity.

D Sketch the sets and analyze.

I 1) The set is the interior of a parabola where we furthermore have the restriction $-5 < y < 0$. Obviously, this set is convex.

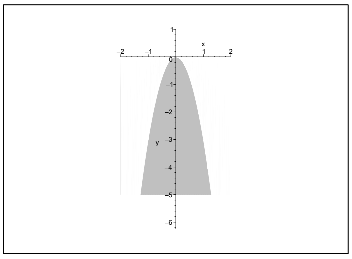


Figure 3.25: The set of Example 3.7.1.

2) The set

$$\{(x,y) \mid x^2+3y^2>2\} = \left\{(x,y) \;\middle|\; \left(\frac{x}{\sqrt{2}}\right)^2 + \left(\frac{y}{\sqrt{\frac{2}{3}}}\right)^2 > 1\right\}$$

is the complementary of an ellipse of centre $(0,0)$ and half axes $\sqrt{2}$ and $\sqrt{\dfrac{2}{3}}$. It is clearly not convex.

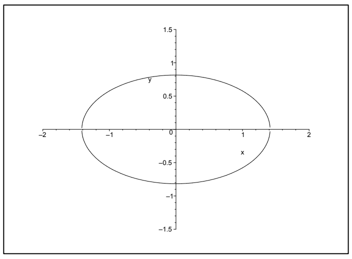


Figure 3.26: The set of Example 3.72.

3) This set is the complementary of a parabola (actually the same set as in Example 3.6.2. It is not star shaped, and therefore not convex either.

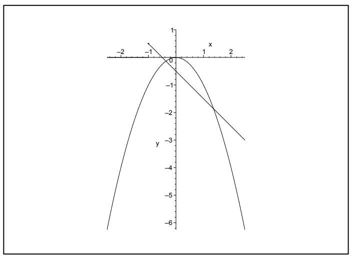


Figure 3.27: The set of Example 3.7.3.

4) This set is the closed fourth quadrant. It is clearly convex. There is no need to sketch it.

Example 3.8 *Let*

$$B = \{(x,y) \in [0,1]\times[0,1] \mid x \text{ is rational and } y \text{ is rational}\}.$$

Find the interior B°*, the boundary* ∂B *and the closure* $\overline{B}$*.*

A Interior, exterior, boundary and closure of a point set. This is the classical “strange” example, which should shock the reader, who has not seen this example before.

D First prove that $B^\circ = \emptyset$, and then $\overline{B]} = [0,1]\times[0,1]$.

I If $(x_0, y_0) \in B$, then $K((x_0, y_0); r)$, $r > 0$, i.e. the solid ball of centre (x_0, y_0) and any positive radius r, will always contain points (x, y), of which at least one of the coordinates is irrational, hence

$$K((x_0, y_0); r) \setminus B \neq \emptyset \qquad \text{for every } r > 0.$$

We conclude from this that $B^\circ = \emptyset$.

Let $(x_0, y_0) \in [0,1] \times [0,1]$ be any point in the bigger set. Then the ball $K((x_0, y_0); r)$ of centre (x_0, y_0) and any radius $r > 0$ will always contain points from B. This means that $(x_0, y_0) \in \overline{B}$, i.e.

$$\overline{B} \supseteq [0,1] \times [0,1].$$

It is on the other hand trivial that $\overline{B} \subseteq [0,1] \times [0,1]$, hence we must have equality,

$$\overline{B} = [0,1] \times [0,1].$$

Finally, the boundary is found by means of the definition,

$$\partial B = \overline{B} \setminus B^\circ = [0,1] \times [0,1] \setminus \emptyset = [0,1] \times [0,1] = \overline{B}.$$

Example 3.9 *In each of the following cases there is given a solid tetrahedron by its four corners. Sketch the tetrahedron T – invisible edges are dotted – and set up equations of the four planes, which bound T. Then derive the inequalities which the points of T must fulfil, and finally set up expressions of the form*

$$T = \{(x, y, z) \mid (x, y) \in B,\ Z_1(x, y) \leq z \leq Z_2(x, y)\}$$

and

$$T = \{(x, y, z) \mid \alpha \leq z \leq \beta,\ (x, y) \in B(z)\};$$

sketch the sets B and $B(z)$.

1) $(0,0,0)$, $(2,0,0)$, $(0,1,0)$, $(0,0,2)$.

2) $(0,0,0)$, $(2,0,2)$, $(0,1,2)$, $(0,0,2)$.

3) $(1,0,0)$, $(0,0,4)$, $(0,2,2)$, $(-1,0,0)$.

4) $(0,0,0)$, $(1,0,0)$, $(1,1,0)$, $(1,0,4)$.

5) $(1,0,0)$, $(0,0,4)$, $(0,2,0)$, $(-1,0,0)$.

A Analysis of tetrahedra.

D The text describes very carefully what should be done. Here we shall deviate a little because figures in space take a very long time to construct in the given programs. There are left to the reader.

I 1) It follows immediately from the missing figure (which the reader should add himself), that three of the planes are described by

$$x = 0, \qquad y = 0 \quad \text{and} \quad z = 0.$$

In fact, the plane $x = 0$ contains the points

$$(0,0,0), \qquad (0,1,0), \qquad (0,0,2),$$

the plane $y = 0$ contains the points

$$(0,0,0), \qquad (2,0,0), \qquad (0,0,2),$$

and the plane $z = 0$ contains the points

$$(0,0,0), \qquad (2,0,0), \qquad (0,1,0).$$

A parametric description of the fourth plane is e.g.

$$\begin{aligned}(x,y,z) &= (2,0,0)+u\{(0,1,0)-(2,0,0)\}+v\{(0,0,2)-(2,0,0)\}\\ &= (2,0,0)+u(-2,1,0)+v(-2,0,2)\\ &= (2-2u-2v,u,2v),\end{aligned}$$

from which $y=u$ and $z=2v$.
When we eliminate u and v, we get

$$x=2-2u-2v=2-2y-z,$$

and the equation of the fourth plane is

$$z=2-x-2y.$$

The points of T must satisfy the inequalities

$$0\le x\,(\le 2),\quad 0\le y\left(\le 1-\frac{x}{2}\right),\quad 0\le z\le 2-x-2y.$$

We immediately get

$$T=\{(x,y,z)\mid (x,y)\in B,\,0\le z\le 2-x-2y\},$$

where

$$B=\{(x,y)\mid 0\le x\le 2,\,0\le y\le 1-\frac{x}{2}\}.$$

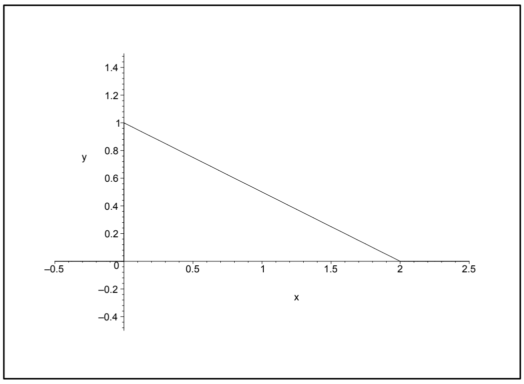


Figure 3.28: The domain B of Example 3.9.1

If we instead keep $z\in[0,2]$ fixed, the tetrahedron is cut into a triangle $B(z)$, bounded by

$$0\le\ (\le 2-z),\qquad 0\le y\le 1-\frac{x}{2}-\frac{z}{2},$$

i.e.

$$B(z) = \left\{(x,y) \;\middle|\; 0 \le x \le 2-z,\, 0 \le y \le 1 - \frac{z}{2} - \frac{x}{2}\right\}, \quad 0 \le x \le z \le 2,$$

and

$$T = \{(x,y,z) \mid (x,y) \in B(z),\, 0 \le z \le 2\}.$$

It follows that $B(z)$ is similar to B above with the factor of similarity $1 - \frac{z}{2}$.

2) We see in the same way as in Example 3.9.1 that three of the planes are described by

$$x = 0, \qquad y = 0 \quad \text{and} \quad z = 2.$$

A parametric description of the fourth plane is e.g.

$$(x,y,z) = (0,0,0) + u(2,0,2) + v(0,1,2) = (2u, v, 2u+2v),$$

from which $x = 2u$ and $y = v$. When u and v are eliminated we get

$$z = 2u + 2v = x + 2y,$$

which is an equation of the fourth plane.

The points of T must satisfy the inequalities

$$0 \le x\,(\le 2), \quad 0 \le y \left(\le 1 - \frac{x}{2}\right), \quad x + 2y \le z \le 2.$$

Hence,

$$T = \{(x,y,z) \mid (x,y) \in B,\, x + 2y \le z \le 2\},$$

where

$$B = \left\{(x,y) \;\middle|\; 0 \le x \le 2,\, 0 \le y \le 1 - \frac{x}{2}\right\}.$$

If we instead keep $z \in [0,2]$ fixed, the tetrahedron is cut into a triangle $B(z)$, bounded by

$$0 \le x \le z, \qquad 0 \le y \le \frac{z}{2} - \frac{x}{2},$$

i.e.

$$B(z) = \left\{(x,y) \;\middle|\; 0 \le x \le z,\, 0 \le y \le \frac{z}{2} - \frac{x}{2}\right\}, \quad 0 \le z \le 2,$$

and

$$T = \{(x,y,z) \mid (x,y) \in (z),\, 0 \le z \le 2\}.$$

We see that $B(z)$ is similar to B with the constant of similarity $\frac{z}{2}$.

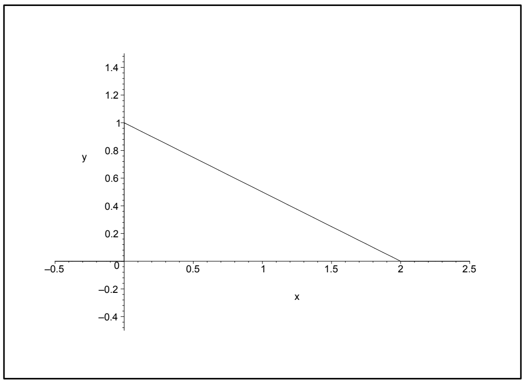


Figure 3.29: The domain B of Example 3.9.2

3) Here a trivial boundary plane is given by $y = 0$.

The points $(1, 0, 0)$, $(0, 2, 2)$, $(0, 0, 4)$ lie in the plane of the parametric description

$$(x, y, z) = (1, 0, 0) + u(-1, 2, 2) + v(-1, 0, 4) = (1 - u - v, 2u, 2u + 4v),$$

i.e.

$$x = 1 - u - v, \qquad y = 2u, \qquad z = 2u + 4v,$$

from which

$$u = \frac{y}{2}, \qquad v = 1 - u - x = 1 - \frac{y}{2} - x,$$

so

$$z = 2u + 4v = y + 4\left(1 - \frac{y}{2} - x\right) = 4 - 4x - y,$$

which is the equation of this plane.

The points $(-1, 0, 0)$, $(0, 2, 2)$, $(0, 0, 4)$ lie in the plane of the parametric description

$$(x, y, z) = (-1, 0, 0) + u(1, 2, 2) + v(1, 0, 4) = (-1 + u + v, 2u, 2u + 4v),$$

i.e.

$$x = -1 + u + v, \qquad y = 2u, \qquad z = 2u + 4v,$$

from which

$$u = \frac{y}{2}, \qquad v = 1 + x - u = 1 + x - \frac{y}{2},$$

hence

$$z = 2u + 4v = y + 4 + 4x - 2y = 4 + 4x - y,$$

which is the equation of this plane.

The points $(1,0,0)$, $(-1,0,0)$, $(0,2,2)$ lie in the plane of the parametric description

$$(x,y,z) = (-1,0,0) + u(2,0,0) + v(1,2,2) = (2u-1, 2v, 2v),$$

from which

$$x = 2u-1, \qquad y = 2v, \qquad z = 2v.$$

We see that the equation of the plane is $z = y$.

Summarizing we have obtained the four planes

$$y = 0, \quad z = 4-4x-y, \quad z = 4+4x-y, \quad z = y.$$

The projection of T onto the XY plane is the triangle B of the corners $(-1,0)$, $(1,0)$, $(0,2)$. This can be described by

$$0 \le y \le 2, \quad \frac{y}{2} - 1 \le x \le 1 - \frac{y}{2} \quad \left(|x| \le 1 - \frac{y}{2}\right),$$

i.e.

$$B = \left\{ (x,y) \;\middle|\; 0 \le y \le 2,\, |x| \le 1 - \frac{y}{2} \right\}.$$

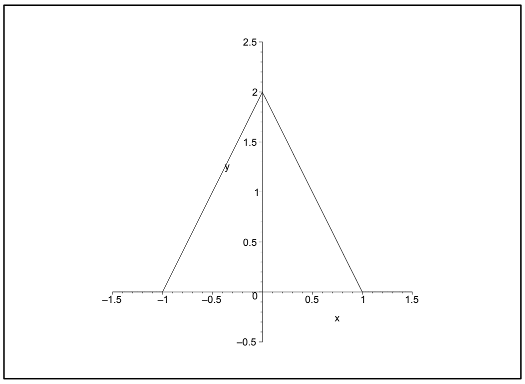


Figure 3.30: The domain B of Example 3.9.3

When $(x,y) \in B$, it is seen from the figure that

$$\begin{cases} y \le z \le 4-4x-y & \text{for } x \ge 0, \\ y \le z \le 4+4x-y & \text{for } x \le 0, \end{cases}$$

i.e.

$$T = \left\{ (x,y,z) \;\middle|\; 0 \le y \le 2,\, |x| \le 1-\frac{y}{2},\, y \le z \le 4-4|x|-y \right\}.$$

The plane $z = \text{constant} \in [2,4]$ cuts T in a triangle $B(z)$ given by

$$0 \le y \le 4-z, \qquad |x| \le 2-\frac{y}{2}-\frac{z}{2},$$

hence

$$B(z) = \left\{ (x,y) \;\middle|\; 0 \le y \le 4-z,\, |x| \le 2-\frac{y}{2}-\frac{z}{2} \right\} \quad \text{for } z \in [2,4].$$

It follows that $B(z)$ is similar to B with the factor of similarity $2-\dfrac{z}{2}$.

Then let $z \in]0,2[$ be fixed. This plane cuts T in a trapeze, which is obtained by cutting a triangle out of B at height z. Thus, for $z \in [0,2[$,

$$B(z) = \left\{ (x,y) \;\middle|\; 0 \le y \le z,\, |x| \le 1-\frac{y}{2} \right\} \quad \text{for } z \in [0,2[.$$

We get the following description of the tetrahedron:

$$\begin{aligned} T &= \left\{ (x,y,z) \;\middle|\; 0 \le z \le 2,\, 0 \le y \le z,\, |x| \le 1-\frac{y}{2} \right\} \\ &\quad \cup \left\{ (x,y,z) \;\middle|\; 2 \le z \le 4,\, 0 \le y \le 4-z,\, |x| \le 2-\frac{y}{2}-\frac{z}{2} \right\}. \end{aligned}$$

4) The obvious planes are here

$y = 0$, [points $(0,0,0)$, $(1,0,0)$, $(1,0,4)$],
$z = 0$, [points $(0,0,0)$, $(1,0,0)$, $(1,1,0)$],
$x = 1$, [points $(1,0,0)$, $(1,1,0)$, $(1,0,4)$].

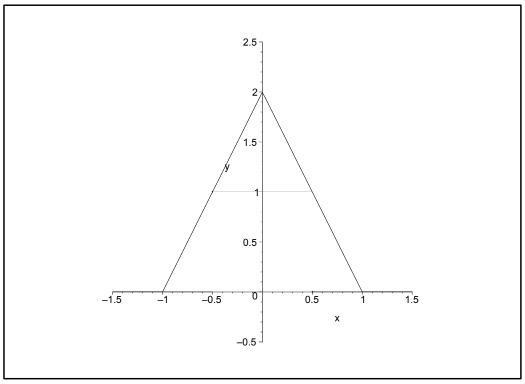


Figure 3.31: The domain $B(z)$ for $z = 1 \in [0, 2[$ in Example 3.9.3

Finally, the points $(0, 0, 0)$, $(1, 0, 4)$, $(1, 1, 0)$ lie in the plane of the parametric description

$$(x, y, z) = u(1, 0, 4) + v(1, 1, 0) = (u + v, v, 4u),$$

from which

$$v = y, \qquad u = x - v = x - y \quad \text{and} \quad z = 4u = 4x - 4y.$$

The points in T must satisfy the inequalities

$$0 \le x \le 1, \qquad 0 \le y \le x, \qquad 0 \le z \le 4x - 4y.$$

In particular, the triangle B is

$$B = \{(x, y) \mid 0 \le x \le 1,\, 0 \le y \le x\},$$

and we get

$$T = \{(x, y, z) \mid 0 \le x \le 1,\, 0 \le y \le x,\, 0 \le z \le 4x - 4y\}.$$

The plane $z = \text{constant} \in [0, 4]$ cuts the tetrahedron in a triangle which is similar to B of the similarity factor $1 - \frac{z}{4}$ for $z \in [0, 4]$), thus

$$B(z) = \left\{(x, y) \;\middle|\; 0 \le x \le 1 - \frac{z}{4},\, 0 \le y \le x\right\},$$

and accordingly,

$$T = \left\{(x, y, z) \;\middle|\; 0 \le z \le 4,\, 0 \le x \le 1 - \frac{z}{4},\, 0 \le y \le x\right\}.$$

5) The obvious planes are

$$\begin{aligned} y &= 0, \qquad [\text{points } (1, 0, 0),\, (-1, 0, 0),\, (0, 0, 4)], \\ z &= 0, \qquad [\text{points } (1, 0, 0),\, (0, 2, 0),\, (-1, 0, 0)]. \end{aligned}$$

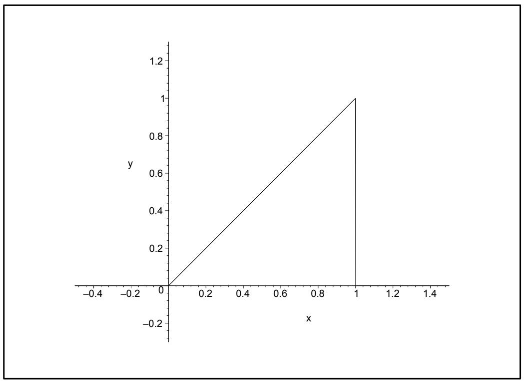


Figure 3.32: The domain B in Example 3.9.4

The points $(1,0,0)$, $(0,2,0)$, $(0,0,4)$ lie in the plane of the parametric description

$$(x,y,z)=(1,0,0)+u(-1,2,0)+v(-1,0,4)=(1-u-v,2u,4v).$$

Hence,

$$u=\frac{y}{2},\quad v=\frac{z}{4},\quad x=1-u-v=1-\frac{y}{2}-\frac{z}{4},$$

and we get the following equation of the plane,

$$z=4-4x-2y.$$

Due to the symmetry the points $(-1,0,0)$, $(0,2,0)$ and $(0,0,4)$ must lie in the plane of the equation

$$z=4+4x-2y.$$

The projection of T onto the XY plane is the triangle

$$B=\left\{(x,y)\;\middle|\;0\le y\le 2,\,|x|\le 1-\frac{y}{2}\right\}.$$

When $(x,y)\in B$, we get for $(x,y,z)\in T$ that

$$\begin{cases}0\le z\le 4-4x-2y, & \text{for } x\ge 0,\\ 0\le z\le 4+4x-2y, & \text{for } x\le 0,\end{cases}$$

i.e.

$$T=\left\{(x,y,z)\;\middle|\;0\le y\le 2,\,|x|\le 1-\frac{y}{2},\,0\le z\le 4-4|x|-2y\right\}.$$

At the height $z\in[0,4]$ the tetrahedron T is cut into a triangle

$$B(z)=\left\{(x,y)\;\middle|\;0\le y\le 2-\frac{z}{2},\,|x|\le 1-\frac{y}{2}-\frac{z}{4}\right\},$$

where $B(z)$ is similar to B of the similarity factor $1-\dfrac{z}{4}$, hence

$$T=\left\{(x,y,z)\;\middle|\;0\le z\le 4,\,0\le y\le 2-\frac{z}{2},\,|x|\le 1-\frac{y}{2}-\frac{z}{4}\right\}.$$

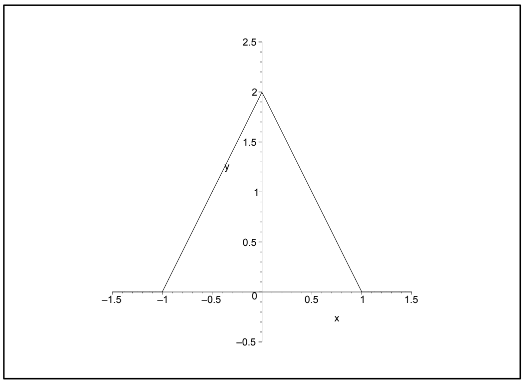


Figure 3.33: The domain B

Example 3.10 *Sketch on a figure the set A, where*

$$A = \{(x,y) \in \mathbb{R}^2 \mid x + 2y \leq 2,\ |x - y| \leq 2\}.$$

On the figure one should indicate the boundary ∂A. Finally, explain why A is not bounded.

A Sketch of a set in the plane.

D Start by analyzing the lines, which bound the set.

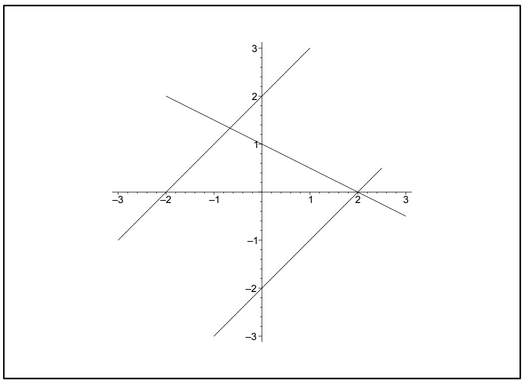


Figure 3.34: The domain A in Example 3.10 is that component of the plane, which contains the point $(0,0)$.

I It follows from the definition of A that we have the three restrictions

$$x + 2y \leq 2, \qquad x - y \leq 2, \qquad y - x \leq 2.$$

We note that $(0,0)$ satisfies all three inequalities. Thus, the domain A is the closed component (the intersection of three closed half planes), which contains $(0,0)$. The boundary ∂A consists of pieces of the lines

$$x + 2y = 2, \quad x - y = 2, \qquad y - x = 2.$$

Now, the unbounded half line

$$\{(x,y) \mid y = x-2, x \le 2\}$$

lies in A, so A must also be unbounded.

3.2 Conics and conical sections

Example 3.11 *A conic $\mathcal{F}$ is given by the equation*

$$2x^2 - 2y^2 + \alpha z^2 = 1,$$

where α is a real constant.

1) *Find the values of α, for which $\mathcal{F}$ is a surface of revolution. Indicate in each of these cases the type of the surface and its axis of symmetry.*

2) *Prove that there is one value of α, for which the surface $\mathcal{F}$ is a cylindric surface. Indicate for this value of α the type of the surface and its axis of symmetry.*

A Conic sections.

D Analyze each of the three cases $\alpha < 0$, $\alpha = 0$ and $\alpha > 0$.

I 1) a) When $\alpha < 0$, the conic is an hyperboloid with two sheets:

$$1 = \left(\frac{x}{1\sqrt{2}}\right)^2 - \left\{\left(\frac{y}{1/\sqrt{2}}\right)^2 + \left(\frac{z}{\sqrt{1/|\alpha|}}\right)^2\right\}.$$

This is an hyperboloid of revolution for $\alpha = -2$, where the X axis is the axis of revolution.

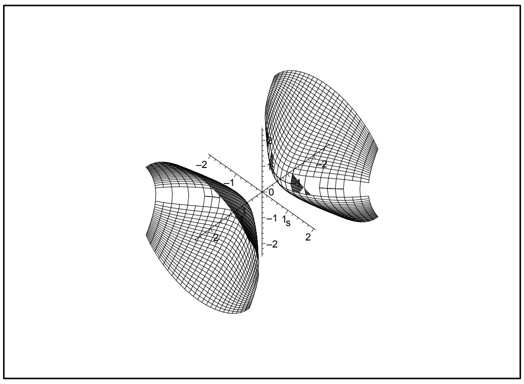


Figure 3.35: The surface of revolution for $\alpha = -2$.

b) When $\alpha > 0$, the conic is an hyperboloid with one sheet:

$$1 = \left\{\left(\frac{x}{\frac{1}{\sqrt{2}}}\right)^2 + \left(\frac{z}{\sqrt{\frac{1}{\alpha}}}\right)^2\right\} - \left(\frac{y}{\frac{1}{\sqrt{2}}}\right)^2.$$

This becomes an hyperboloid of revolution when $\alpha = 2$, with the Y axis as its axis of revolution.

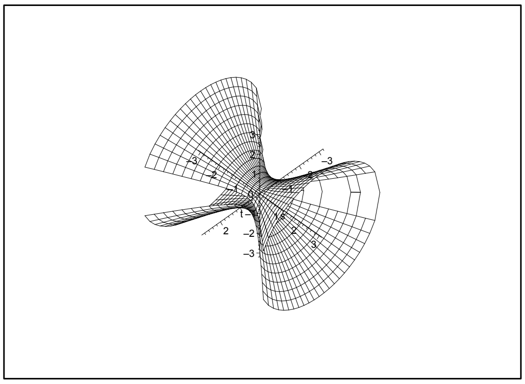


Figure 3.36: The surface of revolution for $\alpha = 2$.

2) When $\alpha = 0$, we get an hyperbolic cylindric surface with the Z axis as its axis of generation,

$$1 = \left(\frac{x}{\frac{1}{\sqrt{2}}} \right)^2 - \left(\frac{y}{\frac{1}{\sqrt{2}}} \right)^2 .$$

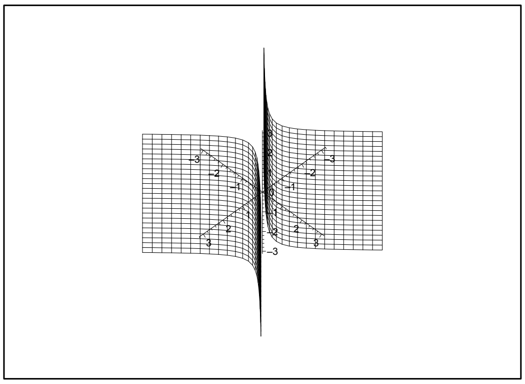


Figure 3.37: The surface for $\alpha = 0$.

Example 3.12 *Find the type and position of the conic of the equation*

$$x^2+2y^2-x+6y+\frac{3}{4}=0.$$

A Conic section.

D Translate the coordinates.

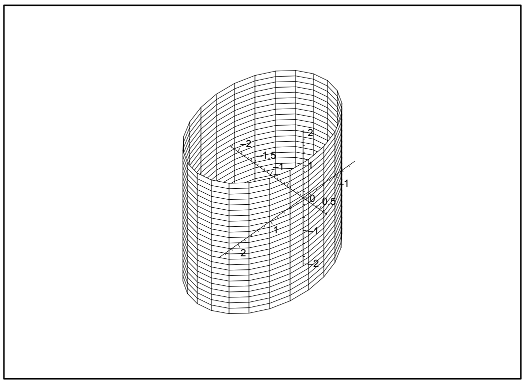
I By a rearrangement we get

$$\begin{aligned} 0 &= x^2+2y^2-x+6y+\frac{3}{4} \\ &= \left(x^2-2\cdot\frac{1}{2}\,x+\frac{1}{4}\right)+2\left(y^2+2\cdot\frac{3}{2}\,y+\frac{9}{4}\right)-2\cdot\frac{9}{4}+\frac{3}{4} \\ &= \left(x-\frac{1}{2}\right)^2+2\left(y+\frac{3}{2}\right)^2-4, \end{aligned}$$

i.e. in the canonical form

$$\left(\frac{x-\frac{1}{2}}{2}\right)^2+\left(\frac{y+\frac{3}{2}}{\sqrt{2}}\right)^2+0\cdot z^2=1,$$

because z does not appear in the equation.



The surface is an *elliptic cylindric surface* with the Z axis as its axis of generation, and with the ellipse of centre $\left(\frac{1}{2}\,,\,-\frac{3}{2}\right)$ and the half axes 2 and $\sqrt{2}$ as generating curve.

Example 3.13 *Let a, b, c be constant different from zero satisfying the equation*

$$a+b+c=0.$$

Prove that the plane of the equation

$$x+y+z=0$$

cuts the conic given by

$$\frac{yz}{a}+\frac{zx}{b}+\frac{xy}{c}=0$$

in two straight lines (generators), which form an angle of $\frac{2\pi}{3}$.

A Intersection of two surfaces.

D Start by e.g. eliminating $z=-x-y$.

I Clearly, $(0,0,0)$ lies in the intersection of the two surfaces. Furthermore, if two of the variables are 0, e.g. $x=y=0$, then we have a point on the conic, no matter the value of the third variable (here z). We conclude that the X, the Y and the Z axes all lie on the conic section. Of course, none of then are contained in the oblique plane $x+y+z=0$.

If we keep off the coordinate planes, i.e. we assume in the following that $xyz \neq 0$, then the equation of the conic can also be written

$$0=\frac{yz}{a}+\frac{zx}{b}+\frac{xy}{x}=xyz\left(\frac{1}{ax}+\frac{1}{by}+\frac{1}{cz}\right),$$

i.e.

$$\frac{1}{ax}+\frac{1}{by}+\frac{1}{cz}=0 \qquad \text{for } xyz \neq 0.$$

Since $z=-(x+y)$ on the plane, we get by insertion into the reduced equation of the conic that

$$0=\frac{1}{ax}+\frac{1}{by}+\frac{1}{cz}=\frac{1}{ax}+\frac{1}{by}-\frac{1}{c(x+y)}.$$

When we put everything here into the same fraction and reduce we get

$$(3.1)\quad 0=\frac{1}{a}\,(x+y)y+\frac{1}{b}\,(x+y)x-\frac{1}{c}\,xy,$$

which is an homogeneous polynomial of second degree in (x,y).

Now $x=0$, if and only if $y=0$, so the solutions must have the structure

$$(3.2)\quad y=\alpha\, x, \qquad \alpha \neq 0.$$

It follows that the intersection of the two surfaces must have the structure

$$\mathbf{r}(t)=(t,\alpha t,-(1+\alpha)t)=t(1,\alpha,-(1+\alpha)), \qquad t \in \mathbb{R},$$

because $z = -x - y$, and because we can trivially continue to $(0,0,0)$.

When (3.2) is put into (3.1), we get that α is a solution of a polynomial of second degree with the roots α_1 and α_2, corresponding to two straight lines. (According to the geometry the solutions exist, so we must necessarily have the the roots α_1 and α_2 are real numbers).

By insertion of $(x,y,z) = (1,\alpha,-(1+\alpha))$ we get for $\alpha \neq -1$ that

$$0 = \frac{1}{ax} + \frac{1}{by} + \frac{1}{cz} = \frac{1}{a} + \frac{1}{b\alpha} - \frac{1}{c(1+\alpha)} = \frac{bc\alpha(1+\alpha) + ac(1+\alpha) - ab\alpha}{abc\alpha(1+\alpha)},$$

which is reduced to

$$0 = \alpha(1+\alpha) + \frac{a}{b}(1+\alpha) - \frac{a}{c}\alpha = \alpha^2 + \left(1 + \frac{a}{b} - \frac{a}{c}\right)\alpha + \frac{a}{b} = \alpha^2 + a\left(\frac{1}{a} + \frac{1}{b} - \frac{1}{c}\right)\alpha + \frac{a}{b},$$

hence

$$\alpha_1 + \alpha_2 = a\left(\frac{1}{c} - \frac{1}{a} - \frac{1}{b}\right) = a\left(\frac{1}{c} - \frac{a+b}{ab}\right) = \frac{a}{c} + \frac{c}{b}$$

and

$$\alpha_1\alpha_2 = \frac{a}{b}.$$

Since $(1, \alpha, -(1+\alpha))$ is of length

$$\sqrt{1+\alpha^2+(1+\alpha)^2} = \sqrt{2(1+\alpha+\alpha^2)},$$

The angle φ between the two lines (which both pass through $(0,0,0)$) is given by

$$\cos\varphi = \frac{(1,\alpha_1,-(1+\alpha_1))}{\sqrt{2(1+\alpha_1+\alpha_1^2)}} \cdot \frac{(1,\alpha_2,-(1+\alpha_2))}{\sqrt{2(1+\alpha_2+\alpha_2^2)}} = \frac{1}{2} \cdot \frac{1+\alpha_1\alpha_2+(1+\alpha_1)(1+\alpha_2)}{\sqrt{(1+\alpha_1+\alpha_1^2)(1+\alpha_2+\alpha_2^2)}}.$$

Here the numerator is

$$\begin{aligned}1+\alpha_1\alpha_2+(1+\alpha_1)(1+\alpha_2) &= 2+(\alpha_1+\alpha_2)+2\alpha_1\alpha_2 = 2+\frac{ab+c^2}{bc}+2\,\frac{a}{b}\\ &= \frac{1}{bc}\left\{2bc-(b+c)b+c^2-2(b+c)c\right\} = -\frac{1}{bc}\,(b^2+bc+c^2),\end{aligned}$$

and the radicand is

$$\begin{aligned}&(1+\alpha_1+\alpha_1^2)(1+\alpha_2+\alpha_2^2)\\ &\quad= 1+\alpha_1+\alpha_2+\alpha_1^2+\alpha_2^2+\alpha_1\alpha_2+\alpha_1+\alpha_2^2+\alpha_1^2\alpha_2+\alpha_1^2\alpha_2^2\\ &\quad= 1+(\alpha_1+\alpha_2)+(\alpha_1+\alpha_2)^2-\alpha_1\alpha_2+\alpha_1\alpha_2(\alpha_1+\alpha_2)+(\alpha_1\alpha_2)^2\\ &\quad= 1+\frac{ab+c^2}{bc}+\left(\frac{ab+c^2}{bc}\right)^2-\frac{a}{b}+\frac{a}{b}\cdot\frac{ab+c^2}{bc}+\frac{a^2}{b^2}\\ &\quad= \frac{1}{b^2c^2}\{b^2c^2+ab^2c+bc^3+a^2b^2+2abc^2+c^4-abc^2+a^2bc+ac^3+a^2c^2\}\\ &\quad= \frac{1}{b^2c^2}\{b^2c^2+bc^3+c^4+a(b^2c+2bc^2-bc^2+c^3)+a^2(b^2+bc+c^2)\}\\ &\quad= \frac{1}{b^2c^2}\{c^2(b^2+bc+c^2)+ac(b^2+bc+c^2)+a^2(b^2+bc+c^2)\}\\ &\quad= \frac{1}{b^2c^2}\,(b^2+bc+c^2)(c^2+ac+a^2) = \frac{1}{b^2c^2}\,(b^2+bc+c^2)(c^2+(-b-c)(-b))\\ &\quad= \frac{1}{b^2c^2}\,(b^2+bc+c^2)^2.\end{aligned}$$

Then by insertion

$$\cos\varphi = \frac{1}{2}\cdot\frac{1+\alpha_1\alpha_2+(1+\alpha_1)(1+\alpha_2)}{\sqrt{(1+\alpha_1+\alpha_1^2)(1+\alpha_2+\alpha_2^2)}} = \frac{1}{2}\cdot\frac{-\frac{1}{bc}\,(b^2+bc+c^2)}{\left|\frac{1}{bc}\,(b^2+bc+c^2)\right|}.$$

Since $b^2+bc+c^2 = \left(b+\frac{1}{2}\,c\right)^2+\frac{3}{4}\,c^2 > 0$, we have

$$\cos\varphi = -\frac{1}{2}\,\frac{|bc|}{bc} = -\frac{1}{2}\,\frac{bc}{|bc|} = \begin{cases}\ \ \frac{1}{2}, & \text{if } bc<0,\\ -\frac{1}{2}, & \text{hvis } bc>0.\end{cases}$$

Hence $\varphi = \frac{\pi}{3}$, if $bc<0$, and $\varphi = \frac{2\pi}{3}$ $\left(\text{or } -\frac{\pi}{3}\right)$, if $bc>0$.

If we do not include the sign of the angle we get $\varphi = \frac{\pi}{3}$.

Example 3.14 *Indicate for each value of the constant k the type of the conic $\mathcal{F}$, which is given by the equation*

$$x^2 + (4-k^2)y^2 + k(2-k)z^2 = 2k,$$

and find in particular those values of k, for which $\mathcal{F}$ is a surface of revolution. Finally, think about if it makes sense to put k equal to $+\infty$ or $-\infty$.

A Conics.

D Discuss the sign of the coefficients and then consider the various cases.

I By considering the signs we get the scheme

	$k<-2$	$k=-2$	$-2<k<0$	$k=0$	$0<k<2$	$k=2$	$k>2$
$4-k^2$	$-$	0	$+$	$+$	$+$	0	$-$
$k(2-k)$	$-$	$-$	$-$	0	$+$	0	$-$
$2k$	$-$	$-$	$-$	0	$+$	$+$	$+$
	1	2	3	4	5	6	7

1) When $k<-2$, we get the canonical form (notice the absolute values)

$$\begin{aligned} 1 &= -\frac{1}{2|k|}\,x^2 + \left|\frac{4-k^2}{2k}\right|\,y^2 + \left|\frac{2-k}{2}\right|\,z^2 \\ &= -\frac{1}{2|k|}\,x^2 + \frac{4-k^2}{2k}\,y^2 + \frac{2-k}{2}\,z^2. \end{aligned}$$

Since we have 2 plus and 1 minus we conclude that we have an *hyperboloid with one sheet.*

2) When $k=-2$, the equation is written

$$x^2 - 8z^2 = -4, \quad \text{dvs.} \quad -\left(\frac{x}{2}\right)^2 + \left(\frac{z}{1/\sqrt{2}}\right)^2 = 1,$$

which describes an *hyperbolic cylindric surface.*

3) When $-2<k<0$, the canonical form becomes

$$-\frac{1}{2|k|}\,x^2 - \left|\frac{4-k^2}{2k}\right|\,y^2 \left|\frac{2-k}{2}\right|\,z^2 = 1.$$

With 1 plus and 2 minus we conclude that we get an *hyperboloid with two sheets.*

4) When $k=0$, the equation is written

$$x^2 + 4y^2 = 0,$$

which is satisfied for the Z *axis.* (Degenerated "surface of revolution").

5) When $0<k<2$, we rewrite to the canonical form

$$\left|\frac{1}{2k}\right|\,x^2 + \left|\frac{4-k^2}{2k}\right|\,y^2 + \left|\frac{2-k}{2}\right|\,z^2 = 1.$$

With 3 plus we get an *ellipsoid.*

6) When $k = 2$, the equation is written

$$x^2 = 4,$$

which describes two planes $x = \pm 2$, parallel to the YZ plane.

7) When $k > 2$, we get

$$\left|\frac{1}{2k}\right| x^2 - \left|\frac{4-k^2}{2k}\right| y^2 - \left|\frac{2-k}{k}\right| z^2 = 1.$$

With 1 plus and 2 minus we see that we get an *hyperboloid with two sheets.*

We obtain surfaces of revolution when

1) $x^2 + (4-k^2)y^2 = x^2 + y^2$, i.e. when $k = \pm\sqrt{3}$.
2) $x^2 + k(2-k)z^2 = x^2 + z^2$, i.e. when $k = 1$.
3) $4 - k^2 = k(2-k)$, i.e. $k = 2$, which however produces degenerated surfaces of revolution.
4) $k = 0$ gives Z axis as the degenerated "surface of revolution".

1) When $k = -\sqrt{3}$ we are in case 3., so we have an *hyperboloid of revolution with two sheets* where the Z axis is the axis of revolution.
2) When $k = 0$ we are in case 4., which is the degenerated case of the Z axis. The Z axis is clearly the axis of revolution.
3) When $k = 1$ we are in case 5., and we get an *ellipsoid of revolution* with the Y axis as the axis of revolution.
4) When $k = \sqrt{3}$ we are again in case 5., so we get an *ellipsoid of revolution* with the Z axis as the axis of revolution.
5) When $k = 2$ we are in the degenerated case 6. The two planes have clearly the X axis as the axis of revolution.

When $k \neq 0$, we get by dividing by $-k^2$ that

$$-\frac{1}{k^2}\, x^2 + \left(1 - \frac{4}{k^2}\right) y^2 + \left(1 - \frac{2}{k}\right) z^2 = -\frac{2}{k}.$$

Then it follows immediately by taking the limits $k \to +\infty$ or $k \to -\infty$,

$$y^2 + z^2 = 0,$$

so $y = z = 0$, while x is free. Therefore, by taking the limits we get the X axis

Example 3.15 *The surfaces $\mathcal{F}_1$ and $\mathcal{F}_2$ are given by the equations*

$$x^2 + 2y^2 = z + 1, \qquad x^2 + 2y^2 = -1 + 3z^2.$$

1) Indicate the type and the top point(s) of both $\mathcal{F}_1$ and $\mathcal{F}_2$.

2) Prove that the intersection $\mathcal{F}_1 \cap \mathcal{F}_2$ consists of two ellipses, lying in planes, which are parallel to the (X, Y) plane.

A Conics and conic sections.

D In 1) we just reformulate the equations to the canonical form. In 2) we first eliminate $x^2 + 2y^2$ in order to get an equation in z. Then insert the solutions in z into one of the original expressions.

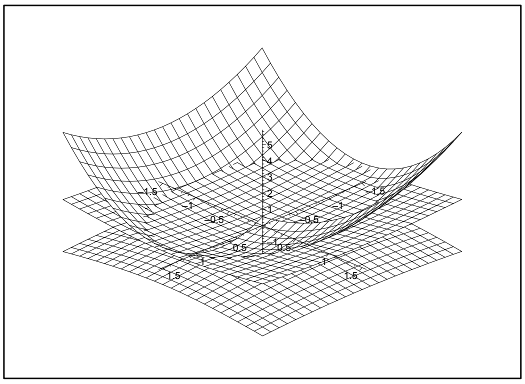


Figure 3.38: The surfaces $\mathcal{F}_1$ and $\mathcal{F}_2$.

I 1) If we put $z_1 = z + 1$, we see that the equation of the surface $\mathcal{F}_1$ can be written in its canonical form

$$\frac{x^2}{1^2} + \frac{y^2}{\left(\frac{1}{\sqrt{2}}\right)^2} = z_1,$$

which shows that $\mathcal{F}_1$ is an elliptic paraboloid with top point $(0, 0, -1)$.

Then the equation of $\mathcal{F}_2$ is written in the following way:

$$-\frac{x^2}{1^2} - \frac{y^2}{\left(\frac{1}{\sqrt{2}}\right)^2} + \frac{z^2}{\left(\frac{1}{\sqrt{3}}\right)^2} = 1.$$

This equation describes an hyperboloid with two sheets. The top points are

$$\left(0, 0, \pm\frac{1}{\sqrt{3}}\right).$$

2) The equation of the intersection is obtained by eliminating the common expression $x^2 + 2y^2$ in (x, y). This gives

$$z + 1 = -1 + 3z^2, \qquad \text{i.e.} \qquad 3z^2 - 2 - 2 = 3(z-1)\left(z + \frac{2}{3}\right) = 0.$$

The solutions are $z = 1$ and $z = -\frac{2}{3}$, so the intersection curves lie in these two planes which are parallel to the (X, Y) plane.

a) When we put $z = 1$, we get $x^2 + 2y^2 = 2$, which in its canonical form becomes

$$\frac{x^2}{(\sqrt{2})^2} + \frac{y^2}{1^2} = 1.$$

This is an equation of an ellipse in the plane $z = 1$ of centrum $(0, 0)$ and half axes $\sqrt{2}$ and 1.

b) If we put $z = -\frac{2}{3}$, we get $x^2 + 2y^2 = \frac{1}{3}$, which is written in its canonical form in the following way

$$\frac{x^2}{\left(\frac{1}{\sqrt{3}}\right)^2} + \frac{y^2}{\left(\frac{1}{\sqrt{6}}\right)^2} = 1.$$

This is an equation of an ellipse in the plane $z = -\frac{2}{3}$ of centre $(0, 0)$ and half axes $\frac{1}{\sqrt{3}}$ and $\frac{1}{\sqrt{6}}$.

4 Formulæ

Some of the following formulæ can be assumed to be known from high school. It is highly recommended that one *learns most of these formulæ in this appendix by heart.*

4.1 Squares etc.

The following simple formulæ occur very frequently in the most different situations.

$$
\begin{array}{ll}
(a+b)^2 = a^2+b^2+2ab, & a^2+b^2+2ab = (a+b)^2, \\
(a-b)^2 = a^2+b^2-2ab, & a^2+b^2-2ab = (a-b)^2, \\
(a+b)(a-b) = a^2-b^2, & a^2-b^2 = (a+b)(a-b), \\
(a+b)^2 = (a-b)^2+4ab, & (a-b)^2 = (a+b)^2-4ab.
\end{array}
$$

4.2 Powers etc.

Logarithm:

$$
\begin{array}{rcl}
\ln|xy| = & \ln|x|+\ln|y|, & x,y \neq 0, \\
\ln\left|\dfrac{x}{y}\right| = & \ln|x|-\ln|y|, & x,y \neq 0, \\
\ln|x^r| = & r\ln|x|, & x \neq 0.
\end{array}
$$

Power function, fixed exponent:

$$(xy)^r = x^r \cdot y^r, x,y>0 \quad \text{(extensions for some } r\text{)},$$

$$\left(\frac{x}{y}\right)^r = \frac{x^r}{y^r}, x,y>0 \quad \text{(extensions for some } r\text{)}.$$

Exponential, fixed base:

$$a^x \cdot a^y = a^{x+y}, \quad a>0 \quad \text{(extensions for some } x,\ y\text{)},$$
$$(a^x)^y = a^{xy}, a>0 \quad \text{(extensions for some } x,\ y\text{)},$$

$$a^{-x} = \frac{1}{a^x}, a>0, \quad \text{(extensions for some } x\text{)},$$

$$\sqrt[n]{a} = a^{1/n}, a \geq 0, \qquad n \in \mathbb{N}.$$

Square root:

$$\sqrt{x^2} = |x|, \qquad x \in \mathbb{R}.$$

Remark 4.1 It happens quite frequently that students make errors when they try to apply these rules. They must be mastered! In particular, as one of my friends once put it: "If you can master the square root, you can master everything in mathematics!" Notice that this innocent looking square root is one of the most difficult operations in Calculus. Do not forget the *absolute value!* $\diamond$

4.3 Differentiation

Here are given the well-known rules of differentiation together with some rearrangements which sometimes may be easier to use:

$$\{f(x) \pm g(x)\}' = f'(x) \pm g'(x),$$

$$\{f(x)g(x)\}' = f'(x)g(x) + f(x)g'(x) = f(x)g(x)\left\{\frac{f'(x)}{f(x)} + \frac{g'(x)}{g(x)}\right\},$$

where the latter rearrangement presupposes that $f(x) \neq 0$ and $g(x) \neq 0$.
If $g(x) \neq 0$, we get the usual formula known from high school

$$\left\{\frac{f(x)}{g(x)}\right\}' = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}.$$

It is often more convenient to compute this expression in the following way:

$$\left\{\frac{f(x)}{g(x)}\right\} = \frac{d}{dx}\left\{f(x) \cdot \frac{1}{g(x)}\right\} = \frac{f'(x)}{g(x)} - \frac{f(x)g'(x)}{g(x)^2} = \frac{f(x)}{g(x)}\left\{\frac{f'(x)}{f(x)} - \frac{g'(x)}{g(x)}\right\},$$

where the former expression often is *much easier* to use in practice than the usual formula from high school, and where the latter expression again presupposes that $f(x) \neq 0$ and $g(x) \neq 0$. Under these assumptions we see that the formulæ above can be written

$$\frac{\{f(x)g(x)\}'}{f(x)g(x)} = \frac{f'(x)}{f(x)} + \frac{g'(x)}{g(x)},$$

$$\frac{\{f(x)/g(x)\}'}{f(x)/g(x)} = \frac{f'(x)}{f(x)} - \frac{g'(x)}{g(x)}.$$

Since

$$\frac{d}{dx}\ln|f(x)| = \frac{f'(x)}{f(x)}, \qquad f(x) \neq 0,$$

we also name these the *logarithmic derivatives.*

Finally, we mention the rule of **differentiation of a composite function**

$$\{f(\varphi(x))\}' = f'(\varphi(x)) \cdot \varphi'(x).$$

We first differentiate the function itself; then the insides. This rule is a 1-dimensional version of the so-called *Chain rule.*

4.4 Special derivatives.

Power like:

$$\frac{d}{dx}(x^\alpha) = \alpha \cdot x^{\alpha-1}, \qquad \text{for } x > 0, \ \text{(extensions for some } \alpha).$$

$$\frac{d}{dx}\ln|x| = \frac{1}{x}, \qquad \text{for } x \neq 0.$$

Exponential like:

$$\frac{d}{dx}\exp x = \exp x, \qquad \text{for } x \in \mathbb{R},$$

$$\frac{d}{dx}(a^x) = \ln a \cdot a^x, \qquad \text{for } x \in \mathbb{R} \text{ and } a > 0.$$

Trigonometric:

$$\frac{d}{dx}\sin x = \cos x, \qquad \text{for } x \in \mathbb{R},$$

$$\frac{d}{dx}\cos x = -\sin x, \qquad \text{for } x \in \mathbb{R},$$

$$\frac{d}{dx}\tan x = 1 + \tan^2 x = \frac{1}{\cos^2 x}, \qquad \text{for } x \neq \frac{\pi}{2} + p\pi, p \in \mathbb{Z},$$

$$\frac{d}{dx}\cot x = -(1 + \cot^2 x) = -\frac{1}{\sin^2 x}, \qquad \text{for } x \neq p\pi, p \in \mathbb{Z}.$$

Hyperbolic:

$$\frac{d}{dx}\sinh x = \cosh x, \qquad \text{for } x \in \mathbb{R},$$

$$\frac{d}{dx}\cosh x = \sinh x, \qquad \text{for } x \in \mathbb{R},$$

$$\frac{d}{dx}\tanh x = 1 - \tanh^2 x = \frac{1}{\cosh^2 x}, \qquad \text{for } x \in \mathbb{R},$$

$$\frac{d}{dx}\coth x = 1 - \coth^2 x = -\frac{1}{\sinh^2 x}, \qquad \text{for } x \neq 0.$$

Inverse trigonometric:

$$\frac{d}{dx}\text{ Arcsin } x = \frac{1}{\sqrt{1 - x^2}}, \qquad \text{for } x \in \,]-1, 1\,[,$$

$$\frac{d}{dx}\text{ Arccos } x = -\frac{1}{\sqrt{1 - x^2}}, \qquad \text{for } x \in \,]-1, 1\,[,$$

$$\frac{d}{dx}\text{ Arctan } x = \frac{1}{1 + x^2}, \qquad \text{for } x \in \mathbb{R},$$

$$\frac{d}{dx}\text{ Arccot } x = \frac{1}{1 + x^2}, \qquad \text{for } x \in \mathbb{R}.$$

Inverse hyperbolic:

$$\frac{d}{dx}\text{ Arsinh } x = \frac{1}{\sqrt{x^2 + 1}}, \qquad \text{for } x \in \mathbb{R},$$

$$\frac{d}{dx}\text{ Arcosh } x = \frac{1}{\sqrt{x^2 - 1}}, \qquad \text{for } x \in \,]1, +\infty\,[,$$

$$\frac{d}{dx}\text{ Artanh } x = \frac{1}{1 - x^2}, \qquad \text{for } |x| < 1,$$

$$\frac{d}{dx}\text{ Arcoth } x = \frac{1}{1 - x^2}, \qquad \text{for } |x| > 1.$$

Remark 4.2 The derivative of the trigonometric and the hyperbolic functions are to some extent exponential like. The derivatives of the inverse trigonometric and inverse hyperbolic functions are power like, because we include the logarithm in this class. $\Diamond$

4.5 Integration

The most obvious rules are dealing with linearity

$$\int \{f(x) + \lambda g(x)\}\,dx = \int f(x)\,dx + \lambda \int g(x)\,dx, \qquad \text{where } \lambda \in \mathbb{R} \text{ is a constant,}$$

and with the fact that differentiation and integration are "inverses to each other", i.e. modulo some arbitrary constant $c \in \mathbb{R}$, which often tacitly is missing,

$$\int f'(x)\,dx = f(x).$$

If we in the latter formula replace $f(x)$ by the product $f(x)g(x)$, we get by reading from the right to the left and then differentiating the product,

$$f(x)g(x) = \int \{f(x)g(x)\}'\,dx = \int f'(x)g(x)\,dx + \int f(x)g'(x)\,dx.$$

Hence, by a rearrangement

The rule of partial integration:

$$\int f'(x)g(x)\,dx = f(x)g(x) - \int f(x)g'(x)\,dx.$$

The differentiation is moved from one factor of the integrand to the other one by changing the sign and adding the term $f(x)g(x)$.

Remark 4.3 This technique was earlier used a lot, but is almost forgotten these days. It must be revived, because MAPLE and pocket calculators apparently do not know it. It is possible to construct examples where these devices cannot give the exact solution, unless you first perform a partial integration yourself. $\Diamond$

Remark 4.4 This method can also be used when we estimate integrals which cannot be directly calculated, because the antiderivative is not contained in e.g. the catalogue of MAPLE. The idea is by a succession of partial integrations to make the new integrand smaller. $\Diamond$

Integration by substitution:

If the integrand has the special structure $f(\varphi(x)) \cdot \varphi'(x)$, then one can change the variable to $y = \varphi(x)$:

$$\int f(\varphi(x)) \cdot \varphi'(x)\,dx = \text{“}\int f(\varphi(x))\,d\varphi(x)\text{”} = \int_{y=\varphi(x)} f(y)\,dy.$$

Integration by a monotonous substitution:

If $\varphi(y)$ is a *monotonous* function, which maps the y-interval *one-to-one* onto the x-interval, then

$$\int f(x)\,dx = \int_{y=\varphi^{-1}(x)} f(\varphi(y))\varphi'(y)\,dy.$$

Remark 4.5 This rule is usually used when we have some "ugly" term in the integrand $f(x)$. The idea is to put this ugly term equal to $y = \varphi^{-1}(x)$. When e.g. x occurs in $f(x)$ in the form $\sqrt{x}$, we put $y = \varphi^{-1}(x) = \sqrt{x}$, hence $x = \varphi(y) = y^2$ and $\varphi'(y) = 2y$. $\Diamond$

4.6 Special antiderivatives

Power like:

$$\int \frac{1}{x}\,dx = \ln|x|, \qquad \text{for } x \neq 0. \quad \text{(Do not forget the numerical value!)}$$

$$\int x^\alpha\,dx = \frac{1}{\alpha+1}x^{\alpha+1}, \qquad \text{for } \alpha \neq -1,$$

$$\int \frac{1}{1+x^2}\,dx = \text{ Arctan } x, \qquad \text{for } x \in \mathbb{R},$$

$$\int \frac{1}{1-x^2}\,dx = \frac{1}{2}\ln\left|\frac{1+x}{1-x}\right|, \qquad \text{for } x \neq \pm 1,$$

$$\int \frac{1}{1-x^2}\,dx = \text{ Artanh } x, \qquad \text{for } |x| < 1,$$

$$\int \frac{1}{1-x^2}\,dx = \text{ Arcoth } x, \qquad \text{for } |x| > 1,$$

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \text{ Arcsin } x, \qquad \text{for } |x| < 1,$$

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = -\text{ Arccos } x, \qquad \text{for } |x| < 1,$$

$$\int \frac{1}{\sqrt{x^2+1}}\,dx = \text{ Arsinh } x, \qquad \text{for } x \in \mathbb{R},$$

$$\int \frac{1}{\sqrt{x^2+1}}\,dx = \ln(x+\sqrt{x^2+1}), \qquad \text{for } x \in \mathbb{R},$$

$$\int \frac{x}{\sqrt{x^2-1}}\,dx = \sqrt{x^2-1}, \qquad \text{for } x \in \mathbb{R},$$

$$\int \frac{1}{\sqrt{x^2-1}}\,dx = \text{ Arcosh } x, \qquad \text{for } x > 1,$$

$$\int \frac{1}{\sqrt{x^2-1}}\,dx = \ln|x+\sqrt{x^2-1}|, \qquad \text{for } x > 1 \text{ eller } x < -1.$$

There is an error in the programs of the pocket calculators TI-92 and TI-89. The *numerical signs* are missing. It is obvious that $\sqrt{x^2-1} < |x|$ so if $x < -1$, then $x + \sqrt{x^2-1} < 0$. Since you cannot take the logarithm of a negative number, these pocket calculators will give an error message.

Exponential like:

$$\int \exp x \, dx = \exp x, \qquad \text{for } x \in \mathbb{R},$$

$$\int a^x \, dx = \frac{1}{\ln a} \cdot a^x, \qquad \text{for } x \in \mathbb{R}, \text{ and } a > 0, a \neq 1.$$

Trigonometric:

$$\int \sin x \, dx = -\cos x, \qquad \text{for } x \in \mathbb{R},$$

$$\int \cos x \, dx = \sin x, \qquad \text{for } x \in \mathbb{R},$$

$$\int \tan x \, dx = -\ln |\cos x|, \qquad \text{for } x \neq \frac{\pi}{2} + p\pi, \quad p \in \mathbb{Z},$$

$$\int \cot x \, dx = \ln |\sin x|, \qquad \text{for } x \neq p\pi, \quad p \in \mathbb{Z},$$

$$\int \frac{1}{\cos x} \, dx = \frac{1}{2} \ln \left(\frac{1 + \sin x}{1 - \sin x} \right), \qquad \text{for } x \neq \frac{\pi}{2} + p\pi, \quad p \in \mathbb{Z},$$

$$\int \frac{1}{\sin x} \, dx = \frac{1}{2} \ln \left(\frac{1 - \cos x}{1 + \cos x} \right), \qquad \text{for } x \neq p\pi, \quad p \in \mathbb{Z},$$

$$\int \frac{1}{\cos^2 x} \, dx = \tan x, \qquad \text{for } x \neq \frac{\pi}{2} + p\pi, \quad p \in \mathbb{Z},$$

$$\int \frac{1}{\sin^2 x} \, dx = -\cot x, \qquad \text{for } x \neq p\pi, \quad p \in \mathbb{Z}.$$

Hyperbolic:

$$\int \sinh x \, dx = \cosh x, \qquad \text{for } x \in \mathbb{R},$$

$$\int \cosh x \, dx = \sinh x, \qquad \text{for } x \in \mathbb{R},$$

$$\int \tanh x \, dx = \ln \cosh x, \qquad \text{for } x \in \mathbb{R},$$

$$\int \coth x \, dx = \ln |\sinh x|, \qquad \text{for } x \neq 0,$$

$$\int \frac{1}{\cosh x} \, dx = \operatorname{Arctan}(\sinh x), \qquad \text{for } x \in \mathbb{R},$$

$$\int \frac{1}{\cosh x} \, dx = 2 \operatorname{Arctan}(e^x), \qquad \text{for } x \in \mathbb{R},$$

$$\int \frac{1}{\sinh x} \, dx = \frac{1}{2} \ln \left(\frac{\cosh x - 1}{\cosh x + 1} \right), \qquad \text{for } x \neq 0,$$

$$\int \frac{1}{\sinh x}\, dx = \ln\left|\frac{e^x-1}{e^x+1}\right|, \qquad \text{for } x \neq 0,$$

$$\int \frac{1}{\cosh^2 x}\, dx = \tanh x, \qquad \text{for } x \in \mathbb{R},$$

$$\int \frac{1}{\sinh^2 x}\, dx = -\coth x, \qquad \text{for } x \neq 0.$$

4.7 Trigonometric formulæ

The trigonometric formulæ are closely connected with circular movements. Thus $(\cos u, \sin u)$ are the coordinates of a point P on the unit circle corresponding to the angle u, cf. figure A.1. This geometrical interpretation is used from time to time.

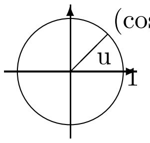


Figure 4.1: The unit circle and the trigonometric functions.

The fundamental trigonometric relation:

$$\cos^2 u + \sin^2 u = 1, \qquad \text{for } u \in \mathbb{R}.$$

Using the previous geometric interpretation this means according to *Pythagoras's theorem*, that the point P with the coordinates $(\cos u, \sin u)$ always has distance 1 from the origo $(0,0)$, i.e. it is lying on the boundary of the circle of centre $(0,0)$ and radius $\sqrt{1} = 1$.

Connection to the complex exponential function:

The *complex exponential* is for imaginary arguments defined by

$$\exp(\mathrm{i}\, u) := \cos u + \mathrm{i}\ \sin u.$$

It can be checked that the usual functional equation for exp is still valid for complex arguments. In other word: The definition above is extremely conveniently chosen.

By using the definition for $\exp(\mathrm{i}\, u)$ and $\exp(-\mathrm{i}\, u)$ it is easily seen that

$$\cos u = \frac{1}{2}(\exp(\mathrm{i}\, u) + \exp(-\mathrm{i}\, u)),$$

$$\sin u = \frac{1}{2i}(\exp(\mathrm{i}\, u) - \exp(-\mathrm{i}\, u)),$$

.

Moivre's formula: We get by expressing $\exp(\mathrm{i}nu)$ in two different ways:

$$\exp(\mathrm{i}nu) = \cos nu + \mathrm{i}\,\sin nu = (\cos u + \mathrm{i}\,\sin u)^n.$$

Example 4.1 If we e.g. put $n = 3$ into Moivre's formula, we obtain the following typical application,

$$\begin{aligned}\cos(3u) + \mathrm{i}\,\sin(3u) &= (\cos u + \mathrm{i}\,\sin u)^3\\ &= \cos^3 u + 3\mathrm{i}\,\cos^2 u\cdot\sin u + 3\mathrm{i}^2\cos u\cdot\sin^2 u + \mathrm{i}^3\sin^3 u\\ &= \{\cos^3 u - 3\cos u\cdot\sin^2 u\} + \mathrm{i}\{3\cos^2 u\cdot\sin u - \sin^3 u\}\\ &= \{4\cos^3 u - 3\cos u\} + \mathrm{i}\{3\sin u - 4\sin^3 u\}\end{aligned}$$

When this is split into the real- and imaginary parts we obtain

$$\cos 3u = 4\cos^3 u - 3\cos u, \qquad \sin 3u = 3\sin u - 4\sin^3 u. \quad \diamondsuit$$

Addition formulæ:

$$\sin(u+v) = \sin u\,\cos v + \cos u\,\sin v,$$
$$\sin(u-v) = \sin u\,\cos v - \cos u\,\sin v,$$
$$\cos(u+v) = \cos u\,\cos v - \sin u\,\sin v,$$
$$\cos(u-v) = \cos u\,\cos v + \sin u\,\sin v.$$

Products of trigonometric functions to a sum:

$$\sin u\,\cos v = \frac{1}{2}\sin(u+v) + \frac{1}{2}\sin(u-v),$$
$$\cos u\,\sin v = \frac{1}{2}\sin(u+v) - \frac{1}{2}\sin(u-v),$$
$$\sin u\,\sin v = \frac{1}{2}\cos(u-v) - \frac{1}{2}\cos(u+v),$$
$$\cos u\,\cos v = \frac{1}{2}\cos(u-v) + \frac{1}{2}\cos(u+v).$$

Sums of trigonometric functions to a product:

$$\sin u + \sin v = 2\sin\left(\frac{u+v}{2}\right)\cos\left(\frac{u-v}{2}\right),$$
$$\sin u - \sin v = 2\cos\left(\frac{u+v}{2}\right)\sin\left(\frac{u-v}{2}\right),$$
$$\cos u + \cos v = 2\cos\left(\frac{u+v}{2}\right)\cos\left(\frac{u-v}{2}\right),$$
$$\cos u - \cos v = -2\sin\left(\frac{u+v}{2}\right)\sin\left(\frac{u-v}{2}\right).$$

Formulæ of halving and doubling the angle:

$$\sin 2u = 2\sin u\,\cos u,$$
$$\cos 2u = \cos^2 u - \sin^2 u = 2\,\cos^2 u - 1 = 1 - 2\,\sin^2 u,$$
$$\sin\frac{u}{2} = \pm\sqrt{\frac{1-\cos u}{2}} \qquad \text{followed by a discussion of the sign,}$$
$$\cos\frac{u}{2} = \pm\sqrt{\frac{1+\cos u}{2}} \qquad \text{followed by a discussion of the sign,}$$

4.8 Hyperbolic formulæ

These are very much like the trigonometric formulæ, and if one knows a little of Complex Function Theory it is realized that they are actually identical. The structure of this section is therefore the same as for the trigonometric formulæ. The reader should compare the two sections concerning similarities and differences.

The fundamental relation:

$$\cosh^2 x - \sinh^2 x = 1.$$

Definitions:

$$\cosh x = \frac{1}{2}\left(\exp(x) + \exp(-x)\right), \qquad \sinh x = \frac{1}{2}\left(\exp(x) - \exp(-x)\right).$$

"Moivre's formula":

$$\exp(x) = \cosh x + \sinh x.$$

This is trivial and only rarely used. It has been included to show the analogy.

Addition formulæ:

$$\sinh(x+y) = \sinh(x)\cosh(y) + \cosh(x)\sinh(y),$$

$$\sinh(x-y) = \sinh(x)\cosh(y) - \cosh(x)\sinh(y),$$

$$\cosh(x+y) = \cosh(x)\cosh(y) + \sinh(x)\sinh(y),$$

$$\cosh(x-y) = \cosh(x)\cosh(y) - \sinh(x)\sinh(y).$$

Formulæ of halving and doubling the argument:

$$\sinh(2x) = 2\sinh(x)\cosh(x),$$

$$\cosh(2x) = \cosh^2(x) + \sinh^2(x) = 2\cosh^2(x) - 1 = 2\sinh^2(x) + 1,$$

$$\sinh\left(\frac{x}{2}\right) = \pm\sqrt{\frac{\cosh(x)-1}{2}} \qquad \text{followed by a discussion of the sign,}$$

$$\cosh\left(\frac{x}{2}\right) = \sqrt{\frac{\cosh(x)+1}{2}}.$$

Inverse hyperbolic functions:

$$\operatorname{Arsinh}(x) = \ln\left(x + \sqrt{x^2+1}\right), \qquad x \in \mathbb{R},$$

$$\operatorname{Arcosh}(x) = \ln\left(x + \sqrt{x^2-1}\right), \qquad x \geq 1,$$

$$\operatorname{Artanh}(x) = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right), \qquad |x| < 1,$$

$$\operatorname{Arcoth}(x) = \frac{1}{2}\ln\left(\frac{x+1}{x-1}\right), \qquad |x| > 1.$$

4.9 Complex transformation formulæ

$$\cos(\mathrm{i}x) = \cosh(x), \qquad \cosh(\mathrm{i}x) = \cos(x),$$

$$\sin(\mathrm{i}x) = \mathrm{i}\,\sinh(x), \qquad \sinh(\mathrm{i}x) = \mathrm{i}\,\sin x.$$

4.10 Taylor expansions

The generalized binomial coefficients are defined by

$$\binom{\alpha}{n} := \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{1\cdot 2\cdots n},$$

with n factors in the numerator and the denominator, supplied with

$$\binom{\alpha}{0} := 1.$$

The Taylor expansions for *standard functions* are divided into *power like* (the radius of convergency is finite, i.e. $= 1$ for the standard series) and*exponential like* (the radius of convergency is infinite).
Power like:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \qquad |x| < 1,$$

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n, \qquad |x| < 1,$$

$$(1+x)^n = \sum_{j=0}^{n} \binom{n}{j} x^j, \qquad n \in \mathbb{N}, x \in \mathbb{R},$$

$$(1+x)^\alpha = \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n, \qquad \alpha \in \mathbb{R}\setminus\mathbb{N}, |x| < 1,$$

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}, \qquad |x| < 1,$$

$$\operatorname{Arctan}(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}, \qquad |x| < 1.$$

Exponential like:

$$\exp(x) = \sum_{n=0}^{\infty} \frac{1}{n!} x^n, \qquad x \in \mathbb{R}$$

$$\exp(-x) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{n!} x^n, \qquad x \in \mathbb{R}$$

$$\sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+1)!} x^{2n+1}, \qquad x \in \mathbb{R},$$

$$\sinh(x) = \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} x^{2n+1}, \qquad x \in \mathbb{R},$$

$$\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n)!} x^{2n}, \qquad x \in \mathbb{R},$$

$$\cosh(x) = \sum_{n=0}^{\infty} \frac{1}{(2n)!} x^{2n}, \qquad x \in \mathbb{R}.$$

4.11 Magnitudes of functions

We often have to compare functions for $x \to 0+$, or for $x \to \infty$. The simplest type of functions are therefore arranged in an hierarchy:

1) logarithms,
2) power functions,
3) exponential functions,
4) faculty functions.

When $x \to \infty$, a function from a higher class will always dominate a function form a lower class. More precisely:

A) A *power function* dominates a *logarithm* for $x \to \infty$:

$$\frac{(\ln x)^\beta}{x^\alpha} \to 0 \qquad \text{for } x \to \infty, \quad \alpha,\, \beta > 0.$$

B) An *exponential* dominates a *power function* for $x \to \infty$:

$$\frac{x^\alpha}{a^x} \to 0 \qquad \text{for } x \to \infty, \quad \alpha,\, a > 1.$$

C) The *faculty function* dominates an *exponential* for $n \to \infty$:

$$\frac{a^n}{n!} \to 0, \qquad n \to \infty, \quad n \in \mathbb{N}, \quad a > 0.$$

D) When $x \to 0+$ we also have that a *power function* dominates the *logarithm:*

$$x^\alpha \ln x \to 0-, \qquad \text{for } x \to 0+, \quad \alpha > 0.$$

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