

Real Functions in Several Variables: Volume I

Point sets in \mathbb{R}^n

Leif Mejlbro



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Real Functions in Several Variables

Volume-I Point sets in \mathbb{R}^n



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Contents

Volume I, Point Sets in \mathbb{R}^n	1
Preface	15
Introduction to volume I, Point sets in \mathbb{R}^n. The maximal domain of a function	19
1 Basic concepts	21
1.1 Introduction	21
1.2 The real linear space \mathbb{R}^n	22
1.3 The vector product	26
1.4 The most commonly used coordinate systems	29
1.5 Point sets in space	37
1.5.1 Interior, exterior and boundary of a set	37
1.5.2 Starshaped and convex sets	40
1.5.3 Catalogue of frequently used point sets in the plane and the space	41
1.6 Quadratic equations in two or three variables. Conic sections	47
1.6.1 Quadratic equations in two variables. Conic sections	47
1.6.2 Quadratic equations in three variables. Conic sectional surfaces	54
1.6.3 Summary of the canonical cases in three variables	66
2 Some useful procedures	67
2.1 Introduction	67
2.2 Integration of trigonometric polynomials	67
2.3 Complex decomposition of a fraction of two polynomials	69
2.4 Integration of a fraction of two polynomials	72
3 Examples of point sets	75
3.1 Point sets	75
3.2 Conics and conical sections	104
4 Formulæ	115
4.1 Squares etc.	115
4.2 Powers etc.	115
4.3 Differentiation	116
4.4 Special derivatives	116
4.5 Integration	118
4.6 Special antiderivatives	119
4.7 Trigonometric formulæ	121
4.8 Hyperbolic formulæ	123
4.9 Complex transformation formulæ	124
4.10 Taylor expansions	124
4.11 Magnitudes of functions	125
Index	127

Volume II, Continuous Functions in Several Variables	133
Preface	147
Introduction to volume II, Continuous Functions in Several Variables	151
5 Continuous functions in several variables	153
5.1 Maps in general	153
5.2 Functions in several variables	154
5.3 Vector functions	157
5.4 Visualization of functions	158
5.5 Implicit given function	161
5.6 Limits and continuity	162
5.7 Continuous functions	168
5.8 Continuous curves	170
5.8.1 Parametric description	170
5.8.2 Change of parameter of a curve	174
5.9 Connectedness	175
5.10 Continuous surfaces in \mathbb{R}^3	177
5.10.1 Parametric description and continuity	177
5.10.2 Cylindric surfaces	180
5.10.3 Surfaces of revolution	181
5.10.4 Boundary curves, closed surface and orientation of surfaces	182
5.11 Main theorems for continuous functions	185
6 A useful procedure	189
6.1 The domain of a function	189
7 Examples of continuous functions in several variables	191
7.1 Maximal domain of a function	191
7.2 Level curves and level surfaces	198
7.3 Continuous functions	212
7.4 Description of curves	227
7.5 Connected sets	241
7.6 Description of surfaces	245
8 Formulæ	257
8.1 Squares etc.	257
8.2 Powers etc.	257
8.3 Differentiation	258
8.4 Special derivatives	258
8.5 Integration	260
8.6 Special antiderivatives	261
8.7 Trigonometric formulæ	263
8.8 Hyperbolic formulæ	265
8.9 Complex transformation formulæ	266
8.10 Taylor expansions	266
8.11 Magnitudes of functions	267
Index	269

Volume III, Differentiable Functions in Several Variables	275
Preface	289
Introduction to volume III, Differentiable Functions in Several Variables	293
9 Differentiable functions in several variables	295
9.1 Differentiability	295
9.1.1 The gradient and the differential	295
9.1.2 Partial derivatives	298
9.1.3 Differentiable vector functions	303
9.1.4 The approximating polynomial of degree 1	304
9.2 The chain rule	305
9.2.1 The elementary chain rule	305
9.2.2 The first special case	308
9.2.3 The second special case	309
9.2.4 The third special case	310
9.2.5 The general chain rule	314
9.3 Directional derivative	317
9.4 C^n -functions	318
9.5 Taylor's formula	321
9.5.1 Taylor's formula in one dimension	321
9.5.2 Taylor expansion of order 1	322
9.5.3 Taylor expansion of order 2 in the plane	323
9.5.4 The approximating polynomial	326
10 Some useful procedures	333
10.1 Introduction	333
10.2 The chain rule	333
10.3 Calculation of the directional derivative	334
10.4 Approximating polynomials	336
11 Examples of differentiable functions	339
11.1 Gradient	339
11.2 The chain rule	352
11.3 Directional derivative	375
11.4 Partial derivatives of higher order	382
11.5 Taylor's formula for functions of several variables	404
12 Formulæ	445
12.1 Squares etc.	445
12.2 Powers etc.	445
12.3 Differentiation	446
12.4 Special derivatives	446
12.5 Integration	448
12.6 Special antiderivatives	449
12.7 Trigonometric formulæ	451
12.8 Hyperbolic formulæ	453
12.9 Complex transformation formulæ	454
12.10 Taylor expansions	454
12.11 Magnitudes of functions	455
Index	457

Volume IV, Differentiable Functions in Several Variables	463
Preface	477
Introduction to volume IV, Curves and Surfaces	481
13 Differentiable curves and surfaces, and line integrals in several variables	483
13.1 Introduction	483
13.2 Differentiable curves	483
13.3 Level curves	492
13.4 Differentiable surfaces	495
13.5 Special C^1 -surfaces	499
13.6 Level surfaces	503
14 Examples of tangents (curves) and tangent planes (surfaces)	505
14.1 Examples of tangents to curves	505
14.2 Examples of tangent planes to a surface	520
15 Formulæ	541
15.1 Squares etc.	541
15.2 Powers etc.	541
15.3 Differentiation	542
15.4 Special derivatives	542
15.5 Integration	544
15.6 Special antiderivatives	545
15.7 Trigonometric formulæ	547
15.8 Hyperbolic formulæ	549
15.9 Complex transformation formulæ	550
15.10 Taylor expansions	550
15.11 Magnitudes of functions	551
Index	553
Volume V, Differentiable Functions in Several Variables	559
Preface	573
Introduction to volume V, The range of a function, Extrema of a Function in Several Variables	577
16 The range of a function	579
16.1 Introduction	579
16.2 Global extrema of a continuous function	581
16.2.1 A necessary condition	581
16.2.2 The case of a closed and bounded domain of f	583
16.2.3 The case of a bounded but not closed domain of f	599
16.2.4 The case of an unbounded domain of f	608
16.3 Local extrema of a continuous function	611
16.3.1 Local extrema in general	611
16.3.2 Application of Taylor's formula	616
16.4 Extremum for continuous functions in three or more variables	625
17 Examples of global and local extrema	631
17.1 MAPLE	631
17.2 Examples of extremum for two variables	632
17.3 Examples of extremum for three variables	668

17.4	Examples of maxima and minima	677
17.5	Examples of ranges of functions	769
18	Formulæ	811
18.1	Squares etc.	811
18.2	Powers etc.	811
18.3	Differentiation	812
18.4	Special derivatives	812
18.5	Integration	814
18.6	Special antiderivatives	815
18.7	Trigonometric formulæ	817
18.8	Hyperbolic formulæ	819
18.9	Complex transformation formulæ	820
18.10	Taylor expansions	820
18.11	Magnitudes of functions	821
Index		823
Volume VI, Antiderivatives and Plane Integrals		829
Preface		841
Introduction to volume VI, Integration of a function in several variables		845
19 Antiderivatives of functions in several variables		847
19.1	The theory of antiderivatives of functions in several variables	847
19.2	Templates for gradient fields and antiderivatives of functions in three variables	858
19.3	Examples of gradient fields and antiderivatives	863
20 Integration in the plane		881
20.1	An overview of integration in the plane and in the space	881
20.2	Introduction	882
20.3	The plane integral in rectangular coordinates	887
20.3.1	Reduction in rectangular coordinates	887
20.3.2	The colour code, and a procedure of calculating a plane integral	890
20.4	Examples of the plane integral in rectangular coordinates	894
20.5	The plane integral in polar coordinates	936
20.6	Procedure of reduction of the plane integral; polar version	944
20.7	Examples of the plane integral in polar coordinates	948
20.8	Examples of area in polar coordinates	972
21 Formulæ		977
21.1	Squares etc.	977
21.2	Powers etc.	977
21.3	Differentiation	978
21.4	Special derivatives	978
21.5	Integration	980
21.6	Special antiderivatives	981
21.7	Trigonometric formulæ	983
21.8	Hyperbolic formulæ	985
21.9	Complex transformation formulæ	986
21.10	Taylor expansions	986
21.11	Magnitudes of functions	987
Index		989

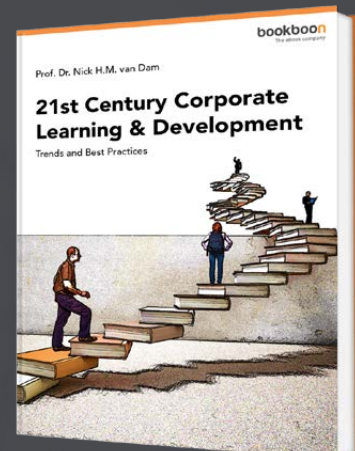
Volume VII, Space Integrals	995
Preface	1009
Introduction to volume VII, The space integral	1013
22 The space integral in rectangular coordinates	1015
22.1 Introduction	1015
22.2 Overview of setting up of a line, a plane, a surface or a space integral	1015
22.3 Reduction theorems in rectangular coordinates	1021
22.4 Procedure for reduction of space integral in rectangular coordinates	1024
22.5 Examples of space integrals in rectangular coordinates	1026
23 The space integral in semi-polar coordinates	1055
23.1 Reduction theorem in semi-polar coordinates	1055
23.2 Procedures for reduction of space integral in semi-polar coordinates	1056
23.3 Examples of space integrals in semi-polar coordinates	1058
24 The space integral in spherical coordinates	1081
24.1 Reduction theorem in spherical coordinates	1081
24.2 Procedures for reduction of space integral in spherical coordinates	1082
24.3 Examples of space integrals in spherical coordinates	1084
24.4 Examples of volumes	1107
24.5 Examples of moments of inertia and centres of gravity	1116
25 Formulæ	1125
25.1 Squares etc.	1125
25.2 Powers etc.	1125
25.3 Differentiation	1126
25.4 Special derivatives	1126
25.5 Integration	1128
25.6 Special antiderivatives	1129
25.7 Trigonometric formulæ	1131
25.8 Hyperbolic formulæ	1133
25.9 Complex transformation formulæ	1134
25.10 Taylor expansions	1134
25.11 Magnitudes of functions	1135
Index	1137
Volume VIII, Line Integrals and Surface Integrals	1143
Preface	1157
Introduction to volume VIII, The line integral and the surface integral	1161
26 The line integral	1163
26.1 Introduction	1163
26.2 Reduction theorem of the line integral	1163
26.2.1 Natural parametric description	1166
26.3 Procedures for reduction of a line integral	1167
26.4 Examples of the line integral in rectangular coordinates	1168
26.5 Examples of the line integral in polar coordinates	1190
26.6 Examples of arc lengths and parametric descriptions by the arc length	1201

27	The surface integral	1227
27.1	The reduction theorem for a surface integral	1227
27.1.1	The integral over the graph of a function in two variables	1229
27.1.2	The integral over a cylindric surface	1230
27.1.3	The integral over a surface of revolution	1232
27.2	Procedures for reduction of a surface integral	1233
27.3	Examples of surface integrals	1235
27.4	Examples of surface area	1296
28	Formulæ	1315
28.1	Squares etc.	1315
28.2	Powers etc.	1315
28.3	Differentiation	1316
28.4	Special derivatives	1316
28.5	Integration	1318
28.6	Special antiderivatives	1319
28.7	Trigonometric formulæ	1321
28.8	Hyperbolic formulæ	1323
28.9	Complex transformation formulæ	1324
28.10	Taylor expansions	1324
28.11	Magnitudes of functions	1325
	Index	1327

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Volume IX, Transformation formulæ and improper integrals	1333
Preface	1347
Introduction to volume IX, Transformation formulæ and improper integrals	1351
29 Transformation of plane and space integrals	1353
29.1 Transformation of a plane integral	1353
29.2 Transformation of a space integral	1355
29.3 Procedures for the transformation of plane or space integrals	1358
29.4 Examples of transformation of plane and space integrals	1359
30 Improper integrals	1411
30.1 Introduction	1411
30.2 Theorems for improper integrals	1413
30.3 Procedure for improper integrals; bounded domain	1415
30.4 Procedure for improper integrals; unbounded domain	1417
30.5 Examples of improper integrals	1418
31 Formulæ	1447
31.1 Squares etc.	1447
31.2 Powers etc.	1447
31.3 Differentiation	1448
31.4 Special derivatives	1448
31.5 Integration	1450
31.6 Special antiderivatives	1451
31.7 Trigonometric formulæ	1453
31.8 Hyperbolic formulæ	1455
31.9 Complex transformation formulæ	1456
31.10 Taylor expansions	1456
31.11 Magnitudes of functions	1457
Index	1459
Volume X, Vector Fields I; Gauß's Theorem	1465
Preface	1479
Introduction to volume X, Vector fields; Gauß's Theorem	1483
32 Tangential line integrals	1485
32.1 Introduction	1485
32.2 The tangential line integral. Gradient fields.	1485
32.3 Tangential line integrals in Physics	1498
32.4 Overview of the theorems and methods concerning tangential line integrals and gradient fields	1499
32.5 Examples of tangential line integrals	1502
33 Flux and divergence of a vector field. Gauß's theorem	1535
33.1 Flux	1535
33.2 Divergence and Gauß's theorem	1540
33.3 Applications in Physics	1544
33.3.1 Magnetic flux	1544
33.3.2 Coulomb vector field	1545
33.3.3 Continuity equation	1548
33.4 Procedures for flux and divergence of a vector field; Gauß's theorem	1549
33.4.1 Procedure for calculation of a flux	1549
33.4.2 Application of Gauß's theorem	1549
33.5 Examples of flux and divergence of a vector field; Gauß's theorem	1551
33.5.1 Examples of calculation of the flux	1551
33.5.2 Examples of application of Gauß's theorem	1580

34 Formulæ	1619
34.1 Squares etc.	1619
34.2 Powers etc.	1619
34.3 Differentiation	1620
34.4 Special derivatives	1620
34.5 Integration	1622
34.6 Special antiderivatives	1623
34.7 Trigonometric formulæ	1625
34.8 Hyperbolic formulæ	1627
34.9 Complex transformation formulæ	1628
34.10 Taylor expansions	1628
34.11 Magnitudes of functions	1629
Index	1631
Volume XI, Vector Fields II; Stokes's Theorem	1637
Preface	1651
Introduction to volume XI, Vector fields II; Stokes's Theorem; nabla calculus	1655
35 Rotation of a vector field; Stokes's theorem	1657
35.1 Rotation of a vector field in \mathbb{R}^3	1657
35.2 Stokes's theorem	1661
35.3 Maxwell's equations	1669
35.3.1 The electrostatic field	1669
35.3.2 The magnostatic field	1671
35.3.3 Summary of Maxwell's equations	1679
35.4 Procedure for the calculation of the rotation of a vector field and applications of Stokes's theorem	1682

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35.5	Examples of the calculation of the rotation of a vector field and applications of Stokes's theorem	1684
35.5.1	Examples of divergence and rotation of a vector field	1684
35.5.2	General examples	1691
35.5.3	Examples of applications of Stokes's theorem	1700
36	Nabla calculus	1739
36.1	The vectorial differential operator ∇	1739
36.2	Differentiation of products	1741
36.3	Differentiation of second order	1743
36.4	Nabla applied on \mathbf{x}	1745
36.5	The integral theorems	1746
36.6	Partial integration	1749
36.7	Overview of Nabla calculus	1750
36.8	Overview of partial integration in higher dimensions	1752
36.9	Examples in nabla calculus	1754
37	Formulæ	1769
37.1	Squares etc.	1769
37.2	Powers etc.	1769
37.3	Differentiation	1770
37.4	Special derivatives	1770
37.5	Integration	1772
37.6	Special antiderivatives	1773
37.7	Trigonometric formulæ	1775
37.8	Hyperbolic formulæ	1777
37.9	Complex transformation formulæ	1778
37.10	Taylor expansions	1778
37.11	Magnitudes of functions	1779
Index		1781
Volume XII, Vector Fields III; Potentials, Harmonic Functions and Green's Identities		1787
Preface		1801
Introduction to volume XII, Vector fields III; Potentials, Harmonic Functions and Green's Identities		1805
38 Potentials		1807
38.1	Definitions of scalar and vectorial potentials	1807
38.2	A vector field given by its rotation and divergence	1813
38.3	Some applications in Physics	1816
38.4	Examples from Electromagnetism	1819
38.5	Scalar and vector potentials	1838
39 Harmonic functions and Green's identities		1889
39.1	Harmonic functions	1889
39.2	Green's first identity	1890
39.3	Green's second identity	1891
39.4	Green's third identity	1896
39.5	Green's identities in the plane	1898
39.6	Gradient, divergence and rotation in semi-polar and spherical coordinates	1899
39.7	Examples of applications of Green's identities	1901
39.8	Overview of Green's theorems in the plane	1909
39.9	Miscellaneous examples	1910

40	Formulæ	1923
40.1	Squares etc.	1923
40.2	Powers etc.	1923
40.3	Differentiation	1924
40.4	Special derivatives	1924
40.5	Integration	1926
40.6	Special antiderivatives	1927
40.7	Trigonometric formulæ	1929
40.8	Hyperbolic formulæ	1931
40.9	Complex transformation formulæ	1932
40.10	Taylor expansions	1932
40.11	Magnitudes of functions	1933
	Index	1935



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Preface

The topic of this series of books on “*Real Functions in Several Variables*” is very important in the description in e.g. *Mechanics* of the real 3-dimensional world that we live in. Therefore, we start from the very beginning, modelling this world by using the coordinates of \mathbb{R}^3 to describe e.g. a motion in space. There is, however, absolutely no reason to restrict ourselves to \mathbb{R}^3 alone. Some motions may be rectilinear, so only \mathbb{R} is needed to describe their movements on a line segment. This opens up for also dealing with \mathbb{R}^2 , when we consider plane motions. In more elaborate problems we need higher dimensional spaces. This may be the case in *Probability Theory* and *Statistics*. Therefore, we shall in general use \mathbb{R}^n as our abstract model, and then restrict ourselves in examples mainly to \mathbb{R}^2 and \mathbb{R}^3 .

For rectilinear motions the familiar *rectangular coordinate system* is the most convenient one to apply. However, as known from e.g. *Mechanics*, circular motions are also very important in the applications in engineering. It becomes natural alternatively to apply in \mathbb{R}^2 the so-called *polar coordinates* in the plane. They are convenient to describe a circle, where the rectangular coordinates usually give some nasty square roots, which are difficult to handle in practice.

Rectangular coordinates and polar coordinates are designed to model each their problems. They supplement each other, so difficult computations in one of these coordinate systems may be easy, and even trivial, in the other one. It is therefore important always in advance carefully to analyze the geometry of e.g. a domain, so we ask the question: Is this domain best described in rectangular or in polar coordinates?

Sometimes one may split a problem into two subproblems, where we apply rectangular coordinates in one of them and polar coordinates in the other one.

It should be mentioned that in *real life* (though not in these books) one cannot always split a problem into two subproblems as above. Then one is really in trouble, and more advanced mathematical methods should be applied instead. This is, however, outside the scope of the present series of books.

The idea of polar coordinates can be extended in two ways to \mathbb{R}^3 . Either to *semi-polar* or *cylindric coordinates*, which are designed to describe a cylinder, or to *spherical coordinates*, which are excellent for describing spheres, where rectangular coordinates usually are doomed to fail. We use them already in daily life, when we specify a place on Earth by its longitude and latitude! It would be very awkward in this case to use rectangular coordinates instead, even if it is possible.

Concerning the contents, we begin this investigation by modelling point sets in an n -dimensional Euclidean space E^n by \mathbb{R}^n . There is a subtle difference between E^n and \mathbb{R}^n , although we often identify these two spaces. In E^n we use *geometrical methods* without a coordinate system, so the objects are independent of such a choice. In the coordinate space \mathbb{R}^n we can use ordinary calculus, which in principle is not possible in E^n . In order to stress this point, we call E^n the “abstract space” (in the sense of calculus; not in the sense of geometry) as a warning to the reader. Also, whenever necessary, we use the colour black in the “abstract space”, in order to stress that this expression is theoretical, while variables given in a chosen coordinate system and their related concepts are given the colours blue, red and green.

We also include the most basic of what mathematicians call *Topology*, which will be necessary in the following. We describe what we need by a function.

Then we proceed with limits and continuity of functions and define continuous curves and surfaces, with parameters from subsets of \mathbb{R} and \mathbb{R}^2 , resp..

Continue with (partial) differentiable functions, curves and surfaces, the chain rule and Taylor's formula for functions in several variables.

We deal with maxima and minima and extrema of functions in several variables over a domain in \mathbb{R}^n . This is a very important subject, so there are given many worked examples to illustrate the theory.

Then we turn to the problems of integration, where we specify four different types with increasing complexity, *plane integral*, *space integral*, *curve (or line) integral* and *surface integral*.

Finally, we consider *vector analysis*, where we deal with vector fields, Gauß's theorem and Stokes's theorem. All these subjects are very important in theoretical Physics.

The structure of this series of books is that each subject is *usually* (but not always) described by three successive chapters. In the first chapter a brief theoretical theory is given. The next chapter gives some practical guidelines of how to solve problems connected with the subject under consideration. Finally, some worked out examples are given, in many cases in several variants, because the standard solution method is seldom the only way, and it may even be clumsy compared with other possibilities.

I have as far as possible structured the examples according to the following scheme:

A *Awareness*, i.e. a short description of what is the problem.

D *Decision*, i.e. a reflection over what should be done with the problem.

I *Implementation*, i.e. where all the calculations are made.

C *Control*, i.e. a test of the result.

This is an ideal form of a general procedure of solution. It can be used in any situation and it is not linked to Mathematics alone. I learned it many years ago in the Theory of Telecommunication in a situation which did not contain Mathematics at all. The student is recommended to use it also in other disciplines.

From high school one is used to immediately to proceed to **I. Implementation**. However, examples and problems at university level, let alone situations in real life, are often so complicated that it in general will be a good investment also to spend some time on the first two points above in order to be absolutely certain of what to do in a particular case. Note that the first three points, **ADI**, can always be executed.

This is unfortunately not the case with **C Control**, because it from now on may be difficult, if possible, to check one's solution. It is only an extra securing whenever it is possible, but we cannot include it always in our solution form above.

I shall on purpose not use the logical signs. These should in general be avoided in Calculus as a shorthand, because they are often (too often, I would say) misused. Instead of \wedge I shall either write "and", or a comma, and instead of \vee I shall write "or". The arrows \Rightarrow and \Leftrightarrow are in particular misunderstood by the students, so they should be totally avoided. They are not telegram short hands, and from a logical point of view they usually do not make sense at all! Instead, write in a plain language what you mean or want to do. This is difficult in the beginning, but after some practice it becomes routine, and it will give more precise information.

When we deal with multiple integrals, one of the possible pedagogical ways of solving problems has been to colour variables, integrals and upper and lower bounds in blue, red and green, so the reader by the colour code can see in each integral what is the variable, and what are the parameters, which

do not enter the integration under consideration. We shall of course build up a hierarchy of these colours, so the order of integration will always be defined. As already mentioned above we reserve the colour black for the theoretical expressions, where we cannot use ordinary calculus, because the symbols are only shorthand for a concept.

The author has been very grateful to his old friend and colleague, the late Per Wennerberg Karlsson, for many discussions of how to present these difficult topics on real functions in several variables, and for his permission to use his textbook as a template of this present series. Nevertheless, the author has felt it necessary to make quite a few changes compared with the old textbook, because we did not always agree, and some of the topics could also be explained in another way, and then of course the results of our discussions have here been put in writing for the first time.

The author also adds some calculations in MAPLE, which interact nicely with the theoretic text. Note, however, that when one applies MAPLE, one is forced first to make a geometrical analysis of the domain of integration, i.e. apply some of the techniques developed in the present books.

The theory and methods of these volumes on “Real Functions in Several Variables” are applied constantly in higher Mathematics, Mechanics and Engineering Sciences. It is of paramount importance for the calculations in *Probability Theory*, where one constantly integrate over some point set in space.

It is my hope that this text, these guidelines and these examples, of which many are treated in more ways to show that the solutions procedures are not unique, may be of some inspiration for the students who have just started their studies at the universities.

Finally, even if I have tried to write as careful as possible, I doubt that all errors have been removed. I hope that the reader will forgive me the unavoidable errors.

Leif Mejlbro
March 21, 2015

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Introduction to volume I, Point sets in \mathbb{R}^n . The maximal domain of a function

In this first volume of the series of books on *Real Functions in Several Variables* we start in Chapter 1 by giving a small theoretical introduction to what is needed in order to get started on the main subject. We shall work in *Euclidean space* E^n , which in rectangular coordinates is similar to the vector space \mathbb{R}^n , also called the *coordinate space*. The difference may at the first glance seem very small, and yet this difference is quite important. If we ever prove something in E^n , then this is done *geometrically* without any coordinate axes. This may be very strange to most younger readers, who have never learned Geometry in school using only ruler and compasses. For that reason I have in lack of better words called objects in E^n for “abstract” or “theoretical”, though they are neither “abstract” nor “purely theoretical”.

Once we have chosen a rectangular coordinate system in E^n , i.e. defined the n orthonormal basic vectors, then we have also defined the *rectangular coordinates* $(x_1, \dots, x_n) \in \mathbb{R}^n$ of an element $\mathbf{x} \in E^n$. The reason for this transformation from the Euclidean space E^n to its corresponding coordinate space \mathbb{R}^n is of course that it is often easier to compute things in \mathbb{R}^n than to argue geometrically in E^n .

Obviously, $E^2 \sim \mathbb{R}^2$ and $E^3 \sim \mathbb{R}^3$ are very important examples of $E^n \sim \mathbb{R}^n$, so the main emphasis is put on these two cases, though we cannot totally rule out higher dimensional spaces.

We introduce the dot product in all \mathbb{R}^n and use it to define the *norm* (or length) and *angle*.

In $E^3 \sim \mathbb{R}^3$ (and only in this space) we also introduce the important *cross product* or *vector product*, which is applied in particular in Physics.

Even if rectangular coordinates may seem natural in the beginning, they are not well suited for all our problems. When we consider Mechanics in the plane E^2 , there are clearly two very important motions, which we should be able to describe in a reasonable way, namely the *rectilinear motion*, where rectangular coordinates clearly are most appropriate, and the *circular motion*, where we in a rectangular description almost always end up with some nasty square roots. To ease matters we instead introduce the *polar coordinates* in the plane. In this case E^2 and the corresponding polar coordinate space $\subset \mathbb{R}^2$ are clearly not of the same geometrical shape. The circular motion is usually easy to describe in polar coordinates, when the coordinate system is put properly.

Once we have started introducing another coordinate system like the polar coordinates instead of the usual rectangular coordinate system, we may of course proceed by introducing other useful coordinate systems, like *semi-polar coordinates* in \mathbb{R}^3 , which are designed to describe bodies of revolution with the z -axis as the axis of revolution, and the *spherical coordinates* in \mathbb{R}^3 , which are convenient, when we are dealing with spheres and balls in E^3 .

All these new coordinate systems are only defined in Chapter 1. However, their applications will be demonstrated over and over again in the following volumes.

We continue with introducing the most basic of what is called *Topology*. We define the interior, exterior, boundary and closure of (abstract) sets. We shall also need all these abstract concepts in the following.

We give some examples of typical sets, which will be used frequently in the following. For the same reason we also include a section on the classical cones and conical sections from Geometry, because we cannot assume that all readers have seen them before.

The short Chapter 2 describes some guidelines of how to solve some typical problems in this book.

Chapter 3 contains a lot of examples describing the theoretical text from Chapter 1.

A short list of useful formulæ is given in Chapter 4.

The table of contents and the index cover all volumes, which are organized with succeeding page numbers. Unfortunately, it has not been possible to organize the index such that the number of the volume is also given.

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1 Basic concepts

1.1 Introduction

We shall start by defining the *model number spaces* \mathbb{R}^n , so they are at hand, when we in the next section consider the corresponding *Euclidean spaces* E^n . There is a bijective correspondence between E^n and its coordinate space \mathbb{R}^n , when we use the obvious orthonormal basis. The subtle difference is that we argue in E^n in an “abstract way” on the geometry of the set, while we set up some rules of computation in the coordinate space \mathbb{R}^n . In other words, E^n contains the abstract geometrical objects, which then are described analytically in the coordinate space \mathbb{R}^n . In rectangular coordinates a point set $A \subseteq E^n$ has the same geometry as its set of coordinates $\tilde{A} \subseteq \mathbb{R}^n$, so one may hardly see the difference. However, whenever it is convenient to use another coordinate system, which is not rectangular, e.g. polar or spherical coordinates, then the set of coordinates $\tilde{A} \subseteq \mathbb{R}^n$ has apparently a different geometry from that of the original set $A \subseteq E^n$.

Whenever there is a need to distinguish between the “abstract space” of $A \subseteq E^n$ and its coordinate set $\tilde{A} \subseteq \mathbb{R}^n$, we shall use the following colour code: black in the “abstract” space E^n , and blue, red, green, etc. in the coordinate space \mathbb{R}^n . This is, however, not needed in the first volumes, and it only becomes convenient, when we are describing plane or space integrals, etc., where we calculate analytically the value of these integrals.

So first we *define* the *model number spaces* \mathbb{R}^n , and then discuss \mathbb{R}^n as a real *vector space*, followed by introducing the most commonly used coordinate systems, i.e. *rectangular coordinates* (in \mathbb{R}^n in general), *polar coordinates* (only in \mathbb{R}^2), *semi-polar coordinates*, also called *cylindric coordinates* (only in \mathbb{R}^3), and finally the *spherical coordinates*. These are here only defined in \mathbb{R}^3 , but it is not hard to prove that *generalized spherical coordinates* can be defined in any number space \mathbb{R}^n , where $n \geq 3$.

In the following sections we turn to *point sets* in the *Euclidean space* E^n . To ease matters for the reader we shall, as already mentioned above, whenever it is felt convenient, identify a point set $A \subseteq E^n$ with its coordinate set $\tilde{A} \subseteq \mathbb{R}^n$ in rectangular coordinates. Note, however, that in principle A and \tilde{A} are *not* the same set, although they may look alike!

We introduce some necessary abstract topological concepts like *open* and *closed sets*, *boundary sets*, *convex* and *starshaped sets*, etc.. These may seem very strange for the unexperienced reader, but they are needed, when we later shall describe limits and continuity of functions.

In the last section of this chapter we describe *functions in several variables*, and extend them to *vector functions*. We also describe how to visualize functions in several variables. Finally, we mention the problem of *implicit given functions*. It is not possible here to give a correct proof of the Theorem of implicit given function, though it clearly is very important.

1.2 The real linear space \mathbb{R}^n

The real number space \mathbb{R}^n is considered as a real *vector space* $(\mathbb{R}^n, +, \cdot, \mathbb{R})$, also called a *linear space*. The elements of \mathbb{R}^n are ordered sets of n real numbers, which are called the *coordinates* of the point. Hence, an element of \mathbb{R}^n is written

$$\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}^n, \quad \text{where } x_1, \dots, x_n \in \mathbb{R}.$$

Although we have not proved it yet, we mention that this is a description of \mathbf{x} in *rectangular coordinates*, so when $\mathbf{x} \in \mathbb{R}^n$ is identified with the corresponding element in the *Euclidean space* E^n , which is also denoted by \mathbf{x} , then \mathbf{x} is interpreted, depending on the actual situation, either as a point $\mathbf{x} \in E^n$, or as a *vector* $\vec{x} \in E^n$ pointing from $\mathbf{0} = (0, \dots, 0)$, or $\vec{0} = (0, \dots, 0)$ to the end point \mathbf{x} .

The *addition* in the vector space \mathbb{R}^n is defined by

$$\mathbf{x} + \mathbf{y} = (x_1, \dots, x_n) + (y_1, \dots, y_n) = (x_1 + y_1, \dots, x_n + y_n),$$

so we add the coordinates at place j , $j = 1, \dots, n$.

The addition is clearly commutative,

$$\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}, \quad \text{for } \mathbf{x}, \mathbf{y} \in \mathbb{R}^n.$$

The *neutral element* is the *zero point* (or *zero vector* $\mathbf{0}$, because

$$\mathbf{x} + \mathbf{0} = (x_1, \dots, x_n) + (0, \dots, 0) = (x_1, \dots, x_n) = \mathbf{x}.$$

The *scalar multiplication* by $\lambda \in \mathbb{R}$ of $\mathbf{x} \in \mathbb{R}^n$ is defined by

$$\lambda \mathbf{x} = \lambda (x_1, \dots, x_n) = (\lambda x_1, \dots, \lambda x_n),$$

so each coordinate is multiplied by the same scalar λ . This can be interpreted as a stretching. We note that we have no notation for the scalar product. In fact, there is no way to misunderstand the concatenation $\lambda \mathbf{x}$, and we shall later use the most obvious notation “ \cdot ” for another important product in \mathbb{R}^n .

A natural *basis* of \mathbb{R}^n is given by the vectors of the coordinates

$$\mathbf{e}_1 = (1, 0, \dots, 0), \dots, \mathbf{e}_n = (0, \dots, 0, 1),$$

where e.g. \mathbf{e}_j has 1 on its j -th coordinate, while all other coordinates are 0. In fact, it is obvious that we have

$$\mathbf{x} = (x_1, \dots, x_n) = x_1 \mathbf{e}_1 + \dots + x_n \mathbf{e}_n,$$

and if

$$x_1 \mathbf{e}_1 + \dots + x_n \mathbf{e}_n = \mathbf{0} = (0, \dots, 0),$$

then necessarily all $x_j = 0$, so the description of \mathbf{x} is unique, and $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ is indeed a basis of \mathbb{R}^n .

One usually adds a so-called *inner product* in \mathbb{R}^n . This is a function denoted by $\cdot : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$. It is in order to avoid confusion that we do not introduce a notation for the scalar product of a scalar and a vector.

The *inner product* of two elements $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ is defined in the following way:

$$\mathbf{x} \cdot \mathbf{y} := (x_1, \dots, x_n) \cdot (y_1, \dots, y_n) = x_1 y_1 + \dots + x_n y_n = \sum_{j=1}^n x_j y_j.$$

This is actually a geometrical concept, which shall be demonstrated in the following. Note in particular that

$$\mathbf{e}_i \cdot \mathbf{e}_j := \delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

The symbol δ_{ij} defined above is called the *Kronecker symbol*. Due to this relation one may say that the vectors $\mathbf{e}_1, \dots, \mathbf{e}_n$ are perpendicular to each other.

Since $\mathbf{e}_j \cdot \mathbf{e}_j = 1$, we call the \mathbf{e}_j *unit vectors*. They form an *orthonormal system*.

We call

$$x_j = \mathbf{x} \cdot \mathbf{e}_j$$

the *projection* of \mathbf{x} onto the line defined by the unit vector \mathbf{e}_j . It is interpreted as the (signed) length of the orthogonal projection of \mathbf{x} onto the line defined by the unit vector \mathbf{e}_j .

Using *Pythagoras's theorem* repeatedly $n - 1$ times we easily derive that

$$\|\mathbf{x}\| := \sqrt{\mathbf{x} \cdot \mathbf{x}} = \sqrt{x_1^2 + \dots + x_n^2} \quad \text{for } \mathbf{x} \in \mathbb{R}^n,$$

is the *length* (also called the *norm*) of the vector $\vec{x} \sim \mathbf{x}$. Hence, whenever we are given an inner product – in general satisfying some conditions, which are not given here – then we can talk about the length of a vector, and even of the angle between two vectors. We shall see below, how this is done.

We mention the following properties of the norm $\|\mathbf{x}\|$ defined above for $\mathbf{x} \in \mathbb{R}^n$. Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$ be given. Then

- 1) $\|\mathbf{x}\| > 0$ for $\mathbf{x} \neq \mathbf{0}$ (and $\|\mathbf{0}\| = 0$)
- 2) $\|\lambda \mathbf{x}\| = |\lambda| \|\mathbf{x}\|$
- 3) $\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$ (triangle inequality)
- 4) $|\mathbf{x} \cdot \mathbf{y}| \leq \|\mathbf{x}\| \|\mathbf{y}\|$ (Cauchy-Schwarz's inequality)

The proofs of the first two claims are straightforward (left to the reader) by using the coordinate description.

Cauchy-Schwarz's inequality is proved in the following way: Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ be given points, and let $\lambda \in \mathbb{R}$ be a scalar. Then

$$\begin{aligned} 0 &\leq \|\lambda \mathbf{x} + \mathbf{y}\|^2 = \sum_{j=1}^n (\lambda x_j + y_j)^2 = \lambda^2 \sum_{j=1}^n x_j^2 + 2\lambda \sum_{j=1}^n x_j y_j + \sum_{j=1}^n y_j^2 \\ &= \lambda^2 \|\mathbf{x}\|^2 + 2\lambda(\mathbf{x} \cdot \mathbf{y}) + \|\mathbf{y}\|^2, \end{aligned}$$

which holds for all $\lambda \in \mathbb{R}$. This is a real polynomial in λ of second degree, and it is nonnegative for all $\lambda \in \mathbb{R}$. Hence, its discriminant is not positive,

$$4(\mathbf{x} \cdot \mathbf{y})^2 - 4\|\mathbf{x}\|^2 \|\mathbf{y}\|^2 \leq 0,$$

so by a rearrangement,

$$|\mathbf{x} \cdot \mathbf{y}| \leq \|\mathbf{x}\| \|\mathbf{y}\|,$$

and the claim is proved.

We prove below, after we have defined the angle between two vectors, that the equality sign holds if and only if \mathbf{x} and \mathbf{y} are proportional, i.e. there exists a $\lambda \in \mathbb{R}$, such that either $\mathbf{x} = \lambda \mathbf{y}$ or $\mathbf{y} = \lambda \mathbf{x}$. (We cannot rule out the possibilities of either $\mathbf{x} = \mathbf{0}$ or $\mathbf{y} = \mathbf{0}$.)

Once we have proved *Cauchy-Schwarz's inequality*, we get the *triangle inequality* in the following way:

$$\begin{aligned} \|\mathbf{x} + \mathbf{y}\|^2 &= (\mathbf{x} + \mathbf{y}) \cdot (\mathbf{x} + \mathbf{y}) = \mathbf{x} \cdot \mathbf{x} + 2\mathbf{x} \cdot \mathbf{y} + \mathbf{y} \cdot \mathbf{y}, \\ &\leq \|\mathbf{x}\|^2 + 2\|\mathbf{x}\| \|\mathbf{y}\| + \|\mathbf{y}\|^2 = (\|\mathbf{x}\| + \|\mathbf{y}\|)^2, \end{aligned}$$

hence, by taking the square root,

$$\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|.$$

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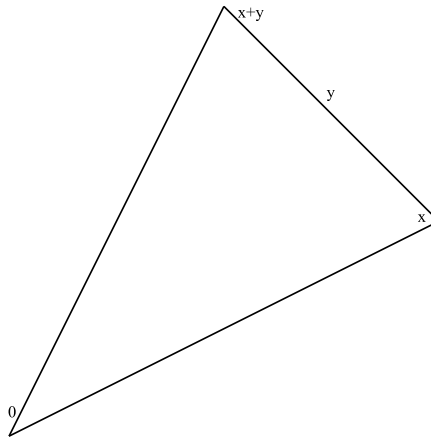


Figure 1.1: The triangle inequality

Remark 1.1 The vectors \vec{x} and $\overrightarrow{x+y}$ form a triangle, if we add the vector \vec{y} from \vec{x} , cf. Figure 1.1. The triangle inequality says that the length from $\mathbf{0}$ to $\mathbf{x} + \mathbf{y}$ is at most equal to the length of the broken path from $\mathbf{0}$ via \mathbf{x} to $\mathbf{x} + \mathbf{y}$. \diamond

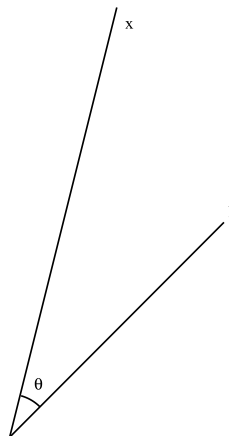


Figure 1.2: The angle between two vectors

Let $\mathbf{x}, \mathbf{y} \neq \mathbf{0}$ be two non-zero vectors from \mathbb{R}^n (or E^n). Then they span an ordinary plane, so we can use the usual geometrical argument of trigonometry in this plane. In fact, we only use Pythagoras's theorem and the high school definition of cosine. In particular, the angle $\theta \in [0, \pi]$ between \mathbf{x} and \mathbf{y} is uniquely determined by the relation

$$\mathbf{x} \cdot \mathbf{y} = \|\mathbf{x}\| \|\mathbf{y}\| \cos \theta,$$

thus

$$\cos \theta = \frac{\mathbf{x} \cdot \mathbf{y}}{\|\mathbf{x}\| \|\mathbf{y}\|} \quad \text{for } \mathbf{x}, \mathbf{y} \neq \mathbf{0},$$

which defines θ uniquely in the interval $[0, \pi]$.

Note in particular that if $\mathbf{x}, \mathbf{y} \neq \mathbf{0}$, and we have equality in *Cauchy-Schwarz's inequality*, then $\cos \theta = \pm 1$, so we have either $\theta = 0$ or $\theta = \pi$. In either cases \mathbf{x} and \mathbf{y} are proportional. When $\mathbf{x} = \mathbf{0}$ or $\mathbf{y} = \mathbf{0}$ this statement is of course trivial.

1.3 The vector product

The three-dimensional case \mathbb{R}^3 has through centuries been thoroughly studied, because it models the daily space which we live in. It was very early realized by physicists and mathematicians that it would be quite convenient to introduce yet another product, denoted $\times : \mathbb{R}^3 \times \mathbb{R}^3 \rightarrow \mathbb{R}^3$. It is *in rectangular coordinates* defined by

$$\begin{aligned} \mathbf{x} \times \mathbf{y} &= (x_1, x_2, x_3) \times (y_1, y_2, y_3) \\ (1.1) \qquad &= (x_2y_3 - x_3y_2, x_3y_1 - x_1y_3, x_1y_2 - y_2x_1) \end{aligned}$$

and it works only in \mathbb{R}^3 !

If the reader is familiar with how to calculate (3×3) -determinants, then (1.1) can also formally be written in the following way,

$$(1.2) \quad \mathbf{x} \times \mathbf{y} = \begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \end{vmatrix},$$

where $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ form an orthonormal basis, and $(x_1, x_2, x_3), (y_1, y_2, y_3) \in \mathbb{R}^3$ are the coordinates of \mathbf{x}, \mathbf{y} , resp., expanded with respect to this basis.

It is easy to remember the structure of this determinant. We put the coordinates of the first factor in the first row, the coordinates of the second factor in the second row, and the three basis vectors in the third row.

By using *Linear Algebra* we immediately get the following results:

- 1) When \mathbf{x} and \mathbf{y} are interchanged, then the first two rows in the determinant are interchanged, so the determinant changes its sign, and we obtain that

$$\mathbf{y} \times \mathbf{x} = -\mathbf{x} \times \mathbf{y}.$$

This means that the vector product is *anticommutative*.

- 2) It is easy to see that

$$(1.3) \quad (\mathbf{x} \times \mathbf{y}) \cdot \mathbf{z} = \begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix}.$$

Since the value of the determinant does not change, when we change the rows cyclically, we immediately get the following result,

$$(\mathbf{x} \times \mathbf{y}) \cdot \mathbf{z} = \begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix} = \begin{vmatrix} y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \\ x_1 & x_2 & x_3 \end{vmatrix} = (\mathbf{y} \times \mathbf{z}) \cdot \mathbf{x} = \mathbf{x} \cdot (\mathbf{y} \times \mathbf{z}),$$

which shows that we can interchange the two products if we only keep the order of the vectors \mathbf{x} , \mathbf{y} , \mathbf{z} . Hence,

$$(\mathbf{x} \times \mathbf{y}) \cdot \mathbf{z} = \mathbf{x} \cdot (\mathbf{y} \times \mathbf{z}).$$

- 3) By choosing $\mathbf{z} = \mathbf{x}$, or $\mathbf{z} = \mathbf{y}$ it also follows from (1.3) that

$$(\mathbf{x} \times \mathbf{y}) \cdot \mathbf{x} = 0 \quad \text{and} \quad (\mathbf{x} \times \mathbf{y}) \cdot \mathbf{y} = 0.$$

This means that $(\mathbf{x} \times \mathbf{y})$ is perpendicular to both \mathbf{x} and \mathbf{y} , and since all vectors lie in \mathbb{R}^3 , the vector $(\mathbf{x} \times \mathbf{y})$ is either $\mathbf{0}$ or normal to the plane spanned by \mathbf{x} and \mathbf{y} .

- 4) The products \cdot and \times are actually geometrically connected with the “abstract” *Euclidean space* E^3 , which means that they are independent of our specific choice of orthonormal basis. This means that we can choose the basis, such that \mathbf{x} and \mathbf{y} lie in the plane spanned by \mathbf{e}_1 and \mathbf{e}_2 , which means that

$$\mathbf{x} = (x_1, x_2, 0) \quad \text{and} \quad \mathbf{y} = (y_1, y_2, 0).$$

Then we get from (1.1) that

$$(\mathbf{x} \times \mathbf{y}) = (0, 0, x_1y_2 - y_1x_2) = \left(0, 0, \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix} \right),$$

and it is well-known that the absolute value of the third coordinate, $|x_1y_2 - y_1x_2|$ is the area of the parallelogram defined by the vectors \mathbf{x} and \mathbf{y} .

When we look closer at the sign of $x_1y_2 - y_1x_2$, it follows that when \mathbf{x} , \mathbf{y} and $\mathbf{x} \times \mathbf{y}$ are all $\neq \mathbf{0}$, then \mathbf{x} , \mathbf{y} and $\mathbf{x} \times \mathbf{y}$ in this order defines a right hand system of vectors. This means that if \mathbf{x} is directed along your right thumb, and \mathbf{y} along your right forefinger, then \mathbf{x} , \mathbf{y} must point along your right middle finger. This is also a way to find out the direction, in which $\mathbf{x} \times \mathbf{y}$ is pointing.

The length of $\mathbf{x} \times \mathbf{y}$ is as noted above equal to the *area* of the parallelogram, which is spanned by \mathbf{x} and \mathbf{y} .

- 5) When we combine 3) and 4) above it follows that $|(\mathbf{x} \times \mathbf{y}) \cdot \mathbf{z}|$ is the volume of the parallelepipedum spanned by the three vectors \mathbf{x} , \mathbf{y} and \mathbf{z} .
- 6) Finally, we shall consider the *double vector product* $\mathbf{x} \times (\mathbf{y} \times \mathbf{z})$, which by 3) must be orthogonal to both \mathbf{x} and $\mathbf{y} \times \mathbf{z}$. It must therefore in particular lie in the plane spanned by \mathbf{y} and \mathbf{z} , so there are real constants α and β , such that

$$\mathbf{x} \times (\mathbf{y} \times \mathbf{z}) = \alpha\mathbf{y} + \beta\mathbf{z}.$$

This is orthogonal to \mathbf{x} , so

$$0 = \mathbf{x} \cdot (\alpha\mathbf{y} + \beta\mathbf{z}) = \alpha(\mathbf{x} \cdot \mathbf{y}) + \beta(\mathbf{x} \cdot \mathbf{z}).$$

This is only possible, if there exists a real constant λ , such that

$$\alpha = \lambda(\mathbf{x} \cdot \mathbf{z}) \quad \text{and} \quad \beta = -\lambda(\mathbf{x} \cdot \mathbf{y}).$$

Finally, by insertion,

$$\mathbf{x} \times (\mathbf{y} \times \mathbf{z}) = \lambda\{(\mathbf{x} \cdot \mathbf{z})\mathbf{y} - (\mathbf{x} \cdot \mathbf{y})\mathbf{z}\}.$$

This shows that $\mathbf{x} \times (\mathbf{y} \times \mathbf{z})$ and $(\mathbf{x} \cdot \mathbf{z})\mathbf{y} - (\mathbf{x} \cdot \mathbf{y})\mathbf{z}$ are proportional. Then use the coordinates of \mathbf{x} , \mathbf{y} and \mathbf{z} with respect to the orthonormal basis $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ and prove that $\lambda = 1$. (Left to the reader.) It follows that

$$\mathbf{x} \times (\mathbf{y} \times \mathbf{z}) = (\mathbf{x} \cdot \mathbf{z})\mathbf{y} - (\mathbf{x} \cdot \mathbf{y})\mathbf{z}.$$

These results on the vector product in \mathbb{R}^3 will later be important in our treatment of e.g. integration in \mathbb{R}^3 .

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1.4 The most commonly used coordinate systems

When we are given the Euclidean space E^n and want to describe it by coordinates in \mathbb{R}^n , it is obvious that the coordinate system can be chosen in many ways. We shall always try to choose the coordinate system in such a way that the calculations become as easy as possible. This is of course a very vague statement, which does not help the reader, so we here list the most commonly used coordinate systems. Concerning the choice of which one, the reader should be guided by e.g. the geometry of the domain, or in case of integration, of the structure of the integrand.

- 1) The *rectangular coordinate system* in \mathbb{R}^n , $n \in \mathbb{N}$ arbitrarily chosen. This is the most obvious coordinate system to start with. As already mentioned previously, its basis is given by the vectors

$$\mathbf{e}_1 = (1, 0, \dots, 0), \dots, \mathbf{e}_n = (0, \dots, 0, 1),$$

in general,

$$\mathbf{e}_j = (\delta_{1j}, \delta_{2j}, \dots, \delta_{nj}),$$

where δ_{ij} is the *Kronecker symbol*, defined by

$$\delta_{ij} = \begin{cases} 1 & \text{for } i = j, \\ 0 & \text{for } i \neq j. \end{cases}$$

The domain in the Euclidean space E^n is congruent with the corresponding coordinate domain in \mathbb{R}^n , and one hardly notices the difference.

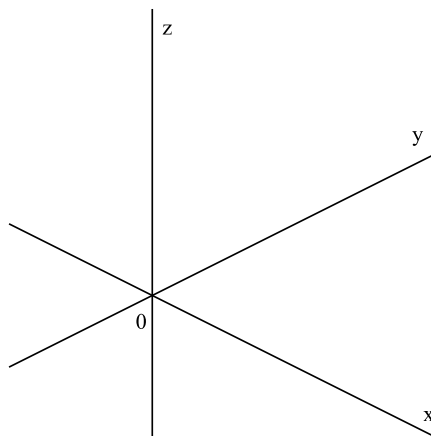


Figure 1.3: The usual way to draw the rectangular coordinate system in \mathbb{R}^3 .

If $\mathbf{x} = (x_1, \dots, x_n)$ and $\mathbf{y} = (y_1, \dots, y_n)$ with respect to the basis above, then the *inner product* of \mathbf{x} and \mathbf{y} is defined by

$$\mathbf{x} \cdot \mathbf{y} = x_1 y_1 + \dots + x_n y_n = \sum_{j=1}^n x_j y_j.$$

In the important special case of \mathbb{R}^3 we also define the *vector product* by

$$\begin{aligned} \mathbf{x} \times \mathbf{y} &= (x_1, x_2, x_3) \times (y_1, y_2, y_3) \\ &= (x_2y_3 - x_3y_2, x_3y_1 - x_1y_3, x_1y_2 - y_2x_1). \end{aligned}$$

The rectangular system is well designed for linear problems, e.g. rectilinear motions. In the case of integration, the domain of integration should be limited by straight lines. If this condition is not satisfied, one may by the following reductions end up with almost incalculable integrals.

2) *Polar coordinates* in the plane. These can only be used in dimension 2.

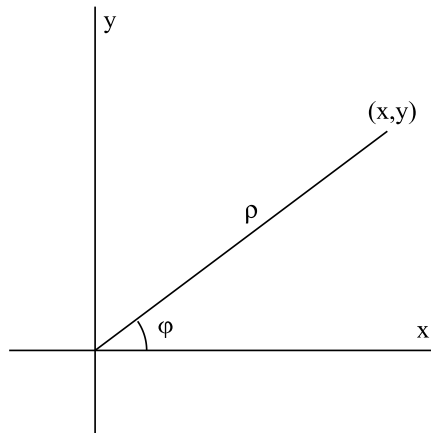


Figure 1.4: The coordinate system in polar coordinates

Assume that the point P in the Euclidean space E^2 has the rectangular coordinates (x, y) , cf. Figure 1.4. The distance ρ from origo $O: (0, 0)$ to $P: (x, y)$ is by *Pythagoras's theorem* given by

$$\rho = \sqrt{x^2 + y^2}.$$

It then follows by high school trigonometry that

$$x = \rho \cos \varphi \quad \text{and} \quad y = \rho \sin \varphi,$$

where φ is the angle measured from the X -axis in the positive sense of the plane.

If $\rho = 0$, i.e. $P = O$, so we are at origo, then the angle φ is undetermined. Every $\varphi \in \mathbb{R}$ will do in this case.

If $x \neq 0$, then

$$\tan \varphi = \frac{y}{x},$$

so we may choose

$$\varphi = \begin{cases} \operatorname{Arctan} \left(\frac{y}{x} \right) & \text{for } x > 0, \\ \operatorname{Arctan} \left(\frac{y}{x} \right) + \pi & \text{for } x < 0. \end{cases}$$

If instead $y \neq 0$, then

$$\cot \varphi = \frac{x}{y},$$

so we may choose

$$\varphi = \begin{cases} \operatorname{Arccot} \left(\frac{x}{y} \right) & \text{for } y > 0, \\ \operatorname{Arccot} \left(\frac{x}{y} \right) + \pi & \text{for } y < 0. \end{cases}$$

Note, however, that when $\varrho > 0$, the angle is only specified modulo 2π , so we can always add a multiple of 2π to the angle φ without changing x and y .

Summing up, we get the following correspondence between rectangular coordinates $(x, y) \in \mathbb{R}^2 \setminus \{(0, 0)\}$ and polar coordinates (ϱ, φ) , where $\varrho > 0$, and φ belongs to some half open interval of length 2π ,

$$(1.4) \quad \begin{cases} x = \varrho \cos \varphi, & y = \varrho \sin \varphi, \\ \varrho = \sqrt{x^2 + y^2} \\ \tan \varphi = \frac{y}{x} \text{ for } x \neq 0, & \text{and } \cot \varphi = \frac{x}{y} \text{ for } y \neq 0. \end{cases}$$

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Experience shows that students are not too happy with the polar coordinates, when they first meet them. This is probably due to the fact that the angle φ is not uniquely determined, in general only modulo 2π . Nevertheless, they are very useful, and when circular motions are considered, they are better than rectangular coordinates, so they are very important in e.g. Mechanics. We shall here illustrate this by the simplest possible example. The *unit circle* is explicitly described in polar coordinates by the simple equation

$$\rho = 1.$$

This unit circle is implicitly described in rectangular coordinates by

$$\sqrt{x^2 + y^2} = 1, \quad \text{or} \quad x^2 + y^2 = 1,$$

so by solving this equation with respect to y we get the more messy explicit expression,

$$y = \begin{cases} \sqrt{1-x^2} & \text{for } x \in [-1, 1] \text{ and } y \geq 0, \\ -\sqrt{1-x^2} & \text{for } x \in [-1, 1] \text{ and } y \leq 0. \end{cases}$$

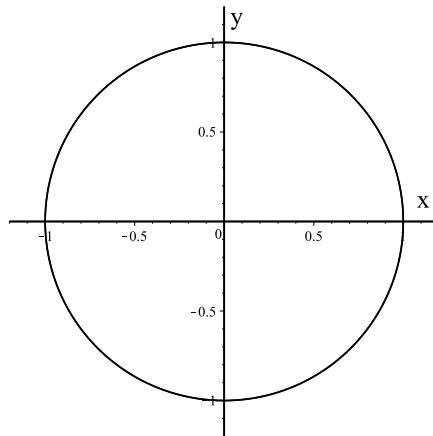


Figure 1.5: The unit circle in E^2 .

When we compare Figure 1.5 and Figure 1.6 it is obvious that although the two sets are in correspondence, they do not look like each other. This means that in polar coordinates the geometry is quite different in the Euclidean plane E^2 and the coordinate plane \mathbb{R}^2 . Therefore, they must not be confused!

The polar coordinates are used, whenever we are dealing with circular motion or domains, which are discs. Also, when the integrand contains expressions which are functions in $\sqrt{x^2 + y^2}$ in the rectangular coordinates, one should rewrite the problem in polar coordinates, because then we may get rid of at least some of these square roots. The drawback is of course that the angle φ in (1.4) is only specified modulo 2π , so we must choose an half-open φ -interval of length 2π , e.g. $] -\pi, \pi]$, or $] 0, 2\pi]$, or more general, $] \alpha, \alpha + 2\pi]$ for some constant α , depending on the geometry of the domain under consideration.

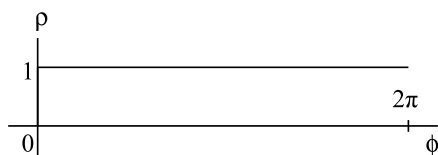


Figure 1.6: The parameter set in polar coordinates of the unit circle in \mathbb{R}^2 .

We note that the description of the *inner product* in polar coordinates is not an easy job, and we shall not derive it.

- 3) *Semi-polar coordinates* in E^3 . These can only be applied in the Euclidean space E^3 . Also in this case, the corresponding domain in the coordinate space \mathbb{R}^3 is distorted compared with the original set in E^3 .

Given the usual rectangular basis $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ in E^n , the idea is to apply the polar coordinates in the plane spanned by \mathbf{e}_1 and \mathbf{e}_2 , and keep the rectangular coordinate along the \mathbf{e}_3 -axis.

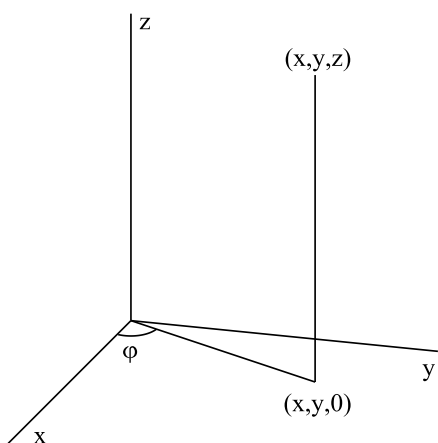


Figure 1.7: The geometry of the definition of the semi-polar coordinates in \mathbb{R}^3 .

It follows from the above that

$$\begin{cases} x = \varrho \cos \varphi, & y = \varrho \sin \varphi, & z = z, \\ \varrho = \sqrt{x^2 + y^2}, \\ \tan \varphi = \frac{y}{x} \text{ for } x \neq 0 & \text{and} & \cot \varphi = \frac{x}{y} \text{ for } y \neq 0. \end{cases}$$

If $(x, y) \neq (0, 0)$, then φ is determined modulo 2π . On the z -axis, where $(x, y) = (0, 0)$, the angle φ is undetermined, and any $\varphi \in \mathbb{R}$ can be used.

When the angle φ is kept fixed, while $\varrho \geq 0$ and $z \in \mathbb{R}$ vary, we describe a half plane, which we call the *meridian half plane*. In such a meridian half plane (ϱ, z) are ordinary rectangular coordinates.

If instead $\varrho > 0$ is kept fixed, while φ and z vary, we describe a *cylindric surface* with the z -axis as its axis of rotation. For that reason the semi-polar coordinates are also called *cylindric coordinates*.

The semi-polar coordinates are typically used, when we are dealing with rotational bodies in E^3 , or, if a rectangular coordinate system in \mathbb{R}^3 e.g. the variables (x, y) only appear in the combined form $\sqrt{x^2 + y^2}$.

- 4) *Spherical coordinates in \mathbb{R}^3* . It was noted above in 3), semi-polar coordinates, that for fixed φ we describe the *meridian half plane* in the rectangular coordinates (ϱ, z) , $\varrho \geq 0$ and $z \in \mathbb{R}$.

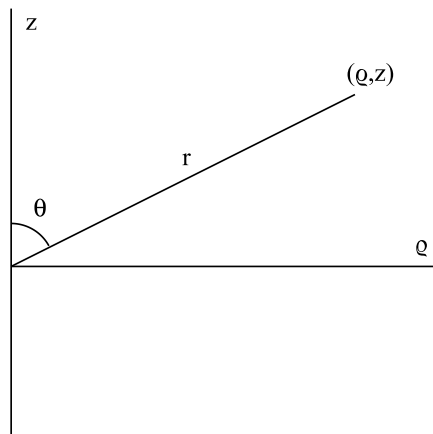


Figure 1.8: The meridian half plane for fixed φ .

Let $r = \sqrt{\varrho^2 + z^2}$ denote the Euclidean distance between $(0, 0)$ and (ϱ, z) , and let $\vartheta \in [0, \pi]$ denote the angle *positive from the z -axis towards the vector of coordinates (ϱ, z)* , cf. Figure 1.8. Then clearly,

$$z = r \cos \theta \quad \text{and} \quad \varrho = r \sin \theta, \quad \text{for } \theta \in [0, \pi] \text{ and } r = \sqrt{z^2 + \varrho^2}.$$

Since we already have

$$x = \varrho \cos \varphi \quad \text{and} \quad y = \varrho \sin \varphi$$

for $\varphi \in I$, where I is some interval of length 2π , where we for convenience here put $I = [0, 2\pi]$, we get by insertion

$$(1.5) \quad \begin{cases} x = r \sin \theta \cos \varphi, \\ y = r \sin \theta \sin \varphi, \\ z = r \cos \theta, \end{cases} \quad \theta \in [0, \pi], \quad \varphi \in [0, 2\pi], \quad r \in [0, +\infty[.$$

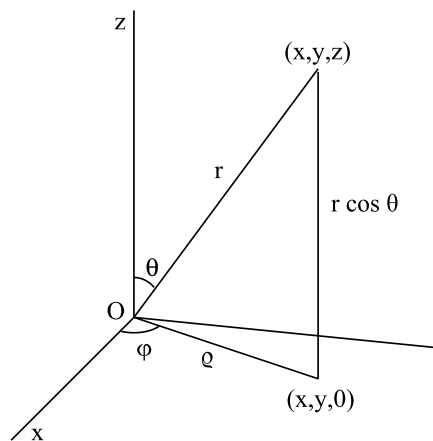


Figure 1.9: The geometry of the definition of the spherical coordinates in \mathbb{R}^3 .

We call (r, θ, φ) the *spherical coordinates* in \mathbb{R}^3 . If $r > 0$ is kept fixed, then (1.5) describes a *sphere* of radius r .

If we let $r =$ the radius of the Earth and specify $\varphi \in [-\pi, \pi] \sim [-180^\circ, 180^\circ]$, and define $\vartheta := \frac{\pi}{2} - \theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \sim [-90^\circ, 90^\circ]$, then φ is the degree of longitude, and ϑ is the degree of latitude. It is well-known that these two spherical coordinates with success have been applied for centuries in Geography and Astronomy.

Spherical coordinates are in particular applied, when we are dealing with a sphere, or when the rectangular coordinates (x, y, z) also appear in the form $\sqrt{x^2 + y^2 + z^2}$.

If instead $\theta \in]0, \pi[$ is kept fixed, then (1.5) describes a *cone*, and – as already seen above – when φ is a constant, then (1.5) describes a *meridian half plane*.

- 5) It is possible to extend this construction of spherical coordinates to \mathbb{R}^n for $n > 3$. In fact, if (x, y, z, t) are the rectangular coordinates in \mathbb{R}^4 , then we can start by using the spherical coordinates above in the variables (x, y, z) . When φ and θ are kept fixed, we again obtain a *meridian half plane*. This time the rectangular coordinates are (r, t) . Let (r, t) be a vector in this half plane, and

define $R = \sqrt{r^2 + t^2}$ and $\vartheta \in [0, \pi]$ as the angle between the t -axis and the vector (r, t) , measured from the t -axis. Then,

$$t = R \cos \vartheta \quad \text{and} \quad r = R \sin \vartheta, \quad \vartheta \in [0, \pi] \quad \text{and} \quad R = \sqrt{r^2 + t^2} = \sqrt{x^2 + y^2 + z^2 + t^2},$$

and we obtain by insertion the rectangular coordinates $(x, y, z, t) \in \mathbb{R}^4$ expressed in the hyperspherical coordinates $(R, \varphi, \theta, \vartheta)$.

Continue this construction to higher dimensions, whenever needed. Note, however, that this construction will not be used in this series of books.

Remark 1.2 The author has actually used this construction in an analysis of solid balls in E^n . These have an unexpected geometry, when $n > 3$, and one cannot just conclude that “they behave as the solid balls in the usual Euclidean space E^3 ”. One example is the following: Choose any small $\varepsilon, \delta \in]0, 1[$, and let B_n denote the unit ball in \mathbb{R}^n of n -dimensional volume $|B_n|$. Let A_n denote the subset of B_n , which is obtained by restricting e.g. the x_1 -coordinate, so

$$A_n := \{(x_1, \dots, x_n) \mid x_1^2 + \dots + x_n^2 \leq 1 \text{ and } -\varepsilon \leq x_1 \leq \varepsilon\}$$

with its n -dimensional volume denoted by $|A_n|$

Then there exists an $N \in \mathbb{N}$, such that for all $n \geq N$ most of the volume of B_n lies the slab A_n , or more precisely,

$$|A_n| \geq (1 - \delta) |B_n|. \quad \diamond$$

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1.5 Point sets in space

We shall in this section introduce the most necessary of what mathematicians call *Topology*. We shall use the Euclidean space E^n as our model space, and whenever necessary we shall choose a rectangular coordinate system and use the equivalent coordinate space \mathbb{R}^n . This means that at least in E^2 and E^3 it should be possible to visualize the sets. In particular, the sets are easily drawn in the Euclidean plane E^2 .

The formal definition of a set A in the Euclidean space E^n is given by

$$A = \{\mathbf{x} \in E^n \mid p(\mathbf{x})\},$$

where p denotes a property, which is satisfied for all $\mathbf{x} \in A$. In plain words this is expressed as “ A is the set of $\mathbf{x} \in \mathbb{R}^n$, for which property $p(\mathbf{x})$ is true”.

If $A \subseteq E^n$ allows some symmetry, it is convenient to introduce the axes, such that these are in harmony with this symmetry. Such a choice will usually have the effect that the corresponding coordinate set $\tilde{A} \subset \mathbb{R}^n$ becomes simple.

In the Euclidean plane $E^2 \sim \mathbb{R}^2$ it is easy to draw the most important sets for the applications. This does not mean that all plane sets can be reasonably drawn. For instance, we have problems in *drawing* the set

$$\{(x, y) \mid x \in [0, 1] \cap \mathbb{Q}, y \in [0, 1] \cap \mathbb{Q}\},$$

which is the set of all points in the square $[0, 1]^2$ of rational coordinates. However, we shall in the following mostly avoid such pathological sets, so in general they are not at problem.

We shall, whenever necessary or convenient, use the following conventions on drawings in $E^2 \sim \mathbb{R}^2$: What is included in a set is marked by

- 1) a hatching (2-dimensional),
- 2) a full-drawn line (1-dimensional),
- 3) a small circle or just a point (0-dimensional).

In particular, a dot-and-dash line is only limiting a hatched set, and the points on such a line do not belong to the set. Cf. Figure 1.10 to the left.

Note that if a closed curve without double points surrounds a set which together with the curve is totally included in the set, we do not hatch the set inside the closed curve. Cf. Figure 1.10 to the right.

1.5.1 Interior, exterior and boundary of a set

Given a Euclidean space E^n with its usual Euclidean distance, which in rectangular coordinates is given by

$$\text{dist}_n(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\| = \sqrt{\sum_{j=1}^n (x_j - y_j)^2}.$$

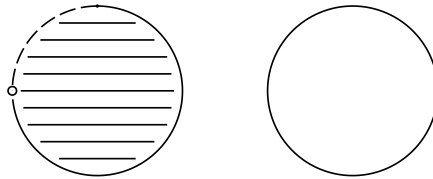


Figure 1.10: Visualization of two discs. On the left disc part of the boundary is not included, so we are forced to hatch the interior. To the right, the full boundary is included, so there is no need to hatch the interior.

where (x_1, \dots, x_n) and $(y_1, \dots, y_n) \in \mathbb{R}^n$ are the coordinates of \mathbf{x} and \mathbf{y} , resp.. Then it is possible to introduce solid balls in $E^n \sim \mathbb{R}^n$ as the points of distance smaller than (or equal to) a given radius from a given centre \mathbf{x}_0 .

The *open ball* $B(\mathbf{x}_0, r)$ of radius $r > 0$ and centre $\mathbf{x}_0 \in E^n \sim \mathbb{R}^n$ is given by

$$B(\mathbf{x}_0, r) := \{\mathbf{x} \in E^n \mid \text{dist}_n(\mathbf{x}, \mathbf{x}_0) < r\} = \{\mathbf{x} \in \mathbb{R}^n \mid \|\mathbf{x} - \mathbf{x}_0\| < r\}.$$

The *closed ball* $B[\mathbf{x}_0, r]$ of radius $r > 0$ and centre $\mathbf{x}_0 \in E^n \sim \mathbb{R}^n$ is given by

$$B[\mathbf{x}_0, r] := \{\mathbf{x} \in E^n \mid \text{dist}_n(\mathbf{x}, \mathbf{x}_0) \leq r\} = \{\mathbf{x} \in \mathbb{R}^n \mid \|\mathbf{x} - \mathbf{x}_0\| \leq r\}.$$

In the latter case we may allow $r = 0$, in which case the closed ball of centre \mathbf{x}_0 and radius 0 is just the centre, $B[\mathbf{x}_0, 0] = \{\mathbf{x}_0\}$. These balls are fundamental in describing more general objects.

- 1) If $\mathbf{x}_1 \in A$, and there exists an $r > 0$, such that $B(\mathbf{x}_1, r) \subseteq A$, then we call \mathbf{x}_1 an *interior point* of A . The set of all interior points of A is called the *interior* of A , and it is denoted by A° .
- 2) If $\mathbf{x}_2 \notin A$, and there exists an $r > 0$, such that $B(\mathbf{x}_2, r) \cap A = \emptyset$, then we call \mathbf{x}_2 an *exterior point* of A . The set of all exterior points of A is called the *exterior* of A . If $\complement A := E^2 \setminus A$ denotes the complementary set of A , then the exterior of A is the interior of the complement of A , i.e. the set $(\complement A)^\circ$. The point \mathbf{x}_2 on Figure 1.11 is exterior.
- 3) The remaining part $E^n \setminus \{A^\circ \cup (\complement A)^\circ\}$ is called the *boundary* of A . It is denoted by ∂A . Due to the ‘‘symmetry’’ it follows that A and $\complement A$ have the same boundary, so

$$\partial A = \partial(\complement A) = E^n \setminus \{A^\circ \cup (\complement A)^\circ\}.$$

On Figure 1.11 the points $\mathbf{x}_3 \in A$ and $\mathbf{x}_4 \notin A$ are both boundary points.

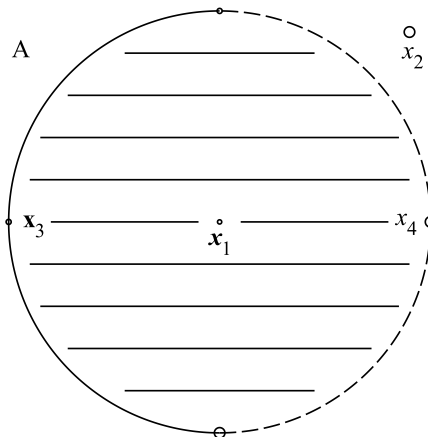


Figure 1.11: A set $A \subseteq E^2$ divides E^2 into three sets, 1) the *interior* S° of A , 2) the *exterior* $(\mathbb{C}A)^\circ$ of A , and 3) the *boundary* ∂A of A , which is the remaining set $E^2 \setminus \{A^\circ \cup (\mathbb{C}A)^\circ\}$.

A boundary point $\mathbf{x} \in \partial A$ is characterized in the following way: For every $r > 0$, the open ball $B(\mathbf{x}, r)$ contains points from both the interior A° and the exterior $(\mathbb{C}A)^\circ$, i.e.

$$B(\mathbf{x}, r) \cap A^\circ \neq \emptyset \quad \text{and} \quad B(\mathbf{x}, r) \cap (\mathbb{C}A)^\circ \neq \emptyset.$$

Note that the boundary point $\mathbf{x} \in \partial A$ may or may not be a point in A .

The *union* of the *interior* and the *boundary* is called the *closure* of A . It is denoted by \overline{A} , hence

$$\overline{A} = A^\circ \cup \partial A = A \cup \partial A.$$

A set A is called *open*, if it does not contain any boundary point, i.e. if

$$A \cap \partial A = \emptyset, \quad \text{or equivalently,} \quad A = A^\circ.$$

Summing up we see that

$$A \text{ is open, if and only if } A \cap \partial A = \emptyset,$$

and

$$A \text{ is closed, if and only if } \partial A \subseteq A.$$

A set A is called a *neighbourhood* of $\mathbf{x} \in A$, if there exists an $r > 0$, such that $B(\mathbf{x}, r) \subseteq A$. In particular, when $A = A^\circ$ is open, then A is a neighbourhood of all its points.

A boundary point P of A is called an *isolated point*, if there exists an $r > 0$, such that $B(P, r) \cap A = \{P\}$, i.e. if P is the only point from A in a neighbourhood of P .

In the rectangular coordinate space \mathbb{R}^n we have already used the distance

$$\text{dist}_n(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|_n := \sqrt{\sum_{j=1}^n (x_j - y_j)^2}.$$

The open/closed balls are written

$$B(\mathbf{x}, r) = \left\{ \mathbf{y} \in \mathbb{R}^n \mid \sum_{j=1}^n (x_j - y_j)^2 < r^2 \right\} \text{ and } B[\mathbf{x}, r] = \left\{ \mathbf{y} \in \mathbb{R}^n \mid \sum_{j=1}^n (x_j - y_j)^2 \leq r^2 \right\}.$$

The importance of these new topological concepts will be demonstrated in connection with limits and continuity in the next volume of this series.

1.5.2 Starshaped and convex sets

Concerning the shapes of the sets under consideration the situation is very simple in the 1-dimensional case of E^1 , where it usually suffices only to consider intervals. However, even in the two-dimensional case of E^2 concerning the shapes of sets, the situation becomes far more complicated, and it is not always obvious which type of sets we should look at.

Clearly, the n -dimensional intervals

$$I_1 \times I_2 \times \cdots \times I_n := \{(x_1, x_2, \dots, x_n) \mid x_1 \in I_1, x_2 \in I_2, \dots, x_n \in I_n\}$$

are obvious candidates, where each I_j is of one of the following four types,

$$I_j =]a_j, b_j[,]a_j, b_j], [a_j, b_j[, [a_j, b_j].$$

The balls defined previously are also often used sets.

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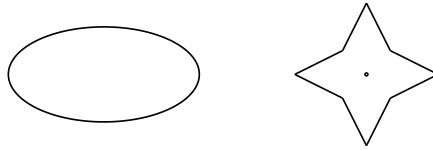


Figure 1.12: A convex and a starshaped set.

We may, however, also be interested in sets having some weaker geometrical properties.

A set $A \subseteq E^n$ is called *starshaped* with respect to a point $\mathbf{x}_0 \in A$, if for every $\mathbf{x} \in A$, the straight *line segment* $[\mathbf{x}_0, \mathbf{x}]$ from \mathbf{x}_0 to \mathbf{x} lies totally in A . The set to the right of Figure 1.12 illustrates why the set is called *starshaped*. Every line segment from the centre to any other point in A lies in A . However, if we choose two points from adjacent arms of the star, it is obvious that the line segment between them is not totally contained in A , so we cannot in general choose the point \mathbf{x}_0 arbitrarily.

If the line segment between any two points of A also lies in A , then we say that this (clearly) starshaped set is *convex*. The set to the left of Figure 1.12 is convex.

Finally, we say that a set $A \subset E^n$ is *bounded*, if there exists an $R > 0$, such that $A \subseteq B(\mathbf{0}, R)$, i.e. A is contained in a ball of finite radius. Any centre \mathbf{x}_0 may of course be used here instead.

1.5.3 Catalogue of frequently used point sets in the plane and the space

We shall in this section give a summary of frequently used point sets in $E^2 \sim \mathbb{R}^2$ and $E^3 \sim \mathbb{R}^3$.

1) If $I, J \subset \mathbb{R}$ are ordinary one-dimensional intervals, we define their *product set* by

$$I \times J := \{(x, y) \in \mathbb{R}^2 \mid x \in I, y \in J\}.$$

If $J = I$, we often write I^2 instead of $I \times I$.

If I and J are bounded, then $I \times J$ is a *rectangle*. In particular, I^2 is a *square*, if I is a bounded interval.

Let I be a bounded interval. Then $I \times \mathbb{R}$ is called a *strip*, and $I \times [a, +\infty[$ and $I \times]-\infty, a]$ are called *half-strips*, cf. Figure 1.13.

The set $\mathbb{R}_+ \times \mathbb{R}_+ = \mathbb{R}_+^2$ is the open first *quadrant*, and $\mathbb{R} \times \mathbb{R}_+$ is the upper *half-plane*, cf. Figure 1.14.

We mention the possibilities of the open right half-plane $\mathbb{R}_+ \times \mathbb{R}$, the open left half-plane $\mathbb{R}_- \times \mathbb{R}$ and the open lower half-plane $\mathbb{R} \times \mathbb{R}_-$ and variants of these.

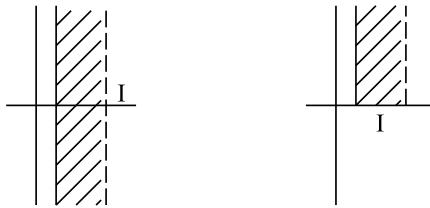


Figure 1.13: A strip and a half-strip.



Figure 1.14: The first quadrant and the upper half-plane.

- 2) Let $A \subset \mathbb{R}^2$ be a bounded plane set, and let $I \subseteq \mathbb{R}$ be an interval. We define a *cylinder* in \mathbb{R}^3 by

$$A \times I := \{(x, y, z) \in \mathbb{R}^3 \mid (x, y) \in A, z \in I\},$$

cf. Figure 1.15.

When the interval I is bounded, then the length $|I|$ of I is called the *height* of the cylinder.

If A is a polygon, we also call the cylinder a *prism*. Special cases are a *parallelepipedum*, where A is a rectangle, and a *cube*, where A is a square.

- 3) Assume that the coordinate system has been chosen, such that the coordinate description of the set A only contains the first two coordinates (x, y) in the form $x^2 + y^2$. Then A is rotational symmetric with respect to the z -axis, and the three-dimensional set A can be fully described by

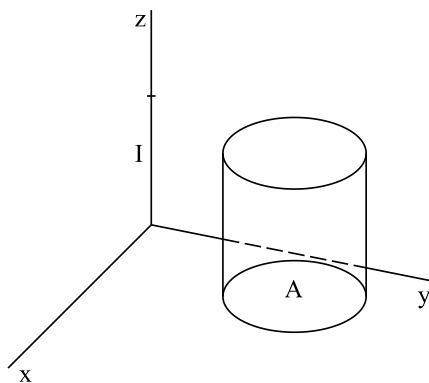


Figure 1.15: A (bounded) cylinder.

one (two-dimensional) *meridian half-plane*, in which we can use either the rectangular coordinates (ϱ, z) or the polar coordinates (r, θ) , as described earlier. Then the point set A can be described as a *body of revolution*, which is obtained by revolving the so-called *meridian section*, cf. Figure 1.16

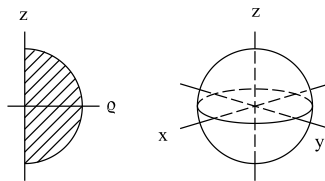


Figure 1.16: The meridian section to the left is a half disc in the right (ϱ, z) -half-plane. The body of revolution is a solid ball

- 4) A *torus* is the body of revolution, which is obtained by revolving a disc with respect to a line, which does not meet the disc. If the coordinate system is placed conveniently with the z -axis as the axis of revolution, then the disc in the meridian half-plane (i.e. the meridian section) is described by the inequality

$$z^2 + (\varrho - a)^2 \leq b^2, \quad \text{where } 0 < b < a,$$

cf. Figure 1.21.

Since $\varrho = \sqrt{x^2 + y^2}$, the *torus* T is then described in \mathbb{R}^3 by

$$T = \left\{ (x, y, z) \in \mathbb{R}^3 \mid \left(\sqrt{x^2 + y^2} - a \right)^2 + z^2 \leq b^2 \right\},$$

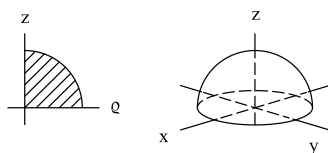


Figure 1.17: The meridian section to the left is a quarter of a disc in the right (ρ, z) -half-plane. The body of revolution is a solid half ball

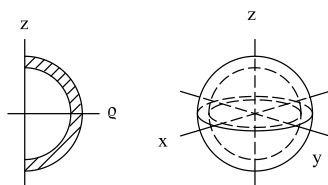


Figure 1.18: The meridian section to the left is the half of a solid ring in the right (ρ, z) -half-plane. The body of revolution is a solid shell of a ball

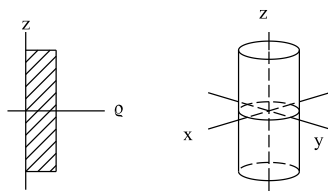


Figure 1.19: The meridian section to the left is a rectangle with one of its sides on the z -axis. The body of revolution is the cylinder to the right.

where $0 < b < a$.

- 5) Consider a (solid) cone of revolution K of height $h > 0$ and radius a of its basis. If the coordinate

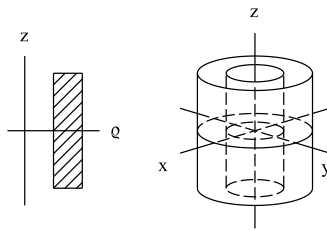


Figure 1.20: The meridian section to the left is a rectangle without one of its sides on the z -axis. The body of revolution to the right is the shell of a cylinder.

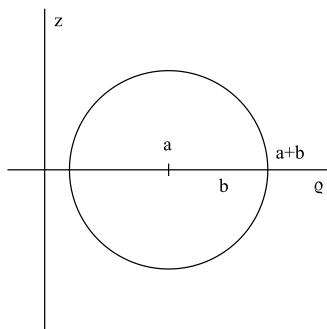


Figure 1.21: The meridian section of a torus.

axes are put as on Figure 1.22, then the cone is described in cylinder coordinates by

$$\frac{z}{h} + \frac{\rho}{a} \leq 1 \quad \text{and} \quad z \geq 0.$$

Using that

$$0 < z < h \left(1 - \frac{\rho}{a}\right) \quad \text{and} \quad \rho = \sqrt{x^2 + y^2} \leq a,$$

we obtain the following rectangular coordinate description of the cone K ,

$$K = \left\{ (x, y, z) \mid x^2 + y^2 \leq a^2, 0 \leq z \leq h \left(1 - \frac{\sqrt{x^2 + y^2}}{a}\right) \right\}.$$

If instead we choose the triangle as in Figure 1.23, then the hypotenuse of the triangle has the equation

$$z = \frac{\rho h}{a}.$$

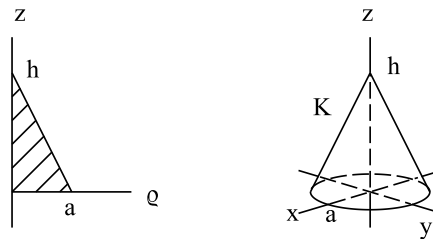


Figure 1.22: A triangle in the meridian half-plane, and the cone K of height h and radius a of its basis, which is the body of revolution of the triangle in the meridian half-plane.

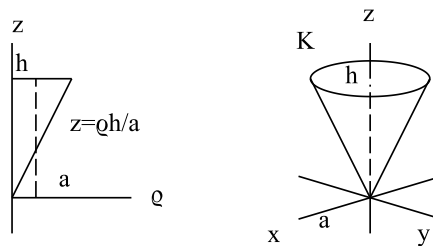


Figure 1.23: A triangle in the meridian half-plane, and the cone K of height h and radius a of its basis, which is the body of revolution of the triangle in the meridian half-plane.

We therefore conclude that the triangle in the meridian half-plane is described by

$$\left\{ (\rho, z) \mid \rho \geq 0, \frac{\rho h}{a} \leq z \leq h \right\}.$$

Since $\rho = \sqrt{x^2 + y^2} \geq 0$, it follows that the cone K in this case is described in rectangular coordinates by

$$\left\{ (x, y, z) \mid x^2 + y^2 \leq a^2, \frac{h}{a} \sqrt{x^2 + y^2} \leq z \leq h \right\}.$$

In particular we see, that the rectangular description contains the ugly looking square root, $\sqrt{x^2 + y^2}$, which may obscure the reader's feeling of what is going on.

Note on Figure 1.23 that we fix ρ (the vertical dashed line) to find the corresponding z -interval. This technique will be used over and over again in this series of books on *Real Functions in Several Variables*.

1.6 Quadratic equations in two or three variables; short theoretical review

1.6.1 Quadratic equations in two variables. Conic sections

The general quadratic equation in two variables is given by

$$(1.6) Ax^2 + By^2 + 2Cxy + 2Dx + 2Ey + F = 0,$$

where $A, B, C, D, E, F \in \mathbb{R}$, and $(A, B, C) \neq (0, 0, 0)$.

If $C \neq 0$, then this equation can also be written in the following way,

$$(x \ y) \begin{pmatrix} A & C \\ C & B \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + 2(D \ E) \begin{pmatrix} x \\ y \end{pmatrix} + F = 0.$$

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If we apply some orthogonal substitution of the form

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} q_{11} & -q_{21} \\ q_{21} & q_{11} \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \quad q_{11}^2 + q_{21}^2 = 1,$$

then we may obtain by a suitable choice of q_{11} and q_{21} above that this equation is reduced to

$$\lambda_1 x_1^2 + \lambda_2 y_1^2 + 2(D \ E) \begin{pmatrix} q_{11} & -q_{21} \\ q_{21} & q_{11} \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + F = 0,$$

hence,

$$\lambda_1 x_1^2 + \lambda_2 y_1^2 + 2D_1 x_1 + 2E_1 y_1 + F = 0,$$

where the term $2C_1 x_1 y_1$ has disappeared, because we have obtained that $C_1 = 0$ for some suitable choice of (q_{11}, q_{21}) , which defines an orthogonal substitution.

We have proved that if we choose a specific orthogonal substitution, then the general quadratic equation (1.6) is reduced to

$$(1.7) \quad Ax^2 + By^2 + 2Dx + 2Ey + F = 0, \quad \text{where } (A, B) \neq (0, 0),$$

and where we for convenience write (x, y) instead of (x_1, y_1) .

I. Both coefficients are $\neq 0$

When both $A \neq 0$ and $B \neq 0$, then the reduced equation (1.7) can be written

$$A \left(x + \frac{D}{A} \right)^2 + B \left(y + \frac{E}{B} \right)^2 = \frac{D^2}{A} + \frac{E^2}{B} - F.$$

This equation is simplified, when we introduce the new variables

$$x_1 = x + \frac{D}{A}, \quad t_1 = y + \frac{E}{B}, \quad \text{and the constant } K = \frac{D^2}{A} + \frac{E^2}{B} - F.$$

Then the reduced equation becomes

$$Ax_1^2 + By_1^2 = K.$$

We have here two possibilities: Either $K \neq 0$ or $K = 0$. If $K \neq 0$, then we “norm” the equation by dividing it by K to get

$$\frac{x_1^2}{K/A} + \frac{y_1^2}{K/B} = 1.$$

It is customary to introduce new constants by

$$a = \sqrt{\left| \frac{K}{A} \right|} \quad \text{and} \quad b = \sqrt{\left| \frac{K}{B} \right|}.$$

Depending on the signs of A , B and K we then get three possibilities,

$$\text{Ellipse} \quad \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} = 1$$

$$\text{Hyperbola} \quad \frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} = 1 \quad \left(\text{and also } -\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} = 1 \right)$$

$$\text{Empty set} \quad -\frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} = 1.$$

If instead $K = 0$, then we put

$$a = \sqrt{\frac{1}{|A|}} \quad \text{and} \quad b = \sqrt{\frac{1}{|B|}}.$$

Then, depending on the signs of A and B we get the following two possibilities:

$$\text{A point} \quad \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} = 0,$$

$$\text{Two straight lines} \quad \frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} = 0.$$

We shall in the following briefly discuss these possibilities. For simplicity we again write (x, y) instead of (x_1, y_1) .

The ellipse. The normed equation of the ellipse is given by

$$(1.8) \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad \text{where } a, b > 0.$$

In the special case where $a = b$, formula (1.8) describes a *circle* of centre $(0, 0)$ and radius $r = a = b$.

In general, (1.8) has the two coordinate axes as *axes of symmetry*. The ellipse cuts the x -axis at the points $A_+ : (a, 0)$ and $A_- : (-a, 0)$, and the y -axis at the points $B_+ : (0, b)$ and $B_- : (0, -b)$. These four points are called the *top point* of the ellipse. The numbers a and b (or more correctly the line segments from $O : (0, 0)$ to $A_+ : (a, 0)$, and from $O : (0, 0)$ to $B_+ : (0, b)$) are called the *semi-axes* of the ellipse. The larger of a and b is called the *major semi-axis*, and the smaller of them is called the *minor semi-axis* of the ellipse. Let us assume in the following that $a > b$. Then we define the *eccentricity* e of the ellipse by

$$e := \sqrt{1 - \frac{b^2}{a^2}}, \quad 0 < e < 1,$$

where we formally may add $e = 0$ in the limiting case $b = a$, when the ellipse becomes a circle.

The *foci* of the ellipse (in singular: *focus*) are when $a > b$ the points

$$F_+ : (ea, 0) \quad \text{and} \quad F_- : (-ea, 0).$$

If $P : (x, y)$ lies on the ellipse, then a small computation shows that

$$\left| \overrightarrow{F_+ P} \right| = a - ex \quad \text{and} \quad \left| \overrightarrow{F_- P} \right| = a + ex,$$

hence by addition,

$$(1.9) \quad \left| \overrightarrow{F_+ P} \right| + \left| \overrightarrow{F_- P} \right| = 2a,$$

i.e. equal twice the major semi-axis. It is possible to prove that a relation like (1.9) only holds for an ellipse of foci F_+ and F_- and the major semi-axis a , where we of course must require that $\left| \overrightarrow{F_- F_+} \right| < 2a$.

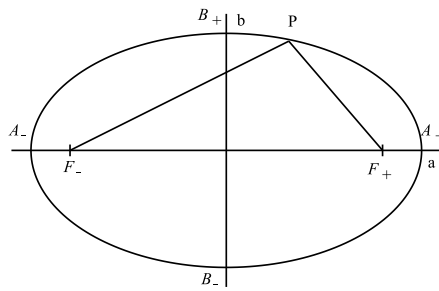



Figure 1.24: An ellipse.

The hyperbola. The normed equation of the hyperbola is for a convenient choice of the variables of the form

$$(1.10) \quad \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

The coordinate axes are the *axes of symmetry*. The hyperbola (1.10) intersects the x -axis at the two *top points* $A_+ : (a, 0)$ and $A_- : (-a, 0)$, and it has no point in common with the y -axis. The positive numbers a and b are called the *semi-axes* of the hyperbola.

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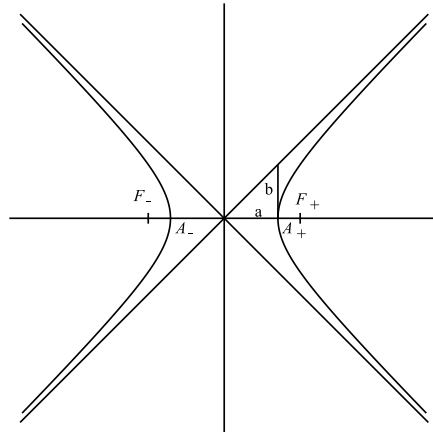


Figure 1.25: An hyperbola.

The lines $y = \pm \frac{b}{a} x$ are the *asymptotes* of the hyperbola. They are found by replacing 1 on the right hand side of (1.10) by 0 and then solving the equation. It is obvious that b is the length of the line segment perpendicular to the x -axis from A to the asymptote in the first quadrant.

The *eccentricity* e of the hyperbola is defined by

$$e := \sqrt{1 + \frac{b^2}{a^2}}, \quad e > 1.$$

The *foci* are defined by their coordinates, i.e.

$$F_+ : (ea, 0) \quad \text{and} \quad F_- : (-ea, 0).$$

If $P : (x, y)$ is a point on the hyperbola in the right half plane (i.e. closest to the focus F_+), then one likewise proves that

$$\left| \overrightarrow{F_+ P} \right| = ex - a \quad \text{and} \quad \left| \overrightarrow{F_- P} \right| = ex + a,$$

hence by subtraction,

$$\left| \overrightarrow{F_- P} \right| - \left| \overrightarrow{F_+ P} \right| = 2a,$$

so in general for $P : (x, y)$ just a point on the hyperbola,

$$(1.11) \quad \left| \left| \overrightarrow{F_+ P} \right| - \left| \overrightarrow{F_- P} \right| \right| = 2a.$$

It is possible to prove that if F_+ and F_- are two fixed points in the plane, then all points P , which satisfy (1.11), describe an hyperbola of foci F_+ and F_- and a one of the semi-axes. The other one, b , is then obtained from the equation

$$\left| \overrightarrow{F_- F_+} \right| = 2ae = 2a\sqrt{1 + \frac{b^2}{a^2}} = 2\sqrt{a^2 + b^2},$$

hence

$$b = \frac{1}{2} \sqrt{|\overrightarrow{F_- F_+}|^2 - 4a^2}.$$

A point. The general equation is here

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 0,$$

where $O : (0, 0)$ is the only solution.

Two lines. The general solution is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0,$$

which is rewritten as

$$\left(\frac{x}{a} - \frac{y}{b}\right) \left(\frac{x}{a} + \frac{y}{b}\right) = 0.$$

The solutions are the two lines

$$bx + ay = 0 \quad \text{and} \quad bx - ay = 0.,$$

which describe two lines through $(0, 0)$.

II. Precisely one of the constants A and B is 0.

We may assume that $A \neq 0$ and $B = 0$. Then (1.7) is written

$$(1.12) \quad Ax^2 + 2Dx + 2Et + F = 0.$$

If also $E \neq 0$, then this equation is rewritten as

$$A \left(x + \frac{D}{A}\right)^2 = -2E \left(y - \frac{1}{2E} \left\{ \frac{D^2}{A} - F \right\}\right),$$

so if we put

$$x_1 = x + \frac{D}{A}, \quad y_1 = y - \frac{1}{2E} \left\{ \frac{D^2}{A} - F \right\} \quad \text{and} \quad a = -\frac{A}{2E},$$

then we get the structure,

$$y_1 = ax_1^2 \quad (\text{a parabola}).$$

If instead $E = 0$, then (1.12) becomes

$$Ax^2 + 2Dx + F = 0,$$

which is written as

$$x_1^2 = k, \quad (\text{no solution, one line, or, two parallel lines}),$$

where

$$x_1 = x + \frac{D}{A} \quad \text{and} \quad k = \frac{1}{A} \left\{ \frac{D^2}{A} - F \right\}.$$

As usual we write in the following for convenience (x, y) instead of (x_1, y_1) . Then the analysis above shows that we have two cases.

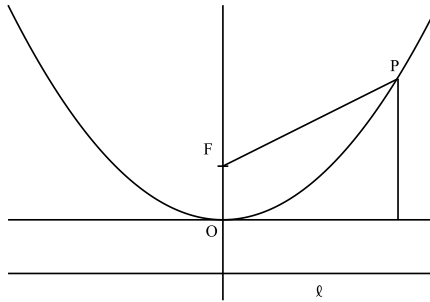


Figure 1.26: A parabola.

The parabola. The normed equation is here

$$(1.13) \quad y = ax^2, \quad a \neq 0.$$

It intersects the coordinate axes only at the origo, $O : (0, 0)$, which is called the *top point* of the parabola, and the y -axis is the only *axis of symmetry*.

One usually instead put $p = \frac{1}{a}$, and then (1.13) is written

$$(1.14) \quad x^2 = py,$$

where p is called the *parameter of the parabola*. The *focus* of the parabola is $F : \left(0, \frac{p}{4}\right)$, and the line ℓ of the equation $y = -\frac{p}{4}$ is called the *directrix* of the parabola. Its geometric meaning is that if P is any point on the parabola, then

$$\left| \overrightarrow{FP} \right| = \text{dist}(P, \ell),$$

i.e. the distance from P to the focus is equal to the distance from P to the directrix ℓ .

The empty set, one line, or two parallel lines. In the case the equation is

$$x^2 = k.$$

- If $k < 0$, then we have no solution.
- If $k = 0$, then the line $x = 0, y \in \mathbb{R}$, is the only solution.
- If $k > 0$, then the two parallel lines $x = \pm\sqrt{k}, y \in \mathbb{R}$, are the solutions.

We call the ellipses, the hyperbolas and the parabolas the (non-degenerated) *conic sections*, because they can be obtained as the intersection of a cone with a plane. The other cases mentioned above are then called the degenerated conic sections.

1.6.2 Quadratic equations in three variables. Conic sectional surfaces.

The *general quadratic equation* in three variables has the form

$$(1.15) \quad Ax^2 + By^2 + Cz^2 + 2Dxy + 2Exz + 2Fyz + 2gx + 2Hy + 2Iz + J = 0,$$

where $A, B, \dots, J \in \mathbb{R}$ are real constants, and where $(A, B, C, D, E, F) \neq (0, 0, 0, 0, 0, 0)$.

As usual, the product terms $2Dxy + 2Exz + 2Fyz$ are a nuisance, when $(D, E, F) \neq (0, 0, 0)$, so the first task is to transform (1.15) into some new variables x_1, y_1, z_1 , such that the new coefficients are all zero, $(D_1, E_1, F_1) = (0, 0, 0)$.

We note that (1.15) can be written

$$(1.16) \quad (x \ y \ z)\mathfrak{A} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + 2(G \ H \ I) \begin{pmatrix} x \\ y \\ z \end{pmatrix} + J = 0, \quad \text{where } \mathfrak{A} := \begin{pmatrix} A & D & E \\ D & B & F \\ E & F & C \end{pmatrix}.$$



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We find by methods described previously a coordinate transformation

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \mathbf{Q} \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}, \quad \text{where } \mathbf{Q} = (\mathbf{q}_1 \ \mathbf{q}_2 \ \mathbf{q}_3), \quad \text{with } \mathbf{q}_3 = \mathbf{q}_1 \times \mathbf{q}_2,$$

such that

$$(x_1 \ y_1 \ z_1) \mathbf{Q}^T \mathfrak{A} \mathbf{Q} \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} = \lambda_1 x_1^2 + \lambda_2 y_1^2 + \lambda_3 z_1^2.$$

Just find the eigenvalues and the corresponding eigenvectors of \mathfrak{A} . (Here MAPLE may be used to ease the computations.)

When we use this coordinate transformation, then (1.16) is reduced to

$$\lambda_1 x_1^2 + \lambda_2 y_1^2 + \lambda_3 z_1^2 + 2(G \ H \ I) \mathbf{Q} \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + J = 0.$$

Then introduce

$$G_1 = (G \ H \ I) \mathbf{q}_1, \quad H_1 = (G \ H \ I) \mathbf{q}_2, \quad I_1 = (G \ H \ I) \mathbf{q}_3,$$

and the equation is reduced to

$$(1.17) \quad \lambda_1 x_1^2 + \lambda_2 y_1^2 + \lambda_3 z_1^2 + 2G_1 x_1 + 2H_1 y_1 + 2I_1 z_1 + J = 0.$$

It follows from the analysis above that it suffices to consider the simpler equation

$$A x^2 + B y^2 + C z^2 + 2G x + 2H y + 2I z + J = 0, \quad (A, B, C) \neq (0, 0, 0),$$

where we again have simplified the notation.

We shall split the investigation into three cases.

I. First case, $A \neq 0$, $B \neq 0$ and $C \neq 0$.

In this case, (1.17) can be rewritten as

$$A \left(x + \frac{G}{A} \right)^2 + B \left(y + \frac{H}{B} \right)^2 + C \left(z + \frac{I}{C} \right)^2 = \frac{G^2}{A} + \frac{H^2}{B} + \frac{I^2}{C} - J.$$

If we put

$$x_1 = x + \frac{G}{A}, \quad y_1 = y + \frac{H}{B}, \quad z_1 = z + \frac{I}{C}, \quad K = \frac{G^2}{A} + \frac{H^2}{B} + \frac{I^2}{C} - J,$$

then the equation (1.17) is reduced to the simpler form

$$A x_1^2 + B y_1^2 + C z_1^2 = K.$$

Let us first assume that $K \neq 0$. Then it is customary to norm the equation by dividing it by K ,

$$\frac{x_1^2}{K/A} + \frac{y_1^2}{K/B} + \frac{z_1^2}{K/C} = 1.$$

We write for short,

$$a := \sqrt{\left|\frac{K}{A}\right|}, \quad b := \sqrt{\left|\frac{K}{B}\right|}, \quad c := \sqrt{\left|\frac{K}{C}\right|}.$$

Then we obtain the canonical form

$$\pm \left(\frac{x_1}{a}\right)^2 \pm \left(\frac{y_1}{b}\right)^2 \pm \left(\frac{z_1}{c}\right)^2 = 1,$$

with all possible choices of the signs, i.e. in principle eight subcases in total, which, however, by some trivial argument of symmetry (where we rename the variables) can be reduced to four. These are

ellipsoid $\left(\frac{x_1}{a}\right)^2 + \left(\frac{y_1}{b}\right)^2 + \left(\frac{z_1}{c}\right)^2 = 1,$

hyperboloid of one sheet $\left(\frac{x_1}{a}\right)^2 + \left(\frac{y_1}{b}\right)^2 - \left(\frac{z_1}{c}\right)^2 = 1,$

hyperboloid of two sheets $\left(\frac{x_1}{a}\right)^2 - \left(\frac{y_1}{b}\right)^2 - \left(\frac{z_1}{c}\right)^2 = 1,$

empty set $-\left(\frac{x_1}{a}\right)^2 - \left(\frac{y_1}{b}\right)^2 - \left(\frac{z_1}{c}\right)^2 = 1.$

If instead $K = 0$, then we put

$$a := \sqrt{\frac{1}{|A|}}, \quad b := \sqrt{\frac{1}{|B|}}, \quad c := \sqrt{\frac{1}{|C|}},$$

from which we get the two possibilities,

a point $\left(\frac{x_1}{a}\right)^2 + \left(\frac{y_1}{b}\right)^2 + \left(\frac{z_1}{c}\right)^2 = 0,$

conic sectional conic surface $\left(\frac{x_1}{a}\right)^2 + \left(\frac{y_1}{b}\right)^2 - \left(\frac{z_1}{c}\right)^2 = 0.$

We shall briefly describe these possibilities in the following, where we again for short write (x, y, z) instead of (x_1, y_1, z_1) .

1. The ellipsoid has the canonical equation

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 = 1.$$

The semi-axes are clearly a , b and c .

2. The hyperboloid with one sheet. The normed equation is

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 - \left(\frac{z}{c}\right)^2 = 1,$$

with one minus sign on the left hand side of the equation. The corresponding surface is connected, i.e. it consists of one surface. (This is only indicated on the figure, because the author has not been clever enough to make the right figure.)

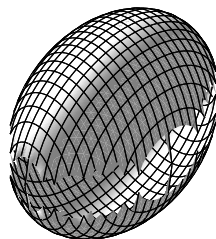


Figure 1.27: An ellipsoid.

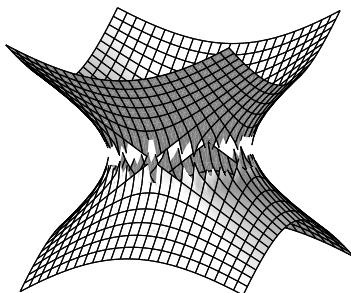


Figure 1.28: An hyperboloid with one sheet.

An important special case is obtained, when $a = b$, in which case

$$\frac{x^2 + y^2}{a^2} - \frac{z^2}{c^2} = 1.$$

This *hyperboloid of one sheet* is obtained by revolving the (two dimensional) hyperbola

$$\left(\frac{x}{a}\right)^2 - \left(\frac{z}{c}\right)^2 = 1, \quad y = 0,$$

in the XZ -plane around the z -axis.

It is possible to prove the following

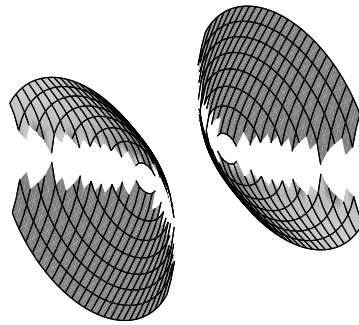


Figure 1.29: An hyperboloid with two sheets.

Theorem 1.1 *An hyperboloid of one sheet contains two systems of straight lines. Two different lines from the same system are always oblique. Two lines, one from each system, always lie in the same plane.*

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3. The hyperboloid with two sheets. The normed equation is

$$\left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2 - \left(\frac{z}{c}\right)^2 = 1.$$

It is characterized by having *two* minus signs on the left hand side of the normed equation. The corresponding surface is split into *two* connected components.

4. The conic sectional conic surface has the canonical equation

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 - \left(\frac{z}{c}\right)^2 = 0.$$

It is clearly a cone with $O : (0, 0, 0)$ as its centrum.

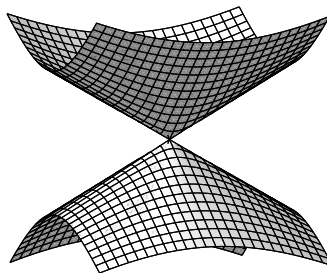


Figure 1.30: A conic sectional conic surface.

5. A point. The equation is

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 = 0,$$

which is only satisfied for $O : (0, 0, 0)$.

II. Second case. Here we assume that $A \neq 0$, $B \neq 0$ and $C = 0$. Then (1.17) is reduced to

$$(1.18) \quad Ax^2 + By^2 + 2Gx + 2Hy + 2Iz + J = 0.$$

First assume that also $I \neq 0$. Then (1.18) can be reformulated as

$$A \left(x + \frac{G}{A}\right)^2 + B \left(y + \frac{H}{B}\right)^2 = -2I \left(z - \frac{1}{2I} \left\{ \frac{G^2}{A} + \frac{H^2}{B} - J \right\}\right).$$

If we put

$$x_1 = x + \frac{G}{A}, \quad y_1 = y + \frac{H}{B}, \quad z_1 = z - \frac{1}{2I} \left\{ \frac{G^2}{A} + \frac{H^2}{B} - J \right\}, \quad L = -2I,$$

then (1.18) is reduced to

$$A x_1^2 + B y_1^2 = L z_1.$$

By assumption, $L = -2I \neq 0$, so when we divide by L , we get

$$\frac{x_1^2}{L/A} + \frac{y_1^2}{L/B} = z_1.$$

Then write for short,

$$a := \sqrt{\left| \frac{L}{A} \right|} \quad \text{and} \quad b := \sqrt{\left| \frac{L}{B} \right|},$$

and we get the two possibilities,

$$\text{elliptic paraboloid} \quad \left(\frac{x_1}{a} \right)^2 + \left(\frac{y_1}{b} \right)^2 = z_1,$$

$$\text{hyperbolic paraboloid} \quad \left(\frac{x_1}{a} \right)^2 - \left(\frac{y_1}{b} \right)^2 = z_1.$$

If instead $I = 0$, then (1.18) is written

$$A \left(x + \frac{G}{A} \right)^2 + B \left(y + \frac{H}{B} \right)^2 = \frac{G^2}{A} + \frac{H^2}{B} - J.$$

We simplify by writing

$$x_1 = x + \frac{G}{A}, \quad y_1 = y + \frac{H}{B}, \quad K = \frac{G^2}{A} + \frac{H^2}{B} - J,$$

because then (1.18) takes the simpler form

$$(1.19) \quad A x_1^2 + B y_1^2 = K.$$

Again we must split into the two cases, $K \neq 0$ and $K = 0$. If $K \neq 0$, then we write for short

$$a := \sqrt{\left| \frac{K}{A} \right|}, \quad b := \sqrt{\left| \frac{K}{B} \right|}.$$

We obtain the following three possibilities,

$$\text{elliptic cylindrical surface} \quad \left(\frac{x_1}{a} \right)^2 + \left(\frac{y_1}{b} \right)^2 = 1,$$

$$\text{hyperbolic cylindrical surface} \quad \left(\frac{x_1}{a} \right)^2 - \left(\frac{y_1}{b} \right)^2 = 1,$$

$$\text{empty set} \quad - \left(\frac{x_1}{a} \right)^2 - \left(\frac{y_1}{b} \right)^2 = 1.$$

If instead $K = 0$, we put

$$a := \sqrt{\frac{1}{|A|}} \quad \text{and} \quad b := \sqrt{\frac{1}{|B|}}.$$

Then we get the two possibilities,

the z_1 -axis $\left(\frac{x_1}{a}\right)^2 + \left(\frac{y_1}{b}\right)^2 = 0,$

two planes through the z_1 -axis $\left(\frac{x_1}{a}\right)^2 - \left(\frac{y_1}{b}\right)^2 = 0.$

We shall in the following briefly sketch the possibilities above. Again we write for short (x, y, z) instead of (x_1, y_1, z_1) .

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1. **The elliptic paraboloid.** The canonical equation is

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = z.$$

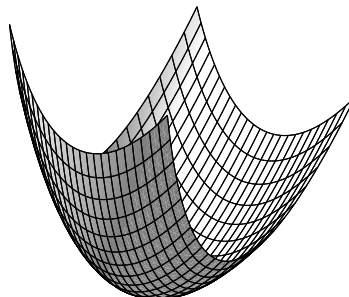


Figure 1.31: An elliptic paraboloid.

2. **The hyperbolic paraboloid.** The canonical equation is

$$\left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2 = z.$$

It is possible to prove the following theorem

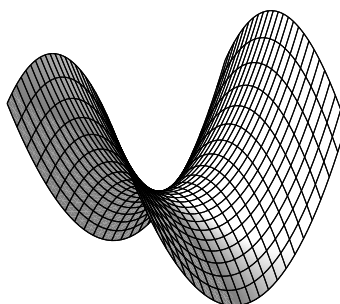


Figure 1.32: An hyperbolic paraboloid.

Theorem 1.2 *An hyperbolic paraboloid contains two systems of straight lines. Two different lines from the same system are always oblique with respect to each other. Two lines from different systems will always intersect each other.*

3. The elliptic cylindrical surface. The canonical equation is

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1.$$

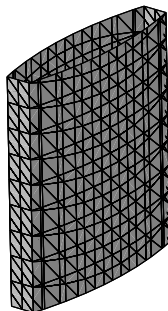


Figure 1.33: An elliptic cylindrical surface.

4. The hyperbolic cylindrical surface. The canonical equation is

$$\left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2 = 1.$$

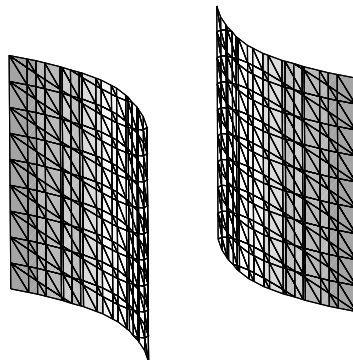


Figure 1.34: An hyperbolic cylindrical surface.

III. The third case. Here we assume that $A \neq 0$, while $B = C = 0$. Then (1.17) is reduced to

$$(1.20) \quad Ax^2 + 2Gx + 2Hy + 2Iz + J = 0.$$

If $(H, I) \neq (0, 0)$, e.g. $I \neq 0$, then (1.20) is reformulated as

$$A \left(x + \frac{G}{A} \right)^2 + 2Hy + 2I \left(z + \frac{1}{2I} \left\{ J - \frac{G^2}{A} \right\} \right) = 0.$$

We put

$$x_1 = x + \frac{G}{A}, \quad y_1 = y \quad \text{and} \quad z_1 = z + \frac{1}{2I} \left\{ J - \frac{G^2}{A} \right\},$$

from which

$$Ax_1^2 + 2Hy_1 + 2Iz_1 = 0.$$

Then apply the orthogonal substitution

$$\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & I/\sqrt{H^2 + I^2} & H/\sqrt{H^2 + I^2} \\ 0 & -H/\sqrt{H^2 + I^2} & I/\sqrt{H^2 + I^2} \end{pmatrix} \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}$$

to reduce the equation above to

$$Ax_2^2 + 2\sqrt{H^2 + I^2} z_2 = 0.$$

This structure invites to put $p := -2\sqrt{H^2 + I^2}/A$, so we get

$$\text{parabolic cylindric surface} \quad x_2^2 = p z_2,$$

which is the canonical equation.

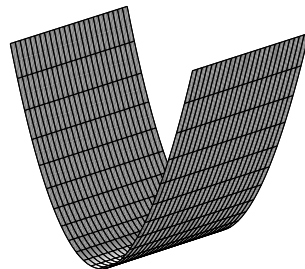


Figure 1.35: A parabolic cylindric surface.

If instead $(H, I) = (0, 0)$, then (1.20) reduces to

$$A \left(x + \frac{G}{A} \right)^2 = \frac{G^2}{A} - J.$$

Writing

$$x_1 = x + \frac{G}{A} \quad \text{and} \quad k = \frac{1}{A} \left\{ \frac{G^2}{A} \cdot J \right\},$$

we see that this case can be written in the form

$$x_1^2 = k^2,$$

i.e. the *empty set*, *one plane*, or *two (parallel) planes*.



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1.6.3 Summary of the canonical cases in three variables

Equation	Name	$(0, 0, 0)$	Generators
$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$	Ellipsoid	Centrum	None
$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$	Hyperboloid of one sheet	Centrum	Two systems of lines
$\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$	Hyperboloid of two sheets	Centrum	None
$-\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$	Empty set	—	—
$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 0$	Point $(0, 0, 0)$	—	—
$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0$	Conic sectional conic surface	Centrum	Lines through the centrum
$\frac{x^2}{a^2} + \frac{y^2}{b^2} = z$	Elliptic paraboloid	Toppoint	None
$\frac{x^2}{a^2} - \frac{y^2}{b^2} = z$	Hyperbolic paraboloid	Toppoint	Two systems of lines
$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$	Elliptic cylindric surface	Centrum	Lines parallel with the z -axis
$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$	Hyperbolic cylindric surface	Centrum	Lines parallel with the z -axis
$-\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$	Empty set	—	—
$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 0$	z - axis	—	—
$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0$	Two planes through the z -axis	—	—
$x^2 = pz$	Parabolic cylindrical surface	Toppoint	Lines parallel with the y -axis
$x^2 = k > 0$	Two planes parallel with the YZ -plane	—	—
$x^2 = 0$	YZ -plane	—	—
$x^2 = k < 0$	Empty set	—	—

2 Some useful procedures

2.1 Introduction

In this chapter we collect some simple and useful practical procedures, like integration of trigonometric polynomials, the technique of partial fractions, when MAPLE is not at hand, integration of a quotient of two polynomials, and how to find the domain of a given function. All these will be important in the following chapter.

2.2 Integration of trigonometric polynomials

Problem 2.1 Calculate the integral

$$\int \sin^m x \cos^n x \, dx \quad \text{for } m, n \in \mathbb{N}_0.$$

Notation: By the *degree* of the product $\sin^m x \cos^n x$ we shall understand the sum $m + n$ of the exponents.

Split the problem into a simpler one: There are two main cases, *odd* and *even* degree. Each of these is again split into two *subcases*:

1) $m + n$ *odd*.

a) m even and n odd, i.e. $m = 2p$ and $n = 2q + 1$, $p, q \in \mathbb{N}_0$,

b) m odd and n even, i.e. $m = 2p + 1$ and $n = 2q$, $p, q \in \mathbb{N}_0$.

2) $m + n$ *even*.

a) m and n are both odd, i.e. $m = 2p + 1$ and $n = 2q + 1$, $p, q \in \mathbb{N}_0$,

b) m and n are both even, i.e. $m = 2p$ and $n = 2q$, $p, q \in \mathbb{N}_0$.

The most difficult case occurs in 2b), where both m and n are even.

Method of solution:

1) a) $m = 2p$ and $n = 2q + 1$.

Apply the substitution $u = \sin x$ corresponding to $m = 2p$ *even*:

$$\begin{aligned} \int \sin^{2p} x \cos^{2q+1} x \, dx &= \int \sin^{2p} x \cdot \cos^{2q} x \cdot \cos x \, dx \\ &= \int \sin^{2p} x \cdot (1 - \sin^2 x)^q \, d \sin x \\ &= \int_{u=\sin x} u^{2p} (1 - u^2)^q \, du, \end{aligned}$$

where the integral is a usual polynomial in u of degree $2p + 2q$.

b) $m = 2p + 1$ and $n = 2q$.

Apply the substitution $u = \cos x$ corresponding to $n = 2q$ even:

$$\begin{aligned} \int \sin^{2p+1} x \cdot \cos^{2q} x \, dx &= \int \sin^{2p} x \cdot \cos^{2q} x \cdot \sin x \, dx \\ &= \int (1 - \cos^2 x)^p \cos^{2q} x \cdot (-1) \, d \cos x \\ &= - \int_{u=\cos x} (1 - u^2)^p u^{2q} \, du, \end{aligned}$$

where the integral is a usual polynomial in u of degree $2p + 2q$.

2) When the degree $m + n$ is even, the trick is to change the problem to a similar one by doubling the angle, thereby halving the degree. Therefore, we use the formulæ

$$\cos^2 x = \frac{1}{2}(1 + \cos 2x), \quad \sin^2 x = \frac{1}{2}(1 - \cos 2x), \quad \sin x \cos x = \frac{1}{2} \sin 2x.$$

a) $m = 2p + 1$ and $n = 2q + 1$ are both odd.

The integrand is transformed in the following way:

$$\begin{aligned} \sin^{2p+1} x \cdot \cos^{2q+1} x &= (\sin^2 x)^p \cdot (\cos^2 x)^q \cdot \sin x \cos x \\ &= \left\{ \frac{1}{2}(1 - \cos 2x) \right\}^p \left\{ \frac{1}{2}(1 + \cos 2x) \right\}^q \cdot \frac{1}{2} \sin 2x. \end{aligned}$$

Hence we are in a special case of 1b), so by the substitution $u = \cos 2x$ we get

$$\begin{aligned} \int \sin^{2p+1} x \cos^{2q+1} x \, dx &= \frac{1}{2^{p+q+1}} \int (1 - \cos 2x)^p (1 + \cos 2x)^q \sin 2x \, dx \\ &= \frac{1}{2^{p+q+1}} \int (1 - \cos 2x)^p (1 + \cos 2x)^q \cdot \left(-\frac{1}{2}\right) \, d \cos 2x \\ &= -\frac{1}{2^{p+q+2}} \int_{u=\cos 2x} (1 - u)^p (1 + u)^q \, du. \end{aligned}$$

b) $m = 2p$ and $n = 2q$ are both even.

In this case there is no final formula, but there is a procedure by which we can reduce the problem to a sum of several problems of the types 1a) and 2b) of lower degree. The result is obtained after a finite number of steps.

The integrand is rewritten in the following way:

$$\sin^{2p} x \cos^{2q} x = \left\{ \frac{1}{2}(1 - \cos 2x) \right\}^p \left\{ \frac{1}{2}(1 + \cos 2x) \right\}^q.$$

The left hand side is a trigonometrical polynomial of degree $2p + 2q$ in the angle x . The right hand side is a trigonometric polynomial of degree $p + q$ in the doubled angle $2x$. Each term of this polynomial must be handled separately, depending on whether the degree j ($\leq p + q$) is odd (case 1a) or 1b)) or even (case 2b)).

Remark 2.1 It is of course in principle possible to create a specific solution formula, but it will be more confusing than the description of the procedure given above. \diamond

MAPLE. When $m, n \in \mathbb{N}_0$ are explicitly given as numbers, an application of MAPLE is of course the easiest method. When either m or $n \in \mathbb{N}_0$ is not specified, one applies the method above.

2.3 Complex decomposition of a fraction of two polynomials

Problem 2.2 Write the quotient $\frac{P(x)}{Q(x)}$ of two polynomials as a sum of elementary fractions.

Remark 2.2 This problem occurs typically in connection with integration, and in courses on series also in telescopic summation. If the denominator has complex conjugated roots of at least order 2, a complex decomposition is usually the easiest method. If the order is 1, then real decomposition may be applied instead. We shall here show the method of *complex decomposition*. \diamond

Procedure.

- 1) If the degree of the numerator is \geq the degree of the denominator, we first perform a *division by the denominator*,

$$\frac{P(x)}{Q(x)} = P_1(x) + \frac{R(x)}{Q(x)},$$

where the residual polynomial $R(x)$ (the new numerator) has a degree smaller than the degree of $Q(x)$. We save the resulting polynomial $P_1(x)$ for the last step.

- 2) The *denominator* $Q(x)$ is then factorized into polynomials of degree one (with *complex* roots):

$$Q(x) = c \cdot (x - a_1)^{p_1} \cdots (x - a_k)^{p_k}.$$

Check that the sum $p_1 + \cdots + p_k$ of all exponents is equal to the degree of $Q(x)$. If $Q(x)$ is a real polynomial, check that the complex conjugated roots occur of the *same degree*.

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- 3) To ease matters, choose the simplest one of the two polynomials $P(x)$ and $R(x)$. The following method gives the same result, whether $P(x)$ or $R(x)$ is used. Since it is here theoretically most correct to use $R(x)$, we shall use $R(x)$ in the rest of this description, and it is left to the reader to write $P(x)$ instead of $R(x)$, whenever this will give us simpler calculations.
- 4) The fraction is rewritten in the following way

$$\frac{R(x)}{Q(x)} = \frac{1}{c} \cdot \frac{R(x)}{(x - a_1)^{p_1} \cdots (x - a_k)^{p_k}}.$$

We get the coefficient of the special simple fraction

$$\frac{1}{(x - a_1)^{p_1}}$$

by “covering by one’s hand” the factor $(x - a_1)^{p_1}$ in the denominator and then putting $x = a_1$ into the remainder part:

$$b_{1,p_1} = \frac{1}{c} \cdot \frac{R(x)}{(x - a_2)^{p_2} \cdots (x - a_k)^{p_k}} \Big|_{x=a_1}.$$

Save the result

$$\frac{b_{1,p_1}}{(x - a_1)^{p_1}}$$

for the last step in this procedure.

- 5) Repeat 4) on any other of the factors

$$(x - a_2)^{p_2}, \quad \cdots \quad (x - a_k)^{p_k},$$

in the denominator and save all the found special fractions.

- 6) Subtract all the found special fractions from $\frac{R(x)}{Q(x)}$ and reduce:

$$\begin{aligned} & \frac{1}{c} \cdot \frac{R(x)}{(x - a_1)^{p_1} \cdots (x - a_k)^{p_k}} - \frac{b_{1,p_1}}{(x - a_1)^{p_1}} - \cdots - \frac{b_{k,p_k}}{(x - a_k)^{p_k}} \\ &= \frac{1}{d} \cdot \frac{R_1(x)}{(x - a_1)^{q_1} \cdots (x - a_k)^{q_k}}. \end{aligned}$$

If the calculations are made without errors, then

$$q_1 < p_1, \quad \cdots \quad q_k < p_k.$$

Check this! (A weak test.)

- 7) Repeat 4), 5) and 6) on the reduced fraction

$$\frac{1}{d} \cdot \frac{R_1(x)}{(x - a_1)^{q_1} \cdots (x - a_k)^{q_k}}.$$

Remember in each step to write down the elementary fractions which have been found. The process must necessarily stop after a finite number of steps, because the degree of the denominator is becoming smaller by each iteration.

8) Finally, collect all the found elementary fractions together with the polynomial from 1).

The description above is the standard procedure. My experience has shown me that one often can find shortcuts, which are impossible to systemize here. I shall therefore here only give one example of many possibilities.

Example 2.1 Let us here try to decompose the fractional function

$$\frac{1}{x^4 - 1}.$$

1) *The standard procedure* as described above. The denominator has the simple roots $1, i, -1, -i$, hence

$$\begin{aligned} \frac{1}{x^4 - 1} &= \frac{1}{(x-1)(x-i)(x+1)(x+i)} \\ &= \frac{1}{(1-i)(1+1)(1+i)} \cdot \frac{1}{x-1} + \frac{1}{(i-1)(i+1)(i+i)} \cdot \frac{1}{x-i} \\ &\quad + \frac{1}{(-1-1)(-1-i)(-1+i)} \cdot \frac{1}{x+1} + \frac{1}{(-i-1)(-i-i)(-i+1)} \cdot \frac{1}{x+i} \\ &= \frac{1}{4} \cdot \frac{1}{x-1} - \frac{1}{4i} \cdot \frac{1}{x-i} - \frac{1}{4} \cdot \frac{1}{x+1} + \frac{1}{4i} \cdot \frac{1}{x+i} \\ &= \frac{1}{4} \cdot \frac{1}{x-1} - \frac{1}{4} \cdot \frac{1}{x+1} - \frac{1}{4i} \left\{ \frac{1}{x-i} - \frac{1}{x+i} \right\} \\ &= \frac{1}{4} \cdot \frac{1}{x-1} - \frac{1}{4} \cdot \frac{1}{x+1} - \frac{1}{2} \cdot \frac{1}{x^2 + 1}. \end{aligned}$$

This is of course fairly tiresome, though it works.

2) *Alternatively* it is seen that

$$x^4 - 1 = (x^2)^2 - 1 = (x^2 + 1)(x^2 - 1),$$

so if we write $u = x^2$, and first decompose with respect to u followed by a decomposition with respect to x , we easily get in two simpler steps that

$$\begin{aligned} \frac{1}{x^4 - 1} &= \frac{1}{u^2 - 1} = \frac{1}{(u-1)(u+1)} = \frac{1}{2} \frac{1}{u-1} - \frac{1}{2} \frac{1}{u+1} \\ &= \frac{1}{2} \frac{1}{x^2 - 1} - \frac{1}{2} \frac{1}{x^2 + 1} = \frac{1}{2} \frac{1}{(x-1)(x+1)} - \frac{1}{2} \frac{1}{x^2 + 1} \\ &= \frac{1}{4} \frac{1}{x-1} - \frac{1}{4} \frac{1}{x+1} - \frac{1}{2} \frac{1}{x^2 + 1}. \quad \diamond \end{aligned}$$

3) **MAPLE**. This is easy here, because the rational function does not contain extra parameters:

$$\begin{aligned} &\text{convert} \left(\frac{1}{x^4 - 1}, \text{parfrac}, x \right) \\ &\quad -\frac{1}{2(x^2 + 1)} - \frac{1}{4(x + 1)} + \frac{1}{4(x - 1)} \end{aligned}$$

Here, “parfrac” is of course a shorthand for “partial fraction”.

In the latter two cases we should of course continue with a *complex decomposition* of $\frac{1}{x^2 + 1}$. The simple details are left to the reader.

2.4 Integration of a fraction of two polynomials

Problem 2.3 Calculate $\int \frac{P(x)}{Q(x)} dx$, where $P(x)$ and $Q(x)$ are (real) polynomials.

Procedure.

1) Decompose $\frac{P(x)}{Q(x)}$ as described in the previous chapter on *complex decomposition*.

Then $\frac{P(x)}{Q(x)}$ is written as a sum of a polynomial $P_1(x)$ and some elementary fractions of the type $\frac{c}{(x-a)^p}$, i.e. we perform a partial fraction construction.

2) The polynomial $P_1(x)$ is integrated in the usual way.

3) The elementary fractions where $p > 1$ are also integrated in the usual way

$$\int \frac{c}{(x-a)^p} dx = -\frac{c}{p-1} \cdot \frac{1}{(x-a)^{p-1}},$$

no matter whether a is real or complex. If $P(x)$ and $Q(x)$ are real, then any complex fraction of the type $\frac{c}{(x-a)^p}$ will be accompanied by its complex conjugated fraction $\frac{\bar{c}}{(x-\bar{a})^p}$. This means that the integration of such a pair of complex conjugated fractions can be reduced to

$$\begin{aligned} \int \left\{ \frac{c}{(x-a)^p} + \frac{\bar{c}}{(x-\bar{a})^p} \right\} dx &= -\frac{1}{p-1} \left\{ \frac{c}{(x-a)^{p-1}} + \frac{\bar{c}}{(x-\bar{a})^{p-1}} \right\} \\ &= -\frac{2}{p-1} \operatorname{Re} \left\{ \frac{c}{(x-a)^{p-1}} \cdot \frac{(x-\bar{a})^{p-1}}{(x-\bar{a})^{p-1}} \right\} \\ &= -\frac{2}{p-1} \cdot \frac{\operatorname{Re}\{c \cdot (x-\bar{a})^{p-1}\}}{\{x^2 - 2 \operatorname{Re} a \cdot x + |a|^2\}^{p-1}} \end{aligned}$$

4) If $p = 1$, and a is real, then of course

$$\int \frac{x}{x-a} dx = c \cdot \ln|x-a|.$$

5) If $p = 1$, and a is complex, then both $\frac{c}{x-a}$ and $\frac{\bar{c}}{x-\bar{a}}$ occur in the decomposition. A direct integration is not possible, unless one is familiar with the theory of *Complex Functions*. Instead we add the two elementary fractions *before* the integration. (Note that when $p > 1$, this is done *after* the integration, cf. 3)).

More precisely we put $a = \alpha + i\beta$ and $c = r + is$. Then

$$\begin{aligned} \frac{c}{x-a} + \frac{\bar{c}}{x-\bar{a}} &= \frac{r+is}{x-\alpha-i\beta} + \frac{r-is}{x-\alpha+i\beta} \\ &= \frac{(r+is)(x-\alpha-i\beta) + (r-is)(x-\alpha+i\beta)}{(x-\alpha)^2 + \beta^2} \\ &= \frac{2r(x-\alpha)}{(x-\alpha)^2 + \beta^2} - \frac{2s\beta}{(x-\alpha)^2 + \beta^2}, \end{aligned}$$

whence

$$\begin{aligned} \int \left\{ \frac{c}{x-a} + \frac{\bar{c}}{x-\bar{a}} \right\} dx &= r \int \frac{2(x-\alpha)}{(x-\alpha)^2 + \beta^2} dx - 2s \int \frac{1}{1 + \left(\frac{x-\alpha}{\beta}\right)^2} \frac{1}{\beta} dx \\ &= r \cdot \ln \{(x-\alpha)^2 + \beta^2\} - 2s \operatorname{Arctan} \left(\frac{x-\alpha}{\beta} \right). \end{aligned}$$

6) The final result is obtained by gathering all the results from 2), 3), 4) and 5).

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Example 2.2 In Example 2.1 we found the decomposition

$$\frac{1}{x^4 - 1} = \frac{1}{4} \frac{1}{x - 1} - \frac{1}{4} \frac{1}{x + 1} - \frac{1}{2} \frac{1}{x^2 + 1},$$

from which we immediately get

$$\int \frac{1}{x^4 - 1} dx = \frac{1}{4} \ln \left| \frac{x - 1}{x + 1} \right| - \frac{1}{2} \operatorname{Arctan} x, \quad x \neq \pm 1. \quad \diamond$$

ALTERNATIVELY it is straightforward here to apply MAPLE instead. The details are left to the reader.

3 Examples of point sets, conics and conical sections

3.1 Point Sets

Example 3.1 Sketch the point set A , the interior A° , the boundary ∂A and the closure \bar{A} in each of the cases below.

Furthermore, examine whether A is open, closed or nothing of that kind.

Finally, check whether A is bounded or unbounded.

- 1) $\{(x, y) \mid xy \neq 0\}$.
- 2) $\{(x, y) \mid 0 < x < 1, 1 \leq y \leq 3\}$.
- 3) $\{(x, y) \mid y \geq x^2, |x| < 2\}$.
- 4) $\{(x, y) \mid x^2 + y^2 - 2x + 6y \leq 15\}$.

A Examination of point sets in the plane.

D Each set is analyzed on a figure.

I 1) The set $A = \{(x, y) \mid xy \neq 0\}$ is the whole plane with the exception of the X and the Y axes. It is obvious that it is *open*,

$$A = A^\circ.$$

The boundary ∂A is the union of the X and the Y axes.

The closure is $\bar{A} = A^\circ \cup \partial A = \mathbb{R}^2$, i.e. the whole plane.

Finally, A is clearly *not bounded*.

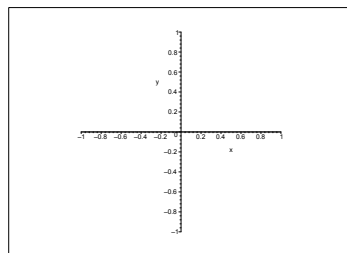


Figure 3.1: The set of Example 3.1.1

2) It is easy to sketch the rectangle $A =]0, 1[\times]1, 3[$. We see that

$$A^\circ =]0, 1[\times]1, 3[.$$

The boundary of the rectangle is rather complicated to describe formally:

$$\begin{aligned} \partial A &= \{(x, y) \mid 0 \leq x \leq 1, y = 1\} \cup \{(x, y) \mid 0 \leq x \leq 1, y = 3\} \\ &\cup \{(x, y) \mid x = 0, 1 \leq y \leq 3\} \cup \{(x, y) \mid x = 1, 1 \leq y \leq 3\}. \end{aligned}$$

This example shows why one shall often prefer a figure instead of a formally correct mathematical description.

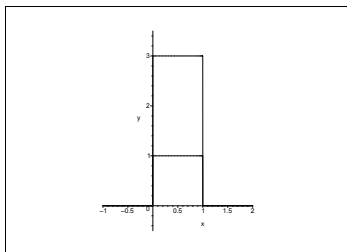


Figure 3.2: The set of Example 3.1.2

The closure is

$$\bar{A} = [0, 1] \times [1, 3].$$

The set A is neither open nor closed.

Obviously, the set is bounded (it is e.g. contained in the disc of centre $(0, 0)$ and radius 4).

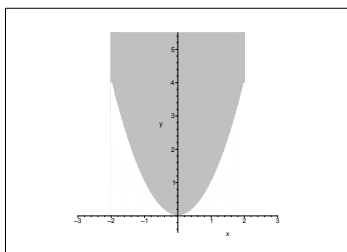


Figure 3.3: The set of Example 3.1.3

3) The set

$$A = \{(x, y) \mid y > x^2, |x| < 2\},$$

is also easily sketched. Here

$$A^\circ = \{(x, y) \mid y > x^2, |x| < 2\}$$

and

$$\partial A = \{(x, y) \mid x = -2, y \geq 4\} \cup \{(x, y) \mid |x| \leq 2, y = x^2\} \cup \{(x, y) \mid x = 2, y \geq 4\},$$

and

$$\bar{A} = \{(x, y) \mid y \geq x^2, |x| \leq 2\}.$$

The set A is neither open nor closed.

The set is clearly not bounded.

4) Since

$$x^2 + y^2 - 2x + 6y \leq 15$$

can be rewritten as

$$x^2 - 2x + 1 + y^2 + 6y + 9 \leq 9 \leq 15 + 1 + 9 = 25 = 5^2,$$

i.e. put into the form

$$(x - 1)^2 + (y + 3)^2 \leq 25 = 5^2,$$

it follows that

$$A = \{(x, y) \mid (x - 1)^2 + (y + 3)^2 \leq 5^2\} = \overline{K}((1, -3); 5).$$

This describes a closed disc of centre $(1, -3)$ and radius 5, thus $A = \overline{A}$.

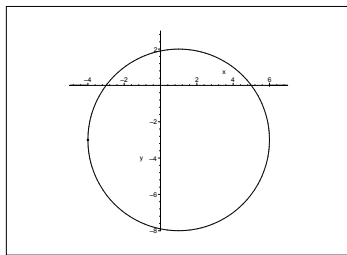


Figure 3.4: The set of Example 3.1.4

Then

$$A^\circ = K((1, -3); 5) = \{(x, y) \mid (x - 1)^2 + (y + 3)^2 < 5^2\}$$

and

$$\partial A = \{(x, y) \mid (x - 1)^2 + (y + 3)^2 = 5^2\},$$

and $A = \overline{A}$ is closed and bounded.

REMARK. Note that whenever a set like the one under consideration is described by an inequality between simple algebraic expressions, one will *usually* obtain the open set A° by only using the inequality signs $<$ or $>$ without equality sign, obtain the closed set by using \leq or \geq everywhere, and finally get the boundary by only using equality sign $=$. This is unfortunately only a rule of thumb, and one must be aware of that there are exceptions from this rule. \diamond

Example 3.2 Sketch in each of the following cases the point set A .
Examine whether A is open or closed or none of the kind.

- 1) $\{(x, y) \mid 3x^2 + 2y^2 < 6\}$.
- 2) $\{(x, y) \mid x^2 + y^2 \leq 1, y > 0\}$.
- 3) $\{(x, y) \mid x^2(1 - x^2 - y^2) > 0\}$.
- 4) $\{(x, y) \mid 0 < x - y \leq 1, y > 4\}$.
- 5) $\{(x, y) \mid x^2 + y^2 \geq \sqrt{x^2 + y^2}\}$.
- 6) $\{(x, y) \mid \max\{|x|, |y|\} \leq 1\}$.
- 7) $\{(x, y) \mid |x| + |y| < 1\}$.
- 8) $\{(x, y) \mid x \leq y \leq 4 - x^2\}$.
- 9) $\{(x, y) \mid (x - 1)(x^2 + y^2) \geq 0\}$.
- 10) $\{(x, y) \mid (y^2 - 1)(y - 3) > 0\}$.

A Examination of point sets in the plane.

D Analyze each set on a figure, e.g. by first examining the function. (Neither \LaTeX nor MAPLE er may be well fit for the sketches in every one of the cases).

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I 1) It follows from the rearrangement

$$A = \{(x, y) \mid 3x^2 + 2y^2 < 6\} = \left\{ (x, y) \mid \left(\frac{x}{\sqrt{2}} \right)^2 + \left(\frac{y}{\sqrt{3}} \right)^2 < 1 \right\}$$

that the set is an open ellipsoidal disc of centre $(0, 0)$ and length of the half axes $\sqrt{2}$ and $\sqrt{3}$. The set is open.

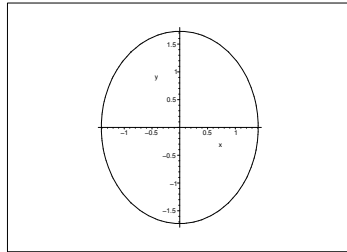


Figure 3.5: The set of Example 3.2.1

2) The set

$$A = \{(x, y) \mid x^2 + y^2 \leq 1, y > 0\}$$

is the intersection of the closed unit disc and the open upper half plane. The set is neither open nor closed.

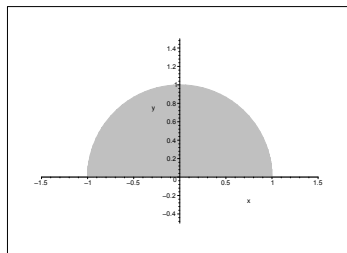


Figure 3.6: The set of Example 3.2.2

3) The set

$$A = \{(x, y) \mid x^2(1 - x^2 - y^2) > 0\} = \{(x, y) \mid x \neq 0, x^2 + y^2 < 1\}$$

is the open unit disc with the exception of the points on the Y axis (where $x = 0$).

The set is open.

4) The set $A = \{(x, y) \mid 0 < x - y \leq 1, y > 4\}$ is the intersection of the three half planes

$$\{(x, y) \mid x > y\}, \quad \{(x, y) \mid y \geq x - 1\}, \quad \{(x, y) \mid y > 4\}.$$

This set is neither open nor closed.

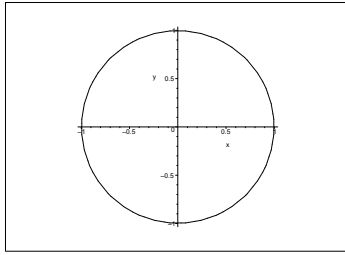


Figure 3.7: The set of Example 3.2.3

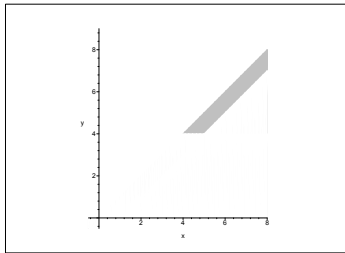


Figure 3.8: The set of Example 3.2.4

5) The set

$$\begin{aligned} A &= \{(x, y) \mid x^2 + y^2 \geq \sqrt{x^2 + y^2} \geq 1\} \\ &= \{(0, 0)\} \cup \{(x, y) \mid \sqrt{x^2 + y^2} \geq 1\} \\ &= \{(0, 0)\} \cup \{(x, y) \mid x^2 + y^2 \geq 1\} \end{aligned}$$

is the complementary set of a disc (centre $(0, 0)$ and radius 1), supplied with the point $(0, 0)$.
The set is closed.

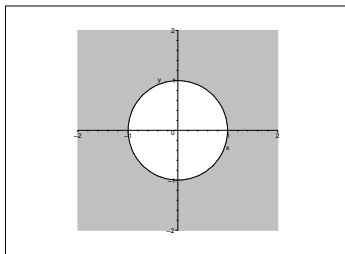


Figure 3.9: The set of Example 3.2.5

6) The set

$$A = \{(x, y) \mid \max\{|x|, |y|\} \leq 1\} = [-1, 1] \times [-1, 1]$$

is a closed square.

7) The set $A = \{(x, y) \mid |x| + |y| < 1\}$ is the open square bounded by the lines

$$x + y = 1, \quad -x + y = 1, \quad x - y = 1, \quad -x - y = 1.$$

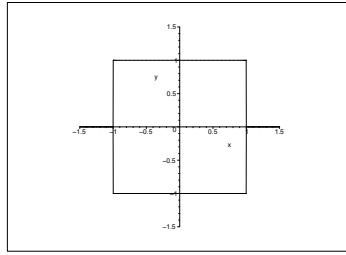


Figure 3.10: The set of Example 3.2.6

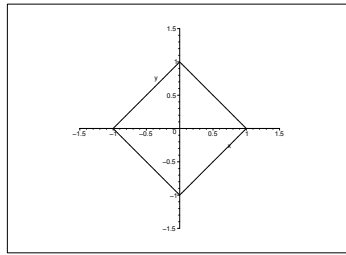


Figure 3.11: The set of Example 3.2.7

- 8) The set $A = \{(x, y) \mid x \leq y \leq 4 - x^2\}$ lies above the line $y = x$ and below the parabola $y = 4 - x^2$. These curves cut each other when $x^2 + x = 4$, i.e. when $x = -\frac{1}{2} \pm \frac{1}{2}\sqrt{17}$.
- 9) Since we always have $x^2 + y^2 \geq 0$ and $x^2 + y^2 = 0$ only for $(x, y) = (0, 0)$, we get that

$$A = \{(x, y) \mid (x - 1)(x^2 + y^2) \geq 0\} = \{(0, 0)\} \cup \{(x, y) \mid x \geq 1\}$$

is the union of a point $(0, 0)$ and a closed half plane $x \geq 1$. It follows that A is closed.

- 10) The set

$$\begin{aligned} A &= \{(x, y) \mid (y^2 - 1)(y - 3) > 0\} = \{(x, y) \mid (y + 1)(y - 1)(y - 3) > 0\} \\ &= \{(x, y) \mid -1 < y < 1\} \cup \{(x, y) \mid 3 < y\} \end{aligned}$$

is open.

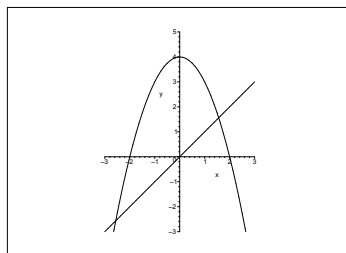


Figure 3.12: The set of Example 3.2.8.

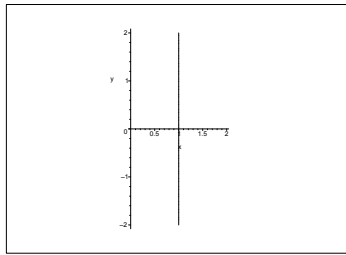


Figure 3.13: The set of Example 3.2.9.

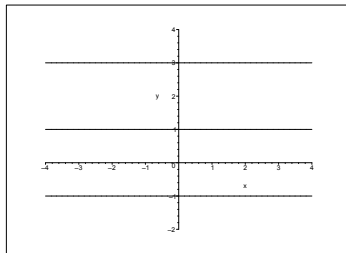


Figure 3.14: The set of Example 3.2.10.

Example 3.3 Examine in each of the following cases, possibly by means of a sketch of a figure, the given point set. Do these sets have names?

- 1) $A = \{(x, y, z) \mid \max\{|x|, |y|, |z|\} \leq 1\}$.
- 2) $A = \{(x, y, z) \mid |x| + |y| + |z| \leq 1\}$.
- 3) $A = \{(x, y, z) \mid x > 0, y > 0, z > 0\}$.
- 4) $A = \{(x, y, z) \mid 0 < x < y\}$.
- 5) $A = \{(x, y, z) \mid 0 < y\}$.
- 6) $A = \{(x, y, z) \mid x^2 + 2y^2 \leq 8\}$.

REMARK. It is difficult in all cases to let L^AT_EX or MAPLE sketch the three dimensional figures. The readers are kindly asked to sketch them themselves. \diamond

A Point sets in the three dimensional space \mathbb{R}^3 .

D Analyze each set, possibly on a figure.

I 1) The set

$$A = \{(x, y, z) \mid \max\{|x|, |y|, |z|\} \leq 1\} = [-1, 1]^3$$

is a closed cube of centre $(0, 0, 0)$ and edge length 2.

2) The set

$$A = \{(x, y, z) \mid |x| + |y| + |z| \leq 1\}$$

is a closed dodecahedron. It is obtained by taking the intersection of the eight half spaces

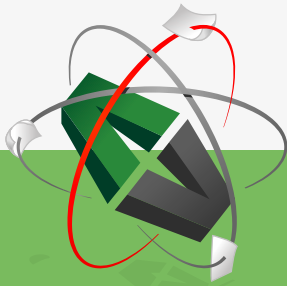
$$\begin{array}{ll} x + y + z \leq 1, & x + y + z \geq -1, \\ x + y - z \leq 1, & x + y - z \geq -1, \\ x - y + z \leq 1, & x - y + z \geq -1, \\ x - y - z \leq 1, & x - y - z \geq -1. \end{array}$$

3) The set

$$A = \{(x, y, z) \mid x > 0, y > 0, z > 0\}$$

is the open first octant.

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- 4) The set $A = \{(x, y, z) \mid 0 < x < y\}$ is the intersection of two open half spaces, hence itself open. The axis of the set is the Z axis, and the projection onto the XY plane in the direction of the Z axis is the angular set which lies between the line $y = x$ and the Y axis in the first quadrant.
- 5) The set $A = \{(x, y, z) \mid 0 < y\}$ is the open half space which is given by the inequality $y > 0$, i.e. bounded by the XZ plane where $y = 0$.
- 6) The set

$$A = \{(x, y, z) \mid x^2 + 2y^2 \leq 8\} = \left\{ (x, y, z) \mid \left(\frac{x}{2\sqrt{2}} \right)^2 + \left(\frac{y}{2} \right)^2 \leq 1 \right\}$$

is the closed cylinder over the ellipse in the XY plane with centre $(0, 0)$ and half axes $2\sqrt{2}$ and 2 . The figure shows the projection of the set onto the XY plane in the direction of the Z axis, hence a cross section.

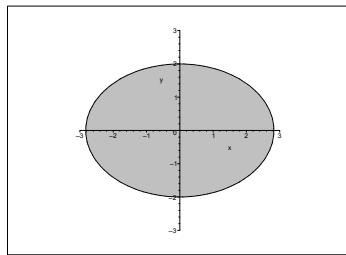


Figure 3.15: The projection onto the XY plane of the set of Example 3.3.6.

Example 3.4 In each of the following cases a plane point set A is given in polar coordinates. Sketch the point set and find a name of it.

- 1) $0 \leq \varphi \leq \frac{\pi}{2}, \quad 0 \leq \rho \leq a \cos \varphi.$
- 2) $0 \leq \varphi \leq \frac{\pi}{4}, \quad 0 \leq \rho \leq a \cos \varphi + a \sin \varphi.$
- 3) $-\pi < \varphi \leq \pi, \quad (\rho - a)^2 \geq |\rho - a|a.$
- 4) $\begin{cases} 0 \leq \varphi \leq \operatorname{Arctan} \frac{b}{a}, & 0 \leq \rho \leq \frac{a}{\cos \varphi}, \\ \operatorname{Arctan} \frac{b}{a} \leq \varphi \leq \frac{\pi}{2}, & 0 \leq \rho \leq \frac{b}{\sin \varphi}. \end{cases}$

A Point sets in the plane given in polar coordinates.

D Analyze the point sets and sketch them.

I 1) When $0 \leq \rho \leq a \cos \varphi$ a multiplication by $\rho \geq 0$ gives

$$0 \leq \rho^2 \leq a\rho \cos \varphi,$$

i.e.

$$x^2 + y^2 \leq ax,$$

and then by a rearrangement

$$\left(x - \frac{a}{2}\right)^2 + y^2 \leq \left(\frac{a}{2}\right)^2.$$

Since $0 \leq \varphi \leq \frac{\pi}{2}$, we get a closed half disc in the first quadrant of centre $\left(\frac{a}{2}, 0\right)$ and radius $\frac{a}{2}$.

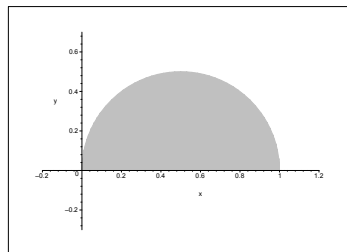


Figure 3.16: The set of Example 3.4.1.

2) By a multiplication by ρ we get

$$\rho^2 \leq a\rho \cos \varphi + a\rho \sin \varphi,$$

thus in rectangular coordinates

$$x^2 + y^2 \leq ax + ay,$$

which is reduced to

$$\left(x - \frac{a}{2}\right)^2 + \left(y - \frac{a}{2}\right)^2 \leq \frac{a^2}{2} = \left(\frac{a}{\sqrt{2}}\right)^2.$$

This expression describes a closed disc of centre $\left(\frac{a}{2}, \frac{a}{2}\right)$ and radius $\frac{a}{\sqrt{2}}$. From the condition $0 \leq \varphi \leq \frac{\pi}{4}$ follows that the set A is that part of the disc, which lies in in this angular set (a circumference angle).

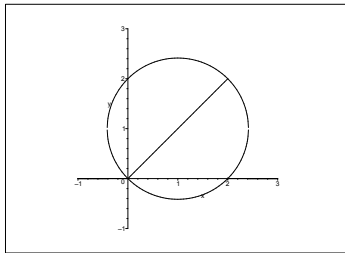


Figure 3.17: The set of Example 3.4.2.

3) It follows from $(\varrho - a)^2 \geq |\varrho - a|a$ that either $\varrho = a$ or $|\varrho - a| \geq a$, hence

$$\varrho - a \geq a \quad \text{or} \quad \varrho - a \leq -a.$$

Summarizing we get

$$\varrho = a \quad \text{or} \quad \varrho \geq 2a \quad \text{or} \quad \varrho = 0,$$

since $\varrho < 0$ is not possible.

The point set is the union of a point $\{(0,0)\}$, a circumference $\varrho = a$ and the closed complementary set of a disc $\varrho \geq 2a$, since we have no restrictions on the angle $-\pi < \varrho \leq \pi$.

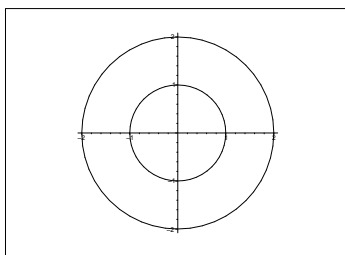


Figure 3.18: The set of Example 3.4.3.

4) Since $\cos \varphi > 0$ for $0 \leq \varphi \leq \text{Arctan } \frac{b}{a}$, the condition $0 \leq \varrho \leq \frac{a}{\cos \varphi}$ is equivalent to

$$0 \leq \varrho \cos \varphi = x \leq a, \quad 0 \leq \varphi \leq \text{Arctan } \frac{b}{a}.$$

Analogously, $\sin \varphi > 0$ for $\text{Arctan} \frac{b}{a} \leq \varphi \leq \frac{\pi}{2}$, thus $0 \leq \varrho \leq \frac{b}{\sin \varphi}$ is equivalent to

$$0 \leq \varrho \sin \varphi = y \leq b, \quad \text{Arctan} \frac{b}{a} \leq \varphi \leq \frac{\pi}{2}.$$

The two cases are described by each their triangle, and the conclusion is that the set in rectangular coordinates is just the rectangle $A = [0, a] \times [0, b]$.

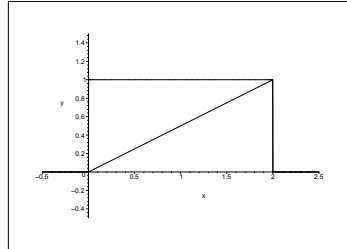
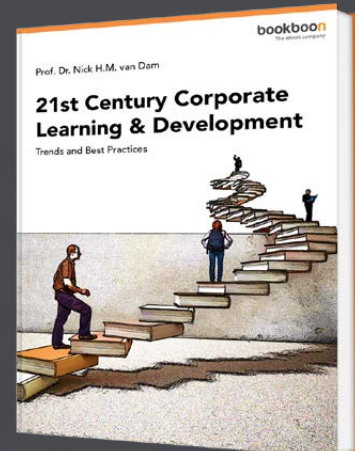


Figure 3.19: The set of Example 3.4.4.

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Example 3.5 Sketch and describe in polar coordinates the set A , where A is given below in rectangular coordinates.

$$1) A = \left\{ (x, y) \mid x \geq 0, (x^2 + y^2)^2 \geq x^2 + y^2 \right\}.$$

$$2) A = \left\{ (x, y) \mid x > 0, \frac{1}{2} + y^2 \leq x^2 \leq 1 - y^2 \right\}.$$

A Point sets in the plane, given in rectangular coordinates should be described in polar coordinates instead.

D Sketch the sets and use that $x = \rho \cos \varphi$ and $y = \rho \sin \varphi$.

I 1) The point set A is the intersection of a closed complementary set of a disc and the closed right half plane supplied by the point $(0, 0)$.

In polar coordinates this is described by

$$-\frac{\pi}{2} \leq \varphi \leq \frac{\pi}{2} \quad \text{and} \quad \rho^2 \geq \rho.$$

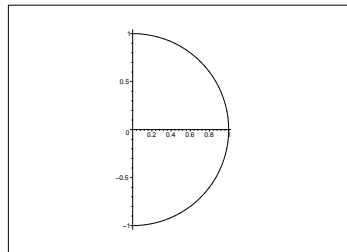


Figure 3.20: The set of Example 3.5.1.

2) Since $x > 0$, the point set lies in the open right half plane. It follows from $x^2 \leq 1 - y^2$ that $x^2 + y^2 \leq 1$, so the point set lies in the unit disc.

Finally, $\frac{1}{2} + y^2 \leq x^2$ describes the interior of a branch of a hyperbola. The two limiting curves

$$x^2 + y^2 = 1 \quad \text{and} \quad x^2 - y^2 = \frac{1}{2}$$

cut each other at the points $\left(\frac{\sqrt{3}}{2}, \pm \frac{1}{2} \right)$, so A lies in the angular set $-\frac{\pi}{6} \leq \varphi \leq \frac{\pi}{6}$.

In polar coordinates the upper is described by $\rho \leq 1$, and the lower bound is given by

$$\frac{1}{2} + \rho^2 \sin^2 \varphi \leq \rho^2 \cos^2 \varphi,$$

hence by a rearrangement,

$$\frac{1}{2} \leq \rho^2 \{ \cos^2 \varphi - \sin^2 \varphi \} = \rho^2 \cos 2\varphi.$$

Summarizing we get the following polar description

$$-\frac{\pi}{6} \leq \varphi \leq \frac{\pi}{6} \quad \text{and} \quad \frac{1}{\sqrt{2 \cos 2\varphi}} \leq \rho \leq 1.$$

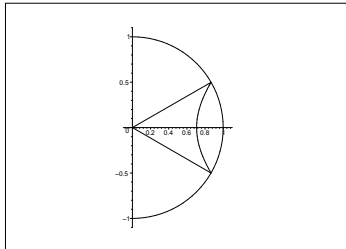


Figure 3.21: The set of Example 3.5.2.

Example 3.6 Sketch the following subsets of \mathbb{R}^2 , and if any of them is star shaped.

- 1) $\{(x, y) \mid y > 3x^2\}$.
- 2) $\{(x, y) \mid x^2 + y^2 > 1\}$.
- 3) $\{(x, y) \mid y > -x^2\}$.
- 4) $\{(x, y) \mid x > 0, y > -x^2\}$.

A Analysis of sets concerning if they are star shaped.

D Start by sketching the sets. In this case I have had problems with the sketching programs, so the sets are only given by their boundaries and not by the more desirable hatching.

I 1) Here, A in the interior of a parabola. Obviously, this set is star shaped (and even convex).

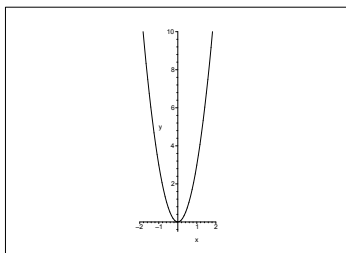


Figure 3.22: The set of Example 3.6.1.

- 2) This set is the complementary of a disc, so it cannot be star shaped. For any point from the set the unit disc shades for some other points.

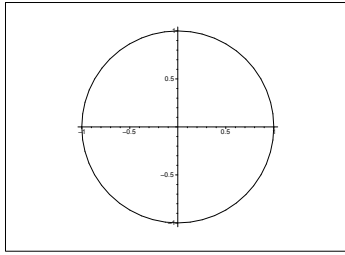


Figure 3.23: The set of Example 3.6.2.

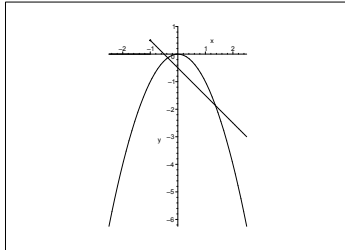


Figure 3.24: The set of Example 3.6.3.

- 3) The set is the exterior of a parabola. If $(x_0, y_0) \in A$ is any point we can always find a straight line through (x_0, y_0) , which cuts the parabola in two different points. The points on the line beyond the most distant intersection point cannot be connected with (x_0, y_0) by a straight line inside A , so A is not star shaped seen from any point.
- 4) This set A is a part of the set in Example 3.6.3, hence it lies in the right half plane. First note that

$$y + \lambda^2 = -2\lambda(x - \lambda)$$

is a tangent of the parabola for every $\lambda > 0$. This can also be written

$$y + 2\lambda x = \lambda^2, \quad \lambda > 0.$$

Indirect proof. Assume that A indeed was star shaped from a point (x, y) . Then

$$y + 2\lambda x \geq \lambda^2 \quad \text{for all } \lambda > 0,$$

which can also be written

$$y \geq \lambda(\lambda - 2x) \quad \text{for all } \lambda > 0.$$

This is of course not possible for any $(x, y) \in A$. In fact, the right hand side of this inequality tends to $+\infty$ for $\lambda \rightarrow +\infty$, while y remains constant, and the inequality is violated.

We conclude from this contradiction that A is not star shaped.

Example 3.7 . Sketch the point sets given below, and indicate which ones are convex.

- 1) $\{(x, y) \mid -5 < y < -3x^2\}$.
- 2) $\{(x, y) \mid x^2 + 3y^2 > 2\}$.
- 3) $\{(x, y) \mid y > -x^2\}$.
- 4) $\{(x, y) \mid x \geq 0, y \leq 0\}$.

A Examination of convexity.

D Sketch the sets and analyze.

I 1) The set is the interior of a parabola where we furthermore have the restriction $-5 < y < 0$. Obviously, this set is convex.

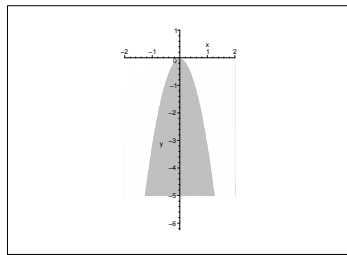


Figure 3.25: The set of Example 3.7.1.

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2) The set

$$\{(x, y) \mid x^2 + 3y^2 > 2\} = \left\{ (x, y) \mid \left(\frac{x}{\sqrt{2}}\right)^2 + \left(\frac{y}{\sqrt{\frac{2}{3}}}\right)^2 > 1 \right\}$$

is the complementary of an ellipse of centre $(0, 0)$ and half axes $\sqrt{2}$ and $\sqrt{\frac{2}{3}}$. It is clearly not convex.

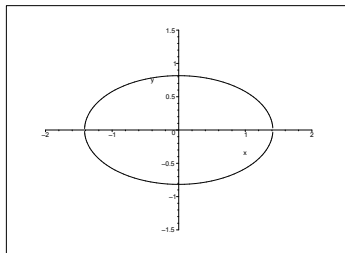


Figure 3.26: The set of Example 3.7.2.

3) This set is the complementary of a parabola (actually the same set as in Example 3.6.2). It is not star shaped, and therefore not convex either.

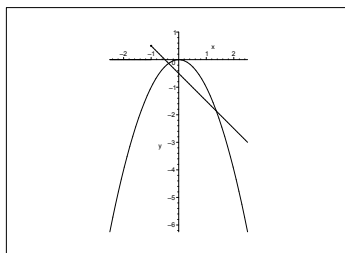


Figure 3.27: The set of Example 3.7.3.

4) This set is the closed fourth quadrant. It is clearly convex. There is no need to sketch it.

Example 3.8 *Let*

$$B = \{(x, y) \in [0, 1] \times [0, 1] \mid x \text{ is rational and } y \text{ is rational}\}.$$

Find the interior B° , the boundary ∂B and the closure \overline{B} .

A Interior, exterior, boundary and closure of a point set. This is the classical “strange” example, which should shock the reader, who has not seen this example before.

D First prove that $B^\circ = \emptyset$, and then $\overline{B} = [0, 1] \times [0, 1]$.

I If $(x_0, y_0) \in B$, then $K((x_0, y_0); r)$, $r > 0$, i.e. the solid ball of centre (x_0, y_0) and any positive radius r , will always contain points (x, y) , of which at least one of the coordinates is irrational, hence

$$K((x_0, y_0); r) \setminus B \neq \emptyset \quad \text{for every } r > 0.$$

We conclude from this that $B^\circ = \emptyset$.

Let $(x_0, y_0) \in [0, 1] \times [0, 1]$ be any point in the bigger set. Then the ball $K((x_0, y_0); r)$ of centre (x_0, y_0) and any radius $r > 0$ will always contain points from B . This means that $(x_0, y_0) \in \overline{B}$, i.e.

$$\overline{B} \supseteq [0, 1] \times [0, 1].$$

It is on the other hand trivial that $\overline{B} \subseteq [0, 1] \times [0, 1]$, hence we must have equality,

$$\overline{B} = [0, 1] \times [0, 1].$$

Finally, the boundary is found by means of the definition,

$$\partial B = \overline{B} \setminus B^\circ = [0, 1] \times [0, 1] \setminus \emptyset = [0, 1] \times [0, 1] = \overline{B}.$$

Example 3.9 In each of the following cases there is given a solid tetrahedron by its four corners. Sketch the tetrahedron T – invisible edges are dotted – and set up equations of the four planes, which bound T . Then derive the inequalities which the points of T must fulfil, and finally set up expressions of the form

$$T = \{(x, y, z) \mid (x, y) \in B, Z_1(x, y) \leq z \leq Z_2(x, y)\}$$

and

$$T = \{(x, y, z) \mid \alpha \leq z \leq \beta, (x, y) \in B(z)\};$$

sketch the sets B and $B(z)$.

- 1) $(0, 0, 0)$, $(2, 0, 0)$, $(0, 1, 0)$, $(0, 0, 2)$.
- 2) $(0, 0, 0)$, $(2, 0, 2)$, $(0, 1, 2)$, $(0, 0, 2)$.
- 3) $(1, 0, 0)$, $(0, 0, 4)$, $(0, 2, 2)$, $(-1, 0, 0)$.
- 4) $(0, 0, 0)$, $(1, 0, 0)$, $(1, 1, 0)$, $(1, 0, 4)$.
- 5) $(1, 0, 0)$, $(0, 0, 4)$, $(0, 2, 0)$, $(-1, 0, 0)$.

A Analysis of tetrahedra.

D The text describes very carefully what should be done. Here we shall deviate a little because figures in space take a very long time to construct in the given programs. There are left to the reader.

I 1) It follows immediately from the missing figure (which the reader should add himself), that three of the planes are described by

$$x = 0, \quad y = 0 \quad \text{and} \quad z = 0.$$

In fact, the plane $x = 0$ contains the points

$$(0, 0, 0), \quad (0, 1, 0), \quad (0, 0, 2),$$

the plane $y = 0$ contains the points

$$(0, 0, 0), \quad (2, 0, 0), \quad (0, 0, 2),$$

and the plane $z = 0$ contains the points

$$(0, 0, 0), \quad (2, 0, 0), \quad (0, 1, 0).$$



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A parametric description of the fourth plane is e.g.

$$\begin{aligned}(x, y, z) &= (2, 0, 0) + u\{(0, 1, 0) - (2, 0, 0)\} + v\{(0, 0, 2) - (2, 0, 0)\} \\ &= (2, 0, 0) + u(-2, 1, 0) + v(-2, 0, 2) \\ &= (2 - 2u - 2v, u, 2v),\end{aligned}$$

from which $y = u$ and $z = 2v$.

When we eliminate u and v , we get

$$x = 2 - 2u - 2v = 2 - 2y - z,$$

and the equation of the fourth plane is

$$z = 2 - x - 2y.$$

The points of T must satisfy the inequalities

$$0 \leq x (\leq 2), \quad 0 \leq y \left(\leq 1 - \frac{x}{2}\right), \quad 0 \leq z \leq 2 - x - 2y.$$

We immediately get

$$T = \{(x, y, z) \mid (x, y) \in B, 0 \leq z \leq 2 - x - 2y\},$$

where

$$B = \{(x, y) \mid 0 \leq x \leq 2, 0 \leq y \leq 1 - \frac{x}{2}\}.$$

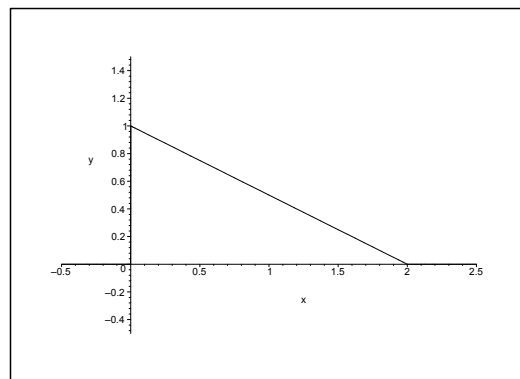


Figure 3.28: The domain B of Example 3.9.1

If we instead keep $z \in [0, 2]$ fixed, the tetrahedron is cut into a triangle $B(z)$, bounded by

$$0 \leq x \leq 2 - z, \quad 0 \leq y \leq 1 - \frac{x}{2} - \frac{z}{2},$$

i.e.

$$B(z) = \left\{ (x, y) \mid 0 \leq x \leq 2 - z, 0 \leq y \leq 1 - \frac{z}{2} - \frac{x}{2} \right\}, \quad 0 \leq x \leq z \leq 2,$$

and

$$T = \{(x, y, z) \mid (x, y) \in B(z), 0 \leq z \leq 2\}.$$

It follows that $B(z)$ is similar to B above with the factor of similarity $1 - \frac{z}{2}$.

2) We see in the same way as in Example 3.9.1 that three of the planes are described by

$$x = 0, \quad y = 0 \quad \text{and} \quad z = 2.$$

A parametric description of the fourth plane is e.g.

$$(x, y, z) = (0, 0, 0) + u(2, 0, 2) + v(0, 1, 2) = (2u, v, 2u + 2v),$$

from which $x = 2u$ and $y = v$. When u and v are eliminated we get

$$z = 2u + 2v = x + 2y,$$

which is an equation of the fourth plane.

The points of T must satisfy the inequalities

$$0 \leq x (\leq 2), \quad 0 \leq y \left(\leq 1 - \frac{x}{2} \right), \quad x + 2y \leq z \leq 2.$$

Hence,

$$T = \{(x, y, z) \mid (x, y) \in B, x + 2y \leq z \leq 2\},$$

where

$$B = \left\{ (x, y) \mid 0 \leq x \leq 2, 0 \leq y \leq 1 - \frac{x}{2} \right\}.$$

If we instead keep $z \in [0, 2]$ fixed, the tetrahedron is cut into a triangle $B(z)$, bounded by

$$0 \leq x \leq z, \quad 0 \leq y \leq \frac{z}{2} - \frac{x}{2},$$

i.e.

$$B(z) = \left\{ (x, y) \mid 0 \leq x \leq z, 0 \leq y \leq \frac{z}{2} - \frac{x}{2} \right\}, \quad 0 \leq z \leq 2,$$

and

$$T = \{(x, y, z) \mid (x, y) \in B(z), 0 \leq z \leq 2\}.$$

We see that $B(z)$ is similar to B with the constant of similarity $\frac{z}{2}$.

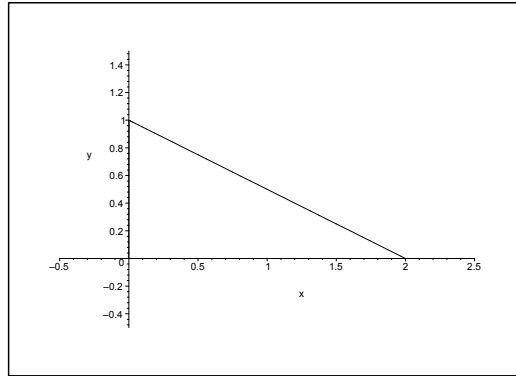


Figure 3.29: The domain B of Example 3.9.2

3) Here a trivial boundary plane is given by $y = 0$.

The points $(1, 0, 0)$, $(0, 2, 2)$, $(0, 0, 4)$ lie in the plane of the parametric description

$$(x, y, z) = (1, 0, 0) + u(-1, 2, 2) + v(-1, 0, 4) = (1 - u - v, 2u, 2u + 4v),$$

i.e.

$$x = 1 - u - v, \quad y = 2u, \quad z = 2u + 4v,$$

from which

$$u = \frac{y}{2}, \quad v = 1 - u - x = 1 - \frac{y}{2} - x,$$

so

$$z = 2u + 4v = y + 4 \left(1 - \frac{y}{2} - x\right) = 4 - 4x - y,$$

which is the equation of this plane.

The points $(-1, 0, 0)$, $(0, 2, 2)$, $(0, 0, 4)$ lie in the plane of the parametric description

$$(x, y, z) = (-1, 0, 0) + u(1, 2, 2) + v(1, 0, 4) = (-1 + u + v, 2u, 2u + 4v),$$

i.e.

$$x = -1 + u + v, \quad y = 2u, \quad z = 2u + 4v,$$

from which

$$u = \frac{y}{2}, \quad v = 1 + x - u = 1 + x - \frac{y}{2},$$

hence

$$z = 2u + 4v = y + 4 + 4x - 2y = 4 + 4x - y,$$

which is the equation of this plane.

The points $(1, 0, 0)$, $(-1, 0, 0)$, $(0, 2, 2)$ lie in the plane of the parametric description

$$(x, y, z) = (-1, 0, 0) + u(2, 0, 0) + v(1, 2, 2) = (2u - 1, 2v, 2v),$$

from which

$$x = 2u - 1, \quad y = 2v, \quad z = 2v.$$

We see that the equation of the plane is $z = y$.

Summarizing we have obtained the four planes

$$y = 0, \quad z = 4 - 4x - y, \quad z = 4 + 4x - y, \quad z = y.$$

The projection of T onto the XY plane is the triangle B of the corners $(-1, 0)$, $(1, 0)$, $(0, 2)$. This can be described by

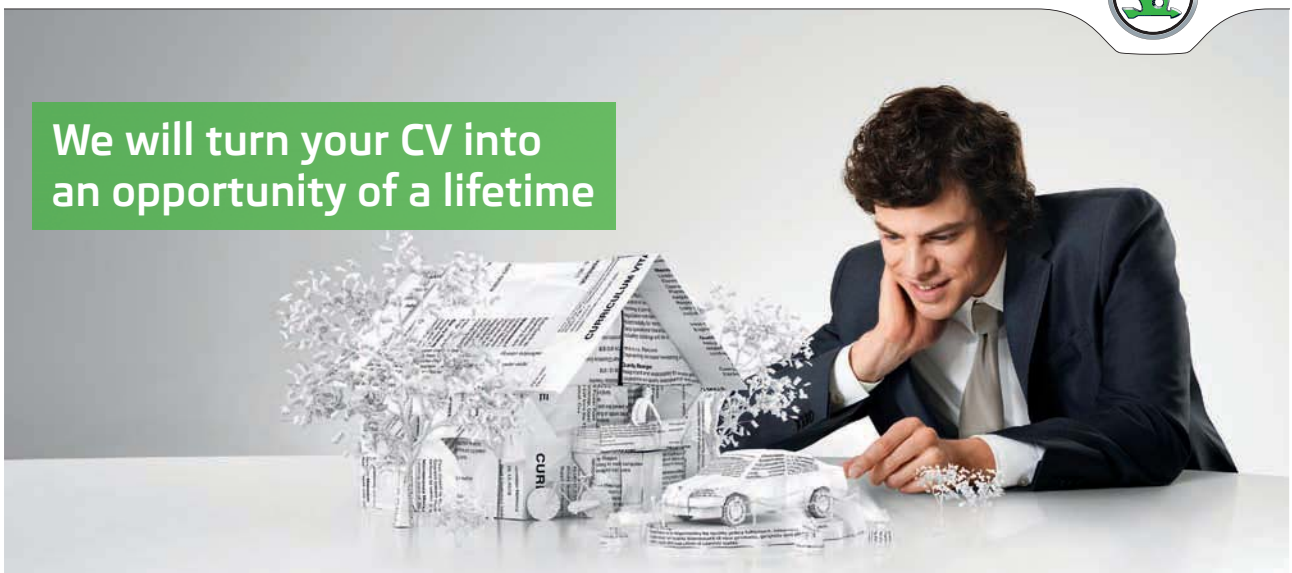
$$0 \leq y \leq 2, \quad \frac{y}{2} - 1 \leq x \leq 1 - \frac{y}{2} \quad \left(|x| \leq 1 - \frac{y}{2} \right),$$

i.e.

$$B = \left\{ (x, y) \mid 0 \leq y \leq 2, |x| \leq 1 - \frac{y}{2} \right\}.$$

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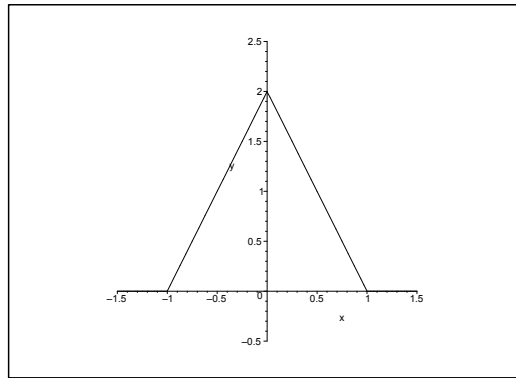


Figure 3.30: The domain B of Example 3.9.3

When $(x, y) \in B$, it is seen from the figure that

$$\begin{cases} y \leq z \leq 4 - 4x - y & \text{for } x \geq 0, \\ y \leq z \leq 4 + 4x - y & \text{for } x \leq 0, \end{cases}$$

i.e.

$$T = \left\{ (x, y, z) \mid 0 \leq y \leq 2, |x| \leq 1 - \frac{y}{2}, y \leq z \leq 4 - 4|x| - y \right\}.$$

The plane $z = \text{constant} \in [2, 4]$ cuts T in a triangle $B(z)$ given by

$$0 \leq y \leq 4 - z, \quad |x| \leq 2 - \frac{y}{2} - \frac{z}{2},$$

hence

$$B(z) = \left\{ (x, y) \mid 0 \leq y \leq 4 - z, |x| \leq 2 - \frac{y}{2} - \frac{z}{2} \right\} \quad \text{for } z \in [2, 4].$$

It follows that $B(z)$ is similar to B with the factor of similarity $2 - \frac{z}{2}$.

Then let $z \in]0, 2[$ be fixed. This plane cuts T in a trapezoid, which is obtained by cutting a triangle out of B at height z . Thus, for $z \in [0, 2[$,

$$B(z) = \left\{ (x, y) \mid 0 \leq y \leq z, |x| \leq 1 - \frac{y}{2} \right\} \quad \text{for } z \in [0, 2[.$$

We get the following description of the tetrahedron:

$$\begin{aligned} T &= \left\{ (x, y, z) \mid 0 \leq z \leq 2, 0 \leq y \leq z, |x| \leq 1 - \frac{y}{2} \right\} \\ &\cup \left\{ (x, y, z) \mid 2 \leq z \leq 4, 0 \leq y \leq 4 - z, |x| \leq 2 - \frac{y}{2} - \frac{z}{2} \right\}. \end{aligned}$$

4) The obvious planes are here

$$\begin{aligned} y = 0, & \quad [\text{points } (0, 0, 0), (1, 0, 0), (1, 0, 4)], \\ z = 0, & \quad [\text{points } (0, 0, 0), (1, 0, 0), (1, 1, 0)], \\ x = 1, & \quad [\text{points } (1, 0, 0), (1, 1, 0), (1, 0, 4)]. \end{aligned}$$

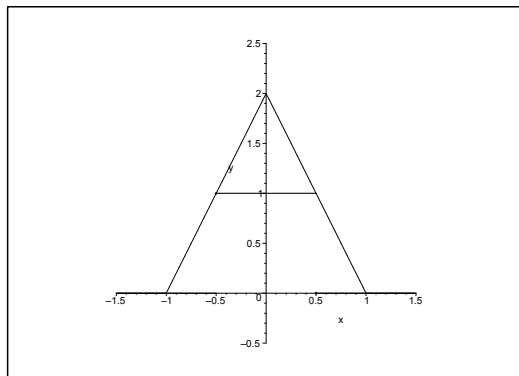


Figure 3.31: The domain $B(z)$ for $z = 1 \in [0, 2[$ in Example 3.9.3

Finally, the points $(0, 0, 0)$, $(1, 0, 4)$, $(1, 1, 0)$ lie in the plane of the parametric description

$$(x, y, z) = u(1, 0, 4) + v(1, 1, 0) = (u + v, v, 4u),$$

from which

$$v = y, \quad u = x - v = x - y \quad \text{and} \quad z = 4u = 4x - 4y.$$

The points in T must satisfy the inequalities

$$0 \leq x \leq 1, \quad 0 \leq y \leq x, \quad 0 \leq z \leq 4x - 4y.$$

In particular, the triangle B is

$$B = \{(x, y) \mid 0 \leq x \leq 1, 0 \leq y \leq x\},$$

and we get

$$T = \{(x, y, z) \mid 0 \leq x \leq 1, 0 \leq y \leq x, 0 \leq z \leq 4x - 4y\}.$$

The plane $z = \text{constant} \in [0, 4]$ cuts the tetrahedron in a triangle which is similar to B of the similarity factor $1 - \frac{z}{4}$ for $z \in [0, 4]$, thus

$$B(z) = \left\{ (x, y) \mid 0 \leq x \leq 1 - \frac{z}{4}, 0 \leq y \leq x \right\},$$

and accordingly,

$$T = \left\{ (x, y, z) \mid 0 \leq z \leq 4, 0 \leq x \leq 1 - \frac{z}{4}, 0 \leq y \leq x \right\}.$$

5) The obvious planes are

$$\begin{aligned} y = 0, & \quad [\text{points } (1, 0, 0), (-1, 0, 0), (0, 0, 4)], \\ z = 0, & \quad [\text{points } (1, 0, 0), (0, 2, 0), (-1, 0, 0)]. \end{aligned}$$

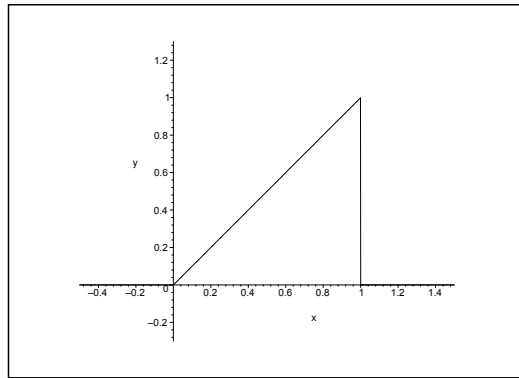


Figure 3.32: The domain B in Example 3.9.4

The points $(1, 0, 0)$, $(0, 2, 0)$, $(0, 0, 4)$ lie in the plane of the parametric description

$$(x, y, z) = (1, 0, 0) + u(-1, 2, 0) + v(-1, 0, 4) = (1 - u - v, 2u, 4v).$$

Hence,

$$u = \frac{y}{2}, \quad v = \frac{z}{4}, \quad x = 1 - u - v = 1 - \frac{y}{2} - \frac{z}{4},$$

and we get the following equation of the plane,

$$z = 4 - 4x - 2y.$$

Due to the symmetry the points $(-1, 0, 0)$, $(0, 2, 0)$ and $(0, 0, 4)$ must lie in the plane of the equation

$$z = 4 + 4x - 2y.$$

The projection of T onto the XY plane is the triangle

$$B = \left\{ (x, y) \mid 0 \leq y \leq 2, |x| \leq 1 - \frac{y}{2} \right\}.$$

When $(x, y) \in B$, we get for $(x, y, z) \in T$ that

$$\begin{cases} 0 \leq z \leq 4 - 4x - 2y, & \text{for } x \geq 0, \\ 0 \leq z \leq 4 + 4x - 2y, & \text{for } x \leq 0, \end{cases}$$

i.e.

$$T = \left\{ (x, y, z) \mid 0 \leq y \leq 2, |x| \leq 1 - \frac{y}{2}, 0 \leq z \leq 4 - 4|x| - 2y \right\}.$$

At the height $z \in [0, 4]$ the tetrahedron T is cut into a triangle

$$B(z) = \left\{ (x, y) \mid 0 \leq y \leq 2 - \frac{z}{2}, |x| \leq 1 - \frac{y}{2} - \frac{z}{4} \right\},$$

where $B(z)$ is similar to B of the similarity factor $1 - \frac{z}{4}$, hence

$$T = \left\{ (x, y, z) \mid 0 \leq z \leq 4, 0 \leq y \leq 2 - \frac{z}{2}, |x| \leq 1 - \frac{y}{2} - \frac{z}{4} \right\}.$$

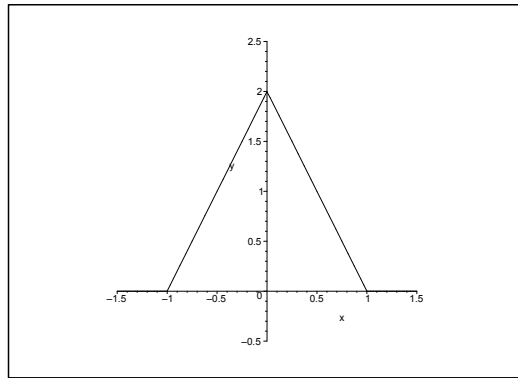


Figure 3.33: The domain B

Example 3.10 Sketch on a figure the set A , where

$$A = \{(x, y) \in \mathbb{R}^2 \mid x + 2y \leq 2, |x - y| \leq 2\}.$$

On the figure one should indicate the boundary ∂A . Finally, explain why A is not bounded.

A Sketch of a set in the plane.

D Start by analyzing the lines, which bound the set.

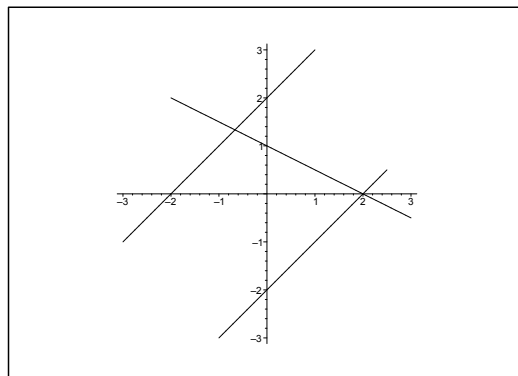


Figure 3.34: The domain A in Example 3.10 is that component of the plane, which contains the point $(0, 0)$.

I It follows from the definition of A that we have the three restrictions

$$x + 2y \leq 2, \quad x - y \leq 2, \quad y - x \leq 2.$$

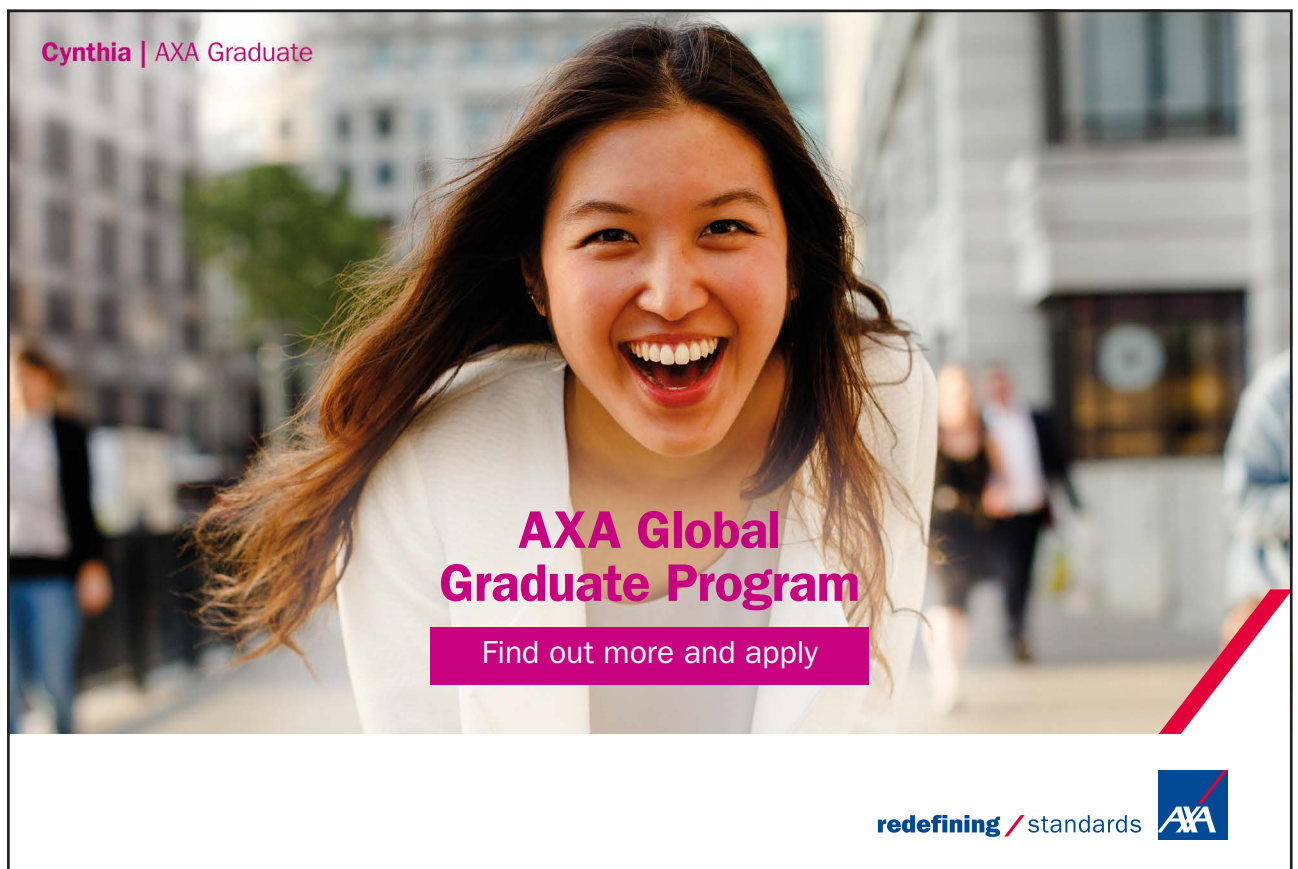
We note that $(0, 0)$ satisfies all three inequalities. Thus, the domain A is the closed component (the intersection of three closed half planes), which contains $(0, 0)$. The boundary ∂A consists of pieces of the lines

$$x + 2y = 2, \quad x - y = 2, \quad y - x = 2.$$

Now, the unbounded half line

$$\{(x, y) \mid y = x - 2, x \leq 2\}$$

lies in A , so A must also be unbounded.



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3.2 Conics and conical sections

Example 3.11 A conic \mathcal{F} is given by the equation

$$2x^2 - 2y^2 + \alpha z^2 = 1,$$

where α is a real constant.

- 1) Find the values of α , for which \mathcal{F} is a surface of revolution. Indicate in each of these cases the type of the surface and its axis of symmetry.
- 2) Prove that there is one value of α , for which the surface \mathcal{F} is a cylindric surface. Indicate for this value of α the type of the surface and its axis of symmetry.

A Conic sections.

D Analyze each of the three cases $\alpha < 0$, $\alpha = 0$ and $\alpha > 0$.

I 1) a) When $\alpha < 0$, the conic is an hyperboloid with two sheets:

$$1 = \left(\frac{x}{1/\sqrt{2}} \right)^2 - \left\{ \left(\frac{y}{1/\sqrt{2}} \right)^2 + \left(\frac{z}{\sqrt{1/|\alpha|}} \right)^2 \right\}.$$

This is an hyperboloid of revolution for $\alpha = -2$, where the X axis is the axis of revolution.

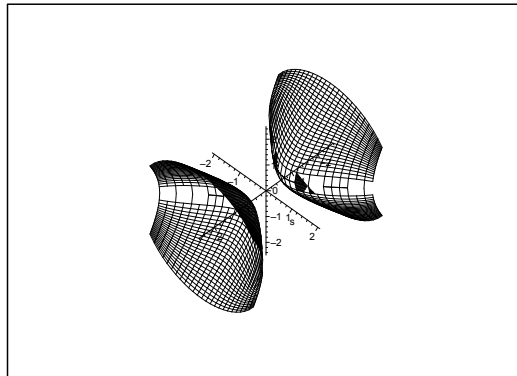


Figure 3.35: The surface of revolution for $\alpha = -2$.

b) When $\alpha > 0$, the conic is an hyperboloid with one sheet:

$$1 = \left\{ \left(\frac{x}{1/\sqrt{2}} \right)^2 + \left(\frac{z}{\sqrt{1/\alpha}} \right)^2 \right\} - \left(\frac{y}{1/\sqrt{2}} \right)^2.$$

This becomes an hyperboloid of revolution when $\alpha = 2$, with the Y axis as its axis of revolution.

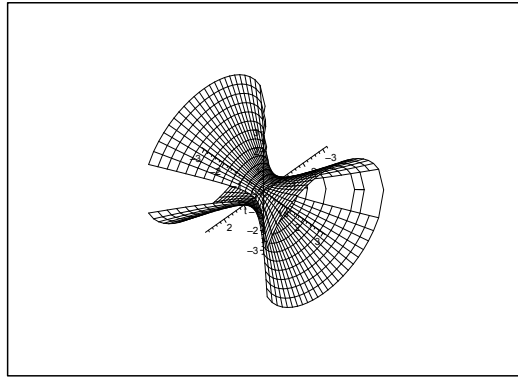


Figure 3.36: The surface of revolution for $\alpha = 2$.

2) When $\alpha = 0$, we get an hyperbolic cylindrical surface with the Z axis as its axis of generation,

$$1 = \left(\frac{x}{\frac{1}{\sqrt{2}}} \right)^2 - \left(\frac{y}{\frac{1}{\sqrt{2}}} \right)^2 .$$

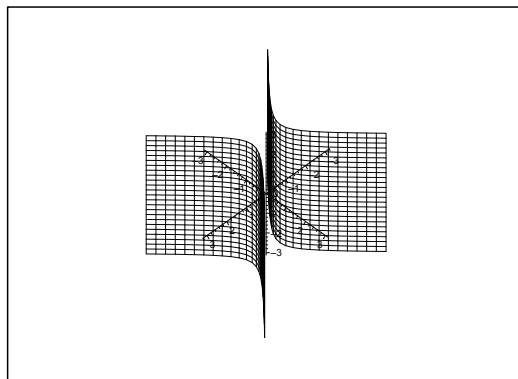


Figure 3.37: The surface for $\alpha = 0$.

Example 3.12 Find the type and position of the conic of the equation

$$x^2 + 2y^2 - x + 6y + \frac{3}{4} = 0.$$

A Conic section.

D Translate the coordinates.

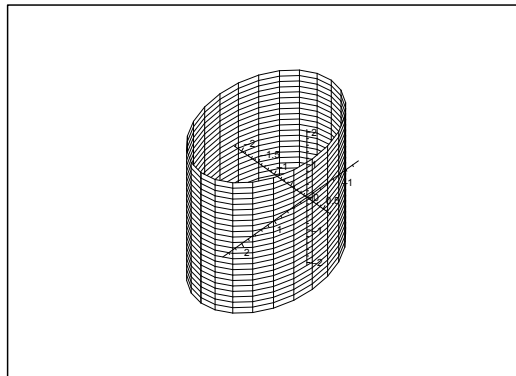
I By a rearrangement we get

$$\begin{aligned} 0 &= x^2 + 2y^2 - x + 6y + \frac{3}{4} \\ &= \left(x^2 - 2 \cdot \frac{1}{2}x + \frac{1}{4}\right) + 2\left(y^2 + 2 \cdot \frac{3}{2}y + \frac{9}{4}\right) - 2 \cdot \frac{9}{4} + \frac{3}{4} \\ &= \left(x - \frac{1}{2}\right)^2 + 2\left(y + \frac{3}{2}\right)^2 - 4, \end{aligned}$$

i.e. in the canonical form

$$\left(\frac{x - \frac{1}{2}}{2}\right)^2 + \left(\frac{y + \frac{3}{2}}{\sqrt{2}}\right)^2 + 0 \cdot z^2 = 1,$$

because z does not appear in the equation.



The surface is an *elliptic cylindrical surface* with the Z axis as its axis of generation, and with the ellipse of centre $\left(\frac{1}{2}, -\frac{3}{2}\right)$ and the half axes 2 and $\sqrt{2}$ as generating curve.

Example 3.13 Let a, b, c be constant different from zero satisfying the equation

$$a + b + c = 0.$$

Prove that the plane of the equation

$$x + y + z = 0$$

cuts the conic given by

$$\frac{yz}{a} + \frac{zx}{b} + \frac{xy}{c} = 0$$

in two straight lines (generators), which form an angle of $\frac{2\pi}{3}$.

A Intersection of two surfaces.

D Start by e.g. eliminating $z = -x - y$.

I Clearly, $(0, 0, 0)$ lies in the intersection of the two surfaces. Furthermore, if two of the variables are 0, e.g. $x = y = 0$, then we have a point on the conic, no matter the value of the third variable (here z). We conclude that the X , the Y and the Z axes all lie on the conic section. Of course, none of them are contained in the oblique plane $x + y + z = 0$.

If we keep off the coordinate planes, i.e. we assume in the following that $xyz \neq 0$, then the equation of the conic can also be written

$$0 = \frac{yz}{a} + \frac{zx}{b} + \frac{xy}{c} = xyz \left(\frac{1}{ax} + \frac{1}{by} + \frac{1}{cz} \right),$$

i.e.

$$\frac{1}{ax} + \frac{1}{by} + \frac{1}{cz} = 0 \quad \text{for } xyz \neq 0.$$

Since $z = -(x + y)$ on the plane, we get by insertion into the reduced equation of the conic that

$$0 = \frac{1}{ax} + \frac{1}{by} + \frac{1}{cz} = \frac{1}{ax} + \frac{1}{by} - \frac{1}{c(x+y)}.$$

When we put everything here into the same fraction and reduce we get

$$(3.1) \quad 0 = \frac{1}{a}(x+y)y + \frac{1}{b}(x+y)x - \frac{1}{c}xy,$$

which is an homogeneous polynomial of second degree in (x, y) .

Now $x = 0$, if and only if $y = 0$, so the solutions must have the structure

$$(3.2) \quad y = \alpha x, \quad \alpha \neq 0.$$

It follows that the intersection of the two surfaces must have the structure

$$\mathbf{r}(t) = (t, \alpha t, -(1 + \alpha)t) = t(1, \alpha, -(1 + \alpha)), \quad t \in \mathbb{R},$$

because $z = -x - y$, and because we can trivially continue to $(0, 0, 0)$.

When (3.2) is put into (3.1), we get that α is a solution of a polynomial of second degree with the roots α_1 and α_2 , corresponding to two straight lines. (According to the geometry the solutions exist, so we must necessarily have the the roots α_1 and α_2 are real numbers).

By insertion of $(x, y, z) = (1, \alpha, -(1 + \alpha))$ we get for $\alpha \neq -1$ that

$$0 = \frac{1}{ax} + \frac{1}{by} + \frac{1}{cz} = \frac{1}{a} + \frac{1}{b\alpha} - \frac{1}{c(1 + \alpha)} = \frac{bc\alpha(1 + \alpha) + ac(1 + \alpha) - ab\alpha}{abc\alpha(1 + \alpha)},$$

which is reduced to

$$0 = \alpha(1 + \alpha) + \frac{a}{b}(1 + \alpha) - \frac{a}{c}\alpha = \alpha^2 + \left(1 + \frac{a}{b} - \frac{a}{c}\right)\alpha + \frac{a}{b} = \alpha^2 + a\left(\frac{1}{a} + \frac{1}{b} - \frac{1}{c}\right)\alpha + \frac{a}{b},$$

hence

$$\alpha_1 + \alpha_2 = a\left(\frac{1}{c} - \frac{1}{a} - \frac{1}{b}\right) = a\left(\frac{1}{c} - \frac{a+b}{ab}\right) = \frac{a}{c} + \frac{c}{b}$$

and

$$\alpha_1\alpha_2 = \frac{a}{b}.$$

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Since $(1, \alpha, -(1 + \alpha))$ is of length

$$\sqrt{1 + \alpha^2 + (1 + \alpha)^2} = \sqrt{2(1 + \alpha + \alpha^2)},$$

The angle φ between the two lines (which both pass through $(0, 0, 0)$) is given by

$$\cos \varphi = \frac{(1, \alpha_1, -(1 + \alpha_1)) \cdot (1, \alpha_2, -(1 + \alpha_2))}{\sqrt{2(1 + \alpha_1 + \alpha_1^2)} \sqrt{2(1 + \alpha_2 + \alpha_2^2)}} = \frac{1}{2} \cdot \frac{1 + \alpha_1 \alpha_2 + (1 + \alpha_1)(1 + \alpha_2)}{\sqrt{(1 + \alpha_1 + \alpha_1^2)(1 + \alpha_2 + \alpha_2^2)}}.$$

Here the numerator is

$$\begin{aligned} 1 + \alpha_1 \alpha_2 + (1 + \alpha_1)(1 + \alpha_2) &= 2 + (\alpha_1 + \alpha_2) + 2\alpha_1 \alpha_2 = 2 + \frac{ab + c^2}{bc} + 2 \frac{a}{b} \\ &= \frac{1}{bc} \{2bc - (b + c)b + c^2 - 2(b + c)c\} = -\frac{1}{bc} (b^2 + bc + c^2), \end{aligned}$$

and the radicand is

$$\begin{aligned} &(1 + \alpha_1 + \alpha_1^2)(1 + \alpha_2 + \alpha_2^2) \\ &= 1 + \alpha_1 + \alpha_2 + \alpha_1^2 + \alpha_2^2 + \alpha_1 \alpha_2 + \alpha_1 + \alpha_2^2 + \alpha_1^2 \alpha_2 + \alpha_1^2 \alpha_2^2 \\ &= 1 + (\alpha_1 + \alpha_2) + (\alpha_1 + \alpha_2)^2 - \alpha_1 \alpha_2 + \alpha_1 \alpha_2 (\alpha_1 + \alpha_2) + (\alpha_1 \alpha_2)^2 \\ &= 1 + \frac{ab + c^2}{bc} + \left(\frac{ab + c^2}{bc}\right)^2 - \frac{a}{b} + \frac{a}{b} \cdot \frac{ab + c^2}{bc} + \frac{a^2}{b^2} \\ &= \frac{1}{b^2 c^2} \{b^2 c^2 + ab^2 c + bc^3 + a^2 b^2 + 2abc^2 + c^4 - abc^2 + a^2 bc + ac^3 + a^2 c^2\} \\ &= \frac{1}{b^2 c^2} \{b^2 c^2 + bc^3 + c^4 + a(b^2 c + 2bc^2 - bc^2 + c^3) + a^2(b^2 + bc + c^2)\} \\ &= \frac{1}{b^2 c^2} \{c^2(b^2 + bc + c^2) + ac(b^2 + bc + c^2) + a^2(b^2 + bc + c^2)\} \\ &= \frac{1}{b^2 c^2} (b^2 + bc + c^2)(c^2 + ac + a^2) = \frac{1}{b^2 c^2} (b^2 + bc + c^2)(c^2 + (-b - c)(-b)) \\ &= \frac{1}{b^2 c^2} (b^2 + bc + c^2)^2. \end{aligned}$$

Then by insertion

$$\cos \varphi = \frac{1}{2} \cdot \frac{1 + \alpha_1 \alpha_2 + (1 + \alpha_1)(1 + \alpha_2)}{\sqrt{(1 + \alpha_1 + \alpha_1^2)(1 + \alpha_2 + \alpha_2^2)}} = \frac{1}{2} \cdot \frac{-\frac{1}{bc} (b^2 + bc + c^2)}{\left| \frac{1}{bc} (b^2 + bc + c^2) \right|}.$$

Since $b^2 + bc + c^2 = \left(b + \frac{1}{2}c\right)^2 + \frac{3}{4}c^2 > 0$, we have

$$\cos \varphi = -\frac{1}{2} \frac{|bc|}{bc} = -\frac{1}{2} \frac{bc}{|bc|} = \begin{cases} \frac{1}{2}, & \text{if } bc < 0, \\ -\frac{1}{2}, & \text{if } bc > 0. \end{cases}$$

Hence $\varphi = \frac{\pi}{3}$, if $bc < 0$, and $\varphi = \frac{2\pi}{3}$ (or $-\frac{\pi}{3}$), if $bc > 0$.

If we do not include the sign of the angle we get $\varphi = \frac{\pi}{3}$.

Example 3.14 Indicate for each value of the constant k the type of the conic \mathcal{F} , which is given by the equation

$$x^2 + (4 - k^2)y^2 + k(2 - k)z^2 = 2k,$$

and find in particular those values of k , for which \mathcal{F} is a surface of revolution. Finally, think about if it makes sense to put k equal to $+\infty$ or $-\infty$.

A Conics.

D Discuss the sign of the coefficients and then consider the various cases.

I By considering the signs we get the scheme

	$k < -2$	$k = -2$	$-2 < k < 0$	$k = 0$	$0 < k < 2$	$k = 2$	$k > 2$
$4 - k^2$	-	0	+	+	+	0	-
$k(2 - k)$	-	-	-	0	+	0	-
$2k$	-	-	-	0	+	+	+
	1	2	3	4	5	6	7

1) When $k < -2$, we get the canonical form (notice the absolute values)

$$\begin{aligned} 1 &= -\frac{1}{2|k|}x^2 + \left|\frac{4 - k^2}{2k}\right|y^2 + \left|\frac{2 - k}{2}\right|z^2 \\ &= -\frac{1}{2|k|}x^2 + \frac{4 - k^2}{2k}y^2 + \frac{2 - k}{2}z^2. \end{aligned}$$

Since we have 2 plus and 1 minus we conclude that we have an *hyperboloid with one sheet*.

2) When $k = -2$, the equation is written

$$x^2 - 8z^2 = -4, \quad \text{dvs.} \quad -\left(\frac{x}{2}\right)^2 + \left(\frac{z}{1/\sqrt{2}}\right)^2 = 1,$$

which describes an *hyperbolic cylindrical surface*.

3) When $-2 < k < 0$, the canonical form becomes

$$-\frac{1}{2|k|}x^2 - \left|\frac{4 - k^2}{2k}\right|y^2 + \left|\frac{2 - k}{2}\right|z^2 = 1.$$

With 1 plus and 2 minus we conclude that we get an *hyperboloid with two sheets*.

4) When $k = 0$, the equation is written

$$x^2 + 4y^2 = 0,$$

which is satisfied for the Z axis. (Degenerated “surface of revolution”).

5) When $0 < k < 2$, we rewrite to the canonical form

$$\left|\frac{1}{2k}\right|x^2 + \left|\frac{4 - k^2}{2k}\right|y^2 + \left|\frac{2 - k}{2}\right|z^2 = 1.$$

With 3 plus we get an *ellipsoid*.

6) When $k = 2$, the equation is written

$$x^2 = 4,$$

which describes two planes $x = \pm 2$, parallel to the YZ plane.

7) When $k > 2$, we get

$$\left| \frac{1}{2k} \right| x^2 - \left| \frac{4 - k^2}{2k} \right| y^2 - \left| \frac{2 - k}{k} \right| z^2 = 1.$$

With 1 plus and 2 minus we see that we get an *hyperboloid with two sheets*.

We obtain surfaces of revolution when

- 1) $x^2 + (4 - k^2)y^2 = x^2 + y^2$, i.e. when $k = \pm\sqrt{3}$.
 - 2) $x^2 + k(2 - k)z^2 = x^2 + z^2$, i.e. when $k = 1$.
 - 3) $4 - k^2 = k(2 - k)$, i.e. $k = 2$, which however produces degenerated surfaces of revolution.
 - 4) $k = 0$ gives Z axis as the degenerated "surface of revolution".
- 1) When $k = -\sqrt{3}$ we are in case 3., so we have an *hyperboloid of revolution with two sheets* where the Z axis is the axis of revolution.
 - 2) When $k = 0$ we are in case 4., which is the degenerated case of the Z axis. The Z axis is clearly the axis of revolution.
 - 3) When $k = 1$ we are in case 5., and we get an *ellipsoid of revolution* with the Y axis as the axis of revolution.
 - 4) When $k = \sqrt{3}$ we are again in case 5., so we get an *ellipsoid of revolution* with the Z axis as the axis of revolution.
 - 5) When $k = 2$ we are in the degenerated case 6. The two planes have clearly the X axis as the axis of revolution.

When $k \neq 0$, we get by dividing by $-k^2$ that

$$-\frac{1}{k^2} x^2 + \left(1 - \frac{4}{k^2}\right) y^2 + \left(1 - \frac{2}{k}\right) z^2 = -\frac{2}{k}.$$

Then it follows immediately by taking the limits $k \rightarrow +\infty$ or $k \rightarrow -\infty$,

$$y^2 + z^2 = 0,$$

so $y = z = 0$, while x is free. Therefore, by taking the limits we get the X axis

Example 3.15 The surfaces \mathcal{F}_1 and \mathcal{F}_2 are given by the equations

$$x^2 + 2y^2 = z + 1, \quad x^2 + 2y^2 = -1 + 3z^2.$$

- 1) Indicate the type and the top point(s) of both \mathcal{F}_1 and \mathcal{F}_2 .
- 2) Prove that the intersection $\mathcal{F}_1 \cap \mathcal{F}_2$ consists of two ellipses, lying in planes, which are parallel to the (X, Y) plane.

A Conics and conic sections.

D In 1) we just reformulate the equations to the canonical form. In 2) we first eliminate $x^2 + 2y^2$ in order to get an equation in z . Then insert the solutions in z into one of the original expressions.

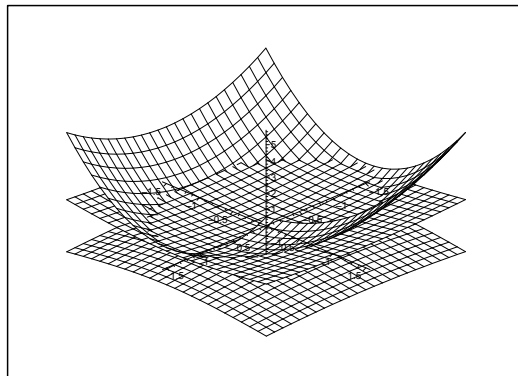


Figure 3.38: The surfaces \mathcal{F}_1 and \mathcal{F}_2 .

I 1) If we put $z_1 = z + 1$, we see that the equation of the surface \mathcal{F}_1 can be written in its canonical form

$$\frac{x^2}{1^2} + \frac{y^2}{\left(\frac{1}{\sqrt{2}}\right)^2} = z_1,$$

which shows that \mathcal{F}_1 is an elliptic paraboloid with top point $(0, 0, -1)$.

Then the equation of \mathcal{F}_2 is written in the following way:

$$-\frac{x^2}{1^2} - \frac{y^2}{\left(\frac{1}{\sqrt{2}}\right)^2} + \frac{z^2}{\left(\frac{1}{\sqrt{3}}\right)^2} = 1.$$

This equation describes an hyperboloid with two sheets. The top points are

$$\left(0, 0, \pm \frac{1}{\sqrt{3}}\right).$$

- 2) The equation of the intersection is obtained by eliminating the common expression $x^2 + 2y^2$ in (x, y) . This gives

$$z + 1 = -1 + 3z^2, \quad \text{i.e.} \quad 3z^2 - 2 - 2 = 3(z - 1) \left(z + \frac{2}{3} \right) = 0.$$

The solutions are $z = 1$ and $z = -\frac{2}{3}$, so the intersection curves lie in these two planes which are parallel to the (X, Y) plane.

- a) When we put $z = 1$, we get $x^2 + 2y^2 = 2$, which in its canonical form becomes

$$\frac{x^2}{(\sqrt{2})^2} + \frac{y^2}{1^2} = 1.$$

This is an equation of an ellipse in the plane $z = 1$ of centrum $(0, 0)$ and half axes $\sqrt{2}$ and 1.

- b) If we put $z = -\frac{2}{3}$, we get $x^2 + 2y^2 = \frac{1}{3}$, which is written in its canonical form in the following way

$$\frac{x^2}{\left(\frac{1}{\sqrt{3}}\right)^2} + \frac{y^2}{\left(\frac{1}{\sqrt{6}}\right)^2} = 1.$$

This is an equation of an ellipse in the plane $z = -\frac{2}{3}$ of centre $(0, 0)$ and half axes $\frac{1}{\sqrt{3}}$ and $\frac{1}{\sqrt{6}}$.

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4 Formulæ

Some of the following formulæ can be assumed to be known from high school. It is highly recommended that one *learns most of these formulæ in this appendix by heart.*

4.1 Squares etc.

The following simple formulæ occur very frequently in the most different situations.

$$\begin{aligned} (a+b)^2 &= a^2 + b^2 + 2ab, & a^2 + b^2 + 2ab &= (a+b)^2, \\ (a-b)^2 &= a^2 + b^2 - 2ab, & a^2 + b^2 - 2ab &= (a-b)^2, \\ (a+b)(a-b) &= a^2 - b^2, & a^2 - b^2 &= (a+b)(a-b), \\ (a+b)^2 &= (a-b)^2 + 4ab, & (a-b)^2 &= (a+b)^2 - 4ab. \end{aligned}$$

4.2 Powers etc.

Logarithm:

$$\begin{aligned} \ln |xy| &= \ln |x| + \ln |y|, & x, y &\neq 0, \\ \ln \left| \frac{x}{y} \right| &= \ln |x| - \ln |y|, & x, y &\neq 0, \\ \ln |x^r| &= r \ln |x|, & x &\neq 0. \end{aligned}$$

Power function, fixed exponent:

$$\begin{aligned} (xy)^r &= x^r \cdot y^r, x, y > 0 & (\text{extensions for some } r), \\ \left(\frac{x}{y} \right)^r &= \frac{x^r}{y^r}, x, y > 0 & (\text{extensions for some } r). \end{aligned}$$

Exponential, fixed base:

$$\begin{aligned} a^x \cdot a^y &= a^{x+y}, a > 0 & (\text{extensions for some } x, y), \\ (a^x)^y &= a^{xy}, a > 0 & (\text{extensions for some } x, y), \\ a^{-x} &= \frac{1}{a^x}, a > 0, & (\text{extensions for some } x), \\ \sqrt[n]{a} &= a^{1/n}, a \geq 0, & n \in \mathbb{N}. \end{aligned}$$

Square root:

$$\sqrt{x^2} = |x|, \quad x \in \mathbb{R}.$$

Remark 4.1 It happens quite frequently that students make errors when they try to apply these rules. They must be mastered! In particular, as one of my friends once put it: “If you can master the square root, you can master everything in mathematics!” Notice that this innocent looking square root is one of the most difficult operations in Calculus. Do not forget the *absolute value!* \diamond

4.3 Differentiation

Here are given the well-known rules of differentiation together with some rearrangements which sometimes may be easier to use:

$$\{f(x) \pm g(x)\}' = f'(x) \pm g'(x),$$

$$\{f(x)g(x)\}' = f'(x)g(x) + f(x)g'(x) = f(x)g(x) \left\{ \frac{f'(x)}{f(x)} + \frac{g'(x)}{g(x)} \right\},$$

where the latter rearrangement presupposes that $f(x) \neq 0$ and $g(x) \neq 0$.

If $g(x) \neq 0$, we get the usual formula known from high school

$$\left\{ \frac{f(x)}{g(x)} \right\}' = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}.$$

It is often more convenient to compute this expression in the following way:

$$\left\{ \frac{f(x)}{g(x)} \right\}' = \frac{d}{dx} \left\{ f(x) \cdot \frac{1}{g(x)} \right\} = \frac{f'(x)}{g(x)} - \frac{f(x)g'(x)}{g(x)^2} = \frac{f(x)}{g(x)} \left\{ \frac{f'(x)}{f(x)} - \frac{g'(x)}{g(x)} \right\},$$

where the former expression often is *much easier* to use in practice than the usual formula from high school, and where the latter expression again presupposes that $f(x) \neq 0$ and $g(x) \neq 0$. Under these assumptions we see that the formulæ above can be written

$$\frac{\{f(x)g(x)\}'}{f(x)g(x)} = \frac{f'(x)}{f(x)} + \frac{g'(x)}{g(x)},$$

$$\frac{\{f(x)/g(x)\}'}{f(x)/g(x)} = \frac{f'(x)}{f(x)} - \frac{g'(x)}{g(x)}.$$

Since

$$\frac{d}{dx} \ln |f(x)| = \frac{f'(x)}{f(x)}, \quad f(x) \neq 0,$$

we also name these the *logarithmic derivatives*.

Finally, we mention the rule of **differentiation of a composite function**

$$\{f(\varphi(x))\}' = f'(\varphi(x)) \cdot \varphi'(x).$$

We first differentiate the function itself; then the insides. This rule is a 1-dimensional version of the so-called *Chain rule*.

4.4 Special derivatives.

Power like:

$$\frac{d}{dx} (x^\alpha) = \alpha \cdot x^{\alpha-1}, \quad \text{for } x > 0, \text{ (extensions for some } \alpha).$$

$$\frac{d}{dx} \ln |x| = \frac{1}{x}, \quad \text{for } x \neq 0.$$

Exponential like:

$$\frac{d}{dx} \exp x = \exp x, \quad \text{for } x \in \mathbb{R},$$

$$\frac{d}{dx} (a^x) = \ln a \cdot a^x, \quad \text{for } x \in \mathbb{R} \text{ and } a > 0.$$

Trigonometric:

$$\frac{d}{dx} \sin x = \cos x, \quad \text{for } x \in \mathbb{R},$$

$$\frac{d}{dx} \cos x = -\sin x, \quad \text{for } x \in \mathbb{R},$$

$$\frac{d}{dx} \tan x = 1 + \tan^2 x = \frac{1}{\cos^2 x}, \quad \text{for } x \neq \frac{\pi}{2} + p\pi, p \in \mathbb{Z},$$

$$\frac{d}{dx} \cot x = -(1 + \cot^2 x) = -\frac{1}{\sin^2 x}, \quad \text{for } x \neq p\pi, p \in \mathbb{Z}.$$

Hyperbolic:

$$\frac{d}{dx} \sinh x = \cosh x, \quad \text{for } x \in \mathbb{R},$$

$$\frac{d}{dx} \cosh x = \sinh x, \quad \text{for } x \in \mathbb{R},$$

$$\frac{d}{dx} \tanh x = 1 - \tanh^2 x = \frac{1}{\cosh^2 x}, \quad \text{for } x \in \mathbb{R},$$

$$\frac{d}{dx} \coth x = 1 - \coth^2 x = -\frac{1}{\sinh^2 x}, \quad \text{for } x \neq 0.$$

Inverse trigonometric:

$$\frac{d}{dx} \operatorname{Arcsin} x = \frac{1}{\sqrt{1-x^2}}, \quad \text{for } x \in]-1, 1[,$$

$$\frac{d}{dx} \operatorname{Arccos} x = -\frac{1}{\sqrt{1-x^2}}, \quad \text{for } x \in]-1, 1[,$$

$$\frac{d}{dx} \operatorname{Arctan} x = \frac{1}{1+x^2}, \quad \text{for } x \in \mathbb{R},$$

$$\frac{d}{dx} \operatorname{Arccot} x = \frac{1}{1+x^2}, \quad \text{for } x \in \mathbb{R}.$$

Inverse hyperbolic:

$$\frac{d}{dx} \operatorname{Arsinh} x = \frac{1}{\sqrt{x^2+1}}, \quad \text{for } x \in \mathbb{R},$$

$$\frac{d}{dx} \operatorname{Arcosh} x = \frac{1}{\sqrt{x^2-1}}, \quad \text{for } x \in]1, +\infty[,$$

$$\frac{d}{dx} \operatorname{Artanh} x = \frac{1}{1-x^2}, \quad \text{for } |x| < 1,$$

$$\frac{d}{dx} \operatorname{Arcoth} x = \frac{1}{1-x^2}, \quad \text{for } |x| > 1.$$

Remark 4.2 The derivative of the trigonometric and the hyperbolic functions are to some extent exponential like. The derivatives of the inverse trigonometric and inverse hyperbolic functions are power like, because we include the logarithm in this class. \diamond

4.5 Integration

The most obvious rules are dealing with linearity

$$\int \{f(x) + \lambda g(x)\} dx = \int f(x) dx + \lambda \int g(x) dx, \quad \text{where } \lambda \in \mathbb{R} \text{ is a constant,}$$

and with the fact that differentiation and integration are “inverses to each other”, i.e. modulo some arbitrary constant $c \in \mathbb{R}$, which often tacitly is missing,

$$\int f'(x) dx = f(x).$$

If we in the latter formula replace $f(x)$ by the product $f(x)g(x)$, we get by reading from the right to the left and then differentiating the product,

$$f(x)g(x) = \int \{f(x)g(x)\}' dx = \int f'(x)g(x) dx + \int f(x)g'(x) dx.$$

Hence, by a rearrangement

The rule of partial integration:

$$\int f'(x)g(x) dx = f(x)g(x) - \int f(x)g'(x) dx.$$

The differentiation is moved from one factor of the integrand to the other one by changing the sign and adding the term $f(x)g(x)$.

Remark 4.3 This technique was earlier used a lot, but is almost forgotten these days. It must be revived, because MAPLE and pocket calculators apparently do not know it. It is possible to construct examples where these devices cannot give the exact solution, unless you first perform a partial integration yourself. \diamond

Remark 4.4 This method can also be used when we estimate integrals which cannot be directly calculated, because the antiderivative is not contained in e.g. the catalogue of MAPLE. The idea is by a succession of partial integrations to make the new integrand smaller. \diamond

Integration by substitution:

If the integrand has the special structure $f(\varphi(x)) \cdot \varphi'(x)$, then one can change the variable to $y = \varphi(x)$:

$$\int f(\varphi(x)) \cdot \varphi'(x) dx = \int f(\varphi(x)) d\varphi(x) = \int_{y=\varphi(x)} f(y) dy.$$

Integration by a monotonous substitution:

If $\varphi(y)$ is a *monotonous* function, which maps the y -interval *one-to-one* onto the x -interval, then

$$\int f(x) dx = \int_{y=\varphi^{-1}(x)} f(\varphi(y))\varphi'(y) dy.$$

Remark 4.5 This rule is usually used when we have some “ugly” term in the integrand $f(x)$. The idea is to put this ugly term equal to $y = \varphi^{-1}(x)$. When e.g. x occurs in $f(x)$ in the form \sqrt{x} , we put $y = \varphi^{-1}(x) = \sqrt{x}$, hence $x = \varphi(y) = y^2$ and $\varphi'(y) = 2y$. \diamond

4.6 Special antiderivatives

Power like:

$$\int \frac{1}{x} dx = \ln |x|, \quad \text{for } x \neq 0. \text{ (Do not forget the numerical value!)}$$

$$\int x^\alpha dx = \frac{1}{\alpha + 1} x^{\alpha+1}, \quad \text{for } \alpha \neq -1,$$

$$\int \frac{1}{1+x^2} dx = \text{Arctan } x, \quad \text{for } x \in \mathbb{R},$$

$$\int \frac{1}{1-x^2} dx = \frac{1}{2} \ln \left| \frac{1+x}{1-x} \right|, \quad \text{for } x \neq \pm 1,$$

$$\int \frac{1}{1-x^2} dx = \text{Artanh } x, \quad \text{for } |x| < 1,$$

$$\int \frac{1}{1-x^2} dx = \text{Arcoth } x, \quad \text{for } |x| > 1,$$

$$\int \frac{1}{\sqrt{1-x^2}} dx = \text{Arcsin } x, \quad \text{for } |x| < 1,$$

$$\int \frac{1}{\sqrt{1-x^2}} dx = -\text{Arccos } x, \quad \text{for } |x| < 1,$$

$$\int \frac{1}{\sqrt{x^2+1}} dx = \text{Arsinh } x, \quad \text{for } x \in \mathbb{R},$$

$$\int \frac{1}{\sqrt{x^2+1}} dx = \ln(x + \sqrt{x^2+1}), \quad \text{for } x \in \mathbb{R},$$

$$\int \frac{x}{\sqrt{x^2-1}} dx = \sqrt{x^2-1}, \quad \text{for } x \in \mathbb{R},$$

$$\int \frac{1}{\sqrt{x^2-1}} dx = \text{Arcosh } x, \quad \text{for } x > 1,$$

$$\int \frac{1}{\sqrt{x^2-1}} dx = \ln |x + \sqrt{x^2-1}|, \quad \text{for } x > 1 \text{ eller } x < -1.$$

There is an error in the programs of the pocket calculators TI-92 and TI-89. The *numerical signs* are missing. It is obvious that $\sqrt{x^2-1} < |x|$ so if $x < -1$, then $x + \sqrt{x^2-1} < 0$. Since you cannot take the logarithm of a negative number, these pocket calculators will give an error message.

Exponential like:

$$\int \exp x \, dx = \exp x, \quad \text{for } x \in \mathbb{R},$$

$$\int a^x \, dx = \frac{1}{\ln a} \cdot a^x, \quad \text{for } x \in \mathbb{R}, \text{ and } a > 0, a \neq 1.$$

Trigonometric:

$$\int \sin x \, dx = -\cos x, \quad \text{for } x \in \mathbb{R},$$

$$\int \cos x \, dx = \sin x, \quad \text{for } x \in \mathbb{R},$$

$$\int \tan x \, dx = -\ln |\cos x|, \quad \text{for } x \neq \frac{\pi}{2} + p\pi, \quad p \in \mathbb{Z},$$

$$\int \cot x \, dx = \ln |\sin x|, \quad \text{for } x \neq p\pi, \quad p \in \mathbb{Z},$$

$$\int \frac{1}{\cos x} \, dx = \frac{1}{2} \ln \left(\frac{1 + \sin x}{1 - \sin x} \right), \quad \text{for } x \neq \frac{\pi}{2} + p\pi, \quad p \in \mathbb{Z},$$

$$\int \frac{1}{\sin x} \, dx = \frac{1}{2} \ln \left(\frac{1 - \cos x}{1 + \cos x} \right), \quad \text{for } x \neq p\pi, \quad p \in \mathbb{Z},$$

$$\int \frac{1}{\cos^2 x} \, dx = \tan x, \quad \text{for } x \neq \frac{\pi}{2} + p\pi, \quad p \in \mathbb{Z},$$

$$\int \frac{1}{\sin^2 x} \, dx = -\cot x, \quad \text{for } x \neq p\pi, \quad p \in \mathbb{Z}.$$

Hyperbolic:

$$\int \sinh x \, dx = \cosh x, \quad \text{for } x \in \mathbb{R},$$

$$\int \cosh x \, dx = \sinh x, \quad \text{for } x \in \mathbb{R},$$

$$\int \tanh x \, dx = \ln \cosh x, \quad \text{for } x \in \mathbb{R},$$

$$\int \coth x \, dx = \ln |\sinh x|, \quad \text{for } x \neq 0,$$

$$\int \frac{1}{\cosh x} \, dx = \operatorname{Arctan}(\sinh x), \quad \text{for } x \in \mathbb{R},$$

$$\int \frac{1}{\cosh x} \, dx = 2 \operatorname{Arctan}(e^x), \quad \text{for } x \in \mathbb{R},$$

$$\int \frac{1}{\sinh x} \, dx = \frac{1}{2} \ln \left(\frac{\cosh x - 1}{\cosh x + 1} \right), \quad \text{for } x \neq 0,$$

$$\int \frac{1}{\sinh x} dx = \ln \left| \frac{e^x - 1}{e^x + 1} \right|, \quad \text{for } x \neq 0,$$

$$\int \frac{1}{\cosh^2 x} dx = \tanh x, \quad \text{for } x \in \mathbb{R},$$

$$\int \frac{1}{\sinh^2 x} dx = -\operatorname{coth} x, \quad \text{for } x \neq 0.$$

4.7 Trigonometric formulæ

The trigonometric formulæ are closely connected with circular movements. Thus $(\cos u, \sin u)$ are the coordinates of a point P on the unit circle corresponding to the angle u , cf. figure A.1. This geometrical interpretation is used from time to time.

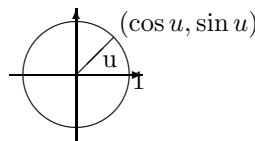


Figure 4.1: The unit circle and the trigonometric functions.

The fundamental trigonometric relation:

$$\cos^2 u + \sin^2 u = 1, \quad \text{for } u \in \mathbb{R}.$$

Using the previous geometric interpretation this means according to *Pythagoras's theorem*, that the point P with the coordinates $(\cos u, \sin u)$ always has distance 1 from the origo $(0, 0)$, i.e. it is lying on the boundary of the circle of centre $(0, 0)$ and radius $\sqrt{1} = 1$.

Connection to the complex exponential function:

The *complex exponential* is for imaginary arguments defined by

$$\exp(iu) := \cos u + i \sin u.$$

It can be checked that the usual functional equation for \exp is still valid for complex arguments. In other word: The definition above is extremely conveniently chosen.

By using the definition for $\exp(iu)$ and $\exp(-iu)$ it is easily seen that

$$\cos u = \frac{1}{2}(\exp(iu) + \exp(-iu)),$$

$$\sin u = \frac{1}{2i}(\exp(iu) - \exp(-iu)),$$

Moivre's formula: We get by expressing $\exp(inu)$ in two different ways:

$$\exp(inu) = \cos nu + i \sin nu = (\cos u + i \sin u)^n.$$

Example 4.1 If we e.g. put $n = 3$ into Moivre's formula, we obtain the following typical application,

$$\begin{aligned} \cos(3u) + i \sin(3u) &= (\cos u + i \sin u)^3 \\ &= \cos^3 u + 3i \cos^2 u \cdot \sin u + 3i^2 \cos u \cdot \sin^2 u + i^3 \sin^3 u \\ &= \{\cos^3 u - 3 \cos u \cdot \sin^2 u\} + i\{3 \cos^2 u \cdot \sin u - \sin^3 u\} \\ &= \{4 \cos^3 u - 3 \cos u\} + i\{3 \sin u - 4 \sin^3 u\} \end{aligned}$$

When this is split into the real- and imaginary parts we obtain

$$\cos 3u = 4 \cos^3 u - 3 \cos u, \quad \sin 3u = 3 \sin u - 4 \sin^3 u. \quad \diamond$$

Addition formulæ:

$$\sin(u + v) = \sin u \cos v + \cos u \sin v,$$

$$\sin(u - v) = \sin u \cos v - \cos u \sin v,$$

$$\cos(u + v) = \cos u \cos v - \sin u \sin v,$$

$$\cos(u - v) = \cos u \cos v + \sin u \sin v.$$

Products of trigonometric functions to a sum:

$$\sin u \cos v = \frac{1}{2} \sin(u + v) + \frac{1}{2} \sin(u - v),$$

$$\cos u \sin v = \frac{1}{2} \sin(u + v) - \frac{1}{2} \sin(u - v),$$

$$\sin u \sin v = \frac{1}{2} \cos(u - v) - \frac{1}{2} \cos(u + v),$$

$$\cos u \cos v = \frac{1}{2} \cos(u - v) + \frac{1}{2} \cos(u + v).$$

Sums of trigonometric functions to a product:

$$\sin u + \sin v = 2 \sin \left(\frac{u + v}{2} \right) \cos \left(\frac{u - v}{2} \right),$$

$$\sin u - \sin v = 2 \cos \left(\frac{u + v}{2} \right) \sin \left(\frac{u - v}{2} \right),$$

$$\cos u + \cos v = 2 \cos \left(\frac{u + v}{2} \right) \cos \left(\frac{u - v}{2} \right),$$

$$\cos u - \cos v = -2 \sin \left(\frac{u + v}{2} \right) \sin \left(\frac{u - v}{2} \right).$$

Formulæ of halving and doubling the angle:

$$\sin 2u = 2 \sin u \cos u,$$

$$\cos 2u = \cos^2 u - \sin^2 u = 2 \cos^2 u - 1 = 1 - 2 \sin^2 u,$$

$$\sin \frac{u}{2} = \pm \sqrt{\frac{1 - \cos u}{2}} \quad \text{followed by a discussion of the sign,}$$

$$\cos \frac{u}{2} = \pm \sqrt{\frac{1 + \cos u}{2}} \quad \text{followed by a discussion of the sign,}$$

4.8 Hyperbolic formulæ

These are very much like the trigonometric formulæ, and if one knows a little of Complex Function Theory it is realized that they are actually identical. The structure of this section is therefore the same as for the trigonometric formulæ. The reader should compare the two sections concerning similarities and differences.

The fundamental relation:

$$\cosh^2 x - \sinh^2 x = 1.$$

Definitions:

$$\cosh x = \frac{1}{2} (\exp(x) + \exp(-x)), \quad \sinh x = \frac{1}{2} (\exp(x) - \exp(-x)).$$

“Moivre’s formula”:

$$\exp(x) = \cosh x + \sinh x.$$

This is trivial and only rarely used. It has been included to show the analogy.

Addition formulæ:

$$\sinh(x + y) = \sinh(x) \cosh(y) + \cosh(x) \sinh(y),$$

$$\sinh(x - y) = \sinh(x) \cosh(y) - \cosh(x) \sinh(y),$$

$$\cosh(x + y) = \cosh(x) \cosh(y) + \sinh(x) \sinh(y),$$

$$\cosh(x - y) = \cosh(x) \cosh(y) - \sinh(x) \sinh(y).$$

Formulæ of halving and doubling the argument:

$$\sinh(2x) = 2 \sinh(x) \cosh(x),$$

$$\cosh(2x) = \cosh^2(x) + \sinh^2(x) = 2 \cosh^2(x) - 1 = 2 \sinh^2(x) + 1,$$

$$\sinh\left(\frac{x}{2}\right) = \pm \sqrt{\frac{\cosh(x) - 1}{2}} \quad \text{followed by a discussion of the sign,}$$

$$\cosh\left(\frac{x}{2}\right) = \sqrt{\frac{\cosh(x) + 1}{2}}.$$

Inverse hyperbolic functions:

$$\operatorname{Arsinh}(x) = \ln\left(x + \sqrt{x^2 + 1}\right), \quad x \in \mathbb{R},$$

$$\operatorname{Arcosh}(x) = \ln\left(x + \sqrt{x^2 - 1}\right), \quad x \geq 1,$$

$$\operatorname{Artanh}(x) = \frac{1}{2} \ln\left(\frac{1+x}{1-x}\right), \quad |x| < 1,$$

$$\operatorname{Arcoth}(x) = \frac{1}{2} \ln\left(\frac{x+1}{x-1}\right), \quad |x| > 1.$$

4.9 Complex transformation formulæ

$$\begin{aligned}\cos(ix) &= \cosh(x), & \cosh(ix) &= \cos(x), \\ \sin(ix) &= i \sinh(x), & \sinh(ix) &= i \sin x.\end{aligned}$$

4.10 Taylor expansions

The generalized binomial coefficients are defined by

$$\binom{\alpha}{n} := \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{1\cdot 2\cdots n},$$

with n factors in the numerator and the denominator, supplied with

$$\binom{\alpha}{0} := 1.$$

The Taylor expansions for *standard functions* are divided into *power like* (the radius of convergency is finite, i.e. = 1 for the standard series) and *exponential like* (the radius of convergency is infinite).

Power like:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \quad |x| < 1,$$

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n, \quad |x| < 1,$$

$$(1+x)^n = \sum_{j=0}^n \binom{n}{j} x^j, \quad n \in \mathbb{N}, x \in \mathbb{R},$$

$$(1+x)^\alpha = \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n, \quad \alpha \in \mathbb{R} \setminus \mathbb{N}, |x| < 1,$$

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}, \quad |x| < 1,$$

$$\text{Arctan}(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}, \quad |x| < 1.$$

Exponential like:

$$\exp(x) = \sum_{n=0}^{\infty} \frac{1}{n!} x^n, \quad x \in \mathbb{R}$$

$$\exp(-x) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{n!} x^n, \quad x \in \mathbb{R}$$

$$\sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+1)!} x^{2n+1}, \quad x \in \mathbb{R},$$

$$\sinh(x) = \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} x^{2n+1}, \quad x \in \mathbb{R},$$

$$\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n)!} x^{2n}, \quad x \in \mathbb{R},$$

$$\cosh(x) = \sum_{n=0}^{\infty} \frac{1}{(2n)!} x^{2n}, \quad x \in \mathbb{R}.$$

4.11 Magnitudes of functions

We often have to compare functions for $x \rightarrow 0+$, or for $x \rightarrow \infty$. The simplest type of functions are therefore arranged in an hierarchy:

- 1) logarithms,
- 2) power functions,
- 3) exponential functions,
- 4) faculty functions.

When $x \rightarrow \infty$, a function from a higher class will always dominate a function from a lower class. More precisely:

A) A *power function* dominates a *logarithm* for $x \rightarrow \infty$:

$$\frac{(\ln x)^\beta}{x^\alpha} \rightarrow 0 \quad \text{for } x \rightarrow \infty, \quad \alpha, \beta > 0.$$

B) An *exponential* dominates a *power function* for $x \rightarrow \infty$:

$$\frac{x^\alpha}{a^x} \rightarrow 0 \quad \text{for } x \rightarrow \infty, \quad \alpha, a > 1.$$

C) The *faculty function* dominates an *exponential* for $n \rightarrow \infty$:

$$\frac{a^n}{n!} \rightarrow 0, \quad n \rightarrow \infty, \quad n \in \mathbb{N}, \quad a > 0.$$

D) When $x \rightarrow 0+$ we also have that a *power function* dominates the *logarithm*:

$$x^\alpha \ln x \rightarrow 0-, \quad \text{for } x \rightarrow 0+, \quad \alpha > 0.$$

Index

- absolute value 162
- acceleration 490
- addition 22
- affinity factor 173
- Ampère-Laplace law 1671
- Ampère-Maxwell's law 1678
- Ampère's law 1491, 1498, 1677, 1678, 1833
- Ampère's law for the magnetic field 1674
- angle 19
- angular momentum 886
- angular set 84
- annulus 176, 243
- anticommutative product 26
- antiderivative 301, 847
- approximating polynomial 304, 322, 326, 336, 404, 488, 632, 662
- approximation in energy 734
- Archimedes's spiral 976, 1196
- Archimedes's theorem 1818
- area 887, 1227, 1229, 1543
- area element 1227
- area of a graph 1230
- asteroid 1215
- asymptote 51
- axial moment 1910
- axis of revolution 181
- axis of rotation 34, 886
- axis of symmetry 49, 50, 53

- barycentre 885, 1910
- basis 22
- bend 486
- bijective map 153
- body of revolution 43, 1582, 1601
- boundary 37–39
- boundary curve 182
- boundary curve of a surface 182
- boundary point 920
- boundary set 21
- bounded map 153
- bounded set 41
- branch 184
- branch of a curve 492
- Brownian motion 884

- cardioid 972, 973, 1199, 1705

- Cauchy-Schwarz's inequality 23, 24, 26
- centre of gravity 1108
- centre of mass 885
- centrum 66
- chain rule 305, 333, 352, 491, 503, 581, 1215, 1489, 1493, 1808
- change of parameter 174
- circle 49
- circular motion 19
- circulation 1487
- circulation theorem 1489, 1491
- circumference 86
- closed ball 38
- closed differential form 1492
- closed disc 86
- closed domain 176
- closed set 21
- closed surface 182, 184
- closure 39
- clothoid 1219
- colour code 890
- compact set 186, 580, 1813
- compact support 1813
- complex decomposition 69
- composite function 305
- conductivity of heat 1818
- cone 19, 35, 59, 251
- conic section 19, 47, 54, 239, 536
- conic sectional conic surface 59, 66
- connected set 175, 241
- conservation of electric charge 1548, 1817
- conservation of energy 1548, 1817
- conservation of mass 1548, 1816
- conservative force 1498, 1507
- conservative vector field 1489
- continuity equation 1548, 1569, 1767, 1817
- continuity 162, 186
- continuous curve 170, 483
- continuous extension 213
- continuous function 168
- continuous surfaces 177
- contraction 167
- convective term 492
- convex set 21, 22, 41, 89, 91, 175, 244
- coordinate function 157, 169
- coordinate space 19, 21

- Cornu's spiral 1219
 Coulomb field 1538, 1545, 1559, 1566, 1577
 Coulomb vector field 1585, 1670
 cross product 19, 163, 169, 1750
 cube 42, 82
 current density 1678, 1681
 current 1487, 1499
 curvature 1219
 curve 227
 curve length 1165
 curved space integral 1021
 cusp 486, 487, 489
 cycloid 233, 1215
 cylinder 34, 42, 43, 252
 cylinder of revolution 500
 cylindrical coordinates 15, 21, 34, 147, 181, 182, 289, 477, 573, 841, 1009, 1157, 1347, 1479, 1651, 1801
 cylindrical surface 180, 245, 247, 248, 499, 1230

 degree of trigonometric polynomial 67
 density 885
 density of charge 1548
 density of current 1548
 derivative 296
 derivative of inverse function 494
 Descartes's leaf 974
 dielectric constant 1669, 1670
 difference quotient 295
 differentiability 295
 differentiable function 295
 differentiable vector function 303
 differential 295, 296, 325, 382, 1740, 1741
 differential curves 171
 differential equation 369, 370, 398
 differential form 848
 differential of order p 325
 differential of vector function 303
 diffusion equation 1818
 dimension 1016
 direction 334
 direction vector 172
 directional derivative 317, 334, 375
 directrix 53
 Dirichlet/Neumann problem 1901
 displacement field 1670
 distribution of current 886
 divergence 1535, 1540, 1542, 1739, 1741, 1742
 divergence free vector field 1543

 dodecahedron 83
 domain 153, 176
 domain of a function 189
 dot product 19, 350, 1750
 double cone 252
 double point 171
 double vector product 27

 eccentricity 51
 eccentricity of ellipse 49
 eigenvalue 1906
 elasticity 885, 1398
 electric field 1486, 1498, 1679
 electrical dipole moment 885
 electromagnetic field 1679
 electromagnetic potentials 1819
 electromotive force 1498
 electrostatic field 1669
 element of area 887
 elementary chain rule 305
 elementary fraction 69
 ellipse 48–50, 92, 113, 173, 199, 227
 ellipsoid 56, 66, 110, 197, 254, 430, 436, 501, 538, 1107
 ellipsoid of revolution 111
 ellipsoidal disc 79, 199
 ellipsoidal surface 180
 elliptic cylindrical surface 60, 63, 66, 106
 elliptic paraboloid 60, 62, 66, 112, 247
 elliptic paraboloid of revolution 624
 energy 1498
 energy density 1548, 1818
 energy theorem 1921
 entropy 301
 Euclidean norm 162
 Euclidean space 19, 21, 22
 Euler's spiral 1219
 exact differential form 848
 exceptional point 594, 677, 920
 expansion point 327
 explicit given function 161
 extension map 153
 exterior 37–39
 exterior point 38
 extremum 580, 632

 Faraday-Henry law of electromagnetic induction 1676
 Fick's first law of diffusion 297

- Fick's law 1818
 field line 160
 final point 170
 fluid mechanics 491
 flux 1535, 1540, 1549
 focus 49, 51, 53
 force 1485
 Fourier's law 297, 1817
 function in several variables 154
 functional matrix 303
 fundamental theorem of vector analysis 1815
- Gaussian integral 938
 Gauß's law 1670
 Gauß's law for magnetism 1671
 Gauß's theorem 1499, 1535, 1540, 1549, 1580, 1718, 1724, 1737, 1746, 1747, 1749, 1751, 1817, 1818, 1889, 1890, 1913
 Gauß's theorem in \mathbb{R}^2 1543
 Gauß's theorem in \mathbb{R}^3 1543
 general chain rule 314
 general coordinates 1016
 general space integral 1020
 general Taylor's formula 325
 generalized spherical coordinates 21
 generating curve 499
 generator 66, 180
 geometrical analysis 1015
 global minimum 613
 gradient 295, 296, 298, 339, 847, 1739, 1741
 gradient field 631, 847, 1485, 1487, 1489, 1491, 1916
 gradient integral theorem 1489, 1499
 graph 158, 179, 499, 1229
 Green's first identity 1890
 Green's second identity 1891, 1895
 Green's theorem in the plane 1661, 1669, 1909
 Green's third identity 1896
 Green's third identity in the plane 1898
- half-plane 41, 42
 half-strip 41, 42
 half disc 85
 harmonic function 426, 427, 1889
 heat conductivity 297
 heat equation 1818
 heat flow 297
 height 42
 helix 1169, 1235
- Helmholtz's theorem 1815
 homogeneous function 1908
 homogeneous polynomial 339, 372
 Hopf's maximum principle 1905
 hyperbola 48, 50, 51, 88, 195, 217, 241, 255, 1290
 hyperbolic cylindrical surface 60, 63, 66, 105, 110
 hyperbolic paraboloid 60, 62, 66, 246, 534, 614, 1445
 hyperboloid 232, 1291
 hyperboloid of revolution 104
 hyperboloid of revolution with two sheets 111
 hyperboloid with one sheet 56, 66, 104, 110, 247, 255
 hyperboloid with two sheets 59, 66, 104, 110, 111, 255, 527
 hysteresis 1669
- identity map 303
 implicit given function 21, 161
 implicit function theorem 492, 503
 improper integral 1411
 improper surface integral 1421
 increment 611
 induced electric field 1675
 induction field 1671
 infinitesimal vector 1740
 infinity, signed 162
 infinity, unspecified 162
 initial point 170
 injective map 153
 inner product 23, 29, 33, 163, 168, 1750
 inspection 861
 integral 847
 integral over cylindrical surface 1230
 integral over surface of revolution 1232
 interior 37–40
 interior point 38
 intrinsic boundary 1227
 isolated point 39
 Jacobian 1353, 1355
- Kronecker symbol 23
- Laplace equation 1889
 Laplace force 1819
 Laplace operator 1743
 latitude 35
 length 23
 level curve 159, 166, 198, 492, 585, 600, 603

- level surface 198, 503
- limit 162, 219
- line integral 1018, 1163
- line segment 41
- Linear Algebra 627
- linear space 22
- local extremum 611
- logarithm 189
- longitude 35
- Lorentz condition 1824

- Maclaurin's trisectrix 973, 975
- magnetic circulation 1674
- magnetic dipole moment 886, 1821
- magnetic field 1491, 1498, 1679
- magnetic flux 1544, 1671, 1819
- magnetic force 1674
- magnetic induction 1671
- magnetic permeability of vacuum 1673
- magnostatic field 1671
- main theorems 185
- major semi-axis 49
- map 153
- MAPLE 55, 68, 74, 156, 171, 173, 341, 345, 350, 352–354, 356, 357, 360, 361, 363, 364, 366, 368, 374, 384–387, 391–393, 395–397, 401, 631, 899, 905–912, 914, 915, 917, 919, 922–924, 926, 934, 935, 949, 951, 954, 957–966, 968, 971–973, 975, 1032–1034, 1036, 1037, 1039, 1040, 1042, 1053, 1059, 1061, 1064, 1066–1068, 1070–1072, 1074, 1087, 1089, 1091, 1092, 1094, 1095, 1102, 1199, 1200
- matrix product 303
- maximal domain 154, 157
- maximum 382, 579, 612, 1916
- maximum value 922
- maximum-minimum principle for harmonic functions 1895
- Maxwell relation 302
- Maxwell's equations 1544, 1669, 1670, 1679, 1819
- mean value theorem 321, 884, 1276, 1490
- mean value theorem for harmonic functions 1892
- measure theory 1015
- Mechanics 15, 147, 289, 477, 573, 841, 1009, 1157, 1347, 1479, 1651, 1801, 1921
- meridian curve 181, 251, 499, 1232
- meridian half-plane 34, 35, 43, 181, 1055, 1057, 1081

- method of indefinite integration 859
- method of inspection 861
- method of radial integration 862
- minimum 186, 178, 579, 612, 1916
- minimum value 922
- minor semi-axis 49
- mmf 1674
- Möbius strip 185, 497
- Moivre's formula 122, 264, 452, 548, 818, 984, 1132, 1322, 1454, 1626, 1776, 1930
- monopole 1671
- multiple point 171

- nabla 296, 1739
- nabla calculus 1750
- nabla notation 1680
- natural equation 1215
- natural parametric description 1166, 1170
- negative definite matrix 627
- negative half-tangent 485
- neighbourhood 39
- neutral element 22
- Newton field 1538
- Newton-Raphson iteration formula 583
- Newton's second law 1921
- non-oriented surface 185
- norm 19, 23
- normal 1227
- normal derivative 1890
- normal plane 487
- normal vector 496, 1229

- octant 83
- Ohm's law 297
- open ball 38
- open domain 176
- open set 21, 39
- order of expansion 322
- order relation 579
- ordinary integral 1017
- orientation of a surface 182
- orientation 170, 172, 184, 185, 497
- oriented half line 172
- oriented line 172
- oriented line segment 172
- orthonormal system 23

- parabola 52, 53, 89–92, 195, 201, 229, 240, 241
- parabolic cylinder 613

- parabolic cylindrical surface 64, 66
- paraboloid of revolution 207, 613, 1435
- parallelepipedum 27, 42
- parameter curve 178, 496, 1227
- parameter domain 1227
- parameter of a parabola 53
- parametric description 170, 171, 178
- parfrac 71
- partial derivative 298
- partial derivative of second order 318
- partial derivatives of higher order 382
- partial differential equation 398, 402
- partial fraction 71
- Peano 483
- permeability 1671
- piecewise C^k -curve 484
- piecewise C^m -surface 495
- plane 179
- plane integral 21, 887
- point of contact 487
- point of expansion 304, 322
- point set 37
- Poisson's equation 1814, 1889, 1891, 1901
- polar coordinates 15, 19, 21, 30, 85, 88, 147, 163, 172, 213, 219, 221, 289, 347, 388, 390, 477, 573, 611, 646, 720, 740, 841, 936, 1009, 1016, 1157, 1165, 1347, 1479, 1651, 1801
- polar plane integral 1018
- polynomial 297
- positive definite matrix 627
- positive half-tangent 485
- positive orientation 173
- potential energy 1498
- pressure 1818
- primitive 1491
- primitive of gradient field 1493
- prism 42
- Probability Theory 15, 147, 289, 477, 573, 841, 1009, 1157, 1347, 1479, 1651, 1801
- product set 41
- projection 23, 157
- proper maximum 612, 618, 627
- proper minimum 612, 613, 618, 627
- pseudo-sphere 1434
- Pythagoras's theorem 23, 25, 30, 121, 451, 547, 817, 983, 1131, 1321, 1453, 1625, 1775, 1929
- quadrant 41, 42, 84
- quadratic equation 47
- range 153
- rectangle 41, 87
- rectangular coordinate system 29
- rectangular coordinates 15, 21, 22, 147, 289, 477, 573, 841, 1009, 1016, 1079, 1157, 1165, 1347, 1479, 1651, 1801
- rectangular plane integral 1018
- rectangular space integral 1019
- rectilinear motion 19
- reduction of a surface integral 1229
- reduction of an integral over cylindrical surface 1231
- reduction of surface integral over graph 1230
- reduction theorem of line integral 1164
- reduction theorem of plane integral 937
- reduction theorem of space integral 1021, 1056
- restriction map 153
- Ricatti equation 369
- Riesz transformation 1275
- Rolle's theorem 321
- rotation 1739, 1741, 1742
- rotational body 1055
- rotational domain 1057
- rotational free vector field 1662
- rules of computation 296
- saddle point 612
- scalar field 1485
- scalar multiplication 22, 1750
- scalar potential 1807
- scalar product 169
- scalar quotient 169
- second differential 325
- semi-axis 49, 50
- semi-definite matrix 627
- semi-polar coordinates 15, 19, 21, 33, 147, 181, 182, 289, 477, 573, 841, 1009, 1016, 1055, 1086, 1157, 1231, 1347, 1479, 1651, 1801
- semi-polar space integral 1019
- separation of the variables 853
- signed curve length 1166
- signed infinity 162
- simply connected domain 849, 1492
- simply connected set 176, 243
- singular point 487, 489
- space filling curve 171
- space integral 21, 1015

- specific capacity of heat 1818
- sphere 35, 179
- spherical coordinates 15, 19, 21, 34, 147, 179, 181, 289, 372, 477, 573, 782, 841, 1009, 1016, 1078, 1080, 1081, 1157, 1232, 1347, 1479, 1581, 1651, 1801
- spherical space integral 1020
- square 41
- star-shaped domain 1493, 1807
- star shaped set 21, 41, 89, 90, 175
- static electric field 1498
- stationary magnetic field 1821
- stationary motion 492
- stationary point 583, 920
- Statistics 15, 147, 289, 477, 573, 841, 1009, 1157, 1347, 1479, 1651, 1801
- step line 172
- Stokes's theorem 1499, 1661, 1676, 1679, 1746, 1747, 1750, 1751, 1811, 1819, 1820, 1913
- straight line (segment) 172
- strip 41, 42
- substantial derivative 491
- surface 159, 245
- surface area 1296
- surface integral 1018, 1227
- surface of revolution 110, 111, 181, 251, 499
- surjective map 153

- tangent 486
- tangent plane 495, 496
- tangent vector 178
- tangent vector field 1485
- tangential line integral 861, 1485, 1598, 1600, 1603
- Taylor expansion 336
- Taylor expansion of order 2, 323
- Taylor's formula 321, 325, 404, 616, 626, 732
- Taylor's formula in one dimension 322
- temperature 297
- temperature field 1817
- tetrahedron 93, 99, 197, 1052
- Thermodynamics 301, 504
- top point 49, 50, 53, 66
- topology 15, 19, 37, 147, 289, 477, 573, 841, 1009, 1157, 1347, 1479, 1651, 1801
- torus 43, 182–184
- transformation formulæ 1353
- transformation of space integral 1355, 1357
- transformation theorem 1354
- trapeze 99

- triangle inequality 23,24
- triple integral 1022, 1053

- uniform continuity 186
- unit circle 32
- unit disc 192
- unit normal vector 497
- unit tangent vector 486
- unit vector 23
- unspecified infinity 162

- vector 22
- vector field 158, 296, 1485
- vector function 21, 157, 189
- vector product 19, 26, 30, 163, 169, 1227, 1750
- vector space 21, 22
- vectorial area 1748
- vectorial element of area 1535
- vectorial potential 1809, 1810
- velocity 490
- volume 1015, 1543
- volumen element 1015

- weight function 1081, 1229, 1906
- work 1498

- zero point 22
- zero vector 22

- (r, s, t) -method 616, 619, 633, 634, 638, 645–647, 652, 655
- C^k -curve 483
- C^n -functions 318
- 1-1 map 153