Elementary Algebra and Calculus

The Whys and Hows

Larissa Fradkin

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Contents

Part I Introduction

These notes are based on the lectures delivered by the author to engineering freshers at London South Bank University over the period of 16 years. This is a University of widening participation, with students coming from all over the world, many with limited English. Most also have limited mathematical background and limited time both to revise the basics and to study new material. A system has been developed to assure efficient learning even under these challenging restrictions. The emphasis is on systematic presentation and explanation of basic abstract concepts. The technical jargon is reduced to the bare minimum.

Nothing gives a teacher a greater satisfaction than seeing a spark of understanding in the students' eyes and genuine pride and pleasure that follows such understanding. The author believes that most people are capable of succeeding in – and therefore enjoying – college mathematics. This belief has been reinforced many times by these subjective signs of success as well as genuine improvement in students' exam pass rates. Interestingly, no correlation had ever been found at the Department where the author worked between the students' entry qualification on entry and the class of their degree.

The book owns a lot to authors' students – too numerous to be named here – who talked to her at length about their difficulties and successes, see e.g. Appendix VII on Teaching Methodology. One former student has to be mentioned though – Richard Lunt – who put a lot of effort into making this book much more attractive than it would have been otherwise.

The author can be contacted through her website www.soundmathematics.com. All comments are welcome and teachers can obtain there a copy of notes with answers to questions suggested in the text as well as detailed Solutions to Home Exercises.

Good luck everyone!

Part II   Concept Maps

Throughout when we first introduce a new **concept** (a technical term or phrase) or make a conceptual point we use the bold red font. We use the bold blue to verbalise or emphasise an important idea. Two major topics are covered in this course, Elementary Algebra and Elementary Calculus.

Here is a **concept map** of Elementary Algebra. It is best to study it before studying any of the Algebra Lectures 1–3 and 10–12 to understand where it is on the map. The more you see the big picture the faster you learn!

Elementary ALGEBRA

Main concepts   Visualisation   Applications

variables   operations on variables   Venn diagram   number line

complex plane (the Argand diagram)

solving algebraic equations (transposing the formulae)

Examples   Full list

…
integer
real
complex

addition
subtraction

multiplication
division

raising to power
extracting roots
taking logs

advanced mathematics, science, engineering, business, finance

Here is a concept map of Elementary Calculus. It is best to study it before studying any of the Calculus Lectures 4 – 9 and Calculus Lectures starting from Lecture 13 to understand where they are on the map.

ELEMENTARY CALCULUS

Main concepts   *Visualisation*   *Applications*

functions
(real
functions
of one real
variable)

operations
on functions

diagram

graphs

using graphs

approximate
calculations

finding
means,
areas,
volumes…

solving
differential
equations

solving
integral
equations

advanced
mathematics,
science,
engineering
business,
finance

Some types

Two types

sketching
methods

algebraic

trig

elementary

advanced

using
tables

using
simple
transforma-
tions

using
pointwise
operations

using
analysis

Examples

Full list

$(\ )^n$

$P_n(\ )$

$e^{(\ )}$

$\ln (\ )$

sin( )

cos( )

tan( )

+-,   */

composition/
decomposition

taking inverse

taking a limit

differentiation

integration

Part III   Lectures

We start by introducing a general framework and general concepts used throughout all analytical subjects you are likely to study.

Lecture 1   ALGEBRA: Addition, Subtraction, Multiplication and Division of Rational Numbers

In Elementary Algebra we study **variables** and **operations on variables** which are called algebraic. These concepts are discussed below.

1.1   Variables

A **variable** is an **abstraction** of a quantity.
In algebra, variables are denoted mostly by $a, b, c, d, i, j, k, l, m, n, x$, y and z.

Abstraction is a general concept formed by extracting common features from specific examples. Specific examples of a quantity are time, distance, magnitude of force, current, speed, concentration, profit and so on.

A variable can take any **value** from an allowed **set of numbers**. If a variable represents a dimensional quantity, that is, a quantity measured in dimensional units s, m, N, A, m/s, kg/m3, £… each value has to be multiplied by the corresponding unit. Otherwise the variable is called non-dimensional.

Diagrammatically any set can be represented (visualised) as a circle (this circle is called a **Venn diagram**). It might help you to think of this circle as a bag containing all elements of the set.

Example: A set of numbers 1, 2, 3, 4, 5 can be represented using the Venn diagram

1, 2, 3, 4, 5
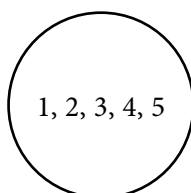
1.2   Variables and operations on variables

1.2.1   Variables: The set of whole numbers

Whole numbers are 1, 2, 3, ..., that is, the numbers used to count (three dots stand for *etc.,* that is, **"and so on"**). **The set of whole numbers can be visualised graphically using the number line:**
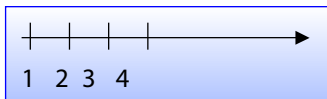
A **number line** is a straight arrowed line. A point representing number 1 is chosen arbitrarily on this line, and so is the unit distance between the points representing numbers 1 and 2. All further neighbouring points are separated by the same unit distance and represent numbers 3, 4 *etc*. The arrow reminds us that the further the point is positioned to the right the greater the number.

Whole numbers are usually denoted by letters *i, j, k ,l ,m, n.*

1.2.2 Operations: Addition

> **Operations on variables** are things you can do with variables.

Addition is the first and simplest algebraic operation. Its symbol is the + sign (read as "plus"). **Addition** of a whole number n to a whole number m **can be visualised using the number line**:

- find a point on the number line that represents number m,
- move along the number line n units to the right.

Addition is called a **direct operation** to emphasise two facts:

1. we just define (declare) what the result of addition is and
2. addition of whole numbers results in a whole number.

We now introduce the laws of addition which are easy to verify (but not prove) by substituting whole numbers. We verbalise the laws in a way that helps us to apply them when required to perform algebraic manipulations.

Law 1: $a + b = b + a$

> **Terminology:**
>
> a and b are called **terms** (expressions that are being added),
> $a + b$ is called a **sum** (expression in which the last operation is addition).

Law 1 verbalised: order of terms does not matter.

Law 2: $(a + b) + c = a + (b + c)$.

Law 2 verbalised: knowing how to add up two terms we can add up three terms, four terms *etc.* (add up any two, add the result to any of the remaining terms, repeat the operation until all terms are used up).

1.2.3 Operations: Subtraction

The symbol of subtraction is the – sign (read as "minus"). Subtraction is an **inverse** (opposite) **operation** to addition. This means that it is defined via addition:

in this context : means **"such that"**

Definition: $a - b = x$: $x + b = a$.

The definition implies that we have the following relation between addition and subtraction:

$$a + b - b = a \text{ (subtraction undoes addition)}$$
$$a - b + b = a \text{ (addition undoes subtraction)}$$

1.2.4   Variables: The Set of Integers

Subtraction is the first inverse operation we encounter and as with many other inverse operations considered later its application might cause a difficulty: subtracting a whole number from a whole number does not always produce a whole number. However, using specific examples of debt and temperature it makes sense to say that subtraction introduces new types of numbers, **0** and **negative whole numbers**:

1. $a - a = 0$
2. $a - b = -(b - a)$ if $b > a$

Note: the – sign between variables or numbers is a symbol of subtraction while the – sign in front of a number tells us that the number is negative. This might be a bit confusing but once the convention is grasped it becomes very ~~easy~~

this symbol means **"greater than"**

Question: is $-a$ positive, negative or zero?
Answer:

We can now introduce wider number sets than the set of whole numbers:

Natural numbers are 0, 1, 2, … The set of natural numbers can be visualised graphically as points on the number line:
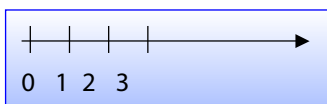
Integers are …, −2, −1, 0, 1, 2, … The set of integers can be visualised graphically in a similar manner:
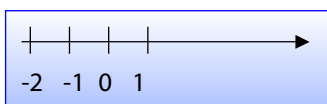
1.2.5   Operations: Addition and Subtraction (ctd.)

We can now introduce further laws of addition:

Law 3: $a + 0 = a$.

Law 3 verbalised: The zero term can always be dropped or put in.

Law 4: for each *a* there exists one **additive inverse** *–a: a + (–a)* = 0.
Law 4 verbalised: every number has an additive inverse.

Laws of addition can be used to justify the rules given below:

Rules

1. $+(b + c) = +b + c$ (since $+(b + c) = 0 + (b + c) = (0 + b) + c = b + c$)

    Rule 1 verbalised: when removing brackets with + in front just copy the terms inside the brackets.

2. $+a + b = a + b$ (since $+a + b = 0 + a + b = (0 + a) + b$)

    Rule 2 verbalised: the + sign in front of the first term can be dropped or put in.
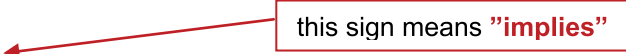
3. $-(a) = -a$ (since additive inverse of a is $-a$)

4. $-(-a) = +a$ (since additive inverse of $-a$ is a)

    Rules 3 and 4 verbalised: when removing brackets with the minus sign in front copy each term inside the brackets but with the opposite sign.

Subtraction of an integer n from a number m can be visualised using the number line:

- choose the point on the number line that represents number m
- if n is a whole number move along the number line to the left by n units
- if n is zero there should be no movement,
- if n is a negative integer move along the number line to the right by $-n$ units (it could not be moving to the left, otherwise there would be no difference between subtracting positive integers and subtracting negative integers!)

Note: Using Rule 1 we can write

this sign means **"implies"**

$a + (-b) = a - b \quad \Rightarrow \quad a - b = a + (-b)$.

that is, a **difference** (expression in which the last operation is subtraction) can be turned into a sum (expression in which the last operation is addition). This becomes useful in some algebraic manipulations.

Question: How many terms are there in expression $3 - 2a$ and what are they?
Answer:

1.2.6   Operations: Multiplication

Multiplication is the second direct algebraic operation. Its symbols are the $\times$ sign or else the $\cdot$ sign. When symbols of variables are put next to each other or else when they are put next to a number, the absence of any operational sign also indicates multiplication. In other words we can write:

$$ab = a \cdot b = a \times b, \ 2b = 2 \cdot b = 2 \times b$$

For whole numbers n, the result of multiplication of a by n is

$$a \cdot n \;=\; \underbrace{a + \ldots\ldots\ldots\ldots + a}_{n \text{ equal terms}}$$

Thus, multiplication by a whole number n is shorthand for addition of n equal terms.

Example*:* $5a$ is a much shorter expression than $a + a + a + a + a$.

Multiplication is a **direct operation** in the sense that we just define (declare) what a product of a number and a whole number is and also, in the sense that a product of integers is an integer. We now introduce the laws of multiplication which are easy to verify (but not prove) by substituting whole numbers. As before, we verbalise them so as to make algebraic manipulations an easier task.

Law 1: $a \times b = b \times a$.

Terminology:

a and b are called **factors** (expressions that are multiplied),
ab is called a **product** (expression in which the last operation is multiplication).

Law 1 verbalised: order of factors does not matter.

Law 2: $(a \times b) \times c = a \times (b \times c)$.

Convention:

$abc = (ab)c$, $a(-bc) = -abc$.

Law 2 verbalised: knowing how to multiply two factors can multiply three, four *etc.*

Law 3: $a(b + c) = ab + ac$

$\longrightarrow$ Removing brackets
$\longleftarrow$ Factoring

Law 3 verbalised:

Left to right: we can turn a product into a sum, that is, can change the order of operation from AM (Addition, Multiplication) to MA (Multiplication, Addition).

Right to left: we can turn a sum into a product, that is, can change the order of operation from MA (Multiplication, Addition) to AM (Addition, Multiplication).

Law 4: $a \times 0 = 0$.

Law 4 verbalised:

Left to right: any number times 0 is 0.

Right to left: 0 can be represented as a product of 0 and any other number.

Law 5: $a \times 1 = a$.

Law 5 verbalised: factor 1 can be dropped or put in.

The multiplication and addition laws can be used to deduce the following useful rules:

Rules

1. $(a + b)(c + d) = ac + ad + bc + bd$ -   **SMILE RULE**

The rule can be proved using Multiplication Law 3.

Question: Is the order of terms and factors important?
Answer:

However, memorising the **SMILE RULE** is advisable, because it makes checking results easier. It can be extended to multiplying any number of sums with any number of terms.

2. $(-1) \times n = -n.$

> **Justification:**
>
> The answer should be n with a sign and it cannot be $+n$, because then there would be no difference between $(-1) \times n$ and $1 \times n$. So the answer should be $-n$. (Note that if n is a whole number, we can produce another proof: if n is a whole number $= -1 + (-1) + \ldots + (-1)$ – sum of n (-1)s – and so $(-1) \times n = -n$.)

Rule 2 verbalised:

Left to right: multiplying by -1 is equivalent to changing sign.

Right to left: minus sign in front of a term is a shorthand for $(-1) \times$.

1.2.7   Operations: Division

If writing is restricted to just one line of text, the symbol of division is the $\div$ sign or else / sign. More often than not one uses the $-$ sign, with one expression on top and another, at the bottom. It should not be confused with the minus sign. Division is an **inverse** (opposite) **operation** to multiplication. This means that it is defined via multiplication:

Definition: $\dfrac{a}{b} = x : \ xb = a$

The definition implies that we have the following relations between multiplication and division:

$\dfrac{ab}{b} = a$ (division undoes multiplication)

$\dfrac{a}{b}b = a$ (multiplication undoes division)

Temrinology:

a – **numerator** (expression divided),

b – **denominator** (expression by which numerator is divided),

$\dfrac{a}{b}$ – **quotient, ratio** (expression in which the last operation is division).

A quotient is called a **proper fraction** if a, b are whole numbers and $a < b$. It is called an **improper fraction** if a, b are whole numbers and $a \geq b$.

this sign means **"does not equal to"**

Convention:

$$2\frac{1}{2} \neq 2 \cdot \frac{1}{2}, \quad 2\frac{1}{2} = 2 + \frac{1}{2}, \quad 2\frac{3}{2} \neq 2 + \frac{3}{2} \text{ but rather } \quad 2\frac{3}{2} = 2 \cdot \frac{1}{2}$$

So, when adding a whole number and a proper fraction just put this fraction next to the whole number. When multiplying a whole number by an improper fraction or adding them up put the appropriate operation sign between them to avoid confusion.

1.2.8   Variables: Rational Numbers

Division is the second inverse operation we encounter and again its application might cause a difficulty: dividing an integer by an integer does not always produce an integer. To give a simple example, dividing 2 pies between 2 people, each gets two pies ($2/2 = 1$, in this case division of two whole numbers produces a whole number). However, dividing 1 pie between two people can be only achieved by cutting this pie into two portions ($1/2$ is not a whole number, but a new type of number called **rational**).

Definition: A **rational number** is a number $\dfrac{m}{n}$, where m and $n \neq 0$ are integers.

Division by zero is not defined!

Proof

Indeed, **assume** that $a \neq 0$ and there exists a number x such that $\frac{a}{0} = x$.

Let $a \neq 0$. Then $0 \times x = a$. The number on the left of the $=$ sign is 0 and the number on the right is not. Hence we arrived at a contradiction. Let now $a = 0$. Then $0 \times x = 0$. However, this is true for all numbers and not just one. Hence the **assumption** (that there exists a number x, such that $\frac{a}{0} = x$) **is invalid**.

Question: Is 2 a rational number? Explain your answer.

Answer:

Rationals can be visualised graphically using the number line:
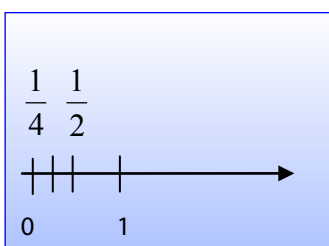
Download free eBooks at bookboon.com

Question: How many rationals are there between 0 and 1?
Answer:

A **rational variable** is a variable that takes rational values.

1.2.9   Operations: Multiplication and Division (ctd.)

Law 6: For each $a \neq 0$ there exists one **multiplicative inverse** $\frac{1}{a}$: $a \cdot \frac{1}{a} = 1$.

Law 6 verbalised: every number but 0 has a multiplicative inverse.

The addition and multiplication laws can be used to deduce useful rules described below:

Rules

1. $\frac{a}{b} \cdot n = \frac{an}{b}$

Rule 1 verbalised: any product (an expression in which the last operation is multiplication) can be turned into a quotient (an expression in which the last operation is division) and ***vice versa*** (that is, **"the other way around"**).

Note: Using Rule 1 right to left, $\frac{a}{b} = a \times \frac{1}{b}$, that is, a quotient (expression in which the last operation is division) can be turned into a product (expression in which the last operation is multiplication).

Question: How many factors are there in expression $\frac{a}{3}$ and what are they?
Answer:

2. $\dfrac{a/b}{n} = \dfrac{a}{bn} = \dfrac{a/n}{b}$

> **Convention:**
>
> 1.   Division prescribed with the slash line / is performed before division prescribed with the horizontal line.
> 2.   If only horizontal lines are used to prescribe division the shorter lines take precedence.

3. $\dfrac{an}{bn} = \dfrac{a \cdot \cancel{n}}{b \cdot \cancel{n}} = \dfrac{a}{b}$ – **cancellation** (in this context, the slash means **cross out** and **not divide**)

Rule 3 verbalised:

Left to right: if first multiplying by a number (expression) and then dividing by it this number (expression) can be cancelled.

Right to left: can multiply both numerator and denominator by the same non-zero number/expression without changing the quotient.

4. $\dfrac{a/b}{c/d} = \dfrac{ad}{cb}$ – **FLIP RULE**

Rules 2 and 4 verbalised:

Left to right: a multi-storey fraction can always be turned into a two-storey fraction.

Right to left: a two-storey fraction can always be turned into a multi-storey fraction.

5. $\dfrac{a}{b} + \dfrac{c}{b} = \dfrac{a+c}{b}$

Note: $\dfrac{a+b}{c+d} = (a+b)/(c+d)$ (if a quotient is presented on one line and there is more than one term in the numerator or denominator they have to be bracketed).

6. $\overset{\times d}{\underset{}{\dfrac{a}{b}}} + \overset{\times b}{\underset{}{\dfrac{c}{d}}} = \dfrac{ad}{bd} + \dfrac{cb}{db} = \dfrac{ad+cb}{bd}$ - **COMMON DENOMINATOR RULE**

This line means **"apply the same operation to top and bottom"**. What follows this line is the operation to be applied.

Law 5 and 6 verbalised: we can turn a sum of quotients into a quotient and *vice versa*. Another way of putting this: we can change the order of operation from DA (Division, Addition) to AD (Addition, Division) and *vice versa*.

1.3   General remarks

1. All the laws and rules introduced above can be checked (but not proved!) by **substituting** whole numbers for letters (that is, by putting whole numbers in place of letters). If you are not sure whether you remember a law or rule correctly, this type of substitution might jog your memory. **Checking by substitution is not 100 % safe, since the "law" you invent might by chance be satisfied by a couple of whole numbers but not all of them.** This means that a check of that nature will minimise the chance of a mistake but not eliminate it.

2. It is easier to remember the above laws and rules from **left to right**. However, you have to be able to use them from **right to left** with similar ease.

3. Note that while the phrases "integer variable" or "rational variable" are quite conventional nobody talks of a "whole variable" or "natural variable". The mathematical language can be just as inconsistent as the natural language!

4. All laws and rules introduced above are first introduced for whole numbers. They can be **generalised** (that is, declared to apply) to wider sets of numbers too. Checking the consistency of these generalisations lies outside the scope of these notes. We just state that above laws and rules of addition and multiplication apply to all rational numbers.

5. If an expression involves more than one operation performing these operations in different order can produce different results. It is important to memorise the following convention:

Order of Operations (OOO)

1. Make implicit (invisible) brackets visible
    (everything raised and everything lowered with respect to the main line is considered to be bracketed)
2. Perform operations in brackets {[()]} first (inside out)

<div align="center">OOO</div>

$$\longrightarrow$$

Operations in Brackets including implicit (invisible) brackets	Multiplication	Addition
	×	+
	÷	-

Note: talking about order of operations we treat both + and – signs as Addition symbols and both × and ÷ signs as Multiplication symbols. Similar rules introduced in school make unnecessary distinctions which suggest that division should be performed before multiplication and addition before subtraction. Not so!

1.4   Glossary of terms introduced in this Lecture

An **abstraction** is a general concept formed by extracting common features from specific examples.

A **diagrammatic representation** is a very general (abstract) visualisation tool, a pictorial representation of a general set or relationship.

A **generalisation** is an act of introducing a general concept or rule by extracting common features from specific examples.

A **graphical representation** is a specific visualisation tool, a pictorial representation of a particular set or relationship.

A **sum** is a mathematical expression where the last operation is addition.

A **difference** is a mathematical expression where the last operation is subtraction.

Terms are expressions you add. They are separated by + sign(s).

A **product** is a mathematical expression where the last operation is multiplication.

A **quotient** is a mathematical expression where the last operation is division.

Factors are expressions you multiply. They are separated by multiplication sign(s).

Note 1: Our usage of words *term* and *factor* is not universal. Mathematicians also use words *addend* and *summand* for the *term*. Some talk of *multipliers* and *multiplicands* rather than *factors*. Engineers use words *term* and *factor* interchangeably – very confusing! In these notes we use them only in the sense described above. This allows us to produce very short explanations.

Note 2: From now own you are expected to create your own glossary for each lecture. Remember – the words and phrases in bold red introduce new concepts and conceptual ideas. Do not forget to keep using the Glossary Appendix.

1.5   Historical notes

"We learn to count at such an early age that we tend to take the notion of abstract numbers for granted. We know the word "two" and the symbol "2" express a quantity that we can attach to apples, oranges, or any other object. We readily forget the mental leap required to go from counting specific things to the abstract concept of number as an expression of quantity.

Abstract numbers are the product of a long cultural evolution. They also apparently played a crucial role in the development of writing in the Middle East. Indeed, numbers came before letters.

http://www.maa.org/mathland/mathland_2_24.html

"Negative numbers appear for the first time in history in the *Nine Chapters on the Mathematical Art* (Jiu zhang suan-shu), which in its present form dates from the period of the *Han Dynasty* (202 B.C.–A.D. 220), but may well contain much older material. The Nine Chapters used red *counting rods* to denote positive *coefficients* and black rods for negative. (This system is the exact opposite of contemporary printing of positive and negative numbers in the fields of *banking*, *accounting*, and *commerce*, wherein red numbers denote negative values and black numbers signify positive values). The Chinese were also able to solve simultaneous equations involving negative numbers.

For a long time, negative solutions to problems were considered "false". In *Hellenistic Egypt*, *Diophantus* in the *third century A.D.* referred to an equation that was equivalent to 4x + 20 = 0 (which has a negative solution) in *Arithmetica*, saying that the equation was absurd.

The use of negative numbers was known in early *India*, and their role in situations like mathematical problems of debt was understood. Consistent and correct rules for working with these numbers were formulated. The diffusion of this concept led the *Arab* intermediaries to pass it to *Europe*.

The ancient Indian *Bakhshali Manuscript*, which Pearce Ian claimed was written some time between 200 B.C. and A.D. 300, while George Gheverghese Joseph dates it to about A.D. 400 and no later than the early 7th century, carried out calculations with negative numbers, using "+" as a negative sign.

During the 7th century A.D., negative numbers were used in *India* to represent debts. The Indian mathematician *Brahmagupta*, in *Brahma-Sphuta-Siddhanta* (written in *A.D. 628*), discussed the use of negative numbers to produce the general form *quadratic formula* that remains in use today. He also found negative solutions of *quadratic equations* and gave rules regarding operations involving negative numbers and *zero*, such as *"A debt cut off from nothingness becomes a credit; a credit cut off from nothingness becomes a debt."* He called positive numbers "fortunes," zero "a cipher," and negative numbers "debts." During the 8th century A.D., the *Islamic world* learned about negative numbers from *Arabic* translations of Brahmagupta's works, and by A.D. 1000 Arab mathematicians were using negative numbers for debts.

In the 12th century A.D. in India, *Bhaskara* also gave negative roots for quadratic equations but rejected them because they were inappropriate in the context of the problem. He stated that a negative value is "*in this case not to be taken, for it is inadequate; people do not approve of negative roots.*"

Knowledge of negative numbers eventually reached Europe through *Latin* translations of Arabic and Indian works. *European* mathematicians, for the most part, resisted the concept of negative numbers until the 17th century, although *Fibonacci* allowed negative solutions in financial problems where they could be interpreted as debits (chapter 13 of *Liber Abaci*, *A.D. 1202*) and later as losses (in *Flos*).

In the 15th century, *Nicolas Chuquet*, a Frenchman, used negative numbers as *exponents* and referred to them as "absurd numbers."

In *A.D. 1759*, Francis Maseres, an English mathematician, wrote that negative numbers "darken the very whole doctrines of the equations and make dark of the things which are in their nature excessively obvious and simple". He came to the conclusion that negative numbers were nonsensical.

In the 18th century it was common practice to ignore any negative results derived from equations, on the assumption that they were meaningless."

http://en.wikipedia.org/wiki/Negative_and_non-negative_numbers#History

"It has been suggested that the concept of irrationality was implicitly accepted by *Indian mathematicians* since the 7th century BC, when *Manava* (c. 750–690 BC) believed that the *square roots* of numbers such as 2 and 61 could not be exactly determined, but such claims are not well substantiated and unlikely to be true.

The first proof of the existence of irrational numbers is usually attributed to a *Pythagorean* (possibly *Hippasus of Metapontum*), who probably discovered them while identifying sides of the *pentagram*. The then-current Pythagorean method would have claimed that there must be some sufficiently small, indivisible unit that could fit evenly into one of these lengths as well as the other. However, Hippasus, in the 5th century BC, was able to deduce that there was in fact no common unit of measure. His reasoning is as follows:

- The ratio of the hypotenuse to an arm of an isosceles right triangle is *a:b* expressed in the smallest units possible
- By the *Pythagorean theorem*: $a^2 = 2b^2$.
- Since a^2 is even, *a* must be even as the square of an odd number is odd.
- Since *a:b* is in its lowest terms, *b* must be odd.
- Since *a* is even, let $a = 2y$.
- Then $a^2 = 4y^2 = 2b^2$
- $b^2 = 2y^2$ so b^2 must be even, therefore *b* is even.
- However we asserted *b* must be odd. *Here is the contradiction."*

http://en.wikipedia.org/wiki/Irrational_number

1.6   Instructions for self-study

- **Study Lecture 1 using the STUDY SKILLS Appendix**
- **Attempt the following exercises:**

Q1. Simplify
   a) 0×1
   b) 0×10
   c) $0 \times x$
   d) $0 \times (x - 1)$
   e) 1×1
   f) 1×10
   g) $1 \times x$
   h) $1 \times (x - 1)$

Q2. Solve

    a) $0 = \dfrac{x}{2}$

    b) $\dfrac{x-2}{2} = 0$

    c) $\dfrac{1}{x} = 1$

    d) $\dfrac{1}{x-2} = 1$

Q3. Remove brackets

    a) $-(2 - a)$
    b) $+(2 + a - b)$
    c) $-(2 + a - b)$
    d) $-(2 - (a - b))$
    e) $(a + b)(a + b)$
    f) $(a + b)(a - b)$

Q4. Factorise

    a) $2 - 6a$

    b) $\dfrac{1}{3} - \dfrac{1}{9}b$

    c) $\dfrac{1}{3} - \dfrac{1}{9}b + 2c$

    d) $(a + b)c + (a + b)d$

Q5. Turn into a one storey fraction:

    a) $\dfrac{2}{6/3}$

    b) $\dfrac{ab}{a/b}$

Q6. Add fractions:

    a) $\dfrac{1}{2} + \dfrac{1}{3}$

    b) $\dfrac{1}{x-1} + \dfrac{1}{x-2}$

    c) $\dfrac{1}{x-1} + \dfrac{1}{x+1}$

If you need more exercises (particularly on addition of fractions) you can use the following sites:
http://helm.lboro.ac.uk/documents/1_4_arthmtic_algebraic_fractns.pdf
http://www.mathtutor.ac.uk/viewdisks.php

Lecture 2   Applications of Elementary ALGEBRA: Solving Simple Equations

2.1   Revision: Factorisation

The Multiplication Law 3 allows us to change the order of operations of multiplication and addition. When going from left to right (that is, removing or expanding the brackets), a product is turned into a sum. When going from right to left (that is, factoring), the sum is turned into a product

$$ab + ac = a(b + c) \tag{2.1}$$

Question: What operations are implied in the LHS (left-hand side) of (2.1) and in what order?

Answer:

Question: What would you call the expression in LHS of (2.1) and why?

Answer:

Question: What are the terms in the LHS of (2.1)?

Answer:

Question: What operations are implied in the RHS (right-hand side) of (2.1) and in what order?

Answer:

Question: What would you call the expression in RHS of (2.1) and why?

Answer:

Question: What are the terms in in the RHS of (2.1)?

Answer:

Note that you can turn a sum into a product using any factor *a*. It does not have to appear as a common factor in all terms in the sum. This is a general recipe for finding *b* and *c*:

$$ab + ac = a(b + c)$$

To verbalise, to get the first term *b* in the RHS of (2.1) take the first term *ab* in the LHS and divide by *a*. To get the second term *c* in the RHS take the second term *ac* in the LHS and divide by *a*.

Examples: Factorise and check your answer by removing brackets

1. $3u + 9v = 3(u + 3v)$
2. $xu + 3xv = x(u + 3v)$
3. $(x - 2)u + 3(x - 2)v = (x - 2)(u + 3v)$
4. $(-x + 1)u + 3(-x + 1)v = (-x + 1)(u + 3v)$
5.   Find x and y in $s - 1 + \dfrac{3}{s + 1} = (s - 1)(x + y)$

Solution

$$x = \frac{s-1}{s-1} = 1, \quad y = \frac{\dfrac{3}{s+1}}{s-1} = \frac{3}{s^2 - 1}$$

2.2   Revision: Adding fractions

Adding fractions we change the order of operations of addition and division, that is, turn a sum of fractions into one fraction.

Examples: Turn into one fraction

> This sign means "**apply the same operaton to both top and bottom**". The opeartion is indicated after the sign.

1. $\dfrac{1}{2}+\dfrac{1}{3}$. **Solution:** $\dfrac{1}{2}+\dfrac{1}{3}=\dfrac{3}{6}+\dfrac{2}{6}=\dfrac{5}{6}$

2. $\dfrac{1}{x-2}+\dfrac{1}{3}$. **Solution:** $\dfrac{1}{x-2}+\dfrac{1}{3}=\dfrac{3}{3(x-2)}+\dfrac{x-2}{3(x-2)}=\dfrac{3+x-2}{3(x-2)}=\dfrac{x+1}{3(x-2)}$

3. $\dfrac{4}{5}-\dfrac{2}{3}$. **Solution:** $\dfrac{4}{5}-\dfrac{2}{3}=\dfrac{12}{15}-\dfrac{10}{15}=\dfrac{2}{15}$

4. $\dfrac{2u}{u-2}-\dfrac{2}{3}$

   Solution:
   $$\dfrac{2u}{u-2}-\dfrac{2}{3}=\dfrac{6u}{3(u-2)}-\dfrac{2(u-2)}{3(u-2)}=\dfrac{6u-2u+4}{3(u-2)}=\dfrac{4u+4}{3(u-2)}=\dfrac{4}{3}\dfrac{u+1}{(u-2)}=1\dfrac{1}{3}\dfrac{u+1}{(u-2)}$$

2.3   Decision Tree for Solving Simple Equations

Equations are mathematical statements that involve at least one **unknown** variable and the equality sign, =. This means that equations state that the mathematical **expression** in the **LHS** (left-hand side of the equation, to the left of the = sign) is the same as the mathematical **expression** in the **RHS** (right-hand side of the equation, to the right of the = sign). For some values of the unknown these statements are false and for others they are true. The equations are different from **mathematical formulae** which also contain variables and the equality sign, =, but are always true.

Algebraic equations involve only algebraic operations on the unknown(s). By convention the first choice for the symbol of an algebraic unknown is *x*. If a **value of** *x* (a number substituted for *x*) turns an equation into a true statement. This value is called **solution of the equation**. **To solve an algebraic equation** means to find all values of *x* that turn this equation into a true statement.

Simple equations are equations that can be simply rearranged so that only one term contains the unknown.

Question: What simple manipulations allow you to rearrange algebraic equations so that only one term contains the unknown?
Answer:

Note: After solving an equation with the unknown in a denominator make sure that the value you found does not turn this denominator into zero. If it does it might not be a solution of the original equation!

Once you rearranged the equation so that only one term contains the unknown – conventionally in the LHS – you can use **The Decision Tree for Solving Equations with Only One Term Containing the Unknown** (figure 2.1). This Decision Tree is based on the idea that to solve such equation we have to keep **applying operations** which are **inverse to the last operation** on the unknown **to both sides** of the equation – **until the unknown becomes the subject of the equation** (stands on its own, no operations are applied to it anymore). The idea is very general and is used to solve all equations that contain the unknown only in one term. If it is present in more than one term we should use a more general Decision Tree presented in figure 2.2.
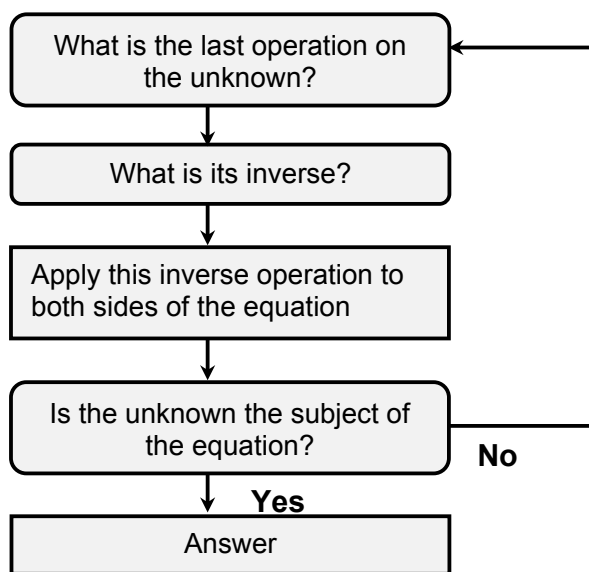

Figure 2.1. Decision Tree for Solving Equations with Only One Term Containing the Unknown

Examples:

Solve the following equations using the Decision Tree for Solving Simple Equations:

1. $x - 1 = 5$

Solution

Using the Decision Tree for Solving Equations with Only One Term Containing the Unknown,
the last operation on the unknown x is subtraction, specifically, –1.
The inverse operation to –1 is +1. Apply this operation to both sides of the equation:

$x - 1 = 5$ $/$ $+1$

$x = 6$

This sign means "**apply the same operation to both sides of the equation**".
The opeartion is indicated after the sign.

The unknown is the subject the equation. Hence the equation has been solved.

We can use substitution to check that this solution is correct.
Consider the LHS, substitute 6 for x, $6 - 1 = 5$.
Since LHS = RHS the solution is correct.

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2. $2x - 3 = x - 5$

Solution

Using the Decision Tree for Solving Simple Equations, we first collect the like terms:
subtracting x from both sides of the equation we get

$x - 3 = -5$

The last operation on the unknown x is subtraction, specifically, –3.
The inverse operation to –3 is +3. Apply this operation to both sides of the equation:

$x - 3 = -5$ / +3
$x = -2$

The unknown is the subject the equation. Hence the equation has been solved.

We can use substitution to check that this solution is correct.
Consider the LHS, substitute –2 for x, $2 \cdot (-2) - 3 = -4 - 3 = -7$.
Consider the RHS, substitute –2 for x, $-2 - 5 = -7$.
Since LHS = RHS the solution is correct.

---

3. $\dfrac{1}{v-2}+\dfrac{1}{3}=0$

Solution

Using the Decision Tree for Solving Simple Equations, we first multiply both sides of the equation by the denominator containing the unknown *v*:

$$1+\dfrac{1}{3}(v-2) \ = \ 0$$

Removing brackets and collecting the like terms,

$$\dfrac{1}{3}v+\dfrac{1}{3}=0$$

The last operation on the unknown *v* is addition, specifically, $+\dfrac{1}{3}$.
The inverse operation to $+\dfrac{1}{3}$ is $-\dfrac{1}{3}$. Apply this operation to both sides of the equation:

$$\dfrac{1}{3}v+\dfrac{1}{3}=0 \ \Big/ \ -\dfrac{1}{3}$$

$$\dfrac{1}{3}v=-\dfrac{1}{3}$$

The last operation on the unknown *v* is multiplication, specifically, $\cdot\dfrac{1}{3}$.
The inverse operation to $\cdot\dfrac{1}{3}$ is $\cdot 3$. Apply this operation to both sides of the equation:

$$v=-1$$

The unknown is the subject the equation. Hence the equation has been solved.

We can use substitution to check that this solution is correct.
Consider the LHS, substitute −1 for *v*, $\dfrac{1}{-1-2}+\dfrac{1}{3}=0$.
Since LHS = RHS the solution is correct.

---

4. $\dfrac{2u}{u-2} + \dfrac{2}{3} = 0$

Solution

Using the Decision Tree for Solving Simple Equations, we first multiply both sides of the equation by the denominator containing the unknown *u*:

$$2u + \frac{2}{3}(u-2) = 0$$

Removing brackets and collecting the like terms,

$$\frac{8}{3}u - \frac{4}{3} = 0$$

The last operation on the unknown u is subtraction, specifically, $-\dfrac{4}{3}$.
The inverse operation to $-\dfrac{4}{3}$ is $+\dfrac{4}{3}$. Apply this operation to both sides of the equation:

$$\frac{8}{3}u - \frac{4}{3} = 0 \quad \Big/ + \frac{4}{3}$$

$$\frac{8}{3}u = \frac{4}{3}$$

The last operation on the unknown *u* is multiplication, specifically, $\cdot\dfrac{8}{3}$. The inverse operation to $\cdot\dfrac{8}{3}$ is $\cdot\dfrac{3}{8}$. Apply this operation to both sides of the equation:

$$\frac{8}{3}u = \frac{4}{3} \quad \Big/ \cdot \frac{3}{8}$$

$$u = \frac{1}{2}$$

The unknown is the subject the equation. Hence the equation has been solved.

We can use substitution to check that this solution is correct.

Consider the LHS, substitute $\dfrac{1}{2}$ for *u*, $\dfrac{2\cdot\frac{1}{2}}{\frac{1}{2}-2} + \dfrac{2}{3} = \dfrac{1}{\frac{1}{2}-\frac{2}{1}} + \dfrac{2}{3} = \dfrac{1}{\frac{1}{2}-\frac{4}{2}} + \dfrac{2}{3} = \dfrac{1}{-\frac{3}{2}} + \dfrac{2}{3} = -\dfrac{2}{3} + \dfrac{2}{3} = 0.$

Since LHS = RHS the solution is correct.

5. $\dfrac{s+1}{s+2} = 0$

Solution

Using the Decision Tree for Solving Simple Equations, we first multiply both sides of
the equation by the denominator containing the unknown *s*:

$$s + 1 = 0$$

The last operation on the unknown s is subtraction, specifically, –1.
The inverse operation to –1 is +1. Apply this operation to both sides of the equation:

$s + 1 = 0$ / –1
$s = -1$

The unknown is the subject the equation. Hence the equation has been solved.

We can use substitution to check that this solution is correct.

Consider the LHS, substitute –1 for *s*, $\dfrac{-1+1}{-1-1} = \dfrac{0}{-2} = 0$.

Since LHS = RHS the solution is correct.

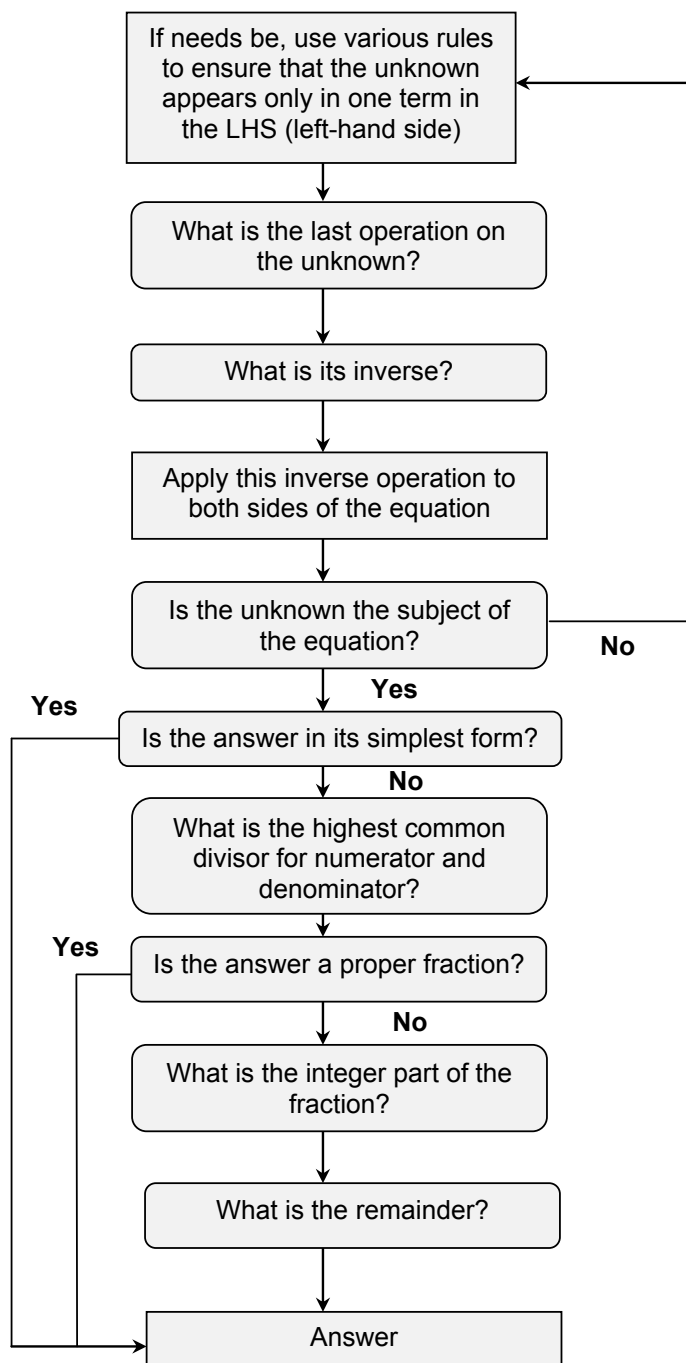Download free eBooks at bookboon.com

Figure 2.2. Decision Tree for Solving Simple Equations.

2.4   Applications of equations

Scientific and engineering systems are described using equations, sometimes referred to in science and engineering as **laws of nature, scientific formulae** or **principles** describing the relationships between various measurable quantities. If you have one equation (law, formula, principle), which contains one unknown, this equation (law, formula, principle) can be solved (**transposed, rearranged**) to find this unknown.

Example: A conker, of mass m=0.2 kg, falls vertically down from a tree in autumn. Whilst it falls it experiences the gravity force $F_g = mg$ and air resistance of magnitude $F_R = 0.4v$, where v is its speed in $m\ s^{-1}$. Calculate the speed at which it is falling when it has an acceleration $a = 1.81\ m\ s^{-2}$ (Take $g = 9.81\ m\ s^{-2}$).

Solution

Step 1. According to the 2nd Law of Newton,

$$F = ma$$

Step 2. The resulting force on the falling conker is

$$F = F_g - F_R \qquad\qquad (*)$$

Step 3. Using the statement of the problem, the right-hand side of the force balance equation (*) is

$$F = 0.2 \times 9.81 - 0.4v = 19.63 - 0.4v$$

Step 4. Using the statement of the problem the left-hand side of the force balance equation (*) is

$$ma = 0.2 \times 1.81 = 3.62$$

Step 5. Thus, the force balance equation (*) for the falling conker can be written as

$$19.63 - 0.4v = 3.62$$

Step 6. Using the Decision Tree for Solving Equations with One Term Containing the Unkown we can find the speed v.

2.5   A historical note

"Al-Khwarizmi (Mohammad ebne Mūsā Khwārazmī was a Persian mathematician, astronomer, astrologer and geographer. He was born around 780 in Khwārizm, then part of the Persian Empire (now Khiva, Uzbekistan) and died around 850. He worked most of his life as a scholar in the House of Wisdom in Baghdad. His Algebra was the first book on the systematic solution of linear and quadratic equations. Consequently he is considered to be the father of algebra, a title he shares with Diophantus. Latin translations of his Arithmetic, on the Indian numerals, introduced the decimal positional number system to the Western world in the twelfth century…

The word algebra is derived from al-jabr, one of the two operations used to solve quadratic equations, as described in his book. "Al-ğabr is the process of…" adding the same quantity to each side. For example, $x^2 = 40x - 4x^2$ is reduced to $5x^2 = 40x$." http://en.wikipedia.org/wiki/Mu%E1%B8%A5ammad_ ibn_M%C5%ABs%C4%81_al%E1%B8%B4w%C4%81rizm%C4%AB

2.6   Instructions for self-study

- **Revise Lecture 1 using the STUDY SKILLS Appendix**
- **Study Lecture 2 using the STUDY SKILLS Appendix**
- **Attempt the following exercises:**

Q1. Remove brackets

a) $(3 + x)(2 + y)$
b) $(x + 3)(x + 3)$
c) $(x + 3)(x - 3)$

Q2. Find the multiplicative inverse of $\dfrac{1}{\dfrac{1}{R_1} + \dfrac{1}{R_2}}$.

Q3. State and explain whether the following is correct:

a) $\dfrac{2 + 4ba}{2} = 1 + 2ab$

b) $\dfrac{2 + 4ab}{b} = 2 + 4a$

Q4. Factorise

a) $6x - 12xy$
b) $4x + 32z + 16y$

Q5. Solve

a) $5 - 2x = 2 + 3x$

b) $\dfrac{x + 3}{2} = 3$

c) $\dfrac{3}{u + 1} = \dfrac{2}{4u + 1}$

d) $\dfrac{3}{t} = \dfrac{1}{2t + 1}$

e) $\dfrac{u + 1}{u - 1} = \dfrac{1}{u - 1}$

If you need more exercises you can use the following site: http://www.mathtutor.ac.uk/viewdisks.php

Lecture 3   ALGEBRA: Exponentiation, Roots and Logarithms of Real Numbers

3.1   Types of variables and operations on variables (ctd.)

3.1.1   OPERATIONS: *n-th* POWER b^n

The symbol of the **n-th power** is a superscript *n*. Raising to power is also called **exponentiation**.

For whole numbers *n*,

$$b^n = \underbrace{b \cdot b \cdot b \cdot \ldots \cdot b}_{n \text{ equal factors}}$$

Thus the symbol $(\ )^n$, where *n* is a whole number is a shorthand for multiplication of *n* equal factors.

Raising to power is a **direct operation** in the sense that we just define the result and raising a rational number to a whole power is a rational number.

Using the whole power indices m and n and multiplication rules the following useful rules can be verified:

<u>**Rules**</u>

1. $a^m \cdot a^n = a^{m+n}$ **product of powers with same base can be turned into a single power – just add indices**

2. $a^n \cdot b^n = (ab)^n$ **product of powers with same index is power of products with this index**

3. $\dfrac{a^m}{a^n} = a^{m-n}$ **quotient of powers with same base can be turned into a single power – subtract indices**

4. $\dfrac{a^m}{b^m} = \left(\dfrac{a}{b}\right)^m$ **quotient of powers is power of quotient**

5. $a^0 = 1$

6. $a^{-n} = \dfrac{1}{a^n}$

7. $(a^m)^n = a^{mn}$ **Convention:** $a^{m^n} = a^{(m^n)}$

3.1.2   OPERATIONS: *n-th* ROOT

When n is a whole number the symbol of the ***n-th* root** is $\sqrt[n]{\ }$. The symbol of the square root is $\sqrt[n]{\ }$. When n is a whole number the *n-th* root is an **inverse operation** to raising to power n. This means that it can be defined via raising to power n:

Definition: $\sqrt[n]{b} = x$**:** $x^n = b$.

The definition implies the following relations between raising to power n and the *n-th* root:

$\sqrt[n]{b^n} = b$ (taking the *n-th* root undoes raising to *n-th* power)

$(\sqrt[n]{b})^n = b$ (raising to *n-th* power undoes taking the *n-th* root)

Therefore, we can use notation $b^{1/n} = \sqrt[n]{b}$. Indeed, it is easy to check using the indices rules that exactly like the operation $\sqrt[n]{\ }$, the operation $(\ )^{1/n}$ undoes $(\ )^n$ and *vice versa.*

3.1.3   Variables: Irrational, Real and Complex Numbers

Extracting an *n-th* root is an operation inverse to raising to power n. It is the third inverse operation we encounter.

Question: what other inverse operations have we covered so far?

Answer:

As with subtraction and division, application of this new inverse operation might cause a difficulty:

1.   First of all, the answer might not be unique.

Question: what is a square root of 4, 9, 16 …?

Answer:

2.   Also, extracting a whole root of a rational number does not always result in a rational number. However, it makes sense to say that extracting a whole root of a rational number introduces a new type of number called **irrational**, for example $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$, $\sqrt[3]{2}$, $\sqrt[3]{3}$ *etc.* We get an idea of what an irrational number $\sqrt{2}$ is by saying that squaring it gives 2.

Question: what is an irrational number? Note that "ir" is Latin for "non".

Answer:

All rational numbers combined with all irrational numbers constitute a set of **real numbers.** A set of real numbers can be visualised as points on a number line. Any real number can be represented by a point on the number line and every point on the number line represents a real number (the proof of this lies outside the scope of these notes).

$$\begin{array}{c} \xrightarrow{\qquad | \quad | \qquad\qquad} \\ 0 \quad 1 \end{array}$$
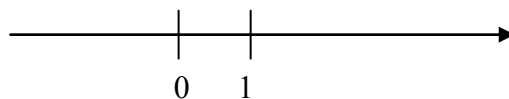
A **real variable** is a variable that takes real values.

The operation of taking a root also introduces **complex numbers** $\sqrt{-1}$ *etc.* Complex numbers are numbers that are not real and are to be discussed later. The main message: **an even root of a negative number cannot be a real number.**

> **Proof:**
>
> Indeed, **assume** that $\sqrt{-1} = x$, where x is real. Then squaring both sides, $-1 = x^2$. But any real number squared is positive or zero. Hence the **assumption** (that there exists a real number x such $\sqrt{-1} = x$) **is invalid.**

A **complex variable** is a variable that takes complex values.

3.1.4   Operations: Logarithm Base b

The symbol of operation **taking a logarithm base b** is $\log_b$. Taking a logarithm base b is an **inverse operation** to raising b to power. Note that the base is always written as a subscript of log. Since the logarithm is inverse to raising b to power it can be defined via this operation:

Definition: $\log_b a = n$**:** $b^n = a$.

The definition implies the following relations between raising b to power and log base b:

$\log_b (b^n) = n$ (taking $\log_b$ undoes taking b to power)
$b^{\log_b n} = n$ (taking b to power undoes $\log_b$)

We can use rules for indices to deduce rules for logs:

Rules

1. $\log_b xy = \log_b x + \log_b y$ **log of a product is sum of logs**
2. $\log_b \dfrac{x}{y} = \log_b x - \log_b y$ **log of a ratio is difference of logs**
3. $\log_b 1 = 0$ **log of 1 is 0**
4. $\log_b b = 1$
5. $\log_b \dfrac{1}{a} = -\log_b a$
6. $\log_b x^n = n \log_b x$ **log of a power is power times log**
7. $\log_b a = \dfrac{\log_c a}{\log_c b}$ **changing base rule**

3.1.5   Variables: Irrational, Real and Complex Numbers

The operation of taking a log is the fourth inverse operation we encounter.

Question: what other inverse operations have we covered so far?
Answer:

As with subtraction, division and taking the whole root, application of a log might cause a difficulty: Taking a log of a rational number does not always result in a rational number and taking a log of a real number does not always result in a real number. For example $\log_{10} 2$, $\log_{10} 3$, $\log_{10} 5$ *etc.* are all irrational numbers.

Question: what is an irrational numbers? "Ir" stands for "non".
Answer:

Also, $\log_{10}(-1)$ is not a real number. The main message: **if the base is positive the logarithm of a negative number cannot be a real number.**

General Comments

1. All laws and rules of addition, multiplication and taking to integer power are first introduced for whole numbers and can be verified (not proved) by substitution of whole numbers. It is then postulated (declared) that they extend to wider and wider number sets.

2. These operations can be applied to real numbers to produce results that are real too. Other algebraic operations when applied to real numbers do not necessarily produce results that are real.

3. We can now extend the Order of Operations convention to

Order of Operations (OOO)

1.   Make implicit (invisible) brackets visible (everything raised and everything lowered with respect to the main line is considered to be bracketed)
2.   Perform operations in brackets {[()]} first (inside out)

OOO

Operations in Brackets (including implicit)	Powers powers roots logs	Multiplication × ÷	Addition + -
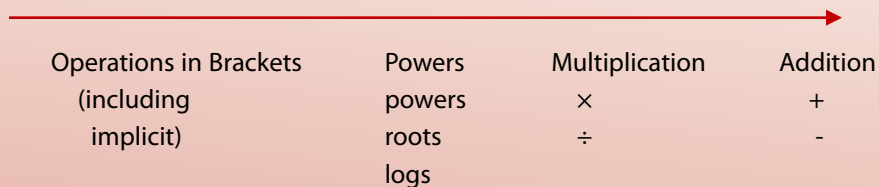
When dealing with expressions involving algebraic operations the convention is to perform operations in brackets first, then the most involved algebraic operations of power and its inverses, then an easier operation of multiplication and its inverse and finally the simplest algebraic operations of addition and its inverse.

3.2   Applications

3.2.1   Solving Quadratic Equations

The general form of a quadratic equation is

$$ax^2 + bx + c = 0,$$

where constants *a, b* and *c* are assumed to be known and *x* is assumed to be an unknown variable.

Question: Why is it an equation?

Answer:

Question: Is it an algebraic equation? Why?

Answer:

Question: Is it a simple algebraic equation?

Answer:

Question: What is a constant?

Answer:

We can solve any quadratic equation by using a trick called "completing the square",

---

Completing the square means re-writing the quadratic expression

$$ax^2 + bx + c = (ax^2 + bx) + c = a(x^2 + \frac{b}{a}x) + c = a[(x + \frac{1}{2}\frac{b}{a})^2 - (\frac{1}{2}\frac{b}{a})^2] + c =$$

$$a(x + \frac{b}{2a})^2 - \frac{b^2}{4a} + c,$$

so that the unknown is present in only one term.

Any simple algebraic equation can be solved using the Decision Tree given in figure 2.1 and therefore completing the square can be used to prove that for any quadratic equation,

$$ax^2 + bx + c = 0$$

we have a formula for its solutions,

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4a}}{2a}$$

The notation means that if we choose the top sign in the RHS, we get solution x_1 and if we choose the bottom sign, we get solution x_2. Let us now produce an (optional) proof of this formula:

Proof: **Optional**

Completing the square we can re-write the quadratic equation $ax^2 + bx + c$ as $a(x + \dfrac{b}{2a})^2 - \dfrac{b^2}{4a} + c = 0$. We can now solve it using the Decision Tree for Solving Equations with the Unknonw in Only One Term.

Step 1: The last specific operation on x in the LHS is adding a constant $-\dfrac{b^2}{4a} + c$

Step 2: Its inverse is $+\dfrac{b^2}{4a} - c$.

Step 3: Applying this inverse operation to both sides of the equation we get $a(x + \dfrac{b}{2a})^2 = \dfrac{b^2}{4a} - c$

which can be re-written using the common denominator method as $a(x + \dfrac{b}{2a})^2 = \dfrac{b^2 - 4ac}{4a}$.

Step 4: x is not the subject of the equation, so we have to go through the solution loop again. The last specific operation on x in the LHS is $\times a$

Step 5: Its inverse is $\times \dfrac{1}{a}$.

Step 6: Applying this inverse to both sides of the equation we get $(x + \dfrac{b}{2a})^2 = \dfrac{b^2 - 4ac}{4a^2}$.

Step 7: x is not the subject of the equation, so we have to go through the solution loop again. The last operation on x in the LHS is squaring.

Step 8: Its inverse is $\sqrt{}$ (the square root or $(\ )^{1/2}$, the power half).

Step 9: Applying this inverse to both sides of the equation we get

$$x + \dfrac{b}{2a} = \dfrac{\sqrt{b^2 - 4ac}}{2a} \text{ or } x_{1,2} + \dfrac{b}{2a} = \pm \dfrac{\sqrt{b^2 - 4ac}}{2a}$$

The latter notation means that if we choose the top sign in the RHS, we get solution x_1 and if we choose the bottom sign, we get solution x_2.

Step 10: The unknown is not the subject of the last equation, we have to go through the solution loop again. The last operation on $x_{1,2}$ in the LHS is $+\dfrac{b}{2a}$.

Step 11: Its inverse is $-\dfrac{b}{2a}$.

Step 12: Applying this inverse to both sides of the equation, we get $x_{1,2} = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

3.2.2   Applications: Factorising Quadratic Expressions

Every quadratic expression $ax^2 + bx + c$ can be factorised as follows:

$$ax^2 + bx + c = a(x - x_1)(x - x_2)$$

where $x_{1,2}$ are

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a},$$

solutions of the corresponding quadratic equation

$$ax^2 + bx + c = 0.$$

$x_{1,2}$ are also known as **roots** of the quadratic.

You can check that factorisation makes sense:

Step 1. Substitute x_1 into the LHS and then RHS. You get 0 in both cases.

Step 2. Substitute x_2 into the LHS and then RHS. You get 0 in both cases.

Step 3. Remove brackets in the RHS. The highest power term is ax^2. It is possible to get the other two terms too.

This factorisation (turning a quadratic into a product) proves extremely useful in many engineering and scientific applications.

3.3   Instructions for self-study

- **Revise Summaries on ALGEBRA and ORDER OF OPERATIONS**
- **Revise Lecture 1 and study Solutions to Exercises in Lecture 1 using the STUDY SKILLS Appendix**
- **Revise Lecture 2 using the STUDY SKILLS Appendix**
- **Study Lecture 3 using the STUDY SKILLS Appendix**
- **Attempt the following exercises:**

Q1. Turn into a single power

a) $2^3 \times 2^5$

b) $c^3 \times c^5$

c) $2^{2+j} \times 2^{2-j}$

d) $\dfrac{s^{1-\sqrt{2}}}{s^{2+\sqrt{2}}}$

e) $\dfrac{e^{1-j\sqrt{2}}}{e^{2+j\sqrt{2}}}$

Q2. Modify into an expression containing a power with a positive index

a) a^{-3}

b) u^{-1}

c) w^{-10}

Q3. Evaluate and check your result:

a) $\sqrt{4}$

b) $\sqrt{9}$

c) $\sqrt{16}$

d) $\sqrt{25}$

e) $36^{1/2}$

f) $49^{1/2}$

g) $8^{1/3}$

h) $27^{1/3}$

i) $\sqrt[3]{64}$

j) $\sqrt[3]{125}$

Q4. Evaluate without a calculator:

    a) $\log_{10} 2 + \log_{10} 5$

    b) $\log_{10} 200 - \log_{10} 2$. Check your result.

    c) $\log_2 8$. Check your result $\log_{10} 10000$.

    d) $\log_2 \dfrac{1}{2}$. Check your result.

    e) $\log_{10} \dfrac{1}{100}$. Check your result.

Q5. Solve the following quadratic equations:

    a) $s^2 + 5s + 6 = 0$

    b) $x^2 + 4x + 4 = 0$

    c) $s^2 - 36 = 0$

Q6. Factorise the following quadratic expressions:

    a) $s^2 + 5s + 6$

    b) $x^2 + 4x + 4$

    c) $s^2 - 36$

Lecture 4   FUNCTIONS

We generalise the concept of an operation on variables as introduced in Algebra by introducing functions.

4.1   Variables

As stated in Section 1.1,

> a **variable** is an **abstraction** of a quantity.
> In calculus, variables are denoted mostly by $x, y, z, p, …, w$.

Abstraction is a general concept formed by extracting common features from specific examples. Specific examples of a quantity are time, distance, magnitude of force, current, speed, concentration, profit and so on.

A variable can take any **value** from an allowed **set of numbers**. If a variable represents a dimensional quantity, that is, a quantity measured in dimensional units $s, m, N, A, m/s, kg/m^3$, £… each value has to be multiplied by the corresponding unit. Otherwise the variable is called non-dimensional.

> **Diagrammatically any** set can be represented (visualised) as a circle (this circle is called a **Venn diagram**). It might help you to think of this circle as a bag containing all elements of the set.
>
> ◯

Examples of commonly used sets of numbers in science and engineering:

I – set of all integers

R – set of all real numbers

All numbers that can be represented **graphically** by points on a number line are called **reals**.

A **real variable** x is a variable that takes only real values, **maybe with some exceptions** (that is, for some reason some real values are excluded)**.** Describing the set of all such values we can say "x real" or "$x \in R$".

> this symbol means **"element of"**

4.2   Functions

A **function** represents a relationship between variables. In elementary calculus we study functional relationships between two variables only, one of them is called an **independent variable**, **input** or a **control variable** and the other, **dependent variable** or **output**. The independent variable is also called an **argument** and dependent variable is also called a **function of** independent variable.

The first choice for a function name is f or f ( ), the second choice is g or g ( ), and the third choice is h or h ( ). The brackets put after the function name stand for the word **of** and are **functional** and **not algebraic**, which means that **no multiplication sign is implied!**

Examples:

1. $f(x) = x + 2$

    This line is verbalised as follows: f of x is x plus 2. The operation $f(\ )$ is "+2".

    $\Rightarrow f(1) = 3$

    This line is verbalised as follows: f at 1 is 3.

2.   A **sequence** $\{x_n\} = x_1, x_2, x_3, \ldots$ is a function of integer n, that is, we can write

    $$x_n = x(n)$$

    A simple example of a sequence is $x_n = n$.

3. $f(x) = |x|$, where the **absolute value** (**modulus**) of x is

    $$|x| \;=\; \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

This definition should be read line by line from top to bottom, but – unusually – each line should be read from right to left: If x is positive or zero, then its absolute value is itself. If x is negative, then its absolute value is its additive inverse, e.g. $|2| = 2$ and $|-2| = 2$.

Note that the **geometrical interpretation** (visualisation) of the modulus based on the number line is the **distance** (from the point representing the number on this line) **to the origin** (the point representing 0).

Question: What is modulus of $-\dfrac{1}{3}$?

Answer:

To specify a function we need to specify a (chain of) operation(s) and **domain** D (a set of allowed values of the independent variable). To each $x \in D$, $f(\ )$ assigns **one and only one** value $y \in R$, where R is **range** (a set of all possible values of dependent variable).

A diagrammatical representation of a function
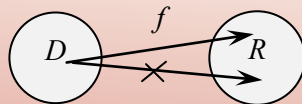
Note that in Elementary Calulus the word *function* has three meanings:

1. $f(\ )$ – **an operation or a chain of operations** on an independent variable;
2. $f(x)$ – a **dependent variable**, that is the variable obtained when $f(\ )$ acts on an independent variable x;
3. Set of pairs $\{(x, f(x)): f(\ )$ assigns one value $f(x)$ to every allowed value of $x\}$.

Which meaning is implied should be understood from the context; if the word *function* is not followed by the word *of* it is most likely to be an operation, otherwise it is most likely to refer to a dependent variable.

In Elementary Calculus we study only real functions of a real variable (that is, of one real argument).

A **real function** has the range that is a set of real numbers. The **natural domain** of any function of a real variable is reals, **with possible exceptions. Exceptions arise due to operations that might not lead to a real number**, such as division by zero, taking the square root (in fact, any even root) or a log. So, a real function of a real variable is a function whose range and domain are both real.

4.3   Elementary operations on functions

1. ADDITION (INCLUDING SUBTRACTION)

$(f \pm g)(x) = f(x) \pm g(x)$ if $D_f = D_g$ (that is, if the functions have the same domain)

2. MULTIPLICATION

$(f \cdot g)(x) = f(x) \cdot g(x)$ if $D_f = D_g$

3. DIVISION

$\dfrac{f}{g}(x) = \dfrac{f(x)}{g(x)}$ if $D_f = D_g$, $g(x) \neq 0$

4. COMPOSITION

$f \circ g(x) \equiv f(g(x))$ if $R_g \subseteq D_f$

symbol of composition

"is subset of"

"is by definition"
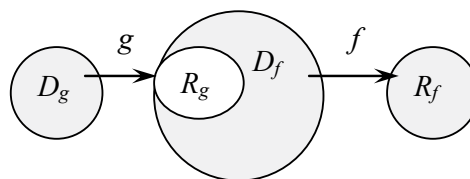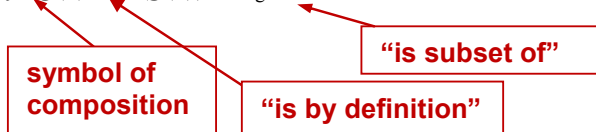
Composition is the chain of operations applied one by one and involves substitution: when performing composition we keep putting in place of successive arguments their actual expressions in terms of the independent variable (in this context, we reserve the name ''the independent variable'' for the argument of the first function in the chain). This means that we keep removing the brackets inside out.

Example: $f(x) = x + 2$, $g(x) = x^2$. **Note: When the natural domain is R it is usually not mentioned.**

1.   Find the composite function $f(g(x))$.

Solution

Step 1.

$f(g(x)) = f(x^2)$ (in place of $g(x)$ we put x^2)

Step 2.

$f(x^2) = x^2 + 2$ (operation f is $+2$, that is *add 2 to whatever you have in brackets*.)

2.   Find the composite function $g(f(x))$.

Solution

Step 1.

$g(f(x)) = g(x+2)$ (in place of $f(x)$ we put $x+2$)

Step 2.

$g(x+2) = (x + 2)^2$ (operation g is $(\ )^2$, that is, *square whatever you have in brackets*.)

We often have to perform an operation of **DECOMPOSITION** (inverse to composition) by sorting out the order of operations in a given expression.

Examples:

1. Decompose the function $h(x) = x^2 + 2$.

Solution

Using the Order of Operations,

  1st operation is $(\ )^2$ (power 2)

  2nd operation is $+2$.

2. Decompose the function $p(x) = (x + 2)^2$

Solution

Using the Order of Operations,

  1st operation is $+2$

  2nd operation is $(\ )^2$ (power 2)

if $R_{f^{-1}} \subseteq D_f$

if $R_f \subseteq D_{f^{-1}}$

5. TAKING INVERSE OF A FUNCTION

$$f^{-1}(x) : f \circ f^{-1}(x) = f^{-1} \circ f(x) = x$$

**symbol of inverse function.
Not to be confused with the reciprocal!**

Note: the inverse function does not always exist (see the diagram below).
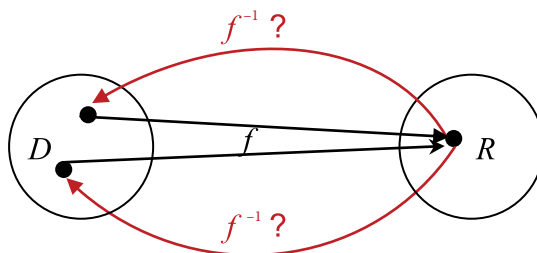
Finding an inverse function (when it exists)

To find an inverse functions use the following **algorithm** (a sequence of steps):

    Step 1.   rename $f(x)$ as y

    Step 2.   solve for x

    Step 3.   rename x as $f^{-1}(y)$
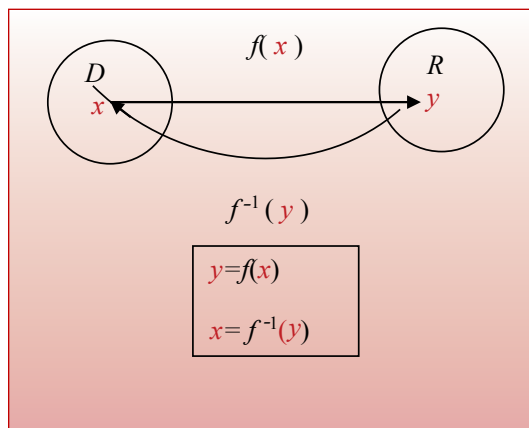
Example: $f(x) = x + 2$. Find $f^{-1}(x)$.

Solution

Step 1.   Rename $f(x)$ as y
           $y = x + 2$
Step 2.   Solve for x
           $x = y - 2$
Step 3.   Rename x as $f^{-1}(y)$
           $f^{-1}(y) = y - 2$
Step 4.   Rename y as x (the independet variable can be denoted by any symbol)
           $f^{-1}(x) = x - 2$

4.4   Order of Operations (extended)

B	$f( )$	P	M	A
including		powers	×	+
implicit		roots	÷	-
		logs		

Note: all operations related to Powers can be considered to be advanced algebraic operations as well as being functions (see below).

4.5   Applications of real functions of real variable and operations on such functions

Any analogue signal representing a physical quantity, such as voltage, current, speed, force, temperature, concentration *etc.* can be considered to be a real function of time (which is a real variable). Many instruments, such as controllers transform signals, that is, perform operations on functions.

Example: Flow-rate (Q) is the rate at which the water, in terms of volume, flows along a channel (see figure 4.1). It is expressed as volume divided by time, in units of m3/s (cubic metres per second). If we assume that flow in a channel in uniform, then the flow-rate is a function of:

- the area of flow in the channel (m^2)
- the hydraulic radius (m)
- the slope of the channel (in the direction of flow)
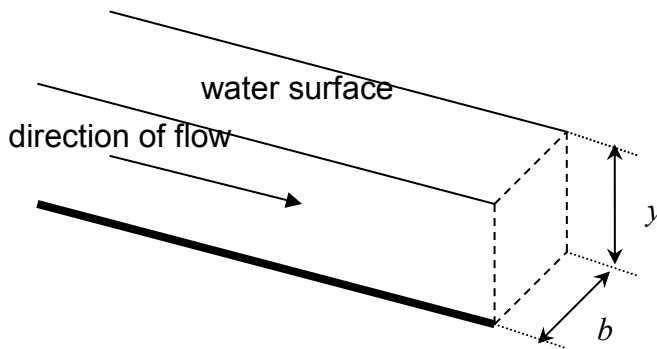- the roughness the channel material that is in contact with the water

Figure 4.1. The longitudinal section of a channel

Specifically this can be represented in the "Manning equation":

$$Q = \frac{A}{n} R^{2/3} S^{1/2}$$

where A is cross-sectional area,

  R is hydraulic radius A / P, with P – 'wetted perimeter' (the length of contact between
   the water and the channel, which is $y + b + y$ for the rectangle),
  S is channel slope in the direction of flow,
  n is Manning coefficient of roughness.

You have been employed on the design of the irrigation scheme intended to provide water to a number of rural villages where clean water for drinking and for agricultural purposes is lacking. It is proposed to provide open 0.9m wide rectangular-section channels lined with concrete with the water flowing to a maximum depth of 1.2 m. It is required to have a capacity of 2.0 $m^3/_s$ (the highest flow-rate it is expected to carry). If the channel is constructed using rough finished concrete ($n = 0.017$), what slope (in the direction of flow) will be needed assuming that the flow is uniform? If the fall of the land is fairly shallow and the maximum fall of the channels for economic construction is restricted to 0.3m per 100 metres is this design adequate?

http://www.raeng.org.uk/education/diploma/maths/pdf/CBE7_Channel_Flow.pdf

4.6   A historical note

The idea of a function dates back to the Persian mathematician, Sharaf al-Dīn al-Tūsī, in the 12th century"
who used this word in the sense of the dependent variable as did Gottfried Leibniz in end of 17th
and beginning of the 18th century and Leonhard Euler in the 18th century. It seems that a more formal
definition of a function as a set of ordered pairs, a relation in which every element from a domain has
a unique element from the range has been given in the 19th century by Lobachevsky and according to
some sources, Dirichlet. The definition of a function as an operation (or a chain of operations) that
assigns to every element of the domain one element from the range was introduced in 1939 by Bourbaki.

4.7   Instructions for self-study

- **Revise Summaries on ALGEBRA, FUNCTIONS and ORDER OF OPERATIONS**
- **Revise ALGEBRA Lecture 2 and study Solutions to Exercises in Lecture 2 using the STUDY SKILLS Appendix**
- **Revise ALGEBRA Lecture 3 using the STUDY SKILLS Appendix**
- **Study Lecture 4 using the STUDY SKILLS Appendix**
- **Attempt the following exercises:**

Q1. Let $f(t) = \dfrac{9}{5}t + 32$. This function converts temperature from ^{0}C to ^{0}F. Evaluate

    a) $f(0)$

    b) $f(100)$

    c) $f(24)$

Q2. The speed $v(t)$ of an object is given by $v(t) = 98t$. Find

    a) $v(t^2)$

    b) $v(t+1)$

Q3. Decompose the following functions (that is, state what is the first operation in $f(x)$ and what is the second). Use the Order of Operations Summary.

    a) $f(x) = \dfrac{1}{x+2}$

    b) $f(x) = \dfrac{1}{x} + 2$

    c) $f(x) = \dfrac{1}{2}(x+2)$

    d) State which of these can be called a sum, a quotient or a product.

    e) What are the factors in this product?

Q4. Given function $f(x) = \dfrac{1}{x+2}$

    a) state what is the natural domain of $f(x)$

    b) find the inverse function $f^{-1}(x)$

    c) what is its natural domain?

Additional Exercises

1. Turn into one power

    a) $2^3 \times 2^5$

    b) $c^3 \times c^5$

    c) $2^{2+j} \times 2^{2-j}$

    d) $\dfrac{s^{1-\sqrt{2}}}{s^{2+\sqrt{2}}}$

    e) $\dfrac{e^{1-j\sqrt{2}}}{e^{2+j\sqrt{2}}}$

2. Modify into a power with positive index

    a) a^{-3}

    b) u^{-1}

    c) w^{-10}

3. Evaluate and check your result:

    a) $\sqrt{4}$

    b) $\sqrt{9}$

    c) $\sqrt{16}$

    d) $\sqrt{25}$

    e) $36^{1/2}$

    f) $49^{1/2}$

    g) $8^{1/3}$

    h) $27^{1/3}$

    i) $\sqrt[3]{64}$

    j) $\sqrt[3]{125}$

4. Evaluate without a calculator:

    a) $\log_{10} 2 + \log_{10} 5$

    b) $\log_{10} 200 - \log_{10} 2$

    c) $\log_{10} 10000$

    d) $\log_{2} 8$

    e) $\log_{2} \dfrac{1}{2}$

    f) $\log_{10} \dfrac{1}{100}$

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Lecture 5   Real FUNCTIONS of One Real Variable: Graphs, Polynomials

5.1   Graphical representation of real functions of one real variable

In the last lecture we gave a diagrammatical representation of a function,
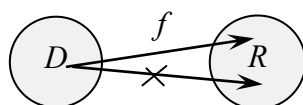
Each particular real function of a real variable can also be visualised using a more specific, **graphical**, representation. On a **graph** of a function $y = f(x)$,

---

domain D is represented by a set of points on a **horizontal axis** (number line), maybe not the whole axis, maybe an interval; it is usually called the x-axis;

range R is represented by a set of points on a **vertical axis** (number line), maybe not the whole axis, maybe an interval; it is usually called the y-axis (note that the point of intersection of the two axes is chosen to represent 0 on both of them);

function f (in the sense of *the collection of all ordered pairs (x, f(x))*) is represented by a set of points, maybe a line on the plane. Each point in this set has **coordinates** x and y (projections onto x- and y-axis, respectively) which are associated by the **functional relationship** $y = f(x)$.

Graphs can be **sketched by using the table** of pairs (x, y).

Examples:

1. $y = 2x$

Table:

x	0	1	...
y	0	2	...

Graph:

2. $y = -2x$

Table:

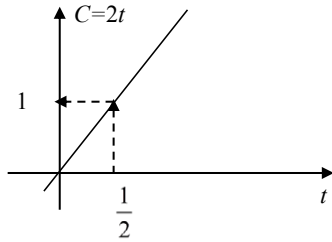| x	0	−1	...
y	0	2	...

Graph:

5.2   How to use graphs of real functions $y = f(x)$ of one real variable

The graph may be used as a rule (an operation or a chain of operations), which, given an x, allows one to find the value of y and *vice versa*.
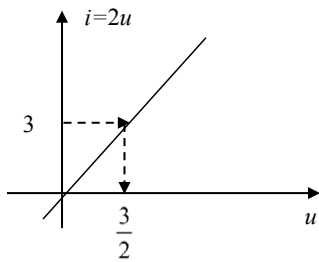
Examples:
1. Below we are given the graph of function $C = 2t$. $C\left(\frac{1}{2}\right)$? (**verbalise**: given t find C)

Answer:

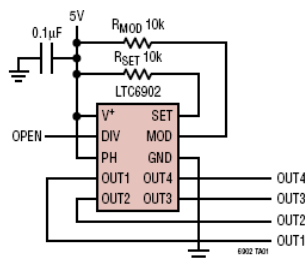2. Below we are given the graph of function $i = 2u$. $u = i^{-1}(3)$? (**verbalise**: given i find u)

Answer:

5.3   Applications of graphs

Specifications of various instruments often contain graphs showing say, the device output at various frequencies (see figure 5.1). Properties of various devices and materials can also often be summarised as graphs (see figure 5.2).
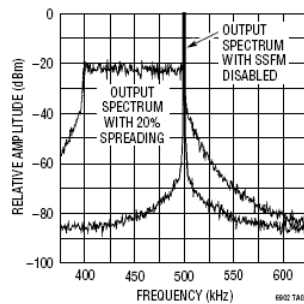
Figure 5.1. Copied from description of an oscillator at
http://www.linear.com/pc/downloadDocument.do?navId=H0,C1,C1010,C1096,P2293,D1568

5.4   Elementary functions: monomials (natural powers) and polynomials

Elementary functions involve only elementary (algebraic and trigonometric) operations.

A **polynomial** is an elementary function, involving only algebraic operations of addition and multiplication
by a constant of **monomials** (natural power functions),

$$P_n(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_0,$$

where n is a natural number and coefficients (factors independent of x) $a_0, a_1, \ldots, a_n$ are constants.
Verbalise: A polynomial is a sum. Each term in this sum is a product of a whole power of x and a
constant (coefficient). If $a_n \neq 0$, n is called the **degree** of $P_n(x)$.

Question: What is a constant?
Answer:

To be precise, we say that $a_0, a_1, \ldots, a_n$ are **constant with respect to** x.

5.4.1   The Main Theorem of Algebra

The Main Theorem of Algebra (without proof)

The polynomial equation $P_n(x) = 0$ has exactly n roots (solutions), $x_1, \ldots, x_n$, which may be equal or complex
(not real).

5.4.2   Factorisation of polynomials

Each polynomial may be factorised

$$P_n(x) = a_n(x - x_1)(x - x_2)...(x - x_n)$$

Note: the derivation of the above formula lies outside the scope of these notes but you can check yourself that it is correct at the roots by substituting x_1 for x, x_2 for x *etc.* You can also check that both LHS (left-hand side) and RHS (right-hand side) of this formula have the same coefficient a_n associated the highest power x^n.

In most cases roots (solutions) of polynomial equations can be found only by using a computer code. However, as discussed in Section 3.2.1, for any quadratic equation $ax^2 + bx + c = 0$ we have a formula for finding roots:

$$ax^2 + bx + c = 0$$

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

where $x_{1,2} = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ is a short–hand for two roots,

$$x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \text{ and } x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

For cubic equations, roots can be found without the help of a computer only in some cases. For example, $x^3 - 3x^2 + 3x - 1 = (x - 1)^3$. Therefore, the equation

$$x^3 - 3x^2 + 3x - 1 = 0$$

has three equal roots, $x_1 = x_2 = x_3 = 1$.

5.4.3   Graphs of Polynomials of Degree 1 (linear), 2 (quadratic) and 3 (cubic)

1.   A polynomial of degree 1, $P_1(x)$.is called a **linear function**. Its graph is called a **straight line.**

Example: Sketch a **monomial** (power function) $y = x$

Table:

x	0	1
y	0	1

Graph:

$D = \mathcal{R}$
$R = \mathcal{R}$

One intersection with the x-axis

NOTE: to draw a straight line all you need is two points. More points are required to sketch other graphs correctly.
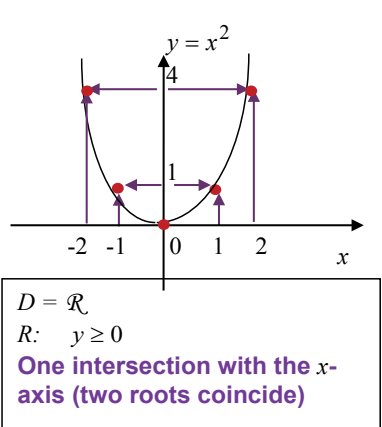
2. A polynomial of degree 2, $P_2(x)$ is called a **quadratic function**. Its graph is called a **parabola**.

Example: Sketch a monomial $y = x^2$

Table:

x	0	1	−1	2	−2
y	0	1	1	4	4

Graph:

$D = \mathcal{R}$
$R: \quad y \geq 0$
One intersection with the x-axis (two roots coincide)

Note: any parabola is symmetric with respect to the vertical line drawn through its tip (a turning point). Three points are enough to sketch a parabola.
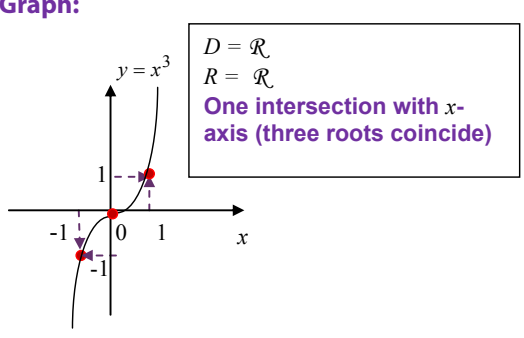
3. A polynomial of degree 3, $P_3(x)$.is called a **cubic function**. Its graph is called a **cubic parabola**.

Example: Sketch a monomial $y = x^3$

Table:

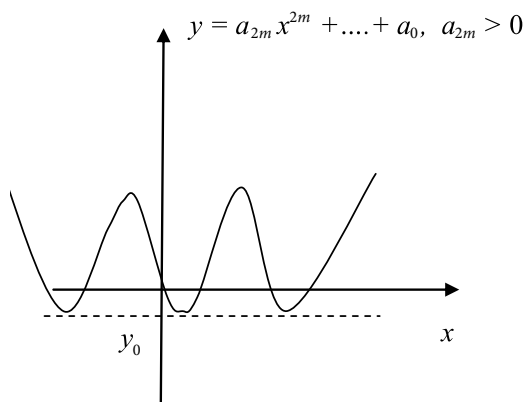| x	0	1	−1	2	−2
y	0	1	1	8	8

Graph:

$D = \mathcal{R}$
$R = \mathcal{R}$
One intersection with x-axis (three roots coincide)

Note: At each intersection with the x-axis, $y=0$. Therefore, the intersections represent roots (solutions) of the equation $y=0$.

5.4.4 Graphs of General Polynomials of Even and Odd Degree

The general polynomial of even degree $2m$**, where** m **is a natural number**
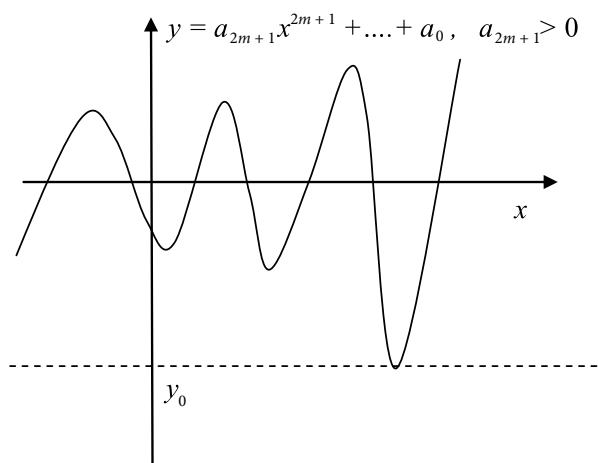
$$y = a_{2m} x^{2m} + ... + a_0, \quad a_{2m} > 0$$

$D = \mathcal{R}$

$R: \quad y \geq y_0$

Both branches up

No more than $2m$
intersections with the x-axis

y_0

x

The general polynomial of odd degree $2m+1$**, where** m **is a natural number**

$$y = a_{2m+1} x^{2m+1} + ... + a_0, \quad a_{2m+1} > 0$$

$D = \mathcal{R}$

$R: \quad y \geq y_0$

Left branch down, right
branch – up

No more than $2m+1$
intersections with the x-axis
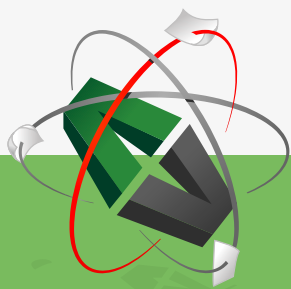
x

y_0

5.5   Instructions for self-study

- **Revise Summaries on FUNCTIONS and ORDER OF OPERATIONS**
- **Revise Lectures 3 and study Solutions to Exercises in Lecture 3 using the STUDY SKILLS Appendix**
- **Revise Lecture 4 using the STUDY SKILLS Appendix**
- **Study Lecture 5 using the STUDY SKILLS Appendix**
- **Attempt the following exercises:**

Q1. If $F(s) = \dfrac{1}{s+1}$, $s \neq -1$, find

a) $F(s-1)$

b) $F(s+1)$

c) $F(\dfrac{1}{s^2 + \omega^2})$

Q2. If $f(t) = \dfrac{1}{t-3}, t \ne 3$ and $g(t) = \dfrac{1}{t+3}, t \ne -3$ what are the following functions (do not forget to describe their natural domains):

    a) $f(g(x))$

    b) $g(f(x))$

    c) $f(f(x))$

Q3. Find the inverse of the function $g(z) = \dfrac{z-1}{z+1}, z \ne -1$. Indicate its natural domain.

Q4.

    a) Plot a graph of the function $T_F = \dfrac{9}{5}T_C + 32, \ 0 \le T_C \le 100$. What is the range of this function?

    b) Sketch the function $f(x) = (x-2)(x+4)$. What is the natural domain of this function? What is the natural range of this function?

Additional Exercises

    1. $f(x) = x^2 - 3x + 2$ (no domain indicated means that domain is R, the set of all reals). Find $g(t) = f(\sqrt{t})$ and indicate the natural domain of $g(t)$.

    2. Find the natural domain of $f(x) = \dfrac{1}{x}$.

    3. Find the natural domain of

    a) $\sqrt{x-2}$

    b) $\sqrt{t^2 - 3t - 2}$

    c) $\sqrt{-t^2 + 3t + 2}$

    4. Sketch $y = 2(x-1)(x-2)(x-3)$ using only information from Lectures 1 – 5.

    5. Sketch $y = \dfrac{1}{2}x^2 + \dfrac{1}{4}x + 5$ using only information from Lectures 1 – 5.

Lecture 6   Real FUNCTIONS of One Real Variable: Exponential Functions, Logarithmic Functions, Inverse Functions

6.1   Exponential functions

In algebra we studied the operation of *raising to power*. It can be used to introduce **exponential functions** $f(x) = b^x$, $b > 0$ (compare to monomial functions $f(x) = x^n$ with n being a whole number, which were covered in Lecture 5). Any other *x*-value could be chosen, you can check that the resulting point would still lie on the same curve.

Note: the position of symbols – that is, whether they are on the main line, lowered or raised – is very important in mathematics.

Often it is convenient to choose the positive number *b* to be $e \approx 2.71$ (**Neper's constant**, an irrational number). Why is it convenient is discussed later.

Question: What is a rational number?
Answer:

Question: What is an irrational number?
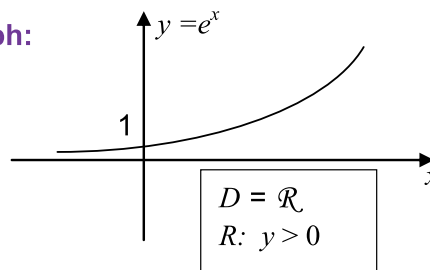**Answer:**

Question: Can we be more specific?
Answer:

Let us use the table to sketch the graph of the exponential function $y = e^x$:

Table:

x	0	1	−1	2	−2
y	1	$e \approx 3$	$\dfrac{1}{e} \approx \dfrac{1}{3}$	$e^2 \approx 9$	$\dfrac{1}{e^2} \approx \dfrac{1}{9}$

Graph:

$$y = e^x$$
$$D = \mathcal{R}$$
$$R: \; y > 0$$

Any other *x*-value could be chosen, you can check that the resulting point would still lie on the same curve.

6.2   Logarithmic functions

In algebra we learned that the inverse operation to *raising b to power* is called $\log_b$. It is defined as follows: $\log_b x = p$: $b^p = x$, so that $\log_b x$ is the power (index, exponent) to which you have to raise *b* to get *x*.

We now introduce a **logarithmic function** $f(x) = \log_b x$. Note that the function (operation) $\log_b (\;)$ is inverse of the exponential function (operation) $b^{(\;)}$. This means that when executed in sequence these two operations cancel each other.

If $b = e$, the logarithmic function is usually written as $f(x) = \ln x$ (this is verbalised as the **natural log of** *x*). Note that the function $\ln (\;)$ is inverse to the function $e^{(\;)}$. We could use a table to sketch the log function and obtain

Table:

x	1	e	e^{-1}	e^2	e^{-2}
y	0	1	−1	2	−2

Graph:

$$y = \ln x$$
$$D: \; x > 0$$
$$R = \mathcal{R}$$
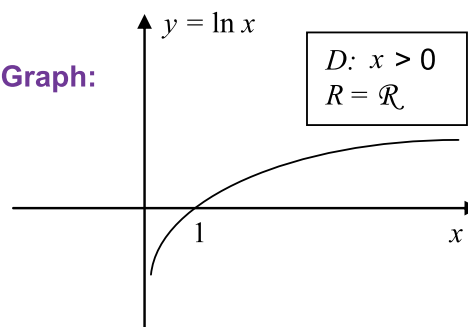
A much better way of sketching the log functions is to use the following general algorithm for sketching inverse functions:

SKETCHING AN INVERSE FUNCTION $y = f^{-1}(x)$

**Step 1.** Sketch the graph of function $y = f(x)$.
   Note that it is also the graph of inverse function $x = f^{-1}(y)$.
Step 2. Sketch the "mirror" line $y = x$.
   Note that every point on this line has y-coordinate the same as x-coordinate.
Step 3. If horizontal and vertical scales are the same, sketch the mirror image
   $y = f^{-1}(x)$ of the original graph $y = f(x)$ with respect to the line $y = x$.

$y = f(x)$, $x = f^{-1}(y)$ *(dashed line)*, $y = f^{-1}(x)$ **(solid line)**
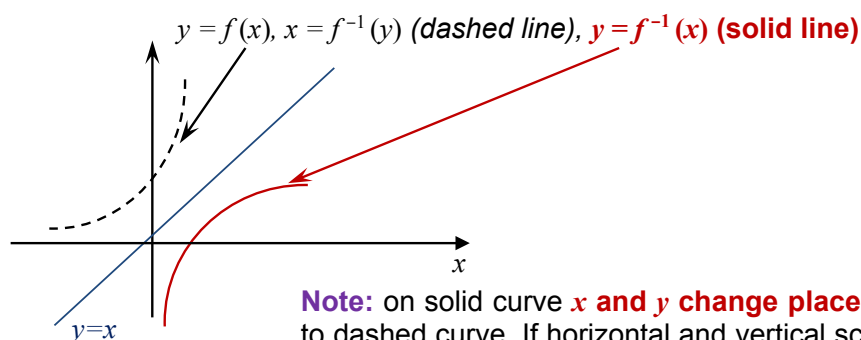
Note: on solid curve x **and** y **change places** compared to dashed curve. If horizontal and vertical scales are the same this corresponds to reflection of the dashed line with respect to the line $y = x$.

6.3   Revision: Trigonometry

6.3.1   Angles, Degrees and Radians

Let us consider a segment of length r and rotate it around a fixed end until it returns to the original position. The angle described after a full rotation is said to be 360° (**Verbalise:** 360 **degrees**). The degree is a **dimensional unit of angle**.

Note: the degree is a dimensional unit in the sense that it has a convenient but arbitrarily chosen size (dimension). For example, we could just as well subdivide the full rotation into 720 units and call the new unit of angle *a degree*.

The line described by the other end of the segment is called **circumference**. If this end does not complete the full rotation the line is called a **circular arc** *s* – see figure 6.1 below**.**
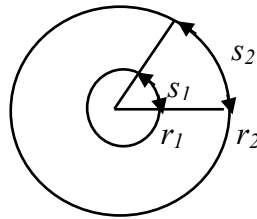

Figure 6.1. Circumferences of radii r_1 and r_2 and circular arcs s_1 and s_2.

It turns out that whatever the length of the segment (which is called the **radius** of this circumference), the quotient

$$\frac{\text{length of circumference}}{\text{radius}}$$

is always the same number called 2π, where π (a Greek letter *pi*) is short hand for an irrational number $\pi \approx 3.14$. By the same token, whatever the radius the quotient

$$\frac{\text{length of circular arc}}{\text{radius}}$$

is always the same number. This number can be taken to be a dimensionless measure of the corresponding angle, without involving any units. Yet, for historical reasons, it is customary to call this number **radian**, the dimensionless unit of angle**.** So, while we *write* that the angle of full rotation is 2π, we sometimes *say* that the angle of full rotation is 2π radians**:**

$$\Rightarrow 2\pi \, (rad) = 360° \Big/ \div 2\pi$$

$$\Rightarrow 1 \, (rad) = \frac{360°}{2\pi} = \frac{180°}{\pi} \approx 57°$$

MEMORISE

Thus, 1 radian is the angle $\approx 57°$ such that whatever the radius, the corresponding circular arc has the same length as this radius – see figure 6.1. Similarly, for any angle,

$$\Rightarrow x \, (rad) = x \, \frac{180°}{\pi} = y^0$$

MEMORISE

where x is the number of radians (angles which are $\approx 57°$) in a given angle and y is the number of degrees in this angle.

> **Note: The radian is a dimensionless (not arbitrary) unit** of angle.

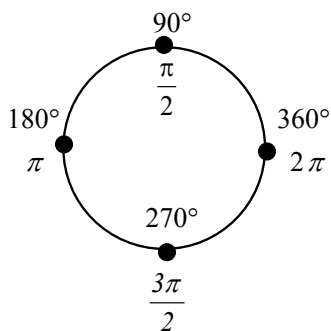**Question:** What is the angle $y = 90°$ in radians and why?
Answer:

Question: What is the angle $y = 180°$ in radians and why?
Answer:

Question: What is the angle y = 270° in radians and why?
Answer:

Here are a few more frequently used angles:

|	
$30° = (\dfrac{30\pi}{180} =) \dfrac{\pi}{6}$	$45° = (\dfrac{90\pi}{180} =) \dfrac{\pi}{4}$
$60° = (\dfrac{60\pi}{180} =) \dfrac{\pi}{3}$	
$120° = (\dfrac{120\pi}{180} =) \dfrac{2\pi}{3}$	

6.3.2   A historical note

"The concept of radian measure, as opposed to the degree of an angle, is normally credited to Roger Cotes in 1714. He had the radian in everything but name, and he recognized its naturalness as a unit of angular measure. The idea of measuring angles by the length of the arc was used already by other mathematicians. For example al-Kashi (c. 1400) used so-called diameter parts as units where one diameter part was 1/60 radian and they also used sexagesimal subunits of the diameter part.

The term radian first appeared in print on 5 June 1873, in examination questions set by James Thomson (brother of Lord Kelvin) at Queen's College, Belfast. He used the term as early as 1871, while in 1869, Thomas Muir, then of the University of St Andrews, vacillated between rad, radial and radian. In 1874, Muir adopted radian after a consultation with James Thomson." http://en.wikipedia.org/wiki/Radian

6.3.3   The Right Angle Triangles, Pythagoras Theorem and Trigonometric Ratios

1. The **Right Angle Triangle**
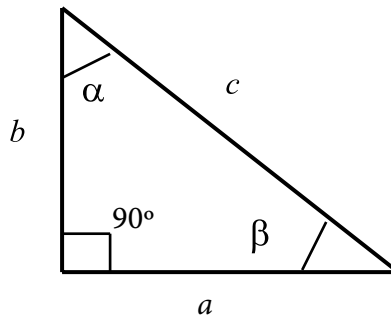
The **right angle** is the angle of 90°. Side c which is opposite the right angle is called the **hypotenuse**.

Note: if $\alpha + \beta = 90°$, then and α and β are called **complementary angles**.

2. **Pythagoras Theorem** states that in any right angle triangle

$$c^2 = a^2 + b^2$$
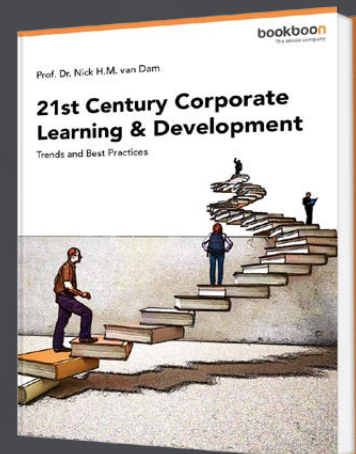
3. **The trigonometric ratios** sin, cos and tan are defined for **acute angles** (that is, the angles that are less than 90º) inside any right angle triangle as follows:

$$\sin \alpha = \frac{\text{lengh of side opposite to } \alpha}{\text{length of hypothenuse}} = \frac{a}{c}$$

$$\cos \alpha = \frac{\text{lengh of side adjacent to } \alpha}{\text{length of hypothenuse}} = \frac{a}{c}$$

$$\tan \alpha = \frac{\text{lengh of side opposite to } \alpha}{\text{lengh of side adjacent to } \alpha} = \frac{a}{b}$$

Interestingly, these ratios do not depend on the length of the triangle sides. The above definitions imply two simple trigonometric **identities** (**formulae**, mathematical statements that are always true):

$$\sin \alpha = \cos \beta$$
$$\sin \beta = \cos \alpha$$

Here are frequently used specific trigonometric ratios:

$\sin \dfrac{\pi}{6} = \dfrac{1}{2}$	$\sin \dfrac{\pi}{4} = \dfrac{\sqrt{2}}{2}$
$\cos \dfrac{\pi}{6} = \dfrac{\sqrt{3}}{2}$	$\cos \dfrac{\pi}{4} = \dfrac{\sqrt{2}}{2}$
	$\tan \dfrac{\pi}{4} = 1$

MEMORISE

Examples:

1. Consider the following right-angle triangle:

a) $\beta = 90° - 30° = 60°$

b) $a = \dfrac{1}{2}b$

$a^2 + b^2 = c^2$
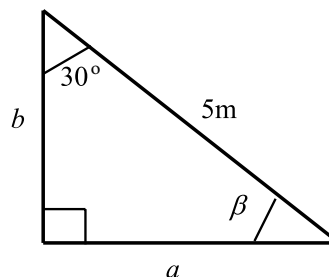
$\dfrac{1}{4}b^2 + b^2 = c^2$

$\dfrac{5}{4}b^2 = 25$

$b^2 = 20$

$b = 2\sqrt{5}$

$a = \sqrt{5}$

2. Consider the following right-angle triangle:

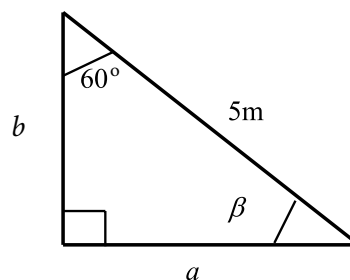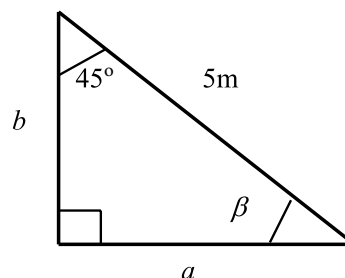a) $\beta = 90° - 30° = 60°$

b) $a = 2b$

$a^2 + b^2 = c^2$

$4b^2 + b^2 = c^2$

$5b^2 = 25$

$b^2 = 20$

$b = \sqrt{5}$

$a = 2\sqrt{5}$

3. Consider the following right-angle triangle:

a) $\beta = 90° - 45° = 45°$

b) a = b

$a^2 + b^2 = c^2$

$2b^2 = c^2$

$2b^2 = 25$

$b^2 = \dfrac{25}{2}$

$b = \dfrac{5}{\sqrt{2}}$
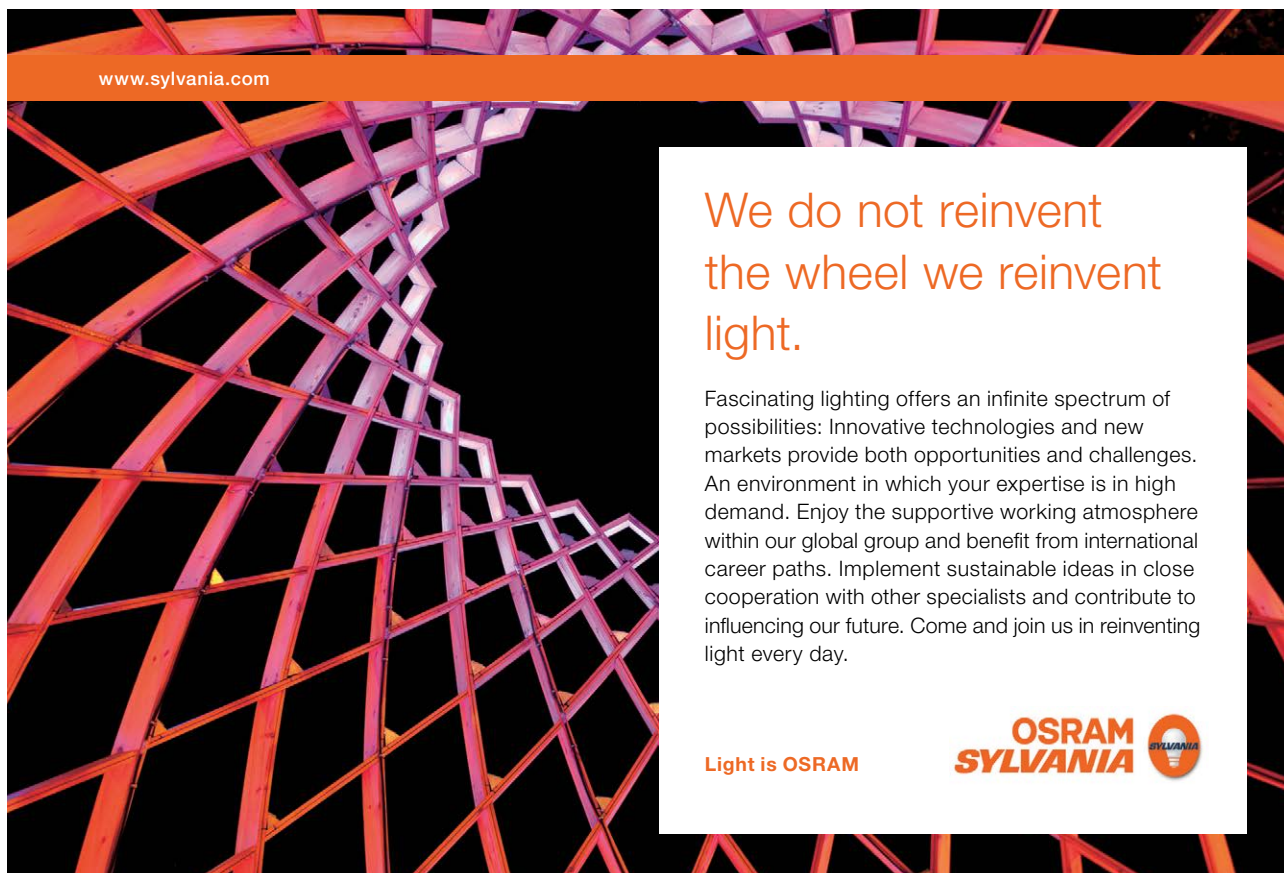
$a = \dfrac{5}{\sqrt{2}}$

82

6.4　A historical note

"The method of logarithms was first publicly propounded in 1614, in a book entitled *Mirifici Logarithmorum Canonis Descriptio,* by John Napier, Baron of Merchiston, in Scotland. (Joost Bürgi independently discovered logarithms; however, he did not publish his discovery until four years after Napier.) Early resistance to the use of logarithms was muted by Kepler's enthusiastic support and his publication of a clear and impeccable explanation of how they worked.

Their use contributed to the advance of science, and especially of astronomy, by making some difficult calculations possible. Prior to the advent of calculators and computers, they were used constantly in surveying, navigation, and other branches of practical mathematics. It supplanted the more involved method of prosthaphaeresis,… a quick method of computing products". http://en.wikipedia.org/wiki/Logarithm#History

6.5　Instructions of self-study

- **Revise Summaries on FUNCTIONS and ORDER OF OPERATIONS**
- **Revise ALGEBRA Lecture 4 and study Solutions to Exercises in Lecture 4 using the STUDY SKILLS Appendix**
- **Revise FUNCTIONS Lecture 5 using the STUDY SKILLS Appendix**
- **Study Lecture 6 using the STUDY SKILLS Appendix**
- **Attempt the following exercises:**

Q1. Simplify using algebraic laws and rules if you can and then evaluate if you can:
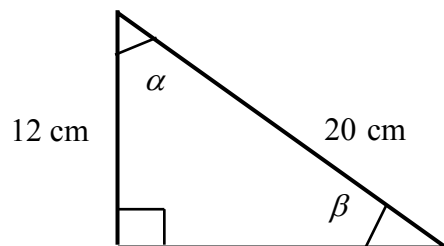
   a) $\dfrac{10^5}{10^3}$

   b) $\dfrac{10^5}{2^3}$

   c) $\dfrac{10^3}{5^5}$

   d) 2^{3^2}

   e) $\sqrt{x^2 + a^2}$

   f) $(\sqrt{x^2})^2$

   g) $\ln e^{10}$

   h) $\ln e^{100}$

   i) $e^{\ln 100}$

   j) $e^{\log_{10} 100}$

Q2.

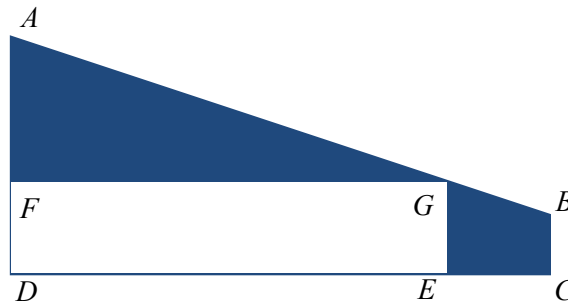   a) Sketch the function $f(x)=x^3$.

   b) Sketch its inverse $f^{-1}(x)$.

Q3.

   a) Given the right angle triangle

   find the length of the side opposite the angle α using Pythagoras Theorem.

b) *ABCD* is the end wall in an attic. Segment *AB* is the sloping ceiling. *AD* = 1600 mm.

Ramona decides to hide a large gift box, *DEGF*, in the attic. If the box is 600 *mm* high, what is its length?

c) Express the following angles in radians:

   i)   120°
   ii)   135°
   iii) 150°
   iv)   210°
   v)   240°
   vi)   300°
   vii) 330°

d) Express the following angles in degrees:

   i) $\dfrac{\pi}{5}$

   ii) $\dfrac{5\pi}{12}$

   iii) 3

Q4.

a) Prove that Pythagoras Theorem can be reformulated as $\sin^2 \alpha + \cos^2 \alpha = 1$.

b) Use Pythagoras Theorem to prove that

$$\sin\frac{\pi}{4} = \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$$

c) Given that $\sin\dfrac{\pi}{6} = \dfrac{1}{2}$ use the Pythagoras Theorem to prove that $\cos\dfrac{\pi}{6} = \dfrac{\sqrt{3}}{2}$.

Lecture 7   Real FUNCTIONS of One Real Variable: Trigonometric Functions, Inverse Trigonometric Functions, Hyperbolic Functions

7.1   Trigonometric functions

We can generalise trigonometric ratios using **URD** (the **Unit Rod Diagram**) to define **trigonometric functions** of *any* angle, not necessarily acute. **URD** is a circumference of radius 1 supplied with the Cartesian coordinate system whose origin lies at the centre of the circumference.
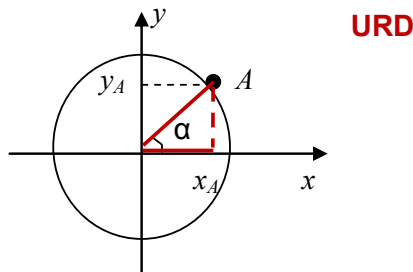
Every point A on URD is described by its polar angle α. We draw a straight segment to connect point A to the origin and project point A onto the x-axis and y-axis.

Question: In the resulting right-angle triangle, what is the relationship between the length x_A of the side adjacent to α and point A?
Answer:

Question: What is the relationship between the length y_A of the side opposite to α and point A?
Answer:

Question: What is cos α here?
Answer:

Question: What is sin α here?

Answer:

**Generalisation**: from now on, for any angle α, acute or not, we say that cos $\alpha = x_A$ and sin $\alpha = y_A$, where A is the point on the URD whose polar angle is α and the Cartesian coordinates are x_A and y_A.

We can sketch the graphs of the newly introduced functions y = sin α and $y = \cos \alpha$ using the following table:

Table:

angle α	point A	$\cos \alpha = x_A$	$\sin \alpha = y_A$
0		1	0
$\dfrac{\pi}{4}$		$\dfrac{\sqrt{2}}{2}$	$\dfrac{\sqrt{2}}{2}$
$\dfrac{\pi}{2}$		0	1
$\dfrac{3\pi}{4}$		$-\dfrac{\sqrt{2}}{2}$	$\dfrac{\sqrt{2}}{2}$
π		-1	0
$\dfrac{5\pi}{4}$		$-\dfrac{\sqrt{2}}{2}$	$-\dfrac{\sqrt{2}}{2}$
$\dfrac{3\pi}{2}$		0	-1
$\dfrac{7\pi}{4}$		$\dfrac{\sqrt{2}}{2}$	$-\dfrac{\sqrt{2}}{2}$
2π		1	0

Graphs of two functions, sin and cos:

$D = \mathcal{R}$ $R: -1 \le y \le 1$	$T = 2\pi$
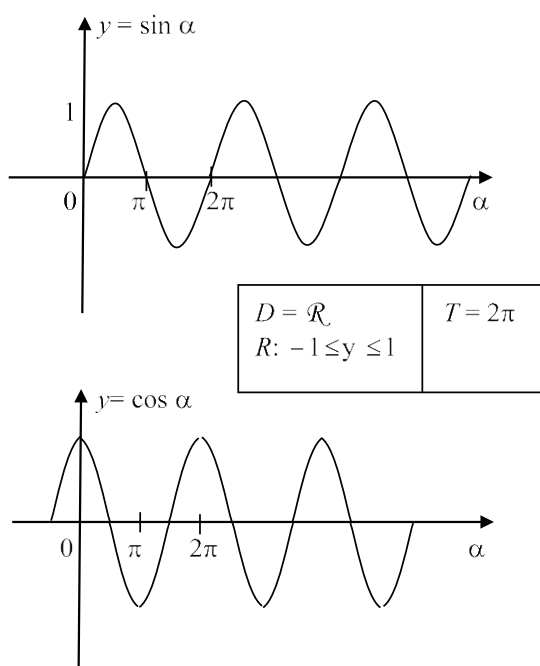
The entry $T = 2\pi$ in the table directly underneath the graph of the sine-function means that both sine and cosine functions are **periodic**, i.e. there exists a number T: $f(t + T) = f(t)$. The smallest positive T is called a **period**.

7.1.1   Useful Trigonometric Identities (Formulae)

1. $\sin x = \cos\left(\dfrac{\pi}{2} - x\right), \qquad \cos x = \sin\left(\dfrac{\pi}{2} - x\right).$

Note: this is true for complementary acute angles x and $\dfrac{\pi}{2} - x$. Prove that this identity holds for all angles x and not just acute angles.

2. $\sin^2 x + \cos^2 x = 1,$
$\sin(-x) = -\sin x,$
$\cos(-x) = \cos x,$

$\tan(-x) = -\tan x,$ **where** $\tan x = \dfrac{\sin x}{\cos x}.$
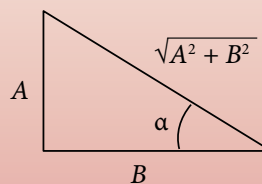
Note: we can check these identities by using URD or else by using graphs of sine and cosine functions.

Extra Trigonometric Identities

$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$

$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$

$A \cos x + B \sin x = \dfrac{1}{\sqrt{A^2 + B^2}} \sin(x + \alpha)$, where $\tan \alpha = \dfrac{A}{B}$
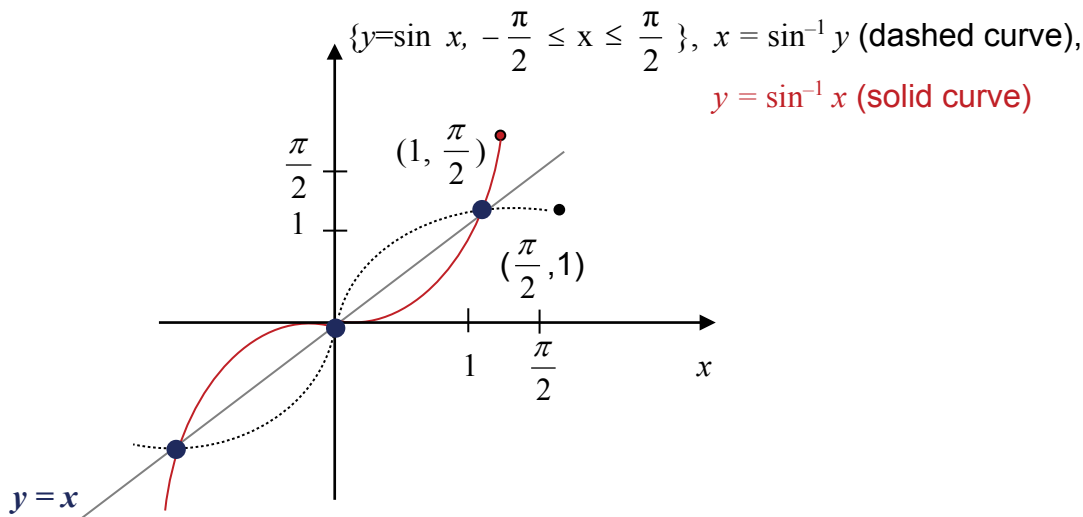

7.2   Inverse trigonometric functions

7.2.1   Inverse Sine ($\sin^{-1}$, arcsin)

$f(x) = \sin x$ has **no inverse on its natural domain**, because this function can be used to assign to every y any number of values of x.

Let us **restrict its domain** to the interval where sin x is only **increasing** or only **decreasing** (goes up, up, up or down, down, down, respectively). Conventionally, we choose the interval $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$ When using notation sin^{-1} or arcsin this is the choice implied.

$$\{y = \sin\ x,\ -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}\},\ x = \sin^{-1} y \text{ (dashed curve)},$$
$$y = \sin^{-1} x \text{ (solid curve)}$$

Note: when sketching the inverse of a function by creating its mirror image with respect to the line $y = x$ divide the graph of the original function into portions utilising special points. Then reflect the original graph portion by portion. In the case of sine-graph the special points are the intersections with the $y = x$ line. These points do not move when performing reflection (nothing changes when x and y exchange places since they are the same!)

7.2.2   Inverse Cosine (cos^{-1}, arcos)

$f(x) = \cos x$ has **no inverse on its natural domain**, because this function can be used to assign to every y any number of values of x.

Let us **restrict its domain** to the interval where cos x is only increasing or only decreasing (goes up, up, up or down, down, down, respectively). Conventionally, we choose the interval $0 \leq x \leq \pi$. When using notation cos^{-1} or arccos this is the choice implied.
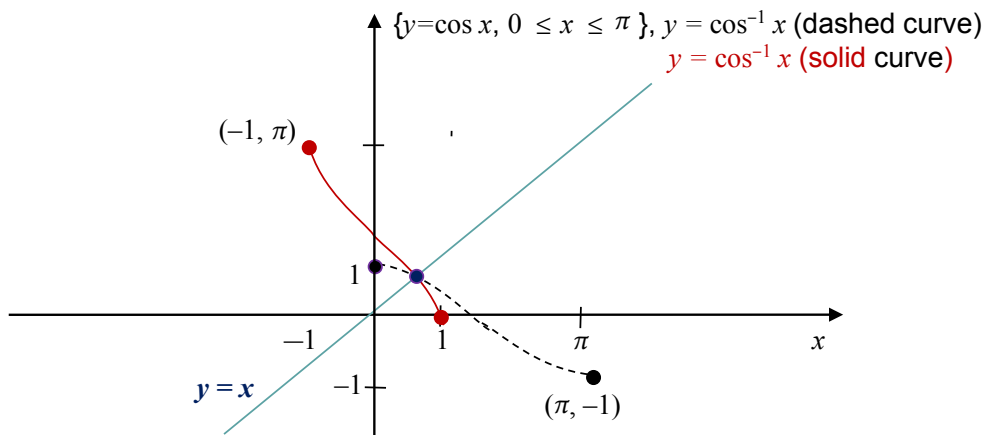
Note: when creating a mirror image with respect to the line $y = x$ divide the cosine-graph into portions by using its intersection with the $y = x$ line and its intersection with the x-axis. Then reflect the original graph portion by portion.

7.3   Hyperbolic functions and their inverses

7.3.1   The Hyperbolic Sine, $\sinh x = \dfrac{e^x - e^{-x}}{2}$

$y = \sinh x, \quad x = \sinh^{-1} y$ (dashed curve)

$y = \sinh^{-1} x$ (solid curve)

x

$y = x$
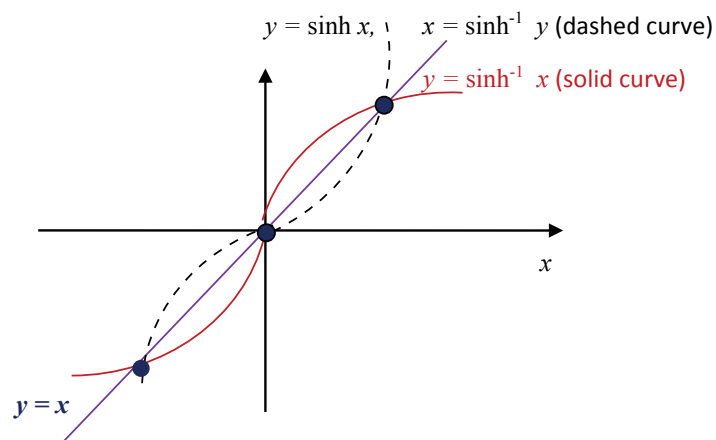
Note: The hyperbolic sin can be **sketched pointwise** (point by point), by using elementary operations on functions (see Section 4.3). Here are the steps to follow:
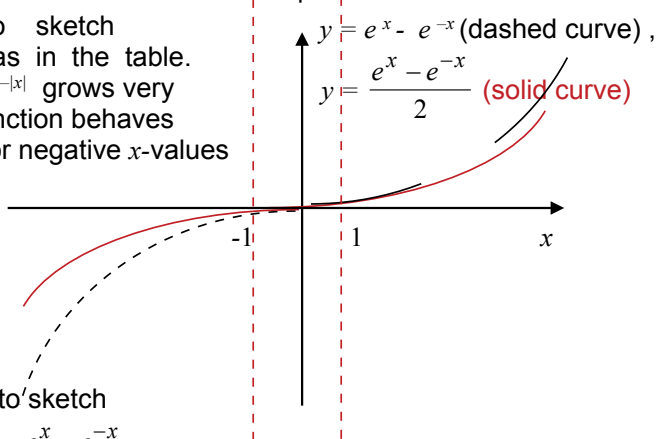
Step 1. Use tables to draw $y=e^x$ and $y= e^{-x}$ underneath one another.

x	0	1	-1	2	-2
e^x	1	$e \approx 3$	$\dfrac{1}{e} \approx \dfrac{1}{3}$	$e^2 \approx 9$	$\dfrac{1}{e^2} \approx \dfrac{1}{9}$

| x	0	-1	1	-2	2
e^{-x}	1	$e \approx 3$	$\dfrac{1}{e} \approx \dfrac{1}{3}$	$e^2 \approx 9$	$\dfrac{1}{e^2} \approx \dfrac{1}{9}$
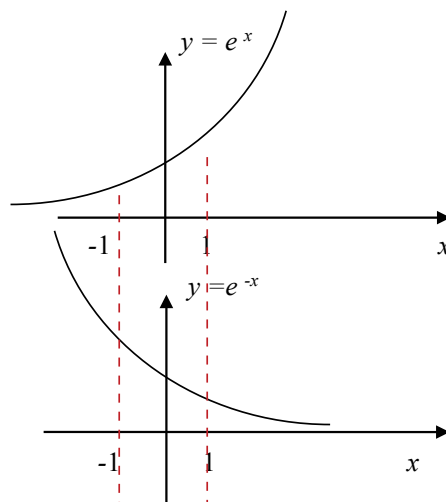
Step 2. Use pointwise subtraction to sketch $y = e^x - e^{-x}$. Use the same x-values as in the table. Then note that when $|x|$ grows large $e^{-|x|}$ grows very small, so that when $|x|$ is large the function behaves as e^x for positive x-values and as $-e^{-x}$ for negative x-values

Step 3. Use the pointwise division by 2 to sketch
$$y=\frac{e^x - e^{-x}}{2}.$$

Note: When sketching the inverse by creating its mirror image with respect to the line $y = x$ divide the graph into portions by using special points. In this case the special points are the intersection of the hyperbolic sine with the $y = x$ line. These points will not move when performing reflection. Then reflect the graph of the hyperbolic sine portion by portion.

7.3.2   The Hyperbolic Cosine, $\cosh x = \dfrac{e^x + e^{-x}}{2}$

$y = \cosh x,\ x = \cosh^{-1} y$ (dashed curve)
$y = \cosh^{-1} x$ (solid curve)
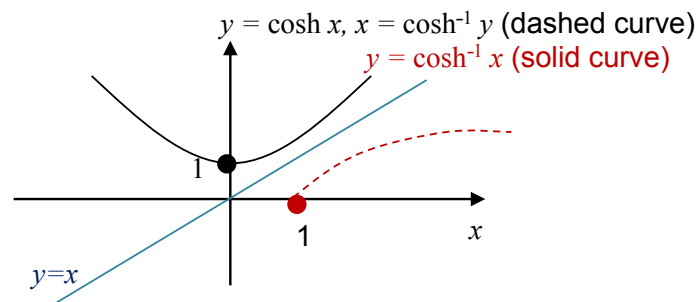
The hyperbolic cosine can be **sketched pointwise**, by using elementary operations on functions as above – first addition, then division by 2 (see Section 4.3).

The hyperbolic function $f(x) = \cosh x$ has **no inverse on its natural domain**, because to every y we can assign two x's. Let us **restrict its domain** to the interval where $\cosh x$ is only increasing or only decreasing (goes up, up, up or down, down, down, respectively). Conventionally, $y = \cosh^{-1} x$ is inverse function to $y = \cosh x,\, x \geq 0$

7.4   Instructions for self-study

- **Revise Summaries on FUNCTIONS and ORDER OF OPERATIONS**
- **Revise Lecture 5 and study Solutions to Exercises in Lecture 5 using the STUDY SKILLS Appendix**
- **Revise Lecture 6 using the STUDY SKILLS Appendix**
- **Study Lecture 7 using the STUDY SKILLS Appendix**
- **Attempt the following exercises:**

Q1.

    a) Sketch $y = \tan x$

    b) What is its natural domain?

    c) What is its period?

    d) Sketch $y = \tan^{-1} x$

Q2. Simplify

    a) $y = \sin(-x)$

    b) $y = \sin\left(\dfrac{\pi}{2} - x\right)$

    c) $y = \cos(-x)$

    d) $y = \cos\left(\dfrac{\pi}{2} - x\right)$

Q3.

    a) Explain why function $y = x^2$ has no inverse.

    b) Suggest how many inverses can you define for it?

    c) Sketch all of them.

Q4.

    a) Sketch the function $y = \tanh x$.

    b) Sketch its inverse.

# Lecture 8   Real FUNCTIONS of One Real Variable: Sketching and Using Graphs, Simple Transformations

In Lecture 5 we have learned how to sketch graphs using a table as well as how to use these graphs to find a value of a dependent variable given a value of an independent variable and *vice versa*. Here is some revision.
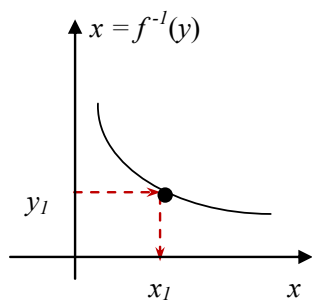
8.1   Sketching graphs using a table
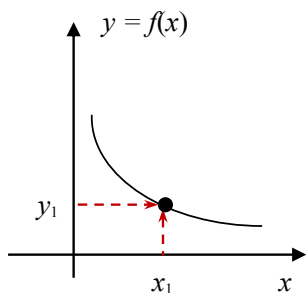
Verbalise: Given a value x_1 of independent variable and a value y_1 of dependent variable find the point with x-coordinate x_1 and y-coordinate y_1.

8.2   Using graphs to find *y* given *x*
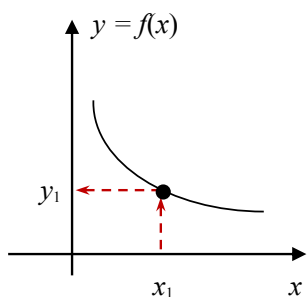
Verbalise: Given a value x_1 of independent variable find a point on the curve with x-coordinate x_1 and project it onto the y-axis to find the value y_1 of dependent variable.

8.3   Using graphs to find *x* given *y*

$x = f^{-1}(y)$

Verbalise: Given a value y_1 of dependent variable find the point on the curve with *y*-coordinate y_1 and project it onto the *x*-axis to find the value x_1 of dependent variable

y_1

x_1 x

Note: the same graph can be used to visualise function $y = f(x)$ and its inverse $x = f^{-1}(y)$ – **provided this inverse exists.**
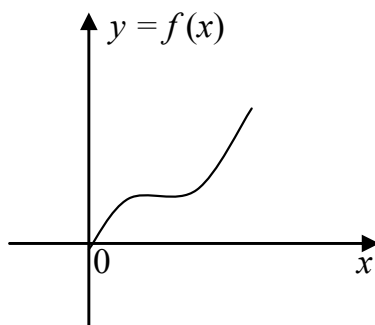
8.4   Sketching graphs using simple transformations

Sketching graphs using the table is advisable when trying to recall the shape of one of elementary functions. However, if you do not know the shape of the graph of a composite function you are likely to create an incomplete table. There are more reliable ways of sketching composite functions.

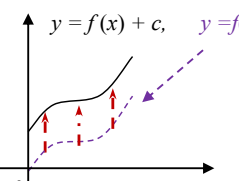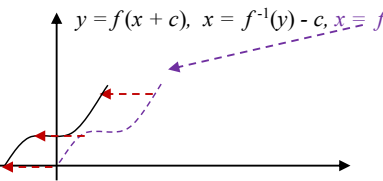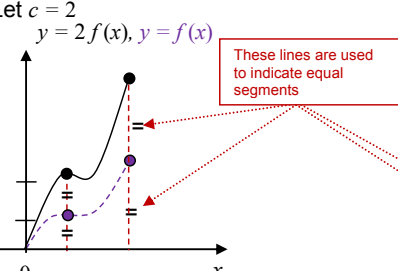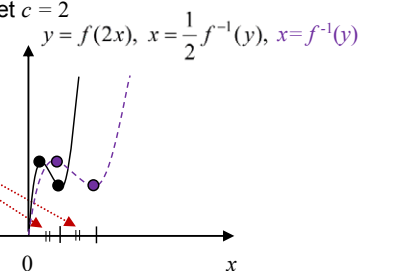**Question:** Which elementary functions have we covered?
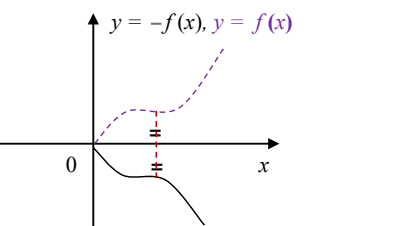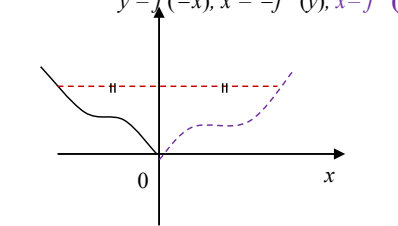
Answer:

Let us assume that we know how to sketch a function $y = f(x)$. Say it looks like that

$y = f(x)$

0   x

Let us consider composite functions which involve the rule $f (\ )$ and addition of a constant or else multiplication by a constant. Such composite functions can be sketched using **simple transformations** (translation, scaling or reflection) of the curve $y = f(x)$. **In order to see what transformation is required**, it is useful to make **the transformed variable the subject of the equation**.

Let us discuss above six transformations in detail (below we use the abbreviation **wrt** for *with respect to*):

Sketch $y = f(x) + c$
Let $c > 0$

$y = f(x) + c,$ $y = f(x)$

Algebraic description:
    $+c$ – **last operation**
    For every x, old $y = f(x)$
                new $y = f(x) + c \Rightarrow$
                new $y =$ old $y + c$

Geometrical description:
    translation wrt y-axis by c
    $c > 0$ - shifting up,
    $c < 0$ - shifting down

Sketch $y = f(x + c)$
Let $c > 0$

$y = f(x + c),$ $x = f^{-1}(y) - c, x = f^{-1}(y)$

Algebraic description:
    $+c$ – **first operation**
    For every y, old $x = f^{-1}(y)$
                new $x = f^{-1}(y) - c \Rightarrow$
                new $x =$ old $x - c$

Geometrical description:
    translation wrt x-axis by $-c$
    $c > 0$ - shifting left,
    $c < 0$ - shifting right

Sketch $y = cf(x)$, $c > 0$

Let $c = 2$
    $y = 2f(x),$ $y = f(x)$

> These lines are used to indicate equal segments

Algebraic description:
    $\times$ +ve constant -- **last operation**
    For every x, old $y = f(x)$
                new $y = c \cdot f(x) \Rightarrow$
                new $y = c \cdot$ old y

Geometrical description:

    scaling wrt y-axis by c

    $c > 1$ - stretching wrt y-axis
    $c < 1$ - squashing wrt y-axis

Sketch $y = cf(x)$, $c > 0$

Let $c = 2$
    $y = f(2x),$ $x = \dfrac{1}{2} f^{-1}(y),$ $x = f^{-1}(y)$

Algebraic description:
    $\times$ +ve constant -- **first operation**
    For every y, old $x = f^{-1}(y)$

                new $x = \dfrac{1}{c} f^{-1}(y) \Rightarrow$

                new $x = \dfrac{1}{c} \cdot$ old x

Geometrical description:

    scaling wrt x-axis by $\dfrac{1}{c}$

    $c > 1$ - squashing wrt x-axis
    $c < 1$ - stretching wrt x-axis

Sketch $y = -f(x)$

$y = -f(x),$ $y = f(x)$

Algebraic description:
    $\cdot(-1)$ - **last operation**
    For every x, old $y = f(x)$,
                new $y = -f(x) \Rightarrow$
                new $y = -$old y
Geometrical description:
    reflection wrt the x-axis

Sketch $y = f(-x)$

$y = f(-x),$ $x = -f^{-1}(y),$ $x = f^{-1}(y)$

Algebraic description:
    $\cdot(-1)$ - **last operation**
    For every y, old $x = f^{-1}(y)$,
                new $x = -f^{-1}(y) \Rightarrow$
                new $x = -$old x
Geometrical description:
    reflection wrt the y-axis

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8.5   Sketching graphs using several simple transformations

If a composite function involves f (whose graph is known) and several operations of addition of a constant or multiplication by a constant, then the graph of this composite function can be sketched by simple transformations of the graph of f using the following recipe:

Step 1. Drop all constant factors and terms.

Step 2. Bring the constants back one by one in **Order of Operations** (not necessary but advisable) and at each step use the **Decision Tree** in figure 8.1 to decide which simple transformation is effected by each constant. Sketch the resulting graphs underneath one another.

Note 1: if c is a **negative factor**, write $c = (-1) * |c|$, so it effects two operations.

Note 2: if c affects **neither the first operation nor last**, the Decision Tree is not applicable. Try algebraic manipulations first.
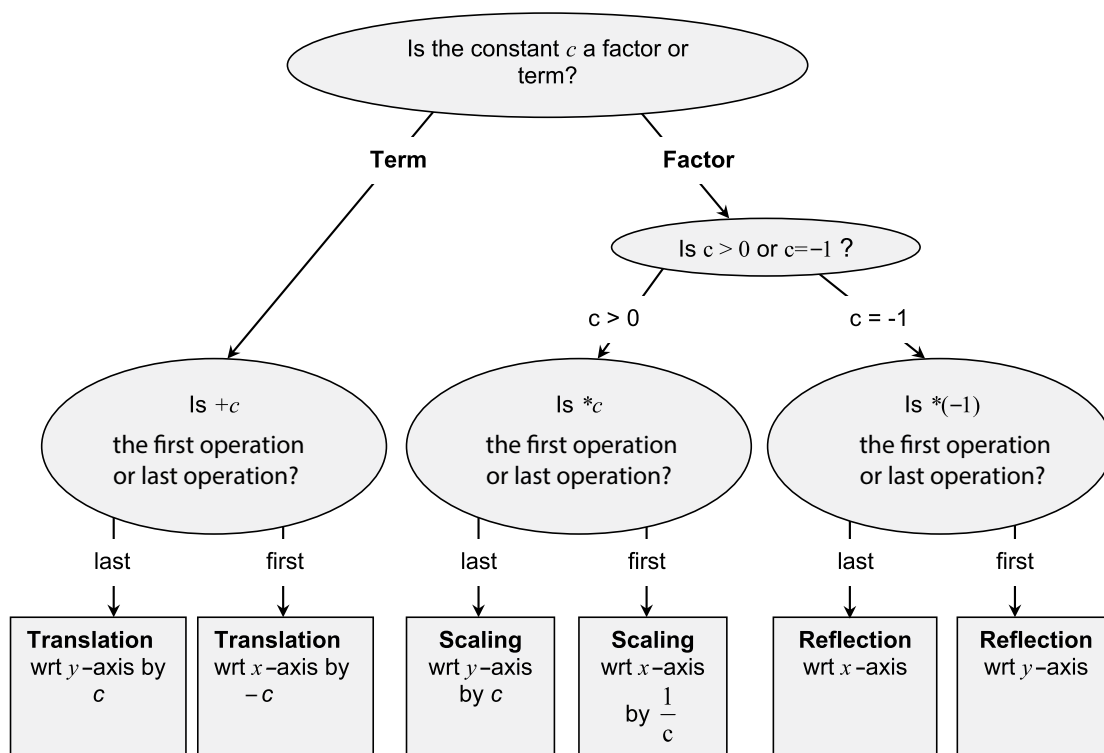

Figure 8.1. Decision Tree for Sketching by Simple Transformations Functions
$y=f(x)+c$, $y=f(x+c)$, $y=cf(x)$ and $y=f(cx)$.

Note 3: If **the same operation** is applied to all x–values or y–values – as for example in $y = \ln(x - 2) + \sin(x - 2)$ – this operation should be treated as one operation.

Note 4: if y is given **implicitly rather than explicitly**, so that the functional equation looks like $f(x,y) = 0$, rather than $y = f(x)$, then in order to see what transformation is effected by a constant c multiplying y or added to y, y should be made the subject of the equation. Therefore, similarly to transformations of x above, the corresponding transformation of y is defined by the inverse operation, $-c$ or $\dfrac{1}{c}$, respectively.

8.6   Completing the square

Completing the square means re-writing the quadratic expression

$$ax^2 + bx + c = (ax^2 + bx) + c = a(x^2 + \frac{b}{a}x) + c = a[(x + \frac{1}{2}\frac{b}{a})^2 - (\frac{1}{2}\frac{b}{a})^2] + c =$$

$$a(x + \frac{1}{2}\frac{b}{a})^2 - \frac{1}{4}\frac{b^2}{a} + c \text{, so that the unknown is present only in one – square – term.}$$

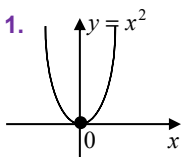It is useful for sketching a parabola by simple transformations.

Example: Sketch $y = x^2 + 6x + 4$ by simple transformations.
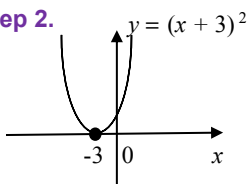
Solution

Completing the square, $y = x^2 + 6x + 4 = (x + 3)^2 - 9 + 4 = (x + 3)^2 - 5$

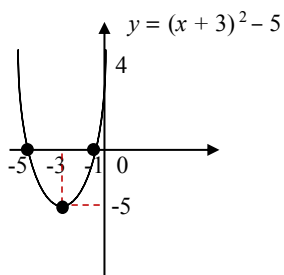Drop all constant factors and terms and bring them back in **Order of Operations**.

Step 1.

$y = x^2$

Step 2.

$y = (x + 3)^2$

+3 – constant term, first operation
$\Rightarrow$ translation wrt x-axis by -3

Step 3.

$y = (x + 3)^2 - 5$

+5 – constant term, last operation
$\Rightarrow$ translation wrt y-axis by -5

Intersection with the x-axis: $y = 0$, $x_{1,2} = -6 \pm \sqrt{9-4} \approx -3 \pm 2 = -1, -5$

Intersection with the y-axis: $x = 0$, $y = 4$

8.7   Applications of simple transformations

To give just one example, any linear dynamic system (any linear electrical, mechanical or chemical device) transforms the input signal (which is a real function of time) into the output signal (which is a real function of time) by effecting multiplication by constants and addition of constants. In particular, if the input is sinusoidal the linear transformation involves scaling of the amplitude and shifting of the phase.

8.8   Instructions for self-study

- **Revise ALGEBRA Summary (addition and multiplication, factorising and smile rule, flip rule)**
- **Revise Summaries on ORDER OF OPERATIONS and FUNCTIONS**
- **Revise Lecture 6 and study Solutions to Exercises in Lecture 6 (sketching elementary and inverse functions) using the STUDY SKILLS Appendix**
- **Revise Lecture 7 using the STUDY SKILLS Appendix**
- **Study Lecture 8 using the STUDY SKILLS Appendix**
- **Do the following exercises:**

Q1. Sketch using simple transformations:

a) $y = \sin 2x$

b) $y = 2 \sin x$

c) $y = \cos \left(\frac{1}{2} x \right)$

d) $y = \frac{1}{2} \cos x$

e) $y = e^{3x}$

f) $y = 3 e^x$

g) $y = \ln \left(\frac{1}{2} x \right)$

h) $y = \frac{1}{2} \ln x$

i) $y = \cos x + 2$

j) $y = \cos x - 2$

k) $y = \sin (x + 2)$
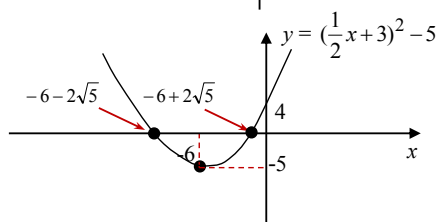
l) $y = \sin (x - 2)$

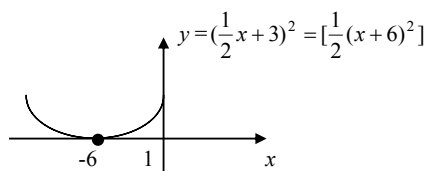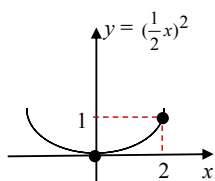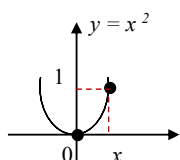m) $y = e^{x + 3}$

n) $y = e^{x - 3}$

o) $y = \frac{1}{2} x^2 + 2x - 3$ (complete the square first).

Lecture 9   Real FUNCTIONS of One Real Variable: Sketching Graphs by Simple Transformations (ctd.)

9.1   Sketching graphs using several simple transformations (ctd.)

Example: Sketch the parabola $y = (\frac{1}{2}x + 3)^2 - 5$

Solution

$y = x^2$

$y = (\frac{1}{2}x)^2$

$y = (\frac{1}{2}x + 3)^2 = [\frac{1}{2}(x + 6)^2]$

$y = (\frac{1}{2}x + 3)^2 - 5$

Intersection with the x-axis: $y = 0$, $(\frac{1}{2}x + 3)^2 - 5 = 0$, $x_{1,2} = -6 \pm 2\sqrt{5}$.

Intersection with the y-axis: $x = 0$, $y = 4$.

9.2   Sketching graphs using pointwise operations

Here is a recipe for sketching graphs using pointwise operations:

Step 1. Decompose the function into simpler ones starting with the last operation.

Step 2. Sketch each simple function separately (you might need to decompose it as well).

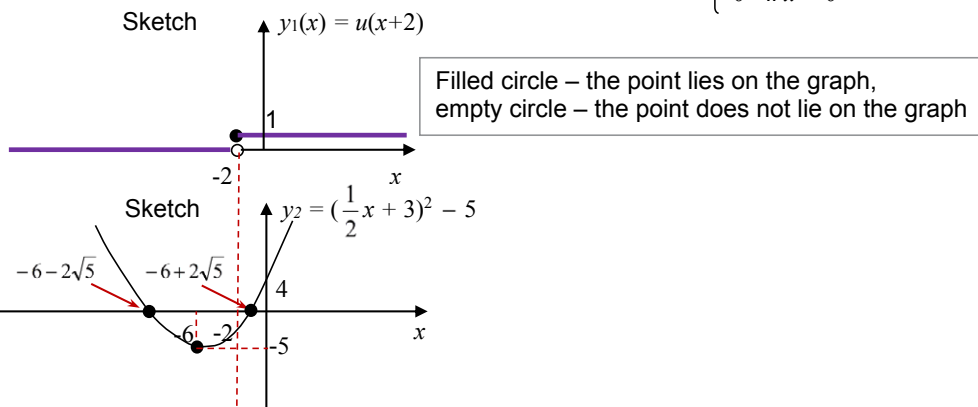**Step 3.** Compose the original function by pointwise operations.

Example: $y = u(x+2)[\,(\frac{1}{2}x+3)^2 - 5\,]$.

Solution

Step 1. Decompose $y(x)$ as $y(x) = y_1(x)\, y_2(x)$, where $y_1(x) = u(x+2)$, $y_2(x) = [\,(\frac{1}{2}x+3)^2 - 5\,]$

Step 2. Sketch y_1 and y_2, where the unit step function $u(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$

Sketch $y_1(x) = u(x+2)$

Filled circle – the point lies on the graph,
empty circle – the point does not lie on the graph

Sketch $y_2 = (\frac{1}{2}x + 3)^2 - 5$

$-6-2\sqrt{5}$ $-6+2\sqrt{5}$

Step 3. Sketch $y = y_1(x)y_2(x) = u(x+2)[(\frac{1}{2}x + 3)^2 - 5]$
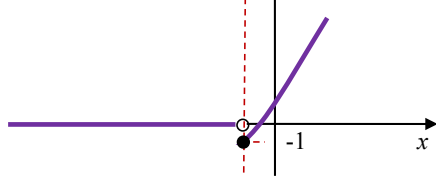
9.3   Instructions for self-study

- **Revise ALGEBRA Summary (addition and multiplication)**
- **Revise Summaries on ORDER OF OPERATIONS and FUNCTIONS**
- **Study Lecture 7 and study Solutions to Exercises in Lecture 7 using the STUDY SKILLS Appendix**
- **Study Lecture 8 using the STUDY SKILLS Appendix**
- **Study Lecture 9 using the STUDY SKILLS Appendix**
- **Do the following exercises:**

Q1. Sketch

a) $y = (2x + 3)^2 - 4$

b) $y = 2\,e^{-x}$

c) $y = \ln(\frac{x}{2} - 3)^{\frac{1}{2}} = \frac{1}{2}\ln(\frac{1}{2}x - 3)$

d) $y = \sin\left(\frac{\pi}{2} - x\right)$

e) $y = u(x - 6)\,[(\frac{1}{2}x + 3)^2 - 5] + u(x - 3)(x^2 + 6x + 4)$, where $u(x)$ is the unit step.

f) $y = -\frac{1}{2}x^2 + \frac{1}{4}x + \frac{1}{8}$

Lecture 10 ALGEBRA: Addition of Complex Numbers, the Argand Diagram, Forms of Complex Numbers

In Elementary Algebra we study **variables** and **operations on variables**. As discussed in Lectures 1–3, **inverse algebraic operations** introduce new types of numbers:

- subtraction introduces **negative** numbers,
- division introduces **rational** numbers,
- roots and logs introduce **irrational** and **complex numbers**.

10.1   Imaginary unity j

Let us introduce a new number j,

$$j \equiv \sqrt{-1}: \quad j^2 = -1 \tag{10.1}$$

It is easy to check that j cannot be real.

Proof:

Assume that $\sqrt{-1} = x$, where x is real. Then squaring both sides, $-1 = x^2$. However, any real number squared is positive or zero. Hence the **assumption** (that there exists a real number x such that $\sqrt{-1} = x$) **is invalid**.

j is a new type of number called the **imaginary unity.**

10.2   Applications of complex numbers: solving quadratic equations

Consider the quadratic equation

$$ax^2 + bx + c = 0, \qquad (10.2)$$

Using Section 3.1.6, if $b^2 - 4ac < 0$ we can write

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-b \pm \sqrt{(-1)(4ac - b^2)}}{2a} \; \frac{-b \pm j\sqrt{4ac - b^2}}{2a}. \qquad (10.3)$$

Note: every quadratic equation has exactly two roots, but sometimes they coincide and sometimes they are not real.

Example: $x^2 + x + 1 = 0$, where $a = 1, \; b = 1, \; c = 1$.

$$x_{1,2} = \frac{-b \pm j\sqrt{4a - b^2}}{2} = \frac{-1 \pm j\sqrt{4 - 1}}{2} = \frac{-1}{2} \pm j\frac{\sqrt{3}}{2}$$

10.3   Operations: the whole powers of j

$j = \sqrt{-1}, \qquad j^2 = -1, \qquad j^3 = j^2 \cdot j = -j, \qquad j^4 = j^2 \cdot j^2 = 1,$

$j^5 = j^4 \cdot j = j, \qquad j^6 = j^4 \cdot j^2 = -1, \qquad j^7 = j^4 \cdot j^3 = -j, \qquad j8 = j^4 \cdot j^4 = 1, \qquad etc. \qquad (10.4)$

10.4   Variables: definition of a complex number z

In Section 10.2, Eq. (10.3) we introduced a **complex number** presented as $z = x + jy$, that is, as a sum of two terms, where one term, x is real and another term, jy is **purely imaginary** (that is, j times a real number y).

x is called the **real part** of z, Re (z).
y is called (illogically!) the **imaginary part** of z, Im (z).

Operations on complex numbers are performed the same way they are performed on any algebraic expression (we just apply the laws and rules of Algebra formally).

10.5   Operations: addition of complex numbers z_1 and z_2

Consider two complex numbers, $z_1 = x_1 + jy_1$ and $z_2 = x_2 + jy_2$. To represent their sum in the same form we just collect the like terms, that is, collect real terms separately and imaginary terms separately:

$$z_1 + z_2 = x_1 + x_2 + j(y_1 + y_2)$$

Examples: $(2 + 3j) + (-3 - 2j) = (2 - 3) + (3 - 2)j = -1 + j$

$(2 + 3j) + (-3 + 2j) = (2 - 3) + (3 + 2)j = -1 + 5j$

Rules of addition (check them yourself using the Laws of Addition of real numbers)

$z_1 + z_2 = z_2 + z_1$

$(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$

10.6   Operations: subtraction of complex numbers

The difference between two complex numbers is introduced in a standard way:

Definition: $z_1 - z_2 \equiv z_3 : z_3 + z_2 = z_1$

$\Rightarrow$ $z_1 - z_2 = x_1 - x_2 + j(y_1 - y_2)$

this symbol means **"implies"**

Example: $(2 + 3j) - (3 + 2j) = (2 - 3) + (3 - 2)j = -1 + j$

$(2 + 3j) - (-3 + 2j) = (2 + 3) + (3 - 2)j = -1 + 5j$

The above rules suggest that a complex number can be thought of as a pair of two real numbers. Since each real number can be represented **(visualised)** as a point on a number line, a complex number can be represented **(visualised)** using two number lines. For simplicity, let them be perpendicular to one other and let number 0 be represented on both number lines by the point of their intersection. In other words, a complex number can be represented **(visualised)** as a point on a number plane otherwise known as **the complex plane** or **the Argand diagram**.

10.7   The Argand diagram and Cartesian form of a complex number
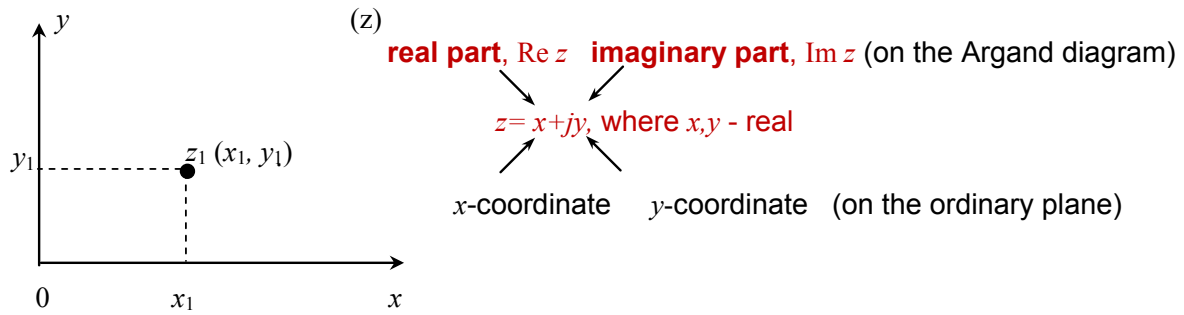
Verbalise: In the **Cartesian representation,** a complex number is a sum of two terms, one real (x) and one purely imaginary (jy).

10.8   The Argand diagram and polar form of a complex number
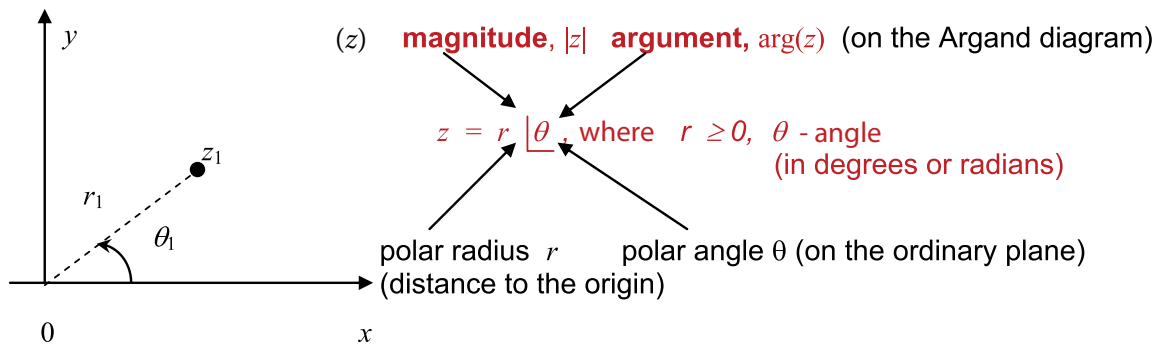
Verbalise: in the **polar representation**, a complex number is a pair of polar coordinates r and θ rather than a pair of Cartesian coordinates x and y.

10.9   The Cartesian-polar and polar-Cartesian transformations

Using Pythagoras Theorem and trigonometric ratios we can always transform Cartesian coordinates into polar coordinates and vice versa:

Cartesian $\rightarrow$ polar	polar $\rightarrow$ Cartesian
$r = \sqrt{x^2 + y^2}$ (Pythagoras Theorem) $$\theta = \begin{cases} \tan^{-1}\dfrac{y}{x} & \text{if } x > 0 \\[2mm] \tan^{-1}\dfrac{y}{x} + \pi & \text{if } x < 0 \\[2mm] \operatorname{sgn}(y)\dfrac{\pi}{2} & \text{if } x = 0 \end{cases}$$ where $\operatorname{sgn}(y)$ is a sign of y, $+$ or $-$	$$x = r\cos\theta$$ $$y = r\sin\theta$$

Example:

1. What is the polar representation of z = -√3 − 1j? Use the Argand diagram.
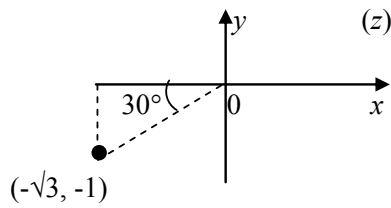
$$r = \sqrt{3+1^2} = 2$$

$$\theta = \tan^{-1} \frac{1}{\sqrt{3}} +180°=30°+180°=210°$$

$$z = 2 \underline{|210°}$$

2. What is the Cartesian representation of z = 1 $\underline{|45°}$? Use the Argand diagram.
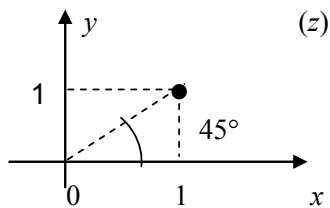
$$x = 1 \cdot \cos 45° = \frac{\sqrt{2}}{2}$$

$$y = 1 \cdot \sin 45° = \frac{\sqrt{2}}{2}$$

$$z = \frac{\sqrt{2}}{2} + j\frac{\sqrt{2}}{2}$$

10.10   The trigonometric form of a complex number

Using the Cartesian – polar transformation we can represent a complex number in the so-called **trigonometric form**:

$$z = x + jy = r(\cos\theta + j\sin\theta)$$

10.11   Euler's formula

There is a remarkable formula that connects the sine function, cosine function and exponential function:

Euler's formula: $\cos \theta + j \sin \theta = e^{j\theta}$ **MEMORISE**

It can be proven later using the Maclaurin series.

10.12   The exponential form of a complex number

Using the Euler's formula, the trigonometric representation can be re-written in the so-called **exponential form**:

$z = r\, e^{j\theta}$, $r \geq 0$ or, using the Argand diagram notation, $z = |z|\, e^{j\theta\ \arg(z)}$,   **MEMORISE**

where $\theta = \arg(z)$ is real (dimensionless number, angle in radians)

Verbalise: In the exponential representation, a complex number is a product of two factors, one positive (r) and one ($e^{j\theta}$) – an imaginary exponent (that is, e to power which is purely imaginary, j times a real number).

Example:

1. Express $z = 1\ \underline{|45°}$ in the exponential form. **Answer:** $z = e^{j\pi/4}$

2. Express $z = 2\ \underline{|210°}$ in the exponential form. **Answer:** $z = 2e^{j7\pi/6}$

3. Express $z = \dfrac{\sqrt{2}}{2}\ (1 + j)$ in the exponential form. **Answer:** using the Argand diagram,

$r = \sqrt{\dfrac{1}{2} + \dfrac{1}{2}} = 1$, $\theta = \tan^{-1} 1 = \dfrac{\pi}{4}$, $z = e^{j\pi/4}$

10.13   Applications

"In electrical engineering, the… treatment of resistors, capacitors, and inductors can … be unified by introducing imaginary, frequency-dependent resistances for the latter two and combining all three in a single complex number called the impedance. This use is also extended into digital signal processing and digital image processing… to transmit, compress, restore, and otherwise process digital audio signals, still images, and video signals."

http://en.wikipedia.org/wiki/Complex_number

10.14   A historical note

"The earliest fleeting reference to square roots of negative numbers perhaps occurred in the work of the Greek mathematician and inventor Heron of Alexandria in the 1st century AD, when he considered the volume of an impossible frustum of a pyramid, though negative numbers were not conceived in the Hellenistic world.

Complex numbers became more prominent in the 16th century, when closed formulas for the roots of cubic and quartic polynomials were discovered by Italian mathematicians… Niccolo Fontana Tartaglia, Gerolamo Cardano…. It was soon realized that these formulas, even if one was only interested in real solutions, sometimes required the manipulation of square roots of negative numbers."

http://en.wikipedia.org/wiki/Complex_number#History

10.15   Instructions for self-study

- **Revise ALGEBRA Summary (addition and subtraction)**
- **Revise Summaries on QUADRATICS and TRIGONOMETRY**
- **Revise Lecture 8 and study Solutions to Exercises in Lecture 8 using the STUDY SKILLS Appendix**
- **Revise Lecture 9 using the STUDY SKILLS Appendix**
- **Revise Lecture 10 using the STUDY SKILLS Appendix**
- **Study Section 4 the STUDY SKILLS Appendix**
- **Do the following exercises:**

Q1.

a) Solve the quadratic equation $5x^2 - 11x + 13 = 0$

b) Given $z_1 = -1 + 2j$ and $z_2 = -1 - 6j$ find

    i) $z_1 + z_2$

    ii) $z_1 - z_2$

    iii) $z_1 - 2z_2$

    iv) $2z_1 + z_2$

Q2. Write down the real and imaginary parts of

a) $3 + 7j$

b) $-0.35j$

c) $\cos \omega t + j \sin \omega t$, where ω and t are real

d) $jV \sin (\omega t + \varphi)$, where V, ω, t and φ are real

Q3. Show the following complex numbers on the Argand diagram

a) $3e^{j\pi/3}$

b) $\sqrt{2}e^{j2\pi}$

c) $3e^{-j\pi}$

d) $5e^{j0}$

Q4. Find the real and imaginary parts of

a) $4e^{j\pi/3}$

b) $e^{j\pi/2}$

c) $3e^{-j\pi/2}$

d) e^{j-1}

Q5. Find $|z|$ and $\arg(z)$ and express z in the exponential form for

a) $z = -1 + 2j$

b) $z = -3 - 6j$

Lecture 11 ALGEBRA: Multiplication and Division of Complex Numbers

11.1 Multiplication of a complex number z by a real number α

Given a complex number $z = x + jy = r\, e^{jq}$ and a real number a, using the *Cartesian form* we can write

$$\alpha z \;=\; \alpha x \;+\; \alpha j\, y$$

and using the *exponential form*, we have

$$\alpha z \;=\; \begin{cases} \alpha\, r\, e^{j\theta} & \text{if } \alpha \geq 0 \\[2mm] \alpha\, r\, e^{j(\theta+\pi)} & \text{if } \alpha < 0 \end{cases}$$

Example:

1. Given $z = \dfrac{1}{\sqrt{2}} + j\,\dfrac{1}{\sqrt{2}}$ and $\alpha = 2$ find $\alpha\,z$. **Answer:** $\alpha\,z \;=\; 1 + j$.

2. Given $z = \dfrac{1}{\sqrt{2}}\,(\cos\theta \;+\; j\sin\theta)$ and $\alpha = \sqrt{2}$ find $\alpha\,z$. **Answer:** $\alpha\,z \;=\; \cos\theta \;+\; j\sin\theta$.

3. Given $z \;=\; e^{j\pi/4}$ and $\alpha \;=\; \sqrt{2}$ find $\alpha\,z$. **Answer:** $\alpha\,z = \sqrt{2}\, e^{j\pi/4}$.

11.2 Operations: multiplication of complex numbers z_1 and z_2

Using the *Cartesian form* we can write

$$z_1 \cdot z_2 \;=\; (x_1 \;+\; jy_1)(x_2 \;+\; jy_2) \;=\; x_1\,x_2 - y_1\,y_2 \;+\; j\,(y_1\,x_2 \;+\; x_1\,y_2)$$

Multiplication is easier to perform in the *exponential form:*

$$z_1 \cdot z_2 \;=\; (r_1 r_2)e^{j(\theta_1+\theta_2)}$$

Examples:

1. Find $(1 + j)(1 - j)$ in the Cartesian form.

Solution

$(1 + j)(1 - j) = 1 + 1 + j(1 - 1) = 2$

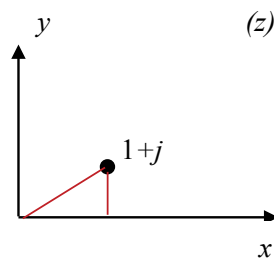2. Find $(1 + j)(1 - j)$ in the exponential form.

Solution

$1 + j = \sqrt{2}e^{j\frac{\pi}{4}} \ (\tan^{-1}\frac{1}{1} = \frac{\pi}{4})$

$1 - j = \sqrt{2}e^{-j\frac{\pi}{4}} \ (\tan^{-1}\frac{1}{1} = \frac{\pi}{4})$

$(1 + j)(1 - j) = 2e^{j(\pi/4 - \pi/4)} = 2$

3. $j \cdot z = e^{j\pi/2}re^{j\theta} = re^{j(\theta + \pi/2)}$

$\Rightarrow$ multiplication by j is equivalent to rotation on the Argand diagram by 90° anti–clockwise.

11.3   Operations: complex conjugate z* of a complex number z

In the *Cartesian form* we define the complex conjugate of z as $z^\star = x - jy$. Some authors use a bar instead of the star.

It follows that in the *exponential form* the complex conjugate of z is $z^\star = r\,e^{-j\theta}$.

Example:
In the Cartesian form, $z\,z^* = x^2 + y^2$.
In the exponential form, $z\,z^* = |r|\,e^{j\theta}\,|r|\,e^{-j\theta} = r^2\,e^{j\theta \cdot j\theta} = r^2\,e^0 = r^2 \times 1$.
$\Rightarrow z\,z^* = r^2$.

11.4   Operations: division of complex numbers z_1 and z_2

In the *Cartesian form,* we can use the complex conjugate of the denominator to represent the ratio of two complex numbers as

$$\frac{z_1}{z_2} = \frac{x_1 + jy_1}{x_2 + jy_2} = \frac{x_1 x_2 + y_1 y_2 + j(x_2 y_1 - x_1 y_2)}{x_2^2 + y_2^2} = \frac{x_1 x_2 + y_1 y_2}{x_2^2 + y_2^2} + j\,\frac{x_2 y_1 - x_1 y_2}{x_2^2 + y_2^2}.$$

Division is much easier to perform in the *exponential form*:

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\,e^{j(\theta_1 - \theta_2)}.$$

Example: Find $\dfrac{1+j}{1-j}$.

Solution

In the *Cartesian form*, we have

$$\frac{1+j}{1-j} = \frac{0+2j}{2} = j.$$

In the *exponential form*, we have

$$\frac{1+j}{1-j} = \frac{\sqrt{2}\,e^{j\pi/4}}{\sqrt{2}\,e^{-j\pi/4}} = e^{j\pi/2} = j.$$

11.5   Integer powers of complex numbers

De Moivre's Theorem

$$z_n = r^n(\cos n\theta + j\sin n\theta)$$

where r – magnitude of z, $|z|$, θ – argument of z, arg(z), and n – integer.

Example:

1. $(1+j)^2 = (\sqrt{2}\, e^{j\pi/4})^2 = 2\, e^{j\pi/2} = 2j$

   (use the Argand diagram to put $1+j$ into the exponential form first)

2. $(1-j)^2 = (\sqrt{2}\, e^{-j\pi/4})^2 = 2\, e^{-j\pi/2} = -2j$

   (use the Argand diagram to put $1+j$ into the exponential form first)

11.6   The fractional powers $\dfrac{1}{k}$ (k-*th* roots) of complex numbers

---

$e^{j2\pi n} = 1$, where n is an integer
(check by putting 1 into the exponential form).

$\Rightarrow e^{j\theta} = e^{j\theta} \times 1 = e^{j\theta} \times e^{j2\pi n} = e^{j(\theta + 2\pi n)}$

Thus, any complex number can be represented as

$$z = r\,e^{j\theta} = r\,e^{j(\theta + 2\pi n)}, \; n - \text{integer}, \; -\pi < \theta_0 \leq \pi.$$

Verbalise: $2\pi n$ is n full rotations around the origin – clockwise or anticlockwise – of the point z. These rotations do not change the point's position on the Argand diagram, and therefore, adding $2\pi n$ to the argument of a complex number does not change z.
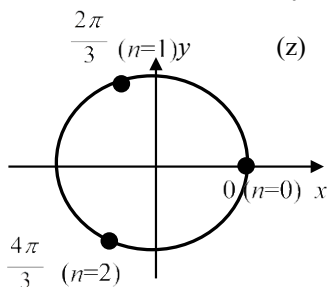
Example:

1. Find $1^{1/3}$.

Solution

$1 = e^{2\pi nj}$, n – integer

$1^{1/3} = \left(e^{2\pi nj}\right)^{1/3} = e^{j2\pi/3 \, n}$

These roots are complex numbers in the exponential form.
Let us visualise them as points on the Argand diagram:

$$n = 0 \;\Rightarrow\; \theta = 0, \qquad n = 3 \qquad \theta = 2\pi$$
$$n = 1 \;\Rightarrow\; \theta = \frac{2\pi}{3}, \qquad n = 4 \qquad \theta = \frac{8\pi}{3}$$
$$n = 2 \;\Rightarrow\; \theta = \frac{4\pi}{3}, \qquad n = 5 \qquad \theta = \frac{10\pi}{3}$$

whatever other n, the corresponding number is represented on the Argand diagram by one of the above three points.

Answer: 3 roots, $\dfrac{2\pi}{3}$ (rad), that is, 120^0 apart.

11.7   Instructions for self-study

- **Revise ALGEBRA Summary (powers and roots)**
- **Revise Summaries on TRIGONOMETRY and COMPLEX NUMBERS**
- **Revise Section 10 and study Solutions to Exercises 10 using the STUDY SKILLS Appendix**
- **Revise Lecture 10 using the STUDY SKILLS Appendix**
- **Study Lecture11 using the STUDY SKILLS Appendix**
- **Do the following exercises:**

Q1.

     a) Find $(1 + 2j)(1 - j)$

     b) Find $(1 + 2j)(1 - j)(3j - 1)$

     c) Find $z_1 z_2$, where $z_1 = \dfrac{1}{3}e^{j\pi/3}$, $z_2 = 3e^{j\pi/3}$

     d) Show that multiplying any complex number by $-j$ means rotating the corresponding point on the Argand diagram around the origin through 90° clockwise.

Q2.

     a) Find $\dfrac{1+j}{1-2j}$

     b) Find $\dfrac{z_1}{z_2}$, where $z_1 = \dfrac{1}{3}e^{j\pi/3}$, $z_2 = 3e^{j\pi/3}$

     c) Find $\dfrac{1}{j}$

     d) Show that $j^* = \dfrac{1}{j}$

Q3.

     a) Use De Moivre's Theorem to prove that $\cos 2x = \cos^2 x - \sin^2 x$

     b) Use De Moivre's Theorem to prove that $\sin 2x = 2 \sin x \cos x$

     c) Find $(3 + j4)^{1/2}$

     d) Solve $z^2 + 2z + - j = 0$

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Additional Exercises
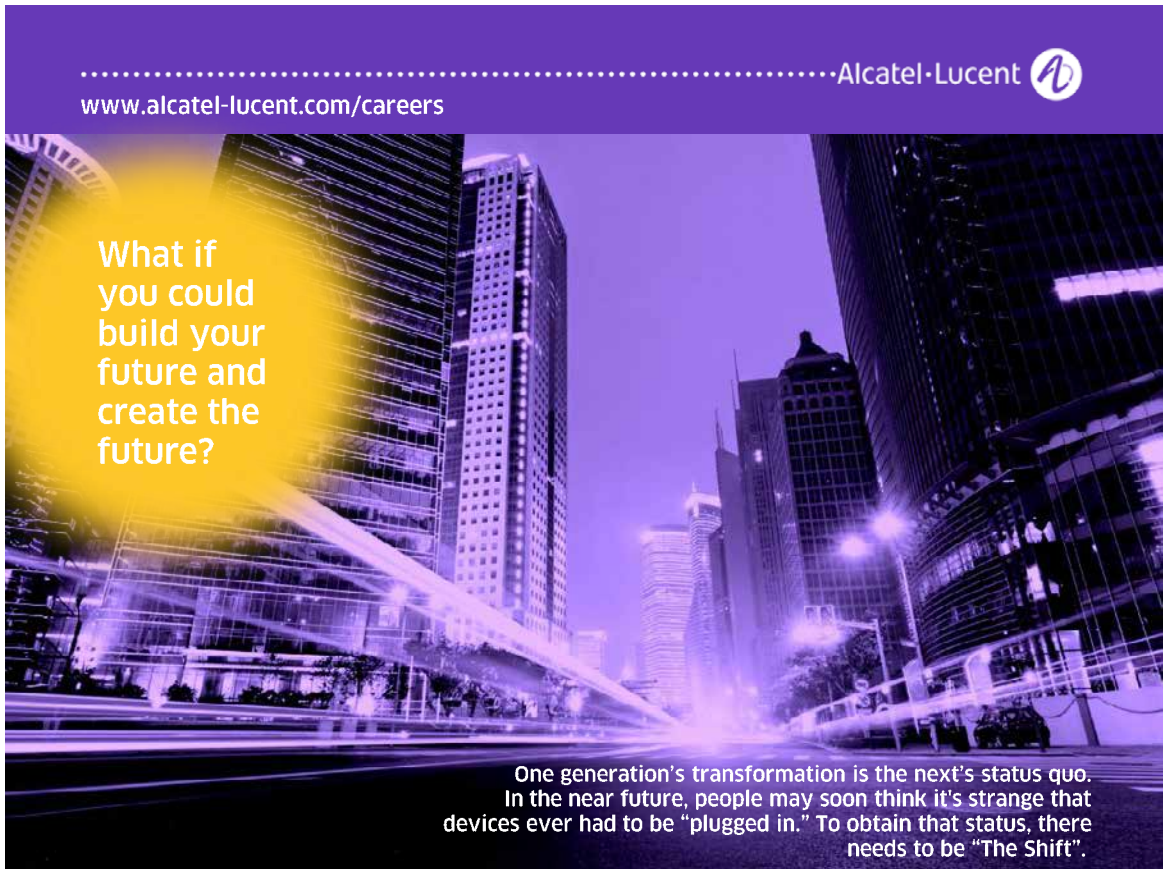
1. Find $z_1 z_2$, where $z_1 = 3 - 2j$, $z_2 = 4 - 3j$

2. Find $(3 - 2j)j(4 - 3j)$

3. Find $z_1 z_2$, where $z_1 = \dfrac{1}{\sqrt{3}} e^{j\pi/6}$, $z_2 = \sqrt{3} e^{-j\pi/3}$

4. Find $\dfrac{1+j}{2-j}$

5. Find $\dfrac{z_1}{z_2}$, where $z_1 = \dfrac{1}{\sqrt{3}} e^{j\pi/6}$, $z_2 = \sqrt{3} e^{-j\pi/3}$

6. Solve $-z^2 + 3z - 4 = 0$

7. Solve $z^3 = -1$

Lecture 12 ALGEBRA: Fractional Powers, Logs and Loci of Complex Numbers

12.1   The fractional powers $\dfrac{1}{k}$ (whole roots k) of complex numbers (ctd.)
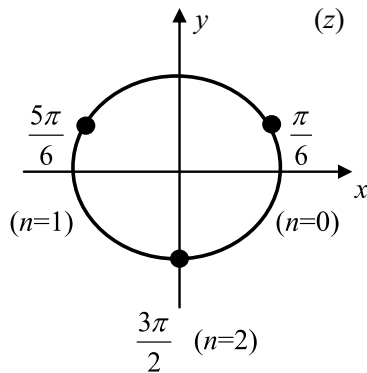
Example:

1.   Find $j^{1/3}$.

Solution

Step 1. Represent j in the exponential form: $j = e^{j(\pi/2 + 2\pi n)}$, n – integer

Step 2. Raise j to the fractional power 1/3: $j^{1/3} = \left[e^{j(\pi/2 + 2\pi n)} \right]^{1/3} = e^{j(\pi/6 + 2\pi/3\, n)}$

Step 3. Visualise all roots as points on the Argand diagram:

$n = 0 \qquad \theta = \dfrac{\pi}{6}$

$n = 1 \qquad \theta = \dfrac{\pi}{6} + \dfrac{2\pi}{3} = \dfrac{5\pi}{6}$

$n = 2 \qquad \theta = \dfrac{\pi}{6} + \dfrac{4\pi}{3} = \dfrac{9\pi}{6} = \dfrac{3\pi}{2}$

- 3 roots, $2\pi/3$ (*rad*) = 120º **apart**

2. Find $(1+j)^{1/4}$.

Solution

Step 1. Represent 1+j in the exponential form: $1 + j = \sqrt{2}\, e^{j\pi/4} = \sqrt{2}\, e^{j(\pi/4 + 2\pi n)}$, n – integer

Step 2. Raise 1+j to the fractional power 1/4:

$(1 + j)^{1/4} = \left[\sqrt{2}\, e^{j(\pi/4 + 2\pi n)} \right]^{1/4} = 2^{1/8}\, e^{j(\pi/16 + \pi/2\, n)}$

Step 3. Visualise all roots as points on the Argand diagram:
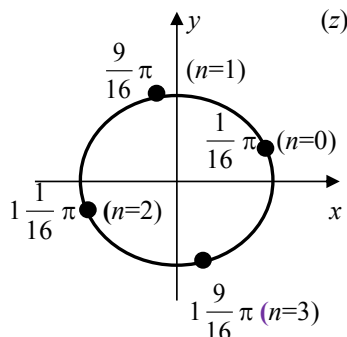
$n = 0 \quad \theta = \dfrac{\pi}{16}$

$n = 1 \quad \theta = \dfrac{\pi}{16} + \dfrac{2\pi}{4} = \dfrac{9\pi}{16}$

$n = 2 \quad \theta = \dfrac{\pi}{16} + \dfrac{2\pi \cdot 2}{4} = \dfrac{17\pi}{16} = 1\dfrac{1}{16}\pi$

$n = 3 \quad \theta = \dfrac{\pi}{16} + \dfrac{2\pi \cdot 3}{4} = \dfrac{25\pi}{16} = 1\dfrac{9}{16}\pi$

- 4 roots, $\pi/2$ (*rad*) = 90⁰ **apart**

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12.2   Logs of complex numbers

Using the exponential form of a complex number z, $\ln z = \ln(re^{j(\theta_0 + 2\pi n)}) = \ln r + j\theta_0 + j2\pi n$, n – integer

---

Example: Find $\ln(1 - j)$.

Solution

Step 1. $1 - j = \sqrt{2}e^{-j\pi/4}$

Step 2. $\ln(1 - j) = \ln(\sqrt{2}e^{-j\pi/4}) = \ln\sqrt{2} + \ln e^{-j(\pi/4 + 2\pi n)} = \ln\sqrt{2} - j\dfrac{\pi}{4} + j2\pi n$, $n = \pm 1, \pm 2, \ldots$

Step 3. Visualise them as points on the Argand diagram:
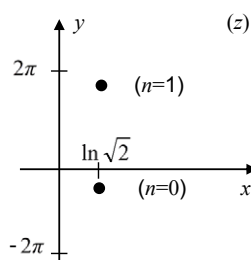
countless number of solutions on the vertical line, $2\pi \approx 6.28$ **units apart**

12.3   Loci on the Argand diagram

Locus is a geometrical location of points with a specific property.

Remember: $z = \cos\theta + j\sin\theta = re^{j\theta}$

---

1. On the Argand diagram, the locus of points with property

$$\text{Re}\,(z) \geq 0$$

is the right half-plane.
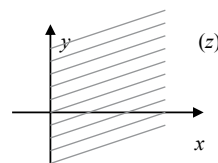
2. On the Argand diagram, the locus of points with property

$$0 \leq \arg(z) \leq \frac{\pi}{2}$$

is a quarter plane which is the 1st quadrant.
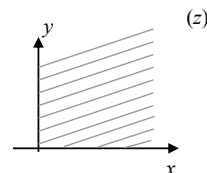
3. On the Argand diagram, the locus of points with property

$$\arg\,(z)\ =\ \frac{\pi}{4}$$

is a ray issuing from the origin at 45˚ to the horizontal axis.
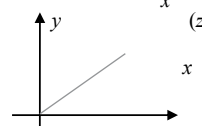
4. On the Argand diagram, the locus of points with property

$$|z|\ =\ r$$

is the circumference of radius of r centred at the origin.
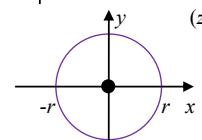
5. On the Argand diagram, the locus of points with property

$$x^2\ +\ y^2 \leq 1$$

is the circle of radius of 1 centred at the origin.
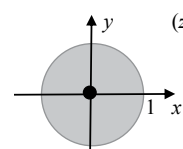
6. On the Argand diagram, the locus of points with property

$$(x - 1)^2\ +\ y^2 \leq 1$$

is the circle of radius of 1 centred at (1,0).
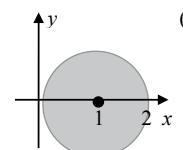
7. On the Argand diagram, the locus of points with property

$$(x - x_0)2\ +\ (y - y_0)^2 \leq R^2$$

is the circle of radius of R centred at (x_0, y_0).
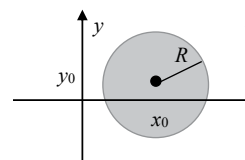
12.4   Applications

In control theory, single input single output (SISO) linear dynamic systems are often characterised by the so-called transfer functions. A transfer function is a ratio of two polynomials in a complex variable *s*. The locus of zeros of these polynomials is a useful tool for analysing *stability* of SISOs. They are stable if all zeros of the denominator are in the left-hand side of the complex s-plane.

12.5   Instructions for self-study

- **Revise ALGEBRA Summary (powers and logs)**
- **Revise Summaries on TRIGONOMETRY and COMPLEX NUMBERS**
- **Revise Section 10 and study Solutions to Exercises 10 using the STUDY SKILLS Appendix**
- **Revise Section 11 the STUDY SKILLS Appendix**
- **Study Section 12 using the STUDY SKILLS Appendix**
- **Do the following exercises:**

Q1.

a) Find the cubic roots of $z = \dfrac{(-3+j)^4}{(2-j)^2}$

b) Solve the equation $z^4 + 25 = 0$.

Q2. Assuming z – complex and x, y – real, indicate on the Argand diagram the following loci:

a) $\text{Re}(z) \le 2$

b) $(x-1)^2 + (y-2)^2 = 2$

c) $\left| \dfrac{1}{1+jy} \right| < 1$ **(advanced)**

d) $\arg \left(\dfrac{1}{1+jy} \right) = 1$ **(advanced).**

Lecture 13 CALCULUS: Sequences, Limits and Series

In Calculus we study **functions and operations on functions** (see Calculus Concept Map). Elementary operations on functions (algebraic operations and composition/decomposition) were covered in Lectures 4–7. We now introduce the first **advanced operation on functions** – **taking a limit**. We introduce it first for discrete functions called sequences.

13.1   Sequences

A **sequence** is an ordered set of numbers x_n, a **discrete function** $x(n)$, where the argument is an integer n. This means that a discrete function is a function whose domain is the set of all integers or else its subset. Such argument is called the **counter**.

There are different ways of writing up a sequence: $\{x_n\}$ or x_1, x_2, x_3, ..., x_N (each symbol n, 1, 2, 3, …, N here can also be called an **index** or subscript). A sequence can be **finite** (having a fixed number of elements) or **infinite** (whatever index you specify there is always an element in the sequence with a greater index). Traditionally sequence elements are called terms but in these notes we reserve the word "term" for addends.

13.1.1   Defining a Sequence as Function of Counter

A sequence can be defined (described) by specifying a relationship between each element and its counter.

Examples:

1. Describe sequence 1, 2, 3, 4, … using the above notation.
Solution
$x_1 = 1, x_2 = 2, x_3 = 3, …$
Question: What is the relationship between the counter and the sequence element?
Answer:

2. Describe sequence 1, 4, 9, 16, … using the above notation.
Solution
$x_1 = 1, x_2 = 4, x_3 = 9, …$
Question: What is the relationship between the counter and the sequence element?
Answer:

13.1.2   Visualising a Sequence as a Function of Counter

If a sequence is defined (described) by specifying a relationship between each element and its counter it can be visualised as a sequence of points on the number line (see figure 13.1) or else as a discrete graph (see figure 13.2).
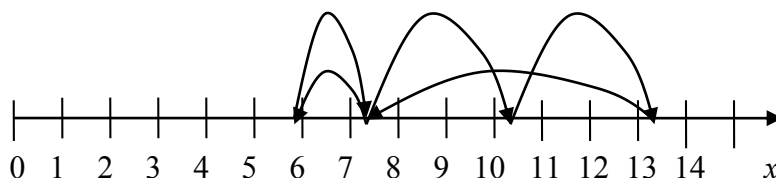
Figure 13.1. An example of a sequence x_n, $n = 0, 1, …, 5$ visualised using the number line. The arrowed arcs are used to indicate the order of elements in the sequence.
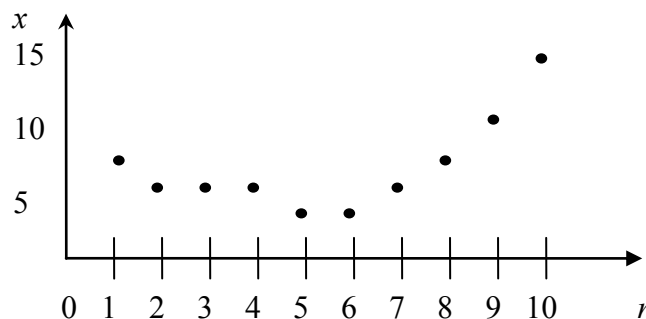
Figure 13.2. An example of a graph of sequence x_n, $n = 0, 1, …, 10$.
http://upload.wikimedia.org/wikipedia/commons/8/88/Sampled.signal.svg

13.1.3   Applications of Sequences

A digital signal x (see figure 13.3 below) has the following characteristics:

1. it holds a fixed value for a specific length of time
2. it has sharp, abrupt changes
3. a preset number of values is allowed. http://www.privateline.com/manual/threeA.html
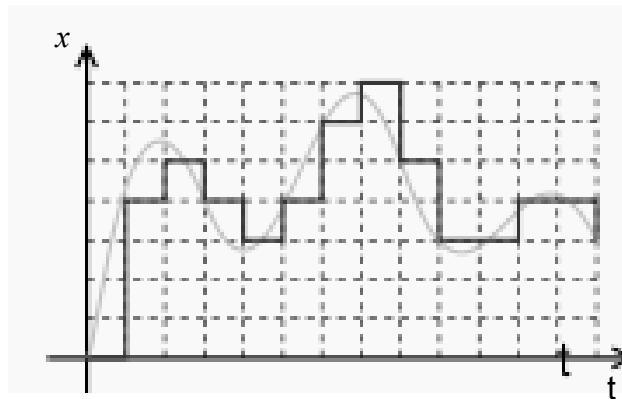
Figure 13.3. A typical digital signal. http://www.privateline.com/manual/threeA.html

Therefore, digital signals are examples of sequences.

13.1.4   Defining a Sequence via a Recurrence Relationship

A sequence element may be defined in another way, via a **recurrence relation**

$$x_{n+1} = f(x_n, x_{n-1}, ..., x_1),$$

where f may be a function of several arguments and not just one argument,

Thus, there are two ways to describe a sequence,

1.   using a functional relation $x(n)$, which specifies how each sequence element is defined by its counter;
2.   using a recurrence relation $x_{n+1} = f(x_n, x_{n-1}, ..., x_1)$, which specifies how each sequence element is defined by previous sequence element(s).

Examples:

1.   Given a sequence $x_n = n^2$ we can change the functional description to a
         recurrence relation
         $x_{n+1} = (n+1)^2 = n^2 + 2n + 1$
         $\Rightarrow x_{n+1} = x_n + 2\sqrt{x_n} + 1$

         When given such a recurrence the first element needs to be specified.
         Only then can we start evaluating other elements.

2.   A **Fibonacci sequence**: 1, 1, 2, 3, 5, 8, 13, 21, … can be described via a
         recurrence relation

$$x_{n+1} = x_n + x_{n-1}.$$

When given such a recurrence the first two elements have to be specified. Only then can we start
evaluating other elements. Let us check that the above recurrence describes the given sequence:

Question: What are x_1 and x_2?
Answer:
Question: Does x_3 satisfy the given recurrence relation and why?
Answer:

Question: Does x_4 satisfy the given recurrence relation and why?
Answer:

Question: Does x_5 satisfy the given recurrence relation and why?
Answer:

Fibonacci sequences in nature

When superimposed over the image of a nautilus shell we can see a Fibonacci sequence in nature:
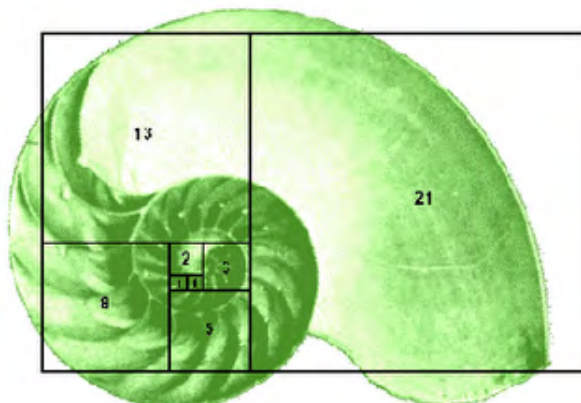
Each of the small spirals of broccoli below follows the Fibonacci's sequence.
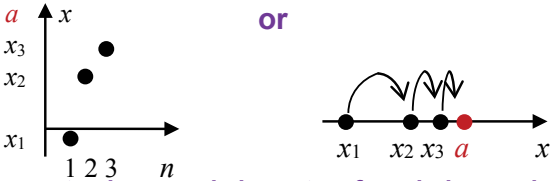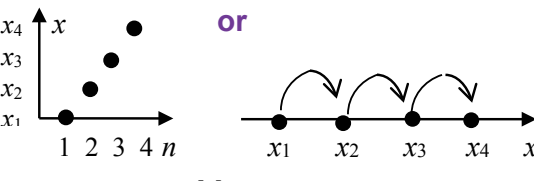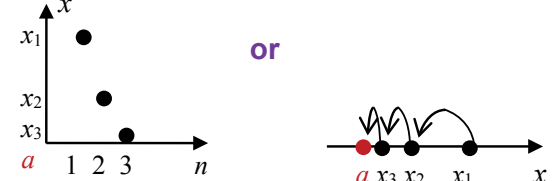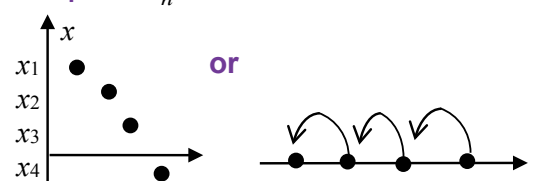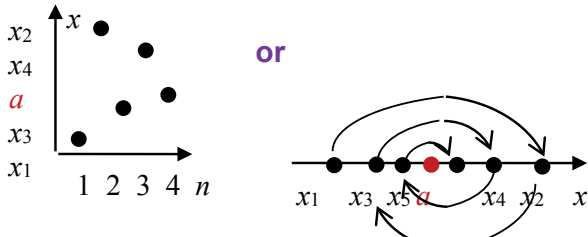
13.2   Limit of a sequence

Taking a limit of a sequence as *n* grows larger and larger without bounds is the first **advanced operation on functions** that we cover. In mathematics, instead of the phrase *n grows larger and larger without bounds* we use the shorthand $n \to \infty$ (verbalised as ***n* tends to infinity**). If it exists the outcome of applying this operation to a (discrete) function x_n is called $\lim x_n$ and is either a number or else $\pm \infty$ (either +**infinity** or -**infinity**). Sometimes instead of $\lim x_n$ we write $\lim_{n \to \infty} x_n$. However, usually, the condition n $\to \infty$ is understood and not mentioned.

Note: ∞ is not a number but a symbol of a specific sequence behavior, $+\infty$ means that the sequence increases without bounds and $-\infty$ means that the sequence decreases without bounds – see the right column in the Table in Section 13.2.1 below.

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13.2.1   Definition of a Limit of a Sequence

Examples of a finite limit $\lim\limits_{(n\to\infty)} x_n = a$ **(can also write $x_n \to a$)** $(n\to\infty)$	**Examples of an infinite limit** $\lim\limits_{(n\to\infty)} x_n = \pm\infty$ **(can also write $x_n \to \pm\infty$)** $(n\to\infty)$
Example 1: $x_n \to a-$ x_n **gets closer and closer to a from below and maybe reaches it.**	**Example 1:** $x_n \to +\infty$ **For any number M there exists $N : x_n > M$ for all $n \geq N$ (sequence increases without bounds**
Example 2: $x_n \to a+$ x_n **gets closer and closer to a from above and maybe reaches it.**	**Example 2:** $x_n \to -\infty$ **For any number M there exists $N : x_n < M$ for all $n \geq N$ (sequence decreases without bounds)**
Example 3: $x_n \to a$ x_n **gets closer and closer to a and maybe reaches it.**	

Examples: Sketch the sequence to find its limit:

1. $x_n = \dfrac{1}{n} \to 0$ **Verbalise:** as n increases, elements x_n (come closer and closer) to 0.

2. $x_n = -\dfrac{1}{n} \to 0$ **Verbalise:** as n increases, elements x_n (come closer and closer) to 0.

3. $x_n = \dfrac{(-1)^{n+1}}{n} \to 0$ **Verbalise:** as n increases, x_n (come closer and closer) to 0.

4. $x_n = n \to \infty$ **Verbalise:** as n increases, x_n tend to ∞ (increase without bounds).

5. $x_n = -n \to -\infty$ **Verbalise:** as n increases, x_n tend to $-\infty$ (increase without bounds).

Sometimes, limits can be found using analytical considerations and sometimes using a Table, Rules and a Decision Tree for Limits:

13.2.2 Table of Limits

x_n	$\lim x_n$
$1/n$	0
$a^{1/n}, a > 0$	1
$q^n, |q| < 1$	0
$(1 + \dfrac{1}{n})^n$	e

easy to find

the proof is rather involved ($e \approx 2.71$, **Neper's constant**, an irrational number).

If $\{x_n\}$ has a finite limit the sequence is said to **converge**.

13.2.3 Rules for (Properties of) Limits

1. The limit is unique.

2. $\lim (\alpha x_n + \beta y_n) = \alpha \lim x_n + \beta \lim y_n, \alpha, \beta$ – constant.
   Verbalise: limit of a sum is a sum of limits and any constant factor can be taken out – **linearity property.**

3. $\lim x_n y_n = \lim x_n \lim y_n$.
   Verbalise: limit of a product is a product of limits.

4. $\lim \dfrac{x_n}{y_n} = \dfrac{\lim x_n}{\lim y_n}$, $\lim y_n \neq 0$.
   Verbalise: limit of a ratio is a ratio of limits.

Laws 2 – 4 give a general recipe for evaluating a limit of a composition of sequences **– instead of the sequence element substitute its limit.** That is, the Laws 2–4 state that if a limit exists the operation of taking the limit and any subsequent algebraic operation can be performed in reverse order. For more complicated operations such reversal is not always justified.

Optional

5. $x_n \geq y_n \Rightarrow \lim x_n \geq \lim y_n$ – **every element of one sequence is greater than or equal to the element with the same index in the other sequence $\Rightarrow$**
   its limit is also greater than or equal to the limit of the other sequence.

6. $cx_n \leq y_n \leq z_n \Rightarrow \lim x_n = \lim z_n \Rightarrow \lim y_n = \lim x_n$
   a sequence squeezed between two others with the same limit has the same limit.

Example: $x_n = (-1)^{n+1}n$. $\lim x_n$?

Solution: $\lim x_n$ does not exist $\Leftarrow$ Rule 1

13.2.4   Indeterminate forms

Sometimes using the above rules leads to an **indeterminate form** (no clear answer) like

$$\frac{0}{0}, \frac{\infty}{\infty}, \ 0 \cdot \infty, \ \infty - \infty, \ 0^0, \ 1^\infty \ etc. \ \text{-} \quad \textbf{MEMORISE}$$

Question: Why do these outcomes present no clear answer?

Answer:

When substitution leads to an indeterminant form we use tricks in an attempt to evaluate the limit or determine it to be non-existent.

Examples:

1. $x_n = \dfrac{1}{n^2}$, $y_n = \dfrac{1}{n} = \lim \dfrac{x_n}{y_n}$ $=$ $\lim \dfrac{\frac{1}{n^2}}{\frac{1}{n}}$ $=$ $\lim \dfrac{1}{n} = 0$

$\dfrac{0}{0}$ ↓ **use algebraic trick – FLIP RULE**

2. $x_n = \dfrac{1}{n^2}$, $y_n = \dfrac{1}{n} \Rightarrow \lim \dfrac{y_n}{x_n}$ $=$ $\lim \dfrac{\frac{1}{n}}{\frac{1}{n^2}}$ $=$ $\lim n = \infty$

$\dfrac{0}{0}$ ↓ **use algebraic trick – FLIP RULE**

3. $x_n = n$, $y_n = n^2$

$\infty - \infty$ ↓ **use algebraic trick - factorisation**

$\Rightarrow \lim(x_n - y_n) = \lim(n - n^2) = \lim n(1-n)$ $=$ $\lim n \lim(1-n) = \infty \cdot (-\infty) = -\infty$

4. $x_n = n^2$, $y_n = n^2 - 1$

$\dfrac{\infty}{\infty}$ ↓ **use algebraic trick – divide top and bottom by the highest power**

$\Rightarrow \lim \dfrac{y_n}{x_n} = \lim \dfrac{n^2 - 1}{n^2}$ $=$ $\lim \dfrac{n^2 - 1 \div n^2}{n^2} = \lim \dfrac{1 - \frac{1}{n^2}}{1} = 1 - 0 = 1$

13.2.5   Decision Tree for Limits

Decision Tree (a Flow Chart) that can be helpful when evaluating limits is given in figure 13.1. Start reading it from the very top and follow the arrows as dictated by your answers. The tricks required to find a limit can be very complicated. However, in standard undergraduate courses they involve only simple algebraic and trigonometric laws and formulae.
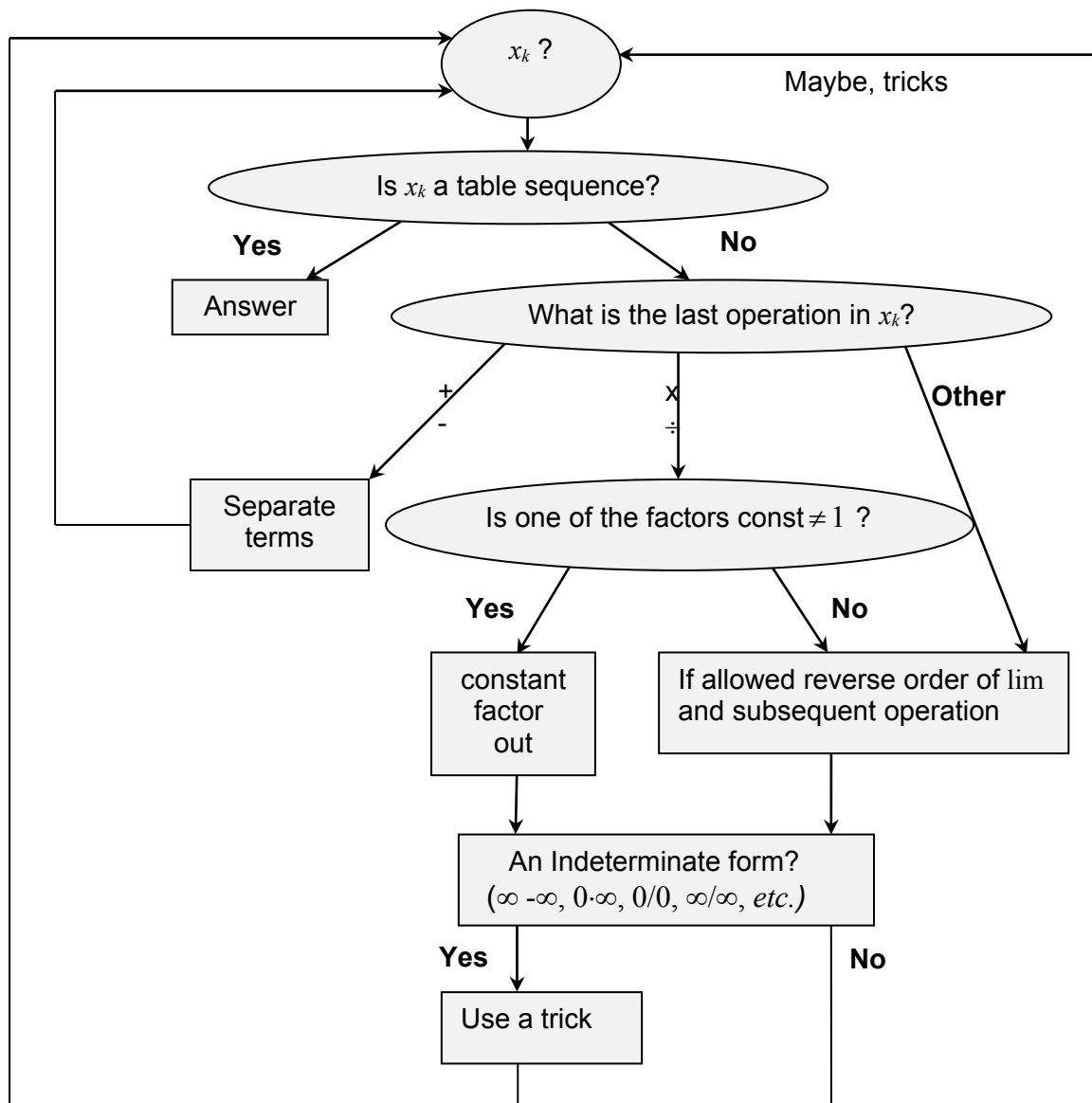
Figure 13.1. Decision Tree for Limits.

13.3   Series

A **series** is a sum of sequence elements.

Question: What is a sum?
Answer:

A series $x_1 + \ldots + x_N$ can be written using the shorthand S_N or $\sum_{n=1}^{N} x_n$. In the latter case we say that we use the **sigma notation**, because Σ is a capital Greek letter called *sigma*.

13.3.1   Arithmetic Progression

An arithmetic progression is a sequence defined by the following recurrence relationship: $x_{n+1} = x_n + a$, where a – constant (with respect to n). a is called the **common difference.**

$$\Rightarrow S_N = x_1 + ... + x_N = N \frac{x_1 + x_N}{2}$$ **MEMORISE**

13.3.2   Geometric Progression

A geometric progression is a sequence defined by the following recurrence relationship:

$x_{n+1} = ax_n$,, where where a is a constant (with respect to n). a is called the **common ratio.**

$$\Rightarrow S_N = \frac{x_1(1 - a^N)}{1 - a}$$

If $|a| < 1$, then $S_N = \frac{x_1}{1-a} - \frac{x_1 a^N}{1-a} \rightarrow \frac{x_1}{1-a} \Rightarrow \lim S_N = \frac{x_1}{1-a}$ **MEMORISE**

13.4   Instructions for self-study

- **Revise Summaries on ALGEBRA, FUNCTIONS and TRIGONOMETRY**
- **Revise Lecture 11 and study Solutions to Exercises in Lecture 11 using the STUDY SKILLS Appendix**
- **Revise Lecture 12 using the STUDY SKILLS Appendix**
- **Study Lecture 13 using the STUDY SKILLS Appendix**
- **Attempt the following exercises:**

Q1. Find the following limits:

a) $\lim \dfrac{n^2 + n + 1}{2n + 1}$;

b) $\lim \dfrac{n^2 + n + 1}{2n^2 + 1}$;

c) $\lim \dfrac{n^2 + n + 1}{2n^3 + 1}$

Q2. Find the sum of a hundred terms of the arithmetic series with first term 1 and common difference 2.

Q3. A steel ball-bearing drops on to a smooth hard surface from a height h. The time to first impact is $T = \sqrt{\dfrac{2h}{g}}$, where g is the acceleration due to gravity. The times between successive bounces are $2eT, 2e^2T, 2e^3T,...$, where e is the coefficient of restitution between the ball and the surface ($0 < e < 1$). Find the total time taken up to the fifth bounce if $T = 1$ and $e = 0.1$.

Q4 (**advanced**). The irrational number π can be defined as the area of a unit circle (circle of radius 1). Thus, it can be defined as a limit of a sequence of known numbers a_n. The method used by Archimedes was to inscribe in the unit circle a sequence of regular polygons (that is, polygons with equal sides – see figure 8.1). As the number of sides increases the polygon "fills" the circle. For any given number n find the area a_n of the regular polygon inscribed in the unit circle. Show, by use of trigonometric identities $\sin 2\theta = 2\sin\theta\cos\theta$, $\cos 2\theta = 1 - 2\sin^2\theta$ and $\sin^2\theta + \cos^2\theta = 1$ (Pythagoras theorem), that the area a_n satisfies the recurrence relation

$$2\left(\frac{a_{2n}}{n}\right)^2 = 1 - \sqrt{\left[1 - \left(\frac{2a_n}{n}\right)^2\right]}, \ n \geq 4.$$

Use trigonometry to show that $a_4 = 2$ and use the above recurrence relation to find a_8.
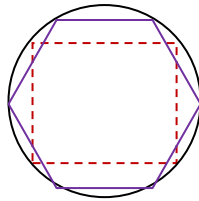
Figure 8.1. A circle with two inscribed regular polygons, one with four sides (a square) and one, with six (a hexagon).

Q5. Explain why $\displaystyle\sum_{k=1}^{\infty} x[k] = \sum_{n=1}^{\infty} x[n] = \sum_{k=0}^{\infty} x[k+1]$.

Lecture 14 DIFFERENTIAL CALCULUS: Limits, Continuity and Differentiation of Real Functions of One Real Variable

14.1   Limits

Taking a limit is **an advanced operation on functions**. The result is a number (more generally, a constant with respect to the independent variable). The operation involves finding a limiting value of the dependent variable when the independent variable approaches (tends to) a specified limiting value of its own or else to ∞ or -∞.

14.1.1 Definition of a Limit of a Real Function of One Real Variable

Four limiting behaviours are possible:

1. $\lim\limits_{x \to a} f(x) = c$

   Verbalise: as x tends to a finite limit a, $f(x)$ tends to a finite limit c.
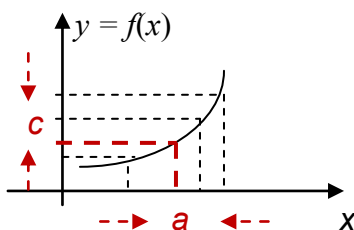
2. $\lim\limits_{x\to\infty} f(x) = c$ or $\lim\limits_{x\to\infty} f(x) = c$

    Verbalise: as x tends (goes off) to infinity, f(x) tends to a finite limit c.

3. $\lim\limits_{x\to\infty} f(x) = \infty$ or   -∞, $\lim\limits_{x\to-\infty} f(x) = \infty$ or   -∞,

    Verbalise: as x tends (goes off) to + or – infinity, $f(x)$ tends to + or – infinity.

4. $\lim\limits_{x\to a} f(x) = \infty$ **or**   -∞

    Verbalise: as x tends to (approaches) a finite limit a, $f(x)$ tends to + or – infinity.

Note: symbols ±∞ are not numbers, they describe a specific function behaviour.

Examples:

1.
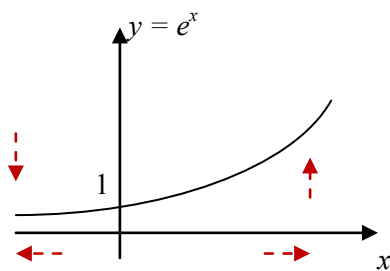
$$\lim\limits_{x\to-\infty} e^x = 0 \qquad\qquad \lim\limits_{x\to\infty} e^x = \infty$$

2.
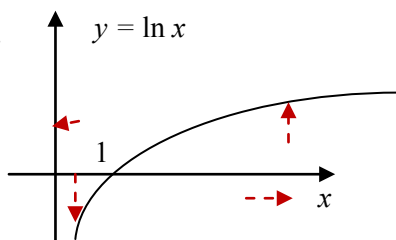
$$\lim\limits_{x\to 0+} \ln x = -\infty \qquad\qquad \lim\limits_{x\to\infty} \ln x = \infty$$

14.1.2   Table of Limits of Functions

$f(x)$	$\lim\limits_{x\to 0} f(x)$
$\dfrac{\sin x}{x}$	1
$\dfrac{e^x - 1}{x}$	1

the proof is rather involved

14.1.3   Rules for (Properties of) Limits

1.   The limit is unique (that is, there can be only one limiting value).

2. $\lim\limits_{x \to a} [f_1 \begin{smallmatrix} + \\ - \\ \times \\ \div \end{smallmatrix} f_2(x)] = \lim\limits_{x \to a} f_1(x) \begin{smallmatrix} + \\ - \\ \times \\ \div \end{smallmatrix} \lim\limits_{x \to a} f_2(x)$ if all limits exist

Using rule 2 is equivalent to just substituting *a* for *x*. If substitution produces an indeterminate form tricks have to be used to establish whether the limit exists and if yes, find it.

Using these rules is equivalent to just substituting *a* for *x*.

Optional

3. $f_1(x) \le f_2(x) \le f_3(x)$ and $\lim\limits_{x \to a} f_1(x) = \lim\limits_{x \to a} f_3(x)$

$\Rightarrow \lim\limits_{x \to a} f_2(x) = \lim\limits_{x \to a} f_1(x)$

Examples:

1. $\lim\limits_{x \to 1} (x^2 - 4x + 8) = 1^2 - 4 \cdot 1 + 8 = 5$ **(instead of *x* we substituted its limiting value)**

2. $\lim\limits_{x \to \pi/2} \sin x = 1$ **(instead of *x* we substituted its limiting value)**

3. $\lim\limits_{x \to \pi/2} \cos x = 0$ **(instead of *x* we substituted its limiting value)**

4. $\lim\limits_{x \to \pi/2} \tan x$ **does not exist, but** $\lim\limits_{x \to \pi/2} \dfrac{\sin x}{\cos x} = \mp \infty$, **which is a shorthand for two statements,**

   $\lim\limits_{x \to \pi/2+} \dfrac{\sin x}{\cos x} = -\infty$ **and** $\lim\limits_{x \to \pi/2-} \dfrac{\sin x}{\cos x} = +\infty$

5. $\lim\limits_{x \to 2} \dfrac{x^2 - 4}{x - 2} \overset{\frac{0}{0}}{=} \lim\limits_{x \to 2} \dfrac{(x - 2)(x + 2)}{x - 2} = 4$

Verbalise: 0 over 0 indeterminate form – we find the limit using an algebraic trick.

Sometimes, limits can be found using analytical considerations and sometimes, using the Table, Rules and a Decision Tree for Limits (practically the same as the one for sequences).

14.1.4   Applications of the Concept of a Limit

1. "Imagine a person walking over a landscape represented by the graph of $y = f(x)$. Her horizontal position is measured by the value of x, much like the position given by a map of the land or by a global positioning system. Her altitude is given by the coordinate y. She is walking towards the horizontal position given by $x = a$. As she does so, she notices that her altitude approaches L. If later asked to guess the altitude over $x = a$, she would then answer L, even if she had never actually reached that position."
   http://en.wikipedia.org/wiki/Limit_of_a_function#Motivation

2. It is often important for engineers to assess how a measured quantity (current, voltage, density, load) behaves with time: does it grow without bounds (then its limit is ∞), tends to a specific value (then its limit is a finite number), oscillates (then it has no limit) *etc.*

14.1.5   Existence of a Limit of a Function

A limit of a function does not always exist.

Examples:

1. $\lim\limits_{x \to \infty} \sin x$ does not exist.

2. $\lim\limits_{x \to \infty} \sin \dfrac{1}{x} = 0.$

14.2   Continuity of a function

A function $f(x)$ is said to be **continuous** at point a if $\lim\limits_{x \to a} f(x) = f(a)$.

Examples:
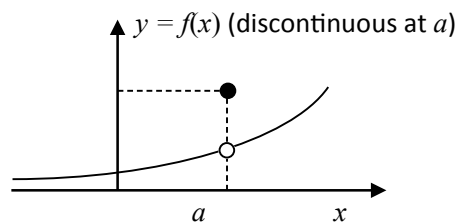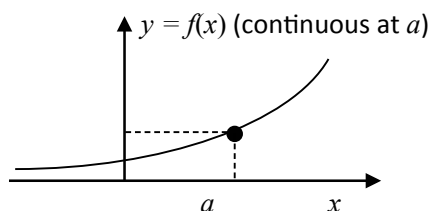
Note: the filled circle indicates the point that belongs to the graph and an empty circle – the point that does not.

If a function is continuous at each point of an interval it is **continuous on** this **interval**. **Intuitively, a continuous function is a function for which, smaller and smaller changes in the input (argument or independent variable) result in smaller and smaller changes in the output (value of a function or dependent variable).** Otherwise, a function is said to be **discontinuous**.

If $f_1(x)$ and $f_2(x)$ are continuous on an interval, so is $f_1(x) \begin{smallmatrix} + \\ - \\ \times \\ \div \end{smallmatrix} f_2(x)$ (provided $f_2(x) \neq 0$ when dividing).

14.2.1 Applications of Continuous Functions

"An analogue signal is a continuous signal for which the time varying feature of the signal is a representation of some other time varying quantity… For example, in sound recording, fluctuations in air pressure (that is to say, sound) strike the diaphragm of a microphone which causes corresponding fluctuations in a voltage or the current in an electric circuit. The voltage or the current is said to be an 'analog' of the sound."
http://en.wikipedia.org/wiki/Analog_signal

14.3   Differentiation of a real function of one real variable

Differentiation is the second advanced operation on functions we cover. The result is a function, which is called **the derivative** of the original function.

14.3.1   A Derivative of a Function

A continuous function $f(x)$ is **differentiable** at $x = a$ if there exists $\lim\limits_{\Delta x \to 0} \dfrac{f(a + \Delta x) - f(a)}{\Delta x} \overset{\frac{0}{0}}{=}$
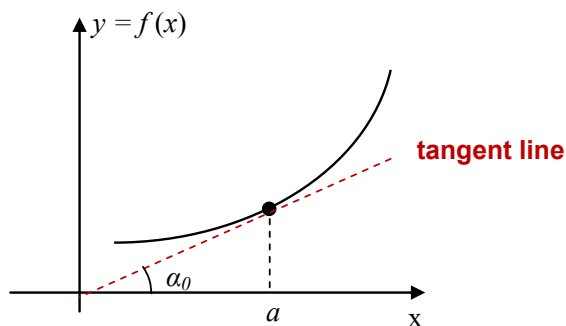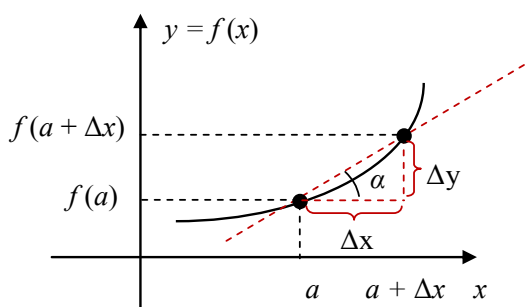
$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \tan \alpha = \tan \alpha_0 = \frac{dy}{dx}(a) \equiv \frac{df}{dx}(a) \equiv f'(a).$$ This limit is called a **derivative**.

Thus, we arrived at the following **definition of the derivative**:

$$\frac{df(x)}{dx} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \, .$$

14.3.2   Geometrical Interpretation of a Derivative

A derivative of a function is a local slope (gradient) of the function at a point. More specifically, it is the slope of the line tangent to the graph of the function at this point.

14.3.3   Mechanical Interpretation of a Derivative

A derivative of a function is a local rate (speed) of change of the function at a point.

14.3.4   Existence of a Derivative

A derivative of a function does not always exist.

Here are graphs of functions which have no derivative at point a of their domain:
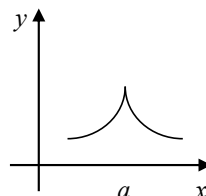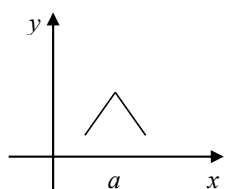
Examples:

1. Differentiate $y = c$ (constant)
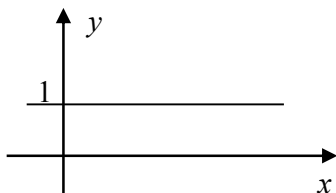
Answer: $\Rightarrow \dfrac{dy}{dx} = 0$

   a) Can differentiate using a graph: the slope is the same everywhere and is zero.
   b) Can differentiate using the basic definition:

2. Differentiate $y = c\,x$
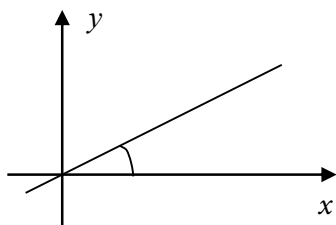
Answer: $\Rightarrow \dfrac{dy}{dx} = c$

   a) Can differentiate using a graph: the slope is the same everywhere – c
   b) Can differentiate using the basic definition:

$$\lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{c(x + \Delta x) - cx}{\Delta x} = c$$

14.4   A historical note

"When Newton and Leibniz first published their results on calculus in the 17the century, there was great controversy over which mathematician (and therefore which country) deserved credit. Newton derived his results first, but Leibniz published first. Newton claimed Leibniz stole ideas from his unpublished notes, which Newton had shared with a few members of the Royal Society. This controversy divided English-speaking mathematicians from continental mathematicians for many years, to the detriment of English mathematics. A careful examination of the papers of Leibniz and Newton shows that they arrived at their results independently, with Leibniz starting first with integration and Newton with differentiation. Today, both Newton and Leibniz are given credit for developing calculus independently. It is Leibniz, however, who gave the new discipline its name. Newton called his calculus 'the science of fluxions'". http://en.wikipedia.org/wiki/Calculus

14.5   Instructions for self-study

- **Revise ALGEBRA Summary (particularly, the words term, sum, factor, product)**
- **Revise Lecture 12 and study Solutions to Exercises in Lecture 12 using the STUDY SKILLS Appendix**
- **Revise Lecture 13 using the STUDY SKILLS Appendix**
- **Study Lecture 14 using the STUDY SKILLS Appendix**
- **Attempt the following exercises:**

Q1. Find

a) $\lim\limits_{x \to 0} \dfrac{1 - \cos x}{x^2}$ (using $\lim\limits_{x \to 0} \dfrac{\sin x}{x} = 1$ and trigonometric identities)

b) $\lim\limits_{x \to 0} \dfrac{e^x - e^{-x}}{2}$

c) $\lim\limits_{x \to 0} \dfrac{\sin^2 3x}{x^3}$

d) $\lim\limits_{x \to 0} \dfrac{x^2 - 1}{2x^2 + 3}$

e) $\lim\limits_{x \to \infty} \dfrac{x^2 - 1}{2x^2 + 3}$

Q2. Discuss continuity of $y = \tan x$.

Q3. Plot $y = e^x$. Differentiate this function approximately using the graph and plot the approximate derivative $\dfrac{d}{dx} e^x$.

Q4. Prove

a) $\dfrac{d}{dx} x^n = n x^{n-1}$ (Hint: use the Binomial Theorem – look it up in books or Google it.)

b) $\dfrac{d}{dx} e^x = e^x$

c) $\dfrac{d}{dx} \sin x = \cos x$

d)
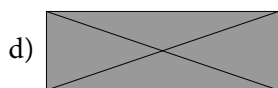
Lecture 15   DIFFERENTIAL CALCULUS: Differentiation (ctd.)

Using **the first principles** (that is, the definition of a derivative) we can create a **table of derivatives of elementary functions**.

15.1   Differentiation Table

$y = f(x)$	$\dfrac{df(x)}{dx}$	
	0	This row means $\dfrac{d\,\text{const}}{dx} = 0$
x^n	nx^{n-1}	This row means $\dfrac{d\,\mathrm{x}^{\mathrm{n}}}{dx} = \mathrm{nx}^{\mathrm{n}-1}$
e^x	e^x	This row means $\underline{}$ x

Examples:

1. $\dfrac{d\cos\pi}{dx} = 0$

> **Differentiation variable?** x
> **Function to differentiate?** constant
> **It is a table function**

2. $\dfrac{d\,t^3}{dt} = 3t^2$

> **Differentiation variable?** t
> **Function to differentiate?** power 3
> **It is a table function of** t

3. $\dfrac{d\,\sin\,v}{dv} = \cos\,v$

> **Differentiation variable?** v
> **Function to differentiate?** $\sin(\,)$
> **It is a table function of** v

4. $\dfrac{d(\ln t)^3}{d(\ln t)} = 3(\ln t)^2$

> **Differentiation variable?** v
> **Function to differentiate?** $\sin(\,)$
> **It is a table function of** v

5. $\dfrac{d\,\cos\,e^v}{de^v} = -\sin\,e^v$

> **Differentiation variable?** e^v
> **Function to differentiate?** $\cos(\,)$
> **It is a table function of** e^v

To differentiate combinations of elementary functions we can use **Differentiation Table, Differentiation Rules** and **Decision Tree** given in figure 15.1.

15.2   Differentiation Rules

1. $\dfrac{d\alpha f(x)}{dx} = \alpha \dfrac{df(x)}{dx}$ - **constant factor out rule**

2. $\dfrac{d[f(x)+g(x)]}{dx} = \dfrac{df(x)}{dx} + \dfrac{dg(x)}{dx}$ **– sum rule**

linearity property

3. $\dfrac{d(f(x)g(x))}{dx} = \dfrac{df(x)}{dx}g(x) + \dfrac{dg(x)}{dx}f(x)$ **– product rule**

4. $\dfrac{d\,\frac{f(x)}{g(x)}}{dx} = \dfrac{\frac{df(x)}{dx}g(x) - \frac{dg(x)}{dx}f(x)}{g^2}$ **– quotient rule**

5. $\dfrac{d(f(g(x)))}{dx} = \dfrac{df(g)}{dg}\dfrac{dg}{dx}$ **– chain rule (decompose, differentiate, multiply).**   Here

$g = g(x)$.

Note: These rules can be proven using the definition of the derivative as a limit and rules for limits.

Examples:

1. $\dfrac{d\,2\,\sin x}{dx} = 2\cos x$

Differentiation variable – x
Function to differentiate – $2\sin x$
Last operation in this function – multiplication
One of the factors constant
Apply "constant factor out" rule

Constant factor $\alpha = 2$

Variable Factor $f(x) = \sin x,\ \dfrac{df(x)}{dx} = \dfrac{d\,\sin x}{dx} = \cos x$

2. $\dfrac{d\,(\,\sin\,x\,+\cos\,x)}{dx} = \cos x\,-\,\sin x$

Differentiation variable – x
Function to differentiate – $3\sin x + 5\cos x$
Last operation in this function – addition
Apply the sum rule

---

f-line: 1st term $f(x) = \sin x$, $\quad \dfrac{d\,f(x)}{dx} = \dfrac{d\,\sin\,x}{dx} = \cos x$

g-line: 2nd term $g(x) = \cos x$, $\quad \dfrac{d\,g(x)}{dx} = \dfrac{d\,\cos\,x}{dx} = -\sin x$
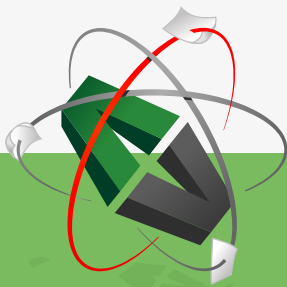
3. $\dfrac{d\,(x^2 + 3x + 2)}{dx} = \dfrac{d\,x^2}{dx} + \dfrac{d\,3x}{dx} + \dfrac{d\,2}{dx} = 2x + 3$

Differentiation variable – x

Function to differentiate – $x^2 + 3x + 2$
Last operation in this function – addition
Apply the sum rule
Then go over the same reasoning for each term

4. $\dfrac{d\,(x\,\sin\,x)}{dx} = \sin\,x + x\,\cos\,x$

Last operation in this function – multiplication
Is one of the factors constant? No
Apply the product rule

f-line: 1st factor $f(x) = x,\ \dfrac{d\,f(x)}{dx} = \dfrac{d\,x}{dx} = 1$

g-line: 2nd factor $g(x) = \sin x\,,\ \dfrac{d\,g(x)}{dx} = \dfrac{d\,\sin\,x}{dx} = \cos x$

5. $\dfrac{d\,(x\,\ln\,x)}{dx} = \ln\,x + 1$ (using the product rule).

f-line: $f(x) = x,\ \dfrac{df(x)}{dx} = \dfrac{dx}{dx} = 1$

g-line: $g(x) = \ln x\,,\ \dfrac{dg(x)}{dx} = \dfrac{d\,\ln x}{dx} = \dfrac{1}{x}$

6. $\dfrac{d\left(\dfrac{\sin x}{x}\right)}{dx} = \dfrac{x\cos x - \sin x}{x^2}$ (using the quotient rule)

f-line: Numerator $f(x) = \sin x,\quad \dfrac{d\,f(x)}{dx} = \dfrac{d\,\sin x}{dx} = \cos x$

g-line: Denominator $g(x) = x,\quad \dfrac{d\,g(x)}{dx} = \dfrac{dx}{dx} = 1$
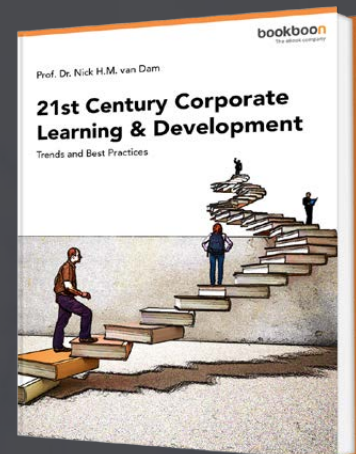
7. $\dfrac{d \left(\dfrac{\ln x}{x} \right)}{dx} = \dfrac{x \cdot \dfrac{1}{x} - \ln x}{x^2} = \dfrac{1 - \ln x}{x^2}$

Differentiation variable? x

f(x) ? $\dfrac{\ln x}{x}$

Last operation in this function ?   **Division**

Apply which rule? quotient rule

---

f-line: Numerator $f(x) = \ln x,$ $\dfrac{d\,f(x)}{dx} = \dfrac{d\,\ln x}{dx} = \dfrac{1}{x}$

g-line: Denominator $g(x) = x,$ $\dfrac{d\,g(x)}{dx} = \dfrac{dx}{dx} = 1$

8. $\dfrac{d \sin 2x}{dx} = 2\cos 2x$

Differentiation variable? x
f(x)? $\sin 2x$
Last operation in this function? $\sin(\ )$
Its argument? $2x$
Apply which rule? The chain rule

---

f-line: Last operation $f(\ ) = \sin(\ ),$

$\dfrac{d\,f(x)}{dx} = \dfrac{d\,\sin(g)}{dg} = \cos g = \cos 2x$

g-line: Its argument $g(x) = 2x,$ $\dfrac{d\,g(x)}{dx} = \dfrac{d\,2x}{dx} = 2$

9. $\dfrac{d\ \ln\ 3x}{dx}\ =\ 3\,\dfrac{1}{3x}=\dfrac{1}{x}$

Differentiation variable – x
Function to differentiate – $\ln 3x$
Last operation in this function – $\ln(\ )$
Its argument – $3\,x$
Apply the chain rule

---

f-line: Last operation $f(\ )=\ln(\ ),\quad \dfrac{d\,f(x)}{dx}=\dfrac{d\,\ln(g)}{dg}=\dfrac{1}{g}=\dfrac{1}{3x}$

g-line: Its argument $g(x)=3x,\quad \dfrac{d\,g(x)}{dx}=\dfrac{d\,3x}{dx}=3$

15.3   Decision Tree for Differentiation

Generally, to differentiate a function we use the **Decision Tree for Differentiation** (see figure 15.1 below) that allows us to decide whether to use the Differentiation Table (containing derivatives of elementary functions) straight away or first use Differentiation Rules (on how to differentiate combinations of elementary functions). Once the rule is applied, even if the problem is not solved, it is reduced to one or several simpler problems which have to be treated using the Decision Tree again.

To use the Decision Tree start at the top and follow the arrows which are associated with the correct answers (if any). Use it as a formula, substituting your differentiation variable for *x* and your function for *f*(*x*).

15.4   The higher order derivatives

If a derivative of *f*(*x*) is a differentiable function its derivative is called **the second derivative** of *f*(*x*).

$$\frac{d^2 f}{dx^2} = \frac{d}{dx}\left(\frac{df}{dx}\right)$$

If the second derivative of *f*(*x*) is differentiable its derivative is called **the third derivative** of *f*(*x*),

$$\frac{d^3 f}{dx^3} = \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \text{ etc.}$$
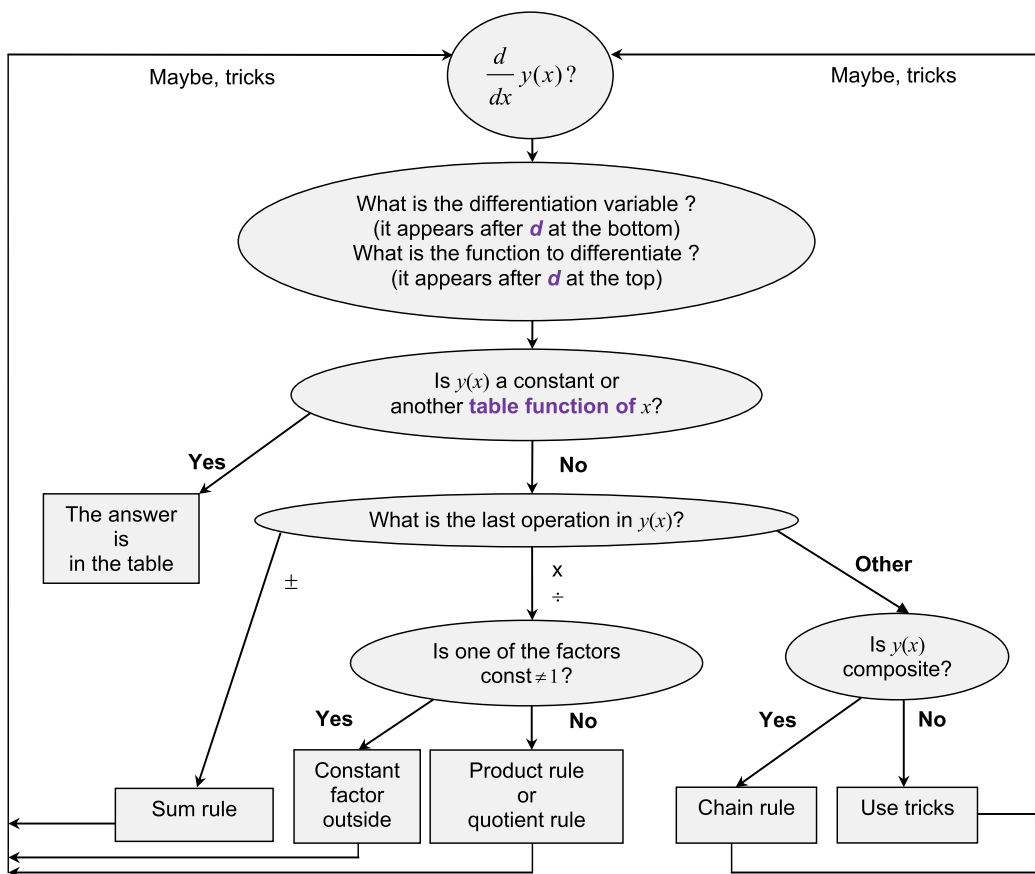
Figure 15.1. Decision Tree for Differentiation.

15.5   The partial derivatives

If we consider a function of say, two variables (two arguments) $f(x,y)$ its **partial derivatives** with respect to x and y are denoted by $\dfrac{\partial f(x,y)}{\partial x}$ and $\dfrac{\partial f(x,y)}{\partial y}$, respectively.

15.6   Applications of differentiation

Later you will find that all simple scientific phenomena and engineering systems that you are going to study are described by differential equations that relate values of measured variables, such as displacement, angle or rotation, concentration, current or voltage to their first or second derivatives. You will be also shown that derivatives and limits are helpful in sketching functions and thus illustrating or even predicting behaviour of the above measurable variables. Finally you will find out how derivatives can help to approximate functions, e.g. how your calculators calculate sin, cos values *etc.*

15.7   Instructions for self-study

- **Revise ALGEBRA Summary (particularly, the words term, sum, factor, product)**
- **Revise Summaries on the ORDER OF OPERATIONS and FUNCTIONS**
- **Revise Lecture 4 and Solutions to Exercises in Lecture 4 (particularly, the operation of composition)**
- **Revise Lecture 13 and study Solutions to Exercises in Lecture 13 using the STUDY SKILLS Appendix**
- **Revise Lecture 14 using the STUDY SKILLS Appendix**
- **Study Lecture 15 using the STUDY SKILLS Appendix**
- **Attempt the following exercises:**

Q1. Find

a) $\dfrac{d}{d\sqrt{t^2+1}}e^{\sqrt{t^2+1}}$

b) $\dfrac{d}{d\sqrt{t^2+1}}e^{\sqrt{x^2+1}}$

Q2. Find the derivatives

a) $\dfrac{d}{du}(u^2+1)\sin u$

b) $\dfrac{d}{du}\dfrac{u^2+1}{\sin u}$

Q3. Find the derivatives

a) $\dfrac{d\cos\sqrt{x^2+1}}{dx}$

b) $\dfrac{d\cos\sqrt{x^2+1}}{dy}$

Q4. Differentiate y with respect to t if $y = (t+\sqrt{t^2+a^2})^n$.

Lecture 16 DIFFERENTIAL CALCUUS: Sketching Graphs Using Analysis

In general, any composite function can be sketched using our knowledge of limits and derivatives. In this Lecture we show how. There are four possible behaviours in the vicinity of the stationary point:

16.1   Stationary points

A **stationary point** is a point x_0 in the function domain where its derivative $f'(x_0) = 0$.
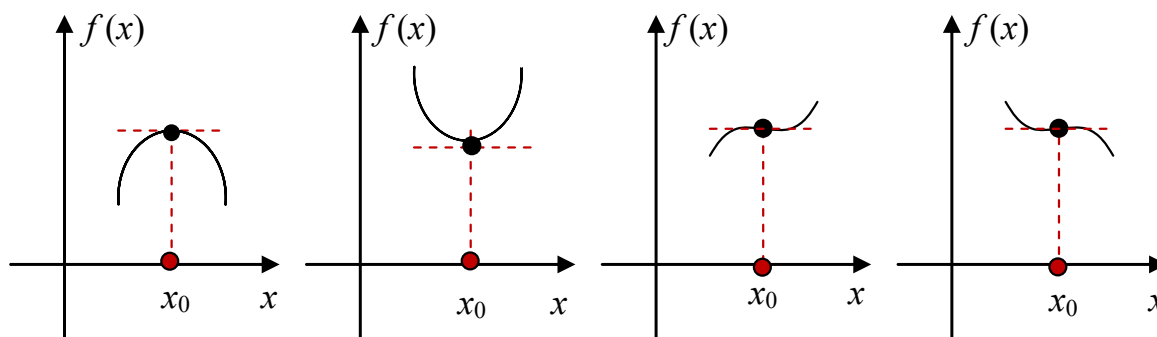
Examples: Sketch the following functions and find their stationary ponts x_0:

1. $y = x^2$

$$\frac{dy}{dx} = 0$$

$$2x = 0$$

$$x_0 = 0$$
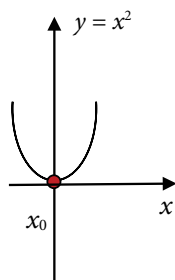
2. $y = x^3$

$$\frac{dy}{dx} = 0$$

$$3x^2 = 0$$

$$x_0 = 0$$
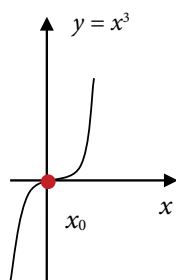
3. $y = e^x$

$$\frac{dy}{dx} = e^x \neq 0$$

No stationary points
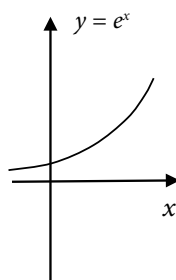
4. $y = \sin x$

$$\frac{dy}{dx} = \cos x = 0$$

$$x_0 = \frac{\pi}{2} n,$$

n – odd integer
$(n = 2\,m + 1,\ m$ – integer$)$
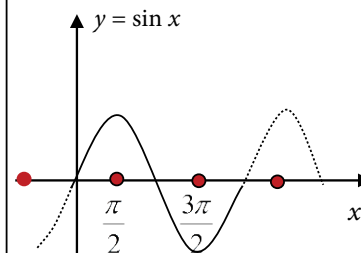
16.2   Increasing and decreasing functions

If derivative (local slope) $f'(x) > 0$ on interval I, function $f(x)$ is said to be **increasing** on I.
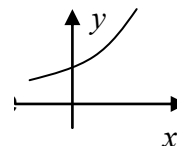
If derivative (local slope) $f'(x) > 0$ on interval I, function $f(x)$ is said to be **decreasing** on I.
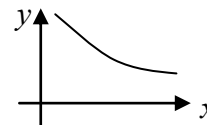
> **Examples:** Functions $y = e^x$ and $y = \ln x$ are increasing everywhere; functions $y = |x|$ and $y = x^2$ are decreasing for $x < 0$ and increasing for $x > 0$.

16.3   Maxima and minima

If derivative (local slope) $f'(x) > 0$ to the left of x_0 and < 0 to the right of x_0, then function $f(x_0)$ is said to have a **maximum** at x_0.
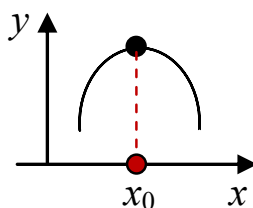
If derivative (local slope) $f'(x) < 0$ to the left of x_0 and > 0 to the right of x_0, then function $f(x_0)$ is said to have a **minimum** at x_0.
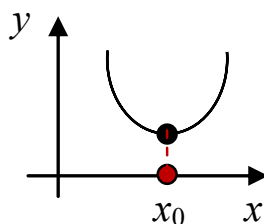
Examples: $y = x^2$ has minimum at 0, x^3 has no minimum or maximum, $\sin x$ has maxima at $x = \dfrac{\pi}{2}m$ when m is odd and minima when m is even.

Examples:

1. Sketch $y = x^2 + 3x + 1$ by completing the square

Solution:

Completing the square, $y = (x + \frac{3}{2})^2 - \frac{9}{4} + 1 = (x + \frac{3}{2})^2 - \frac{5}{4}$.

Step 1. Using the recipe for sketching by simple transformations, we first drop all constant factors and terms and sketch the basic shape of the function that remains:
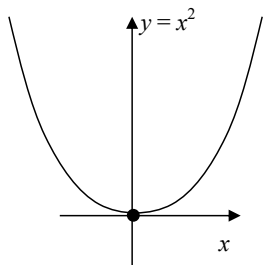
Step 2. We then reintroduce the constant factors and terms that appeared in the original functional equation one by one, in Order of Operations and sketch the resulting functions underneath one another:
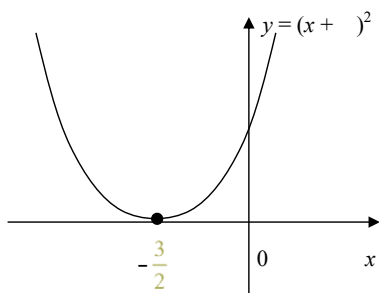
$+ \dfrac{3}{2}$ – constant term, first operation

$\Rightarrow$ translation wrt x-axis by $- \dfrac{3}{2}$
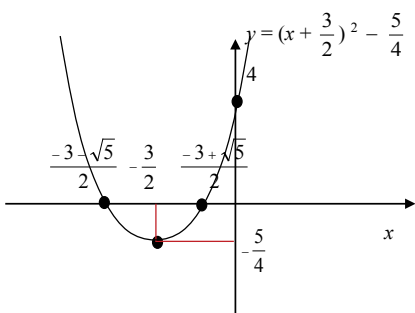
$- \dfrac{5}{4}$ constant term, first operation

$\Rightarrow$ translation wrt y-axis by $- \dfrac{5}{4}$

Step 3.

Intersection with the y-axis: $x = 0$, $y = 4$

Intersection with the x-axis: $y = 0$,

$$x_{1,2} = -\frac{3 \pm \sqrt{9-4}}{2} = -\frac{3 \pm \sqrt{5}}{2} \approx -\frac{3 \pm 2.1}{2} \approx -2\frac{1}{2}, -\frac{1}{2}$$

2. Sketch $y = x^2 + 3x + 1$ by analysis.

Solution:

Step 1. Intersection with the y–axis: $x = 0$, $y = 4$

Intersection with the x–axis: $y = 0$, $x_{1,2} = -\dfrac{3 \pm \sqrt{9-4}}{2} = -\dfrac{3 \pm \sqrt{5}}{2} \approx -\dfrac{3 \pm 2.1}{2}$

Step 2. Find and sketch the first derivative

$y' = 2x + 3$

a) Find where the first derivative is zero: $2x + 3 = 0 \Rightarrow x = -\dfrac{3}{2}$

$\Rightarrow$ Stationary point of the function is $x = -\dfrac{3}{2}$

b) Find where the first derivative is positive and where it is negative:

Can do so by solving the inequalities

$2x + 3 > 0 \quad (y' > 0)$ for $x > -\dfrac{3}{2}$

$2x + 3 < 0 \quad (y' < 0)$ for $x < -\dfrac{3}{2}$

Note: inequalities can be solved the same way as equations but when multiplying by negative factors the inequality sign should be reversed.

or graphically

Step 3: Sketch the function y directly underneath the first derivative y'

Step 4: Find the minimum function value

$y(-\dfrac{3}{2}) = \dfrac{9}{4} - \dfrac{9}{2} + 1 = \dfrac{9 - 18 + 4}{4} = -\dfrac{5}{4}$
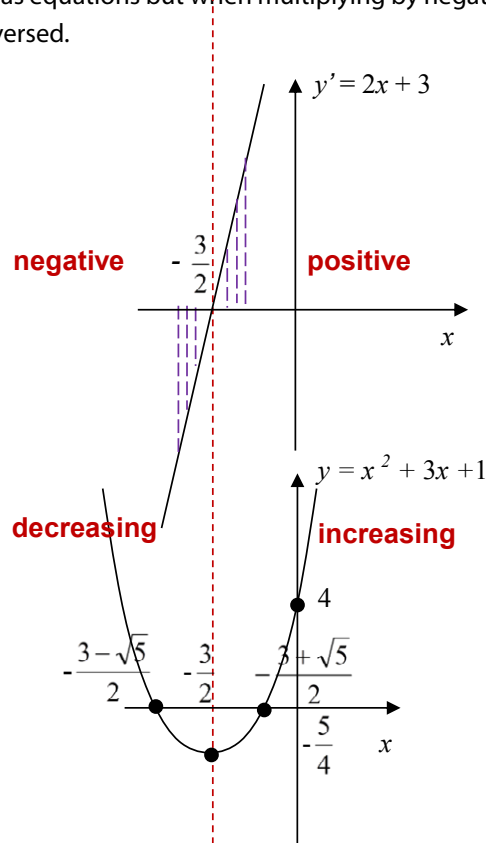
3. Sketch $y = x^3 + 3x^2 + 4x + 1$ by analysis

Solution

Step 1. Find and sketch the first derivative y'

$y' = 3x^2 + 6x + 4 = 3(x^2 + 2x) + 4 = 3[(x + 1)^2 - 1] + 4 = 3(x + 1)^2 + 1 > 0$
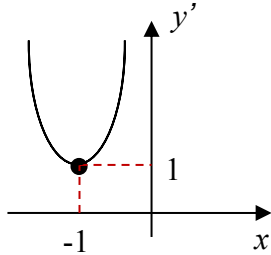
$y' > 0$ everywhere

Step 2. Sketch the function y underneath
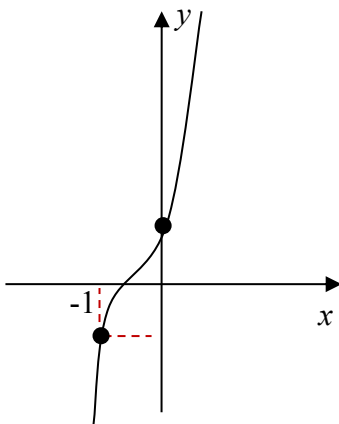
y is increasing everywhere

Special points: y(0) = 1

y(−1) = −1 (not that special – but easy to find)

Optional

16.4   Convex and concave functions

If the second derivative $f''(x) > 0$ on interval I
the first derivative $f'(x)$ is increasing on I, and
function $f(x)$ is said to be **convex** on interval I.

If the second derivative $f'''(x) < 0$ on interval I
the first derivative $f'(x)$ is decreasing on I, and
function $f(x)$ is said to be **concave** on interval I.
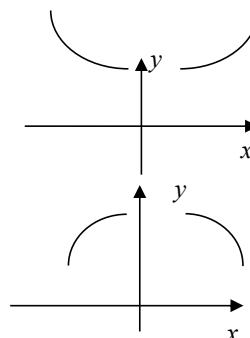
Examples: establish the regions of convexity of the following elementary functions:

$y = x^2$	$y = x^3$	$y = e^x$
$y' = 2x$	$y' = 3x^2$	$y' = e^x$
$y'' = 2 > 0$	$y'' = 6x \begin{cases} > 0 \text{ if } x > 0 \\ < 0 \text{ if } x < 0 \end{cases}$	$y'' = e^x > 0$
$\Rightarrow\ y = x^2$ is convex everywhere	$\Rightarrow\ y = x^3$ is $\begin{cases} \text{convex if } x > 0 \\ \text{convex if } x < 0 \end{cases}$	$\Rightarrow\ y = e^x$ is convex everywhere

16.5   Inflexion points

If the second derivative $f'''(x_0) = 0$ and $f''(x)$ changes sign at the stationary point x_0, then the stationary point x_0 is called **an inflexion point**.

Example: $y = x^3$ has an inflexion point at $x = 0$.

Thus, there is another way of determining whether the stationary point is at a maximum or minimum:

If $f'(x_0) = 0$ and $f''(x_0) > 0$, then the function is convex in the vicinity of x_0 and x_0 **is at a minimum**.

If $f'(x_0) = 0$ and $f''(x_0) < 0$, then the function is concave in the vicinity of x_0 and x_0 **is at a maximum**.

If $f'(x_0) = 0$ and $f''(x_0) = 0$, then further **checks are required**.

16.6   Sketching rational functions using analysis

We illustrate the general principles of sketching by analysis by applying the method to rational functions.

A **rational function** $R(x)$ is

$$R(x) = \frac{P(x)}{Q(x)},$$

where P and Q are polynomials.

Examples:

1. Sketch the rational function $y = \dfrac{x-1}{x+1}$.

Solution

Step 1. Domain: $x \neq -1$ ($x = -1$ is called **a pole**, the dashed line $x = -1$ is called **a vertical asymptote**)

Step 2. Intersection with the y–axis: $x = 0,\ y = -1$
  Intersection with the x–axis: $y = 0,\ x = 1$

Step 3. Behaviour at the domain boundaries:

$x \to \infty, y \to 1$
$x \to -\infty, y \to 1$

The line $y=1$ is called **a horizontal asymptote**.
We know that the function approaches it when $|x|$ grows large but we do not know how.

Step 4. Find the first derivative:

$$\frac{dy}{dx} = \frac{x+1-(x-1)}{(x+1)^2} = \frac{2}{(x+1)^2} > 0$$

$\Rightarrow$ y is increasing everywhere
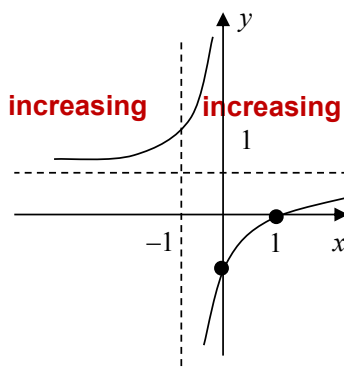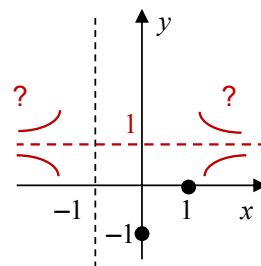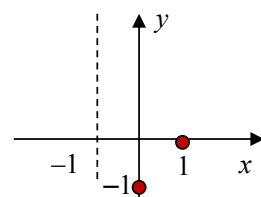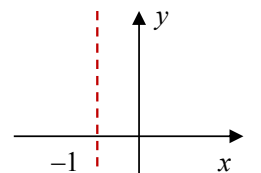
2) Sketch $y = \dfrac{x-1}{x^2 - 4x + 4}$.

Solution

Step 1. Domain: $x \neq 2$ ($x = 2$ is a pole, the line is a **vertical asymptote**)

Step 2. Intersection with the y-axis: $x = 0$, $y = -\dfrac{1}{4}$

  Intersection with the x-axis: $y = 0$, $x = 1$

Step 3. Behaviour at the domain boundaries:

$x \to \pm\infty, y \to \pm 0$

The line $y = 0$ is **a horizontal asymptote**.

Step 4. Find the first derivative:

$$\frac{dy}{dx} = -\frac{x}{(x-2)^3} = \frac{-x^2 + 2x}{(x-2)^4} \begin{cases} = 0 & \text{at } x = 0 \\ > 0 \ (y \text{ is increasing}) & \text{for } 0 < x < 2 \\ < 0 \ (y \text{ is decreasing}) & \text{for } x < 0 \ \& \ x > 2 \end{cases}$$
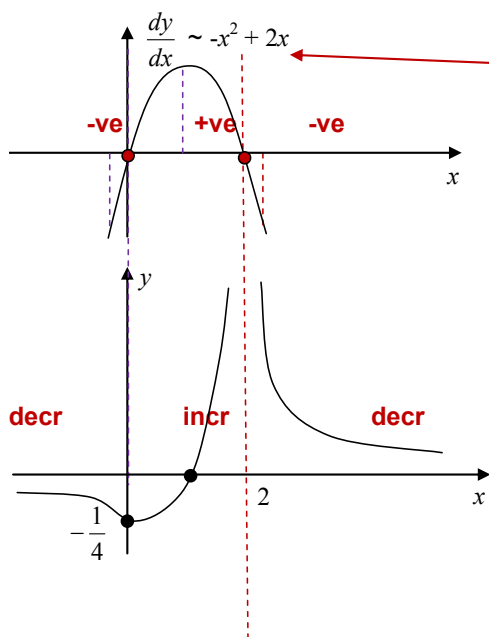
This is not a derivative but it is 0, positive and negative where $\dfrac{dy}{dx}$ is – since $(x-2)^4 > 0$ when $x \neq 2$.

The sign $\sim$ means "behaves as".

Note that $x = 2$ turns the numerator $-x + 2x$ into zero but $\dfrac{dy}{dx}$ is not zero at this point. The "phantom" zero appears because we multiplied both numerator and denominator of $\dfrac{dy}{dx}$ by $(x - 2)$. This was a trick performed to make sure that the denominator is never negative.

Optional

polynomial degree 2

Sketch $y = \dfrac{x^2 + 4x + 2}{x + 1}$ -   **improper rational function**

polynomial degree 1

Difference in degrees = 1 $\Rightarrow$ the whole part is a **polynomial degree** 1

$$\frac{x^2 + 4x + 2}{x + 1} = Ax + B + \frac{C}{x + 1}$$

polynomial degree 0

$\Uparrow$

polynomial degree 1

To find *A, B* and *C* use

1. **long division** or

2. **partial fractions method**

$$\frac{x^2 + 4x + 2}{x + 1} = Ax + B + \frac{C}{x + 1} \Big/ (x + 1)$$

$$x^2 + 4x + 2 = Ax(x + 1) + B(x + 1) + C$$

True for all $x \Rightarrow$ can choose any convenient value of x

Choose values that "kill" terms $x = -1$: $1 - 4 + 2 = C,\ C = -1$
$x = 0$: $2 = B + C, 2 = B - 1,\ B = 3$

Equate coefficients of highest power: $1 = A$

Now we have to sketch $y = x + 3 - \dfrac{1}{x + 1}$

Step 1. Domain: $x \neq -1$ (point $x = -1$ is a pole and line $x = -1$ is a **vertical asymptote**)

Step 2. Intersection with the *y*-axis: $x = 0, y = 2$

  Intersection with the *x*-axis: $y = 0$, $x^2 + 4x + 2 = 0$, $x_{1,2} = -2 \pm \sqrt{2}$

Step 3. Behaviour at the domain boundaries:

  $x \rightarrow \pm\infty, y \rightarrow x + 3$ – **an oblique asymptote** (typical for improper rational functions)

Step 4. Find the first derivative:

$\dfrac{dy}{dx} = \dfrac{x^2 + 2x + 2}{(x + 1)^2} > 0$ because $x^2 + 2x + 2 = 0$ at $x_{1,2} = -1 \pm j$ (no real roots)

$\dfrac{dy}{dx} \sim x^2 + 2x + 2$

positive everywhere

y

$-2-\sqrt{2}$

2

increasing everywhere

-1

$-2+\sqrt{2}$

x

$y = x + 3$
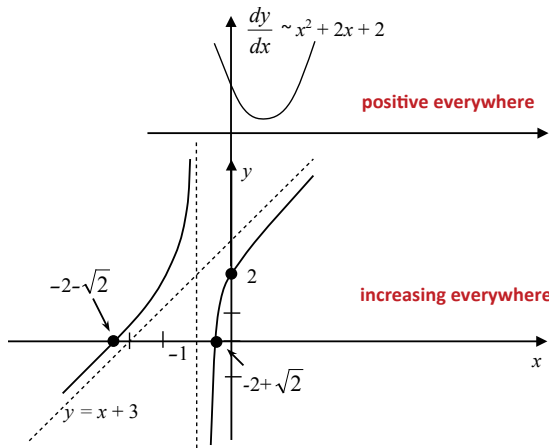
16.7   Applications of rational functions and their graphs

In control theory, single input single output (SISO) linear dynamic systems are often characterised by the so-called transfer functions. Any such transfer function is a rational function of one complex variable. Sketching magnitudes and phases of these functions helps design efficient control systems.

16.8   Instructions for self-study

- **Revise Lecture 9 and study Solutions to Exercises in Lecture 9 (sketching by simple transformations)**
- **Revise ALGEBRA Summary (addition and multiplication, factorising and smile rule, flip rule)**
- **Revise Summaries on the ORDER OF OPERATIONS and FUNCTIONS**
- **Revise Lectures 13–15 (limits and differentiation) and study Solutions to Exercises in Lecture 14 using the STUDY SKILLS Appendix**
- **Study Lecture 16 using the STUDY SKILLS Summary**
- **Do the following exercises:**

Q1. Sketch using analysis

a) $y = -3 \, (x^3 - 3x^2 + x)$

b) $y = -\dfrac{1}{2}x^2 + \dfrac{1}{4}x + \dfrac{1}{8}$

Q2. Sketch $y = \dfrac{x^2 - 5x + 6}{x^2 + 1}$

Q3. Sketch $y = u(x + 2) \, \dfrac{x^2 - 5x + 6}{x^2 + 1}$

Q4. Sketch $y = \dfrac{x^2 - 5x + 6}{x + 1}$

Lecture 17 Application of DIFFERENTIAL CALCULUS to Approximation of Functions: the Taylor and Maclaurin Series

17.1   Approximating a real function of a real variable using its first derivative

Let us discuss how we can **approximate** the value of function $f(x)$ at $a + \Delta x$ if we know its value at a. It is clear from the figure below that

$$f(a + \Delta x) \approx f(a) + \Delta y.$$

How can we approximate Δy? To see this, let us revise the definition of the first derivative given in Lecture 13.
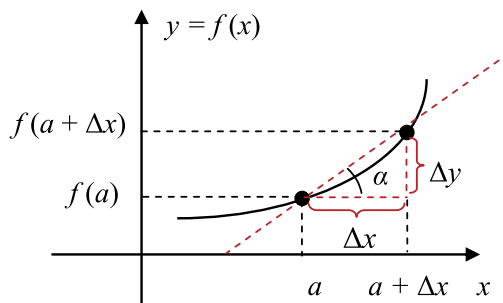
Question: What is the geometrical interpretation of derivative?
Answer:

Question: How can we find its approximate value?
Answer:

Question: How to solve this approximate equality for Δy?
Answer:

$$\Rightarrow f(a + \Delta x) \approx f(a) + f'(a)\,\Delta x$$

Verbalise: The function value near a **approximately equals** the function value at a + derivative at a times distance to a.

This is an approximation *linear* in Δx (in the vicinity of x the graph of $f(x)$ looks like a *straight line*). Sometimes we need approximations based on quadratic, cubic … *n-th* degree polynomials. Before showing how to find such approximations we introduce a new way of representing a polynomial.

17.2   The Maclaurin polynomials

Question: What is a polynomial?
Answer:

Question: Conventionally we write the first term of the polynomial $P_n(x)$ as $a_n x^n$. All coefficients are constant with respect to *x* **and are denoted by the letter** *a*. What will be the next term and the next and the next…?
Answer:

Any polynomial can be re-written in another form as

the Maclaurin polynomial: $P_n(x) = P_n(0) + P'_n(0)x + \dfrac{P''_n(0)}{2!}x^2 + \dfrac{P'''_n(0)}{3!}x^3 + \ldots \dfrac{P_n^{(n)}(0)}{\hbar}x^n$

where n **factorial** $n! = 1 \times 2 \times 3 \times 4 \times \ldots \times n$.

We can show this using the following steps:

$P_n(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_3 x^3 + a_2 x^2 + a_1 x^1 + a_0 \qquad \Rightarrow \quad P_n(0) = a_0 \quad \Rightarrow \quad a_0 = P_n(0)$

$P'_n(x) = a_n n x^{n-1} + a_{n-1}(n-1)x^{n-2} + \ldots + 3a_3 x^2 + 2a_2 x + a_1 \quad \Rightarrow \quad P'_n(0) = a_1 \quad \Rightarrow \quad a_1 = P'_n(0)$

$P''_n(x) = a_n n(n-1)x^{n-2} + a_{n-1}(n-1)(n-2)x^{n-3} + \ldots + 3\cdot 2 \cdot a_3 x + 2a_2 \Rightarrow P''_n(0) = 2a_2 \Rightarrow a_2 = \dfrac{P''_n(0)}{2}$

$P'''_n(x) = a_n n(n-1)(n-2)x^{n-3} + \ldots + 3\cdot 2 \cdot a_3 \qquad\qquad \Rightarrow \quad P'''_n(0) = 2 \cdot 3\; a_3 \Rightarrow a_3 = \dfrac{P'''_n(0)}{3!}$

$\ldots \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \ldots$

$P_n^{(n)}(x) = a_n n(n-1)(n-2)\ldots[n-(n-1)]x^{n-n} \qquad \Rightarrow \quad P_n^{(n)}(0) = n!\; a_n \quad \Rightarrow \quad a_n = \dfrac{P_n^{(n)}(0)}{n!}$

$\Rightarrow P_n(x) = P_n(0) + P'_n(0)x + \dfrac{P''_n(0)}{2!}x^2 + \dfrac{P'''_n(0)}{3!}x^3 + \ldots \dfrac{P_n^{(n)}(0)}{n!}x^n$

Verbalise: The Maclaurin polynomial at x is the polynomial at 0 + 1st derivative at 0 times x + 2nd derivative at 0 times x squared over 2! + ….

17.3   The Taylor polynomials

In the same way we can prove that any polynomial can be re-written as

the Taylor polynomial: $P_n(x) = P_n(a) + P'_n(a)(x-a) + \ldots \dfrac{P_n^{(n)}(a)}{n!}(x-a)^n$

Verbalise: The Taylor polynomial at x is the polynomial at a + derivative at a times the signed distance from x to a + ….

Note: in the Maclaurin series x can be viewed as the signed distance from x to 0.

17.4   The Taylor series

Many differentiable functions $f(x)$ can be represented using the infinite

> **Taylor series:** $f(x) = f(a) + f'(a)(x-a) + \dfrac{f''(a)}{2!}(x-2)^2 + \ldots$

Verbalise: The function value at x is the function value at a + derivative at a times the distance to a + ….

These functions can be **approximated** as polynomials using **truncated Taylor series** (that is, Taylor polynomials).

Optional

Consider the approximation error

$$e_n(a, x) \equiv f(x) - P_n(a, x)$$

If the error $e_n(x) \to 0$ as $n \to \infty$, then $f(x)$ equals its Taylor series at x and $y(x) \approx P_n(a, x)$ is an approximation based on the $n-th$ order Taylor polynomial.

For e^x, $\sin x$, $\cos x$, $e_n(x-a) \to 0$ faster than $(x-a)^n$.

Proving the above results and checking the limiting behaviour of the error e_n is rather involved.

17.5   The Maclaurin series

If $a = 0$, the Taylor series is called the **Maclaurin series.**

Examples:

1. Find the Maclaurin series for e^x.

Solution

Step 1. The general Maclaurin series is

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \ldots$$

Step 2. Function and its derivatives are	**Step 3.** Values of the function and its derivative at 0 are
$f(x) = e^x$	$f(0) = 1$
$f'(x) = e^x$	$f'(0) = 1$
$f''(x) = e^x$	$f''(0) = 1$
…	…
$f^{(n)}(x) = e^x$	$f^{(n)}(0) = 1$

Step 4. Substitute the above values into the general the Maclarin series:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!} \ldots$$

$$\Rightarrow e^x \approx 1 + x, |x| < 1 \text{ (in the vicinity of 0 the graph of } e^x \text{ is almost a straight line)}$$

---

2. Find the Maclaurin series for sin x.

Solution
Step 1. $f(x) = f(0) + f'(0)x + \dfrac{f''(0)}{2!}x^2 + ...$

Step 2. Function and its derivatives are	**Step 3.** Values of the function and its derivative at 0 are
$f(x) = \cos x$	$f(0) = 1$
$f'(x) = \sin x$	$f'(0) = 1$
$f''(x) = \cos x$	$f''(0) = -1$
$f^{(IV)}(x) = \cos x$	$f^{(IV)}(0) = 0$
...	...

Step 4. Substitute the above values into the general Maclarin series:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + ...$$

$\Rightarrow \sin x \approx x, |x| < 1$ (in the vicinity of 0 the graph of sin x is almost a straight line)

3. Find the Maclaurin series for cos x.

Solution
Step 1. $f(x) = f(0) + f'(0)x + \dfrac{f''(0)}{2!}x^2 + ...$

Step 2. Function and its derivatives are	**Step 3.** Values of the function and its derivative at 0 are
$f(x) = \cos x$	$f(0) = 1$
$f'(x) = \sin x$	$f'(0) = 1$
$f''(x) = \cos x$	$f''(0) = -1$
...	...

Step 4.
$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + ... -$$

$\Rightarrow \cos x \approx 1 - \dfrac{x^2}{2}, |x| < 1$ (in the vicinity of 0 the graph of cos x looks like a parabola – reflected **wrt** (with respect to) the horizontal axis shifted up by 1).

Thus, while any polynomial can be **re-written** as a Taylor or MacLaurin polynomial, many other n time differentiable functions can be **approximated** by Taylor or MacLaurin polynomials $P_n(x)$. The discussion on whether such approximation is possible or how to decide the optimal degree of the approximating polynomial lies outside the scope of these notes.

17.6   L'Hospital's rule

Let us find $\lim\limits_{x \to a} \dfrac{f(x)}{g(x)}$ when $\lim\limits_{x \to a} f(x) = \lim\limits_{x \to a} g(x) = 0$, so that we have the 0/0 indeterminate form.

If both $f(x)$ and $g(x)$ have Taylor expansion in the vicinity of a we can write

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(a)(x-a) + \ldots}{g'(a)(x-a) + \ldots} = \lim_{x \to a} \frac{f'(x)}{g'(x)} \,.$$

with the factor $:(x - a)$

The recipe

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)} \quad \text{if} \quad \lim_{x \to a} f(x) = \lim_{x \to a} g(x) = 0$$

is called **L'Hospital's** rule.

If still get the 0/0 indeterminate form – continue applying the rule until arrive at a determinate answer (find the second derivatives of the numerator and denominator, their third derivatives *etc.*)

L'Hospital's rule can be extended to cover functions with just one derivative, $a = +\infty$ or $-\infty$ and ∞/∞ indeterminate forms but these proofs lie outside the scope of these notes.

Examples:

1. Find $\lim\limits_{x \to 0} \dfrac{\sin x}{x}$.

Solution: $\lim\limits_{x \to 0} x = \lim\limits_{x \to 0} \sin x = 0$. Applying L'Hospital's rule, $\lim\limits_{x \to 0} \dfrac{\sin x}{x} = \lim\limits_{x \to 0} \dfrac{\cos x}{1} = 1$

2. Find $\lim\limits_{x \to 0} \dfrac{1 - e^x}{x}$.

Solution: $\lim\limits_{x \to 0} x = \lim\limits_{x \to 0} 1 - e^x = 0$. Applying L'Hospital's rule, $\lim\limits_{x \to 0} \dfrac{1 - e^x}{x} = \lim\limits_{x \to 0} \dfrac{-e^x}{1} = -1$

Examples:

1. Find $\lim\limits_{x\to 0} \dfrac{\sin x}{x}$.

Solution: $\lim\limits_{x\to 0} x = \lim\limits_{x\to 0} \sin x = 0$. Applying L'Hospital's rule, $\lim\limits_{x\to 0} \dfrac{\sin x}{x} = \lim\limits_{x\to 0} \dfrac{\cos x}{1} = 1$

2. Find $\lim\limits_{x\to 0} \dfrac{1-e^x}{x}$.

Solution: $\lim\limits_{x\to 0} x = \lim\limits_{x\to 0} 1-e^x = 0$. Applying L'Hospital's rule, $\lim\limits_{x\to 0} \dfrac{1-e^x}{x} = \lim\limits_{x\to 0} \dfrac{-e^x}{1} = -1$

17.7   Applications

The above series are used by calculators to give approximate values of elementary functions, sin, cos, exponent *etc.*

17.8   Instructions for self-study

- **Revise ALGEBRA Summary (smile rule)**
- **Revise Lecture 5 (polynomials)**
- **Revise FUNCTIONS Summary**
- **Revise Lectures 13 – 14 (limits)**
- **Revise Lectures 14 – 15 (differentiation) and Solutions to Exercises in Lecture 14 using the STUDY SKILLS Appendix**
- **Revise Lecture 16 using the STUDY SKILLS Appendix**
- **Study Lecture 17 using the STUDY SKILLS Appendix**
- **Do the following exercises:**

Q1.

a) Find the first three non-zero terms in the Maclaurin series for $y = \dfrac{1}{1+x}$

b) Use the first two terms of the above Maclaurin series to approximate $y = \dfrac{1}{1+x}$ in the vicinity of $x = 0$

c) Find, without using a calculator, the approximate value of $\dfrac{4}{5}$

d) Find $\displaystyle\lim_{x \to -1} [\dfrac{1}{1+x}(1 - e^{1+x})]$

Q2.

a) Find the first three non-zero terms in the Maclaurin series for $y = \dfrac{1}{1-x}$

b) Use the first two terms of the above Maclaurin series to approximate $y = \dfrac{1}{1-x}$ in the vicinity of $x = 0$

c) Find, without using a calculator, the approximate value of $\dfrac{4}{3}$

d) Find $\displaystyle\lim_{x \to 1} [\dfrac{1}{1-x}\sin(1 - x)]$

Q3.

    a) Find the first three non-zero terms in the Maclaurin series for $y = \sqrt{1+x}$

    b) Use the first two terms of the above Maclaurin series to approximate $y = \sqrt{1+x}$ in the vicinity of $x = 0$

    c) Find, without using a calculator, the approximate value of $\sqrt{\dfrac{4}{3}}$

    d) Find $\displaystyle\lim_{x\to-1} \dfrac{\sqrt{1+x}}{1-x^2}$

Q4.

    a) Find the first three non-zero terms in the Maclaurin series for $y = \sqrt{1-x}$

    b) Use the first two terms of the above Maclaurin series to approximate $y = \sqrt{1-x}$ in the vicinity of $x = 0$

    c) Find, without using a calculator, the approximate value of $\sqrt{\dfrac{4}{5}}$

    d) Find $\displaystyle\lim_{x\to1} \dfrac{\sqrt{1-x}}{1-x^2}$

Lecture 18 INTEGRAL CALCULUS: Integration of Real Functions of One Real Variable (Definite Integrals)

Integration (over a finite interval) of a function is a new advanced operation on functions. If it exists, the outcome is a number (or a constant with respect to the control variable) called **a definite integral**.

18.1   A definite integral

Consider the area A between a curve, which is the graph of a function $f(x)$, the horizontal axis and lines $x = a$ and $x = b$:
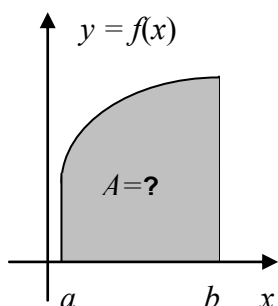
Area A between
the curve and horizontal axis
approximately equals
the sum of areas of inscribed
thin rectangles
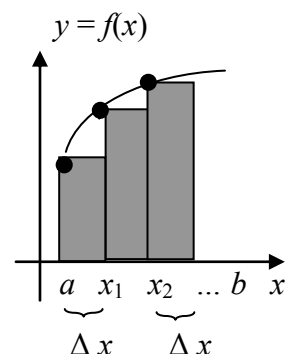
Question: What is the area of first rectangle, second rectangle *etc.*?

Answer:

In other words, this area A approximately equals,

$$A \approx f(x_0)\Delta x + f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + .. + f(x_{n-1})\Delta x = \sum_{i=0}^{n-1} f(x_i)$$

where $x_0 = a$ and $x_n = b$.

Question: How many rectangles do we have here?

Answer:

Taking the limit $\Delta x \to 0$, so that the width of the rectangles gets smaller and smaller, the number of rectangles grows larger and larger ($n \to \infty$) and we can write that this area A exactly equals

if the limit exists

$$A = \lim_{\Delta x \to 0} \sum_{i=0}^{n-1} f(x_i)\Delta x \equiv \int_a^b f(x)dx$$

Thus, we introduced the definition of the **definite integral** of a function *f(x)* over the interval [*a, b*],

The **geometrical interpretation of the definite integral** of a real function of one real variable is a "(**signed**) area between a curve (the graph of this function) and horizontal axis". The area is **signed,** because on the interval where $f(x)$ is negative, all products $f(x_i)\,\Delta x$ are negative. Thus, we introduced a definition of the **definite integral** of the function over the interval [a, b], is negative and equal the area between the curve and horizontal axis with the – sign.

Example: $\displaystyle\int_{0}^{2\pi}\sin x\,dx = ?$ **Answer:** (sketch the graph to see that).

18.2 Notation

$\int$ - integral (an elongated letter S, to remind us that we are talking about the limit of a sum Σ)

$\displaystyle\int_{a}^{b}...dx$ – a definite integral between a and b (to remind us that we are talking about the limit of a sum of areas of thin rectangles of width Δx fitted between the curve, horizontal axis and straight lines $x = a$ and $x = b$).

a, b are called **the lower and upper limits of integration**, respectively.

18.3 Discussion of definitions of a definite Integral and derivative
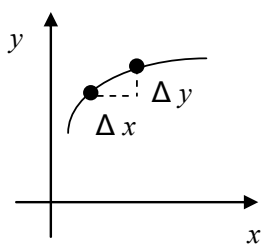
The derivative
approximatlely equals
the slope of local inscribed segment
$$\frac{dy}{dx} \approx \frac{\Delta y}{\Delta x}$$
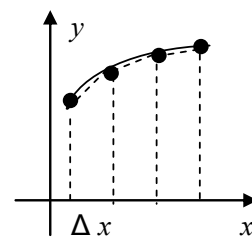
When defining integrals and derivatives we reduce a difficult problem involving a curve to many simple problems involving small straight segments. Ability to reduce complicated unfamiliar pictures to simple familiar patterns is an invaluable engineering skill.

18.4   Examples of integrable functions

1. Any $f(x)$ continuous in $[a, b]$ is **integrable**, that is, can be integrated.

Question: Why?
Answer:
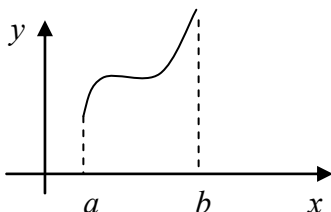
2. Any bounded $f(x)$ with finite number of jumps can be integrated
Question: Why?
Answer:
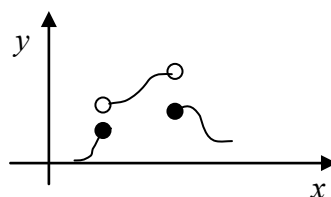
Note: If change $f(x)$ at a finite number of points its **integrability** does not change.
Question: Why?
Answer:

Revision

Let C be a condition.
Let E be an event.

If $C \Rightarrow E$, then the condition C is **sufficient** for event E to take place.
If $E \Rightarrow C$, then the condition C is **necessary** for event E to take place.

Question: Is continuity sufficient for integrability, necessary or both?
Answer:

Note: Continuity is not sufficient for differentiability. Continuity is necessary for differentiability.

18.5   The Mean Value Theorem

If $f(x)$ is continuous, then there exists a value μ, such that $m \leq \mu \leq M$ and

$$\int_a^b f(x)dx = \mu(b-a),$$

where m is the smallest value of $f(x)$ on the interval $[a, b]$ and M is the largest value of $f(x)$ on the interval $[a, b]$. μ is the Greek letter *mu*.

The area under the curve (between a and b)
equals
$\mu(b-a)$, the area of a rectangle (in red, long dashes)
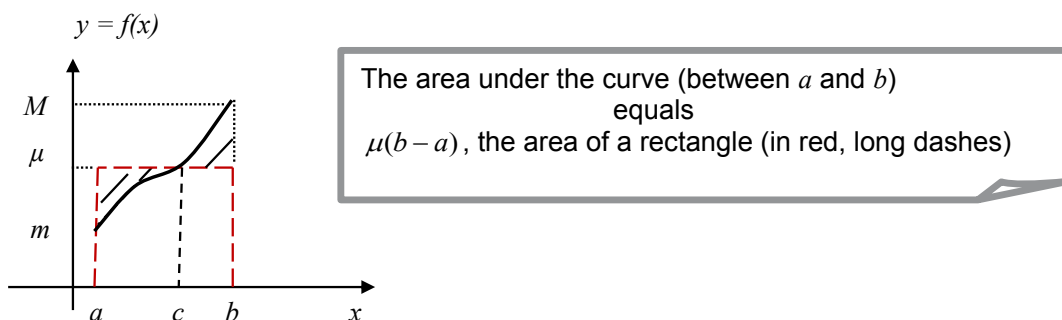
$\mu = \dfrac{1}{b-a}\displaystyle\int_a^b f(x)dx$ is called the **mean value** of $f(x)$ on the interval $[a, b]$.

Why is it called that?

1. The mean value of the sequence $\{1, 3\}$ is $\dfrac{1+3}{2}$

2. The mean value of the sequence $\{x_0, \ldots, x_{n-1}\}$ is $\dfrac{x_0 + x_1 \ldots + x_{n-1}}{n}$

3. The mean value of the sequence $\{f(x_0), \ldots, f(x_{n-1})\}$ is $\dfrac{1}{n}\displaystyle\sum_{i=0}^{n-1} f(x_i)$

Question: With reference to Section 18.1, given a and b and Δx, what is n, the number of rectangles of width Δx used to approximate the area between $f(x)$, the horizontal axis and lines $x = a$ and $x = b$?
Answer:

Question: What is $\displaystyle\lim_{\Delta x \to 0} n\Delta x$?
Answer:

This means that we can write

$$\frac{1}{n}\sum_{i=0}^{n-1}f(x_i) = \frac{1}{n\Delta x}\sum_{i=0}^{n-1}f(x_i)\Delta x \quad \underset{\Delta x \to 0}{\to} \quad \frac{1}{b-a}\int_a^b f(x)dx$$

This is why $\mu = \dfrac{1}{b-a}\displaystyle\int_a^b f(x)dx$ is called the mean value of $f(x)$ on the interval $[a, b]$.

18.6   A definite integral with a variable upper limit – function Φ(*x*)

Let *f*(*x*) be integrable in [*a, b*]. We can introduce a new advanced operation on a function – evaluating a definite integral with a variable upper limit. The result is

$$\Phi(x) = \int_a^x f(t)dt,$$

a **function of** *x*, since if *a* is fixed for every *x* we have one and only one value for the signed area between the curve and horizontal axis.

Question: Does this area depend on *t*?
Answer:
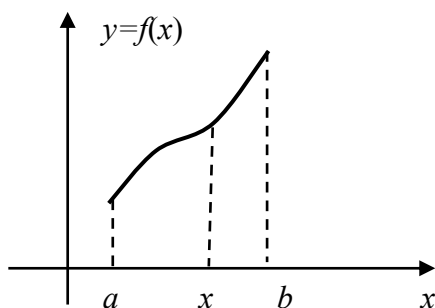
18.7   The Fundamental Theorem of Calculus

The Fundamental Theorem of Calculus: If *f*(*x*) continuous, then $\dfrac{d}{dx}\int_a^x f(t)dt = f(x)$.

Attempt the proof yourself (see Q2.)

18.8   Applications of integration

Later you will find that apart from being useful in finding areas and volumes, integration is necessary to solve differential equations that describe all scientific phenomena and engineering systems that you are going to study.

18.9   A historical note

According to Knowledgewiki, "integration can be traced as far back as ancient Egypt, *circa* 1800 BC, with the Moscow Mathematical Papyrus demonstrating knowledge of a formula for the volume of a pyramidal frustum. The first documented systematic technique capable of determining integrals is the method of exhaustion of Eudoxus (circa 370 BC), which sought to find areas and volumes by breaking them up into an infinite number of shapes for which the area or volume was known. This method was further developed and employed by Archimedes and used to calculate areas for parabolas and an approximation to the area of a circle. Similar methods were independently developed in China around the 3rd Century AD by Liu Hui, who used it to find the area of the circle. This method was later used in the 5th century by Zu Chongzhi to find the volume of a sphere. That same century, the Indian mathematician Aryabhata used a similar method in order to find the volume of a cube.

The next major step in integral calculus came in the 11th century, when the Iraqi mathematician, Ibn al-Haytham (known as *Alhazen* in Europe), devised what is now known as "Alhazen's problem", which leads to an equation of the fourth degree, in his *Book of Optics*. While solving this problem, he performed an integration in order to find the volume of a paraboloid. Using mathematical induction, he was able to generalize his result for the integrals of polynomials up to the fourth degree. He thus came close to finding a general formula for the integrals of polynomials, but he was not concerned with any polynomials higher than the fourth degree. Some ideas of integral calculus are also found in the *Siddhanta Shiromani*, a 13th century astronomy text by Indian mathematician Bhāskara II.

The next significant advances in integral calculus did not begin to appear until the 16th century. At this time the work of Cavalieri with his *method of indivisibles*, and work by Fermat, began to lay the foundations of modern calculus. Further steps were made in the early 17th century by Barrow and Torricelli, who provided the first hints of a connection between integration and differentiation.

The major advance in integration came in the 17th century with the independent discovery of the fundamental theorem of calculus by Newton and Leibniz. The theorem demonstrates a connection between integration and differentiation. This connection, combined with the comparative ease of differentiation, can be exploited to calculate integrals. In particular, the fundamental theorem of calculus allows one to solve a much broader class of problems. Equal in importance is the comprehensive mathematical framework that both Newton and Leibniz developed. Given the name infinitesimal calculus, it allowed for precise analysis of functions within continuous domains. This framework eventually became modern Calculus, whose notation for integrals is drawn directly from the work of Leibniz."

http://www.knowledgewiki.org/article/
Integral?enk=B7FmqUapZqFmGSahZJFkkQaRhoFkiWahBsFGgQapBxkm#History

18.10   Instructions for self-study

- **Revise Lecture 16 and study Solutions to Exercise in Lecture 16 using the STUDY SKILLS Appendix**
- **Revise Lecture 17 using the STUDY SKILLS Appendix**
- **Study Lecture 18 using the STUDY SKILLS Appendix**
- **Attempt the following exercises:**

Q1. Find using the first principles, that is using the definition of definite integral,

a) $\displaystyle\int_0^1 0.2\,dp$

b) $\displaystyle\int_0^1 5\,dq$

c) $\displaystyle\int_0^1 0.5\,du$

d) $\displaystyle\int_0^1 s\,ds$

e) $\displaystyle\int_0^1 3u\,du$

f) $\displaystyle\int_0^1 4v\,dv$

Q2. **(advanced)**. Prove that if $f(x)$ is continuous, then $\dfrac{d}{dx}\displaystyle\int_a^x f(t)\,dt = f(x)$ (using the first principles, that is using the definition of definite integral, and Mean Value Theorem). Describe this formula in words.

Lecture 19 INTEGRAL CALCULUS: Integration of Real Functions of One Real Variable (Indefinite Integrals)

19.1   The relationship between differentiation and integration

According to the Fundamental Theorem of Calculus if $f(x)$ is continuous we have

$$\frac{d}{dx}\int\limits_{a}^{x} f(t)dt = f(x) . \qquad (19.1)$$

Question: What is the order of operations on $f(t)$ in the left hand side above?

Answer:

Thus, differentiation of a continuous function undoes what integration does, that is, the **differentiation is inverse to integration.**

Let us now subdivide the $[a, x]$ interval into small subintervals with $t_0 = a$ and $t_n = x$:

$$\int\limits_{a}^{x}\frac{d}{dt}f(t)dt = \lim_{\Delta t \to 0}\sum_{i=0}^{n-1}\frac{\Delta f(t_i)}{\Delta t}\Delta t = \lim_{\Delta t \to 0}[\Delta f(t_0) + \Delta f(t_1) + ... + \Delta f(t_{n-1})] = \qquad (19.2)$$

$$\lim_{\Delta t \to 0}\{[f(t_1) - f(t_0)] + [f(t_2) - f(t_1)] + ... + [f(t_n) - f(t_{n-1})]\} = f(t_n) - f(t_0) \equiv f(x) + c$$

Question: What is the order of operations on $f(t)$ in the left land side above?

Answer:

Thus, **integration is not a perfect inverse to differentiation**, the definite integral with the variable upper limit depends on the lower limit too.

Another way to see this is as follows: Examining the graph below
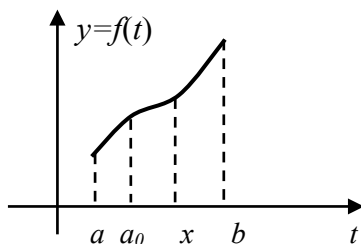
and remembering that the integral can be interpreted as a signed area, we can write

$$\int_a^x f(t)dt = \int_a^{a_0} f(t)dt + \int_{a_0}^x f(t)dt = c + \int_{a_0}^x f(t)dt = \int_{a_0}^x f(t)dt + c. \qquad (19.3)$$

Since $c = \int_a^{a_0} f(t)dt$ is **constant with respect to** x but depends on a_0, Eq. (19.1) implies

$$\frac{d}{dx}(\Phi(x,a_0)+c) = f(x),$$

where instead of $\Phi(x)$ (the notation used in the last Lecture), we introduce a more logical notation

$\Phi(x,a) = \int_a^x f(t)dt$ (a definite integral with a variable upper limit x actually depends on the lower limit

a too).

This means that each function $f(x)$ is associated with many definite integrals $\Phi(x,a)$. For this reason, the integral $\Phi(x,a)$ is called an **antiderivative** of $f(x)$, rather than the *inverse derivative* of $f(x)$ (remember, inverses being functions are supposed to be unique!)

19.2   An indefinite integral

We have shown that for every continuous function there are infinitely many antiderivatives, differing from one another by a constant term. Often instead of the antiderivative we use a somewhat confusing shorthand

the **indefinite integral** $F(x) \equiv \int f(x)dx \equiv \int\limits_{a}^{x} f(t)dt$ (19.4)

Using (19.3)

$$\int\limits_{a}^{x} f(t)td = \int\limits_{a_0}^{x} f(t)td + c$$

Substituting the above decomposition into definition (19.4) we get

$$\int f(x)dx \equiv \int\limits_{a_0}^{x} f(t)dt + c$$ (19.5)

This is a reiteration of the fact that the antiderivative is **defined up to a constant term**, because we do not know (and do not need to know) what is its lower limit of integration.

We can now introduce a very important formula for evaluating definite integrals:

$$\int_a^b f(x)dx = F(b) - F(a) \tag{19.6}$$

This formula implies that

1. $\displaystyle\int_b^a f(x)dx = -\int_a^b f(x)dx$ **(Why?)**

2. if we know an indefinite integral we can **evaluate** (find the value of) any associated definite integral (note that **the c-terms** in $F(b)$ and $F(a)$ in Eq. (19.6) **cancel out**).

Finally, note that using the indefinite integral notation the two Fundamental Theorems of Calculus (19.1) and (19.2) can be re-written as

$$\frac{d}{dx}\int f(x)dx = f(x) \qquad \int \frac{df(x)}{dx}dx = f(x) + c \tag{19.7}$$

Terminology

$f(x)$ – integrand, x – integration variable, $\int ...dx$ is an indefinite integral with respect to x. It has limits but these limits are allowed to vary, that is, are indefinite.

19.3   Finding an indefinite integral

To find an indefinite integral we use the **Integration Table** of elementary functions, **Rules** (or **Methods**) for integrating combinations of functions and **Decision Tree** to decide which rule or Table entry to use.

19.3.1   The Integration Table

We can derive Integration Table for elementary functions using Differentiation Table.

Differentiation Table

$g(x)$	$\dfrac{dg(x)}{dx}$
constant	0
e^x	e^x
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
x^k	kx^{k-1}
$\ln x,\ x>0$	$\dfrac{1}{x}, x \neq 0$

$\Rightarrow$

Integration Table

integrand   **integration variable**

$f(x)$	$F(x) = \int f(x)dx$
0	c (constant)
e^x	$e^x + c$
$\cos x$	$\sin x + c$
$\sin x$	$-\cos x + c$
$x^n,\ n \neq -1$	$\dfrac{x^{n+1}}{n+1} + c$
$\dfrac{1}{x}, x \neq 0$	$\ln\left|x\right| + c$

Optional

Let us check that $\int \dfrac{1}{x}dx = \ln\left|x\right|$, for any $x \neq 0$.

Indeed,

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x \geq 0 \end{cases}$$

$$\Rightarrow \frac{d}{dx}\ln|x| = \begin{cases} \dfrac{1}{x}, & x > 0 \\ \\ -(-\dfrac{1}{x}) & x < 0 \end{cases}$$ where we used the chain rule of differentiation.

Examples:

1. $\int e^{x^2} dx^2 = e^{x^2} + c$

2. $\int \sin e^w de^w = -\cos e^w + c$

3. $\int dx = \int 1dx = \int x^0 dx = x + c \quad \Rightarrow \quad \int dx = \int \frac{dx(t)}{dt} dt = x + c$

19.3.2   Elementary integration rules

1. $\int \alpha \, f(x)dx = \alpha \int f(x)dx$ **constant factor out**

2. $\int [f(x) + g(x)]dx = \int f(x)dx + \int g(x)dx$ **sum rule**

linearity of integration

These rules can be proven using the definition of integral and rules for limits.

Examples:

1. Integrate $\int 2\sin x\, dx$

Solution
Integration variable? x
Integrand? $2\sin x$
It is not a table function of x
The last operation in the integrand? Multiplicaton
Is one of the factors a constant? Yes, 2
What rule to use? The "constant factor out" rule
Constant factor $- \alpha = 2$, variable factor $f(x) = \sin x$
Substituting α and $f(x)$ into the "constant factor out" rule,

$$\int 2\sin x\, dx = 2\int \sin x\, dx = -2\cos x + c$$

2. Integrate $\int (\sin x + \cos x) dx$

Solution
Integration variable? x
Integrand? $\sin x + \cos x$
It is not a table function of x
The last operation in the integrand? Addition
What rule to use? The sum rule
First term $f(x) = \sin x$, second term $g(x) = \cos x$
Substituting $f(x)$ and $g(x)$ into the sum rule,

$$\int (\sin x + \cos x) dx = \int \sin x\, dx + \int \cos x\, dx = -\cos x + c_1 + \sin x + c_2 = \sin x - \cos x + c$$

19.3.3 Integration Decision Tree

Similarly to the Decision Tree for Limits and Decision Tree for Differentiation, the Decision Tree for Integration (figure 19.1) allows you to decide at each step which Integration Rule or Integration Table entry to use when finding an indefinite integral. The decision depends on the last operation in the integrand. Do not forget to start at the top of the Tree and then follow the arrows that are associated with the correct answers (if any). After you have gone through the Decision Tree once check whether the operation of integration has been carried out. If not, hopefully, now you have simpler functions to integrate. In order to do that go over the Decision Tree again, substituting new integrands for $f(x)$ (and your integration variable for x).
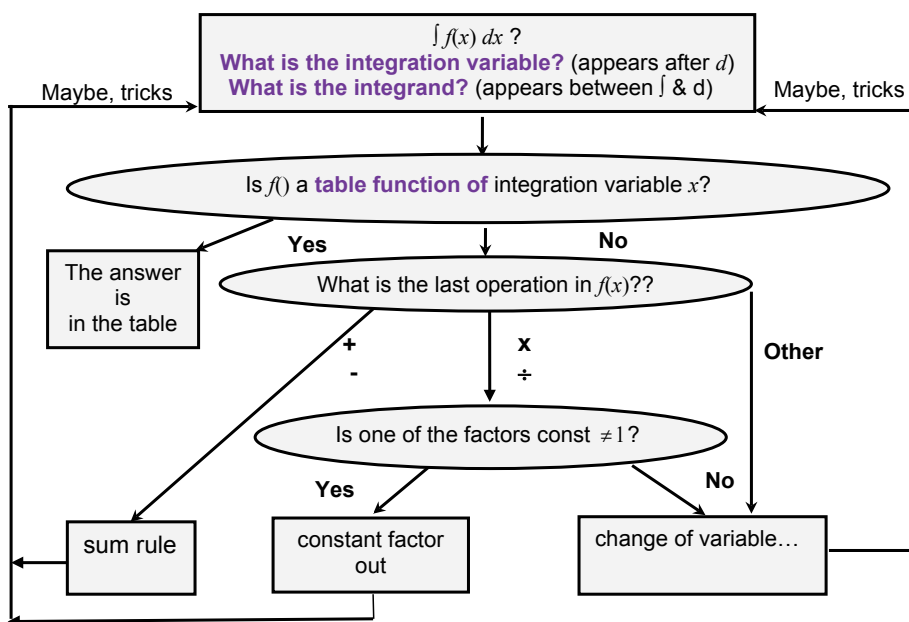
Figure 19.1. Decision Tree for Integration (incomplete).

19.3.4 Integration method (rule) of change of variable (substitution)

There are many other integration rules apart from the elementary ones considered in Section 19.3.1. We will study them later. They all follow from the rules for differentiation but are more complicated. First we introduce the simplest of them. It is called **the change of variable method**.

Example:

1. $\int (3x+1)^{2.7}\, dx = \int u^{2.7}\, \dfrac{du}{3} = \dfrac{1}{3}\dfrac{u^{3.7}}{3.7} + c = \dfrac{1}{11.1}(3x+1)^{3.7} + c$

Integration variable? x
Integrand? $(3x+1)^{2.7}$
The integrand is not a table function of x.
The last operation in the integrand? Power (hence the elementary rules do not apply).
Introduce a new variable that lumps together the first few operations in the integrand

$u = 3x + 1 \ \Big/ \dfrac{\mathrm{d}}{\mathrm{d}x}, \quad \dfrac{du}{dx} = 3 \ \Big/ dx, \ \div 3, \quad dx = \dfrac{du}{3}$

Note: $\dfrac{du}{dx}$ is not an ordinary fraction but a limit of a fraction. Since the limit of a fraction is a fraction of limits it can be proven that **we can formally "multiply"** $\dfrac{du}{dx}$ **by** dx **to get** du.

Question: What is du equal to?
Answer:
Question: What is dx equal to?
Answer:

2. $\displaystyle\int_{2}^{3} t\sin t^2\,dt = -\frac{1}{2}\cos t^2\Big|_{2}^{3} = -\frac{1}{2}(\cos 9 - \cos 4) \approx 0.129$

$$\int t\sin t^2\,dt = \frac{1}{2}\int \sin u\,du = -\frac{1}{2}\cos u + c = -\frac{1}{2}\cos t^2 + c$$

$$u = t^2,\ \frac{du}{dt} = 2t\Big/dt,\ \ tdt = \frac{du}{2}$$

3. $\displaystyle\int_{1}^{2} \sin t\cos^2 t\,dt = -\frac{1}{3}(\cos^3 2 - \cos^3 1) \approx 0.0766$

$$\int \sin t\cos^2 t\,dt = -\int u^2\,du = -\frac{u^3}{3} + c = -\frac{\cos^3 t}{3} + c$$

$$u = \cos t,\ du = -\sin t\,dt$$

4. $\displaystyle\int \frac{e^{\tan x}}{\cos^2 x}\,dx = \int e^u\,du = e^u + c = e^{\tan x} + c$

$$u = \tan x, \quad \frac{du}{dx} = \frac{1}{\cos^2 x}, \quad du = \frac{1}{\cos^2 x}\,dx\,,$$

5. $\displaystyle\int \frac{3x^2 + 1}{x^3 + x + 2}\,dx = \int \frac{du}{u} = \ln\left|x^3 + x + 2\right| + c$

$$u = x^3 + x + 2,\ du = (3x^2 + 1)\,dx$$

6. $\displaystyle\int \frac{4}{5x - 7}\,dx = 4\int \frac{dx}{5x - 7} = \frac{4}{5}\int \frac{du}{u} = \frac{4}{5}\ln\left|5x - 7\right| + c$

$$u = 5x - 7,\ du = 5\,dx$$

7. $\displaystyle\int \frac{t}{t^2 + 1}\,dt = \frac{1}{2}\ln\left(t^2 + 1\right) + c$

8. $\displaystyle\int \frac{e^{\frac{t}{2}}}{e^{\frac{t}{2}} + 1}\,dt = 2\ln\left(e^{\frac{t}{2}} + 1\right) + c$

19.4   Instructions for self-study

- **Revise ALGEBRA Summary (particularly, the words term, sum, factor, product)**
- **Revise Lecture 4 and Solutions to Solutions to Exercises in Lecture 4 (particularly, the operation of composition)**
- **Revise Summaries on the ORDER OF OPERATIONS and FUNCTIONS**
- **Revise Lecture 17 and study Solutions to Exercises in Lecture 17 using the STUDY SKILLS Appendix**
- **Revise Lecture 18 using the STUDY SKILLS Appendix**
- **Study Lecture 19 using the STUDY SKILLS Appendix**
- **Attempt the following exercises:**

Q1. Find

a) $\int \dfrac{1}{u} du$

b) $\int \dfrac{1}{u} dx$

c) $\int \dfrac{1}{u+t} d(u+t)$

d) $\int e^{\sin t} d \sin t$

e) $\int e^{n \sin t} de^{\sin t}$

Q2. Find

a) $\int (\dfrac{1}{u} + \cos u) du$

b) $\int (u + u^2) du$

c) $\int (u + 2u^2 + u^3) du$

d) $\int (2v + 3v^2 + 5v^5) du$

Q3. Find

a) $\int \cos t e^{\sin t} dt$

b) $\int \dfrac{4}{3-t} dt$

c) $\int_{-1}^{1} \dfrac{2 + 6v}{2v + 3v^2 + 5u^5} dv$

Additional Exercises

Q4. Use the change of variable (substitution) to find

    a) $\int (4x+1)^7 dx$

    b) $\int t^2 \sin(t^3 +1)dt.$ Hint: let $u = t^3 +1$

    c) $\int \sin(3x-1)dx$

    d) $\int e^{2x-3}dx$

    e) $\int x(2x^2 +7)^4 dx$

    f) $\int \sin^2 4t \cos 4t dt$

Q5 Evaluate

    a) $\int\limits_3^4 \dfrac{t}{\sqrt{t^2 -3}} dt$

    b) $\int\limits_0^2 \dfrac{\ln z}{z} dz.$ Hint: let $u = \ln z$

Q6. Find

    a) $\int \dfrac{3x^2 +2}{x^3 +2x+5} dx$

    b) $\int\limits_2^3 t^2 \sin t^3 dt$

    c) $\int \dfrac{1}{u-1} du$

    d) $\int \dfrac{1}{1-u} du$

Lecture 20 INTEGRAL CALCULUS: Advanced Integration Methods

In Lecture 19 we have introduced the table of indefinite integrals of elementary functions and two simple linearity rules: one for integrating a sum of two functions and one for integrating a product with one of the factors constant. We now introduce additional rules (or methods) for integrating products and quotients of elementary functions.

20.1   Integration of products of trigonometric functions

Examples:

1. $\int \cos^2 t \ dt =$

> **Question:** Is $\cos^2 t$ a product?
> **Answer:** No, but it can easily be re-written as a product, $\cos^2 t = \cos t \times \cos t$.
> The square of a cosine is simply related to the cosine of a double angle
> $$\cos 2t = \cos^2 t - \sin^2 t ,$$
> see the **Trigonometry Summary**.
> We can eliminate $\sin^2 t$ above using **Pythagoras' Theorem** in terms of angles.
>
> **Question:** What is **Pythagoras' Theorem** in terms of angles?
> **Answer:** $\sin^2 t + \cos^2 t = 1$
> $$\Rightarrow \sin^2 t = 1 - \cos^2 t$$
> Substituting $\sin^2 t$ into the **double angle identity** above gives us
> $$\cos 2t = \cos^2 t - \sin^2 t = \cos^2 t - (1 - \cos^2 t) = \cos^2 t - 1 + \cos^2 t = 2\cos^2 t - 1.$$
>
> We can now use the **Decision Tree for Solving Simple Equations** to express $\cos t$ in terms of $\cos 2t$, a simpler function of t,
> $$\cos^2 t = \frac{1 + \cos 2t}{2} .$$
> The latter expression involves simple operations (on $\cos 2t$) of addition and division by a constant, so we can integrate using the **linearity rules.**

$$\int \frac{1 + \cos 2t}{2} dt = \frac{1}{2} \int (1 + \cos 2t) dt = \frac{1}{2} (\int dt + \int \cos 2t dt) = \frac{1}{2} (t + \int \cos 2t dt) =$$

The last integral is not in the **Integration Table** for elementary functions in Lecture 19, but it can be found in Integration Tables in other textbooks or on the internet. The integral can also be evaluated using the Integration Table for elementary functions and **change of variable method** presented in Lecture 17, with $u = 2t$. Whatever the method is used,

$$\int \cos 2t = \frac{\sin 2t}{2} + c.$$

Question: What is a derivative of $\cos 2t$?
Answer:
Verbalise: Thus, when differentiating $\cos$, a constant factor in the argument becomes a factor in the result and when integrating $\cos$, it appears in the denominator of the result.

$$\frac{1}{2}(t + \frac{\sin 2t}{2}) + c = \frac{1}{2}t + \frac{\sin 2t}{4} + c$$

using Pythagoras' Theorem

2. $\int \sin^2 t\, dt = \int (1 - \cos^2 t)\, dt = \int 1 dt - \int \cos^2 t\, dt = \frac{t}{2} - \frac{\sin 2t}{4} + c,$ where we used the result

obtained in Example 1.

3. $\int \sin mt \sin nt\, dt = \frac{1}{2}\int [\cos(m-n)t - \cos(m+n)t]dt = \frac{1}{2}\left[\frac{\sin(m-n)t}{m-n} - \frac{\sin(m+n)t}{m+n} \right] + c,$

where we used a trig identity for a product of sinuses, which can be obtained by combining identities for cosine of a difference, $\cos(A - B)$, and cosine of a sum, $\cos (A + B)$.

20.2   Integration by parts (integration of products of different types of functions)

The product rule for differentiation gives us a product rule for integration (traditionally called **integration by parts**)

$$\frac{d(u \cdot v)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx} \Big/ dx$$

Question: What is dx?
Answer:

Multiplication by dx is purely formal, its meaning can be clarified by using properties of limits

$$\Rightarrow d\,(u\,v) = u\,dv + v\,du$$

$$\Rightarrow u\,dv = d\,(u\,v) - v\,du \Big/ \int \quad \Rightarrow$$

Integration by Parts formula: $\int u\,dv = uv - \int v\,du$

Examples:

1. $\int x \cos x\,dx = x \sin x - \int \sin x\,dx = x \sin x + \cos x + c$

u - line: $u = x,\; \dfrac{du}{dx} = 1,\; du = dx$
v – line: $dv = \cos dx \Big/ \int,\; v = \int \cos x\,dx = \sin x$

Note: can check that putting $+c$ here would not change the final result. Substitute $u,\,du,\,v$ and dv into the **Integration by Parts formula**.

Note 1: we reduced our task to integrating $\cos x$ – that is, to integrating **a part of** (the old term for a factor in) the original integrand. But note that $\int x \cos x\,dx \neq \int x\,dx \cdot \int \cos x\,dx$ (that is, **an integral of a product is not a product of integrals!**)

Note 2: The only challenge when using integration by parts is deciding which part of the integrand should be denoted by u. The rest is automatic.

2. $\int x^3 \ln x dx = \dfrac{x^4 \ln x}{4} - \int \dfrac{x^4}{4} \dfrac{dx}{x} = \dfrac{x^4 \ln x}{4} - \dfrac{x^4}{16} + c$

> **u - line:** $u = \ln x,\ \dfrac{du}{dx} = \dfrac{1}{x},\ du = \dfrac{dx}{x}$
>
> **v – line:** $dv = x^3\, dx\ /\!\!\int,\ v = \dfrac{x^4}{4}$

Note: $\ln x$ is a good choice for u, because u should be differentiated and derivative of $\ln x$ is $\dfrac{1}{x}$, which leads to simplification when multiplying by the integral of power x^3 (which in itself is a power of x).

3. $\int \ln x dx = x \ln x - \int x \dfrac{dx}{4} = x \ln x - x + c$

> **u - line:** $u = \ln x,\ \dfrac{du}{dx} = \dfrac{1}{x},\ du = \dfrac{dx}{x}$
>
> **v – line:** $dv = dx\ /\!\!\int,\ v = x$

4. $\int x^2 \sin x dx = -x^2 \cos x + \int x \cos x dx =$

> **u - line:** $u = \ln x,\ \dfrac{du}{dx} = \dfrac{1}{x},\ du = \dfrac{dx}{x}$
>
> **v – line:** $dv = dx\ /\!\!\int,\ v = x$
>
> **Note:** x^2 is a good choice for u, because u should be differentiated and derivative of x^2 is $2x$, a lower power, leading to a simpler integrand. Let us continue by evaluating $\int x \cos x dx$ by parts:

$= -x^2 \cos x + x \sin x - \int \sin x\, dx = -x^2 \cos x + x \sin x + \cos x + c = \cos x(1 - x^2) + x \sin x + c$

u - line: $u = x,\ \dfrac{du}{dx} = 1,\ du = dx$

v – line: $dv = \cos x\, dx\ \Big/ \!\int,\ v = \int \cos x\, dx = \sin x$

Substitute u , du, v and dv into the **Integration by Parts formula** to obtain

5. $\displaystyle \int x^2 \mathrm{e}^{3x}\, dx = x^2\, \frac{e^{3x}}{3} - \frac{2}{3}\int x \mathrm{e}^{3x}\, dx =$

$$x^2\,\frac{e^{3x}}{3} - \frac{2}{3}x\,\frac{e^{3x}}{3} + \frac{2}{9}\int \mathrm{e}^{3x}\, dx = x^2\,\frac{e^{3x}}{3} - \frac{2}{3}\,\frac{e^{3x}}{3}\,x + \frac{2}{27}e^{3x} + c = \frac{e^{3x}}{27}(9x^2 - 6x + 2) + c$$

6. $\displaystyle \int \sin x\,\mathrm{e}^{3x}\, dx = e^{3x}(-\cos x) + 3e^{3x}\int \cos x\,\mathrm{e}^{3x}\, dx =$

$$-e^{3x}\cos x + 3(\sin x\, e^{3x} - 3\int e^{3x} \sin x\, dx)$$
$$\Rightarrow\ \int \sin x\, e^{3s}\, dx = e^{3s}(3\sin x - \cos x) - 9\int \sin x\, e^{3s}\, dx$$

$$\Rightarrow I = e^{3x}\,(3\sin x - \cos x) - 9\,I$$
$$\Rightarrow I = \frac{1}{10}e^{3x}\,(3\sin x - \cos x)\ + c \ \text{ **(since indefinite integrals are defined up to a constant term)**}$$

20.3   Partial fractions

The method of partial fraction is an algebraic trick that allows us to re-write a rational function as a sum of the simplest possible rational fractions. It is an inverse (opposite) operation to adding up fractions using the common denominator method.

Consider $\dfrac{1}{x^2 - a^2} = \dfrac{1}{(x-a)(x+a)} = \dfrac{A \ \cancel{(x+a)}}{x-a} + \dfrac{B \ \cancel{(x-a)}}{x+a}$

$\Rightarrow \quad A\,(x+a) + B\,(x-a) = 1$

To solve this equation for A and B use one of the two methods:

I.   **Remove the brackets**

$$A\,x + Aa + B\,x - Ba = 1$$

Collect the like terms

$$(A + B)x + Aa - Ba = 1 \text{ for all } x$$

Equate the like coefficients

$$\begin{cases} A + B = 0 \\ Aa - Ba = 1 \end{cases}$$

$$A = -B \implies -Ba - Ba = 1 \implies B = -\frac{1}{2a}, \ \ A = \frac{1}{2a}$$

OR

II.   **Use the killer instinct (!)**

$x = -a$: $B\,(-2a) = 1 \implies B = -\dfrac{1}{2a}$ (chose x to "kill" the first term)

$x = a$: $2aA = 1 \quad\implies A = \dfrac{1}{2a}$ (chose x to "kill" the second term)

Whatever method you use, you can see that

$$\frac{1}{x^2 - a^2} = \frac{1}{2a}\left(\frac{1}{x-a} - \frac{1}{x+a}\right)$$

Check: $\dfrac{1}{2a}\left(\dfrac{1}{x-a} \cancel{\ (x+a)} - \dfrac{1}{x+a} \cancel{\ (x-a)}\right) = \dfrac{1}{2a}\dfrac{x+a-(x-a)}{x^2-a^2} = \dfrac{1}{2a}\dfrac{2a}{x^2-a^2} = \dfrac{1}{x^2-a^2}$

Representing a rational function as a sum of partial fractions is useful e.g. when integrating rational functions.

20.4   Integration of rational functions

Exampe: $\int \dfrac{1}{(x-2)(x-3)} dx =$

$$\frac{1}{(x-2)(x-3)} = \frac{A}{x-2} \overset{(x-3)}{} + \frac{B}{x-3} \overset{(x-2)}{} = \frac{A(x-3)+B(x-2)}{(x-2)(x-3)}$$

$\Rightarrow (A + B)\, x - (3A + 2B) = 1$

Equating the like coefficients,

$A + B = 0 \qquad \Rightarrow \qquad A = -B$

$3A + 2B = -1 \Rightarrow -3B + 2B = -1 \Rightarrow -B = -1 \Rightarrow B = 1 \Rightarrow A = -1$

Check: $\quad -\dfrac{1}{x-2} + \dfrac{1}{x-3} = \dfrac{-(x-3)+(x-2)}{(x-2)(x-3)} = \dfrac{1}{(x-2)(x-3)}$ ✓

$$= \int -\frac{dx}{x-2} + \int \frac{dx}{x-3} = -\ln|x-2| + \ln|x-3| + c = \ln\left|\frac{x-3}{x-2}\right| + c$$

20.5   Decision Tree for Integration

We can now introduce the full Decision Tree for Finding Indefinite Integrals – see figure 20.1:
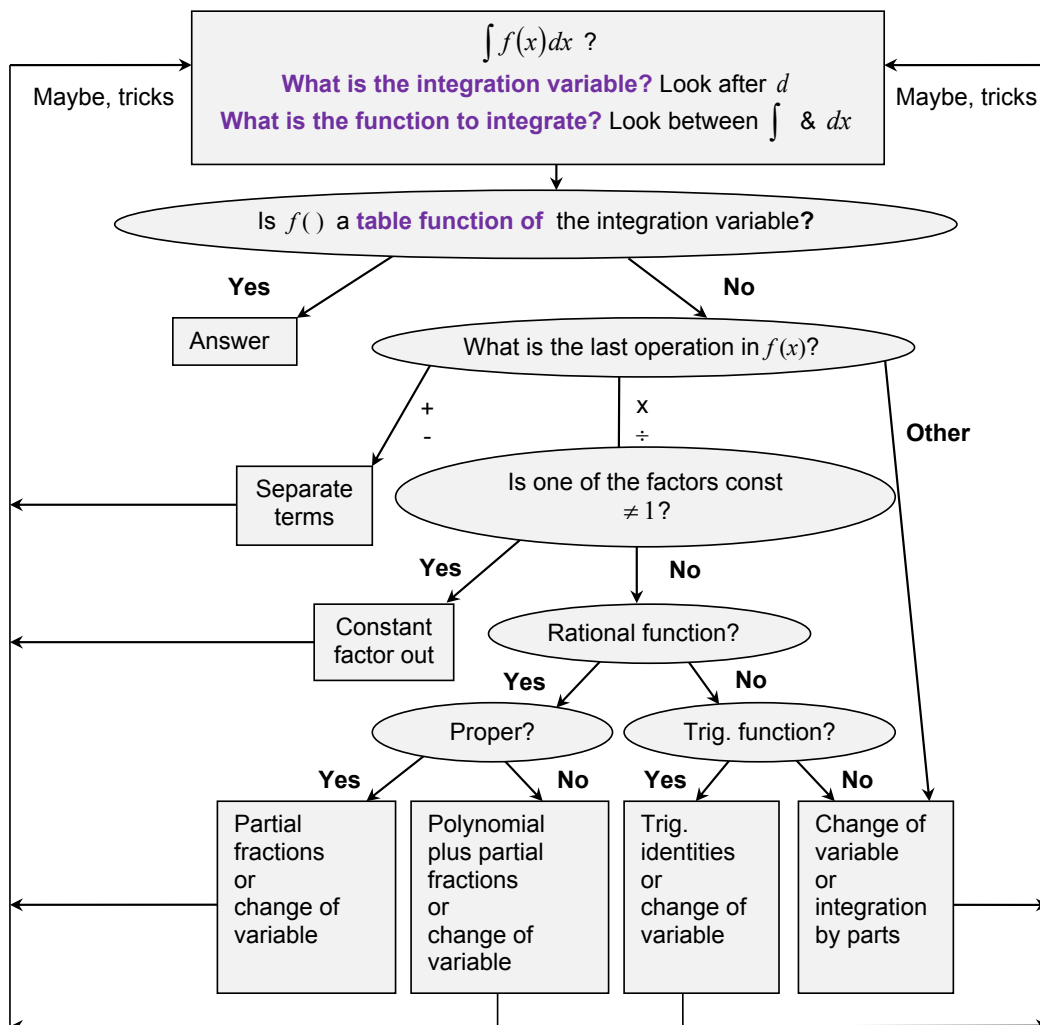
Figure 20.1. Decision Tree for Integration.

20.6   Instructions for self-study

- **Revise ALGEBRA Summary (addition of fractions)**
- **Revise Summaries on FUNCTIONS and TRIGONOMETRY**
- **Revise Lectures 14 – 15 (limits and differentiation)**
- **Revise Lectures 18 – 19 (integration) and study Solutions to Exercises in Lecture 18 using the STUDY SKILLS Appendix**
- **Study Lecture 20 using the STUDY SKILLS Appendix**
- **Do the following exercises:**

Q1. Find

a) $\displaystyle\int_{0}^{2\pi}\sin^2 t\,dt$

b) $\displaystyle\int_{0}^{2\pi}\cos^2 t\,dt$

c) $\displaystyle\int\sin x\cos x\,dx$

d) $\displaystyle\int\cos x\cos 2x\,dx$

Q2. Find

a) $\displaystyle\int x\sin x\,dx$

b) $\displaystyle\int x^2\ln x\,dx$

c) $\displaystyle\int x^2 e^{3x}\,dx$

d) $\displaystyle\int e^{2x}\cos x\,dx$

Q3. Find

a) $\displaystyle\int\frac{1}{(x-1)(x-3)}\,dx$

b) $\displaystyle\int\frac{x+1}{x^2-3x+2}\,dx$

c) $\displaystyle\int\frac{x-1}{x^2-2x-8}\,dx$

d) $\displaystyle\int\frac{x-1}{x^2-2x+8}\,dx$

Lecture 21 INTEGRAL CALCULUS: Applications of Integration

21.1 Mean value of a function

Using **The Mean Value Theorem** (Lecture 18), the mean value of a function on the interval between a and b is

$$M \equiv f(c) = \frac{1}{b-a} \int_a^b f(x)dx \text{ (mean height)}$$

$\Rightarrow$ Area of dashed rectangle = Area under the curve

$$M(b - a) \qquad = \qquad \int_a^b f(x)dx$$

Example: Find mean of $y = 3x^2 + 4x + 1, \ -1 \le x \le 2$

Solution

$a = -1, \ b = 2$

$$M = \frac{1}{b-a} \int_a^b f(x)dx = \frac{1}{2-(-1)} \int_{-1}^{2} (3x^2 + 4x + 1)dx = \frac{1}{3}(3x^3 + 2x + x) = 6$$
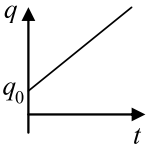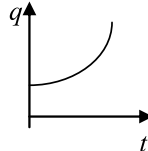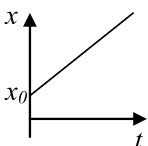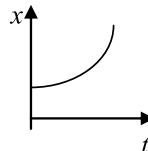
Optional

RMS value of a function (the square root of the mean value of the squares of y)

Example: Find RMS of $y = x^2 + 3, \ 1 \le x \le 3$

$$= \left[\frac{1}{3-1} \int_1^3 y^2 dx \right]^{\frac{1}{2}} = \left[\frac{1}{2} \int_1^3 (x^4 + 6x^2 + 9)dx \right]^{\frac{1}{2}}$$

$$= \left[\frac{1}{2} \left(\frac{x^5}{5} + \frac{x^3}{3} + 9x \right) \Big|_1^3 \right]^{\frac{1}{2}} = \frac{1}{2^{1/2}} \left[\left(\frac{3^5}{5} + \frac{3^3}{3} + 27 \right) - \left(\frac{1^5}{5} + \frac{1^3}{3} + 9 \right) \right]^{\frac{1}{2}} \approx \frac{1}{2^{1/2}} [74]^{\frac{1}{2}} \approx \sqrt{37} \approx 6$$

21.2 Electrical systems

If charge q varies linearly with time t, then current i is constant with time

$$i = \frac{q - q_0}{t}$$

If charge q is non-linear in time t, then instantaneous current i is

$$i = \frac{dq}{dt} \Rightarrow q(t) = \int_0^t i(\tau)\,d\tau + q(0).$$

21.3 Mechanical systems

If displacement x varies linearly with time t, then speed v is constant with time

$$v = \frac{x - x_0}{t}$$

If displacement x is non-linear in time t, then instantaneous speed v is

$$v = \frac{dx}{dt} \Rightarrow x(t) = \int_0^t v(\tau(\tau)) + x(0).$$

Example: Consider a capacitor of capacitance C (in *Faradays*).
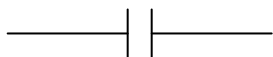
The voltage drop on the capacitor is

$e_c(\text{t}) = \dfrac{q(\text{t})}{C} = \dfrac{1}{C}\int\limits_0^t i(\tau)d\tau + \dfrac{q_0}{C}$. Below we often assume $q_0 = 0$.

Let $i = i_0 \cos \omega t$, $\omega = \dfrac{2\pi}{T}$ - circular frequency, T – period

$\Rightarrow e_c = \int\limits_0^t \dfrac{i_0}{C}\cos\omega\tau d\tau = \dfrac{i_0}{C\omega}\sin\omega t = \dfrac{i_0}{C\omega}\cos(\dfrac{\pi}{2} - \omega t) = \dfrac{i_0}{C\omega}\cos(\omega t - \dfrac{\pi}{2})$

$\Rightarrow$ a capacitor effects a $\dfrac{\pi}{2}$ shift between current and voltage
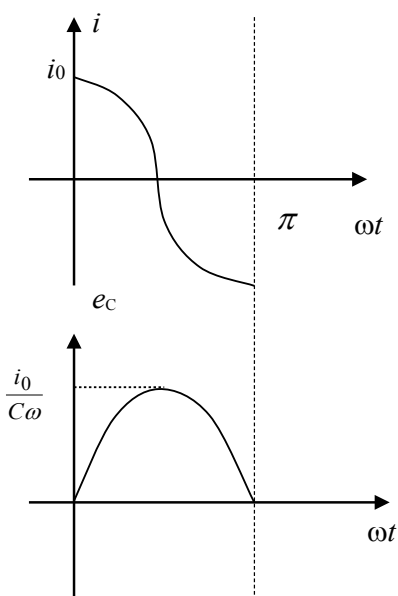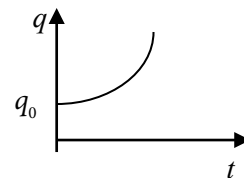
21.4 Rotational systems

21.4.1 Volumes of Solids of Revolution

If the plane figure bounded by the curve $y = f(x)$, the x-axis and the straight lines $x = a$ and $x = b$ rotates through a complete revolution about the x-axis, it generates a solid. Let V be its volume.
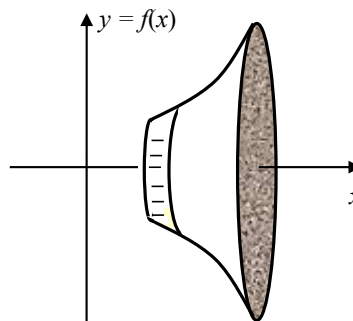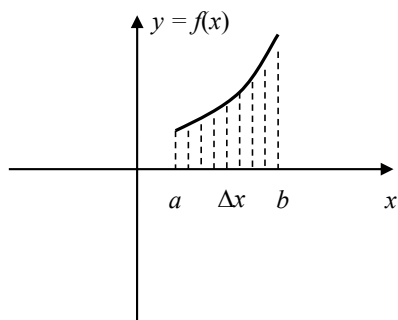
To find *V* we represent the area under the curve as a sum of areas of thin strips of width Δx (see figure above). The volume is approximately a sum of volumes of thin cylinders generated by each of these thin strips,

Question: What is the volume of a cylinder?
Answer:

Question: In our case, what is the radius *r* at the base of each thin cylinder?
Answer:

Question: What is the height *h* of each thin cylinder?
Answer:

$$\Rightarrow V \approx \sum_n \Delta V = \sum_n \pi y^2 \Delta x \rightarrow \pi \int_a^b y^2 \, dx \text{ as } \Delta x \rightarrow 0.$$

Example: Find the volume generated when the plane figure bounded by $y = 5\cos 2x$, *x*-axis and straight lines $x = 0$ and $x = \dfrac{\pi}{4}$ rotates through a complete revolution about the *x*-axis.

$$V = \pi \int_a^b y^2 \, dx = \pi \int_0^{\pi/4} [5\cos(2x)]^2 \, dx = \pi \int_0^{\pi/4} 25\cos^2(2x)\, dx = 25\pi \int_0^{\pi/4} \frac{1 + \cos(4x)}{2} \, dx =$$

$$12.5\pi \left[x + \frac{\sin(4x)}{4} \right]\Bigg|_0^{\frac{\pi}{4}} = 12.5\pi \left[\frac{\pi}{4} + \frac{\sin(4 \cdot \frac{\pi}{4})}{4} \right] \approx 3.1\pi^2 \approx 30$$

21.4.2   The moment of inertia

The moment of inertia I of a point of mass m rotating around a center O is $I = mr^2$, where r is the distance between the point and the centre.
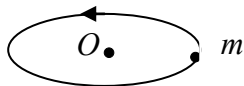
The moment of inertia of a lamina around an axis is approximately the sum of the moments of inertia $\Delta I = r^2 \Delta m$ of elementary points of mass Δm lying at the distance r from the axis of rotation,

$$I \approx \sum_n \Delta\mathrm{I} = \sum_n r_n^2 \Delta\mathrm{m} \rightarrow \int\limits_{surface} r^2 dm \ \ \text{as} \ \ \Delta x \rightarrow 0.$$

Example:

Let a square lamina rotate around the *y*-axis (see figure below). All points in the thin shaded area are approximately the same distance *x* from the axis.
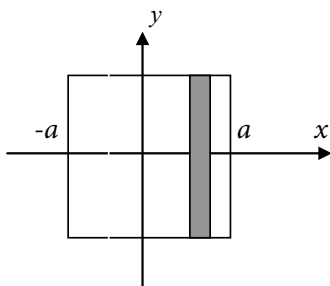
If **mass density** (mass per unit area), ρ, of the lamina is constant (does not change with position), then the mass of the shaded strip is $\Delta m = 2\rho a \Delta x$ (density times area), and the moment of inertia of the lamina is

$$I = \int\limits_{surface} r^2 dm = \int\limits_{-a}^{a} x^2 2\rho a\, dx = 2\rho a \left. \frac{x^3}{3} \right|_{-a}^{a} \frac{2\rho a}{3}(a^3 - (-a)^3) = \frac{4\rho a^4}{3} = \frac{Ma^2}{3}$$

where the lamina mass is $M = 4\rho a^2$.

Question: Why is the lamina mass $4\rho a^2$?
Answer:

21.5   Instructions for self-study

- **Revise Lectures 14–15 (differentiation)**
- **Revise Lectures 19 and study Solutions to Exercises in Lecture 19 using the STUDY SKILLS Appendix**
- **Study Lecture 20 using the STUDY SKILLS Appendix**
- **Study Lecture 21 using the STUDY SKILLS Appendix**
- **Do the following exercises:**

Q1. Given $i(t) = i_0 \sin \omega t$ find e_C and e_L – provided $q_0 = -\dfrac{i_0}{\omega}$, that is, $e_C(0) = \dfrac{q_0}{C} = -\dfrac{i_0}{\omega C}$.

Q2. Find the volume generated when the plane figure bounded by $y = 4x$, the x-axis and straight lines $x = 0$ and $x = 1$.

Q3.

a) The moment of inertia about a diameter of a sphere of radius 1 m and mass 1 kg is found by calculating the integral

$$\frac{3}{8}\int_{-1}^{1}(1-x^2)^2\,dx\,.$$

Show that the moment of inertia of the sphere is $\dfrac{2}{5}$ $kg\ m^2$.

b) Calculate the moment of inertia of a uniform thin rod of mass M and length l about a perpendicular axis of rotation at its end.

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Lecture 22 Ordinary Differential Equations

22.1   Basic concepts

Consider the equation

$$\frac{dy(x)}{dx} = ay(x) \tag{22.1}$$

Eq. (22.1) is an example of an **ordinary differential equation**. It is called an **equation**, because it is a mathematical statement that contains the = sign and that can be true or false depending on which **function** you substitute for the unknown $y(x)$ (compare to the case of an algebraic equation $P_n(x) = 0$: an algebraic equation is a mathematical statement that contains the = sign and can be true or false depending on which **value** you substitute for the unknown x).

Terminology: $y(x)$ is called an unknown (function) or dependent variable; x – an independent variable; a is a (constant) coefficient.

Eq. (22.1) is a **differential** equation, because it contains a derivative of the unknown y. It is called **ordinary,** because the unknown is a function of one variable (the differential equations with unknowns being functions of several variables are called **partial**, because they contain partial derivatives).

To solve a differential equation means to find all **functions** which turn it into a true statement (compare to the case of an algebraic equation: to solve an algebraic equation means to find all **values** of the unknown variable which turn it into a true statement).

Example:

1. In the following equations, state which variable is dependent and which variable is independent:

a) $\dfrac{d^2 y}{dt^2} + \dfrac{dy}{dt} = t$ **Answer:**

b) $RC \dfrac{dv}{dt} + v = v_0$ **Answer:**

c) $y \dfrac{dy}{dt} + c(t)y = 0$.   **Answer:**

2. Solve $\dfrac{dy(x)}{dx} = e^x$

The above equation is simple – only one term contains the unknown function $y(x)$. Therefore, it can be solved using the Decision Tree for Solving Equations with Only One Term Containing the Unknown.

In general, to solve a differential equation it is important to classify it first, because different types of differential equations can be solved using different methods (tricks). The differential equations are classified according to their order, to whether they are linear or non-linear and to whether their coefficients are constant or variable.

22.2   Order of a differential equation

The **order of a differential equation** is the order of its highest derivative.

> **Example:**
> Establish the order of the above differential equations a) – c). **Answers:**

22.3   Linearity or non-linearity of a differential equation

A differential equation is called **linear** if each of its terms either contains no unknown function or is a product of a known coefficient (constant or variable) and the unknown function or one of its derivatives).

> **Example:**
> Establish which of the above differential equations a) – c) is linear and which non-linear. **Answers:**

22.4   Differential equations with constant coefficients

A differential equation has constant coefficients if all coefficients (co-factors) of the dependent variable and its derivatives are constant (with respect to the independent variable).

> **Example:**
> Establish which of the above differential equations a) – c) has constant coefficients. **Answers:**

22.5   Homogeneous and inhomogeneous ODEs

A differential equation is called **homogeneous** if each of its terms contains the unknown or one of its derivatives. Otherwise it is called **inhomogeneous**.

> **Example:**
> Establish which of the differential equations a) – c) is homogeneous. **Answers:**

22.6   The first order linear homogeneous equation with constant coefficients

Eq. (22.1) is a general form of the first order linear homogeneous equation with constant coefficients. We can check by substitution that $y = e^{ax}$ is its solution and so is $y = c\, e^{ax}$ with any arbitrary constant factor c.

$\Rightarrow$ Any **first** order linear homogeneous equation with constant coefficients has infinitely many solutions involving **one** arbitrary constant.

22.7   The initial value problems

In reality, the initial value of an unknown function is often known. If given an initial condition, say $y(0) = y_0$, the solution becomes unique.

The problem

$$\begin{cases} y' = ay \\ y(0) = y_0 \end{cases}$$

is an example of an **Initial Value Problem**.

22.8   Balance equations in chemical engineering

For every element of a chemical system there will be a material balance equation for each chemical species present:

$$\frac{d(\text{quantity})}{dt} = \sum (\text{inflow rates}) - \sum (\text{outflow rates})$$

In this equation *quantity* must be an extensive quantity, e.g. mass (*kg*), energy *(Joules) etc.* Do not try to write material balance equations for temperature or concentration. Flow rates must be measured, respectively, in *kg/s, Watts*, etc.

Example:

Consider a tank containing V *litres* of a solution consisting of x_0 (*kg*) of salt dissolved In water. At the initial moment of time $t = 0$, let us start pumping pure water into the tank at the rate of r (*litres/s*) and let $x(t)$ (*kg*) be the mass of salt in the tank at any moment t. Let us keep the mixture uniform by stirring. Let the outflow rate of mixture be the same as the inflow rate of water, that is, r. Then the volume V of mixture is kept constant throughout. It is easy to check that under these conditions the outflow rate of salt is

$$r\,\frac{x(t)}{V} \quad (kg/s)$$

and the amount of salt $x(t)$ satisfies the ordinary differential equation

$$x'(t) = -r\,\frac{x(t)}{V}$$

The associated Initial Value Problem is

$$\begin{cases} x'(t) = -\dfrac{r}{V}\,x(t) \\ x(0) = x_0 \end{cases}$$

22.9   Ordinary differential equations with complex coefficients

Consider an equation

$$y' = jy \tag{22.2}$$

Using Section 22.6, the function $y = y_0\,e^{it}$ is its solution.

We can also prove this without using Section 22.6 as follows:

Substituting $y = y_1 + jy_2$ into the left-hand side of Eq. (22.2) we obtain:

$$y' = y_1' + jy_2'$$

Substituting $y = y_1 + jy_2$ into the right-hand side of Eq. (22.2) we obtain:

$$jy = jy_1 - y_2$$

This means that Eq. (22.2) can be written as

$$y_1' + jy_2' = jy_1 - y_2 \qquad (22.3)$$

At each moment t, Eq. (22.3) has one complex number on the left-hand side and another, in the right-hand side. This means that their real parts are equal and so are their imaginary parts,

$$y_1' = -y_2$$
$$y_2' = y_1$$

Let $y_2 = y_0 \sin t \Rightarrow y_1 = y_2' = y_0 \cos t.$ This means that

$$y = y_1 + jy_2 = y_0 \cos t + j\, y_0 \sin t = y_0\, e^{jt},$$

where we used the Euler's formula.

Note: When differentiating or integrating, complex coefficients must be treated in the same manner as real.

22.10 Applications

Eq. (22.1) describes many systems in which at any given time, the rate of change of the variable (that is, derivative of the variable) is proportional to the value of this variable.

Ordinary differential equations and associated Initial Value Problems are common in science and engineering, because scientific and engineering systems often effect transformations of signals, which involve their rates of change, that is their derivatives.

In many of your science and engineering courses you will be using linear ordinary differential equations of second order, with constant coefficients.

22.11   Instructions for self-study

- **Revise Decision Tree for Solving Simple Equations**
- **Revise Summaries on COMPLEX NUMBERS and DIFFERENTIATION**
- **Revise Lecture 20 and study Solutions to Exercises in Lecture 20 using the STUDY SKILLS Appendix**
- **Revise Lecture 21 using the STUDY SKILLS Appendix**
- **Study Lecture 22 using the STUDY SKILLS Appendix**
- **Do the following exercises:**

Q1. Verify that

a) $y = 3 \sin 2x$ is a solution of $y'' + 4y = 0$

b) $2e^x$ satisfies equation $y'' + 4y = 0$

c) Ae^x satisfies equation $y'' - 2y' + y = 0$

d) $Ae^x + Bxe^x$ satisfies equation $y'' - 2y' + y = 0$

e) Classify all the above equations.

Q2. Consider a circuit, comprising a capacitor of capacitance C and a resistor R placed in series:
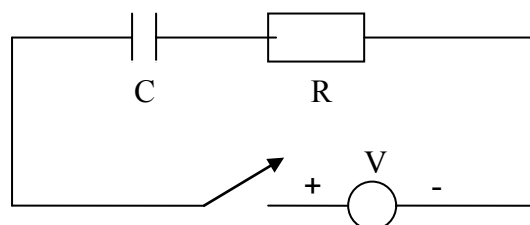
When a constant voltage source, *V*, is applied it can be shown that the current, *I*, through the circuit satisfies the ordinary differential equation.

$$R\frac{dI}{dt} + \frac{1}{C}I = 0$$

Given the initial current values $I(0) = VR$ solve the corresponding initial value problem, that is, find the current *I*(t) through this circuit.

Q3. Consider a tank containing 5 *litres* of a solution consisting of 0.5 (kg) of salt dissolved in water. Let pure water be pumped into the tank at the rate of 1 (*litres*/s) and the mixture, which is kept uniform by stirring, be pumped out at the same rate. Find *x*(*t*), the amount of salt in the tank at any moment *t*.

Q4. Solve the equations

    a) $y' = 2jy$
    b) $y'' = 2\sin x$
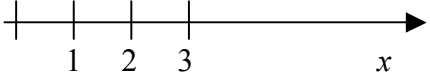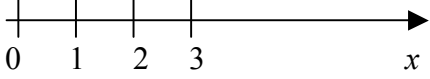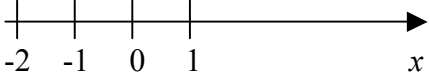
Part IV   Summaries

Algebra Summary

OPERATIONS	TYPES OF VARIABLES

Addition (direct operation)
 	
Addition of whole numbers gives whole number
 	
1. $a + b = b + a$
 	
Terminology: a and b are called **terms**
            $a + b$ is called **sum**
 	
2. $(a + b) + c = a + (b + c)$
	Whole numbers are $1, 2, 3, ...$
	
	    1 2 3 x

Subtraction (inverse operation)
 	
Def: $a - b = x: x + b = a$
 	
Note: $a + b - b = a$ **(subtraction undoes addition)**
    $a - b + b = a$ **(addition undoes subtraction)**
 	
3. $a + 0 = a$
 	
4. for each a there exists one **additive inverse** $-a$:
    $a + (-a) = 0$
 	
Rules (follow from Laws):
 	
$+ (b + c) = + b + c$
$+ a + b = a + b$
$- (- a) = a$
$- (a) = -a$
	introduces **0** and **negative numbers**:
	
	$a - a = 0$
	if $b > a$ $a - b = -(b - a)$
	
	Natural numbers are $0, 1, 2, ...$
	
	0 1 2 3 x
	
	Integers are $..., -2, -1, 0, 1, 2, ...$
	
	-2 -1 0 1 x

<u>**Multiplication**</u> (direct operation)

For whole numbers n

$$a \cdot n = \underbrace{a + \ldots\ldots\ldots\ldots + a}_{n \text{ terms}}$$

Notation: $ab = a \cdot b = a \times b$

$\qquad\qquad 2b = 2 \cdot b = 2 \times b$

$\qquad\qquad 23 \neq 2 \cdot 3, \; 23 = 2 \cdot 10 + 3$

$\qquad\qquad 2\dfrac{1}{2} \neq 2 \cdot \dfrac{1}{2}, \; 2\dfrac{1}{2} = 2 + \dfrac{1}{2}$

$\qquad\qquad 2\dfrac{3}{2} = 2 \cdot \dfrac{3}{2}$

1. $a \cdot b = a \cdot b$

Terminology: a and b are called **factors**

$\qquad\qquad\qquad ab$ – **product**

2. $(a \cdot b) \cdot c = a \cdot (b \cdot c)$

Conventions: $abc = (ab)c$

$\qquad\qquad\qquad a(-bc) = -abc$

3. $a(b+c) = ab + ac$

$\qquad \longrightarrow \qquad$ Removing brackets

$\qquad \longleftarrow \qquad$ Factoring

4. $a \cdot 0 = 0$
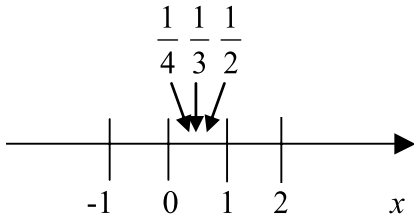
5. $a \cdot 1 = a$
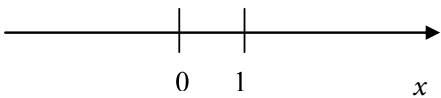
Rules (follow from Laws):

$(a + b)(c + d) = ac + ad + bc + bd$ (**SMILE RULE**)

$(-1) \cdot n = -n$

$(-1) \cdot (-1) = 1$

Division (inverse operation)

Def: $a/b = x: xb = a$

Terminology: a – **numerator**

$\qquad\qquad$ b – **denominator**

$\qquad\qquad$ a/b – **fraction (ratio)**

$\qquad\qquad$ – **proper fraction** if $|a|<|b|$, a, b

$\qquad\qquad$ integers

Note: $ab/b = a$ **(division undoes multiplication)**

$\qquad$ $(a/b)b = a$ **(multiplication undoes**

$\qquad\qquad\qquad$ **division)**

6. $\quad$ For each $a \neq 0$ there exists one **multiplicative inverse** $1/a$: $\ a{\cdot}1/a = 1$

introduces **rational numbers**

Def: Rationals are all numbers $\dfrac{m}{n}$, where m and $n \neq 0$ are integers **(division by zero is not defined)**

$$\frac{1}{4} \quad \frac{1}{3} \quad \frac{1}{2}$$

Rules:

$$\frac{a}{b}\cdot n = \frac{an}{b}$$

$$\frac{a/b}{n} = \frac{a}{bn} = \frac{a/n}{b}$$

$$\frac{an}{bn} = \frac{a\cdot\cancel{n}}{b\cdot\cancel{n}} = \frac{a}{b} \qquad \textbf{CANCELLATION}$$

$$\frac{\frac{1}{n}}{m} = \frac{m}{n} \qquad\qquad \textbf{FLIP RULE}$$

$$\frac{a}{b} + \frac{c}{b} = \frac{a+c}{b} \qquad \textbf{Note:} \ \frac{a+c}{b} = (a+c)/b$$

$$\frac{a}{b} + \frac{c}{d} = \frac{ad}{bd} + \frac{cb}{db} = \frac{ad+cb}{bd} \quad \textbf{COMMON}$$

$$\textbf{DENOMINATOR}$$

$$\textbf{RULE}$$

n-th **power** b^n (direct operation)

For whole numbers n
$$b^n = \underbrace{b \cdot b \cdot b \cdot \ldots \cdot b}_{n \text{ factors}}$$

Rules:

$a^m \cdot a^n = a^{m+n}$

(product of powers with the same base is a power with indices added)

$a^n \cdot b^n = (ab)^n$

(product of powers is power of product)

$$\frac{a^m}{a^n} = a^{m-n}$$

(ratio of powers is power of ratios)

$$\frac{a^m}{b^m} = \left(\frac{a}{b}\right)^m$$

**(ratio of powers with the same base –
 subtract indices)**

$a^0 = 1$

$$a^{-n} = \frac{1}{a^n}$$

$(a^m)^n = a^{mn}$ [Convention: $a^{m^n} = a^{(m^n)}$]

---

n–th **root** (inverse to taking to power *n*)

Def: $\sqrt[n]{b} = x: \ x^n = b$

Note:

$\sqrt[n]{b^n} = b$

(taking *n-th* root undoes taking *n-th* power)

$(\sqrt[n]{b})^n = b$

(taking *n-th* power undoes taking *n-th* root)

Therefore, can use **notation** $b^{1/n} = \sqrt[n]{b}$

(Indeed, $\sqrt[n]{b^n} = (b^n)^{1/n} = b^{n \cdot \frac{1}{n}} = b^1 = b$)

introduces **irrational** (not rational) **numbers** $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$, $\sqrt[3]{2}$, $\sqrt[3]{3}$…

Real numbers are all rationals and all irrationals combined. Corresponding points cover the whole number line (called **real line** for this reason)

<u>**Logarithm base** *b*</u> (inverse to taking *b* to power)

Def: $\log_b a = n$: $b^n = a$

Note:
$\log_b b^n = n$ (check using definition: $b^n = b^n$)
(**taking $\log_b$ undoes taking b to power**)

$b^{\log_b n} = n$
(**taking b to power undoes $\log_b$**)

Rules (follow from Rules for Indices):
$\log_b xy = \log_b x + \log_b y$
(**log of a product is sum of logs**)

$\log_b \dfrac{x}{y} = \log_b x - \log_b y$

(**log of a ratio is difference of logs**)
$\log_b 1 = 0$ (**log of 1 is 0**)
$\log_b b = 1$

$\log_b \dfrac{1}{a} = -\log_b a$

$\log_b x^n = n \log_b x$
(**log of a power is power times log**)

$\log_b a = \dfrac{\log_c a}{\log_c b}$

(**changing base**)

introduces **irrational** (not rational) **numbers**, $\log_{10} 2$, $\log_{10} 3$, *etc.*

Roots and logs also introduce **complex** (not real) **numbers**, $\sqrt{-1}, \log_{10}(-1)$, *etc.*

General remarks

1. $a - b = a + (-b)$ $\rightarrow$ a difference can be re-written as a sum

2. $\dfrac{a}{b} = a \cdot \dfrac{1}{b} = ab^{-1}$ $\rightarrow$ a ratio can be re-written as a product

3. $\sqrt[n]{b} = b^{1/n}$ $\rightarrow$ a root can be re-written as a power

4. All laws and rules of addition, multiplication and taking to integer power operations apply to real numbers.

5. Operations of addition, subtraction, multiplication, division (by non-zero) and taking to integer power when applied to real numbers produce real numbers. Other algebraic operations applied to real numbers do not necessarily produce real numbers.

6. In any algebraic expression operations should be performed using the Order of Operations convention.

Functions Summary

Variables are denoted mostly by *x, y, z, p, ... w.*
A variable can take any value from a set of allowed numbers.

Functions are denoted mostly by *f, g* and *h* or *f* ( ), *g* ( ) and *h*( )
(no multiplication sign is intended!*).*

In mathematics, the word **function** has three meanings:

1. *f*( ) – **an operation or a chain of operations** on an independent variable;
2. *f* (*x*) – a **dependent variable**, that is the variable obtained when *f*( ) acts on an independent variable *x*;
3. Set of pairs {(*x, f(x)*): *f*( ) assigns one value *f* (*x*) to every allowed value of *x*}.

A diagrammatical representation of a function
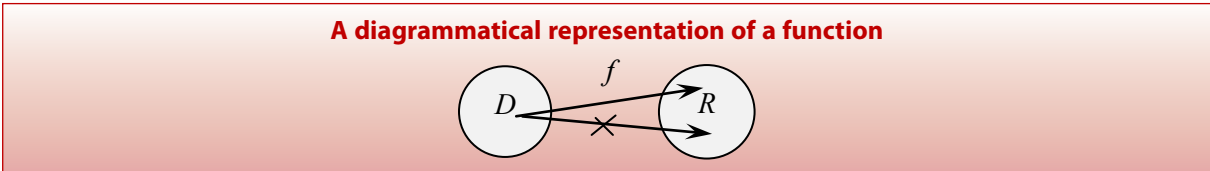
To specify a function we need to specify a (series of) operation(s) and **domain** D (an allowed set of values of the independent variable). To each $x \in D$, *f* (*x*) assigns <u>one and only one</u> value $y \in R$ (**range**, the set of all possible values of the dependent variable).

Inverse functions

$f^{-1}(x)$**:** $f \circ f^{-1}(x) = f^{-1} \circ f(x) = x$ (the function and inverse function undo each other)

symbol of inverse function, not a reciprocal

The inverse function does not always exist!

A Diagrammatic Representation of Inverse Function

$y = f(x)$

$x = f^{-1}(y)$
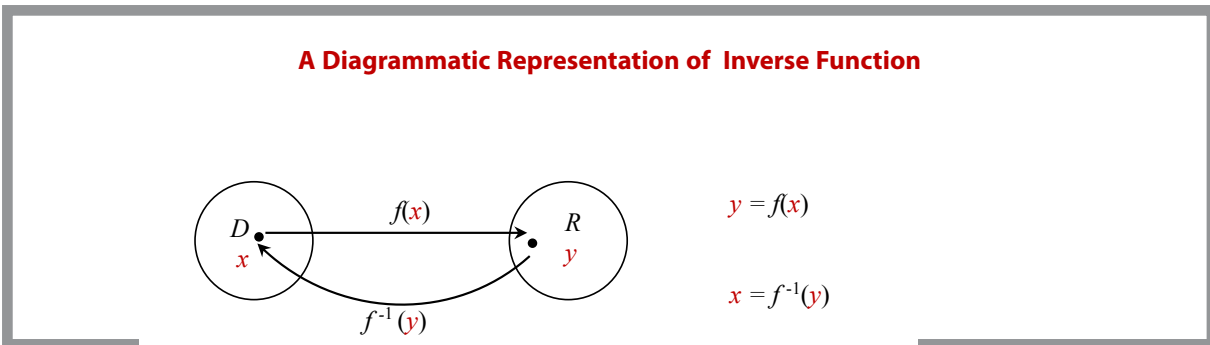
Order of Operations Summary

When **evaluating** a mathematical **expression** it is important to know the order in which the **operations** must be performed. By convention, the **Order of Operations** is as follows:

First, expression in **Brackets** must be **evaluated**. If there are several sets of brackets, e.g. {[( )]}, expressions inside the inner brackets must be **evaluated** first. The rule applies not only to brackets explicitly present, but also to brackets, which are implied. **Everything raised and everything lowered is considered as bracketed**, and some authors do not bracket **arguments** of elementary **functions**, such as exp, log, sin, cos, tan *etc*. In other words, e^x should be understood as exp (*x*), sin *x* as sin (x) *etc*.

Other **operations** must be performed in the order of decreasing complexity, which is

oiB – operations in Brackets (including implicit)
F – Functions f ( )
P- Powers (including inverse operations of roots and logs)
M – Multiplication (including inverse operation of division)
A – Addition (including inverse operation of subtraction)

That is, the more complicated **operations** take precedence. For simplicity, we refer to this convention by the abbreviation **oiBFPMA**.

Order of Operations (OOO)

1. Make **implicit** (invisible) brackets visible
    (everything raised and everything lowered is considered to be bracketed and so are function arguments)
2. Perform operations in brackets {[()]} first (inside out)

OOO

oiB	f()	P	M	A
(including		$( )^{( )}$	×	+
implicit)		roots	÷	-
		logs		

Quadratics Summary

A **quadratic expression** is a general polynomial of degree 2 traditionally written as

$$ax^2 + bx + c,$$

where a is the constant factor in the quadratic term (that is, the term containing the independent variable squared), b is a constant factor in the linear term (that is, the term containing the independent variable) and c is the free term (that is, the term containing no independent variable).

A **quadratic equation** is the polynomial equation

$$ax^2 + bx + c = 0.$$

Its two **roots** (solutions) can be found using the standard formula

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Once the roots are found the quadratic expression can be **factorised** as follows:

$$ax^2 + bx + c = a(x - x_1)(x - x_2).$$

Trigonometry Summary

Conversion between degrees and radians

An angle described by a segment with a fixed end after a full rotation is said to be 360° or 2π (radians)

$$\Rightarrow 2\pi \,(rad) = 360°$$
$$\Rightarrow 1 \,(rad) \approx 57°$$

The radian is a dimensionless unit of angle.

$$\Rightarrow \; x \,(rad) = x \,(rad) \; \frac{180^{\mathrm{o}}}{\pi(\mathrm{rad})} = y^{\mathrm{o}}, \; y^{\mathrm{o}} = y^{\mathrm{o}} \; \frac{\pi(\mathrm{rad})}{180^{\mathrm{o}}} = x \,(rad)$$

Usually, if the angle is given in radians the units are not mentioned (since the radian is a dimensionless unit).

Frequently used angles
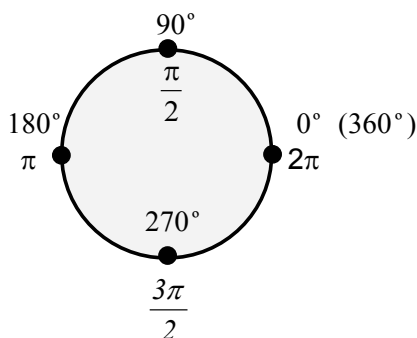
$30° = (\dfrac{30\pi}{180} =) \; \dfrac{\pi}{6}$	$45° = \dfrac{\pi}{4}$
$60° = (\dfrac{60\pi}{180} =) \; \dfrac{\pi}{3}$	
$120° = (\dfrac{120\pi}{180} =) \; \dfrac{2\pi}{3}$	

Right Angle Triangles and Trigonometric Ratios

Trigonometric ratios sin, cos and tan are defined for **acute angles** (that is, angles less than 90⁰) as follows:

$$\sin \alpha = \cos \beta = \frac{a}{c}$$

$$\cos \alpha = \sin \beta = \frac{b}{c}$$

$$\tan a = \cot \beta = \frac{a}{b}$$
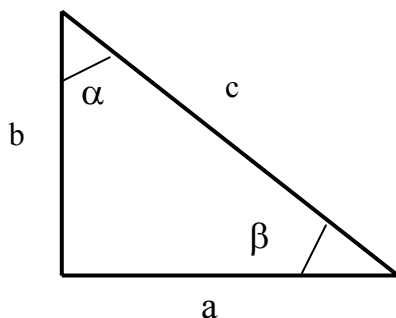
$\alpha + \beta = 90^0$ and α and β are called **complementary angles**

Frequently used trigonometric ratios

$\sin \dfrac{\pi}{6} = \dfrac{1}{2}$	$\sin \dfrac{\pi}{4} = \dfrac{\sqrt{2}}{2}$
$\cos \dfrac{\pi}{6} = \dfrac{\sqrt{3}}{2}$	$\cos \dfrac{\pi}{4} = \dfrac{\sqrt{2}}{2}$
	$\cos \dfrac{\pi}{4} = \dfrac{\sqrt{2}}{2}$

Trigonometric identities

$\sin^2 x + \cos^2 x = 1$

$\sin(-x) = -\sin x$

$\cos(-x) = \cos x$

$\tan(-x) = -\tan x$

$$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$$
$$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$$

$$\cos 2x = \cos^2 x - \sin^2 x$$
$$\sin 2x = 2\sin x \cos x$$
$$A\cos x + B\sin x = \frac{1}{\sqrt{A^2 + B^2}} \sin(x + \alpha), \text{ where } \tan \alpha = \frac{A}{B}$$

Complex Numbers

The Cartesian representation of a complex number

(z)   **real part**, $\text{Re}\, z$ **imaginary part**, $\text{Im}\, z$ (on the Argand diagram)

$$z = x + jy, \text{ where } x, y - \text{ real}, \qquad j = \sqrt{-1}$$

x-coordinate y-coordinate (on the ordinary plane)

The exponential representation of a complex number

(z)   **magnitude**, $|z|$ **argument,** $\arg(z)$ (on the Argand diagram)

$$z = r\, e^{j\theta}, \text{ where } r \geq 0, \quad \theta \text{ - angle}$$

(in degrees or radians)

polar radius r polar angle θ (on the ordinary plane)
(distance to the origin)

Decision Tree For Solving Simple Equations

If needs be, use various rules to ensure that the unknown appears only in one term in the LHS (left-hand side)

What is the last operation on the unknown?

What is its inverse?

Apply this inverse operation to both sides of the equation

Is the unknown the subject of the equation?

No

Yes

Is the answer in its simplest form?

Yes

No

What is the highest common divisor for numerator and denominator?

Is the answer a proper fraction?

Yes

No

What is the integer part of the fraction?

What is the remainder?

Answer

Sketching Graphs by Simple Transformations

1. Drop all constant factors and terms.
2. Bring the constants back one by one in the Order of Operations (not necessary but advisable) and at each Step use the Decision Tree given below to decide which simple transformation is effected by each constant. Sketch the resulting graphs underneath one another.

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DECISION TREE FOR SKETCHING $y=f(x)+c$, $y=f(x+c)$, $y=cf(x)$ **AND** $y=f(cx)$

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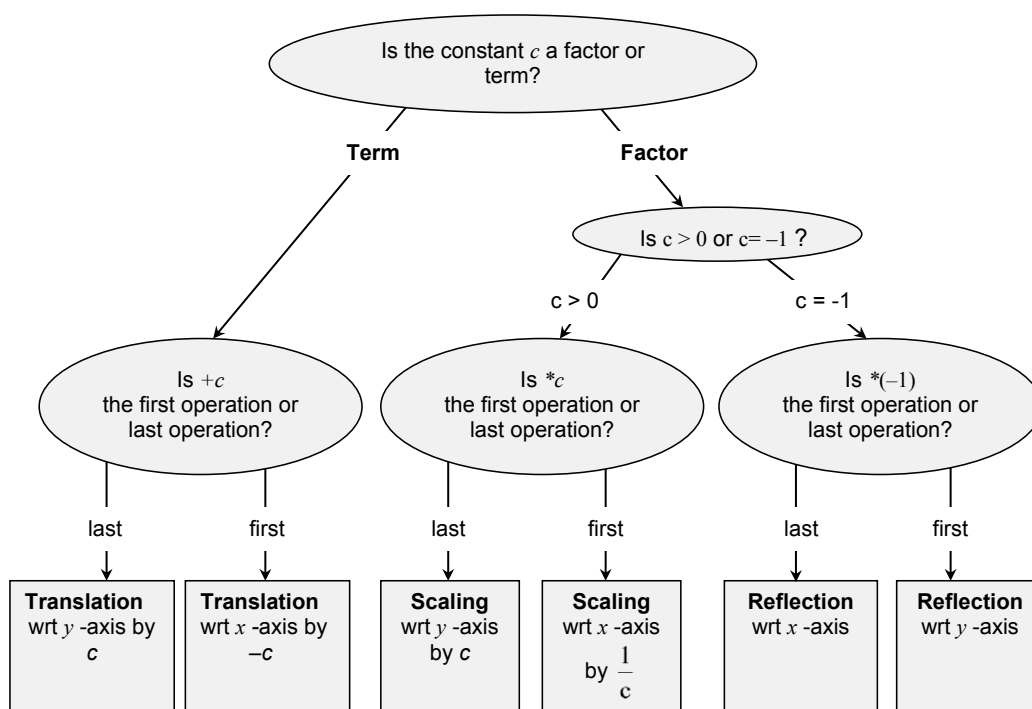
Note 1: if c is a **negative factor**, write $c = (-1)*|c|$, so that c effects two simple transformations and not one.

Note 2: if c is effects **neither first operation nor last**, the Decision Tree is not applicable. Try algebraic manipulations.

Note 3: if **the same operation** is applied to x in all positions in the equation this operation still should be treated as the first operation, as in $\ln(x-2) + \sin(x-2)$.

Note 4: if y is given **implicitly rather than explicitly**, so that the equation looks like $f(x,y) = 0$, then in order to see what transformation is effected by adding a constant c to y or multiplying it by a constant c, y should be made the subject of the functional equation. Therefore, similarly to transformations of x above, the corresponding transformation of y is defined by the inverse operation, $-c$ or $\dfrac{1}{c}$, respectively.

Finding a Limit of a Sequence

To find a limit of a sequence (as $k \to \infty$) use the first principles (graphical representation) or

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TABLE

x_k	$\lim x_k$
constant	constant
k	∞
$\dfrac{1}{k}$	0
$\dfrac{1}{a^n}$ $(a > 0)$	1
q^n $(|q| < 1)$	0

RULES

1. $\lim \alpha x_k = \alpha \lim x_k$ – *constant factor out*
2. $\lim [x_k \pm y_k] = \lim x_k \pm \lim y_k$ – *sum rule*
   (*separate terms*)
3. $\lim (x_k \, y_k) = \lim x_k \lim y_k$ – *product rule*
4. $\lim \dfrac{x_k}{y_k} = \dfrac{\lim x_k}{\lim y_k}$ – *quotient rule*

</div>

DECISION TREE
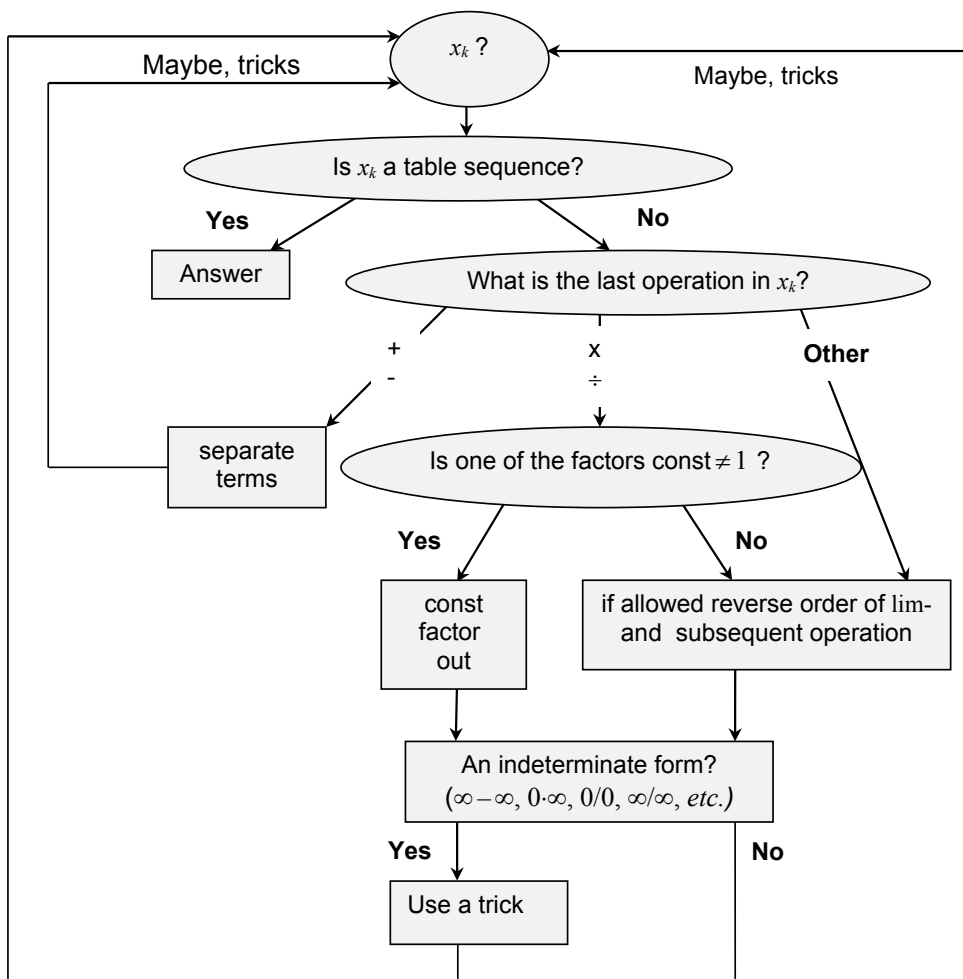
Differentiation Summary

To differentiate a function use the definition

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$
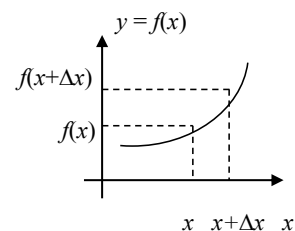
or represent roots and fractions as powers, make invisible brackets visible and use

<div align="center">

TABLE

$f(x)$	$\dfrac{d}{dx} f(x)$
constant	0
x^n	nx^{n-1}
e^x	e^x
$\ln x$	$\dfrac{1}{x}$
$\sin x$	$\cos x$
$\cos x$	$-\sin x$

</div>

RULES

1. $\dfrac{d}{dx}[\alpha f(x)] = \alpha \dfrac{df(x)}{dx}$ – *const factor out*

2. $\dfrac{d}{dx}[f(x) + g(x)] = \dfrac{df(x)}{dx} + \dfrac{dg(x)}{dx}$ – *sum rule (separate terms)*

3. $\dfrac{d}{dx}[f(x)g(x)] = \dfrac{df(x)}{dx} g(x) + \dfrac{dg(x)}{dx} f(x)$ – *product rule*

4. $\dfrac{d}{dx} \dfrac{f(x)}{g(x)} = \dfrac{\dfrac{df(x)}{dx} g(x) - \dfrac{dg(x)}{dx} f(x)}{g^2(x)}$ – *quotient rule*

5. $\dfrac{d}{dx} f(g(x)) = \dfrac{df(g(x))}{dg(x)} \dfrac{dg(x)}{dx}$ – *chain rule*

(*decompose, differentiate, multiply*)

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DECISION TREE

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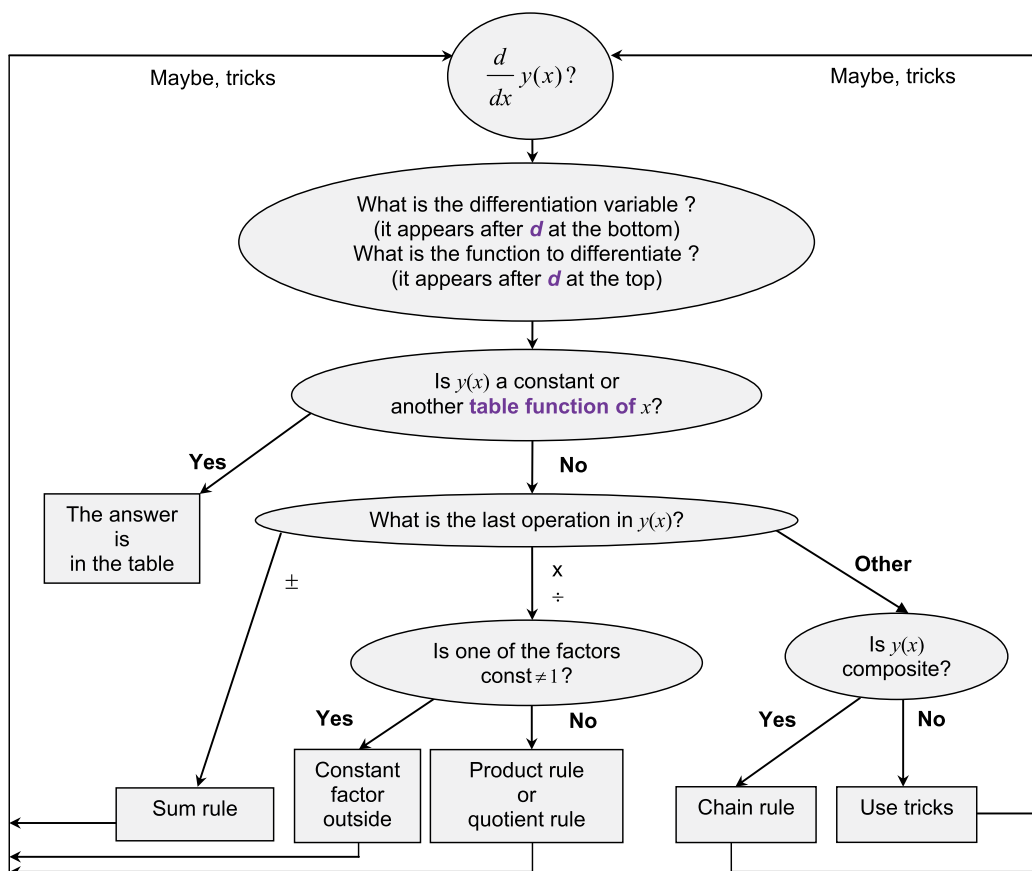
Integration Summary

To find a definite integral of a function use the definition

$$\int_a^b f(x)dx = \lim_{\Delta x \to 0} \sum_{k=0}^{N} f(x_k)\Delta x$$

or represent roots and fractions as powers, make invisible brackets visible and use
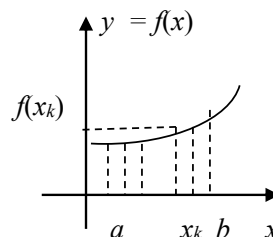
TABLE

$f(x)$	$F(x) = \int f(x)dx$
0	c
e^x	$e^x + c$
$\cos x$	$\sin x + c$
$\sin x$	$-\cos x + c$
$x^n,\ n \neq -1$	$\dfrac{x^{n+1}}{n+1} + c$
$\dfrac{1}{x}, x \neq 0$	$\ln|x| + c$

RULES

1. $\int \alpha f(x)dx = \alpha \int f(x)dx$ – *constant factor out*

2. $\int [f(x)+g(x)]dx = \int f(x)dx + \int g(x)dx$ – *sum rule*

   (separate terms)

DECISION TREE
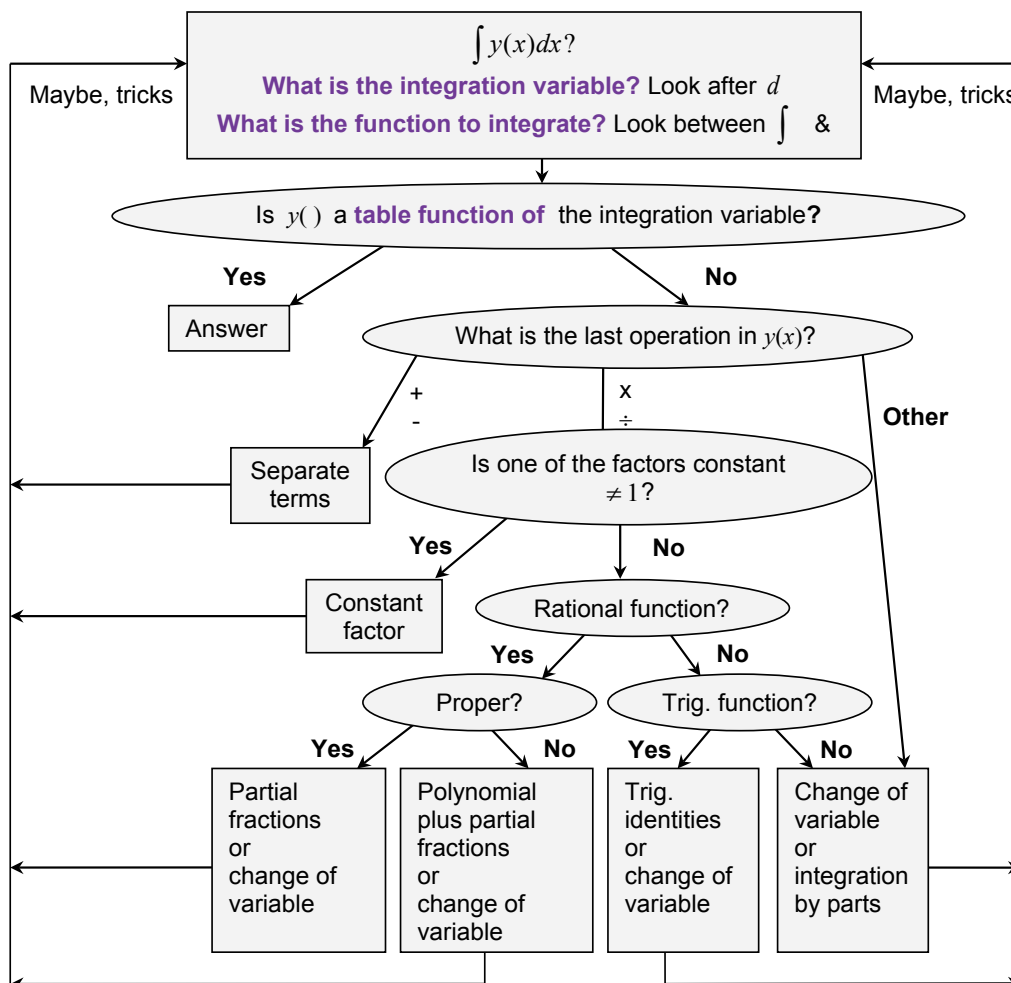
Part V   Glossary

ABSTRACTION – a general concept formed by extracting common features from specific examples.

ACUTE ANGLE – a positive angle which is smaller than 90^0.

ALGEBRAIC OPERATION – OPERATION of addition, subtraction, multiplication, raising to power, extracting a root (surd) or taking a log.

ALGORITHM – a sequence of solution steps.

ARGUMENT – INDEPENDENT VARIABLE, INPUT.

CANCELLATION (in a numerical fraction) – operation of dividing both numerator and denominator by a common divisor.

CIRCLE – a LOCUS of points inside and on a CIRCUMFERENCE.

CIRCUMFERENCE – a LOCUS of points at the same distance from a specific point (called a centre).

COEFFICIENT – a CONSTANT FACTOR multiplying a VARIABLE, e.g. in the EXPRESSION $2ax$, x is normally a VARIABLE and $2a$ is its COEFFICIENT.

COMPOSITION – a combination of two or more functions (operations), with one acting on the output of another, e.g. composition of two functions $f(\ )$ and $g(\ )$ is $f(g(\ ))$.

CONCEPT – a technical word or phrase.

CONSTANT – a number or a mathematical quantity that can take a range of (numerical) values but is independent of the main CONTROL VARIABLE – ARGUMENT or UNKNOWN.

If there is only one algebraic CONSTANT, the preferred choice for its algebraic symbol is a. The second choice is b, and the third, c. If there are more CONSTANTS in the EXPRESSION, then one chooses, in the order of preference, d and e, and then upper case letters in the same order of preference. If more CONSTANTS are required, we make use of subscripts and superscripts and Greek letters.

DIAGRAM – a general (abstract) visualisation tool, a pictorial representation of a general set or relationship.

DIFFERENCE – a mathematical expression in which the last operation is subtraction.

DIMENSIONAL QUANTITY – a quantity measured in arbitrary units chosen for their convenience, such as *s, m, N, A, m/s, kg/m³, £*.

DOMAIN – a set of all allowed values of the ARGUMENT.

EQUATION – a mathematical statement which involvs UNKNOWN(S) and the = sign and which can be true or false, e.g. $2x + 3y = 10$ is true when x = y = 2 and false when $x = y = 1$.

EVALUATE – find the (numerical) value of a (mathematical) EXPRESSION.

EXPLICIT – 1) clearly visible; 2) a subject of equation.

EXPONENTIATION – a mathematical operation of raising to power.

EXPRESSION (mathematical) – a combination of numbers, brackets, symbols for variables and symbols for mathematical operations, e.g. *2(a + b), 2ab*.

FACTOR – a (mathematical) EXPRESSION which multiplies another (mathematical) EXPRESSION, e.g. *ab* is a PRODUCT of two FACTORS, *a* and *b*.

FINAL (SIMPLEST) FORM (of a numerical fraction) – no CANCELLATIONS are possible, and only PROPER FRACTIONS are involved.

FORMULA – a mathematical statement involving VARIABLES and the = sign which and is always true.

FREE TERM – a CONSTANT TERM.

FUNCTION
1. $f( )$ – an operation or a chain of operations on an DEPENDENT VARIABLE.
2. $f(x)$ – a DEPENDENT VARIABLE, that is the variable obtained when $f( )$ acts on an independent variable *x*.
3. Set of pairs $\{(x, f(x))$: $f( )$ assigns one value $f(x)$ to every allowed value of $x\}$.

   To give an example, '$f(x) = 2x + 3$, x – real' describes a FUNCTION whose SYMBOL is 'f' or '$f( )$' (no multiplication is implied) and whose ARGUMENT is called 'x'. Its DOMAIN is 'all reals'. Given any value of x, you can find the corresponding value of this FUNCTION by first multiplying the given value of x by 2 and then by adding 3 to the result.

FUNCTION SYMBOL – usually a Latin letter, usually from the first part of the alphabet. If there is only one FUNCTION, the preferred choice for its symbol is *f*. The second choice is *g*, and the third, *h*. If there are more variables, then one chooses, in order of preference, *u*, *v* and *w*. If more FUNCTIONS are involved, we often make use of subscripts and superscripts, upper case and Greek letters.

GENERALISATION – an act of introducing a general concept or rule by extracting common features from specific examples.

GRAPH – a specific visualisation tool, a pictorial representation of a particular set or relationship.

IDENTITY – the same as FORMULA.

IMPLICIT – not EXPLICIT.

INDEPENDENT VARIABLE – ARGUMENT, INPUT, e.g. given *y = f(x)*, *x* is an INDEPENDENT VARIABLE.

INPUT – (value of) ARGUMENT, (value of) INDEPENDENT VARIABLE, e.g. given '*y = f(x)*', *x* is INPUT; also given '*y = f*(2)', 2 is INPUT.

INTEGER PART – when dividing a positive integer *m* into a positive integer *n, k* is the INTEGER PART if it is the largest positive integer producing k*m ≤ n. The REMAINDER is the difference n – k*m, e.g. when dividing 9 into 2, the INTEGER PART is 4 and the REMAINDER is 1, so that $\frac{9}{2} = 4 + \frac{1}{2} = 4\frac{1}{2}$.

INVERSE (to an) **OPERATION** – operation that undoes what the original OPERATION does.

LAST OPERATION – see ORDER OF OPERATIONS.

LHS – Left Hand Side of the EQUATION or FORMULA, to the left of the '='-sign

LINEAR EQUATION – an EQUATION which involves only FREE TERMS and TERMS which contain the UNKNOWN only as a FACTOR, e.g. '2*x* – 3 = 0' is a LINEAR EQUATION, '2' is a COEFFICIENT in front of the UNKNOWN and '-3' is a FREE TERM.

LOCUS – a set of points with a specific property.

NECESSARY – 'A' is a NECESSARY condition of 'B' if 'B $\Rightarrow$ A' (B implies A), so that 'B' cannot take place unless 'A' is satisfied.

NON-DIMENSIONAL QUANTITY – a quantity taking any value from an allowed set of numbers.

NON-LINEAR EQUATION – an EQUATION which is not LINEAR, e.g. '2 ln (x) – 3 = 0' is a NON-LINEAR EQUATION in x.

OPERATION (mathematical) – action on CONSTANTS and VARIABLES. When all CONSTANTS and VARIABLES entering an EXPRESSION are given values, OPERATIONS are used to EVALUATE this EXPRESSION.

ORDER OF OPERATIONS When EVALUATING a (mathematical) EXPRESSION it is important to know the order in which the OPERATIONS must be performed. By convention, the ORDER OF OPERATIONS is as follows: First, expression in BRACKETS must be EVALUATED. If there are several sets of brackets, e.g. {[( )]}, expressions inside the inner brackets must be EVALUATED first. The rule applies not only to brackets explicitly present, but also to brackets, which are implied. Two special cases to watch for are fractions and functions. Indeed, when $(a + b)/(c + d)$ is presented as a two-storey fraction the brackets are absent, and some authors do not bracket ARGUMENTS of elementary FUNCTIONS, such as exp, log, sin, cos, tan *etc.* In other words, e^x should be understood as exp (x), sin x as sin (x) *etc.* Other OPERATIONS must be performed in the order of decreasing complexity, which is

> **FUNCTIONS** f ( )
> **POWERS** (including inverse operations of roots and logs)
> **MULTIPLICATION** (including inverse operation of division)
> **ADDITION** (including inverse operation of subtraction)

That is, the more complicated OPERATIONS take precedence.

OUTPUT – (value of) DEPENDENT VARIABLE, (value of the) FUNCTION, e.g. given '$y = f(x)$', 'y' is OUTPUT; also given '$f(x) = 2x$ +3' and '$x = 2$', '7' is OUTPUT (indeed, 2*2+3 = 7).

PRODUCT – a (mathematical) EXPRESSION in which the LAST operation (see the ORDER OF OPERATIONS) is multiplication, e.g. ab is a PRODUCT, and so is $(a + b)c$.

QUOTIENT – a mathematical expression where the last operation is division.

RANGE – the set of all possible values of a FUNCTION.

REARRANGE EQUATION, FORMULA, IDENTITY – the same as TRANSPOSE.

REMAINDER – see INTEGER PART.

RHS – Right Hand Side of the EQUATION or FORMULA, to the right of the '=' sign.

ROOT OF THE EQUATION – SOLUTION of the EQUATION.

SEQUENCE – a function with an INTEGER ARGUMENT.

SIMPLE EQUATION – an EQUATION that can be rearranged to contain the unknown in one term only.

SIMPLE TRANSFORMATION – translation, scaling or reflection.

SOLUTION OF AN ALGEBRAIC EQUATION – constant values of the UNKNOWN VARIABLE which turn the EQUATION into a true statement.

SOLVE – find SOLUTION of the EQUATION.

SUBJECT OF THE EQUATION – the unknown is the SUBJECT OF THE EQUATION if it stands alone on one side of the EQUATION, usually, LHS.

SUBSTITUTE – put in place of.

SUFFICIENT – 'A' is a SUFFICIENT condition of 'B' if 'A $\Rightarrow$ B' (A implies B), so that if 'A' is satisfied, then 'B' takes place.

SUM – a (mathematical) EXPRESSION in which the LAST operation (see the ORDER OF OPERATIONS) is addition, e.g. $a + b$ is a SUM, and so is $a(b + c) + ed$.

TERM – a (mathematical) EXPRESSION that is added to another (mathematical) EXPRESSION, e.g. $a + b$ is a SUM of two TERMS, a and b.

TRANSPOSE EQUATION, FORMULA, IDENTITY – make a particular unknown the subject of EQUATION, FORMULA, IDENTITY, so that it stands on its own in the LHS or RHS of the corresponding mathematical statement.

UNKNOWN – a VARIABLE whose value or expression can be found by solving an EQUATION, e.g. in equation $x + 2 = 3$, x is an UNKNOWN.

VALUE – a number.

VARIABLE – a mathematical quantity that can take a range of (numerical) values and is represented by a mathematical symbol, usually a Latin letter, usually from the second part of the alphabet. If there is only one VARIABLE, the preferred choice for its symbol is x. The second choice is y and the third, z. If there are more variables, then one chooses, in the order of preference, letters u, v, w, s, t, r, p and q, then the upper case letters in the same order of preference. If more VARIABLES are required, we make use of subscripts, superscripts and Greek letters.

Part VI   Study Skills for Maths

Assuming that you have **an average background** in mathematics you need to study these notes on your own for **6 hours each week**:

1. **Spend half an hour revising the Summary or Summaries suggested for Self Study.** You should be able to use Order of Operations, algebraic operations and Decision Trees very fast. Do not forget to keep consulting the **Glossary**.
2. **Spend 2.5 hours revising previous Lectures and Solutions to Exercises.**
3. **Spend 1.5 hours studying the latest Lecture** (see tips below on how to do that).
4. **Spend 1.5 hours doing the exercises given in that lecture for self-study.**

Some of you need to study 12 hours a week. Then multiply each of the above figures by two!

This is how to study each new lecture:

1. Write down the topic studied and list all the subtopics covered in the lecture. Create a flow chart of the lecture. This can be easily done by watching the numbers of the subtopics covered.

   For example in Lecture 1 the main topic is ALGEBRA but subtopics on the next level of abstraction are 1.1, 1.2, …, 1.3.

   Thus, you can construct the flow chart which looks like that

ALGEBRA

variables   operations on variables   general remarks   glossary

whole   addition,
natural   subtraction
integer   multiplication
rational   division
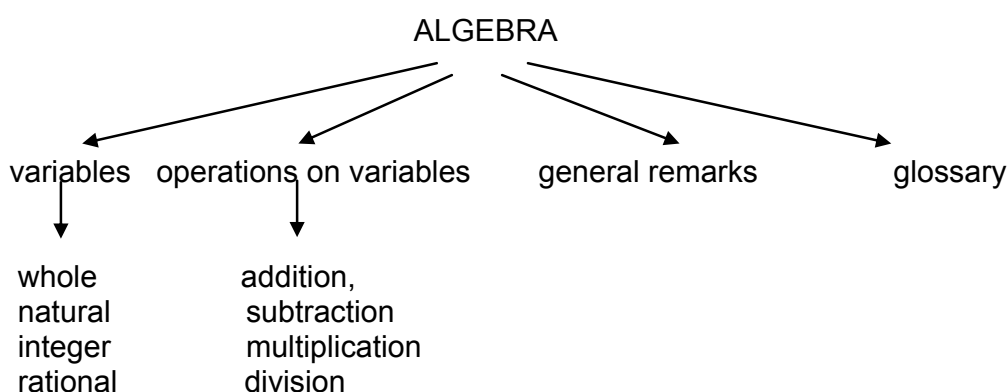
2. Write out the glossary for this lecture; all the new words you are to study are presented in bold red letters.
3. Read the notes on the first subtopic **several times** trying to understand how each problem is solved

4. Copy the first problem, put the notes aside and try to reproduce the solution. **Be sure in your mind that you understand what steps you are doing**. Try this a few times before checking with the notes which step is a problem.
5. Repeat the process for each problem.
6. Repeat the process for each subtopic.
7. Do exercises suggested for self-study

Here are a few **tips on how to revise for a mathematics test or exam:**

1. Please study the **Summaries** first.
2. Keep consulting the **Glossary**.
3. Then study **Lectures and Solutions** to relevant Exercises one by one in a manner suggested above.
4. Then go over **Summaries** again.

Here is your **check list**: you should be thoroughly familiar with

1. The **Order of Operations** Summary and how to "make invisible brackets visible".
2. The **Algebra Summary**, in particular, how to **remove brackets, factorise and add fractions**. You should know that **division by zero is not defined**. Rules for logs are secondary. You should know what are integers and real numbers. You should not "invent rules" with wrong cancellations in fractions. You should not invent rules on changing order of operations, such as addition and power or function. You should all know the precise meaning of the words **factor, term, sum and product**.
3. The concept of **inverse operation**.
4. The **Decision Tree for Solving Simple Equations**.
5. The **formula for the roots of the quadratic equation** and how to use them **to factorise any quadratic**.
6. The diagrammatic representation of the function (see the **Functions Summary**). You should know that a function is an operation (or a chain of operations) plus domain. You should know the meaning of the words argument and domain. You should know what is meant by a real function of real variable (real argument). You should know the precise meaning of the word constant – you should always say "constant with respect to (the independent variable) x or t or whatever…"
7. How to do function composition and decomposition using **Order of Operations**
8. How to use graphs
9. How to sketch elementary functions: the straight line, parabola, exponent, log, sin and cos
10. The **Trigonometry Summary**
11. The approximate values of e (≈ 2.71) and π (≈ 3.14)

12. What is j ($= \sqrt{-1}$) and what is j^2 ($= -1$)

13. The Cartesian and exponential form of a complex number and how to represent a complex number on the Argand digram (the **Complex Numbers Summary**)

14. How to add, multiply, divide complex numbers, raise them to integer and fractional power

15. The **Differentiation Summary**

16. The **Integration Summary**

17. The **Sketching by Simple Transformation Summary**

18. The **Limits Decision Tree** and basic indeterminate forms

19. Sketching by analysis

20. The definition of a **mean of a function on an interval**

Part VII   Teaching Methodology (FAQs)

Here I reproduce a somewhat edited correspondence with one of my students who had a score of about 50 in his Phase Test and 85 in his exam. You might find it instructive.

Dear Student

The difference between stumbling blocks and stepping stones is how you use them!

Your letter is most welcome and helpful. It is extremely important for students to understand the rationale behind every teacher's decision. All your questions aim at the very heart of what constitutes good teaching approach. For this reason I will answer every one of your points in turn in the form of a Question – Answer session:

Q: I have understood the gist of most lectures so far. However there have been a number of lectures that towards the end have been more complicated and more complex methods were introduced. When faced with the homework on these lectures I have really struggled. The only way I have survived have been to look at the lecture notes, Croft's book, Stroud's book and also various websites.

A: Any new topic has to be taught this way: simple basic facts are put across first and then you build on them. If you understand simple facts then the more sophisticated methods that use them seem easy. If they do not this means that you have not reached understanding of basics. While in general, reading books is extremely important, at this stage I would advise you to look at other books only briefly and only as a last resort, spending most of the time going over the lectures over and over again. The problem with the books available at this level is that they do not provide too many explanations. LEARNING IS A CHALLENGIG AND UNINTUITIVE POCESS. IF YOU BELIEVE THAT YOU UNDERSTAND IT DOES NOT MEAN THAT YOU DO!

Q: Many of the homework questions are way beyond the complexity of any examples given in lectures. Some are or seem beyond the examples given in books.

A: None of them are, although some could be solved only by very confident students who are already functioning on the level of the 1st class degree. There are four important points to be aware of here:

1. If you have not reached the 1st class level yet, it does not mean that you cannot reach it in future.

2. 1st class degree is desirable to be accepted for a PhD at elite Universities, others as well as employers are quite happy with 2.1.

3. It is absolutely necessary for students to stretch themselves when they study and attempt more challenging problems than they would at exams, partly because then exams look easy.

4. Even if you cannot do an exercise yourself, you can learn a lot by just trying and then reading a solution.

Q: This has been and continues to be demoralising.

A: A proper educational process is a painful one (no pain no gain!), but it also should be enlightening. One of the things you should learn is how to "talk to yourself" in order to reassure yourself. One of the things that I have been taught as a student and find continually helpful is the following thought: "Always look for contradictions. If you find a contradiction (that is, see that there is something fundamentally flawed in your understanding) – rejoice! Once the contradiction is resolved you jump one level up in your mastery of the subject (problem)." In other words, you should never be upset about not understanding something and teach yourself to see joy in reaching new heights.

Q: Take this week as an example. Exercises suggested in Lecture 1 look to me like 'integration by parts 'which I learned in Croft's book. However in your notes these seem to be 'separate terms'. There were not any examples given in the lecture on ' integration by parts' or by ' separate terms ' and I am not sure if we were meant to be using the 'substitution method', if this is even possible.

A: This example illustrates very well why at this stage reading books brings more harm than good. There are no questions in Lecture 19 which require "integration by parts", only the techniques that have been discussed in that Lecture, that is, "separate terms" rule, "constant factor out" rule and "substitution" method. It is true that there were no examples in the lectures on how to "separate terms". This omission is due to the fact that sometimes when during a lecture, I omit very simple steps. They are included into the written lectures.

Q: I also feel that the pace has been too high; we spent only about an hour on sequences and 2 weeks on integration.

A: We have only 22 weeks in one year to cover the necessary material, and it is spread out accordingly. However, you do not **need** to know more about sequences than was already discussed.

Q: When revising each week it is most unnatural to have to take your mindset back a week or two to try to remember what you have learned at a certain stage. I really do not think that many do it.

A: This question touches on one of the most fundamental aims of education – development of long term memory. Both short-term memory and long-term memory are required to be a successful student and a successful professional. When I ask you to memorise something (and say that this is best done by going over the set piece just before going to bed) I am exercising your short-term memory. How can you develop a long-term memory, so that what we study to-day stays with you – in its essence – for ever? The only way to do that is by establishing the appropriate connections between neural paths in your brain. If you have to memorise a sequence of names, facts or dates there are well established techniques promoted in various books on memory. They suggest that you imagine a Christmas tree or a drive-in to your house, imagine various objects on this tree or along the drive-way and associate the names, facts or dates with these objects. However, this technique will not work with technical information. What you need to establish are much deeper – meaningful – connections. The only way to do this is to go over the same material again and again, always looking at it from a new vantage¬ point. While your first intuitive reaction is that "it is most unnatural to have to take your mindset back a week to try to remember what you have learned at a certain stage", this is the only proper way to learn a technical subject and develop your long term memory.

Q: Related to this is the strange system where only the specified method can be used to derive an answer. At our level I feel that any method which produces the correct answer should be accepted. If you have been used to doing something one way and are forced to change then, for students that are a bit weak anyway, this will be a problem.

A: This is actually a classical educational technique, aiming at two things at once:

1. practicing certain methods and techniques,

2. developing students' ability to "work to specs".

People who do not come to terms with this idea are going to have problems with the exam questions where the desired techniques are specified. They will lose most of the marks if they use another technique.

Q: The principles based on the first principles I feel are out of the question at our level in S1 and should be left to S2 as a minimum.

A: These proofs are above the A-level, but are definitely the 1st year level. Everyone should be able to start these proofs on the right note, namely write out the appropriate definitions and then SUBSTITUTE the appropriate functions. Only the students who reached the 1st class level are expected to finish these proofs using various tricks, but everyone should be able to follow the proofs as given by their teachers in Solutions to Exercises and learn how the tricks work. Also, remember, that in real life you will hardly ever differentiate or integrate using differentiation or integration techniques, but remembering the definitions (a derivative is a local slope or a local rate of change and integral is a signed area between a curve and a horizontal axis) might come very handy in your engineering life.

Q: Techniques should be introduced into the whole system to help build confidence, although I realise that there is a balance to be struck.

A: Techniques are introduced according to the internal logic of the material. But confidence building is important and this is something teachers and students have to work at together. Teachers unfortunately have little time for that, all we can do is keep saying "good, good" when progress is made. You spend more time with yourself, so please keep reminding yourself that Exercises are only there to help to learn. What is important is that you are constantly stretching yourself. Please keep reminding yourself how much you achieved already. Surely, there are lots of things you can do now that you could not even dream of doing before. An extremely important educational point that you are touching upon here is the following: the so-called liberal system of education that was introduced in the 60s (and consequences of which we all suffer now) provided only "instant gratification". What the real education should be aiming at is "delayed gratification". You will see the benefits of what you are learning now – in their full glory – LATER, in year 2 and 3, not to-day.

Hope this helps!