Econodynamics

New Economic Windows

For further volumes:
www.springer.com/series/6901

Vladimir N. Pokrovskii

Econodynamics

The Theory of Social Production

Second Edition

Springer

Professor Vladimir N. Pokrovskii
Institute of Chemical Physics
Russian Academy of Sciences
Kosygin St 4
Moscow 117977
Russia
vpok@comtv.ru

ISSN 2039-411X e-ISSN 2039-4128
New Economic Windows
ISBN 978-94-007-2095-4 e-ISBN 978-94-007-2096-1
DOI 10.1007/978-94-007-2096-1
Springer Dordrecht Heidelberg London New York

Library of Congress Control Number: 2011937695

Cover design: eStudio Calamar S.L.

Printed on acid-free paper

Springer is part of Springer Science+Business Media (www.springer.com)

Preface

While studying and teaching *Methods of Mathematical Modelling of Economic Processes*, I have been confused about some discrepancies between various parts of the economic theory. There was an impression that the economic theory exists in independent fragments. Especially upsetting for me, a person who began the study of economic theory with *Das Kapital*, was the fact that Marx's theory seems to have no concern in mainstream economics.

I realised later that I was not the sole person to feel a deep dissatisfaction with the situation with the economic theory and its ability to describe reality. To say nothing of the numerous papers, there are many books devoted to a critique of mainstream economics (Nelson and Winter, 1982; Kornai, 1975; Beaudreau, 1998; Keen, 2001). There is a special online *Real-World Economics Review* (http://www.paecon.net/) opposing the mainstream theories. The people who are engaged in ecology are traditionally confronting the conventional economic thinking and are looking for physical terms to explain the phenomena of production (Costanza, 1980; Odum, 1996). Some physicists are trying to find new approaches to the analysis of economic situations (Mantegna and Stanley, 1999).

This book contains no critique of any theories. It is devoted to understanding the principles of production and contains a consecutive exposition of the technological theory of social production, which can also be understood as the theory of production of *value*. In the foundation of the theory are laid the achievements of classical political economy. The labour theory of value is completed, after Marx's hints in *Das Kapital*, with *the law of substitution*. The latter states that, when interpreting value, one has to consider that the workers' efforts in the production of things are substituted with the work of production equipment. A new important concept of *substitutive work, as a value-creating production factor*, was introduced and used to formulate the appropriate theory. The adequacy of the theory has been tested by using historical data for the U.S. economy.

The book is written by a physicist for the scientifically literate reader who wishes to understand the principles of the functioning of a national economy. The book contains a discussion of conventional models (Leontief's input–output model, the classical Walras market theory and others) and can be considered as a textbook for students of various specialities who have the necessary preparation in physics and mathematics and a desire to study economic problems. I think the monograph could be interesting for energy specialists, who are engaged in planning and analysing the production and consumption of energy carriers, and for economists, who want to know how energy and technology are affecting economic growth.

The appropriate formulation of the theory has a long history. This monograph was launched, in fact, as a revision and enlargement of my book *Physical Principles in the Theory of Economic Growth*, issued by Ashgate Publishing in 1999. However, it appears that the proper description of the theory has required the text to be completely rewritten and new material to be added, so that I have the opportunity to present a new book with a new title. I have used this edition to clarify the concepts and methods of the theory as far as it was possible for me at the moment.

I am grateful to many people who support and encourage me in my work. I especially would like to separate a few persons, with whom I have had the opportunity to discuss many relevant topics: Robert Ayres, Bernard Beaudreau, Sergio Ulgiati, Andre Maisseu, Michail Gelvanovskii, Grigorii Zuev and Irina Kiselyeva. Some issues became clearer for me after a discussion on the *generalised labour theory of value* with members of the *Socintegrum forum* (http://socintegrum.ru/); I am thankful especially to Valerii Kalyuzhnyi and Grigorii Pushnoi. Finally, I would like to express special thanks to my editors Maria Bellantone and Mieke van der Fluit at Springer.

Moscow, Russia Vladimir N. Pokrovskii

References

Beaudreau, B.C.: Energy and Organization: Growth and Distribution Reexamined, 1st edn. Greenwood Press, Westport (1998)
Costanza, R.: Embodied energy and economic valuation. Science **210**, 1219–1224 (1980)
Keen, S.: Debunking Economics: The Naked Emperor of the Social Sciences. Pluto Press, Sydney (2001)
Kornai, J.: Anti-equilibrium. On Economic Systems Theory and the Tasks of Research, 2nd edn. North-Holland, Amsterdam (1975)
Mantegna, R.N., Stanley, H.E.: An Introduction to Econophysics: Correlations and Complexity in Finance. Cambridge University Press, Cambridge (1999)
Nelson, R.R., Winter, S.A.: An Evolutionary Theory of Economic Change. Belknap Press of Harvard University Press, Cambridge (1982)
Odum, H.T.: Environmental Accounting. Emergy and Environmental Decision Making. Wiley, New York (1996)

Contents

Notation and Conventions[1]

A	input-output matrix with components a_i^j;
B	capital-output matrix with components b_i^j;
$B = \dfrac{Y}{L}$	labour productivity;
E	primary energy used in production;
E_P	primary substitutive work used in production;
I	gross investment in production system;
I_j	gross investment of product j;
I^i	gross investment in sector i;
I_j^i	gross investment of product j in sector i;
K	value of production equipment in production system;
K_j	value of production equipment of kind j in production system;
K^i	value of production equipment in sector i;
K_j^i	value of production equipment of kind j in sector i;
L	labour in production system;
L^i	labour in sector i;
M	amount of circulating money;
M_0	amount of circulating paper money;
N	number of population;
p	price of substitutive work as a production factor;
p_j	price of product j;
P	substitutive work
P^j	substitutive work in sector labelled j;
Q_j	quantity of product j in natural units;
R_j	value of stock of non-material product j;

[1]Latin suffixes take values $1, 2, \ldots, n$ and numerate products and sectors. As a rule, the upper suffix numerates sectors, the lower suffix numerates products. The rule about summation with respect to twice repeated suffixes is sometimes used.

The chapter number and the number of a formula in the chapter are shown in references to formulae.

S entropy;
t time;
$U(\cdot)$ utility function, welfare function;
$u(\cdot)$ subjective utility function;
W value of national wealth;
W_j value of national wealth of kind j;
w price of labour, wage;
X_j gross output of product j;
Y final output, gross domestic product;
Y_j final output of product j;
Z^i production of value in sector i;
α technological index;
α^i technological index in sector i;
$\beta = \dfrac{\Delta Y}{\Delta L}$ marginal productivity of labour at $P = const$;
β_i marginal productivity of labour in sector i;
$\gamma = \dfrac{\Delta Y}{\Delta P}$ marginal productivity of substitutive work at $L = const$;
γ_i marginal productivity of substitutive work in sector i;
$\delta = \dfrac{1}{K}\dfrac{dK}{dt}$ rate of real growth of capital stock;
$\tilde{\delta}$ rate of potential growth of capital stock;
ε substitutive work requirement;
$\bar{\varepsilon} = \varepsilon \dfrac{K}{P}$ non-dimensional technological variable;
ε^i substitutive work requirement in sector i;
$\bar{\varepsilon}^i = \varepsilon^i \dfrac{K^i}{P^i}$ non-dimensional technological variable for sector i;
η rate of real (effective) growth of substitutive work;
$\tilde{\eta}$ rate of potential growth of substitutive work;
Θ index of labour productivity growth;
λ labour requirement;
$\bar{\lambda} = \lambda \dfrac{K}{L}$ non-dimensional technological variable;
λ^i labour requirement in sector i;
$\bar{\lambda}^i = \lambda^i \dfrac{K^i}{L^i}$ non-dimensional technological variable for sector i;
μ rate of capital depreciation;
ν rate of real (effective) growth of labour;
$\tilde{\nu}$ rate of potential growth of labour;
$\xi = \dfrac{\Delta Y}{\Delta K}$ marginal productivity of capital;
$\xi^i = \dfrac{\Delta Z}{\Delta K^i}$ sectoral marginal productivity;
Ξ marginal productivities tensor with component $\xi^i_j = \dfrac{\Delta Y_j}{\Delta K^i}$;
ρ price index;
τ time of technological rearrangement.

Chapter 1
Introduction: Concept of Value and Production Factors

Abstract It is enough to look at the contents of economic courses to become easily convinced that the common thing for all of them is 'a substance' of value. It is convenient to use the name economic dynamics (econodynamics) for the discipline. It investigates the processes of emergence, motion and disappearance of value, just as hydrodynamics investigates processes of motion of liquids; electrodynamics, those of changing electric and magnetic fields; thermodynamics, processes connected with the motion and conversion of heat. In this chapter, the concept of value is reviewed, and the role of basic production equipment, as a set of sophisticated devices which allow human beings to attract energy from natural sources for the production of useful things, is discussed.

1.1 A National Economy at a Glance

The enormous growth of the human population through the centuries is connected with special features of the population. In contrast to any other biological population inhabiting the Earth, humans have invented highly sophisticated artificial means of supporting their own existence, while developing a great level of co-operation of members of their society. Since Palaeolithic times, clothing, shelter and fuel have become necessities of life almost as fundamental as food itself. Since Palaeolithic times the organisation of human society has also been progressing.

Modern society presents itself as a huge hierarchal organisation, including the government, firms, banks, colleges, libraries and so on. It is a very complex organisation, and every one of the members of the society, in some way, is included in the system. The society, as an economic system, produces everything that is needed for survival of the community: both the means for supporting human existence and the means for generating such support.

A huge amount of artificial things are accumulated by societies: buildings, transport routes, bridges, production equipment, energy supply systems, sanitation systems and so on. Aside from the tangible things, a society accumulates a great amount of intangible objects: knowledge of the laws of nature, principles of organisation of society, items of literature and arts and so on. Both the tangible and intangible constituents of *the wealth of the society* are equally important for maintaining the existence of human communities.

V.N. Pokrovskii, *Econodynamics*, New Economic Windows 12,
DOI 10.1007/978-94-007-2096-1_1, © Springer Science+Business Media B.V. 2012

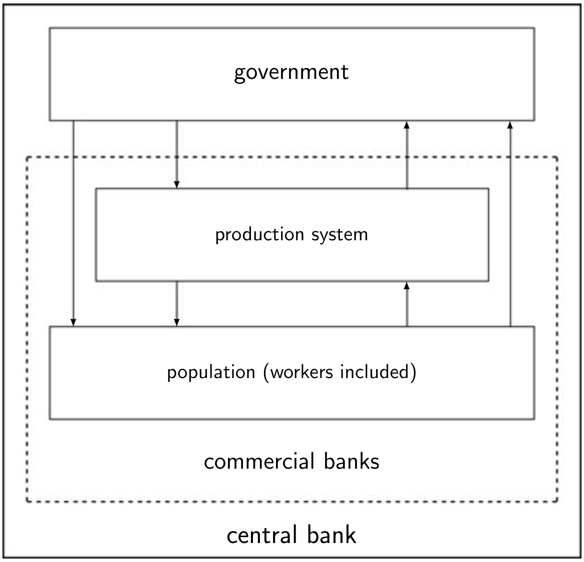

Fig. 1.1 The architecture of a national economy. The central bank and commercial banks create a money medium for the activity of economic agents. The production system creates all products and generates the fluxes of products to workers and between production units. The fluxes of money, depicted in the picture, are moving in opposite directions. Households are buying products, and money is returning to the producers. The government receives its part of produced value in the form of taxes, which, in different amounts, are returning to the economic agents. Each flux of money is a result of negotiation and agreement between corresponding agents

All (tangible and intangible) objects have been created by *the production system* of the society, which includes firms, plants, institutes, schools and so on. The production system takes minerals and ores from the environment, transforms natural substances into finished and semi-finished things, the latter are transformed into other things and so on, until all this is finally consumed, and the substances are returned into the environment as waste. This is the material side of production.

To discuss the mechanism of motion of products, one needs to consider human beings, and their desire to consume and, consequently, to produce. In developed societies, man does not consume only those products which he produces. The exchange of products, which, in fact, is the exchange of efforts, is a general phenomenon in modern societies. Man exchanges his services for an intermediate product—*money*—and then exchanges the money for products he wants. Therefore, simultaneously with the motion of products, one discovers the motion of money, which has to be considered as a separate, special product. The money is circulating in the economy, providing the exchange of products. Modern money is paper money and records on the accounts in the central and commercial banks and, thus, is inherently useless. Modern money is nothing more than a certificate that its owner has a right to get a certain set of products. The value of modern money derives only from the fact that it can be exchanged for the product.

The real production and *the money system* are intervened with each other, thus one can think that an appropriate description can be achieved when these phenomena are studied together. One can consider the production system and population as being immersed in a money system of the society, as shown schematically in Fig. 1.1. The money medium is created by the government, the central bank and many commercial banks. The central bank issues the bank notes and coins—the primary money—which is distributed to the commercial banks. The mechanism of issuing assumes that all paper money is circulating among economic subjects: practically no paper money is contained in commercial banks. The central bank also provides commercial banks with credits, so that the commercial banks can provide the customers with credit money. The records on the accounts of customers are non-paper money, which are created by the commercial banks. The central bank and commercial banks introduce an uncertain amount of the circulating money in coins, bank notes and cashing deposits in the system consisting of the government and the many customers of the commercial banks.

The subject of discussion in the proposed monograph is a theory of the social production system, and the latter is represented by many interacting enterprises. The architecture of the production system appears complex, but in a simple approach the production system can be considered as a set of the interacting pure sectors. In the most elementary case the production system can be considered as the only sector. This heuristic model of the society allows us to develop the theory of the production system in a simple, so-called macroeconomic approach.

1.2 The Concept of Value

The notion of *product* appears to be one of the fundamental concepts of economic theory. It can be defined as something which is produced to be consumed. It does not matter whether the moment of consumption coincides with the moment of production as, for example, in the case of transport services, or does not coincide. In the latter case the product exists for some time in its material or non-material form. Also it is insignificant whether the product is intended to satisfy the needs of the producer or is prepared for sale.[1]

[1]Let us pay attention to the distinction of the concepts of *a product* and *a commodity*. The latter is defined as something that is made for sale that is for an exchange at which value is disposed. From here some people wrongly conclude that the thing made for the producer's consumption does not possess value. This statement has been rejected by Marx [1, Chap. 1, Sect. 4]: "Since Robinson Crusoe's experiences are a favourite theme with political economists, let us take a look at him on his island. Moderate though he be, yet some few wants he has to satisfy, and must therefore do a little useful work of various sorts, such as making tools and furniture, taming goats, fishing and hunting. Of his prayers and the like we take no account, since they are a source of pleasure to him, and he looks upon them as so much recreation. In spite of the variety of his work, he knows that his labour, whatever its form, is but the activity of one and the same Robinson, and consequently, that it consists of nothing but different modes of human labour. Necessity itself compels him to apportion his time accurately between his different kinds of work. Whether one kind occupies a

According to the statements of the researchers,[2] the product can be considered as the unity of *use-value* and *production-value*, which allows products to participate in the processes of exchange. In the exchange, the products oppose each other, and the use-value of one product stands against the use-value of another. Products with various use-values can be compared due to the fact that the production-values of all products differ only in quantity, not in quality. Thus, the property that allows the products to be compared and exchanged is their *exchange value* or just *value*, which is an attribute of a product, just as mass is an attribute of matter.

One believes that the products are exchanged on average according to their values. This is an axiom which gives a relative measure of value, and allows one to ascribe a certain quantity of value to the products and to estimate the value of a set of products. Value is measured in conventional money units, which are set when the recognised means of circulation (money) are introduced into the economic system (see Chap. 3). Due to the overall exchange with the help of the money, all commodities can be evaluated, and this is considered as an estimation of their *value* in arbitrary money units (dollars, pound sterlings, euros, etc.). One can estimate, for example, a multitude of services and things produced by a nation for a year. This quantity is called the Gross Domestic Product (GDP).

The mechanism of exchange has been scrutinised. Some scholars emphasised the demand side of the phenomenon and argued that there is no value without utility, so that value ought to be considered as a market estimate of the utility of a thing. Other scholars argued that there are some things (water and air, for example) which have utility without market value, and thus, to understand the meaning of value, one has to refer to the supply side and take into account the production costs of things. It was understood later (the contributions of Walras [3] and Marshall [4]

---

greater space in his general activity than another, depends on the difficulties, greater or less as the case may be, to be overcome in attaining the useful effect aimed at. This our friend Robinson soon learns by experience, and having rescued a watch, ledger, and pen and ink from the wreck, commences, like a true-born Briton, to keep a set of books. His stock-book contains a list of the objects of utility that belong to him, of the operations necessary for their production; and lastly, of the labour time that definite quantities of those objects have, on an average, cost him. All the relations between Robinson and the objects that form this wealth of his own creation, are here so simple and clear as to be intelligible without exertion, even to Mr. Sedley Taylor. And yet those relations contain all that is essential to the determination of value."

[2]Still Aristotle, analysing the exchange of various things, wrote "... all things that are exchanged must be somehow comparable" [2, Book 5, Sect. 5]. Marx [1, p. 14] wrote: "... when commodities are exchanged, their exchange value manifests itself as something totally independent of their use value. But if we abstract from their use value, there remains their value as defined above. Therefore, the common substance that manifests itself in the exchange value of commodities, whenever they are exchanged, is their value." The brief history and the analysis of concept of value are exposed, for example, by A.N. Usoff in a work "What is value" (http://www.usoff.narod.ru/Us4.htm, in Russian). Having begun with concepts of use-value and production-value, Usoff has shown how it is necessary to introduce the concept of value, free from the pre-prepared interpretations. Everyone who was studying in a higher educational institution in the USSR until 1990 knows the statement that 'value is the expenses of labour.' However, there is no indispensability to reduce concept of value to expenses of labour in advance. Factorial theories of value, that is the reduction of value to labour, capital and other universal factors of production, are considered in the following section.

used to be especially stressed) that both the cost of production (supply) and utility (demand) were interdependent and mutually determinant of the value of things. It had appeared to be fruitless to argue whether demand or supply determines value, as, in Marshall's words, "we might as reasonably dispute whether it is the upper or under blade of a pair of scissors that cuts a piece of paper, as whether value is governed by utility or costs of production."

The motion and transformations of products in an economy can be described as fluxes of value, which appears at the first touching the substances of nature with the hand of a human being, moves together with the material substance of a product, leaving its material form, transfers into other substances, and disappears at final consumption. The study of these processes is a subject of an empirical science that can be called *economic dynamics* (*econodynamics*). Econodynamics itself can be defined as a science which investigates the processes of emerging, moving and disappearing of value, and is hardly interested in its material carriers. The concept of value in econodynamics is as important as the concepts of energy and entropy in physics. Now we have the fragments of this science only, and one of the fragments—the theory of production—is described in this monograph.

Note that, due to some difficulties with the concept of value, modern scholars of economy are trying to avoid that concept; the concept of utility is used instead. The political economy of the nineteenth century has turned into the *economics* of our days, which is defined as "...the study of how societies use scarce resources to produce valuable commodities and distribute them among different groups" [5, p. 5].

Both econodynamics and economics study one and the same object: the national economy, whereas econodynamics, in contrast to economics, gives us the opportunity to restore the scientific traditions of studying the society.

1.3 Production System in More Detail

To create and maintain national wealth, that is, things that are useful for human beings, *a social production system* was invented and maintained by humans, and this is just what distinguishes human populations from other biological populations. The production system consists of many production units, such as enterprises, factories, plants and firms that create all the things that man needs. The investigation of the laws of production is one of the central issues of econodynamics.

From a material point of view, the process of production is a process of transformation of raw materials into finished and semi-finished goods, semi-finished goods into other semi-finished and finished goods and so on, until the finished commodities can finally be used by human beings. The products are always consumed by human beings, so the products always have to be created. Figure 1.2 shows the main constituents of the production-consumption system as it is imagined due to the remarkable achievements of the classical political economy and neo-classical economics. One can refer to Blaug [6] to follow the fascinating history of approaches to understanding and describing the economic production-consumption system.

1.3.1 The Law of Substitution

Any description of the production system of economy assumes that a specific motion takes place. The task of the production system is to change forms of matter, that is, to transform, for example, ores of different chemical elements into an aircraft, which can fly. One can observe how clay transforms into pots, how clay, sand and stone transform into buildings, how ores and raw materials transform into a car. To produce a good or a service, some specific work[3] must be done. Modern technologies assume that this work can be done by a human being himself and/or by some external sources, such as energy sources, simultaneously. To grind corn into flour, for example, one can use a hand mill, a water mill, a wind mill or a steam mill. In these cases, as in many others, the production equipment is some means of attracting external sources of energy (water, wind, coal, oil, etc.) to the production of things; the workers' efforts are substituted by the work of falling water, or wind, or heat. No matter who or what does the work, all of the work must be done to obtain the final result which should be compared with the consumed energy and the workers' efforts.

Different mechanisms and appliances are invented to perform the work. Some of these are handled by a man only, and some of them allow the man to attract energy from external sources. This is a material realisation of technology: production equipment.

It is possible that the first person to write about the functional role of machinery in production was Galileo Galilei. He realised that all machines transmitted and applied force as special cases of the lever and fulcrum principle. A prominent historian of science and technology, Donald Cardwell [7], wrote that Galileo in his notes *On Motion* and *On Mechanics* recognised that "the function of a machine is to deploy and use the powers that nature makes available in the best possible way for man's purposes... the criterion is the amount of work done—however that is evaluated—and not a subjective assessment of the effort put into accomplishing it" (pp. 38–39). The advantage of machines is to harness cheap sources of energy because "the fall of a river costs little or nothing."

The relevance of machinery to economic performance was clearly recognised by Marx [1], who described the functional role of machinery in production processes in Chapter XV, *Machinery and Modern Industry*, of *Das Kapital* as follows:

> On a closer examination of the working machine proper, we find in it, as a general rule, though often, no doubt, under very altered forms, the apparatus and tools used by the handicraftsmen or manufacturing workman: with this difference that instead of being human implements, they are the implements of a mechanism, or mechanical implements (pp. 181–182). The machine proper is therefore a mechanism that, after being set in motion performs with its tools the same operations that were formerly done by the workman with similar tools. Whether the motive power is derived from man or from some other machine, makes no difference in this respect (p. 182). The implements of labour, in the form of machinery,

[3] One can understand work as a process of conversion of energy in technological processes from one form to another, for example, from mechanical into thermal form.

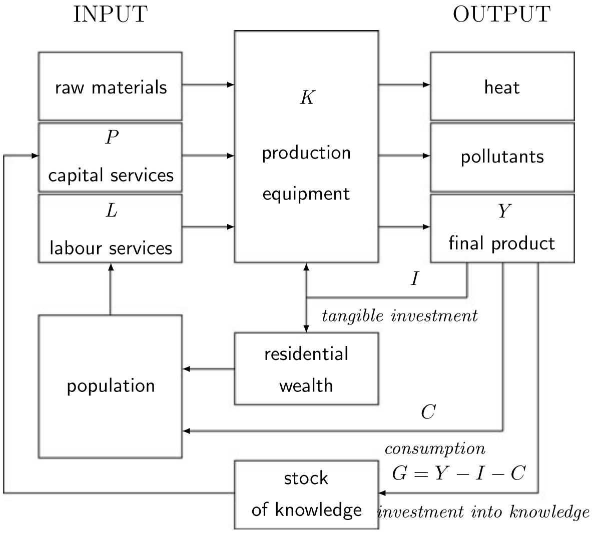

Fig. 1.2 Fluxes in the production–consumption system. To produce a thing or a service, apart from production equipment K, one needs raw materials (ores, water, air, energy carriers and so on), worker efforts L and some factor which can be conventionally called capital services P. The last factor is closely connected with production equipment—capital stock K, but different from it. Though capital services P can be considered formally as an independent production factor, it is hardly possible to find any other interpretation for it different from the amount of work of production equipment, which is done with the help of external energy sources instead of the workers' efforts. The output of the production process is a multitude of things and services, which are measured by their total value Y. A part C of final product Y is directly consumed by human beings, and a part I goes to enhancement of the production system through an increase in the stock of production equipment, so that the production system itself is a subject of evolution. The production processes are accompanied by the emergence of heat and pollutant fluxes, but this is another side of the problem, to which we shall not pay much attention in the monograph

> necessitate the substitution of natural forces for human force, and the conscious application of science instead of rule of thumb (p. 188). After making allowance, both in the case of the machine and of the tool, for their average daily cost, that is, for the value they transmit to the product by their average daily wear and tear, and for their consumption of auxiliary substances such as oil, coal and so on, they each do their work gratuitously, just like the forces furnished by nature without the help of man (p. 189).

Hence, both physicists and political economists recognised the important role of machinery in production processes as having to do with the *substitution of workers' efforts by the work of machines moved by external sources of energy*, while the extent of this substitution depends on the technology per se. It is important to keep in mind that while capital is a necessary factor input, work can only be replaced by work, or put differently, work cannot be replaced by capital.

Note that by contrast with Smith and Marx, who focused on physical labour, here and in the following text, we regard all possible energy-driven activities of workers including supervision of any kind, that is, the extended concept of labour (human capital) is used.

1.3.2 The Generalised Labour Theory of Value

Over the centuries researchers have tried to understand how things get value, or, in other words, to find a certain universal source of wealth, and the first candidate for this role was the land. Benjamin Franklin, known for his works on electricity, was one of the first to formulate the statement that a measure of value is the work spent by labourers [8]. This idea appears to be central in the political economy of the beginning of the nineteenth century and was especially developed in works of Adam Smith [9], David Ricardo [10] and Karl Marx [1]. These great scholars had no doubt that the production-value was equivalent to the employment of labour only, which gave foundation to *the labour theory of value*. According to Smith, "value of any commodity... to the person who processes it and who means not to use or consume it himself, but to exchange it for other commodities, is equal to the quantity of labour which enables him to purchase or command." According to Marx, "all commodities are only definite masses of congealed labour time." Every economist would agree that labour is the most important factor of production, but the situation appears to be more complicated. The production-value, generally speaking, does not reduce to the expenses of labour; something else should be added to the theory.

One can guess that the 'something' that is needed in the theory is Marx's phenomenon of 'the substitution of natural forces for human force.' Indeed, after understanding this phenomenon, Marx could suggest that it affects the mechanism of production of value. To understand how gratuitous work influences the value of the products, he could analyse the performance of two similar enterprises. He could suggest that the first of the enterprises uses a technology which requires some amounts of labour L and substitution work P, and, to produce the same quantity of the same product, the second enterprise uses a technology with the quantities $L - \Delta L$ and $P + \Delta P$ for production factors. So far as the products are considered to be identical, the exchange values of the products of either enterprise on the market are equal, despite the difference in labour consumption. Therefore, Marx could continue to argue, value cannot be determined by labour only, but the properly accounted work of natural forces ought to be considered. To produce the same quantity of value, the decrease in workers' efforts ought to be compensated by an increase in work of external sources, so that one can write the relation

$$-\beta \Delta L + \gamma \Delta P = 0,$$

where productivities β and γ of the corresponding production factors are introduced. Thus, equally with human efforts, the work of natural forces appears to be an important production factor. It is easy to see that the quantity β/γ determines the amount of gratuitous work of external sources which is needed to substitute for the unit of human work to get an equal effect in the production of value.

In the general case, the work performed by labour L and substitutive work P has to correspond to a set of products, which has the exchange value Y, and one can write, assuming that the production system itself remains unchanged, the relation between differentials of the quantities

$$dY = \beta\, dL + \gamma\, dP. \tag{1.1}$$

The coefficients $\beta > 0$ and $\gamma > 0$ correspond to the value produced by the addition of the unit of labour input at constant pure substitutive energy consumption and by the addition of the unit of work of production equipment at constant labour input, respectively; in line with the existing practise, these quantities can be labelled as marginal productivities of the corresponding production factors. The two production factors, the workers' efforts and the work of external sources of energy, can substitute for each other and, in this sense, be equivalent, so that labour is eventually, using Adam Smith's words, "the only universal, as well as the only accurate measure of value, or the only standard by which we can compare the values of different commodities at all times, and at all places."

The discussed mechanism of substitution formalises Marx's statements. Really, by substitution of a labourer's work by forces of nature, that is, by substitution of efforts of people by the work of external forces of nature using production equipment, work operates in a complex of workers' efforts plus work of the equipment. Thus, the work of machines can be appreciated only so far as this work does what people wish, replacing their efforts. Consequently, a measure of value, certainly, can be the labourers' work only. It is possible to say also, according to Marx, that only labourers' work creates value, but Marx, unfortunately, did not complete the theory of substitution to the logical end. Taking into account the effect of substitution, one can say that the only universal and accurate measure of value is the work of labourers or other agents used for production.

1.4 The Law of Production of Value

The material and non-material results of production—buildings and machinery, cars and planes and other things among which human beings live—are characterised by value, so that one can speak about both the production of things and the production of value. The classical and neo-classical traditions relate the production of value Y in money units to quantities of universal value-created factors, the so-called production factors which one needs to create a set of products. According to Smith, Ricardo and Marx, labour ought to be considered as the only value-creating factor, that is, output can be considered as a function of consumption of labour L

$$Y = Y(L). \tag{1.2}$$

However, it appears to be impossible to explain the growth of productivity of labour, that is, the growth of value of commodities produced by units of labour in units of time Y/L using this simple hypothesis. To explain empirical facts, other production factors (land and capital, first) in line with labour have been introduced.

1.4.1 Earlier Neo-classical Formulations

The amount of production equipment (measured by its value K) was taken as an important production factor in the frame of neo-classical economics, and output Y has been considered to be a function of two variables

$$Y = Y(K, L). \tag{1.3}$$

In relation (1.3) capital K and labour L are regarded as perfect substitutes for one another, that is, a given output can be achieved by any combination of the two factors, though, of course, there is a most efficient combination, depending on the prices of production factors. The specific form of function (1.3) was proposed by Cobb and Douglas [11]

$$Y = Y_0 \frac{L}{L_0} \left(\frac{L_0}{L} \frac{K}{K_0} \right)^{\alpha}, \tag{1.4}$$

where index α ought to be considered as a characteristic of the production system itself. Though many particular forms of the function (1.3) are known [12], function (1.4) has the advantage of not depending on the initial values of production factors and is often used for interpretation of phenomena of economic development.

The other tradition [13–16], in accordance with empirical facts, considers the output as a linear function of capital (or generalised capital)

$$Y = AK, \tag{1.5}$$

where 'capital' productivity A, due to empirical evidence, does not depend on production factors.

It is easy to see that the laws (1.4) and (1.5) are compatible only at the value $\alpha = 1$, which leads to exception of the expenses of labour in the law of production of value. Another paradox, described by Solow [17], exists: the theory based on the neo-classic production function (1.3) in any form does not include technological changes. Nevertheless, there has been a clear belief that, in recent centuries, technological progress was ultimately the source of economic growth in developed countries and should be incorporated into the theory of economic growth.

1.4.2 Amendments to the Neo-classical Formulation

To avoid the specified difficulties, it was suggested [17] to modify the concepts of labour and capital in function (1.3). An extra time dependence of function (1.3) (the so-called exogenous technological progress) has to be assumed. Otherwise, the arguments of function (1.3) must be considered, not as capital and expenditures of labour, but as services of the capital K' and work L'

$$Y = Y(K', L'). \tag{1.6}$$

The quantities K' and L' are capital and labour services which are connected with measured quantities of capital stock K and labour L, but are somewhat different from them. The concepts of labour and capital services appeared to be necessary and very useful to explain the observed growth of output [18, 19]. However, the problem of endogenous inclusions of technical progress in the theory remains unsolved, or, taking advantage of Solow's words [20], there remains a question: "*whether one has anything useful to say about the progress, in a form that can be made part of an aggregative growth model*."

In past decades, there have been some attempts to improve the neo-classical theory by including in the production function new variables such as technology, or human capital, or stock of knowledge H [21–25] or energy E [26–28]. It was assumed that output Y can be written as a function of three, or more, variables

$$Y = Y(K, L, H, E, \ldots).$$

There was a belief that the only thing one needs to solve the problem is to find a sufficient number of appropriate variables. However, the econometric investigations of over 90 different variables, proposed as potential growth determinants, did not give a definite result [29]. A review of the latest development of the neo-classical approach can be found in a book by Aghion and Howitt [30].

1.4.3 The Law in the Technological Theory

The theory considered in the monograph, which has been designed to consider the phenomenon of production of value, keeps the main attributes of the neo-classical approach, that is, the concept of value created by production factors (donor value) and the concept of production factors themselves; it can be regarded as a generalisation and extension of the conventional neo-classical approach, while the roles of production factors are revised. In the conventional, neo-classical theory, capital as a variable played two distinctive roles: capital stock as value of production equipment and capital service as a substitute for labour. In the technological theory, capital service is considered as an independent production factor, whereas capital stock is considered to be the means of attracting labour and energy services to the production. Human effort and the work of external energy sources are regarded as the true sources of value.

In line with the conventional production factors: capital stock K and labour L, the theory contains capital service P, as an independent production factor.[4] The

[4] An earlier formulation was presented by Beaudreau [31, 32], who accounts for work of the production equipment W and another factor named by the organisation which is considered as something distinct from work. To exclude a discussion of the process of transformation of consumed power carriers into work and the second-law efficiency of the process, for simplicity (it is mainly a technical problem which is not universal), work W is identical to the substitutive work or true work of the production equipment, which is discussed in this monograph. The organisation can correspond to workers' efforts L, as to the control, which is, apparently, also actual work requiring energy consumption. These two factors, accordingly, are called inanimate and animate work by Beaudreau [31, 32]. The output now can be considered as a function of two factors, $Y = Y(W, L)$,

production of value Y is considered as a function of the three production factors

$$Y = Y(K, L, P). \tag{1.7}$$

The production system itself may be viewed as a collection of equipment: *capital stock* (measured by its value K), acquiring its ability to act from *labour and capital services inputs*, that is, the amount of human effort L and the work of natural sources of energy (wind, water, coal, etc.) P, which substitutes human efforts in the production of commodities.

The third production factor—substitutive work P—provides the consistency in explaining the phenomenon of economic growth. It is important that this approach allow researchers to include characteristics of technology in the description and to formulate a phenomenological (macroeconomical, no price fluctuations are discussed) theory of production as a set of evolutionary equations in one-sector and many-sector approximations. The growth of production is demonstrated to be caused by achievements in technological consumption of labour and energy. This statement corresponds to a clear understanding that, in recent centuries, technological progress is ultimately the source of economic growth in developed countries. The technologies are changing, and these changes have appeared to be incorporated into the theory of economic growth to describe the empirical facts properly. From a physical point of view, the main result of technological progress is substitution of human energy by energy from external sources by means of different types of sophisticated equipment.

1.5 Energy and Production

The relationship between economic growth and energy consumed by the production system[5] is one of the most dramatic issues in economics. The spectrum of opinions on the relationship of energy with value is very broad. The majority of economists, who believe in the productive force of capital, consider energy (or more correctly: energy carriers) to be an ordinary intermediate product that contributes to the value of produced commodities by adding its cost to the price; in other words, consumption of energy is not a source of value. However, one can find many words and arguments in the literature in favour of a universal role of energy in economic processes (see, for example, [28, 31–42]). These researchers have long argued that energy must also be considered as a value-creating factor which must be introduced

but Beaudreau [31, 32] identifies output and primary work of the production equipment, including efficiency in the discussion. The output measured as an added market value depends, apparently, on the chosen unit of value, which should be set independently (see Sect. 10.3).

[5]It is customary to speak about energy consumption, though, for the sake of precision, the word *consumption* should be replaced by the word *conversion*. Energy cannot be *used up* in a production process. It can only be converted into other forms: chemical energy into heat energy, heat energy into mechanical energy, mechanical energy into heat energy and so on. The measure of potentially converted energy (work) is exergy.

into the list of production factors in line with other production factors. Moreover, some biophysicists are arguing that energy must be considered as the only source and measure of value [41, 42], and the concept of value itself can be reduced to the concept of energy.

All approaches to the inclusion of energy into the theory of production are known as *the energy theory of value*, which, nevertheless, does not have an accurate and complete formulation. Reviewing the development of the discussion, Mirovski [43, p. 816] concluded that "... the energy theory of value was never developed with any seriousness or concerted effort by any of the groups...." Despite further arguments and investigations performed in later years [44–46], up to now there are no conventional rules according to which one could calculate 'the energy content of a money unit' and test the hypothesis.

Of course, energy carriers (primary energy)[6] are quite similar to other intermediate products participating in the production process. Nobody can distinguish energy carriers, which are used in the production of aluminium, metallurgical operations and some chemical processes, from other intermediate products. In all these cases, the cost of energy is included in the cost of the final products. But in some cases, apart from being a product, energy from external sources is used to substitute for labour in the technological processes. Energy-driven equipment works in the place of manual labour; genuine work done by the production equipment acquires all the properties of a value-creating production factor, including the property to produce surplus value. The work that corresponds to a part of consumed energy carries—it is convenient to have a special name for it: *substitutive work* or *productive energy*—cannot be considered as an intermediate product only, but must be considered as a value-creating factor which has to be introduced into the list of production factors equally with other production factors.[7] The substitutive work has to be interpreted as genuine work done by production equipment with the help of external sources of energy instead of workers. This quantity can also be considered as capital service provided by capital stock.

At the cost of introducing the third production factor—substitutive work or productive energy—the discussed theory allows one to unravel the proper role of energy in production of value, on one side, and to eliminate the contradictions of conventional neo-classical theory, on the other side. I think that this monograph proposes

[6]Primary energy is the name for the amount of primary energy carriers (oil, coal, running water, wind and so on) measured in energy units. It is convenient to measure huge amounts of energy in a special unit *quad* (1 quad $= 10^{15}$ Btu $\approx 10^{18}$ joules), which is the unit usually used by the U.S. Department of Energy.

[7]Before the year 2000, it was realised that primary energy or total consumption of energy (or exergy) cannot be a proper production factor. In my book [35, pp. 62–63] I refer to a production factor called *final energy*, which, by its definition, is primary energy input times a coefficient of efficiency. The definition is completely equivalent to that of *useful work*, used by Ayres [47]. The growth rate of *final energy* differs from that of *primary energy* by 1.5%; it is the growth rate of efficiency of usage of primary energy. Later I realised [36] that it is substitutive work (not useful work) that has to be exploited as the production factor.

some reconciliation of contrasting points of view on the role of energy in production of value.[8]

1.6 Organisation of the Monograph

The monograph presents a general technological theory of production, which is based on the conventional terms and concepts of classical political economy and neo-classical economics, and on conventional physical principles and methods. Some main fundamentals and concepts of modern economics, illustrated with historical (1900–2000) data for the U.S. economy, are described in Chaps. 2 and 3 to introduce the scientifically literate reader, who has not study economics, into the language and problems of the economic theory and to facilitate him eventually to understand the contents of the book. The next chapters contain consecutive derivations of the theory of production, which is the main topic of this book.

The core of the formal theory itself is contained in Chaps. 5 and 6.[9] In one-sector approximation, the list of production factors contains two production factors of conventional neo-classical economics: capital K and labour L, and a new production factor: capital service or substitutive work P. Capital stock is considered to be the means of attracting labour and substitutive work to production, while human efforts and the work of external energy sources are considered as true sources of value. Due to the definition of the production factors, capital stock K and a certain combination of services L and P are complements to each other, while capital services (substitutive work) P and labour inputs L act as substitutes for each other. The properties of the production factors allow one to specify (see details in Chap. 6) the production function for output Y in the form of the two alternative lines

$$Y = \begin{cases} \xi K, \\ Y_0 \dfrac{L}{L_0}\left(\dfrac{L_0}{L}\dfrac{P}{P_0}\right)^{\alpha}. \end{cases}$$

[8]The introduction of energy can also be justified from a thermodynamic point of view. In terms of modern thermodynamics [48] all the artificial things, as well as all biological organisms and natural structures, ought to be considered as deviations from equilibrium in our environment, the latter being reasonably considered a thermodynamic system, and the process of production of useful things is the process of creation of far-from-equilibrium objects (the dissipative structures), as is explained by Prigogine with collaborators [48, 49] (see also Chap. 10). To create and support these structures in our environment, as in any thermodynamic system, the matter and energy fluxes must run through the system [48, 50]. In our case, energy comes in the form of human effort and the work of external sources which can be obtained by using the appropriate equipment. The system of a social production is the mechanism which involves a huge quantity of energy to transform ‘wild’ substances into useful things. The production of useful things can be connected with an establishment of the order (complexity) in the environment by human activity.

[9]The principles of the theory were discussed earlier in the author’s monograph [35], though some issues have been reformulated here. In particular, the concept of substitutive work was not clearly defined, and the important contribution to production of value from technological and structural changes was erroneously omitted. The correct version is given in the author’s article [36].

The complementary descriptions of production of value can be traced back. The first line in the above formula reminds us of the Harod–Domar approach [13–16], while the function in the second line coincides with the Cobb–Douglas production function (1.4), in which substitutive work P stands in the place of capital stock K. The productivity of capital stock ξ is an internal characteristic of the production system itself and is the 'sum' of the marginal productivities of labour and productive energy, so that capital productivity eventually determines efficiency of 'transformation' of performed work into value.

To complete the theory, one needs equations for the dynamics of the production factors, which allow one to investigate trajectories of development. The equations for the growth rate of capital K, labour L and substitutive work P are formulated in Chap. 5, and some characteristics of technology, namely, labour and energy requirements, λ and ε, that is, amounts of labour and substitutive work needed for unit of production equipment to be launched in action, are introduced. These quantities are combined to create the above introduced index α, thus, connecting it with characteristics of applied technology. The index α appears to be a technological index, which can also be interpreted as a share of capital services in the total expenses for maintenance of production factors.

Chapters 4 and 8 contain a generalisation of the theory for a many-sector system. The basis of the development is the well-known *linear input–output model*, described in Chap. 4. The model represents the production system as a set of coupled sectors, and each of them creates its own specific product. The applicability of the linear input–output model to dynamic situations (Chap. 8) is extended in proposed work, so the restrictions imposed by production factors (labour and substitutive work) and the evolution of the production system itself, that is, structural and technological changes, are taken into account. In fact, a phenomenological version of the evolutionary theory of the production system is formulated in these chapters.

In Chap. 7, the ability of the theory to describe a real situation is illustrated for the example of historical (1900–2000) statistical data for the U.S. economy. To identify the considered model, one needs in the empirical time series of output Y, capital K and labour L. A method of separating substitutive work P from the total primary consumption of energy, as well as a method of calculating the technological index α, appears to be an organic part of the theory. Besides, at the given time series for investment, one can estimate the technological characteristics of the production system. The comparison shows the consistency of the theory and its correspondence to empirical facts. The proposed theory can explain facts of economic growth, especially, the main fact of recent development, that output expansion has outpaced population growth in the 200 years since the industrial revolution. The theory has the means to describe the difference in productivity growth for different countries. Within empirical accuracy, the consistency is perfect, so that one can acquire a feeling that substitutive work or, more generally, capital service is the only missing production factor in the conventional two-factor theory of economic growth, and no other production factors, aside from capital, labour and substitutive work, are needed to describe the path of growth quantitatively. Perhaps the substitutive work is the same production factor that the scholars of the modern endogenous theory of economic growth have been seeking.

Chapters 9 and 10 represent an attempt to understand and interpret the very concept of value, which is a unique specific concept, as frequently used and important in economics as the concepts of ‘energy’ and ‘entropy’ in physics. The relationships among the thermodynamic concepts and economic concepts of value and utility are analysed. Reconciliation of the two points of view on the phenomenon of production leads to a unified picture that enables us to relate some aspects of our observations of economic phenomena to physical principles.

Concluding the description, the monograph investigates one of the main problems of economics—why do economies grow—and reconsiders the theory of production from a physicist’s point of view. The monograph contains a quantitative description of production as a social mechanism, embedded in the environment. The approach allows us to include characteristics of technology into the description and to formulate a phenomenological (macroeconomical, no price fluctuations are discussed) theory of production as a set of evolutionary equations in one-sector and multi-sector approximations.

References

1. Marx, K.: Capital. Encyclopaedia Britannica, Chicago (1952). English translation of Karl Marx, Das Kapital. Kritik der politischen Oekonomie, Otto Meissner, Hamburg, 1867. http://marxists.anu.edu.au/archive/marx/works/cw/index.htm. Accessed 17 March 2011
2. Aristotle (about 350 B.C.) Nicomachean ethics. http://classics.mit.edu/Aristotle/nicomachaen.5.v.html. Accessed 17 March 2011
3. Walras, L.: Elements d’economie politique pure ou theorie de la richesse sociale. Corbaz, Lausanne (1874)
4. Marshall, A.: Principle of Economics, 8th edn. MacMillan, London (1920)
5. Samuelson, P., Nordhaus, W.: Economics, 13th edn. McGraw-Hill, New York (1989)
6. Blaug, M.: Economic Theory in Retrospect, 5th edn. Cambridge University Press, Cambridge (1997)
7. Cardwell, D.S.L.: Turning Points in Western Technology. Neale Watson, Science History Publications, New York (1972)
8. Franklin, B.: A modest enquiry into the nature and necessity of a paper-currency. Electronic Text Center, University of Virginia Library, Philadelphia (1729). http://etext.lib.virginia.edu/modeng/modengB.browse.html. Accessed 17 March 2011
9. Smith, A.: An Inquiry into the Nature and Causes of the Wealth of Nations. Clarendon Press, Oxford (1976). In two volumes
10. Ricardo, D.: On the principles of political economy and taxation (1817)
11. Cobb, G.W., Douglas, P.N.: A theory of production. Am. Econ. Rev., Suppl. **18**(1):139–165 (1928)
12. Ferguson, C.E.: The Neo-classical Theory of Production and Distribution. Cambridge University Press, Cambridge (1969)
13. Harrod, R.F.: An essay in dynamic theory. Economic Journal **49**, 14–23 (1939)
14. Harrod, R.F.: Towards a Dynamic Economics. MacMillan, London (1948)
15. Domar, E.D.: Capital expansion, rate of growth and employment. Econometrica **14**, 137–147 (1946)
16. Domar, E.D.: Expansion and employment. Am. Econ. Rev. **37**, 343–355 (1947)
17. Solow, R.: Technical change and the aggregate production function. Rev. Econ. Stud. **39**, 312–330 (1957)

18. Jorgenson, D.W., Griliches, Z.: The explanation of productivity change. Rev. Econ. Stud. **34**, 249–283 (1967)
19. Jorgenson, D.W., Stiroh, K.: Raising the speed limit: U.S. economic growth in the information age. Brookings Pap. Econ. Act. **1**, 125–211 (2000)
20. Solow, R.: Perspective on growth theory. J. Econ. Perspect. **8**, 45–54 (1994)
21. Griliches, Z.: Issues in assessing the contribution of research and development to productivity growth. Bell J. Econ. **10**, 92–116 (1979)
22. Griliches, Z.: Productivity puzzles and R & D: another nonexplanation. J. Econ. Perspect. **2**, 9–21 (1988)
23. Romer, P.M.: Increasing returns and long-run growth. J. Polit. Econ. **94**, 1002–1037 (1986)
24. Romer, P.M.: Endogenous technological change. J. Polit. Econ. **98**, 71–102 (1990)
25. Lucas, R.E.: On the mechanics of economic development. J. Monet. Econ. **22**, 3–42 (1988)
26. Hudson, E.A., Jorgenson, D.W.: U.S. energy policy and economic growth, 1975–2000. Bell J. Econ. Manag. Sci. **5**, 461–514 (1974)
27. Berndt, E.R., Wood, D.O.: Engineering and econometric interpretations of energy—capital complementarity. Am. Econ. Rev. **69**, 342–354 (1979)
28. Kümmel, R.: The impact of energy on industrial growth. Energy **7**, 189–203 (1982)
29. Durlauf, S., Quah, D.: The new empirics of economic growth. In: Taylor, J., Woodford, M. (eds.) Handbook of Macroeconomics. North-Holland, Amsterdam (1999)
30. Aghion, P., Howitt, P.: The Economics of Growth. MIT Press, Cambridge (2009)
31. Beaudreau, B.C.: Energy and Organization: Growth and Distribution Reexamined, 1st edn. Greenwood Press, New York (1998)
32. Beaudreau, B.C.: Energy and Organisation: Growth and Distribution Reexamined, 2nd edn. Greenwood Press, New York (2008)
33. Cottrell, W.F.: Energy and Society: The Relation Between Energy, Social Change and Economic Development. McGraw-Hill, New York (1955)
34. Maïsseu, A., Voss, A.: Energy entropy and sustainable development. Int. J. Glob. Energy Issues **8**, 201–220 (1995)
35. Pokrovski, V.N.: Physical Principles in the Theory of Economic Growth. Ashgate Publishing, Aldershot (1999)
36. Pokrovski, V.N.: Energy in the theory of production. Energy **28**, 769–788 (2003)
37. Beaudreau, B.C.: Energy Rent. A Scientific Theory of Income Distribution. Lincoln, New York (2005)
38. Hall, C., Lindenberger, D., Kümmel, R., Kroeger, T., Eichhorn, W.: The need to reintegrate the natural sciences with economics. Bioscience **51**, 663–673 (2001)
39. Ayres, R.U., Ayres, L.W., Warr, B.: Exergy, power and work in the U.S. economy, 1900–1998. Energy **28**, 219–273 (2003)
40. Ayres, R.U., Ayres, L.W., Pokrovski, V.N.: On the efficiency of electricity usage since 1900. Energy **30**, 1092–1145 (2005)
41. Costanza, R.: Embodied energy and economic valuation. Science **210**, 1219–1224 (1980)
42. Cleveland, C.J., Costanza, R., Hall, C.A.S., Kaufmann, R.: Energy and the U.S. economy: a biophysical perspective. Science **225**, 890–897 (1984)
43. Mirowski, P.: Energy and energetics in economic theory: a review essay. J. Econ. Issues **22**, 811–830 (1988)
44. Odum, H.T.: Environmental Accounting. EMERGY and Environmental Decision Making. Wiley, New York (1996)
45. Valero, A.: Thermoeconomics as a conceptual basis for energy-ecological analysis. In: Ulgiati, S. (ed.) Advances in Energy Studies Workshop. Energy Flows in Ecology and Economy. Porto Venere, Italy, 1998. MUSIS, Rome, pp. 415–444 (1998)
46. Sciubba, E.: On the possibility of establishing a univocal and direct correlation between monetary price and physical value: the concept of extended exergy accounting. In: Ulgiati, S. (ed.) Advances in Energy Studies Workshop. Exploring Supplies, Constraints, and Strategies. Porto Venere, Italy, 2000. Servizi Grafici Editoriali, Padova, pp. 617–633 (2001)
47. Ayres, R.U.: Sustainability economics: where do we stand? Ecol. Econ. **67**, 281–310 (2008)

48. Prigogine, I.: From Being to Becoming. Time and Complexity in the Physical Sciences. Freeman, New York (1980)
49. Nicolis, G., Prigogine, I.: Self-organisation in Non-equilibrium Systems: From Dissipative Structures to Order Through Fluctuations. Wiley, New York (1977)
50. Morowitz, H.J.: Energy Flow in Biology. Biological Organisation as a Problem in Thermal Physics. Academic Press, New York (1968)

Chapter 2
Empirical Foundation of Input–Output Model

Abstract To describe the performance of a production system, one uses basic terms and notions which were introduced by many researchers during the long period of development of economic theory. In this chapter, the terms needed to describe the phenomenon of social production and economic growth are introduced and discussed. The main chain of definitions is as follows: *product–output–investment–stock of production equipment*. The latter is a set of the real means of production: the collection of tools and all energy-conversion machines, including information processing equipment, plus ancillary structures to contain and move them. The term *capital stock* is applied for the value of the stock of production equipment (the means of production). In this and the following chapters, time series of some quantities for the U.S. economy, which are collected in Appendix B, are used for illustration.

2.1 On the Classification of Products

To be able to describe the internal processes in an economy in some detail, we need to focus on a variety of production units, and we also need some classification of products. Further, we shall use the assumption that all outputs of the production units can be divided into n classes, which allows us to consider n products, circulating in a national economy [1–4]. Following this tradition, one can assume that all enterprises of the production system can be divided into n classes as well. Therefore, we imagine, following Leontief [2], that the production system of an economy consists of n production sectors, each of them producing only one product. In fact, in reality it is not that simple to divide the production system of the economy into production sectors or, more exactly, into *pure production sectors* [2]; nevertheless, the scheme appears to be fruitful for a theoretical analysis.

The division of the economy into sectors can vary; the number of sectors depends on the aims one is pursuing. For the current description and planning, the economy can be divided into no more than a few hundred sectors. An example of a working classification can be found in Appendix A. For research aims, the economy can be divided into a few sectors only [2, 3]. As an example, one can consider a simple model of the production system of an economy consisting of three sectors as follows.

V.N. Pokrovskii, *Econodynamics*, New Economic Windows 12,
DOI 10.1007/978-94-007-2096-1_2,

1. The first sector deals with resources for material production of goods and provides the equipment and all the material products necessary for production. This sector devours natural resources and uses its own products and products of the second sector to produce the means of production. The sector includes extraction of raw material (ore, stone, coal, oil, etc.), construction, transportation, manufacturing of cars, appliances for homework and furniture, etc. One can see that the activities with codes 21 and 23 (Appendix A) must be included in this sector.
2. The second sector produces non-material information products, i.e., general knowledge and various instructions on how to organise a matter for human use. Instructions are partially embodied in the performance of the production system, another part exists in a non-material form as postponed messages, forming the huge collection of *information resources*. This sector includes the scientific and project institutes and deals with creation of principles of organisation: science, research and development, design and experimental works, art, management, a financial system and computer programs, so that the activities with codes 51, 52, 54, 71 and 92 (Appendix A), for example, must be included in this sector. It is necessary to note that non-material production, i.e., principles of organisation, software and results of research works, should be connected with material production, as they are useless if they are not consumed.
3. The third sector produces the things which human beings need directly. This sector includes the food processing industry, agriculture, retail, restaurants, hotels, healthcare and so on. Examples of businesses belonging to this sector are activities with codes 11, 44, 45, 62 and 72 (Appendix A). We can say that one needs the first two sectors only to keep the third sector in action. Strictly speaking, human beings do not need the products of the first two sectors directly.

Note that the first and the third sectors above are those sectors of production which were introduced by Marx [5], as the sector of the means of production and the sector of production of commodities. Following Smith, Marx considered that workers who, according to the above classification, are engaged in the second sector do not create value, so there was no need for him to consider the second sector. The necessity of introducing this sector was recognised by Tougan-Baranovsky [6] and Bortkiewicz [7], who believed that the additional sector makes luxury goods. For a complete description of the production system, it is necessary to consider the interaction of tangible and intangible products. For the description to be complete, all production enterprises should be included in one of these three sectors, although one can see that it is difficult to locate some of the activities listed in Appendix A.

Let us note that, in addition to the sector classification, some groups of products can also be selected according to the aims and modes of their consumption. Some products can be used to produce other products [4]. If things are used for production many times, as, for example, instruments and tools, machinery, means of transport, agricultural land and so on, one speaks of *fixed production assets*. One speaks of *intermediate production consumption*, if products, for example, coal, oil and ore, are disappearing in the production processes. Products for *final consumption* by human beings comprise products which are used as final products many times, e.g., residential buildings, furniture and so on (*residential assets*), and products which disappear at consumption, like food, for example. Sometimes it is difficult to decide whether a product (for example, roads and buildings) ought to be classified as production assets or as residential wealth.

2.2 Motion of Products

Consider an economy as consisting of the production sectors, each of them producing its own product. The important characteristic of a sector is its *output*, that is, the amount of product created by the sector in a time unit

$$dQ_i, \quad i = 1, 2, \ldots, n.$$

These quantities are measured in natural units such as tons, meters, pieces and so on. We do not discuss here the difficulties which appear when many primary natural products are aggregated in the only product of a sector.

To compare the quantities of different products, an empirical estimation of *value of product* is used. Measures or scales of value are conditional monetary units, such as the rouble, dollar and others.[1] Neglecting fluctuations, which are the accidental deviations of quantity from some mean value, one defines the value of a unit of a product in arbitrarily chosen units as its *price*. We assume that the prices, as empirical estimations of value, for all products are known

$$p_i, \quad i = 1, 2, \ldots, n.$$

The price of a product is not an intrinsic characteristic of the product. The price depends on the quantities of all products which are in existence at the moment. As a rule, the price decreases if the quantity of the product increases, though the situation can be more complicated. Note that there are coupled sets of products, such that an increase in the quantity of one product in a couple is followed by an increase (in the case of a couple of complementary products) or a decrease (in the case of a couple of substituting products) in the price of the other product of the couple. Therefore, one ought to consider the price of a product to be a function of quantities of, generally speaking, all products

$$p_i = p_i(Q_1, Q_2, \ldots, Q_n). \tag{2.1}$$

One can define the *gross output* of a sector i as the value of the product created by the sector labelled i for a unit of time

$$X_i = p_i\, dQ_i, \tag{2.2}$$

[1] The assessment and comparison of value of the various products existing in various points in time is complicated by the lack of a constant scale of value. As known financier Lietaer [8, p. 254] writes: ‘The world has been living without an international standard of value for decades, a situation which should be considered as inefficient as operating without standard of length or weight.’ The absence of a constant scale of value is a headache, both for experts and for analysts.

so that the gross output of the economy appears to be a vector with n components

$$\mathsf{X} = \begin{Vmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{Vmatrix}.$$

2.2.1 Balance Equations

To create the product of a sector, apart from fixed production capital, it is necessary to use the products of, generally speaking, all the sectors. For example, to produce bread, apart from an oven, it is necessary to have flour, yeast, fuel and so on. Therefore, the gross output of each sector is distributed among the others

$$X_i = \sum_{j=1}^{n} X_i^j + Y_i, \quad i = 1, 2, \ldots, n, \tag{2.3}$$

where X_i^j is an amount of the product labelled i used for production of the product labelled j. The *intermediate production consumption* of the products is determined by the existing technology and does not include consumption of the basic production assets. The residue Y_i is called the *final output*, which is the value of the products used for productive and non-productive consumption beyond the current production processes. It will be discussed later.

On the other hand, the value of the output of a sector i is the sum of the values of products consumed in the production and an additive term

$$X^i = \sum_{j=1}^{n} X_j^i + Z^i, \quad i = 1, 2, \ldots, n. \tag{2.4}$$

This relation defines the quantity Z^i which is called the *production of value* in sector i. One can consider that every sector creates value. The first terms on the right-hand sides of relations (2.3) and (2.4) represent products which are swallowed up by the acting production sectors.

The final output of the sectors Y_i characterises production achievements of the society. For this purpose, it is convenient to use the sum

$$Y = \sum_{j=1}^{n} Y_j. \tag{2.5}$$

This is the value of all the material and non-material products created by a society per unit of time (year). We call it the *Gross Domestic Product* (*GDP*), if we are considering a national economy.

Table 2.1 Balance of products

Gross output	X^1	X^2	$\dots$	X^n	Final output
X_1	X_1^1	X_1^2	$\dots$	X_1^n	Y_1
X_2	X_2^1	X_2^2	$\dots$	X_2^n	Y_2
$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$
X_n	X_n^1	X_n^2	$\dots$	X_n^n	Y_n
Production of value	Z^1	Z^2	$\dots$	Z^n	Y

One can sum relations (2.3) and (2.4) over the suffixes and compare the results to obtain

$$Y = \sum_{j=1}^{n} Z^j. \tag{2.6}$$

It means that the GDP is equal to the production of value in all production sectors of the economy.

The quantities incorporated in formulae (2.3)–(2.6) can be conventionally represented by a balance table (Table 2.1). All quantities in the table should be replaced by numbers in order for a real economy to be analysed.

When we take into account that some products can be objects of import and export from other countries (international trade), the production balance changes a little. In this case it is necessary to subtract an export part from the gross product of each sector $T_i^{\uparrow}$ and to add import quantity of the product $T_i^{\downarrow}$, so that the balance parity (2.3) is recorded in the modified form

$$X_i + T_i^{\downarrow} - T_i^{\uparrow} = \sum_{j=1}^{n} X_i^j + Y_i, \quad i = 1, 2, \dots, n, \tag{2.7}$$

where X_i^j is the part of the product with index i which is used for production in sector j. The difference between import and export can be used both for intermediate production consumption and for final consumption. The residual Y_i, called the final product, presents the value of the products used beyond current processes for productive and non-productive consumption.

On the other hand, the value of a product X_i can be presented as the sum of value of the products consumed by production, and some additive term, which is presented by (2.4). This parity defines the quantity of value Z^i, created in sector i. One can suppose that each sector creates value, as the production equipment takes part in the production.

Summing up relations (2.4) and (2.7) on indexes i and comparing the results, one obtains, instead of (2.6),

$$Z = Y + T^{\uparrow} - T^{\downarrow}. \tag{2.8}$$

In this case, the value created by the production system $Z = \sum_{j=1}^{n} Z_j$ is referred to as the GDP, which is used for productive and non-productive consumption $Y = \sum_{j=1}^{n} Y_j$ and pure export $T^{\uparrow} - T^{\downarrow}$.

2.2.2 Distribution of the Social Product

To move further, it is necessary to consider the main constituents of both the value of the final products created in sectors Y_i, and the production of value in sectors Z^i. The last quantity was considered by Marx, who called it the social product and supposed that the production of value in each sector can be broken into wages V^j, surplus product M^j and value of the production assets disappearing in the process of production A^j; consequently,

$$Z^j = V^j + M^j + A^j, \quad j = 1, 2, \ldots, n. \tag{2.9}$$

Both the wages V^j and the surplus product M^j can be used for direct consumption or for the further development of production.

One of the major characteristics of the sector functioning is the rate of profit, defined as

$$\frac{M^j}{A^j + V^j}, \quad j = 1, 2, \ldots, n. \tag{2.10}$$

According to Marx, because of the aspiration of separate manufacturers to profit, these quantities tend to accept identical values; however, actually alignments of rates of profit are not observed.

The final product Y_j, defined by the balance equation (2.3), is used both for direct consumption and for maintenance and expansion of the production system of an economy. Consequently we can present a vector of the final product as the sum of three vectors

$$Y_j = I_j + G_j + C_j, \quad j = 1, 2, \ldots, n, \tag{2.11}$$

where C_j stands for the value of products which are consumed by people directly and immediately (one-time consumption), G_j designates the value of intermediate products (material and non-material) not consumed and not used in production and I_j designates gross investments (with inclusion of amortisation expenses) in the basic production equipment (fixed capital). It is believed that all quantities are estimations of values of actual fluxes of products. Certainly, some of the components of fluxes I_j, G_j and C_j can be set equal to zero.

The situation becomes simpler if we refer to the three-sector model described in Sect. 2.1. We assume that storage of intermediate material products can be neglected here, so, instead of relation (2.11), we have

$$Y_1 = I, \qquad Y_2 = G, \qquad Y_3 = C, \tag{2.12}$$

where I is investment in stock of residential and non-residential assets, G is investment in stock of knowledge and C is one-time human consumption. So, the total final output can be represented as a sum of the three components

$$Y = I + G + C. \tag{2.13}$$

2.2.3 Gross Domestic Product

The Gross Domestic Product (GDP) represents a measure of the current achievements of an economy as a whole—a measure of a multitude of fluxes of products. The equations recorded in the previous section show the various methods of calculating the GDP, which can be estimated as the results of production, that is, the value of created products (2.5), or by the account of the use of products (see (2.5) and (2.11)), or by the contribution of separate components of the created value (see (2.6), (2.8) and (2.9)). Using a similar foundation, methods of an assessment of GDP, based on a system of national accounts,[2] have been developed under the patronage of the United Nations.[3]

When an arbitrary monetary unit of value is chosen, the GDP can be estimated for a given point in time in an uncontested way. However, due to possible changes of the money units, there is a question of how to compare the GDPs for various years. Assuming that values of equivalent sets of products for various years are identical, one finds a parity between monetary scales at various points in time [10]. The monetary unit, established in this way, possesses the property to have constant purchasing capacity, but has nothing to do with a parity of value at various points in time. When a monetary unit of constant purchasing capacity is used, inflation is excluded, but with variation of productivity, the value content of the monetary unit changes in due course.

As an illustration, the GDP of the U.S. economy measured in different scales of value is shown in Fig. 2.1. The direct assessment of the progress of a social production is made in current monetary units; for the U.S. economy, the dependence of the directly estimated total product in current monetary units can be approximated by the exponential function

$$\hat{Y} = 19.965 \times 10^9 \cdot e^{0.0518t} \text{ dollar/year.}$$

Here, time t is measured in years, beginning ($t = 0$) at 1900. After some tedious procedures [10], the directly estimated quantity $\hat{Y}$ can be transformed into an assessment of GDP in the monetary scale of constant purchasing capacity Y. In this

[2]The System of National Accounts 1993. http://unstats.un.org/unsd/nationalaccount/sna.asp.

[3]An interesting description of the history of approaches to the estimation of the GDP for various nations was given by Studenski [9].

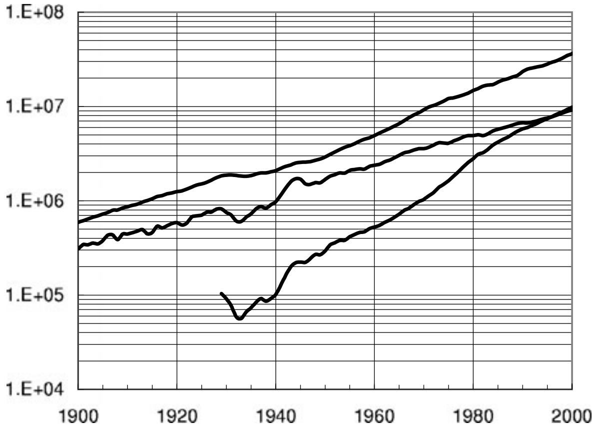

Fig. 2.1 Production of value in the U.S. economy. The *lower curve* depicts GNP in millions of current dollars, the *middle one* in millions of dollars for year 1996. The *latter curve* shows real income of the society in money units of constant purchasing power and can be approximated by an exponential function (2.14). The *upper curve* presents values of GNP measured in millions of energy units, taken as 50000 J (see Sect. 10.3)

case, the time dependence of GDP (the middle curve of Fig. 2.1) can be approximated by the exponential function

$$Y = 1.69 \times 10^{12} \cdot e^{0.0326t} \text{ dollar}(1996)/\text{year}. \tag{2.14}$$

Time t is measured in years, and $t = 0$ corresponds to year 1950. The upper curve of Fig. 2.1 depicts the real change of production of value with a constant money scale, which is introduced in Chap. 10 (Sect. 10.3).

The ratio of the output in the current money units to the output in the constant purchasing power money units defines *the price index*

$$\rho(t) = \hat{Y}/Y.$$

The actual price index is a pulsing quantity, but, with the above assessments, it is possible to see that the average price index for the U.S. has increased (since 1950) as

$$\rho \sim e^{0.0192t}.$$

The purchasing capacity of the monetary unit of the U.S.—the dollar—decreases as an inverse quantity. Each holder of the dollar in 1950–2000 has been losing annually nearly 2% of its purchasing capacity, which is, in fact, an implicit tax in favour of an emitter. In the third chapter, we shall return to the discussion of money units and price index.

Though the time dependence of GDP is smooth, consideration of the rate of growth $\frac{1}{Y}\frac{dY}{dt}$ shows a pulsating character in the progress of production. On the chart of Fig. 2.2 it is possible to see that the period of pulsations of the rate of growth of

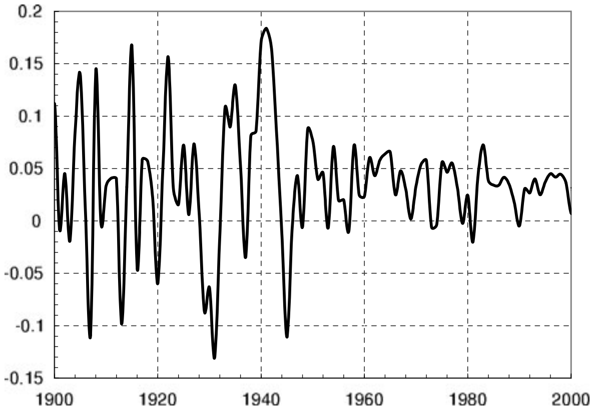

Fig. 2.2 The rate of growth of the U.S. GDP. The rate of growth of the GDP for the U.S. economy shows a pulsating character of production

GDP takes about four years. We shall return to the discussion of the reason for the pulsations in the seventh chapter (Sect. 7.3.2).

2.2.4 *Constituents of Gross Domestic Product*

2.2.4.1 Investments in the Production Equipment

One recognises a set of products as investments, both material and non-material, if the products are not intended for immediate consumption and are kept for use in production. In the material form, the investments are buildings, cars and the various equipment sets in various sectors. A part of a sector output is distributed over sectors, so it is possible to define quantity I_j^i as a part of a product j invested in sector i and to consider investments as a matrix with components I_j^i

$$\mathsf{I} = \begin{Vmatrix} I_1^1 & I_1^2 & \dots & I_1^n \\ I_2^1 & I_2^2 & \dots & I_2^n \\ \dots & \dots & \dots & \dots \\ I_n^1 & I_n^2 & \dots & I_n^n \end{Vmatrix}. \tag{2.15}$$

The quantities I_j^i apparently cannot be chosen arbitrarily, and the society works out the mechanisms of the choice of investments. When the development of an economy is planned, which is possible in the case where all means of production basically belong to the state, the choice has a directive character: the special state body centrally makes decisions about investment that define the future assortment and volumes of goods and services. When the market economy is reined, and the means of production belong to various proprietors, including the state, each proprietor itself defines the investment decision, and therefore the future production is determined spontaneously.

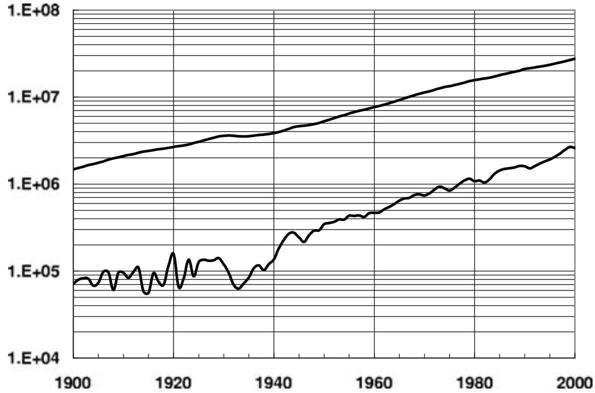

Fig. 2.3 Investment and capital in the U.S. economy. Estimates of value of material national wealth K (*upper curve*) and value of investment I (*lower curve*) are given, according to Appendix B, in million dollars for year 1996. The growth of capital can be approximated by the exponential function (2.29)

One can define investment of type j in all sectors as

$$I_j = \sum_{i=1}^{n} I_j^i, \quad j = 1, 2, \ldots, n.$$

Quite similarly, we can calculate the gross investment of all products in sector i as

$$I^i = \sum_{j=1}^{n} I_j^i, \quad i = 1, 2, \ldots, n.$$

The gross investment in the entire production system is now defined as

$$I = \sum_{i=1}^{n} I^i = \sum_{j=1}^{n} I_j = \sum_{i,j=1}^{n} I_j^i. \tag{2.16}$$

One can find very good estimates of investment I for the U.S. economy (see Appendix B). The time dependence of the gross investment for the entire economy is shown in Fig. 2.3.

2.2.4.2 Personal Consumption

The consumption C is defined as the value of the products which are consumed by humans immediately (one-time consumption). Perhaps a proper estimate of this quantity could be the minimum amount of products which are needed in order for humans to subsist. To characterise the necessary consumption, it is convenient to use the poverty threshold used in the U.S. statistics. The estimates of this quantity for a person in different family situations since year 1959 can be found on the U.S. Census Bureau website.[4] One can consider the poverty threshold per person in a

[4]http://www.census.gov/hhes/poverty/histrov/hstpov1.htm.

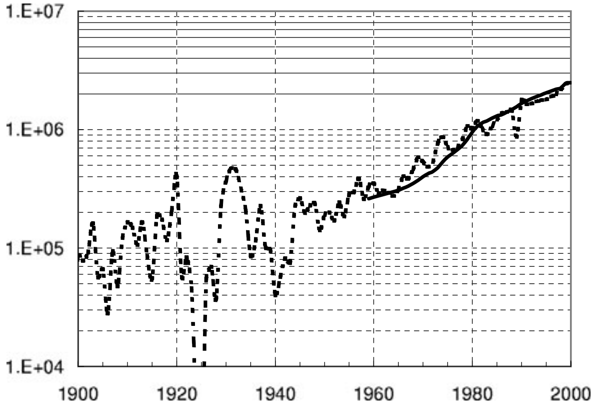

Fig. 2.4 Personal consumption in the U.S. economy. Estimates of value of personal consumption $C = cN$ are given in millions of dollars for year 1996. The *solid line* is based on direct estimates of the poverty threshold by the U.S. Census Bureau; the *dashed line* presents the results of calculation due to (6.33)

one-person family to give a realistic estimate of the current consumption. For year 1996, for example, this quantity is estimated as 7995 dollars per person per year. This quantity ought to be multiplied by the number of population to get the lower estimate of the consumption in year 1996 as $C = 2{,}120$ billion dollars. The time dependence of the personal consumption is depicted in Fig. 2.4. On the other hand, one can use (6.33) for the cost of labour and the estimated (in Sect. 7.1.2) values of the technological index to calculate the personal consumption. The results for the U.S. in the twentieth century are shown in Fig. 2.4 by the dashed line.

One can consider consumption as the most important part of the GDP. 'Every man is rich or poor according to the degree in which he can afford to enjoy the necessaries, conveniences, and amusements of human life.' [11, p. 47].

2.2.4.3 Fluxes of Non-material Products

Many employees in different sectors of the production system create and distribute different messages. But there are some businesses, such as education, science and R&D, publishers, theatres, TV, cinema, post, law services, statistics, consulting companies and so on, for which the main activity is the creation and distribution of different messages. One calls these sectors the information sectors. The product of these sectors is a great amount of messages, informative or not; it depends on the recipient. Therefore, one cannot say that the product of the information sectors is information. Some messages are never read; they are waiting for the recipients in depositories such as libraries. Some of the messages are received by many recipients, and for some of them the messages carry no information. Some messages certainly carry valuable information for the recipients, e.g., instructions on how to use the energy of running water as a work horse, and the instructions on how to organise matter to be used as a transport vehicle or an appliance. Some of the messages lose their value, and some disappear, but for many years society has stored a great deal of messages—information resources.

The total amount of produced services on the creation and distribution of different messages in the U.S. economy was estimated by Machlup [12] as 29% of the

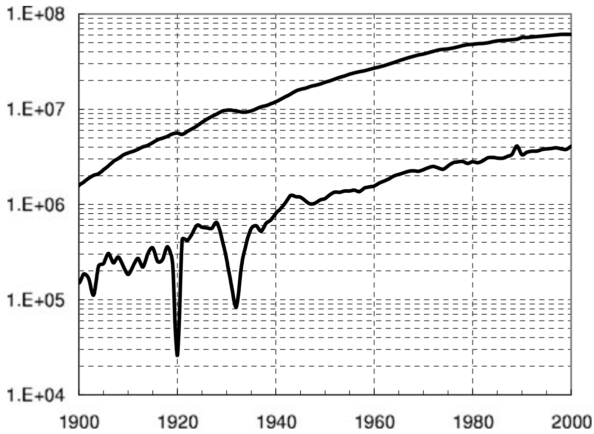

Fig. 2.5 Non-material products in the U.S. economy. Values of non-material products $G = Y - I - C$ (*lower curve*) are calculated from known values of output Y, investment I and averaged values of consumption C (see Figs. 2.1, 2.2, 2.3). Values of non-material national wealth R (*upper curve*) are calculated according to (2.28), whereas depreciation coefficient is assumed to take the same values as for material products. All quantities are given in million dollars for year 1996

Gross National Product (GNP) for year 1959 and as 46% of the GNP for year 1967. For recent times one can easily get an estimate of the non-material information product G from formula (2.13). For example, one has estimates for year 1996: GNP $Y = 7{,}813$, material investment $I = 2{,}054$ and the current consumption $C = 2{,}120$ billion 1996 dollars. Thus, one can get the estimate for the non-material information product $G = Y - I - C = 3{,}638$ billion 1996 dollars, which is about 47% of the GNP. The time dependence of the flux G is depicted in Fig. 2.5.

The value of the achievements of science, research and projects is essential and cannot be ignored. The information products are considered to be important for society (because much effort is spent to produce them), and the share of the information products in the GNP apparently does not decrease.

2.2.4.4 Principles of Distribution of Products

All three parts of the final product for the U.S.: investment, personal consumption and storing of information products, are comparable, and it seems possible that the final output of any society is distributed among the three parts in approximately equal fractions. The distribution certainly experiences some operating influences from the society, and it would be interesting to determine whether there exists a principle which governs such a division. One of the main questions to understand is: What are the rules to determine a splitting of the final output into three parts?

The future amounts of production, consumption and information products depend on today's investments. At any moment of time a society has to decide what part of the final product ought to be consumed and what part ought to be saved for the sake of future consumption. One can imagine two alternative approaches to the problem: one from the side of consumption and the other from the side of production. Some models (see, for example, [13]) determine investment as a result of maximisation of present and future consumption. In Chap. 5, we discuss how investment can be determined from the side of production.

2.3 The National Wealth

Every society holds a huge stock of material and non-material products—the national wealth—which, in a natural form, is a set of objects, both tangible (buildings, networks of supply, machinery, transport means, furniture, home appliances and so on) and intangible (principles of the organisation of the matter and society, works of art and literature and other things).

2.3.1 Assessments of the Stored Products

The value of the material and non-material parts of the national wealth can be estimated, if one estimates *pure investments*, which are *gross investments* minus the value of the products, that disappear for the same unit of time (value of depreciation)

$$\frac{dK_j}{dt} = I_j - \mu K_j, \tag{2.17}$$

$$\frac{dR_j}{dt} = G_j - \mu R_j. \tag{2.18}$$

Here I_j and G_j are gross investments representing the increase of material and non-material wealth per unit of time. We assume that investments become productive instantaneously. The second terms in relations (2.17) and (2.18) describe the depreciation of national wealth due to wearing and ageing.

Equations (2.17) and (2.18) introduce the stocks of products: K_j is the value of the material assets including basic production equipment (production capital); R_j is the value of the storage of intermediate production materials including the stock of knowledge. It is difficult to give an exact estimate of these amounts, because some of these products disappear very quickly, but others keep their value for centuries. Apparently, estimates of the stocks K_j and R_j depend on the choice of the second terms on the right-hand side of (2.17) and (2.18). One can assume, for simplicity, that the depreciation is proportional to the amount of national wealth with one and the same coefficients of depreciation μ for all products in all situations.

The above relations allow one to represent the components of the national wealth in the following form:

$$K_j(t) = \int_0^\infty e^{-\mu x} I_j(t-x)\,dx, \tag{2.19}$$

$$R_j(t) = \int_0^\infty e^{-\mu x} G_j(t-x)\,dx. \tag{2.20}$$

One can see that the national wealth represents accumulated investments, especially investments of the recent past, as the earlier produced commodities disappear. The quantity

$$k_j(t, t-x) = e^{-\mu x} I_j(t-x)$$

is a part of the existing fixed production capital, which was introduced during a unit of time at the moment of time $t - x$. This quantity is the smallest part of capital stock which can be considered in macroeconomic theory.

Relations (2.17), (2.18) and (2.19), (2.20) connect with each other two kinds of quantities: fluxes I_j, G_j and stocks K_j, R_j. Only one set of quantities, namely, fluxes, can be estimated directly. The other quantities, stocks, are usually calculated in value units. But this does not mean that stocks are theoretical constructs; they are realities, which can be measured by natural units of products. However, apparently it is difficult to give a precise direct assessment of value of the stored products, especially non-material products.

The total value of the national wealth is a sum of the quantities which were defined above

$$W = \sum_{j=1}^{n} (K_j + R_j). \tag{2.21}$$

The national wealth consists of products which were produced at different moments of time and under different conditions of production, which implies different bygone current prices. The value of national wealth W is a characteristic of the set of the products which depends of the history of bygone prices. In other words, the value of national wealth cannot be a function of amounts of products. However, we can introduce such a function for a set of products or a function of a state—the utility function—which is closely related to value (see Chap. 10, Sect. 10.2). The utility function U replaces the non-existing value function in theoretical considerations.

2.3.2 Structure of Fixed Production Capital

The national wealth is created by the production system of the economy, which is a real engine of the economic system, and production capital, which was considered very thoroughly by many researchers, appears to be a very important part of national wealth. Note that different approaches to the concept of capital stock can be accepted. In a wider sense, capital stock includes all material national wealth; in a narrower sense, the concept of capital stock can be understood as the value of basic production equipment, one can say, the core production capital. To illustrate application of the theory, we shall apply the wider concept of capital stock.

The accumulation of invested products (2.15) determines the production capital (capital stock) via the equation

$$\frac{dK^i_j}{dt} = I^i_j - \mu K^i_j, \quad i, j = 1, 2, \dots, n, \tag{2.22}$$

where K^i_j stands for value of production equipment of type j in sector i. One can see that the production equipment can be considered as a matrix with components K^i_j

$$\mathsf{K} = \begin{Vmatrix} K^1_1 & K^2_1 & \dots & K^n_1 \\ K^1_2 & K^2_2 & \dots & K^n_2 \\ \dots & \dots & \dots & \dots \\ K^1_n & K^2_n & \dots & K^n_n \end{Vmatrix}. \tag{2.23}$$

The total amount of product of type j in all sectors is defined as

$$K_j = \sum_{i=1}^{n} K^i_j, \quad j = 1, 2, \dots, n.$$

Quite similarly, we can calculate the total amount of the production capital in sector i as

$$K^i = \sum_{j=1}^{n} K^i_j, \quad i = 1, 2, \dots, n.$$

The production capital of the whole economy is now defined as

$$K = \sum_{i=1}^{n} K^i = \sum_{j=1}^{n} K_j = \sum_{i,j=1}^{n} K^i_j. \tag{2.24}$$

One can sum (2.22) over suffixes i or j to obtain equations for the dynamics of the total amount of equipment labelled j and for the dynamics of the fixed capital in sector i, correspondingly,

$$\frac{dK_j}{dt} = I_j - \mu K_j, \quad j = 1, 2, \dots, n, \tag{2.25}$$

$$\frac{dK^i}{dt} = I^i - \mu K^i, \quad i = 1, 2, \dots, n. \tag{2.26}$$

Remember that all dynamic equations in this section are valid for the case where the depreciation is proportional to the amount of national wealth with one and the same coefficients of depreciation μ for all products in all situations. Generally speaking, coefficients of depreciation are different for different equipment in different sectors.

2.3.3 Estimates of Fixed Production Capital

Formulae (2.17), (2.18) give a basis for approximate formulae, according to which the separate parts of the national wealth can be estimated. In a simple case, when one considers the three-sector model described in Sect. 2.1, (2.17) and (2.18) reduce to equations for two components of national wealth: stock of basic equipment K and stock of knowledge and projects R

$$\frac{dK}{dt} = I - \mu K, \tag{2.27}$$

$$\frac{dR}{dt} = G - \mu R. \tag{2.28}$$

It is easy to see that, at the given fluxes I and G, the calculated amounts of components of national wealth must depend on the choice of the value of the depreciation coefficient μ, which is neither a quite arbitrary nor a well-known quantity.

The time series for capital K and investment I for the U.S. economy is known (see Appendix B) and allows us to calculate values of the rate of capital depreciation μ by using (2.27). The results are shown in Fig. 2.6. The website of the U.S. Bureau of Economic Analysis (www.bea.gov) also contains estimates of depreciated capital μK which allow us to calculate the rate of capital depreciation μ in a different way, as the ratio of depreciated amount of capital to the total amount. These results are also depicted in Fig. 2.6. These estimates allow us to consider the depreciation coefficient as an increasing function of time which has value $\mu = 0.026$ in year 1925 and increases linearly from 0.026 to 0.07 over years 1925–2000. However, the results show inconsistency of the primary data: the two estimates from the same source differ from each other; also, the depreciation coefficient cannot be negative. We have chosen to consider the empirical values of investment and capital depicted in Fig. 2.3 to be 'correct' values and to exploit the calculated values of the depreciation coefficient, while using local averaged values (dashed line in Fig. 2.6) instead of negative ones.

The calculated time dependence of capital as well as gross investment for the entire U.S. economy is shown in Fig. 2.3 on p. 28. The time dependence of production capital can be approximated by the exponential function

$$K = 5.49 \times 10^{12} \cdot e^{0.0316t} \text{ dollar(1996)}, \tag{2.29}$$

where time t is measured in years, and $t = 0$ corresponds to year 1950.

The time dependence of the stock of knowledge R can be calculated according to (2.28), assuming the flux G (which was described in Sect. 2.2.4.3 as a quantitative measure of efforts for creating principles of organisation per year, that is, investment in science and in research and developments) is given, as shown in Fig. 2.5, and the rate of depreciation μ of the stock of knowledge can be guessed. The results are demonstrated in Fig. 2.5.

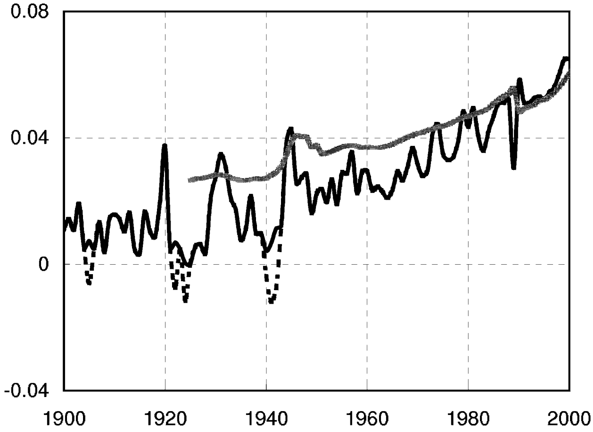

Fig. 2.6 The depreciation coefficient in the U.S. economy. The direct estimates of the quantity as the ratio of depreciated amount of capital stock to the total amount (the *shorter curve*) and estimates due to (2.27) (*pulsating curve*) with use of values of investment and capital. The *dashed lines* represent corrected values

2.4 Labour Force

Work is the most important production factor. Its role in production was thoroughly investigated in systems of concepts of political economy and neo-classical economics. ‘Labour is, in the first place, a process in which both man and Nature participate, and in which man of his own accord starts, regulates, and controls the material reactions between himself and Nature. He opposes himself to Nature as one of her own forces, setting in motion arms and legs, head and hands, the natural forces of his body, in order to appropriate Nature’s productions in a form adapted to his own wants. By thus acting on the external world and changing it, he at the same time changes his own nature. He develops his slumbering powers and compels them to act in obedience to his sway. … At the end of every labour-process, we get a result that already existed in the imagination of the labourer at its commencement. … Besides the exertion of the bodily organs, the process demands that, during the whole operation, the workman’s will be steadily in consonance with his purpose.’ (See [5], vol. 1, Chap. 7, Sect. 1.) ‘… however varied the useful kinds of labour, or productive activities, may be, it is a physiological fact, that they are functions of the human organism, and that each such function, whatever may be its nature or form, is essentially the expenditure of human brain, nerves, muscles, & c.’ (See [5], vol. 1, Chap. 1, Sect. 4.)

2.4.1 Consumption of Labour

Modern technology assumes that man is installed into the production process and works inside it. The true measure of labour is work (in a physical sense, in energy units) done by a labourer, but practically, the labour is measured by working time, so that it is important to estimate the work which can be done by a labourer per hour. In a sedentary state, the human organism (an adult male) requires about 2500 kcal/day

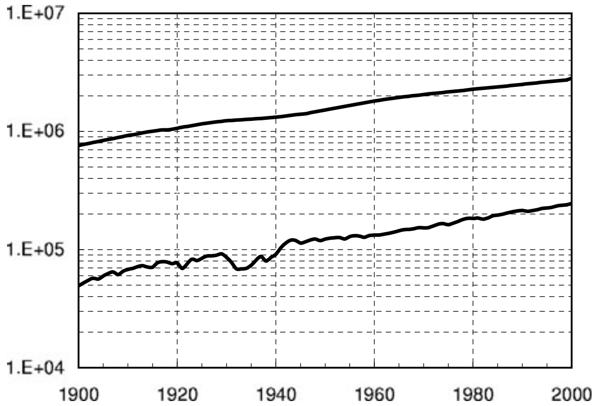

Fig. 2.7 Population and consumption of labour in the U.S. The *upper curve* represents population in hundreds of persons. The *lower curve* represents consumption of labour in millions of man-hours per year. The latter dependence can be approximated by exponential function (2.30)

or about 10^6 kcal/year $\approx 4 \cdot 10^9$ J/year.[5] Extra activity requires an extra supply of energy. The energy needed for a working man can be up to two times more than the energy needed for a resting man (Chap. 26 in [14], [15]). Though some types of work require significant energy consumption, we accept the value of the work done by a labourer to be approximately 100 kcal/hour or 4.18×10^5 J/hour. The possibilities of the human engine were lower in earlier times, as was shown by Fogel and Costa [16] on the basis of historical data for France and Britain for years 1785 and 1790, correspondingly.

Therefore, labour is measured in man-hours, while corrections due to the character of labour (heavy or light), intensity of work and other factors are considered to have been taken into account. For the last statement, I rely on Scott [17], who in his turn refers to other researchers. As an example, according to the data compiled in Appendix B, the amount of man-hours per year (labour consumption) in the economy of the U.S. is shown in Fig. 2.7 as a function of time. The dependence can be approximated by a straight line, especially after year 1950, so that for this period

$$L = 1.23 \times 10^{11} \cdot e^{0.0147t} \text{ man·hour/year}, \tag{2.30}$$

where time t is measured in years, and $t = 0$ at year 1950.

According to Marx [5], labour is a commodity that produces value. The bulk productivity of labour, that is, the value produced per unit of labour, due to formulae (2.14) and (2.30), can be approximated for the U.S. economy as

$$Y/L = 13.74 \cdot e^{0.0179t} \text{ dollar(1996)/man·hour}. \tag{2.31}$$

One can estimate that productivity of labour in the U.S. economy has grown by six times during the past century. This growth of productivity cannot be explained without taking into account that there is another commodity—energy—with a similar property that can substitute labour and produce value. We believe that the increase

[5] 1 cal = 4.18 joules.

in the labour productivity is connected with the use of newer and newer sources of energy by human beings.

2.4.2 *Population and Labour Supply*

The supply of the labour is the potential amount of labour $\tilde{L}$, available at given wage w, in other words, at a given price of labour. The labour supply is conventionally considered to be connected with the whole population N

$$\tilde{L} = f(w)N. \tag{2.32}$$

The population is a reservoir (a pool) from which labour is supplied. The increasing function $f(w)$ changes from zero at $w = 0$ to a certain limiting value, which is usually about 0.5 for developed countries.

The dynamic equation for the change in population can be written as

$$\frac{dN}{dt} = (b - d)N, \tag{2.33}$$

where $b - d$ is the birth rate minus the death rate, i.e., the growth rate of the population.

To obtain an equation for the labour supply, one ought to differentiate relation (2.32) to get

$$\frac{d\tilde{L}}{dt} = \tilde{\nu}\left(N, b - d, w, \frac{dw}{dt}\right)\tilde{L}, \tag{2.34}$$

where the potential growth rate of the labour supply is determined by the growth of the population and changes in the level of the wage

$$\tilde{\nu} = (b - d)f(w) + Nf'(w)\frac{dw}{dt}.$$

Note that the total amount of wages wL also includes, generally speaking, investments in capital, so that the amount of subsistence cL, that is, the amount of expenses which are needed to provide a living for and training of labour, is less than wL (see also Sect. 2.2.4.2).

2.5 Energy Resources in the Production Processes

Energy, as has been discussed repeatedly and for a long time (see, for example, [18, 19]), is vital for the performance of the production system. The socially organised stream of energy begins with identification of primary energy carriers: coal, oil, potential energy of falling water—all that humans find in the nature and that costs nothing, until it is not recognised yet, how to take energy from energy carriers.

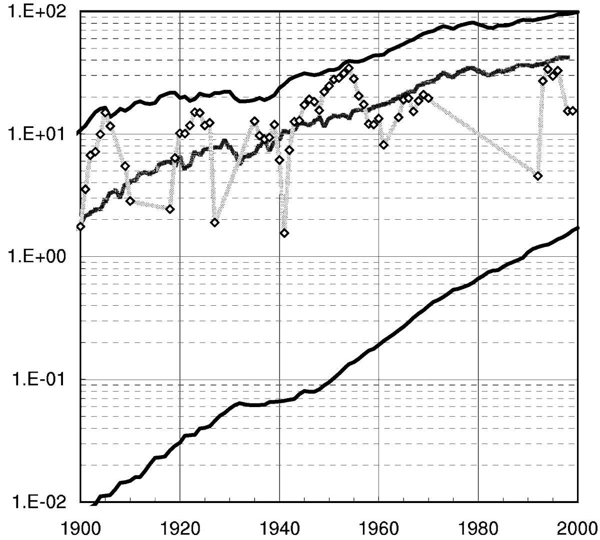

Fig. 2.8 Consumption of energy in the U.S. economy. The *solid lines* represent consumption of energy carriers (primary energy, *top curve*) and productive consumption of energy (substitutive work, *bottom curve*). The *dashed line* depicts primary energy (exergy) needed for work of production equipment, estimated based on the data of Ayres et al. [20] as the sum of half of the net electricity consumption, consumption of energy by other prime movers and non-fuel consumption of oil products. Primary substitutive energy is also calculated (and depicted by symbol ◇) as a part of primary energy, which is anti-correlated with labour (see Sect. 7.1.5). All quantities are estimated in quads per year (1 quad $= 10^{15}$ Btu $\approx 10^{18}$ J). The primary energy and substitutive work from year 1950 can be approximated by exponential function (2.36). Reproduced from [26] with permission of *Elsevier*

2.5.1 Work and Quasi-work in a National Economy

An energy carrier is what we call something that contains potential energy: the chemical energy embodied in fossil fuels (coal, oil and natural gas) or in biomass; the potential energy of a water reservoir; the electromagnetic energy of solar radiation; the energy stored in the nuclei of atoms. The total of the primary energy carriers used by humans and estimated in power units, is listed in handbooks as the quantity of used[6] *primary energy*. The *primary energy* consumption is the consumption of energy carriers as they can be taken from nature.

As an illustration, Fig. 2.8 shows with a solid line the total consumption of primary energy carriers, as shown by official statistics of the U.S. Department of Energy (see Appendix B). Apparently, the primary energy carriers (for simplicity, one

[6]It is customary to speak about the consumption of energy in a national economy. For precision, the word *consumption* should be replaced by the word *conversion*. Energy cannot be *used up* in the production process; it can only be converted into other forms: chemical energy into heat energy, heat energy into mechanical energy, mechanical energy into heat energy and so on. The measure of converted energy (work) is exergy.

speaks about consumption of primary energy E) in public facilities are used for the most variety of tasks. So, for example, 0.55 quad[7] of oil products from the total amount of about 97 quad of primary energy consumed in the U.S. economy in year 1999 was laid on the roads. It is clear that it is not even the energy content that is important in this case, but the property of oil products as specific materials.

For the most part, primary energy is not used directly but is first transformed and converted into fuels and electricity—*final energy*—which can be transported and distributed to the points of final use. The final energy consumption provides energy services for manufacturing, transportation, space heating, cooking and so on.[8] Extensive investigations of the consumption of primary and final energy in the U.S. economy was conducted by Ayres with collaborators [20, 21].

The total of the primary energy carriers can be broken into two parts according to their role in productions. It is possible to allocate a part which is used for operating various adaptations allowing substitution of labour efforts by work of the production equipment. This quantity can be called *primary substitutive work* E_{P}. True *substitutive work* or *productive energy* P, which really replaces workers' efforts, is a small part of the consumed primary productive energy E_{P}, and the coefficient of efficiency P/E_{P} depends on exploited technology. In the United States in the beginning of 60th years, for example, in general consumption nearly $5 \cdot 10^{19}$ J, about a third of all consumed energy, went to substitution of labourers' work. At an efficiency ratio equal to 0.01, true substitutive work made nearly $5 \cdot 10^{17}$ J.

The other part of the socially organised stream of energy, called quasi-work, is used directly in production and in households for illumination, heating, chemical transformations and other tasks.

2.5.2 Direct Estimation of Substitutive Work

Although one can easily find estimates of the total amount of primary energy carriers, the biggest interest for our aims is caused by possible assessments of the quantity of energy going to the substitution of workers' efforts in the production processes. Based on the results of fundamental investigations [20, 21] of the usage of primary and final energy in the U.S. economy, one can estimate the amount of substitutive work in this case.

[7] Primary energy is the name for primary energy carriers (oil, coal, running water, wind and so on) measured in energy units. It is convenient to measure huge amounts of energy in a special unit quad (1 quad $= 10^{15}$ Btu $\approx 10^{18}$ J), which is usually used by the U.S. Department of Energy.

[8] The problems arising in the estimation of the amount of energy which is converted (used up) in production processes to do useful work are discussed by Patterson [22], Nakićenović et al. [23], Zarnikau et al. [24] and Ayres [25]. According to Nakićenović et al. [23], the global average of primary to final efficiency was about 70% in year 1990, while it was higher in developed countries. Data collected by Ayres [25, Table 2] demonstrates that efficiency of energy conversion increased during the last centuries.

The substitutive work or productive energy P could be generally interpreted as capital services. The most important property of this quantity is its ability to substitute labour services, which are different efforts of humans in production processes, and the substitutive work itself should be defined as an amount of work which is done by external energy sources with the help of production equipment instead of workers' efforts. To estimate substitutive work, we have to consider human efforts, which, we assume, can be replaced by the work of production equipment driven by external energy sources. We can divide all efforts into three groups.

2.5.2.1 Efforts on Displacements of Substances and Bodies (Including Human Bodies)

These efforts were substituted by the work of animals, wind and moving steamer engines in the past. Now in the U.S., they are substituted mainly by the work of self-moving machines—automobiles, trucks, aeroplanes and other mobile equipment—driven by the products of oil. Estimates of energy used for this purposes can be obtained for the U.S. economy as the sum of energy of consumed distillate fuel oil, jet fuel and motor gasoline. According to U.S. Department of Energy data (www.eia.gov), the amount was 19.46 quad in year 1998. This is the energy content of fuel; the amount is different from the amount of work (service energy) which is needed to move vehicles. The service delivery efficiency for transportation was analysed by Ayres [25], and the ratio of the energy delivered to wheels to the fuel energy was estimated as 0.06. The ratio of the useful work (substitutive work) to fuel energy is much less; it is close, one can suppose, to the Ayres [25] technical efficiency, which was 0.015 for transportation (much less for farming and construction) in year 1979. According to Ayres et al. [20], efficiency has been improving beginning with 1975, so that one can estimate the contribution to substitutive work from transportation. The genuine work of transportation vehicles due to energy carriers can be calculated as 0.1 quad in year 1998, though the amount of energy carriers needed to provide this work was about 19.46 quad.

2.5.2.2 Efforts on Transformation and Separation of Substances and Bodies

These are efforts in the production of clothes, tools, different appliances and so on—much, if not all, manufacturing. Animal-driven, wind-driven, water-driven and steam engine-driven power were used to do work instead of humans in previous centuries. Nowadays the same work is mainly done by machines with electric drives. According to the U.S. Department of Energy (http://www.eia.gov), motor-driven equipment accounts for about half of the electricity in the manufacturing sector. Non-industrial motors, driving pumps, compressors, washing machines, vacuum cleaners and power tools also account for quite a lot of electricity consumption. Part of the electricity consumed by clothes washers and dish washers provides mechanical movement. So we can account that more than half of the consumed site electricity in the U.S. economy, that is about 6 quad in 2000, is taken by motors. In the best cases, electricity in a machine drive can be recovered into rotational motion with an efficiency of up to 0.8–0.9 [20]. However, the result of the work of a machine tool, for example, is a component or detail of another machine, and one has to consider the whole procedure of making something: installation, stop-start movements, measurement and so on. It is difficult to get an absolute measure of efficiency in this case, but one can imagine that there is a certain amount of work which has to be done to obtain the necessary effect. Presumably, it is the work of a human who can obtain this effect on his own. The efficiency of machine drives was estimated by Ayres [20] as about 0.002 in years 1960–1970. At manual operation the efficiency is low, but automated control and operation allow increases in efficiency. One assumes that the introduction of information processors into the production could affect the efficiency of the processes, which could reach 0.005 in year 2000. This gives an estimate for the contribution to substitutive work from machine drives to be 0.2–0.3 quad per year 2000.

2.5.2.3 Efforts on Sense-based Supervision and Co-ordination, Development of Principles of Organisation

While the human efforts listed in the preceding two groups have been successfully substituted by work of other sources of energy from ancient times, attempts to mechanise the functions of the brain were mainly unsuccessful until the advent of computers (information processors) in the twentieth century. Up until recent times these functions were considered as essentially human functions. Now the work of the brain is being substituted by information processors driven by electricity. According to the U.S. Department of Energy (http://www.eia.gov), the consumption of electricity by computers and office equipment in the commercial sector of the U.S. economy in year 1999 was 0.4 quad. In the residential sector electricity was consumed by computers and electronics in the amount of 0.35 quad in year 1999. There is no data on the consumption of electricity by computers in the industrial sector, though one can hardly have any doubt about the presence of the appliances of information technology in this sector and the sector of transportation. To the sum of the above figures—0.75 quad—one has to add the amount of electricity consumed by other office and communication equipment in all sectors. In total, one can estimate the consumption of electricity by computers, electronics and office equipment to be about 1 quad in year 1999. This figure estimates, at least, a scale of phenomenon. One cannot directly measure the work produced by the devices of information technology to measure the efficiency, but one can see some signs that the useful effect per unit of consumed energy (efficiency) has been increasing. For example, the consumption of electricity by one computer decreased from 299 kWh/yr in 1985 to 213 kWh/yr in 1999 [27, 28]. This means that consumption of electricity by a computer was decreasing with average rate 0.025. Simultaneously, the number of computers and consumption of electricity increased with average rate of growth 0.027 between years 1990 and 1999, as can be calculated from the data of Koomey et al. [27] and Kawamoto et al. [28]. All this means that the useful effect from the consumption of electricity by computers has been growing in recent times with a growth rate of more than 0.052, which is the sum of the rate of growth of consumption of electricity, 0.027, and the rate of decrease of consumption of electricity by one unit, 0.025, plus the estimate of improving the unit performance. Similar considerations can be made for all devices of information technology from the collection of data by Koomey et al. [27] and Kawamoto et al. [28]. The efficiency of computers is certainly less than unity, but they may be more efficient than many other appliances. It is difficult to judge what part of the resulting amount of 1 quad per year can be attributed to substitutive work itself, but, perhaps, an estimate of 0.5 quad per year is realistic. This huge amount of energy was spent usefully in year 1999 to produce instructions to humans and apparatuses in the U.S. economy.

2.5.2.4 Final Remarks

Summing up, the total amount of substitutive work in the U.S. economy in 1999 can be estimated as 1 quad per year. It is approximately one hundred times less than the total (primary) consumption of energy, which was about 97 quad in 1999. However, the amount of primary energy (energy carriers) needed to provide this amount of substitutive work is about 25 quad, which is about 26% of the total primary consumption of energy. This number corresponds to the estimates by Ayres [25, Table 1] who found that the part of energy which can be considered as the primary production factor (machine drive, transport drive, farming and construction) in the U.S. economy was 9% in year 1800, 23% in 1900 and about 32% in 1991.

2.5.3 Energy Carriers as Intermediate Products and Energy as a Production Factor

Energy carriers are consumed now in great amounts in production processes and are considered to be products which are moving in the production system and thus must be included in the balance table (Table 2.1, p. 23). From the conventional economic point of view, all consumed energy carriers can be considered as intermediate or, sometimes, final products.

Electricity as an energy carrier, for example, is the most important intermediate product in the production of aluminium, metallurgical operations and some chemical processes, among others. Electricity consumed for lighting, comfort and process heating must be considered either as a final product (in the residential sector) or as intermediate products (in commercial and other sectors). In all cases of production consumption, the cost of energy is included in the cost of the final products, and energy contributes to the value of produced commodities no more than other intermediate products participating in the production process.

However, it has long been argued [18, 19] that, aside from regarding the energy carriers as intermediate or final products, the delivered energy is universally vital to the performance of the economy and must be included in the theory of production as an important production factor. Apart from being a commodity, in some cases, energy from external sources plays a special role, substituting for efforts of workers in the technological processes. Energy-driven equipment works in the place of workers, and energy can be ascribed all the properties of labour, including the property to produce surplus value. In these cases, work or energy, which apparently is only a part of the total (primary) consumption of energy, has to be specified as a value-creating production factor in the conventional economic terms.

Thus, one can define the different roles of the consumed energy carriers in the production processes. In any case, energy carriers participate in the production processes as usual commodities. However, part of the consumed energy P—it is called *productive energy* or *substitutive work*—has to be considered not only as an ordinary intermediate or final product, but also as a value-creating factor, which has to be introduced in the list of production factors equally with the production factors of conventional neo-classical economics, capital K and labour L. This production factor, substitutive work P, is not primary energy and, moreover, not even energy delivered to production equipment. It has to be considered as genuine work done by production equipment with the help of external sources of energy instead of workers. This quantity can also be considered as capital service provided by capital stock.

The substitutive work P defined in this way has a special price, different from the prices of energy carriers as a usual intermediate or final products. It is clear that the amount of consumed products which are needed to support substitutive work P is valued as μK, so that the price of substitutive work, as a production factor, is

$$p = \frac{\mu K}{P} \tag{2.35}$$

2.5.4 Estimates of Primary Energy and Substitutive Work

There are plenty of data on the total consumption of primary energy E in different countries (in the Energy Statistics Yearbook, for example), but little is known about the productive part of consumption P which is a true value-creating production factor. However, there is a method of estimation of substitutive work P which is based on a relation between the rates of growth of production factors (5.20). This method, which is described in detail in Sect. 7.1.2, allows one to calculate the growth rate η of substitutive work, if one knows the rates of growth of output, capital and labour consumption. Then, one can restore the time dependence of substitutive work if the absolute value of the quantity itself is known in one of the moments of time.

As an illustration, Fig. 2.8 shows the total consumption of primary energy carriers, as shown by official statistics of the U.S. Department of Energy (see Appendix B), and the calculated usage of substitutive work [29] in the U.S. economy according to official estimation of the empirical situation. The method does not allow one to calculate absolute values of substitutive work; it was taken to be about 1 quad at the end of the century, as was estimated in Sect. 2.5.2. The extra growth

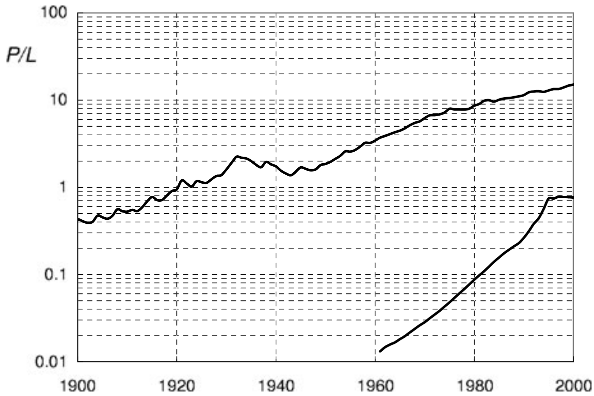

Fig. 2.9 The ratio of substitutive work to workers' efforts. The ratio of substitutive work to estimates of workers' efforts for the U.S. economy (the *upper curve*) and for the Russian economy (the *lower shorter curve*). Reproduced from [30] with permission of *Elsevier*

rate of substitutive work in the U.S. economy in years 1950–2000 in comparison with the primary consumption of energy was about 0.04 per year in the second half of the century. The dependence of the total and productive consumption of energy from year 1950 can be approximated by the functions

$$E = 33.3 \cdot e^{0.0205t} \text{ quad/year}, \tag{2.36}$$

$$P = 1.96 \cdot e^{0.0585t} \text{ quad/year}, \tag{2.37}$$

where, as in previous examples, time t is measured in years, starting from year 1950. It is possible to estimate the productive consumption of energy for a unit of labour. For the U.S. economy, since 1950,

$$P/L = 6.42 \times 10^5 \cdot e^{0.0441t} \text{ J/man·hour}. \tag{2.38}$$

Figure 2.9 shows the ratio of the work executed by the production equipment to energy estimates of the efforts of the workers, taking into account an estimate of an hour of work, obtained in Sect. 2.4.1. One can find that, by the present time, the efforts of every worker in the economy of the U.S. are amplified more than 10 times. This is a rule: consumption of energy from external sources exceeds the work done by man by a few times in all developed countries.

The average productivity of substitutive work for the U.S. economy can be approximated by the function

$$Y/P = 2.14 \times 10^{-5} \cdot e^{-0.0259t} \text{ dollar(1996)/J}. \tag{2.39}$$

The best characteristics of labour and energy productivity are marginal productivities, which will be introduced and estimated in Chap. 7.

2.5.5 *Stock of Knowledge and Supply of Substitutive Work*

While the labour supply $\tilde{L}$ can be related to the population, which can be considered to be a pool from which the labour force emerges (see Sect. 2.4.2), the productive energy supply $\tilde{P}$ can be related to the stock of knowledge which is playing a role of a reservoir (pool) from which applications of energy emerge. Indeed, one ought to have available sources of energy and appliances, which allow the use of energy in production aims. Some devices ought to be invented, made and installed for work. Human imagination provides methods of using energy in production tasks. Therefore, the base for the energy supply lies in a deposit of knowledge which is fallow, unless it is used in a routine production process. This deposit determines the possibility of the society attracting the extra energy to production. The stock of knowledge should be considered as a resource.

To describe the process of evolution of the energy supply, one can refer to the simple three-sector model of the production system introduced in Sect. 2.1. Discovering the principle of organisation and developing projects of technological processes is the content of activity of the second sector. One can consider that this stock of knowledge, that is, fundamental results of science, results of research, project works and so on (stock of principles of organisation), are measured by their total value R, which is governed by (2.28). Alternatively, the stock of knowledge can be measured directly in terms of natural units, that is, by numbers of patents issued, numbers of technical journals, numbers of books in print and so on. The knowledge is embodied in organisations and cultures more than in individuals, although individual skills are also part of this category. Can the value of stock of knowledge R be a measure of the information contained in all this?

Then, the first sector materialises the projects. One can find plenty of brilliant examples of 'transformation' of knowledge into useful work in the history of technology and one can try to formalise this process, considering the stock of knowledge as a resource or as a reservoir (pool) from which applications of energy emerge. One can assume, noting an analogy of (2.28) with (2.33), that an equation for energy supply $\tilde{P}$, that is, the amount of energy which can be used in production processes as substitutive work, can be written similarly to (2.34) in the form

$$\frac{d\tilde{P}}{dt} = \tilde{\eta}(\varepsilon, R)\tilde{P}. \tag{2.40}$$

One can assume that the rate of potential growth of substitutive work $\tilde{\eta}$ depends on the stock of knowledge R and on the price of introducing substitutive work into production $1/\varepsilon$ (see Sect. 5.2, (5.16)). The price of transformation and materialisation of deposited massages, that is, the price of attracting the energy, has been appearing on the stage of materialisation of principles of organisation. The function $\tilde{\eta} = \tilde{\eta}(\varepsilon, R)$ remains unknown; one can assume a simple dependence

$$\tilde{\eta} = g(\varepsilon)R. \tag{2.41}$$

However, in a situation of uncertainty, the growth rate of potential energy, or the energy supply itself, ought to be given.

Though it is indisputable that knowledge makes energy available for humans, the question remains of how to describe it in quantitative terms. Does function (2.41) really exist and, if it exists, which is its asymptotic behaviour? One may think that the current attention to the stock of knowledge, as to the genuine source of economic growth [31–33] (see also the textbook [34]) can help to solve the problem. However, we do not know whether the available energy is limited or not. One can imagine and consider two scenarios of development: the energy supply $\tilde{P}$ as a function of time has or does not have a limit value. There is apparently no question of lack of energy. It is a question of ways of utilisation of energy to get the desired effect. This question is clearly connected with the other question: Can the stock of knowledge be limited?

2.6 Natural Processes in a Human-Designed Production System

The production system is embedded in the natural environment. In the beginning of the production cycle, raw materials are extracted from the natural environment, while at the end of the production cycle, the wastes and useless by-products are thrown out into nature. The flow of substances starts and finishes in the natural environment (see Fig. 1.2 on p. 7), thus one has to consider the interaction of the production system with the environment.

Some industries (agriculture and forestry, for example) use natural processes to provide the production of commodities. Some natural things are even used as production equipment. Soil (land) is used to produce corn, cows are used to produce milk and so on. The natural things are considered as production capital, and their value is estimated in the same way as the value of all other capital products.

The sector theory of production, considered in Sect. 2.2, assumes that some natural processes are included in the production system. To consider the interaction between the environment and the production system in more detail, one has to admit that some of the variables X_j represent amounts of natural products. It is convenient to assume that, in consistency with the definitions of Sect. 2.2, X_j, $j \leq r$ is the gross output of artificial products in money units and N_j, $j > r$ is the gross output of natural products measured in natural units. The gross output X_j both of artificial and natural products can be distributed (similar to (2.3)) as

$$X_i = \sum_{j=1}^{r} X_i^j + \sum_{j=r+1}^{n} X_i^j + Y_i, \quad i = 1, 2, \ldots, r, \tag{2.42}$$

$$N_i = \sum_{j=1}^{r} N_i^j + \sum_{j=r+1}^{n} N_i^j + \frac{Y_i}{p_i}, \quad i = r+1, r+2, \ldots, n, \tag{2.43}$$

where X_i^j is an amount of artificial product labelled i used for the production of product labelled j and, similarly, N_i^j is an amount of natural product labelled i used for the production of product labelled j, while there is a residue Y_i called final output. We assume that the price p_i and money measure might be introduced for those of the natural products which are supported by human activity.

Now, one can write the second set of balance equations, which, as in (2.4), represent the balance of production of value in sectors of production of both artificial

Table 2.2 Balance of artificial and natural products

Gross output	X^1	X^2	...	X^r	X^{r+1}	X^{r+2}	...	X^n	Final output
X_1	MAN-CREATED PROCESSES				X_1^{r+1}	X_1^{r+2}	...	X_1^n	Y_1
X_2					X_2^{r+1}	X_2^{r+2}	...	X_2^n	Y_2
...					...	...	...	...	...
X_r					X_r^{r+1}	X_r^{r+2}	...	X_r^n	Y_r
X_{r+1}	X_{r+1}^1	X_{r+1}^2	...	X_{r+1}^r	NATURAL PROCESSES				Y_{r+1}
X_{r+2}	X_{r+2}^1	X_{r+2}^2	...	X_{r+2}^r					Y_{r+2}
...	...	...	...	...					...
X_n	X_n^1	X_n^2	...	X_n^r					Y_n
Production of value	Z^1	Z^2	...	Z^r	0	0	...	0	Y

and natural products

$$X^j = \sum_{l=1}^{r} X_l^j + \sum_{l=r+1}^{n} p_l N_l^j + Z^j, \quad j = 1, 2, \dots, r, \tag{2.44}$$

$$p_j N^j = \sum_{l=1}^{r} X_l^j + \sum_{l=r+1}^{n} p_l N_l^j, \quad j = r+1, r+2, \dots, n, \tag{2.45}$$

where Z^j is production of value in sector j, and we admit that there is no production of value in the sectors of natural production.

It is convenient to define the amounts of value of gross product and intermediate consumption for products of the natural processes

$$X_j = p_j N_j, \qquad X_l^j = p_l N_l^j$$

to include all considered quantities in the more detailed (in comparison with Table 2.1) balance table (Table 2.2). However, the majority of natural products are traditionally regarded as zero-price products and are not included in the balance scheme.

To determine the production of value Z and components of the gross output Y in this case, we sum (2.42) over index i from 1 to r and also (2.44) over index j from 1 to r. After comparing the results, one obtains

$$Z = \sum_{j=1}^{r} Y_j + \sum_{j=r+1}^{n} \left(\sum_{l=1}^{r} X_l^j - p_j \sum_{l=1}^{r} N_j^l \right). \tag{2.46}$$

The conventional characteristic of efficiency of the production system, the final output $Y = \sum_{j=1}^{n} Y_j$, is considered to be equal to the production of value

$Z = \sum_{j=1}^{n} Z^j$. Thus, the right-hand side of (2.46) can be considered as the sum of components of the vector Y, which can be determined as

$$Y_j = \begin{cases} Y_j, & j = 1, 2, \ldots, r \\ \sum_{l=1}^{r} X_l^j - p_j \sum_{l=1}^{r} N_j^l, & j = r+1, r+2, \ldots, n \end{cases} \tag{2.47}$$

The quantity X_l^j, at $l \leq r$, $j > r$ is the amount of artificial product labelled l supporting the production of natural product j, so that the sum $\sum_{l=1}^{r} X_l^j$ is a total amount of the artificial products supporting the production of the natural product j. On the other hand, N_j^l, at $j > r$, $l \leq r$ is an amount of the natural product labelled j needed for production of the artificial product l, so that the sum $\sum_{l=1}^{r} N_j^l$ is the total amount of natural product j used in production in all sectors. Therefore, one can see that the components of final output (2.47), at $j > r$, are characteristics of our interactions with nature. Values of the characteristics depend on prices of the natural products, so that it is very important to use the right prices for estimation of the interaction characteristics. Because nature does not have a representative agent on the market, there is no market evaluation of the prices of natural products, and one can choose the prices arbitrarily. It is natural to choose the right prices in such a way that in the situation of balance all the components (2.47) at $j > r$ vanish. This requirement is followed by a definition of the balancing price of natural product j as

$$p_j = \frac{\sum_{l=1}^{r} X_l^j}{\sum_{l=1}^{r} N_j^l}, \quad j = r+1, r+2, \ldots, n. \tag{2.48}$$

At these prices, the interaction characteristics can be positive or negative: the former case means that humans invest in the environment, whereas the latter case means that there is a damage to the environment, or this can be interpreted as our debt to nature.

However, whatever the prices of natural products, one always assumes that $Y_j = 0$ at $j > r$. The production of natural sectors is not usually accounted for at all, so the national statistics can show a truncated produced value $Y = \sum_{j=1}^{r} Y_j$ instead of the real amount $Y = \sum_{j=1}^{n} Y_j$. The underestimation of prices of natural products in comparison with the balancing price (2.48) result in a deficiency of gross investment in nature. In both our century and in the previous ones, the production system was contained in the environment, but in previous years the interaction with the environment was not as large in scale as it is in our times, and it was mainly local. Nowadays, ecological problems have appeared, which seem to stem from underestimation of the prices of the natural products. A proper social mechanism of regulation of our interaction with the environment does not exist at the moment; it has to be invented and implemented in reality.

References

1. Leontief, W.W.: Quantitative input and output relations in the economic system of the United States. Rev. Econ. Stat. **18**, 105–125 (1936)
2. Leontief, W.W.: The Structure of the American Economy 1919–1939. Harvard University Press, Cambridge (1941)
3. Leontief, W.W.: Input–Output Economics, 2nd edn. Oxford University Press, New York (1986)
4. Sraffa, P.: Production of Commodities by Means of Commodities. Prelude to a Critique of Economic Theory. Cambridge University Press, Cambridge (1975)
5. Marx, K.: Capital. Encyclopaedia Britannica, Chicago (1952). English translation of Karl Marx, Das Kapital. Kritik der politischen Oekonomie, Otto Meissner, Hamburg, 1867. http://marxists.anu.edu.au/archive/marx/works/cw/index.htm. Accessed 17 March 2011
6. Tougan-Baranovsky, M.: Theoretische Grundlagen des Marxismus. Dunker and Humbolt, Leipzig (1905)
7. Bortkiewicz, L.: Value and price in the Marxian system. Int. Econ. Pap. **2**, 5–60 (1952). English translation of L. von Bortkiewicz (1906) Wertrechnung und Preisrechnung im Marxschen System. Archiv fur Sozialwissenschaft und Sozialpolitik 3, XXIII and XXV
8. Lietaer, B.: The Future of Money. A New Way to Create Wealth, Work, and a Wiser World. Century, London (2001)
9. Studenski, P.: The Income of Nations. Theory, Measurement and Analysis: Past and Present. New York University Press, New York (1961)
10. Afriat, S.N.: The Price Index. Cambridge University Press, Cambridge (1977)
11. Smith, A.: An Inquiry into the Nature and Causes of the Wealth of Nations. Clarendon Press, Oxford (1976). In two volumes
12. Machlup, F.: Knowledge and Knowledge Production. Knowledge, Its Creation, Distribution, and Economic Significance, vol. 1. Princeton University Press, Princeton (1980)
13. Blanchard, O.J., Fisher, S.: Lectures on Macroeconomics. MIT Press, Cambridge (1989)
14. Harrison, G.A., Weiner, J.S., Tanner, I.M., Barnicot, N.A., Reynolds, V.: Human Biology. An Introduction to Human Evolution, Variation, Growth and Ecology. Oxford University Press, Oxford (1977)
15. Rivers, J.P.W., Payne, P.R.: The comparison of energy supply and energy needs: a critique of energy requirements. In: Harrison, G.A. (ed.) Energy and Effort, pp. 85–105. Taylor and Francis, London (1982)
16. Fogel, R.W., Costa, D.L.: A theory of technophysio evolution, with some implications for forecasting population, health care costs, and pension costs. Demography **34**, 49–66 (1997)
17. Scott, M.F.G.: A New View of Economic Growth. Clarendon Press, Oxford (1989)
18. Allen, E.L.: Energy and Economic Growth in the United States. MIT Press, Cambridge (1979)
19. Cottrell, W.F.: Energy and Society: The Relation Between Energy, Social Change and Economic Development. McGraw-Hill, New York (1955)
20. Ayres, R.U., Ayres, L.W., Warr, B.: Exergy, power and work in the U.S. economy, 1900–1998. Energy **28**, 219–273 (2003)
21. Ayres, R.U., Ayres, L.W., Pokrovski, V.N.: On the efficiency of electricity usage since 1900. Energy **30**, 1092–1145 (2005)
22. Patterson, M.G.: What is energy efficiency? Concepts, indicators and methodological issues. Energy Policy **24**, 377–390 (1996)
23. Nakićenović, N., Gilli, P.V., Kurz, R.: Regional and global exergy and energy efficiencies. Energy **21**, 223–237 (1996)
24. Zarnikau, J., Guermouche, S., Schmidt, P.: Can different energy resources be added or compared? Energy **21**, 483–491 (1996)
25. Ayres, R.U.: Technological progress: a proposed measure. Technol. Forecast. Soc. Change **59**, 213–233 (1998)
26. Pokrovski, V.N.: Productive energy in the US economy. Energy **32**, 816–822 (2007)

27. Koomey, J.G., Cramer, M., Piette, M.A., Eto, J.H.: Efficiency improvements in U.S. office equipment: expected policy impacts and uncertainties. Lawrence Berkeley National Laboratory 37383, Berkeley, CA (1995). http://enduse.lbl.gov/projects/offeqpt.html. Accessed 17 March 2011
28. Kawamoto, K., Koomey, J.G., Nordman, B., Brown, R.E., Piette, M.A., Ting, M., Meier, A.K.: Electricity used by office equipment and network equipment in the U.S. Energy **27**, 255–269 (2002)
29. Pokrovski, V.N.: Energy in the theory of production. Energy **28**, 769–788 (2003)
30. Beaudreau, B.C., Pokrovskii, V.N.: On the energy content of a money unit. Phys. A, Stat. Mech. Appl. **389**, 2597–2606 (2010)
31. Romer, P.M.: Increasing returns and long-run growth. J. Polit. Econ. **94**, 1002–1037 (1986)
32. Romer, P.M.: Endogenous technological change. J. Polit. Econ. **98**, 71–102 (1990)
33. Lucas, R.E.: On the mechanics of economic development. J. Monet. Econ. **22**, 3–42 (1988)
34. Aghion, P., Howitt, P.: The Economics of Growth. MIT Press, Cambridge (2009)

Chapter 3
Monetary Side of Social Production

Abstract The input–output model of an economic production system assumes that a specific motion takes place: natural substances transform into finished and semi-finished things, the latter transform into other things and so on, until all this is finally consumed, and the substances return into the environment as waste. Simultaneously with the motion of products, one discovers the motion of money, which has to be considered as a separate, special product. The money is circulating in the economy, providing the exchange of products. To describe the phenomenon of money circulation, in this chapter, we shall consider fluxes of money in a simplified system, consisting of the government and many production firms, and where the sector, which produces money, consists of a central bank and many commercial banks. A system of dynamic equations is formulated to describe the money circulation in the production system. The set of equations is investigated for both steady-state and unsteady situations. The description of money circulation is impossible beyond the description of real production fluxes; therefore, an optimal quantity of money is apparently defined by the social production, though the basic features of real production can be described on their own.

3.1 Architecture of the Social Production System

The production system, which is needed to maintain the existence of the human society, used to be described, due to the works of Leontief [1] (see also Chap. 2), as a system of interacting sectors, each of them creating its own product. In the simplest case, one can consider a system consisting of three sectors, as was described in Sect. 2.1. The first sector creates basic production equipment (K, production funds), the second one creates non-material intermediate products (S), consumed by the other two sectors and stored in warehouses and depositories for future production and non-production consumption, and the third sector creates products for direct consumption by humans (C). We consider here the production system immersed in the money system of the society, as shown schematically in Fig. 3.1.

The main organisers and managers of money circulation are a central bank and commercial banks. The central bank issues the bank notes and coins—the primary money. These bank notes and coins are distributed to the commercial banks, which supply many customers with cash money. The central bank also provides the com-

V.N. Pokrovskii, *Econodynamics*, New Economic Windows 12,
DOI 10.1007/978-94-007-2096-1_3, © Springer Science+Business Media B.V. 2012

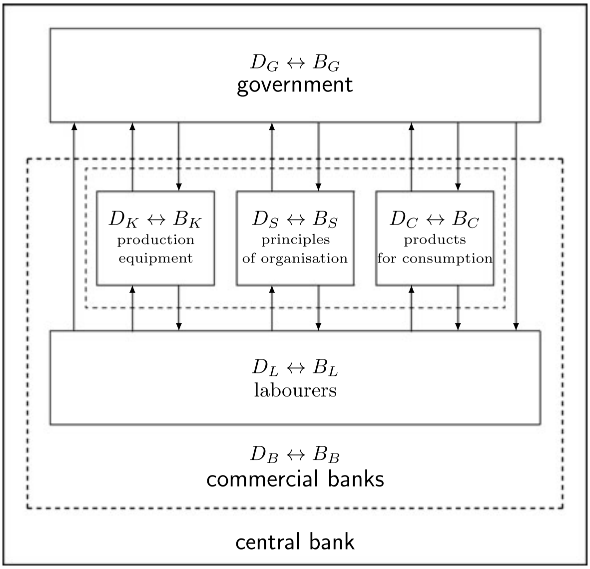

Fig. 3.1 The scheme of money fluxes. The central bank and the commercial banks create a money medium for activity of the economic agents. The three sectors of the production system create all products and originate the fluxes of money between the sectors, which are not shown here, and to workers in the form of wages W_K, W_S, W_C. The workers are buying products, and money is returning to the producers. The government receives its part of produced value in the form of taxes $T = \theta_K Y_K + \theta_S Y_S + \theta_C Y_C + \theta_L W$, which in different amounts are returning to the economic agents. To each *arrow* representing a flux of money there corresponds an *arrow* of opposite direction, presenting fluxes of labour force and products. There is a bargain every time, when money is exchanged for products and labour force

mercial banks with credits, so that the commercial banks can provide the customers with credit money. The records on the accounts of customers are non-paper money, which are created by the commercial banks. So, the central bank and commercial banks introduce an uncertain arbitrary amount of the circulating money in coins, bank notes and cashing deposits into the system consisting of the government and many customers of the commercial banks.

Although the money system contains many commercial banks, each with many customers, for simplicity, we consider all commercial banks together as the only commercial bank; further, instead of many customers, we consider four groups of customers. One can separate all accounts in the commercial banks into groups: a group of producers of main production equipment (K), a group of producers of non-material intermediate products (S), a group of producers of products for immediate consumption (C) and a group of final consumers (L).

Economic subjects interact with each other using money as a tool for the purchase of resources, both for consumption and for production. The major function of the bank system resides in the redistribution of money, in particular, in directing money from investors to firms, which require finances for forthcoming projects. One can assume an elementary diagram of monetary streams, in which only banks are accumulators of expenses and incomes. The system of monetary circulation is described using the assumptions that the central bank has no aim to receive any profit, and the commercial banks are limited to obtaining a reasonable modest percent. Actually the situation is more complex: the commercial banks are aspiring to increase in profit by increasing their own capital, due to a share issue, and are engaging in speculative operations. All of this can lead to an essential discrepancy of the monetary circulation with production needs, which reveals itself as a financial crisis.

Due to its huge practical importance, the phenomenon of money circulation has been studied thoroughly throughout the centuries [2]. In addition to some seminal monographs [3–5], there is a huge amount of books and articles devoted to different aspects of the problem. A paramount understanding of the problem provokes a closed mathematical description of money circulation or a mathematical monetary theory of production, which has been successfully developed [6, 7]. In this chapter, following the method elucidated by Keen [7], a system of equations for the simplest closed system, described above, will be formulated and investigated. The results of the theory depend on the specific assumptions of the system architecture and the preferences of the process participants. The derived parities are a schematisation of the processes of monetary circulation.

3.2 Participants of the Money System

3.2.1 The Customers of the Commercial Banks

For expansion of production and consumption, the clients of commercial banks need money; thus, at each given point in time, clients should determine whether a financial source of possible expenses should be money from their own accounts or a loan from a commercial bank. The customers of commercial banks create a demand for credit money, and they appear to be the basic movers of the progress of the economic system.

3.2.1.1 The Producers

In accordance with the speculations in Chap. 2, one can assume that the output of each sector is needed to maintain the production of, generally speaking, all other sectors, so that the gross products X_K, X_S or X_C are generally distributed among three sectors, and the balance relation for the products can be written as

$$\begin{aligned} X_K &= X_{KK} + X_{KS} + X_{KC} + I, \\ X_S &= X_{SK} + X_{SS} + X_{SC} + G, \\ X_C &= C, \end{aligned} \tag{3.1}$$

where I, G and C are components of final products, which are planned for sale beyond the intermediate production usage; $I = I_K + I_S + I_C$ is the value of the investment products, distributed over the three sectors. For simplicity, it is assumed that the product of the third sector in the amount C is completely consumed.

The product fluxes are accompanied by money fluxes, which are moving in the opposite direction. Each production sector receives money from the sales of its product from the government, workers and all production sectors, including payments from its own sector,

$$\begin{aligned}
&M_{K\to K} + M_{S\to K} + M_{C\to K} + Y_K,\\
&M_{K\to S} + M_{S\to S} + M_{C\to S} + Y_S,\\
&Y_C.
\end{aligned} \tag{3.2}$$

The quantities Y_K, Y_S, Y_C can be considered to be components of the final output—Gross Domestic Product (GDP).

Simultaneously, each production sector pays (symbolised by minus signs) wages to labourers and money for the products of all the sectors,

$$\begin{aligned}
&-M_{K\to K} - M_{K\to S} - I_K - W_K - \theta_K Y_K,\\
&-M_{S\to K} - M_{S\to S} - I_S - W_S - \theta_S Y_S,\\
&-M_{C\to K} - M_{C\to S} - I_C - W_C - \theta_C Y_C.
\end{aligned} \tag{3.3}$$

Here, we take into account that the producers have to pay taxes to the government in the amounts $\theta_K Y_K$, $\theta_S Y_S$ and $\theta_C Y_C$.

Before writing the payment balance for the sectors, note that, though the receiving and payments of the money occur at one and the same time, due to the time involved for production, marketing, transportation, investment, consumption and so on, the symbols in (3.2) and (3.3) present payments for amounts of products produced at different times. For simplicity, one can consider the symbols for intermediate products within the sectors to have identical meaning, so that the payment balance for every production sector can be written as

$$\begin{aligned}
0 &= Y_K + M_{S\to K} + M_{C\to K} - M_{K\to S} - I_K - W_K - \theta_K Y_K,\\
0 &= Y_S + M_{K\to S} + M_{C\to S} - M_{S\to K} - I_S - W_S - \theta_S Y_S,\\
0 &= Y_C - M_{C\to K} - M_{C\to S} - I_C - W_C - \theta_C Y_C.
\end{aligned} \tag{3.4}$$

Regarding the interaction of the production units with banks, for simplicity we assume that the commercial bank is the only source of financing of production activity, not considering the issuing of shares and bonds.[1] Consequently, we consider that

[1]Production units distributing *shares* can receive money to cover expenses directly from consumers. These primary securities are promissory notes on which emitters undertake to pay the cost of the securities and a percentage on them through a certain time and in a certain way. Money from securities is directed by the emitters to cover investment expenses, which after a while results in an additional product.

the financial state of the producers is defined by the amounts of deposits D and debts B (with corresponding superscripts) in the commercial banks. These quantities are connected with the balance equations

$$
\begin{aligned}
\frac{dD_K}{dt} &= r_K D_K + Y_K + M_{S\to K} + M_{C\to K} - M_{K\to S} \\
&\quad - I_K - W_K - \theta_K Y_K - q_K B_K + \frac{dB_K}{dt}, \\
\frac{dD_S}{dt} &= r_S D_S + Y_S + M_{K\to S} + M_{C\to S} - M_{S\to K} \\
&\quad - I_S - W_S - \theta_S Y_S - q_S B_S + \frac{dB_S}{dt}, \\
\frac{dD_C}{dt} &= r_C D_C + Y_C - M_{C\to K} - M_{C\to S} - I_C - W_C \\
&\quad - \theta_C Y_C - q_C B_C + \frac{dB_C}{dt}.
\end{aligned}
\tag{3.5}
$$

Here, we use the designations q_K, q_S, q_C for the norms of payments to banks for credit, and r_K, r_S, r_C for the norms of payments of banks for customer deposits. These quantities are established by the commercial banks to adjust the quantities of deposits and debts.

To exclude the payments for intermediate products from discussion, we introduce notation for the total amount of deposits and debts of the production customers in the commercial banks

$$
\begin{aligned}
D_P &= D_K + D_S + D_C, \qquad & r_P D_P &= r_K D_K + r_S D_S + r_C D_C, \\
B_P &= B_K + B_S + B_C, \qquad & q_P B_P &= q_K B_K + q_S B_S + q_C B_C.
\end{aligned}
$$

Summing up (3.5), we get

$$
\frac{dD_P}{dt} = r_P D_P + Y - I - W_P - \theta_P Y - q_P B_P + \frac{dB_P}{dt}. \tag{3.6}
$$

The loans allow the producers to avoid a disruption between receiving and payment. They are needed to provide the expenditures I and W_P, so one can consider that the amount of the loan is connected with the amount of the expenditures.

3.2.1.2 The Consumers

The financial state of the consumers is defined by the amounts of deposits and debts, D_L and B_L, in the commercial banks. In addition, they are holders of paper money in the amount M_0. The consumers use their money and possible loans to acquire products, so the balance equations for the consumers can be written as

$$
\frac{dD_L}{dt} = r_L D_L + W - C - \theta_L W - q_L B_L + \frac{dB_L}{dt} - \frac{dM_0}{dt}, \tag{3.7}
$$

where $W = W_K + W_S + W_C + W_G$ is a flux of money to workers in the form of wages, which are received from the production sectors (W_K, W_S, W_C) and the government (W_G). The consumers pay money in the amount C to the third sector for products for consumption, which were created some time ago, and to the government in the form of taxes, $\theta_L W$. In reality, the situation can be somewhat more complex; part of the wages can be used to purchase shares of the enterprises, that is, for investment in various sectors, which we do not consider here. The loan B_L is needed to provide the expenditures C. The banks establish norms of payments for debts and deposits of clients q_L and r_L.

The right-hand sides of (3.6) and (3.7) contain payments to and by the commercial banks. The banks establish the payments for debts and deposits of the customers q_P, q_L and r_P, r_L. These quantities can be considered as functions of the amounts of deposits and debts; they are determined by the commercial banks from the requirement to get a profit from bank operations (see the next section). So, as there is a payment for debts, customers try to reduce the quantity of debts as much as possible and to keep some money on the depository accounts in commercial banks.

3.2.2 Commercial Bank as a Customer of the Central Bank and a Producer of Credit Money

One can assume that the commercial bank has the only account with the central bank D_B, on which it holds all its reserve, including the amount of mandatory deposit of the commercial bank $\xi(D_P + D_L)$, where ξ is a norm of the mandatory reserve deposit set up by the central bank. The commercial banks have loans B_B from the central bank to produce credit money, which is needed to facilitate interaction among the production sectors and consumers within the economic system.

The state of the commercial bank is determined by its actives: K_{KB}, $D_B - \xi(D_P + D_L)$, B_P, B_L and passives: B_B, D_P, D_L, so that the income of the bank, neglecting the income from the bank's capital K_{KB}, can be written as

$$r_B\big[D_B - \xi(D_P + D_L)\big] - q_B B_B + q_P B_P + q_L B_L - r_P D_P - r_L D_L. \qquad (3.8)$$

The central bank fixes the payments for deposits and debts of the commercial bank, r_B and q_B, and the commercial bank sets norms of payments for debts and deposits of clients q_P, q_L and r_P, r_L. In any case, it is expected that the value of q with any index will appear greater than the value of r with an appropriating index. Usually the central bank does not pay for mandatory deposits of commercial banks and sets up a high level of the refinancing rate q_B. The norm of the mandatory reserve deposit ξ and the refinancing rate q_B are considered as main regulators of the amount of non-paper money.

The primary activity of commercial banks is connected with crediting the clients. To its disposal, the commercial bank gets the loan from the central bank in quantity $\frac{dB_B}{dt}$ which is used by the bank for crediting the clients. To provide loans, the bank uses the available quantity of money, including client deposits, and, at the same time, excepting a quantity of a mandatory reserve in the central bank. The sum of money provided on loan from the commercial bank, $B_P + B_L$, is connected with the sum of money B_B, provided on loan by the central bank. Indeed, according to the well-known mechanism of multiplication (see, for example, [8, p. 240]), taking an extra amount of credit ΔB_B from the central bank, a commercial bank can lend $(1-\xi)\Delta B_B$ to clients and other banks, whereas a part $\xi\Delta B_B$ of the total amount must be reserved in the central bank. The banks use the amount $(1-\xi)\Delta B_B$ for further lending, leaving the part $\xi(1-\xi)\Delta B_B$ of the amount in the central bank as a reserve. The process is continuing, so that the banks are creating money on customer deposits in the total amount

$$\Delta B_B + (1-\xi)\Delta B_B + (1-\xi)^2\Delta B_B + \cdots = \frac{1}{\xi}\Delta B_B,$$

and one can write the relation

$$\frac{d(B_P + B_L)}{dt} \le \frac{1}{\xi}\frac{dB_B}{dt}. \tag{3.9}$$

Considering the aspiration of commercial banks to expanding credits and some rationality of their behaviour, the inequality (3.9) can be recorded in the form of the equality

$$\frac{d(B_P + B_L)}{dt} = \frac{1}{\xi}\frac{dB_B}{dt}. \tag{3.10}$$

When being credited, the clients increase the amounts of deposits (see (3.6) and (3.7)) in the commercial bank: the bank creates credit money.

The amount of money in the commercial banks is understood as deposit D_B with the central bank. This quantity changes due to its income (3.8), increasing or decreasing the amount of the changes of the debt to the central bank B_B and operations with the customers of commercial banks, so that the balance equation can be written as

$$\begin{aligned}\frac{dD_B}{dt} = {} & r_B\bigl[D_B - \xi(D_P + D_L)\bigr] - q_B B_B + q_P B_P + q_L B_L \\ & - r_P D_P - r_L D_L + \frac{dB_B}{dt} + \frac{d(D_P + D_L)}{dt} - \frac{d(B_P + B_L)}{dt}. \end{aligned} \tag{3.11}$$

Using (3.6) and (3.7), (3.11) can be rewritten in the form

$$\begin{aligned}\frac{dD_B}{dt} = {} & r_B\bigl(D_B - \xi(D_P + D_L)\bigr) - q_B B_B \\ & + Y - I - C + W_G - T - E_0 + E_c, \end{aligned} \tag{3.12}$$

where symbols for emission of paper and credit money are introduced,

$$E_0 = \frac{dM_0}{dt}, \qquad E_c = \frac{dB_B}{dt}. \tag{3.13}$$

The mechanism of issuing assumes that all paper money is circulating among consumers: practically no paper money is contained in commercial banks.

Equations (3.12) describe the main relationship of the commercial bank with the central bank and clients.

The behaviour of the commercial bank is determined by its desire to maximise profits (3.8), and the bank can choose the quantities q_P, q_L, r_P, r_L at every moment of time. To increase profits, the commercial banks are motivated to produce more credits B_P and B_L to their customers and to take as little credit B_B from the central bank as possible, but there are some restrictions on these amounts: only an increase in B_B allows an increase in B_P and B_L. There are apparently some other restrictive relations on the amount of loans to the customers of the commercial bank $B_P + B_L$. The available reserve has to be positive, so the amount of credits cannot be more than the available resources,

$$B_P + B_L \leq D_B + (1 - \xi)(D_P + D_L) + K_{KB}.$$

Within this restriction, the commercial banks can increase loans $B_P + B_L$ to their customers, creating credit money. A bank needs some amount of initial capital K_{KB} to start operations.

3.2.3 The Government as a Customer of the Central Bank and the Central Bank as a Producer of Paper Money

The institution which is crucial in the organisation of money circulation in a society is the central bank, which is a bank of the commercial banks and the bank of the government. The activity of the central bank is closely connected with the activity of the government and is based on the credit to the government and the central bank's assets. To organise the money circulation, the central bank issues money in the form of paper notes (coins) and credits to commercial banks. The central bank creates fiat money that sets up a scale of value.

3.2.3.1 The Balance of the Government

The government wants at its disposal enough money to finance national projects and salary payments to civil servants. The main account of the government with the central bank presents the governmental budget and reflects motion of money to and from the government. The state of the government's budget is defined by the amounts of deposits D_G and debts B_G with the central bank. The motion of money includes a flux of taxes (and other income) T into the budget, which are the payments from the producers and consumers

$$T = \theta_P Y + \theta_L W, \tag{3.14}$$

where $Y = Y_K + Y_S + Y_C$ is the GDP with contributions from the three production sectors and $W = W_K + W_S + W_C + W_G$ is the total amount of wages paid to workers. The government supervises norms of the taxation θ_P and θ_L. As a rule, taxes should provide expenses of the government G, which represents investments in various national projects and wage payments to the civil servants W_G. In the closed system external loans are impossible and, consequently,

$$T \geq G + W_G.$$

The difference between the income and expenses

$$\Delta = T - G - W_G$$

is called a primary proficit, if $\Delta > 0$, or a primary deficit, if $\Delta < 0$, of the state budget.

With the expansion of production, the income and expenses of the government increase, and a possible budget deficit of the government in the closed system becomes covered due to the internal loan B_G and (or) issues of paper money M_0, so that the amount of money at the government's disposal D_G obeys the balance equation

$$\frac{dD_G}{dt} = r_G D_G + T - G - W_G - q_G B_G + E_g + E_0, \tag{3.15}$$

where symbols for emission of paper money and bonds are used

$$E_0 = \frac{dM_0}{dt}, \qquad E_g = \frac{dB_G}{dt}. \tag{3.16}$$

The government can stimulate the production sectors by some money interventions G. For simplicity, it is assumed further that all money is coming to the second sector producing non-material products.

Note that, for a more detailed analysis, one has to take into account that, if the government pays money at moment t, it receives taxes from the earlier activity. The loan is needed to provide the governmental expenditures G and W_G, so that one can assume that the amount of the loan is connected with the amount of the expenditures.

3.2.3.2 The Central Bank

For a closed economy, the only source of loan for the government is the central bank, the state of which is determined by its actives: K_{CB}, B_G, B_B and passives: D_G, D_B, M_0. It is assumed that the central bank is set up to organise the circulation of money in the system, not to receive profit, so that its payments for deposits and the expenses for production of money have to be balanced by receiving payments for loans. One can write the balance relation for the bank's income as

$$q_G B_G + q_B B_B - r_G D_G - r_B\big[D_B - \xi(D_P + D_L)\big] - \chi M_0 = 0, \tag{3.17}$$

where $q_B B_B$ is the payment of the commercial banks for the use of credits of the central bank, and q_B is a refinancing rate. The last term in (3.17) represents the expenses needed to maintain the circulation of the paper money. The expenses for production of a unit of money χ is a characteristic that is determined by the central bank. The norms of payments for debts and deposits of the government, q_G and r_G, in (3.15) and (3.17) are specified by agreement of the central bank with the government. Due to its close relationship with the government, the central bank does not intend to get any profit from the service to the government.

The central bank is set up to credit the government. In any case, the total amount of the credit of the central bank to the government should be restricted by the relation

$$B_G \leq D_G + D_B - B_B + M_0,$$

where M_0 is the total amount of paper money in circulation, and D_B is the sum of all deposits of the commercial banks with the central bank. The above requirement actually does not restrict the credit to the government. To supply the government, the central bank issues extra paper money, so that the total amount of paper money in circulation can be written as

$$M_0 \geq B_G - D_G - D_B + B_B. \tag{3.18}$$

The credit to the government opens the windows of the cash storehouses, and the fluxes of paper money rush into the economy. The extra emission is connected with the deficit of the government's budget.

3.3 Money Circulation and Production

The assumptions about the composition and architecture of the closed system, consisting of the government, the central bank, the commercial banks and many production and consumption units, allow us to describe the situation. The economic subjects are interacting with each other by means of fluxes of money. To create money, the central bank issues coins and paper money in the amount of M_0 and credits the commercial banks in the amount of B_B. The sum of the issued paper and non-paper money $M_0 + B_B$ is called the *monetary base*. The credit of the central bank B_B to the commercial banks provides an opportunity to credit the producers and consumers, thus creating deposits $D_P + D_L$, which can be called non-paper money. The non-paper money can be converted into paper money and, likewise, the paper money can be converted into non-paper money, so that the characteristic quantity is the sum of all deposits in commercial banks $D_P + D_L$ and paper money M_0. The total is called the *monetary mass*, for which the conventional symbol $M_2 = M_0 + D_P + D_L$ is used. The process of introducing and circulating money is described by the equations formulated in Sect. 3.2, and our task now is to calculate the amounts of both paper and non-paper money needed for the proper functioning of the production system.

3.3.1 The Description of Production

In the 'basement' of the economic activity, one can find apparently the real consumption and production. John Maynard Keynes wrote in his *Treatise on Money* that '[h]uman effort and human consumption are the ultimate matters from which alone economic transactions are capable of deriving any significance; and all other forms of expenditure only acquire importance from their having some relationship, sooner or later, to the effort of producers or to the expenditure of consumers' [9, pp. 120–121].

In the considered case the production-consumption processes are characterised with the quantities: I, G, C, W_P, W_G, which are assumed to be given as functions of time. An expression for the important characteristics of the system—the Gross Domestic Product, Y—can be obtained when (3.6), (3.7) and (3.15) have been aggregated,

$$\begin{aligned} Y &= I + G + C + q_P B_P + q_L B_L + q_G B_G - r_P D_P \\ &\quad - r_L D_L - r_G D_G + \frac{d(D_P + D_L + D_G)}{dt} - \frac{d(B_P + B_L + B_G)}{dt}. \end{aligned} \tag{3.19}$$

This formula appears to be a generalisation of (2.13): the GDP Y is a sum of assessments of investments I, output of non-material products G, direct consumption C and services of the bank system. The quantity Y can be determined after calculating the quantities of loans and deposits, according to the equations presented in the next section.

3.3.2 Equations for Money Circulation

The basis of the system of evolutionary equations comprises the balance parities discussed in Sect. 3.2. Equations (3.6), (3.7), (3.10), (3.12) and (3.15) connect the state variable of five interacting economic subjects, each one of which possesses certain financial actives and has its own tactics of behaviour. Below we rewrite these equations in a convenient form for the analysis,

$$\begin{aligned} \frac{dD_B}{dt} &= r_B\big(D_B - \xi(D_P + D_L)\big) - q_B B_B \\ &\qquad + Y - I - C + W_G - T - E_0 + E_c, \\ \frac{dD_P}{dt} &= r_P D_P + Y - I - W_P - \theta_P Y - q_P B_P + \frac{1-f}{\xi} E_c, \\ \frac{dD_L}{dt} &= r_L D_L + W - C - \theta_L W - q_L B_L - E_0 + \frac{f}{\xi} E_c, \\ \frac{dD_G}{dt} &= r_G D_G + T - G - W_G - q_G B_G + E_0 + E_g, \end{aligned} \tag{3.20}$$

$$\frac{dB_P}{dt} = \frac{1-f}{\xi} E_c, \qquad \frac{dB_L}{dt} = \frac{f}{\xi} E_c,$$
$$\frac{dB_B}{dt} = E_c, \qquad \frac{dM_0}{dt} = E_0, \qquad \frac{dB_G}{dt} = E_g,$$

where f defines the consumers' fraction in an increase in the amount of credit money in the commercial banks. The amount of tax is connected with the income of the enterprises and workers by (3.14), that is,

$$T = \theta_P Y + \theta_L (W_P + W_G).$$

The government fixes norms of taxes θ_P, θ_L, to provide the expenses G and W_G. The government and the central bank also determine the emission of paper and credit money E_c, E_0, E_g, which are considered as known functions of time. The central bank establishes the norm of mandatory deduction ξ and the norms of payments r_B, q_B for commercial banks. The commercial banks define norms of payments for deposits and credits r_P, r_L, q_P, q_L for the clients. Certainly, a necessary condition is non-negative profits of the commercial banks (3.8).

Within the determined restrictions, the commercial bank chooses the debt to the central bank, and the clients of commercial banks choose amounts of the deposits and loans. Each client of a commercial bank can choose a parity between debts and deposits, and the government can take money on debt (to accept obligations) or let out paper money at its sole discretion.

The set of equations (3.20) describes the behaviour of the money system as determined with seven variables: D_P, D_L, D_B, D_G, B_B, B_G, M_0. A difficulty is that the quantities, set by the government, the central bank and commercial banks, are not constant but depend on the situation and modes of behaviour of economic agents. Apparently, various models of behaviour of the agents are possible, and it is necessary to analyse an actual situation to record the appropriate equation. Though, one can add a condition (3.17) of non-profitness of the central bank to the system of equations, some variables remain free, and the system of balance equations should be somehow completed.

3.3.3 The Steady-State Situation

In the general case, a trajectory of development of the system is determined by evolutionary equations (3.20), which should be completed with some preferences of the economic agents, but the simplest steady-state situation is determined by the balance equations only, and one does not need any additional assumptions. In a steady-state case, one has to consider all characteristics of the system, as well as variables, to be constant. Economists call a steady-state situation an equilibrium one, but, from the thermodynamic point of view, the system is not in equilibrium, but in a stationary, non-equilibrium state.

3.3.3.1 The Steady-State Conditions

The steady-state situation means that all rates of growth of the variables are equal to zero, providing that the system (3.20) reduces to the equations

$$
\begin{aligned}
0 &= r_B(D_B - \xi(D_P + D_L)) - q_B B_B + q_P B_P + q_L B_L - r_P D_P - r_L D_L,\\
0 &= r_P D_P + Y - I - W_P - \theta_P Y - q_P B_P,\\
0 &= r_L D_L + W - C - \theta_L W - q_L B_L,\\
0 &= r_G D_G + T - G - W_G - q_G B_G.
\end{aligned}
\tag{3.21}
$$

The GDP, Y, defined by (3.19), and other characteristics of production are considered constant. The equations includes streams of taxes T from production and consumption (with indexes P and L) and a total sum of wages received by workers,

$$
T = \theta_P Y + \theta_L W, \qquad W = W_P + W_G. \tag{3.22}
$$

These quantities and other parameters of the system are also considered constant.

In a stationary case, from (3.10) it follows that the money borrowed by commercial banks from the central bank B_B should be defined as

$$
B_B = \xi(B_P + B_L) + \text{const.} \tag{3.23}
$$

The system of equations (3.21)–(3.23) does not define a unique point; the number of variables D_P, D_L, B_P, B_L, D_B, B_B, D_G, B_G appears to be greater than the number of equations.

With the use of (3.6) and (3.7), one can reformulate (3.21) to record the system in the following way:

$$
\begin{aligned}
&D_B = \xi(D_P + D_L) + \frac{q_B}{r_B}\left(B_B - B_B^0\right),\\
&B_B^0 = \frac{1}{q_B}(Y - I - T + W_G - C),\\
&D_P = \frac{q_P}{r_P}\left(B_P - B_P^0\right), \qquad B_P^0 = \frac{1}{q_P}(Y - I - W_P - \theta_P Y),\\
&D_L = \frac{q_L}{r_L}\left(B_L - B_L^0\right), \qquad B_L^0 = \frac{1}{q_L}(W - C - \theta_L W),\\
&D_G = \frac{q_G}{r_G}\left(B_G - B_G^0\right), \qquad B_G^0 = \frac{1}{q_G}(T - G - W_G).
\end{aligned}
\tag{3.24}
$$

These equations determine the quantities of deposits D_P, D_L, D_G and D_B in steady state as functions of loans B_P, B_L, B_G and B_B. The quantities of deposits should be considered non-negative, so that (3.24) determine the quantities of loans which are necessary for the commercial bank to start to function. The quantities of loans with zero indexes B_P^0, B_L^0, B_G^0 and B_B^0 can be interpreted, accordingly, as the minimum quantities of the capital of the commercial bank and the loan from the central bank which allow the bank system to start operations.

The recorded equations allow us to define the constant in (3.23), which now can be expressed in the form

$$B_B - B_B^0 = \xi\big(B_P - B_P^0 + B_L - B_L^0\big). \tag{3.25}$$

3.3.3.2 The Quantity of Circulating Money

This simple schematisation of real production-consumption processes allows one to estimate the amount of paper and non-paper money needed for a stationary functioning of the production system. The quantity of paper money, M_0, circulating in the system can be found from the condition of non-profitness (3.17), which, undoubtedly, should be valid for the central bank in a steady-state situation. This condition leads to the assessment of the quantity of paper money

$$M_0 = \frac{1}{\chi}\big\{T - G - W_G + q_B B_B - r_B\big[D_B - \xi(D_P + D_L)\big]\big\}. \tag{3.26}$$

By using (3.21), one finds other forms of the expression for paper money,

$$\begin{aligned} M_0 &= \frac{1}{\chi}(Y - I - G - C) \\ &= \frac{1}{\chi}(q_P B_P + q_L B_L - r_P D_P - r_L D_L + q_G B_G - r_G D_G). \end{aligned} \tag{3.27}$$

These equations show that the quantity of paper money is determined by the functioning of the production system and is connected with the activity of the bank system: a part of the output is presented in monetary form. The recorded equations, in fact, do not determine an absolute quantity of paper money, but present a relation between quantity of circulating money and value of production output and show that this relation depends on the overall performance of the bank system.

The obtained relations allow one to record expressions for monetary base $M_0 + B_B$, as the sums of (3.25) and (3.26), and for money mass $M_0 + D_P + D_L$, as the sums of expressions (3.24) and (3.26). These quantities are determined, naturally, by the activity of the entire system of consumption-production, not only by the activity of the bank system. The expressions for both monetary base and money mass can be expressed through arbitrary amounts of the loan B_B to a commercial bank from the central bank. We recall that all relations are valid for steady-state situations, or, as the economists say, for situations of equilibrium.

3.3.3.3 Efficiency of Social Production

It is convenient to introduce the characteristic of efficiency of the organisation of social production, considering the ratio of the assessment of services of the bank system to the actual output of the production system $I+G+C$. The expression for the GDP (3.19) can be rewritten in the form

$$Y=(1+\chi\theta)(I+G+C). \tag{3.28}$$

The greater the quantity $\chi\theta$, the more expenses for the maintenance of monetary circulation one has, and the less effective is the social production.

Equations (3.27) and (3.28) allow one to present an expression for the quantity of paper money in the form

$$M_0=\theta(I+G+C), \tag{3.29}$$

where the parameter of efficiency of the social production looks like

$$\theta=\frac{1/\chi}{I+G+C}(q_P B_P+q_L B_L-r_P D_P-r_L D_L+q_G B_G-r_G D_G). \tag{3.30}$$

3.3.4 Unsteady Situations

The estimation of the quantity of money for unsteady situations of the system can be executed on the basis of the system of equations (3.20). We have to note that, considering non-stationary situations, it would be more correct to keep inequality (3.9) in the system of equations, but one can keep the equality sign, assuming that the parameter ξ in (3.10) is some effective quantity, which is not identical with its value installed by the central bank.

3.3.4.1 The Program for Production Development

In the basement of any program of economic development, one can apparently find the program of consumption and production. It is natural to believe that, by studying the actual situation, the producers, the government and the consumers can formulate their programs of expenditure, which can be described by means of the time-dependent rates of growth as

$$\begin{aligned}
&\frac{dI}{dt}=\sigma_I I, \qquad \frac{dG}{dt}=\sigma_G G, \qquad \frac{dC}{dt}=\sigma_C C,\\
&\frac{dW_P}{dt}=\psi_P W_P, \qquad \frac{dW_G}{dt}=\psi_G W_G.
\end{aligned} \tag{3.31}$$

Values of the growth rates σ_I, σ_G, σ_C, ψ_W, ψ_G are, generally speaking, functions of time which are defined or appointed by operating agents.

The mean values of the growth rates

$$\begin{aligned} \sigma &= \frac{1}{I+G+C}(\sigma_I I + \sigma_G G + \sigma_C C), \\ \psi &= \frac{1}{W_P + W_G}(\psi_P W_P + \psi_G W_G) \end{aligned} \tag{3.32}$$

can be used accordingly as some characteristic values of the growth rates of the production fluxes of money and wage payments to the workers.

3.3.4.2 Balanced Growth

To estimate the quantity of money in the developing system at given constant rates of growth, it is necessary to address the system of equations (3.20). For harmonious progress, the plan of issue of paper and credit money E_c, E_0, E_g should conform to the production situation, and the government and the central bank proceed to release emission after the assessment of the situation. Normally, the emission should be connected with the growth rate of production σ which, in this case, is equal to the growth rate of wages $\sigma = \psi$, so that one has

$$E_0 = \sigma M_0, \qquad E_c = \xi \sigma B_B, \qquad E_g = 0. \tag{3.33}$$

It is possible to assume that the plan of emission (3.33) provides the 'correct' balanced growth of paper and credit money, so that relation (3.29), that is, relation

$$M_0 = \theta(I + G + C) \tag{3.34}$$

with constant coefficient of proportionality θ remains valid. The factor of proportionality θ appears to be greater, the less effective the production-distribution system is. This results in a greater share of the bank sector in a total national product.

3.3.4.3 Unbalanced Growth

When estimating the rates of real growth, humans can make involuntary mistakes, and the correspondence of monetary streams to the streams of products can be disturbed. In fact, neither the central bank nor commercial banks can directly adjust the quantity of both paper and non-paper money, and the purchasing capacity of the unit of money (rouble, dollar, euro and so on) can change in due course, as is observed in reality. Besides, the banks, as active participants of economic processes, have a temptation to receive additional profit from their activity, which tends to increase the factor of proportionality in the ratio (3.34).

Generally, conditions for balanced growth are not fulfilled. One can consider the simplest case, when the growth rate of production σ does not coincide with the rate of growth of wages ψ. The emission of paper money is more likely connected with the rate of growth of wages ψ, so that expressions for the emission should be written as

$$E_0 = \psi M_0, \qquad E_c = \xi \sigma B_B, \qquad E_g = 0. \tag{3.35}$$

In the general case, it is possible to retain the form of (3.34), which can be conveniently written as

$$M_0 = \theta \rho (\underline{I + G + C}), \tag{3.36}$$

where the bottom line means that the value of the output is measured in monetary units of constant purchasing capacity. The quantity θ, as a characteristic of the system, is considered to be constant, and the dependence of the factor of proportionality on time is attributed to the price index ρ, which reflects all variations of the monetary unit. Moreover, the money is distributed over the production sectors in a non-uniform manner: some sectors have more money than necessary for normal functioning, others have less, so that one needs to introduce many price indexes. However, for simplicity, one can consider only one price index.

The relation (3.36) is usually discussed as a relation of the quantity theory of money,[2] in which the coefficient θ has been interpreted as the mean time from production to consumption of final product. This quantity is a characteristic of the system and does not depend on the amount of the circulating money M (at large M). The reverse quantity to θ is interpreted as a velocity of money circulation.

3.4 About the Unit of Measurement of Value

The fluxes of products are estimated by means of some arbitrary monetary unit, the value of which is defined in terms of quantities of products that can be bought with this unit. This quantity can be established by parity (3.36), where the quantity of paper money M_0 introduced into the system is compared with production output $\underline{I + G + C}$, measured by some 'physical' measure, that is the quantity of some product. It is known (see Sect. 2.2.2) that, in the practice of assessing of the fluxes of products, the monetary unit of constant purchasing capacity is used. The assessment of current fluxes of products with the monetary unit of constant purchasing capacity excludes any price index and favours balanced development.

Throughout the centuries the role of 'monetary unit' was played by various products, but gold eventually achieved a special advantage, and the monetary unit before the first world war almost everywhere was defined as a quantity of gold (the gold standard). The 'gold' monetary unit is not a constant measure of value, but can be a proxy of the unit of constant purchasing capacity. To be a proper unit of constant

[2] The relation of the quantity theory of money is also known as Fisher's relation [3], though, according to Harrod [10, p. 26] this law was classically exposed in the report of the British Bullion Committee in 1810. Moreover, Harrod notes: 'Of course, the Bullion Committee did not invent the quantity theory. Traces of it may be found in writers dating back for centuries before that.'

purchasing capacity, the changes in the effort involved in producing a unit of gold have to correspond to the changes in effort for production of any other product. Apparently, this condition cannot be fulfilled in practice.

The use of gold as a money unit leads to some difficulties for the government and the central bank, which has caused a movement for imputation of the idea of the gold standard. Thus it appears that modern monetary units do not adhere absolutely to any 'physical content.' The price of gold grows in the conventional monetary units, which testifies that the purchasing capacity of these units falls. The modern and continuously changing money units, which are not connected with any 'physical content,' create severe difficulties, for both the functioning and the analysis of economic systems. Although the 'gold' monetary unit is not an ideal measure of purchasing capacity, it is better than any monetary unit not connected with 'a physical content.' Apparently, projects with a return to the gold standard will appear, though these projects would have many powerful opponents from those who receive income from a manipulation of monetary units.[3]

There is a question of whether some true scale of value similar to the kilogramme or meter for mass and length can be introduced. Is it possible to find an objective basis for the establishment of a monetary unit? The crisis situations of the past years have shown that the problem is worth thinking about. We shall return to this question in Chap. 10.

References

1. Leontief, W.W.: Input–Output Economics, 2nd edn. Oxford University Press, New York (1986)
2. Blaug, M.: Economic Theory in Retrospect, 5th edn. Cambridge University Press, Cambridge (1997)
3. Fisher, I.: The Purchasing Power of Money: Its Determination and Relation to Credit, Interest, and Crises. MacMillan, New York (1911)
4. Keynes, J.M.: The General Theory of Employment, Interest and Money. MacMillan, New York (1936)
5. Graziani, A.: The Monetary Theory of Production. Cambridge University Press, Cambridge (2003)

[3] As known financier Lietaer [11, p. 254] writes: 'The world has been living without an international standard of value for decades, a situation which should be considered as inefficient as operating without standard of length or weight.' To demonstrate the indispensability of a constant unit of measure for production efficiency, we shall imagine a contractor who builds houses. It is possible to utilise any measure of length to build a house, a good house, which is pleasant not only to the customer, but also to another customer who will hasten to place an order. At the construction of the second house the contractor does not notice that its measure of length has decreased a little bit; this will not prevent him from building precisely the same house, but a little bit smaller in size, which allows the contractor to save building materials. And what will occur, if the measure of length changes during construction of the house? Certainly, the cunning contractor has realised for a long time that the skillful manipulation of a measure of length brings good income, and he does not have any interest in changing that. And what do his clients think?

6. Chiarella, C., Flaschel, P.: Keynesian monetary growth dynamics in open economies. Ann. Oper. Res. **89**, 35–59 (1999)
7. Keen, S.: The dynamics of the monetary circuit. In: Rossi, S., Ponsot, J.-F. (eds.) The Political Economy of Monetary Circuits: Tradition and Change, pp. 161–187. MacMillan, London (2009)
8. Samuelson, P., Nordhaus, W.: Economics, 13th edn. McGraw-Hill, New York (1989)
9. Keynes, J.M.: A Treatise on Money. MacMillan, London (1930). Reprinted in: The Collected Writings of John Maynard Keynes, vols. V and VI, MacMillan for the Royal Economic Society, London and Basingstoke (1971)
10. Harrod, R.F.: Money. MacMillan, London (1969)
11. Lietaer, B.: The Future of Money. A New Way to Create Wealth, Work, and a Wiser World. Century, London (2001)

Chapter 4
Many-Sector Model of Production System

Abstract In this chapter, the input–output model, already discussed in Chap. 2, is considered in a linear approximation. This allows us to formulate the static and dynamic equations for the vector of gross output. The model determines the growth rate, which, in the considered case, when only internal restrictions for development are taken into account, is the rate of potential growth. The restrictions imposed by labour and energy will be introduced in the following chapters, and we will return to the many-sector dynamic equations in Chap. 8.

4.1 Linear Approximation

A method of production first determines what one needs to create this or that thing, determining the material side of the production process. Exploiting, due to Leontief [1–3] and Sraffa [4], the many-sector model of the production system of an economy, one describes the transfer of products from one sector to another, which is reflected in the balance equations (2.3) and (2.4), that is,

$$X_i = \sum_{j=1}^{n} X_i^j + Y_i, \qquad X^i = \sum_{j=1}^{n} X_j^i + Z^i. \tag{4.1}$$

These equations contain the gross and final sector outputs X_j and Y_j, intermediate production consumption X_j^i and the sector production of value Z_i. The equations do not allow one to determine the output of the production system of the economy without extra information. A fortunate idea was to reduce the number of variables, by introducing quantities which are characteristics of the production system itself. We shall discuss the simplest case of linear approximation, which was proposed and investigated by Leontief [1, 2]. We will consider some assumptions to connect the intermediate production consumption X_j^i and fixed production capital K_j^i with the gross sector output X_j. This allows one to introduce fundamental technological characteristics of technology: technological matrices (4.5) and (4.13).

V.N. Pokrovskii, *Econodynamics*, New Economic Windows 12,
DOI 10.1007/978-94-007-2096-1_4,

4.1.1 The Input–Output Matrix

Apparently, the greater the gross output of a sector, the more intermediate products are needed for production. This observation allows one to reduce intermediate production consumption X^i_j to gross output X_i by introducing characteristics of technology, which can be done in various ways. In the original description [1, 2], it is assumed that the intermediate production consumption X^i_j is proportional to the gross sector output,

$$X^i_j = a^i_j X_i \tag{4.2}$$

with coefficients of proportionality a^i_j reflecting the exploited technology and comprising a matrix. Otherwise, one can assume that a technological matrix can be introduced as the ratio of the velocities of the quantities

$$\frac{dX^i_j}{dt} = \tilde{a}^i_j \frac{dX_i}{dt}. \tag{4.3}$$

The last relation defines a matrix of intermediate consumption coefficients, which are characteristics of technology introduced in a given moment of time, while (4.2) introduces a matrix of intermediate consumption coefficients, which are average characteristics of all existing technology. It is easy to see that the matrices introduced by the different methods are connected with each other

$$\tilde{a}^i_j = a^i_j + \frac{1}{\delta^i} \frac{da^i_j}{dt}, \tag{4.4}$$

where $\delta^i = \frac{1}{X_i} \frac{dX_i}{dt}$ is the growth rate of the gross output. Of course, in the case where the technology does not depend on time, components of matrices introduced in alternative ways coincide. The difference can be negligible if the technology changes slowly.

We shall follow Leontief [1, 2], who has chosen relation (4.2) to introduce the matrix of intermediate consumption coefficients, or input–output matrix, which in complete form can be written as

$$\mathsf{A} = \begin{Vmatrix} a^1_1 & a^2_1 & \dots & a^n_1 \\ a^1_2 & a^2_2 & \dots & a^n_2 \\ \dots & \dots & \dots & \dots \\ a^1_n & a^2_n & \dots & a^n_n \end{Vmatrix}. \tag{4.5}$$

Matrix A is a phenomenological characteristic of the technological organisation existing in the production system at the moment. It changes if the technology organisation changes during the time. The components of matrix a^i_j represent a mixture of all technologies, old and new. In contrast, the components of the matrix $\tilde{a}^i_j$ represent characteristics of newly introduced technology.

4.1.2 Static Leontief Equation

One can use (4.2) to rewrite the balance relations (4.1) in the form

$$X_j = \sum_{i=1}^{n} a^i_j X_i + Y_j, \tag{4.6}$$

$$X_i = a^i X_i + Z^i, \quad a^i = \sum_{l=1}^{n} a^i_l. \tag{4.7}$$

The first of these equations is known as the Leontief equation. Equations (4.6) and (4.7) connect three vectors: the gross output X_j, the final output Y_j and the sector production of value Z^i, so that only one of them can be considered to be independent. All the variables X_j, Y_j and Z_j in (4.6) and (4.7) are referred to the same moment of time.

The first equation in (4.7) allows one to specify the properties of matrix A. A requirement of productivity is that the sector production of value has to be non-negative,

$$Z^i \geq 0,$$

and taking into account (4.7), this is followed by the property

$$\sum_{l=1}^{n} a^i_l < 1, \quad i = 1, 2, \ldots, n.$$

It is also natural to suppose that all components of matrix A are non-negative, so that one has for each component of matrix A

$$0 \leq a^i_j < 1, \quad i, j = 1, 2, \ldots, n. \tag{4.8}$$

4.1.3 The Planning of Gross Output

To create a final product Y_j, one needs the products of, generally speaking, all sectors, so one has to consider production of the gross output in all sectors. Equation (4.6), which is known as the static Leontief equation, allows one to plan the gross output X_j which is needed to get the final output Y_j at a given technology. It is convenient to rewrite (4.6) in vector form,

$$\mathsf{X} = \mathsf{A}\mathsf{X} + \mathsf{Y}. \tag{4.9}$$

The formal solution of this equation

$$\mathsf{X} = (\mathsf{E} - \mathsf{A})^{-1}\,\mathsf{Y}, \tag{4.10}$$

where E is the unity matrix, can be represented as

$$\mathsf{X} = \left(\mathsf{E} + \sum_{k=1}^{\infty} \mathsf{A}^k \right) \mathsf{Y}. \tag{4.11}$$

From property (4.8) of matrix A,

$$\mathsf{A}^k \to 0,$$

so that the series in (4.11) converges. The non-negativity of components of the matrix ensures non-negativity of the gross output, when the final output is non-negative. It can be tested directly that expression (4.11) satisfies (4.9). Indeed,

$$\begin{aligned}
\mathsf{Y} = (\mathsf{E} - \mathsf{A}) \left(\mathsf{E} + \sum_{k=1}^{\infty} \mathsf{A}^k \right) \mathsf{Y} \\
= \left(\mathsf{E} + \sum_{k=1}^{\infty} \mathsf{A}^k - \mathsf{A} - \sum_{k=1}^{\infty} \mathsf{A}^{k+1} \right) \mathsf{Y} \\
= \left(\mathsf{E} + \sum_{k=1}^{\infty} \mathsf{A}^k - \sum_{i=1}^{\infty} \mathsf{A}^i \right) \mathsf{Y} \; = \; \mathsf{Y}.
\end{aligned}$$

Note that in representation (4.11), matrix A is called the matrix of direct input, matrix A^2 is called the matrix of indirect input of the first order, and so on. The matrix $(\mathsf{E} - \mathsf{A})^{-1}$ is called the matrix of total input.

4.1.4 The Capital-Output Matrix

To produce something, one also needs production equipment (production capital stock); the greater the amount, the more the scale of production. In linear approximation, the fixed production capital (value of basic production equipment) of type j in sector i is proportional to the gross sector output

$$K^i_j = b^i_j X_i. \tag{4.12}$$

The coefficient of proportionality b^i_j represents a mixture of all technologies, old and new, if the technologies are changing during that time.[1]

[1] Instead of matrix B, one can use matrix $\tilde{\mathsf{B}}$ which characterises currently introduced technology. The relations between components of matrices B and $\tilde{\mathsf{B}}$ are the same as those between components of matrices A and $\tilde{\mathsf{A}}$ in (4.4).

The relation (4.12) defines the matrix of fixed capital coefficients (capital-output matrix)

$$\mathsf{B} = \begin{Vmatrix} b_1^1 & b_1^2 & \dots & b_1^n \\ b_2^1 & b_2^2 & \dots & b_2^n \\ \dots & \dots & \dots & \dots \\ b_n^1 & b_n^2 & \dots & b_n^n \end{Vmatrix}. \tag{4.13}$$

In line with the input–output matrix A, matrix B is also a characteristic of the technology used in the production system.

It is easy to get a relation between the amount of production equipment of a certain kind K_j and sector capital K^i. Indeed, from the definitions of the quantities in Sect. 2.3.2 and relation (4.12), one finds that the quantities are connected with each other by means of components of the capital-output matrix

$$K_j = \sum_{i=1}^{n} \bar{b}_j^i K^i. \tag{4.14}$$

Here a non-dimensional matrix of capital coefficients is introduced,

$$\bar{b}_j^i = \left(b^i\right)^{-1} b_j^i, \quad b^i = \sum_{l=1}^{n} b_l^i. \tag{4.15}$$

By definition, components of this matrix are connected by the relations

$$\sum_{j=1}^{n} \bar{b}_j^i = 1, \quad i = 1, 2, \dots, n.$$

It is also possible to get a relation between the investment of product j in all sectors I_j and total investment I^i in sector i. To obtain such a relation, one can refer to the equations for the dynamics of the total amount of equipment K_j and for the dynamics of fixed sectoral capital K^i from Sect. 2.3.2, that is, to the equations

$$\begin{aligned} \frac{dK_j}{dt} &= I_j - \mu K_j, \quad j = 1, 2, \dots, n, \\ \frac{dK^i}{dt} &= I^i - \mu K^i, \quad i = 1, 2, \dots, n. \end{aligned} \tag{4.16}$$

To get a relation between the investment of product j in all sectors I_j and total investment I^i in sector i, one can use relation (4.14) and rewrite (4.16) in the form

$$\sum_{i=1}^{n} \bar{b}^i_j \frac{dK^i}{dt} + \sum_{i=1}^{n} K^i \frac{d\bar{b}^i_j}{dt} = I_j - \mu \sum_{i=1}^{n} \bar{b}^i_j K^i, \quad j = 1, 2, \ldots, n,$$
$$\frac{dK^i}{dt} = I^i - \mu K^i, \quad i = 1, 2, \ldots, n. \tag{4.17}$$

Then, we can exclude the derivatives of sector capital from relations (4.17) to obtain

$$I_j = \sum_{i=1}^{n} \bar{b}^i_j I^i + \sum_{i=1}^{n} K^i \frac{d\bar{b}^i_j}{dt}. \tag{4.18}$$

The approximations made earlier are connected with representations of depreciated capital, so we ought to consider relations (4.18) to be valid within the first terms with respect to the growth rates.

4.2 Effects of Prices

Relations (4.6) and (4.7) are written with the implicit assumption that prices of products are given and constant. To obtain a law of transformation of matrix A when the prices are changing, we assume that there is a reference state with a set of the fixed prices, which are considered to be all equal to unity, and an arbitrary state with a given set of prices, so that components of the output in the reference (with sign ˆ) and the arbitrary states are connected by relations

$$X_i = p_i \hat{X}_i, \qquad Y_i = p_i \hat{Y}_i, \quad i = 1, 2, \ldots, n, \tag{4.19}$$

where p_i is the price index in the arbitrary state. One can assume that the basic balance equations are valid at any system of prices.

4.2.1 The Condition of Consistency

Equation (4.6) can then be rewritten in the form

$$\hat{X}_j = \sum_{i=1}^{n} a^i_j \frac{p_i}{p_j} \hat{X}_i + \hat{Y}_j.$$

The balance equation should have the same form for any system of prices, so a new input–output matrix must be introduced,

$$\hat{a}^i_j = a^i_j \frac{p_i}{p_j}, \quad a^i_j = \hat{a}^i_j \frac{p_j}{p_i}. \tag{4.20}$$

Certainly, the description also must be covariant, if one chose another way of description using the matrix (4.4). In this case, in line with (4.6), one has the equation

$$\frac{dX_j}{dt} = \sum_{i=1}^{n} \tilde{a}^i_j \frac{dX_i}{dt} + \frac{dY_j}{dt}, \tag{4.21}$$

which can be rewritten with the help of (4.19) in the form

$$p_j \frac{d\hat{X}_j}{dt} + \hat{X}_j \frac{dp_j}{dt} = \sum_{i=1}^{n} \tilde{a}^i_j \left(p_i \frac{d\hat{X}_i}{dt} + \hat{X}_i \frac{dp_i}{dt} \right) + p_j \frac{dY_j}{dt} + \hat{Y}_j \frac{dp_j}{dt}.$$

So as balance equations have the same form for any system of prices, one should write two separate relations: a balance equation and an equation for prices,

$$\begin{aligned} \frac{d\hat{X}_j}{dt} &= \sum_{i=1}^{n} \hat{\tilde{a}}^i_j \frac{d\hat{X}_i}{dt} + \frac{d\hat{Y}_j}{dt}, \\ \hat{X}_j \frac{d\ln p_j}{dt} &= \sum_{i=1}^{n} \hat{\tilde{a}}^i_j \hat{X}_i \frac{d\ln p_i}{dt} + \hat{Y}_j \frac{d\ln p_j}{dt}, \end{aligned} \tag{4.22}$$

where, similar to relations (4.20), one has

$$\hat{\tilde{a}}^i_j = \tilde{a}^i_j \frac{p_i}{p_j}, \quad \tilde{a}^i_j = \hat{\tilde{a}}^i_j \frac{p_j}{p_i}.$$

Relation (4.22) can be considered as a set of equations for quantities

$$\frac{d\ln p_i}{dt}, \quad i = 1, 2, \ldots, n$$

which can have a non-trivial solution if the following condition is fulfilled:

$$\left| (\delta^i_j - \tilde{a}^i_j) X_i - Y_j \delta^i_j \right| = 0. \tag{4.23}$$

If $\tilde{a}^i_j$ does not depend on time, relation (4.23) is always valid; otherwise, condition (4.23) presents an equation for the growth rates of components $\tilde{a}^i_j$.

One can see that the prices of products cannot be quite arbitrary quantities. A non-trivial solution of set (4.22) determines a relation between prices of different products. However, the prices are appointed independently, and it is possible to imagine a situation in which the prices are chosen in such a way that (4.22) is not satisfied. This means an infringement of the value balance of products, which cannot last long.

Note that it is convenient to rewrite relations (4.19) in vector form,

$$\mathsf{X} = \mathsf{P}\hat{\mathsf{X}}, \qquad \hat{\mathsf{X}} = \mathsf{P}^{-1}\mathsf{X}, \qquad \mathsf{Y} = \mathsf{P}\hat{\mathsf{Y}}, \qquad \hat{\mathsf{Y}} = \mathsf{P}^{-1}\mathsf{Y}, \tag{4.24}$$

where the transformation matrices are introduced,

$$\mathsf{P} = \begin{Vmatrix} p_1 & 0 & \dots & 0 \\ 0 & p_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & p_n \end{Vmatrix}, \qquad \mathsf{P}^{-1} = \begin{Vmatrix} p_1^{-1} & 0 & \dots & 0 \\ 0 & p_2^{-1} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & p_n^{-1} \end{Vmatrix}. \tag{4.25}$$

Then, the rules (4.20) for transformation of the technological matrix can be written as

$$\mathsf{A} = \mathsf{P}\hat{\mathsf{A}}\mathsf{P}^{-1}, \qquad \hat{\mathsf{A}} = \mathsf{P}^{-1}\mathsf{A}\mathsf{P}. \tag{4.26}$$

It is easy to see that analogous relations are valid for the matrix of capital coefficients,

$$\mathsf{B} = \mathsf{P}\hat{\mathsf{B}}\mathsf{P}^{-1}, \qquad \hat{\mathsf{B}} = \mathsf{P}^{-1}\mathsf{B}\mathsf{P}.$$

One calls matrices $\hat{\mathsf{A}}$ and A, and also matrices $\hat{\mathsf{B}}$ and B, similar matrices [5]. To separate changes of matrices that are connected with technology changes from the changes influenced by price changes, one has to consider some invariant combinations of components of the matrices. It is known that similar matrices have the same eigenvalues that, thus, do not depend on the prices.

4.2.2 Dynamics of Sectoral Production of Value

In line with relation (4.7), an equation for the derivatives of the quantities can be written

$$\frac{dZ^i}{dt} = \left(1 - \tilde{a}^i\right)\frac{dX_i}{dt}. \tag{4.27}$$

This relation, taking the formulae (4.19) into account, can be rewritten in the form

$$\frac{dZ^i}{dt} = \left(1 - \tilde{a}^i\right)\left(p_i\frac{d\hat{X}_i}{dt} + \hat{X}_i\frac{dp_i}{dt}\right). \tag{4.28}$$

The growth rate of the sector production of value in a new set of prices is broken into two parts: a rate connected with a change of gross output in ‘natural’ units and a rate connected with a change of the price.

One can consider the sector as an economic agent that plans its activity and can make decisions about the amount of gross output. The aim of the sector is to obtain a bigger amount of production of value Z^j. One can assume that increase of output is stimulated by increase of production of value in the sector. In the simplest case,

$$\frac{d\hat{X}_i}{dt} = k_i\frac{dZ^i}{dt}, \tag{4.29}$$

where k_i $(i = 1, 2, \ldots, n)$ are sensibility coefficients. The coefficient k_i shows how sector i reacts with an increase of production of value. We consider k_i to be non-negative and limited due to the possibilities of production

$$k_i < \frac{1}{(1-\tilde{a}^i)p_i}. \tag{4.30}$$

Relations (4.28) and (4.29) determine the rate of sector production of value,

$$\frac{dZ^i}{dt} = \frac{(1-\tilde{a}^i)X_i}{[1-k_i(1-\tilde{a}^i)p_i]p_i}\frac{dp_i}{dt}. \tag{4.31}$$

From this equation, using relations (4.21) and (4.27), one can obtain formulae for the growth rates of the gross and final products of the sector,

$$\frac{dX_i}{dt} = \frac{X_i}{[1-k_i(1-\tilde{a}^i)p_i]p_i}\frac{dp_i}{dt}, \quad i = 1, 2, \ldots, n, \tag{4.32}$$

$$\frac{dY_j}{dt} = \sum_{i=1}^{n}\frac{(\delta^i_j - \tilde{a}^i_j)X_i}{[1-k_i(1-\tilde{a}^i)p_i]p_i}\frac{dp_i}{dt}, \quad j = 1, 2, \ldots, n. \tag{4.33}$$

The last expression defines the partial derivatives of the function $Y_j = Y_j(p_1, p_2, \ldots, p_n)$, which is called the supply function, as

$$\frac{\partial Y_j}{\partial p_i} = \frac{(\delta^i_j - \tilde{a}^i_j)X_i}{[1-k_i(1-\tilde{a}^i)p_i]p_i}. \tag{4.34}$$

One can see that the final output of a sector is an increasing function of the price of its own product, and a decreasing function of the prices of all other products.

4.3 Dynamics of Output

The relations given in the previous sections allow one to write the equations for gross and final outputs at given matrices A and B as functions of time. To get a dynamic equation for gross output, we differentiate relation (4.12), obtaining

$$\frac{dK^i_j}{dt} = b^i_j\frac{dX_i}{dt} + X_i\frac{db^i_j}{dt}, \quad i, j = 1, 2, \ldots, n,$$

and refer to (2.22) for the dynamics of value of basic production equipment K^i_j of type j in sector i, that is to the equation

$$\frac{dK^i_j}{dt} = I^i_j - \mu K^i_j, \quad i, j = 1, 2, \ldots, n. \tag{4.35}$$

A comparison of the above equations determines a set of dynamic equations for gross output X_i in the form

$$b^i_j \frac{dX_i}{dt} + X_i \frac{db^i_j}{dt} = I^i_j - \mu b^i_j X_i, \quad i, j = 1, 2, \ldots, n. \tag{4.36}$$

It is assumed that matrix B and gross investment I^i_j are given as functions of time.

4.3.1 Dynamic Leontief Equation

The situation becomes simpler if one assumes that time dependence of the matrix B can be neglected. In this case, one can rewrite (4.36) as

$$\sum_{i=1}^{n} b^i_j \frac{dX_i}{dt} = I_j - \mu \sum_{i=1}^{n} b^i_j X_i, \quad j = 1, 2, \ldots, n. \tag{4.37}$$

The gross investment of product j in all sectors I_j is determined, according to relation (2.11), as

$$I_j = Y_j - C_j - G_j, \quad j = 1, 2, \ldots, n, \tag{4.38}$$

where C_j is the total personal consumption of product j, and G_j is investment in storage of intermediate products.

Because of the non-negativity of the quantities C_j and G_j, the investments I_j are certain parts of the final output, so they can be conveniently written as

$$I_j = s_j Y_j, \quad 0 < s_j < 1, \; j = 1, 2, \ldots, n, \tag{4.39}$$

and, then, from (4.6), as

$$I_j = s_j \sum_{i=1}^{n} \left(\delta^i_j - a^i_j\right) X_i, \quad 0 < s_j < 1, \; j = 1, 2, \ldots, n. \tag{4.40}$$

The quantity s_j is introduced here as the ratio of investment products to final product of sector j.

The last relation allows us to rewrite (4.37) in the following form:

$$\sum_{i=1}^{n} b^i_j \frac{dX_i}{dt} = s_j \sum_{i=1}^{n} \left(\delta^i_j - a^i_j\right) X_i - \mu \sum_{i=1}^{n} b^i_j X_i, \quad j = 1, 2, \ldots, n. \tag{4.41}$$

This is a dynamic equation for gross sectoral output in a form which was originally derived by Leontief [2]. The equation can be conveniently rewritten in vector form:

$$\mathsf{B} \frac{d\mathsf{X}}{dt} = \left[\mathsf{S}(\mathsf{E} - \mathsf{A}) - \mu \mathsf{B} \right] \mathsf{X}, \tag{4.42}$$

where S is a symbol for the diagonal matrix with values s_j on the diagonal.

4.3.2 Balanced Growth

Equation (4.41) is applicable to a real system, if changes of matrices A and B due to technological changes can be neglected. In the considered case, when all parameters in (4.41) are constant, the gross output can be found in the form

$$X_j(t) = X_j(0)e^{\sigma t}, \tag{4.43}$$

where $X_j(0)$ satisfy the set of algebraic equations

$$\big[\mathsf{S}(\mathsf{E}-\mathsf{A}) - (\mu+\sigma)\mathsf{B}\big]\mathsf{X}(0) = 0. \tag{4.44}$$

The solution (4.43) presents a trajectory of the balanced (homothetic) growth, when the growth rates of the gross output in all sectors are the same.

There are non-trivial solutions of (4.44) if the determinator of the system (4.44) is equal to zero, that is,

$$\big|\mathsf{S}(\mathsf{E}-\mathsf{A}) - (\mu+\sigma)\mathsf{B}\big| = 0. \tag{4.45}$$

This is an equation for the growth rate σ. However, it is easy to see that, if there is a sector in the production system which does not create products for investment, the determinant (4.45) is identically equal to zero at any value of σ.

In any case, the growth rate of the output is restricted. Specially developed methods [6] are used (see, for example, [7]) to calculate the growth rate for the homothetic trajectories. To apply these methods, one can consider (4.41) in moments of time $t = 0, 1, 2, \ldots$ and assume

$$X_i = X_i(t), \qquad \frac{dX_i}{dt} = X_i(t+1) - X_i(t),$$

which implies

$$\sum_{i=1}^{n} b_j^i X_i(t+1) = \sum_{i=1}^{n}\big[s_j\big(\delta_j^i - a_j^i\big) + (1-\mu)b_j^i\big]X_i, \quad j = 1, 2, \ldots, n.$$

This equations can be rewritten in the form of the inequality

$$\alpha\mathsf{B}\mathsf{X} \le \big(\mathsf{E} - \mathsf{A} - (1-\mu)\mathsf{B}\big)\mathsf{X}, \tag{4.46}$$

if one can introduce the ratio of growth

$$\alpha = \min_{i=1\div n}\frac{X_i(t+1)}{X_i(t)}.$$

This quantity depends on the vector X which can be chosen in such a way that α would take the greatest value,

$$\hat{\alpha} = \max_{\mathsf{X}} \alpha(\mathsf{X}).$$

This value of α determines the optional balance trajectory of the quickest growth,

$$X_i(t) = \hat{X}_i(0)\hat{\alpha}^t = X_i(0)e^{\hat{\sigma}t}. \tag{4.47}$$

4.3.3 Potential Investment

One can avoid the restriction of the previous section and formulate dynamic equations for the general case, when components of technological matrices A and B are assumed to be functions of time. By combining (4.6) and (4.12), the final output can be defined as a linear function of the sectoral capital stock,

$$Y_j = \sum_{i=1}^{n} \xi_j^i K^i, \quad \xi_j^i = \frac{\delta_j^i - a_j^i}{b^i}, \tag{4.48}$$

where a matrix of marginal productivities of sectoral capital with components ξ_j^i is introduced. Capital stock K^i is governed by the dynamic equation (2.26), that is, by the equation

$$\frac{dK^i}{dt} = I^i - \mu K^i, \tag{4.49}$$

where sectoral investments I^i obey, according to formulae (4.18) and (4.39), a system of algebraic equations

$$\sum_{i=1}^{n} \bar{b}_j^i I^i = s_j Y_j - \sum_{i=1}^{n} K^i \frac{d\bar{b}_j^i}{dt}. \tag{4.50}$$

Equations (4.50) do not determine the sectoral investment I^i in a unique way: the number of equations is less than the number of variables. For given values of parameters, (4.48)–(4.50) determine a set of dynamic trajectories of a many-sector system. To separate a unique trajectory of evolution of the dynamic system, one has to complete the system of (4.50) or use an extra condition.

One can say about any extra condition which one needs to determine a trajectory of evolution as about principle of development. It is convenient to require some criterion to be optimal. For example, the criterion can be connected with the growth rates of the final output,

$$\frac{dY_j}{dt} = \sum_{i=1}^{n} \xi_j^i \frac{dK^i}{dt} + \sum_{i=1}^{n} K^i \frac{d\xi_j^i}{dt}. \tag{4.51}$$

It is enough to consider the rate of the total of the final products and to require it to have a maximum value at the given technology, that is,

$$\max \frac{dY}{dt} = \max \sum_{i=1}^{n} \xi^i \left(I^i - \mu K^i\right), \quad \xi^i = \sum_{j=1}^{n} \xi_j^i, \tag{4.52}$$

while relations (4.50) ought to be considered as restrictions that must be rewritten in the form

$$\sum_{i=1}^{n} \bar{b}^i_j I^i \le s_j Y_j. \tag{4.53}$$

In this way, investments are specified as a function of the parameters of the problem

$$\tilde{I}^i = \tilde{I}^i\left(\xi^i, b^i_j, s_j, Y_j\right).$$

In the considered case, the principle of development means that the society manages its resources in the best way.

When investments are known, dynamic equations (4.48), (4.49) determine a unique trajectory: the trajectory of the largest potential growth. One can imagine that the trajectory of evolution will be sought by numerical methods, while the maximisation problem (4.52), (4.53) has to be solved by standard methods of linear programming in every step of the solution of the Cauchy problem.

Instead of the described procedure for calculating the investment, other procedures can be invented, but in any case, it is assumed here that the final consumption and the storage of intermediate products are given; in this way the *potential investments* $\tilde{I}^i$ can be determined. It was assumed that there is no restriction due to material resources and no other restrictions. The availability of labour and energy impose some strong restrictions on the development of the production system and should be included in the theory. Real investments appear to be no more than the potential investments $\tilde{I}^i$ determined from the described procedure. The final consumption and the storage of intermediate products appear to be consequences of the system evolution. We shall discuss this question in the following chapters and return to the many-sector model in Chap. 8.

In the simplest way, the potential investment can be specified as an investment of homothetic trajectory. In this case, the growth rate of all sectors are equal to the growth rate of the entire system, and, from the expressions for the real investment, which for the entire system, due to (4.39) and (4.48), can be written as

$$I = \sum_{i,j=1}^{n} s_j\, \xi^i_j\, K^i, \qquad \xi^i_j = \frac{\delta^i_j - a^i_j}{b^i},$$

the potential investment can be determined. This allows one to define the potential growth rate of the gross output in any sector l as

$$\tilde{\delta}^l = -\mu + \frac{1}{K} \sum_{i,j=1}^{n} s_j\, \xi^i_j\, K^i, \quad l = 1, 2, \ldots, n. \tag{4.54}$$

4.4 Enterprise and Basic Technological Processes

From a microeconomic point of view, a production system consists of numerous enterprises, each of them including one or more *basic technological processes*. The

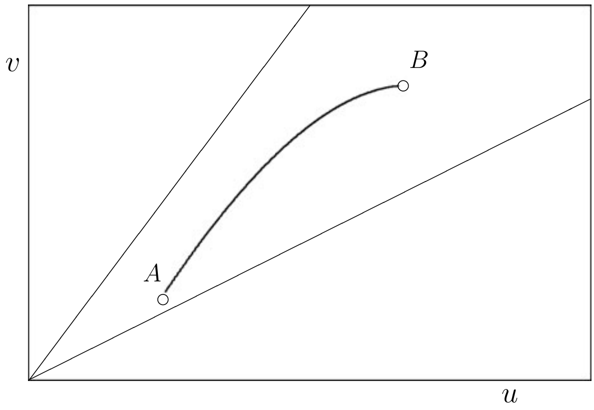

Fig. 4.1 Input–output space. All trajectories are situated (similar to the trajectory $A\ B$) inside sector restricted by straight lines which represent basic technological processes

latter can be considered as *atoms* of the production system. Methods to describe the resources of an enterprise consisting of a finite or an infinite set of basic technological processes were proposed by von Neumann [6] and Gale [8], respectively. We shall consider a finite set of basic technological processes, namely, the von Neumann model of enterprise.

Every enterprise produces one or several products, and it consumes some products. In other words, the enterprise transforms a set of the input products $x_j, x_i, \ldots,$ where the labels of products $j,\ i, \ldots$ are fixed, into an output set $x_l, x_m, \ldots,$ where the labels of products $l,\ m, \ldots$ are also fixed. It is convenient to introduce the input and output vectors u and v with non-negative components u_k and v_k, $k = 1, 2, \ldots, n$. Some of the components are equal to zero.

We assume that the components of the vectors are measured in units of value, so the final output of the enterprise is

$$y = \sum_{j=1}^{n} (v_j - u_j). \tag{4.55}$$

This is a contribution of the enterprise to the gross national product.

A couple of vectors (u, v) characterise the technology used in production or, to put it differently, the technological process is represented as a couple of vectors (u, v). The technological process can be depicted as a point in Euclidean space of dimension $2n$ (Fig. 4.1).

One can assume that there are basic technological processes, each of which cannot be divided or changed, for example, a car assembly line, the characteristics of which are constant. The only action the manager can perform is to switch the line on and off. One can assume that the enterprise consists of a set of basic technological processes, which can be used in different combinations, so that the technological process of the enterprise can be represented as an expansion over the basic technological processes,

$$(\mathsf{u}, \mathsf{v}) = \sum_{j=1}^{r} (\mathsf{a}^j, \mathsf{h}^j) z_j, \tag{4.56}$$

where $z_j \geq 0$ is the intensity of the use of the basic technological process labelled j.

For given basic technological processes and an arbitrary vector $\mathsf{z} = (z_1, z_2, \ldots, z_r)$, relation (4.56) determines a set of possible technological processes of the enterprise, that is, a technological set which is a specific characteristic of the enterprise.

The aim of the enterprise and of its investigator is to find a value of z such that the final output (4.55) takes the greatest value.

It is convenient to use the following representation for the components of the input and output vectors:

$$u_i = \sum_{j=1}^{r} a_i^j z_j, \qquad v_i = \sum_{j=1}^{r} h_i^j z_j, \tag{4.57}$$

where a_i^j and h_i^j are components of input and output matrices A and H, respectively. One can say that the von Neumann model is given if matrices A and H are given.

Then, the final output of the enterprise can be written as

$$y(\mathsf{z}) = \sum_{i=1}^{n} \sum_{j=1}^{r} (h_i^j - a_i^j) z_j. \tag{4.58}$$

One can introduce a potential output–input ratio

$$\alpha(\mathsf{z}) = \min_{j=1,2,\ldots,n;\ v_i \neq 0} \frac{v_i}{u_i} \tag{4.59}$$

which depends on the intensity $\mathsf{z} = (z_1, z_2, \ldots, z_r)$.

Because the final output is non-negative, it follows from relation (4.58) that

$$\mathsf{Az} \leq \mathsf{Hz}$$

or, using definition (4.59),

$$\alpha \mathsf{Az} \leq \mathsf{Hz}. \tag{4.60}$$

To find the maximum value of output (4.55), we have to find the greatest rate of growth α. One can see that this problem is quite similar mathematically to the problem considered in the previous section (compare (4.46) and (4.60)).

One can assume that intensity z is a function of time, so the von Neumann model can describe dynamic processes or, in geometrical terms, the trajectory in input–output space (Fig. 4.1). If the basic technological processes are unchanged during time, vector $\mathsf{z} =$ constant and the production trajectory is a straight line in technological space—the von Neumann ray. Otherwise, more complicated problems appear.

Hundreds of papers have been written about the properties of von Neumann-type growth models, the golden rules of capital accumulation, turnpike theorems and so on. However, all this is concerned with potential trajectories of a production system.

References

1. Leontief, W.W.: Quantitative input and output relations in the economic system of the United States. Rev. Econ. Stat. **18**, 105–125 (1936)
2. Leontief, W.W.: The Structure of the American Economy 1919–1939. Harvard University Press, Cambridge (1941)
3. Leontief, W.W.: Input–Output Economics, 2nd edn. Oxford University Press, New York (1986)
4. Sraffa, P.: Production of Commodities by Means of Commodities. Prelude to a Critique of Economic Theory. Cambridge University Press, Cambridge (1975)
5. Korn, G.A., Korn, T.M.: Mathematical Handbook for Scientists and Engineers. McGraw-Hill, New York (1968)
6. Von Neumann, J.: Über ein okonomisches Gleichungssystem und eine Verallgemeinerung des brouwerschen Fixpunktsatzes. Ergebnisse eines Mathematischen Kolloqueiums **8**, 73–83 (1937)
7. Cheremnykh, Y.N.: Analiz Povedenija Trajectorii Dinamiki Narodno-Khozjaistvennykh Modelei. Nauka, Moscow (1982) (in Russian; Analysis of Trajectories of Economical Models)
8. Gale, D.: The Theory of Linear Economic Models. McGraw-Hill, Auckland (1960)

Chapter 5
Production Factors and Technology

Abstract The description of production processes in the previous chapter assumes that something can be made from something else and that there are tools for making things. Now we are going to look at the production process from the energy side, including the effect of basic production equipment, as a set of sophisticated devices allowing human beings to attract energy from natural sources to the production of commodities. We are introducing two sets of quantities which, in the macroeconomical approach, reflect the level of technology in the economy: (1) potential rates of growth of production factors $\tilde{\nu}$ and $\tilde{\eta}$ and (2) technological coefficients λ and ε, which show how much labour and substitutive work are needed to introduce a unit of investment in the production system. In this chapter, the first set of quantities will be considered as exogenous factors, whereas the second set is connected with internal characteristics of technology. Technological coefficients appear to be appropriate and convenient characteristics of technology instilled in an economy and can be easily estimated considering the performance of production equipment.

5.1 Dynamics of Production Factors

To describe the process of production in a proper way, one should take into account fixed production equipment (fixed production capital stock) and two factors: manpower and the work of production equipment substituting for manpower. The production system is considered as consisting of n sectors. Each of them is characterised by the basic production equipment K^i and production factors L^i and P^i which are needed to animate the basic production equipment of the sector. In this chapter, it will be more convenient to proceed considering the production system as consisting of only one sector, which is characterised by aggregate amounts of production factors

$$K = \sum_{i=1}^{n} K^i, \qquad L = \sum_{i=1}^{n} L^i, \qquad P = \sum_{i=1}^{n} P^i. \tag{5.1}$$

V.N. Pokrovskii, *Econodynamics*, New Economic Windows 12,
DOI 10.1007/978-94-007-2096-1_5, © Springer Science+Business Media B.V. 2012

5.1.1 Dynamics of Capital

To develop a proper description of a technological process, it is necessary to take into account that equipment with different technological characteristics is introduced into the production process at different moments in time. Therefore, the age of existing production capital is different and, similar to formula (2.19), we can write an expression for the structure of fixed production capital

$$K(t) = \int_{-\infty}^{t} k(t,s)\,ds, \tag{5.2}$$

where $k(t,s)$ is the part of the fixed capital existing at time t which was introduced at time s during a unit of time. This part of the capital stock is depreciating according to the law

$$\frac{\partial k(t,s)}{\partial t} = -\mu(t,s)k(t,s) \tag{5.3}$$

from the initial amount

$$k(s,s) = I(s). \tag{5.4}$$

Equation (5.3) can be integrated with respect to s, which gives the expression

$$\frac{dK}{dt} = I - \int_{-\infty}^{t} \mu(t,s)k(t,s)\,ds. \tag{5.5}$$

One can introduce an effective coefficient of depreciation

$$\mu(t) = \frac{1}{K}\int_{-\infty}^{t} \mu(t,s)k(t,s)\,ds$$

to rewrite (5.5) in the conventional [1] form

$$\frac{dK}{dt} = I - \mu K. \tag{5.6}$$

One can see that a solution of this equation, if $\mu = \text{const}$, can be written, in accordance with (5.2), as

$$K(t) = \int_{-\infty}^{t} k(t,s)\,ds, \quad k(t,s) = e^{-\mu(t-s)}I(s). \tag{5.7}$$

5.1.2 Dynamics of Substitutive Work and Labour

The existing technology determines how many workers' efforts and how much extra energy is needed in order for the production mechanism to be in action; therefore, the dynamic equations for labour and substitutive work must contain characteristics of technology. Further, we reconsider the derivation developed in works [2, 3].

Similar to the presentation of capital, the amounts of production factors can be represented by the formulae

$$L(t) = \int_{-\infty}^{t} l(t,s)\,ds, \qquad P(t) = \int_{-\infty}^{t} e(t,s)\,ds, \tag{5.8}$$

where $l(t,s)$ and $e(t,s)$ are the labour and substitutive work which are necessary for part of the fixed capital $k(t,s)$ to be in action, so the following relations can be written:

$$l(t,s) = \lambda(t,s)k(t,s), \qquad e(t,s) = \varepsilon(t,s)k(t,s). \tag{5.9}$$

From relation (5.4),

$$\begin{aligned} l(s,s) &= \lambda(s)I(s), & \lambda(s) &= \lambda(s,s), \\ e(s,s) &= \varepsilon(s)I(s), & \varepsilon(s) &= \varepsilon(s,s), \end{aligned} \tag{5.10}$$

where $I(s)$ is the gross investment at time s.

The coefficients $\lambda(s) > 0$ and $\varepsilon(s) > 0$ determine the required amount of labour and substitutive work, delivered by external energy sources, per unit of increase in capital; therefore, they can be denominated as the labour requirement and energy requirement, respectively. The values of these coefficients are determined by the applied technology, and we call them the technological coefficients.

One can see that, from (5.3), equations for the dynamics of quantities (5.9) can be written as follows:

$$\begin{aligned} \frac{\partial l(t,s)}{\partial t} &= -\mu_{\mathsf{L}}(t,s)l(t,s), & \mu_{\mathsf{L}}(t,s) &= \mu(t,s) - \frac{1}{\lambda(t,s)}\frac{\partial \lambda(t,s)}{\partial t}, \\ \frac{\partial e(t,s)}{\partial t} &= -\mu_{\mathsf{P}}(t,s)e(t,s), & \mu_{\mathsf{P}}(t,s) &= \mu(t,s) - \frac{1}{\varepsilon(t,s)}\frac{\partial \varepsilon(t,s)}{\partial t}. \end{aligned} \tag{5.11}$$

The last terms in the definitions of the depreciation coefficients $\mu_{\mathsf{L}}(t,s)$ and $\mu_{\mathsf{P}}(t,s)$ are connected with the change of quality of the production equipment after it has been installed at time s.

One can use definitions (5.8) to determine the required amount of the production factors,

$$\frac{dL}{dt} = \lambda I - \int_{-\infty}^{t} \mu_{\mathsf{L}}(t,s)l(t,s)\,ds, \qquad \frac{dP}{dt} = \varepsilon I - \int_{-\infty}^{t} \mu_{\mathsf{P}}(t,s)e(t,s)\,ds. \tag{5.12}$$

The first terms on the right side of these relationships describe the increase in the quantities of interest, caused by gross investments I; the second terms reflect the decrease of the corresponding quantities due to both the change of the quality of production equipment after installation and the removal of a part of the production equipment from the service.

We can rewrite (5.12), introducing a special notation for the last parts of the equations, as

$$\frac{dL}{dt} = \lambda I - (\mu + \nu')L, \qquad \frac{dP}{dt} = \varepsilon I - (\mu + \eta')P. \tag{5.13}$$

One can consider the quantities $\mu + \nu'$ and $\mu + \eta'$ as the effective depreciation coefficients of the production factors. Also, one can consider the quantities $\frac{1}{L}\frac{dL}{dt} + \nu'$ and $\frac{1}{P}\frac{dP}{dt} + \eta'$ to be the effective growth rates of labour and productive energy.

If, for example, the installed technological equipment requires more labour during ageing, $\nu' < 0$, which means a decrease in the effective depreciation coefficient. If the technological equipment does not change its quality over time, that is the technological coefficients in (5.9) do not depend on the argument t, the quantities $\nu' = 0$ and $\eta' = 0$ and all the depreciation coefficients in (5.6) and (5.13) will appear to be the same.

5.2 Macroeconomic Characteristics of Technology

Technology is commonly understood as methods of production, so characteristics of tools, machines, materials, techniques and sources of power are needed to describe the technology of production of useful things. Below we consider the introduced phenomenological characteristics of production equipment.

5.2.1 Technological Coefficients

To discuss the meaning and properties of the technological coefficients $\lambda(t)$ and $\varepsilon(t)$ in the dynamic equations (5.13), one can neglect the quantities ν' and η' for simplicity. It is easy to see from dynamic equations (5.6) and (5.13) that the constant technological coefficients can be expressed as

$$\lambda = \frac{L}{K}, \qquad \varepsilon = \frac{P}{K}.$$

For example, in the case when

$$\lambda < \frac{L}{K}, \qquad \varepsilon < \frac{P}{K}$$

labour-saving and energy-saving technology is being introduced. It is convenient to introduce the non-dimensional technological quantities

$$\bar{\lambda} = \lambda\,\frac{K}{L}, \qquad \bar{\varepsilon} = \varepsilon\,\frac{K}{P}. \tag{5.14}$$

These quantities play an essential role in the description of the production system and appear to be independent variables in the set of equations of evolution of the system. It is natural to consider the technological coefficients to be non-negative; one can hardly imagine a situation where one of them would be negative. Apart from this, the requirement of positivity of the marginal productivities (Chap. 6, Sect. 6.1) puts some restrictions (formula (6.10)) on the values of the technological coefficients, namely,

$$\bar{\lambda} < 1 < \bar{\varepsilon} \quad \text{or} \quad \bar{\lambda} > 1 > \bar{\varepsilon}. \tag{5.15}$$

Note that the quantities reciprocal to the technological coefficients can be interpreted as the costs of the equipment needed to introduce a unit of labour or a unit of substitutive work into the production process or, in other words, the *prices of introduction* of the corresponding production factors

$$\frac{1}{\lambda} = \frac{K}{\bar{\lambda}\, L}, \qquad \frac{1}{\varepsilon} = \frac{K}{\bar{\varepsilon}\, P}. \tag{5.16}$$

For an unchangeable technology, the costs of introduction are

$$\frac{1}{\lambda} = \frac{K}{L}, \qquad \frac{1}{\varepsilon} = \frac{K}{P}.$$

It is essential to note that the price of introduction of substitutive work as a production factor is different from the price of use of substitutive work as a production factor, defined by (2.35). The latter is not the price of an energy carrier with the corresponding energy content, but the price of equipment (capital) which allows energy to be converted usefully into other forms.

5.2.2 Technological Index

Equations (5.6) and (5.13) can be considered as relations which determine a demand of the production factors for a given technology and investment. In other words, the values of investment and production factors cannot be quite arbitrary and have to correspond to values of the technological coefficients, which are subjects of the restrictions (5.15). If any discrepancy in empirical data is observed, consistency could be achieved by considering the quantities ν' and η' to be the proper corrections for the growth rates of labour and substitutive work. These quantities could also compensate possible errors.

It is convenient to introduce special notation for the growth rate of capital and the effective rates of growth of labour and substitutive work,

$$\delta = \frac{1}{K}\frac{dK}{dt}, \qquad \nu = \nu' + \frac{1}{L}\frac{dL}{dt}, \qquad \eta = \eta' + \frac{1}{P}\frac{dP}{dt}. \tag{5.17}$$

This allows us to rewrite equations (5.6) and (5.13) in the following form:

$$\delta = \frac{I}{K} - \mu, \qquad \nu = \bar{\lambda}\frac{I}{K} - \mu, \qquad \eta = \bar{\varepsilon}\frac{I}{K} - \mu, \tag{5.18}$$

where the symbols for the non-dimensional technological quantities (5.14) are used.

Equations (5.18) describe exact relations between the rates of real growth of production factors $\delta(t)$, $\nu(t)$, and $\eta(t)$ and technological coefficients in the form

$$\bar{\lambda} = \frac{\nu + \mu}{\delta + \mu}, \qquad \bar{\varepsilon} = \frac{\eta + \mu}{\delta + \mu}. \tag{5.19}$$

The depreciation coefficient μ can be excluded from relations (5.19). Therefore, we can obtain a relation between the different rates of real growth,

$$\delta = \nu + \alpha(\eta - \nu), \quad \alpha = \frac{1 - \bar{\lambda}}{\bar{\varepsilon} - \bar{\lambda}}. \tag{5.20}$$

The technological index α is introduced here. The condition of productivity of the production factors (5.15) imposes a certain restriction on the values of the technological index,

$$0 < \alpha < 1. \tag{5.21}$$

Moreover, some other estimates of this quantity can be made. It will be shown in Chap. 6 that the technological index α has the meaning of the share of expenses for maintenance of substitutive work as a production factor in the total expenses for maintenance of production factors (labour L and substitutive work P).

Let us recall that the quantities ν and η represent the effective growth rates of production factors and can coincide with the rates of real growth, if corrections ν' and η' can be neglected.

5.3 Investment and Dynamics of Technology

5.3.1 Investment and Three Modes of Development

To determine investment, one must take into account the restriction imposed by internal (the limiting output and necessary level of consumption) and external circumstances. According to (4.54), if no restrictions are imposed by the availability of labour and substitutive work, the potential growth rate of capital can be written in terms of a many-sector model as

$$\tilde{\delta} = -\mu + \frac{1}{K} \sum_{j,l=1}^{n} s_j K^l \xi_j^l. \tag{5.22}$$

Because the quantity s_j, which is a share of the investment product in the final product of the sector j, changes from 0 to 1 in every sector, the rate of potential growth of investment $\tilde{\delta}$ is restricted by the inequalities

$$-\mu < \tilde{\delta} < \frac{Y}{K} - \mu.$$

The other restrictions emerge from non-availability of other production factors. One can assume here that there are external sources of labour and energy, so that the amounts of available labour $\tilde{L}$ and substitutive work $\tilde{P}$ are known. It is convenient to consider them solutions of the equations

$$\frac{d\tilde{L}}{dt} = \tilde{\nu}\,\tilde{L}, \qquad \frac{d\tilde{P}}{dt} = \tilde{\eta}\,\tilde{P}. \tag{5.23}$$

Though the rates of potential growth $\tilde{\nu}$ and $\tilde{\eta}$ can be, in principle, calculated as was discussed in Chap. 2 (see Sects. 2.4.2 and 2.5.5), later on they are assumed to be given as functions of time. Thus, we assume the rates of potential growth of production factors to be known as functions of time,

$$\tilde{\delta} = \tilde{\delta}(t), \qquad \tilde{\nu} = \tilde{\nu}(t), \qquad \tilde{\eta} = \tilde{\eta}(t).$$

In any case, the rates of real growth δ, ν and η, defined by (5.18), do not exceed the rates of potential growth $\tilde{\delta}$, $\tilde{\nu}$ and $\tilde{\eta}$, that is,

$$\delta \le \tilde{\delta}, \qquad \nu \le \tilde{\nu}, \qquad \eta \le \tilde{\eta}.$$

This determines the restrictions for investments in the production sector,

$$I \le (\mu + \tilde{\delta})K, \qquad I \le \frac{\mu + \tilde{\nu}}{\lambda} L, \qquad I \le \frac{\mu + \tilde{\eta}}{\varepsilon} P. \tag{5.24}$$

The real investments are determined by a competition between potential investments from one side and labour and energy supplies from the other side. One can assume that the production system tries to swallow up all available production factors. In this case, for investments we should write

$$I = (\delta + \mu)K = \min \begin{cases} (\tilde{\delta} + \mu)K, \\ (\tilde{\nu} + \mu)K/\bar{\lambda}, \\ (\tilde{\eta} + \mu)K/\bar{\varepsilon}. \end{cases} \tag{5.25}$$

The rates of real growth of production factors δ, ν and η are different from the rates of potential growth. According to the three lines of relation (5.25), one can define three modes of economic development for which we have different formulae for calculation. From (5.18) and (5.25), the real rates of growth can be calculated for the three modes as

$$\begin{aligned}
&\delta = \tilde{\delta}, \qquad \nu = (\tilde{\delta} + \mu)\bar{\lambda} - \mu, \qquad \eta = (\tilde{\delta} + \mu)\bar{\varepsilon} - \mu, \\
&\delta = (\tilde{\nu} + \mu)\frac{1}{\bar{\lambda}} - \mu, \qquad \nu = \tilde{\nu}, \qquad \eta = (\tilde{\nu} + \mu)\frac{\bar{\varepsilon}}{\bar{\lambda}} - \mu, \\
&\delta = (\tilde{\eta} + \mu)\frac{1}{\bar{\varepsilon}} - \mu, \qquad \nu = (\tilde{\eta} + \mu)\frac{\bar{\lambda}}{\bar{\varepsilon}} - \mu, \qquad \eta = \tilde{\eta}.
\end{aligned} \tag{5.26}$$

The first set of equations is valid in the case of lack of investment, and abundance of labour, substitutive work and raw materials. The second line is valid in the case of lack of labour, and abundance of investment, substitutive work and raw materials. The last line of equations is valid in the case of lack of substitutive work, and abundance of investment, labour and raw materials.

5.3.2 *Unemployment and Principle of Development*

According to the preceding speculation, the rates of real growth of production factors are not bigger than the potential ones. If, indeed, the production system is trying to devour all available production factors, the growth of one of the production factors coincides with potential growth. This means that the gaps between real and potential amounts of production factors, for example, a gap between labour supply $\tilde{L}$ and labour demand L, will increase. Therefore, one can see that the index of unemployment $u = (\tilde{L} - L)/\tilde{L}$, for example, cannot shrink in a 'natural' way. Considering the situation in a one-sector approach, in order to decrease the gaps between real and potential amounts of production factors, an intervention in the form of governmental investments is needed, and to take it into account, one can rewrite relation (5.26) in the form

$$I = (\delta + \mu)K = \chi(u)\,K + \min \begin{cases} (\tilde{\delta} + \mu)K, \\ (\tilde{\nu} + \mu)K/\bar{\lambda}, \\ (\tilde{\eta} + \mu)K/\bar{\varepsilon}, \end{cases} \tag{5.27}$$

where quantity $\chi(u)$ is designed to regulate the gaps between supply and demand of production factors. One has to reserve some amount of products for investments to be regulated. In this case, three modes of economic development exist as well. In the many-sector approach, the intervention is defined in similar way, but the interpretation can be different (see Sect. 8.1.2).

5.3.3 *Dynamics of Technological Coefficients*

To determine the law of evolution of the technological coefficients in time, one has to consider again the restrictions on investments. According to the three modes defined by relations (5.26), one can obtain three sets of equations for the non-dimensional technological quantities $\bar{\lambda}$, $\bar{\varepsilon}$ and their ratio $\Theta = \bar{\varepsilon}/\bar{\lambda}$

$$1 = \frac{\tilde{\delta}+\mu}{\delta+\mu}, \quad \bar{\lambda} \le \frac{\tilde{\nu}+\mu}{\delta+\mu}, \quad \bar{\lambda} \le \frac{\tilde{\nu}+\mu}{\tilde{\delta}+\mu}, \quad \bar{\varepsilon} \le \frac{\tilde{\eta}+\mu}{\delta+\mu}, \quad \bar{\varepsilon} \le \frac{\tilde{\eta}+\mu}{\tilde{\delta}+\mu},$$

$$1 \le \frac{\tilde{\delta}+\mu}{\delta+\mu}, \quad \bar{\lambda} = \frac{\tilde{\nu}+\mu}{\delta+\mu}, \quad \bar{\lambda} \ge \frac{\tilde{\nu}+\mu}{\tilde{\delta}+\mu}, \quad \bar{\varepsilon} \le \frac{\tilde{\eta}+\mu}{\delta+\mu}, \quad \Theta \le \frac{\tilde{\eta}+\mu}{\tilde{\nu}+\mu}, \tag{5.28}$$

$$1 \le \frac{\tilde{\delta}+\mu}{\delta+\mu}, \quad \bar{\lambda} \le \frac{\tilde{\nu}+\mu}{\delta+\mu}, \quad \bar{\varepsilon} = \frac{\tilde{\eta}+\mu}{\delta+\mu}, \quad \bar{\varepsilon} \ge \frac{\tilde{\eta}+\mu}{\tilde{\delta}+\mu}, \quad \Theta \ge \frac{\tilde{\eta}+\mu}{\tilde{\nu}+\mu},$$

In the first case, there is an internal restriction to growth. In the last two lines, one of the production factors, L or P, is limited.

One can assume that there are internal technological changes, which lead to alteration of the technological coefficients, where the economic system tries to use all available resources. This means that the technological coefficients have tendencies to change such that the inequalities in conditions (5.28) are trending to turn into equalities. These processes are connected with the invention of new technologies and the propagation of known ones.

One can consider the rates of growth

$$\frac{d\Theta}{dt}, \qquad \frac{d\bar{\lambda}}{dt}, \qquad \frac{d\bar{\varepsilon}}{dt}$$

to be functions of differences

$$\Theta - \frac{\tilde{\eta}+\mu}{\tilde{\nu}+\mu}, \qquad \bar{\lambda} - \frac{\tilde{\nu}+\mu}{\tilde{\delta}+\mu}, \qquad \bar{\varepsilon} - \frac{\tilde{\eta}+\mu}{\tilde{\delta}+\mu}.$$

In the first approximation, the tendencies to changes can be described by equations for the quantities

$$\frac{d\Theta}{dt} = -\frac{1}{\tau_\theta}\left(\Theta - \frac{\tilde{\eta}+\mu}{\tilde{\nu}+\mu}\right), \tag{5.29}$$

$$\frac{d\bar{\lambda}}{dt} = -\frac{1}{\tau_\lambda}\left(\bar{\lambda} - \frac{\tilde{\nu}+\mu}{\tilde{\delta}+\mu}\right), \tag{5.30}$$

$$\frac{d\bar{\varepsilon}}{dt} = -\frac{1}{\tau_\varepsilon}\left(\bar{\varepsilon} - \frac{\tilde{\eta}+\mu}{\tilde{\delta}+\mu}\right). \tag{5.31}$$

So as $\Theta = \bar{\varepsilon}/\bar{\lambda}$, only two of (5.29)–(5.31) are independent. For (5.29), (5.30) and (5.31) to be consistent, relaxation times τ_λ and τ_ε should be equated and connected with relaxation time τ_θ as follows:

$$\tau_\lambda = \tau_\varepsilon = \frac{1}{\bar{\lambda}}\,\frac{\tilde{\nu}+\mu}{\tilde{\delta}+\mu}\,\tau_\theta.$$

Equations (5.29)–(5.31) are the first-order approximations to the relaxation equations with respect to the quantities in the brackets, so one should take the zero-order terms of the relaxation times in these equations. This means that, in the considered approximation, all relaxation times are equal to each other, namely,

$$\tau_\lambda = \tau_\varepsilon = \tau_\theta = \tau, \tag{5.32}$$

so that the subscripts to the relaxation times in (5.29)–(5.31) can be omitted in the following exposition. The meaning of τ is the time of crossover from one technological situation to another, when external parameters $\tilde{\nu}$ and $\tilde{\eta}$ change. It is determined by internal processes of developing and attracting the proper technology.

It is easy to see that, if the rates of growth are constant, (5.29), for example, at initial value

$$\Theta(0) = (1 - \Delta)\frac{\tilde{\eta} + \mu}{\tilde{\nu} + \mu}$$

has a simple solution

$$\Theta(t) = \frac{\tilde{\eta} + \mu}{\tilde{\nu} + \mu}\left[1 - \Delta \exp\left(-\frac{t}{\tau}\right)\right]. \tag{5.33}$$

5.3.4 Dynamics of the Technological Index

Now we can directly calculate changes of the technological index

$$\alpha = \frac{1 - \bar{\lambda}}{\bar{\varepsilon} - \bar{\lambda}}.$$

After differentiating the quantity and making use of (5.30) and (5.31), one gets the relation

$$\frac{d\alpha}{dt} = \frac{\tilde{\delta} - \tilde{\nu} - \alpha\,(\tilde{\eta} - \tilde{\nu})}{\tau(\bar{\varepsilon} - \bar{\lambda})(\tilde{\delta} + \mu)}. \tag{5.34}$$

To specify the change of the technological index, we have to compare the rate of capital potential growth $\tilde{\delta}$ with the combination of the rates of potential growth of the other factors $\tilde{\nu} + \alpha\,(\tilde{\eta} - \tilde{\nu})$. If the first quantity is bigger than the second, the technological index grows. One can assume that, in a steady-state situation, there is a relation

$$\tilde{\delta} = \tilde{\nu} + \alpha\,(\tilde{\eta} - \tilde{\nu}), \tag{5.35}$$

which is similar to relation (5.20) for the rates of real growth of production factors. In this case, the technological index α appears to be constant during evolution; in other words, the technological index appears to be the first integral of evolution. One can expect that there are social mechanisms which ensure the validity of (5.35). Of course, this equation should be considered as an approximate equality, which can be violated by disturbances in the social life.

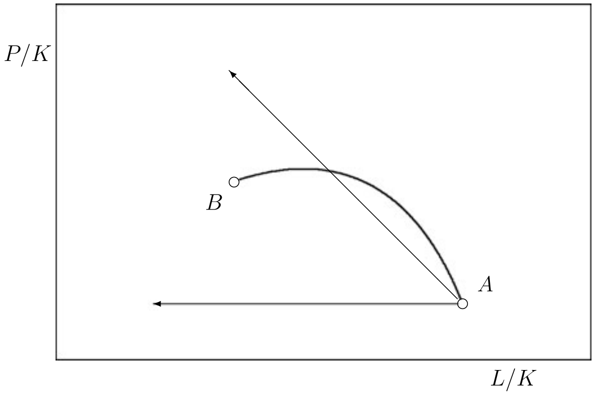

Fig. 5.1 Space of production factors. The area inside the sector presents a set of more effective technologies in comparison with initial technology A

5.4 Mechanism of Evolution of Production System

In this section, we turn to the microeconomic approach and consider the production system to consist of numerous enterprises, each of them including one or more *technological processes*, as was discussed in Sect. 4.4. Each technological process consumes some products in order to output other ones. In other words, the enterprise transforms the input set of products $x_j, x_i, \ldots$, where the labels of products $j,\ i, \ldots$ are fixed, into an output set $x_l, x_m, \ldots$, where the labels of products $l,\ m, \ldots$ are also fixed. It is convenient to use the input and output vectors u and v with non-negative components u_k and v_k, $k = 1, 2, \ldots, n$. This side of the technological process was described in Sect. 4.4.

In addition, each technological process is characterised by equipment with value K and the production factors: the labour L and substitutive work P needed to animate the production equipment. So, the technological process or the enterprise can be given by a set of five quantities

$$\mathsf{u}, \qquad \mathsf{v}, \qquad K, \qquad L, \qquad P$$

or, considering the scale of production as non-essential, by four quantities

$$\frac{\mathsf{u}}{K}, \qquad \frac{\mathsf{v}}{K}, \qquad r_1 = \frac{L}{K}, \qquad r_2 = \frac{P}{K}. \tag{5.36}$$

The technological process can be depicted by a point in a many-dimensional space which consists of input–output space (Fig. 4.1) and production-factor space (Fig. 5.1).

According to Schumpeter [4], the mechanism of evolution of the production system can be imagined as the emergence, growth and disappearance of technological processes. An essential moment in this scheme is the emergence of a new enterprise, which can use a known technology or create a new one. In the first case one speaks of diffusion of the known technology. The values of quantities (5.36) are the same for all enterprises with similar technology. One says that the production expands extensively. In the second case new technological processes appear. The new enterprise (technological process) produces either known products by new methods (process innovation), or completely new products (product innovation). In any case, the values of quantities (5.36) have been arising in new combinations. Some people take the risk of investigating new combinations of production factors and setting up a new technological process. The people running such businesses—the new entrepreneurs, according to Schumpeter [4]—are central figures of economic development.

One can imagine a simple scheme of development of a production system consisting of many enterprises. We do not discuss stages of infancy, adolescence, maturity and senescence of an enterprise. For simplicity, we assume that a technological process remains unchanged until the moment of disappearance and consider elementary acts of evolution of a production system to be emergence and reproduction of a new technological process. We can enumerate all technological processes according to the time of their emergence by a label α ($\alpha = 1, 2, \ldots$). An enterprise (technological process) with characteristics $K^\alpha,\ L^\alpha,\ P^\alpha$ emerges at moment t^α and disappears at moment $t^\alpha + \tau$. For simplicity, the time of existence of the enterprises τ is assumed to be equal for all enterprises. The quantities $K^\alpha,\ L^\alpha,\ P^\alpha$ and t^α are random ones, so an *emergence distribution function* should be introduced. It is convenient to use variables t^α, r_1^α, r_2^α ($\alpha = 1, 2, \ldots$) as arguments of the emergence function.

Then, one can define the technological progress in terms of microeconomical variables r_1 and r_2. The empirical data for the progressively developing U.S. economy, demonstrated with formulae (2.29), (2.30) and (2.37), show that the rate of capital growth exceeds the rates of growth of labour, whereas the rate of substitutive work growth exceeds the rate of capital growth; so, in order for an enterprise to be a partner in the technological progress, the relations between the variables should be

$$r_1^\alpha < r_1^0, \qquad r_2^\alpha > r_2^0, \qquad r_2^\alpha / r_1^\alpha > r_2^0 / r_1^0, \tag{5.37}$$

where the superscript 'zero' denotes values of quantities of the foregoing (initial) technology. It can be seen in Fig. 5.1 that a point corresponding to a progressively new technology falls in a sector of more productive technologies.

Now, expressions for the production factors can be defined. One can neglect deterioration of the equipment and write

$$\begin{aligned}
K(t) &= \sum_\alpha \big\langle K^\alpha \big[\theta\big(t - t^\alpha\big) - \theta\big(t - \tau - t^\alpha\big)\big]\big\rangle, \\
L(t) &= \sum_\alpha \big\langle L^\alpha \big[\theta\big(t - t^\alpha\big) - \theta\big(t - \tau - t^\alpha\big)\big]\big\rangle, \\
P(t) &= \sum_\alpha \big\langle P^\alpha \big[\theta\big(t - t^\alpha\big) - \theta\big(t - \tau - t^\alpha\big)\big]\big\rangle,
\end{aligned} \tag{5.38}$$

where the symmetric step function $\theta(x)$ and, later, the Dirac delta function $\delta(x)$ are used (see, for example, Korn and Korn [5] for an explanation of the properties of these functions). The angle brackets in formula (5.38) denote averaging with respect to the emergence function.

To calculate the technological coefficient on the basis of relations (5.38), it is necessary to know the emergence distribution function or, in other words, the specific mechanism of evolution of production system should be estimated. Without discussing the specific mechanism, one can assume that the emergence function is steady state; that is, it does not depend explicitly on time. This assumption allows one to calculate the derivative of (5.38) and to find expressions for the investment and technological coefficients,

$$
\begin{aligned}
I(t) &= \sum_{\alpha} \big\langle K^{\alpha} \big[\delta\big(t - t^{\alpha}\big) - \delta\big(t - \tau - t^{\alpha}\big) \big] \big\rangle, \\
\lambda(t) &= \frac{1}{I(t)} \sum_{\alpha} \big\langle r_1^{\alpha} K^{\alpha} \big[\delta\big(t - t^{\alpha}\big) - \delta\big(t - \tau - t^{\alpha}\big) \big] \big\rangle, \\
\varepsilon(t) &= \frac{1}{I(t)} \sum_{\alpha} \big\langle r_2^{\alpha} K^{\alpha} \big[\delta\big(t - t^{\alpha}\big) - \delta\big(t - \tau - t^{\alpha}\big) \big] \big\rangle.
\end{aligned}
\tag{5.39}
$$

Time dependence of the technological coefficients is determined by the dependence of variables r_1 and r_2 on index α. To estimate the behaviour of the technological coefficients, one can introduce two non-dimensional functions of the index,

$$
r_1^{\alpha} = r_1^0\big(1 - \varphi_1(\alpha)\big), \qquad r_2^{\alpha} = r_2^0\big(1 + \varphi_2(\alpha)\big), \tag{5.40}
$$

where $\varphi_1(\alpha)$ and $\varphi_2(\alpha)$ are assumed to be small positive quantities in the case of progressive evolution of a production system.

This allows one to see an obvious result: the technological coefficient $\lambda(t)$ is a diminishing function of time in this case. This approach allows us to investigate details of the mechanism of evolution of the production system.

References

1. Blanchard, O.J., Fisher, S.: Lectures on Macroeconomics. MIT Press, Cambridge (1989)
2. Pokrovski, V.N.: Physical Principles in the Theory of Economic Growth. Ashgate, Aldershot (1999)
3. Pokrovski, V.N.: Energy in the theory of production. Energy **28**, 769–788 (2003)
4. Schumpeter, J.A.: Theorie der wirtschaftlichen Entwicklung: Eine Untersuchung über Unternehmergewinn, Kapital, Kredit, Zins und den Kojunkturzyklus. Dunker und Humblot, Berlin (1911)
5. Korn, G.A., Korn, T.M.: Mathematical Handbook for Scientists and Engineers. McGraw-Hill, New York (1968)

Chapter 6
Production of Value

Abstract In this chapter, the relationship between the production of value and the original (primary) sources of value, the production factors, is considered. From the input–output relations (Chap. 4), an increase in production of value is connected with an increase in production equipment (capital stock), and the capital stock is conventionally considered a production factor. On the other hand, production of value can be associated with an increase in technological work, that is, an increase in the efforts of labourers and the work of production equipment (Chap. 1). In all, we have three production factors to consider: machine work and labour inputs act as substitutes for each other, but capital stock and the total technological work are complements to each other. An approximation of the production function allows us to find explicit forms of marginal productivities which are connected with each other. The roles of production factors are different: labour and substitutive work are the true sources of value. Capital stock presents the means by which the labour and energy resources are attracted to the production, allowing workers' efforts to be substituted by a machine's work.

6.1 Output and Production Factors

The final output Y represents the value of the products created by the production system per year. It is assumed that this scale of value is chosen so that the purchasing capacity of the monetary unit remains identical at all times. Otherwise, the price index ρ appears, and the production of value in current money units has to be written as

$$d\hat{Y} = \rho(dY)_\rho + \hat{Y}d\ln\rho,$$

where $(dY)_\rho$ is the production of value at constant prices.

6.1.1 Specification of the Production Function

In Sect. 1.4, we briefly discussed the history of the search for appropriate production factors. One can came to the conviction that functioning of the production system,

V.N. Pokrovskii, *Econodynamics*, New Economic Windows 12,
DOI 10.1007/978-94-007-2096-1_6,

in the most simple approximation, can be correctly described by means of three variables: fixed capital stock K, expenditures of labour L and substitutive work P. The output Y must be considered to depend on the three production factors

$$Y = Y(K, L, P). \tag{6.1}$$

To specify this general form[1] of the law of production of value, one has to take into account two issues. First, as far as there is a relation (5.20) among the growth rates of the production factors, the variables K, L and P appear to be interdependent: only two of the production factors are independent.[2] Then, the technological description assumes that the machine's work and labour inputs act as substitutes to each other, and the amount of production equipment, universally measured by its value K, must be considered to be a complement to the technological work (L and P) of the production equipment.[3] All this motivates one to write the relation between output and production factors in the form of two alternative lines,

$$Y = \begin{cases} Y(K), \\ Y(L, P), \end{cases} \qquad dY - \Delta\, dt = \begin{cases} \xi(K)\, dK, \\ \beta(L, P)\, dL + \gamma(L, P)\, dP, \end{cases} \tag{6.2}$$

where $\Delta\, dt$ is a part of the increment of production of value which is connected with the change of characteristics of the production system (the structural and technological change). When $\Delta = 0$, the marginal productivities ξ, β and γ correspond to the value produced by the addition of a unit of capital, or by the addition of a unit of labour input at constant external energy consumption or by the addition of a unit of energy at constant labour input, respectively. In line with the existing practise, these quantities can be labelled as marginal productivities of the corresponding production factors. The written relation can be considered an expression of a substitution law.

The forms (6.2) seem to be consistent with some different approaches to the theory of production of value [4, 5]. The present theory keeps the main attributes of the neo-classical approach, i.e., the concept of value produced by the production factors (donor value) and the concept of the production factors themselves, and can be considered as a generalisation and extension of the conventional neo-classical approach. In the conventional, neo-classical theory, capital as a variable played two

[1] Note that, in the more general case, it is possible to admit that production of value $Y(t)$ is a function of the arguments $L(t-s)$ and $P(t-s)$ taken at previous points in time, i.e., that production of value is a function of a trajectory of evolution. This possibility was considered earlier [1], though, as is clear from the analysis, there is no indispensability to use this concept of production function.

[2] One can see that (6.8) is an inexplicit relation for the production factors.

[3] The relationship of complementarity and substitutability among capital K, labour L and primary energy E has been analysed by some researchers. For example, Berndt and Wood [2] remark on p. 351 '...that $E - K$ complementarity and $E - L$ substitutability are consistent with the recent high-employment, low-investment recovery path of the U.S. economy.' Patterson [3, p. 382], found 'for New Zealand (1960–1985) that energy and labour inputs acted as mild substitutes to each other, and energy and capital inputs were mild complements to each other.' These research works dealt with the total primary consumption of energy E and could not discover the relationship of exact substitution between labour L and substitutive work P, which is a part of the primary energy.

distinctive roles: capital stock as value of production equipment and capital service as a substitute of labour. We consider capital stock to be the means of attracting labour and energy services to the production, while human work and the work of external energy sources are considered to be the true sources of value.[4] Human work is replaced by the work of external energy sources using different sophisticated appliances. In contrast to the conventional theory, the perfect substitution of labour and energy does not lead to any discrepancies. One can imagine a factory working without energy or without labour, but one cannot imagine a factory without production equipment.

We shall use formulae (6.2) as a starting point for the productivity theory to obtain relations between marginal productivities of capital on one side and of labour and energy on the other side (see formulae (6.8)). Note that, just as capital consists of many parts with their own productivity, labour and energy can be divided into separate parts according to their qualities, so that (6.2) can be generalised. In this chapter, however, we shall use the simplest approach.

6.1.2 Principle of Productivity

One uses production factors to create things and services, and an addition of any of the production factors has to increase production. Therefore, one has to consider all marginal productivities to be positive. This statement is known as the *principle of productivity*.

It was discovered by Marx [8] that the labour force is a commodity that gives surplus value. In our notation, Marx's statement can be written as

$$(\beta - w)\,dL > 0, \tag{6.3}$$

where w is the price of labour (see Sect. 2.4.2). To explain the modern surplus in industrial societies, one should take the other production factor, substitutive work P, into account and write

$$(\beta - w)\,dL + (\gamma - p)\,dP > 0, \tag{6.4}$$

where p is the price of production consumption of energy providing the substitutive work (see (2.35)). Both the first and the second terms in formula (6.4) are expected to be positive. This statement can be considered as the *strong principle of productivity*.

[4]Some argue that, in this case, labour can be reduced to energy, and one has only the argument energy as a source of value [6, 7]. However, labour and energy are measured in different units, and nobody knows how to calculate the real work provided by these production factors and compare them. Besides, even if possible, such a comparison could not be universal, so it is better to deal with the two separate arguments.

6.2 Productivities and Technological Coefficients

To link marginal productivities with technological coefficients, one can refer to differential expressions (6.2) for the production of value. We can rewrite them in the form

$$\frac{dY}{dt} - \Delta = \begin{cases} \xi \dfrac{dK}{dt}, \\ \beta \dfrac{dL}{dt} + \gamma \dfrac{dP}{dt}, \end{cases} \tag{6.5}$$

where ξ, β and γ are marginal productivities of corresponding production factors. The derivatives of the production factors can be written on the basis of (5.6) and (5.13) in the form

$$\begin{aligned} &\frac{dK}{dt} = \left(\frac{I}{K} - \mu\right)K, \qquad \frac{dL}{dt} = \left(\bar{\lambda}\frac{I}{K} - \nu' - \mu\right)L, \\ &\frac{dP}{dt} = \left(\bar{\varepsilon}\frac{I}{K} - \eta' - \mu\right)P, \end{aligned} \tag{6.6}$$

where $\bar{\lambda} = \lambda K/L$ and $\bar{\varepsilon} = \varepsilon K/P$ are the non-dimensional technological variables which characterise the quality of introduced equipment. Both the technological coefficients λ, ε and the non-dimensional technological variables $\bar{\lambda}$, $\bar{\varepsilon}$ are functions of time.

Combining (6.5) and (6.6) and assuming, for the start, that the quantities ν' and η' in (6.6) can be neglected, one rewrites production of value in the form

$$\frac{dY}{dt} - \Delta = \begin{cases} \xi \left(\dfrac{I}{K} - \mu\right)K, \\ (\beta\bar{\lambda}L + \gamma\bar{\varepsilon}P)\dfrac{I}{K} - \mu(\beta L + \gamma P). \end{cases} \tag{6.7}$$

Comparing the right-hand sides of (6.7) on the assumption that the investment I has arbitrary value, one finds relations between the characteristic parameters of the system,

$$\bar{\lambda}\beta L + \bar{\varepsilon}\gamma P = \xi K,$$

$$\beta L + \gamma P = \xi K.$$

The relations ought to be considered as a set of equations for the marginal productivities β and γ. Then, in accordance with [9], one can find the expressions of the marginal productivities through the technological coefficients

$$\xi = \beta\frac{L}{K} + \gamma\frac{P}{K}, \tag{6.8}$$

$$\beta = \xi\frac{\bar{\varepsilon} - 1}{\bar{\varepsilon} - \bar{\lambda}}\frac{K}{L}, \qquad \gamma = \xi\frac{1 - \bar{\lambda}}{\bar{\varepsilon} - \bar{\lambda}}\frac{K}{P}. \tag{6.9}$$

One can see that the marginal productivities are positive, if

$$\bar{\lambda} < 1 < \bar{\varepsilon} \quad \text{or} \quad \bar{\lambda} > 1 > \bar{\varepsilon}. \tag{6.10}$$

However, it is possible that one of the marginal productivities is negative, if the technological coefficients are unrestricted. But they cannot both be negative. It is natural to expect that the mean marginal productivities must be positive for some span of time, that is, that one of the sets of requirements (6.10) is fulfilled, which was discussed in Sect. 5.2.2. One can consider inequalities (6.10) as a formulation of the productivity principle.

In the general case, when $\nu' \neq 0$, $\eta' \neq 0$, that is, the change of technology during the time of exploitation is taken into account, the relations between marginal productivities (6.8) and (6.9) can be generalised as

$$\xi = \beta\left(1 + \frac{\nu'}{\mu}\right)\frac{L}{K} + \gamma\left(1 + \frac{\eta'}{\mu}\right)\frac{P}{K},$$

$$\beta = \xi \frac{\bar{\varepsilon} - (1 + \frac{\eta'}{\mu})}{\bar{\varepsilon}(1 + \frac{\nu'}{\mu}) - \bar{\lambda}(1 + \frac{\eta'}{\mu})} \frac{K}{L}, \qquad \gamma = \xi \frac{(1 + \frac{\nu'}{\mu}) - \bar{\lambda}}{\bar{\varepsilon}(1 + \frac{\nu'}{\mu}) - \bar{\lambda}(1 + \frac{\eta'}{\mu})} \frac{K}{P}.$$

6.3 Approximation of Marginal Productivities

The production function is assumed to satisfy some requirements. One of them—the principle of *productivity*—has been discussed in the previous sections. Then, one should take into account that the arguments of the production function must be relative quantities, so the production function can be written as

$$Y = Y_0 f\left(\frac{L}{L_0}, \frac{P}{P_0}\right),$$

where L_0 and P_0 are values of labour and capital services in the base year. In this section, we shall consider restrictions imposed on the production function by the requirements of *uniformity* and *universality*.

6.3.1 Principle of Universality

This requirement means that a proposed production function can be used not only for a given case, but for many different situations. For example, the initial point can be chosen arbitrarily, but the form of production function must not be affected by this arbitrariness. If, for example, two initial points, t_0 and $t_1 > t_0$, are chosen, one must write for the production of value

$$Y = Y_1 f\left(\frac{L}{L_1}, \frac{P}{P_1}\right), \qquad Y_1 = Y_0 f\left(\frac{L_1}{L_0}, \frac{P_1}{P_0}\right).$$

This can be rewritten as

$$Y = Y_0 f\left(\frac{L_1}{L_0}, \frac{P_1}{P_0}\right) f\left(\frac{L}{L_1}, \frac{P}{P_1}\right),$$

so that, in order for the description to be universal, that is, independent of the arbitrary choice of initial point (to be consistent), the production function must satisfy the following relation:

$$Y_0 f\left(\frac{L}{L_0}, \frac{P}{P_0}\right) = Y_0 f\left(\frac{L_1}{L_0}, \frac{P_1}{P_0}\right) f\left(\frac{L}{L_1}, \frac{P}{P_1}\right).$$

One must choose a form of function $f(\cdot)$ such that values of L_1, P_1 on the right-hand side of the last formula would disappear. This requirement puts some restrictions on the form of the production function. One can see that a production function of the form

$$Y = Y_0 \left(\frac{L}{L_0}\right)^{\alpha} \left(\frac{P}{P_0}\right)^{\beta} \tag{6.11}$$

obeys the above requirement. The parameters α and β do not depend on the initial point and can be considered as characteristics of the production system. Of course, one can consider them functions of production factors, which can be represented by an expansion series in the powers of quantities $\ln \frac{L}{L_0}, \ln \frac{P}{P_0}$. For example, in linear approximation,

$$\alpha = \alpha_0 + a \ln \frac{L}{L_0} + b \ln \frac{P}{P_0},$$

where the parameters a, b are also characteristics of the production system.[5] As for any other forms of the production function in the literature, it is worth testing them for consistency before using them.

6.3.2 *Principle of Uniformity*

One expects the production output for a large system to be proportional to the scale of production. This means that the production function has to be a homogeneous, uniform function of first order, that is,

$$Y(\lambda L, \lambda P) = \lambda Y(L, P).$$

[5]The first-order terms of expansion determine the first-order function

$$Y = Y_0 \left(\frac{L}{L_0}\right)^{\beta_0} \left(\frac{P}{P_0}\right)^{\gamma_0} \exp \frac{1}{2}\left(\beta_l \ln^2 \frac{L}{L_0} + (\beta_e + \gamma_l) \ln \frac{L}{L_0} \ln \frac{P}{P_0} + \gamma_e \ln^2 \frac{P}{P_0}\right),$$

which under the condition of uniformity takes the following form:

$$Y = Y_0 \frac{L}{L_0} \left(\frac{L_0}{L}\frac{P}{P_0}\right)^{\alpha} \exp\left(\frac{1}{2} b \ln^2\left(\frac{L_0}{L}\frac{P}{P_0}\right)\right), \quad b = -\beta_l = \beta_e = \gamma_l = -\gamma_e.$$

Under this condition function (6.11) can be written as

$$Y = Y_0 \frac{L}{L_0} \left(\frac{L_0}{L} \frac{P}{P_0} \right)^{\alpha}. \tag{6.12}$$

The constant Y_0 is determined by the initial conditions, so that the only parameter which remains unknown in the expression for the production function, to say nothing of the initial values of the variables, is the quantity $\alpha = \gamma_0 = 1 - \beta_0$. It is a characteristic of the production system, which, as shown below, coincides with the technological index introduced in Sect. 5.2.2. The productivity principle restricts values of the technological index, $0 < \alpha < 1$.

6.3.3 Marginal Productivities

Function (6.12) has the exact form of the neo-classical, i.e., Cobb–Douglas production function (1.4), in which capital K stands for substitutive work P. This function, in accordance with (6.5), provides the following expressions for marginal productivities and the contribution from technological and structural change:

$$\beta = Y_0 \frac{1-\alpha}{L_0} \left(\frac{L_0}{L} \frac{P}{P_0} \right)^{\alpha}, \qquad \gamma = Y_0 \frac{\alpha}{P_0} \left(\frac{L_0}{L} \frac{P}{P_0} \right)^{\alpha - 1}, \tag{6.13}$$

$$\Delta = Y \ln \left(\frac{L_0}{L} \frac{P}{P_0} \right) \frac{d\alpha}{dt} \tag{6.14}$$

where L_0 and P_0 are values of labour and capital services in the base year.

Comparing expressions (6.9) and (6.13) for the marginal productivities, one obtains

$$\xi = Y_0 \frac{L}{L_0 K} \left(\frac{L_0}{L} \frac{P}{P_0} \right)^{\alpha}, \qquad \alpha = \frac{1 - \bar{\lambda}}{\bar{\varepsilon} - \bar{\lambda}}. \tag{6.15}$$

Thus, the index α in (6.12) is, indeed, the same quantity as the technological index introduced in (5.20). In addition, all available information about the technological performance could be introduced by estimating this quantity. Moreover, a condition regarding the optimal use of production factors enables us to establish a relation between the parameter α on one hand and the shared costs of production factors on the other (see Sect. 6.6). This provides a different means of estimating the technological index.

The condition of productivity $0 < \alpha < 1$ means that the marginal productivities are increasing functions of the ratio of substitutive work to labour, that is,

$$\frac{d\beta}{d(P/L)} > 0, \qquad \frac{d\gamma}{d(P/L)} > 0.$$

6.4 Decomposition of the Growth Rate of Output

The preceding results allow us to record output Y as a function of production factors K, L and P, whereas properties of the production system itself are fixed by the internal characteristics α and ξ. From these findings, formula (6.2) can be specified as

$$Y = \begin{cases} \xi K, \\ Y_0 \dfrac{L}{L_0} \left(\dfrac{L_0}{L} \dfrac{P}{P_0} \right)^{\alpha} \end{cases} \tag{6.16}$$

and the growth rate of the output in terms of the present theory can be written in the form of two alternative expressions,

$$\frac{1}{Y}\frac{dY}{dt} = \begin{cases} \dfrac{1}{K}\dfrac{dK}{dt} + \dfrac{1}{\xi}\dfrac{d\xi}{dt}, \\ (1-\alpha)\dfrac{1}{L}\dfrac{dL}{dt} + \alpha \dfrac{1}{P}\dfrac{dP}{dt} + \ln\left(\dfrac{L_0}{L} \dfrac{P}{P_0} \right) \dfrac{d\alpha}{dt}. \end{cases} \tag{6.17}$$

The first terms in the first and second lines on the right-hand side of this formula represent the contribution to growth due to the growth of production factors: capital, labour and substitutive work. The last ones are due to the change of the production system itself; changes of quantities ξ and α are connected with technological and structural changes. The last terms cannot be reduced to any function of production factors.

Let us note that variations of characteristics of production system ξ and α are connected with each other. To find a formula for a change of capital marginal productivity, we differentiate relation (6.15)

$$\frac{1}{\xi}\frac{d\xi}{dt} = -\frac{1}{K}\frac{dK}{dt} + (1-\alpha)\frac{1}{L}\frac{dL}{dt} + \alpha \frac{1}{P}\frac{dP}{dt} + \ln\left(\frac{L_0}{L} \frac{P}{P_0} \right) \frac{d\alpha}{dt}.$$

To simplify this relation, one can use the dynamic equations for production factors (6.6), on the assumption $\nu' \neq 0$, $\eta' \neq 0$, and obtain

$$\frac{1}{\xi}\frac{d\xi}{dt} = \ln\left(\frac{L_0}{L} \frac{P}{P_0} \right) \frac{d\alpha}{dt}, \tag{6.18}$$

so that, in line with (6.14), one has

$$\Delta = Y \frac{1}{\xi}\frac{d\xi}{dt}. \tag{6.19}$$

Returning now to the expression for the output growth rate (6.17) and using (5.26) for the growth rates of production factors in three possible cases, one can record

$$\frac{1}{Y}\frac{dY}{dt} = \frac{1}{\xi}\frac{d\xi}{dt} + \begin{cases} \delta, & \text{if } \dfrac{d\bar{\lambda}}{dt} > 0 \text{ and } \dfrac{d\bar{\varepsilon}}{dt} > 0, \\ (\nu+\mu)\dfrac{1}{\bar{\lambda}} - \mu, & \text{if } \dfrac{d\bar{\lambda}}{dt} < 0 \text{ and } \dfrac{d\bar{\varepsilon}}{dt} > 0, \\ (\eta+\mu)\dfrac{1}{\bar{\varepsilon}} - \mu, & \text{if } \dfrac{d\bar{\lambda}}{dt} > 0 \text{ and } \dfrac{d\bar{\varepsilon}}{dt} < 0. \end{cases} \tag{6.20}$$

It is assumed that the characteristics of the equipment do not change after its installation, that is, $\nu' = 0$ and $\eta' = 0$; otherwise, the expression for the growth rate takes a more complicated form.

The growth rate of the output is expressed by three different relations for three modes of development. The first line of (6.20) is valid in the case of lack of investment, and abundance of labour, substitutive work and raw materials. The second line is valid in the case of lack of labour, and abundance of investment, substitutive work and raw materials. The last line of equations is valid in the case of lack of substitutive work, and abundance of investment, labour and raw materials.

To simplify relations (6.20), one can use (5.20) to express the energy requirement and the growth rate of substitutive work through the technological index as

$$\bar{\varepsilon} = \frac{1-(1-\alpha)\bar{\lambda}}{\alpha}, \qquad \eta = \frac{\delta-(1-\alpha)\nu}{\alpha}, \quad 0 < \alpha < 1.$$

These relations allow one to identify the expressions in the second and the third lines of (6.20). Moreover, one can see that these relations practically exclude the first line, so that (6.20) reduce to a universal expression for the growth rate of output,

$$\frac{1}{Y}\frac{dY}{dt} = \frac{\nu+(1-\bar{\lambda})\mu}{\bar{\lambda}} + \frac{1}{\xi}\frac{d\xi}{dt}. \tag{6.21}$$

One can see that the growth rate of the output is determined by four quantities:

- Productivity of capital stock ξ. This is a fundamental quantity connected with the fundamental technological matrices (see (8.23)), when one refers to the many-sector approach. Technological and structural changes are introduced through this quantity.
- The non-dimensional technological coefficient $\bar{\lambda}$. If the quantity $\bar{\lambda} < 1$, the consumption (for unit of capital stock) of labour decreases and consumption of productive energy (substitutive work of production equipment) increases. The situation is opposite, if the quantity $\bar{\lambda} > 1$.
- The coefficient of depreciation μ. This quantity does not affect the rate of output growth, if $\bar{\lambda} = 1$.
- The rate of growth of labour ν. The rate of output growth coincides with this quantity, if $\bar{\lambda} = 1$.

All the quantities are characteristics of the method of production, that is, characteristics of technology.

6.5 Productivity of Labour

The scientific and technical progress could be reduced to processes of introduction of innovations, that is, consecutive replacement of instruments, materials, designs, adaptations and other objects with more perfect ones, from some point of view. Among all processes of replacement, the outstanding role is played by the processes of replacement of workers' efforts by machine work with the assistance of forces of nature. The substitution of worker efforts with machine work is a unique process of replacement which influences the labour productivity, which is understood as the ratio of the value of output, measured in value units of constant purchasing capacity, to the expenditures of labour. According to (6.12), the productivity of labour can be written as

$$B = \frac{Y}{L} = \frac{Y_0}{L_0}\left(\frac{L_0}{L}\frac{P}{P_0}\right)^{\alpha}.$$

This quantity depends on the ratio of substitutive work to workers' efforts P/L and is the same labour productivity, the increase of which determines change of one social formation by another, more perfect one. The increase in labour productivity cannot be understood without considering the phenomenon accompanying the progress of production—the attraction of natural energy sources (animals, wind, water, coal, oil and others) for performing work that replaces human efforts in production. One should check that, at the definition of labour productivity, the output is measured in value units of constant purchasing capacity, as one says, to represent a 'physical' measure of output.

The expression for the growth rate of labour productivity follows from (6.21) written on the assumption that characteristics of the equipment do not change after its installation, that is, $\nu' = 0$ and $\eta' = 0$,

$$\frac{1}{B}\frac{dB}{dt} = \frac{(1-\bar{\lambda})(\nu + \mu)}{\bar{\lambda}} + \frac{1}{\xi}\frac{d\xi}{dt}. \tag{6.22}$$

In this equation, besides the known factor of amortisation μ, there is a non-dimensional quantity $\bar{\lambda}$, which characterises the technology introduced into the production. The labour requirement $\bar{\lambda}$ appears to be the most important quantity to determine the growth rate of labour productivity. Below it will be demonstrated that this quantity is a measure of the substitution of labour by energy. If $\bar{\lambda} = 1$, variations in technology do not occur, labour productivity is constant, and all incrementing of a product is connected only with an increase in human efforts. Human efforts are, certainly, the main motive power, but, under the condition $\bar{\lambda} < 1$, the workers' efforts are partially replaced with the work of machines movable by outer energy sources, and the labour productivity increases. This is a general description of the influence of scientific and technological progress, which occurs naturally in a picture of the progress of mankind.

To introduce a global characteristic of labour efficiency in a different form, one can start with expression (6.7) for the production of value, which can be rewritten as

$$\frac{dY}{dt} = (\beta\lambda + \gamma\varepsilon)\big(I - \breve{I}\big), \tag{6.23}$$

where

$$\breve{I} = \mu \frac{\beta L + \gamma P}{\beta \lambda + \gamma \varepsilon} = \mu \frac{\beta L + \gamma P}{\beta \bar{\lambda} L + \gamma \bar{\varepsilon} P} K. \tag{6.24}$$

When output is constant, that is $dY/dt = 0$, according to (6.23) and (6.24), investments are determined as

$$I = \mu \frac{\beta L + \gamma P}{\beta \lambda + \gamma \varepsilon}. \tag{6.25}$$

Though the output does not change, the production factors are changing according to relations (6.6) and (6.25), as

$$\frac{dL}{dt} = -\mu\gamma \frac{\varepsilon L - \lambda P}{\beta \lambda + \gamma \varepsilon}, \qquad \frac{dP}{dt} = \mu\beta \frac{\varepsilon L - \lambda P}{\beta \lambda + \gamma \varepsilon}. \tag{6.26}$$

One can see that, at constant output, the amount of labour decreases, that is, the productivity of labour increases, if

$$\Theta = \frac{\bar{\varepsilon}}{\bar{\lambda}} = \frac{\varepsilon L}{\lambda P} > 1. \tag{6.27}$$

Otherwise, the productivity of labour decreases.

The quantity Θ can be called a general index of labour productivity growth that is connected with technological progress, and relation (6.27) can be considered as a condition of increase in the efficiency of labour. This is a condition of the labour productivity growth.

The index of labour productivity growth obeys (5.29), that is,

$$\frac{d\Theta}{dt} = -\frac{1}{\tau}\left(\Theta - \frac{\tilde{\eta} + \mu}{\tilde{\nu} + \mu}\right). \tag{6.28}$$

One can see that the general index of labour productivity follows the ratio of potential rates of production factors with a lag, which is determined by time τ. This time is a time of introduction of production equipment. We refer to formulae (5.19) to write the index of technical progress as

$$\Theta = \frac{\eta + \mu}{\nu + \mu}. \tag{6.29}$$

It follows from this relation that, when (6.27) is valid,

$$\eta > \nu$$

which means that the growth rate of consumption of productive energy exceeds the growth rate of labour, when technological progress has occurred.

6.6 The Best Utilisation of Production Factors

One can assume that production factors L and P are chosen in such amounts to be the most effective in production, that is, their values maximise the production function at given total expenses for production factors. The question of the choice of production factors can be interpreted as a problem of finding the maximum value of function $Y(L, P)$ at condition

$$cL + pP = V,$$

where c and p are prices of 'consumption' of production factors, and V is a part of the gross output, which goes for maintenance of production factors. The discussion of prices c and p is given in Sects. 2.4.2 and 2.5.3 (see (2.35)).

One can follow a general method of searching a conditional extremum [10], so that we look for the unconditional maximum of the Lagrange function

$$Y(L, P) - \kappa(cL + pP - V), \tag{6.30}$$

where κ is a Lagrange multiplier.

Therefore, we obtain equations for a point of the conditional extremum

$$\frac{\partial Y}{\partial L} = \kappa c, \qquad \frac{\partial Y}{\partial P} = \kappa p,$$

which can be rewritten, remembering the definitions of the marginal productivities, as

$$\beta = \kappa c, \qquad \gamma = \kappa p.$$

From the last relations it follows that the ratio of the prices of production factors is equal to the ratio of marginal productivities or, referring to (6.9), is inversely proportional to the ratio of production factors

$$\frac{c}{p} = \frac{\beta}{\gamma} = \frac{\bar{\varepsilon} - 1}{1 - \bar{\lambda}} \frac{P}{L}.$$

The relation between the prices of production factors, taking definition (5.20) for the technological index into account, can also be written as

$$\frac{c}{p} = \frac{\beta}{\gamma} = \frac{1 - \alpha}{\alpha} \frac{P}{L}.$$

Therefore, the index α can be expressed through the prices and the amounts of production factors,

$$\alpha = \frac{pP}{cL + pP}. \tag{6.31}$$

This relation means that the technological index α in the equilibrium situation can be interpreted as follows: it is a share of the expenses, needed for utilisation of substitutive work as a production factor, within the total expenses for production factors. If the production factors are chosen as optimal, then

$$0 < \alpha < 1,$$

which implies the known restrictions on values of technological variables

$$\bar{\lambda} < 1 < \bar{\varepsilon} \quad \text{or} \quad \bar{\lambda} > 1 > \bar{\varepsilon}.$$

In terms of the growth rates, the restricting conditions are read as

$$\nu < \delta < \eta \quad \text{or} \quad \nu > \delta > \eta. \tag{6.32}$$

These conditions coincide with conditions of positivity of marginal productivities (see formulae (6.10) in Sect. 6.2).

Expression (6.31) allows one to estimate the technological index α due to the estimates of the cost of consumption of production factors. Remembering the definition of the cost of substitutive work (2.35), one can define the cost of labour as

$$c = \mu \frac{\bar{\varepsilon} - 1}{1 - \bar{\lambda}} \frac{K}{L} = \mu \frac{1 - \alpha}{\alpha} \frac{K}{L}. \tag{6.33}$$

Note that the cost of labour c, which is the value of products needed to compensate current living expenses, does not include, in contrast to the price of labour (wage), any accumulation. It is the value of the minimum amount of products which are needed for humans to subsist.

6.7 On the Choice Between Consumption and Saving

The production system of an economy is driven by the desires of economic subjects, first of all by the desires of producers to produce as much as they can. The production system tries to swallow all available resources, and three modes of economic development for which we have different formulae for calculation are possible (see Sect. 5.3.1). In the case of abundance of labour and energy, the desires of producers can meet restrictions from the side of consumers. In this case, a nation must decide how much it should save or consume, taking into account present and future consumption. According to Blanchard and Fisher [11], there are two basic models for solving this problem: the infinite horizon optimising model [12] and the overlapping generations model with finite horizon [13–15].

Frank Ramsay [12] used a simple model consisting of the neo-classical production function (1.4) and the equation for capital dynamics (5.6). According to the conventional conviction of that time, he supposed that a trajectory of evolution can be chosen in such a way that the consumption for the time T is the biggest one. In other words, one ought to maximise the function

$$U(T) = \int_0^T u(c) e^{-\theta s}\, ds, \tag{6.34}$$

where consumption c is a function of time s, $u(c)$ is a concave objective function and T is the horizon of planning.

We have no need to consider the original version of the Ramsay solution (it can be found, for example, in the monograph by Blanchard and Fisher [11]); instead, we consider the problem as applying to evolution equations (5.6) and (5.13) which were formulated in Sect. 5.1. To introduce the consumption into equations, one can take the relation

$$I = Y - cL \tag{6.35}$$

and rewrite the equation of the second line in formula (6.7) for production of value in another form to get the system of equations

$$\begin{aligned}
\frac{dY}{dt} &= (\beta\lambda + \gamma\varepsilon)Y - (\beta\lambda + \gamma\varepsilon)cL - \mu(\beta L + \gamma P),\\
\frac{dK}{dt} &= Y - cL - \mu K, \qquad \frac{dL}{dt} = \lambda Y - (\lambda c + \mu)L,\\
\frac{dP}{dt} &= \varepsilon Y - \varepsilon cL - \mu P.
\end{aligned} \tag{6.36}$$

Further, it is convenient to use the variables

$$y = \frac{Y}{L}, \qquad \epsilon = \frac{P}{L}, \qquad \ell = \frac{L}{K}$$

to write a system of evolutionary equations in the form

$$\begin{aligned}
\frac{dy}{dt} &= \big(\beta(\epsilon) - y\big)\big[\bar{\lambda}(y - c)\ell - \mu\big] + \gamma(\epsilon)\big[\bar{\varepsilon}(y - c)\ell - \mu\big]\epsilon,\\
\frac{d\epsilon}{dt} &= (\bar{\varepsilon} - \bar{\lambda})(y - c)\epsilon\ell,\\
\frac{d\ell}{dt} &= (\bar{\lambda} - 1)(y - c)\ell^2.
\end{aligned} \tag{6.37}$$

The technological variables $\bar{\lambda} = \lambda K/L, \bar{\varepsilon} = \varepsilon K/S$ are determined by (5.30) and (5.31). It is assumed that the marginal productivities are given, for example, by formula (6.13).

One can use the standard procedure to find a trajectory which maximises function (6.34) under restrictions (6.37). The result is a differential equation for consumption as a function of time.

The second model—the overlapping generation model [13–15]—assumes that at any time individuals of different generations are alive and may be interacting with one another. The investment is generated by individuals who save during their lives to ensure their consumption during retirement. In this way, preferences of individuals determine investment into the production system.

The models that we have discussed in this section can be very useful in estimating potential investment and potential rate of capital growth. However, it is necessary to take the availability of labour and energy into account to obtain a real trajectory of evolution.

References

1. Pokrovski, V.N.: Physical Principles in the Theory of Economic Growth. Ashgate, Aldershot (1999)
2. Berndt, E.R., Wood, D.O.: Engineering and econometric interpretations of energy—capital complementarity. Am. Econ. Rev. **69**, 342–354 (1979)

3. Patterson, M.G.: What is energy efficiency? Concepts, indicators and methodological issues. Energy Policy **24**, 377–390 (1996)
4. Odum, H.T.: Environmental Accounting. EMERGY and Environmental Decision Making. Wiley, New York (1996)
5. Valero, A.: Thermoeconomics as a conceptual basis for energy-ecological analysis. In: Ulgiati, S. (ed.) Advances in Energy Studies Workshop. Energy Flows in Ecology and Economy, Porto Venere, Italy, 1998, pp. 415–444. MUSIS, Rome (1998)
6. Costanza, R.: Embodied energy and economic valuation. Science **210**, 1219–1224 (1980)
7. Cleveland, C.J., Costanza, R., Hall, C.A.S., Kaufmann, R.: Energy and the U.S. economy: a biophysical perspective. Science **225**, 890–897 (1984)
8. Marx, K.: Capital. Encyclopaedia Britannica, Chicago (1952). English translation of Karl Marx, Das Kapital. Kritik der politischen Oekonomie, Otto Meissner, Hamburg, 1867. http://marxists.anu.edu.au/archive/marx/works/cw/index.htm. Accessed 17 March 2011
9. Pokrovski, V.N.: Energy in the theory of production. Energy **28**, 769–788 (2003)
10. Korn, G.A., Korn, T.M.: Mathematical Handbook for Scientists and Engineers. McGraw-Hill, New York (1968)
11. Blanchard, O.J., Fisher, S.: Lectures on Macroeconomics. MIT Press, Cambridge (1989)
12. Ramsey, F.P.: A mathematical theory of saving. Econ. J. **38**, 543–559 (1928)
13. Allais, M.: Economie et Interet. Imprimerie Nationale, Paris (1947)
14. Samuelson, P.A.: An exact consumption-loan model of interest with or without the social contrivance of money. J. Polit. Econ. **66**, 467–482 (1958)
15. Diamond, P.A.: National debt in a neo-classical growth model. Am. Econ. Rev. **55**, 1126–1150 (1965)

Chapter 7
Application to the U.S. Economy

Abstract In this chapter, to illustrate and test the theory, we refer to time series for the U.S. economy for years 1890–2009 collected in Appendix B. The choice of this case is justified by the availability and reliability of the data, which can be easily found on webpages of the U.S. Census Bureau and the U.S. Bureau of Economic Analysis. These organisations have been permanently improving methods of estimation of time series, and the data has been permanently revised in order for the numbers to be as accurate as possible. We have used the latest available series to illustrate the methods of estimation of some quantities: substitutive work, technological index, marginal productivities and technological coefficients.

7.1 Production Factors

The empirical time series of output Y, capital K and labour L are usually known, and, for the U.S. economy, are collected in Appendix B. The methods of direct estimation of the third production factor—substitutive work P—are not perfect at the moment (see Sect. 2.5.2), so a method of indirect estimation of substitutive work P is developed in Sect. 7.1.2. It also allows simultaneous calculation of the technological index α. When time series for all production factors are known, the technological characteristics of the production system can be estimated.

7.1.1 Personal Consumption and the Technological Index

The technological index α appears to be a very important characteristic of the production system. An estimate of the technological index can be obtained when the optimal use of production factors is considered. According to Sect. 6.6 (formula (6.31)), the technological index α can be expressed through prices and amounts of production factors as

$$\alpha = \frac{pP}{cL + pP}. \tag{7.1}$$

This means that the technological index α represents the share of expenses needed for utilisation of capital services as a production factor (substitutive work) within

V.N. Pokrovskii, *Econodynamics*, New Economic Windows 12,
DOI 10.1007/978-94-007-2096-1_7,

the total expenses for production factors. Expression (7.1) allows one to estimate the technological index α from estimates of the cost of consumption of production factors.

The current consumption $C = cL$ is defined as the value of the minimum amount of products which are needed for the labour force to maintain. The cost of maintenance of labour can be estimated (see Sect. 2.2.4) through the poverty threshold c^* as $cL = c^*N$, where N is the number of the population. Measured in this way, the consumption in year 1996 was $C = 2{,}120$ billion dollars compared with the expenses for maintenance of consumption of capital services $pP = \mu K = 1{,}378$ billion dollars (1996). So, for the latest decade of the twentieth century, one can obtain $\alpha \approx 0.4$, which means that about 40% of the total expenses for production factors take energy as substitute of labour.

7.1.2 Substitutive Work and the Technological Index

A simple method allows us to calculate both substitutive work P and values of the technological index α, if the empirical time series for output Y, capital K and labour L are known.

The value of the technological index α can be represented, from (6.12), as

$$\alpha = \frac{\ln\left(\frac{Y}{Y_0}\frac{L_0}{L}\right)}{\ln\left(\frac{L_0}{L}\frac{P}{P_0}\right)}. \tag{7.2}$$

However, the amount of substitutive work P itself depends on the value of the technological index α. The growth rate of substitutive work, from (5.20), is calculated as

$$\eta = \frac{\delta - (1-\alpha)\nu}{\alpha}, \quad 0 < \alpha < 1, \tag{7.3}$$

where ν is defined by (5.17) and includes a possible correction determined by relation (7.7). Then, the time dependence of substitutive work can be restored by solving the equation

$$\frac{dP}{dt} = \eta(\alpha)P. \tag{7.4}$$

Equations (7.2)–(7.4) allow one to estimate the technological index α and substitutive work P at given time series for Y, K and L.

The results of calculation for α are depicted on the plot of Fig. 7.1 in line with the values of α calculated due to formula (7.1). Note that the choice of initial value of the technological index allows us to move the whole curve of α up and down, so it is important to have at least one point where the absolute value of α is known, which, according to the estimation in Sect. 7.1.1, is taken as $\alpha \approx 0.4$ in year 1997. The calculated values of the technological index are used to estimate the total personal consumption. The results for the U.S. in the twentieth century are shown in Fig. 2.4 in Sect. 2.2.4.2.

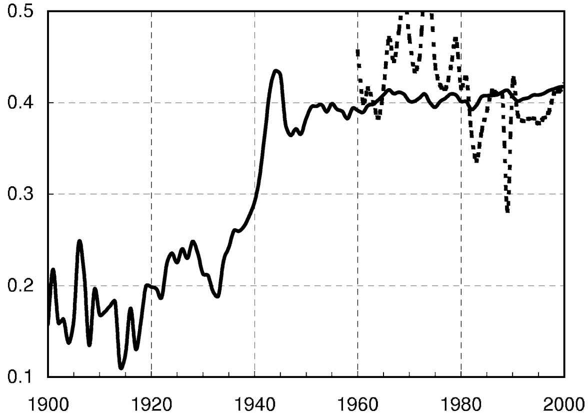

Fig. 7.1 Technological index. The *solid line* represents values of α found according to (7.2)–(7.4). The *dashed line* represents values calculated due to formula (7.1). Reproduced from [1] with permission of Elsevier

The estimated values of the technological index allow us to calculate the growth rate of substitutive work η and to restore the time dependence of the quantity. The results for P are shown in Fig. 2.8 in Sect. 2.5.1 in line with the total (primary) consumption of energy in the U.S. economy. One can see that the substitutive work grew on average faster than the total consumption of energy in years 1900–2000; however, there are some years of recession.

7.1.3 Estimation of the Technological Coefficients

From formula (5.20), the technological index α can also be calculated through the technological coefficients $\bar{\lambda}$ and $\bar{\varepsilon}$, which can be estimated independently. Values of the coefficients of labour and substitutive work requirement can be found from (5.6) and (5.13), if one has time series of the production factors and investment. The equations for the production factors can be rewritten in terms of the non-dimensional technological coefficients as

$$\frac{dL}{dt} = \left(\bar{\lambda}\frac{I}{K} - \nu' - \mu\right)L, \qquad \frac{dP}{dt} = \left(\bar{\varepsilon}\frac{I}{K} - \eta' - \mu\right)P. \tag{7.5}$$

These equations allow one to develop methods of estimation of the technological coefficients $\bar{\lambda}$ and $\bar{\varepsilon}$, whereas one has to take into account the conditions (6.10) of non-negativity of the marginal productivities (principle of productivity), which, in terms of the technological index, can be rewritten as

$$0 < \bar{\lambda} < \frac{1}{1-\alpha}, \qquad 0 < \bar{\varepsilon} < \frac{1}{\alpha}, \quad 0 < \alpha < 1. \tag{7.6}$$

The depreciation coefficient μ in (7.5) is estimated from the time series of I and K; these empirical values are determined in Sect. 2.2.4.1 and shown in Fig. 2.3. The conditions (7.6) allow one to separate the extra depreciation rate of labour and substitutive work ν' and η' on the basis of time series of quantities I/K, L and P.

One can assume that all consumption of labour L is productive, thus values of the labour requirement $\bar{\lambda}$ can be calculated directly from the first equation from the set (7.5). An attempt to exploit this procedure, considering $\nu' = 0$, determines negative values of the technological coefficients (see the solid lines on the top plot in Fig. 7.2), which can be connected with errors in estimating the amount of labour and must be corrected by the quantity ν'. To ensure the fulfilment of relations (7.6), one has to set the amendment ν' as

$$\nu' = -\frac{1}{L}\frac{dL}{dt} - \mu + \begin{cases} \bar{\lambda}_0 \dfrac{I}{K}, & \dfrac{1}{L}\dfrac{dL}{dt} + \mu < \bar{\lambda}_0 \dfrac{I}{K}, \quad \bar{\lambda}_0 < \dfrac{1}{1-\alpha}, \\ \dfrac{1-\alpha\bar{\varepsilon}_0}{1-\alpha}\dfrac{I}{K}, & \dfrac{1}{L}\dfrac{dL}{dt} + \mu > \dfrac{1-\alpha\bar{\varepsilon}_0}{1-\alpha}\dfrac{I}{K}, \quad \bar{\varepsilon}_0 < \dfrac{1}{\alpha}, \end{cases} \tag{7.7}$$

where the positive quantities $\bar{\lambda}_0$ and $\bar{\varepsilon}_0$ are the prescribed bottom values of the technological coefficients.

The corrected results of the technological coefficients $\bar{\lambda}$ are shown by the solid lines on the bottom plot in Fig. 7.2. Values of the extra rate of depreciation ν' appear to be noticeable in the first half of the century, but quite insignificant after year 1950. It is known that estimates of the economic quantities for the first half of the century are less reliable than those for the second half, so the deviation of the quantity ν' from zero can be connected not only with performance of technological equipment, but with some possible errors in estimating economic quantities.

Then, one can turn to the estimation of the second technological coefficient—substitutive work requirement $\bar{\varepsilon}$. The dotted lines on the top plot in Fig. 7.2 shows values of primary energy requirement, calculated on the basis of the second equation from the set (7.5), in which primary energy E stands for substitutive work P. The dotted line on the bottom plot in Fig. 7.2 shows values of substitutive work requirement, calculated, at known values of the labour requirement and technological index α (see the previous section), according to the formula

$$\bar{\varepsilon} = \frac{1 - (1-\alpha)\bar{\lambda}}{\alpha}. \tag{7.8}$$

For a relatively reliable data of period 1950–2000, the technological coefficients do not change much, thus one can estimate mean values

$$\bar{\lambda} = 0.758, \qquad \bar{\varepsilon} = 1.367.$$

The first quantity is less than unity. It means that on average labour-saving technologies are introduced during this span of time. One can see that the technological coefficients are determined by two controversial tendencies. In order to save labour and energy, they both must be less than unity. However, one of the technological coefficients must be greater than unity in order for the marginal productivity to be

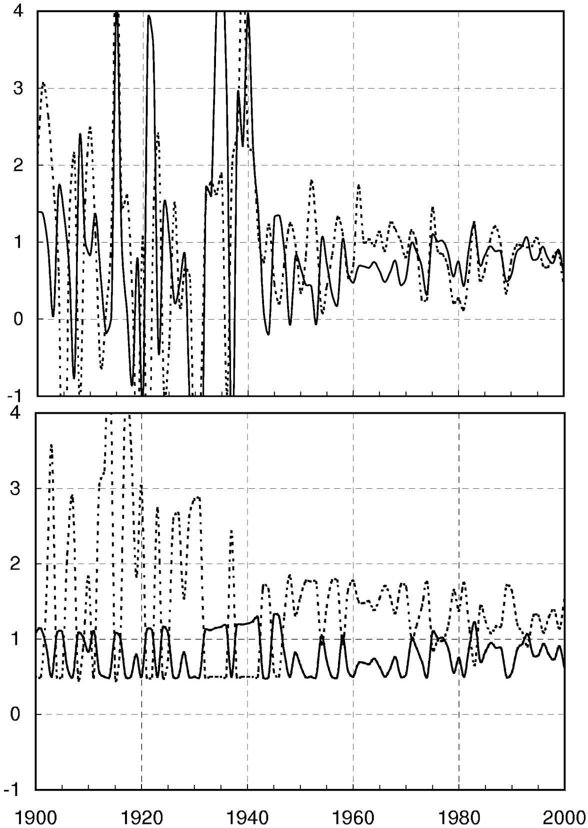

Fig. 7.2 Technological coefficients for the U.S. economy. *Top*: Labour (*solid lines*) and primary energy (*dotted lines*) requirements are calculated according to relations from (7.5) (for labour at $\nu' = 0$) on the basis of time series for capital stock, labour, primary energy and investment. One can see that sometimes inequalities (7.6) are not fulfilled. *Bottom*: Corrected labour (*solid lines*) and substitutive work (*dotted lines*) requirements are calculated due to relations (7.5) and (7.8) (at $\nu' \neq 0$). Values of ν' are estimated due to (7.7) at $\bar{\lambda}_0 = \bar{\varepsilon}_0 = 0.5$

positive. As a result, it appears that the technological coefficients pulsate around unity.

7.1.4 Trajectories of Development

To illustrate the applicability of the theory, in this section we apply the dynamic equations, obtained in Chap. 5, which allow us to calculate trajectories of evolution of the production factors. The empirical values of production factors K, L and P for the period 1900–2000 (see Appendix B), represented on the chart of Fig. 7.3, allow one to test the adequacy of the obtained equations and to extrapolate results for the future.

7.1.4.1 The System of Equations

First of all, we collect the dynamic equations for the production factors, that is, (5.6), (5.13), (5.25), (5.30), (5.31) and (5.34) of Chap. 5,

$$
\begin{aligned}
&\frac{dK}{dt} = I - \mu K, \qquad \frac{dL}{dt} = \left(\bar{\lambda}\frac{I}{K} - \mu\right)L, \qquad \frac{dP}{dt} = \left(\bar{\varepsilon}\frac{I}{K} - \mu\right)P,\\
&\frac{I}{K} = \min\left\{(\tilde{\delta}+\mu),\, (\tilde{\nu}+\mu)\frac{1}{\bar{\lambda}},\, (\tilde{\eta}+\mu)\frac{1}{\bar{\varepsilon}}\right\}, \qquad \frac{d\alpha}{dt} = \frac{\tilde{\delta} - \tilde{\nu} - \alpha(\tilde{\eta} - \tilde{\nu})}{\tau(\bar{\varepsilon} - \bar{\lambda})(\tilde{\delta} + \mu)}, \qquad (7.9)\\
&\frac{d\bar{\lambda}}{dt} = -\frac{1}{\tau}\left(\bar{\lambda} - \frac{\tilde{\nu}+\mu}{\tilde{\delta}+\mu}\right), \qquad \frac{d\bar{\varepsilon}}{dt} = -\frac{1}{\tau}\left(\bar{\varepsilon} - \frac{\tilde{\eta}+\mu}{\tilde{\delta}+\mu}\right), \qquad \alpha = \frac{1-\bar{\lambda}}{\bar{\varepsilon} - \bar{\lambda}}.
\end{aligned}
$$

There are five independent differential equations for seven variables K, L, P, I, $\bar{\lambda}$, $\bar{\varepsilon}$ and α, so one can choose five independent variables. It is assumed that the initial values of the five independent variables as well as the rates of potential growth of production factors

$$\tilde{\delta} = \tilde{\delta}(t), \qquad \tilde{\nu} = \tilde{\nu}(t), \qquad \tilde{\eta} = \tilde{\eta}(t)$$

together with the time of crossover from one technological situation to another τ and the coefficient of amortisation μ are given. These equations allow us to analyse the evolution of an economic system.

7.1.4.2 Exponential Growth

One can see on the plot in Fig. 7.3 that the time dependence of production factors K, L and P for the relatively calm period of years 1950–2000 can be approximately depicted by straight lines, so that, for these years, the growth of production factors can be described by exponential functions, as was demonstrated in Chap. 2 (see formulae (2.29), (2.30) and (2.37)), with the rates of growth (in units of year^{-1})

$$\frac{1}{K}\frac{dK}{dt} = 0.0316, \qquad \frac{1}{L}\frac{dL}{dt} = 0.0147, \qquad \frac{1}{P}\frac{dP}{dt} = 0.0585. \qquad (7.10)$$

The exponential laws are known as the ‘stylised’ facts of economic growth.

In the simplest case, when all rates of growth are given as constant:

$$\delta = \tilde{\delta}, \qquad \nu = \tilde{\nu}, \qquad \eta = \tilde{\eta},$$

the system of equations (7.9) has a simple asymptotic solution

$$
\begin{aligned}
&K = K_0 e^{\delta t}, \qquad L = L_0 e^{\nu t}, \qquad P = P_0 e^{\eta t},\\
&\frac{I}{K} = \delta + \mu, \qquad \bar{\lambda} = \frac{\nu+\mu}{\delta+\mu}, \qquad \bar{\varepsilon} = \frac{\eta+\mu}{\delta+\mu}, \qquad \alpha = \frac{\delta-\nu}{\eta-\nu}.
\end{aligned}
\qquad (7.11)
$$

This solution corresponds to the ‘stylised’ facts of economic growth described by exponential functions. In this case, the rates of potential growth of production factors coincide with the real ones.

Formulae (7.11) allow one to estimate the technological coefficients as well as the technological index, as far as the constant rates of growth of production factors are known. One can consider the rates of growth in the period of 1950–2000 as approximately constant and equal to the mean values defined by (7.10), so that one can estimate a value of the technological index as

$$\alpha = 0.39.$$

This estimate is naturally close to the previous one, which means that the rates of real growth of production factors can be considered approximately constant in the described period.

7.1.4.3 Scenarios of Development

In the general case, the system (7.9) describes development of the production system at the rates of potential growth of the capital, expenditures of labour and substitutive work, $\tilde{\delta}$, $\tilde{\nu}$ and $\tilde{\eta}$, as functions of time. At the given initial values of the variables, the problem reduces to a Cauchy problem, which can be solved by numerical methods. However, the greatest difficulty is that the rates of potential growth remain unknown and should be the object of special research. To test the applicability of the theory to reality, we have considered [1] the past development of the United States economy, when the rates of potential growth can be estimated, and we can specify hypothetical scripts of progress.

Scenarios of development can be obtained, if one sets the rates of potential growth of capital, labour and productive energy $\tilde{\delta}$, $\tilde{\nu}$ and $\tilde{\eta}$ as function of time. Otherwise, empirical values of the technological index α and the rates of potential growth of labour and productive energy can be chosen as exogenous quantities. Before the year 2000, the rates of potential growth of labour and productive energy $\tilde{\nu}$ and $\tilde{\eta}$ are taken to be a little bit more than the rates of real growth to reproduce the empirical dependencies of L and P. One can see that the calculated trajectory approximates the real time dependence of production factors, though some details, for example, the behaviour in the turmoil years 1930–1940, are not reproduced correctly. Beyond the year 2000, we explore two scenarios of development. In both cases, the rate of growth of labour $\tilde{\nu}$ coincides with the rate of population growth, namely, $\tilde{\nu} = 0.01$ for the U.S. The first scenario corresponds to the value $\tilde{\eta} = 0.05$ for all years. The second one shows the effect of diminishing the energy supply for the substitutive work in the economy: the value of $\tilde{\eta} = 0.05$ in year 2000 decreases to zero in year 2010. The thin lines in Fig. 7.3 show the results of calculation of the production factors at values of the depreciation coefficient μ calculated according to cited statistical data ($\mu \approx 0.02$ before year 1925 and increases from 0.026 to 0.068 over years 1925–2000) and time of technological rearrangement $\tau = 1$ year.

The initial values of all variables, apart from the technological variables, are known from empirical data. The initial values of the technological variables can be chosen arbitrarily, because, due to the relaxation equations from the set (7.9), the initial values of the technological variables are forgotten in $\tau \approx 1$ year. However, the choice of the technological variables must correspond to the value of the technological index α.

The rates of potential growth for the past development are chosen such that defined trajectories of production factors conform with the actual ones. The restrictions for growth are interchangeable between lack of humans' work and lack of substitutive work (but not investments). It is possible to assume that this is a typical situation.

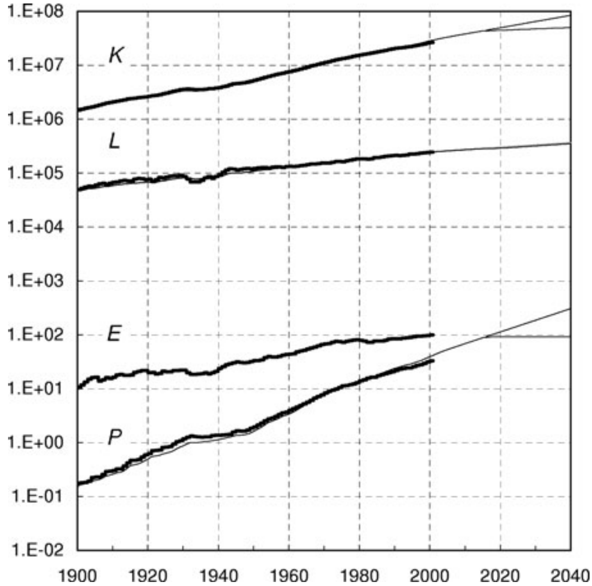

Fig. 7.3 Production factors in the U.S. economy. Basic production equipment (capital stock) K in million dollars (1996); consumption of labour L in million man-hours per year; primary energy E and substitutive work P, correspondingly, in quads (1 quad $= 10^{18}$ joules) per year. The *thick solid lines* represent empirical values, while the *thin lines* show the results of calculation. Reproduced from [1] with permission from Elsevier

One can see that the system of equations (7.9) allows us to draw scenarios of evolution of national economies for possible development of available production factors. However, the potential rates of growth should be considered as endogenous quantities in the problem of evolution of human population on the Earth. This problem was discussed in Chap. 2, but there is no solution yet.

7.1.5 Decomposition of Primary Energy

One can assume that, in accordance with the speculations of Sect. 2.5.2, the total primary energy E can be broken down into two parts,

$$E = E_C + E_P. \tag{7.12}$$

The last part is the amount of energy carriers providing, after some transformation, the pure work of production equipment P, which is only a small part of total primary energy E. The two constituents of primary energy as functions of time behave differently with respect to labour L as a function of time. The property of production factor P to be a substitute for labour allows us to state that an increase in consumption of substitutive work, which corresponds to an increase in consumption of primary substitutive work, can lead to a decrease in consumption of labour and otherwise. One expects the change of the first part E_C to correlate with the change of labour, and the change of the second part E_P to anti-correlate.

To analyse the situation, following [2], one has to consider the growth rates of the production factors, which, as was shown earlier, are connected with the investment I, depreciation coefficient μ and technological characteristics, $\bar{\lambda}$ and $\bar{\varepsilon}$, of a production system by (7.9). The non-dimensional technological coefficients $\bar{\lambda}$ and $\bar{\varepsilon}$ are characteristics of production equipment, which denote the required amount of labour and substitutive work per unit of introduced equipment (measured in units of total amount of capital K), respectively. The technological coefficients apparently have to be considered for characterisation of the process of substitution. From the definition of substitutive work and empirical investigation of these quantities (see Fig. 7.2), one has to define the correlation[1] of the technological coefficients as

$$\operatorname{corr}(\bar{\lambda}, \bar{\varepsilon}) = -1. \tag{7.13}$$

Additionally, one has to consider changes of primary energy E and its parts E_{C} and E_{P}, the last being primary substitutive work. We assume that each quantity is also characterised by its own technological coefficients, so that, in line with (7.9), one can write three more balance equations,

$$\frac{dE}{dt} = \left(\bar{\varepsilon}_{\mathrm{E}} \frac{I}{K} - \mu\right) E, \qquad \frac{dE_{\mathrm{C}}}{dt} = \left(\bar{\varepsilon}_{\mathrm{C}} \frac{I}{K} - \mu\right) E_{\mathrm{C}}, \qquad \frac{dE_{\mathrm{P}}}{dt} = \left(\bar{\varepsilon}_{\mathrm{P}} \frac{I}{K} - \mu\right) E_{\mathrm{P}}. \tag{7.14}$$

The second terms on the right sides of these equations reflect the decrease in the production factors due to the removal of a part of the production equipment from service. For simplicity, it is assumed that the depreciation coefficients of all quantities in (7.14) are equal to the depreciation coefficient μ of production equipment (capital stock). This is true for the case when installed technological equipment does not change its quality during the time of service, which is assumed in the above equations.

One can presume the quantity $\bar{\varepsilon}$ to be a proxy of the quantity $\bar{\varepsilon}_{\mathrm{P}}$, and the quantity $\bar{\varepsilon}_{\mathrm{C}}$ to be proportional to quantity $\bar{\lambda}$,

$$\bar{\varepsilon}_{\mathrm{C}} = \frac{\langle \bar{\varepsilon}_{\mathrm{C}} \rangle}{\langle \bar{\lambda} \rangle} \bar{\lambda}, \qquad \bar{\varepsilon}_{\mathrm{P}} = \bar{\varepsilon}, \tag{7.15}$$

so that some of the correlations of the technological coefficients have to be defined as

$$\operatorname{corr}(\bar{\lambda}, \bar{\varepsilon}_{\mathrm{C}}) = 1, \qquad \operatorname{corr}(\bar{\lambda}, \bar{\varepsilon}_{\mathrm{P}}) = -1, \qquad \operatorname{corr}(\bar{\varepsilon}, \bar{\varepsilon}_{\mathrm{P}}) = 1. \tag{7.16}$$

Note that these relations are the consequences of assumptions (7.15) and, in contrast to relation (7.13), have to be considered as approximate ones.

[1]The correlation and covariance of two quantities a and b are defined as

$$\operatorname{corr}(a, b) = \frac{\operatorname{cov}(a, b)}{\Delta a \Delta b}, \qquad (\Delta a)^2 = \frac{1}{n} \sum_{j=1}^{n} \big(a_j - \langle a \rangle\big)^2,$$

$$\operatorname{cov}(a, b) = \frac{1}{n} \sum_{j=1}^{n} \big(a_j - \langle a \rangle\big)\big(b_j - \langle b \rangle\big), \quad \langle a \rangle = \frac{1}{n} \sum_{j=1}^{n} a_j.$$

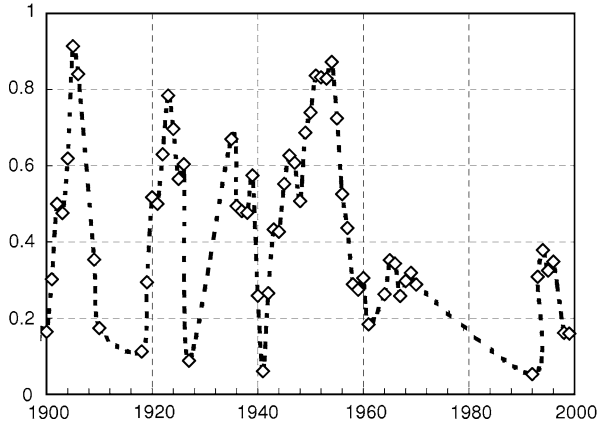

Fig. 7.4 Share of primary substitutive work. The ratio of primary substitutive work to total primary work E_P/E is calculated according to (7.23)

One can see that, due to (7.12) and (7.14), some of the technological coefficients are connected by the relation

$$\bar{\varepsilon}_E = (1-x)\bar{\varepsilon}_C + x\bar{\varepsilon}_P, \quad x = \frac{E_P}{E}. \tag{7.17}$$

This equation can be easily obtained, if one sums the last two equations from (7.14) and compares the result with the first equation from the same set.

To find an equation for the ratio x, we consider statistical characteristics of the technological coefficients. Relation (7.17) is followed by the relations for mean values, covariances and correlations, respectively,

$$\langle\bar{\varepsilon}_E\rangle = (1-x)\langle\bar{\varepsilon}_C\rangle + x\langle\bar{\varepsilon}_P\rangle, \tag{7.18}$$

$$\operatorname{cov}(\bar{\lambda}, \bar{\varepsilon}_E) = (1-x)\operatorname{cov}(\bar{\lambda}, \bar{\varepsilon}_C) + x\operatorname{cov}(\bar{\lambda}, \bar{\varepsilon}_P), \tag{7.19}$$

$$\operatorname{corr}(\bar{\lambda}, \bar{\varepsilon}_E)\Delta\bar{\varepsilon}_E = (1-x)\operatorname{corr}(\bar{\lambda}, \bar{\varepsilon}_C)\Delta\bar{\varepsilon}_C + x\operatorname{corr}(\bar{\lambda}, \bar{\varepsilon}_P)\Delta\bar{\varepsilon}_P. \tag{7.20}$$

The last relation, taking (7.16) into account, can be rewritten as

$$\operatorname{corr}(\bar{\lambda}, \bar{\varepsilon}_E)\Delta\bar{\varepsilon}_E = (1-x)\Delta\bar{\varepsilon}_C - x\Delta\bar{\varepsilon}_P. \tag{7.21}$$

One can use relations (7.15) and (7.18) to find the deviations of the quantities

$$\Delta\bar{\varepsilon}_C = \frac{\langle\bar{\varepsilon}_C\rangle}{\langle\bar{\lambda}\rangle}\Delta\bar{\lambda}, \qquad \Delta\bar{\varepsilon}_P = \Delta\bar{\varepsilon}, \qquad \langle\bar{\varepsilon}_C\rangle = \frac{\langle\bar{\varepsilon}_E\rangle - x\langle\bar{\varepsilon}_P\rangle}{1-x}. \tag{7.22}$$

Equations (7.21) and (7.22) determine a formula for calculation of the ratio of primary substitutive work to total primary energy

$$\frac{E_P}{E} = \frac{\langle\bar{\varepsilon}_E\rangle\Delta\bar{\lambda} - \langle\bar{\lambda}\rangle\operatorname{corr}(\bar{\lambda}, \bar{\varepsilon}_E)\Delta\bar{\varepsilon}_E}{\langle\bar{\varepsilon}\rangle\Delta\bar{\lambda} + \langle\bar{\lambda}\rangle\Delta\bar{\varepsilon}}. \tag{7.23}$$

The formula contains statistical characteristics of the quantities $\bar{\lambda}$, $\bar{\varepsilon}$ and $\bar{\varepsilon}_E$, which can be estimated directly according to (7.9) and (7.14) on the basis of values of substitutive work and time series for capital K, labour L, primary energy E and investment from Table of Appendix B. The values of the ratio for the U.S. economy are represented in Fig. 7.4. According to these results, absolute values of primary substitutive work are shown in Fig. 2.8 of Sect. 2.5.1. The results are realistic (close to the direct estimates of this quantity) and show ups and downs of the quantity in contrast to oversimplified direct estimates. The deviations of the calculated values of primary substitutive work from empirical ones are quite understandable, considering the rather arbitrary assumptions made at the empirical estimation of the quantity. For years 1911–1917, 1927–1934, 1962–1963 and 1971–1988, the calculated values are unrealistically small; one can suppose that assumptions (7.15), which are consequences of assumptions about the rates of depreciation of quality of production equipment, are too coarse in these cases.

7.2 Marginal Productivities

The differential formulae (6.2) and (6.14), that is,

$$dY - \Delta\, dt = \begin{cases} \xi\, dK, \\ \beta\, dL + \gamma\, dP, \end{cases} \qquad \Delta = Y \ln\left(\frac{L_0}{L}\frac{P}{P_0}\right)\frac{d\alpha}{dt} \tag{7.24}$$

allow one to estimate directly the marginal productivities ξ, β and γ due to empirical data. We assume that the time series for output Y, production factors K, L and P and the technological index α are known (the last two quantities obtained by exploiting the method of calculation described in Sect. 7.1.2).

From expressions (6.8), (6.13) and (6.15), the marginal productivities are connected with each other, that is,

$$\xi = \beta\frac{L}{K} + \gamma\frac{P}{K}, \qquad \beta = \xi(1-\alpha)\frac{K}{L}, \qquad \gamma = \xi\alpha\frac{K}{P} \tag{7.25}$$

which allows us to test the theory using alternative estimates of the marginal productivities.

7.2.1 Productivity of Capital Stock

One way to calculate the capital marginal productivity ξ is a direct use of the formula

$$dY - \Delta\, dt = \xi\, dK. \tag{7.26}$$

Another way is to use the bulk productivity of capital Ξ defined by the relation

$$Y = \Xi K \tag{7.27}$$

and calculate the marginal productivity through the bulk productivity as

$$\xi = \Xi + K\frac{d\Xi}{dK}. \tag{7.28}$$

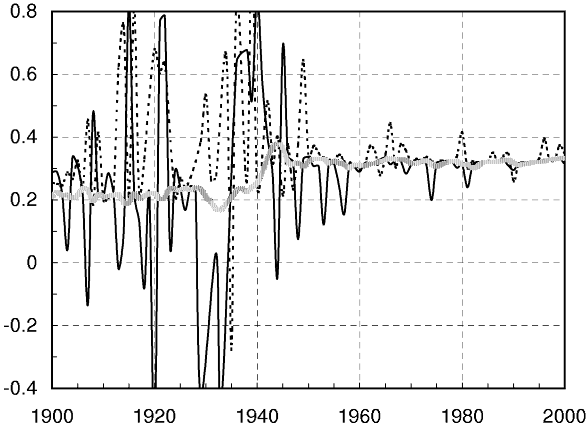

Fig. 7.5 Marginal productivity of capital stock. The *solid line* represents direct estimates of ξ from the empirical data and the equation $dY - \Delta\, dt = \xi\, dK$. The *dashed line* shows the marginal productivity calculated according to (7.25), while β and γ are estimated directly due to the empirical data and the equation $dY - \Delta\, dt = \beta\, dL + \gamma\, dP$. The *dotted line* represents the ratio Y/K

The two methods of calculation give almost identical results, which are shown in Fig. 7.5 by a solid line. For years 1950–2000, the mean values of the marginal productivity and its standard deviation can be estimated as

$$\xi = (0.307 \pm 0.044)\ \text{year}^{-1}. \tag{7.29}$$

Note that the differences dY and dK cannot be determined with great accuracy, thus negative and very large values of the marginal productivity are excluded as erroneous. It is not out of place to remember here the words of Morgenstern [3, p. 4]: 'There are many reasons why one should be deeply concerned with the "accuracy" of quantitative economic data and observations.'

One more way to estimate the capital marginal productivity is to use the first of relations (7.25), assuming that the labour and energy marginal productivities, β and γ, respectively, are directly calculated from empirical data, as will be demonstrated in the next subsection. The quantity ξ, calculated in this way, is shown in Fig. 7.5 by the dashed line. For years 1950–2000, in this case, the mean values of the marginal productivity and its standard deviation can be estimated as

$$\xi = (0.337 \pm 0.039)\ \text{year}^{-1}. \tag{7.30}$$

The values of the marginal productivity (7.29) and (7.30) practically coincide with the averaged bulk productivity Y/K, which is $(0.321 \pm 0.009)\ \text{year}^{-1}$; this is evidence that the capital marginal productivity does not depend on argument K. According to numerous observations [3, 4], the growth of Y is approximately equal to the rate of growth of capital K, which can be confirmed by comparing formulae (2.14) and (2.29), so that the bulk productivity of capital Y/K is approximately constant in the U.S. economy in the second half of the twentieth century.

7.2.2 Productivities of Labour and Substitutive Work

The direct way to calculate the marginal productivities of labour and substitutive work is to use the formula

$$dY - \Delta\, dt = \beta\, dL + \gamma\, dP. \tag{7.31}$$

Otherwise, the marginal productivities β and γ can be expressed through the bulk productivities of labour and substitutive work, B and Γ, which are considered to be functions of the ratio P/L and are defined by the relations

$$Y = B(P/L)L, \qquad Y = \Gamma(P/L)P. \tag{7.32}$$

Indeed, having calculated the total differential of the output from the first equation, one obtains

$$\beta = B + \frac{P}{L}\frac{dB}{d(P/L)}, \qquad \gamma = \frac{dB}{d(P/L)}. \tag{7.33}$$

Similarly, one can obtain, having calculated the total differential of the output from the second equation of the set (7.32),

$$\beta = -\left(\frac{P}{L}\right)^2 \frac{d\Gamma}{d(P/L)}, \qquad \gamma = \Gamma + \frac{P}{L}\frac{d\Gamma}{d(P/L)}. \tag{7.34}$$

Calculations with the use of relations (7.33) or relations (7.34) give slightly different values for the marginal productivities: one can use mean values of the two calculations. In comparison with (7.31), (7.33) and (7.34) give an alternative method of direct estimation of the marginal productivities.

The non-dimensional marginal productivities $\beta L/K$ and $\gamma E/K$, estimated directly according to formula (7.31), are shown in the plot of Fig. 7.6 by the solid lines. The marginal productivities are pulsating functions of time, and a maximum of one of the marginal productivities corresponds to a minimum of the other and vice versa. Averaged values of the quantities for years 1950–2000 are estimated as

$$\begin{aligned} \beta\frac{L}{K} &= (0.211 \pm 0.048)\ \text{year}^{-1}, \\ \gamma\frac{P}{K} &= (0.117 \pm 0.012)\ \text{year}^{-1}. \end{aligned} \tag{7.35}$$

Note that negative and very large values of marginal productivities are omitted as connected with erroneous values of production factors.

Alternatively, the marginal productivities β and γ can be estimated from relations (7.25), assuming that capital productivity ξ and values of the technological index α are known. The dashed lines in Fig. 7.6 show the alternative estimates of marginal productivities at empirical values of ξ and empirical values of the technological index α calculated in Sect. 7.1.2. For years 1950–2000 the alternative mean values of the marginal productivities and their standard deviations can be estimated as

$$\begin{aligned} \beta\frac{L}{K} &= (0.192 \pm 0.025)\ \text{year}^{-1}, \\ \gamma\frac{P}{K} &= (0.129 \pm 0.018)\ \text{year}^{-1}. \end{aligned} \tag{7.36}$$

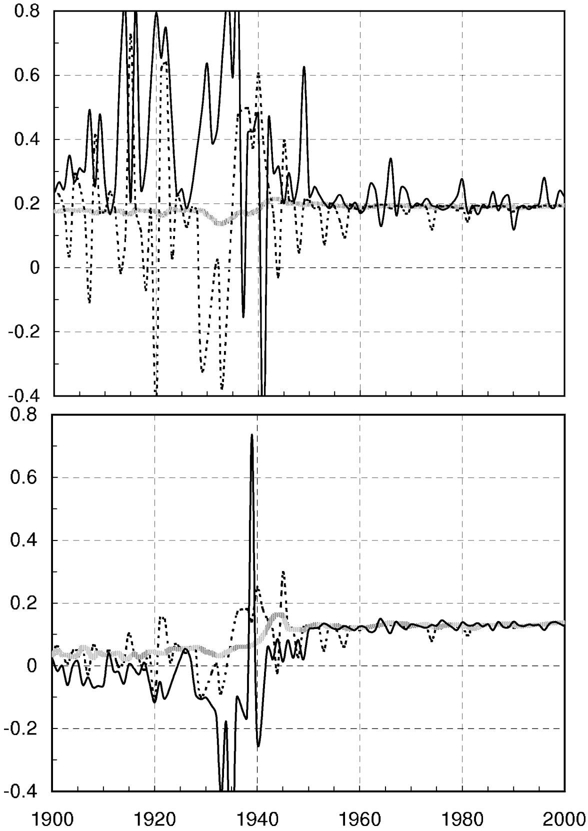

Fig. 7.6 Marginal productivities of labour and substitutive work. *Solid curves* show direct estimate productivities of labour (*top*) and substitutive work (*bottom*) according to (7.31). *Dashed curves* represent results of calculation according to (7.25) at known values of capital stock productivity

These values should be compared with the above estimates (formulae (7.35)) of the same quantities.

The dotted lines in Fig. 7.6 represents the results of calculation of marginal productivities according to formulae (7.25) at empirical values of Y/K and empirical values of the technological index α calculated in Sect. 7.1.2.

7.2.3 What is Productivity of Capital?

The results of estimating the marginal productivities confirm that relations (7.25) are valid for the U.S. economy. Thus, indeed, the marginal productivity of capital stock can be considered as the 'sum' of the marginal productivities of labour and substitutive work, and no other factors need to be included in the production function. Although one needs production equipment (capital stock) to attract an extra amount of external energy to substitute labour, work (labour services) can be replaced only by work (capital services), not by capital stock. Productivity of capital stock is, in fact, productivity of labour and energy, and the main result of technological progress is the substitution of human efforts by the work of external energy sources by means of different sophisticated appliances. The production system of society is a mechanism which attracts a huge amount of energy to transform matter into things that are useful for human beings.

According to expression (6.18), the productivity of capital changes if the technological coefficients and/or the production factors change. The former causes fast pulsation, while the latter provokes slow trends of the capital productivity. This quantity apparently depends on the definition of production capital K. Let us recall that the notion of production capital has to be refined by excluding some commodities from production investment. The share of core production capital in the total production investment remains unknown, but more importantly, the growth rate of the core production capital can differ from the growth rate estimated on available statistical data.

7.3 Production of Value

From the results of the previous chapter, the production of value can be estimated from the production function, which can be rewritten as

$$Y = \begin{cases} \xi K, \\ Y_0 \dfrac{L}{L_0} \left(\dfrac{L_0}{L} \dfrac{P}{P_0} \right)^{\alpha}. \end{cases} \tag{7.37}$$

The function contains both characteristics of the production system: the technological index α and productivity of capital ξ, and the exploited production factors: capital stock K, labour services L and substitutive work P. The second line in relations (7.37) represents the well-known Cobb–Douglas production function in which productive energy P stands in the place of capital stock K.

Empirical values of the Gross Domestic Product (GDP) and production factors K and L for the U.S. economy for years 1900–2000 are known (see Appendix B) and are depicted on the plots of Fig. 2.1 (Sect. 2.2.3), Fig. 2.3 (Sect. 2.2.4.1) and Fig. 2.7 (Sect. 2.4.1). Methods of direct estimates of substitutive work P are feasible (see Sect. 2.5.2), but are not developed sufficiently to estimate the substituting work with enough accuracy. Therefore, substitutive work P and also the technological index α are calculated according to the time series of the GDP Y and production factors K and L. At given time dependence of labour services L and at calculated values of the technological index α and substitutive work P, the time dependence of the output naturally identically coincides with the empirical one.

7.3.1 *Exponential Growth*

There is some interest in writing approximate relations for relatively calm periods of development to describe the ‘stylised’ facts of economic growth, that is, the ex-

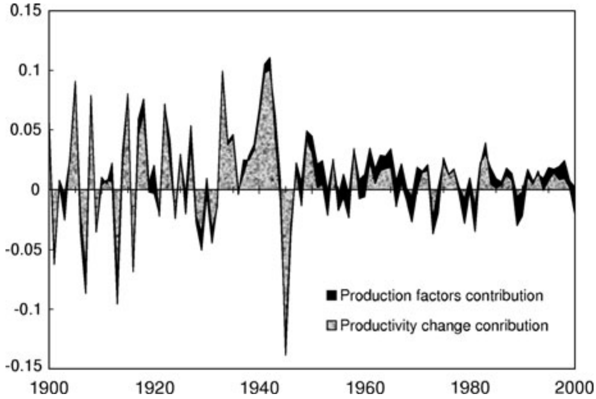

Fig. 7.7 Decomposition of the Solow residual. The conventional 'total factor productivity growth' consists of the difference between the growth rates of capital services and capital stock (*black area*) and the true residual connected with changes of the production system itself (the technological and structural changes)

ponential growth of output when the exponential growth of production factors is given,

$$K = K_0 e^{\delta t}, \qquad L = L_0 e^{\nu t}, \qquad P = P_0 e^{\eta t}.$$

For exponential growth of production factors, an expression for output follows immediately by relations (7.37) and can be written in the following form:

$$Y = Y_0 e^{[\nu + \alpha(\eta - \nu)]t} = Y_0 e^{\delta t}. \tag{7.38}$$

One can see that the theory describes the 'stylised' facts of economic growth, while the growth rate of output is equal to the growth rate of capital and is connected with the growth rates of labour and substitutive work.

Returning to the empirical data for the U.S. economy, one uses the values of the growth rates of production factors given in Sect. 7.1.4.2 (formulae (7.10)), that is,

$$\delta = 0.0316, \qquad \nu = 0.0147, \qquad \eta = 0.0585.$$

For years 1950–2000, the empirical growth of the GDP of the U.S. can be considered to be approximately exponential (see formula (2.14))

$$Y = 1.69 \times 10^{12} \cdot e^{0.0326t} \text{ dollar(1996)/year}.$$

Time t is measured in years, and $t = 0$ corresponds to year 1950.

In accordance with relation (7.38), the empirical averaged growth rate of output 0.0326 is approximately equal to the growth rate of capital $\delta = 0.0316$, which is confirmed by numerous observations [4, 5].[2] The difference between the growth

---

[2]Here the reader can be pertinently reminded of the contradictions arising if one uses for an assessment of output the neo-classical production function in the Cobb–Douglas form,

$$Y = Y_0 \frac{L}{L_0}\left(\frac{L_0}{L}\frac{K}{K_0}\right)^{\alpha'}, \quad 0 < \alpha' < 1.$$

Considering exponential growth (7.10), the expression for output is determined in the form of

$$Y = Y_0 e^{[(1-\alpha')\nu + \alpha'\delta]t}.$$

rates of capital and output seems to be quite unreliable, given the rough estimate of the parameters of the problem, although under more detailed consideration, the difference can be attributed to an intrasectoral technological change and to the difference of the growth rates of sector outputs (see Sect. 8.2.2, formula (8.19)).

The rate of growth of output can be broken into two parts. On average a fraction of the rate $(1-\alpha)\nu \approx 0.0112$ is connected with growth of expenditures of labour, and the other part $\alpha\eta \approx 0.0235$ with growth of substitutive work. The capital is the means of attracting the production factors to production, thus an increase in consumption of the production factors is connected with an increase in capital. One can formally separate the growth rate of capital δ within the growth rate of substitutive work η to get a breakdown of the growth rate of output in conventional terms: the contribution from the labour growth $(1-\alpha)\nu \approx 0.0112$ and the contribution from the capital growth $\alpha\delta \approx 0.0126$. One can see that the Solow residual (total factor productivity) can be expressed through the technological index and the growth rates as

$$\mathit{Solow\ Residual} = \alpha(\eta - \delta) = (1-\alpha)(\delta - \nu) \approx 0.0109. \tag{7.39}$$

Structural and technological changes, if they exist, compensate each other in this simple case of exponential growth. A detailed decomposition of the Solow residual for the U.S. economy is shown in Fig. 7.7.

7.3.2 *Pulsating Character of Production Development*

The empirical data demonstrates the pulsating character of development of production. An example of well-documented dynamics of the U.S. economy (see Fig. 2.2 in Sect. 2.2.3) shows that the period of pulsations of the growth rate of the GDP is about four years. The considered theory of production allows for description of the cyclic character of development naturally. In a simple approximation, when characteristics of the equipment do not change after its installation, the rate of growth of the GDP, according to (6.21) is recorded as

$$\frac{1}{Y}\frac{dY}{dt} = \frac{\nu + (1-\bar{\lambda})\mu}{\bar{\lambda}} + \frac{1}{\xi}\frac{d\xi}{dt}. \tag{7.40}$$

The rate of growth of output is connected with four quantities, while the change of the growth rate of the labour demand is connected with the coefficient of labour requirement $\bar{\lambda}$, which is a strongly pulsating quantity, as is possible to see on the charts of Fig. 7.2. One can see that, if the technological coefficients are equal to

---

It is easy to see that the Cobb–Douglas production function describes empirical data for the U.S. for years 1950–2000 at $\alpha' \approx 1$, which excludes the influence of labour. Moreover, the index α' can be interpreted as a share of capital in total expenses for maintenance of production factors—the quantity that is equal to 0.3–0.4 for the U.S. economy. This is a well-known fact [4, p. 4] which has led to the introduction of *the full factor of productivity* [6] and to numerous modifications of the neo-classical production function [7, 8].

unity, labour productivity is constant and all addition of a product is connected only with an increase in the number of workers.

The condition, when the labour requirement $\bar{\lambda}$ is less than unity, shows that efforts of workers are partially replaced with the work of machines movable by outer energy sources; it is a typical situation for the U.S. economy in the second half of the twentieth century. The existence of pulsations of output can be connected with the existence of the three alternative types of functioning of the production system, which were described in Sect. 5.3. In the considered period, the second and third cases (see Sect. 7.1.4) are realised for production of the U.S., that is, the processes are running at deficiency of labour and abundance of investments, substitutive work and raw materials, when $\frac{d\bar{\lambda}}{dt} < 0$; or at deficiency of substitutive work and abundance of investments, work and raw materials, when $\frac{d\bar{\lambda}}{dt} > 0$.

To describe an ideal cycle, one can start from the point where the coefficient of labour requirement has its minimum value. In the case when $\frac{d\bar{\lambda}}{dt} > 0$, the production system is experiencing a deficiency of substitutive work, whereas the creation of working places is restricted, and, at small values of the coefficient of labour requirement, the index of unemployment starts to increase. The coefficient of labour requirement is also growing, according to (5.30); the production system attracts more labour, but it appears insufficient to decrease the index of unemployment immediately, and the index grows simultaneously with the technological coefficient. The growth of unemployment stops when the production system succeeds in using all the extra supply of labour, and the technological coefficient reaches its potential value at $\frac{d\bar{\lambda}}{dt} = 0$. The situation is being balanced at the peak of unemployment, and, at this point, the change of the growth mode has occurred. Further, when $\frac{d\bar{\lambda}}{dt} < 0$, the production system is functioning at a deficiency of labour and is able to use all available resources of workers: the index of unemployment decreases simultaneously with decrease in labour demand, until at some point in time a new balance occurs at $\frac{d\bar{\lambda}}{dt} = 0$, where the type of functioning of the production system is changing again: a new cycle begins. The period of a cycle in this case is connected with the mechanism of propagation of exploited technologies.

To design a mathematical model of an ideal business cycle, apparently, aside from the output, technological coefficients and labour demand, one must include some other variables, first of all, labour supply and wage, and refer to some relations between output, wages and labour supply. There remain many reasons for the observable real cycles not to be ideal. However, empirical data for the U.S. economy confirm the general patterns of business cycle phenomena: the changes of the coefficient of labour requirement and the index of unemployment correlate.

The considered approximation of a national economy as a uniform sector allows us to describe the dynamics of short cycles in the U.S. economy in the last century from the point of view of production functioning. In reality the national economy consists of many sectors, and each sector is characterised by technological coefficients with different times of propagations of technology. Thus it appears to be possible to have a wider set of modes of development leading to the occurrence of cycles of various durations; these were detected empirically when the functioning of the world economy was investigated [9]. Apparently, it is possible to also explain observable long cycles from the point of view of production and changes of modes of development. Also, the majority of researchers specify introduction of innovations into the production as one of the essential circumstances connected with cycles. For the appropriating analysis it is necessary to consider an empirical situation in the production sectors and, except for time series for the GDP, to involve in the discussion time series for expenditures of labour, substitutive work, capital and investment for the considered sectors.

7.3.3 Trajectories of Output

From relation (7.37), a trajectory of development of output can be easily calculated if values of the production factors are known; for example, as in the case when the exponential growth in Sect. 7.3.1 was investigated. To calculate the evolution of production factors, according to the system of equations (7.9), in which the technological index α is also treated as a variable, one has to know the rates of potential growth of production factors. Due to the difficulties of judging the potential amount of production factors, one can consider the problem in another way, assuming that some other characteristics of the production system are given.

One can see that the growth rate of the output, from (7.40), can be calculated on the assumption that the four quantities: capital stock productivity $\xi(t)$, the rate of labour growth $\nu(t)$, the depreciation coefficient $\mu(t)$ and the non-dimensional technological coefficient $\bar{\lambda}(t)$, are given as functions of time. To find the time dependence of the output, there is no need to know the time dependence of the production factors, though it is convenient to formulate a simple scheme which allows one to calculate trajectories of the production factors as well. The above results allow one to write the system of equations

$$
\begin{aligned}
&Y = \xi K, \quad \xi = \xi(t),\\
&\frac{dK}{dt} = \delta K, \quad \delta = \frac{\nu + (1-\bar{\lambda})\mu}{\bar{\lambda}}, \; \mu = \mu(t), \; \bar{\lambda} = \bar{\lambda}(t),\\
&\frac{dL}{dt} = \nu L, \quad \nu = \nu(t),\\
&\frac{dP}{dt} = \eta P, \quad \eta = \frac{\delta - (1-\alpha)\nu}{\alpha}, \; \alpha = \frac{\ln\left(\frac{Y}{Y_0}\frac{L_0}{L}\right)}{\ln\left(\frac{L_0}{L}\frac{P}{P_0}\right)}.
\end{aligned}
\tag{7.41}
$$

The system determines the time dependence of the variables Y, K, L, P and α, if the four quantities ξ, μ, ν and $\bar{\lambda}$ are given as functions of time. Initial values of all five variables have to be given, and the initial value of capital stock must correspond to the initial values of output and marginal productivity $K_0 = Y_0/\xi(0)$, while the

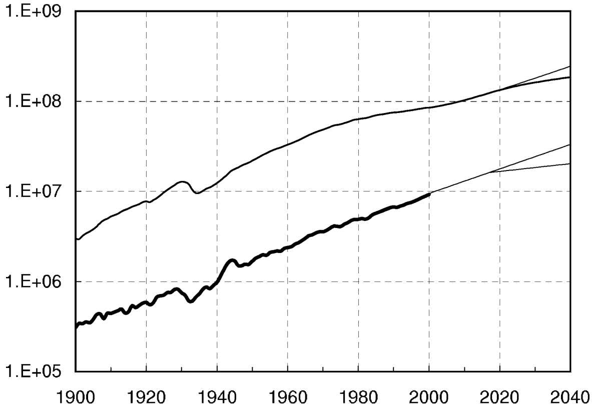

Fig. 7.8 Output and national wealth in the U.S. economy. The empirical (*solid shorter line* for GDP) and calculated values of the total national wealth W and GDP in millions of dollars for year 1996. Two scenarios are shown; the *lower line* after year 2010 corresponds to a diminishing supply of energy

initial values of labour L_0 and substitutive work P_0 can be chosen arbitrarily. Note also that these quantities are connected by (6.15), that is,

$$\xi = Y_0 \frac{L}{L_0 K} \left(\frac{L_0}{L} \frac{P}{P_0} \right)^{\alpha}, \tag{7.42}$$

which shows that one has to require consistency of the solution to the given quantity ξ.

To illustrate the procedure of drawing a scenario and deviations arising due to approximations, we refer to the U.S. data for years 1900–2000. Empirical values of the capital stock productivity $\xi(t)$, the rate of labour growth $\nu(t)$, the depreciation coefficient $\mu(t)$ and the non-dimensional technological coefficient $\bar{\lambda}$ were introduced. The thick solid line in Fig. 7.8 depicts empirical values of the GDP. One can see that the calculated time dependence of output (thin lines) approximates the real-time dependence of the GDP before the year 2000 in all details, which confirms the consistency of the theory.

One can assume that a development will continue beyond year 2000. One can imagine any program of future development of technology after year 2000 in terms of the four quantities ξ, μ, ν and $\bar{\lambda}$ to have an outline of output Y and production factors K, L, P for the U.S. economy. The outputs of two scenarios of development of the U.S. economy for years 2000–2040, which correspond to the growth rates of labour and energy described in Sect. 7.1.4, are presented in Fig. 7.8 by thin lines. In both cases, the growth rate of labour ν coincides with the rate of population growth, namely, $\nu = 0.01$ for the U.S. economy. The first scenario corresponds to the values of μ and $\bar{\lambda}$ in year 2000, that is, $\mu = 0.68$ and $\bar{\lambda} = 0.78$ for all years. The second one shows the effect of diminishing the energy supply in the economy: the value of $\bar{\lambda}$ increases to unity in year 2010. One can see a decrease in the growth rate of the output in the case when the growth rate of productive consumption of energy is decreasing. Of course, these results should be considered as an illustration of a method of forecasting rather than as the forecast itself. One needs to know the future availability of labour and substitutive work to do a real prediction.

We can have the picture of development for any economy, if we set the values of variables Y, L and α in the initial year and the program of development for the future in terms of the four quantities ξ, μ, ν and $\bar{\lambda}$. No procedure of adjustment is needed, but, to set initial values and to imagine a program of development, we must know something about the investigated economy.

7.3.4 National Wealth

In accordance with the definitions of Chap. 2, the total national wealth W (see Sect. 2.3) consists of the capital stock K and the storage of products, which are mostly non-material products of value R,

$$W = K + R.$$

The separate parts of the national wealth, in accordance with (2.27) and (2.28), can be estimated from the equations

$$\frac{dK}{dt} = I - \mu K, \qquad \frac{dR}{dt} = Y - I - C - \mu R,$$

where I is the investment in the stock of production capital and C is the current consumption, which can be calculated, according to (6.33), as the cost of labour,

$$C = cL = \frac{1-\alpha}{\alpha}\mu K. \tag{7.43}$$

It is assumed that the depreciation coefficient has the same value for both tangible and non-tangible stocks.

Then, from the above relations, the total national wealth is determined by the equation

$$\frac{dW}{dt} = Y - C - \mu W. \tag{7.44}$$

Results of calculations of national wealth W for the U.S., in line with GDP Y, for the two above-described scenarios are shown in Fig. 7.8 by thin lines.

References

1. Pokrovski, V.N.: Energy in the theory of production. Energy **28**, 769–788 (2003)
2. Pokrovski, V.N.: Productive energy in the US economy. Energy **32**, 816–822 (2007)
3. Morgenstern, O.: On the Accuracy of Economic Observation, 2nd edn., completely revised. Princeton University Press, Princeton (1973)
4. Blanchard, O.J., Fisher, S.: Lectures on Macroeconomics. MIT Press, Cambridge (1989)
5. Scott, M.F.G.: A New View of Economic Growth. Clarendon Press, Oxford (1989)
6. Solow, R.: Technical change and the aggregate production function. Rev. Econ. Stud. **39**, 312–330 (1957)
7. Ferguson, C.E.: The Neo-classical Theory of Production and Distribution. Cambridge University Press, Cambridge (1969)
8. Brown, M.: On the Theory and Measurement of Technological Change. Cambridge University Press, Cambridge (1966)
9. Korotayev, A.V., Tsirel, S.V.: A spectral analysis of World GDP dynamics: Kondratiev waves, Kuznets swings, Juglar and Kitchin cycles in global economic development, and the 2008–2009 economic crisis. Struct. Dyn. **4**, 3–57 (2010)

Chapter 8
Dynamics of Production in Many-Sector Approach

Abstract In this chapter, we are returning to the many-sector model of production system, discussed in Chap. 4. One can assume that production of value and dynamics of production factors in every sector are described in the same way as for the whole economy, according to the rules set up in Chaps. 5 and 6. This involves, in addition to the internal restriction discussed in Chap. 4, the restrictions imposed by the availability of labour and substitutive work. The schematisation of the production process allows us to formulate the simplest theory including only three production factors in every sector. This allows one to draw a picture of economic growth, taking into account the specific features of each sector.

8.1 Description of the Sector Dynamics

The methods of description of the production system developed by Leontief [1–3] and Sraffa [4] assume the many-sector model of an economic production system. Following the speculations of Chap. 4, we consider the production system of an economy consisting of n production sectors, each of them creating its own product. The sectors are interacting with each other, but to develop a description of the dynamic behaviour of the system, we have to first consider a formulation for each separate sector.

8.1.1 Dynamics of Production Factors

Similar to the entire production system, the sector can be thought of as a collection of production equipment K^i, which is activated by labour L^i and substitutive work P^i [5]. One can assume that all speculations of Chap. 5 can be reproduced for a separate sector, so that (5.6) and (5.13) for the production factors can be rewritten in a simplified form as

$$\frac{dK^i}{dt} = I^i - \mu K^i, \qquad \frac{dL^i}{dt} = \lambda^i I^i - \mu L^i, \qquad \frac{dP^i}{dt} = \varepsilon^i I^i - \mu P^i. \tag{8.1}$$

V.N. Pokrovskii, *Econodynamics*, New Economic Windows 12,
DOI 10.1007/978-94-007-2096-1_8, © Springer Science+Business Media B.V. 2012

The technological coefficients λ^i and ε^i $(i = 1, 2, \ldots, n)$ characterise a quality of the production equipment introduced in sector i. The coefficient of depreciation μ has to be given as a parameter of the problem.

It is also convenient to introduce the non-dimensional technological variables for every sector,

$$\bar{\lambda}^i = \lambda^i K^i / L^i, \qquad \bar{\varepsilon}^i = \varepsilon^i K^i / P^i$$

and assume that the speculations of Sect. 5.3 can be repeated for each sector, so that one supposes the technological variables are determined by equations similar to (5.30) and (5.31), that is,

$$\frac{d\bar{\lambda}^i}{dt} = -\frac{1}{\tau^i}\left(\bar{\lambda}^i - \frac{\tilde{\nu}^i + \mu}{\tilde{\delta}^i + \mu}\right), \qquad \frac{d\bar{\varepsilon}^i}{dt} = -\frac{1}{\tau^i}\left(\bar{\varepsilon}^i - \frac{\tilde{\eta}^i + \mu}{\tilde{\delta}^i + \mu}\right). \tag{8.2}$$

The time of crossover from one technological situation to another τ^i can be different for different sectors. It is determined by internal processes of replacement of technology within the sector. The symbols for the rates of potential growth of capital, labour and substitutive work in every sector, $\tilde{\delta}^i$, $\tilde{\nu}^i$ and $\tilde{\eta}^I$ $(i = 1, 2, \ldots, n)$, are introduced here. These quantities apparently relate to the rates of potential growth of the production factors for the entire production system and are assumed to be given as functions of time.

It is also convenient to write an equation for the sectoral technological index,

$$\frac{d\alpha^i}{dt} = \frac{\tilde{\delta}^i - \tilde{\nu}^i - \alpha^i(\tilde{\eta}^i - \tilde{\nu}^i)}{\tau^i(\bar{\varepsilon}^i - \bar{\lambda}^i)(\tilde{\delta}^i + \mu)}. \tag{8.3}$$

One can see that, if the potential rate of capital growth $\tilde{\delta}^i(t)$ is determined by equation

$$\tilde{\delta}^i = \tilde{\nu}^i + \alpha^i\left(\tilde{\eta}^i - \tilde{\nu}^i\right),$$

the technological index appears to be an integral of evolution and, therefore, can be considered as a very important characteristic of the production system.

8.1.2 Investment

To determine investment I^i in (8.1), one has to take into account internal and external restrictions on the development of the system. The internal restrictions, imposed by the technological structure of the production system and by necessary private consumption, determine the potential growth rate of capital stock $\tilde{\delta}^i$, which is connected with the potential investments in sector i,

$$\tilde{I}^i = (\tilde{\delta}^i + \mu)K^i.$$

It is assumed that there is a method, for example, the method described in Sect. 4.3.3, to calculate the sector potential investments and, consequently, according to the above relation, the rate of potential growth of capital stock.

The external restrictions are imposed by the availability of labour and substitutive work, which is assumed to be described by their rates of potential growth for every sector $\tilde{\nu}^i(t)$ and $\tilde{\eta}^i(t)$. These exogenous characteristics are given functions of time.

Therefore, one can rewrite relation (5.27) for each sector,

$$I^i = \chi^i K^i + \min\{\tilde{I}^i, (\tilde{\nu}^i + \mu)L^i/\lambda^i, (\tilde{\eta}^i + \mu)P^i/\varepsilon^i\}, \tag{8.4}$$

where χ^i is a central-planning intervention in sector i. In the many-sector model, interventions χ^i can be understood as coefficients of re-allocation of investments among different sectors, so that

$$\sum_{i=1}^{n} \chi^i K^i = 0.$$

By relation (8.4), three modes of the development of each sector are determined. The first choice in relation (8.4) corresponds to the internal restrictions. The second choice in (8.4) is valid in the case of abundance of substitutive work and raw materials and a lack of labour. In this case, labour is used completely and there is a possibility of attracting extra substitutive work. The latter is a reason for technological changes. Internal processes lead to decrease of labour and increase of substitutive work in production processes. The last choice in (8.4) is valid in the case of a lack of substitutive work and an abundance of labour and raw materials. Approaching the production system with the many-sector model, one has three modes of development for every sector, and, consequently, there are many possible modes of development of the production system.

8.1.3 Sector Production of Value

8.1.3.1 Sector Production Functions

As was described in Chap. 2, the output of a sector i is characterised by three quantities: gross output $X^i = X_i$, final output Y_i and sector production of value Z^i. These quantities are connected with each other by (4.6) and (4.7), which can be written in the form

$$X_i = \frac{1}{1-a^i} Z^i, \qquad Y_j = \sum_{i=1}^{n} \frac{\delta^i_j - a^i_j}{1-a^i} Z^i, \quad a^i = \sum_{i=1}^{n} a^i_j \tag{8.5}$$

where a^i_j is a component of the matrix of intermediate production consumption (the input–output matrix) introduced by relation (4.2).

Our immediate task is to determine the output vectors as functions of production factors. From relation (4.12) and the above relations, the gross output $X^i = X_i$, final output Y_i and production of value Z^i in each sector $(i = 1, 2, \dots, n)$ are connected with the vector of amount of sectoral production equipment or sectoral capital stock K^i,

$$X_i = \frac{1}{b^i} K^i, \quad b^i = \sum_{j=1}^{n} b^i_j, \tag{8.6}$$

$$Y_j = \sum_{i=1}^{n} \xi^i_j K^i, \quad \xi^i_j = \frac{\delta^i_j - a^i_j}{b^i} = \xi^i \frac{\delta^i_j - a^i_j}{1 - a^i}, \tag{8.7}$$

$$Z^i = \xi^i K^i, \quad \xi^i = \sum_{j=1}^{n} \xi^i_j = \frac{1 - a^i}{b^i}, \; i, j = 1, 2, \ldots, n, \tag{8.8}$$

where coefficients a^i_j comprise a matrix of intermediate production consumption (the input–output matrix) and coefficients b^i_j comprise a matrix of fixed capital (the capital-output matrix). The components of the fundamental technological matrices a^i_j and b^i_j are combined to form components ξ^i_j of a matrix of capital productivities. The quantities ξ^i_j show how an increase in production equipment in sector i affects the final output in sector j. One can see that, to calculate the final product, all the components of the matrix of capital productivity are needed.

Formulae (8.6)–(8.8) connect the vector characteristics of output X_i, Y_j and Z^i with the amount of sector capital K^i. On the other hand, production of value is connected with the sector consumption of production factors: labour L^i and productive energy P^i. In the general case, the sectoral productivities can depend on the production factors consumed in all the other sectors. It is possible to expect that quantities of marginal productivities are determined by a mutual market of production factors. In the simplest case, one can consider production of value Z^i in a sector to be determined by consumption of production factors in the same sector. In this case, we refer to the procedure which was used in Sect. 6.3 to determine

$$Z^i = Z^i_0 \left(\frac{L^i_0}{L^i} \frac{P^i}{P^i_0} \right)^{\alpha^i}, \quad \alpha^i = \frac{1 - \bar{\lambda}^i}{\bar{\varepsilon}^i - \bar{\lambda}^i}. \tag{8.9}$$

The constant Z^i_0 is controlled by initial values of variables. Then, the gross output X_i and the final output Y_i can be easily found with the help of relations (8.5).

8.1.3.2 Marginal Productivities and Technological Change

Equations (8.8) and (8.9) present two complementary expressions for production of value in sector i,

$$Z^i = \begin{cases} \xi^i K^i, & \xi^i = \dfrac{1 - a^i}{b^i}, \\ Z^i_0 \left(\dfrac{L^i_0}{L^i} \dfrac{P^i}{P^i_0} \right)^{\alpha^i}, & \alpha^i = \dfrac{1 - \bar{\lambda}^i}{\bar{\varepsilon}^i - \bar{\lambda}^i}, \end{cases} \qquad i = 1, 2, \ldots, n,$$

so that, comparing them, one can obtain

$$\xi^i = Z_0^i \frac{L^i}{L_0^i K^i} \left(\frac{L_0^i}{L^i} \frac{P^i}{P_0^i} \right)^{\alpha^i}, \quad \alpha^i = \frac{1 - \bar{\lambda}^i}{\bar{\varepsilon}^i - \bar{\lambda}^i}. \tag{8.10}$$

To separate the effects of production factors and technological change, one can consider the differential of production of value

$$dZ^i - \Delta^i dt = \begin{cases} \xi^i \, dK^i, \\ \beta_i \, dL^i + \gamma_i \, dP^i, \end{cases} \qquad i = 1, 2, \ldots, n \tag{8.11}$$

where capital marginal productivity ξ^i is defined above and, from the above definitions,

$$\beta_i = \xi^i \left(1 - \alpha^i\right) \frac{K^i}{L^i}, \quad \gamma_i = \xi^i \alpha^i \frac{K^i}{P^i}, \tag{8.12}$$

$$\Delta^i = -\frac{K^i}{b^i} \frac{da^i}{dt} - \frac{(1 - a^i) K^i}{b^i} \frac{db^i}{dt} = Z^i \ln\left(\frac{L_0^i}{L^i} \frac{P^i}{P_0^i} \right) \frac{d\alpha^i}{dt} = Z^i \frac{1}{\xi^i} \frac{d\xi^i}{dt}. \tag{8.13}$$

The quantity Δ^i is connected with changes of components of the technological matrices A and B and can be called the technological change within the sector labelled i.

Let us note that relations (8.11) are valid for the case when all prices of products do not depend on time. In the opposite case, we ought to use a new quantity $\hat{Z}^i$, namely, production of value measured in the current money unit, for which we have

$$\frac{d\hat{Z}^i}{dt} = p_i \left(\frac{dZ^i}{dt} \right)_{p_i} + \hat{Z}^i \frac{d \ln p_i}{dt}, \tag{8.14}$$

where $(\frac{dZ^i}{dt})_{p_i}$ is the derivation of production of value in sector i at constant prices given by formula (8.11). The indexes of prices of products p_i must be considered to be new variables. One needs extra equations to include them for consideration in this case. Some equations will be discussed in Chap. 9. Further, we restrict ourselves to the case when all prices are constant.

8.2 Rules of Aggregation and Structural Shift

8.2.1 Production Factors

To return to the one-sector description of the system, we can refer to natural definitions (which were discussed in Sect. 2.3.2, (2.24) and in Sect. 5.1, (5.1)) of production factors as sums of corresponding sectoral quantities,

$$K = \sum_{i=1}^{n} K^i, \qquad L = \sum_{i=1}^{n} L^i, \qquad P = \sum_{i=1}^{n} P^i.$$

One can sum (8.1) for the dynamics of sectoral production factors to obtain (5.6) and (5.13) for the dynamics of production factors for the production system as a whole. The procedure defines the technological coefficients for the production system as a whole,

$$\lambda = \sum_{j=1}^{n} \frac{I^j}{I} \lambda^j, \qquad \varepsilon = \sum_{j=1}^{n} \frac{I^j}{I} \varepsilon^j, \tag{8.15}$$

where I^j is the gross investment in sector j, and $I = \sum_{j=1}^{n} I^j$ is the gross investment in the entire economy.

It is convenient to introduce the symbols for the growth rates of sector production factors: capital, labour and substitutive work, δ^i, ν^i, η^i, respectively, and to define the growth rates of production factors for the entire system by the relations

$$\delta = \frac{1}{K} \sum_{i=1}^{n} \delta^i K^i, \qquad \nu = \frac{1}{L} \sum_{i=1}^{n} \nu^i L^i, \qquad \eta = \frac{1}{P} \sum_{i=1}^{n} \eta^i P^i. \tag{8.16}$$

The introduced averaged growth rates depend, generally speaking, on time, even if the sectoral growth rates do not.

In the introduced symbols (8.1)–(8.4) can be rewritten as

$$I^i = \left(\delta^i + \mu\right) K^i, \qquad \lambda^i = \frac{\nu^i + \mu}{\delta^i + \mu} \frac{L^i}{K^i}, \qquad \varepsilon^i = \frac{\eta^i + \mu}{\delta^i + \mu} \frac{P^i}{K^i}, \tag{8.17}$$

and one can easily see that formulae (8.16) and (8.17) allow us to return to expressions (written in Sect. 5.2.2) for investment and the technological coefficients of the production system as a whole,

$$I = (\delta + \mu) K, \qquad \lambda = \frac{\nu + \mu}{\delta + \mu} \frac{L}{K}, \qquad \varepsilon = \frac{\eta + \mu}{\delta + \mu} \frac{P}{K}.$$

8.2.2 Production of Value

The final output of the entire system has to be calculated according to one of the two equations

$$Y = \begin{cases} \displaystyle\sum_{j=1}^{n} Y_j \\ \displaystyle\sum_{i=1}^{n} Z^i \end{cases}$$

while the sectoral production of value Z^i is defined by expressions (8.8) and (8.9). One can use (8.8) to obtain the final output and its derivative

$$Y = \sum_{i=1}^{n} \xi^i K^i, \qquad \frac{dY}{dt} = \sum_{i=1}^{n} Z^i \left(\delta^i + \frac{1}{\xi^i} \frac{d\xi^i}{dt} \right). \tag{8.18}$$

One can separate the growth rate of the total capital stock δ and use (8.13) for the technological change Δ^i to obtain an expression for the growth rate of output in the form

$$\frac{1}{Y}\frac{dY}{dt} = \delta + \frac{1}{Y}\sum_{i=1}^{n}\left[\Delta^i + Z^i\left(\delta^i - \delta\right)\right]. \tag{8.19}$$

The deviations of the growth rate of final output from the growth rate of capital δ are connected with the technological change of the production system and with the non-homogeneity of sectoral development. Comparison of this equation with (6.17) and (6.19) allows us to determine an expression for the total of technological change and structural shift,

$$\Delta = \sum_{i=1}^{n}\left[\Delta^i + Z^i\left(\delta^i - \delta\right)\right]. \tag{8.20}$$

It allows us to calculate the technological index according to the equation

$$\frac{d\alpha}{dt} = \frac{1}{Y}\ln^{-1}\left(\frac{L_0}{L}\frac{P}{P_0}\right)\sum_{i=1}^{n}\left[\Delta^i + Z^i\left(\delta^i - \delta\right)\right]. \tag{8.21}$$

Let us note that the growth rate of the sectoral final output δ_j is, generally speaking, different from the growth rate of the production of value in sector δ^i. The quantities are connected, due to (8.5), by the relation

$$\delta_j = \sum_{i=1}^{n}\delta^i\frac{\delta^i_j - a^i_j}{1-a^i}\frac{Z^i}{Y_j}. \tag{8.22}$$

8.3 Equations of Growth

The preceding results allow one to write a closed set of equations for the dynamics of the production system, which is assumed to consist of n sectors and is characterised by the parameters listed below. The equations can be written in different equivalent forms, by making use of the different sets of assumptions and variables. Here, we refer to the method used for the one-sector approach and considered in Sect. 7.3.3, and we assume that every sector (labelled i, $i = 1, 2, \dots, n$) is described by variables:

Z^i production of value,
K^i value of production equipment,
L^i consumption of labour,
P^i consumption of substitutive work,
α^i technological index.

Additionally, the gross and final outputs can be calculated according to (8.5), that is,

$$X_i = \frac{1}{1-a^i}Z^i, \qquad Y_j = \sum_{i=1}^{n}\frac{\xi^i_j}{\xi^i}Z^i, \quad \xi^i = \sum_{i=1}^{n}\xi^i_j.$$

It is convenient to list the fundamental characteristics of a production system, which are needed for the description:

ξ^i_j the components of the matrix of capital marginal productivity. The components have to be given as functions of time or can be calculated through the components of the fundamental technological matrices: input–output matrix a^i_j and capital-output matrix b^i_j as

$$\xi^i_j = \frac{\delta^i_j - a^i_j}{b^i}, \quad b^i = \sum_{l=1}^{n} b^i_l. \tag{8.23}$$

Components of the matrices a^i_j and b^i_j ought to be estimated empirically from definitions (4.2) and (4.12).

s_j share of investment product in final output of sector j. One can assume that investment sectors are separated, so that for them $s_j = 1$, whereas for others $s_j = 0$.

μ^i coefficient of depreciation of production equipment. As a simplification, it can be accepted that it has the same value for all products in all situations and is constant.

$\tilde{\nu}^i$ the rate of potential growth of labour in every sector should be given as a function of time.

$\bar{\lambda}^i$ the dimensionless technological coefficient should be given as a function of time. If the quantity $\bar{\lambda}^i < 1$, the consumption (for unit of capital stock) of labour decreases and the consumption of substitutive work (work of production equipment) increases. The situation is opposite, if the quantity $\bar{\lambda}^i > 1$.

One can see that these quantities determine the applied technology of the production system, and we can say that the technology in an input–output model is given if we know these parameters.

We refer to the results of Chap. 6 and this chapter to collect the set of equations for the listed variables:

$$\begin{aligned}
&Z^i = \xi^i K^i, \\
&\frac{dK^i}{dt} = \delta^i K^i, \quad \delta^i = -\mu^i + \min \begin{cases} \displaystyle\sum_{l,j=1}^{n} s_j \frac{K^l}{K} \xi^l_j, \quad K = \sum_{l=1}^{n} K^l, \\ \displaystyle\frac{\tilde{\nu}^i + \mu^i}{\bar{\lambda}^i}, \end{cases} \\
&\frac{dL^i}{dt} = \nu^i L^i, \quad \nu^i = \left(\delta^i + \mu^i\right)\bar{\lambda}^i - \mu^i, \\
&\frac{dP^i}{dt} = \eta^i P^i, \quad \eta^i = \frac{\delta^i - (1-\alpha^i)\nu^i}{\alpha^i}, \; \alpha^i = \frac{\ln\left(\frac{Z^i}{Z^i_0}\frac{L^i_0}{L^i}\right)}{\ln\left(\frac{L^i_0}{L^i}\frac{P^i}{P^i_0}\right)}.
\end{aligned} \tag{8.24}$$

It is assumed that there are no other restrictions.

This is a set of evolutionary equations, so the initial values of all the variables, that is,

$$Z^i(0), \qquad K^i(0), \qquad L^i(0), \qquad P^i(0), \qquad \alpha^i(0) \tag{8.25}$$

should be given, while the initial value of capital stock must correspond to the initial values of output and marginal productivity $K(0) = Y(0)/\xi(0)$. The initial values of labour $L(0) = L_0$ and substitutive work $P(0) = P_0$ can be chosen independently.

The problem—in this case one has a Cauchy problem—can be solved by numerical methods. The equations determine a trajectory of evolution of the production system. The final consumption and the storage of intermediate products appear to be the consequence of evolution of the system.

Note that the above system of equations is valid for the case when all prices are considered to be constant during time. The system can be reformulated for the case when prices change. In this case, one has to introduce the new variables for the final output measured in the current money unit $\hat{Z}^i$ instead of variables Z^i (using (8.14)). Some equations for the indexes of prices of the product p_i, $i = 1, 2, \ldots, n$, that must be considered to be new variables, have to be added as well. Some principles of the design of equations for the price indexes will be discussed in Chap. 9.

One supposes that the equations reflect some universal features of production systems. In our theory, economic growth is coupled with growth of consumed energy and technological changes. Understanding of the energy-economy coupling in production systems was considered crucial for the design of proper models to generate scenarios of development, and much effort and money has been spent to create some simulation models of energy-economy systems [6]. Such models provide us with many details of internal processes but require much input information (many empirical parameters). In contrast, phenomenological models, such as the one above, deal with aggregate variables and look simpler. They can be helpful in generating reliable scenarios of the evolution of global and national economies in macroeconomic terms for government use and research.

8.4 Three-Sector System

As a simple example, we consider the dynamics of the production system consisting of the three sectors that were described in Sect. 2.1. The first sector produces the means for production and supplies all sectors with material resources, which are necessary for the production in all sectors. The second sector creates principles of organisation (codification, science, research and development). The third sector produces goods and services for current personal and social consumption. The balance relation for the sectors can be written as

$$\begin{aligned} X_1 &= X_{11} + X_{12} + X_{13} + Y_1, \\ X_2 &= X_{21} + X_{22} + X_{23} + Y_2, \\ X_3 &= Y_3. \end{aligned} \tag{8.26}$$

The final product of the first sector, Y_1, is invested into all the sectors,

$$Y_1 = I^1 + I^2 + I^3. \tag{8.27}$$

The second sector creates an intermediate non-material product, part of which, Y_2, is stored. The final product of the third sector, Y_3, is being totally consumed. Only the first sector produces the products for investment, so, in this situation, $s_1 = 1$, $s_2 = s_3 = 0$ in (8.24). We assume that the other characteristics of the system can also be given as described below.

8.4.1 Fundamental Matrices

The components of the matrices A and B are defined by relations (4.2) and (4.12) and can be estimated using available empirical data and known statistical methods of evaluation. For the considered case of a three-sector system, the numbers below were inspired by the data for the Soviet Union economy in year 1987 [7]. However, the estimates are quite uncertain; thus these numbers can be considered only as an illustrative example.

$$\mathsf{A} = \begin{Vmatrix} 0.581 & 0.207 & 0.0712 \\ 0.0102 & 0.0451 & 0.0134 \\ 0 & 0 & 0 \end{Vmatrix}, \tag{8.28}$$

$$\mathsf{B} = \begin{Vmatrix} 1.28 & 2.34 & 2.14 \\ 0.098 & 0.90 & 0.214 \\ 0 & 0 & 0 \end{Vmatrix} \text{ year}. \tag{8.29}$$

It is seen that all components of the matrices are non-negative. Note that the matrices are degenerate. This is not an exception but the rule.

The components of the matrices A and B combine, according to the rule (8.23), to create the fundamental matrix of capital marginal productivity

$$\Xi = \begin{Vmatrix} 0.304 & -0.0639 & -0.031 \\ -0.0074 & 0.295 & -0.0058 \\ 0 & 0 & 0.418 \end{Vmatrix} \text{ year}^{-1}. \tag{8.30}$$

The matrix Ξ describes the quality of capital, which is improving over time. The components of matrices as functions of time can be approximated, considering that, as a result of technological achievements, one needs less materials and less labour to create products. We consider the case when the component ξ_1^1 of matrix (8.30) increases slowly with time.

8.4.2 Program of Development

As an example, we consider the development of the system from an initial state with values of output and capital in each sector,

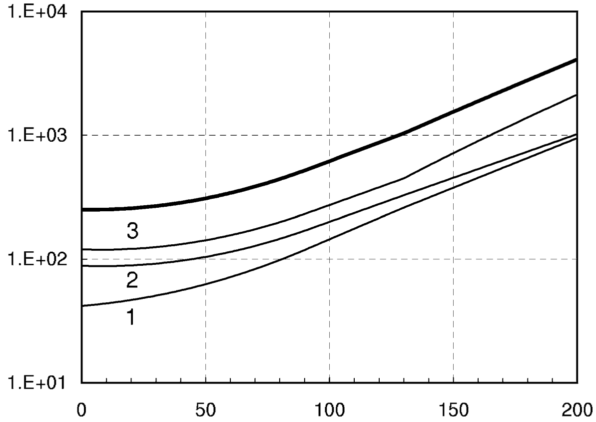

Fig. 8.1 Impact of parameters on the sectoral output. The *labelled lines* depict output in corresponding sectors. The *thick line* represents output for the entire economy

$$Y_1(0) = 40, \qquad Y_2(0) = 90, \qquad Y_3(0) = 120,$$
$$K^1(0) = 226, \qquad K^2(0) = 317, \qquad K^3(0) = 282.$$

The depreciation coefficients, the rates of potential growth of labour and the technological coefficients are considered to be constant until year 140. In year 140, the technological coefficient in sector 3 decreases to 0.7.

$$\mu^1 = 0.05, \qquad \mu^2 = 0.05, \qquad \mu^3 = 0.05,$$

$$\nu^1 = 0.01, \qquad \nu^2 = 0.01, \qquad \nu^3 = 0.01,$$

$$\bar{\lambda}^1 = 0.9, \qquad \bar{\lambda}^2 = 0.9, \qquad \bar{\lambda}^3 = \begin{cases} 0.9, & \text{before year 140,} \\ 0.7, & \text{after year 140.} \end{cases}$$

The results, depicted in Fig. 8.1, show the effects of change of the parameters. The increase in the productivity of the first sector brings a change of modes of development. The initial development of the system is restricted by investment. About year 100 the change of modes occurs: after this year, labour appears to be the limiting factor. The impact of the replacement of labour by substitutive work is illustrated for sector 3. The decrease of the technological coefficient in year 140 provokes an increase in the growth rate of output in this sector, as can be seen in Fig. 8.1.

8.5 Technological Coefficients and Technological Matrices

To characterise the technology of the production system, two pairs of quantities were introduced earlier: the fundamental technological matrices A and B in Chap. 4 and technological coefficients λ and ε in Sect. 5.2. We can guess that these pairs of quantities are connected with each other, or at least we can try to establish some relations in the frame of the many-sector economic model.

One can assume that the production equipment, produced in sector i, is characterised by the primary technological coefficients λ_i and ε_i. The basic production equipment of sector j is a mixture of products with different technological coefficients,

$$\lambda^j = \sum_{i=1}^{n} \frac{I_i^j}{I^j} \lambda_i, \qquad \varepsilon^j = \sum_{i=1}^{n} \frac{I_i^j}{I^j} \varepsilon_i, \qquad I^j = \sum_{i=1}^{n} I_i^j, \tag{8.31}$$

where I_i^j is the gross investment of product i in sector j.

These last relations can be written in another form. First of all, we transform the first of (8.31) by summing up as follows:

$$\sum_{i=1}^{n} \lambda^i I^i = \sum_{l=1}^{n} \lambda_l I_l.$$

Then we use formulae (4.18), assuming that coefficients $\bar{b}_l^i$ are constant, for investment I_l, which gives

$$\sum_{i=1}^{n} \lambda^i I^i = \sum_{i,l=1}^{n} \bar{b}_l^i \lambda_l I^i$$

and, from the arbitrary values of the sectoral investments I^i, we have for the technological coefficients

$$\lambda^j = \sum_{i=1}^{n} \bar{b}_i^j \lambda_i, \qquad \varepsilon^j = \sum_{i=1}^{n} \bar{b}_i^j \varepsilon_i. \tag{8.32}$$

It is clear that the primary technological coefficients λ_i and ε_i depend only on the amount of products used in production. Therefore,

$$\lambda_i = \lambda_i\left(X_1^i, X_2^i, \ldots, X_n^i\right),$$
$$\varepsilon_i = \varepsilon_i\left(X_1^i, X_2^i, \ldots, X_n^i\right).$$

We can assume that the technological coefficients do not depend on a scale of production, so we can consider the technological coefficients to be uniform functions of the zeroth power, that is, their arguments have to be combined in ratios. From (4.2), the technological coefficients can be written as a function of components of technological matrix A,

$$\begin{aligned} \lambda_i &= \lambda_i\left(a_1^i, a_2^i, \ldots, a_n^i\right), \\ \varepsilon_i &= \varepsilon_i\left(a_1^i, a_2^i, \ldots, a_n^i\right), \end{aligned} \tag{8.33}$$

where i is the label of the sectors which create the production equipment.

Thus, one can write for the sectoral technological coefficients

$$\begin{aligned} \lambda^j &= \sum_{i=1}^{n} \bar{b}_i^j \lambda_i\left(a_1^i, a_2^i, \ldots, a_n^i\right), \\ \varepsilon^j &= \sum_{i=1}^{n} \bar{b}_i^j \varepsilon_i\left(a_1^i, a_2^i, \ldots, a_n^i\right). \end{aligned} \tag{8.34}$$

The technological coefficients are determined as functions of the components of the fundamental technological matrices A and B, which are considered to depend on time (see Sect. 4.1). Further on, dependence (8.34) ought to be approximated from empirical data, and one can see that the possible approximation depends on our choice of the sectors. The number of sectors we should take into account must be rather large to describe the technological changes properly.

We can illustrate relations (8.34) by our simple example of the three-sector production system, in which only one sector produces production equipment with technological coefficients λ_1 and ε_1. In this simple case, all basic equipment in all sectors consists of the product of the first sector. Therefore,

$$\lambda^1 = \lambda^2 = \lambda^3 = \lambda_1, \qquad \varepsilon^1 = \varepsilon^2 = \varepsilon^3 = \varepsilon_1.$$

According to expression (8.33), the technological coefficients can be written as a function of components of technological matrix A,

$$\lambda_1 = \lambda_1\big(a_1^1, a_2^1\big), \qquad \varepsilon_1 = \varepsilon_2\big(a_1^1, a_2^1\big).$$

The values of the technological coefficients are determined by technological research and the level of development of science. Therefore, we assume that an increase of products of science, research and development (the products of sector 2) ensures technological progress, so that dependence can be approximated in a simple way by the relations

$$\lambda_1 \sim \left(\frac{a_2^1}{a_1^1}\right)^{-v}, \qquad \varepsilon_1 \sim \left(\frac{a_2^1}{a_1^1}\right)^{-u}.$$

These relations determine a decrease in the values of coefficients λ_1 and ε_1 at positive values of indexes v and u.

References

1. Leontief, W.W.: Quantitative input and output relations in the economic system of the United States. Rev. Econ. Stat. **18**, 105–125 (1936)
2. Leontief, W.W.: The Structure of the American Economy 1919–1939. Harvard University Press, Cambridge (1941)
3. Leontief, W.W.: Input–Output Economics, 2nd edn. Oxford University Press, New York (1986)
4. Sraffa, P.: Production of Commodities by Means of Commodities. Prelude to a Critique of Economic Theory. Cambridge University Press, Cambridge (1975)
5. Pokrovski, V.N.: Physical Principles in the Theory of Economic Growth. Ashgate, Aldershot (1999)
6. Wene, C.-O.: Energy-economy analysis: linking the macroeconomic and systems engineering approaches. Energy **21**, 809–824 (1996)
7. Goskomstat SSSR: Narodnoje Khozjaistvo SSSR v 1987 Godu. Finansi i statistika, Moscow (1988) (in Russian: USSR Statistics Yearbook 1987)

Chapter 9
Mechanism of Social Estimation of Value

Abstract In previous chapters, the notion of price was introduced and understood as an empirical estimate of the value of a product. The price is not an intrinsic characteristic of the product as a thing, but it appears as a result of a bilateral assessment: *the producer* estimates the efforts and expenses necessary to create a thing, and *the consumer* estimates the usefulness of that thing for him. The price emerges as a result of the agreement between the producer and the consumer, and it thus appears connected with features of behaviour of economic agents. However, this does not mean that price is a subjective quantity; the price of a product exceeds expenses (cost) of manufacture for an amount which the consumer can pay willingly, so that the attribution of value of a set of products to the production factors is not unreasonable. The relationship between producers and consumers in a process of exchange of products is a market of products. The theory of prices is a theory of the market. In this chapter, the theory of prices is considered for a simple scheme that can be described in macroeconomic terms.

9.1 The System of Production and Consumption

In the basis of the economic activity of human beings, one can detect explicit or implicit desires of people to satisfy vital needs. In a society, each member needs the services of other persons, which he receives in exchange for his services to other members of the society. Thus, it is possible to assert that the essential content of economic activities of a person is an exchange of services. It is the main thing which unites human beings in society. 'Every man thus lives by exchanging or becomes in some measure a merchant, and the society itself grows to be what is properly a commercial society' [1, Chap. 4]. The exchange of services has the form of exchange of products.

9.1.1 Economic Agents

An economic agent (*actor*) as a carrier of will and a source of decisions on the operation of exchange appears in the theory. Strictly speaking, each active member

V.N. Pokrovskii, *Econodynamics*, New Economic Windows 12,
DOI 10.1007/978-94-007-2096-1_9,

of the society is an economic agent (actor). Moreover, each economic agent is a consumer and a producer simultaneously. As a consumer, he uses a set of products $\check{x}_1, \check{x}_2, \dots, \check{x}_n$. As a producer, the individual spends some efforts of various types $\hat{e}_1, \hat{e}_2, \dots, \hat{e}_m$ to create products. It is assumed that every agent knows what is good and what is bad for him; in other words, he has a certain *system of values*, which can be different for different agents, though some values can be mutual ones. Every economic agent tries to improve his situation, however beautiful it may be.

It is possible to assume that each individual, as an economic agent, tries to spend less effort to get more products. To describe formally the behaviour of an economic agent and the aspiration to improve one's situation, we traditionally introduce, following Walras [2] and many others, the characteristic of the economic agent, namely, the function of utility,

$$u(\hat{\mathsf{e}}, \check{\mathsf{x}}) = u(\hat{e}_1, \hat{e}_2, \dots, \hat{e}_m, \check{x}_1, \check{x}_2, \dots, \check{x}_n). \tag{9.1}$$

It is a decreasing function in relation to coordinates $\hat{e}_1, \hat{e}_2, \dots, \hat{e}_m$ and an increasing function in relation to coordinates $\check{x}_1, \check{x}_2, \dots, \check{x}_n$. It is regarded as more convincing to start with a description of the preferences of the agent to formally determine the agent's subjective utility function. We will return to it in Sect. 9.2.

As a producer, the economic agent tries to reduce his efforts and, as a consumer, he tries to increase the amount of products that he can obtain for the efforts; thus, speaking formally, the agent aspires to reach the greatest value of the function of utility at some restrictions. For a single economic agent, the restriction can be recorded in the form of an inequality,

$$\sum_{i=1}^{n} p_i \check{x}_i \le \sum_{j=1}^{m} w_j \hat{e}_j, \tag{9.2}$$

where p_i is the price of a product with index i, and w_j is a money estimate of a unit of effort of type j. Restriction (9.2) means that the agent cannot spend more than he or she gets for the efforts. Some other restrictions can be taken into account as well.

In modern societies, a single person can be considered as a consumer, but products are usually created by enterprises, which unite the efforts of several (many) individuals and should be considered as economic agents themselves. The behaviour of an enterprise is determined by the tendency to get the greatest profit at given prices p_i and w_j; speaking formally, the aim of the enterprise is to maximise an object function

$$\pi(\check{\mathsf{e}}, \hat{\mathsf{x}}) = \sum_{i=1}^{n} p_i \hat{x}_i - \sum_{j=1}^{m} w_j \check{e}_j \ge 0. \tag{9.3}$$

Here we suppose that among a set of products $\hat{x}_1, \hat{x}_2, \dots, \hat{x}_n$ there can be quantities both with a positive sign (output), and with a negative one (input). The enterprise aims to increase the output (at the same time to reduce expenses) $\hat{x}_1, \hat{x}_2, \dots, \hat{x}_n$ and to decrease the consumed effort $\check{e}_1, \check{e}_2, \dots, \check{e}_m$.

9.1.2 Elementary Economic System

The theory of the market was originally developed by Walras [2] for a system consisting of an assembly of individual consumers and production units, co-operating only through the prices of products and services. A simple heuristic model of the society allowed Walras to consider the mechanism of estimation of value.

To develop an appropriate theory, one has to consider all the economic subjects simultaneously, assuming that economic agents depend on each other through the exchange of efforts and products. According to Walras [2], the economic system can be imagined as consisting of many agents, say, s consumers and r producers. Each consumer ($\alpha = 1, 2, \ldots, s$) is characterised by an offer of efforts $\hat{\mathsf{e}}^{\alpha}$ and a demand of products $\check{\mathsf{x}}^{\alpha}$, the variables being the arguments of the function of utility of type (9.1). Each producer ($\gamma = 1, 2, \ldots, r$) is characterised by a demand of efforts $\check{\mathsf{e}}^{\gamma}$ and a supply of products $\hat{\mathsf{x}}^{\gamma}$, the variables being the arguments of the function of profit of a type (9.3). All economic agents are independently running businesses according to their rules, but, nevertheless, they depend on each other through the exchange of efforts and products.

To calculate the amounts of efforts and products, it is necessary to find the maximum points of s functions of utility of type (9.1) and r functions of profit of type (9.3), considering restrictions which follow from the balance of efforts, products and profits. Walras [2] recorded a system of the algebraic equations for all variables, which, naturally, also appear the equations for the prices of products and wages, $p_1, p_2, \ldots, p_n$ and $w_1, w_2, \ldots, w_m$. Later, Wald [3], McKenzie [4] and Arrow and Debreu [5] showed that the system of equations has a non-negative solution, which confirms the consistency of a considered model. The proof has required powerful and esoteric mathematical tools, fascinating a few generations of mathematical economists who have written thousands of papers.[1] Thus, it was shown that the proposed mechanism of exchanges can be taken as a basis for explaining the mechanism of human estimation of prices and, consequently, of the value of any set of products.

Note that Walras's classical results describe only equilibrium situations,[2] when all variables have constant values. This circumstance limits the area of applicability of the theory, which has been noted by many researchers [7–9]. The real dynamics of the production-consumption system must be represented by a system of stochastic equations, whose structure reflects the real structure of the economic system, and it may be very complex. Examples of construction of the theory for some simple cases are considered by Weidlich [9]; however, there is ample opportunity for research.

---

[1] Mark Blaug [6, p. 17] paradoxically evaluates that 'The result of all this is that we now understand almost less of how actual markets work than did Adam Smith or even Léon Walras.'

[2] Economic equilibrium assumes that all macroeconomic variables which define the economic system are constant, aside from fluctuations. From the point of view of thermodynamics, it is a steady-state situation, where the processes of production and consumption of products are occurring. Economic equilibrium is an idealisation of reality, which has been emphasised many times. However, this concept appears to be a very useful idealisation, just like thermodynamic equilibrium or a steady-state situation in physics.

The theory of Walras and his followers, though valid only for equilibrium situations, has a unique importance. The theory refers to principles of behaviour of economic agents, which, in general form, are universal; they are applicable to any economic system, whether it be capitalism or socialism. These formal principles are laid in the foundation of the theory of exchange, or the theory of the market. At the same time, this theory is a theory of prices.

9.1.3 Problem of Management and Co-ordination

The presentation of the economic system as an assembly of individual consumers and production associations, co-operating only through the prices of products and services, can be considered as a very idealised model. Though some general principles of behaviour of the producers and consumers appear to be universal, the results of the theory of the market appear to depend on assumptions about the architecture of the system, which can be rather complicated and cannot be universal.

Modern economic systems are characterised by a complex hierarchical organisation: production activity is organised in the scale of the whole society, whereas the sovereign government (state) is the highest body governing production and distribution of products. At the same time, numerous economic associations of both producers and consumers take part in the organisation and functioning of the production processes. Both the government and the associations, acting according to their interests and reasons, decide what part of their income should be used, and what part should be kept for future development. There is a problem of the optimum organisation of production and distribution, one that takes into account the interests of both the entire society and the different associations of producers and consumers.

Apparently, various patterns of the organisation of production and the mechanisms of management can be imagined, whereby it is possible to separate the limiting cases about which one speaks in terms of a centrally planned economy and a market economy. In the previous section, we considered an idealised model of a completely disaggregated market economy: interests of producers and consumers are taken into account completely, whereas the possible interests and demands of the society as a whole are entirely ignored. In another idealised model of the organisation of production, the central government on behalf of all society and in the general interest organises production and decides what part of the social product can be used for direct consumption and what part should be saved for investment in the production. None of the models is actually realised in these extreme formulations; the real picture for all countries always appears to be an intermediate version between the two extreme cases.

Even in the post-war Soviet Union, when it was declared that all means of production belonged to the society, the personal property of some things and the possibility of usage of the land areas allowed persons to be engaged in productive activity which could not be controlled by the state. Moreover, there was production activity on a greater scale which was not advertised, so that, to adequately describe the national economy, it was necessary to consider, apparently, that production activity existed outside of the government's control.

On the other hand, when a private property on the means of production flourishes, the government is compelled to keep the decisive influence on the enterprises of those industries, which matter for the country as a whole, but are ignorant by private proprietors. It can be power, transport, communication, protection against epidemics and acts of nature, roads, mail, education, information service, a security, social insurance, care of old people and invalids and more. The state establishes tools to take a part of the income of the enterprises and provides order and protection against external enemies, stability of the national currency and social protection.

Thus, the problem is to investigate systems with complex architectures that provide us with the modern theory of prices. Some works have demonstrated the diversity of architecture of economic systems and the variety of behaviour of economic agents (e.g., [10, 11]).

9.2 Subjective Utility Function

The modern introduction of the utility function [12, 13] starts with a formalisation of human preferences. Introduced in this way, a subjective utility function is a characteristic of an economic agent rather than a characteristic of a fixed set of products, which is an argument of the utility function and can be represented as a vector,

$$\mathsf{x} = \begin{Vmatrix} x_1 \\ \vdots \\ x_n \end{Vmatrix}. \tag{9.4}$$

The first candidate for the function which can characterise the given amount of products in their relation to a human's needs is the value of these products. However, one has to decline this candidate: the value of a set of products does not appear to be a function of amounts of products at all (see Sect. 10.2). Instead of the value function, one uses the utility function, which we introduce here, following the classics [12, 13], in the following way.

To compare two sets of products, that is, vectors x^1 and x^2, one introduces relations between the vectors. A human as a consumer can estimate if he prefers a set x^1 to a set x^2, or, on the contrary, a set x^2 to a set x^1, or if he cannot distinguish between two sets, respectively:

$$\mathsf{x}^1 \succ \mathsf{x}^2, \qquad \mathsf{x}^2 \succ \mathsf{x}^1, \qquad \mathsf{x}^1 \sim \mathsf{x}^2.$$

It was shown [12] that a monotonically increasing function can be defined on the space of dimension n in such a way that

$$\begin{aligned} \mathsf{x}^1 \succ \mathsf{x}^2 \quad &\Rightarrow \quad u(\mathsf{x}^1) > u(\mathsf{x}^2), \\ \mathsf{x}^1 \sim \mathsf{x}^2 \quad &\Rightarrow \quad u(\mathsf{x}^1) = u(\mathsf{x}^2). \end{aligned} \tag{9.5}$$

The arrow $\Rightarrow$ shows that the right-hand side relation follows the left-hand side one. The properties of the utility function $u(\mathsf{x})$ follow simple assumptions.

If the amount of at least one single product in the set x^1 is more than the amount of the same product in the set x^2, then

$$\mathsf{x}^1 \succ \mathsf{x}^2 \quad \Rightarrow \quad u(\mathsf{x}^1) > u(\mathsf{x}^2).$$

This means that all partial derivatives of the utility function are positive,

$$\frac{\partial u}{\partial x_j} > 0. \tag{9.6}$$

Then, it is assumed that a mixture of two sets x^1 and x^2 is preferred to any of the sets; consequently,

$$u(\lambda\mathsf{x}^1 + (1-\lambda)\mathsf{x}^2) > u(\mathsf{x}^1),$$
$$u(\lambda\mathsf{x}^1 + (1-\lambda)\mathsf{x}^2) > u(\mathsf{x}^2), \quad 0 < \lambda < 1.$$

The property is followed by the relation

$$u(\lambda\mathsf{x}^1 + (1-\lambda)\mathsf{x}^2) > \lambda u(\mathsf{x}^1) + (1-\lambda)u(\mathsf{x}^2).$$

This means that the utility function u is strictly convex and a matrix of second partial derivatives,

$$u_{ij} = \frac{\partial^2 u}{\partial x_i \partial x_j}, \tag{9.7}$$

that is, the Hesse matrix is negatively determined.

The described function $u(x_1, x_2, \dots, x_n)$ is called a utility function, or more precisely, a subjective utility function. The properties of the utility function are determined by postulates which are the reflection of empirical evidence. Note that any monotonically increasing transformation of variables of the utility function determines a new utility function with the same properties. These functions should be considered identical.

9.3 Demand Functions

One can consider a separate consumer who has to decide which products are the most necessary ones for him. The consumer has some income M that is obtained in exchange for his efforts e_i, while label i ($i = 1, 2, \dots, m$) enumerates types of efforts. This income has a money form and increases when the agent's efforts increase, so in the simplest case, one can write

$$M = \sum_{i=1}^{m} w_i e_i, \tag{9.8}$$

where w_i is a money estimate of a unit of effort of type i.

The money is spent to acquire products according to the consumer's preferences. One can assume that the main thing is to get money; if it is obtained, there is no difficulty to acquire any thing. This assertion can be formalised in the theory [2] as

$$p_1x_1 + p_2x_2 + \cdots + p_nx_n \le M, \tag{9.9}$$

where p_i is the price of product, and x_i is the quantity of acquiring products. We suppose here that there are no other restrictions.

However, the assumption about the unrestricted market distribution of products is not always valid. In certain societies, there is another mechanism for the distribution of products. In centrally planned societies, the existence of money does not mean that a person can buy any products; it is necessary to have a special right to buy products. This right is reached by efforts to achieve a certain social rank, so the main aim of a person's activity is to increase in social rank [14]. It is not surprising that 'scientific norms of consumption' were elaborated in such societies.

One can then assume that the consumer is characterised by a subjective utility function,

$$u(x_1, x_2, \dots, x_n),$$

where $x_1, x_2, \dots, x_n$ are amounts of products.

On the assumption that the consumer chooses a set of products that will give the utility function the biggest value, one can describe the consumer's behaviour as an attempt to maximise the utility function,

$$\max u(x_1, x_2, \dots, x_n)$$

with restriction (9.9). The amount of money M for acquiring the desirable set of products is fixed, as well as the prices p_i of all products. Note that budget restriction (9.9) can be written in the form of an equality, and the problem of the choice of products can be solved as a problem of searching for a conditional maximum,

$$\max u(x_1, x_2, \dots, x_n), \qquad \sum_{i=1}^{n} p_i x_i = M, \quad \mathsf{x} \ge 0. \tag{9.10}$$

There are no difficulties in adding other restrictions, if any.

To solve the problem, one starts with the Lagrange function

$$\mathcal{L}(\mathsf{x}, \lambda) = u(\mathsf{x}) - \lambda\left(\sum_{i=1}^{n} p_i x_i - M\right),$$

where λ is a Lagrange multiplier. One should equate partial derivatives of the Lagrange function to zero to obtain a set of equations for the unknown quantities,

$$\frac{\partial u}{\partial x_j} - \lambda p_j = 0, \tag{9.11}$$

$$M - \sum_{i=1}^{n} p_i x_i = 0. \tag{9.12}$$

The first of the equations determines the ratio of product prices as a ratio of the marginal utilities of the products, that is, as a ratio of the first derivatives of the utility function,

$$\frac{\partial u}{\partial x_i} : \frac{\partial u}{\partial x_j} = \frac{p_i}{p_j}, \quad i, j = 1, 2, \ldots, n. \tag{9.13}$$

These relations were written by Marshall ([15], Mathematical Appendix II).

Solutions of (9.11) and (9.12) can be represented as functions of the quantities x_i and λ depending on the parameters of the problem,

$$x_i = x_i(\mathsf{p}, M), \qquad \lambda = \lambda(\mathsf{p}, M). \tag{9.14}$$

A change of scale of value does not change the problem, so one can define a demand function as a uniform function of its arguments, that is,

$$x_i = x_i\left(\frac{\mathsf{p}}{M}\right). \tag{9.15}$$

The effects of prices and money on demand can be investigated without knowledge of the utility function in explicit form [16, 17]. If the utility function is given, the demand functions can be found easily. For example, for the utility function

$$u = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}, \qquad \sum_{i=1}^{n} \alpha_i < 1$$

a solution of the problem (9.10) is

$$x_i(\mathsf{p}, M) = \frac{\alpha_i M}{\alpha p_i}, \qquad \lambda(\mathsf{p}, M) = \alpha \frac{u}{M}, \quad \alpha = \sum_{i=1}^{n} \alpha_i. \tag{9.16}$$

The speculations in this section determine the demand functions, which themselves can be set in the foundation of the theory. In fact, one should assume special properties of the utility function in order for the demand functions (9.15) to describe the empirical facts. However, the demand function can be determined from empirical data independently. The demand functions are usually decreasing functions of prices, though there are some exceptions to this rule.

9.4 Welfare Function

The utility function considered in Sect. 9.2 is based on the consumer's subjective preferences and can be called *a subjective utility function*. It can be introduced for each consumer, who tries to choose a situation to maximise this function. To investigate links between these utility functions and the objective utility function, which will be considered in the next chapter, one has to introduce a subjective utility function for the whole community.

Consider a society consisting of s independent consumers, who together hold some amount of products

$$Q_1, Q_2, \dots, Q_n,$$

while the consumer α owns parts of the products

$$x_1^\alpha, x_2^\alpha, \dots, x_n^\alpha, \quad \alpha = 1, 2, \dots, s,$$

where

$$\sum_{\alpha=1}^{s} x_i^\alpha = Q_i, \quad i = 1, 2, \dots, n. \tag{9.17}$$

One assumes that a consumer's utility function

$$u^\alpha\big(\mathsf{x}^1, \mathsf{x}^2, \dots, \mathsf{x}^s\big), \quad \alpha = 1, 2, \dots, s \tag{9.18}$$

depends both on variables with label α and on all other variables, while

$$\frac{\partial u^\alpha}{\partial x_i^\alpha} > 0, \quad \alpha = 1, 2, \dots, s, \; i = 1, 2, \dots, n,$$
$$\frac{\partial u^\alpha}{\partial x_i^\nu} \le 0, \quad \alpha \ne \nu, \; \alpha, \nu = 1, 2, \dots, s, \; i = 1, 2, \dots, n.$$

Every consumer maximises his utility function, as was described in the previous section, but now we should consider maximisation of s functions simultaneously. There is apparently no single point which gives maximum values for all functions simultaneously. However, one can exclude all points where values of the utility functions can be enlarged simultaneously. The remaining points make up what is called the Pareto-optimal set. No consumer in the Pareto-optimal point can improve his/her welfare without diminishing someone's else welfare. The problem of distribution of products among the members of the society was posed and investigated by Pareto [18].

To find a Pareto-optimal set, one can construct the welfare function for the entire community,

$$U\big(\mathsf{x}^1, \mathsf{x}^2, \dots, \mathsf{x}^s\big) = \sum_{\nu=1}^{s} \alpha_\nu u^\nu\big(\mathsf{x}^1, \mathsf{x}^2, \dots, \mathsf{x}^s\big), \quad \alpha_\nu \ge 0, \; \sum_{\nu=1}^{s} \alpha_\nu = 1. \tag{9.19}$$

A point of maximum of function (9.19) at any values of multipliers α_ν belongs to the Pareto-optimal set.

To find Pareto-optimal points, we can use the Lagrange method for solving the problem of maximisation of function (9.19) at restrictions (9.17), and we find that the Pareto-optimal points are obeyed for the set of equations

$$\sum_{\nu=1}^{s} \alpha_\nu \frac{\partial u^\nu}{\partial x_i^\mu} - p_i = 0, \quad \mu = 1, 2, \dots, s; \; i = 1, 2, \dots, n, \tag{9.20}$$

where p_i are the Lagrange multipliers of the problem which are functions of the multipliers α_ν.

One can reasonably assume that the consumer's utility function depends mainly on his own choice, so that relation (9.20) can be rewritten in a simpler form,

$$\alpha_\nu \frac{\partial u^\nu}{\partial x_i^\nu} - p_i = 0, \quad \nu = 1, 2, \ldots, s;\ i = 1, 2, \ldots, n.$$

This gives the relation for every consumer,

$$\frac{\partial u^\nu}{\partial x_i^\nu} : \frac{\partial u^\nu}{\partial x_j^\nu} = \frac{p_i}{p_j}, \quad \nu = 1, 2, \ldots, s;\ i, j = 1, 2, \ldots, n. \tag{9.21}$$

The equation for each consumer has the same form as (9.13), and one can assert that the set of Lagrange multipliers of the problem is a set of prices, which are identical for all the participants.

Welfare function (9.19) is a function of amounts of products distributed among the members of the society. In line with this function, one can consider a function of the entire amounts of products,

$$U(Q_1, Q_2, \ldots, Q_n), \quad Q_i = \sum_{\alpha=1}^{s} x_i^\alpha,\ i = 1, 2, \ldots, n.$$

To relate this function to function (9.19), we assume that all consumers are in identical situations, that is, the amounts of products are distributed equally among all consumers, which means that the quantity x^ν in function (9.19) does not depend on the index ν and can be replaced by the quantity Q/s. Then, for every consumer,

$$\alpha_\nu \frac{\partial u^\nu}{\partial x_j^\nu} = \frac{\partial U}{\partial Q_j}, \quad \nu = 1, 2, \ldots, s;\ j = 1, 2, \ldots, n.$$

One can see that the following relation is valid:

$$\frac{\partial U}{\partial Q_i} : \frac{\partial U}{\partial Q_j} = \frac{p_i}{p_j}, \quad i, j = 1, 2, \ldots, n. \tag{9.22}$$

One can compare the properties of the welfare function as a function of the entire amounts of products with the properties of the objective utility function introduced in Sect. 10.2. The relation (9.22) is exactly relation (10.7) in which U is an objective utility function. Therefore, one can suppose that, as characteristics of a set of products, the objective utility function and the welfare function are indistinguishable. Although the objective utility function does not present the value of a set of products, nevertheless, one expects that the function is connected with the estimations of value.

9.5 The Simplest Markets

Not only a separate individual, but a group of people, or even a society as a whole can be considered to be an economic agent. Further, in this section, we shall consider very simple cases, which will allow us to demonstrate equations for prices. To consider special examples in more detail, we assume that the economy consists of n producers—each sector is a separate producer, which outputs a single product—and only one consumer, the society as a whole. Every one of the $n + 1$ economic agents has its own aims, plans its activity and can make decisions.

The aim of a sector is to obtain a bigger amount of production of value Z^j. To imitate the behaviour of the sector, we solve a maximisation problem, that is, we find the amount of gross output which determines the greatest value of Z^j. We refer to the results of Sect. 4.2.2 to say that the gross output should be planned to be as big as possible at the restriction given by the production factors, so a supply function for each product can be determined as a function of prices,

$$Y_j^{\mathsf{supply}}(\mathsf{p}) = Y_j(p_1, p_2, \ldots, p_n), \quad j = 1, 2, \ldots, n. \tag{9.23}$$

The aim of the entire society is to obtain the greatest usefulness from the products, that is, to find quantities of the output which would maximise a utility function at a restricted amount of money M. In this way, one can determine the demand functions,

$$Y_j^{\mathsf{demand}}(\mathsf{p}, M) = Y_j(p_1, p_2, \ldots, p_n, M), \quad j = 1, 2, \ldots, n. \tag{9.24}$$

9.5.1 Free-Price Market

The problem of simultaneous maximisation of $n + 1$ objective functions can be reduced to the problem of simultaneous consideration of the demand and supply functions for each product. One can consider each product separately and introduce the excess demand function

$$Z_j(\mathsf{p}, M) = Y_j^{\mathsf{demand}}(\mathsf{p}, M) - Y_j^{\mathsf{supply}}(\mathsf{p}), \quad j = 1, 2, \ldots, n, \tag{9.25}$$

where in contrast to what will be considered in the next section, one assumes that there is only one price for each product on the market.

In the situations which we call *economic equilibrium*,[3] demand is equal to supply, so that

$$Z_j(\mathsf{p}, M) = 0, \quad j = 1, 2, \ldots, n. \tag{9.26}$$

This system of equations defines *equilibrium prices*, whereby it is assumed that the demand and supply functions have dependencies of prices such that a stable

[3] Economic equilibrium assumes that all macroeconomic variables which define an economic system are constant, aside from fluctuations. It recalls the definition of equilibrium in thermodynamics: all thermodynamic variables are constant on average, though there is movement of constituent particles of the thermodynamic system. Similarly, the material constituents of an economic system at equilibrium are not in thermodynamic equilibrium: there are processes of production and consumption of products. Economic equilibrium is an idealisation of reality, which has been stressed many times [7]; nevertheless, it is a very useful idealisation, like that of thermodynamic equilibrium in physics.

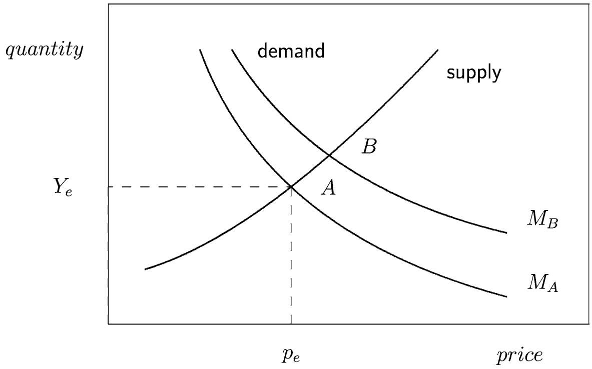

Fig. 9.1 Situations in a free-price market of a single product. The *intersection* of the demand and supply curves determines an equilibrium point (p_e, Y_e). At $M_B > M_A$, the demand increases and a new equilibrium price and a new equilibrium quantity, which are greater than the previous ones, appear

solution of system (9.26) exists. One can consider the demand and supply functions of a chosen product as functions of its own price only. Situations in the market, as have usually been considered [19], are depicted on the plot of Fig. 9.1, which shows demand and supply curves.

The conditions for equilibrium prices can be written in the form of the Walras law,

$$\sum_{i=1}^{n} p_i Z_i(\mathsf{p}, M) = 0. \tag{9.27}$$

The last relation defines equilibrium prices $p_j \geq 0$, if the relation is valid at $Z_i(\mathsf{p}) \leq 0$. The existence of a system of prices is connected with the behaviour of the excess demand functions $Z_i(\mathsf{p})$. So, as we rely on the empirical observation that equilibrium prices exist, the problem is in defining a class of functions which can describe demand and supply. The problem has been studied in many details [19].

The essential component of a real market is money, which is considered as a special product. The amount of money M should correspond to the value of the commodities in the sense discussed in Chap. 3. Relation (9.27) is valid for equilibrium situations and is an analogy of relation (3.29) in Chap. 3. In non-equilibrium situations there is an excess of money, which should be equated to the excess demand,

$$\Delta M = \sum_{i=1}^{n} p_i Z_i(\mathsf{p}, M). \tag{9.28}$$

If money is taken as a separate product, the situation with the demand excess can be considered as an equilibrium, but it is convenient to refer to such situations as non-equilibrium ones. The name is justified, as the system trends to an equilibrium state when relation (9.27) is valid.

Empirical observations assure us that excess demand for a product provokes an increase in the price and vice versa. At small deviations of the demand from the supply, one can write a simple rule for the growth rate of price,

$$\frac{dp_i}{dt} = k_i Z_i(\mathsf{p}, M), \quad k_i \geq 0. \tag{9.29}$$

On the plot of Fig. 9.1, a point A of intersection of curves determines the equilibrium price and quantity of the product. An excess or shortage of money causes a displacement of the demand curve up ($M_B > M_A$) or down ($M_B < M_A$). A new point B of equilibrium appears, whereby the price and quantity of the product take new values. However, the trajectory of the approach to the equilibrium point can differ from a straight line and can resemble a cobweb, as has been described many times [19, 20].

Thus, the money excess provokes changes of prices, which can be determined with the help of relations (9.28) and (9.29). One should know the demand and supply functions to determine these changes. However, when we restrict ourselves to investigating a system which includes only one sector which needs money, the situation is simple. We can assume that, in the case of the discussed three-sector model, money is only needed for products of the third sector. Therefore, (9.28) and (9.29) can be rewritten as

$$\frac{dM}{dt} = p_3 Z_3(p_1, p_2, p_3, M),$$
$$\frac{dp_3}{dt} = k_3 Z_3(p_1, p_2, p_3, M).$$

One can see that these equations are followed by the simple relation

$$\frac{dp_3^2}{dt} = 2k_3 \frac{dM}{dt}. \tag{9.30}$$

The growth rate of the squared price of the commodities for immediate consumption is equal to the amount of emitted money.

9.5.2 Fixed-Price Market

In cases when there is a monopoly on the production of all commodities, fixed prices of the commodities can be set. This is a case of a *centrally planned economy*, designed in countries where the state is the only owner of the whole production system. Analysis shows [14] that the real owner of the production is a *nomenclature class* which governs the economy on the behalf of the state.

As in the previous section, we consider a market where n producing sectors and only one consumer are participating. We assume that the economic agents are characterised by demand and supply functions, but, in contrast to the assumption in the previous section, we suppose that each product has, generally speaking, two prices: a wholesale price p' for the producer and a retail price p'' for the consumer. The existence of the two sets of prices is, according to Polterovich [21, p. 185], 'a phenomenon of centrally planned economy.' To describe a situation on the market, instead of function (9.25), one should introduce an excess demand function which, in contrast to the function in the previous section, depends on the two sets of prices,

$$Z_j(\mathsf{p}', \mathsf{p}'', M) = Y_j^{\text{demand}}(\mathsf{p}'', M) - Y_j^{\text{supply}}(\mathsf{p}'), \quad j = 1, 2, \ldots, n. \tag{9.31}$$

Instead of (9.26), equilibrium situations are defined by the relations

$$Z_j(\mathsf{p}', \mathsf{p}'', M) = 0, \quad j = 1, 2, \ldots, n. \tag{9.32}$$

The profound meaning of the two-price-set market for the owner, the state, is the possibility to obtain income that is equal to the quantity

$$\sum_{i=1}^{n} (p_i'' - p_i') Y_i. \tag{9.33}$$

The state declares (and has the means to persuade its citizens) that the state's (or one can read: the nomenclature's) interests are the most important ones, so that, in addition to the optimisation criteria for each participant of the market considered in the previous section, the situation in the market is determined by the requirement of maximisation of quantity (9.33) as well.

Though the state controls the prices, the fixed prices p_i' and p_i'' cannot be set quite arbitrarily. One can see that system (9.32) can determine a set of equilibrium retail prices, if a set of the wholesale prices is given, and vice versa. Aside from this, the enterprises of the production system keep some independence and interest in obtaining profit. This is provoked by the requirements of finance for self-support of each enterprise. Therefore, the wholesale prices are fixed at such levels that enterprises (at least, some) have to obtain some moderate profit.

The equilibrium quantity of the product, which is being sold and bought, can be less or greater than the coordinate of the point of intersection of the demand and supply curves (the true equilibrium value which could be reached in the free-price market), as is shown on the plots of Fig. 9.2. It is a well-known fact [21] that there are commodities of both types in a centrally planned economy. Initial equilibrium situations for different products are depicted with lines $A'' - A'$ and $A' - A''$. In one case, at $p'' > p'$, the state gets a profit, which is called the turnover tax; in the other case, at $p' > p''$, the state is urged to subsidise the poor sectors.

It is supposed above, that, in the equilibrium situation, there is no excess or deficit of money. In the non-equilibrium case, instead of (9.32), one should write the relation

$$\Delta M = \sum_{i=1}^{n} p_i'' Z_i(\mathsf{p}', \mathsf{p}'', M). \tag{9.34}$$

The excess of money provokes crossover to a new equilibrium situation (line $B'' - B'$ in Fig. 9.2 (top) and $B' - B''$ in Fig. 9.2 (bottom)) which has to be determined by the state, as it controls both production and prices. To begin with, the state usually announces some changes in wholesale prices. This announcement is accompanied by statements that the decision does not concern retail prices. Despite the statements, after some time, the retail prices go up.

The rules of crossover can be understood on the basis of relation (9.32) and the requirement to get the greatest value of criterion (9.33). One can see that the greater the quantity of the profit product and the difference between the retail and wholesale prices $p'' - p'$ are, the greater the profit, which is favourable for the state. However, as one can see from Fig. 9.2 (bottom), these requirements are conflicting for the

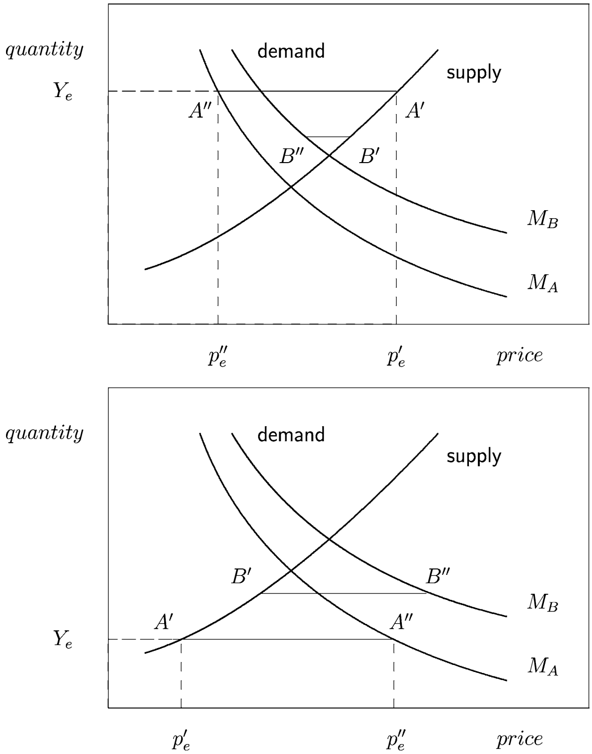

Fig. 9.2 Situations in a fixed-price market of a single product. *Top*: Subsidised product. The wholesale price is greater than the retail price ($p'_e > p''_e$), and the government is urged to pay subsidies $(p'_e - p''_e)Y_e$ to the sector. *Bottom*: Profit product. The wholesale price is less than the retail price ($p'_e < p''_e$), and the government obtains a profit $(p''_e - p'_e)Y_e$

profit products, and it is difficult to forecast the resulting equilibrium situation in this case.

For the subsidised products, in contrast, the lower both the quantity of the output and the difference between the prices are, the better for the state. These are not conflicting requirements, so one expects that the output would not be greater in the new equilibrium situation (Fig. 9.2, top). Therefore, it is favourable for the state interest to have a low level of production and consumption of the subsidised products, which included in the USSR almost all food products except vodka and, perhaps, a few other things. However, there was apparently a 'natural' lower level of consumption; the production system has to provide for the survival and reproduction of the labour force.

References

1. Smith, A.: An Inquiry into the Nature and Causes of the Wealth of Nations. Clarendon Press, Oxford (1976). In two volumes
2. Walras, L.: Elements d'economie politique pure ou theorie de la richesse sociale. Corbaz, Lausanne (1874)
3. Wald, A.: On some system of equations of mathematical economics. Econometrica **19**, 368–403 (1951)

4. McKenzie, L.: On the existence of general equilibrium for a competitive market. Econometrica **27**, 54–71 (1959)
5. Arrow, K.J., Debreu, G.: Existence of an equilibrium for a competitive economy. Econometrica **22**, 265–290 (1954)
6. Blaug, M.: Disturbing currents in modern economics. Challenge **41**, 11–34 (1998)
7. Kornai, J.: Anti-equilibrium. On Economic Systems Theory and the Tasks of Research, 2nd edn. North-Holland, Amsterdam (1975)
8. Soros, G.: The Crisis of Global Capitalism: Open Society Endangered. Public Affairs, New York (1998)
9. Weidlich, W.: Sociodynamics. A Systematic Approach to Mathematical Modelling in the Social Sciences. Harwood Academic Publishers, Amsterdam (2000)
10. Baumol, W.J.: Economic Theory and Operations Analysis, 4th edn. Prentice-Hall, London (1977)
11. Akerlof, G.A.: The market for "lemons": quality uncertainty and the market mechanism. Q. J. Econ. **84**, 488–500 (1970)
12. Von Neumann, J., Morgenstern, O.: Theory of Games and Economic Behaviour, 3rd edn. Princeton University Press, Princeton (1953)
13. Bridges, D.S., Mehta, G.B.: Representation of Preference Orderings. Springer, Berlin (1995)
14. Voslenskii, M.S.: Nomenklatura. The Soviet Ruling Class. Doubleday, Garden City (1984)
15. Marshall, A.: Principle of Economics, 8th edn. MacMillan, London (1920)
16. Slutsky, E.: Sulla teoria del bilancio del consumatore. G. Econ. **60**, 19–23 (1915)
17. Hicks, J.R.: Value and Capital, 2nd edn. Clarendon Press, Oxford (1946)
18. Pareto, V.: Manuel d'economie politique. Girard et Briere, Paris (1909)
19. Intriligator, M.D.: Mathematical Optimisation and Economic Theory. Prentice-Hall, Englewood Cliffs (1971)
20. Allen, R.G.D.: Mathematical Economics, 2nd edn. MacMillan, London (1963)
21. Polterovich V.M.: Ekonomicheskoje Ravnovesije i Khozjaistvennyi Mekanizm. Nauka, Moscow (1990) (in Russian; Equilibrium and Economic Mechanism)

Chapter 10
Value from a Physicist's Point of View

Abstract As a specific concept of economics, value does not need to be reduced to any scientific concepts, but as far as a production process can be considered as a process of transformation of 'wild' forms of matter into forms useful for humans (dwellings, food, clothes, machinery and so on), one can look for analogies in thermodynamics. The phenomenon of production can be considered from a general point of view, assuming all our environment to be *a thermodynamic system*, which is in a far-from-equilibrium state. Thermodynamic laws are quite general and are applicable to any system, no matter how big and complicated it is, while they do not require a knowledge of the structure of the system in all details. A flux of information and work eventually determines a new organisation of matter, which acquires forms of different commodities (complexity), whereby the production process is considered as a process of materialisation of information. The cost of materialisation of information is the work of the production system. To maintain complexity in a thermodynamic system, fluxes of matter and energy must flow through the system.

10.1 Energy Principle of Evolution

10.1.1 Thermodynamics of the Earth

One can consider the upper layers of the Earth as *a thermodynamic system* in a non-equilibrium state. Due to Prigogine [1, 2], we know that stable dissipative structures can exist in such states. All biological organisms and all artificial things on the Earth can be considered dissipative structures which exist due to fluxes of energy [2, 3]. There are also dissipative structures of larger scale: convection flows in the atmosphere and oceans, ecological systems, socio-economic systems, systems of knowledge and others. The human population itself is an example of a large-scale dissipative structure. However, we only begin to recognise and describe large-scale dissipative structures in the upper layers of the Earth.

In the most rough approximation, the Earth, as a thermodynamic system, can be characterised by internal energy E, temperature T and entropy S. These quantities are connected with each other by the first law of thermodynamics, which, assuming that the work of the external gravitational forces, leading to deformation of the

V.N. Pokrovskii, *Econodynamics*, New Economic Windows 12,
DOI 10.1007/978-94-007-2096-1_10,

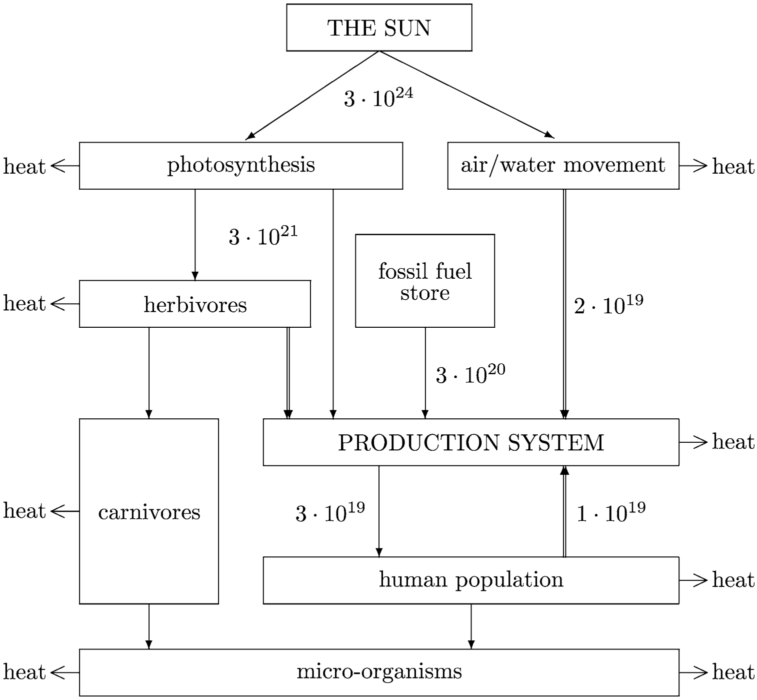

Fig. 10.1 The energy flows in terrestrial systems. The main fluxes of chemical energy (*single lines*) and mechanical energy (*double lines*) are shown. Estimates of fluxes are in joules per year, according to [4]

Earth's form, can be neglected in comparison with other terms, is recorded in the form of

$$dE = T\,dS. \tag{10.1}$$

Values of the internal energy E and entropy S of the Earth, as *an open thermodynamic system*, are influenced by external fluxes and internal processes, so that a variation of the entropy of the Earth is defined by the formula

$$T\,dS = \Delta Q - \sum_i \Xi_i \Delta\xi_i + \sum_{j=1}^{K} \mu_j \Delta N_j. \tag{10.2}$$

This fundamental equation[1] shows that various influences on the system lead to the variation of one universal quantity—entropy—and the variation can be connected both with the fluxes of heat ΔQ and substances ΔN_j through the borders of the system, and with variations of the internal structure of the system. It is assumed

[1] One can note that (10.2) represents a generalisation of the equation for variation of entropy, recorded by Prigogine [5, equation 3.52] for systems with chemical reactions (see also [6]). One of the methods of generalisation can be found in work [7].

that the internal complexity of the system is described with a set of some internal variables $\xi_1, \xi_2, \ldots$.

The non-equilibrium state of the Earth is supported eventually by the energy fluxes: the Earth receives the fluxes of radiant energy from the Sun and re-radiates heat, and a number of transformations take place with the energy on its way through the system (Fig. 10.1). By modern assessments [8, 9], about a third of the total incoming radiation energy, which is $5.5 \cdot 10^{24}$ joules (J) per year, is reflected by clouds and the surface of the Earth, and another part (about $3 \cdot 10^{24}$ J per year) is absorbed by the atmosphere and the surface of the Earth. Assessments show that photosynthesis is the basic mechanism of absorbing the solar energy. The flux of the external radiation energy has caused changes of thermodynamic characteristics of the Earth during the time of its evolution.

In the current epoch, the energy influx from the Sun is balanced by a radiation outflow [8], so the Earth can be considered to be in a steady state, which implies that the thermodynamic characteristics of the Earth do not change. Though the entropy of the stationary Earth is constant, there is internal production of entropy in the system, as in any non-equilibrium thermodynamic system. The production of entropy is connected with the variation of the internal variables,

$$d_{\mathrm{i}}S = -\frac{1}{T}\sum_j \varXi_j \, d\xi_j, \quad d_{\mathrm{i}}S \geq 0. \tag{10.3}$$

To estimate this quantity in the case when the Earth is considered to be in a steady state, we note that the internal production of entropy is compensated by a change of entropy, due to the outgoing fluxes of energy,

$$d_{\mathrm{i}}S = -d_{\mathrm{e}}S = -\frac{1}{T}\left(\Delta Q + \sum_{j=1}^{K}\mu_j \Delta N_j\right). \tag{10.4}$$

The Earth receives a solar flux of high-energy photons (appropriating temperature about 6000 K with chemical potential $\mu = 0$) and radiates heat at a much lower temperature, about 300 K. These fluxes of radiation energy contribute to a change of entropy of the Earth $d_{\mathrm{e}}S$ that has been estimated by many researchers [9, 10] and appears to be negative $d_{\mathrm{e}}S < 0$. From (10.4), this evidences the internal production of entropy, which can be considered a measure of complexity of the Earth. To create and maintain the special complexity (far-from-equilibrium objects or dissipative structures), as in any thermodynamic system [1–3], there is a need for energy fluxes moving through the system.

10.1.2 Human Population and Fluxes of Energy

The life of every biological population is based on energy fluxes which come to the population through food and organisms of species. One can refer to those fluxes as *biologically organised fluxes of energy*. For the human population, for example, the biologically organised flux of energy is about $4 \cdot 10^9$ J/year$\cdot$man $= 4 \cdot 10^6$ Btu/year$\cdot$man in all the centuries of human existence (see the discussion in Sect. 2.4.1).[2]

In line with the biologically organised fluxes of energy, the human population has *socially organised fluxes of energy*, where the *production system* of society plays the role of a mechanism attracting energy from out-of-body sources. Prime sources of energy (the remains of the former biospheres: wood, coal, oil; direct and indirect solar energy in the form of wind, water, tides: energy of fission and fusion of nuclei) are used via different appliances to transform matter of the natural environment into things of the artificial environment that are useful for the human complexity. The ways in which energy has been utilised by humans has been considered in previous chapters in some detail.

The socially organised consumption of energy per capita from traditional sources (mainly firewood and charcoal, animal dung and agricultural wastes) and commercial sources (oil, coal, gas, hydroelectricity, nuclear power and so on) for the world and for some countries is shown in Fig. 10.2. This quantity has reached the amount of the biologically organised flux of energy during the agricultural era. In the middle of the nineteenth century, the amounts of the two fluxes were approximately equal. Nowadays, in developed countries, the socially organised flux exceeds by 50–100 times the biologically organised flux of energy. For the U.S. economy, for example, the consumption of primary energy is equal approximately 4×10^8 Btu $\approx 4 \times 10^{11}$ J per person per year in year 2000. The entire population of the world would need more than 3×10^{21} J per year to live as the U.S. citizens live now. This is only one thousand times less than the amount of energy received by the Earth from the Sun!

10.1.3 Principle of Evolution

According to the *energy principle of evolution*, those populations and their associations (ecosystems) which can utilise the greater amount of energy from their environment have an advantage for survival [11–13]. One can state, taking into account the existence of the two fluxes of energy, that the energy principle of evolution is also valid for the human population. Indeed, as was argued in the previous chapters, the principle according to which the production system is developing can be stated as a principle of the maximal captivation of available resources: the production system tries to swallow up all available production factors. In fact, this principle of progress is a principle of evolution, stated by Lotka [11], Pechurkin [12] and Odum [13, p. 20]: the trajectory of evolution of a system is defined by trends of the system to use the greatest quantity of available energy.

This formal statement is a description of a total of the joint actions of many entrepreneurs trying to obtain the largest profits. The real path of evolution of the

[2] British thermal unit (Btu) $= 252$ cal $= 1053.36$ J $\approx 10^3$ J.

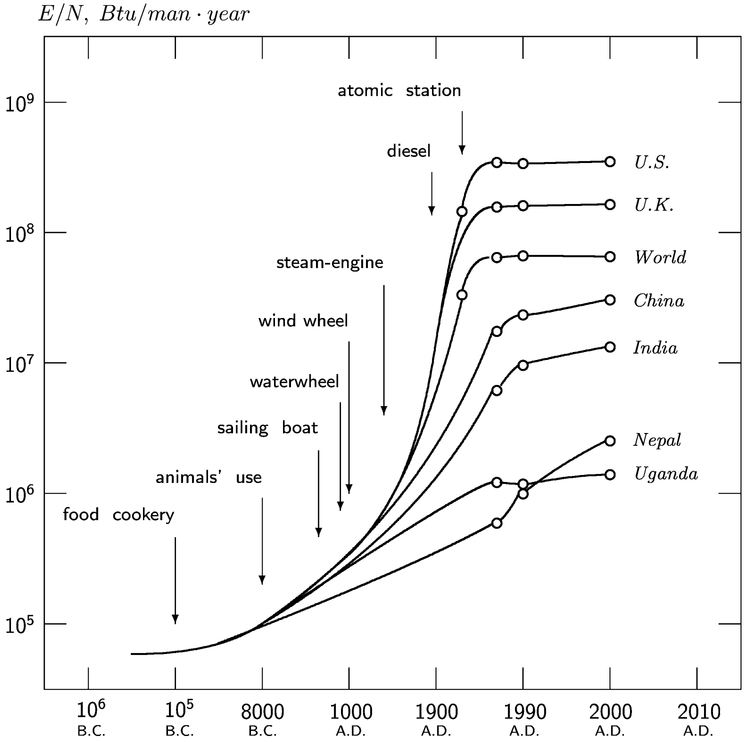

Fig. 10.2 Socially organised flux of energy per capita. An increase in consumption of energy is connected with the invention of more and more sophisticated devices for energy utilisation (e.g., wind, running water, coal, oil). The pictured data does not include the work of animals, which should be added to the fluxes. These corrections are essential for Uganda and Nepal, but can be neglected for other countries. Values for points are taken from Energy Statistics Yearbook (2003 and previous issues)

production system is determined by the availability of labour and energy. The development of the production system is eventually determined by the growth of the labour supply and by possibilities of attracting an extra amount of external energy, so that the set of evolution equations of the production system contains two important quantities: potential growth rates of labour and substitutive work, $\tilde{\nu}(t)$ and $\tilde{\eta}(t)$. These quantities have to be given as exogenous functions of time, but, as we argued in Sects. 2.4.2 and 2.5.5, in fact, they are endogenous characteristics in the problem of evolution of human population on the Earth.

Plenty of energy is used by the human population through the improvements of technology. Managing the huge amount of energy allows the human population to survive in every climate zone of the Earth and expand itself in great measure. Moreover, one can see from the history of mankind that the nations which controlled the available energy had an advantage over other nations. One can refer to the classic example: the industrial revolution and prosperity of Britain began from the invention of the steam engine proper, which allowed utilisation of energy stored in coal in great amounts. The history of nations can be rewritten as the history of the struggle for fluxes of energy.

In the beginning of the twenty-first century, the main primary sources of energy are oil, gas and coal—the remains of the former biospheres. It is assumed that energy needs will increase; however, the peak of the fossil era has passed. Fossil fuel consumption will grow more slowly than total primary energy needs. The future development of mankind is connected with available energy. Only an abundance of available energy would ensure the prosperous development of the human population. Can we then get energy directly from the Sun, which gives to the Earth $3 \cdot 10^{24}$ J per year or $3 \cdot 10^{21}$ Btu per year, or can we find new sources of energy?

10.2 Thermodynamic Interpretation of Value

One can note that all the artificial things around us have special forms and are adjusted for use in special tasks. This means that there is some *complexity* in the environment, a complexity that is created by man and for man. A human being is encircled by artificial products that can be sorted and counted, so one consider the amounts of quantities in natural units of measurement,

$$Q_1, Q_2, \ldots, Q_n.$$

All these objects: buildings, machines, vehicles, sanitation, clothes, home appliances and so on make up the national wealth, which can be characterised from different points of view. However, a general characteristic of artificial objects appears to be value. As a specific concept of economy, value should not be reduced to any other known concepts, but, as processes of production can be considered as processes of transformation of 'wild' forms of substances into forms useful to people (mainly, without variation of internal energy), it is possible to look for analogies in thermodynamics.

10.2.1 Value of a Stock of Products

From a conventional point of view, a stock of products is characterised by its value, and an empirical estimate of the *value of a product* exists. The value of a unit of a product is its *price*. It is assumed that the prices of all products are given,

$$p_i, \quad i = 1, 2, \ldots, n.$$

This allows one to estimate the increase in value of a stock of products,

$$dW = \sum_{j=1}^{n} p_j \, dQ_j. \tag{10.5}$$

It is necessary to take into account that the price of a product is not an intrinsic characteristic of the product. As was already noted earlier in Sect. 2.2, the price depends on the quantities of all products which are in existence at the moment. As a rule, the price decreases if the quantity of the product increases, though the situation can be more complicated. One can observe that there are coupled sets of products, such that an increase in the quantity of one product in a couple is followed by an increase (in the case of a couple of complementary products) or a decrease (in the case of a couple of substituting products) of the price of the other product of the couple. Therefore, one ought to consider the price of a product to be a function of quantities of, generally speaking, all products,

$$p_i = p_i(Q_1, Q_2, \ldots, Q_n).$$

The dependence of prices p_i on the amounts of products is arbitrary, so that one can hardly expect that form (10.5) is a total differential of some function $W(Q_1, Q_2, \ldots, Q_n)$. One cannot say that W is a characteristic of the set of the products which is independent of the history of their creation. In other words, value is not a function of a state of the system. However, a function of a state, which is closely related to value, can be introduced.

10.2.2 Objective Utility Function

Using some assumptions, a function of a state of a system, which is called the utility function, can be introduced on the basis of relation (10.5). Indeed, the linear form (10.5) can be multiplied by a certain function, called the integration factor,

$$\phi = \phi(Q_1, Q_2, \ldots, Q_n),$$

so that, instead of form (10.5), one has a total differential of a new function,

$$dU = \sum_{j=1}^{n} \phi(Q_1, Q_2, \ldots, Q_n) p_j(Q_1, Q_2, \ldots, Q_n) \, dQ_j. \tag{10.6}$$

Requirements on the integrating multiplier are connected with properties of prices as functions of products, so existence of the function U depends on the properties of prices. The integrating multiplier can be taken to be positive, so the linear form (10.6) defines a monotonically increasing function of each variable. It resembles the behaviour of value. One can also expect other properties of function U to relate to the properties of value. In particular, as one could expect for the partial derivatives of the value function (if it exists), the ratio of the partial derivatives of the function U is equal to the ratio of the prices

$$\frac{\partial U}{\partial Q_i} : \frac{\partial U}{\partial Q_j} = \frac{p_i}{p_j}. \tag{10.7}$$

One can see that the important properties of the artificial environment can be characterised by a function U. The introduced function U is called the (*objective*) *utility function*, taking into account that the properties of function U coincide with those of the conventional *utility function* which is introduced as a *subjective* utility function, connected with the sensation of preference of one aggregate of products against another (see Sect. 9.2). As a matter of fact, these functions have to be considered identical, because the utility function is defined with accuracy to monotonic transformation of its variables. Moreover, one can ensure that the utility function exists, as far as the proof was achieved for the subjective utility function. The above transformation of value to utility reminds us of the transformation of heat to entropy. In other terms, an analogy between the theory of utility and the theory of heat was discussed by von Neumann and Morgenstern [14] (see item 3.2.1 of their work).

The understanding of the fact that utility function U can replace a non-existing value function in theoretical considerations was achieved in the second half of the nineteenth century and is considered as a revolution in economic theory [15], completed by the work of prominent investigators.

10.2.3 Thermodynamics of Production

The internal processes which provide an increase and/or a decrease in entropy are connected with structural inhomogeneity. The parts of the system which are in favourable situations to grasp more energy than others can act on the other parts and provoke numerous processes, producing dissipative structures, according to Prigogine [1]. To include processes of production for consideration, it is convenient to consider two subsystems of the entire environment. One can allocate *an artificial environment*, which includes humans and their direct environment: buildings, clothes, cars, sewer networks and all other objects created by humans. The natural characteristics of the artificial environment are quantities of products $Q_1, Q_2, \ldots, Q_n$. The artificial things together with a human population can be considered as one of the subsystems. The other part—*the natural environment*—includes woods, lakes, rocks and similar natural objects. This part is regarded as *a habitat of dwelling* (*the environment*), containing all natural formations which are not touched by the hand of a human, and it represents a subsystem of *natural environment*.

Thus, the set of numbers $Q_1, Q_2, \ldots, Q_n$ defines borders between the two considered subsystems. Each of the subsystems is an open thermodynamic system, exchanging heat and substances with each other and with an outer space. For further consideration we use a simplified schematisation: the natural environment subsystem receives energy from the Sun and passes it to the other subsystem in chemical form. For maintenance and development of the artificial environment, there is a production system that fulfils work, which becomes, after many transformations, the work of energy from the Sun. The results of the production processes are the changes in both the natural and artificial environments that are described by the changes in the amount of products $Q_1, Q_2, \ldots, Q_n$. We note that this set unites both useful products, and also useless, sometimes harmful, but inevitable consequences of production. When new products appear, the borders between the two systems move.

In conformity with the general laws of thermodynamics, it is possible to formulate thermodynamic relations for the considered subsystems. Designating with a prime symbol the characteristic quantities related to the subsystem of the human population and artificial things, one records

$$dE' = T\,dS' - dA, \quad dA = \frac{1}{T}\sum_j a_j\,dQ_j,$$
$$dS' = -\frac{Q'}{T} - \frac{1}{T}\sum_j \Xi'_j\,d\xi'_j + \frac{G}{T}, \tag{10.8}$$

where T is temperature, dA is work of the system of human population and artificial environment on the shift of borders with a natural environment, a_j is work on creation of unit of a product j, Q' is a flux of heat from the considered subsystem in the environment and G is a flux of chemical energy from the natural environment subsystem to the considered subsystem. Work dA is fulfilled by people and external power sources by means of the production equipment and energy sources. This work changes both the natural and the artificial environment.

The subsystem of the natural environment does not fulfil any work, so that for it

$$dE'' = T\,dS'', \quad dS'' = \frac{Q' - Q}{T} - \frac{1}{T}\sum_j \Xi''_j\,d\xi''_j - \frac{G}{T}, \tag{10.9}$$

where Q is a flux of heat disseminated by the Earth, Q' is a flux of heat from the subsystem of artificial things and G is a flux of chemical energy from the subsystem of the natural environment to the subsystem of the artificial environment.

The sum of parities (10.8) and (10.9) give naturally the description of the thermodynamics of the Earth, disposed in Sect. 10.1.1, in particular, relation (10.1), whereby one gets an expression for the increment of entropy,

$$dE = T\,dS, \quad dS = -\frac{Q}{T} - \frac{1}{T}\sum_j \Xi_j\,d\xi_j - \frac{1}{T}\sum_j a_j\,dQ_j. \tag{10.10}$$

The last two terms in the formula for the increment of entropy have an identical form; consequently, this formula shows that quantities of products $Q_1, Q_2, \ldots, Q_n$ can be introduced into the list of internal variables (parameters of internal complexity) of the thermodynamic system of the Earth. Like any internal variables, these variables, left to themselves, disappear, which leads to additional dissipation of energy. The creation of artificial things is connected with the reduction of entropy of the Earth,

$$dS = -\frac{1}{T}\sum_j a_j\,dQ_j, \tag{10.11}$$

where a_j is the work for creation of a unit of product j. It is the work of the production system, and is, eventually after many conversions, the work of the energy flux which the Earth receives from the Sun.

The process of creation of artificial things (the production process) can be interpreted as a process of creation complexity, or negative entropy, of the system. Negative entropy $-S$ is a natural measure of complexity of a non-equilibrium state of the system, in this case, the quantities and complexities of the artificial environment. The total work on the environment, done by humans and by mechanisms which use external sources of energy, is work as it is understood by thermodynamics, so the above relation is the known thermodynamic relation between entropy S and work A,

$$-dS = \frac{1}{T}\,dA. \tag{10.12}$$

This simple scheme becomes complicated: simultaneously with useful products, the production system also creates useless and harmful products (waste and pollution), while all real processes are irreversible. The Earth is not an isolated system, and heat is radiated into space. The production of useful things also stimulates processes of mixing, dispersion and diffusion, so that matter necessary for production will become progressively unavailable [16]. In other words, the chemical elements become increasingly mixed together and thus more and more difficult to separate from each other. Substances necessary for manufacture all become more and more inaccessible. But given the availability of energy, the materials could be recovered from waste, like an ore pile [17]. The Earth is not waiting for a diffusion death: despite some processes of degradation of matter, the essence of production processes is the creation of useful complexity in the environment.

10.2.4 Do Negative Entropy and the Utility Function Coincide?

From relation (10.11), negative entropy $-S$ can be a natural characteristic of complexity of the artificial environment from the thermodynamic point of view, while quantities of products $Q_1, Q_2, \ldots, Q_n$ can be considered as internal (complexity) parameters. Note that an increment of entropy includes both useful complexity and useless, sometimes harmful, but inevitable consequences of production. The other characteristic of a set of products is its utility U, defined by (10.6). In contrast to negative entropy, the utility function conventionally describes only useful complexity, though estimates of damage of the environment are included in prices of the products.

One has two functions: U and $-S$ as characteristics of a set of commodities. Either function is a monotonically increasing function of all variables; the function can be identified if one neglects all by-products: useless and harmful complexity connected with production.[3] One can choose the function U to be a characteristic

[3]The existence of the utility function is justified by the fact that there is a preference relation on the set of products. Similar to that, the existence of entropy is justified by an acceptability relation on the space of thermodynamic variables. The similarity between the utility representation problem in economics and the entropy representation problem in thermodynamics was demonstrated by Candeal et al. [18]. Astonishingly, it seems to be not just a formal analogy: the two functions appear to be different estimates of a set of products.

function which describes useful man-made complexity in the system, and interpret this function as negative entropy of the system of useful artificial things.

Due to (10.6) and (10.11), one might think that value appears to be an analogy to negative entropy which can be considered as a measure of complexity in the system—in our case, the complexity which is useful for human beings. The correspondence is not accurate, because entropy is a function of state, in contrast to value, which is not a function of state. The increase in value dW is close, but might be different from a change of negative entropy dS. From relation (10.12), as far as the change of internal energy of the system can be neglected and the conditions considered to be isothermal, one has proportionality between increase in value of a product and total work done on the system, that is,

$$dW \sim dA. \tag{10.13}$$

Proportionality between increase in value dW and total work dA was found with the analysis of empirical data for the U.S. economy [19, 20]. The authors consider these and similar results as confirmation of what is called the embodied energy theory of value. However, it is not energy but complexity which is left in matter after the work has been done. This complexity exists and can be, in principle, estimated in another way. Besides, in contrast to relation (10.12), there is no exact relation between completed work and produced value. Therefore, the attempts to calculate embodied energy in artificial things remind us of the attempts to calculate the amount of phlogiston in a body. From all points of view, it is better to regard value as something which is very close to negative entropy of our close environment. Fluxes of products should be considered as fluxes of negative entropy, not as fluxes of energy.

10.3 Energy Content of a Monetary Unit

In Marx's theory of value, it is postulated that expenditure of labour is an absolute measure of value, a source of all created wealth (products). When one accounts the effect of substitution of labour with true work of the production equipment, one could expect that the total amount of work, including properly accounted labour work and work of production equipment, could be an absolute measure of value [21]. We shall test this statement, following the work by Beaudreau and Pokrovskii [22].

The total work on the production of value in a unit of time is the sum of the work of substitution P, measured in power units, and the work of humans hL, where $h \approx 4{,}18 \cdot 10^5$ J/hour is an estimate of the work of one person per hour (see Sect. 2.4.1). The total can be recorded as

$$A = P + hL. \tag{10.14}$$

This work fulfils 'useful' changes in our environment (in the form of useful consumer goods and services) which can be estimated by production of value Y (in money units, for year, for example), written, from relations (6.12) and (6.13), as

$$Y = \beta L + \gamma P = \gamma\left(P + \frac{\beta}{\gamma} L\right). \tag{10.15}$$

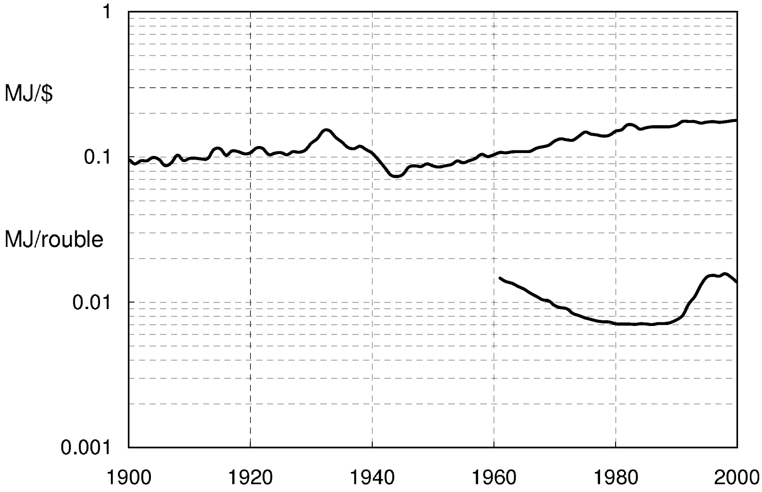

Fig. 10.3 'Energy content' of monetary units. Curves show the amount of work necessary for creation of a product with a value of one dollar of 1996 (*top curve*) and one rouble of 2000 (*bottom curve*) in different years. Reproduced from [22] with permission of Elsevier

If value is estimated by monetary units of constant purchasing capacity, marginal productivities β and γ depend on production factors. For a choice of monetary unit with constant 'energy content,' the marginal productivities appear to be constant.

Expressions (10.14) and (10.15) allow us to determine the true work necessary to complete a thing or service, which costs one monetary unit or, in other words, 'the energy content' of the monetary unit,

$$\frac{A}{Y} = \frac{1}{\gamma}\frac{P + hL}{P + (\beta/\gamma)L} = \frac{1}{Y}(P + hL). \tag{10.16}$$

The assessments of 'energy content' of a monetary unit for the economic systems of the United States and Russian Federations are shown in Fig. 10.3. The 'energy content' of the dollar is (1–2) $\times$ 10^5 J per dollar of 1996; its mean value in the last years of the century (1960–2000) is 1.4×10^5 J. The 'energy content' of the 2000 rouble is less: the mean value for the same years (1960–2000) is 0.1×10^5 J. Pulsations of this quantity (see Fig. 10.3) can be connected with natural variations in the contribution of work (in terms of energy), contrary to our assumption of a constancy of the coefficient h. The 'energy content' of the dollar is 14 times that of the rouble, whereas the exchange rate was about 30 roubles per dollar. These differences can be attributed to the difference of values for h in the American and Russian economies. One could also attribute this to the absence of purchasing power parity (disequilibrium exchange rate). Lastly, it should be noted that the Russian data is, in general, less reliable than the U.S. data.

The values of the 'energy content' shown in Fig. 10.3 appear not to be constant in time: when the contribution of the substitutive work is dominating, which was the case in the U.S. in the second half of the last century (see Chap. 2, Fig. 2.9), the expression (10.16) for 'the energy content' can be written as

$$\frac{A}{Y} \approx \frac{P}{Y}, \quad P \gg hL.$$

So as substitutive work increases faster than output, when the production grows, this relation shows that 'the energy content' is an increasing function of time. At exponential growth, when the variables are given by (2.30) and (2.37), 'the energy content' of a money unit grows as

$$\frac{A}{Y} \sim e^{(\eta-\delta)t}. \tag{10.17}$$

For the U.S. in the second half of the last century, for example, $\eta - \delta = 0.0272$. Indeed, one can see the increase of 'the energy content' on the plot of Fig. 10.3 for the U.S. economy after year 1950.

Here it is necessary to recollect that output in expressions (10.15)–(10.17) is estimated by a special monetary measure with constant purchasing capacity at all times, but this unit of value apparently changes with time. An absolute measure of value is equivalent to some amount of energy. Thus, the expression for output can be written as

$$Y = \frac{A}{\epsilon_{\mathrm{ref}}}, \tag{10.18}$$

where work A is determined by expression (10.14), and ϵ_{ref} represents some standard reference 'energy content' of a monetary unit. To measure production of value in the U.S., it is possible to accept the amount of 10^5 J, but, to draw a distinctive curve in Fig. 2.1, we calculate time dependence of production of value (GDP of the U.S. economy) according to the formula (10.18) at $\epsilon_{\mathrm{ref}} = 50000$ J.

Other approaches [13, 23, 24] are giving the assessments of the total consumption of energy (or exergy) for output, taking into account all 'previous' expenditures of energy needed for production 'from the very beginning.' Such estimates include all losses of energy during production. In this case universality is lost: these estimates of 'energy content' depend on the efficiency of transformation of energy in processes of production. Note that our assessments of 'energy content' of a monetary unit naturally appear to be lower than 'the total exergy or energy content' [13, 23], because substitutive work is a small part of the total consumed energy.

10.4 Thermodynamics of Production Cycle

The performance of thermic machines, which are designed to transform heat into mechanical work, is described by thermodynamics by means of thermodynamic cycles. Thermodynamics itself has emerged from studying this kind of thermodynamic cycle [6]. Investigating the processes of production, one deals with thermodynamic cycles of another kind, i.e., production cycles which are designed to transform the 'wild' forms of substances into 'useful' forms (dwellings, food, clothes, machinery and so on). We shall consider a simple example of production cycle and demonstrate what one needs in order to create a non-equilibrium state of matter artificially.

The main elements of a production process are production equipment, which is able to perform special operations, remaining (excluding wear and tear) unchanged after the cycle, and some substances, the forms of which (one can assume that the change of energy can be neglected) are being changed due to special work of the production equipment. A production cycle can be considered to be a sequence of elementary operations $j_1, j_2, \ldots$, while a set of elementary operations is given. The index j_l is an index of an elementary operation which is fulfilled as number l in the sequence of operations. The unique choice of indexes determines where, when and how forces are allowed to act to perform work which can be calculated as a sum of work at elementary operations,

$$\Delta A = A_{j_1} + A_{j_2} + \cdots .$$

10.4.1 A Simple Production Process

So as not to be too abstract, let us consider, as an example, a system of $2N$ particles (ideal gas) in a container consisting of two compartments of volume V each, as shown in Fig. 10.4. There are some devices which allow the compartments to be connected or isolated (let us call this operation A) and the volume of the second department to be diminished or restored to its previous volume (operation B).

Let us assume that in an initial state each compartment has volume V and the compartments are connected with each other, while the gas is in an equilibrium state, so that each compartment has on average N particles. One can imagine that a deliberate sequence of operations can be applied to the system. We consider isothermal processes consisting of several elementary operations, while the operation is fulfilled in a reversible manner. After one has performed the sequence B—decreasing of volume 2 in ΔV, A—isolating of the compartments, B—increasing of volume 2 in ΔV, and A—connecting the compartments, the configuration of the outer devices is initial, but the gas appears to be in a non-equilibrium state. The mean number of particles in each compartment can be found to be

$$N_1 = N(1+\xi), \qquad N_2 = N(1-\xi), \quad \xi = \frac{(\Delta V/2V)}{1-(\Delta V/2V)}. \tag{10.19}$$

The entropy of the system can be directly estimated according to Boltzmann formulae applied to this case,

$$S = k \ln W, \quad W = \frac{(2N)!}{N_1!\,N_2!},$$

so that the change of entropy of the system can be calculated as

$$\Delta S = -kN\xi^2. \tag{10.20}$$

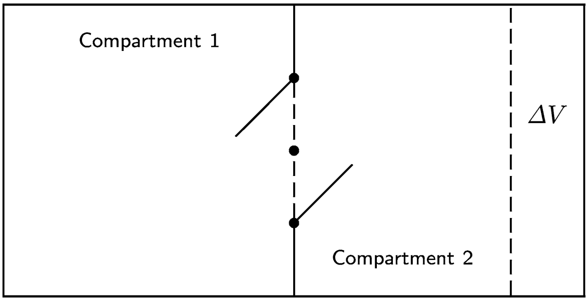

Fig. 10.4 Scheme of the production container. The compartments contain a gas, the state of which can be affected by a sequence of the two production operations: A—the compartments can be connected or isolated and B—the volume of the second department can be diminished or restored to its previous volume

The terms of third order and higher are neglected here and further on.

The work ΔA which is needed to pass the system through the cycle can be calculated as work of/on the ideal gas in every four steps of the cycle. One finds eventually that external forces have to produce extra work during the cycle,

$$\Delta A = kTN\xi^2. \tag{10.21}$$

The internal energy of the system

$$E = 3NkT$$

as the internal energy of the ideal gas does not change in the process, so the first law of thermodynamics can be written, considering every step of the process to be reversible, in the form

$$T\Delta S = -\Delta A,$$

and the change of entropy of the system in the process can be estimated as

$$\Delta S = -kN\xi^2. \tag{10.22}$$

The considered particular non-equilibrium state cannot be created without work of external forces and without the deliberate choice of sequence of elementary operations. Somebody possesses certain sources of energy and has the aim to create a unique non-equilibrium form of matter. To achieve the goal, the creator sends the message in codes of elementary operations: BABA. No other messages can be helpful. The information content of the deliberate message can be estimated if one takes into account that this message is one of 8 possibilities. So, the message carries the information entropy in the amount

$$\Delta I = -\log_2 \frac{1}{8} = 3. \tag{10.23}$$

The information content of the message can be considered to be materialised in non-equilibrium form (complexity) of matter. The cost of materialisation is the work of production system ΔA.

10.4.2 Output of the Production Cycle

Returning to the general case, the inputs of information ΔI and work ΔA eventually determine a new organisation of matter which, due to production processes, acquires forms of different objects. This is a special complexity, which, from a thermodynamic point of view, can be characterised by negative entropy. The decrease in entropy of the entire environment ΔS due to processes of production can be connected with special work of the production system,

$$-\Delta S = \frac{1}{T}\Delta A. \tag{10.24}$$

This is the same relation as (10.12), and one can state that only properly organised work of the production system is needed to transform the natural environment into an artificial environment.

The artificial things created in the process of production can also be estimated by their value, so that a decrease in entropy can correspond to the output of the production system in money units,

$$Y \sim -dS,$$

which means that the output should correspond to work during the production processes,

$$Y \sim \Delta A. \tag{10.25}$$

Useful work ΔA includes muscle work by animals or humans, work by electric motors, mobile engine power as delivered, for example, to the rear wheels of vehicles, but, in this context, not useful heat, as delivered in a space in a building or in a process.

References

1. Prigogine, I.: From Being to Becoming. Time and Complexity in the Physical Sciences. Freeman, New York (1980)
2. Nicolis, G., Prigogine, I.: Self-organisation in Non-equilibrium Systems: From Dissipative Structures to Order Through Fluctuations. Wiley, New York (1977)
3. Morowitz, H.J.: Energy Flow in Biology. Biological Organisation as a Problem in Thermal Physics. Academic Press, New York (1968)
4. Newman, E.I.: Applied Ecology. Blackwell Scientific Publications, London (1993)
5. Prigogine, I.: Introduction to Thermodynamics of Irreversible Processes, 2nd edn. Interscience, New York (1961)
6. Kondepudi, D., Prigogine, I.: Modern Thermodynamics. From Heat Engines to Dissipative Structures. Wiley, Chichester (1999)
7. Pokrovskii, V.N.: Extended thermodynamics in a discrete-system approach. Eur. J. Phys. **26**, 769–781 (2005)
8. Harries, J.E.: Physics of the Earth radiative energy balance. Contemp. Phys. **41**, 309–322 (2000)

9. Kleidon, A.: Nonequilibrium thermodynamics and maximum entropy production in the Earth system. Applications and implications. Naturwissenschaften **96**, 653–677 (2009)
10. Rebane, K.K.: Energija, Entopija, Sreda Obitanija. Znanije, Moscow (1985) (in Russian; Energy, Entropy, Habitat)
11. Lotka, A.J.: Elements of Physical Biology. Williams and Wilkins, Baltimore (1925)
12. Pechurkin, N.S.: Energeticheskije Aspekty Nadorganizmennykh Sistem. Nauka, Novosibirsk (1982) (in Russian; Energy Aspects of Superorganism Systems)
13. Odum, H.T.: Environmental Accounting. EMERGY and Environmental Decision Making. Wiley, New York (1996)
14. Von Neumann, J., Morgenstern, O.: Theory of Games and Economic Behaviour, 3rd edn. Princeton University Press, Princeton (1953)
15. Blaug, M.: Economic Theory in Retrospect, 5th edn. Cambridge University Press, Cambridge (1997)
16. Georgescu-Roegen, N.: The Entropy Law and the Economic Process. Harvard University Press, Cambridge (1971)
17. Ayres, R.U.: Comments on Georgescu-Roegen. Ecol. Econ. **22**, 285–287 (1997)
18. Candeal, J.C., De Miguel, J.R., Induráin, E., Mehta, G.B.: Utility and entropy. Econ. Theory **17**, 233–238 (2001)
19. Costanza, R.: Embodied energy and economic valuation. Science **210**, 1219–1224 (1980)
20. Cleveland, C.J., Costanza, R., Hall, C.A.S., Kaufmann, R.: Energy and the U.S. economy: a biophysical perspective. Science **225**, 890–897 (1984)
21. Beaudreau, B.C.: Energy and Organization: Growth and Distribution Reexamined, 1st edn. Greenwood Press, New York (1998) (2nd edn., 2008)
22. Beaudreau, B.C., Pokrovskii, V.N.: On the energy content of a money unit. Phys. A, Stat. Mech. Appl. **389**, 2597–2606 (2010)
23. Sciubba, E.: On the possibility of establishing a univocal and direct correlation between monetary price and physical value: the concept of extended exergy accounting. In: Ulgiati, S. (ed.) Advances in Energy Studies Workshop. Exploring Supplies, Constraints, and Strategies. Porto Venere, Italy, 2000, pp. 617–633. Servizi Grafici Editoriali, Padova (2001)
24. Valero, A.: Thermoeconomics as a conceptual basis for energy-ecological analysis. In: Ulgiati, S. (ed.) Advances in Energy Studies Workshop. Energy Flows in Ecology and Economy. Porto Venere, Italy, 1998, pp. 415–444. MUSIS, Rome (1998)

Appendix A
NAICS Codes and Titles

The current analysis of the US economy uses a classification system that is based on the 6-digit North American Industry Classification System (NAICS) introduced in year 1997 (http://www.census.gov/eos/www/naics/). The table below shows the 3-digit list of industries (commodities).

Code	Description
11	**Agriculture, Forestry, Fishing and Hunting**
111	Crop Production
112	Animal Production
113	Forestry and Logging
114	Fishing, Hunting and Trapping
115	Support Activities for Agriculture and Forestry
21	**Mining**
211	Oil and Gas Extraction
212	Mining (except Oil and Gas)
213	Support Activities for Mining
22	**Utilities**
221	Utilities
23	**Construction**
233	Building, Developing, and General Contracting
234	Heavy Construction
235	Special Trade Contractors

V.N. Pokrovskii, *Econodynamics*, New Economic Windows 12,
DOI 10.1007/978-94-007-2096-1, © Springer Science+Business Media B.V. 2012

Code	Description
31–33	**Manufacturing**
311	Food Manufacturing
312	Beverage and Tobacco Product Manufacturing
313	Textile Mills
314	Textile Product Mills
315	Apparel Manufacturing
316	Leather and Allied Product Manufacturing
321	Wood Product Manufacturing
322	Paper Manufacturing
323	Printing and Related Support Activities
324	Petroleum and Coal Products Manufacturing
325	Chemical Manufacturing
326	Plastics and Rubber Products Manufacturing
327	Non-metallic Mineral Product Manufacturing
331	Primary Metal Manufacturing
332	Fabricated Metal Product Manufacturing
333	Machinery Manufacturing
334	Computer and Electronic Product Manufacturing
335	Electrical Equipment, Appliance, and Component Manufacturing
336	Transportation Equipment Manufacturing
337	Furniture and Related Product Manufacturing
339	Miscellaneous Manufacturing
42	**Wholesale Trade**
421	Wholesale Trade, Durable Goods
422	Wholesale Trade, Nondurable Goods
44–45	**Retail Trade**
441	Motor Vehicle and Parts Dealers
442	Furniture and Home Furnishings Stores
443	Electronics and Appliance Stores
444	Building Material and Garden Equipment and Supplies Dealers
445	Food and Beverage Stores
446	Health and Personal Care Stores
447	Gasoline Stations
448	Clothing and Clothing Accessories Stores
451	Sporting Goods, Hobby, Book, and Music Stores
452	General Merchandise Stores
453	Miscellaneous Store Retailers
454	Nonstore Retailers

Code	Description
48–49	**Transportation and Warehousing**
481	Air Transportation
482	Rail Transportation
483	Water Transportation
484	Truck Transportation
485	Transit and Ground Passenger Transportation
486	Pipeline Transportation
487	Scenic and Sightseeing Transportation
488	Support Activities for Transportation
491	Postal Service
492	Couriers and Messengers
493	Warehousing and Storage
51	**Information**
511	Publishing Industries
512	Motion Picture and Sound Recording Industries
513	Broadcasting and Telecommunications
514	Information Services and Data Processing Services
52	**Finance and Insurance**
521	Monetary Authorities—Central Bank
522	Credit Intermediation and Related Activities
523	Securities, Commodity Contracts, and Other Financial Investments and Related Activities
524	Insurance Carriers and Related Activities
525	Funds, Trusts, and Other Financial Vehicles
53	**Real Estate and Rental and Leasing**
531	Real Estate
532	Rental and Leasing Services
533	Lessors of Nonfinancial Intangible Assets (except Copyrighted Works)
54	**Professional, Scientific, and Technical Services**
541	Professional, Scientific, and Technical Services
55	**Management of Companies and Enterprises**
551	Management of Companies and Enterprises
56	**Administrative and Support and Waste Management and Remediation Services**
561	Administrative and Support Services
562	Waste Management and Remediation Services

Code	Description
61	**Educational Services**
611	Educational Services
62	**Health Care and Social Assistance**
621	Ambulatory Health Care Services
622	Hospitals
623	Nursing and Residential Care Facilities
624	Social Assistance
71	**Arts, Entertainment, and Recreation**
711	Performing Arts, Spectator Sports, and Related Industries
712	Museums, Historical Sites, and Similar Institutions
713	Amusement, Gambling, and Recreation Industries
72	**Accommodation and Food Services**
721	Accommodation
722	Food Services and Drinking Places
81	**Other Services (except Public Administration)**
811	Repair and Maintenance
812	Personal and Laundry Services
813	Religious, Grantmaking, Civic, Professional, and Similar Organizations
814	Private Households
92	**Public Administration**
921	Executive, Legislative, and Other General Government Support
922	Justice, Public Order, and Safety Activities
923	Administration of Human Resource Programs
924	Administration of Environmental Quality Programs
925	Administration of Housing Programs, Urban Planning, and Community Development
926	Administration of Economic Programs
927	Space Research and Technology
928	National Security and International Affairs

Appendix B
Data on the U.S. Economy

Time series on an annual basis provide the basic empirical data for the test of every theory of economic growth. National statistical compilations contain information about gross national product, labour, energy, and other quantities. High art and hard work are needed to elaborate time series for economic quantities. Sometimes, it needs in courage as well.

The population estimates were found on a website of the U.S. Census Bureau (http://www.census.gov). Values of gross national product Y, gross investment I and capital K are available on a website of the U.S. Bureau of Economic Analysis (http://www.bea.gov). The dollar (1996) values for Y from year 1929 and for I and K from year 1925 are reproduced in the Table. Investment I is understood, in terms of the U.S. Bureau of Economic Analysis, as a sum of investments in private fixed assets, in government fixed assets and in consumer durable goods which make up capital K. The time series for labour L for the latest decades (from year 1948) are found on a website of the U.S. Bureau of Labour Statistics (http://www.stats.bls.gov). The series of quantities compiled by different researchers are used to restore absolute values of quantities Y, I, K and L for the earlier years, whereas there is no need to discuss the discrepancies between series from different sources here, for we use the series not for analysis of economic growth but only for illustration of methods of analysis. Data for total consumption of energy E are taken from a website of the U.S. Department of Energy (http://www.eia.gov) for years from 1949 and from Historical Statistics (Historical Statistics of the United States: Colonial Times to 1970, Parts 1 & 2. U.S. Department of Commerce, Washington, 1975.) for the earlier years. The Table contains also values of substitutive work P estimated in Chap. 7.

V.N. Pokrovskii, *Econodynamics*, New Economic Windows 12,
DOI 10.1007/978-94-007-2096-1,

Year	Population $N \cdot 10^{-3}$	GNP $Y \cdot 10^{-6}$ \$ (1996)	Investment $I \cdot 10^{-6}$ \$ (1996)	Capital $K \cdot 10^{-6}$ \$ (1996)	Labour $L \cdot 10^{-6}$ man-hour	Substitutive work P quad	Primary energy E quad
1891	–	214,663	47,059	967,288	39,927	0.0028	7.04
1892	–	236,357	60,567	1,047,614	41,640	0.0033	7.33
1893	–	222,488	45,865	1,123,236	41,071	0.0037	7.63
1894	–	216,891	43,545	1,172,219	39,234	0.0053	7.93
1895	–	246,560	54,564	1,231,646	42,384	0.0054	8.23
1896	–	241,260	44,133	1,287,255	42,398	0.0065	8.62
1897	–	267,115	53,596	1,331,494	44,119	0.0066	9.02
1898	–	272,514	50,050	1,382,140	44,483	0.0076	9.42
1899	–	305,550	62,951	1,432,732	48,624	0.0077	9.91
1900	76,090	308,472	71,130	1,489,105	49,404	0.0088	10.70
1901	77,580	343,044	79,186	1,543,855	52,181	0.0089	11.69
1902	79,160	340,172	82,638	1,601,086	55,141	0.0090	13.38
1903	80,630	355,526	81,633	1,666,279	57,160	0.0096	15.06
1904	82,170	348,691	67,877	1,716,012	56,213	0.0111	16.06
1905	83,820	378,409	73,291	1,774,784	59,734	0.0113	16.35
1906	85,450	431,208	98,273	1,858,641	62,715	0.0114	13.78
1907	87,010	434,922	96,940	1,947,528	64,516	0.0127	14.67
1908	88,710	387,770	60,558	2,018,638	61,265	0.0144	16.06
1909	90,490	443,541	96,205	2,071,670	65,471	0.0145	15.46
1910	92,410	442,748	95,114	2,137,790	67,694	0.0149	16.35
1911	93,860	457,558	83,589	2,199,302	69,182	0.0159	17.94
1912	95,360	476,230	98,175	2,251,571	71,857	0.0160	18.43
1913	97,230	495,300	109,430	2,326,498	72,900	0.0180	17.74
1914	99,110	446,562	58,553	2,398,262	71,077	0.0205	17.54
1915	100,550	460,975	56,161	2,445,923	70,822	0.0229	17.94
1916	101,960	537,895	94,979	2,494,075	77,346	0.0231	20.12
1917	103,270	513,626	75,380	2,548,021	79,096	0.0236	20.91
1918	103,210	543,839	69,314	2,596,922	78,440	0.0264	21.70
1919	104,510	575,043	116,507	2,643,806	75,961	0.0287	21.70
1920	106,460	587,921	157,481	2,707,472	77,128	0.0305	19.62
1921	108,540	552,754	65,451	2,763,599	69,459	0.0348	20.12
1922	110,050	581,482	82,070	2,814,941	75,597	0.0350	18.63
1923	111,950	672,617	135,455	2,917,966	82,960	0.0353	19.23
1924	114,110	693,915	86,915	3,039,886	80,919	0.0397	21.31
1925	115,830	704,811	127,210	3,161,810	84,418	0.0403	20.70
1926	117,400	755,827	134,742	3,288,677	88,209	0.0415	20.42
1927	119,040	760,285	131,413	3,405,460	88,573	0.0458	21.61
1928	120,510	816,254	133,016	3,515,442	89,667	0.0504	21.61
1929	121,770	822,198	141,193	3,635,742	92,218	0.0532	22.10

Year	Population $N \cdot 10^{-3}$	GNP $Y \cdot 10^{-6}$ \$ (1996)	Investment $I \cdot 10^{-6}$ \$ (1996)	Capital $K \cdot 10^{-6}$ \$ (1996)	Labour $L \cdot 10^{-6}$ man-hour	Substitutive work P quad	Primary energy E quad
1930	123,080	751,500	119,147	3,702,810	85,803	0.0577	22.10
1931	124,040	703,600	95,211	3,719,694	77,930	0.0616	19.62
1932	124,840	611,800	69,100	3,685,926	68,577	0.0641	18.43
1933	125,580	603,300	63,036	3,642,074	68,744	0.0627	18.43
1934	126,370	668,300	72,261	3,628,941	69,262	0.0619	18.63
1935	127,250	728,300	84,404	3,639,025	73,774	0.0615	18.93
1936	128,050	822,500	108,340	3,695,540	81,575	0.0618	19.62
1937	128,820	865,800	116,353	3,762,608	87,334	0.0620	18.83
1938	129,820	835,600	102,012	3,797,784	80,190	0.0655	19.62
1939	130,880	903,500	120,585	3,862,507	85,803	0.0657	20.76
1940	132,120	980,700	136,303	3,945,521	91,125	0.0661	23.69
1941	133,400	1,148,800	183,970	4,096,776	103,518	0.0670	25.47
1942	134,860	1,360,000	229,151	4,329,638	113,359	0.0681	27.85
1943	136,740	1,583,700	269,298	4,591,813	120,357	0.0692	29.14
1944	138,400	1,714,100	276,264	4,804,742	119,483	0.0757	30.13
1945	139,930	1,693,300	242,855	4,896,667	113,651	0.0804	31.22
1946	141,390	1,505,500	215,735	4,929,967	116,640	0.0794	30.62
1947	144,130	1,495,100	257,875	5,019,781	120,868	0.0792	30.13
1948	146,630	1,560,000	291,303	5,140,316	122,909	0.0827	30.62
1949	149,190	1,550,900	294,899	5,285,005	117,809	0.0894	32.01
1950	152,270	1,686,600	345,504	5,494,417	122,919	0.0948	33.30
1951	154,880	1,815,100	356,476	5,716,491	124,413	0.1023	33.30
1952	157,550	1,887,300	367,447	5,937,394	125,663	0.1116	34.29
1953	160,180	1,973,900	392,164	6,188,547	127,280	0.1221	37.56
1954	163,030	1,960,500	391,654	6,414,843	122,270	0.1329	39.35
1955	165,930	2,099,500	433,277	6,684,991	127,953	0.1381	38.85
1956	168,900	2,141,100	430,935	6,924,888	131,174	0.1473	38.85
1957	171,980	2,183,900	435,866	7,157,984	130,269	0.1598	39.84
1958	174,880	2,162,800	418,833	7,340,662	126,309	0.1720	41.72
1959	177,830	2,319,000	463,953	7,594,864	132,272	0.1775	43.51
1960	180,670	2,376,700	466,357	7,836,872	133,578	0.1905	43.71
1961	183,690	2,432,000	470,568	8,072,548	133,225	0.2055	44.30
1962	186,540	2,578,900	514,763	8,354,421	134,940	0.2191	47.83
1963	189,240	2,690,400	546,897	8,663,731	137,276	0.2340	49.65
1964	191,890	2,846,500	587,743	9,012,907	140,097	0.2513	51.83
1965	194,300	3,028,500	645,272	9,411,797	144,126	0.2691	54.02
1966	196,560	3,227,500	683,714	9,835,779	146,916	0.2917	57.02
1967	198,710	3,308,300	693,556	10,231,152	147,620	0.3183	58.91
1968	200,710	3,466,100	747,120	10,653,728	150,607	0.3441	62.41
1969	202,680	3,571,400	760,064	11,071,613	153,556	0.3663	65.63
1970	205,050	3,578,000	739,764	11,424,775	152,147	0.3968	67.86

Year	Population $N \cdot 10^{-3}$	GNP $Y \cdot 10^{-6}$ \$ (1996)	Investment $I \cdot 10^{-6}$ \$ (1996)	Capital $K \cdot 10^{-6}$ \$ (1996)	Labour $L \cdot 10^{-6}$ man-hour	Substitutive work P quad	Primary energy E quad
1971	207,660	3,697,700	783,096	11,804,436	151,990	0.4284	69.31
1972	209,900	3,898,400	863,164	12,258,669	158,838	0.4459	72.76
1973	211,910	4,123,400	935,651	12,756,051	164,395	0.4700	75.81
1974	213,850	4,099,000	894,004	13,155,176	165,966	0.5004	74.08
1975	215,970	4,084,400	840,255	13,466,597	161,152	0.5382	72.04
1976	218,040	4,311,700	907,749	13,838,519	167,209	0.5485	76.07
1977	220,240	4,511,800	1,007,912	14,290,877	173,140	0.5661	78.12
1978	222,590	4,760,600	1,104,643	14,814,054	180,386	0.5864	80.12
1979	225,060	4,912,100	1,150,502	15,341,921	184,244	0.6162	81.04
1980	227,760	4,900,900	1,080,070	15,752,771	182,185	0.6630	78.44
1981	229,940	5,021,000	1,097,206	16,147,910	184,644	0.6970	76.57
1982	232,170	4,919,300	1,041,772	16,447,136	181,172	0.7458	73.44
1983	234,300	5,132,300	1,138,894	16,824,687	184,208	0.7683	73.32
1984	236,370	5,505,200	1,317,892	17,359,824	194,388	0.7752	76.97
1985	238,490	5,717,100	1,430,937	17,943,972	194,400	0.8266	76.78
1986	240,680	5,912,400	1,498,944	18,534,452	199,432	0.8701	77.07
1987	242,840	6,113,300	1,523,250	19,087,645	204,292	0.9017	79.63
1988	245,060	6,368,400	1,554,809	19,636,149	208,570	0.9402	83.07
1989	247,340	6,591,800	1,618,728	20,167,768	212,477	0.9790	84.72
1990	249,910	6,707,900	1,602,374	20,650,376	214,686	1.0840	84.34
1991	252,640	6,676,400	1,515,484	20,931,321	211,162	1.1560	84.52
1992	255,420	6,879,529	1,606,935	21,299,101	213,181	1.1949	85.87
1993	258,140	7,063,412	1,719,528	21,748,774	216,989	1.2295	87.58
1994	260,680	7,347,348	1,817,471	22,246,928	223,330	1.2505	89.25
1995	262,803	7,531,325	1,916,976	22,787,433	225,363	1.3137	91.22
1996	265,229	7,810,009	2,054,615	23,410,780	227,962	1.3851	94.22
1997	267,784	8,161,271	2,232,978	24,089,180	234,445	1.4373	94.73
1998	270,248	8,502,032	2,469,906	24,873,020	237,892	1.5234	95.15
1999	272,691	8,880,300	2,674,882	25,723,677	240,859	1.6329	96.77
2000	282,172	9,205,400	2,607,368	27,655,636	245,567	1.7130	98.91
2001	285,040	9,274,509	2,608,868	28,468,773	243,494	1.8540	96.38
2002	287,727	9,422,759	2,608,868	29,216,898	241,983	1.9884	98.03
2003	290,211	9,659,247	2,715,579	29,983,768	242,761	2.1063	98.16
2004	292,892	10,010,697	2,876,488	30,792,361	245,433	2.2113	100.35
2005	295,561	10,304,854	3,029,802	31,585,756	250,541	2.2839	100.51
2006	298,363	10,591,133	3,110,069	32,444,607	256,064	2.3627	99.86
2007	301,290	10,805,961	3,106,975	33,214,996	258,936	2.4608	101.60
2008	304,060	10,926,080	3,020,894		255,441		99.40

Index

V.N. Pokrovskii, *Econodynamics*, New Economic Windows 12,
DOI 10.1007/978-94-007-2096-1, © Springer Science+Business Media B.V. 2012