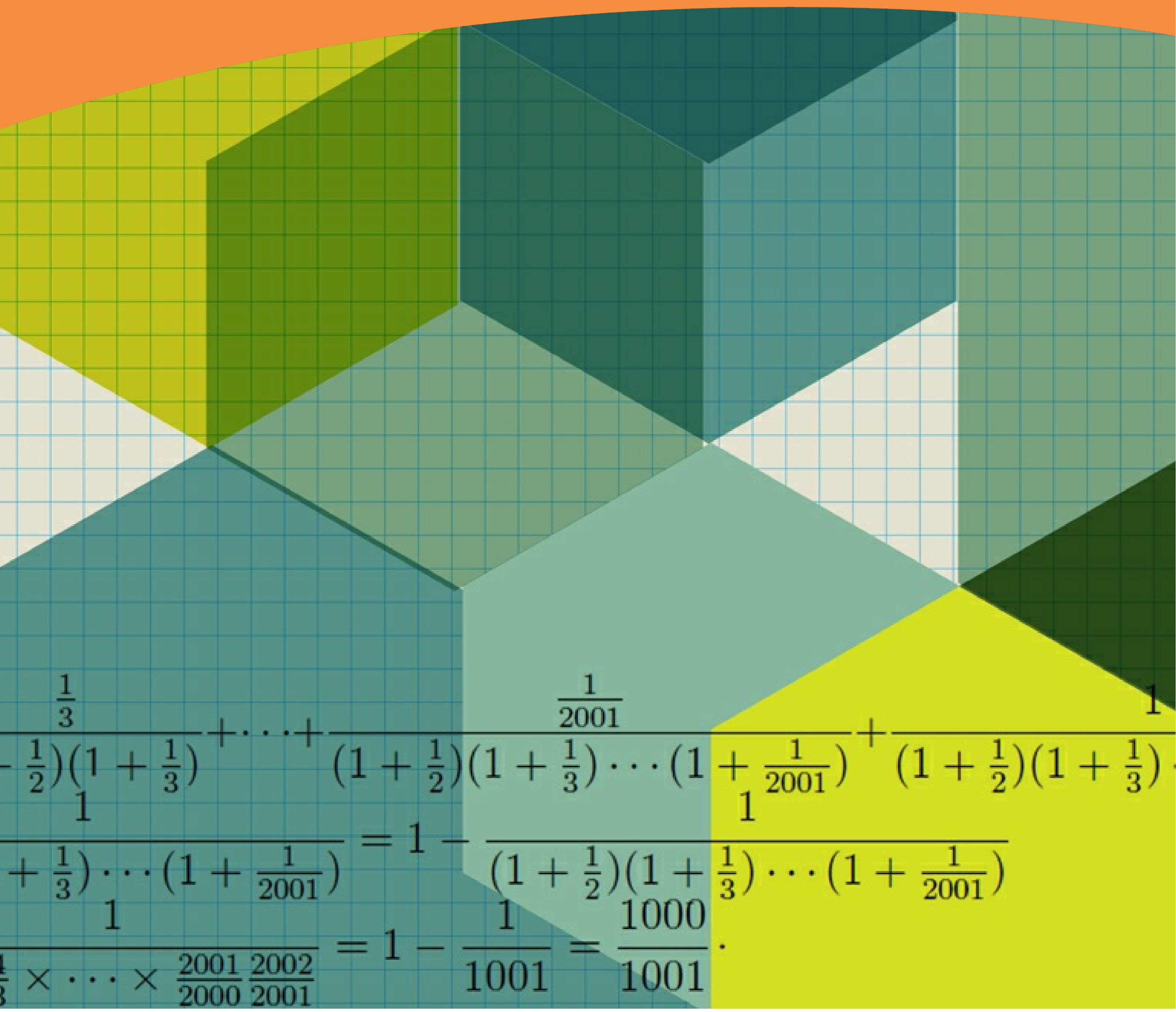
Elementary Algebra Exercise Book I

Hao Wang; Wenlong Wang



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WENLONG WANG AND HAO WANG

ELEMENTARY ALGEBRA EXERCISE BOOK I

Elementary Algebra Exercise Book I
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CONTENTS

AUTHOR BIOGRAPHIES

Mr. Wenlong Wang is a retired mathematics educator in China. He has been working on algebra and geometry problems for many years, and has taught many students in the past few years. He is an expert and a senior researcher in mathematics education.

Professor Hao Wang is a faculty member in the Department of Mathematical & Statistical Sciences at the University of Alberta, an advisory board member of Centre for Mathematical Biology, an associate editor for International Journal of Numerical Analysis & Modeling – Series B, an editor for Nonlinear Dynamics and Systems Theory (an international journal of research and surveys), an editor for a special issue of The Canadian Applied Mathematics Quarterly, and an associate faculty member for Faculty of 1000 Biology. Dr. Wang has strong interests in interdisciplinary research of mathematical biology. His research group is working on areas as diverse as modeling stoichiometry-based ecological interactions, microbiology, infectious diseases, predator-prey interactions, habitat destruction and biodiversity, risk assessment of oil sands pollution. Mathematical models include ordinary differential equations, delay differential equations, partial differential equations, stochastic differential equations, integral differential/difference equations.

PREFACE

The series of elementary algebra exercise books is designed for undergraduate students with any background and senior high school students who like challenging problems. This series should be useful for non-math college students to prepare for GRE general test – quantitative reasoning and GRE subject test – mathematics. All the books in this series are independent and helpful for learning elementary algebra knowledge.

The number of stars represents the difficulty of the problem: the least difficult problem has zero star and the most difficult problem has five stars. With this difficulty indicator, each reader can easily pick suitable problems according to his/her own level and goal.

1 REAL NUMBERS

1.1 Compute $\frac{\frac{1}{2}}{1+\frac{1}{2}}+\frac{\frac{1}{3}}{(1+\frac{1}{2})(1+\frac{1}{3})}+\cdots+\frac{\frac{1}{2001}}{(1+\frac{1}{2})(1+\frac{1}{3})\cdots(1+\frac{1}{2001})}$.

Solution: This quantity is equal to

$$\frac{\frac{1}{2}}{1+\frac{1}{2}}+\frac{\frac{1}{3}}{(1+\frac{1}{2})(1+\frac{1}{3})}+\cdots+\frac{\frac{1}{2001}}{(1+\frac{1}{2})(1+\frac{1}{3})\cdots(1+\frac{1}{2001})}+\frac{1}{(1+\frac{1}{2})(1+\frac{1}{3})\cdots(1+\frac{1}{2001})}$$

$$-\frac{1}{(1+\frac{1}{2})(1+\frac{1}{3})\cdots(1+\frac{1}{2001})}=1-\frac{1}{(1+\frac{1}{2})(1+\frac{1}{3})\cdots(1+\frac{1}{2001})}$$

$$=1-\frac{1}{\frac{3}{2}\times\frac{4}{3}\times\cdots\times\frac{2001}{2000}\frac{2002}{2001}}=1-\frac{1}{1001}=\frac{1000}{1001}.$$

1.2 If p, q are prime numbers and satisfy $5p+3q=19$. Compute the value of $\frac{1}{\sqrt{q}-\sqrt{p}}$.

Solution: The equation $5p+3q=19$ implies that one of p, q is even. Since p, q are prime numbers and the only even prime number is 2, we have two possibilities: if $q=2$, then $p=13/5$, not a prime number, so this case is impossible; if $p=2$, then $q=3$, thus $\frac{1}{\sqrt{q}-\sqrt{p}}=\frac{1}{\sqrt{3}-\sqrt{2}}=\sqrt{3}+\sqrt{2}$.

1.3 Solve $|x|+x+y=10$ and $x+|y|-y=12$ for x, y.

Solution: It is easy to figure out that $x \le 0$ or $y \ge 0$ are impossible. Thus $x>0$ and $y<0$ which lead to $x=32/5, y=-14/5$.

1.4 Given $a=\frac{1+\sqrt{1001}}{2}$, compute the value of $(4a^3-1004a-1001)^{1001}$.

Solution: $a=\frac{1+\sqrt{1001}}{2} \Rightarrow 2a-1=\sqrt{1001} \Rightarrow 4a^2-4a-1000=0 \Rightarrow (4a^3-1004a-1001)^{1001}=$ $[(4a^3-4a^2-1000a)+(4a^2-4a-1000)-1]^{1001}=(0-0-1)^{1001}=-1$.

1.5 If a, b, x are real numbers and $(x^3+\frac{1}{x^3}-a)^2+|x+\frac{1}{x}-b|=0$. Show $b(b^2-3)=a$.

Proof: Since a, b, x are real numbers, the equation implies that $x^3+\frac{1}{x^3}=a$ and $x+\frac{1}{x}=b$. Hence, $a=x^3+\frac{1}{x^3}=(x+\frac{1}{x})(x^2-1+\frac{1}{x^2})=(x+\frac{1}{x})[(x+\frac{1}{x})^2-3]=b(b^2-3)$.

1.6 If the real numbers a, b, c satisfy $a=2b+\sqrt{2}$ and $ab+\frac{\sqrt{3}}{2}c^2+\frac{1}{4}=0$. Evaluate bc/a.

Solution: Substitute $a = 2b + \sqrt{2}$ into $ab + \frac{\sqrt{3}}{2}c^2 + \frac{1}{4} = 0$, then $2b^2 + \sqrt{2}b + (\frac{\sqrt{3}}{2}c^2 + \frac{1}{4}) = 0$. Since b is a real number, $\Delta_b = (\sqrt{2})^2 - 4 \times 2 \times (\frac{\sqrt{3}}{2}c^2 + \frac{1}{4}) = -4\sqrt{3}c^2 \geq 0$, that is $c^2 \leq 0$. On the other hand, c is a real number, thus $c^2 \geq 0$. As a conclusion, $c = 0$, therefore $bc/a = 0$.

1.7 Compute $\frac{1}{1024} + \frac{1}{512} + \frac{1}{256} + \cdots + \frac{1}{2} + 1 + 2 + 4 + \cdots + 1024$.

Solution: Since $\frac{1}{1024} = 1 - \frac{1}{2} - \frac{1}{4} - \frac{1}{8} - \cdots - \frac{1}{1024}$, the original sum is equal to $1 - \frac{1}{2} - \frac{1}{4} - \frac{1}{8} - \cdots - \frac{1}{1024} + \frac{1}{512} + \frac{1}{256} + \cdots + \frac{1}{2} + 1 + 2 + 4 + \cdots + 1024 = 1 - \frac{1}{1024} + 1 + 2 + 4 + \cdots + 1024$.

Let $S = 1 + 2 + 4 + \cdots + 1024$ denoted as (i), then $2S = 2 + 4 + 8 + \cdots + 2048$ denoted as (ii). (ii)-(i) $\Rightarrow S = 2048 - 1 = 2047$. Hence, the original sum is

$$1 - \frac{1}{1024} + 2047 = 2047\frac{1023}{1024}.$$

1.8 If the prime numbers x, y, z satisfy $xyz = 5(x + y + z)$, find the values of x, y, z.

Solution: $xyz = 5(x + y + z)$ implies that at least one of the three prime numbers is five. Without loss of generality, let $x = 5$, then the equation becomes $yz = 5 + y + z$, that is, $(y-1)(z-1) = 6$. Since $6 = 2 \times 3 = 1 \times 6$, there are two possibilities (without considering the order of y and z): (1) $y = 3, z = 4$; (2) $y = 2, z = 7$. The case (1) is inappropriate since $z = 4$ is not a prime number. Therefore, the three prime numbers are 2, 5, 7.

1.9 Simplify $\frac{2\sqrt{6} - 1}{\sqrt{2} + \sqrt{3} + \sqrt{6}}$.

Solution: Let $\sqrt{2} + \sqrt{3} = a$, and take square to obtain $2\sqrt{6} = a^2 - 5$, thus

$$\frac{2\sqrt{6} - 1}{\sqrt{2} + \sqrt{3} + \sqrt{6}} = \frac{a^2 - 5 - 1}{a + \sqrt{6}} = \frac{(a + \sqrt{6})(a - \sqrt{6})}{a + \sqrt{6}} = a - \sqrt{6} = \sqrt{2} + \sqrt{3} - \sqrt{6}.$$

1.10 If $a > 1, b > 0$ and $a^b + a^{-b} = 2\sqrt{2}$, evaluate $a^b - a^{-b}$.

Solution: $a^b + a^{-b} = 2\sqrt{2} \Rightarrow (a^b + a^{-b})^2 = 8 \Rightarrow a^{2b} + a^{-2b} = 6$. Thus
$(a^b - a^{-b})^2 = a^{2b} - 2 + a^{-2b} = 6 - 2 = 4 \Rightarrow a^b - a^{-b} = \pm 2$.

The conditions $a > 1, b > 0$ imply that $a^b - a^{-b} > 0$. As a conclusion, $a^b - a^{-b} = 2$.

1.11 Find the integer part of $A = \frac{11\times70+12\times69+13\times68+\cdots+20\times61}{11\times69+12\times68+13\times67+\cdots+20\times60} \times 100$.

Solution: $A = \frac{11\times69+12\times68+13\times68+\cdots+20\times60}{11\times69+12\times68+13\times67+\cdots+20\times60} \times 100 + \frac{11+12+13+\cdots+20}{11\times69+12\times68+13\times67+\cdots+20\times60} \times 100$

$= 100 + \frac{11+12+13+\cdots+20}{11\times69+12\times68+13\times67+\cdots+20\times60} \times 100$.

Since $1\frac{31}{69} = \frac{11+12+13+\cdots+20}{(11+12+13+\cdots+20)\times 69} \times 100$

$< \frac{11+12+13+\cdots+20}{11\times69+12\times68+13\times67+\cdots+20\times60} \times 100 < \frac{11+12+13+\cdots+20}{(11+12+13+\cdots+20)\times 60} \times 100 = 1\frac{2}{3}$.

Therefore the integer part of A is $100 + 1 = 101$.

1.12 If $a < b < 0$ and $a^2 + b^2 = 4ab$, evaluate $\frac{a+b}{a-b}$.

Solution 1: $a^2 + b^2 = 4ab \Rightarrow (a+b)^2 = 6ab$. Since $a < b < 0$, $a + b = -\sqrt{6ab}$. Similarly, we can obtain $a - b = -\sqrt{2ab}$. Hence, $\frac{a+b}{a-b} = \frac{-\sqrt{6ab}}{-\sqrt{2ab}} = \sqrt{3}$.

Solution 2: Let $a + b = x$, $a - b = y$, then $a = \frac{x+y}{2}, b = \frac{x-y}{2}$. Substitute them into $a^2 + b^2 = 4ab$ to obtain $x^2 = 3y^2$. Since $x, y < 0$, then $x = \sqrt{3}y$, that is $a + b = \sqrt{3}(a-b)$. Thus $\frac{a+b}{a-b} = \sqrt{3}$.

1.13 Given $x = (b^{\frac{n}{n+1}} - a^{\frac{n}{n+1}})^{\frac{n+1}{n}}$, compute the value of $A = \sqrt[n]{x^n + \sqrt[n+1]{a^n x^{n^2}}} + \sqrt[n]{a^n + \sqrt[n+1]{x^n a^{n^2}}} - b$.

Solution: $x = (b^{\frac{n}{n+1}} - a^{\frac{n}{n+1}})^{\frac{n+1}{n}} \Rightarrow x^{\frac{n}{n+1}} = b^{\frac{n}{n+1}} - a^{\frac{n}{n+1}}$. Thus $A = \sqrt[n]{x^{\frac{n^2}{n+1}}(x^{\frac{n}{n+1}} + a^{\frac{n}{n+1}})} + \sqrt[n]{a^{\frac{n^2}{n+1}}(a^{\frac{n}{n+1}} + x^{\frac{n}{n+1}})} - b = \sqrt[n]{x^{\frac{n}{n+1}} + a^{\frac{n}{n+1}}}(\sqrt[n]{x^{\frac{n^2}{n+1}}} + \sqrt[n]{a^{\frac{n^2}{n+1}}}) - b = \sqrt[n]{x^{\frac{n}{n+1}} + a^{\frac{n}{n+1}}}(x^{\frac{n}{n+1}} + a^{\frac{n}{n+1}}) - b = \sqrt[n]{b^{\frac{n}{n+1}} - a^{\frac{n}{n+1}} + a^{\frac{n}{n+1}}}(b^{\frac{n}{n+1}} - a^{\frac{n}{n+1}} + a^{\frac{n}{n+1}}) - b = b^{\frac{1}{n+1}} b^{\frac{n}{n+1}} - b = b - b = 0$.

1.14 If x, y are positive integers and satisfy $2^5 \times x^y = \overline{25xy}$ where $\overline{25xy}$ is one number instead of a multiplication, find the values of x and y.

Solution: Since 2^5 is an even number, $\overline{25xy}$ is even, thus y can only be 2, 4, 6, 8. When $y = 2$, we consider two cases: If $x < 9$, then $2^5 x^2 \leq 2^5 \times 8^2 = 2048$ not in the structure of $\overline{25xy}$; If $x = 9$, we have $2^5 \times 9^2 = 2592$ within the structure of $\overline{25xy}$. Hence, $x = 9, y = 2$ satisfy all the conditions. Similarly we can discuss the cases $y = 4, 6, 8$, and we find that no x value satisfies all the conditions.

1.15 The real numbers a, b, c satisfy $a^2 + b^2 + c^2 = 9$, what is the maximum of $(a-b)^2 + (b-c)^2 + (c-a)^2$?

Solution: $(a-b)^2 + (b-c)^2 + (c-a)^2 = 2(a^2 + b^2 + c^2) - (2ab + 2bc + 2ca) = 3(a^2 + b^2 + c^2) - (a + b + c)^2$. Since a, b, c are real numbers, $(a + b + c)^2 \geq 0$. In addition, $a^2 + b^2 + c^2 = 9$. Thus $(a-b)^2 + (b-c)^2 + (c-a)^2 \leq 3(a^2 + b^2 + c^2) = 3 \times 9 = 27$. The maximal value is 27.

1.16 x, y are positive real numbers and $\frac{1}{x}-\frac{1}{y}-\frac{1}{x+y}=0$, what is the value of $\left(\frac{y}{x}\right)^3+\left(\frac{x}{y}\right)^3$?

Solution: $\frac{1}{x}-\frac{1}{y}-\frac{1}{x+y}=0 \Rightarrow \frac{y-x}{xy}=\frac{1}{x+y} \Rightarrow \frac{y}{x}-\frac{x}{y}=1$, thus $\frac{y}{x}+\frac{x}{y}=$ $\sqrt{\left(\frac{y}{x}-\frac{x}{y}\right)^2+4\frac{y}{x}\frac{x}{y}}=\sqrt{5}$. Therefore, $\left(\frac{y}{x}\right)^3+\left(\frac{x}{y}\right)^3=\left(\frac{y}{x}+\frac{x}{y}\right)\left(\frac{y^2}{x^2}-\frac{y}{x}\frac{x}{y}+\frac{x^2}{y^2}\right)=$ $\left(\frac{y}{x}+\frac{x}{y}\right)\left[\left(\frac{y}{x}+\frac{x}{y}\right)^2-3\frac{y}{x}\frac{x}{y}\right]=\sqrt{5}(5-3)=2\sqrt{5}$.

1.17 Let x, y, z are distinct real numbers, and $x+\frac{1}{y}=y+\frac{1}{z}=z+\frac{1}{x}$, show $x^2y^2z^2=1$.

Proof: The conditions imply that $z\,y=\frac{y-z}{x-y}, xz=\frac{z-x}{y-z}, xy=\frac{x-y}{z-x}$. Multiply them together to obtain $x^2y^2z^2=1$.

1.18 ★ Given $2x+6y\le 15$, $x\ge 0$, $y\ge 0$, find the maximum value of $4x+3y$.

Solution: $2x+6y\le 15 \Rightarrow y\le \frac{5}{2}-\frac{1}{3}x \Rightarrow 4x+3y\le 4x+\frac{15}{2}-x=3x+\frac{15}{2} \Rightarrow \frac{5}{2}-\frac{1}{3}x\ge y\ge 0$, thus $4x+3y\le 3\times\frac{15}{2}+\frac{15}{2}=30$. The maximum value is 30.

1.19 ★ Given $x+y=8$, $xy=z^2+16$, find the value of $3x+2y+z$.

Solution 1: Let $x=4+t, y=4-t$, substitute into $xy=z^2+16$: $16-t^2=z^2+16$, which leads to $t=z=0$, then $x=y=4$, thus $3x+2y+z=12+8+0=20$.

Solution 2: Treat x, y as two roots of the equation $u^2-8u+z^2+16=0$. $\Delta=64-4z^2-64\ge 0 \Rightarrow 4z^2\le 0 \Rightarrow z=0 \Rightarrow u^2-8u+16=0 \Rightarrow (u-4)^2=0 \Rightarrow u_1=u_2=4$, i.e. $x=y=4$.

1.20 ★ Given $x+y+z=0$, find the value of $x(\frac{1}{y}+\frac{1}{z})+y(\frac{1}{x}+\frac{1}{z})+z(\frac{1}{x}+\frac{1}{y})$.

Solution: $x(\frac{1}{y}+\frac{1}{z})+y(\frac{1}{x}+\frac{1}{z})+z(\frac{1}{x}+\frac{1}{y})=x(\frac{1}{x}+\frac{1}{y}+\frac{1}{z})-1+y(\frac{1}{x}+\frac{1}{y}+\frac{1}{z})-1+z(\frac{1}{x}+\frac{1}{y}+\frac{1}{z})-1=$ $(\frac{1}{x}+\frac{1}{y}+\frac{1}{z})(x+y+z)-3=0-3=-3$.

1.21 For a natural number n, let t_n be the sum of all digits in n, for instance, $t_{2009}=2+0+0+9=11$, evaluate $t_1+t_2+\cdots+t_{2009}$.

Solution: Let $T = t_1 + t_2 + \cdots + (2+0+0+8) + (2+0+0+9)$, and then reverse the order of right hand side to obtain $T = (2+0+0+9) + (2+0+0+8) + \cdots + 2 + 1$. Add up these two equalities to obtain
$2T = [1+(2+0+0+9)]+[2+(2+0+0+8)]+\cdots+[(2+0+0+8)+2]+[(2+0+0+9)+1] = 12 \times 2009 \Rightarrow T = 12 \times 2009/2 = 12054$.

1.22 Let a be a positive integer, b and c are prime numbers, and they satisfy $a = bc, \frac{1}{a} + \frac{1}{b} = \frac{1}{c}$, find the value of a.

Solution: $\dfrac{1}{a} + \dfrac{1}{b} = \dfrac{1}{c} \Rightarrow \dfrac{1}{a} = \dfrac{1}{c} - \dfrac{1}{b} = \dfrac{b-c}{bc}$. Since $a = bc$, we have $b - c = 1$, thus c and b are two consecutive prime numbers, which has the only choice $c = 2, b = 3$, thus $a = 6$.

1.23 ★ Let x, y are two natural numbers and they satisfy $x > y, x + y = 667$. Let p be the least common multiple of x and y, let d be the greatest common divisor of x and y, and $p = 120d$. Find the maximum value of $x - y$.

Solution: Let $x = md, y = nd$, then m, n should be coprime since d is the greatest common divisor. $x > y$ implies $m > n$. $p = mnd = 120d \Rightarrow mn = 120$. In addition, $(m+n)d = 667 = 23 \times 29 = 1 \times 667$. Since 23 and 29 are coprime, there are only three possibilities: (1) $m + n = 23, d = 29$; (2) $m + n = 29, d = 23$; (3) $m + n = 667, d = 1$. Together with $mn = 120$, we have (1) $m = 15, n = 8$; (2) $m = 24, n = 5$; (3) no solution. Thus $(m-n)_{max} = 24 - 5 = 19$ which leads to $(x-y)_{max} = (24-5) \times 23 = 437$.

1.24 If x, y, z satisfy $3x + 7y + z = 5$ (i), $4x + 10y + z = 39$ (ii), find the value of $\dfrac{x+y+z}{x+3y}$.

Solution: (i) $\times 3 -$ (ii) $\times 2 \Rightarrow x + y + z = -63$. (ii) $-$ (i) $\Rightarrow x + 3y = 34$.

Hence, $\dfrac{x+y+z}{x+3y} = -\dfrac{63}{34}$.

1.25 Given $a = \sqrt[3]{4} + \sqrt[3]{2} + 1$, find the value of $\dfrac{3}{a} + \dfrac{3}{a^2} + \dfrac{1}{a^3}$.

Solution: $(\sqrt[3]{2} - 1)a = (\sqrt[3]{2} - 1)(\sqrt[3]{4} + \sqrt[3]{2} + 1) = 2 - 1 = 1 \Rightarrow \dfrac{1}{a} = \sqrt[3]{2} - 1$, thus $\dfrac{3}{a} + \dfrac{3}{a^2} + \dfrac{1}{a^3}$
$= \dfrac{3a^2 + 3a + 1}{a^3} = \dfrac{a^3 + 3a^2 + 3a + 1 - a^3}{a^3} = \left(\dfrac{a+1}{a}\right)^3 - 1 = \left(1 + \dfrac{1}{a}\right)^3 - 1 = 2 - 1 = 1$.

1.26 $a \neq 0$ is a real number, and $\dfrac{x}{x^2 + x + 1} = a$, express $\dfrac{x^2}{x^4 + x^2 + 1}$ as a function of a.

Solution: $\dfrac{x}{x^2+x+1} = a \Rightarrow \dfrac{x^2+x+1}{x} = \dfrac{1}{a} \Rightarrow x + \dfrac{1}{x} = \dfrac{1}{a} - 1 \Rightarrow x^2 + \dfrac{1}{x^2} = (\dfrac{1}{a} - 1)^2 - 2$.

Hence, $\dfrac{x^4+x^2+1}{x^2} = x^2 + \dfrac{1}{x^2} + 1 = (\dfrac{1}{a} - 1)^2 - 1 = \dfrac{(1-a)^2 - a^2}{a^2} = \dfrac{1-2a}{a^2} \Rightarrow \dfrac{x^2}{x^4+x^2+1} = \dfrac{a^2}{1-2a}$.

1.27 A nonzero real number a satisfies $a^2 = 3a - 1$, then find the value of $\dfrac{2a^5 - 5a^4 + 2a^3 - 8a^2}{a^2+1}$.

Solution: $a^2 = 3a - 1 \Rightarrow a^2 - 3a + 1 = 0$, and since $\dfrac{3a}{a^2+1} = 1$, we have $\dfrac{2a^5 - 5a^4 + 2a^3 - 8a^2}{a^2+1}$

$= \dfrac{(a^2 - 3a + 1)(2a^3 + a^2 + 3a) - 3a}{a^2+1} = -\dfrac{3a}{a^2+1} = -1$.

1.28 ★ Given $\dfrac{y+z}{ay+bz} = \dfrac{z+x}{az+bx} = \dfrac{x+y}{ax+by} = m$ and $xyz \neq 0$, show $m = \frac{2}{a+b}$ when $a + b \neq 0$.

Proof: $\frac{y+z}{ay+bz} = m \Rightarrow y+z = m(ay+bz) \Rightarrow (1-am)y = (bm-1)z$

(i). Similarly we can obtain $(1-am)z = (bm-1)x$

(ii). $(1-am)x = (bm-1)y$, (iii). $(1-am)y = (bm-1)z$

(i) $\times$ (ii) $\times$ (iii) $\Rightarrow (1-am)^3 xyz = (bm-1)^3 xyz$, which together with $xyz \neq 0$ leads to $(1-am)^3 = (bm-1)^3 \Rightarrow 1-am = bm-1 \Rightarrow m = \frac{2}{a+b}$ when $a+b \neq 0$.

1.29 Given $a+b=2$, find the value of $a^3+6ab+b^3$.

Solution: $a^3+6ab+b^3 = (a+b)(a^2-ab+b^2)+6ab = 2(a^2-ab+b^2)+6ab = 2a^2+4ab+2b^2 = 2(a+b)^2 = 2 \times 2^2 = 8$

1.30 Given $x^2+xy=3$ (i), $xy+y^2=-2$ (ii), find the value of $2x^2-xy-3y^2$.

Solution: (i) $\times 2 -$ (ii) $\times 3 \Rightarrow 2x^2-xy-3y^2 = 12$.

1.31 ★ If the real numbers a, b, c satisfy $\frac{ab}{a+b} = \frac{1}{3}, \frac{bc}{b+c} = \frac{1}{4}, \frac{ca}{c+a} = \frac{1}{5}$, find the value of $\frac{abc}{ab+bc+ca}$.

Solution: $\frac{ab}{a+b} = \frac{1}{3} \Rightarrow \frac{a+b}{ab} = 3 \Rightarrow \frac{1}{a}+\frac{1}{b} = 3$ (i). Similarly, we can obtain $\frac{1}{b}+\frac{1}{c} = 4$ (ii), $\frac{1}{c}+\frac{1}{a} = 5$ (iii). (i)+(ii)+(iii) $\Rightarrow \frac{1}{a}+\frac{1}{b}+\frac{1}{c} = 6 \Rightarrow \frac{abc}{ab+bc+ca} = \frac{1}{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}} = \frac{1}{6}$.

1.32 ★ Given $a^4+b^4+c^4+d^4 = 4abcd$, show $a=b=c=d$.

Proof: $a^4+b^4+c^4+d^4-4abcd = 0 \Rightarrow (a^4-2a^2b^2+b^4)+(c^4-2c^2d^2+d^4)+(2a^2b^2-4abcd+2c^2d^2) = 0 \Rightarrow (a^2-b^2)^2+(c^2-d^2)^2+2(ab-cd)^2 = 0 \Rightarrow a^2=b^2, c^2=d^2, ab=cd \Rightarrow a=b=c=d.$

1.33 Consider two real numbers x, y, find the minimum value of $5x^2-6xy+2y^2+2x-2y+3$ and the associated values of x, y.

Solution: $5x^2 - 6xy + 2y^2 + 2x - 2y + 3 = (x - y + 1)^2 + (2x - y)^2 + 2$ Since $(x - y + 1)^2 \geq 0, (2x - y)^2 \geq 0$, the minimum value of $5x^2 - 6xy + 2y^2 + 2x - 2y + 3$ is 2, and the associated values of x, y are $x = 1, y = 2$ (solved from $x - y + 1 = 0, 2x - y = 0$).

1.34 ★ Factoring $x^4 + x^3 + x^2 + 2$.

Solution: Let $x^4+x^3+x^2+2 = (x^2+ax+1)(x^2+bx+2) = x^4+(a+b)x^3+(ab+3)x^2+(2a+b)x+2$, then equaling the corresponding coefficients to obtain $a + b = 1, ab + 3 = 1, 2a + b = 0 \Rightarrow a = -1, b = 2 \Rightarrow x^4 + x^3 + x^2 + 2 = (x^2 - x + 1)(x^2 + 2x + 2)$.

1.35 Let a, b, c are lengths of three sides of a triangle, and they satisfy $a^2 - 16b^2 - c^2 + 6ab + 10bc = 0$ show $a + c = 2b$.

Proof: $a^2 - 16b^2 - c^2 + 6ab + 10bc = a^2 + 6ab + 9b^2 - 25b^2 + 10bc - c^2 = (a + 3b)^2 - (5b - c)^2 = (a + 3b - 5b + c)(a + 3b + 5b - c) = (a - 2b + c)(a + 8b - c) = 0$. Since a, b, c represent lengths of three sides of a triangle, $a + 8b - c > 0$, thus $a - 2b + c = 0 \Rightarrow a + c = 2b$.

1.36 ★ x, y are prime numbers, $x = yz$, $\dfrac{1}{x} + \dfrac{1}{y} = \dfrac{3}{z}$, find the value of $x + 5y + 2z$.

Solution: $\dfrac{1}{x} + \dfrac{1}{y} = \dfrac{3}{z} \Rightarrow yz + xz = 3xy$. Since $x = yz$, we have $x + xz = 3xy$. Since $x \neq 0$, we have $1 + z = 3y$. Since y, z are prime numbers, the only possibility is $y = 2, z = 5, x = 2 \times 5 = 10$ Hence, $x + 5y + 2z = 10 + 5 \times 2 + 2 \times 5 = 30$.

1.37 ★ Given $\dfrac{a + b}{a - b} = \dfrac{b + c}{2(b - c)} = \dfrac{c + a}{3(c - a)}$ where a, b, c are distinct, show $8a + 9b + 5c = 0$.

Proof: Let $\dfrac{a + b}{a - b} = \dfrac{b + c}{2(b - c)} = \dfrac{c + a}{3(c - a)} = k$, then $a + b = k(a - b)$ (i), $b + c = 2k(b - c)$ (ii), $c + a = 3k(c - a)$ (iii). (i) $\times 6$ + (ii) $\times 3$ + (iii) $\times 2 \Rightarrow 8a + 9b + 5c = 0$.

1.38 ★ The positive integers x, y, z satisfy $x + \frac{y}{z} = 11$, $y + \frac{x}{z} = 14$, and $x + y \neq z$, find a positive integer for $\dfrac{x + y}{z}$ if possible.

Solution: Since $\dfrac{x + y}{z}$ is a positive integer, we can let $\dfrac{x + y}{z} = k$ where k is a positive integer, then $x + y = k$ (i). The sum of $x + \frac{y}{z} = 11$ and $y + \frac{x}{z} = 14$ leads to $x + y + \frac{x+y}{z} = 25$ (ii). Substitute (i) into (ii): $kz + k = 25 \Rightarrow k = \frac{25}{z+1}$. Therefore, $z = 4$ or 24. However when $z = 24$, $k = 1$ which violates $x + y \neq z$. Hence, $z = 4, k = 5$, then $\dfrac{x + y}{z} = 5$.

1.39 ★ If the polynomial $6x^2-5xy-4y^2-11x+22y+m$ can be factored into the product of two linear polynomials, find the value of m and factor the polynomial.

Solution: Let $6x^2-5xy-4y^2-11x+22y+m=(2x+y+k)(3x-4y+l)=6x^2-5xy-4y^2+(3k+2l)x+(l-4k)y+kl$, then equaling the coefficients:
$3k+2l=-11, l-4k=22, kl=m$, which result in $k=-5, l=2, m=-10$.
Hence, $6x^2-5xy-4y^2-11x+22y+m=6x^2-5xy-4y^2-11x+22y-10=(2x+y-5)(3x-4y+2)$.

1.40 ★ Given $|x+4|+|3-x|=10-|y-2|-|1+y|$, find the maximum and minimum values of xy.

Solution: $|x+4|+|3-x|=10-|y-2|-|1+y| \Rightarrow |x+4|+|3-x|+|y-2|+|1+y|=10$.
Since $|x+4|+|3-x| \geq 7$ and $|y-2|+|1+y| \geq 3$. $|x+4|+|3-x|+|y-2|+|1+y|=10$ only if we choose equal sign in both inequalities.

$$|x+4|+|3-x| \geq 7 \Rightarrow -4 \leq x \leq 3;$$

$$|y-2|+|1+y| \geq 3 \Rightarrow -1 \leq y \leq 2.$$

Hence, xy has the maximum value 6 and the minimum value -8.

1.41 ★★ If the real numbers a, b, c satisfy $a+b+c=0, abc=1$, show one of a, b, c should be greater than $3/2$.

Proof: Since a, b, c are real numbers and $abc=1$, we have at least one of a, b, c is greater than zero. Without loss of generality, let $c>0$.

$a+b+c=0 \Rightarrow a+b=-c$ (i); $abc=1 \Rightarrow ab=\frac{1}{c}$ (ii); $ab=\left(\frac{a+b}{2}\right)^2-\left(\frac{a-b}{2}\right)^2$ (iii).

Substitute (i),(ii) into (iii): $\frac{1}{c}=\frac{c^2}{4}-\left(\frac{a-b}{2}\right)^2 \Rightarrow \left(\frac{a-b}{2}\right)^2=\frac{c^2}{4}-\frac{1}{c}=\frac{c^3-4}{4c} \geq 0$, which together with $c>0$ implies $c^3 \geq 4$. Hence, $c \geq \sqrt[3]{4}=\sqrt[3]{32/8}>\sqrt[3]{27/8}=3/2$.

1.42 ★★ Given $m+n+p=30$, $3m+n-p=50$, m, n, p are positive, and $x=5m+4n+2p$, find the range of x.

Solution: $(3m+n+p)-(m+n+p)=50-30 \Rightarrow m-p=10 \Rightarrow m=10+p>10$ since $p>0$.

$(3m+n+p)+(m+n+p)=50+30 \Rightarrow m+\frac{n}{2}=20 \Rightarrow m=20-\frac{n}{2}<20$ since $n>0$.

$n+p=30-m \Rightarrow 10<n+p<20$.

Hence, $x=5m+4n+2p=(4m+2n)+(m+n+p)+n+p=80+30+n+p=110+n+p \in (120,130)$.

1.43 ★★ Given a, b, c are real numbers and satisfy $a+b=4, 2c^2-ab=4\sqrt{3}c-10$, find the values of a, b, c.

Solution: $2c^2-ab=4\sqrt{3}c-10 \Rightarrow 2c^2-4\sqrt{3}c+10=ab=\left(\frac{a+b}{2}\right)^2-\left(\frac{a-b}{2}\right)^2=\left(\frac{4}{2}\right)^2-\left(\frac{a-b}{2}\right)^2=4-\left(\frac{a-b}{2}\right)^2 \Rightarrow 2c^2-4\sqrt{3}c+6+\left(\frac{a-b}{2}\right)^2=0 \Rightarrow 2(c-\sqrt{3})^2+\left(\frac{a-b}{2}\right)^2=0 \Rightarrow c=\sqrt{3}, a=b=2$.

1.44 ★ x is a real number, find the minimum value of $x-\sqrt{x-1}$ and its associated x value.

Solution: Let $y = x - \sqrt{x-1} \Rightarrow x - y = \sqrt{x-1} \Rightarrow x^2 - 2xy + y^2 = x - 1 \Rightarrow x^2 - (2y+1)x + y^2 + 1 = 0$ (i). $\Delta = (2y+1)^2 - 4(y^2+1) = 4y - 3 \geq 0 \Rightarrow y \geq 3/4$. Substitute the minimum value of y, 3/4, into (i): $x^2 - \frac{5}{2}x + \frac{25}{16} = 0 \Rightarrow \frac{1}{16}(16x^2 - 40x + 25) = 0 \Rightarrow (4x-5)^2 = 0 \Rightarrow x = 5/4$. Hence, when $x = 5/4$, $x - \sqrt{x-1}$ has the minimum value $3/4$.

1.45 ★ If a, b, c are nonzero real numbers, and $a + b + c = abc, a^2 = bc$, show $a^2 \geq 3$.

Proof: The conditions $a + b + c = abc, a^2 = bc$ imply that $b + c = abc - a = a^3 - a \Rightarrow$

$$\begin{aligned} b + c &= a^3 - a, \\ bc &= a^2. \end{aligned}$$

Hence, we can treat b, c as two roots of the quadratic equation $x^2 - (a^3 - a)x + a^2 = 0$. Since a, b, c are nonzero real numbers, we have

$\Delta = (a^3 - a)^2 - 4a^2 \geq 0 \Rightarrow a^2(a^2+1)(a^2-3) \geq 0 \Rightarrow a^2 - 3 \geq 0 \Rightarrow a^2 \geq 3$.

1.46 ★★ t is a positive integer, show 2 and 3 are not common factors of $t^2 - t + 1$ and $t^2 + t - 1$.

Proof: t is a positive integer, thus one of the two consecutive integers $t - 1$ and t should be an even number, then $t^2 - t = t(t-1)$ is even, then $t^2 - t + 1$ is odd. Same logic to get $t^2 + t$ is even, $t^2 + t - 1$ is odd. Hence, 2 is not a common factor of $t^2 - t + 1$ and $t^2 + t - 1$.

One of the three consecutive integers $t - 1, t, t + 1$ should be divisible by 3, thus at least one of $t^2 - t = t(t-1)$ and $t^2 + t = t(t+1)$ is divisible by 3. Therefore, at least one of $t^2 - t + 1$ and $t^2 + t - 1$ is not divisible by 3.

1.47 ★ If $3a - b + 2c = 8, a + 4b - c = 2$, evaluate $6a + 11b - c$.

Solution: Let $6a + 11b - c = m(3a - b + 2c) + n(a + 4b - c) = (3m+n)a + (4n-m)b + (2m-n)c$, then equaling the coefficients to obtain

$$\begin{aligned} 3m + n &= 6 \\ 4n - m &= 11 \\ 2m - n &= -1 \end{aligned}$$

$\Rightarrow m = 1, n = 3 \Rightarrow 6a + 11b - c = 1 \times 8 + 3 \times 2 = 14.$

1.48 ★★ Given $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{1}{x+y+z}=1$, show $x=1$ or $y=1$ or $z=1$.

Proof: $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=1 \Rightarrow xy+xz+yz=xyz$ (i);

$\frac{1}{x+y+z}=1 \Rightarrow x+y+z=1 \Rightarrow x=1-y-z$ (ii).

Substitute (ii) into (i): $(1-y-z)y+(1-y-z)z+yz=(1-y-z)yz \Rightarrow (z-1)(y+z)(y-1)=0$ (iii). (ii) is equivalent to $y+z=1-x$ and substitute it into (iii): $(z-1)(1-x)(y-1)=0$, therefore $x=1$ or $y=1$ or $z=1$.

1.49 ★ Show $1+2+2^2+\cdots+2^{5n-1}$ is divisible by 31.

Proof: $1+2+2^2+\cdots+2^{5n-1} = \frac{1-2^{5n}}{1-2} = 2^{5n}-1 = 32^n-1 = (31+1)^n-1 = C_n^0 31^n + C_n^1 31^{n-1}+\cdots+C_n^{n-1}31+C_n^n-1 = 31(C_n^0 31^{n-1}+C_n^1 31^{n-2}+\cdots+C_n^{n-1})$ which is obviously divisible by 31.

1.50 ★ The real numbers a,b,c satisfy $\frac{a}{b}=\frac{b}{c}$, and x,y are mean values of a,b and b,c respectively. Show $\frac{a}{x}+\frac{c}{y}=2$.

Proof: $\frac{a}{b}=\frac{b}{c}\Rightarrow\frac{a}{a+b}=\frac{b}{b+c}$. Since $x=\frac{a+b}{2},y=\frac{b+c}{2}$,

we have $\frac{a}{x}+\frac{c}{y}=\frac{a}{(a+b)/2}+\frac{c}{(b+c)/2}=\frac{2a}{a+b}+\frac{2c}{b+c}=\frac{2b}{b+c}+\frac{2c}{b+c}=2$.

1.51 ★ Given $abc=1$, evaluate $\frac{2012}{1+a+ab}+\frac{2012}{1+b+bc}+\frac{2012}{1+c+ca}$.

Solution:
$$\begin{aligned}&\frac{2012}{1+a+ab}+\frac{2012}{1+b+bc}+\frac{2012}{1+c+ca}\\&=2012\left(\frac{1}{1+a+\frac{1}{c}}+\frac{1}{1+\frac{1}{ac}+\frac{1}{ac}c}+\frac{1}{1+c+ca}\right)\\&=2012\left(\frac{c}{c+ac+1}+\frac{ac}{ac+1+c}+\frac{1}{1+c+ca}\right)=2012\cdot\end{aligned}$$

1.52 ★ Given $x^7+x^6+x=-1$, evaluate $x^{2000}+x^{2001}+\cdots+x^{2012}$.

Solution: $x^7+x^6+x=-1\Rightarrow x^6(x+1)+(x+1)=0\Rightarrow(x+1)(x^6+1)=0\Rightarrow x=-1$ since

$x^6+1>0$. Hence, $x^{2000}+x^{2001}+\cdots+x^{2012}=\underbrace{1+1+\cdots+1}_{\text{seven } 1's}+\underbrace{(-1)+(-1)+\cdots+(-1)}_{\text{six } (-1)'s}=1$.

1.53 ★★ Given $a<b<c$, determine ± sign of $\frac{1}{a-b}+\frac{1}{b-c}+\frac{1}{c-a}$.

Solution: Let $a-b=x,b-c=y,c-a=z$. Since $a<b<c$, we have $x<0,y<0,z>0$.

$x+y+z=a-b+b-c+c-a=0\Rightarrow(x+y+z)^2=0\Rightarrow 2(xy+yz+zx)=$

$-(x^2+y^2+z^2)\Rightarrow xy+yz+zx<0$. Therefore

$\frac{1}{a-b}+\frac{1}{b-c}+\frac{1}{c-a}=\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{yz+zx+xy}{xyz}<0$, that is, $\frac{1}{a-b}+\frac{1}{b-c}+\frac{1}{c-a}$ is negative.

1.54 ★ Factor $(a+b-2ab)(a-2+b)+(1-ab)^2$.

Solution: Let $a+b=x,ab=y$, then

$(a+b-2ab)(a-2+b)+(1-ab)^2=(x-2y)(x-2)+(1-y)^2=x^2-2x-2xy+4y+1-2y+y^2=x^2-2x(y+1)+(y+1)^2=(x-y-1)^2=(a+b-ab-1)^2=[(a-1)-b(a-1)]^2=(a-1)^2(b-1)^2$

1.55 ★ The real numbers m,n,p are not all equal, and $x=m^2-np,y=n^2-pm,z=p^2-mn$

Show at least one of x,y,z is positive.

Proof: $2(x+y+z)=2(m^2+n^2+p^2-mn-np-pm)=(m-n)^2+(n-p)^2+(p-m)^2\geq 0$ In addition, since m,n,p are not all equal, then $m-n,n-p,p-m$ are not all zeros. Thus $x+y+z>0$, which shows at least one of x,y,z is positive.

1.56 ★ a,b,c are nonzero real numbers, and $a+b+c=0$,

evaluate $\frac{1}{b^2+c^2-a^2}+\frac{1}{c^2+a^2-b^2}+\frac{1}{a^2+b^2-c^2}$.

Solution: $a+b+c=0\Rightarrow b+c=-a\Rightarrow (b+c)^2=a^2\Rightarrow b^2+c^2-a^2=-2bc$. Similarly, we can obtain $a^2+b^2-c^2=-2ab, c^2+a^2-b^2=-2ca$. In addition, $-2bc, -2ab, -2ca$ are nonzero. Hence, $\frac{1}{b^2+c^2-a^2}+\frac{1}{c^2+a^2-b^2}+\frac{1}{a^2+b^2-c^2}=-\frac{1}{2bc}-\frac{1}{2ca}-\frac{1}{2ab}=-\frac{a+b+c}{2abc}=0$.

1.57 ★ Find the minimum value of the fraction $\frac{3x^2+6x+5}{\frac{1}{2}x^2+x+1}$.

Solution: $\frac{3x^2+6x+5}{\frac{1}{2}x^2+x+1}=\frac{6x^2+12x+10}{x^2+2x+2}=\frac{6(x^2+2x+2)-2}{x^2+2x+2}=6-\frac{2}{(x+1)^2+1}$ which has the minimum value 4 when $x=-1$.

1.58 ★★ For real numbers x,y, define the operator $x*y=ax+by+cxy$ where a,b,c are constants. We know that $1*2=3, 2*3=4$, and there is a nonzero real number d such that $x*d=x$ holds for any real number x. Find the value of d.

Solution: For any real number x, we have $x*d=ax+bd+cdx=x$, $0*d=bd=0$. Since $d\neq 0$, then $b=0$, thus

$$\begin{aligned} 1*2 &= a+2b+2c=3 \\ 2*3 &= 2a+3b+6c=4 \end{aligned}$$

$\Rightarrow$

$$\begin{aligned} a+2c &= 3 \\ 2a+6c &= 4 \end{aligned}$$

which results in $a=5, c=-1$. In addition, $1*d=a+bd+cd=1$, and substitute $a=5,b=0,c=-1$ into it to obtain $d=4$.

1.59 ★★ Show for any positive integer N, we can find two positive integers a and b such that $N=\frac{b-2a+1}{a^2-b}$.

Proof: $N=\frac{b-2a+1}{a^2-b}=\frac{-a^2+b+a^2-2a+1}{a^2-b}=\frac{-(a^2-b)+(a-1)^2}{a^2-b}\Rightarrow(N+1)(a^2-b)=(a-1)^2$.

Choose $N+1=a-1$, then $a^2-b=a-1$. Thus $a=N+2$,

$b=a^2-a+1=(N+2)^2-(N+2)+1=N^2+4n+4-N-2+1=N^2+3N+3$.

Since N is a positive integer, then a,b are are positive integers as well.

1.60 ★★ Given $x\neq 0$, find the maximum value of $\dfrac{\sqrt{1+x^2+x^4}-\sqrt{1+x^4}}{x}$.

Solution: $\dfrac{\sqrt{1+x^2+x^4}-\sqrt{1+x^4}}{x}=\dfrac{x}{\sqrt{1+x^2+x^4}+\sqrt{1+x^4}}$

$=\dfrac{x}{|x|(\sqrt{x^2+\frac{1}{x^2}+1}+\sqrt{x^2+\frac{1}{x^2}})}=\dfrac{x}{|x|(\sqrt{(x-\frac{1}{x})^2+3}+\sqrt{(x-\frac{1}{x})^2+2})}$ whose maximum

value is $\dfrac{1}{\sqrt{3}+\sqrt{2}}=\sqrt{3}-\sqrt{2}$ when $x=\frac{1}{x}>0$.

1.61 ★★ Given $\dfrac{1}{4}(b-c)^2=(a-b)(c-a)$, $a\neq 0$, evaluate $\dfrac{b+c}{a}$.

Solution 1: When $a=b$, we have $(b-c)^2=0 \Rightarrow b=c$, then $\frac{b+c}{a}=\frac{b+b}{b}=2$.

When $a \neq b$, the given equality becomes
$(b-c)^2=4(a-b)(c-a) \Rightarrow (b+c)^2-4a(b+c)+4a^2=0 \Rightarrow [(b+c)-2a]^2=0 \Rightarrow$
$b+c=2a \Rightarrow \frac{b+c}{a}=2$.

Solution 2: When $a=b$, it is the same as solution 1.

When $a \neq b$, $(b-c)^2-4(a-b)(c-a)=0$. Treat this as the discriminant of the quadratic equation $(a-b)x^2+(b-c)x+(c-a)=0$. Since the sum of all coefficients is 0, then 1 is a root of this quadratic equation. Since $\Delta=(b-c)^2-4(a-b)(c-a)=0$, then $x_1=x_2=1$. Vieta's formulas implies that $x_1x_2=\frac{c-a}{a-b}=1 \Rightarrow b+c=2a \Rightarrow \frac{b+c}{a}=2$.

1.62 ★★ Find the minimum positive integer A and the corresponding positive integer B such that (1) A is divisible by 200 and its quotient divided by 19 has a remainder of 2, divided by 23 has a remainder of 10; (2) $B>A$, $B-A$ has four digits and is divisible by 3,4,17,25.

Solution: (1) $\Rightarrow A/200=19U+2=23V+10$ where U,V are positive integers, then $U=V+\frac{4V+8}{19}$. Since U is a positive integer, then $\frac{4V+8}{19}$ is a positive integer. Let $\frac{4V+8}{19}=p$, then $V=4p-2+\frac{3}{4}p$ in which $\frac{3}{4}p$ should be a positive integer. Since 3 and 4 are coprime, then $p=4n$ (n is a positive integer). To have minimum A, we choose $n=1, p=4, V=17$, then $A=200(23\times 17+10)=80200$.

According to (2) and since 3,4,17,25 are coprime, then $B-A$ should be $3\times 4\times 17\times 25\times k=5100k$ (k is a positive integer). In addition, $B-A$ is a four-digit number, thus $k=1$. Hence, $B=A+5100=85300$.

1.63 ★★ If the real numbers x,y,z satisfy $x+y+z=a$, $x^2+y^2+z^2=a^2/2$ $(a>0)$, show x,y,z are nonnegative and not greater than $2a/3$.

Proof: $x+y+z=a \Rightarrow z=a-(x+y)$ and substitute it into $x^2+y^2+z^2=a^2/2$ to obtain $x^2+y^2+(x+y)^2-2a(x+y)+a^2=a^2/2 \Rightarrow x^2+y^2+xy-ax-ay+a^2/4=0 \Rightarrow$ $y^2+(x-a)y+(x-a/2)^2=0$. Since x, y are real numbers, then $\Delta=(x-a)^2-4(x-a/2)^2$ $\geq 0 \Rightarrow x(2a-3x) \geq 0 \Rightarrow 0 \leq x \leq 2a/3$ Similarly, we can show $0 \leq y \leq 2a/3$, $0 \leq z \leq 2a/3$. Therefore x, y, z are nonnegative and not greater than $2a/3$.

1.64 ★★ Two real numbers x, y satisfy $x^3+y^3=2$. Find the maximum value of $x+y$.

Solution: Let $x+y=t$. $x^3+y^3=2 \Rightarrow (x+y)[(x+y)^2-3xy]=2 \Rightarrow t(t^2-3xy)=2 \Rightarrow xy=\frac{t^3-2}{3t}$ Thus we can treat x, y as the two roots of the quadratic equation $u^2-tu+\frac{t^3-2}{3t}=0$, then $\Delta=t^2-\dfrac{4t^3-8}{3t} \geq 0 \Rightarrow \dfrac{-t^3+8}{3t} \geq 0 \Rightarrow 0<t \leq 2 \Rightarrow 0<x+y \leq 2$. Hence, the maximum value of $x+y$ is 2.

1.65 ★ Write $\dfrac{x+4}{x^3+2x-3}$ as partial fractions.

Solution: $x=1$ is a root of the cubic equation $x^3+2x-3=0$, thus $x-1$ is a factor of x^3+2x-3 . Use polynomial long division to obtain the other factor x^2+x+3. Let $\dfrac{x+4}{x^3+2x-3}=\dfrac{A}{x-1}+\dfrac{Bx+c}{x^2+x+3}=\dfrac{(A+B)x^2+(A-B+2C)x+3A-C}{x^3+2x-3}$. Make the coefficients equal to obtain

$$\begin{aligned} A+B &= 0 \\ A-B+C &= 1 \\ 3A-C &= 4 \end{aligned}$$

$\Rightarrow A=1, B=-1, C=-1$. Hence, $\dfrac{x+4}{x^3+2x-3}=\dfrac{1}{x-1}-\dfrac{x+1}{x^2+x+3}$.

1.66 ★★ Show $a^3+\frac{3}{2}a^2+\frac{1}{2}a-1$ is an integer for any positive integer a, and it has a remainder of 2 when divided by 3.

Proof: $a^3+\frac{3}{2}a^2+\frac{1}{2}a-1=\frac{2a^3+3a^2+a}{2}-1=\frac{a(a+1)(2a+1)}{2}-1$. For any positive integer a, $\dfrac{a(a+1)}{2}$ is an integer, thus $a^3+\frac{3}{2}a^2+\frac{1}{2}a-1$ is an integer.
$a^3+\frac{3}{2}a^2+\frac{1}{2}a-1=\frac{a(a+1)(2a+1)}{2}-1=\frac{2a(2a+1)(2a+2)}{8}-1$. One of $2a, 2a+1, 2a+2$ is a multiple of 3. Since 3 and 8 are coprime, then $\dfrac{2a(2a+1)(2a+2)}{8}$ is divisible by 3. Hence the original expression is a multiple of 3 minus 1, i.e. it has a remainder of 2 when divided by 3.

1.67 ★★ x, y, z are real numbers, $3x, 4y, 5z$ follow a geometric sequence, and $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$ follow an arithmetic sequence, find the value of $\frac{x}{z} + \frac{z}{x}$.

Solution:

$$\begin{aligned} (4y)^2 &= 15xz \qquad \text{(i)} \\ \frac{2}{y} &= \frac{1}{x} + \frac{1}{z} \qquad \text{(ii)} \end{aligned}$$

(ii) $\Rightarrow$ $y = \frac{2xz}{x+z}$, substitute it into (i):

$$16\left(\frac{2xz}{x+z}\right)^2 = 15xz \Rightarrow \frac{(x+z)^2}{xz} = \frac{64}{15} \Rightarrow \frac{x}{z} + 2 + \frac{z}{x} = \frac{64}{15} \Rightarrow \frac{x}{z} + \frac{z}{x} = \frac{34}{15}.$$

1.68 ★★ Given $0 < a < 1$ and $[a + \frac{1}{50}] + [a + \frac{2}{50}] + \cdots + [a + \frac{39}{50}] = 6$, evaluate $[50a]$. Here $[*]$ means the integer part of $*$.

Solution: $0 < a + \frac{1}{50} < a + \frac{2}{50} < \cdots < a + \frac{39}{50} < 2$, thus $[a + \frac{1}{50}], [a + \frac{2}{50}], \cdots, [a + \frac{39}{50}]$ are equal to 0 or 1. The condition $[a + \frac{1}{50}] + [a + \frac{2}{50}] + \cdots + [a + \frac{39}{50}] = 6$ implies that six of $[a + \frac{1}{50}], [a + \frac{2}{50}], \cdots, [a + \frac{39}{50}]$ are equal to 1. Hence, $[a + \frac{1}{50}] = [a + \frac{2}{50}] = \cdots = [a + \frac{33}{50}] = 0$ and $[a + \frac{34}{50}] = [a + \frac{35}{50}] = \cdots = [a + \frac{39}{50}] = 1$. Then $0 < a + \frac{33}{50} < 1$ and $1 \le a + \frac{34}{50} < 2$, which lead to $16 \le 50a < 17 \Rightarrow [50a] = 16$.

1.69 ★ Factoring $x^3 + y^3 + z^3 - 3xyz$.

Solution:

$x^3 + y^3 + z^3 - 3xyz = x^3 + 3x^2y + 3xy^2 + y^3 + z^3 - 3x^2y - 3xy^2 - 3xyz = (x + y)^3 + z^3 - 3xy(x + y + z) = (x + y + z)[(x + y)^2 - (x + y)z + z^2] - 3xy(x + y + z) = (x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx)$.

1.70 ★★ a, b, c are prime numbers, c is a one-digit number, and $ab + c = 1993$, evaluate

$a + b + c$.

Solution: The right hand side of $ab + c = 1993$ is an odd number, thus one of ab and c is an even number. If c is an even prime number which has to be 2, then $ab = 1993 - 2 = 1991 = 11 \times 181$, then one of a, b is 11, and the other one is 181. If ab is an even number, let b be the even prime number 2, then $2a + c = 1993$. Since c is a prime number, then $c = 3$, 5, or 7, and $a = 995$, 944, or 993, all of which are not prime numbers. Hence $a + b + c = 11 + 181 + 2 = 194$.

1.71 ★★ Find the minimum positive fraction such that it is an integer when divided by $54/175$ or multiplied by $55/36$.

Solution: Let the minimum positive fraction be y/x, where x, y are coprime positive integers, then $\frac{y}{x} \div \frac{54}{175} = \frac{y}{x} \times \frac{175}{54}$ and $\frac{y}{x} \times \frac{55}{36}$ are both integers. Thus $175/54$ and $55/36$ are irreducible fractions, then x is a common divisor of 175 and 55, and y is the smallest common multiple of 54 and 36. To minimize y/x, we should maximize x and minimize y, then x should be the largest common divisor of $175 = 5^2 \times 7$ and $55 = 5 \times 11$, which is 5, and y should be the smallest common multiple of $54 = 2 \times 3^3$ and $36 = 2^2 \times 3^2$, which is $2^2 \times 3^3 = 108$. Therefore, the minimum positive fraction $y/x = 108/5$.

1.72 ★★ a, b, c, d are positive integers, and $a^5 = b^4, c^3 = d^2, c - a = 11$, evaluate $d - b$.

Solution: Let $a^5 = b^4 = t^{20}$ where t is a positive integer, then $a = t^4, b = t^5$. Let $c^3 = d^2 = p^6$ where p is a positive integer, then $c = p^2, d = p^3$. In addition, $c - a = 11$, then $p^2 - t^4 = 11 \Rightarrow (p - t^2)(p + t^2) \Rightarrow p - t^2 = 1, p + t^2 = 11 \Rightarrow p = 6, t = \sqrt{5}, b = (\sqrt{5})^5 = 25\sqrt{5}, d = 6^3 = 216, d - 6 = 216 - 25\sqrt{5}$.

1.73 ★★ Given $\frac{x+y-z}{z} = \frac{x-y+z}{y} = \frac{y+z-x}{x}$ and $xyz \neq 0$, evaluate $\frac{(x+y)(y+z)(z+x)}{xyz}$.

Solution: Let $\frac{x+y-z}{z} = \frac{x-y+z}{y} = \frac{y+z-x}{x} = k$, then $x + y - z = kz$ (i), $x - y + z = ky$ (ii), $y + z - x = kx$ (iii). (i)+(ii)+(iii): $x + y + z = k(x + y + z) \Rightarrow (k-1)(x+y+z) = 0$. There are two possibilities $k = 1$ or $x + y + z = 0$. When $k = 1$, then $x + y = 2z, x + z = 2y, y + z = 2x$, then $\frac{(x+y)(y+z)(z+x)}{xyz} = \frac{2z \cdot 2x \cdot 2y}{xyz} = 8$. When $x + y + z = 0$, then $x + y = -z, y + z = -x, z + x = -y$, then $\frac{(x+y)(y+z)(z+x)}{xyz} = \frac{(-z) \cdot (-x) \cdot (-y)}{xyz} = -1$. As a conclusion, $\frac{(x+y)(y+z)(z+x)}{xyz}$ is 8 or -1.

1.74 ★★★ Let

$S = \sqrt{1 + \frac{1}{1^2} + \frac{1}{2^2}} + \sqrt{1 + \frac{1}{2^2} + \frac{1}{3^2}} + \sqrt{1 + \frac{1}{3^2} + \frac{1}{4^2}} + \cdots + \sqrt{1 + \frac{1}{2010^2} + \frac{1}{2011^2}}$, find the integer part of S.

Solution: According to the rule in the terms of S, we obtain the general formula

$a_n = \sqrt{1+\frac{1}{n^2}+\frac{1}{(n+1)^2}} = \sqrt{(1+\frac{1}{n})^2-\frac{2}{n}+\frac{1}{(n+1)^2}} = \sqrt{(\frac{n+1}{n})^2-2\frac{n+1}{n}\frac{1}{n+1}+\frac{1}{(n+1)^2}} =$
$\sqrt{(\frac{n+1}{n}-\frac{1}{n+1})^2} = \frac{n+1}{n}-\frac{1}{n+1} = 1+\frac{1}{n}-\frac{1}{n+1}$. Thus
$S=(1+\frac{1}{1}-\frac{1}{2})+(1+\frac{1}{2}-\frac{1}{3})+(1+\frac{1}{3}-\frac{1}{4})+\cdots+(1+\frac{1}{2009}-\frac{1}{2010})+(1+\frac{1}{2010}-\frac{1}{2011}) = 2010\,\frac{2010}{2011}$

$\in (2010, 2011)$ which implies that S has the integer part 2010.

1.75 ★★★ Given $x=\frac{\sqrt{5}+1}{2}$, evaluate $\dfrac{x^3+x+1}{x^5}$.

Solution: Let $y=\frac{\sqrt{5}-1}{2}$, then $xy=1, x-y=1$.

$$\frac{x^3+x+1}{x^5} = \frac{x^3+x+xy}{x^5} = \frac{x^2+1+y}{x^4} = \frac{x^2+x-y+y}{x^4} = \frac{x+1}{x^3} = \frac{x+xy}{x^3} =$$
$$\frac{1+y}{x^2} = \frac{x}{x^2} = \frac{1}{x} = y = \frac{\sqrt{5}-1}{2}.$$

1.76 ★★★ M is a 2000-digit number and a multiple of 9. M_1 is the sum of all digits of M, M_2 is the sum of all digits of M_1, and M_3 is the sum of all digits of M_2. Find the value of M_3.

Solution: Obviously M_1, M_2, M_3 are multiples of 9. Since M has 2000 digits, the sum of all its digits $M_1 \le 9 \times 2000 = 18000$, then M_1 has at most five digits and the first digit is 0 or 1. Thus $M_2 \le 1 + 4 \times 9 = 37$, which implies that M_2 has at most two digits and the first digit is less than or equal to 3. Thus $M_3 \le 3 + 9 = 12$. In addition, M_3 is divisible by 9 and $M_3 \ne 0$, hence $M_3 = 9$.

1.77 ★★ Let x, y be two distinct positive integers, and $\frac{1}{x} + \frac{1}{y} = \frac{2}{5}$, evaluate $\sqrt{x+y}$.

Solution: Set $\frac{1}{x} = \frac{2a}{5(a+b)}, \frac{1}{y} = \frac{2b}{5(a+b)}$ where a, b are positive integers and coprime, and let $a > b$. Then $2x = \frac{5(a+b)}{a} = 5 + 5 \times \frac{b}{a}$. Since x is a positive integer, then $\frac{b}{a} = \frac{1}{5}$, thus $2x = 6 \Rightarrow x = 3$. On the other hand, $2y = \frac{5(a+b)}{b} = 5 + 5 \times \frac{a}{b}$. Since y is a positive integer, then $\frac{a}{b} = 5$, thus $y = 15$. Therefore, $\sqrt{x+y} = \sqrt{18} = 3\sqrt{2}$.

1.78 ★★ The positive integers a, b, c satisfy $a^2 + 3b^2 + 3c^2 + 13 < 2ab + 4b + 12c$, find the value of $a + b + c$.

Solution: $a^2 + 3b^2 + 3c^2 + 13 < 2ab + 4b + 12c \Rightarrow a^2 + 3b^2 + 3c^2 + 13 - 2ab - 4b - 12c + 1 < 1 \Rightarrow (a-b)^2 + 2(b-1)^2 + 3(c-2)^2 < 1$. $(a-b)^2 \ge 0, (b-1)^2 \ge 0, (c-2)^2 \ge 0$, and a, b, c are positive integers, thus $a = b = 1, c = 2$, hence $a + b + c = 4$.

1.79 ★★★ Find the minimum positive integer n that is a multiple of 75 and has 75 positive integer factors (including 1 and itself).

Solution: $n = 75k = 3 \times 5^2 k$ where k is a positive integer. To minimize n, let $n = 2^\alpha \cdot 3^\beta \cdot 5^\gamma$ $(\gamma \ge 2, \beta \ge 1)$, and $(\alpha+1)(\beta+1)(\gamma+1) = 75$, from which $\alpha+1, \beta+1, \gamma+1$ are all odd numbers, thus α, β, γ are all even numbers. Then $\gamma = 2$, and $(\alpha+1)(\beta+1) = 25 = 5^2 = 1 \times 25$.

1) If $\alpha + 1 = 5, \beta + 1 = 5$, then $\alpha = 4, \beta = 4$, thus $n = 2^4 \cdot 3^4 \cdot 5^2$.

2) If $\alpha + 1 = 1, \beta + 1 = 25$, then $\alpha = 0, \beta = 24$, thus $n = 2^0 \cdot 3^{24} \cdot 5^2$.

According to (1)(2), the minimum positive integer $n = 2^4 \cdot 3^4 \cdot 5^2 = 32400$

1.80 ★★★ Given the sets $M = \{x, xy, \lg(xy)\}, N = \{0, |x|, y\}$, and $M = N$, evaluate $(x + \frac{1}{y}) + (x^2 + \frac{1}{y^2}) + (x^3 + \frac{1}{y^3}) + \cdots + (x^{2001} + \frac{1}{y^{2001}})$.

Solution: $M = N$ implies that one element in M should be 0. The existence of $\lg(xy)$ implies that $xy \neq 0$, thus x, y cannot be 0. Hence, $\lg(xy) = 0 \Rightarrow xy = 1 \Rightarrow y = \frac{1}{x}$.

Thus $M = \{x, 1, 0\}, N = \{0, |x|, \frac{1}{x}\}$. According to $M = N$ again, we have either

$$\begin{aligned} x &= |x| \\ 1 &= \frac{1}{x} \end{aligned}$$

or

$$\begin{aligned} x &= \frac{1}{x} \\ 1 &= |x| \end{aligned}$$

However, $x = 1$ violates the uniqueness of every element in a set. Hence, $x = -1, y = -1$. Then $x^{2k+1} + \frac{1}{y^{2k+1}} = -2 \ (k = 0, 1, 2, \cdots)$; $x^{2k} + \frac{1}{y^{2k}} = 2 \ (k = 1, 2, \cdots)$. In the original summation, the number of $2k+1$ terms is one more than the number of $2k$ terms, therefore $(x + \frac{1}{y}) + (x^2 + \frac{1}{y^2}) + (x^3 + \frac{1}{y^3}) + \cdots + (x^{2001} + \frac{1}{y^{2001}}) = -2$.

1.81 ★★ Find the integer part of the number $(\sqrt{7} + \sqrt{5})^6$.

Solution: Let $\sqrt{7} + \sqrt{5} = x, \sqrt{7} - \sqrt{5} = y$, then

$x + y = 2\sqrt{7}, xy = 2 \Rightarrow x^2 + y^2 = (x+y)^2 - 2xy = (2\sqrt{7})^2 - 2 \times 2 = 24 \Rightarrow$. $x^6 + y^6 = (x^2)^3 + (y^2)^3 = (x^2 + y^2)(x^4 - x^2y^2 + y^4) = (x^2 + y^2)[(x^2 + y^2)^2 - 3x^2y^2] = 24[24^2 - 3 \times 4] = 24 \times 564 = 13536$

Hence, $(\sqrt{7} + \sqrt{5})^6 + (\sqrt{7} - \sqrt{5})^6 = 13536$. Since $0 < \sqrt{7} - \sqrt{5} < 1$,

then $13535 < (\sqrt{7} + \sqrt{5})^6 < 13536$, which implies that the integer part of $(\sqrt{7} + \sqrt{5})^6$ is 13535.

1.82 ★★ The real numbers x, y satisfy $4x^2 - 5xy + 4y^2 = 5$, let $S = x^2 + y^2$, evaluate $\frac{1}{S_{min}} + \frac{1}{S_{max}}$.

Solution 1: Let $x = a + b, y = a - b$ and substitute into $4x^2 - 5xy + 4y^2 = 5$: $4(a+b)^2 - 5(a+b)(a-b) + 4(a-b)^2 = 5 \Rightarrow 3a^2 + 13b^2 = 5 \Rightarrow b^2 = \frac{5 - 3a^2}{13} \geq 0 \Rightarrow$ $5 - 3a^2 \geq 0 \Rightarrow 0 \leq a^2 \leq \frac{5}{3}$. Therefore $S = (a+b)^2 + (a-b)^2 = 2a^2 + 2b^2 = 2a^2 + \frac{10 - 6a^2}{13} = \frac{20}{13}a^2 + \frac{10}{13}$

When $a = 0$, $S_{min} = \frac{10}{13}$. When $a^2 = \frac{5}{3}$, $S_{max} = \frac{10}{3}$. Hence, $\frac{1}{S_{min}} + \frac{1}{S_{max}} = \frac{13}{10} + \frac{3}{10} = \frac{8}{5}$.

Solution 2: Obviously $S = x^2 + y^2 > 0$ (since x, y cannot be both zero due to $4x^2 - 5xy + 4y^2 = 5$). Let $x = \sqrt{S}\cos\theta, y = \sqrt{S}\sin\theta$, and substitute into $4x^2 - 5xy + 4y^2 = 5$:

$4S\cos^2\theta - 5S\cos\theta\sin\theta + 4S\sin^2\theta = 5 \Rightarrow 4S - \frac{5}{2}\sin 2\theta = 5 \Rightarrow \sin 2\theta = \frac{8S-10}{5S}$. Since $|\sin 2\theta| \le 1$, then $\left|\frac{8S-10}{5S}\right| \le 1 \Rightarrow -1 \le \frac{8S-10}{5S} \le 1 \Rightarrow \frac{10}{13} \le S \le \frac{10}{3}$.

1.83 ★★★ For a positive integer n, find the integer part of $(\sqrt{n^2+2n}+n)^2$.

Solution: For a positive integer n, we have $n^2 < n^2 + 2n < n^2 + 2n + 1 = (n+1)^2 \Rightarrow n < \sqrt{n^2+2n} < n+1 \Rightarrow 0 < \sqrt{n^2+2n} - n < 1$. Let $x = (\sqrt{n^2+2n}+n)^2, y = (\sqrt{n^2+2n}-n)^2$, then

$x + y = (\sqrt{n^2+2n}+n)^2 + (\sqrt{n^2+2n}-n)^2 = 4n^2 + 4n$. Since $0 < \sqrt{n^2+2n} - n < 1$,

then $0 < (\sqrt{n^2+2n}-n)^2 < 1$, then

$(\sqrt{n^2+2n}+n)^2 = 4n^2 + 4n - (\sqrt{n^2+2n}-n)^2 \in (4n^2+4n-1, 4n^2+4n)$, thus the integer

part of $(\sqrt{n^2+2n}+n)^2$ is $4n^2+4n-1$.

1.84 ★★★ The positive real numbers p, q satisfy $p^2 + q^2 = 7pq$ and make the polynomial $xy + px + qy + 1$ of x, y be factored into a product of two first-order polynomials, find the values of p, q.

Solution: $p > 0, q > 0, p^2 + q^2 = 7pq \Rightarrow p^2 + 2pq + q^2 = 9pq \Rightarrow p + q = 3\sqrt{pq}$. Since the polynomial $xy + px + qy + 1$ can be factored into a product of two first-order polynomials, we have $xy + px + qy + 1 = (ax + b)(cy + d) = acxy + adx + bcy + bd$. Make the corresponding coefficients equal: $ac = 1, bd = 1, ad = p, bc = q$, thus $pq = abcd = 1, p + q = 3\sqrt{pq} = 3 \Rightarrow p = \frac{3+\sqrt{5}}{2}, q = \frac{3-\sqrt{5}}{2}$ or $p = \frac{3-\sqrt{5}}{2}, q = \frac{3+\sqrt{5}}{2}$.

1.85 ★★ The positive integers a, b, c satisfy $a^2 + b^2 + c^2 + 3 < ab + 3b + 2c$, find the values of a, b, c.

Solution: a, b, c are positive integers, thus $a^2 + b^2 + c^2 + 3$ and $ab + 3b + 2c$ are both integers, then $a^2 + b^2 + c^2 + 3 < ab + 3b + 2c \Rightarrow a^2 + b^2 + c^2 + 4 \leq ab + 3b + 2c \Rightarrow a^2 - ab + \frac{b^2}{4} + \frac{3b^2}{4} - 3b + 3 + c^2 - 2c + 1 \leq 0 \Rightarrow (a - \frac{b}{2})^2 + \frac{3}{4}(b-2)^2 + (c-1)^2 \leq 0 \Rightarrow a - \frac{b}{2} = 0, b - 2 = 0, c - 1 = 0 \Rightarrow a = 1, b = 2, c = 1$.

1.86 ★★★ The positive integers m, n satisfy $(11111 + m)(11111 - n) = 123456789$, show $m - n$ is a multiple of 4.

Proof: Since 123456789 is an odd number, then $11111 + m$ and $11111 - n$ are odd numbers, then m, n are both even numbers.
$(11111 + m)(11111 - n) = 123456789 \Leftrightarrow 11111 \times 11111 - 11111n + 11111m - mn = 123456789 \Leftrightarrow 11111(m - n) = mn + 2468$. Since mn is a multiple of 4 and $2468 = 4 \times 617$ is also a multiple of 4, then $11111(m - n)$ is a multiple of 4. In addition, since 11111 and 4 are coprime, then $m - n$ is a multiple of 4.

1.87 ★★★ The nonzero real numbers a, b, c, x, y, z satisfy $\frac{x}{a} = \frac{y}{b} = \frac{z}{c}$, evaluate $\frac{xyz(a+b)(b+c)(c+a)}{abc(x+y)(y+z)(z+x)}$.

Solution: Let $\frac{x}{a}=\frac{y}{b}=\frac{z}{c}=t \Rightarrow \frac{xyz}{abc}=t^3$, and
$\frac{x+y}{a+b}=\frac{y+z}{b+c}=\frac{z+x}{c+a}=t \Rightarrow \frac{(x+y)(y+z)(z+x)}{(a+b)(b+c)(c+a)}=t^3 \Rightarrow \frac{(a+b)(b+c)(c+a)}{(x+y)(y+z)(z+x)}=\frac{1}{t^3}$.

Hence,

$$\frac{xyz(a+b)(b+c)(c+a)}{abc(x+y)(y+z)(z+x)}=t^3\frac{1}{t^3}=1.$$

1.88 ★★★ Given $2007x^2=2009y^2=2011z^2, x>0, y>0, z>0$, and $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=1$, show $\sqrt{2007x+2009y+2011z}=\sqrt{2007}+\sqrt{2009}+\sqrt{2011}$.

Proof: Let $2007x^2=2009y^2=2011z^2=k \ \ (k>0)$, then
$2007x=k/x, 2009y=k/y, 2011z=k/z$. Since $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=1$, then
$\sqrt{2007x+2009y+2011z}=\sqrt{k/x+k/y+k/z}=\sqrt{k(1/x+1/y+1/z)}=\sqrt{k}$.

On the other hand,
$2007=k/x^2, 2009=k/y^2, 2011=k/z^2 \Rightarrow \sqrt{2007}+\sqrt{2009}+\sqrt{2011}=\sqrt{k}/x+\sqrt{k}/y+\sqrt{k}/z=\sqrt{k}$. Hence the aimed equality holds.

1.89 ★★ If x,y,z are nonzero real numbers, and $x+y+z=xyz, x^2=yz$, show $x^2\geq 3$.

Proof: $x+y+z=xyz \Leftrightarrow y+z=xyz-x=x^3-x$ since $yz=x^2$. Then we can treat y,z as two roots of the quadratic equation $u^2-(x^3-x)u+x^2=0$.. Since x,y,z are real numbers, then
$\Delta=(x^3-x)^2-4x^2\geq 0 \Rightarrow x^6-2x^4-3x^2\geq 0 \Rightarrow x^2(x^4-2x^2-3)\geq 0 \Rightarrow x^2(x^2+1)(x^2-3)\geq 0$. Since $x\neq 0$, then $x^2>0, x^2+1>0$, thus $x^2-3\geq 0$, i.e. $x^2\geq 3$.

1.90 ★★ Given a,b,c are nonzero real numbers, and $a^2+b^2+c^2=1$, $a\left(\frac{1}{b}+\frac{1}{c}\right)+b\left(\frac{1}{c}+\frac{1}{a}\right)+c\left(\frac{1}{a}+\frac{1}{b}\right)+3=0$, find all possible values of $a+b+c$.

Solution:
$a(\frac{1}{b}+\frac{1}{c})+b(\frac{1}{c}+\frac{1}{a})+c(\frac{1}{a}+\frac{1}{b})+3=0 \Rightarrow \frac{ac+ab}{bc}+\frac{ab+bc}{ac}+\frac{bc+ca}{ab}+3=0 \Rightarrow (a+b+c)(ab+bc+ca)=0 \Rightarrow a+b+c=0$ or $ab+bc+ca=0$. For the case $ab+bc+ca=0$, since $a^2+b^2+c^2=(a+b+c)^2-2(ab+bc+ca)=1$, then
$(a+b+c)^2=1 \Rightarrow a+b+c=\pm 1$. Hence, $a+b+c$ can be -1, 0, or 1.

1.91 ★★ If the sum of two consecutive natural numbers n and $n+1$ is the square of another natural number m, show n is divisible by 4.

Proof: $n+n+1=m^2$, i.e. $m^2=2n+1$. $2n+1$ is an odd number, then m^2 is also an odd number, then m has to be odd. Let $m=2k+1$ (k is a nonnegative integer). $n=\frac{m^2-1}{2}=\frac{(m-1)(m+1)}{2}=2k(k+1)$. Since $k(k+1)$ is obviously an even number, then $n=2k(k+1)$ is divisible by 4.

1.92 ★★ If x^3-2x^2+ax-6 and x^3+5x^2+bx+8 have a second order common factor, determine the values of a,b.

Solution: Let

$$x^3-2x^2+ax-6=(x^2+px+q)(x+c)=x^3+(c+p)x^2+(cp+q)x+cq$$
$$x^3+5x^2+bx+8=(x^2+px+q)(x+d)=x^3+(d+p)x^2+(dp+q)x+dq$$

Make the corresponding coefficients equal to have $p+c=-2, cp+q=a, cq=-6, d+p=5, dp+q=6, dq=8$. From these six algebraic equations, we obtain $a=-1, b=6, c=-3, d=4, p=1, q=2$.

1.93 ★★★ In the Cartesian plane XOY, all coordinates of the points $A(x_1, y_1)$ and $B(x_2, y_2)$ are one-digit positive integers. The angle between OA and the positive part of x axis is greater than 45^0, and the angle between OB and the positive part of x axis is less than 45^0. Denote $B' = (x_2, 0), A' = (0, y_1)$. The area of $\triangle OB'B$ is 33.5 larger than the area of $\triangle OA'A$. Find the four-digit number $\overline{x_1x_2y_2y_1}$ where x_1, x_2, y_2, y_1 are the four digits.

Solution: $S_{\triangle OB'B} = S_{\triangle OA'A} + 33.5 \Rightarrow \frac{1}{2}x_2y_2 = \frac{1}{2}x_1y_1 + 33.5 \Rightarrow x_2y_2 = x_1y_1 + 67$. Since $x_1y_1 > 0$, then $x_2y_2 > 67$. In addition, x_2, y_2 are one-digit positive integers, then $x_2y_2 = 72$ or 81. $\angle BOB' < 45^0$, then the point B is below the diagonal line $y = x$, then $x_2 > y_2$, thus $x_2y_2 \neq 81$, then $x_2y_2 = 72$, which implies $x_2 = 9, y_2 = 8$. Hence, $x_1y_1 = 5$. Since $\angle AOB' > 45^0$, then the point A is above the diagonal line $y = x$, then $x_1 < y_1$. Since x_1, y_1 are one-digit positive integers, then $x_1 = 1, y_1 = 5$. Therefore the four-digit number $\overline{x_1x_2y_2y_1} = 1985$.

1.94 ★★ Given a positive integer $n > 30$ and $2002n$ is divisible by $4n-1$, find the value of n.

Solution: Let $\dfrac{2002n}{4n-1} = k$, then $k = 500 + \frac{2(n+250)}{4n-1}$. Since $4n-1$ is an odd number, then $2(n+250)$ is divisible by $4n-1$. Let $\dfrac{n+250}{4n-1} = p$ (p is a positive integer), then $4p = \dfrac{4n+1000}{4n-1} = 1 + \dfrac{1001}{4n-1}$, thus 1001 is divisible by $4n-1$. Since $n \geq 30$ and $1001 = 7 \times 11 \times 13$, then we should have $4n-1 = 143$, which implies $n = 36,\ p = 2$.

1.95 ★★ How many integers satisfying the inequality $|x-2000| + |x| \leq 9999$?

Solution: If $x \geq 2000$, then the inequality becomes $(x-2000) + x \leq 9999 \Leftrightarrow 2000 \leq x \leq 5999.5$ There are 4000 integers satisfying the inequality. If $0 \leq x < 2000$, then the inequality becomes $(2000-x) + x \leq 9999 \Leftrightarrow 2000 \leq 9999$ that is always true, then there are 2000 integers satisfying the inequality. If $x < 0$, then the inequality becomes $(2000-x) + (-x) \leq 9999 \Leftrightarrow -3999.5 \leq x < 0$ There are additionally 3999 integers satisfying the inequality. Hence, totally there are $4000 + 2000 + 3999 = 9999$ integers satisfying the inequality.

1.96 ★★★ The real numbers x, y, z satisfy $x+y+z=3$ (i), $x^2+y^2+z^2=29$ (ii), $x^3+y^3+z^3=45$ (iii). Evaluate xyz and $x^4+y^4+z^4$.

Solution: (i)^2-(ii): $xy+yz+zx=-10$. Since $x^3+y^3+z^3-3xyz = x^3+3x^2y+3xy^2+y^3+z^3-3x^2y-3xy^2-3xyz = (x+y)^3+z^3-3xy(x+y+z) = (x+y+z)[(x+y)^2-(x+y)z+z^2]-3xy(x+y+z) = (x+y+z)(x^2+y^2+z^2-xy-yz-zx)$, then $45-3xyz = 3(29+10) \Rightarrow xyz=-24$.
Since $(xy+yz+zx)^2=100 \Rightarrow x^2y^2+y^2z^2+z^2x^2+2xyz(x+y+z)=100$, then $x^2y^2+y^2z^2+z^2x^2 = 100-2\cdot(-24)\cdot 3 = 244$.
Hence, $x^4+y^4+z^4 = (x^2+y^2+z^2)^2-2(x^2y^2+y^2z^2+z^2x^2) = 29^2-2\times 244 = 353$.

1.97 ★ Let $S=1+\frac{1}{2^2}+\frac{1}{3^2}+\cdots+\frac{1}{2009^2}$, find $[S]$.

Solution: $1 < S = 1+\dfrac{1}{2^2}+\dfrac{1}{3^2}+\cdots+\dfrac{1}{2009^2} < 1+\dfrac{1}{1\times 2}+\dfrac{1}{2\times 3}+\cdots+\dfrac{1}{2008\times 2009} = 1+1-\dfrac{1}{2}+\dfrac{1}{2}-\dfrac{1}{3}+\cdots+\dfrac{1}{2008}-\dfrac{1}{2009} = 2-\dfrac{1}{2009} = 1\dfrac{2008}{2009}$. Hence, $[S]=1$.

1.98 ★★★ Let $S=1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\cdots+\frac{1}{\sqrt{994009}}$, find $[S]$.

Solution: Let k be a positive integer, we have $\dfrac{1}{\sqrt{k+1}+\sqrt{k}} < \dfrac{1}{2\sqrt{k}} < \dfrac{1}{\sqrt{k}+\sqrt{k-1}} \Leftrightarrow \sqrt{k+1}-\sqrt{k} < \dfrac{1}{2\sqrt{k}} < \sqrt{k}-\sqrt{k-1}$. Thus we have $\sqrt{2}-1 < \dfrac{1}{2\sqrt{1}} < 1$, $\sqrt{3}-\sqrt{2} < \dfrac{1}{2\sqrt{2}} < \sqrt{2}-1, \cdots, \sqrt{994010}-\sqrt{994009} < \dfrac{1}{2\sqrt{994009}} < \sqrt{994009}-\sqrt{994008}$. Add them up to get $\sqrt{994010}-1 < \dfrac{1}{2}(1+\dfrac{1}{\sqrt{2}}+\dfrac{1}{\sqrt{3}}+\cdots+\dfrac{1}{\sqrt{994009}}) < \sqrt{994009}-\dfrac{1}{2} \Rightarrow 997-1 < \dfrac{1}{2}S < 997-\dfrac{1}{2} \Rightarrow 1992 < 2\sqrt{994010}-2 < S < 1993 \Rightarrow [S]=1992$.

1.99 ★★★ Given x, y, z, a, b, c are distinct rewal numbers, and

$$\begin{aligned}
\frac{1}{x+a}+\frac{1}{y+a}+\frac{1}{z+a} &= \frac{1}{a},\\
\frac{1}{x+b}+\frac{1}{y+b}+\frac{1}{z+b} &= \frac{1}{b},\\
\frac{1}{x+c}+\frac{1}{y+c}+\frac{1}{z+c} &= \frac{1}{c},
\end{aligned}$$

Evaluate $\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{c}$.

Solution: The three equalities imply that we can treat a, b, c as three distinct roots of the equation $\frac{1}{x+t}+\frac{1}{y+t}+\frac{1}{z+t}=\frac{1}{t}$, which is equivalent to $2t^3+(x+y+z)t^2-xyz=0$. Vieta's formulas lead to $ab+bc+ca=0$, thus $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=\frac{ab+bc+ca}{abc}=0$.

1.100 ★★★ If m is a natural number, S_m represents the sum of all digits of m, and the largest common divisor of S_m and S_{m+1} is a prime greater than 2, find the minimum value of m.

Solution: $(S_m, S_{m+1})>2 \Rightarrow S_{m+1}-S_m \neq 1$. Assume m has 9's as the last n digits ($n \geq 0$), then $S_{m+1}=S_m-9n+1$. Let $(S_m, S_{m+1})=d$, then $d=(S_m, 9n-1)$, $d|9n-1$, thus $n \neq 0,1$ (since $d>2$). If $n=2$, then $d|17$, $d=17$, S_m has the minimum value 34 (since $S_m \geq 18$) and m has the minimum value 8899. If $n=3$, then $d|26$, $d=13$, S_m has the minimum value 39 (since $S_m \geq 27$) and m has the minimum value 48999. If $n \geq 4$, then $m \geq 9999$ when d exists. Hence, m has the minimum value 8899.

1.101 ★★★ Given $ax+by=7, ax^2+by^2=49, ax^3+by^3=133, ax^4+by^4=406$, evaluate $2002(x+y)+2002xy+\frac{a+b}{21}$.

Solution: $(ax+by)(x+y)=ax^2+axy+bxy+by^2=(ax^2+by^2)+(a+b)xy$,
$(ax^2+by^2)(x+y)=ax^3+ax^2y+bxy^2+by^3=(ax^3+by^3)+(ax+by)xy$,
$(ax^3+by^3)(x+y)=ax^4+ax^3y+bxy^3+by^4=(ax^4+by^4)+(ax^2+by^2)xy$.
Substitute $ax+by=7, ax^2+by^2=49, ax^3+by^3=133, ax^4+by^4=406$ into the above equalities to obtain

$$\begin{aligned} 7(x+y) &= 49+(a+b)xy \quad \text{(i)} \\ 49(x+y) &= 133+7xy \quad \text{(ii)} \\ 133(x+y) &= 406+49xy \quad \text{(iii)} \end{aligned}$$

(ii) $\times 7-$ (iii) $\Rightarrow x+y=2.5$.
(ii) $\times 19-$ (iii) $\times 7 \Rightarrow xy=-1.5$.
Substitute $x+y=2.5, xy=-1.5$ into (i): $a+b=21$.
Therefore, $2002(x+y)+2002xy+\frac{a+b}{21}=2002\times 2.5+2002\times(-1.5)+\frac{21}{21}=2003$.

1.102 ★★★ If $, q, \frac{2p-1}{q}, \frac{2q-1}{p}$ are integers, and $p>1, q>1$, find the value of $p+q$.

Solution 1: If $p=q$, then $\frac{2p-1}{q}=\frac{2p-1}{p}=2-\frac{1}{p}$. Since $p>1$ is an integer, then $\frac{2p-1}{q}=2-\frac{1}{p}$ is not an integer, a contradiction to the given problem. Hence, $p\neq q$. Without loss of generality, Let $p>q$ and let $\frac{2q-1}{p}=m$ (m is a positive integer). Since $mp=2q-1<2p-1<2p$ then $m=1$, then $p=2q-1$, then $\frac{2p-1}{q}=\frac{4q-3}{q}=4-\frac{3}{q}$. Additionally since $\frac{2p-1}{q}$ is also a positive integer and $q>1$, then $q=3$, then $p=2q-1=5$, thus $p+q=8$.

Solution 2: Starting from $p>q$, let $\frac{2p-1}{q}=m$ (i), $\frac{2q-1}{p}=n$ (ii). m,n are both positive integers and $m>n$. (ii) is equivalent to $q=\frac{np+1}{2}$, substitute it into (i): $2p-1=m\frac{np+1}{2}$, thus $(4-mn)p=m+2$, thus $4-mn$ is a positive integer, i.e. $mn=1$ or $mn=2$ or $mn=3$. Recall that $m>n$, then we only have two possibilities $m=2,n=1$ or $m=3,n=1$. When $m=2,n=1$, (i)(ii) lead to $p=2,q=3/2$ (q is not an integer). When $m=3,n=1$, (i)(ii) lead to $p=5,q=3$, hence $p+q=8$.

1.103 ★★★★ If the real numbers a, b, c, d are all distinct, and $a+\frac{1}{b} = b + \frac{1}{c} = c + \frac{1}{d} = d + \frac{1}{a} = x$ find the value of x .

Solution: $a+\frac{1}{b} = b + \frac{1}{c} = c + \frac{1}{d} = d + \frac{1}{a} = x \Rightarrow a+\frac{1}{b} = x$ (i), $b+\frac{1}{c} = x$ (ii), $c+\frac{1}{d} = x$ (iii), $d+\frac{1}{a} = x$ (iv). (i) implies $b = \frac{1}{x-a}$, substitute it into (ii): $c = \frac{x-a}{x^2-ax-1}$, substitute it into (iii): $\dfrac{x-a}{x^2-ax-1} + \dfrac{1}{d} = x$, that is, $dx^3 - (ad+1)x^2 - (2d-a)x + ad + 1 = 0$ (v).

(iv) implies $ad + 1 = ax$, substitute it into (v):

$dx^3 - ax^3 - 2dx + ax + ax = 0 \Rightarrow (d-a)x^3 - (d-a)2x = 0 \Rightarrow (d-a)(x^3 - 2x) = 0.$

Since $d - a \neq 0$, then $x^3 - 2x = 0$. If $x = 0$, then $c = \frac{x-a}{x^2-ax-1} \Rightarrow a = c$, a contradiction. Hence, $x^2 - 2 = 0 \Rightarrow x^2 = 2 \Rightarrow x = \pm\sqrt{2}$.

1.104 ★★★ Consider a group of natural numbers $a_1, a_2, \cdots, a_n$, in which there are K_i numbers equal to i $(i = 1, 2, \cdots, m)$. Let $S = a_1 + a_2 + \cdots + a_n$, $S_j = K_1 + K_2 + \cdots + K_j$ $(1 \leq j \leq m)$. Show $S_1 + S_2 + \cdots + S_m = (m+1)S_m - S$.

Proof: $S = a_1 + a_2 + \cdots + a_n = K_1 \cdot 1 + K_2 \cdot 2 + \cdots + K_n \cdot m = (K_1 + K_2 + \cdots + K_m) + \cdot (K_2 + K_3 + \cdots + K_m) + \cdots + K_m = S_m + (S_m - S_1) + \cdots + (S_m - S_{m-1}) + S_m - S_m = (m+1)S_m - (S_1 + S_2 + \cdots + S_m)$

Hence, $S_1 + S_2 + \cdots + S_m = (m+1)S_m - S$.

1.105 ★★★ There are ten distinct rational numbers, and the sum of any nine of them is an irreducible proper fraction whose denominator is 22, find the sum of these ten rational numbers.

Solution: Let these ten distinct rational numbers be $a_1 < a_2 < \cdots < a_10$. We have $(a_1 + a_2 + \cdots + a_{10}) - a_k = \dfrac{m}{22}$, where $k = 1, 2, \cdots, 10$. m is an odd number and $1 \leq m \leq 21, m \neq 11$. Additionally because $a_1, a_2, \cdots, a_{10}$ are all distinct, then $10(a_1+a_2+\cdots+a_{10}) - (a_1+a_2+\cdots+a_{10}) = \dfrac{1+3+5+7+9+13+15+17+19+21}{22}$.

Hence, $a_1 + a_2 + \cdots + a_{10} = 5/9$.

1.106 ★★ Given $a + b + c = abc \neq 0$, evaluate
$\dfrac{(1-b^2)(1-c^2)}{bc} + \dfrac{(1-a^2)(1-c^2)}{ac} + \dfrac{(1-a^2)(1-b^2)}{ab}$.

Solution: $a+b+c = abc \neq 0 \Rightarrow ab = \frac{a+b+c}{c} \Rightarrow \frac{a+b}{c} = ab - 1$.

Similarly, $\dfrac{b+c}{a} = bc - 1$, $\dfrac{a+c}{b} = ac - 1$. We have

$$\frac{(1-b^2)(1-c^2)}{bc}+\frac{(1-a^2)(1-c^2)}{ac}+\frac{(1-a^2)(1-b^2)}{ab}=\frac{1-b^2-c^2+b^2c^2}{bc}+\frac{1-a^2-c^2+a^2c^2}{ac}+$$
$$\frac{1-a^2-b^2+a^2b^2}{ab}=(\frac{1}{bc}+\frac{1}{ac}+\frac{1}{ab})-\frac{b+c}{a}-\frac{a+c}{b}-\frac{a+b}{c}+ab+ac+bc=\frac{a+b+c}{abc}-$$
$$(bc-1)-(ac-1)-(ab-1)+ab+ac+bc=1-bc+1-ac+1-ab+1+ab+ac+bc=4 \quad .$$

1.107 ★★★ Let a, b, c be distinct positive integers, show at least one of $a^3b-ab^3, b^3c-bc^3, c^3a-ca^3$ is divisible by 10.

Proof: Because $a^3b-ab^3=ab(a^2-b^2), b^3c-bc^3=bc(b^2-c^2), c^3a-ca^3=ca(c^2-a^2)$, then if a, b, c has at least one even number or they are all odd numbers, $a^3b-ab^3, b^3c-bc^3, c^3a-ca^3$ are divisible by 2.

If one of a, b, c is a multiple of 5, then the conclusion is proven.

If a, b, c are not divisible by 5, then the last digits of a^2, b^2, c^2 can only be 1,4,6,9. Thus the last digits of $a^2 - b^2, b^2 - c^2, c^2 - a^2$ should have 0 or ± 5, that is, at least one of $a^2 - b^2, b^2 - c^2, c^2 - a^2$ is divisible by 5. Since 2 and 5 are coprime, thus at least one of $a^3b - ab^3 = ab(a^2 - b^2)$, $b^3c - bc^3 = bc(b^2 - c^2), c^3a - ca^3 = ca(c^2 - a^2)$ is divisible by 10.

1.108 ★★★ Let a, b, c are positive integers and follow a geometric sequence, and $b - a$ is a perfect square, $\log_6 a + \log_6 b + \log_6 c = 6$, find the value of $a + b + c$.

Solution: $\log_6 a + \log_6 b + \log_6 c = 6 \Rightarrow \log_6 abc = 6 \Rightarrow abc = 6^6$. In addition, $b^2 = ac$, then $b = 6^2 = 36, ac = 36^2$. In order to make $36 - a$ a perfect square, a can only be 11,27,32,35, and a is a divisor of 36^2, thus $a = 27$, then $c = 48$. Therefore, $a + b + c = 27 + 36 + 48 = 111$.

1.109 ★★★ The real numbers a, b, c, d satisfy $a + b = c + d, a^3 + b^3 = c^3 + d^3$, show $a^{2011} + b^{2011} = c^{2011} + d^{2011}$.

Proof: If $a + b = c + d = 0$, then the conclusion is obviously true.

If $a + b = c + d \neq 0$, then

$a^3 + b^3 = c^3 + d^3 \Rightarrow (a + b)(a^2 - ab + b^2) = (c + d)(c^2 - cd + d^2) \Rightarrow a^2 - ab + b^2 = c^2 - cd + d^2 \Rightarrow (a + b)^2 - 3ab = (c + d)^2 - 3cd \Rightarrow ab = cd \Rightarrow (a + b)^2 - 4ab = (c + d)^2 - 4cd \Rightarrow (a - b)^2 = (c - d)^2 \Rightarrow a - b = \pm(c - d)$. Hence,

$$\begin{aligned} a - b &= c - d \quad \text{(i)} \\ a + b &= c + d \quad \text{(ii)} \end{aligned}$$

or

$$\begin{aligned} a - b &= d - c \quad \text{(iii)} \\ a + b &= c + d \quad \text{(iv)} \end{aligned}$$

(i)+(ii): $a = c$; (i)-(ii): $b = d$.

(iii)+(iv): $a = d$; (iii)-(iv): $b = c$.

For either case, we have $a^{2011} + b^{2011} = c^{2011} + d^{2011}$.

1.110 ★★★ The real numbers a, b, c, d satisfy $a + b + c + d = 0$, show $a^3 + b^3 + c^3 + d^3 = 3(abc + bcd + cda + dab)$.

Proof: $a+b+c+d=0 \Rightarrow a+b=-(c+d) \Rightarrow 0=(a+b)^3+(c+d)^3=a^3+b^3+3a^2b+3ab^2+c^3+d^3+3c^2d+3cd^2 \Rightarrow a^3+b^3+c^3+d^3=-3(a^2b+ab^2+c^2d+cd^2) \Rightarrow a^3+b^3+c^3+d^3-3(abc+bcd+cda+dab)=-3(a^2b+ab^2+c^2d+cd^2)-3(abc+bcd+cda+dab)=-3(a^2b+ab^2+c^2d+cd^2+abc+bcd+cda+dab)=-3[(a^2b+ab^2+abc+abd)+(acd+bcd+c^2d+cd^2)]=-3[ab(a+b+c+d)+cd(a+b+c+d)]=0 \Rightarrow a^3+b^3+c^3+d^3=3(abc+bcd+cda+dab)$.

1.111 ★★★ Consider a $2n \times 2n$ square grid chessboard, each grid can only have one piece, and there are $3n$ grids having pieces, show we can always find n rows and n columns such that these $3n$ pieces are within these n rows or these n columns.

Proof: Denote the number of pieces in each row or column as $p_1, p_2, \cdots, p_n, p_{n+1}, \cdots, p_{2n}$ with the order $p_1 \geq p_2 \geq \cdots \geq p_n \geq p_{n+1} \geq \cdots \geq p_{2n}$. The given condition implies that $p_1+p_2+\cdots+p_n+p_{n+1}+\cdots+p_{2n}=3n$ (i). If $p_1+p_2+\cdots+p_n \leq 2n-1$ (ii), then at least one of $p_1, p_2, \cdots, p_n$ is not greater than 1. (i)-(ii): $p_{n+1}+\cdots+p_{2n} \geq n+1$, then at least one of $p_{n+1}, \cdots, p_{2n}$ is greater than 1, a contradiction. Hence, we have $p_1+p_2+\cdots+p_n \geq 2n$. Hence, we choose not less than $2n$ pieces from the n rows and then choose the remaining pieces from the n columns to include all $3n$ pieces.

1.112 ★★★★ Find a positive number such that its fractional part, its integer part, and itself are geometric.

Solution: Denote the number as $x>0$, its integer part $[x]$, and its fractional part $x-[x]$. The given condition implies that $x(x-[x])=[x]^2 \Rightarrow x^2-[x]x-[x]^2=0$, where $[x]>0, 0<x-[x]<1$. The solution is $x=\frac{1+\sqrt{5}}{2}[x]$. Since $0<x-[x]<1$, then $0<\frac{1+\sqrt{5}}{2}[x]<1 \Rightarrow 0<[x]<\frac{1+\sqrt{5}}{2}<2 \Rightarrow [x]=1, x=\frac{1+\sqrt{5}}{2}[x]$.

1.113 ★★★★ Consider a sequence $a_1, a_2, a_3, \cdots, a_n$ satisfying $a_1+a_2+\cdots+a_n=n^3$ for any positive integer n, evaluate $\frac{1}{a_2-1}+\frac{1}{a_3-1}+\cdots+\frac{1}{a_{100}-1}$.

Solution: When $n \geq 2$, we have $a_1+a_2+\cdots+a_n=n^3$ (i),

$a_1+a_2+\cdots+a_{n-1}=(n-1)^3$ (ii). (i)-(ii): $a_n=n^3-(n-1)^3=3n^2-3n+1$. Thus $\frac{1}{a_n-1}=\frac{1}{3n^2-3n}=\frac{1}{3n(n-1)}=\frac{1}{3}\left(\frac{1}{n-1}-\frac{1}{n}\right)$, $n=1,2,3,\cdots,100$.

Hence, $\frac{1}{a_2-1}+\frac{1}{a_3-1}+\cdots+\frac{1}{a_{100}-1}=\frac{1}{3}(1-\frac{1}{2})+\frac{1}{3}(\frac{1}{2}-\frac{1}{3})+\cdots+\frac{1}{3}(\frac{1}{99}-\frac{1}{100})=\frac{1}{3}(1-\frac{1}{100})=\frac{1}{3}\times\frac{99}{100}=\frac{33}{100}$.

1.114 ★★★★ x_1, x_2, x_3, x_4, x_5 are distinct positive odd numbers and satisfy $(2005 - x_1)(2005 - x_2)(2005 - x_3)(2005 - x_4)(2005 - x_5) = 576$, what is the last digit of $x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2$?

Solution: Since x_1, x_2, x_3, x_4, x_5 are distinct positive odd numbers, then

$2005 - x_1, 2005 - x_2, 2005 - x_3, 2005 - x_4, 2005 - x_5$ are distinct even numbers, thus

576 needs to be factored into the product of five distinct even numbers, which has a unique form: $576 = 24^2 = 2 \times (-2) \times 4 \times 6 \times (-6)$. Hence,

$(2005-x_1)^2+(2005-x_2)^2+(2005-x_3)^2+(2005-x_4)^2+(2005-x_5)^2 = 2^2+(-2)^2+4^2+6^2+(-6)^2 = 96 \Rightarrow 5\times 2005^2-4010(x_1+x_2+x_3+x_4+x_5)+(x_1^2+x_2^2+x_3^2+x_4^2+x_5^2) = 96 \Rightarrow x_1^2+x_2^2+x_3^2+x_4^2+x_5^2 = 96 - 5\times 2005^2 + 4010(x_1+x_2+x_3+x_4+x_5) \equiv 1$

(mod 10), that is, the last digit of $x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2$ is 1.

1.115 ★★★★ There are 95 numbers $a_1, a_2, \cdots, a_{95}$, which can only be +1 or -1. Find the minimum value of the sum of all products of any two, $S = a_1a_2 + a_1a_3 + \cdots + a_{94}a_{95}$; also determine how many (+1)'s and how many (-1)'s in the 95 numbers such that the minimum S is obtained.

Proof: Assume there are m (+1)'s and n (-1)'s in $a_1, a_2, \cdots, a_{95}$, then $m + n = 95$ (i). $a_1a_2 + a_1a_3 + \cdots + a_{94}a_{95} = S$, multiply it by 2 plus $a_1^2 + a_2^2 + \cdots + a_{95}^2 = 95$: $(a_1 + a_2 + \cdots + a_{95})^2 = 2S + 95$. $a_1 + a_2 + \cdots + a_{95} = m - n$, then $(m-n)^2 = 2S + 95$. The minimum value of S to make $2S + 95$ a perfect square is $S_{min} = 13$. When $S = S_{min}$, $(m-n)^2 = 11^2$, that is, $m - n = \pm 11$ (ii). (i)(ii) imply that $m + n = 95, m - n = 11$ or $m + n = 95, m - n = -11$, from which we have $m = 53, n = 42$ or $m = 42, n = 53$. Hence, when there are 53 (+1)'s and 42 (-1)'s, or there are 42 (+1)'s and 53 (-1)'s, $S = S_{min} = 13$.

1.116 ★★★★ Let $p = n(n+1)(n+2)\cdots(n+7)$, where n is a positive integer, show $[\sqrt[4]{p}] = n^2 + 7n + 6$.

Proof: Let $a = n^2 + 7n + 6$, then

$$p = n(n+7)(n+1)(n+6)(n+2)(n+5)(n+3)(n+4) = (n^2+7n)(n^2+7n+6)(n^2+7n+10)(n^2+7n+12) = (a-6)a(a+4)(a+6) = a^4+4a(a^2-9a-36) = a^4+4a(a+3)(a-12)$$

When $n \geq 1$, $a > 12$, then $a^4 < p$. On the other hand,

$(a+1)^4 - p = a^4+4a^2+1+4a^3+2a^2+4a-a^4-4a^3+36a^2+144a = 42a^2+148a+1 > 0 \Rightarrow p < (a+1)^4$. Hence, $a^4 < p < (a+1)^4 \Rightarrow a < \sqrt[4]{p} < a+1 \Rightarrow [\sqrt[4]{p}] = a = n^2 + 7n + 6$.

1.117 ★★★★★ The real numbers a, b, c, d, e satisfy $a + b + c + d + e = 8, a^2 + b^2 + c^2 + d^2 + e^2 = 16$, find the maximum value of e.

Solution: Substitute $a = 8 - b - c - d - e$ into $a^2 + b^2 + c^2 + d^2 + e^2 = 16$:

$(8 - b - c - d - e)^2 + b^2 + c^2 + d^2 + e^2 = 16 \Rightarrow 2b^2 - 2(8 - c - d - e)b + (8 - c - d - e)^2 + c^2 + d^2 + e^2 - 16 = 0$. Since b is a real number, then

$\Delta_b = 4(8 - c - d - e)^2 - 8[(8 - c - d - e)^2 + c^2 + d^2 + e^2 - 16] \geq 0 \Rightarrow 3c^2 - 2(8 - d - e)c + (8 - d - e)^2 - 2(16 - d^2 - e^2) \leq 0$. There are real values c satisfying this inequality if and only if
$\Delta_c = 4(8 - d - e)^2 - 12[(8 - d - e)^2 - 2(16 - d^2 - e^2)] \geq 0 \Rightarrow 4d^2 - 2(8 - e)d + (8 - e)^2 - 3(16 - e^2) \leq 0$. There are real values d satisfying this inequality if and only if $\Delta_d = 4(8 - e)^2 - 16[(8 - e)^2 - 3(16 - e^2)] \geq 0 \Rightarrow 5e^2 - 16e \leq 0 \Rightarrow e(5e - 16) \leq 0 \Rightarrow 0 \leq e \leq 16/5$. Hence, the maximum value of e is $16/5$.

1.118 ★★★★★ Let a positive integer d not equal to 2,5,13, show we can find two elements a, b from the set $\{2, 5, 13, d\}$ such that $ab - 1$ is not a perfect square.

Proof: $2 \times 5 - 1 = 3^2, 2 \times 13 - 1 = 5^2, 5 \times 13 - 1 = 8^2$, thus we need to show at least one of $2d - 1, 5d - 1, 13d - 1$ is not a perfect square. We prove this by contradiction. Suppose these three numbers are perfect squares, that is, $2d - 1 = x^2$ (i), $5d - 1 = y^2$ (ii), $13d - 1 = z^2$ (iii), where x, y, z are positive integers. (i) implies $2d - 1 \equiv 1 \pmod 2$, then x is an odd number, thus $2d - 1 \equiv 1 \pmod 4$, thus $d \equiv 1 \pmod 2$, that is, d is an odd number. Similarly (ii)(iii) imply y, z are even numbers. Let $y = 2y_1, z = 2z_1$, where y_1, z_1 are positive integers. Substitute them into (ii)(iii) and subtract the two resulting equalities:

$z_1^2 - y_1^2 = 2d \Rightarrow (z_1 - y_1)(z_1 + y_1) = 2d$ (iv). The right hand side of (iv) is divisible by 2, but the left hand side $(z_1 - y_1) + (z_1 + y_1) = 2z_1$ is an even number, then $z_1 - y_1$ and $z_1 + y_1$ are multiples of 2, thus the left hand side of (iv) is divisible by 2^2. However, d is an odd number, thus the right hand side of (iv) is not divisible by 2^2, a contradiction to the assumption.

1.119 ★★★★ Let x, y, z be nonnegative real numbers, and $x + y + z = 1$, find the maximum value of $xy + yz + zx - 2xyz$.

Solution:

$(1 - 2x)(1 - 2y)(1 - 2z) = (1 - 2y - 2x + 4xy)(1 - 2z) = 1 - 2y - 2x + 4xy - 2z + 4yz + 4zx - 8xyz = 1 - 2(x + y + z) + 4(xy + yz + zx - 2xyz)$, thus xy

$+yz + zx - 2xyz = \frac{1}{4}[(1-2x)(1-2y)(1-2z)+1]$.

Since $x+y+z=1$, then at most one of $1-2x, 1-2y, 1-2z$ is less than zero, thus $(1-2x)(1-2y)(1-2z) \le \left(\frac{1-2x+1-2y+1-2z}{3}\right)^3 = \left[\frac{3-2(x+y+z)}{3}\right]^3 =$ $\left(\frac{3-2}{3}\right)^3 = \frac{1}{27}$. Hence, $xy + yz + zx - 2xyz \le \frac{1}{4}(\frac{1}{27}+1) = \frac{7}{27}$. Therefore, the maximum value of $xy+yz+zx-2xyz$ is $7/27$.

1.120 ★★★★★ Let $a, b, c \in \mathcal{R}, a+b+c=0$, show

$$\frac{a^5+b^5+c^5}{5} = \frac{a^2+b^2+c^2}{2} \cdot \frac{a^3+b^3+c^3}{3}.$$

Proof: Let $F(n) = a^n + b^n + c^n$. Obviously a, b, c are roots of the equation $(x-a)(x-b)(x-c) = 0$. This equation is equivalent to $x^3 = (a+b+c)x^2 - (ab+bc+ca)x + abc$.

When $n \geq 4$, we have $x^n = (a+b+c)x^{n-1} - (ab+bc+ca)x^{n-2} + (abc)x^{n-3}$.

Thus $a^n = (a+b+c)a^{n-1} - (ab+bc+ca)a^{n-2} + (abc)a^{n-3}$.

Similarly, $b^n = (a+b+c)b^{n-1} - (ab+bc+ca)b^{n-2} + (abc)b^{n-3}$

and $c^n = (a+b+c)c^{n-1} - (ab+bc+ca)c^{n-2} + (abc)c^{n-3}$. Add the above three equalities together: $F(n) = (a+b+c)F(n-1) - (ab+bc+ca)F(n-2) + (abc)F(n-3)$.

In addition, we know $a^2 + b^2 + c^2 + 2ab + 2bc + 2ca = (a+b+c)^2$

and $a^3 + b^3 + c^3 - 3abc = (a+b+c)(a^2 + b^2 + c^2 - ab - bc - ca)$. When $a+b+c = 0$, then $F(1) = 0, ab + bc + ca = -\frac{a^2+b^2+c^2}{2} = -\frac{1}{2}F(2), abc = \frac{1}{3}(a^3 + b^3 + c^3) = \frac{1}{3}F(3), F(n) = \frac{1}{2}F(2)F(n-2) + \frac{1}{3}F(3)F(n-3)$. Choose $n = 4$, we have $F(4) = \frac{1}{2}F^2(2)$. Choose $n = 5$, we have $F(5) = \frac{1}{2}F(2)F(3) + \frac{1}{3}F(3)F(2) = \frac{5}{6}F(2)F(3)$. Hence,

$\frac{F(5)}{5} = \frac{F(2)}{2} \cdot \frac{F(3)}{3}$, that is, $\frac{a^5 + b^5 + c^5}{5} = \frac{a^2 + b^2 + c^2}{2} \cdot \frac{a^3 + b^3 + c^3}{3}$.

1.121 ★★★★★ Given $x = by + cz, y = cz + ax, z = ax + by$, find the value of $\frac{a}{a+1} + \frac{b}{b+1} + \frac{c}{c+1}$.

Solution: From the given conditions, we have

$x - y = by + cz - cz - ax \Rightarrow (a+1)x = (b+1)y$;

$y - z = cz + ax - ax - by \Rightarrow (b+1)y = (c+1)z$;

$z - x = ax + by - by - cz \Rightarrow (c+1)z = (a+1)x$.

Hence, $(a+1)x = (b+1)y = (c+1)z$. Let $(a+1)x = (b+1)y = (c+1)z = k$, then

$(ax + by + cz) + (x + y + z) = 3k$ (i). Add up the given equalities in the problem:

$x + y + z = 2(ax + by + cz)$ (ii). (i)(ii) lead to $ax + by + cz = k$, thus

$$\frac{a}{a+1} + \frac{b}{b+1} + \frac{c}{c+1} = \frac{ax}{(a+1)x} + \frac{by}{(b+1)y} + \frac{cz}{(c+1)z} = \frac{ax + by + cz}{k} = \frac{k}{k} = 1.$$

1.122 ★★★★★ Consider a cuboid whose length, width, height are positive integers m, n, r and $m \leq n \leq r$. We paint red color on the surface of the cuboid completely and then chop it into cubes with side length 1. If we know that the number of cubes without red face plus the number of cubes with two red faces minus the number of cubes with one red face is 1985, find the values of m, n, r.

Solution: We have three cases, separated by the value of m, to discuss.

(1) If $m > 2$, then the number of cubes without red face is $k_0 = (m-2)(n-2)(r-2)$, the number of cubes with one red face is

$k_1 = 2(m-2)(n-2) + 2(m-2)(r-2) + 2(n-2)(r-2)$, the number of cubes with two red faces is $k_2 = 4(m-2) + 4(n-2) + 4(r-2)$. We have

$k_0 + k_2 - k_1 = 1985 \Rightarrow (m-2)(n-2)(r-2) + 4[(m-2) + (n-2) + (r-2)] - 2[(m-2)(n-2) + (m-2)(r-2) + (n-2)(r-2)] = 1985 \Rightarrow (m-2)(n-2)[(r-2)-2] - 2(m-2)[(r-2)-2] - 2(n-2)[(r-2)-2] + 4(r-2) = 1985 \Rightarrow (m-2)(n-2)[(r-2)-2] - 2(m-2)[(r-2)-2] - 2(n-2)[(r-2)-2] + 4[(r-2)-2] = 1977 \Rightarrow [(r-2)-2][(m-2)(n-2) - 2(m-2) - 2(n-2) + 4] = 1977 \Rightarrow [(r-2)-2]\{(m-2)[(n-2)-2] - 2[(n-2)-2]\} = 1977 \Rightarrow (m-4)(n-4)(r-4) = 1977$

Because $1977 = 1 \times 3 \times 659 = 1 \times 1 \times 1977 = (-1)(-1) \cdot 1977$, then $m-4=1, n-4=3, r-4=659$, or $m-4=1, n-4=1, r-4=1977$, or $m-4=-1, n-4=-1, r-4=1977$. Therefore, $m=5, n=7, r=663$, or $m=5, n=5, r=1981$, or $m=3, n=3, r=1981$.

(2) If $m = 1$, then $n = 1$ leads to no solution, thus $n \geq 2$. In this case, the number of cubes without red face $k_0 = 0$, the number of cubes with one red face $k_1 = 0$, and the number of cubes with two red faces $k_2 = (n-2)(r-2)$. We have $k_0 + k_2 - k_1 = k_2 = 1985$, thus $(n-2)(r-2) = 1985 = 5 \times 397 = 1 \times 1985$, from which we obtain $n-2=5, r-2=397$, or $n-2=1, r-2=1985$. Therefore, $m=1, n=7, r=399$, or $m=1, n=3, r=1987$.

(3) If $m = 2$, then $k_0 = 0$, k_1 and k_2 are even numbers. In this case, obviously $k_0 + k_2 - k_1 \neq 1985$.

As a conclusion, there are five possibilities:

$$m = 5, n = 7, r = 663;$$

$$m = 5, n = 5, r = 1981;$$

$$m = 3, n = 3, r = 1981;$$

$$m = 1, n = 7, r = 399;$$

$$m = 1, n = 3, r = 1987.$$

2 EQUATIONS

2.1 Given the equation $\frac{5}{2}x - b = \frac{8}{5}x + 142$, find the smallest positive integer b such that the solution x is a positive integer.

Solution: $\frac{5}{2}x - b = \frac{8}{5}x + 142 \Rightarrow b = \frac{9}{10}x - 142$. Since b is a positive integer, then $\frac{9}{10}x$ should be a positive integer and greater than 142. Thus x should be a multiple of 10. To minimize b, $x = 160$, then $b = \frac{9}{10} \times 160 - 142 = 2$, that is, the smallest positive integer b is 2.

2.2 Solve $\frac{x-a-b}{c} + \frac{x-b-c}{a} + \frac{x-c-a}{b} = 3$.

Solution 1: The equation implies $a, b, c \neq 0$. Multiply abc on both sides of the equation: $(x-a-b)ab + (x-b-c)bc + (x-c-a)ca = 3abc \Leftrightarrow (ab+bc+ca)x = 3abc + ab(a+b) + bc(b+c) + ca(c+a) \Leftrightarrow (ab+bc+ca)x = (a+b+c)(ab+bc+ca)$. When $ab+bc+ca \neq 0$, $x = a+b+c$; When $ab+bc+ca = 0$, x can be any real number.

Solution 2: $\frac{x-a-b}{c} + \frac{x-b-c}{a} + \frac{x-c-a}{b} = 3 \Leftrightarrow \frac{x-a-b}{c} - 1 + \frac{x-b-c}{a} - 1 + \frac{x-c-a}{b} - 1 = 0 \Leftrightarrow \frac{x-(a+b+c)}{c} + \frac{x-(a+b+c)}{a} + \frac{x-(a+b+c)}{b} = 0 \Leftrightarrow [x-(a+b+c)](\frac{1}{a} + \frac{1}{b} + \frac{1}{c}) = 0 \Leftrightarrow [x-(a+b+c)]\frac{ab+bc+ca}{abc} = 0$. When $ab+bc+ca \neq 0$, $x = a+b+c$; When $ab+bc+ca = 0$, x can be any real number.

2.3 Find the condition for a such that the equation $|ax-2y-3| + |5x+9| = 0$ has the solution (x, y) where x, y have the same sign.

Solution: $|ax-2y-3| + |5x+9| = 0 \Rightarrow ax-2y-3 = 0, 5x+9 = 0 \Rightarrow x = -\frac{9}{5} < 0, y = \frac{ax}{2} - \frac{3}{2} = -\frac{9}{10}a - \frac{3}{2}$. Since x, y have the same sign, $y < 0 \Rightarrow a > -\frac{5}{3}$.

2.4 Find all positive integer solutions of the equation $123x + 57y = 531$.

Solution: $123x + 57y = 531 \Leftrightarrow 41x + 19y + 177 \Leftrightarrow y = 9 - 2x + \frac{6-3x}{19}$. Thus $x = 2, y = 5$ is a specific solution, then all positive integer solutions should have the form $x = 2 - 19t, y = 5 + 41t$, where t is an integer.
$2 - 19t > 0, t \;\; 5 + 41t > 0 \Rightarrow -\frac{5}{41} < t < \frac{2}{19}$, thus the only integer $t = 0$. Hence, the original equation only has one positive integer solution $x = 2, y = 5$.

2.5 Solve the equation $x^4 - 12x^3 + 47x^2 - 60x = 0$.

Solution:$x^4 - 12x^3 + 47x^2 - 60x = 0 \Leftrightarrow x(x^3 - 3x^2 - 9x^2 + 27x + 20x - 60) = 0 \Leftrightarrow x[x^2(x-3) - 9x(x-3) + 20(x-3)] = 0 \Leftrightarrow x(x-3)(x-4)(x-5) = 0$, which leads to four solutions: $x = 0$ or 3 or 4 or 5.

2.6 Given $|x-2| < 3$, solve the equation $|x+1| + |x-3| + |x-5| = 8$.

Solution 1: $|x-2| < 3 \Rightarrow -1 < x < 5$.

Then $|x+1| + |x-3| + |x-5| = 8 \Rightarrow x + 1 + |x-3| - x + 5 = 8 \Rightarrow |x-3| = 2$.

When $x \geq 3$, $x - 3 = 2 \Rightarrow x = 5$ which does not satisfy the given inequality;

When $x < 3$, $-(x-3) = 2 \Rightarrow x = 1$ which satisfies the given inequality.

Solution 2: $|x-2| < 3 \Rightarrow -1 < x < 5$.
When $-1 < x < 3$, $|x+1| + |x-3| + |x-5| = 8 \Rightarrow (x+1) - (x-3) - (x-5) = 8 \Rightarrow x = 1$;
When $3 \leq x < 5$, $|x+1| + |x-3| + |x-5| = 8 \Rightarrow (x+1) + (x-3) - (x-5) = 8 \Rightarrow x = 5$,
a contradiction to $3 \leq x < 5$, or $x = 5$ does not satisfy the given inequality.

2.7 Solve the equation $x|x| - 3|x| - 4 = 0$.

Solution: When $x \geq 0$, $x|x| - 3|x| - 4 = 0 \Rightarrow x^2 - 3x - 4 = 0 \Rightarrow (x+1)(x-4) = 0 \Rightarrow x = -1$ (deleted since $x \geq 0$) or $x = 4$.

When $x < 0$, $x|x| - 3|x| - 4 = 0 \Rightarrow -x^2 + 3x - 4 = 0 \Rightarrow x^2 - 3x + 4 = 0$ which has no solution since $\Delta = 9 - 16 < 0$.

As a conclusion, the original equation has a unique solution $x = 4$.

2.8 We know that the equation system

$$\begin{aligned} 3x + my - 5 &= 0 \\ x + ny - 4 &= 0 \end{aligned}$$

has no solution, and m, n are integers whose absolute values less than 7, find the values of m, n.

Solution: The equation system has no solution, then $\frac{3}{1} = \frac{m}{n} \neq \frac{5}{4}$, thus $m = 3n$ and $4m \neq 5n$. Additionally since $|m| = |3n| < 7$, thus $-\frac{7}{3} < n < \frac{7}{3}$. Since n is an integer, then $n = -2, -1, 0, 1, 2$, then $m = -6, -3, 0, 3, 6$. Hence, when $m = -6, n = -2$ or $m = -3, n = -1$ or $m = 0, n = 0$ or $m = 3, n = 1$ or $m = 6, n = 2$, the original equation system has no solution.

2.9 Assume the equation $2x^2 + x + a = 0$ has the solution set A, and the equation $2x^2 + bx + 2 = 0$ has the solution set B, and $A \cap B = \{1/2\}$, find $A \cup B$.

Solution: Let $A = \{1/2, x_1\}, B = \{1/2, x_2\}$.
Vieta's formulas lead to $x_1 + 1/2 = -1/2, x_2/2 = 1 \Rightarrow x_1 = -1, x_2 = 2$.
Hence, $A \cup B = \{1/2, -1\} \cup \{1/2, 2\} = \{-1, 1/2, 2\}$.

2.10 Find real valued solutions of the equation $\sqrt{x}+\sqrt{y-1}+\sqrt{z-2}=(x+y+z)/2$.

Solution 1:
$\sqrt{x}+\sqrt{y-1}+\sqrt{z-2}=(x+y+z)/2 \Leftrightarrow x-2\sqrt{x}+1+(y-1)-2\sqrt{y-1}+(z-2)-2\sqrt{z-2}+1=0 \Leftrightarrow (\sqrt{x}-1)^2+(\sqrt{y-1}-1)^2+(\sqrt{z-2}-1)^2=0 \Rightarrow \sqrt{x}-1=0, \sqrt{y-1}-1=0, \sqrt{z-2}-1=0 \Rightarrow x=1, y=2, z=3$.

Solution 2: Let $\sqrt{x}=t \quad (t\geq 0)$, $\sqrt{y-1}=u \quad (u\geq 0)$, $\sqrt{z-2}=v \quad (v\geq 0)$. Then $x=t^2, y=u^2+1, z=u^2+2$, substitute it into the original equation to obtain $t+u+v=(t^2+u^2+1+v^2+2)/2 \Leftrightarrow t^2+u^2+v^2-2t-2u-2v+3=0 \Leftrightarrow (t-1)^2+(u-1)^2+(v-1)^2=0 \Rightarrow t=u=v=1 \Rightarrow x=1, y=2, z=3$.

2.11 Solve the equation $\frac{x+1}{x+4}-\frac{x+4}{x+1}+\frac{x+1}{x-2}-\frac{x-2}{x+1}=\frac{2}{3}$.

Solution: The equation is equivalent to
$1-\frac{3}{x+4}-1-\frac{3}{x+1}+1-\frac{3}{x-2}-1+\frac{3}{x+1}=\frac{2}{3} \Leftrightarrow \frac{3}{x-2}=\frac{2}{3}+\frac{3}{x+4} \Leftrightarrow$ $x^2+2x-35=0$ and $x\neq 2, x\neq -4$. We can factor it to be $(x-5)(x+7)=0$, which leads to solutions $x=5, x=-7$.

2.12 If the equation $x^2-2x-4y=5$ has real valued solutions, find the maximum value of $x-2y$.

Solution: Let $x-2y=t$, then we have a system of equations: $x^2-2x-4y=5$ (i) and $x-2y=t$ (ii). (i)-(ii)×2: $x^2-4x=5-2t \Leftrightarrow x^2-4x+2t-5=0$ This quadratic equation has real valued solutions, thus $\Delta=16-4(2t-5)=4(9-2t)\geq 0 \Leftrightarrow t\leq 9/2$, that is, the maximum value of $x-2y$ is $9/2$.

2.13 Solve the equation $\lg x+\lg x^3+\lg x^5+\cdots+\lg x^{2n-1}=n \quad (n\in\mathcal{N})$.

Solution: $\lg x+\lg x^3+\lg x^5+\cdots+\lg x^{2n-1}=n \Leftrightarrow \lg x^{1+3+5+\cdots+(2n-1)}=n \Leftrightarrow \lg x^{(1+2n-1)n/2}=n \Leftrightarrow n^2\lg x=n \Leftrightarrow \lg x=1/n$ (since $n\in\mathcal{N}$), whose solution is $x=\sqrt[n]{10}$.

2.14 Solve the equation $5^{x+1}=3^{x^2-1}$.

Solution: $5^{x+1}=3^{x^2-1} \Leftrightarrow (x+1)\lg 5=(x^2-1)\lg 3 \Leftrightarrow (x+1)[\lg 5-(x-1)\lg 3]=0$ thus $x+1=0$ or $\lg 5-(x-1)\lg 3=0$, which imply two solutions $x=-1,\ x=\log_3 15$.

2.15 Solve the equation $x^4 - 4x^2 + 1 = 0$.

Solution: Let $y = x^2$, then the equation becomes
$y^2 - 4y + 1 = 0 \Leftrightarrow (y-2)^2 = 3 \Rightarrow y = 2 \pm \sqrt{3} \Rightarrow x^2 = 2 + \sqrt{3}$ or $x^2 = 2 - \sqrt{3}$,
the first of which implies $x = \pm\sqrt{2+\sqrt{3}} = \pm\frac{\sqrt{3}+1}{\sqrt{2}}$, the second of which implies
$x = \pm\sqrt{2-\sqrt{3}} = \pm\frac{\sqrt{3}-1}{\sqrt{2}}$. Hence, the four solutions are $\frac{\sqrt{3}+1}{\sqrt{2}}, -\frac{\sqrt{3}+1}{\sqrt{2}}, \frac{\sqrt{3}-1}{\sqrt{2}}, -\frac{\sqrt{3}-1}{\sqrt{2}}$.

2.16 For any real number k, the equation $(k^2+k+1)x^2 - 2(a+k)^2x + k^2 + 3ak + b = 0$ always has the root $x = 1$. Find (1) the real numbers a, b; (2) the range of the other root when k is a random real number.

Solution: (1) $x = 1$ is always a root, then $(k^2+k+1) - 2(a+k)^2 + k^2 + 3ak + b = 0$ is always valid for any k, that is, $(1-a)k + (1-2a+b) = 0$ for any k, thus $1-a = 0, 1-2a+b = 0$, which lead to $a = b = 1$.

(2) Let the other root be x_2, then Vieta's formulas imply

$$1 \cdot x_2 = \frac{k^2+3ak+b}{k^2+k+1} = \frac{k^2+3k+1}{k^2+k+1} \Leftrightarrow (x_2-1)k^2 + (x_2-3)k + (x_2-1) = 0.$$

$\Delta = (x_2-3)^2 - 4(x_2-1)^2 = -3x_2^2 + 2x_2 + 5 \geq 0$ which implies $-1 \leq x_2 \leq 5/3$.

2.17 Solve $|3x - |1-2x|| = 2$.

Solution: $|3x - |1-2x|| = 2 \Rightarrow 3x - |1-2x| = \pm 2$.
When $3x - |1-2x| = 2$, $|1-2x| = 3x - 2$, then $3x - 2 \geq 0 \Rightarrow x \geq 2/3$, and
$1 - 2x = \pm(3x-2)$ which leads to $x = 1$ or $x = 3/5 < 2/3$ (deleted).
When $3x - |1-2x| = -2$, $|1-2x| = 3x + 2$, then $3x + 2 \geq 0 \Rightarrow x \geq -2/3$, and
$1 - 2x = \pm(3x+2)$ which leads to $x = -1/5$ or $x = -3 < -2/3$ (deleted).
Hence, the original equation has solutions $x = 1$ or $x = -1/5$.

2.18 The equation $7x^2 - (k+13)x + k^2 - k - 2 = 0$ (k is a real number) has two real roots α, β, and $0 < \alpha < 1, 1 < \beta < 2$. Fine the range of k.

Solution: Let $f(x) = 7x^2 - (k+13)x + k^2 - k - 2$, since $0 < \alpha < 1, 1 < \beta < 2$ are two roots of $f(x) = 0$, then

$$\begin{aligned} f(0) &= k^2 - k - 2 > 0 \\ f(1) &= k^2 - 2k - 8 < 0 \\ f(2) &= k^2 - 3k > 0 \end{aligned}$$

$$\Rightarrow \begin{cases} k > 2 \quad \text{or} \quad k < -1 \\ -2 < k < 4 \\ k > 3 \quad \text{or} \quad k < 0 \end{cases} \Rightarrow 3 < k < 4 \text{ or } -2 < k < -1.$$

2.19 Solve the equation $\sqrt{x^2 + 2x - 63} + \sqrt{x+9} - \sqrt{7-x} + x + 13 = 0$.

Solution: The equation has real roots if and only if $x^2 + 2x - 63 \geq 0, x + 9 \geq 0, 7 - x \geq 0 \Rightarrow x \geq 7$ or $x \leq -9, x \geq -9, x \leq 7 \Rightarrow x = -9$ or $x = 7$. It is easy to obtain that $x = -9$ is a root of the original equation, but $x = 7$ is not. Hence, the original equation has a unique root $x = -9$.

2.20 The equation $k \lg^2 x + 3(k-1)\lg x + 2k = 0$ has the variable x and the parameter k, if the equation has two roots, one less than 100, one greater than 100, Find the range of k.

Solution: Let $t = \lg x$, and $x_1 < 100, x_2 > 100$, then the original equation becomes $kt^2 + 3(k-1)t + 2k = 0$. Because $x_1 < 100 < x_2$, we have $\lg x_1 < 2 < \lg x_2 \Rightarrow t_1 < 2 < t_2 \Rightarrow kf(2) < 0 \Rightarrow k[4k + 6(k-1) + 2k] < 0 \Rightarrow k(2k-1) < 0 \Rightarrow 0 < k < 1/2$.

2.21 Given $y-\sqrt{ab}=a\sqrt{bx-a}+b\sqrt{a-bx}$ $(a>0,b>0)$, show $\log_a(xy^2)=2$.

Proof: The equation makes sense if and only if $bx-a\geq 0, a-bx\geq 0$, i.e. $x\geq a/b, x\leq a/b$, then $x=a/b$, substitute it into the original equation to obtain $y=\sqrt{ab}$. Hence, $\log_a(xy^2)=\log_a[\frac{a}{b}(\sqrt{ab})^2]=\log_a(\frac{a}{b}\cdot ab)=\log_a a^2=2$.

2.22 For what values of k, the quadratic equation $(k^2-1)x^2-6(3k-1)x+72=0$ with variable x has two distinct positive integer roots.

Solution: $\Delta=36(3k-1)^2-4\times 72(k^2-1)=36(k-3)^2>0\Rightarrow k\neq 3$. The quadratic formula implies $x=\frac{6(3k-1)\pm 6(k-3)}{2(k^2-1)}$, that is, $x_1=\frac{12}{k+1}, x_2=\frac{6}{k-1}$. Since x_1,x_2 are positive integer roots and $k\neq 3$, then $k=2$. When $k=2$, $x_1=4, x_2=6$. Hence, $k=2$ is the only value of k such that the equation has two distinct positive integer roots.

2.23 ★ Solve the equation $P_4^2\cdot C_{x+3}^4=(C_8^5-1)P_{x+1}^2$.

Solution:

$$P_4^2\cdot C_{x+3}^4=(C_8^5-1)P_{x+1}^2$$

$\Leftrightarrow$

$$4\times 3\times\frac{(x+3)(x+2)(x+1)x}{4\times 3\times 2\times 1}=\left(\frac{8\times 7\times 6\times 5\times 4}{5\times 4\times 3\times 2\times 1}-1\right)(x+1)x$$

$\Leftrightarrow$

$$\frac{(x+3)(x+2)(x+1)x}{2}=55(x+1)x.$$

Since $x+1\geq 2, x+3\geq 4$, then $x\geq 1$, then $x\neq 0, x\neq -1$. We can divide $(x+1)x/2$ on both sides: $(x+3)(x+2)=110\Leftrightarrow x^2+5x-104=0\Leftrightarrow(x+13)(x-8)=0$ which leads to $x=8$ or $x=-13$ (deleted). Hence, the original equation has the root $x=8$.

2.24 ★ The three roots of the equation $3x^3+px^2+qx-4=0$ are the side length, the radius of the inscribed circle, the radius of the circumcircle, of a same equilateral triangle. Find the values of p,q.

Solution: Let the equilateral triangle has the side length a, then the radius of the inscribed circle and the radius of the circumcircle are $\frac{\sqrt{3}}{6}a$ and $\frac{\sqrt{3}}{3}a$, respectively. Vieta's formulas imply $a+\frac{\sqrt{3}}{6}a+\frac{\sqrt{3}}{3}a=-\frac{p}{3}$ (i), $a\frac{\sqrt{3}}{6}a+a\frac{\sqrt{3}}{3}a+\frac{\sqrt{3}}{6}a\cdot\frac{\sqrt{3}}{3}a=\frac{q}{3}$ (ii), $a\cdot\frac{\sqrt{3}}{6}a\cdot\frac{\sqrt{3}}{3}a=\frac{4}{3}$ (iii). (iii) leads to $a=2$, substitute it into (i)(ii) to obtain $p=-6-3\sqrt{3}, q=2+6\sqrt{3}$.

2.25 Solve the equation system

$$\begin{aligned}\log_2 x+\log_y 8 &= 2,\\ \log_y 2+\log_8 x^2 &= 1.\end{aligned}$$

Solution: The system is equivalent to

$$\begin{aligned}\log_2 x+\frac{3}{\log_2 y} &= 2 \quad \text{(i)}\\ \frac{1}{\log_2 y}+\frac{2\log_2 x}{3} &= 1 \quad \text{(ii)}\end{aligned}$$

(ii) $\times 3$-(i): $\log_2 x=1\Rightarrow x=2$. Substitute it into (i): $y=8$. We can easily verify $x=2, y=8$ is a solution of the original system.

2.26 Given $f(x)=x-\frac{1}{x}$, solve the equation $f[f(x)]=x$.

Solution: $f[f(x)]=x-\frac{1}{x}-\frac{1}{x-\frac{1}{x}}=\frac{x^4-3x^2+1}{x^3-x}$,
thus $f[f(x)]=x\Leftrightarrow\frac{x^4-3x^2+1}{x^3-x}=x\Rightarrow x^2=\frac{1}{2}\Rightarrow x=\pm\frac{\sqrt{2}}{2}$.

2.27 ★★ n is a positive integer, and denote a_n as the number of nonnegative integer solutions (x,y,z) to the equation $x+y+2z=n$. Find the values of a_3 and a_{2001}.

Solution: When $n=3$, we have $x+y+2z=3$. Since $x\geq 0, y\geq 0, z\geq 0$, we have $0\leq z\leq 1$. When $z=1$, then $x+y=1$, then $(x,y)=(0,1)$ or $(1,0)$. When $z=0$, then $x+y=3$, then there are four possibilities of (x,y). Hence, $a_3=2+4=6$.
When $n=2001$, we have $x+y+2z=2001$, thus $0\leq z\leq 1000$. When $z=0$, then $x+y=2001$, then there are 2002 possibilities of (x,y). When $z=1$, then $x+y=1999$, then there are 2000 possibilities of (x,y). ...... When $z=1000$, then $x+y=1$, then there are two possibilities of (x,y). As a conclusion, $a_{2001}=2002+2000+1998+\cdots+4+2=1003002$.

2.28 Solve the equation $x^2+x-2x\sqrt{x-2}-6=0$.

Solution: $x^2+x-2x\sqrt{x-2}-6=0 \Leftrightarrow x^2-2x\sqrt{x-2}+x-2=4 \Leftrightarrow (x-\sqrt{x-2})^2=4 \Leftrightarrow$ $x-\sqrt{x-2}=\pm 2$.

When $x-\sqrt{x-2}=2$, then $x-2-\sqrt{x-2}=0 \Leftrightarrow \sqrt{x-2}(\sqrt{x-2}-1)=0 \Rightarrow x=2$ or $x=3$.

When $x-\sqrt{x-2}=-2$, then $x+2=\sqrt{x-2} \Rightarrow x^2+3x+6=0$ which has no solution since $\Delta=3^2-4\times 6<0$.

It is easy to check that $x=2, x=3$ are the solutions of the original equation.

2.29 Solve the equation $\log_2(9^{x-1}+7)=2+\log_2(3^{x-1}+1)$.

Solution: The equation is equivalent to

$\log_2[(3^{x-1})^2+7]=\log_2 4(3^{x-1}+1) \Leftrightarrow (3^{x-1})^2+7=4(3^{x-1}+1)$. Let $y=3^{x-1}>0$, then $y^2-4y+3=0 \Leftrightarrow (y-1)(y-3)=0 \Rightarrow y=1$ or $y=3$.

When $y=1$, $3^{x-1}=1 \Rightarrow x-1=0 \Rightarrow x=1$.

When $y=3$, $3^{x-1}=3 \Rightarrow x-1=1 \Rightarrow x=2$.

It is easy to verify that $x=1, x=2$ are the solutions of the original equation.

2.30 ★ Find all prime number solutions of the equation $x(x+y) = z + 120$.

Solution: When $z = 2$, then $x(x+y) = 122$, then $x + y = 122/x$ is an integer and since x is a prime number, then $x = 2$ or 61. When $x = 2$, then $y = 59$; When $x = 61$, then $y = -59$ (deleted).

When z is an odd number, then x and $x + y$ are both odd numbers. Thus y has to be the only even prime number, i.e. $y = 2$. Then $x(x+2) = z + 120 \Leftrightarrow z = (x-10)(x+12)$. Since z is a prime number, then $x - 10 = 1$, then $x = 11, z = 23$.

As a conclusion, there are two possibilities: $x = 2, y = 59, z = 2$ or $x = 11, y = 2, z = 23$.

2.31 Solve the equation $\sqrt{2x^2 - 7x + 1} - \sqrt{2x^2 - 9x + 4} = 1$ (i).

Solution: Multiply both sides by $\sqrt{2x^2 - 7x + 1} + \sqrt{2x^2 - 9x + 4}$:
$\sqrt{2x^2 - 7x + 1} + \sqrt{2x^2 - 9x + 4} = 2x - 3$ (ii). (i)+(ii): $\sqrt{2x^2 - 7x + 1} = x - 1$, taking square to obtain $x^2 - 5x = 0 \Rightarrow x(x-5) = 0 \Rightarrow x = 0$ or $x = 5$. We can verify these two possible solutions via the original equation (i): $x = 5$ is indeed a root of (i), but $x = 0$ is a extraneous root generated by taking square.

2.32 ★ Find positive integers m, n such that the quadratic equation $4x^2 - 2mx + n = 0$ has two real roots both of which are between 0 and 1.

Solution: The equation has two real roots, thus $\Delta = 4m^2 - 16n \geq 0 \Rightarrow n \leq m^2/4$. Since both roots are between 0 and 1, then $f(0) = n > 0, f(1) = 4 - 2m + n > 0$, then $n > 2m - 4$ $(m, n \in \mathcal{N})$. Hence, $2m - 4 < n \leq m^2/4$, which implies a unique choice: $m = 2, n = 1$.

2.33 ★★ Solve the system of equations

$$\begin{aligned} (1+y)^x &= 100 \quad \text{(i)}, \\ (y^4 - 2y^2 + 1)^{x-1} &= \frac{(y-1)^{2x}}{(y+1)^2} \quad (y > 1) \quad \text{(ii)}. \end{aligned}$$

Solution: (ii) $\Leftrightarrow \dfrac{(y-1)^{2x}(y+1)^{2x}}{(y-1)^2(y+1)^2} = \dfrac{(y-1)^{2x}}{(y+1)^2}$. Since $y \neq \pm 1$, then
$(y-1)^{2x}(y+1)^{2x} - (y-1)^{2x}(y-1)^2 = 0 \Leftrightarrow (y-1)^{2x}[(y+1)^{2x} - (y-1)^2] = 0 \Rightarrow y = 1$ or $(y+1)^x = \pm(y-1)$. The second case together with (i) leads to
$\pm(y-1) = 100 \Rightarrow y = 101$ or $y = -99$ (deleted since $y > 1$).
When $y = 1$, then (i) implies $2^x = 100 \Rightarrow x = \dfrac{2}{\lg 2}$.

When $y = 101$, then (i) implies $102^x = 100 \Rightarrow x = \dfrac{2}{\lg 102}$.

We can verify that $x = \frac{2}{\lg 2}, y = 1$ and $x = \frac{2}{\lg 102}, y = 101$ are the two solutions of the original system of equations.

2.34 ★ If a, b, c are nonzero real numbers, solve the system of equations

$$\begin{aligned} \frac{xy}{ay+bx} &= c, \\ \frac{yz}{bz+cy} &= a, \\ \frac{zx}{az+cx} &= b. \end{aligned}$$

Solution:

$$\begin{aligned} \frac{xy}{ay+bx} &= c \\ \frac{yz}{bz+cy} &= a \\ \frac{zx}{az+cx} &= b \end{aligned}$$

$\Leftrightarrow$

$$\begin{aligned} \frac{ay+bx}{xy} &= \frac{1}{c} \\ \frac{bz+cy}{yz} &= \frac{1}{a} \\ \frac{az+cx}{zx} &= \frac{1}{b} \end{aligned}$$

$\Leftrightarrow$

$$\begin{aligned} \frac{a}{x}+\frac{b}{y} &= \frac{1}{c} \quad \text{(i)} \\ \frac{b}{y}+\frac{c}{z} &= \frac{1}{a} \quad \text{(ii)} \\ \frac{a}{x}+\frac{c}{z} &= \frac{1}{b} \quad \text{(iii)} \end{aligned}$$

(i)+(ii)+(iii): $\dfrac{a}{x}+\dfrac{b}{y}+\dfrac{c}{z} = \dfrac{1}{2}\left(\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{c}\right)$ (iv). Then (iv)-(ii), (iv)-(iii), (iv)-(i):

$= \frac{2a^2bc}{ab+ac-bc}, y = \frac{2b^2ca}{bc+ab-ac}, z = \frac{2c^2ab}{ca+bc-ab}$.

2.35 The real numbers x, y satisfy the equation $x^2 - 2xy + y^2 - \sqrt{2}x - \sqrt{2}y + 6 = 0$. Find the minimum value of $x + y$.

Solution: Let $x + y = k$, then $y = k - x$. Substitute it into the equation:
$x^2 - 2x(k-x) + (k-x)^2 - \sqrt{2}x - \sqrt{2}(k-x) + 6 = 0 \Leftrightarrow 4x^2 - 4kx + (k^2 - \sqrt{2}k + 6) = 0$,
then $\Delta = (4k)^2 - 16(k^2 - \sqrt{2}k + 6) = 16\sqrt{2}k - 96 \geq 0 \Leftrightarrow k \geq 3\sqrt{2}$,
thus $k = x + y$ has the minimum value $3\sqrt{2}$.

2.36 ★ Solve the equation $(\sqrt{2+\sqrt{3}})^x + (\sqrt{2-\sqrt{3}})^x = 4$.

Solution: The equation is equivalent to $(\sqrt{2+\sqrt{3}})^x + \frac{1}{(\sqrt{2+\sqrt{3}})^x} = 4$. Let $y = (\sqrt{2+\sqrt{3}})^x$, then $y + \frac{1}{y} = 4 \Rightarrow y^2 - 4y + 1 = 0$ whose roots are $y = 2 \pm \sqrt{3}$.
When $y = 2 + \sqrt{3}$, $(\sqrt{2+\sqrt{3}})^x = 2 + \sqrt{3} = (\sqrt{2+\sqrt{3}})^2$, thus $x = 2$.
When $y = 2 - \sqrt{3}$, $(\sqrt{2+\sqrt{3}})^x = 2 - \sqrt{3} = \frac{1}{2+\sqrt{3}} = (\sqrt{2+\sqrt{3}})^{-2}$, thus $x = -2$.
We can verify that $x = 2, x = -2$ are indeed roots of the original equation.

2.37 If the equation $x^2 - kx + k^2 - 4 = 0$ has two positive roots, find the range of k.

Solution: The condition of two positive roots (denoted by x_1, x_2) implies $x_1 + x_2 = k > 0, x_1x_2 = k^2 - 4 > 0, \Delta = k^2 - 4(k^2 - 4) = -3k^2 + 16 \geq 0$. From these three inequalities, we can easily obtain $2 < k \leq 4\sqrt{3}/3$.

2.38 Solve the system of equations

$$\begin{aligned} 2^{\sqrt{x^2-x-2}} &= 4y, \\ \lg(1+y) &= 2\lg y + \lg 2. \end{aligned}$$

Solution: The second equation leads to $\lg(1+y) = \lg 2y^2 \Rightarrow 1 + y = 2y^2 \Rightarrow 2y^2 - y - 1 = 0 \Rightarrow y = 1$ or $-1/2$ (deleted since $y > 0$). Substitute $y = 1$ into the first equation:

$2^{\sqrt{x^2-x-2}} = 2^2 \Rightarrow \sqrt{x^2 - x - 2} = 2 \Rightarrow x^2 - x - 6 = 0 \Rightarrow (x-3)(x+2) = 0 \Rightarrow x = 3$ or $x = -2$. Hence, $(3, 1), (-2, 1)$ are solutions of the original equation system.

2.39 Solve the equation $\sqrt{4x^2 + 2x + 7} = 12x^2 + 6x - 119$.

Solution: Write the equation as $\sqrt{4x^2 + 2x + 7} = 3(4x^2 + 2x + 7) - 140$. Let $\sqrt{4x^2 + 2x + 7} = t \ \ (t \geq 0)$, then
$t = 3t^2 - 140 \Rightarrow 3t^2 - t - 140 = 0 \Rightarrow (3t + 20)(t - 7) = 0 \Rightarrow t = -20/3$ (deleted) or $t = 7$.
Thus $\sqrt{4x^2 + 2x + 7} = 7 \Rightarrow 4x^2 + 2x + 7 = 49 \Rightarrow 2x^2 + x - 21 = 0 \Rightarrow (x-3)(2x+7) = 0 \Rightarrow x = 3$ or $x = -7/2$. We can verify that both $x = 3, x = -7/2$ are roots of the original equation.

2.40 ★ Let S be the sum of reciprocals of two real roots of the equation $(a^2 - 4)x^2 + (2a - 1)x + 1 = 0$ where a is a real number, find the range of S.

Solution: Let x_1, x_2 be the two roots, then $x_1 + x_2 = \frac{1-2a}{a^2-4}, x_1x_2 = \frac{1}{a^2-4}$. The quadratic equation has real roots, thus $a^2 - 4 \neq 0, \Delta = (2a-1)^2 - 4(a^2 - 4) \geq 0$, thus $a \neq \pm 2, a \leq 17/4$. Hence, $S = \frac{1}{x_1} + \frac{1}{x_2} = \frac{x_1+x_2}{x_1x_2} = 1 - 2a$ should satisfy $S \neq -3, S \neq 5, S \geq -15/2$.

2.41 ★ Let a, b be two real numbers, $|a| > 0$, and the equation $||x - a| - b| = 5$ has three distinct roots, find the value of b.

Solution: The equation $||x-a|-b|=5$ is equivalent to $|x-a|-b=\pm 5 \Leftrightarrow |x-a|=b\pm 5$. The equation has three distinct roots if and only if $b-5=0$, that is $b=5$ and the roots are $x=a, x=a\pm 10$.

2.42 ★ Solve the system of equations

$$\begin{aligned} x^2+xy+y^2 &= 84, \\ x+\sqrt{xy}+y &= 14. \end{aligned}$$

Solution: The second equation is equivalent to $x+y=14-\sqrt{xy} \Rightarrow x^2+xy+y^2=196-28\sqrt{xy}$ The left hand side is 84 due to the first equation, then $84=196-28\sqrt{xy} \Rightarrow \sqrt{xy}=4$. Substitute it into the second equation to obtain $x+y=10$. Hence, we can treat x, y as roots of the quadratic equation $z^2-10z+16=0$. The roots are $z=2$ or $z=8$. Therefore, the original system of equations has two solutions $(2,8)$, $(8,2)$.

2.43 ★★ The real numbers a, b, c satisfy $a \neq b$ and $2009(a-b)+\sqrt{2009}(b-c)+(c-a)=0$ find the value of $\frac{(c-b)(c-a)}{(a-b)^2}$.

Solution: Let $\sqrt{2009}=x$, then $(a-b)x^2+(b-c)x+(c-a)=0$. $a \neq b$ implies that this equation is a quadratic equation. Obviously, $x=\sqrt{2009}$ and 1 are two roots of this quadratic equation, then $\sqrt{2009}+1=\frac{c-b}{a-b}, \sqrt{2009}\times 1=\frac{c-a}{a-b}$.

Hence, $\frac{(c-b)(c-a)}{(a-b)^2}=\frac{c-b}{a-b}\times\frac{c-a}{a-b}=(\sqrt{2009}+1)\sqrt{2009}=2009+\sqrt{2009}$.

2.44 ★★ Find all functions $f(x)$ that satisfy the equation $2f(1-x)+1=xf(x)$.

Solution: Replace x with $1-x$ in the equation: $2f(x)+1=(1-x)f(1-x)$ (i).

Rewrite the original equation as $f(1-x)=\frac{1}{2}[xf(x)-1]$ (ii). Substitute (ii) into (i):

$2f(x)+1=(1-x)\frac{1}{2}[xf(x)-1] \Leftrightarrow 4f(x)+2=xf(x)-1-x^2f(x)+x \Leftrightarrow$ $(x^2-x+4)f(x)=x-3 \Leftrightarrow f(x)=\frac{x-3}{x^2-x+4}$.

2.45 ★★ If the equality $ab=2(c+d)$ is always valid, show at least one of the equations $x^2+ax+c=0$ and $x^2+bx+d=0$ has real root(s).

Proof: $\Delta_1=a^2-4c, \Delta_2=b^2-4d$. Assume $\Delta_1<0$, then $a^2-4c<0$, then $a^2<4c$. $ab=2(c+d) \Leftrightarrow ab-2c=2d$, thus $\Delta_2=b^2-4d=b^2-2ab+4c>b^2-2ab+a^2=(b-a)^2\geq 0$. Similarly if we assume $\Delta_2<0$, then we will obtain $\Delta_1\geq 0$.

2.46 Solve the equation $\log_a x+\log_x b=1$ where $a>1, b>1$.

Solution: $\log_a x + \log_x b = 1 \Rightarrow \dfrac{\lg x}{\lg a} + \dfrac{\lg b}{\lg x} = 1 \Rightarrow \lg^2 x - \lg a \lg x + \lg a \lg b = 0$. To guarantee the existence of real valued solutions, we need $\Delta = \lg^2 a - 4 \lg a \lg b = \lg a (\lg a - 4 \lg b) \geq 0$. Since $a > 1$, $\lg a > 0$, thus $\lg a \geq \lg b^4$. Hence, $a \geq b^4$. When $a \geq b^4$, we have $\lg x = \dfrac{\lg a \pm \sqrt{\lg^2 a - 4 \lg a \lg b}}{2}$, thus $x = 10^{(\lg a \pm \sqrt{\lg^2 a - 4 \lg a \lg b})/2}$. When $a < b^4$, the original equation has no root.

2.47 ★★ The real number x satisfies the equation $= \sqrt{x - \frac{1}{x}} + \sqrt{1 - \frac{1}{x}}$, find the value of $[2x]$.

Solution: Let $a = \sqrt{x - \frac{1}{x}}, b = \sqrt{1 - \frac{1}{x}}$, then $x = a + b$ (i), $a^2 - b^2 = x - 1$, then $a - b = \frac{a^2 - b^2}{a+b} = \frac{x-1}{x} = 1 - \frac{1}{x}$ (ii). (i)+(ii):
$2a = x - \frac{1}{x} + 1 = a^2 + 1 \Rightarrow a^2 - 2a + 1 = 0 \Rightarrow a = 1 \Rightarrow \sqrt{x - \frac{1}{x}} = 1 \Rightarrow x^2 - x - 1 = 0 \Rightarrow x = \frac{1 \pm \sqrt{5}}{2}$. Since $x > 0$, then $x = \frac{1+\sqrt{5}}{2} \Rightarrow 2x = \sqrt{5} + 1 \Rightarrow 3 < 2x < 4 \Rightarrow [2x] = 3$.

2.48 ★ Solve the equation $a^{2x}(a^2+1)=(a^{3x}+a^x)a$.

Solution: $a^{2x}(a^2+1)=(a^{3x}+a^x)a \Leftrightarrow a^{2x+2}+a^{2x}=a^{3x+1}+a^{x+1} \Leftrightarrow a^{3x+1}-a^{2x+2}-a^{2x}+a^{x+1}=0$. Since $a\neq 0$ by the definition of an exponential function, then we can divide both sides by a to obtain $a^{3x}-a^{2x+1}-a^{2x-1}+a^x=0 \Leftrightarrow (a^x-a)(a^{2x}-a^{x-1})=0 \Rightarrow a^x=a$ or $a^{2x}=a^{x-1}$, which imply $x=1$ or $x=-1$. We can verify that both $x=1, x=-1$ are roots of the original equation.

2.49 ★ Solve the equation $(x-1)^4+(x+3)^4=82$.

Solution: Let $y=x+1$, then the original equation becomes

$(y-2)^4+(y+2)^4=82 \Leftrightarrow (y^2-4y+4)^2+(y^2+4y+4)^2=82 \Leftrightarrow y^4+24y^2-25=0 \Leftrightarrow (y^2+25)(y^2-1)=0$. Since $y^2+25>0$,
then $y^2-1=0 \Rightarrow y=\pm 1 \Rightarrow x+1=\pm 1 \Rightarrow x=0$ or $x=-2$.
Hence, the original equation has two roots $x=0, x=-2$.

2.50 ★ Solve the equation $\dfrac{1}{x^2+2x-3}+\dfrac{18}{x^2+2x+2}-\dfrac{18}{x^2+2x+1}=0$.

Solution: Let $x^2+2x+1=y$, then the original equation becomes

$\dfrac{1}{y-4}+\dfrac{18}{y+1}-\dfrac{18}{y}=0 \Rightarrow \dfrac{1}{y-4}=\dfrac{18}{y(y+1)} \Rightarrow y^2-17y+72=0 \Rightarrow (y-8)(y-9)=0 \Rightarrow y=8$ or $y=9$.
When $y=8$, we have $x^2+2x+1=8 \Rightarrow x=-1+2\sqrt{2}$ or $x=-1-2\sqrt{2}$.
When $y=9$, we have $x^2+2x+1=9 \Rightarrow x=2$ or $x=-4$.
We can easily verify that $x=-1+2\sqrt{2}, x=-1-2\sqrt{2}, x=2, x=-4$ are roots of the original equation.

2.51 ★ Let x_1, x_2 be the two real roots of the quadratic equation $x^2+x-3=0$, find the value of $x_1^3-4x_2^2+19$.

Solution: $x_1^2+x_1-3=0, x_2^2+x_2-3=0$, thus $x_1^2=3-x_1, x_2^2=3-x_2$. Vieta's formulas imply $x_1+x_2=-1$. Hence,
$x_1^3-4x_2^2+19=x_1(3-x_1)-4(3-x_2)+19=3x_1-x_1^2+4x_2+7=3x_1-(3-x_1)+4x_2+7=4(x_1+x_2)+4=4\times(-1)+4=0$.

2.52 ★★ If x, y, z are real roots of the equation system

$$\begin{aligned} x^2 - yz - 8x + 7 &= 0, \\ y^2 + z^2 + yz - 6x + 6 &= 0, \end{aligned}$$

find the range of x .

Solution: The system is equivalent to

$$\begin{aligned} yz &= x^2 - 8x + 7 \quad \text{(i)} \\ y^2 + z^2 + yz &= 6x - 6 \quad \text{(ii)} \end{aligned}$$

(ii)-(i) $\times 3$: $y^2+z^2-2yz = -3x^2+30x-27 \Rightarrow (y-z)^2 = -3(x-1)(x-9) \geq 0 \Rightarrow (x-1)(x-9) \leq 0 \Rightarrow 1 \leq x \leq 9$ ·

2.53 ★ If a, b, k are rational numbers, and $b = ak + \frac{c}{k}$, show the equation $ax^2 + bx + c = 0$ has two rational roots.

Proof: The discriminant $\Delta = b^2 - 4ac = (ak + \frac{c}{k})^2 - 4ac = (ak - \frac{c}{k})^2$, thus $\sqrt{\Delta} = \pm(ak - \frac{c}{k})$. In addition, $ak - \frac{c}{k} = ak - (b - ak) = 2ak - b$. Since a, b, k are rational numbers, $ak - \frac{c}{k}$ is also a rational number, thus $\sqrt{\Delta}$ is a rational number, therefore the two roots of the quadratic equation $x = \frac{-b \pm \sqrt{\Delta}}{2a}$ are rational numbers.

2.54 ★★ If x_1, x_2 are the two real roots of the equation $x^2 + ax + a - \frac{1}{2} = 0$, find the value of a such that $(x_1 - 3x_2)(x_2 - 3x_1)$ reaches the maximum value.

Solution: Vieta's formulas imply $x_1 + x_2 = -a, x_1 x_2 = a - \frac{1}{2}$, thus $(x_1 - 3x_2)(x_2 - 3x_1) = 10x_1x_2 - 3(x_1^2 + x_2^2) = 16x_1x_2 - 3(x_1 + x_2)^2 = 16a - 8 - 3a^2 = -3(a - \frac{8}{3})^2 + \frac{40}{3}$. Since the quadratic equation has two real roots, the discriminant $\Delta = a^2 - 4(a - \frac{1}{2}) = [(a-2) + \sqrt{2}][(a-2) - \sqrt{2}] \geq 0$ which leads to $a \geq 2 + \sqrt{2}$ or $a \leq 2 - \sqrt{2}$. Since $8/3 \in (2 - \sqrt{2}, 2 + \sqrt{2})$, the extreme values should be obtained at boundaries.

When $a = 2 + \sqrt{2}$, $(x_1 - 3x_2)(x_2 - 3x_1) = 4\sqrt{2} + 6$.

When $a = 2 - \sqrt{2}$, $(x_1 - 3x_2)(x_2 - 3x_1) = -4\sqrt{2} + 6$.

Hence, when $a = 2 + \sqrt{2}$, $(x_1 - 3x_2)(x_2 - 3x_1)$ reaches the maximum value $4\sqrt{2} + 6$.

2.55 ★★ Given $y=\frac{x^2-2x+4}{x^2-3x+3}$, find all values of x such that y is an integer.

Solution: $y=\frac{x^2-2x+4}{x^2-3x+3}=1+\frac{x+1}{x^2-3x+3}$ Let $\lambda=\dfrac{x+1}{x^2-3x+3}$, then $\lambda x^2-(3\lambda+1)x+3\lambda-1=0$ ($\divideontimes$). The discriminant $\Delta\geq 0\Rightarrow(3\lambda+1)^2-4\lambda(3\lambda-1)\geq 0\Rightarrow 3\lambda^2-10\lambda-1\leq 0$ $\Rightarrow\frac{5-2\sqrt{7}}{3}\leq\lambda\leq\frac{5+2\sqrt{7}}{3}$. To make y an integer, λ should be an integer, thus $\lambda=0,1,2,3$, substitute them into ($\divideontimes$) to obtain $x=-1,2+\sqrt{2},2-\sqrt{2},1,5/2,2,4/3$.

2.56 ★★ Solve the equation $\sqrt{12-\frac{12}{x^2}}+\sqrt{x^2-\frac{12}{x^2}}=x^2$.

Solution: Squaring both sides to obtain $12-\frac{12}{x^2}+x^2-\frac{12}{x^2}+2\sqrt{12x^2-\frac{144}{x^2}+\frac{144}{x^4}-12}=x^4\Leftrightarrow 2\sqrt{12}\sqrt{x^4-x^2-12+\frac{12}{x^2}}=$ $|x|(x^4-x^2-12+\frac{24}{x^2})$. Let $x^4-x^2-12=t$ and substitute it into the above equality: $4\times 12(t+\frac{12}{x^2})=x^2(t^2+\frac{24^2}{x^4}+\frac{48t}{x^2}\Leftrightarrow 48t+\frac{24^2}{x^2}=x^2t^2+\frac{24^2}{x^2}+48t\Leftrightarrow x^2t^2=0$ Since $x\neq 0$, then $t=0$, then $x^4-x^2-12=0$, then $x^2=\frac{1\pm 7}{2}$ which should be nonnegative, thus $x^2=4$, that is $x=\pm 2$. We can easily verify that $x=\pm 2$ are roots of the original equation.

2.57 ★★ The real numbers α, β, γ are roots of the cubic equation $2x^3 + x^2 - 4x + 1 = 0$. Evaluate (1) $\alpha^2 + \beta^2 + \gamma^2$, (2) $\frac{1}{\beta\gamma} + \frac{1}{\gamma\alpha} + \frac{1}{\alpha\beta}$, (3) $\alpha^3 + \beta^3 + \gamma^3$.

Solution: (1) Vieta's formulas imply $\alpha + \beta + \gamma = -\frac{1}{2}$ (i), $\alpha\beta + \beta\gamma + \gamma\alpha = -2$ (ii), $\alpha\beta\gamma = -\frac{1}{2}$ (iii). (i)^2-(ii)$\times 2$:
$\alpha^2 + \beta^2 + \gamma^2 + 2\alpha\beta + 2\beta\gamma + 2\gamma\alpha - 2\alpha\beta - 2\beta\gamma - 2\gamma\alpha = (-\frac{1}{2})^2 - (-2) \times 2 \Rightarrow$
$\alpha^2 + \beta^2 + \gamma^2 = 4\frac{1}{4}$.

(2) $\frac{1}{\beta\gamma} + \frac{1}{\gamma\alpha} + \frac{1}{\alpha\beta} = \frac{\alpha+\beta+\gamma}{\alpha\beta\gamma} = \frac{-1/2}{-1/2} = 1$.

(3) The original equation is equivalent to $x^3 = \frac{-x^2+4x-1}{2}$, substitute α, β, γ into it and add them up to obtain $\alpha^3+\beta^3+\gamma^3 = \frac{-\alpha^2+4\alpha-1}{2} + \frac{-\beta^2+4\beta-1}{2} + \frac{-\gamma^2+4\gamma-1}{2} = \frac{-(\alpha^2+\beta^2+\gamma^2)+4(\alpha+\beta+\gamma)-3}{2}$
$= \frac{-4\frac{1}{4}-2-3}{2} = -4\frac{5}{8}$.

2.58 ★ Solve the system of equations

$$\begin{aligned} x^2 - xy + y^2 - 19x - 19y &= 0, \\ xy &= -6. \end{aligned}$$

Multiply the second equation by 3 and add it to the first equation:
$(x+y)^2 - 19(x+y) + 18 = 0 \Leftrightarrow (x+y-1)(x+y-18) = 0 \Rightarrow x+y = 1$or

$x + y = 18$. We discuss these two cases separately.

When $x + y = 1$, we can treat x, y as two roots of the quadratic equation $z^2 - z - 6 = 0 \Leftrightarrow$ $(z-3)(z+2) = 0 \Rightarrow z = 3$ or $z = -2$, thus we obtain two solutions $(3, -2), (-2, 3)$.

When $x + y = 18$, we can treat x, y as two roots of the quadratic equation $z^2 - 18z - 6 = 0$ $\Rightarrow z = 9 \pm 2\sqrt{97}$ thus we obtain two solutions $(9+2\sqrt{97}, 9-2\sqrt{97}), (9-2\sqrt{97}, 9+2\sqrt{97})$.
Hence the original system has four solutions $(3,-2), (-2,3), (9+2\sqrt{97}, 9-2\sqrt{97})$, $(9-2\sqrt{97}, 9+2\sqrt{97})$.

2.59 ★★ Solve the equation $x^x + 85x^{-x} - 100x^{-2x} = -14$.

Solution: Let $y = x^x$, then the equation becomes
$y + \frac{85}{y} - \frac{100}{y^2} = -14 \Leftrightarrow y^3 + 14y^2 + 85y - 100 = 0$. Obviously $y = 1$ is one root, that is, $x^x = 1$ whose root is $x = 1$. Let α, β be the other two roots, then Vieta's formulas imply $1 + \alpha + \beta = -14, \alpha + \beta + \alpha\beta = 85, \alpha\beta = 100$, from which we can obtain $\alpha^2 + 15\alpha + 100 = 0, \beta^2 + 15\beta + 100 = 0$. The discriminant $\Delta = 15^2 - 400 < 0$, thus α, β do not exist. Hence, $x = 1$ is the only root of the original equation.

2.60 ★★ The equation $5x^2-(10\cos\alpha)x+7\cos\alpha+6=0$ has two identical roots, α is one angle of a parallelogram, and the sum of two adjacent sides is 6, find the maximal area of the parallelogram.

Solution: The quadratic equation has two identical roots, thus the discriminant $\Delta=100\cos^2\alpha-140\cos\alpha-120=0\Leftrightarrow 5\cos^2\alpha-7\cos\alpha-6=0\Rightarrow\cos\alpha=\frac{7\pm13}{10}$. Since $|\cos\alpha|\leq1$, then $\cos\alpha=\frac{7-13}{10}=-\frac{3}{5}$. The angle of a parallelogram, α, is between 0^0 and 180^0, and since $\cos\alpha=-\frac{3}{5}<0$, thus $\alpha\in(90^0,180^0)$, then $\sin\alpha=\sqrt{1-\cos^2\alpha}=\frac{4}{5}$. Let one side of parallelogram has length u, then one adjacent side has length $6-u$. The area $S=u(6-u)\sin\alpha=u(6-u)\frac{4}{5}=-\frac{4}{5}(u-3)^2+\frac{36}{5}$. Hence, the maximal area $S_{max}=\frac{36}{5}$ when $u=3$.

2.61 ★★ Find all positive integer solutions (x,y) of the equation

$x\sqrt{y}+y\sqrt{x}-\sqrt{2011x}-\sqrt{2011y}+\sqrt{2011xy}=2011$.

Solution: The equation is equivalent to $\sqrt{xy}\sqrt{x}+\sqrt{xy}\sqrt{y}-\sqrt{2011x}-\sqrt{2011y}+\sqrt{2011xy}-(\sqrt{2011})^2=0\Leftrightarrow\sqrt{xy}(\sqrt{x}+\sqrt{y})-\sqrt{2011}(\sqrt{x}+\sqrt{y})+\sqrt{2011}\sqrt{xy}-(\sqrt{2011})^2=0\Leftrightarrow$ $(\sqrt{xy}-\sqrt{2011})(\sqrt{x}+\sqrt{y}+\sqrt{2011})=0$.

Since $\sqrt{x}+\sqrt{y}+\sqrt{2011}>0$, then $\sqrt{xy}-\sqrt{2011}=0\Rightarrow xy=2011$. Since 2011 is a prime, then $x=1,y=2011$ or $x=2011,y=1$. Hence the original equation has two positive integer solutions $(1,2011),(2011,1)$.

2.62 ★ x_1,x_2 are two roots of the quadratic equation $x^2-(k-2)x+k^2+3k+5=0$ where k is a real number, find the maximum value of $x_1^2+x_2^2$.

Solution: According to Vieta's formulas, we have $x_1+x_2=k-2,x_1x_2=k^2+3k+5$, thus $x_1^2+x_2^2=(x_1+x_2)^2-2x_1x_2=(k-2)^2-2(k^2+3k+5)=-(k+5)^2+19$.

Since the equation has real roots, then the discriminant $\Delta=(k-2)^2-4(k^2+3k+5)\geq0\Leftrightarrow 3k^2+16k+16\leq0\Rightarrow-4\leq k\leq-\frac{4}{3}$. The function $f(k)=-(k+5)^2+19$ is a monotonically decreasing function on the interval $[-4,-\frac{4}{3}]$, thus the maximum value is $f(-4)=18$ which is also the maximum value of $x_1^2+x_2^2$.

2.63 ★ Solve the equation $\frac{x^2}{4} - \frac{3}{2}x + \frac{6}{x} + \frac{4}{x^2} = 0$.

Solution:

$$\frac{x^2}{4} - \frac{3}{2}x + \frac{6}{x} + \frac{4}{x^2} = 0 \Leftrightarrow x^2 - 6x + \frac{24}{x} + \frac{16}{x^2} = 0 \Leftrightarrow (x - \frac{4}{x})^2 - 6(x - \frac{4}{x}) + 8 = 0.$$

Let $x - \frac{4}{x} = y$, then $y^2 - 6y + 8 = 0 \Leftrightarrow (y-2)(y-4) = 0 \Rightarrow y = 2$ or $y = 4$.

When $y = 2$, we have $x - \frac{4}{x} = 2 \Rightarrow x^2 - 2x - 4 = 0 \Rightarrow x = 1 \pm \sqrt{5}$.

When $y = 4$, we have $x - \frac{4}{x} = 4 \Rightarrow x^2 - 4x - 4 = 0 \Rightarrow x = 2 \pm 2\sqrt{2}$.

Therefore, the original equation has four roots $1 \pm \sqrt{5}, 2 \pm 2\sqrt{2}$.

2.64 ★★ m, n are positive integers, $m \neq n$, the equation $(m-1)x^2 - (m^2+2)x + (m^2+2m) = 0$ and the equation $(n-1)x^2 - (n^2+2)x + (n^2+2n) = 0$ has a common root. Find the value of $\frac{m^n + n^m}{m^{-n} + n^{-m}}$.

Solution: The quadratic formula together with $m > 1, n > 1, m \neq n$ gives us the following: the first equation has roots $x = m, \frac{m+2}{m-1}$, and the second equation has roots $x = n, \frac{n+2}{n-1}$. Since $m \neq n$, then $m = \frac{n+2}{n-1}, n = \frac{m+2}{m-1}$. Both of these two equalities give us the same result: $mn - m - n - 2 = 0 \Leftrightarrow (m-1)(n-1) = 3$.

Since m, n are both positive integers, then we only have two possibilities:

$m-1=1, n-1=3$ or $m-1=3, n-1=1$, which lead to $m=2, n=4$ or $m=4, n=2$, thus $\dfrac{m^n+n^m}{m^{-n}+n^{-m}}=m^n\cdot n^m=4^2\cdot 2^4=256$.

2.65 ★★ We usually use $[x]$ to represent the integer part of the real number x, here we define $\{x\}=x-[x]$ which is the decimal part of the real number x. (1) Find a real number x to satisfy $\{x\}+\{\frac{1}{x}\}=1$. (2) Show that all x satisfying the equation in (1) are not rational numbers.

Solution: (1) Let $x=m+\alpha, \frac{1}{x}=n+\beta$ (m, n are integers, $0\le\alpha,\beta\le 1$).
$\{x\}+\{\frac{1}{x}\}=1\Leftrightarrow\alpha+\beta=1$, thus $x+\frac{1}{x}=m+\alpha+n+\beta=m+n+1$ is an integer.
Let $x+\frac{1}{x}=k$ (k is an integer), that is $x^2-kx+1=0$ whose roots are
$x=\frac{1}{2}(k\pm\sqrt{k^2-4})$.
When $|k|=2$, $|x|=1$ which does not satisfy the equation $\{x\}+\{\frac{1}{x}\}=1$.
When $|k|\ge 3$, $x=\frac{1}{2}(k\pm\sqrt{k^2-4})$ which satisfies $\{x\}+\{\frac{1}{x}\}=1$.

(2) k^2-4 is not a perfect square (if it is, then $k^2-4=h^2$, i.e. $k^2-h^2=4$, but when $|k|\ge 3$ the difference between two perfect squares is not less than 5), thus x is an irrational number.

2.66 ★★ The equation $(x^2-1)(x^2-4)=k$ has four nonzero real roots, and these roots form an arithmetic sequence, find the value of k.

Solution: Let $y=x^2$, then the equation becomes $y^2-5y+4-k=0$. Let α,β $(0<\alpha<\beta)$ are roots of $y^2-5y+4-k=0$, then the original equation has four roots $\pm\sqrt{\alpha},\pm\sqrt{\beta}$. They form an arithmetic sequence, then $\sqrt{\beta}-\sqrt{\alpha}=\sqrt{\alpha}-(-\sqrt{\alpha})$, then $\beta=9\alpha$. In addition, Vieta's formulas imply $\alpha+\beta=5$, then we can obtain $\alpha=\frac{1}{2},\beta=\frac{9}{2}$, thus $4-k=\alpha\beta=\frac{9}{4}$, therefore $k=4-\frac{9}{4}=\frac{7}{4}$.

2.67 ★★ Given a real number d and $|d|\le 1/4$, solve the equation
$x^4-2x^3+(2d-1)x^2+2(1-d)x+2d+d^2=0$.

Solution: Rewrite the equation as $d^2+(2x^2-2x+2)d+x^4-2x^3-x^2+2x=0$ and treat it a quadratic equation for d, then the quadratic formula implies $d=-x^2-x$ or $d=-x^2+3x-2$. Both are quadratic equations for x. Solve them to obtain four roots of the original equation:

$x=\frac{-1+\sqrt{1-4d}}{2},\frac{-1-\sqrt{1-4d}}{2},\frac{3+\sqrt{4d+1}}{2},\frac{3-\sqrt{4d+1}}{2}$. All these roots exist since $|d|\le 1/4$.

2.68 ★ Show that the x, y-dependent equation $x^2-y^2+dx+ey+f=0$ represents two straight lines if and only if $d^2-e^2-4f=0$.

Proof: The equation represents two straight lines, then we should have
$x^2-y^2+dx+ey+f=(x-y+k_1)(x+y+k_2)=x^2-y^2+(k_1+k_2)x+(k_1-k_2)y+k_1k_2$.
Make the corresponding coefficients equal: $k_1+k_2=d, k_1-k_2=e, k_1k_2=f$. The first two equations lead to $k_1=\frac{d+e}{2}, k_2=\frac{d-e}{2}$, and substitute them into the third equation: $\frac{d+e}{2}\cdot\frac{d-e}{2}=f$, which is equivalent to $d^2-e^2-4f=0$.

2.69 ★ Solve the equation $\log_8(x^2+1)^3-\log_2 xy+\log_{\sqrt{2}}\sqrt{y^2+4}=3$.

Solution:

$\log_8(x^2+1)^3-\log_2 xy+\log_{\sqrt{2}}\sqrt{y^2+4}=3 \Leftrightarrow \log_2(x^2+1)-\log_2 xy+\log_2(y^2+4)=3 \Leftrightarrow \log_2\frac{(x^2+1)(y^2+4)}{xy}=3 \Leftrightarrow \frac{(x^2+1)(y^2+4)}{xy}=8$.

Since $x,y\neq 0$, we have

$x^2y^2+4x^2+y^2+4=8xy \Leftrightarrow (2x-y)^2+(xy-2)^2=0 \Rightarrow 2x-y=0, xy-2=0$.

Solve these two equations to obtain two solutions of the original equation:
$(1,2),(-1,-2)$.

2.70 ★ Solve the equation $\sqrt{x}+\sqrt{x+7}+2\sqrt{x^2+7x}=35-2x$.

Solution:

$\sqrt{x}+\sqrt{x+7}+2\sqrt{x^2+7x}=35-2x \Leftrightarrow x+2\sqrt{x(x+7)}+x+7+\sqrt{x}+\sqrt{x+7}-42=0 \Leftrightarrow (\sqrt{x}+\sqrt{x+7})^2+(\sqrt{x}+\sqrt{x+7})-42=0 \Leftrightarrow (\sqrt{x}+\sqrt{x+7}+7)(\sqrt{x}+\sqrt{x+7}-6)=0$.

Since $\sqrt{x}+\sqrt{x+7}+7>0$, then $\sqrt{x}+\sqrt{x+7}=6$. Squaring both sides to obtain $2\sqrt{x^2+7x}=29-2x$, and squaring again to obtain $144x=841$, thus $x=841/144$ which is the root of the original equation.

2.71 ★★★ The x-dependent equation $x^2+p|x|=qx-1$ has four distinct real roots, show that $p+|q|<-2$.

Proof: When $x>0$, the equation becomes $x^2+(p-q)x+1=0$ (i); When $x<0$, the equation becomes $x^2-(p+q)x+1=0$ (ii). We need two positive roots from (i) and two negative roots from (ii). Hence, the smaller root of (i) is greater than zero, and the larger root of (ii) is less than zero, that is $\frac{q-p-\sqrt{(p-q)^2-4}}{2}>0$ and $\frac{p+q+\sqrt{(p+q)^2-4}}{2}<0$ (obviously both discriminants need to be positive, $(p-q)^2-4>0, (p+q)^2-4>0$). Therefore, $q-p>\sqrt{(p-q)^2-4}>0$ (iii) and $0<\sqrt{(p+q)^2-4}<-(p+q)$ (iv). (iii) implies $q>p$, and since $(p-q)^2-4>0$, then $q-p>2$, then $p-q<-2$. (iv) implies $p+q<0$, and since $(p+q)^2-4>0$, then $p+q<-2$. As a conclusion, $p+|q|<-2$.

2.72 ★★★ Solve the functional equation $f(x) + f(\frac{x-1}{x}) = 1 + x \ \ (x \neq 0, x \neq 1)$ (i).

Solution: Replace x with $\frac{x-1}{x}$ in (i): $f(\frac{x-1}{x}) + f(\frac{-1}{x-1}) = \frac{2x-1}{x}$ (ii). Replace x with $\frac{-1}{x-1}$ in (i): $f(\frac{-1}{x-1}) + f(x) = \frac{x-2}{x-1}$ (iii). (i)+(iii)-(ii) $\Rightarrow f(x) = \frac{x^3-x^2-1}{2x(x-1)} = \frac{x^3-x^2-1}{2x^2-2x}$, which is the only solution of the original functional equation (i).

2.73 ★★ Solve the equation $(\sqrt{3})^{\tan 2x} - \dfrac{9\sqrt{3}}{3^{\tan 2x}} = 0$.

Solution: Let $(\sqrt{3})^{\tan 2x} = y \ \ (y > 0)$, then the equation becomes $y - \frac{9\sqrt{3}}{y^2} = 0 \Rightarrow \frac{y^3-9\sqrt{3}}{y^2} = 0 \Rightarrow y^3 - 9\sqrt{3} = 0 \Rightarrow y = 3^{5/6} \Rightarrow (\sqrt{3})^{\tan 2x} = 3^{5/6} \Rightarrow \frac{\tan 2x}{2} = \frac{5}{6} \Rightarrow \tan 2x = \frac{5}{3} \Rightarrow 2x = k\pi + \arctan\frac{5}{3} \ \ (k \in \mathcal{N}) \Rightarrow x = \frac{k\pi}{2} + \frac{1}{2}\arctan\frac{5}{3} \ \ (k \in \mathcal{N})$. Hence, the solution set of the original equation is $\{x | x = \dfrac{k\pi}{2} + \dfrac{1}{2}\arctan\dfrac{5}{3}, k \in \mathcal{N}\}$.

2.74 ★ Solve the system of equations

$$\begin{aligned} \lg|x+y| &= 1, \\ \lg y - \lg|x| &= \frac{1}{\log_4 100}. \end{aligned}$$

Solution: The system is equivalent to

$$\begin{aligned} \lg|x+y| &= \lg 10, \\ \lg\frac{y}{|x|} &= \lg 2, \end{aligned}$$

which lead to

$$\begin{aligned} |x+y| &= 10, \\ y &= 2|x|. \end{aligned}$$

$y > 0$ is always true since $y = 2|x|$ and $x \neq 0$.

When $x > 0$, the system becomes

$$\begin{aligned} x+y &= 10, \\ y &= 2x, \end{aligned}$$

whose solution is $x = 10/3, y = 20/3$.

When $x < 0$, the system become

$$\begin{aligned} x+y &= 10, \\ y &= -2x, \end{aligned}$$

whose solution is $x = -10, y = 20$.

We can verify that $(10/3, 20/3), (-10, 20)$ indeed are solutions of the original system.

2.75 ★ Solve the equation $2x + \sqrt{x} + \sqrt{x+2} + 2\sqrt{x^2+2x} - 4 = 0$.

Solution: The equation is equivalent to
$x+2\sqrt{x}\sqrt{x+2}+x+2+\sqrt{x}+\sqrt{x+2}-6 = 0 \Leftrightarrow (\sqrt{x}+\sqrt{x+2})^2+(\sqrt{x}+\sqrt{x+2})-6 = 0$.
Let $y = \sqrt{x} + \sqrt{x+2}$ $(y > 0)$, then $y^2 + y - 6 = 0 \Rightarrow (y-2)(y+3) = 0 \Rightarrow y = 2$ or $y = -3$ (deleted). Hence,
$\sqrt{x} + \sqrt{x+2} = 2 \Rightarrow \sqrt{x+2} = 2 - \sqrt{x} \Rightarrow x + 2 = 4 - 4\sqrt{x} + x \Rightarrow \sqrt{x} = 1/2 \Rightarrow x = 1/4$, which is the root of the original equation.

2.76 ★★ Solve the system of equations

$$\begin{aligned} x+y+z &= 3, \\ x^2+y^2+z^2 &= 3, \\ x^5+y^5+z^5 &= 3. \end{aligned}$$

Solution: $x+y+z=3 \Leftrightarrow x+y=3-z$ (i), $x^2+y^2+z^2=3 \Leftrightarrow x^2+y^2=3-z^2$ (ii). $xy=(\frac{x+y}{2})^2-(\frac{x-y}{2})^2=(\frac{3-z}{2})^2-(\frac{x-y}{2})^2$ (iii). (i)^2-(ii): $xy=\frac{(3-z)^2}{2}-\frac{3-z^2}{2}$ (iv). (iii)&(iv) $\Rightarrow (\frac{3-z}{2})^2-(\frac{x-y}{2})^2=\frac{(3-z)^2}{2}-\frac{3-z^2}{2} \Rightarrow 3(z-1)^2+(x-y)^2=0 \Rightarrow z=1, x=y\cdot$ Substitute them into (i) to obtain $x=y=1$. Obviously $x=y=z=1$ satisfies $x^5+y^5+z^5=3$. Hence, the original system has the solution $x=1, y=1, z=1$.

2.77 ★★ Solve the equation $4x^4+12x^3-47x^2+12x+4=0$.

Solution: Obviously $x=0$ is not a root, so we assume $x\neq 0$, then we can divide both sides by x^2: $4x^2+12x-47+\frac{12}{x}+\frac{4}{x^2}=0$, then $4(x^2+\frac{1}{x^2})+12(x+\frac{1}{x})-47=0$ (i). Let $x+\frac{1}{x}=u$, then $x^2+\frac{1}{x^2}=u^2-2$.

Substitute them into (i) to obtain $4(u^2-2)+12u-47=0 \Rightarrow 4u^2+12u-55=0 \Rightarrow u=5/2$ or $u=-11/2$. When $u=5/2$, $x+\frac{1}{x}=\frac{5}{2} \Rightarrow 2x^2-5x+2=0 \Rightarrow x=2$ or $x=1/2$. When $u=-11/2, x+\frac{1}{x}=-\frac{11}{2} \Rightarrow 2x^2+11x+2=0 \Rightarrow x=\frac{-11\pm\sqrt{105}}{4}$. Hence, the original equation has four roots: $x=2, x=1/2, x=\frac{-11+\sqrt{105}}{4}, x=\frac{-11-\sqrt{105}}{4}$.

2.78 ★★ Solve the equation $\sqrt[3]{10-2x}+\sqrt[3]{2x-1}=3$.

Solution: Let $\sqrt[3]{10-2x}=a$, $\sqrt[3]{2x-1}=b$, then $a+b=3$. $\sqrt[3]{10-2x}=a \Rightarrow a^3=10-2x$ (i). $\sqrt[3]{2x-1}=b \Rightarrow b^3=2x-1$ (ii). (i)+(ii)$\Rightarrow a^3+b^3=9 \Rightarrow (a+b)(a^2-ab+b^2)=9 \Rightarrow a^2-ab+b^2=3$ (iii). Substitute $a=3-b$ into (iii): $(3-b)^2-(3-b)b+b^2=3 \Rightarrow b^2-3b+2=0 \Rightarrow b=1$ or $b=2$.

When $b=1$, (ii)$\Rightarrow x=1$.

When $b=2$, (ii)$\Rightarrow x=9/2$.

We can verify that $x=1, x=9/2$ are indeed two roots.

2.79 ★★ The real coefficient equation $x^3+2kx^2+9x+5k=0$ has an imaginary root whose modulus is $\sqrt{5}$, find the value of k and solve the equation.

Solution: The equation should have two imaginary roots and one real root: $a \pm bi, c$. Vieta's formulas and the modulus $\sqrt{5}$ lead to

$$\begin{aligned} a+bi+a-bi+c &= -2k \\ (a+bi)(a-bi)+(a+bi)c+(a-bi)c &= 9 \\ (a+bi)(a-bi)c &= -5k \\ a^2+b^2 &= 5 \end{aligned}$$

$\Rightarrow$

$$\begin{aligned} 2a+c &= -2k \\ a^2+b^2+2ac &= 9 \\ (a^2+b^2)c &= -5k \\ a^2+b^2 &= 5 \end{aligned}$$

$\Rightarrow a = \pm 1, b = \pm 2, c = \pm 2, k = \pm 2.$

When $k = 2$, the equation becomes $x^3 + 4x^2 + 9x + 10 = 0$ and its roots are $x = -1 \pm 2i, x = -2$

When $k = -2$, the equation becomes $x^3 - 4x^2 + 9x - 10 = 0$ and its roots are $x = 1 \pm 2i, x = 2$

2.80 ★ Solve the equation $x+\log_9 3^x=\log_9(4-5\cdot 9^x)$.

Solution: The equation is equivalent to $\log_9 9^x+\log_9 3^x=\log_9(4-5\cdot 9^x)\Rightarrow \log_9 3^{3x}=\log_9(4-5\cdot 9^x)\Rightarrow 3^{3x}=4-5\cdot 3^{2x}$. Let $3^x=y$, then the equation becomes $y^3+5y^2-4=0\Rightarrow y^3+y^2+4y^2-4=0\Rightarrow (y+1)(y^2+4y-4)=0\Rightarrow y=-1$ or $y=-2(1+\sqrt{2})$ or $y=2(\sqrt{2}-1)$. Since $y=3^x>0$, then $y=-1, y=-2(1+\sqrt{2})$ are incorrect. Hence, $y=2(\sqrt{2}-1)\Rightarrow 3^x=2(\sqrt{2}-1)\Rightarrow x=\log_3 2(\sqrt{2}-1)$, which is the only root.

2.81 ★★★ Solve the system of equations

$$\begin{aligned}\frac{4x^2}{1+4x^2}&=y,\\ \frac{4y^2}{1+4y^2}&=z,\\ \frac{4z^2}{1+4z^2}&=x.\end{aligned}$$

Solution: Obviously $x\geq 0, y\geq 0, z\geq 0$. The first equation together with $1+4x^2=(1-2x)^2+4x\geq 4x$ leads to $y=\frac{4x^2}{1+4x^2}\leq\frac{4x^2}{4x}=x$. Similarly, the second and the third equations lead to $z\leq y, x\leq z$. Hence, $x=y=z$, then $\frac{4x^2}{1+4x^2}=x\Rightarrow 4x^3-4x^2+x=0\ \Rightarrow x(2x-1)^2=0\Rightarrow x=0$ or $x=1/2$. Therefore, $(0,0,0),(1/2,1/2,1/2)$ are the solutions.

2.82 ★★★ Find all distinct real roots of the equation $(x^3-3x^2+x-2)(x^3-x^2-4x+7)+6x^2-15x+18=0$.

Solution: Let $A=x^3-2x^2+\frac{3}{2}x+\frac{5}{2}, B=x^2-\frac{5}{2}x+\frac{9}{2}$, then the equation becomes $(A-B)(A+B)+6B-9=0\Rightarrow A^2-(B-3)^2=0\Rightarrow (A+B-3)(A-B+3)=0\Rightarrow A+B-3=0$ or $A-B+3=0$.

If $A+B-3=0$, then $x^3-x^2-4x+4=0\Rightarrow (x-1)(x-2)(x+2)=0\Rightarrow x=1$ or $x=\pm 2$.
If $A-B+3=0$, then $x^3-3x^2+x+1=0\Rightarrow (x-1)(x^2-2x-1)=0\Rightarrow x=1$ or $x=1\pm\sqrt{2}$.
As a conclusion, the equation has four distinct real roots: $x=1, x=\pm 2, x=1\pm\sqrt{2}$.

2.83 ★★★ If a,b,c are real numbers, $ac<0, \sqrt{2}a+\sqrt{3}b+\sqrt{5}c=0$, show the quadratic equation $ax^2+bx+c=0$ has a root within the interval $(\frac{3}{4},1)$.

Proof: Let $f(x) = ax^2 + bx + c$, then

$f(\frac{3}{4}) \cdot f(1) = (\frac{9}{16}a + \frac{3}{4}b + c)(a + b + c) = \frac{1}{16}(9a + 12b + 16c)(a + b + c)$. Since

$\sqrt{2}a + \sqrt{3}b + \sqrt{5}c = 0,\ b = \frac{-\sqrt{6}a - \sqrt{15}c}{3}$, then $(9a + 12b + 16c)(a + b + c) = (9a - 4\sqrt{6}a$

$- 4\sqrt{15}c + 16c)(a - \frac{\sqrt{6}}{3}a - \frac{\sqrt{15}}{3}c + c) = [(\sqrt{81} - \sqrt{96})a + (\sqrt{256} - \sqrt{240})c][\frac{3-\sqrt{6}}{3}a +$

$\frac{3-\sqrt{15}}{3}c] = c^2[(\sqrt{81} - \sqrt{96})\frac{a}{c} + (\sqrt{256} - \sqrt{240})][\frac{3-\sqrt{6}}{3}\frac{a}{c} + \frac{3-\sqrt{15}}{3}] < 0$, thus

$f(\frac{3}{4}) \cdot f(1) < 0$, which implies that one root is within $(\frac{3}{4}, 1)$.

2.84 ★★★ The real numbers a, b satisfy

$$\begin{aligned} ax + by &= 3, \\ ax^2 + by^2 &= 7, \\ ax^3 + by^3 &= 16, \\ ax^4 + by^4 &= 42, \end{aligned}$$

compute $ax^5 + ay^5$ and x, y.

Solution 1: We have

$(ax + by)(x + y) = ax^2 + axy + bxy + by^2 = (ax^2 + by^2) + (a + b)xy$;

$(ax^2 + by^2)(x + y) = ax^3 + ax^2y + bxy^2 + by^3 = (ax^3 + by^3) + (ax + by)xy$;

$(ax^3 + by^3)(x + y) = ax^4 + ax^3y + bxy^3 + by^4 = (ax^4 + by^4) + (ax^2 + by^2)xy$;

$(ax^4 + by^4)(x + y) = ax^5 + ax^4y + bxy^4 + by^5 = (ax^5 + by^5) + (ax^3 + by^3)xy$.

Substitute the given equations into them:

$$\begin{aligned} 3(x + y) &= 7 + (a + b)xy \quad \text{(i)}, \\ 7(x + y) &= 16 + 3xy \quad \text{(ii)}, \\ 16(x + y) &= 42 + 7xy \quad \text{(iii)}, \\ 42(x + y) &= (ax^5 + by^5) + 16xy \quad \text{(iv)}. \end{aligned}$$

(ii)$\times 7-$ (iii)$\times 3$: $x + y = -14$, substitute it into (ii): $xy = -38$. Substitute

$x + y = -14, xy = -38$ into (iv): $ax^5 + by^5 = 42(-14) - 16(-38) = 20$.

In addition, $x + y = -14, xy = -38 \Rightarrow x = -7 - \sqrt{87}, y = -7 + \sqrt{87}$ or

$x = -7 + \sqrt{87}, y = -7 - \sqrt{87}$.

Solution 2: Let $a_n = ax^n + by^n$, then $a_1 = 3, a_2 = 7, a_3 = 16, a_4 = 42$. Let x, y be the two roots of the quadratic equation $t^2 - pt - q = 0$, then $x^2 - px - q = 0 \Rightarrow ax^{n+2} = pax^{n+1} + qax^n$. Similarly, $by^{n+2} = pby^{n+1} + qby^n$. Add them up to obtain $ax^{n+2} + by^{n+2} = p(ax^{n+1} + by^{n+1})$ $+ q(ax^n + by^n) \Rightarrow a_{n+2} = pa_{n+1} + qa_n$.

When $n = 1$, $7p + 3q = 16$.

When $n = 2$, $16p + 7q = 42$.

Solve $7p + 3q = 16$ and $16p + 7q = 42$ to obtain $p = -14, q = 38$, thus $a_{n+2} = -14a_{n+1} + 38a_n$.

Hence, $ax^5 + bx^5 = a_5 = -14 \times 42 + 38 \times 16 = 20$.

Substitute $p = -14, q = 38$ into the equation $x^2 - px - q = 0$: $x^2 + 14x - 38 = 0 \Rightarrow x = -7 \pm \sqrt{87}$.

Substitute $p = -14, q = 38$ into the equation $t^2 - pt - q = 0$: $t^2 + 14t - 38 = 0$. Since x, y are the two roots, then Vieta's formulas imply $x + y = -14$, thus $y = -14 - x = -7 \mp \sqrt{87}$. Hence, the system has two solutions: $(-7 + \sqrt{87}, -7 - \sqrt{87}), (-7 - \sqrt{87}, -7 + \sqrt{87})$.

2.85 ★★ Given $f(x)=\lg(x^2+1)$, solve the equation $f(100^x-10^{x+1})-f(24)=0$.

Solution: The function $f(x)=\lg(x^2+1)$ has the domain $(-\infty,+\infty)$, and it is decreasing on $(-\infty,0)$ and increasing on $(0,+\infty)$. In addition, it is an even function. Hence, $f(100^x-10^{x+1})-f(24)=0 \Leftrightarrow f(100^x-10^{x+1})=f(24) \Leftrightarrow 100^x-10^{x+1}=\pm 24$.

When $100^x-10^{x+1}=24$, we have $(10^x)^2-10\cdot 10^x-24=0 \Rightarrow (10^x+2)(10^x-12)$ $=0 \Rightarrow 10^x=12 \Rightarrow x=\lg 12$ since $10^x+2>0$.

When $100^x-10^{x+1}=-24$, we have $(10^x)^2-10\cdot 10^x+24=0 \Rightarrow (10^x-4)(10^x-6)=0 \Rightarrow 10^x=4$ or $10^x=6 \Rightarrow x=\lg 4$ or $x=\lg 6$. Therefore, the original equation has three roots: $x=\lg 12, x=\lg 4, x=\lg 6$.

2.86 ★★★ The equation $x^4+ax^3+bx^2+ax+1=0$ has at least one real root, where a,b are real numbers. Find the minimum value of a^2+b^2.

Solution: $x=0$ is not a root, so we assume $x\neq 0$ and divide both sides by x^2 to obtain $(x+\frac{1}{x})^2+a(x+\frac{1}{x})+b-2=0$ (i). $(x+\frac{1}{x})^2=x^2+2+\frac{1}{x^2}=(x-\frac{1}{x})^2+4\geq 4$, thus $|x+\frac{1}{x}|\geq 2$. Let $y=x+\frac{1}{x}$, then (i) becomes $y^2+ay+b-2=0$ $(|y|\geq 2)$ (ii). (ii) needs to have a real root and $|y|\geq 2$, then $|\frac{a}{2}|+|\frac{\sqrt{a^2-4(b-2)}}{2}|\geq|\frac{-a\pm\sqrt{a^2-4(b-2)}}{2}|\geq 2$, thus $\sqrt{a^2-4(b-2)}\geq 4-|a|$. Now we are ready to find the minimum value of a^2+b^2. Without loss of generality, assume $a>0$.

(1) When $a\leq 4$, we have $\sqrt{a^2-4(b-2)}\geq 4-a\geq 0$, taking square to obtain $2a\geq b+2$. When $b+2\geq 0$, $b\geq -2$, $4a^2\geq b^2+4b+4$, then $a^2+b^2\geq\frac{1}{4}(b^2+4b+4)+b^2=\frac{5}{4}(b+\frac{2}{5})^2+\frac{4}{5}$. Hence, a^2+b^2 has the minimum value $\frac{4}{5}$ when $b=-\frac{2}{5}$. When $b+2\leq 0$, $b\leq -2$, then $a^2+b^2\geq b^2\geq 4>\frac{4}{5}$.

(2) When $a>4$, we have $a^2+b^2>a^2>16>\frac{4}{5}$.

As a conclusion from (1)(2), a^2+b^2 has the minimum value $\frac{4}{5}$.

2.87 ★★★★ If a,b are distinct prime numbers, show the x,y-dependent equation $\sqrt{x}+\sqrt{y}=\sqrt{ab}$ has no positive integer solution.

Proof: We prove the result by contraction. Assume the equation has a positive solution x,y such that $\sqrt{x}+\sqrt{y}=\sqrt{ab}$ holds. Taking square to obtain $x+y+2\sqrt{xy}=ab$, thus $\sqrt{xy}$ is a rational number. xy is a positive integer whose square root is either a positive integer or a irrational number. Hence, $\sqrt{xy}$ has to be a positive integer.

On the other hand, multiply $\sqrt{x}+\sqrt{y}=\sqrt{ab}$ by $\sqrt{x}$: $x+\sqrt{xy}=\sqrt{abx}$, thus $\sqrt{abx}$ is a positive integer. Since a, b are distinct prime numbers, then $x=abt^2$, $t\in\mathcal{N}$. Same logic follows for y: $y=abs^2$, $s\in\mathcal{N}$. Therefore, $\sqrt{x}+\sqrt{y}=\sqrt{ab}$ becomes $\sqrt{ab}(t+s)=\sqrt{ab}\Rightarrow t+s=1$, a contradiction to $t+s\geq 2$. As a result, $\sqrt{x}+\sqrt{y}=\sqrt{ab}$ has no positive integer solution.

2.88 ★★★ The real numbers x, y, z satisfy the equations

$$\begin{aligned} x+y+z &= 2, \\ xyz &= 4. \end{aligned}$$

(1) Find the minimum value of the largest one of x, y, z; (2) Find the minimum value of $|x|+|y|+|z|$.

Solution: (1) Without loss of generality, assume x is the largest one among x, y, z, that is, $x\geq y, x\geq z$. The first equation implies that $x>0$ and $y+z=2-x$, and the second equation implies $yz=\frac{4}{x}$, thus y, z are the two roots of the quadratic equation $u^2-(2-x)u+\frac{4}{x}=0$. The discriminant $\Delta=(2-x)^2-4\cdot\frac{4}{x}\geq 0\Rightarrow x^3-4x^2+4x-16\geq 0\Rightarrow (x^2+4)(x-4)\geq 0\Rightarrow x-4\geq 0\Rightarrow x\geq 4$. Hence, $x=4$ is the minimum value of the largest one of x, y, z. At this time, $y=z=-1$.

(2) Since $xyz>0$, then x, y, z are all positive, or they are one positive two negative. If x, y, z are all positive, (1) implies $x\geq 4$, a contradiction to $x+y+z=2$. If x, y, z are one positive two negative, without loss of generality we assume $x>0, y<0, z<0$, then $|x|+|y|+|z|=x-y-z=x-(y+z)=x-(2-x)=2x-2$. (1) implies $x\geq 4$, thus $2x-2\geq 6$. $x=4, y=z=-1$ satisfy all conditions and the equal sign is obtained in the inequality. Hence, the minimum value of $|x|+|y|+|z|$ is 6.

2.89 ★★★ a, b, c are nonzero real numbers, solve the system of equations

$$\begin{aligned} (x+y)(x+z) &= a^2, \quad \text{(i)} \\ (y+z)(x+y) &= b^2, \quad \text{(ii)} \\ (x+z)(y+z) &= c^2. \quad \text{(iii)} \end{aligned}$$

Solution 1: (i)$\times$ (ii)/(iii), (i)$\times$ (iii)/(ii), (ii)$\times$ (iii)/(i) $\Rightarrow$

$$
\begin{aligned}
(x+y)^2 &= \frac{a^2b^2}{c^2} \\
(x+z)^2 &= \frac{a^2c^2}{b^2} \\
(y+z)^2 &= \frac{b^2c^2}{a^2}
\end{aligned}
$$

$\Rightarrow$

$$
\begin{aligned}
x+y &= \pm\frac{ab}{c} \quad \text{(iv)} \\
x+z &= \pm\frac{ac}{b} \quad \text{(v)} \\
y+z &= \pm\frac{bc}{a} \quad \text{(vi)}
\end{aligned}
$$

$[\text{(iv)} + \text{(v)} - \text{(vi)}]/2 \Rightarrow x = \pm\frac{a^2b^2+a^2c^2-b^2c^2}{2abc}$.

$[\text{(iv)} + \text{(vi)} - \text{(v)}]/2 \Rightarrow y = \pm\frac{a^2b^2+b^2c^2-a^2c^2}{2abc}$.

$[\text{(v)} + \text{(vi)} - \text{(iv)}]/2 \Rightarrow z = \pm\frac{a^2c^2+b^2c^2-a^2b^2}{2abc}$.

Obviously (iv)(v)(iv) should have the same sign on the right hand side. Hence, the original system has two solutions: $(\frac{a^2b^2+a^2c^2-b^2c^2}{2abc}, \frac{a^2b^2+b^2c^2-a^2c^2}{2abc}, \frac{a^2c^2+b^2c^2-a^2b^2}{2abc})$,

$(-\frac{a^2b^2+a^2c^2-b^2c^2}{2abc}, -\frac{a^2b^2+b^2c^2-a^2c^2}{2abc}, -\frac{a^2c^2+b^2c^2-a^2b^2}{2abc})$.

Solution 2: (i) × (ii) × (iii): $(x+y)^2(x+z)^2(y+z)^2 = a^2b^2c^2 \Rightarrow (x+y)(x+z)(y+z) = \pm abc$ (iv). (iv)/(i),(iv)/(ii),(iv)/(iii) $\Rightarrow y+z = \pm\frac{bc}{a}, x+z = \pm\frac{ac}{b}, x+y = \pm\frac{ab}{c}$. The right hand side should have the same sign, thus

$$\begin{aligned} y+z &= \frac{bc}{a} \\ x+z &= \frac{ac}{b} \\ x+y &= \frac{ab}{c} \end{aligned}$$

or

$$\begin{aligned} y+z &= -\frac{bc}{a} \\ x+z &= -\frac{ac}{b} \\ x+y &= -\frac{ab}{c} \end{aligned}$$

They lead to the two solutions same as Solution 1.

2.90 ★★★★ Nonnegative real numbers x, y, z satisfy $4^{\sqrt{5x+9y+4z}} - 68 \times 2^{\sqrt{5x+9y+4z}} + 256 = 0$ Find the maximum and minimum values of $x+y+z$.

Solution: Let $2^{\sqrt{5x+9y+4z}} = t$, then $t^2 - 68t + 256 = 0 \Rightarrow (t-4)(t-64) = 0 \Rightarrow t = 4$ or $t = 64$.

When $t = 4$, $2^{\sqrt{5x+9y+4z}} = 4 = 2^2 \Rightarrow \sqrt{5x+9y+4z} = 2 \Rightarrow 5x+9y+4z = 4$.

When $t = 64$, $2^{\sqrt{5x+9y+4z}} = 64 = 2^6 \Rightarrow \sqrt{5x+9y+4z} = 6 \Rightarrow 5x+9y+4z = 36$.

Since x, y, z are nonnegative real numbers, $4(x+y+z) \le 5x+9y+4z \le 9(x+y+z)$.

When $5x+9y+4z = 36$, $x+y+z \le 9$, thus $x+y+z$ has the maximum value 9, which can be obtained when $x = y = 0, z = 9$.

When $5x+9y+4z = 4$, $x+y+z \ge 4/9$, thus $x+y+z$ has the minimum value 4/9, which can be obtained when $x = z = 0, y = 4/9$.

2.91 ★★★★ Solve the equation $x^3 - [x] = 3$.

Solution: $x = [x] + \{x\} \Rightarrow [x] = x - \{x\}$, then the equation is equivalent to $x^3 - (x - \{x\}) = 3 \Leftrightarrow x^3 - x = 3 - \{x\}$. Since $0 \leq \{x\} < 1$, then $2 < x^3 - x \leq 3 \Leftrightarrow 2 < (x-1)x(x+1) \leq 3$ (✠). When $x \leq -1$, $(x-1)x(x+1) < 0$, (✠) has no solution. When $x \geq 2$, $x^3 - x = x(x^2-1) \geq 2(2^2-1) = 6$, (✠) has no solution. When $1 < x < 2$, $[x] = 1$, then the original equation becomes $x^3 - 1 = 3 \Rightarrow x^3 = 4 \Rightarrow x = \sqrt[3]{4}$, which is the root of the original equation.

2.92 ★★★★ The x-relevant equation $(a^2-1)(\frac{x}{x-1})^2 - (2a+7)(\frac{x}{x-1} + 1 = 0$ has real roots. (1) Find the range of the parameter a. (2) If the equation has two real roots x_1, x_2, and $\frac{x_1}{x_1-1} + \frac{x_2}{x_2-1} = \frac{3}{11}$, find the value of a.

Solution: (1) Let $\frac{x}{x-1} = t$, $t \neq 1$, then the equation becomes $(a^2-1)t^2 - (2a+7)t + 1 = 0$.

When $a^2 - 1 = 0$, $a = \pm 1$, the equation is equivalent to $-9t + 1 = 0$ or $-5t + 1 = 0$, thus $t = \frac{1}{9}$ or $t = \frac{1}{5}$. When $t = \frac{1}{9}$, $\frac{x}{x-1} = \frac{1}{9}$ whose root is $x = -\frac{1}{8}$. When $t = \frac{1}{5}$, $\frac{x}{x-1} = \frac{1}{5}$ whose root is $x = -\frac{1}{4}$. Hence, the original equation has real roots when $a = \pm 1$.

When $a \neq \pm 1$, the equation $(a^2-1)t^2 - (2a+7)t + 1 = 0$ has real roots if and only if $\Delta = (2a+7)^2 - 4(a^2-1) = 28a + 53 \geq 0$ which implies $a \geq -\frac{53}{28}$. When $a = -\frac{53}{28}$, the equation $(a^2-1)t^2 - (2a+7)t + 1 = 0$ has two identical roots which are not one. Hence, when $a \geq -\frac{53}{28}$, the original equation has real roots, that is, the range of a is $[-\frac{53}{28}, +\infty)$.

(2) Since $\frac{x_1}{x_1-1}$, $\frac{x_2}{x_2-1}$ are the two roots of $(a^2-1)t^2 - (2a+7)t + 1 = 0$, Vieta's formulas imply $\frac{x_1}{x_1-1} + \frac{x_2}{x_2-1} = \frac{2a+7}{a^2-1}$. On the other hand, we have $\frac{x_1}{x_1-1} + \frac{x_2}{x_2-1} = \frac{3}{11}$, thus $\frac{2a+7}{a^2-1} = \frac{3}{11} \Rightarrow 3a^2 - 22a \quad - 80 = 0 \Rightarrow (a-10)(3a+8) = 0 \Rightarrow a = 10$ or $a = -\frac{8}{3}$. Since $a \geq -\frac{53}{28}$, then $a = 10$ is the only possibility.

2.93 ★★★ Solve the equation $2\log_x a + \log_{ax} a + 3\log_{a^2x} a = 0$.

Solution: If $a = 1$, then the equation becomes $6\log_x 1 = 0$ whose solution set is $x > 0$ but $x \neq 1$.

If $a>0$ but $a\neq 1$, then $x>0, x\neq 1, x\neq \frac{1}{a}, x\neq \frac{1}{a^2}$ and $\log_x a = \frac{1}{\log_a x}, \log_{ax} a = \frac{\log_a a}{\log_a a+\log_a x} = \frac{1}{1+\log_a x}, \log_{a^2x} a = \frac{\log_a a}{2\log_a a+\log_a x} = \frac{1}{2+\log_a x}$. The original equation is equivalent to $\frac{2}{\log_a x} + \frac{1}{1+\log_a x} + \frac{3}{2+\log_a x} = 0$. Let $t=\log_a x$, then $\frac{2}{t}+\frac{1}{1+t}+\frac{3}{2+t}=0 \Rightarrow \frac{6t^2+11t+4}{t(1+t)(2+t)}=0$ $\Rightarrow 6t^2+11t+4=0 \Rightarrow (3t+4)(2t+1)=0 \Rightarrow t=-\frac{4}{3}$ or $t=-\frac{1}{2}$. When $t=-\frac{4}{3}$, then $\log_a x = -\frac{4}{3} \Rightarrow x=a^{-\frac{4}{3}}$. When $t=-\frac{1}{2}$, then $\log_a x=-\frac{1}{2} \Rightarrow x=a^{-\frac{1}{2}}$. It is not difficult to verify that $x=a^{-\frac{4}{3}}, x=a^{-\frac{1}{2}}$ are roots of the original equation.

2.94 ★★★ The coefficients of the last three terms of the expansion of $(x^{\lg x}+1)^n$ are positive integer roots of the equation $3^{y^2}\cdot 9^{-10y}\cdot 81^{-11}=1$. The middle term of the expansion is the root of the equation $3\sqrt{\frac{m}{2}}=0.1^{-2}+\sqrt{2m}$, find the value of x.

Solution:

$3^{y^2}\cdot 9^{-10y}\cdot 81^{-11}=1\Leftrightarrow 3^{y^2}\cdot 3^{-20y}\cdot 3^{-44}=3^0\Rightarrow y^2-20y-44=0\Rightarrow (y+2)(y-22)=0\Rightarrow y=-2$ or $y=22$. We only need positive integer roots, so $y=22$. The coefficients of the last three terms are $C_n^{n-2}+C_n^{n-1}+C_n^n=22$, then

$C_n^2+C_n^1+1=2\Rightarrow \frac{n(n-1)}{2}+n=21\Rightarrow n^2+n-42=0\Rightarrow (n+7)(n-6)=0\Rightarrow n=6$ since $n+7>0$.

$3\sqrt{\frac{m}{2}}=0.1^{-2}+\sqrt{2m}\Leftrightarrow \frac{3}{2}\sqrt{2m}=100+\sqrt{2m}\Rightarrow \sqrt{2m}=200\Rightarrow m=20000$.
Since $n=6$, the middle term of the expansion is
$T_4=C_6^3(x^{\lg x})^3=\frac{6\times5\times4}{3\times2\times1}(x^{\lg x})^3=20x^{3\lg x}$. According to the condition of the problem, we have $20x^{3\lg x}=20000\Rightarrow x^{3\lg x}=1000\Rightarrow \lg x^{3\lg x}=\lg 1000\Rightarrow 3(\lg x)^2=3\Rightarrow \lg x=\pm1\Rightarrow x=10$ or $x=1/10$.

2.95 ★★★★ Let p be an odd prime number, find all positive integer roots of the equation $x^2=y(y+p)$.

Solution: $x^2=y(y+p)\Leftrightarrow (x+y)(x-y)=py$. Since p is a prime number, we have $p|x-y$ or $p|x+y$. If $p|x-y$, then $x-y\geq p$ (note that $x>y$), thus we should have $x+y\leq y$, impossible. Thus $p|x+y$. Let $x+y=pn$ (i), where n is a positive integer, then the original equation becomes $n(x-y)=y$, thus $n|y$ and $x=\frac{n+1}{n}y$. Hence, $x+y=\frac{2n+1}{n}y$ (ii). (i)&(ii) lead to $(2n+1)y=n^2p$. Since $(n^2,2n+1)=1$, we have $n^2|y$. $(2n+1)\frac{y}{n^2}=p$ and p is a prime, thus $\frac{y}{n^2}=1$, then $p=2n+1\Rightarrow n=\frac{p-1}{2}$, then $y=n^2=(\frac{p-1}{2})^2$, $x=\frac{n+1}{n}y=\frac{n+1}{n}n^2=n(n+1)=\frac{p-1}{2}\cdot\frac{p+1}{2}=\frac{p^2-1}{2}$. Therefore, the original equation has only one positive integer root: $x=\frac{p^2-1}{2}, y=(\frac{p-1}{2})^2$.

2.96 ★★★★ Consider the real coefficient equations

$$\begin{aligned} ax_1^2+bx_1+c &= x_2\\ ax_2^2+bx_2+c &= x_3\\ &\vdots\\ ax_{n-1}^2+bx_{n-1}+c &= x_n\\ ax_n^2+bx_n+c &= x_1 \end{aligned}$$

where $a\neq 0$, show that when $\Delta=(b-1)^2-4ac=0$, this equation system has a unique solution.

Proof: The system is equivalent to

$$\begin{array}{rcl} ax_1^2+(b-1)x_1+c &=& x_2-x_1 \\ ax_2^2+(b-1)x_2+c &=& x_3-x_2 \\ &\vdots& \\ ax_{n-1}^2+(b-1)x_{n-1}+c &=& x_n-x_{n-1} \\ ax_n^2+(b-1)x_n+c &=& x_1-x_n \end{array}$$

and we can observe that $\Delta=(b-1)^2-4ac$ is the discriminant of the quadratic equation $ax^2+(b-1)x+c=0$. When $\Delta=0$, $ax_i^2+(b-1)x_i+c$ is a perfect square, i.e. $ax_i^2+(b-1)x_i+c=a(x_i+\frac{b-1}{2a})^2$ $(i=1,2,3,\cdots,n)$. If $a<0$, then $a(x_i+\frac{b-1}{2a})^2\leq 0$, thus $x_2-x_1\leq 0, x_3-x_2\leq 0,\cdots,x_n-x_{n-1}\leq 0, x_1-x_n\leq 0$, which is equivalent to $x_1\geq x_2\geq x_3\geq\cdots\geq x_{n-1}\geq x_n\geq x_1$. Hence, we can only choose equal sign in all these inequalities, that is, $x_1=x_2=x_3=\cdots=x_n=\frac{1-b}{2a}$. If $a>0$, same logic follows to obtain $x_1=x_2=x_3=\cdots=x_n=\frac{1-b}{2a}$. As a conclusion, the equation system has a unique solution when $\Delta=0$.

2.97 ★★★★★ Given $f(1)=\frac{1}{5}$ and when $n>1$, $\frac{f(n-1)}{f(n)}=\frac{2nf(n-1)+1}{1-2f(n)}$, find $f(n)$.

Solution: Multiply $\frac{f(n-1)}{f(n)}=\frac{2nf(n-1)+1}{1-2f(n)}$ by $f(n)[1-2f(n)]$ to obtain $f(n-1)-2f(n-1)f(n)=2nf(n-1)f(n)+f(n)$, which is equivalent to $f(n-1)-f(n)=2(n+1)f(n)f(n-1)$. Divide both sides by $f(n)f(n-1)$ to obtain $\frac{1}{f(n)}-\frac{1}{f(n-1)}=2(n+1)$. Replace n with $2,3,\cdots,n$ successively to obtain $\frac{1}{f(2)}-\frac{1}{f(1)}=2\times 3, \frac{1}{f(3)}-\frac{1}{f(2)}=2\times 4,\cdots,\frac{1}{f(n)}-\frac{1}{f(n-1)}=2(n+1)$. Add them up to obtain $\frac{1}{f(n)}-\frac{1}{f(1)}=2[3+4+\cdots+(n+1)]=2\times\frac{(3+n+1)(n-1)}{2}=(n-1)(n+4)$. Hence, $\frac{1}{f(n)}=\frac{1}{f(1)}+(n-1)(n+4)=5+n^2+3n-4=n^2+3n+1$.

As a conclusion, $f(n)=\frac{1}{n^2+3n+1}$.

2.98 ★★★★ Solve the equation $\frac{1}{2}(a^x+a^{-x})=m$.

Solution: Multiply the equation by $2a^x$ and reorganize it to obtain $a^{2x} - 2ma^x + 1 = 0$. Let $t = a^x$ $(t > 0)$, then $t^2 - 2mt + 1 = 0$. When $\Delta = 4m^2 - 4 \geq 0$, i.e. $m \geq 1$ or $m \leq -1$, the t-dependent equation has real roots: $t_1 = m - \sqrt{m^2-1}, t_2 = m + \sqrt{m^2-1}$. If $m = 1$, then $t_1 = t_2 = 1$, thus $a^x = 1$, the original equation has a unique root $x = 0$. If $m > 1$, $m + \sqrt{m^2-1} > m - \sqrt{m^2-1} > 0$, then $a^x = m \pm \sqrt{m^2-1}$, that is, the original equation has two distinct real roots: $x = \log_a(m \pm \sqrt{m^2-1})$. If $m < 1$, since $|m| > \sqrt{m^2-1}$, then $m < -\sqrt{m^2-1}$, then $m - \sqrt{m^2-1} \leq m + \sqrt{m^2-1} < 0$. $a^x = m \pm \sqrt{m^2-1}$ has no solution since $a^x > 0$. As a conclusion, when $m < 1$, the original equation has no root; when $m = 1$, the original equation has a unique root $x = 0$; when $m > 1$, the original equation has two distinct roots $x = \log_a(m \pm \sqrt{m^2-1})$.

2.99 ★★★★ Solve the system of equations

$$\begin{aligned} x^4 + y^2 + z &= 18 \\ x^2y - yz^{1/2} &= -3 \\ z^{1/2}x^2 &= 4 \end{aligned}$$

to obtain real solutions.

Solution: Let $x^2 = u, y = v, z^{1/2} = t$, then the system becomes

$$\begin{aligned} u^2 + v^2 + t^2 &= 18 \quad \text{(i)} \\ uv - vt &= -3 \quad \text{(ii)} \\ ut &= 4 \quad \text{(iii)} \end{aligned}$$

(i)+(ii)×2-(iii)×2: $u^2+v^2+t^2+2(uv-vt-ut) = 4 \Leftrightarrow (u+v-t)^2 = 4 \Rightarrow u+v-t = \pm 2 \Rightarrow$ $u-t = -v\pm 2$, substitute it into (ii): $v^2 \pm 2v - 3 = 0$ whose roots are $v = \pm 1$ or $v = \pm 3$. Substitute the values of v into (ii)(iii):

$$\begin{aligned} u - t &= -3 \\ ut &= 4 \end{aligned}$$

$$\begin{aligned} t - u &= -3 \\ ut &= 4 \end{aligned}$$

$$\begin{aligned} 3u - 3t &= -3 \\ ut &= 4 \end{aligned}$$

$$\begin{aligned} 3t - 3u &= -3 \\ ut &= 4 \end{aligned}$$

Solve them to obtain $(u, v, t) = (1, 1, 4), (-4, 1, -1), (4, -1, 1), (-1, -1, -4),$ $(\frac{\sqrt{17}-1}{2}, 3, \frac{\sqrt{17}+1}{2}), (\frac{\sqrt{17}+1}{2}, 3, \frac{\sqrt{17}-1}{2}), (\frac{\sqrt{17}+1}{2}, -3, \frac{\sqrt{17}-1}{2}), (\frac{\sqrt{17}-1}{2}, -3, \frac{\sqrt{17}+1}{2})$. Notice that the second, fourth, sixth, eighth solutions have negative $u = x^2$ which is impossible, therefore all possible solutions of the original system are

x	± 1	± 2	$\pm\sqrt{\frac{\sqrt{17}-1}{2}}$	$\pm\sqrt{\frac{\sqrt{17}+1}{2}}$
y	1	-1	3	-3
z	16	1	$\frac{9+\sqrt{17}}{2}$	$\frac{9-\sqrt{17}}{2}$

2.100 ★★★★ Find the polynomial $p(x)$ defined on a set of real numbers such that $p(0) = 0$ and $p(x^2 + 1) = [p(x)]^2 + 1$.

Solution: Let $x = 0$ and substitute into $p(x^2 + 1) = [p(x)]^2 + 1$to obtain $p(1) = [p(0)]^2 + 1 = 1$ since $p(0) = 0$. Choose $x = 1, 2$ to obtain $p(2) = [p(1)]^2 + 1 = 2, p(5) = [p(2)]^2 + 1 = 5$. Keep going, we have $p(26) = [p(5)]^2 + 1 = 26, p(26^2 + 1) = [p(26)]^2 + 1 = 26^2 + 1, \cdots$. Hence, the equation $p(x) - x = 0$ has infinitely many roots: $0, 1, 2, 5, 26, 26^2 + 1, \cdots$. Since $p(x) - x$ is a polynomial, $p(x) - x = 0$ always holds, that is, $p(x) = x$.

3 INEQUALITIES

3.1 Determine the order of the numbers $\frac{4}{9}, \log_5 2, \frac{2}{5}$.

Solution: $\frac{4}{9} - \log_5 2 = \frac{4}{9} - \frac{\lg 2}{\lg 5} = \frac{4\lg 5 - 9\lg 2}{9\lg 5} = \frac{\lg 5^4 - \lg 2^9}{9\lg 5} = \frac{\lg 625 - \lg 512}{9\lg 5} > 0$, thus $\frac{4}{9} > \log_5 2$. $\log_5 2 - \frac{2}{5} = \frac{\lg 2}{\lg 5} - \frac{2}{5} = \frac{5\lg 2 - 2\lg 5}{5\lg 5} = \frac{\lg 32 - \lg 25}{5\lg 5} > 0$, thus $\log_5 2 > \frac{2}{5}$. Hence, $\frac{4}{9} > \log_5 2 > \frac{2}{5}$.

3.2 Solve the inequality $\frac{1}{100} < \log_{0.1}^2 x < 1$.

Solution: $\frac{1}{100} < \log_{0.1}^2 x < 1 \Rightarrow 1 < \log_{0.1} x < \frac{1}{10}$ or $-1 < \log_{0.1} x < -\frac{1}{10} \Rightarrow 0.1 < x < \sqrt[10]{0.1}$ or $\sqrt[10]{10} < x < 10$.

3.3 a, b, c, d are positive numbers, show $\sqrt{(a+c)(b+d)} \geq \sqrt{ab} + \sqrt{cd}$.

Proof: $ad + bc > 2\sqrt{abcd} \Leftrightarrow ad + bc + ab + cd > ab + 2\sqrt{abcd} + cd \Leftrightarrow (a+c)(b+d) > (\sqrt{ab} + \sqrt{cd})^2 \Leftrightarrow \sqrt{(a+c)(b+d)} \geq \sqrt{ab} + \sqrt{cd}$ since $a, b, c, d > 0$.

3.4 ★ Given $-1 \leq u + v \leq 1, 1 \leq u - 2v \leq 3$, find the range of $2u + 5v$.

Solution: Let $2u + 5v = \lambda_1(u+v) + \lambda_2(u - 2v) = (\lambda_1 + \lambda_2)u + (\lambda_1 - 2\lambda_2)v$, then $\lambda_1 + \lambda_2 = 2$ and $\lambda_1 - 2\lambda_2 = 5$. Solve them to obtain $\lambda_1 = 3, \lambda_2 = -1$. Hence, $2u + 5v = 3(u+v) - 1(u - 2v) \in [(-1) \times 3 - 3, 1 \times 3 - 1] = [-6, 2]$.

3.5 Given $|h| < \frac{\varepsilon}{4}, |k| < \frac{\varepsilon}{6}$, show $|2h - 3k| < \varepsilon$.

Proof: $|h| < \frac{\varepsilon}{4} \Leftrightarrow -\frac{\varepsilon}{4} < h < \frac{\varepsilon}{4} \Leftrightarrow -\frac{\varepsilon}{2} < 2h < \frac{\varepsilon}{2}$.
$|k| < \frac{\varepsilon}{6} \Leftrightarrow -\frac{\varepsilon}{6} < k < \frac{\varepsilon}{6} \Leftrightarrow -\frac{\varepsilon}{2} < 3k < \frac{\varepsilon}{2}$.
Hence, $-\varepsilon < 2h - 3k < \varepsilon \Leftrightarrow |2h - 3k| < \varepsilon$.

3.6 Show the inequality $\frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdot \cdots \cdot \frac{99}{100} < \frac{1}{10}$.

Proof: $\frac{1}{2} < \frac{2}{3}, \frac{3}{4} < \frac{4}{5}, \cdots, \frac{97}{98} < \frac{98}{99}, \frac{99}{100} < \frac{100}{101}$. Multiply all these inequalities: $\frac{1}{2} \cdot \frac{3}{4} \cdot \cdots \cdot \frac{97}{98} \cdot \frac{99}{100} < \frac{2}{3} \cdot \frac{4}{5} \cdot \cdots \cdot \frac{98}{99} \cdot \frac{100}{101}$. Multiply this inequality by $\frac{1}{2} \cdot \frac{3}{4} \cdot \cdots \cdot \frac{97}{98} \cdot \frac{99}{100}$: $(\frac{1}{2} \cdot \frac{3}{4} \cdot \cdots \cdot \frac{97}{98} \cdot \frac{99}{100})^2 < \frac{1}{101}$. Take the square root to obtain $\frac{1}{2} \cdot \frac{3}{4} \cdot \cdots \cdot \frac{97}{98} \cdot \frac{99}{100} < \frac{1}{\sqrt{101}} < \frac{1}{10}$.

3.7 Solve the inequality $\lg(x^2-x-6)<\lg(2-3x)$.

Solution: The inequality holds if and only if

$$\begin{array}{rcl} x^2-x-6 & > & 0 \\ 2-3x & > & 0 \\ x^2-x-6 & < & 2-3x \end{array}$$

$\Rightarrow$

$$\begin{array}{c} x<-2 \quad \text{or} \quad x>3 \\ x<\frac{2}{3} \\ -4<x<2 \end{array}$$

$\Rightarrow -4<x<-2$, which is the solution of the original inequality.

3.8 ★ a, b are real numbers and $a^3+b^3=2$, show $a+b\le 2$.

Proof: Suppose $a+b>2$, then $b>2-a \Rightarrow b^3>(2-a)^3=8-12a+6a^2-a^3 \Rightarrow a^3+b^3>8-12a+6a^2=6a^2-12a+6+2=6(a-1)^2+2>2$, a contradiction to $a^3+b^3=2$. Hence, $a+b\le 2$.

3.9 The rational numbers a, b, c, d satisfy $d > c$ (i), $a + b = c + d$ (ii), $a + d < b + c$ (iii), determine the order of these four numbers.

Solution: (i) $\Rightarrow b + d > b + c$. This together with (iii) implies $a + d < b + d$, thus $a < b$. (iii)-(ii) $\Rightarrow d - b < b - d \Rightarrow d < b$. (ii) $\Rightarrow b - d = c - a$, since $b - d > 0$, then $c - a > 0$, i.e. $c > a$. As a conclusion, we obtain the order $a < c < d < b$.

3.10 If the inequality $ax^2 + bx - 6 < 0$ has the solution set $\{x| - 2 < x < 3\}$, find the values of a and b.

Solution: The condition implies that the equation $ax^2 + bx - 6 = 0$ has two roots $x = -2, x = 3$. Vieta's formulas imply

$$\begin{aligned} -2 + 3 &= -\frac{a}{b} \\ (-2) \times 3 &= -\frac{6}{a} \end{aligned}$$

from which we can obtain $a = 1, b = -1$.

3.11 Given $2x + 6y \leq 15, x \geq 0, y \geq 0$, find the maximum value of $4x + 3y$.

Solution: $2x + 6y \leq 15 \Leftrightarrow y \leq \frac{15-2x}{6} = \frac{5}{2} - \frac{1}{3}x$, thus $4x + 3y \leq 4x + \frac{15}{2} - x = 3x + \frac{15}{2}$. $y \geq 0 \Rightarrow \frac{5}{2} - \frac{1}{3}x \geq 0 \Rightarrow x \leq \frac{15}{2}$. Hence, $4x + 3y \leq 3 \times \frac{15}{2} + \frac{15}{2} = 30$, which implies that the maximum value of $4x + 3y$ is 30.

3.12 ★ Given $(m + 1)x^2 - 2(m - 1)x + 3(m - 1) < 0$, find all real values of m such that the inequality has no solution.

Solution: The inequality has no solution if and only if

$$\begin{aligned} \Delta = 4(m-1)^2 - 12(m+1)(m-1) &\leq 0 \\ m + 1 &> 0 \end{aligned}$$

$\Rightarrow$

$$\begin{aligned} m^2 + m - 2 &\geq 0 \\ m + 1 &> 0 \end{aligned}$$

$\Rightarrow$

$$\begin{gathered} m \leq -2 \quad \text{or} \quad m \geq 1 \\ m > -1 \end{gathered}$$

$\Rightarrow m \geq 1$.

3.13 The inequality $\sqrt{x} > ax + \frac{3}{2}$ has the solution set $\{x|4 < x < b\}$, find the values of a and b.

Solution: $\sqrt{x} > ax + \frac{3}{2} \Leftrightarrow a(\sqrt{x})^2 - \sqrt{x} + \frac{3}{2} < 0$. The solution set $\{x|4 < x < b\}$ is equivalent to $\{x|2 < \sqrt{x} < \sqrt{b}\}$. Vieta's formulas imply

$$\begin{aligned} 2 + \sqrt{b} &= \frac{1}{a} \\ 2\sqrt{b} &= \frac{3}{2a} \\ a &> 0 \end{aligned}$$

$\Rightarrow a = \frac{1}{8}, b = 36.$

3.14 If the inequality $x^2 - ax - 6a \leq 0$ has solutions, and the two roots x_1, x_2 of $x^2 - ax - 6a = 0$ satisfy $|x_1 - x_2| \leq 5$. Find the range of the real number a.

Solution: The inequality has solutions if and only if $\Delta = a^2 + 24a \geq 0 \Leftrightarrow a \geq 0$ or $a \leq -24$. Vieta's formulas imply

$$\begin{aligned} x_1 + x_2 &= a \\ x_1 x_2 &= -6a \end{aligned}$$

$|x_1 - x_2| = \sqrt{(x_1 - x_2)^2} = \sqrt{(x_1 + x_2)^2 - 4x_1x_2} = \sqrt{a^2 + 24a} \leq 5$, that is
$a^2 + 24a - 25 \leq 0 \Rightarrow (a + 25)(a - 1) \leq 0 \Rightarrow -25 \leq a \leq 1$.
As a conclusion, a has the range: $-25 \leq a \leq -24$ or $0 \leq a \leq 1$.

3.15 ★ If $0 < a < 1, 0 < b < 1, 0 < c < 1$, show it is impossible that $(1 - a)b, (1 - b)c, (1 - c)a$ are all greater than $1/4$.

Proof 1: We prove the conclusion by contraction. Suppose
$(1 - a)b > \frac{1}{4}, (1 - b)c > \frac{1}{4}, (1 - c)a > \frac{1}{4}$. Multiply them to obtain
$abc(1 - a)(1 - b)(1 - c) > \frac{1}{64}$. On the other hand, $0 < (1 - a)a \leq [\frac{1-a+a}{2}]^2 = \frac{1}{4}$, similarly we have $0 < (1 - b)b \leq \frac{1}{4}$, $0 < (1 - c)c \leq \frac{1}{4}$. Multiply them to obtain
$abc(1 - a)(1 - b)(1 - c) \leq \frac{1}{64}$, a contractions.

Proof 2: Since $0 < a < 1, 0 < b < 1, 0 < c < 1$, we let $a = \sin^2 \alpha, b = \sin^2 \beta, c = \sin^2 \gamma$, then $(1-a)b \cdot (1-b)c \cdot (1-c)a = abc(1-a)(1-b)(1-c) = \sin^2 \alpha \sin^2 \beta \sin^2 \gamma \cos^2 \alpha \cos^2 \beta \cos^2 \gamma =$ $\frac{1}{64} \sin^2 2\alpha \sin^2 2\beta \sin^2 2\gamma \leq \frac{1}{64}$, thus it is impossible that $(1 - a)b, (1 - b)c, (1 - c)a$ are all greater than $1/4$.

3.16 If $-1 < x < 1, -1 < y < 1$, show $|\frac{x+y}{1+xy}| < 1$.

Proof: $|\frac{x+y}{1+xy}| < 1 \Leftrightarrow (\frac{x+y}{1+xy})^2 < 1 \Leftrightarrow (x+y)^2 < (1+xy)^2 \Leftrightarrow x^2+y^2 < 1+x^2y^2 \Leftrightarrow (x^2-1)(1-y^2) < 0$, which is obviously valid since $-1 < x < 1, -1 < y < 1$.

3.17 ★★ Given $f(x) = \lg \frac{1+2^x+a\cdot 4^x}{3}$ $(a \in \mathcal{R})$, (1) $f(x)$ is well defined when $x \le 1$, find the range of a, (2) if $0 < a \le 1$, show $2f(x) < f(2x)$ when $x \ne 1$.

Solution: (1) Since $1+2^x+a\cdot 4^x > 0$, $a > -[(\frac{1}{4})^x + (\frac{1}{2})^x]$. Since $(\frac{1}{4})^x, (\frac{1}{2})^x$ are decreasing functions on the interval $(-\infty, 1]$, then $-[(\frac{1}{4})^x + (\frac{1}{2})^x]$ reaches the maximum value $-(\frac{1}{4}+\frac{1}{2}) = -\frac{3}{4}$ at $x = 1$, thus $a > -\frac{3}{4}$.

(2) Use the inequality $\frac{a+b+c}{3} < \sqrt{\frac{a^2+b^2+c^2}{3}}$ to obtain $(1+2^x+a\cdot 4^x)^2 < 3(1+4^x+a^2\cdot 16^x) <$ $3(1+4^x+a\cdot 16^x) \Rightarrow \frac{1+4^x+a\cdot 16^x}{3} > (\frac{1+2^x+a\cdot 4^x}{3})^2$, that is $f(2x) > 2f(x)$.

3.18 ★ If $a, b, c > 0$, show $2(\frac{a+b}{2} - \sqrt{ab}) \le 3(\frac{a+b+c}{3} - \sqrt[3]{abc})$.

Proof: $2(\frac{a+b}{2} - \sqrt{ab}) \le 3(\frac{a+b+c}{3} - \sqrt[3]{abc}) \Leftrightarrow a+b-2\sqrt{ab} \le a+b+c-3\sqrt[3]{abc} \Leftrightarrow c+2\sqrt{ab} \ge 3\sqrt[3]{abc}$. We only need to show the last inequality. $c + 2\sqrt{ab} = c + \sqrt{ab} + \sqrt{ab} \ge 3\sqrt[3]{c \cdot \sqrt{ab} \cdot \sqrt{ab}} = 3\sqrt[3]{abc}$.

3.19 ★ Given the function $f(x) = \frac{2^{x+3}}{4^x+8}$, (1) find the maximum value of $f(x)$, (2) show $f(a) < b^2 - 4b + \frac{11}{2}$ for any real numbers a, b.

Solution: (1) $f(x) = \frac{2^{x+3}}{4^x+8} = \frac{8}{2^x+\frac{8}{2^x}} \le \frac{8}{2\sqrt{2^x \cdot \frac{8}{2^x}}} = \frac{8}{4\sqrt{2}} = \sqrt{2}$, thus $f(x)_{max} = \sqrt{2}$.

(2) Since $f(a) \le \sqrt{2}$ and $b^2 - 4b + \frac{11}{2} = (b-2)^2 + \frac{3}{2} \ge \frac{3}{2} > \sqrt{2}$, we have $f(a) < b^2 - 4b + \frac{11}{2}$ for any real numbers a, b.

3.20 ★ Show $2(\sqrt{n+1} - 1) < 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} < 2\sqrt{n}$ for any $n \in \mathcal{N}$.

Proof: $\frac{1}{\sqrt{k}} = \frac{2}{2\sqrt{k}} > \frac{2}{\sqrt{k}+\sqrt{k+1}} = 2(\sqrt{k+1} - \sqrt{k})$. Let $m = 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}}$,
then $m > 2(\sqrt{2} - 1 + \sqrt{3} - \sqrt{2} + \cdots + \sqrt{n+1} - \sqrt{n}) = 2(\sqrt{n+1} - 1)$,
and $m < 2(1 - 0 + \sqrt{2} - 1 + \sqrt{3} - \sqrt{2} + \cdots + \sqrt{n} - \sqrt{n-1}) = 2\sqrt{n}$.
Hence, $2(\sqrt{n+1} - 1) < 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} < 2\sqrt{n}$

3.21 ★ Given $a^2 + b^2 + c^2 = 1$, show $-\frac{1}{2} \le ab + bc + ca \le 1$.

Proof: $a^2 + b^2 \ge 2ab, b^2 + c^2 \ge 2bc, c^2 + a^2 \ge 2ca$, add them up to obtain $ab + bc + ca \le$ $\le a^2 + b^2 + c^2 = 1$. Since $(a+b+c)^2 \ge 0$, $a^2 + b^2 + c^2 + 2(ab + bc + ca) \ge 0$, then $ab + bc + ca \ge -\frac{1}{2}(a^2 + b^2 + c^2) = -\frac{1}{2}$, thus $-\frac{1}{2} \le ab + bc + ca \le 1$.

3.22 ★★ Given $a, b, c > 0$, show $\frac{c}{a+b} + \frac{a}{b+c} + \frac{b}{c+a} \ge \frac{3}{2}$.

Proof: $\frac{c}{a+b} + \frac{a}{b+c} + \frac{b}{c+a} = \frac{a+b+c}{a+b} + \frac{a+b+c}{b+c} + \frac{a+b+c}{c+a} - 3 = (a+b+c)(\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a}) - 3 = \frac{1}{2}[(a+b) + (b+c) + (c+a)](\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a}) - 3 \ge \frac{1}{2} \cdot 3\sqrt[3]{(a+b)(b+c)(c+a)} \cdot 3\sqrt[3]{\frac{1}{a+b} \cdot \frac{1}{b+c} \cdot \frac{1}{c+a}} - 3 = \frac{9}{2} - 3 = \frac{3}{2}$.

3.23 Solve the inequality $\sqrt{2x+5} > x + 1$.

Solution: To make the square root valid, we need $2x + 5 \ge 0 \Leftrightarrow x \ge -\frac{5}{2}$. When $x + 1 < 0$, i.e. $x < -1$, we have $\sqrt{2x+5} \ge 0 > x + 1$, thus the original inequality has the solution $-\frac{5}{2} \le x < -1$. When $x \ge -1$, the original inequality has the solution $-1 \le x < 2$. The union of these two solution sets provides the solution of the original inequality: $\{x | -\frac{5}{2} \le x < 2\}$.

3.24 ★ Solve the inequality $x^{\log_a x} > \frac{x^4 \cdot \sqrt{x}}{a^2}$ $(a > 0, a \neq 1)$.

Solution: When $a > 1$, take the log with base a on both sides to obtain $(\log_a x)^2 > \frac{9}{2}\log_a x - 2 \Rightarrow 2(\log_a x)^2 - 9\log_a x + 4 > 0 \Rightarrow (2\log_a x - 1)(\log_a x - 4) > 0 \Rightarrow \log_a x < \frac{1}{2}$ or $\log_a x > 4 \Rightarrow 0 < x < \sqrt{a}$ or $x > a^4$. When $0 < a < 1$, $(\log_a x)^2 < \frac{9}{2}\log_a x - 2 \Rightarrow (2\log_a x - 1)(\log_a x - 4) < 0 \Rightarrow \frac{1}{2} < \log_a x < 4 \Rightarrow a^4 < x < \sqrt{a}$.

3.25 ★★ Given $a > 0, b > 0, c > 0, a + b + c = 1$, show $(1 + \frac{1}{a})(1 + \frac{1}{b})(1 + \frac{1}{c}) \geq 64$.

Proof: Since $a, b, c > 0, a + b + c = 1$, then $1 + \frac{1}{a} = 1 + \frac{a+b+c}{a} = 2 + \frac{b+c}{a} \geq 2 + \frac{2\sqrt{bc}}{a} \geq$ $\geq 2\sqrt{2 \cdot \frac{2\sqrt{bc}}{a}} = 4\sqrt{\frac{\sqrt{bc}}{a}}$. Similarly we can obtain $1 + \frac{1}{b} \geq 4\sqrt{\frac{\sqrt{ca}}{b}}, 1 + \frac{1}{c} \geq 4\sqrt{\frac{\sqrt{ab}}{c}}$. Multiply these three inequalities to obtain $(1 + \frac{1}{a})(1 + \frac{1}{b})(1 + \frac{1}{c}) \geq 4\sqrt{\frac{\sqrt{bc}}{a}} \cdot 4\sqrt{\frac{\sqrt{ca}}{b}} \cdot 4\sqrt{\frac{\sqrt{ab}}{c}} = 64$.

3.26 ★ Show $\frac{1}{n} + \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n^2} > 1$ for $n \in \mathcal{N}, n \geq 2$.

Proof: $\frac{1}{n} + \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n^2} > \frac{n}{n^2} + \frac{1}{n^2} + \frac{1}{n^2} + \cdots + \frac{1}{n^2} = \frac{n+n(n-1)}{n^2} = 1$.

3.27 ★ Given $x \geq 0, y \geq 0$, show $\frac{1}{2}(x+y)^2 + \frac{1}{4}(x+y) \geq x\sqrt{y} + y\sqrt{x}$.

Proof: $\frac{1}{2}(x+y)^2 + \frac{1}{4}(x+y) = \frac{1}{2}(x+y)[(x+y) + \frac{1}{2}] = \frac{1}{2}(x+y)[(x+\frac{1}{4}) + (y+\frac{1}{4})] \geq \sqrt{xy}[(x+\frac{1}{4}) + (y+\frac{1}{4})] \geq \sqrt{xy}[2\sqrt{\frac{1}{4}}x + 2\sqrt{\frac{1}{4}}y] = \sqrt{xy}(\sqrt{x} + \sqrt{y}) = x\sqrt{y} + y\sqrt{x}$.

3.28 ★ Show $\frac{1}{3} \leq \frac{x^2-x+1}{x^2+x+1} \leq 3$.

Proof: Let $y = \frac{x^2-x+1}{x^2+x+1}$, then $yx^2 + yx + y - x^2 + x - 1 = 0 \Leftrightarrow (y-1)x^2 + (y+1)x + y - 1 = 0$. Consider the discriminant $\Delta = (y+1)^2 - 4(y-1)^2 = -3y^2 + 10y - 3 \geq 0 \Rightarrow 3y^2 - 10y + 3 \leq 0 \Rightarrow (3y-1)(y-3) \leq 0 \Rightarrow \frac{1}{3} \leq y \leq 3$.

3.29 ★ a, b, x, y are positive numbers and satisfy $a + b = 10, \frac{a}{x} + \frac{b}{y} = 1$, and $x + y$ has the minimum value 18, find the values of a, b.

Solution: The conditions imply that $x + y = (\frac{a}{x} + \frac{b}{y})(x+y) = a + b + \frac{ay}{x} + \frac{bx}{y} = 10 + \frac{ay}{x} + \frac{bx}{y} \geq 10 + 2\sqrt{\frac{ay}{x} \cdot \frac{bx}{y}} = 10 + 2\sqrt{ab}$. Since $x + y$ has the minimum value 18, then $10 + 2\sqrt{ab} = 18 \Rightarrow \sqrt{ab} = 4 \Rightarrow ab = 16$. Solve

$$\begin{aligned} a + b &= 10 \\ ab &= 16 \end{aligned}$$

to obtain $a = 2, b = 8$ or $a = 8, b = 2$.

3.30 ★ If $x, y > 0$, find the maximum value of $\frac{\sqrt{x}+\sqrt{y}}{\sqrt{x+y}}$.

Solution: $f(x,y) = \frac{\sqrt{x}+\sqrt{y}}{\sqrt{x+y}}$, $f^2(x,y) = \frac{x+y+2\sqrt{xy}}{x+y} = 1 + \frac{2\sqrt{xy}}{x+y} \le 1 + \frac{2\sqrt{xy}}{2\sqrt{xy}} = 2$, thus $f(x,y) \le \sqrt{2}$, which means the maximum value of $\frac{\sqrt{x}+\sqrt{y}}{\sqrt{x+y}}$ is $\sqrt{2}$.

3.31 ★★ Given $x > 0$, show $x + \frac{1}{x} + \frac{1}{x+\frac{1}{x}} \ge \frac{5}{2}$.

Solution: Let $f(x) = x + \frac{1}{x}(x > 0)$, then $x + \frac{1}{x} \ge 2$. Let $2 \le \alpha < \beta$, then $f(\alpha) - f(\beta) = (\alpha + \frac{1}{\alpha}) - (\beta + \frac{1}{\beta}) = (\alpha - \beta) + (\frac{1}{\alpha} - \frac{1}{\beta}) = \frac{(\alpha-\beta)(\alpha\beta-1)}{\alpha\beta} < 0$, that is, $f(x)$ is an increasing function on $[2, +\infty)$. Hence, $f(x + \frac{1}{x}) \ge f(2) = \frac{5}{2}$.

3.32 ★ Real numbers a, b, c satisfy $a + b + c = 0, abc = 2$, show that at least one of a, b, c is not less than 2.

Proof: Obviously at least one of a, b, c is positive. Without loss of generality, let $a > 0$, then $b + c = -a, bc = 2/a$, that is, b, c are the two roots of the quadratic equation $x^2 + ax + \frac{2}{a} = 0$. Consider the discriminant $\Delta \ge 0 \Rightarrow a^2 - \frac{8}{a} \ge 0 \Rightarrow a^3 \ge 8 \Rightarrow a \ge 2$.

3.33 ★ $|x^2-4|<1$ holds whenever $|x-2|<a$ holds, find the range of the positive number a.

Solution: Let $A=\{x:|x-2|<a,a>0\},B=\{x:|x^2-4|<1\}$,
then $A=\{x:2-a<x<2+a,a>0\},B=\{x:-\sqrt{5}<x<-\sqrt{3},\sqrt{3}<x<\sqrt{5}\}$.
Since $A\subseteq B$, we have

$$\begin{aligned} 2-a &> -\sqrt{5} \\ 2+a &< -\sqrt{3} \end{aligned}$$

or

$$\begin{aligned} 2-a &> \sqrt{3} \\ 2+a &< \sqrt{5} \end{aligned}$$

$\Leftrightarrow$

$$\begin{aligned} a &< 2+\sqrt{5} \\ a &< -2-\sqrt{3} \end{aligned}$$

or

$$\begin{aligned} a &< 2-\sqrt{3} \\ a &< \sqrt{5}-2 \end{aligned}$$

which implies $0<a<\sqrt{5}-2$ since $a>0$.

3.34 Solve the inequality $\sqrt{x^2-3x+2}>x-3$.

Solution: The inequality is equivalent to

$$\begin{aligned} x-3 &< 0 \\ x^2-3x+2 &\geq 0 \end{aligned}$$

or

$$\begin{aligned} x-3 &\geq 0 \\ x^2-3x+2 &> (x-3)^2 \end{aligned}$$

$\Rightarrow$

$$\begin{gathered} x < 3 \\ x\leq 1 \quad \text{or} \quad x\geq 2 \end{gathered}$$

or

$$\begin{aligned} x &\geq 3 \\ x &> 7/3 \end{aligned}$$

$\Rightarrow x\leq 1$ or $2\leq x<3$ or $x\geq 3$.

3.35 ★★ Given $|a| < 1, |b| < 1, |c| < 1$, show (1) $|1 - abc| > |ab - c|$; (2) $a + b + c < abc + 2$

Proof: (1) The given conditions imply $1 - a^2b^2 > 0, 1 - c^2 > 0$. Multiply them together to obtain $1 + a^2b^2c^2 > a^2b^2 + c^2 \Rightarrow 1 - 2abc + a^2b^2c^2 > a^2b^2 - 2abc + c^2 \Rightarrow (1 - abc)^2 > (ab - c)^2 \Rightarrow |1 - abc| > |ab - c|$.

(2) $(a-1)(b-1) > 0 \Rightarrow a + b < ab + 1$ (i). $(ab-1)(c-1) > 0 \Rightarrow ab + c < abc + 1$ (ii). (i)+(ii)$\Rightarrow a + b + c < abc + 2$.

3.36 The smaller root of the quadratic equation $x^2 - 5x\log_8 k + 6\log_8^2 k = 0$ is in the interval $(1, 2)$. Find the range of the parameter k.

Solution: The parabola opens upward, and the smaller root is within $(1, 2)$, then

$$\begin{array}{rcl} f(1) & > & 0 \\ f(2) & < & 0 \end{array}$$

$$\Rightarrow \begin{array}{c} \log_8 k > 1/2 \quad \text{or} \quad \log_8 k < 1/3 \\ 2/3 < \log_8 k < 1 \end{array} \Rightarrow 2/3 < \log_8 k < 1 \Rightarrow 4 < k < 8.$$

3.37 ★ The inequality $ax^2 + bx + c > 0$ has the solution set $\{x|\alpha < x < \beta\}$ where $0 < \alpha < \beta$. Find the solution set of the inequality $cx^2 + bx + a < 0$.

Solution: The given condition implies that $\begin{cases} \alpha + \beta = -b/a > 0 \\ \alpha\beta = c/a > 0 \\ a < 0 \end{cases}$ and let the quadratic equation

$cx^2 + bx + a = 0$ has two roots x_1, x_2. Then $x_1 + x_2 = -\frac{b}{c} = \frac{\alpha+\beta}{\alpha\beta} = \frac{1}{\alpha} + \frac{1}{\beta}$; $x_1x_2 = \frac{a}{c} = \frac{1}{\alpha\beta} = \frac{1}{\alpha} \cdot \frac{1}{\beta}$. $0 < \alpha < \beta \Rightarrow \frac{1}{\beta} < \frac{1}{\alpha}$, in addition $c < 0$, then $cx^2 + bx + a < 0$ has the solution set $\{x|x < \frac{1}{\beta}$ or $x > \frac{1}{\alpha}\}$.

3.38 ★ Real numbers a, b, x, y satisfy $a^2 + b^2 = 1, x^2 + y^2 = 1$, show $|ax + by| \leq 1$.

Proof 1: $(|a| - |x|)^2 \geq 0 \Rightarrow a^2 + x^2 \geq 2|ax|$. Similarly we have $b^2 + y^2 \geq 2|by|$. Therefore, $a^2 + b^2 + x^2 + y^2 \geq 2(|ax| + |by|)$. Since $a^2 + b^2 = 1, x^2 + y^2 = 1$, then $2 \geq 2(|ax| + |by|) \geq 2|ax + by|$, thus $|ax + by| \leq 1$.

Proof 2: Since $a^2 + b^2 = 1, x^2 + y^2 = 1$, let $a = \sin\theta, b = \cos\theta, x = \sin\varphi, y = \cos\varphi$, then $ax + by = \sin\theta\sin\varphi + \cos\theta\cos\varphi = \cos(\theta - \varphi)$. Thus $|ax + by| = |\cos(\theta - \varphi)| \leq 1$.

3.39 ★ If a, b, c are distinct positive numbers, show $\frac{1}{a}+\frac{1}{b}+\frac{1}{c} > \frac{1}{\sqrt{bc}}+\frac{1}{\sqrt{ca}}+\frac{1}{\sqrt{ab}}$.

Proof 1: $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}-(\frac{1}{\sqrt{bc}}+\frac{1}{\sqrt{ca}}+\frac{1}{\sqrt{ab}}) = \frac{bc+ca+ab-(a\sqrt{bc}+b\sqrt{ca}+c\sqrt{ab})}{abc} = \frac{2(bc+ca+ab)-2(a\sqrt{bc}+b\sqrt{ca}+c\sqrt{ab})}{2abc} = \frac{(\sqrt{ab}-\sqrt{bc})^2+(\sqrt{bc}-\sqrt{ca})^2+(\sqrt{ca}-\sqrt{ab})^2}{abc} > 0.$

Proof 2: $\frac{1}{a}+\frac{1}{b}+\frac{1}{c} > \frac{1}{\sqrt{bc}}+\frac{1}{\sqrt{ca}}+\frac{1}{\sqrt{ab}} \Leftrightarrow \frac{bc+ca+ab}{abc} > \frac{a\sqrt{bc}+b\sqrt{ca}+c\sqrt{ab}}{abc} \Leftrightarrow bc+ca+ab > a\sqrt{bc}+b\sqrt{ca}+c\sqrt{ab} \Leftrightarrow 2(bc+ca+ab) > 2(a\sqrt{bc}+b\sqrt{ca}+c\sqrt{ab}) \Leftrightarrow (\sqrt{ab}-\sqrt{bc})^2+(\sqrt{bc}-\sqrt{ca})^2+(\sqrt{ca}-\sqrt{ab})^2 \geq 0$ which is obviously valid.

Proof 3: Since a, b, c are distinct positive numbers, then $(\sqrt{ab}-\sqrt{bc})^2 > 0, (\sqrt{bc}-\sqrt{ca})^2 > 0,$ $(\sqrt{ca}-\sqrt{ab})^2 > 0$. Add them up to obtain $2(ab+bc+ca)-2(a\sqrt{bc}+b\sqrt{ca}+c\sqrt{ab}) > 0 \Rightarrow ab+bc+ca > a\sqrt{bc}+b\sqrt{ca}+c\sqrt{ab} \Rightarrow \frac{1}{a}+\frac{1}{b}+\frac{1}{c} > \frac{1}{\sqrt{bc}}+\frac{1}{\sqrt{ca}}+\frac{1}{\sqrt{ab}}$.

3.40 ★ Real numbers x, y, z satisfy the inequalities $|x| \geq |y+z|, |y| \geq |z+x|, |z| \geq |x+y|$. Show $x+y+z=0$.

Proof: If one of x, y, z is zero, without loss of generality, assume $x = 0$, then $|y+z| = 0$, thus $y+z=0$, which implies $x+y+z=0$. If x, y, z are all nonzero, then there are four possibilities:

1) If x, y, z are all positive, then $y+z \le x, z+x \le y, x+y \le z$, impossible.
2) If x, y, z have two positive one negative, without loss of generality, assume $x>0, y>0, z<0$. $|y+z| \le x \Rightarrow -x \le y+z \le x \Rightarrow x+y+z \ge 0$. On the other hand $|x+y| \le |z| \Rightarrow x+y \le -z \Rightarrow x+y+z \le 0$. As a conclusion, $x+y+z=0$.
3) If x, y, z have one positive two negative, without loss of generality, assume $x>0, y<0, z<0$. $|y+z| \le x \Rightarrow y+z \ge -x \Rightarrow x+y+z \ge 0$. On the other hand, $|x+y| \le |z| \Rightarrow x+y \le -z \Rightarrow x+y+z \le 0$. As a conclusion, $x+y+z=0$.
4) If x, y, z are all negative, then $x \le y+z \le -x, y \le z+x \le -y, z \le x+y \le -z$, then $x+y+z \le 2(x+y+z)$, thus $x+y+z \ge 0$, a contradiction to the assumed negativity condition.

3.41 ★ If $x>y>0$, show $\sqrt{x^2-y^2}+\sqrt{2xy-y^2}>x$.

Proof 1: $x>y>0 \Rightarrow xy>y^2, 2xy-y^2>y^2 \Rightarrow x^2-y^2>x^2-2xy+y^2=(x-y)^2 \Rightarrow \sqrt{x^2-y^2}>x-y$, and $\sqrt{2xy-y^2}>y$, thus $\sqrt{x^2-y^2}+\sqrt{2xy-y^2}>x-y+y=x$.

Proof 2: $x>y>0 \Rightarrow y>-y \Rightarrow x+y>x-y \Rightarrow x^2-y^2>(x-y)^2 \Rightarrow \sqrt{x^2-y^2}>x-y$ (i). $2xy>2y^2 \Rightarrow 2xy-y^2>y^2 \Rightarrow \sqrt{2xy-y^2}>y$ (ii).
(i)+(ii)$\Rightarrow \sqrt{x^2-y^2}+\sqrt{2xy-y^2}>x$.

Proof 3: $\sqrt{x^2-y^2}+\sqrt{2xy-y^2}>x \Leftrightarrow x^2-y^2+2\sqrt{(x^2-y^2)(2xy-y^2)}+2xy-y^2>x^2 \Leftrightarrow \sqrt{(x^2-y^2)(2xy-y^2)}>y^2-xy$. The left hand side is greater than zero, while he right hand side $y^2-xy=y(y-x)<0$, thus $\sqrt{(x^2-y^2)(2xy-y^2)}>y^2-xy$ always holds.

3.42 ★ Given $x>0, y>0, \frac{1}{x}+\frac{9}{y}=1$, show $x+y \ge 12$.

Proof: Since $x>0, y>0$, we have $\frac{1}{x}+\frac{9}{y} \ge 2\sqrt{\frac{1}{x}\cdot\frac{9}{y}}=\frac{6}{\sqrt{xy}}$. Since $\frac{1}{x}+\frac{9}{y}=1$, we have $\frac{6}{\sqrt{xy}} \le 1$, which is equivalent to $\sqrt{xy} \ge 6$. $x+y \ge 2\sqrt{xy} \ge 12$.

3.43 ★ a, b, c are real numbers and $a+b+c=1$, show $a^2+b^2+c^2 \ge \frac{1}{3}$.

Proof 1: $a+b+c=1 \Rightarrow c=1-a-b$, then $a^2+b^2+c^2-\frac{1}{3}=a^2+b^2+(1-a-b)^2-\frac{1}{3}=a^2+b^2+1+a^2+b^2-2a-2b+2ab-\frac{1}{3}=2(a^2+b^2+ab-a-b+\frac{1}{3})=2[a^2+(b-1)a+(\frac{b-1}{2})^2-(\frac{b-1}{2})^2+b^2-b+\frac{1}{3}]=2[(a+\frac{b-1}{2})^2+\frac{(3b-1)^2}{12}] \ge 0$.

Proof 2: $a+b+c=1 \Rightarrow (a+b+c)^2=1 \Rightarrow a^2+b^2+c^2=1-2(ab+bc+ca)$ (i).

$a^2+b^2 \geq 2ab, b^2+c^2 \geq 2bc, c^2+a^2 \geq 2ca$, add them up to obtain $2(a^2+b^2+c^2) \geq$

$\geq 2(ab+bc+ca)$ (ii).
(i)+(ii)$\Rightarrow 3(a^2+b^2+c^2) \geq 1 \Rightarrow a^2+b^2+c^2 \geq \frac{1}{3}$.

3.44 ★ The function $f(x)$ is defined on $[0,1]$, and $f(0)=f(1)$. For any distinct $x_1, x_2 \in [0,1]$, we have $|f(x_2)-f(x_1)| < |x_2-x_1|$. Show $|f(x_2)-f(x_1)| < \frac{1}{2}$.

Proof: Let $0 \leq x_1 < x_2 \leq 1$. We consider two cases:

1) If $x_2 - x_1 \leq \frac{1}{2}$, then $|f(x_2)-f(x_1)| < |x_2-x_1| \leq \frac{1}{2}$.
2) If $x_2 - x_1 > \frac{1}{2}$, then from $f(0)=f(1)$ we obtain
$|f(x_2)-f(x_1)| = |f(x_2)-f(1)+$
$f(0)-f(x_1)| \leq |f(x_2)-f(1)|+|f(0)-f(x_1)|$. Hence, $(1-x_2)+(x_1-0)=$
$1-(x_2-x_1) < \frac{1}{2}$.

3.45 ★ The equation $|x| = ax+1$ has one negative root but no positive root, find the range of the parameter a.

Solution 1: Let x be the negative root of the equation $|x|=ax+1$, then $-x=ax+1 \Rightarrow x=\frac{-1}{a+1} < 0$, thus $a+1>0$. Equivalently, when $a>-1$, the equation has a negative root.
Suppose the equation has a positive root x, then $x=ax+1 \Rightarrow x=\frac{1}{1-a} > 0$, thus $a<1$.

As a conclusion, the condition that the equation has one negative root but no positive root is equivalent to $a>-1$ holds but $a<1$ fails, that is, $a \geq 1$.

Solution 2: Another approach is to plot the functions $y=|x|$ and $y=ax+1$ on the Cartesian plane. This will directly give us the same conclusion.

3.46 Solve the inequality $\frac{3x^2-4x-23}{x^2-9} > 2$.

Solution: $\frac{3x^2-4x-23}{x^2-9} > 2 \Leftrightarrow \frac{x^2-4x-5}{x^2-9} > 0 \Leftrightarrow \frac{(x+1)(x-5)}{(x+3)(x-3)} > 0$. This inequality is equivalent to $(x+3)(x+1)(x-3)(x-5) > 0$ whose solution set is $(-\infty,-3) \cup (-1,3) \cup (5,+\infty)$.

3.47 ★ Consider the inequality $x+2>m(x^2-1)$, (1) if the inequality holds for any real number x, find the range of m; (2) if for any $m\in[-2,2]$ the inequality holds, find the range of x.

Solution: (1) $x+2>m(x^2-1)\Leftrightarrow m(x^2-1)-(x+2)<0\Leftrightarrow mx^2-x-m-2<0$. This inequality holds for any real number x, then $\begin{cases} m<0 \\ \Delta=1+4m(m+2)<0 \end{cases}\Leftrightarrow\begin{cases} m<0 \\ 4m^2+8m+1<0 \end{cases}$ $\Leftrightarrow\begin{cases} m<0 \\ -1-\frac{\sqrt{3}}{2}<m<-1+\frac{\sqrt{3}}{2} \end{cases}\Leftrightarrow -1-\frac{\sqrt{3}}{2}<m<-1+\frac{\sqrt{3}}{2}$.

(2) For any $m\in[-2,2]$ the inequality holds. Let $f(m)=(x^2-1)m-(x+2)$ which is a linear function of m, and $f(m)<0$ should hold for any $m\in[-2,2]$, equivalently we should have $\begin{cases} -2(x^2-1)-(x+2)<0 \\ 2(x^2-1)-(x+2)<0 \end{cases}\Leftrightarrow\begin{cases} 2x^2+x>0 \\ 2x^2-x-4<0 \end{cases}\Leftrightarrow\begin{cases} x>0 \text{ or } x<-\frac{1}{2} \\ \frac{1-\sqrt{33}}{4}<x<\frac{1+\sqrt{33}}{4} \end{cases}\Leftrightarrow$ $0<x<\frac{1+\sqrt{33}}{4}$ or $\frac{1-\sqrt{33}}{4}<x<-\frac{1}{2}$.

3.48 ★ x,y,z are positive numbers, and $xyz(x+y+z)=1$. Find the minimum value of $(x+y)(x+z)$.

Solution: The given conditions imply that $(x+y)(x+z) = yz + x(x+y+z) \geq 2\sqrt{yz \cdot x(x+y+z)} = 2$. When $x = \sqrt{2}-1, y = z = 1$, the equal sign is reached in the above inequality, thus the minimum value of $(x+y)(x+z)$ is 2.

3.49 ★★ Real numbers a, b, c satisfy $a^2+b^2+c^2=1$. Show that one of $|a-b|, |b-c|, |c-a|$ is not greater than $\frac{\sqrt{2}}{2}$.

Proof: Without loss of generality, we assume $a \leq b \leq c$, and let m be the minimum one of $|a-b|, |b-c|, |c-a|$. Then $b-a \geq m, c-b \geq m, c-a = (c-b)+(b-a) \geq 2m$. On one hand, $(a-b)^2+(b-c)^2+(c-a)^2 = 2(a^2+b^2+c^2) - 2(ab+bc+ca) = 3(a^2+b^2+c^2) - (a+b+c)^2 \leq 3$. On the other hand, $(a-b)^2+(b-c)^2+(c-a)^2 \geq m^2+m^2+(2m)^2 = 6m^2$ $(a-b)^2+(b-c)^2+(c-a)^2 \geq m^2+m^2+(2m)^2 = 6m^2$. Hence, $6m^2 \leq 3 \Rightarrow m \leq \frac{\sqrt{2}}{2}$.

3.50 ★ Let α, β be real numbers, show $\log_2(2^\alpha+2^\beta) \geq \frac{\alpha+1}{2}+\frac{\beta+1}{2}$.

Proof: To show $\log_2(2^\alpha+2^\beta) \geq \frac{\alpha+1}{2}+\frac{\beta+1}{2}$, we only need to show $2^\alpha+2^\beta \geq 2^{\frac{\alpha+\beta+2}{2}}$, we only need to show $2^{2\alpha}+2^{\alpha+\beta+1}+2^{2\beta} \geq 2^{\alpha+\beta+2}$, we only need to show $2^{\alpha-\beta-2}+2^{-1}+2^{\beta-\alpha-2} \geq 1$, we only need to show $2^{\alpha-\beta}+2+\frac{1}{2^{\alpha-\beta}} \geq 4$, we only need to show $(2^{\frac{\alpha-\beta}{2}} - \frac{1}{2^{\frac{\alpha-\beta}{2}}})^2 \geq 0$ which is obviously valid.

3.51 ★★ Given the function $f(x) = ax^2 - c$ that satisfies $-4 \leq f(1) \leq -1, -1 \leq f(2) \leq 5$. Find the range of $f(3)$.

Solution: From $f(x) = ax^2 - c$, we know $f(1) = a - c, f(2) = 4a - c$, thus $a = \frac{1}{3}[f(2)-f(1)]$, $c = \frac{1}{3}[f(2)-4f(1)]$. Then $f(3) = 9a - c = 3[f(2)-f(1)] - \frac{1}{3}[f(2)-4f(1)] = \frac{8}{3}f(2) - \frac{5}{3}f(1)$. Hence, $\frac{8}{3} \times (-1) + (-\frac{5}{3}) \times (-1) \leq f(3) \leq \frac{8}{3} \times 5 + (-\frac{5}{3}) \times (-4)$, that is, $-1 \leq f(3) \leq 20$.

3.52 ★ Given real numbers $a > 0, b > 0, c > 0$ and $a+b+c = 1$, show $\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \geq 9$.

Proof 1: The conditions together with Cauchy's Inequality imply $\frac{a+b+c}{3} \geq \sqrt[3]{abc} \Rightarrow \frac{1}{\sqrt[3]{abc}} \geq \frac{3}{a+b+c} = 3$ Apply Cauchy's Inequality again to obtain $\frac{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}}{3} \geq \sqrt[3]{\frac{1}{a} \cdot \frac{1}{b} \cdot \frac{1}{c}} = \frac{1}{\sqrt[3]{abc}} \geq 3 \Rightarrow \frac{1}{a}+\frac{1}{b}+\frac{1}{c} \geq 9$.

Proof 2: $\frac{1}{a}+\frac{1}{b}+\frac{1}{c} - 9 = \frac{bc+ac+ab}{abc} - 9 = \frac{(a+b+c)(bc+ac+ab)-9abc}{abc} = \frac{a^2c+a^2b+b^2c+ab^2+ac^2+bc^2-6abc}{abc} = \frac{a(b-c)^2+b(c-a)^2+c(a-b)^2}{abc} \geq 0$.

3.53 ★★ Let $a>0, b>0$ and $a+b=1$, show $(a+\frac{1}{a})^2+(b+\frac{1}{b})^2 \geq \frac{25}{2}$.

Proof 1: $1=a+b \geq 2\sqrt{ab} \Rightarrow \sqrt{ab} \leq \frac{1}{2} \Rightarrow ab \leq \frac{1}{4} \Rightarrow \frac{1}{ab} \geq 4$.
And $\frac{(a+\frac{1}{a})^2+(b+\frac{1}{b})^2}{2} \geq [\frac{a+\frac{1}{a}+b+\frac{1}{b}}{2}]^2 = \frac{1}{4}(1+\frac{1}{a}+\frac{1}{b})^2 = \frac{1}{4}(1+\frac{1}{ab})^2 \geq \frac{25}{4}$,
thus $(a+\frac{1}{a})^2+(b+\frac{1}{b})^2 \geq \frac{25}{2}$.

Proof 2: Let $a=\sin^2\alpha, b=\cos^2\alpha$, then

$(a+\frac{1}{a})^2+(b+\frac{1}{b})^2 = (\sin^2\alpha+\csc^2\alpha)^2+(\cos^2\alpha+\sec^2\alpha)^2 \geq \frac{1}{2}(\sin^2\alpha+\csc^2\alpha+\cos^2\alpha+\sec^2\alpha)^2 = \frac{1}{2}(1+\frac{1}{\sin^2\alpha}+\frac{1}{\cos^2\alpha})^2 = \frac{1}{2}(1+\frac{1}{\sin^2\alpha\cos^2\alpha})^2 = \frac{1}{2}(1+\frac{4}{\sin^2 2\alpha})\ \frac{1}{2}(1+4\csc^2 2\alpha)^2 \geq \frac{1}{2}(1+4)^2 = \frac{25}{2}$.

3.54 ★ Let a,b,c,d,m,n be positive real numbers,
$P=\sqrt{ab}+\sqrt{cd}, Q=\sqrt{ma+nc}\sqrt{\frac{b}{m}+\frac{d}{n}}$. Compare P and Q.

Solution: $P^2=ab+cd+2\sqrt{abcd}, Q^2=(ma+nc)(\frac{b}{m}+\frac{d}{n})=ab+cd+\frac{nbc}{m}+\frac{mad}{n}$. Since $\frac{nbc}{m}+\frac{mad}{n} \geq 2\sqrt{\frac{nbc}{m}\cdot\frac{mad}{n}}=2\sqrt{abcd}$, then $P^2 \leq Q^2$. Because P,Q are positive, we have $P \leq Q$.

3.55 ★★ Show the inequality $(a+b)^8 \leq 128(a^8+b^8)$.

Proof: $(a-b)^2 \geq 0 \Rightarrow a^2+b^2 \geq 2ab$. Similarly we have $a^4+b^4 \geq 2a^2b^2, a^8+b^8 \geq 2a^4b^4$. Add $a^2+b^2, a^4+b^4, a^8+b^8$ to the above three inequalities respectively to obtain $2(a^2+b^2) \geq (a+b)^2,\ 2(a^4+b^4) \geq (a^2+b^2)^2, 2(a^8+b^8) \geq (a^4+b^4)^2$. The last inequality leads to $128(a^8+b^8) \geq 64(a^4+b^4)^2 = 16[2(a^4+b^4)]^2 \geq 16[(a^2+b^2)^2]^2 = \{[2(a^2+b^2)]^2\}^2 \geq \{[(a+b)^2]^2\}^2 = (a+b)^8$.

3.56 ★★ Given the function $f(x^2-3)=\log_a\frac{x^2}{6-x^2}$ $(a>0, a\neq 1)$ that satisfies $f(x) \geq \log_a 2x$. Find the domain of the function x.

Solution: Let $x^2-3=t$, then $x^2=3+t$. Substitute it into the function: $f(t)=\log_a\frac{3+t}{3-t}$, thus $f(x)=\log_a\frac{3+x}{3-x}$. Then the inequality $f(x) \geq \log_a 2x$ is equivalent to $\log_a\frac{3+x}{3-x} \geq \log_a 2x$.

If $a>1$, then $\begin{cases} \frac{3+x}{3-x}>0 \\ \frac{3+x}{3-x} \geq 2x \\ x>0 \end{cases} \Rightarrow x \in (0,1)\cup[-\frac{3}{2},3)$.

If $0<a<1$, then $\begin{cases} \frac{3+x}{3-x}>0 \\ \frac{3+x}{3-x} \leq 2x \\ x>0 \end{cases} \Rightarrow x \in [1,\frac{3}{2})$.

3.57 ★★ Given $a < -1$, and x satisfies $x^2 + ax \le -x$, and $x^2 + ax$ has the minimum value $-\frac{1}{2}$, find the value of a.

Solution: $a < -1, x^2 + ax \le -x \Rightarrow x[x + (a + 1)] \le 0 \Rightarrow 0 \le x \le -(a + 1)$. Let $f(x) = x^2 + ax = (x + \frac{a}{2})^2 - \frac{a^2}{4}$.

If $-(a + 1) < -\frac{a}{2} \Leftrightarrow -2 < a < -1$, then $f(x)$ reaches its minimum value $f(-a - 1) = a + 1$ at $x = -(a + 1)$, thus $a + 1 = -\frac{1}{2} \Rightarrow a = -\frac{3}{2}$.

If $-(a + 1) \ge -\frac{a}{2} \Leftrightarrow a \le -2$, then $f(x)$ reaches it minimum value $-\frac{a^2}{4}$ at $x = -\frac{a}{2}$, thus $-\frac{a^2}{4} = -\frac{1}{2} \Rightarrow a = \pm\sqrt{2}$ both of which violate $a \le -2$.

As a conclusion, $a = -\frac{3}{2}$.

3.58 ★★ $a_1, a_2, \cdots, a_n$ are positive numbers and satisfy $a_1 a_2 \cdots a_n = 1$, show $(2+a_1)(2+a_2)\cdots(2+a_n) \geq 3^n$.

Proof: Use an arithmetic mean-geometric mean inequality $a+b+c \geq 3\sqrt[3]{abc}$ (a,b,c are positive numbers) to obtain $2+a_i = 1+1+a_i \geq 3\sqrt[3]{a_i}$ ($i=1,2,\cdots,n$). Then $(2+a_1)(2+a_2)\cdots(2+a_n) \geq 3^n \cdot \sqrt[3]{a_1 a_2 \cdots a_n} = 3^n$.

3.59 ★★★ If a,b,c are side lengths of a triangle, show $a^2b(a-b)+b^2c(b-c)+c^2a(c-a) \geq 0$ and determine when the equal sign is reached.

Proof: Let $a=y+z, b=z+x, c=x+y$ where x,y,z are positive numbers. Substitute them into the inequality:
$(y+z)^2(x+z)(y-x)+(z+x)^2(x+y)(z-y)+(x+y)^2(y+z)(x-z) \geq 0 \Leftrightarrow$
$x^3z+y^3x+z^3y-xyz(x+y+z) \geq 0$. Divide both sides by xyz to obtain
$\frac{x^2}{y}+\frac{y^2}{z}+\frac{z^2}{x} \geq x+y+z$ which can be proven by the inequalities
$\frac{x^2}{y}+y \geq 2x, \frac{y^2}{z}+z \geq 2y, \frac{z^2}{x}+x \geq 2z$.

These inequalities have the equal sign if and only if $x=y=z$, that is, the original inequality has the equal sign if and only if $a=b=c$.

3.60 ★★ a,b are real numbers, show $\frac{|a+b|}{1+|a+b|} \leq \frac{|a|}{1+|a|}+\frac{|b|}{1+|b|}$.

Proof: Since $|a+b| \leq |a|+|b|$, we have $\frac{|a+b|}{1+|a+b|} = \frac{1+|a+b|-1}{1+|a+b|} = 1-\frac{1}{1+|a+b|} \leq 1-\frac{1}{1+|a|+|b|} =$
$= \frac{|a|+|b|}{1+|a|+|b|} = \frac{|a|}{1+|a|+|b|}+\frac{|b|}{1+|a|+|b|} \leq \frac{|a|}{1+|a|}+\frac{|b|}{1+|b|}$.

3.61 ★★ Given $0<a<1, x^2+y=0$, show $\log_a(a^x+a^y) \leq \log_a 2+\frac{1}{8}$.

Proof: $a^x+a^y \geq 2\sqrt{a^xa^y} = 2a^{\frac{x+y}{2}}$. Since $0<a<1$, we have $\log_a(a^x+a^y) \leq \log_a(2a^{\frac{x+y}{2}}) =$
$\log_a 2+\frac{x+y}{2} = \log_a 2+\frac{x-x^2}{2} = \log_a 2+\frac{1}{2}x(1-x) \leq \log_a 2+\frac{1}{2}(\frac{x+1-x}{2})^2 = \log_a 2+\frac{1}{8}$.

3.62 ★★★ The system of inequalities

$$\begin{aligned} \sqrt{x^2-2x-8} &< 8-x \\ x^2+ax+b &< 0 \end{aligned}$$

has the solution $4 \leq x < 5$, find the conditions a and b should satisfy.

Solution: $\sqrt{x^2-2x-8} < 8-x \Rightarrow \begin{cases} x^2-2x-8 \geq 0 \\ 8-x>0 \\ x^2-2x-8<(8-x)^2 \end{cases} \Rightarrow \begin{cases} x \leq -2 \text{ or } x \geq 4 \\ x<8 \\ x<\frac{36}{7} \end{cases} \Rightarrow x \leq -2$ or $4 \leq x \leq \frac{36}{7}$.

The solution of the inequality $x^2 + ax + b < 0$ should have the form $\alpha < x < \beta$. Since the solution of the inequality system is $4 \leq x < 5$, then $\beta = 5, -2 \leq \alpha < 4$. Since $\beta = 5$, then $25 + 5a + b = 0$ Since $\alpha + \beta = -a$, then $3 \leq -a < 9$, that is, $-9 < a \leq -3$. As a conclusion, a, b should satisfy $\begin{cases} -9 < a \leq -3 \\ 5a + b + 25 = 0. \end{cases}$

3.63 ★★★ If $x, y, z \geq 1$, show $(x^2 - 2x + 2)(y^2 - 2y + 2)(z^2 - 2z + 2) \leq (xyz)^2 - 2xyz \cdot + 2$.

Proof: Since $x \geq 1, y \geq 1$, we have $(x^2 - 2x + 2)(y^2 - 2y + 2) - [(xy)^2 - 2xy + 2] = (-2y + 2)x^2 + (6y - 2y^2 - 4)x + (2y^2 - 4y + 2) = -2(y-1)x^2 - 2(y-1)(y-2) + 2(y-1)^2 = -2(y-1)[x^2 + (y-2)x + 1 - y] = -2(y-1)(x-1)(x+y-1) \leq 0$, then $(x^2 - 2x + 2)(y^2 - 2y + 2) \leq (xy)^2 - 2xy + 2$(i). Similarly, since $xy \geq 1, z \geq 1$, we have $[(xy)^2 - 2xy + 2](z^2 - 2z + 2) \leq \leq (xyz)^2 - 2xyz + 2$(ii). From (i) and (ii), we can obtain $(x^2 - 2x + 2)(y^2 - 2y + 2)(z^2 - 2z + 2) \leq (xyz)^2 - 2xyz + 2$.

3.64 ★★ Given natural numbers $a < b < c$, m is an integer, and $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = m$, find a, b, c.

Solution: Since a, b, c are natural numbers and $a < b < c$, we have $a \geq 1, b \geq 2, c \geq 3, 0 < m \leq \frac{1}{1} + \frac{1}{2} + \frac{1}{3} = 1\frac{5}{6}$. Since m is an integer, we have $m = 1$ and $a \neq 1$. If $a \geq 3$, then $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \leq \frac{1}{3} + \frac{1}{4} + \frac{1}{5} = \frac{47}{60} < 1$. Hence, $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \neq m = 1$. Therefore, $a = 2$. Then $\frac{1}{b} + \frac{1}{c} = 1 - \frac{1}{2} = \frac{1}{2}$. If $b \geq 4$, then $\frac{1}{b} + \frac{1}{c} \leq \frac{1}{4} + \frac{1}{5} = \frac{9}{20} < \frac{1}{2}$, thus $b = 3$. Then $\frac{1}{c} = 1 - \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$, thus $c = 6$.

3.65 ★★★ Given $1 < a < 2, x \geq 1$, and $f(x) = \frac{a^x + a^{-x}}{2}, g(x) = \frac{2^x + 2^{-x}}{2}$. (1) Compare $f(x)$ and $g(x)$. (2) Let $n \in N, n \geq 1$, show $f(1) + f(2) + \cdots + f(2n) < 4^n - \frac{1}{2^{2n}}$.

Solution: (1) $f(x) - g(x) = \frac{a^x + a^{-x}}{2} - \frac{2^x + 2^{-x}}{2} = \frac{1}{2}\left(\frac{a^{2x}+1}{a^x} - \frac{2^{2x}+1}{2^x}\right) = \frac{2^x a^{2x} + 2^x - 2^{2x} a^x - a^x}{2^{x+1} a^x} = \frac{(a^x - 2^x)(2^x a^x - 1)}{2^{x+1} a^x}$. Since $1 < a < 2, x \geq 1$, then $2^x a^x > 1, a^x < 2^x$, thus $\frac{(a^x - 2^x)(2^x a^x - 1)}{2^{x+1} a^x} < 0$, that is, $f(x) - g(x) < 0$. Hence, $f(x) < g(x)$.

(2) Since $f(x) < g(x)$, then $f(1) + f(2) + \cdots + f(2n) < g(1) + g(2) + \cdots + g(2n) = \frac{1}{2}(2 + 2^2 + \cdots + 2^{2n}) + \frac{1}{2}\left(\frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{2n}}\right) = (1 + 2 + 2^2 + \cdots + 2^{2n-1}) + \frac{1}{2}\left(1 - \frac{1}{2^{2n}}\right) < 4^n - 1 + 1 - \frac{1}{2^{2n}} = 4^n - \frac{1}{2^{2n}}$.

3.66 ★★ a, b, c are real numbers, and $a+b+c<0$, show $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} \geq 0$.

Proof: $a^2+b^2 \geq 2ab, b^2+c^2 \geq 2bc, c^2+a^2 \geq 2ca$, and add them up to obtain $2(a^2+b^2+c^2) \geq 2(ab+bc+ca)$, thus $(ab+bc+ca)-(a^2+b^2+c^2) \leq 0$.

$$\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = \begin{vmatrix} a+b+c & b & c \\ a+b+c & c & a \\ a+b+c & a & b \end{vmatrix} = (a+b+c)\begin{vmatrix} 1 & b & c \\ 1 & c & a \\ 1 & a & b \end{vmatrix} = (a+b+c)\begin{vmatrix} 1 & b & c \\ 0 & c-b & a-c \\ 0 & a-b & b-c \end{vmatrix} =$$

$$(a+b+c)[-(b-c)^2-(a-b)(a-c)] = (a+b+c)[(ab+bc+ca)-(a^2+b^2+c^2)] \geq 0.$$

3.67 ★★★ If the system of inequalities $\begin{cases} x^2-x-2>0 \\ 2x^2+(5+2k)x+5k<0 \end{cases}$ has only one integer solution -2, find the range of k.

Solution: The solution of $x^2-x-2>0$ is $x<-1$ or $x>2$. The second inequality is equivalent to $(2x+5)(x+k)<0$. When $-k<-\frac{5}{2}$, i.e. $k>\frac{5}{2}$, the second inequality has the solution $-k<x<-\frac{5}{2}$, in which -2 is not included. When $-k>-\frac{5}{2}$, i.e. $k<\frac{5}{2}$, the second inequality has the solution $-\frac{5}{2}<x<-k$, then the solution of the inequality system is $\begin{cases} x<-1 \\ -\frac{5}{2}<x<-k \end{cases}$ or $\begin{cases} x>2 \\ -\frac{5}{2}<x<-k \end{cases}$. To have only one integer solution -2, we should have $-k\le 3$ and $-k>-2$, that is, $-3\le k<2$. When $-k=-\frac{5}{2}$, i.e. $k=\frac{5}{2}$, the second inequality has no solution. As a conclusion, $k\in[-3,2)$.

3.68 ★★ Let a,b,c are positive numbers, show $a^ab^bc^c\ge(abc)^{\frac{a+b+c}{3}}$.

Proof: Without loss of generality, let $a\ge b\ge c>0$. To show $a^ab^bc^c\ge(abc)^{\frac{a+b+c}{3}}$, we only need to show $a^{3a}b^{3b}c^{3c}\ge(abc)^{a+b+c}$, we only need to show $\frac{a^{a-b}}{a^{c-a}}\frac{b^{b-a}}{b^{c-a}}\frac{c^{c-a}}{c^{b-c}}\ge 1$, we only need to show $\frac{a^{a-b}}{b^{a-b}}\frac{b^{b-c}}{c^{b-c}}\frac{c^{a-c}}{a^{a-c}}\ge 1$. Since $a-b\ge 0, b-c\ge 0, a-c\ge 0$, we have $\frac{a}{b}\ge 1, \frac{b}{c}\ge 1, \frac{a}{c}\ge 1$, thus the last inequality holds.

3.69 ★★★ If a,b,c,x,y,z are all real numbers, and $a^2+b^2+c^2=25, x^2+y^2+z^2=36$, $ax+by+cz=30$, find the value of $\frac{a+b+c}{x+y+z}$.

Solution: Cauchy's Inequality implies
$25\times 36=(a^2+b^2+c^2)(x^2+y^2+z^2)\ge(ax+by+cz)^2=30^2$. The equal sign is obtained since $25\times 36=30^2$. Thus there exist λ,μ (not both zero) such that
$\lambda a=\mu x, \lambda b=\mu y, \lambda c=\mu z$. Therefore
$\lambda^2(a^2+b^2+c^2)=\mu^2(x^2+y^2+z^2)\Rightarrow 25\lambda^2=36\mu^2 \Rightarrow 5\lambda=\pm 6\mu$. However,
$ax+by+cz=30$, thus $5\lambda=6\mu\Rightarrow\frac{\mu}{\lambda}=\frac{5}{6}\Rightarrow\frac{a+b+c}{x+y+z}=\frac{\mu}{\lambda}=\frac{5}{6}$.

3.70 ★★★ Let r, s, t satisfy $1 \le r \le s \le t \le 4$, find the minimum value of $(r-1)^2 + (\frac{s}{t}-1)^2 + (\frac{t}{s}-1)^2 + (\frac{4}{r}-1)^2$.

Solution: $(r-1)^2+(\frac{s}{t}-1)^2+(\frac{t}{s}-1)^2+(\frac{4}{r}-1)^2 \ge [\frac{(r-1)+(\frac{s}{t}-1)+(\frac{t}{s}-1)+(\frac{4}{r}-1)}{2}]^2 \Rightarrow 4[(r-1)^2+(\frac{s}{t}-1)^2+(\frac{t}{s}-1)^2+(\frac{4}{r}-1)^2] \ge [(r-1)+(\frac{s}{t}-1)+(\frac{t}{s}-1)+(\frac{4}{r}-1)]^2 = [(r+\frac{s}{t}+\frac{t}{s}+\frac{4}{r})-4]^2$. Cauchy's Inequality implies that $r+\frac{s}{t}+\frac{t}{s}+\frac{4}{r} \ge 4\sqrt[4]{r\cdot\frac{s}{t}\cdot\frac{t}{s}\cdot\frac{4}{r}} = 4\sqrt[4]{4}$, thus $4[(r-1)^2+(\frac{s}{t}-1)^2+(\frac{t}{s}-1)^2+(\frac{4}{r}-1)^2] \ge [4\sqrt[4]{4}-4]^2 \Rightarrow (r-1)^2+(\frac{s}{t}-1)^2+(\frac{t}{s}-1)^2+(\frac{4}{r}-1)^2 \ge 4(\sqrt{2}-1)^2$. The equal sign is obtained if and only if $r=\sqrt{2}, s=2, t=2\sqrt{2}$. Hence, the minimum value of $(r-1)^2+(\frac{s}{t}-1)^2+(\frac{t}{s}-1)^2+(\frac{4}{r}-1)^2$ is $4(\sqrt{2}-1)^2$.

3.71 ★★★ Real numbers a_1, a_2 satisfy $a_1^2+a_2^2 \le 1$, show that for any real numbers b_1, b_2, $(a_1b_1+a_2b_2-1)^2 \ge (a_1^2+a_2^2-1)(b_1^2+b_2^2-1)$ always holds.

Proof: If $b_1^2+b_2^2-1>0$, since $a_1^2+a_2^2 \le 1$, we have $(a_1^2+a_2^2-1)(b_1^2+b_2^2-1) \le 0$, then obviously $(a_1b_1+a_2b_2-1)^2 \ge (a_1^2+a_2^2-1)(b_1^2+b_2^2-1)$. If $b_1^2+b_2^2-1 \le 0$, Mean Inequality implies that $a_1b_1 \le \frac{a_1^2+b_1^2}{2}, a_2b_2 \le \frac{a_2^2+b_2^2}{2}$. Thus $a_1b_1+a_2b_2 \le \frac{1}{2}(a_1^2+a_2^2+b_1^2+b_2^2) \le 1 \Rightarrow 1-a_1b_1-a_2b_2 \ge \frac{(1-a_1^2-a_2^2)+(1-b_1^2-b_2^2)}{2} \Rightarrow (1-a_1b_1-a_2b_2)^2 \ge [\frac{(1-a_1^2-a_2^2)+(1-b_1^2-b_2^2)}{2}]^2 \ge \frac{1}{2}[(1-a_1^2-a_2^2)^2+(1-b_1^2-b_2^2)^2] \ge (a_1^2+a_2^2-1)(b_1^2+b_2^2-1)$.

3.72 ★★★ Let $A=\{x|1+\frac{1}{\log_3 x}-\frac{1}{\log_5 x}<0\}; B=\{x|(\frac{1}{3})^{a\log_3 2}<(\frac{1}{2})^{x(x-a+1)}, a\in\mathcal{R}\}$, find the range of a such that $A \subseteq B$.

Solution: $1+\frac{1}{\log_3 x}-\frac{1}{\log_5 x}<0 \Rightarrow 1+\log_x 3-2\log_x 5<0 \Rightarrow \log_x\frac{3}{25}<\log_x x^{-1}$. Thus $x>1$, then $\frac{1}{x}>\frac{3}{25}$, then $1<x<\frac{25}{3}$. Hence, $A=\{x|1<x<\frac{25}{3}\}$. $(\frac{1}{3})^{a\log_3 2} < (\frac{1}{2})^{x(x-a+1)} \Rightarrow 3^{\log_3 2^{-a}} < 2^{-x(x-a+1)} \Rightarrow 2^{-a} < 2^{-x(x-a+1)} \Rightarrow -a < -x(x-a+1) \Rightarrow (x-a)(x+1)<0$ (✠).

When $a=-1$, (✠) has no solution.

When $a>-1$, (✠) has the solution $-1<x<a$.

When $a<-1$, (✠) has the solution $a<x<-1$.

Hence, $B=\begin{cases}\phi & (a=-1)\\ \{x|-1<x<a\} & (a>-1)\\ \{x|a<x<-1\} & (a<-1)\end{cases}$ from which we know that when $a \ge \frac{25}{3}$, $A \subseteq B$.

3.73 ★★★★ x, y, z are positive numbers, show $\frac{(x+1)^3}{y} + \frac{(y+1)^3}{z} + \frac{(z+1)^3}{x} \geq \frac{81}{4}$.

Proof: Since $x, y, z > 0$,Mean Inequality implies that $\frac{(x+1)^3}{y} + \frac{27}{2}y + \frac{27}{4} \geq 3\sqrt[3]{\frac{(x+1)^3}{y} \cdot \frac{27}{2}y \cdot \frac{27}{4}} = \frac{27}{2}(x+1) \Rightarrow \frac{(x+1)^3}{y} \geq \frac{27}{2}(x-y) + \frac{27}{4}$. Similarly, we can obtain $\frac{(y+1)^3}{z} \geq \frac{27}{2}(y-z) + \frac{27}{4}$, $\frac{(z+1)^3}{x} \geq \frac{27}{2}(z-x) + \frac{27}{4}$. Add them up to obtain the aimed inequality.

3.74 ★★★★ m, n are positive numbers, show $\sqrt{m} + 1 > \sqrt{n}$ holds if and only if for any $x > 1$, $mx + \frac{x}{x-1} > \sqrt{n}$.

Proof: $m, n > 0, x - 1 > 0$, then $mx + \frac{x}{x-1} = mx - m + m + \frac{x-1+1}{x-1} = [m(x-1) + \frac{1}{x-1}] + m + 1 \geq 2\sqrt{m} + m + 1 = (\sqrt{m}+1)^2$. If and only if $m(x-1) = \frac{1}{x-1}$, i.e. $x = 1 + \frac{1}{\sqrt{m}}$, $mx + \frac{x}{x-1}$ has the minimum value $(\sqrt{m}+1)^2$. Hence, $mx + \frac{x}{x-1} > \sqrt{n}$ for any $x > 1$ if and only if $(\sqrt{m}+1)^2 > n$, i.e. $\sqrt{m} + 1 > \sqrt{n}$.

3.75 ★★★ Given $f(x) = ax^2 + bx$, and $1 \le f(-1) \le 3, 2 \le f(1) \le 4$, find the range of $f(-3)$.

Solution: $f(-1) = a - b, f(1) = a + b, f(-3) = 9a - 3b$. Let $f(-3) = mf(-1) + nf(1)$ where m, n are parameters ready to be determined. $9a - 3b = m(a-b) + n(a+b) = (m+n) \cdot a - (m-n)b$. Comparing the coefficients to obtain $\begin{cases} m+n=9 \\ m-n=3 \end{cases} \Rightarrow m = 6, n = 3$. Thus $f(-3) = 6f(-1) + 3f(1)$. Since $1 \le f(-1) \le 3, 2 \le f(1) \le 4$, we have $12 \le 6f(-1) + 3f(1) \le 30$, then $12 \le f(-3) \le 30$. Therefore the range of $f(-3)$ is $[12, 30]$.

3.76 ★★★ Given $0 < b < 1, 0 < a < \frac{\pi}{4}$, and $x = (\sin\alpha)^{\log_b \sin\alpha}, y = (\cos\alpha)^{\log_b \cos\alpha}, z = (\sin\alpha)^{\log_b \cos\alpha}$, determine the order of x, y, z.

Solution: $0 < b < 1$, thus $f(x) = \log_b x$ is a decreasing function. $0 < a < \frac{\pi}{4}$, thus
$0 < \sin\alpha < \cos\alpha < 1$. Therefore, $\log_b \sin\alpha > \log_b \cos\alpha > 0$, then
$(\sin\alpha)^{\log_b \sin\alpha} < (\sin\alpha)^{\log_b \cos\alpha}$,
i.e. $x < z$. And $(\sin\alpha)^{\log_b \cos\alpha} < (\cos\alpha)^{\log_b \cos\alpha}$, i.e. $z < y$. Hence,
we obtain the order $x < z < y$.

3.77 ★★★ Consider a triangle with side lengths a, b, c, and its area is $1/4$, the radius of its circumcircle is 1. If $s = \sqrt{a} + \sqrt{b} + \sqrt{c}, t = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$. Compare s and t.

Solution: Let C be the angle whose opposite side length is c, and the radius of circumcircle $R = 1$, then $c = 2R\sin C = 2\sin C$. In addition, $\frac{1}{2}ab\sin C = \frac{1}{4}$. Therefore $abc = 1$. Then $t = \frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \frac{1}{2}(\frac{1}{a} + \frac{1}{b}) + \frac{1}{2}(\frac{1}{b} + \frac{1}{c}) + \frac{1}{2}(\frac{1}{c} + \frac{1}{a}) \ge \sqrt{\frac{1}{ab}} + \sqrt{\frac{1}{bc}} + \sqrt{\frac{1}{ca}} = \frac{\sqrt{c}+\sqrt{a}+\sqrt{b}}{\sqrt{abc}} = \sqrt{a} + \sqrt{b} + \sqrt{c} = s$. The equal sign can only be obtained if $a = b = c = R = 1$, which is impossible. Hence, $s < t$.

3.78 ★★★ a, b, c are positive numbers and $a + b + c \le 3$, show $\frac{3}{2} \le \frac{1}{a+1} + \frac{1}{b+1} + \frac{1}{c+1} < 3$.

Proof: Since $a, b, c > 0$, we have $\frac{1}{a+1} < 1, \frac{1}{b+1} < 1, \frac{1}{c+1} < 1$, then $\frac{1}{a+1} + \frac{1}{b+1} + \frac{1}{c+1} < 3$.
Mean Inequality implies

$\frac{1}{a+1}+\frac{1}{b+1}+\frac{1}{c+1} \geq 3\sqrt[3]{\frac{1}{(a+1)(b+1)(c+1)}}$, $(a+1)+(b+1)+(c+1) \geq 3\sqrt[3]{(a+1)(b+1)(c+1)}$ Therefore, $(\frac{1}{a+1}+\frac{1}{b+1}+\frac{1}{c+1})[(a+1)+(b+1)+(c+1)] \geq 3\sqrt[3]{\frac{1}{(a+1)(b+1)(c+1)}} \cdot 3\sqrt[3]{(a+1)(b+1)(c+1)} = 9$. Since $0 < a+b+c \leq 3$, then $\frac{1}{a+1}+\frac{1}{b+1}+\frac{1}{c+1} \geq \frac{9}{(a+1)+(b+1)+(c+1)} \geq \frac{9}{3+3} = \frac{3}{2}$.

3.79 ★★★ Given $a, b, c, m, n, p > 0$, and $a+m = b+n = c+p = R$, show $an+bp+cm < R^2$.

Proof: Construct an equilateral triangle ABC with side length R . Choose points D, E, F on sides AB, BC, CA respectively such that $AD = a, DB = m, BE = c, EC = p, CF = b, FA = n$. In this way, three side lengths are $a+m, c+p, b+n$, and $a+m = c+p = b+n = R$. Connect D with E , connect E with F , and connect F with D . Let $S_{\triangle ADF} = S_1, S_{\triangle BDE} = S_2, S_{\triangle CEF} = S_3, S_{\triangle ABC} = S$. Then $S_1+S_2+S_3 = \frac{1}{2}an\sin 60^0 + \frac{1}{2}cm\sin 60^0 + \frac{1}{2}bp\sin 60^0 = \frac{\sqrt{3}}{4}(an+cm+bp)$. $S = \frac{1}{2}R^2\sin 60^0 = \frac{\sqrt{3}}{4}R^2$. $S = S_1+S_2+S_3+S_{\triangle DEF} > S_1+S_2+S_3$, thus $\frac{\sqrt{3}}{4}(an+cm+bp) < \frac{\sqrt{3}}{4}R^2$, that is, $an+bp+cm < R^2$.

3.80 ★★★★ Let $x_1, x_2, \cdots, x_n$ are positive numbers, show $\frac{x_1^2}{x_2}+\frac{x_2^2}{x_3}+\cdots+\frac{x_{n-1}^2}{x_n}+\frac{x_n^2}{x_1} \geq x_1+x_2+\cdots+x_n$.

Proof 1: Since $x_1, x_2, \cdots, x_n > 0$, we can do the following: $(x_1-x_2)^2 \geq 0 \Rightarrow x_1^2+x_2^2 \geq 2x_1x_2 \Rightarrow \frac{x_1^2}{x_2}+x_2 \geq 2x_1$. Similarly, we can obtain $\frac{x_2^2}{x_3}+x_3 \geq 2x_2, \cdots, \frac{x_{n-1}^2}{x_n}+x_n \geq 2x_{n-1}, \frac{x_n^2}{x_1}+x_1 \geq 2x_n$. Add them up to obtain $(\frac{x_1^2}{x_2}+\frac{x_2^2}{x_3}+\cdots+\frac{x_{n-1}^2}{x_n}+\frac{x_n^2}{x_1})+(x_1+x_2+\cdots+x_n) \geq 2(x_1+x_2+\cdots+x_n) \geq 2(x_1+x_2+\cdots+x_n) \Rightarrow \frac{x_1^2}{x_2}+\frac{x_2^2}{x_3}+\cdots+\frac{x_{n-1}^2}{x_n}+\frac{x_n^2}{x_1} \geq x_1+x_2+\cdots+x_n$.

Proof 2: Mean Inequality implies that $\frac{x_1^2}{x_2}+x_2 \geq 2\sqrt{\frac{x_1^2}{x_2}\cdot x_2} = 2x_1$. Similarly, we have $\frac{x_2^2}{x_3}+x_3 \geq 2x_2, \cdots, \frac{x_{n-1}^2}{x_n}+x_n \geq 2x_{n-1}, \frac{x_n^2}{x_1}+x_1 \geq 2x_n$. Add them up to obtain the result.

Proof 3: $\frac{x_n^2}{x_1}+\frac{x_1^2}{x_1}+\frac{x_1^2}{x_2}+\frac{x_2^2}{x_2}+\cdots+\frac{x_{n-1}^2}{x_n}+\frac{x_n^2}{x_n} = \frac{x_n^2+x_1^2}{x_1}+\frac{x_1^2+x_2^2}{x_2}+\cdots+\frac{x_{n-1}^2+x_n^2}{x_n} \geq 2(\frac{x_1x_n}{x_1}+\frac{x_1x_2}{x_2}+\cdots+\frac{x_{n-1}x_n}{x_n}) = 2(x_1+x_2+\cdots+x_n)$ which implies the result.

Proof 4: $\frac{x_1^2}{x_2}+\frac{x_2^2}{x_3}+\cdots+\frac{x_{n-1}^2}{x_n}+\frac{x_n^2}{x_1} = \frac{[x_2-(x_2-x_1)]^2}{x_2}+\frac{[x_3-(x_3-x_2)]^2}{x_3}+\cdots+\frac{[x_n-(x_n-x_{n-1})]^2}{x_n}+.\frac{[x_1-(x_1-x_n)]^2}{x_1} = (x_2+x_3+\cdots+x_n+x_1)-2(x_2-x_1+x_3-x_2+\cdots+x_n-x_{n-1}+x_1-x_n)+\frac{(x_2-x_1)^2}{x_2}+\frac{(x_3-x_2)^2}{x_3}+\cdots+\frac{(x_n-x_{n-1})^2}{x_n}+\frac{(x_1-x_n)^2}{x_1} = (x_1+x_2+\cdots+x_n)+\frac{(x_2-x_1)^2}{x_2}+\frac{(x_3-x_2)^2}{x_3}+\cdots+\frac{(x_1-x_n)^2}{x_1} \geq x_1+x_2+\cdots+x_n$

Proof 5: Since $x_1, x_2, \cdots, x_n > 0$, let $a_1 = \sqrt{x_2}, a_2 = \sqrt{x_3}, \cdots, a_{n-1} = \sqrt{x_n}, a_n = \sqrt{x_1}, b_1 = \frac{x_1}{\sqrt{x_2}}, b_2 = \frac{x_2}{\sqrt{x_3}}, b_3 = \frac{x_3}{\sqrt{x_4}}, \cdots, b_{n-1} = \frac{x_{n-1}}{\sqrt{x_n}}, b_n = \frac{x_n}{\sqrt{x_1}}$. Cauchy Inequality implies that $(a_1^2 + a_2^2 + \cdots + a_n^2)(b_1^2 + b_2^2 + \cdots + b_n^2) \geq (a_1b_1 + a_2b_2 + \cdots + a_nb_n)^2$, then $[(\sqrt{x_2})^2 + (\sqrt{x_3})^2 + \cdots + (\sqrt{x_n})^2 + (\sqrt{x_1})^2] \cdot [(\frac{x_1}{\sqrt{x_2}})^2 + (\frac{x_2}{\sqrt{x_3}})^2 + \cdots + (\frac{x_{n-1}}{\sqrt{x_n}})^2 + (\frac{x_n}{\sqrt{x_1}})^2] \geq [\sqrt{x_2}\frac{x_1}{\sqrt{x_2}} + \sqrt{x_3}\frac{x_2}{\sqrt{x_3}} + \cdots + \sqrt{x_n}\frac{x_{n-1}}{\sqrt{x_n}} + \sqrt{x_1}\frac{x_n}{\sqrt{x_1}}]^2 \Rightarrow (x_2 + x_3 + \cdots + x_n + x_1)(\frac{x_1^2}{x_2} + \frac{x_2^2}{x_3} + \cdots + \frac{x_{n-1}^2}{x_n} + \frac{x_n^2}{x_1} \geq (x_1 + x_2 + \cdots + x_{n-1} + x_n)^2$. Divide both sides by $x_1 + x_2 + \cdots + x_{n-1} + x_n \geq 0$ to obtain $\frac{x_1^2}{x_2} + \frac{x_2^2}{x_3} + \cdots + \frac{x_{n-1}^2}{x_n} + \frac{x_n^2}{x_1} \geq x_1 + x_2 + \cdots + x_{n-1} + x_n$.

3.81 ★★★★ If x, y are real numbers, and $y \geq 0, y(y+1) \leq (x+1)^2$, show $y(y-1) \leq x^2$.

Proof: If $0 \leq y \leq 1$, obviously $y(y-1) \leq 0 \leq x^2$. If $y > 1$, then $y(y+1) \leq (x+1)^2 \Rightarrow y^2 + y + \frac{1}{4} \leq (x+1)^2 + \frac{1}{4} \Rightarrow (y+\frac{1}{2})^2 \leq (x+1)^2 + \frac{1}{4} \Rightarrow 1 < y \leq \sqrt{(x+1)^2 + \frac{1}{4}} - \frac{1}{2}$ The inequality to prove $y(y-1) \leq x^2 \Leftrightarrow y^2 - y + \frac{1}{4} \leq x^2 + \frac{1}{4} \Leftrightarrow (y-\frac{1}{2})^2 \leq x^2 + \frac{1}{4} \Leftrightarrow y \leq \sqrt{x^2+\frac{1}{4}} + \frac{1}{2} \Leftrightarrow \sqrt{(x+1)^2+\frac{1}{4}} - \frac{1}{2} \leq \sqrt{x^2+\frac{1}{4}} + \frac{1}{2} \Leftrightarrow \sqrt{(x+1)^2+\frac{1}{4}} \leq \sqrt{x^2+\frac{1}{4}} + 1 \Leftrightarrow (x+1)^2 + \frac{1}{4} \leq x^2 + \frac{1}{4} + 2\sqrt{x^2+\frac{1}{4}} + 1 \Leftrightarrow x^2 + 2x + 1\frac{1}{4} \leq x^2 + 2\sqrt{x^2+\frac{1}{4}} + 1\frac{1}{4} \Leftrightarrow x \leq \sqrt{x^2+\frac{1}{4}}$ which is obviously valid.

3.82 ★★★★ If real numbers x, y, z satisfy $x^2 + y^2 + z^2 = 2$, show $x + y + z \leq xyz + 2$.

Proof: If one (or more) of x, y, z is not positive, without loss of generality let $z \leq 0$. Since $x+y \leq \sqrt{2(x^2+y^2)} \leq \sqrt{2(x^2+y^2+z^2)} = 2, xy \leq \frac{1}{2}(x^2+y^2) \leq \frac{1}{2}(x^2+y^2+z^2) = 1$, then $2 + xyz - (x+y+z) = [2-(x+y) - z(xy-1)] \geq 0$, that is, $x+y+z \leq xyz+2$.

If x, y, z are all positive, let $0 < x \leq y \leq z$.

When $z \leq 1$, $2 + xyz - (x+y+z) = 1 - x - y + xy + 1 - xy - z + xyz = (1-x) - y(1-x) + (1-xy) - z(1-xy) = (1-x)(1-y) + (1-xy)(1-z) \geq 0$, that is, $x + y + z \leq xyz + 2$.

When $z > 1$, $x + y + z \leq \sqrt{2[z^2+(x+y)^2]} = \sqrt{2(2+2xy)} = 2\sqrt{1+xy} \leq 2 + xy < 2 + xyz$.

As a conclusion, $x + y + z \leq xyz + 2$ holds.

3.83 ★★★★ Given the function $f(x) = ax^2 + bx + c$ $(a > 0)$, and the two roots of the equation $f(x) - x = 0$ satisfy $0 < x_1 < x_2 < \frac{1}{a}$. (1) When $x \in (0, x_1)$, show $x < f(x) < x_1$; (2) Assume the curve of the function $f(x)$ is symmetric about the straight line $x = x_0$, show $x_0 < \frac{x_1}{2}$.

Proof: (1) Let $G(x) = f(x) - x$. Since x_1, x_2 are the two roots of the equation $f(x) - x = 0$, then $G(x) = a(x - x_1)(x - x_2)$. When $x \in (0, x_1)$, since $x_1 < x_2, a > 0$, then $G(x) = a(x - x_1)(x - x_2) > 0 \Rightarrow f(x) - x > 0 \Rightarrow f(x) > x$. $x_1 - f(x) = x_1 - [x + G(x)] = x_1 - x - a(x - x_1)(x - x_2) = (x_1 - x)[1 + a(x - x_2)]$. Since $0 < x_1 < x_2 < \frac{1}{a}$, we have $x_1 - x > 0, 1 + a(x - x_2) = 1 + ax - ax_2 > 1 - ax_2 > 0$, thus $x_1 > f(x)$.

(2) $x_0 = -\frac{b}{2a}$. Since x_1, x_2 are the roots of the equation $f(x) - x = 0$, that is, x_1, x_2 are the roots of the equation $ax^2 + (b-1)x + c = 0$. Vieta's formula implies $x_1 + x_2 = -\frac{b-1}{a}$, thus $b = 1 - a(x_1 + x_2)$, then $x_0 = -\frac{b}{2a} = \frac{a(x_1+x_2)-1}{2a} = \frac{ax_1+ax_2-1}{2a}$. Since $ax_2 < 1$, that is, $ax_2 - 1 < 0$, then $x_0 = \frac{ax_1+ax_2-1}{2a} < \frac{ax_1}{2a} = \frac{x_1}{2}$.

3.84 ★★★★★ Let a, b, c are positive numbers, show $a + b + c \le \frac{a^2+b^2}{2c} + \frac{b^2+c^2}{2a} + \frac{c^2+a^2}{2b} \le \frac{a^3}{bc} + \frac{b^3}{ca} + \frac{c^3}{ab}$.

Proof: Without loss of generality, assume $a \ge b \ge c > 0$, then $a^2 \ge b^2 \ge c^2, \frac{1}{c} \ge \frac{1}{b} \ge \frac{1}{a}$, then $a^2 \cdot \frac{1}{a} + b^2 \cdot \frac{1}{b} + c^2 \cdot \frac{1}{c} \le a^2 \cdot \frac{1}{b} + b^2 \cdot \frac{1}{c} + c^2 \cdot \frac{1}{a}, a^2 \cdot \frac{1}{a} + b^2 \cdot \frac{1}{b} + c^2 \cdot \frac{1}{c} \le a^2 \cdot \frac{1}{c} + b^2 \cdot \frac{1}{a} + c^2 \cdot \frac{1}{b}$. Add these two inequalities up to obtain $a + b + c \le \frac{a^2+b^2}{2c} + \frac{b^2+c^2}{2a} + \frac{c^2+a^2}{2b}$. $a^3 \ge b^3 \ge c^3, \frac{1}{bc} \ge \frac{1}{ca} \ge \frac{1}{ab}$, then $a^3 \cdot \frac{1}{bc} + b^3 \cdot \frac{1}{ca} + c^3 \cdot \frac{1}{ab} \ge a^3 \cdot \frac{1}{ab} + b^3 \cdot \frac{1}{bc} + c^3 \frac{1}{ca}, a^3 \cdot \frac{1}{bc} + b^3 \cdot \frac{1}{ca} + c^3 \cdot \frac{1}{ab} \ge a^3 \cdot \frac{1}{ca} + b^3 \cdot \frac{1}{ab} + c^3 \frac{1}{bc}$ Add these two inequalities up to obtain $\frac{a^3}{bc} + \frac{b^3}{ca} + \frac{c^3}{ab} \ge \frac{a^2+b^2}{2c} + \frac{b^2+c^2}{2a} + \frac{c^2+a^2}{2b}$. Hence, $a + b + c \le \frac{a^2+b^2}{2c} + \frac{b^2+c^2}{2a} + \frac{c^2+a^2}{2b} \le \frac{a^3}{bc} + \frac{b^3}{ca} + \frac{c^3}{ab}$.

3.85 ★★★★★ Solve the inequality $\log_2(x^{12} + 3x^{10} + 5x^8 + 3x^6 + 1) > 1 + \log_2(x^4 + 1)$.

Solution: The inequality is equivalent to $\log_2(x^{12} + 3x^{10} + 5x^8 + 3x^6 + 1) > \log_2(2x^4 + 2) \Leftrightarrow x^{12} + 3x^{10} + 5x^8 + 3x^6 + 1 > 2x^4 + 2 \Leftrightarrow 2x^4 + 1 < x^{12} + 3x^{10} + 5x^8 + 3x^6$. Obviously $x = 1$ does not satisfy the inequality, thus we can divide both sides by x^6 to obtain $\frac{2}{x^2} + \frac{1}{x^6} < x^6 + 3x^4 + 5x^2 + 3 = x^6 + 3x^4 + 3x^2 + 1 + 2x^2 + 2 = (x^2+1)^3 + 2(x^2+1) \Leftrightarrow (\frac{1}{x^2})^3 + 2(\frac{1}{x^2}) < (x^2+1)^3 + 2(x^2+1)$. Let $g(t) = t^3 + 2t$, then the inequality becomes $g(\frac{1}{x^2}) < g(x^2+1)$. Since $g(t) = t^3 + 2t$ is an increasing function, then the inequality is equivalent to $\frac{1}{x^2} < x^2 + 1 \Leftrightarrow (x^2)^2 + x^2 - 1 > 0$ whose solution is $x^2 > \frac{\sqrt{5}-1}{2}$ (the other part $x^2 < -\frac{\sqrt{5}+1}{2}$ is deleted). Hence, the original inequality has the solution set $(-\infty, -\sqrt{\frac{\sqrt{5}-1}{2}}) \cup (\sqrt{\frac{\sqrt{5}-1}{2}}, +\infty)$.

3.86 ★★★★★ Let a, b, c are integers and at least one of them is nonzero, and their absolute values are not greater than 10^6, show $|a + b\sqrt{2} + c\sqrt{3}| > 10^{-21}$.

Proof: When $b = 0, c = 0$, the conclusion is obviously valid. When one of b, c is nonzero, we consider the following four numbers: $t_1 = a + b\sqrt{2} + c\sqrt{3}, t_2 = a + b\sqrt{2} - c\sqrt{3}, t_3 = a - b\sqrt{2} + c\sqrt{3}, t_4 = a - b\sqrt{2} - c\sqrt{3}$. They are all irrational numbers, and $t = t_1t_2t_3t_4 = [(a + b\sqrt{2})^2 - 3c^2][(a - b\sqrt{2})^2 - 3c^2] = (a^2 + 2\sqrt{2}ab + 2b^2 - 3c^2)(a^2 - 2\sqrt{2}ab + 2b^2 - 3c^2) = [(a^2 + 2b^2 - 3c^2) + 2\sqrt{2}ab][(a^2 + 2b^2 - 3c^2) - 2\sqrt{2}ab] = [(a^2 + 2b^2 - 3c^2)^2 - 8a^2b^2] \in \mathcal{Z}$. Thus $|t| \geq 1$, which implies that $|t_1| \geq \frac{1}{|t_2| \cdot |t_3| \cdot |t_4|}$. In addition, since $1 + \sqrt{2} + \sqrt{3} < 10$ and $|a|, |b|, |c| \leq 10^6$, we have $|t_i| \leq (1 + \sqrt{2} + \sqrt{3}) \cdot 10^6 < 10^7$, thus $|t_1| > \frac{1}{10^7 \cdot 10^7 \cdot 10^7} = 10^{-21}$.

3.87 ★★★★★ For $k \in \mathcal{N}$, show $16 < \sum_{k=1}^{80} \frac{1}{\sqrt{k}} < 17$.

Proof: From $\sqrt{k-1} < \sqrt{k} < \sqrt{k+1}$, we have $\sqrt{k} + \sqrt{k-1} < 2\sqrt{k} < \sqrt{k+1} + \sqrt{k}$.
$k \in \mathcal{N}$, then
$\frac{1}{\sqrt{k+1}+\sqrt{k}} < \sqrt{1}2\sqrt{k} < \frac{1}{\sqrt{k}+\sqrt{k-1}} \Rightarrow 2(\sqrt{k+1} - \sqrt{k}) < \frac{1}{\sqrt{k}} < 2(\sqrt{k} - \sqrt{k-1}) \Rightarrow$
$2(\sqrt{n+1} - \sqrt{m}) < \sum_{k=m}^{n} \frac{1}{\sqrt{k}} < 2(\sqrt{n} - \sqrt{m-1})$, where $1 \le m \le n$, and $m, n \in \mathcal{N}$. Choose $n = 80, m = 1$, then $16 < \sum_{k=1}^{80} \frac{1}{\sqrt{k}}$. Choose $n = 80, m = 2$, then
$1 + \sum_{k=2}^{80} \frac{1}{\sqrt{k}} < 2(\sqrt{80} - 1) + 1 < 2\sqrt{81} - 1 = 17$. Hence, $16 < \sum_{k=1}^{80} \frac{1}{\sqrt{k}} < 17$.

3.88 ★★★★★ For $n \in \mathcal{N}, n > 1$, show $n! < (\frac{n+1}{2})^n$.

Proof 1: (applying mean inequality)

$n! = 1 \cdot 2 \cdot \cdots \cdot k \cdot \cdots \cdot (n-1) \cdot n$ (i),

$n! = n \cdot (n-1) \cdot \cdots \cdot (n-k+1) \cdot \cdots \cdot 2 \cdot 1$ (ii).

(i)$\times$ (ii): $(n!)^2 = (1 \cdot n)[2(n-1)] \cdot \cdots \cdot [k(n-k+1)] \cdot \cdots \cdot [(n-1)2](n \cdot 1)$. Since $1 \cdot n \le (\frac{1+n}{2})^2, 2(n-1) \le (\frac{2+n-1}{2})^2 = (\frac{1+n}{2})^2, \cdots, k(n-k+1) \le (\frac{k+n-k+1}{2})^2 = (\frac{1+n}{2})^2, \cdots, (n-1)2 \le (\frac{n-1+2}{2})^2 = (\frac{1+n}{2})^2, n \cdot 1 \le (\frac{1+n}{2})^2$. There are n terms. Thus $(n!)^2 \le [(\frac{1+n}{2})^2]^n \Leftrightarrow (n!)^2 \le (\frac{n+1}{2})^{2n} \Leftrightarrow n! \le (\frac{n+1}{2})^n$. Since $n > 1$, then $n! < (\frac{n+1}{2})^n$.

Proof 2: (applying mathematical induction)

When $n = 2$, LHS$= 2! = 2$, RHS$= (\frac{2+1}{2})^2 = \frac{9}{4}$. The inequality holds.
Assume the inequality holds for $n = k$, that is, $k! < (\frac{k+1}{2})^k$. Then
$k!(k+1) < (\frac{k+1}{2})^k(k+1) \Leftrightarrow (k+1)! < (\frac{k+1}{2})^k(k+1)$.
$(\frac{k+1}{2})^k(k+1) < [\frac{(k+1)+1}{2}]^{k+1} \Rightarrow 2(\frac{k+1}{2})^k(\frac{k+1}{2}) < [\frac{(k+1)+1}{2}]^{k+1} \Rightarrow$
$2(\frac{k+1}{2})^{k+1} < [\frac{(k+1)+1}{2}]^{k+1} \Rightarrow 2 < [\frac{(k+1)+1}{2}]^{k+1}(\frac{2}{k+1})^{k+1}$. Binomial theorem implies that $(1 + \frac{1}{k+1})^{k+1} = 1 + (k+1)\frac{1}{k+1} + \cdots > 2$. Thus $(\frac{k+1}{2})^k(k+1) < [\frac{(k+1)+1}{2}]^{k+1}$ holds.
Hence, the original inequality $n! < (\frac{n+1}{2})^n$ holds according to mathematical induction above.

3.89 ★★★★★ Let $a_1, a_2, \cdots, a_n$ be a permutation of $1, 2, \cdots, n$, show $\frac{1}{2} + \frac{2}{3} + \cdots + \frac{n-1}{n} \le \frac{a_1}{a_2} + \frac{a_2}{a_3} + \cdots + \frac{a_{n-1}}{a_n}$.

Proof: Since $a_1, a_2, \cdots, a_n$ is a permutation of $1, 2, \cdots, n$, we have $(a_1+1)(a_2+1)\cdot\cdots\cdot(a_{n-1}+1) \ge (1+1)(2+1)\cdot\cdots\cdot(n-1+1) = a_1a_2\cdots a_n$. Thus $\frac{a_1}{a_2}+\frac{a_2}{a_3}+\cdots+\frac{a_{n-1}}{a_n}+\frac{1}{1}+\frac{1}{2}+\cdots+\frac{1}{n} = \frac{a_1}{a_2}+\frac{a_2}{a_3}+\cdots+\frac{a_{n-1}}{a_n}$ $+\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n} = \frac{1}{a_1}+\frac{a_1+1}{a_2}+\frac{a_2+1}{a_3}+\cdots+\frac{a_{n-1}+1}{a_n} \ge n\sqrt[n]{\frac{(a_1+1)(a_2+1)\cdot\cdots\cdot(a_{n-1}+1)}{a_1a_2\cdots a_n}} \ge n$. In addition, $n = (\frac{1}{1}+\frac{1}{2}+\cdots+\frac{1}{n}) + (\frac{1}{2}+\frac{2}{3}+\cdots+\frac{n-1}{n})$. Hence, $\frac{1}{2}+\frac{2}{3}+\cdots+\frac{n-1}{n} \le \frac{a_1}{a_2}+\frac{a_2}{a_3}+\cdots+\frac{a_{n-1}}{a_n}$.

3.90 ★★★★★ If real numbers a, b, c satisfy $a+b+c=3$, show $\frac{1}{5a^2-4a+11}+\frac{1}{5b^2-4b+11}+\frac{1}{5c^2-4c+11} \le \frac{1}{4}$.

Proof: If a, b, c are all less than $\frac{9}{5}$, then we can show $\frac{1}{5a^2-4a+11} \le \frac{1}{24}(3-a)$ (✠). Actually (✠) $\Leftrightarrow (3-a)(5a^2-4a+11) \ge 24 \Leftrightarrow 5a^3-19a^2+23a-9 \le 0 \Leftrightarrow (a-1)^2(5a-9) \le 0 \Leftrightarrow$ $a < \frac{9}{5}$. Similarly, we can obtain $\frac{1}{5b^2-4b+11} \le \frac{1}{24}(3-b)$, $\frac{1}{5c^2-4c+11} \le \frac{1}{24}(3-c)$. Add these three inequalities up to obtain $\frac{1}{5a^2-4a+11}+\frac{1}{5b^2-4b+11}+\frac{1}{5c^2-4c+11} \le \frac{1}{24}(3-a)+\frac{1}{24}(3-b)+\frac{1}{24}(3-c) = \frac{1}{24}[9-(a+b+c)] = \frac{1}{24}[9-3] = \frac{1}{4}$.

If at least one of a, b, c is not less than $\frac{9}{5}$, without loss of generality, assume $a \ge \frac{9}{5}$, then $5a^2-4a+11 = 5a(a-\frac{4}{5})+11 \ge 5\cdot\frac{9}{5}\cdot(\frac{9}{5}-\frac{4}{5})+11 = 20$. Thus $\frac{1}{5a^2-4a+11} \le \frac{1}{20}$. Since $5b^2-4b+11 \ge 5\cdot(\frac{2}{5})^2-4\cdot(\frac{2}{5})+11 = 11-\frac{4}{5} > 10$, then $\frac{1}{5b^2-4b+11} < \frac{1}{10}$. Similarly, we have $\frac{1}{5c^2-4c+11} < \frac{1}{10}$. Hence, $\frac{1}{5a^2-4a+11}+\frac{1}{5b^2-4b+11}+\frac{1}{5c^2-4c+11} < \frac{1}{20}+\frac{1}{10}+\frac{1}{10} = \frac{1}{4}$.

3.91 ★★★★★ Given a natural number $n > 1$, show $C_n^1 + C_n^2 + C_n^3 + \cdots + C_n^n > n \times 2^{\frac{n-1}{2}}$.

Proof: According to Binomial theorem, we have $2^n = (1+1)^n = 1 + C_n^1 + C_n^2 + \cdots + C_n^n$, thus $C_n^1 + C_n^2 + C_n^3 + \cdots + C_n^n = 2^n - 1$. On the other hand, the geometric series with first term 1 and common ratio 2 is $S_n = \frac{1\cdot(1-2^n)}{1-2} = 2^n - 1$, i.e. $2^n - 1 = 1 + 2 + 2^2 + 2^3 + \cdots + 2^{n-1}$. Therefore, $\frac{2^n-1}{n} = \frac{1+2+2^2+2^3+\cdots+2^{n-1}}{n} > \sqrt[n]{1\times 2\times 2^2\times 2^3\times\cdots\times 2^{n-1}} = \sqrt[n]{2^{1+2+3+\cdots+(n-1)}} = \sqrt[n]{2^{\frac{n(n-1)}{2}}} = 2^{\frac{n-1}{2}}$, that is, $2^n - 1 > n \times 2^{\frac{n-1}{2}}$. Hence, $C_n^1 + C_n^2 + C_n^3 + \cdots + C_n^n > n \times 2^{\frac{n-1}{2}}$.

3.92 ★★★★★ Positive numbers x, y, z satisfy $x^2+y^2+z^2 = 1$, find the minimum value of $\frac{x}{1-x^2}+\frac{y}{1-y^2}+\frac{z}{1-z^2}$.

Solution: (applying mean inequality)

$x^5+\frac{2}{3\sqrt{3}}x^2=x^5+\frac{1}{3\sqrt{3}}x^2+\frac{1}{3\sqrt{3}}x^2\geq 3\sqrt[3]{x^5\cdot\frac{1}{3\sqrt{3}}x^2\cdot\frac{1}{3\sqrt{3}}x^2}=x^3$.

Similarly, we can obtain $y^5+\frac{2}{3\sqrt{3}}y^2\geq y^3, z^5+\frac{2}{3\sqrt{3}}z^2\geq z^3$.

Add these three inequalities up to obtain $x^5+y^5+z^5+\frac{2}{3\sqrt{3}}(x^2+y^2+z^2)\geq x^3+y^3+z^3$.

Since $x^2+y^2+z^2=1$, then $x^5+y^5+z^5+\frac{2}{3\sqrt{3}}\geq x^3+y^3+z^3$,

then $x^3(1-x^2)+y^3(1-y^2)+z^3(1-z^2)\leq\frac{2}{3\sqrt{3}}$ (i).

$[x^3(1-x^2)+y^3(1-y^2)+z^3(1-z^2)](\frac{x}{1-x^2}+\frac{y}{1-y^2}+\frac{z}{1-z^2})\geq(\sqrt{x^3(1-x^2)}\sqrt{\frac{x}{1-x^2}}+\sqrt{y^3(1-y^2)}\sqrt{\frac{y}{1-y^2}}+\sqrt{z^3(1-z^2)}\sqrt{\frac{z}{1-z^2}}=x^2+y^2+z^2=1$. Thus

$\frac{x}{1-x^2}+\frac{y}{1-y^2}+\frac{z}{1-z^2}\geq\frac{1}{x^3(1-x^2)+y^3(1-y^2)+z^3(1-z^2)}$(ii). From (i) and (ii), we obtain

$\frac{x}{1-x^2}+\frac{y}{1-y^2}+\frac{z}{1-z^2}\geq\frac{3\sqrt{3}}{2}$.

When $x=y=z=\frac{1}{\sqrt{3}}$, $\frac{x}{1-x^2}+\frac{y}{1-y^2}+\frac{z}{1-z^2}$ reaches the minimum value $\frac{3\sqrt{3}}{2}$.

3.93 ★★★★★ Positive numbers $a_1,a_2,\cdots,a_n$ and $b_1,b_2,\cdots,b_n$ satisfy $a_1+a_2+\cdots+a_n\leq 1, b_1+b_2+\cdots+b_n\leq n$, show $(\frac{1}{a_1}+\frac{1}{b_1})(\frac{1}{a_2}+\frac{1}{b_2})\cdots(\frac{1}{a_n}+\frac{1}{b_n})\geq(n+1)^n$.

Proof: The given conditions together with Mean Inequality result in $a_1a_2\cdots a_n\leq(\frac{a_1+a_2+\cdots+a_n}{n})^n\leq\frac{1}{n^n}$ (i), and $b_1b_2\cdots b_n\leq(\frac{b_1+b_2+\cdots+b_n}{n})=1$ (ii). In addition, $\frac{1}{a_i}+\frac{1}{b_i}=\underbrace{\frac{1}{na_i}+\cdots+\frac{1}{na_i}}_{\text{n terms}}+\frac{1}{b_i}\geq(n+1)\sqrt[n+1]{\left(\frac{1}{na_i}\right)^n\left(\frac{1}{b_i}\right)}$ $(i=1,2,\cdots,n)$ (iii).

From (i),(ii),(iii), we can obtain

$$\begin{aligned}\left(\frac{1}{a_1}+\frac{1}{b_1}\right)\left(\frac{1}{a_2}+\frac{1}{b_2}\right)\cdots\left(\frac{1}{a_n}+\frac{1}{b_n}\right) &\geq (n+1)^n\sqrt[n+1]{\frac{1}{(n^n)^n}\cdot\left(\frac{1}{a_1a_2\cdots a_n}\right)^n\cdot\frac{1}{b_1b_2\cdots b_n}}\\ &\geq (n+1)^n\sqrt[n+1]{\frac{1}{(n^n)^n}\cdot(n^n)^n\cdot 1}\\ &= (n+1)^n.\end{aligned}$$