Introductory Algebra

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Contents

Introduction

The purpose of this text is to serve as a primer for those who want to prepare for a high school or college first semester algebra course.

The text is not considered in any manner to be a comprehensive course of algebra.

The strength of the text is my step-by-step method of problem solving demonstrated in each chapter.

Another strength of the text is a comprehensive description of the governing principles of algebra.

The step-by-step approach to problem solving explains in full, sometimes multiple, sentences each step used to solve a problem. As additional identical problems are solved their solutions are described using the exact words used to describe the previous problems.

In some places in the text an entire page is used to describe the solution to a single problem.

At the end of chapters 3 through 11 there are problem sets where the problems are of the exact same degree of difficulty as the examples in the chapters. The answers to these end-of-chapter exercises are listed in Appendix A at the end of the text.

The repetition of instructions in using my step-by-step problem solving method generates a positive feedback that helps make students better problem solvers.

I predict that any student who completes my text will experience a sense of confidence that will help them in their academic pursuit of algebra and higher forms of mathematics.

1 The Real Number System

1.1 Natural Numbers

The earliest form of counting was to make a single mark for each item being counted. These marks could be notches on a stick or rod or could be marks on bark or a stone slab.

One important thing to note is that these counting systems did not contain a zero.

In keeping with this simple form of counting let's define ***natural numbers*** as a set of numbers containing all whole positive numbers.

The following set notation denotes all natural numbers.

N {1,2,3,4,5 …..}

Natural numbers are represented on the number line as follows.

The following are things to note about the set of all natural numbers.

- Zero is not a member of the set.
- There are no negative numbers in the set.
- There are no fractions in the set.

Keep in mind that if you were to start writing out all of the natural numbers you would use a zero as part of a number as soon as you reached the number ten.

This is due to the fact that we presently are not limited to the use of natural numbers.

1.2 Whole Numbers

There is some ambiguity associated with whether or not zero should be included in the set of natural numbers.

For the purpose of this text let's leave zero out of the set of natural numbers and define ***whole numbers*** as the set of all whole positive numbers including zero.

The following set notation denotes all whole numbers.

N^0 {0,1,2,3,4,5 …..}

Whole numbers are represented on the number line as follows.

It is important to note that very few early number systems used the zero. Civilizations as advanced as the ancient Greeks and Romans did not use the zero in their number systems.

The importance of the use of zero in our present day decimal number system is as a place holder.

There are numerous references in historical documents of the use of the zero as far back as 700 BC.

A famous Italian mathematician named Fibonacci is credited with introducing an Arabic numbering system (including the zero) to Europe around 1200 AD.

1.3 Integers

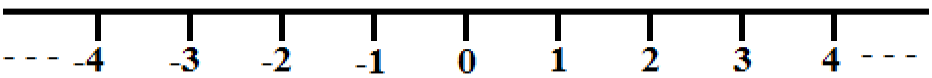
Integers are a system of whole numbers plus their negative counterparts.

Integers are often referred to as whole integers which should serve to remind one that there are no fractions in the integer system of numbers.

The following set notation denotes all integers.

Z {…-4,-3,-2,-1,0,1,2,3,4…}

Integers are represented on the number line as follows.



In the manner of introduction, a negative number is defined as a number that will produce zero when added to its positive counterpart.

Example 1.3.1

$4 + (-4) = 0$

1.4 Rational Numbers

So far fractions have been absent from all the number systems introduced.

The ***rational*** number system contains all numbers that are, or could be, expressed as a ***ratio***.

If you want to convey the idea of one half of something as a ratio of numbers you must write 1/2.

Also, if you wish to express the number four as a ratio of numbers you can write 4/1.

The rational number system is defined by the following formula.

$$Q = a/b \quad (b \neq 0)$$

The denominator (***b***) cannot equal zero. Division by zero is undefined.

1.5 Irrational Numbers

A rational number is rational because it can be solved.

Example 1.5.1

$1/4 = 0.25$

An ***irrational number*** is irrational because it cannot be solved.

Example 1.5.2 The square root of 2.

$\sqrt{2} = 1.41421356237\ldots$

The dots at the end of the above value for the square root of two are meant to indicate the numbers would go on forever without any repeating patterns.

1.6 Real Numbers

The ***real number system*** is simply the inclusion of all of the number systems in sections 1.1 through 1.5 of this chapter.

Figure 1.6.1 demonstrates the real number system.

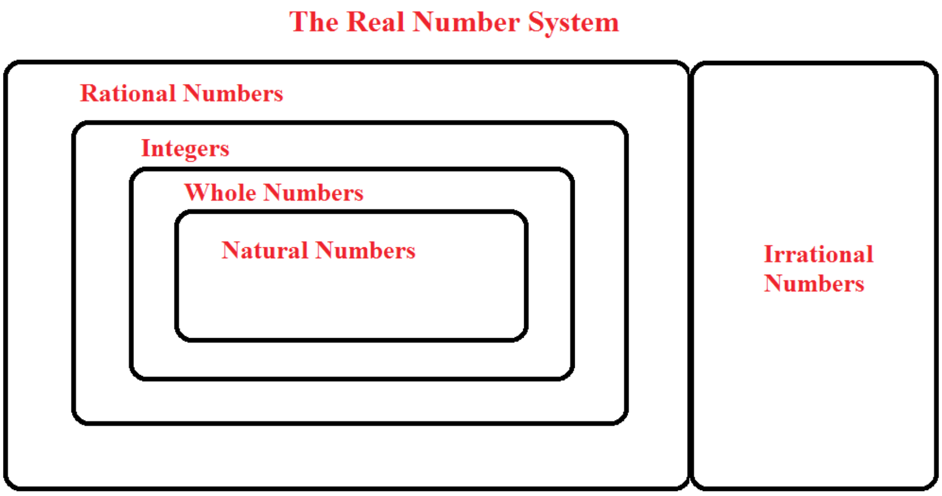


Figure 1.6.1 The Real Number System

2 Mathematical Properties

2.1 The Distributive Property

The ***distributive property*** of mathematics states that when a single number is used to multiply a series of numbers in parentheses the multiplication by that single number is ***distributed*** to each number in parentheses.

$$a(b + c) = ab + ac$$

Note: Numbers or expressions written next to each other are to be multiplied.

Example 2.1.1

$$5(3 + 2)$$
$$5(3 + 2) = (5 \times 3) + (5 \times 2) = 15 + 10 = 25$$

Example 2.1.2

$$4\,(2 + 3 - 1)$$
$$4\,(2 + 3 - 1) = (4 \times 2) + (4 \times 3) - (4 \times 1) = 8 + 12 - 4 = 16$$

Example 2.1.3

$$6\,(4 - 2 + 3 - 1)$$
$$6\,(4 - 2 + 3 - 1) = (6 \times 4) - (6 \times 2) + (6 \times 3) - (6 \times 1) = 24 - 12 + 18 - 6 = 24$$

2.2 The Associative Properties

The ***associative properties*** of addition and multiplication deal with grouping or regrouping of elements within expressions of addition or multiplication.

The ***associative property of addition*** is expressed as follows.

$$\boldsymbol{a + (b + c) = (a + b) + c}$$

Example 2.2.1

$$2 + (4 + 6) = (2 + 4) + 6$$
$$2 + 10 = 6 + 6$$
$$12 = 12$$

Example 2.2.2

$$3 + (2 + 5) = (3 + 2) + 5$$
$$3 + 7 = 5 + 5$$
$$10 = 10$$

The ***associative property of multiplication*** is expressed as follows.

$$\boldsymbol{a \times (b \times c) = (a \times b) \times c}$$

It is perfectly permissible to write the above expression as **a(bc) = (ab)c.**

Example 2.2.3

$$5 \times (2 \times 4) = (5 \times 2) \times 4$$
$$5 \times 8 = 10 \times 4$$
$$40 = 40$$

Example 2.2.4

$$3 \times (3 \times 4) = (3 \times 3) \times 4$$
$$3 \times 12 = 9 \times 4$$
$$36 = 36$$

2.3 The Commutative Properties

The ***commutative properties*** deal with the order in which we add or multiply numbers. As long as we are adding or multiplying the order in which we do so does not matter.

The ***commutative property of addition*** allows us to add a series of numbers in any order we choose.

$$\boldsymbol{a + b = b + a}$$

Example 2.3.1

$$2 + 3 = 3 + 2$$
$$5 = 5$$

Example 2.3.2

$$6 + 4 = 4 + 6$$
$$10 = 10$$

The ***commutative property of multiplication*** allows us to multiply a series of numbers in any order we choose.

$$\boldsymbol{a \times b = b \times a}$$

Example 2.3.3

$$3 \times 6 = 6 \times 3$$
$$18 = 18$$

Example 2.3.4

$$5 \times 10 = 10 \times 5$$
$$50 = 50$$

2.4 The Identity Property

The ***identity property of multiplication*** simply states that any number multiplied by 1 equals that number.

$$\boldsymbol{a \times 1 = a}$$

This may seem rather simplistic but it is a mathematical property and as such needs to be stated.

Example 2.4.1

$34 \times 1 = 34$

This operation could be looked at as one 34 or thirty four 1's.

2.5 A Statement on Mathematical Properties

Mathematical properties can seem very simple and easy to understand when read in a textbook.

The examples I prepared for these properties are very simplistic examples of the stated properties.

However, be careful not to underestimate the importance of these properties. No matter how complicated an algebraic problem may appear, you can be certain that these mathematical properties lie at the heart of the problem's solution.

3 Arithmetic Operations

3.1 Orders of Operations

Operations are all of the things we do to numbers. Addition, subtraction, multiplication, and division are all examples of operations. Raising a number to a power and taking a root of a number are also operations that will be covered in this text.

If in a single arithmetic expression more than one of these operations is utilized, it can make a difference in which order they are applied.

Often portions of these arithmetic expressions are emphasized inside parenthetic groupings.

Use the following rules to ensure the use of the proper orders of operations.

1) Perform operations inside **p**arentheses first.
2) Perform the operations of **e**xponents and roots prior to multiplication and division.
3) Perform **m**ultiplication and **d**ivision, from left to right, prior to addition and subtraction.
4) Perform **a**ddition and **s**ubtraction from left to right.

In order to remember these orders of operations use the acronym formed by the emboldened red letters from the rules above.

PEMDAS

(parentheses, exponents, multiplication, division, addition, subtraction)

Perhaps some examples of each rule would be in order.

Example 3.1.1

$2 + (5 \times 3)$

Rule 1 tells you to resolve what's inside the parentheses and then add.

$2 + (5 \times 3) = 2 + 15 = 17$

Example 3.1.2

6×3^2

Rule 2 tells you to resolve the exponent and then multiply.

$6 \times 3^2 = 6 \times 9 = 54$

Example 3.1.3

$5 \times 2 + 3$

Rule 3 tells you to multiply before you add.

$5 \times 2 + 3 = 10 + 3 = 13$

Example 3.1.4

$2 + 4 - 3 + 1$

Rule 4 tells you to add and subtract from left to right.

$2 + 4 - 3 + 1 = 6 - 3 + 1 = 3 + 1 = 4$

A few examples of more complex orders of operations problems are also in order.

Example 3.1.5

$5 + (2 \times 4^2) - 3$

Using PEMDAS we get the following orders of operations.

$5 + (2 \times 4^2) - 3$

$5 + (2 \times 16) - 3$

$5 + 32 - 3$

$37 - 3$

34

Example 3.1.6

$5 \times 2(3 + 3^2) - 5$

Using PEMDAS we get the following orders of operations.

$5 \times 2(3 + 3^2) - 5$

$5 \times 2(3 + 9) - 5$

$5 \times 2 \times 12 - 5$

$120 - 5$

115

Example 3.1.7

$16 - (\sqrt{16} + 5) + 3$

Using PEMDAS we get the following orders of operations.

$16 - (\sqrt{16} + 5) + 3$
$16 - (4 + 5) + 3$
$16 - 9 + 3$
$7 + 3$
10

3.2 Fractions, Decimals, and Percentages

Fractions are numbers, that are in between whole numbers, expressed as $a/b = c$ $(b \neq 0)$.

In other words ***c*** is expressed as the ***ratio*** of ***a*** to ***b*** (a ***ratio***nal number).

A fraction consists of a ***numerator*** above a ***denominator*** where these are separated by a ***divisor line***.

Figure 3.2.1 demonstrates the form of a fraction.

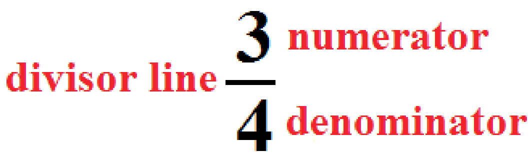


Figure 3.2.1 A Fraction

There are three types of fractions.

1) A ***proper*** fraction is a fraction where the numerator is smaller than the denominator.
   For example 2/3 is a proper fraction.
   All proper fractions have a value less than one.

2) An ***improper*** fraction is a fraction where the numerator is larger than the denominator.
   For example 4/3 is an improper fraction.
   All improper fractions have a value greater than one.

3) A ***mixed*** fraction is a combination of a whole number and a fraction.
   For example 3 ½ is a mixed fraction.

Decimals are the expressions of numbers, in between whole numbers, in a base ten place value format.

Figure 3.2.2 shows the place values of a decimal number.

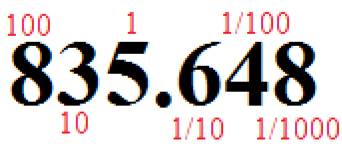


Figure 3.2.2 Decimal Place Values

The places of a decimal number can extend infinitely in either direction.

The three types of decimal expressions are ***exact, repeating,*** and, ***nonrepeating.***

An example of an ***exact*** decimal would be 0.5.

An example of a ***repeating*** decimal would be $0.5555555\overline{5}$ where the horizontal bar over the last 5 indicates that the 5 will repeat indefinitely.

An example of a ***non-repeating*** decimal would be the number pi (π) where $\pi = 3.14159265358979323846$….. without ending or repeating.

Percentage is a measurement of a part of a whole based on an equal 100 parts of that whole.

For example, if I had 500 apples and divided them into 100 parts of 5 apples each, then 5 apples would represent 1 percent of all the apples.

The symbol for percentage is **%**.

A percent value is easily calculated by multiplying the decimal form of a number by 100. The following example shows the relationship between fractions, decimals, and percentages.

$$1/2 = 0.50 = 50\%$$

Obviously the 0.50 is the result of 1 divided by 2 and the 50% is the result of 0.50 multiplied by 100.

To convert a decimal to a fraction write the decimal over one. Multiply that expression by 10/10 if there is one number after the decimal, 100/100 if there are two numbers after the decimal, 1,000/1,000 if there are three numbers after the decimal and so on.

$$0.5/1 \times 10/10 = 5/10 = 1/2$$

Example 3.2.1

Convert 3/8 to decimal and percentage forms.

$$3 \div 8 = 0.375 \qquad 0.375 \times 100 = 37.5\%$$

Example 3.2.2

Convert 0.25 to fraction and percentage forms.

$$0.25/1 \times 100/100 = 25/100 = 1/4$$
$$0.25 \times 100 = 25\%$$

Example 3.2.3

Convert 75% to decimal and fraction forms.

$$75\%/100 = 0.750.75/1 \times 100/100 = 75/100 = 3/4$$

3.3 Common Denominators

It is most common to hear common denominators referred to as ***least common denominators***. However, in this text, there are some examples of problems where a common denominator, that is not the least common denominator, is necessary to solve the problem.

Common denominators of fractions are denominators that are the same for each of the fractions.

For example it is easy enough to see that 1/5 + 3/5 = 4/5 because the denominators are the same or common.

But what if the denominators of two or more fractions are not the same? Can they be expressed with common denominators? Yes! They can.

The simplest way to calculate a common denominator of two fractions is to multiply both the numerator and the denominator of each fraction by the denominator of the other fraction.

Some examples are definitely needed here.

Example 3.3.1

Add the two fractions below.

$$1/3 + 2/5 = 1/3(5/5) + 2/5(3/3) = 5/15 + 6/15 = \boldsymbol{11/15}$$

The reason we are allowed to do this operation is that 5/5 and 3/3 are both equal to one. For this reason they do not change the values of the fractions.

Example 3.3.2

Subtract the two fractions below.

$$3/4 - 3/8 = 3/4(8/8) - 3/8(4/4) = 24/32 - 12/32 = \boldsymbol{12/32}$$

The answer should be simplified to ***3/8***

Another approach to common denominators, with evenly divisible denominators, is to use what I like to call the ***clockwise method***.

The clockwise method divides the smaller denominator into the larger denominator and moves, in a clockwise manner, up to the numerator above the smaller denominator and multiplies the numerator by that value.

Applying the clockwise method to Example 3.3.2 we get the following.

$$3/4 - 3/8 = (3 \times 2)/8 - 3/8 = 6/8 - 3/8 = \boldsymbol{3/8}$$

In other words $8 \div 4 = 2$ and 2×3 in the numerator of 3/4 is 6 for a converted value of 6/8

Example 3.3.3

Add the two fractions below.

$$3/5 + 3/10 = (2 \times 3)/10 + 3/10 = 6/10 + 3/10 = \boldsymbol{9/10}$$

Example 3.3.4

Add the two fractions below.

$$2/7 + 3/14 = (2 \times 2)/14 + 3/14 = 4/14 + 3/14 = 7/14 = \boldsymbol{1/2}$$

3.4 Positive and Negative Numbers

As was shown in Chapter 1, the real number system contains the set of integers. Integers are defined as the set of whole numbers plus their negative counterparts.

At this point in the text I would like to establish a set of rules that govern the addition, subtraction, multiplication, and division of positive and negative numbers. The following tables give the signs of problem solutions in multiplying, dividing, adding, and subtracting,positive and negative numbers.

1st Value	Operation	2nd Value	Equals	Final Value
+	x	+	=	+
+	x	-	=	-
-	x	-	=	+
-	x	+	=	-
+	÷	+	=	+
+	÷	-	=	-
-	÷	-	=	+
-	÷	+	=	-

Table 3.4.1 Multiplying and Dividing Positive and Negative Numbers

Let's look at examples of the eight multiplication and division rules listed in Table 3.4.1.

Example 3.4.1

$4 \times 2 = \boldsymbol{8}$

Example 3.4.2

$3 \times (-2) = \boldsymbol{-6}$

Example 3.4.3

$-3 \times (-2) = \boldsymbol{6}$

Example 3.4.4

$-8 \times 4 = \boldsymbol{-32}$

Example 3.4.5

$18 \div 3 = \boldsymbol{6}$

Example 3.4.6

16 ÷ -4 = ***-4***

Example 3.4.7

-8 ÷ -2 = ***4***

Example 3.4.8

-4 ÷ 2 = ***-2***

1^{st} Value	Operation	2^{nd} Value	Equals	Final Value
+	+	+	=	+
+	+	smaller -	=	+
+	+	larger -	=	-
-	+	-	=	-
-	+	smaller +	=	-
-	+	larger +	=	+
-	-	smaller -	=	-
-	-	larger -	=	+

Table 3.4.2 Adding and Subtracting Positive and Negative Numbers

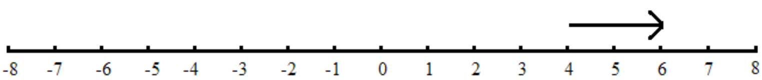
Note: The terms "smaller" and "larger" indicate the 2^{nd} value being smaller or larger than the 1^{st} value.

Let's look at examples of the eight addition and subtraction rules listed in Table 3.4.2.

The distance and direction of movement along the number line is indicated for each problem.

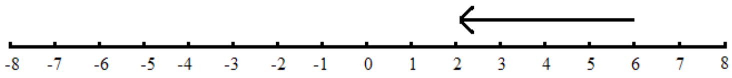
Example 3.4.9

4 + 2 = **6**



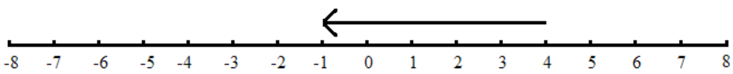
Example 3.4.10

$6 + (-4) = \boldsymbol{2}$



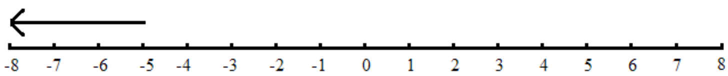
Example 3.4.11

$4 + (-5) = \boldsymbol{-1}$



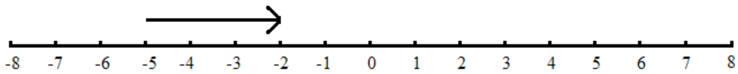
Example 3.4.12

$-5 + (-3) = \boldsymbol{-8}$



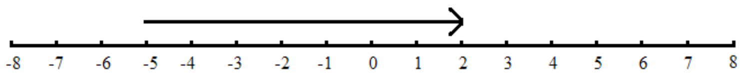
Example 3.4.13

$-5 + 3 = \boldsymbol{-2}$



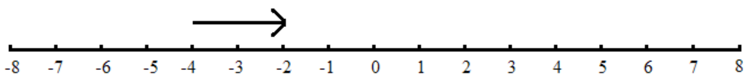
Example 3.4.14

$-5 + 7 = \boldsymbol{2}$



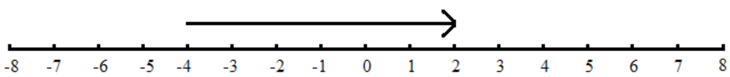
Example 3.4.15

$-4 - (-2) = \boldsymbol{-2}$



Example 3.4.16

$-4 - (-6) = \boldsymbol{2}$



3.5 Exponents

When an exponent is positioned as a superscript to a number it is a direction to multiply that number by itself the number of times of the value of the exponent.

For example if I wanted to represent three 5's multiplied together, I would write 5^3 and say 5 to the 3rd power or 5 cubed.

$$5^3 = 5 \times 5 \times 5 = 125$$

Second and third power exponents are named squared and cubed respectively. No other exponents have these type names. So you would say to the 4th power, to the 5th power, and so on.

There are three ***operations with exponents*** that I will present here.

The first operation with an exponent is simply ***raising a number to a power*** or exponent.

Simply multiply the number indicated by itself the number of times indicated by the exponent.

Example 3.5.1

$10^3 = 10 \times 10 \times 10 = \boldsymbol{1{,}000}$

The second operation is ***multiplying or dividing a number raised to an exponent by that same number raised to an exponent***.

When you multiply two exponential numbers, with the same base, the exponents are additive.

$10^m \times 10^n = 10^{m+n}$

Example 3.5.2

$10^2 \times 10^3 = 10^{2+3} = 10^5 = \boldsymbol{100{,}000}$

When you divide two exponential numbers, with the same base, the exponents are subtractive.

$10^m \div 10^n = 10^{m-n}$

Example 3.5.3

$10^4 \div 10^2 = 10^{4-2} = 10^2 = \boldsymbol{100}$

The third operation is ***raising a power to another power***.

When you raise a power to another power the exponents are multiplicative.

$(10^m)^n = 10^{m \times n}$

Example 3.5.4

$(10^2)^4 = 10^{2 \times 4} = 10^8 = \boldsymbol{100{,}000{,}000}$

3.6 Chapter 3 End of Chapter Exercises

Answer the following questions as directed. The answers are in Appendix A at the end of the text.

3.1. Perform the indicated operations using the proper ***orders of operations***.

1) $3 + (6 \times 2)$
2) 4×2^3
3) $6 \times 2 + 4$
4) q4 + 2 -3 +5
5) $6 + (2 \times 4^2) - 4$
6) $15 - (2 \times 2^2) + 4$
7) $14 - (\sqrt{9} + 4) + 2$
8) $\sqrt{16} + (\sqrt{25} + 5) + 9$

3.2. Identify the following fractions as ***proper, improper, or mixed***.

1) 3/16
2) 4/3
3) 3/4
4) 1 1/2
5) 7/6
6) 7/8
7) 3 1/3
8) 3/2

Identify the ***place value*** for the numbers in red for the following decimals.

1) 102
2) 46.2
3) 15.001
12) 105,238
13) 0.00346
14) 1,237,843

Write the ***decimal and percentage forms*** for the following fractions.

1) 2/3
2) 1/2
3) 5/8
18) 2/7
19) 11/14
20) 4/9

3.3. Add or subtract the following fractions using ***common denominators***. Express your answer in the simplest form.

1) 1/2 + 1/3
2) 2/3 – 3/5
3) 7/8 + 7/16
4) 4/5 – 3/10
5) 4/9 – 2/18
6) 4/3 + 2/9
7) 3/2 – 7/8
8) 1/3 + 1/9

3.4. Perform the indicated arithmetic operations on the following ***positive and negative numbers***.

1) $6 \times (-2)$	5) $4 + (-2)$
2) $-3 \times (-4)$	6) $-8 + 6$
3) -2×5	7) $-3 - (-5)$
4) $16 \times (-1)$	8) $4 - (-2)$

3.5. Perform the indicated operations for the following exponential expressions.

1) 10^4	6) $5^3 + 5^2$
2) 2^3	7) $16^5 - 16^4$
3) 5^4	8) $4^4 \div 4^2$
4) $2^3 \times 2^2$	9) $(4^2)^2$
5) $10^2 \times 10^3$	10) $(10^2)^3$

4 Expressions, Variables, and Equations

4.1 Algebraic Expressions

What is an ***algebraic expression***?

An algebraic expression is an expression where one or more of the components of that expression can ***change value***.

Ask yourself this ***very important*** question. If the value of acomponent, in an expression, can change how do I represent it in the expression?

The answer to the question is that you represent it with a ***place holder***.

The most common of these place holders are "***x***" and "***y***". In algebra these changeable values are called ***variables***.

An example is in order.

Example 4.1.1

Mary receives \$3 for every book sale she makes plus a \$20 salary each day.

An algebraic expression for this is ***3x + 20*** where ***x*** represents the number of books sold.

Example 4.1.2

Fred needs to purchase twelve times the length of one side of a three board square corral fence. He also needs an extra one foot per side for overlap.

An algebraic expression for this is ***12y + 4*** where y represents the length, in feet, of one side of the corral.

4.2 Algebraic Variables and Equations

As shown in section 4.1, a ***variable*** is the place holder used to represent components of an algebraic expression that can change.

An ***equation*** is an algebraic expression set equal to a number or another variable.

Let's look at Mary's book sales in Example 4.1.1 with two variables. The x still represents the number of books Mary sells. The second variable y will represent Mary's daily pay.

Example 4.2.1

Write and solve an equation for Mary's daily pay if she sold twenty books.

Mary's pay is represented by $y = 3x + 20$. Substituting twenty book sales for x the equation is solved as follows.

$y = 3(20) + 20 = 60 + 20 =$ ***80 dollars***

Example 4.2.2

How many board feet of lumber must Fred, from Example 4.1.2, purchase if one side of his square corral is 30 feet in length?

$y = 12x + 4 = 12(30) + 4 = 360 + 4 =$ ***364 feet***

Please notice in the two previous examples y always depended on the value of x.

Now that we've seen a few equations let's identify the separate parts of that equation.

The x and y variables are identified as follows. In both of the previous examples x was used to represent an ***independent variable*** and y was used to represent ***a dependent variable***.

The numbers preceding the variables in an equation are known as ***coefficients***.

The lone numbers in an equation are known as ***constants***.

Figure 4.2.1 shows a typical algebraic equation.

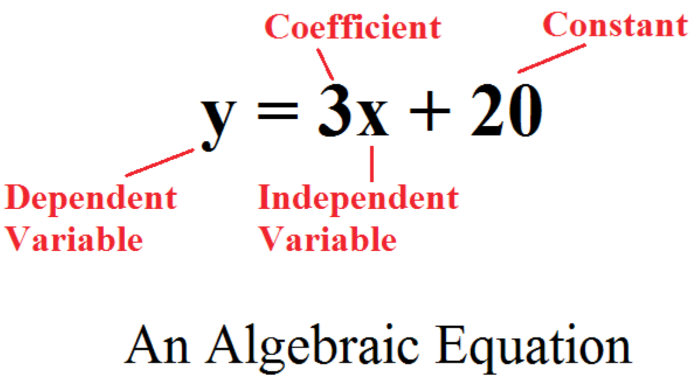


Figure 4.2.1 An Algebraic Equation

4.3 The Language of Algebraic Equations

An algebraic equation is a fairly simple form of representing a condition or state of being. In fact an algebraic equation can be looked at as a form of simplification of the verbal expression of said condition or state of being.

Most beginning algebra students would not agree with the above assessment of algebraic equations. This is due, solely I hope, to their unfamiliarity with algebraic equations and problem solving.

The best way to demonstrate the ***language of algebra*** is to work examples.

Example 4.3.1

A number is twice the size of another number. The sum of the two numbers is six. What are the two numbers?

Let x be one of the numbers and $2x$ be the other number. The word ***twice*** means multiply by 2. The word ***sum*** tells us to add. The word ***is*** in the second sentence means equal to

$$x + 2x = 6 \qquad 3x = 6 \qquad x = 6/3 \qquad x = 2$$
$$2x = 2(2) = 4$$

Example 4.3.2

A number is three times another number. The difference of the two numbers is six. What are the two numbers?

Let x be one of the numbers and $3x$ be the other number. The word ***times*** means multiplication. The word ***difference*** tells us to subtract. The word ***is*** in the second sentence means equal to.

$$3x - x = 6 \qquad 2x = 6 \qquad x = 6/2 \qquad x = 3$$
$$3x = 3(3) = 9$$

Example 4.3.3

Two times a number plus five times that number is twenty eight. What is that number?

Let the number equal x. The word ***times*** means multiplication. The word ***plus*** means addition. The word ***is*** means equal to.

$$2x + 5x = 28 \qquad 7x = 28 \qquad x = 28/7 \qquad x = 4$$

Example 4.3.4

A number is four more than another number. The sum of the two numbers is eight. What are the numbers?

Let one number equal x. The word ***more*** means add. Let the other number equal $x + 4$. The word ***sum*** means add. The word ***is*** in the second sentence means equal to. Add the two numbers and set them equal to eight.

$$x + (x + 4) = 8 \qquad 2x + 4 = 8 \qquad 2x = 8 - 4 \qquad 2x = 4$$
$$x = 4/2 \qquad x = 2 \qquad x + 4 = 2 + 4 = 6$$

Example 4.3.5

A number is three less than another number. The sum of the two numbers is five. What are the numbers?

Let one number equal x. The term ***less than*** means subtraction. Let the other number equal $x - 3$. The word ***sum*** means addition. The word ***is*** in the second sentence means equal to. Add the two numbers and set them equal to five.

$x + (x - 3) = 5 \qquad 2x - 3 = 5 \qquad 2x = 8 \qquad x = 8/2$

$x = 4 \qquad x - 3 = 4 - 3 = 1$

Example 4.3.6

A number is one half another number. The sum of the two numbers is six. What are the numbers?

Let one number equal x. The term ***one half*** means division by two or a fraction. Let the other number equal $1/2\ x$. The word ***sum*** means addition. The word ***is*** in the second sentence means equal to. Add the two numbers and set them equal to six.

$x + 1/2\ x = 6 \qquad 3/2\ x = 6 \qquad 2(3/2\ x = 6) \qquad 3x = 12$

$x = 12/3 \qquad x = 4 \qquad 1/2\ x = 1/2\ (4) = 2$

4.4 Solving Linear Equations

A ***linear equation*** is an equation defining a straight line. In fact, all of the equations you have seen so far in this text have been linear.

We will cover the graphing of equations in Chapter 9 in great detail. I will, however, use some graphing to illustrate the properties of equations in this chapter.

For the purpose of this text, linear equations will be presented as equations consisting of ***two variables*** (or multiples of variables) that are ***no greater than the first order*** with or without a ***constant***.

A first order variable is a variable with an exponent of one. An exponent of one is never written. It is understood that no exponent on a variable is the same as an exponent of one.

Figure 4.4.1 illustrates the components of a linear equation.

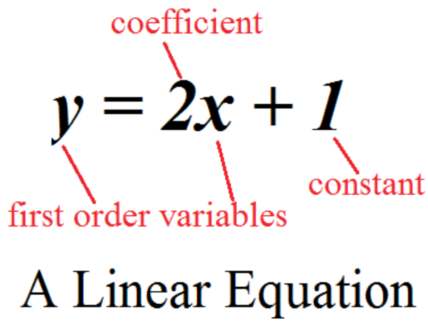


Figure 4.4.1 A Linear Equation

Let's see some examples.

Example 4.4.1

Solve the equation below for y when x is 2.

$$y = 2x + 1 \qquad y = 2(2) + 1 \qquad y = 4 + 1 \qquad y = 5$$

Example 4.4.2

Solve the equation from Example 4.4.1 when x is

$$y = 2x + 1 \qquad y = 2(3) + 1 \qquad y = 6 + 1 \qquad y = 7$$

Example 4.4.3

Solve the equation from Example 4.4.1 when x is

$$y = 2x + 1 \qquad y = 2(4) + 1 \qquad y = 8 + 1 \qquad y = 9$$

We have just seen the same equation solved for y three times with a different value for x each time.

Let's graph these results on a simple rectangular coordinate system graph with x and y axes.

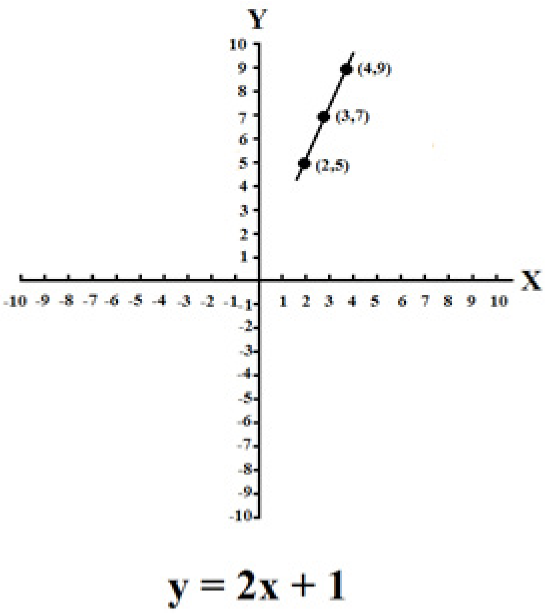


Figure 4.4.2 Graph of $y = 2x + 1$

Please notice that each point on the graph of $y = 2x + 1$ is accompanied by ***set notations*** of their positions on the graph. Set notations are always expressed as (x,y). In other words, a set notation of (2,5) means that position on the graph has an x value of 2 and a y value of 5.

This is the same as saying that when x equals 2 y equals 5 for the equation $y = 2x + 1$.

Much more detailed graphing in the rectangular coordinate system will be covered in Chapter 9.

For now let it suffice that a graph has shown us that the equation $y = 2x + 1$ is ***linear***.

Solution Strategies for Linear Equations

The following strategies will help you manipulate linear equations that are not written in the convenient format used in the previous examples.

1) Isolate a single y on the left hand side of the equation.
2) Substitute the stated value for x.
3) Perform the remaining arithmetic

Example 4.4.4

Solve the equation below for y and express the value of y when x is 3.

$$3x - y = 4$$

Add y to both sides of the equation.	$3x = 4 + y$
Subtract 4 from both sides of the equation.	$3x - 4 = y$
Reverse the order of the equation.	$y = 3x - 4$
Substitute 3 for the value of x.	$y = 3(3) - 4$
Do the remaining arithmetic.	$y = 9 - 4 = 5$

Example 4.4.5

Solve the equation below for y and express the value of y when x is 1.

$$4x + y = 8$$

Subtract $4x$ from both sides of the equation.	$y = 8 - 4x$
Substitute 1 for the value of x.	$y = 8 - 4(1)$
Do the remaining arithmetic.	$y = 8 - 4 = 4$

Example 4.4.6

Solve the equation below for y and express the value of y when x is 3.

$$12 = 2x - y$$

Add y to both sides of the equation.	$12 + y = 2x$
Subtract 12 from both sides of the equation.	$y = 2x - 12$
Substitute 3 for the value of x.	$y = 2(3) - 12$
Do the remaining arithmetic.	$y = 6 - 12 = -6$

4.5 Chapter 4 End of Chapter Exercises

Answer the following questions as directed. The answers are in Appendix A at the end of the text.

4.1. Write algebraic expressions for the following examples.

1) Mark is paid one dollar for every adobe brick that he makes. He is also paid a daily salary of twenty five dollars.
2) Amala is paid 100 rupee per day plus 10 rupee per foot of silk material that she weaves.
3) Jérôme is paid 50 euro per day plus a 5 euro commission for each pair of shoes he sells.
4) Anne earns $50 per day in salary and earns a $10 dollar commission on each magazine subscription she sells.

4.2. Solve the following algebraic equations referring to the four algebraic expressions in 4.1 problem set.

1) How much does Mark earn if he makes 100 adobe bricks in one day?
2) How many rupee will Amala earn if she weaves 400 feet of silk fabric in one day?
3) How much will Jérôme earn, in euro, if he sells six pairs of shoes in one day?
4) How many dollars will Anne earn if she sells 12 magazine subscriptions in one day?

4.3. Solve for the unknown values in the following examples.

1) One number is twice another number. The sum of the two numbers is twelve.
2) One number is four numbers greater than the first number. The sum of the two numbers is sixteen.
3) One number is one half of another number. The difference of the two numbers is ten.
4) One number is five numbers less that another number. The sum of the two numbers is nine.
5) One number is four times another number. The difference between the two numbers is twelve.
6) One number is six times another number. The sum of the two numbers is fourteen.
7) One number is twice another number. The sum of the two numbers is fifteen.

4.4. Solve the following linear equations using the given values for the variables in the equations.

1) $y = 3x - 2$ where $x = 3$
2) $2x -3 = y$ where $x = 2$
3) $y +2 = 2x$ where $x = 3$
4) $2y + 4 = 2x$ where $x = 3$
5) $2y - 6 = 4x$ where $x = 2$
6) $x = 2y + 2$ where $x = 4$
7) $x + 4y = 16$ where $x = 4$
8) $3y = 3 + x$ where $x = 3$
9) $6x = 7y - 2$ where $x = 2$
10) $2y - 6x = 4$ where $x = 1$

5 Polynomial Expressions

5.1 Polynomial Expressions of Variables

What is a polynomial?

A ***polynomial*** is an expression that consists of ***coefficients***, ***variables***, ***integer exponents greater than zero, constants***, and is limited to the mathematical operations of ***addition***, ***subtraction***, and ***multiplication***.

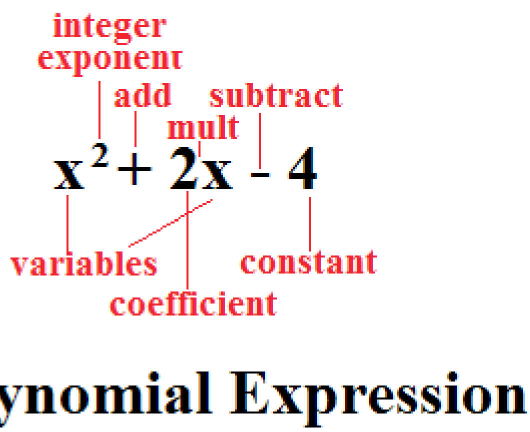


Figure 5.1.1 Polynomial Expression

The following are examples of what are and are not polynomial expressions.

Example 5.1.1

$x^{-2} + 3x - 5$ This is ***not a polynomial***. No exponents less than zero are allowed.

Example 5.1.2

$x^2 - 4x + 7$ This is ***a polynomial***. No rules were broken.

Example 5.1.3

$1/(4x^2) - 5x - 6$ This is ***not a polynomial***. Division is not allowed.

Example 5.1.4

$x^2 + 2x + 5$ This is ***a polynomial***. No rules were broken.

Example 5.1.5

$6 + 3x^2 - 4x$ This is ***a polynomial***. But should be expressed as follows. $3x^2 - 4x + 6$

Example 5.1.6

$x^{1/2} + 3x - 4$ This is ***not a polynomial***. The exponent is not an integer.

Example 5.1.7

$x^4 + 3x^3 - x^2 + 2$This is ***a polynomial***. No rules were broken.

Example 5.1.8

$3y^3 - 4x^2 + 3x$ This is ***a polynomial***. The constant value is zero.

In all of the polynomials listed in the previous examples, there were different portions of the polynomial separated by addition or subtraction.

In the language of mathematics these multiple variable components are known as ***terms***.

5.2 Degrees of Polynomials

According to the rules for the structure of polynomials all terms in a polynomial must have an integer exponent greater than zero.

Remember that a term with no exponent actually is expressing an exponent of one. That is to say that $x = x^1$ but is always written as just x.

The ***degree of a term*** in a polynomial is the highest degree exponent of that term in the polynomial.

The ***degree of a polynomial*** is the highest degree expressed for any of the terms in the polynomial.

Some examples are definitely in order at this point.

Example 5.2.1

State the degree for each term of the polynomial and the degree of the polynomial.

$$x^3 + 2x^2 + 3x -1$$

x^3 is a ***third degree term***, $2x^2$ is a ***second degree term***, 3x is a ***first degree term***. The polynomial is a ***third degree polynomial*** because x^3 is the highest degree term in the polynomial.

Example 5.2.2

State the degree for each term of the polynomial and the degree of the polynomial.

$$y^5 + 4y^4 - 3y^3 + y^2 + 3y + 2$$

y^5 is a ***fifth degree term***, $4y^4$ is a ***fourth degree term***, $3y^3$ is a ***third degree term***, y^2 is a ***second degree term***, $3y$ is a ***first degree term***. The polynomial is a ***fifth degree polynomial*** because y^5 is the highest degree term in the polynomial.

If a particular term has two or more multiplicative components the degree of that term is the sum of all the components exponents.

So the degree of the term x^2y is ***three*** since the degree of x^2 is two and the degree of y is ***one***.

Example 5.2.3

State the degree for each term of the polynomial and the degree of the polynomial.

$$x^3y^2 + 2x^2y - y^2x + 3x + 2y - 4$$

x^3y^2 is a ***fifth degree term***, $2x^2y$ is a ***third degree term***, y^2x is a ***third degree term***, $3x$ is a ***first degree term***, $2y$ is a ***first degree term***. The polynomial is a ***fifth degree polynomial*** because x^3y^2 is the highest degree term in the polynomial.

Example 5.2.4

State the degree for each term of the polynomial and the degree of the polynomial.

$$x^2y^4z^2 + y^2z^2 - 3x^2z + 3x + y - z + 1$$

$x^2y^4z^2$ is an ***eighth degree term***, y^2z^2 is a ***fourth degree term***, $3x^2z$ is a ***third degree term***, 3x is a ***first degree*** term, y is a ***first degree term***, z is a ***first degree term***. The polynomial is an ***eighth degree polynomial*** because $x^2y^4z^2$ is the highest degree term in the polynomial.

5.3 Addition and Subtraction of Polynomials

Now that we have some rules and nomenclature for polynomials established let's do some operations.

There is only one rule for ***adding*** or ***subtracting*** (also known as ***combining***) polynomials as follows.

You may only combine polynomials by ***adding or subtracting the coefficients of identical terms*** in the polynomials.

Some examples will make all of this much clearer.

Example 5.3.1

Add the two following polynomials.

$3x^2y + xy + 5x + 4y + 4$ plus $6xy - 4y + x + 2$

$3x^2y$ has no identical term in the second polynomial so write it down.

$$3x^2y$$

xy from the first polynomial plus 6xy from the second equals 7xy.

$$3x^2y + 7xy$$

5x from the first polynomial plus x from the second equals 6x.

$$3x^2y + 7xy + 6x$$

4y from the first polynomial minus 4y from the second equals 0y.

$$3x^2y + 7xy + 6x$$

4 from the first polynomial plus 2 from the second equals 6.

$$\boldsymbol{3x^2y + 7xy + 6x + 6}$$

Example 5.3.2

Subtract the two following polynomials.

$4x^2y^2 + 3x^2y + xy + 2x - 3y + 4$ minus $x^2y^2 + 2xy + \times + y + 2$

$4x^2y^2$ from the first polynomial minus x^2y^2 from the second equals $3x^2y^2$.

$$3x^2y^2$$

There is no x^2y term in the second polynomial so write in $3x^2y$.

$$3x^2y^2 + 3x^2y$$

xy from the first polynomial minus 2xy from the second is –xy.

$$3x^2y^2 + 3x^2y - xy$$

2x from the first polynomial minus x from the second is x.

$$3x^2y^2 + 3x^2y - xy + x$$

-3y from the first polynomial minus y from the second is -4y.

$$3x^2y^2 + 3x^2y - xy + \times - 4y$$

4 from the first polynomial minus 2 from the second is 2.

$$\boldsymbol{3x^2y^2 + 3x^2y - xy + \times - 4y + 2}$$

Please note one difference between the last two steps of this problem. I referred to the coefficient y term as ***-4*** and it showed up in construction of the polynomial as ***-4***.

I referred to the constant value as ***2*** and in showed up in final answer of the polynomials as ***+2***.

A number or coefficient without a sign in front of it is ***understood to be positive***.

Example 5.3.3

Add the two following polynomials.

$-4a^2b^3 + 6ab + 5a - 3b + 4$ plus $3a^2b^3 + ab^2 - a + b - 2$

$-4a^2b^3$ from the first polynomial plus $3a^2b^3$ from the second equals $-a^2b^3$.

$-a^2b^3$

There is no ab^2 term in the first polynomial so write in ab^2.

$-a^2b^3 + ab^2$

There is no ab term in the second polynomial so write in 6ab.

$-a^2b^3 + ab^2 + 6ab$

5a from the first polynomial plus –a from the second equals 4a.

$-a^2b^3 + ab^2 + 6ab + 4a$

-3b from the first polynomial plus b from the second equals -2b.

$-a^2b^3 + ab^2 + 6ab + 4a - 2b$

4 from the first polynomial plus -2 from the second is +2.

$-a^2b^3 + ab^2 + 6ab + 4a - 2b + 2$

Example 5.3.4

Subtract the two following polynomials.

$4x^2y + 3xy^2 - xy + x + 4$ minus $x^2y + xy^2 + xy - y - 2$

$4x^2y$ minus x^2y is $3x^2y$.

$3x^2y$

$3xy^2$ minus xy^2 is $2xy^2$.

$3x^2y + 2xy^2$

-xy minus xy is -2xy.

$3x^2y + 2xy^2 - 2xy$.

There is no x term in the second polynomial so write x.

$3x^2y + 2xy^2 - 2xy + x$

There is no y term in the first polynomial so 0y minus –y is y.

$3x^2y + 2xy^2 - 2xy + x + y$

4 minus a negative 2 is 6.

$3x^2y + 2xy^2 - 2xy + x + y + 6$

Example 5.3.5

Add the two following polynomials.

$3s^2t – 4st^2 + x – 2y+ 1$ plus $2s^2t + 3st^2 -3x + 4y +6$

$3s^2t$ plus $2s^2t$ is $5s^2t$.

$5s^2t$

$– 4st^2$ plus $3st^2$ is $–st^2$.

$5s^2t – st^2$

x plus -3x is -2x.

$5s^2t – st^2 -2x$

-2y plus 4y is 2y.

$5s^2t – st^2 -2x + 2y$

A constant value of 1 in the first polynomial plus 6 is 7.

$\boldsymbol{5s^2t – st^2 -2x + 2y + 7}$

5.4 Multiplication and Division of Monomials

This section does not cover the multiplication and division of polynomials. The multiplication and division of polynomials is essentially the concept of ***factoring*** which will be covered in Chapter 6 along with long division of polynomials.

This section deals with the multiplication and division of ***monomials***. Monomials are algebraic expressions that do not contain addition or subtraction of terms.

In the multiplication and division of monomials we will see that terms ***do not have to be identical*** to be multiplied. However, only like terms, or like elements of terms, will realize actual new products of multiplication.

Some examples seem necessary here.

Example 5.4.1

The product of $x \bullet y$ is xy. Not much change here.

Example 5.4.2

The product of $x \bullet x$ is x^2. This is a new form.

Example 5.4.3

The product of $3 \bullet 2$ is 6. This is a new number.

Some more examples will continue to demonstrate these forms of multiplication.

Example 5.4.4

Multiply $3xy$ times $4xy$.

Set the problem up as such. $(3xy)(4xy)$. Multiply like elements within each term.

$$(3xy)(4xy) = 12x^2y^2$$

It is important to note that even though the x and y terms do not appear to be multiplied directly, they are still being multiplied in the answer.

Example 5.4.5

Multiply $2x^2y$ times $3xy^2$.

The rule evoked by Example 3.5.3 states $10^m \bullet 10^n = 10^{m+n}$

For unknown values use the same idea expressed as follows.

$$x^m \bullet x^n = x^{m+n}.$$

$$(2x^2y)(3xy^2) = 6x^3y^3$$

Remember that the x and the y can carry an unexpressed 1 exponent.

Example 5.4.6

Multiply $-2xy^2$ times $3xy$.

$$(-2xy^2)(3xy) = -6x^2y^3$$

Example 5.4.7

Divide $4xy$ by $2x$.

One good habit to develop when dividing monomials is to use fractional expression of division rather than the conventional division sign (÷).

$$\frac{4xy}{2x} = \frac{4\cancel{x}y}{2\cancel{x}} = 2y$$

This works because $4 \div 2 = 2$ and $x \div x = 1$.

Example 5.4.8

Divide $6x^2y$ by $3xy$.

The rule evoked in Example 3.5.4 states that $\frac{10^m}{10^n} = 10^{m\text{-}n}$.

For unknown values use the same idea expressed as follows.

$$\frac{x^m}{x^n} = x^{m-n}$$

$$\frac{6x^2y}{3xy} = \frac{6x^2\cancel{y}}{3x\cancel{y}} = 2x$$

This works because 6 ÷ 3 = 2, $x^2 \div x = x$, and y ÷ y = 1.

Example 5.4.9

Divide $6x^3y^4$ by $3x^2y^2$

$$\frac{(6x^3y^4)}{(3x^2y^2)} = 2xy^2$$

This is true because 6 ÷ 3 = 2, $x^3 \div x^2 = x$, and $y^4 \div y^2 = y^2$.

Example 5.4.10

Divide $4r^6s^4t^2$ by $2r^3s^2t^4$

$$\frac{4r^6s^4t^2}{2r^3s^2t^4} = \frac{2r^3s^2}{t^2}$$

This is true because 4 ÷ 2 = 2, $r^6 \div r^3 = r^3$, $s^4 \div s^2 = s^2$, $t^2 \div t^4 = 1/t^2$.

5.5 Chapter 5 End of Chapter Exercises

Answer the following questions as directed. The answers are in Appendix A at the end of the text.

5.1. Determine which of the following are polynomials and which are not.

1) $x^2 + x - 2$
2) $x^3 + 2x^2 - 3x$
3) $2x^2 + 13x + 15$
4) $4x^2 + 6x - 4$
5) $x^{-3} + 3x^2 -4x + 10$
6) $x^2 + 2x -15$
7) $3x^2 + 10x - 8$
8) $x^{-2} + 4x^2 - 8$
9) $x^2/2 + 4x - 3$
10) $5x^2 + 9x - 2$

5.2. State the degree for each of the polynomials.

1) $xy^3 + xy^2 + xy + 4$
2) $r^3st^2 + r^2st - rs^2t + 12$
3) $4xy^3z^2 - x^2y^2z + 6xyz - 14$
4) $6a^2b^2c^3 + 4a^2b^2 - 5ac + 10$
5) $5x^2y^3z^3 - x^3z^3 + x^2y^2z - 5xyz + 6$

5.3. Add the following polynomials

1) $2x^2y + 4xy^2 - xy + 6$ plus $x^2y - 3xy^2 + 2xy + 4$
2) $a^3b^2c + 4a^2b + 6ab + 2$ plus $3a^3b^2c + 3ab^2 + ab - 5$
3) $3x^3y^2z - x^2y^2 + xyz - 2$ plus $2x^3yz^2 + 4x^2y^2 - 2xyz + 4$
4) $5r^5s^2t - r^4s + 4r^3s + 5$ plus $r^5s^2t + rs^4 + r^3s - 6$
5) $3x^2yz + xy^2z + 2xyz^2 + xyz - 2$ plus $x^2yz + 4xyz^2 - 2xyz + 6$

Subtract the following polynomials.

6) $4q^3r^2s - q^2r + 2qr + 3$ minus $2q^3r^2s + 2q^2r - qr + 4$
7) $7x^3y^2z + 4x^2yz - 6xz + 4$ minus $4x^3y^2z + 2x^2yz - xy + 2$
8) $4a^4b^3c^2 - 6a^3b^2c + abc - 2$ minus $a^4b^3c^2 + a^2b^3c - abc + 4$
9) $3x^5y^4z^3 - 4x^4y^2z + 3xyz - 2$ minus $x^5y^4z^3 + xyz - 6$
10) $5r^4s^3t - r^3s^2t + rt - 4$ minus $2r^4s^3t + 3r^3s^2t - rs + 5$

5.4. Perform the following multiplications and divisions of monomials.

1) $(2x)(3x)$
2) $(4r)(-2s)$
3) $(3y^2)(2y^3)$
4) $(3x^2y)(4xy^3)$
5) $(-r^2s^3t)(2r^3s^2t^2)$
6) $(x^2y) \div (xy)$
7) $(3x^3y^2z) \div (3xyz)$
8) $(2m^5n^4o^2) \div (m^2n^2o)$
9) $(4r^4s^3t) \div (2r^2t^3)$
10) $(6x^4y^3z^2) \div (3x^5y^4z)$

6 Factoring of Polynomials

6.1 The Concept of Common Factors

Factoring is essentially a process of deconstructing a number or algebraic expression into ***common factors***. When these common factors are multiplied they must equal the original number or expression.

Examples are definitely needed at this point.

Example 6.1.1

What are some common factors of 32?

Two positive, whole integer, sets of common factors would be 2 and 16 and 4 and 8.

This is to say that $2 \times 16 = 4 \times 8 = 32$.

Example 6.1.2

What are the common factors of $y^2 + y$?

The unknown y is **common** to both the y^2 term and y term of the expression. We will now ***factor out***, or remove the common factor as follows.

$y(y + 1)$ (Check that $y(y + 1) = y^2 + y$.)

So both the expression "y" and "y + 1" are common factors of $y^2 + y$.

Example 6.1.3

What are the common factors of $x^2 + xy$?

The only thing common to both terms of the expression is x.

$x(x + y)$ (Check that $x(x + y) = x^2 + xy$.)

Example 6.1.4

What are the common factors of $2rs - 4rt$?

Note that both a 2 and an r are common to both terms.

$2r(s - 2t)$ (Check that $2r(s - 2t) = 2rs - 4rt$.)

Example 6.1.5

What are the common terms of $x^2y^2z - x^2yz$?

Note that x^2, y, and z are common to both terms.

$x^2yz(y - 1)$ (Check that $x^2yz(y - 1) = x^2y^2z - x^2yz$.)

6.2 Factoring by Grouping

Often it is convenient to gather terms with like factors, in an algebraic expression, for the express purpose of being able to factor them.

Once again examples are called for.

Example 6.2.1

Regroup and factor the following expression.

$$2xy + xz + 4y + 2z$$

Gather the two terms containing y and the two terms containing z as follows.

$$(2xy + 4y) + (xz + 2z)$$

Determine the common factors of the two separate expressions.

$$2y(x + 2) + z(x + 2)$$

Since both expressions now contain an (x + 2) factor that out as follows.

$$\boldsymbol{(x + 2)(2y + z)}$$

Multiply the two terms to check your result.

$$(x + 2)(2y + z) = 2xy + xz + 4y + 2z$$

Example 6.2.2

Regroup and factor the following expression.

$$6ab + 3ac + 8b + 4c$$

Gather the two terms containing b and the two terms containing c as follows.

$$(6ab + 8b) + (3ac + 4c)$$

Determine the common factors of the two separate expressions.

$$2b(3a + 4) + c(3a + 4)$$

Since both expressions now contain a (3a + 4) factor that out as follows.

$$\boldsymbol{(3a + 4)(2b + c)}$$

Multiply the two terms to check your result.

$$(3a + 4)(2b + c) = 6ab + 3ac + 8b + 4c$$

Example 6.2.3

Regroup and factor the following expression.

$$8xy + 4xz - 16y - 8z$$

Gather the two terms containing y and the two terms containing z as follows.

$$(8xy - 16y) + (4xz - 8z)$$

Determine the common factors of the two separate expressions.

$$8y(x - 2) + 4z(x - 2)$$

Since both expressions now contain a $(x - 2)$ factor that out as follows.

$$\boldsymbol{(x - 2)(8y + 4z)}$$

Multiply the two terms to check your result.

$$(x - 2)(8y + 4z) = 8xy + 4xz - 16y - 8z$$

Example 6.2.4

Regroup and factor the following expression.

$$12rs + 6rt - 8s - 4t$$

Gather the two terms containing s and the two terms containing t as follows.

$$(12rs - 8s) + (6rt - 4t)$$

Determine the common factors of the two separate expressions.

$$4s(3r - 2) + 2t(3r - 2)$$

Since both expressions now contain a $(3r - 2)$ factor that out as follows.

$$\boldsymbol{(4s + 2t)(3r - 2)}$$

Note: A 2 could be factored out of the expression $(4s + 2t)$ resulting in $2(2s + t)$ but is not considered absolutely necessary.

Multiply the two terms to check your result.

$$(4s + 2t)(3r - 2) = 12rs + 6rt - 8s - 4t$$

Example 6.2.5

Regroup and factor the following expression.

$$8xy + 16xz + 4y + 8z$$

Gather the two terms containing y and the two terms containing z as follows.

$$(8xy + 4y) + (16xz + 8z)$$

Determine the common factors of the two separate expressions.

$$4y(2x + 1) + 8z(2x + 1)$$

Since both expressions now contain a $(2x + 1)$ factor that out as follows.

$(4y + 8z)(2x + 1)$

Multiply the two terms to check your result.

$(4y + 8z)(2x + 1) = 8xy + 16xz + 4y + 8z$

6.3 Factoring Second Degree Polynomials

Before we learn to factor second degree polynomials let's review what the terms "polynomial" and "second degree" mean.

A ***polynomial*** is an expression that consists of ***coefficients***, ***variables***, ***integer exponents greater than zero, constants***, and is limited to the mathematical operations of ***addition***, ***subtraction***, and ***multiplication***.

A second degree polynomial is a polynomial whose highest degreed term is second degree or squared.

More examples!

Example 6.3.1

Factor the following second degree polynomial.

$x^2 + 4x + 3$

Start with the following structure for your factoring.

$(x \quad)(x \quad)$

Since the ***polynomial has two positive signs*** the two terms must both have ***positive signs***.

$(x + \ )(x + \ )$

Think of two numbers whose sum is 4 and whose product is 3.

$(x + 1)(x + 3)$

Multiply these two expressions to check your answer.

$(x + 1)(x + 3) = x^2 + 4x + 3$

Example 6.3.2

Factor the following second degree polynomial.

$$x^2 - \times - 6$$

Start with the following structure for your factoring.

$$(x \quad)(x \quad)$$

Since the ***polynomial has at least one negative sign*** the two factors must have a positive sign and a negative sign.

$$(x + \ )(x - \ )$$

Think of two numbers whose sum is -1 and whose product is – 6.

$$\boldsymbol{(x + 2)(x - 3)}$$

Multiply these two expressions to check your answer.

$$x^2 - x - 6$$

Example 6.3.3

Factor the following second degree polynomial.

$a^2 - 6a + 8$

Start with the following structure for your factoring.

(a)(a)

Since the ***polynomial has a negative first order term and a positive constant term*** the two terms must both have a negative sign.

(a –)(a –)

Think of two numbers whose sum is – 6 and whose product is 8.

(a – 2)(a – 4)

Multiply these two expressions to check your answer.

$a^2 - 6a + 8$

Example 6.3.4

Factor the following second degree polynomial.

$y^2 + 3y - 4$

Start with the following structure for your factoring.

(y)(y)

Since the ***constant term sign is negative***, there must be both a positive and negative sign for the factors.

(y +)(y –)

Think of two numbers whose sum is 3 and whose product is – 4.

(y – 1)(y + 4)

Multiply these two expressions to check your answer.

$y^2 + 3y - 4$

Note that for each of the first four examples a rule concerning the signs of the two factored terms based on the signs in the polynomial was stated. They are as follows.

1) If the polynomial contains ***two positive signs***, the signs of both factors are ***positive***.
2) If the polynomial contains ***two negative signs***, the signs of the factors are ***one positive*** and ***one negative***.
3) If the polynomial contains a ***negative first order term sign*** and a ***positive constant term sign***, the signs of the factors are ***both negative***.
4) If the polynomial's ***constant term is negative***, there must a ***positive*** and a ***negative*** sign in the two terms.

There are very few rules governing factoring. With the exception of rule 4, these rules are only always true when the coefficient of the second degree term is one.

These rules are helpful only in a limited capacity. You are much better off using my structuring of a factoring problem and reasoning your way through the problem.

Rather than looking at them as rules look at them as examples of good reasoning.

You should always multiply your two factors together to see if they equal the original polynomial.

Let's continue with some more examples.

Example 6.3.5

Factor the following second degree polynomial.

$2x^2 + 8x + 6$

Start with the following structure for your factoring.

$(2x\ )(x\ )$

Since the ***polynomial has two positive signs*** the two terms must both have ***positive signs***.

$(2x + \ )(x + \ )$

Think of two numbers whose product is 6 and the sum of the multiplication of their coefficients equals 8.

$$(2x + 2)(x + 3)$$

Here the multiplication of the x coefficients worked because $(3 \bullet 2) + (2 \bullet 1) = 8$.

Multiply these two expressions to check your answer.

$$2x^2 + 8x + 6$$

Example 6.3.6

Factor the following second degree polynomial.

$$4s^2 + 10s + 6$$

Here we have the possibility of two different starting structures.

$$(4s\ \ )(s\ \ ) \text{ or } (2s\ \ )(2s\ \ )$$

Since the coefficients of the first order term and the constant are not very large, start with the following structure for your factoring.

(2s)(2s)

Since the polynomial has two positive signs the two terms must both have positive signs.

(2s +)(2s +)

Think of two numbers where twice their sum is 10 and whose product is 6.

(2s + 2)(2s + 3)

Multiply these two expressions to check your answer.

$4s^2 + 10s + 6$

Example 6.3.7

Factor the following second degree polynomial.

$3z^2 - 7z + 2$

Start with the following structure for your factoring.

(3z)(z)

Since the polynomial has a positive constant term sign and a negative first term sign, the two terms must both have negative signs.

(3z –)(z –)

Think of two numbers whose product is 2 and place them so the ***first order term coefficient is** -7*.

(3z – 1)(z – 2)

Multiply these two expressions to check your answer.

$3z^2 - 7z + 2$

Example 6.3.8

Factor the following second degree polynomial.

$8x^2 + 26x + 6$

Here we have the possibility of two different starting structures.

$(8x\ )(x\ )$ or $(4x\ )(2x\ )$

Since the coefficients of the first order term is so large, start with the following structure for your factoring.

$(8x\ )(x\ )$

Since the polynomial has two positive signs the two terms must both have positive signs.

$(8x + \ )(x + \ )$

Think of two numbers whose product is 6 and place them so the ***first order term coefficient is 26***.

$\boldsymbol{(8x + 2)(x + 3)}$

Multiply these two expressions to check your answer.

$8x^2 + 26x + 6$

Example 6.3.10

Factor the following second degree polynomial.

$x^2 + 3x - 1$

Start with the following structure for your factoring.

$(x\ )(x\ )$

Since the polynomial has a negative constant sign the two terms must have a positive sign and a negative sign.

$(x + \ )(x - \ )$

Think of two numbers whose sum is 3 and product is -1.

There are no such two numbers. The polynomial cannot be factored.

The only two factors possible are the polynomial itself and the number 1. The polynomial is only divisible by itself and 1.

The polynomial is ***prime***.

6.4 Special Factors

Special factors are those factors of algebraic expressions that have special properties. These properties are special in that they render the algebraic expression extremely easy to factor. This ease of factoring manifests itself in extremely simple, or special, factors.

There are three categories of special factors that will be covered in this text. They are as follows.

1) The difference of two squares.
2) The difference of two cubes.
3) The sum of two cubes.

The following generalized equation describes the ***difference of two squares***.

$$a^2 - b^2 = (a - b)(a + b)$$

The fact that there are no "ab" terms after factoring is what makes the difference of squares so special.

Some examples will illustrate this fact.

Example 6.4.1

Factor the following algebraic expression.

$$x^2 - y^2$$

Simply follow the form of the generalized equation.

$$(x - y)(x + y)$$

Multiply the two factors to check your answer.

$$(x - y)(x + y) = x^2 + xy - xy - y^2 = x^2 - y^2$$

Notice that the + xy and – xy terms cancel one another.

Example 6.4.2

Factor the following algebraic expression.

$$4x^2 - 9$$

Here you must select which expressions compare to the "a" and "b" terms of the generalized equation. They are essentially the square roots of the two terms of the expression.

2x compares to "a" and 3 compares to "b".

Simply follow the form of the generalized equation.

$$(2x - 3)(2x + 3)$$

Multiply the two factors to check your answer.

$$(2x - 3)(2x + 3) = 4x^2 + 6x - 6x - 9 = 4x^2 - 9$$

Example 6.4.3

Factor the following algebraic expression.

$16s^2 - 25t$

Here you must select which expressions compare to the "a" and "b" terms of the generalized equation. They are essentially the square roots of the two terms of the expression.

4s compares to "a" and 5t compares to "b".

Simply follow the form of the generalized equation.

(4s – 5t)(4s + 5t)

Multiply the two factors to check your answer.

$(4s - 5t)(4s + 5t) = 16s^2 + 20st - 20st - 25t = 16s^2 - 25t^2$

No matter how complicated a difference of squares looks, it can be easily factored as long as it follows the generalized form.

In other words, the coefficients of the variables must factor into ***whole integer factors***. Also, the exponents of the variables must factor into ***whole integer exponents***.

Example 6.4.4

Factor the following algebraic expression.

$100x^2y^2 - 81z^2$

Here you must select which expressions compare to the "a" and "b" terms of the generalized equation. They are essentially the square roots of the two terms of the expression.

10xy compares to "a" and 9z compares to "b".

Simply follow the form of the generalized equation.

(10xy – 9z)(10xy + 9z)

Multiply the two factors to check your answer.

$$(10xy - 9z)(10xy + 9z) = 100x^2y^2 + 90xyz - 90xyz - 81z^2 = 100x^2y^2 - 81z^2$$

Example 6.4.5

Factor the following algebraic expression.

$$49q^4r^2 - 16s^2t^2$$

Here you must select which expressions compare to the "a" and "b" terms of the generalized equation. They are essentially the square roots of the two terms of the expression.

$7q^2r$ compares to "a" and $4st$ compares to "b".

Simply follow the form of the generalized equation.

$$\boldsymbol{(7q^2r - 4st)\ (7q^2r + 4st)}$$

Multiply the two factors to check your answer.

$$(7q^2r - 4st)(7q^2r + 4st) = 49q^4r^2 + 28q^2rst - 28q^2rst - 16s^2t^2 = 49q^4r^2 - 16s^2t^2$$

The following generalized equation describes the ***difference of two cubes***.

$$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$$

The very complicated expression to the right of the equals sign, in the above equation, can be seen as equal to the left of the equals sign once all of the "a" and "b" relationships are certified.

Note this process of certification in the following example.

Example 6.4.6

Solve the following algebraic expression.

$$(x - y)(x^2 + xy + y^2)$$

x compares to a, y compares to b, x^2 compares to a^2, xy compares to ab, and y^2 compares to b^2.

All these direct comparisons between the expression given and the generalized expression of the difference of cubes shows that the expression given is a difference of cubes.

Simply follow the form of the generalized equation.

$$(x^3 - y^3)$$

Multiplying the two terms of the complicated algebraic expression will result in the simple difference of two cubes. However, the usefulness of the special factor equation for the difference of cubes is that you don't have to do that.

It is important to note that with the difference of squares problems we went from the difference of squares to find the factors. With the difference of cube problems we are going from the factors to find the difference of cubes.

Example 6.4.7

Solve the following algebraic expression.

$$(2x - 3y)(4x^2 + 6xy + 9y^2)$$

The first term of the expression shows that the "a" component is 2x and the "b" component is 3y.

Since $(2x)^2 = 4x^2$, $(2x)(3y) = 6xy$, and $(3y)^2 = 9y^2$ the form of the difference of cubes is certified.

Substituting the certified terms into the generalized equation results in the following.

$[(2x)^3 - (3y)^3] = \boldsymbol{8x^3 - 27y^3}$

Example 6.4.8

Solve the following algebraic expression.

$(3s - 4t)(9s^2 + 12st + 16t^2)$

The first term of the expression shows that the "a" component is 3s and the "b" component is 4t.

Since $(3s)^2 = 9s^2$, $(3s)(4t) = 12st$, and $(4t)^2 = 16t^2$ the form of the difference of cubes is certified.

Substituting the certified terms into the generalized equation results in the following.

$[(3s)^3 - (4t)^3] = \boldsymbol{27s^3 - 64t^3}$

Example 6.4.9

Solve the following algebraic expression.

$(4y - z)(16y^2 + 4yz + z^2)$

The first term of the expression shows that the "a" component is 4y and the "b" component is 1z. Remember that coefficients of 1 are rarely expressed. Also remember that 1 cubed is 1.

Since $(4y)^2 = 16y^2$, $(4y)(z) = 4yz$, and $(z)^2 = z^2$ the form of the difference of cubes is certified.

Substituting the certified terms into the generalized equation results in the following.

$[(4y)^3 - (z)^3] = \boldsymbol{64y^3 - z^3}$

Example 6.4.10

Solve the following algebraic expression.

$(2x - 5y)(4x^2 + 10xy + 25y^2)$

The first term of the expression shows that the "a" component is 2x and the "b" component is 5y.

Since $(2x)^2 = 4x^2$, $(2x)(5y) = 10xy$, and $(5y)^2 = 25y^2$ the form of the difference of cubes is certified.

Substituting the certified terms into the generalized equation results in the following.

$$[(2x)^3 - (5y)^3] = \boldsymbol{16x^3 - 125y^3}$$

The following generalized equation describes the ***sum of two cubes***.

$$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$$

This equation appears similar in structure to the equation for the difference of two cubes. Take note where the signs are different in the two equations.

The following examples will make things clear.

Example 6.4.11

Solve the following algebraic expression.

$$(x + y)(x^2 - xy + y^2)$$

The first term of the expression shows that the "a" component is x and the "b" component is y.

Since $(x)^2 = x^2$, $(x)(y) = xy$, and $(y)^2 = y^2$ the form of the sum of two cubes is certified.

Substituting the certified terms into the generalized equation results in the following.

$$[(x)^3 + (y)^3] = \boldsymbol{x^3 + y^3}$$

Example 6.4.12

Solve the following algebraic expression.

$$(2r + 3s)(4r^2 - 6rs + 9s^2)$$

The first term of the expression shows that the "a" component is 2r and the "b" component is 3s.

Since $(2r)^2 = 4r^2$, $(2r)(3s) = 6rs$, and $(3s)^2 = 9s^2$ the form of the sum of two cubes is certified.

Substituting the certified terms into the generalized equation results in the following.

$$[(2r)^3 + (3s)^3] = \boldsymbol{8r^3 + 27s^3}$$

Example 6.4.13

Solve the following algebraic expression.

$$(4y + 5z)(16y^2 - 20yz + 25z^2)$$

The first term of the expression shows that the "a" component is 4y and the "b" component is 5z.

Since $(4y)^2 = 16y^2$, $(4y)(5z) = 20yz$, and $(5z)^2 = 25z^2$ the form of the sum of two cubes is certified.

Substituting the certified terms into the generalized equation results in the following.

$$[(4y)^3 + (5z)^3] = \boldsymbol{64y^3 + 125z^3}$$

Example 6.4.14

Solve the following algebraic expression.

$$(3x + y)(9x^2 - 3xy + y^2)$$

The first term of the expression shows that the "a" component is 3x and the "b" component is y.

Since $(3x)^2 = 9x^2$, $(3x)(y) = 3xy$, and $(y)^2 = y^2$ the form of the sum of two cubes is certified.

Substituting the certified terms into the generalized equation results in the following.

$$[(3x)^3 + (y)^3] = \boldsymbol{27x^3 + y^3}$$

Example 6.4.15

Solve the following algebraic expression.

$$(5m + 2n)(25m^2 - 10mn + 4n^2)$$

The first term of the expression shows that the "a" component is 5m and the "b" component is 2n.

Since $(5m)^2 = 25m^2$, $(5m)(2n) = 10mn$, and $(2n)^2 = 4n^2$ the form of the sum of two cubes is certified.

Substituting the certified terms into the generalized equation results in the following.

$$[(5m)^3 + (2n)^3] = \boldsymbol{125m^3 + 8n^3}$$

6.5 Long Division of Polynomials

The process of ***long division of polynomials*** is exactly what it sounds like. It is dividing polynomials by factors just like long division of numbers in arithmetic.

Figure 6.5.1 shows an example of long division of a polynomial along with labeled parts of the problem.

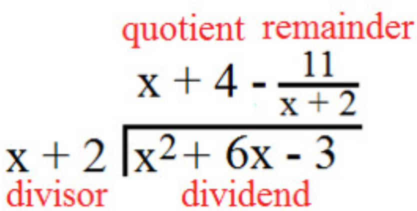


Figure 6.5.1 Long Division of Polynomials

As our first example, let's work the problem from Figure 6.5.1.

Example 6.5.1

Perform the following long division.

$$x + 2\,\overline{)\,x^2 + 6x - 3}$$

The x of the x + 2 divisor goes into the x^2 of the dividend x times.

$$\begin{array}{r l} & \quad x \\ x + 2 & \overline{)\,x^2 + 6x - 3} \end{array}$$

Multiply the x + 2 by x and subtract that product from the dividend.

$$\begin{array}{r l} & \quad x \; + \\ x + 2 & \overline{)\,x^2 + 6x - 3} \\ & -(x^2 + 2x) \\ \cline{2-2} & \qquad\quad 4x \end{array}$$

Bring down the – 3. The x of the x + 2 divisor goes into the 4x of the 4x – 3 term four times. Multiply the divisor by 4 and subtract that product from the 4x -3 term.

$$
\begin{array}{r|l}
 & x + 4 \\ \hline
x + 2 & x^2 + 6x - 3 \\
 & -(x^2 + 2x) \\ \cline{2-2}
 & \qquad 4x - 3 \\
 & \qquad -(4x + 8) \\ \cline{2-2}
 & \qquad\quad -11
\end{array}
$$

The -11 difference is expressed as the -11/x+2 remainder.

$$
\begin{array}{r|l}
 & x + 4 - \frac{11}{x+2} \\ \hline
x + 2 & x^2 + 6x - 3 \\
 & -(x^2 + 2x) \\ \cline{2-2}
 & \qquad 4x - 3 \\
 & \qquad -(4x + 8) \\ \cline{2-2}
 & \qquad\quad -11
\end{array}
$$

Example 6.5.2

Perform the following long division.

$$
x + 2 \overline{\big)\, x^3 + 4x^2 + 8x + 8}
$$

The x of the x + 2 divisor goes into the x^3 of the dividend x^2 times. Multiply the divisor by x^2 and subtract that product from the dividend.

$$
\begin{array}{r|l}
 & x^2 \\ \hline
x + 2 & x^3 + 4x^2 + 8x + 8 \\
 & -(x^3 + 2x^2) \\ \cline{2-2}
 & \qquad 2x^2
\end{array}
$$

Bring down the + 8x. The x of the x + 2 divisor goes into the $2x^2$ of the $2x^2 + 8x$ term 2x times. Multiply the divisor by 2x and subtract that product from the dividend.

$$
\begin{array}{r l}
 & x^2 + 2x \\
x + 2 \big) & \overline{x^3 + 4x^2 + 8x + 8} \\
 & \underline{-(x^3 + 2x^2)} \\
 & \qquad 2x^2 + 8x \\
 & \qquad \underline{-(2x^2 + 4x)} \\
 & \qquad\qquad\quad 4x
\end{array}
$$

Bring down the + 8. The x of the x + 2 divisor goes into 4x of the 4x + 8 term 4 times. Multiply the divisor by 4 and subtract that product from the 4x + 8 term. The zero remainder is not expressed in the quotient.

$$
\begin{array}{r l}
 & x^2 + 2x + 4 \\
x + 2 \big) & \overline{x^3 + 4x^2 + 8x + 8} \\
 & \underline{-(x^3 + 2x^2)} \\
 & \qquad 2x^2 + 8x \\
 & \qquad \underline{-(2x^2 + 4x)} \\
 & \qquad\qquad\quad 4x + 8 \\
 & \qquad\qquad \underline{-(4x + 8)} \\
 & \qquad\qquad\qquad\quad 0
\end{array}
$$

Example 6.5.3

Perform the following long division.

$$
s - 5 \big) \overline{s^3 + 3s^2 - 4s + 2}
$$

The s of the s – 5 divisor goes into the s^3 of the dividend s^2 times. Multiply the divisor by s^2 and subtract that product from the dividend.

$$
\begin{array}{r l}
 & s^2 \\
s - 5 \big) & \overline{s^3 + 3s^2 - 4s + 2} \\
 & \underline{-(s^3 - 5s^2)} \\
 & \qquad 8s^2
\end{array}
$$

Bring down the – 4s. The s of the s – 5 divisor goes into the $8s^2$ of the $8s^2$ -4s term 8s times. Multiply the divisor by 8s and subtract that product from the dividend.

$$
\begin{array}{r l}
 & s^2 + 8s \\
s - 5 & \overline{\big)\, s^3 + 3s^2 - 4s + 2} \\
 & -(s^3 - 5s^2) \\
\hline
 & \quad 8s^2 - 4s \\
 & \quad -(8s^2 - 40s) \\
\hline
 & \qquad 36s
\end{array}
$$

Bring down the + 2. The s of the s – 5 divisor goes into 36s of the 36s + 2 term 36 times. Multiply the divisor by 36 and subtract that product from the 36s + 2 term. The 182/(s – 5) is expressed as the remainder.

$$
\begin{array}{r|l}
 & s^2 + 8s + 36 + \frac{182}{(s-5)} \\
\hline
s - 5 & s^3 + 3s^2 - 4s + 2 \\
 & -(s^3 - 5s^2) \\
\cline{2-2}
 & \quad 8s^2 - 4s \\
 & \quad -(8s^2 - 40s) \\
\cline{2-2}
 & \qquad 36s + 2 \\
 & \qquad -(36s - 180) \\
\cline{2-2}
 & \qquad\qquad 182
\end{array}
$$

Example 6.5.4

Perform the following long division.

$$
y + 2 \overline{\big)\, 2y^3 + y^2 - 2y + 8}
$$

The y of the y + 2 divisor goes into the $2y^3$ of the dividend $2y^2$ times. Multiply the divisor by $2y^2$ and subtract that product from the dividend.

$$
\begin{array}{r|l}
 & 2y^2 \\
\hline
y + 2 & 2y^3 + y^2 - 2y + 8 \\
 & -(2y^3 + 4y^2) \\
\cline{2-2}
 & \qquad -3y^2
\end{array}
$$

Bring down the – 2y. The y of the y + 2 divisor goes into the – $3y^2$ of the – $3y^2$ – 2y term – 3y times. Multiply the divisor by – 3y and subtract that product from the dividend.

$$
\begin{array}{r|l}
 & 2y^2 - 3y \\
\hline
y + 2 & 2y^3 + y^2 - 2y + 8 \\
 & -(2y^3 + 4y^2) \\
\cline{2-2}
 & \qquad -3y^2 - 2y \\
 & \qquad -(3y^2 - 6y) \\
\cline{2-2}
 & \qquad\qquad 4y
\end{array}
$$

Bring down the + 8. The y of the y + 2 divisor goes into 4y of the 4y + 8 term 4 times. Multiply the divisor by 4 and subtract that product from the 4y + 8 term. The zero remainder is not expressed in the quotient.

$$\begin{array}{r|l} & 2y^2 - 3y + 4 \\ \hline y+2 & 2y^3 + y^2 - 2y \\ & -(2y^3 + 4y^2) \\ \hline & \quad -3y^2 - 2y \\ & \quad -(3y^2 - 6y) \\ \hline & \qquad 4y + 8 \\ & \qquad -(4y + 8) \\ \hline & \qquad\qquad 0 \end{array}$$

Example 6.5.5

Perform the following long division.

$$z - 1 \overline{\big)\, 3z^3 + z^2 - 6z + 2}$$

The z of the z – 2 divisor goes into the $3z^3$ of the dividend $3z^2$ times. Multiply the divisor by $3z^2$ and subtract that product from the dividend.

$$\begin{array}{r|l} & 3z^2 \\ \hline z-1 & 3z^3 + z^2 - 6z + 2 \\ & -(3z^3 - 3z^2) \\ \hline & \qquad 4z^2 \end{array}$$

Bring down the – 6z. The z of the z – 1 divisor goes into the $4z^2$ of the $4z^2$ – 6z term 4z times. Multiply the divisor by 4z and subtract that product from the dividend.

$$\begin{array}{r|l} & 3z^2 + 4z \\ \hline z-1 & 3z^3 + z^2 - 6z + 2 \\ & -(3z^3 - 3z^2) \\ \hline & \qquad 4z^2 - 6z \\ & \quad -(4z^2 - 4z) \\ \hline & \qquad\quad -2z \end{array}$$

Bring down the + 2. The z of the z – 2 divisor goes into – 2z of the -2z + 2 term -2 times. Multiply the divisor by – 2 and subtract that product from the -2z + 2 term. The zero remainder is not expressed in the quotient.

$$
\begin{array}{r|l}
 & 3z^2 + 4z - 2 \\
\cline{2-2}
z - 1 & 3z^3 + z^2 - 6z + 2 \\
 & -(3z^3 - 3z^2) \\
\cline{2-2}
 & \quad 4z^2 - 6z \\
 & \quad -(4z^2 - 4z) \\
\cline{2-2}
 & \qquad -2z + 2 \\
 & \qquad -(-2z + 2) \\
\cline{2-2}
 & \qquad\qquad 0
\end{array}
$$

6.6 Chapter 6 End of Chapter Exercises

Answer the following questions as directed. The answers are in Appendix A at the end of the text.

6.1. Factor the following expressions.

1) $y^3 + y^2$
2) $4x^2 - 2x$
3) $5s^3 - 15s^2 + 10s$
4) $x^4 + x^3$
5) $3z^2 + z$
6) $10x^3 + 60x^2 + 50x$
7) $2r^2s - rs$
8) $x^3y + x^2y^2 - xy$
9) $3r^3s^2t - r^2s - r$
10) $4x^2y^2z^3 + 4x^2y^2z + 8xyz$

6.2. Regroup and factor the following expressions.

1) $4rs + 2rt - 6s - 3t$
2) $4pq + 12pr - 2q - 6r$
3) $xy + 2xz + 4y + 8z$
4) $6xy + 3xz - 2y - z$
5) $15st + 20su - 9t - 12u$
6) $4xy + 6xz + 6y + 9z$
7) $27ab + 9ac + 18b + 6c$
8) $24ab + 16ac - 12b - 8c$
9) $42kn + 12mn + 35k + 10m$
10) $24xz - 16x + 6yz - 4y$

6.3. Factor the following second degree polynomials.

1) $x^2 - 2x - 8$
2) $z^2 - 5z + 4$
3) $x^2 - \times - 12$
4) $y^2 - 5y + 6$
5) $m^2 + 3m - 4$
6) $2s^2 - 6s - 8$
7) $3x^2 - 5x - 2$
8) $6x^2 - 3x - 3$
9) $12y^2 + 10y - 2$
10) $50z^2 - 15z - 2$

6.4. Factor the following expressions using a special factors formula.

1) $4x^2 - 16$
2) $y^2 - 9$
3) $9r^2 - 4s^2$
4) $81u^2 - 49v^2$

Solve the following expressions using a special factors formula.

5) $(x - y)(x^2 + xy + y^2)$
6) $(2r - 3s)(4r^2 + 6rs + 9s^2)$
7) $(4m - 3n)(16m^2 + 12mn + 9n^2)$
8) $(5x + 4y)(25x^2 - 20xy + 16y^2)$
9) $(2a + 3b)(4a^2 - 6ab + 9b^2)$
10) $(10x + 6y)(100x^2 - 60xy + 36y^2)$

6.5. Perform the following long divisions of polynomials.

1) $x^2 + 6x + 8 \div x + 4$
2) $y^2 - y - 6 \div y + 2$
3) $2s^2 + s - 6 \div s + 2$
4) $2z^2 - 3z + 2 \div z + 1$
5) $3s^2 + s - 2 \div s + 1$
6) $4x^3 - 10x^2 - 4x - 6 \div x - 3$
7) $3y^3 - 2y^2 + 4y - 3 \div y + 1$
8) $6x^3 + 16x^2 - 10x - 12 \div 3x + 2$
9) $6z^4 - 15z^3 + 30z^2 - 30z - 9 \div 6z - 3$
10) $14r^4 - 22r^3 + 9r^2 - 5r - 6 \div 7r + 3$

7 Rational Expressions

7.1 Simplification of Rational Expression

A ***rational expression*** is any type of algebraic expression that can be expressed as a ***ratio***.

Just like 6/8 can be simplified to 3/4, 6x/8y can be simplified to 3x/4y.

This is true because the multiplication in 6x and 8y is compatible with the division of the rational expression. That is to say such an expression as 6x-1/8y+1 cannot be simplified because the subtraction and addition are not compatible with multiplication and division.

Let's do some examples!

Example 7.1.1

Simplify the rational expression.

$$\frac{10a^2b^3}{5ab^2}$$

Let's rewrite this as follows.

$$\frac{10a \cdot a \cdot b \cdot b \cdot b}{5a \cdot b \cdot b}$$

Since only multiplication and division are in play, eliminate identical terms that appear in the numerator and denominator.

$$\frac{10a \cdot \cancel{a} \cdot \cancel{b} \cdot \cancel{b} \cdot b}{5\cancel{a} \cdot \cancel{b} \cdot \cancel{b}}$$

With the simplification of 10/5 this can be written as follows.

2ab

Example 7.1.2

$$\frac{4x^3y^2}{8x^2y^3}$$

Without drawing the rational expression out, the following observations should be obvious.

The 4 and the 8 simplify to a 2 in the denominator.

The x^3 and x^2 simplify to an x in the numerator.

The y^2 and y^3 simplify to a y in the denominator.

$$\frac{x}{2y}$$

Example 7.1.3

Simplify the following rational expression.

$$\frac{15r^4s^3}{20r^3s^4}$$

Without drawing the rational expression out, the following observations should be obvious.

The 15 and the 20 simplify to a 3 in the numerator and a 4 in the denominator.

The r^4 and r^3 simplify to an r in the numerator.

The s^3 and s^4 simplify to an s in the denominator.

$$\frac{3r}{4s}$$

Example 7.1.4

Simplify the following rational expression.

$$\frac{6(x+2)^3}{3(x+2)^2}$$

Since the (x + 2) expressions are both in parentheses they can be treated as if the addition inside the parentheses is suspended and only multiplication and division is occurring.

The 6 and the 3 simplify to a 2 in the numerator.

The $(x + 2)^3$ and $(x + 2)^2$ simplify to an (x + 2) in the numerator.

$$\boldsymbol{2(x+2)}$$

Example 7.1.5

Simplify the following rational expression.

The subtraction in the (n – 1) expressions is suspended so that only multiplication and division is occurring.

The 4 and 8 simplify to a 2 in the denominator.

The m and m^2 simplify to an m in the denominator.

The $(n - 1)^4$ and $(n - 1)^3$ simplify to an (n -1) in the numerator.

$$\boldsymbol{\frac{(n-1)}{2m}}$$

7.2 Addition and Subtraction of Rational Expressions

Adding and subtracting rational expressions must always be done with the use of a ***common denominator***.

Often times the lowest common denominator of two algebraic expressions is the product of the two denominators.

Some examples will make this clear.

Example 7.2.1

Add the two following rational expressions.

Here the lowest common denominator is x^2. This is because x^2 and x will divide into x^2. Set up the problem as follows.

$$\frac{4}{x^2} + \frac{2}{x} = \frac{\qquad}{x^2}$$

To decide the portion of the numerator for the $4/x^2$ term realize that x^2 divides into the common denominator once times it's numerator 4 for a product of 4 in the new numerator.

$$\frac{4}{x^2} + \frac{2}{x} = \frac{4 + \qquad}{x^2}$$

To decide the portion of the numerator for the 2/x term realize that x divides into the common denominator x times it's numerator 2 for a product of 2x in the new numerator.

$$\frac{4}{x^2} + \frac{2}{x} = \frac{4 + 2x}{x^2}$$

Since this expression will not simplify further the final answer is as follows.

$$\boldsymbol{\frac{4 + 2x}{x^2}}$$

Note: It is this authors opinion that $2(2 + x)/x^2$ is not a simpler answer.

Example 7.2.2

Add the two following rational expressions.

$$\frac{4}{2m} + \frac{3}{6n}$$

Here the lowest common denominator is 6mn. This is because 2m and 6n will both divide into 6mn. Set up the problem as follows.

$$\frac{4}{2m} + \frac{3}{6n} = \frac{\qquad}{6mn}$$

To decide the portion of the numerator for the 4/2m term realize that 2m divides into the common denominator 3n times it's numerator 4 for a product of 12n in the new numerator.

$$\frac{4}{2m}+\frac{3}{6n}=\frac{12n+}{6mn}$$

To decide the portion of the numerator for the 3/6n term realize that 6n divides into the common denominator m times it's numerator 3 for a product of 3m in the new numerator.

$$\frac{4}{2m}+\frac{3}{6n}=\frac{12n+3m}{6mn}$$

A 3 will factor out of the numerator which is divisible into the 6 of the denominator.

$$\frac{4}{2m}+\frac{3}{6n}=\frac{3(4n+m)}{6mn}$$

This expression simplifies further as follows.

$$\frac{4n+m}{2mn}$$

Example 7.2.3

Subtract the two following rational expressions.

$$\frac{6}{x+1} - \frac{3}{x^2+3x+2}$$

Here the lowest common denominator is the polynomial $x^2 + 3x + 2$. Looking at the polynomial in its factored form, $(x + 1)(x + 2)$, it is easy to see that $(x +1)$ and $(x + 1)(x + 2)$ divide into $(x +1)(x + 2)$. Set up the problem as follows.

$$\frac{6}{x+1} - \frac{3}{(x+1)(x+2)} = \frac{}{(x+1)(x+2)}$$

To decide the portion of the numerator for the $\frac{6}{x+1}$ term realize that $x + 1$ divides into the common denominator $x + 2$ times its numerator 6 for a product of $6x + 12$ in the new numerator.

$$\frac{6}{x+1} - \frac{3}{(x+1)(x+2)} = \frac{(6x+12)-}{(x+1)(x+2)}$$

To decide the portion of the numerator for the $\frac{3}{(x+1)(x+2)}$ term realize that $(x +1)(x + 2)$ divides into the common denominator 1 times it's numerator 3 for a product of 3 in the new numerator.

$$\frac{6}{x+1} - \frac{3}{(x+1)(x+2)} = \frac{(6x+12)-3}{(x+1)(x+2)}$$

Collecting terms in the numerator gives the following answer.

$$\boldsymbol{\frac{6x+9}{(x+1)(x+2)}}$$

Example 7.2.4

Subtract the two following rational expressions.

$$\frac{x+2}{5} - \frac{x+3}{7}$$

Here the lowest common denominator is the product of the two denominators, 35. Because the common denominator was obtained by multiplying the original two denominators, it is obvious that 5 and 7 will divide into the common denominator. Set up the problem as follows.

$$\frac{x+2}{5} - \frac{x+3}{7} = \frac{}{35}$$

To decide the portion of the numerator for the $\frac{x+2}{5}$ term realize that 5 divides into the common denominator 7 times it's numerator x + 2 for a product of 7x + 14 in the new numerator.

$$\frac{x+2}{5} - \frac{x+3}{7} = \frac{(7x+14) -\qquad\qquad}{35}$$

To decide the portion of the numerator for the $\frac{x+3}{7}$ term realize that 7 divides into the common denominator 5 times it's numerator x + 3 for a product of 5x + 15 in the new numerator.

$$\frac{x+2}{5} - \frac{x+3}{7} = \frac{(7x+14) - (5x+15)}{35}$$

Collecting terms in the numerator gives the following answer.

$$\frac{2x - 1}{35}$$

Example 7.2.5

Perform the subtraction and addition on the following rational expressions.

$$\frac{1}{2x-1} - \frac{1}{2x+2} + \frac{1}{4x^2+2x-2}$$

Here the lowest common denominator is the polynomial $4x^2 + 2x - 2$. Looking at the polynomial in its factored form, (2x – 1)(2x + 2), it is easy to see that (2x -1), (2x + 2), and (2x – 1)(2x + 2) divide into (2x – 1)(2x + 2). Set up the problem as follows.

$$\frac{1}{2x-1} - \frac{1}{2x+2} + \frac{1}{4x^2+2x-2} = \frac{\qquad\qquad}{(2x-1)(2x+2)}$$

To decide the portion of the numerator for the $\frac{1}{2x-1}$ term realize that 2x – 1 divides into the common denominator 2x + 2 times its numerator 1 for a product of 2x + 2 in the new numerator.

$$\frac{1}{2x-1} - \frac{1}{2x+2} + \frac{1}{4x^2+2x-2} = \frac{(2x+2) -\qquad\qquad}{(2x-1)(2x+2)}$$

To decide the portion of the numerator for the $\frac{1}{2x+2}$ term realize that 2x + 2 divides into the common denominator 2x – 1 times it's numerator 1 for a product of 2x -1 in the new numerator.

$$\frac{1}{2x-1} - \frac{1}{2x+2} + \frac{1}{4x^2+2x-2} = \frac{(2x+2) - (2x-1) +\qquad}{(2x-1)(2x+2)}$$

To decide the portion of the numerator for the $\frac{1}{4x^2+2x-2}$ term realize that $4x^2 + 2x - 2$ divides into the common denominator 1 times the numerator 1 for a product of 1 in the new numerator.

$$\frac{1}{2x-1} - \frac{1}{2x+2} + \frac{1}{4x^2+2x-2} = \frac{(2x+2)-(2x-1)+1}{(2x-1)(2x+2)}$$

Collecting terms in the numerator gives the following answer.

$$\frac{4}{(2x-1)(2x+2)} \text{ Which reduces to } \frac{2}{(2x-1)(x+1)}$$

7.3 Multiplication and Division of Rational Expressions

Multiplication of two rational expressions is simply the product of the two numerators divided by the product of the two denominators.

Division of two rational expressions is generally accomplished by multiplying one rational expression by the ***reciprocal*** of the other rational expression.

Again, detailed examples are the best teacher.

Example 7.3.1

Multiply the following rational expressions.

$$\frac{2x^2}{5} \bullet \frac{3x}{6}$$

Multiply the two numerators together and multiply the two denominators together.

$$\frac{2x^2}{5} \bullet \frac{3x}{6} = \frac{6x^3}{30}$$

This answer simplifies as follows.

$$\boldsymbol{\frac{x^3}{5}}$$

Example 7.3.2

Multiply the following rational expressions.

$$\frac{3y^2z}{3z} \bullet \frac{6yz}{3y}$$

Multiply the two numerators together and multiply the two denominators together.

$$\frac{18y^3z^2}{9yz}$$

This answer simplifies as follows.

$$\boldsymbol{2y^2z}$$

Example 7.3.3

Multiply the following rational expressions.

$$\frac{3(x + 1)}{x - 2} \bullet \frac{2(x - 3)}{x^2 - 2x - 3}$$

Multiply only the coefficients times each other. The other terms in the numerator are the factors of the polynomial in the denominator. Factor the polynomial in the denominator.

$$\frac{3(x + 1)}{x - 2} \bullet \frac{2(x - 3)}{x^2 - 2x - 3} = \frac{6(x + 1)(x - 3)}{(x - 2)(x + 1)(x - 3)}$$

The answer simplifies as follows.

$$\boldsymbol{\frac{6}{x - 2}}$$

Example 7.3.4

Multiply the following rational expressions.

$$\frac{25r^2 - 16s^2}{r^2 + r} \bullet \frac{r + 1}{5r - 4s}$$

Factor the numerator and denominator of the first expression. Multiply the two expressions.

$$\frac{25r^2 - 16s^2}{r^2 + r} \bullet \frac{r + 1}{5r - 4s} = \frac{(5r+4s)(5r-4s)(r+1)}{r(r+1)(5r-4s)}$$

The answer simplifies as follows.

$$\frac{5r + 4s}{r}$$

Example 7.3.5

Multiply the following rational expressions.

$$\frac{z(3z + 1)}{9z^2 - 1} \bullet \frac{6(3z - 1)}{3z + 1}$$

Multiply the coefficients of the terms in parentheses. Factor the denominator of the first expressions.

$$\frac{z(3z + 1)}{9z^2 - 1} \bullet \frac{6(3z - 1)}{3z + 1} = \frac{6z(3z+1)(3z-1)}{(3z+1)(3z-1)(3z+1)}$$

The answer simplifies as follows.

$$\frac{6z}{3z + 1}$$

Example 7.3.6

Divide the following rational expressions.

$$\frac{x^2}{x} \div \frac{2x}{x}$$

Multiply the first expression by the reciprocal of the second expression.

$$\frac{x^2}{x} \bullet \frac{x}{2x}$$

Multiply the numerators together. Multiply the denominators together.

$$\frac{x^3}{2x^2}$$

The answer simplifies as follows.

$$\frac{x}{2}$$

Example 7.3.7

Divide the following rational expressions.

$$\frac{y^2+y}{z^2} \div \frac{y+1}{z}$$

Multiply the first expression by the reciprocal of the second expression.

$$\frac{y^2+y}{z^2} \bullet \frac{z}{y+1}$$

Multiply the numerators together. Multiply the denominators together.

$$\frac{y^2+y}{z^2} \bullet \frac{z}{y+1} = \frac{z(y^2+y)}{z^2(y+1)}$$

Factor the numerator of the new rational expression.

$$\frac{z(y^2+y)}{z^2(y+1)} = \frac{z\,y(y+1)}{z^2(y+1)}$$

The answer simplifies as follows.

$$\frac{y}{z}$$

Example 7.3.8

Divide the following rational expressions.

$$\frac{4a^2 - 9}{a(2a - 3)} \div \frac{2a + 3}{6a^2}$$

Multiply the first expression by the reciprocal of the second expression.

$$\frac{4a^2 - 9}{a(2a - 3)} \bullet \frac{6a^2}{2a + 3}$$

Factor the numerator of the first rational expression.

$$\frac{(2a - 3)(2a + 3)}{a(2a - 3)} \bullet \frac{6a^2}{2a + 3}$$

Multiply the two rational expressions.

$$\frac{(2a - 3)(2a + 3)6a^2}{a(2a - 3)(2a + 3)}$$

This expression simplifies as follows.

$$6a$$

Example 7.3.9

Divide the following rational expressions.

$$\frac{y^3 + y^2}{16z^2 - 4} \div \frac{y^2 + 3y + 2}{4z + 2}$$

Multiply the first expression by the reciprocal of the second expression.

$$\frac{y^3 + y^2}{16z^2 - 4} \bullet \frac{4z + 2}{y^2 + 3y + 2}$$

Factor the numerator and denominator of the first term and the denominator of the second term.

$$\frac{y^2(y + 1)}{(4z - 2)(4z + 2)} \bullet \frac{4z + 2}{(y + 1)(y + 2)}$$

Multiply the two terms and simplify.

$$\frac{y^2}{(4z - 2)(y + 2)}$$

Example 7.3.10

Divide the following rational expressions.

$$\frac{s^2 - t^2}{q^4 - q^3} \div \frac{3s + 3t}{q^2 - 1}$$

Multiply the first expression by the reciprocal of the second expression.

$$\frac{s^2 - t^2}{q^4 - q^3} \bullet \frac{q^2 - 1}{3s + 3t}$$

Factor both numerators and both denominators.

$$\frac{(s - t)(s + t)}{q^3(q - 1)} \bullet \frac{(q + 1)(q - 1)}{3(s + t)}$$

Multiply the two terms and simplify.

$$\frac{(s - t)(q + 1)}{3q^3}$$

7.4 Simplifying Complex Fractions

A ***complex fraction*** is a fraction that contains a fraction in either or both the numerator and denominator.

Figure 7.4.1 shows an example of a complex fraction.

$$\frac{3 + \frac{1}{z}}{2 - \frac{1}{z}}$$

Figure 7.4.1 A Complex Fraction

The method to resolve these complex fractions is to multiply the complex fraction by ***some value divided by itself***. This results in multiplying the fraction by 1 and thus does not change the value of the fraction. Choosing the right value will simplify the complex fraction.

Let's work some examples.

Example 7.4.1

Reduce the complex fraction to its simplest form.

$$\frac{3+\frac{1}{z}}{2-\frac{1}{z}}$$

To resolve the fractions within the complex fraction multiply the complex fraction by z/z.

$$\frac{3+\frac{1}{z}}{2-\frac{1}{z}}\bullet\frac{z}{z}$$

Multiply each term by *z*.

$$\frac{3+\frac{1}{z}}{2-\frac{1}{z}}\bullet\frac{z}{z}=\frac{3z+\frac{z}{z}}{2z-\frac{z}{z}}$$

This simplifies to the following expression.

$$\frac{3z+1}{2z-1}$$

Example 7.4.2

Reduce the complex fraction to its simplest form.

$$\frac{4x - \frac{9}{x}}{2 - \frac{3}{x}}$$

Multiply the complex fraction by x/x.

$$\frac{4x - \frac{9}{x}}{2 - \frac{3}{x}} \bullet \frac{x}{x} = \frac{4x^2 - 9}{2x - 3}$$

Factor the numerator of the new fraction.

$$\frac{4x - \frac{9}{x}}{2 - \frac{3}{x}} \bullet \frac{x}{x} = \frac{4x^2 - 9}{2x - 3} = \frac{(2x - 3)(2x + 3)}{2x - 3}$$

This simplifies to the following expression.

$$\boldsymbol{2x + 3}$$

Example 7.4.3

Reduce the complex fraction to its simplest form.

$$\frac{\frac{16}{y} - y}{4 + y}$$

Multiply the complex fraction by y/y.

$$\frac{\frac{16}{y} - y}{4 + y} \bullet \frac{y}{y} = \frac{16 - y^2}{4y + y^2}$$

Factor both the numerator and denominator of the new fraction.

$$\frac{\frac{16}{y} - y}{4 + y} \bullet \frac{y}{y} = \frac{16 - y^2}{4y + y^2} = \frac{(4 - y)(4 + y)}{y(4 + y)}$$

This simplifies to the following expression.

$$\boldsymbol{\frac{4 - y}{y}}$$

Example 7.4.4

Reduce the complex fraction to its simplest form.

$$\frac{\frac{4}{s} + 1}{\frac{16}{s^2} - 1}$$

Multiply the complex fraction by s^2/s^2.

$$\frac{\frac{4}{s}+1}{\frac{16}{s^2}-1}\bullet\frac{s^2}{s^2}=\frac{4s+s^2}{16-s^2}$$

Factor both the numerator and denominator.

$$\frac{\frac{4}{s}+1}{\frac{16}{s^2}-1}\bullet\frac{s^2}{s^2}=\frac{4s+s^2}{16-s^2}=\frac{s(4+s)}{(4+s)(4-s)}$$

This simplifies to the following expression.

$$\boldsymbol{\frac{s}{4-s}}$$

Example 7.4.5

Reduce the complex fraction to its simplest form.

$$\frac{x-\frac{4}{x}}{1-\frac{2}{x}}$$

Multiply the complex fraction by x/x.

$$\frac{x-\frac{4}{x}}{1-\frac{2}{x}}\bullet\frac{x}{x}=\frac{x^2-4}{x-2}$$

Factor the numerator of the new fraction.

$$\frac{x-\frac{4}{x}}{1-\frac{2}{x}}\bullet\frac{x}{x}=\frac{x^2-4}{x-2}=\frac{(x-2)(x+2)}{x-2}$$

This simplifies to the following expression.

$$\boldsymbol{x+2}$$

7.5 Solving Equations with Fractions

Solving equations with fractions involves developing a ***common denominator*** for all of the rational components of the equation.

Precisely defined examples still remains the best method of demonstrating this technique.

Example 7.5.1

Solve the equation. Express it in its simplest form.

$$\frac{4}{y-2} + 2 = \frac{6}{y-2}$$

Considering the 2 as 2/1 the least common denominator is 1(y – 2) or simply y – 2. The y – 2 denominator of the first term goes into the common denominator 1 time multiplied by 4 equals 4. The 1 denominator of the second term goes into the common denominator y – 2 times multiplied by 2 equals 2(y – 2).

$$\frac{4 + 2(y-2)}{y-2} = \frac{6}{y-2}$$

If the entire left hand side of an equation has the exact same denominator as the entire right hand side of the equation the two numerators are equal.

$$4 + 2(y-2) = 6$$

Multiply through the 2(y – 2) term.

$$4 + 2y - 4 = 6$$

This then rearranges to the following.

$$2y = 6$$

The final answer is as follows.

$$\boldsymbol{y = 3}$$

Example 7.5.2

Solve the equation. Express it in its simplest form.

$$\frac{9x - 4}{3x - 2} - 2 = \frac{6}{3x - 2}$$

Considering the 2 as 2/1 the least common denominator is 1(3x – 2) or simply 3x – 2. The 3x – 2 denominator of the first term goes into the common denominator 1 time multiplied by 9x – 4 equals 9x – 4. The 1 denominator of the second term goes into the common denominator 3x – 2 times multiplied by – 2 equals -2(3x – 2).

$$\frac{9x - 4 -2(3x - 2)}{3x - 2} = \frac{6}{3x - 2}$$

If the entire left hand side of an equation has the exact same denominator as the entire right hand side of the equation the two numerators are equal.

$$9x - 4 -2(3x - 2) = 6$$

Multiply through the -2(3x – 2) term.

$$9x - 4 - 6x + 4 = 6$$

The collected terms result in the following.

$$3x = 6$$

The final answer is as follows.

$$\boldsymbol{x = 2}$$

Example 7.5.3

Solve the equation. Express it in its simplest form.

$$\frac{3}{z^2 - 9} + \frac{4}{z - 3} = \frac{2}{z - 3}$$

Knowing that z – 3 is one of the two factors of the difference of squares expression $z^2 - 9$, the least common denominator is $z^2 - 9$. The $z^2 - 9$ denominator of the first term goes into the common denominator 1 time multiplied by 3 equals 3. The z – 3 denominator of the second term goes into the common denominator z + 3 times multiplied by 4 equals 4(z + 3). The z – 3 denominator of the third term goes into the common denominator z + 3 times multiplied by 2 equals 2(z + 3).

$$\frac{3 + 4(z + 3)}{z^2 - 9} = \frac{2(z + 3)}{z^2 - 9}$$

If the entire left hand side of an equation has the exact same denominator as the entire right hand side of the equation the two numerators are equal.

$3 + 4\ (z + 3) = 2\ (z + 3)$

Multiply through the two z + 3 terms.

$3 + 4z + 12 = 2z + 6$

Combine like terms and simplify.

$2z = -9$

This simplifies as follows.

***z* = – 9/2**

Example 7.5.4

Solve the equation. Express it in its simplest form.

$$\frac{4}{s^2 + 6s + 8} - \frac{2}{s + 4} = \frac{6}{s + 4}$$

Knowing that s + 4 is one of the two factors of the second degree polynomial expression $s^2 + 6s + 8$, the least common denominator is $s^2 + 6s + 8$. The $s^2 + 6s + 8$ denominator of the first term goes into the common denominator 1 time multiplied by 4 equals 4. The s + 4 denominator of the second term goes into the common denominator s + 2 times multiplied by – 2 equals – 2(s + 2). The s + 4 denominator of the third term goes into the common denominator s + 2 times multiplied by 6 equals 6(s + 2).

$$\frac{4 - 2(s + 2)}{s^2 + 6s + 8} = \frac{6(s + 2)}{s^2 + 6s + 8}$$

If the entire left hand side of an equation has the exact same denominator as the entire right hand side of the equation the two numerators are equal.

$$4 - 2(s + 2) = 6(s + 2)$$

Multiply through the s + 2 terms.

$$4 - 2s - 4 = 6s + 12$$

Combine like terms and simplify.

$$-8s = 12$$

Divide both sides of the equation by – 8.

$$s = -12/8$$

This simplifies to the following expression.

$$\boldsymbol{s = -3/2}$$

Example 7.5.5

Solve the equation. Express it in its simplest form.

$$\frac{2}{4r^2 - 9} + \frac{2}{2r + 3} = \frac{4}{2r - 3}$$

Knowing that the 2r + 3 of the second term and the 2r – 3 of the third term are the two factors of the $4r^2 - 9$ of the first term, the lowest common denominator is $4r^2 - 9$. The $4r^2 - 9$ of the first term goes into the common denominator 1 time multiplied by 2 equals 2. The 2r + 3 of the second term goes into the common denominator 2r – 3 times multiplied by 2 equals 2(2r – 3). The 2r – 3 of the third term goes into the common denominator 2r + 3 times multiplied by 4 equals 4(2r + 3).

$$\frac{2 + 2(2r - 3)}{4r^2 - 9} = \frac{4(2r + 3)}{4r^2 - 9}$$

If the entire left hand side of an equation has the exact same denominator as the entire right hand side of the equation the two numerators are equal.

$$2 + 2(2r - 3) = 4(2r + 3)$$

Multiply through the (2r – 3) and (2r + 3) terms.

$$2 + 4r - 6 = 8r + 12$$

Combine like terms and simplify.

$$-4r = 16$$

Divide both sides of the equation by – 4.

$$\boldsymbol{r = -4}$$

7.6 Ratios and Proportions

A ***ratio and proportion*** is the expressions of the quantity of one thing in relation to another thing. This relationship is expressed as the quotient of those two things.

Cross multiplication is used to solve for the unknown.

Example 7.6.1

There are three Toyotas sold in the US for every 1 Kia sold in the US. Last year 33 million Toyotas were sold in the US. How many Kias were sold in the US last year?

The ratio of Toyotas to Kia is three to one. The symbol for this is 3:1. The proportion is written as follows.

$$\frac{33\text{ mill}}{3} = \frac{x\text{ mill}}{1}$$

By cross multiplying we arrive at the following equation.

$$3x = 33$$

Here x is the number of Kia sold last year in millions of cars.

$$\boldsymbol{x = 11}$$

Example 7.6.2

A 150 pound dog consumes 60 pounds of dog food in a 30 day period. How much dog food does a 100 pound dog consume for the same 30 day period, assuming the same dog to dog food ratio?

We are not given the ratio in its most reduced form. However, 150 pound dog to 60 pounds dog food is a valid ratio. Set the problem up as follows.

$$\frac{150}{60} = \frac{100}{x}$$

Here x is the amount of dog food consumed by the 100 pound dog in a thirty day period.

$$150x = 6{,}000$$

This reduces to the following.

$$\boldsymbol{x = 40\text{ pounds}}$$

Example 7.6.3

The sum of two numbers is 21. The ratio of the larger to the smaller number is 2:1. What are the two numbers?

Set up the relationship of the two numbers as follows.

$$\frac{2}{21 - x} = \frac{1}{x}$$

Here x represents one number and 21 – x represents the other number. Through cross multiplication we get the following.

$$2x = 21 - x \; ; \; 3x = 21 \; ; \; x = 7$$

The two numbers are as follows.

7 and 14

Example 7.6.4

A car uses 10 gallons of gasoline on a 300 mile trip. How many miles would a trip be where 12 gallons of gasoline are used?

Use the ratio of 300 miles to 10 gallons as follows.

$$\frac{300}{10} = \frac{x}{12}$$

Here x represents the number of miles for the trip where 12 gallons of gasoline were consumed.

$$10x = 3{,}600$$

This reduced as follows.

x = 360 gallons

Example 7.6.5

If a production line robot can weld 1,200 widgets in 60 minutes, how many widgets can it weld in 72 minutes?

Use the ratio of 1,200 widgets to 60 minutes as follows.

$$\frac{1{,}200}{60} = \frac{x}{72}$$

Here x represents the number of widgets welded in 72 minutes.

$$60x = 86{,}400$$

This reduces as follows.

x = 1,440

7.7 Chapter 7 End of Chapter Exercises

Answer the following questions as directed. The answers are in Appendix A at the end of the text.

7.1. Simplify the following rational expressions.

1) $\frac{m^3n^2}{m^2n}$

2) $\frac{3x^2y}{6xy}$

3) $\frac{4s^4t^2}{2s^2t}$

4) $\frac{18y^4z^2}{3y^2z^4}$

5) $\frac{6x^4y^2z}{3x^2y^3z^2}$

6) $\frac{4(x+1)^2}{(x+1)}$

7) $\frac{10(x+y)^4}{5(x+y)^3}$

8) $\frac{4(s+2)^2(s-2)^4}{2(s+2)^4(s-2)^2}$

9) $\frac{16(x-4)^2(x+4)^3}{(x-4)(x+4)^3}$

10) $\frac{18(y+z)^2(y-z)^3}{2(y-z)^4(y+z)^5}$

7.2. Add the following rational expressions.

1) $\frac{2}{z^2}+\frac{4}{z}$

2) $\frac{4}{2s}+\frac{3}{6t}$

3) $\frac{5}{y^2}+\frac{2}{y^3}$

4) $\frac{z+4}{6}+\frac{z+2}{8}$

5) $\frac{5}{x-1}+\frac{2}{x^2+x-2}$

Subtract the following rational expressions.

6) $\frac{6}{y^2} - \frac{3}{y}$

7) $\frac{8}{3z} - \frac{6}{4z}$

8) $\frac{4}{s^2} - \frac{2}{s^3}$

9) $\frac{x-1}{5} - \frac{x-2}{3}$

10) $\frac{4}{y^2 + y - 6} - \frac{5}{y-2}$

7.3. Multiply the following rational expressions.

1) $2r^3 \bullet 3r$

2) $\dfrac{4x^2y}{2y} \bullet \dfrac{6xy}{3x}$

3) $\dfrac{3y^3z^2}{yz^2} \bullet \dfrac{9}{y^2}$

4) $\dfrac{3(x+2)}{x-3} \bullet \dfrac{2(x-3)}{x^2-x-6}$

5) $\dfrac{16x^2-9y^2}{4x-3y} \bullet \dfrac{4x+3y}{5}$

Divide the following rational expressions.

6) $\dfrac{y^3}{y} \div \dfrac{3y}{y}$

7) $\dfrac{x^2+x}{3y^2} \div \dfrac{x+1}{y}$

8) $\dfrac{16r^2-4}{r(4r-2)} \div \dfrac{4r+2}{4r}$

9) $\dfrac{z^2-z-2}{9y^2-9} \div \dfrac{z-2}{3y+3}$

10) $\dfrac{4x^2-16y^2}{a^4+a^2} \div \dfrac{2x+4y}{a^2+1}$

7.4. Reduce each complex fraction to its simplest form.

1) $\dfrac{4+\frac{1}{y}}{2-\frac{1}{y}}$

2) $\dfrac{6x-\frac{6}{x}}{3x-\frac{3}{x}}$

3) $\dfrac{\frac{9}{z}-z}{3+z}$

4) $\dfrac{4r+\frac{4}{r}}{9r+\frac{9}{r}}$

5) $\dfrac{y+3+\frac{2}{y}}{1+\frac{1}{y}}$

6) $\dfrac{x-4+\frac{4}{x}}{1-\frac{2}{x}}$

7) $\dfrac{z-\frac{4}{z}}{1+\frac{2}{z}}$

8) $\dfrac{y-1-\frac{6}{y}}{1-\frac{3}{y}}$

9) $\dfrac{x+1-\frac{2}{x}}{x+2}$

10) $\dfrac{s^2+s}{s+4+\frac{3}{s}}$

7.5. Solve the following equations. Express them in the simplest form.

1) $\dfrac{8}{y+1}+4=\dfrac{4}{y+1}$

2) $\dfrac{7}{x-2}-4=\dfrac{3}{x-2}$

3) $\dfrac{9z-6}{4z+2}-4=\dfrac{7}{4z+2}$

4) $\dfrac{4s-2}{2s+2}-4=\dfrac{2}{2s+2}$

5) $\dfrac{4}{t^2-9}+\dfrac{3}{t+3}=\dfrac{2}{t+3}$

6) $\dfrac{4}{x^2-4}-\dfrac{2}{x-2}=\dfrac{2}{x+2}$

7) $\dfrac{4}{t^2-9}+\dfrac{6}{t-3}=\dfrac{2}{t+3}$

8) $\dfrac{4}{z^2-z-6}+\dfrac{6}{z-3}=\dfrac{4}{z+2}$

9) $\dfrac{4}{2x^2 + 10x + 8} - \dfrac{6}{2x + 2} = \dfrac{2}{x + 4}$

10) $\dfrac{4}{6x^2 - 4x - 2} + \dfrac{6}{2x - 2} = \dfrac{2}{3x + 1}$

7.6. Solve the following ratio to proportion problems.

1) The sum of two numbers is 12. The ratio of the larger number to the smaller number is 3:1. What are the two numbers?
2) A car can make a 300 mile trip on 10 gallons of gasoline. How long of a trip could the car make on 13 gallons of gasoline?
3) For every 3 MCCorpK motorcycles sold in the US there are 5 MCCorpH motorcycles sold. Last year there were 150,000 MCCorpK motorcycles sold in the US. How many MCCorpH motorcycles were sold in the US last year?
4) Last year during the World Series, Radio Hut agreed to donate 50 ToddlerComp computers to the US Marine Corps Toys for Tots program for each home run hit during the series. There were 12 home runs hit during the series. How many ToddlerComp computers did Radio Hut donate to Toys for Tots?
5) For every 100 people born in the US eight of them are born left handed. If there are 350,000,000 people in the US, how many of them are left handed?

8 Inequalities and Absolute Values

8.1 Inequality Expressions

So far we have worked with expressions, like x + 5, or equations like x + 5 = 10.

Now we have to consider whether an unknown, such as x, is in a relationship, other than equality with a given number.

Table 8.1.1 gives the four relationships of inequality along with the accepted symbol for each relationship.

Relationship	Symbol
greater than	>
less than	<
greater than or equal to	≥
less than or equal to	≤

Table 8.1.1 Relationships of Inequality

Some examples of inequalities follow.

Example 8.1.1

Solve the following inequality

$$x + 4 \leq 12$$

Even though this is an inequality you treat it the same as you would if it were an equation. Subtract 4 from both sides of the inequality.

$$\boldsymbol{x \leq 8}$$

Example 8.1.2

Solve the following inequality.

$$y - 3 > 6$$

Add 3 to both sides of the inequality.

$$\boldsymbol{y > 9}$$

Example 8.1.3

Solve the following inequality.

$s + 4 < 16$

Subtract 4 from both sides of the inequality.

$\boldsymbol{s < 12}$

Example 8.1.4

Solve the following inequality.

$3z < z + 6$

Subtract z from both sides of the inequality.

$2z < 6$

Divide both sides of the inequality by 2.

$\boldsymbol{z < 3}$

Example 8.1.5

Solve the following inequality.

$6 \geq x + 2$

Subtract 2 from both sides of the inequality.

$4 \geq x$

This cannot be the final answer. We want to know the inequality relationship of x not 4. Simply read the expression from right to left. Observe that this ***reverses the inequality sign***.

$\boldsymbol{x \leq 4}$

Example 8.1.6

Solve the following inequality.

$-4 \leq y - 2 \leq 0$

Add 2 to each of the three terms of the inequality.

$\boldsymbol{-2 \leq y \leq 2}$

Here the inequality expression tells us that y represents ***all numbers*** that are less than and equal to 2 and greater than and equal to -2.

Example 8.1.7

Solve the following inequality.

$3 < x + 4 < 5$

Subtract 4 from all three terms of the inequality expression.

$\mathbf{-1 < x < 1}$

Here the inequality expression tells us that x represents ***all numbers*** in between -1 and 1 not including -1 or 1.

Example 8.1.8

Solve the following inequality.

$$y - 1 < 2y + 4 < y + 2$$

Subtract y from all three terms of the inequality.

$$-1 < y + 4 < 2$$

Subtract 4 from all three terms of the inequality.

$$\boldsymbol{-5 < y < -2}$$

Here the inequality expression tells that y represents all numbers between – 5 and – 2 not including – 5 and – 2.

Example 8.1.9

Solve the following inequality.

$$2 \geq y + 4 \geq 5$$

Subtract 4 from all three terms of the inequality.

$$\boldsymbol{-2 \geq y \geq 1}$$

Here the inequality expression tells us that y is ***all numbers*** less than and equal to -2 and greater than and equal to 1.

Example 8.1.10

Solve the following inequality.

$$4 > x - 3 > -2$$

Add 3 to all three terms of the inequality.

$$\boldsymbol{7 > x > 1}$$

Here the inequality expression tells us that x is ***all numbers*** less than 7 and greater than 1 but not equal to 7 or 1.

8.2 Graphing Inequalities on the Number Line

In mathematics it is always advantageous to be able to see the mathematical relationship with which you are dealing.

When positioning inequality values on the number line you must distinguish between greater than and greater than and equal to. Also you must distinguish between less than and less than and equal to. This is accomplished by using ***opened dots*** for greater than and less than and ***filled dots*** for greater than and equal to and less than and equal to.

Always remember that the opened dot means that the number ***is not*** included in the inequality expression.

Some examples will make this clear.

Example 8.2.1

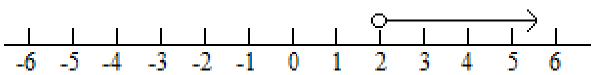
Solve the following inequality and graph on the number line.

$$y + 2 > 4$$

Subtract 2 from both sides of the inequality.

$$\boldsymbol{y > 2}$$

Graph the inequality as follows.



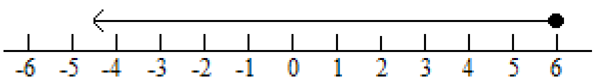
Example 8.2.2

Solve the following inequality and graph on the number line.

$$x - 4 \leq 2$$

Add 4 to both sides of the inequality

$$\boldsymbol{x \leq 6}$$



Example 8.2.3

Solve the following inequality and graph on the number line.

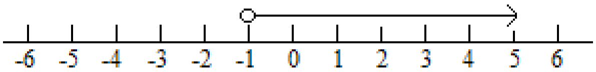
$3 < z + 4$

Subtract 4 from both sides of the inequality.

$-1 < z$

Read this from right to left as the following.

$z > -1$



Example 8.2.4

Solve the following inequality and graph on the number line.

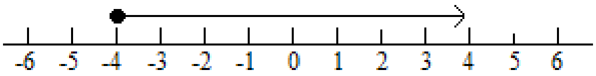
$2x + 8 \geq x + 4$

Subtract x from both sides of the inequality.

$x + 8 \geq 4$

Subtract 8 from both sides of the inequality.

$\boldsymbol{x \geq -4}$



Example 8.2.5

Solve the following inequality and graph on the number line.

$y + 4 < 2y + 2$

Subtract y from both sides of the inequality.

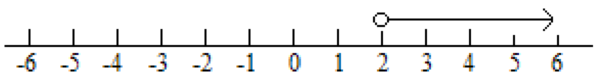
$4 < y + 2$

Subtract 2 from both sides of the inequality.

$2 < y$

Read this from right to left as the following.

$\boldsymbol{y > 2}$



Example 8.2.6

Solve the following inequality and graph on the number line.

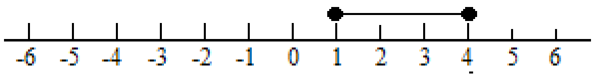
$4 \geq z \geq 1$

Read from left to right the following.

$\boldsymbol{z \geq 1}$

Read from right to left the following.

z* ≤ *4



Example 8.2.7

Solve the following inequality and graph on the number line.

$- 2 < x + 2 \leq 4$

Subtract 2 from the three terms of the inequality.

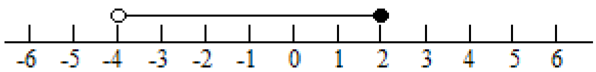
$- 4 < x \leq 2$

Read from left to right the following.

x* ≤ *2

Read from right to left the following.

x* > *-4



Example 8.2.8

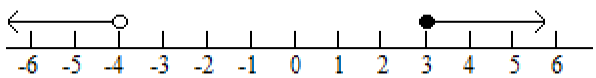
Solve the following inequality and graph on the number line.

$y + 3 < -1$ or $y + 3 \geq 6$

Subtract 3 from each side of both the expressions.

y* < *-4 or y* ≥ *3

The inequality could not be written in the continuous form $-1 > y + 3 \geq 6$ since -1 is not greater than or equal to 6.



Example 8.2.9

Solve the following inequality and graph on the number line.

$$-1 \leq z - 2 < 3$$

Add 2 to the three elements of the inequality.

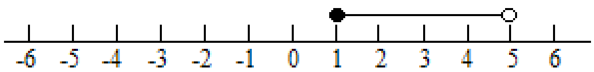
$$1 \leq z < 5$$

Read from left to right the following.

$$z < 5$$

Read from right to left the following.

$$z \geq 1$$



Example 8.2.10

Solve the following inequality and graph on the number line.

$5 > 2x + 1 > 3$

Subtract 1 from the three terms of the inequality.

$4 > 2x > 2$

Divide each of the three terms of the inequality by 2.

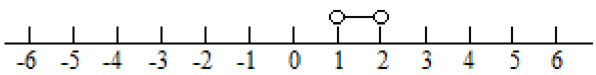
$2 > x > 1$

Read from left to right the following.

$\boldsymbol{x > 1}$

Read from right to left the following.

$\boldsymbol{x < 2}$



8.3 Absolute Values

In keeping with the simple and fundamental approach of this text an ***absolute value*** of a number will be described as ***the distance of that number from zero*** on the number line.

Figure 8.3.1

An Absolute Value Expression

$\boldsymbol{|-3|}$

The resolution of an absolute value is gained by stating the absolute value of the number as the positive value of that number.

Figure 8.3.2

Resolving an Absolute Value

$\boldsymbol{|-3| = 3}$

This can be looked at graphically as in the following figure.

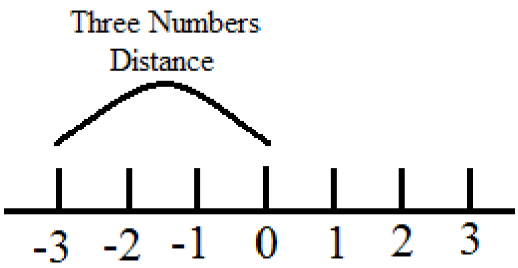


Figure 8.3.3

In order to perform operations with absolute values you must first resolve all absolute value expressions in the expression or equation you are working with. After this perform the operation according to the order of operation rules (PEMDAS) in section 3.1 of this text.

Example 8.3.1

Find the value of the following equation.

$| -4 | -2 =$

Resolve the absolute value as 4 and subtract 2.

4 – 2 = 2

Example 8.3.2

Find the value of the following equation.

$4 \times | 2 | =$

Resolve the absolute value as 2 and multiply by 4.

4 × 2 = 8

Example 8.3.3

Find the value of the following equation.

$| 3 | - | -2 | =$

Resolve the two absolute values as 3 and 2, respectively and subtract 2 from 3.

$$\boldsymbol{3 - 2 = 1}$$

Example 8.3.4

Find the value of the following equation.

$$| -3 | \times | -4 | =$$

Resolve the two absolute values as 3 and 4, respectively and multiply 3 by 4.

$$\boldsymbol{3 \times 4 = 12}$$

Example 8.3.5

Find the value of the following equation.

$$| -4 | \div | 2 | =$$

Resolve the absolute values as 4 and 2, respectively and divide 4 by 2.

$$4 \div 2 = \boldsymbol{2}$$

Example 8.3.6

Find the value of the following equation.

$$| x - 3 | = (x = 1)$$

Substitute 1 as the value of x and subtract 3.

$$|1 - 3 | = | -2 |$$

Resolve the absolute value as 2.

$$| -2 | = \boldsymbol{2}$$

Example 8.3.7

Find the value of the following equation.

$$| -y + 2 | = (y = 3)$$

Substitute 3 as the value of y and add 2.

$$| -3 + 2 | = | -1 |$$

Resolve the absolute value as 1.

$$| -1 | = \boldsymbol{1}$$

Example 8.3.8

Find the value of the following equation.

$$| x + y | = \qquad (x = 1, y = -3)$$

Substitute 1 as the value of x and -3 as the value of y.

$$| 1 + (-3) | =$$

Subtract 3 from 1.

$$|1 - 3| = |-2|$$

Resolve the absolute value as 2.

$$|-2| = \boldsymbol{2}$$

Example 8.3.9

Find the value of the following equation.

$$|s - t| + t = \qquad (s = 3, t = 1)$$

Substitute 3 as the value of s and 1 as the value for each t.

$$|3 - 1| + 1 =$$

Subtract 1 from three.

$$|2| + 1 =$$

Resolve the absolute value as 2 and add 1.

$$2 + 1 = \boldsymbol{3}$$

Example 8.3.10

Find the value of the following equation.

$$|x - y| \div |x| \qquad (x = 4, y = 2)$$

Substitute 4 as the value of each x and 2 as the value of y.

$$|4 - 2| \div |4|$$

Subtract 2 from 4.

$$|2| \div |4|$$

Resolve the two absolute values as 2 and 4, respectively and divide 2 by 4.

$$2 \div 4 = \boldsymbol{1/2}$$

8.4 Graphing Absolute Values of Inequalities on the Number Line

Since the number line is essential in defining ***absolute values of inequalities*** the definition, solving, and graphing of them will be handled in a single section.

Any absolute value, whether for an equation or an inequality, denotes two different possibilities.

If I say $|x| \le 5$, I am defining x as all values between -5 and 5 on the number line including -5 and 5.

This is true because the $|5| = |-5| = 5$

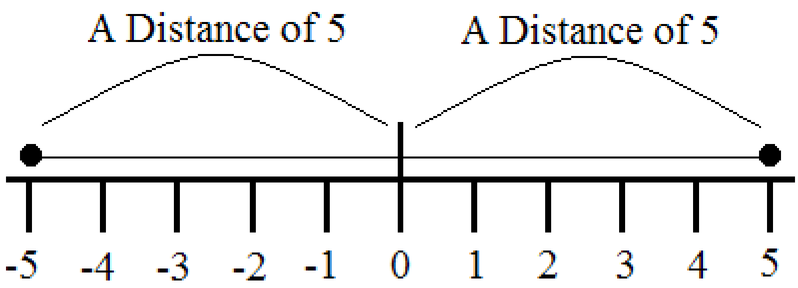


Figure 8.4.1 Graphing the Absolute Inequality $|x| \le 5$

So the solution to $|x| \le 5$ is $x \le 5$ ***and*** $x \ge -5$.

Since the line is continuous between the two values the inequality can be written as follows.

$$-5 \le x \le 5$$

Please note that the $x \ge -5$ must be read from right to left in this form.

This form holds for less than (<) relationships as well.

The form for greater than or equal (≥) or greater than (>) is markedly different.

If I say $|x| \ge 2$, I am defining x as allvalues greater than 2 ***or*** less than -2 on the number line including -2 and 2.

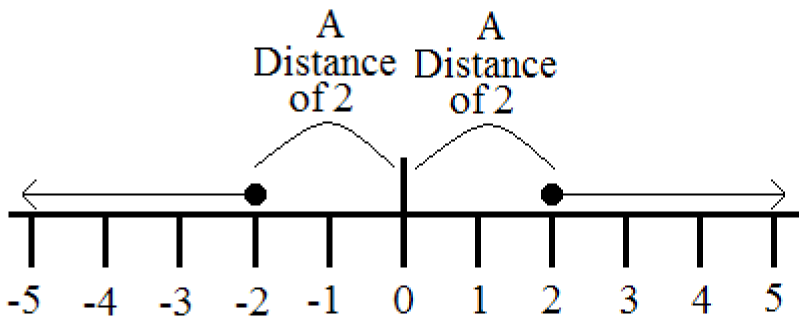


Figure 8.4.2 Graphing the Absolute Inequality | x | ≥ 2

So the solution to $|x| \ge 2$ is as follows.

$$x \ge 2 \text{ or } x \le -2$$

The solution to this absolute inequality is not continuous between the two points on the graph. For this reason the solution must be stated as two separate inequalities.

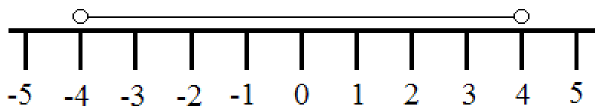
Example 8.4.1

Solve the absolute inequality and graph on the number line.

$$|y| < 4$$

The value y is 4 numbers away from 0 on the number line in both the positive and negative directions.
y < 4 and y > -4

-4 < y < 4



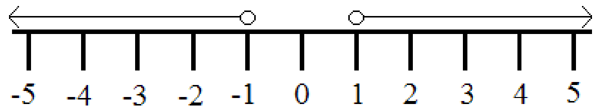
Example 8.4.2

Solve the absolute inequality and graph on the number line.

| x | > 1

The value x is greater than 1 in the positive direction or less than -1 in the negative direction.

x < -1 or x > 1



Example 8.4.3

Solve the absolute inequality and graph on the number line.

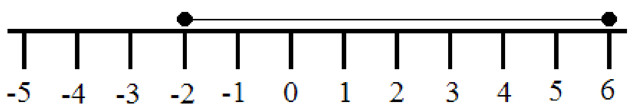
| x -2 | ≤ 4

This inequality is resolvable as a continuous graph on the number line.

-4 ≤ | x – 2 | ≤ 4

Add 2 to both sides of the inequality.

-2 ≤ x ≤ 6



Example 8.4.4

Solve the absolute inequality and graph on the number line.

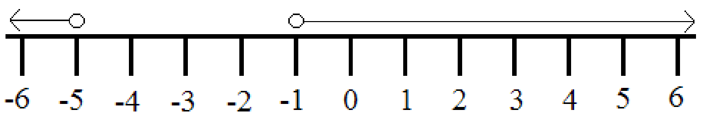
$$| y + 3 | > 2$$

Resolve the absolute inequality into the following two inequalities.

$$y + 3 > 2 \text{ and } y + 3 < -2$$

Subtract 3 from both sides of each inequality.

$$\boldsymbol{y > -1 \text{ or } y < -5}$$



Example 8.4.5

Solve the absolute inequality and graph on the number line.

$3 < | y + 2 |$

Establish the proper order of expression for the absolute inequality.

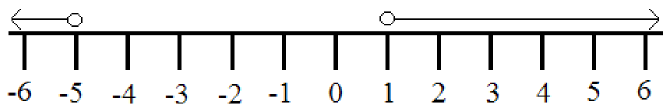
$| y + 2 | > 3$

Resolve the absolute inequality into the following two inequalities.

$y + 2 > 3$ and $y + 2 < -3$

Subtract 2 from both sides of each inequality.

y > 1 or y < -5



8.5 Chapter 8 End of Chapter Exercises

Answer the following questions as directed. The answers are in Appendix A at the end of the text.

8.1. Solve the following inequalities.

1) $x + 2 \leq 4$
2) $y - 3 > 2$
3) $t + 4 < 3$
4) $3s \geq s + 3$
5) $2 - y < 2y$
6) $10 - x > 2x + 1$
7) $-1 < z + 3 < 6$
8) $x - 1 \leq -2$ or $x - 1 \geq 4$
9) $8 \leq y + 1 \leq 14$
10) $4 > y/2 > 2$

8.2. Solve the following inequalities and graph on the number line.

1) $x + 3 > -2$
2) $t - 2 \leq 3$
3) $y + 2 < 4$
4) $5 \geq x - 2$
5) $z + 1 > 2z$
6) $s + 2 \leq 3$
7) $8 \geq x + 1 \geq 4$
8) $6 > y + 4 > 2$
9) $8 \geq 3x - 4 > 2$
10) $20 \geq 6t + 2 \geq 14$

8.3. Solve the following equations of absolute values.

1) $|-4| =$
2) $4 - |6| =$
3) $|-4| + |6|$
4) $|-2| \bullet |-4|$
5) $|x + 1| =$ $(x = -3)$
6) $|y + 2| + 3 =$ $(y = 2)$
7) $|x| \div |y|$ $(x = -1, y = 3)$
8) $|s + t| - t =$ $(s = -1, t = 4)$
9) $|q - r| \bullet |r|$ $(q = 2, r = 4)$
10) $|x - 2| \div |y + 1|$ $(x = 3, y = 2)$

8.4. Solve the absolute inequalities and graph on the number line.

1) $|x| \geq 4$
2) $|y| < < 2$
3) $|t + 3| > 2$
4) $|r - 4| \leq 6$
5) $|s + 2| \geq 4$
6) $|x + 2| + 3 > 6$
7) $|y - 3| + 4 \leq 8$
8) $|z + 3| + 2 \geq 6$
9) $|y - 2| \div 2 < 2$
10) $|x + 5| \div 3 > 2$

9 Graphing in the Rectangular Coordinate System

9.1 The Rectangular Coordinate System

The ***Rectangular Coordinate System***, also known as the Cartesian Coordinate System, is a system where all points in the system are identified by a pair of values known as ***coordinates***.

Each coordinate value is represented by a signed (positive or negative) value that measures the distance from the intersection of two perpendicular axes. This intersection of these axes is known as the ***origin***.

The measurement of distances along these axes are generally measured using the same values on each axis.

The axes of the rectangular coordinate system are characteristically referred to as the ***x*** and ***Y*** axes. The x axis is always horizontal and the Y axis is always vertical.

With all of these definitions in place it is definitely time to visually examine the rectangular coordinate system.

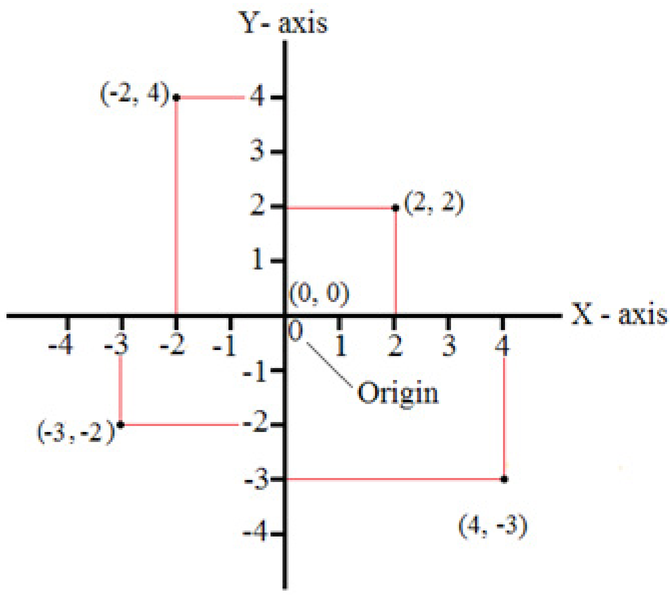


Figure 9.1.1 The Rectangular Coordinate System

Let's review the elements of the graph in Figure 9.1.1.

- The Y axis is the vertical axis.
- The x axis is the horizontal axis.
- The two axes are perpendicular.
- The origin is valued at 0 for both axes.
- The number values on both axes are the same.
- Each set of numbers for points on the graph are in the form of (x, y).

The points on the graph in Figure 9.1.1 were randomly assigned for display purposes. Typically points on a graph describe a specific geometric shape such as a line or a curve.

Figure 9.1.2 shows a graph of three points that, when connected, form a straight line. The set of numbers for each point were included on this graph. In general they are not included on graphs.

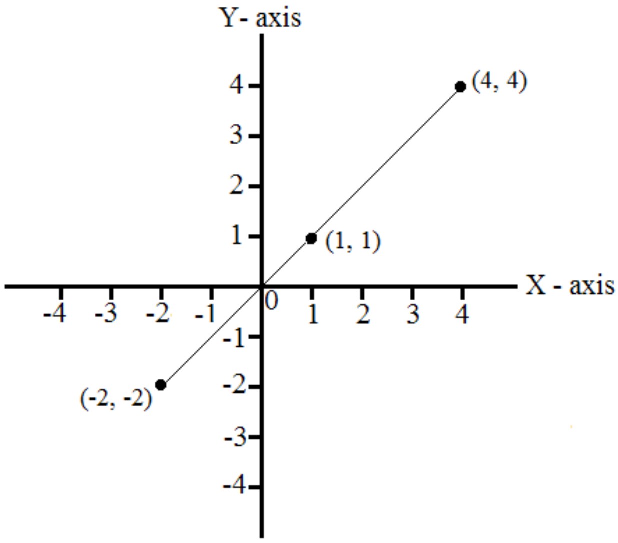


Figure 9.1.2 The Graph of A Straight Line

Along with the numerical values for each point there is additional information that can be extracted from the graph.

First let's inspect the nature of the slope of the line formed by the three points. As you move from left to right on the x-axis the line is pointing upward. This indicates the line has a ***positive slope***.

Additionally, you can observe where the line intersects the y-axis. For this graph the line intersects the y-axis at y = 0. This value is known as the ***y-intercept***.

Sections 9.2 and 9.3 will demonstrate what can be done with this information.

9.2 Linear Equations: The Slope Intercept Form

If one could derive an equation from a graph then the geometry of the graph could be expressed algebraically.

The equation of a straight line is as follows.

$$Y = mX + b$$

The elements of this equation are defined below.

- Y is the value on the y-axis.
- X is the value on the x-axis.
- m is the slope of the line.
- b is the y-intercept of the line.

To fully understand the concept of relating an algebraic equation to a geometric figure we will convert graphs to equations and equations to graphs.

The ***slope calculation*** of the graph of a straight line is as follows.

$$m = (Y_2 - Y_1)/(X_2 - X_1)$$

In this formula the subscript 1's and 2's simply indicate a first and second number on the line of the graph.

The y-intercept is where the line of the graph intersects with the y-axis and, for the purposes of this text, is observed visually.

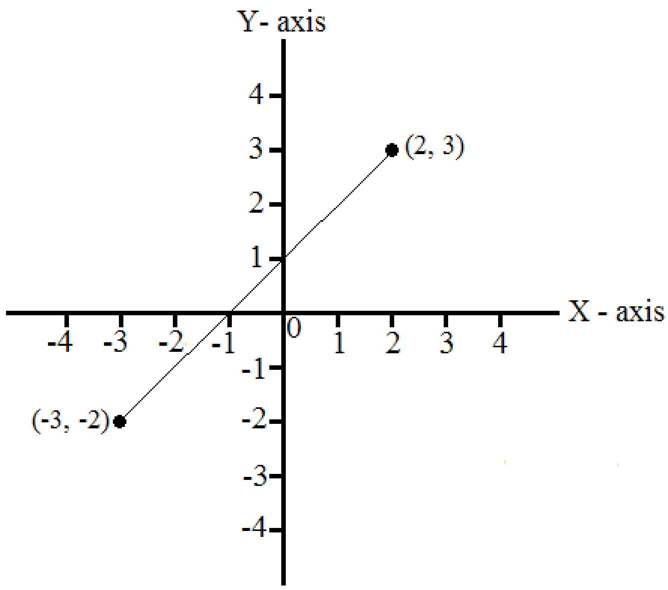


Figure 9.2.1 Extracting an Equation from a Graph

The convention for assigning an order to the two points on the graph is as an increasing order from left to right. But right to left will work.

Therefore, $X_1 = -3$, $Y_1 = -2$, $X_2 = 2$, and $Y_2 = 3$.

The slope calculation is as follows.

$$m = [3-(-2)] / [2-(-3)] = 5/5 = 1$$

The y-intercept is visually observed to be at y = 1.

Therefore, the equation extracted from the line on this graph is as follows.

$$\boldsymbol{y = x + 1}$$

Please note that the slope of 1 is not written in front of the x.

Let's graph the equation of a straight line. The algebraic relationship of the equation and the graph of the line do not change in this problem. The order of applying the relationship is the only thing that changes.

Draw the graph for the equation y = -2x + 1. The simplest way to approach this is to choose two or three simple x values, substitute them into the equation, and record the y values.

Remember, simpler x values result in a simpler graph.

x	y
-1	3
0	1
1	-1

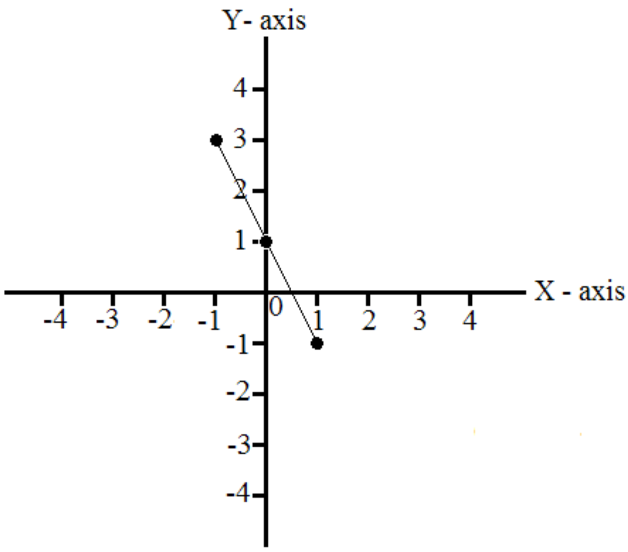


Figure 9.2.2 Graphing the Equation of a Straight Line

Please note that the ***negative slope*** of the line in the graph agrees with -2 coefficient of the x in the equation. That is to say that m = -2.

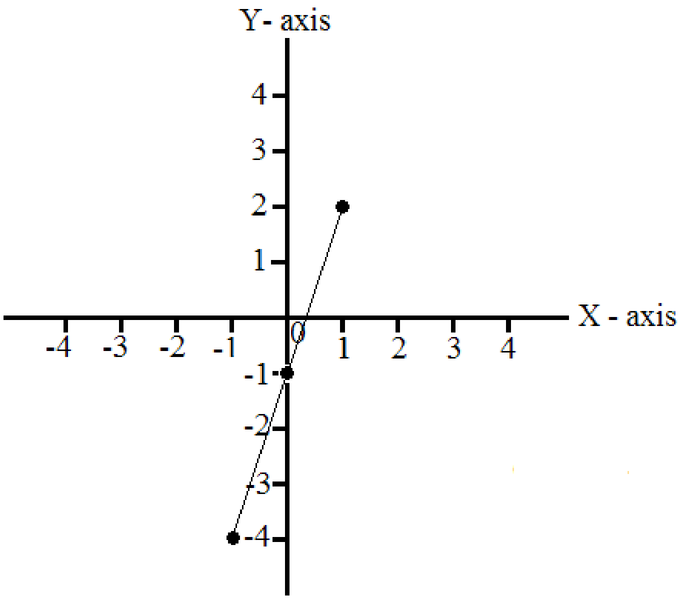
The line of the graph intersects the y-axis at y = 1. This agrees with y-intercept in the equation of one. That is to say that b = 1.

Let's do some examples!

Example 9.2.1

Write the equation of the graph.

x	y
-1	-4
0	-1
1	2



Please note that the set notation for each point has been omitted so you can extract them from the graph.

The points from left to right are (-1, -4), (0, -1), and (1, 2).

The slope calculation of first and second points is as follows.

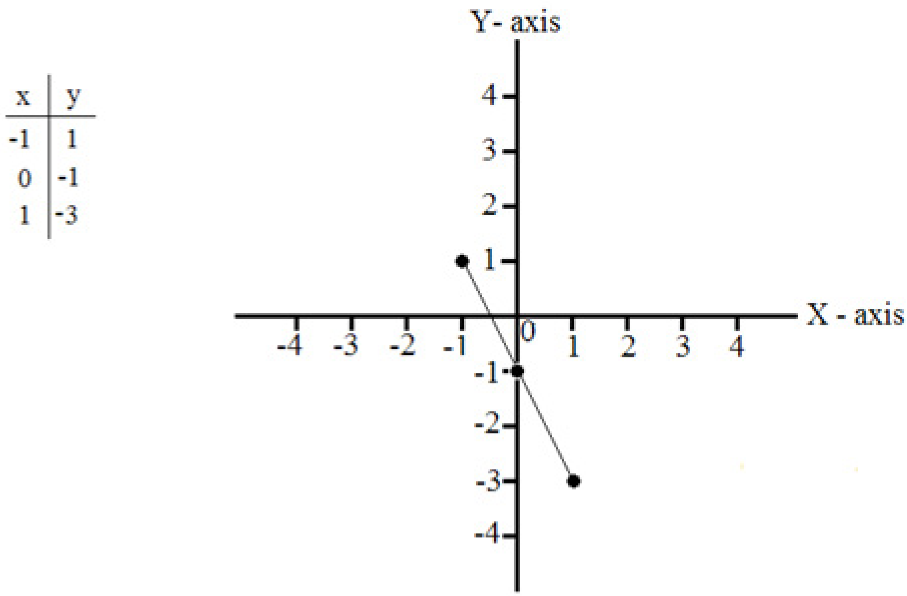
m = -1-(-4)/0-(-1) = 3/1 = 3

The y-intercept is visually observed at y = -1.

y = 3x – 1

Example 9.2.2

Write the equation of the graph.



The points from left to right are (-1, 1), (0, -1), and (1, -3).

The slope calculation of the second and third points is as follows.

m = -3-(-1)/1-0 = -2/1 = -2

The y-intercept is visually observed as -1.

y = -2x – 1

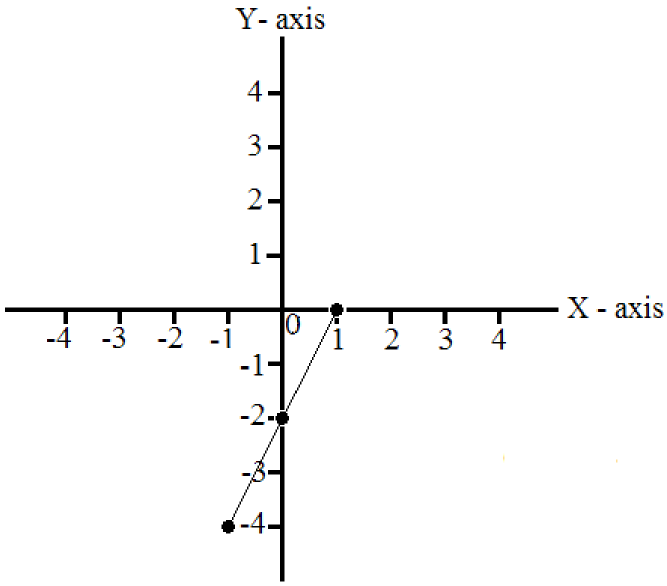
Please note the slope calculation was done with the second and third points on the graph. The slope calculation doesn't demand the use of ***the*** first and second points rather the use of ***a*** first point and ***a*** second point.

To do the slope calculation pick any pair of points and number them from left to right or right to left. I do advise that you choose a convention and then stick to it.

Example 9.2.3

Write the equation of the graph.

x	y
-1	-4
0	-2
1	0



The points from left to right are (-1, -4), (0, -2), and (1, 0).

The slope calculation of the first and third points is as follows.

m = 0-(-4)/1-(-1) = 4/2 = 2

The y-intercept is visually observed as -2.

$$y = 2x - 2$$

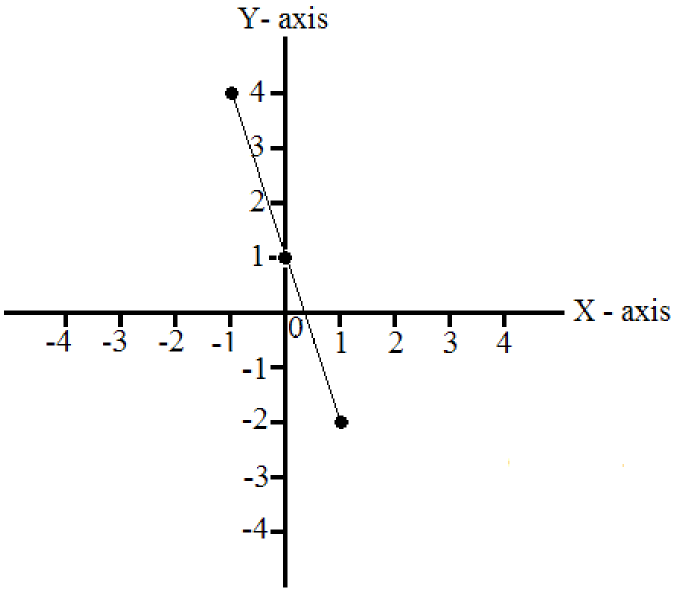
Please note that any of the graphs we have been drawing can be extended infinitely in the positive and negative directions. In this graph by graphing points that were on or below the x-axis we still calculated a positive slope.

Again, the slope calculation can be done with any two sets of points.

Example 9.2.4

Write the equation of the graph.

x	y
-1	4
0	1
1	-2



The points from left to right are (-1, 4), (0, 1), and (1, -2).

The slope calculation of the first and second points is as follows.

$$m = 1\text{-}4/0\text{-}(-1) = -3/1 = -3$$

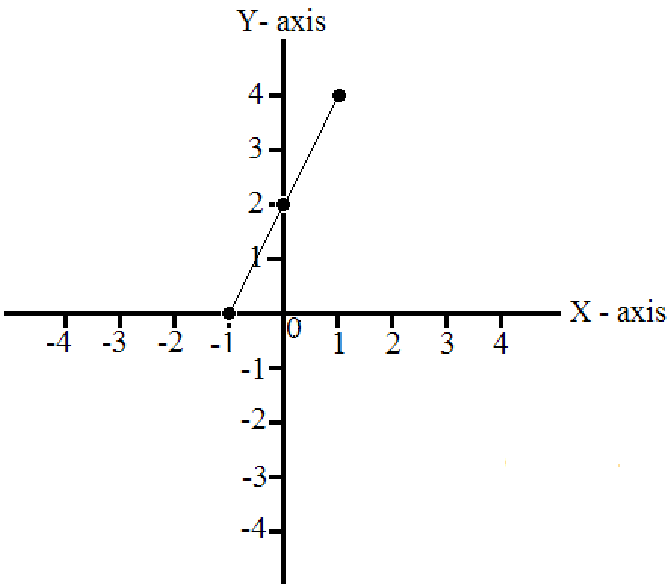
The y-intercept is visually observed as 1.

$$\boldsymbol{y = -3x + 1}$$

Example 9.2.5

Write the equation of the graph.

x	y
-1	0
0	2
1	4



The points from left to right are (-1, 0), (0, 2), and (1, 4).

The slope calculation of the second and third points is as follows.

$$m = 4\text{-}2/1\text{-}0 = 2/1 = 2$$

The y-intercept is visually observed as 2.

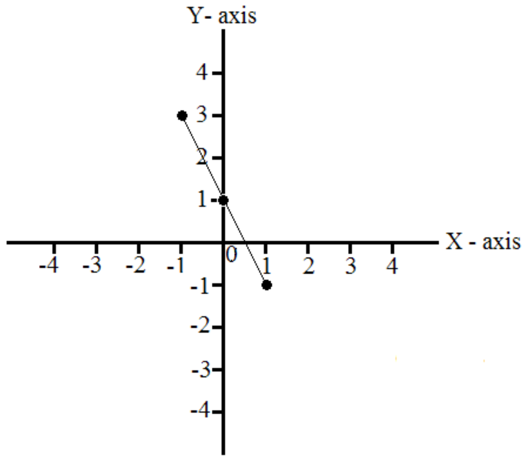
$$\boldsymbol{y = 2x + 2}$$

Example 9.2.6

Graph the equation y = -2x +1.

When x = -1, y = 3 so (-1, 3). When x = 0, y = 1 so (0, 1). When x = 1, y = -1.

x	y
-1	3
0	1
1	-1



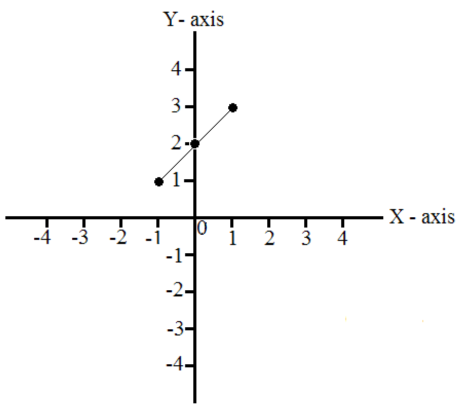
Example 9.2.7

Graph the equation x = y – 2.

Solve the equation for y. x = y – 2 to x + 2 = y to y = x +2

When x = -1, y = 1 so (-1, 1). When x = 0, y = 2 so (0, 2). When x = 1, y = 3 so (1, 3).

x	y
-1	1
0	2
1	3



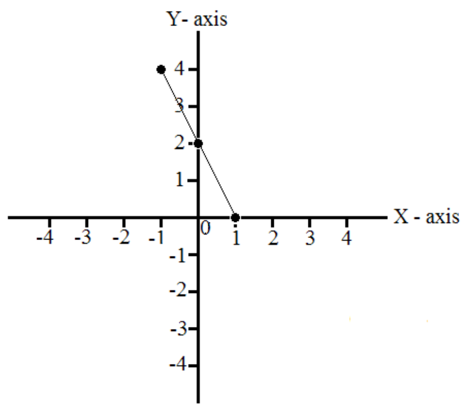
Example 9.2.8

Graph the equation 2x = 2 – y.

Solve the equation for y. 2x = 2 – y to 2x + y = 2 to y = 2 – 2x to y = -2x +2

When x = -1, y = 4 so (-1, 4). When x = 0, y = 2 so (0, 2). When x = 1, y = 0.

x	y
-1	4
0	2
1	0



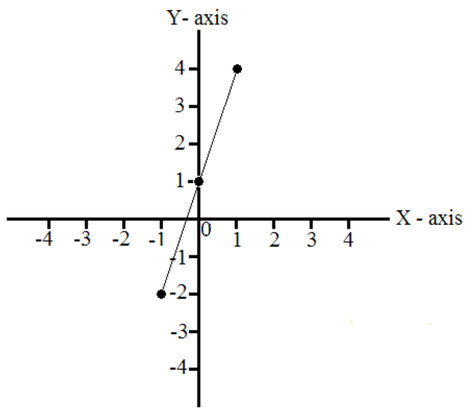
Example 9.2.9

Graph the equation 3y = 9x + 3.

Solve the equation for y. 3y = 9x + 3 to y = 3x + 1

When x = -1, y = -2 so (-1, -2). When x = 0, y= 1 so (0, 1). When x = 1, y = 4 so (1, 4).

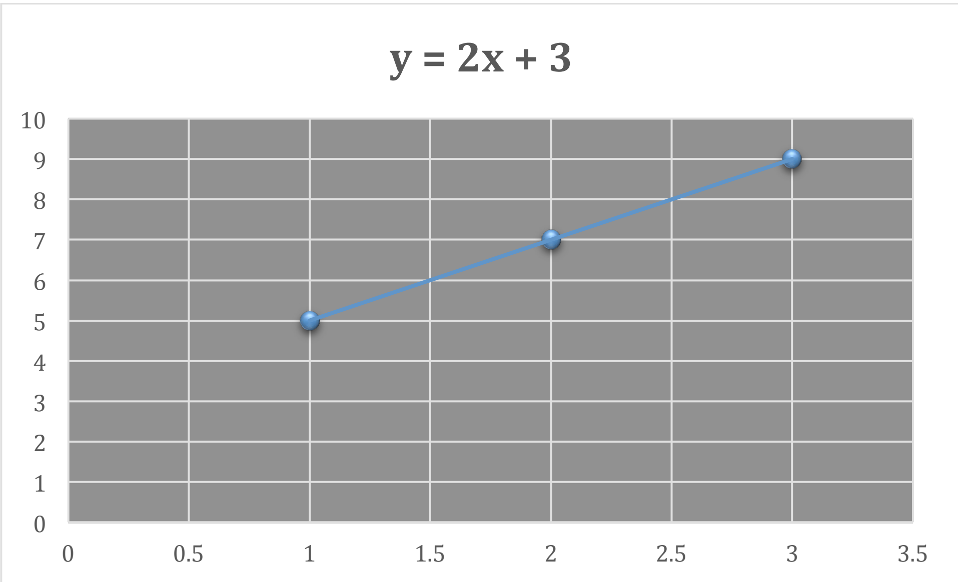
x	y
-1	-2
0	1
1	4



Example 9.2.10

Graph the equation y = 2x +3

When x = 1, y = 5 so (1, 5). When x = 2, y = 7 so (2, 7). When x = 3, y = 9 so (3, 9).



All of the graphs presented so far were painstakingly created using Paint. This problem is an example of using XL to graph an equation.

In XL insert your x values into the A column and your y values into the B column.

Highlight your data.

Select the x,y scatter without a line option from the graph wizard.

Connect your data points by adding a trend line.

9.3 Linear Equations: The Point-Slope Form

The ***point-slope*** form is another algebraic expression of a straight line.

$$y - y_1 = m(x - x_1)$$

To solve the above equation for a specific equation of a line you must be given a point on that line ***(x₁, y₁)*** and the slope ***(m)***.

Example 9.3.1

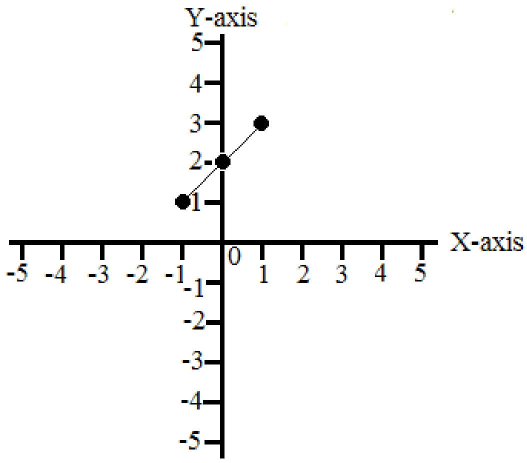
Given the point (1, 3) and a slope of 1, solve for and graph the equation of the straight line.

$$y - 3 = 1(x - 1)$$

Solve the equation for y in terms of x.

$$y - 3 = x - 1 \; ; \; \boldsymbol{y = x + 2}$$

x	y
-1	1
0	2
1	3



Example 9.3.2

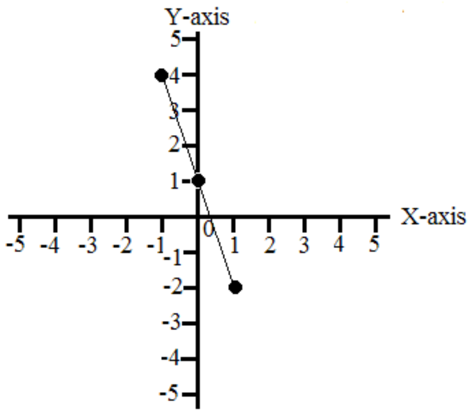
Given the point (1, -2) and a slope of -3, solve for and graph the equation of the straight line.

$$y - (-2) = -3(x - 1)$$

Solve the equation for y in terms of x.

$$y + 2 = -3x + 3; \; \boldsymbol{y = -3x + 1}$$

x	y
-1	4
0	1
1	-2



Example 9.3.3

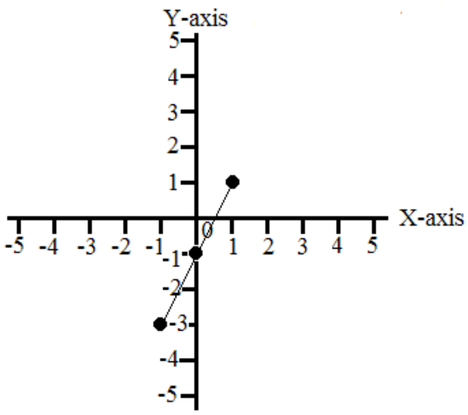
Given the point (1, 1) and a slope of 2, solve for and graph the equation of the straight line.

$$y - 1 = 2(x - 1)$$

Solve the equation for y in terms of x.

$$y - 1 = 2x - 2; \; \boldsymbol{y = 2x - 1}$$

x	y
-1	-3
0	-1
1	1



Example 9.3.4

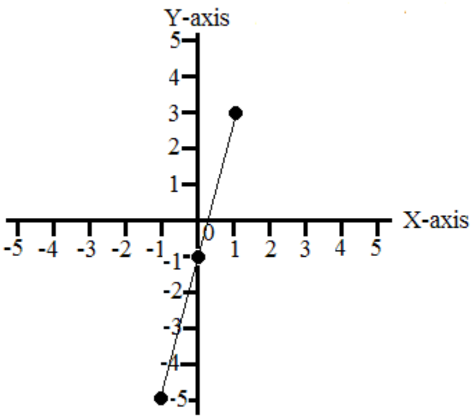
Given the point (1, 3) and a slope of 4, solve for and graph the equation of the straight line.

$y - 3 = 4(x - 1)$

Solve for y in terms of x.

$y - 3 = 4x - 4$; $\boldsymbol{y = 4x - 1}$

x	y
-1	-5
0	-1
1	3



Example 9.3.5

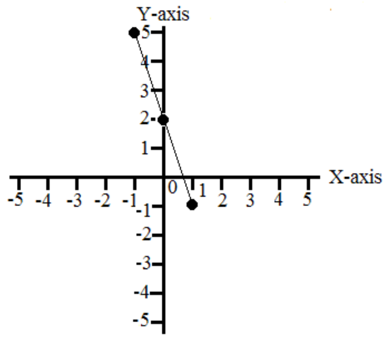
Given the point (0, 2) and a slope of -3, solve for and graph the equation of the straight line.

$y - 2 = -3(x - 0)$

Solve for y in terms of x.

$y - 2 = -3x$; $\boldsymbol{y = -3x + 2}$

x	y
-1	5
0	2
1	-1



9.4 Horizontal and Vertical Graphs

A ***horizontal graph*** of a line is a line that is parallel to the x-axis.

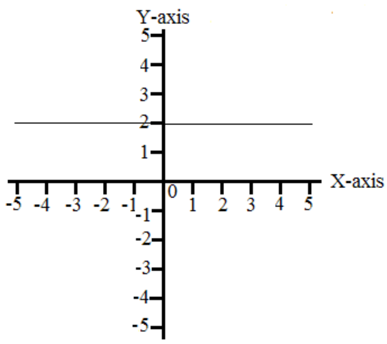


Figure 9.4.1 A Horizontal Graph

No matter what x value you choose on the above graph, the value for y is 2. The reason for this is that the graph has nothing to do with x values.

With no x values and y always equal to 2 only one equation can apply.

$$y = 2$$

A ***vertical graph*** of a line is a line that is parallel to the y-axis.

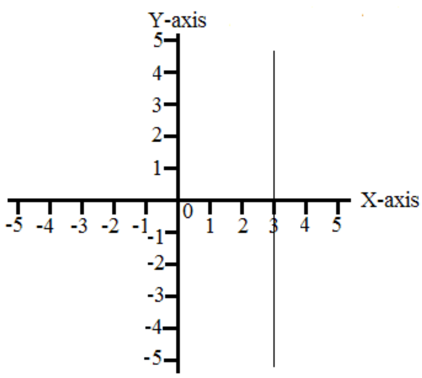


Figure 9.4.2 A Vertical Graph

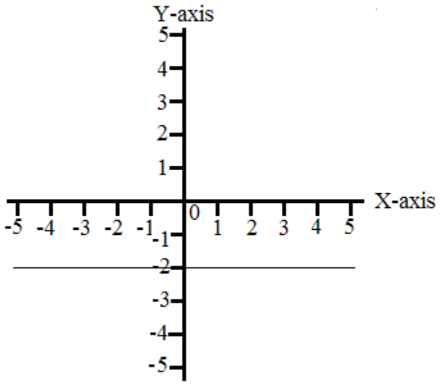
No matter what y value you choose on the above graph, the value for x is 3. The reason for this is that the graph has nothing to do with y values.

With no y values and x always equal to 3 only one equation can apply.

$x = 3$

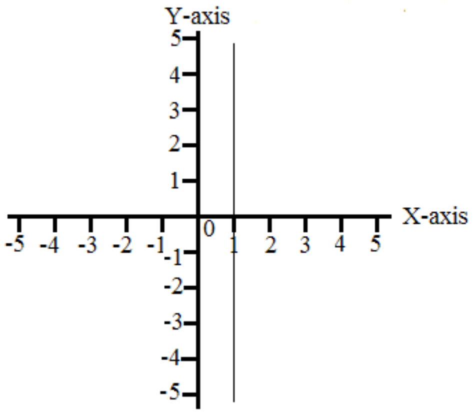
Example 9.4.1

Graph the equation y = -2.



Example 9.4.2

Graph the equation x = 1.

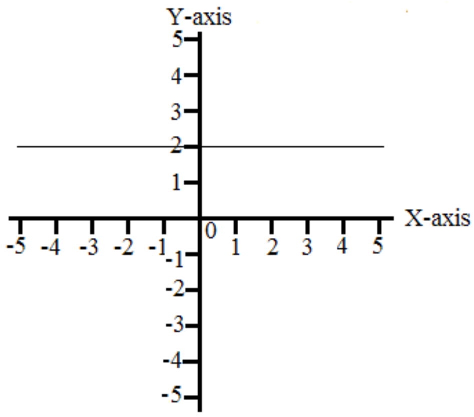


Example 9.4.3

Graph the equation y + 2x = 2(x + 1).

Solve for y in terms of x.

y + 2x = 2(x + 1); y + 2x = 2x +2; ***y = 2***

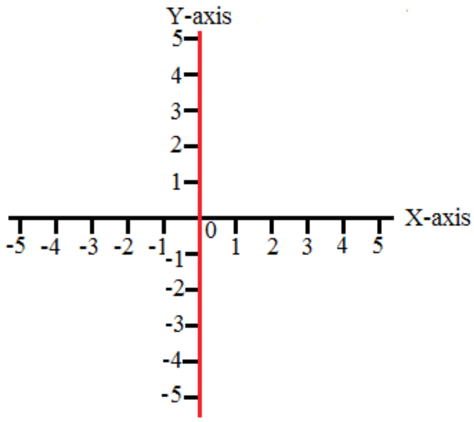


Example 9.4.4

Graph the equation x + y + 4 = -(-4 – y).

Solve for x in terms of y.

x + y +4 = 4 + y; ***x = 0***



x is equal to the y-axis.

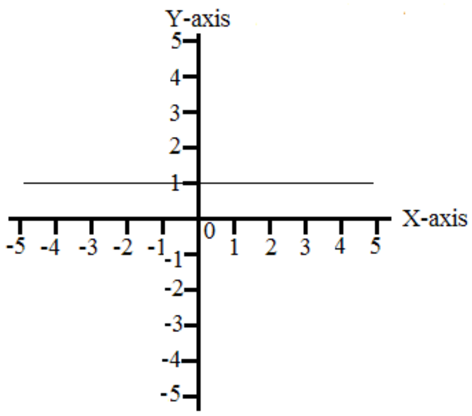
No matter what value of y that you choose x = 0.

Example 9.4.5

Graph the equation 2y + 2x = 2(x + 1)

Solve for y in terms of x.

$2y + 2x = 2x + 2; 2y = 2;$ $\boldsymbol{y = 1}$



9.5 Parallel and Perpendicular Graphs

Parallel graphs are graphs that have the same slopes.

$$m_1 = m_2$$

Figure 9.5.1 shows the graphs of y = 2x + 1 and y = 2x – 2.

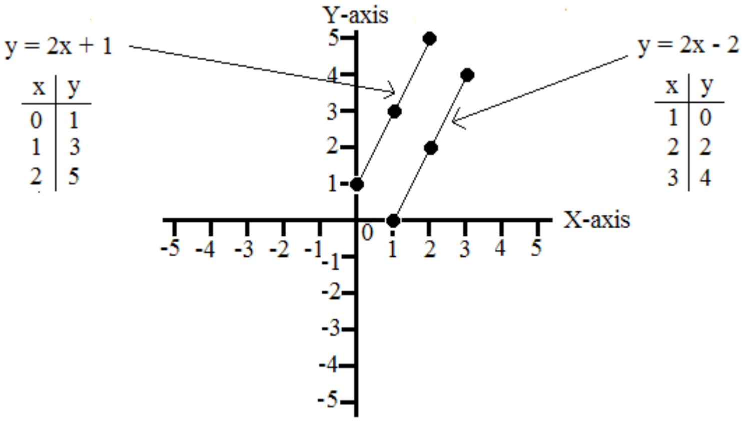
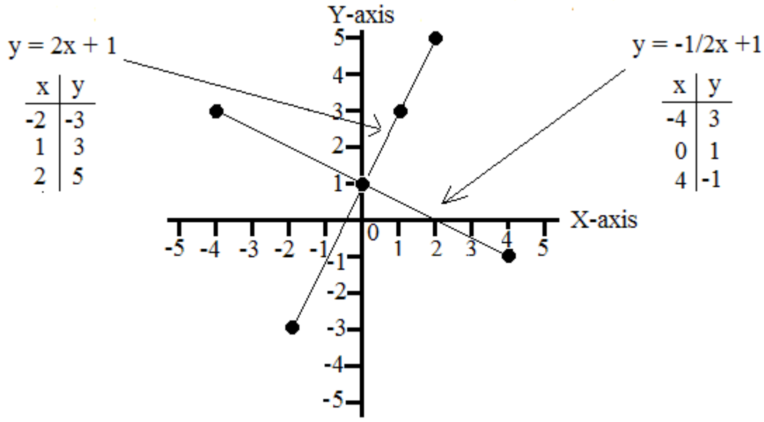


Figure 9.5.1 Parallel Graphs

Perpendicular graphs have slopes that are negative reciprocals.

$m_1 = -1/m_2$

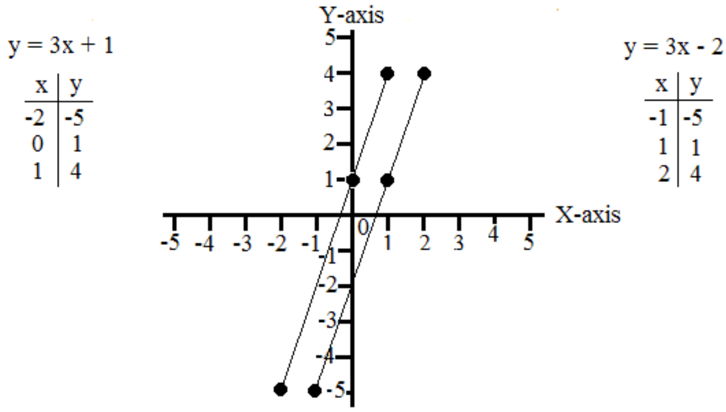
Figure 9.5.2 Shows the graphs of y = 2x + 1 and y = -1/2x + 1



Example 9.5.1

Graph the two equations to determine if they are parallel, perpendicular, or neither.

y = 3x + 1 and y = 3x -2

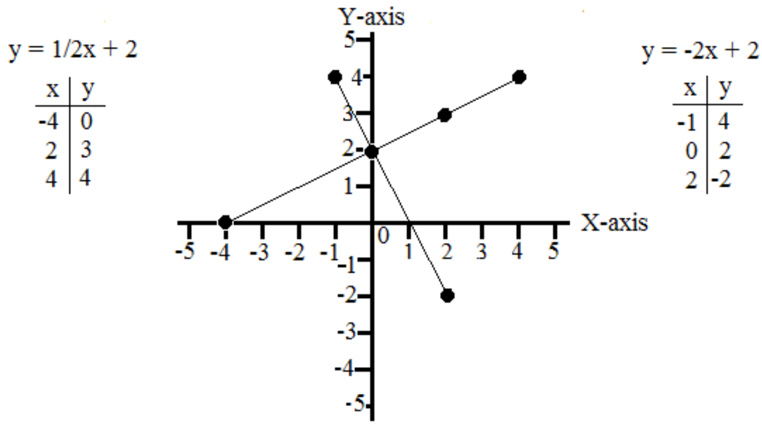


Parallel

Example 9.5.2

Graph the two equations to determine if they are parallel, perpendicular, or neither.

y = 1/2x + 2 and y = -2x +2

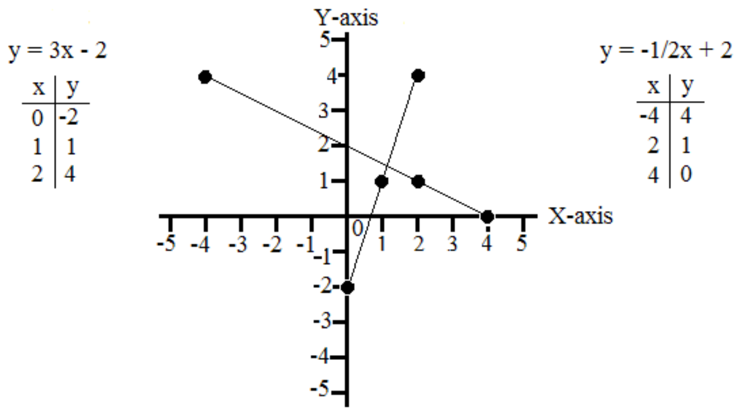


Perpendicular

Example 9.5.3

Graph the two equations to determine if they are parallel, perpendicular, or neither.

$y = 3x - 2$ and $y = -1/2x + 2$



Neither

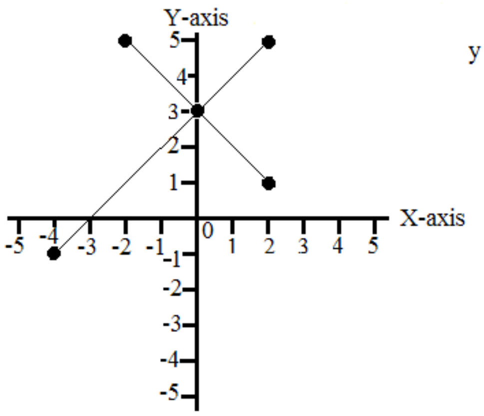
Example 9.5.4

Graph the two equations to determine if they are parallel, perpendicular, or neither.

y = x + 3 and y = -x + 3

y = x + 3

x	y
-4	-1
0	3
2	5



y = -x + 3

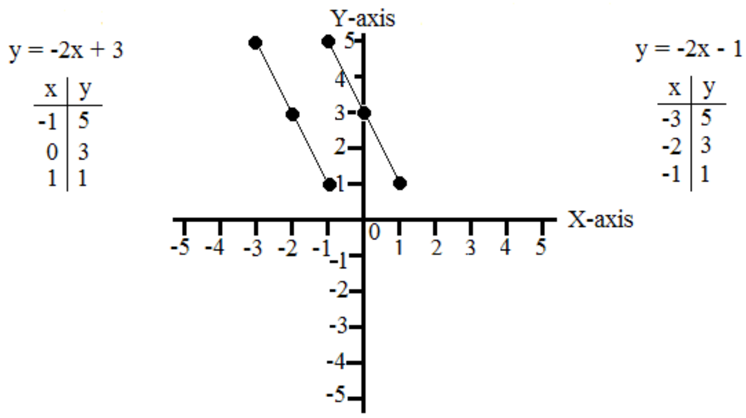
x	y
2	1
0	3
-2	5

Perpendicular (The negative reciprocal of 1 is -1.)

Example 9.5.5

Graph the two equations to determine if they are parallel, perpendicular, or neither.

$y = -2x + 3$ and $y = -2x - 1$



Parallel

9.6 Graphing Linear Inequalities

Graphing linear inequalities with two unknowns is similar to graphing an equation with two unknowns.

The graph of an inequality with one unknown is a dot on the number line and a line drawn from that dot in either a negative or positive direction.

The dots would be open to signify that the number was not in the solution or solid to signify that the number was in the solution.

The graph of a linear inequality with two unknowns would be a ***solid line*** if the line was included in the solution or a ***dashed line*** if the line was not included in the solution.

The graph is shaded ***above the line for $>$ and $\geq$ graphs and below the line for $<$ and $\leq$ graphs***.

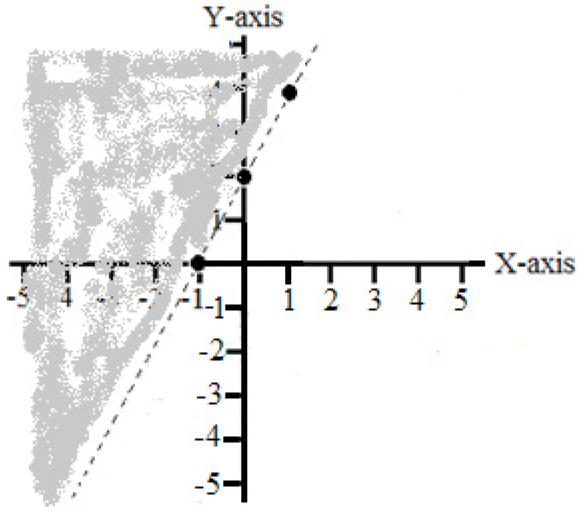
The following examples will demonstrate these graph conditions.

Example 9.6.1

Graph the inequality y > 2x + 2.

The small data table used with equations can still be used here since no sign of equality is inherent to the data table.

x	y
-1	0
0	2
1	4

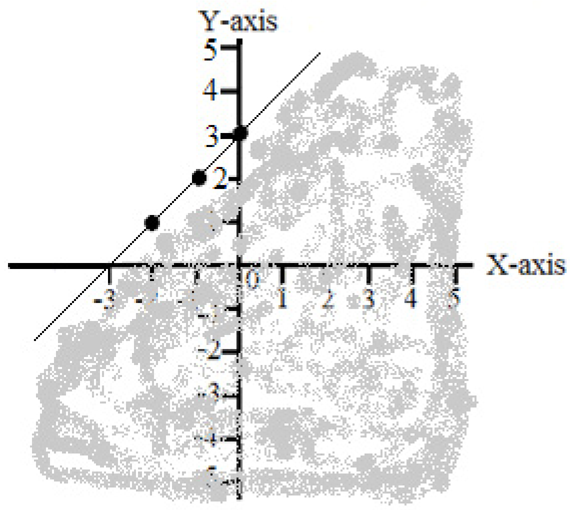


A greater than (>) requires a dashed line and shading above that line.

Example 9.6.2

Graph the inequality y ≤ x + 3.

x	y
-2	1
-1	2
0	3

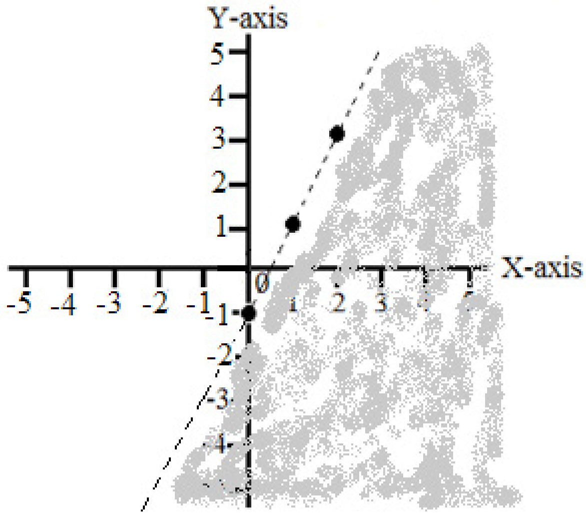


The less than or equal to (≤) requires a solid line and shading below that line.

Example 9.6.3

Graph the inequality y < 2x – 1.

x	y
2	3
1	1
0	-1

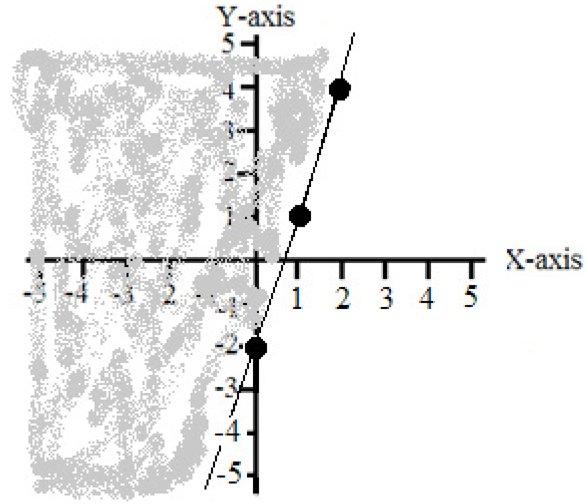


The less than (<) sign requires a dashed line and shading below that line.

Example 9.6.4

Graph the inequality y ≥ 3x – 2

x	y
0	-2
1	1
2	4

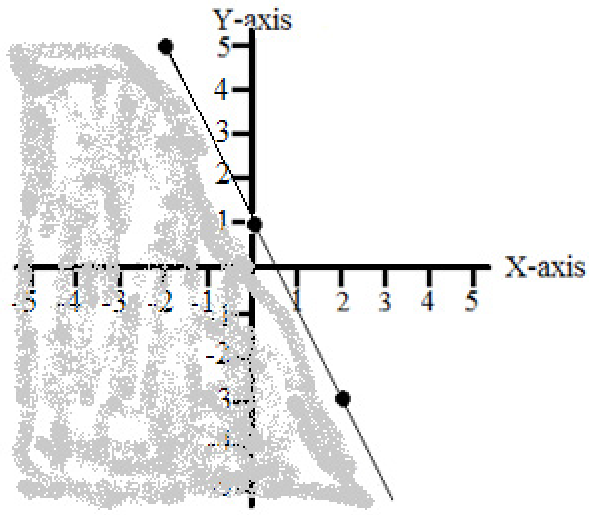


The greater than or equal to (≥) requires a solid line and shading above that line.

Example 9.6.5

Graph the inequality y ≤ -2x + 1.

x	y
-2	5
0	1
2	-3

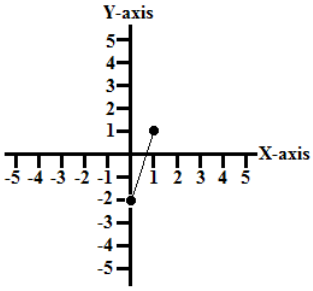


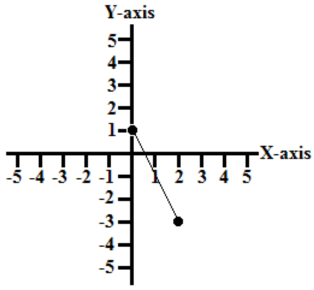
The less than or equal to (≤) requires a solid line and shading below that line.

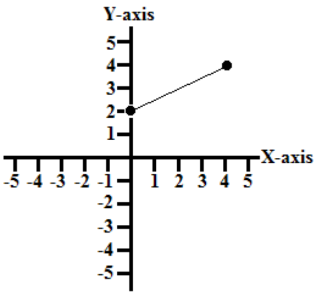
9.7 Chapter 9 End of Chapter Exercises

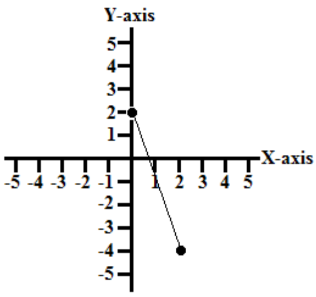
Answer the following questions as directed. The answers are in Appendix A at the end of the text.

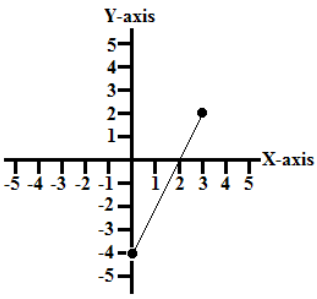
9.2. Write the equations for the following graphs.

1) 

2) 

3) 

4) 

5) 

Graph the following equations.

6. $y = -x + 4$
7. $y = 2x - 2$
8. $y = 4x - 1$
9. $y = -1/2x + 1$
10. $y = -2x - 2$

9.3. Given the following points and slopes solve for and graph the equations for the following straight lines.

1) $(1, -2)$ $m = -2$
2) $(0, 3)$ $m = 2$
3) $(2, -2)$ $m = 3$
4) $(1, 3)$ $m = -1$
5) $(2, -3)$ $m = -2$

9.4. Solve and graph the following equations. Declare whether the graphs are horizontal, vertical, or neither.

1) $y + 3x = 3(x + 1)$
2) $x - 2y = -2(y + 3)$
3) $x + 3y = 2(y + 2) + y$
4) $y = 3(x - 1) - x$
5) $y + 2x = 4(x + 1) - 2x$

9.5. Graph the following sets of equations to determine if they are parallel, perpendicular, or neither.

1) $y = 2x + 3$	and	$y = 2x$
2) $y = -1/2x + 2$	and	$y = 2x - 1$
3) $y = -2x + 2$	and	$y = -1/2x + 1$
4) $y = 3x + 1$	and	$y = 3x - 1$
5) $y = \times + 4$	and	$y = -x + 4$

9.6. Graph the following inequalities. Be certain to include proper shading and solid or dashed lines with each graph.

1) $y \geq \times + 1$
2) $y < 2x - 4$
3) $2y - 4 > -x + 2$
4) $3y \leq y + \times - 2$
5) $y \geq \times$

10 Exponents and Radicals

10.1 Whole Integer Positive Exponents

In section 3.5 of this text the concept of ***exponents*** was introduced. In that section the consideration of exponents was restricted to ***whole integer positive exponents***.

Please review that section with special emphasis on the rules for using exponents. As we observe other forms of exponents these rules will not change.

10.2 Negative and Zero Exponents

There are three special conditions for exponents that I will introduce at this time. The first special condition is raising a number to the ***exponent of zero***.

Any number raised to the zeroth power is equal to one.

$2^0 = 1$, $200^0 = 1$, $5{,}000^0 = 1$, $x^0 = 1$

The second special condition is raising a number to the ***exponent of one***.

Any number raised to an exponent of one is equal to that number.

$4^1 = 4$, $67^1 = 67$, $582^1 = 582$, $x^1 = x$

The third special condition is raising a number to a ***negative exponent***.

Any number raised to a negative exponent is equal to the reciprocal of the positive value of that exponent.

$4^{-2} = 1/4^2 = 1/16$, $5^{-3} = 1/5^3 = 1/125$, $x^{-4} = 1/x^4$

As always numerous examples is the best way to familiarize one with these concepts.

Example 10.2.1

Evaluate the expression 3^{-3}.

$3^{-3} = 1/3^3 = 1/27$

Example 10.2.2

Evaluate the expression 64^0.

$$64^0 = 1$$

Example 10.2.3

Evaluate the expression -32^1.

$$-32^1 = -32$$

Example 10.2.4

Evaluate the expression x^{-3}.

$$x^{-3} = 1/x^3$$

Example 10.2.5

Evaluate the expression y^0.

$$y^0 = 1$$

Example 10.2.6

Evaluate the expression z^1.

$z^1 = z$

Example 10.2.7

Evaluate the expression $(4^{-2})^0$.

$(4^{-2})^0 = (1/4^2)^0 = (1/16)^0 = 1$

Example 10.2.8

Evaluate the expression $(3^{-2})^1$.

$(3^{-2})^1 = (1/3^2)^1 = (1/9)^1 = 1/9$

Example 10.2.9

Evaluate the expression $(2^{-3})^{-2}$.

$(2^{-3})^{-2} = (1/2^3)^{-2} = (1/8)^{-2} = 1/(1/8)^2 = 1/(1/64) = 64$

Example 10.2.10

Evaluate the expression.

$(3^{-2})^{-2} = (1/3^2)^{-2} = (1/9)^{-2} = 1/(1/9)^2 = 1/(1/81) = 81$

10.3 Rational Exponents: The nth Root

A ***rational exponent*** is an exponent in the form of a fraction.

Consider the following two expressions.

$$a^{1/n} = \sqrt[n]{a} \qquad a^{m/n} = \left(\sqrt[n]{a}\right)^m$$

Now consider the second expression using actual numbers.

$$4^{3/2} = \left(\sqrt[2]{4}\right)^3$$

In all of the above expressions the ***exponential*** expression is to the left of the equals sign and the ***radical*** expression is to the right of the equals sign.

Now let's solve both the exponential and the radical expressions from the above numerical expression.

$$4^{3/2} = (4^{1/2})^3 = (\sqrt{4})^3 = 2^3 = 8$$

Now for some more examples.

Example 10.3.1

Evaluate the following expression.

$$a^{3/2}$$

Express as "a" cubed times "a" to the 1/2.

$$(a^3)^{1/2}$$

Express as the square root of "a" cubed.

$$\boldsymbol{\sqrt{a^3}}$$

Example 10.3.2

Evaluate the following expression.

$$(8x^3)^{2/3}$$

Express as the cubed root of $8x^3$ squared.

$$\left(\sqrt[3]{8x^3}\right)^2$$

Express as the cubed root of 8 times the cubed root of x^3 quantity squared.

$$(2x)^2$$

Express as 2 squared times x squared.

$$\boldsymbol{4x^2}$$

Example 10.3.3

Evaluate the following expression.

$$(16y^4)^{1/4}$$

Express as the fourth root of $16y^4$.

$$\sqrt[4]{16y^4}$$

Express as the fourth root of 16 times the fourth root of y^4.

$$2y$$

Example 10.3.4

Evaluate the following expression.

$$(3^3)^{1/3}$$

Express as the cubed root of 3^3.

$$\sqrt[3]{3^3}$$

Express as the cubed root of 27.

$$\sqrt[3]{27}$$

Express as 3.

3

Example 10.3.5

Evaluate the following expression.

$(125z^3)^{1/3}$

Express as the cubed root of $125z^3$.

$$\sqrt[3]{125\,z^3}$$

Express as the cubed root of 5^3z^3.

$$\sqrt[3]{5^3z^3}$$

Express as 5z.

5z

Example 10.3.6

Evaluate the following expression.

$(64^{1/3})^{1/2}$

Express as the squared root of $64^{1/3}$.

$$\sqrt[2]{64^{1/3}}$$

Express as the squared root of 4.

$$\sqrt[2]{4}$$

Express as 2.

2

Example 10.3.7

Evaluate the following expression.

$(64^{1/6}x^2)^{1/2}$

Express as $(2x^2)^{1/2}$.

$(2x^2)^{1/2}$

Express as the squared root of $(2x^2)$.

$$\sqrt{2}\,x$$

Example 10.3.8

Evaluate the following expression.

$(125y^6x^9)^{1/3}$

Express as the cubed root of $125y^6x^9$.

$$\sqrt[3]{125y^6x^9}$$

Express as the cubed root of $5^3y^3y^3x^3x^3x^3$.

$$\sqrt[3]{5^3y^3y^3x^3x^3x^3}$$

Express as $5 \cdot y \cdot y \cdot x \cdot x \cdot x$.

$5y^2x^3$.

Example 10.3.9

Evaluate the following expression.

$$(64x^4y^5)^{1/3}$$

Express as the cubed root of $64x^4y^5$.

$$\sqrt[3]{64x^4y^5}$$

Express as the cubed root of $4^3x^3xy^3y^2$.

$$\sqrt[3]{4^3x^3xy^3y^2}$$

Express as $4xy$ times the cubed root of xy^2.

$$4xy\sqrt[3]{xy^2}$$

Example 10.3.10

Evaluate the following expression.

$(16x^6y^5z^9)^{1/4}$

Express as the fourth root of $16x^6y^5z^9$.

$$\sqrt[4]{16x^6y^5z^9}$$

Express as the fourth root of $2^4x^4x^2y^4yz^4z^4z$.

$$\sqrt[4]{2^4x^4x^2y^4yz^4z^4z}$$

Express as $2xyz^2$ times the fourth root of x^2yz.

$$2xyz^2\sqrt[4]{x^2yz}$$

10.4 Mathematical Operations with Exponents and Radicals

Addition and ***subtraction*** of exponents and radicals is done only with the coefficients of the exponent and radical.

Example 10.4.1

$3x^2 + 5x^2 = 8x^2$

These are additive because the base " ***x*** " and the exponent " ***2*** " are the same for each expression.

Example 10.4.2

$4\sqrt{5} - 2\sqrt{5} = 2\sqrt{5}$

These are subtractive because the same number "***5*** " is taken to the same root " ***2*** " for each expression.

Multiplication of exponents is governed by the following equation.

$$a^m \cdot a^n = a^{m+n}$$

Division of exponents is governed by the following equation.

$$a^m \div a^n = a^{m-n}$$

Raising an exponent to a power is governed by the following equation.

$$(a^m)^n = a^{m \cdot n}$$

When using the previous operations on radicals express the radicals as ***rational exponents***.

Now let's do some more examples!

Example 10.4.3

Solve the following equation.

$$x^2 \cdot x^4 =$$

Express as $\times\ ^{2+4} = x^6$.

$$x^6$$

Example 10.4.4

Solve the following equation.

$$3y^3 \cdot 4y^2 =$$

Express as $(3{\cdot}4)y^{3+2} = 12y^5$.

$$12y^5$$

Example 10.4.5

Solve the following equation.

$$a^6 \div a^4 =$$

Express as $a^{6-4} = a$

$$a^2$$

Example 10.4.6

Solve the following equation.

$$4x^5 \div 2x^3 =$$

Express as $4/2x^{5-3} = 2x^2$.

$$\boldsymbol{2x^2}$$

Example 10.4.7

Solve the following equation.

$$(y^2)^3 =$$

Express as $y^{2 \cdot 3} = y^6$.

$$\boldsymbol{y^6}$$

Example 10.4.8

Solve the following equation.

$$(3x^4)^3 =$$

Express as $3^3x^{4 \cdot 3} = 27x^{12}$.

$\boldsymbol{27x^{12}}$

Example 10.4.9

Solve the following equation.

$$(\sqrt[3]{x})^2 =$$

Express as $x^{1/3 \cdot 2} = x^{2/3}$.

$\boldsymbol{x^{2/3}}$

Example 10.4.10

Solve the following equation.

$$(\sqrt[3]{8y})^2 =$$

Express as $(\sqrt[3]{8})^2(\sqrt[3]{y})^2$.

This simplifies to $(2^2)(y^{1/3 \cdot 2})$.

$\boldsymbol{4y^{2/3}}$

10.5 Chapter 10 End of Chapter Exercises

Answer the following questions as directed. The answers are in Appendix A at the end of the text.

10.2. Evaluate the following expressions.

1) 2^{-2}
2) 49^0
3) -18^{-1}
4) y^{-4}
5) z^0
6) $(4^{-3})^0$
7) s^{-1}
8) $(3^{-3})^1$
9) $(2^{-2})^{-2}$
10) $(4^{-2})^{-1}$

10.3. Evaluate the following expressions.

1) $(49^{1/2})^2$
2) $(x^{1/3})^2$
3) $(27y^3)^{1/3}$
4) $(125z^3)^{2/3}$
5) $(16^{1/4}z^2)^{1/2}$
6) $(125x^6y^9)^{1/3}$
7) $(16s^8t^{12})^{1/4}$
8) $(27x^4y^6z^9)^{1/3}$
9) $(64p^4q^5r^6)^{1/3}$
10) $(32x^6y^{10}z^{12})^{1/5}$

10.4. Solve the following equations.

1) $y^3 \bullet y^4 =$
2) $3x^2 \bullet 4x^3 =$
3) $t^6 \div t^4 =$
4) $6s^3 \div 2s$
5) $12x^6 \div 4x^3 =$
6) $(y^4)^2 =$
7) $(3z^3)^3 =$
8) $(7x^4)^2 =$
9) $(\sqrt[3]{27y})^2$
10) $(\sqrt[4]{16x})^2$

11 Systems of Equations

11.1 Systems of Linear Equations in Two Unknowns: Solutions by Graphing

A ***system of linear equations*** is a set of equations that when graphed on the same coordinate axes show one of the following three outcomes.

A Unique Solution: The equations are defined by graphs that intersect at one unique point and show that the solution for the equations are ***consistent***.

Coincident Solutions: The equations are defined by graphs that coincide with one another and show an infinite number of ***consistent*** solutions.

Parallel Lines: The equations are defined by graphs that never intersect with one another. This defines an ***inconsistent*** system with no solutions.

For the purpose of this chapter we will be solving only unique solutions to consistent linear systems.

Example 11.1.1

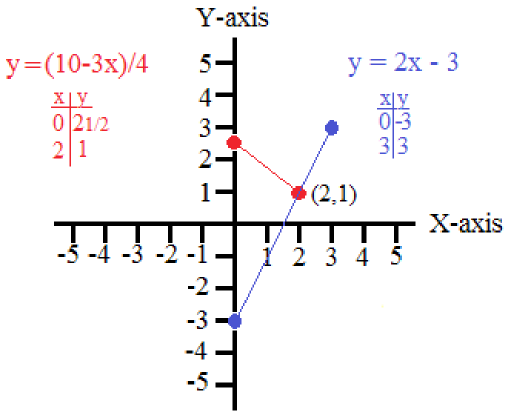
Show that the following two equations have a unique solution by graphing.

$3x + 4y = 10$ and $2x - y = 3$

Arrange the two equations in the $y = mx + b$ format.

$y = (10 - 3x)/4$ and $y = 2x - 3$

Graph the two equations and find the point of intersection.



The two graphs intersect at (2,1) so the unique solution for the system of equations is $\boldsymbol{x = 2}$ and $\boldsymbol{y = 1}$.

Example 11.1.2

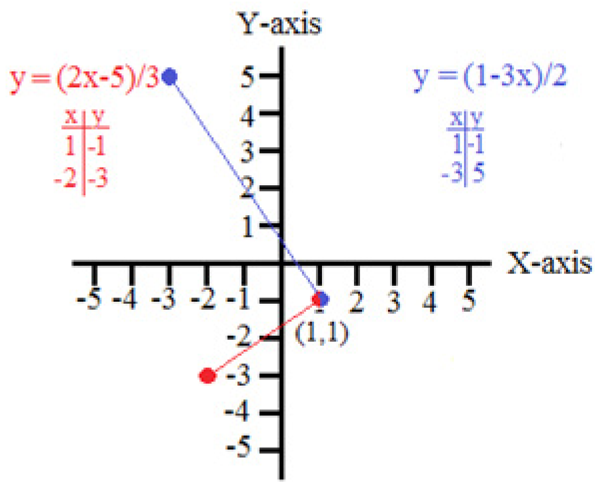
Show that the following two equations have a unique solution by graphing.

$2x - 3y = 5$ and $3x + 2y = 1$

Arrange the two equations in the $y = mx + b$ format.

$y = (2x-5)/3$ and $y = (1-3x)/2$

Graph the two equations and find the point of intersection.



The two graphs intersect at (1,-1) so the unique solution for the system of equations is ***x* = *1*** and ***y* = *-1***.

Example 11.1.3

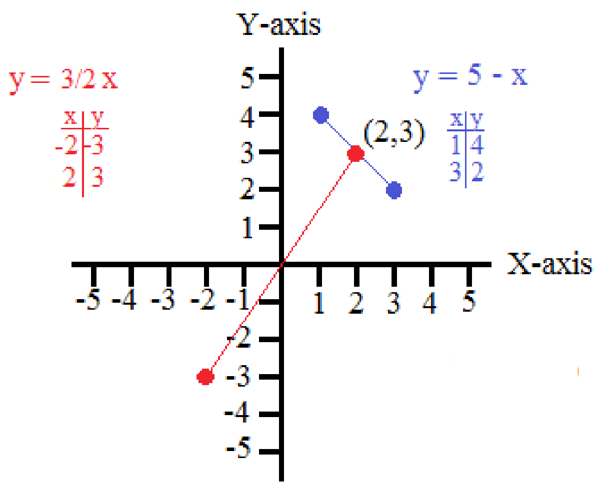
Show that the following two equations have a unique solution by graphing.

$3x - 2y = 0$ and $x + y = 5$

Arrange the two equations in the $y = mx + b$ format.

$y = 3/2x$ and $y = 5 - x$

Graph the two equations and find the point of intersection.



The two graphs intersect at (2, 3) so the unique solution for the system of equations is ***x* = *2*** and ***y* = *3***.

Example 11.1.4

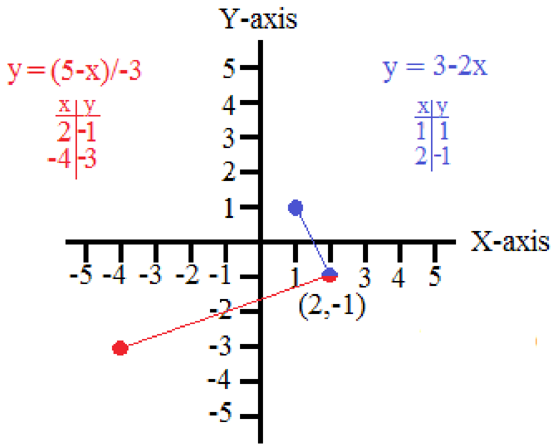
Show that the following two equations have a unique solution by graphing.

$$x - 3y = 5 \qquad \text{and} \qquad 2x + y = 3$$

Arrange the two equations in the $y = mx + b$ format.

$$y = (5-x)/-3 \qquad \text{and} \qquad y = 3 - 2x$$

Graph the two equations and find the point of intersection.



The two graphs intersect at (2, -1) so the unique solution for the system of equations is ***x* = 2** and ***y* = -1**.

Example 11.1.5

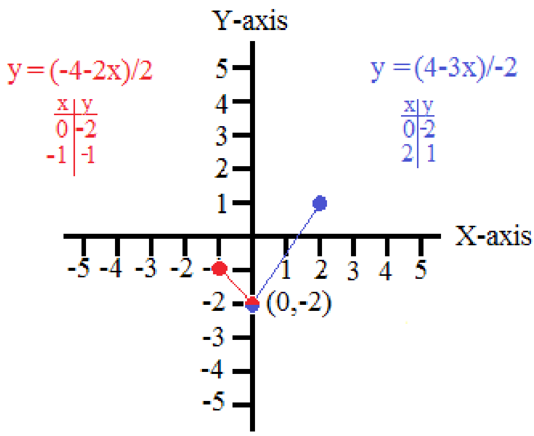
Show that the following two equations have a unique solution by graphing.

$2x + 2y = -4$ and $3x - 2y = 4$

Arrange the two equations in the $y = mx + b$ format.

$y = (-4-2x)/2$ and $y = (4-3x)/-2$

Graph the two equations and find the point of intersection.



The two graphs intersect at (0, -2) so the unique solution for the system of equations is ***x* = 0** and ***y* = -2**.

11.2 Systems of Linear Equations in Two Unknowns: Solutions by Substitution

Substitution is the method for solving systems of linear equations in two unknowns by expressing one of the unknowns in an equation in terms of the other unknown in that equation.

This expression of the unknown is then ***substituted*** into the other equation resulting in an equation with a single unknown.

The value for this unknown is then ***substituted*** into any equation with both unknowns to solve for the other unknown.

Example 11.2.1

Show that the following two equations have a unique solution by substitution.

$2x - y = 5$ and $x + 3y = -1$

Express y in the first equation in terms of x.

$y = 2x - 5$

Substitute $2x - 5$ for y in the second equation.

$x + 3(2x - 5) = -1$
$x + 6x - 15 = -1$
$7x = 14$
$\boldsymbol{x = 2}$

Substitute 2 for x in the equation $y = 2x - 5$.

$y = 2(2) - 5$
$\boldsymbol{y = -1}$

Example 11.2.2

Show that the following two equations have a unique solution by substitution.

$y + 3x = 3$ and $2x - y = 2$

Express y in the first equation in terms of x.

$y = 3 - 3x$

Substitute 3 – 3x for y in the second equation.

$2x - (3 - 3x) = 2$
$2x + 3x - 3 = 2$
$5x = 5$
$\boldsymbol{x = 1}$

Substitute 1 for x in the equation y = 3 – 3x.

$y = 3 - 3(1)$
$\boldsymbol{y = 0}$

Example 11.2.3

Show that the following two equations have a unique solution by substitution.

$2x + 3y = 0$ and $2y - x = -7$

Express y in the first equation in terms of x.

$y = -2/3\ x$

Substitute -2/3 x for y in the second equation.

$2(-2/3\ x) - x = -7$

$-4/3\ x - x = -7$

$-7/3\ x = -7$

x = 3

Substitute 3 for x in the equation y = -2/3 x.

$y = -2/3\ (3)$

y = -2

Example 11.2.4

Show that the following two equations have a unique solution by substitution.

$3x + y = 4$ and $2y - 3x = -10$

Express y in the first equation in terms of x.

$y = 4 - 3x$

Substitute 4 – 3x for y in the second equation.

$2(4 - 3x) - 3x = -10$

$8 - 6x - 3x = -10$

$-9x = -18$

x = 2

Substitute 2 for x in the equation y = 4 – 3x.

$y = 4 - 3(2)$

y = -2

Example 11.2.5

Show that the following two equations have a unique solution by substitution.

$x - 3y = -10$ and $3x + 2y = 3$

Express x in the first equation in terms of y.

$x = -10 + 3y$

Substitute -10 + 3y for x in the second equation.

$3(-10 + 3y) + 2y = 3$
$-30 + 9y + 2y = 3$
$11y = 33$
$\boldsymbol{y = 3}$

Substitute 3 for y in the equation x = -10 + 3y.

$x = -10 + 3(3)$
$\boldsymbol{x = -1}$

11.3 Systems of Linear Equations in Two Unknowns: Solutions by Elimination

Elimination is the method of solving systems of equations of two unknowns by adding or subtracting multiples of one equation to the other equation to ***eliminate*** one of the unknowns.

The value for this unknown is then entered into any other equation containing both unknowns to solve for the other unknown.

Example 11.3.1

Show that the following two equations have a unique solution by elimination.

$2x - y = 4$ and $x + y = -1$

Add the two equations to one another to eliminate y.

$2x - y = 4$
$+(\underline{x + y = -1})$
$3x = 3$
$\boldsymbol{x = 1}$

Solve the second equation for y.

$y = -1 - x$
$y = -1 - 1$
$\boldsymbol{y = -2}$

Example 11.3.2

Show that the following two equations have a unique solution by elimination.

$$3x + y = 5 \qquad \text{and} \qquad x - 3y = 5$$

Subtract three times the second equation from the first to eliminate x.

$$3x + y = 5$$
$$-3(\underline{x - 3y = 5)}$$
$$10\,y = -10$$
$$\boldsymbol{y = -1}$$

Solve the second equation for x.

$$x = 5 + 3y$$
$$x = 5 - 3$$
$$\boldsymbol{x = 2}$$

Example 11.3.3

Show that the following two equations have a unique solution by elimination.

$3x + y = 2$ and $2x - 2y = -4$

Add two times the first equation to the second equation to eliminate y.

$2x - 2y = -4$
$+2(\underline{3x + y = 2})$
$8x = 0$
$\boldsymbol{x = 0}$

Solve the first equation for y.

$y = 2 - 3x$
$y = 2 - 0$
$\boldsymbol{y = 2}$

Example 11.3.4

Show that the following two equations have a unique solution by elimination.

$x + 2y = 4$ and $2x + 2y = 2$

Subtract two times the first equation from the second equation to eliminate x.

$2x + 2y = 2$
$-2(\underline{x + 2y = 4})$
$-2y = -6$
$\boldsymbol{y = 3}$

Solve the first equation for x.

$x = 4 - 2y$
$x = 4 - 6$
$\boldsymbol{x = -2}$

Example 11.3.5

Show that the following two equations have a unique solution by elimination.

$2x - y = -7$ and $x + 3y = 7$

Add three times the first equation to the second equation to eliminate y.

$$\begin{aligned} x + 3y &= 7 \\ +3(\underline{2x - y = -7}&) \\ 7x &= -14 \\ \boldsymbol{x} &\boldsymbol{= -2} \end{aligned}$$

Solve the first equation for y.

$$\begin{aligned} y &= 2x + 7 \\ y &= -4 + 7 \\ \boldsymbol{y} &\boldsymbol{= 3} \end{aligned}$$

11.4 Systems of Linear Equations in Three Unknowns: Solutions by Elimination and Back Substitution

Elimination using ***back substitution*** is a method of solving systems of three equations in three unknowns by eliminating a single unknown in two of the three equations of the system.

Now with a system of two equations in two unknowns, a single unknown can be solved for using elimination or substitution. The remaining two unknowns can then be solved for using substitution.

Please note that, in systems of equations in three unknowns, elimination must be used first to obtain a value for the first unknown. At this point substitution or elimination can be used to solve for the remaining two unknowns.

In the following examples E1, E2, and E3 stand for equations 1, 2, and 3 respectively. The operations of elimination are shown above the arrows between the bracketed systems of three equations.

Example 11.4.1

Show that the following three equations have unique solutions by elimination and back substitution.

$2x - 3y + z = 8$ $x + 4y - z = -3$ $x + y - 2z = -1$

$$\begin{array}{c} E1\\ E2\\ E3 \end{array}\begin{bmatrix} 2x - 3y + z = 8 \\ x + 4y - z = -3 \\ x + y - 2z = -1 \end{bmatrix} \xrightarrow{E1+E2} \begin{array}{c} E1\\ E2\\ E3 \end{array}\begin{bmatrix} 3x + y \quad = 5 \\ x + 4y - z = -3 \\ x + y - 2z = -1 \end{bmatrix} \xrightarrow{E3-2E2} \begin{array}{c} E1\\ E2\\ E3 \end{array}\begin{bmatrix} 3x + y \quad = 5 \\ x + 4y - z = -3 \\ -x - 7y \quad = 5 \end{bmatrix}$$

y = 5 – 3x; 5 = -x – 7(5 – 3x); 5 = -x – 35 + 21x; 20x = 40; **_x = 2_**

y = 5 – 3(2); **_y = -1_**; z = x + 4y + 3; z = 2 – 4 + 3; **_z = 1_**

Example 11.4.2

Show that the following three equations have unique solutions by elimination and back substitution.

2x – y + z = 5 x + y – z = -22 x + y + z = 3

$$\begin{array}{c} E1\\ E2\\ E3 \end{array}\begin{bmatrix} 2x - y + z = 5 \\ x + y - z = -2 \\ 2x + y + z = 3 \end{bmatrix} \xrightarrow{E1+E2} \begin{array}{c} E1\\ E2\\ E3 \end{array}\begin{bmatrix} 3x \quad = 3 \\ x + y - z = -2 \\ 2x + y + z = 3 \end{bmatrix} \xrightarrow{E2+E3} \begin{array}{c} E1\\ E2\\ E3 \end{array}\begin{bmatrix} 3x \quad = 3 \\ 3x + 2y \quad = 1 \\ 2x + y + z = 3 \end{bmatrix}$$

3x = 3; **_x = 1_**; 2y = 1 – 3x; 2y = 1 – 3; 2y = -2; **_y = -1_**;

z = 3 – 2x – y; z = 3 – 2 + 1; **_z = 2_**

Example 11.4.3

Show that the following three equations have unique solutions by elimination and back substitution.

$x - 2y + z = -1$ $2x + y + 2z = 3$ $x + y + 2z = 1$

$$\begin{array}{c} E1 \\ E2 \\ E3 \end{array}\begin{bmatrix} x - 2y + z = -1 \\ 2x + y + 2z = 3 \\ x + y + 2z = 1 \end{bmatrix} \xrightarrow{E2-E3} \begin{array}{c} E1 \\ E2 \\ E3 \end{array}\begin{bmatrix} x - 2y + z = -1 \\ x \qquad = 2 \\ x + y + 2z = 1 \end{bmatrix} \xrightarrow{E3-2E1} \begin{array}{c} E1 \\ E2 \\ E3 \end{array}\begin{bmatrix} x - 2y + z = -1 \\ x \qquad = 2 \\ -x + 5y \qquad = 3 \end{bmatrix}$$

$\underline{\boldsymbol{x = 2}}$; $5y = 3 + x$; $5y = 3 + 2$; $5y = 5$; $\underline{\boldsymbol{y = 1}}$;

$z = -1 - x + 2y$; $z = -1 - 2 + 2$; $\underline{\boldsymbol{z = -1}}$

Example 11.4.4

Show that the following three equations have unique solutions by elimination and back substitution.

$x + 2y - z = 2$ $2x - 2y + z = 1$ $x - y - z = -4$

$$\begin{array}{c} E1 \\ E2 \\ E3 \end{array}\begin{bmatrix} x + 2y - z = 2 \\ 2x - 2y + z = 1 \\ x - y - z = -4 \end{bmatrix} \xrightarrow{E1+E2} \begin{array}{c} E1 \\ E2 \\ E3 \end{array}\begin{bmatrix} 3x \qquad = 3 \\ 2x - 2y + z = 1 \\ x - y - z = -4 \end{bmatrix} \xrightarrow{E2+E3} \begin{array}{c} E1 \\ E2 \\ E3 \end{array}\begin{bmatrix} 3x \qquad = 3 \\ 3x - 3y \qquad = -3 \\ x - y - z = -4 \end{bmatrix}$$

$3x = 3$; $\underline{\boldsymbol{x = 1}}$; $3y = 3x + 3$; $3y = 3 + 3$; $3y = 6$; $\underline{\boldsymbol{y = 2}}$

$z = x - y + 4$; $z = 1 - 2 + 4$; $\underline{\boldsymbol{z = 3}}$

Example 11.4.5

Show that the following three equations have unique solutions by elimination and back substitution.

$2x - y + z = -3$ $x + y - 2z = -1$ $3x + 2y - z = 0$

$$\begin{array}{c} E1 \\ E2 \\ E3 \end{array}\begin{bmatrix} 2x - y + z = -3 \\ x + y - 2z = -1 \\ 3x + 2y - z = 0 \end{bmatrix} \xrightarrow{E1+E2} \begin{array}{c} E1 \\ E2 \\ E3 \end{array}\begin{bmatrix} 3x \qquad - z = -4 \\ x + y - 2z = -1 \\ 3x + 2y - z = 0 \end{bmatrix} \xrightarrow{E3-2E2} \begin{array}{c} E1 \\ E2 \\ E3 \end{array}\begin{bmatrix} 3x \qquad - z = -4 \\ x + y - 2z = -1 \\ x \qquad + 3z = 2 \end{bmatrix}$$

$z = 3x + 4$, $x + 3(3x + 4) = 2$, $x + 9x + 12 = 2$, $10x = -10$, $\underline{\boldsymbol{x = -1}}$,

$z = 3(-1) + 4$, $\underline{\boldsymbol{z = 1}}$, $y = -1 + 2z - x$, $y = -1 + 2 + 1$, $\underline{\boldsymbol{y = 2}}$

Please note that in some examples the result of elimination is a single equation in one unknown. This works as an immediate substitution into a single equation in two unknowns.

11.5 Chapter 11 End of Chapter Exercises

Answer the following questions as directed. The answers are in Appendix A at the end of the text.

11.1. Show that the following systems of two equations have unique solutions by graphing.

1) $2x - y = -1$ and $x - 2y = -5$
2) $x + 3y = 1$ and $2x - y = -5$
3) $x + 2y = -1$ and $2x + 2y = 2$
4) $2x + y = 1$ and $y + x = -1$
5) $2x + y = 2$ and $2y + x = -2$

11.2. Show that the following systems of two equations have unique solutions by substitution.

1) $2x + y = 1$ and $x - 2y = 8$
2) $x - 3y = 0$ and $2x + 3y = 9$
3) $x + 4y = -1$ and $2x - 3y = -2$
4) $y + 2x = -3$ and $2x - 2y = -6$
5) $x + 3y = 1$ and $3x + 4y = -2$

11.3. Show that the following systems of two equations have unique solutions by elimination.

1) $x + 2y = 4$ and $2x - 2y = -10$
2) $3x + y = -8$ and $2x - y = -7$
3) $x - y = 3$ and $2x - 3y = 8$
4) $2x - y = -1$ and $x + y = 4$
5) $2x - y = 6$ and $3x + 2y = 2$

11.4. Show that the following systems of three equations have unique solutions by elimination and back substitution.

1) $x + 2y -2z = -4$, $2x - y + z = 7$, and $x + y + z = 1$
2) $2x + y - z = 1$, $x - y + z = 5$, and $x + 2y + 2z = 0$
3) $x + 2y - z = -3$, $2x + y - z = -4$, and $x + 2y - 2z = -5$
4) $x - 2y + z = 0$, $2x + y - 2z = 6$, and $x + y -z = 4$
5) $2x - y + z = 9$, $x + 2y - 2z = -3$, and $3x + 4y - 2z = -1$

12 Appendix A

12.1 Answers to End of Chapter Exercises

3.1	1)	15	5)	34
	2)	32	6)	11
	3)	16	7)	9
	4)	8	8)	23
3.2	1)	proper	11)	1/1,000
	2)	improper	12)	1,000
	3)	proper1	3)	1/10,000
	4)	mixed	14)	100,000
	5)	improper	15)	0.667 ; 66.7%
	6)	proper	16)	0.500 ; 50.0%
	7)	mixed	17)	0.625 ; 62.5%
	8)	improper	18)	0.286 ; 28.6%
	9)	10	19)	0.786 ; 78.6%
	10)	1/10	20)	0.444 ; 44.4%

3.3	1)	5/6	5)	1/3
	2)	1/15	6)	14/9
	3)	21/16	7)	5/8
	4)	1/2	8)	4/9
3.4	1)	-12	5)	2
	2)	12	6)	-2
	3)	-10	7)	2
	4)	-16	8)	6
3.5	1)	10,000	6)	150
	2)	8	7)	983,040
	3)	625	8)	16
	4)	32	9)	256
	5)	100,000	10)	1,000,000
4.1	1)	$x + 25$	3)	$5x + 50$
	2)	$10x + 100$	4)	$10x + 50$
4.2	1)	125 dollars	3)	80 euros
	2)	4,100rupee	4)	170 dollars
4.3	1)	$x = 4, 2x = 8$	5)	$x = 4, 4x = 16$
	2)	$x = 6, \times + 4 = 10$	6)	$x = 2, 6x = 12$
	3)	$x = 20, x/2 = 10$	7)	$x = 5, 2x = 10$
	4)	$x = 7, \times - 5 = 2$		
4.4	1)	$y = 7$	6)	$y = 1$
	2)	$y = 1$	7)	$y = 3$
	3)	$y = 4$	8)	$y = 2$
	4)	$y = 1$	9)	$y = 2$
	5)	$y = 7$	10)	$y = 5$
5.1	1)	polynomial	6)	polynomial
	2)	polynomial	7)	polynomial
	3)	polynomial	8)	not a polynomial
	4)	polynomial	9)	polynomial
	5)	not a polynomial	10)	polynomial

5.2	1)	fourth	4)	seventh
	2)	sixth	5)	eighth
	3)	sixth		

5.3	1)	$3x^2y + xy^2 + xy + 10$
	2)	$4a^3b^2c + 4a^2b + 3ab^2 + 7ab - 3$
	3)	$3x^3y^2z + 2x^3yz^2 + 3x^2y^2 - xyz + 2$
	4)	$6r^5s^2t - r^4s + rs^4 + 5r^3s - 1$
	5)	$4x^2yz + xy^2z + 6xyz^2 - xyz + 4$
	6)	$2q^3r^2s - 3q^2r + 3qr - 1$
	7)	$3x^3y^2z + 2x^2yz - 6xz + xy + 2$
	8)	$3a^4b^3c^2 - 6a^3b^2c - a^2b^3c + 2abc - 6$
	9)	$2x^5y^4z^3 - 4x^4y^2z + 2xyz + 4$
	10)	$3r^4s^3t - 4r^3s^2t + rt + rs - 9$

5.4	1)	$6x^2$
	2)	$-8rs$
	3)	$6y^5$
	4)	$12x^3y^4$
	5)	$-2r^5s^5t^3$
	6)	x
	7)	x^2y
	8)	$2m^3n^2o$
	9)	$(2r^2s^3) / t^2$
	10)	$2z/xy$

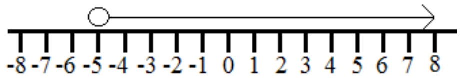
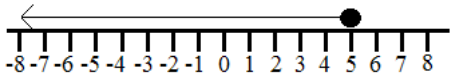
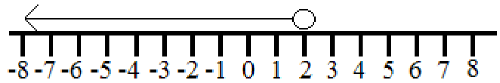
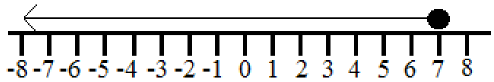
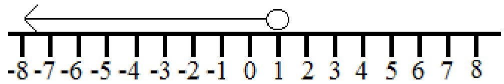
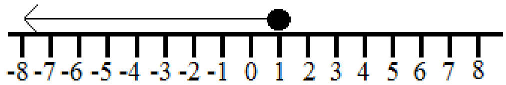
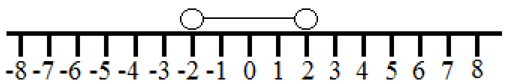
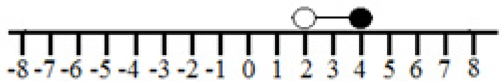
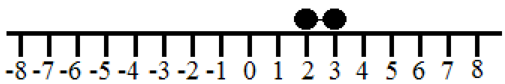
6.1	1)	$y^2(y + 1)$
	2)	$2x(2x - 1)$
	3)	$5s(s - 2)(s - 1)$
	4)	$x^3(x + 1)$
	5)	$z(3z+1)$
	6)	$10x(x + 1)(x + 5)$
	7)	$rs(2r - 1)$
	8)	$xy(x^2 + xy - 1)$
	9)	$r(3r^2s^2t - rs -1)$
	10)	$4xyz(xyz^2 + xy + 2)$

6.2	1)	(2r – 3)(2s + t)
	2)	(2q + 6r)(2p – 1) or 2(q + 3r)(2p – 1)
	3)	(y + 2z)(x + 4)
	4)	(2y + z)(3x – 1)
	5)	(3t + 4u)(5s – 3)
	6)	(2y + 3z)(2x + 3)
	7)	(9b + 3c)(3a + 2) or 3(3b + c)(3a + 2)
	8)	(12b + 8c)(2a – 1) or 4(3b + 2c)(2a – 1)
	9)	(7k + 2m)(6n + 5)
	10)	(6z – 4)(4x + y) or 2(3z – 2)(4x + y)
6.3	1)	(x – 4)(x + 2)
	2)	(z – 1)(z – 4)
	3)	(x + 3)(x – 4)
	4)	(y – 2)(y – 3)
	5)	(m – 1)(m + 4)
	6)	(2s +2)(s – 4) or 2(s + 1)(s -4)
	7)	(3x + 1)(x – 2)
	8)	(3x – 3)(2x + 1) or 3(x – 1)(2x + 1)
	9)	(6y – 1)(2y + 2) or 2(6y – 1)(y + 1)
	10)	(10z + 1)(5z – 2)

6.4	1)	$(2x - 4)(2x + 4)$ or $4(x - 2)(x + 2)$
	2)	$(y - 3)(y + 3)$
	3)	$(3r - 2s)(3r + 2s)$
	4)	$(9u - 7v)(9u + 7v)$
	5)	$x^3 - y^3$
	6)	$8r^3 - 27s^3$
	7)	$64m^3 - 27n^3$
	8)	$125x^3 + 64y^3$
	9)	$8a^3 + 27b^3$
	10)	$1{,}000x^3 + 216y^3$
6.5	1)	$x + 2$
	2)	$y - 3$
	3)	$2s - 3$
	4)	$2z - 5 + 7/(z + 1)$
	5)	$3s - 2$
	6)	$4x^2 + 2x + 2$
	7)	$3y^2 - 5y + 9 - 12/(y + 1)$
	8)	$2x^2 + 4x - 6$
	9)	$z^3 - 2z^2 + 4z -3 - 18/(6z - 3)$
	10)	$2r^3 - 4r^2 + 3r - 2$
7.1	1)	mn
	2)	$x/2$
	3)	$2s^2t$
	4)	$6y^2/z^2$
	5)	$2x^2/yz$
	6)	$4(x + 1)$
	7)	$2(x + y)$
	8)	$2(s - 2)^2/(s + 2)^2$
	9)	$16(x - 4)$
	10)	$9/(y + z)^3(y - z)$
7.2	1)	$(2 + 4z)/z^2$
	2)	$(4t + s)/2st$
	3)	$(5y + 2)/y^3$
	4)	$(7z + 22)/24$
	5)	$(5x + 12)/(x - 1)(x + 2)$
	6)	$(6 - 3y)/y^2$

	7)	$7/6z$
	8)	$(4s - 2)/s^3$
	9)	$(-2x + 7)/15$
	10)	$(-5y - 11)/(y - 2)(y + 3)$
7.3	1)	$(3r^4)/4$
	2)	$4x^2y$
	3)	27
	4)	$6/(x - 3)$
	5)	$(4x + 3y)^2/5$
	6)	$y^2/3$
	7)	$x/3y$
	8)	4
	9)	$(z + 1)/(3y - 3)$
	10)	$(2x - 4y)/a^2$
7.4	1)	$(4y + 1)/(2y - 1)$
	2)	2
	3)	$(3 - z)/z$
	4)	$4/9$
	5)	$y + 2$
	6)	$x - 2$
	7)	$z - 2$
	8)	$y + 2$
	9)	$(x - 1)/x$
	10)	$s^2/(s + 3)$
7.5	1)	$y = -2$
	2)	$x = 3$
	3)	$z = -3$
	4)	$s = -3$
	5)	$t = -1$
	6)	$x = 1$
	7)	$t = -7$
	8)	$z = -14$
	9)	$x = -12/5$
	10)	$x = -1$

7.6	1)	3 and 9
	2)	390 miles
	3)	250,000 motorcycles
	4)	600 computers
	5)	28,000,000 people
8.1	1)	$x \leq 2$
	2)	$y > 5$
	3)	$t < -1$
	4)	$s \geq 3/2$
	5)	$y > 2/3$
	6)	$x < 3$
	7)	$-4 < z < 3$
	8)	$x \leq -1$ or $\times \geq 5$
	9)	$7 \leq y \leq 13$
	10)	$8 > y > 4$

8.2	1)	$x > -5$	
	2)	$t \le 5$	
	3)	$y < 2$	
	4)	$x \le 7$	
	5)	$z < 1$	
	6)	$s \le 1$	
	7)	$7 \ge x \ge 3$	
	8)	$2 > y > -2$	
	9)	$4 \ge x > 2$	
	10)	$3 \ge t \ge 2$	
8.3	1)	4	
	2)	-2	
	3)	10	
	4)	8	
	5)	2	
	6)	7	
	7)	1/3	
	8)	-1	
	9)	8	
	10)	1/3	

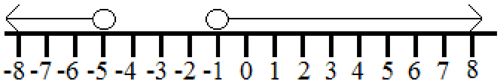
8.4

1) $x \geq 4$ or

$x \leq -4$

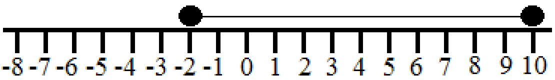
2) $-2 < y < 2$

3) $t < -5$ or

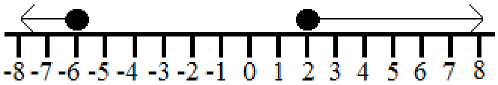


$t > -1$

4) $-2 \leq r \leq 10$

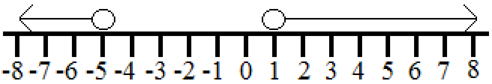


5) $s \leq -6$ or



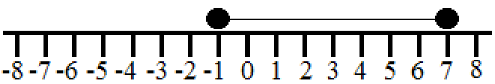
$s \geq 2$

6) $x < -5$ or

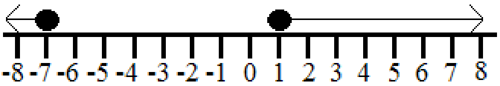


$x > 1$

7) $-1 \leq y \leq 7$

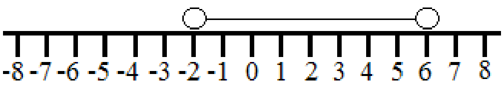


8) $z \leq -7$ or

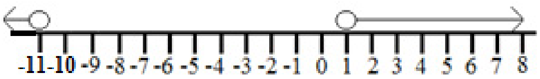


$z \geq 1$

9) $-2 < y < 6$

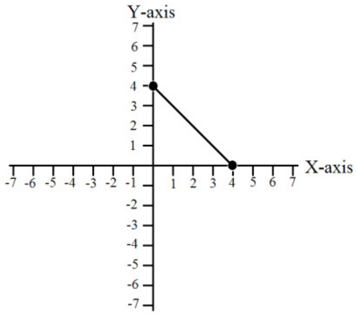


10) $x < -11$ or

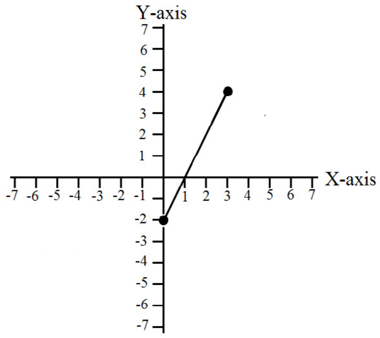


$x > 1$

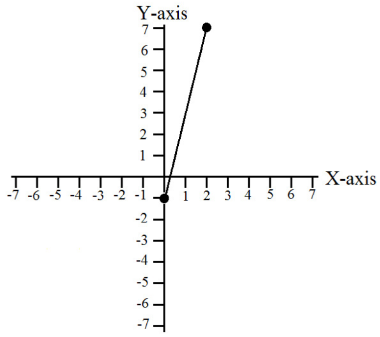
9.2	1)	y = 3x – 2
	2)	y = -2x + 1
	3)	y = 1/2x + 2
	4)	y = -3x + 2
	5)	y = 2x – 4
	6)	(y = -x + 4)



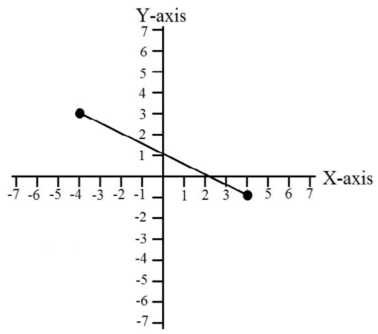
7) (y = 2x – 2)



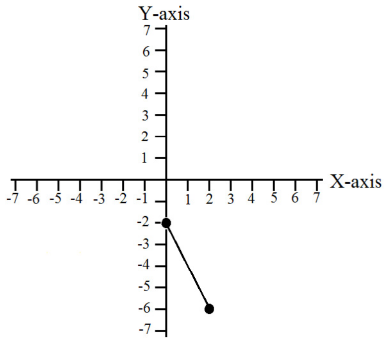
8) (y = 4x – 1)



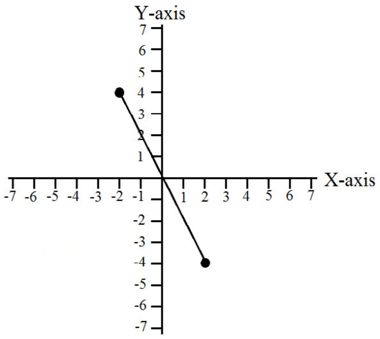
9) (y = -1/2x + 1)



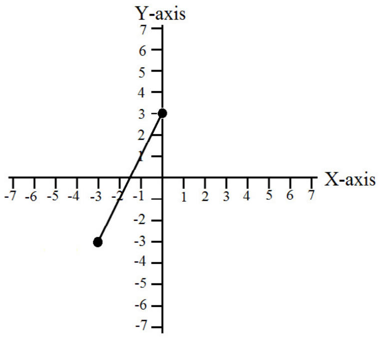
10) (y = -2x – 2)



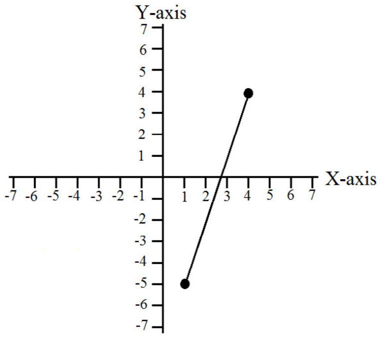
9.3 1) y = -2x



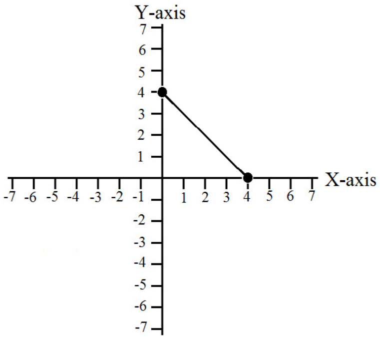
2) y = 2x + 3



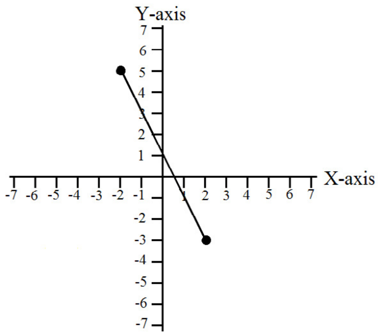
3) $y = 3x - 8$



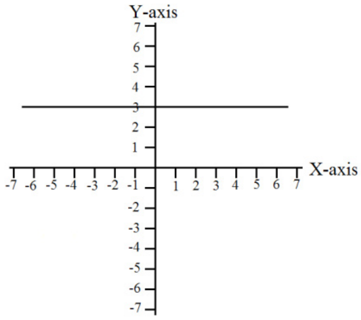
4) $y = -x + 4$



5) $y = -2x + 1$

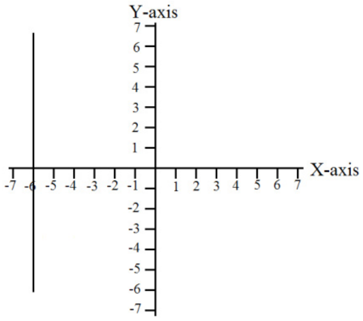


9.4 1) y = 3



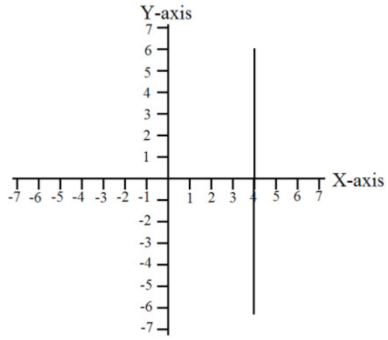
Horizontal

2) x = -6



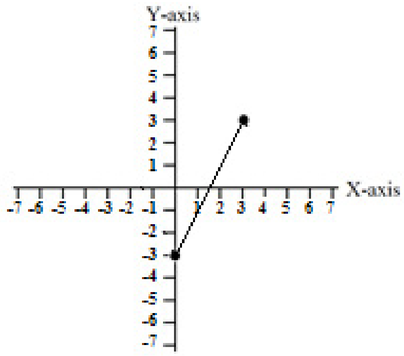
Vertical

3) x = 4



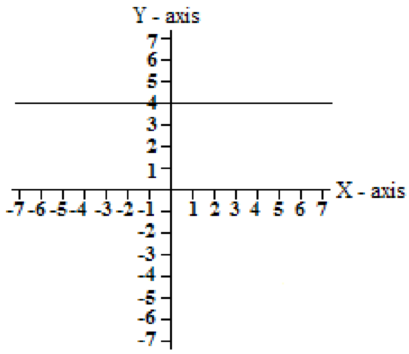
Vertical

4) y = 2x − 3



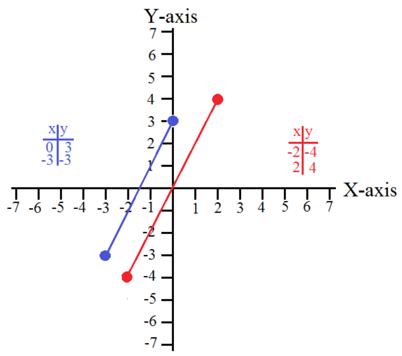
Neither

5) y = 4



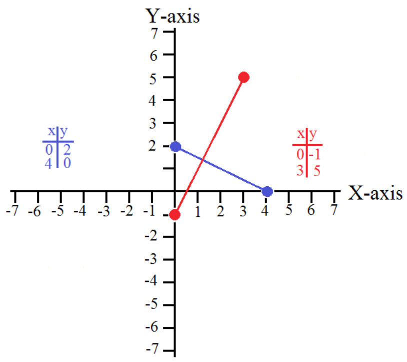
Horizontal

9.5 1) (y = 2x + 3, y = 2x)



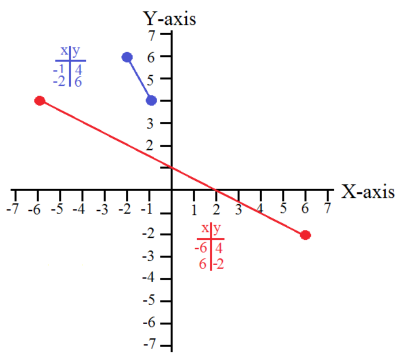
Parallel

2) (y = -1/2x + 2, y = 2x – 1)



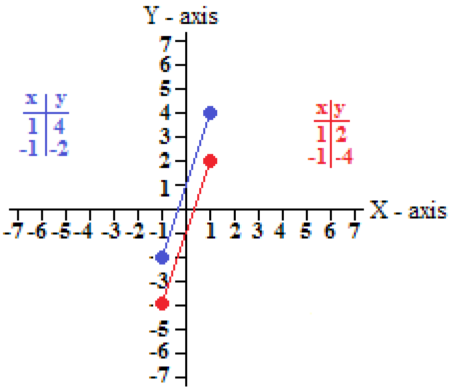
Perpendicular

3) y = -2x + 2, y = -1/2x +1



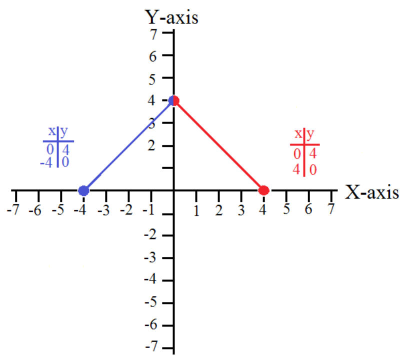
Neither

4) $y = 3x + 1$, $y = 3x - 1$



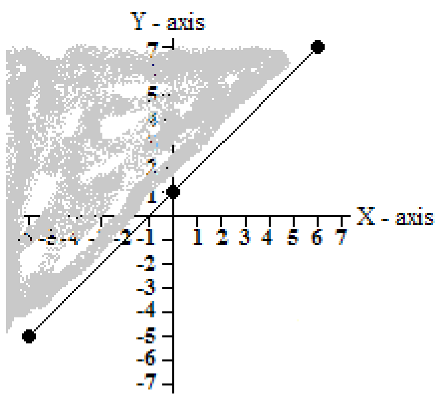
Parallel

5) $y = x + 4$, $y = -x + 4$

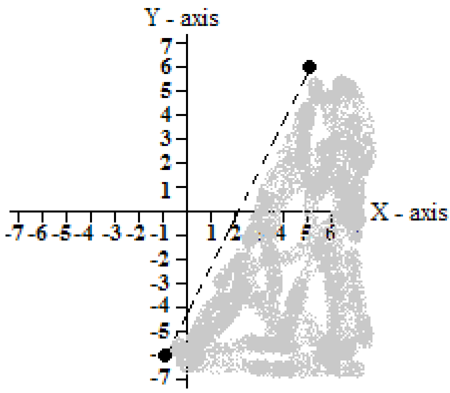


Perpendicular

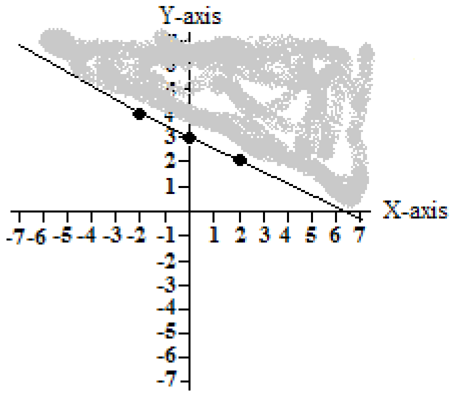
9.6 1) $y \geq x + 1$



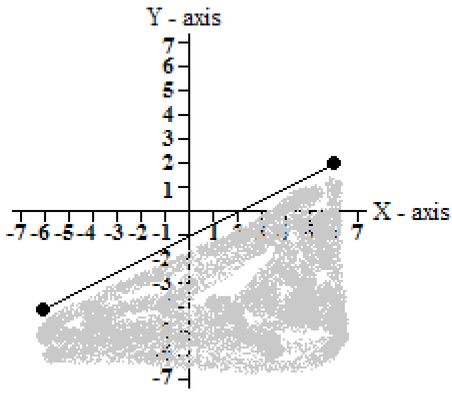
2) $y < 2x - 4$



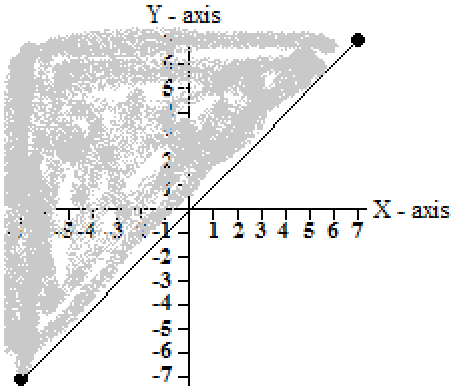
3) $y > (-x + 6)/2$



4) $y \le (x - 2)/2$

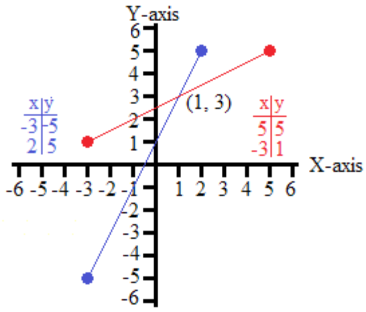


	5)	$y \geq x$



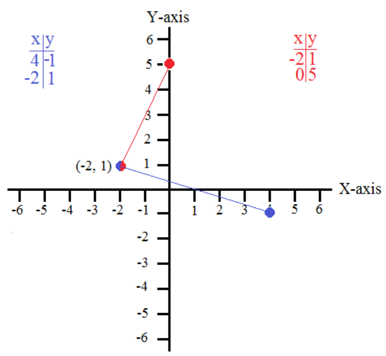
10.2	1)	1/4
	2)	1
	3)	-1/18
	4)	$1/y^4$
	5)	1
	6)	1
	7)	1/s
	8)	1/27
	9)	16
	10	16
10.3	1)	49
	2)	$x^{2/3}$
	3)	3y
	4)	$25z^2$
	5)	$\sqrt{2}\,z$
	6)	$5x^2y^3$
	7)	$2s^2t^3$
	8)	$3xy^2z^3\sqrt[3]{x}$
	9)	$4pqr^2\sqrt[3]{pq^2}$
	10)	$2xy^2z^2\sqrt[5]{xz^2}$

10.4	1)	y^7
	2)	$12x^5$
	3)	t^2
	4)	$3s^2$
	5)	$3x^3$
	6)	y^8
	7)	$27z^9$
	8)	$49x^8$
	9)	$9y^{2/3}$
	10)	$4x^{1/2}$
11.1	1)	$y = 2x + 1$, $y = (x + 5)/2$



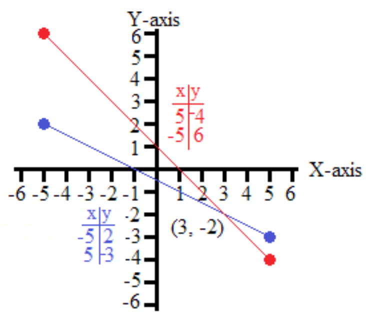
Unique solution at (1, 3).

2) $y = (1- x)/3$, $y = 2x - 5$



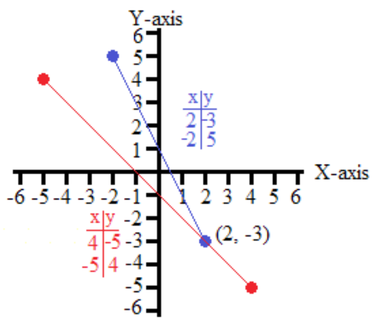
Unique solution at (-2,1).

3) $y = (-1 - x)/2$, $y = 1 - x$



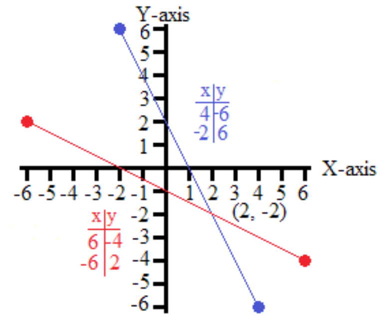
Unique solution at (3,-2).

4) $y = 1 - 2x$, $y = -1 - x$



Unique solution at (2,-3).

5) y = 2 – 2x, y = (-2 – x)/2



Unique solution at (2,-2).

11.2
1) x = 2, y = -3
2) x = 3, y = 1
3) x = -1, y = 0
4) x = -2, y = 1
5) x = -2, y = 1

11.3
1) x = -2, y = 3
2) x = -3, y = 1
3) x = 1, y = -2
4) x = 1, y = 3
5) x = 2, y = -2

11.4
1) x = 2, y = -2, z = 1
2) x = 2, y = -2, z = 1
3) x = -1, y = 0, z = 2
4) x = 3, y = 2, z = 1
5) x = 3, y = -2, z = 1

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His specialty is the pedagogical research of chemistry and mathematics for college students. Within this specialty he concentrates on chemistry and mathematics courses for non-science and non-mathematics students.

This text book is a reflection of his lessons plans from several introductory mathematics courses he has taught.