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Continuous Average Control of Piecewise Deterministic Markov Processes

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Preface

The intent of this book is to present recent results in the control theory for the long-run average continuous control problem of Piecewise Deterministic Markov Processes (PDMPs). This is neither a textbook nor a complete account of the state-of-the-art on the subject. Instead, we attempt to provide a systematic framework for an understanding of the main concepts and tools associated with this problem, based on previous works of the authors. The limited size of a book makes it unfeasible to cover all the aspects in this field and therefore some degree of specialization was inevitable. Due to that, the book focuses mainly on the long-run average cost criteria and tries to extend to the PDMPs some well-known techniques related to discrete-time and continuous-time Markov decision processes, including the so-called "average inequality approach," "vanish discount technique," and "policy iteration algorithm."

Most of the material presented in this book was scattered throughout a variety of sources, which included journal articles and conference proceedings papers. This motivated the authors to write this text, putting together systematically these results. Although the book is mainly intended to be a theoretically oriented text, it also contains some motivational examples. The notation is mostly standard although, in some cases, it is tailored to meet specific needs. A glossary of symbols and conventions can be found at the end of the book.

The book is targeted primarily for advanced students and practitioners of control theory. In particular, we hope that the book will be a valuable source for experts in the field of Markov decision processes. Moreover, we believe that the book should be suitable for certain advanced courses or seminars. As background, one needs an acquaintance with the theory of Markov decision processes and some knowledge of stochastic processes and modern analysis.

The authors are indebted to many people and institutions which have contributed in many ways to the writing of this book. We gratefully acknowledge the support of the IMB, Institut Mathématiques de Bordeaux, INRIA Bordeaux Sud Ouest, team CQFD, and the Laboratory of Automation and Control—LAC/USP at the University of São Paulo. This book owes much to our research partners, to whom we are immensely grateful. Many thanks go also to our former Ph.D. students. We acknowledge with great pleasure the efficiency and support of Donna

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Last, but not least, we are very grateful to our families for their continuing and unwavering support. To them we dedicate this book.

São Paulo, Brazil — O. L. V. Costa
Bordeaux, France — F. Dufour

Contents

Notation and Conventions

As a general rule, lowercase greek and roman letters are used for functions while uppercase greek are used for selectors. Sets and spaces are denoted by capital roman letters. Blackboard and calligraphic letters represent Borel measurable spaces or a σ-algebra. Sometimes it is not possible or convenient to adhere completely to this rule, but the exceptions should be clearly perceived based on their specific context.

The following lists present the main symbols and general notation used throughout the book, followed by a brief explanation and the number of the page of their definition or first appearance.

Symbol	**Description**
$\square$	End of proof
$\mathbb{N}$	The set of natural numbers
$\mathbb{N}_*$	The set of positive real numbers
$\mathbb{R}$	The real numbers
$\mathbb{R}_+$	The positive real numbers
$\mathbb{R}^d$	The d-dimensional euclidian space
$\overline{\mathbb{R}}_+$	$= \mathbb{R}_+ \cup \{+\infty\}$
$\mathcal{B}(X)$	σ-algebra generated by the open sets of X
$\mathbb{B}(X;Y)$	Borel bounded functions from X into Y
$\mathbb{B}(X)$	$= \mathbb{B}(X;\mathbb{R})$
$\mathbb{B}(X)^+$	$= \mathbb{B}(X;\mathbb{R}^+)$
$\mathcal{M}(X)$	The set of all finite measures on $(X, \mathcal{B}(X))$
$\mathbb{M}(X;Y)$	Borel measurable functions from X into Y
$\mathbb{M}(X)$	$= \mathbb{M}(X;\mathbb{R})$
$\mathbb{M}(X)^+$	$= \mathbb{M}(X;\mathbb{R}^+)$
$\mathbb{B}_g(X)$	Functions υ such that $\sup_{x\in X} \frac{\lvert v(x)\rvert}{g(x)} < +\infty$
$\mathbb{C}(X)$	Continuous functions from X to $\mathbb{R}$
$\mathcal{P}(X)$	Set of all probability measures on $(X, \mathcal{B}(X))$
h^+	The positive part of a function h

h^-	The negative part of a function h
η	The Lebesgue measure on the real numbers
I_A	The indicator function of the set A
E	The state space of a PDMP: an open subset of $\mathbb{R}^n$
∂E	The boundary of the state space E
$\overline{E}$	The closure of the state space E
$\phi(x,t)$	The flow of a PDMP
$t_*(x)$	The time the flow ϕ takes to reach the boundary ∂E starting from x
$\mathcal{X}$	Vector field associated with the flow ϕ
$\mathbb{U}$	The set of control actions
$\mathbb{U}(x)$	The set of feasible control actions that can be taken when the state process is in $x \in \overline{E}$
λ	The jump rate
Q	The transition measure Q
$\mathbb{M}^{ac}(E)$	Functions absolutely continuous along the flow with limit toward the boundary
$\overline{\lambda}$	A upper bound of λ with respect to the control variable
Δ	An arbitrary fixed point in ∂E
K	The set of feasible state/action pairs
$\mathcal{U}$	The class of admissible control strategies
$\widehat{\phi}$	The flow of the controlled PDMP
$\widehat{\lambda}^U$	The jump rate of the controlled PDMP
$\widehat{Q}^U$	The transition measure of the controlled PDMP
$P^U_{\hat{x}}$	The probability of the probability space on which the PDMP is defined
$E^U_{\hat{x}}$	The expectation under the probability $P^U_{\hat{x}}$
$\widehat{X}^U(t)$	The controlled PDMP
$X(t)$	The state of the system
$Z(t)$	The value of $X(t)$ at the last jump time before t
$\tau(t)$	The time elapsed between the last jump and time t
$N(t)$	The number of jumps of the process $\{X(t)\}$ at time t
$(T_n)_{n\in\mathbb{N}}$	The sequence of jump times of the PDMP
f, r	The running and boundary costs
$\mathbf{J}^\alpha(U,t)$	The finite horizon cost function
$\mathbf{J}(U,t)$	$= \mathbf{J}^0(U,t)$
$p^*(t)$	The counting process associated with the number of times the process hits the boundary up to time t
$\mathcal{A}(U,x)$	The long-run average cost function
$\mathcal{J}_{\mathcal{A}}(x)$	The value function associated with the long-run average cost function
$\mathcal{D}^\alpha(U,x)$	The α-discounted cost function

$\mathcal{J}^{\alpha}_{\mathcal{D}}(x)$	The value function associated with the α-discounted cost function
$\mathcal{D}^{\alpha}_{m}(U, x)$	The truncated version of the α-discounted cost function
$L^{1}(\mathbb{R}_{+}; \mathbb{C}(\mathbb{U}))$	The set of Bochner integrable functions with values in $\mathbb{C}(\mathbb{U})$
$L^{\infty}(\mathbb{R}_{+}; \mathcal{M}(\mathbb{U}))$	The space of bounded measurable functions from $\mathbb{R}_{+}$ to $\mathcal{M}(\mathbb{U})$
$\mathcal{V}^{r}, \mathcal{V}^{r}(x), \mathcal{V}(x)$	See Definition 2.12
$\mathbb{V}^{r}$, $\mathbb{V}^{r}(x)$	The set of relaxed controls
$\mathbb{V}, \mathbb{V}(x)$	The set of ordinary controls
$[\Theta]_{t}$	Shifted control strategy (see Definition 2.7)
$\mathcal{K}$	The set of feasible state/relaxed-control pairs
$w(x, \mu)$	See equation (2.8)
$Qh(x, \mu)$	See equation (2.9)
$\lambda Qh(x, \mu)$	See equation (2.10)
$\Lambda^{\mu}(x, t)$	See equation (2.11)
$G_{\alpha}(x, \Theta; A)$	See equation (2.12)
$G_{\alpha}h(x, \Theta)$	See equation (2.13)
$L_{\alpha}v(x, \Theta)$	See equation (2.14)
$H_{\alpha}w(x, \Theta)$	See equation (2.15)
$\mathcal{L}_{\alpha}(x, \Theta)$	See equation (2.16)
G, L, H, $\mathcal{L}$	$G = G_{0}, L = L_{0}, H = H_{0}, \mathcal{L} = \mathcal{L}_{0}$
$\mathcal{T}_{\alpha}(\rho, h)(x)$	The one-stage optimization operator
$\mathcal{R}_{\alpha}(\rho, h)(x)$	The relaxed one-stage optimization operator
$\mathcal{T}$ and $\mathcal{R}$	$\mathcal{T} = \mathcal{T}_{0}$ and $\mathcal{R} = \mathcal{R}_{0}$
$\mathcal{S}_{\mathbb{U}}, \mathcal{S}_{\mathbb{V}}, \mathcal{S}_{\mathbb{V}^{r}}$	The sets of measurable selectors
u_{ϕ}	See Definition 2.22
U_{ϕ}	See Definition 2.23
$J^{U}_{m}(t, x, k)$	The truncated version of finite horizon α-discounted cost function
$(f_{j})_{j \in \mathbb{N}}$	The approximating sequence of the running cost f
$(r_{j})_{j \in \mathbb{N}}$	The approximating sequence of the running cost r
$\underline{\lambda}$	A lower bound of λ with respect to the control variable
$\overline{f}$	A upper bound of f with respect to the control variable
$\widehat{u}(w, h) \in \mathcal{S}_{\mathbb{U}}$	See definition 3.12
$\widehat{u}_{\phi}(w, h) \in \mathcal{S}_{\mathbb{V}}$	See definition 3.12
$\widehat{U}_{\phi}(w, h) \in \mathcal{U}$	See definition 3.12
$\mathcal{W}g$	$= \mathcal{R}_{\alpha}(0, g)$
$h_{\alpha}(x)$	$= \mathcal{J}^{\alpha}_{\mathcal{D}}(x) - \mathcal{J}^{\alpha}_{\mathcal{D}}(x_{0})$, the relative difference of the α-discount value functions $\mathcal{J}^{\alpha}_{\mathcal{D}}$
K_{h}	A lower bound for h_{α}
$\overline{h}$	A upper bound for h_{α}
g	Test function for the so-called expected growth condition
$\overline{r}$	Test function for the so-called expected growth condition
ν_{u}	Invariant probability measure of the kernel $G(., u_{\phi}; .)$
κ	See equation (4.18)

Abbreviation	**Description**
MDP(s)	Markov Decision Process(es)
PDMP(s)	Piecewise Deterministic Markov Process(es)
PIA	Policy Iteration Algorithm

Chapter 1
Introduction

1.1 Preliminaries

Dynamical systems that are subject to abrupt changes have been a theme of increasing investigation in recent years. For instance, complex technological processes must maintain an acceptable behavior in the event of random structural perturbations, such as failures or component degradation. Aerospace engineering provides numerous examples of such situations: an aircraft has to pursue its mission even if some gyroscopes are out of order, a space shuttle has to succeed in its reentry with a failed on-board computer. Failed or degraded operating modes are parts of an embedded system history and should therefore be accounted for during the control synthesis. These few basic examples show that complex systems like embedded systems are inherently vulnerable to failure of components, and their reliability has to be improved through a control process. Complex systems require mathematical representations that are in essence dynamic, multimodel, and stochastic.

Different approaches have emerged over the last decades to analyze multimodel stochastic processes. A particularly interesting one, and the main theme of this book, is the application of piecewise deterministic Markov processes (PDMPs), introduced in [23, 25] as a general family of nondiffusion stochastic models, suitable for formulating many optimization problems in queuing and inventory systems, maintenance–replacement models, and many other areas of engineering and operations research.

PDMPs are determined by three local characteristics: the flow ϕ, the jump rate λ, and the transition measure Q. Starting from x, the motion of the process follows the flow $\phi(x,t)$ until the first jump time T_1, which occurs either spontaneously in a Poisson-like fashion with rate λ or when the flow $\phi(x,t)$ hits the boundary of the state space. In either case, the location of the process at the jump time T_1 is selected by the transition measure $Q(\phi(x,T_1),.)$, and the motion restarts from this new point as before. As shown in [25], a suitable choice of the state space and the local characteristics ϕ, λ, and Q provides stochastic models covering a great number of problems of engineering and operations research.

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As pointed out by Davis in [25, p. 134], there exist two types of control for PDMPs: *continuous control*, in which the control variable acts at all times on the process through the characteristics (ϕ, λ, Q), and *impulse control*, used to describe control actions that intervene in the process by moving it to a new point of the state space at some specific times. This book is devoted to the long-run average continuous control problem of PDMPs taking values in a general Borel space. At each point x of the state space, a control variable is chosen from a compact action set $\mathbb{U}(x)$ and is applied to the jump parameter λ and transition measure Q. The goal is to minimize the long-run average cost, which is composed of a running cost and a boundary cost (which is added each time the PDMP touches the boundary). Both costs are assumed to be positive but not necessarily bounded.

The main approach in this book is, using the special features of PDMPs, to trace a parallel with the general theory of discrete-time Markov decision processes (MDPs) (see, for instance, [45, 49]) rather than the continuous-time case (see, for instance, [44, 70]). The two main reasons for doing that are to use the powerful tools developed in the discrete-time framework (see, for example, the references [6, 34, 49, 51]) and to avoid working with the infinitesimal generator associated with a PDMP, which in most cases has a domain of definition that is difficult to characterize. We follow a similar approach to that used for studying continuous-time MDPs, which consists in reducing the original continuous-time control problem to a semi-Markov or discrete-time MDP [5, 36, 59, 64, 66]. For a detailed discussion about these reduction techniques, the reader is referred to the reference [36]. The reduction method proposed in [36] consists of two steps. First, the original continuous-time MDP is converted into a semi-Markov decision process (SMDP) in which the decisions are selected only at the jump epoch. Second, within the discounted cost context, the SMDP is reduced to a discrete-time MDP. Regarding PDMPs, the idea developed by Davis in [24] is somehow related to the reduction technique previously described in the context of MDPs. It consists in reformulating the optimal control problem of a PDMP for a discounted cost as an equivalent discrete-time Markov decision model in which the stages are the jump times T_n of the PDMP.

A somewhat different approach to the problem of controlling a PDMP through an embedded discrete-time MDP is also considered in [1], in which the decision function space is made compact by permitting piecewise construction of an open-loop control function. It must be stressed that one of the key points in the development of these methods is that the control problem under consideration is concerned with discounted cost criteria. Discounted cost criteria are usually easier to deal with since, as pointed out in [36], it is well known that an SMDP with discounted cost criteria can be reduced to an MDP with discounted cost. The approach adopted in [24] for PDMPs with discounted cost is somehow related to these ideas, since the key point in [24] is to rewrite the integral cost as a sum of integrals between two consecutive jump times of the PDMP and by doing this, to obtain naturally the one-step cost function for the discrete-time Markov decision model. However, this decomposition for the long-run average cost is no longer possible, and therefore, a more specific approach has to be developed. This is one of the goals of the present book. It must be pointed out that there exists another framework for studying continuous-time MDPs in which the controller can choose *continuously* in time the actions to be applied to the process. There exists an extensive literature within this context; see, for example, [42–44, 58] and the references therein. This could be another way of studying the control problem for PDMPs with average cost. However, as far as the authors are aware, it is an open problem to convert a control problem for a PDMP into a continuous-time MDP. In particular, the main problem is how to write explicitly the transition rate of a PDMP in terms of its parameters: the state space E, its boundary ∂E, and (ϕ, λ, Q).

We assume in this book that the control acts only on (λ, Q). The main difficulty in considering the control acting also on the flow comes from the fact that in such a situation, the time $t_*(x)$ that the flow takes to hit the boundary starting from x and the first-order differential operator $\mathcal{X}$ associated to the flow would depend on the control. Under these conditions, it is not obvious how to write an optimality equation for the long-run average cost in terms of a discrete-time optimality equation related to the embedded Markov chain given by the postjump location of the PDMP. This step is easier to derive in the situation studied in [24], which considers the control acting on all the local characteristics (ϕ, λ, Q) of the PDMP, since, as noted previously, for a discounted cost, it is very natural to rewrite the integral cost as a sum of integrals between two consecutive jump times of the PDMP, obtaining naturally the one-step cost function for the discrete-time Markov decision model. However, this decomposition for the long-run average cost is no longer possible, and consequently, due to this technical difficulty, the present approach may be applied only to PDMPs in which the control acts on the jump rate and transition measure.

1.2 Overview of the Chapters

The book is organized in the following way.

In Chap. 2, we introduce some notation, basic assumptions, and the control problems to be considered. The definitions of ordinary and relaxed control spaces as well as some operators required for characterizing the optimality equation are also presented in this chapter. The results of this chapter are based on the papers [17] and [20].

Chapter 3 presents the main characterization results regarding the optimality equation for the long-run average cost. The first main result in presented in Sect. 3.2, which obtains an optimality equation for the long-run average cost in terms of a discrete-time optimality equation. The second main result of this chapter presents conditions to guarantee the existence of a feedback measurable selector (that is, a selector that depends on the present value of the state variable) for this optimality equation. This is done by establishing a link between the discrete-time optimality equation and an integrodifferential equation (using the weaker concept of absolute continuity along the flow of the value function). The common approach for proving the existence of a measurable selector is to impose semicontinuity properties of the cost function and to introduce the class of relaxed controls to get a compactness property for the action space. It should be pointed out that other approaches without the compactness assumption would also be possible; see, for instance, [35]. By doing this, one obtains an existence result, but within the class of relaxed controls. However, what it is desired is to show the existence of an optimal control in the class of ordinary controls. Combining the existence result within the class of relaxed controls with the connection between the integrodifferential equation and the discrete-time equation, we can show that the optimal control is nonrelaxed and in fact an ordinary feedback control. In Sect. 3.3, we introduce continuity assumptions, while in Sect. 3.4, we derive sufficient conditions for the existence of an ordinary feedback optimal control and establish a connection between the discrete-time optimality equation and an integrodifferential equation.

In general, it is a hard task to obtain equality in the solution of the discrete-time optimality equation and verify the extra condition. A common approach to avoiding this is to consider an inequality instead of an equality for the optimality equation and to use an Abelian result to get the reverse inequality (see, for instance, [49]). Chapter 4 is devoted to deriving sufficient conditions for the existence of an optimal control strategy for the long-run average continuous control problem of PDMPs by applying the so-called vanishing discount approach (see [49]). Combining our result with the link between the integrodifferential equation and the discrete-time equation, we obtain the existence of an ordinary optimal feedback control for the long-run average cost (see Theorem 4.10). In order to do this, we need first to establish, in Sect. 4.2, an optimality equation for the discounted control problem. Two sets of assumptions are considered in Chap. 4. The first one is presented in Sect. 4.3 and is mainly expressed in terms of the relative difference of the α-discount value functions. From a practical point of view, this result is not completely satisfactory, due to the fact that these conditions depend on the α-discount value function, which may be difficult to obtain explicitly even for simple examples. The second set of assumptions is presented in Sect. 4.4.2. These are written in terms of some integrodifferential inequalities related to the so-called expected growth condition and geometric convergence of the postjump location kernel associated to the PDMP. The results of this chapter are based on the papers [16] and [17].

Chapter 5 seeks to apply the so-called policy iteration algorithm (PIA) to the long-run average continuous control problem of PDMPs. In order to do this, we first derive some important properties for a pseudo-Poisson equation associated with the problem. In the sequel, it is shown that the convergence of the PIA to a solution satisfying the optimality equation holds under some classical hypotheses and that this optimal solution yields an optimal control strategy for the average control problem for the continuous-time PDMP in a feedback form. The results of this chapter are based on the paper [18].

Examples are presented in Chap. 6 illustrating the possible applications of the results developed in Chap. 4.

1.3 General Comments and Historical Remarks

There is by now an extensive theory surrounding PDMPs. For a general overview on PDMPs, the reader is referred to the book [25] and the references therein. Some stability topics for PDMPs are dealt with in [11, 15, 33]. For optimal stopping and impulse control problems for PDMPs, the reader is referred to [10, 21, 22, 30, 38–41, 46]. For numerical approximations related to these problems, we can mention [12–14, 27, 28]. For continuous control problems with discounted cost criteria the reader is referred to [1, 24, 29, 31, 37, 62, 67, 68]. The expected discounted continuous control of PDMPs using a singular perturbation approach for dealing with rapidly oscillating parameters was considered in [19]. Applications of PDMPs can be found in [3, 26, 62].

Regarding MDPs, we recommend, without attempting to be exhaustive, the surveys [2, 43] and the books [6, 49, 51, 59, 63] and the references therein to get a rather complete view of this research field.

As mentioned above, we have considered in this book some compactness conditions for the existence of measurable selectors, but other approaches, without the compactness assumption, would also be possible (see, for instance, [35]).

A paper closely related to the approach adopted in Chap. 3, but which considers the discounted control case, is the paper by Forwick et al. [37], which also considers unbounded costs and relaxed controls and obtains sufficient conditions for the existence of ordinary feedback controls. However, in [37], the authors consider neither the long-run average cost case nor the related limit problem associated with the vanishing discount approach. Moreover, in contrast to [37], we consider here boundary jumps and the control action space depending on the state variable. Note, however, that control on the flow is not considered here, while it was studied in [37].

The idea of using the vanishing discount approach to get an optimality condition (i.e., a condition for the existence of an average policy) has been widely developed in the literature. Different methods have been proposed based on conditions for ensuring the existence of a solution to the average cost optimality equality (see, for example, [2, 52]) and to the average cost optimality inequality (see, for example, [44, 45, 47–49, 57]).

The PIA has received considerable attention in the literature and consists of three steps: initialization; policy evaluation, which is related to the Poisson equation (PE) associated with the transition law defining the MDP; and policy improvement. Without attempting to present here an exhaustive panorama of the literature for the PIA, we can mention the surveys [2, 8, 51, 52, 59] and the references therein and more specifically the references [50, 55], which analyze in detail the PIA for general MDPs and provide conditions that guarantee its convergence.

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Chapter 2
Average Continuous Control of PDMPs

2.1 Outline of the Chapter

This chapter is devoted to the presentation of some notation, basic assumptions, and the control problems to be considered in this book. In Sect. 2.2, we present some standard notation and some basic definitions related to the motion of a piecewise deterministic Markov process (PDMP). In Sect. 2.3, the definitions of ordinary and relaxed control spaces as well as some operators required for characterizing the optimality equation are presented. Some proofs of auxiliary results are presented in Sect. 2.4.

2.2 Notation, Assumptions, and Problem Formulation

In this section, we present some standard notation, basic definitions, and some assumptions related to the motion of a PDMP $\{X(t)\}$, and the control problems we will consider throughout the book. For further details and properties, the reader is referred to [25]. The following notation will be used in this book: $\mathbb{N}$ is the set of natural numbers, and $\mathbb{N}_* = \mathbb{N} - \{0\}$. Also, $\mathbb{R}$ denotes the set of real numbers, $\mathbb{R}_+$ the set of positive real numbers, and $\overline{\mathbb{R}}_+ = \mathbb{R}_+ \cup \{+\infty\}$, $\mathbb{R}^d$ d-dimensional Euclidian space, and η the Lebesgue measure on $\mathbb{R}$. For X a metric space, we denote by $\mathcal{B}(X)$ the σ-algebra generated by the open sets of X; $\mathcal{M}(X)$ (respectively $\mathcal{P}(X)$) denotes the set of all finite (respectively probability) measures on $(X, \mathcal{B}(X))$. Let X and Y be metric spaces. The set of all Borel measurable (respectively bounded) functions from X into Y is denoted by $\mathbb{M}(X; Y)$ (respectively $\mathbb{B}(X; Y)$). Moreover, for notational simplicity, $\mathbb{M}(X)$ (respectively $\mathbb{B}(X)$, $\mathbb{M}(X)^+$, $\mathbb{B}(X)^+$) denotes $\mathbb{M}(X; \mathbb{R})$ (respectively $\mathbb{B}(X; \mathbb{R})$, $\mathbb{M}(X; \mathbb{R}_+)$, $\mathbb{B}(X; \mathbb{R}_+)$). Also, $\mathbb{C}(X)$ denotes the set of continuous functions from X to $\mathbb{R}$. For $g \in \mathbb{M}(X)$ with $g(x) > 0$ for all $x \in X$, $\mathbb{B}_g(X)$ is the set of functions $v \in \mathbb{M}(X)$ such that $||v||_g = \sup_{x\in X} \dfrac{|v(x)|}{g(x)} < +\infty$. For $h \in \mathbb{M}(X)$,

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h^+ (respectively h^-) denotes the positive (respectively negative) part of h; I_A is the indicator function of the set A: $I_A(x) = 1$ if $x \in A$ and $I_A(x) = 0$ otherwise.

For the definition of the state space of the PDMP, we will consider for notational simplicity that E is an open subset of $\mathbb{R}^n$ with boundary ∂E and closure $\overline{E}$. This definition could be easily generalized to include some boundary points and countable union of sets as in [25, Section 24]. In the examples of Chap. 6, we illustrate how the state space could be generalized in this direction.

Let us introduce some data that will be used to define a PDMP:

- The flow $\phi(x, t)$ is a function $\phi : \mathbb{R}^n \times \mathbb{R}_+ \longrightarrow \mathbb{R}^n$ continuous in (x, t) and such that $\phi(x, t + s) = \phi(\phi(x, t), s)$.
- For each $x \in E$, the time the flow takes to reach the boundary starting from x is defined as
$$t_*(x) \doteq \inf\{t > 0 : \phi(x, t) \in \partial E\}.$$
For $x \in E$ such that $t_*(x) = \infty$ (that is, the flow starting from x never touches the boundary), we set $\phi(x, t_*(x)) = \Delta$, where Δ is a fixed point in ∂E.
- The set $\mathbb{U}$ of control actions is a Borel space. For each $x \in \overline{E}$, we define the subsets $\mathbb{U}(x)$ of $\mathbb{U}$ as the set of feasible control actions that can be taken when the state process is in $x \in \overline{E}$.
- $\rightarrow$ The jump rate λ is a mapping in $\mathbb{M}(\overline{E} \times \mathbb{U})^+$.
- The transition measure Q is a stochastic kernel on E given $\overline{E} \times \mathbb{U}$.

Definition 2.1 *We define $\mathbb{M}^{ac}(E)$ as the space of functions absolutely continuous along the flow with limit toward the boundary:*

$$\mathbb{M}^{ac}(E) = \Big\{ g \in \mathbb{M}(E);\ g(\phi(x, t)) : [0, t_*(x)) \mapsto \mathbb{R} \textit{ is absolutely continuous for each } x \in E \textit{ and when } t_*(x) < \infty \textit{ the limit } \lim_{t \to t_*(x)} g(\phi(x, t)) \textit{ exists in } \mathbb{R} \Big\}.$$

For $g \in \mathbb{M}^{ac}(E)$ and $z \in \partial E$ for which there exists $x \in E$ such that $z = \phi(x, t_*(x))$, where $t_*(x) < \infty$, we define $g(z) = \lim_{t \to t_*(x)} g(\phi(x, t))$ (note that the limit exists by assumption).

We introduce in the next lemma the function $\mathcal{X}g$ for every function $g \in \mathbb{M}^{ac}(E)$; $\mathcal{X}$ can be seen as the vector field associated to the flow ϕ. The proof of this lemma can be found in Sect. 2.4.

Lemma 2.2 *Assume that $w \in \mathbb{M}^{ac}(E)$. Then there exists a function $\mathcal{X}w$ in $\mathbb{M}(E)$ such that for all $x \in E$ and $t \in [0, t_*(x))$,*

$$w(\phi(x, t)) - w(x) = \int_0^t \mathcal{X}w(\phi(x, s))ds. \tag{2.1}$$

A controlled PDMP is determined by its local characteristics (ϕ, λ, Q), to be presented in the sequel. The following assumptions, based on the standard theory of MDPs (see [49]), will be made throughout the book.

Assumption 2.3 *For all $x \in \overline{E}$, $\mathbb{U}(x)$ is a compact subspace of a compact set $\mathbb{U}$.*

Assumption 2.4 *The set $K = \{(x, a) : x \in \overline{E}, a \in \mathbb{U}(x)\}$ is a Borel subset of $\overline{E} \times \mathbb{U}$.*

The following assumption will also be required throughout the book.

Assumption 2.5 *There exist $\overline{\lambda} \in \mathbb{M}(\overline{E})^+$, $\underline{\lambda} \in \mathbb{M}(\overline{E})^+$, and $K_\lambda \in \mathbb{R}_+$ such that for every $(x, a) \in K$,*

(a) $\lambda(x, a) \leq \overline{\lambda}(x)$, *and for* $t \in [0, t_*(x))$, $\int_0^t \overline{\lambda}(\phi(x, s))ds < \infty$, *and if* $t_*(x) < \infty$, *then* $\int_0^{t_*(x)} \overline{\lambda}(\phi(x, s))ds < \infty$.
(b) $\lambda(x, a) \geq \underline{\lambda}(x)$ *and* $\int_0^{t_*(x)} e^{-\int_0^t \underline{\lambda}(\phi(x,s))ds} dt \leq K_\lambda$.

Since $\mathbb{U}(x)$ is not defined for $x \notin E$, we introduce the following notation.

Definition 2.6 *Consider $x \in E$ with $t_*(x) < +\infty$. Then with a slight abuse of notation, $\mathbb{U}(\phi(x, t))$ is defined for $t > t_*(x)$ by $\{\Delta_u\}$, where Δ_u is an arbitrary fixed point in $\mathbb{U}$.*

We present next the definition of an admissible control strategy and the associated motion of a controlled process.

Definition 2.7 *A control strategy U is a pair of functions*

$$(u, u_\partial) \in \mathbb{M}(\mathbb{N} \times E \times \mathbb{R}_+; \mathbb{U}) \times \mathbb{M}(\mathbb{N} \times E; \mathbb{U}).$$

It is admissible if for every $(n, x, t) \in \mathbb{N} \times E \times \mathbb{R}_+$,

$$u(n, x, t) \in \mathbb{U}(\phi(x, t)) \text{ and } u_\partial(n, x) \in \mathbb{U}(\phi(x, t_*(x))).$$

The class of admissible control strategies will be denoted by $\mathcal{U}$.

Given a control strategy $U = (u, u_\partial) \in \mathcal{U}$, the motion of a piecewise deterministic process $X(t)$ is described in the following manner. Define $T_0 = 0$ and $X(0) = x$. Assume that the process $\{X(t)\}$ is located at Z_n at the nth jump time T_n. Then select a random variable S_n having distribution

$$F(t) = 1 - I_{\{t<t_*(Z_n)\}} e^{-\int\limits_0^t \lambda(\phi(Z_n,t),u(n,Z_n,s))ds}.$$

Define $T_{n+1} = T_n + S_n$, and for $t \in [T_n, T_{n+1})$,

$$X(t) = \phi(Z_n, t - T_n).$$

Let Z_{n+1} be a random variable having distribution

$$Q(\phi(Z_n, T_{n+1}), u(n, Z_n, S_n)); .)$$

if $\phi(Z_n, T_{n+1}) \in E$ and

$$Q(\phi(Z_n, T_{n+1}), u_\partial(n, Z_n); .)$$

if $\phi(Z_n, T_{n+1}) \in \partial E$. At time T_{n+1}, the process $\{X(t)\}$ is defined by

$$X(T_{n+1}) = Z_{n+1}.$$

Now we give a more precise definition of the controlled piecewise deterministic Markov process described above. Consider the state space $\widehat{E} = E \times E \times \mathbb{R}_+ \times \mathbb{N}$. For a control strategy $U = (u, u_\partial)$, let us introduce the following parameters for $\hat{x} = (x, z, s, n) \in \widehat{E}$:

- the flow $\widehat{\phi}(\hat{x}, t) = (\phi(x, t), z, s + t, n)$,
- the jump rate $\widehat{\lambda}^U(\hat{x}) = \lambda(x, u(n, z, s))$,
- the transition measure

$$\widehat{Q}^U(\hat{x}, A \times B \times \{0\} \times \{n+1\}) = \begin{cases} Q(x, u(n, z, s); A \cap B) & \text{if } x \in E, \\ Q(x, u_\partial(n, z); A \cap B) & \text{if } x \in \partial E, \end{cases}$$

for A and B in $\mathcal{B}(E)$.

From [25, section 25], it can be shown that for every control strategy $U = (u, u_\partial) \in \mathcal{U}$, there exists a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \{P^U_{\hat{x}}\}_{\hat{x}\in\widehat{E}})$ such that the PDMP $\{\widehat{X}^U(t)\}$ with local characteristics $(\widehat{\phi}, \widehat{\lambda}^U, \widehat{Q}^U)$ may be constructed as follows. For notational simplicity, the probability $P^U_{\hat{x}_0}$ will be denoted by $P^U_{(x,k)}$ for $\hat{x}_0 = (x, x, 0, k) \in \widehat{E}$. Moreover, $E^U_{\hat{x}_0}$ denotes the expectation under the probability $P^U_{\hat{x}_0}$, and $E^U_{\hat{x}_0}$ will be denoted by $E^U_{(x,k)}$ for $\hat{x}_0 = (x, x, 0, k) \in \widehat{E}$.

Take a random variable T_1 such that

$$P^U_{(x,k)}(T_1 > t) \doteq \begin{cases} e^{-\Lambda^U(x,k,t)} & \text{for } t < t_*(x), \\ 0 & \text{for } t \geq t_*(x), \end{cases}$$

where for $x \in E$ and $t \in [0, t_*(x)[$, $\Lambda^U(x, k, t) \doteq \int_0^t \lambda(\phi(x, s), u(k, x, s))ds$. If T_1 is equal to infinity, then for $t \in \mathbb{R}_+$, $\widehat{X}^U(t) = \big(\phi(x, t), x, t, k\big)$. Otherwise, select independently an $\widehat{E}$-valued random variable (labeled $\widehat{X}^U_1$) having distribution

$$P^U_{(x,k)}(\widehat{X}^U_1 \in A \times B \times \{0\} \times \{k+1\} | \sigma\{T_1\})$$

(where $\sigma\{T_1\}$ is the σ-field generated by T_1) defined by

$$\begin{cases} Q(\phi(x,T_1), u(k,x,T_1); A \cap B) & \text{if } \phi(x,T_1) \in E, \\ Q(\phi(x,T_1), u_\partial(k,x); A \cap B) & \text{if } \phi(x,T_1) \in \partial E. \end{cases}$$

The trajectory of $\{\widehat{X}^U(t)\}$ starting from $(x, x, 0, k)$, for $t \leq T_1$, is given by

$$\widehat{X}^U(t) \doteq \begin{cases} \big(\phi(x,t), x, t, k\big) & \text{for } t < T_1, \\ \widehat{X}_1^U & \text{for } t = T_1. \end{cases}$$

Starting from $\widehat{X}^U(T_1) = \widehat{X}_1^U$, we now select the next interjump time $T_2 - T_1$ and postjump location $\widehat{X}^U(T_2) = \widehat{X}_2^U$ in a similar way. The sequence of jump times of the PDMP is denoted by $(T_n)_{n \in \mathbb{N}}$. Let us define the components of the PDMP $\{\widehat{X}^U(t)\}$ by

$$\widehat{X}^U(t) = \big(X(t), Z(t), \tau(t), N(t)\big). \tag{2.2}$$

From the previous construction of the PDMP $\{\widehat{X}^U(t)\}$, it is easy to see that $X(t)$ corresponds to the trajectory of the system, $Z(t)$ is the value of $X(t)$ at the last jump time before t, $\tau(t)$ is time elapsed between the last jump and time t, and $N(t)$ is the number of jumps of the process $\{X(t)\}$ at time t. As in Davis [25], we consider the following assumption to avoid accumulation points of the jump times.

Assumption 2.8 *For every $x \in E$, $U = (u, u_\partial) \in \mathcal{U}$, and $t \geq 0$, we have*

$$E^U_{(x,0)}\left[\sum_{i=1}^{\infty} I_{\{T_i \leq t\}}\right] < \infty.$$

Remark 2.9 In particular, a consequence of Assumption 2.8 is that $T_m \to \infty$ as $m \to \infty$, $P^U_{(x,0)}$ for all $x \in E$, $U \in \mathcal{U}$.

The costs of our control problem will contain two terms: a running cost f and a boundary cost r, satisfying the following properties.

Assumption 2.10 *$f \in \mathbb{M}(\overline{E} \times \mathbb{U})^+$ and $r \in \mathbb{M}(\partial E \times \mathbb{U})^+$.*

Define for $\alpha \geq 0$, $t \in \mathbb{R}_+$, and $U \in \mathcal{U}$, the finite-horizon α-discounted cost function

$$\begin{aligned}\mathbf{J}^\alpha(U,t) = &\int_0^t e^{-\alpha s} f\big(X(s), u(N(s), Z(s), \tau(s))\big) ds \\ &+ \int_0^t e^{-\alpha s} r\big(X(s-), u_\partial(N(s-), Z(s-))\big) dp^*(s),\end{aligned}$$

where $p^*(t) = \sum_{i=1}^{\infty} I_{\{T_i \leq t\}} I_{\{X(T_i-) \in \partial E\}}$ counts the number of times the process hits the boundary up to time t, and for notational simplicity, set $\mathbf{J}(U,t) = \mathbf{J}^0(U,t)$.

The long run expected average cost to minimize over $\mathcal{U}$ is given by

$$\mathcal{A}(U, x) = \overline{\lim_{t \to +\infty}} \frac{1}{t} E^U_{(x,0)}[\mathbf{J}(U, t)],$$

and the average value function is defined by

$$\mathcal{J}_{\mathcal{A}}(x) = \inf_{U \in \mathcal{U}} \mathcal{A}(U, x).$$

For the infinite-horizon expected α-discounted case, with $\alpha > 0$, the cost we want to minimize is given by

$$\mathcal{D}^{\alpha}(U, x) = E^U_{(x,0)}[\mathbf{J}^{\alpha}(U, \infty)], \tag{2.3}$$

and the α-discount value function is

$$\mathcal{J}^{\alpha}_{\mathcal{D}}(x) = \inf_{U \in \mathcal{U}} \mathcal{D}^{\alpha}(U, x). \tag{2.4}$$

We also consider a truncated version of the discounted problem defined, for each $m \in \mathbb{N}$, as

$$\mathcal{D}^{\alpha}_m(U, x) = E^U_{(x,0)}[\mathbf{J}^{\alpha}(U, T_m)]. \tag{2.5}$$

We need the following assumption to avoid infinite costs for the discounted case.

Assumption 2.11 *For all* $\alpha > 0$ *and all* $x \in E$, $\mathcal{J}^{\alpha}_{\mathcal{D}}(x) < \infty$.

It is clear that for all $x \in E$, $0 \leq \inf_{U \in \mathcal{U}} \mathcal{D}^{\alpha}_m(U, x) \leq \mathcal{J}^{\alpha}_{\mathcal{D}}(x) < \infty$.

2.3 Discrete-Time Markov Control Problem

The class of strategies denoted by $\mathcal{U}$ was introduced in the previous section as the set of admissible control strategies for a PDMP. As mentioned in the introduction, we will study in Sect. 3.2 how the original continuous-time control problem can be associated to an optimality equation of a discrete-time problem related to the embedded Markov chain given by the postjump location of the PDMP. In this section, we first present the definitions of the discrete-time ordinary and relaxed control sets used in the formulation of the optimality equation of the discrete-time Markov control problem as well as the characterization of some topological properties of these sets. In particular, using a result of the theory of multifunctions (see the book by Castaing and Valadier [9]), it is shown that the set of relaxed controls is compact. In what follows, we present some important operators associated to the optimality equation of the discrete-time problem as well as some measurability properties.

2.3.1 Discrete-Time Ordinary and Relaxed Controls

We present in this subsection the set of discrete-time relaxed controls and the subset of ordinary controls. Let $\mathbb{C}(\mathbb{U})$ be a Banach space equipped with the topology of uniform convergence. Consider the Banach space $L^1(\mathbb{R}_+; \mathbb{C}(\mathbb{U}))$ of Bochner integrable functions with values in the Banach space $\mathbb{C}(\mathbb{U})$, see [32]. Let $\mathcal{M}(\mathbb{U})$ be equipped with the weak* topology $\sigma(\mathcal{M}(\mathbb{U}), \mathbb{C}(\mathbb{U}))$. The subset $\mathcal{P}(\mathbb{U})$ of $\mathcal{M}(\mathbb{U})$ is endowed with the induced topology. We denote by $L^\infty(\mathbb{R}_+; \mathcal{M}(\mathbb{U}))$ the linear space of equivalence classes of bounded measurable functions from $\mathbb{R}_+$ to $\mathcal{M}(\mathbb{U})$. It is the dual of $L^1(\mathbb{R}_+; \mathbb{C}(\mathbb{U}))$ (see [4, pp. 94–95]). We make the following definitions:

Definition 2.12 *Define $\mathcal{V}^r$ as the set of all measurable functions μ defined on $\mathbb{R}_+$ with values in $\mathcal{P}(\mathbb{U})$ such that $\mu(t, \mathbb{U}) = 1$ for all $t \in \mathbb{R}_+$. For $x \in E$, $\mathcal{V}^r(x)$ is defined as the set of all measurable functions μ defined on $\mathbb{R}_+$ with values in $\mathcal{P}(\mathbb{U})$ such that $\mu(t, \mathbb{U}(\phi(x, t))) = 1$.*

We have the following proposition.

Proposition 2.13 *For every fixed $x \in E$,*

(*i*) *the multifunction Γ_x from $\mathbb{R}_+$ to $\mathbb{U}$ defined by $\Gamma_x(t) = \mathbb{U}(\phi(x, t))$ is measurable.*
(*ii*) *$\mathcal{V}^r$ and $\mathcal{V}^r(x)$ are compact metric spaces with metric compatible with the weak* topology $\sigma(L^\infty(\mathbb{R}_+; \mathcal{M}(\mathbb{U})), L^1(\mathbb{R}_+; \mathbb{C}(\mathbb{U})))$.*

Proof Let us prove (i). Considering any open set G in $\mathbb{U}$, we have from Assumption 2.4 that $\mathbb{U}^{-1}[G] \doteq \{y \in E : \mathbb{U}(y) \cap G \neq \emptyset\} \in \mathcal{B}(E)$. Consequently, if $t_*(x) = +\infty$, then

$$\{t \in \mathbb{R}_+ : \mathbb{U}(\phi(x, t)) \cap G \neq \emptyset\} = \{t \in \mathbb{R}_+ : \phi(x, t) \in \mathbb{U}^{-1}[G]\} \in \mathcal{B}(\mathbb{R}_+),$$

since ϕ is continuous, so that (i) holds. Now if $t_*(x) < +\infty$ and $\Delta_u \notin G$, then

$$\{t \in \mathbb{R}_+ : \mathbb{U}(\phi(x, t)) \cap G \neq \emptyset\} = \{t \in [0, t_*(x)] : \phi(x, t) \in \mathbb{U}^{-1}[G]\} \in \mathcal{B}(\mathbb{R}_+),$$

since ϕ is continuous, giving (i). Finally, if $t_*(x) < +\infty$ and $\Delta_u \in G$, then

$$\begin{aligned} &\{t \in \mathbb{R}_+ : \mathbb{U}(\phi(x, t)) \cap G \neq \emptyset\} \\ &\qquad = \{t \in [0, t_*(x)] : \phi(x, t) \in \mathbb{U}^{-1}[G]\} \cup]t_*(x), +\infty[\in \mathcal{B}(\mathbb{R}_+), \end{aligned}$$

since ϕ is continuous, so that (i) also holds for this case.

Let us now prove (ii). From Theorem V-2 in [9], it follows that $\mathcal{V}^r$ and $\mathcal{V}^r(x)$ for $x \in E$ are compact sets with respect to the weak* topology $\sigma(L^\infty(\mathbb{R}_+; \mathcal{M}(\mathbb{U})), L^1(\mathbb{R}_+; \mathbb{C}(\mathbb{U})))$. Notice that $L^1(\mathbb{R}_+; \mathbb{C}(\mathbb{U}))$ is separable, since $\mathbb{U}$ is compact. Consequently, from Bishop's theorem (see Theorem I.3.11 in [65]), there exists a metric compatible with the topology $\sigma(L^\infty(\mathbb{R}_+; \mathcal{M}(\mathbb{U})), L^1(\mathbb{R}_+; \mathbb{C}(\mathbb{U})))$ on $\mathcal{V}^r$.

This shows that $\mathcal{V}^r$ is a compact metric space and therefore a Borel space and that $\mathcal{V}^r(x)$ is a compact set of $\mathcal{V}^r$ for all $x \in E$, completing the proof of (ii). □

Note that a sequence $(\mu_n)_{n\in\mathbb{N}}$ in $\mathcal{V}^r(x)$ converges to μ if and only if

$$\lim_{n\to\infty} \int_{\mathbb{R}_+} \int_{\mathbb{U}(\phi(x,t))} g(t,u)\mu_n(t,du)dt = \int_{\mathbb{R}_+} \int_{\mathbb{U}(\phi(x,t))} g(t,u)\mu(t,du)dt, \quad (2.6)$$

for all $g \in L^1(\mathbb{R}_+; \mathbb{C}(\mathbb{U}))$.
Therefore, the set of relaxed controls are defined as follows. For $x \in E$,

$$\mathbb{V}^r(x) = \mathcal{V}^r(x) \times \mathcal{P}\big(\mathbb{U}(\phi(x, t_*(x)))\big),$$
$$\mathbb{V}^r = \mathcal{V}^r \times \mathcal{P}\big(\mathbb{U}\big).$$

The set of ordinary controls, denoted by $\mathbb{V}$ (or $\mathbb{V}(x)$ for $x \in E$), is defined as above except that it is composed of deterministic functions instead of probability measures. More specifically, we have

$$\mathcal{V}(x) = \Big\{\nu \in \mathbb{M}(\mathbb{R}_+, \mathbb{U}) : (\forall t \in \mathbb{R}_+), \nu(t) \in \mathbb{U}(\phi(x,t))\Big\},$$
$$\mathbb{V}(x) = \mathcal{V}(x) \times \mathbb{U}(\phi(x, t_*(x))),$$
$$\mathbb{V} = \mathbb{M}(\mathbb{R}_+, \mathbb{U}) \times \mathbb{U}.$$

Consequently, the set of ordinary controls is a subset of the set of relaxed controls $\mathbb{V}^r$ (or $\mathbb{V}^r(x)$ for $x \in E$) by identifying every control action $u \in \mathbb{U}$ with the Dirac measure concentrated on u. Thus we can write that $\mathbb{V} \subset \mathbb{V}^r$ (or $\mathbb{V}(x) \subset \mathbb{V}^r(x)$ for $x \in E$), and from now, on we will consider that $\mathbb{V}$ (or $\mathbb{V}(x)$ for $x \in E$) will be endowed with the topology generated by $\mathbb{V}^r$.
The necessity to introduce the class of relaxed controls $\mathbb{V}^r$ is justified by the fact that in general, there does not exist a topology for which $\mathbb{V}$ and $\mathbb{V}(x)$ are compact sets. However, from the previous construction, it follows that $\mathbb{V}^r$ and $\mathbb{V}^r(x)$ are compact sets.
We present next the definition of a shifted control strategy, which will be useful in the following sections.

Definition 2.14 *For every $x \in E$, $t \in [0, t_*(x))$, and $\Theta = (\mu, \mu_\partial) \in \mathbb{V}^r(x)$, define*

$$\big[\Theta\big]_t = \big(\mu(.+t), \mu_\partial\big). \quad (2.7)$$

Clearly, $\big[\Theta\big]_t \in \mathbb{V}^r(\phi(x,t))$.

As in [49], page 14, we need that the set of feasible state/relaxed-control pairs is a measurable subset of $\mathcal{B}(E) \times \mathcal{B}(\mathbb{V}^r)$; that is, we need the following assumption.

Assumption 2.15

$$\mathcal{K} \doteq \big\{(x,\Theta) : \Theta \in \mathbb{V}^r(x), x \in E\big\} \textit{ is a Borel subset of } E \times \mathbb{V}^r.$$

2.3.2 Discrete-Time Operators and Measurability Properties

In this section, we present some important operators associated with the optimality equation of the discrete-time problem as well as some measurability properties.

We consider the following notation:

$$w(x,\mu) \doteq \int_{\mathbb{U}} w(x,u)\mu(du), \tag{2.8}$$

$$Qh(x,\mu) \doteq \int_{\mathbb{U}} \int_E h(z)Q(x,u;dz)\mu(du), \tag{2.9}$$

$$\lambda Qh(x,\mu) \doteq \int_{\mathbb{U}} \lambda(x,u) \int_E h(z)Q(x,u;dz)\mu(du), \tag{2.10}$$

for $x \in \overline{E}$, $\mu \in \mathcal{P}(\mathbb{U})$, $h \in \mathbb{M}(E)^+$, and $w \in \mathbb{M}(\overline{E} \times \mathbb{U})^+$.

The following operators will be associated with the optimality equations of the discrete-time problems that will be presented in the following sections. For $\alpha > 0$, $\Theta = (\mu, \mu_\partial) \in \mathbb{V}^r$, $(x, A) \in E \times \mathcal{B}(E)$, define

$$\Lambda^\mu(x,t) \doteq \int_0^t \lambda(\phi(x,s),\mu(s))ds, \tag{2.11}$$

$$\begin{aligned} G_\alpha(x,\Theta;A) \doteq &\int_0^{t_*(x)} e^{-\alpha s - \Lambda^\mu(x,s)} \lambda Q I_A(\phi(x,s),\mu(s))ds \\ &+ e^{-\alpha t_*(x) - \Lambda^\mu(x,t_*(x))} Q(\phi(x,t_*(x)),\mu_\partial;A). \end{aligned} \tag{2.12}$$

For $x \in E$, $\Theta = (\mu, \mu_\partial) \in \mathbb{V}^r$, $h \in \mathbb{M}(E)^+$, $v \in \mathbb{M}(E \times \mathbb{U})^+$, $w \in \mathbb{M}(\partial E \times \mathbb{U})^+$, $\alpha > 0$, set

$$G_\alpha h(x,\Theta) \doteq \int_E h(y)G_\alpha(x,\Theta;dy), \tag{2.13}$$

$$L_\alpha v(x,\Theta) \doteq \int_0^{t_*(x)} e^{-\alpha s - \Lambda^\mu(x,s)} v(\phi(x,s),\mu(s))ds, \tag{2.14}$$

$$H_\alpha w(x,\Theta) \doteq e^{-\alpha t_*(x) - \Lambda^\mu(x,t_*(x))} w(\phi(x,t_*(x)),\mu_\partial). \tag{2.15}$$

For $h \in \mathbb{M}(E)$ (respectively $v \in \mathbb{M}(E \times \mathbb{U})$), $G_\alpha h(x, \Theta) = G_\alpha h^+(x, \Theta) - G_\alpha h^-(x, \Theta)$ (respectively $L_\alpha v(x, \Theta) = L_\alpha v^+(x, \Theta) - L_\alpha v^-(x, \Theta)$), provided the difference has a meaning.
It will be useful in what follows to define the function $\mathcal{L}_\alpha(x, \Theta)$ as

$$\mathcal{L}_\alpha(x, \Theta) \doteq L_\alpha I_{E\times\mathbb{U}}(x, \Theta). \tag{2.16}$$

Similarly, for $\alpha = 0$, $\Theta = \big(\mu, \mu_\partial\big) \in \mathbb{V}^r$, $(x, A) \in E \times \mathcal{B}(E)$, we define

$$G_0(x, \Theta; A) \doteq \int_0^{t_*(x)} e^{-\Lambda^\mu(x,s)} \lambda Q I_A(\phi(x, s), \mu(s))ds + e^{-\Lambda^\mu(x,t_*(x))} Q(\phi(x, t_*(x)), \mu_\partial; A),$$

and for $x \in E$, $\Theta = \big(\mu, \mu_\partial\big) \in \mathbb{V}^r$, $h \in \mathbb{M}(E)^+$, $v \in \mathbb{M}(E\times\mathbb{U})^+$, $w \in \mathbb{M}(\partial E\times\mathbb{U})^+$, set

$$G_0 h(x, \Theta) \doteq \int_E h(y) G_0(x, \Theta; dy),$$
$$L_0 v(x, \Theta) \doteq \int_0^{t_*(x)} e^{-\Lambda^\mu(x,s)} v(\phi(x, s), \mu(s))ds,$$
$$H_0 w(x, \Theta) \doteq e^{-\Lambda^\mu(x,t_*(x))} w(\phi(x, t_*(x)), \mu_\partial).$$

For $h \in \mathbb{M}(E)$ (respectively $v \in \mathbb{M}(E \times \mathbb{U})$), $G_0 h(x, \Theta) = G_0 h^+(x, \Theta) - G_0 h^-(x, \Theta)$ (respectively $L_0 v(x, \Theta) = L_0 v^+(x, \Theta) - L_0 v^-(x, \Theta)$), provided the difference has a meaning. Moreover, $\mathcal{L}_0(x, \Theta) \doteq L_0 I_{E\times\mathbb{U}}(x, \Theta)$. We write for simplicity $G_0 = G$, $L_0 = L$, $H_0 = H$, $\mathcal{L}_0 = \mathcal{L}$.

Remark 2.16 A consequence of item (b) of Assumption 2.5 is that for every $\alpha \in \mathbb{R}_+$ and $x \in E$ with $t_*(x) = +\infty$, $\lim_{t\to+\infty} e^{-\alpha t - \int_0^t \underline{\lambda}(\phi(x,s))ds} = 0$. Therefore, for every $x \in E$ with $t_*(x) = +\infty$, $A \in \mathcal{B}(E)$, $\alpha \in \mathbb{R}_+$, $\Theta = (\mu, \mu_\partial) \in \mathbb{V}^r(x)$, $w \in \mathbb{M}(\partial E \times \mathbb{U})^+$, we have $G_\alpha(x, \Theta; A) = \int_0^{t_*(x)} e^{-\alpha s - \Lambda^\mu(x,s)} \lambda Q I_A(\phi(x, s), \mu(s))ds$ and $H_\alpha w(x, \Theta) = 0$.

The next proposition presents some important measurability properties of the operators G_α, L_α, and H_α (defined in (2.12), (2.14), and (2.15)), and its proof can be found in Sect. 2.4.

Proposition 2.17 *Let $\alpha \in \mathbb{R}_+$, $w_0 \in \mathbb{M}(E)$ be bounded from below, $w_1 \in \mathbb{M}(E \times \mathbb{U})^+$, and $w_2 \in \mathbb{M}(\partial E \times \mathbb{U})^+$. Then the mappings $G_\alpha w_0(x, \Theta)$, $L_\alpha w_1(x, \Theta)$, and $H_\alpha w_2(x, \Theta)$ defined on $E\times\mathbb{V}^r$ with values in $\mathbb{R}\cup\{+\infty\}$ are $\mathcal{B}(E\times\mathbb{V}^r)$-measurable.*

We now present the definitions of the one-stage optimization operators in terms of the running cost f and the boundary cost r introduced in Sect. 2.2.

Definition 2.18 *For $\alpha \in \mathbb{R}_+$, the (ordinary) one-stage optimization function associated with the pair (ρ, h) is defined by*

$$\mathcal{T}_\alpha(\rho, h)(x) = \inf_{\Upsilon \in \mathbb{V}(x)} \Big\{ -\rho \mathcal{L}_\alpha(x, \Upsilon) + L_\alpha f(x, \Upsilon) + H_\alpha r(x, \Upsilon) + G_\alpha h(x, \Upsilon) \Big\},$$

and the relaxed one-stage optimization function associated with (ρ, h) is defined by

$$\mathcal{R}_\alpha(\rho, h)(x) = \inf_{\Theta \in \mathbb{V}^r(x)} \Big\{ -\rho \mathcal{L}_\alpha(x, \Theta) + L_\alpha f(x, \Theta) + H_\alpha r(x, \Theta) + G_\alpha h(x, \Theta) \Big\},$$

for $\rho \in \mathbb{R}$ and $h \in \mathbb{M}(E)$ bounded from below.

In particular, for $\alpha = 0$, we write for simplicity $\mathcal{T}_0 = \mathcal{T}$ and $\mathcal{R}_0 = \mathcal{R}$.

Definition 2.19 *Let us introduce the following sets of measurable selectors associated with $\big(\mathbb{U}(x)\big)_{x \in E}$ (respectively $\big(\mathbb{V}(x)\big)_{x \in E}$, $\big(\mathbb{V}^r(x)\big)_{x \in E}$):*

$$\begin{aligned}
\mathcal{S}_{\mathbb{U}} &= \Big\{ u \in \mathbb{M}(\overline{E}, \mathbb{U}) : \quad \textit{for all } x \in \overline{E},\ u(x) \in \mathbb{U}(x) \Big\}, \\
\mathcal{S}_{\mathbb{V}} &= \Big\{ (\nu, \nu_\partial) \in \mathbb{M}(E, \mathbb{V}) : \quad \textit{for all } x \in E,\ \big(\nu(x), \nu_\partial(x)\big) \in \mathbb{V}(x) \Big\}, \\
\mathcal{S}_{\mathbb{V}^r} &= \Big\{ (\mu, \mu_\partial) \in \mathbb{M}(E, \mathbb{V}^r) : \quad \textit{for all } x \in E,\ \big(\mu(x), \mu_\partial(x)\big) \in \mathbb{V}^r(x) \Big\}.
\end{aligned}$$

Remark 2.20 The set $\mathcal{S}_{\mathbb{U}}$ characterizes a control law $u(x)$ that depends only on the value of the state variable x. On the other hand, $(\nu, \nu_\partial) \in \mathcal{S}_{\mathbb{V}}$ characterizes an ordinary control law for the control problem associated with the one-stage optimization operator. Indeed, starting from x, it defines the control law for all $t \in [0, t_*(x))$ through the function $\nu(x, t)$, and at $t = t_*(x)$ (if $t_*(x) < \infty$) through ν_∂. Finally, $(\mu, \mu_\partial) \in \mathcal{S}_{\mathbb{V}^r}$ characterizes a relaxed control law for the control problem associated with the relaxed one-stage optimization operator. Since it starts from x, it defines a probability over the feasible control actions for all $t \in [0, t_*(x))$ through the probability measure $\mu(x, t)$, and at $t = t_*(x)$ (if $t_*(x) < \infty$) through the probability measure μ_∂.

Since for $u \in \mathcal{S}_{\mathbb{U}}$, $u(x)$ is not defined for $x \notin \overline{E}$, we introduce the following notation.

Definition 2.21 *Consider $u \in \mathcal{S}_{\mathbb{U}}$, $x \in E$ with $t_*(x) < +\infty$. Then with a slight abuse of notation, $u(\phi(x, t))$ is defined for $t > t_*(x)$ by Δ_u.*

For $\alpha \in \mathbb{R}_+$, $\rho \in \mathbb{R}$, and $h \in \mathbb{M}(E)$ bounded from below, the one-stage optimization problem associated with the operator $\mathcal{T}_\alpha(\rho, h)$ (respectively $\mathcal{R}_\alpha(\rho, h)$) consists in finding a measurable selector $\Upsilon \in \mathcal{S}_{\mathbb{V}}$ (respectively $\Theta \in \mathcal{S}_{\mathbb{V}^r}$) such that for all $x \in E$,

$$\mathcal{T}_\alpha(\rho, h)(x) = -\rho\mathcal{L}_\alpha(x, \Upsilon(x)) + L_\alpha f(x, \Upsilon(x)) + H_\alpha r(x, \Upsilon(x)) + G_\alpha h(x, \Upsilon(x)),$$

and respectively

$$\mathcal{R}_\alpha(\rho, h)(x) = -\rho\mathcal{L}_\alpha(x, \Theta(x)) + L_\alpha f(x, \Theta(x)) + H_\alpha r(x, \Theta(x)) + G_\alpha h(x, \Theta(x)).$$

Finally, we conclude this section by showing that there exist two natural mappings from $\mathcal{S}_\mathbb{U}$ to $\mathcal{S}_\mathbb{V}$ and from $\mathcal{S}_\mathbb{U}$ to $\mathcal{U}$.

Definition 2.22 *For $u \in \mathcal{S}_\mathbb{U}$, define the mapping*

$$u_\phi : x \to \big(u(\phi(x, .)), u(\phi(x, t_*(x)))\big)$$

of the space E into $\mathbb{V}$.

Definition 2.23 *For $u \in \mathcal{S}_\mathbb{U}$, define the mapping*

$$U_{u_\phi} : (n, x, t) \to \big(u(\phi(x, t)), u(\phi(x, t_*(x)))\big)$$

of the space $\mathbb{N} \times E \times \mathbb{R}_+$ into $\mathbb{U}$.

We have the following proposition, proved in Sect. 2.4.

Proposition 2.24 *If $u \in \mathcal{S}_\mathbb{U}$, then $u_\phi \in \mathcal{S}_\mathbb{V}$. If $u \in \mathcal{S}_\mathbb{U}$, then $U_{u_\phi} \in \mathcal{U}$.*

Remark 2.25 The measurable selectors of type u_ϕ as in Definition 2.22 are called feedback measurable selectors in the class $\mathcal{S}_\mathbb{V} \subset \mathcal{S}_{\mathbb{V}^r}$, and the control strategies of type U_{u_ϕ} as in Definition 2.23 are called feedback control strategies in the class $\mathcal{U}$.

Remark 2.26 For $u \in \mathcal{S}_\mathbb{U}$ and $h \in \mathbb{M}(E)^+$, we have the following identity:

$$\begin{aligned} E^{U_{u_\phi}}_{(x,0)}\Big(\int_0^{T_1} & e^{-\alpha s} f(\phi(x, s), u(\phi(x, s)))ds + e^{-\alpha T_1} h(X(T_1)) \\ &+ e^{-\alpha T_1} r(\phi(x, t_*(x)), u(\phi(x, t_*(x)))I_{\{T_1 = t_*(x)\}}\Big) \\ = {} & L_\alpha f(x, u_\phi) + H_\alpha r(x, u_\phi) + G_\alpha h(x, u_\phi). \end{aligned} \tag{2.17}$$

Iterating (2.17) and using the strong Markov property of the process $\{\widehat{X}^{U_{u_\phi}}(t) = (X(t), Z(t), \tau(t), N(t))\}$, it follows that for $m \in \mathbb{N}_*$,

$$\begin{aligned} \sum_{k=0}^{m-1} G^k_\alpha(L_\alpha f + H_\alpha r)(x, u_\phi) = E^{U_{u_\phi}}_{(x,0)}\Bigg[\int_0^{T_m} e^{-\alpha s} f\big(X(s), u(N(s), Z(s), \tau(s))\big)ds \\ + \int_0^{T_m} e^{-\alpha s} r\big(X(s-), u_\partial(N(s-), Z(s-))\big)dp^*(s)\Bigg]. \end{aligned}$$

2.4 Proofs of the Results of Section 2.3

Proof of Lemma 2.2: Define

$$w^{sup}(x) \doteq \overline{\lim_{n\to+\infty}}\, n\left[w\left(\phi\left(x, t_*(x) \wedge \frac{1}{n+1}\right)\right) - w(x)\right]$$
$$w^{inf}(x) \doteq \underline{\lim_{n\to+\infty}}\, n\left[w\left(\phi\left(x, t_*(x) \wedge \frac{1}{n+1}\right)\right) - w(x)\right].$$

Since $\left\{w\left(\phi\left(x, t_*(x) \wedge \frac{1}{n+1}\right)\right)\right\}_{n\in\mathbb{N}}$ is a sequence in $\mathbb{M}(E)$, it follows that $w^{sup}(x)$ and $w^{inf}(x)$ are Borel measurable functions from E into $\mathbb{R} \cup \{\infty\} \cup \{-\infty\}$. Consequently, the set $\mathcal{D}_w \doteq \left\{x \in E : w^{sup}(x) = w^{inf}(x)\right\} \cap \left\{x \in E : w^{sup}(x) \in \mathbb{R}\right\}$ belongs to $\mathcal{B}(E)$.
Define the function $\mathcal{X}w(x)$ by

$$\mathcal{X}w(x) = \begin{cases} w^{sup}(x), & \text{if } x \in \mathcal{D}_w, \\ g(x), & \text{otherwise,} \end{cases}$$

where g is any function in $\mathbb{M}(E)$.

Clearly, $\mathcal{X}w$ belongs to $\mathbb{M}(E)$. Since $w \in \mathbb{M}^{ac}(E)$, we have that for every $x \in E$, there exists a set $T_x^w \in \mathcal{B}([0, t_*(x)))$ such that $\eta((T_x^w)^c \cap [0, t_*(x))) = 0$ and $w(\phi(x, .))$ admits derivatives in T_x^w. Let us show now that for every $x \in E$ and $t_0 \in T_x^w$, we have that $\phi(x, t_0) \in \mathcal{D}_w$ and $\mathcal{X}w(\phi(x, t_0)) = \frac{dw(\phi(x,t))}{dt}\big|_{t=t_0}$. First, recall that $w(\phi(x, t_0 + \epsilon)) = w(\phi(\phi(x, t_0), \epsilon))$. Since $w(\phi(t, x))$ admits derivative in $t = t_0$, we have that

$$\begin{aligned}\frac{dw(\phi(x,t))}{dt}\Big|_{t=t_0} &= \lim_{\substack{\epsilon\to 0\\ \epsilon>0}} \frac{1}{\epsilon}\big[w(\phi(x, t_0 + \epsilon)) - w(\phi(x, t_0))\big] \\ &= \lim_{\substack{\epsilon\to 0\\ \epsilon>0}} \frac{1}{\epsilon}\big[w(\phi(\phi(x, t_0), \epsilon)) - w(\phi(x, t_0))\big] \\ &= w^{sup}(\phi(x, t_0)) = w^{inf}(\phi(x, t_0)) = \mathcal{X}w(\phi(x, t_0)).\end{aligned}$$

Therefore, $\mathcal{X}w$ satisfies (2.1), which establishes the result. □

We proceed to the proof of Proposition 2.17. First we need the following lemmas.

Lemma 2.27 *Let $h \in \mathbb{B}(\mathbb{R}_+ \times \mathbb{U})$ be such that $h(t, .) \in \mathbb{C}(\mathbb{U})$ for every $t \in \mathbb{R}_+$. Then $\overline{h} : \mathbb{R}_+ \to \mathbb{C}(\mathbb{U})$ defined by $\overline{h}(t) = h(t, .)$ is measurable.*

Proof Since $\mathbb{C}(\mathbb{U})$ is separable, there exists a countable family $(c_i)_{i\in\mathbb{N}}$ dense in $\mathbb{C}(\mathbb{U})$. From Proposition D.5 in [49], the $\mathbb{R}_+$-valued mapping defined on $\mathbb{R}_+$ by $t \to \sup_{u\in\mathbb{U}} \big|c_i(u) - h(t, u)\big|$ is $\mathcal{B}(\mathbb{R}_+)$-measurable. Consequently, $\big\{t \in \mathbb{R}_+ : \sup_{u\in\mathbb{U}} \big|c_i(u) - h(t, u)\big| < \frac{1}{n+1}\big\}$ belongs to $\mathcal{B}(\mathbb{R}_+)$ for every $(i, n) \in \mathbb{N}^2$. Therefore,

$\{t \in \mathbb{R}_+ : h(t,.) \in B(c_i, \frac{1}{n+1})\} \in \mathcal{B}(\mathbb{R}_+)$, where $B(c_i, \frac{1}{n+1}) \doteq \{f \in \mathbb{C}(\mathbb{U}) : \|f - c_i\| < \frac{1}{n+1}\}$ for every $(i,n) \in \mathbb{N}^2$. Clearly, the family $\big(B(c_i, \frac{1}{n+1})\big)_{(i,n)\in\mathbb{N}^2}$ is a base for the topology of $\mathbb{C}(\mathbb{U})$, which proves the result. □

Lemma 2.28 *Let $h \in \mathbb{B}(\mathbb{R}_+ \times \mathbb{U})$. Then the real-valued mapping defined on $\mathcal{V}^r$ by*

$$\mu \to \int_{\mathbb{R}_+}\int_{\mathbb{U}} e^{-s} h(s,u)\mu(s,du)ds$$

is $\mathcal{B}(\mathcal{V}^r)$-measurable.

Proof Let $\mathcal{H}$ be the class of functions $h \in \mathbb{B}(\mathbb{R}_+ \times \mathbb{U})$ such that the real-valued mapping defined on $\mathcal{V}^r$ by

$$\mu \to \int_{\mathbb{R}_+}\int_{\mathbb{U}} e^{-s} h(s,u)\mu(s,du)ds$$

is $\mathcal{B}(\mathcal{V}^r)$-measurable. The set $\mathcal{H}$ is a closed linear subspace of $\mathbb{B}(\mathbb{R}_+ \times \mathbb{U})$ equipped with the supremum norm. Moreover, the limit of every increasing uniformly bounded sequence of nonnegative functions of $\mathcal{H}$ belongs to $\mathcal{H}$. Let $\mathcal{H}_0$ denote the set of functions $g \doteq g_1 g_2$, $g_1 \geq 0$ belongs to $\mathbb{B}(\mathbb{R}_+)$ and $g_2 \geq 0$ is a lower semicontinuous function defined on $\mathbb{U}$. Clearly, $\mathcal{H}_0$ is closed with respect to multiplication. Moreover, every $g \in \mathcal{H}_0$ can be written as an increasing sequence of functions $f = f_1 f_2$, where $f_1 \geq 0$ belongs to $\mathbb{B}(\mathbb{R}_+)$ and $f_2 \geq 0$ belongs to $\mathbb{C}(\mathbb{U})$. For every function f satisfying the above decomposition, we have from Lemma 2.27 that the mapping $h \in \mathbb{B}(\mathbb{R}_+ \times \mathbb{U})$ defined by $h(t,u) = e^{-t} f(t,u)$ is in $L^1(\mathbb{R}_+; \mathbb{C}(\mathbb{U}))$, and so the real-valued mapping defined on $\mathcal{V}^r$ by

$$\mu \to \int_{\mathbb{R}_+}\int_{\mathbb{U}} e^{-s} f_1(s) f_2(u)\mu(s,du)ds$$

is continuous (by definition of the topology of $\mathcal{V}^r$) and so $\mathcal{B}(\mathcal{V}^r)$-measurable. This shows that $\mathcal{H}_0$ is a subset of $\mathcal{H}$. Finally, notice that the σ-algebra generated by $\mathcal{H}_0$ is given by $\mathcal{B}(\mathbb{R}_+ \times \mathbb{U})$. Now by applying the functional monotone class theorem (see Theorem 2.12.9 in [7]), the result follows. □

Lemma 2.29 *Let $h \in \mathbb{M}(\mathbb{R}_+ \times E \times \mathcal{V}^r \times \mathbb{R}_+ \times \mathbb{U})^+$. Then the $\overline{\mathbb{R}}_+$-valued mapping defined on $\mathbb{R}_+ \times E \times \mathcal{V}^r$ by*

$$(t,x,\mu) \to \int_{\mathbb{R}_+}\int_{\mathbb{U}} h(t,x,\mu,s,u)\mu(s,du)ds$$

is $\mathcal{B}(\mathbb{R}_+ \times E \times \mathcal{V}^r)$-measurable.

Proof Let $\mathcal{H}$ be the class of functions $h \in \mathbb{B}(\mathbb{R}_+ \times E \times \mathcal{V}^r \times \mathbb{R}_+ \times \mathbb{U})$ such that the mapping

$$(t, x, \mu) \in \mathbb{R}_+ \times E \times \mathcal{V}^r \to \int_{\mathbb{R}_+} e^{-s} \int_{\mathbb{U}} h(t, x, \mu, s, u)\mu(s, du)ds \in \mathbb{R}$$

is $\mathcal{B}(\mathbb{R}_+ \times E \times \mathcal{V}^r)$-measurable. The set $\mathcal{H}$ is a closed linear subspace of $\mathbb{B}(\mathbb{R}_+ \times E \times \mathcal{V}^r \times \mathbb{R}_+ \times \mathbb{U})$ equipped with the supremum norm. Moreover, the limit of every increasing uniformly bounded sequence of nonnegative functions of $\mathcal{H}$ belongs to $\mathcal{H}$. Consider the class $\mathcal{H}_0$ of functions h such that $h = h_1 h_2$ with $h_1 = \mathbb{B}(\mathbb{R}_+ \times E \times \mathcal{V}^r)$ and $h_2 = \mathbb{B}(\mathbb{R}_+ \times \mathbb{U})$. Then from Lemma 2.28, $\mathcal{H}_0 \subset \mathcal{H}$. Moreover, $\mathcal{H}_0$ is closed relative to multiplication. Finally, notice that the σ-algebra generated by $\mathcal{H}_0$ is given by $\mathcal{B}(\mathbb{R}_+ \times E \times \mathcal{V}^r \times \mathbb{R}_+ \times \mathbb{U})$. Now applying the functional monotone class theorem (see Theorem 2.12.9 in [7]), it follows that $\mathcal{H}$ contains $\mathbb{B}(\mathbb{R}_+ \times E \times \mathcal{V}^r \times \mathbb{R}_+ \times \mathbb{U})$. Consider $g \in \mathbb{M}(\mathbb{R}_+ \times E \times \mathcal{V}^r \times \mathbb{R}_+ \times \mathbb{U})^+$ and define $g_k(t, x, \mu, s, u) = \big(e^s g(t, x, \mu, s, u)\big) \wedge k$. Therefore, the mapping

$$(t, x, \mu) \in \mathbb{R}_+ \times E \times \mathcal{V}^r \to \int_{\mathbb{R}_+} e^{-s} \int_{\mathbb{U}} g_k(t, x, \mu, s, u)\mu(s, du)ds \in \mathbb{R}_+$$

is $\mathcal{B}(\mathbb{R}_+ \times E \times \mathcal{V}^r)$-measurable. Using the monotone convergence theorem, it follows that

$$\lim_{k\to\infty} \int_{\mathbb{R}_+} e^{-s} \int_{\mathbb{U}} g_k(t, x, \mu, s, u)\mu(s, du)ds = \int_{\mathbb{R}_+} \int_{\mathbb{U}} g(t, x, \mu, s, u)\mu(s, du)ds,$$

which yields the result. □

Proof of Proposition 2.17 Applying Lemma 2.29 to $h(t, x, \mu, s, u) = I_{\{s\leq t\}} \lambda(\phi(x, s), u)$ implies that the mapping $\Lambda^\mu(x, t)$ defined on $\mathbb{R}_+ \times E \times \mathcal{V}^r$ with value in $\overline{\mathbb{R}}_+$ is measurable with respect to $\mathcal{B}(\mathbb{R}_+ \times E \times \mathcal{V}^r)$. Notice that the integral $\int_0^{t_*(x)} e^{-\alpha s - \Lambda^\mu(x,s)} \lambda(\phi(x, s), \mu(s))ds$ is finite for every $(x, \mu) \in E \times \mathcal{V}^r$ and $\alpha \in \mathbb{R}_+$. First assume that w_0 is positive and bounded. Clearly, $Qw_0(., .)$ is measurable with respect to $\mathcal{B}(\overline{E} \times \mathbb{U})$. Therefore, for the function

$$h(t, x, \mu, s, u) = I_{\{s\leq t_*(x)\}} e^{-\alpha s - \Lambda^\mu(x,s)} \lambda(\phi(x, s), u) Qw_0(\phi(x, s), u),$$

Lemma 2.29 shows that the $\overline{\mathbb{R}}_+$-valued mapping defined on $E \times \mathcal{V}^r$ by

$$(x, \mu) \to \int_0^{t_*(x)} e^{-\alpha s - \Lambda^\mu(x,s)} \lambda Qw_0(\phi(x, s), \mu(s))ds$$

is $\mathcal{B}(E \times \mathcal{V}^r)$-measurable. Moreover, the mapping

$$(x, \mu, u) \in E \times \mathcal{V}^r \times \mathbb{U} \to e^{-\alpha t_*(x) - \Lambda^\mu(x, t_*(x))} Qw_0(\phi(x, t_*(x)), u) \in \mathbb{R}_+$$

is clearly $\mathcal{B}(E \times \mathcal{V}^r \times \mathbb{U})$-measurable. Consequently, it follows that for w_0 bounded and positive, the function $G_\alpha w_0(x, \Theta)$ defined on $E \times \mathbb{V}^r$ is $\mathcal{B}(E \times \mathbb{V}^r)$-measurable. If w_0 is positive but not bounded, we can easily get the result using the sequence of functions $w_0^k \doteq w_0 \wedge k \uparrow w_0$ and the monotone convergence theorem. Finally, if w_0 is bounded below by a constant $-c \leq 0$, we can get the result by observing that $G_\alpha c(x, \Theta) \in \mathbb{R}_+$, and for $w_{0c} \doteq w_0 + c$, we have $G_\alpha w_0(x, \Theta) = G_\alpha w_{0c}(x, \Theta) - G_\alpha c(x, \Theta)$. Using the same arguments, the same property can be proved for the mappings $L_\alpha w_1(x, \Theta)$ and $H_\alpha w_2(x, \Theta)$. □

Proof of Proposition 2.24 From item (2) of Theorem I.5.25 in [65], the space $L^1(\mathbb{R}_+, \mathbb{C}(\mathbb{U}))$ is isometrically isomorphic to the space of real-valued functions f defined on $\mathbb{R}_+ \times \mathbb{U}$ satisfying $f(t, .) \in \mathbb{C}(\mathbb{U})$ for every $t \in \mathbb{R}_+$, $f(., u)$ is measurable for every $u \in \mathbb{U}$, and $\int_{\mathbb{R}_+} \sup_{u \in \mathbb{U}} |f(t, u)| dt < \infty$. Consequently, the topology of $\mathcal{V}$ induced by the weak* topology $\sigma(L^\infty(\mathbb{R}_+; \mathcal{M}(\mathbb{U})), L^1(\mathbb{R}_+; \mathbb{C}(\mathbb{U})))$ is identical to the Young topology as defined in [37, p. 255]. Having this in mind, we can use Lemma A.3 in [37] to get that the $\mathcal{V}$-valued mapping h defined on E by $x \to \delta_{u(\phi(x,.))}$ is measurable if and only if the $\mathbb{R}$-valued mappings defined on E by $x \to \int_{\mathbb{R}_+} e^{-s} g(s, u(\phi(x, s)))ds$ are measurable for every $g \in \mathbb{B}(\mathbb{R}_+ \times \mathbb{U})$. Using Lemma 2.29, it follows that h is measurable. Moreover, for all $(x, t) \in E \times \mathbb{R}_+$, $u(\phi(x, t)) \in \mathbb{U}(\phi(x, t))$. Therefore, u_ϕ belongs to $\mathcal{S}_\mathbb{V}$. The second statement is a straightforward consequence of the measurability properties of u and ϕ and the fact that $u(x) \in \mathbb{U}(x)$. □

Chapter 3
Optimality Equation for the Average Control of PDMPs

3.1 Outline of the Chapter

This chapter presents the main characterization results regarding the optimality equation for long-run average cost. It is shown in Sect. 3.2 (see Theorem 3.1) that if there exist a measurable function h, a parameter ρ, and a measurable selector satisfying a discrete-time optimality equation related to the embedded Markov chain given by the postjump location of a PDMP, and if also an extra condition involving the function h is satisfied, then an optimal control can be obtained from the measurable selector, and ρ is the optimal cost. The hypothesis of the existence of a measurable selector in Theorem 3.1 is removed in Sect. 3.4 (see Theorem 3.15). This is done by establishing a link (see the proof of Theorem 3.14) between the discrete-time optimality equation and an integrodifferential equation (using the weaker concept of absolute continuity along the flow of the value function). This yields the existence of a feedback measurable selector (that is, a selector that depends on the present value of the state variable; see Remark 2.25), provided that the function h and parameter ρ satisfy the optimality equation.

In order to establish the existence of a measurable selector, we follow a common approach adopted in the literature that consists in assuming semicontinuity properties for the cost function (see Sect. 3.3) and introducing the class of relaxed controls to get a compactness property for the action space (it should be noticed that other approaches without the compactness assumption would also be possible as presented, for instance, in [35]). This yields the existence result for the measurable selector, but within the class of relaxed controls instead of the desired class of ordinary controls. By going one step further, we combine the existence result within the class of relaxed controls with the connection between the integrodifferential equation and the discrete-time equation to obtain that the optimal control is nonrelaxed and in fact, is an ordinary feedback control.

O. L. V. Costa and F. Dufour, *Continuous Average Control of Piecewise Deterministic Markov Processes*, SpringerBriefs in Mathematics,
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3.2 Discrete-Time Optimality Equation for the Average Control

In this section, we obtain an optimality equation for the long-run average cost problem defined in Sect. 2.2 in terms of a discrete-time optimality equation related to the embedded Markov chain given by the postjump location of the PDMP, and an additional condition on a limit over the solution of the optimality equation divided by the time t. Note that in this section, we assume the existence of a measurable selector for the optimality equation, in particular the existence of an ordinary (discrete-time) optimal control.

The main result of this section reads as follows:

Theorem 3.1 *Suppose that there exists a pair* $(\rho, h) \in \mathbb{R}_+ \times \mathbb{M}(E)$ *with* h *bounded from below satisfying the discrete-time optimality equation*

$$\mathcal{T}(\rho, h)(x) = h(x), \tag{3.1}$$

and for all $U \in \mathcal{U}$,

$$\overline{\lim_{t\to+\infty}} \frac{1}{t} \overline{\lim_{m\to+\infty}} E^{U}_{(x,0)}\Big[h\big(X(t \wedge T_m)\big)\Big] = 0. \tag{3.2}$$

Moreover, assume that there exists a solution to the one-stage optimization function associated to (ρ, h)*, that is, the existence of an optimal measurable selector* $\hat{\Gamma} = (\hat{\gamma}, \hat{\gamma}_\partial)$ *in* $\mathcal{S}_\mathbb{V}$ *such that for all* $x \in E$,

$$\mathcal{T}(\rho, h)(x) = -\rho\mathcal{L}(x, \hat{\Gamma}(x)) + Lf(x, \hat{\Gamma}(x)) + Hr(x, \hat{\Gamma}(x)) + Gh(x, \hat{\Gamma}(x)). \tag{3.3}$$

Define the control strategy $\widehat{U}$ *by* $(\hat{u}, \hat{u}_\partial)$ *with* $\hat{u}(n, x, t) = \hat{\gamma}(x, t)$, $\hat{u}_\partial(n, x) = \hat{\gamma}_\partial(x)$ *for* $(n, x, t) \in \mathbb{N} \times E \times \mathbb{R}_+$*, and assume that* $\widehat{U}$ *belongs to* $\mathcal{U}$*. Then* $\widehat{U}$ *is optimal. Moreover,* $\rho = \mathcal{J}_\mathcal{A}(x) = \mathcal{A}(\widehat{U}, x)$.

The proof of this theorem is presented at the end of this section.

In Theorem 3.1, notice that (3.1) can be seen as the optimality equation of a discrete-time problem related to the embedded Markov chain given by the postjump location of the PDMP with transition kernel G, and (3.2) as an additional technical condition.

The problem of existence of an optimal measurable selector for the optimality Eq. (3.1) will be considered in Sect. 3.4.

In order to prove the previous theorem, we first need to present several intermediate results. These results will be written in terms of an extra parameter $\alpha \geq 0$ that will be useful for the discounted control problem, to be analyzed in Sects. 4.2, 4.3, and 4.4. The proofs of these intermediate results can be found in Sect. 3.5.

The next proposition presents some important properties of the one-stage optimality function (see Eq. (3.4)). It is shown that it has a special time representation displayed in Eq. (3.6). As a consequence, it follows that it is absolutely continuous along trajectories with a limit on the boundary (that is, it belongs to $\mathbb{M}^{ac}(E)$).

Proposition 3.2 *Let $\rho \in \mathbb{R}_+$ and $h \in \mathbb{M}(E)$ be bounded from below by $-K_h$ with $K_h \in \mathbb{R}_+$. For $\alpha \geq 0$ and $x \in E$, define*

$$w(x) = \mathcal{T}_\alpha(\rho, h)(x). \tag{3.4}$$

Assume that $w \in \mathbb{M}(E)$ and that there exists $\hat{\Gamma} \in \mathcal{S}_{\mathbb{V}}$ such that

$$w(x) = -\rho\mathcal{L}_\alpha(x, \hat{\Gamma}(x)) + L_\alpha f(x, \hat{\Gamma}(x)) + H_\alpha r(x, \hat{\Gamma}(x)) + G_\alpha h(x, \hat{\Gamma}(x)). \tag{3.5}$$

Then $w \in \mathbb{M}^{ac}(E)$, and for all $x \in E$ and $t \in [0, t_*(x))$,

$$w(x) = \int_0^t e^{-\alpha s - \Lambda^{\hat{\gamma}(x)}(x,s)}\Big[-\rho + f(\phi(x,s), \hat{\gamma}(x,s)) + \lambda Q h(\phi(x,s), \hat{\gamma}(x,s))\Big]ds$$
$$+ e^{-\alpha t - \Lambda^{\hat{\gamma}(x)}(x,t)} w(\phi(x,t)) \tag{3.6}$$
$$= \inf_{\nu \in \mathcal{V}(x)} \Bigg\{ \int_0^t e^{-\alpha s - \Lambda^{\nu}(x,s)}\Big[-\rho + f(\phi(x,s), \nu(s)) + \lambda Q h(\phi(x,s), \nu(s))\Big]ds$$
$$+ e^{-\alpha t - \Lambda^{\nu}(x,t)} w(\phi(x,t)) \Bigg\}, \tag{3.7}$$

where $\hat{\Gamma}(x) = \big(\hat{\gamma}(x), \hat{\gamma}_\partial(x)\big)$.

The next two propositions deal with two inequalities of opposite directions for the one-stage optimality equation. Roughly speaking, these two results show that if h is a solution for an optimality inequality (see Eq. (3.8) or (3.10)), then this inequality is preserved, in one case for every control strategy and in the other case for a specific control strategy, along the jump time iterations for a cost conveniently defined; see (3.9) or (3.11).

Proposition 3.3 *Let $h \in \mathbb{M}(E)$ be bounded from below. For $\rho \in \mathbb{R}_+$ and $\alpha \in \mathbb{R}_+$, assume that $\mathcal{T}_\alpha(\rho, h) \in \mathbb{M}(E)$ and that there exists $\hat{\Gamma} = (\hat{\gamma}, \hat{\gamma}_\partial) \in \mathcal{S}_{\mathbb{V}}$ such that for all $x \in E$,*

$$h(x) \leq \mathcal{T}_\alpha(\rho, h)(x)$$
$$= -\rho\mathcal{L}_\alpha(x, \hat{\Gamma}(x)) + L_\alpha f(x, \hat{\Gamma}(x)) + H_\alpha r(x, \hat{\Gamma}(x)) + G_\alpha h(x, \hat{\Gamma}(x)). \tag{3.8}$$

For $U \in \mathcal{U}$, $m \in \mathbb{N}$, *and* $(t, x, k) \in \mathbb{R}_+ \times E \times \mathbb{N}$, *define*

$$J_m^U(t, x, k) = E_{(x,k)}^U\Big[\int_0^{t\wedge T_m} e^{-\alpha s}\Big[f\big(X(s), u(N(s), Z(s), \tau(s))\big) - \rho\Big]ds$$
$$+ \int_0^{t\wedge T_m} e^{-\alpha s} r\big(X(s-), u_\partial(N(s-), X(s-))\big)dp^*(s)$$
$$+ e^{-\alpha(t\wedge T_m)}\mathcal{T}_\alpha(\rho, h)\big(X(t\wedge T_m)\big)\Big].$$

Then for all $m \in \mathbb{N}$ *and* $(t, x, k) \in \mathbb{R}_+ \times E \times \mathbb{N}$,

$$J_m^U(t, x, k) \geq h(x). \tag{3.9}$$

The next proposition considers the reverse inequality referred to in the literature as the average cost optimality inequality (ACOI).

Proposition 3.4 *Let* $h \in \mathbb{M}(E)$ *be bounded from below. For* $\rho \in \mathbb{R}_+$ *and* $\alpha \in \mathbb{R}_+$, *assume that* $\mathcal{T}_\alpha(\rho, h) \in \mathbb{M}(E)$ *and that there exists* $\hat{\Gamma} = (\hat{\gamma}, \hat{\gamma}_\partial) \in \mathcal{S}_{\mathbb{V}}$ *such that for all* $x \in E$,

$$\begin{aligned} h(x) &\geq \mathcal{T}_\alpha(\rho, h)(x) \\ &= -\rho\mathcal{L}_\alpha(x, \hat{\Gamma}(x)) + L_\alpha f(x, \hat{\Gamma}(x)) + H_\alpha r(x, \hat{\Gamma}(x)) + G_\alpha h(x, \hat{\Gamma}(x)). \end{aligned} \tag{3.10}$$

Introduce the control strategy $\widehat{U}$ *by* $(\hat{u}, \hat{u}_\partial)$ *with* $\hat{u}(n, x, t) = \hat{\gamma}(x, t)$, *and* $\hat{u}_\partial(n, x) = \hat{\gamma}_\partial(x)$ *for* $(n, x, t) \in \mathbb{N} \times E \times \mathbb{R}_+$, *and assume that* $\widehat{U}$ *belongs to* $\mathcal{U}$. *Moreover, defining*

$$J_m^{\widehat{U}}(t, x, k) = E_{(x,k)}^{\widehat{U}}\Big[\int_0^{t\wedge T_m} e^{-\alpha s}\Big[f\big(X(s), \hat{u}(N(s), Z(s), \tau(s))\big) - \rho\Big]ds$$
$$+ \int_0^{t\wedge T_m} e^{-\alpha s} r\big(X(s-), \hat{u}_\partial(N(s-), X(s-))\big)dp^*(s)$$
$$+ e^{-\alpha(t\wedge T_m)}\mathcal{T}_\alpha(\rho, h)\big(X(t\wedge T_m)\big)\Big],$$

we have, for all $m \in \mathbb{N}$ *and* $(t, x, k) \in \mathbb{R}_+ \times E \times \mathbb{N}$, *that*

$$J_m^{\widehat{U}}(t, x, k) \leq h(x). \tag{3.11}$$

Combining the previous propositions with $\alpha = 0$, we get the proof of Theorem 3.1.

Proof of Theorem 3.1 From Proposition 3.3, it follows that

$$E^U_{(x,0)}\Big[\mathbf{J}(U, t \wedge T_m)\Big] + E^U_{(x,0)}\Big[h\big(X(t \wedge T_m)\big)\Big] \geq \rho E^U_{(x,0)}\Big[t \wedge T_m\Big] + h(x).$$

Consequently, we have from Assumption 2.8 (which implies that $T_m \to \infty$ $P^U_{(x,0)}$ a.s.) that

$$\overline{\lim_{t\to+\infty}} \frac{1}{t} E^U_{(x,0)}\Big[\mathbf{J}(U, t)\Big] + \overline{\lim_{t\to+\infty}} \frac{1}{t} \overline{\lim_{m\to+\infty}} E^U_{(x,0)}\Big[h\big(X(t \wedge T_m)\big)\Big] \geq \rho,$$

showing that $\rho \leq \mathcal{J}_{\mathcal{A}}(x)$, where we have used (3.2). Since h is bounded below, it is easy to show that

$$\overline{\lim_{t\to+\infty}} \frac{1}{t} \overline{\lim_{m\to+\infty}} -E^U_{(x,0)}\Big[h\big(X(t \wedge T_m)\big)\Big] \leq 0. \tag{3.12}$$

Applying Proposition 3.4, we obtain from Eq. (3.11) that

$$E^{\widehat{U}}_{(x,0)}\Big[\mathbf{J}(\widehat{U}, t \wedge T_m)\Big] \leq \rho E^{\widehat{U}}_{(x,0)}\Big[t \wedge T_m\Big] + h(x) - E^{\widehat{U}}_{(x,0)}\Big[h\big(X(t \wedge T_m)\big)\Big].$$

Using Eq. (3.12), this gives $\rho \geq \mathcal{A}(\widehat{U}, x)$, completing the proof. □

3.3 Convergence and Continuity Properties of the Operators: $\mathcal{L}_\alpha$, L_α, G_α, H_α

In the previous section, we assumed the existence of an ordinary optimal measurable selector for the one-stage optimization problem associated with $\mathcal{T}(\rho, h)$ (see (3.3)) for (ρ, h) satisfying the optimality equation $\mathcal{T}(\rho, h)(x) = h(x)$. In the following sections, we will suppress this hypothesis. In order to do so, we need to consider relaxed controls so that we can take advantage of the compactness property of the sets $\mathbb{V}^r(x)$ and $\mathbb{V}^r$ as presented in Sect. 2.3.1. Note, however, that we also need the cost function to be lower semicontinuous. Thus in this section, we present the assumptions and results that will guarantee some convergence and lower semicontinuity properties of the operators G_α, L_α, and H_α that appear in the one-stage optimization function with respect to the topology defined in (2.6). Combining the compactness of the sets $\mathbb{V}^r(x)$ with the lower semicontinuity of the operators G_α, L_α, and H_α, we can use a selection theorem like that presented in Proposition D.5 of [49] to get the existence of a relaxed optimal control and measurability of the one-stage optimization equation. Moreover, in parallel, we get some important convergence properties that will be applied with the vanishing discount approach in Sect. 4.3.

From now on, we will consider the following assumptions on the parameters of the PDMP (ϕ, λ, Q) and on the running cost f and the boundary cost r introduced in Sect. 2.2.

Assumption 3.5 *For each $x \in E$, the restriction of $\lambda(x, .)$ to $\mathbb{U}(x)$ is continuous.*

Assumption 3.6 *There exists a sequence of measurable functions $(f_j)_{j\in\mathbb{N}}$ in $\mathbb{M}(\overline{E}\times\mathbb{U})^+$ such that for all $y \in \overline{E}$, $f_j(y, .) \uparrow f(y, .)$ as $j \to \infty$ and the restriction of $f_j(y, .)$ to $\mathbb{U}(y)$ is continuous.*

Assumption 3.7 *There exists a sequence of measurable functions $(r_j)_{j\in\mathbb{N}}$ in $\mathbb{M}(\partial E\times\mathbb{U})^+$ such that for all $z \in \partial E$, $r_j(z, .) \uparrow r(z, .)$ as $j \to \infty$ and the restriction of $r_j(z, .)$ to $\mathbb{U}(z)$ is continuous.*

Assumption 3.8 *For all $x \in \overline{E}$ and $\ell \in \mathbb{B}(E)$, the restriction of $Q\ell(x, .)$ to $\mathbb{U}(x)$ is continuous.*

Assumption 3.9 *There exists $\overline{f} \in \mathbb{M}(\overline{E})^+$ such that for every $(x, a) \in K$, $f(x, a) \leq \overline{f}(x)$ and $\int_0^{t_*(x)} e^{-\int_0^t \lambda(\phi(x,s))ds}\,\overline{f}(\phi(x, t))dt < \infty$.*

The next proposition presents convergence results of the operators G_α, L_α, and H_α with respect to the topology defined in Sect. 2.3.1. Note that the convergence is taken not only with respect to a sequence of controls but also with respect to some functions and the parameter α. This is justified by the fact that we will need this convergence for the vanishing discount approach in Sects. 4.3 and 4.4. The proof of the proposition is in Sect. 3.5.

Proposition 3.10 *Consider $\alpha \in \mathbb{R}_+$ and a nonincreasing sequence of nonnegative numbers $\{\alpha_k\}$, $\alpha_k \downarrow \alpha$, and a sequence of functions $h_k \in \mathbb{M}(E)$ uniformly bounded from below by a positive constant K_h (that is, $h_k(y) \geq -K_h$ for all $y \in E$). Set $h = \underline{\lim}_{k\to\infty} h_k$. For $x \in E$, consider $\Theta_n = \big(\mu_n, \mu_{\partial,n}\big) \in \mathbb{V}^r(x)$ and $\Theta = \big(\mu, \mu_\partial\big) \in \mathbb{V}^r(x)$ such that $\Theta_n \to \Theta$. We have the following results:*

(a) $\lim_{n\to\infty} \mathcal{L}_{\alpha_n}(x, \Theta_n) = \mathcal{L}_\alpha(x, \Theta)$.
(b) $\underline{\lim}_{n\to\infty} L_{\alpha_n} f(x, \Theta_n) \geq L_\alpha f(x, \Theta)$.
(c) $\underline{\lim}_{n\to\infty} H_{\alpha_n} r(x, \Theta_n) \geq H_\alpha r(x, \Theta)$.
(d) $\underline{\lim}_{n\to\infty} G_{\alpha_n} h_n(x, \Theta_n) \geq G_\alpha h(x, \Theta)$.

The lower semicontinuity properties mentioned at the beginning of this section follow easily from this proposition, as stated in the next corollary.

Corollary 3.11 *Consider $h \in \mathbb{M}(E)$ bounded from below. We have the following results:*

(a) *$\mathcal{L}_\alpha(x, \Theta)$ is continuous on $\mathbb{V}^r(x)$.*
(b) *$L_\alpha f(x, \Theta)$ is lower semicontinuous on $\mathbb{V}^r(x)$.*
(c) *$H_\alpha r(x, \Theta)$ is lower semicontinuous on $\mathbb{V}^r(x)$.*
(d) *$G_\alpha h(x, \Theta)$ is lower semicontinuous on $\mathbb{V}^r(x)$.*

Proof By taking $\alpha_k = \alpha \geq 0$, $h_{\alpha_k} = h$ in Proposition 3.10, it is easy to obtain the result. □

3.4 Existence of an Ordinary Optimal Feedback Control

The main result of this section is Theorem 3.15, which strengthens Theorem 3.1 by assuming only that the discrete-time optimality equation $\mathcal{T}(\rho, h) = h$ has a solution in order to ensure the existence of an optimal control strategy for the long-run average control problem. Moreover, it is shown that this optimal control strategy is in the feedback class and can be characterized as in item (D3) of Definition 3.12. These results are obtained by establishing a connection (see the proof of Theorem 3.14) between the discrete-time optimality equation and an integrodifferential equation (using the weaker concept of absolute continuity along the flow of the value function). The basic idea is to use the set of relaxed controls $\mathbb{V}^r(x)$. The advantage of considering $\mathbb{V}^r(x)$ is that it is compact, so that together with the assumptions we have made in Sect. 3.3, we can apply a measurable selector theorem to guarantee the existence of an optimal measurable selector (see Proposition 3.13). The price to pay is that this measurable selector belongs to the space of relaxed controls. However, we can show that in fact there exists a nonrelaxed feedback selector for the discrete-time optimality equation $\mathcal{T}(\rho, h) = h$ by establishing a connection between the discrete-time optimality equation and the integrodifferential equation (see the proof of Theorem 3.14).

Definition 3.12 Consider $w \in \mathbb{M}(E)$ and $h \in \mathbb{M}(E)$ bounded from below.

(D1) Denote by $\widehat{u}(w, h) \in \mathcal{S}_{\mathbb{U}}$ the measurable selector satisfying

$$\begin{aligned}\inf_{a\in\mathbb{U}(x)} &\big\{f(x,a) - \lambda(x,a)\big[w(x) - Qh(x,a)\big]\big\} \\ &= f(x,\widehat{u}(w,h)(x)) - \lambda(x,\widehat{u}(w,h)(x))\big[w(x) - Qh(x,\widehat{u}(w,h)(x))\big], \\ &\inf_{a\in\mathbb{U}(z)} \{r(z,a) + Qh(z,a)\} = r(z,\widehat{u}(w,h)(z)) + Qh(z,\widehat{u}(w,h)(z)).\end{aligned}$$

(D2) $\widehat{u}_\phi(w, h) \in \mathcal{S}_{\mathbb{V}}$ is the measurable selector derived from $\widehat{u}(w, h)$ through Definition 2.22.

(D3) $\widehat{U}_\phi(w, h) \in \mathcal{U}$ is the control strategy derived from $\widehat{u}(w, h)$ through Definition 2.23.

The existence of $\widehat{u}(w, h)$ follows from Assumptions 3.5–3.8 and Proposition D.5 in [49], and the fact that $\widehat{u}_\phi(w, h) \in \mathcal{S}_{\mathbb{V}}$ and $\widehat{U}_\phi(w, h) \in \mathcal{U}$ comes from Proposition 2.24. Notice that $\widehat{u}_\phi(w, h)$ is a feedback measurable selector and $\widehat{U}_\phi(w, h)$ is a feedback control strategy.

The proof of the next proposition is presented in Sect. 3.5. It shows the existence of an optimal relaxed measurable selector for the relaxed one-stage optimization function $\mathcal{R}_\alpha(\rho, h)(x)$ and that $\mathcal{R}_\alpha(\rho, h) \in \mathbb{M}^{ac}(E)$.

Proposition 3.13 *Let $\alpha \geq 0$, $\rho \in \mathbb{R}_+$, and $h \in \mathbb{M}(E)$ be bounded from below. For $x \in E$, define $w(x) = \mathcal{R}_\alpha(\rho, h)(x)$. Assume that for all $x \in E$, $w(x) \in \mathbb{R}$. Then there exists $\hat{\Theta} \in \mathcal{S}_{\mathbb{V}^r}$ such that*

$$w(x) = -\rho\mathcal{L}_\alpha(x, \hat{\Theta}(x)) + L_\alpha f(x, \hat{\Theta}(x)) + H_\alpha r(x, \hat{\Theta}(x)) + G_\alpha h(x, \hat{\Theta}(x)). \tag{3.13}$$

Moreover, $w \in \mathbb{M}^{ac}(E)$ and satisfies, for all $x \in E$ and $t \in [0, t_*(x))$,

$$w(x) = \inf_{\mu \in \mathcal{V}^r(x)} \left\{ \int_0^t e^{-\alpha s - \Lambda^\mu(x,s)} \Big[-\rho + f(\phi(x,s), \mu(s)) + \lambda Q h(\phi(x,s), \mu(s)) \Big] ds + e^{-\alpha t - \Lambda^\mu(x,t)} w(\phi(x,t)) \right\} \tag{3.14}$$

$$= \int_0^t e^{-\alpha s - \Lambda^{\hat{\mu}(x)}(x,s)} \Big[-\rho + f(\phi(x,s), \hat{\mu}(x,s)) + \lambda Q h(\phi(x,s), \hat{\mu}(x,s)) \Big] ds + e^{-\alpha t - \Lambda^{\hat{\mu}(x)}(x,t)} w(\phi(x,t)), \tag{3.15}$$

where $\hat{\Theta}(x) = (\hat{\mu}(x), \hat{\mu}_\partial(x))$.

The following theorem shows the existence of a feedback measurable selector for the one-stage optimization problems associated with $\mathcal{T}_\alpha(\rho, h)$ and $\mathcal{R}_\alpha(\rho, h)$. Its proof is presented in Sect. 3.5.

Theorem 3.14 *Let $\alpha \geq 0$, $\rho \in \mathbb{R}_+$, and $h \in \mathbb{M}(E)$ be bounded from below by $-K_h$ with $K_h \in \mathbb{R}_+$. Define*

$$w(x) = \mathcal{R}_\alpha(\rho, h)(x), \tag{3.16}$$

for $x \in E$, and suppose that $w(x) \in \mathbb{R}$ for all $x \in E$. Then $w \in \mathbb{M}^{ac}(E)$, and the feedback measurable selector $\widehat{u}_\phi(w, h) \in \mathcal{S}_{\mathbb{V}}$ satisfies the following one-stage optimization problems:

$$\mathcal{R}_\alpha(\rho, h)(x) = \mathcal{T}_\alpha(\rho, h)(x) = -\rho\mathcal{L}_\alpha(x, \widehat{u}_\phi(w,h)(x)) + L_\alpha f(x, \widehat{u}_\phi(w,h)(x)) + H_\alpha r(x, \widehat{u}_\phi(w,h)(x)) + G_\alpha h(x, \widehat{u}_\phi(w,h)(x)). \tag{3.17}$$

The main result of this section is as follows:

Theorem 3.15 *Suppose that there exists a pair $(\rho, h) \in \mathbb{R} \times \mathbb{M}(E)$ with h bounded from below satisfying the discrete-time optimality equation*

$$\mathcal{T}(\rho, h)(x) = h(x),$$

and for all $U \in \mathcal{U}$*,*

$$\overline{\lim_{t \to +\infty}} \frac{1}{t} \overline{\lim_{m \to +\infty}} E^U_{(x,0)}\big[h\big(X(t \wedge T_m)\big)\big] = 0.$$

Then $h \in \mathbb{M}^{ac}(E)$*, the feedback optimal control strategy* $\widehat{U}_\phi(\rho, h)$ *(see item* (D3) *of Definition 3.12) is optimal, and*

$$\rho = \mathcal{J}_{\mathcal{A}}(x) = \mathcal{A}(\widehat{U}_\phi(\rho, h), x).$$

Proof The proof of this result is straightforward, obtained by combining Theorem 3.1 of Sect. 3.2 and Theorem 3.14. □

3.5 Proof of Auxiliary Results

In this section, we present the proof of some auxiliary results needed in this chapter.

3.5.1 Proofs of the Results of Sect. 3.2

The following lemma applies the semigroup property of the flow ϕ into the operators G_α, L_α, and H_α (defined in (2.12), (2.14), and (2.15)). Recall also the definition of $[\Theta]_t$ in Definition 2.14.

Lemma 3.16 *For every* $\alpha \geq 0$*,* $x \in E$*,* $t \in [0, t_*(x))$*,* $\Theta = \big(\mu, \mu_\partial\big) \in \mathbb{V}^r(x)$*, and* $g \in \mathbb{M}(E)$ *bounded from below, we have that*

$$\mathcal{L}_\alpha(x, \Theta) = \int_0^t e^{-\alpha s - \Lambda^\mu(x,s)} ds + e^{-\alpha t - \Lambda^\mu(x,t)} \mathcal{L}_\alpha\big(\phi(x,t), \big[\Theta\big]_t\big), \tag{3.18}$$

$$\begin{aligned} L_\alpha f(x, \Theta) = &\int_0^t e^{-\alpha s - \Lambda^\mu(x,s)} f(\phi(x,s), \mu(s)) ds \\ &+ e^{-\alpha t - \Lambda^\mu(x,t)} L_\alpha f\big(\phi(x,t), \big[\Theta\big]_t\big), \\ H_\alpha r(x, \Theta) = &e^{-\alpha t - \Lambda^\mu(x,t)} H_\alpha r\big(\phi(x,t), \big[\Theta\big]_t\big), \end{aligned}$$

$$G_\alpha g(x,\Theta) = \int_0^t e^{-\alpha s - \Lambda^\mu(x,s)} \lambda Q g(\phi(x,s),\mu(s)) ds + e^{-\alpha t - \Lambda^\mu(x,t)} G_\alpha g((\phi(x,t)), [\Theta]_t).$$

Proof For every $x \in E$ and $\Theta = \big(\mu, \mu_\partial\big) \in \mathbb{V}^r(x)$, using the semigroup property of ϕ, we have for $t + s < t_*(x)$,

$$\begin{aligned}\Lambda^\mu(x, t+s) &= \int_0^t \lambda(\phi(x,\ell),\mu(\ell))d\ell + \int_t^{t+s} \lambda(\phi(x,\ell),\mu(\ell))d\ell \\ &= \Lambda^\mu(x,t) + \int_0^s \lambda(\phi(\phi(x,t),\ell),\mu(\ell+t))d\ell.\end{aligned}$$

Notice that $t_*(x) - t = t_*(\phi(x,t))$. Consequently, combining the previous equation and Definition 2.14, we obtain for $t \in [0, t_*(x))$ that

$$\begin{aligned}\mathcal{L}_\alpha(x,\Theta) &= \int_0^t e^{-\alpha s - \Lambda^\mu(x,s)} ds + \int_0^{t_*(\phi(x,t))} e^{-\alpha(t+s) - \Lambda^\mu(x,t+s)} ds, \\ &= \int_0^t e^{-\alpha s - \Lambda^\mu(x,s)} ds \\ &\quad + e^{-\alpha t - \Lambda^\mu(x,t)} \int_0^{t_*(\phi(x,t))} e^{-\alpha s - \int_0^s \lambda(\phi(\phi(x,t),\ell),\mu(\ell+t))d\ell} ds \\ &= \int_0^t e^{-\alpha s - \Lambda^\mu(x,s)} ds + e^{-\alpha t - \Lambda^\mu(x,t)} \mathcal{L}_\alpha(\phi(x,t), \big[\Theta\big]_t),\end{aligned}$$

proving Eq. (3.18). The other equalities can be obtained by similar arguments. □

Next we present the proof of Proposition 3.2.

Proof of Proposition 3.2 From Lemma 3.16, it follows that for every $x \in E$, $t \in [0, t_*(x))$, and $\Upsilon = \big(\nu, \nu_\partial\big) \in \mathbb{V}(x)$,

$$\begin{aligned}&-\rho\mathcal{L}_\alpha(x,\Upsilon) + L_\alpha f(x,\Upsilon) + H_\alpha r(x,\Upsilon) + G_\alpha h(x,\Upsilon) \\ &\quad = \Big[-\rho\mathcal{L}_\alpha\big(\phi(x,t),\big[\Upsilon\big]_t\big) + L_\alpha f\big(\phi(x,t),\big[\Upsilon\big]_t\big) \\ &\qquad + H_\alpha r\big(\phi(x,t),\big[\Upsilon\big]_t\big) + G_\alpha h(\phi(x,t),[\Upsilon]_t)\Big] e^{-\alpha t - \Lambda^\nu(x,t)}\end{aligned}$$

$$+ \int_0^t e^{-\alpha s - \Lambda^{\nu}(x,s)} \Big[-\rho + f(\phi(x,s), \nu(s)) + \lambda(\phi(x,s), \nu(s)) Qh(\phi(x,s), \nu(s)) \Big] ds. \tag{3.19}$$

Notice now that we must have

$$w(\phi(x,t)) = -\rho \mathcal{L}_\alpha\big(\phi(x,t), \big[\hat{\Gamma}(x)\big]_t\big) + L_\alpha f\big(\phi(x,t), \big[\hat{\Gamma}(x)\big]_t\big) + H_\alpha r\big(\phi(x,t), \big[\hat{\Gamma}(x)\big]_t\big) + G_\alpha h(\phi(x,t), [\hat{\Gamma}(x)]_t), \tag{3.20}$$

since otherwise, we would have a contradiction to the fact that the infimum is reached in (3.4) for $\hat{\Gamma}(x)$. Indeed, if Eq. (3.20) were not satisfied, then there would exist $\Psi = \big(\beta, \beta_\partial\big) \in \mathbb{V}(\phi(x,t))$ such that

$$-\rho \mathcal{L}_\alpha\big(\phi(x,t), \big[\hat{\Gamma}(x)\big]_t\big) + L_\alpha f\big(\phi(x,t), \big[\hat{\Gamma}(x)\big]_t\big) + H_\alpha r\big(\phi(x,t), \big[\hat{\Gamma}(x)\big]_t\big) + G_\alpha h(\phi(x,t), [\hat{\Gamma}(x)]_t) > -\rho \mathcal{L}_\alpha\big(\phi(x,t), \Psi\big) + L_\alpha f\big(\phi(x,t), \Psi\big) + H_\alpha r\big(\phi(x,t), \Psi\big) + G_\alpha h(\phi(x,t), \Psi). \tag{3.21}$$

Now defining $\nu(s) = I_{[0,t[}(s)\hat{\gamma}(x,s) + I_{[t,\infty[}(s)\beta(s)$, we would get that $\hat{\Upsilon} = (\nu, \beta_\partial) \in \mathbb{V}(x)$, and by equation (3.21),

$$w(x) > -\rho \mathcal{L}_\alpha(x, \hat{\Upsilon}) + L_\alpha f(x, \hat{\Upsilon}) + H_\alpha r(x, \hat{\Upsilon}) + G_\alpha h(x, \hat{\Upsilon}),$$

in contradiction to Eq. (3.4). Consequently, by taking $\Upsilon = \hat{\Gamma}(x)$ in (3.19), we obtain (3.6).

From Assumption 2.5, we have that for all $x \in E$ and $t \in [0, t_*(x))$, $e^{-\Lambda^{\hat{\gamma}(x)}(x,t)} > 0$, and so (3.6) implies that for all $x \in E$, $w(\phi(x,t))$ is absolutely continuous on $[0, t_*(x))$. Recalling that $r \in \mathbb{M}(\partial E \times \mathbb{U})^+$ (see Assumption 2.10), it is easy to obtain from Eq. (3.5) that

$$\int_0^{t_*(x)} e^{-\alpha s - \Lambda^{\hat{\gamma}(x)}(x,s)} [-\rho + f(\phi(x,s), \hat{\gamma}(x,s)) + \lambda Qh(\phi(x,s), \hat{\gamma}(x,s))] ds \le w(x) + K_h. \tag{3.22}$$

If $t_*(x) < \infty$, we have by Assumption 2.5 that for all $0 \le t < t_*(x)$,

$$e^{\alpha t + \Lambda^{\hat{\gamma}(x)}(x,t)} \le e^{\alpha t_*(x) + \int_0^{t_*(x)} \overline{\lambda}(\phi(x,s)) ds} < \infty. \tag{3.23}$$

From (3.6), we get that

$$w(\phi(x,t)) = e^{\alpha t+\Lambda^{\hat{\gamma}(x)}(x,t)}\Big(w(x) \\ - \int_0^t e^{-\alpha s-\Lambda^{\hat{\gamma}(x)}(x,s)}\Big[-\rho + f(\phi(x,s),\hat{\gamma}(x,s)) + \lambda Qh(\phi(x,s),\hat{\gamma}(x,s))\Big]ds\Big). \tag{3.24}$$

Combining (3.22), (3.23), and (3.24), we have that the limit of $w(\phi(x,t))$ as $t \to t_*(x)$ exists in $\mathbb{R}$, showing that $w \in \mathbb{M}^{ac}(E)$.

Let $\nu \in \mathcal{V}(x)$. Define $\tilde{\nu}$ by $\tilde{\nu}(s) = I_{[0,t[}(s)\nu(s) + I_{[t,\infty[}(s)\hat{\gamma}(x,s)$. Then $\tilde{\nu} \in \mathcal{V}(x)$, and Υ defined by $\big(\tilde{\nu}(x), \hat{\gamma}_\partial(x)\big)$ belongs to $\mathbb{V}(x)$ and satisfies $\big[\Upsilon\big]_t = \big[\hat{\Gamma}\big]_t$. Consequently, combining (3.4), (3.19), and (3.20), it follows that for all $\nu \in \mathcal{V}(x)$,

$$\begin{aligned} w(x) &\le -\rho\mathcal{L}_\alpha(x,\Upsilon) + L_\alpha f(x,\Upsilon) + H_\alpha r(x,\Upsilon) + G_\alpha h(x,\Upsilon) \\ &= e^{-\alpha t-\Lambda^\nu(x,t)}w(\phi(x,t)) + \int_0^t e^{-\alpha s-\Lambda^\nu(x,s)}\Big[-\rho + f(\phi(x,s),\nu(s)) \\ &\quad + \lambda(\phi(x,s),\nu(s))Qh(\phi(x,s),\nu(s))\Big]ds. \end{aligned}$$

Now from the previous inequality and (3.6), we obtain (3.7). □

Next we present the proof of Proposition 3.3.

Proof of Proposition 3.3 Set $w = \mathcal{T}_\alpha(\rho,h)$. By hypothesis, w is bounded from below, and so $J_m^U(t,x,k)$ is well defined. For $U = (u,u_\partial) \in \mathcal{U}$, defined for $\hat{y} = (x,z,s,n) \in \widehat{E}$, $\widehat{f}^U(\hat{y}) = f(x,u(n,z,s))$, $\widehat{r}^U(\hat{y}) = r(x,u_\partial(n,z))$, $\widehat{h}(\hat{y}) = h(x)$, $\widehat{w}(\hat{y}) = w(x)$, and for $t \in [0,t_*(x)]$, $\widehat{\Lambda}^U(y,t) = \Lambda^U(x,n,t)$. Clearly, we have that $J_0^U(t,x,k) = w(x) \ge h(x)$ for all $(t,x,k) \in \mathbb{R}_+ \times E \times \mathbb{N}$. Now assume that for $m \in \mathbb{N}$, $J_m^U(t,x,k) \ge h(x)$ for all $(t,x,k) \in \mathbb{R}_+ \times E \times \mathbb{N}$. Defining $\hat{x} = (x,x,0,k)$, we have that

$$\begin{aligned} J_{m+1}^U(t,x,k) = E_{(x,k)}^U\Big[& I_{\{t<T_1\}}\Big(\int_0^t e^{-\alpha s}\Big[\widehat{f}^U(\widehat{\phi}(\hat{x},s)) - \rho\Big]ds + e^{-\alpha t}\widehat{w}(\widehat{\phi}(\hat{x},t))\Big) \\ & + I_{\{t\ge T_1\}}\Big(\int_0^{t\wedge T_{m+1}} e^{-\alpha s}\Big[\widehat{f}^U(\widehat{X}^U(s)) - \rho\Big]ds \\ & + \int_0^{t\wedge T_{m+1}} e^{-\alpha s}\widehat{r}^U\big(\widehat{X}^U(s-)\big)dp^*(s) \\ & + e^{-\alpha(t\wedge T_{m+1})}\widehat{w}\big(\widehat{X}^U(t\wedge T_{m+1})\big)\Big)\Big]. \end{aligned}$$

Therefore,

$$
\begin{aligned}
J_{m+1}^U(t,x,k) = E_{(x,k)}^U\Bigg[I_{\{t<T_1\}}\Bigg(\int_0^t e^{-\alpha s}\Big[\widehat{f}^U(\widehat{\phi}(\hat{x},s))-\rho\Big]ds + e^{-\alpha t}\widehat{w}(\widehat{\phi}(\hat{x},t))\Bigg) \\
+ I_{\{t\ge T_1\}}\Bigg(\int_0^{T_1} e^{-\alpha s}\Big[\widehat{f}^U(\widehat{\phi}(\hat{x},s))-\rho\Big]ds \\
+ I_{\{T_1=t_*(x)\}}e^{-\alpha t_*(x)}\,\widehat{r}^U\big(\widehat{\phi}(\hat{x},t_*(x))\big)\Bigg) \\
+ I_{\{t\ge T_1\}}\Bigg(\int_{T_1}^{t\wedge T_{m+1}} e^{-\alpha s}\Big[\widehat{f}^U(\widehat{X}^U(s))-\rho\Big]ds \\
+ \int_{T_1}^{t\wedge T_{m+1}} e^{-\alpha s}\widehat{r}^U\big(\widehat{X}^U(s-)\big)dp^*(s) \\
+ e^{-\alpha(t\wedge T_{m+1})}\widehat{w}\big(\widehat{X}^U(t\wedge T_{m+1})\big)\Bigg)\Bigg].
\end{aligned}
\tag{3.25}
$$

However, by the strong Markov property of the process $\{\widehat{X}^U(t)\}$, it follows that

$$
\begin{aligned}
& I_{\{t\ge T_1\}}e^{-\alpha T_1}J_m^U(t-T_1,\widehat{X}_1^U,k+1) \\
&= E_{(x,k)}^U\Bigg[I_{\{t\ge T_1\}}\Bigg(\int_{T_1}^{t\wedge T_{m+1}} e^{-\alpha s}\Big[\widehat{f}^U(\widehat{X}^U(s))-\rho\Big]ds \\
&\quad + \int_{T_1}^{t\wedge T_{m+1}} e^{-\alpha s}\widehat{r}^U\big(\widehat{X}^U(s-)\big)dp^*(s) + e^{-\alpha(t\wedge T_{m+1})}\widehat{w}\big(\widehat{X}^U(t\wedge T_{m+1})\big)\Bigg)|\mathcal{F}_{T_1}^{\widehat{X}^U}\Bigg].
\end{aligned}
\tag{3.26}
$$

Combining Eqs. (3.25) and (3.26) and the fact that

$$
I_{\{t\ge T_1\}}J_m^U(t-T_1,\widehat{X}_1^U,k+1) \ge I_{\{t\ge T_1\}}\widehat{h}(\widehat{X}_1^U),
$$

we obtain that

$$
J_{m+1}^U(t,x,k) \ge E_{(x,k)}^U\Bigg[\int_0^{t\wedge T_1} e^{-\alpha s}\Big[\widehat{f}^U(\widehat{\phi}(\hat{x},s))-\rho\Big]ds
$$

$$+ I_{\{t<T_1\}} e^{-\alpha t} \widehat{w}(\widehat{\phi}(\hat{x}, t)) + I_{\{t \geq T_1 = t_*(x)\}} e^{-\alpha t_*(x)} \widehat{r}^U \big(\widehat{\phi}(\hat{x}, t_*(x))\big)$$
$$+ I_{\{t \geq T_1\}} e^{-\alpha T_1} \widehat{h}(\widehat{X}_1^U)\bigg]. \tag{3.27}$$

However,

$$E_{(x,k)}^U \bigg[\int_0^{t \wedge T_1} e^{-\alpha s} \Big[\widehat{f}^U(\widehat{\phi}(\hat{x}, s)) - \rho \Big] ds + I_{\{t<T_1\}} e^{-\alpha t} \widehat{w}(\widehat{\phi}(\hat{x}, t)) \bigg]$$
$$= \int_0^{t \wedge t_*(x)} \Big[\widehat{f}^U(\widehat{\phi}(\hat{x}, s)) - \rho \Big] e^{-\alpha s - \widehat{\Lambda}^U(\hat{x}, s)} ds$$
$$+ I_{\{t<t_*(x)\}} e^{-\alpha t - \widehat{\Lambda}^U(\hat{x}, t)} \widehat{w}(\widehat{\phi}(\hat{x}, t)), \tag{3.28}$$

and

$$E_{(x,k)}^U \Big[I_{\{t \geq T_1\}} e^{-\alpha T_1} \widehat{h}(\widehat{X}_1^U) + I_{\{t \geq T_1 = t_*(x)\}} e^{-\alpha t_*(x)} \widehat{r}^U \big(\widehat{\phi}(\hat{x}, t_*(x))\big) \Big]$$
$$= e^{-\alpha t_*(x) - \widehat{\Lambda}^U(\hat{x}, t_*(x))} \widehat{r}(\widehat{\phi}(\hat{x}, t_*(x))) I_{\{t \geq t_*(x)\}}$$
$$+ \int_0^{t \wedge t_*(x)} \widehat{Q}^U \widehat{h}(\widehat{\phi}(\hat{x}, s)) \widehat{\lambda}^U(\widehat{\phi}(\hat{x}, s)) e^{-\alpha s - \widehat{\Lambda}^U(\hat{x}, s)} ds$$
$$+ e^{-\alpha t_*(x) - \widehat{\Lambda}^U(\hat{x}, t_*(x))} \widehat{Q}^U \widehat{h}(\widehat{\phi}(\hat{x}, t_*(x))) I_{\{t \geq t_*(x)\}}. \tag{3.29}$$

Combining Eqs. (3.27)–(3.29), it follows that for $t \in \mathbb{R}_+$,

$$J_{m+1}^U(t, x, k) \geq \int_0^{t \wedge t_*(x)} \Big[\widehat{f}^U(\widehat{\phi}(\hat{x}, s)) - \rho$$
$$+ \widehat{Q}^U \widehat{h}(\widehat{\phi}(\hat{x}, s)) \widehat{\lambda}^U(\widehat{\phi}(\hat{x}, s)) \Big] e^{-\alpha s - \widehat{\Lambda}^U(\hat{x}, s)} ds$$
$$+ I_{\{t \geq t_*(x)\}} e^{-\alpha t_*(x) - \widehat{\Lambda}^U(\hat{x}, t_*(x))} \Big[\widehat{Q}^U \widehat{h}(\widehat{\phi}(\hat{x}, t_*(x))) + \widehat{r}(\widehat{\phi}(\hat{x}, t_*(x))) \Big]$$
$$+ I_{\{t<t_*(x)\}} e^{-\alpha t - \widehat{\Lambda}^U(\hat{x}, t)} \widehat{w}(\widehat{\phi}(\hat{x}, t))$$
$$= \int_0^{t \wedge t_*(x)} e^{-\alpha s - \Lambda^{\nu_k}(x, s)} \Big[-\rho + f(\phi(x, s), \nu_k(s))$$
$$+ \lambda(\phi(x, s), \nu_k(s)) Qh(\phi(x, s), \nu_k(s)) \Big] ds$$
$$+ I_{\{t \geq t_*(x)\}} e^{-\alpha t_*(x) - \Lambda^{\nu_k}(x, t_*(x))} \Big[Qh(\phi(x, t_*(x)), u_\partial(k, x))$$

$$+ r(\phi(x, t_*(x)), u_\partial(k, x))\Big]$$
$$+ I_{\{t<t_*(x)\}} e^{-\alpha t - \Lambda^{\nu_k}(x,t)} w(\phi(x, t)), \tag{3.30}$$

with $\nu_k(.) = u(k, x, .)$. Clearly, $\nu_k(.) \in \mathcal{V}(x)$.

Now if $t < t_*(x)$, then

$$J^U_{m+1}(t, x, k) \geq \int_0^t e^{-\alpha s - \Lambda^{\nu_k}(x,s)} \Big[-\rho + f(\phi(x, s), \nu_k(s)) + \lambda(\phi(x, s), \nu_k(s)) Q h(\phi(x, s), \nu_k(s))\Big] ds + e^{-\alpha t - \Lambda^{\nu_k}(x,t)} w(\phi(x, t)),$$

and by applying Proposition 3.2, it follows that $J^U_{m+1}(t, x, k) \geq w(x) \geq h(x)$.

If $t \geq t_*(x)$, then by defining $\Upsilon = (\nu_k, u_\partial(k, x)) \in \mathbb{V}(x)$ and using Eq. (3.8), we have that

$$\begin{aligned} J^U_{m+1}(t, x, k) &\geq e^{-\alpha t_*(x) - \Lambda^{\nu_k}(x, t_*(x))} \Big[Q h(\phi(x, t_*(x)), u_\partial(k, x)) \\ &+ r(\phi(x, t_*(x)), u_\partial(k, x))\Big] + \int_0^{t_*(x)} e^{-\alpha s - \Lambda^{\nu_k}(x,s)} \Big[-\rho + f(\phi(x, s), \nu_k(s)) \\ &+ \lambda(\phi(x, s), \nu_k(s)) Q h(\phi(x, s), \nu_k(s))\Big] ds \\ &= -\rho \mathcal{L}_\alpha(x, \Upsilon) + L_\alpha f(x, \Upsilon) + H_\alpha r(x, \Upsilon) + G_\alpha h(x, \Upsilon) \geq w(x) \geq h(x), \end{aligned}$$

proving the result. □

Next we present the proof of Proposition 3.4.

Proof of Proposition 3.10 Set $w = \mathcal{T}_\alpha(\rho, h)$. It is easy to check that w is bounded from below, and so $J^{\widehat{U}}_m(t, x, k)$ is well defined. Note that $J^{\widehat{U}}_0(t, x, k) = w(x) \leq h(x)$ for all $(t, x, k) \in \mathbb{R}_+ \times E \times \mathbb{N}$. Now assume that for $m \in \mathbb{N}$, $J^{\widehat{U}}_m(t, x, k) \leq h(x)$ for all $(t, x, k) \in \mathbb{R}_+ \times E \times \mathbb{N}$; then from this hypothesis, it follows that the inequalities in (3.30) can be inverted for the control process given by $\widehat{U}$. Consequently, if $t < t_*(x)$, the last statement of Proposition 3.2 implies that

$$J^{\widehat{U}}_{m+1}(t, x, k) \leq w(x) \leq h(x).$$

If $t \geq t_*(x)$, then

$$\begin{aligned} J^{\widehat{U}}_{m+1}(t, x, k) &\leq -\rho \mathcal{L}_\alpha(x, \hat{\Gamma}(x)) + L_\alpha f(x, \hat{\Gamma}(x)) + H_\alpha r(x, \hat{\Gamma}(x)) \\ &+ G_\alpha h(x, \hat{\Gamma}(x)) = w(x) \leq h(x), \end{aligned}$$

establishing the desired result. □

3.5.2 Proofs of the Results of Sect. 3.3

Next we present the proof of Proposition 3.10.

Proof of Proposition 3.10 Item (a) Consider $x \in E$ and $t \in [0, t_*(x))$ if $t_*(x) = \infty$, and $t \in [0, t_*(x)]$ if $t_*(x) < \infty$. We have from Assumptions 2.5 and 3.5 that the mapping $\widehat{\lambda}$ defined on K by $\widehat{\lambda}(y, u) = \frac{\lambda(y,u)}{\overline{\lambda}(y)}$ is a Carathéodory function on K (that is, measurable and bounded on K and continuous for u in $\mathbb{U}(x)$; see ([69], p. 459) for the definition of a Carathéodory function). By Theorem 2 in [69], $\widehat{\lambda}$ can be extended to a Carathéodory function on $E \times \mathbb{U}$, denoted by $\widetilde{\lambda}$, such that $\max_{a\in A} \widetilde{\lambda}(x, a) = \max_{a\in A(x)} \widehat{\lambda}(x, a)$. Consequently, the mapping $s \to \widetilde{\lambda}(\phi(x, s), .)\overline{\lambda}(\phi(x, s))I_{\{s\leq t\}}$ belongs to $L^1(\mathbb{R}_+; \mathbb{C}(\mathbb{U}))$. Therefore,

$$
\begin{aligned}
&\lim_{n\to\infty} \int_0^t \int_{\mathbb{U}(\phi(x,s))} \lambda(\phi(x, s), u)\mu_n(s, du)ds \\
&= \lim_{n\to\infty} \int_0^t \int_{\mathbb{U}(\phi(x,s))} \widehat{\lambda}(\phi(x, s), u)\overline{\lambda}(\phi(x, s))\mu_n(s, du)ds \\
&= \lim_{n\to\infty} \int_0^t \int_{\mathbb{U}(\phi(x,s))} \widetilde{\lambda}(\phi(x, s), u)\overline{\lambda}(\phi(x, s))\mu_n(s, du)ds \\
&= \int_0^t \int_{\mathbb{U}(\phi(x,s))} \widetilde{\lambda}(\phi(x, s), u)\overline{\lambda}(\phi(x, s))\mu(s, du)ds \\
&= \int_0^t \int_{\mathbb{U}(\phi(x,s))} \lambda(\phi(x, s), u)\mu(s, du)ds,
\end{aligned}
$$

or in other words,

$$\lim_{n\to\infty} \Lambda^{\mu_n}(x, t) = \Lambda^{\mu}(x, t).$$

From item (b) of Assumption 2.5, we have that

$$e^{-\alpha_n t - \Lambda^{\mu_n}(x,t)} \leq e^{-\int_0^t \underline{\lambda}(\phi(x,s))ds}$$

and

$$\int_0^{t_*(x)} e^{-\int_0^t \underline{\lambda}(\phi(x,s))ds} dt < \infty.$$

Consequently, by the dominated convergence theorem, we obtain that

$$\lim_{n\to\infty} \mathcal{L}_{\alpha_n}(x, \Theta_n) = \int_0^{t_*(x)} \lim_{n\to\infty} e^{-\alpha_n t - \Lambda^{\mu_n}(x,t)} dt$$
$$= \int_0^{t_*(x)} e^{-\alpha t - \Lambda^{\mu}(x,t)} dt = \mathcal{L}_{\alpha}(x, \Theta),$$

proving item (a).

Item (b) We have from Assumption 3.6 that there exists a sequence of measurable functions $(f_j)_{j\in\mathbb{N}}$ such that for all $y \in \overline{E}$, $f_j(y, .) \uparrow f(y, .)$ and $f_j(y, .) \in \mathbb{C}(\mathbb{U}(y))$. We have for $(n, j) \in \mathbb{N}^2$, $x \in E$,

$$L_{\alpha_n} f_j(x, \Theta_n) = \int_0^{t_*(x)} \left[e^{-\alpha_n t - \Lambda^{\mu_n}(x,t)} - e^{-\alpha t - \Lambda^{\mu}(x,t)}\right] f_j(\phi(x, t), \mu_n(t)) dt$$
$$+ \int_0^{t_*(x)} e^{-\alpha t - \Lambda^{\mu}(x,t)} f_j(\phi(x, t), \mu_n(s)) dt. \tag{3.31}$$

However, Assumptions 2.5 and 3.9 give

$$|e^{-\alpha_n t - \Lambda^{\mu_n}(x,t)} - e^{-\alpha t - \Lambda^{\mu}(x,t)}| f_j(\phi(x, t), \mu_n(t)) \leq 2e^{-\int_0^t \underline{\lambda}(\phi(x,s))ds}\, \overline{f}(\phi(x, t)).$$

By item (a) and the dominated convergence theorem, we obtain that

$$\lim_{n\to\infty} \int_0^{t_*(x)} \left[e^{-\alpha_n t - \Lambda^{\mu_n}(x,t)} - e^{-\alpha t - \Lambda^{\mu}(x,t)}\right] f_j(\phi(x, t), \mu_n(t)) dt = 0. \tag{3.32}$$

Now by the fact that $\sup_{a\in\mathbb{U}(x)} f_j(x, a) \leq \overline{f}(x)$ (see Assumptions 3.6 and 3.9), we can proceed as for item (a) to show that

$$\lim_{n\to\infty} \int_0^{t_*(x)} e^{-\alpha t - \Lambda^{\mu}(x,t)} f_j(\phi(x, t), \mu_n(s)) dt$$
$$= \int_0^{t_*(x)} e^{-\alpha t - \Lambda^{\mu}(x,t)} f_j(\phi(x, t), \mu(s)) dt. \tag{3.33}$$

Therefore, from Eqs. (3.31)–(3.33), it follows that

$$\lim_{n\to\infty} L_{\alpha_n} f_j(x, \Theta_n) = L_\alpha f_j(x, \Theta).$$

However, notice that $L_{\alpha_n} f(x, \Theta_n) \geq L_{\alpha_n} f_j(x, \Theta_n)$, and the result follows by the monotone convergence theorem.

Item (c) Let us consider first that $t_*(x) = \infty$. From item (b) of Assumption 2.5 and Remark 2.16, it follows that

$$\begin{aligned} e^{-\Lambda^{\mu_n}(x,t_*(x))} &\leq e^{-\int_0^{t_*(x)} \underline{\lambda}(\phi(x,s))ds} = 0, \text{ and} \\ e^{-\Lambda^{\mu}(x,t_*(x))} &\leq e^{-\int_0^{t_*(x)} \underline{\lambda}(\phi(x,s))ds} = 0, \end{aligned}$$

and the result follows immediately, since $H_{\alpha_n} r(x, \Theta_n) = Hr(x, \Theta) = 0$. Suppose now that $t_*(x) < \infty$ and set $z = \phi(x, t_*(x))$. We have from Assumption 3.7 that there exists a sequence of measurable functions $(r_j)_{j\in\mathbb{N}}$ such that for all $y \in \partial E$, $r_j(y, .) \uparrow r(y, .)$ and $r_j(y, .) \in \mathbb{C}(\mathbb{U}(y))$. Consequently, $r(z, \mu_{\partial,n}) \geq r_i(z, \mu_{\partial,n})$, and so $\underline{\lim}_{n\to\infty} r(z, \mu_{\partial,n}) \geq r_i(z, \mu_\partial)$. From the monotone convergence theorem, we obtain (c).

Item (d) From Remark 2.16 it is easy to show that

$$L_{\alpha_n}(\lambda + \alpha_n)(x, \Theta_n) = \begin{cases} 1 - e^{-\alpha_n t_*(x) - \Lambda^{\mu_n}(x,t_*(x))} & \text{if } t_*(x) < \infty, \\ 1 & \text{otherwise.} \end{cases}$$

Consequently, if $t_*(x) < \infty$, then

$$\begin{aligned} \lim_{n\to\infty} L_{\alpha_n}\lambda(x, \Theta_n) &= \lim_{n\to\infty} L_{\alpha_n}(\lambda + \alpha_n)(x, \Theta_n) - \alpha\mathcal{L}_\alpha(x, \Theta) \\ &= 1 - e^{-\alpha t_*(x) - \Lambda^{\mu}(x,t_*(x))} - \alpha\mathcal{L}_\alpha(x, \Theta) \\ &= L_\alpha(\lambda + \alpha)(x, \Theta) - \alpha\mathcal{L}_\alpha(x, \Theta) \\ &= L_\alpha\lambda(x, \Theta). \end{aligned}$$

If $t_*(x) = \infty$, then $\lim_{n\to\infty} L_{\alpha_n}\lambda(x, \Theta_n) = L_\alpha\lambda(x, \Theta)$. In any case, we have that

$$\lim_{n\to\infty} L_{\alpha_n}\lambda(x, \Theta_n) = L\lambda(x, \Theta). \tag{3.34}$$

Set $\widetilde{h}_k = h_k + K_h$, $\widetilde{h} = h + K_h$, and $\widehat{h}_k = \inf_{j\geq k} \widetilde{h}_j$. Therefore, $\widehat{h}_k \uparrow \widetilde{h}$ and $\widehat{h}_k \leq \widetilde{h}_n$ for $n \geq k$. By hypothesis, $\widehat{h}_k(y) \geq 0$ for all $y \in E$. Define $\lambda_m = m \wedge \lambda$ and $\widehat{h}_{k,m} = m \wedge \widehat{h}_k$. From Assumptions 3.5 and 3.8, we have that for each k and m, $\lambda_m Q\widehat{h}_{k,m} \in \mathbb{B}(E \times A)^+$, and for every $y \in E$, $\lambda_m Q\widehat{h}_{k,m}(y, .)$ is continuous on $\mathbb{U}(y)$. Consequently, we can proceed as for the proof of item (b) to show that

$$\underline{\lim}_{n\to\infty} L_{\alpha_n}(\lambda_m Q\widehat{h}_{k,m})(x, \Theta_n) \geq L_\alpha(\lambda_m Q\widehat{h}_{k,m})(x, \Theta).$$

Recalling that $\lambda \geq \lambda_m$ and $\tilde{h}_n \geq \widehat{h}_k \geq \widehat{h}_{k,m}$ for $n \geq k$, we have

$$\underline{\lim}_{n\to\infty} L_{\alpha_n}(\lambda Q\tilde{h}_n)(x, \Theta_n) \geq \underline{\lim}_{n\to\infty} L_{\alpha_n}(\lambda_m Q\widehat{h}_{k,m})(x, \Theta_n),$$

and so

$$\underline{\lim}_{n\to\infty} L_{\alpha_n}(\lambda Q\tilde{h}_n)(x, \Theta_n) \geq L_{\alpha}(\lambda_m Q\widehat{h}_{k,m})(x, \Theta).$$

From the monotone convergence theorem and taking the limit over m and k, we get that

$$\underline{\lim}_{n\to\infty} L_{\alpha_n}(\lambda Q\tilde{h}_n)(x, \Theta_n) \geq L_{\alpha}(\lambda Q\tilde{h})(x, \Theta). \tag{3.35}$$

Notice that $L_{\alpha_n}\lambda(x, \Theta_n)$ and $L_{\alpha}\lambda(x, \Theta)$ are finite by Assumption 2.5, and thus

$$L_{\alpha_n}(\lambda Q\tilde{h}_{\alpha_n})(x, \Theta_n) = L_{\alpha_n}(\lambda Qh_{\alpha_n})(x, \Theta_n) + K_h L_{\alpha_n}\lambda(x, \Theta_n), \tag{3.36}$$

and similarly,

$$L_{\alpha}(\lambda Q\tilde{h}_{\alpha})(x, \Theta) = L_{\alpha}(\lambda Qh_{\alpha})(x, \Theta) + K_h L_{\alpha}\lambda(x, \Theta). \tag{3.37}$$

By combining Eqs. (3.34)–(3.37), we get that

$$\underline{\lim}_{n\to\infty} L_{\alpha_n}(\lambda Qh_{\alpha_n})(x, \Theta_n) \geq L_{\alpha}(\lambda Qh)(x, \Theta).$$

Using similar arguments as above and (c), we can show that

$$\underline{\lim}_{n\to\infty} H_{\alpha_n} Qh_n(x, \Theta_n) \geq HQh(x, \Theta),$$

which completes the proof of (d). □

3.5.3 *Proofs of the Results of Sect. 3.4*

We present first the proof of Proposition 3.13.

Proof of Proposition 3.13 From Assumption 2.15 and Proposition 2.17, it follows that the mapping V defined on $\mathcal{K}$ by

$$V(x, \Theta) = -\rho\mathcal{L}_{\alpha}(x, \Theta) + L_{\alpha}f(x, \Theta) + H_{\alpha}r(x, \Theta) + G_{\alpha}h(x, \Theta)$$

is measurable. Moreover, by Corollary 3.11, it follows that for all $x \in E$, $V(x, .)$ is lower semicontinuous on $\mathbb{V}^r(x)$. Recalling that $\mathbb{V}^r(x)$ is a compact subset of $\mathbb{V}^r$

and by Assumption 2.15, we obtain from Proposition D.5 in [49] that there exists $\hat{\Theta} \in \mathcal{S}_{\mathbb{V}^r}$ such that (3.13) is satisfied.

The rest of the proof is similar to the proof of Proposition 3.2 and is therefore omitted. □

Before presenting the proof of Theorem 3.14, we need the following auxiliary results.

Lemma 3.17 *For every $\mu \in \mathcal{P}(\mathbb{U}(x))$ and $x \in \overline{E}$, $\lambda(x, \mu) < \infty$.*

Proof From Assumption 2.3, $\mathbb{U}(x)$ is a compact subspace of $\mathbb{U}$, and from Assumption 3.5, $\lambda(x, .) : \mathbb{U}(x) \mapsto \mathbb{R}_+$ is continuous. Therefore, there exists $\hat{a} \in \mathbb{U}(x)$ such that $\max_{a \in \mathbb{U}(x)} \lambda(x, a) = \lambda(x, \hat{a})$, and thus

$$0 \leq \lambda(x, \mu) = \int_{\mathbb{U}(x)} \lambda(x, a)\mu(da) \leq \lambda(x, \hat{a}).$$

□

Lemma 3.18 *Suppose that $w \in \mathbb{M}(E)$ and $h \in \mathbb{M}(E)$ is bounded from below $-K_h$ with $K_h \in \mathbb{R}_+$. Then*

$$\inf_{a \in \mathbb{U}(x)} \Big\{ f(x, a) + \lambda(x, a)Qh(x, a) - \lambda(x, a)w(x) \Big\} = \inf_{\mu \in \mathcal{P}(\mathbb{U}(x))} \Big\{ f(x, \mu) - \lambda(x, \mu)w(x) + \lambda Qh(x, \mu) \Big\}, \tag{3.38}$$

$$\inf_{a \in \mathbb{U}(\phi(x, t_*(x)))} \{r(\phi(x, t_*(x)), a) + Qh(\phi(x, t_*(x)), a)\} = \inf_{\mu \in \mathcal{P}(\mathbb{U}(\phi(x, t_*(x))))} \{r(\phi(x, t_*(x)), \mu) + Qh(\phi(x, t_*(x)), \mu)\}. \tag{3.39}$$

Proof For simplicity, set $\vartheta(x, a) = f(x, a) + \lambda(x, a)Qh(x, a) - \lambda(x, a)w(x)$. Notice that from Lemma 3.17, for every $\mu \in \mathcal{P}(\mathbb{U}(x))$, $\lambda(x, \mu) < \infty$, and thus, recalling that f and $h + K_h$ are positive,

$$\begin{aligned} f(x, \mu) &- \lambda(x, \mu)w(x) + \lambda Qh(x, \mu) \\ &= f(x, \mu) + \lambda Q(h + K_h)(x, \mu) - \lambda(x, \mu)(w(x) + K_h) \\ &= \int_{\mathbb{U}(x)} \big[f(x, a) + \lambda(x, a)Q(h + K_h)(x, a) - \lambda(x, a)(w(x) + K_h) \big]\mu(da) \\ &= \int_{\mathbb{U}(x)} \vartheta(x, a)\mu(da). \end{aligned} \tag{3.40}$$

However, as in Lemma 5.7 of [37], we have that

$$\inf_{a\in\mathbb{U}(x)} \vartheta(x,a) = \inf_{\mu\in\mathcal{P}(\mathbb{U}(x))} \int_{\mathbb{U}(x)} \vartheta(x,a)\mu(da). \tag{3.41}$$

Combining (3.40) and (3.41), we get (3.38). Similarly, we have (3.39). □

Next we present the proof of Theorem 3.14.

Proof of Theorem 3.14 According to Proposition 3.13, $w \in \mathbb{M}^{ac}(E)$, and there exists $\hat{\Theta} \in \mathcal{S}_{\mathbb{V}^r}$ such that for all $x \in E$ and $t \in [0, t_*(x))$, we have

$$\begin{aligned} &e^{-\alpha t-\Lambda^{\hat{\mu}(x)}(x,t)} w(\phi(x,t)) - w(x) \\ &\quad = \int_0^t e^{-\alpha s-\Lambda^{\hat{\mu}(x)}(x,s)}\Big[\rho - f(\phi(x,s),\hat{\mu}(x,s)) - \lambda Qh(\phi(x,s),\hat{\mu}(x,s))\Big]ds, \end{aligned} \tag{3.42}$$

where $\hat{\Theta}(x) = (\hat{\mu}(x), \hat{\mu}_\partial(x))$. Since $w \in \mathbb{M}^{ac}(E)$, it follows from Lemma 2.2 that there exists a function $\mathcal{X}w$ in $\mathbb{M}(E)$ satisfying Eq. (2.1). Therefore, we obtain from Eq. (3.42) that

$$\begin{aligned} \mathcal{X}w(\phi(x,t)) - [\alpha + \lambda(\phi(x,t),\hat{\mu}(x,t))]w(\phi(x,t)) = \rho - f(\phi(x,t),\hat{\mu}(x,t)) \\ - \lambda Qh(\phi(x,t),\hat{\mu}(x,t)), \end{aligned}$$

$\eta - a.s.$ on $[0, t_*(x))$, implying that

$$\begin{aligned} &-\mathcal{X}w(\phi(x,t)) + \alpha w(\phi(x,t)) \\ &\geq \inf_{\mu\in\mathcal{P}(\mathbb{U}(\phi(x,t)))} \Big\{ f(\phi(x,t),\mu) - \lambda(\phi(x,t),\mu)w(\phi(x,t)) + \lambda Qh(\phi(x,t),\mu)\Big\} - \rho. \end{aligned}$$

From Lemma 3.18, we obtain that

$$\begin{aligned} &\inf_{\mu\in\mathcal{P}(\mathbb{U}(\phi(x,t)))} \Big\{ f(\phi(x,t),\mu) - \lambda(\phi(x,t),\mu)w(\phi(x,t)) + \lambda Qh(\phi(x,t),\mu)\Big\} - \rho \\ &= \inf_{a\in\mathbb{U}(\phi(x,t))} \Big\{ f(\phi(x,t),a) - \lambda(\phi(x,t),a)\big[w(\phi(x,t)) - Qh(\phi(x,t),a)\big]\Big\} - \rho. \end{aligned}$$

Consequently, by considering the measurable selector $\overline{u} \in \mathcal{S}_{\mathbb{U}}$ given by $\overline{u} = \widehat{u}(w,h)$ (see Definition 3.12, (D1)), we have

$$\begin{aligned} &-\mathcal{X}w(\phi(x,t)) + \alpha w(x) \geq -\rho + f(\phi(x,t),\overline{u}(\phi(x,t))) \\ &- \lambda(\phi(x,t),\overline{u}(\phi(x,t)))\big[w(\phi(x,t)) - Qh(\phi(x,t),\overline{u}(\phi(x,t)))\big], \end{aligned} \tag{3.43}$$

η-a.s. on $[0, t_*(x))$. Let Ξ be the set in $\mathcal{B}([0, t_*(x)))$ such that the previous inequality is strict. If $\eta(\Xi) > 0$, then there would exist $t \in [0, t_*(x))$ such that

$$w(x) - e^{-(\alpha t+\overline{\Lambda}(x,t))}w(\phi(x,t)) > \int_0^t e^{-(\alpha s+\overline{\Lambda}(x,s))}\big[f(\phi(x,s),\overline{u}(\phi(x,s)))$$
$$+ \lambda(\phi(x,s),\overline{u}(\phi(x,s)))Qh(\phi(x,s),\overline{u}(\phi(x,s))) - \rho\big]ds,$$

where $\overline{\Lambda}(x,t)$ denotes $\int_0^t \lambda(\phi(x,s),\overline{u}(\phi(x,s)))ds$. However, this would lead to a contradiction with (3.14). Thus we have that

$$\begin{aligned}-\mathcal{X}w(\phi(x,t)) + \alpha w(\phi(x,t)) &= -\rho + f(\phi(x,t),\overline{u}(\phi(x,t)))\\ &\quad - \lambda(\phi(x,t),\overline{u}(\phi(x,t)))\big[w(\phi(x,t)) - Qh(\phi(x,t),\overline{u}(\phi(x,t)))\big],\end{aligned}$$

$\eta - a.s.$ on $[0, t_*(x))$. Consequently, for all $t \in [0, t_*(x))$,

$$\begin{aligned}w(x) = e^{-(\alpha t+\overline{\Lambda}(x,t))}w(\phi(x,t)) &+ \int_0^t e^{-(\alpha s+\overline{\Lambda}(x,s))}\big[f(\phi(x,s),\overline{u}(\phi(x,s)))\\ &+ \lambda(\phi(x,s),\overline{u}(\phi(x,s)))Qh(\phi(x,s),\overline{u}(\phi(x,s))) - \rho\big]ds. \qquad (3.44)\end{aligned}$$

First, consider the case in which $t_*(x) < \infty$. Recalling that $w \in \mathbb{M}^{ac}(E)$, we obtain, by taking the limit as t tends to $t_*(x)$ in the previous equation, that the feedback measurable selector $\widehat{u}_\phi(w,h) \in \mathcal{S}_\mathbb{V}$ (see item (D2) of Definition 3.12) satisfies

$$\begin{aligned}w(x) = e^{-(\alpha t_*(x)+\overline{\Lambda}(x,t_*(x)))}&w(\phi(x,t_*(x))) - \rho\mathcal{L}_\alpha(x,\widehat{u}_\phi(w,h)(x))\\ &+ L_\alpha f(x,\widehat{u}_\phi(w,h)(x))\\ &+ \int_0^{t_*(x)} e^{-(\alpha s+\overline{\Lambda}(x,s))}\lambda(\phi(x,s),\overline{u}(\phi(x,s)))Qh(\phi(x,s),\overline{u}(\phi(x,s)))ds.\end{aligned} \qquad (3.45)$$

Define the control $\Theta(x)$ by $(\hat{\mu}(x),\mu)$ for $\mu \in \mathcal{P}\big(\mathbb{U}(\phi(x,t_*(x)))\big)$, where $\hat{\mu}$ was introduced in Proposition 3.13 as the first component of $\hat{\Theta}$. From Eq. (3.16), we obtain that

$$w(x) \le -\rho\mathcal{L}_\alpha(x,\Theta(x)) + L_\alpha f(x,\Theta(x)) + H_\alpha r(x,\Theta(x)) + G_\alpha h(x,\Theta(x)),$$

and from the definition of $\Theta(x)$ and $\hat{\Theta}(x)$, it follows that

$$\begin{aligned}w(x) \le -\rho\mathcal{L}_\alpha(x,\hat{\Theta}(x)) &+ L_\alpha f(x,\hat{\Theta}(x))\\ &+ \int_0^{t_*(x)} e^{-\alpha s-\Lambda^{\hat{\mu}(x)}(x,s)}\lambda Qh(\phi(x,s),\hat{\mu}(x,s))ds\end{aligned}$$

$$+ e^{-\alpha t_*(x) - \Lambda^{\hat{\mu}(x)}(x, t_*(x))} \big[Qh(\phi(x, t_*(x)), \mu) + r(\phi(x, t_*(x)), \mu) \big]. \quad (3.46)$$

From Eq. (3.15), we have that

$$w(x) = \int_0^t e^{-\alpha s - \Lambda^{\hat{\mu}(x)}(x,s)} \Big[-\rho + f(\phi(x, s), \hat{\mu}(x, s)) + \lambda Qh(\phi(x, s), \hat{\mu}(x, s)) \Big] ds$$
$$+ e^{-\alpha t - \Lambda^{\hat{\mu}(x)}(x,t)} w(\phi(x, t)).$$

Since $w \in \mathbb{M}^{ac}(E)$ and $t_*(x) < \infty$, we obtain

$$\begin{aligned} w(x) &= \lim_{t \to t_*(x)} \int_0^t e^{-\alpha s - \Lambda^{\hat{\mu}(x)}(x,s)} \Big[-\rho + f(\phi(x, s), \hat{\mu}(x, s)) \\ &\quad + \lambda Qh(\phi(x, s), \hat{\mu}(x, s)) \Big] ds + \lim_{t \to t_*(x)} e^{-\alpha t - \Lambda^{\hat{\mu}(x)}(x,t)} w(\phi(x, t)) \\ &= -\rho \mathcal{L}_\alpha(x, \hat{\Theta}(x)) + L_\alpha f(x, \hat{\Theta}(x)) \\ &\quad + \int_0^{t_*(x)} e^{-\alpha s - \Lambda^{\hat{\mu}(x)}(x,s)} \lambda Qh(\phi(x, s), \hat{\mu}(x, s)) ds \\ &\quad + e^{-\alpha t_*(x) - \Lambda^{\hat{\mu}(x)}(x, t_*(x))} w(\phi(x, t_*(x))). \end{aligned} \quad (3.47)$$

From Assumption 2.5, we have that $e^{-\Lambda^{\hat{\mu}(x)}(x, t_*(x))} > 0$. Therefore, combining (3.46) and (3.47), it follows that for all $x \in E$ and $\mu \in \mathcal{P}\big(\mathbb{U}(\phi(x, t_*(x)))\big)$,

$$w(\phi(x, t_*(x))) \leq Qh(\phi(x, t_*(x)), \mu) + r(\phi(x, t_*(x)), \mu).$$

Clearly, by (3.13), it can be claimed that the previous inequality becomes an equality for $\mu = \hat{\mu}_\partial(x)$, and from Lemma 3.18 we obtain that

$$\begin{aligned} w(\phi(x, t_*(x))) &= \inf_{\mu \in \mathcal{P}(\mathbb{U}(\phi(x, t_*(x))))} \{ r(\phi(x, t_*(x)), \mu) + Qh(\phi(x, t_*(x)), \mu) \} \\ &= \inf_{a \in \mathbb{U}(\phi(x, t_*(x)))} \{ r(\phi(x, t_*(x)), a) + Qh(\phi(x, t_*(x)), a) \}. \end{aligned}$$

Consequently, we have that

$$w(\phi(x, t_*(x))) = r(\phi(x, t_*(x)), \overline{u}(\phi(x, t_*(x)))) + Qh(\phi(x, t_*(x)), \overline{u}(\phi(x, t_*(x)))). \quad (3.48)$$

Combining Eqs. (3.16), (3.45), and (3.48), it follows that

$$\mathcal{R}_\alpha(\rho, h)(x) = -\rho\mathcal{L}_\alpha(x, \widehat{u}_\phi(w, h)(x)) + L_\alpha f(x, \widehat{u}_\phi(w, h)(x)) \\ + H_\alpha r(x, \widehat{u}_\phi(w, h)(x)) + G_\alpha h(x, \widehat{u}_\phi(w, h)(x)).$$

Consider now the case in which $t_*(x) = \infty$. From item (b) of Assumption 2.5, we obtain that the limit as t tends to infinity of

$$\int_0^t e^{-(\alpha s+\overline{\Lambda}(x,s))}\big[f(\phi(x,s), \overline{u}(\phi(x,s))) \\ + \lambda(\phi(x,s), \overline{u}(\phi(x,s)))Qh(\phi(x,s), \overline{u}(\phi(x,s))) - \rho\big]ds$$

exists in $\mathbb{R} \cup \{+\infty\}$ and that $w(\phi(x,t)) \geq -\rho K_\lambda - K_h$ for all $t \in [0, +\infty)$. Therefore, by (3.44), we obtain that

$$w(x) \geq -e^{-(\alpha t+\overline{\Lambda}(x,t))}[\rho K_\lambda + K_h] + \int_0^t e^{-(\alpha s+\overline{\Lambda}(x,s))}\big[f(\phi(x,s), \overline{u}(\phi(x,s))) \\ + \lambda(\phi(x,s), \overline{u}(\phi(x,s)))Qh(\phi(x,s), \overline{u}(\phi(x,s))) - \rho\big]ds.$$

Consequently, from Remark 2.16, the feedback measurable selector $\widehat{u}_\phi(w, h) \in \mathcal{S}_{\mathbb{V}}$ satisfies

$$w(x) \geq -\rho\mathcal{L}_\alpha(x, \widehat{u}_\phi(w, h)(x)) + L_\alpha f(x, \widehat{u}_\phi(w, h)(x)) \\ + \int_0^{t_*(x)} e^{-(\alpha s+\overline{\Lambda}(x,s))}\lambda(\phi(x,s), \overline{u}(\phi(x,s)))Qh(\phi(x,s), \overline{u}(\phi(x,s)))ds \\ = -\rho\mathcal{L}_\alpha(x, \widehat{u}_\phi(w, h)(x)) + L_\alpha f(x, \widehat{u}_\phi(w, h)(x)) + H_\alpha r(x, \widehat{u}_\phi(w, h)(x)) \\ + G_\alpha h(x, \widehat{u}_\phi(w, h)(x)).$$

From Eq. (3.16), this yields

$$\mathcal{R}_\alpha(\rho, h)(x) = -\rho\mathcal{L}_\alpha(x, \widehat{u}_\phi(w, h)(x)) + L_\alpha f(x, \widehat{u}_\phi(w, h)(x)) \\ + H_\alpha r(x, \widehat{u}_\phi(w, h)(x)) + G_\alpha h(x, \widehat{u}_\phi(w, h)(x)).$$

In any case,

$$\mathcal{R}_\alpha(\rho, h)(x) = -\rho\mathcal{L}_\alpha(x, \widehat{u}_\phi(w, h)(x)) + L_\alpha f(x, \widehat{u}_\phi(w, h)(x)) \\ + H_\alpha r(x, \widehat{u}_\phi(w, h)(x)) + G_\alpha h(x, \widehat{u}_\phi(w, h)(x)).$$

Since $\mathbb{V}(x) \subset \mathbb{V}^r(x)$, it follows that

$$\mathcal{R}_\alpha(\rho, h)(x) \leq \mathcal{T}_\alpha(\rho, h)(x).$$

However, $\widehat{u}_\phi(w, h) \in \mathcal{S}_{\mathbb{V}}$, proving the desired result. □

Chapter 4
The Vanishing Discount Approach for PDMPs

4.1 Outline of the Chapter

This chapter is devoted to the existence of an optimal control strategy for the long-run average continuous control problem of PDMPs using the vanishing discount approach. This is done by first establishing in Sect. 4.2 an optimality equation for the discounted control problem. In the sequel, two sets of assumptions are considered. In Sect. 4.3, the first one is presented, expressed mainly in terms of the relative difference $h_\alpha(x) = \mathcal{J}_{\mathcal{D}}^{\alpha}(x) - \mathcal{J}_{\mathcal{D}}^{\alpha}(x_0)$ of the α-discount value functions $\mathcal{J}_{\mathcal{D}}^{\alpha}$ (see Assumption 4.7). Roughly speaking, it is shown that if there exists a fixed state x_0 such that $\alpha\mathcal{J}_{\mathcal{D}}^{\alpha}(x_0)$ is bounded in a neighborhood of $\alpha = 0$ and if the relative difference h_α satisfies $-K_h \leq h_\alpha(x) \leq b(x)$ for a nonnegative constant K_h and a measurable function b, then there exists an optimal control. Two examples will be presented in Sects. 6.3 and 6.4, illustrating the possible applications of this first set of assumptions. The second set of assumptions is presented in Sect. 4.4; they are written in terms of some integrodifferential inequalities related to the so-called expected growth condition and geometric convergence of the postjump location kernel associated with the PDMP. More precisely, in Sect. 4.4, the assumptions are based on integrodifferential inequalities related to a positive test function g and $\overline{r}$ (see Assumption 4.13), and on the geometric convergence of the postjump location kernel associated with the PDMP (see Assumption 4.14), so that under these hypotheses, we can show that $\alpha\mathcal{J}_{\mathcal{D}}^{\alpha}(x_0)$ is bounded in a neighborhood of $\alpha = 0$ and that the relative difference of the α-discount value function h_α belongs to a weighted-norm space of functions, denoted by $\mathbb{B}_g(E)$. Notice that an important difference with respect to Sect. 4.3 is that in this case, h_α is not necessarily bounded below by a constant. An example is presented in Sect. 6.5, illustrating the possible applications of this second set of assumptions.

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4.2 Optimality Equation for the Discounted Case

In this section, we consider the discounted optimal control problem posed in Sect. 2.2, Eqs. (2.3), (2.4), and under the assumptions made in Chaps. 2 and 3, we derive an optimality equation for this problem. As is usual in this kind of problem, we characterize first the optimality equation for the problems truncated on the jump time T_m and then take the limit as $m \to \infty$.

Throughout this section, we consider $\alpha > 0$ fixed. For every $g \in \mathbb{M}(E)^+$, we set $\mathcal{W}g$ as the function on E defined as $\mathcal{W}g(x) = \mathcal{R}_\alpha(0, g)(x)$ for $x \in E$. The following proposition is an immediate consequence of the results derived in Sect. 3.4.

Proposition 4.1 *For $g \in \mathbb{M}(E)^+$, consider $w = \mathcal{W}g$, and suppose that for all $x \in E$, $w(x) \in \mathbb{R}$. Under these conditions, $w \in \mathbb{M}(E)^+$ and $\widehat{u}_\phi(w, g) \in \mathcal{S}_\mathbb{V}$. Moreover, w satisfies*

$$w(x) = L_\alpha f(x, \widehat{u}_\phi(w, g)(x)) + H_\alpha r(x, \widehat{u}_\phi(w, g)(x)) + G_\alpha g(x, \widehat{u}_\phi(w, g)(x)). \tag{4.1}$$

Proof Notice that $f \in \mathbb{M}(\overline{E} \times \mathbb{U})^+$ and $r \in \mathbb{M}(\partial E \times \mathbb{U})^+$, and so $\mathcal{W}g \in \mathbb{M}(E)^+$. The proof of this proposition is then a direct consequence of Theorem 3.14. □

Define the sequence of functions $(v_m)_{m\in\mathbb{N}}$ as

$$v_{m+1} = \mathcal{W}v_m, \quad v_0 = 0.$$

We have the following proposition (recall the definition of $\mathcal{D}^\alpha_m(U, x)$ in (2.5)).

Proposition 4.2 *For all $x \in E$ and $m \in \mathbb{N}$, we have that*

$$v_m(x) = \inf_{U\in\mathcal{U}} \mathcal{D}^\alpha_m(U, x).$$

Proof From Proposition 4.1, we have for every $m \in \mathbb{N}_*$, that $v_m \in \mathbb{M}(E)^+$, $\widehat{u}_\phi(v_m, v_{m-1}) \in \mathcal{S}_\mathbb{V}$, and for every $\Gamma = (\gamma, \gamma_\partial) \in \mathcal{S}_\mathbb{V}$, that

$$\begin{aligned} v_m(x) &= L_\alpha f(x, \widehat{u}^m_\phi(x)) + H_\alpha r(x, \widehat{u}^m_\phi(x)) + G_\alpha v_{m-1}(x, \widehat{u}^m_\phi(x)) \\ &\le L_\alpha f(x, \Gamma(x)) + H_\alpha r(x, \Gamma(x)) + G_\alpha v_{m-1}(x, \Gamma(x)), \end{aligned} \tag{4.2}$$

where $\widehat{u}^m_\phi = \widehat{u}_\phi(v_m, v_{m-1})$. Using similar arguments as in the proof of Propositions 3.3 and 3.4, we get that for an arbitrary $U = (u, u_\partial) \in \mathcal{U}$,

$$\mathcal{D}^\alpha_m(U, x) = L_\alpha f(x, \Gamma(x)) + H_\alpha r(x, \Gamma(x)) + G_\alpha \mathcal{D}^\alpha_{m-1}(T(U), .)(x, \Gamma(x)), \tag{4.3}$$

with $\Gamma(x) = (u(0, x, .), u_\partial(0, x))$ and $T(U) = (T(u), T(u_\partial))$, where for $n \in \mathbb{N}_*$, $T(u)(n, ., .) = u(n+1, ., .)$ and $T(u_\partial)(n, .) = u_\partial(n+1, .)$. For $m \in \mathbb{N}$, set $\widehat{U}^m = (u^m, u^m_\partial) \in \mathcal{U}$, where

$$(u^m(k,.,.), u^m_\partial(k,.)) = \begin{cases} \widehat{u}_\phi^{m+1-k}, & \text{if } k \in \{1,\ldots,m\} \\ (\nu(.,.), \nu_\partial(.)), & \text{otherwise,} \end{cases}$$

for an arbitrary $(\nu, \nu_\partial) \in \mathbb{M}(E \times \mathbb{R}_+; \mathbb{U}) \times \mathbb{M}(E; \mathbb{U})$ satisfying $\nu(x,t) \in \mathbb{U}(\phi(x,t))$ and $\nu_\partial(x) \in \mathbb{U}(\phi(x, t_*(x)))$. Notice that

$$T(\widehat{U}^m) = \widehat{U}^{m-1}. \tag{4.4}$$

Let us show by induction on $m \in \mathbb{N}$ that

$$v_m(x) = \inf_{U \in \mathcal{U}} \mathcal{D}_m^\alpha(U, x) = \mathcal{D}_m^\alpha(\widehat{U}^m, x). \tag{4.5}$$

Clearly, (4.5) holds for $m = 0$. Suppose it holds for $m-1$, so that by the induction hypothesis, $\mathcal{D}_{m-1}^\alpha(T(U),.) \geq v_{m-1}(.)$. From (4.3) and (4.2), we get that

$$\begin{aligned}\mathcal{D}_m^\alpha(U,x) &\geq L_\alpha f(x, \Gamma(x)) + H_\alpha r(x, \Gamma(x)(x)) + G_\alpha v_{m-1}(x, \Gamma(x)) \\ &\geq v_m(x). \end{aligned} \tag{4.6}$$

From the induction hypothesis for $m-1$ and (4.4), we have that

$$v_{m-1}(x) = \mathcal{D}_{m-1}^\alpha(\widehat{U}^{m-1}, x) = \mathcal{D}_{m-1}^\alpha(T(\widehat{U}^m), x). \tag{4.7}$$

From (4.3), (4.2), and (4.7), we have that

$$\begin{aligned}\mathcal{D}_m^\alpha(\widehat{U}^m, x) &= L_\alpha f(x, \widehat{u}_\phi^m(x)) + H_\alpha r(x, \widehat{u}_\phi^m(x)) + G_\alpha \mathcal{D}_{m-1}^\alpha(\widehat{U}^{m-1}, .)(x, \widehat{u}_\phi^m(x)) \\ &= L_\alpha f(x, \widehat{u}_\phi^m(x)) + H_\alpha r(x, \widehat{u}_\phi^m(x)) + G_\alpha v_{m-1}(x, \widehat{u}_\phi^m(x)) \\ &= v_m(x). \end{aligned} \tag{4.8}$$

Combining (4.6) and (4.8), we obtain (4.5), completing the proof. □

Notice that $v_m(x) = \inf_{U \in \mathcal{U}} \mathcal{D}_m^\alpha(U,x) \leq \mathcal{J}_\mathcal{D}^\alpha(x)$ and that the functions $v_m \in \mathbb{M}(E)^+$ are nondecreasing. Consequently, there exists $v \in \mathbb{M}(E)^+$ such that $v_m \uparrow v$, and it follows that $v \leq \mathcal{J}_\mathcal{D}^\alpha$. We need the following propositions.

Proposition 4.3 *If $h \in \mathbb{M}(E)^+$ is such that $h(x) \geq \mathcal{W}h(x)$, then $h(x) \geq \mathcal{J}_\mathcal{D}^\alpha(x)$.*

Proof By using Theorem 3.14 with $\rho = 0$, we obtain that

$$h(x) \geq \mathcal{T}_\alpha(0, g)(x) = L_\alpha f(x, \widehat{u}_\phi(x)) + H_\alpha r(x, \widehat{u}_\phi(x)) + G_\alpha h(x, \widehat{u}_\phi(x)),$$

where for notational convenience, $\widehat{u}_\phi$ denotes $\widehat{u}_\phi(\mathcal{W}h, h)$ and $\widehat{u}_\phi \in \mathcal{S}_\mathbb{V}$. Define

$$w(x) = L_\alpha f(x, \widehat{u}_\phi(x)) + H_\alpha r(x, \widehat{u}_\phi(x)) + G_\alpha h(x, \widehat{u}_\phi(x)).$$

Clearly $w(x) \geq 0$. Moreover, $\widehat{U}_{\phi}(\mathcal{W}h, h)$, denoted by $\widehat{U}$, belongs to $\mathcal{U}$ and therefore the hypotheses of Proposition 3.4 are satisfied with $\rho = 0$. Consequently, it follows that for all $m \in \mathbb{N}$, $(t, x) \in \mathbb{R}_+ \times E$,

$$E^{\widehat{U}}_{(x,0)}\Big[\int_0^{t\wedge T_m} e^{-\alpha s}\Big[f\big(X(s), \hat{u}(N(s), Z(s), \tau(s))\big)\Big]\mathrm{d}s + \int_0^{t\wedge T_m} e^{-\alpha s} r\big(X(s-), \hat{u}_\partial(N(s-), X(s-))\big)\mathrm{d}p^*(s)\Big] \leq h(x),$$

or in other words, $E^{\widehat{U}}_{(x,0)}\big[\mathbf{J}^\alpha(\widehat{U}, t \wedge T_m)\big] \leq h(x)$. From Assumption 2.8 (which implies that $T_m \to \infty$, $P^{\widehat{U}}_{(x,0)}$ a.s.) and the monotone convergence theorem, we have, by taking the limit as $m \to \infty$, that

$$E^{\widehat{U}}_{(x,0)}\Big[\int_0^{t} e^{-\alpha s}\Big[f\big(X(s), \hat{u}(N(s), Z(s), \tau(s))\big)\Big]\mathrm{d}s + \int_0^{t} e^{-\alpha s} r\big(X(s-), \hat{u}_\partial(N(s-), X(s-))\big)\mathrm{d}p^*(s)\Big] \leq h(x).$$

Finally, we obtain the result $h(x) \geq \mathcal{J}^\alpha_{\mathcal{D}}(x)$ by taking the limit as $t \to \infty$. □

Proposition 4.4 *The following equality holds:* $v(x) = \mathcal{W}v(x)$.

Proof Let us show first that $v(x) \leq \mathcal{W}v(x)$. By the definition of $\mathcal{W}$, we have for every $\Upsilon \in \mathbb{V}^r(x)$ that

$$v_{m+1}(x) \leq L_\alpha f(x, \Upsilon) + H_\alpha r(x, \Upsilon) + G_\alpha v_m(x, \Upsilon).$$

Recalling that $v_m \uparrow v$, we get by taking the limit as $m \uparrow \infty$ that

$$v(x) \leq L_\alpha f(x, \Upsilon) + H_\alpha r(x, \Upsilon) + \lim_{m\to\infty} G_\alpha v_m(x, \Upsilon).$$

Now by the monotone convergence theorem, it follows that

$$v(x) = L_\alpha f(x, \Upsilon) + H_\alpha r(x, \Upsilon) + G_\alpha v(x, \Upsilon),$$

showing that $v(x) \leq \mathcal{W}v(x)$.

From Proposition 4.1, there exists for every $m \in \mathbb{N}$, $u_\phi^m \in \mathcal{S}_{\mathbb{V}}$ such that

$$\mathcal{W}v_m(x) = L_\alpha f(x, u_\phi^m(x)) + H_\alpha r(x, u_\phi^m(x)) + G_\alpha v_m(x, u_\phi^m(x)). \tag{4.9}$$

Fix $x \in E$. Since $u_\phi^m(x) \in \mathbb{V}(x) \subset \mathbb{V}^r(x)$ and $\mathbb{V}^r(x)$ is compact, we can find a further subsequence, still written as $u_\phi^m(x)$ for notational simplicity, such that $u_\phi^m(x) \to \hat{\Theta} \in \mathbb{V}^r(x)$. From Proposition 3.10,

$$\begin{aligned} v(x) &= \lim_{m\to\infty} v_{m+1}(x) \\ &= \varliminf_{m\to\infty} \left\{ L_\alpha f(x, u_\phi^m(x)) + H_\alpha r(x, u_\phi^m(x)) + G_\alpha v_m(x, u_\phi^m(x)) \right\}, \end{aligned}$$

and so

$$v(x) \geq L_\alpha f(x, \hat{\Theta}) + H_\alpha r(x, \hat{\Theta}) + G_\alpha v(x, \hat{\Theta}) \geq \mathcal{R}_\alpha(0, g)(x) = \mathcal{W}v(x), \tag{4.10}$$

giving the result. □

Finally, we have the following theorem characterizing the (discrete-time) optimality equation for the discounted optimal control problem and showing the convergence of the truncated problems.

Theorem 4.5 *We have that* $v_n \uparrow \mathcal{J}_{\mathcal{D}}^\alpha$ *and* $\mathcal{J}_{\mathcal{D}}^\alpha(x) = \mathcal{W}\mathcal{J}_{\mathcal{D}}^\alpha(x)$.

Proof All we need to show is that $\mathcal{J}_{\mathcal{D}}^\alpha(x) \leq v(x)$. However, this is immediate from Propositions 4.4 and 4.3. □

4.3 The Vanishing Discount Approach: First Case

In general, it is hard to obtain a solution for the discrete-time optimality equation (see 3.1) for the long-run average cost. A common approach is to deal with an optimality inequality of the kind $h \geq \mathcal{T}(\rho, h)$. In the literature, this equation is referred to as the average cost optimality inequality (ACOI). We present sufficient conditions, mainly expressed in terms of the relative difference of the α-discount value functions, for the existence of a solution to this inequality, using the so-called vanishing discount approach (see Theorem 4.10). Combining this result with the connection between the integrodifferential equation and the discrete-time equation, we obtain the existence of an ordinary optimal feedback control for the long-run average cost (see Theorem 4.10). Moreover, a proposition is presented showing the existence of a solution to the discrete-time optimality equation (3.1). In Sect. 4.4, the vanishing discount approach is revisited, but under conditions directly related to the primitive data of the PDMP.

First we have the following result, which traces a parallel with an Abelian theorem (see [49]).

Proposition 4.6 *We have that* $\overline{\lim}_{\alpha\downarrow 0}\, \alpha\mathcal{J}_{\mathcal{D}}^{\alpha}(x) \leq \mathcal{J}_{\mathcal{A}}(x)$.

Proof Let us show that for every $U \in \mathcal{U}$,

$$\overline{\lim_{\alpha\downarrow 0}}\, \alpha\mathcal{D}^{\alpha}(U, x) \leq \mathcal{A}(U, x). \tag{4.11}$$

Clearly, the previous equation is satisfied if $\mathcal{A}(U, x) = +\infty$. Let us assume now that $\mathcal{A}(U, x) < +\infty$. For notational convenience, let us denote $\mathbf{J}(U, t)$ by $\beta(t)$ for $t \in \mathbb{R}_+$. By assumption, we have that there exist $M \in \mathbb{R}_+$ and $T \in \mathbb{R}_+$ such that $0 \leq \beta(t) \leq tM$ for every $t \geq T$. This implies that for every $\alpha > 0, \lim_{t\to\infty} e^{-\alpha t}\beta(t) = 0$. Now by the integration by parts formula, we obtain that

$$\int_0^t e^{-\alpha s}\mathrm{d}\beta(s) = e^{-\alpha t}\beta(t) + \alpha\int_0^t e^{-\alpha s}\beta(s)\mathrm{d}s.$$

Therefore, for every $K \in \mathbb{R}_+$,

$$\begin{aligned}\alpha\int_0^\infty e^{-\alpha s}\mathrm{d}\beta(s) &= \alpha^2\int_0^\infty e^{-\alpha s}\beta(s)\mathrm{d}s \\ &\leq \alpha^2\int_0^K e^{-\alpha s}\beta(s)\mathrm{d}s + \sup_{K\leq s<\infty}\frac{\beta(s)}{s},\end{aligned}$$

and thus $\overline{\lim}_{\alpha\downarrow 0}\, \alpha\int_0^\infty e^{-\alpha s}\mathrm{d}\beta(s) \leq \sup_{K\leq s<\infty}\frac{\beta(s)}{s}$. Consequently, we have that

$$\overline{\lim_{\alpha\downarrow 0}}\, \alpha\int_0^\infty e^{-\alpha s}\mathrm{d}\beta(s) \leq \overline{\lim_{t\to\infty}}\,\frac{\beta(t)}{t} = \mathcal{A}(U, x).$$

Now notice that $\beta(0) = 0$ and β is right continuous, which implies that for every $t_1 < t_2$ in $[0, t]$,

$$\begin{aligned}\int_0^t I_{]t_1,t_2[}(s)\mathrm{d}\beta(s) &= E_{(x,0)}^U\Big[\int_0^t I_{]t_1,t_2[}(s)f\,(X(s), u(N(s), Z(s), \tau(s)))\,\mathrm{d}s\Big] \\ &\quad + E_{(x,0)}^U\Big[\int_0^t I_{]t_1,t_2[}(s)r\,(X(s-), u_\partial(N(s-), Z(s-)))\,\mathrm{d}p^*(s)\Big],\end{aligned}$$

showing that for every $\alpha > 0$,

$$\int_0^t e^{-\alpha s} \mathrm{d}\beta(s) = E^U_{(x,0)}\left[\int_0^t e^{-\alpha s} f\,(X(s), u(N(s), Z(s), \tau(s)))\,\mathrm{d}s\right]$$
$$+ E^U_{(x,0)}\left[\int_0^t e^{-\alpha s} r\,(X(s-), u_\partial(N(s-), Z(s-)))\,\mathrm{d}p^*(s)\right].$$

Therefore it follows that for every $\alpha > 0$,

$$\int_0^\infty e^{-\alpha s} \mathrm{d}\beta(s) = \mathcal{D}^\alpha(U, x),$$

giving (4.11). Since $\mathcal{J}^\alpha_{\mathcal{D}}(x) \leq \mathcal{D}^\alpha(U, x)$, we obtain that for every $U \in \mathcal{U}$, $\overline{\lim}_{\alpha\downarrow 0}\, \alpha \mathcal{J}^\alpha_{\mathcal{D}}(x) \leq \mathcal{A}(U, x)$, and thus $\overline{\lim}_{\alpha\downarrow 0}\, \alpha \mathcal{J}^\alpha_{\mathcal{D}}(x) \leq \mathcal{J}_{\mathcal{A}}(x)$, proving the result. □

We add the following assumptions in addition to those presented in Chaps. 2 and 3, for discounted problems.

Assumption 4.7 *There exist a state $x_0 \in E$, numbers $\beta > 0$, $C \geq 0$, $K_h \geq 0$, and a nonnegative function $\overline{h}(.)$ such that for all $x \in E$ and $\alpha \in (0, \beta]$, $\rho_\alpha \leq C$, where*

$$\rho_\alpha = \alpha \mathcal{J}^\alpha_{\mathcal{D}}(x_0)$$

and

$$-K_h \leq h_\alpha(x) \leq \overline{h}(x),$$

where

$$h_\alpha(x) = \mathcal{J}^\alpha_{\mathcal{D}}(x) - \mathcal{J}^\alpha_{\mathcal{D}}(x_0)$$

is the so-called relative difference of the α-discount value functions.

We have the following propositions.

Proposition 4.8 *There exists a decreasing sequence of positive numbers $\alpha_k \downarrow 0$ such that $\rho_{\alpha_k} \to \rho$ and for all $x \in E$, $\lim_{k\to\infty} \alpha_k \mathcal{J}^{\alpha_k}_{\mathcal{D}}(x) = \rho$.*

Proof See the lemma in [49, p. xx]. □

Proposition 4.9 *Set $h = \underline{\lim}_{k\to\infty} h_{\alpha_k}$. Then for all $x \in E$, $h(x) \geq -K_h$ and $h(x) \geq \mathcal{T}(\rho, h)(x)$.*

Proof From Proposition 4.1 and Theorem 4.5, we have that the following equation is satisfied for each $\alpha > 0$ and $x \in E$:

$$\mathcal{J}^\alpha_{\mathcal{D}}(x) = L_\alpha f(x, u^\alpha_\phi(x)) + H_\alpha r(x, u^\alpha_\phi(x)) + G_\alpha \mathcal{J}^\alpha_{\mathcal{D}}(x, u^\alpha_\phi(x))$$

for $u_\phi^\alpha \in \mathcal{S}_{\mathbb{V}}$. Moreover, notice that for every constant γ, we have

$$G_\alpha \gamma(x, u_\phi^\alpha(x)) = \gamma[1 - \alpha \mathcal{L}_\alpha(x, u_\phi^\alpha(x))].$$

Therefore, choosing $\gamma = \mathcal{J}_{\mathcal{D}}^\alpha(x_0)$, we get

$$\begin{aligned} h_\alpha(x) = &- \rho_\alpha \mathcal{L}_\alpha(x, u_\phi^\alpha(x)) + L_\alpha f(x, u_\phi^\alpha(x)) + H_\alpha r(x, u_\phi^\alpha(x)) \\ &+ G_\alpha h_\alpha(x, u_\phi^\alpha(x)). \end{aligned} \tag{4.12}$$

For $x \in E$ fixed and for all $k \in \mathbb{N}$, $u_\phi^{\alpha_k}(x) \in \mathbb{V}(x) \subset \mathbb{V}^r(x)$, since $\mathbb{V}^r(x)$ is compact, we can find a further subsequence, still written as $u_\phi^{\alpha_k}(x)$ for notational simplicity, such that $u_\phi^{\alpha_k}(x) \to \hat{\Theta} \in \mathbb{V}^r(x)$. Combining Proposition 3.10 and Eq. (4.12) yields

$$\begin{aligned} h(x) &= \varliminf_{k\to\infty} h_{\alpha_k}(x) \\ &= \varliminf_{k\to\infty} \Big\{ -\rho_{\alpha_k} \mathcal{L}_{\alpha_k}(x, u_\phi^{\alpha_k}(x)) + L_{\alpha_k} f(x, u_\phi^{\alpha_k}(x)) + H_{\alpha_k} r(x, u_\phi^{\alpha_k}(x)) \\ &\quad + G_{\alpha_k} h_{\alpha_k}(x, u_\phi^{\alpha_k}(x)) \Big\} \\ &\geq -\rho \mathcal{L}(x, \hat{\Theta}) + Lf(x, \hat{\Theta}) + Hr(x, \hat{\Theta}) + Gh(x, \hat{\Theta}). \end{aligned} \tag{4.13}$$

Therefore, from Theorem 3.14, we have that

$$h(x) \geq \mathcal{R}(\rho, h)(x) = \mathcal{T}(\rho, h)(x),$$

establishing the result. □

Our next result establishes the existence of an optimal feedback control strategy for the long-run average cost problem. Let h and ρ be as in Propositions 4.8 and 4.9, and $w = \mathcal{T}(\rho, h)$.

Theorem 4.10 *The feedback control strategy $\widehat{U}_\phi(w, h) \in \mathcal{U}$ as defined in* (D3) *of Definition 3.12 is such that*

$$\rho = \mathcal{J}_{\mathcal{A}}(x) = \mathcal{A}(\widehat{U}_\phi(w, h), x).$$

Proof From Proposition 4.9, it follows that h is bounded below by $-K_h$. Therefore, applying Theorem 3.14, we obtain that the hypotheses of Proposition 3.4 are satisfied for $\alpha = 0$. Setting for simplicity $\widehat{U} = \widehat{U}_\phi(w, h)$, we obtain that

$$E_{(x,0)}^{\widehat{U}} \left[\mathbf{J}(\widehat{U}, t \wedge T_m) \right] + E_{(x,0)}^{\widehat{U}} \left[w\left(X(t \wedge T_m)\right) \right] \leq \rho E_{(x,0)}^{\widehat{U}} \left[t \wedge T_m \right] + w(x).$$

Combining Proposition 4.9 and item (b) of Assumption 2.5, we obtain that $w(x) \geq -\rho K_\lambda - K_h$. Moreover, we have from Assumption 2.8 that $T_m \to \infty$, $P_{(x,0)}^{\widehat{U}}$ a.s.

Consequently,

$$E_{(x,0)}^{\widehat{U}}\left[\mathbf{J}(\widehat{U},t)\right] \leq \rho t + \rho K_\lambda + K_h + w(x),$$

showing that $\rho \geq \mathcal{A}(\widehat{U}, x)$. From Propositions 4.6 and 4.8, we have $\rho \leq \mathcal{J}_{\mathcal{A}}(x)$, completing the proof. □

Under the following hypothesis, the next proposition shows the existence of a solution to the discrete-time optimality equation (3.1).

Assumption 4.11 *The function $\overline{h}$ introduced in Assumption 4.7 is measurable and satisfies $\int_0^{t_*(x)} e^{-\Lambda^\nu(x,s)} \lambda(\phi(x,s), \nu(s)) Q\overline{h}(\phi(x,s), \nu(s)) \mathrm{d}s < \infty$ for all $x \in E$ and $\nu \in \mathcal{V}(x)$, and the sequence $(h_{\alpha_k})_{k\in\mathbb{N}}$ is equicontinuous.*

Proposition 4.12 *There exists a pair $(\rho, h) \in \mathbb{R}_+ \times \mathbb{M}(E)$ such that for all $x \in E$, $h(x) \geq -K_h$ and $h(x) = \mathcal{T}(\rho, h)(x)$.*

Proof From Assumptions 4.7 and 4.11, it follows that there exist a function h and a subsequence of $(\alpha_k)_{k\in\mathbb{N}}$, still denoted by $(\alpha_k)_{k\in\mathbb{N}}$, such that for all $x \in E$, $\lim_{k\to\infty} h_{\alpha_k}(x) = h(x)$, $\lim_{k\to\infty} \rho_{\alpha_k} = \rho$, and $\lim_{k\to\infty} \alpha_k \mathcal{J}_{\mathcal{D}}^{\alpha_k}(x) = \rho$. Consequently, from Proposition 4.9, we have $h(x) \geq \mathcal{T}(\rho, h)(x)$ for all $x \in E$.

Let us prove the reverse inequality. For all $k \in \mathbb{N}$, $x \in E$, $\Upsilon = (\nu, \nu_\partial) \in \mathbb{V}(x)$,

$$\begin{aligned} h_{\alpha_k}(x) &= \mathcal{T}_{\alpha_k}(\rho_{\alpha_k}, h_{\alpha_k})(x) \\ &\leq -\rho_{\alpha_k}\mathcal{L}_{\alpha_k}(x,\Upsilon) + L_{\alpha_k} f(x,\Upsilon) + H_{\alpha_k} r(x,\Upsilon) + G_{\alpha_k} h_{\alpha_k}(x,\Upsilon). \end{aligned}$$

From item (a) of Proposition 3.10, $\lim_{k\to\infty} \mathcal{L}_{\alpha_k}(x,\Upsilon) = \mathcal{L}(x,\Upsilon)$. Moreover, we clearly have that $\lim_{k\to\infty} H_{\alpha_k} r(x,\Upsilon) = Hr(x,\Upsilon)$. By the monotone convergence theorem, $\lim_{k\to\infty} L_{\alpha_k} f(x,\Upsilon) = Lf(x,\Upsilon)$. Using Assumption 4.11 and the dominated convergence theorem, it is easy to show that $\lim_{k\to\infty} G_{\alpha_k} h_{\alpha_k}(x,\Upsilon) = Gh(x,\Upsilon)$. Consequently, we obtain that for all $x \in E$, $\Upsilon = (\nu, \nu_\partial) \in \mathbb{V}(x)$,

$$h(x) \leq -\rho\mathcal{L}(x,\Upsilon) + Lf(x,\Upsilon) + Hr(x,\Upsilon) + Gh(x,\Upsilon),$$

showing that $h(x) \leq \mathcal{T}(\rho, h)(x)$ and giving the result. □

4.4 The Vanishing Discount Approach: Second Case

As previously mentioned, we want to obtain sufficient conditions for the existence of an optimal control for the long-run average control problem of a PDMP posed in Sect. 2.2. In this section, this is done by assuming hypotheses directly on the parameters of the PDMP, instead of, as in Assumption 4.7, considering assumptions based on the α-discount value functions $\mathcal{J}_{\mathcal{D}}^{\alpha}$. The purpose of the next subsection is to introduce these several assumptions. The basic idea is that they will yield to tractable conditions that may be easier to check in practice.

4.4.1 Assumptions on the Parameters of the PDMP

The following assumption is somehow related to the so-called expected growth condition (see, for instance, Assumption 3.1 in [45] for the discrete-time case, or Assumption A in [44] for the continuous-time case). It will be used, among other things, to guarantee the uniform boundedness of $\alpha\mathcal{J}_D^\alpha(x)$ with respect to α (see Theorem 4.20).

Assumption 4.13 *There exist $b \geq 0$, $c > 0$, $\delta > 0$, $M \geq 0$ and $g \in \mathbb{M}^{ac}(E)$, $g \geq 1$, and $\overline{r} \in \mathbb{M}(\partial E)^+$ satisfying for all $x \in E$,*

$$\sup_{a\in\mathbb{U}(x)} \Big\{\mathcal{X}g(x) + cg(x) - \lambda(x,a)\,[g(x) - Qg(x,a)]\Big\} \leq b, \tag{4.14}$$

$$\sup_{a\in\mathbb{U}(x)} \Big\{f(x,a)\Big\} \leq Mg(x), \tag{4.15}$$

and for all $x \in E$ with $t_*(x) < \infty$,

$$\sup_{a\in\mathbb{U}(\phi(x,t_*(x)))} \{\overline{r}(\phi(x,t_*(x))) + Qg(\phi(x,t_*(x)),a)\} \leq g(\phi(x,t_*(x))), \tag{4.16}$$

$$\sup_{a\in\mathbb{U}(\phi(x,t_*(x)))} \Big\{r(\phi(x,t_*(x)),a)\Big\} \leq \frac{M}{c+\delta}\overline{r}(\phi(x,t_*(x))). \tag{4.17}$$

In the next assumption, notice that for every $u \in \mathcal{S}_{\mathbb{U}}$, $G(.,u_\phi;.)$ can be seen as the stochastic kernel associated with the postjump location of a PDMP. This assumption is related to geometric ergodic properties of the operator G (see, for example, the comments on p. xxx in [51] or Lemma 3.3 in [45] for more details on this kind of assumption). This assumption is very important, because it will be used in particular to ensure that the relative difference of the α-discount value functions h_α, defined by $h_\alpha(x) = \mathcal{J}_{\mathcal{D}}^\alpha(x) - \mathcal{J}_{\mathcal{D}}^\alpha(x_0)$, belong to the weighted-norm space of functions $\mathbb{B}_g(E)$ (see Theorem 4.21).

Assumption 4.14 *For each $u \in \mathcal{S}_{\mathbb{U}}$, there exists a probability measure ν_u such that $\nu_u(g) < +\infty$ and*

$$|G^k h(x,u_\phi) - \nu_u(h)| \leq a\|h\|_g \kappa^k g(x), \tag{4.18}$$

for all $h \in \mathbb{B}_g(E)$ and $k \in \mathbb{N}$, with $a > 0$ and $0 < \kappa < 1$ independent of u.

Throughout this section, we consider again all the assumptions of Chap. 2 and Assumptions 3.5, 3.6, 3.7, 3.8, and 3.9 introduced in Sect. 3.3. As seen in Sect. 3.3, they are needed to guarantee some convergence and semicontinuity properties of the one-stage optimization operators, the equality between the operators $\mathcal{T}_\alpha$ and $\mathcal{R}_\alpha$, and the existence of an ordinary feedback measurable selector (see Sect. 4.5.3 and in particular Theorem 4.35). Furthermore, we need the following assumptions:

Assumption 4.15 *For the function g as in Assumption 4.13,*

(i) the restriction of $Qg(x, .)$ *to* $\mathbb{U}(x)$ *is continuous for every* $x \in \overline{E}$;
(ii) there exists $\overline{g} \in \mathbb{M}(\overline{E})^+$ *such that* $\lambda Qg(y, a) \leq \overline{g}(y)$ *for every* $y \in E$, $a \in \mathbb{U}(y)$ *and* $\int_0^{t_*(x)} e^{-\int_0^t \underline{\lambda}(\phi(x,s))\mathrm{d}s}\overline{g}(\phi(x, t))\mathrm{d}t < \infty$, *for all* $x \in E$.

The next result extends Proposition 3.10 to the case of functions that are not necessarily bounded below but belong to $\mathbb{B}_g(E)$.

Proposition 4.16 *Consider* $\alpha \in \mathbb{R}_+$ *and a nonincreasing sequence of nonnegative numbers* $(\alpha_k)_{k\in\mathbb{N}}$, $\alpha_k \downarrow \alpha$, *and a sequence of functions* $(h_k)_{k\in\mathbb{N}} \in \mathbb{B}_g(E)$ *such that* $\|h_k\|_g \leq K_h$ *for some positive constant* K_h. *Set* $h = \underline{\lim}_{k\to\infty} h_k$. *For* $x \in E$, *consider* $\Theta_n = \big(\mu_n, \mu_{\partial,n}\big) \in \mathbb{V}^r(x)$ *and* $\Theta = \big(\mu, \mu_\partial\big) \in \mathbb{V}^r(x)$ *such that* $\Theta_n \to \Theta$. *We have the following results:*

(i) $\lim_{n\to\infty} G_{\alpha_n} g(x, \Theta_n) = G_\alpha g(x, \Theta)$;
(ii) $\underline{\lim}_{n\to\infty} G_{\alpha_n} h_n(x, \Theta_n) \geq G_\alpha h(x, \Theta)$.

Proof Item (a) We have for $n \in \mathbb{N}$, $x \in E$,

$$\begin{aligned} G_{\alpha_n} g(x, \Theta_n) = &\int_0^{t_*(x)} \big[e^{-\alpha_n t - \Lambda^{\mu_n}(x,t)} - e^{-\alpha t - \Lambda^{\mu}(x,t)}\big]\lambda Qg(\phi(x, t), \mu_n(t))\mathrm{d}t \\ &+ \int_0^{t_*(x)} e^{-\alpha t - \Lambda^{\mu}(x,t)}\lambda Qg(\phi(x, t), \mu_n(s))\mathrm{d}t \\ &+ H_{\alpha_n} Qg(x, \Theta_n). \end{aligned} \tag{4.19}$$

From Assumption 2.5 and Assumption 4.15 (i), we have that

$$\begin{aligned} |e^{-\alpha_n t - \Lambda^{\mu_n}(x,t)} - e^{-\alpha t - \Lambda^{\mu}(x,t)}|\lambda Qg(\phi(x, t), \mu_n(t)) \\ \leq 2e^{-\int_0^t \underline{\lambda}(\phi(x,s))ds}\overline{g}(\phi(x, t)). \end{aligned}$$

Using the same arguments as in the proof of item (a) in Proposition 3.10 and the dominated convergence theorem, we obtain that

$$\lim_{n\to\infty} \int_0^{t_*(x)} \big[e^{-\alpha_n t - \Lambda^{\mu_n}(x,t)} - e^{-\alpha t - \Lambda^{\mu}(x,t)}\big]\lambda Qg(\phi(x, t), \mu_n(t))\mathrm{d}t = 0. \tag{4.20}$$

Now by the fact that $\sup_{a\in\mathbb{U}(x)} \lambda Qg(x, a) \leq \overline{g}(x)$ (see Assumption 4.15), we can proceed as in item (a) of Proposition 3.10 to show that

$$\lim_{n\to\infty}\int_0^{t_*(x)} e^{-\alpha t-\Lambda^{\mu}(x,t)}\lambda Qg(\phi(x,t),\mu_n(s))\mathrm{d}t$$
$$=\int_0^{t_*(x)} e^{-\alpha t-\Lambda^{\mu}(x,t)}\lambda Qg(\phi(x,t),\mu(s))\mathrm{d}t. \tag{4.21}$$

Now by item (i) of Assumption 4.15, it follows easily that

$$\lim_{n\to\infty} H_{\alpha_n}Qg(x,\Theta_n)=H_{\alpha}Qg(x,\Theta). \tag{4.22}$$

Therefore, combining Eqs. (4.19)–(4.22), it follows that

$$\lim_{n\to\infty} G_{\alpha_n}g(x,\Theta_n)=G_{\alpha}g(x,\Theta),$$

proving item (a).

Item (b) Set $\widetilde{h}_n=h_n+K_h g$, $\widetilde{h}=h+K_h g$. We can apply item (d) of Proposition 3.10 to get

$$\underline{\lim}_{n\to\infty} G_{\alpha_n}\widetilde{h}_n(x,\Theta_n)\geq G_{\alpha}\widetilde{h}(x,\Theta).$$

Combining the previous equation with item (a), we get the result. □

The objective of the next remark is to extend the definition of $\widehat{u}(w,h)$ in item (D1) of Definition 3.12 the case in which the function is not only bounded below but belongs to $\mathbb{B}_g(E)$. This generalization will be important in characterizing the optimal feedback measurable selector and optimal control strategy for our problem.

Remark 4.17 Consider $w\in\mathbb{M}(E)$ and $h\in\mathbb{B}_g(E)$. We define:

(D1) $\widehat{u}(w,h)\in\mathcal{S}_{\mathbb{U}}$ as the measurable selector satisfying

$$\inf_{a\in\mathbb{U}(x)}\Big\{f(x,a)-\lambda(x,a)\Big[w(x)-Qh(x,a)\Big]\Big\}$$
$$=f(x,\widehat{u}(w,h)(x))-\lambda(x,\widehat{u}(w,h)(x))\Big[w(x)-Qh(x,\widehat{u}(w,h)(x))\Big],$$

$$\inf_{a\in\mathbb{U}(z)}\{r(z,a)+Qh(z,a)\}=r(z,\widehat{u}(w,h)(z))+Qh(z,\widehat{u}(w,h)(z));$$

(D2) $\widehat{u}_{\phi}(w,h)\in\mathcal{S}_{\mathbb{V}}$ as the measurable selector derived from $\widehat{u}(w,h)$ through the Definition 2.22.

Notice that from Assumptions 3.8 and 4.15 and Lemma 8.3.7 in [51], we have that $Qh(x,.)$ is continuous in $\mathbb{U}(x)$. From this and Assumptions 3.5, 3.6, and 3.7, we get that the existence of $\widehat{u}(w,h)$ in (D1) follows from Proposition D.5 in [49].

The final assumption is the following.

Assumption 4.18 *Consider $\underline{\lambda}$, as in Assumption 2.5. Then*

(i) $\lim_{t\to+\infty} e^{ct-\int_0^t \underline{\lambda}(\phi(x,s))\mathrm{d}s} = 0$*, for all* $x \in E$ *with* $t_*(x) = +\infty$*;*

(ii) $\lim_{t\to+\infty} e^{-\int_0^t \underline{\lambda}(\phi(x,s))\mathrm{d}s} g(\phi(x,t)) = 0$*, for all* $x \in E$ *with* $t_*(x) = \infty$*;*

(iii) $\int_0^{t_*(x)} e^{ct-\int_0^t \underline{\lambda}(\phi(x,s))\mathrm{d}s}\mathrm{d}t \leq K_\lambda$*, for all* $x \in E$.

Remark 4.19 In this section, the definitions of the operators G_α, L_α, and H_α are extended to $\alpha \geq -c$, where c was introduced in Assumption 4.13. Notice the following consequences of Assumptions 4.18:

(i) Assumption 4.18 (i) implies that

$$G_\alpha(x, \Theta; A) = \int_0^{t_*(x)} e^{-\alpha s - \Lambda^\mu(x,s)} \lambda Q I_A(\phi(x,s), \mu(s))\mathrm{d}s,$$

and $H_\alpha w(x, \Theta) = 0$, for every $x \in E$ with $t_*(x) = +\infty$, $A \in \mathcal{B}(E)$, $\alpha \geq -c$, $\Theta = (\mu, \mu_\partial) \in \mathbb{V}^r(x)$, $w \in \mathbb{M}(\partial E \times \mathbb{U})$.

(ii) Assumption 4.18 (iii) implies that $\mathcal{L}_\alpha(x, \Theta) \leq K_\lambda$ for every $\alpha \geq -c$, $x \in E$, $\Theta \in \mathbb{V}^r(x)$.

4.4.2 Main Results

In this subsection we present the main results related to the vanishing discount approach, supposing that the assumptions on the parameters of the PDMP presented in Sect. 4.4.1 are satisfied. Our first main result, Theorem 4.20, consists in showing that there exists a fixed state x_0 such that $\alpha \mathcal{J}_{\mathcal{D}}^\alpha(x_0)$ is bounded in a neighborhood of $\alpha = 0$. Notice that this property was considered to be an assumption in Sect. 4.3 (Assumption 4.7), while in the present section, this is a consequence of the assumptions made on the primitive data of the PDMP. Our second main result, Theorem 4.21, states that the relative difference $h_\alpha(x) = \mathcal{J}_{\mathcal{D}}^\alpha(x) - \mathcal{J}_{\mathcal{D}}^\alpha(x_0)$ of the α-discount value function $\mathcal{J}_{\mathcal{D}}^\alpha$ belongs to $\mathbb{B}_g(E)$. This is a major difference with respect to the results developed in Sect. 4.3, where we assumed in Assumption 4.7 the stronger hypothesis that $h_\alpha(x)$ is bounded below by a constant. Next, as a consequence of Theorems 4.20 and 4.21, it is shown in Proposition 4.22 that there exists a pair (ρ, h) satisfying the ACOI $h \geq \mathcal{T}(\rho, h)$, where $\rho \in \mathbb{R}_+$ and $h \in \mathbb{B}_g(E)$. Here we have again some important differences with respect to the results obtained in Sect. 4.3, since the hypothesis imposed in Assumption 4.7, that $h_\alpha(x)$ is bounded below by a constant, implied that a solution for the ACOI was also bounded below by a constant. From this, one could easily prove the existence of an optimal control for the average cost control problem of PDMP. Under the hypotheses of this section, we have only that $h(x)$ is bounded below by $-Cg(x)$ for some $C > 0$, and consequently, the approach developed previously in Sect. 4.3 cannot be used. The idea to overcome this difficulty is to show in Proposition 4.24 that under the assumptions presented in Sect. 4.4.1, one can get that for $\widehat{u} \in \mathcal{S}_{\mathbb{U}}$,

$$\overline{\lim_{t\to+\infty}} \frac{1}{t} \overline{\lim_{m\to\infty}} -E^{U_{\widehat{u}_\phi}}_{(x,0)}\Big[\mathcal{T}(\rho, h)\big(X(t \wedge T_m)\big)\Big] \leq 0.$$

This technical result will lead in Theorem 4.25 to the main result of this section, which is the existence of an optimal ordinary feedback control strategy for the long-run average cost problem of a PDMP.

In order to prove the first two main theorems, several intermediate and technical results are required. For the sake of clarity in the exposition, the proofs of these intermediate results and of these two main theorems are presented in Sect. 4.5. The following theorem states that for every discount factor α and for every state $x \in E$, $\alpha\mathcal{J}^{\alpha}_{D}(\cdot)$ is bounded.

Theorem 4.20 *For every $\alpha > 0$ and $x \in E$,*

$$\mathcal{J}^{\alpha}_{D}(x) \leq \frac{M}{c+\alpha} g(x) + \frac{Mb}{c\alpha}. \tag{4.23}$$

Proof The proof of this result can be found in Sect. 4.5.1. □

Now it is shown that for every state y fixed in E, the difference $\mathcal{J}^{\alpha}_{D}(\cdot)$-$\mathcal{J}^{\alpha}_{D}(y)$ belongs to $\mathbb{B}_g(E)$.

Theorem 4.21 *For every $\alpha > 0$, $x \in E$, and $y \in E$, there exists M' such that*

$$|\mathcal{J}^{\alpha}_{D}(x) - \mathcal{J}^{\alpha}_{D}(y)| \leq \frac{aM'}{1-\kappa}(1 + g(y))g(x). \tag{4.24}$$

Proof The proof of this result can be found in Sect. 4.5.2. □

The purpose of the next result is to show that by combining the two previous theorems, there exists a pair (ρ, h) in $\mathbb{R}_+ \times \mathbb{B}_g(E)$ satisfying the ACOI $h \geq \mathcal{T}(\rho, h)$. A crucial intermediate result is Theorem 4.35, presented in Sect. 4.5.3 for the sake of clarity in the exposition, which states that for every function $h \in \mathbb{B}_g(E)$, the one-stage optimization functions $\mathcal{R}_\alpha(\rho, h)(x)$ and $\mathcal{T}_\alpha(\rho, h)(x)$ are equal, and that there exists an ordinary feedback measurable selector for the one-stage optimization problems associated with these operators. This theorem can be seen as an extension of the results obtained in Sect. 3.4, Theorem 3.14, to the case in which the functions under consideration are not necessarily bounded below, as was supposed in Sect. 3.4, but instead belong to $\mathbb{B}_g(E)$.

Proposition 4.22 *Set $\rho_\alpha = \alpha\mathcal{J}^{\alpha}_{\mathcal{D}}(x_0)$ and $h_\alpha(\cdot) = \mathcal{J}^{\alpha}_{\mathcal{D}}(\cdot) - \mathcal{J}^{\alpha}_{\mathcal{D}}(x_0)$ for a fixed state $x_0 \in E$. Then,*

(i) *there exists a decreasing sequence of positive numbers* $\alpha_k \downarrow 0$ *such that* $\rho_{\alpha_k} \to \rho$ *and for all* $x \in E$, $\lim_{k\to\infty} \alpha_k \mathcal{J}_{\mathcal{D}}^{\alpha_k}(x) = \rho$;

(ii) h *defined by* $h = \underline{\lim}_{k\to\infty} h_{\alpha_k}$ *belongs to* $\mathbb{B}_g(E)$ *and satisfies for every* $x \in E$,

$$\begin{aligned} h(x) &\geq \mathcal{T}(\rho, h)(x) \\ &= -\rho\mathcal{L}(x, \widehat{u}_\phi(x)) + Lf(x, \widehat{u}_\phi(x)) + Hr(x, \widehat{u}_\phi(x)) + Gh(x, \widehat{u}_\phi(x)), \end{aligned}$$

where $\widehat{u} = \widehat{u}(\mathcal{T}(\rho, h), h)$ *and* $\widehat{u}_\phi = \widehat{u}_\phi(\mathcal{T}(\rho, h), h)$ *(see Remark 4.17).*

Proof Item (i) is a straightforward consequence of Theorem 4.20.
From Theorem 4.21, one now obtains that there exists a constant C independent of k and x such that $|h_{\alpha_k}(x)| \leq Cg(x)$. This implies that $h = \underline{\lim}_{k\to\infty} h_{\alpha_k}$ belongs to $\mathbb{B}_g(E)$. By using Theorem 4.35 and Proposition 4.16, it can be shown following the same arguments as in the proof of Proposition 4.9 that $h(x) \geq \mathcal{R}(\rho, h)(x)$. Applying Theorem 4.35, it follows that

$$\begin{aligned} h(x) &\geq \mathcal{R}(\rho, h)(x) = \mathcal{T}(\rho, h)(x) \\ &\geq -\rho\mathcal{L}(x, \widehat{u}_\phi(x)) + Lf(x, \widehat{u}_\phi(x)) + Hr(x, \widehat{u}_\phi(x)) + Gh(x, \widehat{u}_\phi(x)). \end{aligned}$$

where $\widehat{u}$ is defined by $\widehat{u}(\mathcal{T}(\rho, h), h)$. This gives item (ii), yielding the result. □

In what follows, recall the definition of $U_{\widehat{u}_\phi}$ in Definition 2.23. Next we need to derive a technical result, which is

$$\overline{\lim_{t\to+\infty}} \frac{1}{t} \overline{\lim_{m\to\infty}} -E_{(x,0)}^{U_{\widehat{u}_\phi}}\Big[\mathcal{T}(\rho, h)\big(X(t \wedge T_m)\big)\Big] \leq 0,$$

in order to get the existence of an optimal control for the PDMP (this would be easily obtained if h were bounded from below). Proposition 4.24 provides this result, but first we need to prove the following lemma:

Lemma 4.23 *Consider an arbitrary* $u \in \mathcal{S}_{\mathbb{U}}$ *and let* u_ϕ *and* U_{u_ϕ} *be as in Definitions 2.22 and 2.23 respectively. For all* $x \in E$, *define* $\widehat{g}(x) = -b\mathcal{L}_{-c}(x, u_\phi(x)) + G_{-c}g(x, u_\phi(x))$. *Then* $\widehat{g} \in \mathbb{B}_g(E)$, *and* U_{u_ϕ} *satisfies*

$$\begin{aligned} E_{(x,0)}^{U_{u_\phi}}\Big[\widehat{g}\big(X(t \wedge T_m)\big)\Big] &\leq e^{-ct}g(x) + \frac{b}{c}\big[1 - e^{-ct}\big] + a\|\widehat{g}\|_g g(x)\kappa^m \\ &\quad + \|\widehat{g}\|_g \nu_u(g) + bK_\lambda. \end{aligned} \tag{4.25}$$

Proof From Corollary 4.28 with $\alpha = -c$ and the fact that $\overline{r}(z) \geq 0$, we obtain that $-b\mathcal{L}_{-c}(x, u_\phi(x)) + G_{-c}g(x, u_\phi(x)) \leq g(x)$. Clearly, $\widehat{g} \in \mathbb{M}(E)$ is bounded from below by $-bK_\lambda$ from Assumption 4.18 (iii) (see Remark 4.19), and thus $\widehat{g} \in \mathbb{B}_g(E)$. Since $\widehat{g} \in \mathbb{M}(E)$ is bounded from below, it is easy to show that

$$-bE^{U_{u_\phi}}_{(x,0)}\Big[\int_0^{t\wedge T_m} e^{cs}\mathrm{d}s\Big] + E^{U_{u_\phi}}_{(x,0)}\Big[e^{c(t\wedge T_m)}\widehat{g}\big(X(t\wedge T_m)\big)\Big] \leq g(x),$$

using the same arguments as in the proof of Proposition 3.4. Combining the monotone convergence theorem and Assumption 2.8, we obtain that

$$E^{U_{u_\phi}}_{(x,0)}\Big[\widehat{g}\big(X(t)\big)\Big] \leq e^{-ct}g(x) + \frac{b}{c}\big[1-e^{-ct}\big]. \tag{4.26}$$

Clearly, we have

$$E^{U_{u_\phi}}_{(x,0)}\Big[\widehat{g}\big(X(t\wedge T_m)\big)\Big] = E^{U_{u_\phi}}_{(x,0)}\Big[I_{\{t<T_m\}}\widehat{g}\big(X(t)\big)\Big] + E^{U_{u_\phi}}_{(x,0)}\Big[I_{\{t\geq T_m\}}\widehat{g}\big(X(T_m)\big)\Big].$$

Consequently, we get

$$E^{U_{u_\phi}}_{(x,0)}\Big[\widehat{g}\big(X(t\wedge T_m)\big)\Big] \leq E^{U_{u_\phi}}_{(x,0)}\Big[\widehat{g}\big(X(t)\big)\Big] + G^m\widehat{g}(x,u_\phi(x)) + bK_\lambda$$

by recalling that $\widehat{g}$ is bounded from below by $-bK_\lambda$. The result follows by Assumption 4.14 and Eq. (4.26). □

As a consequence of Lemma 4.23, we get the following result:

Proposition 4.24 *For all $x \in E$, $E^{U_{\widehat{u}_\phi}}_{(x,0)}\Big[\mathcal{T}(\rho,h)\big(X(t\wedge T_m)\big)\Big]$ is well defined and satisfies*

$$\overline{\lim_{t\to+\infty}}\,\frac{1}{t}\,\overline{\lim_{m\to\infty}}\,-E^{U_{\widehat{u}_\phi}}_{(x,0)}\Big[\mathcal{T}(\rho,h)\big(X(t\wedge T_m)\big)\Big] \leq 0. \tag{4.27}$$

Proof By definition, we have that $\mathcal{T}(\rho,h)(x) \geq -\rho\mathcal{L}(x,\widehat{u}_\phi(x)) + Gh(x,\widehat{u}_\phi(x))$. Therefore, using the definition of $\widehat{g}$ in Lemma 4.23 with $u=\widehat{u}$, we obtain that

$$-\mathcal{T}(\rho,h)(x) \leq (\rho + b\|h\|_g)K_\lambda + \|h\|_g\widehat{g}(x). \tag{4.28}$$

Consequently, combining Lemma 4.23 and Eq. (4.28), we obtain the result. □

The next theorem, which is the main result of this section, shows that the feedback control strategy $U_{\widehat{u}_\phi}$ is optimal for the long-run average cost problem of a PDMP.

Theorem 4.25 *For all $x \in E$,*

$$\rho = \mathcal{J}_{\mathcal{A}}(x) = \mathcal{A}(U_{\widehat{u}_\phi}, x).$$

Proof Define

$$J_m^{U_{\widehat{u}_\phi}}(t,x) = E_{(x,0)}^{U_{\widehat{u}_\phi}}\Big[\int_0^{t\wedge T_m}\Big[f\big(X(s),\hat{u}(X(s))\big)-\rho\Big]\mathrm{d}s$$
$$+\int_0^{t\wedge T_m} r\big(X(s-),\hat{u}_\partial(X(s-))\big)\mathrm{d}p^*(s)+\mathcal{T}(\rho,h)\big(X(t\wedge T_m)\big)\Big].$$

From Proposition 4.24, we have that $E_{(x,0)}^{U_{\widehat{u}_\phi}}\Big[\mathcal{T}(\rho,h)\big(X(t\wedge T_m)\big)\Big]$ is well defined. Consequently, by Proposition 4.22, we can show that $J_m^{U_{\widehat{u}_\phi}}(t,x)\leq h(x)$ for all $m\in\mathbb{N}$, $(t,x)\in\mathbb{R}_+\times E$. Therefore,

$$E_{(x,0)}^{U_{\widehat{u}_\phi}}\Big[\int_0^{t\wedge T_m}\Big[f\big(X(s),\hat{u}(X(s))\big)\Big]\mathrm{d}s+\int_0^{t\wedge T_m} r\big(X(s-),\hat{u}_\partial(X(s-))\big)\mathrm{d}p^*(s)\Big]$$
$$\leq \rho E_{(x,0)}^{U_{\widehat{u}_\phi}}\Big[t\wedge T_m\Big]+h(x)-E_{(x,0)}^{U_{\widehat{u}_\phi}}\Big[\mathcal{T}(\rho,h)\big(X(t\wedge T_m)\big)\Big],$$

and so

$$\overline{\lim_{t\to+\infty}}\,\frac{1}{t}\,\overline{\lim_{m\to\infty}}\,E_{(x,0)}^{U_{\widehat{u}_\phi}}\Big[\int_0^{t\wedge T_m}\Big[f\big(X(s),\hat{u}(X(s))\big)\Big]\mathrm{d}s$$
$$+\int_0^{t\wedge T_m} r\big(X(s-),\hat{u}_\partial(X(s-))\big)\mathrm{d}p^*(s)\Big]$$
$$\leq \rho+\overline{\lim_{t\to+\infty}}\,\frac{1}{t}\,\overline{\lim_{m\to\infty}}\,-E_{(x,0)}^{U_{\widehat{u}_\phi}}\Big[\mathcal{T}(\rho,h)\big(X(t\wedge T_m)\big)\Big].$$

From the monotone convergence theorem and Proposition 4.24, it follows that

$$\overline{\lim_{t\to\infty}}\,\frac{1}{t}E_{(x,0)}^{U_{\widehat{u}_\phi}}\Big[\int_0^{t}\Big[f\big(X(s),\hat{u}(X(s))\big)\Big]\mathrm{d}s+\int_0^{t} r\big(X(s-),\hat{u}_\partial(X(s-))\big)\mathrm{d}p^*(s)\Big]\leq\rho,$$

showing that $\mathcal{J}_{\mathcal{A}}(x)\leq\mathcal{A}(U_{\widehat{u}_\phi},x)\leq\rho$. From Proposition 4.6 and item (i) of Proposition 4.22, we get easily the reverse inequality, completing the proof. □

4.5 Proof of the Results of Section 4.4.2

In this section, we present the proofs of the main results in Sect. 4.4.2.

4.5.1 *Proof of Theorem 4.20*

The next two propositions establish a connection between a general integrodifferential inequality (respectively equality) related to the local characteristics of the PDMP and an inequality (respectively equality) related to the operators G_α, L_α, and H_α. They will be crucial for the boundedness results on $\mathcal{J}_D^\alpha(\cdot)$ to be developed in the sequel.

Proposition 4.26 *Suppose that there exist $v \in \mathbb{M}^{ac}(E, \mathbb{R}_+)$, $\ell \in \mathbb{M}(E)$, $k \in \mathbb{M}(E)^+$, $p \in \mathbb{M}(\partial E)^+$, $\Theta = (\mu, \mu_\partial) \in \mathcal{S}_{\mathbb{V}^r}$, $d \geq 0$, and $\alpha \geq -c$ such that $L_\alpha \ell(x, \Theta(x))$ is well defined with values in $\mathbb{R} \cup \{+\infty\}$ for every $x \in E$ and*

$$\mathcal{X}v(\phi(x,t)) - [\alpha + \lambda(\phi(x,t), \mu(x,t))]\, v(\phi(x,t)) + \ell(\phi(x,t)) + \lambda Qk(\phi(x,t), \mu(x,t)) \leq d, \tag{4.29}$$

for all $x \in E$, $t \in [0, t_*(x))$, and

$$v(\phi(x, t_*(x))) \geq p(\phi(x, t_*(x))) + Qk(\phi(x, t_*(x)), \mu_\partial(\phi(x, t_*(x)))), \tag{4.30}$$

for all $x \in E$ with $t_*(x) < \infty$. Then

$$v(x) \geq -d\mathcal{L}_\alpha(x, \Theta(x)) + L_\alpha \ell(x, \Theta(x)) + H_\alpha p(x, \Theta(x)) + G_\alpha k(x, \Theta(x)). \tag{4.31}$$

Proof Since $L_\alpha \ell(x, \Theta(x))$ is well defined with values in $\mathbb{R} \cup \{+\infty\}$ for every $x \in E$, then for every $s \in [0, t_*(x))$, $\int_0^s e^{-\alpha t - \Lambda^{\mu(x)}(x,t)} \ell(\phi(x,t)) \mathrm{d}t \in \mathbb{R} \cup \{+\infty\}$. Consequently, multiplying both sides of Eq. (4.29) by $e^{-\alpha t - \Lambda^{\mu(x)}(x,t)}$ and integrating over $[0, s]$ for $s \in [0, t_*(x))$, we get that

$$\mathrm{d}\int_0^s e^{-\alpha t - \Lambda^{\mu(x)}(x,t)} \mathrm{d}t \geq e^{-\alpha s - \Lambda^{\mu(x)}(x,s)} v(\phi(x,s)) - v(x) + \int_0^s e^{-\alpha t - \Lambda^{\mu(x)}(x,t)} \big[\ell(\phi(x,t)) + \lambda Qk(\phi(x,t), \mu(x,t))\big] \mathrm{d}t. \tag{4.32}$$

Consider the case in which $t_*(x) < \infty$. From the fact that $v \in \mathbb{M}^{ac}(E)$, we obtain from Remark 4.19 (ii) and Eq. (4.32) that

$$v(x) \geq -\,\mathrm{d}\mathcal{L}_\alpha(x,\Theta(x)) + L_\alpha\ell(x,\Theta(x)) + e^{-\alpha t_*(x)-\Lambda^{\mu(x)}(x,t_*(x))}v(\phi(x,t_*(x))) + \int_0^{t_*(x)} e^{-\alpha t-\Lambda^{\mu(x)}(x,t)}\lambda Qk(\phi(x,t),\mu(x,t))\mathrm{d}t. \tag{4.33}$$

However, from Eq. (4.30), it follows that

$$v(x) \geq -\mathrm{d}\mathcal{L}_\alpha(x,\Theta(x)) + L_\alpha\ell(x,\Theta(x)) + H_\alpha p(x,\Theta(x)) + G_\alpha k(x,\Theta(x)).$$

Now consider the case in which $t_*(x) = +\infty$. From Eq. (4.32) (and recalling that v is positive), we have that

$$\mathrm{d}\int_0^s e^{-\alpha t-\Lambda^{\mu(x)}(x,t)}\mathrm{d}t \geq -v(x) + \int_0^s e^{-\alpha t-\Lambda^{\mu(x)}(x,t)}\big[\ell(\phi(x,t)) + \lambda Qk(\phi(x,t),\mu(x,t))\big]\mathrm{d}t,$$

and so by taking the limit as s tends to infinity in the previous equation, we obtain

$$v(x) \geq -\mathrm{d}\mathcal{L}_\alpha(x,\Theta(x)) + L_\alpha\ell(x,\Theta(x)) + \int_0^{t_*(x)} e^{-\alpha t-\Lambda^{\mu(x)}(x,t)}\lambda Qk(\phi(x,t),\mu(x,t))\mathrm{d}t.$$

However, by the fact that $t_*(x) = +\infty$ and Remark 4.19 (i), we have that $H_\alpha p(x,\Theta(x)) = 0$ and

$$G_\alpha k(x,\Theta(x)) = \int_0^{t_*(x)} e^{-\alpha t-\Lambda^{\mu(x)}(x,t)}\lambda Qk(\phi(x,t),\mu(x,t))\mathrm{d}t,$$

establishing the result. □

If the inequalities in (4.29) and (4.30) are replaced by equalities, then the hypotheses of Proposition 4.26 must be restricted to $\alpha \geq 0$ to show that the inequality in (4.31) becomes an equality. More specifically, we have the following result:

Proposition 4.27 *Suppose that there exist $v \in \mathbb{M}^{ac}(E,\mathbb{R}_+) \cap \mathbb{B}_g(E)$, $\ell \in \mathbb{M}(E)$, $k \in \mathbb{M}(E)^+$, $p \in \mathbb{M}(\partial E)^+$, $\Theta = (\mu,\mu_\partial) \in \mathcal{S}_{\mathbb{V}^r}$, $d \geq 0$, and $\alpha \geq 0$ such that $L_\alpha\ell(x,\Theta(x)) \in \mathbb{R}\cup\{+\infty\}$ for every $x \in E$ and*

$$\mathcal{X}v(\phi(x,t)) - [\alpha + \lambda(\phi(x,t),\mu(x,t))]\,v(\phi(x,t)) + \ell(\phi(x,t)) + \lambda Qk(\phi(x,t),\mu(x,t)) = d, \tag{4.34}$$

for all $x \in E$, $t \in [0, t_*(x))$, *and*

$$v(\phi(x, t_*(x))) = p(\phi(x, t_*(x))) + Qk(\phi(x, t_*(x)), \mu_\partial(\phi(x, t_*(x)))), \tag{4.35}$$

for all $x \in E$ *with* $t_*(x) < \infty$. *Then*

$$v(x) = -\mathrm{d}\mathcal{L}_\alpha(x, \Theta(x)) + L_\alpha \ell(x, \Theta(x)) + H_\alpha p(x, \Theta(x)) + G_\alpha k(x, \Theta(x)). \tag{4.36}$$

Proof By following the same steps as in the first part of the proof of Proposition 4.26, we have that for all $s \in [0, t_*(x))$,

$$\begin{aligned} \mathrm{d}\int_0^s e^{-\alpha t - \Lambda^{\mu(x)}(x,t)}\mathrm{d}t = {} & e^{-\alpha s - \Lambda^{\mu(x)}(x,s)} v(\phi(x, s)) - v(x) \\ & + \int_0^s e^{-\alpha t - \Lambda^{\mu(x)}(x,t)}\big[\ell(\phi(x, t)) + \lambda Qk(\phi(x, t), \mu(x, t))\big]\mathrm{d}t. \end{aligned} \tag{4.37}$$

The case in which $t_*(x) < \infty$ can be treated in the same manner as in the proof of Proposition 4.26. However, the case in which $t_*(x) = +\infty$ is different. By Assumption 4.18 (ii) and the fact that $0 \leq v \leq \|v\|_g g$, we have that for every $\alpha \geq 0$,

$$\lim_{s \to +\infty} e^{-\alpha s - \Lambda^{\mu(x)}(x,s)} v(\phi(x, s)) \leq \|v\|_g \lim_{s \to +\infty} e^{-\int_0^{t_*(x)} \lambda(\phi(x,t))\mathrm{d}t} g(\phi(x, s)) = 0.$$

Therefore, taking the limit as s tends to infinity in Eq. (4.37), we have that

$$\begin{aligned} \mathrm{d}\mathcal{L}_\alpha(x, \Theta(x)) = {} & - v(x) + L_\alpha \ell(x, \Theta(x)) \\ & + \int_0^s e^{-\alpha t - \Lambda^{\mu(x)}(x,t)} \lambda Qk(\phi(x, t), \mu(x, t))\mathrm{d}t, \end{aligned}$$

and this proves Eq. (4.36) on account of Remark 4.19 (i). □

Applying Proposition 4.26 to the inequalities (4.14) and (4.16), we obtain the following corollary:

Corollary 4.28 *For every* $u \in \mathcal{S}_{\mathbb{U}}$, $\alpha \geq -c$, *and* $x \in E$,

$$\begin{aligned} g(x) \geq {} & -b\mathcal{L}_\alpha(x, u_\phi(x)) + (c + \alpha)L_\alpha g(x, u_\phi(x)) + H_\alpha \overline{r}(x, u_\phi(x)) \\ & + G_\alpha g(x, u_\phi(x)), \end{aligned} \tag{4.38}$$

and for all $\Theta \in \mathcal{S}_{\mathbb{V}^r}$,

$$(c+\alpha)L_\alpha g(x,\Theta(x)) + H_\alpha \overline{r}(x,\Theta(x)) + G_\alpha g(x,\Theta(x)) \leq bK_\lambda + g(x). \quad (4.39)$$

Proof Clearly, from Proposition 2.24 and Remark 2.25, it follows that $u_\phi \in \mathcal{S}_{\mathbb{V}^r}$. Consequently, by Assumption 4.13 and setting $d = b$, $v = g$, $\ell = (c+\alpha)g$, $p = \overline{r}$, $k = g$, and $\Theta = u_\phi$ in Proposition 4.26, we get Eq. (4.38). Similarly, from Remark 4.19 (ii) and Assumption 4.13, the inequality (4.39) is a straightforward consequence of the inequality (4.31). □

The next theorem provides bounds in terms of α and g for a sequence of functions defined by a general recursive equation and for the functions Lf, Hr and Lg.

Theorem 4.29 *Define the sequence* $(q_m^\alpha(x))_{m\in\mathbb{N}}$ *by*

$$\begin{aligned} q_0^\alpha(x) &= 0,\\ q_{m+1}^\alpha(x) &= L_\alpha f(x,u_\phi^m(x)) + H_\alpha r(x,u_\phi^m(x)) + G_\alpha q_m^\alpha(x,u_\phi^m(x)), \end{aligned} \quad (4.40)$$

where $x \in E$, $(u^m)_{m\in\mathbb{N}} \in \mathcal{S}_{\mathbb{U}}$, *and* $\alpha > 0$.
Then the following assertions hold:

(i) For every $x \in E$, $m \in \mathbb{N}$ *and* $\alpha \in [0,\delta)$, *we have that*

$$q_m^\alpha(x) \leq \frac{M}{c+\alpha} g(x) + \frac{Mb}{c\alpha}. \quad (4.41)$$

(ii) For every $x \in E$, $u \in \mathcal{S}_{\mathbb{U}}$,

$$0 \leq Lf(x,u_\phi(x)) + Hr(x,u_\phi(x)) \leq \frac{M(1+bK_\lambda)}{c} g(x), \quad (4.42)$$

$$0 \leq Lg(x,u_\phi(x)) \leq \frac{(1+bK_\lambda)}{c} g(x). \quad (4.43)$$

Proof Let us prove (4.41) by induction. For $m = 0$, it is immediate, since $q_0^\alpha = 0$. Suppose it holds for m. Combining (4.40) and (4.41), we have

$$\begin{aligned} q_{m+1}^\alpha(x) \leq & L_\alpha f(x,u_\phi^m(x)) + H_\alpha r(x,u_\phi^m(x)) + \frac{M}{c+\alpha} G_\alpha g(x,u_\phi^m(x))\\ &+ \frac{Mb}{c\alpha} G_\alpha 1(x,u_\phi^m(x)). \end{aligned} \quad (4.44)$$

Moreover, from Eqs. (4.38) and (4.39), we obtain that

$$\begin{aligned} G_\alpha g(x,u_\phi^m(x)) \leq & g(x) + b\mathcal{L}_\alpha(x,u_\phi^m(x)) - (c+\alpha)L_\alpha g(x,u_\phi^m(x))\\ &- H_\alpha \overline{r}(x,u_\phi^m(x)). \end{aligned} \quad (4.45)$$

Substituting (4.45) into (4.44), we get

$$q_{m+1}^{\alpha}(x) \leq L_{\alpha}(f - Mg)(x, u_{\phi}^{m}(x)) + H_{\alpha}(r - \frac{M}{c+\alpha}\overline{r})(x, u_{\phi}^{m}(x)) + \frac{M}{c+\alpha}g(x)$$
$$+ Mb\Big(\frac{1}{c\alpha}G_{\alpha}1(x, u_{\phi}^{m}(x)) + \frac{1}{c+\alpha}\mathcal{L}_{\alpha}(x, u_{\phi}^{m}(x))\Big),$$

and so by (4.15) and (4.17),

$$\begin{aligned} q_{m+1}^{\alpha}(x) &\leq \frac{M}{c+\alpha}g(x) + \frac{Mb}{c\alpha}\Big(G_{\alpha}1(x, u_{\phi}^{m}(x)) + \alpha\mathcal{L}_{\alpha}(x, u_{\phi}^{m}(x))\Big) \\ &\leq \frac{M}{c+\alpha}g(x) + \frac{Mb}{c\alpha}, \end{aligned} \tag{4.46}$$

since $G_{\alpha}1(x, u_{\phi}^{m}(x)) + \alpha\mathcal{L}_{\alpha}(x, u_{\phi}^{m}(x)) = 1$.

Let us now prove (4.42) and (4.43). For $\alpha = 0$, it follows from Remark 4.19 (ii) and Eq. (4.38) that

$$\begin{aligned} g(x) + bK_{\lambda} \geq g(x) + b\mathcal{L}(x, u_{\phi}(x)) &\geq cLg(x, u_{\phi}(x)) \\ &+ H\overline{r}(x, u_{\phi}(x)) + Gg(x, u_{\phi}(x)), \end{aligned} \tag{4.47}$$

proving equation (4.43), since $g \geq 1$ and $\overline{r} \geq 0$. Combining Eqs. (4.15), (4.17) and (4.47), we get (4.42), proving the last part of the result. □

Proof of Theorem 4.20 By Propositions 4.1 and 4.4, it can be shown that there exists $u_{\phi}^{m} \in \mathcal{S}_{\mathbb{V}}$ such that the sequence $\big(v_{m}^{\alpha}(x)\big)_{m\in\mathbb{N}}$ defined by

$$v_{m+1}^{\alpha}(x) = L_{\alpha}f(x, u_{\phi}^{m}(x)) + H_{\alpha}r(x, u_{\phi}^{m}(x)) + G_{\alpha}v_{m}^{\alpha}(x, u_{\phi}^{m}(x))$$

and $v_{0}^{\alpha}(x) = 0$ satisfies $v_{m+1}^{\alpha} \uparrow \mathcal{J}_{D}^{\alpha}(x)$ as $m \uparrow \infty$. Therefore, considering $q_{m}^{\alpha} = v_{m}^{\alpha}$ in Theorem 4.29 and taking the limit as $m \uparrow \infty$, we get (4.23). □

4.5.2 Proof of Theorem 4.21

The following technical lemma shows that $\mathcal{J}_{D}^{\alpha}(x)$ can be written as an infinite sum of iterates of the stochastic kernel G_{α}. Using this result, $\mathcal{J}_{D}^{\alpha}(x)$ is characterized in terms of the Markov kernel G in Proposition 4.31. This is an important property. Indeed, by classical hypotheses on G such as the geometric ergodic condition in Assumption 4.14, it will be shown in Theorem 4.21 that the mapping defined by $\mathcal{J}_{D}^{\alpha}(\cdot)$-$\mathcal{J}_{D}^{\alpha}(y)$ for y fixed in E belongs to $\mathbb{B}_{g}(E)$.

Lemma 4.30 *For each $\alpha > 0$, there exists $u^{\alpha} \in \mathcal{S}_{\mathbb{U}}$ such that*

$$\mathcal{J}_{D}^{\alpha}(x) = \sum_{k=0}^{\infty} G_{\alpha}^{k}(L_{\alpha}f + H_{\alpha}r)(x, u_{\phi}^{\alpha}(x)). \tag{4.48}$$

Proof As shown in Theorem 4.5, $\mathcal{J}_{\mathcal{D}}^{\alpha} \in \mathbb{M}(\mathbb{E})$ and $\mathcal{J}_{\mathcal{D}}^{\alpha}(x) = \mathcal{R}_{\alpha}(0, \mathcal{J}_{\mathcal{D}}^{\alpha})(x)$. Moreover, from Theorem 3.14, there exists $u^{\alpha} \in \mathcal{S}_{\mathbb{U}}$ such that the ordinary feedback measurable selector $u_{\phi}^{\alpha} \in \mathcal{S}_{\mathbb{V}}$ satisfies

$$\begin{aligned} \mathcal{J}_{D}^{\alpha}(x) &= \mathcal{R}_{\alpha}(0, \mathcal{J}_{D}^{\alpha})(x) = \mathcal{T}_{\alpha}(0, \mathcal{J}_{D}^{\alpha})(x) \\ &= L_{\alpha} f(x, u_{\phi}^{\alpha})(x) + H_{\alpha} r(x, u_{\phi}^{\alpha}) + G_{\alpha} \mathcal{J}_{\mathcal{D}}^{\alpha}(x, u_{\phi}^{\alpha}). \end{aligned} \tag{4.49}$$

Iterating (4.49) and recalling that $\mathcal{J}_{\mathcal{D}}^{\alpha}(y) \geq 0$ for every y yields for every $m \in \mathbb{N}_*$ that

$$\begin{aligned} \mathcal{J}_{D}^{\alpha}(x) &= \sum_{k=0}^{m-1} G_{\alpha}^{k}(L_{\alpha} f + H_{\alpha} r)(x, u_{\phi}^{\alpha}(x)) + G_{\alpha}^{m} \mathcal{J}_{\mathcal{D}}^{\alpha}(x, u_{\phi}^{\alpha}(x)) \\ &\geq \sum_{k=0}^{m-1} G_{\alpha}^{k}(L_{\alpha} f + H_{\alpha} r)(x, u_{\phi}^{\alpha}(x)). \end{aligned} \tag{4.50}$$

For the control $U_{u_{\phi}^{\alpha}} \in \mathcal{U}$ (see Definition 2.23), it follows from Remark 2.26 that

$$\begin{aligned} &\sum_{k=0}^{m-1} G_{\alpha}^{k}(L_{\alpha} f + H_{\alpha} r)(x, u_{\phi}^{\alpha}(x)) \\ &= E_{(x,0)}^{U_{u_{\phi}^{\alpha}}}\bigg[\int_{0}^{T_m} e^{-\alpha s} f\big(X(s), u(N(s), Z(s), \tau(s))\big) \mathrm{d}s \\ &\quad + \int_{0}^{T_m} e^{-\alpha s} r\big(X(s-), u_{\partial}(N(s-), Z(s-))\big) \mathrm{d}p^{*}(s)\bigg]. \end{aligned} \tag{4.51}$$

From Assumption 2.8, $T_m \to \infty$, $P_{(x,0)}^{U_{u_{\phi}^{\alpha}}}$ a.s. Therefore, from the monotone convergence theorem, Eq. (4.51) implies that

$$\sum_{k=0}^{\infty} G_{\alpha}^{k}(L_{\alpha} f + H_{\alpha} r)(x, u_{\phi}^{\alpha}(x)) = \mathcal{D}^{\alpha}(U_{u_{\phi}^{\alpha}}, x),$$

and from Eq. (4.50),

$$\mathcal{J}_{D}^{\alpha}(x) \geq \sum_{k=0}^{\infty} G_{\alpha}^{k}(L_{\alpha} f + H_{\alpha} r)(x, u_{\phi}^{\alpha}(x)) = \mathcal{D}^{\alpha}(U_{u_{\phi}^{\alpha}}, x). \tag{4.52}$$

But since $U_{u_{\phi}^{\alpha}} \in \mathcal{U}$ and $\mathcal{J}_{D}^{\alpha}(x) = \inf_{U \in \mathcal{U}} \mathcal{D}^{\alpha}(U, x)$ it is clear that $\mathcal{D}^{\alpha}(U_{u_{\phi}^{\alpha}}, x) \geq \mathcal{J}_{D}^{\alpha}(x)$, so that (4.52) yields (4.49). □

The next proposition gives a characterization of $\mathcal{J}_D^\alpha(x)$ in terms of G.

Proposition 4.31 *For $\alpha > 0$ and u_ϕ^α as in Lemma 4.30, define the sequence $\big(s_m^\alpha(x)\big)_{m\in\mathbb{N}}$ for $x \in E$ by $s_0^\alpha(x) = 0$ and*

$$s_{m+1}^\alpha(x) = L_\alpha f(x, u_\phi^\alpha(x)) + H_\alpha r(x, u_\phi^\alpha(x)) + G_\alpha s_m^\alpha(x, u_\phi^\alpha(x)).$$

Then

$$\mathcal{J}_D^\alpha(x) = \lim_{m\to\infty} \sum_{k=0}^{m} G^k (L(f - \alpha s_{m+1-k}^\alpha) + Hr)(x, u_\phi^\alpha(x)). \tag{4.53}$$

Proof By definition, for all $m \in \mathbb{N}$, $s_m^\alpha \in \mathbb{M}(E)$ and

$$s_{m+1}^\alpha(x) = \sum_{k=0}^{m} G_\alpha^k (L_\alpha f + H_\alpha r)(x, u_\phi^\alpha(x)),$$

and clearly from Lemma 4.30, we have that $s_m^\alpha \uparrow \mathcal{J}_D^\alpha$ as $m \uparrow \infty$. Applying Lemma 3.16, it can be shown that for all $x \in E$ and $t \in [0, t_*(x))$,

$$\begin{aligned} s_{m+1}^\alpha(x) = \int_0^t & e^{-\alpha s - \int_0^s \lambda(\phi(x,\theta), u^\alpha(\phi(x,\theta)))\mathrm{d}\theta} \Big[f(\phi(x,s), u^\alpha(\phi(x,s))) \\ & + \lambda(\phi(x,s), u^\alpha(\phi(x,s))) Q s_m^\alpha(\phi(x,s), u^\alpha(\phi(x,s))) \Big] \mathrm{d}s \\ & + e^{-\alpha t - \int_0^t \lambda(\phi(x,s), u^\alpha(\phi(x,s)))\mathrm{d}s} s_{m+1}^\alpha(\phi(x,t)). \end{aligned} \tag{4.54}$$

Notice that we have $s_m^\alpha \in \mathbb{B}_g(E)$ by Theorem 4.20, and so from Assumption 4.14 and Eq. 4.42, we get that

$$\begin{aligned} \int_0^{t_*(x)} & e^{-\alpha s - \int_0^s \lambda(\phi(x,\theta), u^\alpha(\phi(x,\theta)))\mathrm{d}\theta} \Big[f(\phi(x,s), u^\alpha(\phi(x,s))) \\ & + \lambda(\phi(x,s), u^\alpha(\phi(x,s))) Q s_m^\alpha(\phi(x,s), u^\alpha(\phi(x,s))) \Big] \mathrm{d}s < \infty. \end{aligned}$$

Moreover, from Assumption 2.5, we have that $e^{-\int_0^{t_*(x)} \lambda(\phi(x,s), u^\alpha(\phi(x,s)))\mathrm{d}s} > 0$ for $t_*(x) < \infty$, and so from Eq. (4.54), we have $s_m^\alpha \in \mathbb{M}^{ac}(E)$.
Again by Eq. (4.54), it follows that

$$\begin{aligned} \mathcal{X} s_{m+1}^\alpha(x) - \big[\alpha + \lambda(x, u^\alpha(x))\big] s_{m+1}^\alpha(x) + f(x, u^\alpha(x)) \\ + \lambda(x, u^\alpha(x)) Q s_m^\alpha(x, u^\alpha(x)) = 0. \end{aligned} \tag{4.55}$$

Consider the case in which $t_*(x) < \infty$. Since $s^\alpha_{m+1} \in \mathbb{M}^{ac}(E)$, this yields that

$$\begin{aligned} s^\alpha_{m+1}(x) =& L_\alpha f(x, u^\alpha_\phi(x)) \\ &+ e^{-\alpha t_*(x) - \int_0^{t_*(x)} \lambda(\phi(x,s), u^\alpha(\phi(x,s)))\mathrm{d}s} s^\alpha_{m+1}(\phi(x, t_*(x))) \\ &+ \int_0^{t_*(x)} e^{-\alpha s - \int_0^s \lambda(\phi(x,\theta), u^\alpha(\phi(x,\theta)))\mathrm{d}\theta} \times \\ &\lambda(\phi(x,s), u^\alpha(\phi(x,s))) Q s^\alpha_m(\phi(x,s), u^\alpha(\phi(x,s)))\mathrm{d}s. \end{aligned} \tag{4.56}$$

From Assumption 2.5, we have that $e^{-\int_0^{t_*(x)} \lambda(\phi(x,s), u^\alpha(\phi(x,s)))\mathrm{d}s} > 0$. Therefore, combining the definition of $s^\alpha_m(x)$ and Eq. (4.56), we obtain

$$\begin{aligned} s^\alpha_{m+1}(\phi(x, t_*(x))) = & Q s^\alpha_m(\phi(x, t_*(x)), u(\phi(x, t_*(x)))) \\ &+ r(\phi(x, t_*(x)), u(\phi(x, t_*(x)))). \end{aligned} \tag{4.57}$$

Notice that $f(., u^\alpha(.)) - \alpha s^\alpha_{m+1}(.) \in \mathbb{B}_g(E)$, and so from Eq. (4.43), we have that $L(f - \alpha s^\alpha_{m+1})(x, u^\alpha_\phi(x))$ is finite. Consequently, by Proposition 4.27, we get from (4.55), (4.57) that

$$s^\alpha_{m+1}(x) = L(f - \alpha s^\alpha_{m+1})(x, u^\alpha_\phi(x)) + Hr(x, u^\alpha_\phi(x)) + G s^\alpha_m(x, u^\alpha_\phi(x)). \tag{4.58}$$

Iteration of (4.58) over m yields (4.53). □

Before showing that the mapping defined by $\mathcal{J}^\alpha_D(\cdot)$-$\mathcal{J}^\alpha_D(y)$ for y fixed in E belongs to $\mathbb{B}_g(E)$, we need to prove that the mapping $L(f - \alpha s^\alpha_{m+1})(., u^\alpha_\phi(.)) + Hr(., u^\alpha_\phi(.))$ belongs to $\mathbb{B}_g(E)$.

Lemma 4.32 *Define $M' = \frac{M(1+\frac{b}{c})(1+bK_\lambda)}{c}$. For $\alpha > 0$, u^α_ϕ as in Lemma 4.30, s^α_m as in Lemma 4.31, and $x \in E$, we have that*

$$|L(f - \alpha s^\alpha_{m+1})(x, u^\alpha_\phi(x)) + Hr(x, u^\alpha_\phi(x))| \le M' g(x). \tag{4.59}$$

Proof Notice that

$$\begin{aligned} -\alpha L s^\alpha_{m+1}(x, u^\alpha_\phi(x)) &\le L(f - \alpha s^\alpha_{m+1})(x, u^\alpha_\phi(x)) + Hr(x, u^\alpha_\phi(x)) \\ &\le Lf(x, u^\alpha_\phi(x)) + Hr(x, u^\alpha_\phi(x)). \end{aligned} \tag{4.60}$$

Considering $q^\alpha_m = s^\alpha_m$ in Theorem 4.29 and recalling that $g \ge 1$, we get from Eq. (4.41) that

$$s^\alpha_m(x) \le \frac{M}{c+\alpha} g(x) + \frac{Mb}{c\alpha} \le \frac{M(1+\frac{b}{c})}{\alpha} g(x). \tag{4.61}$$

Therefore, from (4.61), we have that $\alpha s_m^\alpha \leq M(1+\frac{b}{c})g$, and thus from (4.43),

$$\alpha L s_{m+1}^\alpha(x, u_\phi^\alpha(x)) \leq \frac{M(1+\frac{b}{c})(1+bK_\lambda)}{c} g(x). \tag{4.62}$$

By combining Eqs. (4.42), (4.60), and (4.62), the result follows. □

Proof of Theorem 4.21 From Assumption 4.14 and Lemma 4.32, we get that for all $x \in E$,

$$|G^k(L(f-\alpha s_{m+1-k}^\alpha) + Hr)(x, u_\phi^\alpha(x)) - \pi_{u^\alpha}\big(L(f-\alpha s_{m+1-k}^\alpha) + Hr\big)| \\ \leq aM'\kappa^k g(x).$$

Consequently,

$$\Big|\sum_{k=0}^{m} G^k(L(f-\alpha s_{m+1-k}^\alpha) + Hr)(x, u_\phi^\alpha(x)) \\ - G^k(L(f-\alpha s_{m+1-k}^\alpha) + Hr)(y, u_\phi^\alpha(y))\Big| \\ \leq aM'(g(x)+g(y))\frac{1-\kappa^{m+1}}{1-\kappa}.$$

Taking the limit as $m \uparrow \infty$ in the previous equation and recalling that $g \geq 1$, we get the desired result from Proposition 4.31. □

4.5.3 Existence of an Ordinary Feedback Measurable Selector

The main goal of this subsection is to show that for every function $h \in \mathbb{B}_g(E)$, the one-stage optimization operators $\mathcal{R}_\alpha(\rho, h)(x)$ and $\mathcal{T}_\alpha(\rho, h)(x)$ are equal and that there exists an ordinary feedback measurable selector for the one-stage optimization problems associated with these operators (see Theorem 4.35). This theorem is an extension of a result obtained in Chap. 3, Theorem 3.14, to the case in which the functions under consideration are not necessarily bounded below, as was supposed in Chap. 3, but instead, belong to $\mathbb{B}_g(E)$. The next two technical lemmas will be used to derive Theorem 4.35.

Lemma 4.33 *Let $\alpha \geq 0$, $\rho \in \mathbb{R}_+$, $h \in \mathbb{B}_g(E)$, and set $w = \mathcal{R}_\alpha(\rho, h)$. Then there exists $\hat{\Theta} \in \mathcal{S}_{\mathbb{V}^r}$ such that*

$$w(x) = -\rho\mathcal{L}_\alpha(x, \hat{\Theta}(x)) + L_\alpha f(x, \hat{\Theta}(x)) + H_\alpha r(x, \hat{\Theta}(x)) + G_\alpha h(x, \hat{\Theta}(x)). \tag{4.63}$$

Moreover, $w \in \mathbb{M}^{ac}(E)$, and for all $x \in E$ and $t \in [0, t_*(x))$, we have

$$w(x) = \inf_{\mu \in \mathcal{V}^r(x)} \left\{ \int_0^t e^{-\alpha s - \Lambda^{\mu}(x,s)} \Big[-\rho + f(\phi(x,s), \mu(s)) + \lambda Q h(\phi(x,s), \mu(s)) \Big] \mathrm{d}s + e^{-\alpha t - \Lambda^{\mu}(x,t)} w(\phi(x,t)) \right\} \tag{4.64}$$

$$= \int_0^t e^{-\alpha s - \Lambda^{\hat{\mu}(x)}(x,s)} \Big[-\rho + f(\phi(x,s), \hat{\mu}(x,s)) + \lambda Q h(\phi(x,s), \hat{\mu}(x,s)) \Big] \mathrm{d}s + e^{-\alpha t - \Lambda^{\hat{\mu}(x)}(x,t)} w(\phi(x,t)), \tag{4.65}$$

where $\hat{\Theta}(x) = (\hat{\mu}(x), \hat{\mu}_{\partial}(x))$.

Proof For every $h \in \mathbb{B}_g(E)$, we have that $h = h^+ - h^-$ with $0 \le h^+ \le \|h\|_g g$, $0 \le h^- \le \|h\|_g g$. From Corollary 4.28, we have that $G_\alpha g(x, \Theta) < \infty$, and therefore $G_\alpha h^+(x, \Theta) < \infty$, $G_\alpha h^-(x, \Theta) < \infty$, and we can conclude that $G_\alpha h(x, \Theta) = G_\alpha h^+(x, \Theta) - G_\alpha h^-(x, \Theta)$ takes values in $\mathbb{R}$. From Proposition 2.17, it follows that $G_\alpha h^+(x, \Theta)$ and $G_\alpha h^-(x, \Theta)$ are measurable, and thus $G_\alpha h(x, \Theta)$ is measurable. From Corollary 4.28 again, the mapping V defined on $\mathcal{K}$ by

$$V(x, \Theta) = -\rho \mathcal{L}_\alpha(x, \Theta) + L_\alpha f(x, \Theta) + H_\alpha r(x, \Theta) + G_\alpha h(x, \Theta)$$

takes values in $\mathbb{R}$, and from Proposition 2.17, it is measurable. Furthermore, combining Corollary 3.11 and Proposition 4.16, it follows that for all $x \in E$, $V(x, .)$ is lower semicontinuous on $\mathbb{V}^r(x)$. Recalling that $\mathbb{V}^r(x)$ is a compact subset of $\mathbb{V}^r$ and using Proposition D.5 in [49], we obtain that there exists $\hat{\Theta} \in \mathcal{S}_{\mathbb{V}^r}$ such that Eq. (4.63) is satisfied. The rest of the proof is similar to the proof of Proposition 3.2 and it is therefore omitted. □

Lemma 4.34 *Let* $\alpha \ge 0$, $\rho \in \mathbb{R}_+$, *and* $h \in \mathbb{B}_g(E)$. *Then for all* $x \in E$,

$$\mathcal{R}_\alpha(\rho, h)(x) \ge -(\rho + b\|h\|_g) K_\lambda - \|h\|_g g(x), \tag{4.66}$$

and for all $x \in E$ *such that* $t_*(x) = \infty$ *and* $\Theta = (\mu, \mu_\partial) \in \mathbb{V}^r(x)$,

$$-\rho \mathcal{L}_\alpha(x, \Theta) + L_\alpha f(x, \Theta) + H_\alpha r(x, \Theta) + G_\alpha h(x, \Theta) = \lim_{t \to +\infty} \int_0^t e^{-\alpha s - \Lambda^{\mu}(x,s)} \Big[-\rho + f(\phi(x,s), \mu(s)) + \lambda Q h(\phi(x,s), \mu(s)) \Big] \mathrm{d}s. \tag{4.67}$$

Proof From Eq. (4.39), we have

$$G_\alpha g(x, \Theta) \le b K_\lambda + g(x), \tag{4.68}$$

for all $x \in E$ and $\Theta \in \mathbb{V}^r$. Consequently, by Eq. (4.63) and the fact that $f \geq 0$ and $r \geq 0$, it follows that

$$\mathcal{R}_\alpha(\rho, h)(x) \geq -\rho\mathcal{L}_\alpha(x, \hat{\Theta}(x)) + G_\alpha h(x, \hat{\Theta}(x)) \geq -(\rho + b\|h\|_g)K_\lambda - \|h\|_g g(x),$$

establishing the first part of the result.
From Assumptions 2.5 and 3.9, we have that

$$\lim_{t\to+\infty} \int_0^t e^{-\alpha s - \Lambda^\mu(x,s)}\big[-\rho + f(\phi(x,s), \mu(s))\big]\mathrm{d}s$$

exists in $\mathbb{R}$, and from Assumption 4.15,

$$\lim_{t\to+\infty} \int_0^t e^{-\alpha s - \Lambda^\mu(x,s)}\lambda Qg(\phi(x,s), \mu(s))\mathrm{d}s$$

exists in $\mathbb{R}$. By the fact that $h \in \mathbb{B}_g(E)$, it follows that the limit on the right-hand side of Eq. (4.67) exists in $\mathbb{R}$. Finally, from Remark 4.19 (i), we get the last part of the result. □

The next result shows that for every function $h \in \mathbb{B}_g(E)$, the one-stage optimization operators $\mathcal{R}_\alpha(\rho, h)(x)$ and $\mathcal{T}_\alpha(\rho, h)(x)$ are equal and that there exists an ordinary feedback measurable selector for the one-stage optimization problems associated with these operators.

Theorem 4.35 *Let $\alpha \geq 0$, $\rho \in \mathbb{R}_+$, $h \in \mathbb{B}_g(E)$, and set*

$$w = \mathcal{R}_\alpha(\rho, h). \tag{4.69}$$

Then $w \in \mathbb{M}^{ac}(E)$, and the ordinary feedback measurable selector $\widehat{u}_\phi(w, h) \in \mathcal{S}_\mathbb{V}$ (see item (D2) of Remark 4.17) satisfies the one-stage optimization problems

$$\begin{aligned}\mathcal{R}_\alpha(\rho, h)(x) &= \mathcal{T}_\alpha(\rho, h)(x)\\ &= -\rho\mathcal{L}_\alpha(x, \widehat{u}_\phi(w, h)(x)) + L_\alpha f(x, \widehat{u}_\phi(w, h)(x))\\ &\quad + H_\alpha r(x, \widehat{u}_\phi(w, h)(x)) + G_\alpha h(x, \widehat{u}_\phi(w, h)(x)).\end{aligned}$$

Proof According to Lemma 4.33, there exists $\hat{\Theta} \in \mathcal{S}_{\mathbb{V}^r}$ such that for all $x \in E$ and $t \in [0, t_*(x))$, we have

$$e^{-\alpha t - \Lambda^{\hat{\mu}(x)}(x,t)} w(\phi(x,t)) - w(x) = \int_0^t e^{-\alpha s - \Lambda^{\hat{\mu}(x)}(x,s)} \times \big[\rho - f(\phi(x,s), \hat{\mu}(x,s)) - \lambda Q h(\phi(x,s), \hat{\mu}(x,s))\big] \mathrm{d}s, \tag{4.70}$$

where $\hat{\Theta}(x) = (\hat{\mu}(x), \hat{\mu}_{\partial}(x))$. Since $w \in \mathbb{M}^{ac}(E)$, we obtain from Eq. (4.70) that

$$- \mathcal{X} w(\phi(x,t)) + \alpha w(\phi(x,t)) \geq \inf_{\mu \in \mathcal{P}\left(\mathbb{U}(\phi(x,t))\right)} \Big\{ f(\phi(x,t), \mu) - \lambda(\phi(x,t), \mu) w(\phi(x,t)) + \lambda Q h(\phi(x,t), \mu) \Big\} - \rho$$

$\eta - a.s.$ on $[0, t_*(x))$. However, notice that

$$\inf_{\mu \in \mathcal{P}\left(\mathbb{U}(\phi(x,t))\right)} \Big\{ f(\phi(x,t), \mu) - \lambda(\phi(x,t), \mu) w(\phi(x,t)) + \lambda Q h(\phi(x,t), \mu) \Big\} - \rho$$
$$= \inf_{a \in \mathbb{U}(\phi(x,t))} \Big\{ f(\phi(x,t), a) - \lambda(\phi(x,t), a)\big[w(\phi(x,t)) - Qh(\phi(x,t), a)\big] \Big\} - \rho.$$

Consequently, by considering the measurable selector $\overline{u} \in \mathcal{S}_{\mathbb{U}}$ given by $\overline{u} = \widehat{u}(w, h)$ (see Remark 4.17, (D1)), we have that

$$-\mathcal{X} w(\phi(x,t)) + \alpha w(\phi(x,t)) = -\rho + f(\phi(x,t), \overline{u}(\phi(x,t))) - \lambda(\phi(x,t), \overline{u}(\phi(x,t)))\big[w(\phi(x,t)) - Qh(\phi(x,t), \overline{u}(\phi(x,t)))\big],$$

$\eta - a.s.$ on $[0, t_*(x))$. Otherwise, this would lead to a contradiction with Eq. (4.64) as in the proof of Theorem 3.14. Consequently, for all $t \in [0, t_*(x))$, it follows that

$$w(x) = e^{-(\alpha t + \overline{\Lambda}(x,t))} w(\phi(x,t)) + \int_0^t e^{-(\alpha s + \overline{\Lambda}(x,s))} \Big[f(\phi(x,s), \overline{u}(\phi(x,s))) + \lambda(\phi(x,s), \overline{u}(\phi(x,s))) Q h(\phi(x,s), \overline{u}(\phi(x,s))) - \rho \Big] \mathrm{d}s, \tag{4.71}$$

where we set $\overline{\Lambda}(x,t) = \int_0^t \lambda(\phi(x,s), \overline{u}(\phi(x,s)))\mathrm{d}s$.

First consider the case in which $t_*(x) < \infty$. Following similar steps as in the proof of Theorem 3.14, we obtain, by taking the limit as t tends to $t_*(x)$ in the previous equation, that the ordinary feedback measurable selector $\widehat{u}_{\phi}(w, h) \in \mathcal{S}_{\mathbb{V}}$ (see item (D2) of Remark 4.17) satisfies

$$\begin{aligned} w(x) = e^{-(\alpha t_*(x)+\overline{\Lambda}(x,t_*(x)))} w(\phi(x, t_*(x))) \\ - \rho\mathcal{L}_\alpha(x, \widehat{u}_\phi(w, h)(x)) + L_\alpha f(x, \widehat{u}_\phi(w, h)(x)) \\ + \int_0^{t_*(x)} e^{-(\alpha s+\overline{\Lambda}(x,s))} \lambda(\phi(x, s), \overline{u}(\phi(x, s))) Qh(\phi(x, s), \overline{u}(\phi(x, s)))\mathrm{d}s. \end{aligned} \tag{4.72}$$

Define the control $\Theta(x)$ by $(\hat{\mu}(x), \mu)$ for $\mu \in \mathcal{P}\big(\mathbb{U}(\phi(x, t_*(x)))\big)$. Therefore, we have by definition of w (see Eq. (4.69)) that

$$\begin{aligned} w(x) \le -\rho\mathcal{L}_\alpha(x, \hat{\Theta}(x)) + L_\alpha f(x, \hat{\Theta}(x)) \\ + \int_0^{t_*(x)} e^{-\alpha s - \Lambda^{\hat{\mu}(x)}(x,s)} \lambda Qh(\phi(x, s), \hat{\mu}(x, s))\mathrm{d}s \\ + e^{-\alpha t_*(x) - \Lambda^{\hat{\mu}(x)}(x,t_*(x))} \big[Qh(\phi(x, t_*(x)), \mu) + r(\phi(x, t_*(x)), \mu)\big]. \end{aligned} \tag{4.73}$$

From Eq. (4.65) and since $w \in \mathbb{M}^{ac}(E)$, we have that

$$\begin{aligned} w(x) = -\rho\mathcal{L}_\alpha(x, \hat{\Theta}(x)) + L_\alpha f(x, \hat{\Theta}(x)) \\ + \int_0^{t_*(x)} e^{-\alpha s - \Lambda^{\hat{\mu}(x)}(x,s)} \lambda Qh(\phi(x, s), \hat{\mu}(x, s))\mathrm{d}s \\ + e^{-\alpha t_*(x) - \Lambda^{\hat{\mu}(x)}(x,t_*(x))} w(\phi(x, t_*(x))). \end{aligned} \tag{4.74}$$

By Assumption 2.5, we have that $e^{-\Lambda^{\hat{\mu}(x)}(x,t_*(x))} > 0$. Therefore, combining Eqs. (4.73) and (4.74), we obtain that for all $x \in E$ and $\mu \in \mathcal{P}\big(\mathbb{U}(\phi(x, t_*(x)))\big)$,

$$w(\phi(x, t_*(x))) \le Qh(\phi(x, t_*(x)), \mu) + r(\phi(x, t_*(x)), \mu).$$

Clearly, using Eq. (4.63), it can be claimed that the previous inequality becomes an equality for $\mu = \hat{\mu}_\partial(x)$, implying that

$$\begin{aligned} w(\phi(x, t_*(x))) &= \inf_{\mu\in\mathcal{P}(\mathbb{U}(\phi(x,t_*(x))))} \{r(\phi(x, t_*(x)), \mu) + Qh(\phi(x, t_*(x)), \mu)\} \\ &= \inf_{a\in\mathbb{U}(\phi(x,t_*(x)))} \{r(\phi(x, t_*(x)), a) + Qh(\phi(x, t_*(x)), a)\}. \end{aligned}$$

Consequently, we have that

$$w(\phi(x, t_*(x))) = r(\phi(x, t_*(x)), \overline{u}(\phi(x, t_*(x)))) + Qh(\phi(x, t_*(x)), \overline{u}(\phi(x, t_*(x)))). \tag{4.75}$$

Combining Eqs. (4.69), (4.72), and (4.75), it follows that

$$\begin{aligned}\mathcal{R}_\alpha(\rho, h)(x) =& -\rho\mathcal{L}_\alpha(x, \widehat{u}_\phi(w, h)(x)) + L_\alpha f(x, \widehat{u}_\phi(w, h)(x)) \\ &+ H_\alpha r(x, \widehat{u}_\phi(w, h)(x)) + G_\alpha h(x, \widehat{u}_\phi(w, h)(x)).\end{aligned}$$

Consider now the case in which $t_*(x) = \infty$. From Eqs. (4.71) and (4.66), we obtain that

$$\begin{aligned}w(x) \geq& -e^{-(\alpha t+\overline{\Lambda}(x,t))}\big[(\rho + b\|h\|_g)K_\lambda + \|h\|_g g(\phi(x, t))\big] \\ &+ \int_0^t e^{-(\alpha s+\overline{\Lambda}(x,s))}\Big[f(\phi(x, s), \overline{u}(\phi(x, s))) \\ &+ \lambda(\phi(x, s), \overline{u}(\phi(x, s)))Qh(\phi(x, s), \overline{u}(\phi(x, s))) - \rho\Big]\mathrm{d}s. \qquad (4.76)\end{aligned}$$

However, from item (b) of Assumption 2.5 and Assumption 4.18 (i) and (ii), we obtain that

$$\lim_{t\to+\infty} e^{-(\alpha t+\overline{\Lambda}(x,t))}\big[(\rho + b\|h\|_g)K_\lambda + \|h\|_g g(\phi(x, t))\big] = 0. \qquad (4.77)$$

Consequently, combining Eqs. (4.67), (4.76), and (4.77), we see that the ordinary feedback measurable selector $\widehat{u}_\phi(w, h) \in \mathcal{S}_{\mathbb{V}}$ satisfies

$$\begin{aligned}w(x) \geq& -\rho + \mathcal{L}_\alpha(x, \widehat{u}_\phi(w, h)(x)) + L_\alpha f(x, \widehat{u}_\phi(w, h)(x)) + H_\alpha r(x, \widehat{u}_\phi(w, h)(x)) \\ &+ G_\alpha h(x, \widehat{u}_\phi(w, h)(x)).\end{aligned}$$

From Eq. (4.69), it follows that the inequality in the previous equation is in fact an equality.

Finally, in every case,

$$\begin{aligned}\mathcal{R}_\alpha(\rho, h)(x) =& -\rho\mathcal{L}_\alpha(x, \widehat{u}_\phi(w, h)(x)) + L_\alpha f(x, \widehat{u}_\phi(w, h)(x)) \\ &+ H_\alpha r(x, \widehat{u}_\phi(w, h)(x)) + G_\alpha h(x, \widehat{u}_\phi(w, h)(x)).\end{aligned}$$

Since $\mathbb{V}(x) \subset \mathbb{V}^r(x)$, it follows that $\mathcal{R}_\alpha(\rho, h)(x) \leq \mathcal{T}_\alpha(\rho, h)(x)$. However, we have shown that $\widehat{u}_\phi(w, h) \in \mathcal{S}_{\mathbb{V}}$, implying $\mathcal{R}_\alpha(\rho, h)(x) = \mathcal{T}_\alpha(\rho, h)(x)$, which gives the desired result. □

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Chapter 5
The Policy Iteration Algorithm for PDMPs

5.1 Outline of the Chapter

The main goal of this chapter is to apply the so-called policy iteration algorithm (PIA) for the long run average continuous control problem of PDMPs. The first step in this direction is to derive some important properties for a pseudo-Poisson equation associated with the problem. The next step is to show that the convergence of the PIA to a solution satisfying the optimality equation holds and that this optimal solution yields an optimal control strategy for the average control problem for the continuous-time PDMP in feedback form. To derive this result, we need to assume the same integrodifferential inequalities related to the so-called expected growth condition and geometric convergence of the postjump location kernel associated with the PDMP as seen in Sect. 4.13 (see Assumptions 4.13 and 4.14). Moreover, we need to assume a Lyapunov-like inequality condition (Assumption 5.2) and a convergence condition on the relative difference of the α-discount value functions (see Assumption 5.8). Section 5.2 introduces the assumptions that will be needed later on in this chapter and derives some important properties for a pseudo-Poisson equation associated with the problem. In Sect. 5.3, it is shown that the convergence of the PIA to a solution satisfying the optimality equation holds. Moreover, it is shown that this optimal solution yields an optimal control strategy for the average continuous control problem of a PDMP in feedback form.

5.2 Assumptions and a Pseudo-Poisson Equation

The two main results of this chapter, to be presented in Sect. 5.3 (Theorems 5.9 and 5.14), will rely on the assumptions of Chap. 2, Assumptions 3.5, 3.8, and 3.9 considered in Sect. 3.3, and Assumptions 4.13, 4.14, 4.15, and 4.18 considered in Sect. 4.4.1. Moreover, we assume the following:

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Assumption 5.1 *For all* $y \in \overline{E}$*, the restriction of* $f(y, .)$ *to* $\mathbb{U}(y)$ *is continuous and for all* $z \in \partial E$*, the restriction of* $r(z, .)$ *to* $\mathbb{U}(z)$ *is continuous.*

Notice that this assumption is a strengthened version of Assumptions 3.6 and 3.7. The following hypothesis is given by a Lyapunov-like inequality yielding an expected growth condition on the function g with respect to G (for further comments on this kind of assumption, see, for example, Sect. 10.2 in [51, p. 121]).

Assumption 5.2 *Let* g *be as in Assumption 4.13. There exist* $0 < k_g < 1$ *and* $K_g \geq 0$ *such that for all* $x \in E$, $\Gamma \in \mathbb{V}(x)$,

$$Gg(x, \Gamma) \leq k_g g(x) + K_g. \tag{5.1}$$

We introduce in Definition 5.3 a pseudo-Poisson equation associated with the stochastic kernel G. Proposition 5.5 shows that there exists a solution for such an equation. Moreover, it is proved in Proposition 5.6 that this equation has the important characteristic of ensuring the policy improvement property in the set $\mathcal{S}_{\mathbb{U}}$.

Definition 5.3 *Consider* $u \in \mathcal{S}_{\mathbb{U}}$*. A pair* $(\rho, h) \in \mathbb{R} \times \mathbb{B}_g(E)$ *is said to satisfy the pseudo-Poisson equation associated with* u *if*

$$h(x) = -\rho\mathcal{L}(x, u_\phi(x)) + Lf(x, u_\phi(x)) + Hr(x, u_\phi(x)) + Gh(x, u_\phi(x)). \tag{5.2}$$

Remark 5.4 This equation is clearly different from a classical Poisson equation encountered in the literature on discrete-time Markov control processes; see, for example, Eq. (2.13) in [50]. In particular, the constant ρ, which will be shown to be the optimal cost, appears here as a multiplicative factor of the mapping $\mathcal{L}(x, u_\phi(x))$, and the costs f and r appear through the terms $Lf(x, u_\phi(x))$ and $Hr(x, u_\phi(x))$. However, it will be shown in the following propositions that this pseudo-Poisson equation still has the good properties that we might expect it to satisfy in order to guarantee the convergence of the policy iteration algorithm.

Proposition 5.5 *For arbitrary* $u \in \mathcal{S}_{\mathbb{U}}$*, the following assertions hold:*

(a) *Set* $D_u = \int_E \mathcal{L}(y, u_\phi(y))\nu_u(dy)$*. Then* $0 < D_u \leq K_\lambda$.

(b) *If* $v \in \mathbb{B}_g(E)$ *and* $b \in \mathbb{R}$ *are such that for all* $x \in E$,

$$v(x) = b\mathcal{L}(x, u_\phi(x)) + Gv(x, u_\phi(x)), \tag{5.3}$$

then $b = 0$*, and for some* $c_0 \in \mathbb{R}$, $v(x) = c_0$ *for all* $x \in E$.

(c) *Define* (ρ_u, h_u) *by*

$$\rho_u = \frac{\int_E \big[Lf(y, u_\phi(y)) + Hr(y, u_\phi(y))\big]\nu_u(dy)}{D_u} \geq 0, \tag{5.4}$$

$$h_u(x) = \sum_{k=0}^{\infty} G^k w_u(x, u_\phi(x)), \tag{5.5}$$

where the mapping w_u in $\mathbb{M}(E)$ is given by

$$w_u(x) = Lf(x, u_\phi(x)) + Hr(x, u_\phi(x)) - \rho_u \mathcal{L}(x, u_\phi(x))$$

for $x \in E$. Then $(\rho_u, h_u) \in \mathbb{R} \times \mathbb{B}_g(E)$, and it is the unique solution to the Poisson equation (5.2) associated with u that satisfies

$$\nu_u(h_u) = 0. \tag{5.6}$$

Moreover,

$$\|h_u\|_g \le \frac{aM_u}{1-\kappa}, \text{ with } M_u := \max\left\{\rho_u K_\lambda, \frac{M(1+bK_\lambda)}{c}\right\}. \tag{5.7}$$

Proof Item (a) is easy to obtain. Indeed, from Remark 4.19 (ii), we get that $\mathcal{L}(x, u_\phi(x)) \le K_\lambda$, and from Assumption 2.5, it follows that $0 < \mathcal{L}(x, u_\phi(x))$ for every $x \in E$.

For (b), let us suppose that $b \ge 0$. Since $0 < \mathcal{L}(x, u_\phi(x))$ for all $x \in E$, it follows from (5.3) that $v(x) \ge Gv(x, u_\phi(x))$ for all $x \in E$, and from Lemma 4.1 (a) in [50], $v(x) = c_0$ ν_u-a.s. for some $c_0 \in \mathbb{R}$. Returning to (5.3) and integrating with respect to ν_u, we have that $0 = bD_u$, and so $b = 0$. Therefore, from (5.3), $v(x) = Gv(x, u_\phi(x))$, that is, v is a ν_u-harmonic function, and therefore $v(x) = c_0$ for all $x \in E$ (see Lemma 4.1 (a) in [50]). If $b < 0$, then from (5.3), it follows that $v(x) \le Gv(x, u_\phi(x))$ for all $x \in E$, and from Lemma 4.1 (a) in [50], $v(x) = c_0$ ν_u-a.s. for some $c_0 \in \mathbb{R}$. Returning to (5.3) and integrating with respect to ν_u, we have that $0 = bD_u$, and since $D_u > 0$, we have a contradiction.

For (c), we first note that from Theorem 4.29,

$$0 \le Lf(x, u_\phi(x)) + Hr(x, u_\phi(x)) \le \frac{M(1+bK_\lambda)}{c} g(x),$$

so that clearly,

$$\int_E \big[Lf(y, u_\phi(y)) + Hr(y, u_\phi(y))\big] \nu_u(dy) < \infty,$$

and thus ρ_u is well defined by item (a). Moreover, $0 \le \rho_u \mathcal{L}(x, u_\phi(x)) \le \rho_u K_\lambda$, and thus from Eq. 4.42, we have $w_u \in \mathbb{B}_g(E)$ with $\|w_u\|_g \le M_u$, where M_u is defined in (5.7). We also have from (5.4) that

$$\int_E w_u(y)\nu_u(dy) = \int_E \big[Lf(y, u_\phi(y)) + Hr(y, u_\phi(y))\big]\nu_u(dy) - \rho_u D_u$$
$$= 0, \tag{5.8}$$

and thus from (4.18),

$$|G^k w_u(x, u_\phi(x))| = |G^k w_u(x, u_\phi(x)) - \nu_u(w_u)| \leq aM_u\kappa^k g(x), \tag{5.9}$$

for all $x \in E$ and $k \in \mathbb{N}$. From (5.5) and (5.9), it is clear that

$$|h_u(x)| \leq \frac{aM_u}{1-\kappa} g(x), \tag{5.10}$$

showing that h_u is in $\mathbb{B}_g(E)$ and satisfies (5.6) and (5.7). We also have from (5.5) that

$$h_u(x) - w_u(x) = \sum_{k=1}^{\infty} G^k w_u(x, u_\phi(x)) = G_u h_u(x, u_\phi(x)),$$

showing that $(\rho_u, h_u) \in \mathbb{R} \times \mathbb{B}_g(E)$ satisfies (5.2).

If $(\rho_i, h_i) \in \mathbb{R} \times \mathbb{B}_g(E)$, $i = 1, 2$, are two solutions to the Poisson equation (5.2), then setting $v = h_1 - h_2$ and $b = \rho_2 - \rho_1$, we get that (5.3) is satisfied, and uniqueness follows from (b). □

From now on, (ρ_u, h_u) will denote the unique solution of the pseudo-Poisson equation (5.2) that satisfies $\nu_u(h_u) = 0$.

The properties given in the following proposition are important for showing the convergence of the PIA.

Proposition 5.6 *Consider $u \in \mathcal{S}_{\mathbb{U}}$. Then there exists $\widehat{u} \in \mathcal{S}_{\mathbb{U}}$ such that*

$$\mathcal{R}(\rho_u, h_u)(x) = -\rho_u \mathcal{L}(x, \widehat{u}_\phi(x)) + Lf(x, \widehat{u}_\phi(x)) + Hr(x, \widehat{u}_\phi(x))$$
$$+ Gh_u(x, \widehat{u}_\phi(x)), \tag{5.11}$$

and $\rho_{\widehat{u}} \leq \rho_u$.

Proof From Theorem 4.35, we have that there exists $\widehat{u} \in \mathcal{S}_{\mathbb{U}}$ such that (5.11) holds. Clearly, we have for every $x \in E$ that $h_u(x) \geq \mathcal{R}(\rho_u, h_u)(x)$, that is, from (5.11),

$$h_u(x) \geq -\rho_u \mathcal{L}(x, \widehat{u}_\phi(x)) + Lf(x, \widehat{u}_\phi(x)) + Hr(x, \widehat{u}_\phi(x)) + Gh_u(x, \widehat{u}_\phi(x)).$$

Integrating the previous equation with respect to $\nu_{\widehat{u}}$ and recalling the definition of D_u (see item (a) in Proposition 5.5) and

$$\int_E Gh_u(y, \widehat{u}_\phi(y))\nu_{\widehat{u}}(dy) = \int_E h_u(y)\nu_{\widehat{u}}(dy),$$

we get that

$$\int_E h_u(y)\nu_{\widehat{u}}(dy) \geq -\rho_u D_{\widehat{u}} + \rho_{\widehat{u}} D_{\widehat{u}} + \int_E h_u(y)\nu_{\widehat{u}}(dy),$$

that is, $\rho_u D_{\widehat{u}} \geq \rho_{\widehat{u}} D_{\widehat{u}}$, and since $D_{\widehat{u}} > 0$, we get that $\rho_u \geq \rho_{\widehat{u}}$. □

5.3 The Policy Iteration Algorithm

Having studied the pseudo-Poisson equation defined in Sect. 5.2, we are now in a position to analyze the policy iteration algorithm. In the first part of this section, it is shown that the convergence of the policy iteration algorithm holds under a classical hypothesis (see, for example, assumption (H1) of Theorem 4.3 in [50]). Roughly speaking, this means that if the PIA computes a solution (ρ_n, h_n) at the nth step, then $(\rho_n, h_n) \to (\rho, h)$ and (ρ, h) satisfies the optimality equation (5.15). However, it is far from obvious that ρ is actually the optimal cost for the long-run average cost problem of the PDMP $\{X(t)\}$ and that there exists an optimal control. In the second part of this section, these two issues are studied. In particular, we show that $\rho = \inf_{U \in \mathcal{U}} \mathcal{A}(U, x)$ and that the measurable selector $\widehat{u}_\phi$ of the optimality equation (5.15) provides an optimal control strategy of the feedback form $U_{\widehat{u}_\phi}$ for the process $\{X(t)\}$: $\inf_{U \in \mathcal{U}} \mathcal{A}(U, x) = \mathcal{A}(U_{\widehat{u}_\phi}, x)$.
The policy iteration algorithm performs the following steps:

S1: Initialize with an arbitrary $u_0 \in \mathcal{S}_{\mathbb{U}}$, and set $n = 0$.

S2: Policy Evaluation—At the nth iteration, consider $u_n \in \mathcal{S}_{\mathbb{U}}$ and evaluate $(\rho_n, h_n) \in \mathbb{R} \times \mathbb{B}_g(E)$, the (unique) solution of the Poisson equations (5.2), (5.6) given by (5.4) and (5.5), replacing u by u_n. Thus we have that

$$\begin{aligned} h_n(x) = &-\rho_n \mathcal{L}(x, (u_n)_\phi(x)) + Lf(x, (u_n)_\phi(x)) + Hr(x, (u_n)_\phi(x)) \\ &+ Gh_n(x, (u_n)_\phi(x)), \end{aligned} \tag{5.12}$$

with $\nu_{u_n}(h_n) = 0$.

S3: Policy Improvement—Determine $u_{n+1} \in \mathcal{S}_{\mathbb{U}}$ such that

$$\begin{aligned} \mathcal{R}(\rho_n, h_n)(x) = &-\rho_n \mathcal{L}(x, (u_{n+1})_\phi(x)) + Lf(x, (u_{n+1})_\phi(x)) \\ &+ Hr(x, (u_{n+1})_\phi(x)) + Gh_n(x, (u_{n+1})_\phi(x)). \end{aligned} \tag{5.13}$$

Notice that from Propositions 5.5 and 5.6, the sequences $(\rho_n, h_n) \in \mathbb{R}_+ \times \mathbb{B}_g(E)$ and $u_n \in \mathcal{S}_{\mathbb{U}}$ are well defined and moreover, $\rho_n \geq \rho_{n+1} \geq 0$. We set $\rho = \lim_{n\to\infty} \rho_n$.

5.3.1 Convergence of the PIA

First, we present in the next result some convergence properties of G, H, L, and $\mathcal{L}$.

Proposition 5.7 *Consider $h \in \mathbb{B}_g(E)$ and a sequence of functions $\big(h_k\big)_{k\in\mathbb{N}} \in \mathbb{B}_g(E)$ such that for all $x \in E$, $\lim_{k\to\infty} h_k(x) = h(x)$ and there exists K_h satisfying $|h_k(x)| \leq K_h g(x)$ for all k and all $x \in E$. For $x \in E$, consider $\Theta_n = \big(\mu_n, \mu_{\partial,n}\big) \in \mathbb{V}^r(x)$ and $\Theta = \big(\mu, \mu_\partial\big) \in \mathbb{V}^r(x)$ such that $\Theta_n \to \Theta$. We have the following results:*

$$(a) \lim_{n\to\infty} \mathcal{L}(x, \Theta_n) = \mathcal{L}(x, \Theta), \qquad (b) \lim_{n\to\infty} Lf(x, \Theta_n) = Lf(x, \Theta),$$
$$(c) \lim_{n\to\infty} Hr(x, \Theta_n) = Hr(x, \Theta), \qquad (d) \lim_{n\to\infty} Gh_n(x, \Theta_n) = Gh(x, \Theta).$$

Proof These results can be easily obtained by similar arguments as in Propositions 3.10 and 4.16. □

We shall consider now the following assumption.

Assumption 5.8 *There exists $h \in \mathbb{M}(E)$ such that for each $x \in E$,*

$$\lim_{n\to\infty} h_n(x) = h(x). \tag{5.14}$$

The following theorem is the first main result of this chapter. It shows the convergence of the PIA and ensures the existence of a measurable selector for the optimality equation.

Theorem 5.9 *We have that $(\rho, h) \in \mathbb{R} \times \mathbb{B}_g(E)$ satisfies the optimality equation*

$$h(x) = \mathcal{R}(\rho, h)(x). \tag{5.15}$$

Moreover, there exists $\widehat{u} \in \mathcal{S}_{\mathbb{U}}$ such that

$$h(x) = -\rho\mathcal{L}(x, \widehat{u}_\phi(x)) + Lf(x, \widehat{u}_\phi(x)) + Hr(x, \widehat{u}_\phi(x)) + Gh(x, \widehat{u}_\phi(x)). \tag{5.16}$$

Proof From (5.7) and recalling that $\rho_n \geq \rho_{n+1}$, we get that for all n,

$$\|h_n\|_g \leq \widetilde{M} := \frac{aM_{u_0}}{1-\kappa}, \quad M_{u_0} := \max\left\{\rho_0 K_\lambda, \frac{M(1+bK_\lambda)}{c}\right\}. \tag{5.17}$$

From (5.17), we get that $h \in \mathbb{B}_g(E)$, where h is as in (5.14). Consider $u_n \in \mathcal{S}_{\mathbb{U}}$, the measurable selector associated with (ρ_n, h_n) as in (5.12), that is,

$$h_n(x) = -\rho_n\mathcal{L}(x, (u_n)_\phi(x)) + Lf(x, (u_n)_\phi(x)) + Hr(x, (u_n)_\phi(x)) \\ + Gh_n(x, (u_n)_\phi(x)). \tag{5.18}$$

We have that for each $x \in E$, $\mathbb{V}^r(x)$ is compact and $\{(u_n)_\phi\}$ is a sequence in $\mathcal{S}_{\mathbb{V}^r}$. Then according to Proposition 8.3 in [50] (see also [61]), there exists $\Theta \in \mathcal{S}_{\mathbb{V}^r}$ such that $\Theta(x) \in \mathbb{V}^r(x)$ is an accumulation point of $\{(u_n)_\phi(x)\}$ for each $x \in E$. Therefore, for every $x \in E$, there exists a subsequence $n_i = n_i(x)$ such that $\lim_{i\to\infty}(u_{n_i})_\phi(x) = \Theta(x)$. We fix now $x \in E$, and we consider the subsequence $n_i = n_i(x)$ as above. From Proposition 5.7 and taking the limit in (5.18) for n_i as $i \to \infty$, we have that

$$h(x) = -\rho\mathcal{L}(x, \Theta(x)) + Lf(x, \Theta(x)) + Hr(x, \Theta(x)) + Gh(x, \Theta(x)), \quad (5.19)$$

and thus clearly $h(x) \geq \mathcal{R}(\rho, h)(x)$. On the other hand, from (5.12) and (5.13), we have that

$$\begin{aligned}\mathcal{R}(\rho_{n-1}, h_{n-1})(x) + (\rho_{n-1} - \rho_n)\mathcal{L}(x, (u_n)_\phi(x)) + G(h_n - h_{n-1})(x, (u_n)_\phi(x))\\ = -\rho_n\mathcal{L}(x, (u_n)_\phi(x)) + Lf(x, (u_n)_\phi(x)) + Hr(x, (u_n)_\phi(x))\\ + Gh_n(x, (u_n)_\phi(x)) = h_n(x). \qquad (5.20)\end{aligned}$$

From (5.20), it is immediate that for every $\widetilde{\Theta} \in \mathcal{S}_{\mathbb{V}^r}$,

$$\begin{aligned}h_n(x) \leq -\rho_{n-1}\mathcal{L}(x, \widetilde{\Theta}(x)) + Lf(x, \widetilde{\Theta}(x)) + Hr(x, \widetilde{\Theta}(x)) + Gh_{n-1}(x, \widetilde{\Theta}(x))\\ + (\rho_{n-1} - \rho_n)\mathcal{L}(x, (u_n)_\phi(x)) + G(h_n - h_{n-1})(x, (u_n)_\phi(x)). \qquad (5.21)\end{aligned}$$

Fix x and $n_i = n_i(x)$ as before, and notice that $\lim_{i\to\infty}(h_{n_i}(y) - h_{n_i-1}(y)) = 0$ for every $y \in E$ and that from (5.17), $\|h_{n_i} - h_{n_i-1}\|_g \leq \widetilde{M}$. Applying Proposition 5.7 to Eq. (5.21), replacing n by n_i and taking the limit as $i \to \infty$, yields that

$$h(x) \leq -\rho\mathcal{L}(x, \widetilde{\Theta}(x)) + Lf(x, \widetilde{\Theta}(x)) + Hr(x, \widetilde{\Theta}(x)) + Gh(x, \widetilde{\Theta}(x)), \quad (5.22)$$

and from (5.22), we get that $h(x) \leq \mathcal{R}(\rho, h)(x)$. Thus we have (5.15). □

Remark 5.10 A possible way of getting Assumption 5.8 would be to show that under some conditions, the sequence of functions $\{h_n\}$, or a modification thereof, is monotone or "nearly monotone" (see [55], p. 1667), so that $\{h_n\}$ itself satisfies (5.14). This approach was adopted for MDPs in [55] and [59], and it is possible that in some cases, similar arguments could be applied to PDMP.

5.3.2 Optimality of the PIA

Let us recall that the PDMP $\{\widehat{X}^U(t)\}$ and its associated components $X(t)$, $Z(t)$, $N(t)$, $\tau(t)$ were introduced in Sect. 2.2 (see in particular Eq. (2.2)). We need several auxiliary results (Propositions 5.11, 5.12 and Corollary 5.13) to show that the PIA actually provides an optimal solution for the average cost problem of the PDMP $\{X(t)\}$. With a slight abuse of notation, the definition of a shifted control strategy (see Definition 2.14) will be extended to every $\Theta = \big(u, u_\partial\big) \in \mathbb{V}$ by setting $\big[\Theta\big]_t = \big(u(. + t), u_\partial\big)$ for $s \in \mathbb{R}_+$ and every $\Theta = \big(u, u_\partial\big) \in \mathbb{V}$.

Proposition 5.11 *For* $\hat{y} = (y, z, s, n) \in \widehat{E}$ *and* $U = (u, u_\partial) \in \mathbb{M}(\mathbb{N} \times E \times \mathbb{R}_+; \mathbb{U}) \times \mathbb{M}(\mathbb{N} \times E; \mathbb{U})$*, define* $\Gamma^U(n, z) = \big(u(n, z, .), u_\partial(n, z)\big) \in \mathbb{V}$*. For* $\epsilon \in (0, c)$*, introduce*

$$\widehat{w}^U(\hat{y}) = \overline{c} L_{-\epsilon} f(y, \big[\Gamma^U(n, z)\big]_s) + H_{-\epsilon}\overline{r}(y, \big[\Gamma^U(n, z)\big]_s) + G_{-\epsilon} g(y, \big[\Gamma^U(n, z)\big]_s) - b\mathcal{L}_{-\epsilon}(y, \big[\Gamma^U(n, z)\big]_s), \tag{5.23}$$

where $\overline{c} = c - \epsilon$ *and* $\mathcal{L}_{-\epsilon}(y, \big[\Gamma^U(n, z)\big]_s)$ *is finite. Then for all* $x \in E$*,* $U \in \mathcal{U}$*, we have*

$$E^U_{(x,0)}\Big[\widehat{w}^U\big(\widehat{X}^U(t)\big)\Big] \le e^{-\epsilon t} g(x) + \frac{b}{\epsilon}\big[1 - e^{-\epsilon t}\big]. \tag{5.24}$$

Proof First for notational convenience, we introduce the following definition: $\widehat{f}^U(\hat{y}) = f(y, u(n, z, s))$, $\widehat{r}^U(\hat{y}) = \overline{r}(y, u_\partial(n, z))$, $\widehat{g}(\hat{y}) = g(y)$, and $\widehat{\Lambda}^U(y, t) = \Lambda^U(x, n, t)$, for $\hat{y} = (y, z, s, n) \in \widehat{E}$, $U = (u, u_\partial) \in \mathbb{M}(\mathbb{N} \times E \times \mathbb{R}_+; \mathbb{U}) \times \mathbb{M}(\mathbb{N} \times E; \mathbb{U})$, and $t \in \mathbb{R}_+$.

It can be shown that $\widehat{w}^U \in \mathbb{M}(\widehat{E})$. Indeed, for $U = (u, u_\partial) \in \mathbb{M}(\mathbb{N} \times E \times \mathbb{R}_+; \mathbb{U}) \times \mathbb{M}(\mathbb{N} \times E; \mathbb{U})$, the $\mathbb{V}$-valued mapping defined on $\widehat{E}$ by $\hat{y} = (y, z, s, n) \to \big[\Gamma^U(n, z)\big]_s$ is measurable, and so by Proposition 2.17, we have the desired measurability property of $\widehat{w}^U$. Consider $\hat{y} = (y, z, s, n) \in \widehat{E}$ and $U = (u, u_\partial) \in \mathbb{M}(\mathbb{N} \times E \times \mathbb{R}_+; \mathbb{U}) \times \mathbb{M}(\mathbb{N} \times E; \mathbb{U})$, satisfying $\big[\Gamma^U(n, z)\big]_s \in \mathbb{V}(y)$. Notice that $\mathcal{L}_{-\epsilon}(y, \big[\Gamma^U(n, z)\big]_s)$ is finite, since from Remark 4.19 (ii),

$$0 < \mathcal{L}_{-\epsilon}(y, \big[\Gamma^U(n, z)\big]_s) \le \mathcal{L}_{-c}(y, \big[\Gamma^U(n, z)\big]_s) \le K_\lambda. \tag{5.25}$$

Moreover, we have by similar arguments as in Corollary 4.28 that

$$\overline{c} L_{-\epsilon} f(y, \big[\Gamma^U(n, z)\big]_s) + H_{-\epsilon}\overline{r}(y, \big[\Gamma^U(n, z)\big]_s) + G_{-\epsilon} g(y, \big[\Gamma^U(n, z)\big]_s) - b\mathcal{L}_{-\epsilon}(y, \big[\Gamma^U(n, z)\big]_s) \le g(y). \tag{5.26}$$

From now on, consider $U = (u, u_\partial) \in \mathcal{U}$. For every $\hat{x} = (x, x, 0, k) \in \widehat{E}$, we get from Eq. (5.25) that $\mathcal{L}_{-\epsilon}(x, \big[\Gamma^U(k, x)\big]_0)$ is finite, since $\big[\Gamma^U(k, x)\big]_0 = (u(k, x, .), u_\partial(k, x)) \in \mathbb{V}(x)$ and so $\widehat{w}^U(\widehat{x})$ is well defined by

$$\begin{aligned}
\widehat{w}^U(\widehat{x}) &= \overline{c} L_{-\epsilon} f(x, \Gamma^U(k, x)) + H_{-\epsilon}\overline{r}(y, \Gamma^U(k, x)) \\
&\quad + G_{-\epsilon} g(x, \Gamma^U(k, x)) - b\mathcal{L}_{-\epsilon}(x, \Gamma^U(k, x)) \\
&= \int_0^{t_*(x)} e^{\epsilon s - \Lambda^{\nu_k}(x,s)} \Big[-b + \overline{c} f(\phi(x, s), \nu_k(s))
\end{aligned}$$

$$+ \lambda(\phi(x,s),\nu_k(s))Qg(\phi(x,s),\nu_k(s))\Big]ds$$
$$+ e^{\epsilon t_*(x)-\Lambda^{\nu_k}(x,t_*(x))}\Big[Qg(\phi(x,t_*(x)),u_\partial(k,x))$$
$$+\overline{r}(\phi(x,t_*(x)),u_\partial(k,x))\Big], \tag{5.27}$$

with $\nu_k(.) = u(k,x,.)$. Since for all $k \in \mathbb{N}$, $x \in E$, $\Gamma^U(k,x) \in \mathbb{V}(x)$, it follows from Eqs. (5.26) and (5.25) that

$$-bK_\lambda \leq \widehat{w}^U(\widehat{x}) \leq g(x). \tag{5.28}$$

Moreover, since $\big[\Gamma^U(N(t),Z(t))\big]_{\tau(t)} \in \mathbb{V}(X(t))$, the inequality (5.25) implies that

$$0 \leq \mathcal{L}_{-\epsilon}\big(X(t),\big[\Gamma^U(N(t),Z(t))\big]_{\tau(t)}\big) \leq K_\lambda,$$

and so $\widehat{w}^U\big(\widehat{X}^U(t\wedge T_m) \geq -bK_\lambda$

$$J_m^U(t,\hat{x}) := E^U_{(x,k)}\Bigg[\int_0^{t\wedge T_m} e^{\epsilon s}\Big[\overline{c}\widehat{f}^U(\widehat{X}^U(s)) - b\Big]ds$$
$$+\int_0^{t\wedge T_m} e^{\epsilon s}\widehat{r}^U\big(\widehat{X}^U(s-)\big)dp^*(s) + e^{\epsilon(t\wedge T_m)}\widehat{w}^U\big(\widehat{X}^U(t\wedge T_m)\big)\Bigg],$$

is well defined for every $\hat{x} = (x,x,0,k) \in \widehat{E}$.

Let us show by induction on $m \in \mathbb{N}$ that $J_m^U(t,\hat{x}) \leq g(x)$ for all $t \in \mathbb{R}_+$, $\hat{x} = (x,x,0,k) \in \widehat{E}$. Clearly, we have that $J_0^U(t,\hat{x}) = \widehat{w}^U(\widehat{x})$. Consequently, from Eq. (5.28), we have that $J_0^U(t,\hat{x}) \leq g(x)$ for all $t \in \mathbb{R}_+$, $\hat{x} = (x,x,0,k) \in \widehat{E}$. Now assume that for $m \in \mathbb{N}$, we have that $J_m^U(t,\hat{x}) \leq g(x)$ for all $t \in \mathbb{R}_+$, $\hat{x} = (x,x,0,k) \in \widehat{E}$. Following the same arguments as in the proof of Proposition 3.3, it is easy to show that for $t \in \mathbb{R}_+$,

$$J_{m+1}^U(t,\hat{x}) \leq \int_0^{t\wedge t_*(x)} e^{\epsilon s-\Lambda^{\nu_k}(x,s)}\Big[-b+\overline{c}f(\phi(x,s),\nu_k(s))$$
$$+ \lambda(\phi(x,s),\nu_k(s))Qg(\phi(x,s),\nu_k(s))\Big]ds$$
$$+ I_{\{t\geq t_{*(x)}\}}e^{\epsilon t_*(x)-\Lambda^{\nu_k}(x,t_*(x))}\Big[Qg(\phi(x,t_*(x)),u_\partial(k,x))$$
$$+\overline{r}(\phi(x,t_*(x)),u_\partial(k,x))\Big] + I_{\{t<t_{*(x)}\}}e^{\epsilon t-\Lambda^{\nu_k}(x,t)}\widehat{w}^U(\widehat{\phi}(\hat{x},t)). \tag{5.29}$$

Now if $t < t_*(x)$, then by the fact that $\widehat{\phi}(\hat{x}, t) = \big(\phi(x,t), x, t, k\big)$, we get by the definition of $\widehat{w}^U$ (see Eq. (5.23)) that

$$\widehat{w}^U(\widehat{\phi}(\hat{x}, t)) = \overline{c} L_{-\epsilon} f(\phi(x,t), \big[\Gamma^U(k,x)\big]_t) + H_{-\epsilon}\overline{r}(\phi(x,t), \big[\Gamma^U(k,x)\big]_t)$$
$$+ G_{-\epsilon} g(\phi(x,t), \big[\Gamma^U(k,x)\big]_t) - b\mathcal{L}_{-\epsilon}(\phi(x,t), \big[\Gamma^U(k,x)\big]_t),$$

and it follows by Lemma 3.16 that

$$\widehat{w}^U(\hat{x}) = \int_0^t e^{\epsilon s - \Lambda^{\nu_k}(x,s)} \Big[-b + \overline{c} f(\phi(x,s), \nu_k(s))$$
$$+ \lambda(\phi(x,s), \nu_k(s)) Q g(\phi(x,s), \nu_k(s)) \Big] ds$$
$$+ e^{\epsilon t - \Lambda^{\nu_k}(x,t)} \widehat{w}^U(\widehat{\phi}(\hat{x}, t)). \tag{5.30}$$

Therefore, combining Eqs. (5.29) and (5.30), we get that

$$J^U_{m+1}(t, \hat{x}) \le \widehat{w}^U(\hat{x}),$$

and by Eq. (5.28), we have that $J^U_{m+1}(t, \hat{x}) \le g(x)$.

If $t \ge t_*(x)$, then Eqs. (5.27) and (5.29) yield $J^U_m(t, \hat{x}) \le \widehat{w}^U(\hat{x})$. By Eq. (5.28), we have $J^U_m(t, \hat{x}) \le g(x)$, showing that for all $m \in \mathbb{N}$, $J^U_m(t, \hat{x}) \le g(x)$ for all $t \in \mathbb{R}_+$, $\hat{x} = (x, x, 0, k) \in \widehat{E}$.

Consequently, this implies that

$$-bE^U_{(x,0)}\left[\int_0^{t \wedge T_m} e^{\epsilon s} ds \right] + E^U_{(x,0)}\Big[e^{\epsilon(t \wedge T_m)} \widehat{w}^U\big(\widehat{X}^U(t \wedge T_m)\big)\Big] \le g(x).$$

Similarly to the proof of Lemma 4.23, we obtain that

$$-\frac{b}{\epsilon}\big[e^{\epsilon t} - 1\big] + e^{\epsilon t} E^U_{(x,0)}\Big[\widehat{w}^U\big(\widehat{X}^U(t)\big)\Big] \le g(x),$$

establishing the result. □

Proposition 5.12 *For all $x \in E$, $U \in \mathcal{U}$, we have that*

$$E^U_{(x,0)}\Big[\widehat{w}^U\big(\widehat{X}^U(t \wedge T_m)\big)\Big]$$

exists in $\mathbb{R}_+$ for every $(t, m) \in \mathbb{R}_+ \times \mathbb{N}$ and

$$\overline{\lim_{t \to \infty}} \frac{1}{t} \overline{\lim_{m \to \infty}} E^U_{(x,0)}\Big[\widehat{w}^U\big(\widehat{X}^U(t \wedge T_m)\big)\Big] = 0. \tag{5.31}$$

Proof Clearly, we have

$$E^U_{(x,0)}\Big[\widehat{w}^{\,U}\big(\widehat{X}^U(t\wedge T_m)\big)\Big] = E^U_{(x,0)}\Big[I_{\{t<T_m\}}\widehat{w}^{\,U}\big(\widehat{X}^U(t)\big)\Big] + E^U_{(x,0)}\Big[I_{\{t\geq T_m\}}\widehat{w}^{\,U}\big(\widehat{X}^U(T_m)\big)\Big],$$

and thus by Remark 4.19 (ii),

$$0 \leq E^U_{(x,0)}\Big[\widehat{w}^{\,U}\big(\widehat{X}^U(t\wedge T_m)\big)\Big] \leq E^U_{(x,0)}\Big[\widehat{w}^{\,U}\big(\widehat{X}^U(t)\big)\Big] + E^U_{(x,0)}\Big[\widehat{w}^{\,U}\big(\widehat{X}^U(T_m)\big)\Big] + bK_\lambda. \tag{5.32}$$

Iterating Assumption 5.2, we obtain that for all $m \in \mathbb{N}$,

$$E^U_{(x,0)}\Big[\widehat{w}^{\,U}\big(\widehat{X}^U(T_m)\big)\Big] \leq g(x) + \frac{K_g}{1-k_g}.$$

The result follows by combining Eqs. (5.24) and (5.32) with the previous inequality. □

Corollary 5.13. *For all $U \in \mathcal{U}$,*

$$\overline{\lim_{t\to\infty}}\frac{1}{t}\overline{\lim_{m\to\infty}} E^U_{(x,0)}\Big[h\big(X(t\wedge T_m)\big)\Big] \leq 0 \tag{5.33}$$

and

$$\overline{\lim_{t\to\infty}}\frac{1}{t}\overline{\lim_{m\to\infty}} -E^{U_{\widehat{u}_\phi}}_{(x,0)}\Big[h\big(X(t\wedge T_m)\big)\Big] \leq 0. \tag{5.34}$$

Proof From Eq. (5.16), it follows that for all $x \in E$, $\Gamma \in \mathbb{V}(x)$,

$$-\rho\mathcal{L}(x,\widehat{u}_\phi(x)) + Gh(x,\widehat{u}_\phi(x)) \leq h(x) \leq Lf(x,\Gamma) + Hr(x,\Gamma) + Gh(x,\Gamma). \tag{5.35}$$

Consequently, by Remark 4.19 (ii), the definition of $\widehat{w}$, and Assumption 4.13, we obtain that there exists $M_1 > 0$ such that for every $U \in \mathcal{U}$,

$$h\big(X(t\wedge T_m)\big) \leq M_1\Big[\widehat{w}^{\,U}\big(\widehat{X}^U(t\wedge T_m)\big) + bK_\lambda\Big].$$

Consequently, combining the previous equation and (5.31), we obtain Eq. (5.33). Moreover, notice that $\big[\Gamma^{U_{\widehat{u}_\phi}}(N(t), Z(t))\big]_{\tau(t)} = \widehat{u}_\phi(X(t)) \in \mathbb{V}(X(t))$ and so Eq. (5.35) and the definition of $\widehat{w}^{\,U}$ (see Eq. (5.23)) give that

$$\|h\|_g\Big[\widehat{w}^{U_{\widehat{u}_\phi}}\big(\widehat{X}^{U_{\widehat{u}_\phi}}(t\wedge T_m)\big) + bK_\lambda\Big] + \rho K_\lambda \geq -h\big(X(t\wedge T_m)\big),$$

and so

$$\overline{\lim_{t\to\infty}} \frac{1}{t} \overline{\lim_{m\to\infty}} -E^{U_{\widehat{u}_\phi}}_{(x,0)}\Big[h\big(X(t\wedge T_m)\big)\Big]$$
$$\leq \|h\|_g \overline{\lim_{t\to\infty}} \frac{1}{t} \overline{\lim_{m\to\infty}} E^{U_{\widehat{u}_\phi}}_{(x,0)}\Big[\widehat{w}^{U_{\widehat{u}_\phi}}\big(\widehat{X}^{U_{\widehat{u}_\phi}}(t\wedge T_m)\big)\Big].$$

The result follows by combining the previous inequality with (5.31). □

Finally, we can now present the second main result of this chapter. It states that the measurable selector $\widehat{u}_\phi$ of the optimality equation (5.15) associated with (ρ, h) gives an optimal feedback control strategy $U_{\widehat{u}_\phi}$ for the process $\{X(t)\}$.

Theorem 5.14 *The control $U_{\widehat{u}_\phi}$ is an optimal control strategy for the long-run average control problem*

$$\rho = \inf_{U\in\mathcal{U}} \mathcal{A}(U, x) = \mathcal{A}(U_{\widehat{u}_\phi}, x),$$

for all $x \in E$.

Proof From Proposition 5.12, we have that $E^{U}_{(x,0)}\Big[h\big(X(t\wedge T_m)\big)\Big]$ is well defined. Therefore, following the same arguments as in Proposition 3.3, it can be shown that

$$E^{U}_{(x,0)}\Bigg[\int_0^{t\wedge T_m} f\big(X(s), u(N(s), Z(s), \tau(s))\big)ds$$
$$+ \int_0^{t\wedge T_m} r\big(X(s-), u_\partial(N(s-), X(s-))\big)dp^*(s)\Bigg]$$
$$+ E^{U}_{(x,0)}\Big[h\big(X(t\wedge T_m)\big)\Big] \geq E^{U}_{(x,0)}\Big[\rho[t\wedge T_m]\Big] + h(x),$$

where $U = \big(u, u_\partial\big) \in \mathcal{U}$. From Eq. (5.33), it follows that

$$\overline{\lim_{t\to\infty}} \frac{1}{t} E^{U}_{(x,0)}\Bigg[\int_0^{t} f\big(X(s), u(N(s), Z(s), \tau(s))\big)ds$$
$$+ \int_0^{t} r\big(X(s-), u_\partial(N(s-), X(s-))\big)dp^*(s)\Bigg] \geq \rho,$$

showing that $\inf_{U\in\mathcal{U}} \mathcal{A}(U, x) \geq \rho$.

By the same arguments as in the proof of Proposition 3.4, we obtain that

$$E_{(x,0)}^{U_{\widehat{u}_\phi}}\left[\int_0^{t\wedge T_m} f\big(X(s),\widehat{u}(X(s))\big)ds + \int_0^{t\wedge T_m} r\big(X(s-),\widehat{u}(X(s-))\big)dp^*(s)\right]$$
$$\leq E_{(x,0)}^{U_{\widehat{u}_\phi}}\Big[\rho[t\wedge T_m]\Big] + h(x) - E_{(x,0)}^{U_{\widehat{u}_\phi}}\Big[h\big(X(t\wedge T_m)\big)\Big].$$

Now from Eq. (5.34), we get that

$$\overline{\lim_{t\to\infty}}\,\frac{1}{t}E_{(x,0)}^{U_{\widehat{u}_\phi}}\left[\int_0^{t} f\big(X(s),\widehat{u}(X(s))\big)ds + \int_0^{t} r\big(X(s-),\widehat{u}(X(s-))\big)dp^*(s)\right]$$
$$\leq \rho + \overline{\lim_{t\to\infty}}\,\frac{1}{t}\,\overline{\lim_{m\to\infty}}\,-E_{(x,0)}^{U_{\widehat{u}_\phi}}\Big[h\big(X(t\wedge T_m)\big)\Big] \leq \rho,$$

and so $\inf\limits_{U\in\mathcal{U}}\mathcal{A}(U,x)\leq\rho$. Therefore, it follows that

$$\rho = \inf_{U\in\mathcal{U}}\mathcal{A}(U,x) = \mathcal{A}(U_{\widehat{u}_\phi},x)$$

for all $x\in E$. □

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Chapter 6
Examples

6.1 Outline of the Chapter

This chapter will present three examples illustrating the possible applications of the results in Chap. 4. They are based on the capacity expansion model, analyzed in [26] and [25, Example (34.45)], and by the authors from the stability point of view in [15, 33]. Typical examples are electrical power generating stations, water resource facilities, computer and communication systems, and large manufacturing facilities. The interested reader may consult the references [26, 53] for a survey on capacity expansion including theoretical results and applications. For the first example, it is shown that from Proposition 4.9, there exists a solution for the discrete-time optimality inequality, and from Theorem 4.10, there exists an ordinary optimal feedback control for the long-run average cost problem. Moreover, it illustrates how the setup developed in this section could cover some problems in which it is desired to control a flow. For the second example, it is shown that Assumption 4.11 is satisfied, so that from Proposition 4.12, there exists a solution for the discrete-time optimality equation. The third example satisfies the assumptions of Sect. 4.4, so that the results obtained in Sect. 4.4.2 can be applied. Some numerical procedures are also presented.

6.2 The Capacity Expansion Problem

Capacity expansion consists in general processes of adding facilities to meet a rising demand. This demand is met by consecutive construction of expansion projects. A point process models the arrivals of the demand with intensity λ, which could depend on the present level of demand and on the control variable. At each arrival, the demand increases by one unit. The construction of a new project is accomplished at a rate γ (which could also depend on the control variable), and it is completed after the cumulative investment in the current project reaches a value τ. On completion, the present level of demand is reduced by κ units. We will assume in the next examples

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$\kappa = 1$ (without loss of generality) and that τ does not depend on the present level of demand. We consider that when there is no demand, no construction will take place.

We will present now the modeling of the control problem that will be considered in Sect. 4.3. Another capacity expansion control problem will be considered in Sect. 4.4, with the modeling following very closely the example that will be presented next.

In the example of Sect. 4.3, we will assume that λ is constant and that the construction of a new project is done at one of the possible rates γ_j per unit of time, $j = 1, \ldots, \iota$. The PDMP $\{X(t)\}$ with the state space $E \subset \mathbb{R}^3$ is defined as follows. We take $x_3 = 0$ to denote that no construction is taking place and the process just waits for a new arrival. In this situation, the vector $x = (x_1, x_2, x_3) \in E$ has the following interpretation: $x_1 = 0$ (this is just for notational convenience), $x_2 \in \mathbb{N}$ represents the demand level, and $x_3 = 0$ represents that no construction is taking place. The other possibility is that construction proceeds at the rate γ_j, and we denote this situation again by a 3-dimensional vector $x = (x_1, x_2, x_3) \in E$, with $x_1 \in [0, \tau)$ denoting the amount of cumulative investment in the current project, $x_2 \in \{1, 2, 3, \ldots\}$ representing the demand level (note that by assumption, no construction takes place whenever there is no demand level, that is, $x_2 = 0$), and $x_3 = j$ representing that construction is taking place at the rate γ_j. On completion of a new project, the control variable a will act on the transition measure at the boundary to send the process either to the "no construction" space (in this case $a = x_3 = 0$) or to the "construction space" at the rate γ_j (in this case $a = x_3 = j$). We assume that once a project starts, it cannot be interrupted or to have the rate changed.

On the "no construction" space, at the arrival of a new demand, the control variable a will act on the transition measure either to start a new project at the rate γ_j ($a = x_3 = j$) or not start ($a = x_3 = 0$) a new project. We consider that a running cost per unit of time $C(i) \geq 0$ is paid whenever the demand level is at $i \in \mathbb{N}$, with $C(i+1) \geq C(i)$ and $C(0) = 0$ (this represents no loss of generality, since the addition of a constant to the running cost does not alter the optimal solution of the problem). Moreover, a "construction" cost $K(j) \geq 0$ per unit of time is paid whenever construction is proceeding at the rate γ_j. For convenience, we set $K(0) = 0$. We also assume that there is an upper bound for the costs, that is, $C(i) \leq \bar{C}$ for some $\bar{C} \geq 0$ and all $i \in \mathbb{N}$, and that the stability condition $\lambda \frac{\tau}{\gamma_j} < 1$, $j = 1, \ldots, \iota$, is satisfied (see Proposition 34.46 of [25]).

The goal is to choose the optimum demand level at which it is worth constructing at a rate in the set $\{\gamma_j; j = 1, \ldots, \iota\}$. By deciding through the transition measure Q whether either $a = 0 = x_3$ (no construction) or $a = j = x_3$ (start construction at the rate γ_j), we are actually controlling the flow in a "bang-bang" fashion. It is easy to see that for this problem, whenever $x_3 = 0$, we have $t_*(x) = \infty$, $\phi(x,t) = x$, and whenever $x_3 = j$, we have $t_*(x) = \frac{\tau - x_1}{\gamma_j}$, $\phi(x,t) = (x_1 + \gamma_j t, x_2, j)$. When $x_3 = j$, $j \in \{1, \ldots, \iota\}$, the control variable acts on the transition measure Q at the frontier, so that at the point $x = (\tau, i, j)$ and with action $a \in \{0, 1, \ldots, \iota\}$, we have $Q(x, a; (0, i-1, a)) = 1$, except for the case $i = 1$, in which the only possible action is $a = 0$ (by hypothesis, no construction takes place when there is no demand). Since a running project cannot be interrupted, we have that there is no control for the points $x = (\ell, i, j)$, $0 \le \ell < \tau$, and in this case, $Q(x; (\ell, i+1, j)) = 1$. When $x_3 = 0$, the control variable acts on the transition measure Q so that at the point $x = (0, i, 0)$ and with action $a \in \{0, 1, \ldots, \iota\}$, we have $Q(x, a; (0, i+1, a)) = 1$.

Consequently, E is defined by $\{x \in \mathbb{R}^3 : x = (0, i, 0)$ or $x = (\ell, i', j)$ for $i \in \mathbb{N}$, $i' \in \mathbb{N}_*$, $\ell \in [0, \tau)\}$, $j \in \{1, \ldots, \iota\}$, $\phi\big((0, i, 0), t\big) = (0, i, 0)$, $t_*\big((0, i, 0)\big) = \infty$ for $i \in \mathbb{N}$, and

$$\phi\big((\ell, i', j), t\big) = (\ell + \gamma_j t, i', j), \quad t_*\big((\ell, i', j)\big) = \frac{\tau - \ell}{\gamma_j}$$

for $i' \in \mathbb{N}_*$, $\ell + \gamma_j t \in [0, \tau)$. Moreover, λ is a constant, and for every $A \in \mathcal{B}(E)$, $i \in \mathbb{N}$, we have for $j \in \{1, \ldots, \iota\}$,

$$Q\big((0, i, 0), a; A\big) = I_{\{a=0\}}\delta_{\{(0,i+1,0)\}}(A) + I_{\{a=j\}}\delta_{\{(0,i+1,j)\}}(A),$$

and for $\ell \in [0, \tau)$, $i \in \mathbb{N}_*$,

$$Q\big((\ell, i, j), j; A\big) = \delta_{\{(\ell,i+1,j)\}}(A),$$

for $\ell = \tau$, $i \ge 2$,

$$Q\big((\tau, i, j), a; A\big) = \delta_{\{(0,i-1,a)\}}(A),$$

and finally, for $\ell = \tau$, $i = 1$,

$$Q\big((\tau, i, j), a; A\big) = \delta_{\{(0,0,0)\}}(A).$$

Furthermore, we have $f(x, a) = C(x_2) + K(x_3)$ and $r(x, a) = 0$.

6.3 First Example

6.3.1 Verification of the Assumptions in Sect. 4.3

The goal of this subsection is to show that the assumptions in Sect. 4.3 are satisfied for the capacity expansion problem posed in Sect. 6.2. It is easy to check that the assumptions of Chaps. 2 and 3 are satisfied. One needs only to check that Assumption 4.7 is satisfied in order to show that there exist a solution to the discrete-time optimality inequality and an ordinary optimal feedback control for the long-run average cost problem.

Let the functions v_m^α be as defined in Sect. 4.2, where we write the superscript α to highlight the dependence on the discount factor $\alpha > 0$. Set $K = \max\{K(j); j = 1, \ldots, \iota\}$ and $\mathcal{I} = \{0, 1, \ldots, \iota\}$. Clearly, we have that

$$\mathcal{J}_{\mathcal{D}}^{\alpha}(x) \leq \frac{1}{\alpha}(\bar{C} + K), \tag{6.1}$$

and as seen in Sect. 4.2, $v_m^\alpha \uparrow v^\alpha = \mathcal{J}_{\mathcal{D}}^{\alpha}$. It is also easy to see that (recall that $v_0^\alpha(x) = 0$) for $m = 0, 1, \ldots,$

$$v_{m+1}^\alpha((0, i, 0)) = \frac{C(i)}{\lambda + \alpha} + \left(\frac{\lambda}{\lambda + \alpha}\right) \min_{a \in \mathcal{I}} v_m^\alpha((0, i+1, a)), i = 0, 1, \ldots, \tag{6.2}$$

$$\begin{aligned} v_{m+1}^\alpha((\ell, i, j)) =& \frac{C(i) + K(j)}{\lambda + \alpha}(1 - e^{-(\lambda+\alpha)\left(\frac{\tau-\ell}{\gamma_j}\right)}) \\ &+ \lambda \int_0^{\frac{\tau-\ell}{\gamma_j}} e^{-(\lambda+\alpha)s} v_m^\alpha((\ell + \gamma_j s, i+1, j)) \mathrm{d}s \\ &+ e^{-(\lambda+\alpha)\left(\frac{\tau-\ell}{\gamma_j}\right)} \min_{a \in \mathcal{I}} v_m^\alpha((0, i-1, a)), i = 2, 3, \ldots, \end{aligned} \tag{6.3}$$

$$\begin{aligned} v_{m+1}^\alpha((\ell, 1, j)) =& \frac{C(1) + K(j)}{\lambda + \alpha}(1 - e^{-(\lambda+\alpha)\left(\frac{\tau-\ell}{\gamma_j}\right)}) \\ &+ \lambda \int_0^{\frac{\tau-\ell}{\gamma_j}} e^{-(\lambda+\alpha)s} v_m^\alpha((\ell + \gamma_j s, 2, j)) \mathrm{d}s \\ &+ e^{-(\lambda+\alpha)\left(\frac{\tau-\ell}{\gamma_j}\right)} v_m^\alpha((0, 0, 0)). \end{aligned} \tag{6.4}$$

Proposition 6.1 *For all $x = (x_1, x_2, x_3) \in E$, $j = 1, \ldots, \iota$, and $m = 0, 1, \ldots,$ we have that*

$$v_m^\alpha((x_1, x_2 + 1, x_3)) \geq v_m^\alpha((x_1, x_2, x_3)), \quad v_m^\alpha((0, 1, j)) \geq v_m^\alpha((0, 0, 0)),$$

and consequently,

$$v^\alpha((x_1, x_2 + 1, x_3)) \geq v^\alpha((x_1, x_2, x_3)), \quad v^\alpha((0, 1, j)) \geq v^\alpha((0, 0, 0)).$$

Proof For $m = 0$, the result is immediate. Suppose it holds for m. We notice from (6.2) and $C(0) = 0$ that

$$v_{m+1}^\alpha((0, 0, 0)) \leq v_{m+2}^\alpha((0, 0, 0)) \leq v_{m+1}^\alpha((0, 1, j)).$$

By the induction hypothesis, it is easy to see that

$$v_{m+1}^\alpha((x_1, x_2 + 1, x_3)) \geq v_{m+1}^\alpha((x_1, x_2, x_3))$$

for all $x = (x_1, x_2, x_3) \in E$, completing the proof. □

Proposition 6.2 *For* $i = 1, 2, \ldots$, $j = 1, \ldots, \iota$, *and* $0 \le s \le t < \tau$,

$$v^{\alpha}((t, i, j)) - v^{\alpha}((s, i, j)) \le \frac{1}{\gamma_j}\Big((\bar{C} - C(i)) + (K - K(j))\Big)(t - s). \qquad (6.5)$$

Proof As seen in Sect. 4.2, v^{α} satisfies the equation $v^{\alpha}(x) = \mathcal{R}_{\alpha}(0, v^{\alpha})(x)$. Since v^{α} is positive, it follows from Theorem 3.14 that $v^{\alpha} \in \mathbb{M}^{ac}(E)$, and there exists a feedback measurable selector $\widehat{u}_{\phi}(w, v^{\alpha}) \in \mathcal{S}_{\mathbb{V}}$ such that (3.5) is satisfied for $\hat{\Gamma}(x) = \widehat{u}_{\phi}(w, v^{\alpha})(x)$. From Proposition 3.2 and by differentiating (3.6), it follows that

$$\mathcal{X} v^{\alpha}((t, i, j)) - (\lambda + \alpha) v^{\alpha}((t, i, j)) + K(j) + C(i) + \lambda v^{\alpha}((t, i + 1, j)) = 0. \quad (6.6)$$

Recalling that $\alpha v^{\alpha}(x) \le \bar{C} + K$ and from Proposition 6.1 that

$$v^{\alpha}((\ell, i, j)) - v^{\alpha}((\ell, i + 1, j)) \le 0,$$

we have from (6.6) that

$$\begin{aligned}
v^{\alpha}((\gamma_j t', i, j)) - v^{\alpha}((\gamma_j s', i, j)) &= \int_{s'}^{t'} \mathcal{X} v^{\alpha}((\gamma_j \ell, i, j)) \mathrm{d}\ell \\
&= \int_{s'}^{t'} \Big(\alpha v^{\alpha}((\gamma_j \ell, i, j)) - (C(i) + K(j))\Big) \mathrm{d}\ell \\
&\quad + \lambda \int_{s'}^{t'} \Big(v^{\alpha}((\gamma_j \ell, i, j)) - v^{\alpha}((\gamma_j \ell, i + 1, j))\Big) \mathrm{d}\ell \\
&\le \int_{s'}^{t'} \Big((\bar{C} - C(i)) + (K - K(j))\Big) \mathrm{d}\ell \\
&= \Big((\bar{C} - C(i)) + (K - K(j))\Big)(t' - s'), \qquad (6.7)
\end{aligned}$$

and setting $t = \gamma_j t'$, $s = \gamma_j s'$, we obtain from (6.7) that

$$v^{\alpha}((t, i, j)) - v^{\alpha}((s, i, j)) \le \frac{1}{\gamma_j}\Big((\bar{C} - C(i)) + (K - K(j))\Big)(t - s),$$

establishing the result. □

In what follows, set $\widehat{C} = \max\left\{\frac{1}{\gamma_j};\ j = 1, \ldots, \iota\right\}(\bar{C} + K)$. We have the following proposition.

Proposition 6.3 *For all* $x = (x_1, x_2, x_3) \in E$,

$$v^\alpha((0,0,0)) \leq v^\alpha((x_1, x_2, x_3)) + \widehat{C}\tau.$$

Proof For $x_3 = 0$, the result is evident, since from Proposition 6.1,

$$v^\alpha((0,0,0)) \leq v^\alpha((0,i,0)).$$

From (6.4), we have that $\lim_{t\to\tau} v^\alpha((t,1,j)) = v^\alpha((0,0,0))$. From this and (6.5), we have for every $0 \leq s < \tau$ that

$$v^\alpha((0,0,0)) - v^\alpha((s,1,j)) \leq \widehat{C}\tau.$$

Recalling from Proposition 6.1 that $v^\alpha((s,i,j)) - v^\alpha((s,i+1,j)) \leq 0$, we have that for every $i = 1, 2, \ldots,$

$$v^\alpha((0,0,0)) \leq v^\alpha((s,1,j)) + \widehat{C}\tau \leq v^\alpha((s,i,j)) + \widehat{C}\tau,$$

completing the proof. □

Consider a control strategy U_1 such that we always choose $a = 1$ and the stopping time $S = \inf\{t \geq 0;\ X(t) = (0,0,0)\}$. As shown in Example 34.50 in [25], under the stability assumption ($\lambda\frac{\tau}{\gamma_1} < 1$), we have that $E_x^{U_1}(S) < \infty$ for all $x \in E$. From this, it follows that

$$\begin{aligned} v^\alpha(x) &\leq E_x^{U_1}\left(\int_0^S e^{-\alpha t}(\bar{C}+K)ds + e^{-\alpha S}v^\alpha((0,0,0))\right) \\ &\leq E_x^{U_1}(S)(\bar{C}+K) + v^\alpha((0,0,0)). \end{aligned} \tag{6.8}$$

Defining $h_\alpha(x) = v^\alpha(x) - v^\alpha((0,0,0))$ and $\rho_\alpha = \alpha v^\alpha((0,0,0))$, it follows from Proposition 6.3 and (6.8) that for all $\alpha > 0$ and $x \in E$,

$$-\widehat{C}\tau \leq h_\alpha(x) \leq E_x^{U_1}(S)(\bar{C}+K) \tag{6.9}$$

and $\rho_\alpha \leq \bar{C} + K$, so that Assumption 4.7 is satisfied. It is easy to see that all the other conditions in Sect. 4.3 are satisfied, so that from Proposition 4.9, there exists a solution for the discrete-time optimality inequality, and from Theorem 4.10, there exists an ordinary optimal feedback control for the long-run average cost problem.

6.3.2 Numerical Example

Let us consider now a numerical example. To solve the optimization problem numerically, we must restrict ourselves to a finite-dimensional state space. Consider a high level of demand N. For the case in which no construction is taking place and the demand level is $N-1$ (that is, $x=(0, N-1, 0)$), we assume that if there is a new arrival, then the next decision must be to start a new construction (that is, the process must move to one of the "construction" states $(0, N, j)$, with $j=1, \ldots, \iota$). Furthermore, when the process is at one of the "construction" states with level of demand N (that is, $x=(t, N, j)$, with $j=1, \ldots, \iota$), we assume that no new arrivals are allowed (that is, $\lambda=0$ in these states), and on completion of the present project, the process must move to the state $x=(0, N-1, j)$. For N sufficiently large, this truncation of the state space should have negligible influence on the value function, since the stationary probability should be concentrated on moderate levels of demand. With this truncation, we have from (6.2) and (6.3) that

$$v_{m+1}^{\alpha}((0, N-1, 0)) = \frac{C(N-1)}{\lambda+\alpha} + \left(\frac{\lambda}{\lambda+\alpha}\right) \min_{a\in\{1,\ldots,\iota\}} v_m^{\alpha}((0, N, a)), \tag{6.10}$$

$$\begin{aligned} v_{m+1}^{\alpha}((\ell, N, j)) = &\frac{C(N)+K(j)}{\alpha}(1-e^{-\alpha\left(\frac{\tau-\ell}{\gamma_j}\right)}) \\ &+ e^{-\alpha\left(\frac{\tau-\ell}{\gamma_j}\right)} v_m^{\alpha}((0, N-1, j)). \end{aligned} \tag{6.11}$$

Set

$$\widehat{v}_m^{\alpha}((\ell, i, j)) = e^{(\lambda+\alpha)\ell} v_m^{\alpha}((\tau-\gamma_j\ell, i, j)). \tag{6.12}$$

After some algebraic manipulations, we get from (6.3) and (6.4) that for $\ell \in [0, \frac{\tau}{\gamma_j})$,

$$\begin{aligned} \widehat{v}_{m+1}^{\alpha}((\ell, i, j)) = &\frac{C(i)+K(j)}{\lambda+\alpha}(e^{(\lambda+\alpha)\ell}-1) + \lambda\int_0^{\ell} \widehat{v}_m^{\alpha}((s, i+1, j))\mathrm{d}s \\ &+ \min_{a\in\mathcal{I}} v_m^{\alpha}((0, i-1, a)), i=2, \ldots, N-1, \end{aligned} \tag{6.13}$$

$$\begin{aligned} \widehat{v}_{m+1}^{\alpha}((\ell, 1, j)) = &\frac{C(1)+K(j)}{\lambda+\alpha}(e^{(\lambda+\alpha)\ell}-1) + \lambda\int_0^{\ell} \widehat{v}_m^{\alpha}((s, 2, j))\mathrm{d}s \\ &+ v_m^{\alpha}((0, 0, 0)), \end{aligned} \tag{6.14}$$

and from (6.11) that

$$\widehat{v}_{m+1}^{\alpha}((\ell, N, j)) = \frac{C(N)+K(j)}{\alpha}(e^{(\lambda+\alpha)\ell}-e^{\lambda\ell}) + e^{\lambda\ell} v_m^{\alpha}((0, N-1, j)). \tag{6.15}$$

For $j \in \{1, \dots, \iota\}$, define

$$\begin{aligned} a^j(N) &= \frac{C(N)+K(j)}{\alpha}, \\ a^j(i) &= \frac{C(i)+K(j)}{\lambda+\alpha}, \qquad i = 1, \dots, N-1, \\ b^j_m(N) &= 0, \\ b^j_m(i) &= \min_{a\in\mathcal{I}} v^\alpha_m((0, i-1, a)) - \frac{C(i)+K(j)}{\lambda+\alpha}, \qquad i = 2, \dots, N-1, \end{aligned} \tag{6.16}$$

$$b^j_m(1) = v^\alpha_m((0,0,0)) - \frac{C(1)+K(j)}{\lambda+\alpha}, \tag{6.17}$$

$$c^j_m = v^\alpha_m((0, N-1, j)) - \frac{C(N)+K(j)}{\alpha}. \tag{6.18}$$

Notice that from (6.13)–(6.15) and for $i = 1, \dots, N-1$,

$$\widehat{v}^\alpha_{m+1}((\ell, N, j)) = a^j(N)e^{(\lambda+\alpha)\ell} + c^j_m e^{\lambda\ell}, \tag{6.19}$$

$$\widehat{v}^\alpha_{m+1}((\ell, i, j)) = a^j(i)e^{(\lambda+\alpha)\ell} + b^j_m(i) + \lambda \int_0^\ell \widehat{v}^\alpha_m((s, i+1, j))\mathrm{d}s. \tag{6.20}$$

Set recursively for $i = N, \dots, 2$,

$$\begin{aligned} \widehat{a}^j(N) &= a^j(N) = \frac{C(N)+K(j)}{\alpha}, \\ \widehat{a}^j(i-1) &= a^j(i-1) + \frac{\lambda}{\lambda+\alpha}\widehat{a}^j(i), \\ \widehat{b}^j_m(N) &= 0, \\ \widehat{b}^j_m(i-1, 0) &= b^j_m(i-1) - \frac{\lambda}{\lambda+\alpha}\widehat{a}^j_m(i) - c^j_m. \end{aligned} \tag{6.21}$$

Set also for $i = N-1, \dots, 2$ and $k = 1, \dots, N-i$,

$$\widehat{b}^j_m(i-1, k) = \frac{\lambda}{k}\widehat{b}^j_m(i, k-1). \tag{6.22}$$

We have the following proposition.

Proposition 6.4 *For $i = N, \dots, 1$ and $j = 1, \dots, \iota$,*

$$\widehat{v}^\alpha_{m+1}((\ell, i, j)) = \widehat{a}^j(i)e^{(\lambda+\alpha)\ell} + \sum_{k=0}^{N-i-1} \widehat{b}^j_m(i,k)\ell^k + c^j_m e^{\lambda\ell}. \tag{6.23}$$

Proof We can prove (6.23) by induction on i. For $i = N$, we have from (6.19) that (6.23) holds. Suppose that (6.23) holds for i. From (6.19), it follows that

$$\begin{aligned}
\widehat{v}^{\alpha}_{m+1}((\ell, i-1, j)) &= a^j(i-1)e^{(\lambda+\alpha)\ell} + b^j_m(i-1) \\
&\quad + \lambda \int_0^{\ell} \Big[\widehat{a}^j(i)e^{(\lambda+\alpha)s} + \sum_{k=0}^{N-i-1} \widehat{b}^j_m(i,k)s^k + c^j_m e^{\lambda s}\Big] \mathrm{d}s \\
&= a^j(i-1)e^{(\lambda+\alpha)\ell} + b^j_m(i-1) + \frac{\lambda}{\lambda+\alpha}\widehat{a}^j(i)\Big(e^{(\lambda+\alpha)\ell} - 1\Big) \\
&\quad + \lambda \sum_{k=0}^{N-i-1} \widehat{b}^j_m(i,k)\frac{\ell^{k+1}}{k+1} + c^j_m\Big(e^{\lambda s} - 1\Big) \\
&= \Big(a^j(i-1) + \frac{\lambda}{\lambda+\alpha}\widehat{a}^j(i)\Big)e^{(\lambda+\alpha)\ell} + \Big(b^j_m(i-1) \\
&\quad - \frac{\lambda}{\lambda+\alpha}\widehat{a}^j(i) - c^j_m\Big) + \lambda \sum_{k=1}^{N-i} \widehat{b}^j_m(i,k-1)\frac{\ell^k}{k} + c^j_m e^{\lambda\ell} \\
&= \widehat{a}^j(i-1)e^{(\lambda+\alpha)\ell} + \sum_{k=0}^{N-i} \widehat{b}^j_m(i-1,k)\ell^k + c^j_m e^{\lambda\ell},
\end{aligned}$$

establishing the result. □

From (6.12) and (6.23), we get that

$$v^{\alpha}_{m+1}((0, i, j)) = \widehat{a}^j(i) + e^{-(\lambda+\alpha)\left(\frac{\tau}{\gamma_j}\right)} \sum_{k=0}^{N-i-1} \widehat{b}^j_m(i,k)\left(\frac{\tau}{\gamma_j}\right)^k + c^j_m e^{-\alpha\left(\frac{\tau}{\gamma_j}\right)}. \quad (6.24)$$

The algorithm to obtain ρ_α goes as follows:

Algorithm 1 (1) Start with $v^{\alpha}_0(x) = 0$.

(2) Suppose that we have $v^{\alpha}_m((0, i, 0))$, $i = 0, \ldots, N-1$, and $v^{\alpha}_m((0, i, j))$, $i = 1, \ldots, N-1$, $j = 1, \ldots, \iota$. Calculate $b^j_m(i)$, c^j_m, $\widehat{b}^j_m(i-1, k)$ from (6.16), (6.17), (6.18), (6.21), (6.22).

(3) Evaluate $v^{\alpha}_{m+1}((0, i, 0))$, $i = 0, \ldots, N-1$, from (6.2), and $v^{\alpha}_{m+1}((0, i, j))$, $i = 1, \ldots, N-1$, $j = 1, \ldots, \iota$, from (6.24).

(4) If $\|v^{\alpha}_{m+1} - v^{\alpha}_m\| < \epsilon$, where ϵ is a specified threshold level, stop and set $\rho_\alpha = \alpha v^{\alpha}_{m+1}((0, 0, 0))$. Otherwise, return to step 2).

We have implemented the above algorithm for the following numerical example:

(a) $\lambda = 0.15$, $\tau = 6$.

(b) $\iota = 2$, $K(1) = 5$, $\gamma_1 = 1$, $K(2) = 17$, $\gamma_2 = 2$.

(c) $C(i) = 0.5i$.

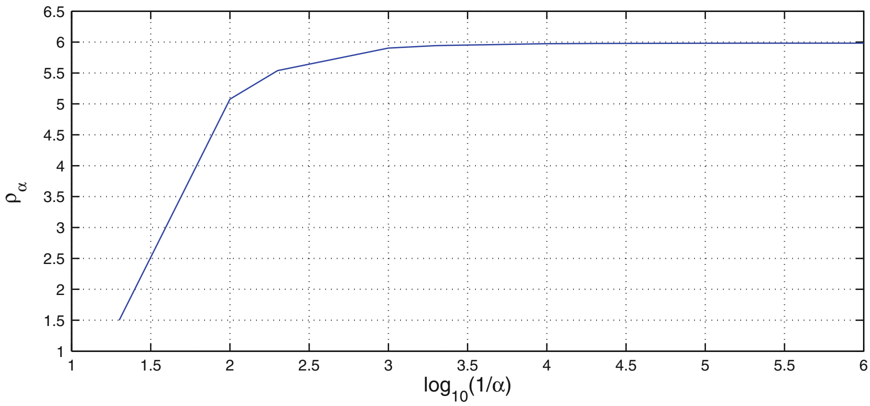

Fig. 6.1 Convergence of ρ_α

(d) $N = 40$.

We have varied α from 0.05 to 0.000001. In Fig. 6.1, we present the values of $\log_{10}\left(\frac{1}{\alpha}\right)$ on the x-axis and ρ_α on the y-axis. We can see, as expected, that $\rho_\alpha \to 5.9825$ as $\alpha \to 0$. The optimal policy is to construct at the rate $\gamma_1 = 1$ whenever the demand is less than or equal to 3, and to construct at the rate $\gamma_2 = 2$ whenever the demand is greater than or equal to 4.

6.4 Second Example

6.4.1 Verification of the Assumptions in Sect. 4.3

Our second example is based on the one presented in [60, pp. 164–165] and has a setup very close to the capacity expansion problem presented in Sects. 6.2 and 6.3, so for this reason, we will omit the details and will use the same notation and definitions as presented in Sects. 6.2 and 6.3. Suppose that letters arrive at a post office according to a Poisson process with rate λ. At any time, the post office may, at a cost of K per unit of time, summon a truck to pick up all letters presently in the post office. The truck takes τ units of time to arrive, and every letter arriving during this time interval will also be collected by the truck. The post office incurs a running cost $C(i)$ per unit of time when there are i letters waiting to be picked up. The decision is to select the optimum moment to call the truck in order to minimize the long-run average cost. It is easy to see that this problem has a very similar setup to that of the previous one with $\iota = 1$, $\gamma_1 = 1$, the difference being (6.3), which in this case, is given by

$$
\begin{aligned}
v_{m+1}^{\alpha}((\ell, i, 1)) = & \frac{C(i)+K}{\lambda+\alpha}(1-e^{-(\lambda+\alpha)(\tau-\ell)}) \\
& + \lambda \int_0^{\tau-\ell} e^{-(\lambda+\alpha)s} v_m^{\alpha}((\ell+s, i+1, 1))\mathrm{d}s \\
& + e^{-(\lambda+\alpha)(\tau-\ell)} v_m^{\alpha}((0,0,0)), i = 2, 3, \ldots.
\end{aligned} \tag{6.25}
$$

The reason for this is that the truck takes all the letters from the post office when leaving, bringing the process to the state (0, 0, 0). We assume the same conditions as in the previous example and set, for $i \in \mathbb{N}$, $c_i = C(i+1) - C(i) \geq 0$. We also assume that $c_{i+1} \leq c_i$ for each $i \in \mathbb{N}$. Define $\xi_\alpha = 1 - e^{(\lambda+\alpha)\tau}$, $\xi = 1 - e^{\lambda\tau}$, and $a_{m,i} = \left(\frac{1-\xi_\alpha^m}{1-\xi_\alpha}\right)\left(\frac{c_i}{\lambda}\right)$ for $m \in \mathbb{N}$ and $i \in \mathbb{N}$. By the assumption that $c_{i+1} \leq c_i$, it follows that $a_{m,i+1} \leq a_{m,i}$. We have the following proposition.

Proposition 6.5 *For each $m \in \mathbb{N}$ and $i \in \mathbb{N}$, we have that*

$$
|v_m^{\alpha}((t, i+1, 1)) - v_m^{\alpha}((t, i, 1))| \leq a_{m,i}. \tag{6.26}
$$

Proof We prove this by induction on m. For $m = 0$, this is clearly true, since $a_{0,i} = 0$ and $v_0^{\alpha} = 0$. Suppose it holds for m. Then by the induction hypothesis and (6.25), it is easy to see that

$$
\begin{aligned}
|v_{m+1}^{\alpha}((t, i+1, 1)) - v_{m+1}^{\alpha}((t, i, 1))| &\leq \frac{\xi_\alpha c_i}{\lambda} + \frac{\lambda}{\lambda+\alpha}\xi_\alpha a_{m,i+1} \leq \frac{c_i}{\lambda} + \xi_\alpha a_{m,i} \\
&\leq \frac{c_i}{\lambda}\left(1 + \xi_\alpha \frac{1-\xi_\alpha^m}{1-\xi_\alpha}\right) = a_{m+1,i},
\end{aligned}
$$

establishing the result. □

From (6.26) and taking the limit as $m \to \infty$, it is immediate that $|v^{\alpha}((t, i+1, 1)) - v^{\alpha}((t, i, 1))| \leq a_i$, where $a_i = \frac{c_i}{\lambda(1-\xi)}$. From this and (6.7), it follows that for some constant c independent of α, we have that $|v^{\alpha}((t, i, 1)) - v^{\alpha}((s, i, 1))| \leq c|t-s|$, and from this, it follows that the sequence $h_\alpha(x)$ is equicontinuous. Moreover, considering, as in the previous example, U_1 such that we always choose $a = 1$ and the stopping time $S = \inf\{t \geq 0;\ X(t) = (0,0,0)\}$, we have, as before, that $\overline{h}(x) = E_x^{U_1}(S) < \infty$ for all $x \in E$. Furthermore,

$$
\overline{h}(x) = \int_0^{\tau-t} \lambda e^{-\lambda s}(s + \overline{h}((t+s, i+1, 1)))\mathrm{d}s + e^{-\lambda(\tau-t)}(\tau - t)
$$

for $x = (t, i, 1) \in E$. Consequently,

$$\int_{0}^{\tau-t} \lambda e^{-\lambda s}\overline{h}((t+s, i+1, 1))\mathrm{d}s = \int_{0}^{t_*(x)} e^{-\Lambda(x,s)}\lambda(\phi(x,s))Q\overline{h}(\phi(x,s))\mathrm{d}s < \infty.$$

Therefore, Assumption 4.11 is satisfied, so that from Proposition 4.12, there exists a solution for the discrete-time optimality equation.

6.4.2 Numerical Example

For the numerical example, we consider again a truncation at a high level of letters N, as was done in Sect. 6.3.2. For the case in which there are $N-1$ letters (that is, $x = (0, N-1, 0)$), we assume that if there is a new arrival, then the next decision must be to collect the letters (that is, the process must move to state $(0, N, 1)$). Furthermore, while the truck is picking up all the N letters (that is, the process is in state $x = (t, N, 1)$), we assume that no new arrivals are allowed (that is, $\lambda = 0$ in these states), and on completion of the trip, the process must move to state $x = (0, 0, 0)$. For N sufficiently large, this truncation of the state space should have negligible influence on the value function, since the stationary probability should be concentrated on moderate levels of wanting letters.

Algorithm 2 can also be used to solve this problem. Notice that we can suppress the superscript j, since $\iota = 1$. The update for $b_m(i)$ and c_m in step (2) would be as follows:

$$b_m(i) = v_m^{\alpha}((0,0,0)) - \frac{C(i)+K}{\lambda+\alpha}, \quad i = 1, \ldots, N-1,$$
$$c_m = v_m^{\alpha}((0,0,0)) - \frac{C(N)+K}{\alpha}.$$

We have implemented the above algorithm for the following numerical example:

(a) $\lambda = 0.15$, $\tau = 6$.
(b) $K = 15$.
(c) $C(i) = 0.5i$.
(d) $N = 40$.

We have varied α from 0.01 to 0.000001. In Fig. 6.2, we present the values of $\log_{10}\left(\frac{1}{\alpha}\right)$ on the x-axis and ρ_α on the y-axis. We can see, as expected, that $\rho_\alpha \to 3.4621$ as $\alpha \to 0$. The optimal policy is to call the truck whenever the number of letters is greater than or equal to 7.

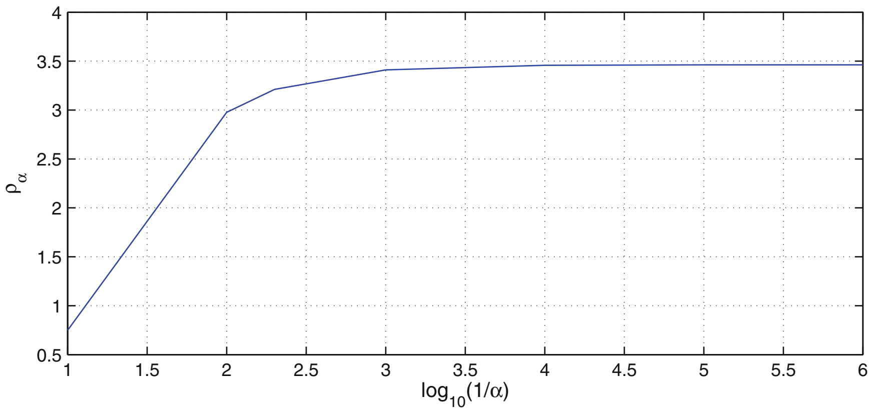

Fig. 6.2 Convergence of ρ_α

6.5 Third Example

In this section, we present an example that satisfies the assumptions of Sect. 4.4, so that the results obtained in Sect. 4.4.2 can be applied. This example is also based on the capacity expansion model described in Sect. 6.2. As posed there, the demand for some utility is modeled as a random point process, i.e., it increases by one unit at random times. This demand is met by consecutive construction of identical expansion projects. In this subsection, the intensity λ of the point process is supposed to be the controlled variable, and it can assume values in a compact set $[\lambda_a, \lambda_b]$ with $\lambda_b \geq \lambda_a > 0$ (notice that in Sect. 6.3, λ was assumed to be constant). The construction of a new project is accomplished at a rate γ per unit of time, and it is completed after the cumulative investment in the current project reaches a value τ. On completion, the present level of demand is reduced κ units. We will consider for simplicity in this example that $\kappa = 1$, $\gamma = 1$, and that τ does not depend on the present level of demand. Moreover, we suppose that no construction will take place whenever there is no demand, and we say in this case that the system is in a standby situation. In this case, it is assumed for simplicity that a new demand will occur according to a point process with a fixed uncontrolled rate equal to μ. Regarding the cost, roughly speaking, the idea is to penalize high levels of demand and the controller whenever it slows down the demand by choosing a lower intensity. It is also supposed that an immediate (boundary) cost is incurred whenever a project is finished, and that there are no costs when the system is in the standby situation.

As the state space of the process we take $E = \cup_{i\in\mathbb{N}}\big[\{i\} \times [0, \tau)\big]$. A point $(0, t) \in E$ denotes that the system is in standby. A point $(i, t) \in E$ with $i \geq 1$ indicates that the level of demand for the system is i, and the amount of elapsed time of the present project is t. For all $x \in E$, $\mathbb{U}(x) = \mathbb{U} = [\lambda_a, \lambda_b]$. The costs are defined for $x = (i, t) \in E$ and $a \in \mathbb{U}$ by $f(x, a) = \alpha_i + h_i(a)$ and $r(x, a) = r_i$. It is assumed that $(\alpha_i)_{i\in\mathbb{N}}$ is an increasing sequence of nonnegative real numbers satisfying for each $i \in \mathbb{N}$, $\alpha_i \leq i\alpha$ for a constant α, $h_i(a)$ is a decreasing mapping in a with nonnegative values that satisfies $h_i(a) \leq i\psi$ for a constant ψ, and $(r_i)_{i\in\mathbb{N}}$ is a sequence of nonnegative real numbers satisfying $r_i \leq i\xi$ for a constant ξ. The flow ϕ of the PDMP is defined by $\phi(x, s) = (i, t + s)$ for $x = (i, t) \in E$, $s \leq \tau - t$. For $t \in [0, \tau)$, $a \in \mathbb{U}$, the intensity of the jump is given by $\lambda((i, t), a) = a$ when $i \geq 1$ and by $\lambda(x, a) = \mu$ when $i = 0$. Finally, the transition measure is defined by

$$Q\big((0, t), a; A\big) = \delta_{\{(1,0)\}}(A), \qquad Q\big((i, t), a; A\big) = \delta_{\{(i+1,t)\}}(A)$$

when $t \in [0, \tau)$ and by

$$Q\big((0, \tau), a; A\big) = \delta_{\{(0,0)\}}(A), \qquad Q\big((i, \tau), a; A\big) = \delta_{\{(i-1,0)\}}(A).$$

The standby situation of the system is represented by the set $\{0\} \times [0, \tau)$ for mathematical convenience. It should be noted that it does not affect the optimization problem due to the memoryless property of the exponential distribution. Finally, we assume the classical stability condition $\lambda_b \tau < 1$ (see [25, Proposition 34.36]).

We show next that all the assumptions of Sect. 4.4 are satisfied for this example. Assumptions 2.3, 2.4, and 2.10 are trivially satisfied. Moreover:

(i) There exists a set $A = \{y \in E : t_*(y) = \tau\}$ such that for all $z \in \partial E$ and for all $a \in \mathbb{U}(z)$, $Q(z, a; A) = 1$. Moreover, for all $x \in E$, $a \in \mathbb{U}(x)$, $\lambda(x, a) \leq \lambda_b$. Consequently, the hypotheses of Proposition 24.6 in [25] are satisfied, implying that Assumption 2.8 is satisfied.

(ii) Since $\mathbb{U}(x)$ does not depend on x, Assumption 2.15 is clearly satisfied.

(iii) Now notice that for all $x \in E$, $t_*(x) \leq \tau < \infty$. For all $x \in E$, the mapping $\lambda(x, .)$ is continuous on $\mathbb{U}(x) = [\lambda_a, \lambda_b]$. Taking for $x \in E$, $\overline{\lambda}(x) = \lambda_b$, we have that $\int_0^t \overline{\lambda}(\phi(x, s))ds \leq \int_0^{t_*(x)} \overline{\lambda}(\phi(x, s))\mathrm{d}s = \lambda_b \tau < \infty$. Define $\underline{\lambda}(x) = \lambda_a$, $\overline{f}(x) = (\alpha + \psi)i$ for $x = (i, t) \in E$, and $K_\lambda = \tau e^{c\tau}$. Then we have $\lambda(x, a) \geq \underline{\lambda}(x)$, $f(x, a) \leq \overline{f}(x)$, and

$$\int_0^{t_*(x)} e^{ct - \int_0^t \underline{\lambda}(\phi(x,s))\mathrm{d}s} dt \leq \tau e^{c\tau} = K_\lambda,$$

$$\int_0^{t_*(x)} e^{-\int_0^t \underline{\lambda}(\phi(x,s))ds} \overline{f}(\phi(x, t))\mathrm{d}t \leq \overline{f}(x)\tau < \infty.$$

Recalling that $t_*(x) \leq \tau < \infty$ for all $x \in E$, it follows that Assumptions 2.5, 3.5, 3.9, and 4.18 are satisfied.

(iv) By definition, $f(x, .)$, $r(x, .)$ are continuous on $\mathbb{U}(x) = [\lambda_a, \lambda_b]$ for all $x \in E$, implying that Assumptions 3.6 and 3.7 are satisfied.

(v) Q does not depend on the control, and consequently, Assumptions 3.8 and 4.15 (i) are trivially satisfied.

It remains to show that Assumptions 4.13, 4.14, and 4.15 (ii) are satisfied. In order to verify these assumptions, we need to define appropriately the test functions g and $\overline{r}$. We need first the following proposition.

Proposition 6.6 *There exist $d_1 > 0$, $d_2 > 0$, and $c > 0$ such that*

$$e^{d_1}\frac{\lambda_b}{\lambda_b + d_2} < e^{-d_1+d_2\tau} < 1, \tag{6.27}$$

$$c < d_2 - \lambda_b(e^{d_1} - 1), \tag{6.28}$$

$$e^{d_1}(1 - e^{-\mu\tau}) - e^{-a_1} \leq 0. \tag{6.29}$$

Proof Consider the function $c(z) = \frac{z}{\tau} - \lambda_b(e^z - 1)$. Then under the assumption that $\lambda_b\tau < 1$, we can find $z_0 > 0$ such that $c(z_0) > 0$ and $1 - e^{-\mu\tau} \leq e^{-2z_0}$ by observing that $c(0) = 0$ and $c'(0) > 0$. This implies that we can find $0 < \epsilon < \frac{z_0}{2}$ such that

$$c(z_0) - \frac{2\epsilon}{\tau} = \frac{z_0 - 2\epsilon}{\tau} - \lambda_b(e^{z_0} - 1) > 0.$$

We set $d_1 = z_0 - \epsilon > 0$ and $d_2 = \frac{z_0-2\epsilon}{\tau} > 0$. Since $1 - e^{-\mu\tau} \leq e^{-2z_0}$, we have $1 - e^{-\mu\tau} \leq e^{-2z_0+2\epsilon} = e^{-2a_1}$, giving (6.28). Notice that

$$-d_1 + d_2\tau = -(z_0 - \epsilon) + z_0 - 2\epsilon = -\epsilon,$$

and thus $e^{-d_1+d_2\tau} = e^{-\epsilon} < 1$. From $c(z_0) - \frac{2\epsilon}{\tau} > 0$, we get that $d_2 + \lambda_b > \lambda_b e^{d_1+\epsilon}$ and thus $e^{d_1}\frac{\lambda_b}{\lambda_b+d_2} < e^{-d_1+d_2\tau}$, proving (6.27). Moreover, choosing $c > 0$ such that $c < c(z_0) - \frac{2\epsilon}{\tau}$, we get that

$$c < d_2 - \lambda_b(e^{d_1+\epsilon} - 1) < d_2 - \lambda_b(e^{d_1} - 1),$$

yielding (6.29). □

We can now define the test functions $g, \overline{r}$ and the parameters required in Assumptions 4.13 and 4.14. Consider d_1, d_2, c satisfying Eqs. (6.27) and (6.28) and define $b = e^{d_2\tau}(\mu[e^{d_1} - 1] + c) > 0$, $\delta = c$, $d_3 = 1 - e^{-d_1+d_2\tau} > 0$, $M = \max(\frac{\alpha+\psi}{d_1}, \frac{2c\xi}{d_3 d_1})$, and

$$g(x) = \begin{cases} e^{d_2\tau} & \text{if } x = (0, t) \in E, \\ e^{d_1 i + d_2(\tau - t)} & \text{if } x = (i, t) \in E, \text{ and } i \geq 1, \end{cases}$$

$$\overline{r}(z) = \begin{cases} 0 & \text{if } z = (0, \tau), \\ d_3 e^{d_1 i} & \text{if } z = (i, \tau), \text{ and } i \geq 1. \end{cases}$$

By taking $\overline{g}(i,t) = \lambda_b e^{d_1(i+1)+d_2\tau}$, Assumption 4.15 (ii) is satisfied. For the constants b, c, δ, M defined previously, we have the following result:

Proposition 6.7 *The functions g and $\overline{r}$ satisfy Assumption 4.13.*

Proof Consider first $x = (0,t) \in E$ and $z = (0,\tau)$. Then in this case, $\mathcal{X}g(x) = 0$ and

$$\mathcal{X}g(x) + cg(x) - \lambda(x,a)\,[g(x) - Qg(x,a)] = e^{d_2\tau}(\mu[e^{d_1} - 1] + c) = b,$$

and so (4.14) holds. Clearly, $f(x,a) = 0 \le Mg(x)$, and so Eq. (4.15) is satisfied. Since $\overline{r}(z) = 0$, we have $\overline{r}(z) + Qg(z,a) = e^{d_2\tau} = g(z)$, which proves Eq. (4.16). Finally, Eq. (4.17) is trivially satisfied, since $r(z,a) = 0$.
Consider now $x = (i,t) \in E$ and $z = (i,\tau)$ with $i \ge 1$. Notice that $\mathcal{X}g(x) = -d_2 g(x)$ and

$$\mathcal{X}g(x) + cg(x) - \lambda(x,a)\,[g(x) - Qg(x,a)] = e^{d_1 i + d_2(\tau - t)}\big(c - d_2 + a(e^{d_1} - 1)\big).$$

From Eq. (6.28), it follows that

$$\begin{aligned}\max_{a\in[\lambda_a,\lambda_b]}&\Big\{e^{d_1 i + d_2(\tau-t)}\Big(c - d_2 + a(e^{d_1} - 1)\Big)\Big\}\\ &\le e^{d_1 i + d_2(\tau-t)}\Big(c - d_2 + \lambda_b(e^{d_1} - 1)\Big)\\ &\le 0,\end{aligned}$$

yielding (4.14). Moreover, $f(x,a) \le (\alpha + \psi)i \le Mg(x)$, proving Eq. (4.15). Equation (4.16) also holds, since

$$\overline{r}(z) + Qg(z,a) = d_3 e^{d_1 i} + e^{d_1(i-1)+d_2\tau} = e^{d_1 i} \le g(z).$$

Finally, $r(z,a) = r_i \le \xi i \le M\overline{r}(z)$, implying Eq. (4.17) and establishing the result. □

Next we want to show that Assumption 4.14 is satisfied. We need first the following proposition.

Proposition 6.8 *Set $\beta = e^{-d_1 + d_2\tau} < 1$ (see (6.27)). For every $u \in \mathcal{S}_{\mathbb{U}}$, we have that*

$$Gg(x,u_\phi) \le \beta g(x), \;\; \textit{for } x = (i,t) \in E, \;\; \textit{with } i \ge 1, \tag{6.30}$$

$$Gg(x,u_\phi) \le \beta g(x) + l(x)g(0,0), \;\; \textit{for } x = (0,t) \in E, \tag{6.31}$$

where $l(x) = e^{-\mu(\tau - t)}$ for $x = (0,t)$, and $l(x) = 0$ otherwise.

Proof Let us show first that (6.30) holds. Consider $u \in \mathcal{S}_{\mathbb{U}}$ and $x = (i, t) \in E$, with $i \geq 1$ fixed. For notational simplicity, we write $\lambda_i(t+s) = \lambda(\phi(x,s), u(\phi(x,s)))$ and $\Lambda_i(t,s) = \int_0^s \lambda(\phi(x,v), u(\phi(x,v)))dv$. Noticing that $\frac{\lambda_i(t+s)}{\lambda_b} \leq 1$, we have from Eq. (6.27) that

$$
\begin{aligned}
Gg(x, u_\phi) &= e^{d_1 i}\Big\{e^{d_1+d_2(\tau-t)}\int_0^{\tau-t}\lambda_i(t+s)e^{-(\Lambda_i(t,s)+d_2 s)}\mathrm{d}s \\
&\quad + e^{-\Lambda_i(t,\tau-t)-d_2(\tau-t)}e^{-d_1+d_2\tau}\Big\} \\
&= g(i,t)\Big\{e^{d_1}\frac{\lambda_b}{\lambda_b+d_2}\int_0^{\tau-t}(1+\frac{d_2}{\lambda_b})\lambda_i(t+s)e^{-(\Lambda_i(t,s)+d_2 s)}\mathrm{d}s \\
&\quad + e^{-\Lambda_i(t,\tau-t)-d_2(\tau-t)}e^{-d_1+d_2\tau}\Big\} \\
&\leq \beta g(i,t)\Big\{\int_0^{\tau-t}(\lambda_i(t+s)+d_2)e^{-(\Lambda_i(t,s)+d_2 s)}\mathrm{d}s \\
&\quad + e^{-\Lambda_i(t,\tau-t)-d_2(\tau-t)}\Big\} \\
&= \beta g(i,t),
\end{aligned}
$$

proving Eq. (6.30).
Now, for $x = (0, t) \in E$, we have

$$
\begin{aligned}
Gg(x, u_\phi) &= e^{d_1+d_2\tau}[1-e^{-\mu(\tau-t)}] + e^{d_2\tau}e^{-\mu(\tau-t)} \\
&= \beta g(x) - e^{-d_1+d_2\tau}e^{d_2\tau} + e^{d_1+d_2\tau}[1-e^{-\mu(\tau-t)}] + l(x)g(0,0),
\end{aligned}
$$

since $e^{d_2\tau}e^{-\mu(\tau-t)} = l(x)\nu(g)$. From Eq. (6.29), we get

$$
e^{d_1+d_2\tau}[1-e^{-\mu(\tau-t)}] - e^{-d_1+d_2\tau}e^{d_2\tau} \leq 0,
$$

proving Eq. (6.31) and completing the proof. □

The next proposition shows that Assumption 4.14 is satisfied.

Proposition 6.9 *For every $u \in \mathcal{S}_{\mathbb{U}}$, there exists a probability measure ν_u such that Assumption 4.14 is satisfied.*

Proof Clearly, it is easy to see that for fixed $u \in \mathcal{S}_{\mathbb{U}}$, the Markov kernel $G(., u_\phi; .)$ is irreducible. Moreover, we have $G(x, u_\phi; A) \geq l(x)\delta_{(0,0)}(A)$, where the function l was defined in Proposition 6.8. Define the set $C = \{(0,t) : 0 \leq t < \tau\}$. Therefore, we have for all $x \in C$, $G(x, u_\phi; A) \geq e^{-\mu\tau}\delta_{(0,0)}(A)$, implying that the set C is a petite set; see [56, p. 121]. Now since $\beta < 1$, we get from Proposition 6.8 that there exists a constant K such that

$$Gg(x, u_\phi) \leq g(x) + K I_C(x).$$

By Theorem 4.1(i) in [54] and Theorem 11.0.1 in [56], the previous inequality shows that the Markov kernel $Gg(., u_\phi; .)$ is positive Harris recurrent. Consequently, there exists a unique invariant probability measure for $Gg(., u_\phi; .)$. Notice that $g(x) \geq 1$, $0 \leq l(x) \leq 1$. Moreover, for every $u \in \mathcal{S}_{\mathbb{U}}$ and $x \in E$, we have $G(x, u_\phi, A) \geq l(x)\delta_{(0,0)}(A)$, $\int_E l(y)\delta_{(0,0)}(dy) > 0$, and that $\int_E g(y)\delta_{(0,0)}(dy) = g((0,0)) < \infty$. Now from Proposition 6.8, it follows that for every $u \in \mathcal{S}_{\mathbb{U}}$ and $x \in E$,

$$Gg(x, u_\phi) \leq \beta g(x) + l(x) \int_E g(y)\delta_{(0,0)}(dy), \tag{6.32}$$

implying that the hypotheses and items (i)–(iv) of Proposition 10.2.5 in [51] are satisfied. Consequently, by Proposition 10.2.5 in [51], the result follows. □

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