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Design Sensitivitya Analysis of Structural Systems

Adward j. Haug
Kyung K.Choi
Vadim Komkov

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EDWARD J. HAUG
Center for Computer Aided Design
and Department of Mechanical Engineering
The University of Iowa
Iowa City, Iowa

KYUNG K. CHOI
Department of Mechanical Engineering
and Center for Computer Aided Design
The University of Iowa
Iowa City, Iowa

VADIM KOMKOV
Department of Mathematics
Winthrop College
Rock Hill, South Carolina

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Dedicated to our wives

Carol

Ho-Youn

Joyce

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Preface

Structural design sensitivity analysis concerns the relationship between design variables available to the engineer and structural response or state variables that are determined by the laws of mechanics. The dependence of structural response measures such as displacement, stress, natural frequency, and buckling load on design variables such as truss member cross-sectional area, plate thickness, and component shape is implicitly defined through the state equations of structural mechanics. Attention is restricted in this text to linear structural mechanics; i.e., to structures whose governing equations (matrix, ordinary differential, or partial differential) are linear in the state variables, once the design variable is fixed. Since design variables appear in the coefficients of linear operators, however, the state equations are nonlinear as functions of state and design. The mathematical challenge is to treat the nonlinear problem of design sensitivity analysis with methods that take advantage of mathematical properties of the linear (for fixed design) state operators.

A substantial literature on the technical aspects of structural design sensitivity analysis exists. Some is devoted directly to the subject, but most is imbedded in papers devoted to structural optimization. The premise of this text is that a comprehensive theory of structural design sensitivity analysis for linear elastic structures can be treated in a unified way. The objective of the text is to provide a complete treatment of the theory and numerical methods of structural design sensitivity analysis. The theory supports optimality criteria methods of structural optimization and serves as the foundation for iterative methods of structural optimization. One of the most common methods of structural design involves decisions made by the designer, based on experience and intuition. This conventional mode of structural design can be substantially enhanced if the designer is provided with design sensitivity information that explains what the influence of design changes will be, without requiring trial and error.

The advanced state of the art of finite element structural analysis provides a reliable tool for evaluation of structural designs. In its present form, however, it is used to identify technical problems, but it gives the designer little help in identifying ways to modify the design to avoid problems or improve desired qualities. Using design-sensitivity information that can be generated by methods that exploit the finite-element formulation, the designer can carry out systematic trade-off analysis and improve the design. The numerical efficiency of finite-element-based design sensitivity analysis and the emergence of interactive graphics technology and CAD systems are factors that suggest the time is right for interactive computer-aided design of structures, using design sensitivity information and any of a variety of modern optimization methods. It is encouraging to note that as of September 1983, the MacNeal–Schwendler Corporation introduced one of the design sensitivity analysis methods presented in Chapter 1 in NASTRAN finite element code. It is hoped that this represents a trend that will be followed by other codes.

In addition to developing design sensitivity formulas and numerical methods, the authors have attempted to present a unified and relatively complete mathematical theory of structural design sensitivity analysis. Recent developments in functional analysis and linear operator theory provide a foundation for rigorous mathematical analysis of the problem. In addition to fulfilling one's instincts to be accurate, complete, and pure of heart, the mathematical theory provides valuable insights and some surprisingly practical results. The mathematical theory shows that positive definiteness (actually strong ellipticity) of the operators of structural mechanics for stable elastic structures is the property that provides most of the theoretical results and makes numerical methods work. In the case of repeated eigenvalues, which are now known to arise systematically in optimized designs, the theory shows that repeated eigenvalues are not generally differentiable with respect to design, but are only directionally differentiable. Erroneous results that have appeared in the literature under the assumption of differentiability can now be corrected. Since such pathological problems and dangers lurk in broad classes of optimum structures, it is hoped that the mathematical tools presented in this text will help in constructing truly optimum structures that do not have mathematically induced flaws.

The authors have attempted to write this text to meet the needs of both the engineer who is interested in applications and the mathematician and theoretically inclined engineer who are interested in the mathematical subtleties of the subject. To accomplish this objective, each chapter has been written to first present formulation, examples, method development, and illustrations, with theoretical foundations contained in the last section of each chapter. The intent is that the first sections of each chapter will meet the needs of the applications-oriented engineer and will clearly define the structure of problems for the mathematician. The theoretically oriented sections presented at the end of each chapter provide proofs of results that are cited and used in earlier sections.

The book is organized into four chapters. The first three treat distinct types of design variables, and the fourth presents a built-up structure formulation that combines the other three. The first chapter treats finite-dimensional problems, in which the state variable is a finite-dimensional vector of structural displacements and the design variable is a finite-dimensional vector of design parameters. The structural state equations are matrix equations for static response, vibration, and buckling of structures and matrix differential equations for transient dynamic response of structures, with design variables appearing in the coefficient matrices. Examples treated include trusses, frames, and finite-element models of more complex structures. Both direct design differentiation and adjoint variable methods of design sensitivity analysis are presented. Computational aspects of implementing the methods in conjunction with finite-element analysis codes are treated in some detail. The mathematical complexity of this class of finite-dimensional problems is minimal, with the exception of the repeated eigenvalue case, in which some technically sophisticated issues arise.

The second chapter treats infinite-dimensional problems, in which the state and design variables are functions (displacement field and material distribution) and the structural state equations are boundary-value problems of ordinary or partial differential equations. Examples treated include beams, plates, and plane-elastic-solid structural elements. The adjoint variable method of design sensitivity analysis is developed, and design derivatives of eigenvalues are derived. Computational aspects of design sensitivity analysis, using the finite-element method for solution of both state and adjoint equations, are considered. Analytical solutions of simple examples and numerical solutions of more complex examples are presented. Proofs of differentiability of displacement and eigenvalues with respect to design are given, using methods of functional analysis and operator theory.

Structural components, in which shape of the elastic body is the design, are treated in Chapter 3. The material derivative idea of continuum mechanics is used to predict the effect of a change in shape on functionals that define structural response. The adjoint variable method is used to derive expressions for differentials of structural response as boundary integrals that involve normal perturbation of the boundary. A similar treatment of shape design sensitivity of eigenvalues is presented. Examples that involve the length of a beam and the shapes of membranes, plane and spatial elastic solids, and plates are studied. Numerical methods that are based on parameterization of shape and the finite-element method of structural analysis are presented. Finally, proofs of structural response differentiability with respect to shape are presented.

The fourth amd final chapter treats built-up structures that are composed of the coupled finite-dimensional components treated in Chapter 1, the distributed components with design functions treated in Chapter 2, and the components with variable shape treated in Chapter 3. Hamilton's principle is used as the underlying basis of mechanics, directly yielding the variational formulation of structural state equations. Design sensitivity analysis methods of the first three chapters are employed to predict the effect of design variations in the three respective design variable types. The adjoint variable method is used to develop a unified formulation for representing response variations in terms of variations in design. Numerical methods for calculating design sensitivities, using the finite element of structural analysis, are presented. Examples that involve coupled plate – beam – trusses, variable shape and thickness components, and a spatial structure that is composed of plane elastic sheets are studied.

A final comment on notation used in this text is in order here. The structural engineer may be frustrated to find that conventional notation of structural mechanics has not always been adhered to. The field of design sensitivity analysis presents a dilemma regarding notation since it draws from fields such as structural mechanics, differential calculus, calculus of variations, control theory, differential operator theory, and functional analysis. Unfortunately, the literature in each of these fields assigns the same symbol for a different meaning. For example, the symbol δ is used to designate virtual displacement, total differential, variation, Dirac measure, and other unrelated qualities. Since two or more of these uses will often be required in the same equation, some notational compromise is required. The authors have adhered to standard notation except where ambiguity would arise, in which case the notation employed is defined. Principal notation is defined in Appendix A.4.

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1

Finite-Dimensional Structural Systems

Development of finite element methods for structural analysis during the 1960s was preceded by a more physically based theory of matrix structural analysis pioneered by Pipes [1], Langhaar [2], and a group of engineers concerned with applications. A formal distinction that can be drawn between finite element theory [3–6] and the theory of matrix structural analysis is the viewpoint taken in modeling the structure. In the case of matrix structural analysis, the structure is imagined to be dissected into bite-size pieces, each of which is characterized by a set of nodal displacements and an associated force–displacement relationship. In contrast, if a continuum viewpoint is adopted, the displacement field associated with the structure is characterized by a set of differential equations of equilibrium and applied loads. The finite element technique is based then on piecewise polynomial approximation of the displacement field and application of variational methods for approximating solutions of the governing boundary-value problems. The finite element approach is employed throughout this monograph.

This chapter concentrates on a class of structures that can be described readily by finite element matrix equations. Basic ideas of the finite element structural analysis method are presented in Section 1.1, which includes a discussion of variational principles upon which derivation of structural equations is based. These basic ideas are used in the following sections to carry out design sensitivity analysis of static response, eigenvalues, and dynamic response of structures. Both first- and second-order design sensitivity analysis techniques are developed, and examples are presented. An important problem associated with repeated eigenvalues that arise in design optimization is analyzed. It is shown that for this class of problems the eigenvalue is not a differentiable function of the design, but is only directionally differentiable. Finally, first-order information derived from design sensitivity analysis is employed to obtain projected gradients that allow the designer to do trade-off analysis.

1.1 FINITE-ELEMENT STRUCTURAL EQUATIONS

The finite-element structural analysis approach is introduced in this section, using beam, truss, and plate elements as models. Apart from more intricate algebra associated with more complex elements, the basic approach in deriving element equations is identical to that illustrated in this section.

1.1.1 Element Analysis

Finite-element methods of structural analysis require a knowledge of the behavior of each element in the structure. Once each element is described, the governing equations of the entire structure may be derived. Energy methods are used to obtain the governing equations. In order to apply energy theorems for analysis of a structure, the strain energy, kinetic energy, and change in external dimensions due to bending must be described. The basic idea is to select element displacement functions that are of the form expected in structural deformation and that are uniquely specified when displacements at the nodes of the element are known.

BEAM-ELEMENT STIFFNESS MATRIX

A typical planar beam element, with its displacement sign convention, is shown in Fig. 1.1.1. This displacement coordinates q_1, q_2, q_4, and q_5 are components of endpoint displacement, and q_3 and q_6 are endpoint rotations. The longitudinal displacement of a point x on the beam $(0 \leq x \leq l)$ due to longitudinal strain is approximated by

$$s(x) = -q_1\frac{(x - l)}{l} + q_4\frac{x}{l} \tag{1.1.1}$$

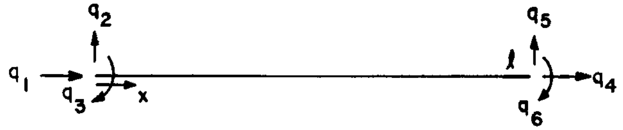

Fig. 1.1.1 Planar beam element.

which is exact for a beam element with constant cross section and no axial distributed load. It should be emphasized that the longitudinal displacement $s(x)$ is due to only longitudinal strain in the beam and not to the change in length caused by the lateral displacement $w(x)$.

Lateral displacement of the beam at a point x is approximated by

$$
\begin{aligned}
w(x) = {} & \frac{q_2}{l^3}(2x^3 - 3lx^2 + l^3) - \frac{q_5}{l^3}(2x^3 - 3lx^2) \\
& + \frac{q_3}{l^2}(x^3 - 2lx^2 + l^2x) + \frac{q_6}{l^2}(x^3 - lx^2)
\end{aligned}
\tag{1.1.2}
$$

which is exact for beam elements with constant cross section and no lateral distributed load. The strain energy SE due to deformation of the beam is [1–6]

$$
\begin{aligned}
\mathrm{SE} &= \tfrac{1}{2}\int_0^l hE\left(\frac{ds}{dx}\right)^2 dx + \tfrac{1}{2}\int_0^l EI\left(\frac{d^2w}{dx^2}\right)^2 dx \\
&= \tfrac{1}{2}\int_0^l hE\left(\frac{q_1}{l} - \frac{q_4}{l}\right)^2 dx + \tfrac{1}{2}\int_0^l EI\left[\frac{q_2}{l^3}(12x - 6l) - \frac{q_5}{l^3}(12x - 6l)\right. \\
&\quad \left. + \frac{q_3}{l^2}(6x - 4l) + \frac{q_6}{l^2}(6x - 2l)\right]^2 dx
\end{aligned}
\tag{1.1.3}
$$

where h is the cross-sectional area of the beam, I the second moment of the cross-sectional area about its centroidal axis, and E Young's modulus of the material. Carrying out the integrations in Eq. (1.1.3), the following quadratic form in $q = [q_1\ q_2\ \cdots\ q_6]^T$ is obtained:

$$
\mathrm{SE} = \tfrac{1}{2} q^T k_B q \tag{1.1.4}
$$

where k_B is the beam *element stiffness matrix*,

$$
k_B = \frac{E}{l^3}\begin{bmatrix}
hl^2 & 0 & 0 & hl^2 & 0 & 0 \\
 & 12I & 6lI & 0 & -12I & 6lI \\
 & & 4l^2I & 0 & -6lI & 2l^2I \\
 & & & hl^2 & 0 & 0 \\
 & \text{symmetric} & & & 12I & -6lI \\
 & & & & & 4l^2I
\end{bmatrix}
\tag{1.1.5}
$$

TRUSS-ELEMENT STIFFNESS MATRIX

If bending effects are neglected, a truss element is obtained for which only coordinates q_1 and q_4 of Fig. 1.1.1 influence the strain energy. In this case, the strain energy is as given in Eq. (1.1.4), but with the truss-element stiffness matrix

$$k_{\mathrm{T}} = \frac{Eh}{l}\begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 0 \\ & 0 & 0 & 0 & 0 & 0 \\ & & 0 & 0 & 0 & 0 \\ & & & 1 & 0 & 0 \\ & \text{symmetric} & & & 0 & 0 \\ & & & & & 0 \end{bmatrix} \tag{1.1.6}$$

Note that only q_1 and q_4 have any effect on truss-element strain energy. While q_2 and q_5 need to be retained in the analysis, since the truss element does not bend, $q_3 = q_6 = (q_5 - q_2)/l$, and these rotation variables may be suppressed. Note also that k_{T} is only of rank 1.

PROPERTIES OF BEAM- AND TRUSS-ELEMENT STIFFNESS MATRICES

It is important to note that the beam-element stiffness matrix of Eq. (1.1.5) depends on the length l of the beam element, the cross-sectional area h, and the moment of inertia I of the cross-sectional area. If cross-sectional dimensions of the beam element are taken as design variables, as is the case when structural-element sizes are regarded as design variables, the element stiffness matrix depends on the design variables. If geometry of the structure is varied, then the length l of the element depends on the design variables and is also involved in a nonlinear way in the element stiffness matrix.

Somewhat more tedious computations will show that the element stiffness matrix of Eq. (1.1.5) is positive semidefinite and of rank 3. The rank of the matrix is associated with the physical observation that there are three rigid-body degrees of freedom of the element shown in Fig. 1.1.1. That is, it is possible to move the element in the plane with three kinematic degrees of freedom, yielding no deformation and hence no strain energy. On the other hand, if the left end of the beam element shown in Fig. 1.1.1 were fixed (i.e., $q_1 = q_2 = q_3 = 0$), then the strain energy calculated by Eq. (1.1.4) would be positive definite in the variables q_4, q_5, and q_6. This simple observation has a nontrivial analog in analysis of more complex structures. As will be shown throughout this text, positive definiteness of the system strain energy when no rigid-body degrees of freedom exist plays a crucial role in the mathemat ical theory of design sensitivity analysis.

BEAM- AND TRUSS-ELEMENT MASS MATRICES

The kinetic energy of the beam element, neglecting rotatory inertia of the beam cross section, is [1–6]

$$
\begin{aligned}
\mathrm{KE} &= \tfrac{1}{2}\int_0^l \rho h\left[\left(\frac{ds}{dt}\right)^2 + \left(\frac{dw}{dt}\right)^2\right]dx \\
&= \tfrac{1}{2}\int_0^l \rho h\left\{\left[-\dot{q}_1\left(\frac{x-l}{l}\right) + \dot{q}_4\frac{x}{l}\right]^2\right. \\
&\quad + \left[\frac{\dot{q}_2}{l^3}(2x^3 - 3lx^2 + l^3) + \frac{\dot{q}_5}{l^3}(2x^3 - 3lx^2)\right. \\
&\quad \left.\left. + \frac{\dot{q}_3}{l^2}(x^3 - 2lx^2 + lx) + \frac{\dot{q}_6}{l^2}(x^3 - lx^2)\right]^2\right\}dx
\end{aligned}
\tag{1.1.7}
$$

where ρ is mass density of beam material and the dot over the variable $(\dot{\ })$ denotes time derivative. Carrying out the integration,

$$
\mathrm{KE} = \tfrac{1}{2}\dot{q}^{\mathrm{T}} m_{\mathrm{B}}\dot{q} \tag{1.1.8}
$$

where m_{B} is the *beam-element mass matrix*,

$$
m_{\mathrm{B}} = \frac{\rho h l}{420}\begin{bmatrix}
140 & 0 & 0 & 70 & 0 & 0 \\
 & 156 & 22l & 0 & 54 & -13l \\
 & & 4l^2 & 0 & 13l & -3l^2 \\
 & & & 140 & 0 & 0 \\
 & \text{symmetric} & & & 156 & -22l \\
 & & & & & 4l^2
\end{bmatrix} \tag{1.1.9}
$$

Since kinetic energy of the beam element is positive if any $\dot{q}_i \neq 0$, it is expected that m_{B} is positive definite, hence nonsingular. These properties can be verified analytically.

To obtain a truss element, bending is neglected and $w(x) = q_2 + (q_5 - q_2)x/l$. Integration of Eq. (1.1.7) yields a quadratic form, as in Eq. (1.1.8), but with the truss-element mass matrix

$$
m_{\mathrm{T}} = \frac{\rho h l}{6}\begin{bmatrix}
2 & 0 & 0 & 1 & 0 & 0 \\
 & 2 & 0 & 0 & 1 & 0 \\
 & & 0 & 0 & 0 & 0 \\
 & & & 2 & 0 & 0 \\
 & \text{symmetric} & & & 2 & 0 \\
 & & & & & 0
\end{bmatrix} \tag{1.1.10}
$$

Note that $\dot{q}_3$ and $\dot{q}_6$ are suppressed in the kinetic energy expression and are not needed in truss analysis. The velocities $\dot{q}_2$ and $\dot{q}_5$, however, play an important role. Since the third and sixth columns of m_{T} are zero, the matrix is singular. In fact, m_{T} is only of rank 4. Note also, as in the case of strain energy, the area h and length l may depend on design variables.

BEAM-ELEMENT GEOMETRIC STIFFNESS MATRIX

Shortening Δl of the beam element due to lateral bending is [1–6]

$$
\begin{aligned}
\Delta l &= l - \int_0^l \left[1 - \left(\frac{dw}{dx}\right)^2\right]^{1/2} dx \\
&\approx \int_0^l \tfrac{1}{2}\left(\frac{dw}{dx}\right)^2 dx \\
&= \tfrac{1}{2}\int_0^l \left[\frac{q_2}{l^3}(6x^2 - 6lx) - \frac{q_5}{l^3}(6x^2 - 6lx)\right. \\
&\qquad \left. + \frac{q_3}{l^2}(3x^2 - 4lx + l^2) + \frac{q_6}{l^2}(3x^2 - 2lx)\right]^2 dx
\end{aligned}
\tag{1.1.11}
$$

Carrying out the integration,

$$
\Delta l = q^{\mathrm{T}} d_{\mathrm{B}} q \tag{1.1.12}
$$

where d_{B} is the *beam-element geometric stiffness matrix*,

$$
d_{\mathrm{B}} = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
 & \frac{3}{5l} & -\frac{1}{20} & 0 & -\frac{3}{5l} & \frac{1}{20} \\
 & & \frac{l}{50} & 0 & -\frac{1}{20} & -\frac{l}{60} \\
 & & & 0 & 0 & 0 \\
 & \text{symmetric} & & & \frac{3}{5l} & -\frac{1}{20} \\
 & & & & & \frac{l}{15}
\end{bmatrix}
\tag{1.1.13}
$$

PLATE-ELEMENT STIFFNESS AND MASS MATRICES

Similar arguments can be applied to plate bending elements to obtain element stiffness and mass matrices. One of the displacement functions that is commonly used to calculate stiffness properties of rectangular plates in bending is of the form

$$w(x, y) = N(x, y)q \tag{1.1.14}$$

where the sign convention of the nodal displacement coordinates

$$q = [q_1 \ q_2 \ \cdots \ q_{12}]^{\mathrm{T}} \tag{1.1.15}$$

is illustrated in Fig. 1.1.2 and the vector shape function $N(x, y)$ is given by

$$N^{\mathrm{T}} = \begin{bmatrix} 1 - \xi\eta - (3 - 2\xi)\xi^2(1 - \eta) - (1 - \xi)(3 - 2\eta)\eta^2 \\ (1 - \xi)\eta(1 - \eta)^2\beta \\ -\xi(1 - \xi)^2(1 - \eta)\alpha \\ (1 - \xi)(3 - 2\eta)\eta^2 + \xi(1 - \xi)(1 - 2\xi)\eta \\ -(1 - \xi)(1 - \eta)\eta^2\beta \\ -\xi(1 - \xi)^2\eta\alpha \\ (3 - 2\xi)\xi^2\eta - \xi\eta(1 - \eta)(1 - 2\eta) \\ -\xi(1 - \eta)\eta^2\beta \\ (1 - \xi)\xi^2\eta\alpha \\ (3 - 2\xi)\xi^2(1 - \eta) + \xi\eta(1 - \eta)(1 - 2\eta) \\ \xi\eta(1 - \eta)^2\beta \\ (1 - \xi)\xi^2(1 - \eta)\alpha \end{bmatrix} \tag{1.1.16}$$

where

$$\xi = x/\alpha, \quad \eta = y/\beta \tag{1.1.17}$$

The displacement coordinates q_1, q_4, q_7, and q_{10} are components of corner-node displacements, while q_2, q_3, q_5, q_6, and q_8, q_9, q_{11}, and q_{12} are corner-node rotations. The displacement function represented by Eqs. (1.1.14) and (1.1.16) ensures that the boundary displacements of adjacent plate elements are compatible. However, rotations of the element

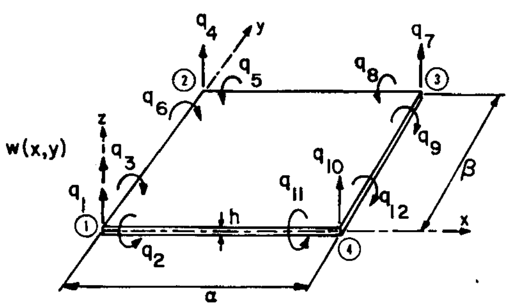

Fig. 1.1.2 Rectangular plate element.

edges on a common boundary are not compatible. Consequently, discontinuities in first derivatives of displacement exist across element boundaries.

Integration of specific strain energy over the plate element results in the following plate-element stiffness matrix [7]:

$$k_P = \frac{Eh^3}{12(1 - \nu^2)\alpha\beta} \begin{bmatrix} k_{I,I} & \text{symmetric} \\ k_{II,I} & k_{II,II} \end{bmatrix}_{12 \times 12} \tag{1.1.18}$$

where E is Young's modulus, ν Poisson's ratio, and the 6×6 submatrices $k_{I,I}$, $k_{II,I}$, and $k_{II,II}$ are presented in Tables 1.1.1–1.1.3. Similarly, the element mass matrix of the plate bending element is given by Eq. (1.1.19) (see page 10), where ρ is mass density of plate material. Note that both stiffness and mass matrices depend on material properties and plate thickness, which may be taken as design parameters.

STRESS IN ELEMENTS

Once element-generalized coordinates are determined, the displacement shape functions employed in derivation of the element matrices yield deformation throughout the element. Knowing deformation, strain can be calculated, and stress can be computed. This is important, since one of the principal constraints in structural design involves bounds on stress in the structure. Following elementary beam theory, at the neutral axis of a beam element no bending strain occurs, and axial strain is simply

$$\varepsilon = ds/dx = (q_4 - q_1)/l \tag{1.1.20}$$

where the definition of axial deformation s is given in Eq. (1.1.1). Using the linear stress–strain law for simple axial deformation,

$$\sigma = E\varepsilon \tag{1.1.21}$$

where E is Young's modulus. Direct stress on the neutral axis of the beam, from Eqs. (1.1.20) and (1.1.21), is

$$\sigma_d = E(q_4 - q_1)/l \tag{1.1.22}$$

Note that direct stress depends only on material properties, dimensions, and displacements. It does not depend explicitly on cross-sectional properties.

From elementary beam bending theory, bending stress at the extreme fiber of the beam element [2] is

$$\sigma_b(x) = Ed\frac{d^2w(x)}{dx^2} \tag{1.1.23}$$

where w is lateral deflection of the element and d the half-depth of the beam.

Table 1.1.1.

Submatrix $k_{I,I}$ for Rectangular Plates in Bending, Based on Noncompatible Deflections

$4(\gamma^2 + \gamma^{-2}) + \frac{1}{5}(14 - 4\nu)$			symmetric		
$[2\gamma^{-2} + \frac{1}{5}(1 + 4\nu)]\beta$	$[\frac{4}{3}\gamma^{-2} + \frac{4}{15}(1 - \nu)]\beta^2$				
$-[2\gamma^2 + \frac{1}{5}(1 + 4\nu)]\alpha$	$-\nu\alpha\beta$	$[\frac{4}{3}\gamma^2 + \frac{4}{15}(1 - \nu)]\alpha^2$			
$2(\gamma^2 - 2\gamma^{-2}) - \frac{1}{5}(14 - 4\nu)$	$-[2\gamma^{-2} + \frac{1}{5}(1 - \nu)]\beta$	$[-\gamma^2 + \frac{1}{5}(1 + 4\nu)]\alpha$	$4(\gamma^2 + \gamma^{-2}) + \frac{1}{5}(14 - 4\nu)$		
$[2\gamma^{-2} + \frac{1}{5}(1 - \nu)]\beta$	$[\frac{2}{3}\gamma^{-2} - \frac{1}{15}(1 - \nu)]\beta^2$	0	$-[2\gamma^{-2} + \frac{1}{5}(1 + 4\nu)]\beta$	$[\frac{4}{3}\gamma^{-2} + \frac{4}{15}(1 - \nu)]\beta^2$	
$[-\gamma^2 + \frac{1}{5}(1 + 4\nu)]\alpha$	0	$[\frac{2}{3}\gamma^2 - \frac{4}{15}(1 - \nu)]\alpha^2$	$-[2\gamma^2 + \frac{1}{5}(1 + 4\nu)]\alpha$	$\nu\alpha\beta$	$[\frac{4}{3}\gamma^2 + \frac{4}{15}(1 - \nu)]\alpha^2$

Table 1.1.2

Submatrix $k_{II,I}$ for Rectangular Plates in Bending, Based on Noncompatible Deflection

$-2(\gamma^2 + \gamma^{-2}) + \frac{1}{5}(14 - 4\nu)$	$[-\gamma^{-2} + \frac{1}{5}(1 - \nu)]\beta$	$[\gamma^2 - \frac{1}{5}(1 - \nu)]\alpha$	$-2(2\gamma^2 - \gamma^{-2}) - \frac{1}{5}(14 - 4\nu)$	$[-\gamma^{-2} + \frac{1}{5}(1 + 4\nu)]\beta$	$[2\gamma^2 + \frac{1}{5}(1 - \nu)]\alpha$
$[\gamma^{-2} - \frac{1}{5}(1 - \nu)]\beta$	$[\frac{1}{3}\gamma^{-2} + \frac{1}{15}(1 - \nu)]\beta^2$	0	$[-\gamma^{-2} + \frac{1}{5}(1 + 4\nu)]\beta$	$[\frac{2}{3}\gamma^{-2} - \frac{4}{15}(1 - \nu)]\beta^2$	0
$[-\gamma^2 + \frac{1}{5}(1 - \nu)]\beta$	0	$[\frac{1}{3}\gamma^2 + \frac{1}{15}(1 - \nu)]\alpha^2$	$-[2\gamma^2 + \frac{1}{5}(1 - \nu)]\alpha$	0	$[\frac{2}{3}\gamma^2 - \frac{1}{15}(1 - \nu)]\alpha^2$
$-2(2\gamma^2 - \gamma^{-2}) - \frac{1}{5}(14 - 4\nu)$	$[\gamma^{-2} - \frac{1}{5}(1 + 4\nu)]\beta$	$[2\gamma^2 + \frac{1}{5}(1 - \nu)]\alpha$	$-2(\gamma^2 + \gamma^{-2}) + \frac{1}{5}(14 - 4\nu)$	$[\gamma^{-2} - \frac{1}{5}(1 - \nu)]\beta$	$[\gamma^2 - \frac{1}{5}(1 - \nu)]\alpha$
$[\gamma^{-2} - \frac{1}{5}(1 + 4\nu)]\beta$	$[\frac{2}{3}\gamma^{-2} - \frac{4}{15}(1 - \nu)]\beta^2$	0	$[-\gamma^{-2} + \frac{1}{5}(1 - \nu)]\beta$	$[\frac{1}{3}\gamma^{-2} + \frac{1}{15}(1 - \nu)]\beta^2$	0
$-[2\gamma^2 + \frac{1}{5}(1 - \nu)]\alpha$	0	$[\frac{2}{3}\gamma^2 - \frac{1}{15}(1 - \nu)]\alpha^2$	$[-\gamma^2 + \frac{1}{5}(1 - \nu)]\alpha$	0	$[\frac{1}{3}\gamma^2 + \frac{1}{15}(1 - \nu)]\alpha^2$

Table 1.1.3

Submatrix $k_{\text{II,II}}$ for Rectangular Plates in Bending, Based on Noncompatible Deflections

$4(\gamma^2 + \gamma^{-2}) + \frac{1}{5}(14 - 4\nu)$			symmetric		
$-[2\gamma^{-2} + \frac{1}{5}(1 + 4\nu)]\beta$	$[\frac{4}{3}\gamma^{-2} + \frac{4}{15}(1 - \nu)]\beta^2$				
$[2\gamma^2 + \frac{1}{5}(1 + 4\nu)]\alpha$	$-\nu\alpha\beta$	$[\frac{4}{3}\gamma^2 + \frac{4}{15}(1 - \nu)]\alpha^2$			
$2(\gamma^2 - 2\gamma^{-2}) - \frac{1}{5}(14 - 4\nu)$	$[2\gamma^{-2} + \frac{1}{5}(1 - \nu)]\beta$	$[\gamma^2 - \frac{1}{5}(1 + 4\nu)]\alpha$	$4(\gamma^2 + \gamma^{-2}) + \frac{1}{5}(14 - 4\nu)$		
$-[2\gamma^{-2} + \frac{1}{5}(1 - \nu)]\beta$	$[\frac{2}{3}\gamma^{-2} - \frac{1}{15}(1 - \nu)]\beta^2$	0	$[2\gamma^{-2} + \frac{1}{5}(1 + 4\nu)]\beta$	$[\frac{4}{3}\gamma^{-2} + \frac{4}{15}(1 - \nu)]\beta^2$	
$[\gamma^2 - \frac{1}{5}(1 + 4\nu)]\alpha$	0	$[\frac{2}{3}\gamma^2 - \frac{4}{15}(1 - \nu)]\alpha^2$	$[2\gamma^2 + \frac{1}{5}(1 + 4\nu)]\alpha$	$\nu\alpha\beta$	$[\frac{4}{3}\gamma^2 + \frac{4}{15}(1 - \nu)]\alpha^2$

$$m_P = \frac{\rho\alpha\beta h}{176{,}400}\begin{bmatrix}
24{,}178 & & & & & & & & & & \\
3227\beta & 560\beta^2 & & & & & & & & & \\
-3227\alpha & -441\alpha\beta & 560\alpha^2 & & & & & & & \\
8582 & 1918\beta & -1393\alpha & 24{,}178 & & & & & \text{symmetric} & \\
-1918\beta & -420\beta^2 & 294\alpha\beta & -3227\beta & 560\beta^2 & & & & & \\
-1393\alpha & -294\alpha\beta & 280\alpha^2 & -3227\alpha & 441\alpha\beta & 560\alpha^2 & & & & \\
2758 & 812\beta & -812\alpha & 8582 & -1393\beta & -1918\alpha & 24{,}178 & & & \\
-812\beta & -210\beta^2 & 196\alpha\beta & -1393\beta & 280\beta^2 & 294\alpha\beta & -3227\beta & 560\beta^2 & & \\
812\alpha & 196\alpha\beta & -210\alpha^2 & 1918\alpha & -294\alpha\beta & -420\alpha^2 & 3227\alpha & -441\alpha\beta & 560\alpha^2 & \\
8582 & 1393\beta & -1918\alpha & 2758 & -812\beta & -812\alpha & 8582 & -1918\beta & 1393\alpha & 24{,}178 \\
1393\beta & 280\beta^2 & -294\alpha\beta & 812\beta & -210\beta^2 & -196\alpha\beta & 1918\beta & -420\beta^2 & 294\alpha\beta & 3227\beta \\
1918\alpha & 294\alpha\beta & -420\alpha^2 & 812\alpha & -196\alpha\beta & -210\alpha^2 & 1393\alpha & -294\alpha\beta & 280\alpha^2 & 3227\alpha
\end{bmatrix} \tag{1.1.19}$$

Using Eq. (1.1.2), the bending stress may be written explicitly as

$$\sigma_b(x) = Ed[q_2(12x - 6l)/l^3 - q_5(12x - 6l)/l^3 + q_3(6x - 4l)/l^2 + q_6(6x - 2l)/l^2] \quad (1.1.24)$$

Note that bending stress depends on displacements and on the half-depth of the beam, which generally will be explicitly dependent on design variables. Thus, bending stress depends both on displacements and on design variables.

Presuming that the half-depth of the beam is the same on both the positive and negative sides of the neutral axes, stresses at the extreme fibers may be written, using superposition, from Eqs. (1.1.22) and (1.1.24) as

$$\sigma = \sigma_d \pm \sigma_b \quad (1.1.25)$$

where the sign depends on whether the extreme fiber is at the top or the bottom of the beam. Using Eq. (1.1.25), the maximum stress arising in a beam element may be calculated and a constraint placed on its magnitude for design. Note also that in the absence of bending (i.e., in a truss element), only the direct stress given by Eq. (1.1.22) arises.

1.1.2 Global Stiffness and Mass Matrices

The total strain and kinetic energies of a structure may be obtained by summing the strain and kinetic energies of all elements that make up the structure. Before a meaningful expression for total system strain and kinetic energy may be written, it is first necessary to define a system of global displacements of all nodes in the structure, relative to a global coordinate system. Let $z_g \in R^n$ denote this *global displacement vector*. (Use of the symbol z for structural displacement is selected here and throughout the text, rather than the more conventional symbol u. This is due to the use of u as a design variable function later in the text, a convention that is adopted from control and optimization theory.)

TRANSFORMATION FROM LOCAL TO GLOBAL COORDINATES

Since the individual elements of the structure have their own inherent displacement coordinates relative to a body-fixed coordinate system, as in Figs. 1.1.1 and 1.1.2, displacements must first be transformed from the element's body-fixed coordinate system to a coordinate system parallel to the global coordinates. Let q^i denote the vector of nodal displacement coordinates of the ith element in its body-fixed system. A *rotation matrix* S^i may be defined to define these local displacement coordinates in terms of global coordinates, denoted $\hat{q}^i$; that is,

$$q^i = S^i \hat{q}^i \tag{1.1.26}$$

The transformed element displacements now coincide with components of the global displacement vector z_g. Therefore, a *Boolean transformation matrix* β^i may be defined, consisting of only zeros and ones, that gives the relation

$$\hat{q}^i = \beta^i z_g \tag{1.1.27}$$

Note that if $\hat{q}^i$ is an r-vector and z_g is an n-vector ($n > r$), then β^i is an $r \times n$ matrix that consists only of r unit components, with zeros as the remaining entries.

GENERALIZED GLOBAL STIFFNESS MATRIX

Denoting the ith element stiffness matrix as k^i, strain energy in the ith element may be written as

$$\text{SE}^i = \tfrac{1}{2} q^{i\text{T}} k^i q^i \tag{1.1.28}$$

Substituting from Eqs. (1.1.26) and (1.1.27), this is

$$\text{SE}^i = \tfrac{1}{2} \hat{q}^{i\text{T}} S^{i\text{T}} k^i S^i \hat{q}^i = \tfrac{1}{2} z_g^{\text{T}} \beta^{i\text{T}} S^{i\text{T}} k^i S^i \beta^i z_g \tag{1.1.29}$$

The strain energy of the entire structure is now obtained by summing the strain energy over all NE elements in the structure, to obtain

$$\begin{aligned} \text{SE} &= \tfrac{1}{2} z_g^{\text{T}} \left[\sum_{i=0}^{\text{NE}} \beta^{i\text{T}} S^{i\text{T}} k^i S^i \beta^i \right] z_g \\ &\equiv \tfrac{1}{2} z_g^{\text{T}} K_g z_g \end{aligned} \tag{1.1.30}$$

where K_g is the *generalized global stiffness matrix*,

$$K_g = \sum_{i=1}^{\text{NE}} \beta^{i\text{T}} S^{i\text{T}} k^i S^i \beta^i \tag{1.1.31}$$

REDUCED GLOBAL STIFFNESS MATRIX

If all boundary conditions associated with the structure have been imposed so that no rigid-body degrees-of-freedom exist, then the generalized global stiffness matrix K_g is positive definite, denoted simply by K, and is called the *reduced global stiffness matrix*. However, if the generalized global stiffness matrix is assembled without consideration of boundary conditions, it will generally not be positive definite. It is important to make this distinction, as will be seen later, since many formulations and computer codes use matrix methods that employ the generalized global stiffness matrix and impose constraint conditions during the solution process. They do not explicitly eliminate dependent displacement coordinates, so the positive definite reduced global stiffness matrix K is not constructed and thus is not available for design sensitivity calculations.

GENERALIZED GLOBAL MASS MATRIX

As in the case of strain energy, the kinetic energy of the ith element may be written in terms of generalized velocities. Since the matrices S^i and β^i do not depend on generalized coordinates,

$$\dot{q} = S^i \dot{\hat{q}}^i \tag{1.1.32}$$

$$\dot{\hat{q}} = \beta^i \dot{z}_g \tag{1.1.33}$$

Using these relationships, the kinetic energy of the ith element may be written as

$$\mathrm{KE}^i = \tfrac{1}{2}\dot{q}^{i^{\mathrm{T}}} m^i \dot{q}^i = \tfrac{1}{2}\dot{\hat{q}}^{i^{\mathrm{T}}} S^{i^{\mathrm{T}}} m^i S^i \dot{\hat{q}}^i = \tfrac{1}{2}\dot{z}_g^{\mathrm{T}} \beta^{i^{\mathrm{T}}} S^{i^{\mathrm{T}}} m^i S^i \beta^i \dot{z}_g \tag{1.1.34}$$

Summing the kinetic energy over all elements, the total kinetic energy for the system is

$$\mathrm{KE} = \tfrac{1}{2}\dot{z}_g^{\mathrm{T}} \left[\sum_{i=1}^{\mathrm{NE}} \beta^{i^{\mathrm{T}}} S^{i^{\mathrm{T}}} m^i S^i \beta^i \right] \dot{z}_g$$

$$\equiv \tfrac{1}{2}\dot{z}_g^{\mathrm{T}} M_g \dot{z}_g \tag{1.1.35}$$

where M_g is the *generalized global mass matrix*,

$$M_g = \sum_{i=1}^{\mathrm{NE}} \beta^{i^{\mathrm{T}}} S^{i^{\mathrm{T}}} m^i S^i \beta^i \tag{1.1.36}$$

Presuming that all structural elements have mass, it is impossible to obtain a nonzero velocity without investing a finite amount of kinetic energy. Therefore, a global system mass matrix will always be positive definite.

REDUCED GLOBAL MASS MATRIX

If boundary conditions have been taken into account before the global displacement vector is defined, the *reduced global mass matrix* will be denoted M, as in the case of the corresponding reduced global stiffness matrix K.

Note that the global stiffness and mass matrices depend on design variables that appear in the element stiffness and mass matrices, in the case of member size design variables, and on geometrical design variables that appear in the rotation matrices S^i. It is clear that the dependence of global stiffness and mass matrices on geometric variables is much more complex than the dependence on member size design variables.

ELEMENTARY EXAMPLE

As a simple example to illustrate use of the foregoing transformations, consider the two-bar truss of Fig. 1.1.3. Since rotations at the ends of the truss elements do not arise in either the strain or kinetic energy expressions, they are simply suppressed. The transformations between body-fixed and globally oriented element displacement coordinates are

$$q^1 \equiv \begin{bmatrix} \sin\theta & \cos\theta & 0 & 0 \\ -\cos\theta & \sin\theta & 0 & 0 \\ 0 & 0 & \sin\theta & \cos\theta \\ 0 & 0 & -\cos\theta & \sin\theta \end{bmatrix} \hat{q}^1$$

$$\equiv S^1 \hat{q}^1$$

$$q^2 = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ -\sin\theta & -\cos\theta & 0 & 0 \\ 0 & 0 & \cos\theta & -\sin\theta \\ 0 & 0 & -\sin\theta & -\cos\theta \end{bmatrix} \hat{q}^2$$

$$\equiv S^2 \hat{q}^2$$

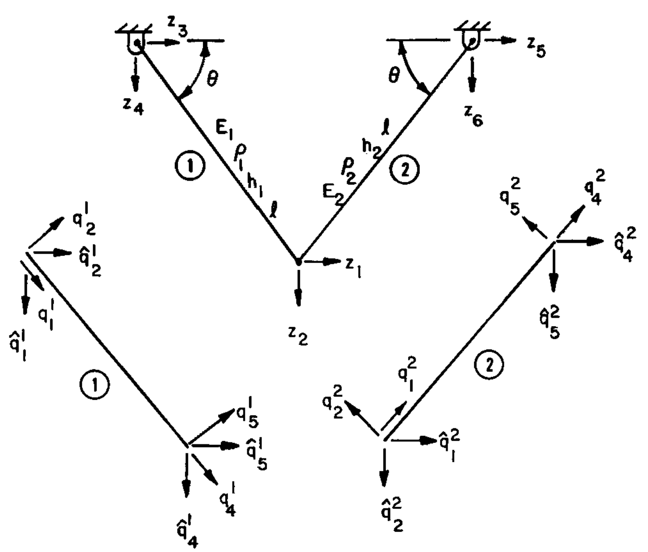

Fig. 1.1.3 Two-bar truss.

The transformations between globally oriented element coordinates and global coordinates are

$$\hat{q}^1 = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} z_g$$

$$\equiv \beta^1 z_g$$

$$\hat{q}^2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} z_g$$

$$\equiv \beta^2 z_g$$

Using these transformations and the expressions of Eqs. (1.1.31) and (1.1.36) for the generalized global stiffness and mass matrices [see page 16 for Eq. (1.1.37)],

$$M_g = \frac{l}{6} \begin{bmatrix} 2(\rho_1 h_1 + \rho_2 h_2) & 0 & \rho_1 h_1 & 0 & \rho_2 h_2 & 0 \\ & 2(\rho_1 h_1 + \rho_2 h_2) & 0 & \rho_1 h_1 & 0 & \rho_2 h_2 \\ & & 2\rho_1 h_1 & 0 & 0 & 0 \\ & & & 2\rho_2 h_2 & 0 & 0 \\ & \text{symmetric} & & & 2\rho_2 h_2 & 0 \\ & & & & & 2\rho_2 h_2 \end{bmatrix} \tag{1.1.38}$$

If the pin joints at the top of the truss are fixed, then boundary conditions for this structure are $z_3 = z_4 = z_5 = z_6 = 0$. Imposing these boundary conditions, the strain and kinetic energies are obtained in terms of only two displacement coordinates, z_1 and z_2. This amounts to deleting rows and columns corresponding to specified displacement coordinates in the generalized global stiffness and mass matrices of Eqs. (1.1.37) and (1.1.38). As a result, the reduced stiffness and mass matrices are obtained as

$$K = \frac{1}{l} \begin{bmatrix} (E_1 h_1 + E_2 h_2)\cos^2\theta & (E_1 h_1 - E_2 h_2)\sin\theta\cos\theta \\ (E_1 h_1 - E_2 h_2)\sin\theta\cos\theta & (E_1 h_1 + E_2 h_2)\sin^2\theta \end{bmatrix} \tag{1.1.39}$$

$$M = \frac{l}{3} \begin{bmatrix} \rho_1 h_1 + \rho_2 h_2 & 0 \\ 0 & \rho_1 h_1 + \rho_2 h_2 \end{bmatrix} \tag{1.1.40}$$

$$
K_g = \frac{1}{l}\begin{bmatrix}
(E_1h_1 + E_2h_2)\cos^2\theta & (E_1h_1 - E_2h_2)\sin\theta\cos\theta & -E_1h_1\cos^2\theta & -E_1h_1\sin\theta\cos\theta & -E_2h_2\cos^2\theta & E_2h_2\sin\theta\cos\theta \\
 & (E_1h_1 + E_2h_2)\sin^2\theta & -E_1h_1\sin\theta\cos\theta & -E_1h_1\sin^2\theta & -E_2h_2\sin\theta\cos\theta & -E_2h_2\sin^2\theta \\
 & & E_1h_1\cos^2\theta & E_1h_1\sin\theta\cos\theta & 0 & 0 \\
 & \text{symmetric} & & E_1h_1\sin^2\theta & 0 & 0 \\
 & & & & E_2h_2\cos^2\theta & -E_2h_2\sin\theta\cos\theta \\
 & & & & & E_2h_2\sin^2\theta
\end{bmatrix}
\qquad (1.1.37)
$$

Note that while the generalized global stiffness matrix K_g of Eq. (1.1.37) is singular (in fact it has rank deficiency 4), the reduced stiffness matrix K of Eq. (1.1.39) is positive definite.

While this two-bar truss is a trivial example, it illustrates a systematic procedure for assembling global stiffness and mass matrices. Since this assembly process is systematic, numerous computer codes have been developed to automate the process of constructing K_g and M_g. Depending on the nature of the boundary conditions in the problem, it is possible to systematically collapse the generalized global stiffness and mass matrices to reduced global stiffness and mass matrices K and M, as was done in this example. In many applications, however, more complicated constraints among generalized coordinates arise (e.g., multipoint constraints), making the reduction process nontrivial. Numerical techniques, including systematic reduction and application of Lagrange multipliers, are used in such problems [4, 5, 7].

1.1.3 Variational Principles of Mechanics

POTENTIAL ENERGY

Structural systems considered in this chapter are *conservative* in nature; that is, the work done by a system of applied forces in traversing any closed path in displacement space must be equal to zero. Denoting by F_g a vector of force components that are consistent with the global displacement vector z_g, this condition is

$$\int_C F_g^T \, dz_g = 0 \tag{1.1.41}$$

where C is any closed path in the space of displacement-generalized coordinates. As is well known [2], an analytical condition for a force field $F_g(z_g) = [F_{g1} \ \ldots \ F_{gn}]^T$ to be conservative is that

$$\partial F_{gi}/\partial z_{gj} = \partial F_{gj}/\partial z_{gi}, \qquad i, j = 1, \ldots, n \tag{1.1.42}$$

Presuming that the force field $F_g(z_g)$ is conservative, there exists a *potential energy* function $\mathrm{PE}(z_g)$ such that

$$F_{gi}(z_g) = -\partial \mathrm{PE}/\partial z_{gi}, \qquad i = 1, \ldots, n \tag{1.1.43}$$

For a constant applied force F_g, the condition of Eq. (1.1.42) is trivially satisfied, and the potential energy may be written as

$$\mathrm{PE} = -F_g^T z_g \tag{1.1.44}$$

It may be verified that Eq. (1.1.43) holds in this case.

Consider next a situation that arises in the case of buckling of structures, in which displacement at the point of application of a constant applied load P is given as a quadratic form in displacement z_g. This is indicated in Eq. (1.1.12) for a beam element, where

$$\Delta = \tfrac{1}{2}z_g^T D_g z_g \tag{1.1.45}$$

It is presumed that the *global geometric stiffness matrix* D_g has been transformed to symmetric form, which is always possible for a quadratic form. The potential energy of the load P is thus

$$\mathrm{PE} = -P\Delta = -\frac{P}{2}z_g^T D_g z_g \tag{1.1.46}$$

where the sign convention has P and Δ measured positive in the same direction.

For conservative mechanical systems, it is possible to obtain a potential energy function associated with all applied loads. The *total potential energy* of a structural system is defined as the sum of the strain energy of the structure and the potential energy of the applied loads; that is,

$$\mathrm{TPE} = \mathrm{SE} + \mathrm{PE} \tag{1.1.47}$$

For linear structural systems, the strain energy is given by the quadratic form of Eq. (1.1.30), and the potential energy of the applied loads is the sum of terms arising in Eqs. (1.1.44) and (1.1.46). The total potential energy can thus be written as

$$\mathrm{TPE} = \tfrac{1}{2}z_g^T K_g z_g - F_g^T z_g - \frac{P}{2}z_g^T D_g z_g \tag{1.1.48}$$

THEOREM OF MINIMUM TOTAL POTENTIAL ENERGY

Denoting by Z the vector space (see Appendix A.2) of *kinematically admissible displacements* for the structural system (presuming homogeneous boundary and interface conditions), the following theorem of minimum total potential energy [2, 7] is true.

THEOREM 1.1.1 (minimum total potential energy) The displacement $z_g \in Z$ that occurs due to an externally applied conservative load acting on an elastic structure minimizes the total potential energy of the structural system, over all kinematically admissible displacements.

It is important to note that this statement of the theorem of minimum total potential energy does not require that the displacement coordinates z_{gi}

($i = 1, \ldots, n$) be independent. However, it is presumed that they are related by homogeneous linear equations. While this limitation is not essential in the theory of structural mechanics, it is adequate for the purposes of this text.

LAGRANGE'S EQUATIONS OF MOTION

The second major variational principle of structural mechanics employed here provides a variational form of the equations of motion of a dynamic system. Presuming that the applied forces F_g depend only on time [i.e., $F_g = F_g(t)$], the *Lagrangian* of a dynamic system may be defined as

$$L = T(\dot{z}_g, \dot{z}_g) - \mathrm{TPE}(z_g) \tag{1.1.49}$$

where $T(\dot{z}_g, \dot{z}_g)$ is the kinetic energy of the system, which is a quadratic form in $\dot{z}_g$. The Lagrangian for a linear structural system, neglecting the effect of the last term in Eq. (1.1.48), is

$$L = \tfrac{1}{2}\dot{z}_g^{\mathrm{T}} M_g \dot{z}_g - \tfrac{1}{2} z_g^{\mathrm{T}} K_g z_g + z_g^{\mathrm{T}} F_g \tag{1.1.50}$$

In terms of the Lagrangian, the motion of a conservative structural system with a subspace Z of kinematically admissible displacements may be characterized by the following theorem [8].

THEOREM 1.1.2 (variational form of Lagrange's equations) The equations of motion of a conservative structural system, for $z_g(t)$ in the space Z of kinematically admissible displacements, may be written in the form

$$\sum_{i=1}^{n} \left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{z}_{gi}}\right) - \frac{\partial L}{\partial z_{gi}} \right] \bar{z}_{gi}(t) = 0 \tag{1.1.51}$$

which is valid for all *virtual displacements* $\bar{z}_g(t)$ that are consistent with constraints [i.e., $\bar{z}_g(t) \in Z$].

The notation $\bar{z}_g$ as a virtual displacement is used here in place of the more conventional δz_g. This and related departures from conventional structural mechanics notation are selected to avoid ambiguity and excessive use of the symbol δ, which appears later as a total differential, the Dirac-δ operator, and in other mathematical contexts.

This variational form of Lagrange's equations of motion is valid even for dependent state variables. In case kinematic admissibility conditions have been employed to algebraically reduce the global displacement vector z_g to independent form (of dimension m), so that $M_g = M$ and $K_g = K$ are the reduced global mass and stiffness matrices for the system, Eq. (1.1.51) may be written in the reduced form

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{z}_i}\right) - \frac{\partial L}{\partial z_i} = 0, \qquad i = 1, \ldots, m \tag{1.1.52}$$

It is critical to verify that the displacement coordinates z_i are independent before Eq. (1.1.52) is employed. This form of Lagrange's equations of motion is invalid if the displacement coordinates are dependent.

1.1.4 Reduced Matrix Equations of Structural Mechanics

DISPLACEMENT DUE TO STATIC LOAD

Consider the case of a linear structural system described by reduced global stiffness and mass matrices K and M and externally applied loads F. For such a system, kinematic constraints have been used to eliminate dependent displacement coordinates, thus yielding an independent *reduced displacement vector* z. In this case, the theorem of minimum total potential energy requires that at the position of equilibrium, the gradient of the total potential energy must be equal to zero. Using Eq. (1.1.48), with $P = 0$,

$$Kz = F \tag{1.1.53}$$

Further, with boundary conditions and interface conditions explicitly eliminated, the reduced global stiffness matrix K for a structure is positive definite and Eq. (1.1.53) is both necessary and sufficient for stable equilibrium.

BUCKLING

In buckling of structures, a potential energy term of the form given in Eq. (1.1.46) arises, and no other externally applied forces are considered. In such a case, the theorem of minimum total potential energy for stable equilibrium yields the condition

$$Kz - PDz = 0 \tag{1.1.54}$$

For a positive definite reduced global stiffness matrix K, if $P = 0$, then the only solution of Eq. (1.1.54) is the trivial solution $z = 0$; that is, the only stable equilibrium state of the system with no externally applied load is zero displacement. As P increases, particularly since D is generally positive semidefinite, a point will be reached at which the matrix $K - PD$ ceases to be positive definite, hence it becomes singular. The smallest load P for which this occurs is called the fundamental *buckling load* of the structure.

Since the coefficient matrix of z in Eq. (1.1.54) becomes singular, a nontrivial solution exists, but it is not unique. The solution is, therefore, an eigenvector corresponding to the eigenvalue P. In order to distinguish the eigenvector associated with buckling from the static displacement state, the eigenvector is denoted as y (called a *buckling mode*), rather than z, yielding the *generalized eigenvalue problem*

$$Ky = PDy \tag{1.1.55}$$

The matrix K is taken to be positive definite, and D is positive semidefinite. Thus, all eigenvalues of Eq. (1.1.55) are strictly positive.

DYNAMIC RESPONSE

Consider next the case of dynamic response of a structure with no boundary or interface conditions; that is, with independent generalized coordinates. Lagrange's equations [Eq. (1.1.52)] apply in this case and may be written in matrix form, using Eq. (1.1.50) with $F = F_g$, $M = M_g$ and $K = K_g$, as

$$M\ddot{z} + Kz - F = 0 \tag{1.1.56}$$

Initial conditions of motion for such a system consist of specifying the position and velocity of the system at some initial time, say $t = 0$; that is,

$$\begin{aligned} z(0) &= z^0 \\ \dot{z}(0) &= \dot{z}^0 \end{aligned} \tag{1.1.57}$$

NATURAL VIBRATION

Natural vibration of a structure is defined as harmonic motion of the structural system, with no applied load. A *natural frequency* ω is sought such that the solution $z(t)$ of Eq. (1.1.56), with $F = 0$, is harmonic; that is,

$$z(t) = y \sin(\omega t + \alpha) \tag{1.1.58}$$

where y is a constant vector defining a *mode shape* of vibration. Substituting $z(t)$ of Eq. (1.1.58) into Eq. (1.1.56), with $F = 0$,

$$[-\omega^2 My + Ky] \sin(\omega t + \alpha) = 0 \tag{1.1.59}$$

which must hold for all time t. Therefore, the generalized eigenvalue problem is

$$Ky = \zeta My \tag{1.1.60}$$

where $\zeta = \omega^2$. Equation (1.1.60) is an eigenvalue problem for natural frequency ω and associated mode shape y, just as Eq. (1.1.55) was an eigenvalue problem for buckling load P and mode shape y. In both cases, the reduced global stiffness matrix K is positive definite, and both D and M are at least positive semidefinite. These mathematical properties of the matrices arising in structural equations play a key role in both theoretical properties of solutions and computational methods for constructing solutions.

1.1.5 Variational Equations of Structural Mechanics

VARIATIONAL EQUILIBRIUM EQUATION

It is not necessary to eliminate explicitly the dependent displacement coordinates in order to obtain the governing equations of a structural system. Let Z be the vector space of kinematically admissible displacements. Consider first a structure with externally applied load F_g and potential energy given by Eq. (1.1.44). The theorem of minimum total potential energy is still valid for displacement in the vector space Z. Let z_g be the equilibrium position that minimizes TPE of Eq. (1.1.47) over the vector space Z. Consider an arbitrary *virtual displacement* $\bar{z}_g \in Z$, and evaluate the total potential energy at an arbitrary point neighboring z_g; that is, for ε small and $\bar{z}_g$ fixed,

$$\mathrm{TPE}(z_g + \varepsilon \bar{z}_g) \equiv H(\varepsilon) \tag{1.1.61}$$

Since the total potential energy has a minimum at z_g, the function $H(\varepsilon)$ defined by Eq. (1.1.61) has a minimum at $\varepsilon = 0$ for any $\bar{z}_g \in Z$. It is therefore required that the derivative of H with respect to ε be zero at $\varepsilon = 0$. Using the expression of Eq. (1.1.48) for the total potential energy, with the last term deleted,

$$\bar{z}_g^{\mathrm{T}} K_g z_g - \bar{z}^{\mathrm{T}} F_g = 0 \qquad \text{for all} \quad \bar{z}_g \in Z \tag{1.1.62}$$

This is called the *variational equation of equilibrium* of the structure.

In order to take advantage of the mathematical form of this problem, define the *energy bilinear form*

$$a(z_g, \bar{z}_g) = \bar{z}_g^{\mathrm{T}} K_g z_g \tag{1.1.63}$$

and the *load linear form* defined by the load F_g as

$$l(\bar{z}_g) = \bar{z}_g^{\mathrm{T}} F_g \tag{1.1.64}$$

Using this notation, the variational equation of Eq. (1.1.62) can be written as

$$a(z_g, \bar{z}_g) = l(\bar{z}_g) \qquad \text{for all} \quad \bar{z}_g \in Z \tag{1.1.65}$$

Under the hypothesis that the strain energy quadratic form is positive definite on the vector space Z of kinematically admissible displacements, the following theorem is true.

THEOREM 1.1.3 (theorem of virtual work) Assume that

$$a(z_g, z_g) > 0 \qquad \text{for all} \quad z_g \in Z, \quad z_g \neq 0 \tag{1.1.66}$$

Then Eq. (1.1.65) has a unique solution $z_g \in Z$.

PROOF This theorem follows directly from the Lax–Milgram theorem of functional analysis [9] and the positive definite property of $a(z_g, z_g)$. An alternative proof uses the fact that $a(z_g, z_g)$ is convex on Z and the result from optimization theory [10] that Eq. (1.1.65) is necessary and sufficient for z_g to be the minimum point. ■

The unique solution of Eq. (1.1.65), which is guaranteed by Theorem 1.1.3, is exactly the solution that would be obtained by first eliminating dependent displacement coordinates, constructing the reduced global stiffness matrix, and solving Eq. (1.1.53). The latter procedure is executed numerically in finite element computer codes. The variational form of the structural equations of Eq. (1.1.65) will be shown to have substantial theoretical advantage in design sensitivity analysis.

REDUCTION OF VARIATIONAL EQUILIBRIUM EQUATION TO MATRIX FORM

Equation (1.1.65) can be used to generate a matrix equation for constructing a numerical solution. Let $\phi^i \in Z \subset R^n$ $(i = 1, \ldots, m; m < n)$ be a *basis* of the vector space Z of kinematically admissible displacements (i.e., a *linearly independent* set of vectors that *span* Z). Then the solution of Eq. (1.1.65) may be written as

$$z_g = \sum_{i=1}^{m} c_i \phi^i = \Phi c \tag{1.1.67}$$

where $\Phi = [\phi^1 \cdots \phi^m]$ and the coefficients c_i are uniquely determined. Substituting this representation for z into Eq. (1.1.65) and evaluating Eq. (1.1.65) at $\bar{z}_g = \phi^j$ $(j = 1, \ldots, m)$ gives the equations

$$\sum_{i=1}^{m} a(\phi^i, \phi^j) c_i = l(\phi^j), \qquad j = 1, \ldots, m \tag{1.1.68}$$

Defining

$$\begin{aligned} \check{K} &\equiv [a(\phi^i, \phi^j)]_{m \times m} = [\phi^{i\mathrm{T}} K_g \phi^j]_{m \times m} = \Phi^{\mathrm{T}} K_g \Phi \\ \check{F} &\equiv [l(\phi^i)]_{m \times 1} = [\phi^{i\mathrm{T}} F]_{m \times 1} = \Phi^{\mathrm{T}} F_g \\ c &\equiv [c_i]_{m \times 1} \end{aligned} \tag{1.1.69}$$

Eq. (1.1.68) may be written in matrix form as

$$\check{K} c = \check{F} \tag{1.1.70}$$

This equation has a unique solution c since $\check{K}$ is positive definite (due to the assumption of positive definiteness of the energy bilinear form on Z). It is also clear that the matrices $\check{K}$ and $\check{F}$ depend on the choice of basis of the space Z of kinematically admissible displacements. Different choices of bases yield different matrices, but the resulting solution is unique.

It is an interesting exercise to show that the foregoing argument can be reversed to construct a proof of Theorem 1.1.3.

VARIATIONAL EQUATION OF BUCKLING

Consider now the problem of buckling of a structure, in which the potential energy is given by Eq. (1.1.46). Just as in the preceding discussion, the total potential energy must be minimized over the space Z of kinematically admissible displacements. Using the total potential energy expression of Eq. (1.1.48) with $F_g = 0$ and Eq. (1.1.61), the derivative with respect to ε must be equal to zero, yielding the *variational equation of buckling*

$$\bar{y}_g^T K_g y_g = P \bar{y}_g^T D_g y_g \qquad \text{for all} \quad \bar{y}_g \in Z \tag{1.1.71}$$

where the solution is denoted by the vector y_g. Defining the bilinear form d as

$$d(y_g, \bar{y}_g) = \bar{y}_g^T D_g y_g \tag{1.1.72}$$

Eq. (1.1.71) may be written in the more compact form

$$a(y_g, \bar{y}_g) = P d(y_g, \bar{y}_g) \qquad \text{for all} \quad \bar{y}_g \in Z \tag{1.1.73}$$

This is the variational form of the eigenvalue problem for buckling of the structure.

REDUCTION OF THE VARIATIONAL EQUATION OF BUCKLING TO MATRIX FORM

Just as in the case of equilibrium of the structure, the variational equation of Eq. (1.1.73) can be reduced to a matrix equation, using a basis for the space Z of kinematically admissible displacements. This yields a generalized eigenvalue problem

$$\check{K} c = P \check{D} c \tag{1.1.74}$$

where components of the vector c are coefficients of Eq. (1.1.67). Expanding the eigenvector y_g in terms of the basis ϕ^i and the matrix D_g gives

$$\check{D} \equiv [d(\phi^i, \phi^j)]_{m \times m} = [\phi^{iT} D_g \phi^j]_{m \times m} = \Phi^T D_g \Phi \tag{1.1.75}$$

As will normally be the case, the matrix D_g is positive definite on the vector space Z of kinematically admissible displacements, so the matrices $\check{D}$ and $\check{K}$ are positive definite, yielding important theoretical and computational properties.

VARIATIONAL EQUATION OF VIBRATION

Consider now the variational form of the Lagrange equations of motion in Eq. (1.1.51), with the Lagrangian defined by Eq. (1.1.50). In vector form, Eq. (1.1.51) is

$$\bar{z}_g^T(t)[M_g \ddot{z}_g(t) + K_g z_g(t) - F_g(t)] = 0 \qquad \text{for all} \quad \bar{z}_g(t) \in Z \tag{1.1.76}$$

This equation must hold for all values of time t.

If harmonic motion with $F_g = 0$ is of interest, a solution of Eq. (1.1.76) of the form given in Eq. (1.1.58) is sought. Substituting into Eq. (1.1.76) (with $\bar{y}_g$ in place of $\bar{z}_g$) gives

$$[-\omega^2 \bar{y}_g^T M_g y_g + \bar{y}_g^T K_g y_g] \sin(\omega t + \alpha) = 0 \qquad \text{for all} \quad \bar{y}_g \in Z \tag{1.1.77}$$

which must hold for all time t. Thus, it is required that

$$a(y_g, \bar{y}_g) = \zeta d(y_g, \bar{y}_g) \qquad \text{for all} \quad \bar{y}_g \in Z \tag{1.1.78}$$

where $\zeta = \omega^2$. The bilinear form $a(\cdot, \cdot)$ is as given in Eq. (1.1.63), and the bilinear form $d(\cdot, \cdot)$ is defined as

$$d(y_g, \bar{y}_g) = \bar{y}_g^T M_g y_g \tag{1.1.79}$$

Since the generalized mass matrix M_g is positive definite and the strain energy bilinear form $a(\cdot, \cdot)$ is normally positive definite on Z, attractive mathematical properties are associated with the variational equation given by Eq. (1.1.78).

REDUCTION OF THE VARIATIONAL EQUATION OF VIBRATION TO MATRIX FORM

Just as in the foregoing analysis of the buckling eigenvalue problem, a matrix equation of the form of Eq. (1.1.74) may be obtained for the vibration problem. Thus, vibration and buckling problems have very similar form and similar mathematical properties.

While it is clear that one can always reduce the finite-dimensional structural analysis problem to matrix equation form, it will be shown in Section 1.2.4 that the variational form given here is better suited for structural design sensitivity analysis.

1.2 STATIC RESPONSE DESIGN SENSITIVITY

1.2.1 Statement of the Problem

As explained in Section 1.1, when member size and geometric variables are taken as design variables, the generalized global stiffness matrix and load vector are functions of the design variables; that is,

$$\begin{aligned} K_g &= K_g(b) \\ F_g &= F_g(b) \end{aligned} \tag{1.2.1}$$

where the vector $b = [b_1, \ldots, b_k]^T$ is the vector of member-size design variables and variables that locate selected nodes in the structure. It is presumed here that kinematic admissibility conditions (boundary conditions and interface conditions) are not explicit functions of design. The case in which kinematic admissibility conditions are functions of design is included in the shape design sensitivity formulation of Chapter 3.

Since the generalized global stiffness matrix and load vector are design dependent, the bilinear and linear forms of Eqs. (1.1.63) and (1.1.64) are also design dependent. They are denoted here as

$$\begin{aligned} a_b(z_g, \bar{z}_g) &= \bar{z}_g^T K_g(b) z_g \\ l_b(\bar{z}_g) &= \bar{z}_g^T F_g(b) \end{aligned} \tag{1.2.2}$$

Recall that there exists a unique solution z_g of Eq. (1.1.65), or equivalently of Eq. (1.1.70). Since these equations depend explicitly on design, it is clear that the solution z_g is design dependent; that is,

$$z_g = z_g(b) \tag{1.2.3}$$

In most structural design problems, some cost function is to be minimized (or an objective function is to be maximized), subject to constraints on stress, displacement, and design variables. Consider now a general function that may represent any of these performance measures for a structure, written in the form

$$\psi = \psi(b, z_g(b)) \tag{1.2.4}$$

The dependence of this function on design arises in two ways: (1) explicit design dependence; and (2) implicit dependence through the solution z_g of the state equations. The objective of *design sensitivity analysis* is to determine the total dependence of such functions on design (i.e., to find $d\psi/db$). In this connection, two questions should be asked: (1) Given that the function ψ is differentiable in its arguments, is the total dependence of ψ on design differentiable? (2) If the solution z_g of the state equations is differentiable with respect to design, how can the derivative of ψ with respect to design be calculated?

1.2.2 Design Sensitivity Analysis with Reduced Global Stiffness Matrix

Consider first the structural formulation in which dependent state variables have been removed through direct elimination with boundary conditions and a set of structural equations of the form of Eq. (1.1.53) arise, in the form

$$K(b)z = F(b) \tag{1.2.5}$$

where $K(b)$ is the reduced global stiffness matrix and $F(b)$ the reduced load. Recall that the reduced global stiffness matrix $K(b)$ is positive definite, hence nonsingular. Assume that all entries in $K(b)$ and $F(b)$ are s times continuously differentiable with respect to design. The implicit function theorem [11] thus guarantees that the solution $z = z(b)$ of Eq. (1.2.5) is also s times continuously differentiable. Thus, the foregoing question concerning differentiability of z with respect to design is answered. The problem of computing total derivatives of the function ψ of Eq. (1.2.4) with respect to design remains to be solved.

DIRECT DIFFERENTIATION METHOD

Using the chain rule of differentiation and the matrix calculus notation of Appendix A.3, the total derivative of ψ with respect to b may be calculated as

$$\frac{d\psi}{db} = \frac{\partial\psi}{\partial b} + \frac{\partial\psi}{\partial z}\frac{dz}{db} \tag{1.2.6}$$

Differentiating both sides of Eq. (1.2.5) with respect to b,

$$K(b)\frac{dz}{db} = -\frac{\partial}{\partial b}(K(b)\tilde{z}) + \frac{\partial F(b)}{\partial b} \tag{1.2.7}$$

where the tilde ($\tilde{\ }$) indicates a variable that is to be held constant for the process of partial differentiation. Since the matrix $K(b)$ is nonsingular, Eq. (1.2.7) may be solved for dz/db as

$$\frac{dz}{db} = K^{-1}(b)\left[\frac{\partial F(b)}{\partial b} - \frac{\partial}{\partial b}(K(b)\tilde{z})\right] \tag{1.2.8}$$

This result may now be substituted into Eq. (1.2.6) to obtain

$$\frac{d\psi}{db} = \frac{\partial\psi}{\partial b} + \frac{\partial\psi}{\partial z}K^{-1}(b)\frac{\partial}{\partial b}[F(b) - (K(b)\tilde{z})] \tag{1.2.9}$$

The usefulness of Eq. (1.2.9) is dubious, since in realistic applications direct computation of $K^{-1}(b)$ is impractical. Two alternatives may be used to overcome this difficulty. First, Eq. (1.2.7) may be numerically solved for dz/db and substituted into Eq. (1.2.6) to obtain the desired result. This is known as the *direct differentiation method*, which has been used extensively in structural optimization. Computational aspects of this approach are discussed in Section 1.2.4.

ADJOINT VARIABLE METHOD

An alternative approach is to define as *adjoint variable* λ as

$$\lambda \equiv \left[\frac{\partial\psi}{\partial z}K^{-1}(b)\right]^{\mathrm{T}} = K^{-1}(b)\frac{\partial\psi^{\mathrm{T}}}{\partial z} \tag{1.2.10}$$

where symmetry of the matrix K has been used. Rather than evaluating λ directly from Eq. (1.2.10), which involves $K^{-1}(b)$, both sides of Eq. (1.2.10) may be multiplied by the matrix $K(b)$ to obtain the following *adjoint equation in* λ:

$$K(b)\lambda = \partial\psi^{\mathrm{T}}/\partial z \tag{1.2.11}$$

Equation (1.2.11) may be solved for λ and the result substituted, using Eq. (1.2.10), into Eq. (1.2.9) to obtain

$$\frac{d\psi}{db} = \frac{\partial\psi}{\partial b} + \lambda^{\mathrm{T}}\left[\frac{\partial F(b)}{\partial b} - \frac{\partial}{\partial b}(K(b)\tilde{z})\right] \tag{1.2.12}$$

A somewhat more convenient form for derivative calculation is

$$\frac{d\psi}{db} = \frac{\partial\psi}{\partial b} + \frac{\partial}{\partial b}[\tilde{\lambda}^{\mathrm{T}}F(b) - \tilde{\lambda}^{\mathrm{T}}K(b)\tilde{z}] \tag{1.2.13}$$

This approach is called the *adjoint variable method* of design sensitivity analysis. Computational aspects of this approach are discussed in Section 1.2.4

1.2.3 Design Sensitivity Analysis with Generalized Global Stiffness Matrix

If the reduced global stiffness matrix $K(b)$ and reduced applied force vector $F(b)$ are readily available, either one of the foregoing methods yields a complete solution of the design sensitivity analysis problem. However, for nontrivial kinematic admissibility conditions (boundary conditions), particularly multipoint constraints involving linear combinations of several state variables, the matrices $K(b)$ and $F(b)$ are not explicitly generated. Thus, computation of the partial derivatives with respect to design on the right-hand side of Eq. (1.2.7) or in Eq. (1.2.13) is nontrivial. It is therefore desirable to develop a formulation for design sensitivity analysis that works directly with the singular generalized global stiffness matrix.

DIFFERENTIABILITY OF GLOBAL DISPLACEMENT

Consider an explicit form for the vector space Z of kinematically admissible displacements given by

$$Z = \{z_g \in R^n\colon\ Gz_g = 0\} \tag{1.2.14}$$

where G is an $(n - m) \times n$ matrix that defines boundary conditions and does not depend on design. With a basis ϕ^i $(i = 1,\ldots,m)$ of Z, which is independent of design, the solution z_g of the structural variational equation of

Eq. (1.1.62) may be represented in the form of Eq. (1.1.67), where the vector c of coefficients is determined by Eq. (1.1.70), written in the form

$$K(b)c = F(b) \tag{1.2.15}$$

Note that the dependence of K and F on design b is explicitly defined in terms of $K_g(b)$ and $F_g(b)$ in Eq. (1.1.69). Therefore, $K(b)$ and $F(b)$ are differentiable with respect to design, and $K(b)$ is nonsingular in a neighbourhood of the nominal design. The derivative of c with respect to design can thus be obtained by either of the foregoing methods. Once dc/db is determined, Eq. (1.1.67) may be used to obtain

$$dz_g/db = \Phi\, dc/db \tag{1.2.16}$$

since Φ does not depend on b. Thus, the question of differentiability is resolved. Computation of the required derivatives may be carried out using the variational formulation of Eq. (1.1.62), written here using the notation of Eq. (1.2.2) as

$$a_b(z_g, \bar{z}_g) = l_b(\bar{z}_g) \qquad \text{for all} \quad \bar{z}_g \in Z \tag{1.2.17}$$

DIRECTIONAL DERIVATIVES

In order to take advantage of the variational equation of Eq. (1.2.17), it is helpful to introduce directional derivative notation that will be used throughout the remainder of the text. Consider a nominal design b and neighboring designs described by arbitrary design variations δb and a small parameter $\tau > 0$ as

$$b_\tau = b + \tau\, \delta b \tag{1.2.18}$$

Similar to the first variation of the calculus of variations, the following *directional derivative* notation is employed (see Appendix A.3):

$$\begin{aligned}
z'_g = z'_g(b, \delta b) &\equiv \frac{d}{d\tau} z_g(b + \tau\, \delta b)\Big|_{\tau=0} = \frac{dz_g}{db}\, \delta b \\
a'_{\delta b}(z_g, \bar{z}_g) &\equiv \frac{d}{d\tau} a_{b+\tau\delta b}(z_g(b), \bar{z}_g)\Big|_{\tau=0} \\
&= \frac{\partial}{\partial b}(\bar{z}_g^T K_g(b) \tilde{z}_g)\, \delta b \\
l'_{\delta b}(\bar{z}_g) &\equiv \frac{d}{d\tau} l_{b+\tau\delta b}(\bar{z}_g)\Big|_{\tau=0} \\
&= \frac{\partial}{\partial b}(\bar{z}_g^T F_g(b))\, \delta b
\end{aligned} \tag{1.2.19}$$

where the prime (′) denotes *differential* (or *variation*) of a function of b in the direction δb. If the result is linear in δb, then the function whose differential has been taken is *differentiable*. Otherwise, it is only *directionally differentiable* (in the sense of Gateaux, Appendix A.3). With this notation, the prime may be employed with explicit inclusion of the argument δb to emphasize dependence on design variation.

Note that since the matrix G in Eq. (1.2.14), which defines the vector space Z of kinematically admissible displacements, does not depend on design, the arbitrary vector $\bar{z}_g \in Z$ in Eq. (1.2.17) need not depend on design. Taking the total variation of both sides of Eq. (1.2.17) and using the chain rule of differentiation,

$$a_b(z'_g, \bar{z}_g) = -a'_{\delta b}(z_g, \bar{z}_g) + l'_{\delta b}(\bar{z}_g) \qquad \text{for all} \quad \bar{z}_g \in Z \qquad (1.2.20)$$

where the arguments of all variations are b and δb.

Note that for $z_g(b) \in Z$, $Gz_g(b) = 0$. Taking the variation of both sides of this equation,

$$Gz'_g(b, \delta b) = Gz'_g = 0 \qquad (1.2.21)$$

Thus, z'_g is in the space Z of kinematically admissible displacements for any design variation δb. Equation (1.2.20) thus has a unique solution for z'_g.

DIRECT DIFFERENTIATION METHOD

By taking δb as a unit vector in the ith design coordinate direction, Eq. (1.2.20) may be solved for $\partial z_g/\partial b_i$. Repeating this process with $i = 1, 2, \ldots, k$ yields all the partial derivatives of z_g with respect to b. Specifically, Eq. (1.2.20) may be written in terms of the ith component of b,

$$\bar{z}_g^{\mathrm{T}} K_g(b) \frac{\partial z_g}{\partial b_i} = -\frac{\partial}{\partial b_i}(\bar{z}_g^{\mathrm{T}} K_g(b) \tilde{z}_g) + \frac{\partial}{\partial b_i}(\bar{z}_g^{\mathrm{T}} F_g(b)), \qquad i = 1, \ldots, k \qquad (1.2.22)$$

This may be interpreted as solving the original structural equation with an artificial applied load that is the coefficient of $\bar{z}_g^{\mathrm{T}}$ on the right side of Eq. (1.2.22).

ADJOINT VARIABLE METHOD

Consider the last term $(\partial\psi/\partial z_g)(dz_g/db)$ of Eq. (1.2.6), which is to be written without evaluation of the matrix dz_g/db. The "recipe" for the adjoint variable method is to regard the coefficient $\partial\psi/\partial z_g$ of dz_g/db as the transpose of a load vector, called the *adjoint load* $\partial\psi^{T}/\partial z_g$. The *adjoint variable* $\lambda_g \in Z$ associated with this load is to be found; that is,

$$a_b(\lambda_g, \bar{\lambda}_g) = \bar{\lambda}_g^T \frac{\partial\psi^T}{\partial z_g} = \frac{\partial\psi}{\partial z_g}\bar{\lambda}_g \qquad \text{for all} \quad \bar{\lambda}_g \in Z \tag{1.2.23}$$

Note that this is just the structural equation for a displacement λ_g due to an applied load vector $(\partial\psi/\partial z_g)^T$. Therefore, it may be readily solved.

Evaluating Eq. (1.2.23) at $\bar{\lambda}_g = z'_g$ (recall that $z'_g \in Z$) and using the notation introduced in the first line of Eq. (1.2.19),

$$\frac{\partial\psi}{\partial z_g} z'_g = \frac{\partial\psi}{\partial z_g}\frac{dz_g}{db}\,\delta b = a_b(\lambda_g, z'_g) \tag{1.2.24}$$

Similarly, evaluating Eq. (1.2.20) at $\bar{z}_g = \lambda_g$,

$$a_b(z'_g, \lambda_g) = -a'_{\delta b}(z_g, \lambda_g) + l'_{\delta b}(\lambda_g) \tag{1.2.25}$$

Noting that the energy bilinear form $a_b(\cdot, \cdot)$ is symmetric, Eqs. (1.2.24) and (1.2.25) yield

$$\frac{\partial\psi}{\partial z_g}\frac{dz_g}{db}\,\delta b = -a'_{\delta b}(z_g, \lambda_g) + l'_{\delta b}(\lambda_g) \tag{1.2.26}$$

Writing the total differential of the function ψ of Eq. (1.2.4),

$$\frac{d\psi}{db}\,\delta b = \left[\frac{\partial\psi}{\partial b} + \frac{\partial\psi}{\partial z_g}\frac{dz_g}{db}\right]\delta b \tag{1.2.27}$$

Substituting for the second term on the right of Eq. (1.2.27), using the expression from Eq. (1.2.26) and employing the second and third lines of Eq. (1.2.19), gives

$$\frac{d\psi}{db}\,\delta b = \left[\frac{\partial\psi}{\partial b} - \frac{\partial}{\partial b}(\tilde{\lambda}_g^T K_g(b)\tilde{z}_g) + \frac{\partial}{\partial b}(\tilde{\lambda}_g^T F_g(b))\right]\delta b \tag{1.2.28}$$

for any design variation δb. Since Eq. (1.2.28) holds for all design variations δb,

$$\frac{d\psi}{db} = \frac{\partial\psi}{\partial b} - \frac{\partial}{\partial b}[\tilde{\lambda}_g^T K_g(b)\tilde{z}_g - \tilde{\lambda}_g^T F_g(b)] \tag{1.2.29}$$

It is interesting to note that even though the generalized global stiffness matrix K_g is singular, the load vectors that are used in the direct differentiation approach of Eq. (1.2.22) are of the same form that arise in computation with the reduced global stiffness matrix in Eq. (1.2.7). Similarly, in the adjoint variable method, the single load vector that is employed in adjoint computation in Eq. (1.2.23) is of exactly the same form as in the matrix adjoint equation of Eq. (1.2.11), using the reduced global stiffness matrix. Computational considerations associated with this observation will be discussed in Section 1.2.4.

1.2.4 Computational Considerations

In most problems of structural design, numerous loading conditions must be accounted for in the design process. Therefore, instead of the single load discussed in the preceding sections, a family of loads arises that is denoted as F_g^j ($j = 1, \ldots, \mathrm{NL}$). The same structural stiffness matrix is applicable for all load conditions, but the structural equations yield different displacement vectors z_g^j ($j = 1, \ldots, \mathrm{NL}$) associated with different applied force vectors.

Further, in realistic design problems there are numerous performance constraints that must be accounted for in the design process. Even though there may be a multitude of constraints under consideration, the designer normally evaluates constraints at a trial design and wishes to obtain design sensitivity information for only those constraints that are violated or are nearly active. For the discussion here, designate the active constraints under consideration by the designer as ψ_i ($i = 1, \ldots, \mathrm{NC}$). It is further presumed that some contraint is active for each load condition. Otherwise, load conditions having no influence on any active constraint may be suppressed for purposes of design sensitivity analysis. Computations required for design sensitivity analysis by the direct differentiation approach and the adjoint variable approach may now be summarized, for both the matrix and variational analysis methods of Sections 1.2.2 and 1.2.3.

DIRECT DIFFERENTIATION METHOD

Consider first the direct analysis method of Section 1.2.2. To calculate the total derivative of each constraint ψ^i using the direct differentiation approach, Eq. (1.2.7) must be solved for each load condition, yielding the following system of equations:

$$K(b)\frac{dz^j}{db} = -\frac{\partial}{\partial b}(K(b)\tilde{z}^j) + \frac{\partial F^j(b)}{\partial b}, \qquad j = 1, \ldots, \mathrm{NL} \qquad (1.2.30)$$

Since each of the equations in Eq. (1.2.30) represents k equations for dz^j/db_i ($i = 1, \ldots, k$), there are $k \times \mathrm{NL}$ equations to be solved. Their solution is quite efficient since the reduced global stiffness matrix K has been factored previously in structural analysis. With all state derivatives with respect to design calculated in Eq. (1.2.30), design sensitivities may now be directly calculated from Eq. (1.2.6).

ADJOINT VARIABLE METHOD

Consider next the adjoint variable method in which Eq. (1.2.11) must be solved for each constraint under consideration; that is,

$$K(b)\lambda^i = \partial\psi_i^{\mathrm{T}}/\partial z^i, \qquad i = 1, \ldots, \mathrm{NC} \tag{1.2.31}$$

where it is presumed that each constraint $\psi_i(z^i)$ involves only the displacement z^i corresponding to the ith load. Thus, there are exactly NC equations to be solved for the vectors λ^i ($i = 1, \ldots, \mathrm{NC}$). Once this computation is complete, design sensitivities of each of the constraints are calculated directly from Eq. (1.2.13), requiring only a moderate amount of computation. Note that the coefficient matrix in Eq. (1.2.31) is the reduced global stiffness matrix, which was factored during structural analysis. Therefore, the amount of computational effort required to solve Eq. (1.2.31) is also moderate.

COMPARISON OF THE DIRECT DIFFERENTIATION AND ADJOINT VARIABLE METHODS

In determining which of the two approaches discussed above is to be employed, only the number of equations to be solved by the two approaches and the number of vectors to be stored and operated on in design sensitivity analysis need be compared. If $k \times \mathrm{NL} < \mathrm{NC}$, then the direct differentiation method of Eq. (1.2.30) is preferred. On the other hand, if $k \times \mathrm{NL} > \mathrm{NC}$, then the adjoint variable method of Eq. (1.2.31) is preferred. Since in structural optimization the number of active constraints NC must be no greater than the number of design variables k, the adjoint variable approach will be most efficient, even for a single loading condition. With multiple loading conditions, NC is normally much smaller than $k \times \mathrm{NL}$, leading to the conclusion that in most structural optimization applications the adjoint variable method will be more efficient. However, there may be applications in preliminary design in which the designer is considering a small number of design variables and trade-offs involve a large number of constraints. In such cases, the direct design differentiation approach of Eq. (1.2.30) is preferred.

Precisely the same counting process is applicable to the variational analysis approach of Eq. (1.2.17). In this approach, exactly $k \times \mathrm{NL}$ equations are solved in Eq. (1.2.22) for derivatives of the state variables with respect to design, for each loading condition. Similarly, using the adjoint variable technique exactly NC adjoint equations of Eq. (1.2.23) are solved for adjoint variables associated with each of the active constraints. Thus, precisely the same criteria are involved in determining which of the two approaches is best suited for the design problem under consideration.

COMPUTATION OF DESIGN DERIVATIVES

A comparison of the practicality of the reduced matrix equation and variational equation design sensitivity analysis approaches of Sections 1.2.2 and 1.2.3, respectively, is also possible. Since most structural analysis computer codes numerically construct or completely avoid the reduced global stiffness matrix $K(b)$, an explicit form of $K(b)$ is not generally available. Therefore, computations of the derivatives of $K(b)$ with respect to design required in Eq. (1.2.9) for the direct differentiation approach and in Eq. (1.2.13) for the adjoint variable approach lead to some difficulty. While transformations may be written that reduce the generalized global stiffness matrix K_g to a reduced global stiffness matrix K and inserted in the appropriate equations, the transformations differ from one computer code to another. Therefore, implementation of design sensitivity analysis using the reduced global stiffness matrix becomes code dependent and may be numerically inefficient.

If the variational equation formulation is employed, then the derivatives of the generalized global stiffness matrix $K_g(b)$ with respect to design that are required in Eq. (1.2.22) for the direct differentiation approach and in Eq. (1.2.29) for the adjoint variable approach can be calculated without difficulty. In fact, using Eq. (1.1.31) the derivative required in Eq. (1.2.29) may be written as the sum of derivatives of element matrices as

$$\begin{aligned}\frac{\partial}{\partial b}(\tilde{\lambda}_g^T K_g(b)\tilde{z}_g) &= \frac{\partial}{\partial b}\left[\sum_{i=1}^{NE} \tilde{\lambda}_g^T \beta^{iT} S^{iT}(b) k^i(b) S^i(b) \beta^i \tilde{z}_g\right] \\ &= \frac{\partial}{\partial b}\left[\sum_{i=1}^{NE} \tilde{\lambda}^{iT} S^{iT}(b) k^i(b) S^i(b) \tilde{z}^i\right] \qquad (1.2.32)\end{aligned}$$

where λ^i and z^i are components of the global adjoint and generalized coordinate vectors associated with the ith element. The practicality of this computation follows from two observations. First, for each element, the element stiffness matrix $k^i(b)$ and geometric matrix $S^i(b)$ will depend on only a small number of design variables that are associated with the given element and its nodes. Thus, only a few terms in the sum of Eq. (1.2.32) will be different from zero. Second, evaluation of design derivatives of the element bilinear forms in Eq. (1.2.32) require calculation of only a moderate number of terms.

A similar argument may be associated with computation of design derivatives in Eq. (1.2.22) for the direct differentiation approach, with the following exception: The entire set of k vectors $\partial z_g/\partial b_i$ is now required for complete design sensitivity analysis. However, the summation form of the generalized global stiffness matrix of Eq. (1.2.32), with $\bar{\lambda}$ replaced by $\bar{z}$, can be employed to somewhat reduce the computational burden in evaluating the right-hand side of Eq. (1.2.22). These are important practical considerations in adapting large-scale matrix structural analysis codes for computation of derivatives that are required in design sensitivity analysis. The directness with which these computations are performed with the variational analysis approach favors it for both generality and numerical effectiveness.

Another practical consideration that should not be overlooked involves calculating design derivatives of element matrices that are implicitly generated [3, 4]. Many modern finite element formulations carry out numerical integration to evaluate element stiffness and mass matrices, rather than using closed form expressions in terms of design variables, such as those presented in Section 1.1.1. For implicitly generated element matrices, the design differentiation can be carried through the sequence of calculations used to generate the element matrices, thus leading to implicit design derivative routines.

An alternative approach is simply to perturb one design variable at a time and use finite differences to approximate element matrix derivatives. For example,

$$\frac{\partial k^i}{\partial b_j} \approx \frac{k^i(b + \tau e^j) - k^i(b)}{\tau}$$

where e^j has a one in the jth position and zeros elsewhere and τ is a small perturbation in b_j.

Computational methods of design sensitivity analysis with implicitly generated elements have not yet been fully investigated and justifiably require future work owing to increasing use of implicit elements.

1.2.5 Second-Order Design Sensitivity Analysis

As shown in Sections 1.2.2 and 1.2.3, if the applied load vector and system stiffness matrix has s continuous derivatives with respect to design, then the state z has s continuous derivatives with respect to design. Presuming that a function ψ also has s continuous derivatives, calculation of up to s partial derivatives of ψ with respect to the design variables can be considered. Consider the specific case $s = 2$, that is, second-order design sensitivity.

DIRECT DIFFERENTIATION METHOD WITH REDUCED GLOBAL STIFFNESS MATRIX

Consider first the case of the structural formulation with a reduced global stiffness matrix (i.e., kinematic constraints have been explicitly eliminated and all components of the displacement vector z are independent). The chain rule of differentiation may then be used to obtain two derivatives of ψ with respect to components of the design variable,

$$\begin{aligned}\frac{d^2\psi}{db_i\,db_j} &= \frac{d}{db_j}\left[\frac{\partial\psi}{\partial b_i} + \frac{\partial\psi}{\partial z}\frac{dz}{db_i}\right] \\ &= \frac{\partial^2\psi}{\partial b_i\,\partial b_j} + \frac{\partial^2\psi}{\partial b_i\,\partial z\,db_j}\frac{dz}{} + \frac{\partial^2\psi}{\partial z\,\partial b_j}\frac{dz}{db_i} \\ &\quad + \frac{dz^{\mathrm{T}}}{db_j}\frac{\partial^2\psi}{\partial z^2}\frac{dz}{db_i} + \frac{\partial\psi}{\partial z}\frac{d^2 z}{db_i\,db_j}\end{aligned} \tag{1.2.33}$$

The notation used here needs some explanation. The derivative of ψ with respect to b_i on the left side of Eq. (1.2.33) is in reality a partial derivative of ψ with respect to b_i, accounting for dependence of ψ on b directly and on $z(b)$. Terms on the right side of Eq. (1.2.33) include partial derivatives of ψ with respect to its explicit dependence on b_i.

Since both first- and second-order derivatives of z with respect to design arise in Eq. (1.2.33), consider calculating them by using the structural equation

$$K(b)z = F(b) \tag{1.2.34}$$

Using the direct design differentiation approach, differentiate Eq. (1.2.34) with respect to b_i to obtain

$$K(b)\frac{dz}{db_i} = \frac{\partial F}{\partial b_i} - \frac{\partial}{\partial b_i}(K(b)\tilde{z}) \tag{1.2.35}$$

Since the reduced global stiffness matrix $K(b)$ is nonsingular, Eq. (1.2.35) may be solved numerically to obtain the first derivative of z with respect to design. Differentiating Eq. (1.2.35) with respect to b_j,

$$\begin{aligned}K(b)\frac{d^2 z}{db_i\,db_j} &= \frac{\partial^2 F}{\partial b_i\,\partial b_j} - \frac{\partial^2}{\partial b_i\,\partial b_j}(K(b)\tilde{z}) \\ &\quad - \frac{\partial}{\partial b_i}\left[K(b)\frac{d\tilde{z}}{db_j}\right] - \frac{\partial}{\partial b_j}\left[K(b)\frac{d\tilde{z}}{db_i}\right]\end{aligned} \tag{1.2.36}$$

Again, note that the coefficient matrix of the second derivatives of state with respect to design is nonsingular, so the second derivatives may be computed numerically.

Consider next solving Eqs. (1.2.35) and (1.2.36) for both the first and second derivatives of the state vector z and substituting them into Eq. (1.2.33) to obtain second derivatives of ψ. While this approach is conceptually simple, it leads to a massive amount of computation. If k is the dimension of the design variable, then $1 + 3k/2 + k^2/2$ equations must be solved, all having the same coefficient matrix. As will be seen in the following paragraph, considerably better results are achieved with the adjoint variable approach for second-order design sensitivity analysis.

ADJOINT VARIABLE METHOD WITH REDUCED GLOBAL STIFFNESS MATRIX

From Eq. (1.2.13), the derivative of ψ with respect to b_i may be written as

$$\frac{d\psi}{db_i} = \frac{\partial\psi}{\partial b_i} - \lambda^{\mathrm{T}} \frac{\partial K(b)}{\partial b_i} z + \lambda^{\mathrm{T}} \frac{\partial F}{\partial b_i} \tag{1.2.37}$$

It is important to note that in order for Eq. (1.2.37) to be valid, z must be the solution of Eq. (1.2.34) and λ the solution of

$$K(b)\lambda = \frac{\partial\psi^{\mathrm{T}}}{\partial z} \tag{1.2.38}$$

This follows from Eq. (1.2.11). Thus, both z and λ in Eq. (1.2.37) depend on design. Therefore, in calculation of second derivatives of ψ with respect to design, dependence of both z and λ on design must be accounted for. By chain rule calculation, the second derivative of ψ with respect to design is

$$\begin{aligned} \frac{d^2\psi}{db_i\, db} = \frac{\partial^2\psi}{\partial b_i\, \partial b} - \frac{\partial}{\partial b}\left[\tilde{\lambda}^{\mathrm{T}} \frac{\partial K(b)}{\partial b_i} \tilde{z}\right] + \frac{\partial}{\partial b}\left[\tilde{\lambda}^{\mathrm{T}} \frac{\partial F}{\partial b_i}\right] \\ + \left[\frac{\partial^2\psi}{\partial b_i\, \partial z} - \lambda^{\mathrm{T}} \frac{\partial K(b)}{\partial b_i}\right]\frac{dz}{db} + \left[\frac{\partial F^{\mathrm{T}}}{\partial b_i} - z^{\mathrm{T}} \frac{\partial K(b)}{\partial b_i}\right]\frac{d\lambda}{db} \end{aligned} \tag{1.2.39}$$

In order to evaluate the second derivatives in Eq. (1.2.39), dz/db and $d\lambda/db$ must be accounted for. Differentiating both sides of Eq. (1.2.38) with respect to design and premultiplying by $K^{-1}(b)$,

$$\frac{d\lambda}{db} = K^{-1}(b)\left[\frac{\partial^2\psi}{\partial z\, \partial b} - \frac{\partial}{\partial b}(K(b)\tilde{\lambda}) + \frac{\partial^2\psi}{\partial z^2}\frac{dz}{db}\right] \tag{1.2.40}$$

This result may be inserted into Eq. (1.2.39) and an adjoint variable γ^i can be defined as the solution of

$$K(b)\gamma^i = \frac{\partial F}{\partial b_i} - \frac{\partial K(b)}{\partial b_i} z. \tag{1.2.41}$$

Substituting the result of Eq. (1.2.41) into Eq. (1.2.39) and using Eq. (1.2.40) [the same procedure used in obtaining Eq. (1.2.12)],

$$\frac{d^2\psi}{db_i\,db} = \frac{\partial^2\psi}{\partial b_i\,\partial b} - \frac{\partial}{\partial b}\left[\tilde{\lambda}^{\mathrm{T}}\frac{\partial K(b)}{\partial b_i}\tilde{z}\right] + \frac{\partial}{\partial b}\left[\tilde{\lambda}^{\mathrm{T}}\frac{\partial F}{\partial b_i}\right]$$
$$+ \gamma^{i\mathrm{T}}\left[\frac{\partial^2\psi}{\partial z\,\partial b} - \frac{\partial}{\partial b}(K(b)\tilde{\lambda})\right] + \left[\frac{\partial^2\psi}{\partial b_i\,\partial z} - \lambda^{\mathrm{T}}\frac{\partial K(b)}{\partial b_i} + \gamma^{i\mathrm{T}}\frac{\partial^2\psi}{\partial z^2}\right]\frac{dz}{db} \tag{1.2.42}$$

The preceding sequence of computations may now be repeated, defining a new adjoint variable η^i as the solution of

$$K(b)\eta^i = \frac{\partial^2\psi}{\partial z\,\partial b_i} - \frac{\partial K(b)}{\partial b_i}\lambda + \frac{\partial^2\psi}{\partial z^2}\gamma^i \tag{1.2.43}$$

Using Eqs. (1.2.35) and (1.2.43) to replace the last term in Eq. (1.2.42) by directly computable expressions gives the desired result

$$\frac{d^2\psi}{db_i\,db_j} = \frac{\partial^2\psi}{\partial b_i\,\partial b_j} - \frac{\partial^2}{\partial b_i\,\partial b_j}[\tilde{\lambda}^{\mathrm{T}}K(b)\tilde{z}] + \frac{\partial^2}{\partial b_i\,\partial b_j}(\tilde{\lambda}^{\mathrm{T}}F)$$
$$+ \gamma^{i\mathrm{T}}\frac{\partial^2\psi}{\partial z\,\partial b_j} - \frac{\partial}{\partial b_j}[\tilde{\gamma}^{i\mathrm{T}}K(b)\tilde{\lambda}] + \eta^{i\mathrm{T}}\frac{\partial F}{\partial b_j} - \frac{\partial}{\partial b_j}[\tilde{\eta}^{i\mathrm{T}}K(b)\tilde{z}] \tag{1.2.44}$$

where only the jth component of the second derivative of ψ is included.

Equation (1.2.44) provides an explicit formula for all second derivatives of ψ with respect to design, requiring solution of a total of only $2k + 2$ equations, which normally is considerably less than the $1 + 3k/2 + k^2/2$ equation solutions required in the direct differentiation approach of Eq. (1.2.33).

A HYBRID DIRECT DIFFERENTIATION–ADJOINT VARIABLE METHOD

Haftka [12] introduced a refinement that combines the direct differentiation and adjoint methods to realize a computational advantage of a factor of two. From Eq. (1.2.36),

$$\frac{d^2z}{db_i\,db_j} = K^{-1}\left[\frac{\partial^2 F}{\partial b_i\,\partial b_j} - \frac{\partial^2}{\partial b_i\,\partial b_j}(K(b)\tilde{z}) - \frac{\partial}{\partial b_i}\left(K(b)\frac{d\tilde{z}}{db_j}\right) - \frac{\partial}{\partial b_j}\left(K(b)\frac{d\tilde{z}}{db_i}\right)\right] \tag{1.2.45}$$

Recall from Eq. (1.2.38) that

$$\lambda = K^{-1}\frac{\partial\psi^{\mathrm{T}}}{\partial z}$$

or

$$\lambda^{\mathrm{T}} = \frac{\partial\psi}{\partial z}K^{-1} \tag{1.2.46}$$

Now substitute the second derivatives of z from Eqs. (1.2.45) and (1.2.46) into Eq. (1.2.33) to obtain

$$\begin{aligned}\frac{d^2\psi}{db_i\,db_j} &= \frac{\partial^2\psi}{\partial b_i\,\partial b_j} + \frac{\partial^2\psi}{\partial b_i\,\partial z}\frac{dz}{db_j} + \frac{\partial^2\psi}{\partial z\,\partial b_j}\frac{dz}{db_i} + \frac{dz^{\mathrm{T}}}{db_j}\frac{\partial^2\psi}{\partial z^2}\frac{dz}{db_i}\\ &\quad + \lambda^{\mathrm{T}}\left[\frac{\partial^2 F}{\partial b_i\,\partial b_j} - \frac{\partial^2}{\partial b_i\,\partial b_j}(k(b)\tilde{z}) - \frac{\partial}{\partial b_i}\left(K(b)\frac{d\tilde{z}}{db_j}\right) - \frac{\partial}{\partial b_j}\left(K(b)\frac{d\tilde{z}}{db_i}\right)\right]\end{aligned} \tag{1.2.47}$$

If the direct differentiation method is employed to solve Eq. (1.2.35) for dz/db_i $(i = 1, \ldots, k)$, all terms on the right side of Eq. (1.2.47) can be evaluated. Note that z, k vectors dz/db_i, and λ are now needed, for a total of $k + 3$ solutions, or about half the $2k + 2$ solutions in the pure adjoint variable method.

COMPUTATIONAL CONSIDERATION

The practicality of computations involved in Eqs. (1.2.44) and (1.2.47) should be evaluated. Consider here only the case of member size design variables (fixed geometry). The last four terms on the right side of Eq. (1.2.44) are identical in form to terms arising in Eq. (1.2.13) for the first-order design sensitivity result. Of course, the first term on the right side of Eq. (1.2.44) must be calculated directly. The second term may be calculated, using the summation form for the reduced global stiffness matrix of Eq. (1.2.5), as

$$\frac{\partial^2}{\partial b_i\,\partial b_j}(\tilde{\lambda}^{\mathrm{T}}K(b)\tilde{z}) = \sum_{l=1}^{\mathrm{NE}} \lambda^{\mathrm{T}}\beta^{l\mathrm{T}}S^{l\mathrm{T}}\frac{\partial^2 k^l(b)}{\partial b_i\,\partial b_j}S^l\beta^l z \tag{1.2.48}$$

Note that most terms in the sum of derivatives of element stiffness matrices in Eq. (1.2.48) will be equal to zero. The third term on the right side of Eq. (1.2.44) involves second derivatives of the load vector with respect to design. If the load vector is constant, all these derivatives are zero. If the load vector depends on design, then expressions for second derivatives of components of the load vector must be calculated. Similar observations follow for evaluation of terms in Eq. (1.2.47).

ANALYSIS WITH GENERALIZED GLOBAL STIFFNESS MATRIX

The foregoing analysis requires explicit computation of the reduced global stiffness matrix and its first and second derivatives with respect to design. In applications involving nontrivial kinematic admissibility conditions, difficulties may arise in such computations.

The second derivatives of ψ with respect to design may be written, as in Eq. (1.2.33), as

$$\frac{d^2\psi}{db_i\,db_j} = \frac{\partial\psi}{\partial z_g}\frac{d^2 z_g}{db_i\,db_j} + \frac{\partial^2\psi}{\partial z_g\,\partial b_j}\frac{dz_g}{db_i} + \frac{dz_g^T}{db_j}\frac{\partial^2\psi}{\partial z_g^2}\frac{dz_g}{db_i} + \frac{\partial^2\psi}{\partial b_i\,\partial z_g}\frac{dz_g}{db_j} + \frac{\partial^2\psi}{\partial b_i\,\partial b_j} \tag{1.2.49}$$

where total derivative notation on the left is used to emphasize the inclusion of design dependence of z_g that appears in the performance measure. In order to treat the first term on the right side of Eq. (1.2.49), consider the ith component of Eq. (1.2.22) and differentiate both sides with respect to b_j to obtain the identity

$$\begin{aligned}\bar{z}_g^T K_g(b)\frac{d^2 z_g}{db_i\,db_j} &= -\frac{\partial^2}{\partial b_i\,\partial b_j}(\bar{z}_g^T K_g(b)\tilde{z}_g) - \frac{\partial}{\partial b_i}\left(\bar{z}_g^T K_g(b)\frac{d\tilde{z}_g}{db_j}\right)\\ &\quad - \frac{\partial}{\partial b_j}\left(\bar{z}_g^T K_g(b)\frac{d\tilde{z}_g}{db_i}\right) + \bar{z}_g^T\frac{\partial^2 F_g}{\partial b_i\,\partial b_j} \qquad \text{for all} \quad \bar{z}_g \in Z\end{aligned} \tag{1.2.50}$$

Observe that Eq. (1.2.23) may be evaluated at $\bar{\lambda}_g = d^2 z_g/db_i\,db_j$ and Eq. (1.2.50) at $\bar{z}_g = \lambda_g$ to obtain an expression for the first term on the right side of Eq. (1.2.49). Making substitutions into Eq. (1.2.49) gives

$$\begin{aligned}\frac{d^2\psi}{db_i\,db_j} &= -\frac{\partial^2}{\partial b_i\,\partial b_j}(\tilde{\lambda}_g^T K_g(b)\tilde{z}_g) - \frac{\partial}{\partial b_i}\left(\tilde{\lambda}_g^T K_g(b)\frac{d\tilde{z}_g}{db_j}\right)\\ &\quad - \frac{\partial}{\partial b_j}\left(\tilde{\lambda}_g^T K_g(b)\frac{d\tilde{z}_g}{db_i}\right) + \lambda_g^T\frac{\partial^2 F_g}{\partial b_i\,\partial b_j} + \frac{\partial^2\psi}{\partial z_g\,\partial b_j}\frac{dz_g}{db_i}\\ &\quad + \frac{dz_g^T}{db_j}\frac{\partial^2\psi}{dz_g^2}\frac{dz_g}{db_i} + \frac{\partial^2\psi}{\partial b_i\,\partial z_g}\frac{dz_g}{db_j} + \frac{\partial^2\psi}{\partial b_i\,\partial b_j}\end{aligned} \tag{1.2.51}$$

Note that the forms of the second derivatives calculated in Eqs. (1.2.51) and (1.2.47) are identical. It is important to note, however, that Eq. (1.2.51) is valid for even a singular global stiffness matrix $K_g(b)$, whereas the derivation of Eq. (1.2.47) relied heavily on the existence of an inverse of the reduced global stiffness matrix $K(b)$. Computational considerations associated with constructing terms in Eq. (1.2.51) are identical to those associated with constructing terms in Eq. (1.2.47). However, Eq. (1.2.51) has the desirable property that design derivatives of only the generalized global stiffness matrix must be computed, and not those of the reduced global stiffness matrix.

1.2.6 Examples

BEAM

Consider a clamped–clamped beam of unit length that is subjected to externally applied load $f(x)$ and self-weight $\gamma h(x)$, where γ is weight density of the beam material. Assume that all dimensions of the cross section vary with the same ratio (i.e., all are geometrically similar). Thus, the moment of inertia of the cross-sectional area is $I(x) = \alpha h^2(x)$, where α is a positive constant that depends on the shape of the cross section. If a stepped beam, is considered, as shown in Fig. 1.2.1,

$$h(x) = b_i, \qquad (i-1)/n < x < i/n \tag{1.2.52}$$

where the beam has been subdivided into n sections, each with a constant cross-sectional area b_i. The areas b_i $(i = 1, 2, \ldots, n)$ and Young's modulus $E = b_{n+1}$ may be viewed as design variables.

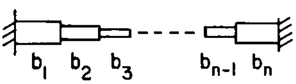

Fig. 1.2.1 Stepped beam.

Consider compliance as a response functional, given as

$$\begin{aligned}\psi &= \int_0^1 (f + \gamma h) w(x)\, dx \\ &= \sum_{i=1}^{n} \int_{i-1/n}^{i/n} (f + \gamma b_i) w(x)\, dx\end{aligned} \tag{1.2.53}$$

Using the shape function of Eq. (1.1.2) for each element, for the ith element,

$$w(x) = Nq^i = NS^i\beta^i z_g \tag{1.2.54}$$

where N, S^i, and β^i are the shape function, rotation matrix, and Boolean matrix, respectively. For the beam problem, S^i can be identity matrix and from Eq. (1.2.53)

$$\begin{aligned}\psi &= \left(\sum_{i=1}^{n} \int_{i-1/n}^{i/n} (f + \gamma b_i) N\beta^i\, dx \right) z_g \\ &= F_g(b)^{\mathrm{T}} z_g\end{aligned} \tag{1.2.55}$$

For the structural equation

$$\bar{z}_g^{\mathrm{T}} K_g z_g = \bar{z}_g^{\mathrm{T}} F_g \qquad \text{for all} \quad \bar{z}_g \in Z \tag{1.2.56}$$

where elements of Z satisfy clamped boundary conditions. Using the adjoint variable method of Section 1.2.3, Eq. (1.2.23) gives

$$\bar{\lambda}_g^{\mathrm{T}} K_g \lambda_g = \bar{\lambda}_g^{\mathrm{T}} F_g \qquad \text{for all} \quad \bar{\lambda}_g \in Z \tag{1.2.57}$$

Note that the generalized load on the right side of Eq. (1.2.57) is precisely the same as the generalized load for the original beam problem of Eq. (1.2.56). In this special case, λ is the displacement of an adjoint beam, which is identical to the original beam, and $\lambda_g = z_g$.

The sensitivity formula is, from Eq. (1.2.29),

$$\begin{aligned}
\frac{d\psi}{db_i} &= \frac{\partial \psi}{\partial b_i} + \frac{\partial}{\partial b_i}[\tilde{z}_g^{\mathrm{T}} F_g(b) - \tilde{z}_g^{\mathrm{T}} K_g(b) \tilde{z}_g] \\
&= \int_{i-1/n}^{i/n} 2\gamma w \, dx - \frac{\partial}{\partial b_i}\left[\sum_{i=1}^{n} \tilde{z}_g^{\mathrm{T}} \beta^{i\mathrm{T}} k^i \beta^i \tilde{z}_g\right] \\
&= \int_{i-1/n}^{i/n} 2\gamma w \, dx - \int_{i-1/n}^{i/n} 2E\alpha b_i q_i^{\mathrm{T}} N_{xx}^{\mathrm{T}} N_{xx} q_i \, dx \\
&= \int_{i-1/n}^{i/n} [2\gamma w - 2E\alpha b_i (w_{xx})^2] \, dx
\end{aligned} \tag{1.2.58}$$

for $i = 1, 2, \ldots, n$, where, from Eq. (1.1.3),

$$k^i = \int_{i-1/n}^{i/n} N_{xx}^{\mathrm{T}} E\alpha b_i^2 N_{xx} \, dx \tag{1.2.59}$$

Also,

$$\begin{aligned}
\frac{d\psi}{db_{n+1}} &= -\frac{\partial}{\partial b_{n+1}}[\tilde{z}_g^{\mathrm{T}} K_g(b) \tilde{z}_g] \\
&= -\sum_{i=1}^{n} \int_{i-1/n}^{i/n} \alpha b_i^2 q_i^{\mathrm{T}} N_{xx}^{\mathrm{T}} N_{xx} q_i \, dx \\
&= -\sum_{i=1}^{n} \int_{i-1/n}^{i/n} \alpha b_i^2 (w_{xx})^2 \, dx
\end{aligned} \tag{1.2.60}$$

Hence,

$$\begin{aligned}
\psi' = &\sum_{i=1}^{n} \left(\int_{i-1/n}^{i/n} [2\gamma w - 2E\alpha b_i (w_{xx})^2] \, dx \right) \delta b_i \\
&- \left(\sum_{i=1}^{n} \int_{i-1/n}^{i/n} \alpha b_i^2 (w_{xx})^2 \, dx \right) \delta E
\end{aligned} \tag{1.2.61}$$

THREE-BAR TRUSS

Consider next a simple three-bar truss with multipoint boundary conditions, as shown in Fig. 1.2.2. Design variables for the structure are the cross-sectional areas b_i of the truss members. The generalized global stiffness matrix is

$$K_g(b) = \frac{E}{l}\begin{bmatrix} b_3c^2s & b_3cs^2 & 0 & 0 & -b_3c^2s & -b_3cs^2 \\ b_3cs^2 & b_1 + b_3s^2 & 0 & -b_1 & -b_3cs^2 & -b_3s^3 \\ 0 & 0 & \frac{b_2s}{c} & 0 & -\frac{b_2s}{c} & 0 \\ 0 & -b_1 & 0 & b_1 & 0 & 0 \\ -b_3c^2s & -b_3cs^2 & -\frac{b_2s}{c} & 0 & \frac{b_2s}{c} + b_3c^2s & b_3cs^2 \\ -b_3cs^2 & -b_3s^3 & 0 & 0 & b_3cs^2 & b_3s^3 \end{bmatrix} \tag{1.2.62}$$

where $c = \cos\theta$ and $s = \sin\theta$. In this problem, the space Z of kinematically admissible displacements is

$$Z = \{z_g \in R^6 \colon\; z_3 = z_4 = 0,\; z_5\cos\alpha + z_6\sin\alpha = 0\} \tag{1.2.63}$$

and $K_g(b)$ is positive definite on Z, even though it is not positive definite on all of R^6.

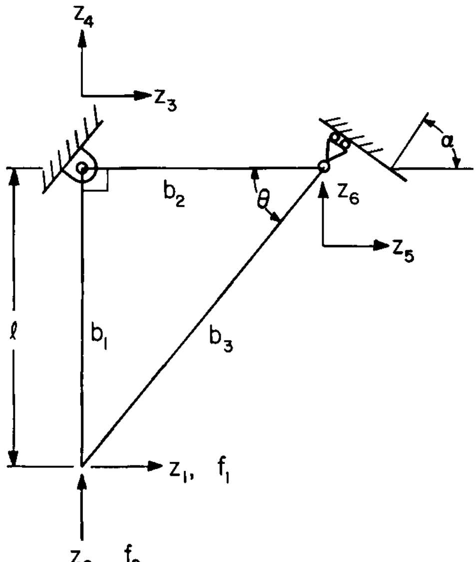

Fig. 1.2.2 Three-bar truss.

If $\theta = 45°$ and $\alpha = 30°$, then with $z = [z_1 \ z_2 \ z_5]^T$ the reduced stiffness matrix in this elementary example is

$$K(b) = \frac{E}{2\sqrt{2}l}\begin{bmatrix} b_3 & b_3 & (\sqrt{3}-1)b_3 \\ b_3 & 2\sqrt{2}b_1 + b_3 & (\sqrt{3}-1)b_3 \\ (\sqrt{3}-1)b_3 & (\sqrt{3}-1)b_3 & 2\sqrt{2}b_2 + (4-2\sqrt{3})b_3 \end{bmatrix} \tag{1.2.64}$$

If $f_1 = f_2 = 1$ and $l = 1$, then the solution of the reduced stiffness matrix formulation of Eq. (1.2.5) is

$$z = \left[\frac{4-2\sqrt{3}}{Eb_2} + \frac{2\sqrt{2}}{Eb_3} \quad 0 \quad \frac{1-\sqrt{3}}{Eb_2}\right]^T \tag{1.2.65}$$

If $\psi = z_1$, then the adjoint equation of (Eq. 1.2.11) is

$$K(b)\lambda = \partial\psi^T/\partial z = [1 \quad 0 \quad 0]^T \tag{1.2.66}$$

with solution

$$\lambda = \left[\frac{1}{Eb_1} + \frac{4-2\sqrt{3}}{Eb_2} + \frac{2\sqrt{2}}{Eb_3} \quad -\frac{1}{Eb_1} \quad \frac{1-\sqrt{3}}{Eb_2}\right]^T \tag{1.2.67}$$

The reduced stiffness matrix design sensitivity formula of Eq. (1.2.13) gives, using z and λ from Eqs. (1.2.65) and (1.2.67),

$$\frac{d\psi}{db} = -\frac{\partial}{\partial b}(\tilde{\lambda}^T K(b)\tilde{z}) = \left[0 \quad \frac{2\sqrt{3}-4}{Eb_2^2} \quad -\frac{2\sqrt{2}}{Eb_3^2}\right] \tag{1.2.68}$$

This can be verified by taking the derivative of z_1 in Eq. (1.2.65) with respect to design parameter b.

If the generalized global formulation is employed, the solution z_g of Eq. (1.2.17) must be found, which is

$$z_g = \left[\frac{4-2\sqrt{3}}{Eb_2} + \frac{2\sqrt{2}}{Eb_3} \quad 0 \quad 0 \quad 0 \quad \frac{1-\sqrt{3}}{Eb_2} \quad \frac{3-\sqrt{3}}{Eb_2}\right]^T \tag{1.2.69}$$

For $\psi = z_1$, the adjoint equation of Eq. (1.2.23) is

$$\lambda_g^T K_g(b)\bar{\lambda}_g = [1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0]^T\bar{\lambda}_g \qquad \text{for all} \quad \bar{\lambda}_g \in Z \tag{1.2.70}$$

with solution

$$\lambda_g = \left[\frac{1}{Eb_1} + \frac{4-2\sqrt{3}}{Eb_2} + \frac{2\sqrt{2}}{Eb_3} \quad -\frac{1}{Eb_1} \quad 0 \quad 0 \quad \frac{1-\sqrt{3}}{Eb_2} \quad \frac{3-\sqrt{3}}{Eb_2}\right]^T \tag{1.2.71}$$

Then the design sensitivity formula of Eq. (1.2.29) gives

$$\frac{d\psi}{d\mathrm{b}} = -\frac{\partial}{\partial b}(\tilde{\lambda}_g^{\mathrm{T}} K_g(b)\tilde{z}_g) = \left[0 \quad \frac{2\sqrt{3}-4}{Eb_2^2} \quad -\frac{2\sqrt{2}}{Eb_3^2} \right] \tag{1.2.72}$$

which is identical to the result obtained in Eq. (1.2.68).

For second-order design sensitivity, solving Eq. (1.2.22) for dz_g/db_i ($i = 1, 2, 3$) gives

$$\begin{aligned} \frac{dz_g}{db_1} &= [0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0]^T \\ \frac{dz_g}{db_2} &= \left[\frac{2\sqrt{3}-4}{Eb_2^2} \quad 0 \quad 0 \quad 0 \quad \frac{\sqrt{3}-1}{Eb_2^2} \quad \frac{\sqrt{3}-3}{Eb_2^2} \right]^{\mathrm{T}} \\ \frac{dz_g}{db_3} &= \left[-\frac{2\sqrt{2}}{Eb_3^2} \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \right]^{\mathrm{T}} \end{aligned} \tag{1.2.73}$$

For $\psi = z_1$, from Eq. (1.2.51),

$$\frac{d^2\psi}{db_2^2} = -2\frac{\partial}{\partial b_2}\left[\tilde{\lambda}_g^{\mathrm{T}} K_g(b) \frac{d\tilde{z}_g}{db_2} \right] = \frac{8-4\sqrt{3}}{Eb_2^3} \tag{1.2.74}$$

and

$$\frac{d^2\psi}{db_3^2} = -2\frac{\partial}{\partial b_3}\left[\tilde{\lambda}_g^{\mathrm{T}} K_g(b) \frac{d\tilde{z}_g}{db_3} \right] = \frac{4\sqrt{2}}{Eb_3^3} \tag{1.2.75}$$

The remaining second derivatives are zero. Hence, the Hessian of ψ is a diagonal matrix. These results can be verified by taking the derivative of $d\psi/db$ of Eq. (1.2.72) with respect to the design parameter b.

TEN-MEMBER CANTILEVER TRUSS

In order to illustrate the foregoing method, a ten-member cantilever truss shown in Fig. 1.2.3 is considered. Young's modulus of elasticity of the truss is $E = 1.0 \times 10^7$ psi and weight density is $\gamma = 0.1$ lb/in.^{3}.

This problem has been used in the literature [9] to compare various techniques of optimal design. The problem is to choose the cross-sectional area of each member of the truss to minimize its weight, subject to stress, displacement, and member-size constraints. The cost function in the present case is a linear function of the design variables,

$$\psi_0 = \sum_{i=1}^{m} \gamma_i l_i b_i \tag{1.2.76}$$

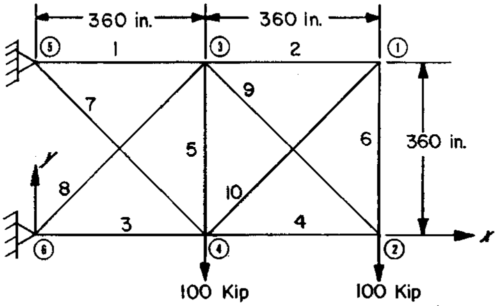

Fig. 1.2.3 Ten-member cantilever truss.

where γ_i, ℓ_i, and b_i are the weight density, length, and cross-sectional area of the ith member, respectively. Stress and displacement constraints for the problem are expressed as

$$\psi_i \; = |\sigma_i|/\sigma_i^a - 1.0 \le 0, \qquad i = 1, 2, \ldots, m \tag{1.2.77}$$

$$\psi_{j+m} = |z_j|/z_j^a - 1.0 \le 0, \qquad j = 1, 2, \ldots, n \tag{1.2.78}$$

where σ_i and σ_i^a are the calculated and allowable stresses for the ith member and z_j and z_j^a the calculated and allowable jth nodal displacements. Allowable stresses and displacements are given as $\sigma_i^a = 2.5 \times 10^4$ psi and $z_j^a = 2.0$ in., respectively.

For the cost function, direct calculation of design derivatives yield

$$\frac{d\psi_0}{db_i} = \gamma_i l_i \tag{1.2.79}$$

and no adjoint problem needs to be defined. For stress constraints, Eq. (1.1.22) gives

$$\sigma_i = \frac{E\,\Delta l_i}{l_i}, \qquad i = 1, 2, \ldots, m \tag{1.2.80}$$

where Δl_i is the change in l_i, which must be expressed in terms of the nodal displacements z. The adjoint equation of Eq. (1.2.11) is then

$$K\lambda = \frac{\partial \psi_i^{\mathrm{T}}}{\partial z} = \frac{E}{l_i \sigma_i^a} \frac{\partial |\Delta l_i|^{\mathrm{T}}}{\partial z}, \qquad i = 1, 2, \ldots, m \tag{1.2.81}$$

which is just the structural equation for a displacement λ due to a generalized load vector $\partial \psi_i^{\mathrm{T}}/\partial z$. Therefore, the solution $\lambda^{(i)}$ may be found, where superscript (i) denotes association of λ with the constraint ψ_i. The reduced stiffness matrix design sensitivity formula of Eq. (1.2.13) gives

$$\frac{d\psi_i}{db} = -\frac{\partial}{\partial b}[\tilde{\lambda}^{(i)\mathrm{T}} K(b) \tilde{z}] \tag{1.2.82}$$

For displacement constraints ψ_{j+m}, the adjoint equation is

$$K\lambda = \frac{\partial \psi_{j+m}^{\mathrm{T}}}{\partial z} = \operatorname{sgn}(z_j)\begin{bmatrix} 0 & \ldots & 0 & \dfrac{1}{z_j^a} & 0 & \ldots & 0 \end{bmatrix}^{\mathrm{T}} \tag{1.2.83}$$

where

$$\operatorname{sgn}(z_j) = \begin{cases} +1, & \text{if } z_j > 0 \\ -1, & \text{if } z_j < 0 \end{cases}$$

Note that the adjoint load of Eq. (1.2.83) is a point load of magnitude $\pm 1/z_j^a$ in the jth nodal displacement direction. As before, the solution $\lambda^{(j+m)}$ of Eq. (1.2.83) may be found. Then the sensitivity formula of Eq. (1.2.13) gives

$$\frac{d\psi_{j+m}}{db} = -\frac{\partial}{\partial b}[\tilde{\lambda}^{(j+m)\mathrm{T}} K(b)\tilde{z}] \tag{1.2.84}$$

Using these sensitivity formulas, design derivatives of some constraints are calculated and given in Table 1.2.1 for the initial design given in the second column of Table 1.2.1. The vectors $d\psi_1^{\mathrm{T}}/db$ and $d\psi_2^{\mathrm{T}}/db$ are design derivatives of the normalized stresses in members 5 and 7, respectively, and $d\psi_3^{\mathrm{T}}/db$ is the derivative of the normalized displacement in the y direction at node 2.

Define ψ_i^1 and ψ_i^2 as the constraint function values for the initial design b and modified design $b + \delta b$, respectively. Let $\Delta\psi_i$ be the difference between ψ_i^1 and ψ_i^2, and let $\psi_i' = (d\psi_i/db)\,\delta b_i$ be the difference predicted by design sensitivity calculations. The ratio of ψ_i' and $\Delta\psi_i$ times 100 is used as a measure of accuracy of the derivative (i.e., 100% means that the predicted change is exactly the same as the actual change). Notice that this accuracy measure will not give correct information when $\Delta\psi_i$ is very small compared

Table 1.2.1
Design Derivatives of Constraints for Ten-Member Cantilever Truss

Number	Design	$d\psi_1^{\mathrm{T}}/db$	$d\psi_2^{\mathrm{T}}/db$	$d\psi_3^{\mathrm{T}}/db$
1	28.6	0.0082	−0.0009	−0.0093
2	0.2	−0.0696	−0.0284	0.0109
3	23.6	−0.0104	0.0012	−0.0062
4	15.4	−0.0006	−0.0003	−0.0076
5	0.2	−2.3520	−0.9601	0.1402
6	0.2	−0.0696	−0.0284	0.0109
7	3.0	−0.8369	−0.4398	−0.0177
8	21.0	0.0231	−0.0026	−0.0128
9	21.8	−0.0009	−0.0004	−0.0108
10	0.2	−0.1968	−0.0803	0.0308

Table 1.2.2

Comparison of Sensitivity Calculation

Constraint	ψ_i^1	ψ_i^2	$\Delta\psi_i = \psi_i^2$	ψ_i'	$(\psi_i'/\Delta\psi_i \times 100)\%$
ψ_1	1.6038	1.4798	−0.1240	−0.1302	105.0
ψ_2	0.6094	0.5325	-0.7688×10^{-1}	-0.8072×10^{-1}	105.0
ψ_3	0.4722×10^{-1}	-0.265×10^{-2}	-0.4987×10^{-1}	-0.5236×10^{-1}	105.0

to ψ_i^1 and ψ_i^2, because the difference $\Delta\psi_i$ may not have meaning in this case. Numerical results with a 5% design change, $\delta b = 0.05b$, are given in Table 1.2.2.

As a second numerical example, consider the same ten-member cantilever truss, but with the multipoint boundary conditions shown in Fig. 1.2.4. In this problem, the space Z of kinematically admissible displacements is

$$Z = \{z_g \in R^{12}: z_9 = z_{10} = 0, z_{11}\cos\alpha + z_{12}\sin\alpha = 0\} \tag{1.2.85}$$

where $\alpha = 30°$. The same constraints given Eqs. (1.2.77) and (1.2.78) are considered in this problem. For stress constraints of Eq. (1.2.77), the adjoint equation of Eq. (1.2.23) is

$$\lambda_g^T K_g \bar{\lambda}_g = \frac{\partial\psi_i}{\partial z_g}\bar{\lambda}_g = \frac{E}{l_i\sigma_i^a}\frac{\partial|\Delta l_i|}{\partial z_g}\bar{\lambda}_g \quad \text{for all} \quad \bar{\lambda}_g \in Z \tag{1.2.86}$$

with solution $\lambda_g^{(i)}$ $(i = 1, 2, \ldots, m)$. The design sensitivity formula of Eq. (1.2.29) becomes

$$\frac{d\psi_i}{db} = -\frac{\partial}{\partial b}[\tilde{\lambda}_g^{(i)T} K_g(b)\tilde{z}_g] \tag{1.2.87}$$

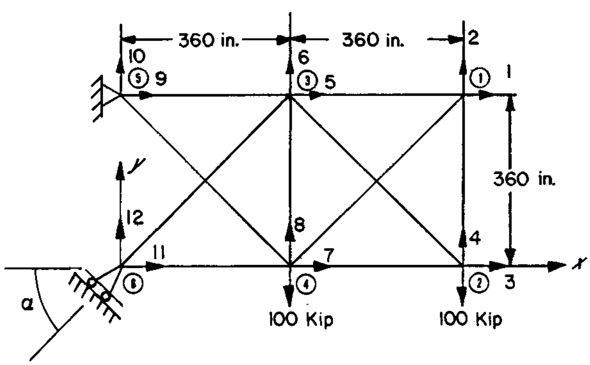

Fig. 1.2.4 Ten-member cantilever truss with multipoint boundary condition.

For displacement constraints ψ_{j+m}, the adjoint equation is

$$\bar{\lambda}_g^T K_g \lambda_g = \frac{\partial \psi_{j+m}}{\partial z_g} \bar{\lambda}_g$$
$$= \operatorname{sgn}(z_j)[0 \quad \ldots \quad 0 \quad 1/z_j^a \quad 0 \quad \ldots \quad 0]\bar{\lambda}_g \qquad \text{for all} \quad \bar{\lambda}_g \in Z \tag{1.2.88}$$

with solution $\lambda_g^{(j+m)}$ $(j = 1, 2, \ldots, n)$. The design sensitivity formula of Eq. (1.2.29) becomes

$$\frac{d\psi_{j+m}}{db} = -\frac{\partial}{\partial b}[\tilde{\lambda}_g^{(j+m)T} K_g(b)\tilde{z}_g] \tag{1.2.89}$$

Comparison of design sensitivities between the actual changes and the predictions by the sensitivity formulas of constraint values, with 5% overall changes of design variables, is presented in Table 1.2.3.

Table 1.2.3
Comparison of Sensitivity Calculation (Multipoint Boundary Condition)

Constraint	ψ_i^1	ψ_i^2	$\Delta\psi_i = \psi_i^2 - \psi_i^1$	ψ_i'	$(\psi_i'/\Delta\psi_i \times 100)\%$
ψ_1	11.1476	10.6936	−0.5848	−0.6140	105.0
ψ_2	−0.4176	−0.4488	-0.3124×10^{-1}	-0.3296×10^{-1}	105.6
ψ_3	1.1134	2.0127	−0.1006	−0.1057	105.0

1.3 EIGENVALUE DESIGN SENSITIVITY

As shown in Section 1.1, the natural frequency of vibration and buckling load of a structure are eigenvalues of a generalized eigenvalue problem, hence they depend on design. It is the purpose of this section to obtain design derivatives of such eigenvalues and to explore an important exceptional case in which repeated eigenvalues occur as solutions of optimal design problems. Due to singularity of the characteristic matrix associated with an eigenvalue, some technical complexities arise in eigenvalue and eigenvector design sensitivity analysis that do not appear in design sensitivity of response of a structure to static load presented in Section 1.2.

1.3.1 First-Order Eigenvalue Design Sensitivity with Reduced Global Stiffness and Mass Matrices

Consider first the eigenvalue formulation for natural frequency or buckling [for buckling problems, $M(b)$ is the geometric stiffness matrix] described by the eigenvalue problem

$$K(b)y = \zeta M(b)y \tag{1.3.1}$$

where the eigenvector y is normalized by the condition

$$y^{T}M(b)y = 1 \tag{1.3.2}$$

It is presumed here that the reduced global stiffness and mass (geometric stiffness) matrices are positive definite and differentiable with respect to design. Under these hypotheses, the following theorem is true.

THEOREM 1.3.1 If the symmetric, positive definite matrices $K(b)$ and $M(b)$ in Eq. (1.3.1) are continuously differentiable with respect to design and if an eigenvalue ζ is simple (not repeated), then the eigenvalue and associated eigenvector of Eqs. (1.3.1) and (1.3.2) are continuously differentiable with respect to design.

PROOF For a direct proof, see section II.6 of Kato [13]. A more general theorem, which specializes to the results stated here, is proved in Section 1.3.6 of this text. ■

Premultiplying Eq. (1.3.1) by the transpose of an arbitrary vector $\bar{y}$, one obtains the identity

$$\bar{y}^{T}K(b)y = \zeta\bar{y}^{T}M(b)y \qquad \text{for all} \quad \bar{y} \in R^{m} \tag{1.3.3}$$

Consider now a perturbation δb of the nominal design b of the form

$$b_{\tau} = b + \tau\,\delta b \tag{1.3.4}$$

Substituting b_{τ} into Eq. (1.3.3) and differentiating both sides with respect to τ, one obtains the identity

$$\begin{aligned} &\frac{\partial}{\partial b}[\bar{y}^{T}K(b)\tilde{y}]\,\delta b + \bar{y}^{T}K(b)y' \\ &\quad = \zeta'\bar{y}^{T}M(b)y + \zeta\frac{\partial}{\partial b}[\bar{y}^{T}M(b)\tilde{y}]\,\delta b + \zeta\bar{y}^{T}M(b)y' \qquad \text{for all} \quad \bar{y} \in R^{m} \end{aligned} \tag{1.3.5}$$

where, as in Eq. (1.2.19),

$$\begin{aligned} y' &= y'(b, \delta b) \equiv \frac{d}{d\tau}y(b + \tau\,\delta b)\bigg|_{\tau=0} = \frac{dy}{db}\,\delta b \\ \zeta' &= \zeta'(b, \delta b) \equiv \frac{d}{d\tau}\zeta(b + \tau\,\delta b)\bigg|_{\tau=0} = \frac{d\zeta}{db}\,\delta b \end{aligned} \tag{1.3.6}$$

Since Eq. (1.3.5) must hold for an arbitrary vector $\bar{y}$, substitute $\bar{y} = y$ in Eq. (1.3.5), using Eq. (1.3.2), to obtain

$$\zeta' = \left\{\frac{\partial}{\partial b}[\tilde{y}^{\mathrm{T}}K(b)\tilde{y}] - \zeta\frac{\partial}{\partial b}[\tilde{y}^{\mathrm{T}}M(b)\tilde{y}]\right\}\delta b + y'^{\mathrm{T}}[K(b)y - \zeta M(b)y] \tag{1.3.7}$$

Note that the last term of Eq. (1.3.7) is zero since y is an eigenvector of Eq. (1.3.1). Thus, Eq. (1.3.7) reduces to the desired result,

$$\frac{d\zeta}{db} = \frac{\partial}{\partial b}[\tilde{y}^{\mathrm{T}}K(b)\tilde{y}] - \zeta\frac{\partial}{\partial b}[\tilde{y}^{\mathrm{T}}M(b)\tilde{y}] \tag{1.3.8}$$

It is interesting to note that this vector of derivatives of the eigenvalue with respect to design may be calculated without the solution of an adjoint equation or a derivative of the eigenvector. Thus, once the eigenvalue problem has been solved for a simple (nonrepeated) eigenvalue, the eigenvalue derivatives are directly calculated using Eq. (1.3.8). In this sense, differentiation of eigenvalues is simpler than differentiation of structural performance functions that involve response to a static load. This statement is false if multiple (repeated) eigenvalues are encountered.

1.3.2 First-Order Eigenvalue Design Sensitivity with Generalized Global Stiffness and Mass Matrices

Consider the variational form of the eigenvalue problem of Eqs. (1.1.73) and (1.1.78), written in the form

$$a_b(y_g, \bar{y}_g) \equiv \bar{y}_g^{\mathrm{T}}K_g(b)y_g = \zeta\bar{y}_g^{\mathrm{T}}M_g(b)y_g \equiv \zeta d_b(y_g, \bar{y}_g) \qquad \text{for all} \quad \bar{y}_g \in Z \tag{1.3.9}$$

Recall that the energy bilinear forms $a_b(\cdot, \cdot)$ and $d_b(\cdot, \cdot)$ are positive definite on the space $Z \subset R^n$ of kinematically admissible displacements; that is,

$$\begin{aligned} a_b(y_g, y_g) > 0, \\ d_b(y_g, y_g) > 0, \end{aligned} \qquad \text{for all} \quad y_g \in Z, \quad y_g \neq 0 \tag{1.3.10}$$

In order to obtain the simplest possible derivation of eigenvalue design sensitivity in this setting, a basis ϕ^i ($i = 1, \ldots, m$) of Z may be introduced. It is presumed here that kinematic constraints do not depend explicitly on design, so the vectors ϕ^i are independent of design. Recall that the dimension of the space $Z \subset R^n$ of kinematically admissible displacements is $m < n$. Any vector $y_g \in Z$ may be written as a linear combination of the ϕ^i, that is,

$$y_g = \sum_{i=1}^{m} c_i\phi^i = \Phi c \tag{1.3.11}$$

where $\Phi = [\phi^1 \; \phi^2 \; \cdots \; \phi^m]$ and the coefficients c_i are to be determined. Substituting this expression for y into Eq. (1.3.9) and evaluating Eq. (1.3.9) with $\bar{y}_g = \phi^j$ $(j = 1, \ldots, m)$ gives the following system of equations for the coefficients c_i:

$$\sum_{i=1}^{m} a_b(\phi^i, \phi^j)c_i = \zeta \sum_{i=1}^{m} d_b(\phi^i, \phi^j)c_i, \qquad j = 1, \ldots, m \tag{1.3.12}$$

In matrix form, these equations may be written as

$$\begin{aligned} \breve{K}(b)c &= \zeta \breve{M}(b)c \\ c^{\mathrm{T}}\breve{M}(b)c &= 1 \end{aligned} \tag{1.3.13}$$

where

$$\begin{aligned} \breve{K}(b) &= \Phi^{\mathrm{T}} K_g(b)\Phi \\ \breve{M}(b) &= \Phi^{\mathrm{T}} M_g(b)\Phi \end{aligned} \tag{1.3.14}$$

Note that since the matrix Φ does not depend on design, the matrices $\breve{K}(b)$ and $\breve{M}(b)$ are differentiable with respect to design if $K_g(b)$ and $M_g(b)$ are.

Using the conditions of Eq. (1.3.10), the matrices $K(b)$ and $M(b)$ may be shown to be positive definite. Thus, the result of Section 1.3.1 applies [Eq. (1.3.8)] to obtain the derivative of the eigenvalue with respect to design as

$$\frac{d\zeta}{db} = \frac{\partial}{\partial b}[\tilde{c}^{\mathrm{T}} K(b)\tilde{c}] - \zeta \frac{\partial}{\partial b}[\tilde{c}^{\mathrm{T}} M(b)\tilde{c}] \tag{1.3.15}$$

In order to use this result, first note that the second equation of Eq. (1.3.13) and Eq. (1.3.11) yield

$$1 = c^{T}\breve{M}(b)c = c^{\mathrm{T}}\Phi^{\mathrm{T}} M_g(b)\Phi c = y_g^{\mathrm{T}} M_g(b) y_g = d_b(y_g, y_g) \tag{1.3.16}$$

Furthermore, substituting for the matrices $\breve{K}(b)$ and $\breve{M}(b)$ from Eq. (1.3.14) into Eq. (1.3.15) gives

$$\frac{d\zeta}{db} = \frac{\partial}{\partial b}[\tilde{c}^{\mathrm{T}}\Phi^{\mathrm{T}} K_g(b)\Phi\tilde{c}] - \zeta \frac{\partial}{\partial b}[\tilde{c}^{\mathrm{T}}\Phi^{\mathrm{T}} M_g(b)\Phi\tilde{c}]$$

With Eq. (1.3.11), the desired result is obtained as

$$\frac{d\zeta}{db} = \frac{\partial}{\partial b}[\tilde{y}_g^{\mathrm{T}} K_g(b)\tilde{y}_g] - \zeta \frac{\partial}{\partial b}[\tilde{y}_g^{\mathrm{T}} M_g(b)\tilde{y}_g] \tag{1.3.17}$$

Note that the form of Eq. (1.3.17) is identical to that obtained with the reduced global stiffness matrix in Eq. (1.3.8). Computational advantages associated with Eq. (1.3.17), however, are considerable. The generalized global stiffness and mass matrices can be used for calculating design sensitivity of a simple eigenvalue, without resorting to matrix manipulations that transform the generalized global matrices to reduced form.

1.3.3 First-Order Design Sensitivity of Eigenvectors Corresponding to Simple Eigenvalues

As in the static response case, since Φ in Eq. (1.3.11) does not depend on b, $y'_g = (dy_g/db)\,\delta b = \Phi(dc/db)\,\delta b$. Thus, eigenvectors y_g corresponding to simple eigenvalues are differentiable with respect to design. In this section, only the case that the bilinear form $d(\cdot,\cdot)$ is independent of design will be considered. In order to obtain the directional derivative y'_g of the eigenvector y_g corresponding to the smallest simple eigenvalue of Eq. (1.3.9), take the total variation of both sides of Eq. (1.3.9) and use the chain rule of differentiation to obtain

$$\begin{aligned} a_b(y'_g, \bar{y}_g) - \zeta d(y'_g, \bar{y}_g) &= -a'_{\delta b}(y_g, \bar{y}_g) + \zeta' d(y_g, \bar{y}_g) \\ &= -a'_{\delta b}(y_g, \bar{y}_g) + a'_{\delta b}(y_g, y_g) d(y_g, \bar{y}_g) \qquad \text{for all} \quad \bar{y}_g \in Z \end{aligned} \tag{1.3.18}$$

where Eq. (1.3.17), written in terms of energy bilinear forms, has been used to evaluate the directional derivative of the eigenvalue.

The bilinear form on the left side of Eq. (1.3.18) need no longer be positive definite on Z, since it is a difference of positive definite forms. Therefore, it is not clear that a unique solution of Eq. (1.3.18) exists. However, note that Eq. (1.3.18) is trivially satisfied for $\bar{y}_g = y_g$. A subspace W of Z that is d-orthogonal to y_g may be defined and Z may be written as the direct sum of W and y_g; that is,

$$Z = W \oplus \{y_g\}$$

where $\{y_g\}$ is the one-dimensional subspace of Z spanned by y_g and

$$W = \{v \in Z : d(v, y_g) = 0\} \tag{1.3.19}$$

The notation $\oplus$ means that since $d(\cdot,\cdot)$ is positive definite on Z, every vector $w \in Z$ can be written uniquely in the form

$$w = v + \alpha y_g, \qquad v \in W, \quad \alpha \in R^1$$

Since Eq. (1.3.18) is valid for all $\bar{y}_g \in Z$, every element of Z can be written uniquely as the sum of elements from W and $\{y_g\}$, and Eq. (1.3.18) is trivially satisfied for $\bar{y}_g = y_g$, Eq. (1.3.18) reduces to

$$a_b(y'_g, \bar{y}_g) - \zeta d(y'_g, \bar{y}_g) = -a'_{\delta b}(y_g, \bar{y}_g) \qquad \text{for all} \quad \bar{y}_g \in W \tag{1.3.20}$$

It now remains to show that the bilinear form on the left side of Eq. (1.3.20) is positive definite on W.

Using the Rayleigh quotient representation of eigenvalues of Eq. (1.3.9), it is well known [14] that the second eigenvalue of the problem minimizes the Rayleigh quotient over all vectors $v \in W$. Since the second eigenvalue is strictly larger than the smallest simple eigenvalue ζ,

$$\zeta < \frac{a_b(v, v)}{d(v, v)} \qquad \text{for all} \quad v \in W, \quad v \neq 0$$

or

$$a_b(v, v) - \zeta d(v, v) > 0 \qquad \text{for all} \quad v \in W, \quad v \neq 0 \tag{1.3.21}$$

This shows that the bilinear form on the left side of Eq. (1.3.20) is positive definite on W. Thus, Eq. (1.3.20) has a unique solution $y_g' \in W$ for the directional derivative of the eigenvector y_g corresponding to the smallest simple eigenvalue. This argument can be extended to any simple eigenvalue, replacing W by the subspace of Z that is d-orthogonal to all eigenvectors corresponding to eigenvalues smaller than ζ.

Letting δb be a vector with a one in the jth position and zeros elsewhere, y_g' becomes dy_g/db_j, and Eq. (1.3.20) becomes

$$\bar{y}_g^{\mathrm{T}}[K_g(b) - \zeta M_g] \frac{dy_g}{db_j} = -\frac{\partial}{\partial b_j}[\bar{y}_g^{\mathrm{T}} K_g(b) \tilde{y}_g] \qquad \text{for all} \quad \bar{y}_g \in W \tag{1.3.22}$$

Several numerical techniques exist for solving Eq. (1.3.20) for y_g' or Eq. (1.3.22) for dy_g/db_j. Nelson [15] presented a direct computational technique that uses the reduced global stiffness matrix and is effective for computations in which the reduced system matrices are known. Potential exists for direct application of numerical techniques such as subspace iteration [16] to construct a solution of Eq. (1.3.20), in conjunction with solution of the basic eigenvalue problem.

1.3.4 Second-Order Design Sensitivity of a Simple Eigenvalue

The ith component of the gradient of the smallest eigenvalue ζ with respect to design may be written from Eq. (1.3.17) as

$$\frac{d\zeta}{db_i} = \frac{\partial}{\partial b_i}[\tilde{y}_g^{\mathrm{T}} K_g(b) \tilde{y}_g] - \zeta \frac{\partial}{\partial b_i}[\tilde{y}_g^{\mathrm{T}} M_g(b) \tilde{y}_g] \tag{1.3.23}$$

Differentiating with respect to b_j gives

$$\begin{aligned}\frac{d^2\zeta}{db_i\,db_j} &= \frac{\partial^2}{\partial b_i\,\partial b_j}[\tilde{y}_g^T K_g(b)\tilde{y}_g] - \zeta\frac{\partial^2}{\partial b_i\,\partial b_j}[\tilde{y}_g^T M_g(b)\tilde{y}_g] \\ &\quad - \left\{\frac{\partial}{\partial b_j}[\tilde{y}_g^T K_g(b)\tilde{y}_g] - \zeta\frac{\partial}{\partial b_j}[\tilde{y}_g^T M_g(b)\tilde{y}_g]\right\}\frac{\partial}{\partial b_i}[\tilde{y}_g^T M_g(b)\tilde{y}_g] \\ &\quad + 2\frac{\partial}{\partial b_i}[\tilde{y}_g^T K_g(b)]\frac{dy_g}{db_j} - 2\zeta\frac{\partial}{\partial b_i}[\tilde{y}_g^T M_g(b)]\frac{dy_g}{db_j} \qquad (1.3.24)\end{aligned}$$

In order to evaluate the second derivative of τ in Eq. (1.3.24), dy_g/db_i and dy_g/db_j must be calculated. This may be done by solving Eq. (1.3.22). Once Eq. (1.3.22) is solved, the result may be substituted into Eq. (1.3.24) to obtain the second design derivative of ζ with respect to design components b_i and b_j.

Note that computation of all second design derivatives of ζ requires solution of Eq. (1.3.23) for $j = 1, \ldots, k$. These results may be substituted into Eq. (1.3.24) and the partial derivatives with respect to b_i ($i = 1, \ldots, k$) may be calculated. Thus, all $k^2/2 + k/2$ distinct derivatives of ζ are obtained with respect to design. In doing so, k sets of equations in Eq. (1.3.22) must be dealt with and numerical computation performed to evaluate the right side of Eq. (1.3.24). While this is a substantial amount of computation, availability of second design derivative of eigenvalues with respect to design can be of value in iterative design optimization.

1.3.5 Systematic Occurrence of Repeated Eigenvalues in Structural Optimization

In carrying out vibration and buckling analysis of structures, it is well known that computational difficulties can arise if *repeated eigenvalues* (natural frequencies or buckling loads) arise. Occurrence of repeated eigenvalues has often been dismissed on practical grounds, since it is felt that a precisely repeated eigenvalue is an extremely unlikely accident.

While repeated eigenvalues may indeed be unlikely in randomly specified structures, they become far more likely in optimized structures. Thompson and Hunt [17] have devoted considerable attention to designs that are constructed with simultaneous buckling failure modes (i.e., repeated eigenvalues). More recently, Olhoff and Rasmussen [18] showed that a repeated buckling load may occur in an optimized clamped–clamped column. Their result corrected an erroneous solution published much earlier [19]. Subsequent to the Olhoff–Rasmussen finding, Masur and Mroz [20] gave an elegant treatment of optimality criteria for structures in which repeated eigenvalues occur. They showed that a singular (nondifferentiable) optimization problem arises. Prager and Prager [21] demonstrated that singular behavior associated with repeated eigenvalues may arise for even a very simple finite-dimensional column model of the distributed parameter column of Olhoff and Rasmussen [18]. Simple vibration and buckling problems are introduced here to show how repeated eigenvalues arise in structural optimization.

VIBRATION EXAMPLE

Consider first the spring–mass system shown in Fig. 1.3.1. The eigenvalue equation for small-amplitude vibration of the rigid body is derived simply as

$$K(b)y \equiv \begin{bmatrix} 4b_1 + b_2 & b_2 \\ b_2 & 4b_1 + b_2 \end{bmatrix} y = \zeta \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} y \equiv \zeta M y \quad (1.3.25)$$

where $\zeta = 2\omega^2 m/3$, m is mass of the bar, and $I = ml^2/12$ is moment of inertia of the bar. Horizontal motion of the bar is ignored and the spring constants are regarded as design variables. Note that they do not appear in the mass matrix M.

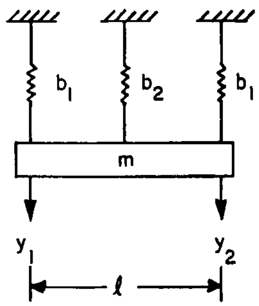

Fig. 1.3.1 Spring–mass system with two degrees of freedom.

The optimal design objective is to find design parameters b_1 and b_2 to minimize weight of the spring supports, which is presumed to be of the form

$$\psi_0 = c_1 b_1 + c_2 b_2 \quad (1.3.26)$$

where c_1 and c_2 are known constants. The minimization is to be carried out, subject to constraints that the eigenvalues are not lower than $\zeta_0 > 0$ and the spring constants are nonnegative. These constraints are given in inequality constraint form as

$$\psi_1 = \zeta_0 - \zeta_1 \le 0$$

$$\psi_2 = \zeta_0 - \zeta_2 \le 0$$

$$\psi_3 = -b_1 \le 0$$

$$\psi_4 = -b_2 \le 0$$

Since the eigenvalues of Eq. (1.3.25) are $\zeta_1 = (4b_1 + 2b_2)/3$ and $\zeta_2 = 4b_1$, these constraints become

$$\begin{aligned} \psi_1 &= \zeta_0 - (4b_1 + 2b_2)/3 \le 0 \\ \psi_2 &= \zeta_0 - 4b_1 \le 0 \\ \psi_3 &= -b_2 \le 0 \\ \psi_4 &= -b_4 \le 0 \end{aligned} \tag{1.3.27}$$

Equations (1.3.26) and (1.3.27) define a linear programming problem. The feasible set is shown graphically in Fig. 1.3.2. Note that the slope of the line connecting points A and B in Fig. 1.3.2 is -2. The level lines of the cost function of Eq. (1.3.26) are straight, with slope equal to $-c_1/c_2$. The cost function decreases as level lines of cost move to the lower left. Thus, it is clear that point A (repeated eigenvalue) is optimum if $c_1/c_2 > 2$ and point B (simple eigenvalue) is optimum if $c_1/c_2 < 2$.

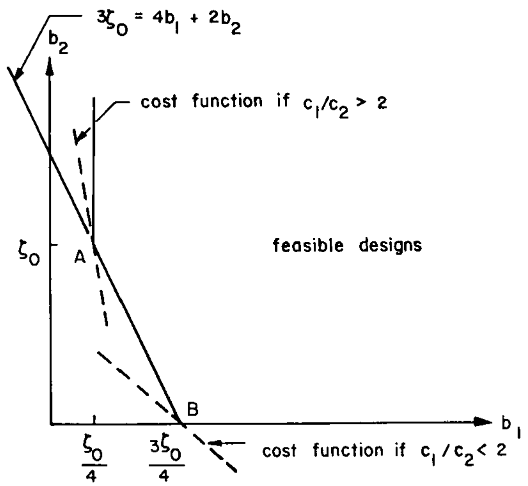

Fig. 1.3.2 Feasible region in design space for systems with two degrees of freedom.

COLUMN BUCKLING EXAMPLE

Next consider the column shown in Fig. 1.3.3, with elastically clamped ends. The column has five rigid segments of length l and six elastic hinges, the hinges at the ends of the column having bending stiffness b_0^2. That is, the rotation of the end sections of the column by an angle θ_0 is opposed by a clamping moment $M_0 = b_0^2\theta_0$, where b_0 is a given constant. The cases $b_0 = 0$ and $b_0 = \infty$ correspond to pin-supported or rigidly clamped ends, respectively. Because boundary conditions at the ends are identical, the bending stiffnesses of the optimum design will be symmetric with respect to

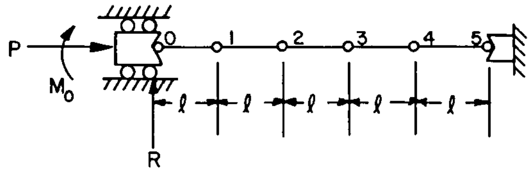

Fig. 1.3.3 Elastically supported column.

the center of the column, and the buckling modes are either symmetric or antisymmetric with respect to the center. A column design is specified by the bending stiffness b_1^2 of hinges 1 and 4 and b_2^2 of hinges 2 and 3 in Fig. 1.3.3. A buckling mode that is known to be symmetric or antisymmetric is specified by the deflection y_1 of nodes 1 and 4 and y_2 of nodes 2 and 3. Upward deflections are regarded as positive.

At the left end, the column is subject to an axial load P, a reaction force R, and a clamping moment M_0. The bending moment at the ith hinge is

$$M_i = M_0 - ilR - Py_i, \qquad i = 0, \ldots, 4 \tag{1.3.28}$$

where $y_0 = 0$. If θ_i denotes the relative rotation of the segments meeting at the ith hinge, considered positive if counterclockwise rotation of the segment to the right of the ith hinge exceeds that of the segment to the left,

$$M_i = b_i^2\theta_i = b_i^2(y_{i+1} - 2y_i + y_{i-1})/l, \qquad i = 0, \ldots, 4 \tag{1.3.29}$$

where $y_{-1} = y_0 = 0$. It is convenient to introduce a reference stiffness b^{*2} and define the dimensionless variables

$$\hat{P} = Pl/b^{*2}, \qquad \hat{R} = Rl/b^{*2}, \qquad \hat{M}_i = M_i/b^{*2}, \qquad \hat{y}_i = y_i/l, \qquad \hat{b}_i = b_i/b^* \tag{1.3.30}$$

Note that with these dimensionless variables, Eqs. (1.3.28) and (1.3.29) yield (after deleting ˆ for notational simplicity),

$$M_0 - iR - Py_i = b_i^2(y_{i+1} - 2y_i + y_{i-1}), \qquad i = 0, 1, 2 \tag{1.3.31}$$

For a symmetric buckling mode, $y_3 = y_2$ and $R = 0$. For $i = 0, 1, 2$, Eq. (1.3.31) yields

$$\begin{aligned} M_0 &= b_0^2 y_1 \\ M_0 - P_s y_1 &= b_1^2(y_2 - 2y_1) \\ M_0 - P_s y_2 &= b_2^2(-y_2 + y_1) \end{aligned} \tag{1.3.32}$$

where P_s is the buckling load of the symmetric mode. When the value of M_0 from the first of these equations is substituted into the other two, linear homogeneous equations are obtained for y_1 and y_2 that admit a nontrivial solution only if

$$P_s^2 - (b_0^2 + 2b_1^2 + b_2^2)P_s + b_0^2(b_1^2 + b_2^2) + b_1^2 b_2^2 = 0 \tag{1.3.33}$$

The smaller root of this equation is the symmetric buckling load.

Let the cost of the design $[b_1, b_2]^T$ be fixed by the relation

$$b_1 + b_2 = 1 \tag{1.3.34}$$

and find a design that has the greatest buckling load. In view of Eq. (1.3.34), Eq. (1.3.33) reduces to

$$P_s^2 - (1 + b_0^2 + 3b_1^2 - 2b_1)P_s + b_0^2(2b_1^2 - 2b_1 + 1) + b_1^2(b_1^2 - 2b_1 + 1) = 0 \tag{1.3.35}$$

For an antisymmetric buckling mode, $y_3 = -y_2$ and $R = 2M_0/(5l)$ because both the bending moment and the deflection vanish at the center of the column. Proceeding as above, a quadratic equation is obtained for the buckling load P_a of the antisymmetric mode, in the form

$$P_a^2 - (3 + 0.6b_0^2 + 5b_1^2 - 6b_1)P_a + b_0^2(2b_1^2 - 3.6b_1 + 1.8) + 5b_1^2(b_1^2 - 2b_1 + 1) = 0 \tag{1.3.36}$$

The smaller root of this equation is the antisymmetric buckling load.

In Fig. 1.3.4, the smaller of the buckling loads P_s and P_a is shown as a function of b, for fixed values of b_0. To indicate important features of this relation, consider the case $b_0 = 0.3$, for which the variation of the buckling

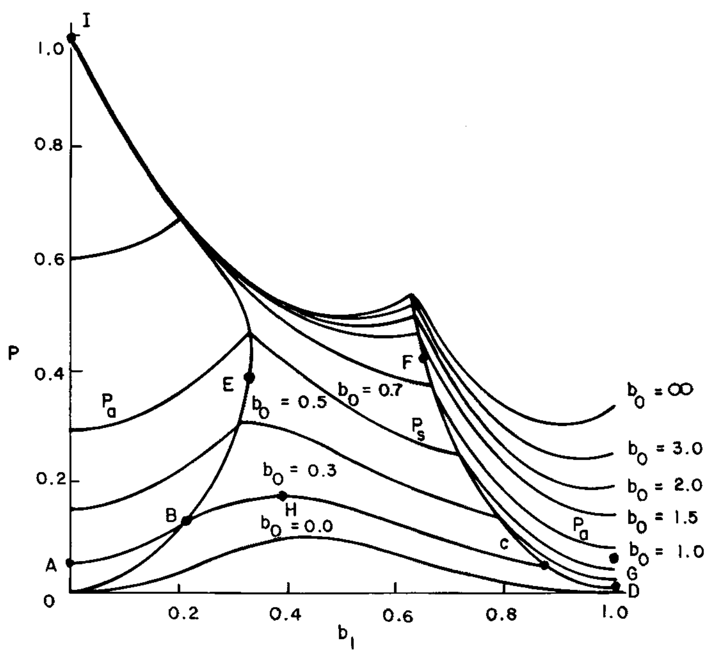

Fig. 1.3.4 Buckling loads for optimum columns.

load is shown by the line ABCD. The arcs AB and CD correspond to antisymmetric buckling, while the arc BC corresponds to symmetric buckling. At point B in Fig. 1.3.4, both symmetric and antisymmetric buckling are possible, and the buckling load is given by a repeated eigenvalue. A similar observation applies to point C. The arc BC, however, has its greatest ordinate at point H, so the optimum design for $b_0 = 0.3$ corresponds to $b_1 = 0.39$, which causes buckling in a symmetric mode, the buckling load being a simple eigenvalue. However, the buckling load is a repeated eigenvalue if b_0 is larger than the value $b_0 = 0.57$ that corresponds to point E, and the optimum design may buckle in a symmetric or an antisymmetric mode, or in any linear combination of the two.

Note that a second local maximum occurs as a repeated eigenvalue for $b_0 \geq 1.31$, corresponding to point F. Another local maximum occurs as a simple eigenvalue for $b_0 \geq 0.94$, corresponding to point G. Also, note that the curve of Fig. 1.3.4 is not concave; indeed several relative maxima occur, and when a repeated eigenvalue occurs at an optimum, the eigenvalue is not differentiable with respect to design. Thus, if $dP/db_1 = 0$ were to be used as an optimality criteria, serious errors would result from the outcome of such computations.

1.3.6 Directional Derivatives of Repeated Eigenvalues

ANALYSIS WITH REDUCED GLOBAL STIFFNESS AND MASS MATRICES

Consider first the eigenvalue problems that arise in vibration or buckling of structures, using the reduced global stiffness and mass (or geometrical stiffness) matrices,

$$K(b)y = \zeta M(b)y \tag{1.3.37}$$

where $y \in R^m$. In the problems considered here, $K(b)$ and $M(b)$ are symmetric, positive definite matrices.

The derivation of design derivatives in Section 1.3.1 is valid only under the assumption that the eigenvalues and eigenvectors are differentiable with respect to design, which is true if the eigenvalue is simple. However, even a repeated eigenvalue is *directionally differentiable* (see Appendix A.3), which is to be shown in the theorem that follows.

Let the eigenvalue $\zeta(b)$ of Eq. (1.3.37) have multiplicity $s \geq 1$ at b and define an $s \times s$ matrix $\mathscr{M}$ with elements

$$\mathscr{M}_{ij} = \frac{\partial}{\partial b}[\tilde{y}^{i\mathrm{T}}K(b)\tilde{y}^j]\,\delta b - \zeta(b)\frac{\partial}{\partial b}[\tilde{y}^{i\mathrm{T}}M(b)\tilde{y}^j]\,\delta b, \qquad i,j = 1,\ldots,s \tag{1.3.38}$$

where $\{y^i\}$ $(i = 1, 2, \ldots, s)$ is any $M(b)$-orthonormal basis of the eigenspace associated with $\zeta(b)$. Note that $\mathscr{M}$ depends on the direction δb of design change (i.e., $\mathscr{M} = \mathscr{M}(b, \delta b)$). The following theorem characterizes the directional derivatives of repeated eigenvalues.

THEOREM 1.3.2 If the matrices $K(b)$ and $M(b)$ are symmetric, positive definite, and differentiable, then the directional derivatives $\zeta_i'(b, \delta b)$ $(i = 1, 2, \ldots, s)$ of a repeated eigenvalue $\zeta(b)$ in the direction δb exist and are equal to the eigenvalues of the matrix $\mathscr{M}$.

PROOF Since matrices $K(b)$ and $M(b)$ are positive definite, hence nonsingular, Eq. (1.3.37) may be rewritten as

$$C(b)y^i \equiv [K^{-1}(b)M(b)]y^i = \frac{1}{\zeta}y^i, \qquad i = 1, \ldots, s \tag{1.3.39}$$

where $(y^i, My^i) = \delta_{ij}$ and δ_{ij} is the Kronecker delta, which is one if $i = j$ and zero otherwise. Since $K(b)$ and $M(b)$ are differentiable with respect to b, so is $C(b)$. In particular,

$$C(b + \tau\,\delta b) = C(b) + \tau\left[\sum_l \frac{\partial C}{\partial b_l}\,\delta b_l\right] + o(\tau) \tag{1.3.40}$$

where $o(\tau)$ denotes a quantity such that

$$\lim_{\tau \to 0} o(\tau)/\tau = 0.$$

By theorem 5.11, chapter 2, of Kato [13],

$$\zeta_i(b + \tau\,\delta b) = \zeta + \tau\zeta_i'(b, \delta b) + o(\tau), \qquad i = 1, \ldots, s \tag{1.3.41}$$

where $\zeta_i'(b, \delta b) = -[\zeta(b)]^2\alpha_i'(b, \delta b)$ and $\alpha_i'(b, \delta b)$ are eigenvalues of the operator (suppressing argument b for notational simplification)

$$N = P\left[\sum_l \frac{\partial C}{\partial b_l}\,\delta b_l\right]P \tag{1.3.42}$$

where P is the M-orthogonal projection matrix that maps R^m onto the eigenspace

$$Y \equiv \left\{y \in R^m\colon\ y = \sum_{i=1}^{s} a_i y^i,\ a_i \text{ real}\right\}$$

That is, for any $y \in R^m$,

$$Py = \sum_{i=1}^{s} (My, y^i)y^i \tag{1.3.43}$$

and the scalar product $(\cdot, \cdot)$ is defined as $(v, y) \equiv v^{\mathrm{T}}y = \sum_{i=1}^{s} v_i y_i$.

The eigenvalues of the operator N must now be found. Each eigenvector of N can be expressed as

$$\bar{y} = \sum_{j=1}^{s} a_j y^j \tag{1.3.44}$$

where not all values of a_j are zero. Hence,

$$P\left[\sum_l \frac{\partial C}{\partial b_l}\delta b_l\right]P\bar{y} = \alpha'(b, \delta b)\bar{y}$$

or

$$\sum_{j=1}^{s} a_j P\left[\sum_l \frac{\partial C}{\partial b_l}\delta b_l\right]Py^j = \alpha' \sum_{j=1}^{s} a_j y^i \tag{1.3.45}$$

Taking the scalar product of Eq. (1.3.45) with My^i gives

$$\begin{aligned}\sum_{j=1}^{s} a_j\left(P\left[\sum_l \frac{\partial C}{\partial b_l}\delta b_l\right]Py^j, My^i\right) &= \alpha' \sum_{j=1}^{s} a_j(y^j, My^i)\\ &= \alpha' a_i, \qquad i = 1, 2, \ldots, s \end{aligned}\tag{1.3.46}$$

To have a nontrivial solution a_j, α' must be an eigenvalue of the matrix

$$\hat{N}_{ij} = [(Ny^j, My^i)]_{s\times s} \tag{1.3.47}$$

By definition of $C = K^{-1}M$ in Eq. (1.3.39),

$$\frac{\partial C}{\partial b_l} = K^{-1}\frac{\partial M}{\partial b_l} - K^{-1}\frac{\partial K}{\partial b_l}K^{-1}M \tag{1.3.48}$$

Thus,

$$\hat{N} = \left[\left(My^i, PK^{-1}\sum_l\left(\frac{\partial M}{\partial b_l} - \frac{\partial K}{\partial b_l}C\right)\delta b_l Py^j\right)\right]_{s\times s} \tag{1.3.49}$$

Since $Py^j = y^j$ and $Cy^j = (1/\zeta)y^j$,

$$\hat{N} = \left[\left(My^i, PK^{-1}\sum_l\left(\frac{\partial M}{\partial b_l}y^j - \frac{1}{\zeta}\frac{\partial K}{\partial b_l}y^j\right)\delta b_l\right)\right]_{s\times s} \tag{1.3.50}$$

Note that for any vector $v \in R^m$,

$$\begin{aligned}(My^i, Pv) &= \sum_{j=1}^{s}(My^i, (Mv, y^j)y^j)\\ &= (Mv, y^i) = (v, My^i) = \frac{1}{\zeta}(v, Ky^i)\end{aligned}\tag{1.3.51}$$

Applying this result to Eq. (1.3.50) gives

$$\hat{N} = \left[\left(\frac{1}{\zeta}Ky^i, K^{-1}\sum_l\left(\frac{\partial M}{\partial b_l}y^j - \frac{1}{\zeta}\frac{\partial K}{\partial b_l}y^j\right)\delta b_l\right)\right]$$

$$= \left[\frac{1}{\zeta}\frac{\partial}{\partial b}(\tilde{y}^i, M(b)\tilde{y}^j)\,\delta b - \frac{1}{\zeta^2}\frac{\partial}{\partial b}(\tilde{y}^i, K(b)\tilde{y}^j)\,\delta b\right]_{s\times s} \tag{1.3.52}$$

Noting that $\alpha_i'(b, \delta b)$ are the eigenvalues of $\hat{N}$ and $\zeta_i'(b, \delta b) = -(\zeta(b))^2\alpha_i'(b, \delta b)$, it is concluded that $\zeta_i'(b,\delta b)$ are the eigenvalues of $\mathcal{M} = -(\zeta(b))^2\hat{N}$, which gives

$$\mathcal{M} = \left[\frac{\partial}{\partial b}(\tilde{y}^i, K(b)\tilde{y}^j)\,\delta b - \zeta\frac{\partial}{\partial b}(\tilde{y}^i, M(b)\tilde{y}^j)\,\delta b\right]_{s\times s} \tag{1.3.53}$$

Since this is the matrix defined in Eq. 1.3.38, the proof of the theorem is complete. ■

The notation $\zeta_i'(b, \delta b)$ is selected in Theorem 1.3.2 to emphasize dependence of the directional derivative on δb. It is not surprising that in the neighborhood of a design for which the eigenvalue is repeated s times that there may be s distinct eigenvalues. The remarkable fact implied by the preceding result is that the eigenvalues of the matrix $\mathcal{M}$ of Eq. (1.3.38) do not depend on the $M(b)$-orthonormal basis that is selected for the eigenspace. Moreover, if the eigenvalues $\zeta_i(b + \delta b)$ are ordered by increasing magnitude, their directional derivatives are the eigenvalues of $\mathcal{M}$, in the order of increasing magnitude.

To see that the directional derivatives of the eigenvalue are not generally linear in δb, consider the case of a double eigenvalue (i.e., $s = 2$). The characteristic equation for determining the eigenvalues of $\mathcal{M}$ may be written, for the case $s = 2$, as

$$\begin{vmatrix} \mathcal{M}_{11} - \zeta' & \mathcal{M}_{12} \\ \mathcal{M}_{21} & \mathcal{M}_{22} - \zeta' \end{vmatrix} = \mathcal{M}_{11}\mathcal{M}_{22} - \mathcal{M}_{12}^2 - (\mathcal{M}_{11} + \mathcal{M}_{22})\zeta' + (\zeta')^2 = 0 \tag{1.3.54}$$

where the fact that $\mathcal{M}_{12} = \mathcal{M}_{21}$ has been used. Solving this characteristic equation for ζ' gives a pair of roots that provide the directional derivatives of the eigenvalue as

$$\zeta_i'(b, \delta b) = \tfrac{1}{2}\{(\mathcal{M}_{11} + \mathcal{M}_{22}) \pm [(\mathcal{M}_{11} + \mathcal{M}_{22})^2 - 4(\mathcal{M}_{11}\mathcal{M}_{22} - \mathcal{M}_{12}^2)]^{1/2}\}, \qquad i = 1, 2 \tag{1.3.55}$$

where $i = 1$ corresponds to the $-$ sign and $i = 2$ corresponds to the $+$ sign.

This equation orders the directional derivatives of the repeated eigenvalue according to magnitude. It is clear that with this ordering, even though the $\mathscr{M}_{ij}$ are linear in δb, the resulting formula for $\zeta'(b, \delta b)$ is not linear in δb, hence it is not a Fréchet derivative (Appendix A.3). This situation may arise because the eigenvalues have been ordered by magnitude. As indicated by the

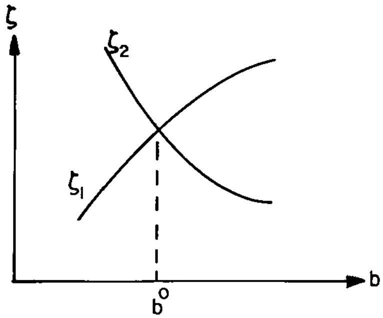

Fig. 1.3.5 Schematic of eigenvalue crossing.

schematic diagram in Fig. 1.3.5, even if a smooth ordering of the eigenvalue versus design curves exists, ordering by magnitude leads to a derivative discontinuity at the point of a repeated eigenvalue.

COMPUTATION OF DIRECTIONAL DERIVATIVES OF A REPEATED EIGENVALUE

There exists an ordering of $\zeta_i(b + \tau\,\delta b)$ such that the mapping $\tau \to \zeta_i(b + \tau\,\delta b)$ is differentiable at $\tau = 0$. In general, however, this ordering depends on δb. To see this, let $s = 2$. A method for determining the directional derivatives was introduced by Masur and Mroz [22]. They used an orthogonal transformation of eigenvectors, beginning with a given $\mathscr{M}(b)$-orthonormal set y^1 and y^2 and defined a "rotated" set

$$\begin{aligned} \hat{y}^1 &= y^1 \cos\phi + y^2 \sin\phi \\ \hat{y}^2 &= -y^1 \sin\phi + y^2 \cos\phi \end{aligned} \tag{1.3.56}$$

where ϕ is a rotation parameter. An easy calculation shows that if y^1 and y^2 are $M(b)$-orthonormal, then so are $\hat{y}^1$ and $\hat{y}^2$. The transformed eigenvectors may thus be used in evaluating the matrix $\mathscr{M}$ of Eq. (1.3.38), denoted as $\hat{\mathscr{M}}$. Since the eigenvalues of $\hat{\mathscr{M}}$ are the same as those for $\mathscr{M}$, a rotation parameter ϕ may be chosen to cause the matrix $\mathscr{M}$ to be diagonal. If such a ϕ can be found, then the diagonal elements of $\hat{\mathscr{M}}$ will be the eigenvalues of the original matrix, hence the directional derivatives of the repeated eigenvalue. Thus, it is required that

$$\begin{aligned}0 = \hat{\mathscr{M}}_{12} &= \frac{\partial}{\partial b}[\tilde{\hat{y}}^{1\mathrm{T}}K(b)\tilde{\hat{y}}^{2}]\,\delta b - \zeta(b)\frac{\partial}{\partial b}[\tilde{\hat{y}}^{1\mathrm{T}}\mathscr{M}(b)\tilde{\hat{y}}^{2}]\,\delta b \\ &= -\cos\phi\sin\phi\,\mathscr{M}_{11} + (\cos^2\phi - \sin^2\phi)\mathscr{M}_{12} + \sin\phi\cos\phi\,\mathscr{M}_{22} \\ &= \tfrac{1}{2}\cos 2\phi\,(\mathscr{M}_{22} - \mathscr{M}_{11}) + \cos 2\phi\,\mathscr{M}_{12} \end{aligned} \tag{1.3.57}$$

Equation (1.3.57) may be solved for

$$\phi = \phi(\delta b) = \tfrac{1}{2}\,\mathrm{Arctan}\left[\frac{2\mathscr{M}_{12}(y^1, y^2, \delta b)}{\mathscr{M}_{11}(y^1, y^1, \delta b) - \mathscr{M}_{22}(y^2, y^2, \delta b)}\right] \tag{1.3.58}$$

where the notation is chosen to emphasize that ϕ depends on the direction of design change δb. Even though the $\mathscr{M}_{ij}$ depend linearly on δb, their ratio on the right side of Eq. (1.3.58) is not linear in δb. Furthermore, the Arctan function is nonlinear.

This angle ϕ may be used in evaluating $\hat{\mathscr{M}}_{11}$ and $\hat{\mathscr{M}}_{22}$ to obtain the directional derivatives of the repeated eigenvalue; that is,

$$\begin{aligned}\zeta_1'(b, \delta b) = \hat{\mathscr{M}}_{11} &= \cos^2\phi\,(\delta b)\mathscr{M}_{11}(\delta b) \\ &\quad + \sin 2\phi\,(\delta b)\mathscr{M}_{12}(\delta b) + \sin^2\phi\,(\delta b)\mathscr{M}_{22}(\delta b)\end{aligned} \tag{1.3.59}$$

$$\begin{aligned}\zeta_2'(b, \delta b) = \hat{\mathscr{M}}_{22} &= \sin^2\phi\,(\delta b)\mathscr{M}_{11}(\delta b) \\ &\quad - \sin 2\phi\,(\delta b)\mathscr{M}_{12}(\delta b) + \cos^2\phi\,(\delta b)\mathscr{M}_{22}(\delta b)\end{aligned} \tag{1.3.60}$$

where the notation $\mathscr{M}_{ij}(\delta b)$ is used to emphasize dependence on design change. Note that even though $\mathscr{M}_{ij}(\delta b)$ is linear in δb since the trigonometric multipliers depend on δb, the directional derivatives appearing in Eqs. (1.3.59) and (1.3.60) are in general nonlinear in δb. Thus, the directional derivatives of a repeated eigenvalue are indeed not linear in δb. Hence, ζ is nondifferentiable. Only if $\mathscr{M}_{12}(\delta b)$ is identically equal to zero for all δb, with some pair of $\underset{\sim}{M}(b)$-orthonormal eigenvectors, can the repeated eigenvalues be ordered in such a way that they are Fréchet differentiable.

It may be noted in Eq. (1.3.58) that for $\tau \neq 0$, $\phi(b, \tau\,\delta b) = \phi(b, \delta b)$; that is, $\phi(b, \delta b)$ is homogeneous of degree zero in δb. Thus, since $\mathscr{M}_{ij}(\delta b)$ are linear in δb,

$$\zeta_i'(b, \tau\,\delta b) = \tau\zeta_i'(b, \delta b) \tag{1.3.61}$$

That is, the directional derivatives of a repeated eigenvalue are homogeneous of degree one in δb. This implies that once δb is fixed, the eigenvalues can be ordered such that the repeated eigenvalue is differentiable with respect to τ.

While the foregoing approach could be used to treat a triple eigenvalue, the analysis would be much more delicate. In this case, the matrix is 3×3, and a cubic characteristic equation would have to be solved. The alternative is to use a three-parameter family of $M(b)$-orthonormal eigenfunctions and to choose the three rotation parameters to cause the offdiagonal terms of $\hat{\mathcal{M}}$ to be zero. This will be a complicated task, since three trigonometric equations in three unknowns must be solved. While analytical solution of directional derivatives for eigenvalues of multiplicity greater than 2 may be difficult, the same basic ideas may be employed for numerical calculation.

ANALYSIS WITH GENERALIZED GLOBAL STIFFNESS AND MASS MATRICES

Consider the reduced formulation of the generalized eigenvalue problem of Eq. (1.3.9), given by Eq. (1.3.13), for the repeated eigenvalue problem with nonsingular reduced stiffness and mass matrices $\check{K}(b)$ and $\check{M}(b)$ in Eq. (1.3.14). Let c^i ($i = 1, \ldots, s$) be $\check{M}(b)$-orthonormal eigenvectors, with

$$\check{K}(b)c^i = \zeta\check{M}(b)c^i, \qquad i = 1, \ldots, s \tag{1.3.62}$$

Thus, vectors $y_g^i = \Phi c^i$ satisfy the relation

$$\delta_{ij} = c^{iT}\check{M}(b)c^j = c^{iT}\Phi^T M_g(b)\Phi c^j = y_g^{iT} M_g(b) y_g^j \tag{1.3.63}$$

where δ_{ij} is the Kronecker delta, so y_g^i are $M_g(b)$-orthonormal.

For the reduced eigenvalue equation of Eq. (1.3.62), use Eq. (1.3.38) to define

$$\check{\mathcal{M}}_{ij} = \frac{\partial}{\partial b}[\tilde{c}^{iT}\check{K}(b)\tilde{c}^j]\,\delta b - \zeta(b)\frac{\partial}{\partial b}[\tilde{c}^{iT}\check{M}(b)\tilde{c}^j]\,\delta b \tag{1.3.64}$$

By Theorem 1.3.2, the directional derivatives $\zeta_i'(b, \delta b)$ ($i = 1, \ldots, s$) of the repeated eigenvalue ζ of Eq. (1.3.9), or equivalently Eq. (1.3.62), are the eigenvalues of $\check{\mathcal{M}}$. Using $y_g^i = \Phi c^i$ and Eq. (1.3.14),

$$\begin{aligned}\check{\mathcal{M}}_{ij} &= \frac{\partial}{\partial b}[\tilde{c}^{iT}\Phi^T K_g(b)\Phi\tilde{c}^j]\,\delta b - \zeta(b)\frac{\partial}{\partial b}[\tilde{c}^{iT}\Phi^T M_g(b)\Phi\tilde{c}^j]\,\delta b \\ &= \frac{\partial}{\partial b}[\tilde{y}_g^{iT}K_g(b)\tilde{y}_g^j]\,\delta b - \zeta(b)\frac{\partial}{\partial b}[\tilde{y}_g^{iT}M_g(b)\tilde{y}_g^j]\,\delta b, \qquad i,j = 1, \ldots, s\end{aligned} \tag{1.3.65}$$

Thus, Eqs. (1.3.58)–(1.3.60) are valid for the directional derivatives of a repeated eigenvalue, with $\mathcal{M}_{ij}$ replaced by $\check{\mathcal{M}}_{ij}$ [i.e., written in terms of the generalized global stiffness and mass matrices in Eq. (1.3.65)].

1.3.7 Examples

THREE-BAR TRUSS

To illustrate results of previous sections, consider the three bar truss considered in Section 1.2.6. To simplify the calculation, consider the generalized global lumped mass matrix give as

$$M_g(b) = \frac{\rho l}{2}\begin{bmatrix} b_1+\sqrt{2}b_3 & & & & & \\ & b_1+\sqrt{2}b_3 & & & 0 & \\ & & b_1+b_2 & & & \\ & 0 & & b_1+b_2 & & \\ & & & & b_2+\sqrt{2}b_3 & \\ & & & & & b_2+\sqrt{2}b_3 \end{bmatrix} \tag{1.3.66}$$

where ρ is mass density. The space of kinematically admissible displacements is

$$Z = \{y_g \in R^6:\ y_3 = y_4 = 0,\ y_5 \cos\alpha + y_6 \sin\alpha = 0\} \tag{1.3.67}$$

and $K_g(b)$ of Eq. (1.2.63) is positive definite on Z. If $\theta = 45°$ and $\alpha = 30°$, then with $y = [y_1 \quad y_2 \quad y_3]^T$, the reduced mass matrix in this example is

$$M(b) = \frac{\rho l}{2}\begin{bmatrix} b_1+\sqrt{2}b_3 & 0 & 0 \\ 0 & b_1+\sqrt{2}b_3 & 0 \\ 0 & 0 & 4(b_2+\sqrt{2}b_3) \end{bmatrix} \tag{1.3.68}$$

For the eigenvalue problem, assume $E = 1$, $\rho = 1$, $b_1 = b_2 = 1$, and $b_3 = 2\sqrt{2}$. Then the fundamental eigenvalue is $\zeta = 0.08038$ and the $M(b)$-normalized eigenvector is

$$y = [y_1 \quad y_2 \quad y_5]^T = [-0.3496 \quad 0.08451 \quad 0.2601]^T \tag{1.3.69}$$

The reduced eigenvalue design sensitivity may now be evaluated from Eq. (1.3.8) as

$$\begin{aligned} \frac{d\zeta}{db} &= \frac{\partial}{\partial b}(\tilde{y}^T K(b)\tilde{y}) - \zeta\frac{\partial}{\partial b}(\tilde{y}^T M(b)\tilde{y}) \\ &= [0.001944 \quad 0.05678 \quad -0.02076] \end{aligned} \tag{1.3.70}$$

In case the generalized global formulation is employed, the same eigenvalue as in the reduced formulation is computed. The $M_g(b)$-normalized eigenvector is

$$y_g = [-0.3496 \quad 0.08451 \quad 0 \quad 0 \quad 0.2601 \quad -0.45051]^T \tag{1.3.71}$$

The eigenvalue design sensitivity formula of Eq. (1.3.17), with the variational formulation, gives

$$\begin{aligned}\frac{d\zeta}{db} &= \frac{\partial}{\partial b}(\tilde{y}_g^{\mathrm{T}} K_g(b)\tilde{y}_g) - \zeta\frac{\partial}{\partial b}(\tilde{y}_g^{\mathrm{T}} M_g(b)\tilde{y}_g) \\ &= [0.001944 \quad 0.05678 \quad -0.02076]\end{aligned} \tag{1.3.72}$$

which is the same as in Eq. (1.3.70).

Since there is no evidence of designs leading to repeated eigenvalues in this example, repeated eigenvalue sensitivity formulas are not written.

PORTAL FRAME

As an example in which repeated eigenvalues occur at a given design, consider the portal frame shown in Fig. 1.3.6. The structure is modeled using beam elements of lengths l_i and uniform cross-sectional areas b_i, as shown in Fig. 1.3.6. No axial deformation is considered in this example. The design problem is to find $b \in R^n$ that minimizes the weight

$$\psi_0(b) = \gamma \sum_{i=1}^{n} l_i b_i \tag{1.3.73}$$

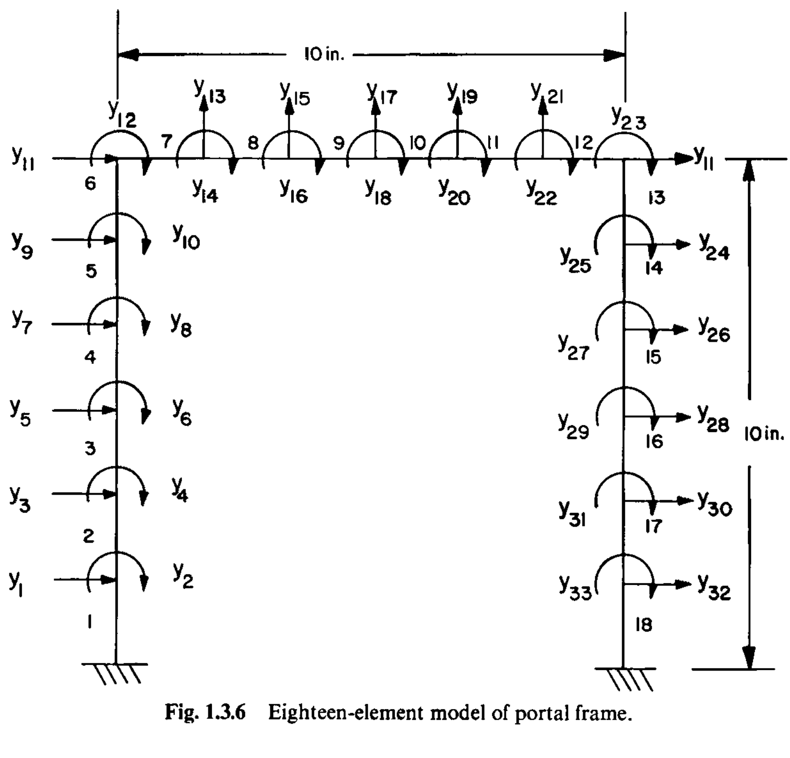

Fig. 1.3.6 Eighteen-element model of portal frame.

subject to natural frequency constraints

$$\psi_i = \zeta_0 - \zeta_i \le 0, \qquad i = 1, 2 \tag{1.3.74}$$

and constraints on cross-sectional area

$$\psi_{j+2} = c_j - b_j \le 0, \qquad j = 1, 2, \ldots, n$$

where γ is weight density of the material and $\zeta_i = \omega_i^2$.

The numerical results presented are based on the following data:

1. Length of each member of portal frame is 10 in.
2. Moment of inertia of cross-sectional area is $I_i = \alpha b_i^2$.
3. Geometry of a cross section is circular ($\alpha = 0.08$).
4. Young's modulus of elasticity is $E = 10.3 \times 10^6$ psi.
5. Mass density of the material is $\gamma = 0.26163 \times 10^{-3}$ lb-sec^2/in.^{4}.

The eighteen-finite-element model of Fig. 1.3.6 is used in computation, and the current design, which gives repeated eigenvalues $\zeta_1 = 3.360591 \times 10^7$ and $\zeta_2 = 3.364971 \times 10^7$, is given in Table 1.3.1 (column a).

The perturbation direction δb to be used in the calculation of directional derivatives $\zeta_i'(b, \delta b)$ in Eqs. (1.3.59) and (1.3.60) is given in Table 1.3.1 (column b). A comparison of design sensitivity between the actual changes

Table 1.3.1

Current Design and Perturbation

(a) Current Design		(b) Perturbation	
i	b_i	i	$\delta b(i)$
1	0.6614E + 01	1	0.6960E − 01
2	0.4626E + 01	2	0.4933E − 01
3	0.2747E + 01	3	0.2921E − 01
4	0.1602E + 01	4	0.7251E − 02
5	0.9134E + 00	5	0.4467E − 02
6	0.3709E + 00	6	0.1841E − 02
7	0.3500E + 00	7	0.0000E + 00
8	0.3500E + 00	8	0.0000E + 00
9	0.3500E + 00	9	0.0000E + 00
10	0.3500E + 00	10	0.0000E + 00
11	0.3500E + 00	11	0.0000E + 00
12	0.3500E + 00	12	0.0000E + 00
13	0.3709E + 00	13	−0.2025E − 02
14	0.9134E + 00	14	−0.5360E − 02
15	0.1602E + 01	15	−0.9426E − 01
16	0.2747E + 01	16	−0.4090E − 01
17	0.4626E + 01	17	−0.7400E − 01
18	0.6614E + 01	18	−0.1114E + 00

Table 1.3.2

Comparison of Sensitivity

Constraint	$\Delta\psi_i$	ψ_i'	$(\psi_i'/\Delta\psi_i \times 100)\%$
ψ_1	-0.1875×10^6	-0.2016×10^6	107.5
ψ_2	0.8397×10^5	0.9968×10^5	118.7

and predictions by the sensitivity formulas of Eqs. (1.3.59) and (1.3.60) is presented in Table 1.3.2. Since $\psi_i'(b, \delta b)$ $(i = 1, 2)$ are nonlinear in δb for the current design, $d\psi_i/db$ can not be found to calculate $\psi_i' = (d\psi_i/db)\,\delta b$.

1.4 DYNAMIC RESPONSE DESIGN SENSITIVITY

Thus far in this chapter, only static response and eigenvalues that represent steady-state motion and buckling of structures have been treated. Under time-varying loads or nonzero initial conditions, transient dynamic response of the structure must be considered. Design sensitivity analysis of structural response measures in a transient dynamic environment is treated in this section, first including the effect of damping and then specializing the results to undamped structures, yielding substantial computational simplification.

1.4.1 Design Sensitivity Analysis of Damped Elastic Structures

Consider first the case of a structure in which the generalized global stiffness and mass matrices have been reduced by accounting for boundary conditions. Further, let the damping force in the structure be represented in the form $-C(b)\dot{z}$. Under these conditions, the *Lagrange equations of motion* become

$$M(b)\ddot{z} + C(b)\dot{z} + K(b)z = F(t, b) \tag{1.4.1}$$

with initial conditions

$$z(0) = z^0, \qquad \dot{z}(0) = \dot{z}^0 \tag{1.4.2}$$

Consider a structural response functional of the general form

$$\psi = g(z(T), b) + \int_0^T G(z, b)\, dt \tag{1.4.3}$$

where the final time T is determined by a condition of the form

$$\Omega(z(T), \dot{z}(T), b) = 0 \tag{1.4.4}$$

That is, given a design b, the equations of motion of Eqs. (1.4.1) and (1.4.2) can be integrated to monitor the value of $\Omega(z(t), \dot{z}(t), b)$. The time at which this quantity goes to zero is defined as the time T. The functional in Eq. (1.4.3) can then be evaluated. It is presumed that Eq. (1.4.4) determines T uniquely, at least locally. This requires that

$$\dot{\Omega} \equiv \frac{\partial \Omega}{\partial z} \dot{z}(T) + \frac{\partial \Omega}{\partial \dot{z}} \ddot{z}(T) \neq 0 \tag{1.4.5}$$

It is clear from Eq. (1.4.1) that the solution $z = z(t; b)$ of the initial-value problem of Eqs. (1.4.1) and (1.4.2) depends on the design variable b. The nature of this dependence is characterized by a well-known theorem from ordinary differential equations [23].

THEOREM 1.4.1 If the matrices $M(b)$, $C(b)$, and $K(b)$ and vector $F(t, b)$ are s times continuously differentiable with respect to b and if the matrix $M(b)$ is nonsingular, then the solution $z = z(t; b)$ is s times continuously differentiable with respect to b.

Theorem 1.4.1 guarantees that the dynamic response of a structural system is essentially as smooth as the dependence on b in the equations of motion.

Consider now a variation in design of the form

$$b_\tau = b + \tau\, \delta b \tag{1.4.6}$$

Substituting b_τ into Eq. (1.4.3), the derivative of both sides of Eq. (1.4.3) can be evaluated with respect to τ at $\tau = 0$. Leibnitz's rule of differentiation of an integral [24] may be used to obtain

$$\psi' = \frac{\partial g}{\partial b} \delta b + \frac{\partial g}{\partial z} [z'(T) + \dot{z}(T)T'] + G(z(T), b)T' + \int_0^T \left[\frac{\partial G}{\partial z} z' + \frac{\partial G}{\partial b} \delta b \right] dt \tag{1.4.7}$$

where

$$z' = z'(b, \delta b) \equiv \frac{d}{d\tau} z(t, b + \tau\, \delta b) \bigg|_{\tau=0} = \frac{d}{db} [z(t, b)]\, \delta b$$

$$T' = T'(b, \delta b) \equiv \frac{d}{d\tau} T(b + \tau\, \delta b) \bigg|_{\tau=0} = \frac{dT}{db} \delta b$$

Note that since the expression in Eq. (1.4.4) that determines T depends on design, T itself will depend on design. Thus, terms arise in Eq. (1.4.7) that involve the derivative of T with respect to design. In order to eliminate these terms, take the derivative of Eq. (1.4.4) with respect to τ and evaluate at $\tau = 0$ to obtain

$$\frac{\partial \Omega}{\partial z} [z'(T) + \dot{z}(T)T'] + \frac{\partial \Omega}{\partial \dot{z}} [\dot{z}'(T) + \ddot{z}(T)T'] + \frac{\partial \Omega}{\partial b} \delta b = 0 \tag{1.4.8}$$

This equation may be rewritten as

$$\dot{\Omega}T' \equiv \left[\frac{\partial\Omega}{\partial z}\dot{z}(T) + \frac{\partial\Omega}{\partial\dot{z}}\ddot{z}(T)\right]T' = -\left(\frac{\partial\Omega}{\partial z}z'(T) + \frac{\partial\Omega}{\partial\dot{z}}\dot{z}'(T) + \frac{\partial\Omega}{\partial b}\delta b\right) \tag{1.4.9}$$

Since it is presumed that $\dot{\Omega} \neq 0$ [see Eq. (1.4.5)],

$$T' = -\left\{\left(\frac{\partial\Omega}{\partial z}\Big/\dot{\Omega}\right)z'(T) + \left(\frac{\partial\Omega}{\partial\dot{z}}\Big/\dot{\Omega}\right)\dot{z}'(T) + \left(\frac{\partial\Omega}{\partial b}\Big/\dot{\Omega}\right)\delta b\right\} \tag{1.4.10}$$

Substituting the result of Eq. (1.4.10) into Eq. (1.4.7),

$$\begin{aligned}\psi' = &\left[\frac{\partial g}{\partial z} - \frac{\partial g}{\partial z}\dot{z}(T) + G(z(T), b)\left(\frac{\partial\Omega}{\partial z}\Big/\dot{\Omega}\right)\right]z'(T)\\ &- \left[\frac{\partial g}{\partial z}\dot{z}(T) + G(z(T), b)\right]\left(\frac{\partial\Omega}{\partial b}\Big/\dot{\Omega}\right)\dot{z}'(T) + \int_0^T \left[\frac{\partial G}{\partial z}z' + \frac{\partial G}{\partial b}\delta b\right]dt\\ &+ \frac{\partial g}{\partial b}\delta b - \left[\frac{\partial g}{\partial z}\dot{z}(T) + G(z(T), b)\right]\left(\frac{\partial\Omega}{\partial b}\Big/\dot{\Omega}\right)\delta b\end{aligned} \tag{1.4.11}$$

Note that the differential of ψ in Eq. (1.4.11) depends on the differential of the state z and velocity $\dot{z}$ at T, as well as on the differential of z in the integral.

In order to write the variation of ψ in Eq. (1.4.11) explicitly in terms of a variation in design, the adjoint variable technique employed in Sections 1.2.2 and 1.2.3 may be used. In the case of a dynamic system, one may multiply all terms in Eq. (1.4.1) by the transpose of $\lambda = \lambda(t)$ and integrate over the interval $[0, T]$ to obtain the following identity in λ:

$$\int_0^T \lambda^{\mathrm{T}}[M(b)\ddot{z} + C(b)\dot{z} + K(b)z - F(t, b)]\,dt = 0 \tag{1.4.12}$$

Since this equation must hold for arbitrary λ, which is taken at this point to be independent of design, substitute b_τ of Eq. (1.4.6) and take the differential of Eq.(1.4.12) to obtain the relationship, using Leibnitz's rule and noting that the integrand is zero at T,

$$\int_0^T \left[\lambda^{\mathrm{T}}M(b)\ddot{z}' + \lambda^{\mathrm{T}}C(b)\dot{z}' + \lambda^{\mathrm{T}}K(b)z' - \frac{\partial R}{\partial b}\delta b\right]dt = 0 \tag{1.4.13}$$

where

$$R \equiv \tilde{\lambda}^{\mathrm{T}}F(t, b) - \tilde{\lambda}^{\mathrm{T}}M(b)\ddot{\tilde{z}} - \tilde{\lambda}^{\mathrm{T}}C(b)\dot{\tilde{z}} - \tilde{\lambda}^{\mathrm{T}}K(b)\tilde{z} \tag{1.4.14}$$

with denoting variables that are to be held constant for the purposes of taking partial derivatives with respect to design in Eq. (1.4.13).

Integrating the first two terms under the integral in Eq. (1.4.13) by parts yields

$$\lambda^{\mathrm{T}}(T)M(B)\dot{z}'(T) - \dot{\lambda}^{\mathrm{T}}(T)M(b)z'(T) + \lambda^{\mathrm{T}}(T)C(b)z'(T)$$
$$+ \int_0^T \left\{[\ddot{\lambda}^{\mathrm{T}}M(b) - C(b)\dot{\lambda}^{\mathrm{T}}(b) + \lambda^{\mathrm{T}}K(b)]z' - \frac{\partial R}{\partial b}\,\delta b\right\} dt = 0 \tag{1.4.15}$$

Since Eq. (1.4.15) must hold for arbitrary functions $\lambda(t)$, λ may be chosen so that the coefficients of terms involving $z'(T)$, $\dot{z}'(T)$, and z' in Eqs. (1.4.11) and (1.4.15) are equal. If such a function $\lambda(t)$ can be found, then the unwanted terms in Eq. (1.4.11) involving $z'(T)$, $\dot{z}'(T)$, and z' can be replaced by terms that depend explicitly on δb in Eq. (1.4.15). To be more specific, require that

$$M(b)\lambda(T) = -\left[\frac{\partial g}{\partial z}\dot{z}(T) + G(z(t), b)\right]\frac{\partial \Omega^{\mathrm{T}}}{\partial \dot{z}}\Big/\dot{\Omega} \tag{1.4.16}$$

$$M(b)\dot{\lambda}(T) = C^{\mathrm{T}}(b)\lambda(T) - \frac{\partial g^{\mathrm{T}}}{\partial z} + \left[\frac{\partial g}{\partial z}\dot{z}(T) + G(z(T), b)\right]\frac{\partial \Omega^{\mathrm{T}}}{\partial z}\Big/\dot{\Omega} \tag{1.4.17}$$

$$M(b)\ddot{\lambda} - C^{\mathrm{T}}(b)\dot{\lambda} + K(b)\lambda = \frac{\partial G^{\mathrm{T}}}{\partial z}, \qquad 0 \le t < T \tag{1.4.18}$$

Note that once the state equations of Eqs. (1.4.1) and (1.4.2) are solved and Eq. (1.4.4) is used to determine T, then $z(T)$, $\dot{z}(T)$, $\partial\Omega/\partial z$, $\partial\Omega/\partial\dot{z}$, and $\dot{\Omega}$ may be evaluated. Equation (1.4.16) may then be solved for $\lambda(T)$ since the mass matrix $M(b)$ is nonsingular. Having determined $\lambda(T)$, all terms on the right of Eq. (1.4.17) may be evaluated, and this equation may be solved for $\dot{\lambda}(T)$. Thus, a set of terminal conditions on λ has been determined. Since $M(b)$ is nonsingular, Eq. (1.4.18) may then be integrated from T to 0, yielding a unique solution $\lambda(t)$. The system of Eqs. (1.4.16)–(1.4.18) may be thought of as a *terminal-value problem*.

Since terms involving variation of the state variable in Eqs. (1.4.11) and (1.4.15) are identical, substitute from Eq. (1.4.15) into Eq. (1.4.11) to obtain

$$\psi' = \left\{\frac{\partial g}{\partial b} + \int_0^T \left[\frac{\partial G}{\partial b} + \frac{\partial R}{\partial b}\right] dt \right.$$
$$\left. - \left[\frac{\partial g}{\partial z}\dot{z}(T) + G(z(T), b)\right]\left(\frac{\partial \Omega}{\partial b}\Big/\dot{\Omega}\right)\right\}\delta b \equiv \frac{d\psi}{db}\,\delta b \tag{1.4.19}$$

All terms in this equation can now be calculated. The first term outside the integral represents an explicit partial derivative with respect to design, as does the first term inside the integral. The second term inside the integral, however, must be evaluated from Eq. (1.4.14), hence requiring $\lambda(t)$. Note also that since the design variation δb does not depend on time, it is taken outside the integral in Eq. (1.4.19).

Since Eq. (1.4.19) must hold for all δb, the vector of design derivatives of ψ with respect to b is

$$\frac{d\psi}{db} = \frac{\partial g}{\partial b}(z(T), b) + \int_0^T \left[\frac{\partial G}{\partial b}(z, b) + \frac{\partial R}{\partial b}(\lambda(t), z(t), \dot{z}(t), \ddot{z}(t), b)\right] dt$$
$$- \left[\frac{\partial g}{\partial z}\dot{z}(T) + G(z(T), b)\right]\left(\frac{\partial \Omega}{\partial b}\Big/\dot{\Omega}\right) \qquad (1.4.20)$$

The computational algorithm leading to determination of $d\psi/db$ requires that the state initial-value problem of Eqs. (1.4.1) and (1.4.2) be integrated forward in time from 0 to T. Then the adjoint terminal-value problem of Eqs. (1.4.18), (1.4.16), and (1.4.17) must be integrated backward in time from T to 0. Both sets of calculations can be done with well-known numerical integration algorithms [25]. Once these initial- and terminal-value problems have been solved, the design derivatives of ψ in Eq. (1.4.20) can be calculated using a numerical integration formula [25]. Thus, while substantial numerical computation is required, it is clear that design derivatives of dynamic response can be computed.

1.4.2 Design Sensitivity Analysis of Undamped Structures

Consider now the case in which damping in the structure can be neglected. In this case, the state initial-value problem is reduced to

$$\begin{aligned} M(b)\ddot{z} + K(b)z &= F(t, b) \\ z(0) = 0, \qquad \dot{z}(0) &= 0 \end{aligned} \qquad (1.4.21)$$

While theoretical considerations in this case are identical to those in Section 1.4.1, an essential computational advantage arises.

Consider the generalized eigenvalue problem associated with the differential equation in Eq. (1.4.21), written as

$$K(b)\phi^i = \zeta_i M(b)\phi^i, \qquad i = 1, \ldots, r \le m \qquad (1.4.22)$$

where the number r of eigenvectors calculated is generally substantially less than the number of independent degrees of freedom of the system. Further, it is presumed that the eigenvectors ϕ^i are normalized by the condition

$$\phi^{j\mathrm{T}} M(b)\phi^i = \delta_{ij}, \qquad i, j = 1, \ldots, r \qquad (1.4.23)$$

Using these eigenvectors, approximate the solution $z(t)$ of Eq. (1.4.21) by the eigenvector expansion

$$z(t) \approx \sum_{i=1}^{r} c_i(t)\phi^i = \Phi c(t) \tag{1.4.24}$$

where $\Phi = [\phi^1, \ldots, \phi^r]$. Note that if $r = m$, the solution $z(t)$ can be precisely represented by Eq. (1.4.24). On the other hand, it is conventional in structural dynamics to select a number r of eigenvectors strictly less than m in order to carry out an approximate solution efficiently. For a discussion of the number of eigenvectors to retain, the reader is referred to Bathe [16].

Substituting from Eq. (1.4.24) into the differential equation of Eq. (1.4.21) and premultiplying by ϕ^{T}, the following system of differential equations for $c(t)$ is obtained:

$$\Phi^{\mathrm{T}}M(b)\Phi\ddot{c} + \Phi^{\mathrm{T}}K(b)\Phi c = \Phi^{\mathrm{T}}F(t,b) \equiv \check{F}(t,b) \tag{1.4.25}$$

where it is required that

$$c(0) = 0, \qquad \dot{c}(0) = 0 \tag{1.4.26}$$

Using the normalizing condition of Eq. (1.4.23) and the eigenvalue equation of Eq. (1.4.22), Eq. (1.4.25) reduces to

$$\ddot{c} + \Lambda c = \check{F}(t,b) \tag{1.4.27}$$

where

$$\Lambda = \mathrm{diag}[\zeta_1, \ldots, \zeta_r] \tag{1.4.28}$$

Since Λ is diagonal, Eqs. (1.4.27) are decoupled and may be written in scalar form with the initial conditions of Eq. (1.4.26). This system is given by

$$\left.\begin{aligned} &\ddot{c}_i + \zeta_i c_i = \check{F}_i(t,b), \\ &c_i(0) = 0, \qquad \dot{c}_i(0) = 0, \end{aligned}\right. \qquad i = 1, \ldots, r \tag{1.4.29}$$

An explicit solution of each of these decoupled initial-value problems may be written as

$$c_i(t) = \frac{1}{\sqrt{\zeta_i}} \int_0^T \sin[\sqrt{\zeta_i}(t - \tau)]\check{F}_i(\tau,b)\,d\tau, \qquad i = 1, \ldots, r \tag{1.4.30}$$

This may be verified by differentiation and substitution in Eq. (1.4.29). Thus, in the case of a structure without damping and with homogeneous initial conditions, an explicit solution of the equations of motion may be obtained by evaluating $c_i(t)$ from Eq. (1.4.30) and substituting their values into Eq. (1.4.24).

The homogeneous initial condition of Eq. (1.4.21) is not restrictive, since nonhomogeneous initial conditions $z(0) = z^0$ and $t\dot{z}(0) = \dot{z}^0$ can be treated by defining a particular solution $z_p = z^0 + t\dot{z}^0$ of Eqs. (1.4.1) and (1.4.2) and substituting it into the initial-value problem to obtain a problem of the form of Eq. (1.4.21), with only an additional term $-K(b)z_p$ appearing on the right side of the differential equation.

Consider now the special case of a functional of a form of the Eq. (1.4.3), with $g = 0$ and an explicitly given terminal time T. In this special case, the right sides of Eqs. (1.4.16) and (1.4.17) are zero and the terminal-value problem becomes

$$\begin{aligned} M(b)\ddot{\lambda} + K(b)\lambda &= \frac{\partial G^{\mathrm{T}}}{\partial z}(t, z(t), b) \\ \lambda(T) = 0, \qquad \dot{\lambda}(T) &= 0 \end{aligned} \tag{1.4.31}$$

These limiting assumptions are not restrictive, since in the general case nonhomogeneous terminal conditions $\lambda(T) = \lambda^0$ and $\dot{\lambda}(T) = \dot{\lambda}^0$ can be found from Eqs. (1.4.16) and 1.4.17) and variables changed, using a particular solution $\lambda = \lambda^0 + (t - T)\dot{\lambda}^0$, to obtain homogeneous terminal conditions of Eq. (1.4.31) with an additional term $-K(b)\lambda_p$ on the right side of the differential equation. The special case is treated here to avoid the algebra associated with this transformation.

Note that the left side of the differential equation in Eq. (1.4.30) is identical to the left side of the differential equation in Eq. (1.4.21). Thus, the eigenvector expansion technique, using precisely the same set of eigenvectors determined from Eqs. (1.4.22) and (1.4.23), may be employed. Thus, the adjoint variable is approximated as

$$\lambda(t) \approx \sum_{i=1}^{r} e_i(t)\phi^i = \Phi e(t) \tag{1.4.32}$$

Substituting this formula into Eq. (1.4.31) and premultiplying by Φ^{T}, the uncoupled terminal-value problems are

$$\begin{aligned} \ddot{e}_i + \zeta_i e_i &= \frac{\partial G}{\partial z}(t, z(t), b)\phi^i, \\ e_i(T) = 0, \qquad \dot{e}_i(T) &= 0, \end{aligned} \qquad i = 1, \ldots, r \tag{1.4.33}$$

These equations may be solved in closed form to obtain

$$e_i(t) = \frac{1}{\sqrt{\zeta_i}} \int_t^T \sin[\sqrt{\zeta_i}(t - \tau)] \frac{\partial G}{\partial z}(\tau, z(\tau), b)\phi^i \, d\tau, \qquad i = 1, \ldots, r \tag{1.4.34}$$

The adjoint variable $\lambda(t)$ may now be constructed from Eq. (1.4.32), and design derivatives may be evaluated from Eq. (1.4.20). These results are of substantial practical importance, since structural damping may be neglected in many important elastic structures, yielding a practical and computationally efficient design sensitivity analysis algorithm. To further generalize this result, structural damping effects are often approximated so that the damping matrix $C(b)$ is proportional to either the stiffness or the mass matrix [3, 4]. Using such an approximation, the foregoing design sensitivity analysis method can be extended to treat structures with this special form of damping.

1.4.3 Functionals Arising in Structural Dynamic Design

The general form of the cost or constraint functional of Eq. (1.4.3) can be used to represent or approximate most quantities that measure structural response in practice. Consider first the case of a constraint on response and design that must hold for all time; that is,

$$\eta(z(t), b) \leq 0, \qquad 0 \leq t \leq T \tag{1.4.35}$$

Such constraints may be approximated in several ways.

$$\psi_i = \int_0^T [\phi(z, b) + |\phi(z, b)|]\, dt = 0 \tag{1.4.36}$$

Equivalence of Eqs. (1.4.35) and (1.4.36) for continuous functions is easily demonstrated. Use of the functional of Eq. (1.4.36) has provided a capability for reducing constraint errors in Eq. (1.4.35) to near zero [10]. However, as the error approaches zero, the domain over which the integrand in Eq. (1.4.36) is defined reduces to zero length, and a singular functional occurs. This behavior limits the precision with which convergence can be obtained in structural optimization calculations.

An alternative treatment of the constraint of Eq. (1.4.34) is to define the time t_1 at which the maximum value of the left side of Eq. (1.4.35) occurs. It must satisfy the condition

$$\Omega(z(t_1), \dot{z}(t_1), b) \equiv \frac{\partial \eta}{\partial z}(z(t_1), b)\dot{z}(t_1) = 0 \tag{1.4.37}$$

The constraint of Eq. (1.4.35) may now be replaced by the equivalent constraint

$$\psi_2 = \eta(z(t_1), b) \leq 0 \tag{1.4.38}$$

This functional is of the form of Eq. (1.4.3), with the terminal time t_1 determined by Eq. (1.4.36). Thus, the algorithm of the preceding section can be directly applied.

Finally, an averaging multiplier technique may be used in which a characteristic function $m(t, t_1)$ is defined to be symmetric about the point $t_1 < T$ and to have a unit integral. The function m is defined on a small subdomain of the interval from 0 to T in such a way that as the length of the subdomain approaches 0, m approaches the Dirac δ-function (more properly the Dirac measure). The value of $\eta(z(t_1), b)$ may thus be approximated as

$$\psi_3 = \int_0^T m(t, t_1)\eta(z(t), b)\, dt \leq 0 \tag{1.4.39}$$

where

$$\int_0^T m(t, t_1)\, dt = 1 \tag{1.4.40}$$

While some error is involved in the approximation of Eq. (1.4.39), quite good numerical results can be obtained using a function m that is defined on a finite subdomain about the time t_1 at which the actual maximum displacement occurs for the nominal design. This formulation has the advantage that sensitivity of the time t_1 with respect to design need not be considered in the approximate computations. Thus, only an integral constraint is involved in actual iterative calculation.

1.5 PROJECTED GRADIENT FOR TRADE-OFF DETERMINATION

1.5.1 Constrained Design Sensitivity Analysis

While it is necessary to know the cost and each constraint function ψ_i $(i = 0, 1, \ldots, q)$ depends on each design variable, this is not sufficient for large-scale system design. More valuable information for the designer would be derivatives of one of these functions, say the cost ψ_0, subject to the conditions that

$$\hat{\psi}' = \frac{d\hat{\psi}}{db}\,\delta b = 0 \tag{1.5.1}$$

where $\hat{\psi}$ is a vector of the constraint functions that are at their critical values and for which the designer decides no change is desired. For example, if stresses are large in several elements of a machine or structure and displacements of a few points are near their allowable limits, then the designer may wish to consider design changes that leave these quantities unchanged—hence Eq. (1.5.1).

Geometrically, the goal is to project the design gradient of ψ_0 onto the surface in design space defined by Eq. (1.5.1). Since the design space may be rather complicated, this operation must be accomplished analytically. Noting that the gradient of a functional is simply the direction of steepest ascent in design space, a vector δb may be sought in design space such that $-\psi_0'$ is as large as possible (hence a direction of steepest descent for ψ_0), consistent with the constraint of Eq. (1.5.1) and with a quadratic step-size limitation,

$$\delta b^{\mathrm{T}} W_b \, \delta b = \xi^2 \tag{1.5.2}$$

which retains the validity of linear approximations. Here W_b is a positive definite design-weighting matrix and ξ is small.

This problem is solved [10] using necessary conditions of optimality. The result is a scalar multiple of the *projected gradient*, or *constrained derivative*, of $-\psi_0$,

$$\delta b = -W_b^{-1}\left[\frac{d\psi_0^{\mathrm{T}}}{db} - \frac{d\hat{\psi}^{\mathrm{T}}}{db} M_{\hat{\psi}\hat{\psi}}^{-1} M_{\hat{\psi}\psi_0}\right] \tag{1.5.3}$$

where

$$M_{\hat{\psi}\hat{\psi}} = \frac{d\hat{\psi}}{db} W_b^{-1} \frac{d\hat{\psi}^{\mathrm{T}}}{db} \tag{1.5.4}$$

$$M_{\hat{\psi}\psi_0} = \frac{d\hat{\psi}}{db} W_b^{-1} \frac{d\psi_0^{\mathrm{T}}}{db} \tag{1.5.5}$$

Once the sensitivity vectors $d\psi_i^{\mathrm{T}}/db$ are known, it is a simple and numerically efficient matter to calculate the constrained steepest descent vector δb in Eq. (1.5.3). As noted in the foregoing, this information can be generated with minor additional computer time compared to that required for analysis of response of a trial design. The value of this constrained design sensitivity information to the designer is perhaps even greater than response data (structural analysis output) with no trend information. Further, the designer can redefine the vector $\hat{\psi}$ of active constraints and get a new set of constrained design derivatives from Eq. (1.5.3), rapidly and with trivial computer cost.

1.5.2 Example

In order to illustrate the foregoing method, consider the ten-member cantilever truss of Section 1.2.6. To illustrate the idea of design sensitivity analysis and constrained derivatives, consider the design given in the first column of Table 1.5.1. For this design, three design sensitivity vectors were

Table 1.5.1

Design Derivatives and Constrained Gradients for 10-Member Cantilever Truss

Number	Design	$d\psi_1^T/db$	$d\psi_2^T/db$	$d\psi_3^T/db$	$\delta b^{(1)}$	$\delta b^{(2)}$	$\delta b^{(3)}$	$\delta b^{(4)}$
1	28.6	0.0082	−0.0009	−0.0093	−36.19	−31.27	−31.91	−32.85
2	0.2	−0.0696	−0.0284	0.0109	−34.44	−38.50	−37.68	−36.53
3	23.6	−0.0104	0.0012	−0.0062	−35.77	−32.03	−33.46	−35.44
4	15.4	−0.0006	−0.0003	−0.0076	−35.99	−31.73	−32.69	−34.03
5	0.2	−2.3520	−0.9601	0.1402	16.90	6.17	5.48	4.38
6	0.2	−0.0696	−0.0284	0.0109	−34.44	−38.50	−37.68	−36.53
7	3.0	−0.8369	−0.4398	−0.0177	−32.09	1.62	3.37	5.03
8	21.0	0.0231	−0.0026	−0.0128	−51.43	−45.00	−45.14	−45.41
9	21.8	−0.0009	−0.0004	−0.0108	−50.89	−44.89	−46.24	−48.12
10	0.2	−0.1968	−0.0803	0.0308	−46.49	−57.97	−55.66	−52.38

calculated in Section 1.2.6. The vectors $d\psi_1^T/db$ and $d\psi_2^T/db$ are design derivatives of the normalized stresses in members 5 and 7, respectively, and $d\psi_3^T/db$ is the derivative of the normalized displacement in the y direction at node 2. The vector $d\psi_1^T/db$ indicates that in order to decrease the stress level in member 5, the areas of members 2–7, 9, and 10 should be increased, whereas areas of members 1 and 8 should be decreased. The vector $d\psi_2^T/db$ indicates that for reduction of stress level in member 7, areas of all members except the third should be increased. This indicates that there is some conflict between the constraints ψ_1 and ψ_2. Similar trends may be observed for the vector $d\psi_3^T/db$.

The constrained derivative of Eq. (1.5.3) resolves this conflict. The constrained derivative is the direction of most rapid decrease of the cost function, subject to the condition that constraints in $\hat{\psi} = [\psi_1\ \psi_2\ \psi_3]^T$ remain at their current values. From the designer's point of view, this information is useful because it tells which of the variables can be adjusted to obtain a desired reduction in cost, consistent with the constraints. Three such constrained derivatives are given in Table 1.5.1. The vector δb is computed from Eq. (1.5.3) by imposing the condition that ψ_1' must be equal to zero. Similarly, $\delta b^{(2)}$ and $\delta b^{(3)}$ are computed from the conditions that $\psi_1' = \psi_3' = 0$ and $\psi_2' = \psi_3' = 0$, respectively. Finally, $\delta b^{(4)}$ is calculated from conditions that $\psi_1' = \psi_2' = \psi_3' = 0$. It is observed that different δb vectors result, depending on the constraints that are included in the analysis.

1.5.3 Interactive Computer-Aided Design

For approximately two decades, substantial effort has been devoted to development of structural optimization techniques, using iterative methods of nonlinear programming and optimality criteria [26, 27]. More recently, attempts to document this theory have appeared in book form [10, 28]. In spite of substantial progress in structural optimization, there is a feeling of despair among workers in the field that fruits of their labor have not seen extensive application. The purpose of this subsection is to offer possible reasons for this dilemma and to suggest an avenue of pursuit that may encourage greater use of structural optimization theory and techniques.

Design sensitivity analysis techniques, leading to the theory and numerical methods presented in this chapter, have heretofore been embedded in iterative optimization algorithms and have not normally been given primary emphasis. In particular, most structural optimization work has tended toward iterative algorithms that contains design sensitivity analysis, with the objective of automated structural optimization (i.e., turning iterative control over to the computer rather than keeping the design engineer in the loop). This approach can suffer from two pitfalls. First, it requires that the designer identify all important constraints, the levels of constraint limits, and the cost function precisely and enter this information into the structural optimization algorithm. Very often in applications, numerous trade-offs exist, and in fact constraints may become important that the designer did not foresee during initial attempts at formulation of the optimal design problem. Recent literature on multicriteria optimization [29] suggests that this dilemma is the rule rather than the exception. If this is indeed the case, it is important to provide the structural engineer with an interactive tool that can be used to refine the formulation of the problem, while proceeding toward an optimum design.

Even if the structural optimization problem is precisely defined, most iterative optimization methods require considerable judgment on the part of the user in selecting parameters that influence convergence of the algorithm. Since structural optimization problems are highly nonlinear, there is always the threat of divergence or convergence to a poor relative optimum. In many cases, the initial design formulation may be so optimistic in its demands that no feasible design satisfying the initially specified constraints exists, so that convergence of an optimization algorithm is impossible. Even when an optimum design exists, the more robust modern optimization methods tend to require a large number of iterations, each of which requires numerous reanalyses and hence a massive amount of computing time. Even the most experienced design engineer who tries to use an automated iterative optimization algorithm and has it fail to converge or has it expend a large amount of computing time, only to find a poor local relative minimum will soon become frustrated and conclude that "tried-and-true conventional methods" are superior to these "new-fangled" optimization methods.

A promising approach to alleviate the foregoing dilemmas may be to resort to interactive trade-off analysis of the kind outlined in Section 1.5.2.

Initial redesign computations can be carried out under the control of the experienced design engineer. In fact, extension of the preceding gradient projection computation, using a sequential quadratic programming approach [30], may yield even better results than the simple algorithm presented here. Using such a technique with an interactive computer-aided design system, the design engineer can explore alternatives and play a role in refining the formulation of the design problem during the early stages of iterative design improvement. In this way, the engineer will more readily identify a situation in which no feasible solution to the originally formulated problem exists and take action to modify the formulation to achieve acceptable trade-offs. Furthermore, the engineer will be more likely to gain confidence in the iterative algorithm as it progresses toward steadily improved designs. The engineer may then be willing to turn iterative control of the optimization algorithm over to the computer for a limited number of redesign iterations and may in fact speed adoption of modern nonlinear programming methods in structural optimization.

In addition to speeding adoption of structural optimization techniques, the interactive computer-aided design technique suggested here need not interfere with global convergence properties of iterative optimization algorithms. Since the designer will be involved in only a finite number of initial redesign iterations, any globally convergent algorithm will still converge when it is begun from the design invented by the engineer at the design station. In fact, intermediate output of sequential quadratic programming algorithms [30] provides information needed by the engineer to override the automated algorithm, if desired. If it is chosen not to override the algorithm, then the desired global convergence properties are retained, and no theoretical harm is done. In fact, if such designer intervention in the progress of the algorithm improves confidence in the algorithm, there is every reason to believe that progress will be made in accelerating the adoption of such techniques and in improving understanding and facility in their use.

2

Distributed-Parameter Structural Components

In contrast to the matrix equation development of design sensitivity theory in Chapter 1, a distributed-parameter (continuum) approach is presented in this chapter. The principal distinction between the two approaches lies in the use of displacement fields that satisfy boundary-value problems of elasticity to characterize structural deformation, rather than nodal displacements that are determined by matrix equations.

While the finite-dimensional and distributed-parameter approaches are related (the former is an approximation of the latter), advantages and disadvantages accrue to the two approaches. The principal disadvantage of the distributed-parameter approach, from an engineering viewpoint, is the higher level of mathematical sophistication associated with the infinite-dimensional function spaces of displacements and designs. As will be seen in this and the following two chapters, however, symmetry and positive definiteness of energy forms associated with elastic structures yield a complete theory that parallels the matrix theory of Chapter 1. The only real penalty associated with the distributed-parameter formulation is the level of complexity of technical proofs that are required. In order to minimize frustration of the reader who is interested primarily in applications, the authors have organized material in this and the following chapters to begin with a treatment of techniques and examples, stating results as they are needed and citing proofs that are given later in the chapter.

Two principal advantages of the distributed-parameter approach to structural design sensitivity analysis are

1. a rigorous mathematical theory is obtained, without the uncertainty associated with finite-dimensional approximation error, and

2. explicit relations for design sensitivity are obtained in terms of physical quantities, rather than in terms of sums of derivatives of element matrices.

The former feature is of importance in development of the theory of structural optimization, which has provided the principal motivation for development of the theory. The latter feature has not yet been fully exploited. Use of the results of this chapter in numerical calculations is discussed in Sections 2.2.4 and 2.3.3.

A final note on the variational (virtual work) viewpoint adopted in this and the following chapters is in order, prior to launching into the details. Both the matrix and variational approaches were seen to be viable in treating finite-dimensional systems in Chapter 1. In this distributed-parameter setting, only the variational approach is acceptable. Use of linear operator theory may be considered to parallel matrix theory, but even the operator theory needed is based on a reduction of each problem to variational form [6, 9, 13, 14, 31–35]. In fact, the elegance and practicality of the variational approach become apparent as design sensitivity theory is developed.

2.1 VARIATIONAL FORMULATION OF STATIC AND EIGENVALUE PROBLEMS

The mathematical theory of boundary-value problems that describe deformation, buckling, and harmonic vibration of elastic structures has recently turned to a powerful variational approach [9, 31]. This theory begins with a statement of the classical boundary-value problem and proceeds to reduce it to a variational, or energy-related, formulation. The result is a rigorous existence and uniqueness theory and a formulation that provides the foundation for a rigorous and practical theory of finite element analysis [5, 6]. In retrospect, the variational formulation obtained may be viewed as the principle of virtual work or the Galerkin method for solution of boundary-value problems [14, 32].

Much as with the generalized global stiffness matrix formulation in Chapter 1, in order to take advantage of the power of the variational formulation for design sensitivity analysis it is essential to work with "kinematically admissible displacement fields". Readers who are interested primarily in applications can restrict their attention to classes of smooth functions and need not be concerned with the more general function spaces presented. Extensions to more general spaces of functions that are needed in later sections to prove the validity of the design sensitivity analysis method are introduced here, however. In order to be concrete, specific examples that are treated later in the text are presented and analyzed in this section.

2.1.1 Static Elastic Systems

BENDING OF A BEAM

Consider the beam of Fig. 2.1.1, with a normalized axial coordinate x, clamped supports, and variable cross-sectional area $h(x)$. The area distribution $h(x)$ may be taken as a smooth function that is bounded above and below, $0 < h_0 \leq h(x) \leq h_1$, or it may be taken in the larger space of essentially bounded functions, $L^\infty(0, 1)$. Here, $L^\infty(a, b)$ is the space of Lebesgue measurable functions $h(x)$ on the interval $a \leq x \leq b$, such that $\|h\|_\infty \equiv \inf\{M > 0: |h(x)| \leq M \text{ a.e. in } [a, b]\}$ (where "almost everywhere" is abbreviated a.e.). For an introductory treatment of such function spaces, the reader is referred to Appendix A.2. Readers who are interested primarily in applications and who are willing to restrict their attention to continuous designs may use the space of continuous designs; $C^0(a, b) = \{h(x): h(x) \text{ is continuous on } [a, b]\}$, with the norm $\|h\|_0 \equiv \max_{a \leq x \leq b} |h(x)|$. The larger space $L^\infty(a, b)$ of designs is included here for mathematical completeness.

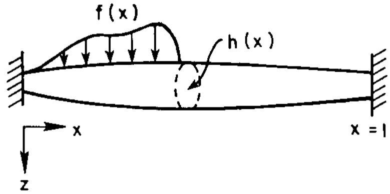

Fig. 2.1.1 Clamped beam of variable cross-sectional area $h(x)$.

It is presumed that all dimensions of the cross section vary with the same ratio (i.e., all are geometrically similar) so the moment of inertia of the cross-sectional area is $I(x) = \alpha h^2(x)$, where α is a positive constant that depends on the shape of the cross section. The boundary-value problem for displacement $z(x)$ is written formally as

$$
\begin{aligned}
(E\alpha h^2(x) z_{xx})_{xx} &= f(x) \\
z(0) = z_x(0) = z(1) &= z_x(1) = 0
\end{aligned}
\tag{2.1.1}
$$

where E is Young's modulus, $f(x)$ distributed load, and subscript notation is used to indicate derivatives with respect to x.

The material constant E and material distribution function $h(x)$ may be viewed as design variables since they serve to specify the structure and may be selected by the designer. To simplify notation, they are denoted as a design vector $u = [E, h(x)]^T \in U \equiv R \times L^\infty(0, 1)$, or $R \times C^0(0, 1)$. This notation simply means that E is real (in R) and $h(x)$ is in $L^\infty(0, 1)$ or $C^0(0, 1)$, as the engineer wishes. Presuming a solution $z(x)$ of Eq. (2.1.1) exists, it is clear that it will depend on the design vector u. This dependence may be denoted by

$z(x; u)$, that is, a displacement function defined on $0 \leq x \leq 1$ that depends on the design u. The form of dependence of z on x and u is very different. If design u is changed, the displacement will generally change at all x.

In linear operator notation, the boundary-value problem of Eq. (2.1.1) is

$$\bar{A}_u z \equiv (E\alpha h^2(x) z_{xx})_{xx} = f \tag{2.1.2}$$

where the subscript u denotes dependence of the differential operator on the design vector u and z must satisfy the boundary conditions of Eq. (2.1.1). For $h \in L^\infty(0, 1)$, or even $C^0(0, 1)$, the boundary-value problem of Eq. (2.1.1) is only formal. If h is twice continuously differentiable [i.e., $h \in C^2(0, 1)$] and if z is four times continuously differentiable [i.e., $z \in C^4(0, 1)$], then the problem of Eq. (2.1.1) has a classical meaning. Considering the classical case, in which all functions are sufficiently smooth, both sides of Eq. (2.1.2) can be multiplied by an arbitrary function $\bar{z}(x)$ and integrated to obtain

$$\int_0^1 [(E\alpha h^2(x) z_{xx})_{xx} - f]\bar{z}\, dx = 0 \tag{2.1.3}$$

which must hold for any integrable function $\bar{z}$. Conversely, if Eq. (2.1.3) holds for all twice continuously differentiable functions $\bar{z}$ that satisfy the boundary conditions of Eq. (2.1.1) and if $z \in C^4(0, 1)$, then the differential equation of Eq. (2.1.1) is satisfied. This is true since the space of kinematically admissible displacements

$$\hat{Z} = \{\bar{z} \in C^2(0, 1)\colon\ \bar{z}(0) = \bar{z}_x(0) = \bar{z}(1) = \bar{z}_x(1) = 0\},$$

which may be viewed in classical mechanics as virtual displacements, is dense in $L^2(0, 1)$ [9, 31].

Two integrations by parts can now be carried out in the first term in Eq. (2.1.3) to obtain

$$\begin{aligned} 0 &= \int_0^1 E\alpha h^2 z_{xx} \bar{z}_{xx}\, dx - \int_0^1 f\bar{z}\, dx + [(E\alpha h^2 z_{xx})_x \bar{z} - E\alpha h^2 z_{xx} \bar{z}_{xx}]\Big|_0^1 \\ &= \int_0^1 E\alpha h^2 z_{xx} \bar{z}_{xx}\, dx - \int_0^1 f\bar{z}\, dx \end{aligned} \tag{2.1.4}$$

where the boundary terms vanish because $\bar{z} \in \hat{Z}$ is required to satisfy the boundary conditions of Eq. (2.1.1). Defining the energy bilinear form

$$a_u(z, \bar{z}) = \int_0^1 E\alpha h^2 z_{xx} \bar{z}_{xx}\, dx \tag{2.1.5}$$

Eq. (2.1.4) is just

$$a_u(z, \bar{z}) = (f, \bar{z}) \equiv l_u(\bar{z}) \qquad \text{for all} \quad \bar{z} \in \hat{Z} \tag{2.1.6}$$

where $(\cdot, \cdot)$ denotes the $L^2(0, 1)$ scalar product, defined as

$$(w, v) \equiv \int_0^1 w(x)v(x)\,dx$$

and $l_u(\bar{z})$ is the load linear form, or virtual work of force f and virtual displacement $\bar{z}$. Note that Eq. (2.1.6) could have been written directly from variational principles of elasticity [33, 34]; the principle of virtual work in this case. This approach is used in Chapter 4 for more complicated structures. The boundary-value formulation is used here for structural elements since it is more commonly encountered in the engineering literature.

It is important to note that the restriction of h to $C^2(0, 1)$ and z to $C^4(0, 1)$ is not only unnatural, but also unnecessary. The bilinear form $a_u(z, \bar{z})$ of Eq. (2.1.5) is well defined for $h \in L^\infty(0, 1)$ or $C^0(0, 1)$ and for any $z(x)$ and $\bar{z}(x)$ that have second derivatives that are in $L^2(0, 1)$, that is, for which $\int_0^1 (z_{xx})^2\,dx$ and $\int_0^1 (\bar{z}_{xx})^2\,dx$ are finite (see Appendix A.2). Thus, the variational equation of Eq. (2.1.6) may be satisfied by a function z that has only one continuous derivative, with a possibly irregular second derivative that is only required to be in $L^2(0, 1)$ and satisfying the boundary conditions of Eq. (2.1.1). Such a function is called the variational or *generalized solution* of the boundary-value problem of Eq. (2.1.6).

An alternative view of the variational formulation of the beam equation may be obtained from the minimum total potential energy characterization of beam bending. That is, the displacement $z(x) \in \hat{Z}$ is to minimize

$$\text{PE} = \int_0^1 \left[\frac{1}{2}E\alpha h^2(z_{xx})^2 - fz\right] dx$$

It is clear that the potential energy is well defined as long as $z_{xx} \in L^2(0, 1)$ and that it does not require z to be $C^4(0, 1)$. Equating the first variation of PE to zero, with the variation $\bar{z}(x)$ having two derivatives, $\bar{z}_{xx} \in L^2(0, 1)$, and $\bar{z}$ satisfying the boundary conditions of Eq. (2.1.1),

$$\delta\text{PE} \equiv \frac{d}{d\tau}\int_0^1 \left[\frac{1}{2}E\alpha h^2(z_{xx} + \tau\bar{z}_{xx})^2 - f(z + \tau\bar{z})\right] dx\bigg|_{\tau=0}$$

$$= \int_0^1 [E\alpha h^2 z_{xx}\bar{z}_{xx} - f\bar{z}]\,dx = 0$$

But this is just Eq. (2.1.6).

Recovery of the differential equation of Eq. (2.1.1) is only possible if integration by parts can be justified, hence requiring either restrictive and physically unjustifiable assumptions on differentiability of z and h or introduction of the idea of distributional derivatives [9, 31, 35], which in reality make the boundary-value problem of Eq. (2.1.1) the equivalent of the variational equation of Eq. (2.1.6). Thus, the variational formulation is more natural from the point of view of mechanics than the fourth-order differential equation of Eq. (2.1.1).

The variational formulation of the problem yields a greater degree of generality if solutions are defined in *Sobolev spaces* (Appendix A.2) of functions. For functions of one variable x, the Sobolev space $H^m(0, 1)$ is the collection of all functions that may be obtained as limits of functions in $\{z \in C^m(0, 1) \colon \|z\|_{H^m} < \infty\}$, where

$$\|z\|_{H^m} = \left[\sum_{i=0}^{m} \int_0^1 \left(\frac{d^i z(x)}{dx^i}\right)^2 dx\right]^{1/2} = \left(\sum_{i=0}^{m} \left\|\frac{d^i z}{dx^i}\right\|_{L^2}^2\right)^{1/2} \tag{2.1.7}$$

Define a function of compact support on the open interval $[0, 1]$ as a function $z(x)$ that is zero outside an interval $[\varepsilon, 1 - \varepsilon]$ for some $\varepsilon > 0$. The space of functions that are $H^m(0, 1)$ limits of $C^\infty(0, 1)$ functions with compact support (Appendix A.2) is denoted $H_0^m(0, 1)$. It is known [8, 36] that the set of all functions in $H^2(0, 1)$ that satisfy the boundary conditions of Eq. (2.1.1) is precisely the space $H_0^2(0, 1) = Z$, which is an extension of the space $\hat{Z}$ of smooth functions that satisfy the boundary conditions defined earlier.

Of particular importance in mathematical analysis of beams by the variational method is the *Sobolev imbedding theorem* (Appendix A.2). Existence theory for variational equations of the form of Eq. (2.1.6) [9, 35] guarantees that there will be a solution $z \in H^2(0, 1)$. The natural question now is, How smooth is $z \in H^2(0, 1)$? For functions of one variable, the Sobolev imbedding theorem asserts that $z \in C^1[0, 1]$ and that there is a constant $C > 0$ such that

$$\max_{i=0,1} \max_{0 \le x \le 1} \left|\frac{d^i z(x)}{dx^i}\right| \le C\|z\|_{H^2(0,1)}$$

This result explains why kinematic boundary conditions are preserved when H^2 limits are taken of smooth functions that satisfy kinematic boundary conditions. For a compact introduction to Sobolev spaces and their application to structural mechanics, the reader is referred to the outstanding article of Fichera [35]. For a comprehensive treatment of the subject, see the book by Adams [36]. The engineering reader who is interested primarily in applications need not be concerned with these functional analysis generalizations of the problem. They are included here to show that extensions of the formulation are possible and as preparation for mathematical proofs of differentiability of displacement with respect to design that are presented in Section 2.4.

In a Sobolev space setting, the variational formulation of the boundary-value problem of Eq. (2.1.1) is to find a function $z \in Z$ such that Eq. (2.1.6) [with $h \in L^\infty(0, 1)$] is satisfied for all $\bar{z} \in Z$. This problem may be reformulated with an operator A_u, called the *Friedrichs extension* of $\bar{A}_u$, such that

$$(A_u z, \bar{z}) = a_u(z, \bar{z}) \qquad \text{for all} \quad \bar{z} \in Z \tag{2.1.8}$$

The domain of the extended operator A_u is the subspace $D(A_u)$ of Z such that $A_u z \in L^2(0, 1)$. It was shown by Aubin [9] that $D(A_u)$ is dense in $L^2(0, 1)$, and due to the Sobolev imbedding theorem, the identity map from $D(A_u) \subset Z$ to $L^2(0, 1)$ is compact [36]. Further, it was shown by Aubin [9] and Fichera [35] that the operator equation

$$A_u z = f \tag{2.1.9}$$

has a unique solution in $D(A_u)$ for each $f \in L^2(0, 1)$. Thus, the operator A_u defined by Eq. (2.1.8) is an extension of the operator $\bar{A}_u$ in Eq. (2.1.2), and the operator equation of Eq. (2.1.9) is a generalization of Eq. (2.1.1) in the sense that any classical solution of Eq. (2.1.1) is a solution of Eq. (2.1.9). Furthermore, even when Eq. (2.1.2) fails to have a classical solution, Eq. (2.1.9) will have a generalized solution, which is in fact the "natural solution" of the structural mechanics problem. For a proof of existence and uniqueness, the reader is referred to the article of Fichera [35] or to Aubin [9].

In addition to the existence properties of Eq. (2.1.9), it was shown by Fichera [35] that the energy bilinear form $a_u(z, \bar{z})$ satisfies the following inequalities:

$$a_u(z, \bar{z}) \leq K \|z\|_{H^2} \|\bar{z}\|_{H^2} \qquad \text{for all} \quad z, \bar{z} \in Z \tag{2.1.10}$$

$$a_u(z) \equiv a_u(z, z) \geq \gamma \|z\|_{H^2}^2 \qquad \text{for all} \quad z \in Z \tag{2.1.11}$$

where $K < \infty$ and $\gamma > 0$, provided $E \geq E_0 > 0$ and $h(x) \geq h_0 > 0$ a.e. Equation (2.1.10) states a form of upper bound on the bilinear form, while Eq. (2.1.11) is a lower bound on strain energy, which may also be written as

$$(A_u z, z) \geq \gamma \|z\|_{H^2}^2 \qquad \text{for all} \quad z \in Z \tag{2.1.12}$$

This is the *strong ellipticity* or *Z-ellipticity* property of the operator A_u.

Since Eq. (2.1.9) has a unique solution in $H_0^2(0, 1)$, write

$$z(x; u) = A_u^{-1} f \tag{2.1.13}$$

to emphasize dependence on u. From Eq. (2.1.12),

$$(A_u z, z) = (f, A_u^{-1} f) \geq \gamma \|A_u^{-1} f\|_{H^2}^2 \tag{2.1.14}$$

By the Schwartz inequality [9, 31],

$$\|f\|_{L^2} \|A_u^{-1} f\|_{L^2} \geq |(f, A_u^{-1} f)|$$

Since $\|v\|_{H^2} \geq \|v\|_{L^2}$ for all $v \in H^2(0, 1)$, from Eq. (2.1.14),

$$\|A_u^{-1} f\|_{H^2} \leq \frac{1}{\gamma} \|f\|_{L^2} \qquad \text{for all} \quad f \in L^2(0, 1) \tag{2.1.15}$$

Thus, the operator A_u^{-1} is bounded and hence continuous. It remains only to determine the regularity of dependence of A_u^{-1} on u.

While the foregoing analysis has been carried out with the clamped–clamped beam of Fig. 2.1.1 with boundary conditions of Eq. (2.1.1), the same results are valid for many other boundary conditions, to include the following support conditions and associated boundary conditions [9, 35]:

1. simply supported

$$z(0) = z_{xx}(0) = z(1) = z_{xx}(1) = 0 \tag{2.1.16}$$

2. cantilevered

$$z(0) = z_x(0) = z_{xx}(1) = [E\alpha h^2(1) z_{xx}(1)]_x = 0 \tag{2.1.17}$$

3. clamped-simply supported

$$z(0) = z_x(0) = z(1) = z_{xx}(1) = 0 \tag{2.1.18}$$

The reader may note that since boundary terms in Eq. (2.1.4) vanish if z and $\bar{z}$ satisfy these boundary conditions, the bilinear form $a_u(z, \bar{z})$ of Eq. (2.1.5) is applicable for all boundary conditions of Eqs. (2.1.16)–(2.1.18). It was shown by Aubin [9] and Fichera [35] that the variational characterization of the solution in Eq. (2.1.6) is valid if z and $\bar{z}$ satisfy only *kinematic boundary conditions* of Eqs. (2.1.16), (2.1.17), or (2.1.18) that involve derivatives of order one or less. That is, boundary conditions involving derivatives of order two or three are *natural boundary conditions* and need not be satisfied by z and $\bar{z}$ in Eq. (2.1.6). It was further shown by Fichera [35] that Eqs. (2.1.10) and (2.1.11) are valid for this class of functions. Hence, bounded invertibility of A_u is retained for the boundary conditions of Eqs. (2.1.16)–(2.1.18).

BENDING OF A PLATE

Consider now the clamped plate of variable thickness $h(x) \geq h_0 > 0$ ($h \in L^\infty(\Omega)$ or $C^0(\Omega)$) shown in Fig. 2.1.2. The formal boundary-value problem for displacement z is written in operator form as

$$\begin{aligned} \bar{A}_u z &= f, && \text{in } \Omega \\ z = 0, \partial z/\partial n &= 0, && \text{on } \Gamma \end{aligned} \tag{2.1.19}$$

where the operator $\bar{A}_u$ is defined as

$$\bar{A}_u z = [\hat{D}(u)(z_{11} + \nu z_{22})]_{11} + [\hat{D}(u)(z_{22} + \nu z_{11})]_{22} + 2(1 - \nu)[\hat{D}(u) z_{12}]_{12} \tag{2.1.20}$$

with a subscript i denoting the operation $\partial/\partial x_i$,

$$\hat{D}(u) = Eh^3/[12(1 - \nu^2)]$$

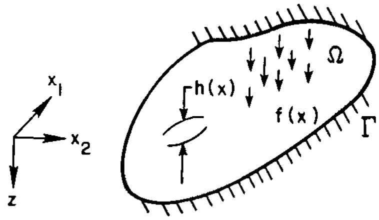

Fig. 2.1.2 Clamped plate of variable thickness $h(x)$.

$E > E_0 > 0$ is Young's modulus, ν Poisson's ratio, and $u = [E, h(x)]^T$. The variational formulation is obtained by multiplying both sides of the differential equation of Eq. (2.1.19) by an arbitrary function $\bar{z} \in C^4(0, 1)$ that satisfies the boundary conditions and integrating by parts

$$
\begin{aligned}
0 &= \iint_\Omega (\bar{A}_u z - f)\bar{z}\, d\Omega \\
&= \iint_\Omega \hat{D}(u)[z_{11}\bar{z}_{11} + \nu z_{22}\bar{z}_{11} + z_{22}\bar{z}_{22} + \nu z_{11}\bar{z}_{22} + 2(1-\nu)z_{12}\bar{z}_{12}]\, d\Omega \\
&\quad - \iint_\Omega f\bar{z}\, d\Omega + \int_\Gamma \{[\hat{D}(u)(z_{11} + \nu z_{22})]_1 \bar{z} n_1 - \hat{D}(u)(z_{11} + \nu z_{22})\bar{z}_1 n_1 \\
&\quad + [\hat{D}(u)(z_{22} + \nu z_{11})]_2 \bar{z} n_2 - \hat{D}(u)(z_{22} + \nu z_{11})\bar{z}_2 n_2 \\
&\quad + (1-\nu)[\hat{D}(u)z_{12}]_1 \bar{z} n_2 + (1-\nu)[\hat{D}(u)z_{12}]_2 \bar{z} n_1 - (1-\nu)\hat{D}(u)z_{12}\bar{z}_1 n_2 \\
&\quad - (1-\nu)\hat{D}(u)z_{12}\bar{z}_2 n_1\}\, d\Gamma \\
&= \iint_\Omega \hat{D}(u)[z_{11}\bar{z}_{11} + \nu z_{22}\bar{z}_{11} + z_{22}\bar{z}_{22} + \nu z_{11}\bar{z}_{22} \\
&\quad + 2(1-\nu)z_{12}\bar{z}_{12}]\, d\Omega - \iint_\Omega f\bar{z}\, d\Omega \\
&\equiv a_u(z, \bar{z}) - l_u(\bar{z})
\end{aligned}
\tag{2.1.21}
$$

for all kinematically admissible "virtual displacements" $\bar{z}$, where n_1 and n_2 are components of the outward unit normal vector to the boundary Γ and the boundary terms vanish because z and $\bar{z}$ satisfy the boundary conditions of Eq. (2.1.19). For a smooth boundary, $\partial\bar{z}/\partial s = 0$, where s is arc length on Γ. Since $\bar{z} = 0$ on Γ, $\bar{z}_1 = \bar{z}_2 = 0$ on Γ. The bilinear form $a_u(z, \bar{z})$ for the plate is the *energy bilinear form*

$$
\begin{aligned}
a_u(z, \bar{z}) = \iint_\Omega \hat{D}(u)[z_{11}\bar{z}_{11} + \nu z_{22}\bar{z}_{11} + z_{22}\bar{z}_{22} + \nu z_{11}\bar{z}_{22} \\
+ 2(1-\nu)z_{12}\bar{z}_{12}]\, d\Omega
\end{aligned}
$$

and the *load linear form* or virtual work of the applied load is

$$l_u(\bar{z}) = \iint_\Omega f\bar{z}\, d\Omega$$

The variational equation of Eq. (2.1.21) is valid for the plate with $h \in L^\infty(\Omega)$. Just as in the case of the beam, it is unnatural and unnecessary to restrict consideration of solutions of the variational equation to $C^4(\Omega)$. Admissible solutions may be defined in $H^2(\Omega)$, which is the completion of $\{z \in C^2(\Omega)\colon \|z\|_{H^2} < \infty\}$, with the Sobolev norm

$$\begin{aligned}\|z\|_{H^2} &= \left[\iint_\Omega [|z|^2 + |z_1|^2 + |z_2|^2 + |z_{11}|^2 + |z_{12}|^2 + |z_{22}|^2]\, d\Omega\right]^{1/2}\\ &= \left[\sum_{\substack{i+j\le 2\\ i,j\ge 0}} \left\|\frac{\partial^{i+j} z}{\partial x_1^i\, \partial x_2^j}\right\|^2_{L^2(\Omega)}\right]^{1/2}\end{aligned} \tag{2.1.22}$$

Further, functions in $H^2(\Omega)$ that satisfy the clamped boundary conditions of Eq. (2.1.19) are limits in the norm of Eq. (2.1.22) of functions in $C^\infty(\Omega)$ that are zero outside compact subsets of the interior of Ω. This space Z of ***kinematically admissible displacements*** is denoted as the Sobolev space $H_0^2(\Omega) = Z$ [9, 35, 36].

Just as in the case of the beam, the *Friedrichs extension* A_u of the $\bar{A}_u$, for all $z \in D(A_u) \subset Z$, may now be defined such that $A_u z \in L^2(\Omega)$. It was shown by Fichera [35] that this operator satisfies the bounds of Eqs. (2.1.10)–(2.1.12). Thus, Eq. (2.1.13) is valid and A_u^{-1} is bounded, as in Eq. (2.1.15). The problem of determining the dependence of the inverse plate operator on u is of the same form as that for the beam. While the calculation is not as trivial as in the case of the beam, it was shown by Fichera [35] that the foregoing results are also valid for a simply supported plate, that is, for the boundary conditions

$$z = 0, \qquad \frac{\partial^2 z}{\partial n^2} + \nu\left(\frac{1}{r}\frac{\partial z}{\partial n} + \frac{\partial^2 z}{\partial s^2}\right) = 0, \qquad \text{on } \Gamma \tag{2.1.23}$$

where r is the radius of curvature of the boundary Γ.

LINEAR ELASTICITY

Consider the three-dimensional linear elasticity problem for a body of arbitrary shape shown in Fig. 2.1.3. Three components of displacement $z = [z^1\ z^2\ z^3]^{\mathrm{T}}$ characterize the displacement at each point in the elastic body.

The strain tensor is defined here as [34]

$$\varepsilon^{ij}(z) = \tfrac{1}{2}(z_j^i + z_i^j), \qquad i = 1, 2, 3 \tag{2.1.24}$$

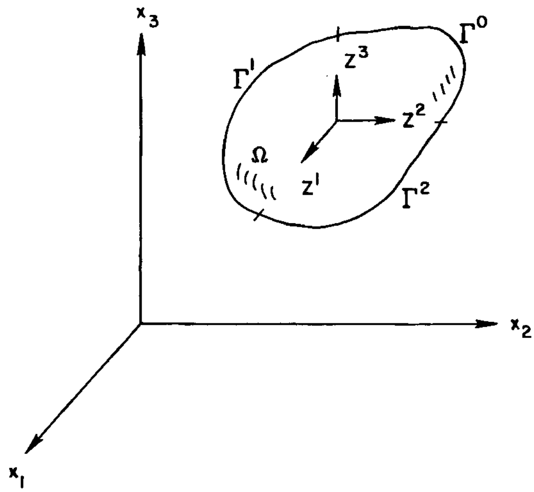

Fig. 2.1.3 Three-dimensional elastic solid.

where a subscript i denotes derivative with respect to x_i. The stress–strain relation is given as [34]

$$\sigma^{ij}(z) = \hat{\lambda}\left(\sum_{k=1}^{3} \varepsilon^{kk}(z)\right)\delta_{ij} + 2\mu\varepsilon^{ij}(z) \qquad (2.1.25)$$

where $\hat{\lambda}$ and μ are positive *Lamé's constants* of the homogeneous material and δ_{ij} is one when $i = j$ and is zero otherwise (the Kronecker delta).

With this notation, equations of equilibrium for the elastic body are [34]

$$-\sum_{j=1}^{3} \sigma_j^{ij}(z) = f^i, \qquad i = 1, 2, 3 \qquad (2.1.26)$$

where f^i is the external force per unit volume exerted on the solid in the x_i direction. Boundary conditions for the body may be given in several different forms. First, displacement may be prescribed on a subset Γ^0 of the boundary in the form

$$z^i = 0, \qquad i = 1, 2, 3, \quad x \in \Gamma^0 \qquad (2.1.27)$$

A second subset Γ^1 of the boundary is traction free, and on a third subset Γ^2 of the boundary, *surface tractions* may be specified in the form

$$T^{n_i}(z) \equiv \sum_{j=1}^{3} \sigma^{ij}(z) n_j = T^i, \qquad i = 1, 2, 3, \quad x \in \Gamma^2 \qquad (2.1.28)$$

where n_j is the jth component of the outward unit normal to the surface Γ^2 and T^{n_i} is the x_i component of surface traction specified on Γ^2.

The foregoing boundary-value problem may be reduced to variational form by multiplying both sides of Eq. (2.1.26) by an arbitrary displacement vector $\bar{z} = [\bar{z}^1 \ \bar{z}^2 \ \bar{z}^3]^T$ in the space Z of kinematically admissible displacements satisfying the boundary condition of Eq. (2.1.27) and then integrating by parts to obtain the variational equation

$$a_u(z, \bar{z}) \equiv \iiint_\Omega \sum_{i,j=1}^{3} \sigma^{ij}(z)\varepsilon^{ij}(\bar{z})\, d\Omega$$

$$= \iiint_\Omega \sum_{i=1}^{3} f^i\bar{z}^i\, d\Omega + \iint_{\Gamma^2} \sum_{i=1}^{3} T^i\bar{z}^i\, d\Gamma \equiv l_u(\bar{z}) \qquad \text{for all} \quad \bar{z} \in Z \tag{2.1.29}$$

Equation (2.1.29) is a generalization of the boundary-value problem of Eqs. (2.1.26)–(2.1.28), in the sense that if a solution of the boundary-value problem exists, it satisfies Eq. (2.1.29) for all displacement fields $\bar{z}$ satisfying Eq. (2.1.27). Conversely, the solution z of Eq. (2.1.29) for all displacement fields $\bar{z}$ satisfying Eq. (2.1.27), solves the boundary-value problem, if a solution of the boundary-value problem exists. Otherwise, it is a *generalized solution* in the Sobolev space $[H^1(\Omega)]^3 \equiv H^1(\Omega) \times H^1(\Omega) \times H^1(\Omega)$.

The three-dimensional elasticity problem specializes to a lower-dimensional problem in certain situations. For example, in thin elastic solids, stress components normal to the plane in which the solid lies are often essentially zero (*plane stress*). In other problems, all components of strain in a given direction are constrained to be zero (*plane strain*). In still other problems, axisymmetry of bodies leads to a special form of the governing equations of elasticity that involve only two independent variables.

Consider the plane stress problem, in which all components of stress in the x_3 direction are zero. From Eq. (2.1.24), this yields

$$\begin{aligned} \sigma^{13} &= 2\mu\varepsilon^{13} = 0 \\ \sigma^{23} &= 2\mu\varepsilon^{23} = 0 \\ \sigma^{33} &= \hat{\lambda}(\varepsilon^{11} + \varepsilon^{22} + \varepsilon^{33}) + 2\mu\varepsilon^{33} = 0 \end{aligned} \tag{2.1.30}$$

so

$$\varepsilon^{33} = -\frac{\hat{\lambda}}{\hat{\lambda} + 2\mu}(\varepsilon^{11} + \varepsilon^{22})$$

Substituting these formulas into the general stress–strain relation of Eq. (2.1.25) yields the plane stress–strain relations

$$\begin{aligned} \sigma^{ii}(z) &= \frac{2\hat{\lambda}\mu}{\hat{\lambda} + 2\mu}(\varepsilon^{11}(z) + \varepsilon^{22}(z)) + 2\mu\varepsilon^{ii}(z), \qquad i = 1, 2 \\ \sigma^{12}(z) &= 2\mu\varepsilon^{12}(z) \end{aligned} \tag{2.1.31}$$

Employing this notation, Eq. (2.1.29) remains valid as the variational equation of elasticity, with limits of summation running only from 1 to 2. Note that even though no dependence on x_3 arises in the problem, the stress–strain relation of Eq. (2.1.31) is not obtained by simply suppressing the third index in Eq. (2.1.25).

Consider the variable thickness, thin elastic slab with in-plane loading and fixed edges shown in Fig. 2.1.4, where Ω is a subset of R^2 and Γ is its boundary. Defining f^i ($i = 1, 2$) as the body force per unit volume, one can integrate over the x_3 coordinate in Eq. (2.1.29), using Eq. (2.1.30), to obtain the variational equation

$$\begin{aligned} a_u(z, \bar{z}) &= \iint_\Omega h(x) \sum_{i,j=1}^{2} \sigma^{ij}(z)\varepsilon^{ij}(\bar{z})\, d\Omega \\ &= \iint_\Omega h(x) \sum_{i=1}^{2} f^i \bar{z}^i\, d\Omega \equiv l_u(\bar{z}) \qquad \text{for all} \quad \bar{z} \in \hat{Z} \end{aligned} \tag{2.1.32}$$

where $\hat{Z}$ is the space $[C^2(\Omega)]^2$ of kinematically admissible displacements (i.e., with $\bar{z} = 0$ on Γ) and the design variable $u = h(x)$ is the variable thickness of the slab.

Using Eqs. (2.1.24) and (2.1.31), Eq. (2.1.32) can be written explicitly in terms of displacements as

$$\begin{aligned} a_u(z, \bar{z}) &\equiv \iint_\Omega h(x)[(2\hat{\lambda}\mu/(\hat{\lambda} + 2\mu))(z_1^1 + z_2^2)(\bar{z}_1^1 + \bar{z}_2^2) \\ &\qquad + 2\mu(z_1^1\bar{z}_1^1 + z_2^2\bar{z}_2^2) + \mu(z_2^1 + z_1^2)(\bar{z}_2^1 + \bar{z}_1^2)]\, d\Omega \\ &= \iint_\Omega h(x)[f^1\bar{z}^1 + f^2\bar{z}^2]\, d\Omega \equiv l_u(\bar{z}) \qquad \text{for all} \quad \bar{z} \in \hat{Z} \end{aligned} \tag{2.1.33}$$

Note from Eq. (2.1.33) that the energy bilinear form $a_u(z, \bar{z})$ is symmetric; that is,

$$a_u(z, \bar{z}) = a_u(\bar{z}, z) \qquad \text{for all} \quad z, \bar{z} \in \hat{Z} \tag{2.1.34}$$

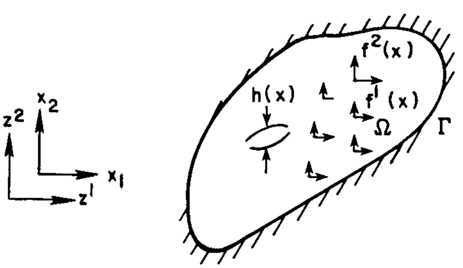

Fig. 2.1.4 Clamped elastic solid of variable thickness $h(x)$.

This result is actually valid for even broader classes of boundary conditions [34]. For future reference, the following relations hold between the Lamé's constants and the more conventional Young's modulus E and Poisson's ratio ν [34]:

$$\nu = \frac{\hat{\lambda}}{2(\hat{\lambda} + \mu)}, \qquad E = \frac{\mu(3\hat{\lambda} + 2\mu)}{\hat{\lambda} + \mu}$$
$$\hat{\lambda} = \frac{E\nu}{(1 + \nu)(1 - 2\nu)}, \qquad \mu = \frac{E}{2(1 + \nu)} \tag{2.1.35}$$

and in order that $\hat{\lambda} > 0$, it is clear that $0 < \nu < \frac{1}{2}$ is required.

The variational equation of Eq. (2.1.33) gives a *generalized solution*, provided only that $z \in H^1(\Omega) \times H^1(\Omega)$, $h \in L^\infty(\Omega)$, and $f \in L^2(\Omega)$. Here, the Sobolev norm on $H^1(\Omega) \times H^1(\Omega)$ is

$$\begin{aligned} \|z\|_{H^1 \times H^1} &= \left[\iint_\Omega (|z^1|^2 + |z^2|^2 + |z_1^1|^2 + |z_2^1|^2 + |z_1^2|^2 + |z_2^2|^2)\, d\Omega \right]^{1/2} \\ &= \left[\sum_{i=1}^{2} \sum_{\substack{j+k \le 1 \\ j,k \ge 0}} \left\| \frac{\partial^{j+k} z^i}{\partial x_1^j\, \partial x_2^k} \right\|_{L^2(\Omega)}^2 \right]^{1/2} \end{aligned} \tag{2.1.36}$$

The subspace Z of kinematically admissible displacements in $H^1(\Omega) \times H^1(\Omega)$ that satisfy the boundary conditions $z^1 = z^2 = 0$ on Γ is $H_0^1(\Omega) \times H_0^1(\Omega)$, which is the completion, in the norm of Eq. (2.1.36), of $C^\infty(\Omega) \times C^\infty(\Omega)$ functions that vanish outside compact subsets of the open set Ω.

Friedrichs extension A_u of the plane elasticity operator can now be defined by

$$a_u(z, \bar{z}) = (A_u z, \bar{z}) \qquad \text{for all} \quad z, \bar{z} \in Z$$

All of the bounds and hence the bounded invertibility of A_u follow just as for the beam and plate, where $h(x) \ge h_0 > 0$. This result is proved for a variety of boundary conditions by Fichera [35]. It is noted that for even this class of complex elastic systems, symmetry and strong ellipticity properties of the operator hold. As in the preceding examples, the design variable u appears in the energy bilinear form. Again it is desirable to determine the regularity of dependence of the state variable vector $z(x; u)$ on the design variable u.

The example problems of this section have been selected to illustrate classes of distributed-parameter structural components in which design dependence arises in a consistent way. In each case, Dirichlet boundary conditions are treated in detail. Selection of these boundary conditions is a convenience rather than a requirement. If the *trace boundary operator* theory were to be used in its full generality [9, 36], Neumann and mixed boundary conditions that arise naturally in applications could also be treated with only a penalty in analytic and algebraic messiness.

GENERAL FORM OF STATIC VARIATIONAL EQUATIONS

In each of the foregoing examples, the boundary-value problem for deformation due to applied load was written as it appears in the mechanics literature. For classical solutions to make sense, a high degree of smoothness of design and state (displacement) functions must be assumed. In each example, however, both sides of the differential equation could be multiplied by an arbitrary *virtual displacement* $\bar{z}$ that satisfies *kinematic boundary conditions*, integrate over the domain of the component, and integrate by parts to decrease the order of derivatives of z that appear, so that z and $\bar{z}$ are differentiated to the same order. The result is a *variational equation* of the form

$$a_u(z, \bar{z}) = l_u(\bar{z}) \tag{2.1.37}$$

which must hold for all *kinematically admissible* smooth virtual displacements $\bar{z} \in \hat{Z}$. As noted in each example, Eq. (2.1.37) can be viewed as the principle of virtual work and could have been derived directly from variational principles of mechanics (as is done in Chapter 4 for more complex systems).

Specific forms of Eq. (2.1.37) are given for a beam in Eq. (2.1.4), for a plate in Eq. (2.1.21), and for a linear elastic solid in Eq. (2.1.29). While the specific formulas differ in detail, they are all of the form of Eq. (2.1.37). For direct engineering design sensitivity analysis by the adjoint variable method, presented in Section 2.2, this form is adequate. From a mathematical point of view, however, it is noted in each case that the state z need not be restricted to a smooth space $\hat{Z}$ of displacements, but can extend this space to a subspace Z of an appropriate *Sobolev space* of functions that satisfy only *kinematic boundary conditions*. Likewise, the design space can be extended to a nonsmooth design space U. This is important if one wishes to admit nonsmooth designs and is also valuable from a theoretical point of view.

In each of the examples studied, it is observed that there exist positive constants K and γ such that

$$a_u(z, \bar{z}) \leq K \|z\|_Z \|\bar{z}\|_Z \qquad \text{for all} \quad z, \bar{z} \in Z \tag{2.1.38}$$

and

$$a_u(z, z) \geq \gamma \|z\|_Z^2 \qquad \text{for all} \quad z \in Z \tag{2.1.39}$$

where $u \in U$ is restricted to be uniformly nonzero. Here, $\|\cdot\|_Z$ denotes the appropriate *Sobolev norm*. To see the physical significance of these inequalities, put $\bar{z} = z$ in Eq. (2.1.38), and using Eq. (2.1.39), note that

$$\gamma \|z\|_Z^2 \leq a_u(z, z) \leq K \|z\|_Z^2 \qquad \text{for all} \quad z \in Z \tag{2.1.40}$$

Since $a_u(z, z)$ is twice the *strain energy* in each example, Eq. (2.1.40) shows that the strain energy defines an *energy norm* that is equivalent to the Sobolev norm. This important fact has been used to advantage by Mikhlin [14, 32] and other authors to develop powerful variational methods in mechanics. Any stronger or weaker norm would destroy the bounds of either Eq. (2.1.38) or Eq. (2.1.39), hence spoiling the equivalence between the energy and function space norms.

Furthermore, the inequalities of Eqs. (2.1.38) and (2.1.39) and the *Lax–Milgram Theorem* [9] of functional analysis guarantee existence of a unique solution $z(x; u)$ of Eq. (2.1.37). Again, a stronger or weaker norm on Z would spoil either the existence or uniqueness result. Thus, one sees that the Sobolev space setting is "just right," from both physical and mathematical points of view.

For design sensitivity analysis, the variational formulation of Eq. (2.1.37) and the inequalities of Eqs. (2.1.38) and (2.1.39) form the foundation for a proof in Section 2.4 that the solution $z(x; u)$ is differentiable with respect to design. More important for applications, knowing that $z(x; u)$ is differentiable with respect to design, the variational equation of Eq. (2.1.37) can be differentiated with respect to design and the result used to write variations of cost and constraint functionals explicitly. An adjoint variable method for implementing this technique is presented and illustrated in Section 2.2. While its theoretical foundations require use of the Sobolev space setting, the method is implemented and calculations are carried out without the formalism of functional analysis. Proofs required for theoretical completeness are given in Section 2.4.

2.1.2 Vibration and Buckling of Elastic Systems

The conventional differential operator formulation of prototype eigenvalue problems is now presented and extended to a more flexible and rigorous variational formulation. Technical justification of the variational formulation follows in a similar way as for the static problems treated in Section 2.1.1.

In each problem formulated here, the formal operator eigenvalue problem is of the form

$$\bar{A}_u y = \zeta \bar{B}_u y, \qquad y \neq 0 \tag{2.1.41}$$

where $\bar{A}_u$ is the formal differential operator encountered in the static response problem and $\bar{B}_u$ is a much simpler continuous operator, except for buckling problems. The symbol y denotes an eigenfunction, to distinguish it from the static response z, and ζ is the associated eigenvalue. If a high degree of differentiability of the eigenfunction y and the design variable u is assumed, the L^2 scalar product of both sides of Eq. (2.1.41) with a smooth function $\bar{y}$ that satisfies the same boundary conditions as y may be formed to obtain the variational equation

$$a_u(y, \bar{y}) \equiv (\bar{A}_u y, \bar{y}) = \zeta(\bar{B}_u y, \bar{y}) \equiv \zeta d_u(y, \bar{y}) \tag{2.1.42}$$

Conversely, if Eq. (2.1.42) holds for all $\bar{y}$ in a smooth class of functions and if y and u are sufficiently regular, then y and ζ constitute the solution of the eigenvalue problem of Eq. (2.1.41) [9, 35, 37].

As in Section 2.1.1, a subspace Z of an appropriate Sobolev space $H^m(\Omega)$ is now defined as generalized candidate solutions of Eq. (2.1.41). The *generalized solution* $y \in Z$ $(y \neq 0)$ is then characterized by the variational equation

$$a_u(y, \bar{y}) = \zeta d_u(y, \bar{y}) \qquad \text{for all} \quad \bar{y} \in Z \tag{2.1.43}$$

where the design variable u is now required only to be in $L^\infty(\Omega)$. As in Section 2.1.1, *Friedrichs extensions* A_u and B_u are defined for the formal operators $\bar{A}_u$ and $\bar{B}_u$, such that for $y \in D(A_u) \subset Z$,

$$\begin{aligned} (A_u y, \bar{y}) &= a_u(y, \bar{y}), \\ (B_u y, \bar{y}) &= d_u(y, \bar{y}), \end{aligned} \qquad \text{for all} \quad \bar{y} \in Z \tag{2.1.44}$$

The domain of B_u is such that $D(A_u) \subset D(B_u)$. Thus, yielding the generalized operator eigenvalue problem

$$A_u y = \zeta B_u y, \qquad y \in D(A_u), \quad y \neq 0 \tag{2.1.45}$$

which is valid for physically meaningful designs $u \in L^\infty(\Omega)$. The regularity conditions associated with functions in $D(A_u)$ are as in Section 2.1.1, which are more physically meaningful than the extreme smoothness conditions associated with the formal operators of Eq. (2.1.41).

Since the extension of candidate solutions to the space $D(A_u)$ is dictated completely by the operator A_u, technical definition of generalized solutions is exactly as in Section 2.1.1. In this section, the operator eigenvalue equation is stated for each problem studied, the bilinear forms $a_u(y, \bar{y})$ and $d_u(y, \bar{y})$ are defined, and the space Z is identified.

VIBRATION OF A STRING

A perfectly flexible string of variable mass density per unit length, $h \in L^\infty(0, 1)$ or $C^0(0, 1)$ $(h(x) \geq h_0 > 0)$ and tension $\hat{T} \geq \hat{T}_0 > 0$, is shown in Fig. 2.1.5. The operator eigenvalue equation is

$$\bar{A}_u y \equiv -\hat{T} y_{xx} = \zeta h y \equiv \zeta \bar{B}_u y \tag{2.1.46}$$

where $\zeta = \omega^2$, ω being the natural frequency. The boundary conditions are

$$y(0) = y(1) = 0 \tag{2.1.47}$$

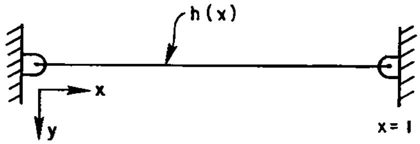

Fig. 2.1.5 Vibrating string with linear mass density $h(x)$.

Here, the design vector is $u = [h(x)\ \hat{T}]^T$, and the bilinear forms of Eq. (2.1.42) are obtained by integration by parts as

$$\begin{aligned}(\bar{A}_u y, \bar{y}) &= a_u(y, \bar{y}) = \hat{T}\int_0^1 y_x \bar{y}_x\, dx \\ (\bar{B}_u y, \bar{y}) &= d_u(y, \bar{y}) = \int_0^1 hy\bar{y}\, dx\end{aligned} \tag{2.1.48}$$

where y and $\bar{y}$ satisfy boundary conditions of Eq. (2.1.47). Since only first-order derivatives appear in the formula for $a_u(y, \bar{y})$, it is logical to select $Z \subset H^1(0, 1)$. The boundary conditions of Eq. (2.1.47) are satisfied in a generalized sense [9, 36] if the space Z of kinematically admissible displacements is restricted to $Z = H_0^1(0, 1)$. It is readily verified [9, 35] that the form $a_u(y, y)$ is Z-elliptic, so all the theory of Section 2.1.1 concerning A_u holds for this problem.

VIBRATION OF A BEAM

For a beam of variable cross-sectional area $h(x)$, let $h \in L^\infty(0, 1)$ or $C^0(0, 1)$ ($h(x) \geq h_0 > 0$) [such that the second moment of the cross-sectional area is $I(x) = \alpha h^2(x)$]. Young's modulus $E \geq E_0 > 0$ and mass density $\rho \geq \rho_0 > 0$ also play the role of design variables. A beam with clamped–clamped supports is as shown in Fig. 2.1.6. The formal operator eigenvalue equation is

$$\bar{A}_u y \equiv (E\alpha h^2 y_{xx})_{xx} = \zeta\rho h y \equiv \zeta \bar{B}_u y \tag{2.1.49}$$

where $\zeta = \omega^2$, ω being the natural frequency. Boundary conditions for the clamped–clamped beam are

$$y(0) = y_x(0) = y(1) = y_x(1) = 0 \tag{2.1.50}$$

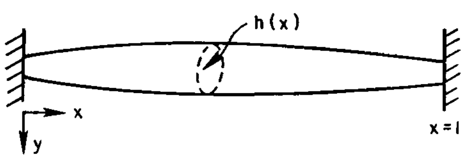

Fig. 2.1.6 Clamped–clamped vibrating beam with variable cross-sectional area $h(x)$.

Here, the design vector is $u = [h(x) \quad E \quad \rho]^T$ and the bilinear forms of Eqs. (2.1.42) are obtained, through integration by parts, as

$$\begin{aligned}(\bar{A}_u y, \bar{y}) &= a_u(y, \bar{y}) = E\alpha \int_0^1 h^2 y_{xx} \bar{y}_{xx}\, dx \\ (\bar{B}_u y, \bar{y}) &= d_u(y, \bar{y}) = \rho \int_0^1 h y \bar{y}\, dx\end{aligned} \tag{2.1.51}$$

where y and $\bar{y}$ satisfy boundary conditions of Eq. (2.1.50). Since only second derivatives arise in $a_u(y, \bar{y})$, it is logical to select $Z \subset H^2(0, 1)$. The boundary conditions of Eq. (2.1.50) are satisfied in a generalized sense [9, 35] if the space Z of kinematically admissible displacements is selected to be $Z = H_0^2(0, 1)$. All properties of $a_u(y, \bar{y})$ that are of interest here are demonstrated in Section 2.1.1. As noted in Section 2.1.1, the bilinear forms of Eq. (2.1.51) are valid for other boundary conditions given in Eqs. (2.1.16)–(2.1.18).

BUCKLING OF A COLUMN

If a column is subjected to an axial load P, as shown in Fig. 2.1.7, then buckling can occur if P is larger than a critical load ζ. With the same design variables as in beam vibration, the formal operator eigenvalue equation is

$$\bar{A}_u y \equiv (E\alpha h^2 y_{xx})_{xx} = -\zeta y_{xx} \equiv \zeta \bar{B}_u y \tag{2.1.52}$$

with boundary conditions as in Eq. (2.1.50). Since mass density does not arise in column buckling, the design vector is $u = [h(x) \quad E]^T$. Integration by parts with these operators in Eq. (2.1.42) yields the bilinear forms

$$\begin{aligned}(\bar{A}_u y, \bar{y}) &= a_u(y, \bar{y}) = \int_0^1 E\alpha h^2 y_{xx} \bar{y}_{xx}\, dx \\ (\bar{B}_u y, \bar{y}) &= d_u(y, \bar{y}) = \int_0^1 y_x \bar{y}_x\, dx\end{aligned} \tag{2.1.53}$$

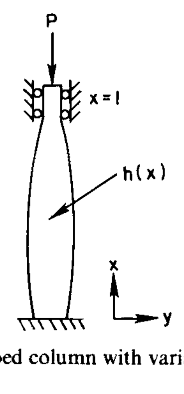

Fig. 2.1.7 Clamped-clamped column with variable cross-sectional area $h(x)$.

where y and $\bar{y}$ satisfy the boundary conditions of Eq. (2.1.50). Since $a_u(y, \bar{y})$ and $d_u(y, \bar{y})$ involve derivatives of y and $\bar{y}$ no higher than second order and the boundary conditions are the same as in the case of the vibrating beam, the space Z of kinematically admissible displacements may again be selected as $Z = H_0^2(0, 1)$.

VIBRATION OF A MEMBRANE

Consider a vibrating membrane with variable mass density $h(x)$ per unit area $h \in L^\infty(\Omega)$ or $C^0(\Omega)$ ($h(x) \geq h_0 > 0$) and membrane tension $\hat{T}$ (force per unit length), as shown in Fig. 2.1.8. The formal operator eigenvalue problem is

$$\bar{A}_u y \equiv -\hat{T}\nabla^2 y = \zeta h y \equiv \zeta \bar{B}_u y \tag{2.1.54}$$

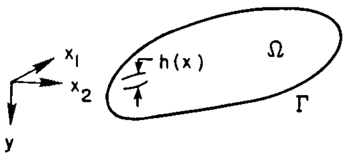

Fig. 2.1.8 Membrane of variable mass density $h(x)$.

where $\zeta = \omega^2$, ω being natural frequency, and the boundary condition is

$$y = 0, \qquad \text{on} \quad \Gamma \tag{2.1.55}$$

Here, $u = [h(x)\ \hat{T}]^{\mathrm{T}}$ is the design variable and the bilinear forms of Eq. (2.1.42) are

$$\begin{aligned}(\bar{A}_u y, \bar{y}) &= a_u(y, \bar{y}) = \hat{T}\iint_\Omega (y_1\bar{y}_1 + y_2\bar{y}_2)\, d\Omega \\ (\bar{B}_u y, \bar{y}) &= d_u(y, \bar{y}) = \iint_\Omega h y\bar{y}\, d\Omega\end{aligned} \tag{2.1.56}$$

where a subscript i denotes $\partial/\partial x_i$ ($i = 1, 2$) and y and $\bar{y}$ satisfy the boundary condition of Eq. (2.1.55). As in the case of the vibrating string, $Z = H_0^1(\Omega)$, and the bilinear form $a_u(y, \bar{y})$ is Z-elliptic [9, 35].

VIBRATION OF A PLATE

Consider a clamped vibrating plate of variable thickness $h \in L^\infty(\Omega)$ or $C^0(\Omega)$ ($h(x) \geq h_0 > 0$), Young's modulus $E \geq E_0 > 0$, and mass density $\rho \geq \rho_0 > 0$, as shown in Fig. 2.1.9. The formal operator eigenvalue equation is

$$\begin{aligned}\bar{A}_u y &\equiv [\hat{D}(u)(y_{11} + \nu y_{22})]_{11} + [\hat{D}(u)(y_{22} + \nu y_{11})]_{22} + 2(1-\nu)[\hat{D}(u)y_{12}]_{12} \\ &= \zeta\rho h y \equiv \zeta \bar{B}_u y\end{aligned} \tag{2.1.57}$$

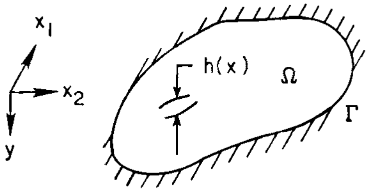

Fig. 2.1.9 Clamped plate of variable thickness $h(x)$.

where

$$y_{ij} \equiv \frac{\partial^2 y}{\partial x_i\, \partial y_j}, \qquad \hat{D}(u) = \frac{Eh^3}{12(1 - \nu^2)}, \qquad \zeta = \omega^2$$

ω is natural frequency, $0 < \nu < 0.5$ is Poisson's ratio, and the boundary conditions for a clamped plate are

$$y = \partial y/\partial n = 0, \qquad \text{on} \quad \Gamma \tag{2.1.58}$$

where $\partial y/\partial n$ is the normal derivative of y on Γ. Here the design vector is $u = [h(x)\;\; E\;\; \rho]^{\mathrm{T}}$.

Multiplying Eq. (2.1.54) by $\bar{y}$, integrating over Ω, and integrating by parts yields the bilinear forms of Eq. (2.1.42) as

$$(\bar{A}_u y, \bar{y}) = a_u(y, \bar{y}) = \iint_\Omega \hat{D}(u)[y_{11}\bar{y}_{11} + \nu(y_{22}\bar{y}_{11} + y_{11}\bar{y}_2)$$
$$+ y_{22}\bar{y}_{22} + 2(1 - \nu)y_{12}\bar{y}_{12}]\, d\Omega$$
$$(\bar{B}_u y, \bar{y}) = d_u(y, \bar{y}) = \zeta \iint_\Omega \rho h y \bar{y}\, d\Omega \tag{2.1.59}$$

where y and $\bar{y}$ satisfy the boundary conditions of Eq. (2.1.58). As in the case of the vibrating beam, the natural domain of the energy bilinear form $a_u(y, \bar{y})$ is $Z = H_0^2(\Omega)$.

With these bilinear forms, the variational formulation presented at the beginning of this section characterizes the eigenvalue behavior of each of the five problems discussed. They all have the same basic variational structure, and all the bilinear forms share the same degree of regularity of design dependence. In each of the problems studied in this section, the eigenvalue ζ depends on the design u since the differential equations and variational equations depend on u [i.e., $\zeta = \zeta(u)$]. The objective is to determine how ζ depends on u. Analysis of sensitivity of ζ to changes in u is somewhat more complicated than in the case of the static displacement problem of Section 2.1.1 since the eigenvector y also depends on u [i.e., $y = y(x; u)$] and its sensitivity must also be considered.

GENERAL FORM OF EIGENVALUE VARIATIONAL EQUATIONS

Much as in the case of static response in Section 2.1.1, a unified variational form of each eigenvalue problem is obtained in the form of Eq. (2.1.43). While detailed expressions for the bilinear forms are different in each example, the same general properties of the forms hold in each case. The most general function space setting is given in each example, but the engineer interested primarily in applications may presume the design and state variables are as smooth as desired. The more detailed Sobolev space settings are used in Section 2.5 to prove differentiability of eigenvalues and to derive formulas that are used in Section 2.3 to calculate derivatives of eigenvalues with respect to design.

2.2 ADJOINT VARIABLE METHOD FOR STATIC DESIGN SENSITIVITY ANALYSIS

As noted in Section 2.1.1, the solution of static structural equations depends on design. Differentiability of the state with respect to design, proved in Section 2.4, is employed in this section to derive an adjoint variable method for design sensitivity analysis of quite general functionals. An adjoint problem that is closely related to the original structural problem is obtained, and explicit formulas for structural response design sensitivity are obtained. Numerical methods for efficiently calculating design sensitivity coefficients, using the finite element method, are obtained and illustrated. The applications-oriented reader will note (happily) that virtually no Sobolev space theory is required in implementing the method.

2.2.1 Differentiability of Energy Bilinear Forms and Static Response

Basic design differentiability results for energy bilinear forms and the solution of the static structural equations are proved in Section 2.4 for each of the examples treated in Section 2.1.1. These differentiability results are cited here for use in developing useful design sensitivity formulas. This order of presentation was selected because technical aspects of existence of design derivatives of the structural state do not contribute insight into the adjoint variable technique that yields computable design sensitivity expressions. It is important to realize, however, that the delicate question of existence of design derivatives should not be ignored. Formal calculations with directional derivatives that may not exist are sure to lead to erroneous results. The occurrence of repeated eigenvalues and their lack of differentiability, discussed in the finite-dimensional case in Chapter 1 and in Sections 2.3 and 2.5, provide a graphic illustration of a very real structural problem in which structural response is indeed not differentiable. Thus, the reader is cautioned to be careful in verifying regularity properties of solutions to structural equations before using results of formal calculations.

As shown technically by Theorem 2.4.1 (Section 2.4), each of the energy bilinear forms encountered in Section 2.1.1 is differentiable with respect to design. That is,

$$a'_{\delta u}(z, \bar{z}) \equiv \frac{d}{d\tau} a_{u+\tau\delta u}(\tilde{z}, \bar{z})\bigg|_{\tau=0} \tag{2.2.1}$$

exists, where $\tilde{z}$ denotes the state z with dependence on τ suppressed and $\bar{z}$ is independent of τ. The prime notation here plays precisely the same role as in Chapter 1 and is, in fact, the first variation of the calculus of variations [38] with respect to explicit dependence of the energy bilinear form a_u on design u. As shown in Theorem 2.4.1, this first variation is continuous and linear in δu, hence it is the Fréchet derivative (Appendix A.3) of a_u with respect to design, evaluated in the direction δu. For proof of this result, the reader is referred to Section 2.4.1.

The load linear form for the problems of Section 2.1.1 is also differentiable with respect to design. More specifically,

$$l'_{\delta u}(\bar{z}) \equiv \frac{d}{d\tau}[l_{u+\tau\delta u}(\bar{z})]\bigg|_{\tau=0} \tag{2.2.2}$$

exists. As in the case of the energy bilinear form, the variation of the load linear form is linear in δu. For proof of validity of this result for the problems of Section 2.1.1, the reader is referred to Section 2.4. As in Chapter 1, the prime will be employed to denote variation of the energy bilinear and load linear forms of Eqs. (2.2.1) and (2.2.2), with explicit inclusion of the argument δu to emphasize dependence on design variation.

A substantially more powerful result, from Theorem 2.4.3 (Section 2.4.3), is that the solution of the state equations of Section 2.1, given here in the form

$$a_u(z, \bar{z}) = l_u(\bar{z}) \qquad \text{for all} \quad \bar{z} \in Z \tag{2.2.3}$$

where Z is the space of kinematically admissible displacements, is differentiable with respect to design. That is

$$z' = z'(x; u, \delta u) \equiv \frac{d}{d\tau} z(x; u + \tau\,\delta u)\bigg|_{\tau=0} \tag{2.2.4}$$

exists and is the first variation of the solution of Eq. (2.2.3) at design u and in the direction δu of design change. Note that z' is a function of the independent variable x that depends on the design u at which the variation is evaluated and on the direction δu of variation in design. As shown in Theorem 2.4.3, z' is linear in δu and in fact is the Fréchet derivative of the state z with respect to design, evaluated in the direction δu. Proof of validity of this result is not trivial, although it might be expected intuitively that the state of a system should be smoothly dependent upon design. For details of the proof, the reader is referred to Section 2.4.

An important property of the variation of state defined in Eq. (2.2.4) is the fact that the order of taking variation and partial differentiation with respect to the independent variable can be interchanged. For displacement states in $H^1(\Omega)$ and $H^2(\Omega)$ or in spaces of smoother functions, this means that

$$\begin{aligned} (z_i)' &= (z')_i, && z \in H^1(\Omega) \\ (z_{ij})' &= ((z_i)')_j = (z')_{ij}, && z \in H^2(\Omega) \end{aligned} \tag{2.2.5}$$

This property is a direct extension of the well-known property in the calculus of variations that the order of variation and partial differentiation can be interchanged.

It is presumed throughout this chapter that boundary conditions are homogeneous and do not depend on design; that is, boundary conditions are of the form $Gz = 0$, where G is a differential operator that does not depend on design. Using Eq. (2.2.5), one obtains $(Gz)' = Gz' = 0$. Thus, for the solution $z(x; u) \in Z$ of Eq. (2.2.3), $z' \in Z$. This important fact will be used often in the following development.

Note that the energy bilinear form $a_u(z, \bar{z})$ is linear in z and involves one or two derivatives of z, depending on whether the Sobolev space of generalized solutions is $H^1(\Omega)$ or $H^2(\Omega)$, respectively. Using these properties, one may use the chain rule of differentiation and the definitions of Eqs. (2.2.1) and (2.2.4) to obtain

$$\frac{d}{d\tau}[a_{u+\tau\delta u}(z(x; u + \tau\,\delta u), \bar{z})]\bigg|_{\tau=0} = a'_{\delta u}(z, \bar{z}) + a_u(z', \bar{z}) \tag{2.2.6}$$

As a first application of the foregoing definitions, for any fixed virtual displacement $\bar{z} \in Z$, one may take the variation of both sides of Eq. (2.2.3) and use Eq. (2.2.6) to obtain

$$a_u(z', \bar{z}) = l'_{\delta u}(\bar{z}) - a'_{\delta u}(z, \bar{z}) \qquad \text{for all} \quad \bar{z} \in Z \tag{2.2.7}$$

Presuming that the state z is known as the solution of Eq. (2.2.3), Eq. (2.2.7) is a variational equation with the same energy bilinear form for the first variation z'. Noting that the right side of Eq. (2.2.7) is a linear form in $\bar{z}$ and that the energy bilinear form on the left is Z-elliptic, Eq. (2.2.7) has a unique solution for z'. The fact that there is a unique solution of Eq. (2.2.7) agrees with the previously stated result that the design derivative of the solution of the state equation exists. Furthermore, if one selects a direction δu of design change, Eq. (2.2.7) may be numerically solved using the finite element method, just as the basic state equation of Eq. (2.2.3) would be solved with the finite element method, to numerically construct z'. Construction of such a solution depends on the direction of design change δu, however, since δu appears on the right side of Eq. (2.2.7). This calculation is therefore not of interest if one seeks explicit forms of design derivatives as a function of δu.

2.2.2 Adjoint Variable Design Sensitivity Analysis

Consider now a measure of structural performance that may be written in integral form as

$$\psi = \int_{\Omega} g(z, \nabla z, u)\, d\Omega \tag{2.2.8}$$

where for the present $z \in H^1(\Omega)$, $\nabla z = [z_1 \;\; z_2 \;\; z_3]^{\mathrm{T}}$, and the function g is continuously differentiable with respect to its arguments. This functional can be extended to functions $z \in H^2(\Omega)$, in which case second derivatives of z may appear in the integrand. This case will be treated as specific applications arise. Functionals of the form of Eq. (2.2.8) represent a wide variety of structural performance measures. For example, the volume of a structural element can be written with g depending only on u, average stress over a subset of a plane elastic solid can be written in terms of u and the gradient of z (defining stress), and displacement at a point in a beam or plate can be written formally using the Dirac δ function times the displacement function in the integrand. These and other examples will be treated in more detail in Section 2.2.3.

Taking the variation of the functional of Eq. (2.2.8) gives

$$\begin{aligned}\psi' &\equiv \frac{d}{d\tau}\left[\int_{\Omega} g(z(x; u + \tau\,\delta u), \nabla z(x; u + \tau\,\delta u), u + \tau\,\delta u)\, d\Omega\right]\Bigg|_{\tau=0} \\ &= \int_{\Omega} [g_z z' + g_{\nabla z} \nabla z' + g_u\,\delta u]\, d\Omega \end{aligned} \tag{2.2.9}$$

where the matrix calculus notation of Appendix A.1 is used, specifically

$$g_{\nabla z} \equiv \begin{bmatrix} \dfrac{\partial g}{\partial z_1} & \dfrac{\partial g}{\partial z_2} & \dfrac{\partial g}{\partial g_3} \end{bmatrix}$$

Leibnitz's rule allows the derivative with respect to τ to be taken inside the integral, and the chain rule of differentiation and Eq. (2.2.5) have been used in calculating the integrand of Eq. (2.2.9). Recall that z' and $\nabla z'$ depend on the direction δu of change in design. The objective here is to obtain an explicit expression for ψ' in terms of δu, which requires rewriting the first two terms under the integral on the right of Eq. (2.2.9) explicitly in terms of δu.

Much as in the case of finite-dimensional structures in Section 1.2.3, an adjoint equation is introduced by replacing z' in Eq. (2.2.9) by a virtual displacement $\bar{\lambda}$ and equating terms involving $\bar{\lambda}$ in Eq. (2.2.9) to the energy bilinear form $a_u(\lambda, \bar{\lambda})$, yielding the *adjoint equation* for the *adjoint variable* λ

$$a_u(\lambda, \bar{\lambda}) = \int_\Omega [g_z \bar{\lambda} + g_{\nabla z} \nabla \bar{\lambda}]\, d\Omega \qquad \text{for all} \quad \bar{\lambda} \in Z \tag{2.2.10}$$

where a solution $\lambda \in Z$ is desired. A simple application of the Schwartz inequality to the right side of Eq. (2.2.10) shows that it is a bounded linear functional of $\bar{\lambda}$ in the $H^1(\Omega)$ norm. Thus, by the Lax–Milgram theorem [9], there exists a unique solution for λ of Eq. (2.2.10), called the adjoint variable associated with the constraint of Eq. (2.2.8).

To take advantage of the adjoint equation, Eq. (2.2.10) may be evaluated at $\bar{\lambda} = z'$, since $z' \in Z$, to obtain

$$a_u(\lambda, z') = \int_\Omega [g_z z' + g_{\nabla z} \nabla z']\, d\Omega \tag{2.2.11}$$

which is just the terms in Eq. (2.2.9) that it is desired to write explicitly in terms of δu. Similarly, the identity of Eq. (2.2.7) may be evaluated at $\bar{z} = \lambda$, since both are in Z, to obtain

$$a_u(z', \lambda) = l'_{\delta u}(\lambda) - a'_{\delta u}(z, \lambda) \tag{2.2.12}$$

Recalling that the energy bilinear form $a_u(\cdot, \cdot)$ is symmetric in its arguments, the left sides of Eqs. (2.2.11) and (2.2.12) are equal, thus yielding the desired result

$$\int_\Omega [g_z z' + g_{\nabla z} \nabla z']\, d\Omega = l'_{\delta u}(\lambda) - a'_{\delta u}(z, \lambda) \tag{2.2.13}$$

where the right side is linear in δu and can be evaluated once the state z and adjoint variable λ are determined as solutions of Eqs. (2.2.3) and (2.2.10), respectively. Substituting this result into Eq. (2.2.9), the explicit design sensitivity of ψ is

$$\psi' = \int_\Omega g_u\, \delta u\, d\Omega + l'_{\delta u}(\lambda) - a'_{\delta u}(z, \lambda) \tag{2.2.14}$$

where the form of the last two terms on the right depend on the problem under investigation. This formula is applicable to any of the examples of Section 2.1.

Equation (2.2.14) will serve as the principal tool throughout the remainder of this section and in later applications for analysis of design sensitivity of functionals that represent response of elastic structures under static load. This powerful result forms the basis for both analytical expressions of functional derivatives and numerical methods for calculating design sensitivity coefficients, using the finite element method.

2.2.3 Analytical Examples of Static Design Sensitivity

The beam, plate, and plane elasticity problems of Section 2.1.1 are used here as examples with which to calculate design sensitivity formulas, using the adjoint variable method. Computational considerations will be discussed in subsequent sections.

BENDING OF A BEAM

Consider the clamped beam of Fig. 2.1.1, with design vector $u = [E\ \ h(x)]^{\mathrm{T}}$, $I(x) = \alpha h^2(x)$ as the moment of inertia of the cross-sectional area about its neutral axis, and α a positive constant. In this formulation, $h(x)$ is the cross-sectional area of the beam, and the load appearing in the beam equation of Eq. (2.1.1) is taken to reflect both externally applied load $F(x)$ and self-weight $\gamma h(x)$, where γ is weight density of the beam material. For these components of loading, the applied load is

$$f(x) = F(x) + \gamma h(x) \tag{2.2.15}$$

From Eqs. (2.1.5) and (2.1.6), the energy bilinear form and load linear form is defined as

$$a_u(z, \bar{z}) = \int_0^1 E\alpha h^2 z_{xx} \bar{z}_{xx}\, dx \tag{2.2.16}$$

$$l_u(\bar{z}) = \int_0^1 [F + \gamma h]\bar{z}\, dx \tag{2.2.17}$$

Calculating the variations of the energy bilinear form and load linear form from Eqs. (2.2.1) and (2.2.2),

$$\begin{aligned} a'_{\delta u}(z, \bar{z}) &= \frac{d}{d\tau}\left[\int_0^1 (E + \tau\,\delta E)\,\alpha(h + \tau\,\delta h)^2 \tilde{z}_{xx} \bar{z}_{xx}\, dx \right]\Bigg|_{\tau=0} \\ &= \int_0^1 [\delta E\,\alpha h^2 + 2E\alpha h\,\delta h] z_{xx} \bar{z}_{xx}\, dx \end{aligned} \tag{2.2.18}$$

$$\begin{aligned} l'_{\delta u}(\bar{z}) &= \frac{d}{d\tau}\left[\int_0^1 [F + \gamma(h + \tau\,\delta h)]\bar{z}\, dx \right]\Bigg|_{\tau=0} \\ &= \int_0^1 \gamma\,\delta h\,\bar{z}\, dx \end{aligned} \tag{2.2.19}$$

Several alternative forms may now be considered for structural response functionals. Consider first the weight of the beam, given as

$$\psi_1 = \int_0^1 \gamma h\, dx \tag{2.2.20}$$

A direct calculation of the variation yields

$$\psi_1' = \int_0^1 \gamma\, \delta h\, dx \tag{2.2.21}$$

Note that the direct variation calculation gives the explicit form of variation of structural weight in terms of variation of design. Thus, for this functional, no adjoint problem needs to be defined.

Consider a second functional that represents compliance of the structure, defined as

$$\psi_2 = \int_0^1 fz\, dx = \int_0^1 [F + \gamma h] z\, dx \tag{2.2.22}$$

Taking the variation, using the definition of Eq. (2.2.9),

$$\psi_2' = \int_0^1 [(F + \gamma h)z' + \gamma z\, \delta h]\, dx \tag{2.2.23}$$

The adjoint equation of Eq. (2.2.10) may be defined, which in this case is

$$a_u(\lambda, \bar{\lambda}) = \int_0^1 (F + \gamma h)\bar{\lambda}\, dx \qquad \text{for all} \quad \bar{\lambda} \in Z \tag{2.2.24}$$

Note that the load functional on the right side of Eq. (2.2.24) is precisely the same as the load functional for the original beam problem of Eq. (2.2.17). Since the original bilinear form $a_u(\cdot, \cdot)$ is Z-elliptic, Eq. (2.2.24) and the basic beam equation of Eq. (2.1.6) have identical solutions. In this special case, λ is the displacement of an *adjoint beam* that is identical to the original beam and is in fact subjected to the identical load, so $\lambda = z$. Thus, the beam and load are self-adjoint and there is no need to solve an additional adjoint problem. The explicit design sensitivity result of Eq. (2.2.14), using Eqs. (2.2.18) and (2.2.19) with $z = \lambda$, is thus

$$\psi_2' = \int_0^1 [2\gamma z - 2E\alpha h(z_{xx})^2]\, \delta h\, dx - \left[\int_0^1 \alpha h^2 (z_{xx})^2\, dx\right] \delta E \tag{2.2.25}$$

The effect of variations can thus be accounted for in cross-sectional area and Young's modulus of the system. It is interesting to note that the variation δE in Young's modulus may be taken outside the integral in Eq. (2.2.25).

As an example that can be calculated analytically, consider a uniform clamped–clamped beam with $h = h_0 = 0.005\,\mathrm{m}^2$, $E = E_0 = 2 \times 10^5\,\mathrm{MPa}$, $\alpha = \frac{1}{6}$, $F = 49.61\,\mathrm{kN/m}$, and $\gamma = 77{,}126\,\mathrm{N/m}^3$. Displacement under the given load is $z(x) = 2.5 \times 10^{-3}[x^2(1-x)^2]$. Compliance sensitivity in Eq. (2.2.25) may thus be evaluated as

$$\psi_2' = \int_0^1 [385.6x^2(1-x)^2 - \tfrac{25000}{3}(6x^2 - 6x + 1)^2]\,\delta h\,dx - 2.08 \times 10^{-11}\,\delta E$$

The graph of the coefficient of δh in the integral (Fig. 2.2.1) shows how addition or deletion of material affects compliance. In order to decrease compliance most effectively, material should be removed from the vicinity of points $x = 0.2$ and 0.8 and added to the ends of the beam.

The general result of Eq. (2.2.25) is applicable for arbitrary variations $\delta h(x)$ of cross-sectional area along the beam. If, however, a parameterized distribution of material is considered along the beam, such as a stepped beam shown in Fig. 1.2.1, then as in Section 1.2.6 the cross-sectional area function may be written in the form

$$h(x) = b_i, \qquad (i-1)/n < x < i/n \tag{2.2.26}$$

where the beam has subdivided into n sections, each with a constant cross-sectional area. The variation of the design function may thus be written directly in terms of variations in the design parameters b_i as

$$\delta h(x) = \delta b_i, \qquad (i-1)/n < x < i/n \tag{2.2.27}$$

This result may now be substituted directly into Eq. (2.2.25) to obtain explicit design sensitivities associated with the individual design parameters,

$$\begin{aligned}\psi_2' = {} & \sum_{i=1}^{n}\left(\int_{(i-1)/n}^{i/n} [2\gamma z - 2E\alpha b_i (z_{xx})^2]\,dx\right)\delta b_i \\ & - \left(\sum_{i=1}^{n}\int_{(i-1)/n}^{i/n} \alpha b_i^2 (z_{xx})^2\,dx\right)\delta E\end{aligned} \tag{2.2.28}$$

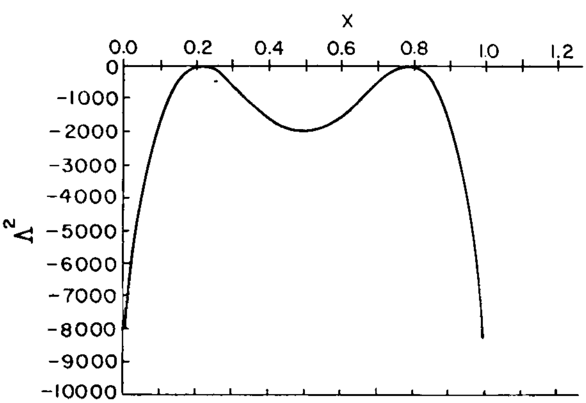

Fig. 2.2.1 Compliance sensitivity $\Lambda^2 = 2\gamma z - 2E\alpha h(z_{xx})^2$.

Design sensitivity coefficients are thus obtained associated with the design parameters, evaluated by numerically calculating integrals that depend only on the solution of the displacement equation.

Note that the sensitivity result of Eq. (2.2.28) is the same as the result of Eq. (1.2.61), which was obtained using a finite-dimensional structural design sensitivity method. That is, the sensitivity result of Eq. (1.2.61) is an approximation of the sensitivity result of Eq. (2.2.25).

Another important functional arising in design of beams is associated with strength constraints, normally stated in terms of allowable stresses in the beam. Since with an arbitrary load distribution there may not be continuous second derivatives of displacement in the beam, pointwise constraints on stress may not be meaningful. Therefore, constraints on average stress over small subintervals of the beam often are imposed. From elementary beam theory [39], the formula for bending stress is given as

$$\sigma(x) = \beta h^{1/2}(x) E z_{xx}(x) \tag{2.2.29}$$

where $\beta h^{1/2}$ is the half-depth of the beam. Defining a characteristic function $m_p(x)$ as an averaging multiplier that is nonzero only on a small open subinterval $(x_a, x_b) \subset (0, 1)$ and whose integral is 1, the average value of stress over this small subinterval (x_a, x_b) *is*

$$\psi_3 = \int_0^1 \beta h^{1/2}(x) E z_{xx}(x) m_p(x)\, dx \tag{2.2.30}$$

Note that if the stress is smooth and if the interval over which m_p is different from zero approaches zero length, m_p plays the role of the Dirac measure (Dirac δ function) and, in the limit, ψ_3 is the stress evaluated at a point. Note also that in the stress constraint formulation, the integrand involves a second derivative of state, which was not covered in the general derivation of Section 2.2.2. To illustrate the ease of extending the adjoint method to the case in which second derivatives arise in the integrand, first repeat the calculation leading to Eq. (2.2.9) to obtain

$$\psi_3' = \int_0^1 [\beta h^{1/2} E z'_{xx} m_p + \beta h^{1/2} z_{xx} m_p\, \delta E + \tfrac{1}{2}(\beta h^{-1/2} E z_{xx} m_p)\, \delta h]\, dx \tag{2.2.31}$$

Using the same argument that led to definition of the adjoint equation of Eq. (2.2.10), replace the state variation term z' on the right of Eq. (2.2.31) by a virtual displacement $\bar{\lambda}$ to obtain the adjoint problem

$$a_u(\lambda, \bar{\lambda}) = \int_0^1 \beta h^{1/2} E \bar{\lambda}_{xx} m_p\, dx \qquad \text{for all} \quad \bar{\lambda} \in Z \tag{2.2.32}$$

In the Sobolev space $H^2(0, 1)$, the functional on the right side of Eq. (2.2.32) is a bounded linear functional. By the Lax–Milgram theorem [9], Eq. (2.2.32) has a unique solution, denoted here as $\lambda^{(3)}$, where superscript (i) denotes association of λ with functional ψ_i. A direct repetition of the argument associated with Eqs. (2.2.11)–(2.2.14) yields

$$\psi_3' = \int_0^1 [\tfrac{1}{2}\beta h^{-1/2} E z_{xx} m_p + \gamma\lambda^{(3)} - 2E\alpha h z_{xx}\lambda_{xx}^{(3)}]\,\delta h\,dx + \left(\int_0^1 [\beta h^{1/2} z_{xx} m_p - \alpha h^2 z_{xx}\lambda_{xx}^{(3)}]\,dx\right)\delta E \tag{2.2.33}$$

It may be helpful to rewrite the adjoint equation of Eq. (2.2.32) more explicitly, using Eq. (2.2.16) for $a_u(\cdot,\cdot)$, as

$$\int_0^1 E\alpha h^2\{\lambda_{xx} - [\beta/\alpha(h(x))^{3/2}]m_p\}\bar{\lambda}_{xx}\,dx = 0 \qquad \text{for all} \quad \bar{\lambda} \in Z$$

This is just the equation of virtual work for deflection λ of an *adjoint beam* with initial curvature $[\beta/\alpha(h(x))^{3/2}]m_p$ and no externally applied load. This interpretation of the adjoint equation of Eq. (2.2.32) as an *adjoint structure* may be helpful in understanding the significance of λ from a physical point of view. As will be seen in Section 2.2.4, efficient solution of Eq. (2.2.32) can be carried out using the finite element method of structural analysis, without using the idea of an adjoint structure. The concept of adjoint structure was recently introduced by Dems and Mroz [40] in a variety of structural optimization problems.

Note that Eq. (2.2.33) provides a linear first variation of the locally averaged stress functional in terms of variations of the cross-sectional area distribution function h and Young's modulus. A parameterization of the cross-sectional area variation $h(x)$, such as the one shown in Fig. 1.2.1, could now be introduced in the sensitivity formula of Eq. (2.2.33), which would then be reduced to parameter variations only.

Consider next a special functional that defines the value of the displacement at an isolated point $\hat{x}$, that is,

$$\psi_4 \equiv z(\hat{x}) = \int_0^1 \hat{\delta}(x - \hat{x})\,z(x)\,dx \tag{2.2.34}$$

where $\hat{\delta}(x)$ is the Dirac measure at zero. By the Sobolev imbedding theorem [36], this functional is continuous, and the preceding analysis may be directly applied interpreting $\hat{\delta}$ as a function (called the Dirac δ function in mechanics). The variation of this functional is thus written as

$$\psi_4' = \int_0^1 \hat{\delta}(x - \hat{x})\,z'(x)\,dx \tag{2.2.35}$$

The adjoint equation of Eq. (2.2.10) in this case is

$$a_u(\lambda, \bar{\lambda}) = \int_0^1 \hat{\delta}(x - \hat{x})\, \bar{\lambda}\, dx \qquad \text{for all} \quad \bar{\lambda} \in Z \tag{2.2.36}$$

Since the right side of this equation defines a bounded linear functional on $H^2(0, 1)$, there exists a unique solution of Eq. (2.2.36), denoted here as $\lambda^{(4)}$. Interpreting the Dirac δ function as a unit load applied at point $\hat{x}$, physical interpretation of $\lambda^{(4)}$ is immediately obtained as the displacement of the beam due to a positive unit load at $\hat{x}$. Thus, the *adjoint beam* in this case is just the original beam with a different load.

Direct evaluation of design sensitivity, using Eqs. (2.2.14), (2.2.18), and (2.2.19), yields

$$\psi_4' = \int_0^1 [\gamma \lambda^{(4)} - 2E\alpha h z_{xx} \lambda_{xx}^{(4)}]\, \delta h\, dx - \left[\int_0^1 \alpha h^2 z_{xx} \lambda_{xx}^{(4)}\, dx \right] \delta E \tag{2.2.37}$$

To illustrate the use of this result, consider the clamped–clamped beam studied earlier in this section. The solution of the state equation is $z = 2.5 \times 10^{-3}[x^2(1 - x)^2]$. If design sensitivity of the displacement at the center of the beam is desired, $\hat{x} = \frac{1}{2}$. Thus, the adjoint load from Eq. (2.2.36) is just a unit point load at the center of the beam. The adjoint variable is thus obtained by solving the beam equation with this load to obtain

$$\lambda^{(4)} = 2.5 \times 10^{-8}[8\langle x - \tfrac{1}{2}\rangle^3 - 4x^3 + 3x^2]$$

where

$$\langle x - \tfrac{1}{2}\rangle = \begin{cases} 0 & \text{for} \quad 0 \le x \le \frac{1}{2} \\ x - \frac{1}{2} & \text{for} \quad \frac{1}{2} \le x \le 1 \end{cases}$$

These expressions may be substituted into Eq. (2.2.37) to obtain the displacement sensitivity as

$$\begin{aligned} \psi_4' = \int_0^1 & [1.93 \times 10^{-3}(8\langle x - \tfrac{1}{2}\rangle^3 - 4x^3 + 3x^2) \\ & - 2.5 \times 10^{-1}(6x^2 - 6x + 1)(8\langle x - \tfrac{1}{2}\rangle - 4x + 1)]\, \delta h\, dx \\ & - 7.81 \times 10^{-16}\, \delta E \end{aligned}$$

To see how material added to or deleted from the beam influences displacement at the center, the coefficient of δh may be graphed (Fig. 2.2.2). To decrease $z(\frac{1}{2})$ most effectively, for example, material should be removed near $x = 0.22$ and 0.78 and added near $x = 0$ and 1.

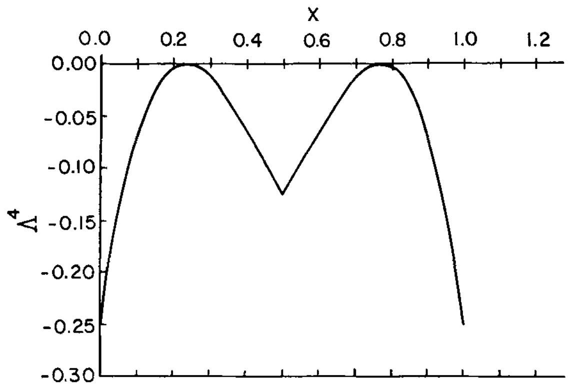

Fig. 2.2.2 Displacement sensitivity $\Lambda^4 = \gamma\lambda^{(4)} - 2E\alpha h z_{xx}\lambda^{(4)}_{xx}$.

As a final beam example, consider the slope of the beam at an isolated point $\hat{x}$ defined as the functional

$$\begin{aligned}\psi_5 \equiv z_x(\hat{x}) &= \int_0^1 \hat{\delta}(x - \hat{x})\, z_x(x)\, dx \\ &= -\int_0^1 \hat{\delta}_x(x - \hat{x})\, z(x)\, dx \end{aligned} \tag{2.2.38}$$

Due to the Sobolev imbedding theorem [36], this is a continuous linear functional on $H^2(0, 1)$, so the preceding theory may be applied. The last equality in Eq. (2.2.38) represents an integration by parts that defines the derivative of the Dirac measure. In beam theory it is well known that the derivative of the Dirac measure is a point moment applied at the point $\hat{x}$. The preceding analysis may now be directly repeated with $\hat{\delta}$ replaced by $-\hat{\delta}_x$, defining the adjoint equation

$$a_u(\lambda, \bar{\lambda}) = -\int_0^1 \hat{\delta}_x(x - \hat{x})\, \bar{\lambda}\, dx \qquad \text{for all} \quad \bar{\lambda} \in Z \tag{2.2.39}$$

where the unique solution is denoted as $\lambda^{(5)}$. Physically, $\lambda^{(5)}$ is the displacement in an *adjoint beam* that is the original beam with a negative unit moment applied at the point $\hat{x}$. As in the preceding, next evaluate Eq. (2.2.14) to obtain

$$\psi_5' = \int_0^1 [\gamma\lambda^{(5)} - 2E\alpha h z_{xx}\lambda^{(5)}_{xx}]\, \delta h\, dx - \left[\int_0^1 \alpha h^2 z_{xx}\lambda^{(5)}_{xx}\, dx\right] \delta E \tag{2.2.40}$$

It is interesting to note that for other boundary conditions in Eqs. (2.1.16)–(2.1.18), the sensitivity formulas for ψ_1–ψ_5 are valid because, as mentioned in Section 2.1.1, the variational equation of Eq. (2.2.3) is valid for all other boundary conditions.

To illustrate the use of Eq. (2.2.40), consider the clamped–clamped beam studied earlier in this section. If design sensitivity of the slope at the center of the beam is desired, the adjoint load from Eq. (2.2.39) is just a negative unit moment at the center of the beam. Thus, the adjoint variable is obtained as

$$\lambda^{(5)} = 1.5 \times 10^{-7}[-4\langle x - \tfrac{1}{2}\rangle^2 + 2x^3 - x^2]$$

Equation (2.2.40) may now be evaluated to obtain

$$\psi_5' = \int_0^1 [1.16 \times 10^{-2}(-4\langle x - \tfrac{1}{2}\rangle^2 + 2x^3 - x^2) - 0.5(6x^2 - 6x + 1)(-4\langle x - \tfrac{1}{2}\rangle^0 + 6x - 1)]\,\delta h\,dx$$

where $\langle x - \frac{1}{2}\rangle^0 = 0$ if $x < \frac{1}{2}$ and $\langle x - \frac{1}{2}\rangle^0 = 1$ if $x > \frac{1}{2}$. One interesting aspect of the above sensitivity result is that the slope at the center of the beam, with the present uniform design $h = 0.005\,\mathrm{m}^2$, is independent of the variation δE of Young's modulus. To see how material added to or deleted from the beam influences the slope at the center, the coefficient of δh may be graphed (Fig. 2.2.3). Figure 2.2.3 indicates that if material is added or removed symmetrically with respect to $\hat{x} = \frac{1}{2}$, then the slope remains at zero value, which is physically clear. Adding material to the left of $\hat{x} = \frac{1}{2}$ increases the slope, while adding to the right decreases the slope.

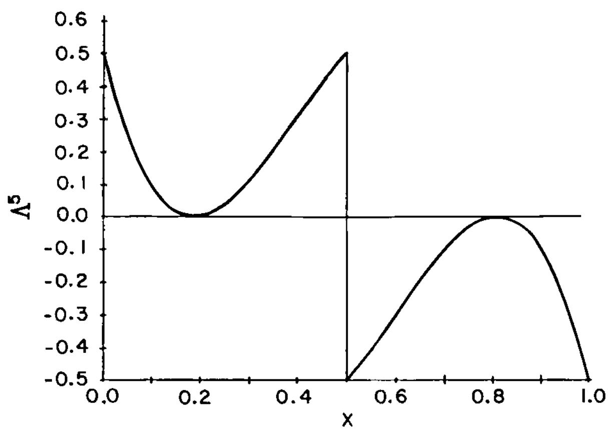

Fig. 2.2.3 Slope sensitivity $\Lambda^5 = \gamma\lambda^{(5)} - 2E\alpha h z_{xx}\lambda_{xx}^{(5)}$.

BENDING OF A PLATE

Consider now the clamped plate of Fig. 2.1.2, with variable thickness $h(x)$ and variable Young's modulus E. Consider a distributed load that consists of externally applied pressure $F(x)$ and self-weight, given by

$$f(x) = F(x) + \gamma h(x) \tag{2.2.41}$$

where γ is weight density of the plate. For this design-dependent loading, the energy bilinear form for the plate and the load linear form, given in Eq. (2.1.12) and following, are

$$a_u(z, \bar{z}) = \iint_\Omega \hat{D}(u)[z_{11}\bar{z}_{11} + z_{22}\bar{z}_{22} + \nu(z_{22}\bar{z}_{11} + z_{11}\bar{z}_{22}) + 2(1 - \nu)z_{12}\bar{z}_{12}]\, d\Omega \tag{2.2.42}$$

$$l_u(\bar{z}) = \iint_\Omega [F + \gamma h]\bar{z}\, d\Omega \tag{2.2.43}$$

where $u = [E \;\; h(x)]^T$ and

$$\hat{D}(u) = Eh^3/[12(1 - \nu^2)] \tag{2.2.44}$$

Consider first the functional defining weight of the plate,

$$\psi_1 = \iint_\Omega \gamma h\, d\Omega \tag{2.2.45}$$

Taking a direct variation yields

$$\psi_1' = \iint_\Omega \gamma\, \delta h\, d\Omega \tag{2.2.46}$$

Since no variation of state arises in this expression, no adjoint problem needs to be defined, and the explicit design derivative of weight is obtained.

Consider next the compliance functional for the plate,

$$\psi_2 = \iint_\Omega [F + \gamma h]z\, d\Omega \tag{2.2.47}$$

Taking the first variation yields

$$\psi_2' = \iint_\Omega [(F + \gamma h)z' + \gamma z\, \delta h]\, d\Omega \tag{2.2.48}$$

Following Eq. 2.2.10, one defines the adjoint equation as

$$a_u(\lambda, \bar{\lambda}) = \iint_\Omega (F + \gamma h)\, \bar{\lambda}\, d\Omega \qquad \text{for all} \quad \bar{\lambda} \in Z \tag{2.2.49}$$

Note that Eq. (2.2.49) is identical to the plate equation of Eq. (2.1.21) for displacement. Therefore, the *adjoint plate* and load are identical to the original, $\lambda = z$, and Eq. (2.2.49) need not be solved separately.

In preparation for evaluating design sensitivity, the definitions of Eqs.

(2.2.1) and (2.2.2) are followed for the plate problem to obtain

$$
\begin{aligned}
a'_{\delta u}(z,\bar{z}) = \iint_\Omega \{&Eh^2[z_{11}\bar{z}_{11} + z_{22}\bar{z}_{22} + \nu(z_{22}\bar{z}_{11} + z_{11}\bar{z}_{22}) \\
&+ 2(1-\nu)z_{12}\bar{z}_{12}]/[4(1-\nu^2)]\}\,\delta h\, d\Omega \\
+ \Big\{ \iint_\Omega &h^3[z_{11}\bar{z}_{11} + z_{22}\bar{z}_{22} + \nu(z_{22}\bar{z}_{11} + z_{11}\bar{z}_{22}) \\
&+ 2(1-\nu)z_{12}\bar{z}_{12}]/[12(1-\nu^2)]\, d\Omega \Big\}\,\delta E
\end{aligned}
\tag{2.2.50}
$$

$$
l'_{\delta u}(\bar{z}) = \iint_\Omega \gamma\bar{z}\,\delta h\, d\Omega \tag{2.2.51}
$$

Direct application of Eq. (2.2.14) for sensitivity of ψ_2, with $\lambda = z$, yields

$$
\begin{aligned}
\psi_2' = &\iint_\Omega \left\{2\gamma z - Eh^2 \frac{z_{11}^2 + z_{22}^2 + 2\nu z_{11}z_{22} + 2(1-\nu)z_{12}^2}{4(1-\nu^2)}\right\} \delta h\, d\Omega \\
&- \left\{\iint_\Omega h^3 \frac{z_{11}^2 + z_{22}^2 + 2\nu z_{11}z_{22} + 2(1-\nu)z_{12}^2}{12(1-\nu^2)}\, d\Omega\right\} \delta E
\end{aligned}
\tag{2.2.52}
$$

As in the case of the beam, note that this sensitivity result consists of a first term, which accounts for the effect of a variation $\delta h(x)$ of the plate shape function $h(x)$, and a second term, which is a scalar times the variation δE.

Consider application of Eq. (2.2.52) to the case of a plate of piecewise constant thickness (Fig. 2.2.4) where b_i is the constant thickness of the ith rectangular element. The thickness function is thus parameterized as

$$
h(x) = b_i, \qquad x \in \Omega_i \tag{2.2.53}
$$

where Ω_i is the ith rectangular element in Fig. 2.2.4. The variation δh in thickness is thus $\delta h(x) = \delta b_i$ $(x \in \Omega_i)$, and Eq. (2.2.52) becomes

$$
\begin{aligned}
\psi_2' = &\sum_{i=1}^{n} \left[\iint_{\Omega_i} \left\{2\gamma z - 3Eb_i^2 \frac{z_{11}^2 + z_{22}^2 + 2\nu z_{11}z_{22} + 2(1-\nu)z_{12}^2}{12(1-\nu^2)}\right\} d\Omega\right] \delta b_i \\
&- \left[\sum_{i=1}^{n} \iint_{\Omega_i} \left\{b_i^3 \frac{z_{11}^2 + z_{22}^2 + 2\nu z_{11}z_{22} + 2(1-\nu)z_{12}^2}{12(1-\nu^2)}\right\} d\Omega\right] \delta E
\end{aligned}
\tag{2.2.54}
$$

Consider the plate response functional defined as displacement at a discrete point $\hat{x}$,

$$
\psi_3 \equiv z(\hat{x}) = \iint_\Omega \hat{\delta}(x - \hat{x})\, z(x)\, d\Omega \tag{2.2.55}
$$

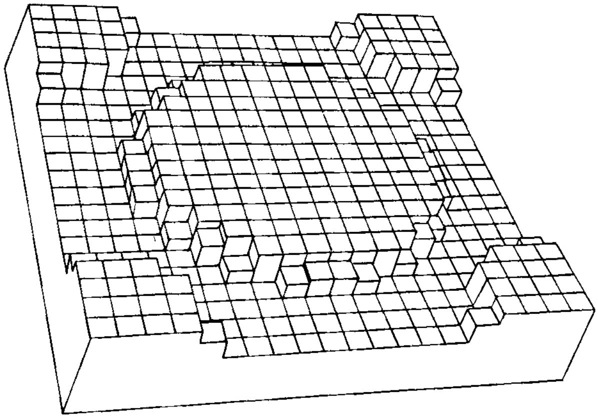

Fig. 2.2.4 Piecewise uniform plate.

where $\hat{\delta}(x)$ is the Dirac measure in the plane acting at the origin. By the Sobolev imbedding theorem [36], this functional is continuous, and the foregoing theory applies. Taking the first variation of Eq. (2.2.55) yields

$$\psi_3' = \iint_\Omega \hat{\delta}(x - \hat{x})\, z'\, d\Omega \tag{2.2.56}$$

Following the general adjoint formulation of Eq. (2.2.10), the adjoint equation is defined as

$$a_u(\lambda, \bar{\lambda}) = \iint_\Omega \hat{\delta}(x - \hat{x})\, \bar{\lambda}\, d\Omega \qquad \text{for all} \quad \bar{\lambda} \in Z \tag{2.2.57}$$

This equation has a unique solution, denoted $\lambda^{(3)}$. Since the load functional on the right side of Eq. (2.2.57) is physically interpreted as a unit point load acting at the point $\hat{x}$, the solution $\lambda^{(3)}$ of the *adjoint plate* problem is simply the displacement of the original plate due to this load.

With $\lambda^{(3)}$ determined, the general result of Eq. (2.2.14) may now be applied, with the variations in bilinear and linear forms defined in Eqs. (2.2.50) and (2.2.51), to obtain

$$\begin{aligned} \psi_3' = \iint_\Omega \{\gamma\lambda^{(3)} - Eh^2[z_{11}\gamma_{11}^{(3)} + z_{22}\lambda_{22}^{(3)} \\ + \nu(z_{22}\lambda_{11}^{(3)} + z_{11}\lambda_{22}^{(3)}) + 2(1-\nu)z_{12}\lambda_{12}^{(3)}]/[4(1-\nu^2)]\}\, \delta h\, d\Omega \\ - \left\{\iint_\Omega h^3[z_{11}\lambda_{33}^{(3)} + z_{22}\lambda_{22}^{(3)} + \nu(z_{22}\lambda_{11}^{(3)} + z_{11}\lambda_{22}^{(3)}) \right. \\ \left. + 2(1-\nu)z_{12}\lambda_{12}^{(3)}]/[12(1-\nu^2)]\, d\Omega\right\} \delta E \end{aligned} \tag{2.2.58}$$

The maximum stress for a thin plate occurs on the surface of the plate and is given in the form [33]

$$\begin{aligned}
\sigma^{11} &= -\frac{Eh}{2(1-\nu^2)}(z_{11} + \nu z_{22}) \\
\sigma^{22} &= -\frac{Eh}{2(1-\nu^2)}(z_{22} + \nu z_{11}) \\
\sigma^{12} &= -\frac{Eh}{2(1+\nu)} z_{12}
\end{aligned} \tag{2.2.59}$$

The von Mises stress is [33]

$$g(\sigma) = [(\sigma^{11} + \sigma^{22})^2 + 3(\sigma^{11} - \sigma^{22})^2 + 12(\sigma^{12})^2]^{1/2} \tag{2.2.60}$$

For simplicity, assume the stress σ^{11} in Eq. (2.2.59) is taken as a strength constraint instead of the von Mises stress. With this done, the idea can be extended to the von Mises stress. As in the beam problem, define a characteristic function $m_p(x)$ as an averaging multiplier, which is nonzero only on an open small region Ω_p of Ω and whose integral is 1. Then, the average value of σ^{11} over this small region is

$$\begin{aligned}
\psi_4 &= \iint_\Omega \sigma^{11} m_p \, d\Omega \\
&= -\frac{1}{2(1-\nu^2)} \iint_\Omega Eh(z_{11} + \nu z_{22}) m_p \, d\Omega
\end{aligned} \tag{2.2.61}$$

As in the beam stress functional case, take the variation of the functional ψ_4 to obtain

$$\begin{aligned}
\psi_4' = -\frac{1}{2(1-\nu^2)} \iint_\Omega & [Eh(z_{11}' + \nu z_{22}') m_p + h(z_{11} + \nu z_{22}) m_p \, \delta E \\
& + E(z_{11} + \nu z_{22}) m_p \, \delta h] \, d\Omega
\end{aligned} \tag{2.2.62}$$

Replace the state variation term z' on the right side of Eq. (2.2.62) by a virtual displacement $\bar{\lambda}$ to obtain

$$a_u(\lambda, \bar{\lambda}) = -\frac{1}{2(1-\nu^2)} \iint_\Omega Eh(\bar{\lambda}_{11} + \nu \bar{\lambda}_{22}) m_p \, d\Omega \qquad \text{for all} \quad \bar{\lambda} \in Z \tag{2.2.63}$$

Using the norm in $H^2(\Omega)$ of Eq. (2.1.22), it is shown that the functional on the right side of Eq. (2.2.63) is a bounded linear functional. Hence, by the

Lax–Milgram Theorem [9], Eq. (2.2.63) has a unique solution $\lambda^{(4)}$. Using the same procedure as in Eqs. (2.2.11)–(2.2.14) gives

$$\psi_4' = \iint_\Omega \left\{ -\frac{E}{2(1-\nu^2)}(z_{11} + \nu z_{22})m_p + \gamma\lambda^{(4)} - Eh^2[z_{11}\lambda_{11}^{(4)} + z_{22}\lambda_{22}^{(4)} + \nu(z_{22}\lambda_{11}^{(4)} + z_{11}\lambda_{22}^{(4)}) + 2(1-\nu)z_{12}\lambda_{12}^{(4)}]/[4(1-\nu^2)] \right\} \delta h \, d\Omega$$
$$- \iint_\Omega \left\{ \frac{h}{2(1-\nu^2)}(z_{11} + \nu z_{22})m_p + h^3[z_{11}\lambda_{11}^{(4)} + z_{22}\lambda_{22}^{(4)} + \nu(z_{22}\lambda_{11}^{(4)} + z_{11}\lambda_{22}^{(4)}) + 2(1-\nu)z_{12}\lambda_{12}^{(4)}]/[12(1-\nu^2)] \right\} d\Omega \, \delta E \qquad (2.2.64)$$

PLANE ELASTICITY

Consider now the plane elastic slab treated in Section 2.1.1. The energy bilinear form and load linear form for this problem are given in Eq. (2.1.32) as

$$a_u(z, \bar{z}) = \iint_\Omega h \left[\sum_{i,j=1}^{2} \sigma^{ij}(z)\varepsilon^{ij}(\bar{z}) \right] d\Omega \qquad (2.2.65)$$

$$l_u(\bar{z}) = \iint_\Omega h[f^1\bar{z}^1 + f^2\bar{z}^2] \, d\Omega \qquad (2.2.66)$$

where $h(x)$ is the thickness of the plane elastic slab, the displacement vector is $z = [z^1 \;\; z^2]^T$, and $\sigma^{ij}(z)$ and $\varepsilon^{ij}(\bar{z})$ are the stress and strain fields associated with displacement z and virtual displacement $\bar{z}$, respectively, given in terms of displacements as

$$\varepsilon^{ij}(\bar{z}) = \tfrac{1}{2}(\bar{z}_j^i + \bar{z}_i^j), \qquad i,j = 1,2 \qquad (2.2.67)$$

$$\sigma^{ii}(z) = \frac{2\hat{\lambda}\mu}{(\hat{\lambda} + 2\mu)}[z_1^1 + z_2^2] + 2\mu z_i^i, \qquad i = 1,2$$
$$\sigma^{12}(z) = \mu(z_2^1 + z_1^2) \qquad (2.2.68)$$

where $\hat{\lambda}$ and μ are Lamé's constants. The design variable is taken here only as the variable thickness $h(x)$ of the elastic slab.

Consider first the functional defining weight of the slab,

$$\psi_1 = \iint_\Omega \gamma h \, d\Omega \qquad (2.2.69)$$

Since this functional does not involve z, its variation is calculated simply as

$$\psi_1' = \iint_\Omega \gamma\, \delta h \, d\Omega \tag{2.2.70}$$

which requires no adjoint solution.

Consider next a locally averaged stress functional, which might involve principal stresses, von Mises stress, or some other material failure criteria. Defining a characteristic averaging function $m_p(x)$ that is nonzero and constant over a small open subset $\Omega_p \subset \Omega$, zero outside of Ω_p, and whose integral is 1, the average stress functional is written in the general form

$$\psi_2 = \iint_\Omega g(\sigma(z)) m_p \, d\Omega \tag{2.2.71}$$

where σ denotes the stress tensor. While this expression could be written explicitly in terms of the gradient of z, it will be seen in the following that it is more effective to continue with the present notation. Since components of the stress tensor given by Eq. (2.2.68) are linear in z and the order of taking variation and partial derivative can be changed, as shown in Eq. (2.2.5), the variation of the functional of Eq. (2.2.71) may be written in the form

$$\psi_2' = \iint_\Omega \left[\sum_{i,j=1}^{2} g_{\sigma^{ij}}(z) \sigma^{ij}(z') \right] m_p \, d\Omega \tag{2.2.72}$$

As in the general derivation of the adjoint equation of Eq. (2.2.10), the variation in state z' may be replaced by a virtual displacement $\bar{\lambda}$ on the right side of Eq. (2.2.72) to define a load functional for the adjoint equation to obtain, as in Eq. (2.2.10),

$$a_u(\lambda, \bar{\lambda}) = \iint_\Omega \left[\sum_{i,j=1}^{2} g_{\sigma^{ij}}(z) \sigma^{ij}(\bar{\lambda}) \right] m_p \, d\Omega \qquad \text{for all} \quad \bar{\lambda} \in Z \tag{2.2.73}$$

It may be shown directly that the linear form in $\bar{\lambda}$ on the right side of Eq. (2.2.73) is bounded in $H^1(\Omega)$ so Eq. (2.2.73) has a unique solution for a displacement field $\lambda^{(2)}$, with the right side of Eq. (2.2.73) defining the load functional. Integrating by parts could be considered on the right side of Eq. (2.2.73) to derive a formula for a distributed load that could be interpreted as acting on the elastic solid. This calculation, however, causes considerable practical and theoretical difficulty, since $g_{\sigma^{ij}}(z)$ depends on stress and derivatives of stress will not generally exist in $L^2(\Omega)$. Thus, the linear form on the right side of Eq. (2.2.73) is left as defined.

Using symmetry of the energy bilinear form $a_u(\cdot, \cdot)$, Eq. (2.2.73) may be written in the form

$$\int_\Omega h \left\{ \sum_{i,j=1}^{2} [\varepsilon^{ij}(\lambda) - (g_{\sigma^{ij}}(z) m_p)/h] \sigma^{ij}(\bar{\lambda}) \right\} d\Omega = 0 \qquad \text{for all} \quad \bar{\lambda} \in Z$$

This is just the equation of elasticity for displacement $\lambda^{(2)}$ of a slab with an initial strain field $(g_{\sigma^{ij}}(z)m_p)/h$ and no externally applied load. Thus, this is a physical interpretation of the *adjoint plate* problem, which may assist in interpreting the significance and properties of the adjoint displacement $\lambda^{(2)}$.

In order to eliminate z' in Eq. (2.2.72), define variations of the energy bilinear form and load linear form of Eqs. (2.2.65) and (2.2.66), using the definitions of Eqs. (2.2.1) and (2.2.3), to obtain

$$a'_{\delta u}(z, \bar{z}) = \iint_\Omega \left[\sum_{i,j=1}^{2} \sigma^{ij}(z)\varepsilon^{ij}(\bar{z}) \right] \delta h \, d\Omega \tag{2.2.74}$$

$$l'_{\delta u}(\bar{z}) = \iint_\Omega [f^1\bar{z}^1 + f^2\bar{z}^2] \, \delta h \, d\Omega \tag{2.2.75}$$

Using these results, symmetry of the energy bilinear form $a_u(\cdot, \cdot)$, and repeating the sequence of calculations in Eqs. (2.2.11)–(2.2.14) gives

$$\psi'_2 = \iint_\Omega \left[f^1\lambda^{(2)^1} + f^2\lambda^{(2)^2} - \sum_{i,j=1}^{2} \sigma^{ij}(z)\varepsilon^{ij}(\lambda^{(2)}) \right] \delta h \, d\Omega \tag{2.2.76}$$

This gives the desired explicit sensitivity of the stress functional of Eq. (2.2.71) in terms of the solution z of the structural problem and $\lambda^{(2)}$ of the adjoint problem in Eq. (2.2.73).

The analytical examples considered in this section show that for each of the static elastic problems studied in Section 2.1.1, direct calculation leads to explicit formulas for design sensitivity of functionals treated, requiring in most cases the solution of an *adjoint problem* that can be interpreted as the original elasticity problem with an artificially defined applied load or initial strain field. This interpretation can be valuable in taking advantage of existing finite element structural analysis codes, as will be discussed in Section 2.2.4, and for visualizing properties of the adjoint displacement.

2.2.4 Numerical Considerations

Before proceeding from analytical derivations to numerical examples, it is helpful to consider numerical aspects of computing design sensitivity expressions. Since functions must be approximated in finite-dimensional subspaces of the associated function space for digital computation, it is important first to define the parameterization that is to be used in design sensitivity analysis. Second, in carrying out actual computations, the finite element method of structural analysis is the most commonly employed computational tool. Therefore, relationships between design sensitivity calculations and the finite element method for solving boundary-value problems should be established.

PARAMETERIZATION OF DESIGN

The piecewise uniform beam and plate shown in Figs. 1.2.1 and 2.2.4, respectively, represent the simplest examples of parameterizing the design of a structure. More generally, consider a beam with appropriate boundary conditions, in which the family of designs being considered is characterized by a finite-dimensional parameter vector $b = [b_1, \cdots b_m]^{\mathrm{T}}$. The moment of inertia of the cross section and its area as functions of these parameters may be written in the form

$$\begin{aligned} I &= I(x; b) \\ h &= h(x; b) \end{aligned} \tag{2.2.77}$$

The energy bilinear form and load linear form for the beam are then expressed as

$$\begin{aligned} a_b(z, \bar{z}) &= \int_0^1 EI(x; b) z_{xx} \bar{z}_{xx}\, dx \\ l_b(\bar{z}) &= \int_0^1 [F(x) + \gamma h(x; b)] \bar{z}\, dx \end{aligned} \tag{2.2.78}$$

The notation used here illustrates that the energy forms are functions of the design parameter b rather than a design function u. Using the definition of variation of these forms in Eqs. (2.2.1) and (2.2.2),

$$\begin{aligned} a'_{\delta b}(z, \bar{z}) &\equiv \frac{d}{d\tau} a_{b+\tau\delta b}(\tilde{z}, \bar{z}) \Big|_{\tau=0} = \left[\int_0^1 EI_b z_{xx} \bar{z}_{xx}\, dx \right] \delta b \\ l'_{\delta b}(\bar{z}) &\equiv \frac{d}{d\tau} l_{b+\tau\delta b}(\bar{z}) \Big|_{\tau=0} = \left[\int_0^1 \gamma h_b \bar{z} \right] \delta b \end{aligned} \tag{2.2.79}$$

where the variation δb can be taken outside the integrals since it is constant.

Consider now a general response functional of the form

$$\psi = \int_0^1 g(z, z_x, z_{xx}, b)\, dx \tag{2.2.80}$$

The variation of this functional may be taken to obtain

$$\psi' = \int_0^1 [g_z z' + g_{z_x} z'_x + g_{z_{xx}} z'_{xx}]\, dx + \left[\int_0^1 g_b\, dx \right] \delta b \tag{2.2.81}$$

Now define an adjoint variable as the solution of the adjoint variational equation

$$a_b(\lambda, \bar{\lambda}) = \int_0^1 [g_z \bar{\lambda}_x + g_{z_{xx}} \bar{\lambda}_{xx}]\, dx \qquad \text{for all} \quad \bar{\lambda} \in Z \tag{2.2.82}$$

Since the right side of Eq. (2.2.82) is continuous in $H^2(\Omega)$, this equation has a unique solution for λ. Repeating the sequence of calculations of Eqs. (2.2.11)–(2.2.14), the final result is

$$\psi' = \left\{ \int_0^1 [g_b + \gamma h_b \lambda - EI_b z_{xx} \lambda_{xx}] \, dx \right\} \delta b \tag{2.2.83}$$

This expression gives sensitivity coefficients of ψ associated with variations in design. It is interesting to note that evaluation of this design sensitivity result requires only numerical calculation of the integral in Eq. (2.2.83) once the state and adjoint variables have been determined. Furthermore, the form of dependence of beam cross-sectional area and moment of inertia on design can be selected by the designer, and only partial derivatives h_b of the cross-sectional area and I_b of the moment of inertia need to be calculated.

Consider, for example, the stepped beam of Fig. 1.2.1, where each uniform segment of the beam is made up of an **I** beam with section properties shown in Fig. 2.2.5. Here, the superscript i ($i = 1, \ldots, n$) denotes the numbering of uniform segments of the beam, and the subscript denotes the four design parameters of each segment, for a total of $4n$ design parameters. For the ith segment,

$$\begin{aligned} I^i(b^i) &= \tfrac{1}{12}[b_3^i(8b_4^{i3} + 6b_1^{i2}b_4^i + 12b_1^i b_4^{i2}) + b_1^{i3} b_2^i] \\ h^i(b^i) &= 2b_3^i b_4^i + b_1^i b_2^i \end{aligned} \tag{2.2.84}$$

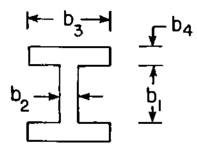

Fig. 2.2.5 I-section beam element.

The integral of Eq. (2.2.83) may also be written as a sum of integrals over each of the segments, yielding

$$\psi' = \sum_{i=1}^{n} \left\{ \int_{(i-1)/n}^{i/n} [g_{b^i} + \gamma h_{b^i}^i \lambda - EI_{b^i}^i z_{xx} \lambda_{xx}] \, dx \right\} \delta b^i \tag{2.2.85}$$

where $\delta b^i = [\delta b_1^i \;\; \delta b_2^i \;\; \delta b_3^i \;\; \delta b_4^i]^T$. This simple formula, evaluated with the aid of numerical integration, yields the design sensitivity of a general functional with respect to all section properties associated with the beam.

Equations (2.2.83) and (2.2.85) show potential for automating design sensitivity computations in terms of design shape functions. Equation (2.2.26) illustrates the simplest possible form of a design shape function, namely piecewise-constant design shape. Piecewise-linear or piecewise-polynomial design shape functions could be considered, describing distribution of material in the structure in terms of the design parameters. Using the general design sensitivity results of Section 2.2.3 and parameterizations of the type introduced here, results of the form of Eq. (2.2.85) are expected. Given this expression, an algorithm can be written for evaluating the integrals appearing in Eq. (2.2.85) over a typical segment, yielding a form for total design sensitivity. This systematic approach to design sensitivity analysis, using distributed parameter sensitivity results and design shape functions, appears to be very promising, particularly as regards its coupling with the finite element method of structural analysis. If design sensitivity is calculated using this approach, the need for calculating and storing the design derivative of the system stiffness matrix that appeared in Chapter 1 is eliminated.

COUPLING DESIGN SENSITIVITY AND FINITE ELEMENT STRUCTURAL ANALYSIS

From a mathematical point of view, the finite element method of structural analysis may be viewed as an application of the Galerkin method [5, 6] for solution of boundary-value problems, with *coordinate functions* defined as piecewise polynomials over segments (*elements*) of the domain. That is, let $\phi^i(x) \in Z$ be linearly independent coordinate functions. For finite element analysis, the domain of the structure is partitioned into subdomains called elements. Functions defined as polynomials on elements, associated with nodal values of the structural state variable and vanishing off elements not adjacent to a given node, are defined for a variety of element shapes, polynomial orders, and smoothness characteristics. For penetrating expositions of this approach to the finite element method, the reader is referred to the texts by Strang and Fix [5] and Ciarlet [6]. For a more engineering-oriented introduction to these ideas, see the text by Mitchell and Wait [41].

Letting $\phi^i(x)$ $(i = 1, \ldots, n)$ denote the coordinate functions, it is desired to approximate a solution for the structural state in the form

$$z(x) = \sum_{i=1}^{n} c_i \phi^i(x) \tag{2.2.86}$$

Recall that the structural state z must satisfy a variational equation of the form

$$a_b(z, \bar{z}) = l_b(\bar{z}) \qquad \text{for all} \quad \bar{z} \in Z \tag{2.2.87}$$

Substituting the approximation of Eq. (2.2.86) into this variational equation,

$$\sum_{j=1}^{n} a_b(\phi^j, \bar{z})c_j = l_b(\bar{z}) \tag{2.2.88}$$

Since the actual solution of Eq. (2.2.87) cannot be written exactly in the form of Eq. (2.2.86), with a finite number of coordinate functions, Eq. (2.2.88) cannot be satisfied for all $\bar{z} \in Z$. Therefore, it is desired to find the coefficients c_i in the approximate solution of Eq. (2.2.86) such that Eq. (2.2.88) holds for $\bar{z}$ equal to each of the coordinate functions. That is, it is required that

$$\sum_{j=1}^{n} a_b(\phi^i, \phi^j)c_j = l_b(\phi^i), \qquad i = 1, \ldots, n \tag{2.2.89}$$

If a matrix associated with the left side of Eq. (2.2.89) is defined as

$$A = [a_b(\phi^i, \phi^j)]_{n \times n} \tag{2.2.90}$$

and a column vector associated with the right side of Eq. (2.2.89) as

$$B = [l_b(\phi^i)]_{n \times 1} \tag{2.2.91}$$

then with $c = [c_1 \ \cdots \ c_n]^T$, Eq. (2.2.89) may be written in matrix form as

$$Ac = B \tag{2.2.92}$$

The matrix A is precisely the stiffness matrix from Chapter 1, and the column vector B represents a load vector applied to the structural system. Without going into detail, it should be recalled that the entries in the stiffness matrix of Eq. (2.2.90) require integration over only elements adjacent to nodes in which both ϕ^i and ϕ^j are nonzero. This fact immediately eliminates integration over all but a small subset of the domain of the structure for evaluation of terms contributing to the system stiffness matrix. Furthermore, because the energy bilinear form is Z-elliptic, if the coordinate functions are linearly independent, the matrix A is positive definite, hence nonsingular.

The idea of using design shape functions in evaluating Eq. (2.2.85) now begins to materialize. Let each of the uniform segments of the beam play the role of a finite element. Coordinate functions ϕ^i are used to represent both the state z, as in Eq. (2.2.86), and the adjoint variable λ as

$$\lambda(x) = \sum_{j=1}^{n} d_j \phi^j(x) \tag{2.2.93}$$

Substituting Eqs. (2.2.86) and (2.2.93) for the state and adjoint variables into Eq. (2.2.85),

$$\begin{aligned} \psi' = \sum_{i=1}^{n} \Bigg[\int_{(i-1)/n}^{i/n} g_{b^i}^i \, dx + \gamma \sum_{j=1}^{n} d_j \int_{(i-1)/n}^{i/n} h_{b^i}^i \phi^j \, dx \\ + E \sum_{j=1}^{n} \sum_{k=1}^{n} c_k d_j \int_{(i-1)/n}^{i/n} I_{b^i}^i \phi_{xx}^k \phi_{xx}^j \, dx \Bigg] \delta b^i \end{aligned} \tag{2.2.94}$$

While many integrals appear in evaluating coefficients in Eq. (2.2.94), the reader who is familiar with finite element methods will note that these calculations are of precisely the kind done routinely in finite element analysis. Incorporating a standard design shape function, represented by the functions h^i and I^i for the beam, and a set of piecewise-polynomial shape functions ϕ^i for displacement approximation, the integrations required in Eq. (2.2.94) may be efficiently carried out. In many cases, piecewise-linear polynomials will be adequate, and the order of polynomials appearing in the integration over elements will be very low, allowing closed-form evaluation of the integrals and tabulation of terms in Eq. (2.2.94) as *design sensitivity finite elements*. In the case of the beam, Hermite bicubics are commonly used as displacement shape functions, as discussed in Section 1.1.1. In this case, if linear variation cross-sectional area and quadratic variation of moment of inertia are incorporated in the beam design shape function, the degree of polynomials arising in the Eq. (2.2.94) is no higher than four. Closed-form integration to obtain and tabulate design sensitivity finite elements appears to be a practical objective. In more complex structures, such as plates and plane elastic solids, higher-order polynomials in more than one variable may be required, hence leading to the need for numerical generation of the design sensitivity finite elements. These calculations, however, are not essentially more tedious than calculations that are now carried out in any finite element code. There thus appears to exist potential for a systematic finite element design sensitivity analysis formulation, employing both design shape functions and displacement shape functions.

An essential advantage that may accrue in an integrated design finite element formulation is associated with the ability to identify the effect of numerical error associated with finite element gridding. It has been observed in calculations that use of distributed-parameter design sensitivity formulas and the finite element method for analysis leads to numerical errors in sensitivity coefficients that may be identified during the process of iterative redesign and reanalysis. The effect of a design change that is to be implemented with the design sensitivity analysis method can be predicted. When reanalysis is carried out, the predicted change in structural response can be compared with the change realized. If disagreement arises, then error has crept into the finite element approximation. While this might appear to be a problem, in fact it can be a blessing in disguise. If the approach of Chapter 1 is followed, in which the structure is discretized and the design variables are imbedded into the global stiffness matrix, then any error inherent in the finite element model is consistently parameterized and will never be reported to the user. Therefore, precise design sensitivity coefficients of the matrix model of the structure are obtained without realizing that there may be substantial inherent error in the original model. In fact, as optimization is carried out, the optimization algorithm may systematically exploit this error and lead to erroneous designs. In the current formulation, the design sensitivity formulas derived from the distributed-parameter theory and the finite element model can be used to obtain a warning that approximation error is creeping into the calculation.

2.2.5 Numerical Examples

BEAM

Consider a simply supported beam with rectangular cross section and a point load of $f(x) = 100\ \delta(x - \hat{x})$ lb (Fig. 2.2.6). Material properties are given as $E = 30 \times 10^6$ psi and $\nu = 0.25$. Weight density γ of the material is ignored. The rectangular beam has unit width, and the depth b_i of element i is taken as a design variable, $i = 1, \ldots, n$.

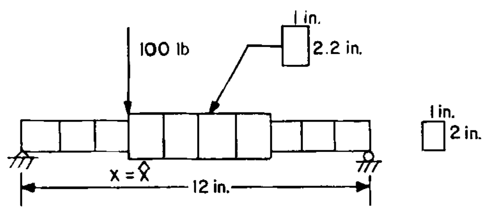

Fig. 2.2.6 Simply supported beam.

Consider a stress constraint of the form

$$\psi_i = -\int_0^l \frac{b_i}{2} E z_{xx}(x) m_i\, dx \tag{2.2.95}$$

where $b_i/2$ is the half-depth of element i and m_i is the characteristic function applied to finite element i. Referring to Eq. (2.2.82), the adjoint equation can be defined as

$$a_b(\lambda, \bar{\lambda}) = -\int_0^l \frac{b_i}{2} E\bar{\lambda}_{xx} m_i\, dx \qquad \text{for all} \quad \bar{\lambda} \in Z \tag{2.2.96}$$

and denote the solution as $\lambda^{(i)}$. Then, from Eq. (2.2.83) the first variation of the functional ψ_i is

$$\psi_i' = -\left(\int_0^l \frac{E}{2} z_{xx} m_i\, dx\right) \delta b_i - \sum_{k=1}^{n} \left[\int_{[(k-1)/n]l}^{(k/n)l} \frac{b_k^2}{4} E z_{xx} \lambda_{xx}^{(i)}\, dx\right] \delta b_k \tag{2.2.97}$$

Note that constant thickness over a single element is assumed in the above equations.

Table 2.2.1
Comparison of Sensitivity for Beam

Element Number	ψ_i^1	ψ_i^2	$\Delta\psi_i$	ψ_i'	$(\psi_i'/\Delta\psi_i \times 100)\%$
1	6.3000E + 01	5.7143E + 01	−5.8571E + 00	−6.3000E + 00	107.6
2	1.8900E + 02	1.7143E + 02	−1.7571E + 01	−1.8900E + 01	107.6
3	3.1500E + 02	2.8571E + 02	−2.9286E + 01	−3.1500E + 01	107.6
4	2.9008E + 02	2.6311E + 02	−2.6969E + 01	−2.9008E + 01	107.6
5	2.4545E + 02	2.2263E + 02	−2.2820E + 01	−2.4545E + 01	107.6
6	2.0083E + 02	1.8216E + 02	−1.8671E + 01	−2.0083E + 01	107.6
7	1.5620E + 02	1.4168E + 02	−1.4522E + 01	−1.5620E + 01	107.6
8	1.3500E + 02	1.2245E + 02	−1.2551E + 01	−1.3500E + 01	107.6
9	8.1000E + 01	7.3469E + 01	−7.5306E + 01	−8.1000E + 00	107.6
10	2.7000E + 01	2.4490E + 01	−2.5102E + 00	−2.7000E + 00	107.6

A 10-element finite element model of the beam shown in Fig. 2.2.6, with a cubic shape function, is employed for design sensitivity calculation. Uniform and good design sensitivity estimates are obtained, as shown in Table 2.2.1, for the average stress on each element with 5% overall changes of design variables.

PLATE

Consider application of Eqs. (2.2.52) and (2.2.58) to account for the effect of variations in plate thickness on the compliance and displacement, at a discrete point $\hat{x}$ respectively. As an example, a clamped square plate of dimension 1 m and uniform thickness $h = 0.05$ m, with E = 200 GPa, $v = 0.3$, F = 2.22 MPa, and $\gamma = 7.71 \times 10^4$ N/m^3 is considered. If piecewise-constant thickness is assumed, with b_i the constant thickness of the ith element, instead of Eq. (2.2.52), Eq. (2.2.54) can be used for the compliance functional.

For numerical calculations, a nonconforming rectangular plate element with 12 degrees of freedom [7] is used. The graph of the coefficient of δb_i in Fig. 2.2.7 shows how addition or deletion of material affects compliance. The maximum value of the coefficient of δb_i is $\Lambda^2_{\max} = -1.305$ at the corner elements. The minimum value occurs at the middle of edge elements, with the value $\Lambda^2_{\min} = -5.625 \times 10^2$. Thus, in order to decrease compliance most effectively, material should be removed from the vicinity of the four corners and added near the middle of the four edges.

If design sensitivity of displacement at the center of the plate is desired, $\hat{x} = [\frac{1}{2}\ \frac{1}{2}]$, and the adjoint load from Eq. (2.2.57) is just a unit load at the

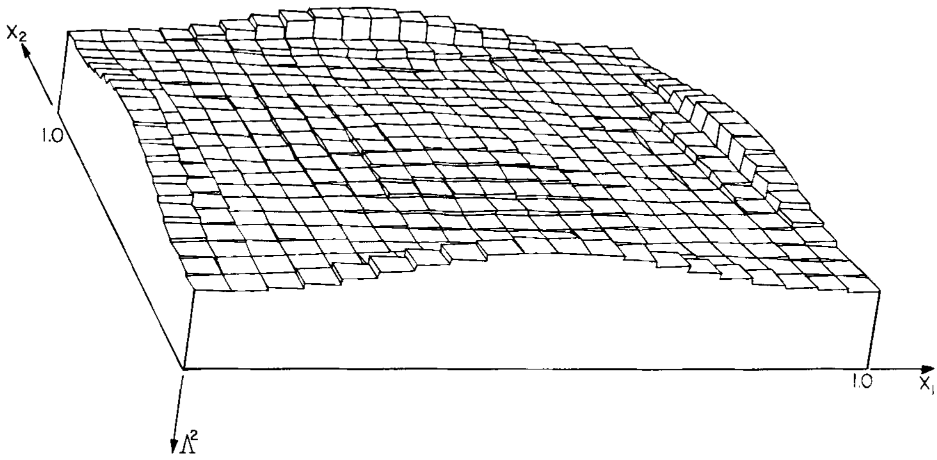

Fig. 2.2.7 Compliance sensitivity Λ^2 for plate.

center of the plate. To see how material added to or deleted from the plate influences displacement at the center, the coefficient of δb_i may be graphed, as in the compliance case, using Eq. (2.2.58) to obtain the result shown in Fig. 2.2.8. The maximum value of the coefficient of δb_i is $\Lambda^3_{\max} = -3.678 \times 10^{-8}$ at the corner elements, while the minimum value occurs at the center elements with the value $\Lambda^3_{\min} = -1.167 \times 10^3$. To decrease $z(\hat{x})$ most effectively, material should be removed near the four corners and added to the center of the plate.

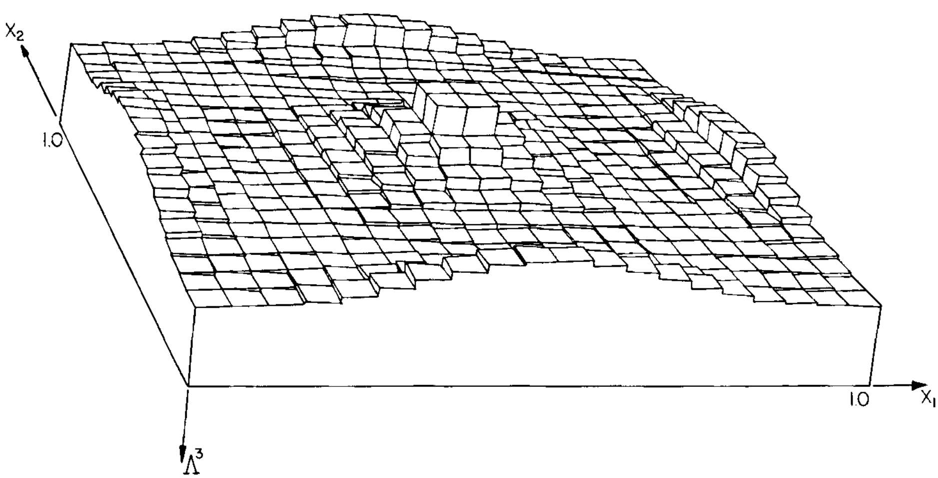

Fig. 2.2.8 Displacement sensitivity Λ^3 for plate.

For sensitivity of stress in the plate, consider the simply supported square plate shown in Fig. 2.2.9, with $E = 30 \times 10^6$ psi and $\nu = 0.25$. Let plate thickness $h(x)$ be a design variable and assume γ is ignorable, so that the load linear form is independent of design.

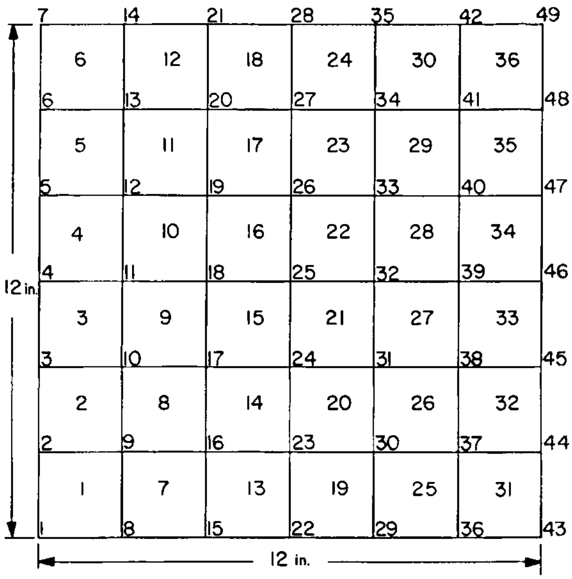

Fig. 2.2.9 Finite element of simply supported plate.

Consider a stress functional of the form

$$\psi_i = \iint_\Omega E z_{11} m_i \, d\Omega \tag{2.2.98}$$

where m_i is the characteristic function applied on finite element i. Referring to Eq. (2.2.63), the adjoint equation is defined as

$$a_u(\lambda, \bar{\lambda}) = \iint_\Omega E \bar{\lambda}_{11} m_i \, d\Omega \qquad \text{for all} \quad \bar{\lambda} \in Z \tag{2.2.99}$$

with solution $\lambda^{(i)}$. From Eq. (2.2.64), the first variation of Eq. (2.2.98) is obtained as

$$\psi_i' = -\iint_\Omega \frac{Eh^2}{4(1 - \nu^2)} [z_{11}\lambda_{11}^{(i)} + z_{22}\lambda_{22}^{(i)} + \nu(z_{22}\lambda_{11}^{(i)} + z_{11}\lambda_{22}^{(i)}) + 2(1 - \nu) z_{12}\lambda_{12}^{(i)}] \, \delta h \, d\Omega \tag{2.2.100}$$

If piecewise-constant thickness is assumed for each finite element, Eq. (2.2.100) can be rewritten as

$$\psi_i' = -\sum_{k=1}^{n} \left\{ \frac{Eb_k^2}{4(1-\nu^2)} \iint_{\Omega_k} [z_{11}\lambda_{11}^{(i)} + z_{22}\lambda_{22}^{(i)} + \nu(z_{22}\lambda_{11}^{(i)} + z_{11}\lambda_{22}^{(i)}) + 2(1-\nu)z_{12}\lambda_{12}^{(2)}]\, d\Omega \right\} \delta b_k \qquad (2.2.101)$$

As before, a nonconforming rectangular plate element with 12 degrees of freedom [7] is employed for numerical calculation. The geometrical configuration and finite element grid used are shown in Fig. 2.2.9. The length of each side of the square plate is 12 in., and its thickness is 0.1 in., uniformly. The model has 36 elements, 49 nodal points, and 95 degrees of freedom. Applied loads consist of a point load of 100 lb at the center and a uniformly distributed load of 100 psi. Results given in Table 2.2.2 show that the design sensitivity for each element is excellent, with 0.1% overall change of design variables. Note that due to symmetry, only sensitivity results for one quarter of the plate are given in Table 2.2.2.

Table 2.2.2

Comparison of Sensitivity for Plate

Element Number	ψ_i^1	ψ_i^2	$\Delta\psi_i$	ψ_i'	$(\psi_i'/\Delta\psi_i \times 100)\%$
1	−7.7010E + 05	−7.6779E + 05	2.3057E + 03	2.3030E + 03	99.9
2	−1.7690E + 06	−1.7637E + 06	5.2965E + 03	5.3094E + 03	100.2
3	−2.2571E + 06	−2.2503E + 06	6.7576E + 03	6.7702E + 03	100.2
7	−1.3671E + 06	−1.3630E + 06	4.0930E + 03	4.0869E + 03	99.9
8	−3.6338E + 06	−3.6229E + 06	1.0880E + 04	1.0906E + 04	100.2
9	−4.8362E + 06	−4.8217E + 06	1.4480E + 04	1.4508E + 04	100.2
13	−1.5622E + 06	−1.5575E + 06	4.6772E + 03	4.6859E + 03	100.2
14	−4.2347E + 06	−4.2220E + 04	1.2679E + 04	1.2706E + 04	100.2
15	−5.7639E + 06	−5.7466E + 06	1.7257E + 04	1.7293E + 04	100.2

TORQUE ARM

As an example involving a plane elastic component, consider the automotive rear suspension torque arm shown in Fig. 2.2.10. For simplicity, a single, nonsymmetric, static traction load is considered. Zero-displacement constraints are applied around the larger hole on the right in order to simulate attachment to a solid rear axle. Thickness of the torque arm is

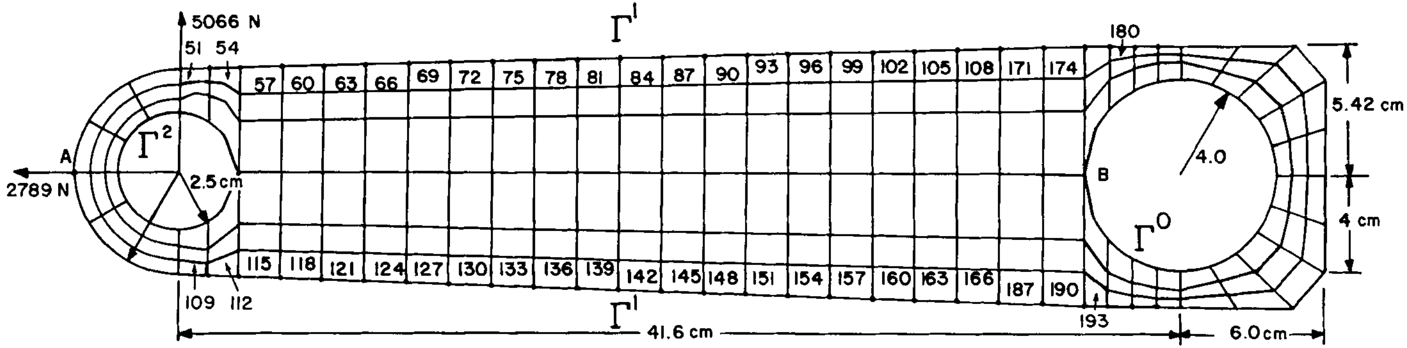

Fig. 2.2.10 Geometry and finite element of torque arm.

chosen as a design variable. The variational equation of the torque arm is

$$a_u(z, \bar{z}) \equiv \iint_\Omega h(x) \sum_{i,j=1}^{2} \sigma^{ij}(z)\varepsilon^{ij}(\bar{z})\, d\Omega$$

$$= \int_{\Gamma^2} \sum_{i=1}^{2} T^i \bar{z}^i \, d\Gamma \equiv l_u(\bar{z}) \qquad \text{for all} \quad \bar{z} \in Z \qquad (2.2.102)$$

where

$$Z = \{z = [z^1 \quad z^2]^{\mathrm{T}} \in [H^1(\Omega)]^2 \colon z = 0 \text{ on } \Gamma^0\} \qquad (2.2.103)$$

Consider a von Mises stress functional of the form

$$\psi_k = \iint_\Omega \frac{(\sigma_y - \sigma^{\mathrm{a}})}{\sigma^{\mathrm{a}}} m_k \, d\Omega = \iint_\Omega g m_k \, d\Omega \qquad (2.2.104)$$

where $g = (\sigma_y - \sigma^{\mathrm{a}})/\sigma^{\mathrm{a}}$, σ^{a} is the allowable stress, m_k is the characteristic function defined on finite element k, and σ_y is the von Mises yield stress defined as

$$\sigma_y = [(\sigma^{11})^2 + (\sigma^{22})^2 + 3(\sigma^{12})^2 - \sigma^{11}\sigma^{22}]^{1/2} \qquad (2.2.105)$$

For this stress functional, the adjoint equation from Eq. (2.2.73) is

$$a_u(\lambda, \bar{\lambda}) = \iint_\Omega \left[\sum_{i,j=1}^{2} g_{\sigma^{ij}}(z)\sigma^{ij}(\bar{\lambda}) \right] m_k \, d\Omega \qquad \text{for all} \quad \bar{\lambda} \in Z \qquad (2.2.106)$$

with solution $\lambda^{(k)}$. The first variation of the functional ψ_k, from Eq. (2.2.76), is

$$\psi_k' = \iint_\Omega \left[\sum_{i,j=1}^{2} \sigma^{ij}(z)\varepsilon^{ij}(\lambda^{(k)}) \right] \delta h \, d\Omega \qquad (2.2.107)$$

If piecewise-constant thickness b_l is assumed for finite element l, then Eq. (2.2.107) can be rewritten as

$$\psi_k' = -\sum_{l=1}^{n} \left[\iint_{\Omega_l} \sum_{i,j=1}^{2} \sigma^{ij}(z)\varepsilon^{ij}(\lambda^{(k)}) \, d\Omega \right] \delta b_l \qquad (2.2.108)$$

The finite element model shown in Fig. 2.2.10, including 204 elements, 707 nodal points, 1332 degrees of freedom, and an 8-noded isoparametric element, is used for numerical calculation. Applied loads and dimensions are also shown in Fig. 2.2.10. Young's modulus, Poisson's ratio, and allowable stress are 207.4 GPa, 0.25, and 81 MPa, respectively. A uniform thickness of 1 cm is used as the initial design. Numerical results for stress in selected boundary elements are shown in Table 2.2.3. With a 0.1% uniform change of design variables, excellent sensitivity results are obtained.

Table 2.2.3

Comparison of Sensitivity for Torque Arm

Element Number	ψ_k^1	ψ_k^2	$\Delta\psi_k$	ψ_k'	$(\psi_k'/\Delta\psi_K \times 100)\%$
54	−9.7690E − 01	−9.7693E − 01	−2.3075E − 05	−2.3098E − 05	100.1
66	−9.6734E − 01	−9.6737E − 01	−3.2632E − 05	−3.2665E − 05	100.1
75	−9.5025E − 01	−9.5030E − 01	−4.9699E − 05	−4.9748E − 05	100.1
87	−9.3080E − 01	−9.3087E − 01	−6.9130E − 05	−6.9199E − 05	100.1
96	−9.1860E − 01	−9.1868E − 01	−8.1317E − 05	−8.1398E − 05	100.1
105	−9.0812E − 01	−9.0821E − 01	−9.1786E − 05	−9.1878E − 05	100.1
115	−9.7021E − 01	−9.7024E − 01	−2.9756E − 05	−2.9786E − 05	100.1
127	−9.5415E − 01	−9.5420E − 01	−4.5805E − 05	−4.5850E − 05	100.1
145	−9.2374E − 01	−9.2382E − 01	−7.6183E − 05	−7.6259E − 05	100.1
160	−9.0483E − 01	−9.0493E − 01	−9.5073E − 05	−9.5169E − 05	100.1
171	−9.0491E − 01	−9.0500E − 01	−9.4997E − 05	−9.4997E − 05	100.1
180	−9.2579E − 01	−9.2587E − 01	−7.4134E − 05	−7.4208E − 05	100.1
187	−8.9958E − 01	−8.9968E − 01	−1.0032E − 04	−1.0042E − 04	100.1
193	−9.1117E − 01	−9.1126E − 01	−8.8743E − 05	−8.8831E − 05	100.1

2.3 EIGENVALUE DESIGN SENSITIVITY

Examples presented in Section 2.2.2 show clearly that eigenvalues that represent natural frequencies and buckling loads of structures depend on the design of the structure. The objective in this section is to obtain design sensitivity of eigenvalues. For conservative systems, it happens that no adjoint equations are necessary, and eigenvalue sensitivities are obtained directly in terms of the eigenvectors of the eigenvalue problem and variations in the eigenvalue bilinear forms. Theorems that establish differentiability of simple eigenvalues and directional differentiability of repeated eigenvalues are first stated, and their significance is discussed. Using these differentiability results, explicit formulas for design variations of eigenvalues, both simple and repeated, are obtained. Analytical calculations with the examples of Section 2.1.2 are carried out to illustrate use of the method. Numerical considerations associated with computation of eigenvalue design sensitivity are discussed, and numerical examples are presented.

2.3.1 Differentiability of Energy Bilinear Forms and Eigenvalues

Basic results concerning differentiability of eigenvalues are developed in detail in Section 2.5. The purpose of this section is to summarize key results that are needed for eigenvalue design sensitivity. In particular, treatment of the repeated eigenvalue case illustrates the need for care in establishing and utilizing properties of functionals involved since it is shown that repeated eigenvalues are in fact not differentiable.

As shown in Section 2.1.2, eigenvalue problems for vibration and buckling of elastic systems are best described by variational equations of the form

$$a_u(y, \bar{y}) = \zeta d_u(y, \bar{y}) \qquad \text{for all} \quad \bar{y} \in Z \tag{2.3.1}$$

where Z is the space of kinematically admissible displacements. Since Eq. (2.3.1) is homogeneous in y, a normalizing condition must be added to define uniquely the eigenfunction. The normalizing condition employed is

$$d_u(y, y) = 1 \tag{2.3.2}$$

The energy bilinear form on the left side of Eq. (2.3.1) is the same as that occurring in static problems treated in Section 2.2. Therefore, it has the same properties discussed there. The bilinear form $d_u(\cdot, \cdot)$ on the right side of Eq. (2.3.1) represents mass effects in vibration problems and geometric effects in buckling. In most cases, it is even more regular in its dependence on design u and eigenfunction y than is the energy bilinear form $a_u(\cdot, \cdot)$. In the exceptional case of buckling of a column, it involves derivatives of the eigenfunction and must be treated somewhat more carefully. As shown in Section 2.5.1, in all cases of interest here, the design derivative of $d_u(\cdot, \cdot)$ is given by

$$d'_{\delta u}(y, \bar{y}) \equiv \frac{d}{d\tau}[d_{u+\tau\delta u}(\tilde{y}, \bar{y})]\bigg|_{\tau=0} \tag{2.3.3}$$

where $\tilde{y}$ denotes holding y constant for purposes of the differentiation with respect to τ.

SIMPLE EIGENVALUES

In the case of a simple eigenvalue (i.e., an eigenvalue with only one independent eigenfunction) it is shown in Section 2.5 that the eigenvalue ζ is differentiable. Kato [13] showed that the corresponding eigenfunction y is also differentiable. Thus, the following variations are well defined:

$$\begin{aligned} \zeta' &= \zeta'(u, \delta u) \equiv \frac{d}{d\tau}[\zeta(u + \tau\, \delta u)]\bigg|_{\tau=0} \\ y' &= y'(x; u, \delta u) \equiv \frac{d}{d\tau}[y(x; u + \tau\, \delta u)]\bigg|_{\tau=0} \end{aligned} \tag{2.3.4}$$

In fact, both eigenvalue and eigenfunction variations are linear in δu, hence they are Fréchet derivatives (Appendix A.3) of the eigenvalue and eigenfunction. Proof of these results is far from trivial; details are given in Section 2.5.

Given this differentiability result, the variation of both sides of Eq. (2.3.1) can be taken to obtain

$$a_u(y', \bar{y}) + a'_{\delta u}(y, \bar{y}) = \zeta' d_u(y, \bar{y}) + \zeta d_u(y', \bar{y}) + \zeta d'_{\delta u}(y, \bar{y}) \qquad \text{for all} \quad \bar{y} \in Z \tag{2.3.5}$$

Since Eq. (2.3.5) holds for all $\bar{y} \in Z$, this equation may be evaluated with $\bar{y} = y$, using symmetry of the bilinear forms $a_u(\cdot, \cdot)$ and $d_u(\cdot, \cdot)$, to obtain

$$\zeta' d_u(y, y) = a'_{\delta u}(y, y) - \zeta d'_{\delta u}(y, y) - [a_u(y, y') - \zeta d_u(y, y')] \tag{2.3.6}$$

Noting that $y' \in Z$, it can be seen that the term in brackets on the right side of Eq. (2.3.6) is zero. Furthermore, due to the normalizing condition of Eq. (2.3.2), a simplified equation is obtained

$$\zeta' = a'_{\delta u}(y, y) - \zeta d'_{\delta u}(y, y) \tag{2.3.7}$$

For precise proof of this result, see Corollary 2.5.1 (Section 2.5).

This result, obtained with little effort, forms the foundation of a large body of work on structural optimization with constraints on eigenvalues. It is a remarkably simple result, showing clearly that the directional derivative of the eigenvalue is indeed linear in δu since the variations of the bilinear forms on the right side of Eq. (2.3.7) are linear in δu. It should be emphasized, however, that validity of this result rests on the existence of variations of the eigenvalue and eigenfunction defined in Eq. (2.3.4). As will be seen in the following, formal extension of this analysis to repeated eigenvalues would lead to an erroneous result.

REPEATED EIGENVALUES

Consider now a situation in which an eigenvalue ζ has associated with it s linearly independent eigenfunctions, that is,

$$\begin{aligned} a_u(y^i, \bar{y}) &= \zeta d_u(y^i, \bar{y}) \qquad \text{for all} \quad \bar{y} \in Z, \\ d_u(y^i, y^i) &= 1, \end{aligned} \qquad i = 1, \ldots, s \tag{2.3.8}$$

It is an easy exercise to show that any linear combination of eigenfunctions of y^i of Eq. (2.3.8) is also an eigenfunction. Therefore, an infinite variety exists of choices for the basis of the eigenspace associate with the repeated eigenvalue ζ. One practical limitation on the family of eigenfunctions employed is to require that they be orthonormal with respect to the bilinear form $d_u(\cdot, \cdot)$, that is,

$$d_u(y^i, y^j) = \delta_{ij}, \qquad i,j = 1, \ldots, s \tag{2.3.9}$$

It is presumed throughout this text that such an orthonormalization of eigenfunctions corresponding to a repeated eigenvalue has been carried out. Nonetheless, there still remains an infinite choice of such families.

It is shown by Theorem 2.5.1 (Section 2.5) that the repeated eigenvalue ζ is a continuous function of design, but that the eigenfunctions are not. While the eigenvalue is continuous, it is shown not to be Fréchet differentiable, but only directionally differentiable (Appendix A.3). It is shown by Theorem 2.5.2 and Corollary 2.5.2 (Section 2.5) that at a design u for which the eigenvalue ζ is repeated s times, for a perturbation of design to $u + \tau\,\delta u$, the eigenvalue may branch into s eigenvalues given by

$$\zeta_i(u + \tau\,\delta u) = \zeta(u) + \tau\zeta_i'(u, \delta u) + o(\tau), \qquad i = 1, \ldots, s \tag{2.3.10}$$

where the directional derivatives $\zeta_i'(u, \delta u)$ are the eigenvalues of the matrix

$$\mathscr{M} = [a_{\delta u}'(y^i, y^j) - \zeta d_{\delta u}'(y^i, y^j)]_{s \times s} \tag{2.3.11}$$

The notation $\zeta_i'(u, \delta u)$ is selected here to emphasize dependence of the directional derivative on δu. The term $o(\tau)$ in Eq. (2.3.10) is defined as a quantity that approaches zero more rapidly than τ [i.e., $\lim_{\tau \to 0} o(\tau)/\tau = 0$]. All the characteristics of repeated eigenvalues discussed in Section 1.3.6 hold true in the distributed-parameter case. Moreover, the directional derivatives of twice-repeated eigenvalues are given in Eqs. (1.3.59) and (1.3.60), which is rewritten here as

$$\begin{aligned} \zeta_1'(u, \delta u) = \hat{\mathscr{M}}_{11} &= \cos^2\phi(\delta u)\,\mathscr{M}_{11}(\delta u) \\ &\quad + \sin 2\phi(\delta u)\,\mathscr{M}_{12}(\delta u) + \sin^2\phi(\delta u)\,\mathscr{M}_{22}(\delta u) \end{aligned} \tag{2.3.12}$$

$$\begin{aligned} \zeta_2'(u, \delta u) = \hat{\mathscr{M}}_{22} &= \sin^2\phi(\delta u)\,\mathscr{M}_{11}(\delta u) \\ &\quad - \sin 2\phi(\delta u)\,\mathscr{M}_{12}(\delta u) + \cos^2\phi(\delta u)\,\mathscr{M}_{22}(\delta u) \end{aligned} \tag{2.3.13}$$

where ϕ is the rotation parameter, given as

$$\phi = \phi(\delta u) = \frac{1}{2}\arctan\left[\frac{2\mathscr{M}_{12}(y^1, y^2, \delta u)}{\mathscr{M}_{11}(y^1, y^1, \delta u) - \mathscr{M}_{22}(y^2, y^2, \delta u)}\right] \tag{2.3.14}$$

and $\mathscr{M}_{ij}(\delta u)$ is the component of the matrix $\mathscr{M}$ given in Eq. (2.3.11).

2.3.2 Analytical Examples of Eigenvalue Design Sensitivity

To illustrate the preceding results, design sensitivity analysis of the eigenvalue problems presented in Section 2.1.2 are now studied. Numerical considerations in the use of the resulting formulas will be discussed in Section 2.3.3.

VIBRATION OF A STRING

Consider the string of Fig. 2.1.5, with variable mass density $h(x)$ and tension $\hat{T}$. The energy and mass bilinear forms of Eq. (2.1.48) are

$$a_u(y, \bar{y}) = \hat{T} \int_0^1 y_x \bar{y}_x \, dx \qquad d_u(y, \bar{y}) = \int_0^1 h y \bar{y} \, dx \tag{2.3.15}$$

Variations of these bilinear forms yield

$$a'_{\delta u}(y, \bar{y}) = \delta\hat{T} \int_0^1 y_x \bar{y}_x \, dx \qquad d'_{\delta u}(y, \bar{y}) = \int_0^1 \delta h \, y \bar{y} \, dx \tag{2.3.16}$$

Since for Sturm–Liouville problems only simple eigenvalues can occur [23], only the variation of a simple eigenvalue is of interest. Direct application of Eq. (2.3.7), with Eq. (2.3.16), yields

$$\zeta' = \left[\int_0^1 (y_x)^2 \, dx \right] \delta\hat{T} - \zeta \int_0^1 y^2 \, \delta h \, dx \tag{2.3.17}$$

It is interesting to note that since the coefficient of the variation in string tension is positive, it is clear that the frequency increases with increasing tension. Similarly, since the coefficient of the variation δh in mass density of the string is positive, any increase in density decreases the frequency of vibration. While both of these results are obvious on an intuitive basis, a plot of $(y(x))^2$ for a uniform string $h_0 = 2.0$ and $\hat{T}_0 = 1.0$ (Fig. 2.3.1) shows that a unit increase in $h(x)$ near the center of the string has substantially more effect

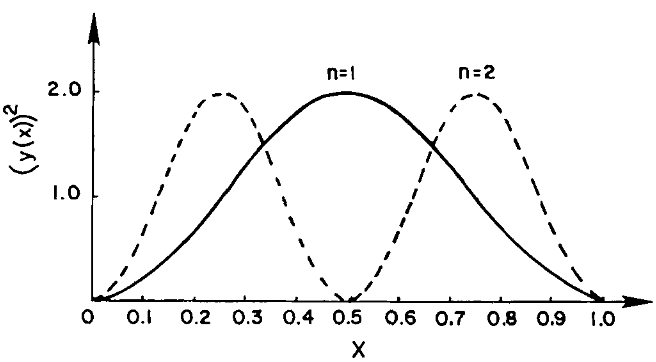

Fig. 2.3.1 First two eigenfunctions of vibrating string.

on the smallest eigenvalue than unit increases elsewhere in the string. Thus, an indication is obtained of the most profitable areas for changing design.

VIBRATION OF A BEAM

For the beam with variable cross section, Young's modulus, and mass density (shown in Fig. 2.1.6), the strain and kinetic energy bilinear forms of Eq. (2.1.51) are

$$
\begin{aligned}
a_u(y, \bar{y}) &= \int_0^1 E\alpha h^2 y_{xx} \bar{y}_{xx}\, dx \\
d_u(y, \bar{y}) &= \rho \int_0^1 hy\bar{y}\, dx
\end{aligned}
\tag{2.3.18}
$$

The design variations of these bilinear forms are

$$
\begin{aligned}
a'_{\delta u}(y, \bar{y}) &= \delta E \int_0^1 \alpha h^2 y_{xx} \bar{y}_{xx}\, dx + \int_0^1 2E\alpha h\, \delta h\, y_{xx} \bar{y}_{xx}\, dx \\
d'_{\delta u}(y, \bar{y}) &= \delta\rho \int_0^1 hy\bar{y}\, dx + \rho \int_0^1 \delta h\, y\bar{y}\, dx
\end{aligned}
\tag{2.3.19}
$$

For a simple eigenvalue, Eq. (2.3.7), with (Eq. 2.3.19), yields

$$
\begin{aligned}
\zeta' = &\left[\int_0^1 \alpha h^2 (y_{xx})^2\, dx\right] \delta E - \left[\zeta \int_0^1 hy^2\, dx\right] \delta\rho \\
&+ \int_0^1 [2E\alpha h (y_{xx})^2 - \zeta\rho y^2]\, \delta h\, dx
\end{aligned}
\tag{2.3.20}
$$

As in the static response case, the sensitivity formula of Eq. (2.3.20) is valid for other boundary conditions in Eqs. (2.1.16)–(2.1.18). This result clearly shows that increasing Young's modulus increases natural frequency, and increasing the density of material decreases the natural frequency, both of which are clear physically. However, since the coefficient of δh in the integral may have either sign, it is not clear how a change in cross-sectional area will influence natural frequency of vibration. Consider, for example, a uniform cantilever beam (Fig. 2.3.2) and nominal properties $E = 2 \times 10^5$ MPa,

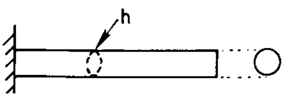

Fig. 2.3.2 Uniform cantilever beam.

$\alpha = \frac{1}{6}$, $\rho = 7.87$ Mg/m^3, and $h = 0.005$ m^2. For this case, the smallest eigenvalue is $\zeta = 0.00157$, and the eigenfunction is

$$y(x) = 0.159\left[\cosh k_n x - \cos k_n x - \frac{\cos k_n + \cosh k_n}{\sin k_n + \sinh k_n}(\sinh k_n x - \sin k_n x)\right]$$

where $k_1 = 1.875$, $k_2 = 4.694, \ldots$. Evaluating the coefficient of δh in the integral of Eq. (2.3.20), a curve is obtained of the form shown in Fig. 2.3.3. The design sensitivity coefficient of Fig. 2.3.3 shows that a unit change in cross-sectional area at the clamped end of the beam has substantially more effect on the smallest eigenvalue than a unit change at the free end.

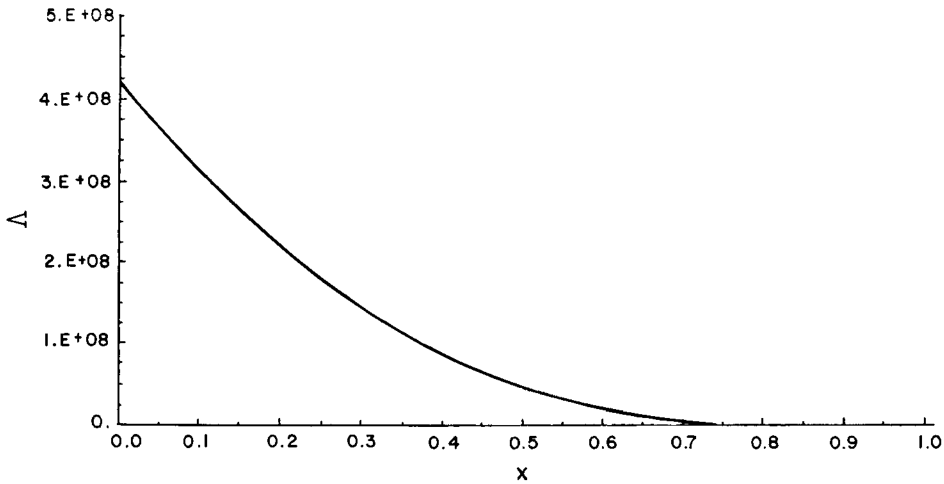

Fig. 2.3.3 Design sensitivity coefficient of δh for cantilever beam.

BUCKLING OF A COLUMN

Consider now the problem of buckling of a column, with design variables being the distribution of cross-sectional area along the column and Young's modulus of the material. The energy and geometric bilinear forms of Eq. (2.1.53) are

$$\begin{aligned} a_u(y, \bar{y}) &= \int_0^1 E\alpha h^2 y_{xx}\bar{y}_{xx}\,dx \\ d_u(y, \bar{y}) &= \int_0^1 y_x\bar{y}_x\,dx \end{aligned} \tag{2.3.21}$$

The variations of these bilinear forms are

$$\begin{aligned} a'_{\delta u}(y, \bar{y}) &= \delta E\int_0^1 \alpha h^2 y_{xx}\bar{y}_{xx}\,dx + \int_0^1 2E\alpha h\,\delta h\; y_{xx}\bar{y}_{xx}\,dx \\ d'_{\delta u}(y, \bar{y}) &= 0 \end{aligned} \tag{2.3.22}$$

The variation of the buckling load for a simple eigenvalue is given by Eq. (2.3.7), with Eq. (2.3.22), as

$$\zeta' = \left[\int_0^1 \alpha h^2 (y_{xx})^2 \, dx \right] \delta E + \int_0^1 [2E\alpha h (y_{xx})^2] \, \delta h \, dx \qquad (2.3.23)$$

Clearly, increasing Young's modulus increases the buckling load, and any increase in cross-sectional area similarly increases the buckling load. Both of these results are to be expected.

Consider, for example, the clamped–clamped column of Fig. 2.1.7, with uniform cross section, $h_0 = 0.005 \text{ m}^2$, $\alpha = \frac{1}{6}$, and $E = 2 \times 10^5$ MPa. A plot of the first and second mode shapes and their second derivatives is shown in Fig. 2.3.4a. Using these functions, the coefficient of δh may be evaluated in the integral of Eq. (2.3.23), obtaining the curve shown in Fig. 2.3.4b. Note that to redistribute material in the column to increase the buckling load in the first mode, material in the vicinity of points a and c, where the sensitivity coefficient of δh for ζ_1 is zero, may be removed and the material added at point b, or at the ends, where the sensitivity coefficient is a maximum. This process, however, may decrease the buckling load in the second mode since its sensitivity coefficient is positive at points a and c and zero at point b. In fact, it has been shown by Olhoff and Rasmussen [18] and others that when attempting to maximize the fundamental buckling load for a clamped–clamped column, systematic occurrence of a repeated eigenvalue may be forced, much as shown in the examples presented in Section 1.3.5. It is therefore of interest to obtain expressions for the directional derivatives of this column for a repeated eigenvalue.

If y^1 and y^2 are eigenfunctions corresponding to a repeated eigenvalue, then from Eq. (2.3.11),

$$\mathcal{M}_{ij} = \int_0^1 2E\alpha h y^i_{xx} y^j_{xx} \, \delta h \, dx \qquad (2.3.24)$$

where the effect of variation of Young's modulus has been suppressed. Note that if attention is limited to designs $h(x)$ that are symmetric about the center of the column and if $\delta h(x)$ is symmetric about the center, then as indicated in Fig. 2.3.4a, the second derivatives of the first and second eigenfunctions will be symmetric and antisymmetric, respectively, about the center of the column. Thus, the product $h \, \delta h \, y^1_{xx}$ is an even function about the center and y^2_{xx} is an odd function about the center. Therefore, $\mathcal{M}_{12} = 0$. Since this is true for all design variations in the class of designs that are symmetric about the center, the directional derivatives of the repeated eigenvalue for symmetric columns are given by Eq. (2.3.28), with symmetric and antisymmetric modes y^1 and y^2. Since the resulting expression is linear in the design variation, this shows that the repeated eigenvalues, ordered by symmetric and antisymmetric modes, for symmetric columns are differentiable and that the

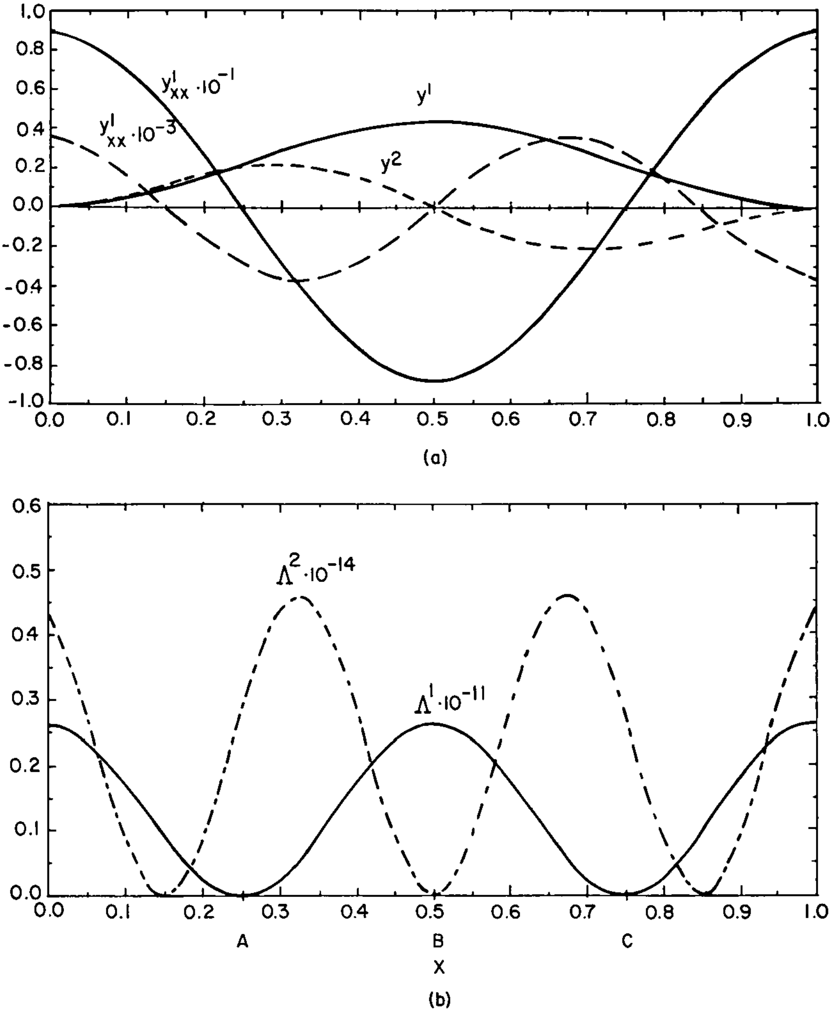

Fig. 2.3.4 Mode and sensitivities for uniform clamped–clamped column.

derivatives may be obtained by using symmetric and antisymmetric modes in the simple formula of Eq. (2.3.23).

On the other hand, if asymmetric designs are allowed, $\mathscr{M}_{12}$ in general will not be zero, and indeed the eigenvalue is only directionally differentiable and not Fréchet differentiable. In this case, the angle of rotation required is obtained from Eq. (2.3.14) as

$$\phi(\delta h) = \frac{1}{2}\arctan\left[\frac{2\int_0^1 hy^1_{xx}y^2_{xx}\,\delta h\,dx}{\int_0^1 h[(y^1_{xx})^2 - (y^2_{xx})^2]\,\delta h\,dx}\right] \tag{2.3.25}$$

The directional derivatives of the repeated eigenvalue are then given by Eqs. (2.3.12) and (2.3.13) as

$$\begin{aligned}\zeta_1'(h, \delta h) &= 2E\alpha \int_0^1 h[\cos^2 \phi(\delta h)\,(y_{xx}^1)^2 + \sin 2\phi(\delta h)\, y_{xx}^1 y_{xx}^2 \\ &\qquad + \sin^2 \phi(\delta h)\,(y_{xx}^2)^2]\, \delta h\, dx \\ \zeta_2'(h, \delta h) &= 2E\alpha \int_0^1 h[\sin^2 \phi(\delta h)\,(y_{xx}^1)^2 - \sin 2\phi(\delta h)\, y_{xx}^1 y_{xx}^2 \\ &\qquad + \cos^2 \phi(\delta h)\,(y_{xx}^2)^2]\, \delta h\, dx\end{aligned} \tag{2.3.26}$$

It is clear from this equation that the directional derivatives of the repeated eigenvalues in general are not linear in δh, hence they are not differentiable.

VIBRATION OF A MEMBRANE

Consider a vibrating membrane with variable mass density $h(x)$ and $\hat{T}$ as design variables. Without repeating the definition of the energy and mass bilinear forms of Eq. (2.1.56), write the first variations of these bilinear forms, evaluated at $\bar{y} = y$, as

$$\begin{aligned} a'_{\delta u}(y, y) &= \delta\hat{T} \iint_\Omega (y_1^2 + y_2^2)\, d\Omega \\ d'_{\delta u}(y, y) &= \iint_\Omega y^2\, \delta h\, d\Omega \end{aligned} \tag{2.3.27}$$

For a simple eigenvalue, Eqs. (2.3.7) and (2.3.27) yield

$$\zeta' = \delta\hat{T} \iint_\Omega (y_1^2 + y_2^2)\, d\Omega - \zeta \iint_\Omega y^2\, \delta h\, d\Omega \tag{2.3.28}$$

As in the vibrating string problem, it is clear that the frequency increases with increasing tension and decreases with increasing density.

For the repeated eigenvalue case [42], if y^1 and y^2 are eigenfunctions corresponding to a repeated eigenvalue ζ, then from Eq. (2.3.11)

$$\mathcal{M}_{ij} = \delta\hat{T} \iint_\Omega \nabla y^{i^T} \nabla y^j\, d\Omega - \zeta \iint_\Omega y^i y^j\, \delta h\, d\Omega, \qquad i, j = 1, 2 \tag{2.3.29}$$

The angle of rotation required is obtained from Eqs. (2.3.14) and (2.3.29) as

$$\phi(\delta h) = \frac{1}{2} \arctan\left[\frac{2[\delta\hat{T} \iint_\Omega \nabla y^{1^T} \nabla y^2\, d\Omega - \zeta \iint_\Omega y^1 y^2\, \delta h\, d\Omega]}{\delta\hat{T} \iint [(\nabla y^1)^2 - (\nabla y^2)^2]\, d\Omega - \zeta \iint_\Omega [(y^1)^2 - (y^2)^2]\, \delta h\, d\Omega}\right] \tag{2.3.30}$$

The directional derivatives of the repeated eigenvalue are then given by Eqs. (2.3.12) and (2.3.13) as

$$
\begin{aligned}
\zeta_1'(u, \delta u) = \delta\hat{T} \iint_\Omega & [(\nabla y^1)^2 \cos^2\phi(\delta u) + (\nabla y^{1^T} \nabla y^2) \sin 2\phi(\delta u) \\
& + (\nabla y^2)^2 \sin^2\phi(\delta u)]\, d\Omega \\
- \zeta \iint_\Omega & [(y^1)^2 \cos^2\phi(\delta u) + (y^1 y^2) \sin 2\phi(\delta u) \\
& + (y^2)^2 \sin^2\phi(\delta u)]\, \delta h\, d\Omega \\
\zeta_2'(u, \delta u) = \delta\hat{T} \iint_\Omega & [(\nabla y^1)^2 \sin^2\phi(\delta u) - (\nabla y^{1^T} \nabla y^2) \sin 2\phi(\delta u) \\
& + (\nabla y^2)^2 \cos^2\phi(\delta u)]\, d\Omega \\
- \zeta \iint_\Omega & [(y^1)^2 \sin^2\phi(\delta u) - (y^1 y^2) \sin 2\phi(\delta u) \\
& + (y^2)^2 \cos^2\phi(\delta u)]\, \delta h\, d\Omega
\end{aligned}
\tag{2.3.31}
$$

VIBRATION OF A PLATE

Consider, as a final example, the variable thickness vibrating plate of Fig. 2.1.8, with thickness variation $h(x)$, Young's modulus E, and mass density ρ as design variables. Without repeating the definition of the energy and mass bilinear forms of Eq. (2.1.59), the first variations of these bilinear forms, evaluated at $\bar{y} = y$, may be written as

$$
\begin{aligned}
a'_{\delta u}(y, y) &= \frac{\delta E}{12(1 - \nu^2)} \iint_\Omega h^3 [y_{11}^2 + 2\nu y_{11} y_{22} + y_{22}^2 + 2(1 - \nu) y_{12}^2]\, d\Omega \\
&\quad + \frac{E}{4(1 - \nu^2)} \iint_\Omega h^2 [y_{11}^2 + 2\nu y_{11} y_{22} + y_{22}^2 + 2(1 - \nu) y_{12}^2]\, \delta h\, d\Omega \\
d'_{\delta u}(y, y) &= \delta\rho \iint_\Omega h y^2\, d\Omega + \rho \iint_\Omega y^2\, \delta h\, d\Omega
\end{aligned}
\tag{2.3.32}
$$

The derivative of a simple eigenvalue is therefore given by Eq. (2.3.7) as

$$
\begin{aligned}
\zeta' &= \left\{ \frac{1}{12(1 - \nu^2)} \iint_\Omega h^3 [y_{11}^2 + 2\nu y_{11} y_{22} + y_{22}^2 + 2(1 - \nu) y_{12}^2]\, d\Omega \right\} \delta E \\
&\quad - \left[\zeta \iint_\Omega h y^2\, d\Omega \right] \delta\rho \\
&\quad + \iint_\Omega \left\{ \frac{E h^2}{4(1 - \nu^2)} [y_{11}^2 + 2\nu y_{11} y_{22} + y_{22}^2 + 2(1 - \nu) y_{12}^2] - \zeta \rho y^2 \right\} \delta h\, d\Omega
\end{aligned}
\tag{2.3.33}
$$

It is clear that increasing Young's modulus increases natural frequency, and increasing density decreases natural frequency, as is expected intuitively. The effect of a thickness variation, however, is not obvious, since the coefficient of δh in the integral may be either positive or negative. Numerical examples of the effect of thickness variation are considered in Section 2.3.4.

For the repeated eigenvalue case [43], if y^1 and y^2 are eigenfunctions corresponding to a repeated eigenvalue ζ, then as in the membrane problem, Eqs. (2.3.12) and (2.3.13) give the directional derivatives of the repeated eigenvalues, where the rotation parameter ϕ is given by Eq. (2.3.14) and

$$\begin{aligned}\mathscr{M}_{ij} = {} & \frac{\delta E}{12(1-\nu^2)} \iint_\Omega h^3[(y^i_{11} + \nu y^i_{22})y^j_{11} + (y^i_{22} + \nu y^i_{11})y^j_{22} \\ & \qquad + 2(1-\nu)y^i_{12}y^j_{12}]\, d\Omega \\ & + \frac{E}{4(1-\nu^2)} \iint_\Omega h^2[(y^i_{11} + \nu y^i_{22})y^j_{11} + (y^i_{22} + \nu y^i_{11})y^j_{22} \\ & \qquad + 2(1-\nu)y^i_{12}y^j_{12}]\, \delta h\, d\Omega \\ & - \zeta\left[\delta\rho \iint_\Omega h y^i y^j\, d\Omega + \rho \iint_\Omega y^i y^j\, \delta h\, d\Omega\right], \qquad i,j = 1,2\end{aligned} \tag{2.3.34}$$

2.3.3 Numerical Considerations

Numerical aspects of evaluating design sensitivity formulas in Eq. (2.3.7) for the simple eigenvalue, or Eqs. (2.3.12) and (2.3.13) for the repeated eigenvalue case, follow the same pattern as considerations presented in Section 2.2.4 on computational aspects of static design sensitivity. In the case of conservative eigenvalue problems, however, the attractive feature arises that an adjoint variable need not be calculated as the solution of a separate problem. This feature of the eigenvalue problem allows direct computation of design sensitivity once the analysis problem has been solved. It may be noted that for nonconservative problems this is not the case [43, 44].

Much as in the case of static response presented in Section 2.2.4, the design can be parameterized and explicit design derivatives of eigenvalues obtained with respect to design parameters. Since the functionals arising in the present formulation are identical to those appearing in Section 2.2, the reader is referred to Section 2.2.4 for details on numerical aspects of evaluating parameterized design sensitivity.

2.3.4 Numerical Examples

PLATE

Consider application of Eq. (2.3.33) to account for the effect of variations in plate thickness on natural frequency of the plate. As an example, a clamped square plate of dimension 1 m and uniform thickness $h = 0.05$ m, with $E = 200$ GPa, $\nu = 0.3$, and $\rho = 7870$ kg/m^3, is considered. For the given design, the eigenvalue is $\zeta = 0.3687 \times 10^6$ (rad/sec)^2. If thickness of the plate is considered as a design variable and piecewise-constant thickness is assumed, as in the plate example in Section 2.2.5, then Eq. (2.3.33) can be rewritten as

$$\zeta' = \sum_{i=1}^{n} \left[\iint_{\Omega_i} \left\{ \frac{Eb_i^2}{4(1-\nu^2)} [y_{11}^2 + 2\nu y_{11} y_{22} + y_{22}^2 + 2(1-\nu) y_{12}^2] - \zeta \rho y^2 \right\} d\Omega \right] \delta b_i \tag{2.3.35}$$

As in the plate example in Section 2.2.5, a nonconforming rectangular plate element with 12 degrees of freedom is used. The graph of the coefficient of δb_i presented in Fig. 2.3.5 shows how addition or deletion of material effects the eigenvalue. The maximum value of the coefficient of δb_i is $\Lambda_{max} = 7.949 \times 10^6$ at the middle of edge elements, while its minimum value occurs at the corner elements, with the value $\Lambda_{min} = 3.365 \times 10^3$. Thus, in order to increase the eigenvalue most effectively, material should be removed from the vicinity of the four corners and added to the middle of the four edges.

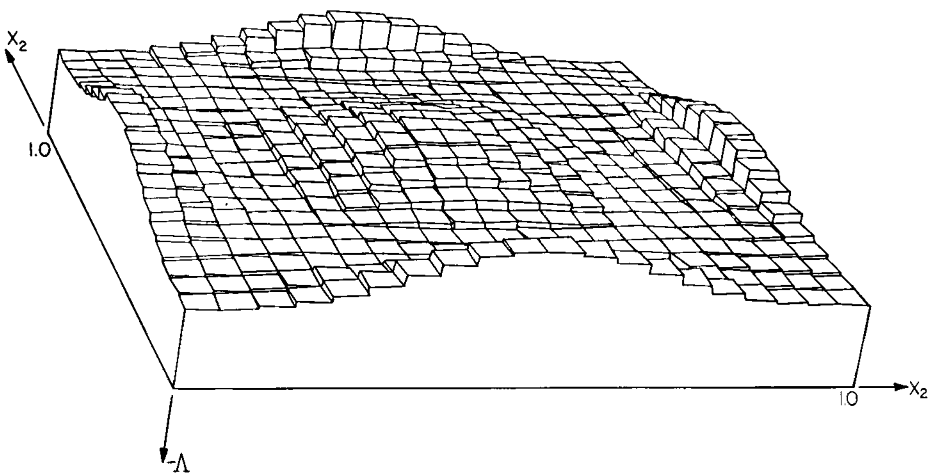

Fig. 2.3.5 Eigenvalue sensitivity Λ for plate.

2.4 FRÉCHET DIFFERENTIABILITY OF THE INVERSE STATE OPERATOR WITH RESPECT TO DESIGN

The purpose of this section is to prove that the inverse state operator, hence the solution of the equations of structural mechanics, is differentiable with respect to design. This section and Section 2.5 are more mathematically technical than others in the text. They are intended for the mathematician or mathematically oriented engineer who loses sleep worrying about technical matters such as differentiability. The engineer interested only in using the methods and results of Sections 2.2 and 2.3 need not go through the details of this section.

2.4.1 Differentiability of Energy Bilinear Forms

In each of the problems of Section 2.1.1, a displacement state variable $z(x; u)$ is determined as the solution of an operator equation of the form

$$A_u z = f, \qquad z \in D(A_u) \tag{2.4.1}$$

where $D(A_u)$ is a subspace of an appropriate Sobolev space Z that is defined by boundary conditions of the problem: $H_0^2(\Omega)$ for the beam and plate with Dirichlet boundary conditions and $H_0^1(\Omega) \times H_0^1(\Omega)$ for the fixed boundary, plane elasticity problem. More generally [35], the subspace Z of the appropriate Sobolev space consists of those displacement fields that satisfy *kinematic boundary conditions* and not necessarily *natural boundary conditions* [32]. The subspace $D(A_u)$, called the *domain of the operator* A_u, is just the restriction to $z \in Z$ such that $A_u z \in L^2(\Omega)$. The forcing function f is in $L^2(\Omega)$. In each problem of Section 2.1.1, the following properties hold:

1. The domain $D(A_u)$ is a subspace of a Sobolev space Z, inheriting its scalar product and norm.
2. The domain $D(A_u)$ is a subspace of $L^2(\Omega)$, and by the Sobolev imbedding theorem [36, Theorem 5.4], the identity operator from $D(A_u)$ into $L^2(\Omega)$ is compact.
3. The operator A_u: $D(A_u) \to L^2(\Omega)$ is self-adjoint and strongly elliptic; that is, there is a constant $\gamma > 0$ such that

$$(A_u z, z) \geq \gamma \|z\|_Z^2 \qquad \text{for all} \quad z \in D(A_u)$$

This property holds for all values of design variables and domains under consideration.

Under these hypotheses, it was proved by Aubin [9] that the solution of Eq. (2.4.1) is the unique solution of the variational (or virtual work) equation

$$a_u(z, \bar{z}) = (f, \bar{z}) \qquad \text{for all} \quad \bar{z} \in Z \tag{2.4.2}$$

where $a_u(\cdot, \cdot)$ is the symmetric energy bilinear form that defines the *Friedrichs extension* A_u of the formal operator $\bar{A}_u$ of Section 2.1.1. It is also the virtual work of the internal forces associated with the displacement field $z \in Z$ due to a virtual displacement $\bar{z} \in Z$.

THEOREM 2.4.1* (Differentiability Theorem for Bilinear Forms) Let $a_u(z, z) \equiv a_u(z)$ be the energy quadratic form associated with $a_u(\cdot, \cdot)$ (twice the strain energy due to $z \in Z$). Each bilinear form $a_u(\cdot, \cdot)$ of Section 2.1.1 is *Fréchet differentiable* with respect to u, in the sense of relatively bounded perturbations; that is, the *Fréchet differential* of $a_u(\cdot, \cdot)$ is a linear form $a'_{\delta u}(\cdot, \cdot)$ in δu such that with $a'_{\delta u}(z) \equiv a'_{\delta u}(z, z)$,

$$|a'_{\delta u}(z)| \leq e_1(u) \, \|\delta u\| \, a_u(z) \qquad \text{for all} \quad \textit{small} \ \|\delta u\| \tag{2.4.3}$$

where $e_1(u)$ is a constant that depends on u, and with the remainder defined as

$$a^{\mathrm{r}}_{\delta u}(z) \equiv a_{u+\delta u}(z) - a_u(z) - a'_{\delta u}(z) \tag{2.4.4}$$

the bound

$$|a^{\mathrm{r}}_{\delta u}(z)| \leq e_2(u, \delta u) \, \|\delta u\| \, a_u(z) \tag{2.4.5}$$

is valid, with $e_2(u, \delta u) \to 0$ as $\|\delta u\| \to 0$.

PROOF To prove the theorem, each structural component is considered individually.

Beam. Consider the beam with admissible design space

$$U = \{u = [E \quad h]^{\mathrm{T}} \in R \times L^{\infty}(\Omega) \colon \ E \geq E_0 > 1, h(x) \geq h_0 > 0 \text{ a.e. in } (0, 1)\}$$

A formal first variation calculation, as in Eq. (2.2.18), yields

$$a'_{\delta u}(z, \bar{z}) = (\delta E)\alpha \int_0^1 h^2 z_{xx} \bar{z}_{xx} \, dx + E\alpha \int_0^1 2h \, \delta h \, z_{xx} \bar{z}_{xx} \, dx \tag{2.4.6}$$

With $\|\delta u\| = |\delta E| + \|\delta h\|_{L^\infty}$, one has

$$\begin{aligned} |a'_{\delta u}(z)| \leq{}& \frac{1}{E_0}\left[E\alpha \int_0^1 h^2 (z_{xx})^2 \, dx\right] |\delta E| \\ &+ \frac{2}{h_0}\left[E\alpha \int_0^1 h^2 (z_{xx})^2 \, dx\right] \|\delta h\|_{L^\infty} \leq \max\left(\frac{1}{E_0}, \frac{2}{h_0}\right) \|\delta u\| \, a_u(z) \end{aligned}$$

Writing the remainder term of Eq. (2.4.4),

$$|a^{\mathrm{r}}_{\delta u}(z)| = \left|\alpha \int_0^1 [E(\delta h)^2 + 2\,\delta E\, h\, \delta h + \delta E\,(\delta h)^2](z_{xx})^2\, dx\right|$$

$$\le \left[\frac{\|(\delta h)^2\|_{L^\infty}}{h_0^2} + \frac{2|\delta E|\,\|\delta h\|_{L^\infty}}{E_0 h_0} + \frac{|\delta E|\,\|(\delta h)^2\|_{L^\infty}}{E_0 h_0^2}\right]\int_0^1 E\alpha h^2 (z_{xx})^2\, dx$$

$$\le \|\delta h\|_{L^\infty}\left[\frac{2\,\|\delta h\|_{L^\infty}}{h_0^2} + \frac{2|\delta E|}{E_0 h_0}\right] a_u(z)$$

if $\|\delta u\| < 1$ and $E_0 > 1$. Finally,

$$|a^{\mathrm{r}}_{\delta u}(z)| \le 2\,\|\delta h\|_{L^\infty} \max\left(\frac{1}{h_0^2}, \frac{1}{E_0 h_0}\right) \|\delta u\|\; a_u(z)$$

This establishes Eqs. (2.4.3) and (2.4.5), since

$$2\,\|\delta h\|_{L^\infty} \max\left(\frac{1}{h_0^2}, \frac{1}{E_0 h_0}\right) \to 0 \quad \text{as} \quad \|\delta u\| \to 0$$

Plate. Consider the plate with admissible design space

$$U = \{u = [E \quad h]^{\mathrm{T}} \in R \times L^\infty(\Omega)\colon\ E \ge E_0 > 0,\ h(x) \ge h_0 > 0 \text{ a.e. in } \Omega\}$$

A formal variational calculation, as in Eq. (2.2.50), leads to

$$a'_{\delta u}(z, \bar{z}) = \iint_\Omega \left[\delta E \frac{h^3}{12(1 - \nu^2)} + \delta h \frac{3Eh^2}{12(1 - \nu^2)}\right]$$
$$\times [z_{11}\bar{z}_{11} + \nu(z_{22}\bar{z}_{11} + \bar{z}_{22}z_{11}) + z_{22}\bar{z}_{22} + 2(1 - \nu)z_{12}\bar{z}_{12}]\, d\Omega \tag{2.4.7}$$

with $\|\delta u\| = |\delta E| + \|\delta h\|_{L^\infty}$, and since a_u is linear in E,

$$|a'_{\delta u}(z)| \le \max\left(\frac{1}{E_0}, \frac{3}{h_0}\right) a_u(z)\,\|\delta u\|$$

and

$$|a^{\mathrm{r}}_{\delta u}(z)| = \left|\iint_\Omega \frac{1}{12(1 - \nu^2)}[3Eh(\delta h)^2 + E(\delta h)^3 + 3h^2\,\delta h\,\delta E + 3h\,\delta E\,(\delta h)^2 + \delta E\,(\delta h)^3] \times [(z_{11})^2 + 2\nu z_{22}z_{11} + (z_{22})^2 + 2(1 - \nu)(z_{12})^2]\, d\Omega\right|$$

$$\le (C\,\|\delta h\|^2_{L^\infty}) a_u(z)$$

$$\le C\,\|\delta h\|_{L^\infty}\,\|\delta u\|\, a_u(z)$$

for sufficiently small $\|\delta u\|$. Since $\|\delta h\|_{L^\infty} \to 0$ as $\|\delta u\| \to 0$, Eqs. (2.4.3) and (2.4.5) are verified.

Plane Elasticity. Consider a plane elastic slab with admissible design space $U = \{u = h \in L^\infty(\Omega): h(x) \geq h_0 > 0 \text{ a.e. in } \Omega\}$. A formal variational calculation, as in Eq. (2.2.74), leads to

$$a'_{\delta u}(z, \bar{z}) = \iint_\Omega \left[\sum_{i,j=1}^{2} \sigma^{ij}(z)\varepsilon^{ij}(\bar{z}) \right] \delta h \, d\Omega \tag{2.4.8}$$

$$|a'_{\delta u}(z)| \leq \frac{1}{h_0} \|\delta h\|_{L^\infty} a_u(z)$$

so Eq. (2.4.3) holds. Also, since $a_u(z)$ is linear in u, $a^{r}_{\delta u}(z) = 0$, and Eq. (2.4.5) holds trivially. This completes the proof. ■

In Section 2.1.1 and above, the consistency of operator and bilinear functional properties for a class of structural systems governed by linear elliptic boundary-value problems is apparent. In the next section, these properties are used to prove that the inverse operator associated with Eq. (2.4.1) is Fréchet differentiable (Appendix A.3) with respect to the design vector u.

2.4.2 Differentiability of Inverse State Operator

THEOREM 2.4.2 (Fréchet Differentiability of the Inverse State Operator) Let the operator A_u and bilinear form $a_u(\cdot, \cdot)$ correspond through the identity

$$a_u(z, \bar{z}) = (A_u z, \bar{z})_{L^2} \tag{2.4.9}$$

for all $z \in D(A_u)$ and all $\bar{z} \in Z$, where the space Z is dense in $L^2(\Omega)$. Further, let

$$a_u(z) \geq \gamma \|z\|_Z^2 \tag{2.4.10}$$

for $\gamma > 0$ and for all $z \in Z$. Let $a'_{\delta u}(z)$ be the *Fréchet differential* of $a_u(z)$ with respect to u, with the property

$$|a'_{\delta u}(z)| \leq e_1(u) \|\delta u\| \, a_u(z) \tag{2.4.11}$$

where $e_1(u)$ is a constant that depends on u, and let $a^{r}_{\delta u}(z)$ be the remainder term with the property

$$\begin{aligned} |a^{r}_{\delta u}(z)| &= |a_{u+\delta u}(z) - a_u(z) - a'_{\delta u}(z)| \\ &\leq e_2(u, \delta u) \|\delta u\| \, a_u(z) \end{aligned} \tag{2.4.12}$$

where $e_2(u, \delta u) \to 0$ as $\|\delta u\| \to 0$. Then the *Fréchet derivative* of A_u^{-1} with respect to u exists and is given as

$$A_{\delta u}^{-1'} = -G_u^{-1}C_1(u, \delta u)G_u^{-1} \tag{2.4.13}$$

where the self-adjoint and invertible operator G_u and continuous operator $C_1(u, \delta u)$ are defined as part of the proof that follows.

PROOF As a result of Eq. (2.4.10) (Kato, [13], Thm. V.3.35), there is a nonnegative, self-adjoint invertible operator G_u (the square root of A_u) such that $G_u G_u = A_u$ and

$$a_u(z, \bar{z}) = (G_u z, G_u \bar{z}) \tag{2.4.14}$$

The domain of G_u is exactly the domain of the bilinear form $a_u(\cdot, \cdot)$, which contains $D(A_u)$. Further, as a result of Eqs. (2.4.11) and (2.4.12) and Lemma VI.3.1 of Kato [13], there are continuous operators $C_1(u, \delta u)$ and $C_2(u, \delta u)$ from $L^2(\Omega)$ into $L^2(\Omega)$ such that

$$a'_{\delta u}(z, \bar{z}) = (C_1(u, \delta u)G_u z, G_u \bar{z}) \tag{2.4.15}$$

$$a^r_{\delta u}(z, \bar{z}) = (C_2(u, \delta u)G_u z, G_u \bar{z}) \tag{2.4.16}$$

where $C_1(u, \delta u)$ is linear in δu, since $a'_{\delta u}(z, \bar{z})$ is linear in δu, and the norms of $C_1(u, \delta u)$ and $C_2(u, \delta u)$ are bounded by

$$\begin{aligned} \|C_1(u, \delta u)\| &\le e_1(u)\,\|\delta u\| \\ \|C_2(u, \delta u)\| &\le e_2(u, \delta u)\,\|\delta u\| \end{aligned} \tag{2.4.17}$$

Note that for every z in the domain of $A_{u+\delta u}$,

$$\begin{aligned} (A_{u+\delta u} z, \bar{z}) &= a_{u+\delta u}(z, \bar{z}) = a_u(z, \bar{z}) + a'_{\delta u}(z, \bar{z}) + a^r_{\delta u}(z, \bar{z}) \\ &= ((G_u + C_1 G_u + C_2 G_u)z, G_u \bar{z}) \\ &= (G_u(I + C_1 + C_2)G_u z, \bar{z}) \end{aligned} \tag{2.4.18}$$

for all $\bar{z} \in Z$, where the arguments of the operators $C_1(u, \delta u)$ and $C_2(u, \delta u)$ have been suppressed for notational convenience. Since Z is dense in $L^2(\Omega)$ and from the definition of Z,†

$$A_{u+\delta u} = G_u(I + C_1 + C_2)G_u \tag{2.4.19}$$

Since

$$\|C_1 + C_2\| \le \|C_1\| + \|C_2\| \le [e_1(u) + e_2(u, \delta u)]\,\|\delta u\|$$

† From Eq. (2.4.18), $D(A_{u+\delta u}) \subset D(G_u(I + C_1 + C_2)G_u)$, but $A_{u+\delta u}$ is defined to be the maximal operator such that the first equality in Eq. (2.4.18) holds. Thus, the domains are in fact equal.

for sufficiently small $\|\delta u\|$, $I + C_1 + C_2$ has an inverse. Thus,

$$A_{u+\delta u}^{-1} = G_u^{-1}(I + C_1 + C_2)^{-1}G_u^{-1} \tag{2.4.20}$$

Note that

$$A_{u+\delta u}^{-1} - A_u^{-1} + G_u^{-1}C_1^{-1}G_u^{-1} = G_u^{-1}[(I + C_1 + C_2)^{-1} - I]G_u^{-1} + G_u^{-1}C_1G_u^{-1} \tag{2.4.21}$$

Thus

$$\|A_{u+\delta u}^{-1} - A_u^{-1} + G_u^{-1}C_1G_u^{-1}\| \leq \|G_u^{-1}\|^2 \|(I + C_1 + C_2)^{-1} - I + C_1\| \tag{2.4.22}$$

Manipulating and applying the triangle inequality,

$$\begin{aligned}\|(I + C_1 + C_2)^{-1} - I + C_1\| &= \|(I + C_1 + C_2)^{-1} - I + C_1 + C_2 - C_2\| \\ &\leq \|(I + C_1 + C_2)^{-1} - I + C_1 + C_2\| + \|C_2\|\end{aligned} \tag{2.4.23}$$

For $\|\delta u\|$ sufficiently small, $\|C_1 + C_2\| < \frac{1}{2}$, and the theory of Neumann series [13] can be applied to obtain the bound

$$\begin{aligned}\|(I + C_1 + C_2)^{-1} - I + C_1 + C_2\| &\leq \frac{\|C_1 + C_2\|^2}{1 - \|C_1 + C_2\|} \\ &\leq 2\|C_1 + C_2\|^2\end{aligned}$$

The triangle inequality now yields

$$\|(I + C_1 + C_2)^{-1} - I + C_1 + C_2\| \leq 2\|C_1\|^2 + 4\|C_1\|\,\|C_2\| + 2\|C_2\|^2 \tag{2.4.24}$$

Thus, Eqs. (2.4.11), (2.4.12), (2.4.16), (2.4.21), and (2.4.23) yield

$$\begin{aligned}&\|A_{u+\delta u}^{-1} - A_u^{-1} + G_u^{-1}C_1(u, \delta u)G_u^{-1}\| \\ &\quad \leq \|G_u^{-1}\|^2\{2[e_1^2(u) + 2e_1(u)e_2(u, \delta u) + e_2^2(u, \delta u)]\,\|\delta u\|^2 + e_2(u, \delta u)\,\|\delta u\|\}\end{aligned} \tag{2.4.25}$$

$$\begin{aligned}&= \|G_u^{-1}\|^2\{2[e_1^2(u) + 2e_1(u)e_2(u, \delta u) + e_2^2(u, \delta u)]\,\|\delta u\| + e_2(u, \delta u)\}\,\|\delta u\| \\ &\equiv e_3(u, \delta u)\,\|\delta u\|\end{aligned} \tag{2.4.26}$$

and $e_3(u, \delta u) \to 0$ as $\|\delta u\| \to 0$. Thus, A_u^{-1} is Fréchet differentiable, with the differential $-G_u^{-1}C_1(u, \delta u)G_u^{-1}$, and the proof is complete. ■

2.4.3 Differentiability of Static Response

As shown in Section 2.1.1, the solution of the structural equation is

$$z(x; u) = A_u^{-1}f(u) \tag{2.4.27}$$

where $f \in L^2(\Omega)$. Theorem 2.4.2 establishes differentiability of A_u^{-1}. If f depends on u in a differentiable way, then the product rule of differentiation establishes differentiability of z. Since $f(u)$ is a mapping from the space of designs, $f(u)$ is Fréchet differentiable if there exists a mapping $f'_{\delta u}$ that is linear in δu and

$$\|f(u + \delta u) - f(u) - f'_{\delta u}\|_{L^2} \leq e_4(u, \delta u)\ \|\delta u\| \tag{2.4.28}$$

where $e_4(u, \delta u) \to 0$ as $\|\delta u\| \to 0$.

To see that load dependence can be Fréchet differentiable, consider the self-weight loads on the beam in Eq. (2.2.15) and on the plate in Eq. (2.2.41), both of the form

$$f(u) = F(x) + \gamma h(x) \tag{2.4.29}$$

where $u = h$. Formally, one obtains the first variation as

$$f'_{\delta h} = \frac{d}{d\tau}[F + \gamma(h + \tau\,\delta h)]\bigg|_{\tau=0} = \gamma\,\delta h \tag{2.4.30}$$

To see that Eq. (2.4.28) is satisfied, note that

$$\|f(h + \delta h) - f(h) - f'_{\delta h}\| = \|F + \gamma(h + \delta h) - F - \gamma h - \gamma\,\delta h\| = 0$$

The chain rule of differentiation establishes the following result.

THEOREM 2.4.3 (Fréchet Differentiability of Static Response) Let the operator A_u and associated energy bilinear form $a_u(\cdot, \cdot)$ satisfy the hypotheses of Theorem 2.4.2, and let $f(u)$ be Fréchet differentiable. Then $z(x; u)$ is Fréchet differentiable, with differential

$$\begin{aligned} z' &= A_{\delta u}^{-1\prime} f + A_u^{-1} f'_{\delta u} \\ &= -G_u^{-1} C_1(u, \delta u) G_u^{-1} f + A_u^{-1} f'_{\delta u} \end{aligned} \tag{2.4.31}$$

where operators G_u and C_1 are defined in the proof of Theorem 2.4.2.

The importance of this result is theoretical at this point, since the explicit forms of G_u, G_u^{-1}, and $C_1(u, \delta u)$ are not known and in fact may not be readily computable. Computation of explicit design derivatives of functionals involved in a variety of structural problems is carried out using the adjoint variable method in Section 2.2.2. An alternative derivation is given in Section 2.4.4.

An extension of the results presented here to forcing functions $f(x)$ that are not in $L^2(\Omega)$ is presented by Haug and Rousselet [46]. This is of value for problems in which applied loads are modeled as concentrated loads and moments. These forcing functions must be viewed as distributions, or bounded linear functionals, acting on displacement fields in $H_0^1(\Omega)$ or $H_0^2(\Omega)$.

These spaces of distributions are denoted $H^{-1}(\Omega)$ and $H^{-2}(\Omega)$, so the operators A_u must be viewed as mappings from $H^i(\Omega)$ to $H^{-i}(\Omega)$ ($i = 1$ or 2). While the analysis presented by Haug and Rousselet [46] for this extension is technically more complex, the same results proved here for $f \in L^2(\Omega)$ are shown to be valid. For illustrations, see Section 2.2.3.

2.4.4 An Alternative Derivation of Design Sensitivity

Consider a typical functional

$$\psi = \psi(u, z) \tag{2.4.32}$$

that is a differentiable mapping from $U \times Z$ into the reals, where the L^2 Hilbert space structure is employed on U and Z. The differential of ψ is

$$\psi' = (\psi_u, \delta u) + (\psi_z, z') \tag{2.4.33}$$

Using the result of Eq. (2.4.31),

$$\psi' = (\psi_u, \delta u) - (\psi_z, G_u^{-1} C_1(u, \delta u) G_u^{-1} f) + (\psi_z, A_u^{-1} f'_{\delta u})$$

Using self-adjointness of G_u and the fact that $G_u G_u = A_u$, manipulation yields

$$\begin{aligned} \delta\psi &= (\psi_u, \delta u) - (G_u^{-1}\psi_z, C_1(u, \delta u) G_u^{-1} f) + (G_u^{-1}\psi_z, G_u^{-1} f'_{\delta u}) \\ &= (\psi_u, \delta u) - (G_u A_u^{-1}\psi_z, C_1(u, \delta u) G_u A_u^{-1} f) + (G_u^{-1} G_u^{-1} \psi_z, f'_{\delta u}) \\ &= (\psi_u, \delta u) - (G_u A_u^{-1}\psi_z, C_1(u, \delta u) G_u z) + (A_u^{-1}\psi_z, f'_{\delta u}) \end{aligned} \tag{2.4.34}$$

Defining $A_u^{-1}\psi_z = \lambda$, or equivalently, λ as the solution of the adjoint equation

$$A_u \lambda = \psi_z \tag{2.4.35}$$

and using Eq. (2.4.15), Eq. (2.4.34) may be rewritten as

$$\psi' = (\psi_u, \delta u) - a'_{\delta u}(z, \lambda) + (\lambda, f'_{\delta u}) \tag{2.4.36}$$

which is written explicitly in terms of δu. Since the load functional in Section 2.1.1 is $l_u(\bar{z}) = (f(u), \bar{z})$, then $l'_{\delta u}(\lambda) = (f'_{\delta u}, \lambda)$, and Eq. (2.4.36) is exactly the same as the result obtained in Eq. (2.2.14) by a different approach.

2.5 DIFFERENTIABILITY OF EIGENVALUES

Consider now the variational eigenvalue problem with an eigenvalue ζ that is repeated m times; that is, there exist $y^i \in Z$ ($i = 1, \ldots, m$) such that

$$a_u(y^i, \bar{y}) = \zeta d_u(y^i, \bar{y}) \qquad \text{for all} \quad \bar{y} \in Z, \quad i = 1, \ldots, m \tag{2.5.1}$$

and

$$d_u(y^i, y^j) = \delta_{ij}, \qquad i, j = 1, \ldots, m \tag{2.5.2}$$

where $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$.

It is clear that changing u changes $\zeta = \zeta(u)$. The question addressed here is twofold: How regular is ζ as a function of u? And how can derivatives of ζ be calculated, presuming they exist?

2.5.1 Differentiability of Energy Bilinear Forms

Differentiability of the bilinear form $a_u(y, \bar{y})$ for the beam, plate, and plane elasticity operators is demonstrated in Section 2.4, where formulas for the differential $a'_{\delta u}(y, \bar{y})$ are given. These derivations may be easily repeated for $a_u(\cdot, \cdot)$ associated with the string and membrane and for $d_u(\cdot, \cdot)$ in each of the examples discussed in Section 2.1.2. Formulas for $a'_{\delta u}(y, \bar{y})$ and $d'_{\delta u}(y, \bar{y})$ are as follows:

Vibrating string, $u = [h(x) \quad \hat{T}]^{\mathrm{T}}$:

$$a'_{\delta u}(y, \bar{y}) = \delta\hat{T} \int_0^1 y_x \bar{y}_x \, dx \tag{2.5.3}$$

$$d'_{\delta u}(y, \bar{y}) = \int_0^1 (\delta h) \, y\bar{y} \, dx \tag{2.5.4}$$

Vibrating beam, $u = [h(x) \quad E \quad \rho]^{\mathrm{T}}$:

$$a'_{\delta u}(y, \bar{y}) = (\delta E) \, \alpha \int_0^1 h^2 y_{xx} \bar{y}_{xx} \, dx + E\alpha \int_0^1 2h \, (\delta h) \, y_{xx} \bar{y}_{xx} \, dx \tag{2.5.5}$$

$$d'_{\delta u}(y, \bar{y}) = \delta\rho \int_0^1 hy\bar{y} \, dx + \rho \int_0^1 (\delta h) \, y\bar{y} \, dx \tag{2.5.6}$$

Buckling column, $u = [h(x) \quad E]^{\mathrm{T}}$:

$$a'_{\delta u}(y, \bar{y}) = (\delta E) \, \alpha \int_0^1 h^2 y_{xx} \bar{y}_{xx} \, dx + E\alpha \int_0^1 2h \, (\delta h) \, y_{xx} \bar{y}_{xx} \, dx \tag{2.5.7}$$

$$d'_{\delta u}(y, \bar{y}) = 0 \tag{2.5.8}$$

Vibrating membrane, $u = [h(x) \quad \hat{T}]^{\mathrm{T}}$:

$$a'_{\delta u}(y, \bar{y}) = \delta\hat{T} \iint_\Omega (y_1 \bar{y}_1 + y_2 \bar{y}_2) \, d\Omega \tag{2.5.9}$$

$$d'_{\delta u}(y, \bar{y}) = \iint_\Omega (\delta h) \, y\bar{y}\Omega \tag{2.5.10}$$

Vibrating plate, $u = [h(x) \quad E \quad \rho]^{\mathrm{T}}$:

$$a'_{\delta u}(y, \bar{y}) = \iint_{\Omega} \left[\frac{h^3\, \delta E}{12(1 - \nu^2)} + \frac{Eh^2\, \delta h}{4(1 - \nu^2)} \right] \times [y_{11}\bar{y}_{11} + \nu(y_{22}\bar{y}_{11} + y_{11}\bar{y}_{22}) + y_{22}\bar{y}_{22} + 2(1 - \nu)y_{12}\bar{y}_{12}]\, d\Omega \tag{2.5.11}$$

$$d'_{\delta u}(y, \bar{y}) = \delta\rho \iint hy\bar{y}\, d\Omega + \rho \iint (\delta h)\, y\bar{y}\, d\Omega \tag{2.5.12}$$

Apart from the algebraic structure of the bilinear forms and their differentials, the mathematical properties of each of the examples is the same. Eigenvalue differentiability is proved using only a common set of operator theoretic properties. The result is applicable to each of the examples discussed here and to any other problem that can be put in the same variational form. As shown by Fichera [35], virtually all problems of linear elasticity fall into this category, as do many other partial differential equations arising in mathematical physics.

The eigenvalue problem is stated in variational form in Eqs. (2.5.1) and (2.5.2). In order to bring to bear powerful results on perturbation of linear operators [13], it is helpful to state the eigenvalue problem in an equivalent operator form. Let A_u and B_u be Friedrichs extensions of the formal operators $\bar{A}_u$ and $\bar{B}_u$ introduced in Section 2.1.2, defined by

$$\begin{aligned} (A_u y, \bar{y}) &= a_u(y, \bar{y}) \qquad \text{for all} \quad \bar{y} \in Z \\ (B_u y, \bar{y}) &= d_u(y, \bar{y}) \qquad \text{for all} \quad \bar{y} \in Z \end{aligned} \tag{2.5.13}$$

where $D(A_u) \subset D(B_u)$ is the domain of the operators. With these operators, the eigenvalue problem of Eqs. (2.5.1) and (2.5.2) can be equivalently written as

$$\begin{aligned} A_u y^i &= \zeta(u) B_u y^i, \qquad i = 1, \ldots, m \\ (B_u y^i, y^j) &= \delta_{ij}, \qquad i, j = 1, \quad j = 1, \ldots, m \end{aligned} \tag{2.5.14}$$

Using the foregoing formulas, it was shown in Section 2.4 that the inverse A_u^{-1} of operator A_u, as a mapping from u to an element of $\mathscr{B}(L^2)$,† is Fréchet differentiable with respect to design. In all examples except the buckling column, the operator B_u is in $\mathscr{B}(L^2)$ and is trivially Fréchet differentiable with respect to design, so that $A_u^{-1} B_u$ is Fréchet differentiable. The operator B_u associated with the column requires a separate analysis, which was presented by Haug and Rousselet [46].

† $\mathscr{B}(L^2)$ denotes the space of bounded operators from L^2 into L^2.

2.5.2 Regularity of the Operator versus Regularity of the Eigenvalues

The repeated eigenvalues occurring in Chapter 1 and the following matrix example (borrowed from Kato [13]), show that analysis of regularity of eigenvalues needs some care and precaution. Consider the symmetric matrix

$$A(x_1, x_2) = \begin{bmatrix} x_1 & x_2 \\ x_2 & -x_1 \end{bmatrix}$$

where $[x_1\ x_2]^T \in R^2$. The matrix A is analytic with respect to $[x_1\ x_2]^T$. However, the eigenvalues $\zeta_\pm = \pm\sqrt{x_1^2 + x_2^2}$ are not Fréchet differentiable at $[0\ 0]^T$, even though A is symmetric. The difficulty arises because A depends on two real parameters. Putting $x_2 = 0$, the eigenvalues of $A(x_1, 0)$ are found to be $\zeta_1 = x_1$ and $\zeta_2 = -x_1$, which are analytic functions of x_1. The partial derivatives with respect to x_1 of these eigenvalues are 1 and -1. Note, however, that by selecting $\zeta_1 = |x_1|$ and $\zeta_2 = -|x_1|$, (ζ_1, ζ_2) are still equal to the eigenvalues of $A(x_1, 0)$, but they are no longer differentiable at $[0\ 0]^T$.

This example shows that more than directional differentiability of repeated eigenvalues can not be expected, which means that differentiable functions of a real parameter can be selected that are equal to the eigenvalues of a self-adjoint operator but which are not differentiable if they are ordered according to increasing magnitude. Setting

$$\hat{\zeta} = \frac{\zeta_+ + \zeta_-}{2} = 0$$

(as introduced by Kato [13]) this is the weighted mean of a clustered group of eigenvalues, which is differentiable at zero (the result is general). This latter property may be considered, together with the fact that $\hat{\zeta}$ is an approximation of the two eigenvalues (for small values of the parameter), to be of numerical interest.

The foregoing considerations suggest first showing the continuity and then the directional differentiability of the eigenvalues.

2.5.3 Continuity of Eigenvalues

THEOREM 2.5.1 (Continuity of eigenvalues) Let $\{\zeta_1, \ldots, \zeta_k\}$ be n eigenvalues (counted with multiplicity such that $n \geq k$) of the generalized eigenvalue problem of Eq. (2.5.14) [equivalently, Eqs. (2.5.1) and (2.5.2)], with operators A_u and B_u. For every neighborhood W of $\{\zeta_1, \ldots, \zeta_k\}$ in the real line that contains no other eigenvalue, there exists a neighborhood S of u (in the space U of the design variables) such that for every $u + \delta u \in S$ there are exactly n eigenvalues of $(A_{u+\delta u}, B_{u+\delta u})$ (counted with their multiplicity) in W.

PROOF Since the mapping of $u \to A_u^{-1}B_u$ is Fréchet differentiable, it is also continuous, in the sense of bounded operators and thus in the sense of generalized convergence of closed operators (Theorem IV.2.23 of Kato [13]). Continuity of eigenvalues of $A_u^{-1}B_u$, as stated in the theorem, then follows from Section IV.3.5 of Kato [13]. The continuity of the eigenvalues of (A_u, B_u) is proved by noting that they are reciprocals of the eigenvalues of $A_u^{-1}B_u$, which are never zero. ■

2.5.4 Differentiability of Eigenvalues

THEOREM 2.5.2 (Differentiability of Eigenvalues) Let the operator A_u have a bounded inverse that is Fréchet differentiable with respect to $u \in U$ and let the operator B_u either be bounded and Fréchet differentiable with respect to u or be unbounded with $B = B_u$ independent of u and $A_u^{-1}B$ bounded. Let $\zeta(u)$ be an eigenvalue of Eq. (2.5.14) [equivalently, of Eqs. (2.5.1) and (2.5.2)] of multiplicity m. Then the group of m eigenvalues of Eq. (2.5.14) associated with operators $A_{u+\delta u}$ and $B_{u+\delta u}$, for $\|\delta u\|$ small enough, is directionally differentiable at u in the direction δu (for any δu) and there exist representations of the eigenvalues $\zeta_j(u + \tau\,\delta u)$ $(j = 1, \ldots, m)$ such that

$$\zeta_j(u + \tau\,\delta u) = \zeta(u) + \tau\zeta_j'(u, \delta u) + o(\tau), \qquad j = 1, \ldots, m \tag{2.5.15}$$

where $\zeta_j'(u, \delta u) = -(\zeta(u))^2\alpha_j'(u, \delta u)$ and $\alpha_j'(u, \delta u)$ are the eigenvalues of

$$P_u\left[\left(\frac{d}{ds}\hat{C}_{u+s\,\delta u}\right)_{s=0}\right]P_u$$

restricted to the subspace of Z that is spanned by the eigenfunctions of $\hat{C}_u$; where $\hat{C}_u = A_u^{-1}B_u$ and P_u is its spectral projector† associated with $\alpha(u) = 1/\zeta(u)$. If $\zeta_j(u + t\,\delta u)$ $(j = 1, \ldots, m)$ are the eigenvalues written in increasing order of magnitude, then Eq. (2.5.15) is valid only for $\tau \geq 0$. Moreover, the directional derivative of the smallest eigenvalue is the smallest of all the directional derivatives of the group of m eigenvalues.

The proof of this theorem is rather technical, involving use of the resolvent $R(\zeta, \hat{C}_u) = (\hat{C}_u - \zeta)^{-1}$, for ζ in the complex plane, and the Dunford integral representation. It is given in Section 2.5.5.

COROLLARY 2.5.1 (Fréchet Derivative of Simple Eigenvalue) Under the hypothesis of Theorem 2.5.2, if $\zeta(u)$ is a simple eigenvalue, it remains simple for τ small enough (by Theorem 2.5.1) and its derivative at $\tau = 0$ is given by

$$\zeta' = a_{\delta u}'(y, y) - \zeta(u)d_{\delta u}'(y, y) \tag{2.5.16}$$

† The spectral projector is the operator projecting Z onto the subspace of Z spanned by eigenfunctions of $\hat{C}_u$ associated with the eigenvalue $\alpha(u)$. Since $\hat{C}_u$ is self-adjoint for the scalar product d_u, this projection is orthogonal with respect to d_u.

where y satisfies

$$\begin{aligned} A_u y &= \zeta(u) B_u y \\ (B_u y, y) &= 1 \end{aligned} \tag{2.5.17}$$

PROOF Note first that

$$A_{u+\tau\delta u} y_{u+\tau\delta u} = \zeta(u + \tau\,\delta u) B_{u+\tau\delta u} y_{u+\tau\delta u}$$

if and only if

$$A_{u+\tau\delta u}^{-1} B_{u+\tau\delta u} y_{u+\tau\delta u} = \frac{1}{\zeta(u + \tau\,\delta u)} y_{u+\tau\delta u}$$

Since A_u and B_u are self-adjoint, $A_u^{-1} B_u$ is self-adjoint for the scalar product $(B_u y, y)$. Thus, for a simple eigenvalue, P_u is the orthogonal projector for the scalar product $(B_u y, y)$ on the line spanned by y; that is, for any $\bar{y} \in Z$,

$$P_u \bar{y} = (B_u \bar{y}, y) y$$

Since the range of P_u is a scalar multiple of y, it is clear that y is an eigenvector of $P_u[(d\hat{C}_{u+s\delta u}/ds)_{s=0}]P_u$; that is,

$$P_u\left[\left(\frac{d}{ds}\hat{C}_{u+s\delta u}\right)_{s=0}\right]P_u y = \lambda y$$

Thus, since $(B_u y, y) = 1$, the eigenvalue of $P_u[(d\hat{C}_{u+s\delta u}/ds)_{s=0}]P_u$, which by Theorem 2.5.2 is $\alpha'(u, \delta u)$, is equal to

$$\alpha'(u, \delta u) = \left(B_u P_u\left[\left(\frac{d}{ds}\hat{C}_{u+s\delta u}\right)_{s=0}\right]P_u y, y\right)$$

Now, Eq. (2.4.13) for the derivative of A_u^{-1} (i.e., $A_{\delta u}^{-1'} = -G_u^{-1} C_1 G_u^{-1}$), noting that $A_u^{-1} G_u = G_u^{-1}$ and that

$$\left(\frac{d}{ds}\hat{C}_{u+s\delta u}\right)_{s=0} = (A_{\delta u}^{-1'}) B_u + (A_u^{-1} B'_{\delta u})$$

leads to the conclusion that,[†]

$$\alpha'(u, \delta u) = (P_u[-A_u^{-1} G_u C_1(u, \delta u) G_u A_u^{-1} B_u + A_u^{-1} B'_{\delta u}]y, B_u y) \tag{2.5.18}$$

Now, for all $\bar{y} \in Z$,

$$(P_u^* B_u y, \bar{y}) \equiv (B_u y, P_u \bar{y}) = (B_u \bar{y}, y)(B_u y, y) = (B_u \bar{y}, y) = (B_u y, \bar{y})$$

[†] For the buckling problem, a generalization presented by Haug and Rousselet [46] is employed, but the same result holds.

and since Z is dense in L^2,

$$P_u^* B_u y = B_u y$$

Thus, Eq. (2.5.18) may be written in the form

$$\begin{aligned}\alpha'(u, \delta u) &= ([-A_u^{-1} G_u C_1(u, \delta u) G_u A_u^{-1} B_u + A^{-1} B'_{\delta u}]y, B_u y) \\ &= (-G_u C_1(u, \delta u) G_u A_u^{-1} B_u y, A_u^{-1} B_u y) + (B'_{\delta u} y, A_u^{-1} B_u y)\end{aligned}$$

Using $A_u^{-1} B_u y = (1/\zeta(u))y$ yields

$$\alpha'(u, \delta u) = \frac{1}{(\zeta(u))^2}(-G_u C_1(u, \delta u) G_u y, y) + \frac{1}{\zeta(u)}(B'_{\delta u} y, y)$$

or in terms of design derivatives of the bilinear forms,

$$\alpha'(u, \delta u) = -\frac{1}{(\zeta(u))^2} a'_{\delta u}(y, y) + \frac{1}{\zeta(u)} d'_{\delta u}(y, y)$$

and Eq. (2.5.16) follows from this last formula, noting that

$$\zeta' = \zeta'(u, \delta u) = -(\zeta(u))^2 \alpha'(u, \delta u) \quad \blacksquare$$

COROLLARY 2.5.2 (Directional Derivatives of Repeated Eigenvalue) Under the hypothesis of Theorem 2.5.2, if $\zeta(u)$ is an m-fold eigenvalue of Eq. (2.5.14) [equivalently, Eqs. (2.5.1) and (2.5.2)], its directional derivatives are the m eigenvalues of the matrix $\mathcal{M}$ with general term

$$\mathcal{M}_{ij} = a'_{\delta u}(y^i, y^j) - \zeta(u) d'_{\delta u}(y^i, y^j), \qquad i, j = 1, \ldots, m \qquad (2.5.19)$$

where y^i satisfies

$$\left.\begin{aligned} A_u y^i &= \zeta(u) B_u y^i, \\ (B_u y^i, y^j) &= \delta_{ij}, \end{aligned}\right\} \qquad i, j = 1, \ldots, m \qquad (2.5.20)$$

PROOF As for simple eigenvalues, P_u is the orthogonal projector for the scalar product $(B_u y, y)$ on the eigenspace associated with $\zeta(u)$ of multiplicity m, where y^i $(i = 1, \ldots, m)$ denotes a basis of this eigenspace that is orthonormal with respect to the $(B_u y, \bar{y})$ scalar product. Thus, for any $\bar{y} \in Z$,

$$P_u \bar{y} = \sum_{i=1}^{m} (B_u \bar{y}, y^i) y^i$$

The eigenvalues of the operator $P_u[(d\hat{C}_{u+s\delta u}/ds)_{s=0}]P_u$ must be found to use Theorem 2.5.2. Note that the range of the operator has dimension m. Hence, the eigenvector $\bar{y}$ corresponding to the eigenvalue $\alpha'(u, \delta u)$ of the operator can be expressed as

$$\bar{y} = \sum_{j=1}^{m} a^j y^j$$

where not all of a^j are zero. Hence,

$$\sum_{j=1}^{m} a^j P_u\left[\left(\frac{d}{ds}\hat{C}_{u+s\delta u}\right)_{s=0}\right]P_u y^j = \alpha'(u, \delta u)\sum_{j=1}^{m} a^j y^j \tag{2.5.21}$$

Taking the scalar product of Eq. (2.5.21) with $B_u y^i$ ($i = 1, 2, \ldots$) gives

$$\sum_{j=1}^{m} a^j\left(B_u P_u\left[\left(\frac{d}{ds}\hat{C}_{u+s\delta u}\right)_{s=0}\right]P_u y^j, y^i\right) = \alpha'(u, \delta u)a^i, \qquad i = 1, 2, \ldots, m \tag{2.5.22}$$

For the above system of equations to have a nontrivial solution $a = [a^1\ a^2\ \cdots\ a^m]^{\mathrm{T}}$, $\alpha'(u, \delta u)$ must be an eigenvalue of the matrix N with components

$$N_{ij} = \left(B_u P_u\left[\left(\frac{d}{ds}\hat{C}_{u+s\delta u}\right)_{s=0}\right]P_u y^j, y^i\right), \qquad i, j = 1, \ldots, m$$

As for the simple eigenvalue case,

$$\left(\frac{d}{ds}\hat{C}_{u+s\delta u}\right)_{s=0} = -G_u^{-1}C_1(u, \delta u)G_u^{-1}B_u + A_u^{-1}B'_{\delta u}$$

so that

$$N_{ij} = (B_u P_u[-G_u^{-1}C_1(u, \delta u)G_u^{-1}B_u + A_u^{-1}B'_{\delta u}]P_u y^j, y^i)$$

For all $\bar{y} \in Z$,

$$\begin{aligned}(P_u^* B_u y^i, \bar{y}) &= (B_u y^i, P_u \bar{y})\\ &= \sum_{j=1}^{m}(B_u y^i, y^j)(B_u \bar{y}, y^j) = (B_u \bar{y}, y^i) = (\bar{y}, B_u y^i)\end{aligned}$$

so that $P_u^* B_u y^i = B_u y^i$, which is clear since P_u is self-adjoint with respect to the $(B_u y, \bar{y})$ scalar product. Thus,

$$\begin{aligned}N_{ij} &= ([-G_u^{-1}C_1(u, \delta u)G_u^{-1}B_u + A_u^{-1}B'_{\delta u}]y^j, B_u y^i)\\ &= (-A_u^{-1}G_u C_1(u, \delta u)G_u A_u^{-1}B_u y^j, B_u y^i) + (A_u^{-1}B'_{\delta u}y^j, B_u y^i)\end{aligned}$$

Using the self-adjointness of A_u^{-1} and

$$A_u^{-1}B_u y^j = \frac{1}{\zeta(u)}y^j$$

yields

$$N_{ij} = \frac{1}{\zeta(u)^2}(-G_u C_1(u, \delta u)G_u y^j, y^i) + \frac{1}{\zeta(u)}(B'_{\delta u}y^j, y^i)$$

By recalling $\zeta' = \zeta'(u, \delta u) = -\zeta(u)^2\alpha'(u, \delta u)$, it is concluded that $\zeta'(u, \delta u)$ are the eigenvalues of the matrix with elements $\mathcal{M}_{ij} = -\zeta(u)^2 N_{ij}$, which gives Eq. (2.5.19). ■

2.5.5 Proof of Theorem 2.5.2

The purpose of this section is to state and prove a proposition that is used to prove Theorem 2.5.2. In the rather technical proof, $\mathscr{B}(X)$ is the set of bounded operators from a Hilbert space X into itself, $\hat{C}(u)$ is an element of $\mathscr{B}(X)$ that depends on an element u of a Banach space U, and $M(u)$ is the invariant subspace associated with $\hat{C}(u)$ and $\zeta(u)$, that is, the subspace of X spanned by the eigenvectors of $\hat{C}(u)$ that are associated with $\zeta(u)$.

PROPOSITION Let W be a zero† neighborhood in a Banach space U and X be a Hilbert space. Consider a Fréchet differentiable mapping from W to $\mathscr{B}(X)$: $u \to \hat{C}(u)$. Let α_0 be an eigenvalue of $\hat{C}(0)$ of multiplicity m and $M(0)$ be the associated invariant subspace. Then there exists a neighborhood $W' \subset W$ and a mapping from W' to $\mathscr{B}(X)$: $u \to \check{C}(u)$ that is Fréchet differentiable at $u = 0$, such that the following hold:

1. $M(0)$ is an invariant subspace associated with $\check{C}(u)$ for all $u \in W'$;
2. the eigenvalues of the restriction of the operator $\check{C}(u)$ to $M(0)$ are equal to the m eigenvalues of $\hat{C}(u)$ neighboring α_0; and $\hat{C}'(0)u = \check{C}'(0)u$, where a prime denotes Fréchet derivative.

PROOF The first step is to show that the spectral projector associated with the eigenvalues neighboring α_0 is continuous and is Fréchet differentiable at zero. $\mathrm{Rs}(\hat{C})$ denotes the resolvent set of the operator $\hat{C}$, and $R(\xi, u)$ is the resolvent of $\hat{C}(u)$. Here, $R(\xi) = R(\xi, 0)$ and $R(\xi, u) = (\hat{C}(u) - \xi I)^{-1}$ for $\xi \in \mathrm{Rs}(\hat{C}(u))$, which is a subset of the complex plane.

It is first to be shown that

$$\|R(\xi, u) - R(\xi) + R(\xi, u)(\hat{C}'(0), u)R(\xi)\| = o(\|u\|_U) \qquad (2.5.23)$$

where $\lim_{\|u\|_U \to 0} o(\|u\|_U)/\|u\|_U = 0$, uniformly in ξ in a compact set of $\mathrm{Rs}(C(0))$.

To show this, note that

$$R(\xi) - R(\xi, u) = R(\xi, u)(\hat{C}(u) - \xi I)R(\xi) - R(\xi, u)(\hat{C}(0) - \xi I)R(\xi)$$

or

$$R(\xi) - R(\xi, u) = R(\xi, u)(\hat{C}(u) - \hat{C}(0))R(\xi) \qquad (2.5.24)$$

† For convenience and without loss of generality, the design perturbation is taken about zero and is denoted as u ($\|u\|$ small), rather than δu.

so

$$R(\xi) = R(\xi, u)[I + (\hat{C}(u) - \hat{C}(0))R(\xi)]$$

and if

$$\|\hat{C}(u) - \hat{C}(0)\| < \frac{1}{\|R(\xi)\|}$$

the inverse of the operator $[I + (\hat{C}(u) - \hat{C}(0))R(\xi)]$ exists, so

$$R(\xi, u) = R(\xi)[I + (\hat{C}(u) - \hat{C}(0))R(\xi)]^{-1}$$

The operator $R(\xi)$ is uniformly bounded on every compact subset of $\mathrm{Rs}(\hat{C}(0))$. Recalling the formula (Eq. 4.23, Section I.4.4.4 of Kato [13])

$$\|(I + T)^{-1} - I\| \leq \frac{\|T\|}{1 - \|T\|}$$

where T is any bounded operator with $\|T\| < 1$,

$$\begin{aligned} \|R(\xi, u) - R(\xi)\| &\leq \|R(\xi)\| \, \|[I + (\hat{C}(u) - \hat{C}(0))R(\xi)]^{-1} - I\| \\ &\leq \|R(\xi)\| \frac{\|\hat{C}(u) - \hat{C}(0)\| \, \|R(\xi)\|}{1 - \|\hat{C}(u) - \hat{C}(0)\| \, \|R(\xi)\|} \end{aligned}$$

which approaches zero uniformly for ξ in a compact subset of $\mathrm{Rs}(\hat{C}(0))$ as $\|u\|_U \to 0$.

The use of $\hat{C}(u) = \hat{C}(0) + \hat{C}'(0)u + \hat{C}_2(u)$, with $\hat{C}_2(u) = o(\|u\|_U)$, leads to the conclusion from Eq. (2.5.24) that

$$R(\xi, u) - R(\xi) = -R(\xi, u)(\hat{C}'(0)u)R(\xi) - R(\xi, u)\hat{C}_2(u)R(\xi)$$

so Eq. (2.5.23) follows.

The continuity and Fréchet differentiability of the spectral projector is now to be shown. Let γ be a simple closed (smooth) curve enclosing α_0 (and no other eigenvalue), which is included in $\mathrm{Rs}(\hat{C}(0))$ and positively oriented. The spectral projector (for $\|u\|_U$ small enough) associated with the m eigenvalues of $\hat{C}(u)$ neighboring α_0 is given by (see the proof of Theorem III.6.17 of Kato [13])

$$P(u) = -\frac{1}{2\pi i} \int_\gamma R(\xi, u) \, d\xi$$

Since $R(\xi, u)$ approaches $R(\xi)$ uniformly for ξ in a compact subset of $\mathrm{Rs}(\hat{C}(0))$, $P(u)$ is continuous at zero (a similar argument could show the continuity at every point of W, but this will not be needed). The Fréchet differentiability at zero follows, in the same manner, from Eq. (2.5.23).

The final step is to construct $\check{C}(u)$ with the required properties, mapping $W' \to \mathscr{B}(X): u \to \check{C}(u)$, where W' is included in W. The key is to construct a bijective mapping $H(u)$ that sends $M(0)$ onto $M(u)$.

Set $Q(u) = (P(u) - P(0))^2$. For $\|u\|_U$ small enough, the following operator is well defined:

$$H(u) = [P(u)P(0) + (I - P(u))(I - P(0))](I - Q(u))^{-1/2}$$

and it can be verified that (see Section I.4.6 of Kato [13])

$$H(u)^{-1} = [P(0)P(u) + (I - P(0))(I - P(u))](I - Q(u))^{-1/2}$$

and

$$P(u) = H(u)P(0)H(u)^{-1}$$

Define

$$\check{C}(u) = H(u)^{-1}\hat{C}(u)H(u) \tag{2.5.25}$$

Since $P(u)$ commutes with $\hat{C}(u)$ (because so does the resolvent),

$$\begin{aligned}\check{C}(u)P(0) &= H(u)^{-1}\hat{C}(u)H(u)P(0) \\ &= H(u)^{-1}\hat{C}(u)P(u)H(u) \\ &= H(u)^{-1}P(u)\hat{C}(u)H(u) \\ &= P(0)H(u)^{-1}\hat{C}(u)H(u) \\ &= P(0)\check{C}(u)\end{aligned}$$

Thus, $\check{C}(u)$ commutes with $P(0)$, and the two subspaces $M(0) = P(0)X$ and $M'(0) = (I - P(0))X$ are $\check{C}(u)$-invariant. Thus, eigenvalues of $\hat{C}(u)$, restricted to $M(u)$, are equivalent to those of $\check{C}(u)$, restricted to $M(0)$. Moreover, the eigenvalues of $\hat{C}(u)|_{M(u)}$ and $\check{C}(u)|_{M(0)}$ are the same, and the spectral projectors and the spectral nilpotent are connected through the formulas (Section VII.1.3 of Kato [13]

$$\check{P}_i(u) = H(u)^{-1}P_i(u)H(u)$$

$$\check{D}_i(u) = H(u)^{-1}D_i(u)H(u)$$

To complete the proof, regularity of the mapping $u \to \check{C}(u)$ is to be shown. Continuity follows from continuity of $u \to P(u)$ (shown above) and from continuity of $u \to (I - Q(u))^{-1/2}$. The argument used is similar to that used to show differentiability. The formulas $(2P(0) - I)(P'(0)u) = 0$ and $2P(0)P'(0)u = P'(0)u$ are derived from $P^2(u) = P(u)$. If $F(u) = (I - Q(u))^{-1/2}$, then $F'(0)u = \frac{1}{2}(I - Q(u))Q'(0)u$, but $Q'(0)u = 0$, so that

$$H'(0)u = 0$$

Similarly,

$$(H^{-1})'(0)u = 0$$

and

$$\check{C}'(0)u = \hat{C}'(0)u$$

This completes the proof of the proposition. ■

To complete the proof of Theorem 2.5.2 now requires a proof that the eigenvalues of the restriction of $\check{C}(u)$ to $M(0)$ are directionally differentiable. Since $M(0)$ is finite-dimensional and invariant with $\check{C}(u)$, Theorem II.5.4 of Kato [13] need only be applied to get the desired result.

2.6 TRANSIENT DYNAMIC RESPONSE DESIGN SENSITIVITY

Design sensitivity of transient dynamic response of distributed parameter systems has had little attention on the literature [47], in contrast to the massive literature in design sensitivity analysis and optimization of dynamic control systems. A development by Rousselet [42] provided the beginning foundation for design sensitivity analysis of this class of problems, but work remains to be done. Since the theory of dynamic, distributed-parameter structural design sensitivity analysis is not as well developed as that for static response and eigenvalues, a more formal treatment of the subject is presented in this section. A variational formulation of structural dynamics problems is initially outlined, and examples are given. Under assumptions of design differentiability of the state, the adjoint variable method presented in Section 2.2 is extended to the transient dynamic response problem and analytical examples are presented.

2.6.1 Variational Formulation of Structural Dynamics Problems

Equations of structural dynamics can be written in the form

$$m(u)z_{tt} + c(u)z_t + \tilde{A}_u z = f(x, t, u) \tag{2.6.1}$$

where $m(u)$ and $c(u)$ represent mass and damping effects in the structure, both taken to be design dependent, and $f(x, t, u)$ is the dynamic applied load. The operator $\tilde{A}_u$ is a spatial differential operator of the form encountered in static and eigenvalue behavior of elastic systems. The design $u(x)$ is independent of time, but may be a function of the spatial variable x. The state variable $z = z(x, t; u)$ is a function of both space and time, and since the differential equation of Eq. (2.6.1) depends on design, so does the solution z. Boundary conditions for the problem are left unspecified at the present time, since they depend on the nature of the specific structural system. Initial conditions are of the form

$$\begin{aligned} z(x, 0; u) &= z^0(x), \\ z_t(x, 0; u) &= \dot{z}^0(x), \end{aligned} \qquad x \in \Omega \tag{2.6.2}$$

While more general settings can be considered, attention here is limited to structural components that involve a scalar displacement variable z, such as strings, beams, membranes, and plates. A variational equation can be derived associated with Eq. (2.6.1) by multiplying through by an arbitrary virtual displacement $\bar{z}$ and integrating over both space and time to obtain

$$\int_0^T \iint_\Omega [\bar{z}m(u)z_{tt} + \bar{z}c(u)z_t]\, d\Omega\, dt + \int_0^T \iint_\Omega \bar{z}\bar{A}_u z\, d\Omega\, dt = \int_0^T \iint_\Omega \bar{z}f\, d\Omega\, dt \tag{2.6.3}$$

Since the integral involving the operator $\bar{A}_u$ on the left side of Eq. (2.6.3) is defined as the bilinear form of the structure, Eq. (2.6.3) may be written in the form

$$\int_0^T \left\{ \iint_\Omega [\bar{z}m(u)z_{tt} + \bar{z}c(u)z_t]\, d\Omega + a_u(z, \bar{z}) \right\} dt = \int_0^T \iint_\Omega \bar{z}f\, d\Omega\, dt \tag{2.6.4}$$

which must hold for all $\bar{z} \in Z$, the space of kinemtically admissible displacements for the structure. A rigorous mathematical theory of such variational equations may be found in the pioneering text by Lions and Magenes [48]. Roughly speaking, the variational form of Eq. (2.6.4), with the initial conditions of Eq. (2.6.2), is equivalent to the initial-boundary-value problem originally posed. To be more concrete, it is helpful to consider specific examples.

STRING

The equation of motion of a vibrating string in a viscous medium is given in the form

$$\begin{aligned} m(u)z_{tt} + c(u)z_t - \hat{T}z_{xx} &= f(x, t, u), \qquad 0 < t < T, \quad x \in \Omega \\ z(x, t; u) &= 0, \qquad 0 \le t \le T, \quad x \in \Gamma \end{aligned} \tag{2.6.5}$$

where $m(u)$ is the mass per unit length along the string, $c(u)$ the damping coefficient per unit length, and $\hat{T}$ tension in the string. Initial conditions are as in Eq. (2.6.2), and the space Z of kinematically admissible displacements is

$H_0^1(0, 1)$, that is, the set of all functions in Sobolev space $H^1(0, 1)$ that vanish at the endpoints of the interval. The energy bilinear form for this problem is given in Eq. (2.1.48) as

$$a_u(z, \bar{z}) = \hat{T} \int_0^1 z_x \bar{z}_x \, dx \tag{2.6.6}$$

BEAM

The equation of motion of a beam in a viscous fluid is

$$m(u)z_{tt} + c(u)z_t + (EI(u)z_{xx})_{xx} = f(x, t, u) \tag{2.6.7}$$

where $m(u)$ is the mass per unit length of the beam and $c(u)$ the damping coefficient per unit length. Initial conditions are as in Eq. (2.6.2), and boundary conditions may be any reasonable set of boundary conditions in Eqs. (2.1.16)–(2.1.18). For the clamped–clamped beam, $Z = H_0^2(0, 1)$. The energy bilinear form for the beam is given in Eq. (2.1.51) as

$$a_u(z, \bar{z}) = \int_0^1 EI(u) z_{xx} \bar{z}_{xx} \, dx \tag{2.6.8}$$

MEMBRANE

The equation of motion of a membrane in a viscous fluid is

$$\begin{aligned} m(u)z_{tt} + c(u)z_t - \hat{T}\nabla^2 z &= f(x, t, u), \qquad 0 < t < T, \quad x \in \Omega \\ z(x, t; u) &= 0, \qquad 0 \le t \le T, \quad x \in \Gamma \end{aligned} \tag{2.6.9}$$

where $m(u)$ is the mass per unit area of the membrane and $c(u)$ the damping coefficient per unit area. Initial conditions are as in Eq. (2.6.2), and the space Z of kinematically admissible displacements is $H_0^1(\Omega)$. The energy bilinear form for this problem is given in Eq. (2.1.56) as

$$a_u(z, \bar{z}) = \hat{T} \iint_\Omega (z_1 \bar{z}_1 + z_2 \bar{z}_2) \, d\Omega \tag{2.6.10}$$

PLATE

The equation of motion of a plate in a viscous fluid is

$$\begin{aligned} m(u)z_{tt} + c(u)z_t + [\hat{D}(u)(z_{11} + \nu z_{22})]_{11} + [\hat{D}(u)(z_{22} + \nu z_{11})]_{22} \\ + 2(1 - \nu)[\hat{D}(u) z_{12}]_{12} = f(x, t, u) \end{aligned} \tag{2.6.11}$$

where $m(u)$ is the mass per unit area of the plate and $c(u)$ the damping coefficient per unit area. Initial conditions are as in Eq. (2.6.2), and boundary conditions are as in either Eq. (2.1.19) or Eq. (2.1.23). For a clamped plate, $Z = H_0^2(\Omega)$. The energy bilinear form for the plate is given in Eq. (2.1.59) as

$$a_u(z, \bar{z}) = \iint_\Omega \hat{D}(u)[z_{11}\bar{z}_{11} + \nu z_{22}\bar{z}_{11} + z_{22}\bar{z}_{22} + \nu z_{11}\bar{z}_{22} + 2(1 - \nu)z_{12}\bar{z}_{12}]\, d\Omega \tag{2.6.12}$$

Equations for other structural systems can be written in the same general form, with the energy bilinear form arising in static response and eigenvalue problems also appearing in the variational dynamic formulation. For each instant in time, the energy bilinear form $a_u(\cdot, \cdot)$ is positive definite. Considering the complete bilinear form in z and $\bar{z}$ on the left side of Eq. (2.6.4), however, it is not positive definite.

2.6.2 Adjoint Variable Design Sensitivity Analysis

Consider a general integral functional of the form

$$\psi = \int_0^T \iint_\Omega g(z, \nabla z, u)\, d\Omega\, dt \tag{2.6.13}$$

Since the solution z of the structural equations is design dependent, dependence on design in such a functional appears both explicitly and through the argument z. As in the static response and eigenvalue problems, something must be known about the nature of the dependence of state on design [i.e., $z = z(x, t; u)$]. Under slightly more restrictive hypotheses than employed in Section 2.4 for the static response problem, it has been shown by Rousselet [42] that z is Fréchet differentiable with respect to u. This fact will be used in this section to develop explicit expressions for sensitivity of a functional ψ of Eq. (2.6.13) with respect to design u, much as was the case in development of similar sensitivity results for static problems in Section 2.2.

To begin, take the variation of Eq. (2.6.13) to obtain

$$\begin{aligned} \psi' &= \frac{d}{d\tau}\left[\int_0^T \iint_\Omega g(z(x, t; u + \tau\, \delta u), \nabla z(x, t; u + \tau\, \delta u), u + \tau\, \delta u)\, d\Omega\, dt\right]\Bigg|_{\tau=0} \\ &= \int_0^T \iint_\Omega [g_z z' + g_{\nabla z}\, \nabla z' + g_u\, \delta u]\, d\Omega\, dt \end{aligned} \tag{2.6.14}$$

The objective is to rewrite the first two terms on the right side of Eq. (2.6.14) explicitly in terms of variation in design.

Presuming that $m(u)$, $c(u)$, $f(u)$, and $a_u(\cdot, \cdot)$ are differentiable with respect to design and that the solution z of the dynamics problem is differentiable with respect to design, take the variation of both sides of Eq. (2.6.4) to obtain

$$\int_0^T \left\{ \iint_\Omega [\bar{z}m_u z_{tt} + \bar{z}c_u z_t - \bar{z}f_u]\, \delta u \, d\Omega + a'_{\delta u}(z, \bar{z}) \right\} dt$$
$$+ \int_0^T \left\{ \iint_\Omega [\bar{z}mz'_{tt} + \bar{z}cz'_t]\, d\Omega \, dt + a_u(z', \bar{z}) \right\} dt = 0 \qquad \text{for all} \quad \bar{z} \in Z \tag{2.6.15}$$

To take advantage of this equation, terms in the second integral may be integrated by parts to move time derivatives from z' over to $\bar{z}$. To carry out this calculation, interchange the order of integration, carry out the integration by parts with respect to time, and again change the order of integration to obtain

$$\int_0^T \left\{ \iint_\Omega [z'm\bar{z}_{tt} - z'c\bar{z}_t]\, d\Omega + a_u(\bar{z}, z') \right\} dt + \iint_\Omega [\bar{z}mz'_t - \bar{z}_t mz' - \bar{z}cz'] \Big|_0^T d\Omega$$
$$= \int_0^T \left\{ \iint_\Omega [\bar{z}f_u - \bar{z}m_u z_{tt} - \bar{z}c_u z_t]\, \delta u \, d\Omega - a'_{\delta u}(\bar{z}, z) \right\} dt \qquad \text{for all} \quad \bar{z} \in Z \tag{2.6.16}$$

Note that as a result of the initial conditions of Eq. (2.6.2), for which the right side does not depend on u, the variation yields

$$\begin{aligned} z'(x, 0; u) &= 0, \\ z'_t(x, 0; u) &= 0, \end{aligned} \qquad x \in \Omega \tag{2.6.17}$$

which eliminates initial terms in Eq. (2.6.16) that arose due to integration by parts.

To take advantage of the identity of Eq. (2.6.16), which must hold for all $\bar{z} \in Z$, define an adjoint variational equation by replacing z' by an arbitrary virtual displacement $\bar{\lambda} \in Z$ in Eqs. (2.6.16) and (2.6.14), defining the variational adjoint equation for $\lambda \in Z$ as

$$\int_0^T \left\{ \iint_\Omega [\bar{\lambda}m\lambda_{tt} - \bar{\lambda}c\lambda_t]\, d\Omega + a_u(\lambda, \bar{\lambda}) \right\} dt$$
$$= \int_0^T \iint_\Omega [g_z \bar{\lambda} + g_{\nabla z} \nabla \bar{\lambda}]\, d\Omega \, dt \qquad \text{for all} \quad \bar{\lambda} \in Z \tag{2.6.18}$$

where the additional terminal condition on λ is defined as

$$\begin{aligned} \lambda(x, T; u) &= 0, \\ \lambda_t(x, T; u) &= 0, \end{aligned} \qquad x \in \Omega \tag{2.6.19}$$

The terminal conditions of Eq. (2.6.19) are introduced to eliminate terms that due to integration by parts had arisen at $t = T$ in Eq. (2.6.16).

Since Eq. (2.6.16) must hold for all $\bar{z} \in Z$, this equation may be evaluated at $\bar{z} = \lambda$, using Eq. (2.6.19), to obtain

$$\int_0^T \left\{ \iint_\Omega [z' m \lambda_{tt} - z' c \lambda_t]\, d\Omega + a_u(\lambda, z') \right\} dt = \int_0^T \left\{ \iint_\Omega [\lambda f_u - \lambda m_u z_{tt} - \lambda c_u z_t]\, \delta u\, d\Omega - a'_{\delta u}(\lambda, z) \right\} dt \tag{2.6.20}$$

Similarly, Eq. (2.6.18) must hold for all $\bar{\lambda} \in Z$, so evaluate this equation at $\bar{\lambda} = z'$ to obtain

$$\int_0^T \left\{ \iint_\Omega [z' m \lambda_{tt} - z' c \lambda_t]\, d\Omega + a_u(\lambda, z') \right\} dt = \int_0^T \iint_\Omega [g_z z' + g_{\nabla z} \nabla z']\, d\Omega\, dt \tag{2.6.21}$$

Note that the right side of Eq. (2.6.21) consists of exactly the terms in Eq. (2.6.14) that are to be rewritten in terms of δu. Furthermore, the left sides of Eqs. (2.6.20) and (2.6.21) are identical, so

$$\int_0^T \iint_\Omega [g_z z' + g_{\nabla z} \nabla z']\, d\Omega\, dt = \int_0^T \left\{ \iint_\Omega [\lambda f_u + \lambda_t m_u z_t - \lambda c_u z_t]\, \delta u\, d\Omega - a'_{\delta u}(\lambda, z) \right\} dt + \iint_\Omega \lambda(x, 0; u) m_u \dot{z}^0(x)\, \delta u\, d\Omega \tag{2.6.22}$$

where an integration by parts has been carried out and the initial conditions of Eq. (2.6.2) have been used to reduce the order of differentiation of z with respect to t that is required in the evaluation. Substituting this result into Eq. (2.6.14) yields

$$\psi' = \int_0^T \left\{ \iint_\Omega [g_u + \lambda f_u + \lambda_t m_u z_t - \lambda c_u z_t]\, \delta u\, d\Omega - a'_{\delta u}(\lambda, z) \right\} dt + \iint_\Omega \lambda(x, 0; u) m_u \dot{z}^0(x)\, \delta u\, d\Omega \tag{2.6.23}$$

Note that the variational adjoint equation of Eqs. (2.6.18) and (2.6.19) is not the same as the variational equation for the state in Eqs. (2.6.3) and (2.6.2). Two fundamental differences arise. First, while the state equations include initial conditions of Eq. (2.6.2), the adjoint equation has terminal conditions of Eq. (2.6.19). Second, the sign of the damping term in Eq. (2.6.3) is different from that of the damping term in Eq. (2.6.18). These facts somewhat complicate calculations associated with dynamic design sensitivity analysis. That is, the adjoint dynamic problem is different from the original dynamics problem. As will be seen in examples of the following section, however, there exists more similarity of form than meets the eye.

2.6.3 Analytical Examples

STRING

Consider first the elementary example of a vibrating string with mean square displacement as the functional; that is,

$$\psi_1 = \frac{1}{T} \int_0^T \int_0^1 z^2 \, dx \, dt \tag{2.6.24}$$

The adjoint equations for this problem, from Eqs. (2.6.18) and (2.6.19), are

$$\begin{aligned} \int_0^T \left\{ \int_0^1 [\bar{\lambda} h \lambda_{tt} - \bar{\lambda} \beta \sqrt{h} \lambda_t] \, dx + \hat{T} \int_0^1 \lambda_x \bar{\lambda}_x \, dx \right\} dt \\ = \frac{1}{T} \int_0^T \int_0^1 2z\bar{\lambda} \, dx \, dt, \qquad \bar{\lambda} \in Z \end{aligned} \tag{2.6.25}$$

and

$$\begin{aligned} \lambda(x, T) = \lambda_t(x, T) = 0, \qquad 0 \le x \le 1 \\ \lambda(0, t) = \lambda(1, t) = 0, \qquad 0 \le t \le T \end{aligned} \tag{2.6.26}$$

where the mass density m is taken as h and the damping coefficient is proportional to the square root of h. The energy bilinear form for the string of Eq. (2.6.6) has been employed. An integration by parts in Eq. (2.6.25), using the boundary condition of Eq. (2.6.26) and the fact that $\bar{\lambda}$ satisfies the same boundary conditions, yields

$$\int_0^T \int_0^1 \left\{ h\lambda_{tt} - \beta\sqrt{h}\lambda_t - \hat{T}\lambda_{xx} - \frac{2}{T} z \right\} \bar{\lambda} \, dx \, dt = 0 \qquad \text{for all } \bar{\lambda} \in Z \tag{2.6.27}$$

Since $\bar{\lambda}$ is arbitrary, except for boundary conditions, its coefficient in Eq. (2.6.25) must be zero, yielding the differential equation

$$h\lambda_{tt} - \beta\sqrt{h}\lambda_t - \hat{T}\lambda_{xx} = \frac{2}{T} z \tag{2.6.28}$$

Note that this differential equation differs in form from Eq. (2.6.5) only by the algebraic sign of the damping term and the load.

To see that the adjoint problem of Eqs. (2.6.26) and (2.6.28) can be rewritten in a form closer to that of the physical structure [Eqs. (2.6.2) and (2.6.5)], a backward time $\tau \equiv T - t$ may be defined. With this variable, $d/dt = -d/d\tau$, and the terminal conditions of Eq. (2.6.26) for the t variable become initial conditions in the τ variable. Thus, the backward time initial-boundary-value problem for the $\tilde{\lambda}(x, \tau) = \lambda(x, T - t)$ is

$$\begin{aligned} h\tilde{\lambda}_{\tau\tau} + \beta\sqrt{h}\tilde{\lambda}_{\tau} - \hat{T}\tilde{\lambda}_{xx} &= \frac{2}{T}z(x, T - \tau), && 0 < \tau < T, \quad 0 < x < 1 \\ \tilde{\lambda}(0, \tau) = \tilde{\lambda}(1, \tau) &= 0, && 0 \le \tau \le T \\ \tilde{\lambda}(x, 0) = -\tilde{\lambda}_{\tau}(x, 0) &= 0, && 0 < x < 1 \end{aligned} \tag{2.6.29}$$

Thus, the *adjoint structure* is physically the same as the original structure, but with a backward clock and an applied load $2z(x, T - \tau)/T$.

If the load f in Eq. (2.6.5) is self-weight of the string plus an excitation $F(t, x)$, then $f = gh + F(t, x)$, where g is the acceleration of gravity, and Eq. (2.6.23) yields the sensitivity of the functional ψ_1 of Eq. (2.6.24) as

$$\begin{aligned} \psi_1' = &\int_0^T \left\{ \int_0^1 \left[\lambda g + \lambda_t z_t - \frac{\beta}{2} \frac{\lambda z_t}{\sqrt{h}} \right] \delta h \, dx - \delta\hat{T} \int_0^1 \lambda_x z_x \, dx \right\} dt \\ &+ \int_0^1 \lambda(x, 0) \dot{z}^0(x) \, \delta h \, dx \end{aligned} \tag{2.6.30}$$

Since δh depends on x and not on t, the order of integration in the first term of Eq. (2.6.27) may be reversed, yielding the explicit relation

$$\begin{aligned} \psi_1' = &\int_0^1 \left\{ \left(\int_0^T \left[g\lambda + \lambda_t z_t - \frac{\beta}{2} \frac{\lambda z_t}{\sqrt{h}} \right] dt + \lambda(x, 0) \dot{z}^0(x) \right) \right\} \delta h \, dx \\ &- \left[\int_0^T \int_0^1 \lambda_x z_x \, dx \, dt \right] \delta\hat{T} \end{aligned} \tag{2.6.31}$$

Note that the sensitivity coefficient of δh is explicitly a function of x since the time variable has been integrated in calculating the coefficient of δh. This fortunate circumstance now permits collapsing time from the design sensitivity formula, which is natural since the design vector $u = [h(x)\ \hat{T}]^{\mathrm{T}}$ is dependent only on x.

BEAM

Consider next dynamics of a clamped–clamped beam, with the functional ψ_2 being the mean over time of the square of displacement at a given point $\hat{x}$,

$$\psi_2 = \frac{1}{T}\int_0^T z^2(\hat{x}, t)\,dt = \frac{1}{T}\int_0^T \int_0^1 \hat{\delta}(x - \hat{x})\,z^2(x, t)\,dx\,dt \qquad (2.6.32)$$

with cross-sectional area h as design, $m = \rho h$, $c = \beta\sqrt{h}$, $I = \alpha h^2$, and $f = \gamma h + F(t, x)$. In this case, the adjoint equation of Eq. (2.6.18) is

$$\int_0^T \left\{ \int_0^1 [\rho\bar{\lambda} h \lambda_{tt} - \beta\sqrt{h}\,\bar{\lambda}\lambda_t]\,dx + \int_0^1 E\alpha h^2 \lambda_{xx}\bar{\lambda}_{xx}\,dx \right\} dt$$

$$= \int_0^T \int_0^1 \frac{2}{T}\,\hat{\delta}(x - \hat{x})\,z\bar{\lambda}\,dx\,dt \qquad \text{for all} \quad \bar{\lambda} \in Z \qquad (2.6.33)$$

with terminal and boundary conditions for a clamped–clamped beam,

$$\begin{aligned} &\lambda(x, T) = \lambda_t(x, T) = 0, && 0 < x < 1 \\ &\lambda(0, t) = \lambda_x(0, t) = \lambda(1, t) = \lambda_x(1, t) = 0, && 0 \le t \le T \end{aligned} \qquad (2.6.34)$$

To reduce the variational equation of Eq. (2.6.33) to a differential equation, carry out integration by parts, using the boundary conditions of Eq. (2.6.34), to obtain

$$\int_0^T \int_0^1 \bar{\lambda} \left\{ \rho h \lambda_{tt} - \beta\sqrt{h}\,\lambda_t + (E\alpha h^2 \lambda_{xx})_{xx} - \frac{2}{T}\,\hat{\delta}(x - \hat{x})\,z \right\} dx\,dt = 0$$

$$\text{for all} \quad \bar{\lambda} \in Z \qquad (2.6.35)$$

Since this equation must hold for all $\bar{\lambda}$ satisfying boundary conditions, its coefficient must be zero, leading to the differential equation

$$\rho h \lambda_{tt} - \beta\sqrt{h}\,\lambda_t + (E\alpha h^2 \lambda_{xx})_{xx} = \frac{2}{T} z\,\hat{\delta}(x - \hat{x}) \qquad (2.6.36)$$

which, except for the sign of the damping term, is just the beam equation with a point load $2z(\hat{x}, t)/T$ applied at the point $\hat{x}$. As in the case of the string, a backward time τ could be defined and the equations rewriten as in Eq. (2.6.29), to obtain the equations for the adjoint structure with a backward time variable, in exactly the same form as the basic structural equations.

The design sensitivity result from Eq. (2.6.23) may thus be directly written as

$$\psi_2' = \int_0^T \left\{ \int_0^1 \left[\gamma\lambda + \rho\lambda_t z_t - \frac{\beta}{2}\frac{\lambda z_t}{\sqrt{h}} \right] \delta h\,dx - \int_0^1 2E\alpha h\,\delta h\,\lambda_{xx} z_{xx}\,dx \right\} dt$$

$$+ \int_0^1 \lambda(x, 0)\rho\dot{z}^0(x)\,\delta h\,dx \qquad (2.6.37)$$

Interchanging the order of integration in the first integral of Eq. (2.6.37) yields

$$\psi_2' = \int_0^1 \left\{ \int_0^T \left[\gamma\lambda + \rho\lambda_t z_t - \frac{\beta}{2} \frac{\lambda z_t}{\sqrt{h}} - 2E\alpha h \lambda_{xx} z_{xx} \right] dt + \lambda(x, 0)\rho \dot{z}^0(x) \right\} \delta h \, dx \tag{2.6.38}$$

which again provides the design sensitivity coefficient of δh as a function of x only.

MEMBRANE

As a third example, consider the vibrating membrane with mean square displacement as the functional

$$\psi_3 = \frac{1}{T} \int_0^T \iint_\Omega z^2 \, d\Omega \, dt \tag{2.6.39}$$

In this case, the adjoint variational equation of Eq. (2.6.18) is

$$\int_0^T \left\{ \iint_\Omega [\bar{\lambda} h \lambda_{tt} - \bar{\lambda}\beta\sqrt{h}\lambda_t] \, d\Omega + \hat{T} \iint_\Omega (\lambda_1 \bar{\lambda}_1 + \lambda_2 \bar{\lambda}_2) \, d\Omega \right\} dt$$
$$= \frac{1}{T} \int_0^T \iint_\Omega 2z\bar{\lambda} \, d\Omega \, dt \qquad \text{for all} \quad \bar{\lambda} \in Z \tag{2.6.40}$$

with terminal and boundary conditions

$$\begin{aligned} &\lambda(x, T) = \lambda_t(x, T) = 0, \qquad && x \in \Omega \\ &\lambda(x, t) = 0, && 0 \le t \le T, \quad x \in \Gamma \end{aligned} \tag{2.6.41}$$

where $m = h$, $c = \beta\sqrt{h}$, and $u = h(x)$.

To reduce the variational equation of Eq. (2.6.40) to a differential equation, carry out integration by parts, using the boundary conditions of Eq. 2.6.41, to obtain

$$\int_0^T \iint_\Omega \bar{\lambda} \left\{ h\lambda_{tt} - \beta\sqrt{h}\lambda_t - \hat{T}\nabla^2\lambda - \frac{2}{T} z \right\} d\Omega \, dt = 0 \qquad \text{for all} \quad \bar{\lambda} \in Z \tag{2.6.42}$$

Since $\bar{\lambda}$ is arbitrary, except for boundary conditions, its coefficient must be zero, yielding

$$h\lambda_{tt} - \beta\sqrt{h}\lambda_t - \hat{T}\nabla^2\lambda = \frac{2}{T} z \tag{2.6.43}$$

This is the membrane equation, except for the sign of the damping term and a different load. As in the case of a string, a backward time τ could be defined to get a backward time initial-boundary-value problem, which is exactly the same membrane equation but with a different load.

For a given load $f(x, t)$, one may obtain design sensitivity for ψ_3 from Eq. (2.6.23) as

$$\psi_3' = \int_0^T \iint_\Omega \left[\lambda_t z_t - \frac{\beta}{2} \frac{\lambda z_t}{\sqrt{h}}\right] \delta h \, d\Omega \, dt + \iint_\Omega \lambda(x, 0)\dot{z}^0(x) \, \delta h \, d\Omega \tag{2.6.44}$$

Since δh is independent of t, interchanging the order of integration in the first term of Eq. (2.6.44) yields

$$\psi_3' = \iint_\Omega \left\{\int_0^T \left[\lambda_t z_t - \frac{\beta}{2} \frac{\lambda z_t}{\sqrt{h}}\right] dt + \lambda(x, 0)\dot{z}^0(x)\right\} \delta h \, dx \tag{2.6.45}$$

PLATE

As a last example, consider the dynamics of a clamped plate with damping coefficient zero, a given load $f(x, t)$, variable thickness h, and $m = \rho h$. The functional ψ_4 considered is the work done by the applied loads during motion of the plate; that is,

$$\psi_4 = \int_0^T \iint_\Omega f z_t \, d\Omega \, dt \tag{2.6.46}$$

Presuming that the load function f is differentiable with respect to time and that $f(x, 0) = f(x, T) = 0$, integrate the term on the right side of Eq. (2.6.46) by parts with respect to time to get

$$\psi_4 = \int_0^T \iint_\Omega f_t z \, d\Omega \, dt \tag{2.6.47}$$

The adjoint variational problem of Eq. (2.6.18) becomes, in this case,

$$\begin{aligned}\int_0^T \bigg\{\iint_\Omega \bar{\lambda} \rho h \lambda_{tt} \, d\Omega &+ \iint_\Omega \hat{D}(u)[\lambda_{11}\bar{\lambda}_{11} + \lambda_{22}\bar{\lambda}_{22} + \nu(\lambda_{22}\bar{\lambda}_{11} + \lambda_{11}\bar{\lambda}_{22}) \\ &+ 2(1 - \nu)\lambda_{12}\bar{\lambda}_{12}] \, d\Omega\bigg\} \, dt \\ &= \int_0^T \iint_\Omega f_t \bar{\lambda} \, d\Omega \, dt \qquad \text{for all} \quad \bar{\lambda} \in Z\end{aligned} \tag{2.6.48}$$

with boundary conditions for the clamped plate and terminal conditions

$$\begin{aligned}\lambda(x, t) &= \frac{\partial \lambda}{\partial n}(x, t) = 0, \qquad 0 \le t \le T, \quad x \in \Gamma \\ \lambda(x, T) &= \lambda_t(x, T) = 0, \qquad x \in \Omega\end{aligned} \tag{2.6.49}$$

Using the definition of the plate operator and spatial integration by parts on the left of Eq. (2.6.48) yields

$$\int_0^T \iint_\Omega \bar{\lambda}\{\rho h \lambda_{tt} + \bar{A}_u \lambda + f_t\}\, d\Omega\, dt = 0 \tag{2.6.50}$$

which must hold for arbitrary virtual displacements $\bar{\lambda}$ that satisfy the boundary conditions. Therefore, the differential equation for λ is

$$\rho h \lambda_{tt} + \bar{A}_u \lambda = -f_t \tag{2.6.51}$$

which is of essentially the same form as the basic plate equation without a damping term.

Using the solution λ of the adjoint equations, write the design sensitivity for ψ_4 directly from Eq. (2.6.23) as

$$\psi_4' = \iint_\Omega \Bigg[\int_0^T \{\rho \lambda_t z_t - Eh^2[z_{11}\lambda_{11} + z_{22}\lambda_{22} + \nu(z_{22}\lambda_{11} + z_{11}\lambda_{22}) + 2(1-\nu)z_{12}\lambda_{12}]/[4(1-\nu^2)]\}\, dt + \lambda(x, 0)\rho \dot{z}^0(x) \Bigg] \delta h \, d\Omega \tag{2.6.52}$$

3

Structural Components with Shape as the Design

Chapter 2 treats design sensitivity analysis of structural components whose shapes are defined by cross-sectional area and thickness variables. In such systems, a function that specifies the shape of a structural component is defined on a fixed physical domain. This design function, or design variable u, then appears explicitly in the variational equation of the problem and may appear explicitly in a performance functional of the form given in Eq. (2.2.8), where integration is taken over a fixed domain Ω.

There is an important class of structural design problems in which the shape of a two- or three-dimensional structural component (the domain it occupies) is to be determined, subject to constraints on natural frequencies, displacements, and stresses in the structure. Such problems cannot always be reduced to a formulation that characterizes shape with a design function appearing explicitly in the formulation. For such problems, it is the shape of the physical domain Ω of the structural component that must be treated as the design variable. The material derivative idea of continuum mechanics and the adjoint variable method of design sensitivity analysis (similar to that presented in Chapter 2) are applied in this chapter to obtain a computable expression for the effect of shape variation on functionals that arise in the design problem. In order to alleviate technical complexities and to give a clear idea of shape design sensitivity, variation of the conventional design variable u treated in Chapter 2 is suppressed. The effects of simultaneous shape and conventional design variable variation are treated in Chapter 4.

3.1 PROBLEMS OF SHAPE DESIGN

In order to be specific about the properties of shape design sensitivity analysis, it is helpful to formulate the variational equations for typical problems, as in Section 2.1. However, in shape design sensitivity analysis, expressions for the effect of shape variation on functionals are given as boundary integrals, using integration by parts and boundary and/or interface conditions. Hence, instead of variational equations that hold for all kinematically admissible displacements, variational identities without regard to boundary conditions will be formulated. These variational identities will then be used to transform a domain integral to a boundary integral and obtain shape design sensitivity expressions in terms of a shape perturbation of the boundary.

Numerical calculation of shape design sensitivity expressions in terms of the resulting boundary integrals requires stresses, strains, and/or normal derivatives of state and adjoint variables on the boundary. Hence, accurate evaluation of this information on the boundary is crucial. For systems with non-smooth loads and interface problems, results of finite element analysis on the boundary may not be satisfactory. To overcome this difficulty, a domain method is developed in which design sensitivity information is expressed as domain integrals instead of boundary integrals. Results obtained with the domain method are analytically equivalent to the boundary expressions. However, when numerically evaluated, these expressions may give quite different results.

BEAM

Bending and vibration of a beam were considered in Section 2.1. The formal operator equation for bending is given as

$$\bar{A}z \equiv (E\alpha h^2(x)z_{xx})_{xx} = f, \qquad x \in \Omega = (0, \hat{l}) \tag{3.1.1}$$

where E is Young's modulus, $h \in C^1[0, \hat{l}]$, $h(x) \geq h_0 > 0$, and $f \in C^1[0, \hat{l}]$ is distributed load. Results obtained in this chapter require that coefficients and right sides of differential equations for static response be smooth.

For vibration, the formal operator eigenvalue equation is

$$\bar{A}y \equiv (E\alpha h^2(x)y_{xx})_{xx} = \zeta\rho h(x)y \equiv \zeta\bar{B}y, \qquad x \in \Omega = (0, \hat{l}) \tag{3.1.2}$$

where $\zeta = \omega^2$, ω is natural frequency, and ρ is mass density.

As in Section 2.1, both sides of Eq. (3.1.1) can be multiplied by an arbitrary function $\bar{z}(x)$ that is twice continuously differentiable and integrated by parts to obtain

$$\int_0^{\hat{l}} E\alpha h^2 z_{xx}\bar{z}_{xx}\,dx - \int_0^{\hat{l}} f\bar{z}\,dx = [E\alpha h^2 z_{xx}\bar{z}_x - (E\alpha h^2 z_{xx})_x\bar{z}]\Big|_0^{\hat{l}} \tag{3.1.3}$$

Note that in Eq. (3.1.3), z and $\bar{z}$ are not required to satisfy kinematic boundary conditions. As in Section 2.1, Eq. (3.1.3) can be extended to a variational formulation in which Eq. (3.1.3) holds for all $\bar{z} \in H^2(0, \hat{l})$. In a variational formulation, derivatives that appear in Eq. (3.1.3) must be interpreted as distributional derivatives [35, 36]. For the mathematically oriented reader, the variational formulation follows from Green's formula for bilinear forms [9]. If a boundary-value problem that contains the kinematic boundary conditions given in Eqs. (2.1.1), (2.1.16), (2.1.17), or (2.1.18) (note that beam length $\hat{l}$ is not normalized in this chapter) is treated, the variational equation is

$$a_\Omega(z, \bar{z}) \equiv \int_0^{\hat{l}} E\alpha h^2 z_{xx} \bar{z}_{xx}\, dx = \int_0^{\hat{l}} f\bar{z}\, dx \equiv l_\Omega(\bar{z}) \qquad \text{for all} \quad \bar{z} \in Z \tag{3.1.4}$$

where the elements of Z satisfy the kinematic boundary conditions.

For the eigenvalue problem, the variational identity is

$$\int_0^{\hat{l}} E\alpha h^2 y_{xx} \bar{y}_{xx}\, dx - \zeta \int_0^{\hat{l}} \rho h y \bar{y}\, dx = [E\alpha h^2 y_{xx} \bar{y}_x - (E\alpha h^2 y_{xx})_x \bar{y}]\Big|_0^{\hat{l}} \qquad \text{for all} \quad \bar{y} \in H^2(0, \hat{l}) \tag{3.1.5}$$

If kinematic boundary conditions are given as in Eqs. (2.1.1), (2.1.16), (2.1.17), or (2.1.18), the variational eigenvalue equation is

$$a_\Omega(y, \bar{y}) \equiv \int_0^{\hat{l}} E\alpha h^2 y_{xx} \bar{y}_{xx}\, dx = \zeta \int_0^{\hat{l}} \rho h y \bar{y}\, dx \equiv \zeta d_\Omega(y, \bar{y}) \qquad \text{for all} \quad \bar{y} \in Z \tag{3.1.6}$$

where elements of Z satisfy the kinematic boundary conditions.

BUCKLING OF A COLUMN

Buckling of a column was considered in Section 2.1.2. The formal operator eigenvalue equation is

$$\bar{A}y \equiv (E\alpha h^2(x) y_{xx})_{xx} = -\zeta y_{xx} \equiv \zeta \bar{B} y, \qquad x \in \Omega = (0, \hat{l}) \tag{3.1.7}$$

where E and h are the same as in the beam problem. As in beam vibration, both sides of Eq. (3.1.7) may be multiplied by an arbitrary element $\bar{y} \in H^2(0, \hat{l})$ and integrated by parts to obtain the variational identity

$$\int_0^{\hat{l}} E\alpha h^2 y_{xx} \bar{y}_{xx}\, dx - \zeta \int_0^{\hat{l}} y_x \bar{y}_x\, dx = [E\alpha h^2 y_{xx} \bar{y}_x - (E\alpha h^2 y_{xx})_x \bar{y} - \zeta y_x \bar{y}]\Big|_0^{\hat{l}} \qquad \text{for all} \quad \bar{y} \in H^2(0, \hat{l}) \tag{3.1.8}$$

If kinematic boundary conditions such as in Eqs. (2.1.1), (2.1.16), (2.1.17), or (2.1.18) are given, Eq. (3.1.8) becomes the variational eigenvalue equation

$$a_\Omega(y, \bar{y}) \equiv \int_0^l E\alpha h^2 y_{xx} \bar{y}_{xx}\, dx = \zeta \int_0^l y_x \bar{y}_x\, dx \equiv \zeta d_\Omega(y, \bar{y}) \qquad \text{for all} \quad \bar{y} \in Z \tag{3.1.9}$$

where elements of Z satisfy the kinematic boundary conditions.

MEMBRANE

Consider the membrane shown in Fig. 3.1.1, with uniform tension $\hat{T}$, mass density $h(x) \in C^1(\bar{\Omega})$ per unit area, and applied lateral load $f \in C^1(\bar{\Omega})$, where $\bar{\Omega}$ is the closure of Ω. The formal operator equation for membrane deflection is

$$\bar{A}z \equiv -\hat{T}\nabla^2 z = f(x), \qquad x \in \Omega \tag{3.1.10}$$

For harmonic vibration of the membrane, the formal operator eigenvalue equation is

$$\bar{A}y \equiv -\hat{T}\nabla^2 y = \zeta h(x) y \equiv \zeta \bar{B} y, \qquad x \in \Omega \tag{3.1.11}$$

where $\zeta = \omega^2$, ω being the natural frequency.

The variational identities for these formal operator equations are

$$\hat{T} \iint_\Omega \nabla z^{\mathrm{T}}\, \nabla \bar{z}\, d\Omega - \iint_\Omega f \bar{z}\, d\Omega = \hat{T} \int_\Gamma \frac{\partial z}{\partial n} \bar{z}\, d\Gamma \qquad \text{for all} \quad \bar{z} \in H^1(\Omega) \tag{3.1.12}$$

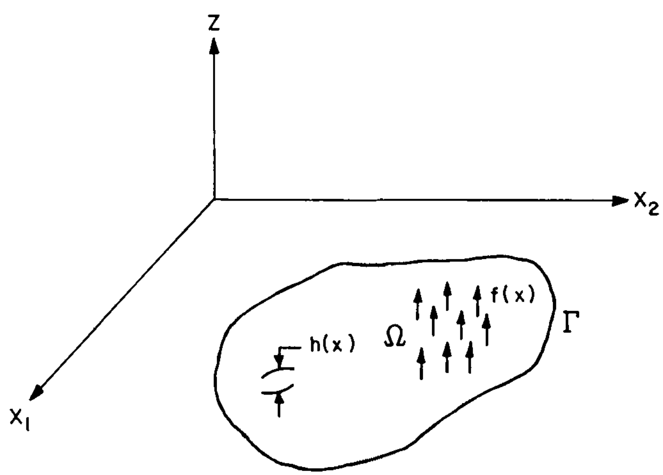

Fig. 3.1.1 Membrane of variable mass density $h(x)$.

for the static response problem, where n is the outward unit normal to Ω, and

$$\hat{T}\iint_{\Omega} \nabla y^{\mathrm{T}}\, \nabla \bar{y}\, d\Omega - \zeta \iint_{\Omega} hy\bar{y}\, d\Omega = \hat{T}\int_{\Gamma} \frac{\partial y}{\partial n}\bar{y}\, d\Gamma \qquad \text{for all} \quad \bar{y} \in H^1(\Omega) \tag{3.1.13}$$

for the eigenvalue problem. As in Section 2.1, if the kinematic boundary conditions $z = 0$ on Γ and $y = 0$ on Γ are given, Eqs. (3.1.12) and (3.1.13) become the variational equations

$$a_{\Omega}(z, \bar{z}) \equiv \hat{T}\iint_{\Omega} \nabla z^{\mathrm{T}}\, \nabla \bar{z}\, d\Omega = \iint_{\Omega} f\bar{z}\, d\Omega \equiv l_{\Omega}(\bar{z}) \qquad \text{for all} \quad \bar{z} \in Z = H_0^1(\Omega) \tag{3.1.14}$$

for static response and

$$a_{\Omega}(y, \bar{y}) \equiv \hat{T}\iint_{\Omega} \nabla y^{\mathrm{T}}\, \nabla \bar{y}\, d\Omega = \zeta \iint_{\Omega} hy\bar{y}\, d\Omega \equiv \zeta d_{\Omega}(y, \bar{y})$$
$$\text{for all} \quad \bar{y} \in Z = H_0^1(\Omega) \tag{3.1.15}$$

for eigenvalue response.

TORSION OF AN ELASTIC SHAFT

Consider the problem of torsion of the elastic shaft shown in Fig. 3.1.2. A torque T is applied to the shaft at its free end, resulting in a unit angle of twist θ. From the St. Venant theory of torsion [34], elastic deformation of the system is governed by the formal boundary-value problem

$$\bar{A}z \equiv -\nabla^2 z = 2, \qquad x \in \Omega \tag{3.1.16}$$

$$z = 0, \qquad x \in \Gamma \tag{3.1.17}$$

where z is the Prandtl stress function. The torque–angular deflection relation is given by $T = GJ\theta$, where G is the shear modulus of the shaft material and J

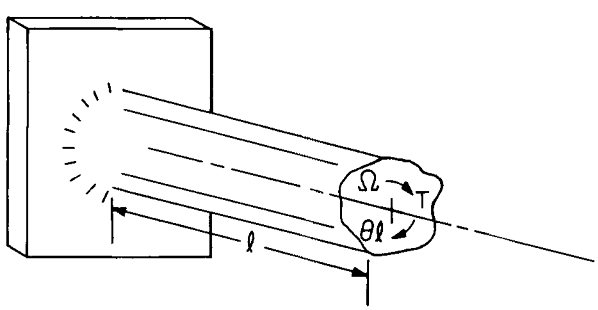

Fig. 3.1.2 Torsion of elastic shaft.

is torsional rigidity of the shaft, given by [34]

$$J = 2 \iint_{\Omega} z \, d\Omega \tag{3.1.18}$$

Comparing Eqs. (3.1.10) and (3.1.16), note that they are exactly the same if $f/T = 2$, which is the basis for the membrane analogy [34]. Hence, the variational identity for the shaft is

$$\iint_{\Omega} \nabla z^{\mathrm{T}} \bar{z} \, d\Omega - 2 \iint_{\Omega} \bar{z} \, d\Omega = \iint_{\Gamma} \frac{\partial z}{\partial n} \bar{z} \, d\Gamma \qquad \text{for all} \quad \bar{z} \in H^1(\Omega) \tag{3.1.19}$$

If the kinematic boundary condition given in Eq. (3.1.17) is imposed, the variational equation is

$$a_{\Omega}(z, \bar{z}) \equiv \iint_{\Omega} \nabla z^{\mathrm{T}} \nabla \bar{z} \, d\Omega = 2 \iint_{\Omega} \bar{z} \, d\Omega \equiv l_{\Omega}(\bar{z}) \qquad \text{for all} \quad \bar{z} \in Z = H_0^1(\Omega) \tag{3.1.20}$$

PLATE

Bending and vibration of a plate were considered in Section 2.1. With thickness $h(x) \geq h_0 > 0$, the operator form of the boundary-value problem is given in Eq. (2.1.19) as

$$\begin{aligned} \bar{A}z \equiv {} & [\hat{D}(z_{11} + \nu z_{22})]_{11} + [\hat{D}(z_{22} + \nu z_{11})]_{22} \\ & + 2(1 - \nu)[\hat{D} z_{12}]_{12} = f, \qquad x \in \Omega \end{aligned} \tag{3.1.21}$$

where $\hat{D} = Eh^3/[12(1 - \nu^2)]$ is the flexural rigidity of the plate, E is Young's modulus, ν is Poisson's ratio, and $f \in C^1(\bar{\Omega})$. For vibration, the formal operator form of the eigenvalue problem is given in Eq. (2.1.57) as

$$\begin{aligned} \bar{A}y \equiv {} & [\hat{D}(y_{11} + \nu y_{22})]_{11} + [\hat{D}(y_{22} + \nu y_{11})]_{22} + 2(1 - \nu)[\hat{D} y_{12}]_{12} \\ = {} & \zeta \rho h y \equiv \zeta \bar{B} y, \qquad x \in \Omega \end{aligned} \tag{3.1.22}$$

where $\zeta = \omega^2$, ω is natural frequency, and ρ is mass density.

As in Section 2.1, both sides of Eq. (3.1.21) may be multiplied by an arbitrary $\bar{z} \in H^2(\Omega)$ and integrated by parts to obtain the variational identity [35, 49]

$$\begin{aligned} & \iint_{\Omega} \hat{D}[(z_{11} + \nu z_{22})\bar{z}_{11} + (z_{22} + \nu z_{11})\bar{z}_{22} + 2(1 - \nu) z_{12} \bar{z}_{12}] \, d\Omega - \iint_{\Omega} f \bar{z} \, d\Omega \\ & = \int_{\Gamma} \bar{z} N z \, d\Gamma + \int_{\Gamma} \frac{\partial \bar{z}}{\partial n} M z \, d\Gamma \qquad \text{for all} \quad \bar{z} \in H^2(\Omega) \end{aligned} \tag{3.1.23}$$

where

$$Mz = \hat{D}\left[\frac{\partial^2 z}{\partial n^2} + \nu\left(\frac{1}{r}\frac{\partial z}{\partial n} + \frac{\partial^2 z}{\partial s^2}\right)\right] \tag{3.1.24}$$

r is the radius of curvature of the boundary Γ and

$$Nz = -\{[\hat{D}(z_{11} + \nu z_{22})]_1 n_1 + [\hat{D}(z_{22} + \nu z_{11})]_2 n_2 + (1-\nu)(\hat{D}z_{12})_2 n_1 + (1-\nu)(\hat{D}z_{12})_1 n_2\} - (1-\nu)\frac{\partial}{\partial s}\left(\hat{D}\frac{\partial^2 z}{\partial n\, \partial s}\right) \tag{3.1.25}$$

Given the boundary conditions

$$z = 0, \qquad \partial z/\partial n = 0, \qquad \text{on} \quad \Gamma \tag{3.1.26}$$

for a clamped plate, or

$$z = 0, \qquad Mz = 0, \qquad \text{on} \quad \Gamma \tag{3.1.27}$$

for a simply supported plate, or

$$Mz = 0, \qquad Nz = 0, \qquad \text{on} \quad \Gamma \tag{3.1.28}$$

for a free edge, the variational equation is

$$a_\Omega(z, \bar{z}) \equiv \iint_\Omega \hat{D}[(z_{11} + \nu z_{22})\bar{z}_{11} + (z_{22} + \nu z_{11})\bar{z}_{22} + 2(1-\nu)z_{12}\bar{z}_{12}]\, d\Omega = \iint_\Omega f\bar{z}\, d\Omega \equiv l_\Omega(\bar{z}) \qquad \text{for all} \quad \bar{z} \in Z \tag{3.1.29}$$

where elements of Z satisfy the kinematic boundary conditions.

For vibration of a plate, the variational identity is

$$\iint_\Omega \hat{D}[(y_{11} + \nu y_{22})\bar{y}_{11} + (y_{22} + \nu y_{11})\bar{y}_{11} + 2(1-\nu)y_{12}\bar{y}_{12}]\, d\Omega - \zeta\iint_\Omega \rho h y\bar{y}\, d\Omega = \int_\Gamma \bar{y}Ny\, d\Gamma + \int_\Gamma \frac{\partial \bar{y}}{\partial n} My\, d\Gamma \qquad \text{for all} \quad \bar{y} \in H^2(\Omega) \tag{3.1.30}$$

If y and $\bar{y}$ satisfy any pair of boundary conditions given in Eqs. (3.1.26)–(3.1.28), the variational eigenvalue equation is

$$a_\Omega(y, \bar{y}) \equiv \iint_\Omega \hat{D}[(y_{11} + \nu y_{22})\bar{y}_{11} + (y_{22} + \nu y_{11})\bar{y}_{22} + 2(1-\nu)y_{12}\bar{y}_{12}]\, d\Omega = \zeta\iint_\Omega \rho h y\bar{y} \equiv \zeta d_\Omega(y, \bar{y}) \qquad \text{for all} \quad \bar{y} \in Z \tag{3.1.31}$$

where elements of Z satisfy the kinematic boundary conditions.

LINEAR ELASTICITY

The three-dimensional linear elasticity problem for a body of arbitrary shape, shown in Fig. 3.1.3, was discussed in Section 2.1.1. The strain tensor was defined as

$$\varepsilon^{ij}(z) = \tfrac{1}{2}(z^i_j + z^j_i), \qquad i,j = 1,2,3, \quad x \in \Omega \tag{3.1.32}$$

where $z = [z^1 \; z^2 \; z^3]^{\mathrm{T}}$ is displacement. The stress–strain relation (generalized Hooke's law) is given as [34]

$$\sigma^{ij}(z) = \sum_{k,l=1}^{3} C^{ijkl}\varepsilon^{kl}(z), \qquad i,j,k,l = 1,2,3, \quad x \in \Omega \tag{3.1.33}$$

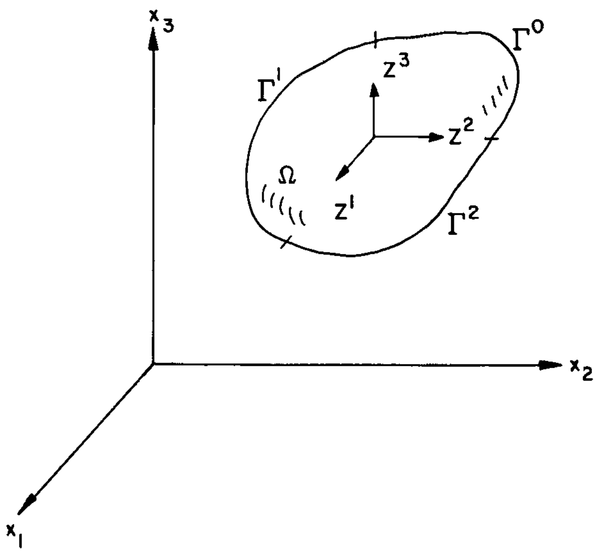

Fig. 3.1.3 Three-dimensional elastic solid.

where C is the elastic modulus tensor, satisfying symmetry relations $C^{ijkl} = C^{jikl}$ and $C^{ijkl} = C^{ijlk}$ $(i,j,k,l = 1,2,3)$. The equilibrium equations are [34]

$$-\sum_{j=1}^{3} \sigma^{ij}_j(z) = f^i, \qquad i = 1,2,3, \quad x \in \Omega \tag{3.1.34}$$

with boundary conditions

$$z^i = 0, \qquad i = 1,2,3, \quad x \in \Gamma^0 \tag{3.1.35}$$

$$T^{n_i}(z) \equiv \sum_{j=1}^{3} \sigma^{ij}(z)n_j = T^i, \qquad i = 1,2,3, \quad x \in \Gamma^2 \tag{3.1.36}$$

and boundary segment Γ^1 is traction free, where n_j is the jth component of the outward unit normal, $f = [f^1\ f^2\ f^3]^T \in [C^1(\bar{\Omega})]^3$, and $T = [T^1\ T^2\ T^3]^T \in [C^1(\Gamma)]^3$.

The foregoing formal operator equation (3.1.34) may be reduced to a variational identity by multiplying both sides of Eq. (3.1.34) by an arbitrary displacement vector $\bar{z} = [\bar{z}^1\ \bar{z}^2\ \bar{z}^3]^T \in [H^1(\Omega)]^3$ and integrating by parts to obtain

$$\iiint_\Omega \left[\sum_{i,j=1}^{3} \sigma^{ij}(z)\varepsilon^{ij}(\bar{z}) \right] d\Omega - \iiint_\Omega \left[\sum_{i=1}^{3} f^i \bar{z}^i \right] d\Omega = \iint_\Gamma \left[\sum_{i,j=1}^{3} \sigma^{ij}(z) n_j \bar{z}^i \right] d\Gamma \quad \text{for all} \quad \bar{z} \in [H^1(\Omega)]^3 \tag{3.1.37}$$

If the boundary conditions given in Eqs. (3.1.35) and (3.1.36) are imposed and Γ^1 is traction free, the variational equation is

$$a_\Omega(z, \bar{z}) \equiv \iiint_\Omega \left[\sum_{i,j=1}^{3} \sigma^{ij}(z)\varepsilon^{ij}(\bar{z}) \right] d\Omega = \iiint_\Omega \left[\sum_{i=1}^{3} f^i \bar{z}^i \right] d\Omega + \iint_{\Gamma^2} \left[\sum_{i=1}^{3} T^i \bar{z}^i \right] d\Gamma \equiv l_\Omega(\bar{z}) \quad \text{for all} \quad \bar{z} \in Z \tag{3.1.38}$$

where Z is the space of kinematically admissible displacements; that is,

$$Z = \{z \in [H^1(\Omega)]^3:\ z^i = 0, \quad i = 1, 2, 3, \quad x \in \Gamma^0\} \tag{3.1.39}$$

As shown in Section 2.1.1, for plane elasticity problems in which either all components of stress in the x_3 direction are zero or all components of strain in the x_3 direction are zero, Eqs. (3.1.37) and (3.1.38) remain valid with limits of summation running from 1 to 2 and an appropriate modification of the generalized Hooke's law of Eq. (3.1.33), as given in Eqs. (2.1.30) and (2.1.31).

INTERFACE PROBLEM OF LINEAR ELASTICITY

Consider two elastic bodies with different elastic modulii in three-dimensional space, as shown in Fig. 3.1.4, where body 1 occupies domain Ω^1 and body 2 occupies Ω^2. Here, $\Omega = \Omega^1 \cup \gamma \cup \Omega^2$, γ is the boundary of Ω^1, and Γ is the boundary of Ω. Therefore, the boundary of Ω^2 is $\gamma \cup \Gamma$.

Denote displacement as $z = [z^1\ z^2\ z^3]^T$ for $x \in \Omega$ and let the restriction of z to Ω^1 be z^* and the restriction to Ω^2 be z^{**}; that is, $z^* = z|_{\Omega^1}$ and $z^{**} = z|_{\Omega^2}$. The strain tensors are then defined to be

$$\varepsilon^{ij}(z^*) = \tfrac{1}{2}(z_j^{*i} + z_i^{*j}), \qquad i,j = 1, 2, 3, \quad x \in \Omega^1 \tag{3.1.40}$$

$$\varepsilon^{ij}(z^{**}) = \tfrac{1}{2}(z_j^{**i} + z_i^{**j}), \qquad i,j = 1, 2, 3, \quad x \in \Omega^2 \tag{3.1.41}$$

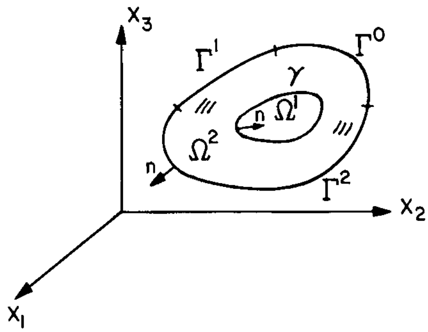

Fig. 3.1.4 Interface problem.

and the stress–strain relations (generalized Hooke's law) are

$$\sigma^{ij}(z^*) = \sum_{k,l=1}^{3} C^{*ijkl}\varepsilon^{kl}(z^*), \qquad i,j,k,l = 1,2,3, \quad x \in \Omega^1 \tag{3.1.42}$$

$$\sigma^{ij}(z^{**}) = \sum_{k,l=1}^{3} C^{**ijkl}\varepsilon^{kl}(z^{**}), \qquad i,j,k,l = 1,2,3, \quad x \in \Omega^2 \tag{3.1.43}$$

where C^* and C^{**} are elastic modulus tensors in Ω^1 and Ω^2, respectively.

The equilibrium equations are

$$-\sum_{j=1}^{3} \sigma_j^{ij}(z^*) = f^{*i}, \qquad i = 1,2,3, \quad x \in \Omega^1 \tag{3.1.44}$$

$$-\sum_{j=1}^{3} \sigma_j^{ij}(z^{**}) = f^{**i}, \qquad i = 1,2,3, \quad x \in \Omega^2 \tag{3.1.45}$$

where $f = [f^1\ f^2\ f^3]^{\mathrm{T}} \in [C^1(\bar{\Omega})]^3$ is the body force, with $f^* = f|_{\Omega^1}$ and $f^{**} = f|_{\Omega^2}$. Boundary conditions are

$$z^{**i} = 0, \qquad i = 1,2,3, \qquad x \in \Gamma^0 \tag{3.1.46}$$

$$T^{n_i}(z^{**}) \equiv \sum_{j=1}^{3} \sigma^{ij}(z^{**})n_j = T^i, \qquad x \in \Gamma^2 \tag{3.1.47}$$

and Γ^1 is traction free, where n_j is the jth component of the outward unit normal to Ω^2, as shown in Fig. 3.1.4, and $T = [T^1\ T^2\ T^3]^{\mathrm{T}} \in [C^1(\Gamma)]^3$. The interface condition on γ is that displacement and traction are continuous; that is,

$$z^{*i} = z^{**i}, \qquad i = 1,2,3, \quad x \in \gamma \tag{3.1.48}$$

$$\sum_{j=1}^{3} \sigma^{ij}(z^*)n_j = \sum_{j=1}^{3} \sigma^{ij}(z^{**})n_j, \qquad i = 1,2,3, \quad x \in \gamma \tag{3.1.49}$$

As in the linear elasticity problem, the foregoing formal operator equations (3.1.44) and (3.1.45) may be reduced to variational identities by multiplying both sides of Eqs. (3.1.44) and (3.1.45) by arbitrary displacement vectors $\bar{z}^* \in [H^1(\Omega^1)]^3$ and $\bar{z}^{**} \in [H^1(\Omega^2)]^3$, respectively, and integrating by parts to obtain

$$\iiint_{\Omega^1} \left[\sum_{i,j=1}^{3} \sigma^{ij}(z^*)\varepsilon^{ij}(\bar{z}^*) \right] d\Omega - \iiint_{\Omega^1} \left[\sum_{i=1}^{3} f^{*i}\bar{z}^{*i} \right] d\Omega$$
$$= -\iint_{\gamma} \left[\sum_{i,j=1}^{3} \sigma^{ij}(z^*)n_j\bar{z}^{*i} \right] d\Gamma \qquad \text{for all} \quad \bar{z}^* \in [H^1(\Omega^1)]^3 \tag{3.1.50}$$

and

$$\iiint_{\Omega^2} \left[\sum_{i,j=1}^{3} \sigma^{ij}(z^{**})\varepsilon^{ij}(\bar{z}^{**}) \right] d\Omega - \iiint_{\Omega^2} \left[\sum_{i=1}^{3} f^{**i}\bar{z}^{**i} \right] d\Omega$$
$$= \iint_{\gamma \cup \Gamma} \left[\sum_{i,j=1}^{3} \sigma^{ij}(z^{**})n_j\bar{z}^{**i} \right] d\Gamma \qquad \text{for all} \quad \bar{z}^{**} \in [H^1(\Omega^2)]^3 \tag{3.1.51}$$

If the boundary conditions given in Eqs. (3.1.46) and (3.1.47) are imposed, with Γ^1 traction free and the interface conditions given in Eqs. (3.1.48) and (3.1.49), adding Eqs. (3.1.50) and (3.1.51) yields the variational equation

$$\iiint_{\Omega^1} \left[\sum_{i,j=1}^{3} \sigma^{ij}(z^*)\varepsilon^{ij}(\bar{z}^*) \right] d\Omega + \iiint_{\Omega^2} \left[\sum_{i,j=1}^{3} \sigma^{ij}(z^{**})\varepsilon^{ij}(\bar{z}^{**}) \right] d\Omega$$
$$\equiv \iiint_{\Omega^1} \left[\sum_{i=1}^{3} f^{*i}\bar{z}^{*i} \right] d\Omega + \iiint_{\Omega^2} \left[\sum_{i=1}^{3} f^{**i}\bar{z}^{**i} \right] d\Omega + \iint_{\Gamma^2} \left[\sum_{i=1}^{3} T^i\bar{z}^{**i} \right] d\Gamma$$

which can be rewritten as

$$a_\Omega(z, \bar{z}) \equiv \iiint_{\Omega^1} \left[\sum_{i,j=1}^{3} \sigma^{ij}(z^*)\varepsilon^{ij}(\bar{z}^*) \right] d\Omega + \iiint_{\Omega^2} \left[\sum_{i,j=1}^{2} \sigma^{ij}(z^{**})\varepsilon^{ij}(\bar{z}^{**}) \right] d\Omega$$
$$= \iiint_{\Omega} \left[\sum_{i=1}^{3} f^i\bar{z}^i \right] d\Omega + \iint_{\Gamma^2} \left[\sum_{i=1}^{3} T^i\bar{z}^i \right] d\Gamma \qquad \text{for all} \quad \bar{z} \in Z. \tag{3.1.52}$$

where

$$Z = \{z = \{z^*, z^{**}\} \in [H^1(\Omega^1)]^3 \times [H^1(\Omega^2)]^3 : \quad z^{*i} = z^{**i},$$
$$i = 1, 2, 3, \quad x \in \gamma \quad \text{and} \quad z^{**i} = 0, \quad i = 1, 2, 3, \quad x \in \Gamma^0\} \tag{3.1.53}$$

is the space of kinematically admissible displacements.

The interface problem in plane elasticity can be defined by limiting the summation in Eqs. (3.1.50)–(3.1.52) on i and j from 1 to 2 and an appropriate modification of generalized Hooke's law of Eqs. (3.1.42) and (3.1.43).

As in Section 2.1, the Freidrichs extension A of the elasticity operators of Eqs. (3.1.44) and (3.1.45) can be defined by

$$a_{\Omega}(z, \bar{z}) = (Az, \bar{z}) \qquad \text{for all} \quad \bar{z} \in Z$$

The symmetry, strong ellipticity, and bounded invertibility of A follow, as in Section 2.1.

It is important to note that even though a variety of physical problems have been discussed in this section, they all have the same basic variational form. The importance of the variational formulation of problems of linear mechanics will be seen in the shape design sensitivity analysis carried out in subsequent sections. The variational formulation of prototype problems discussed in this section serves as the principal tool in developing a broadly applicable and rigorous shape design sensitivity analysis method.

3.2 MATERIAL DERIVATIVE FOR SHAPE DESIGN SENSITIVITY ANALYSIS

The first step in shape design sensitivity analysis is the development of relationships between a variation in shape and the resulting variations in functionals that arise in the shape design problems of Section 3.1. Since the shape of the domain a structural component occupies is treated as the design variable, it is convenient to think of Ω as a continuous medium and utilize the material derivative idea of continuum mechanics. In this section, the basic definition of material derivative is introduced, and several material derivative formulas that will be used in later sections are derived.

3.2.1 Material Derivative

Consider a domain Ω in one, two, or three dimensions, shown schematically in Fig. 3.2.1. Suppose that only one parameter τ defines the transformation T, as shown in Fig. 3.2.1. The mapping T: $x \to x_\tau(x)$, $x \in \Omega$, is given by

$$\begin{aligned} x_\tau &= T(x, \tau) \\ \Omega_\tau &\equiv T(\Omega, \tau) \end{aligned} \tag{3.2.1}$$

The process of deforming Ω to Ω_τ by the mapping of Eq. (3.2.1) may be viewed as a dynamic process of deforming a continuum, with τ playing the role of time. At the initial time $\tau = 0$, the domain is Ω. The trajectories of

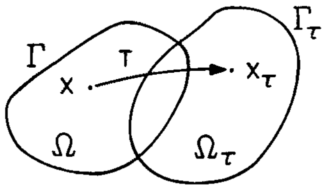

Fig. 3.2.1 One-parameter family of mappings.

points $x \in \Omega$, beginning at $\tau = 0$ can now be followed. The initial point moves to $x_\tau = T(x, \tau)$. Thinking of τ as time, a *design velocity* can be defined as [50]

$$V(x_\tau, \tau) \equiv \frac{dx_\tau}{d\tau} = \frac{dT(x, \tau)}{d\tau} = \frac{\partial T(x, \tau)}{\partial \tau} \tag{3.2.2}$$

since the initial point x does not depend on τ. This velocity can also be expressed in terms of position of the particle at time τ. If it is assumed that T^{-1} exists, that is, $x = T^{-1}(x_\tau, \tau)$, then the design velocity at $x_\tau = T(x, \tau)$ is

$$V(x_\tau, \tau) = \frac{dx_\tau}{d\tau} = \frac{\partial T}{\partial \tau}(T^{-1}(x_\tau, \tau), \tau) \tag{3.2.3}$$

The *design trajectory* of the particle that was at x at $\tau = 0$ is now defined by the initial-value problem

$$\begin{aligned} \dot{x}_\tau &= V(x_\tau, \tau) \\ x_0 &= x \end{aligned} \tag{3.2.4}$$

where $\dot{x}_\tau = dx_\tau/d\tau$. Thus, if T is given, the design velocity V can be constructed. Conversely, if the design velocity field $V(x_\tau, \tau)$ is given, T can be defined by

$$T(x, \tau) = x_\tau(x)$$

where x_τ is the solution of the initial-value problem of Eq. (3.2.4).

In a neighborhood of $\tau = 0$, under certain regularity hypothesis,

$$T(x, \tau) = T(x, 0) + \tau \frac{\partial T}{\partial \tau}(x, 0) + \cdots = x + \tau V(x, 0) + \cdots$$

Ignoring higher-order terms,

$$T(x, \tau) = x + \tau V(x) \tag{3.2.5}$$

where $V(x) \equiv V(x, 0)$. In this text, the transformation T of Eq. (3.2.5) will be considered, the geometry of which is shown in Fig. 3.2.2. Variations of the domain Ω by the design velocity field $V(x)$ are denoted as $\Omega_\tau = T(\Omega, \tau)$, and the boundary of Ω_τ is denoted as Γ_τ. Henceforth, the term "design velocity" will be referred to simply as "velocity."

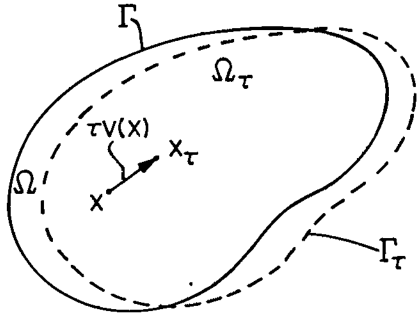

Fig. 3.2.2 Variation of domain.

Let Ω be a C^k-regular open set; that is, its boundary Γ is a compact manifold of C^k in R^n ($n = 2$ or 3), so that the boundary Γ is closed and bounded in R^n and can be locally represented by a C^k function [51]. Let $V(x) \in R^n$ in Eq. (3.2.5) be a vector defined on a neighborhood U of the closure $\bar{\Omega}$ of Ω and $V(x)$ and let its derivatives up to order $k \geq 1$ be continuous. With these hypotheses, it has been shown [52] that for small τ, $T(x, \tau)$ is a homeomorphism (a one-to-one, continuous map with a continuous inverse) from U to $U_\tau \equiv T(U, \tau)$ and that $T(x, \tau)$ and its inverse mapping mapping $T^{-1}(x_\tau, \tau)$ are C^k regular and Ω_τ is C^k regular.

Suppose $z_\tau(x_\tau)$ is a smooth classical solution of the following formal operator equation on the deformed domain Ω_τ:

$$\begin{aligned} \bar{A}z_\tau &= f, \qquad x \in \Omega_\tau \\ z_\tau &= 0, \qquad z \in \Gamma_\tau \end{aligned} \tag{3.2.6}$$

Then the mapping $z_\tau(x_\tau) \equiv z_\tau(x + \tau V(x))$ is defined on Ω, and $z_\tau(x_\tau)$ in Ω_τ depends on τ in two ways. First, it is the solution of the boundary-value problem on Ω_τ. Second, it is evaluated at a point x_τ that moves with τ. The *pointwise material derivative* (if it exists) at $x \in \Omega$ is defined as

$$\dot{z} = \dot{z}(x; \Omega, V) \equiv \frac{d}{d\tau} z_\tau(x + \tau V(x)) \bigg|_{\tau=0} = \lim_{\tau \to 0} \left[\frac{z_\tau(x + \tau V(x)) - z(x)}{\tau} \right] \tag{3.2.7}$$

If z_τ has a regular extension to a neighborhood U_τ of $\bar{\Omega}_\tau$, denoted again as z_τ, then

$$\dot{z}(x) = z'(x) + \nabla z^{\mathrm{T}} V(x) \tag{3.2.8}$$

where

$$z' = z'(X; \Omega, V) \equiv \lim_{\tau \to 0} \left[\frac{z_\tau(x) - z(x)}{\tau} \right] \tag{3.2.9}$$

is the *partial derivative* of z and $\nabla z = [z_1 \;\; z_2 \;\; z_3]^{\mathrm{T}}$.

If $z_\tau(x_\tau)$ is the solution of the variational equation on the deformed domain Ω_τ,

$$a_{\Omega_\tau}(z_\tau, \bar{z}_\tau) = l_{\Omega_\tau}(\bar{z}_\tau) \qquad \text{for all} \quad \bar{z}_\tau \in Z_\tau \tag{3.2.10}$$

where $Z_\tau \subset H^m(\Omega_\tau)$ is the space of kinematically admissible displacements, then $z_\tau \in Z_\tau \subset H^m(\Omega_\tau)$. For $z_\tau \in H^m(\Omega_\tau)$, the *material derivative* $\dot{z}$ (if it exists) at Ω is defined as

$$\lim_{\tau \to 0} \left\| \frac{z_\tau(x + \tau V(x)) - z(x)}{\tau} - \dot{z}(x) \right\|_{H^m(\Omega)} = 0 \tag{3.2.11}$$

Note that for $z_\tau \in H^m(\Omega_\tau)$, the pointwise derivative of Eq. (3.2.7) is meaningless. It was shown by Zolesio [52] that since $T(x, \tau)$ is a C^k homeomorphism, the Sobolev space $H^m(\Omega)$, for $m \le k$, is preserved by $T(x, \tau)$; that is,

$$H^m(\Omega) = \{z_\tau(x + \tau V(x)): \quad z_\tau \in H^m(\Omega_\tau)\} \tag{3.2.12}$$

This fact is used in Section 3.5 to prove the existence of the material derivative $\dot{z}$ in the problems treated in Section 3.1.

If $m > n/2$, then by the Sobolev imbedding theorem (Appendix A.2), the vector space $H^m(\Omega_\tau)$ is a topological subspace of $C^0(\bar{\Omega}_\tau)$, and the pointwise material derivative can be defined. However if $m \le n/2$, then z_τ is only defined almost everywhere on Ω_τ, and the pointwise derivative makes no sense.

For $z_\tau \in H^m(\Omega_\tau)$, Adams [36] showed that for a C^k-regular open set Ω_τ and for k large enough, there exists an extension of z_τ in a neighborhood U_τ of $\bar{\Omega}_\tau$, and hence the partial derivative z' is defined as in Eq. (3.2.9). In this case, the equality in Eq. (3.2.9) must be interpreted in the $H^m(\Omega)$ norm, as in Eq. (3.2.11). The reader who is interested in the exact condition on k to have an extension of z_τ in a neighborhood U_τ of $\bar{\Omega}_\tau$ is referred to Adams [36].

One attractive feature of the partial derivative is that, with smoothness assumptions, it commutes with the derivative with respect to x because they are derivatives with respect to independent variables; that is,

$$\left(\frac{\partial z}{\partial x_i}\right)' = \frac{\partial}{\partial x_i}(z'), \qquad i = 1, 2, 3 \tag{3.2.13}$$

3.2.2 Basic Material Derivative Formulas

A number of technical material derivative formulas that are used throughout the remainder of the text are derived in this section. The reader who is interested primarily in applications may wish to concentrate on the results rather than the derivations. The most important results obtained are stated as lemmas.

Let J be the *Jacobian* matrix of the mapping $T(x, \tau)$; that is,

$$J \equiv \left[\frac{\partial T_i}{\partial x_j}\right] = I + \tau\left[\frac{\partial V_i}{\partial x_j}\right]$$
$$= I + \tau DV(x) \tag{3.2.14}$$

where I is the identity matrix and $DV(x)$ is the Jacobian of $V(x)$. Then, $J|_{\tau=0} = J^{-1}|_{\tau=0} = I$. From Eq. (3.2.14),

$$\left.\frac{dJ}{d\tau}\right|_{\tau=0} = DV(x)$$
$$\left.\frac{dJ^{\mathrm{T}}}{d\tau}\right|_{\tau=0} = DV(x)^{\mathrm{T}} \tag{3.2.15}$$

where superscript T denotes transpose of a matrix. By taking the derivative of $JJ^{-1} = I$,

$$0 = \left.\frac{d}{d\tau}(JJ^{-1})\right|_{\tau=0} = \left.\frac{dJ}{d\tau}J^{-1}\right|_{\tau=0} + \left.J\frac{dJ^{-1}}{d\tau}\right|_{\tau=0}$$

Since $J|_{\tau=0} = J^{-1}|_{\tau=0} = I$, Eq. (3.2.15) and the above equation give

$$\left.\frac{dJ^{-1}}{d\tau}\right|_{\tau=0} = -DV(x) \tag{3.2.16}$$

Similarly,

$$\left.\frac{dJ^{-\mathrm{T}}}{d\tau}\right|_{\tau=0} = -DV(x)^{\mathrm{T}} \tag{3.2.17}$$

where $J^{-\mathrm{T}} = (J^{-1})^{\mathrm{T}} = (J^{\mathrm{T}})^{-1}$. Denoting $|J|$ as the determinant of J, it can be verified by direct calculation that

$$\left.\frac{d}{d\tau}|J|\right|_{\tau=0} = \operatorname{div} V(x) \tag{3.2.18}$$

where $\operatorname{div} V \equiv \sum_{i=1}^{3} \partial V_i/\partial x_i$. Taking the derivative of $|JJ^{-1}| = 1$,

$$0 = \left.\frac{d}{d\tau}|JJ^{-1}|\right|_{\tau=0} = \left.|J^{-1}|\frac{d}{d\tau}|J|\right|_{\tau=0} + \left.|J|\frac{d}{d\tau}|J^{-1}|\right|_{\tau=0}$$

Since $|J||_{\tau=0} = |J^{-1}||_{\tau=0} = 1$, Eq. (3.2.18) and the above equation give

$$\left.\frac{d}{d\tau}|J^{-1}|\right|_{\tau=0} = -\operatorname{div} V(x) \tag{3.2.19}$$

Let n be the unit normal to the infinitesimal area $d\Gamma$ of the parallelogram shown in Fig. 3.2.3, with two edges dx and δx in the undeformed surface Γ.

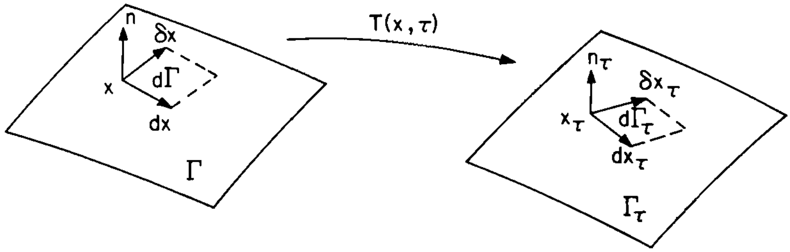

Fig. 3.2.3 Transformation of area.

Let n_τ be the unit normal in the deformed surface Γ_τ to the infinitesimal area $d\Gamma_\tau$ of the parallelogram, with edges dx_τ and δx_τ (Fig. 3.2.3). Since Ω and Ω_τ are C^k regular, n and n_τ are C^{k-1} regular.

Here, dx_τ and δx_τ are given as

$$\begin{aligned} dx_\tau &= J\,dx \\ \delta x_\tau &= J\,\delta x \end{aligned} \tag{3.2.20}$$

so

$$\begin{aligned} dx &= J^{-1}\,dx_\tau \\ \delta x &= J^{-1}\,\delta x_\tau \end{aligned} \tag{3.2.21}$$

Then, using the vector product,

$$\begin{aligned} n\,d\Gamma &= dx \times \delta x \\ n_\tau\,d\Gamma_\tau &= dx_\tau \times \delta x_\tau \end{aligned} \tag{3.2.22}$$

or, in cartesian rectangular components,

$$\begin{aligned} n_i\,d\Gamma &= e_{ijk}\,dx_j\,\delta x_k \\ n_{\tau_r}\,d\Gamma_\tau &= e_{rst}\,dx_{\tau_s}\,\delta x_{\tau_t} \end{aligned} \tag{3.2.23}$$

where summation is taken over all repeated indices and e_{ijk} is a *permutation symbol*, defined as

$$e_{ijk} = \begin{cases} 0 & \text{when any two indices are equal} \\ +1 & \text{when } i, j, k \text{ are } 1, 2, 3 \text{ or an even permutation of } 1, 2, 3 \\ -1 & \text{when } i, j, k \text{ are an odd permutation of } 1, 2, 3 \end{cases} \tag{3.2.24}$$

From the first equation of Eq. (3.2.23), using Eq. (3.2.21),

$$n_i\,d\Gamma = e_{ijk}\frac{\partial x_j}{\partial x_{\tau_s}}\frac{\partial x_k}{\partial x_{\tau_t}}\,dx_{\tau_s}\,\delta x_{\tau_t} \tag{3.2.25}$$

Multiplying both sides of Eq. (3.2.25) by $\partial x_i/\partial x_{\tau_r}$ and summing on i,

$$\frac{\partial x_i}{\partial x_{\tau_r}} n_i \, d\Gamma = e_{ijk} \frac{\partial x_i}{\partial x_{\tau_r}} \frac{\partial x_j}{\partial x_{\tau_s}} \frac{\partial x_k}{\partial x_{\tau_t}} \, dx_{\tau_s} \, \delta x_{\tau_t} \tag{3.2.26}$$

For any 3×3 matrix with elements a_{mn},

$$e_{rst} \det[a_{mn}] = e_{ijk} a_{ir} a_{js} a_{kt} \tag{3.2.27}$$

Hence, for the Jacobian J,

$$\begin{aligned} e_{ijk}|J| &= e_{rst} \frac{\partial x_{\tau_r}}{\partial x_i} \frac{\partial x_{\tau_s}}{\partial x_j} \frac{\partial x_{\tau_t}}{\partial x_k} \\ e_{rst}|J^{-1}| &= e_{ijk} \frac{\partial x_i}{\partial x_{\tau_r}} \frac{\partial x_j}{\partial x_{\tau_s}} \frac{\partial x_k}{\partial x_{\tau_t}} \end{aligned} \tag{3.2.28}$$

Using the second equation of Eq. (3.2.28) in Eq. (3.2.26) and the fact that $|J^{-1}| = |J|^{-1}$,

$$\frac{\partial x_i}{\partial x_{\tau_r}} n_i \, d\Gamma = |J|^{-1} e_{rst} \, dx_{\tau_s} \, \delta x_{\tau_t}$$

which can be rewritten, using (Eq. 3.2.23), as

$$n_\tau \, d\Gamma_\tau = |J| J^{-\mathrm{T}} n \, d\Gamma \tag{3.2.29}$$

Normalizing n_τ,

$$n_\tau = \frac{J^{-\mathrm{T}}(x_\tau) n(x)}{\|J^{-\mathrm{T}}(x_\tau) n(x)\|} \tag{3.2.30}$$

where $\|a\| = (a^{\mathrm{T}} a)^{1/2}$ is the Euclidean norm. Applying Eq. (3.2.30) to Eq. (3.2.29),

$$d\Gamma_\tau = |J| \, \|J^{-\mathrm{T}}(x_\tau) n(x)\| \, d\Gamma \tag{3.2.31}$$

Using Eq. (3.2.17),

$$\frac{d}{d\tau} \|J^{-\mathrm{T}}(x_\tau) n(x)\| \bigg|_{\tau=0} \equiv \frac{d}{d\tau} (J^{-\mathrm{T}} n, J^{-\mathrm{T}} n)^{1/2} \bigg|_{\tau=0} = -(DVn, n) \tag{3.2.32}$$

where $(a, b) \equiv a^{\mathrm{T}} b$. Equations (3.2.17), (3.2.30), and (3.2.32) and $J^{-\mathrm{T}}|_{\tau=0} = I$ now give

$$\begin{aligned} \dot{n} \equiv \frac{dn_\tau}{d\tau}\bigg|_{\tau=0} &= \frac{(dJ^{-\mathrm{T}}/d\tau) n \, \|J^{-\mathrm{T}} n\| - J^{-\mathrm{T}} n \, (d/d\tau) \|J^{-\mathrm{T}} n\|}{\|J^{-\mathrm{T}} n\|^2} \bigg|_{\tau=0} \\ &= (DVn, n) n - DV^{\mathrm{T}} n = (n, DV^{\mathrm{T}} n) n - DV^{\mathrm{T}} n \end{aligned} \tag{3.2.33}$$

Also, using Eqs. (3.2.18) and (3.2.32),

$$\left.\frac{d}{d\tau}(|J|\,\|J^{-T}n\|)\right|_{\tau=0} = \operatorname{div} V - (DVn, n) \tag{3.2.34}$$

LEMMA 3.2.1 Let ψ_1 be a domain functional, defined as an integral over Ω_τ,

$$\psi_1 = \iint_{\Omega_\tau} f_\tau(x_\tau)\, d\Omega_\tau \tag{3.2.35}$$

where f_τ is a regular function defined on Ω_τ. If Ω is C^k regular, then the material derivative of ψ_1 at Ω is

$$\psi_1' = \iint_{\Omega} f'(x)\, d\Omega + \int_{\Gamma} f(x)(V^T n)\, d\Gamma \tag{3.2.36}$$

PROOF By transforming variables of integration in Eq. (3.2.35) [53],

$$\psi_1 = \iint_{\Omega_\tau} f_\tau(x_\tau)\, d\Omega_\tau = \iint_{\Omega} f_\tau(x + \tau V(x))\,|J|\, d\Omega$$

The material derivative of ψ_1 at Ω is, using Eqs. (3.2.8) and (3.2.18),

$$\begin{aligned}
\psi_1' &\equiv \left.\frac{d}{d\tau}\iint_{\Omega} f_\tau(x + \tau V(x))\,|J|\, d\Omega\right|_{\tau=0} \\
&= \iint_{\Omega} [\dot{f}(x) + f(x)\operatorname{div} V(x)]\, d\Omega \\
&= \iint_{\Omega} [f'(x) + \nabla f(x)^T V(x) + f(x)\operatorname{div} V(x)]\, d\Omega \\
&= \iint_{\Omega} [f'(x) + \operatorname{div}(f(x)V(x))]\, d\Omega
\end{aligned} \tag{3.2.37}$$

If Ω is C^k regular, the divergence theorem [53] yields Eq. (3.2.36). ■

It is interesting and important to note that it is only the normal component $(V^T n)$ of the *boundary velocity* appearing in Eq. (3.2.36) that is of importance in accounting for the effect of domain variation. In fact, it is shown by Theorem 3.5.3 (Section 3.5.7) that if a general domain functional ψ has a gradient at Ω and if Ω is C^{k+1} regular, then only the normal component $(V^T n)$ of the velocity field on the boundary needs to be considered for derivative calculations. The basic idea behind this result is that $\Gamma_\tau(V_s) = \Gamma$ for all τ, where V_s is the component of the velocity field V of Eq. (3.2.3) that is tangent to the boundary Γ. That is, the tangential component V_s of the velocity field does not deform the domain Ω.

Next, consider a functional defined as an integration over Γ_τ,

$$\psi_2 = \int_{\Gamma_\tau} g_\tau(x_\tau)\, d\Gamma_\tau \tag{3.2.38}$$

where g_τ is a regular function defined on Γ_τ. Using Eq. (3.2.31),

$$\psi_2 = \int_{\Gamma_\tau} g_\tau(x_\tau)\, d\Gamma_\tau = \int_\Gamma g_\tau(x + \tau V(x))|J|\,\|J^{-\mathrm{T}}n\|\, d\Gamma \tag{3.2.39}$$

and the material derivative of ψ_2 at Ω is, using Eq. (3.2.34),

$$\begin{aligned} \psi_2' &\equiv \frac{d}{d\tau} \int_\Gamma g_\tau(x + \tau V(x))|J|\,\|J^{-\mathrm{T}}n\|\, d\Gamma \bigg|_{\tau=0} \\ &= \int_\Gamma [\dot{g}(x) + g(x)(\operatorname{div} V(x) - (DVn, n))]\, d\Gamma \end{aligned} \tag{3.2.40}$$

Suppose the mapping $V \to \dot{g}$ is linear and continuous. Then Eq. (3.2.40) implies that ψ_2 has a gradient at Ω and by Theorem 3.5.3 (Section 3.5.7), if Ω is C^{k+1} regular, only $V = vn$ needs to be considered, where v is a scalar C^k-regular function. If Ω is C^{k+1} regular, then n is C^k regular and $V = vn$ is C^k regular. For $V = vn$, on Γ

$$DV = n\,\nabla v^{\mathrm{T}} + vDn \tag{3.2.41}$$

Since n is the unit normal,

$$0 = \tfrac{1}{2}\,\nabla(n^{\mathrm{T}}n) = Dn^{\mathrm{T}}n \tag{3.2.42}$$

Hence, from Eq. (3.2.41),

$$DV^{\mathrm{T}}n = \nabla v n^{\mathrm{T}}n + vDn^{\mathrm{T}}n = \nabla v \tag{3.2.43}$$

From Eqs. (3.2.33) and (3.2.43), with normal velocity $V = vn$,

$$\dot{n} = \frac{dn_\tau}{d\tau}\bigg|_{\tau=0} = (n, DV^{\mathrm{T}}n)n - DV^{\mathrm{T}}n = (\nabla v^{\mathrm{T}}n)n - \nabla v \tag{3.2.44}$$

and

$$\operatorname{div} V - (DVn, n) = \nabla v^{\mathrm{T}}n + v \operatorname{div} n - \nabla v^{\mathrm{T}}n = v \operatorname{div} n \tag{3.2.45}$$

It is now to be shown that

$$\operatorname{div} n = H \tag{3.2.46}$$

where H is the curvature of Γ in R^2 and twice the mean curvature of Γ in R^3.

For the proof, consider first R^2, where Γ is locally the graph of a regular function f, say $x_2 = f(x_1)$. Suppose Ω lies below the graph of f. The normal is given by

$$n(x_1, f(x_1)) = (1 + f'^2)^{-1/2}[-f' \quad 1]^T$$

A direct calculation gives

$$\operatorname{div} n(x_1, f(x_1)) = -(1 + f'^2)^{-3/2} f'' \tag{3.2.47}$$

which is the curvature of Γ. This verifies Eq. (3.2.46) for R^2.

In R^3, Γ is a regular surface. For a point $x \in \Gamma$, consider the R^3-orthonormal basis $\{e_1, e_2, n\}$ shown in Fig. 3.2.4, where e_1 and e_2 are vectors tangent to Γ at x, such that [54]

$$Dne_i = -k_i e_i, \qquad i = 1, 2 \tag{3.2.48}$$

The parameters k_1 and k_2 are principal normal curvatures of Γ at x, and the vectors e_1 and e_2 are unit vectors in principal directions. In a neighborhood of x, with x taken as the origin, Γ may be represented by the graph of $w = f(y_1, y_2)$ in the (y_1, y_2, w) coordinate system, as shown in Fig. 3.2.4.

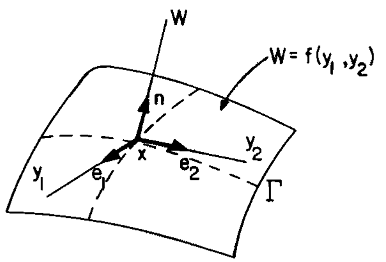

Fig. 3.2.4 Local representation of the boundary Γ.

Since the divergence operator is invariant under translation and rotation [55], div n can be written in (y_1, y_2, w) coordinates; that is,

$$\operatorname{div} n = \sum_{i=1}^{2} (Dne_i, e_i) + (Dnn, n)$$

Thus, using Eqs. (3.2.42) and (3.2.48),

$$\operatorname{div} n = -(k_1 + k_2) \tag{3.2.49}$$

which is twice the mean curvature of Γ [54]. This completes the proof of Eq. (3.2.46).

If the velocity is normal to Γ, $V = vn$. Then, from Eq. (3.2.45),

$$\operatorname{div} V - (DVn, n) = vH \tag{3.2.50}$$

The choice of n, as directed outward from the domain Ω, defines the orientation of the boundary Γ. If the orientation of Γ is changed, then n is changed to $-n$ and H must be changed to $-H$.

From Eq. (3.2.40), using Eq. (3.2.50),

$$\begin{aligned}\psi_2' &= \int_\Gamma [\dot{g}(x) + Hg(x)(V^{\mathrm{T}}n)]\, d\Gamma \\ &= \int_\Gamma [g'(x) + \nabla g^{\mathrm{T}}V + Hg(x)(V^{\mathrm{T}}n)]\, d\Gamma \\ &= \int_\Gamma [g'(x) + (\nabla g^{\mathrm{T}}n + Hg(x))(V^{\mathrm{T}}n)]\, d\Gamma\end{aligned}$$

This proves the following lemma:

LEMMA 3.2.2 Suppose g_τ in Eq. (3.2.38) is a regular function defined on Γ_τ, and the mapping $V \to \dot{g}$ is linear and continuous. If Ω is C^{k+1} regular, the material derivative of ψ_2 in Eq. (3.2.38) at Ω is

$$\psi_2' = \int_\Gamma [g'(x) + (\nabla g^{\mathrm{T}}n + Hg(x))(V^{\mathrm{T}}n)]\, d\Gamma \tag{3.2.51}$$

Equating the right sides of Eqs. (3.2.40) and (3.2.51) and using $\dot{g}(x) = g'(x) + \nabla g^{\mathrm{T}}V$, the following useful relationship for any regular vector field V and regular function g on R^n is obtained:

$$\begin{aligned}\int_\Gamma \nabla g^{\mathrm{T}}V\, d\Gamma = &-\int_\Gamma g(x)[\operatorname{div} V(x) - (DVn, n)]\, d\Gamma \\ &+ \int_\Gamma (\nabla g^{\mathrm{T}}n + Hg(x))(V^{\mathrm{T}}n)\, d\Gamma\end{aligned} \tag{3.2.52}$$

Finally, consider a special functional that is defined as an integration over Γ_τ as

$$\psi_3 = \int_{\Gamma_\tau} h_\tau(x_\tau)^{\mathrm{T}}n_\tau\, d\Gamma_\tau \tag{3.2.53}$$

where h_τ is a regular field defined on Γ_τ, hence $h_\tau^{\mathrm{T}}n_\tau$ is a regular function on Γ_τ. From Eq. (3.2.40), using $h_\tau^{\mathrm{T}}n_\tau$ instead of g_τ,

$$\psi_3' = \int_\Gamma [\dot{h}(x)^{\mathrm{T}}n + h(x)^{\mathrm{T}}\dot{n} + h(x)^{\mathrm{T}}n(\operatorname{div} V(x) - (DVn, n))]\, d\Gamma \tag{3.2.54}$$

where $\dot{n}$ is given by Eq. (3.2.33). If the mapping $V \to \dot{h}$ is linear and continuous, then Eq. (3.2.54) implies that ψ_3 has a gradient at Ω and, by Theorem 3.5.3 (Section 3.5.7), if Ω is C^{k+1} regular, only $V = vn$ needs to be considered, with a C^k-regular scalar function v. Then, using Eqs. (3.2.44) and (3.2.50), Eq. (3.2.54) becomes

$$\psi_3' = \int_\Gamma [\dot{h}(x)^{\mathrm{T}} n + h(x)^{\mathrm{T}}((\nabla v^{\mathrm{T}} n)n - \nabla v) + (h(x)^{\mathrm{T}} n) H v] \, d\Gamma \tag{3.2.55}$$

Using Eq. (3.2.52) and substituting h for V and v for g,

$$\int_\Gamma h^{\mathrm{T}} \nabla v \, d\Gamma = -\int_\Gamma v[\operatorname{div} h - (Dhn, n)] \, d\Gamma + \int_\Gamma (\nabla v^{\mathrm{T}} n + Hv)(h^{\mathrm{T}} n) \, d\Gamma$$

Hence, Eq. 3.2.55 becomes

$$\begin{aligned}
\psi_3' &= \int_\Gamma [\dot{h}(x)^{\mathrm{T}} n + (\operatorname{div} h - (Dhn, n))v] \, d\Gamma \\
&= \int_\Gamma [(h'(x) + DhV)^{\mathrm{T}} n + (\operatorname{div} h - (Dhn, n))v] \, d\Gamma \\
&= \int_\Gamma [h'(x)^{\mathrm{T}} n + v(Dhn, n) + (\operatorname{div} h - (Dhn, n))v] \, d\Gamma \\
&= \int_\Gamma [h'(x)^{\mathrm{T}} n + \operatorname{div} h(V^{\mathrm{T}} n)] \, d\Gamma
\end{aligned}$$

Thus, the following lemma has been proved:

LEMMA 3.2.3 Suppose h_τ in Eq. (3.2.53) is a regular field defined on Γ_τ and the mapping $V \to \dot{h}$ is linear and continuous. If Ω is C^{k+1} regular, the material derivative of ψ_3 in Eq. (3.2.53) at Ω is

$$\psi_3' = \int_\Gamma [h'(x)^{\mathrm{T}} n + \operatorname{div} h(V^{\mathrm{T}} n)] \, d\Gamma \tag{3.2.56}$$

3.3 STATIC-RESPONSE SHAPE DESIGN SENSITIVITY ANALYSIS

As sèen in Section 3.2, the static response of a structure depends on the shape of the domain. Existence of the material derivative $\dot{z}$, which is proved in Section 3.5, and material derivative formulas derived in Section 3.2 are used in this section to derive an adjoint variable method for design sensitivity analysis of general functionals. As in Chapter 2, an adjoint problem that is closely related to the original structural problem is obtained and explicit formulas for shape design sensitivity analysis are obtained. Numerical methods for parameterizing boundary shape and calculating shape design sensitivity coefficients are obtained and illustrated.

3.3.1 Differentiability of Bilinear Forms and Static Response

Basic design differentiability results for energy bilinear forms and static response, for the problems treated in Section 3.1, are proved in Section 3.5. These differentiability results are used here to develop shape design sensitivity formulas. This order of presentation was selected, as in Chapter 2, because technical aspects of existence of design derivatives of the structural state do not contribute insight into the adjoint variable technique. However, as noted in Chapter 2, the delicate question of existence of design derivatives should not be ignored.

The variational equations of the problems of Section 3.1, on a deformed domain, are of the form

$$a_{\Omega_\tau}(z_\tau, \bar{z}_\tau) \equiv \iint_{\Omega} c(z_\tau, \bar{z}_\tau)\, d\Omega_\tau = \iint_{\Omega_\tau} f\bar{z}_\tau\, d\Omega_\tau \equiv l_{\Omega_\tau}(\bar{z}_\tau) \qquad \text{for all} \quad \bar{z}_\tau \in Z_\tau \tag{3.3.1}$$

where $Z_\tau \subset H^m(\Omega_\tau)$ is the space of kinematically admissible displacements and $c(\cdot, \cdot)$ is a bilinear mapping that is defined by the integrand of the variational equation. It is shown in Section 3.5.4 that the load linear forms $l_{\Omega_\tau}(\bar{z}_\tau)$ for the problems of Section 3.1 are also differentiable with respect to design.

A powerful result from Section 3.5.4 is that the solution of Eq. (3.3.1) is differentiable with respect to design. That is, the material derivative $\dot{z}$ defined in Eq. (3.2.11) exists. Note that the material derivative $\dot{z}$ depends on the direction V (velocity field). As shown in Eq. (3.5.36), $\dot{z}$ is linear in V and in fact is the Fréchet derivative with respect to design, evaluated in the direction V. This linearity and continuity of the mapping $V \to \dot{z}$ justify, by Theorem 3.5.3 (Section 3.5.7), use of only the normal component $(V^{\mathrm{T}}n)$ of the velocity field V in the derivation of the material derivative, as in Eqs. (3.2.51) and (3.2.56).

Taking the material derivative of both sides of Eq. (3.3.1), using Eq. (3.2.36), and noting that the partial derivatives with respect to τ and x commute with each other,

$$[a_\Omega(z, \bar{z})]' \equiv a'_V(z, \bar{z}) + a_\Omega(\dot{z}, \bar{z}) = l'_V(\bar{z}) \qquad \text{for all} \quad \bar{z} \in Z, \tag{3.3.2}$$

where, using Eq. (3.2.8),

$$
\begin{aligned}
[a_\Omega(z, \bar{z})]' &= \iint_\Omega [c(z, \bar{z}') + c(z', \bar{z})]\, d\Omega + \int_\Gamma c(z, \bar{z})(V^{\mathrm{T}} n)\, d\Gamma \\
&= \iint_\Omega [c(z, \dot{\bar{z}} - \nabla \bar{z}^{\mathrm{T}} V) + c(\dot{z} - \nabla z^{\mathrm{T}} V, \bar{z})]\, d\Omega + \int_\Gamma c(z, \bar{z})(V^{\mathrm{T}} n)\, d\Gamma
\end{aligned}
\tag{3.3.3}
$$

and

$$
\begin{aligned}
l_V'(\bar{z}) &= \iint_\Omega f\bar{z}'\, d\Omega + \int_\Gamma f\bar{z}(V^{\mathrm{T}} n)\, d\Gamma \\
&= \iint_\Omega f(\dot{\bar{z}} - \nabla \bar{z}^{\mathrm{T}} V)\, d\Omega + \int_\Gamma f\bar{z}(V^{\mathrm{T}} n)\, d\Gamma
\end{aligned}
\tag{3.3.4}
$$

The fact that the partial derivatives of the coefficients, which depend on cross-sectional area and thickness, in the bilinear mapping $c(\cdot, \cdot)$ are zero has been used in Eq. (3.3.3), and $f' = 0$ has been used in Eq. (3.3.4). For $\bar{z}_\tau$, let $\bar{z}_\tau(x + \tau V(x)) = \bar{z}(x)$, that is, choose $\bar{z}$ as constant on the line $x_\tau = x + \tau V(x)$. Then, since $H^m(\Omega)$ is preserved by $T(x, \tau)$ [Eq. (3.2.12)], if $\bar{z}$ is an arbitrary element of $H^m(\Omega)$ that satisfies kinematic boundary conditions on Γ, $\bar{z}_\tau$ is an arbitrary element of $H^m(\Omega_\tau)$ that satisfies kinematic boundary conditions on Γ_τ. In this case, using Eq. (3.2.8),

$$
\dot{\bar{z}} = \bar{z}' + \nabla \bar{z}^{\mathrm{T}} V = 0 \tag{3.3.5}
$$

From Eqs. (3.3.2), (3.3.3), and (3.3.4), Eq. (3.3.5) may be used to obtain

$$
a_V'(z, \bar{z}) = -\iint_\Omega [c(z, \nabla \bar{z}^{\mathrm{T}} V) + c(\nabla z^{\mathrm{T}} V, \bar{z})]\, d\Omega + \int_\Gamma c(z, \bar{z})(V^{\mathrm{T}} n)\, d\Gamma \tag{3.3.6}
$$

and

$$
l_V'(\bar{z}) = -\iint_\Omega f(\nabla \bar{z}^{\mathrm{T}} V)\, d\Omega + \int_\Gamma f\bar{z}(V^{\mathrm{T}} n)\, d\Gamma \tag{3.3.7}
$$

Then, Eq. (3.3.2) can be rewritten to provide the result

$$
\begin{aligned}
a_\Omega(\dot{z}, \bar{z}) &= l_V'(\bar{z}) - a_V'(z, \bar{z}) \\
&= \iint_\Omega [c(z, \nabla \bar{z}^{\mathrm{T}} V) + c(\nabla z^{\mathrm{T}} V, \bar{z}) - f(\nabla \bar{z}^{\mathrm{T}} V)]\, d\Omega \\
&\quad + \int_\Gamma [f\bar{z} - c(z, \bar{z})](V^{\mathrm{T}} n)\, d\Gamma \qquad \text{for all} \quad \bar{z} \in Z
\end{aligned}
\tag{3.3.8}
$$

Note the similarity of Eqs. (3.3.8) and (2.2.7). As noted in Chapter 2, if the state z is known as the solution of Eq. (3.3.1) at Ω and if the velocity field V is known, Eq. (3.3.8) is a variational equation with the same energy bilinear form for $\dot{z} \in H^m(\Omega)$, which satisfies kinematic boundary conditions. Indeed, for second-order problems (membrane, shaft, elasticity), kinematic boundary conditions are imposed only on z, so if $z_\tau = 0$ on Γ_τ, then $\dot{z} = 0$ on Γ and $\dot{z}$ satisfies kinematic boundary conditions. For higher-order problems, such as clamped plates, $\dot{z}$ can be shown to satisfy kinematic boundary conditions. Indeed, for the clamped-plate problem, the boundary condition $z = 0$ on Γ implies $\dot{z} = 0$ on Γ. Also, $z = \partial z/\partial n = 0$ on Γ implies $\nabla z = (\partial z/\partial n)n + (\partial z/\partial s)s = 0$ on Γ, which in turn implies $(\nabla z)^{\cdot} = 0$ on Γ. By Eqs. (3.2.8) and (3.2.13),

$$
\begin{aligned}
\left(\frac{\partial z}{\partial x_i}\right)^{\cdot} &= \frac{\partial}{\partial x_i}(z') + \sum_{j=1}^{2} \frac{\partial^2 z}{\partial x_j\,\partial x_i} V^j \\
&= \frac{\partial}{\partial x_i}\left(\dot{z} - \sum_{j=1}^{2} \frac{\partial z}{\partial x_j} V^j\right) + \sum_{j=1}^{2} \frac{\partial^2 z}{\partial x_j\,\partial x_i} V^j \\
&= \frac{\partial}{\partial x_i}(\dot{z}) - \nabla z^{\mathrm{T}}\left(\frac{\partial V}{\partial x_i}\right), \qquad i = 1, 2
\end{aligned} \tag{3.3.9}
$$

Hence, $(\partial z/\partial x_i)^{\cdot} = 0$ $(i = 1, 2)$ on Γ implies that $\nabla \dot{z} = 0$ on Γ. Thus, $\partial \dot{z}/\partial n = 0$ on Γ and $\dot{z}$ satisfies kinematic boundary conditions.

Note that the right side of Eq. (3.3.8) is linear in $\bar{z}$, and the energy bilinear form on the left is Z elliptic. Thus, Eq. (3.3.8) has a unique solution $\dot{z} \in Z$ [9]. The fact that there is a unique solution of Eq. (3.3.8) agrees with the previously stated result that the design derivative of the solution of the state equation exists. As in Chapter 2, Eq. (3.3.8) can be used in the adjoint variable method of design sensitivity analysis.

3.3.2 Adjoint Variable Design Sensitivity Analysis

Consider a general functional that may be written in integral form as

$$
\psi = \iint_{\Omega_\tau} g(z_\tau, \nabla z_\tau)\, d\Omega_\tau \tag{3.3.10}
$$

where $z \in H^1(\Omega)$, $\nabla z = [z_1 \;\; z_2 \;\; z_3]^{\mathrm{T}}$, and the function g is continuously differentiable with respect to its arguments. In the case $z \in H^2(\Omega)$, second derivatives of z may appear in the integrand of Eq. (3.3.10). This case will be treated as specific applications arise. Note that ψ depends on Ω in two ways. First, there is the obvious dependence of the integral on its domain of integration. Second, the state z_τ depends on the domain Ω_τ, through the variational equation of Eq. (3.3.1).

Taking the variation of the functional of Eq. (3.3.10), using the material derivative formula of Eq. (3.2.13) and Eq. (3.2.36),

$$\psi' = \iint_\Omega [g_z z' + g_{\nabla z} \nabla z'] \, d\Omega + \int_\Gamma g(V^{\mathrm{T}} n) \, d\Gamma \tag{3.3.11}$$

where $g_{\nabla z} = [\partial g/\partial z_1 \;\; \partial g/\partial z_2 \;\; \partial g/\partial z_3]$. Using Eq. (3.2.8), Eq. (3.3.11) can be rewritten as

$$\psi' = \iint_\Omega [g_z \dot{z} + g_{\nabla z} \nabla \dot{z} - g_z(\nabla z^{\mathrm{T}} V) - g_{\nabla z} \nabla(\nabla z^{\mathrm{T}} V)] \, d\Omega + \int_\Gamma g(V^{\mathrm{T}} n) \, d\Gamma \tag{3.3.12}$$

Note that $\dot{z}$ and $\nabla \dot{z}$ depend on the velocity field V. The objective here is to obtain an explicit expression for ψ' in terms of the velocity field V, which requires rewriting the first two terms of the first integral on the right side of Eq. (3.3.12) explicitly in terms of V, that is, eliminating $\dot{z}$.

Much as in Chapter 2, an adjoint equation is introduced by replacing $\dot{z} \in Z$ in Eq. (3.3.12) by a virtual displacement $\bar{\lambda} \in Z$ and equating the sum of terms involving $\bar{\lambda}$ to the energy bilinear form, yielding the *adjoint equation* for the *adjoint variable* λ,

$$a_\Omega(\lambda, \bar{\lambda}) = \iint_\Omega [g_z \bar{\lambda} + g_{\nabla z} \nabla \bar{\lambda}] \, d\Omega \qquad \text{for all} \quad \bar{\lambda} \in Z \tag{3.3.13}$$

Note that the adjoint equation of Eq. (3.3.13) is the same as the one in Eq. (2.2.10). This fact is advantageous when both conventional design and shape design variation are considered simultaneously, as will be done in Chapter 4. As noted in Section 2.2., the Lax–Milgram theorem [9] guarantees that Eq. (3.3.13) has a unique solution λ, which is called the adjoint variable associated with the constraint of Eq. (3.3.10).

To take advantage of the adjoint equation, evaluate Eq. (3.3.13) at $\bar{\lambda} = \dot{z} \in Z$ to obtain the expression

$$a_\Omega(\lambda, \dot{z}) = \iint_\Omega [g_z \dot{z} + g_{\nabla z} \nabla \dot{z}] \, d\Omega \tag{3.3.14}$$

Similarly, the identity of Eq. (3.3.8) may be evaluated at $\bar{z} = \lambda$, since both are in Z, to obtain

$$a_\Omega(\dot{z}, \lambda) = l'_V(\lambda) - a'_V(z, \lambda) \tag{3.3.15}$$

Recalling that the energy bilinear form $a_\Omega(\cdot, \cdot)$ is symmetric in its arguments, the left sides of Eqs. (3.3.14) and (3.3.15) are equal, thus yielding

$$\iint_\Omega [g_z \dot{z} + g_{\nabla z} \nabla \dot{z}] \, d\Omega = l'_V(\lambda) - a'_V(z, \lambda) \tag{3.3.16}$$

Using Eqs. (3.3.8) and (3.3.16), Eq. (3.3.12) yields

$$\begin{aligned}\psi' &= l'_V(\lambda) - a'_V(z, \lambda) - \iint_\Omega [g_z(\nabla z^{\mathrm{T}} V) + g_{\nabla z}\, \nabla(\nabla z^{\mathrm{T}} V)]\, d\Omega + \int_\Gamma g(V^{\mathrm{T}} n)\, d\Gamma \\ &= \iint_\Omega [c(z, \nabla\lambda^{\mathrm{T}} V) - f(\nabla\lambda^{\mathrm{T}} V) + c(\nabla z^{\mathrm{T}} V, \lambda) - g_z(\nabla z^{\mathrm{T}} V) - g_{\nabla z}\, \nabla(\nabla z^{\mathrm{T}} V)]\, d\Omega \\ &\quad + \int_\Gamma [g + f\lambda - c(z, \lambda)](V^{\mathrm{T}} n)\, d\Gamma \end{aligned} \tag{3.3.17}$$

where the integral over Ω can be transformed to a boundary integral by using the variational identities given in Section 3.1 for each structural component and boundary and/or interface conditions. This will be done for each structural component type encountered. The fact that the design sensitivity ψ' can be expressed as a boundary integral gives significant advantages in numerical calculations, if accurate boundary information can be calculated.

Note that evaluation of the design sensitivity formula of Eq. (3.3.17) requires solution of the variational equation of Eq. (3.3.1) for z. Similarly, the variational adjoint equation of Eq. (3.3.13) must be solved for the adjoint variable λ. This is an efficient calculation, using finite element analysis, if the boundary-value problem for z has already been solved, in which case it requires only evaluation of the solution of the same set of finite element equations with a different right side (*adjoint load*).

3.3.3 Analytical Examples of Static Design Sensitivity

The beam, membrane, torsion, and plate problems of Section 3.1 are used here as examples with which to calculate design sensitivity formulas, using the adjoint variable method. Linear elasticity problems will be considered in Section 3.3.4, and computational considerations will be discussed in subsequent sections. The variation of a conventional design variable u (cross-sectional area or thickness, considered in Chapter 2) is suppressed in the discussions of this chapter, and even though there is self-weight in addition to externally applied load, the total applied load will be expressed as $f(x)$.

BENDING OF A BEAM

Consider the beam of Section 3.1, with $\Omega = (0, l) \subset R^1$ and $I(x) = \alpha h^2(x)$. Several structural response functionals are of concern. Consider first the weight of the beam, given as

$$\psi_1 = \int_0^l \gamma h\, dx \tag{3.3.18}$$

Taking the variation, using Eq. (3.2.36) with $(\gamma h)' = 0$,

$$\psi_1' = \gamma h V \Big|_0^{\hat{l}} = \gamma h(\hat{l})V(\hat{l}) - \gamma h(0)V(0) \tag{3.3.19}$$

where $V(0)$ and $V(\hat{l})$ are perturbations of endpoint locations for the beam, positive if $V(0)$ and $V(\hat{l})$ cause the endpoints to move in the positive x direction. Note that this direct variation gives the explicit form of variation of structural weight in terms of variation of shape. Thus, no adjoint problem needs to be defined.

Consider as a second functional the compliance of the structure, defined as

$$\psi_2 = \int_0^{l} fz \, dx \tag{3.3.20}$$

Note that the integrand of Eq. (3.3.20) depends on the load f. However, since $f' = 0$, Eq. (3.3.20) can be treated as the functional form of Eq. (3.3.10). Hence the adjoint equation of Eq. (3.3.13) is

$$a_\Omega(\lambda, \bar{\lambda}) = \int_0^{l} f\bar{\lambda}\, dx \qquad \text{for all} \quad \bar{\lambda} \in Z \tag{3.3.21}$$

Since the load functional on the right side of Eq. (3.2.21) is precisely the same as the load functional for the original beam problem of Eq. (3.1.5), in this special case $\lambda = z$, and from Eq. (3.3.17) with $g = fz$,

$$\psi_2' = \int_0^{l} [2E\alpha h^2 (z_x V)_{xx} z_{xx} - 2f(z_x V)]\, dx + [2fz - E\alpha h^2 (z_{xx})^2] V \Big|_0^{l} \tag{3.3.22}$$

The variational identity of Eq. (3.1.3) may be used, identifying $(z_x V)$ in the integral of Eq. (3.3.22) with $\bar{z}$ in Eq. (3.1.3), to obtain

$$\psi_2' = 2E\alpha h^2 z_{xx}(z_x V)_x \Big|_0^{l} - 2(E\alpha h^2 z_{xx})_x (z_x V) \Big|_0^{l} + [2fz - E\alpha h^2 (z_{xx})^2] V \Big|_0^{l} \tag{3.3.23}$$

For a clamped–clamped beam, using boundary conditions of Eq. (2.1.1) (note that beam length $\hat{l}$ is not normalized in this chapter), Eq. (3.3.23) becomes

$$\psi_2' = E\alpha h^2 (z_{xx})^2 V \Big|_0^{l} \tag{3.3.24}$$

As noted in Section 3.3.2, the design sensitivity in Eq. (3.3.24) is expressed as a boundary evaluation and is given explicitly in terms of design velocity (perturbation) of the boundary.

As an example that can be calculated analytically, consider a uniform clamped–clamped beam with $h = h_0$ and uniform load f_0. The displacement under this load is

$$z(x) = \frac{f_0}{24E\alpha h_0^2}[x^2(\hat{l} - x)^2]$$

The compliance of the beam may be calculated from Eq. (3.3.20) as

$$\psi_2 = \frac{f_0^2 \hat{l}^5}{720E\alpha h_0^2}$$

Consider beam length $\hat{l}$ as a design variable. Since the beam has uniform cross-sectional area h_0 and uniform load f_0, varying either endpoint $x = 0$ or $x = \hat{l}$ will have the same effect on compliance. Hence, the variation of compliance with respect to $\hat{l}$ is

$$\psi_2' = \frac{f_0^2 \hat{l}^4}{144E\alpha h_0^2}\,\delta\hat{l}$$

Using Eq. (3.3.24), $V(\hat{l}) = \delta\hat{l}/2$, and $V(0) = -\delta\hat{l}/2$,

$$\begin{aligned}\psi_2' &= E\alpha h_0^2\left[\frac{f_0}{12E\alpha h_0^2}(\hat{l}^2 - 6\hat{l}x + 6x^2)\right]^2 V\Big|_0^{\hat{l}}\\ &= \frac{f_0^2 \hat{l}^4}{144E\alpha h_0^2}\,\delta\hat{l}\end{aligned}$$

which is the correct result.

For other boundary conditions in Eqs. (2.1.16)–(2.1.18), the shape design sensitivity formula in Eq. (3.3.23) for compliance is valid because (as mentioned in Section 2.1.1) the variational equation of Eq. (3.1.4) is valid for all $\bar{z}$ satisfying corresponding kinematic boundary conditions. To obtain a sensitivity formula for the simply supported case, boundary conditions of Eq. (2.1.16) in Eq. (3.3.23) can be used to obtain

$$\psi_2' = -2E\alpha h^2 z_{xxx} z_x V\Big|_0^{\hat{l}} \tag{3.3.25}$$

For a cantilevered beam, applying boundary conditions of Eq. (2.1.17) to Eq. (3.2.23),

$$\psi_2' = -E\alpha h^2 (z_{xx})^2 V\Big|_{x=0} + 2fzV\Big|_{x=\hat{l}} \tag{3.3.26}$$

For a clamped–simply supported beam, applying boundary conditions of Eq. (2.1.18) to Eq. (3.2.23),

$$\psi_2' = -E\alpha h^2(z_{xx})^2 V\Big|_{x=0} - 2E\alpha h^2 z_{xxx} z_x V\Big|_{x=l} \tag{3.3.27}$$

Consider next a functional that defines the value of displacement at an isolated fixed point $\hat{x} \in (0, l)$; that is,

$$\psi_3 \equiv z(\hat{x}) = \int_0^l \hat{\delta}(x - \hat{x}) z \, dx \tag{3.3.28}$$

For the purpose of evaluating the functional ψ_3, the point $\hat{x}$ does not move, and ψ_3 on the deformed domain Ω_τ is the value of displacement at the same point $\hat{x}$. Since $m = 2$ and $n = 1$, $m > n/2$ and by the Sobolev imbedding theorem [36], $z_\tau \in C^0(\Omega_\tau)$. The functional of Eq. (3.3.28) is thus continuous, and the foregoing theory applies.

Since $\hat{\delta}(x - \hat{x})$ is defined on a neighborhood of $[0, l]$ by zero extension and $\hat{x}$ is a fixed point, $\hat{\delta}'(x - \hat{x}) = 0$. Thus, Eq. (3.3.28) can be treated as the functional form of Eq. (3.3.10) and the adjoint equation is, from Eq. (3.3.13),

$$a_\Omega(\lambda, \bar{\lambda}) = \int_0^l \hat{\delta}(x - \hat{x}) \bar{\lambda} \, dx \qquad \text{for all} \quad \bar{\lambda} \in Z \tag{3.3.29}$$

As noted in Section 2.2.3, since the right side of this equation defines a bounded linear functional on $H^2(0, l)$, Eq. (3.3.28) has a unique solution $\lambda^{(3)}$, where superscript (i) associates λ with the constraint ψ_i. Note that $\lambda^{(3)}$ is the displacement due to a unit load at $\hat{x}$. That is, with smoothness assumptions, the variational equation of Eq. (3.3.28) is equivalent to the formal operator equation

$$(E\alpha h^2 \lambda_{xx})_{xx} = \hat{\delta}(x - \hat{x}), \qquad x \in (0, l) \tag{3.3.30}$$

with λ satisfying the same boundary conditions as the original structural response z. From Eq. (3.3.17) with $g = \hat{\delta}(x - \hat{x})z$,

$$\begin{aligned}\psi_3' = &\int_0^l [E\alpha h^2 z_{xx}(\lambda_x^{(3)} V)_{xx} - f(\lambda_x^{(3)} V) + E\alpha h^2 (z_x V)_{xx} \lambda_{xx}^{(3)} - \hat{\delta}(x - \hat{x})(z_x V)] \, dx \\ &+ [f\lambda^{(3)} - E\alpha h^2 z_{xx} \lambda_{xx}^{(3)}] V\Big|_0^l \end{aligned} \tag{3.3.31}$$

The variational identity of Eq. (3.1.3) may be used twice to transform the integral in Eq. (3.3.31) to a boundary integral (boundary evaluation in the one-dimensional case). To do so, first identify $(\lambda_x^{(3)} V)$ in the first two terms of the integral in Eq. (3.3.31) with $\bar{z}$ in Eq. (3.1.3). Next, identify $\lambda^{(3)}$, $(z_x V)$, and

$\hat{\delta}(x - \hat{x})$ in the second two terms of the integral in Eq. (3.3.31) with z, $\bar{z}$, and f in Eq. (3.1.3), respectively. Then, from Eq. (3.3.31),

$$\psi_3' = [E\alpha h^2 z_{xx}(\lambda_x^{(3)}V)_x - (E\alpha h^2 z_{xx})_x(\lambda_x^{(3)}V)]\Big|_0^{\hat{l}} + [E\alpha h^2 \lambda_{xx}^{(3)}(z_x V)_x - (E\alpha h^2 \lambda_{xx}^{(3)})_x(z_x V)]\Big|_0^{\hat{l}} + [f\lambda^{(3)} - E\alpha h^2 z_{xx}\lambda_{xx}^{(3)}]V\Big|_0^{\hat{l}} \tag{3.3.32}$$

For a clamped beam, using the boundary conditions of Eq. (2.1.1), Eq. (3.3.32) becomes

$$\psi_3' = E\alpha h^2 z_{xx}\lambda_{xx}^{(3)}V\Big|_0^{\hat{l}} \tag{3.3.33}$$

To illustrate the use of this result, consider the clamped–clamped beam studied earlier in this section. If design sensitivity of displacement at the center of the beam is desired, $\hat{x} = \hat{l}/2$. From Eq. (3.3.28) and

$$z(x) = \frac{f_0}{24E\alpha h_0^2}[x^2(\hat{l} - x)^2]$$

the functional ψ_3 is

$$\psi_3 = \frac{f_0\hat{l}^4}{384E\alpha h_0^2}$$

and the shape design sensitivity is

$$\psi_3' = \frac{f_0\hat{l}^3}{96E\alpha h_0^2}\,\delta\hat{l}$$

The adjoint load from Eq. (3.3.29) is just a unit point load at the center of the beam. The adjoint variable is thus obtained as

$$\lambda^{(3)} = \frac{1}{48E\alpha h_0^2}\left[8\left\langle x - \frac{\hat{l}}{2}\right\rangle^3 - 4x^3 + 3\hat{l}x^2\right]$$

where

$$\left\langle x - \frac{\hat{l}}{2}\right\rangle = \begin{cases} 0 & \text{for} \quad 0 \le x < \dfrac{\hat{l}}{2} \\ x - \dfrac{\hat{l}}{2} & \text{for} \quad \dfrac{\hat{l}}{2} \le x \le \hat{l} \end{cases}$$

Using this information, the shape design sensitivity of ψ_3 is obtained, using Eq. (3.3.33), as

$$\psi_3' = \frac{f_0}{576E\alpha h_0^2}\left[(\hat{l}^2 - 6\hat{l}x + 6x^2)(24x - 6\hat{l})V\bigg|_{x=0} + (\hat{l}^2 - 6\hat{l}x + 6x^2)(24x - 18\hat{l})V\bigg|_{x=\hat{l}}\right]$$

$$= \frac{f_0\hat{l}^3}{96E\alpha h_0^2}\,\delta\hat{l}$$

since $V(0) = -\delta\hat{l}/2$ and $V(\hat{l}) = \delta\hat{l}/2$. Note that this is the same result as before.

For simply supported, cantilevered, or clamped–simply supported beams, the sensitivity formula in Eq. (3.3.32) is valid, where z and $\lambda^{(3)}$ are solutions of Eqs. (3.1.4) and (3.3.29), respectively, and Z is the appropriate space of kinematically admissible displacements. Appropriate boundary conditions for z and $\lambda^{(3)}$ can then be applied to Eq. (3.3.32) to obtain useful sensitivity formulas. For a simply supported beam, boundary conditions in Eq. (2.1.16) can be used for both z and $\lambda^{(3)}$ in Eq. (3.3.32), to obtain

$$\psi_3' = -[E\alpha h^2(\lambda_{xxx}^{(3)} z_x + z_{xxx}\lambda_x^{(3)})]V\bigg|_0^{\hat{l}} \tag{3.3.34}$$

For a cantilevered beam, applying boundary conditions in Eq. (2.1.17) for both z and $\lambda^{(3)}$ to Eq. (3.3.32),

$$\psi_3' = -E\alpha h^2 z_{xx}\lambda_{xx}^{(3)} V\bigg|_{x=0} + f\lambda^{(3)}V\bigg|_{x=\hat{l}} \tag{3.3.35}$$

For a clamped–simply supported beam, applying boundary conditions of Eq. (2.1.18) to Eq. (3.2.32),

$$\psi_3' = E\alpha h^2 z_{xx}\lambda_{xx}^{(3)} V\bigg|_{x=0} - [E\alpha h^2(\lambda_{xxx}^{(3)} z_x + z_{xxx}\lambda_x^{(3)})]V\bigg|_{x=\hat{l}} \tag{3.3.36}$$

The shape design sensitivity results in Eq. (3.3.32) or Eqs. (3.3.33)–(3.3.36) for each boundary condition are valid for the functional ψ_3 that defines the value of displacement at a fixed point $\hat{x}$. The case in which ψ_3 on a deformed domain Ω_τ is the displacement at point $\hat{x}_\tau = \hat{x} + \tau V(\hat{x})$ will be considered in Section 3.3.6.

Consider another functional that is associated with strength constraints,

$$\psi_4 = \int_0^{\hat{l}} \beta h^{1/2} E z_{xx} m_p \, dx \tag{3.3.37}$$

where $\beta h^{1/2}$ is the half-depth of the beam and m_p is a characteristic function defined on a small open subinterval (x_a, x_b), such that $[x_a, x_b] \subset (0, \hat{l})$. The characteristic function m_p is positive and constant on (x_a, x_b), zero outside of (x_a, x_b), and its integral is 1. Consider the average stress on the fixed interval (x_a, x_b); that is, m_p in Eq. (3.3.37) does not change with τ. It is possible to extend m_p on R^1 by extending it to zero value outside $(0, \hat{l})$. Then, $m_p' = 0$.

Taking the variation of Eq. (3.3.37), using Eqs. (3.2.13) and (3.2.36) and $h' = 0$,

$$\begin{aligned} \psi_4' &= \int_0^{\hat{l}} \beta h^{1/2} E z_{xx}' m_p \, dx + \beta h^{1/2} E z_{xx} m_p V \Big|_0^{\hat{l}} \\ &= \int_0^{\hat{l}} \beta h^{1/2} E[(\dot{z})_{xx} - (z_x V)_{xx}] m_p \, dx \end{aligned} \tag{3.3.38}$$

since $m_p(0) = m_p(\hat{l}) = 0$. As in the general derivation of the adjoint equation of Eq. (3.3.13), the adjoint equation may be defined by replacing $\dot{z}$ in the first term on the right side of Eq. (3.3.38) by $\bar{\lambda}$, to define a load functional of the adjoint equation, obtaining

$$a_\Omega(\lambda, \bar{\lambda}) = \int_0^{\hat{l}} \beta h^{1/2} E \bar{\lambda}_{xx} m_p \, dx \qquad \text{for all} \quad \bar{\lambda} \in Z \tag{3.3.39}$$

As in the adjoint equation of Eq. (3.3.13), the adjoint equation of Eq. (3.3.39) is the same as Eq. (2.2.32). As noted in Section 2.2.3, since the right side of this equation is a bounded linear functional on $H^2(0, l)$, Eq. (3.3.39) has a unique solution $\lambda^{(4)}$. With smoothness assumptions, the variational equation of Eq. (3.3.39) is equivalent to the formal operator equation

$$(E\alpha h^2 \lambda_{xx})_{xx} = (\beta h^{1/2} E m_p)_{xx}, \qquad x \in (0, \hat{l}) \tag{3.3.40}$$

with λ satisfying the same boundary conditions as the original structural response z. As in Eq. (3.3.30), the derivative on the right side of Eq. (3.3.40) is a derivative in the sense of the theory of distributions [35, 36, 56]. Expanding the derivative,

$$\begin{aligned} (\beta h^{1/2} E m_p)_{xx} = \beta E\{(h^{1/2})_{xx} m_p + m_p[&(h^{1/2})_x(x_a)\, \delta(x - x_a) \\ &- (h^{1/2})_x(x_b)\, \delta(x - x_b) + h^{1/2}(x_a)\, \delta_x(x - x_a) \\ &- h^{1/2}(x_b)\, \delta_x(x - x_b)]\} \end{aligned}$$

Thus, the adjoint load consists of a distributed load on the interval (x_a, x_b), point loads at x_a and x_b, and point moments at x_a and x_b. By the same method used in Section 3.1, a variational identity is obtained for the adjoint system from Eq. (3.3.40) as

$$\int_0^l E\alpha h^2 \lambda_{xx} \bar{\lambda}_{xx}\, dx - \int_0^l \beta h^{1/2} E \bar{\lambda}_{xx} m_p\, dx$$

$$= [E\alpha h^2 \lambda_{xx} \bar{\lambda}_x - (E\alpha h^2 \lambda_{xx})_x \bar{\lambda}]\Big|_0^l + [(\beta h^{1/2} E m_p)_x \bar{\lambda} - \beta h^{1/2} E m_p \bar{\lambda}_x]\Big|_0^l$$

$$\text{for all} \quad \bar{\lambda} \in H^2(0, l) \tag{3.3.41}$$

Since $\dot{z} \in Z$, Eq. (3.3.39) may be evaluated at $\bar{\lambda} = \dot{z}$ to obtain

$$a_\Omega(\lambda^{(4)}, \dot{z}) = \int_0^l \beta h^{1/2} E(\dot{z})_{xx} m_p\, dx \tag{3.3.42}$$

Similarly, the identity of Eq. (3.3.8) may be evaluated at $\bar{z} = \lambda^{(4)}$, since both are in Z, to obtain

$$a_\Omega(\dot{z}, \lambda^{(4)}) = l'_V(\lambda^{(4)}) - a'_V(z, \lambda^{(4)}) \tag{3.3.43}$$

Since the energy bilinear form $a_\Omega(\cdot, \cdot)$ is symmetric, Eqs. (3.3.38), (3.3.42), and (3.3.43) yield

$$\psi'_4 = l'_V(\lambda^{(4)}) - a'_V(z, \lambda^{(4)}) - \int_0^l \beta h^{1/2} E(z_x V)_{xx} m_p\, dx$$

which can be rewritten, using Eq. (3.3.8), as

$$\begin{aligned}\psi'_4 = \int_0^l & [E\alpha h^2 z_{xx} (\lambda_x^{(4)} V)_{xx} - f(\lambda_x^{(4)} V) + E\alpha h^2 (z_x V)_{xx} \lambda_{xx}^{(4)} \\ & - \beta h^{1/2} E(z_x V)_{xx} m_p]\, dx \\ & + [f\lambda^{(4)} - E\alpha h^2 z_{xx} \lambda_{xx}^{(4)}] V\Big|_0^l \end{aligned} \tag{3.3.44}$$

The variational identities of Eq. (3.1.3) and (3.3.41), identifying $(\lambda_x^{(4)} V)$ in the first two terms of the integral in Eq. (3.3.44) with $\bar{z}$ in Eq. (3.1.3) and $\lambda^{(4)}$ and $(z_x V)$ in the second two terms of the integral in Eq. (3.3.44) with λ and $\bar{\lambda}$ in Eq. (3.3.41), respectively, may be used to obtain

$$\begin{aligned}\psi'_4 = & [E\alpha h^2 z_{xx} (\lambda_x^{(4)} V)_x - (E\alpha h^2 z_{xx})_x (\lambda_x^{(4)} V)]\Big|_0^l \\ & + [E\alpha h^2 \lambda_{xx}^{(4)} (z_x V)_x - (E\alpha h^2 \lambda_{xx}^{(4)})_x (z_x V)]\Big|_0^l \\ & + [(\beta h^{1/2} E m_p)_x (z_x V) - \beta h^{1/2} E m_p (z_x V)_x]\Big|_0^l + [f\lambda^{(4)} - E\alpha h^2 z_{xx} \lambda_{xx}^{(4)}] V\Big|_0^l \end{aligned} \tag{3.3.45}$$

Since $[x_a, x_b] \subset (0, \hat{l})$ and $m_p = 0$ in neighborhoods of $x = 0$ and $x = \hat{l}$, Eq. (3.3.45) becomes

$$\psi_4' = [E\alpha h^2 z_{xx}(\lambda_x^{(4)}V)_x - (E\alpha h^2 z_{xx})_x(\lambda_x^{(4)}V)]\Big|_0^{\hat{l}} + [E\alpha h^2 \lambda_{xx}^{(4)}(z_x V)_x - (E\alpha h^2 \lambda_{xx}^{(4)})_x(z_x V)]\Big|_0^{\hat{l}} + [f\lambda^{(4)} - E\alpha h^2 z_{xx}\lambda_{xx}^{(4)}]V\Big|_0^{\hat{l}} \tag{3.3.46}$$

For a clamped beam, using the boundary conditions of Eq. (2.1.1) and the fact that $\lambda^{(4)}$ satisfies the same boundary conditions, Eq. (3.3.46) becomes

$$\psi_4' = E\alpha h^2 z_{xx}\lambda_{xx}^{(4)}V\Big|_0^{\hat{l}} \tag{3.3.47}$$

As before, for simply supported, cantilevered, or clamped–simply supported beams, the shape design sensitivity formula of Eq. (3.3.46) is valid, where z and $\lambda^{(4)}$ are solutions of Eqs. (3.1.4) and (3.3.39), respectively, and Z is the appropriate space of kinematically admissible displacements. Appropriate boundary conditions for z and $\lambda^{(4)}$ can be applied in Eq. (3.3.46) to obtain useful shape design sensitivity formulas.

For a simply supported beam, the boundary conditions in Eq. (2.1.16) can be used for both z and $\lambda^{(4)}$ in Eq. (3.3.46) to obtain

$$\psi_4' = -[E\alpha h^2(\lambda_{xxx}^{(4)} z_x + z_{xxx}\lambda_x^{(4)})]V\Big|_0^{\hat{l}} \tag{3.3.48}$$

For a cantilevered beam, applying the boundary conditions in Eq. (2.1.17) for both z and $\lambda^{(4)}$ to Eq. (3.3.46) yields

$$\psi_4' = E\alpha h^2 z_{xx}\lambda_{xx}^{(4)}V\Big|_{x=0} + f\lambda^{(4)}V\Big|_{x=\hat{l}} \tag{3.3.49}$$

For a clamped–simply supported beam, applying boundary conditions of Eq. (2.1.18) for both z and $\lambda^{(4)}$ in Eq. (3.3.46) yields

$$\psi_4' = -E\alpha h^2 z_{xx}\lambda_{xx}^{(4)}V\Big|_{x=0} - [E\alpha h^2(\lambda_{xxx}^{(4)} z_x + z_{xxx}\lambda_x^{(4)})]V\Big|_{x=\hat{l}} \tag{3.3.50}$$

As in the displacement functional ψ_3, the shape design sensitivity results in Eq. (3.3.46) or Eqs. (3.3.47)–(3.3.50) for each boundary condition are valid for the functional ψ_4 that defines the average stress on a fixed interval (x_a, x_b). The case in which ψ_4 is averaged stress on the moving interval (x_{a_τ}, x_{b_τ}) in a deformed domain Ω_τ will be considered in Section 3.3.6. Another assumption that is used for the average stress functional is that the interval (x_a, x_b), on which stress is averaged, is taken such that $[x_a, x_b] \subset (0, \hat{l})$. Hence, $x_a \neq 0$ and $x_b \neq \hat{l}$. The case in which either $x_a = 0$ or $x_b = \hat{l}$ will be considered in Section 3.3.6.

DEFLECTION OF A MEMBRANE

Consider the membrane of Fig. 3.1.1, with mass density h. The area of the membrane is

$$\psi_1 = \iint_\Omega d\Omega \tag{3.3.51}$$

Taking the variation and using Eq. (3.2.36),

$$\psi_1' = \int_\Gamma (V^{\mathrm{T}}n)\, d\Gamma \tag{3.3.52}$$

Note that this direct variation calculation gives the explicit form of variation of area in terms of the normal velocity $(V^{\mathrm{T}}n)$ of the boundary. Thus, for this functional, no adjoint problem needs to be defined.

Consider a second functional that respresents strain energy of the membrane,

$$\psi_2 = \frac{\hat{T}}{2} \iint_\Omega \nabla z^{\mathrm{T}}\, \nabla z\, d\Omega \tag{3.3.53}$$

From Eq. (3.1.14), the strain energy ψ_2 is equal to half of the compliance, so that

$$\psi_2 = \frac{1}{2} \iint_\Omega fz\, d\Omega \tag{3.3.54}$$

Note that the integrand of Eq. (3.3.54) depends on the load f. However, since $f' = 0$, Eq. (3.3.54) can be treated as the functional form of Eq. (3.3.10). Hence, the adjoint equation of Eq. (3.3.13) is

$$a_\Omega(\lambda, \bar{\lambda}) = \frac{1}{2} \iint_\Omega f\bar{\lambda}\, d\Omega \qquad \text{for all} \quad \bar{\lambda} \in Z \tag{3.3.55}$$

The load functional on the right side of Eq. (3.3.55) is precisely half the load functional for the original membrane problem of Eq. (3.1.14). Hence, in this special case, $\lambda = z/2$, and from Eq. (3.3.17) with $z = 0$ on Γ,

$$\psi_2' = \iint_\Omega [\hat{T}(\nabla z^{\mathrm{T}}\, \nabla(\nabla z^{\mathrm{T}} V)) - f(\nabla z^{\mathrm{T}} V)]\, d\Omega - \frac{\hat{T}}{2} \int_\Gamma (\nabla z^{\mathrm{T}}\, \nabla z)(V^{\mathrm{T}}n)\, d\Gamma \tag{3.3.56}$$

The variational identity of Eq. (3.1.12), identifying $(\nabla z^{\mathrm{T}} V)$ in the integral of Eq. (3.3.56) with $\bar{z}$ in Eq. (3.1.12), may be used to obtain

$$\psi_2' = \hat{T} \int_\Gamma \frac{\partial z}{\partial n} (\nabla z^{\mathrm{T}} V)\, d\Gamma - \frac{\hat{T}}{2} \int_\Gamma (\nabla z^{\mathrm{T}}\, \nabla z)(V^{\mathrm{T}} n)\, d\Gamma \tag{3.3.57}$$

Since $z = 0$ on Γ, $\nabla z = (\partial z/\partial n)n$ on Γ, which yields the simplified result

$$\psi_2' = \frac{\hat{T}}{2} \int_\Gamma \left(\frac{\partial z}{\partial n}\right)^2 (V^{\mathrm{T}} n)\, d\Gamma \tag{3.3.58}$$

As noted in Section 3.3.2, the shape design sensitivity in Eq. (3.3.58) is expressed as a boundary integral, and only normal movements $(V^{\mathrm{T}} n)$ of the boundary appear.

TORSION OF AN ELASTIC SHAFT

Consider torsion of the elastic shaft of Fig. 3.1.2. The torsional rigidity in Eq. (3.1.18) can be considered as a response functional; that is,

$$\psi = 2 \iint_\Omega z\, d\Omega \tag{3.3.59}$$

The adjoint equation is, from Eq. (3.3.13),

$$a_\Omega(\lambda, \bar{\lambda}) = 2 \iint_\Omega \bar{\lambda}\, d\Omega \qquad \text{for all} \quad \bar{\lambda} \in Z \tag{3.3.60}$$

Thus, in this special case $\lambda = z$. Comparing this with the membrane problem,

$$\psi' = \int_\Gamma \left(\frac{\partial z}{\partial n}\right)^2 (V^{\mathrm{T}} n)\, d\Gamma \tag{3.3.61}$$

As an example that can be calculated analytically, consider the elastic shaft with circular cross section (Fig. 3.3.1) and radius a as a design parameter. The Prandtl stress function z for a circular cross section is [57]

$$z = \tfrac{1}{2}(a^2 - x_1^2 - x_2^2) = \tfrac{1}{2}(a^2 - r^2)$$

Using polar coordinates, the torsional rigidity of Eq. (3.3.59) is

$$\psi = \int_0^{2\pi} \int_0^a (a^2 - r^2) r\, dr\, d\theta = \frac{\pi a^4}{2}$$

Considering the radius a as a design parameter, the variation of torsional rigidity with respect to a is

$$\psi' = 2\pi a^3\, \delta a$$

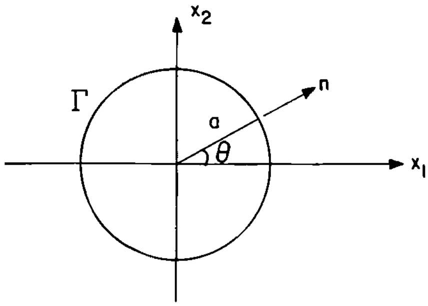

Fig. 3.3.1 Circular cross section of an elastic shaft.

Using polar coordinates, on the boundary Γ,

$$n = [\cos\theta \quad \sin\theta]^{\mathrm{T}}$$

and

$$V = [V^1 \quad V^2]^{\mathrm{T}} = [\delta r \cos\theta \quad \delta r \sin\theta]^{\mathrm{T}}$$

Also,

$$\nabla z^{\mathrm{T}} n = \partial z/\partial r = -r$$

Hence, from Eq. (3.3.61),

$$\psi' = \int_0^{2\pi} (-r)^2 \, \delta r \, r \, d\theta \bigg|_{r=a} = 2\pi a^3 \, \delta a$$

which is the correct result.

BENDING OF A PLATE

Consider the plate of Section 3.1 with thickness $h(x) \geq h_0 > 0$ and constant Young's modulus E. The functional defining weight of the plate is

$$\psi_1 = \iint_\Omega \gamma h \, d\Omega \tag{3.3.62}$$

where γ is weight density of the material. Taking variation, using Eq. (3.2.36) with $(\gamma h)' = 0$,

$$\psi_1' = \int_\Gamma \gamma h (V^{\mathrm{T}} n) \, d\Gamma \tag{3.3.63}$$

Thus, no adjoint variable is necessary and the explicit design derivative of weight is obtained.

Consider next the compliance functional for the plate,

$$\psi_2 = \iint_\Omega fz \, d\Omega \tag{3.3.64}$$

As in Eq. (3.3.52), since $f' = 0$, Eq. (3.3.64) can be treated as the functional form of Eq. (3.3.10), so from Eq. (3.3.13), the adjoint equation is

$$a_\Omega(\lambda, \bar{\lambda}) = \iint_\Omega f\bar{\lambda} \, d\Omega \qquad \text{for all} \quad \bar{\lambda} \in Z \tag{3.3.65}$$

In this special case, $\lambda = z$, and from Eq. (3.3.17) with $g = fz$,

$$\begin{aligned} \psi_2' = \iint_\Omega \{2\hat{D}[(z_{11} + \nu z_{22})(\nabla z^{\mathrm{T}} V)_{11} + (z_{22} + \nu z_{11})(\nabla z^{\mathrm{T}} V)_{22} \\ + 2(1 - \nu) z_{12}(\nabla z^{\mathrm{T}} V)_{12}] - 2f(\nabla z^{\mathrm{T}} V)\} \, d\Omega \\ + \int_\Gamma \{2fz - \hat{D}[(z_{11} + \nu z_{22})z_{11} + (z_{22} + \nu z_{11})z_{22} \\ + 2(1 - \nu) z_{12}^2]\}(V^{\mathrm{T}} n) \, d\Gamma \end{aligned} \tag{3.3.66}$$

The variational identity of Eq. (3.1.23) may be used, identifying $(\nabla z^{\mathrm{T}} V)$ in the domain integral of Eq. (3.3.66) with $\bar{z}$ in Eq. (3.1.23), to obtain

$$\begin{aligned} \psi_2' = 2\int_\Gamma (\nabla z^{\mathrm{T}} V) Nz \, d\Gamma + 2\int_\Gamma \frac{\partial}{\partial n}(\nabla z^{\mathrm{T}} V) Mz \, d\Gamma \\ + \int_\Gamma \{2fz - \hat{D}[(z_{11} + \nu z_{22})z_{11} + (z_{22} + \nu z_{11})z_{22} \\ + 2(1 - \nu) z_{12}^2]\}(V^{\mathrm{T}} n) \, d\Gamma \end{aligned} \tag{3.3.67}$$

For the clamped part $\Gamma_C \subset \Gamma$ of the boundary, using boundary conditions of Eq. (3.1.26), $\nabla z = 0$ on Γ_C. Also,

$$\frac{\partial}{\partial n}(\nabla z^{\mathrm{T}} V) = \sum_{i,j=1}^{2} (V^i z_{ij} n_j + V_j^i z_i n_j)$$

Since $\nabla z = 0$ on Γ_C, this becomes

$$\frac{\partial}{\partial n}(\nabla z^{\mathrm{T}} V) = \sum_{i,j=1}^{2} V^i z_{ij} n_j = \frac{\partial^2 z}{\partial n^2}(V^{\mathrm{T}} n) + \frac{\partial^2 z}{\partial n \, \partial s}(V^{\mathrm{T}} s) \tag{3.3.68}$$

where the second equality of Eq. (3.3.68) can be verified by expanding the last term of Eq. (3.3.68). Since $\partial z/\partial n = 0$ on Γ_C, $(\partial/\partial s)(\partial z/\partial n) = 0$ on Γ_C, and Eq. (3.3.68) becomes

$$\frac{\partial}{\partial n}(\nabla z^{\mathrm{T}} V) = \frac{\partial^2 z}{\partial n^2}(V^{\mathrm{T}} n), \qquad x \in \Gamma_C \tag{3.3.69}$$

Also, since $\partial z/\partial s = 0$ on Γ_C, $\partial^2 z/\partial s^2 = 0$ on Γ_C, so

$$Mz = \hat{D}\left[\frac{\partial^2 z}{\partial n^2} + \nu\left(\frac{1}{r}\frac{\partial z}{\partial n} + \frac{\partial^2 z}{\partial s^2}\right)\right] = \hat{D}\left(\frac{\partial^2 z}{\partial n^2}\right), \qquad x \in \Gamma_C \quad (3.3.70)$$

where r is the radius of curvature of the boundary Γ_C. Using boundary conditions of Eqs. (3.1.26) and (3.3.67), the sensitivity formula due to a variation of the clamped boundary Γ_C is

$$\psi_2' = \int_{\Gamma_C} \hat{D}\left\{2\left(\frac{\partial^2 z}{\partial n^2}\right)^2 - [(z_{11} + \nu z_{22})z_{11} + (z_{22} + \nu z_{11})z_{22} + 2(1-\nu)z_{12}^2]\right\}(V^T n)\, d\Gamma \qquad (3.3.71)$$

which is valid for variable thickness $h(x)$. As before, the design sensitivity in Eq. (3.3.71) is expressed as a boundary integral, and only the normal movement $(V^T n)$ of the boundary Γ_C appears.

It was shown by Mikhlin [32] that if the boundary conditions of Eq. (3.1.26) are satisfied, then

$$\iint_\Omega (z_{12}^2 - z_{11}z_{22})\, d\Omega = 0$$

Hence, if the thickness $h(x)$ of the plate is constant, then the variational equation of Eq. (3.1.29) is simplified to

$$a_\Omega(z, \bar{z}) \equiv \hat{D} \iint_\Omega (\nabla^2 z)(\nabla^2 \bar{z})\, d\Omega = \iint_\Omega f\bar{z}\, d\Omega \equiv l_\Omega(\bar{z}) \qquad \text{for all} \quad \bar{z} \in Z$$

Proceeding exactly as before, instead of Eq. (3.3.71), a simplified expression is obtained:

$$\begin{aligned}\psi_2' &= \int_{\Gamma_C} \hat{D}\left[2\left(\frac{\partial^2 z}{\partial n^2}\right)^2 - (\nabla^2 z)^2\right](V^T n)\, d\Gamma \\ &= \hat{D}\int_{\Gamma_C}\left(\frac{\partial^2 z}{\partial n^2}\right)^2 (V^T n)\, d\Gamma \end{aligned} \qquad (3.3.72)$$

As an analytical example, consider a clamped circular plate with constant thickness h, radius a, and concentrated load $f = p\,\hat{\delta}(x)$ at the center of the plate, as shown in Fig. 3.3.2. The displacement of the plate is given as [58]

$$z = \frac{p}{16\pi\hat{D}}\left[a^2 - r^2\left(1 + 2ln\frac{a}{r}\right)\right]$$

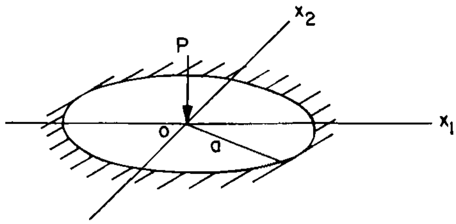

Fig. 3.3.2 Circular plate with concentrated load.

where $r^2 = x_1^2 + x_2^2$. From Eq. 3.3.64 the compliance functional of the plate is

$$\psi_2 = \iint_\Omega \frac{p^2\hat{\delta}(x)}{16\pi\hat{D}}\left[a^2 - r^2\left(1 + 2ln\frac{a}{r}\right)\right] d\Omega = \frac{p^2a^2}{16\pi\hat{D}}$$

Considering the radius a of the plate as a design parameter, the variation of compliance with respect to a is

$$\psi_2' = \frac{p^2a}{8\pi\hat{D}}\,\delta a$$

Expressing Eq. (3.3.72) in polar coordinates, with $(V^Tn) = \delta r$ and $\partial^2 z/\partial n^2 = \partial^2 z/\partial r^2$ on the boundary of the circle, gives

$$\begin{aligned}\psi_2' &= \hat{D}\int_0^{2\pi}\left(\frac{\partial^2 z}{\partial r^2}\right)^2 \delta r\, r\, d\theta\bigg|_{r=a} \\ &= \hat{D}\left[\frac{p}{16\pi\hat{D}}\left(-4ln\frac{a}{r} + 4\right)\right]^2 2\pi r\,\delta r\bigg|_{r=a} \\ &= \frac{p^2a}{8\pi\hat{D}}\,\delta a\end{aligned}$$

which is the same result.

For other boundary conditions in Eqs. (3.1.27) and (3.1.28), the sensitivity formula in Eq. (3.3.67) for compliance is valid because the variational equation of Eq. (3.1.29) is valid for all $\bar{z}$ that satisfy corresponding kinematic boundary conditions. To obtain a sensitivity formula for variation of the simply supported part $\Gamma_S \subset \Gamma$ of the boundary, from boundary conditions of Eq. (3.1.27), $z = 0$ on Γ_S implies $\partial z/\partial s = 0$ on Γ_S, so $\nabla z = (\partial z/\partial n)n$. Thus, from Eq. (3.3.67),

$$\begin{aligned}\psi_2' = \int_{\Gamma_S}\Big\{2\left(\frac{\partial z}{\partial n}\right)Nz &- \hat{D}[(z_{11} + \nu z_{22})z_{11} \\ &+ (z_{22} + \nu z_{11})z_{22} + 2(1-\nu)z_{12}^2]\Big\}(V^Tn)\, d\Gamma \qquad (3.3.73)\end{aligned}$$

For the free edge $\Gamma_F \subset \Gamma$ of the boundary, applying boundary conditions of Eq. (3.1.28) to Eq. (3.3.67),

$$\psi_2' = \int_{\Gamma_F} \{2fz - \hat{D}[(z_{11} + vz_{22})z_{11} + (z_{22} + vz_{11})z_{22} + 2(1 - v)z_{12}^2]\}(V^T n)\, d\Gamma \tag{3.3.74}$$

If $\Gamma = \Gamma_C \cup \Gamma_S \cup \Gamma_F$, the complete shape design sensitivity formula is obtained by adding terms from Eqs. (3.3.71), (3.3.73), and (3.3.74).

Consider next displacement at a discrete point $\hat{x}$, written as

$$\psi_3 = \iint_\Omega \hat{\delta}(x - \hat{x})\, z\, d\Omega \tag{3.3.75}$$

where $\hat{x} \in \Omega$ is a fixed point and $\hat{\delta}(x)$ is the Dirac measure in the plane, acting at the origin. Since $m = 2$ and $n = 2$, $m > n/2$. By the Sobolev imbedding theorem [36], $z_\tau \in C^0(\Omega_\tau)$. The functional of Eq. (3.3.75) is thus continuous, and the foregoing theory applies.

Since $\hat{\delta}(x - \hat{x})$ is defined on a neighborhood of $\bar{\Omega}$ by zero extension and $\hat{x}$ is fixed, $\hat{\delta}'(x - \hat{x}) = 0$. Thus, Eq. (3.3.75) can be treated as the functional form of Eq. (3.3.10), and from Eq. (3.3.13) the adjoint equation is

$$a_\Omega(\lambda, \bar{\lambda}) = \iint_\Omega \hat{\delta}(x - \hat{x})\, \bar{\lambda}\, d\Omega \qquad \text{for all} \quad \bar{\lambda} \in Z \tag{3.3.76}$$

Equation (3.3.76) has a unique solution $\lambda^{(3)}$, which is the displacement due to a unit load at $\hat{x}$. With smoothness assumptions, the variational equation of Eq. (3.3.76) is equivalent to the formal operator equation

$$[\hat{D}(\lambda_{11} + v\lambda_{22})]_{11} + [\hat{D}(\lambda_{22} + v\lambda_{11})]_{22} + 2(1 - v)[\hat{D}\lambda_{12}]_{12} = \hat{\delta}(x - \hat{x}), \qquad x \in \Omega \tag{3.3.77}$$

with λ satisfying the same boundary conditions as the original structural response z. From Eq. (3.3.17), with $g = \hat{\delta}(x - \hat{x})\, z$,

$$\begin{aligned}\psi_3' = \iint_\Omega \{&\hat{D}[(z_{11} + vz_{22})(\nabla\lambda^{(3)T}V)_{11} + (z_{22} + vz_{11})(\nabla\lambda^{(3)T}V)_{22} \\ &+ 2(1 - v)z_{12}(\nabla\lambda^{(3)T}V)_{12}] - f(\nabla\lambda^{(3)T}V) \\ &+ \hat{D}[(\lambda_{11}^{(3)} + v\lambda_{22}^{(3)})(\nabla z^T V)_{11} + (\lambda_{22}^{(3)} + v\lambda_{11}^{(3)})(\nabla z^T V)_{22} \\ &+ 2(1 - v)\lambda_{12}^{(3)}(\nabla z^T V)_{12}] - \hat{\delta}(x - \hat{x})(\nabla z^T V)\}\, d\Omega \\ &+ \int_\Gamma \{f\lambda^{(3)} - \hat{D}[(z_{11} + vz_{22})\lambda_{11}^{(3)} + (z_{22} + vz_{11})\lambda_{22}^{(3)} \\ &+ 2(1 - v)z_{12}\lambda_{12}^{(3)}]\}(V^T n)\, d\Gamma\end{aligned} \tag{3.3.78}$$

As in the beam problem [Eq. (3.3.32)], the variational identity of Eq. (3.1.23) may be used twice to transform the domain integral of Eq. (3.3.78) to a boundary integral, obtaining

$$\begin{aligned}\psi_3' = &\int_\Gamma \left[(\nabla\lambda^{(3)T}V)Nz + \frac{\partial}{\partial n}(\nabla\lambda^{(3)T}V)\, Mz\right] d\Gamma \\ &+ \int_\Gamma \left[(\nabla z^{T}V)N\lambda^{(3)} + \frac{\partial}{\partial n}(\nabla z^{T}V)M\lambda^{(3)}\right] d\Gamma \\ &+ \int_\Gamma \{f\lambda^{(3)} - \hat{D}[(z_{11} + \nu z_{22})\lambda_{11}^{(3)} + (z_{22} + \nu z_{11})\lambda_{22}^{(3)} \\ &\qquad + 2(1-\nu)z_{12}\lambda_{12}^{(3)}]\}(V^{T}n)\, d\Gamma \end{aligned} \tag{3.3.79}$$

Using boundary conditions of Eq. (3.1.26) and the fact that Eqs. (3.3.69) and (3.3.70) hold for $\lambda^{(3)}$ as well as z, the sensitivity formula due to a variation of the clamped boundary Γ_C is

$$\begin{aligned}\psi_3' = \int_{\Gamma_C} \hat{D}\Bigg\{&2\left(\frac{\partial^2 z}{\partial n^2}\right)\left(\frac{\partial^2 \lambda^{(3)}}{\partial n^2}\right) - [(z_{11} + \nu z_{22})\lambda_{11}^{(3)} + (z_{22} + \nu z_{11})\lambda_{22}^{(3)} \\ &+ 2(1-\nu)z_{12}\lambda_{12}^{(3)}]\Bigg\}(V^{T}n)\, d\Gamma \end{aligned} \tag{3.3.80}$$

which is valid for variable thickness $h(x)$. As in the compliance functional, if the thickness $h(x)$ of the plate is constant, a simplified expression is obtained:

$$\psi_3' = \hat{D}\int_{\Gamma_C} \left(\frac{\partial^2 z}{\partial n^2}\right)\left(\frac{\partial^2 \lambda^{(3)}}{\partial n^2}\right)(V^{T}n)\, d\Gamma \tag{3.3.81}$$

As an analytical example, consider a clamped circular plate of constant thickness h, radius a, and linearly increasing axi-symmetric load $f = (q/a_0)r$, as shown in Fig. 3.3.3, where a_0 is the present design.

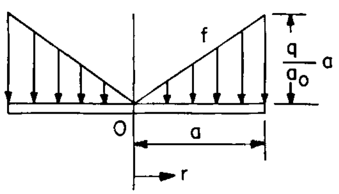

Fig. 3.3.3 Circular plate with axisymmetric load.

The displacement of the plate with radius a is given as [58]

$$z_a = \frac{qa}{450\hat{D}a_0}\left(3a^4 - 5a^2r^2 + 2\frac{r^5}{a}\right)$$

Taking $\hat{x}$ as the center of the plate, ψ_3 in Eq. (3.3.75) is

$$\psi_3 = \frac{qa^5}{150\hat{D}a_0}$$

Considering the radius a of the plate as a design parameter, the variation of ψ_3 with respect to a, evaluated at the present design a_0, is

$$\psi_3' = \frac{\partial \psi_3}{\partial a}\,\delta a\bigg|_{a=a_0} = \frac{qa_0^3}{30\hat{D}}\,\delta a_0$$

To use Eq. (3.3.81), the adjoint response of Eq. (3.3.71) must be found, where the adjoint load is a unit point load at the center of the plate. The adjoint displacement is for a plate with radius a [58]

$$\lambda^{(3)} = \frac{1}{16\pi\hat{D}}\left[a^2 - r^2\left(1 + 2ln\frac{a}{r}\right)\right]$$

Expressing Eq. (3.3.81) in polar coordinates,

$$\begin{aligned}
\psi_3' &= \hat{D}\int_0^{2\pi}\left(\frac{\partial^2 z}{\partial r^2}\right)\left(\frac{\partial^2 \lambda^{(3)}}{\partial r^2}\right)\delta r\, r\, d\theta\bigg|_{r=a_0,\,a=a_0} \\
&= \hat{D}\left[\frac{qa}{(450)\hat{D}a_0}\left(-10a^2 + 40\frac{r^3}{a}\right)\right]\left[\frac{1}{16\pi\hat{D}}\left(-4ln\frac{a}{r} + 4\right)\right]2\pi r\,\delta r\bigg|_{r=a_0,\,a=a_0} \\
&= \frac{qa_0^3}{30\hat{D}}\,\delta a_0
\end{aligned}$$

which is the correct result.

For other boundary conditions in Eqs. (3.3.27) and (3.1.28), proceeding as in the compliance functional yields

$$\begin{aligned}
\psi_3' = \int_{\Gamma_S}\bigg\{&\left(\frac{\partial \lambda^{(3)}}{\partial n}\right)Nz + \left(\frac{\partial z}{\partial n}\right)N\lambda^{(3)} - \hat{D}[(z_{11} + \nu z_{22})\lambda_{11}^{(3)} \\
&+ (z_{22} + \nu z_{11})\lambda_{22}^{(3)} + 2(1-\nu)z_{12}\lambda_{12}^{(3)}]\bigg\}(V^{T}n)\,d\Gamma
\end{aligned} \tag{3.3.82}$$

for the variation of a simply supported part $\Gamma_S \subset \Gamma$ of the boundary. For the variation of a free edge $\Gamma_F \subset \Gamma$ of the boundary, from Eq. (3.3.79),

$$\psi_3' = \int_{\Gamma_F} \{f\lambda^{(3)} - \hat{D}[(z_{11} + \nu z_{22})\lambda_{11}^{(3)} + (z_{22} + \nu z_{11})\lambda_{22}^{(3)} + 2(1 - \nu)z_{12}\lambda_{12}^{(3)}]\}(V^T n)\, d\Gamma \tag{3.3.83}$$

If $\Gamma = \Gamma_C \cup \Gamma_S \cup \Gamma_F$, the complete shape design sensitivity formula can be obtained by adding terms in Eqs. (3.3.80), (3.3.82), and (3.3.83).

As in a beam displacement functional, the sensitivity results in Eq. (3.3.79) or Eqs. (3.3.80)–(3.3.83) for each boundary condition are valid for displacement at a fixed point $\hat{x}$. The case in which ψ_3 is the displacement at a moving point $\hat{x}_\tau = \hat{x} + \tau V(\hat{x})$ in a deformed domain Ω_τ will be considered in Section 3.3.6.

The maximum stress in a thin plate occurs on the surface of the plate and is given in the form [33]

$$\begin{aligned} \sigma^{11} &= -\frac{Eh}{2(1 - \nu^2)}(z_{11} + \nu z_{22}) \\ \sigma^{22} &= -\frac{Eh}{2(1 - \nu^2)}(z_{22} + \nu z_{11}) \\ \sigma^{12} &= -\frac{Eh}{2(1 + \nu)}z_{12} \end{aligned} \tag{3.3.84}$$

The von Mises failure criterion is [33]

$$g(\sigma) = [(\sigma^{11} + \sigma^{22})^2 + 3(\sigma^{11} - \sigma^{22})^2 + 12(\sigma^{12})^2]^{1/2} - 4\sigma_p \leq 0 \tag{3.3.85}$$

where σ_p is a given yield stress. Instead of the von Mises failure criterion, for simplicity assume that the stress σ^{11} in Eq. (3.3.84) is taken as a strength constraint. With this done, the idea can be extended to the von Mises failure criterion.

As in the case of the beam, since a pointwise stress constraint is not meaningful, the characteristic function approach of Eq. (3.3.36) may be used. That is, define a function $m_p(x)$ that is positive and constant on a small open subset Ω_p such that $\bar{\Omega}_p \subset \Omega$, zero outside of Ω_p, and its integral is 1. Then, the average value of σ^{11} over this small region is

$$\begin{aligned} \psi_4 &= \iint_\Omega \sigma^{11} m_p\, d\Omega \\ &= -\iint_\Omega \frac{Eh}{2(1 - \nu^2)}(z_{11} + \nu z_{22}) m_p\, d\Omega \end{aligned} \tag{3.3.86}$$

If the average stress on the fixed region Ω_p is of concern, m_p in Eq. (3.3.86) does not change with τ. It is possible to extend m_p to R^2 by defining it to be zero outside Ω. Then, $m_p' = 0$.

Taking the variation of Eq. (3.3.86), using Eqs. (3.2.13) and (3.2.36) and $h' = 0$, yields

$$\begin{aligned}\psi_4' &= -\iint_\Omega \frac{Eh}{2(1-v^2)}[(z')_{11} + v(z')_{22}]m_p\, d\Omega \\ &\quad - \int_\Gamma \frac{Eh}{2(1-v^2)}(z_{11} + vz_{22})m_p(V^{\mathrm{T}}n)\, d\Gamma \\ &= -\iint_\Omega \frac{Eh}{2(1-v^2)}((\dot{z})_{11} + v(\dot{z})_{22} - (\nabla z^{\mathrm{T}}V)_{11} - v(\nabla z^{\mathrm{T}}V)_{22})m_p\, d\Omega\end{aligned} \tag{3.3.87}$$

since $m_p = 0$ on Γ. As in the general derivation of the adjoint equation of Eq. (3.3.13), the adjoint equation may be defined by replacing $\dot{z}$ in the first two terms on the right side of Eq. (3.3.87) by $\bar{\lambda}$ to define a load functional for the adjoint equation, obtaining

$$a_\Omega(\lambda, \bar{\lambda}) = -\iint_\Omega \frac{Eh}{2(1-v^2)}(\bar{\lambda}_{11} + v\bar{\lambda}_{22})m_p\, d\Omega \qquad \text{for all} \quad \bar{\lambda} \in Z \tag{3.3.88}$$

Note that the adjoint equation of Eq. (3.3.88) is the same as Eq. (2.2.63). Recalling the norm in $H^2(\Omega)$ of Eq. (2.1.22), it can be shown that the linear form in $\bar{\lambda}$ on the right side of Eq. (3.3.88) is bounded in $H^2(\Omega)$. Hence, by the Lax–Milgram theorem [9], Eq. (3.3.88) has a unique solution $\lambda^{(4)}$.

With smoothness assumptions, the variational equation of Eq. (3.3.88) is equivalent to the formal operator equation

$$\begin{aligned}&[\hat{D}(\lambda_{11} + v\lambda_{22})]_{11} + [\hat{D}(\lambda_{22} + v\lambda_{11})]_{22} + 2(1-v)[\hat{D}\lambda_{12}]_{12} \\ &\quad = -\left[\frac{Eh}{2(1-v^2)}m_p\right]_{11} - v\left[\frac{Eh}{2(1-v^2)}m_p\right]_{22}\end{aligned} \tag{3.3.89}$$

with λ satisfying the same boundary conditions as the original structural response z. As in the adjoint equation of Eq. (3.3.40) for the beam problem, the derivatives on the right side of Eq. (3.3.89) are in the sense of distributions. Moreover, the distributional derivatives m_{p_i} and $m_{p_{ij}}$ $(i, j = 1, 2)$ depend on the equation that represents the boundary Ω_p. (The reader is referred to Kecs and Teodorescu [56] for a detailed treatment of the distributional derivative.) By the same method as in Section 3.1, a variational identity for the adjoint system can be obtained from Eq. (3.3.89) as

$$\iint_\Omega \hat{D}[(\lambda_{11} + \nu\lambda_{22})\bar{\lambda}_{11} + (\lambda_{22} + \nu\lambda_{11})\bar{\lambda}_{22} + 2(1-\nu)\lambda_{12}\bar{\lambda}_{12}]\, d\Omega$$

$$+ \iint_\Omega \frac{Eh}{2(1-\nu^2)}(\bar{\lambda}_{11} + \nu\bar{\lambda}_{22})m_p\, d\Omega$$

$$= \int_\Gamma \bar{\lambda}N\lambda\, d\Gamma + \int_\Gamma \frac{\partial\bar{\lambda}}{\partial n}M\lambda\, d\Gamma$$

$$+ \int_\Gamma \left\{\frac{Eh}{2(1-\nu^2)}m_p\bar{\lambda}_1 n_1 - \left[\frac{Eh}{2(1-\nu^2)}m_p\right]_1 \bar{\lambda}n_1 + \frac{Eh}{2(1-\nu^2)}m_p\nu\bar{\lambda}_2 n_2\right.$$

$$\left. - \left[\frac{Eh}{2(1-\nu^2)}m_p\right]_2 \nu\bar{\lambda}n_2\right\} d\Gamma \qquad \text{for all} \quad \bar{\lambda} \in H^2(\Omega) \qquad (3.3.90)$$

Since $\dot{z} \in Z$, Eq. (3.3.88) may be evaluated at $\bar{\lambda} = \dot{z}$ to obtain

$$a_\Omega(\lambda^{(4)}, \dot{z}) = -\iint_\Omega \frac{Eh}{2(1-\nu^2)}[(\dot{z})_{11} + \nu(\dot{z})_{22}]m_p\, d\Omega \qquad (3.3.91)$$

Similarly, evaluating the identity of Eq. (3.3.8) at $\bar{z} = \lambda^{(4)}$, since both are in Z, yields

$$a_\Omega(\dot{z}, \lambda^{(4)}) = l'_V(\lambda^{(4)}) - a'_V(z, \lambda^{(4)}) \qquad (3.3.92)$$

Since the energy bilinear form $a_\Omega(\cdot, \cdot)$ is symmetric, Eqs. (3.3.87), (3.3.91), and (3.3.92) yield

$$\psi'_4 = l'_V(\lambda^{(4)}) - a'_V(z, \lambda^{(4)}) + \iint_\Omega \frac{Eh}{2(1-\nu^2)}[(\nabla z^{\mathrm{T}}V)_{11} + \nu(\nabla z^{\mathrm{T}}V)_{22}]m_p\, d\Omega$$

which can be rewritten, using Eq. (3.3.8), as

$$\psi'_4 = \iint_\Omega \left\{\hat{D}[(z_{11} + \nu z_{22})(\nabla\lambda^{(4)\mathrm{T}}V)_{11} + (z_{22} + \nu z_{11})(\nabla\lambda^{(4)\mathrm{T}}V)_{22}\right.$$

$$+ 2(1-\nu)z_{12}(\nabla\lambda^{(4)\mathrm{T}}V)_{12}]$$

$$- f(\nabla\lambda^{(4)\mathrm{T}}V) + \hat{D}[(\lambda^{(4)}_{11} + \nu\lambda^{(4)}_{22})(\nabla z^{\mathrm{T}}V)_{11} + (\lambda^{(4)}_{22} + \nu\lambda^{(4)}_{11})(\nabla z^{\mathrm{T}}V)_{22}$$

$$+ 2(1-\nu)\lambda^{(4)}_{12}(\nabla z^{\mathrm{T}}V)_{12}]$$

$$\left. + \frac{Eh}{2(1-\nu^2)}[(\nabla z^{\mathrm{T}}V)_{11} + \nu(\nabla z^{\mathrm{T}}V)_{22}]m_p\right\} d\Omega$$

$$+ \int_\Gamma \{f\lambda^{(4)} - \hat{D}[(z_{11} + \nu z_{22})\lambda^{(4)}_{11} + (z_{22} + \nu z_{11})\lambda^{(4)}_{22}$$

$$+ 2(1-\nu)z_{12}\lambda^{(4)}_{12}]\}(V^{\mathrm{T}}n)\, d\Gamma \qquad (3.3.93)$$

As in the beam problem [Eq. (3.3.45)], the variational identities of Eqs. (3.1.23) and (3.3.90) may be used to transform the domain integral of Eq. (3.3.93) to a boundary integral, obtaining

$$\begin{aligned}\psi_4' &= \int_\Gamma \left[(\nabla\lambda^{(4)\mathrm{T}}V)Nz + \frac{\partial}{\partial n}(\nabla\lambda^{(4)\mathrm{T}}V)Mz\right] d\Gamma \\ &\quad + \int_\Gamma \left[(\nabla z^{\mathrm{T}}V)N\lambda^{(4)} + \frac{\partial}{\partial n}(\nabla z^{\mathrm{T}}V)M\lambda^{(4)}\right] d\Gamma \\ &\quad + \int_\Gamma \left\{\frac{Eh}{2(1-\nu^2)}m_p(\nabla z^{\mathrm{T}}V)_1 n_1 - \left[\frac{Eh}{2(1-\nu^2)}m_p\right]_1 (\nabla z^{\mathrm{T}}V)n_1 \right. \\ &\qquad \left. + \frac{Eh}{2(1-\nu^2)}m_p \nu(\nabla z^{\mathrm{T}}V)_2 n_2 - \left[\frac{Eh}{2(1-\nu^2)}m_p\right]_2 \nu(\nabla z^{\mathrm{T}}V)n_2\right\} d\Gamma \\ &\quad + \int_\Gamma \{f\lambda^{(4)} - \hat{D}[(z_{11} + \nu z_{22})\lambda_{11}^{(4)} + (z_{22} + \nu z_{11})\lambda_{22}^{(4)} \\ &\qquad + 2(1-\nu)z_{12}\lambda_{12}^{(4)}]\}(V^{\mathrm{T}}n)\, d\Gamma \end{aligned} \tag{3.3.94}$$

Since $\bar{\Omega}_p \subset \Omega$, $m_p = 0$ in a neighborhood of Γ, and Eq. (3.3.94) becomes

$$\begin{aligned}\psi_4' &= \int_\Gamma \left[(\nabla\lambda^{(4)\mathrm{T}}V)Nz + \frac{\partial}{\partial n}(\nabla\lambda^{(4)\mathrm{T}}V)Mz\right] d\Gamma \\ &\quad + \int_\Gamma \left[(\nabla z^{\mathrm{T}}V)N\lambda^{(4)} + \frac{\partial}{\partial n}(\nabla z^{\mathrm{T}}V)M\lambda^{(4)}\right] d\Gamma \\ &\quad + \int_\Gamma \{f\lambda^{(4)} - \hat{D}[(z_{11} + \nu z_{22})\lambda_{11}^{(4)} + (z_{22} + \nu z_{11})\lambda_{22}^{(4)} \\ &\qquad + 2(1-\nu)z_{12}\lambda_{12}^{(4)}]\}(V^{\mathrm{T}}n)\, d\Gamma \end{aligned} \tag{3.3.95}$$

As in the displacement functional case, the sensitivity formulas due to variation of clamped, simply supported, and free edges of the boundary are

$$\begin{aligned}\psi_4' = \int_{\Gamma_C} \hat{D}\left\{2\left(\frac{\partial^2 z}{\partial n^2}\right)\left(\frac{\partial^2 \lambda^{(4)}}{\partial n^2}\right) - [(z_{11} + \nu z_{22})\lambda_{11}^{(4)} + (z_{22} + \nu z_{11})\lambda_{22}^{(4)} \right. \\ \left. + 2(1-\nu)z_{12}\lambda_{12}^{(4)}]\right\}(V^{\mathrm{T}}n)\, d\Gamma \end{aligned} \tag{3.3.96}$$

$$\begin{aligned}\psi_4' = \int_{\Gamma_S} \left\{\left(\frac{\partial \lambda^{(4)}}{\partial n}\right)Nz + \left(\frac{\partial z}{\partial n}\right)N\lambda^{(4)} - \hat{D}[(z_{11} + \nu z_{22})\lambda_{11}^{(4)} + (z_{22} + \nu z_{11})\lambda_{22}^{(4)} \right. \\ \left. + 2(1-\nu)z_{12}\lambda_{12}^{(4)}]\right\}(V^{\mathrm{T}}n)\, d\Gamma \end{aligned} \tag{3.3.97}$$

and

$$\psi_4' = \int_{\Gamma_F} \{f\lambda^{(4)} - \hat{D}[(z_{11} + \nu z_{22})\lambda_{11}^{(4)} + (z_{22} + \nu z_{11})\lambda_{22}^{(4)} + 2(1 - \nu)z_{12}\lambda_{12}^{(4)}]\}(V^T n)\, d\Gamma \tag{3.3.98}$$

respectively. For $\Gamma = \Gamma_C \cup \Gamma_S \cup \Gamma_F$, the complete shape design sensitivity formula is obtained by adding terms in Eqs. (3.3.96)–(3.3.98).

As in the beam problem, the shape design sensitivity results of Eqs. (3.3.95)–(3.3.98) for average stress are valid for the fixed region Ω_p. The case in which ψ_4 is average stress on the moving region $\Omega_{p_\tau} = T(\Omega_p, \tau)$ will be considered in Section 3.3.6. It has been assumed that $\bar{\Omega}_p \subset \Omega$, so that the boundary Γ_p of Ω_p does not meet the boundary Γ of Ω. The case in which Γ_p intersects Γ will be considered in Section 3.3.6.

3.3.4 Elasticity Problems

Shape design sensitivity analysis of the linear elasticity and interface problems of Section 3.1 is carried out in this section using the adjoint variable method. For plane stress or plane strain problems, the formulas derived in Section 3.1 remain valid, with limits of summation running from 1 to 2 and an appropriate modification of generalized Hooke's Law.

LINEAR ELASTICITY

Consider the three-dimensional elasticity problem of Section 3.1, with a mean stress constraint over a fixed test volume Ω_p, such that $\bar{\Omega}_p \subset \Omega$,

$$\psi = \iiint_{\Omega} g(\sigma(z)) m_p \, d\Omega \tag{3.3.99}$$

where σ denotes the stress tensor, Ω_p is an open set, and m_p is a characteristic function that is constant on Ω_p, zero outside of Ω_p, and whose integral is 1. The function g is assumed to be continuously differentiable with respect to its arguments. Note that $g(\sigma(z))$ might involve principal stresses, von Mises failure criterion, or some other material failure criteria. While the integrand in Eq. (3.3.99) could be written explicitly in terms of the gradient of z, as in the plate problem of Section 3.3.3, it will be seen that it is more effective to continue with the present notation.

For boundary perturbation in the elasticity problem, it is supposed that the boundary $\Gamma = \Gamma^0 \cup \Gamma^1 \cup \Gamma^2$ is varied, except that the curve $\partial\Gamma^2$ that bounds the loaded surface Γ^2 is fixed, so the velocity field V at $\partial\Gamma^2$ is zero. For the case in which $\partial\Gamma^2$ is not fixed, variation of the traction term in Eq. (3.1.38) (given as an integral over Γ^2) gives an additional term that was not discussed in Section 3.2.2. For this case, the interested reader is referred to

Zolesio [59]. Two kinds of boundary loads may be considered. One is a conservative load that depends on position but not the shape of the boundary. The other is a more general nonconservative load that depends not only on position but also on the shape of the boundary.

Consider first the conservative loading case in which the traction T^i in Eq. (3.1.36) depends on position only. Taking the variation of Eq. (3.1.38), using Eqs. (3.2.13), (3.2.36), and (3.2.51) and the fact that $f^{i'} = T^{i'} = 0$,

$$\iiint_\Omega \sum_{i,j=1}^{3} [\sigma^{ij}(z')\varepsilon^{ij}(\bar{z}) + \sigma^{ij}(z)\varepsilon^{ij}(\bar{z}') \, d\Omega + \iint_\Gamma \left[\sum_{i,j=1}^{3} \sigma^{ij}(z)\varepsilon^{ij}(\bar{z})\right](V^{\mathrm{T}}n)\, d\Gamma$$
$$= \iiint_\Omega \left[\sum_{i=1}^{3} f^i \bar{z}^{i'}\right] d\Omega + \iint_{\Gamma^1 \cup \Gamma^2} \left[\sum_{i=1}^{3} f^i \bar{z}^i\right](V^{\mathrm{T}}n)\, d\Gamma$$
$$+ \iint_{\Gamma^2} \left[\sum_{i=1}^{3} T^i \bar{z}^{i'}\right] d\Gamma + \iint_{\Gamma^2} \sum_{i=1}^{3} [\nabla(T^i\bar{z}^i)^{\mathrm{T}} n + H(T^i\bar{z}^i)](V^{\mathrm{T}}n)\, d\Gamma$$
$$\text{for all} \quad z \in \bar{Z} \qquad (3.3.100)$$

Using Eqs. (3.2.8) and (3.3.5), Eq. (3.3.100) can be rewritten as

$$a_\Omega(\dot{z}, \bar{z}) \equiv \iiint_\Omega \left[\sum_{i,j=1}^{3} \sigma^{ij}(\dot{z})\varepsilon^{ij}(\bar{z})\right] d\Omega$$
$$= \iiint_\Omega \sum_{i,j=1}^{3} [\sigma^{ij}(z)\varepsilon^{ij}(\nabla\bar{z}^{\mathrm{T}}V) + \sigma^{ij}(\nabla z^{\mathrm{T}}V)\varepsilon^{ij}(\bar{z})]\, d\Omega$$
$$- \iiint_\Omega \left[\sum_{i=1}^{3} f^i(\nabla\bar{z}^{i\mathrm{T}}V)\right] d\Omega - \iint_\Gamma \left[\sum_{i,j=1}^{3} \sigma^{ij}(z)\varepsilon^{ij}(\bar{z})\right](V^{\mathrm{T}}n)\, d\Gamma$$
$$+ \iint_{\Gamma^1 \cup \Gamma^2} \left[\sum_{i=1}^{3} f^i\bar{z}^i\right](V^{\mathrm{T}}n)\, d\Gamma$$
$$+ \iint_{\Gamma^2} \sum_{i=1}^{3} \{-T^i(\nabla\bar{z}^{i\mathrm{T}}V) + [\nabla(T^i\bar{z}^i)^{\mathrm{T}}n + H(T^i\bar{z}^i)](V^{\mathrm{T}}n)\}\, d\Gamma$$
$$\text{for all} \quad \bar{z} \in Z \qquad (3.3.101)$$

As in Eq. (3.3.8), Eq. (3.3.101) is a variational equation for $\dot{z} \in Z$. That is, $\dot{z} \in [H^1(\Omega)]^3$, and $\dot{z}$ satisfies kinematic boundary conditions.

Taking the variation of the functional of Eq. (3.3.99), using material derivative formulas of Eqs. (3.2.8) and (3.2.36) and $m_p' = 0$,

$$\psi' = \iiint_\Omega \left[\sum_{i,j=1}^{3} g_{\sigma^{ij}}(z)\sigma^{ij}(z')\right] m_p\, d\Omega + \iint_\Gamma g(\sigma(z)) m_p (V^{\mathrm{T}}n)\, d\Gamma$$
$$= \iiint_\Omega \sum_{i,j=1}^{3} g_{\sigma^{ij}}(z)[\sigma^{ij}(\dot{z}) - \sigma^{ij}(\nabla z^{\mathrm{T}}V)] m_p\, d\Omega \qquad (3.3.102)$$

because $m_p = 0$ on Γ.

As in the general derivation of Eq. (3.3.13), the material derivative of state $\dot{z} \in Z$ may be replaced by a virtual displacement $\bar{\lambda}$ in the first term on the right side of Eq. (3.3.102), to define a load functional for the adjoint equation, as in Eq. (3.3.13), obtaining

$$a_\Omega(\lambda, \bar{\lambda}) = \iiint_\Omega \left[\sum_{i,j=1}^{3} g_{\sigma^{ij}}(z)\sigma^{ij}(\bar{\lambda}) \right] m_p \, d\Omega \qquad \text{for all} \quad \bar{\lambda} \in Z \tag{3.3.103}$$

The linear form in $\bar{\lambda}$ on the right side of Eq. (3.3.103) is bounded in $[H^1(\Omega)]^3$. By the Lax–Milgram theorem [9], Eq. (3.3.103) has a unique solution for a displacement field λ, with the right side of Eq. (3.3.103) defining the load functional.

With smoothness assumptions, Eq. (3.3.103) is equivalent to the formal operator equation

$$-\sum_{j=1}^{3} \sigma_j^{ij}(\lambda) = -\sum_{j=1}^{3} \left(\sum_{k,l=1}^{3} g_{\sigma^{kl}}(z) C^{klij} m_p \right)_j, \qquad i = 1, 2, 3, \quad x \in \Omega \tag{3.3.104}$$

with boundary conditions

$$\lambda^i = 0, \qquad i = 1, 2, 3, \qquad x \in \Gamma^0 \tag{3.3.105}$$

$$\sum_{j=1}^{3} \sigma^{ij}(\lambda) n_j = 0, \qquad i = 1, 2, 3, \qquad x \in \Gamma^1 \cup \Gamma^2 \tag{3.3.106}$$

As in the adjoint equation of Eq. (3.3.89) for the plate problem, the derivative on the right side of Eq. (3.3.104) is in the sense of distributions. The distributional derivatives m_{p_j} ($j = 1, 2, 3$) depend on the equation that represents the boundary of Ω_p [56]. By the same method used in Section 3.1, a variational identity can be obtained for the adjoint system from Eq. (3.3.104). That is, by multiplying Eq. (3.3.104) by $\bar{\lambda} \in [H^1(\Omega)]^3$ and integrating by parts,

$$\begin{aligned} &\iiint_\Omega \left[\sum_{i,j=1}^{3} \sigma^{ij}(\lambda) \bar{\lambda}_j^i \right] d\Omega - \iint_\Gamma \left[\sum_{i,j=1}^{3} \sigma^{ij}(\lambda) n_j \bar{\lambda}^i \right] d\Gamma \\ &\quad = \iiint_\Omega \sum_{i,j=1}^{3} \left[\sum_{k,l=1}^{3} g_{\sigma^{kl}}(z) C^{klij} m_p \right] \bar{\lambda}_j^i \, d\Omega \\ &\qquad - \iint_\Gamma \sum_{i,j=1}^{3} \left[\sum_{k,l=1}^{3} g_{\sigma^{kl}}(z) C^{klij} m_p \right] n_j \bar{\lambda}^i \, d\Gamma \end{aligned}$$

Since $\sigma^{ij}(\lambda) = \sigma^{ji}(\lambda)$ and $C^{klij} = C^{klji}$, the above equation becomes, using Eqs. (3.1.32) and (3.1.33) for λ, a variational identity,

$$\iiint_\Omega \left[\sum_{i,j=1}^{3} \sigma^{ij}(\lambda)\varepsilon^{ij}(\bar{\lambda}) \right] d\Omega - \iiint_\Omega \left[\sum_{i,j=1}^{3} g_{\sigma^{ij}}(z)\sigma^{ij}(\bar{\lambda}) \right] m_p \, d\Omega$$
$$= \iint_\Gamma \left[\sum_{i,j=1}^{3} \sigma^{ij}(\lambda) n_j \bar{\lambda}^i \right] d\Gamma - \iint_\Gamma \sum_{i,j=1}^{3} \left[\sum_{k,l=1}^{3} g_{\sigma^{kl}}(z) C^{klij} m_p \right] \bar{\lambda}^i n_j \, d\Gamma$$
$$\text{for all} \quad \bar{\lambda} \in [H^1(\Omega)]^3 \qquad (3.3.107)$$

Note that by imposing the boundary conditions of Eqs. (3.3.105) and (3.3.106) and using the fact that $m_p = 0$ on Γ, the variational equation of Eq. (3.3.103) is obtained.

Since $\dot{z} \in Z$, Eq. (3.3.103) may be evaluated at $\bar{\lambda} = \dot{z}$ to obtain

$$a_\Omega(\lambda, \dot{z}) = \iiint_\Omega \left[\sum_{i,j=1}^{3} g_{\sigma^{ij}}(z)\sigma^{ij}(\dot{z}) \right] m_p \, d\Omega \qquad (3.3.108)$$

Similarly, since $\bar{z} \in Z$ and $\lambda \in Z$, Eq. (3.3.101) may be evaluated at $\bar{z} = \lambda$ to obtain

$$a_\Omega(\dot{z}, \lambda) = \iiint_\Omega \sum_{i,j=1}^{3} [\sigma^{ij}(z)\varepsilon^{ij}(\nabla\lambda^{\mathrm{T}} V) + \sigma^{ij}(\nabla z^{\mathrm{T}} V)\varepsilon^{ij}(\lambda)] \, d\Omega$$
$$- \iiint_\Omega \left[\sum_{i=1}^{3} f^i(\nabla\lambda^{i\mathrm{T}} V) \right] d\Omega - \iint_\Gamma \left[\sum_{i,j=1}^{3} \sigma^{ij}(z)\varepsilon^{ij}(\lambda) \right] (V^{\mathrm{T}} n) \, d\Gamma$$
$$+ \iint_{\Gamma^1 \cup \Gamma^2} \left[\sum_{i=1}^{3} f^i \lambda^i \right] (V^{\mathrm{T}} n) \, d\Gamma$$
$$+ \iint_{\Gamma^2} \sum_{i=1}^{3} \{ -T^i(\nabla\lambda^{i\mathrm{T}} V) + [\nabla(T^i\lambda^i)^{\mathrm{T}} n + H(T^i\lambda^i)](V^{\mathrm{T}} n) \} \, d\Gamma$$
$$(3.3.109)$$

By the Betti's reciprocal theorem [34],

$$a_\Omega(z, \bar{z}) \equiv \iiint_\Omega \left[\sum_{i,j=1}^{3} \sigma^{ij}(z)\varepsilon^{ij}(\bar{z}) \right] d\Omega$$
$$= \iiint_\Omega \left[\sum_{i,j=1}^{3} \sigma^{ij}(\bar{z})\varepsilon^{ij}(z) \right] d\Omega \equiv a_\Omega(\bar{z}, z) \qquad \text{for all} \quad z, \bar{z} \in [H^1(\Omega)]^3$$
$$(3.3.110)$$

Thus, $a_\Omega(\dot{z}, \lambda) = a_\Omega(\lambda, \dot{z})$, and Eqs. (3.3.102), (3.3.108), and (3.3.109) yield

$$\begin{aligned}\psi' &= \iiint_\Omega \sum_{i,j=1}^{3} [\sigma^{ij}(z)\varepsilon^{ij}(\nabla\lambda^{\mathrm{T}}V) + \sigma^{ij}(\lambda)\varepsilon^{ij}(\nabla z^{\mathrm{T}}V)]\, d\Omega \\ &\quad - \iiint_\Omega \left[\sum_{i=1}^{3} f^i(\nabla\lambda^{i\mathrm{T}}V)\right] d\Omega - \iiint_\Omega \sum_{i,j=1}^{3} [g_{\sigma^{ij}}(z)\sigma^{ij}(\nabla z^{\mathrm{T}}V)]m_p\, d\Omega \\ &\quad - \iint_\Gamma \left[\sum_{i,j=1}^{3} \sigma^{ij}(z)\varepsilon^{ij}(\lambda)\right](V^{\mathrm{T}}n)\, d\Gamma + \iint_{\Gamma^1\cup\Gamma^2} \left[\sum_{i=1}^{3} f^i\lambda^i\right](V^{\mathrm{T}}n)\, d\Gamma \\ &\quad + \iint_{\Gamma^2} \sum_{i=1}^{3} \{-T^i(\nabla\lambda^{i\mathrm{T}}V) + [\nabla(T^i\lambda^i)^{\mathrm{T}}n + H(T^i\lambda^i)](V^{\mathrm{T}}n)\}\, d\Gamma\end{aligned} \tag{3.3.111}$$

As before, the variational identities of Eqs. (3.1.37) and (3.3.107) may be used to transform the domain integrals of Eq. (3.3.111) to boundary integrals by identifying $\bar{z}$ in Eq. (3.1.37) and $\bar{\lambda}$ in Eq. (3.3.107) with $(\nabla\lambda^{\mathrm{T}}V)$ and $(\nabla z^{\mathrm{T}}V)$ in Eq. (3.3.111), respectively, obtaining

$$\begin{aligned}\psi' &= \iint_\Gamma \left[\sum_{i,j=1}^{3} \sigma^{ij}(z)n_j(\nabla\lambda^{i\mathrm{T}}V)\right] d\Gamma \\ &\quad + \iint_\Gamma \left\{\sum_{i,j=1}^{3} \sigma^{ij}(\lambda)n_j(\nabla z^{i\mathrm{T}}V) - \sum_{i,j=1}^{3}\left[\sum_{k,l=1}^{3} g_{\sigma^{kl}}(z)C^{klij}m_p\right]n_j(\nabla z^{i\mathrm{T}}V)\right\} d\Gamma \\ &\quad - \iint_\Gamma \left[\sum_{i,j=1}^{3} \sigma^{ij}(z)\varepsilon^{ij}(\lambda)\right](V^{\mathrm{T}}n)\, d\Gamma - \iint_{\Gamma^1\cup\Gamma^2} \left[\sum_{i=1}^{3} f^i\lambda^i\right](V^{\mathrm{T}}n)\, d\Gamma \\ &\quad + \iint_{\Gamma^2} \sum_{i=1}^{3} \{-T^i(\nabla\lambda^{i\mathrm{T}}V) + [\nabla(T^i\lambda^i)^{\mathrm{T}}n + H(T^i\lambda^i)](V^{\mathrm{T}}n)\}\, d\Gamma\end{aligned} \tag{3.3.112}$$

Since $\bar{\Omega}_p \subset \Omega$, $m_p = 0$ on Γ. Using boundary conditions of Eqs. (3.1.36) and (3.3.106), Eq. (3.3.112) becomes

$$\begin{aligned}\psi' &= \iiint_{\Gamma^0} \sum_{i,j=1}^{3} [\sigma^{ij}(z)n_j(\nabla\lambda^{i\mathrm{T}}V) + \sigma^{ij}(\lambda)n_j(\nabla z^{i\mathrm{T}}V)]\, d\Gamma \\ &\quad - \iint_\Gamma \left[\sum_{i,j=1}^{3} \sigma^{ij}(z)\varepsilon^{ij}(\lambda)\right](V^{\mathrm{T}}n)\, d\Gamma + \iint_{\Gamma^1\cup\Gamma^2} \left[\sum_{i=1}^{3} f^i\lambda^i\right](V^{\mathrm{T}}n)\, d\Gamma \\ &\quad + \iint_{\Gamma^2} \sum_{i=1}^{3} [\nabla(T^i\lambda^i)^{\mathrm{T}}n + H(T^i\lambda^i)](V^{\mathrm{T}}n)\, d\Gamma\end{aligned} \tag{3.3.113}$$

On Γ^0, $z = \lambda = 0$ implies $\nabla z^i = (\nabla z^{i\mathrm{T}}n)n$ and $\nabla\lambda^i = (\nabla\lambda^{i\mathrm{T}}n)n$. Hence, Eq.

(3.3.113) becomes

$$\psi' = \iint_{\Gamma^0} \sum_{i,j=1}^{3} [\sigma^{ij}(z)n_j(\nabla\lambda^{i\mathrm{T}}n) + \sigma^{ij}(\lambda)n_j(\nabla z^{i\mathrm{T}}n)](V^{\mathrm{T}}n)\, d\Gamma$$
$$- \iint_{\Gamma} \left[\sum_{i,j=1}^{3} \sigma^{ij}(z)\varepsilon^{ij}(\lambda)\right](V^{\mathrm{T}}n)\, d\Gamma + \iint_{\Gamma^1 \cup \Gamma^2} \left[\sum_{i=1}^{3} f^i\lambda^i\right](V^{\mathrm{T}}n)\, d\Gamma$$
$$+ \iint_{\Gamma^2} \sum_{i=1}^{3} [\nabla(T^i\lambda^i)^{\mathrm{T}}n + H(T^i\lambda^i)](V^{\mathrm{T}}n)\, d\Gamma \qquad (3.3.114)$$

which is the desired result.

As in the plate problem, the stress sensitivity result of Eq. (3.3.114) for average stress is valid for a fixed region Ω_p, with $\bar{\Omega}_p \subset \Omega$. The case in which ψ is average stress on the moving region $\Omega_{p_\tau} = T(\Omega_p, \tau)$ will be considered in Section 3.3.6. The case in which a part of Γ_p intersects Γ will also be considered there.

Next, consider the more general nonconservative loading case. For example, in pressure loading traction is given as

$$T^i(x) = -p(x)n_i(x), \qquad x \in \Gamma^2 \qquad (3.3.115)$$

Substituting T^i into Eq. (3.1.38) and taking the variation of both sides, using Eqs. (3.2.13), (3.2.36), and (3.2.56) and the fact $p' = 0$ and $f^{i'} = 0$,

$$\iiint_{\Omega} \sum_{i,j=1}^{3} [\sigma^{ij}(z')\varepsilon^{ij}(\bar{z}) + \sigma^{ij}(z)\varepsilon^{ij}(\bar{z}')]\, d\Omega + \iint_{\Gamma} \left[\sum_{i,j=1}^{3} \sigma^{ij}(z)\varepsilon^{ij}(\bar{z})\right](V^{\mathrm{T}}n)\, d\Gamma$$
$$= \iiint_{\Omega} \left[\sum_{i=1}^{3} f^i\bar{z}^{i'}\right] d\Omega + \iint_{\Gamma^1 \cup \Gamma^2} \left[\sum_{i=1}^{3} f^i\bar{z}^i\right](V^{\mathrm{T}}n)\, d\Gamma$$
$$- \iint_{\Gamma^2} \left[\sum_{i=1}^{3} pn^i\bar{z}^{i'}\right] d\Gamma - \iint_{\Gamma^2} [\mathrm{div}(p\bar{z})](V^{\mathrm{T}}n)\, d\Gamma \qquad (3.3.116)$$

Comparing Eq. (3.3.116) with Eq. (3.3.100), with the same adjoint equation of Eq. (3.3.103), Eq. (3.3.114) yields

$$\psi' = \iint_{\Gamma^0} \sum_{i,j=1}^{3} [\sigma^{ij}(z)n_j(\nabla\lambda^{i\mathrm{T}}n) + \sigma^{ij}(\lambda)n_j(\nabla z^{i\mathrm{T}}n)](V^{\mathrm{T}}n)\, d\Gamma$$
$$- \iint_{\Gamma} \left[\sum_{i,j=1}^{3} \sigma^{ij}(z)\varepsilon^{ij}(\lambda)\right](V^{\mathrm{T}}n)\, d\Gamma + \iint_{\Gamma^1 \cup \Gamma^2} \left[\sum_{i=1}^{3} f^i\lambda^i\right](V^{\mathrm{T}}n)\, d\Gamma$$
$$- \iint_{\Gamma^2} [\mathrm{div}(p\lambda)](V^{\mathrm{T}}n)\, d\Gamma \qquad (3.3.117)$$

which is the desired result. As in the conservative loading case, the stress sensitivity result of Eq. (3.3.117) is valid for a fixed region Ω_p such that $\bar{\Omega}_p \subset \Omega$.

INTERFACE PROBLEM OF LINEAR ELASTICITY

Consider the interface problem of linear elasticity of Section 3.1. Let $\Gamma = \Gamma^0 \cup \Gamma^1 \cup \Gamma^2$ be fixed and the interface boundary γ be varied. Extension to the case in which Γ is varying can be done easily. The mean stress over a fixed test volume Ω_p, such that $\bar{\Omega}_p \subset \Omega^1$, is

$$\psi = \iiint_{\Omega^1} g(\sigma(z^*))m_p \, d\Omega \tag{3.3.118}$$

where $m_p(x)$ is a characteristic function that is positive on Ω_p, zero outside Ω_p, and has integral equal to 1. Similarly, for the case $\Omega_p \subset \Omega^2$, since the idea is exactly the same, derivation will be carried out only for Eq. (3.3.118).

Taking the variation of Eq. (3.1.52), using Eqs. (3.2.13), (3.2.36), and (3.2.51) and the fact that $V = 0$ on Γ and $f^{i'} = T^{i'} = 0$,

$$\begin{aligned}
&\iiint_{\Omega^1} \sum_{i,j=1}^{3} [\sigma^{ij}(z^{*\prime})\varepsilon^{ij}(\bar{z}^*) + \sigma^{ij}(z^*)\varepsilon^{ij}(\bar{z}^{*\prime})] \, d\Omega \\
&- \iint_{\gamma} \left[\sum_{i,j=1}^{3} \sigma^{ij}(z^*)\varepsilon^{ij}(\bar{z}^*) \right] (V^{\mathrm{T}}n) \, d\Gamma \\
&+ \iiint_{\Omega^2} \sum_{i,j=1}^{3} [\sigma^{ij}(z^{**\prime})\varepsilon^{ij}(\bar{z}^{**}) + \sigma^{ij}(z^{**})\varepsilon^{ij}(\bar{z}^{**\prime})] \, d\Omega \\
&+ \iint_{\gamma} \left[\sum_{i,j=1}^{3} \sigma^{ij}(z^{**})\varepsilon^{ij}(\bar{z}^{**}) \right] (V^{\mathrm{T}}n) \, d\Gamma \\
&\quad = \iiint_{\Omega} \left[\sum_{i=1}^{3} f^{i}\bar{z}^{i'} \right] d\Omega + \iint_{\Gamma^2} \left[\sum_{i=1}^{3} T^{i}\bar{z}^{i'} \right] d\Gamma \qquad \text{for all} \quad \bar{z} \in Z
\end{aligned} \tag{3.3.119}$$

where Z is given in Eq. (3.1.53). From Eq. (3.3.5), $\dot{\bar{z}} = \bar{z}' + \nabla\bar{z}^{\mathrm{T}}V = 0$. Hence, $\dot{\bar{z}} = \bar{z}' = 0$ on Γ, since $V = 0$ on Γ. Thus, Eq. (3.3.119) can be rewritten, using Eq. (3.2.8), as

$$\begin{aligned}
a_\Omega(\dot{z}, \bar{z}) &\equiv \iiint_{\Omega^1} \left[\sum_{i,j=1}^{3} \sigma^{ij}(\dot{z}^*)\varepsilon^{ij}(\bar{z}^*) \right] d\Omega + \iiint_{\Omega^2} \left[\sum_{i,j=1}^{3} \sigma^{ij}(\dot{z}^{**})\varepsilon^{ij}(\bar{z}^{**}) \right] d\Omega \\
&= \iiint_{\Omega^1} \sum_{i,j=1}^{3} [\sigma^{ij}(z^*)\varepsilon^{ij}(\nabla\bar{z}^{*\mathrm{T}}V) + \sigma^{ij}(\nabla z^{*\mathrm{T}}V)\varepsilon^{ij}(\bar{z}^*)] \, d\Omega \\
&\quad + \iiint_{\Omega^2} \sum_{i,j=1}^{3} [\sigma^{ij}(z^{**})\varepsilon^{ij}(\nabla\bar{z}^{**\mathrm{T}}V) + \sigma^{ij}(\nabla z^{**\mathrm{T}}V)\varepsilon^{ij}(\bar{z}^{**})] \, d\Omega \\
&\quad - \iiint_{\Omega} \left[\sum_{i=1}^{3} f^{i}(\nabla\bar{z}^{i\mathrm{T}}V) \right] d\Omega + \iint_{\gamma} \sum_{i,j=1}^{3} [\sigma^{ij}(z^*)\varepsilon^{ij}(\bar{z}^*) \\
&\quad - \sigma^{ij}(z^{**})\varepsilon^{ij}(\bar{z}^{**})](V^{\mathrm{T}}n) \, d\Gamma \qquad \text{for all} \quad \bar{z} \in Z
\end{aligned} \tag{3.3.120}$$

As in Eq. (3.3.8), Eq. (3.3.120) is a variational equation for $\dot{z} \in Z$.

Taking the variation of the functional of Eq. (3.3.118), using material derivative formulas of Eqs. (3.2.8) and (3.2.36) and $m_p' = 0$,

$$\psi' = \iiint_{\Omega^1} \left[\sum_{i,j=1}^{3} g_{\sigma^{ij}}(z^*)\sigma^{ij}(z^{*\prime}) \right] m_p \, d\Omega - \iint_{\gamma} g(\sigma(z^*)) m_p (V^{\mathrm{T}} n) \, d\Gamma$$

$$= \iiint_{\Omega^1} \sum_{i,j=1}^{3} g_{\sigma^{ij}}(z^*)[\sigma^{ij}(\dot{z}^*) - \sigma^{ij}(\nabla z^{*\mathrm{T}} V)] m_p \, d\Omega \tag{3.3.121}$$

since $m_p(x) = 0$ on γ.

As in the general derivation of Eq. (3.3.13), by replacing the material derivative $\dot{z}$ of state in the first term on the right side of Eq. (3.3.121) by a virtual displacement $\bar{\lambda}$, to define a load functional for the adjoint equation,

$$a_\Omega(\lambda, \bar{\lambda}) = \iiint_{\Omega^1} \left[\sum_{i,j=1}^{3} g_{\sigma^{ij}}(z^*)\sigma^{ij}(\bar{\lambda}^*) \right] m_p \, d\Omega \qquad \text{for all} \quad \bar{\lambda} \in Z \tag{3.3.122}$$

As in the linear elasticity case, it may be shown that the linear form in $\bar{\lambda}$ on the right side of Eq. (3.3.122) is bounded in $[H^1(\Omega^1)]^3 \times [H^1(\Omega^2)]^3$. Hence, by the Lax–Milgram theorem [9], Eq. (3.3.122) has a unique solution λ.

With smoothness assumptions, as in the linear elasticity problem, it can be shown that Eq. (3.3.122) is equivalent to the formal operator equation

$$-\sum_{j=1}^{3} \sigma_j^{ij}(\lambda^*) = -\sum_{j=1}^{3} \left(\sum_{k,l=1}^{3} g_{\sigma^{kl}}(z^*) C^{*klij} m_p \right)_j, \qquad i = 1, 2, 3, \quad x \in \Omega^1 \tag{3.3.123}$$

$$-\sum_{j=1}^{3} \sigma_j^{ij}(\lambda^{**}) = 0, \qquad i = 1, 2, 3, \quad x \in \Omega^2 \tag{3.3.124}$$

with boundary conditions

$$\lambda^{**i} = 0, \qquad i = 1, 2, 3, \qquad x \in \Gamma^0 \tag{3.3.125}$$

$$\sum_{j=1}^{3} \sigma^{ij}(\lambda^{**}) n_j = 0, \qquad i = 1, 2, 3, \qquad x \in \Gamma^1 \cup \Gamma^2 \tag{3.3.126}$$

and interface conditions

$$\lambda^{*i} = \lambda^{**i}, \qquad i = 1, 2, 3, \qquad x \in \gamma \tag{3.3.127}$$

$$\sum_{j=1}^{3} \sigma^{ij}(\lambda^*) n_j = \sum_{j=1}^{3} \sigma^{ij}(\lambda^{**}) n_j, \qquad i = 1, 2, 3, \qquad x \in \gamma \tag{3.3.128}$$

where $\lambda^* = \lambda|_{\Omega^1}$ and $\lambda^{**} = \lambda|_{\Omega^2}$.

The derivative on the right side of Eq. (3.3.123) is, as in the linear elasticity case, in the sense of distributions. As before, variational identities can be obtained by multiplying both sides of Eqs. (3.3.123) and (3.3.124) by arbitrary displacement vectors $\bar{\lambda}^* \in [H^1(\Omega^1)]^3$ and $\bar{\lambda}^{**} \in [H^1(\Omega^2)]^3$, respectively, and integrating by parts, to obtain

$$\iiint_{\Omega^1} \left[\sum_{i,j=1}^{3} \sigma^{ij}(\lambda^*)\varepsilon^{ij}(\bar{\lambda}^*) \right] d\Omega - \iiint_{\Omega^1} \left[\sum_{i,j=1}^{3} g_{\sigma^{ij}}(z^*)\sigma^{ij}(\bar{\lambda}^*) \right] m_p \, d\Omega$$
$$= - \iint_{\gamma} \left[\sum_{i,j=1}^{3} \sigma^{ij}(\lambda^*) n_j \bar{\lambda}^{*i} \right] d\Gamma$$
$$+ \iint_{\gamma} \sum_{i,j=1}^{3} \left[\sum_{k,l=1}^{3} g_{\sigma^{kl}}(z^*) C^{*klij} m_p \right] n_j \bar{\lambda}^{*i} \, d\Gamma$$
$$\text{for all} \quad \bar{\lambda}^* \in [H^1(\Omega^1)]^3 \qquad (3.3.129)$$

$$\iiint_{\Omega^2} \left[\sum_{i,j=1}^{3} \sigma^{ij}(\lambda^{**})\varepsilon^{ij}(\bar{\lambda}^{**}) \right] d\Omega = \iint_{\gamma \cup \Gamma} \left[\sum_{i,j=1}^{3} \sigma^{ij}(\lambda^{**}) n_j \bar{\lambda}^{**i} \right] d\Gamma$$
$$\text{for all} \quad \bar{\lambda}^{**} \in [H^1(\Omega^2)]^3 \qquad (3.3.130)$$

Imposing boundary and interface conditions of Eqs. (3.3.125)–(3.3.128), using the fact that $m_p = 0$ on γ, and adding Eqs. (3.3.129) and (3.3.130), yields the variational equation of Eq. (3.3.122).

Since $\dot{z} \in Z$, Eq. (3.3.122) may be evaluated at $\bar{\lambda} = \dot{z}$ to obtain

$$a_{\Omega}(\lambda, \dot{z}) = \iiint_{\Omega^1} \left[\sum_{i,j=1}^{3} g_{\sigma^{ij}}(z^*)\sigma^{ij}(\dot{z}^*) \right] m_p \, d\Omega \qquad (3.3.131)$$

Similarly, since $\bar{z}$ and λ are in Z, Eq. (3.3.120) may be evaluated at $\bar{z} = \lambda$ to obtain an expression for $a_{\Omega}(\dot{z}, \lambda)$, which is equal to $a_{\Omega}(\lambda, \dot{z})$ in Eq. (3.3.131), due to symmetry of the bilinear form $a_{\Omega}(\cdot, \cdot)$. Hence, from Eqs. (3.3.120), (3.3.121), and (3.3.131),

$$\psi' = \iiint_{\Omega^1} \sum_{i,j=1}^{3} [\sigma^{ij}(z^*)\varepsilon^{ij}(\nabla\lambda^{*T}V) + \sigma^{ij}(\lambda^*)\varepsilon^{ij}(\nabla z^{*T}V)] \, d\Omega$$
$$+ \iiint_{\Omega^2} \sum_{i,j=1}^{3} [\sigma^{ij}(z^{**})\varepsilon^{ij}(\nabla\lambda^{**T}V) + \sigma^{ij}(\lambda^{**})\varepsilon^{ij}(\nabla z^{**T}V)] \, d\Omega$$
$$- \iiint_{\Omega} \left[\sum_{i=1}^{3} f^i(\nabla\lambda^{iT}V) \right] d\Omega - \iiint_{\Omega^1} \sum_{i,j=1}^{3} [g_{\sigma^{ij}}(z^*)\sigma^{ij}(\nabla z^{*T}V)] m_p \, d\Omega$$
$$+ \iint_{\gamma} \sum_{i,j=1}^{3} [\sigma^{ij}(z^*)\varepsilon^{ij}(\lambda^*) - \sigma^{ij}(z^{**})\varepsilon^{ij}(\lambda^{**})](V^T n) \, d\Gamma \qquad (3.3.132)$$

The variational identities of Eqs. (3.1.50), (3.1.51), (3.3.129), and (3.3.130) can be used to transform the domain integrals in Eq. (3.3.132) to boundary integrals by identifying $\bar{z}^*$ and $\bar{\lambda}^{**}$ of Eqs. (3.1.50) and (3.1.51) with $(\nabla\lambda^{*\mathrm{T}}V)$ and $(\nabla\lambda^{**\mathrm{T}}V)$, respectively, and $\bar{\lambda}^*$ and $\bar{\lambda}^{**}$ of Eqs. (3.3.129) and (3.3.130) with $(\nabla z^{*\mathrm{T}}V)$ and $(\nabla z^{**\mathrm{T}}V)$, respectively, obtaining

$$\begin{aligned}\psi' = &-\iint_\gamma \sum_{i,j=1}^{3} [\sigma^{ij}(z^*)n_j(\nabla\lambda^{*i^\mathrm{T}}V) + \sigma^{ij}(\lambda^*)n_j(\nabla z^{*i^\mathrm{T}}V)]\, d\Gamma \\ &+ \iint_{\gamma\cup\Gamma} \sum_{i,j=1}^{3} [\sigma^{ij}(z^{**})n_j(\nabla\lambda^{**i^\mathrm{T}}V) + \sigma^{ij}(\lambda^{**})n_j(\nabla z^{**i^\mathrm{T}}V)]\, d\Gamma \\ &+ \iint_\gamma \sum_{i,j=1}^{3} \left[\sum_{k,l=1}^{3} g_{\sigma^{kl}}(z^*)C^{*klij}m_p\right] n_j(\nabla z^{*i^\mathrm{T}}V)\, d\Gamma \\ &+ \iint_\gamma \sum_{i,j=1}^{3} [\sigma^{ij}(z^*)\varepsilon^{ij}(\lambda^*) - \sigma^{ij}(z^{**})\varepsilon^{ij}(\lambda^{**})](V^\mathrm{T}n)\, d\Gamma \end{aligned} \quad (3.3.133)$$

On γ, interface conditions of Eqs. (3.1.48) and (3.3.127) imply

$$\left.\begin{aligned}(\nabla z^{**i} - \nabla z^{*i})^\mathrm{T}V &= (\nabla z^{**i} - \nabla z^{*i})^\mathrm{T}n(V^\mathrm{T}n),\\ (\nabla\lambda^{**i} - \nabla\lambda^{*i})^\mathrm{T}V &= (\nabla\lambda^{**i} - \nabla\lambda^{*i})^\mathrm{T}n(V^\mathrm{T}n),\end{aligned}\right\}\quad i = 1, 2, 3$$

because the directional derivatives of z^{**i} and z^{*i} along the tangent to γ are the same for $i = 1, 2, 3$. The same is true for λ^{**i} and λ^{*i}. Hence, Eq. (3.3.133) becomes, using $V = 0$ on Γ and $m_p = 0$ on γ,

$$\begin{aligned}\psi' = \iint_\gamma \sum_{i,j=1}^{3} [&\sigma^{ij}(z^*)n_j(\nabla\lambda^{**i} - \nabla\lambda^{*i})^\mathrm{T}n + \sigma^{ij}(\lambda^*)n_j(\nabla z^{**i} - \nabla z^{*i})^\mathrm{T}n \\ &+ \sigma^{ij}(z^*)\varepsilon^{ij}(\lambda^*) - \sigma^{ij}(z^{**})\varepsilon^{ij}(\lambda^{**})](V^\mathrm{T}n)\, d\Gamma\end{aligned} \quad (3.3.134)$$

which is the desired sensitivity formula due to the variation of interface boundary γ. If the boundary Γ is varied, then the results of Eqs. (3.3.114) or (3.3.117), can be added to Eq. (3.3.134), depending on the loading case, by replacing z and λ with z^{**} and λ^{**}.

As before, the stress sensitivity result of Eq. (3.3.134) is valid for a fixed region Ω_p, such that $\bar{\Omega}_p \subset \Omega$. The case in which ψ is defined on a deformed domain Ω_τ as the average stress on the moving region $\Omega_{p_\tau} = T(\Omega_p, \tau)$ will be considered in Section 3.3.6. The case in which a part of Γ_p intersects γ will also be considered there.

3.3.5 Parameterization of Boundaries

As in Chapter 2, before proceeding from analytical design sensitivity formulas to numerical implementation, it is helpful to consider numerical aspects of computations. It is important to define an effective parameterization of the boundary for use in shape design sensitivity analysis. Presume that points on the boundary Γ are specified by a position vector

$x(\alpha; b) \in R^n$ $(n = 2, 3)$ from the origin of the coordinate system to point x on the boundary, as shown in Fig. 3.3.4. Here, $\alpha \in R^n$ $(n = 2, 3)$ is a parameter vector that defines points on Γ.

When the vector $b = [b_1\ b_2\ \cdots\ b_m]^T$ of design parameters has been defined, shape design sensitivity formulas can be expressed in terms of a variation δb. To do this, first denote

$$b_\tau = b + \tau\, \delta b \tag{3.3.135}$$

where b defines the boundary Γ of Ω and b_τ defines the boundary Γ_τ of the deformed domain Ω_τ. The velocity field at the boundary is defined as [Eq. (3.2.2)]

$$V(x) = \frac{d}{d\tau}[x(\alpha; b_\tau)]\bigg|_{\tau=0} = \frac{\partial x}{\partial b}(\alpha; b)\, \delta b \tag{3.3.136}$$

Then the shape design sensitivity formula can be expressed as

$$\begin{aligned} \psi' &= \iint_\Gamma G(z, \lambda)(V^T n)\, d\Gamma \\ &= \left[\iint_\Gamma G(z, \lambda) n^T \frac{\partial x}{\partial b}(\alpha; b)\, d\Gamma\right] \delta b \end{aligned} \tag{3.3.137}$$

where the variation δb can be taken outside the integral since it is constant. This expression gives design sensitivity coefficients of ψ associated with variations in design parameters. Hence, only numerical calculation of the integral in Eq. (3.3.137) is required, once the state and adjoint variables have been determined.

A piecewise-linear boundary represents the simplest example of boundary parameterization. There are two principle disadvantages to this boundary

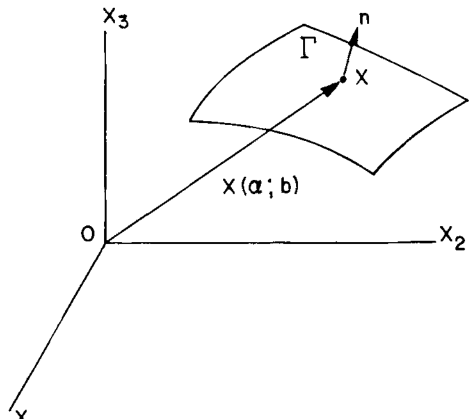

Fig. 3.3.4 Parametric definition of Γ.

representation. One is that for manufacturing, piecewise-linear segments are not practical. Instead, parameterized curves may be desired. The second problem is inaccuracy of finite element analysis, as predicted by the so called Babuska paradox [5, 60–64].

The Babuska paradox states that when a straight-line element is used to approximate a curved boundary, the solution for displacements, strains, and stresses normal to the boundary may not be accurate. Strang and Fix [5] clearly showed that a "boundary layer effect" exists in such cases and that solutions in the direction normal to the boundary will almost always converge to the wrong answer. On the other hand, to numerically calculate shape design sensitivity results of Eq. (3.3.137), stresses, strains, and/or normal derivatives of state variables and adjoint variables must be used on the boundary. Accurate evaluation of this information on the boundary is crucial for calculation of accurate shape design sensitivity information. Krauthammer [64] showed that when isoparametric elements are used, even a simple element configuration will yield results that may be of acceptable accuracy. Boundary stresses and strains can be calculated by linearly extrapolating values at optimal Gauss points to the boundary, to obtain accurate values on the boundary [65, 66].

3.3.6 Shape Design Sensitivity Analysis of Displacement and Stress

In Section 3.3.3, analytical shape design sensitivity formulas for the functional that defines displacement at an isolated fixed point $\hat{x} \in \Omega$ was derived for beams and plates. For the purpose of shape design sensitivity analysis of this displacement functional, it was assumed that the point does not move, that is, the displacement functional on the deformed domain Ω_τ is the value of the displacement at the same point $\hat{x}$.

For numerical implementation of shape design sensitivity analysis, the finite element method can be employed as a computational tool. Nodal points are the natural choice for evaluation of displacement. If the shape (geometry) of domain Ω is perturbed, then the finite element grid will be perturbed and nodal points will move. For this case, a new design sensitivity formula must be derived.

Shape design sensitivity of the mean stress functional over a fixed small test region Ω_p, where $\bar{\Omega}_p \subset \Omega$, was considered in Sections 3.3.3 and 3.3.4 for beam, plate, and elasticity problems. To define the mean stress functional, it was assumed that the functional value on the deformed domain Ω_τ is the mean stress value at the same region Ω_p and the boundary Γ_p of Ω_p does not intersect the boundary Γ of the domain Ω. As in the displacement case, when the finite element method is used for analysis, finite elements are a natural choice for Ω_p. Then Ω_p will move as the finite element grid moves, due to domain perturbation, and the boundary Γ_p of Ω_p may meet the boundary Γ of Ω. For this case, new design sensitivity formulas must be derived.

DISPLACEMENT FUNCTIONAL

Consider the displacement functional

$$\psi \equiv z(\hat{x}) = \iint_\Omega \hat{\delta}(x - \hat{x}) z \, d\Omega \tag{3.3.138}$$

where point $\hat{x}$ is moving to $\hat{x}_\tau = \hat{x} + \tau V(\hat{x})$. By taking the material derivative of Eq. (3.3.138),

$$\psi' = z'(\hat{x}) + \nabla z(\hat{x})^{\mathrm{T}} V(\hat{x}) = \iint_\Omega \hat{\delta}(x - \hat{x}) z' \, d\Omega + \nabla z(\hat{x})^{\mathrm{T}} V(\hat{x}) \tag{3.3.139}$$

Note that the first term on the right side of Eq. (3.3.139) is the one used in Section 3.3.3 to derive Eq. (3.3.32) for the beam and Eq. (3.3.79) for the plate. Thus, if point $\hat{x}$ is considered to be moving, the second term on the right side of Eq. (3.3.139) can be added to Eqs. (3.3.32) and (3.3.79). This additional term represents the contribution from movement of $\hat{x}$. Thus, even though the shape of the physical domain is not changing, if point $\hat{x}$ is moved, a contribution from the new additional term appears.

To illustrate the use of Eq. (3.3.139), consider the clamped beam studied in Section 3.3.3, with a displacement functional. Considering beam length $\hat{l}$ as a design variable, $V(0) = 0$ and $V(\hat{l}) = \delta\hat{l}$. In the domain, it is possible to select $V(x) = x \, \delta\hat{l}/\hat{l}$ $(0 \le x \le \hat{l})$; that is, points on the beam move to the right proportionally. If design sensitivity of the displacement of point $\hat{x} = \hat{l}/4$ is desired, since $\hat{x}$ moves to $\hat{x}_\tau = \hat{x} + \tau V(\hat{x}) = (\hat{l} + \tau\,\delta\hat{l})/4$, from Eq. (3.3.138) and $z(x) = (f_0/24E\alpha h_0^2)[x^2(\hat{l} - x)^2]$,

$$\psi(\tau) = z_\tau(\hat{x}_\tau) = \frac{f_0}{24E\alpha h_0^2}[\hat{x}_\tau^2(\hat{l} + \tau\,\delta\hat{l} - \hat{x}_\tau)^2] \tag{3.3.140}$$

Taking the variation of the displacement functional of Eq. (3.3.140) with respect to τ and evaluating the result at $\tau = 0$,

$$\psi' = \frac{3f_0\hat{l}^3}{512E\alpha h_0^2}\,\delta\hat{l}$$

The adjoint load, from Eq. (3.3.29), is a unit point load at $\hat{x} = \hat{l}/4$. The adjoint variable is thus obtained as

$$\lambda(x) = \frac{1}{384E\alpha h_0^2}\left[64\left\langle x - \frac{\hat{l}}{4}\right\rangle^3 - 54x^3 + 27\hat{l}x^2\right]$$

Using these results, from Eq. (3.3.33) for a clamped beam and Eq. (3.3.139),

$$\psi' = E\alpha h_0^2 z_{xx}\lambda_{xx}V\Big|_0^{\hat{l}} + z_x(\hat{x})V(\hat{x}) = \frac{3f_0\hat{l}^3}{512E\alpha h^2}\,\delta\hat{l}$$

which is the same result obtained before.

From this example it is clear that if different velocity fields are used in the domain, each with $V(0) = 0$ and $V(\hat{l}) = \delta\hat{l}$, different sensitivity results will be obtained, since the second term on the right side of Eq. (3.3.139) depends on the velocity $V(\hat{x})$. This is different from compliance and eigenvalue functionals that depend only on $V(0) = 0$ and $V(\hat{l}) = \delta\hat{l}$.

This additional fictitious perturbation of design (velocity field in the domain) can be eliminated if a local maximum displacement functional is considered. If a local maximum of displacement occurs at the interior point $\hat{x}$, then $\nabla z(\hat{x}) = 0$ in Eq. (3.3.139), and the sensitivity result of Eq. (3.3.139) does not depend on the velocity $V(\hat{x})$. On the other hand, if a local maximum occurs at the boundary point $\hat{x}$, then $\nabla z(\hat{x})$ may not be equal to zero. However, in this case, the velocity $V(\hat{x})$ of point $\hat{x}$ is included in the velocity of the boundary.

STRESS FUNCTIONAL

The mean stress functional over a small test region Ω_p is

$$\psi = \iiint_\Omega g(\sigma(z))m_p\,d\Omega \tag{3.3.141}$$

where m_p is a characteristic function that has the constant value $\bar{m}_p = (\iiint_{\Omega_p} d\Omega)^{-1}$ on Ω_p and is zero outside Ω_p.

Consider first the case in which Ω_p moves. From Eq. (3.3.141),

$$\psi = \frac{\iiint_{\Omega_p} g(\sigma(z))\,d\Omega}{\iiint_{\Omega_p} d\Omega} \tag{3.3.142}$$

Taking the material derivative of Eq. (3.3.142), using Eq. (3.2.36),

$$\begin{aligned}\psi' &= \left[\left(\iiint_{\Omega_p} g'(\sigma(z))\,d\Omega + \iint_{\Gamma_p} g(\sigma(z))(V^{\mathrm{T}}n)\,d\Gamma\right)\iiint_{\Omega_p} d\Omega \right.\\ &\qquad \left. - \iiint_{\Omega_p} g(\sigma(z))\,d\Omega \iint_{\Gamma_p}(V^{\mathrm{T}}n)\,d\Gamma\right]\Big/\left(\iiint_{\Omega_p} d\Omega\right)^2 \\ &= \iiint_\Omega g'(\sigma(z))m_p\,d\Omega + \bar{m}_p\iint_{\Gamma_p}[g(\sigma(z)) - \psi](V^{\mathrm{T}}n)\,d\Gamma \end{aligned} \tag{3.3.143}$$

Note that the first term on the right side of Eq. (3.3.143) is the one used in Eqs. (3.3.38), (3.3.87), (3.3.102), and (3.3.121) for beam, plate, linear elasticity, and interface problems, respectively. If the region Ω_p is considered to be moving, the second term on the right side of Eq. (3.3.143) may be added to the results of Eqs. (3.3.45), (3.3.94), (3.3.112), and (3.3.133) for each problem. This additional term is the contribution due to movement of Ω_p. Thus, movement of Ω_p, even without a change in the shape of the domain, will give a nonzero sensitivity term. The effect of the additional term due to a fictitious perturbation of design can be eliminated if Ω_p is a sufficiently small region that contains an interior point $\hat{x}$ where the stress function $g(\sigma(z))$ has a local maximum. That is, if $\hat{x} \in \Omega_p$ and $\bar{\Omega}_p \subset \Omega$, then the second term on the right side of Eq. (3.3.143) can be ignored since the value of $g(\sigma(z))$ is very close to ψ on Γ_p. On the other hand, if a local maximum of $g(\sigma(z))$ occurs at a point on the boundary Γ, Γ_p will intersect Γ, as shown schematically in Fig. 3.3.4, and the design sensitivity result of Eq. (3.3.143) will be expressed in terms of normal velocity ($V^{\mathrm{T}}n$) of boundaries Γ_p and Γ.

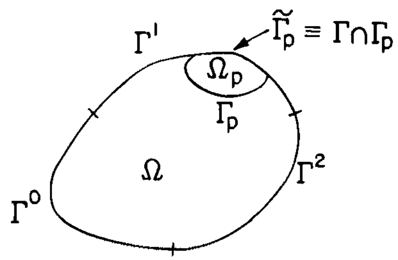

Fig. 3.3.5 Intersection of Γ_p and Γ.

For the case in which a part of Γ_p intersects Γ (Fig. 3.3.5), $m_p = 0$ cannot be permitted on Γ_p; specifically, on $\tilde{\Gamma}_p \equiv \Gamma \cap \Gamma_p$ in Eqs. (3.3.45), (3.3.94), (3.3.112), and (3.3.133). Instead, $m_p = \bar{m}_p$ must be used on $\tilde{\Gamma}_p$, and distributional derivatives m_{p_i} $(i = 1, 2)$ arise on $\tilde{\Gamma}_p \subset \Gamma$. Moreover, even though kinematic boundary conditions for the adjoint response λ in this case are the same as in the case $\bar{\Omega}_p \subset \Omega$, traction boundary conditions will be different on $\tilde{\Gamma}_p$, since $m_p = \bar{m}_p$ and distributional derivatives m_{p_i} $(i = 1, 2)$ must be used on $\tilde{\Gamma}_p$ in the variational identities for the adjoint system given in Eqs. (3.3.41), (3.3.90), (3.3.107), (3.3.129), and (3.3.130). A procedure for deriving shape design sensitivity formulas in this case will be considered for each problem.

BEAM

Consider the stress functional of Eq. (3.3.37), where $x_a = a\hat{l}$ and $x_b = \hat{l}$ $(0 < a < 1)$. That is, average stress is taken on the interval $(a\hat{l}, \hat{l})$ shown in Fig. (3.3.6). The variation of ψ_4 can be obtained by adding the second term

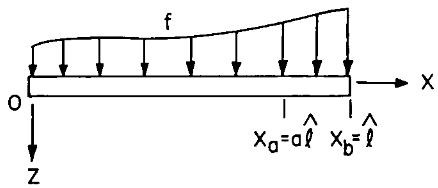

Fig. 3.3.6 Beam with $x_b = \hat{l}$.

on the right side of Eq. (3.3.143) to the result of Eq. (3.3.45), dropping the superscript notation for λ, to obtain

$$\begin{aligned}\psi_4' = {} & [E\alpha h^2 z_{xx}(\lambda_x V)_x - (E\alpha h^2 z_{xx})_x(\lambda_x V)]\Big|_0^{\hat{l}} \\ & + [E\alpha h^2 \lambda_{xx}(z_x V)_x - (E\alpha h^2 \lambda_{xx})_x(z_x V)]\Big|_0^{\hat{l}} \\ & + [(\beta h^{1/2} E m_p)_x(z_x V) - \beta h^{1/2} E m_p (z_x V)_x]\Big|_0^{\hat{l}} + [f\lambda - E\alpha h^2 z_{xx}\lambda_{xx}]V\Big|_0^{\hat{l}} \\ & + \bar{m}_p[\beta h^{1/2} E z_{xx} - \psi_4]V\Big|_{a\hat{l}}^{\hat{l}} \end{aligned} \tag{3.3.144}$$

where λ is the solution of Eq. (3.3.39), with m_p being the characteristic function on $(a\hat{l}, \hat{l})$.

From the variational identity of the adjoint system of Eq. (3.3.41), with smoothness assumptions, a boundary-value problem equivalent to the variational equation of Eq. (3.3.39) may be obtained as

$$(E\alpha h^2 \lambda_{xx})_{xx} = (\beta h^{1/2} E m_p)_{xx}, \qquad x \in (0, \hat{l}) \tag{3.3.145}$$

with boundary conditions

$$\lambda(0) = \lambda_x(0) = \lambda(\hat{l}) = \lambda_x(\hat{l}) = 0 \tag{3.3.146}$$

for a clamped beam,

$$\begin{aligned} \lambda(0) = \lambda_{xx}(0) = \lambda(\hat{l}) = 0 \\ (E\alpha h^2 \lambda_{xx})(\hat{l}) = \bar{m}_p(\beta h^{1/2} E)(\hat{l}) \end{aligned} \tag{3.3.147}$$

for a simply supported beam, and

$$\begin{aligned} \lambda(0) = \lambda_x(0) = 0 \\ (E\alpha h^2 \lambda_{xx})(\hat{l}) = \bar{m}_p(\beta h^{1/2} E)(\hat{l}) \\ (E\alpha h^2 \lambda_{xx})_x(\hat{l}) = (\beta h^{1/2} E m_p)_x(\hat{l}) \end{aligned} \tag{3.3.148}$$

for a cantilevered beam. Note that with these boundary conditions, the variational identity of Eq. (3.3.41) becomes the variational equation of Eq.

(3.3.39) for each beam problem. Also note that the traction boundary conditions of Eqs. (3.3.147) and (3.3.148) are different from the case $[x_a, x_b] \subset (0, \hat{l})$.

For a clamped beam, using boundary conditions of Eq. (2.1.1) for z and Eq. (3.3.146) for λ, Eq. (3.3.144) becomes

$$\psi_4' = E\alpha h^2 z_{xx} \lambda_{xx} V \Big|_0^{\hat{l}} - \beta h^{1/2} E \overline{m}_p z_{xx} V \Big|_{x = a\hat{l}} - \overline{m}_p \psi_4 V \Big|_{a\hat{l}}^{\hat{l}} \tag{3.3.149}$$

For a simply supported beam, applying boundary conditions of Eq. (2.1.16) for z and Eq. (3.3.147) for λ, Eq. (3.3.144) becomes

$$\begin{aligned} \psi_4' = {} & [(\beta h^{1/2} E m_p)_x - (E\alpha h^2 \lambda_{xx})_x] z_x V \Big|_{x = \hat{l}} + E\alpha h^2 \lambda_{xxx} z_x V \Big|_{x = 0} \\ & - E\alpha h^2 z_{xxx} \lambda_x V \Big|_0^{\hat{l}} - \beta h^{1/2} E \overline{m}_p z_{xx} V \Big|_{x = a\hat{l}} - \overline{m}_p \psi_4 V \Big|_{a\hat{l}}^{\hat{l}} \end{aligned} \tag{3.3.150}$$

For a cantilevered beam, using boundary conditions of Eq. (2.1.17) for z and Eq. (3.3.148) for λ, Eq. (3.3.144) becomes

$$\psi_4' = -E\alpha h^2 z_{xx} \lambda_{xx} V \Big|_{x = 0} + f \lambda V \Big|_{x = \hat{l}} - \beta h^{1/2} E \overline{m}_p z_{xx} V \Big|_{x = a\hat{l}} - \overline{m}_p \psi_4 V \Big|_{a\hat{l}}^{\hat{l}} \tag{3.3.151}$$

Comparing Eqs. (3.3.47)–(3.3.49) with Eqs. (3.3.149)–(3.3.151), respectively, additional terms arising in Eqs. (3.3.149)–(3.3.151) can be identified.

To illustrate the use of these results, consider the clamped beam studied earlier in this section. As in the displacement case, consider beam length $\hat{l}$ as a design variable, with $V(0) = 0$ and $V(\hat{l}) = \delta \hat{l}$. In the domain, it is possible to select $V(x) = x\, \delta\hat{l}/\hat{l}\ (0 \le x \le \hat{l})$. Since $\hat{l}$ and $a\hat{l}$ move to $\hat{l} + \tau\, \delta\hat{l}$ and $a(\hat{l} + \tau\, \delta\hat{l})$ respectively, $m_p = 1/[(1 - a)(\hat{l} + \tau\, \delta\hat{l})]$, and from Eq. (3.3.37) and $z(x) = (f_0/24E\alpha h_0^2)[x^2(\hat{l} - x)^2]$,

$$\begin{aligned} \psi_4(\tau) &= \int_0^{\hat{l} + \tau\,\delta\hat{l}} \beta h_0^{1/2} E z_{\tau_{xx}} m_{p_\tau}\, dx \\ &= \frac{\beta f_0}{12\alpha h_0^{3/2}(1 - a)(\hat{l} + \tau\,\delta\hat{l})} \\ &\quad \times \int_{a(\hat{l} + \tau\,\delta\hat{l})}^{\hat{l} + \tau\,\delta\hat{l}} [(\hat{l} + \tau\,\delta\hat{l} - x)(\hat{l} + \tau\,\delta\hat{l} - 5x) + x^2]\, dx \end{aligned} \tag{3.3.152}$$

Taking the variation of the functional ψ_4 with respect to τ and evaluating at $\tau = 0$,

$$\psi_4' = \frac{\beta f_0 a(2a - 1)\hat{l}}{6\alpha h_0^{3/2}}\, \delta\hat{l}$$

For a uniform load f_0 and uniform cross section h_0, from Eq. (3.3.145) the adjoint load is a point moment at $x = a\hat{l}$ with magnitude $M_0 = \beta h_0^{1/2}E/[(1-a)\hat{l}]$ (Fig. 3.3.7). Hence, the adjoint response is [58]

$$\lambda(x) = \frac{\beta}{\alpha h_0^{3/2}\hat{l}}\left[\frac{1}{(1-a)\hat{l}}\langle x - a\hat{l}\rangle^2 - \frac{a}{\hat{l}}x^3 - \frac{(1-3a)}{2}x^2\right]$$

Using these results, from Eq. (3.3.149),

$$\psi_4' = \frac{\beta f_0 a(2a-1)\hat{l}}{6\alpha h_0^{3/2}}\,\delta\hat{l}$$

which is the same result obtained before. As in the displacement case, it is clear that if different velocity fields are used in the domain, each with $V(0) = 0$ and $V(\hat{l}) = \delta\hat{l}$, different sensitivity results will be obtained, since the second and third terms on the right side of Eq. (3.3.149) depend on the velocity $V(a\hat{l})$.

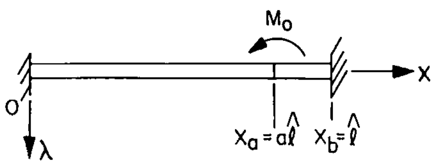

Fig. 3.3.7 Adjoint load for beam.

LINEAR ELASTICITY

Consider the stress functional of Eq. (3.3.99), where Γ_p intersects Γ, as shown in Fig. 3.3.5, and Ω_p moves as the domain Ω is perturbed. In this section, only the conservative loading case will be considered. Once the conservative loading case is done, it can easily be extended to the nonconservative loading case. The variation of ψ in Eq. (3.3.99) can be obtained by adding the second term on the right side of Eq. (3.3.143) to the result of Eq. (3.3.112) to obtain

$$\begin{aligned}
\psi' = &\iint_\Gamma \left[\sum_{i,j=1}^{3} \sigma^{ij}(z)n_j(\nabla\lambda^{i\mathrm{T}}V)\right] d\Gamma \\
&+ \iint_\Gamma \left\{\sum_{i,j=1}^{3} \sigma^{ij}(\lambda)n_j(\nabla z^{i\mathrm{T}}V) - \sum_{i,j=1}^{3}\left[\sum_{k,l=1}^{3} g_{\sigma^{kl}}(z)C^{klij}m_p\right]n_j(\nabla z^{i\mathrm{T}}V)\right\} d\Gamma \\
&- \iint_\Gamma \left[\sum_{i,j=1}^{3} \sigma^{ij}(z)\varepsilon^{ij}(\lambda)\right](V^\mathrm{T}n)\, d\Gamma - \iint_{\Gamma^1\cup\Gamma^2}\left[\sum_{i=1}^{3} f^i\lambda^i\right](V^\mathrm{T}n)\, d\Gamma \\
&+ \iint_{\Gamma^2}\sum_{i=1}^{3}\{-T^i(\nabla\lambda^{i\mathrm{T}}V) + [\nabla(T^i\lambda^i)^\mathrm{T}n + H(T^i\lambda^i)](V^\mathrm{T}n)\}\, d\Gamma \\
&+ \overline{m}_p\iint_{\Gamma_p}[g(\sigma(z)) - \psi](V^\mathrm{T}n)\, d\Gamma
\end{aligned} \tag{3.3.153}$$

where λ is the solution of Eq. (3.3.103).

Two cases may now be considered. In the first case, the boundary Γ_p intersects $\Gamma^1 \cup \Gamma^2$, as shown in Fig. 3.3.5. In the second case, Γ_p intersects Γ^0.

Consider the first case in which Γ_p intersects $\Gamma^1 \cup \Gamma^2$. From the variational identity of the adjoint system of Eq. (3.3.107), with smoothness assumptions, it can be shown that the variational adjoint equation of Eq. (3.3.103) is equivalent to the formal operator equation

$$-\sum_{j=1}^{3} \sigma_j^{ij}(\lambda) = -\sum_{j=1}^{3}\left(\sum_{k,l=1}^{3} g_{\sigma^{kl}}(z)C^{klij}m_p\right)_j, \qquad i = 1, 2, 3, \quad x \in \Omega \tag{3.3.154}$$

with boundary conditions

$$\begin{aligned}
&\lambda^i = 0, \qquad i = 1, 2, 3, \quad x \in \Gamma^0 \\
&\sum_{j=1}^{3} \sigma^{ij}(\lambda)n_j = 0, \qquad i = 1, 2, 3, \quad x \in (\Gamma^1 \cup \Gamma^2), x \notin \tilde{\Gamma}_p \\
&\sum_{j=1}^{3} \sigma^{ij}(\lambda)n_j = \sum_{j=1}^{3}\left(\sum_{k,l=1}^{3} g_{\sigma^{kl}}(z)C^{klij}m_p\right)n_j, \qquad i = 1, 2, 3, \\
&\qquad x \in \tilde{\Gamma}_p \equiv \Gamma_p \cap (\Gamma^1 \cup \Gamma^2)
\end{aligned} \tag{3.3.155}$$

Note that the traction boundary conditions of Eq. (3.3.155) are different from those of Eq. (3.3.106) on the boundary $\tilde{\Gamma}_p$. Using boundary conditions of Eq. (3.1.35) and (3.1.36) for z and Eq. (3.3.155) for λ in Eq. (3.3.153) yields

$$\begin{aligned}
\psi' = &\iint_{\Gamma^0} \sum_{i,j=1}^{3} [\sigma^{ij}(z)n_j(\nabla\lambda^{i\mathrm{T}}n) + \sigma^{ij}(\lambda)n_j(\nabla z^{i\mathrm{T}}n)](V^{\mathrm{T}}n)\, d\Gamma \\
&- \iint_{\Gamma}\left[\sum_{i,j=1}^{3} \sigma^{ij}(z)\varepsilon^{ij}(\lambda)\right](V^{\mathrm{T}}n)\, d\Gamma + \iint_{\Gamma^1 \cup \Gamma^2}\left[\sum_{i=1}^{3} f^i\lambda^i\right](V^{\mathrm{T}}n)\, d\Gamma \\
&+ \iint_{\Gamma^2} \sum_{i=1}^{3} [\nabla(T^i\lambda^i)^{\mathrm{T}}n + H(T^i\lambda^i)](V^{\mathrm{T}}n)\, d\Gamma \\
&+ \overline{m}_p \iint_{\Gamma_p} [g(\sigma(z)) - \psi](V^{\mathrm{T}}n)\, d\Gamma
\end{aligned} \tag{3.3.156}$$

which is the desired result.

Next, consider the case in which Γ_p intersects Γ^0. From Eq. (3.3.107), it can be shown that the variational adjoint equation of Eq. (3.3.103) is equivalent to the formal operator equation of Eq. (3.3.154), but with following boundary conditions:

$$\begin{aligned} \lambda^i &= 0, \qquad i = 1,2,3, \quad x \in \Gamma^0 \\ \sum_{j=1}^{3} \sigma^{ij}(\lambda)n_j &= 0, \qquad i = 1,2,3, \quad x \in \Gamma^1 \cup \Gamma^2 \end{aligned} \tag{3.3.157}$$

Using boundary conditions of Eqs. (3.1.35) and (3.1.36) for z and Eq. (3.3.157) for λ in Eq. (3.3.153),

$$\begin{aligned} \psi' = &\iint_{\Gamma^0} \sum_{i,j=1}^{3} [\sigma^{ij}(z)n_j(\nabla\lambda^{i\mathrm{T}}n) + \sigma^{ij}(\lambda)n_j(\nabla z^{i\mathrm{T}}n)](V^{\mathrm{T}}n)\, d\Gamma \\ &- \iint_{\Gamma} \left[\sum_{i,j=1}^{3} \sigma^{ij}(z)\varepsilon^{ij}(\lambda)\right](V^{\mathrm{T}}n)\, d\Gamma + \iint_{\Gamma^1 \cup \Gamma^2} \left[\sum_{i=1}^{3} f^i\lambda^i\right](V^{\mathrm{T}}n)\, d\Gamma \\ &+ \iint_{\Gamma^2} \sum_{i=1}^{3} [\nabla(T^i\lambda^i)^{\mathrm{T}}n + H(T^i\lambda^i)](V^{\mathrm{T}}n)\, d\Gamma \\ &- \iint_{\Gamma_p} \sum_{i,j=1}^{3} \left(\sum_{k,l=1}^{3} g_{\sigma^{kl}}(z)C^{klij}m_p\right)n_j(\nabla z^{i\mathrm{T}}n)(V^{\mathrm{T}}n)\, d\Gamma \\ &+ \bar{m}_p \iint_{\Gamma_p} [g(\sigma(z)) - \psi](V^{\mathrm{T}}n)\, d\Gamma \end{aligned} \tag{3.3.158}$$

which is the desired result.

INTERFACE PROBLEM OF LINEAR ELASTICITY

Consider the stress functional of Eq. (3.3.118), where Γ_p intersects γ (Fig. 3.3.8) and Ω_p moves as the domain Ω^1 is perturbed. The variation of ψ in Eq.

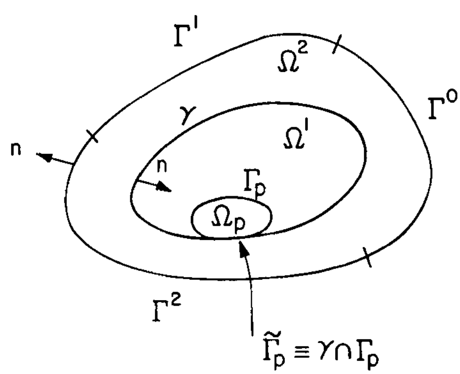

Fig. 3.3.8 Intersection of Γ_p and γ.

(3.3.118) can be obtained by adding the second term on the right side of Eq. (3.3.143) to the result of Eq. (3.3.133):

$$\begin{aligned}\psi' = &-\iint_{\gamma} \sum_{i,j=1}^{3} [\sigma^{ij}(z^*)n_j(\nabla\lambda^{*i\mathrm{T}}V) + \sigma^{ij}(\lambda^*)n_j(\nabla z^{*i\mathrm{T}}V)]\, d\Gamma \\ &+ \iint_{\gamma \cup \Gamma} \sum_{i,j=1}^{3} [\sigma^{ij}(z^{**})n_j(\nabla\lambda^{**i\mathrm{T}}V) + \sigma^{ij}(\lambda^{**})n_j(\nabla z^{**i\mathrm{T}}V)]\, d\Gamma \\ &+ \iint_{\gamma} \left[\sum_{k,l=1}^{3} g_{\sigma^{kl}}(z^*)C^{*klij}m_p\right] n_j(\nabla z^{*i\mathrm{T}}V)\, d\Gamma \\ &+ \iint_{\gamma} \sum_{i,j=1}^{3} [\sigma^{ij}(z^*)\varepsilon^{ij}(\lambda^*) - \sigma^{ij}(z^{**})\varepsilon^{ij}(\lambda^{**})](V^{\mathrm{T}}n)\, d\Gamma \\ &+ \overline{m}_p \iint_{\Gamma_p} [g(\sigma(z^*)) - \psi](V^{\mathrm{T}}n)\, d\Gamma \end{aligned} \tag{3.3.159}$$

From the variational identities of the adjoint system of Eqs. (3.3.129) and (3.3.130), with smoothness assumptions, the variational adjoint equation of Eq. (3.3.122) is equivalent to the formal operator equation

$$-\sum_{j=1}^{3} \sigma_j^{ij}(\lambda^*) = -\sum_{j=1}^{3} \left(\sum_{k,l=1}^{3} g_{\sigma^{kl}}(z^*)C^{*klij}m_p\right)_j, \qquad i = 1, 2, 3, \quad x \in \Omega^1 \tag{3.3.160}$$

$$-\sum_{j=1}^{3} \sigma_j^{ij}(\lambda^{**}) = 0, \qquad i = 1, 2, 3, \quad x \in \Omega^2 \tag{3.3.161}$$

with boundary conditions

$$\begin{aligned} \lambda^{**i} &= 0, & i &= 1, 2, 3, \quad x \in \Gamma^0 \\ \sum_{j=1}^{3} \sigma^{ij}(\lambda^{**})n^j &= 0, & i &= 1, 2, 3, \quad x \in \Gamma^1 \cup \Gamma^2 \end{aligned} \tag{3.3.162}$$

and interface conditions

$$\begin{aligned} &\lambda^{*i} = \lambda^{**i}, \qquad i = 1. 2, 3, \quad x \in \gamma \\ &\sum_{j=1}^{3} [\sigma^{ij}(\lambda^*)n_j - \sigma^{ij}(\lambda^{**})n_j] = \sum_{j=1}^{3} \left(\sum_{k,l=1}^{3} g_{\sigma^{kl}}(z^*)C^{*klij}m_p\right)n_j, \\ &\qquad\qquad i = 1, 2, 3, \quad x \in \tilde{\Gamma}_p \\ &\sum_{j=1}^{3} [\sigma^{ij}(\lambda^*)n_j - \sigma^{ij}(\lambda^{**})n_j] = 0, \qquad i = 1, 2, 3, \quad x \in \gamma, x \notin \tilde{\Gamma}_p \end{aligned} \tag{3.3.163}$$

Using boundary and interface conditions of Eqs. (3.1.46)–(3.1.49) for z and Eqs. (3.3.162) and (3.3.163) for λ in Eq. (3.3.159), with $V = 0$ on Γ,

$$\begin{aligned}\psi' &= \iint_{\gamma} \sum_{i,j=1}^{3} [\sigma^{ij}(z^*)n_j(\nabla\lambda^{**i} - \nabla\lambda^{*i})^{\mathrm{T}}n + \sigma^{ij}(\lambda^*)n_j(\nabla z^{**i} - \nabla z^{*i})^{\mathrm{T}}n](V^{\mathrm{T}}n)\, d\Gamma \\ &+ \iint_{\gamma} \sum_{i,j=1}^{3} [\sigma^{ij}(z^*)\varepsilon^{ij}(\lambda^*) - \sigma^{ij}(z^{**})\varepsilon^{ij}(\lambda^{**})](V^{\mathrm{T}}n)\, d\Gamma \\ &+ \iint_{\Gamma_p} \sum_{i,j=1}^{3} \left(\sum_{k,l=1}^{3} g_{\sigma^{kl}} C^{*klij} m_p\right) n_j(\nabla z^{*i} - \nabla z^{**i}) n (V^{\mathrm{T}}n)\, d\Gamma \\ &+ \bar{m}_p \iint_{\Gamma_p} [g(\sigma(z^*)) - \psi](V^{\mathrm{T}}n)\, d\Gamma \end{aligned} \tag{3.3.164}$$

which is the desired result.

Comparing Eq. (3.3.164) with Eq. (3.3.134), note that the third integral on the right side of Eq. (3.3.164) is due to intersection of Γ_p and γ and the last integral is due to the movement of Ω_p.

INTERPRETATION OF RESULTS

From shape design sensitivity results derived in this section, it is clear that unlike functionals that define global measures such as compliance and eigenvalues, shape design sensitivity of local functionals may involve fictitious perturbations of design (velocity field in the domain). That is, once a perturbed shape of the domain is given, there is only one way to evaluate global functionals, in terms of an integration over the entire perturbed domain. On the other hand, perturbations of local functionals may or may not involve fictitious perturbations of design, depending on the cases considered.

To predict perturbations of local functionals on fixed interior points or regions, results of Sections 3.3.3 and 3.3.4 can be used. That is, the second terms on the right side of Eqs. (3.3.139) and (3.3.143) can be ignored. If the predictions of perturbations of local functionals on moving interior points or regions are desired, the domain velocity field must be considered, as in Eqs. (3.3.139) and (3.3.143). In this case, the perturbation prediction accounts for movement of the point or region on which the functional is defined.

For a local maximum displacement at an interior point and a local maximum mean stress on a sufficiently small interior region, even in the cases of moving points and regions, perturbations of the functionals will not depend on the domain velocity field, since the second terms on the right sides of Eqs. (3.3.139) and (3.3.143) are either zero or ignorable. Finally, consider functionals that define displacement at a point on the boundary and mean stress over a small region that intersects the boundary. This is a very important case in shape design problems since, unlike conventional design problems in which component shapes are defined by cross-section and thickness variables, maximum displacement and stress are very likely to occur on the boundary. In this case the point and small region must be considered to be moving, and the second terms on the right sides of Eqs. (3.3.139) and (3.3.143) must be used. For a displacement functional at a point $\hat{x}$ on the boundary, the velocity $V(\hat{x})$ in the second term on the right side of Eq. (3.3.139) is included, and there is no need to introduce a fictitious design velocity. For a stress functional, consideration cannot be limited to the velocity of the boundary, because the second term on the right side of Eq. (3.3.143) depends on the velocity $(V^{\mathrm{T}}n)$ of Γ_p, even though a part of Γ_p coincides with a part of Γ. However, it is necessary to either hold $\Gamma_p - \tilde{\Gamma}_p$ fixed or, if Ω_p is sufficiently small, express the velocity of $\Gamma_p - \tilde{\Gamma}_p$ in terms of the velocity of $\tilde{\Gamma}_p$, without introducing a fictitious design velocity.

3.3.7 Domain Shape Design Sensitivity Method

To calculate design sensitivity information of Eq. (3.3.137) numerically, stresses, strains, and/or normal derivatives of state and adjoint variables on the boundary must be used. Hence, accurate evaluation of this information on the boundary is crucial. Thus, when a numerical method such as the finite element method is used for analysis, the accuracy of finite element results must be checked for state and adjoint variables on the boundary. It is well known [67] that results of finite element analysis on the boundary may not be satisfactory for a system with nonsmooth load and for interface problems. Note that the adjoint load for an average stress constraint is a concentrated load on an element Ω_p, over which stress is averaged.

There are several methods that might be considered to overcome this difficulty. The first choice is to use a finite element method that gives accurate results on the boundary. A second choice is to use a different numerical method, such as the boundary element method [68, 69]. In the finite element method, the unknown function, (e.g., displacement) is approximated by trial functions that do not satisfy the governing equations but usually satisfy kinematic boundary conditions. Nodal parameters z^i (e.g., nodal displacements) are then determined by approximate satisfaction of both differential equations and nonkinematic boundary conditions, in a domain integral mean sense. On the other hand, in the boundary element method, approximating functions satisfy the governing equations in the domain, but not the boundary conditions. Nodal parameters are determined by approximate satisfaction of boundary conditions in a weighted boundary integral sense.

An important advantage of the boundary element method in shape design sensitivity analysis is that it better represents boundary conditions and is usually more accurate in determining stress at the boundary.

Another method to be investigated is the use of domain information to best utilize the basic character of finite element analysis. To develop a domain method, consider the basic material derivative formula of Lemma 3.2.1. Instead of using Eq. (3.2.36), the result given in Eq. (3.2.37), which requires information on the domain rather than on the boundary may be used. The detailed procedure of the domain method will be explained using the linear elasticity problem. The reader is invited to carry out similar calculations for other problems.

Consider the linear elasticity problem of Section 3.3.4. Suppose the curve $\partial\Gamma^2$ that bounds the loaded surface Γ^2 is fixed and $T = [T^1\ T^2\ T^3]^{\mathrm{T}}$ is a conservative loading. Taking the variation of Eq. (3.1.38), using Eqs. (3.2.13), (3.2.37), and (3.2.51) and the fact that $f^{i'} = T^{i'} = 0$,

$$\begin{aligned}
&\iiint_\Omega \sum_{i,j=1}^{3} [\sigma^{ij}(z')\varepsilon^{ij}(\bar{z}) + \sigma^{ij}(z)\varepsilon^{ij}(\bar{z}')]\, d\Omega \\
&\quad + \iiint_\Omega \nabla\left[\sum_{i,j=1}^{3} \sigma^{ij}(z)\varepsilon^{ij}(\bar{z})\right]^{\mathrm{T}} V\, d\Omega + \iiint_\Omega \left[\sum_{i,j=1}^{3} \sigma^{ij}(z)\varepsilon^{ij}(\bar{z})\right] \operatorname{div} V\, d\Omega \\
&= \iiint_\Omega \left[\sum_{i=1}^{3} f^i \bar{z}^{i'}\right] d\Omega + \iiint_\Omega \nabla\left[\sum_{i=1}^{3} f^i \bar{z}^i\right]^{\mathrm{T}} V\, d\Omega \\
&\quad + \iiint_\Omega \left[\sum_{i=1}^{3} f^i \bar{z}^i\right] \operatorname{div} V\, d\Omega + \iint_{\Gamma^2} \left[\sum_{i=1}^{3} T^i \bar{z}^{i'}\right] d\Gamma \\
&\quad + \iint_{\Gamma^2} \left\{\nabla\left[\sum_{i=1}^{3} T^i \bar{z}^i\right]^{\mathrm{T}} n + H\left[\sum_{i=1}^{3} T^i \bar{z}^i\right]\right\}(V^{\mathrm{T}}n)\, d\Gamma \qquad \text{for all} \quad \bar{z} \in Z
\end{aligned} \tag{3.3.165}$$

Using Eqs. (3.2.8) and (3.3.5), Eq. (3.3.165) can be rewritten as

$$\begin{aligned}
&\iiint_\Omega \sum_{i,j=1}^{3} [\sigma^{ij}(\dot{z})\varepsilon^{ij}(\bar{z}) - \sigma^{ij}(z)\varepsilon^{ij}(\nabla\bar{z}^{\mathrm{T}}V) - \sigma^{ij}(\bar{z})\varepsilon^{ij}(\nabla z^{\mathrm{T}}V)]\, d\Omega \\
&\quad + \iiint_\Omega \nabla\left[\sum_{i,j=1}^{3} \sigma^{ij}(z)\varepsilon^{ij}(\bar{z})\right]^{\mathrm{T}} V\, d\Omega + \iiint_\Omega \left[\sum_{i,j=1}^{3} \sigma^{ij}(z)\varepsilon^{ij}(\bar{z})\right] \operatorname{div} V\, d\Omega \\
&= \iiint_\Omega \sum_{i=1}^{3} \bar{z}^i(\nabla f^{i\mathrm{T}} V)\, d\Omega + \iiint_\Omega \left[\sum_{i=1}^{3} f^i \bar{z}^i\right] \operatorname{div} V\, d\Omega \\
&\quad + \iint_{\Gamma^2} \left\{-\sum_{i=1}^{3} T^i(\nabla\bar{z}^{i\mathrm{T}} V) + \left(\nabla\left[\sum_{i=1}^{3} T^i \bar{z}^i\right]^{\mathrm{T}} n + H\left[\sum_{i=1}^{3} T^i \bar{z}^i\right]\right)(V^{\mathrm{T}}n)\right\} d\Gamma \\
&\qquad \text{for all} \quad \bar{z} \in Z
\end{aligned} \tag{3.3.166}$$

It can be verified that

$$\sum_{i,j=1}^{3} \sigma^{ij}(z)\varepsilon^{ij}(\nabla \bar{z}^{\mathrm{T}} V) = \sum_{i,j=1}^{3} \sigma^{ij}(z)(\nabla \bar{z}_j^{i\mathrm{T}} V + \nabla \bar{z}^{i\mathrm{T}} V_j) \tag{3.3.167}$$

and

$$\nabla\left[\sum_{i,j=1}^{3} \sigma^{ij}(z)\varepsilon^{ij}(\bar{z})\right]^{\mathrm{T}} V = \sum_{i,j=1}^{3} [\sigma^{ij}(z)(\nabla \bar{z}_j^{i\mathrm{T}} V) + \sigma^{ij}(\bar{z})(\nabla z_j^{i\mathrm{T}} V)] \tag{3.3.168}$$

where $V_j = [V_j^1 \ \ V_j^2 \ \ V_j^3]^{\mathrm{T}}$. Using these results, Eq. (3.3.166) becomes

$$\begin{aligned}
a_\Omega(\dot{z}, \bar{z}) &\equiv \iiint_\Omega \left[\sum_{i,j=1}^{3} \sigma^{ij}(\dot{z})\varepsilon^{ij}(\bar{z})\right] d\Omega \\
&= \iiint_\Omega \sum_{i,j=1}^{3} [\sigma^{ij}(z)(\nabla \bar{z}^{i\mathrm{T}} V_j) + \sigma^{ij}(\bar{z})(\nabla z^{i\mathrm{T}} V_j)]\, d\Omega \\
&\quad - \iiint_\Omega \left[\sum_{i,j=1}^{3} \sigma^{ij}(z)\varepsilon^{ij}(\bar{z})\right] \operatorname{div} V\, d\Omega + \iiint_\Omega \sum_{i=1}^{3} \bar{z}^i(\nabla f^{i\mathrm{T}} V)\, d\Omega \\
&\quad + \iiint_\Omega \left[\sum_{i=1}^{3} f^i \bar{z}^i\right] \operatorname{div} V\, d\Omega \\
&\quad + \iint_{\Gamma^2} \left\{-\sum_{i=1}^{3} T^i(\nabla \bar{z}^{i\mathrm{T}} V) + \left(\nabla\left[\sum_{i=1}^{3} T^i \bar{z}^i\right]^{\mathrm{T}} n + H\left[\sum_{i=1}^{3} T^i \bar{z}^i\right]\right)(V^{\mathrm{T}} n)\right\} d\Gamma \\
&\qquad \text{for all} \quad \bar{z} \in Z
\end{aligned} \tag{3.3.169}$$

As in Eq. (3.3.101), Eq. (3.3.169) is a variational equation for $\dot{z} \in Z$.

Consider the mean stress functional of Eq. (3.3.99),

$$\psi = \iiint_\Omega g(\sigma(z)) m_p\, d\Omega = \frac{\iiint_{\Omega_p} g(\sigma(z))\, d\Omega}{\iiint_{\Omega_p} d\Omega} \tag{3.3.170}$$

Taking the material derivative of Eq. (3.3.170) and using Eq. (3.2.37).

$$\begin{aligned}
\psi' &= \left[\iiint_{\Omega_p} (g' + \nabla g^{\mathrm{T}} V + g \operatorname{div} V)\, d\Omega \iiint_{\Omega_p} d\Omega \right. \\
&\quad \left. - \iiint_{\Omega_p} g\, d\Omega \iiint_{\Omega_p} \operatorname{div} V\, d\Omega\right] \Big/ \left(\iiint_{\Omega_p} d\Omega\right)^2 \\
&= \iiint_\Omega \sum_{i,j=1}^{3} g_{\sigma^{ij}}(z)[\sigma^{ij}(\dot{z}) - \sigma^{ij}(\nabla z^{\mathrm{T}} V)] m_p\, d\Omega \\
&\quad + \iiint_\Omega \sum_{k=1}^{3} \left[\sum_{i,j=1}^{3} g_{\sigma^{ij}}(z)\sigma_k^{ij}(z) V^k\right] m_p\, d\Omega + \iiint_\Omega g \operatorname{div} V m_p\, d\Omega \\
&\quad - \iiint_\Omega g m_p\, d\Omega \iiint_\Omega m_p \operatorname{div} V\, d\Omega
\end{aligned} \tag{3.3.171}$$

It can be shown that

$$\sigma^{ij}(\nabla z^{\mathrm{T}} V) = \sum_{k,l=1}^{3} C^{ijkl}(\nabla z_l^{k^{\mathrm{T}}} V + \nabla z^{k^{\mathrm{T}}} V_l) \tag{3.3.172}$$

and

$$\sum_{k=1}^{3} \sigma_k^{ij}(z) V^k = \sum_{k,l=1}^{3} C^{ijkl}(\nabla z_l^{k^{\mathrm{T}}} V) \tag{3.3.173}$$

Using the above results, Eq. (3.3.171) becomes

$$\begin{aligned}\psi' = &\iiint_\Omega \left[\sum_{i,j=1}^{3} g_{\sigma^{ij}}(z)\sigma^{ij}(\dot{z}) \right] m_p \, d\Omega \\ &- \iiint_\Omega \sum_{i,j=1}^{3} \left[\sum_{k,l=1}^{3} g_{\sigma^{ij}}(z) C^{ijkl}(\nabla z^{k^{\mathrm{T}}} V_l) \right] m_p \, d\Omega \\ &+ \iiint_\Omega g \operatorname{div} V m_p \, d\Omega - \iiint_\Omega g m_p \, d\Omega \iiint_\Omega m_p \operatorname{div} V \, d\Omega \end{aligned} \tag{3.3.174}$$

As in the linear elasticity problem of Section 3.3.3, the adjoint equation of Eq. (3.3.103) can be defined. By the same method used in Section 3.3.4, the sensitivity formula is obtained as

$$\begin{aligned}\psi' = &\iiint_\Omega \sum_{i,j=1}^{3} [\sigma^{ij}(z)(\nabla \lambda^{i^{\mathrm{T}}} V_j) + \sigma^{ij}(\lambda)(\nabla z^{i^{\mathrm{T}}} V_j)] \, d\Omega \\ &- \iiint_\Omega \left[\sum_{i,j=1}^{3} \sigma^{ij}(z)\varepsilon^{ij}(\lambda) \right] \operatorname{div} V \, d\Omega + \iiint_\Omega \sum_{i=1}^{3} \lambda^i (\nabla f^{i^{\mathrm{T}}} V) \, d\Omega \\ &+ \iiint_\Omega \left[\sum_{i=1}^{3} f^i \lambda^i \right] \operatorname{div} V \, d\Omega \\ &+ \iint_{\Gamma^2} \left\{ - \sum_{i=1}^{3} T^i (\nabla \lambda^{i^{\mathrm{T}}} V) + \left(\nabla \left[\sum_{i=1}^{3} T^i \lambda^i \right]^{\mathrm{T}} n + H \left[\sum_{i=1}^{3} T^i \lambda^i \right] \right) (V^{\mathrm{T}} n) \right\} d\Gamma \\ &- \iiint_\Omega \sum_{i,j=1}^{3} \left[\sum_{k,l=1}^{3} g_{\sigma^{ij}}(z) C^{ijkl}(\nabla z^{k^{\mathrm{T}}} V_l) \right] m_p \, d\Omega + \iiint_\Omega g \operatorname{div} V m_p \, d\Omega \\ &- \iiint_\Omega g m_p \, d\Omega \iiint_\Omega m_p \operatorname{div} V \, d\Omega \end{aligned} \tag{3.3.175}$$

There are several comments to be made about advantages and disadvantages of this *domain method*. A disadvantage is that a velocity field must be defined in the domain that satisfies regularity properties. There is no unique way of defining domain velocity fields for a given normal velocity field ($V^{\mathrm{T}}n$) on the boundary. Also, numerical evaluation of the sensitivity result of Eq. (3.3.175) is more complicated than evaluation of Eqs. (3.3.156) and (3.3.158) since Eq. (3.3.175) requires domain integration over the entire domain, whereas Eqs. (3.3.156) and (3.3.158) require integration over only the variable boundary. However, this problem can be overcome by introducing a *boundary layer* of finite elements that vary during perturbation of the shape of structural components. This approach is illustrated schematically in Fig. 3.3.9. The domain Ω is divided into subdomains Ω_1 and Ω_2, with Ω_1 held fixed and only boundary layer Ω_2 modified. In this way, the velocity field may be defined only on Ω_2. The thickness of the boundary layer Ω_2 will depend on tradeoffs between numerical accuracy and numerical efficiency.

There are several advantages associated with the domain method, in addition to numerical accuracy. Note that, as in the conventional design case of Chapter 2, variational identities are not required to transform domain integrals to boundary integrals. Thus, for a mean stress functional, there is no need to treat the special case in which Γ_p intersects Γ, as in Section 3.3.6. The result of Eq. (3.3.175) is valid for both cases. The biggest advantage of the domain method is obtained in built-up structures, which are treated in Chapter 4. Built-up structures are made up of combinations of a variety of structural components, with interface conditions that are generalizations of the interface problem of linear elasticity in Section 3.3.4. In applying the domain method, interface conditions are not required to obtain shape design sensitivity formulas. This greatly simplifies the derivation since contributions from each component are simply added. As for numerical accuracy, results of finite element analysis on interface boundaries are often unsatisfactory for built-up structures due to abrupt changes of boundary conditions. Using the domain method and careful finite element analysis, stress evaluation at interfaces may be avoided and accurate sensitivity results obtained. Moreover, as will be seen in Chapter 4, often interface boundaries for built-up structures are straight lines and/or plane sections. Thus, a domain velocity field can easily be defined for a given normal velocity field ($V^T n$) on the boundary.

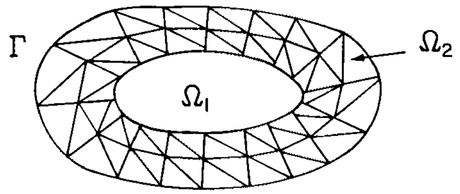

Fig. 3.3.9 Boundary layer.

3.3.8 Numerical Examples

To illustrate numerical implementation of shape design sensitivity formulas derived in the previous sections, several example problems are considered in this section.

FILLET

Selection of the best shape of a fillet in a tension bar so that no yielding occurs has long attracted the attention of engineers. Dimensions and notations of the bar and fillet are shown in Fig. 3.3.10. With symmetry, only the upper half of the bar is considered. Boundary segment Γ^1 is to be varied, with fixed points at A and B. The segment Γ^3 is the central line of the bar, and Γ^4 and Γ^2 are uniformly loaded edges.

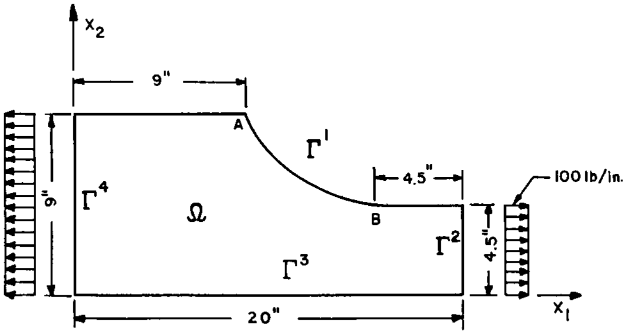

Fig. 3.3.10 Geometric configuration of fillet.

The variational equation of elasticity is

$$a_\Omega(z, \bar{z}) \equiv \iint_\Omega \left[\sum_{i,j=1}^{3} \sigma^{ij}(z)\varepsilon^{ij}(\bar{z}) \right] d\Omega$$

$$= \int_{\Gamma^2} \left[\sum_{i=1}^{2} T^i \bar{z}^i \right] d\Gamma \equiv l_\Omega(\bar{z}) \qquad \text{for all} \quad \bar{z} \in Z \qquad (3.3.176)$$

where

$$Z = \{z \in [H^1(\Omega)]^2 : \ z^1 = 0, \ \ x \in \Gamma^4 \quad \text{and} \quad z^2 = 0, \ \ x \in \Gamma^3\} \qquad (3.3.177)$$

with no body force acting on the fillet.

Consider now the von Mises yield stress functional, averaged over a small region Ω_k, as

$$\psi_k = \iint_\Omega g m_k \, d\Omega \qquad (3.3.178)$$

where $g = (\sigma_y - \sigma^a)/\sigma^a$, σ_y is the von Mises yield stress, defined as

$$\sigma_y = [(\sigma^{11})^2 + (\sigma^{22})^2 + 3(\sigma^{12})^2 - \sigma^{11}\sigma^{22}]^{1/2} \qquad (3.3.179)$$

and σ^a is the given allowable stress. In Eq. (3.3.178), m_k is a characteristic function on a small region Ω_k. From Eq. (3.3.103) an adjoint equation is obtained as

$$a_\Omega(\lambda, \bar{\lambda}) = \iint_\Omega \left[\sum_{i,j=1}^{2} g_{\sigma^{ij}}(z)\sigma^{ij}(\bar{\lambda}) \right] m_k \, d\Omega \qquad \text{for all} \quad \bar{\lambda} \in Z \tag{3.3.180}$$

and the variation of ψ_k is [Eq. (3.3.156)]

$$\psi_k' = -\int_{\Gamma^1} \left[\sum_{i,j=1}^{2} \sigma^{ij}(z)\varepsilon^{ij}(\lambda^{(k)}) \right] (V^T n)\, d\Gamma + \bar{m}_k \int_{\Gamma_k} [g(z) - \psi_k](V^T n)\, d\Gamma \tag{3.3.181}$$

where $\lambda^{(k)}$ is the solution of the adjoint equation of Eq. (3.3.180), $\bar{m}_k$ is the value of the characteristic function on Ω_k, and Γ_k is the boundary of the finite element Ω_k.

Consider the variable boundary Γ^1 of the fillet shown in Fig. 3.3.10, which can be characterized as a curve $x_2 = f(x_1)$, with a small vertical variation $\delta f(x_1)$ (Fig. 3.3.11). From the geometry of the curve, if only a small vertical change $\delta f(x_1)$ is allowed, the normal movement of the boundary can be written as

$$(V^T n) = \delta f \, n_2 = \delta f \left(\frac{dx_1}{ds} \right) \tag{3.3.182}$$

where s is arc length on Γ^1. Thus, the sensitivity formula of Eq. (3.3.181) can be rewritten as

$$\begin{aligned} \psi_k' &= -\int_{\Gamma^1} \left[\sum_{i,j=1}^{2} \sigma^{ij}(z)\varepsilon^{ij}(\lambda^{(k)}) \right] (V^T n)\, ds + \bar{m}_k \int_{\Gamma_k} [g(z) - \psi_k](V^T n)\, d\Gamma \\ &= -\int_A^B \left[\sum_{i,j=1}^{2} \sigma^{ij}(z)\varepsilon^{ij}(\lambda^{(k)}) \right] \delta f \, dx_1 + \bar{m}_k \int_{\Gamma_k} [g(z) - \psi_k](V^T n)\, d\Gamma \end{aligned} \tag{3.3.183}$$

In Eq. (3.3.183), δf can be related easily to δb once the curve Γ^1 defined by $x_2 = f(x_1, b)$ is parameterized by a design variable vector b. If heights of selected boundary points are chosen as design variables and if the boundary

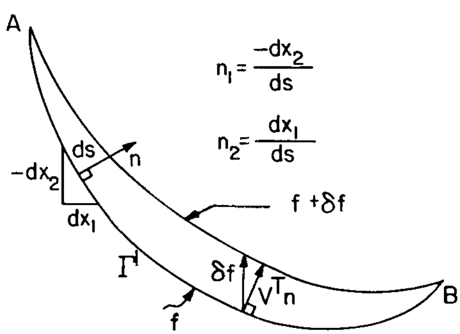

Fig. 3.3.11 Geometry of boundary curve.

is piecewise-linear, the boundary can be expressed as

$$f(x_1) = \left(\frac{x_1^{i+1} - x_1}{h_i}\right)b_i + \left(\frac{x_1 - x_1^i}{h_i}\right)b_{i+1}, \qquad x_1^i \le x_1 \le x_1^{i+1}, \qquad i = 1, 2, \ldots, N \tag{3.3.184}$$

where $h_i = x_1^{i+1} - x_1^i$, $f(x_1^i) = b_i$, and N denotes the number of partitions. Then, δf can be obtained by direct differentiation as

$$\delta f(x_1) = \left(\frac{x_1^{i+1} - x_1}{h_i}\right)\delta b_i + \left(\frac{x_1 - x_1^i}{h_i}\right)\delta b_{i+1}, \qquad x_1^i \le x_1 \le x_1^{i+1}, \qquad i = 1, 2, \ldots, N \tag{3.3.185}$$

When a cubic spline function is employed to parameterize Γ^1, with $f(x_1^i) = b_i$, the boundary can be expressed as [25]

$$f(x_1) = \frac{M_{i+1}}{6h_i}(x_1 - x_1^i)^3 + \frac{M_i}{6h_i}(x_1^{i+1} - x_1)^3 + \left(\frac{b_{i+1}}{h_i} - \frac{M_{i+1}h_i}{6}\right)(x_1 - x_1^i) + \left(\frac{b_i}{h_i} - \frac{M_i h_i}{6}\right)(x_1^{i+1} - x_1), \qquad x_1^i \le x_1 \le x_1^{i+1}, \quad i = 1, 2, \ldots, N \tag{3.3.186}$$

where $M_i = f''(x_1^i)$ is obtained by solving a system of equations in M_i $(i = 1, 2, \ldots, N + 1)$ [25]. Then the variation of f is

$$\begin{aligned}\delta f(x_1) = \sum_{j=1}^{N+1} \Bigg\{ & \left[\frac{(x_1 - x_1^i)^3}{6h_i} - \frac{h_i}{6}(x_1 - x_1^i)\right]\frac{\partial M_{i+1}}{\partial b_j} \\ & + \left[\frac{(x_1^{i+1} - x_1)^3}{6h_i} - \frac{h_i}{6}(x_1^{i+1} - x_1)\right]\frac{\partial M_i}{\partial b_j} \\ & + \delta_{i+1,j}\left(\frac{x_1 - x_1^i}{h_i}\right) + \delta_{i,j}\left(\frac{x_1^{i+1} - x_1}{h_i}\right)\Bigg\}\,\delta b_j, \\ & x_1^i \le x_1 \le x_1^{i+1}, \quad i = 1, 2, \ldots, N \end{aligned} \tag{3.3.187}$$

where $\delta_{i,j}$ is one if $i = j$ and otherwise is zero. The cubic spline function has two continuous derivatives everywhere and a minimum mean curvature property [25]. It also possesses globally controlled properties. Unlike Eq. (3.3.185) for a piecewise-linear function, with a cubic spline function [from Eq. (3.3.187)] perturbation of any design variable b will perturb $f(x_1)$ globally.

Using the result in Eqs. (3.3.185) or (3.3.187) and expressing the boundary of the finite element Ω_k in terms of b by the same method, Eq. (3.3.183) can be expressed as

$$\psi_k' = l_k^{\mathrm{T}}\,\delta b \tag{3.3.188}$$

where l_k is the desired design sensitivity coefficient for the constraint ψ_k.

For numerical calculation of shape design sensitivity, several different finite elements are used for comparison. Constant stress triangular (CST), linear stress triangular (LST), and eight-noded isoparametric (ISP) elements are used to calculate design sensitivity. For the ISP element, stresses and strains are evaluated at the Gauss points, and boundary stresses and strains are calculated by linearly extrapolating from optimal Gauss points [65, 66].

Configurations of triangular and quadrilateral finite elements are shown in Fig. 3.3.12. Height of the varied boundary Γ^1 is chosen as the design variable, and a piecewise-linear boundary parameterization is used for all cases. A cubic spline function is used for the ISP model. For the CST model, 190 elements, 117 nodal points, and 214 degrees of freedom are used. The LST model contains 190 elements, 423 nodal points, and 808 degrees of freedom, while the ISP model contains 111 elements, 384 nodal points, and 716 degrees of freedom. Young's modulus, Poisson's ratio, and allowable stress are $E = 30.0 \times 10^6$ psi, $\nu = 0.293$, and $\sigma^a = 120$ psi, respectively. The nominal design is

$$b = [5.55 \quad 5.1 \quad 4.65 \quad 4.2 \quad 3.75 \quad 3.3 \quad 2.85 \quad 2.4 \quad 1.95]^T$$

which gives a straight boundary for Γ^1, as shown in Fig. 3.3.12.

In order to compare the accuracy of results obtained with different finite

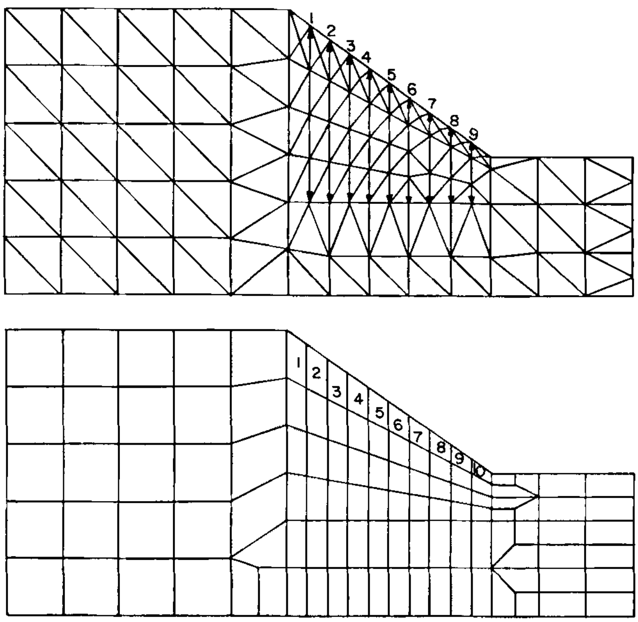

Fig. 3.3.12 Finite element model of fillet. (a) Triangular element model; numbers denote node height as design parameters. (b) Isoparametric element model; numbers denote the region where sensitivities are checked.

elements, the same small region should be used for stress functional evaluation. The small regions selected, shown in Fig. 3.3.12(b), are located next to the variable boundary Γ^1 where high stress occurs. The characteristic function is applied to each quadrilateral element for the ISP model and to four triangular elements for other models.

Numerical results with a 0.1% design change (i.e., $\delta b = b \times 10^{-3}$) are shown in Table 3.3.1. The abbreviation ISPS stands for isoparametric elements with cubic spline function representation for the variable boundary Γ^1. In Table 3.3.1, the LST model gives good sensitivity results, except at region 10, whereas ISP and ISPS models give good results except at region 1. Regions 1 and 10 correspond to low- and high-stress regions, respectively. Results of ISP or ISPS models are preferable to results of the LST model when using these results for optimization. As expected, the CST model yields the worst accuracy, since it cannot give accurate stress and strain on the boundary Γ^1.

Table 3.3.1

Comparison of Design Sensitivity $(\psi_k'/\Delta\psi_k \times 100)\%$

Region	CST	LST	ISP	ISPS
1	1402.9	108.9	43.3	65.9
2	45.3	99.6	104.6	105.9
3	57.9	99.2	103.2	101.9
4	64.2	99.2	103.4	103.6
5	67.5	99.2	102.8	102.6
6	68.6	99.2	101.8	101.7
7	68.3	99.1	100.0	100.4
8	70.1	99.1	98.4	97.4
9	79.3	98.3	105.2	104.9
10	183.6	87.0	102.8	104.1

TORQUE ARM

The automotive rear suspension torque arm, discussed in Section 2.2.5 as a conventional design example, is employed in this section for shape design sensitivity analysis. The finite element grid, geometry, loading conditions, and dimensions are shown in Fig. 2.2.10. Thickness, which is treated as the design variable in Chapter 2, is kept constant at 0.3 cm. The shapes of Γ^1, both upper and lower portions, are varied. The other boundary segments are kept fixed.

Consider the von Mises yield stress functional, averaged over a finite element Ω_k,

$$\psi_k = \iint_\Omega g m_k \, d\Omega, \qquad k = 1, 2, \ldots, \mathrm{NE} \tag{3.3.189}$$

Table 3.3.2

Design Sensitivity of Torque Arm

Element number	ψ_k^1	ψ_k^2	$\Delta\psi_k$	ψ_k'	$(\psi_k'/\Delta\psi_k \times 100)\%$
51	−9.5708E − 01	−9.5701E − 01	7.0455E − 05	8.0886E − 05	114.8
54	−9.2402E − 01	−9.2420E − 01	−1.8128E − 04	−2.2258E − 04	122.8
57	−9.2972E − 01	−9.2999E − 01	−2.7086E − 04	−4.1001E − 04	151.4
60	−9.2627E − 01	−9.2648E − 01	−2.1854E − 04	−2.7725E − 04	126.9
63	−9.1186E − 01	−9.1206E − 01	−2.0094E − 04	−2.3872E − 04	118.8
66	−8.9297E − 01	−8.9322E − 01	−2.5072E − 04	−2.9208E − 04	116.5
69	−8.7337E − 01	−8.7369E − 01	−3.2049E − 04	−3.7399E − 04	116.7
72	−8.5399E − 01	−8.5437E − 01	−3.7765E − 04	−4.4074E − 04	116.7
75	−8.3518E − 01	−8.3560E − 01	−4.2081E − 04	−4.9135E − 04	116.8
78	−8.1703E − 01	−8.1749E − 01	−4.5704E − 04	−5.3355E − 04	116.7
81	−7.9959E − 01	−8.0009E − 01	−4.9640E − 04	−5.7962E − 04	116.8
84	−7.8288E − 01	−7.8342E − 01	−5.4279E − 04	−6.3434E − 04	116.9
87	−7.6689E − 01	−7.6748E − 01	−5.8987E − 04	−6.8981E − 04	116.9
90	−7.5163E − 01	−7.5226E − 01	−6.2698E − 04	−7.3382E − 04	117.0
93	−7.3713E − 01	−7.3777E − 01	−6.4126E − 04	−7.5014E − 04	117.0
96	−7.2330E − 01	−7.2394E − 01	−6.3673E − 04	−7.4326E − 04	116.7
99	−7.0995E − 01	−7.1059E − 01	−6.3961E − 04	−7.4548E − 04	116.5
102	−6.9685E − 01	−6.9754E − 01	−6.8766E − 04	−8.0008E − 04	116.3
105	−6.8397E − 01	−6.8479E − 01	−8.2378E − 04	−9.6213E − 04	116.8
108	−6.7274E − 01	−6.7375E − 01	−1.0156E − 03	−1.1987E − 03	118.0
171	−6.6857E − 01	−6.6968E − 01	−1.1116E − 03	−1.3095E − 03	117.8
174	−6.8065E − 01	−6.8155E − 01	−9.0037E − 04	−1.0709E − 03	118.9
177	−7.0737E − 01	−7.0786E − 01	−4.8863E − 04	−3.9665E − 04	81.2
180	−7.5279E − 01	−7.5278E − 01	7.6247E − 06	1.4928E − 04	—
183	−8.1493E − 01	−8.1461E − 01	3.2090E − 04	3.7072E − 04	115.5
186	−8.8122E − 01	−8.8092E − 01	3.0111E − 04	3.4105E − 04	113.3
109	−9.1133E − 01	−9.1123E − 01	1.0189E − 04	1.2514E − 04	122.8
112	−8.8768E − 01	−8.8799E − 01	−3.1006E − 04	−3.7596E − 04	121.3
115	−9.0411E − 01	−9.0447E − 01	−3.5887E − 04	−4.9377E − 04	137.6
118	−9.0615E − 01	−9.0641E − 01	−2.5739E − 04	−3.0335E − 04	117.9
121	−8.9031E − 01	−8.9054E − 01	−2.2516E − 04	−2.5650E − 04	113.9
124	−8.6974E − 01	−8.7002E − 01	−2.7628E − 04	−3.1554E − 04	114.2
127	−8.4921E − 01	−8.4956E − 01	−3.5956E − 04	−4.0299E − 04	115.3
130	−8.2950E − 01	−8.2991E − 01	−4.0647E − 04	−4.7017E − 04	115.7
133	−8.1069E − 01	−8.1114E − 01	−4.4761E − 04	−5.1876E − 04	115.9
136	−7.9270E − 01	−7.9318E − 01	−4.8184E − 04	−5.5878E − 04	116.0
139	−7.7547E − 01	−7.7599E − 01	−5.2042E − 04	−6.0397E − 04	116.1
142	−7.5898E − 01	−7.5955E − 01	−5.6726E − 04	−6.5914E − 04	116.2
145	−7.4322E − 01	−7.4383E − 01	−6.1502E − 04	−7.1531E − 04	116.3
148	−7.2819E − 01	−7.2884E − 01	−6.5203E − 04	−7.5925E − 04	116.4
151	−7.1392E − 01	−7.1458E − 01	−6.6442E − 04	−7.7358E − 04	116.4
154	−7.0035E − 01	−7.0101E − 01	−6.5677E − 04	−7.6338E − 04	116.2
157	−6.8732E − 01	−6.8798E − 01	−6.5744E − 04	−7.6315E − 04	116.1
160	−6.7466E − 01	−6.7537E − 01	−7.0674E − 04	−8.1905E − 04	115.9

Table 3.3.2 (cont.)

Element number	ψ_k^1	ψ_k^2	$\Delta\psi_k$	ψ_k'	$(\psi_k'/\Delta\psi_k \times 100)\%$
163	−6.6250E − 01	−6.6335E − 01	−8.4965E − 04	−9.8803E − 04	116.3
166	−6.5250E − 01	−6.5355E − 01	−1.0497E − 03	−1.2324E − 03	117.4
187	−6.5021E − 01	−6.5136E − 01	−1.1460E − 03	−1.3410E − 03	117.0
190	−6.6461E − 01	−6.6553E − 01	−9.2317E − 04	−1.0905E − 03	118.1
193	−6.9341E − 01	−6.9391E − 01	−4.9726E − 04	−3.9789E − 04	80.0
196	−7.4070E − 01	−7.4068E − 01	1.5295E − 05	1.6390E − 04	—
199	−8.0535E − 01	−8.0501E − 01	3.4298E − 04	3.9514E − 04	115.2
202	−8.7489E − 01	−8.7457E − 01	3.2183E − 04	3.6356E − 04	113.0

where $g = (\sigma_y - \sigma^a)/\sigma^a$, σ_y is the von Mises yield stress defined in Eq. (3.3.179), m_k a characteristic function on finite element k, and NE the total number of elements. The formulation for shape design sensitivity analysis is the same as in the fillet problem. The only difference is that instead of one boundary changing, two boundaries are varied. Because the moving boundaries are traction free for both problems, Eq. (3.3.181) can be used for design sensitivity coefficient calculation.

Design variables are shown in Fig. 3.3.13, where upper and lower portions of the boundary each have heights specified at seven selected boundary points as design variables. The finite element model includes 204 elements, 707 nodal points, and 1342 degrees of freedom. Design sensitivity analysis results for average stresses on elements next to the variable boundary, with 0.1% uniform design change, are shown in Table 3.3.2. It is observed that most elements have good accuracy. For elements 180 and 196, poor accuracy may result from small differences in functional values.

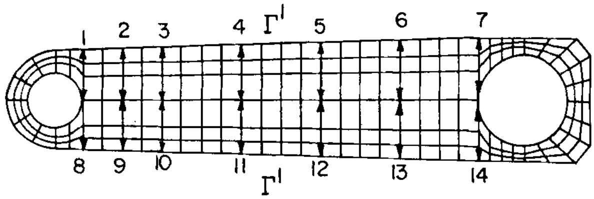

Fig. 3.3.13 Shape design parameters of torque arm; numbers denote node heights as design parameters.

TWO-DIMENSIONAL ELASTIC CONCRETE DAM

Consider a concrete dam shown in Fig. 3.3.14, modeled as a two-dimensional plane strain problem. It is assumed that the length of the dam is infinite and that the height of the water level, which is equal to the height of the dam, is given. The boundary of the dam is composed of four parts; Γ^1 is

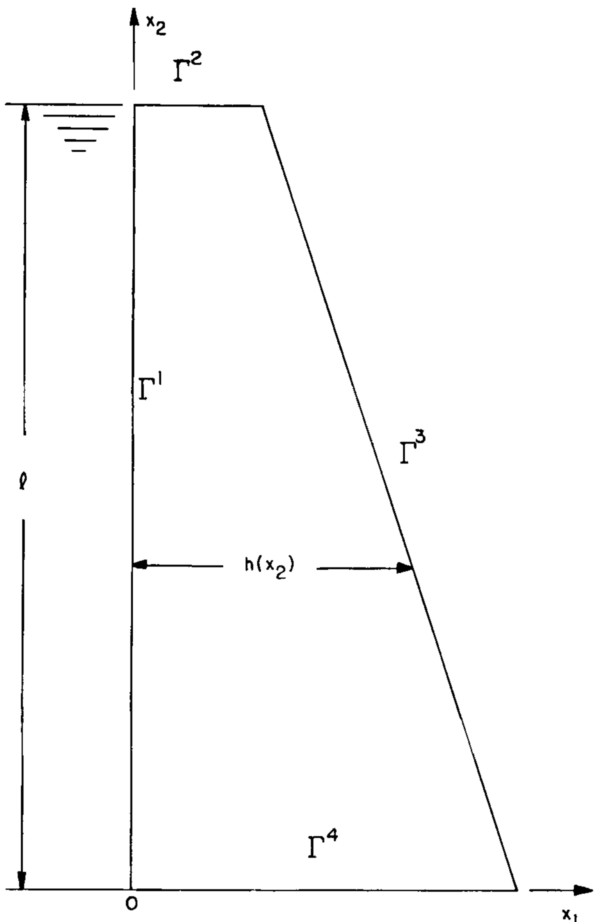

Fig. 3.3.14 Cross section of dam.

the boundary along which hydrostatic pressure acts, Γ^2 and Γ^3 are the top and side, which are load-free boundaries, and Γ^4 is the boundary that is in contact with earth, where homogeneous kinematic boundary conditions are imposed. The shapes of Γ^1 and Γ^3 are to be varied. Since the height of the dam is not changed, every point in the domain is allowed to move only in the x_1 direction (i.e., $V_2 = 0$). Therefore, Γ^2 and Γ^4 are horizontal straight lines.

Consider the principal stress functional, averaged over a finite element Ω_k, as

$$\left.\begin{aligned} \psi_k &= \iint_\Omega g^1 m_k \, d\Omega, \\ \psi_{k+\mathrm{NE}} &= \iint_\Omega g^2 m_k \, d\Omega, \end{aligned}\right\} \qquad k = 1, 2, \ldots, \mathrm{NE} \tag{3.3.190}$$

where $g^1 = (s\sigma_1 - \sigma_U)/\sigma_U$, $g^2 = (\sigma_L - s\sigma_2)/\sigma_L$, and σ_L, σ_U, and s denote lower and upper bounds on stress in concrete and safety factor, respectively. In Eq. (3.3.190), σ_1 and σ_2 are the two principal stresses, given as

$$\begin{aligned}\sigma_1 &= \frac{\sigma^{11} + \sigma^{22}}{2} + \sqrt{\left(\frac{\sigma^{11} - \sigma^{22}}{2}\right)^2 + (\sigma^{12})^2} \\ \sigma_2 &= \frac{\sigma^{11} + \sigma^{22}}{2} - \sqrt{\left(\frac{\sigma^{11} - \sigma^{22}}{2}\right)^2 + (\sigma^{12})^2}\end{aligned} \tag{3.3.191}$$

The variational equation is

$$\begin{aligned} a_\Omega(z, \bar{z}) &\equiv \iint_\Omega \left[\sum_{i,j=1}^{2} \sigma^{ij}(z)\varepsilon^{ij}(\bar{z}) \right] d\Omega \\ &= \iint_\Omega \left[\sum_{i=1}^{2} f^i \bar{z}^i \right] d\Omega + \int_{\Gamma^1} \left[\sum_{i=1}^{3} T^i \bar{z}^i \right] d\Gamma \qquad \text{for all} \quad \bar{z} \in Z \end{aligned} \tag{3.3.192}$$

where

$$Z = \{z \in [H^1(\Omega)]^2\colon \ z^1 = z^2 = 0, \quad x \in \Gamma^4\} \tag{3.3.193}$$

with the weight of the dam applied as body force f. In this example, the traction $T = [T^1 \ T^2]^T$ is due to hydrostatic pressure normal to the boundary, given as

$$T^i = -\gamma_w(l - x_2)n_i, \qquad i = 1, 2, \quad x \in \Gamma^1 \tag{3.3.194}$$

where γ_w is specific weight of water. Note that T is nonconservative loading, as in Eq. (3.3.115). Using the method of Section 3.3.4, the adjoint equations are obtained as

$$a_\Omega(\lambda, \bar{\lambda}) = \iint_\Omega \left[\sum_{i,j=1}^{2} g^1_{\sigma^{ij}}(z)\sigma^{ij}(\bar{\lambda}) \right] m_k \, d\Omega \qquad \text{for all} \quad \bar{\lambda} \in Z \tag{3.3.195}$$

and

$$a_\Omega(\lambda, \bar{\lambda}) = \iint_\Omega \left[\sum_{i,j=1}^{2} g^2_{\sigma^{ij}}(z)\sigma^{ij}(\bar{\lambda}) \right] m_k \, d\Omega \qquad \text{for all} \quad \bar{\lambda} \in Z \tag{3.3.196}$$

The variation of ψ_k is, from Eqs. (3.3.117) and (3.3.143),

$$\begin{aligned} \psi_k' = &- \int_\Gamma \left[\sum_{i,j=1}^{2} \sigma^{ij}(z)\varepsilon^{ij}(\lambda^{(k)}) \right] (V^T n) \, d\Gamma + \int_\Gamma \left[\sum_{i=1}^{2} f^i \lambda^{(k)i} \right] (V^T n) \, d\Gamma \\ &- \gamma_w \int_{\Gamma^1} [(l - x_2) \operatorname{div} \lambda^{(k)} - \lambda^{(k)2}](V^T n) \, d\Gamma \\ &+ \bar{m}_k \int_{\Gamma_k} [g^1(z) - \psi_k](V^T n) \, d\Gamma, \qquad k = 1, 2, \ldots, \text{NE} \end{aligned} \tag{3.3.197}$$

and

$$\psi'_{k+\mathrm{NE}} = -\int_{\Gamma}\left[\sum_{i,j=1}^{2} \sigma^{ij}(z)\varepsilon^{ij}(\lambda^{(k+\mathrm{NE})})\right](V^{\mathrm{T}}n)\, d\Gamma + \int_{\Gamma}\left[\sum_{i=1}^{2} f^{i}\lambda^{(k+\mathrm{NE})i}\right](V^{\mathrm{T}}n)\, d\Gamma$$
$$- \gamma_{\mathrm{w}}\int_{\Gamma^{1}} [(l - x_2)\operatorname{div}\lambda^{(k+\mathrm{NE})} - \lambda^{(k+\mathrm{NE})2}](V^{\mathrm{T}}n)\, d\Gamma$$
$$+ \overline{m}_k\int_{\Gamma_k} [g^2(z) - \psi_{k+\mathrm{NE}}](V^{\mathrm{T}}n)\, d\Gamma, \qquad k = 1, 2, \ldots, \mathrm{NE} \tag{3.3.198}$$

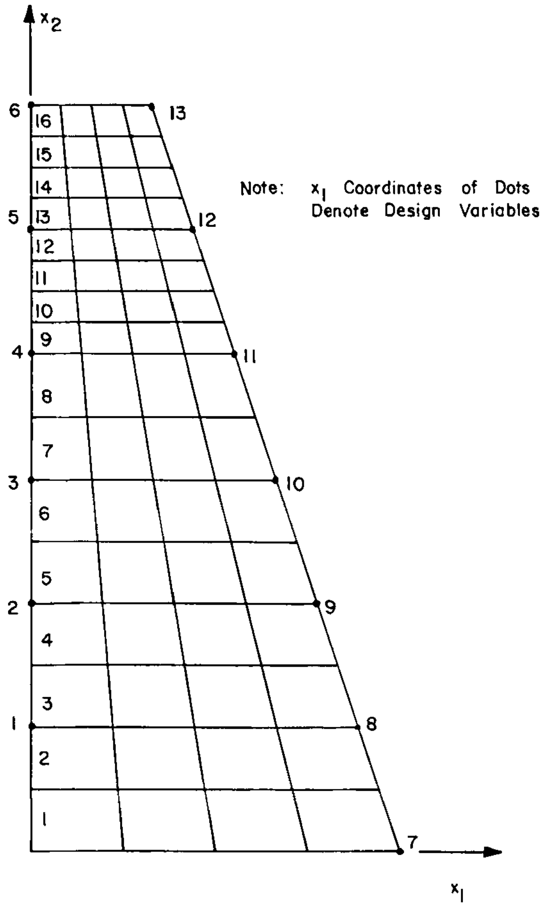

Fig. 3.3.15 Finite element model of dam; x_1 coordinates of dots denote design variables.

where $\lambda^{(k)}$ and $\lambda^{(k+NE)}$ are solutions of Eqs. (3.3.195) and (3.3.196), respectively, and Γ_k is the boundary of finite element Ω_k.

The finite element model of the dam, with optimal eight-node ISP elements [65, 66], is shown in Fig. 3.3.15. It contains 64 elements, 233 nodal points, and 448 degrees of freedom. The x_1 coordinates of 13 points on Γ_1 and Γ_3 (see Fig. 3.1.15) are chosen as design parameters (b_1 to b_{13}). Numerical results are based on the following data: $l = 100$ ft, $E = 3.64 \times 10^6$ psi, $\nu = 0.2$, $\sigma_L = -4000$ psi, $\sigma_U = 255$ psi, $s = 3$, $\gamma_w = 62.4$ lb/ft^3, and $h_L = 150$ in. Specific weight of concrete is chosen as 150 lb/ft^3.

The initial design is chosen as

$$[1.0 \quad 1.0 \quad 1.0 \quad 1.0 \quad 1.0 \quad 1.0 \quad 601.0 \quad 534.33 \quad 467.67 \quad 401.0 \quad 334.33 \quad 267.67 \quad 201.0]^T$$

for design sensitivity analysis. The reason for choosing b_1 to b_6 equal to 1, instead of 0, is convenience for design sensitivity analysis purposes (i.e., in considering percentage change of design variables). Design sensitivity results for stress in elements adjacent to Γ^1, for upper and lower bound stress constraints, are shown in Tables 3.3.3 and 3.3.4, respectively. A 0.1% uniform design change is selected, and the percentage of accuracy is close to 100% (see Tables 3.3.3 and 3.3.4). Only element 16 appears to have a poor accuracy percentage. However, the difference between perturbed and unperturbed constraint values is small, so accuracy of the difference is questionable.

Table 3.3.3

Sensitivity of Upper Principal Stress Constraints

Element number	ψ_k^1	ψ_k^2	$\Delta\psi_k$	ψ_k'	$(\psi_k'/\Delta\psi_k \times 100)\%$
1	−7.9053E − 01	−7.9128E − 01	−7.5253E − 04	−7.4310E − 04	98.7
2	−9.0381E − 01	−9.0452E − 01	−7.1246E − 04	−7.1183E − 04	99.9
3	−9.8040E − 01	−9.8101E − 01	−6.1205E − 04	−6.1903E − 04	101.1
4	−1.0179E + 00	−1.0185E + 00	−5.1437E − 04	−5.2495E − 04	102.1
5	−1.0433E + 00	−1.0438E + 00	−4.0794E − 04	−4.1971E − 04	102.9
6	−1.0586E + 00	−1.0589E + 00	−2.8703E − 04	−2.9766E − 04	103.7
7	−1.0633E + 00	−1.0634E + 00	−1.3293E − 04	−1.3872E − 04	104.3
8	−1.0575E + 00	−1.0575E + 00	−4.9233E − 05	−5.1360E − 05	104.3
9	−1.0500E + 00	−1.0500E + 00	−2.5452E − 05	−2.6611E − 05	104.6
10	−1.0442E + 00	−1.0442E + 00	−1.6348E − 05	−1.7121E − 05	104.7
11	−1.0380E + 00	−1.0380E + 00	−1.0088E − 05	−1.0579E − 05	104.9
12	−1.0315E + 00	−1.0315E + 00	−5.9007E − 06	−6.1863E − 06	104.8
13	−1.0248E + 00	−1.0247E + 00	−3.2191E − 06	−3.3446E − 06	103.9
14	−1.0178E + 00	−1.0178E + 00	−1.6441E − 06	−1.6693E − 06	101.5
15	−1.0108E + 00	−1.0108E + 00	−8.7791E − 07	−9.1872E − 07	104.6
16	−1.0039E + 00	−1.0039E + 00	−2.4626E − 07	−3.8964E − 07	158.2

Table 3.3.4

Sensitivity of Lower Principal Stress Constraints

Element number	$\psi^1_{k+\text{NE}}$	$\psi^2_{k+\text{NE}}$	$\Delta\psi_{k+\text{NE}}$	$\psi'_{k+\text{NE}}$	$(\psi'_{k+\text{NE}}/\Delta\psi_{k+\text{NE}} \times 100)\%$
1	−9.8986E − 01	−9.8985E − 01	9.2975E − 06	9.6684E − 06	104.0
2	−9.8847E − 01	−9.8847E − 01	1.1408E − 06	1.3977E − 06	122.5
3	−9.8993E − 01	−9.8993E − 01	6.3124E − 07	7.8423E − 07	124.2
4	−9.9078E − 01	−9.9078E − 01	5.3598E − 07	6.3913E − 07	119.2
5	−9.9171E − 01	−9.9171E − 01	1.0747E − 06	1.1831E − 06	110.1
6	−9.9251E − 01	−9.9251E − 01	2.6957E − 06	2.8459E − 06	105.6
7	−9.9308E − 01	−9.9307E − 01	6.8458E − 06	7.2127E − 06	105.4
8	−9.9344E − 01	−9.9343E − 01	7.1735E − 06	7.6427E − 06	106.5
9	−9.9385E − 01	−9.9384E − 01	5.4365E − 06	5.8612E − 06	107.8
10	−9.9426E − 01	−9.9426E − 01	4.1916E − 06	4.5704E − 06	109.0
11	−9.9480E − 01	−9.9480E − 01	3.0283E − 06	3.3564E − 06	110.8
12	−9.9548E − 01	−9.9548E − 01	2.0107E − 06	2.2845E − 06	113.6
13	−9.9630E − 01	−9.9629E − 01	1.1840E − 06	1.3998E − 06	118.2
14	−9.9725E − 01	−9.9725E − 01	5.7540E − 07	7.2897E − 07	126.7
15	−9.9831E − 01	−9.9831E − 01	1.9254E − 07	2.7604E − 07	143.4
16	−9.9943E − 01	−9.9943E − 01	2.1519E − 08	4.5401E − 08	211.0

PLANE STRESS INTERFACE PROBLEM

A thin elastic solid that is composed of two different materials and subjected to simple tension is now considered. The finite element configuration, dimensions, material properties of each body, and loading conditions are shown in Fig. 3.3.16. Body i occupies domain Ω^i ($i = 1, 2$), and AB is the interface boundary γ. Design variable b controls the position of the interface boundary γ, while the overall dimensions of the structure are fixed.

Consider the von Mises yield stress functional, averaged over finite element Ω_p,

$$\psi_p = \iint_\Omega g(\sigma(z))m_p \, d\Omega \tag{3.3.199}$$

where $g = \sigma_y$ is the von Mises yield stress, defined in Eq. (3.3.179). For numerical comparison, two methods are used for shape design sensitivity analysis: the boundary method of Section 3.3.6 and the domain method of Section 3.3.7.

For the boundary method, if $\Omega_p \subset \Omega^1$ and Γ_p intersects γ, Eq. (3.3.164) can be used, with limits of summation running from 1 to 2 and an appropriate modification of the generalized Hookes law of Eqs. (3.1.42) and (3.1.43). For the adjoint equation, Eq. (3.3.122) can be used. On the other hand, if $\bar{\Omega}_p \subset \Omega^1$, then the third integral on the right side of Eq. (3.3.164) becomes

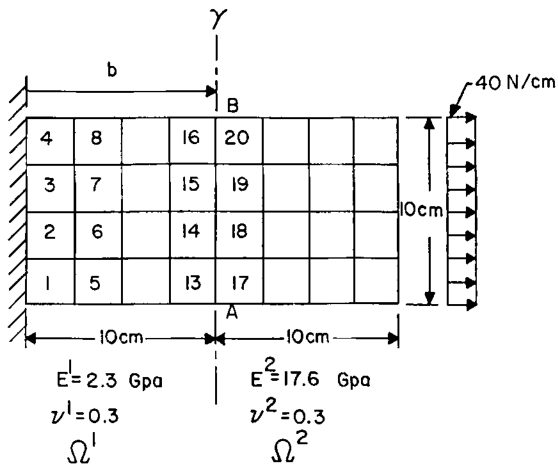

Fig. 3.3.16 Plane stress interface problem.

zero. In Eq. (3.3.164), n is the outward unit normal to Ω^2. Similar results may be obtained for the case $\Omega_p \subset \Omega^2$.

For the domain method, the results of Section 3.3.7 can be used. Suppose again that $\Omega_p \subset \Omega^1$. Since there is no body force applied and the externally loaded boundary is not moving, adding contributions from each component yields

$$\begin{aligned}
\psi_p' = {} & \sum_{l=1}^{2} \iiint_{\Omega^l} \sum_{i,j=1}^{2} [\sigma^{ij}(z)(\nabla\lambda^{i^{\mathrm{T}}} V_j) + \sigma^{ij}(\lambda)(\nabla z^{i^{\mathrm{T}}} V_j)]\, d\Omega \\
& - \sum_{l=1}^{2} \iiint_{\Omega^l} \left[\sum_{i,j=1}^{2} \sigma^{ij}(z)\varepsilon^{ij}(\lambda) \right] \operatorname{div} V \, d\Omega \\
& - \iiint_{\Omega^1} \sum_{i,j=1}^{2} \left[\sum_{k,l=1}^{2} g_{\sigma^{ij}}(z) C^{ijkl} (\nabla z^{k^{\mathrm{T}}} V_l) \right] m_p \, d\Omega \\
& + \iiint_{\Omega^1} g(z) \operatorname{div} V m_p \, d\Omega - \iiint_{\Omega^1} g(z) m_p \, d\Omega \iiint_{\Omega^1} m_p \operatorname{div} V \, d\Omega
\end{aligned} \tag{3.3.200}$$

where λ is the solution of the adjoint equation of Eq. (3.3.122). In Eq. (3.3.200), the asterisk notations for z and λ are dropped. For the integrand, the domain Ω^l ($l = 1, 2$) of integration will indicate which variables are to be used.

The finite element model shown in Fig. 3.3.16 contains 32 elements, 121 nodal points, and 233 degrees of freedom. The optimal eight-noded ISP element [65, 66] is employed for design sensitivity analysis. For the boundary method, stresses and strains are obtained at Gauss points and extrapolated to the boundary.

Numerical results with a 3% design change (i.e., $\delta b = 0.03b$) are shown in Table 3.3.5 for the boundary method and in Table 3.3.6 for the domain method. These results show that the domain method gives excellent results, whereas accuracy of the boundary method is not acceptable. For elements 22, 23, 29, and 32, the predicted values are less accurate than others. However, the magnitude of actual differences $\Delta\psi_p$ for those elements are smaller than others, so $\Delta\psi_p$ may lose precision.

Table 3.3.5

Boundary Method for Interface Problem

Element number	ψ_p^1	ψ_p^2	$\Delta\psi_p$	ψ_p'	$(\psi_p'/\Delta\psi_p \times 100)\%$
1	393.01304	393.17922	0.16618	0.20403	122.8
2	364.37867	364.76664	0.38796	0.67218	173.3
3	364.37867	364.76664	0.38796	0.67218	173.3
4	393.01304	393.17922	0.16618	0.20403	122.8
5	388.07514	388.36215	0.28701	0.56684	197.5
6	402.26903	402.83406	0.56503	0.42080	74.5
7	402.26903	402.83406	0.56503	0.42080	74.5
8	388.07514	388.36215	0.28701	0.56684	197.5
9	386.43461	386.84976	0.41515	−0.08520	−20.5
10	407.14612	407.48249	0.33637	0.14159	42.1
11	407.14612	407.48249	0.33637	0.14159	42.1
12	386.43461	386.84976	0.41515	−0.08520	−20.5
13	388.59634	388.95414	0.35780	−0.53089	−148.4
14	379.04276	379.25247	0.20971	−1.90134	−906.6
15	379.04276	379.25247	0.20971	−1.90134	−906.6
16	388.59634	388.95414	0.35780	−0.53089	−148.4
17	441.68524	442.25032	0.56507	−13.85905	−2452.6
18	424.05820	425.22910	1.17089	−13.63066	−1164.1
19	424.05820	425.22910	1.17089	−13.63066	−1164.1
20	441.68524	442.25032	0.56507	−13.85905	−2452.6
21	424.19015	424.70840	0.51825	−0.21408	−41.3
22	378.85433	378.97497	0.12064	0.76770	636.4
23	378.85433	378.97497	0.12064	0.76770	636.4
24	424.19015	424.70840	0.51825	−0.21408	−41.3
25	407.71528	408.23368	0.51840	0.49878	96.2
26	387.87304	387.32342	−0.54962	−0.48837	88.9
27	387.87304	387.32342	−0.54962	−0.48837	88.9
28	407.71528	408.23368	0.51840	0.49878	96.2
29	400.61014	400.60112	−0.00903	0.01423	−157.7
30	394.61705	394.00702	−0.61003	−0.57794	94.7
31	394.61705	394.00702	−0.61003	−0.57794	94.7
32	400.61014	400.60112	−0.00903	0.01423	−157.7

Table 3.3.6

Domain Method For Interface Problem

Element number	ψ_p^1	ψ_p^2	$\Delta\psi_p$	ψ_p'	$(\psi_p'/\Delta\psi_p \times 100)\%$
1	393.01304	393.17922	0.16618	0.17954	108.0
2	364.37867	364.76664	0.38796	0.37840	97.5
3	364.37867	364.76664	0.38796	0.37840	97.5
4	393.01304	393.17922	0.16618	0.17954	108.0
5	388.07514	388.36215	0.28701	0.28671	99.9
6	402.26903	402.83406	0.56503	0.59634	105.5
7	402.26903	402.83406	0.56503	0.59634	105.5
8	388.07514	388.36215	0.28701	0.28671	99.9
9	386.43461	386.84976	0.41515	0.41748	100.6
10	407.14612	407.48249	0.33637	0.36857	109.6
11	407.14612	407.48249	0.33637	0.36857	109.6
12	386.43461	386.84976	0.41515	0.41748	100.6
13	388.59634	388.95414	0.35780	0.37548	104.9
14	379.04276	379.25247	0.20971	0.20159	96.1
15	379.04276	379.25247	0.20971	0.20159	96.1
16	388.59634	388.95414	0.35780	0.37548	104.9
17	441.68524	442.25032	0.56507	0.57069	101.0
18	424.05820	425.22910	1.17089	1.12871	96.4
19	424.05820	425.22910	1.17089	1.12871	96.4
20	441.68524	442.25032	0.56507	0.57069	101.0
21	424.19015	424.70840	0.51825	0.53919	104.0
22	378.85433	378.97497	0.12064	0.06396	53.0
23	378.85433	378.97497	0.12064	0.06396	53.0
24	424.19015	424.70840	0.51825	0.53919	104.0
25	407.71528	408.23368	0.51840	0.51710	99.7
26	387.87304	387.32342	−0.54962	−0.56083	102.0
27	387.87304	387.32342	−0.54962	−0.56083	102.0
28	407.71528	408.23368	0.51840	0.51710	99.7
29	400.61014	400.60112	−0.00903	−0.00298	33.0
30	394.61705	394.00702	−0.61003	−0.58529	95.9
31	394.61705	394.00702	−0.61003	−0.58529	95.9
32	400.61014	400.60112	−0.00903	−0.00298	33.0

3.4 EIGENVALUE SHAPE DESIGN SENSITIVITY ANALYSIS

Examples presented in Section 3.1 show that eigenvalues such as natural frequencies of vibration depend on the shape of the structure. As in Section 2.3, the objective in this section is to obtain sensitivity of eigenvalues with respect to shape variation. As in Chapter 2, for conservative systems, no adjoint equations are necessary, and eigenvalue shape design sensitivity can be expressed directly in terms of eigenvectors associated with the eigenvalues and the eigenvalue bilinear forms. Differentiability of simple eigenvalues and directional differentiability of repeated eigenvalues are used to obtain explicit formulas, utilizing the material derivative formulas of Section 3.2, for both simple and repeated eigenvalue design sensitivity analysis. Numerical examples of computation of eigenvalue sensitivity are presented.

3.4.1 Differentiability of Bilinear Forms and Eigenvalues

Basic results concerning differentiability of eigenvalues for problems treated in Section 3.1 are proved in Section 3.5. The purpose of this section is to summarize key results that are needed for eigenvalue design sensitivity. The case of repeated eigenvalues is more subtle. It is shown that repeated eigenvalues are only directionally differentiable.

As shown in Section 3.1, eigenvalues for vibration of elastic systems on a deformed domain are determined by variational equations of the form

$$\begin{aligned} a_{\Omega_\tau}(y_\tau, \bar{y}_\tau) &\equiv \iint_{\Omega_\tau} c(y_\tau, \bar{y}_\tau)\, d\Omega_\tau \\ &= \zeta_\tau \iint_{\Omega_\tau} e(y_\tau, \bar{y}_\tau)\, d\Omega_\tau \equiv \zeta_\tau\, d_{\Omega_\tau}(y_\tau, \bar{y}_\tau) \qquad \text{for all} \quad \bar{y}_\tau \in Z_\tau \end{aligned} \tag{3.4.1}$$

where $Z_\tau \subset H^m(\Omega_\tau)$ is the space of kinematically admissible displacements and $c(\cdot, \cdot)$ and $e(\cdot, \cdot)$ are symmetric bilinear mappings. Since Eq. (3.4.1) is homogeneous in y_τ, a normalizing condition must be used to define unique eigenfunctions. The normalizing condition is

$$d_{\Omega_\tau}(y_\tau, y_\tau) = 1 \tag{3.4.2}$$

The energy bilinear form on the left side of Eq. (3.4.1) is the same as the bilinear form in static problems treated in Section 3.3. Therefore, it has the same differentiability properties as discussed there. The bilinear form $d_{\Omega_\tau}(\cdot, \cdot)$ on the right side of Eq. (3.4.1) represents mass effects in vibration problems and geometric effects in buckling problems. In most cases, except for buckling of a column, it is even more regular than the energy bilinear form in its dependence on design and eigenfunction.

SIMPLE EIGENVALUES

It is shown in Section 3.5.5 that a simple eigenvalue ζ is differentiable. It was shown by Kato [13] that the corresponding eigenfunction y is also differentiable. In fact, material derivatives of both the eigenvalue and eigenfunction are linear in V, hence they are Fréchet derivatives of the eigenvalue and eigenfunction. As in the static response case, linearity and continuity of the mapping $V \to \dot{y}$ allows, by Theorem 3.5.3 of Section 3.5.7, use of only the normal component $(V^T n)$ of the velocity field V in derivation of the material derivative, as in Eq. (3.2.36).

Taking the material derivative of both sides of Eq. (3.4.1), using Eq. (3.2.36) and noting that partial derivatives with respect to τ and x commute with each other,

$$\begin{aligned} [a_\Omega(y, \bar{y})]' &\equiv a'_V(y, \bar{y}) + a_\Omega(\dot{y}, \bar{y}) \\ &= \zeta' d_\Omega(y, \bar{y}) + \zeta[d'_V(y, \bar{y}) + d_\Omega(\dot{y}, \bar{y})] \\ &\equiv \zeta' d_\Omega(y, \bar{y}) + \zeta[d_\Omega(y, \bar{y})]' \qquad \text{for all} \quad \bar{y} \in Z \end{aligned} \tag{3.4.3}$$

where, using Eq. (3.2.8),

$$\begin{aligned} [a_\Omega(y, \bar{y})]' &= \iint_\Omega [c(y', \bar{y}) + c(y, \bar{y}')]\, d\Omega + \int_\Gamma d(y, \bar{y})(V^T n)\, d\Gamma \\ &= \iint_\Omega [c(\dot{y} - \nabla y^T V, \bar{y}) + c(y, \dot{\bar{y}} - \nabla \bar{y}^T V)]\, d\Omega + \int_\Gamma c(y, \bar{y})(V^T n)\, d\Gamma \end{aligned} \tag{3.4.4.}$$

and

$$\begin{aligned} [d_\Omega(y, \bar{y})]' &= \iint_\Omega [e(y', \bar{y}) + e(y, \bar{y}')]\, d\Omega + \int_\Gamma e(y, \bar{y})(V^T n)\, d\Gamma \\ &= \iint_\Omega [e(\dot{y} - \nabla y^T V, \bar{y}) + e(y, \dot{\bar{y}} - \nabla \bar{y}^T V)]\, d\Omega + \int_\Gamma e(y, \bar{y})(V^T n)\, d\Gamma \end{aligned} \tag{3.4.5}$$

As in Eq. (3.3.3), the fact that the partial derivatives of the coefficients in the bilinear mappings $c(\cdot, \cdot)$ and $e(\cdot, \cdot)$ are zero has been used in Eqs. (3.4.4) and (3.4.5). As in the static response case, for $\bar{y}_\tau$ select $\bar{y}_\tau(x + \tau V(x)) = \bar{y}(x)$. Since $H^m(\Omega)$ is preserved by $T(x, \tau)$ [Eq. (3.2.12)], if $\bar{y} \in Z$ is arbitrary, then $\bar{y}_\tau$ is an arbitrary element of Z_τ. Also, from Eq. (3.2.8),

$$\dot{\bar{y}} = \bar{y}' + \nabla \bar{y}^T V = 0 \tag{3.4.6}$$

and from Eqs. (3.4.3), (3.4.4), and (3.4.5), using Eq. (3.4.6),

$$a'_V(y, \bar{y}) = -\iint_\Omega [c(\nabla y^T V, \bar{y}) + c(y, \nabla \bar{y}^T V)]\, d\Omega + \int_\Gamma c(y, \bar{y})(V^T n)\, d\Gamma \tag{3.4.7}$$

and

$$d'_V(y, \bar{y}) = -\iint_\Omega [e(\nabla y^T V, \bar{y}) + e(y, \nabla \bar{y}^T V)]\, d\Omega + \int_\Gamma e(y, \bar{y})(V^T n)\, d\Gamma \tag{3.4.8}$$

Since $\bar{y} \in Z$, Eq. (3.4.3) may be evaluated with $\bar{y} = y$, using symmetry of the bilinear forms, to obtain

$$\zeta' d_\Omega(y, \bar{y}) = a'_V(y, \bar{y}) - \zeta d'_V(y, \bar{y}) - [a_\Omega(y, \dot{y}) - \zeta d_\Omega(y, \dot{y})] \tag{3.4.9}$$

Since $\dot{y} \in Z$ [see the paragraph following Eq. (3.3.8)], the term in brackets on the right side of Eq. (3.4.9) is zero. Furthermore, due to the normalizing condition of Eq. (3.4.2) a simplified equation may be used,

$$\begin{aligned} \zeta' &= a'_V(y, y) - \zeta d'_V(y, y) \\ &= 2 \iint_\Omega [-c(y, \nabla y^T V) + \zeta e(y, \nabla y^T V)]\, d\Omega + \int_\Gamma [c(y, y) - \zeta e(y, y)](V^T n)\, d\Gamma \end{aligned} \tag{3.4.10}$$

where, as in the static response case, the integral over Ω can be transformed to boundary integrals by using the variational identities given in Section 3.1 for each structural component and boundary and/or interface conditions. This will be done for each class of problem encountered.

Note that the directional derivative of the eigenvalue is linear in V, since the variations of the bilinear forms on the right side of Eq. (3.4.10) are linear in V. As noted in Section 2.3, validity of this result rests on the existence of derivatives of eigenvalues and eigenfunctions.

REPEATED EIGENVALUES

Consider now the situation in which an eigenvalue ζ has multiplicity $s > 1$ at Ω; that is,

$$\left.\begin{aligned} a_\Omega(y^i, \bar{y}) &= \zeta\, d_\Omega(y^i, \bar{y}) \quad \text{for all} \quad \bar{y} \in Z \\ d_\Omega(y^i, y^j) &= \delta_{ij} \end{aligned}\right\} \quad i,j = 1, 2, \ldots, s \tag{3.4.11}$$

It is shown in Section 3.5 that the repeated eigenvalue ζ is a continuous function of design but that the corresponding eigenfunctions are not. Moreover, as in Section 2.3, it is shown in Section 3.5 that at Ω, where the eigenvalue ζ is repeated s times, it is only directionally differentiable and the directional derivatives $\zeta'_i(V)$ in the direction V are the eigenvalues of the $s \times s$ matrix $\mathcal{M}$ with elements

$$\begin{aligned} \mathcal{M}_{ij} &= a'_V(y^i, y^j) - \zeta\, d'_V(y^i, y^j) \\ &= \iint_\Omega [-c(\nabla y^{i\mathrm{T}} V, y^j) - c(y^i, \nabla y^{j\mathrm{T}} V) + \zeta e(\nabla y^{i\mathrm{T}} V, y^j) + \zeta e(y^i, \nabla y^{j\mathrm{T}} V)]\, d\Omega \\ &\quad + \int_\Gamma [c(y^i, y^j) - \zeta e(y^i, y^j)](V^T n)\, d\Gamma, \qquad i,j = 1, 2, \ldots, s \end{aligned} \tag{3.4.12}$$

The notation $\zeta'_i(V)$ is used here to emphasize dependence of the directional derivative on V. As in the simple eigenvalue case, the integral over Ω in Eq.

(3.4.12) can be transformed to boundary integrals by using the variational identities given in Section 3.1 for each structural component and boundary and/or interface conditions. This will be done for each class of problem encountered.

If the d_Ω-orthonormal basis $\{y^i\}_{i=1,\ldots,s}$ of the eigenspace is changed, then the matrix $\mathscr{M}$ changes, but the eigenvalues of $\mathscr{M}$ remain the same. As mentioned in Section 2.3.1, the directional derivatives $\zeta_i'(V)$ are not generally linear in V, even though each $\mathscr{M}_{ij}$ is linear in V. Other results in Section 2.3.1 on directional derivatives of repeated eigenvalues remain valid in this section. For $s = 2$, directional derivatives of a double eigenvalue are

$$\zeta_i'(V) = \{(\mathscr{M}_{11} + \mathscr{M}_{22}) \pm [(\mathscr{M}_{11} + \mathscr{M}_{22})^2 - 4(\mathscr{M}_{11}\mathscr{M}_{22} - \mathscr{M}_{12}^2)]^{1/2}\}/2, \quad i = 1, 2 \tag{3.4.13}$$

where $i = 1$ corresponds to the minus sign, $i = 2$ corresponds to the plus sign, and $\mathscr{M}_{ij}$ is given in Eq. (3.4.12) $(i, j = 1, 2)$. Another expression for directional derivatives is

$$\zeta_1'(V) = \cos^2\phi(V)\,\mathscr{M}_{11} + \sin 2\phi(V)\,\mathscr{M}_{12} + \sin^2\phi(V)\,\mathscr{M}_{22} \tag{3.4.14}$$

$$\zeta_2'(V) = \sin^2\phi(V)\,\mathscr{M}_{11} - \sin 2\phi(V)\,\mathscr{M}_{12} + \cos^2\phi(V)\,\mathscr{M}_{22} \tag{3.4.15}$$

where the eigenvector rotation angle ϕ is given as

$$\phi(V) = \frac{1}{2}\arctan\left[\frac{2\mathscr{M}_{12}}{\mathscr{M}_{11} - \mathscr{M}_{22}}\right] \tag{3.4.16}$$

3.4.2 Analytical Examples of Eigenvalue Design Sensitivity

The beam, column, membrane, and plate problems of Section 3.1 are used here as examples for eigenvalue design sensitivity analysis.

VIBRATION OF A BEAM

Consider the vibrating beam of Section 3.1, with cross-sectional area $h(x) \geq h_0 \geq 0$, $I(x) = \alpha h^2(x)$, and Young's modulus E. Using Eq. (3.4.10),

$$\zeta' = 2\int_0^l [-E\alpha h^2 y_{xx}(y_x V)_{xx} + \zeta\rho h y(y_x V)]\,dx + [E\alpha h^2(y_{xx})^2 - \zeta\rho h y^2]V\Big|_0^l \tag{3.4.17}$$

The variational identity of Eq. (3.1.5) may be used, identifying $(y_x V)$ in the domain integral of Eq. (3.4.17) with $\bar{y}$ in Eq. (3.1.5), to obtain

$$\zeta' = -2[E\alpha h^2 y_{xx}(y_x V)_x - (E\alpha h^2 y_{xx})_x(y_x V)]\Big|_0^l + [E\alpha h^2(y_{xx})^2 - \zeta\rho h y^2]V\Big|_0^l \tag{3.4.18}$$

Using boundary conditions of Eq. (2.1.1) (note beam length $\hat{l}$ is not normalized in this chapter) for the clamped–clamped beam, Eq. (3.4.18) becomes

$$\zeta' = -E\alpha h^2 (y_{xx})^2 V \Big|_0^{\hat{l}} \tag{3.4.19}$$

It is interesting to note that since the coefficient of the velocity V is negative, frequency decreases as the boundary moves outward, which is clear physically. Moreover, by moving the end of the beam with larger $E\alpha h^2(y_{xx})^2$ outward, the fundamental frequency can be decreased most effectively.

For other boundary conditions in Eqs. (2.1.16)–(2.1.18), as in the static case, the design sensitivity formula of Eq. (3.4.18) is valid. To obtain a design sensitivity formula for the simply supported case, use of boundary conditions of Eq. (2.1.16) in Eq. (3.4.18) yields

$$\zeta' = 2E\alpha h^2 y_{xxx} y_x V \Big|_0^{\hat{l}} \tag{3.4.20}$$

For a cantilevered beam, applying boundary conditions of Eq. (2.1.17) to Eq. (3.4.18) yields

$$\zeta' = E\alpha h^2 (y_{xx})^2 V \Big|_{x=0} - \zeta \rho h y^2 V \Big|_{x=\hat{l}} \tag{3.4.21}$$

For a clamped–simply supported beam, applying boundary conditions of Eq. (2.1.18) to Eq. (3.4.18) yields

$$\zeta' = E\alpha h^2 (z_{xx})^2 V \Big|_{x=0} + 2E\alpha h^2 y_{xxx} y_x V \Big|_{x=\hat{l}} \tag{3.4.22}$$

BUCKLING OF A COLUMN

Consider buckling of the column of Section 3.1, with cross-sectional area h, $I(x) = \alpha h^2(x)$, and Young's modulus E. Using Eq. (3.4.10),

$$\zeta' = 2\int_0^{\hat{l}} [-E\alpha h^2 y_{xx}(y_x V)_{xx} + \zeta y_x (y_x V)_x]\, dx + [E\alpha h^2 (y_{xx})^2 - \zeta (y_x)^2] V \Big|_0^{\hat{l}} \tag{3.4.23}$$

Using the variational identity of Eq. (3.1.8), and identifying $(y_x V)$ in the domain integral of Eq. (3.4.23) with $\bar{y}$ in Eq. (3.1.8),

$$\zeta' = -2[E\alpha h^2 y_{xx}(y_x V)_x - (E\alpha h^2 y_{xx})_x (y_x V) - \zeta y_x (y_x V)] \Big|_0^{\hat{l}} + [E\alpha h^2 (y_{xx})^2 - \zeta (y_x)^2] V \Big|_0^{\hat{l}} \tag{3.4.24}$$

For a clamped–clamped column, using boundary conditions of Eq. (2.1.1), Eq. (3.4.24) becomes

$$\zeta' = -E\alpha h^2 (y_{xx})^2 V \Big|_0^l \tag{3.4.25}$$

As in the case of vibration of a beam, the coefficient of the variation V is negative. Hence, the buckling load decreases as the boundary moves outward.

For a simply supported column, using boundary conditions of Eq. (2.1.16), Eq. (3.4.24) becomes

$$\zeta' = [2E\alpha h^2 y_{xxx} y_x + \zeta (y_x)^2] V \Big|_0^l \tag{3.4.26}$$

For a cantilevered column, using boundary conditions of Eq. (2.1.17), Eq. (3.4.24) becomes

$$\zeta' = E\alpha h^2 (y_{xx})^2 V \Big|_{x=0} - \zeta (y_x)^2 V \Big|_{x=l} \tag{3.4.27}$$

For a clamped–simply supported column, using boundary conditions of Eq. (2.1.18), Eq. (3.4.24) becomes

$$\zeta' = E\alpha h^2 (y_{xx})^2 V \Big|_{x=0} + [2E\alpha h^2 y_{xxx} y_x + \zeta (y_x)^2] V \Big|_{x=l} \tag{3.4.28}$$

For an s-times repeated eigenvalue, using Eq. (3.4.12),

$$\mathscr{M}_{ij} = \int_0^l [-E\alpha h^2 y_{xx}^j (y_x^i V)_{xx} - E\alpha h^2 y_{xx}^i (y_x^j V)_{xx} + \zeta y_x^j (y_x^i V)_x + \zeta y_x^i (y_x^j V)_x] \, dx$$

$$+ [E\alpha h^2 y_{xx}^i y_{xx}^j - \zeta y_x^i y_x^j] V \Big|_0^l, \qquad i,j = 1, 2, \ldots, s \tag{3.4.29}$$

Using the variational identity of Eq. (3.1.8) twice in Eq. (3.4.29),

$$\mathscr{M}_{ij} = -[E\alpha h^2 y_{xx}^j (y_x^i V)_x - (E\alpha h^2 y_{xx}^j)_x (y_x^i V) - \zeta y_x^j (y_x^i V)] \Big|_0^l$$

$$- [E\alpha h^2 y_{xx}^i (y_x^j V)_x - (E\alpha h^2 y_{xx}^i)_x (y_x^j V) - \zeta y_x^i (y_x^j V)] \Big|_0^l$$

$$+ [E\alpha h^2 y_{xx}^i y_{xx}^j - \zeta y_x^i y_x^j] V \Big|_0^l, \qquad i,j = 1, 2, \ldots, s$$

(3.4.30)

As in the simple eigenvalue case, the result of Eq. (3.4.30) is valid for the boundary conditions given in Eqs. (2.1.1) and (2.1.16)–(2.1.18). To obtain $\mathscr{M}_{ij}$ for each case, these boundary conditions may be applied in Eq. (3.4.30). For the case of a double eigenvalue ($s = 2$), the directional derivatives of the repeated eigenvalue can be obtained from Eqs. (3.4.14) and (3.4.15), where rotation angle ϕ is given in Eq. (3.4.16).

VIBRATION OF A MEMBRANE

Consider the membrane of Fig. 3.1.1, with mass density h. For a simple eigenvalue, using Eq. (3.4.10) and the fact that $y = 0$ on Γ,

$$\zeta' = 2\iint_\Omega [-\hat{T}\nabla y^{\mathrm{T}}\nabla(\nabla y^{\mathrm{T}}V) + \zeta h y(\nabla y^{\mathrm{T}}V)]\, d\Omega + \hat{T}\int_\Gamma (\nabla y^{\mathrm{T}}\nabla y)(V^{\mathrm{T}}n)\, d\Gamma \tag{3.4.31}$$

Applying the variational identity of Eq. (3.1.13) to Eq. (3.4.31) and identifying $(\nabla y^{\mathrm{T}}V)$ in the domain integral of Eq. (3.4.31) with $\bar{y}$ in Eq. (3.1.13),

$$\zeta' = -2\hat{T}\int_\Gamma \frac{\partial y}{\partial n}(\nabla y^{\mathrm{T}}V)\, d\Gamma + \hat{T}\int_\Gamma (\nabla y^{\mathrm{T}}\nabla y)(V^{\mathrm{T}}n)\, d\Gamma \tag{3.4.32}$$

Since $y = 0$ on Γ, $\nabla y = (\partial y/\partial n)n$ on Γ, and

$$\zeta' = -\hat{T}\int_\Gamma \left(\frac{\partial y}{\partial n}\right)^2 (V^{\mathrm{T}}n)\, d\Gamma \tag{3.4.33}$$

As noted in Section 3.4.1, the eigenvalue design sensitivity in Eq. (3.4.33) is expressed as a boundary integral, and only the normal movement $(V^{\mathrm{T}}n)$ of the boundary appears.

It is interesting to note that since the coefficient of $(V^{\mathrm{T}}n)$ is negative, the frequency decreases as the boundary moves outward, which is clear physically. Moreover, moving the boundary outward in the vicinity of a high normal derivative decreases the fundamental frequency most effectively.

For an s-times repeated eigenvalue, using Eq. (3.4.12) and the fact that $y^i = 0$ on Γ ($i = 1, 2, \ldots, s$),

$$\begin{aligned}\mathscr{M}_{ij} = \iint_\Omega [&-\hat{T}\nabla y^{j\mathrm{T}}\nabla(\nabla y^{i\mathrm{T}}V) - \hat{T}\nabla y^{i\mathrm{T}}\nabla(\nabla y^{j\mathrm{T}}V) + \zeta h y^j(\nabla y^{i\mathrm{T}}V)\\ &+ \zeta h y^i(\nabla y^{j\mathrm{T}}V)]\, d\Omega + \hat{T}\int_\Gamma (\nabla y^{i\mathrm{T}}\nabla y^j)(V^{\mathrm{T}}n)\, d\Gamma, \qquad i,j = 1, 2, \ldots, s\end{aligned} \tag{3.4.34}$$

Applying the variational identity of Eq. (3.1.13) twice to Eq. (3.4.34)

$$\mathscr{M}_{ij} = -\hat{T}\int_{\Gamma}\left[\frac{\partial y^j}{\partial n}(\nabla y^{i\mathrm{T}}V) + \frac{\partial y^i}{\partial n}(\nabla y^{j\mathrm{T}}V)\right]d\Omega + \hat{T}\int_{\Gamma}(\nabla y^{i\mathrm{T}}\nabla y^j)(V^{\mathrm{T}}n)\,d\Gamma \tag{3.4.35}$$

Since $y^i = 0$ on Γ, $\nabla y^i = (\partial y^i/\partial n)n$ on Γ, and

$$\mathscr{M}_{ij} = -\hat{T}\int_{\Gamma}\left(\frac{\partial y^i}{\partial n}\right)\left(\frac{\partial y^j}{\partial n}\right)(V^{\mathrm{T}}n)\,d\Gamma, \qquad i,j = 1,2,\ldots,s \tag{3.4.36}$$

Consider now the case of a double eigenvalue at Ω (i.e., $s = 2$). The directional derivatives of the repeated eigenvalue are given by Eqs. (3.4.14) and (3.4.15) as

$$\begin{aligned}\zeta_1'(V) &= -\hat{T}\int_{\Gamma}\left[\cos^2\phi(V)\left(\frac{\partial y^1}{\partial n}\right)^2 + \sin 2\phi(V)\left(\frac{\partial y^1}{\partial n}\right)\left(\frac{\partial y^2}{\partial n}\right)\right.\\ &\qquad\left. + \sin^2\phi(V)\left(\frac{\partial y^2}{\partial n}\right)^2\right](V^{\mathrm{T}}n)\,d\Gamma\\ \zeta_2'(V) &= -\hat{T}\int_{\Gamma}\left[\sin^2\phi(V)\left(\frac{\partial y^1}{\partial n}\right)^2 - \sin 2\phi(V)\left(\frac{\partial y^1}{\partial n}\right)\left(\frac{\partial y^2}{\partial n}\right)\right.\\ &\qquad\left. + \cos^2\phi(V)\left(\frac{\partial y^2}{\partial n}\right)^2\right](V^{\mathrm{T}}n)\,d\Gamma\end{aligned} \tag{3.4.37}$$

where the rotation angle ϕ is obtained from Eq. (3.4.17) as

$$\phi(V) = \frac{1}{2}\arctan\left[\frac{2\int_{\Gamma}(\partial y^1/\partial n)(\partial y^2/\partial n)(V^{\mathrm{T}}n)\,d\Gamma}{\int_{\Gamma}[(\partial y^1/\partial n)^2 - (\partial y^2/\partial n)^2](V^{\mathrm{T}}n)\,d\Gamma}\right] \tag{3.4.38}$$

It is clear from Eq. (3.4.37) that the directional derivatives of the repeated eigenvalues are not linear in V, hence they are not Fréchet differentiable.

VIBRATION OF A PLATE

Consider the vibrating plate of Section 3.1, with thickness h, Young's modulus E, and mass density ρ. Using Eq. (3.4.10),

$$\begin{aligned}\zeta' &= 2\iint_{\Omega}\{-\hat{D}[(y_{11} + \nu y_{22})(\nabla y^{\mathrm{T}}V)_{11} + (y_{22} + \nu y_{11})(\nabla y^{\mathrm{T}}V)_{22}\\ &\qquad + 2(1-\nu)y_{12}(\nabla y^{\mathrm{T}}V)_{12}] + \zeta\rho h y(\nabla y^{\mathrm{T}}V)\}\,d\Omega\\ &\quad + \int_{\Gamma}\{\hat{D}[(y_{11} + \nu y_{22})y_{11} + (y_{22} + \nu y_{11})y_{22}\\ &\qquad + 2(1-\nu)y_{12}^2] - \zeta\rho h y^2\}(V^{\mathrm{T}}n)\,d\Gamma\end{aligned} \tag{3.4.39}$$

Using the variational identity of Eq. (3.1.30), and identifying $(\nabla y^{\mathrm{T}}V)$ in the domain integral of Eq. (3.4.39) with $\bar{y}$ in Eq. (3.1.30),

$$\begin{aligned}\zeta' = &-2\int_{\Gamma}(\nabla y^{\mathrm{T}}V)Ny\,d\Gamma - 2\int_{\Gamma}\frac{\partial}{\partial n}(\nabla y^{\mathrm{T}}V)\,My\,d\Gamma \\ &+\int_{\Gamma}\{\hat{D}[(y_{11}+\nu y_{22})y_{11}+(y_{22}+\nu y_{11})y_{22} \\ &+2(1-\nu)y_{12}^{2}]-\zeta\rho h y^{2}\}(V^{\mathrm{T}}n)\,d\Gamma \end{aligned} \tag{3.4.40}$$

As in the static response case, sensitivity formulas due to the variation of clamped, simply supported, and free-edge parts of the boundary are

$$\begin{aligned}\zeta' = \int_{\Gamma_{\mathrm{C}}}\hat{D}\bigg\{-2\left(\frac{\partial^{2}y}{\partial n^{2}}\right)^{2} &+ [(y_{11}+\nu y_{22})y_{11}+(y_{22}+\nu y_{11})y_{22} \\ &+2(1-\nu)y_{12}^{2}]\bigg\}(V^{\mathrm{T}}n)\,d\Gamma\end{aligned} \tag{3.4.41}$$

$$\begin{aligned}\zeta' = \int_{\Gamma_{\mathrm{S}}}\bigg\{-2\left(\frac{\partial y}{\partial n}\right)Ny &+ \hat{D}[(y_{11}+\nu y_{22})y_{11}+(y_{22}+\nu y_{11})y_{22} \\ &+2(1-\nu)y_{12}^{2}]\bigg\}(V^{\mathrm{T}}n)\,d\Gamma\end{aligned} \tag{3.4.42}$$

and

$$\begin{aligned}\zeta' = \int_{\Gamma_{\mathrm{F}}}&\{\hat{D}[(y_{11}+\nu y_{22})y_{11}+(y_{22}+\nu y_{11})y_{22}+2(1-\nu)y_{12}^{2}] \\ &-\zeta\rho h y^{2}\}(V^{\mathrm{T}}n)\,d\Gamma\end{aligned} \tag{3.4.43}$$

respectively. For $\Gamma = \Gamma_{\mathrm{C}} \cup \Gamma_{\mathrm{S}} \cup \Gamma_{\mathrm{F}}$, the complete design sensitivity formula can be obtained by adding terms in Eqs. (3.4.41)–(3.4.43).

For an s-times repeated eigenvalue ζ at Ω [43], the directional derivatives $\zeta_i'(V)$ in the direction V are the eigenvalues of the $s \times s$ matrix $\mathscr{M}$ with elements

$$\begin{aligned}\mathscr{M}_{ij} = \iint_{\Omega}&\{-\hat{D}[(y_{11}^{j}+\nu y_{22}^{j})(\nabla y^{i\mathrm{T}}V)_{11}+(y_{22}^{j}+\nu y_{11}^{j})(\nabla y^{i\mathrm{T}}V)_{22} \\ &+2(1-\nu)y_{12}^{j}(\nabla y^{i\mathrm{T}}V)_{12}]+\zeta\rho h y^{j}(\nabla y^{i\mathrm{T}}V) \\ &-\hat{D}[(y_{11}^{i}+\nu y_{22}^{i})(\nabla y^{j\mathrm{T}}V)_{11}+(y_{22}^{i}+\nu y_{11}^{i})(\nabla y^{j\mathrm{T}}V)_{22} \\ &+2(1-\nu)y_{12}^{i}(\nabla y^{j\mathrm{T}}V)_{12}]+\zeta\rho h y^{i}(\nabla y^{j\mathrm{T}}V)\}\,d\Omega \\ +\int_{\Gamma}&\{\hat{D}[(y_{11}^{i}+\nu y_{22}^{i})y_{11}^{j}+(y_{22}^{i}+\nu y_{11}^{i})y_{22}^{j}+2(1-\nu)y_{12}^{i}y_{12}^{j}] \\ &-\zeta\rho h y^{i}y^{j}\}(V^{\mathrm{T}}n)\,d\Gamma\end{aligned} \tag{3.4.44}$$

Using the variational identity of Eq. (3.1.30) twice in Eq. (3.4.44),

$$\begin{aligned}\mathscr{M}_{ij} = &-\int_\Gamma \left[(\nabla y^{i\mathrm{T}} V) N y^j + \frac{\partial}{\partial n}(\nabla y^{i\mathrm{T}} V) M y^j\right] d\Gamma \\ &-\int_\Gamma \left[(\nabla y^{j\mathrm{T}} V) N y^i + \frac{\partial}{\partial n}(\nabla y^{j\mathrm{T}} V) M y^i\right] d\Gamma \\ &+\int_\Gamma \{\hat{D}[(y^i_{11} + \nu y^i_{22}) y^j_{11} + (y^i_{22} + \nu y^i_{11}) y^j_{22} + 2(1-\nu) y^i_{12} y^j_{12}] \\ &\quad - \zeta \rho h y^i y^j\}(V^{\mathrm{T}} n)\, d\Gamma \end{aligned} \tag{3.4.45}$$

As in the simple eigenvalue case, the result of Eq. (3.4.45) is valid for boundary conditions given in Eqs. (3.1.26)–(3.1.28). To obtain $\mathscr{M}_{ij}$ for each case, these boundary conditions can be applied in Eq. (3.4.45).

3.5 DIFFERENTIABILITY OF STATIC RESPONSE AND EIGENVALUES WITH RESPECT TO SHAPE

The purpose of this section is to characterize dependence of static response and eigenvalues of the structures of Section 3.1 on their shapes. A transformation function is defined that uniquely determines the shape of a body. Differential operator properties and transformation techniques of integral calculus of Section 3.2 are employed to show that static response and eigenvalues of the system depend in a continuous and differentiable way on shape of the body. This section is more mathematically technical than others in the text. The reader who is interested only in using the methods and results of Sections 3.3 and 3.4 need not go through the details of this section. In order to be mathematically complete, differentiability of the membrane problem will be considered in detail. For differentiability in other problems of Section 3.1, the reader is referred to Rousselet and Haug [70].

Another important result given in this section considers the variation of a domain functional. If the gradient of the domain functional exists, only the normal component ($V^{\mathrm{T}}n$) of the velocity field V is of importance. This result was used in Section 3.2.2 to find the derivatives of several domain functionals.

3.5.1 Characterization of Shape

It is essential to first state precisely how structural operators are related to the domain Ω, which amounts to defining the dependence of the coefficients of the differential operators on the geometric domain Ω. The following hypotheses are made:

HYPOTHESIS H_1:

1. The physical domain Ω is a bounded open set of R^n.
2. For every $\bar{x} \in \Gamma$, there exists a system of local coordinates $(x_1, x_2, \ldots, x_n)$ and a cube $Q = (-a, a)^n \equiv (-a, a) \times \cdots \times (-a, a)$ (open neighborhood of $\bar{x}$), as shown in Fig. 3.5.1 for $n = 2$, such that points in $\Omega \cap Q$ satisfy $x_n \leq \Phi(x_1, \ldots, x_{n-1})$ for $(x_1, \ldots, x_{n-1}) \in (-a, a)^{n-1}$, where $\Phi \in C^1[(-a, a)^{n-1}]$ for second-order problems and $\Phi \in C^2[(-a, a)^{n-1}]$ for fourth-order problems.

HYPOTHESIS H_2 All coefficients involved in the operators defined in Section 3.1 are assumed to be $C^1(\Omega)$. Moreover, it is assumed that there exists $h_{\min} > 0$ such that $h(x) \geq h_{\min} > 0$ in Ω.

A difficulty in defining shape as a design variable is that shapes of geometrical domains are not usually considered as a vector space, so that the question arises, How can differentiability with respect to the shape of Ω be defined? Courant and Hilbert [71] proved that the eigenvalues of the Laplace operator are continuous when two open sets Ω and Ω_F of R^n are considered as neighbors, if and only if there exists a C^1 function F such that $\Omega_F = (I + F)(\Omega)$. This point of view has been systemized by Micheletti [72] for regular domains under the so-called Courant topology. In fact, it turns out that this is sufficient to define derivatives relative to F [42] (see also Murat and Simon [73] for a detailed treatment of the subject).

A domain is considered in Section 3.5.2 that satisfies hypothesis H_1 for every $F \in C^1(\Omega)$ or $C^2(\Omega)$, as appropriate, such that $\|F\| \leq C < 1$ and $I + F$ is a homeomorphism of a neighborhood of $\Omega_F = (I + F)(\Omega)$. Let a_F and b_F (respectively, A_F and B_F) be the bilinear forms (respectively, the Friedrichs extension of the differential operators) associated with Ω_F. Then a_F and b_F satisfy H_2. It will be proved that the forms a_F and b_F are Fréchet

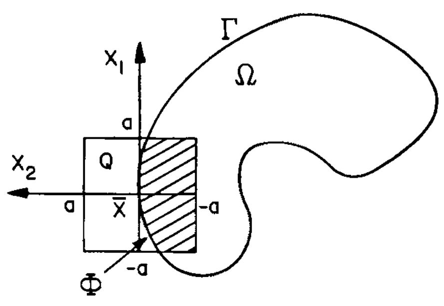

Fig. 3.5.1 Boundary regularity.

differentiable in the sense of relatively bounded perturbations. The differentiability of static response and eigenvalues will then be derived through the use of results proved in Sections 2.4 and 2.5. Once Fréchet differentiability is shown, it is easier to use Gateaux derivatives (material derivatives) of Section 3.2 to find derivative formulas.

3.5.2 Reduction of Variational Equations to Fixed Domains

In order to bring the perturbation methods and results of Sections 2.4 and 2.5 to bear on the problem of domain variation, it is most convenient to transform variational equations for the problem on the perturbed domain Ω_F to variational equations on the domain Ω. The linear and bilinear forms that result will depend on the transformation function F that defines the perturbed domain Ω_F. It will then be possible in the following section to apply results of Sections 2.4 and 2.5 to demonstrate existence of Fréchet derivatives of the forms with respect to domain and to calculate the derivative.

The modified domain Ω_F is prescribed by the transformation $\Omega \to \Omega_F$, given by $x \to x + F(x)$; or $x \to \phi(x)$, where $\phi(x) = x + F(x)$. A function f defined on Ω_F can thus be written as a function on Ω, $\tilde{f}(x) = f(x + F(x))$; or, $\tilde{f} = f \circ \phi$ and $f = \tilde{f} \circ \phi^{-1}$, since ϕ is a homeomorphism from Ω to Ω_F. Since static response and eigenvalues of a system defined on the perturbed domain Ω_F are solutions of variational equations, it is first necessary to transform the variational equation to the fixed domain Ω. In general, linear and bilinear forms $l_F(\bar{z})$, $a_F(z, \bar{z})$, and $d_F(y, \bar{y})$ that are defined on Z_F must be transformed. These transformations will be carried out here for the membrane problem only (for other problems, the reader is referred to Rousselet and Haug [70]).

Static and eigenvalue behavior of a membrane Ω_F are governed by the variational equations (3.1.14) and (3.1.15), defined on Ω_F. The following lemma provides a redefinition of these linear and bilinear forms and the variational equations on Ω.

LEMMA 3.5.1 Consider the bilinear and linear forms with domains $Z = H_0^1(\Omega)$, given by

$$\tilde{a}_F(\tilde{z}, \bar{\tilde{z}}) = \hat{T} \iint_\Omega (D\phi^{-\mathrm{T}}\nabla\tilde{z}, D\phi^{-\mathrm{T}}\nabla\bar{\tilde{z}})|D\phi|\, d\Omega \tag{3.5.1}$$

$$\tilde{d}_F(\tilde{y}, \bar{\tilde{y}}) = \iint_\Omega \tilde{h}\tilde{y}\bar{\tilde{y}}|D\phi|\, d\Omega \tag{3.5.2}$$

$$\tilde{l}_F(\bar{\tilde{z}}) = \iint_\Omega \tilde{f}\bar{\tilde{z}}|D\phi|\, d\Omega \tag{3.5.3}$$

where $\phi(x) = x + F(x)$ and its inverse are assumed to be $C^1(\bar{\Omega})$, $D\phi$ is the Jacobian matrix of ϕ, $|D\phi| = |\det(D\phi)|$, and $D\phi^{-T} = (D\phi^{T})^{-1} = (D\phi^{-1})^{T}$ Then the solution $\tilde{z} \in Z$ of

$$\tilde{a}_F(\tilde{z}, \bar{z}) = \tilde{l}_F(\bar{z}) \qquad \text{for all} \quad \bar{z} \in Z, \tag{3.5.4}$$

is such that $z = \tilde{z} \circ \phi^{-1}$ is the static solution of the membrane equation on Ω_F. Furthermore, the real numbers $\zeta = \zeta(\Omega_F)$ such that there exists $\tilde{y} \neq 0$ satisfying

$$\tilde{a}_F(\tilde{y}, \bar{y}) = \zeta \tilde{d}_F(\tilde{y}, \bar{y}) \qquad \text{for all} \quad \bar{y} \in Z \tag{3.5.5}$$

are the eigenvalues of the membrane over the domain Ω_F, and $y = \tilde{y} \circ \phi^{-1}$ are the associated eigenfunctions.

PROOF Solutions of the static and eigenvalue problems on Ω_F are given by the variational equations

$$a_F(z, \bar{z}) = l_F(\bar{z}), \qquad z \in Z_F \qquad \text{for all} \quad \bar{z} \in Z_F \tag{3.5.6}$$

and

$$a_F(y, \bar{y}) = \zeta d_F(y, \bar{y}), \qquad y \in Z_F \qquad \text{for all} \quad \bar{y} \in Z_F \tag{3.5.7}$$

Transforming from Ω_F to Ω and noting that $D_{\tilde{x}} z = D_x \tilde{z}(D_{\tilde{x}}\phi^{-1})$ and $\nabla_{\tilde{x}} z = (D_{\tilde{x}}\phi)^{-T}\nabla_x \tilde{z}$ where $\tilde{x} = \phi(x)$,

$$\begin{aligned} a_F(z, \bar{z}) &= \hat{T} \iint_{\Omega_F} (\nabla z, \nabla \bar{z})\, d\Omega_F \\ &= \hat{T} \iint_{\Omega} (D\phi^{-T}\nabla \tilde{z}, D\phi^{-T}\nabla \tilde{\bar{z}})|D\phi|\, d\Omega = \tilde{a}_F(\tilde{z}, \tilde{\bar{z}}) \end{aligned} \tag{3.5.8}$$

$$l_F(\bar{z}) = \iint_{\Omega_F} f\bar{z}\, d\Omega_F = \iint_{\Omega} \tilde{f}\tilde{\bar{z}}|D\phi|\, d\Omega = \tilde{l}_F(\tilde{\bar{z}}) \tag{3.5.9}$$

$$d_F(y, \bar{y}) = \iint_{\Omega_F} hy\bar{y}\, d\Omega_F = \iint_{\Omega} \tilde{h}\tilde{y}\tilde{\bar{y}}|D\phi|\, d\Omega = \tilde{d}_F(\tilde{y}, \tilde{\bar{y}}) \tag{3.5.10}$$

Since ϕ is a $C^1(\bar{\Omega})$ homeomorphism, there is a one-to-one correspondence between functions in $H^1(\Omega_F)$ and $H^1(\Omega)$, as in Section 3.2. Thus, by Eqs. (3.5.6), (3.5.8), and (3.5.9),

$$a_F(z, \bar{z}) = \tilde{a}_F(\tilde{z}, \tilde{\bar{z}}) = \tilde{l}_F(\tilde{\bar{z}}) = l_F(\bar{z}) \qquad \text{for all} \quad \bar{z} \in Z_F$$

is the same as for all $\tilde{\bar{z}} \in Z$. Since $\tilde{\bar{z}} \in Z$ is arbitrary, it may be denoted as $\bar{z}$ and the first result of the lemma follows. The second result follows in the same way, using Eqs. (3.5.7), (3.5.8), and (3.5.10). Thus, the lemma is proved. ■

It should be noted that the change of variable holds not only if the homeomorphism ϕ and its inverse are $C^1(\bar{\Omega})$; it is also valid providing ϕ and its inverse are Lipschitzian (see lemma 3.2, chapter 2, of Necas [74]).

3.5.3 Differentiability of Bilinear Forms

In this section, differentiability (in the sense of bounded perturbations) of the bilinear forms of Section 3.1 with respect to the shape variation function F is proved. The development uses expressions for the bilinear forms on fixed domains derived in Section 3.5.2, which explicitly involve the function F. The development also uses expressions for the bilinear forms $\tilde{a}_F(\cdot, \cdot)$ on fixed domains Ω, considered as mappings from $Z \times Z$ and depending on F. The bilinear form $\tilde{a}_F(\cdot, \cdot)$ and its derivatives depend on F, much as the bilinear forms $a_u(\cdot, \cdot)$ of Chapter 2 depend on the design variable u. The objective here is to analytically characterize dependence of $\tilde{a}_F(\cdot, \cdot)$ on F, as the dependence of $a_u(\cdot, \cdot)$ on u was characterized in Chapter 2.

In this section, z and $\bar{z}$ denote arbitrary functions in Z. There is no connection between this z and the solution of the static problems of Section 3.1.

For the problem considered, a formal calculation is used to obtain the desired derivative, much as was done for the static problem in Section 2.4 and for the eigenvalue problem in Section 2.5. Rigorous proofs of the validity of these formulas are then given.

From Eq. (3.5.1), the bilinear form $\tilde{a}_F(z, \bar{z})$ of the membrane problem is obtained explicitly in terms of ϕ, where $\tilde{x} = \phi(x) = x + F(x)$. The derivative of $\tilde{a}_F(z, \bar{z})$ may be formally calculated, for some fixed z and $\bar{z} \in Z$, with respect to F by expanding it, and the validity of the formula may then be proved. The formal calculation is illustrated here in detail, to serve as a guide to a procedure that can be used in other problems.

Since $\phi(x) = x + F(x)$, $D\phi = I + DF$. The formula $(1 + \varepsilon)^{-1} = 1 - \varepsilon + o(\varepsilon)$ for small ε is also valid in the algebra of matrices [see Eq. (3.5.53)], so

$$(D\phi)^{-1} = (I + DF)^{-1} = I - DF + o(|DF|_2) \qquad (3.5.11)$$

where $o(|DF|_2)$ means $o(|DF|_2)/|DF|_2 \to 0$ as $|DF|_2 \to 0$ and

$$|DF|_2 = \left[\sum_{i,j=1}^{2} (F_j^i)^2 \right]^{1/2} \qquad (3.5.12)$$

is the Euclidean norm of matrix DF. On R^2,

$$\det(I + DF) = \begin{vmatrix} 1 + F_1^1 & F_2^1 \\ F_1^2 & 1 + F_2^2 \end{vmatrix} = 1 + \operatorname{div} F + F_1^1 F_2^2 - F_2^1 F_1^2$$

so

$$\det(I + DF) = 1 + \operatorname{div} F + o(|DF|_2) \tag{3.5.13}$$

and for DF small enough,

$$|\det(I + DF)| = 1 + \operatorname{div} F + o(|DF|_2) \tag{3.5.14}$$

Applying these computations to Eq. (3.5.1) for $\tilde{a}_F(z, \bar{z})$ and expanding yields

$$\begin{aligned} \tilde{a}_F(z, \bar{z}) &= \hat{T} \iint_\Omega ([I - DF + o(|DF|_2)]^{\mathrm{T}} \nabla z, [I - DF + o(|DF|_2)]^{\mathrm{T}} \nabla \bar{z}) \\ &\qquad \times (1 + \operatorname{div} F + o(|DF|_2))\, d\Omega \\ &= \hat{T} \iint_\Omega (\nabla z, \nabla \bar{z})\, d\Omega + \hat{T} \iint_\Omega (\nabla z, \nabla \bar{z}) \operatorname{div} F\, d\Omega \\ &\quad - \hat{T} \iint_\Omega ([DF^{\mathrm{T}} + DF] \nabla z, \nabla \bar{z})\, d\Omega + \iint_\Omega o(|DF|_2)\, d\Omega \end{aligned} \tag{3.5.15}$$

This formula can be written to first order as

$$\tilde{a}_F(z, \bar{z}) = a_0(z, \bar{z}) + a'_{\Omega,F}(z, \bar{z}) + \iint_\Omega o(|DF|_2)\, d\Omega \tag{3.5.16}$$

where $a_0(\cdot, \cdot)$ is $a_F(\cdot, \cdot)$ at $F = 0$ and

$$a'_{\Omega,F}(z, \bar{z}) = \hat{T} \iint_\Omega (\nabla z, \nabla \bar{z}) \operatorname{div} F\, d\Omega - \hat{T} \iint_\Omega ([DF^{\mathrm{T}} + DF] \nabla z, \nabla \bar{z})\, d\Omega \tag{3.5.17}$$

Note that Eq. (3.5.17) can also be obtained by taking the material derivative of the bilinear form in Eq. (3.1.14) and using Eqs. (3.2.37), (3.3.2), and (3.3.9),

$$\begin{aligned} [a(z, \bar{z})]'_{\Omega,F} &= a'_{\Omega,F}(z, \bar{z}) + a_0(\dot{z}, \bar{z}) \\ &= \hat{T} \iint_\Omega [((\nabla z)^{\cdot}, \nabla \bar{z}) + (\nabla z, (\nabla \bar{z})^{\cdot})]\, d\Omega + \hat{T} \iint_\Omega (\nabla z, \nabla \bar{z}) \operatorname{div} F\, d\Omega \\ &= \hat{T} \iint_\Omega [(\nabla \dot{z} - DF \nabla z, \nabla \bar{z}) + (\nabla z, \nabla \dot{\bar{z}} - DF \nabla \bar{z})]\, d\Omega \\ &\quad + \hat{T} \iint_\Omega (\nabla z, \nabla \bar{z}) \operatorname{div} F\, d\Omega \end{aligned} \tag{3.5.18}$$

which gives the same result for $a'_{\Omega,F}(z, \bar{z})$ as in Eq. (3.5.17) because $\dot{\bar{z}} = 0$ by Eq. (3.3.5). As noted in Section 3.5.1, it is easier to use the material derivative formula than the expansion method, as in Eq. (3.5.15). Thus, $a'_{\Omega,F}(z, \bar{z})$ is the derivative of $\tilde{a}_F(z, \bar{z})$ with respect to F, if the above calculations are accurate.

These computations are formal, so it must be shown that the foregoing pointwise developments are valid in the appropriate function space norms. This is a technical task that is outlined in the following, with detailed proofs given in Section 3.5.6.

Let $\|F\|$ be the norm in $C^1(\bar{\Omega})$,

$$\|F\| = \sup_{x \in \bar{\Omega}} (|F(x)|_2 + |DF(x)|_2) \tag{3.5.19}$$

where $|F(x)|_2$ denotes the Euclidean norm of $F(x)$ and $|DF(x)|_2$ is the matrix norm given in Eq. (3.5.12).

LEMMA 3.5.2 For the membrane, let

$$a'_{\Omega,F}(z, \bar{z}) = \hat{T} \iint_\Omega (\nabla z, \nabla \bar{z}) \operatorname{div} F \, d\Omega - \hat{T} \iint_\Omega ([DF^{\mathrm{T}} + DF]\nabla z, \nabla \bar{z}) \, d\Omega \tag{3.5.20}$$

Then, $a'_{\Omega,F}$ is linear in F, and for $\|F\|$ small enough, the form

$$a^r_{\Omega,F}(z, z) = \tilde{a}_F(z, z) - a_0(z, z) - a'_{\Omega,F}(z, z)$$

satisfies the inequality

$$|a^r_{\Omega,F}(z, z)| \le c_2(F) \|z\|_Z^2 \tag{3.5.21}$$

where $c_2(F) \ge 0$ and $c_2(F) = o(\|F\|)$. Thus, $a'_{\Omega,F}(z, z)$ is the Fréchet derivative of $\tilde{a}_F$ with respect to F at $F = 0$. Moreover,

$$|a'_{\Omega,F}(z, z)| \le c_1 \|z\|_Z^2 \|F\| \tag{3.5.22}$$

where $c_1 \ge 0$.

PROOF By definition and after some manipulation,

$$\begin{aligned} a^r_{\Omega,F}(z, z) &\equiv \tilde{a}_F(z, z) - a_0(z, z) - a'_{\Omega,F}(z, z) \\ &= \hat{T} \iint_\Omega ((D\phi^{-\mathrm{T}}\nabla z, D\phi^{-\mathrm{T}} \nabla z) - (\nabla z, \nabla z) + 2(DF^{\mathrm{T}} \nabla z, \nabla z)) \, d\Omega \\ &\quad + \hat{T} \iint_\Omega (\nabla z, \nabla z)[|D\phi| - 1 - \operatorname{div} F] \, d\Omega \\ &\quad + \hat{T} \iint_\Omega ((D\phi^{-\mathrm{T}}\nabla z, D\phi^{-\mathrm{T}}\nabla z) - (\nabla z, \nabla z))[|D\phi| - 1] \, d\Omega \end{aligned}$$

which may also be written as

$$
\begin{aligned}
a'_{\Omega,F}(z,z) &= \hat{T}\iint_{\Omega} ((D\phi^{-\mathrm{T}} - I + DF^{\mathrm{T}})\,\nabla z, D\phi^{-\mathrm{T}}\,\nabla z)\,d\Omega \\
&\quad + \hat{T}\iint_{\Omega} (\nabla z, (D\phi^{-\mathrm{T}} - I + DF^{\mathrm{T}})\,\nabla z)\,d\Omega \\
&\quad + \hat{T}\iint_{\Omega} (DF^{\mathrm{T}}\,\nabla z, (I - D\phi^{-\mathrm{T}})\,\nabla z)\,d\Omega \\
&\quad + \hat{T}\iint_{\Omega} (\nabla z, \nabla z)[|D\phi| - 1 - \operatorname{div} F]\,d\Omega \\
&\quad + \hat{T}\iint_{\Omega} ((D\phi^{-\mathrm{T}} - I)\,\nabla z, D\phi^{-\mathrm{T}}\nabla z)[|D\phi| - 1]\,d\Omega \\
&\quad + \hat{T}\iint_{\Omega} (\nabla z, (D\phi^{-\mathrm{T}} - I)\,\nabla z)[|D\phi| - 1]\,d\Omega
\end{aligned}
$$

Taking absolute values and bounds yields

$$
\begin{aligned}
|a'_{\Omega,F}(z,z)| &\le \hat{T}\sup_{x\in\bar{\Omega}}|D\phi^{-\mathrm{T}}(\phi(x)) - I + DF^{\mathrm{T}}(x)|_2\,|D\phi^{-\mathrm{T}}(\phi(x))|_2 \iint_{\Omega} |\nabla z|_2^2\,d\Omega \\
&\quad + \hat{T}\sup_{x\in\bar{\Omega}}|D\phi^{-1\mathrm{T}}(\phi(x)) - I + DF^{\mathrm{T}}(x)|_2 \iint_{\Omega} |\nabla z|_2^2\,d\Omega \\
&\quad + \hat{T}\sup_{x\in\bar{\Omega}}|DF^{\mathrm{T}}(x)|_2\,|I - D\phi^{-\mathrm{T}}(\phi(x))|_2 \iint_{\Omega} |\nabla z|_2^2\,d\Omega \\
&\quad + \hat{T}\sup_{x\in\bar{\Omega}}\big||D\phi(x)| - 1 - \operatorname{div} F(x)\big| \iint_{\Omega} |\nabla z|_2^2\,d\Omega \\
&\quad + \hat{T}\sup_{x\in\bar{\Omega}}|D\phi^{-\mathrm{T}}(\phi(x)) - I|_2\,|D\phi^{-\mathrm{T}}(\phi(x))|_2\,\big||D\phi(x)| - 1\big| \\
&\quad \times \iint_{\Omega} |\nabla z|_2^2\,d\Omega \\
&\quad + \hat{T}\sup_{x\in\bar{\Omega}}|D\phi^{-\mathrm{T}}(\phi(x)) - I|_2\,\big||D\phi(x)| - 1\big| \iint_{\Omega} |\nabla z|_2^2\,d\Omega
\end{aligned}
\tag{3.5.23}
$$

Equation (3.5.23) yields Eq. (3.5.21) if it is shown that every term involving a supremum over $x \in \bar{\Omega}$ is of order $o(\|F\|)$. These results are shown in Section 3.5.6, hence completing the proof. ■

For other problems of Section 3.1, differentiability of the bilinear forms with respect to the shape variation function ($F \in C^1(\bar{\Omega})$ for second-order problems, and $F \in C^2(\bar{\Omega})$ for fourth-order problems) was proved by Rousselet and Haug [70]. With the result of Lemma 3.5.2 and similar results of Rousselet and Haug [70], the following result can be stated:

THEOREM 3.5.1 All the forms of the examples studied in Section 3.1 are Fréchet differentiable in the sense of relatively bounded perturbations. That is,

$$|a'_{\Omega,F}(z, z)| \leq c_1 \|F\| a_0(z, z) \tag{3.5.24}$$

$$|a^r_{\Omega,F}(z, z)| \leq c_2(F) a_0(z, z) \tag{3.5.25}$$

where $c_1 \geq 0$, $c_2(F) \geq 0$, and $c_2(F) = o(\|F\|)$, with $\|F\|$ denoting norm either in $C^1(\bar{\Omega})$ or in $C^2(\bar{\Omega})$.

For proof, it is necessary only to note that these inequalities follow from the inequalities obtained for each example and then to use the strong ellipticity of the form $a_0(z, z)$. ■

For the eigenvalue problem, applying the same expansion as in Eq. (3.5.15) to the bilinear form of the membrane, the formal derivative of $\tilde{d}_F(y, \bar{y})$ may be calculated. The derivative may be found by taking the material derivative of the bilinear form. The next lemma proves differentiability of the bilinear form $\tilde{d}_F(\cdot, \cdot)$,

LEMMA 3.5.3 For the membrane, let

$$d'_{\Omega,F}(y, \bar{y}) = \iint_\Omega (\nabla h(x), F(x)) y\bar{y}\, d\Omega + \iint_\Omega h y \bar{y} \operatorname{div} F\, d\Omega \tag{3.5.26}$$

Then, $d'_{\Omega,F}(y, \bar{y})$ is linear in F, and for $\|F\|$ small enough, the form

$$d^r_{\Omega,F}(y, y) = d_F(y, y) - d_0(y, y) - d'_{\Omega,F}(y, y)$$

satisfies the inequality

$$|d^r_{\Omega,F}(y, y)| \leq c_4(F) \|y\|^2_{L^2(\Omega)} \tag{3.5.27}$$

where $c_4(F) = o(\|F\|)$ and $c_4(F) \geq 0$. Moreover, there is a $c_3 \geq 0$ such that

$$|d'_{\Omega,F}(y, y)| \leq c_3 \|y\|^2_{L^2(\Omega)} \|F\| \tag{3.5.28}$$

PROOF Details of the proof are omitted, since it is much simpler than the proof of Lemma 3.5.2, since only bounded operators are involved. Note, however, that the regularity assumptions on h are used in this lemma, as was pointed out at the beginning of Section 3.5.1.

The bound on $d^r_{\Omega,F}$ is straightforward, since

$$d^r_{\Omega,F}(y, y) = \iint_\Omega \tilde{h}(x)y^2|D\phi|\, d\Omega - \iint_\Omega h(x)y^2\, d\Omega - \iint_\Omega (\nabla h(x), F(x))y^2\, d\Omega - \iint_\Omega hy^2 \operatorname{div} F\, d\Omega$$

may be written

$$\begin{aligned} d^r_{\Omega,F}(y, y) &= \iint_\Omega (\tilde{h}(x) - h(x) - (\nabla h(x), F(x)))y^2|D\phi|\, d\Omega \\ &+ \iint_\Omega h(x)y^2(|D\phi| - 1 - \operatorname{div} F)\, d\Omega \\ &+ \iint_\Omega (\nabla h(x), F(x))y^2(|D\phi| - 1)\, d\Omega \end{aligned}$$

The second term on the right is of the same kind as one of the terms encountered in the proof of Lemma 3.5.2. With this observation, the detailed proof is written in Section 3.5.6 (Eqs. 3.5.59 and 3.5.60). ■

Note that the bilinear forms $\tilde{d}_F(\cdot, \cdot)$ for the beam and plate are similar to the bilinear form for the membrane, so Lemma 3.5.3 may be utilized for proofs of their differentiability.

3.5.4 Differentiability of Static Response

In this section, the shape derivatives of bilinear forms found in Section 3.5.3 are used to extend the operator derivative formulas found in Section 2.4 to derivatives with respect to shape.

As in Chapter 2, define the operator $\tilde{A}_F$ by

$$\tilde{a}_F(z, \bar{z}) = (\tilde{A}_F z, \bar{z})_{L^2(\Omega)} \tag{3.5.29}$$

for all $\bar{z} \in Z$ and for all $z \in D(\tilde{A}_F)$, where the domain $D(\tilde{A}_F)$ of the operator $\tilde{A}_F$ is the subspace of $L^2(\Omega)$ such that $\bar{z} \to \tilde{a}_F(z, \bar{z})$ is continuous for $\bar{z} \in Z$, with $\bar{z}$ considered as an element of $L^2(\Omega)$ [75]. Then, differentiability of $\tilde{A}_F^{-1}$ with respect to F, as a continuous operator on $L^2(\Omega)$, follows from Theorem 2.4.2 of Section 2.4.2.

For the membrane problem, as noted from Eq. (3.5.3), the right side is of the form $\tilde{f}|D\phi|$. In fact for the problems treated in Section 3.1, the right side is of the form $\tilde{f}|D\phi|$. Thus, the static response is $\tilde{z}_F = \tilde{A}_F^{-1}\tilde{f}|D\phi|$, and differentiability of static response follows from differentiability of $\tilde{f}|D\phi|$, as an element of $L^2(\Omega)$, with respect to $F \in C^1(\bar{\Omega})$ for second-order problems and $F \in C^2(\bar{\Omega})$ for fourth-order problems. Showing Fréchet differentiability of $F \to \tilde{f}|D\phi|$ amounts to proving that

$$\left[\iint_\Omega (\tilde{f}|D\phi| - f - (\nabla f, F) - f \operatorname{div} F)^2 \, d\Omega\right]^{1/2} = o(\|F\|) \quad (3.5.30)$$

with $\|F\|$ denoting norm either in $C^1(\bar{\Omega})$ or in $C^2(\bar{\Omega})$. Equation (3.5.30) is implied by

$$\sup_{x \in \bar{\Omega}} |\tilde{f}|D\phi| - f - (\nabla f, F) - f \operatorname{div} F| = o(\|F\|)$$

which in turn is implied by

$$\sup_{x \in \bar{\Omega}} |(\tilde{f} - f - (\nabla f, F))|D\phi|| + \sup_{x \in \bar{\Omega}} |f(|D\phi| - 1 - \operatorname{div} F)| + \sup_{x \in \bar{\Omega}} |(\nabla f, F)(|D\phi| - 1)| = o(\|F\|) \quad (3.5.31)$$

The first term in Eq. (3.5.31) has the proper estimate by using Eq. (3.5.59) of Section 3.5.6 and the second term by using Eq. (3.5.58) of Section 3.5.6. Equation (3.5.58) also implies that

$$\sup_{x \in \bar{\Omega}} ||D\phi| - 1| = C\|F\| \quad (3.5.32)$$

which yields the required estimate of the third term of Eq. (3.5.31).

The derivative of static response can now be written as in Theorem 2.4.3 for the distributed design case. Denote by $C_1(\Omega, F)$ and $C_2(\Omega, F)$ $L^2(\Omega)$-bounded operators such that

$$a'_{\Omega,F}(z, \bar{z}) = (C_1(\Omega, F)G_\Omega z, G_\Omega \bar{z}) \quad (3.5.33)$$

and

$$a^r_{\Omega,F}(z, \bar{z}) = (C_2(\Omega, F)G_\Omega z, G_\Omega \bar{z}) \quad (3.5.34)$$

As in the proof of Theorem 2.4.2, the Fréchet derivative of $\tilde{A}_F^{-1}$ is

$$F \to -G_\Omega^{-1} C_1(\Omega, F) G_\Omega^{-1} \quad (3.5.35)$$

The derivative of static response is then given by

$$\dot{z}_{\Omega,F} = -G_\Omega^{-1} C_1(\Omega, F) G_\Omega^{-1} f + A_\Omega^{-1}[(\nabla f, F) + f \operatorname{div} F] \quad (3.5.36)$$

As in the distributed design case, the importance of this result is theoretical at this point, since the explicit forms of G_Ω, G_Ω^{-1}, and $C^1(\Omega, F)$ are not known and in fact may not be readily computable. Computation of explicit design derivatives of functionals involved in a variety of structural problems is carried out using the adjoint variable method in Section 3.3.2.

An alternative derivation of the design sensitivity equations of Section 3.3.2 may also be given. Consider a typical integral functional

$$\psi = \iint_\Omega g(z)\, d\Omega \tag{3.5.37}$$

The total differential of ψ is

$$\delta_F \psi = \iint_\Omega g_z \dot{z}_{\Omega,F}\, d\Omega + \iint_\Omega g \operatorname{div} F\, d\Omega \tag{3.5.38}$$

Using the $L^2(\Omega)$ scalar product and Eq. (3.5.36), Eq. (3.5.38) becomes

$$\begin{aligned}\delta_F \psi = &-\left(\frac{\partial g}{\partial z}, G_\Omega^{-1} C_1(\Omega, F) G_\Omega^{-1} f\right)_{L^2} \\ &+ \left(\frac{\partial g}{\partial z}, A_\Omega^{-1}[(\nabla f, F) + f \operatorname{div} F]\right)_{L^2} + (g, \operatorname{div} F)_{L^2}\end{aligned}$$

Using self-adjointness of G_Ω and the fact that $A_\Omega^{-1} = G_\Omega^{-1} G_\Omega^{-1}$, direct manipulation yields

$$\begin{aligned}\delta_F \psi &= -\left(G_\Omega A_\Omega^{-1} \frac{\partial g}{\partial z}, C_1(\Omega, F) G_\Omega A_\Omega^{-1} f\right)_{L^2} \\ &\quad + \left(A_\Omega^{-1} \frac{\partial g}{\partial z}, [(\nabla f, F) + f \operatorname{div} F]\right)_{L^2} + (g, \operatorname{div} F)_{L^2} \\ &= -\left(G_\Omega A_\Omega^{-1} \frac{\partial g}{\partial z}, C_1(\Omega, F) G_\Omega z\right)_{L^2} \\ &\quad + \left(A_\Omega^{-1} \frac{\partial g}{\partial z}, [(\nabla f, F) + f \operatorname{div} F]\right)_{L^2} + (g, \operatorname{div} F)_{L^2}\end{aligned} \tag{3.5.39}$$

Defining $A_\Omega^{-1}(\partial g/\partial z) = \lambda$, or equivalently λ, as the solution of the operator equation

$$A_\Omega \lambda = \frac{\partial g}{\partial z} \tag{3.5.40}$$

and using Eq. (3.5.33), Eq. (3.5.39) may be rewritten as

$$\delta_F \psi = -a'_{\Omega,F}(z, \lambda) + (\lambda, [(\nabla f, F) + f \operatorname{div} F])_{L^2} + (g, \operatorname{div} F)_{L^2} \tag{3.5.41}$$

From Eqs. (3.5.3) and (3.5.30),

$$l'_{\Omega,F}(\bar{z}) = \iint_\Omega [(\nabla f, F) + f \operatorname{div} F]\bar{z}\, d\Omega \tag{3.5.42}$$

Integrating the equality

$$[(\nabla f, F) + f \operatorname{div} F]\bar{z} = \operatorname{div}(f\bar{z}F) - f(\nabla\bar{z}, F) \tag{3.5.43}$$

Eq. (3.5.42) becomes, using the divergence theorem [53],

$$l'_{\Omega,F}(\bar{z}) = \int_{\Gamma} f\bar{z}(F, n)\, d\Gamma - \iint_{\Omega} f(\nabla\bar{z}, F)\, d\Omega \tag{3.5.44}$$

which is the same as Eq. (3.3.7) if $F = V$. Using Eq. (3.5.42), Eq. (3.5.41) becomes

$$\delta_F \psi = l'_{\Omega,F}(\lambda) - a'_{\Omega,F}(z, \lambda) + \iint_{\Omega} g \operatorname{div} F\, d\Omega \tag{3.5.45}$$

Integrating the equality

$$\operatorname{div}(gF) = g_z(\nabla z, F) + g \operatorname{div} F \tag{3.5.46}$$

Eq. (3.5.45) becomes, using the divergence theorem [53],

$$\delta_F \psi = l'_{\Omega,F}(\lambda) - a'_{\Omega,F}(z, \lambda) - \iint_{\Omega} g_z(\nabla z, F)\, d\Omega + \int_{\Gamma} g(F, n)\, d\Gamma \tag{3.5.47}$$

Equation (3.5.47) is the same as the result obtained in Eq. (3.3.17) if $F = V$ and g depend on z only.

3.5.5 Differentiability of Eigenvalues

In this section the derivative formulas of Section 2.5 are extended to derivatives of eigenvalues with respect to shape, using the shape derivatives of bilinear forms found in Section 3.5.3.

Define, as in Section 3.5.4, the operator $\tilde{B}_F$ by

$$\tilde{d}_F(y, \bar{y}) = (\tilde{B}_F y, \bar{y})_{L^2(\Omega)} \tag{3.5.48}$$

for all $\bar{y} \in Z$ and all $y \in D(\tilde{A}_F)$. As in Section 2.5, $F \to \tilde{A}_F^{-1}\tilde{B}_F$ is an $L^2(\Omega)$-bounded operator that is Fréchet differentiable with respect to F. Further, the eigenvalues of $\tilde{A}_F^{-1}\tilde{B}_F$ are the inverse of $\zeta(\Omega_F)$ of Eq. (3.5.5). Thus, the continuity and differentiability follow from Section 2.5, as follows:

THEOREM 3.5.2 Let $\zeta(\Omega)$ be an m-fold eigenvalue of Eq. (3.5.5) at $F = 0$. Then, in any neighborhood W of $\zeta(\Omega)$ that contains no other eigenvalue and for $\|F\|$ small enough, there are exactly m eigenvalues $\zeta_1(\Omega_F), \ldots, \zeta_m(\Omega_F)$ (counted with their multiplicity) in W. If $m = 1$, then $\zeta(\Omega_F)$ is Fréchet differentiable at $F = 0$, and

$$\zeta'_{\Omega,F} = a'_{\Omega,F}(y, y) - \zeta(\Omega) d'_{\Omega,F}(y, y) \tag{3.5.49}$$

where y and $\zeta(\Omega)$ are the eigenfunction and associated eigenvalue with $d_0(y, y) = 1$. If $m > 1$, then $\zeta_i(\Omega_F)$ are directionally differentiable at $F = 0$, and the directional derivatives are the m eigenvalues of the matrix $\mathcal{M}$ with general term

$$\mathcal{M}_{ij} = a'_{\Omega,F}(y^i, y^j) - \zeta(\Omega) d'_{\Omega,F}(y^i, y^j), \qquad i, j = 1, 2, \ldots, m \qquad (3.5.50)$$

where $\{y^i\}_{i=1,2,\ldots,m}$ is a basis of the eigenspace associated with the eigenvalue $\zeta(\Omega)$ and $d_0(y^i, y^j) = \delta_{ij}$ $(i, j = 1, 2, \ldots, m)$ where δ_{ij} is one if $i = j$ and otherwise is zero.

3.5.6 Proof of Lemma 3.5.2

It is shown in this section that the pointwise developments carried out in Lemma 3.5.2 are in fact uniform.

The expression

$$D\phi^{-\mathrm{T}}(\phi(x)) - I + DF^{\mathrm{T}}(x) \qquad (3.5.51)$$

is first considered. Recall that

$$D\phi^{-1}(\phi(x)) = (D\phi(x))^{-1} = (I + DF)^{-1} \qquad (3.5.52)$$

It is a standard result of the algebra of linear operators or matrices (section I.4.4 of Kato [13]) that if $w \in X$, where X is a normed algebra of linear operators, then

$$\|(I + w)^{-1} - I + w\|_X \le 2\|w\|_X^2, \qquad \|w\|_X < \tfrac{1}{2} \qquad (3.5.53)$$

Setting $w = DF^{\mathrm{T}}(x)$, then Eq. (3.5.53) yields a bound for Eq. (3.5.51) using Eq. (3.5.52),

$$|D\phi^{-\mathrm{T}}(\phi(x)) - I + DF^{\mathrm{T}}(x)|_2 \le 2|DF(x)|_2^2 \qquad (3.5.54)$$

where $|\cdot|_2$ denotes the matrix norm associated with the usual Euclidean norm for vectors. Taking supremum over $x \in \bar{\Omega}$ of both sides yields the required bound for this term.

The same kind of bound is now to be shown for

$$|D\phi(x)| - 1 - \operatorname{div} F(x)$$

The following result is first established: If w is a linear mapping from R^n into R^n, then

$$|\det(I + w) - 1 - \operatorname{tr} w| \le \sum_{k=2}^{n} \binom{n}{k} |w|_2^k \qquad (3.5.55)$$

where $\operatorname{tr} w$ is the trace of w and $\binom{n}{k}$ is the binomial coefficient. After choice of a basis, denoting by w the matrix formed from the column vectors w_i in this basis, and $e_i = (0 \cdots 1 \cdots 0)^T$, the 1 being in the ith position,

$$\det(I + w) = \det(e_1 + w_1 \quad e_2 + w_2 \quad \cdots \quad e_i + w_i \quad \cdots \quad e_n + w_n)$$

Using the multilinear property of the determinant, this may be expanded as

$$\begin{aligned}\det(I + w) = {} & \det(e_1 \quad \cdots \quad e_n) + \det(w_1 \quad e_2 \quad \cdots \quad e_n) + \cdots \\ & + \det(e_1 \quad \cdots \quad w_n) + \det(w_1 \quad w_2 \quad e_3 \quad \cdots \quad e_n) + \cdots \\ & + \det(w_1 \quad \cdots \quad w_n)\end{aligned} \tag{3.5.56}$$

To get the bound of Eq. (3.5.55), note that there are $\binom{n}{k}$ terms in Eq. (3.5.56), involving k factors w_i (and $n - k$ factors e_j). Since $|w_i|_2 \le |w|_2$, this yields Eq. (3.5.55).

Equation (3.5.55) yields also

$$|\det(I + w) - 1 - \operatorname{tr} w| \le |w|_2^2 \sum_{k=2}^{n} \binom{n}{k} |w|_2^k$$

and for $|w|_2 \le 1$,

$$|\det(I + w) - 1 - \operatorname{tr} w| \le |w|_2^2 \sum_{k=2}^{n} \binom{n}{k} = |w|_2^2(2^n - n - 1) \tag{3.5.57}$$

Now set $w = DF(x)$ and note that $\operatorname{tr} DF(x) = \operatorname{div} F(x)$, so Eq. (3.5.57) yields

$$\big||\det(I + DF(x))| - 1 - \operatorname{div} F(x)\big| \le C|DF(x)|_2^2 \tag{3.5.58}$$

For DF small enough, $|\det(I + DF(x))| = \det(I + DF(x))$. Taking the supremum of both sides of Eqs. (3.5.51), (3.5.54), and (3.5.58), for $x \in \bar{\Omega}$, yields the desired bounds in Eq. (3.5.23). Equations (3.5.54) and (3.5.58) are the two uniform bounds needed to complete the proof of Lemma 3.5.2.

Two results needed for the proof of Lemma 3.5.3 are now to be shown. First, if $h \in C^1(R^n)$,

$$\sup_{x \in \bar{\Omega}} |h(x + F(x)) - h(x) - (\nabla h(x), F(x))| = o(\|F\|) \tag{3.5.59}$$

In fact,

$$h(x + F(x)) - h(x) = \int_0^1 (\nabla h(x + tF(x)), F(x))\, dt$$

so that

$$\begin{aligned}&\sup_{x \in \bar{\Omega}} |h(x + F(x)) - h(x) - (\nabla h(x), F(x))| \\ &\qquad \le \sup_{t \in [0,1]} \sup_{x \in \bar{\Omega}} |\nabla h(x + tF(x)) - \nabla h(x)|\, \|F\|\end{aligned}$$

If $\|F\| \leq 1$, $x + tF(x)$ belongs, for almost all $x \in \Omega$ and every $t \in [0, 1]$, to a compact neighborhood of Ω. On this compact neighborhood, ∇h, which is continuous, is uniformly continuous. Therefore for every $\varepsilon > 0$, there exists $\alpha \in [0, 1]$ such that if $\|F\| \leq \alpha$. Then,

$$\sup_{t \in [0,1]} \sup_{x \in \bar{\Omega}} |\nabla h(x + tf(x)) - \nabla h(x)| \leq \varepsilon$$

which verifies Eq. (3.5.59). The second estimate needed in the proof of Lemma 3.5.3 is more straightforward,

$$\left| \iint (\nabla h(x), F(x)) y^2(|D\phi| - 1)\, d\Omega \right| = o(\|F\|) \tag{3.5.60}$$

Since ∇h is continuous,

$$\sup_{x \in \bar{\Omega}} |(\nabla h(x), F(x))| \leq \sup_{x \in \bar{\Omega}} |\nabla h(x)| \, \|F\|$$

From Eq. (3.5.58), it follows that $\sup_{x \in \bar{\Omega}} \big| |D\phi(x)| - 1 \big| \leq C\|F\|$. Equation (3.5.60) is easily deduced from these two estimates.

3.5.7 Derivatives of Domain Functionals

An important result to be given in this section, without proof, is that when the variation of a domain functional is considered, if the gradient of the domain functional exists, only the normal component $(V^T n)$ of the velocity field V has any influence. (The reader who is interested in a detailed proof is referred to Zolesio [52].)

Let ψ be a functional defined for any regular domain Ω. The material upper semiderivative of ψ at Ω, in the direction of the velocity field V, is the real number (finite or infinite) given by

$$\bar{d}_V \psi = \lim_{\tau \to 0} \sup \frac{\psi(\tau) - \psi(0)}{\tau} \tag{3.5.61}$$

If the limit exists and is finite in Eq. (3.5.61), it defines the material semiderivative of $\psi(\tau)$ at Ω, in the direction of V, and writes it as $d_V \psi$.

Consider now the case in which the mapping $V \to d_V \psi$ is linear and continuous. Then this mapping defines a vector distribution G which is the gradient of ψ at Ω. That is,

$$d_V \psi = \langle G, V \rangle_{\mathscr{D}^k(U,R^n)' \times \mathscr{D}^k(U,R^n)} \tag{3.5.62}$$

where $\mathscr{D}^k(U, R^n)$ is the vector space of k-times continuously differentiable functions with compact support, $\mathscr{D}^k(U, R^n)'$ is the dual space of $\mathscr{D}^k(U, R^n)$, and $\langle \cdot, \cdot \rangle$ denotes the duality pairing.

For any test function V belonging $\mathscr{D}^k(U, R^n)$, V can be decomposed on the boundary Γ as

$$V = V_n + V_s \tag{3.5.63}$$

where $V_n = (V^T n)n$ is the normal component of the velocity field V and $V_s = V - V_n$ is tangent to the boundary. Hence, Γ is the integral surface [54] (or curve) for that field V_s, and $\Gamma_\tau(V_s) = \Gamma$ for all τ. Thus, $\Omega_\tau(V_s)$ has the same boundary for all τ and $\langle G, V_s \rangle = 0$ and

$$d_V \psi = \langle G, V \rangle = \langle G, (V^T n)n \rangle \tag{3.5.64}$$

for any V. Using these results, the following theorem was proved by Zolesio [52].

THEOREM 3.5.3 Suppose ψ has a gradient at any domain Ω of regularity C^k and let Ω be a C^{k+1}-regular domain. Then there exists a scalar distribution g_n on the manifold Γ, of order less than k, such that

$$d_V \psi = \int_\Gamma g_n(V^T n)\, d\Gamma \tag{3.5.65}$$

holds for any field belonging to $C^k(U, R^n)$, where the integral on Γ in Eq. (3.6.65) is, by extension, the bilinear duality form on Γ.

4

Design Sensitivity Analysis of Built-Up Structures

The treatment of design sensitivity analysis of distributed-parameter structures in Chapters 2 and 3 is limited to single structural components. This restriction is common in the distributed-parameter structural optimization literature [76] but needs to be relaxed when modern complex structures are considered. Virtually all aircraft, vehicles, machines, and other mechanical structures are in fact made up of combinations of a variety of interacting structural components. Combinations of truss, beam, plate, and solid elastic structural components make up most real engineering structures. Such built-up structures may be considered to be composed of a complex of structural components, each of which has mathematical properties considered in Chapters 2 and 3. The purpose of this chapter is to provide a summary of the current state of the art of design sensitivity analysis and optimization of built-up structures, a field that is still developing.

It is shown in Chapter 2, that in comparison with the differential equation characterization of structural deformation, a variational formulation is more practical for design sensitivity analysis. Furthermore, the variational formulation obtained mathematically in Chapter 2 can be rigorously related to a virtual work or energy principle in mechanics. This result allows direct extension of the energy ideas employed in Chapter 1 for structures described by finite element methods to a distributed-parameter formulation of built-up structures.

The approach taken in this chapter begins with an energy characterization of structural performance, namely Hamilton's principle. Hamilton's principle results in a unified variational formulation of the governing structural equations that is employed for design sensitivity analysis. Strong ellipticity properties of energy bilinear forms have been proved for individual structural components [35], yielding existence and uniqueness results for the associated variational equations and forming the foundation for a rigorous proof of differentiability of structural response with respect to design variables and shape. Proofs of strong ellipticity for built-up structures are not generally available in the literature, however. The approach taken in this chapter is somewhat more formal than the treatments given in Chapters 2 and 3. Here, it is presumed that strong ellipticity hypotheses are satisfied, and direct variational analysis techniques are used for design sensitivity analysis of built-up structures.

4.1 VARIATIONAL EQUATIONS OF BUILT-UP STRUCTURES

To make the foregoing discussion more concrete, Hamilton's principle is first employed to obtain a general variational formulation of structural equations. Prototype problems involving truss, beam, plate, and plane elastic structural components are then formulated to illustrate the use of the variational method and to provide a foundation for subsequent design sensitivity analysis.

4.1.1 Hamilton's Principle Formulation for Built-Up Structures

Consider a general structure that is made up of a collection of structural components. Each component, except trusses, occupues a domain Ω^i with boundary Γ^i ($i = 1, 2, \ldots, r$). These domains are interconnected by kinematic constraints at their boundaries; that is, structural components are interfaced by joints that connect them to adjacent components and constrain admissible displacement fields at the interfaces. Displacement fields in structural components are said to be *kinematically admissible* if they satisfy kinematic constraints at the joints. The definition of kinematic constraints at an interface depends on the nature of the components that are connected by the joint. The axial displacement of the end of a truss component, for example, must be equal to the projection of the displacement of the point of attachment in an adjacent component onto the axis of the truss component. In the case of a beam component, kinematic boundary conditions at each end may involve displacement, slope, and twist. In the case of plate components, kinematic interface conditions may likewise involve both displacement and slope. In the case of an elastic component of general shape, kinematic interface conditions involve displacement on the interface surface.

In a general setting, let z denote a composite vector of displacement fields in the components making up the built-up structure; that is, $z \equiv [w^1 \ w^2 \ \dots \ w^r \ q]^T$, where $w^i \in [H^{m_i}(\Omega^i)]^{l_i}$ represent displacements of beam, plate, and elastic components and $q \in R^k$ represents displacements of trusses. The space of *kinematically admissible displacement fields* is defined as the set of displacement fields that satisfy homogeneous boundary and interface conditions between components and the ground reference frame and kinematic interface conditions between components. Symbolically, this is

$$Z = \{z \in W\colon \quad \gamma z = 0 \text{ on } \Gamma^0 \qquad \text{and} \qquad \gamma^i z = \gamma^j z \text{ on } \Gamma^{ij}\} \tag{4.1.1}$$

where the product space $W = \prod_{i=1}^{r} [H^{m_i}(\Omega^i)]^{l_i} \times R^k$ is the space of displacement fields that satisfy the required degree of smoothness, γ is a boundary operator (the trace operator [Appendix A.1]) that gives the projection of structural displacements and perhaps their derivatives onto the exterior boundary Γ^0, and γ^i and γ^j are interface operators (also trace operators) that project displacement fields and perhaps their derivatives from within components i and j onto their common boundary Γ^{ij}.

In order to state Hamilton's principle [33, 77, 78] for built-up structures, it is first necessary to define energy quantities that are associated with the structure. First, let the *strain energy* of the built-up structure be denoted

$$\begin{aligned} U(z) &\equiv \frac{1}{2} a_{u,\Omega}(z, z) \\ &= \frac{1}{2}\left[\sum_{i=1}^{r} a_{u^i,\Omega^i}(w^i, w^i) + a_b(q, q)\right] \end{aligned} \tag{4.1.2}$$

where $\frac{1}{2}a_{u^i,\Omega^i}$ is the strain energy of component i and $\frac{1}{2}a_b$ is the strain energy of truss components. The design variable is $u = [u^1 \ u^2 \ \dots \ u^r \ b]^T$, where u^i is the design variable of component i, which may consist of design functions and parameters introduced in Chapter 2, and b is the design parameter vector of the trusses.

The dependence of the strain energy quadratic form on design u and shape Ω of the built-up structure is indicated. It is presumed that the quadratic strain energy in Eq. (4.1.2) is defined for all displacements in the space Z of kinematically admissible displacements. The strain energy quadratic form is defined as the sum of strain energies of components that make up the built-up structure, each involving a matrix or integral quadratic form in its displacement field.

Next, define the *kinetic energy* of the built-up structure as

$$T\left(\frac{dz}{dt}\right) \equiv \frac{1}{2} d_{u,\Omega}\left(\frac{dz}{dt}, \frac{dz}{dt}\right)$$
$$= \frac{1}{2}\left[\sum_{i=1}^{r} d_{u^i,\Omega^i}\left(\frac{dw^i}{dt}, \frac{dw^i}{dt}\right) + d_b\left(\frac{dq}{dt}, \frac{dq}{dt}\right)\right] \tag{4.1.3}$$

where $\frac{1}{2}d_{u^i,\Omega^i}$ is the kinetic energy of component i and $\frac{1}{2}d_b$ is the kinetic energy of the trusses. Here, dz/dt denotes time derivative of the displacement field z, and the kinetic energy quadratic form depends on the design variable and shape of the structure. As in the case of strain energy, kinetic energy is obtained by summing energies of each of the structural components, each involving its own matrix quadratic form or integral over the component domain Ω^i. It is presumed that the kinetic energy in Eq. (4.1.3) is well defined for all kinematically admissible displacement fields.

Finally, let the *virtual work* of all externally applied forces be defined as

$$\bar{L}(\bar{z}) \equiv l_{u,\Omega}(\bar{z}) = \sum_{i=1}^{r} l_{u^i,\Omega^i}(\bar{w}^i) + f_b(\bar{q}) \tag{4.1.4}$$

where l_{u^i,Ω^i} is the virtual work of the applied forces that act on component i and f_b is the virtual work of applied forces that act on trusses, with time held constant, in undergoing a small virtual displacement $\bar{z}$ that satisfies the kinematic admissibility conditions (i.e., for $\bar{z} \in Z$). The virtual work of the externally applied forces that act on a built-up structure is obtained by summing the virtual work of external forces applied to each of the structural components. This virtual work functional is linear in the virtual displacement $\bar{z}$.

Since the displacement of a structural system will in general be time dependent, each of the functionals defined in Eqs. (4.1.2)–(4.1.4) is evaluated at a particular time t. In anticipation of employing Hamilton's principle, it is helpful to define the first variation of the strain and kinetic-energy quadratic forms of Eqs. (4.1.2) and (4.1.3). For any kinematically admissible virtual displacement $\bar{z}$, these variations or differentials (Appendix A.3) are defined as

$$\bar{U} \equiv \frac{d}{d\tau} U(z + \tau\bar{z})\bigg|_{\tau=0} = a_{u,\Omega}(z, \bar{z}) \tag{4.1.5}$$

$$\bar{T} \equiv \frac{d}{d\tau} T\left(\frac{dz}{dt} + \tau\frac{d\bar{z}}{dt}\right)\bigg|_{\tau=0} = d_{u,\Omega}\left(\frac{dz}{dt}, \frac{d\bar{z}}{dt}\right) \tag{4.1.6}$$

where the symmetric strain and kinetic-energy bilinear forms on the right side of Eqs. (4.1.5) and (4.1.6) are obtained by calculating the first variation of the strain and kinetic-energy quadratic forms of Eqs. (4.1.2) and (4.1.3).

With this notation, a general form of Hamilton's principle can be stated that is suitable for design sensitivity analysis of built-up structures. Following the classical literature [33, 77, 78], the *variational form of Hamilton's principle* requires that

$$\int_{t_0}^{t_1} (\bar{U} - \bar{T})\, dt = \int_{t_0}^{t_1} \bar{L} dt \tag{4.1.7}$$

for all times t_0 and t_1 and for all kinematically admissible virtual displacements $\bar{z}$ that satisfy the additional conditions

$$\bar{z}(t_0) = \bar{z}(t_1) = 0 \tag{4.1.8}$$

In terms of the virtual work linear form of Eq. (4.1.4) and the strain and kinetic-energy bilinear forms of Eqs. (4.1.5) and (4.1.6), Eq. (4.1.7) may be written as

$$\int_{t_0}^{t_1} \left\{ a_{u,\Omega}(z, \bar{z}) - d_{u,\Omega}\left(\frac{dz}{dt}, \frac{d\bar{z}}{dt}\right) \right\} dt = \int_{t_0}^{t_1} l_{u,\Omega}(\bar{z})\, dt \tag{4.1.9}$$

for all kinematically admissible virtual displacements $\bar{z}$ that satisfy Eq. (4.1.8).

This general formulation of Hamilton's principle provides the variational equations of structural dynamics, which can be used to extend the theory presented in Section 2.6 for transient dynamic design sensitivity analysis of structures. This extension, however, will not be presented here. The foregoing formulation directly specializes to the cases of static response and natural vibration of the built-up structure. Using the theorem of minimum total potential energy, it is similarly possible to extend the variational formulation for buckling of a built-up structure. This topic, however, will not be pursued here.

4.1.2 Principle of Virtual Work for Built-Up Structures

Consider now the case of static response of a structure to loads that do not depend on time. In this case, time is suppressed completely from the problem, and Hamilton's principle of Eq. (4.1.9) reduces to

$$a_{u,\Omega}(z, \bar{z}) = l_{u,\Omega}(\bar{z}) \qquad \text{for all} \quad \bar{z} \in Z \tag{4.1.10}$$

which may be viewed simply as a statement of the *principle of virtual work*. Note that this equation generalizes the variational formulation of boundary-value problems treated in Chapter 2 for individual structural components. Note also that if the load linear form on the right side of Eq. (4.1.10) is continuous on the space Z and if the energy bilinear form on the left side of

Eq. (4.1.10) is strongly elliptic on Z, then by the Lax–Milgram theorem [9], Eq. (4.1.10) has a unique solution $z \in Z$.

4.1.3 Free Vibration of Built-Up Structures

Consider next the special case in which there are no externally applied loads and in which one wishes to consider *harmonic vibration* of the built-up structure. Harmonic motion of the built-up structure is defined as a displacement field that can be written as the product of a time-independent mode function $y \in Z$ and a harmonic function $\sin(\omega t + \alpha)$; that is,

$$z(x, t) = y(x) \sin(wt + \alpha), \qquad y \in Z \tag{4.1.11}$$

Before substituting this harmonic displacement field into Eq. (4.1.9), it is helpful to transform Eq. (4.1.9) using integration by parts. Since the kinetic-energy bilinear form is linear in its individual factors,

$$\frac{d}{dt}\left[d_{u,\Omega}\left(\frac{dz}{dt}, \bar{z}\right)\right] = d_{u,\Omega}\left(\frac{d^2z}{dt^2}, \bar{z}\right) + d_{u,\Omega}\left(\frac{dz}{dt}, \frac{d\bar{z}}{dt}\right) \tag{4.1.12}$$

Integrating both sides of this equation from t_0 to t_1, recalling that $\bar{z}$ must satisfy Eq. (4.1.8),

$$0 = d_{u,\Omega}\left(\frac{dz}{dt}, \bar{z}\right)\Bigg|_{t_0}^{t_1} = \int_{t_0}^{t_1} \left\{d_{u,\Omega}\left(\frac{d^2z}{dt^2}, \bar{z}\right) + d_{u,\Omega}\left(\frac{dz}{dt}, \frac{d\bar{z}}{dt}\right)\right\} dt \tag{4.1.13}$$

Substituting for the second term in the integrand on the right side of Eq. (4.1.13) into Eq. (4.1.9), with the load linear form equal to zero,

$$\int_{t_0}^{t_1} \left\{a_{u,\Omega}(z, \bar{z}) + d_{u,\Omega}\left(\frac{d^2z}{dt^2}, \bar{z}\right)\right\} dt = 0 \qquad \text{for all} \quad \bar{z} \in Z \tag{4.1.14}$$

Substituting z from Eq. (4.1.11) and $\bar{z}$ in the form $\bar{z} = \bar{y}f(t)$, where $\bar{y}$ is an arbitrary time independent displacement field in Z and $f(t)$ is an arbitrary function of time that vanishes at t_0 and t_1,

$$\{a_{u,\Omega}(y, \bar{y}) - \omega^2 d_{u,\Omega}(y, \bar{y})\} \int_{t_0}^{t_1} \sin(\omega t + \alpha) f(t)\, dt = 0 \qquad \text{for all} \quad \bar{y} \in Z \tag{4.1.15}$$

Since the integral in Eq. (4.1.15) is not zero for all functions f vanishing at t_0 and t_1, its coefficient must be zero. Defining $\zeta = \omega^2$, leads to the *variational eigenvalue equation*

$$a_{u,\Omega}(y, \bar{y}) = \zeta d_{u,\Omega}(y, \bar{y}) \qquad \text{for all} \quad \bar{y} \in Z \tag{4.1.16}$$

Note that this is the form of the variational eigenvalue problem considered at length in Chapter 2 for individual structural components.

4.1.4 Prototype Problems

In order to illustrate the variational formulation of Sections 4.1.1–4.1.3, four prototype problems are considered in this section, involving truss, beam, plate, and elastic solid components.

BEAM–TRUSS

As a first example of a built-up structure, consider the elementary beam–truss structure shown in Fig. 4.1.1. Applied loads and dimensions of the structure are taken as given. The design variables are the cross-sectional area $h(x)$ of the beam and the constant cross-sectional areas b_i $(i = 1, 2, 3, 4)$ of the four truss members. The composite design variable is

$$u = [h(x) \quad b_1 \quad b_2 \quad b_3 \quad b_4]^{\mathrm{T}} \in L^{\infty}(0, l_4) \times R^4$$

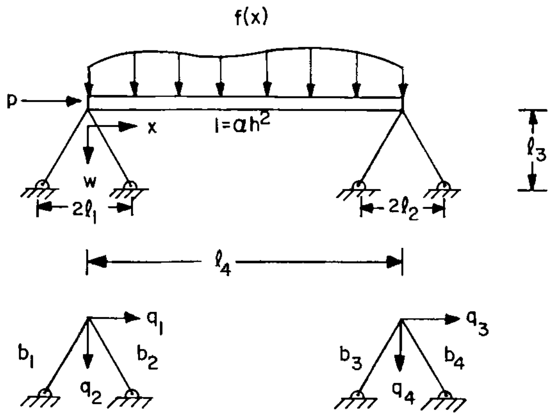

Fig. 4.1.1 Beam–truss built-up structure.

The state variables for this built-up structure consist of the beam displacement function $w(x)$ and the four nodal displacement coordinates q_1 to q_4 of the trusses. In vector form, the state variable is

$$z \equiv [w(x) \quad q_1 \quad q_2 \quad q_3 \quad q_4]^{\mathrm{T}} \in W \equiv H^2(0, l_4) \times R^4 \tag{4.1.17}$$

Kinematically admissible displacements are those for which lateral displacements of the beam at its ends are equal to vertical displacements of the trusses, and since axial deformation of the beam is presumed to be zero, horizontal displacement of the truss nodes must be equal. The set Z of kinematically admissible displacements is thus

$$Z = \{z = [w, q^{\mathrm{T}}]^{\mathrm{T}} \in H^2(0, l_4) \times R^4: \quad w(0) = q_2, \quad w(l_4) = q_4, \quad q_1 = q_3\} \tag{4.1.18}$$

The strain energy of the system may be written as the sum of strain energy of the beam and strain energies of the four truss members. In this case, the total strain energy is

$$\begin{aligned}\frac{1}{2}a_{u,\Omega}(z, z) &= \frac{1}{2}\int_0^{l_4} E\alpha h^2(w_{xx})^2\, dx \\ &+ \frac{1}{2}\frac{E}{(l_1^2 + l_3^2)^{3/2}}[b_1(q_1l_1 - q_2l_3)^2 + b_2(q_1l_1 + q_2l_3)^2] \\ &+ \frac{1}{2}\frac{E}{(l_2^2 + l_3^2)^{3/2}}[b_3(q_3l_2 - q_4l_3)^2 + b_4(q_3l_2 + q_4l_3)^2]\end{aligned} \tag{4.1.19}$$

The virtual work of the externally applied loads is written simply as

$$l_{u,\Omega}(\bar{z}) = \int_0^{l_4} f\bar{w}\, dx + p\bar{q}_1 \tag{4.1.20}$$

The condition of equilibrium of Eq. (4.1.10) requires that the total variation of strain energy with respect to state must equal the virtual work of the applied loads, for all virtual displacements that are consistent with constraints. That is, it is required by Eq. (4.1.10) that

$$\begin{aligned}a_{u,\Omega}(z, \bar{z}) &= \int_0^{l_4} E\alpha h^2 w_{xx}\bar{w}_{xx}\, dx \\ &+ \frac{E}{(l_1^2 + l_3^2)^{3/2}}[b_1(q_1l_1 - q_2l_3)(l_1\bar{q}_1 - l_3\bar{q}_2) \\ &\qquad + b_2(q_1l_1 + q_2l_3)(l_1\bar{q}_1 + l_3\bar{q}_2)] \\ &+ \frac{E}{(l_2^2 + l_3^2)^{3/2}}[b_3(q_3l_2 - q_4l_3)(l_2\bar{q}_3 - l_3\bar{q}_4) \\ &\qquad + b_4(q_3l_2 + q_4l_3)(l_2\bar{q}_3 + l_3\bar{q}_4)] \\ &= \int_0^{l_4} f\bar{w}\, dx + p\bar{q}_1 = l_{u,\Omega}(\bar{z}) \qquad \text{for all} \quad \bar{z} \in Z\end{aligned} \tag{4.1.21}$$

CONNECTING ROD

As a second example, consider the connecting rod shown in Fig. 4.1.2, whose three-dimensional shape is to be determined to minimize weight, subject to stress constraints. In Fig. 4.1.2, T_F denotes firing load during the combustion cycle, and T_I denotes inertia load during the suction cycle of the exhaust stroke. This three-dimensional structure is loaded in a plane, so

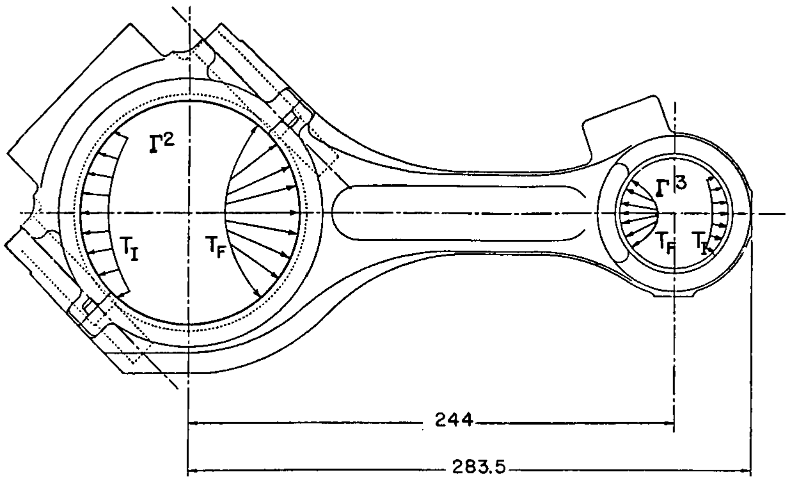

Fig. 4.1.2 Engine connecting rod.

it is reasonable to assume that out-of-plane stresses are zero (i.e., $\sigma_{13} = \sigma_{23} = \sigma_{33} = 0$). Thus, a three-dimensional elasticity problem that describes displacement in the structure may be reduced to a plane problem, under plane stress assumptions. However, the variable thickness $h(x)$ of the connecting rod (the out-of-plane dimension) plays the role of a design variable, while the shape of the domain Ω of the plane cross section of the connecting rod is also a design characteristic. Thus, coupled effects of variations in shape and a conventional design variable are involved.

In the performance of an engine, the connecting rod is acted upon by a large force at the piston pin and a corresponding reaction force at the connecting rod bearing. The actual dynamics of the system is transient, but for purposes of analysis a quasi-static model of structural displacement is employed. In order to preclude rigid-body degrees of freedom of the system, it is necessary to define kinematic boundary conditions that restrict two components of displacement and rotation at a single point within the body, the point at which these constraints are imposed being unimportant. Thus, the space Z of kinematically admissible displacements is

$$Z = \{[z^1(x) \quad z^2(x)]^{\mathrm{T}} \in H^1(\Omega): \quad z^1(\hat{x}) = z^2(\hat{x}) = z^2(\bar{x}) = 0\} \tag{4.1.22}$$

where $\hat{x}$ and $\bar{x}$ are two distinct points in the domain Ω of the connecting rod and the last condition is specified to preclude rotation.

The strain energy of this system is defined as

$$\frac{1}{2}a_{u,\Omega}(z, z) = \frac{1}{2}\iint_{\Omega}\left[\sum_{i,j=1}^{2} \sigma^{ij}(z)\varepsilon^{ij}(z)\right]h(x)\, d\Omega \tag{4.1.23}$$

Note that the strain energy depends on both the thickness design variable $h(x)$ and the shape Ω of the connecting rod.

Using the traction acting on the loaded boundary Γ^2, the virtual work functional is

$$l_{u,\Omega}(\bar{z}) = \int_{\Gamma^2 \cup \Gamma^3} \left[\sum_{i=1}^{2} T^i \bar{z}^i \right] d\Gamma \tag{4.1.24}$$

where $T = [T^1 \quad T^2]^T$ is the given compressive or tensile normal traction force acting on the boundary $\Gamma^2 \cup \Gamma^3$.

Taking the variation of strain energy in Eq. (4.1.23) the variational equilibrium equations of Eq. (4.1.10) can be written as

$$\begin{aligned} a_{u,\Omega}(z, \bar{z}) &= \iint_{\Omega} \left[\sum_{i,j=1}^{2} \sigma^{ij}(z)\varepsilon^{ij}(\bar{z}) \right] h(x) \, d\Omega \\ &= \int_{\Gamma^2 \cup \Gamma^3} \left[\sum_{i=1}^{2} T^i \bar{z}^i \right] d\Gamma \\ &= l_{u,\Omega}(\bar{z}) \qquad \text{for all} \quad \bar{z} \in Z \end{aligned} \tag{4.1.25}$$

This example is presented as a built-up structure because it involves both shape and design variables. As will be seen in the following, the variational formulation of state equations allows direct extension of the methods of Chapter 2 and 3 to this more general problem.

SIMPLE BOX

As a third example, consider the simple box shown in Fig. 4.1.3, in which five plane elastic solids are welded together and attached to a wall. A distributed line load is applied on top of the two side plates and on the end plate. The design variables are the length b_1, width b_2, and height b_3 of the

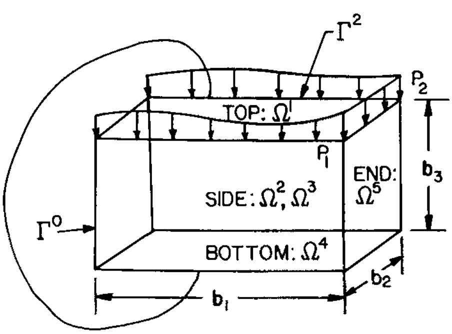

Fig. 4.1.3 Simple box.

box. The five plane elastic solids remain plane and orthogonal to each other for all design perturbations. Note that $b = [b_1 \ b_2 \ b_3]^T$ is a shape design variable since variations of b cause domain variations for each plane elastic solid. Only shape design variables are considered in this example.

Let Ω^i ($i = 1, 2, 3, 4, 5$) denote each plane elastic solid (as shown in Fig. 4.1.3) and Γ^{ij} denote interface boundaries. The state variables for this built-up structure consist of in-plane displacement of each plane elastic solid. For adjacent plates, components of kinematically admissible displacements that are tangent to the common interface boundary Γ^{ij} are equal at the interface. Also, kinematically admissible displacements are equal to zero on Γ^0.

The strain energy of the built-up structure may be written as the sum of strain energies of the plane elastic solids. In this case, the total strain energy is

$$\frac{1}{2}a_{u,\Omega}(z, z) = \frac{1}{2}\sum_{l=1}^{5} \iint_{\Omega^l} \sum_{i,j=1}^{2} \sigma^{ij}(z)\varepsilon^{ij}(\bar{z})\, d\Omega \tag{4.1.26}$$

where the domains Ω^l of integration will indicate which variables are to be used in the integrand.

The virtual work of the externally applied load on Γ^2 is written simply as

$$l_{u,\Omega}(\bar{z}) = \int_{\Gamma^2} \sum_{i=1}^{2} T^i \bar{z}^i \, d\Gamma \tag{4.1.27}$$

where $T = [T^1 \ T^2]^T$ is given traction force acting on the boundary Γ^2.

Taking the variation of strain energy in Eq. (4.1.26) the variational equilibrium equation of Eq. (4.1.10) can be written as

$$\begin{aligned} a_{u,\Omega}(z, \bar{z}) &= \sum_{l=1}^{5} \iint_{\Omega^l} \sum_{i,j=1}^{2} \sigma^{ij}(z)\varepsilon^{ij}(\bar{z})\, d\Omega \\ &= \int_{\Gamma^2} \sum_{i=1}^{2} T^i \bar{z}^i \, d\Gamma \\ &= l_{u,\Omega}(\bar{z}) \qquad \text{for all} \quad \bar{z} \in Z \end{aligned} \tag{4.1.28}$$

where Z is the space of kinematically admissible displacements. Even though only shape design variables are considered in this example, the subscript u in the energy bilinear form and load linear form will be kept, as in Eq. (4.1.28).

TRUSS–BEAM–PLATE

Consider next the truss–beam–plate built-up structure shown in Fig. 4.1.4. Thin flat plates are stiffened by m longitudinal and n transverse beams. The entire structure is supported by four four-bar trusses. A distributed vertical load is applied to the plates. The points supported by the trusses are

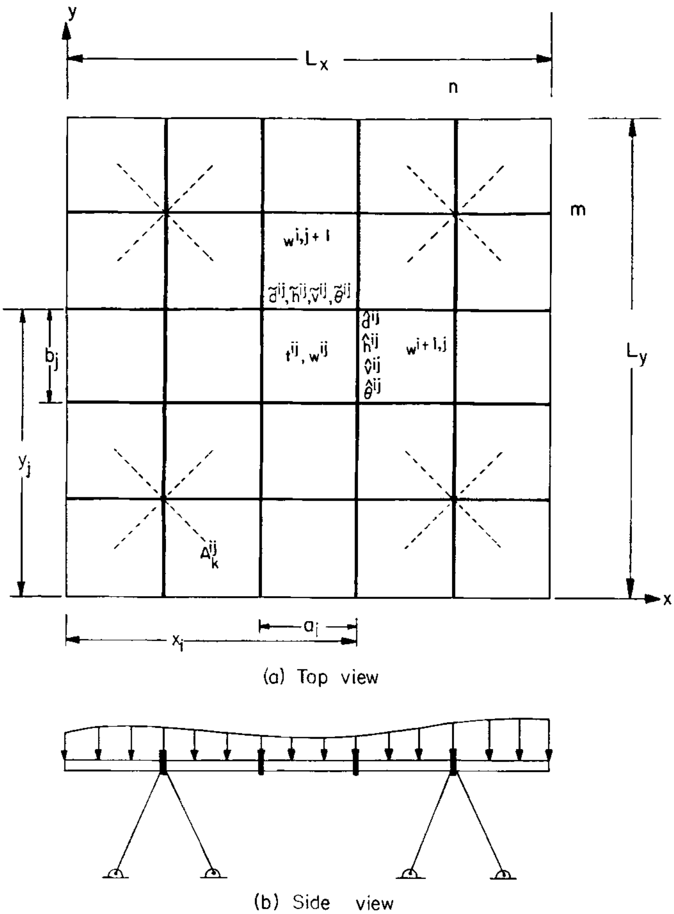

Fig. 4.1.4 Truss–beam–plate built-up structure.

at the intersections of two crossing beams nearest to the free edges of the structure. No external loads are applied to the truss and beam components, and no external torques are applied to the beams. The plates and beams are assumed to be welded together. No dissipation of energy between plate and beam components is presumed to occur during bending and torsion.

The design variables for this built-up structure are the thickness $t^{ij}(x, y)$ of each plate component, the width $\tilde{d}^{ij}(x)$ and height $\tilde{h}^{ij}(x)$ of each longitudinal beam component, the width $\hat{d}^{ij}(y)$ and the height $\hat{h}^{ij}(y)$ of each transverse beam component, the cross-sectional areas A_k^{ij} ($i = 1$ and n, $j = 1$ and m, $k = 1$ to 4) of the four-bar truss members, the positions x_i ($i = 1, \ldots, n$) of transverse beams, and the positions y_j ($j = 1, \ldots, m$) of longitudinal beams.

In vector form this is

$$u = [t^{ij} \quad \tilde{d}^{ij} \quad \tilde{h}^{ij} \quad \hat{d}^{ij} \quad \hat{h}^{ij} \quad A_k^{ij} \quad x_i \quad y_j]^{\mathrm{T}}$$
$$\in C^1(\bar{\Omega}_1^{ij}) \times C^1(\bar{\Omega}_2^{ij}) \times C^1(\bar{\Omega}_2^{ij}) \times C^1(\bar{\Omega}_3^{ij}) \times C^1(\bar{\Omega}_3^{ij}) \times (R^4)^4 \times R^n \times R^m$$

The lengths of the trusses are fixed, but they may change their ground positions, and the outside boundary of the entire structure is fixed; (i.e., only the locations x_i and y_j of beams are shape variables).

Dimensions of the structure and the numbering and spacing of beams in both directions are shown in Fig. 4.1.4. Coordinates of intersection points of beams and plates are supposed to be in the midplanes of the plates and neutral axes of the beams. The coordinates of intersection points are then

$$x_i = x_{i-1} + a_i, \qquad i = 1, \ldots, n \tag{4.1.29}$$

$$y_j = y_{j-1} + b_j, \qquad j = 1, \ldots, m \tag{4.1.30}$$

$$x_0 = y_0 = 0 \tag{4.1.31}$$

$$x_{n+1} = L_x, \; y_{m+1} = L_y \tag{4.1.32}$$

where $a_i(b_j)$ is the distance from the $(i-1)$th to the ith transverse beam (from the $(j-1)$th to the jth longitudinal beam), and $L_x(L_y)$ is the dimension of the entire structure in the $x(y)$ direction.

Applied loads are

$$f^{ij} \in C^1(\bar{\Omega}_1^{ij}), \qquad i = 1, \ldots, n+1, \quad j = 1, \ldots, m+1 \tag{4.1.33}$$

where f^{ij} are defined as distributed loads on the plate and

$$\Omega_1^{ij} = (x_{i-1}, x_i) \times (y_{j-1}, y_j), \qquad i = 1, \ldots, n+1, \quad j = 1, \ldots, m+1 \tag{4.1.34}$$

$$\Omega_2^{ij} = (x_{i-1}, x_i) \times y_j, \qquad i = 1, \ldots, n+1, \quad j = 1, \ldots, m \tag{4.1.35}$$

$$\Omega_3^{ij} = x_i \times (y_{j-1}, y_j), \qquad i = 1, \ldots, n, \quad j = 2, \ldots, m+1 \tag{4.1.36}$$

are domains of plates, longitudinal beams, and transverse beams, respectively.

The state variables for this built-up structure consist of the displacement $w^{ij}(x, y)$ of each plate component, the displacement $\tilde{v}^{ij}(x)$ and the torsion angle $\tilde{\theta}^{ij}(x)$ of each longitudinal beam component, the displacement $\hat{v}^{ij}(y)$ and the torsion angle $\hat{\theta}^{ij}(y)$ of each transverse beam component, and 12 nodal displacement coordinates q_k^{ij} ($i = 1$ and n, $j = 1$ and m, and $k = 1$ to 3) of truss members. In vector form, the state variable is thus

$$z \equiv [w^{ij} \quad \tilde{v}^{ij} \quad \tilde{\theta}^{ij} \quad \hat{v}^{ij} \quad \hat{\theta}^{ij} \quad q_k^{ij}]^{\mathrm{T}} \tag{4.1.37}$$

To use the principle of virtual work of Eq. (4.1.10), the space Z of kinematically admissible displacements must be defined. Hence kinematic boundary and interface conditions must be identified.

Consider first the kinematic boundary conditions at the interfaces. At interfaces between plate and beam components, lateral deflections of plate and beam components are the same, That is, for longitudinal beams,

$$\tilde{v}^{ij} = w^{ij} = w^{i,j+1}, \qquad i = 1, \ldots, n+1, \quad j = 1, \ldots, m \tag{4.1.38}$$

and for transverse beams,

$$\hat{v}^{ij} = w^{ij} = w^{i+1,j}, \qquad i = 1, \ldots, n, \quad j = 1, \ldots, m+1 \tag{4.1.39}$$

The normal slopes of plate components are the same as the torsion angles of beams that are attached at the interfaces. For plates and longitudinal beams,

$$\tilde{\theta}^{ij} = w_y^{ij} = w_y^{i,j+1}, \qquad i = 1, \ldots, n+1, \quad j = 1, \ldots, m \tag{4.1.40}$$

and for plates and transverse beams,

$$\hat{\theta}^{ij} = w_x^{ij} = w_x^{i+1,j}, \qquad i = 1, \ldots, n, \quad j = 1, \ldots, m+1 \tag{4.1.41}$$

The torsion angles of transverse beams and the axial slopes of longitudinal beams must be the same at the intersections of beams; that is,

$$\hat{\theta}^{ij} = \tilde{v}_x^{ij} = \tilde{v}_x^{i+1,j}, \qquad i = 1, \ldots, n, \quad j = 1, \ldots, m \tag{4.1.42}$$

Similarly, the torsion angles of longitudinal beams and the axial slopes of transverse beams must be the same at the intersections of beams; that is,

$$\tilde{\theta}^{ij} = \hat{v}_y^{ij} = \hat{v}_y^{i,j+1}, \qquad i = 1, \ldots, n, \quad j = 1, \ldots, m \tag{4.1.43}$$

It is assumed that each lateral displacement is evaluated at the middle planes of the plates and at the neutral axes of the beams. Then the lateral deflections of two crossing beams and trusses must be the same at the intersection points; that is,

$$\left.\begin{aligned} \tilde{v}^{ij} &= \tilde{v}^{i+1,j} \\ \hat{v}^{ij} &= \hat{v}^{i,j+1} \\ \tilde{v}^{ij} &= \hat{v}^{ij} \end{aligned}\right\}, \qquad i = 1, \ldots, n, \quad j = 1, \ldots, m \tag{4.1.44}$$

$$\hat{v}^{ij} = \tilde{v}^{ij} = q_3^{ij}, \qquad i = 1 \text{ and } n, \quad j = 1 \text{ and } m \tag{4.1.45}$$

With the assumption that there are no in-plane (or axial) deformations in the plates (or beams), the plate–beam structure that rests on the four four-bar trusses must move as a rigid body in the plane of the plates. Referring to Fig. 4.1.5, relationships between horizontal displacements can be obtained.

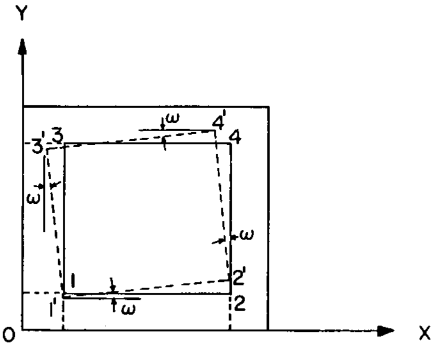

Fig. 4.1.5 Horizontal displacement of a truss–beam–plate built-up structure.

Defining the position of point 1 in Fig. 4.1.5 after deformation as $[(x_1 + q_1^{11}), (y_1 + q_2^{11})]$ and the rotation angle as ω, the coordinates of points 2, 3, and 4 in Fig. 4.1.5 can be identified as follows:

For point 2,

$$\begin{aligned} x_1 + q_1^{11} + (x_n - x_1)\cos\omega &= x_n + q_1^{n1} \\ y_1 + q_2^{11} + (x_n - x_1)\sin\omega &= y_1 + q_2^{n1} \end{aligned} \tag{4.1.46}$$

For point 3,

$$\begin{aligned} x_1 + q_1^{11} - (y_m - y_1)\sin\omega &= x_1 + q_1^{1m} \\ y_1 + q_2^{11} + (y_m - y_1)\cos\omega &= y_m + q_2^{1m} \end{aligned} \tag{4.1.47}$$

For point 4,

$$\begin{aligned} x_1 + q_1^{11} + (x_n - x_1)\cos\omega - (y_m - y_1)\sin\omega &= x_n + q_1^{nm} \\ y_1 + q_2^{11} + (x_n - x_1)\sin\omega + (y_m - y_1)\cos\omega &= y_m + q_2^{nm} \end{aligned} \tag{4.1.48}$$

Assuming that the rotation angle ω is small, $\sin\omega \approx \omega$ and $\cos\omega \approx 1$. With these approximations, Eqs. (4.1.46)–(4.1.48) yield the following geometric relationships between horizontal displacements, in terms of the unknown parameters q_1^{11}, q_2^{11}, and q_1^{1m}:

$$q_1^{n1} = q_1^{11} \tag{4.1.49}$$

$$q_2^{n1} = q_2^{11} + (x_n - x_1)(q_1^{1m} - q_1^{11})/(y_m - y_1) \tag{4.1.50}$$

$$q_2^{1m} = q_2^{11} \tag{4.1.51}$$

$$q_1^{nm} = 2q_1^{11} - q_1^{1m} \tag{4.1.52}$$

$$q_2^{nm} = q_2^{11} + (x_n - x_1)(q_1^{1m} - q_1^{11})/(y_m - y_1) \tag{4.1.53}$$

Finally, the displacement of the bottom of each truss component is zero since it is fastened to a rigid foundation.

The space Z of kinematically admissible displacement fields is defined as

$$Z = \{z = [w^{ij} \quad \tilde{v}^{ij} \quad \tilde{\theta}^{ij} \quad \hat{v}^{ij} \quad \hat{\theta}^{ij} \quad q_k^{11} \quad q_k^{n1} \quad q_k^{1m} \quad q_k^{nm}]^{\mathrm{T}} \\ \in H^2(\Omega_1^{ij}) \times H^2(\Omega_2^{ij}) \times H^1(\Omega_2^{ij}) \times H^2(\Omega_3^{ij}) \times H^1(\Omega_3^{ij}) \\ \times R^3 \times R^3 \times R^3 \times R^3\text{: satisfying interface conditions} \\ \text{of Eqs. (4.1.38)–(4.1.45)}\} \tag{4.1.54}$$

The strain energy of the entire system may be written as the sum of strain energies of the plates, beams, and truss components. In this case, the total strain energy is

$$\begin{aligned} \frac{1}{2}a_{u,\Omega}(z, z) = & \sum_{i=1}^{n+1}\sum_{j=1}^{m+1}\frac{1}{2}\iint_{\Omega_1^{ij}} \hat{D}^{ij}(t)[(w_{xx}^{ij} + \nu w_{yy}^{ij})w_{xx}^{ij} + (w_{yy}^{ij} + \nu w_{xx}^{ij})w_{yy}^{ij} \\ & + 2(1-\nu)w_{xy}^{ij}w_{xy}^{ij}]\, d\Omega_1 \\ & + \sum_{i=1}^{n+1}\sum_{j=1}^{m}\frac{1}{2}\int_{\Omega_2^{ij}} [E\tilde{I}^{ij}(\tilde{v}_{xx}^{ij})^2 + G\tilde{J}^{ij}(\tilde{\theta}_x^{ij})^2]\, d\Omega_2 \\ & + \sum_{i=1}^{n}\sum_{j=1}^{m+1}\frac{1}{2}\int_{\Omega_3^{ij}} [E\hat{I}^{ij}(\hat{v}_{yy}^{ij})^2 + G\hat{J}^{ij}(\hat{\theta}_y^{ij})^2]\, d\Omega_3 \\ & + \frac{1}{2}q_k^{11^{\mathrm{T}}}K(A_l^{11})q_k^{11} + \frac{1}{2}q_k^{1m^{\mathrm{T}}}K(A_l^{1m})q_k^{1m} \\ & + \frac{1}{2}q_k^{n1^{\mathrm{T}}}K(A_l^{n1})q_k^{n1} + \frac{1}{2}q_k^{nm^{\mathrm{T}}}K(A_l^{nm})q_k^{nm} \end{aligned} \tag{4.1.55}$$

where $\hat{D}^{ij}(t) = Et^{ij^3}/12(1 - \nu^2)$, E is Young's modulus, G is shear modulus, $\tilde{J}^{ij}$ and $\hat{J}^{ij}$ are torsion constants (different from polar moment of inertia for circular cross sections) depending on the form and dimensions of the cross sections of beams, $\tilde{I}^{ij}$ and $\hat{I}^{ij}$ are moments of intertia of beams, and $K(A_l^{ij})$ is the stiffness matrix of trusses.

The virtual work of the externally applied loads for the entire system is

$$l_{u,\Omega}(\bar{z}) = \sum_{i=1}^{n+1}\sum_{j=1}^{m+1}\iint_{\Omega_1^{ij}} f^{ij}\bar{w}^{ij}\, d\Omega_1 \tag{4.1.56}$$

For the case of static response of the structure to load that does not depend on time, the condition of equilibrium of Eq. (4.1.10) requires that the total variation of the strain energy with respect to state must equal the virtual work of the applied loads for all virtual displacements that are consistent with constraints. That is, it is required by Eq. (4.1.10) that

$$\begin{aligned}
a_{u,\Omega}(z, \bar{z}) = \sum_{i=1}^{n+1}\sum_{j=1}^{m+1} \iint_{\Omega_1^{ij}} \hat{D}^{ij}(t)[(w_{xx}^{ij} + \nu w_{yy}^{ij})\bar{w}_{xx}^{ij} + (w_{yy}^{ij} + \nu w_{xx}^{ij})\bar{w}_{yy}^{ij} \\
+ 2(1-\nu) w_{xy}^{ij}\bar{w}_{xy}^{ij}]\, d\Omega_1 \\
+ \sum_{i=1}^{n+1}\sum_{j=1}^{m} \int_{\Omega_2^{ij}} [E\tilde{I}^{ij}\tilde{v}_{xx}^{ij}\bar{\tilde{v}}_{xx}^{ij} + G\tilde{J}^{ij}\tilde{\theta}_x^{ij}\bar{\tilde{\theta}}_x^{ij}]\, d\Omega_2 \\
+ \sum_{i=1}^{n}\sum_{j=1}^{m+1} \int_{\Omega_3^{ij}} [E\hat{I}^{ij}\hat{v}_{yy}^{ij}\bar{\hat{v}}_{yy}^{ij} + G\hat{J}^{ij}\hat{\theta}_y^{ij}\bar{\hat{\theta}}_y^{ij}]\, d\Omega_3 \\
+ q_k^{11^{\mathrm{T}}} K(A_l^{11})\bar{q}_k^{11} + q_k^{1m^{\mathrm{T}}} K(A_l^{1m})\bar{q}_k^{1m} \\
+ q_k^{n1^{\mathrm{T}}} K(A_l^{n1})\bar{q}_k^{n1} + q_k^{nm^{\mathrm{T}}} K(A_l^{nm})\bar{q}_k^{nm} \\
= \sum_{i=1}^{n+1}\sum_{j=1}^{m+1} \iint_{\Omega_1^{ij}} f^{ij}\bar{w}^{ij}\, d\Omega_1 = l_{u,\Omega}(\bar{z}) \\
\text{for all} \quad \bar{z} \in Z \qquad (4.1.57)
\end{aligned}$$

where

$$\bar{z} = [\bar{w}^{ij} \quad \bar{\tilde{v}}^{ij} \quad \bar{\tilde{\theta}}^{ij} \quad \bar{\hat{v}}^{ij} \quad \bar{\hat{\theta}}^{ij} \quad \bar{q}_k^{11} \quad \bar{q}_k^{n1} \quad \bar{q}_k^{1m} \quad \bar{q}_k^{nm}]^{\mathrm{T}}$$

Similarly, for the dynamics problem, the kinetic-energy bilinear form is

$$\begin{aligned}
d_{u,\Omega}\left(\frac{dz}{dt}, \frac{d\bar{z}}{dt}\right) \equiv \sum_{i=1}^{n+1}\sum_{j=1}^{m+1} \iint_{\Omega_1^{ij}} t^{ij}\rho\left(\frac{dw^{ij}}{dt}\right)\left(\frac{d\bar{w}^{ij}}{dt}\right) d\Omega_1 \\
+ \sum_{i=1}^{n+1}\sum_{j=1}^{m} \int_{\Omega_2^{ij}} \left[\rho\tilde{d}^{ij}\tilde{h}^{ij}\left(\frac{d\tilde{v}^{ij}}{dt}\right)\left(\frac{d\bar{\tilde{v}}^{ij}}{dt}\right) + \tilde{I}_G^{ij}\left(\frac{d\tilde{\theta}^{ij}}{dt}\right)\left(\frac{d\bar{\tilde{\theta}}^{ij}}{dt}\right)\right] d\Omega_2 \\
+ \sum_{i=1}^{n}\sum_{j=1}^{m+1} \int_{\Omega_3^{ij}} \left[\rho\hat{d}^{ij}\hat{h}^{ij}\left(\frac{d\hat{v}^{ij}}{dt}\right)\left(\frac{d\bar{\hat{v}}^{ij}}{dt}\right) + \hat{I}_G^{ij}\left(\frac{d\hat{\theta}^{ij}}{dt}\right)\left(\frac{d\bar{\hat{\theta}}^{ij}}{dt}\right)\right] d\Omega_3 \\
+ \left(\frac{dq_k^{11}}{dt}\right)^{\mathrm{T}} M(A_l^{11})\left(\frac{d\bar{q}_k^{11}}{dt}\right) + \left(\frac{dq_k^{1m}}{dt}\right)^{\mathrm{T}} M(A_l^{1m})\left(\frac{d\bar{q}_k^{1m}}{dt}\right) \\
+ \left(\frac{dq_k^{n1}}{dt}\right)^{\mathrm{T}} M(A_l^{n1})\left(\frac{d\bar{q}_k^{n1}}{dt}\right) + \left(\frac{dq_k^{nm}}{dt}\right)^{\mathrm{T}} M(A_l^{nm})\left(\frac{d\bar{q}_k^{nm}}{dt}\right) \\
(4.1.58)
\end{aligned}$$

where ρ is the mass density, $\tilde{I}_G^{ij}$ and $\hat{I}_G^{ij}$ are the mass moments of inertia about the centroidal axes of the beams, and $M(A_l^{ij})$ is the mass matrix of the trusses.

The variational eigenvalue problem may now be written, from Eq. (4.1.16), as

$$a_{u,\Omega}(y, \bar{y}) = \zeta d_{u,\Omega}(y, \bar{y}) \qquad \text{for all} \quad \bar{y} \in Z \qquad (4:1.59)$$

where

$$y \equiv [w^{ij} \quad \tilde{v}^{ij} \quad \tilde{\theta}^{ij} \quad \hat{v}^{ij} \quad \hat{\theta}^{ij} \quad q_k^{11} \quad q_k^{n1} \quad q_k^{1m} \quad q_k^{nm}]^T$$

Here y denotes the eigenfunctions, even though the notation for the components of y is borrowed from static response z, to avoid introduction of new variables.

4.2 STATIC DESIGN SENSITIVITY

The variational methods presented in Chapters 2 and 3 for design sensitivity analysis with respect to conventional design variables and shape may now be combined, using the general variational formulation presented in Section 4.1, to obtain expressions for design sensitivity of functionals with respect to combined design variation. The adjoint variable method used in Chapters 1–3 is seen to extend directly to built-up structures. As mentioned in Section 3.3.7, the domain method of shape design sensitivity analysis is used for built-up structures.

4.2.1 Calculation of First Variations

Consider the variational form of the built-up structure equation of Eq. (4.1.10), repeated here as

$$a_{u,\Omega}(z, \bar{z}) = l_{u,\Omega}(\bar{z}) \qquad \text{for all} \quad \bar{z} \in Z \tag{4.2.1}$$

The objective is to use this variational equation to obtain a relationship between variations in conventional design variables and shape and the resulting variation in the state of the system. To simplify notation, consider the deformed domain due to a design velocity field V, written as

$$\Omega_\tau \equiv \{x_\tau \in R^n\colon \quad x_\tau = x + \tau V(x), \quad x \in \Omega\} \tag{4.2.2}$$

Defining the first variation in the same way as in Chapters 2 and 3, but with variation of both shape and conventional design variables,

$$\begin{aligned}[a_{u,\Omega}(z, \bar{z})]' &\equiv a'_{\delta u}(z, \bar{z}) + a'_V(z, \bar{z}) + a_{u,\Omega}(\dot{z}, \bar{z}) \\ &= \left[\sum_{i=1}^{r} a'_{\delta u^i}(w^i, \overline{w}^i) + a'_{\delta b}(q, \bar{q})\right] + \sum_{i=1}^{r} a'_{V^i}(w^i, \bar{w}^i) \\ &\quad + \sum_{i=1}^{r} a_{u^i,\Omega^i}(\dot{w}^i, \overline{w}^i) + a_b(\dot{q}, \bar{q})\end{aligned} \tag{4.2.3}$$

where V^i is the design velocity field on Ω^i and $\dot{z}$ is the total variation of z due to conventional design and shape changes. Note that the first and second terms on the right side of Eq. (4.2.3) for the trusses and distributed components can be obtained from Chapters 1 and 2. This notation is chosen to clearly display which variables are held fixed and which are varying in the terms that arise. The third term on the right side of Eq. (4.2.3) is due to shape variation. For shape design sensitivity of built-up structures, the domain method will be used. For this method, $a'_{V^i}(w^i, \bar{w}^i)$, instead of Eq. (3.3.6), must be written in terms of domain integrals. Using Eq. (3.2.37), instead of Eq. (3.2.36), and proceeding as in the derivation of Eq. (3.3.6),

$$a'_{V^i}(w^i, \bar{w})^i = \iint_{\Omega^i} \{-c_i(w^i, \nabla\bar{w}^{iT}V^i) - c_i(\nabla w^{iT}V^i, \bar{w}^i) + \operatorname{div}\,[c_i(w^i, \bar{w}^i)V^i]\}\,d\Omega \tag{4.2.4}$$

where $c_i(\cdot\,,\cdot)$ is the bilinear function in the integrand of the bilinear form $a_{u^i,\Omega^i}(\cdot\,,\cdot)$.

Similarly, the first variation of the load linear form is

$$\begin{aligned}[l_{u,\Omega}(\bar{z})]' &\equiv l'_{\delta u}(\bar{z}) + l'_V(\bar{z}) \\ &= \left[\sum_{i=1}^{r} l'_{\delta u^i}(\bar{w}^i) + f'_{\delta b}(\bar{q})\right] + \sum_{i=1}^{r} l'_{V^i}(\bar{w}^i)\end{aligned} \tag{4.2.5}$$

where the first and second terms on the right side of Eq. (4.2.5) can be obtained from Chapters 1 and 2. As in Eq. (4.2.3), the third term on the right side of Eq. (4.2.5) is due to shape variation. For the domain method,

$$l'_{V^i}(\bar{w}^i) = \iint_{\Omega^i} [-f^{iT}(\nabla\bar{w}^{iT}V^i) + \operatorname{div}(f^{iT}\bar{w}^iV^i)]\,d\Omega \tag{4.2.6}$$

where $l_{u^i,\Omega^i}(\bar{w}^i) = \iint_{\Omega^i} f^{iT}\bar{w}^i\,d\Omega$ and $f^{i\prime} = 0$ have been used.

With this notation, and denoting the solution of Eq. (4.2.1) on the deformed domain and varied design as z_τ, evaluating the first variation of both sides of Eq. (4.2.1) yields

$$a_{u,\Omega}(\dot{z}, \bar{z}) + a'_{\delta u}(z, \bar{z}) + a'_V(z, \bar{z}) = l'_{\delta u}(\bar{z}) + l'_V(\bar{z}) \qquad \text{for all} \quad \bar{z} \in Z \tag{4.2.7}$$

Note that this equation is valid for arbitrary virtual displacements that are consistent with constraints, so if the energy bilinear form is indeed strongly elliptic, Eq. (4.2.5) uniquely determines $\dot{z}$ once δu and $V$are specified. Explicit solution of this equation for $\dot{z}$ as a function of δu and V, however, is not generally possible

Consider a general functional that defines performance of a built-up structure, of the form

$$\psi \equiv \psi_{u,\Omega}(z) = \sum_{i=1}^{r} \iint_{\Omega^i} g^i(w^i, \nabla w^i, u^i)\,d\Omega + h(b, q) \tag{4.2.8}$$

Taking the total variation of this functional and using Eqs. (2.2.9) and (3.3.12) for each component of the structure,

$$\psi' = \frac{d}{d\tau}\psi_{u+\tau\delta u,\Omega_\tau}(z_\tau)\bigg|_{\tau=0}$$

$$= \sum_{i=1}^{r} \iint_{\Omega^i} \left\{ g^i_{w^i}\dot{w}^i + \sum_{j=1}^{l_i} g^i_{\nabla w^i_j}\nabla\dot{w}^i_j - g^i_{w^i}(\nabla w^{i\mathrm{T}} V^i) - \sum_{j=1}^{l_i} g^i_{\nabla w^i_j}\nabla(\nabla w^{i\mathrm{T}}_j V^i) \right.$$

$$\left. + \operatorname{div}(g^i V^i) + g^i_{u^i}\,\delta u^i \right\} d\Omega + \frac{\partial h}{\partial b}\delta b + \frac{\partial h}{\partial q}\dot{q} \tag{4.2.9}$$

where $w^i = [w^i_1 \ w^i_2 \ \ldots \ w^i_{l^i}]^{\mathrm{T}}$. In order to take advantage of this result, the term on the right side of Eq. (4.2.9) must be written explicitly in terms of δu and V. Since $\dot{z}$ cannot generally be determined explicitly from Eq. (4.2.7), a technique such as the adjoint variable method must be used to achieve the desired result.

4.2.2 The Adjoint Variable Method

In order to treat the term on the right side of Eq. (4.2.9), an adjoint variational equation is defined by replacing $\dot{z}$ in the term on the right side of Eq. (4.2.9) by a virtual displacement $\bar{\lambda}$ and equating the result to the energy bilinear form evaluated at the adjoint variable λ; that is,

$$a_{u,\Omega}(\lambda, \bar{\lambda}) = \sum_{i=1}^{r} \iint_{\Omega^i} \left[g^i_{w^i}\bar{\gamma}^i + \sum_{j=1}^{l_i} g^i_{\nabla w_j}\nabla\bar{\gamma}^i_j \right] d\Omega + \frac{\partial h}{\partial q}\bar{p} \qquad \text{for all} \quad \bar{\lambda} \in Z \tag{4.2.10}$$

where $\lambda = [\gamma^1 \ \gamma^2 \ \ldots \ \gamma^r \ p]^{\mathrm{T}}$. Presuming that the energy bilinear form is strongly elliptic and that the term on the right side is a continuous linear form in $\bar{\lambda}$, this equation uniquely determines λ.

Since $\dot{z}$ satisfies the kinematic admissibility conditions, Eq. (4.2.10) may be evaluated at $\bar{\lambda} = \dot{z}$ and Eq. (4.2.7) at $\bar{z} = \lambda$, to obtain

$$\sum_{i=1}^{r} \iint_{\Omega^i} \left[g^i_{w^i}\dot{w}^i + \sum_{j=1}^{l_i} g^i_{\nabla w_j}\nabla\dot{w}^i_j \right] d\Omega + \frac{\partial h}{\partial q}\dot{q} = a_{u,\Omega}(\lambda, \dot{z}) = a_{u,\Omega}(\dot{z}, \lambda)$$

$$= l'_{\delta u}(\lambda) + l'_V(\lambda) - a'_{\delta u}(z, \lambda) - a'_V(z, \lambda) \tag{4.2.11}$$

Substituting this result into Eq. (4.2.9) and collecting terms associated with variations in the conventional design variable and the design velocity field, the total differential of the functional of Eq. (4.2.8) is written explicitly in terms of design function variation and shape variation as

$$\psi' = \left\{ \sum_{i=1}^{r} \iint_{\Omega^i} g^i_{u^i}\, \delta u^i \, d\Omega + \frac{\partial h}{\partial b} \delta b + l'_{\delta u}(\lambda) - a'_{\delta u}(z, \lambda) \right\}$$
$$+ \left\{ \sum_{i=1}^{r} \iint_{\Omega^i} \left[-g^i_{w^i}(\nabla w^{iT} V^i) - \sum_{j=1}^{l_i} g^i_{\nabla w^i_j} \nabla(\nabla w^{iT}_j V^i) \right.\right.$$
$$\left.\left. + \operatorname{div}(g^i V^i) \right] d\Omega + l'_V(\lambda) - a'_V(z, \lambda) \right\} \qquad (4.2.12)$$

where Eqs. (4.2.3)–(4.2.6) and the results of Chapters 1 and 2 for the truss and distributed components can be used to obtain explicit formulas.

Note that evaluation of this explicit design sensitivity formula requires a solution of Eq. (4.2.10) for the adjoint variable λ and evaluation of functionals involving both state z and adjoint variable λ. As will be seen in examples, these calculations are direct and take full advantage of the finite element method for solving both the state and adjoint equations of the built-up structure.

4.2.3 Examples

BEAM–TRUSS

As a check on the foregoing calculations, consider a concentrated load f applied at the midpoint of a beam, as shown in Fig. (4.2.1). Assume for the moment that the cross-sectional area h is constant and can be only uniformly varied. In this example, for simplicity, the shape (length l_4) of the beam and the lengths of the trusses are held fixed. Consider the displacement of the

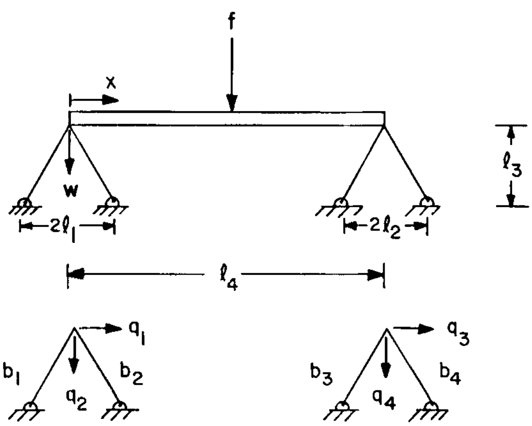

Fig. 4.2.1 Beam–truss built-up structure with point load.

midpoint of the beam, which can be expressed as

$$\psi = \int_0^{l_4} w\,\hat{\delta}\left(x - \frac{l_4}{2}\right) dx \tag{4.2.13}$$

where $\hat{\delta}$ is the Dirac-δ distribution. From Eq. (4.1.21), the variational equation of equilibrium is

$$\begin{aligned}
&\int_0^{l_4} E\alpha h^2 w_{xx}\bar{w}_{xx}\, dx \\
&\quad + \frac{E}{(l_1^2 + l_3^2)^{3/2}}[b_1(q_1 l_1 - q_2 l_3)(l_1\bar{q}_1 - l_3\bar{q}_2) \\
&\qquad\qquad + b_2(q_1 l_1 + q_2 l_3)(l_1\bar{q}_1 + l_3\bar{q}_2)] \\
&\quad + \frac{E}{(l_2^2 + l_3^2)^{3/2}}[b_3(q_3 l_2 - q_4 l_3)(l_2\bar{q}_3 - l_3\bar{q}_4) \\
&\qquad\qquad + b_4(q_3 l_2 + q_4 l_3)(l_2\bar{q}_3 + l_3\bar{q}_4)] \\
&\quad - \int_0^{l_4} f\hat{\delta}\left(x - \frac{l_4}{2}\right)\bar{w}\, dx = 0 \\
&\qquad\qquad\qquad\qquad \text{for all} \quad \bar{z} \in Z
\end{aligned} \tag{4.2.14}$$

where

$$Z = \{[w \quad q^{\mathrm{T}}]^{\mathrm{T}} \in H^2(0, l_4) \times R^4\colon \quad w(0) = q_2, \quad w(l_4) = q_4, \quad q_1 = q_3\}$$

Observe that the adjoint equation defined in Eq. (4.2.10) for the displacement constraint of Eq. (4.2.13) has the same form as Eq. (4.2.14), where the force f is replaced by a unit load. Therefore, only the state equation of Eq. (4.2.14) needs to be solved.

After integrating by parts and applying the interface conditions, $w(0) = q_2$, $w(l_4) = q_4$, and $q_1 = q_3$ and the condition that the joints between the beam and the trusses are hinged, $w_{xx}(0) = w_{xx}(l_4) = 0$, q_1, q_2, q_4, and w can be calculated as

$$\begin{aligned}
q_1 &= q_3 = \frac{f}{2El_3}X \\
q_2 &= \frac{f}{2EAl_3^2}\{l + l_1 BX\} \\
q_4 &= \frac{f}{2ECl_3^2}\{m + l_2 DX\}
\end{aligned} \tag{4.2.15}$$

$$w = \frac{f\langle x - (l_4/2)\rangle^3}{6E\alpha h^2} - \frac{fx^3}{12E\alpha h^2} + \frac{fl_4^2 x}{16E\alpha h^2} - \frac{q_2 - q_4}{l_4}x + q_2 \tag{4.2.16}$$

where $\langle x - (l_4/2)\rangle^3$ is the singularity function, defined as

$$\langle x - (l_4/2)\rangle^3 = \begin{cases} 0 & \text{when} \quad x \le (l_4/2) \\ (x - (l_4/2))^3 & \text{when} \quad x > (l_4/2) \end{cases}$$

and

$$\begin{aligned} l &= (l_1^2 + l_3^2)^{3/2} \\ m &= (l_2^2 + l_3^2)^{3/2} \\ A &= b_1 + b_2 \\ B &= b_1 - b_2 \\ C &= b_3 + b_4 \\ D &= b_3 - b_4 \\ X &= \frac{(ADl_2 + BCl_1)ml}{l_1^2 mC(A^2 - B^2) + l_2^2 Al(C^2 - D^2)} \end{aligned} \tag{4.2.17}$$

In the same way, the adjoint equations can be solved to obtain $\lambda = [\gamma(x) \ p_1 \ p_2 \ p_3 \ p_4]^T$ as

$$\begin{aligned} p_1 = p_3 &= \frac{1}{2EI_3} X \\ p_2 &= \frac{1}{2EAl_3^2}\{l + l_1 BX\} \\ p_4 &= \frac{1}{2ECl_3^2}\{m + l_2 DX\} \end{aligned} \tag{4.2.18}$$

$$\gamma = \frac{\langle x - (l_4/2)\rangle^3}{6E\alpha h^2} - \frac{x^3}{12E\alpha h^2} + \frac{l_4^2 x}{16E\alpha h^2} - \frac{(p_2 - p_4)}{l_4} x + p_2 \tag{4.2.19}$$

From Eq. (4.1.12) the design sensitivity can be calculated as

$$\psi' = \Lambda_1 \, \delta b_1 + \Lambda_2 \, \delta b_2 + \Lambda_3 \, \delta b_3 + \Lambda_4 \, \delta b_4 + \Lambda_5 \, \delta h \tag{4.2.20}$$

where

$$\Lambda_1 = -\frac{f}{4Ell_3^2}\left[\frac{l_1(A - B)}{A} X - \frac{l}{A}\right]^2$$

$$\Lambda_2 = -\frac{f}{4EIl_3^2}\left[\frac{l_1(A+B)}{A}X + \frac{l}{4}\right]^2$$

$$\Lambda_3 = -\frac{f}{4Eml_3^2}\left[\frac{l_2(C-D)}{C}X - \frac{m}{C}\right]^2$$

$$\Lambda_4 = -\frac{f}{4Eml_3^2}\left[\frac{l_2(C-D)}{C}X - \frac{m}{C}\right]^2$$

$$\Lambda_5 = -\frac{fl_4^3}{24E\alpha h^3} \tag{4.2.21}$$

The sensitivity coefficients of Eq. (4.2.21) can be verfied by differentiating the constraint ψ directly, using Eq. (4.2.16), to obtain exact sensitivity coefficients as

$$\begin{aligned}
\Lambda_1' &= \frac{\partial \psi}{\partial b_1} = \frac{\partial}{\partial b_1}\left(\frac{q_2+q_4}{2}\right) \\
\Lambda_2' &= \frac{\partial \psi}{\partial b_2} = \frac{\partial}{\partial b_2}\left(\frac{q_2+q_4}{2}\right) \\
\Lambda_3' &= \frac{\partial \psi}{\partial b_3} = \frac{\partial}{\partial b_3}\left(\frac{q_2+q_4}{2}\right) \\
\Lambda_4' &= \frac{\partial \psi}{\partial b_4} = \frac{\partial}{\partial b_4}\left(\frac{q_2+q_4}{2}\right) \\
\Lambda_5' &= \frac{\partial \psi}{\partial h} = \left.\frac{\partial w}{\partial h}\right|_{x=l_4/2} = -\frac{fl_4^3}{24E\alpha h^3}
\end{aligned} \tag{4.2.22}$$

In Eqs. (4.2.21) and (4.2.22), $\Lambda_5' = \Lambda_5$. After substituting q_2 and q_4 from Eq. (4.2.15) into Eq. (4.2.22) and manipulating,

$$\begin{aligned}
\Lambda_1' &= \Lambda_1 \\
\Lambda_2' &= \Lambda_2 \\
\Lambda_3' &= \Lambda_3 \\
\Lambda_4' &= \Lambda_4
\end{aligned} \tag{4.2.23}$$

In this simple example, the sensitivity coefficients calculated from the adjoint method are exactly the same as the true design sensitivities. Barring errors in computation, this will be true in all applications (i.e., the adjoint variable method gives exact design derivatives, not approximations).

CONNECTING ROD

Consider the connecting rod of Section 4.1.4, with the variational equilibrium equation of Eq. (4.1.25) repeated here as

$$\begin{aligned} a_{u,\Omega}(z, \bar{z}) &= \iint_\Omega h(x)\left[\sum_{i,j=1}^{2} \sigma^{ij}(z)\varepsilon^{ij}(\bar{z})\right] d\Omega \\ &= \int_{\Gamma^2 \cup \Gamma^3} \sum_{i=1}^{2} T^i \bar{z}^i \, d\Gamma \qquad (4.2.24) \\ &= l_{u,\Omega}(\bar{z}) \qquad \text{for all} \quad \bar{z} \in Z \end{aligned}$$

The boundary Γ is composed of four parts, Γ^1 through Γ^4, as shown in Fig. 4.2.2. The boundary segments Γ^2 and Γ^3 are boundaries at which the rod touches the crankshaft and piston pin, respectively. Their shapes are kept unchanged, and Γ^1 is the boundary segment of the shank and neck regions of the rod, whose shapes are to be determined through the design process. Since the main interest in this example is on the shank and neck regions, the shape of other boundary segments can be kept fixed, namely Γ^4. It is also assumed that the traction T on $\Gamma^2 \cup \Gamma^3$ is not changed.

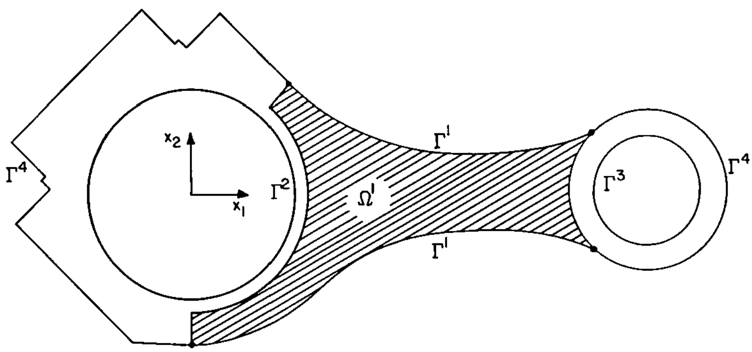

Fig. 4.2.2 Variable domain Ω^1 and boundary Γ^1.

In shape design of the rod, the thickness distribution, which varies independently of the domain variation is to be determined. The thickness distribution in the hatched segment Ω^1 of the domain of Fig. 4.2.2 is to be determined through the design process.

To satisfy the conditions that the distance between the piston pin and crankshaft is constant, it is required that points in the rod remain fixed in the x_1 direction, hence points on Γ^1 are allowed to move only in the x_2 direction. For design sensitivity analysis due to shape variation, the boundary method of Chapter 3, instead of the domain method, is used here.

Consider a stress functional of the form

$$\psi_p = \iint_\Omega g(\sigma(z))m_p \, d\Omega \tag{4.2.25}$$

where Ω_p is a finite element and m_p is a characteristic function on Ω_p. In this problem, several different forms of $g(\sigma(z))$ are considered; that is,

$$g(\sigma(z)) = \frac{\sigma_1}{\sigma_{\text{UI}}} - 1 \tag{4.2.26}$$

or

$$g(\sigma(z)) = 1 - \frac{\sigma_2}{\sigma_{\text{LI}}} \tag{4.2.27}$$

for the inertia load and

$$g(\sigma(z)) = \frac{\sigma_1}{\sigma_{\text{UF}}} - 1 \tag{4.2.28}$$

or

$$g(\sigma(z)) = 1 - \frac{\sigma_2}{\sigma_{\text{LF}}} \tag{4.2.29}$$

for the firing load, where σ_{LI} and σ_{UI} denote lower and upper bounds on principal stress for the inertia load, σ_{LF} and σ_{UF} correspond to those for the firing load, and σ_1 and σ_2 denote principle stresses that are given as

$$\sigma_1 = \frac{\sigma^{11} + \sigma^{22}}{2} + \sqrt{\left(\frac{\sigma^{11} + \sigma^{22}}{2}\right)^2 + (\sigma^{12})^2} \tag{4.2.30}$$

$$\sigma_2 = \frac{\sigma^{11} + \sigma^{22}}{2} - \sqrt{\left(\frac{\sigma^{11} + \sigma^{22}}{2}\right)^2 + (\sigma^{12})^2} \tag{4.2.31}$$

Employing the idea of calculating first variations in Section 4.2.1, the variation of Eq. (4.2.25) can be obtained by adding contributions due to variations of each design variable. Thus, from Eqs. (2.2.76), (3.3.114), and (3.3.156),

$$\begin{aligned}\psi'_p = &-\iint_\Omega \left[\sum_{i,j=1}^{2} \sigma^{ij}(z)\varepsilon^{ij}(\lambda)\right] \delta h \, d\Omega - \int_{\Gamma^1} \left[\sum_{i,j=1}^{2} \sigma^{ij}(z)\varepsilon^{ij}(\lambda)\right](V^{\text{T}}n) \, d\Gamma \\ &+ \bar{m}_p \int_{\Gamma p} [g(\sigma(z)) - \psi_p](V^{\text{T}}n) \, d\Gamma \end{aligned} \tag{4.2.32}$$

where Γ_p is the boundary of Ω_p and $\bar{m}_p$ is the value of the characteristic

Table 4.2.1

Normal Boundary Force Data on Γ^2 (Inertia Loading)

θ_{c1} (deg)	Force (N)	θ_{c1} (deg)	Force (N)	θ_{c1} (deg)	Force (N)
65	1963	15	889	−35	1025
60	3034	10	370	−40	1644
55	3081	5	273	−45	1630
50	2895	0	609	−50	1220
45	2683	−5	1109	−55	2318
40	2534	−10	1472	−60	2933
35	2450	−15	1584	−65	3236
30	2302	−20	1540	−70	2963
25	1999	−25	1391	−75	689
20	1586	−30	1140		

Table 4.2.2

Normal Boundary Force Data on Γ^3 (Inertia Loading)

θ_{p1} (deg)	Force (N)	θ_{p1} (deg)	Force (N)	θ_{p1} (deg)	Force (N)
70	208	20	1392	−30	1157
65	2524	15	1336	−35	1942
60	3164	10	1149	−40	2301
55	2932	5	743	−45	2809
50	2607	0	522	−50	2985
45	2288	−5	877	−55	2619
40	1861	−10	1352	−60	1483
35	1798	−15	1552	−65	2987
30	1717	−20	1564	−70	158
25	1489	−25	1424		

Table 4.2.3

Normal Boundary Force Data on Γ^2 (Firing Loading)

θ_{c2} (deg)	Force (N)	θ_{c2} (deg)	Force (N)	θ_{c2} (deg)	Force (N)
−40	0	−10	19587	20	15741
−35	978	−5	20374	25	10335
−30	6868	0	21237	30	6103
−25	11210	5	22243	35	1402
−20	14689	10	20395	40	0
−15	17816	15	17426		

Table 4.2.4
Normal Boundary Force Data on Γ^3 (Firing Loading)

θ_{p2} (deg)	Force (N)	θ_{p2}	Force (N)	θ_{c1} (deg)	Force (N)
−40	2234	−10	17499	20	15056
−35	6727	−5	17245	25	12622
−30	9808	0	16488	30	9803
−25	12536	5	17365	35	6409
−20	14917	10	17711	40	1055
−15	16308	15	16502		

function. In Eq. (4.2.32), λ is the solution of the adjoint equation of Eq. (2.2.73), repeated here as

$$a_{u,\Omega}(\lambda,\bar{\lambda}) = \iint_{\Omega} \left[\sum_{i,j=1}^{2} g_{\sigma^{ij}}(z)\sigma^{ij}(\bar{\lambda}) \right] m_p \, d\Omega \qquad \text{for all} \quad \bar{\lambda} \in Z \tag{4.2.33}$$

Two loading cases are considered, inertia and firing loads. The load vector for finite element analysis was generated from boundary force data supplied by the manufacturer of the connecting rod, as shown in Tables 4.2.1–4.2.4 and Fig. 4.2.3. Directions of the forces are normal to the boundaries.

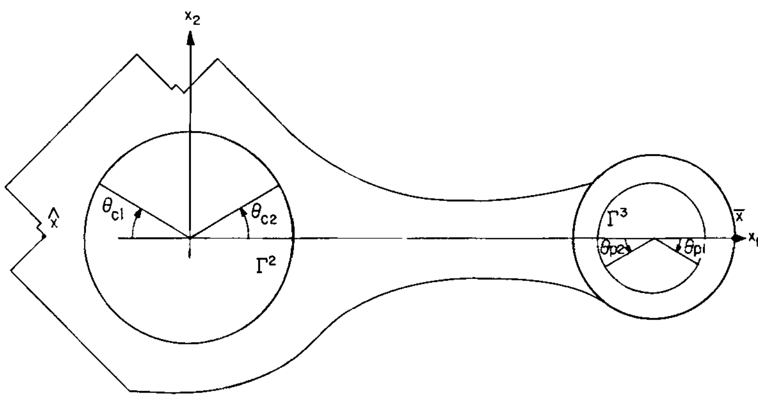

Fig. 4.2.3 Nomenclature for Tables 4.2.1–4.2.4.

In order to eliminate rigid-body translation and rotation, an arbitrary point ($\hat{x}$ in Fig. 4.2.3) is fixed in the x_1 and x_2 directions, and another point ($\bar{x}$ in Fig. 4.2.3) is fixed in the x_2 direction. This is a reasonable procedure, because the loads acting on the rod are in self-equilibrium.

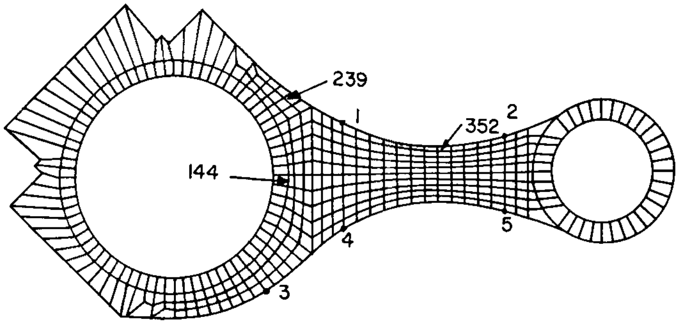

Fig. 4.2.4 Finite element model of engine connecting rod. The x_2 coordinates of dots 1–5 denote shape parameters.

For numerical calculations, an eight-noded ISP element is used for analysis. A finite element model that includes 422 elements, 1493 nodes, and 2983 degrees of freedom is employed, as shown in Fig. 4.2.4. Two and three design variables are used to parameterize the upper and lower boundaries with spline functions, respectively, as shown in Fig. 4.2.4. For thickness distribution, 43 design parameters are used, as shown in Fig. 4.2.5. Each

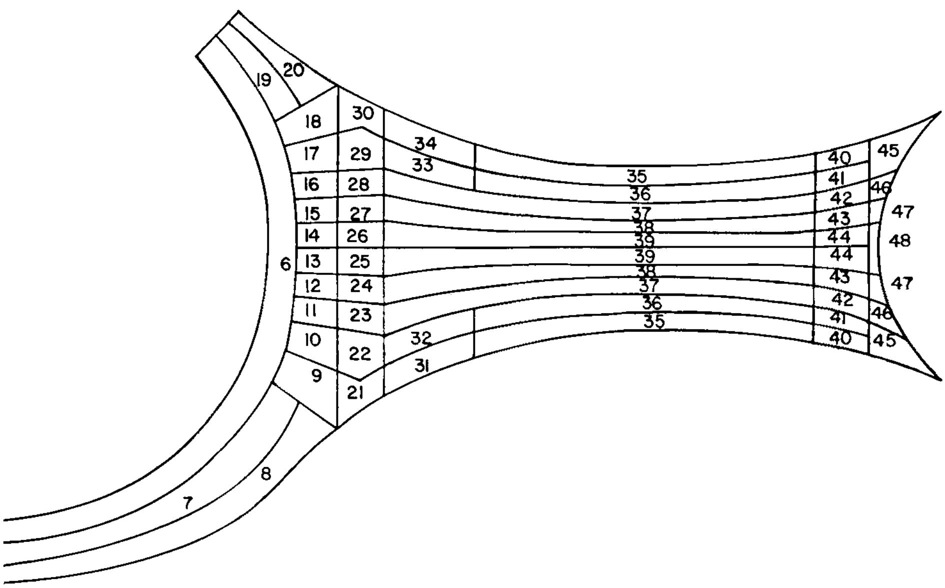

Fig. 4.2.5 Thickness design parameters in Ω^1. Numbers denote conventional design parameters.

number in Fig. 4.2.5 identifies the constant thickness of that area. Numerical results presented are based on the following input data: $E = 2.07 \times 10^5$ MPa, $\nu = 0.298$, $\sigma_{UI} = 136$ MPa, $\sigma_{LI} = -80$ MPa, $\sigma_{UF} = 37$ MPa, and $\sigma_{LF} = -279$ MPa. Dimensions of the connecting rod at the nominal design are $b_1 = 28.1$, $b_2 = 20.0$, $b_3 = 63.4$, $b_4 = 30.5$, $b_5 = 21.2$, and $b_6 = \cdots = b_{40} = 40.0$, all with units in millimeters.

Table 4.2.5

Design Sensitivity of Engine Connecting Rod for Element 144

Case[a]	ψ^1	ψ^2	$\Delta\psi$	ψ'	$(\psi'/\Delta\psi \times 100)\%$
			(a) 0.1% change of b_1 to b_2 (upper boundary)		
1	−1.0259E + 00	−1.0259E + 00	−1.0346E − 05	−1.1052E − 05	106.8
2	−8.2274E − 01	−8.2280E − 01	−6.0366E − 05	−6.0397E − 05	100.1
3	−1.8560E − 01	−1.8602E − 01	−4.1959E − 04	−4.1637E − 04	99.2
4	−6.6577E − 01	−6.6578E − 01	−9.1231E − 06	−9.1305E − 06	100.1
			(b) 0.1% change of b_3 to b_5 (lower boundary)		
1	−1.0259E + 00	−1.0259E + 00	5.0509E − 05	4.8758E − 05	96.5
2	−8.2274E − 01	−8.2269E − 01	5.2476E − 05	4.9116E − 05	93.6
3	−1.8560E − 01	−1.8593E − 01	−3.2836E − 04	−3.2459E − 04	98.9
4	−6.6577E − 01	−6.6578E − 01	−1.1829E − 05	−1.1561E − 05	97.7
			(c) 0.1% change of b_6 to b_{48} (thickness change)		
1	−1.0259E + 00	−1.0259E + 00	1.3192E − 05	1.3190E − 05	100.0
2	−8.2274E − 01	−8.2273E − 01	6.0020E − 06	6.0592E − 06	101.0
3	−1.8560E − 01	−1.8577E − 01	−1.6485E − 04	−1.6475E − 04	99.9
4	−6.6577E − 01	−6.6610E − 01	−3.3765E − 04	−3.3799E − 04	100.1

[a] Case 1: upper principal stress constraint due to inertia load.
Case 2: lower principal stress constraint due to inertia load.
Case 3: upper principal stress constraint due to firing load.
Case 4: lower principal stress constraint due to firing load.

In Tables 4.2.5–4.2.7, design sensitivity accuracy results are given for elements 144, 239, and 352 (see Fig. 4.2.4), due to a 0.1% change in design variables. Observe that agreement between predictions ψ' and actual changes $\Delta\psi$ are excellent, except that in Table 4.2.6(a) design sensitivity of the lower principal stress for the inertia load in element 239 is not good. However, the difference $\Delta\psi$ is small compared to other differences. Since it is the difference between two approximate values, numerical precision is reduced, and its value is of suspect accuracy.

Table 4.2.6

Design Sensitivity of Engine Connecting Rod for Element 239

Case[a]	ψ^1	ψ^2	$\Delta\psi$	ψ'	$(\psi'/\Delta\psi \times 100)\%$
		(a) 0.1% change of b_1 to b_2 (upper boundary)			
1	−5.9578E − 01	−5.9606E − 01	−2.7667E − 04	−2.9248E − 04	105.7
2	−9.9779E − 01	−9.9779E − 01	2.2452E − 06	6.2944E − 05	2803.5
3	−9.6546E − 01	−9.6557E − 01	−1.0657E − 04	−1.2251E − 04	115.0
4	−9.8686E − 01	−9.8683E − 01	3.0799E − 05	3.9093E − 05	126.9
		(b) 0.1% change of b_3 to b_5 (lower boundary)			
1	−5.9578E − 01	−5.9565E − 01	1.3803E − 04	1.3588E − 04	98.4
2	−9.9779E − 01	−9.9780E − 01	−2.5153E − 06	−2.5001E − 06	99.4
3	−9.6546E − 01	−9.6553E − 01	−7.1355E − 05	−7.6392E − 05	107.1
4	−9.8686E − 01	−9.8685E − 01	8.1675E − 06	8.6792E − 06	106.3
		(c) 0.1% change of b_6 to b_{48} (thickness change)			
1	−5.9578E − 01	−5.9607E − 01	−2.8199E − 04	−2.8222E − 04	100.1
2	−9.9779E − 01	−9.9779E − 01	−1.5866E − 06	−1.5886E − 06	100.1
3	−9.6546E − 01	−9.6542E − 01	4.1725E − 05	4.1822E − 05	100.2
4	−9.8686E − 01	−9.8687E − 01	−1.3715E − 05	−1.3736E − 05	100.2

[a] Case 1: upper principal stress constraint due to inertia load.
Case 2: lower principal stress constraint due to inertia load.
Case 3: upper principal stress constraint due to firing load.
Case 4: lower principal stress constraint due to firing load.

Table 4.2.7

Design Sensitivity of Engine Connecting Rod for Element 352

Case[a]	ψ^1	ψ^2	$\Delta\psi$	ψ'	$(\psi'/\Delta\psi \times 100)\%$
		(a) 0.1% change of b_1 to b_2 (upper boundary)			
1	−7.7013E − 01	−7.7113E − 01	−9.9819E − 04	−1.0021E − 03	100.4
2	−1.0042E + 00	−1.0043E + 00	−2.1690E − 05	−2.1984E − 05	101.4
3	−1.0530E + 00	−1.0532E + 00	−2.8030E − 04	−2.7927E − 04	99.6
4	−3.6239E − 01	−3.6511E − 01	−2.7273E − 03	−2.7391E − 03	100.4
		(b) 0.1% change of b_3 to b_5 (lower boundary)			
1	−7.7013E − 01	−7.6984E − 01	2.9207E − 04	2.9245E − 04	100.1
2	−1.0042E + 00	−1.0043E + 00	−5.2670E − 06	−5.2913E − 06	100.5
3	−1.0530E + 00	−1.0530E + 00	−5.8413E − 05	−5.8409E − 05	100.0
4	−3.6239E − 01	−3.6164E − 01	7.4899E − 04	7.4955E − 04	100.1
		(c) 0.1% change of b_6 to b_{48} (thickness change)			
1	−7.7013E − 01	−7.7036E − 01	−2.3032E − 04	−2.3026E − 04	100.0
2	−1.0042E + 00	−1.0042E + 00	4.2592E − 06	4.2407E − 06	99.6
3	−1.0530E + 00	−1.0529E + 00	5.2783E − 05	5.2282E − 05	99.1
4	−3.6239E − 01	−3.6302E − 01	−6.3713E − 04	−6.3663E − 04	99.9

[a] Case 1: upper principal stress constraint due to inertia load.
Case 2: lower principal stress constraint due to inertia load.
Case 3: upper principal stress constraint due to firing load.
Case 4: lower principal stress constraint due to firing load.

SIMPLE BOX

Consider the simple box of Section 4.1.4. Its variational equilibrium equation of Eq. (4.1.28) is repeated here as

$$
\begin{aligned}
a_{u,\Omega}(z, \bar{z}) &= \sum_{l=1}^{5} \iint_{\Omega^l} \sum_{i,j=1}^{2} \sigma^{ij}(z)\varepsilon^{ij}(\bar{z})\, d\Omega \\
&= \int_{\Gamma^2} \sum_{i=1}^{2} T^i \bar{z}^i \, d\Gamma \\
&= l_{u,\Omega}(\bar{z}) \qquad \text{for all} \quad \bar{z} \in Z
\end{aligned} \tag{4.2.34}
$$

Consider the von Mises yield stress functional, averaged over finite element $\Omega_p \subset \Omega^q$, as

$$
\psi_p = \iint_{\Omega^q} g(\sigma(z)) m_p \, d\Omega \tag{4.2.35}
$$

where $g = \sigma_y$ is the von Mises yield stress defined in Eq. (3.3.179) and m_p is a characteristic function on finite element Ω_p.

In this problem, a perturbation of the design variable $[b_1 \;\; b_2 \;\; b_3]^{\mathrm{T}}$ will move the externally loaded boundary. Assume that traction is constant along Γ^2 and independent of position. By taking the variation of the right side of Eq. (4.2.34),

$$
\begin{aligned}
[l_{u,\Omega}(\bar{z})]' = {} & \int_{\Gamma^2} \left[\sum_{i=1}^{2} T^i \bar{z}^{i'} \right] d\Gamma + \int_{\Gamma^2} \left[\sum_{i=1}^{2} \nabla (T^i \bar{z}^i)^{\mathrm{T}} n \right] (V^{\mathrm{T}} n)\, d\Gamma \\
& + \left(\sum_{i=1}^{2} T^i \bar{z}^i V_T(p_1) \right) \Bigg|_{\Omega^2} + \left(\sum_{i=1}^{2} T^i \bar{z}^i V_T(p_2) \right) \Bigg|_{\Omega^3} \\
& + \left(\sum_{i=1}^{2} T^i \bar{z}^i V_T(p_1) \right) \Bigg|_{\Omega^5} + \left(\sum_{i=1}^{2} T^i \bar{z}^i V_T(p_2) \right) \Bigg|_{\Omega^5}
\end{aligned} \tag{4.2.36}
$$

The last four terms on the right side of Eq. (4.2.36) denote corner terms due to movement of points p_1 and p_2 [59].

Define the adjoint equation as

$$
a_{u,\Omega}(\lambda, \bar{\lambda}) = \iint_{\Omega^q} \left[\sum_{i,j=1}^{2} g_{\sigma^{ij}}(z) \sigma^{ij}(\bar{\lambda}) \right] m_p \, d\Omega \qquad \text{for all} \quad \bar{\lambda} \in Z \tag{4.2.37}
$$

Employing the idea of calculating first variations in Section 4.2.1, the variation of Eq. (4.2.35) can be obtained by adding contributions from each component and the four corner terms from Eq. (4.2.36). For each component, the result of Eq. (3.3.175) can be used to obtain the design sensitivity formula

$$
\begin{aligned}
\psi'_p = {} & \sum_{l=1}^{5} \iint_{\Omega^l} \sum_{i,j=1}^{2} [\sigma^{ij}(z)(\nabla \lambda^{i^T} V_j) + \sigma^{ij}(\lambda)(\nabla z^{i^T} V_j)]\, d\Omega \\
& - \sum_{l=1}^{5} \iint_{\Omega^l} \left[\sum_{i,j=1}^{2} \sigma^{ij}(z)\varepsilon^{ij}(\lambda) \right] \operatorname{div} V d\Omega \\
& - \iint_{\Omega^q} \sum_{i,j=1}^{2} \left[\sum_{k,l=1}^{2} g_{\sigma^{ij}}(z) C^{ijkl}(\nabla z^{k^T} V_l) \right] m_p\, d\Omega \\
& + \iint_{\Omega^q} g \operatorname{div} V m_p\, d\Omega - \iint_{\Omega^q} g m_p\, d\Omega \iint_{\Omega^q} m_p \operatorname{div} V d\Omega \\
& - \int_{\Gamma^2} \sum_{i=1}^{2} T^i(\nabla \lambda^{i^T} V)\, d\Gamma + \int_{\Gamma^2} \left[\sum_{i=1}^{2} \nabla (T^i \lambda^i)^{\mathrm{T}} n \right] (V^{\mathrm{T}} n)\, d\Gamma \\
& + \left(\sum_{i=1}^{2} T^i \lambda^i V_T(p_1) \right) \Bigg|_{\Omega^2} + \left(\sum_{i=1}^{2} T^i \lambda^i V_T(p_2) \right) \Bigg|_{\Omega^3} \\
& + \left(\sum_{i=1}^{2} T^i \lambda^i V_T(p_1) \right) \Bigg|_{\Omega^5} + \left(\sum_{i=1}^{2} T^i \lambda^i V_T(p_2) \right) \Bigg|_{\Omega^5}
\end{aligned} \tag{4.2.38}
$$

In Eq. (4.2.38), since shape design variables are given as $[b_1 \;\; b_2 \;\; b_3]^{\mathrm{T}}$, the velocity field can be assumed to be linear on each plate. Thus, div V is constant on each plate.

For numerical calculations, an eight-noded ISP element model with 320 elements, 993 nodes, and 1886 degrees of freedom is used. For numerical data, Young's modulus and Poisson's ratio are 1.0×10^7 psi and 0.316, respectively. Dimensions of the structure at the nominal design are $b_1 = b_2 = b_3 = 8$ in., and the thickness of each plate is 0.1 in. The uniform external load is 4.77 lb/in.

In Table 4.2.8, sensitivity accuracy results are given for typical elements due to a 3% change in design variables. Results of Table 4.2.8(a) and (b) are for 3% changes in b_1 and b_3, respectively. Due to symmetry, results of one side only are given in Table 4.2.8. Results given in Table 4.2.8 show excellent agreement between predictions ψ'_p and actual changes $\Delta\psi_p$, except in elements 261, 266, 271, 306, 311, and 316 of Table 4.2.8(a). However, those elements are in the low-stress region, and $\Delta\psi_p$ are small compared to others. Since they are differences between approximate stresses, they are not accurate. The boundary method that was applied to the plane stress interface problem was tested with the same simple box problem, with unacceptable results.

Table 4.2.8
Domain Method for Simple Box Problem

Element number	ψ_p^1	ψ_p^2	$\Delta\psi_p$	ψ_p'	$(\psi_p'/\Delta\psi_p \times 100)\%$
		(a) 3% perturbation of length b_1 [a]			
1	100.63443	103.78530	3.15087	3.07549	97.6
6	39.87734	42.20731	2.32996	2.29028	98.3
11	38.09229	40.50634	2.41405	2.37384	98.3
16	84.75172	86.95996	2.20825	2.14499	97.1
21	29.10066	31.38476	2.28411	2.25004	98.5
26	47.77615	49.51865	1.74250	1.72446	99.0
31	47.77615	49.51865	1.74250	1.72446	99.0
36	21.28009	22.79503	1.51493	1.50010	99.0
41	52.16973	53.22842	1.05869	1.04999	99.2
46	26.27852	27.32058	1.04206	1.04612	100.4
51	23.75135	24.28852	0.53717	0.53220	99.1
56	43.70809	44.61852	0.91043	0.92291	101.4
61	31.15718	31.30541	0.14823	0.15058	101.6
66	49.20760	51.70151	2.49391	2.44280	98.0
71	49.20760	51.70151	2.49391	2.44280	98.0
76	26.96837	29.18948	2.22111	2.18423	98.3
81	65.59970	67.47185	1.87214	1.83760	98.2
86	31.42032	33.59012	2.16979	2.13633	98.5
91	29.11188	30.98445	1.87257	1.85249	98.9
96	59.01335	60.55661	1.54326	1.52421	98.8
101	17.77022	19.23274	1.46252	1.44528	98.8
106	34.66051	35.85638	1.19587	1.20787	101.0
111	34.66051	35.85638	1.19587	1.20787	101.0
116	20.66130	20.81388	0.15259	0.14922	97.8
121	21.39204	21.86005	0.46801	0.47243	100.9
126	31.19434	31.31165	0.11731	0.12479	106.4
131	127.95324	130.60632	2.65307	2.55823	96.4
136	127.43242	130.94177	3.50974	3.43456	97.9
141	115.99465	118.74030	2.74565	2.68257	97.7
146	117.30361	118.88574	1.58213	1.53328	96.9
151	99.14657	101.16379	2.01721	1.97823	98.1
156	104.97155	106.62767	1.65611	1.61890	97.8
161	84.56467	85.55652	0.99185	0.94271	95.0
166	84.06349	85.20942	1.14593	1.11992	97.7
171	81.94871	82.70473	0.75602	0.72246	95.6
176	56.29105	57.23964	0.94859	0.94555	96.7
181	67.85988	68.09515	0.23527	0.22166	94.2
186	55.58785	55.81064	0.22279	0.29350	131.7
191	45.66123	45.77325	0.11203	0.11082	98.9
261	52.27702	52.27824	0.00122	0.00657	537.8
266	45.12722	45.11601	−0.01121	−0.02510	223.8
271	29.52699	29.53403	0.00704	0.01035	147.0
276	36.62302	36.66663	0.04361	0.04665	107.0
281	45.66577	45.78814	0.12237	0.12677	103.6
286	20.18210	20.19824	0.01614	0.01924	119.2

(*continues*)

Table 4.2.8 *(continued)*

Domain Method for Simple Box Problem

Element number	ψ_p^1	ψ_p^2	$\Delta\psi_p$	ψ_p'	$(\psi_p'/\Delta\psi_p \times 100)\%$
291	35.13625	35.16403	0.02779	0.02644	95.1
296	34.96003	35.17924	0.21920	0.23069	105.2
301	31.97706	32.01814	0.04108	0.04544	110.6
306	45.12722	45.11601	−0.01121	−0.02510	223.8
311	29.52699	29.53403	0.00704	0.01035	147.0
316	54.32027	54.31034	−0.00992	−0.00222	22.4
		(b) 3% perturbation of height b_3[a]			
1	100.63443	97.71343	−2.92100	−3.02560	103.6
6	39.87734	38.60680	−1.27054	−1.31552	103.5
11	38.09229	36.88886	−1.20343	−1.24584	103.5
16	84.75172	82.13030	−2.62142	−2.71628	103.6
21	29.10066	28.22262	−0.87803	−0.90871	103.5
26	47.77615	46.21547	−1.56068	−1.61636	103.6
31	47.77615	46.21547	−1.56068	−1.61636	103.6
36	21.28009	20.67423	−0.60586	−0.62693	103.5
41	52.16973	50.35392	−1.81580	−1.87975	103.5
46	26.27852	25.44055	−0.83797	−0.86795	103.6
51	23.75135	22.92610	−0.82525	−0.85558	103.7
56	43.70809	42.24665	−1.46144	−1.51226	103.5
61	31.15718	29.77414	−1.38304	−1.42616	103.9
66	49.20760	47.31834	−1.88926	−1.95116	103.3
71	49.20760	47.31834	−1.88926	−1.95115	103.3
76	26.96837	25.93788	−1.03049	−1.06359	103.2
81	65.59970	63.06451	−2.53519	−2.61931	103.3
86	31.42032	30.21688	−1.20345	−1.24241	103.2
91	29.11188	27.99637	−1.11551	−1.15172	103.2
96	59.01335	56.72553	−2.28782	−2.36378	103.3
101	17.77022	17.10474	−0.66548	−0.68615	103.1
106	34.66051	33.31319	−1.34731	−1.39214	103.3
111	34.66051	33.31319	−1.34741	−1.39214	103.3
116	20.66130	19.83577	−0.82552	−0.85490	103.6
121	21.39204	20.56917	−0.82287	−0.84901	103.2
126	31.19434	29.91584	−1.27850	−1.32590	103.7
131	127.95324	124.23535	−3.71789	−3.83449	103.1
136	127.43202	122.52440	−4.90762	−5.06634	103.2
141	115.99465	112.13318	−3.86147	−3.98045	103.1
146	117.30361	114.56425	−2.73937	−2.82433	103.1
151	99.14657	95.92302	−3.22355	−3.31679	102.9
156	104.97155	102.10465	−2.86690	−2.94150	102.6
161	84.56467	82.59761	−1.96706	−2.04206	103.8
166	84.06349	81.71003	−2.35346	−2.41896	102.8
171	81.94871	80.05344	−1.89527	−1.95133	103.0
176	56.29105	54.29555	−1.99550	−2.05872	103.2

Table 4.2.8 (*continued*)
Domain Method for Simple Box Problem

Element number	ψ_p^1	ψ_p^2	$\Delta\psi_p$	ψ'_p	$(\psi'_p/\Delta\psi_p \times 100)\%$
181	67.85988	66.31273	−1.54715	−1.59164	102.9
186	55.58785	54.55671	−1.03114	−1.07550	104.3
191	45.66123	44.39744	−1.26379	−1.30200	103.0
261	52.27702	50.89411	−1.38291	−1.42855	103.3
266	45.12722	44.95884	−0.16838	−0.18060	107.3
271	29.52699	28.63060	−0.89639	−0.93170	103.9
276	36.62302	36.21113	−0.41188	−0.42241	102.6
281	45.66577	45.01216	−0.65361	−0.70665	108.1
286	20.18210	19.53853	−0.64357	−0.67564	105.0
291	35.13625	35.28274	0.14750	0.14909	101.1
296	34.96003	33.27873	−1.68131	−1.74729	103.9
301	31.97706	31.34971	−0.62736	−0.64641	103.0
306	45.12722	44.95884	−0.16838	−0.18060	107.3
311	29.52699	28.63060	−0.89639	−0.93170	103.9
316	54.32027	53.05260	−1.26766	−1.31371	103.6

[a] Top, element number 1–64; bottom, 65–128; sides, 129–256; end, 257–320.

TRUSS–BEAM–PLATE

Consider the truss–beam–plate of Section 4.1.4, with the variational equilibrium equation of Eq. (4.1.57) repeated here as

$$\begin{aligned}
a_{u,\Omega}(z,\bar{z}) = \sum_{i=1}^{n+1}\sum_{j=1}^{m+1} \iint_{\Omega_1^{ij}} & \hat{D}^{ij}(t)[(w_{xx}^{ij} + \nu w_{yy}^{ij})\bar{w}_{xx}^{ij} \\
& + (w_{yy}^{ij} + \nu w_{xx}^{ij})\bar{w}_{yy}^{ij} + 2(1-\nu)w_{xy}^{ij}\bar{w}_{xy}^{ij}]\, d\Omega_1 \\
& + \sum_{i=1}^{n+1}\sum_{j=1}^{m} \int_{\Omega_2^{ij}} [E\tilde{I}^{ij}\tilde{v}_{xx}^{ij}\bar{\tilde{v}}_{xx}^{ij} + G\tilde{J}^{ij}\tilde{\theta}_x^{ij}\bar{\tilde{\theta}}_x^{ij}]\, d\Omega_2 \\
& + \sum_{i=1}^{n}\sum_{j=1}^{m+1} \int_{\Omega_3^{ij}} [E\hat{I}^{ij}\hat{v}_{yy}^{ij}\bar{\hat{v}}_{yy}^{ij} + G\hat{J}^{ij}\hat{\theta}_y^{ij}\bar{\hat{\theta}}_y^{ij}]\, d\Omega_3 \\
& + q_k^{11^{\mathrm{T}}}K(A_l^{11})\bar{q}_k^{11} + q_k^{1m^{\mathrm{T}}}K(A_l^{1m})\bar{q}_k^{1m} \\
& + q_k^{n1^{\mathrm{T}}}K(A_l^{n1})\bar{q}_k^{n1} + q_k^{nm^{\mathrm{T}}}K(A_l^{nm})\bar{q}_k^{nm} \\
= \sum_{i=1}^{n+1}\sum_{j=1}^{m+1} \iint_{\Omega_1^{ij}} & f^{ij}\bar{w}^{ij}\, d\Omega_1 = l_{u,\Omega}(\bar{z}) \qquad \text{for all} \quad \bar{z} \in Z \qquad (4.2.39)
\end{aligned}$$

To obtain a design sensitivity formula using Eq. (4.2.12) expressions for $l'_{\delta u}(\lambda)$, $a'_{\delta u}(z,\lambda)$, $l'_V(\lambda)$, and $a'_V(z,\lambda)$ must be obtained. Expressions for $l'_{\delta u}(\lambda)$ and

$a'_{\delta u}(z, \lambda)$ can be found in Chapters 1 and 2 for truss, beam, and plate components. For $l'_V(\lambda)$ and $a'_V(z, \lambda)$, Eqs. (4.2.4) and (4.2.6) can be used. Using Eqs. (1.2.19), (2.2.1), and (4.2.39),

$$
\begin{aligned}
a'_{\delta u}(z, \bar{z}) = \sum_{i=1}^{n+1} \sum_{j=1}^{m+1} \iint_{\Omega_1^{ij}} & [(w_{xx}^{ij} + \nu w_{yy}^{ij})\bar{w}_{xx}^{ij} + (w_{yy}^{ij} + \nu w_{xx}^{ij})\bar{w}_{yy}^{ij} \\
& + 2(1 - \nu) w_{xy}^{ij} \bar{w}_{xy}^{ij}](\hat{D}^{ij}(t))_{t^{ij}}\, \delta t^{ij}\, d\Omega_1 \\
& + \sum_{i=1}^{n} \sum_{j=1}^{m+1} \int_{\Omega_2^{ij}} [\tilde{v}_{xx}^{ij} \bar{\tilde{v}}_{xx}^{ij}(E\tilde{I}^{ij})_{\tilde{b}^{ij}} + \theta_x^{ij} \bar{\tilde{\theta}}_x^{ij}(G\tilde{J}^{ij})_{\tilde{b}^{ij}}]\, \delta \tilde{b}^{ij}\, d\Omega_2 \\
& + \sum_{i=1}^{n+1} \sum_{j=1}^{m} \int_{\Omega_3^{ij}} [\hat{v}_{yy}^{ij} \bar{\hat{v}}_{yy}^{ij}(E\hat{I}^{ij})_{\hat{b}^{ij}} + \hat{\theta}_y^{ij} \bar{\hat{\theta}}_y^{ij}(G\hat{J}^{ij})_{\hat{b}^{ij}}]\, \delta \hat{b}^{ij}\, d\Omega_3 \\
& + [q_k^{11^T} K(A_l^{11}) \bar{q}_k^{11}]_{A_l^{11}}\, \delta A_l^{11} + [q_k^{1m^T} K(A_l^{1m}) \bar{q}_k^{1m}]_{A_l^{1m}}\, \delta A_l^{1m} \\
& + [q_k^{n1^T} K(A_k^{n1}) \bar{q}_k^{n1}]_{A_l^{n1}}\, \delta A_l^{n1} + [q_k^{nm^T} K(A_l^{nm}) \bar{q}_k^{nm}]_{A_l^{nm}}\, \delta A_l^{nm}
\end{aligned}
\tag{4.2.40}
$$

where $\tilde{b}^{ij} = [\tilde{d}^{ij}\ \tilde{h}^{ij}]^T$ and $\hat{b}^{ij} = [\hat{d}^{ij}\ \hat{h}^{ij}]^T$. Using Eqs. (1.2.19), (2.2.2), and (4.2.39),

$$
l'_{\delta u}(\bar{z}) = 0 \tag{4.2.41}
$$

Also, using Eqs. (4.2.3), (4.2.4), and (4.2.39),

$$
\begin{aligned}
a'_V(z, \bar{z}) = \sum_{i=1}^{n+1} \sum_{j=1}^{m+1} \iint_{\Omega_1^{ij}} & \{-\hat{D}^{ij}(t)[(w_{xx}^{ij} + \nu w_{yy}^{ij})(\nabla \bar{w}^{ij^T} V)_{xx} \\
& + (w_{yy}^{ij} + \nu w_{xx}^{ij})(\nabla \bar{w}^{ij^T} V)_{yy} + 2(1 - \nu) w_{xy}^{ij} (\nabla \bar{w}^{ij^T} V)_{xy} \\
& + (\bar{w}_{xx}^{ij} + \nu \bar{w}_{yy}^{ij})(\nabla w^{ij^T} V)_{xx} + (\bar{w}_{yy}^{ij} + \nu \bar{w}_{xx}^{ij})(\nabla w^{ij^T} V)_{yy} \\
& + 2(1 - \nu) \bar{w}_{xy}^{ij} (\nabla w^{ij^T} V)_{xy}] \\
& + \operatorname{div} [\hat{D}^{ij}(t)((w_{xx}^{ij} + \nu w_{yy}^{ij})\bar{w}_{xx}^{ij} \\
& + (w_{yy}^{ij} + \nu w_{xx}^{ij})\bar{w}_{yy}^{ij} + 2(1 - \nu) w_{xy}^{ij} \bar{w}_{xy}^{ij}) V]\}\, d\Omega_1 \\
+ \sum_{i=1}^{n+1} \sum_{j=1}^{m} \int_{\Omega_2^{ij}} & \{-E\tilde{I}^{ij}[\tilde{v}_{xx}^{ij}(\bar{\tilde{v}}_x^{ij} V)_{xx} + \bar{\tilde{v}}_{xx}^{ij}(\tilde{v}_x^{ij} V)_{xx}] \\
& + (E\tilde{I}^{ij} \tilde{v}_{xx}^{ij} \bar{\tilde{v}}_{xx}^{ij} V)_x - G\tilde{J}^{ij}[\tilde{\theta}_x^{ij}(\bar{\tilde{\theta}}_x^{ij} V)_x \\
& + \bar{\tilde{\theta}}_x^{ij}(\tilde{\theta}_x^{ij} V)_x] + (G\tilde{J}^{ij} \tilde{\theta}_x^{ij} \bar{\tilde{\theta}}_x^{ij} V)_x\}\, d\Omega_2 \\
+ \sum_{i=1}^{n} \sum_{j=1}^{m+1} \int_{\Omega_3^{ij}} & \{-E\hat{I}^{ij}[\hat{v}_{yy}^{ij}(\bar{\hat{v}}_y^{ij} V)_{yy} + \bar{\hat{v}}_{yy}^{ij}(\hat{v}_y^{ij} V)_{yy} \\
& + (E\hat{I}^{ij} \hat{v}_{yy}^{ij} \bar{\hat{v}}_{yy}^{ij} V)_y - G\hat{J}^{ij}[\hat{\theta}_y^{ij}(\bar{\hat{\theta}}_y^{ij} V)_y \\
& + \bar{\hat{\theta}}_y^{ij}(\hat{\theta}_y^{ij} V)_y] + (G\hat{J}^{ij} \hat{\theta}_y^{ij} \bar{\hat{\theta}}_y^{ij} V)_y\}\, d\Omega_3
\end{aligned}
\tag{4.2.42}
$$

If the conventional design variables are constant for each component, Eq. (4.2.42) is simplified to

$$
\begin{aligned}
a'_V(z, \bar{z}) = -\sum_{i=1}^{n+1}\sum_{j=1}^{m+1} \iint_{\Omega_1^{ij}} \hat{D}^{ij}(t)\{4(w_{xx}^{ij}\bar{w}_{xx}^{ij}V_x^x + w_{yy}^{ij}\bar{w}_{yy}^{ij}V_y^y) \\
+ [\nu(w_{xx}^{ij}\bar{w}_{yy}^{ij} + w_{yy}^{ij}w_{xx}^{ij}) - (w_{xx}^{ij}\bar{w}_{xx}^{ij} + w_{yy}^{ij}\bar{w}_{yy}^{ij}) \\
+ 2(1-\nu)w_{xy}^{ij}\bar{w}_{xy}^{ij}](V_x^x + V_y^y) \\
+ [w_x^{ij}\bar{w}_{xx}^{ij} + w_{xx}^{ij}\bar{w}_x^{ij} + \nu(w_x^{ij}\bar{w}_{yy}^{ij} + w_{yy}^{ij}\bar{w}_x^{ij})]V_{xx}^x \\
+ [w_y^{ij}\bar{w}_{yy}^{ij} + w_{yy}^{ij}\bar{w}_y^{ij} \\
+ \nu(w_y^{ij}\bar{w}_{xx}^{ij} + w_{xx}^{ij}\bar{w}_y^{ij})]V_{yy}^y\}\, d\Omega_1 \\
-\sum_{i=1}^{n+1}\sum_{j=1}^{m} \int_{\Omega_2^{ij}} \{E\tilde{I}^{ij}[3\tilde{v}_{xx}^{ij}\bar{\tilde{v}}_{xx}^{ij}V_x + (\tilde{v}_x^{ij}\bar{\tilde{v}}_{xx}^{ij} \\
+ \tilde{v}_{xx}^{ij}\bar{\tilde{v}}_x^{ij})V_{xx}] + G\tilde{J}^{ij}\tilde{\theta}_x^{ij}\bar{\tilde{\theta}}_x^{ij}V_x\}\, d\Omega_2 \\
-\sum_{i=1}^{n}\sum_{j=1}^{m+1} \int_{\Omega_3^{ij}} \{E\hat{I}^{ij}[3\hat{v}_{yy}^{ij}\bar{\hat{v}}_{yy}^{ij}V_y + (\hat{v}_y^{ij}\bar{\hat{v}}_{yy}^{ij} \\
+ \hat{v}_{yy}^{ij}\bar{\hat{v}}_y^{ij})V_{yy}] + G\hat{J}^{ij}\hat{\theta}_y^{ij}\bar{\hat{\theta}}_y^{ij}V_y\}\, d\Omega_3
\end{aligned}
\qquad (4.2.43)
$$

where $V = [V^x \quad V^y]^{\mathrm{T}}$ on the plate component and V on each beam is design velocity. Using Eqs. (4.2.5), (4.2.6), and (4.2.39),

$$
\begin{aligned}
l'_V(\bar{z}) &= \sum_{i=1}^{n+1}\sum_{j=1}^{m+1} \iint_{\Omega_1^{ij}} [-f^{ij}(\nabla\bar{w}^{ij\mathrm{T}}V) + \operatorname{div}(f^{ij}\bar{w}^{ij}V)]\, d\Omega_1 \\
&= \sum_{i=1}^{n+1}\sum_{j=1}^{m+1} \iint_{\Omega_1^{ij}} (\bar{w}^{ij}\nabla f^{ij\mathrm{T}}V + f^{ij}\bar{w}^{ij}\operatorname{div} V)\, d\Omega_1
\end{aligned}
\qquad (4.2.44)
$$

Consider first the compliance functional for the structure,

$$
\psi_1 = \sum_{i=1}^{n+1}\sum_{j=1}^{m+1} \iint_{\Omega_1^{ij}} f^{ij}w^{ij}\, d\Omega_1 \qquad (4.2.45)
$$

Since $f^{ij\prime} = 0$, Eq. (4.2.45) can be treated as the functional form of Eq. (4.2.8), so the adjoint equation is, from Eq. (4.2.10),

$$
a_{u,\Omega}(\lambda, \bar{\lambda}) = \sum_{i=1}^{n+1}\sum_{j=1}^{m+1} \iint_{\Omega_1^{ij}} f^{ij}\bar{\gamma}^{ij}\, d\Omega_1 \qquad \text{for all} \quad \bar{\lambda} \in Z \qquad (4.2.46)
$$

where

$$
\lambda = [\gamma^{ij} \quad \tilde{\eta}^{ij} \quad \tilde{\xi}^{ij} \quad \hat{\eta}^{ij} \quad \hat{\xi}^{ij} \quad p_k^{11} \quad p_k^{n1} \quad p_k^{1m} \quad p_k^{nm}]^{\mathrm{T}}
$$

In this special case, $\lambda = z$, and from Eq. (4.2.12)

$$\psi_1' = l'_{\delta u}(z) - a'_{\delta u}(z, z) + \sum_{i=1}^{n+1} \sum_{j=1}^{m+1} \iint_{\Omega_1^{ij}} [-f^{ij}(\nabla w^{ij\mathrm{T}} V) + \mathrm{div}(f^{ij} w^{ij} V)]\, d\Omega_1$$
$$+ l'_V(z) - a'_V(z, z)$$
$$= l'_{\delta u}(z) - a'_{\delta u}(z, z) + 2l'_V(z) - a'_V(z, z) \qquad (4.2.47)$$

where Eqs. (4.2.40), (4.2.41), (4.2.43), and (4.2.44) provide the form of terms in Eq. (4.2.47).

Next, consider displacement at a discrete point $\hat{x} \in \Omega_1^{ij}$, written as

$$\psi_2 = \iint_{\Omega_1^{ij}} \hat{\delta}(x - \hat{x}) w^{ij}\, d\Omega_1 \qquad (4.2.48)$$

where point $\hat{x}$ moves as the shape is modified. Using Eq. (4.2.10), the adjoint equation is obtained as

$$a_{u,\Omega}(\lambda, \bar{\lambda}) = \iint_{\Omega_1^{ij}} \hat{\delta}(x - \hat{x}) \bar{\gamma}^{ij}\, d\Omega_1 \qquad \text{for all} \quad \bar{\lambda} \in Z \qquad (4.2.49)$$

From Eqs. (4.2.12) and (3.3.139),

$$\psi_2' = l'_{\delta u}(\lambda^{(2)}) - a'_{\delta u}(z, \lambda^{(2)}) + l'_V(\lambda^{(2)}) - a'_V(z, \lambda^{(2)}) \qquad (4.2.50)$$

where Eqs. (4.2.40), (4.2.41), (4.2.43), and (4.2.44) provide the form of terms in Eq. (4.2.50) and $\lambda^{(2)}$ is the solution of Eq. (4.2.49).

Finally, consider the mean stress functional over a finite element $\Omega_p \subset \Omega_1^{ij}$ of the plate component,

$$\psi_3 = \iint_{\Omega_1^{ij}} g(t^{ij}, w_{xx}^{ij}, w_{xy}^{ij}, w_{yy}^{ij}) m_p\, d\Omega_1 = \frac{\iint_{\Omega_p} g(t^{ij}, w_{kl}^{ij})\, d\Omega_1}{\iint_{\Omega_p} d\Omega_1} \qquad (4.2.51)$$

where g might involve principal stresses, von Mises failure criterion, or some other material failure criteria. For simplicity of notation, $w_{xx}^{ij} \equiv w_{11}^{ij}$, $w_{xy}^{ij} \equiv w_{12}^{ij}$, and $w_{yy}^{ij} \equiv w_{22}^{ij}$ are used. As before, m_p is a characteristic function defined on Ω_p.

Taking the variation of Eq. (4.2.51), using Eq. (3.2.37),

$$\psi_3' = \iint_{\Omega_1^{ij}} g_{t^{ij}} m_p\, \delta t^{ij}\, d\Omega_1 + \iint_{\Omega_1^{ij}} \sum_{k,l=1}^{2} g_{w_{kl}^{ij}} [\dot{w}_{kl}^{ij} - (\nabla w^{ij\mathrm{T}} V)_{kl}] m_p\, d\Omega_1$$
$$+ \iint_{\Omega_1^{ij}} \mathrm{div}(gV) m_p\, d\Omega_1 - \iint_{\Omega_1^{ij}} g m_p\, d\Omega_1 \iint_{\Omega_1^{ij}} m_p\, \mathrm{div}\, V\, d\Omega_1 \qquad (4.2.52)$$

As in the general derivation of the adjoint equation of Eq. (4.2.10), the adjoint equation may be defined by replacing $\dot{w}^{ij}$ in the integrand of Eq. (4.2.52) by $\bar{\lambda}$ to define a load functional for the adjoint equation, obtaining

$$a_{u,\Omega}(\lambda, \bar{\lambda}) = \iint_{\Omega_1^{ij}} \left[\sum_{k,l=1}^{2} g_{w_{kl}^{ij}} \bar{\gamma}_{kl}^{ij} \right] m_p \, d\Omega_1 \qquad \text{for all} \quad \bar{\lambda} \in Z \tag{4.2.53}$$

Proceeding as in the derivation of Eq. (4.2.12), the design sensitivity formula is obtained as

$$\begin{aligned} \psi_3' = {} & l_{\delta u}'(\lambda^{(3)}) - a_{\delta u}'(z, \lambda^{(3)}) + \iint_{\Omega_1^{ij}} g_{t^{ij}} m_p \, \delta t^{ij} \, d\Omega_1 \\ & + l_V'(\lambda^{(3)}) - a_V'(z, \lambda^{(3)}) - \iint_{\Omega_1^{ij}} \left[\sum_{k,l=1}^{2} g_{w_{kl}^{ij}} (\nabla w^{ij\,T} V)_{kl} \right] m_p \, d\Omega_1 \\ & + \iint_{\Omega_1^{ij}} \operatorname{div}(gV) m_p \, d\Omega_1 - \iint_{\Omega_1^{ij}} g m_p \, d\Omega_1 \iint_{\Omega_1^{ij}} m_p \operatorname{div} V \, d\Omega_1 \end{aligned} \tag{4.2.54}$$

where Eqs. (4.2.40), (4.2.41), (4.2.43), and (4.2.44) provide the form of terms in Eq. (4.2.54) and $\lambda^{(3)}$ is the solution of Eq. (4.2.53). The last two terms in Eq. (4.2.54) are due to the movement of Ω_p.

For numerical calculations, conventional and shape design sensitivity calculations are carried out separately. For plate components, 12 degrees-of-freedom nonconforming rectangular elements [7] are used. For beam components, hermite cubic shape functions are used. The finite element model used for conventional design sensitivity calculation is shown in Fig. 4.2.6. A total of 196 finite elements with 363 degrees of freedom are used to model the built-up structure, including 100 rectangular plate elements, 80 beam elements, and 16 truss elements. The 196 elements are linked to six kinds of independent conventional design variables, such as thickness of plate components, height and width of longitudinal beam components, height and width of transverse beam components, and cross-sectional area of truss components.

For numerical data, Young's modulus and Poisson's ratio are 3.0×10^7 psi and 0.3, respectively. The overall dimensions are $L_x \times L_y = 15$ in. $\times$ 15 in. Beam components are located so that the spaces between them are $a_i = b_j = 3$ in. $(i, j = 1, 2, 3, 4)$. Dimensions of the built-up structure at the nominal design are as follows: uniform thickness $t = 0.1$ in. for plate components, uniform height $h = 0.5$ in. and width $d = 0.15$ in. for beam components, and length $l = 5.364$ in. and cross-sectional area $A = 0.1$ in.^{2} for truss components. A uniform distributed load $f = 0.1$ lb/in.^{2} is applied on the plate components, and mass density for the entire structure is taken as $\rho = 0.1$ lb m/in.^{3} for the eigenvalue problem.

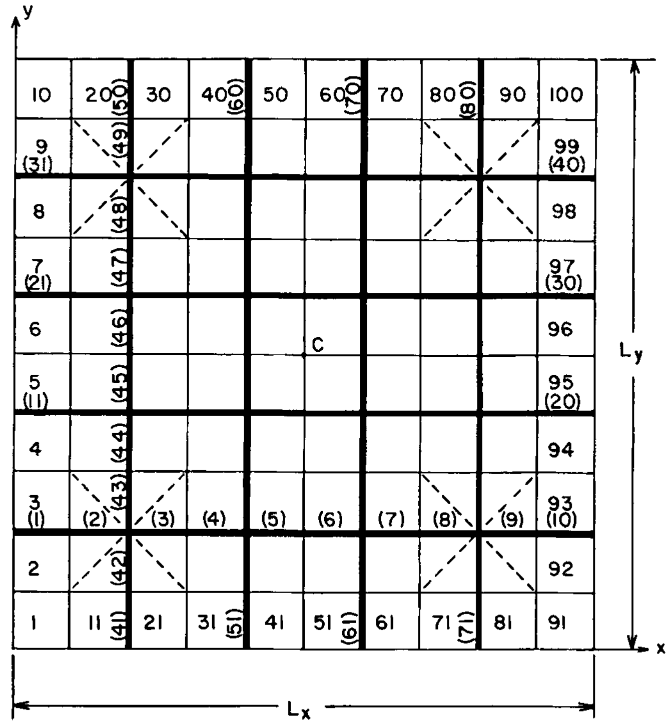

Fig. 4.2.6 Finite element model of a truss-beam-plate built-up structure for conventional design variation.

In Table 4.2.9, sensitivity accuracy results are given for several functionals, with a 5% uniform change in all conventional design variables except the cross-sectional areas of truss components. Sensitivity results of von Mises yield stress

$$g(\sigma) = (\sigma_{xx}^2 + \sigma_{yy}^2 + 3\sigma_{xy}^2 - \sigma_{xx}\sigma_{yy})^{1/2} \tag{4.2.55}$$

for plate components and normal stresses σ_{xx} and σ_{yy} for longitudinal and transverse beam components, respectively, are given in Table 4.2.9. Due to symmetry, sensitivity results for only one quarter of the structure is given in Table 4.2.9. Results given in Table 4.2.9 show good agreement between predictions ψ_p' and actual changes $\Delta\psi_p$.

During numerical calculations it was found that the finite element model of Fig. 4.2.6, which was used for conventional design sensitivity calculation, was not adequate for shape design sensitivity calculations, because of the coarse grid. A finer grid finite element model for shape design sensitivity calculation is shown in Fig. 4.2.7. Only one quarter of the entire structure is

Table 4.2.9

Conventional Design Sensitivity of Truss–Beam–Plate Built-Up Structure

Functional	Element number	ψ_p^1	$\Delta\psi_p$	ψ_p'	$(\psi_p'/\Delta\psi_p \times 100)\%$
Displacement	C	0.4775E − 03	−0.8052E − 04	0.9071E − 04	112.7
Stress on plate element	1	0.1484E + 02	−0.7100E + 00	−0.6750E + 00	95.1
	2	0.5829E + 02	−0.5980E + 01	−0.6780E + 01	113.4
	3	0.5263E + 02	−0.5220E + 01	−0.5810E + 01	111.3
	4	0.5256E + 02	−0.5760E + 01	−0.6320E + 01	109.7
	5	0.8497E + 02	−0.1028E + 02	−0.1126E + 02	109.5
	11	0.5829E + 02	−0.5980E + 01	−0.6780E + 01	113.4
	12	0.6780E + 02	−0.7870E + 01	−0.8630E + 01	109.7
	13	0.5827E + 02	−0.6720E + 01	−0.7580E + 01	112.8
	14	0.5269E + 02	−0.6240E + 01	−0.6830E + 01	109.5
	15	0.7658E + 02	−0.9360E + 01	−0.1034E + 02	110.5
	21	0.5263E + 02	−0.5220E + 01	−0.5810E + 01	111.3
	22	0.5827E + 02	−0.6720E + 01	−0.7580E + 01	112.8
	23	0.5450E + 02	−0.6690E + 01	−0.7300E + 01	109.1
	24	0.5850E + 02	−0.6990E + 01	−0.8060E + 01	115.3
	25	0.6155E + 02	−0.7740E + 01	−0.8500E + 01	109.8
	31	0.5256E + 02	−0.5760E + 01	−0.6320E + 01	109.7
	32	0.5269E + 02	−0.6240E + 01	−0.6830E + 01	109.5
	33	0.5850E + 02	−0.6990E + 01	−0.8060E + 01	115.3
	34	0.4697E + 02	−0.6030E + 01	−0.6340E + 01	105.1
	35	0.4621E + 02	−0.5880E + 01	−0.6770E + 01	115.1
	41	0.8497E + 02	−0.1028E + 02	−0.1126E + 02	109.5
	42	0.7658E + 02	−0.9360E + 01	−0.1034E + 02	110.5
	43	0.6155E + 02	−0.7740E + 01	−0.8500E + 01	109.8
	44	0.4621E + 02	−0.5880E + 01	−0.6770E + 01	115.1
	45	0.3975E + 02	−0.5250E + 01	−0.5980E + 01	113.9
Stress on beam element	1	0.2956E + 02	−0.3640E + 01	−0.3960E + 01	108.8
	2	0.1850E + 03	−0.2428E + 02	−0.2672E + 02	110.0
	3	0.1200E + 03	−0.1608E + 02	−0.1764E + 02	109.7
	4	0.2041E + 03	−0.2552E + 02	−0.2792E + 02	109.4
	5	0.3549E + 03	−0.4444E + 02	−0.4872E + 02	109.6
	11	0.1656E + 02	−0.2360E + 01	−0.2520E + 01	106.8
	12	0.6312E + 02	−0.7920E + 01	−0.8680E + 01	109.6
	13	0.2192E + 02	−0.2400E + 01	−0.2640E + 01	110.0
	14	0.7964E + 02	−0.1088E + 02	−0.1192E + 02	109.2
	15	0.1454E + 03	−0.1960E + 02	−0.2140E + 03	109.2

analyzed, due to symmetry. A total of 484 elements with 1281 degrees of freedom are used to model the built-up structure, including 400 rectangular plate elements, 80 beam elements, and 4 truss elements. The design variables for shape variation are the locations x^i and y^j $(i, j = 1, 2)$, of transverse and

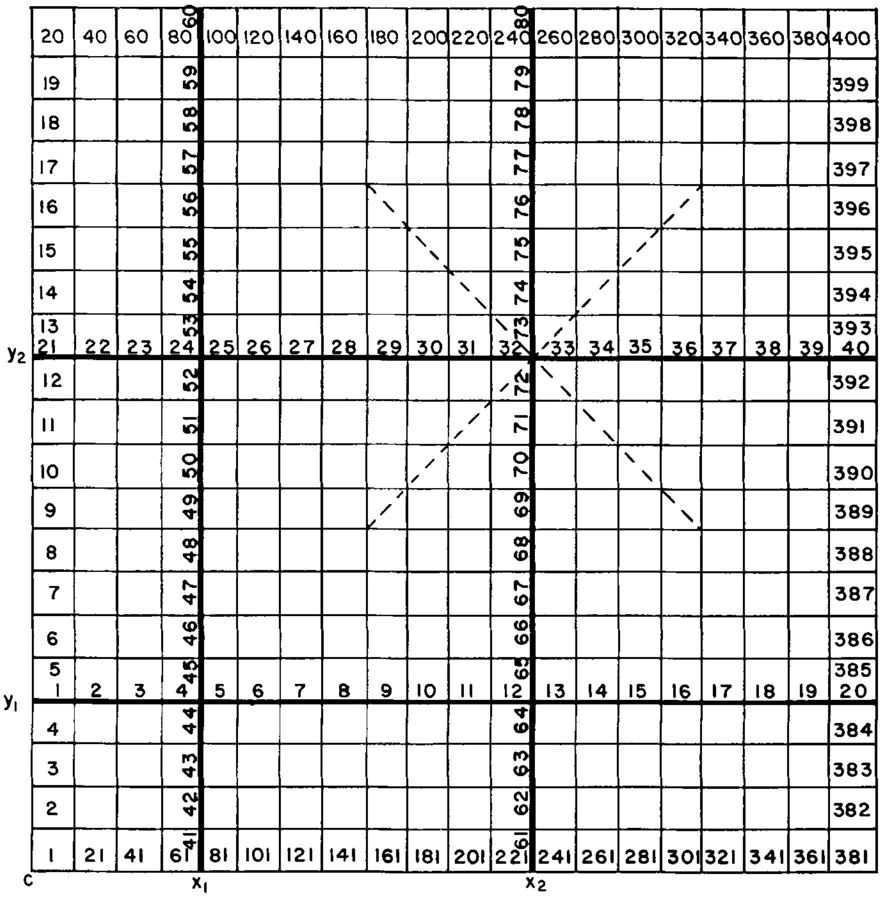

Fig. 4.2.7 Finite element model of a truss–beam–plate built-up structure for shape variation.

longitudinal beams, respectively, measured from center c of the built-up structure. During the shape variation, it is presumed that the outside boundary is fixed and the lengths of truss components are held constant. That is, the ground supports for the trusses move according to the change of beam positions. The same numerical data that are used in conventional design sensitivity calculations are employed. For the nominal design, beam components are located at $x_1 = y_1 = 1.5\,\text{in.}$, and $x_2 = y_2 = 4.5\,\text{in.}$

For shape design sensitivity calculations, a velocity field must be defined that satisfies regularity conditions. For regularity of the velocity field, consider the terms $a'_V(z, z)$, $a'_V(z, \lambda^{(2)})$, and $a'_V(z, \lambda^{(3)})$ in Eqs. (4.2.47), (4.2.50), and (4.2.54), respectively. To evaluate these terms, Eq. (4.2.43) is used, which involves second derivatives of the velocity field $V = [V^x \ \ V^y]^T$ with respect to x and y. Hence, the velocity field must have C^1 regularity. If a velocity field with C^0 regularity is used, second derivatives of the velocity field become Dirac delta measures, which must be integrated to obtain sensitivity results. Note that C^0 regular velocity fields can be used for elasticity problems, such as the simple box example considered before, since the highest order of derivative of the velocity field in the sensitivity formula is one [see Eq. (4.2.38)].

To avoid Dirac delta measures, C^1 regular velocity fields are used in this example. The beam components are allowed to move in transverse directions only. Hence, V^x is a function of x only and V^y is a function of y only. The velocity field in each plate component is represented by hermite cubic functions in each direction. That is, $V^x(x)$ and $V^y(y)$ are represented by hermite cubic functions. To see the velocity field representation graphically, consider Fig. 4.2.8, in which the shape functions for $V^x(x)$ are plotted. In Fig. 4.2.8, δx_1 and δx_2 denote perturbations of locations of transverse beams. From Fig. 4.2.8 note that $V^x(x) = \phi^1(x) + \phi^2(x)$. That is,

$$V^x(x) = \begin{cases} -\dfrac{2x^2}{x_1^3}\left(x - \dfrac{3x_1}{2}\right)\delta x_1, & 0 \le x \le x_1 \\[2ex] \dfrac{2(x - x_1)^2}{(x_2 - x_1)^3}\left[(x - x_1) - \dfrac{3(x_2 - x_1)}{2}\right](\delta x_1 - \delta x_2) + \delta x_1, & x_1 \le x \le x_2 \\[2ex] \dfrac{2(x - x_2)^2}{\left(\dfrac{L_x}{2} - x_2\right)^3}\left[(x - x_2) - \dfrac{3\left(\dfrac{L_x}{2} - x_2\right)}{2}\right]\delta x_2 + \delta x_2, & x_2 \le x \le \dfrac{L_x}{2} \end{cases} \tag{4.2.56}$$

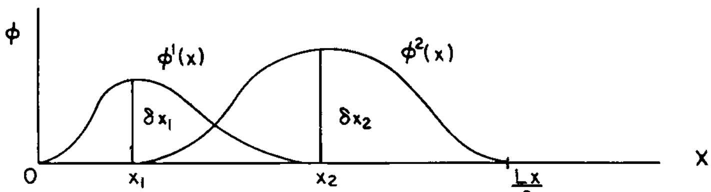

Fig. 4.2.8 Shape functions for the velocity $V^x(x)$.

Similarily,

$$V^y(y) = \begin{cases} -\dfrac{2y^2}{y_1^3}\left(y - \dfrac{3y_1}{2}\right)\delta y_1, & 0 \le y \le y_1 \\[2ex] \dfrac{2(y - y_1)^2}{(y_2 - y_1)^3}\left[(y - y_1) - \dfrac{3(y_2 - y_1)}{2}\right](\delta y_1 - \delta y_2) + \delta y_1, & \\ & y_1 \le y \le y_2 \\[2ex] \dfrac{2(y - y_2)^2}{\left(\dfrac{L_y}{2} - y_2\right)^3}\left[(y - y_2) - \dfrac{3\left(\dfrac{L_y}{2} - y_2\right)}{2}\right]\delta y_2 + \delta y_2, & \\ & y_2 \le y \le \dfrac{L_y}{2} \end{cases} \tag{4.2.57}$$

In Table 4.2.10, sensitivity accuracy results are given for several functionals with a 0.25% uniform change in shape design parameters. Results given in Table 4.2.10 show excellent agreement between predictions ψ_p' and actual changes $\Delta\psi_p$, except for von Mises yield stress in plate element 219. However, note that $\Delta\psi_p$ is small compared to others. The boundary method applied to this problem produced unacceptable results.

Table 4.2.10
Shape Design Sensitivity of Truss–Beam–Plate Built-Up Structure

Functional	Element number	ψ_p^1	$\Delta\psi_p$	ψ_p'	$(\psi_p'/\Delta\psi_p \times 100)\%$
Displacement	C	0.4776E − 03	0.1188E − 04	0.1148E − 04	96.6
Compliance		0.9775E − 03	0.1995E − 04	0.1965E − 04	98.5
Stress on plate element	1	0.4913E + 02	0.915	0.195	100.1
	3	0.4043E + 02	0.860	0.861	100.1
	5	0.3428E + 02	0.602	0.600	99.6
	7	0.5033E + 02	0.858	0.858	100.0
	9	0.6214E + 02	0.925	0.924	99.9
	11	0.6784E + 02	0.924	0.923	99.9
	13	0.7708E + 02	0.873	0.871	99.8
	15	0.8361E + 02	0.994	0.994	100.0
	17	0.9276E + 02	1.088	1.090	100.2
	19	1.0291E + 02	1.251	1.253	100.2
	22	0.4426E + 02	0.891	0.891	100.0
	24	0.3303E + 02	0.776	0.775	99.9

Table 4.2.10 (*continued*)
Shape Design Sensitivity of Truss–Beam–Plate Built-Up Structure

Functional	Element number	ψ_p^1	$\Delta\psi_p$	ψ_p'	$(\psi_p'/\Delta\psi_p \times 100)\%$
	26	0.4266E + 02	0.778	0.776	99.8
	28	0.5594E + 02	0.882	0.880	99.9
	30	0.6352E + 02	0.914	0.913	99.8
	32	0.6965E + 02	0.891	0.889	99.7
	34	0.7793E + 02	0.961	0.960	99.9
	36	0.8552E + 02	1.060	1.060	100.0
	38	0.9187E + 02	1.171	1.172	100.1
	40	1.0464E + 02	1.367	1.374	100.6
	42	0.3954E + 02	0.859	0.858	99.9
	44	0.3333E + 02	0.811	0.810	99.9
	46	0.4428E + 02	0.765	0.763	99.7
	48	0.5250E + 02	0.831	0.830	99.9
	50	0.5853E + 02	0.862	0.861	99.8
	52	0.6753E + 02	0.883	0.881	99.7
	54	0.7550E + 02	0.979	0.978	99.9
	56	0.7905E + 02	1.104	1.104	100.0
	58	0.8127E + 02	1.169	1.170	100.1
	60	0.8983E + 02	1.334	1.340	100.5
	65	0.3836E + 02	0.754	0.749	99.4
	67	0.4488E + 02	0.744	0.744	99.9
	69	0.4606E + 02	0.743	0.742	99.9
	71	0.5603E + 02	0.801	0.800	99.9
	73	0.7075E + 02	0.914	0.911	99.7
	75	0.6838E + 02	1.112	1.112	100.0
	77	0.6713E + 02	1.207	1.209	100.1
	79	0.6501E + 02	1.185	1.189	100.4
	85	0.3830E + 02	0.740	0.738	99.8
	87	0.4058E + 02	0.670	0.669	99.8
	89	0.4786E + 02	0.665	0.663	99.7
	91	0.6018E + 02	0.748	0.744	99.4
	93	0.6787E + 02	0.892	0.889	99.6
	95	0.6411E + 02	1.101	1.100	99.9
	97	0.6272E + 02	1.293	1.294	100.1
	99	0.6443E + 02	1.468	1.471	100.2
	106	0.4305E + 02	0.726	0.725	99.9
	108	0.5160E + 02	0.721	0.719	99.7
	110	0.5914E + 02	0.749	0.745	99.6
	112	0.6223E + 02	0.749	0.745	99.3
	114	0.5604E + 02	0.928	0.924	99.7
	116	0.5721E + 02	1.124	1.124	100.0
	118	0.5927E + 02	1.283	1.284	100.1
	120	0.6532E + 02	1.487	1.490	100.2
	128	0.5512E + 02	0.759	0.756	99.7
	130	0.5826E + 02	0.750	0.746	99.5

(*continues*)

Table 4.2.10 (*continued*)

Shape Design Sensitivity of Truss–Beam–Plate Built-Up Structure

Functional	Element number	ψ_p^1	$\Delta\psi_p$	ψ_p'	$(\psi_p'/\Delta\psi_p \times 100)\%$
	132	0.5914E + 02	0.587	0.581	99.0
	134	0.4801E + 02	0.834	0.830	99.4
	136	0.4705E + 02	1.057	1.056	99.9
	138	0.4975E + 02	1.203	1.203	100.0
	140	0.5682E + 02	1.405	1.408	100.2
	149	0.5550E + 02	0.747	0.744	99.5
	151	0.5322E + 02	0.575	0.569	99.0
	153	0.5555E + 02	0.437	0.430	98.5
	155	0.3700E + 02	0.868	0.863	99.4
	157	0.3514E + 02	1.078	1.077	99.9
	159	0.3790E + 02	1.226	1.226	100.0
	169	0.5208E + 02	0.687	0.683	99.3
	171	0.4887E + 02	0.372	0.365	98.0
	173	0.5629E + 02	0.215	0.208	96.7
	175	0.3270E + 02	0.618	0.608	98.4
	177	0.2338E + 02	0.915	0.907	99.1
	179	0.2327E + 02	1.062	1.056	99.5
	190	0.4529E + 02	0.368	0.360	97.4
	192	0.6133E + 02	−0.068	−0.075	109.9
	194	0.4776E + 02	0.129	0.120	93.1
	196	0.3201E + 02	0.255	0.240	94.0
	198	0.2704E + 02	0.283	0.263	93.0
	200	0.3481E + 02	0.377	0.364	96.5
	211	0.5317E + 02	−0.101	−0.109	108.0
	212	0.6768E + 02	−0.220	−0.226	102.7
	213	0.6722E + 02	−0.162	−0.167	103.3
	214	0.6017E + 02	−0.118	−0.125	106.1
	216	0.5335E + 02	−0.109	−0.118	107.7
	217	0.5372E + 02	−0.099	−0.108	108.9
	218	0.5529E + 02	−0.062	−0.071	114.8
	219	0.5763E + 02	−0.007	−0.016	241.3
	220	0.6573E + 02	0.083	0.077	93.3
	229	0.5827E + 02	0.142	0.135	94.9
	231	0.6768E + 02	−0.220	−0.226	102.7
	233	0.8148E + 02	−0.304	−0.312	102.5
	235	0.8270E + 02	−0.280	−0.285	101.6
	237	0.9004E + 02	−0.187	−0.189	101.4
	239	1.0098E + 02	0.077	0.075	97.5
	249	0.5629E + 02	0.215	0.208	96.7
	251	0.6726E + 02	−0.162	−0.167	103.3
	253	0.8205E + 02	−0.377	−0.382	101.4
	255	0.7951E + 02	−0.366	−0.369	100.9
	257	0.9004E + 02	−0.350	−0.352	100.4
	259	1.0166E + 02	−0.362	−0.364	100.6

Table 4.2.10 *(continued)*
Shape Design Sensitivity of Truss–Beam–Plate Built-Up Structure

Functional	Element number	ψ_p^1	$\Delta\psi_p$	ψ_p'	$(\psi_p'/\Delta\psi_p \times 100)\%$
	269	0.4201E + 02	0.419	0.410	97.8
	271	0.6017E + 02	−0.118	−0.125	106.1
	273	0.7804E + 02	−0.374	−0.379	101.3
	274	0.6736E + 02	−0.364	−0.368	101.2
	275	0.6230E + 02	−0.338	−0.342	101.1
	276	0.6130E + 02	−0.313	−0.316	100.9
	277	0.6268E + 02	−0.297	−0.300	100.8
	278	0.6485E + 02	−0.288	−0.290	100.7
	279	0.6750E + 02	−0.279	−0.281	100.7
	280	0.7533E + 02	−0.289	−0.293	101.6
	286	0.5614E + 02	1.032	1.030	99.8
	288	0.3700E + 02	0.868	0.863	99.4
	300	0.4934E + 02	−0.201	−0.205	102.2
	317	0.2977E + 02	−0.210	−0.214	102.0
	319	0.2789E + 02	−0.177	−0.182	102.5
	337	0.2185E + 02	−0.225	−0.230	102.2
	339	0.1811E + 02	−0.243	−0.248	102.0
	358	0.1456E + 02	−0.309	−0.313	101.2
	360	0.1494E + 02	−0.299	−0.302	101.0
	380	0.1219E + 02	−0.255	−0.255	100.1
	400	0.0847E + 02	−0.156	−0.154	99.1

4.3 EIGENVALUE DESIGN SENSITIVITY

Eigenvalue sensitivity of a built-up structure can be determined, due to variations in conventional design variables and shape. As in the case of separate variations of conventional design variables and shape in Chapters 2 and 3, no adjoint variable is required in eigenvalue design sensitivity calculation. The approach used in this section parallels that employed in Section 4.2. A direct variational analysis is carried out, supported by existence results presented in Chapters 2 and 3.

4.3.1 Calculation of First Variations

Consider the variational form of the built-up structure eigenvalue equations of Eq. (4.1.16), repeated here as

$$\begin{aligned} a_{u,\Omega}(y, \bar{y}) &= \zeta d_{u,\Omega}(y, \bar{y}) \qquad \text{for all} \quad \bar{y} \in Z \\ d_{u,\Omega}(y, y) &= 1 \end{aligned} \tag{4.3.1}$$

Since both bilinear forms in Eq. (4.3.1) depend on the conventional design variable u and shape Ω, it is clear that the eigenvalue ζ also depends on these quantities. The objective here is to use this variational formulation to obtain sensitivity of ζ to variations in the design function and shape.

Using the notation of Eq. (4.2.2) for perturbation of the domain Ω, the design variation of the bilinear form on the right side of Eq. (4.3.1) can be calculated as

$$\begin{aligned}[d_{u,\Omega}(y, \bar{y})]' &\equiv d'_{\delta u}(y, \bar{y}) + d'_V(y, \bar{y}) + d_{u,\Omega}(\dot{y}, \bar{y}) \\ &= \left[\sum_{i=1}^{r} d'_{\delta u^i}(w^i, \bar{w}^i) + d'_{\delta b}(q, \bar{q})\right] + \sum_{i=1}^{r} d'_{V^i}(w^i, \bar{w}^i) \\ &\quad + \sum_{i=1}^{r} d_{u^i,\Omega^i}(\dot{w}^i, \bar{w}^i) + d_b(\dot{q}, \bar{q})\end{aligned} \tag{4.3.2}$$

where $y = [w^1 \; w^2 \; \cdots \; w^r \; q]^{\mathrm{T}}$ denotes an eigenfunction, even though the notation for the component of y is borrowed from static response z, to avoid introduction of new variables. This notation parallels that of Eq. (4.2.3), which remains valid for design variation of the energy bilinear form on the left side of Eq. (4.3.1).

Note that the first and second terms on the right side of Eq. (4.3.2) for the trusses and for each distributed component can be obtained from Chapters 1 and 2. The third term on the right side of Eq. (4.3.2) is due to shape variation. For the domain method, the expression $d'_{V^i}(w^i, \bar{w}^i)$ might be evaluated, instead of Eq. (3.4.8), in terms of domain integrals. Using Eq. (3.2.37) instead of Eq. (3.2.36) and proceeding as in the derivation of Eq. (3.4.8),

$$\begin{aligned}d'_{V^i}(w^i, \bar{w}^i) = \iint_{\Omega^i} \{ &-e_i(\nabla w^{i\mathrm{T}} V^i, \bar{w}^i) - e_i(w^i, \nabla \bar{w}^{i\mathrm{T}} V^i) \\ &+ \operatorname{div}[e_i(w^i, \bar{w}^i) V^i]\} \, d\Omega\end{aligned} \tag{4.3.3}$$

where $e_i(\cdot, \cdot)$ is a bilinear function in the integrand of the bilinear form $d_{u^i,\Omega^i}(\cdot, \cdot)$.

4.3.2 Eigenvalue Design Sensitivity

SIMPLE EIGENVALUE

Presuming differentiability of a simple eigenvalue ζ and the associated eigenfunction y with respect to design variables and shape, supported by the proofs presented in Sections 2.5 and 3.5, the first variation of both sides of Eq. (4.3.1) yields the formal relationship

$$\begin{aligned}a_{u,\Omega}(\dot{y}, \bar{y}) + a'_{\delta u}(y, \bar{y}) + a'_V(y, \bar{y}) &= \zeta' d_{u,\Omega}(y, \bar{y}) + \zeta d_{u,\Omega}(\dot{y}, \bar{y}) \\ &\quad + \zeta d'_{\delta u}(y, \bar{y}) + \zeta d'_V(y, \bar{y}) \qquad \text{for all} \quad \bar{y} \in Z\end{aligned} \tag{4.3.4}$$

This equation may be evaluated at $\bar{y} = y$, using the second equation in Eq. (4.3.1), to obtain

$$\zeta' = [a'_{\delta u}(y, y) - \zeta d'_{\delta u}(y, y)] + [a'_V(y, y) - \zeta d'_V(y, y)] + [a_{u,\Omega}(\dot{y}, y) - \zeta d_{u,\Omega}(\dot{y}, y)] \tag{4.3.5}$$

Using symmetry of the two bilinear forms and $\dot{y} \in Z$, Eq. (4.3.1) implies that the third term on the right side of Eq. (4.3.5) is zero, yielding

$$\zeta' = [a'_{\delta u}(y, y) - \zeta d'_{\delta u}(y, y)] + [a'_V(y, y) - \zeta d'_V(y, y)] \tag{4.3.6}$$

The differentials of the bilinear forms on the right side of of Eq. (4.3.6) may be evaluated using the expansions of Eqs. (4.2.3), (4.2.4), (4.3.2), and (4.3.3) and the results of Chapters 1 and 2 for the trusses and distributed components to obtain explicit formulas.

Note that evaluation of the design sensitivity of a simple eigenvalue given by Eq. (4.3.6) is explicit in terms of the eigenfunction y and does not require solution of a separate adjoint problem.

REPEATED EIGENVALUE

Consider next the case of a repeated eigenvalue ζ with s independent eigenfunctions; i.e.,

$$\left.\begin{aligned} a_{u,\Omega}(y^i, \bar{y}) &= \zeta d_{u,\Omega}(y^i, \bar{y}) \qquad \text{for all} \quad \bar{y} \in Z, \\ d_{u,\Omega}(y^i, y^j) &= \delta_{ij}, \end{aligned}\right\} \quad i, j = 1, \ldots, s \tag{4.3.7}$$

Using the directional derivative theorem for repeated eigenvalues in Sections 2.5 and 3.5, the s directional derivatives of the repeated eigenvalue ζ in Eq. (4.3.7) may be characterized as the eigenvalues of the matrix

$$\mathcal{M} = [(a'_{\delta u}(y^i, y^j) - \zeta d'_{\delta u}(y^i, y^j)) + (a'_V(y^i, y^j) - \zeta d'_V(y^i, y^j))]_{s \times s} \tag{4.3.8}$$

where i is row index and j is column index. As presented in detail in Sections 2.3.1 and 3.4.1, for a twice-repeated eigenvalue, an explicit expression may be obtained for the directional derivatives of the eigenvalue in terms of a rotation parameter derived from the set of orthonormal eigenfunctions selected. More specifically,

$$\zeta'_1(\delta u, V) = \cos^2 \phi(\delta u, V)\mathcal{M}_{11} + \sin 2\phi(\delta u, V)\mathcal{M}_{12} + \sin^2 \phi(\delta u, V)\mathcal{M}_{22} \tag{4.3.9}$$

$$\zeta'_2(\delta u, V) = \sin^2 \phi(\delta u, V)\mathcal{M}_{11} - \sin 2\phi(\delta u, V)\mathcal{M}_{12} + \cos^2 \phi(\delta u, V)\mathcal{M}_{22} \tag{4.3.10}$$

where the rotation angle ϕ is given as

$$\phi(\delta u, V) = \frac{1}{2}\arctan\left[\frac{2\mathscr{M}_{12}}{\mathscr{M}_{11} - \mathscr{M}_{22}}\right] \tag{4.3.11}$$

The notations $\zeta_i'(\delta u, V)$ and $\phi(\delta u, V)$ are used here to emphasize dependence of the directional derivative on δu and V.

4.3.3 Example

Consider the truss–beam–plate of Section 4.1.4 with the variational eigenvalue equation of Eq. (4.1.59). To obtain a design sensitivity formula using Eq. (4.3.6), expressions must be obtained for $a'_{\delta u}(y, y)$, $d'_{\delta u}(y, y)$, $a'_V(y, y)$, and $d'_V(y, y)$. For $a'_{\delta u}(y, y)$ and $a'_V(y, y)$, Eqs. (4.2.40) and (4.2.43) can be used, respectively. Using Eqs. (1.2.19), (2.2.1), and (4.1.58)

$$\begin{aligned} d'_{\delta u}(y, y) = {} & \sum_{i=1}^{n+1}\sum_{j=1}^{m+1}\iint_{\Omega_1^{ij}} \rho w^{ij^2}\,\delta t^{ij}\, d\Omega_1 \\ & + \sum_{i=1}^{n+1}\sum_{j=1}^{m}\int_{\Omega_2^{ij}} [\rho\tilde{v}^{ij^2}(\tilde{d}^{ij}\tilde{h}^{ij})_{\tilde{b}^{ij}} + \tilde{\theta}^{ij^2}(\tilde{I}_G^{ij})_{\tilde{b}^{ij}}]\,\delta\tilde{b}^{ij}\, d\Omega_2 \\ & + \sum_{i=1}^{n}\sum_{j=1}^{m+1}\int_{\Omega_3^{ij}} [\rho\hat{v}^{ij^2}(\hat{d}^{ij}\hat{h}^{ij})_{\hat{b}^{ij}} + \hat{\theta}^{ij^2}(\hat{I}_G^{ij})_{\hat{b}^{ij}}]\,\delta\hat{b}^{ij}\, d\Omega_3 \\ & + [q_k^{11^{\mathrm{T}}} M(A_l^{11}) q_k^{11}]_{A_l^{11}}\,\delta A_l^{11} + [q_k^{1m^{\mathrm{T}}} M(A_l^{1m}) q_k^{1m}]_{A_l^{1m}}\,\delta A_l^{1m} \\ & + [q_k^{n1^{\mathrm{T}}} M(A_l^{n1}) q_k^{n1}]_{A_l^{n1}}\,\delta A_l^{n1} + [q_k^{nm^{\mathrm{T}}} M(A_l^{nm}) q_k^{nm}]_{A_l^{nm}}\,\delta A_l^{nm} \end{aligned} \tag{4.3.12}$$

Also, using Eqs. (4.3.2), (4.3.3), and (4.1.58),

$$\begin{aligned} d'_V(y, y) = {} & \sum_{i=1}^{n+1}\sum_{j=1}^{m+1}\iint_{\Omega_1^{ij}} [-2t^{ij}\rho w^{ij}(\nabla w^{ij^{\mathrm{T}}} V) + \operatorname{div}(t^{ij}\rho w^{ij^2} V)]\, d\Omega_1 \\ & + \sum_{i=1}^{n+1}\sum_{j=1}^{m}\int_{\Omega_2^{ij}} [-2\rho\tilde{d}^{ij}\tilde{h}^{ij}\tilde{v}^{ij}(\tilde{v}_x^{ij} V) + (\rho\tilde{d}^{ij}\tilde{h}^{ij}\tilde{v}^{ij^2} V)_x \\ & \qquad - 2\tilde{I}_G^{ij}\tilde{\theta}^{ij}(\tilde{\theta}_x^{ij} V) + (\tilde{I}_G^{ij}\tilde{\theta}^{ij^2} V)_x]\, d\Omega_2 \\ & + \sum_{i=1}^{n}\sum_{j=1}^{m+1}\int_{\Omega_3^{ij}} [-2\rho\hat{d}^{ij}\hat{h}^{ij}\hat{v}^{ij}(\hat{v}_y^{ij} V) + (\rho\hat{d}^{ij}\hat{h}^{ij}\hat{v}^{ij^2} V)_y \\ & \qquad - 2\hat{I}_G^{ij}\hat{\theta}^{ij}(\hat{\theta}_y^{ij} V) + (\hat{I}_G^{ij}\hat{\theta}^{ij^2} V)_y]\, d\Omega_3 \end{aligned} \tag{4.3.13}$$

If the conventional design variables are assumed to be constant for each component, the above equation is simplified to

$$
\begin{aligned}
d'_V(y, y) = & \sum_{i=1}^{n+1} \sum_{j=1}^{m+1} \iint_{\Omega_1^{ij}} [t^{ij} \rho w^{ij2} (V_x^x + V_y^y)] \, d\Omega_1 \\
& + \sum_{i=1}^{n+1} \sum_{j=1}^{m} \int_{\Omega_2^{ij}} (\rho \tilde{d}^{ij} \tilde{h}^{ij} \tilde{v}^{ij2} + \tilde{I}_G^{ij} \tilde{\theta}^{ij2}) V_x \, d\Omega_2 \\
& + \sum_{i=1}^{n} \sum_{j=1}^{m+1} \int_{\Omega_3^{ij}} (\rho \hat{d}^{ij} \hat{h}^{ij} \hat{v}^{ij2} + \hat{I}_G^{ij} \hat{\theta}^{ij2}) V_y \, d\Omega_3
\end{aligned}
\tag{4.3.14}
$$

where $V = [V^x \;\; V^y]^T$ on the plate component and V on each beam is velocity in the axial direction. The design sensitivity expression for a simple eigenvalue, given by Eq. (4.3.6), is rewritten as

$$
\zeta' = [a'_{\delta u}(y, y) - \zeta d'_{\delta u}(y, y)] + [a'_V(y, y) - \zeta d'_V(y, y)] \tag{4.3.15}
$$

where Eqs. (4.2.40), (4.2.43), (4.3.12), and (4.3.14) can be used to obtain explicit expressions for the terms in Eq. (4.3.15).

For numerical calculations, consider the numerical example of Section 4.2.3. The same numerical data are used for eigenvalue design sensitivity calculation. As in Section 4.2.3, conventional and shape design sensitivity calculations are carried out separately. The finite element model of Fig. 4.2.6 is used for conventional design variation, and the finite element model of Fig. 4.2.7 is used for shape design variation. The same nominal designs are used here. In Table 4.3.1, sensitivity accuracy results are given for the fundamental eigenvalue, with a uniform 5% change in conventional and shape design variables, separately. Results given in Table 4.3.1 show excellent agreement between predictions ζ' and actual changes $\Delta\zeta$.

Table 4.3.1

Design Sensitivity of Fundamental Eigenvalue of Truss–Beam–Plate Built-Up Structure

Design Variable	ζ^1	$\Delta\zeta$	ζ'	$(\zeta'/\Delta\zeta \times 100)\%$
Conventional	0.1242E + 04	0.2408E + 03	0.2199E + 03	91.3
Shape	0.1215E + 04	−0.2220E + 02	−0.2119E + 02	95.5

Appendix

A.1 MATRIX CALCULUS NOTATION

In dealing with systems that are described by many variables, it is essential that a precise matrix calculus notation be employed. To introduce the notation used in this text, let x be a k vector of real variables, y be an m vector of real variables, $a(x, y)$ be a scalar differentiable function of x and y, and $g(x, y) = [g_1(x, y) \cdots g_n(x, y)]^T$ be an n vector of differentiable functions of x and y. Using i as row index and j as column index, define

$$a_x \equiv \frac{\partial a}{\partial x} \equiv \left[\frac{\partial a}{\partial x_j}\right]_{1 \times k} \tag{A.1.1}$$

$$g_x \equiv \frac{\partial g}{\partial x} \equiv \left[\frac{\partial g_i}{\partial x_j}\right]_{n \times k} \tag{A.1.2}$$

$$a_{xy} \equiv \left[\frac{\partial^2 a}{\partial x_i\, \partial y_j}\right]_{k \times m} = \frac{\partial}{\partial y}\left[\frac{\partial a^T}{\partial x}\right] = \frac{\partial}{\partial y}[a_x^T] = [a_x^T]_y \tag{A.1.3}$$

Note that the derivative of a scalar function with respect to a vector variable in Eq. (A.1.1) gives a row vector. This is one of the few vector symbols in the text that is a row vector, rather than the more common column vector. In order to take advantage of this notation, it is important that the correct vector definition of matrix derivatives be used. Note also that the derivative of a vector function with respect to a vector variable in Eq. (A.1.2) gives a matrix. No attempt is made here to define the derivative of a matrix function with respect to a vector variable. Similarly, the second derivative of a scalar function with respect to a vector variable can be defined as in Eq. (A.1.3), but the second derivative of a vector function with respect to a vector variable is not defined.

As an example of the use of this matrix calculus notation, let δx and δy be small perturbations in x and y. Using the total differential formula of calculus [11] gives

$$\begin{aligned} a(x + \delta x, y + \delta y) - a(x, y) \approx \delta a &\equiv \sum_{j=1}^{k} \frac{\partial a}{\partial x_j} \delta x_j + \sum_{j=1}^{m} \frac{\partial a}{\partial y_j} \delta y_j \\ &= \frac{\partial a}{\partial x} \delta x + \frac{\partial a}{\partial y} \delta y \\ &= a_x \, \delta x + a_y \, \delta y \end{aligned} \tag{A.1.4}$$

This is just one example of an application of matrix calculus that avoids cumbersome summation notation. Note that both terms in Eq. (A.1.4) are scalars, since a_x is a row vector and δx is a column vector. It is clear that

$$\delta a \neq \delta x \, a_x + \delta y \, a_y$$

since the left side is a scalar and the two terms on the right side are $k \times k$ and $m \times m$ matrices, respectively.

Similarly, matrix calculus extensions of ordinary calculus rules can be derived, such as the product rule of differentiation. For example, if A is an $n \times n$ constant matrix,

$$\begin{aligned} \frac{\partial}{\partial x}(Ag(x, y)) &= \left[\frac{\partial}{\partial x_j}\left(\sum_{l=1}^{n} A_{il} g_l\right)\right] \\ &= \left[\sum_{l=1}^{n} A_{il} \frac{\partial g_l}{\partial x_j}\right] \\ &= A \frac{\partial g}{\partial x} = A g_x \end{aligned} \tag{A.1.5}$$

A second example, which gives a result that might not be expected, involves two n-vector functions $h(x, y)$ and $g(x, y)$. By careful manipulation,

$$\begin{aligned} \frac{\partial}{\partial x}(g^{\mathrm{T}} h) &= \left[\frac{\partial}{\partial x_j}\left(\sum_{l=1}^{n} g_l h_l\right)\right] \\ &= \left[\sum_{l=1}^{n}\left(\frac{\partial g_l}{\partial x_j} h_l + g_l \frac{\partial h_l}{\partial x_j}\right)\right] \\ &= \left[\sum_{l=1}^{n} h_l \frac{\partial g_l}{\partial x_j}\right] + \left[\sum_{l=1}^{n} g_l \frac{\partial h_l}{\partial x_j}\right] \\ &= h^{\mathrm{T}} \frac{\partial g}{\partial x} + g^{\mathrm{T}} \frac{\partial h}{\partial x} \\ &= h^{\mathrm{T}} g_x + g^{\mathrm{T}} h_x \end{aligned} \tag{A.1.6}$$

To see that Eq. (A.1.6) is reasonable, note that $g^T h = h^T g$ and that in fact interchanging g and h does not change either side of Eq. (A.1.6). Note also that what might have intuitively appeared to be the appropriate product rule of differentiation is not even defined, much less valid; that is,

$$\frac{\partial}{\partial x}(g^T h) \neq \left(\frac{\partial g}{\partial x}\right)^T h + g^T \left(\frac{\partial h}{\partial x}\right)$$

In boundary-value problems, derivatives with respect to the independent variable $x \in R^3$ (or R^2) often arise. In these instances, it is convenient to use the gradient notation

$$\nabla a(x) \equiv \begin{bmatrix} \frac{\partial a}{\partial x_1} & \frac{\partial a}{\partial x_2} & \frac{\partial a}{\partial x_3} \end{bmatrix}^T \tag{A.1.7}$$

That is,

$$\nabla a = a_x^T \tag{A.1.8}$$

Very often in structural mechanics, quadratic forms $x^T A x$ ($x \in R^n$) arise, where A is an $n \times n$ constant matrix, presumed initially not to be symmetric. Using the foregoing definitions,

$$\begin{aligned} \frac{\partial}{\partial x}(x^T A x) &= \left[\frac{\partial}{\partial x_i}\left(\sum_{j,k} x_k a_{kj} x_j\right)\right] \\ &= \left[\sum_k x_k a_{ki} + \sum_j a_{ij} x_j\right] \\ &= \left[\sum_k x_k a_{ki} + \sum_j x_j a_{ji}^T\right] \\ &= x^T(A + A^T) \end{aligned} \tag{A.1.9}$$

In particular, if A is symmetric,

$$\frac{\partial}{\partial x}(x^T A x) = 2x^T A \tag{A.1.10}$$

If a scalar valued function $a(x)$ ($x \in R^n$) is twice continuously differentiable, the first-order approximation of Eq. (A.1.4) can be extended to second order. Using Taylor's formula [11],

$$\begin{aligned} a(x + \delta x) - a(x) &\approx \sum_i \frac{\partial a(x)}{\partial x_i} \delta x_i + \frac{1}{2} \sum_i \sum_j \frac{\partial^2 a(x)}{\partial x_i \partial x_j} \delta x_i \delta x_j \\ &= a_x \delta x + \tfrac{1}{2} \delta x^T a_{xx} \delta x \end{aligned} \tag{A.1.11}$$

A.2 BASIC FUNCTION SPACES

The purpose of this section is to summarize definitions and properties of function spaces that are used throughout the text. The mathematical validity of developments presented in the text rest upon fundamental results associated with these spaces, which in many cases are nontrivial to prove. Basic ideas are discussed in this section to assist the engineer in understanding the nature of the spaces and their properties, with references to the literature given for proofs.

A.2.1 R^k; k-Dimensional Euclidean Space

The simplest space encountered in multidimensional analysis is k-dimensional Euclidean space, denoted here as R^k. This is actually a space of column matrices, rather than a function space. The space R^k is quite important in its own right and serves to introduce basic ideas of vector spaces and their properties, prior to introduction of function spaces. The k-dimensional Euclidean space is defined as

$$R^k = \{x = [x_1 \quad \cdots \quad x_k]^{\mathrm{T}}: \quad x_i \text{ real}, \quad i = 1,\ldots,k\} \tag{A.2.1}$$

Note that R^k is simply the collection of all $k \times 1$ matrices (column vectors) whose components are real numbers.

In order to be useful for analyses of finite dimensional structural systems, algebra must be defined on this space to allow for systematic manipulation. Addition of two vectors is defined as in matrix notation as

$$x + y \equiv [x_1 + y_1 \quad \cdots \quad x_k + y_k]^{\mathrm{T}} \tag{A.2.2}$$

and multiplication of a vector x by a scalar α is defined as

$$\alpha x \equiv [\alpha x_1 \quad \cdots \quad \alpha x_k]^{\mathrm{T}} \tag{A.2.3}$$

These operations have the properties

$$x + y = y + x \tag{A.2.4}$$

$$(x + y) + z = x + (y + z) \tag{A.2.5}$$

There is a unique zero vector $0 = [0 \cdots 0]$ such that

$$0 + x = x \tag{A.2.6}$$

there is also a unique negative vector $-x$ such that

$$x + (-x) = 0 \tag{A.2.7}$$

Additional properties of the operations are

$$\alpha(x + y) = \alpha x + \alpha y \tag{A.2.8}$$

$$(\alpha + \beta)x = \alpha x + \beta x \tag{A.2.9}$$

$$\alpha(\beta x) = (\alpha\beta)x \tag{A.2.10}$$

$$1x = x \tag{A.2.11}$$

where x, y, and z are arbitrary vectors in R^k and α and β are arbitrary real constants.

The set of vectors R^k defined in Eq. (A.2.1), with the operations of addition and multiplication by a scalar defined by Eqs. (A.2.2) and (A.2.3), which satisfy Eqs. (A.2.4)–(A.2.11), constitute a *vector space*. As will be seen in Sections A.2.2–A.2.6, sets of functions that have properties of addition and multiplication by a scalar also obey the properties of Eqs. (A.2.4)–(A.2.11) and define a function space, which is a vector space. The value in such a definition is that functions may be dealt with using an algebra that parallels the arithmetic that is normally used in manipulation of column vectors.

Having defined an algebra on the vector space R^k, it is now helpful to define geometric properties that extend the usual ideas of scalar product and length of a physical vector. The *scalar product* of two vectors in R^k is defined as

$$(x, y) \equiv x^{\mathrm{T}}y \tag{A.2.12}$$

Much as in the case of the properties of Eqs. (A.2.4)–(A.2.11) for vector addition and multiplication by a scalar, it may be verified that the scalar product defined by Eq. (A.2.12) satisfies

$$(x, y) = (y, x) \tag{A.2.13}$$

$$(x, y + z) = (x, y) + (x, z) \tag{A.2.14}$$

$$(\alpha x, y) = \alpha(x, y) \tag{A.2.15}$$

$$(x, x) \geq 0 \tag{A.2.16}$$

$$(x, x) = 0 \quad \text{implies} \quad x = 0 \tag{A.2.17}$$

where x, y, and z are arbitrary vectors in R^k and α is an arbitrary scalar.

Having defined a scalar product of two vectors, the *norm* of a vector in R^k may be defined as

$$\|x\| \equiv (x, x)^{1/2} \tag{A.2.18}$$

It is not difficult to verify that the norm defined by Eq. (A.2.18) has the following properties:

$$\|\alpha x\| = |\alpha|\,\|x\| \tag{A.2.19}$$

$$|(x, y)| \le \|x\|\,\|y\| \tag{A.2.20}$$

$$\|x + y\| \le \|x\| + \|y\| \tag{A.2.21}$$

where x and y are arbitrary vectors and α is an arbitrary scalar. The norm of a vector abstracts the concept of length of a physical vector and allows for extension of the idea of two vectors x and y being close to one another if the norm of their difference $\|x - y\|$ is small.

It is interesting to note that if the norm is defined by Eq. (A.2.18) in terms of a scalar product, it automatically has properties of Eqs. (A.2.19)–(A.2.21). There are situations in which a norm can be defined on a vector space that has no scalar product. In such a case, an abstract norm is defined as a functional operating on a vector, having the properties of Eqs. (A.2.19)–(A.2.21) and $\|x\| > 0$ for all $x \neq 0$. This last property follows automatically by the definition of Eq. (A.2.18), using the scalar product properties of Eqs. (A.2.16) and (A.2.17). In case there exists no scalar product, this latter property must be verified in order to assure properties of the norm.

In addition to allowing definition of two vectors being close, the norm can be used to define *convergence* of a sequence of vectors $\{x^i\}$ $(i = 1, 2, \ldots)$ in R^k as follows:

$$\lim_{i \to \infty} x^i = x \qquad \text{if and only if} \qquad \lim_{i \to \infty} \|x - x^i\| = 0 \tag{A.2.22}$$

The concept of convergence in R^k can be shown to be equivalent to convergence of individual components of the vector. This simple property, however, does not carry over to infinite-dimensional vector spaces, such as function spaces that are encountered in the study of boundary-value problems.

A sequence of vectors that cluster near one another as their index i grows large is called a Cauchy sequence. More precisely, a sequence $\{x^i\}$ is a *Cauchy sequence* if

$$\lim_{m, n \to \infty} \|x^m - x^n\| = 0 \tag{A.2.23}$$

A vector space for which every Cauchy sequence is convergent to a limit in the space is called a *complete vector space.* It is not difficult to show that R^k is a complete vector space under this definition. In fact, any vector space that is complete in the norm defined by a scalar product is called a *Hilbert space.* With this definition, R^k is a Hilbert space.

A *functional* is a mapping from a vector space to a real number. Examples of functionals on R^k include $\|x\|$ and (x, y) for a given y in R^k. A functional l is said to be a *linear functional* if

$$l(x + y) = l(x) + l(y) \tag{A.2.24}$$

$$l(\alpha x) = \alpha l(x) \tag{A.2.25}$$

for all x and y in R^k and all scalars α. A linear functional is said to be *bounded*, or *continuous*, if there exists a positive constant γ such that

$$|l(x)| \leq \gamma \|x\| \tag{A.2.26}$$

for all x in R^k.

It is interesting to note that the functional $\|x\|$ is not linear, as is easily verified using the properties of Eqs. (A.2.19)–(A.2.21). The functional $l(x) = (x, y)$ for a fixed y in R^k can be verified to be linear, using the properties of scalar product given in Eqs. (A.2.14) and (A.2.15). Using Eq. (A.2.20), it is also seen to be bounded.

One of the principal reasons that Hilbert spaces are valuable in structural analysis is that any bounded linear functional on a Hilbert space has a very special representation, defined by the *Reisz representation theorem*; that is, any bounded linear functional $l(x)$ on R^k can be represented as

$$l(x) = (y, x) \tag{A.2.27}$$

for some vector y in R^k. The Reisz representation theorem guarantees existence of the vector y associated with the bounded linear functional l. While this statement may not sound like a commonly used idea in mechanics, in fact it is. The concept of generalized force in mechanics follows from the Reisz representation theorem, in which the bounded linear functional l is the virtual work associated with a virtual displacement x, and the vector y is defined as the generalized force of the system.

The rather obvious algebra, norm, and convergence properties of the finite-dimensional vector space R^k have been formalized in this section in some detail to prepare for the definition of similar properties in spaces of functions that are needed in the study of boundary-value problems. The reader who is unfamiliar with concepts of function spaces should recognize the similarity between operations and properties of function spaces and the more intuitively clear properties of the finite-dimensional vector space R^k.

A.2.2 $C^m(\Omega)$; m-Times Continuously Differentiable Functions on Ω

Consider an open set Ω in R^k, with closure $\bar{\Omega}$ in the norm of R^k. Considerations are limited in this section and in the text to *bounded sets*, that is, sets of points whose distances from the origin are bounded by some finite constant. Restriction is limited to bounded sets, since most structural applications occur on bounded sets in R^1 through R^3. Furthermore, restriction to bounded sets has the attractive property that every continuous function on a closed and bounded set in R^k is bounded.

The set of all *m-times continuously differentiable functions* on a set Ω is defined as the *function space*

$$C^m(\Omega) \equiv \left\{ u(x), x \in \Omega \subset R^k: \ \frac{\partial^{|j|}u(x)}{\partial x_1^{j_1} \cdots \partial x_k^{j_k}} \text{ is continuous for } |j| = 1, 2, \ldots, m \right\} \tag{A.2.28}$$

where j is a vector of indices $j = (j_1, \ldots, j_k)$ and $|j| = \sum_{i=1}^{k} j_i$. For simplification of notation in the following, the derivative $\partial^{|j|}u(x)/\partial x_1^{j_1} \cdots \partial x_k^{j_k}$ will be denoted simply as $\partial^{|j|}u(x)/\partial x^j$. The space of m-times continuously differentiable functions on the closed set $\bar{\Omega}$ is simply defined by replacing Ω in Eq. (A.2.28) by $\bar{\Omega}$. The space $C^m(\Omega)$ is viewed at this point simply as the collection of all possible m-times continuously differentiable functions defined on the set Ω, with no concept of algebra or geometry defined.

To make use of the space of m-times continuously differentiable functions, it is essential to define an algebra on this space. Consider two m-times continuously differentiable functions u and v defined on Ω. The sum of these two functions is defined as

$$(u + v)(x) \equiv u(x) + v(x) \tag{A.2.29}$$

which must hold for all $x \in \Omega$; that is, addition of functions is carried out in the natural way of adding their values at points in physical space. Similarly, a scalar α times a function u is defined as

$$(\alpha u)(x) \equiv \alpha u(x) \tag{A.2.30}$$

for all $x \in \Omega$.

Defining the zero function as

$$0(x) \equiv 0 \tag{A.2.31}$$

and the negative of a function as

$$(-u)(x) \equiv -u(x) \tag{A.2.32}$$

it is easy to show that properties of Eqs. (A.2.4)–(A.2.11) follow for addition and multiplication of functions defined in Eqs. (A.2.29) and (A.2.30). Before concluding that $C^m(\Omega)$ is a vector space, however, it must be demonstrated that given two functions u and v in the space and a scalar α, then $u + v$ and αu are again in the space (that is, they are m-times continuously differentiable functions). This conclusion follows directly from the following elementary properties of differentiation:

$$\frac{\partial^{|j|}}{\partial x^j}[(u + v)(x)] = \frac{\partial^{|j|}}{\partial x^j}[u(x) + v(x)] = \frac{\partial^{|j|}u(x)}{\partial x^j} + \frac{\partial^{|j|}v(x)}{\partial x^j} \qquad \text{(A.2.33)}$$

$$\frac{\partial^{|j|}(\alpha u)(x)}{\partial x^j} = \frac{\partial^{|j|}\alpha u(x)}{\partial x^j} = \alpha\frac{\partial^{|j|}u(x)}{\partial x^j} \qquad \text{(A.2.34)}$$

Since the sum of two continuous functions and the product of a scalar times a continuous function are continuous, the space $C^m(\Omega)$ is closed under the operations of addition and multiplication by a scalar. It is therefore a vector space. The elements of this space may now be viewed as vectors in the same sense that column matrices are viewed as vectors in R^k. It should not be too surprising that this concept of a vector does not correlate completely with the physical idea of a vector in three-dimensional space as something with magnitude and direction, since for k different from 3, these concepts break down even for R^k.

It is possible to make direct definition of a *norm* on the space $C^m(\bar{\Omega})$ as

$$\|u\|_{C^m} \equiv \max_{\substack{x \in \bar{\Omega} \\ 0 \leq |j| \leq m}} \left| \frac{\partial^{|j|}u(x)}{\partial x^j} \right| \qquad \text{(A.2.35)}$$

It can be verified that this is a norm with the properties given in Eqs. (A.2.19)–(A.2.21) and that $\|u\|_{C^m} > 0$ if $u \neq 0$. In fact, it can be shown that the space $C^m(\bar{\Omega})$ is complete in this norm but that this norm is not generated by any scalar product. Therefore, the space $C^m(\bar{\Omega})$ is a complete vector space with a norm, but it is not a Hilbert space. Such spaces are called *Banach spaces* and have a rather rich mathematical theory. The distinction between Banach and Hilbert spaces, however, will not be required in the analysis presented in this text, since an adequate theory can be developed using Hilbert space properties almost exclusively.

A final space of continuously differentiable functions that is often encountered in applications is the space of functions having all derivatives continuously differentiable; that is,

$$C^\infty(\Omega) \equiv \{u(x), x \in \Omega: \quad u \in C^m(\Omega) \qquad \text{for all} \quad m\} \qquad \text{(A.2.36)}$$

It is somewhat remarkable, and nontrivial to prove, that $C^\infty(\Omega)$ is dense in most of the function spaces that are dealt with in this text, many of which are composed of functions that have no continuous derivatives. To say that one space is *dense* in another means that the first space is a subset of the second and that every function in the second can be approximated arbitrarily closely in its own norm by a function in the first space.

A.2.3 $L^2(\Omega)$; Space of Lebesgue Square Integrable Functions

The concept of the Lebesgue integral is a technical extension of the well-known Riemann integral that is introduced in basic calculus and is used throughout the theory of structural mechanics. The distinction between the definitions of the two integrals is illustrated in Fig. A.2.1. In defining the Riemann integral of a function, the horizontal axis is partitioned by a grid of points and the sum of the areas of the rectangles shown in Fig. A.2.1(a) approximates the area beneath the curve defined by the function. It is shown mathematically that for certain classes of regular functions, as the spacing of the grid points approaches zero, hence approaching an infinite number of grid points on the horizontal axis, the sum of areas converges and is defined as the value of the *Riemann integral.*

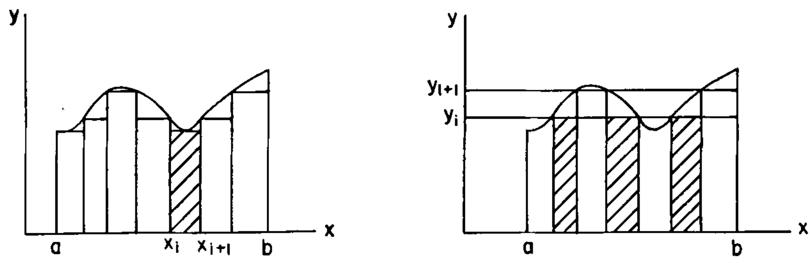

Fig. A.2.1 Integral of a function.

In contrast to the definition of the Riemann integral, the Lebesgue integral is defined by placing a grid of points on the vertical axis and drawing a set of horizontal lines that cut the graph of the function being integrated, as shown in Fig. A.2.1(b). The collection of subintervals on the horizontal axis is associated with a range of values of the function between y_i and y_{i+1}, and a lower bound on the contribution of the area beneath the curve over these subintervals is calculated as y_i times the sum of the lengths of these intervals. Summing over all grid segments along the vertical axis yields a lower bound to the area beneath the curve defined by the function. A limit is then taken as the spacing of grid points on the vertical axis approaches zero. This limit, if it converges, is called the *Lebesgue integral* of the function and the function is declared to be Lebesgue integrable [31, 79].

The value of the Lebesgue integral is equal to the value of the Riemann integral if the Riemann integral of the function under consideration exists. There are, however, pathological functions that do not have Riemann integrals but which do have Lebesgue integrals. Therefore, the Lebesgue integral is an extension of the Riemann integral, with values coinciding for any function that has a Riemann integral. The mathematician takes great delight and substantial pain defining functions that have a Lebesgue integral but do not have a Riemann integral. For purposes of this text, however, such studies in pathology are not necessary. The structural engineer should feel quite comfortable that virtually any function he encounters will have a Riemann integral, which therefore must agree with the value of the Lebesgue integral.

The power of the Lebesgue integral, however, should not be dismissed, since it provides a powerful tool for establishing mathematical properties of the function spaces in which engineers regularly work. Of particular value are properties of the Lebesgue integral in which sequences of functions that are Lebesgue integrable and satisfy certain basic properties have limits that are also Lebesgue integrable. It is shown in the mathematical literature that many sequences of functions that have Riemann integrals either fail to converge or converge to functions for which the Riemann integral is not defined. Thus, if completeness of function spaces is of concern, then the Lebesgue integral is an essential tool. In particular, using the principle of minimum total potential energy of structural mechanics, Lebesgue integration theory can predict exactly what properties the minimizing function should be expected to have, hence defining the mathematical properties of solutions of mechanics problem. This is particularly important in structural mechanics, where minimizing sequences are often defined for total potential energy (i.e., functions that yield successively lower values of the total potential energy). It is desired that such minimizing sequences converge and give solutions to the structural problem. Using the theory of Lebesgue integration and associated function spaces, the mathematician has proved that such sequences do converge and in fact has provided a clear definition of mathematical properties of the solutions.

Lest the engineer dismiss all this as mathematical formalities, it is wise to reflect on the fact that limits of minimizing sequences exist in structural analysis and have well-defined mathematical properties. However, if the engineer is seeking to optimize design of a structure, a minimizing sequence of designs may be obtained, each of which is regular and physically meaningful, and it may be discovered that the limiting function falls outside the class of designs of interest. This dilemma is of very real practical concern if the engineer seeks to use optimality criterion for discovering optimum designs. It is well known in the structural optimization literature that certain problems, such as the problem of finding optimum thickness variation for a plate, may lead to a solution that involves an infinite number of infinitesimal ribs, which perhaps approximate a fiber composite structure. Thus, the solution of the plate optimization problem does not exist in the class of smooth thickness distributions. If the engineer writes down necessary conditions of optimality that would have to hold if there were a smooth solution and attempts to find an optimum design based on these necessary conditions, a rude shock is forthcoming since no solution exists.

It is interesting to ponder such questions in a pseudotheological setting; that is, existence of solutions of structural analysis problems are "God given" (with the help of the mathematician) in spaces of Lebesgue integrable functions, whereas the "Deity in charge of design" has not been so kind as to provide us with existence of smooth optimal designs.

Without going into a detailed treatment of Lebesgue integration theory, it is still possible to provide an intuitive introduction to technical results that are obtainable with Lebesgue integration. For example, the space of *Lebesgue square integrable functions* may be defined as

$$L^2(\Omega) \equiv \left\{ u(x),\, x \in \Omega\colon \quad \iint_\Omega (u(x))^2\, dx < \infty \right\} \tag{A.2.37}$$

where the integral over Ω is the Lebesgue integral, which as noted above coincides with the Riemann integral when it exists.

It is possible in this space to define a *scalar product* as the integral of the product of two functions; that is,

$$(u, v)_{L^2(\Omega)} \equiv \iint_\Omega u(x)v(x)\, dx \tag{A.2.38}$$

where the integral is in the Lebesgue sense. Using Lebesgue integration theory, it is possible to show that properties given by Eqs. (A.2.13)–(A.2.17) are valid [79]. Therefore, a natural *norm* is defined on this space as

$$\|u\|_{L^2(\Omega)} \equiv (u, u)^{1/2}_{L^2(\Omega)} = \left[\iint_\Omega (u(x))^2\, dx \right]^{1/2} \tag{A.2.39}$$

which automatically satisfies the properties of Eqs. (A.2.19)–(A.2.21), in particular the important inequality known as the *Schwartz inequality*,

$$|(u, v)_{L^2(\Omega)}| \leq \|u\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)} \tag{A.2.40}$$

The reader who has studied Fourier series will recognize these ideas as providing the foundation for the theory of construction of series approximations of functions and their convergence properties.

Using properties of the Lebesgue integral, it is shown that the space $L^2(\Omega)$ is *complete* [79]; that is, Cauchy sequences in the L^2 norm converge to square integrable functions. Since the space $L^2(\Omega)$ has a scalar product, it is a *Hilbert space* and has all the desirable properties of Hilbert spaces.

Consider the functional

$$l(u) = \iint_{\Omega} f(x)u(x)\, d\Omega \tag{A.2.41}$$

defined by a given function f in $L^2(\Omega)$. For any function u in $L^2(\Omega)$, the product of f and u is Lebesgue integrable, and the right side of Eq. (A.2.41) creates a real number. Therefore, $l(u)$ is a functional. To see that this is a linear functional, standard properties of integration yield

$$l(\alpha u) = \iint_{\Omega} \alpha f u\, d\Omega = \alpha \iint f u\, d\Omega = \alpha l(u) \tag{A.2.42}$$

$$l(u + v) = \iint_{\Omega} f(u + v)\, d\Omega = \iint_{\Omega} f u\, d\Omega + \iint_{\Omega} f v\, d\Omega = l(u) + l(v) \tag{A.2.43}$$

To see that the functional is bounded, the Schwartz inequality of Eq. (A.2.40) may be applied to obtain

$$|l(u)| = |(f, u)_{L^2(\Omega)}| \leq \|f\|_{L^2(\Omega)} \|u\|_{L^2(\Omega)} \tag{A.2.44}$$

Thus, the scalar product of an arbitrary function u with a fixed function f in $L^2(\Omega)$ [that is, the right side of Eq. (A.2.41)] defines a bounded linear functional on $L^2(\Omega)$.

Since $L^2(\Omega)$ is a Hilbert space, the Reisz representation theorem guarantees that every bounded linear functional on the space can be represented as the scalar product of u with some function in the space; that is, every linear functional $l(u)$ can be written in the form

$$l(u) = (g, u)_{L^2(\Omega)} \tag{A.2.45}$$

for some function g in $L^2(\Omega)$.

A.2.4 $L^\infty(\Omega)$; Space of Essentially Bounded, Lebesgue-Measurable Functions

In Lebesgue integration theory, the *measure* of a set (its length in R^1, area in R^2, or volume in R^3) is defined for very general sets of points. Sets whose measure is zero (e.g., sets of discrete points, line segments in R^2 or R^3, and plane segments in R^3), play key rolls in analysis. A function that has a property that holds everywhere in the space except on a *set of measure zero* is said to have that property almost everywhere (abbreviated a.e.). Functions in spaces such as $L^2(\Omega)$ are defined based on properties that are expressed in terms of integral relations. Their values at discrete points do not influence the integrals. Hence, such functions may have irregular properties at discrete points or on sets of measure zero.

As an extension of a collection of integrable functions that are bounded by some finite constant, *essentially bounded functions* are defined as

$$L^\infty(\Omega) = \{u(x), x \in \Omega\colon \quad |u(x)| \le k < \infty, \quad \text{a.e. in} \quad \Omega\} \tag{A.2.46}$$

A *norm* on $L^\infty(\Omega)$ may be defined as

$$\|u\|_{L^\infty(\Omega)} = \inf\{K\colon \quad |u(x)| \le K \quad \text{a.e. in} \quad \Omega\} \tag{A.2.47}$$

where the term inf denotes least upper bound. It is shown in Lebesgue integration theory [79] that this defines a norm on $L^\infty(\Omega)$ and the space is complete in this norm; that is, Cauchy sequences in this norm converge to functions in the space. It is also shown that it is impossible to define a scalar product on this space, hence that the space is not a Hilbert space, even though it is a Banach space.

Note that for a bounded set Ω, $C^m(\bar{\Omega})$ is a subset of $L^\infty(\Omega)$. However, piecewise-continuous functions are also in $L^\infty(\Omega)$. The property of $L^\infty(\Omega)$ that makes it valuable in considering design problems is that minimizing sequences of functions that define mechanical properties, such as cross-sectional area of a beam or thickness of a plate, have the property that if they converge in the space $L^\infty(\Omega)$, they remain essentially bounded, which is a physical property that must be preserved. Once such a limiting function is defined, it can be modified on only a set of measure zero to cause it to be finite everywhere.

A.2.5 $H^m(\Omega)$; Sobolev Space of Order m

Because strain energies in structural components are written as integrals of quadratic expressions in first or second derivatives of displacement fields and since strain energy must be finite for any physically meaningful displacement field, it is natural to define spaces of functions that can be displacement fields in such a way that strain energy is guaranteed to be finite. Since derivatives of displacement fields define strain, and strain must be integrable, the regularity of such functions must at least allow for evaluation of strain energy. These considerations then make it natural to define a *Sobolev space of order m* as

$$H^m(\Omega) \equiv \left\{u \in L^2(\Omega)\colon \quad \frac{\partial^{|k|}u}{\partial x^k} \in L^2(\Omega), \quad |k| \le m\right\} \tag{A.2.48}$$

Such a space may be considered as a space of candidate displacement fields in elasticity for $m = 1$ and for displacement of a beam or plate with $m = 2$.

A *scalar product* may be defined on this Sobolev space as

$$(u, v)_{H^m(\Omega)} \equiv \sum_{|k| \le m} \iint_\Omega \frac{\partial^{|k|}u}{\partial x^k} \frac{\partial^{|k|}v}{\partial x^k} \, d\Omega \tag{A.2.49}$$

It is reasonably direct to show that this bilinear functional has the properties of Eqs. (A.2.13)–(A.2.17) and is therefore a scalar product [36]. A *norm* on the Sobolev space can therefore be naturally defined as

$$\|u\|_{H^m(\Omega)} = \left[\sum_{|k| \leq m} \iint_\Omega \left(\frac{\partial^{|k|} u}{\partial x^k} \right)^2 d\Omega \right]^{1/2} \tag{A.2.50}$$

It is proved in the literature on Sobolev spaces [36] that an equivalent definition of the Sobolev space can be given in terms of Cauchy sequences of functions in $\{u \in C^m(\Omega)\colon \ \|u\|_{H^m(\Omega)} < \infty\}$ as follows:

$$\begin{aligned} H^m(\Omega) = \{u\colon \ & \text{for some Cauchy sequences} \\ & \{\phi^i\} \ \text{ in } \ \{u \in C^m(\Omega)\colon \ \|u\|_{H^m(\Omega)} < \infty\}, \\ & \lim_{i \to \infty} \|\phi^i - u\|_{H^m(\Omega)} = 0\} \end{aligned} \tag{A.2.51}$$

Thus,

$$\{u \in C^m(\Omega)\colon \quad \|u\|_{H^m(\Omega)} < \infty\}$$

is *dense* in $H^m(\Omega)$. It is also shown in the literature [36] that $H^m(\Omega)$ is *complete*, hence it is a Hilbert space.

Since convergence of a sequence of functions in the $H^m(\Omega)$ norm involves $L^2(\Omega)$ convergence of derivatives up through order m, it appears reasonable that such convergence should preserve m derivatives of the limit function. As will be seen later, this is indeed the case and provides a natural setting for the study of boundary-value problems using modern variational techniques.

A.2.6 $H_0^m(\Omega)$; Sobolev m-Space with Compact Support

A function $u(x)$ is said to have *compact support* on Ω if there is a compact set $S \subset \Omega$ such that $u(x) = 0$ for $x \notin S$. Much as in the alternative definition of Sobolev space of Eq. (A.2.51), a new space may be defined as a similar limit of Cauchy sequences of functions that have compact support; that is,

$$\begin{aligned} H_0^m(\Omega) = \{u \in H^m(\Omega)\colon \ & \text{for some Cauchy sequence} \ \{\phi^i\} \ \text{ of } \ C^\infty(\Omega) \\ & \text{functions with compact support} \ \lim_{i \to \infty} \|\phi^i - u\|_{H^m(\Omega)} = 0\} \end{aligned} \tag{A.2.52}$$

Since, as noted above, it might be expected that limits of functions in Sobolev space preserve properties of derivatives, functions and some of their derivatives that appear in $H_0^m(\Omega)$ should be zero on the boundary of Ω. As will be shown later, this and more is true.

A.2.7 The Sobolev Imbedding Theorem

Although the proof is not easy, it is shown in the literature [36] that if Ω is a bounded domain in R^n with a smooth boundary and if $2m > n$, then

$$H^{j+m}(\Omega) \subset C^j(\bar{\Omega}) \tag{A.2.53}$$

Furthermore, the identity mapping from $H^{j+m}(\Omega)$ to $C^j(\bar{\Omega})$ is continuous; that is, there exist constants $K_j < \infty$ such that for all u in $H^{j+m}(\Omega)$,

$$\|u\|_{C^j(\bar{\Omega})} \leq K_j \|u\|_{H^{j+m}(\Omega)} \tag{A.2.54}$$

This theorem gives valuable information concerning properties of functions in Sobolev spaces. In particular, it was noted earlier that functions that are defined as limits of sequences in the L^2 norm need not have finite values at isolated points. The Sobolev imbedding theorem, however, guarantees that in Sobolev spaces these functions are continuous and in many cases continuously differentiable due to the introduction of L^2-norm convergence of the derivatives of such functions in the Sobolev norm.

As an example, consider the displacement of a string on the interval [0, 1] in R^1. To assure finite strain energy, it must be in $H^1(0, 1)$. By the Sobolev imbedding theorem, Eq. (A.2.53) guarantees that

$$H^1(0, 1) \subset C^0[0, 1] \tag{A.2.55}$$

and boundary conditions such as $u(0) = u^0$ and $u(1) = u^1$ will be preserved in convergence of sequences of functions in $H^1(0, 1)$.

Similarly, in the case of a beam on the interval [0, 1], finiteness of strain energy demands that displacement functions be in $H^2(0, 1)$. Thus, by the Sobolev imbedding theorem,

$$H^2(0, 1) \subset C^1[0, 1] \tag{A.2.56}$$

Thus, admissible beam displacements must be continuously differentiable, and boundary conditions of the form $u(0) = u^0$ and $(du/dx)(0) = u'^0$ will be preserved if limits of sequences of such functions are taken in the H^2 norm.

If $2m > n$ and if $u \in H_0^{j+m}(\Omega)$, it is a $C^j(\bar{\Omega})$ limit of smooth functions that are zero on the boundary Γ of Ω. Thus,

$$\frac{\partial^{|k|} u}{\partial x^k} = 0 \qquad k \leq j \quad \text{on} \quad \Gamma \tag{A.2.57}$$

For example, if $u \in H_0^2(0, 1)$, then since

$$H_0^2(0, 1) \subset H^2(0, 1) \subset C^1[0, 1] \tag{A.2.58}$$

u must be a $C^1[0, 1]$ limit of functions that are zero in the boundary. Hence,

$$u(0) = u(1) = \frac{du(0)}{dx} = \frac{du(1)}{dx} = 0 \tag{A.2.59}$$

A.2.8 Trace Operator

The operation of projecting a function defined on the interior of a set Ω to its boundary Γ is the process of evaluating the function on the boundary, if the function has a regular extension to the boundary. In general, such a projection is called the *trace* of the function. In particular, for $u \in H^m(\Omega)$, the trace is defined as

$$\gamma u = [\gamma_0(u) \quad \cdots \quad \gamma_{m-1}(u)] \qquad \text{on} \quad \Gamma \tag{A.2.60}$$

That is, it contains the projection of the function and its first $m - 1$ derivatives to the boundary Γ of Ω, where $\gamma_j(u) = \partial^j u/\partial n^j$ and n is the outward normal to Γ.

The nature of functions projected onto the boundary is somewhat more complicated than has been encountered in spaces of functions on the domain Ω. In particular, it is shown in the literature [36] that γ is a mapping from $H^m(\Omega)$ to a product space (see Section A.2.9) of boundary values of the function, which are *fractional-order Sobolev spaces* on the boundary; that is,

$$\gamma: H^m(\Omega) \to \prod_{j=1}^{m-1} H^{m-1-1/2}(\Gamma) \tag{A.2.61}$$

Due to technical complexity associated with even defining the fractional-ordered spaces on the boundary, no attempt to describe these spaces is given here (see Adams [36]). This theory, however, makes precise the regularity properties required of functions appearing in boundary conditions of boundary-value problems [9].

Of specific interest here is the anticipated result that boundary evaluations of functions appearing in $H_0^m(\Omega)$ are zero. Even more, it is shown that every function in $H_0^m(\Omega)$ is of this kind; that is,

$$H_0^m(\Omega) = \{u \in H^m(\Omega): \quad \gamma u = 0\} \tag{A.2.62}$$

Thus, the space $H_0^m(\Omega)$ is exactly the space of candidate solutions of Dirichlet boundary-value problems in which homogeneous boundary conditions are specified for a differential operator equation of order $2m$ to include zero values of the function and its first $m - 1$ derivatives on the boundary. This precisely defines the space of candidate solutions of such a boundary-value problem and provides substantial information on the nature of solutions.

A.2.9 Product Spaces

As a final topic in considering function spaces, it is helpful to define a function space whose elements are groupings of functions of quite different character. For example, consider two function spaces denoted X and Y.

Their *product space* is defined as the collection of all pairs of functions, one from X and one from Y,

$$X \times Y = \{[u, v]: \quad u \in X, v \in Y\} \tag{A.2.63}$$

A *norm* on this product space can be defined as

$$\|[u, v]\|_{X \times Y} \equiv \|u\|_X + \|v\|_Y \tag{A.2.64}$$

As an example of a product space, consider the design of a plate of variable thickness, in which the function h, defining the thickness in $L^\infty(\Omega)$, and Young's modulus $E \in R^1$ are the design variables. The design space can be defined as the product space of these two spaces of different types of design variable as

$$U \equiv L^\infty(\Omega) \times R^1 = \{[h, E]: \quad h \in L^\infty(\Omega), \quad E \in R^1\} \tag{A.2.65}$$

and will have the norm

$$\|[h, E]\|_U = \|h\|_{L^\infty(\Omega)} + |E| \tag{A.2.66}$$

Use of this product space idea is essential in establishing the regularity of dependence of solutions of boundary-variable problems on design variables.

A.3 DIFFERENTIALS AND DERIVATIVES IN NORMED SPACES

The purpose of this section is to summarize the definitions of properties of differentials and derivatives of nonlinear mappings or functions, which extend the classical idea of differential and derivative to the calculus of variations and its generalizations. The value of these abstract differentials and derivatives is both practical and theoretical. Practically, the theory allows for first-order approximation or "linearization" of nonlinear functionals that arise in structural design. From a theoretical point of view, differentials and derivatives are used heavily throughout the text to prove existence results and properties of dependence of structural response measures on design variables.

A.3.1 Mappings in Normed Spaces

Consider vector spaces X and Y, with norms $\|\cdot\|_X$ and $\|\cdot\|_Y$, respectively. These spaces may be any of the normed spaces that are discussed in Section A.2. A function $\Phi(x)$ that defines a vector in Y, once a vector x in X is specified, may be viewed as a mapping from X into Y denoted as

$$\Phi: \quad X \to Y \tag{A.3.1}$$

A special case is $X = R^1$ and $Y = R^1$, in which Φ is a real-valued function of a single real variable. If, on the other hand, $X = L^2(\Omega)$ is a space of designs and $Y = [H^1(\Omega)]^3$ is the Sobolev space of displacements of an elastic solid, then Φ may be defined as a mapping from the space X of designs to the space Y of solutions of boundary-value problems of elasticity, where $\Phi(x)$ is the solution of the boundary value problem for design x.

The concept of continuity of a mapping between normed spaces is a direct extension of the concept of continuity of scalar functions of scalar variables. More specifically, the mapping Φ is *continuous* at x if, for every $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$\|\Phi(x + \eta) - \Phi(x)\|_Y \leq \varepsilon \tag{A.3.2}$$

for all $\eta \in X$ such that

$$\|\eta\|_X < \delta \tag{A.3.3}$$

If Φ is continuous at every $x \in X$, then it is said to be continuous on X.

An algebraic property of the mappings that is of some importance in design sensitivity analysis concerns linearity. A mapping Φ is said to be *homogeneous of degree n* if

$$\Phi(\alpha x) = \alpha^n \Phi(x) \tag{A.3.4}$$

where α is any real number. If Eq. (A.3.4) holds only for $\alpha \geq 0$, then Φ is said to be *positively homogeneous of degree n*. A more important concept is linearity of a mapping. More specifically, Φ is said to be a linear mapping if

$$\Phi(\alpha x + \beta z) = \alpha\Phi(x) + \beta\Phi(z) \tag{A.3.5}$$

for all x and z in X and for all real α and β. Note that a linear mapping is homogeneous of degree one.

A.3.2 Variations and Directional Derivatives

The idea of derivative or differential of a scalar function of a scalar variable can be profitably extended to general mappings. First, one may define the *one-sided Gateaux differential* as

$$\Phi'_+(x, \eta) \equiv \lim_{\substack{\tau \to 0 \\ \tau > 0}} \frac{1}{\tau}[\Phi(x + \tau\eta) - \Phi(x)] \tag{A.3.6}$$

providing the limit on the right side exists. The term $\Phi'_+(x, \eta)$ is called the "one-sided Gateaux differential of Φ at point x in the direction η." This differential exists for large classes of mappings, but it may not possess some of the nice properties usually attributed to derivatives in ordinary calculus. A direct calculation shows that for all $\alpha > 0$,

$$\begin{aligned}\Phi'_+(x, \alpha\eta) &= \lim_{\substack{\tau \to 0 \\ \tau > 0}} \frac{1}{\tau}[\Phi(x + \tau\alpha\eta) - \Phi(x)] \\ &= \alpha \lim_{\substack{\alpha\tau \to 0 \\ \tau > 0}} \frac{1}{\alpha\tau}[\Phi(x + \alpha\tau\eta) - \Phi(x)] \\ &= \alpha\Phi'_+(x, \eta) \end{aligned} \tag{A.3.7}$$

which verifies that the one-sided Gateaux differential is positively homogeneous of degree one.

To relate this idea of the differential to a simple function, consider the real-valued function of a single real variable x,

$$\Phi(x) = |x| \tag{A.3.8}$$

A simple check will show that while this function is continuous, it does not have an ordinary derivative at $x = 0$. The one-sided Gateaux differential, however, is defined using Eq. (A.3.6) as

$$\Phi'_+(0, \eta) = \lim_{\substack{\tau \to 0 \\ \tau > 0}} \frac{1}{\tau}[|\tau\eta| - 0] = |\eta| \tag{A.3.9}$$

Note that

$$\Phi'_+(0, -\eta) = |-\eta| = |\eta| \neq -\Phi'_+(0, \eta) \tag{A.3.10}$$

so the one-sided Gateaux differential is not linear in η and in fact is not homogeneous of degree one. Nevertheless, it predicts the change in the function Φ due to a perturbation η in the independent variable x.

If the limit in Eq. (A.3.6) exists for both $\tau > 0$ and $\tau < 0$, then Φ is said to have a *Gateaux differential* (often called the *differential* or *variation*) at x in the direction η, given by

$$\Phi'(x, \eta) = \lim_{\tau \to 0} \frac{1}{\tau}[\Phi(x + \tau\eta) - \Phi(x)] \tag{A.3.11}$$

where the limit may be taken with τ either positive or negative. In this case, the calculations of Eq. (A.3.7) are valid for both positive and negative α, hence the Gateaux differential is homogeneous of degree one.

An example of the Gateaux differential that often arises in structural design sensitivity analysis and in the calculus of variations involves mapping Φ from the space $L^2(\Omega)$ into the real numbers (a functional), defined as

$$\Phi(x) = \iint_\Omega F(x)\, d\Omega \tag{A.3.12}$$

where the scalar-valued function F is presumed to be continuously differentiable. The Gateaux differential of this functional may be calculated as

$$\begin{aligned}\Phi'(x, \eta) &= \lim_{\tau \to 0} \frac{1}{\tau} \iint_\Omega [F(x + \tau\eta) - F(x)]\, d\Omega \\ &= \iint_\Omega \lim_{\tau \to 0} \frac{1}{\tau} [F(x + \tau z) - F(x)]\, d\Omega \\ &= \iint_\Omega \frac{dF}{dx} \eta \, d\Omega \end{aligned} \tag{A.3.13}$$

which may be recognized as the *first variation* of the functional Φ in the calculus of variations. Note that in this special case, $\Phi'(x, \cdot)$ is a linear mapping from $L^2(\Omega)$ to the real numbers.

As will often be the case, the mapping $\Phi'(x, \cdot)$ from X into Y may be continuous and linear, in which case it is called the *Gateaux derivative* of Φ at x.

A.3.3 Fréchet Differential and Derivative

Let the mapping Φ be given as in Eq. (A.3.1). Then Φ is said to be Fréchet differentiable at x if these exists a continuous linear operator $\Phi'(x, \cdot)$: $X \to Y$ such that

$$\lim_{\|\eta\|_X \to 0} [\|\Phi(x + \eta) - \Phi(x) - \Phi'(x, \eta)\|_Y / \|\eta\|_X] = 0 \tag{A.3.14}$$

holds for any $\eta \in X$. The operator $\Phi'(x, \eta)$ in Eq. (A.3.14) is called the Fréchet differential of Φ at x. The mapping $\Phi'(x, \cdot)$ from X into Y is called the Fréchet derivative of Φ at x and is a continuous linear mapping from X to Y.

It is obvious that if Φ is Fréchet differentiable at x, then Φ is Gateaux differentiable at x. It is interesting to note that the Gateaux and Fréchet derivatives are equivalent for functions defined on R^1, but are not equivalent on higher-dimensional spaces. To see this, consider an example with $X = R^2$ and $Y = R^1$. Define Φ: $R^2 \to R^1$ as $\Phi(x_1, 0) = 0$ and $\Phi(x_1, x_2) = (x_1/x_2)(x_1^2 + x_2^2)$, if $x_2 \neq 0$. It is easy to check that the Gateaux derivative exists at $(0, 0)$ and is the zero operator. However, a Fréchet derivative does not exist at $(0, 0)$. In fact, Φ is not even continuous at $(0, 0)$.

Dieudonne [80] showed that if the Gateau derivative $\Phi'(w, \cdot)$ exists for all w in a neighborhood of x and

$$\lim_{w \to x} \|\Phi'(w, \cdot) - \Phi'(x, \cdot)\| = 0, \tag{A.3.15}$$

then the Fréchet derivative exists. Note that the norm in Eq. (A.3.15) is for the space of continuous linear mappings $\mathscr{BL}(X, Y)$ [81].

Consider again the mapping of Eq. (A.3.12) from $L^2(\Omega)$ to the real numbers, with the Gateaux differential defined by Eq. (A.3.13). In order to check whether Φ is Fréchet differentiable, for evaluation of Eq. (A.3.14),

$$\Phi(x + \eta) - \Phi(x) - \Phi'(x, \eta) = \iint_\Omega \left[F(x + \eta) - F(x) - \frac{dF}{dx} \eta \right] d\Omega \tag{A.3.16}$$

By the remainder form of Taylor's formula,

$$F(x + \eta) - F(x) = \frac{dF}{dx} \eta + \frac{1}{2} \frac{d^2F}{dx^2}(\bar{x}) \eta^2 \tag{A.3.17}$$

where $\bar{x} = x + \alpha\eta$ and $0 < \alpha < 1$. If the second derivative of F is bounded by some finite constant K, that is, if

$$\left| \frac{d^2F}{dx^2} \right| < K \tag{A.3.18}$$

then from Eqs. (A.3.16)–(A.3.18),

$$|\Phi(x + \eta) - \Phi(x) - \Phi'(x, \eta)| \leq \frac{K}{2} \iint_\Omega \eta^2 \, d\Omega = \frac{K}{2} \|\eta\|^2_{L^2} \tag{A.3.19}$$

Dividing both sides by $\|\eta\|_{L^2}$ and taking the limit as $\|\eta\|_{L^2}$ goes to zero, it is seen that Eq. (A.3.14) is satisfied and that Φ is Fréchet differentiable.

A.3.4 Partial Derivatives and the Chain Rule of Differentiation

Very often in structural design sensitivity analysis, several variables appear in the same expression. Consider a mapping of Φ that depends on a variable from normed space X and a variable from normed space Z, denoted as $\Phi: X \times Z \to Y$. As in ordinary calculus, $z \in Z$ may be held fixed and the Gateaux differential of Φ calculated as a function of $x \in X$ and similarly hold $x \in X$ fixed and calculate the Gateaux differential of Φ as a function of $z \in Z$ to obtain

$$\begin{aligned} \Phi'_x(x, \eta; z) &\equiv \lim_{\tau \to 0} \frac{1}{\tau} [\Phi(x + \tau\eta; z) - \Phi(x; z)] \\ \Phi'_z(x; z, v) &\equiv \lim_{\tau \to 0} \frac{1}{\tau} [\Phi(x; z + \tau v) - \Phi(x; z)] \end{aligned} \tag{A.3.20}$$

which are called *partial Gateaux differentials* of Φ.

An important result (proved by Dieudonne [80] and Nashed [81]) relates the Gateaux differential of Φ to its partial Gateaux differentials. More specifically, if Φ'_x and Φ'_z of Eq. (A.3.20) exist and are continuous and linear in η and v, then Φ is Fréchet differentiable on $X \times Z$ and

$$\Phi'(x, \eta; z, v) = \Phi'_x(x, \eta; z) + \Phi'_z(x; z, v) \tag{A.3.21}$$

This powerful result permits calculations with individual variables and, providing the hypotheses are checked, yields the Gateaux differential of a mapping as the sum of its partial Gateaux differentials.

A related concept extends the classical chain rule of differentiation. Consider a mapping Θ: $X \to Z$ and a mapping Ψ: $Z \to Y$, both of which are Fréchet differentiable. Then, the composite mapping $\Phi(x) = \Psi(\Theta(x))$ is Fréchet differentiable and

$$\Phi'(x, \eta) = \psi'(\Theta(x))\Theta'(x, \eta) \tag{A.3.22}$$

This result was proved by Dieudonne [80] and its properties were developed and analyzed by Nashed [81]. The chain rule, however, is not valid for Gateaux derivatives [81]. The concept of chain rule differentiation is used extensively in structural design sensitivity analysis, since structural performance measures are often stated as functionals involving the displacement field, which is itself a function of design.

A.4 NOTATION

$a_u(z, \bar{z})$ or $a_\Omega(z, \bar{z})$	Energy bilinear form; $a_u(z, z)$ or $a_\Omega(z, z)$ is twice the strain energy associated with displacement z
A_u	Friedrichs extension of the operator $\bar{A}_u$
$\bar{A}_u$	Formal linear differential operator
b	Design variable vector $b = [b_1 \ \cdots \ b_k]^T$, whose components are parameters (real constants)
B_u	Friedrichs extension of the operator $\bar{B}_u$
$\bar{B}_u$	Formal linear operator for eigenvalue problem
$c(z, \bar{z})$	Bilinear mapping such that $a_\Omega(z, \bar{z}) = \iiint_\Omega c(z, \bar{z})\, d\Omega$
C^{ijkl}	Modulus tensor for linear elasticity
$C^m(\Omega)$	m-times continuously differentiable functions on Ω
$d_u(y, \bar{y})$ or $d_\Omega(y, \bar{y})$	Bilinear form associated with geometric stiffness and mass matrices
D (D_g)	Reduced (generalized) geometric stiffness matrix
$D(A_u)$	Domain of the operator A_u
$D(B_u)$	Domain of the operator B_u
$\hat{D}(u)$	Flexural rigidity of the plate
DV	Jacobian matrix of the velocity field $V(x)$
$e(y, \bar{y})$	Bilinear mapping such that $d_\Omega(y, \bar{y}) = \iiint_\Omega e(y, \bar{y})\, d\Omega$

E	Young's modulus
$F\ (F_g)$	Reduced (generalized) force vector or function
H	Curvature of the boundary Γ
$H^m(\Omega)$	Sobolev space of order m
$H_0^m(\Omega)$	Sobolev m space with compact support
J	Jacobian of the mapping $T(x, \tau)$
$K\ (K_g)$	Reduced (generalized) global stiffness matrix
$l_u(\bar{z})$ or $l_\Omega(\bar{z})$	Virtual work due to virtual displacement $\bar{z}$
$L^2(\Omega)$	Space of Lebesgue square integrable functions on Ω
$L^\infty(\Omega)$	Space of essentially bounded, Lebesgue measurable functions on Ω
m_p	Characteristic function of the set Ω_p
$M\ (M_g)$	Reduced (generalized) global mass matrix
n	External unit normal to the boundary Γ
R^k	k-dimensional Euclidean space
s	Unit tangent to the boundary Γ
$T(x, \tau)$	Transformation mapping
U	Space of the design variables u
$V(x)$	Design velocity field
$\lVert x \rVert$	Norm of the vector x
(x, y)	Scalar product of the vectors x and y
$y\ (y_g)$	Mode (generalized mode) vector or function for vibration and buckling
$z\ (z_g)$	Displacement (generalized displacement) vector or function
$\bar{z}\ (\bar{z}_g)$	Kinematically admissible virtual displacement (generalized displacement)
Z	Space of kinematically admissible displacements
γ	Weight density of the material
δ_{ij}	Kronecker delta
$\hat{\delta}(x)$	Dirac measure at $x = 0$
$\varepsilon^{ij}(z)$	Strain tensor due to the displacement z
ζ	Eigenvalue (the square of natural frequency or buckling load)
$\lambda\ (\lambda_g)$	Adjoint variable (generalized adjoint variable)
$\hat{\lambda}, \mu$	Lamé's constants
ν	Poisson's ratio
ρ	Mass density of the material
$\sigma^{ij}(z)$	Stress tensor due to the displacement z
ψ	Cost or constraint functional
$\Delta\psi = \psi^2 - \psi^1$	Finite difference of the functional ψ

Ω	Open domain in R^n ($n = 1, 2, 3$) with boundary Γ
$\Omega_F = (I + F)\Omega$	Transformation of domain by mapping $F(x)$
Ω_p	Open subset of Ω with boundary Γ_p
$\Omega_\tau = T(\Omega, \tau)$	Transformation of domain Ω by mapping $T(x, \tau)$
$\sim$	Denotes a variable that is held constant for a partial differentiation
	Differential (or variation) of a function or a functional (in Chapters 3 and 4, this notation is used for partial derivative with respect to τ)
	Material or total derivative

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