

Jaap Wijker

Solid Mechanics
and its Applications

Random Vibrations in Spacecraft Structures Design

Theory and Applications

 Springer

Random Vibrations in Spacecraft Structures Design

SOLID MECHANICS AND ITS APPLICATIONS

Volume 165

Series Editor: G.M.L. GLADWELL

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Jaap Wijker

Random Vibrations in Spacecraft Structures Design

Theory and Applications

 Springer

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ISSN 0925-0042

ISBN 978-90-481-2727-6

e-ISBN 978-90-481-2728-3

DOI 10.1007/978-90-481-2728-3

Springer Dordrecht Heidelberg London New York

Library of Congress Control Number: 2009926170

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To my children Léonie and Maarten
and my grandchildren, Noortje, Diede, Neeltje, Job and Pepijn,
who hopefully later will aspire to a job in engineering.

Preface

This book entitled “Random Vibration in Spacecraft Structures Design: Theory and Applications” is based on the lecture notes “*Spacecraft structures*” and “*Special topics about vibration in spacecraft structures*”. The author is lecturer to the graduate students at the Delft University of Technology, faculty of Aerospace Engineering, chair Aerospace Structures. Besides lecturing, the author is employed at Dutch Space BV in The Netherlands, where he gained practical experience applying random vibration analysis techniques in spacecraft design. Both the scientific environment at the University and the practical approach in the course of spacecraft related projects in industry provide a good foundation to compile this book.

This book on low and high frequency mechanical, acoustic random vibrations is of interest to graduate students and engineers working in aerospace engineering, particularly in spacecraft and launch vehicle structures design.

I would like to express my admiration for the patient showed by my wife Wil during the preparation of this manuscript.

Velserbroek, 2009

Jaap Wijker

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Introduction

1.1 General

The various mechanical loads are not all equally important and depend on the type of the mechanical structure: i.e. does it concern a primary structure, the spacecraft structure or other secondary structures (such as solar panels, antennas, instruments and electronic boxes). Requirements are specified to cover loads encountered by handling, testing, during the launch phase and operations in transfer and final orbit, such as [167]:

- natural frequencies
- steady-state (semi-static) acceleration
- sine excitation
- random excitation
- acoustic noise
- transient loads
- shock loads
- temperatures

Natural frequencies: The location of natural frequencies is a primary design requirement for all parts of the spacecraft. This requirement is imposed in order to limit the dynamic coupling of the spacecraft with the launch vehicle

Semi-static and low frequency sinusoidal loads: The design of the primary structure is determined to a large extent by the semi-static and low frequency sinusoidal loads (up to approximately 50 Hz)

Sinusoidal and random loads: To a large extent, the sinusoidal and random loads determine the design of secondary structures (solar panels, antennas, electronic boxes).

Acoustic loads: Light structural parts with relatively large surface areas (such as solar panels and spacecraft antennas) are more sensitive to acoustic loads than sinusoidal and random base excitation.

Shock loads: Deployable structures experience high shock loads; for example during latch-up of hinges in the required final position of these mechanisms. This is especially the case when the deployment speeds are too high.

Temperatures: Temperature variations usually cause high thermal stresses in structures. In general, the various coefficients of expansion are accounted for in the choice of the structural materials. Thermal deformations are taken into account when working with structures that must be aligned with each other.

Random Loads: The design of instruments and electronic boxes is determined by the random base excitation.

All these different types of load are described in detail in [224, 225].

In this book the random mechanical and acoustical vibrations of deterministic and statistical dynamic systems, in the low and high frequency range, are considered, and the following topics will be discussed in great detail:

- Vibrations of deterministic linear mechanical dynamic systems exposed to mechanical random loads and or enforced motion (acceleration)
- Vibrations of deterministic linear mechanical dynamic systems exposed to random acoustic loads (sound pressures)
- Random vibration of statistically defined mechanical systems and loads using Statistical Energy Analysis (SEA)
- Non-linear structures excited to random (white noise) mechanical loads analyzed by using the Fokker-Planck-Kolmogorov (FPK) equation

The theory of random vibration is strongly related to the design of spacecraft structures and will be illustrated with simple and more difficult worked examples; each section ends with posed problems; usually answers are provided.

Figure 1.1 shows a cross section of a typical spacecraft. This may be a communication, scientific or other spacecraft. For this spacecraft, the acoustic and the mechanical random vibration environment outside and inside the spacecraft structure will be discussed. The spacecraft structure is an assembly of structural elements: shells of revolution, panels, shear panels, struts, etc. The spacecraft structure provides strength and stiffness properties to the spacecraft in order to survive test and launch loads.

Among other systems on the outside of the spacecraft there are the antenna reflector and both solar wings, constituting the spacecraft solar array. Both the antenna reflector and the solar array are in folded or stowed configuration, because:

- the folded systems fit better under the fairing of the launch vehicle, and
- the folded systems can carry the launch loads better.

The central structure of the spacecraft is called the primary structure of the structure, and forms the backbone (load path) of the structure. In general,

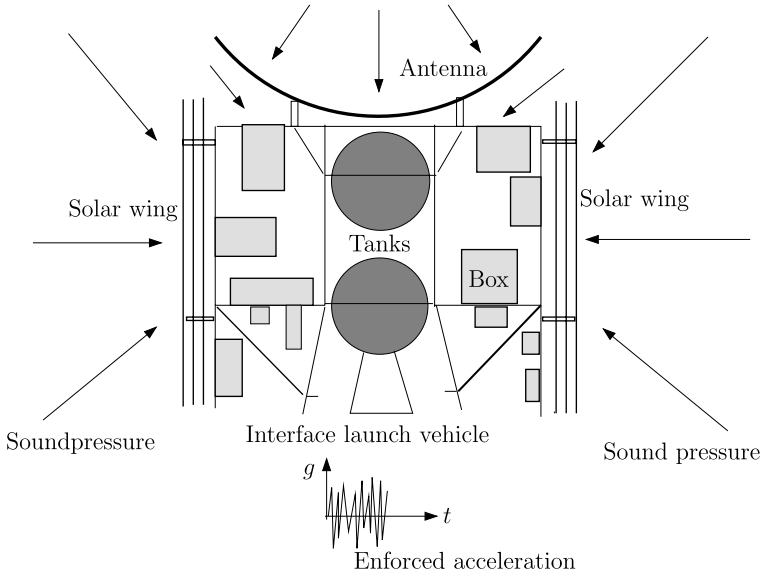


Fig. 1.1. Complete spacecraft

fuel tanks are needed for propulsion. The attitude control systems are fixed to the central body. These tanks are relatively heavy.

Spacecraft quadrilateral sandwich platforms are supported by the central body and side panels used to mount payload and equipment boxes. In Fig. 1.1 we see the top and lower platform. The antenna is mounted to the top platform. The payload and equipment is distributed so as to fulfill center of gravity requirements posed by the launch vehicle authority.

The spacecraft side panels will close the structure box. Solar wings and part of the equipment are mounted to the side panels.

For launch, the spacecraft is placed on the launch vehicle payload adapter structure.

In the liftoff phase of the launch, the exhaust streams of the engines and solid rocket boosters will produce sound waves propagating to the launch vehicle, and will impinge on the launch vehicle structure and fairing. The sound pressures (acoustic load) will excite the launch vehicle structure, which will transfer the vibrations to the interface spacecraft launch vehicle. The acoustic loads are random in nature, hence the derived mechanical vibrations are random too. In the *ECSS*¹ standard [56], general “Qualification” acoustic loads are specified and given in Table 1.1.

The vibrating fairing transfers acoustic loads under the fairing. In general, these have a reverberant nature; they are denoted *sound pressure level* (SPL) and are given in decibels (dB) with a reference pressure $p_{ref} = 2 \times 10^{-5}$ Pa,

¹ European Corporation of Space Standardization.

Table 1.1. Acoustic qualification test levels and duration [56]

Center Frequency (Hz)	Sound Pressure Level (SPL) (dB)
Octave band	0 dB = 2×10^{-5} Pa
31.5	130
63	135.5
125	139
250	143
500	138
1000	132
2000	128
4000	124
8000	120
OASPL	147
Duration:	2 min

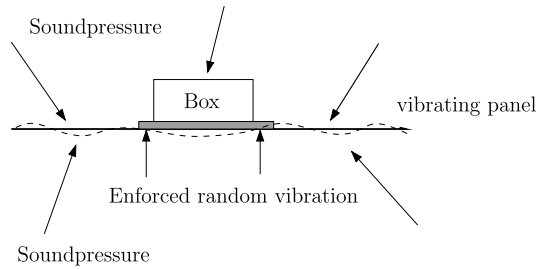


Fig. 1.2. Acoustic loads converted into random mechanical loads

which is ostensibly the audible limit of the human ear [133]. The sound pressure is relatively low with respect to the static atmospheric pressure of 10^5 Pa, 1 Bar, but large areal light weight structures are very sensitive to dynamic sound pressures.

The sound pressure will excite the outside equipment and the outside spacecraft structure, especially the external panels. The random mechanical vibration of the external panels will (a) excite the fixed equipment (see Fig. 1.2) and (b) will generate acoustic loads in the inside cavities of the spacecraft, which in turn will excite the internal load-carrying structures like the central structure and the lower platform. Summarizing, it can be said that the sound pressures will cause random mechanical vibrations in the spacecraft structure, creating a rather heavy random vibration environment for the spacecraft payload, tanks, equipment, etc. Even the direct transfer of vibro-acoustic energy to an unit (experiment, instrument, box, . . .) structure cannot be neglected due to the large unit surfaces [196].

In the ECSS standard [56] “Qualification” mass dependent random enforced accelerations are specified and given in Tables 1.2 and 1.3.

Table 1.2. Random vibration test levels and duration for equipment with mass $M \leq 50$ kg [56]

Location	Duration	Frequency range (Hz)	Levels (g^2/Hz)
Equipment located on external panel or with unknown location	Vertical	20–100	3 dB/oct.
	2.5	100–300	$0.12 \frac{M+20}{M+1}$
	min/axis	300–2000	–5 dB/oct.
	Lateral	20–100	3 dB/oct.
	2.5	100–300	$0.15 \frac{M+20}{M+1}$
Equipment not located on external panel	min/axis	300–2000	–5 dB/oct.
	All axes	20–100	3 dB/oct.
	2.5	100–300	$0.05 \frac{M+20}{M+1}$
	min/axis	300–2000	–5 dB/oct.

Table 1.3. Random vibration test levels and duration for equipment with mass $M > 50$ kg [56]

Frequency range (Hz)	Levels (g^2/Hz)	Remark
20–100	3 dB/oct.	
100–300	0.09	11.12 G_{rms}
300–2000	–3 dB/oct.	
Duration:	all axes	2.5 min/axis

1.2 Random Mechanical Vibration

In part I we discuss the aspects of random vibrations of deterministic mechanical structures. Predictions made about the random vibrations levels are limited to the low frequency domain because the vibration theory is based on simple single degree of freedom (SDOF) systems. These forms the basis for the modal displacement method (MDM), which is frequently used in the finite element applications.

1.3 Random Acoustic Vibration

Large-area light-weight mechanical structures are very sensitive to random acoustic loads. The same procedures as discussed in part I are applied, however, now the applied loads are distributed over the surface of the mechanical structure. The distributed load application is discussed in part II. Plane waves, rain on the roof and reverberant (diffuse) sound fields are considered.

In the first part of the chapter 3 fluid structure interaction (FSI) is ignored. The exposed pressure field cause structural responses, but the influence of the vibrating structure on the pressure field is neglected. Later on, the full FSI is discussed in detail, e.g. radiation, which will introduce radiation damping. Both analytical and approximate methods will be discussed.

1.4 Statistical Energy Analysis

Complementary to low frequency mechanical vibrations, the *statistical energy analysis* (SEA) method is discussed in part III. The basis of the SEA method is the power flow between oscillators or groups of oscillators. The structures and loads are described in a statistical manner that in contrast with the deterministic description of structures using the finite element method. The number of modes per frequency dictates the application of the SEA method in the higher frequency bands. Both random mechanical and acoustical loads can be considered within the framework of the SEA method.

1.5 Fokker-Planck-Kolmogorov Equation

Part IV is more or less based on Gaussian, white noise processes, leading to a Markoff process in which each event is dependent only on the event before it. The *Fokker-Planck-Kolmogorov* (FPK) partial differential diffusion equation is derived from the Markoff process. The unknown in this FPK equation is the transition probability density function, and after integration, the joint probability function. Mean values and correlation functions (second moments), up-crossings and first passage statistics can be obtained from the FPK equation.

The stochastic differential equations (SDE), either in Itô or Stratonovich sense (definition of integration) are closely linked to the FPK equation.

To solve nonlinear random vibrational problems we can use the FPK equation; analytical and numerical methods are discussed.

Huge computer power is needed to solve the FPK equation numerically. In general, the applications of the FPK equation is restricted to nonlinear dynamic systems with a few DOFs.

Random Mechanical Vibration

Linear Random Vibration Systems

2.1 Introduction

By random vibration of a linear dynamic system we mean the vibration of a deterministic linear system exposed to random (stochastic) loads. Random processes are characterized by the fact that their behavior cannot be predicted in advance and therefore can be treated only in a statistical manner. An example of a *micro-stochastic process* is the “Brownian motion” of particles and molecules [218]. A macro-stochastic process example is the motion of the earth during an earthquake. During the launch of a spacecraft, it will be exposed to random loads of mechanical and acoustic nature. The random mechanical loads are the base acceleration excitation at the interface between the launch vehicle and the spacecraft. The random loads are caused by several sources, e.g. the interaction between the launch-vehicle structure and the engines, exhaust noise, combustion. Turbulent boundary layers will introduce random loads. In this chapter we review the theory of random vibrations of linear systems. For further study on the theory of random vibration see [16, 115, 136, 154].

2.2 Probability

The *cumulative probability function* $F(x)$, that $x(t) \leq X$, is (c.d.f.) given by

$$F(X) = \int_{-\infty}^X f(x)dx \quad (2.1)$$

where

- $f(x)$ is the *probability density function* (p.d.f.) with the following properties
- $f(x) \geq 0$
- $\int_{-\infty}^{\infty} f(x)dx = 1$

- $F(X + dx) - F(X) = \int_X^{X+dx} f(x)dx = f(X)dx, X \leq x(t) \leq X + dx$

The cumulative probability function has the following properties:

- $F(-\infty) = 0$
- $F(\infty) = 1$
- $0 \leq F(x) \leq 1$
- $f(x) = \frac{dF(x)}{dx}$

Examples of probability density functions are:

- The constant distribution $U(a, b)$; X is called equally distributed over the interval $[a, b]$, $X \sim G(a, b)$, $f(x) = \frac{1}{b-a}$, $a \leq x \leq b$, $f(x) = 0$ elsewhere.
- The normally distribution¹ $N(\mu, \sigma)$, $\sigma > 0$. X is normally distributed with the parameters μ and σ , $X \sim N(\mu, \sigma)$ when $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$.
- The log normal distribution $LN(\mu, \sigma)$, $\sigma > 0$. X is log normal distributed with the parameters μ and σ , $X \sim LN(\mu, \sigma)$, $x > 0$, when $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(\ln(x)-\mu)^2}{2\sigma^2}}$.
- The Rayleigh distribution $R(\sigma)$, $\sigma > 0$. X is Rayleigh distributed with the parameter σ , $X \sim R(\sigma)$, $x > 0$, when $f(x) = (\frac{2x}{\sigma^2})e^{-\frac{x^2}{\sigma^2}}$.

For an ergodic random process, the term $f(x)dx$ may be approximated by

$$f(x)dx \approx \lim_{T \rightarrow \infty} \frac{1}{T} \sum_i \delta t_i, \tag{2.2}$$

where the δt_i are the lingering periods of $x(t)$ between $\alpha \leq x \leq \beta$. This is illustrated in Fig. 2.1.

The mode is defined as the peak of the p.d.f. $f(x)$, and the *mean value* μ has an equal moment to the left and to the right of it

$$\int_{-\infty}^{\infty} (x - \mu)f(x)dx = 0. \tag{2.3}$$

This means that the *average value* (mean value, mathematical expectation) of x can be calculated from

$$E(x) = \mu = \frac{\int_{-\infty}^{\infty} x f(x)dx}{\int_{-\infty}^{\infty} f(x)dx} = \int_{-\infty}^{\infty} x f(x)dx. \tag{2.4}$$

The definition of the n -th moment about the mean value is as follows

$$\mu_n = \int_{-\infty}^{\infty} (x - \mu)^n f(x)dx. \tag{2.5}$$

¹ The normal distribution was discussed in 1733 by De Moivre. It was afterwards treated by Gauss and Laplace, and is often referred to as the Gauss or Gauss-Laplace distribution [41].

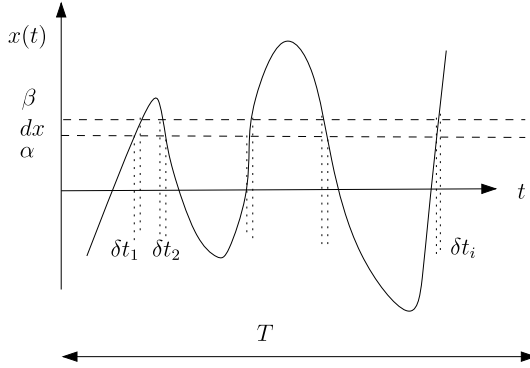


Fig. 2.1. Transient signal

The second moment is called the *variance* of a signal $x(t)$

$$\sigma^2 = \mu_2 = \int_{-\infty}^{\infty} (x - \mu)^2 f(x) dx, \tag{2.6}$$

and σ is called the *standard deviation*.

Example. Suppose a sinusoidal signal $x(t) = A \sin \omega t$. Over one period T the signal $x(t)$ will cross a certain level twice when $X \leq x(t) \leq X + dx$, with a total time $2\delta t$. The p.d.f. can be estimated from $f(x) dx = \frac{2\delta t}{T} = \frac{\omega \delta t}{\pi}$ and with $\delta x = \omega A \cos \omega t \delta t$ the p.d.f. becomes $f(x) = \frac{1}{\pi A \cos \omega t} = \frac{1}{\pi A \sqrt{1 - (\frac{x(t)}{A})^2}}$, $x < A$. The mean value is in accordance with (2.4)

$$E(x) = \mu = \int_{-\infty}^{\infty} x f(x) dx = 0,$$

and the variance σ^2 is in accordance with 2.6

$$\sigma^2 = \mu_2 = \int_{-\infty}^{\infty} (x - \mu)^2 f(x) dx = \frac{A^2}{2}.$$

In general, within the framework of linear vibrations we may assume that the averaged (mean) value μ , of the response of a linear systems exposed to dynamic loads will be zero. So the second moment about the mean, the variance σ^2 , is equal to the mean square value $E(x^2) = \sigma^2 = \int_{-\infty}^{\infty} x^2 f(x) dx$.

Example. A random process x is randomly distributed between $0 \leq x \leq 1$ with a p.d.f.

$$f(x) = \begin{cases} 1 & 0 \leq x \leq 1, \\ 0 & x < 0, x > 1. \end{cases}$$

Calculate the mean value, the mean square value, the variance and the standard deviation of x :

- The mean value $E(x) = \mu = \int_{-\infty}^{\infty} x f(x) dx = \frac{1}{2}$
- The mean square value $E(x^2) = \int_{-\infty}^{\infty} x^2 f(x) dx = \frac{1}{3}$
- The variance $\sigma^2 = E(x^2) - \mu^2 = \int_{-\infty}^{\infty} (x - \mu)^2 f(x) dx = \frac{1}{12}$
- The standard deviation $\sigma = \sqrt{\frac{1}{12}} = 0.289$

The definition of a *cross probability function* or second order *probability distribution function* of two random processes $x(t)$ and $y(t)$ is given by

$$P(X, Y) = \text{Prob}[x(t) \leq X; y(t) \leq Y], \quad (2.7)$$

or, in terms of the specific probability density function

$$P(X, Y) = \int_{-\infty}^X \int_{-\infty}^Y f(x, y) dx dy. \quad (2.8)$$

Therefore, we can conclude that

$$\text{Prob}[X_1 \leq x(t) \leq X_2; Y_1 \leq y(t) \leq Y_2] = \int_{X_1}^{X_2} \int_{Y_1}^{Y_2} f(x, y) dx dy. \quad (2.9)$$

The specific probability density function $f(x, y)$ has the following properties

- $f(x, y) \geq 0$
- $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) dx dy = 1.$

The probability density function of the first order can be obtained from the specific probability density function of the second order because

$$\begin{aligned} \text{Prob}[X_1 \leq x(t) \leq X_2; -\infty \leq y(t) \leq \infty] &= \int_{X_1}^{X_2} \left[\int_{-\infty}^{\infty} f(x, y) dy \right] dx \\ &= \int_{X_1}^{X_2} f(x) dx, \end{aligned} \quad (2.10)$$

where

$$f(x) = \int_{-\infty}^{\infty} f(x, y) dy. \quad (2.11)$$

In a similar manner it is found that

$$f(y) = \int_{-\infty}^{\infty} f(x, y) dx. \quad (2.12)$$

The probability density functions $f(x)$ and $f(y)$ are also called *marginal density functions*, [140].

If random variables $x(t)$ and $y(t)$ are statistically independent, then $f(x, y)$ satisfies

$$f(x, y) = f(x)f(y). \quad (2.13)$$

The mean value or *mathematical expectation* of a continuous function $g(x, y)$ is given by

$$E\{g(x, y)\} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x, y)f(x, y)dxdy. \quad (2.14)$$

The mean values of $x(t)$ and $y(t)$ can be obtained as follows

$$E\{x(t)\} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xf(x, y)dxdy = \int_{-\infty}^{\infty} xf(x)dx, \quad (2.15)$$

$$E\{y(t)\} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} yf(x, y)dxdy = \int_{-\infty}^{\infty} yf(y)dy. \quad (2.16)$$

The *n-dimensional Gaussian probability density function* with the random variables $x_1(t), x_2(t), \dots, x_n(t)$ is given by, [149],

$$f(x_1, x_2, \dots, x_n) = \frac{1}{\sigma_1\sigma_2 \cdots \sigma_n \sqrt{(2\pi)^n \sigma}} e^{-\frac{1}{2\sigma} \sum_{k,l=1}^n \{\sigma_{kl} \frac{(x_k - m_k)(x_l - m_l)}{\sigma_k \sigma_l}\}}, \quad (2.17)$$

where

$$m_i = E\{x_i\}, \quad i = 1, 2, \dots, n$$

represents the mean value, and

$$\sigma_i^2 = E\{(x_i(t) - m_i)^2\}, \quad i = 1, 2, \dots, n$$

is the variance. In addition, the standard deviation σ is given by

$$\sigma = \begin{vmatrix} 1 & \varrho_{12} & \cdots & \varrho_{1n} \\ \varrho_{21} & 1 & \cdots & \varrho_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \varrho_{n1} & \varrho_{n2} & \cdots & 1 \end{vmatrix},$$

where

$$\varrho_{ij} = \frac{E\{(x_i - m_i)(x_j - m_j)\}}{\sigma_i \sigma_j}, \quad i, j = 1, 2, \dots, n,$$

is the correlation coefficient of the two random variables x_i and x_j .

Characteristic Function

The *characteristic function* of a random variable x is defined as the Fourier transform of the probability density function [202]

Table 2.1. Characteristic functions

Distribution	$f(x)$	$E(x) = \mu_x$	σ_x^2	$M_x(\theta)$
$U(a, b)$	$\frac{1}{b-a}$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$	$\frac{1}{j\theta(b-a)}(e^{j\theta b} - e^{j\theta a})$
$N(\mu, \sigma^2)$	$\frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$	μ	σ	$e^{j\mu\theta - \frac{\sigma^2\theta^2}{2}}$

$$M_x(\theta) = E\{e^{j\theta x}\} = \int_{-\infty}^{\infty} e^{j\theta x} f(x) dx. \tag{2.18}$$

Expanding the exponential term $e^{j\theta x}$ in power series will yield

$$M_x(\theta) = 1 + \sum_{n=1}^{\infty} \frac{(j\theta)^n}{n!} E\{x^n\}. \tag{2.19}$$

The moments of the random variable can be calculated from the characteristic function:

$$E\{x^n\} = \frac{1}{j^n} \left. \frac{d^n M_x(\theta)}{d\theta^n} \right|_{\theta=0}. \tag{2.20}$$

The n th *cumulant function* can also be derived from the characteristic function

$$k_n(x) = \frac{1}{j^n} \left. \frac{d^n \ln M_x(\theta)}{d\theta^n} \right|_{\theta=0}. \tag{2.21}$$

The first cumulant function is the same as the first moment, and the second and third cumulant functions are identical with the second and third central moments m_n

$$m_n = k_n(x) = \int_{-\infty}^{\infty} (x - \mu)^n f(x) dx,$$

where $m_1 = \mu$.

Table 2.1 shows two examples of the characteristic function.

Example. For a zero mean Gaussian random variable x , $\mu = 0$, the following expression can be derived $E\{x^4\} = 3(E\{x^2\})^2 = 3\sigma^4$. This can be proved using (2.20)

$$E\{x^4\} = \frac{1}{j^4} \left. \frac{d^4 M_x(\theta)}{d\theta^4} \right|_{\theta=0, \mu=0} = 3\sigma^4.$$

A general recurrent expression for $E\{x^n\}$ is the subject of problem 2.5.

The cumulant functions can be calculated using (2.21)

$$k_1 = \mu = 0, \quad k_2 = \sigma^2, \quad k_n = 0, \quad n > 2.$$

Problems

2.1. The simply supported beam AB shown in Fig. 2.2 is carrying a load of 1000 N that may be placed anywhere along the span of the beam. This

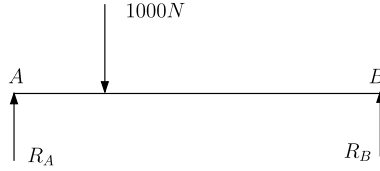


Fig. 2.2. Simply supported beam AB

problem is taken from [5]. The reaction force at support A , R_A , can be any value between 0 and 1000 N depending on the position of the load on the beam. What is the probability density function of the reaction force R_A ? Calculate the probability that

- $Prob(100 \leq R_A \leq 200)$,
- $Prob(R_A \geq 600)$.

Answers: $f(x) = 1/1000$, $0 \leq x \leq 1000$, 0.10, 0.40

2.2. A random variable X is uniformly distributed over the interval (a, b) , $a < b$) and otherwise zero.

- Define the probability density function $f(x)$ such that $\int_{-\infty}^{\infty} f(x)dx = 1$.
- Calculate $E(X)$.
- Calculate $E(X^2)$.
- Calculate the variance $Var(X)$.
- Calculate the standard deviation σ_X , and
- Calculate the distribution function $F(x) = P(X \leq x) = \int_{-\infty}^x f(x)dx$, $a < x < b$.

Answers: $f(x) = \frac{1}{b-a}$, $E(X) = \frac{a+b}{2}$, $E(X^2) = \frac{a^2+b^2+ab}{3}$, $Var(X) = \frac{(b-a)^2}{12}$, $\sigma_X = \frac{(b-a)}{\sqrt{12}}$, and $F(x) = \frac{x-a}{b-a}$.

2.3. A continuous random variable X is said to have *gamma distribution* if the probability density function of X is

$$f(x, \alpha, \beta) = \begin{cases} \frac{1}{\beta^\alpha \Gamma(\alpha)} x^{\alpha-1} e^{-\frac{x}{\beta}}, & x \geq 0; \\ 0, & \text{otherwise,} \end{cases}$$

where the parameters α and β satisfy $\alpha > 0$, $\beta > 0$. Show that the mean and variance of such a random variable X satisfy

$$E(X) = \alpha\beta, \quad Var(X) = \alpha\beta^2.$$

The gamma function is defined by

$$\Gamma(k) = \int_0^{\infty} e^{-u} u^{k-1} du.$$

Show that

$$\Gamma(k) = (k - 1)!$$

for integer k .

2.4. Each front tire on a particular type of vehicle is supposed to be filled to a pressure of 2.5 Bar. Suppose the actual pressure in each tire is a random variable, X for the right tire and Y for the left tire, with joint p.d.f.

$$f(x, y) = \begin{cases} K(x^2 + y^2), & 2.0 \leq x \leq 3.0, \quad 2.0 \leq y \leq 3.0; \\ 0, & \text{otherwise.} \end{cases}$$

- What is the value of K ?
- What is the probability that both tires are underinflated?
- What is the probability that the difference in air pressure between the two tires is at most 0.2 Bar?
- Are X and Y independent random variables?

2.5. This problem is taken from [112]. Let X be a Gaussian random variable with a characteristic function

$$M_x(\theta) = e^{j\mu\theta - \frac{\sigma^2\theta^2}{2}}.$$

Show that

$$E\{X^n\} = \mu E\{X^{n-1}\} + (n - 1)\sigma^2 E\{X^{n-2}\}.$$

2.6. The gamma probability density function is defined by

$$f(x) = \frac{\lambda(\lambda x)^{k-1} e^{-\lambda x}}{\Gamma(k)},$$

where λ and k are distribution parameters. The function $\Gamma(k)$ is the gamma function, which is given by

$$\Gamma(k) = \int_0^\infty e^{-u} u^{k-1} du.$$

Show that the mean and the variance are as follows:

$$\begin{aligned} \mu_x &= \frac{k}{\lambda}, \\ \sigma_x^2 &= \frac{k}{\lambda^2}. \end{aligned}$$

2.3 Random Process

A random process is random in time. The probability can be described with the aid of probabilistic theory of random processes [146]. The mean and the

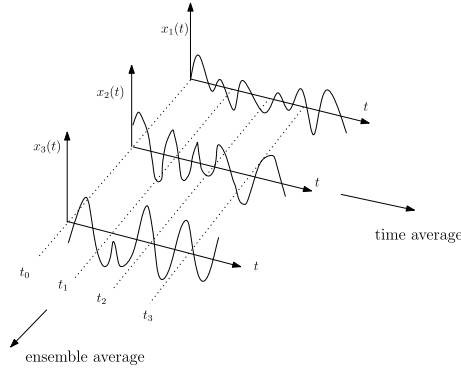


Fig. 2.3. Time history of a random process

mean square values are of great importance for random processes. We can make a distinction between *ensemble average* and *time average*. In this section we review the properties of random processes. The ensemble average $E\{\}$ of a collection of sampled records $x_j(t_i)$, $j = 1, 2, \dots, n$ at certain times t_i is defined as

$$E\{x(t_i)\} = \frac{1}{n} \sum_{j=1}^n x_j(t_i), \quad i = 0, 1, 2, \dots \tag{2.22}$$

This is illustrated in Fig. 2.3.

A random or stochastic process $x(t)$ is said to be stationary in the strict² sense if the set of finite dimensional joint probability distributions of the process is invariant under a linear translation $t \rightarrow t + a$.

$$\begin{aligned} F_x(x_1, t_1) &= F_x(x_1, t_1 + a), \\ F_x(x_1, t_1; x_2, t_2) &= F_x(x_1, t_1 + a; x_2, t_2 + a), \\ &\vdots \end{aligned} \tag{2.23}$$

$$F_x(x_1, t_1; x_2, t_2; \dots; x_n, t_n) = F_x(x_1, t_1 + a; x_2, t_2 + a; \dots; x_n, t_n + a).$$

If (2.23) holds only for $n = 1$ and $n = 2$ the process is stationary in the weak sense or simply weakly stationary [203].

The time average (temporal mean) value of a record $x(t)$, over a very long sampling time T , is given by,

$$\langle x \rangle = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T x(t) dt. \tag{2.24}$$

An *ergodic process* is a stationary process in which ensemble and time averages are constant and equal to one another $E\{x\} = \langle x \rangle$.

In Table 2.2 a qualification of random processes is shown.

² Also mentioned strictly stationary process or strongly stationary process.

Table 2.2. Qualification of random process

Random process	Stationary	Ergodic
	Stationary	Non ergodic
	Non stationary	Non ergodic

For our purposes we will assume that all random processes are stationary and ergodic.

For a stationary and ergodic random process $x(t)$ there are the following relations for the mean value:

$$\mu_x = \langle x \rangle = E(x) = \int_{-\infty}^{\infty} x f(x) dx = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T x(t) dt, \tag{2.25}$$

and for the mean square value

$$\langle x^2 \rangle = E\{x^2\} = \int_{-\infty}^{\infty} x^2 f(x) dx = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T x^2(t) dt = \sigma_x^2 + \mu_x^2. \tag{2.26}$$

The variance of stationary random process $x(t)$ is given by

$$\sigma_x^2 = E\{(x(t) - \mu_x)^2\} = E\{x^2\} - 2\mu_x E\{x\} + \mu_x^2 = E\{x^2\} - \mu_x^2. \tag{2.27}$$

This explains (2.26).

$$\begin{aligned} \sigma_x^2 &= E\{(x(t) - \mu_x)^2\} \\ &= \lim_{T \rightarrow \infty} \frac{1}{4T^2} \int_{-T}^T \int_{-T}^T E\{x(t_1)x(t_2)\} dt_1 dt_2 - \mu_x^2 \\ &= \lim_{T \rightarrow \infty} \frac{1}{4T^2} \int_{-T}^T \int_{-T}^T R_{xx}(t_2 - t_1) dt_1 dt_2 - \mu_x^2, \end{aligned} \tag{2.28}$$

where $R_{xx}(t_2 - t_1)$ is the auto correlation function, which will be discussed later in the next section. The autocorrelation function describes the correlation of the random process $x(t)$ at different points t_1 and t_2 in time.

To put this result in a simpler form, consider the change of variables according to $\tau_1 = t_2 + t_1$ and $\tau_2 = t_2 - t_1$. The Jacobian of this transformation is

$$\left| \frac{\partial(t_1, t_2)}{\partial(\tau_1, \tau_2)} \right| = \frac{1}{2}. \tag{2.29}$$

In terms of the new variables (2.28) becomes,

$$\sigma_x^2 = \lim_{T \rightarrow \infty} \frac{1}{4T^2} \int_0^T \int_0^{T-\tau_2} \frac{1}{2} R_{xx}(\tau_2) d\tau_1 d\tau_2 - \mu_x^2, \tag{2.30}$$

where the domain of integration is a square shown in Fig. 2.4. It is seen that the integrand is an even function of τ_2 and is not a function of τ_1 . Hence, the

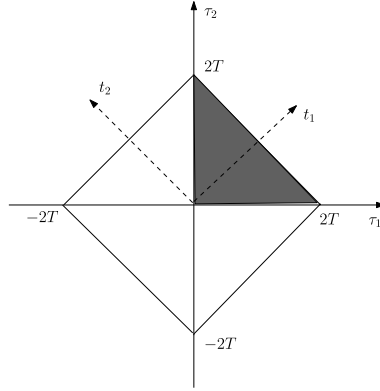


Fig. 2.4. Domain of integration

value of the integral is four times the value of the integral over the shaded area. Thus

$$\begin{aligned}
 \sigma_x^2 &= \lim_{T \rightarrow \infty} \frac{1}{2T^2} \int_0^{2T} \int_0^{2T-\tau_2} R_{xx}(\tau_2) d\tau_1 d\tau_2 - \mu_x^2 \\
 &= \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^{2T} R_{xx}(\tau_2) \left(1 - \frac{\tau_2}{2T}\right) d\tau_2 - \mu_x^2 \\
 &= \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^{2T} \left(1 - \frac{\tau_2}{2T}\right) [R_{xx}(\tau_2) - \mu_x^2] d\tau_2. \tag{2.31}
 \end{aligned}$$

The random variable $x(t)$ is ergodic in the mean if and only if [188]

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^{2T} \left(1 - \frac{\tau}{2T}\right) [R_{xx}(\tau) - \mu_x^2] d\tau = 0. \tag{2.32}$$

Example. A random signal $x(t)$ with zero mean has the following correlation function

$$R(\tau) = e^{-\lambda|\tau|}.$$

Show that this signal is ergodic in the mean using (2.32).

$$\begin{aligned}
 &\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^{2T} \left(1 - \frac{\tau}{2T}\right) [R(\tau)] d\tau \\
 &= \lim_{T \rightarrow \infty} \frac{1}{\lambda T} \left(1 - \frac{1 - e^{-\lambda T}}{2\lambda T}\right) = 0.
 \end{aligned}$$

Normally, all vibration testing and analysis is carried out under the assumption that the random vibration is Gaussian. The primary reasons for this assumption are twofold [143]:

1. The Gaussian process is one of the few processes which have been mathematically defined, and
2. Many physical processes have been found to be at least approximately Gaussian (central limit theorem³).

2.3.1 Power Spectral Density

The *autocorrelation function* (*auto variance function*) of a stationary and an ergodic random process $x(t)$ is illustrated in Fig. 2.5. It expresses the correlation of a function with itself (auto) at points separated by various times τ . The autocorrelation function is defined by

$$\begin{aligned}
 R_{xx}(\tau) = E\{x(t)x(t + \tau)\} &= \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T x(t)x(t + \tau)dt \\
 &= \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T x(t)x(t + \tau)dt, \quad (2.33)
 \end{aligned}$$

with the following properties:

- $\lim_{\tau \rightarrow \infty} R_{xx}(\tau) = \mu_x^2$, $x(t)$ and $x(t + \tau)$ become independent [84]
- $R_{xx}(\tau)$ is a real function
- $R_{xx}(\tau)$ is a symmetric function, $R_{xx}(\tau) = R_{xx}(-\tau)$, $R_{xx}(-\tau) = E\{x(t - \tau)x(t)\}$
- $R_{xx}(0) = E(x^2) = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T x^2(t)dt = \sigma_x^2 + \mu_x^2$
- $R_{xx}(0) \geq |R_{xx}(\tau)|$, which can be proven with the relation $\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T [x(t) \pm x(t + \tau)]^2 dt = E\{[x(t) \pm x(t + \tau)]^2\} \geq 0$, $E\{[x(t)]^2\} + E\{[x(t + \tau)]^2\} \pm 2E\{x(t)x(t + \tau)\} \geq 0$, thus, $2R(0) \pm 2R(\tau) \geq 0$, finally, $R(0) \geq |R(\tau)|$. It should be emphasized, however, that the equality may hold, [209].

³ Let x_1, x_2, \dots, x_n be a sequence of independent random variables with the means $\mu_1, \mu_2, \dots, \mu_n$ and the variances $\sigma_1^2, \sigma_2^2, \dots, \sigma_n^2$. Let S_n be the sum of the sequence

$$S_n = \sum_{i=1}^n x_i, \quad \mu_{sn} = \sum_{i=1}^n \mu_i, \quad \sigma_{sn}^2 = \sum_{i=1}^n \sigma_i^2.$$

As $n \rightarrow \infty$ the normalized variable z_n , with mean $\mu_z = 0$ and $\sigma_z = 1$ is given by

$$z_n = \frac{S_n - \mu_{sn}}{\sigma_{sn}}.$$

The variable z_n has the following normalized distribution

$$f_{sn}(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}.$$

For any individual distribution of x_i , the distribution of the sum converges to a normalized Gaussian distribution.

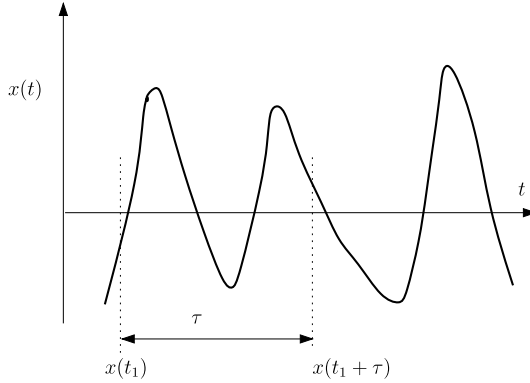


Fig. 2.5. Autocorrelation

- The correlation between $x(t)$ and $\dot{x}(t)$ is $R_{x\dot{x}}(\tau) + R_{\dot{x}x}(\tau) = 0$, because $\frac{\partial R_{xx}(\tau)}{\partial t} = 0$, and therefore $R_{x\dot{x}}(0) = -R_{\dot{x}x}(0) = 0$
- If $x(t) = \alpha y(t) + \beta z(t)$ then $R_{xx}(\tau) = \alpha^2 R_{yy}(\tau) + \alpha\beta R_{yz}(\tau) + \alpha\beta R_{zy}(\tau) + \beta^2 R_{zz}(\tau) = \alpha^2 R_{yy}(\tau) + 2\alpha\beta R_{yz}(\tau) + \beta^2 R_{zz}(\tau)$
- The Fourier transform requirement is satisfied for the autocorrelation function when $\int_{-\infty}^{\infty} |R_{xx}(\tau)| d\tau < \infty$.
- The *normalized correlation coefficient* is defined as $r(\tau) = \frac{R_{xx}(\tau)}{R_{xx}(0)}$. The normalized correlation coefficient of many real physical stochastic processes can be approximated by the formula [181] $e^{-\alpha|\tau|}(\cos \gamma\tau + \frac{\alpha}{\gamma} \sin \gamma|\tau|)$, where α and γ are constants.
- The *correlation time* τ_c is defined as $\tau_c = \frac{1}{R_{xx}(0)} \int_{-\infty}^{\infty} |R_{xx}(\tau)| d\tau = \int_{-\infty}^{\infty} |r(\tau)| d\tau$.

The *cross-correlation function* $R_{xy}(\tau)$ is defined as

$$R_{xy}(\tau) = E\{x(t)y(t + \tau)\} = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T x(t)y(t + \tau) dt. \tag{2.34}$$

It can be proven that $|R_{xy}(0)| = \frac{1}{2}[R_{xx}(0) + R_{yy}(0)]$, and $|R_{xy}(0)|^2 \leq R_{xx}(0)R_{yy}(0)$.

Example. Calculate the autocorrelation function of the function $x(t) = A \sin \omega t$. In accordance with (2.33) the autocorrelation function becomes $R_{xx}(\tau) = \frac{1}{T} \int_0^T x(t)x(t + \tau) dt = \frac{\omega A^2}{2\pi} \int_0^{\frac{2\pi}{\omega}} \sin \omega t \sin \omega(t + \tau) dt = \frac{A^2}{2} \cos \omega\tau$. The mean square value of $x(t)$ can be easily calculated $E\{x^2\} = R_{xx}(0) = \frac{A^2}{2}$.

The *covariance function* $C_{xx}(\tau)$ is defined as

$$C_{xx}(\tau) = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \{x(t) - \mu_x\} \{x(t + \tau) - \mu_x\} dt, \tag{2.35}$$

and that function is related to the autocorrelation function as follows

$$R_{xx}(\tau) = C_{xx}(\tau) + \mu_x^2. \quad (2.36)$$

And if the average value of $x(t)$ is $\mu_x = 0$, then we see that

$$R_{xx}(\tau) = C_{xx}(\tau). \quad (2.37)$$

It can be proved that

$$|C_{xx}(\tau)| \leq \sigma_x^2. \quad (2.38)$$

The *Fourier transform* of a function $x(t)$ is defined in [89, 145] as

$$F\{x(t)\} = X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t} dt, \quad (2.39)$$

and the inverse of the Fourier transform

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega)e^{j\omega t} d\omega, \quad (2.40)$$

assuming that $\int_{-\infty}^{\infty} |x(t)| dt < \infty$, and any discontinuities are finite.

Example. Calculate the Fourier transform of a rectangular pulse:

$$f(t) = \begin{cases} A & |t| \leq T, \\ 0 & |t| > T. \end{cases}$$

From (2.39) we have

$$F(\omega) = \int_{-T}^T Ae^{-j\omega t} dt = \left[-\frac{A}{j\omega} e^{-j\omega t} \right]_{-T}^T = \frac{2A}{\omega} \sin \omega T.$$

The Fourier transform of the autocorrelation function $R_{xx}(\tau)$ is called the *power spectral density function* $S_{xx}(\omega)$ (also called *auto spectral density*)

$$S_{xx}(\omega) = \int_{-\infty}^{\infty} R_{xx}(\tau)e^{-j\omega\tau} d\tau = 2 \int_0^{\infty} R_{xx}(\tau) \cos \omega\tau d\tau, \quad (2.41)$$

and

$$R_{xx}(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{xx}(\omega)e^{j\omega\tau} d\omega = \frac{1}{\pi} \int_0^{\infty} S_{xx}(\omega) \cos \omega\tau d\omega. \quad (2.42)$$

Use has been made of Euler's identity, namely $e^{j\omega t} = \cos \omega t + j \sin \omega t$.

Table 2.3 contains the spectral densities $S(\omega)$ for various correlation functions $R(\tau)$.

The power spectral density function $S_{xx}(\omega)$ quantifies the distribution of power of signal $x(t)$ with respect to the frequency. In the expression of the

Table 2.3. Correlation function versus spectral density [181, 203]

$R(\tau)$	$S(\omega)$
$C\delta(\tau)$	$C = \text{constant}$
$\sum_{k=1}^n C_k \delta^{(k)}(\tau)$	$\sum_{k=1}^n C_k (j\omega)^k$
$Ce^{-\alpha \tau }$	$\frac{2\alpha C}{\alpha^2 + \omega^2}$
$Ce^{-\alpha \tau } \cos \beta\tau$	$\alpha C \left(\frac{1}{\alpha^2 + (\beta + \omega)^2} + \frac{1}{\alpha^2 + (\beta - \omega)^2} \right)$
$Ce^{-\alpha \tau } (\cos \beta\tau - \frac{\alpha}{\beta} \sin \beta \tau)$	$\frac{C\omega^2}{(\omega^2 - \alpha^2 - \beta^2)^2 + 4\alpha^2\omega^2}$
$Ce^{-\alpha \tau } (\cos \beta\tau + \frac{\alpha}{\beta} \sin \beta \tau)$	$\frac{4C(\alpha^2 + \omega^2)}{(\omega^2 - \alpha^2 - \beta^2)^2 + 4\alpha^2\omega^2}$
$Ce^{-(\alpha\tau)^2} \cos \beta\tau$	$\frac{C\sqrt{\pi}}{2\alpha} \left[e^{-\frac{(\omega + \beta)^2}{4\alpha^2}} + e^{-\frac{(\omega - \beta)^2}{4\alpha^2}} \right]$
$\sum_{k=0}^n C_k \cos \frac{k\pi\tau}{T}$ for $ \tau \leq T$ 0 for $ \tau > T$	$2T \sum_{k=0}^n (-1)^k C_k \frac{\omega T \sin \omega T}{(\omega T)^2 - (k\pi)^2}$
$C(1 - \frac{ \tau }{T})$ for $ \tau \leq T$ 0 for $ \tau > T$	$CT \left(\frac{\sin \frac{\omega T}{2}}{\frac{\omega T}{2}} \right)^2$
$-C \sum_{j=1}^n \frac{e^{s_j \tau }}{s_j} \prod_{k=1, k \neq j}^n \frac{B(s_k)B(-s_k)}{(s_k^2 - s_j^2)}$ $\Re\{s_j\} < 0$	$C \frac{B(j\omega)B(-j\omega)}{A(j\omega)A(-j\omega)}$ $B(s) = b_0 s^m + b_1 s^{m-1} + \dots + b_m$ $A(s) = a_0 s^n + a_1 s^{n-1} + \dots + a_n = \prod_{j=1}^n (s - s_j)$ $n > m, s_j \text{ are roots of } A(s) = 0$

power spectral density, **spectral** indicates a measure of the frequency content, and the **power** is the quantity to which the various frequency components contributes in the mean square value of the variable $x(t)$. **Density** tells us that the frequencies are not discrete but continuously distributed, so we cannot speak of the contribution of a single frequency ω but only of the contribution of a band of frequencies between ω and $\omega + d\omega$.

Both the autocorrelation function $R_{xx}(\tau)$ and the power spectral density function $S_{xx}(\omega)$ are symmetric functions about $\tau = 0$ and $\omega = 0$.

The pair of (2.41) and (2.42) is called the Wiener-Khintchine (in German Wiener-Chintschin [81]) relationship. It is evident that for processes monotonically decreasing the integral (2.42) exists. Therefore $S_{xx}(\omega)$ is for large ω of the following order of magnitude

$$S_{xx}(\omega) \sim O\left(\frac{1}{\omega^{1+\varepsilon}}\right), \quad \text{where } \varepsilon > 0. \tag{2.43}$$

If the range of frequency ω , in which the spectral density does not vanish, is much smaller than a certain frequency ω_o belonging to this range, this process is called a *narrow band process* (Fig. 2.6). Thus, a narrow band process is that one that satisfies the condition $\frac{\Delta}{\omega_o} \ll 1$, where Δ is the band width at the half power points. Otherwise the process is called a *wide band process*.

Example. The correlation function $R_{xx}(\tau)$ of a random binary wave is given by:

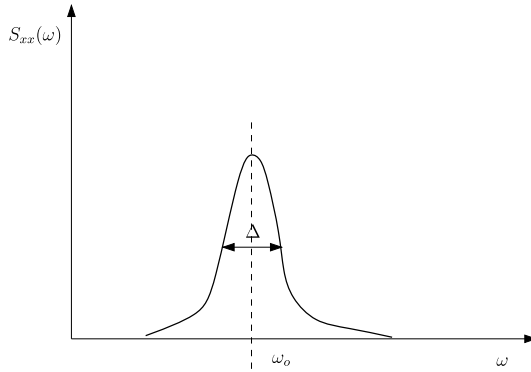


Fig. 2.6. Narrow band process

$$R_{xx}(\tau) = \sigma^2 \begin{cases} 1 - \frac{|t|}{\epsilon} & |t| \leq \epsilon, \\ 0 & |t| > \epsilon. \end{cases}$$

The power spectral density function $S_{xx}(\omega)$ becomes

$$S_{xx}(\omega) = 2 \int_0^\infty R_{xx}(\tau) \cos \omega\tau d\tau = 2 \int_0^\epsilon R_{xx}(\tau) \cos \omega\tau d\tau = \frac{4\sigma^2}{\epsilon\omega^2} \sin^2\left(\frac{\omega\epsilon}{2}\right).$$

The *total energy* E of the signal $x(t)$ is given by [89]

$$E = \int_{-\infty}^\infty \{x(t)\}^2 dt. \tag{2.44}$$

The (average) *power* P of the signal $x(t)$ is given by [89]

$$P = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T \{x(t)\}^2 dt = R_{xx}(0). \tag{2.45}$$

Using (2.40) we can rewrite (2.44) as

$$E = \int_{-\infty}^\infty \{x(t)\}^2 dt = \int_{-\infty}^\infty x(t) dt \left[\frac{1}{2\pi} \int_{-\infty}^\infty X(\omega) e^{j\omega t} d\omega \right]. \tag{2.46}$$

By changing the order of the integration (2.46) becomes

$$E = \frac{1}{2\pi} \int_{-\infty}^\infty X(\omega) \left[\int_{-\infty}^\infty x(t) e^{j\omega t} dt \right] d\omega = \frac{1}{2\pi} \int_{-\infty}^\infty X(\omega) X^*(\omega) d\omega, \tag{2.47}$$

hence,

$$\begin{aligned} E &= \int_{-\infty}^\infty \{x(t)\}^2 dt = \frac{1}{2\pi} \int_{-\infty}^\infty X(\omega) X^*(\omega) d\omega \\ &= \frac{1}{2\pi} \int_{-\infty}^\infty |X(\omega)|^2 d\omega. \end{aligned} \tag{2.48}$$

The resulting equation (2.48) is called *Parseval's theorem* [89], with $|X(\omega)|^2$ the *energy spectral density* (ESD).⁴ The ESD is an even function. Parseval's theorem describes how the energy in the signal is distributed along the frequency axis by the function $|X(\omega)|^2$.

Example. Consider the signal

$$x(t) = e^{-\alpha t}, \quad \alpha, t \geq 0.$$

Calculate the total energy E of the signal $x(t)$ using both sides of Parseval's theorem (2.48). Start with the left hand side (LHS) of (2.48), thus

$$E = \int_{-\infty}^{\infty} \{x(t)\}^2 dt = \int_0^{\infty} e^{-2\alpha t} dt = \frac{1}{2\alpha}.$$

The right hand side (RHS) of (2.48) is obtained. The Fourier transform of $x(t)$ is given by

$$X(\omega) = \frac{1}{\alpha + j\omega}.$$

The spectral density $|X(\omega)|^2$ becomes

$$|X(\omega)|^2 = \frac{1}{\alpha^2 + \omega^2}.$$

The total energy of the signal $x(t)$ is

$$E = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{\alpha^2 + \omega^2} d\omega = \frac{1}{2\pi} \left[\arctan \frac{\omega}{\alpha} \right]_{-\infty}^{\infty} = \frac{1}{2\alpha}.$$

The definition of the energy of a signal relies on the *time domain* representation of the signal $x(t)$. Parseval's theorem gives a second way to compute the total energy based on the Fourier transform of the signal. That means the calculation of the total energy is done in the *frequency domain*. Parseval's theorem relates a time domain representation of the energy in a signal to the frequency domain description.

Equation (2.45), using Parseval's theorem, can be written as

$$P = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T \{x(t)\}^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} \lim_{T \rightarrow \infty} \frac{1}{2T} |X(\omega)|^2 d\omega, \quad (2.49)$$

where $\lim_{T \rightarrow \infty} \frac{1}{2T} |X(\omega)|^2$ is the *power spectral density*⁵ (PSD) of $x(t)$. Parseval's theorem is a relation that states an equivalence between the power P of a signal computed in the time domain and that computed in the frequency domain.

⁴ If $z = x + jy$ and $z^* = x - jy$ then $zz^* = x^2 + y^2 = |z|^2$.

⁵ Also called autospectral density or autospectrum.

Equation (2.41) can be written, after multiplying by $e^{j\omega t}e^{-j\omega t}$, as follows

$$\begin{aligned} S_{xx}(\omega) &= \int_{-\infty}^{\infty} \lim_{T \rightarrow \infty} \frac{1}{2T} \left[\int_{-T}^T x(t)x(t+\tau) dt \right] e^{j\omega t} e^{-j\omega(t+\tau)} d\tau \\ &= \lim_{T \rightarrow \infty} \frac{1}{2T} |X(\omega)|^2. \end{aligned} \quad (2.50)$$

The average power P , using (2.49) and (2.42), can be expressed as

$$P = \frac{1}{2\pi} \int_{-\infty}^{\infty} \lim_{T \rightarrow \infty} \frac{1}{2T} |X(\omega)|^2 d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{xx}(\omega) d\omega = R_{xx}(0), \quad (2.51)$$

hence

$$R_{xx}(0) = E\{x^2\} = \langle x^2 \rangle = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{xx}(\omega) d\omega. \quad (2.52)$$

$S_{xx}(\omega)$ has the following properties:

- $S_{xx}(\omega) = S_{xx}(-\omega)$
- $S_{xx}(\omega) \geq 0$.

The *spectral moment* m_i of a stationary random process $X(t)$ is defined as [154]

$$m_i = \frac{1}{2\pi} \int_{-\infty}^{\infty} |\omega|^i S_{xx}(\omega) d\omega. \quad (2.53)$$

For a process $x(t)$ with $\mu_x = 0$ we may use the Wiener-Khintchine relations to find

$$\sigma_x^2 = m_0 = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{xx}(\omega) d\omega, \quad (2.54)$$

and

$$\sigma_{\dot{x}}^2 = m_2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} |\omega|^2 S_{xx}(\omega) d\omega, \quad (2.55)$$

and

$$\sigma_{\ddot{x}}^2 = m_4 = \frac{1}{2\pi} \int_{-\infty}^{\infty} |\omega|^4 S_{xx}(\omega) d\omega. \quad (2.56)$$

A *normalized moment* with the dimension of circular frequency can be defined as

$$\gamma_n = \left(\frac{m_n}{m_0} \right)^{\frac{1}{n}}, \quad (2.57)$$

where γ_1 is the central frequency and has a geometrical meaning of being the centroid of the spectral distribution $S_{xx}(\omega)$, and γ_2 has the geometrical meaning of the radius of gyration of $S_{xx}(\omega)$ about the origin. A variance parameter δ describing the dispersion of $S_{xx}(\omega)$ around the central frequency is defined as

$$\delta = \frac{\sqrt{(m_0 m_2 - m_1^2)}}{m_1} = \sqrt{\frac{m_0 m_2}{m_1^2} - 1}, \quad (2.58)$$

where the range of δ is $0 \leq \delta < \infty$. For a harmonic process, $m_1 = \omega_0 m_0$ and $m_2 = \omega_0^2 m_0$, $\delta = 0$, hence small values of δ indicate a narrow band process. Another bandwidth parameter γ is defined by

$$\gamma = \sqrt{1 - \frac{m_2^2}{m_0 m_4}} = \sqrt{1 - \alpha_2^2}, \quad (2.59)$$

where the *irregularity factor* α_2 is defined to be

$$\alpha_2 = \frac{m_2}{\sqrt{m_0 m_4}}. \quad (2.60)$$

The PSD function $S_{xx}(\omega)$ is *two-sided*. It is more practical to replace ω (rad/s) with f (Hz, cycles/s) and to replace the two-sided PSD function $S_{xx}(\omega)$ with a *one-sided* PSD function $W_{xx}(f)$ and then (2.52) becomes

$$R_{xx}(0) = E\{x^2\} = \langle x^2 \rangle = \frac{4\pi}{2\pi} \int_0^\infty S_{xx}(\omega) df = \int_0^\infty W_{xx}(f) df, \quad (2.61)$$

where $W_{xx}(f) = 2S_{xx}(\omega)$.

In the narrow frequency band Δf it is assumed that W_{xx} is constant and therefore

$$W_{xx} \Delta f = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T x^2(t) dt, \quad (2.62)$$

and

$$W_{xx} = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \frac{x^2(t)}{\Delta f} dt = \frac{\langle x^2 \rangle}{\Delta f}, \quad (2.63)$$

where T is the averaging time, $x^2(t)$ the instantaneous square of the signal within Δf , $\langle x^2 \rangle$ the mean square value, and $\Delta f \rightarrow 0$.

With use of (2.41) $W_{xx}(f)$ becomes as follows

$$W_{xx}(f) = 4 \int_0^\infty R_{xx}(\tau) \cos \omega \tau d\tau, \quad (2.64)$$

and (2.50) can be written as

$$W_{xx}(f) = \lim_{T \rightarrow \infty} \frac{1}{T} |X(\omega)|^2. \quad (2.65)$$

We will apply (2.64) and (2.65) later in this book to estimate the PSD values numerically.

The definition of the PSD function of $x(t)$ in relation with the Wiener-Khinchine relations and the mean square value will be recapitulated hereafter [142], because several definitions exist.

First Definition of Auto Spectral Density Function

- $S_{xx}(\omega) = \lim_{T \rightarrow \infty} \frac{1}{2T} |X(\omega)|^2$

Wiener-Khintchine theorem $R_{xx}(\tau) \leftrightarrow S_{xx}(\omega)$

- $S_{xx}(\omega) = \int_{-\infty}^{\infty} R_{xx}(\tau) e^{-j\omega\tau} d\tau = 2 \int_0^{\infty} R_{xx}(\tau) \cos(\omega\tau) d\tau$
- $R_{xx}(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{xx}(\omega) e^{j\omega\tau} d\omega = \frac{1}{\pi} \int_0^{\infty} S_{xx}(\omega) \cos(\omega\tau) d\omega$

Relation between spectral density function and average energy (mean square value, variance)

- $\int_0^{\infty} S_{xx}(\omega) d\omega = \pi R_{xx}(0) = \pi \text{Var}\{x(t)\} = \pi \sigma_x^2$

Second Definition of Auto Spectral Density Function

- $S_{xx}(\omega) = \lim_{T \rightarrow \infty} \frac{1}{2\pi T} |X(\omega)|^2$

Wiener-Khintchine theorem $R_{xx}(\tau) \leftrightarrow S_{xx}(\omega)$

- $S_{xx}(\omega) = \frac{1}{\pi} \int_{-\infty}^{\infty} R_{xx}(\tau) e^{-j\omega\tau} d\tau = \frac{2}{\pi} \int_0^{\infty} R_{xx}(\tau) \cos(\omega\tau) d\tau$
- $R_{xx}(\tau) = \frac{1}{2} \int_{-\infty}^{\infty} S_{xx}(\omega) e^{j\omega\tau} d\omega = \int_0^{\infty} S_{xx}(\omega) \cos(\omega\tau) d\omega$

Relation between spectral density function and average energy (mean square value, variance)

- $\int_0^{\infty} S_{xx}(\omega) d\omega = R_{xx}(0) = \text{Var}\{x(t)\} = \sigma_x^2$

Third Definition of Auto Spectral Density Function

- $W_{xx}(f) = \lim_{T \rightarrow \infty} \frac{1}{T} |X(2\pi f)|^2$

Wiener-Khintchine theorem $R_{xx}(\tau) \leftrightarrow W_{xx}(f)$

- $W_{xx}(f) = 2 \int_{-\infty}^{\infty} R_{xx}(\tau) e^{-j\omega\tau} d\tau = 4 \int_0^{\infty} R_{xx}(\tau) \cos(\omega\tau) d\tau$
- $R_{xx}(\tau) = \int_0^{\infty} W_{xx}(f) e^{j2\pi f\tau} d\omega = \int_0^{\infty} W_{xx}(f) \cos(2\pi f\tau) df$

Relation between spectral density function and average energy (mean square value, variance)

- $\int_0^{\infty} W_{xx}(f) d\omega = R_{xx}(0) = \text{Var}\{x(t)\} = \sigma_x^2$

White Noise

White noise contains equal amounts of energy at all frequencies. If the power spectral density function of a signal $x(t)$ is constant over the complete frequency range, $W_{xx}(f) = W_0, 0 \leq x(t) \leq \infty$ we talk about white noise.

The power spectral density function $S_{xx}(\omega) = \frac{W_0}{2}, -\infty \leq \omega \leq \infty$. The autocorrelation function $R_{xx}(\tau)$ can be calculated as follows

$$R_{xx}(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{xx}(\omega) e^{j\omega\tau} d\omega = \frac{W_0}{4\pi} \int_{-\infty}^{\infty} e^{j\omega\tau} d\omega = \frac{W_0}{2} \delta(\tau),$$

where $\int_{-\infty}^{\infty} \delta(\tau) d\tau = 1$, and $\delta(\tau)$ is the *Dirac delta function*. A random process with a constant (white noise by analogy to white light in optics) PSD function between two frequencies (band-limited) is considered. Calculate the associated autocorrelation function $R_{xx}(\tau)$ if

- $W(f) = W_0, f_1 \leq f \leq f_2$
- $W(f) = 0, f < f_1$ and $f > f_2$

The autocorrelation function $R_{xx}(\tau)$ becomes

$$R_{xx}(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{xx}(\omega) e^{j\omega\tau} d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{xx}(\omega) \cos \omega\tau d\omega,$$

and

$$R_{xx}(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{xx}(\omega) \cos \omega\tau d\omega = \frac{1}{\pi} \int_0^{\infty} S_{xx}(\omega) \cos \omega\tau d\omega.$$

Hence

$$R_{xx}(\tau) = \frac{W_0}{2\pi} \int_{2\pi f_1}^{2\pi f_2} \cos \omega\tau d\omega = \frac{W_0}{2\pi\tau} [\sin 2\pi f_2\tau - \sin 2\pi f_1\tau], \quad (2.66)$$

so that

$$R_{xx}(0) = \lim_{\tau \rightarrow 0} \frac{W_0}{2\pi\tau} [\sin 2\pi f_2\tau - \sin 2\pi f_1\tau] = W_0[f_2 - f_1].$$

Assume a very narrow bandwidth $[f_2 - f_1] = \Delta f$. Then (2.66) becomes

$$R_{xx}(\tau) = \frac{W_0}{2\pi\tau} [\sin 2\pi(f_1 + \Delta f)\tau - \sin 2\pi f_1\tau].$$

Using Taylor series

$$f(x + \Delta x) = f(x) + \frac{f'(x)}{1!} \Delta x + \frac{f''(x)}{2!} \Delta x^2 + \dots, \quad (2.67)$$

for a sinus expansion of $\sin(f + \Delta f)$ we obtain

$$\sin\{2\pi(f_1 + \Delta f)\tau\} \approx \sin(2\pi f_1\tau) + \frac{\cos(2\pi f_1\tau)}{1!} 2\pi \Delta f\tau. \quad (2.68)$$

The autocorrelation function $R_{xx}(\tau)$, with (2.67, 2.68), can now be calculated

$$R_{xx}(\tau) = \frac{W_0}{2\pi\tau} [\cos\{2\pi(f_1\tau)\} 2\pi \Delta f\tau] = W_0 \Delta f \cos(2\pi f_1\tau), \quad (2.69)$$

and

$$R_{xx}(0) = W_0 \Delta f. \quad (2.70)$$

Table 2.4. Properties of autocorrelation and power spectral density function [146]

$x(t)$	$R_{xx}(\tau)$	$S_{xx}(\omega)$
$\frac{dx(t)}{dt}$	$-\frac{d^2 R_{xx}(\tau)}{d\tau^2}$	$\omega^2 S_{xx}(\omega)$
$\frac{d^n x(t)}{dt^n}$	$(-1)^n \frac{d^{2n} R_{xx}(\tau)}{d\tau^{2n}}$	$\omega^{2n} S_{xx}(\omega)$
$x(t)e^{\pm\omega_0 t}$	$R_{xx}e^{\pm\omega_0 t}$	$S_{xx}(\omega \mp \omega_0)$

Table 2.4 gives some useful relations between the autocorrelation function $R_{xx}(\tau)$ and the power spectral density function $S_{xx}(\omega)$. In addition to Table 2.4, the $n + m$ derivative of the correlation function $R_{xx}(\tau)$ is given by

$$R_{x^{(m)}x^{(n)}}(\tau) = (-1)^m \frac{d^{m+n}}{d\tau^{m+n}} \int_{-\infty}^{\infty} S_{xx}(\omega) e^{j\omega\tau} d\omega, \tag{2.71}$$

and

$$S_{x^{(m)}x^{(n)}}(\omega) = (-1)^m (j\omega)^{(m+n)} S_{xx}(\omega). \tag{2.72}$$

2.3.2 Measurement of PSD

The PSD function is obtained by averaging the mean square value \bar{x}^2 of the signal $x(t)$ in a narrow bandwidth Δf over a finite period of time T . The approximation of the PSD value $W_x(f_k)$ is

$$W_x(f_k) = \frac{\bar{x}^2(f_k)}{\Delta f_k}, \quad k = 1, 2, \dots, n, \tag{2.73}$$

where $\bar{x}^2(f_k)$ is the mean square value in the center frequency f_k with bandwidth Δf_k of the bandpass filter.

A schematic of the steps required for the PSD analysis (2.73) is presented in Fig. 2.7. The input to the bandpass filter is the random signal $x(t)$ to be analyzed. The bandpass filter allows selection of the desired filter bandwidth Δf_k for the analysis. Also, the center frequency of the filter can be set at

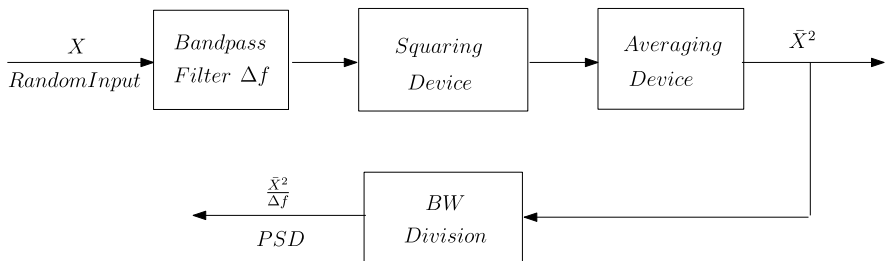


Fig. 2.7. Schematic flow for power spectral density analysis

specific frequencies. The instantaneous pulses from the bandpass filter are then squared, averaged and divided by the bandwidth of the filter. The value, after division by the bandwidth, is the magnitude of the PSD function $W_x(f_k)$ at the specific center frequency f_k of the bandpass filter.

The error e [143] in terms of the standard deviation σ_W of the true PSD function $W(f)$ and is given by

$$e = \frac{\sigma_W}{W(f)} = \frac{1}{\sqrt{\Delta f T}}, \quad (2.74)$$

where T is the averaging time and Δf the analyzer bandwidth. Equation (2.74) is valid for $e \leq 0.2$. Resolution is essentially a measure of the ability to resolve the true resonances and anti-resonances (peak and valleys) when plotting the PSD spectrum. A small error could be produced by making the analyzer bandwidth very large, however, this will lower the resolution!

Example. Calculate the error function e when the averaging time $T = 2$ s and $\Delta f = 20$ Hz.

$$100e = \frac{100}{\sqrt{\Delta f T}} = 15.8\%.$$

2.3.3 Discrete Fourier Transform

For finite-duration, discrete-time signals, the discrete Fourier transform (DFT) [197, 199] is given by

$$X_m = \sum_{n=0}^{N-1} x_n e^{-j2\pi n \frac{m}{N}}, \quad m = 0, 1, 2, \dots, N-1, \quad (2.75)$$

where N is the number of samples in the time series, T is the span in seconds of the time series, $f_s = \frac{1}{\Delta t} = 2f_{max}$ is the sample rate in samples/second (Hz) (Shannon-Kotelnikov, Nyquist criterion [25]), Δf is the frequency resolution or spacing between consecutive points (Hz). The sampling frequency is usually chosen as $f_s = (2.5 \dots 3.0)f_{max}$. f_{max} is the maximum frequency in the process being analyzed.

For $N = 2^m$, a high speed algorithm that exploits the symmetry is used to compute the DFT. This algorithm is called the *Fast Fourier transform* (FFT).

In (2.75) the following notation is used:

- X_m is at frequency f_m
- $f_m = \frac{m}{NT}$, $m = 0, 1, \dots, N-1$ (Hz)
- $\Delta f = \frac{1}{T} = \frac{f_s}{N}$
- $\Delta t = \frac{1}{f_s}$
- $x_n = x(n\Delta t)$ is at time $n\Delta t$
- $T = N\Delta t$
- $j = \sqrt{-1}$

The vector $[X]$ without a subscript represents the DFT vector with the components $[X] = [X_0, X_1, \dots, X_{N-1}]$ which are the DFT of the components $[x] = [x_0, x_1, \dots, x_{N-1}]$.

The inverse DFT is defined as follows

$$x_n = \frac{1}{N} \sum_{m=0}^{N-1} X_m e^{-j2\pi n \frac{k}{N}}, \quad n = 0, 1, \dots, N-1. \quad (2.76)$$

For more information about the redundancy and periodicity of the DFT the reader is advised to read [199].

Parseval's theorem provides an important insight and link between the time and frequency domains

$$\sum_{n=0}^{N-1} x_n^2 = \frac{1}{N} \sum_{m=0}^{N-1} |X_m|^2. \quad (2.77)$$

The average power is given by

$$\frac{1}{N} \sum_{n=0}^{N-1} x_n^2 = \frac{1}{N^2} \sum_{m=0}^{N-1} |X_m|^2. \quad (2.78)$$

2.3.4 Evaluation of the Autocorrelation Function

The autocorrelation function $R_{xx}(r)$ of a random process expresses a measure of the relation of the ordinates to their mutual distance apart (time delay, time lag). From the sampled discrete ordinates $x_i = x(i\Delta t)$, $i = 0, 1, 2, \dots, m$ the discrete autocorrelation function $R_{xx}(r)$ is obtained from [25]

$$R_{xx}(r) = \frac{1}{N-r} \sum_{i=1}^{N-r} x_i x_{i+r}, \quad r = 0, 1, 2, \dots, m, \quad (2.79)$$

where m determines the number of correlation lag values in the products $x_i x_{i+r}$.

The total number of discrete samples x_i is N . The total sampling time is $T = N\Delta T$. A good empirical rule therefore requires that the maximum lag satisfy

$$m < \frac{N}{10}. \quad (2.80)$$

The maximum time delay or lag is given by $\tau_{max} = m\Delta t$.

2.3.5 Evaluation of the PSD

The PSD $W_{xx}(f)$ of a stationary random process may be obtained either using the Wiener-Khintchine relationship, i.e. the Fourier transform of the autocorrelation function $R_{xx}(\tau)$ in the known form

$$W_{xx}(f) = 4 \int_0^\infty R_{xx}(\tau) \cos(2\pi f\tau) d\tau, \quad f \geq 0, \tau \geq 0, \quad (2.81)$$

or directly from the definition of the PSD

$$W_{xx}(f) = \lim_{T \rightarrow \infty} \frac{1}{T} |X(\omega)|^2, \quad (2.82)$$

where $X(\omega)$ is the Fourier transform of $x(t)$.

The smoothed or weighted PSD estimate at harmonics $k = 0, 1, 2, \dots, m$ will have the form [25]

$$W_{xx}\left(\frac{kf_{max}}{m}\right) = \frac{1}{f_{max}} \left[R_{xx}(0) + 2 \sum_{r=1}^{m-1} W(r) R_{xx}(r) \cos\left(\frac{\pi rk}{m}\right) + W(m) R_{xx}(m) (-1)^k \right], \quad k = 0, 1, 2, \dots, m, \quad (2.83)$$

where $R_{xx}(r)$ is obtained from (2.79) and $f_{max} = \frac{1}{2\Delta t}$ (the Shannon-Kotelnikov criterion $\Delta t \leq \frac{1}{2f_{max}}$) for a good reconstruction of W_{xx} .

The possibilities of the time lag windows $W(r)$ are [25]:

- General lag window $W(r) = (1 - |\frac{r}{m}|) e^{-(\frac{r}{m})^2}$
- Hann window $W(r) = \frac{1}{2} (1 + \cos \frac{\pi r}{m})$
- Barlett window $W(r) = 1 - |\frac{r}{m}|$
- Exponential window $W(r) = e^{-|\frac{r}{m}|}$
- Gaussian window $W(r) = e^{-(\frac{r}{m})^2}$

Computation of the PSD directly from (2.82) meets considerably difficulties, because the DFT computations cannot be realized in an economic time by standard methods.

One of the methods is a procedure based on segmentation of a long process, i.e. its division into shorter overlapping or non-overlapping parts. This is done as follows. Consider a sufficiently large set of N process ordinates which is divided into K segments $x_k(j), k = 1, 2, \dots, K$ with an equal number of L ordinates. Because these segments may overlap, beginning of the second segment starts with the D th ordinate, the beginning of the third segment with $2D$ th ordinate, etc. as illustrated in Fig. 2.8. In this way the set of K segments becomes

$$\begin{aligned} x_1(j) &= x(j), \quad j = 1, 2, \dots, L, \\ x_2(j) &= x(j + D), \quad j = 1, 2, \dots, L \end{aligned}$$

and finally

$$x_k(j) = x(j + (K - 1)D), \quad j = 1, 2, \dots, L,$$

where $(K - 1)D = N$.

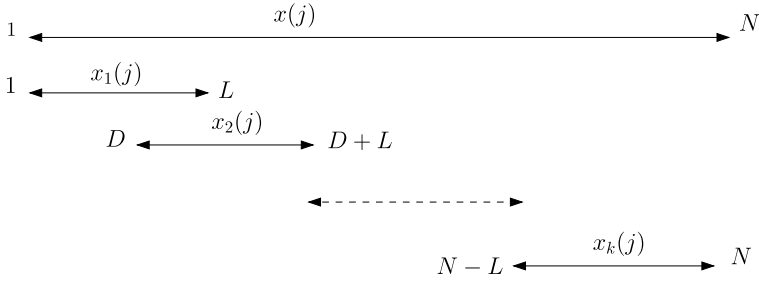


Fig. 2.8. Process segmentation for application of the Welch method [25]

For each of these segment the modified Fourier transform $A_1(n), A_2(n), \dots, A_k(n)$ are computed

$$A_k(n) = \frac{1}{L-1} \sum_{j=1}^L x_k(j)W(j)e^{-2\pi j \frac{n}{L}}, \quad n = 0, 1, \dots, \frac{L}{2} \quad (2.84)$$

where $W(j)$ is the modifying window. According to Welch, the modifying window $W(j)$ may be taken either as

$$W(j) = 1 - \left(\frac{j - \frac{L}{2}}{1 + \frac{L}{2}} \right)^2 \quad (2.85)$$

or the Hann window

$$W(j) = 1 - \left| \frac{j - \frac{L}{2}}{1 + \frac{L}{2}} \right|. \quad (2.86)$$

From these functions $A_k(n)$ we further compute modified periodograms⁶ (PSD) $I_k(f_n)$ using (2.82)

$$I_k(f_n) = \frac{L-1}{U} |A_k(n)|^2, \quad k = 1, 2, \dots, K, \quad (2.87)$$

where

$$f_n = \frac{n}{L}, \quad n = 0, 1, \dots, \frac{L}{2}, \quad (2.88)$$

and

$$U = \frac{1}{L-1} \sum_{j=1}^L W^2(j). \quad (2.89)$$

The estimate of the resulting PSD $W_{xx}(f_n)$ of the process $x(j)$ is then obtained from the expression

⁶ A periodogram is an estimate of the spectral density of a signal.

$$W_{xx}(f_n) = \frac{1}{K} \sum_{k=1}^K I_k(f_n) = \frac{L-1}{UK} \sum_{k=1}^K |A_k(n)|^2, \quad (2.90)$$

or in other words $W_{xx}(f_n)$ is given the average of the partial PSD of each segment.

2.3.6 Digital Simulation of a Random Process

In [25, 129, 170, 179, 180] methods are discussed to simulate a stationary random process, expressed in a PSD function, in the time domain. Time domain random processes are quite convenient for example for

- the numerical analysis of the dynamic response of non-linear structures to random excitations,
- time domain analysis of the dynamic response of linear structures under random excitations in order to obtain a kind of information that is not obtainable from frequency domain analysis.

The signal $f(t)$ can be simulated as follows

$$f(t) = \sqrt{2} \sum_{k=1}^N \sqrt{[W(f_k)\Delta f]} \cos(2\pi \hat{f}_k t + \phi_k), \quad (2.91)$$

where N is the number of frequency intervals and is chosen large enough to simulate the random process which is approximately ergodic, $\hat{f}_k \approx f_k + \frac{\Delta f}{20}$, ϕ_k is a random phase angle uniformly distributed between 0 and 2π with the probability of occurrences $\frac{1}{2\pi}$ and $f_k = f_{min} + (k - \frac{1}{2})\Delta f$, $k = 1, 2, \dots, N$.

Another simulation is discussed in [179, 180] and is given by

$$f(t) = \sigma \sqrt{\frac{2}{N}} \sum_{k=1}^N \cos(2\pi f_k t + \phi_k), \quad (2.92)$$

where $\sigma^2 = \int_0^\infty W(f)df$, ϕ_k is a random phase angle uniformly distributed between 0 and 2π and $f_k = f_{min} + (k - \frac{1}{2})\Delta f$, $k = 1, 2, \dots, N$. It can be shown that for $N \rightarrow \infty$ the process $f(t)$ is normal and ergodic. It is suggested to use $\Delta t = \frac{1}{10f_{max}}$, where f_{max} is the maximum frequency of the PSD. The number of frequency intervals N is taken as 200 [231].

More background of the simulation of random time series is given in Appendix A.

Problems

2.7. Let a stationary, zero mean, Gaussian process $X(t)$ denote a stress in a particular structural member of a linear structure responding under a Gaussian random load. The standard deviation of $X(t)$ is $\sigma_X = 100$ MPa.

- Find $Prob[|X(t)| \geq 100 \text{ MPa}]$ at particular time t .
- Find $Prob[|X(t)| \geq 200 \text{ MPa}]$ at particular time t .
- Find $Prob[|X(t)| \geq 300 \text{ MPa}]$ at particular time t .

Answers: 0.3174, 0.0456 and 0.0026.

2.8. The correlation coefficient $\rho(\tau)$ is given by [109]

$$\rho(\tau) = \frac{R_{xx}(\tau) - \mu_x^2}{\sigma_x^2} = \frac{C_{xx}(\tau)}{\sigma_x^2}.$$

- Prove that $|\rho(\tau)| \leq 1$
- Prove that $R_{xx}(\tau \rightarrow \infty) = \mu_x^2$

2.9. Calculate for the function

$$f(x) = a_0 + \sum_{k=1}^n \sin(\omega_k t + \varphi_k)$$

the autocorrelation function $R(\tau)$.

Answer: $R(\tau) = a_0^2 + \sum_{k=1}^n \frac{a_k^2}{2} \cos \omega_k \tau$ [187].

2.10. Suppose that the autocorrelation function of the process $x(t)$ is given by

$$R_{xx}(\tau) = 4\pi a \tau^{-1} \sin(b\tau).$$

Define the corresponding power spectral density function(s)?

Answer: $S_{xx}(\omega) = a$ for $\omega \in [-b, b]$ and elsewhere $S_{xx}(\omega) = 0$.

2.11. Suppose that the autocorrelation function of the process $x(t)$ is given by

$$R_{xx}(\tau) = \sigma^2 e^{-\alpha|\tau|}.$$

Define the corresponding power spectral density function?

Answer: $S_{xx}(\omega) = 2\sigma^2 \frac{\alpha}{\alpha^2 + \omega^2}$.

2.12. Consider an exponentially correlated noise of a scalar stationary process $x(t)$ with an autocorrelation function

$$R_{xx}(\tau) = \sigma^2 e^{-\frac{|\tau|}{\theta}},$$

with $\theta > 0$. Apply the Fourier transformation to calculate the power spectral density function $S_{xx}(\omega)$.

Answer: $S_{xx}(\omega) = \frac{2\sigma^2\theta}{1+\omega^2\theta^2}$.

2.4 Deterministic Linear Dynamic Systems

A linear deterministic system with the load application $F(t), F(\omega)$ and associated response $x(t), X(\omega)$ are illustrated in Fig. 2.9.

The system shows no random properties and the properties will not change with time. The system is excited with a random load $F(t)$ and the response (output) of the system is denoted by $x(t)$. The random responses $x(t)$ are very generalized and may be displacements, velocities, accelerations, forces, stresses, etc. The linear system will be characterized using the *impulse response function*. Linear means that doubling the loads $F(t)$ will lead to twice as much response $x(t)$. We may present the forces and responses in the time domain or in the frequency domain.

A linear system may be represented either simply, as a single degree of freedom system (SDOF), or multiple with degrees of freedom (MDOF), or even as a continuum. But in the solution of the responses the modal superposition will be applied many times and the problem will be reduced to solve many uncoupled SDOF dynamic systems. The response $x(t)$ of the linear system, due to the force $F(t)$, and the impulse response function $h(t)$ is given by the convolution of $h(t)$ and $F(t)$

$$x(t) = h(t) * F(t) = \int_{-\infty}^{\infty} h(\tau)F(t - \tau)d\tau, \tag{2.93}$$

where $F(t)$ is the generalized force in the time domain, $h(t)$ the damped impulse response function.

$$h(t) = e^{-\zeta\omega_n t} \frac{\sin(\omega_n \sqrt{1 - \zeta^2} t)}{\omega_n \sqrt{1 - \zeta^2}} \quad t \geq 0.$$

This damped impulse response function can be derived from the SDOF dynamic system

$$\ddot{x}(t) + 2\zeta\omega_n \dot{x}(t) + \omega_n^2 x(t) = \delta(t),$$

with ζ the damping ratio, ω_n the natural frequency and $\delta(t)$ the Dirac delta function.

The Fourier transform $X(\omega)$ of the convolution of the two functions $h(t)$ and $f(t)$ equals the product of the Fourier transforms $H(\omega)$ and $F(\omega)$ of these functions [145]. Thus

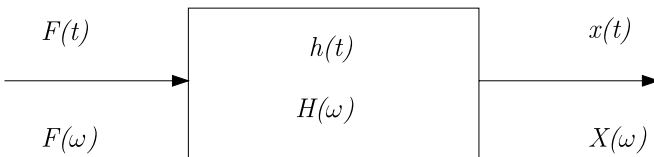


Fig. 2.9. Deterministic dynamic system

$$X(\omega) = \int_{-\infty}^{\infty} e^{-j\omega t} \left[\int_{-\infty}^{\infty} h(\tau) f(t - \tau) d\tau \right] dt. \quad (2.94)$$

Changing the order of integration $X(\omega)$ becomes

$$X(\omega) = \int_{-\infty}^{\infty} h(\tau) \left[\int_{-\infty}^{\infty} e^{-j\omega t} f(t - \tau) dt \right] d\tau.$$

From the time-shifting theorem [145] it can be concluded that the bracket above equals $F(\omega)e^{-j\omega\tau}$, therefore

$$X(\omega) = \int_{-\infty}^{\infty} h(\tau) F(\omega) e^{-j\omega\tau} d\tau = H(\omega) F(\omega). \quad (2.95)$$

The pair of Fourier transforms between the impulse response function $h(t)$ and the frequency response function $H(\omega)$ is given by

$$H(\omega) = \int_{-\infty}^{\infty} h(t) e^{-j\omega t} dt, \quad h(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} H(\omega) e^{j\omega t} d\omega. \quad (2.96)$$

The PSD function of $x(t)$ is given by (2.50)

$$S_{xx}(\omega) = \lim_{T \rightarrow \infty} \frac{1}{2T} X(\omega) X^*(\omega) = \lim_{T \rightarrow \infty} \frac{1}{2T} |X(\omega)|^2,$$

where $X^* = \int_{-\infty}^{\infty} x(t) e^{j\omega t} dt$, the conjugate of $X(\omega)$. Equation (2.96) is very important in analyzing the response characteristics of linear dynamic systems.

The PSD function of the random response $x(t)$ can be expressed in terms of the PSD function of the random loads, applying (2.95)

$$S_{xx}(\omega) = \lim_{T \rightarrow \infty} \frac{1}{2T} X(\omega) X^*(\omega) = \lim_{T \rightarrow \infty} \frac{1}{2T} H(\omega) H^*(\omega) F(\omega) F^*(\omega), \quad (2.97)$$

or

$$S_{xx}(\omega) = \lim_{T \rightarrow \infty} \frac{1}{2T} |H(\omega)|^2 |F(\omega)|^2 = |H(\omega)|^2 S_{FF}(\omega). \quad (2.98)$$

The *cross-power spectral density function* $S_{xF}(\omega)$ is defined by

$$S_{xF}(\omega) = \lim_{T \rightarrow \infty} \frac{1}{2T} X(\omega) F^*(\omega), \quad (2.99)$$

with

$$S_{xF}(\omega) = S_{F_x}^*(\omega), \quad (2.100)$$

where

- $\Re\{S_{xF}(\omega)\}$ the real part of the cross PSD function is called the *co-spectral density function* (CSD).
- $\Im\{S_{xF}(\omega)\}$ the imaginary part of the cross PSD function is called the *quad-spectral density function* (QSD).

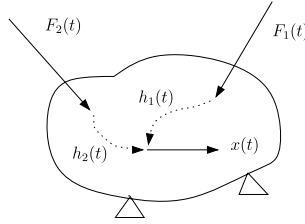


Fig. 2.10. Dynamic system, loads F_1 and F_2 applied

Further the following relationship can be derived:

$$S_{xF}(\omega) = \lim_{T \rightarrow \infty} \frac{1}{2T} X(\omega) F^*(\omega) = \lim_{T \rightarrow \infty} \frac{1}{2T} H(\omega) F(\omega) F^*(\omega) = H(\omega) S_{FF}(\omega). \quad (2.101)$$

The cross PSD function $S_{xF}(\omega)$ is generally a complex-valued function.

Suppose a linear dynamic system is loaded by two dynamic forces $F_1(t)$ and $F_2(t)$ and the interest is in the response $x(t)$ at certain location. The response $x(t)$ of a linear system can be obtained by superposing the convolutions of the impulse response function $h_1(t)$ and $h_2(t)$ with respectively the forces $F_1(t)$ and $F_2(t)$

$$\begin{aligned} x(t) &= h_1(t) * F_1(t) + h_2(t) * F_2(t) \\ &= \int_{-\infty}^{\infty} h_1(\tau) F_1(t - \tau) d\tau + \int_{-\infty}^{\infty} h_2(\tau) F_2(t - \tau) d\tau. \end{aligned} \quad (2.102)$$

The response of $x(t)$ is illustrated in Fig. 2.10. In the frequency domain the response $X(\omega)$ can be expressed as follows

$$X(\omega) = H_1(\omega) F_1(\omega) + H_2(\omega) F_2(\omega). \quad (2.103)$$

The power spectral density of $x(t)$ can now be written as follows

$$\begin{aligned} S_{xx}(\omega) &= |H_1(\omega)|^2 S_{F_1 F_1}(\omega) + H_1(\omega) H_2^*(\omega) S_{F_1 F_2}(\omega) \\ &\quad + H_2(\omega) H_1^*(\omega) S_{F_2 F_1}(\omega) + |H_2(\omega)|^2 S_{F_2 F_2}(\omega). \end{aligned} \quad (2.104)$$

2.4.1 Force Loaded SDOF System

The mass-spring-damper system is loaded by the force $F(t)$ at the mass m . The mass is suspended by a linear spring with spring stiffness k and a damper with damping constant c . The SDOF system is illustrated in Fig. 2.11. More about vibration of a SDOF system can be found in [208].

The equation of motion of the SDOF system is

$$m\ddot{x}(t) + c\dot{x}(t) + kx(t) = F(t). \quad (2.105)$$

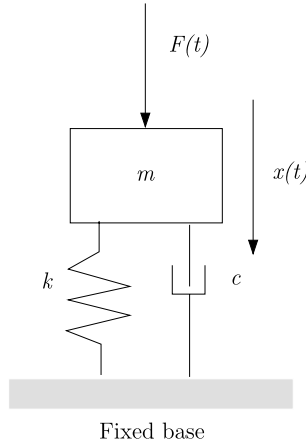


Fig. 2.11. SDOF system loaded with a force $F(t)$

Dividing (2.105) by the mass m we find the equation of motion of the damped SDOF system:

$$\ddot{x}(t) + 2\zeta\omega_n\dot{x}(t) + \omega_n^2x(t) = \frac{F(t)}{m} = f(t), \tag{2.106}$$

where $\omega_n = \sqrt{\frac{k}{m}}$ is the natural frequency and $\zeta = \frac{c}{2\sqrt{km}}$ the damping ratio.

The harmonic motion and force can be written

$$x(t) = X(\omega)e^{j\omega t} \quad \text{and} \quad f(t) = f(\omega)e^{j\omega t}. \tag{2.107}$$

A time derivative is equivalent to multiplying by $j\omega$ in the frequency domain.⁷ Substituting (2.107) into (2.106) we obtain the equation of motion in the frequency domain:

$$\{(j\omega)^2 + 2j\zeta\omega\omega_n + \omega_n^2\}X(\omega) = f(\omega), \tag{2.108}$$

or (2.108) can be written as

$$X(\omega) = H(\omega)f(\omega), \tag{2.109}$$

where $H(\omega)$ is the frequency response function (FRF) and is given by

$$H(\omega) = \frac{1}{(j\omega)^2 + \omega_n^2 + 2j\zeta\omega\omega_n} = \frac{1}{-\omega^2 + \omega_n^2 + 2j\zeta\omega\omega_n}. \tag{2.110}$$

The square of the modulus of the FRF $H(\omega)$ is

⁷ j is the *rotation operator*, multiplying by j in the frequency domain is a rotation of $\frac{\pi}{2}$ radians in the complex plane.

$$|H(\omega)|^2 = H(\omega)H^*(\omega) = \frac{1}{(\omega_n^2 - \omega^2)^2 + (2\zeta\omega\omega_n)^2}, \quad (2.111)$$

thus

$$|H(\omega)| = \frac{1}{\sqrt{(\omega_n^2 - \omega^2)^2 + (2\zeta\omega\omega_n)^2}}. \quad (2.112)$$

Equation (2.98) gives the mean square response of $x(t)$, due to the random load $f(t) = \frac{F(t)}{m}$ with the PSD function $S_{FF}(\omega)$:

$$E\{x^2(t)\} = R_{xx}(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} |H(\omega)|^2 S_{ff}(\omega) d\omega. \quad (2.113)$$

If the forcing function $f(t)$ has a constant PSD function (white noise) $S_{ff}(\omega) = \frac{W_{ff}(f)}{2} = S_f$, then

$$E\{x^2(t)\} = R_{xx}(0) = \frac{S_f}{2\pi} \int_{-\infty}^{\infty} |H(\omega)|^2 d\omega. \quad (2.114)$$

Remember that

$$S_{ff}(\omega) = \frac{S_{FF}(\omega)}{m^2}, \quad W_{ff}(f) = \frac{W_{FF}(f)}{m^2}. \quad (2.115)$$

The integral in (2.114) has a closed-form solution [171] (see Appendix A). We have the following coefficients for $G_2(j\omega)$ and $H_2(j\omega)$:

- $b_0 = 0$
- $b_1 = 1$
- $a_0 = 1$
- $a_1 = 2\zeta\omega_n$
- $a_2 = \omega_n^2$

The integral in (2.114) becomes

$$E\{x^2(t)\} = R_{xx}(0) = \frac{S_f}{2\pi} \int_{-\infty}^{\infty} |H(\omega)|^2 d\omega = \frac{S_f}{4\zeta\omega_n^3} = \frac{S_F}{4\zeta\omega_n^3 m^2}. \quad (2.116)$$

The PSD function of the velocity $\dot{x}(t)$ is

$$E\{\dot{x}^2(t)\} = R_{\dot{x}\dot{x}}(0) = \frac{S_f}{2\pi} \int_{-\infty}^{\infty} (j\omega)^2 |H(\omega)|^2 d\omega. \quad (2.117)$$

We have the following coefficients for $G_n(j\omega)$ and $H_n(j\omega)$:

- $b_0 = 1$
- $b_1 = 0$
- $a_0 = 1$
- $a_1 = 2\zeta\omega_n$
- $a_2 = \omega_n^2$

The integral (2.117) can now be obtained

$$E\{\dot{x}^2(t)\} = R_{\dot{x}\dot{x}}(0) = \frac{S_f}{2\pi} \int_{-\infty}^{\infty} (j\omega)^2 |H(\omega)|^2 d\omega = \frac{S_f}{4\zeta\omega_n} = \frac{S_F}{4\zeta\omega_n m^2}. \quad (2.118)$$

The mean square of the acceleration $\ddot{x}(t)$, $E\{\ddot{x}^2(t)\}$, with a white noise forcing does not exist, because

$$\int_{-\infty}^{\infty} (j\omega)^4 |H(\omega)|^2 d\omega$$

has no solution.

Example. A SDOF system with a natural frequency $f_n = 25$ Hz, a mass $m = 1$ kg, a damping ratio $\zeta = 0.01$ is applied by a force with constant PSD value $W_{FF} = 2S_F = 200$ N²/Hz. The rms value of the displacement x_{rms} can be calculated using (2.116)

$$x_{rms} = \sqrt{E\{x^2(t)\}} = \sqrt{\frac{S_F}{4\zeta\omega_n^3 m^2}} = \sqrt{\frac{W_{FF}}{64\pi^3 \zeta f_n^3 m^2}} = 0.00254 \text{ m.}$$

The rms value of the velocity \dot{x}_{rms} can be calculated using (2.118)

$$\dot{x}_{rms} = \sqrt{E\{\dot{x}^2(t)\}} = \sqrt{\frac{S_F}{4\zeta\omega_n m^2}} = \sqrt{\frac{W_{FF}}{16\pi\zeta f_n m^2}} = 0.390 \text{ m/s.}$$

Example. A SDOF system is shown in Fig. 2.12 as well as the values of the SDOF parameters. The white noise PSD function of the applied force $F(t)$ is $W_F = 3000$ N²/Hz. The mean value of the applied force is $\mu_F = 0$. The applied load is stationary and Gaussian. The SDOF system is linear and therefore the response $x(t)$ is Gaussian too. Calculate the probability that the displacement $x(t) \leq 0.0015$ m and the probability that $|x(t)| \leq 0.0015$ m.

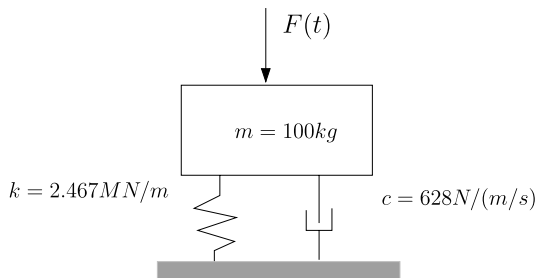


Fig. 2.12. Random force applied to SDOF

At first the natural frequency f_n will be calculated and after that the modal damping ratio ζ . The mean value of the displacement $x(t)$ is $\mu_x = \frac{\mu_F}{k} = 0$. The natural frequency of the SDOF system is

$$f_n = \frac{1}{2\pi} \sqrt{\frac{k}{m}} = 25 \text{ Hz},$$

and the damping ratio ζ is

$$\zeta = \frac{c}{2\sqrt{km}} = 0.02.$$

The standard deviation of the displacement $x(t)$ can be obtained from

$$\sigma_x = \sqrt{\frac{W_F}{64\pi^3 \zeta f_n^3 m^2}} = 6.955 \times 10^{-4} \text{ m}.$$

The probability that the displacement $x(t) \leq 0.0015$ m can be obtained using the standardized normal distribution with $z = \frac{x(t) - \mu_x}{\sigma_x} = \frac{0.0015 - 0}{6.955 \times 10^{-4}} = 2.157$

$$\text{Prob}\{x(t) \leq 0.0015\} = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{2.157} e^{-z^2/2} dz = 0.5 + 0.4845 = 0.9845,$$

and

$$\text{Prob}\{|x(t)| \leq 0.0015\} = \frac{1}{\sqrt{\pi}} \int_{-2.157}^{2.157} e^{-z^2/2} dz = 0.9690.$$

The probability $\text{Prob}\{x(t) > 0.0015\} = 1 - 0.9846 = 0.0155$ and the probability that $\text{Prob}\{|x(t)| > 0.0015\} = 1 - 0.9690 = 0.0310$.

We now study an example where a continuous beam loaded with a running load q will be converted to a SDOF system, using the assumed mode approach.

Example. A simply supported beam with bending stiffness EI , and length $L = 1.25$ m, and mass $m = 0.5$ kg/m must have a minimum natural frequency $f_n = 125$ Hz. The beam is illustrated in Fig. 2.13. Calculate the required bending stiffness EI . To do so the deflection $w(x, t)$ of the beam will be written as

$$w(x, t) = \sin\left(\frac{\pi x}{L}\right) \eta(t),$$

where $\eta(t)$ is called the generalized coordinate or modal participation. Calculate the generalized mass m_g and stiffness k_g . After that the associated natural frequency can be expressed as

$$f_n = \frac{1}{2\pi} \sqrt{\frac{k_g}{m_g}}.$$

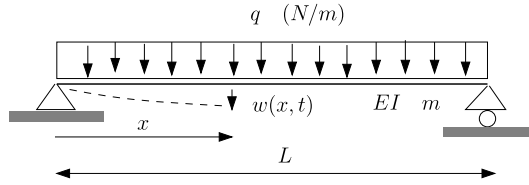


Fig. 2.13. Bending beam load with constant running load q

The generalized mass can be obtained with

$$m_g = m \int_0^L \sin^2\left(\frac{\pi x}{L}\right) dx = \frac{mL}{2}$$

and the generalized stiffness is

$$k_g = \int_0^L EI \left(\frac{\pi}{L}\right)^4 \sin^2\left(\frac{\pi x}{L}\right) dx = \frac{\pi^4 EI}{2 L^3}.$$

The natural frequency associated with the assumed mode $\sin\left(\frac{\pi x}{L}\right)$ will be

$$f_n = \frac{\pi}{2} \sqrt{\frac{EI}{mL^4}}.$$

The required minimum bending stiffness is $EI = 7.73 \times 10^3 \text{ Nm}^2$.

The bending beam is manufactured from an Al-alloy with a Young's modulus $E = 70 \times 10^9 \text{ Pa}$.

The cross section of the beam is a square tube with a height (width) h and a wall thickness $t = \frac{h}{20}$. Calculate the associated h and t from the required second moment of area I . The second moment of area of the cross section is

$$I = \frac{h^4}{30}.$$

From that it follows that the height and width of the square tube is $h = 43 \text{ mm}$ and the wall thickness $t = 2.133 \text{ mm}$.

The constant running load q , has a white noise PSD $W_q = 20\,000 \text{ N}^2/\text{Hz}$ and has been derived from the acoustic loads. The generalized load is

$$F_g = \int_0^L q \sin\left(\frac{\pi x}{L}\right) dx = \frac{2L}{\pi} q.$$

With the *structural damping coefficient* g the equation of motion of the SDOF system is

$$m_g \ddot{\eta}(t) + k_g(1 + jg)\eta(t) = F_g,$$

or

$$\ddot{\eta}(t) + \left(\frac{\pi}{L}\right)^4 \frac{EI}{m}(1 + jg)\eta(t) = \frac{4q}{\pi m}.$$

The adapted PSD function of the running load becomes

$$\bar{W}_q = \left(\frac{4}{\pi m}\right)^2 W_q.$$

The frequency response function

$$H(\omega) = \frac{1}{-\omega^2 + \omega_n^2(1 + jg)},$$

peaks at the resonance frequency with a amplification factor $Q = \frac{1}{g}$ with a low structural damping $g = 0.1$. At resonance, the structural damping is twice the modal damping ratio $g = 2\zeta$ [208]. The rms value of $\eta(t)$ is (see Appendix A)

$$\eta_{rms} = \sqrt{\frac{\bar{W}_q}{32\pi^3 g f_n^3}} = 0.026 \text{ m.}$$

The bending moment $M(x) = -EIw''(x)$ so that

$$\left| M_{rms} \left(\frac{L}{2} \right) \right| = EI \left(\frac{\pi}{L} \right)^2 \eta_{rms} = 1263 \text{ Nm.}$$

The bending resistance is given by

$$W = \frac{2I}{h} = \frac{h^3}{15} = 5.177 \times 10^{-7} \text{ m}^3.$$

The maximum bending stress at the mid of the bending beam is

$$\sigma_{bending,rms} = \frac{M_{rms} \left(\frac{L}{2} \right)}{W} = 2.44 \times 10^8 \text{ Pa.}$$

Problems

2.13. A circular simply supported plate, with radius $r = a$, is loaded by a uniform random pressure $p(t)$ with zero mean and an one-sided PSD $W_p(f) = W_0 \text{ Pa}^2/\text{Hz}$. The undamped axi-symmetric equation of motion is given by

$$\frac{D}{r} \frac{\partial}{\partial r} \left\{ r \frac{\partial}{\partial r} \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial w(r,t)}{\partial r} \right) \right] \right\} + m \frac{\partial^2 w(r,t)}{\partial t^2} = p(t),$$

where D is the bending stiffness, $w(r,t)$ is the deflection, and m is the mass per unit of area of the plate. The deflection $w(t)$ is expressed as follows

$$w(r, t) = \eta(t) \frac{a^4}{64D} \left(1 - \frac{r^2}{a^2}\right)^2 = \eta(t)\phi(r),$$

where $\eta(t)$ is the generalized coordinate, and $\phi(r)$ the assumed mode. The damped equation of motion for $\eta(t)$ is

$$m_g \ddot{\eta}(t) + 2\zeta\omega_0 m_g \dot{\eta}(t) + k_g \eta(t) = \gamma p(t).$$

Calculate the generalized mass m_g , the generalized stiffness k_g , the natural frequency ω_0 , and the participation factor γ .

Calculate the rms value generalized coordinate η_{rms} .

Calculate the rms displacement $w_{rms}(0)$ at the center of the plate. The modal radial bending moment in the center of the plate is given by

$$Mr(0, t) = \frac{\eta(t)}{16} [a^2(1 + \nu)].$$

Calculate the rms value of $Mr_{rms}(0)$.

Answers: $m_g = 2\pi m \int_0^a r \phi^2(r) dr$, $k_g = 2\pi D \int_0^a \frac{\partial}{\partial r} \left\{ r \frac{\partial}{\partial r} \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \phi(r)}{\partial r} \right) \right] \right\} \phi(r) dr$,

$$\gamma = 2\pi \int_0^a r \phi(r) dr = \frac{106.67D}{a^4 m}, \omega_0 = \sqrt{k_g/m_g} = 10.33 \sqrt{\frac{D}{a^4 m}}.$$

Hint: Theory of Plates and Shells [207].

2.4.2 SDOF with Enforced Acceleration

A SDOF system with a discrete mass m , damper element c and spring element k is placed on a moving base with an acceleration $\ddot{u}(t)$. The resulting displacement of the mass is $x(t)$. A relative motion $z(t)$ will be introduced which is the displacement of the mass with respect to the base. The relative displacement is

$$z(t) = x(t) - u(t). \tag{2.119}$$

The equation of motion of the SDOF system, illustrated in Fig. 2.14, is

$$m\ddot{x}(t) + c\{\dot{x} - \dot{u}(t)\} + k\{x(t) - u(t)\} = 0. \tag{2.120}$$

Using (2.119) we can write the equation of motion (2.120) of the SDOF system as

$$\ddot{z}(t) + 2\zeta\omega_n \dot{z}(t) + \omega_n^2 z(t) = -\ddot{u}(t). \tag{2.121}$$

The enforced acceleration of the SDOF system is transformed into an external force.

The absolute displacement $x(t)$ can be calculated from

$$\ddot{x}(t) = \ddot{z}(t) + \ddot{u}(t) = -2\zeta\omega_n \dot{z}(t) - \omega_n^2 z(t). \tag{2.122}$$

The Fourier transform of (2.121) will leads to the following equation of motion in the frequency domain

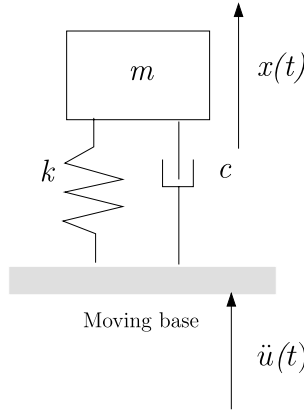


Fig. 2.14. Enforced acceleration of a damped SDOF system

$$[-\omega^2 + 2j\zeta\omega\omega_n + \omega_n^2] Z(\omega) = -\ddot{U}(\omega), \tag{2.123}$$

or

$$Z(\omega) = \frac{-\ddot{U}(\omega)}{-\omega^2 + 2j\zeta\omega\omega_n + \omega_n^2} = -H(\omega)\ddot{U}(\omega). \tag{2.124}$$

The PSD function of $z(t)$ is given by

$$S_{zz}(\omega) = \lim_{T \rightarrow \infty} \frac{1}{T} Z(\omega) Z^*(\omega), \tag{2.125}$$

and the PSD function of $\ddot{u}(t)$ is given by the following equation

$$S_{\ddot{u}\ddot{u}}(\omega) = \lim_{T \rightarrow \infty} \frac{1}{T} \ddot{U}(\omega) \ddot{U}^*(\omega). \tag{2.126}$$

This is in accordance with (2.50).

Using (2.98) we can write the PSD function $S_{zz}(\omega)$ of the relative motion $z(t)$ as

$$S_{zz}(\omega) = |H(\omega)|^2 S_{\ddot{u}\ddot{u}}(\omega). \tag{2.127}$$

We shall derive of the mean square values both in the time and frequency domain starting from the autocorrelation function $R_{zz}(\tau)$. Using (2.42)

$$R_{zz}(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{zz}(\omega) e^{j\omega\tau} d\omega, \tag{2.128}$$

and inserting (2.127) we find the following expression:

$$R_{zz}(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} |H(\omega)|^2 S_{\ddot{u}\ddot{u}}(\omega) e^{j\omega\tau} d\omega, \tag{2.129}$$

or with a constant PSD function $S_{\ddot{u}}$

$$R_{zz}(\tau) = \frac{S_{\ddot{u}}}{2\pi} \int_{-\infty}^{\infty} |H(\omega)|^2 e^{j\omega\tau} d\omega, \quad (2.130)$$

Equation (2.124) gives

$$|H(\omega)|^2 = \frac{1}{(\omega_n^2 - \omega^2)^2 + (2\zeta\omega\omega_n)^2} = \frac{1}{\omega_n^4 \left[\left(1 - \frac{\omega^2}{\omega_n^2}\right)^2 + \left(2\zeta\frac{\omega}{\omega_n}\right)^2 \right]}. \quad (2.131)$$

Finally the autocorrelation function $R_{zz}(\tau)$ is obtained (2.129) (see Appendix B)

$$R_{zz}(t) = \frac{S_{\ddot{u}} e^{-\zeta\omega_n t}}{4\zeta\omega_n^3} \left\{ \cos(\omega_n \sqrt{1 - \zeta^2}) + \frac{\zeta}{\sqrt{1 - \zeta^2}} \sin(\omega_n \sqrt{1 - \zeta^2}) \right\}, \quad t \geq 0. \quad (2.132)$$

The mean square response of the relative displacement $z(t)$ is

$$E\{z(t)^2\} = R_{zz}(0) = \frac{S_{\ddot{u}}}{4\zeta\omega_n^3} = \frac{W_{\ddot{u}}}{8\zeta(2\pi f_n)^3}. \quad (2.133)$$

Also, the autocorrelation functions of the velocity $\dot{z}(t)$ and $\ddot{z}(t)$ can be calculated by differentiating the autocorrelation function of the relative displacement $z(t)$,

$$R_{\dot{z}\dot{z}}(t) = -\frac{d^2 R_{zz}(t)}{dt^2}, \quad (2.134)$$

and

$$R_{\ddot{z}\ddot{z}}(t) = \frac{d^4 R_{zz}(t)}{dt^4}. \quad (2.135)$$

The derivation of the mean square calculation of $z(t)$ in the frequency domain is repeated. With reference to (2.129) the mean square response of $z(t)$, due to the random excitation $\ddot{u}(t)$, with the PSD function $S_{\ddot{u}\ddot{u}}(\omega)$, becomes

$$E\{z(t)^2\} = R_{zz}(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} |H(\omega)|^2 S_{\ddot{u}\ddot{u}}(\omega) d\omega. \quad (2.136)$$

If the enforced acceleration $\ddot{u}(t)$ has a constant PSD function (white noise) $S_{\ddot{u}\ddot{u}}(\omega) = S_{\ddot{u}} = \frac{W_{\ddot{u}}}{2}$, then

$$E\{z(t)^2\} = R_{zz}(0) = \frac{S_{\ddot{u}}}{2\pi} \int_{-\infty}^{\infty} |H(\omega)|^2 d\omega. \quad (2.137)$$

This integral (2.137) has a known solution (see (2.116))

$$E\{z(t)^2\} = R_{zz}(0) = \frac{S_{\ddot{u}}}{2\pi} \int_{-\infty}^{\infty} |H(\omega)|^2 d\omega = \frac{S_{\ddot{u}}}{4\zeta\omega_n^3} = \frac{W_{\ddot{u}}}{8\zeta(2\pi f_n)^3}. \quad (2.138)$$

Example. Suppose the system in Fig. 2.14 has a natural frequency $f_n = 30$ Hz and a damping ratio $\zeta = 0.01$. It is desired to estimate the rms relative displacement of z when the excitation is a stationary random acceleration with a uniform spectral density $W_{\ddot{u}} = 0.10$ g²/Hz.

$$E\{z(t)^2\} = \frac{W_{\ddot{u}}}{8\zeta(2\pi f_n)^3} = \frac{0.10 \times 9.81^2}{8 \times 0.01 \times (2\pi 30)^3} = 17.96 \times 10^{-6} \text{ m}^2.$$

The square root of this is the rms value $z_{rms} = 4.24 \times 10^{-3}$ m. This example is taken from [83], page 29.

The mean square value of the velocity $\dot{z}(t)$ is given by

$$\begin{aligned} E\{\dot{z}(t)^2\} &= R_{\dot{z}\dot{z}}(0) = \frac{S_{\ddot{u}}}{2\pi} \int_{-\infty}^{\infty} (j\omega)^2 |H(\omega)|^2 d\omega = \frac{S_{\ddot{u}}}{4\zeta\omega_n} \\ &= \frac{W_{\ddot{u}}}{8\zeta(2\pi f_n)}. \end{aligned} \quad (2.139)$$

Thus the relationship between the expectation of the velocity and the expectation of the displacement is given by $E\{\dot{z}^2\} = (2\pi f)^2 E\{z^2\}$.

To calculate the mean square value of the acceleration $\ddot{x}(t)$ we use (2.122). The autocorrelation function $R_{\ddot{x}\ddot{x}}(t)$ for the acceleration is

$$R_{\ddot{x}\ddot{x}}(t) = (2\zeta\omega_n)^2 R_{\dot{z}\dot{z}}(t) + \omega_n^4 R_{zz}(t) + 2\zeta\omega_n^3 R_{\dot{z}z}(t) + 2\zeta\omega_n^3 R_{z\dot{z}}(t). \quad (2.140)$$

Differentiating (2.33) with respect to t we find

$$\frac{dR_{xx}(\tau)}{dt} = R_{\dot{x}x}(\tau) + R_{x\dot{x}}(\tau) = 0, \quad (2.141)$$

thus

$$R_{\dot{x}x}(\tau) = -R_{x\dot{x}}(\tau). \quad (2.142)$$

Therefore (2.140) with $\tau = 0$ becomes

$$R_{\ddot{x}\ddot{x}}(0) = (2\zeta\omega_n)^2 R_{\dot{z}\dot{z}}(0) + \omega_n^4 R_{zz}(0), \quad (2.143)$$

and (2.143) can be expressed using the mean square of $\ddot{x}(t)$

$$E\{\ddot{x}^2(t)\} = (2\zeta\omega_n)^2 E\{\dot{z}^2(t)\} + \omega_n^4 E\{z^2(t)\}. \quad (2.144)$$

Substituting (2.138) and (2.139) in (2.144) we find the mean square of $\ddot{x}(t)$:

$$E\{\ddot{x}^2(t)\} = \frac{\omega_n S_{\ddot{u}}}{4\zeta} (1 + 4\zeta^2) = \frac{\pi f_n W_{\ddot{u}}}{4\zeta} (1 + 4\zeta^2). \quad (2.145)$$

Equation (2.145) can be approximated when $4\zeta^2 \ll 1$ ⁸ as follows

⁸ In spacecraft structures, the damping ratio $\zeta \approx 0.01 \dots 0.05$.

$$E\{\ddot{x}^2(t)\} = \ddot{x}_{rms}^2 = \frac{\pi f_n W_{\ddot{u}}}{4\zeta} (1 + 4\zeta^2) \approx \frac{\pi f_n W_{\ddot{u}}}{4\zeta} = \frac{\pi}{2} f_n Q W_{\ddot{u}}, \quad (2.146)$$

where $Q = \frac{1}{2\zeta}$ is the *amplification factor* (quality factor) and f_n the natural frequency (Hz).

In general, the mean value of the acceleration $\ddot{x}(t)$ is zero, $\mu_{\ddot{x}} = 0$. The variance $\sigma_{\ddot{x}}$ of the acceleration $\ddot{x}(t)$ is

$$\sigma_{\ddot{x}} = E\{\ddot{x}^2(t)\} - \mu_{\ddot{x}}^2 = \ddot{x}_{rms}^2, \quad (2.147)$$

where *rms* stands for root mean square.

The modulus of the frequency response function $|H(\omega)|$, (2.131)

$$|H(\omega_n)| = \frac{Q}{\omega_n^2}.$$

The bandwidth $\Delta\omega$ between the half power points, i.e. $|H(\omega_n)|^2 = \frac{Q^2}{2\omega_n^4}$, is

$$\Delta\omega = 2\zeta\omega_n \quad \Delta f = 2\zeta f_n.$$

The mean square of the acceleration $\ddot{x}(t)$ is

$$E\{\ddot{x}^2(t)\} = \ddot{x}_{rms}^2 \approx \frac{\pi}{2} f_n Q W_{\ddot{u}} = \frac{\pi}{2} \Delta f_n Q^2 W_{\ddot{u}}. \quad (2.148)$$

Most of the contribution of the power to the mean square value $E\{\ddot{x}^2(t)\}$ is stored in a very peaked area with bandwidth Δf_n and a height $\frac{\pi}{2} Q^2$. The contribution to the power from outside the bandwidth Δf_n is much less.

If the PSD function of the enforced acceleration $W_{\ddot{u}\ddot{u}}(f)$ is rather constant in the bandwidth Δf_n and $E\{\ddot{x}^2(t)\}$ may be than approximated by

$$E\{\ddot{x}^2(t)\} = \frac{\pi}{2} f_n Q W_{\ddot{u}\ddot{u}}(f_n). \quad (2.149)$$

Equation (2.149) is called Miles' equation [127] and is normally written when the average value of the acceleration $\mu_{\ddot{x}} = 0$ as

$$\sigma_{\ddot{x}} = \ddot{x}_{rms} = \sqrt{\frac{\pi}{2} f_n Q W_{\ddot{u}\ddot{u}}(f_n)}. \quad (2.150)$$

The *rms* value of the acceleration $\ddot{x}(t)$ in (2.150) is in practice often denoted by G_{rms} . Within the frame of spacecraft structures, instruments, equipment, etc., an amplification factor $Q = 10$ is frequently applied.

The equation of motion of the SDOF system as shown in Fig. 2.14 will now be expressed in terms of the response $x(t)$

$$m\ddot{x}(t) + c\{\dot{x}(t) - \dot{u}(t)\} + k\{x(t) - u(t)\} = 0. \quad (2.151)$$

Dividing (2.151) divided by the mass m gives

$$\ddot{x}(t) + 2\zeta\omega_n\dot{x}(t) + \omega_n^2x(t) = 2\zeta\omega_n\dot{u}(t) + \omega_n^2u(t) = F(t). \quad (2.152)$$

The PSD function of the forcing function $f(t)$ is defined as

$$\begin{aligned} S_{FF}(\omega) &= \lim_{T \rightarrow \infty} \frac{1}{T} F(\omega) F^*(\omega) \\ &= \lim_{T \rightarrow \infty} \frac{1}{T} |[2j\zeta\omega_n\omega U(\omega) + \omega_n^2 U(\omega)]|^2, \end{aligned} \quad (2.153)$$

or

$$\begin{aligned} S_{FF}(\omega) &= \lim_{T \rightarrow \infty} \frac{1}{T} |2j\zeta\omega_n\omega + \omega_n^2|^2 |U(\omega)|^2 \\ &= [(2\zeta\omega_n\omega)^2 + \omega_n^4] S_{uu}(\omega). \end{aligned} \quad (2.154)$$

The PSD function of the response $x(t)$ can now be written as follows;

$$S_{xx}(\omega) = |H(\omega)|^2 S_{FF}(\omega) = |H(\omega)|^2 [(2\zeta\omega_n\omega)^2 + \omega_n^4] S_{uu}(\omega). \quad (2.155)$$

We assume a constant PSD function $S_{uu}(\omega) = S_u$; the autocorrelation function $R_{xx}(t)$ is

$$\begin{aligned} R_{xx}(t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{xx}(\omega) e^{j\omega t} d\omega \\ &= \frac{S_u}{2\pi} \int_{-\infty}^{\infty} [(2\zeta\omega_n\omega)^2 + \omega_n^4] |H(\omega)|^2 d\omega. \end{aligned} \quad (2.156)$$

The solution of (2.156) is found in [14]:

$$\begin{aligned} R_{xx}(t) &= \frac{\omega_n S_u (1 + 4\zeta^2)}{4\zeta} e^{-\zeta\omega_n t} \left[\cos(\omega_n \sqrt{1 - \zeta^2} t) \right. \\ &\quad \left. + \frac{\zeta(1 - 4\zeta^2)}{(1 + 4\zeta^2)\sqrt{1 - \zeta^2}} \sin(\omega_n \sqrt{1 - \zeta^2} t) \right], \quad t \geq 0. \end{aligned} \quad (2.157)$$

The PSD function of the acceleration $\ddot{x}(t)$ can be written as (2.155)

$$S_{\ddot{x}\ddot{x}}(\omega) = |H(\omega)|^2 S_{FF}(\omega) = |H(\omega)|^2 [(2\zeta\omega_n\omega)^2 + \omega_n^4] S_{\ddot{u}\ddot{u}}(\omega). \quad (2.158)$$

The relations between the PSD and cross PSD functions for $x(t)$, $\dot{x}(t)$ and $\ddot{x}(t)$ are given in Table 2.5.

The mean square of $\ddot{x}(t)$ due to the random enforced acceleration $\ddot{u}(t)$ with a constant PSD function $S_{\ddot{u}\ddot{u}}(\omega) = S_{\ddot{u}} = \frac{W_{\ddot{u}}}{2}$ is

$$E\{\ddot{x}^2(t)\} = R_{\ddot{u}\ddot{u}}(0) = \frac{S_{\ddot{u}}}{2\pi} \int_{-\infty}^{\infty} |H(\omega)|^2 [(2\zeta\omega_n\omega)^2 + \omega_n^4] d\omega. \quad (2.159)$$

Using the results achieved in Appendix B we can extract the following constants from (2.159):

Table 2.5. PSD and cross PSD functions of $x(t)$, $\dot{x}(t)$ and $\ddot{x}(t)$

	$x(t)$	$\dot{x}(t)$	$\ddot{x}(t)$
$x(t)$	$S_{xx}(\omega)$	$S_{x\dot{x}}(\omega) = j\omega S_{xx}(\omega)$	$S_{x\ddot{x}}(\omega) = (j\omega)^2 S_{xx}(\omega)$
$\dot{x}(t)$	$S_{\dot{x}\dot{x}}(\omega) = -j\omega S_{xx}(\omega)$	$S_{\dot{x}\ddot{x}}(\omega) = \omega^2 S_{xx}(\omega)$	$S_{\dot{x}\ddot{x}}(\omega) = j\omega^3 S_{xx}(\omega)$
$\ddot{x}(t)$	$S_{\ddot{x}\ddot{x}}(\omega) = (-j\omega)^2 S_{xx}(\omega)$	$S_{\ddot{x}\dot{x}}(\omega) = j\omega^3 S_{xx}(\omega)$	$S_{\ddot{x}\ddot{x}}(\omega) = \omega^4 S_{xx}(\omega)$

- $n = 2$
- $a_0 = -1$
- $a_1 = 2\zeta\omega_n$
- $a_2 = \omega_n^2$
- $b_0 = -4\zeta^2\omega_n^2$
- $b_1 = \omega_n^4$

The integral (2.159) now becomes

$$\begin{aligned}
 E\{\ddot{x}^2(t)\} &= R_{\ddot{x}\ddot{x}}(0) = \frac{S_{\ddot{u}}}{2\pi} \int_{-\infty}^{\infty} |H(\omega)|^2 [(2\zeta\omega_n\omega)^2 + \omega_n^4] d\omega \\
 &= \frac{\omega_n S_{\ddot{u}}}{4\zeta} (1 + 4\zeta^2) \approx \frac{\pi}{2} f_n Q W_{\ddot{u}}.
 \end{aligned}
 \tag{2.160}$$

This is again Miles' equation (2.146).

From (2.155), the expectation of the mean square of the displacement $x(t)$ can be found:

$$E\{x^2(t)\} = R_{xx}(0) = \frac{\omega_n S_u}{4\zeta} (1 + 4\zeta^2) \approx \frac{\pi}{2} f_n Q W_u.
 \tag{2.161}$$

The PSD function of the enforced acceleration $\ddot{u}(t)$ is given by $W_{\ddot{u}\ddot{u}}(f) = (2\pi f)^4 W_{uu}(f)$, so that the expectation of the displacement $x(t)$ may be expressed as

$$E\{x^2(t)\} = \frac{E\{\ddot{x}^2(t)\}}{(2\pi f)^4}.
 \tag{2.162}$$

Example. A given SDOF system has a natural frequency $f_n = 100$ Hz and a damping ratio $\zeta = 0.05$, $Q = 10$. The white noise PSD function of the base excitation is $W_{\ddot{u}} = 0.1$ g²/Hz. Calculate the rms acceleration of the SDOF system.

The rms acceleration response of the SDOF system can be calculated using Miles' equation (2.150)

$$\ddot{x}_{rms} = \sqrt{\frac{\pi}{2} f_n Q W_{\ddot{u}\ddot{u}}(f_n)} = \sqrt{\frac{\pi}{2} 100 \times 10 \times 0.1} = 12.53 G_{rms}.$$

2.4.3 Multi-Inputs and Single Output (MISO)

In Fig. 2.15 a SDOF system is shown with both an enforced acceleration $\ddot{u}(t)$ at the base and a direct force $F(t)$. The PSD function of the response $x(t)$ will be calculated.

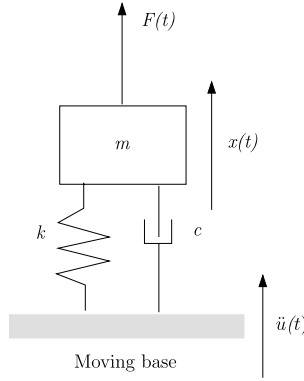


Fig. 2.15. Multi-inputs $F(t)$ and $\ddot{u}(t)$, single output $x(t)$

The equation of motion of the SDOF is

$$m\ddot{x}(t) + c\{\dot{x}(t) - \dot{u}(t)\} + k\{x(t) - u(t)\} = F(t), \tag{2.163}$$

or

$$\ddot{x}(t) + 2\zeta\omega_n\dot{x}(t) + \omega_n^2x(t) = 2\zeta\omega_n\dot{u}(t) + \omega_n^2u(t) + \frac{F(t)}{m} = q(t). \tag{2.164}$$

The PSD function of $q(t)$ with $f(t) = \frac{F(t)}{m}$ becomes

$$S_{qq}(\omega) = \left[\left(\frac{2\zeta\omega_n}{\omega} \right)^2 + \left(\frac{\omega_n}{\omega} \right)^2 \right] S_{\ddot{u}\ddot{u}}(\omega) + S_{ff}(\omega) + 2 \left(\frac{\omega_n}{\omega} \right)^2 S_{f\ddot{u}}(\omega), \tag{2.165}$$

where $S_{ff}(\omega) = \frac{W_{FF}(\omega)}{m^2}$ the PSD function of $f(t)$, $S_{\ddot{u}\ddot{u}}(\omega)$ the PSD function of the enforced acceleration $\ddot{u}(t)$ and $S_{f\ddot{u}}(\omega) = S_{\ddot{u}f}^*(\omega)$ the cross PSD function of $f(t)$ and $\ddot{u}(t)$ and is in general a complex valued function.

2.4.4 Unit Load Random Vibration Responses

The random vibration in a structure, caused by a particular random load $F(t)$, may be approximated by the response of that structure due a static unit load representation F_{ul} of the random load multiplied by a dynamic factor. This approach will be demonstrated with the aid of the SDOF system shown in Fig. 2.16. The static deformation caused by the unit load F_{ul} must have about the same deformation pattern as the mode shape $\{\phi_n\}$ considered with the natural frequency f_n .

The PSD function of the random load $F(t)$ is indicated by $W_{FF}(f)$, and the rms displacement x_{rms} of the response $x(t)$ is given by

$$x_{rms} = \sqrt{\frac{W_{FF}(f_n)}{64\pi^3\zeta f_n^3 m^2}}. \tag{2.166}$$

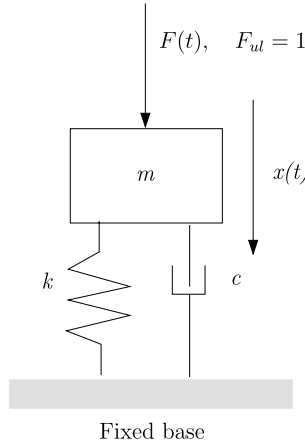


Fig. 2.16. Random and unit load application

The displacement x_{ul} of the SDOF system caused by the static unit load $F_{ul} = 1$ can be easily calculated

$$x_{ul} = \frac{F_{ul}}{k} = \frac{1}{k} = \frac{1}{m(2\pi f_n)^2}, \tag{2.167}$$

where the natural frequency f_n of the SDOF system is given by $f_n = \frac{1}{2\pi} \sqrt{\frac{k}{m}}$.

The term $m(2\pi f_n)^2$ is extracted from (2.166); x_{rms} becomes

$$x_{rms} = \frac{1}{m(2\pi f_n)^2} \sqrt{\frac{\pi f_n W_{FF}(f_n)}{4\zeta}} = \frac{1}{m(2\pi f_n)^2} \sqrt{\frac{\pi}{2} f_n Q W_{FF}(f_n)} \tag{2.168}$$

substituting (2.167) in (2.168) to obtain the following final expression for x_{rms} :

$$x_{rms} = x_{ul} \sqrt{\frac{\pi}{2} f_n Q W_{FF}(f_n)}. \tag{2.169}$$

Equation (2.169) can be written very generally, the displacements may for example be replaced by stresses σ

$$\frac{x_{rms}}{x_{ul}} = \frac{\sigma_{rms}}{\sigma_{ul}} = \sqrt{\frac{\pi}{2} f_n Q W_{FF}(f_n)}. \tag{2.170}$$

The approximate *rms* responses obtained from (2.170) will give conservative results!

Problems

2.14. Consider a weakly stationary random process with the PSD function of order $(m, n) = (0, 2)$

$$S(\omega) = \frac{S_0}{(\omega_0^2 - \omega^2)^2 + (2\zeta\omega_0\omega)^2}, \quad \zeta < 1.$$

Define the corresponding autocorrelation function $R(\tau)$ using Table 2.3, page 23.

Answer: $R(\tau) = \frac{S_0}{2\zeta\omega_0^3} e^{-\zeta\omega_0|\tau|} (\cos(\omega_d\tau) + \frac{\zeta}{\sqrt{1-\zeta^2}} \sin(\omega_d\tau)), \omega_d = \omega_0\sqrt{1-\zeta^2}.$

2.15. A damped SDOF linear system is excited by a random ergodic external force F having a PSD function given by $W_F(f) = W_0, f \in (0, f_0]$ elsewhere $W_F(f) = 0$. The sdof system has a damping ratio $\zeta = 0.05$ and a natural undamped frequency $f_n = \frac{f_0}{2}$ (Hz). The spring stiffness is k . Find the PSD function $W_x(f)$ of the displacement x as a function of $\frac{f}{f_0}, f \in (0, f_0)$. Calculate the PSD values of the displacement x , the velocity \dot{x} , the acceleration \ddot{x} and for the force F_{spring} in the spring at the natural frequency f_n .

Answers: $W_x(f_n) = 100\frac{W_0}{k^2}, W_{\dot{x}}(f_n) = 100(2\pi f_n)^2\frac{W_0}{k^2}, W_{\ddot{x}}(f_n) = 100(2\pi f_n)^4\frac{W_0}{k^2}, W_{F_{spring}}(f_n) = 100W_0.$

2.16. A SDOF system has a natural frequency $f_n = 40$ Hz. The mass $m = 5$ kg. The damping ratio $\zeta = 0.05$. The base acceleration is specified in Table 2.22.

- Calculate the PSD value $W_{\ddot{u}\ddot{u}}(f_n)$,
- Calculate the rms reaction force $F_{base,rms}$ at the base ($g = 9.81 \text{ m/s}^2$).

Answers:

- $W_{\ddot{u}\ddot{u}}(f_n) = 0.1024 \text{ g}^2/\text{Hz}$
- $F_{base,rms} = 393.4 \text{ N}$

2.17. This problem is taken from [188]. A “band-limited” white noise enforced acceleration \ddot{u} is one whose PSD $S_{\ddot{u}}$ has the form shown in Fig. 2.17. Let \ddot{u} be the enforced acceleration to the system

$$\ddot{z} + 2\zeta\omega_0\dot{z} + \omega_0^2z = -\ddot{u},$$

where z is the relative displacement, and $\ddot{x} = -\omega_0^2z - 2\zeta\omega_0\dot{z}$ the absolute displacement. Derive analytical expressions for the rms response of \ddot{z}, \dot{z}, z

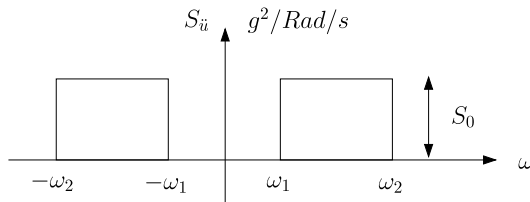


Fig. 2.17. “Band-limited” white noise enforced acceleration $S_{\ddot{u}}$

and \ddot{x} . Under what condition can the rms response of the \ddot{z} be approximated by one subject to an ideal white-noise excitation (Miles equation).

The mean square or the zero-lag correlation function $R(0)$ can be obtained from the following expression:

$$R(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S(\omega) d\omega.$$

Calculate the numerical values of the rms responses when

- the natural frequency is $\omega_0 = 2\pi f_0$, $f_0 = 30$ Hz,
- the damping ratio is $\zeta = 0.05$,
- the PSD function is $S_0 = 0.05 \text{ g}^2/\text{rad/s}$ ($g = 9.81 \text{ m/s}^2$) and
- the radian frequencies are $\omega_1 = 0.5\omega_0$ and $\omega_2 = 1.5\omega_0$.

Answers: $\ddot{z}_{rms} = 66.3571 \text{ m/s}^2$, $\dot{z}_{rms} = 0.3520 \text{ m/s}$, $z_{rms} = 0.0019 \text{ m}$, $\ddot{x}_{rms} = 65.9195 \text{ m/s}^2$, $\ddot{z}_{rms, Miles} = 67.3426 \text{ m/s}^2$.

2.5 Deterministic MDOF Linear Dynamic Systems

Multi degrees of freedom (MDOF) dynamic systems may be exposed to random forces and or random enforced motions (i.e. acceleration at the base). Both kinds of random loads will be discussed in the following sections. The matrix equations of motions are provided by (general purpose) finite element programs or otherwise.

2.5.1 Random Forces

In general, the matrix equations of motion of a discrete MDOF dynamic system can be written as

$$[M]\{\ddot{x}(t)\} + [C]\{\dot{x}(t)\} + [K]\{x(t)\} = \{F(t)\}, \tag{2.171}$$

and consists of the following matrices and vectors:

- the mass matrix $[M]$
- the stiffness matrix $[K]$
- the damping matrix $[C]$
- the force vector $\{F(t)\}$
- the displacement, velocity and acceleration vectors $\{x(t)\}$, $\{\dot{x}(t)\}$ and $\{\ddot{x}(t)\}$

For linear MDOF systems the mass, stiffness and damping matrix do not vary with time and are deterministic, however, the displacement, velocity, and acceleration and force vector do usually change with time and are random.

Using the *modal displacement method*⁹ (MDM) [126, 208], we can express the physical displacement vector $\{x(t)\}$ in terms of the independent set of eigenvectors of the *modal matrix* $[\Phi]$

$$\{x(t)\} = [\Phi]\{\eta(t)\}, \quad (2.172)$$

where $\{\eta(t)\}$ is the vector of generalized coordinates or modal participations.

The modal matrix $[\Phi] = [\phi_1, \phi_2, \dots, \phi_n]$ has the following *orthogonality properties* with respect to the mass and the stiffness matrix

$$[\Phi]^T[M][\Phi] = [m], \quad [\Phi]^T[K][\Phi] = [Am], \quad (2.173)$$

where $[m]$ is the diagonal matrix of generalized masses and $[Am]$ the diagonal matrix of the eigenvalues Λ multiplied by the generalized masses.

Using (2.172) and pre-multiplying by the transpose of the modal matrix $[\Phi]^T$ we find

$$[\Phi]^T[M][\Phi]\{\ddot{\eta}(t)\} + [\Phi]^T[C][\Phi]\{\dot{\eta}(t)\} + [\Phi]^T[K][\Phi]\{\eta(t)\} = [\Phi]^T\{F(t)\}. \quad (2.174)$$

Making use of the orthogonality relation of the modal matrix $[\Phi]$, (2.173), the equation of motions are expressed in *generalized coordinates, generalized masses, eigenvalues* and *generalized forces*

$$[m]\{\ddot{\eta}(t)\} + [c]\{\dot{\eta}(t)\} + [Am]\{\eta(t)\} = \{q(t)\}, \quad (2.175)$$

where $\{q(t)\} = [\Phi]^T\{F(t)\}$ is the vector of generalized forces, $[c]$ the diagonal matrix of the generalized damping. This means that the damping matrix $[C]$ consists of *proportional damping*¹⁰ [208], which is, in general, not the case. The influence of modal coupling of non-classically damped linear systems is discussed in [147]. Generally, the modal viscous damping will be added on an ad hoc basis to the uncoupled equations of the generalized coordinates, $c_i = \{\phi_i\}^T[C]\{\phi_i\}$ and $\frac{c_i}{m_i} = 2\zeta_i\omega_i$, with $\omega_i = \sqrt{\lambda_i}$. $m_i = \{\phi_i\}^T[M]\{\phi_i\}$ is the generalized mass associated with the mode $\{\phi_i\}$, $\zeta_i = \frac{c_i}{2\sqrt{k_i m_i}}$ is the modal damping ratio, $k_i = \{\phi_i\}^T[K]\{\phi_i\}$ is the generalized stiffness and $\lambda_i = \omega_i^2$ is the eigenvalue of the eigenvalue problem $([K] - \lambda_i[M])\{\phi_i\} = \{0\}$.

In general, the modal damping ratio is taken to be $\zeta = 0.05$ when random vibrations are involved. In [1] an alternative value for the modal damping ratio has been given

$$\zeta_i = \frac{1}{10 + 0.05f_i}, \quad (2.176)$$

where f_i (Hz) is the i th natural frequency.

Finally, the uncoupled equations of motion of the m generalized coordinates with the generalized forces $\{q(t)\}$ become

⁹ Modal superposition, the orthogonal modal method, or the principal coordinate method [111].

¹⁰ Proportional to the stiffness and mass matrix.

$$\ddot{\eta}_i(t) + 2\zeta_i\omega_i\dot{\eta}_i(t) + \omega_i^2\eta_i(t) = \frac{q_i(t)}{m_i}, \quad i = 1, 2, \dots, m. \quad (2.177)$$

To find the PSD of a structural response in the physical coordinates, all the modal PSD (direct- and cross) should be summed. The modal cross PSD may be neglected if, [47]:

- the system is lightly damped, and
- the eigenfrequencies of the system are well separated.

The conditions can be satisfied when

$$\max(\omega_j\zeta_j, \omega_k\zeta_k) \ll |\omega_j^2 - \omega_k^2|. \quad (2.178)$$

The PSD function of the external generalized forces $q_i(t)$ is defined as

$$\begin{aligned} S_{q_i q_j}(\omega) &= \lim_{T \rightarrow \infty} \frac{1}{2T} Q_i(\omega) Q_j^*(\omega) \\ &= \lim_{T \rightarrow \infty} \frac{1}{2T} \{\phi_i\}^T \{F_i(\omega)\} \{F_j^*(\omega)\} \{\phi_j\}, \end{aligned} \quad (2.179)$$

or

$$S_{q_i q_j}(\omega) = \{\phi_i\}^T [S_{F_i F_j}(\omega)] \{\phi_j\}. \quad (2.180)$$

The matrix of the PSD and cross PSD functions $S_{F_i F_j}(\omega)$ can be written as

$$[S_{F_i F_j}(\omega)] = \begin{bmatrix} S_{F_1 F_1}(\omega) & S_{F_1 F_2}(\omega) & \cdots & S_{F_1 F_n}(\omega) \\ S_{F_2 F_1}(\omega) & S_{F_2 F_2}(\omega) & \cdots & \vdots \\ \vdots & \vdots & \ddots & S_{F_{n-1} F_n}(\omega) \\ S_{F_n F_1}(\omega) & \cdots & S_{F_n F_{n-1}}(\omega) & S_{F_n F_n}(\omega) \end{bmatrix}, \quad (2.181)$$

where n is the number of applied random forces.

If there is no correlation between the individual forces F_i and F_j the cross PSD function $S_{F_i F_j}(\omega) = 0$, $i, j = 1, 2, \dots, n$.

In the frequency domain (2.177) can be written with $\eta(t) = \Pi(\omega)e^{j\omega t}$ and $q(t) = Q(\omega)e^{j\omega t}$

$$\Pi_i(\omega) = \frac{1}{-\omega^2 + \omega_i^2 + 2j\zeta_i\omega\omega_i} \frac{Q_i(\omega)}{m_i} = H_i(\omega) \frac{Q_i(\omega)}{m_i}. \quad (2.182)$$

The general cross PSD function $S_{\eta_i \eta_j}(\omega)$ of the generalized coordinates $\eta_i(t)$ can be expressed as follows

$$S_{\eta_i \eta_j}(\omega) = \frac{H_i(\omega)}{m_i} S_{Q_i Q_j}(\omega) \frac{H_j^*(\omega)}{m_j}, \quad (2.183)$$

or we can write using (2.180)

$$S_{\eta_i \eta_j}(\omega) = \frac{H_i(\omega)}{m_i} \{\phi_i\}^T [S_{F_i F_j}(\omega)] \{\phi_j\} \frac{H_j^*(\omega)}{m_j}. \quad (2.184)$$

The matrix of PSD and cross PSD functions of the generalized coordinates $[S_{\eta_i \eta_j}(\omega)]$ becomes

$$[S_{\eta_i \eta_j}(\omega)] = \left[\frac{H_i(\omega)}{m_i} \right] [\Phi]^T [S_{F_i F_j}(\omega)] [\Phi] \left[\frac{H_j^*(\omega)}{m_j} \right]. \quad (2.185)$$

The matrix $[S_{\eta_i \eta_j}(\omega)]$ ¹¹ is the PSD and cross PSD function of m generalized coordinates $\eta_i(t)$ given by

$$[S_{\eta_i \eta_j}(\omega)] = \begin{bmatrix} S_{\eta_1 \eta_1}(\omega) & S_{\eta_1 \eta_2}(\omega) & \cdots & S_{\eta_1 \eta_m}(\omega) \\ S_{\eta_2 \eta_1}(\omega) & S_{\eta_2 \eta_2}(\omega) & \cdots & \vdots \\ \vdots & \vdots & \ddots & S_{\eta_{m-1} \eta_m}(\omega) \\ S_{\eta_m \eta_1}(\omega) & \cdots & S_{\eta_m \eta_{m-1}}(\omega) & S_{\eta_m \eta_m}(\omega) \end{bmatrix}. \quad (2.186)$$

The matrix of PSD and cross PSD functions of the physical displacements is $\{x(t)\} = [\Phi]\{\eta(t)\}$ with $\{\eta(t)\} = \Pi(\omega)e^{j\omega t}$ is

$$\begin{aligned} [S_{x_i x_j}(\omega)] &= \lim_{T \rightarrow \infty} \frac{1}{T} X_i(\omega) X_j^*(\omega) \\ &= \lim_{T \rightarrow \infty} \frac{1}{T} [\Phi_i] \{ \Pi_i(\omega) \} \{ \Pi_j^*(\omega) \} [\Phi_j]^T, \end{aligned} \quad (2.187)$$

or

$$[S_{x_i x_j}] = [\Phi] S_{\eta_i \eta_j}(\omega) [\Phi]^T. \quad (2.188)$$

Finally, it ends with

$$[S_{x_i x_j}(\omega)] = [\Phi] \left[\left[\frac{H_i(\omega)}{m_i} \right] [\Phi]^T [S_{F_i F_j}(\omega)] [\Phi] \left[\frac{H_j^*(\omega)}{m_j} \right] \right] [\Phi]^T. \quad (2.189)$$

Equation (2.189) can also be written as

$$[W_{x_i x_j}(f)] = [\Phi] \left[\left[\frac{H_i(2\pi f)}{m_i} \right] [\Phi]^T [W_{F_i F_j}(f)] [\Phi] \left[\frac{H_j^*(2\pi f)}{m_j} \right] \right] [\Phi]^T. \quad (2.190)$$

The matrix of the mean square values of $x(t)$ can now be calculated using

$$[E(x_i x_j)] = [R_{x_i x_j}(0)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} [S_{x_i x_j}(\omega)] d\omega = \int_0^{\infty} [W_{x_i x_j}(f)] df. \quad (2.191)$$

Random enforced accelerations will be discussed in the following section.

2.5.2 MDOF System Loaded by Random Enforced Acceleration

A MDOF dynamic system will have an acceleration base excitation as illustrated in Fig. 2.18.

¹¹ The power spectral density matrix $[S(\omega)]$ is in general a complex matrix that has the following properties:

- $[S(-\omega)] = [S(\omega)]$ for all ω
- $[S^*(\omega)] = [S(\omega)]$ for all ω .

The * denotes the complex conjugate transpose.

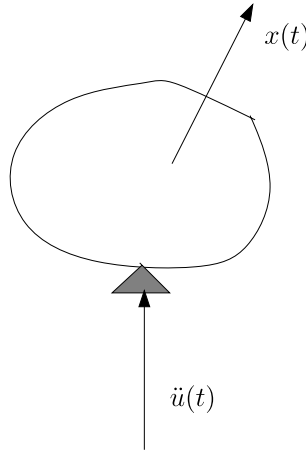


Fig. 2.18. Base excitation

The undamped equations of motion of the dynamic system can be written as

$$[M] \{\ddot{x}(t)\} + [K] \{x(t)\} = \{0\}. \tag{2.192}$$

A relative motion $\{z(t)\} = \{x(t)\} - \{T\}u(t)$ is introduced and $[K]\{T\}u(t) = \{0\}$, because $\{T\}$ is the rigid body mode in the direction of the enforced displacement $u(t)$. In fact there are six rigid body modes, three translational and three rotational. Equation (2.192) becomes

$$[M] \{\ddot{z}(t)\} + [K] \{z(t)\} = -[M] \{T\} \ddot{u}(t). \tag{2.193}$$

The PSD function of the relative motion $\{z(t)\}$ has the same structure as (2.190), thus

$$[W_{z_i z_j}] = [\Phi] \left[\left[\frac{H_i(2\pi f)}{m_i} \right] [\Phi]^T [W_{F_i F_j}(f)] [\Phi] \left[\frac{H_j^*(2\pi f)}{m_j} \right] \right] [\Phi]^T, \tag{2.194}$$

where

- $[W_{F_i F_j}(f)] = (-1)^2 [M] \{T\} \{T\}^T [M] W_{\ddot{u}\ddot{u}}(f)$ the PSD function of the applied random loads (base excitation)
- $H_i(2\pi f) = \frac{1}{(2\pi)^2 (f_i^2 - f^2 + 2j\zeta_i f f_i)}$ the frequency response function (FRF)
- ζ_i the modal damping ratio associated with mode $\{\phi_i\}$
- $m_i = \{\phi_i\}^T [M] \{\phi_i\}$ the generalized mass
- $[\Phi] = [\phi_1, \phi_2, \dots, \phi_n]$ the modal matrix and
- $\{(-2\pi f_i)^2 [M] + [K]\} \{\phi_i\} = \{0\}$ the eigenvalue problem.

Equation (2.194) can be written as

$$[W_{z_i z_j}(f)] = [[H_{z\ddot{u}}(f)]^2 W_{\ddot{u}\ddot{u}}(f) = [H_{z\ddot{u}}(f)] [H_{z\ddot{u}}^*(f)]^T W_{\ddot{u}\ddot{u}}(f), \tag{2.195}$$

or

$$\begin{aligned} [W_{z_i z_j}(f)] &= \left([\Phi] \left[\left[\frac{H_i(2\pi f)}{m_i} \right] [\Phi]^T [M] \{T\} \right. \right. \\ &\quad \left. \left. \times \{T\}^T [M] [\Phi] \left[\frac{H_j^*(2\pi f)}{m_j} \right] \right] [\Phi]^T \right) W_{\ddot{u}\ddot{u}}(f). \end{aligned} \quad (2.196)$$

Thus, the FRF $[H_{z\ddot{u}}(f)]$ can now be written

$$[H_{z\ddot{u}}(f)] = -[\Phi] \left[\left[\frac{H_i(2\pi f)}{m_i} \right] [\Phi]^T [M] \{T\} \right]. \quad (2.197)$$

The minus sign reflects the negative RHS of (2.193).

The FRF $[H_{zu}(\omega)]$ between the displacement $U(\omega)$ and the relative displacement vector $\{Z(\omega)\}$ is given by

$$\{Z(\omega)\} = [H_{zu}(\omega)]U(\omega) = -\omega^2[H_{z\ddot{u}}(\omega)]U(\omega), \quad (2.198)$$

because $\ddot{U}(\omega) = (j\omega)^2 U(\omega) = -\omega^2 U(\omega)$, thus the FRF $[H_{zu}(\omega)]$ becomes

$$[H_{zu}(\omega)] = \omega^2 [\Phi] \left[\left[\frac{H_i(2\pi f)}{m_i} \right] [\Phi]^T [M] \{T\} \right]. \quad (2.199)$$

The absolute displacement vector $\{x(t)\} = \{z(t)\} + \{T\}u(t)$. The PSD function of the absolute displacement component $\{x(t)\}$ is defined as

$$[S_{x_i x_j}(\omega)] = \lim_{T \rightarrow \infty} \frac{1}{2T} \{X_i(\omega)\} \{X_j^*(\omega)\}^T, \quad (2.200)$$

or

$$[S_{x_i x_j}(\omega)] = \lim_{T \rightarrow \infty} \frac{1}{2T} [\{Z(\omega)\} + \{T\}U(\omega)] [\{Z^*(\omega)\} + \{T\}U^*(\omega)]^T, \quad (2.201)$$

and further

$$[S_{x_i x_j}(\omega)] = [[H_{zu}(\omega)] + \{T\}] [[H_{zu}^*(\omega)] + \{T\}]^T \lim_{T \rightarrow \infty} \frac{1}{2T} U(\omega) U^*(\omega), \quad (2.202)$$

and, finally,

$$\begin{aligned} [S_{x_i x_j}(\omega)] &= ([H_{zu}(\omega)][H_{zu}^*(\omega)]^T + \{T\}[H_{zu}^*(\omega)]^T + [H_{zu}(\omega)]\{T\}^T \\ &\quad + \{T\}\{T\}^T) S_{uu}(\omega). \end{aligned} \quad (2.203)$$

The PSD function of the accelerations $\{\ddot{x}(t)\}$ can now be easily obtained, because $[S_{\ddot{x}\ddot{x}}(\omega)] = \omega^4 [S_{xx}(\omega)]$ and $S_{\ddot{u}\ddot{u}}(\omega) = \omega^4 S_{uu}(\omega)$, and therefore

$$\begin{aligned} [S_{\ddot{x}_i \ddot{x}_j}(\omega)] &= ([H_{zu}(\omega)][H_{zu}^*(\omega)]^T + \{T\}[H_{zu}^*(\omega)]^T + [H_{zu}(\omega)]\{T\}^T \\ &\quad + \{T\}\{T\}^T) S_{\ddot{u}\ddot{u}}(\omega). \end{aligned} \quad (2.204)$$

In the frequency (Hz) domain (2.204) can be written

$$[W_{\ddot{x}_i \ddot{x}_j}(f)] = ([H_{z\ddot{u}}(2\pi f)][H_{z\ddot{u}}^*(2\pi f)]^T) W_{\ddot{u}\ddot{u}}(f), \quad (2.205)$$

where

- $[H_{\ddot{z}\ddot{u}}(2\pi f)] = [H_{zu}(2\pi f)] + \{T\}$
- $[H_{zu}(2\pi f)] = (2\pi f)^2 [\Phi] \left[\frac{H_i(2\pi f)}{m_i} \right] [\Phi]^T [M] \{T\}$
- $m_i = \{\phi_i\}^T [M] \{\phi_i\}$
- $H_i(2\pi f) = \frac{1}{(2\pi)^2 (f_i^2 - f^2 + 2j\zeta_i f f_i)}$
- ζ_i the modal damping ratio associated with mode $\{\phi_i\}$, and
- $\{T\}$ the rigid body vector in the excitation direction $u(t)$ and $\ddot{u}(t)$

The matrix of the mean square values of $\ddot{x}(t)$ can now be calculated using

$$[E(\ddot{x}_i \ddot{x}_j)] = [R_{\ddot{x}_i \ddot{x}_j}(0)] = \int_0^\infty [W_{\ddot{x}_i \ddot{x}_j}(f)] df. \quad (2.206)$$

The rigid body vector $\{T\}$ can be extracted from the free-free stiffness matrix $[K]$ of an elastic system. Divide the degrees of freedom (DOF) into a set of *internal dofs*, called the *i*-set and 6 *external dofs* in one node in the external (boundary) set, *b*-set, and applying unit displacements and rotations in the *b*-set, we can find the following static equilibrium equation

$$\begin{bmatrix} K_{ii} & K_{ib} \\ K_{bi} & K_{bb} \end{bmatrix} \begin{Bmatrix} x_i \\ x_b \end{Bmatrix} = \begin{Bmatrix} 0 \\ F_b \end{Bmatrix}. \quad (2.207)$$

From the first equation of (2.207) we can express $\{x_i\}$ in terms of $\{x_b\}$

$$\begin{aligned} \{x_i\} &= -[K_{ii}]^{-1} [K_{ib}] \{x_b\} = -[K_{ii}]^{-1} [K_{ib}] [I] \\ &= -[K_{ii}]^{-1} [K_{ib}] = [\Psi_{i,rigid}]. \end{aligned} \quad (2.208)$$

The complete matrix of the six *rigid body vectors* $[\Phi_{rigid}]$ is given by

$$[\Phi_{rigid}] = \begin{pmatrix} [\Psi_{i,rigid}] \\ [I] \end{pmatrix}. \quad (2.209)$$

The six rigid body vectors are respectively translations in the *x*-, *y*-, *z*-direction and rotations about the *x*-, *y*- and *z*-axis. The rigid body vector $\{T\}$ is one of the rigid body vectors of $[\Phi_{rigid}]$.

2.5.3 Random Forces and Stresses

Besides the responses of the DOFs the internal forces and associated stresses should be calculated to predict the strength characteristics of the linear structure itself.

The matrix of cross-PSD of the forces or stresses, say $[S_{\sigma_i \sigma_j}(\omega)]$ can be calculated

$$[S_{\sigma_i \sigma_j}(\omega)] = [\Phi_\sigma] [S_{\eta_i \eta_j}(\omega)] [\Phi_\sigma]^T, \quad (2.210)$$

where $[\Phi_\sigma]$ is the matrix of force or stress modes associated with the modal matrix $[\Phi]$ and $[S_{\eta_i \eta_j}(\omega)]$ the matrix of auto-PSD functions (main diagonal) and cross-PSD functions of the generalised coordinates $\eta_i(t)$.

The *stress modes* $[\Phi_\sigma]$ can be calculated using the mode shapes $[\Phi]$ and the so-called force or *stress matrix*¹² $[D_\sigma]$. The stress modes are defined as

$$[\Phi_\sigma] = [D_\sigma][\Phi]. \quad (2.211)$$

When random loads are applied to a deterministic linear system (2.210) can be written as

$$[W_{\sigma_i\sigma_j}(f)] = [H_{\sigma F}(2\pi f)][W_{FF}(f)][H_{\sigma F}^*(2\pi f)]^T, \quad (2.212)$$

and when random base excitation is applied the matrix $[W_{\sigma_i\sigma_j}(f)]$ (2.212) can be written

$$[W_{\sigma_i\sigma_j}(f)] = [H_{\sigma\ddot{u}}(2\pi f)][W_{\ddot{u}\ddot{u}}(f)][H_{\sigma\ddot{u}}^*(2\pi f)]^T, \quad (2.213)$$

where

- $[W_{FF}(f)]$ the PSD matrix of applied forces
- $[W_{\ddot{u}\ddot{u}}(f)]$ the PSD matrix of enforced accelerations
- $[H_{\sigma F}(2\pi f)] = [D_\sigma][\Phi]\left[\frac{H_i(2\pi f)}{m_i}\right][\Phi]^T$
- $[H_{\sigma\ddot{u}}(2\pi f)] = -[D_\sigma][\Phi]\left[\frac{H_i(2\pi f)}{m_i}\right][\Phi]^T[M]\{T\}$
- $H_i(2\pi f) = \frac{1}{(2\pi)^2(f_i^2 - f^2 + 2j\zeta_i f f_i)}$
- ζ_i the modal damping ratio associated with mode $\{\phi_i\}$, and
- $\{T\}$ the rigid body vector in the excitation direction $u(t)$ and $\ddot{u}(t)$.

Example. A 3 mass-spring dynamic system, as shown in Fig. 2.19, is excited at the base with a constant band limited random enforced acceleration $W_{\ddot{u}} = 0.01 \text{ g}^2/\text{Hz}$ in a frequency range $f \in [5, 500] \text{ Hz}$. The discrete masses are $m_1 = 200 \text{ kg}$, $m_2 = 250 \text{ kg}$ and $m_3 = 300 \text{ kg}$. The spring stiffness of the springs is given by $k_1 = 10^8 \text{ N/m}$, $k_2 = 2 \times 10^8 \text{ N/m}$ and $k_3 = 3 \times 10^8 \text{ N/m}$. The modal damping ratio for all modes is $\zeta = 0.05$ or the amplification factor (transmissibility) $Q = 10$. Calculate the accelerations $\{\ddot{x}\}$ and the internal loads.

The undamped equations of motion are

$$\begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{Bmatrix} + \begin{bmatrix} k_1 & -k_1 & 0 \\ -k_1 & k_1 + k_2 & -k_2 \\ 0 & -k_2 & k_2 + k_3 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix},$$

or previous equation can be expressed in the relative motion $\{z(t)\} = \{x(t)\} - \{T\}u(t)$, thus

$$\begin{aligned} & \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix} \begin{Bmatrix} \ddot{z}_1 \\ \ddot{z}_2 \\ \ddot{z}_3 \end{Bmatrix} + \begin{bmatrix} k_1 & -k_1 & 0 \\ -k_1 & k_1 + k_2 & -k_2 \\ 0 & -k_2 & k_2 + k_3 \end{bmatrix} \begin{Bmatrix} z_1 \\ z_2 \\ z_3 \end{Bmatrix} \\ & = - \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix} \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix} \ddot{u}(t), \end{aligned}$$

¹² The stress matrix is the assembly of element force or stress matrices.

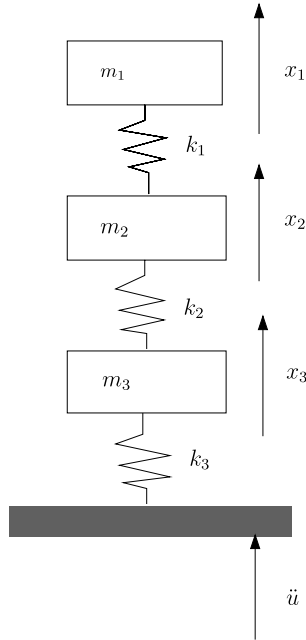


Fig. 2.19. 3 mass-spring system, enforced acceleration

where

$$\{T\} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

The natural frequencies and associated mode shapes of the MDOF system are

$$\{f_n\} = \left\{ \begin{array}{l} 69.5745 \\ 153.7834 \\ 238.3038 \end{array} \right\} \text{ Hz}, \quad [\Phi] = \begin{bmatrix} 0.0560 & 0.0417 & 0.0111 \\ 0.0346 & -0.0362 & -0.0387 \\ 0.0156 & -0.0329 & 0.0448 \end{bmatrix}.$$

The generalized mass matrix $[m]$ is given by

$$[m] = [\Phi]^T [M] [\Phi] = \begin{bmatrix} 1.0000 & 0.0000 & 0.0000 \\ 0.0000 & 1.0000 & 0.0000 \\ 0.0000 & 0.0000 & 1.0000 \end{bmatrix}.$$

The *modal effective mass* associated with the vibration mode $\{\phi_i\}$ with respect to the base is defined as [224] (see also Appendix D)

$$M_{eff,i} = \left(\frac{\{\phi_i\}^T [M] \{T\}}{\sqrt{m_i}} \right)^2$$

with $[M]$ the mass matrix. The modal effective masses calculated for the dynamic system are as follows

$$\{M_{eff}\} = \begin{Bmatrix} 602.2987 \\ 111.7754 \\ 35.9259 \end{Bmatrix} \text{ kg.}$$

The most important mode is that with the maximum modal effective mass. The sum of the modal effective masses is equal to the total mass of the system $\sum_{i=1}^3 M_{eff,i} = M_{tot} = 750 \text{ kg}$.

The integrals of the 3×3 matrix of PSD values $W_{\ddot{x}_i \ddot{x}_j}(f)$ are the mean square values. The matrix of the mean square (expected) values of the accelerations $E\{\ddot{x}_i \ddot{x}_j\} \text{ (m/s}^2\text{)}^2$ can be calculated by

$$\begin{aligned} E\{\ddot{x}_i \ddot{x}_j\} &= \int_5^{500} W_{\ddot{x}_i \ddot{x}_j}(f) df \\ &= 10^3 \begin{bmatrix} 2.4262 & 0.8029 - 0.1640j & 0.2655 - 0.0887j \\ 0.8029 + 0.1640j & 1.2838 & 0.4506 - 0.1925j \\ 0.2655 + 0.0887j & 0.4506 + 0.1925j & 0.7414 \end{bmatrix}. \end{aligned}$$

The integration of the previous equation is carried out by the trapezoidal rule with a frequency increment $\Delta f = 0.2 \text{ Hz}$. The rms values of the accelerations [g] on the main diagonal are

$$\sqrt{E\{\ddot{x}_i \ddot{x}_i\}} = \{\ddot{x}_{rms}\} = \begin{Bmatrix} 5.0211 \\ 3.6524 \\ 2.7757 \end{Bmatrix} \text{ g.}$$

The internal force matrix $D_{\bar{F}}$ is set up as follows

$$D_{\bar{F}} = \begin{bmatrix} k_1 & -k_1 & 0 \\ 0 & k_2 & -k_2 \\ 0 & 0 & k_3 \end{bmatrix}.$$

The mean square values of internal force matrix $E\{\bar{F}_i \bar{F}_j\} \text{ (N}^2\text{)}$ is established with a $\Delta f = 0.2 \text{ Hz}$.

$$\begin{aligned} E\{\bar{F}_i \bar{F}_j\} &= \int_5^{500} W_{\sigma_i \sigma_j}(f) df \\ &= 10^8 \begin{bmatrix} 0.9672 & 1.3640 - 0.0814j & 1.5169 - 0.1321j \\ 1.3640 + 0.0814j & 2.5564 & 3.0372 - 0.1923j \\ 1.5169 + 0.1321j & 3.0372 + 0.1923j & 4.1681 \end{bmatrix}. \end{aligned}$$

The rms values of the internal forces on the main diagonal are

$$\sqrt{E\{F_i F_i\}} = \{F_{ii,rms}\} = 10^4 \begin{Bmatrix} 0.9835 \\ 1.5989 \\ 2.0416 \end{Bmatrix} \text{ N.}$$

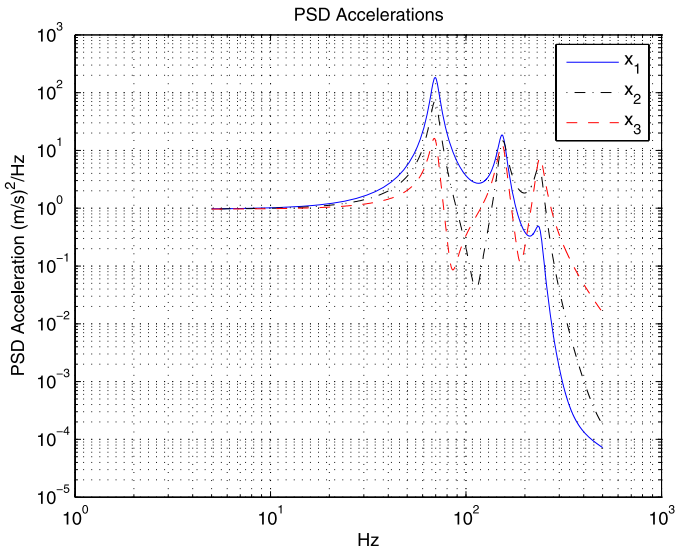


Fig. 2.20. PSD accelerations of masses

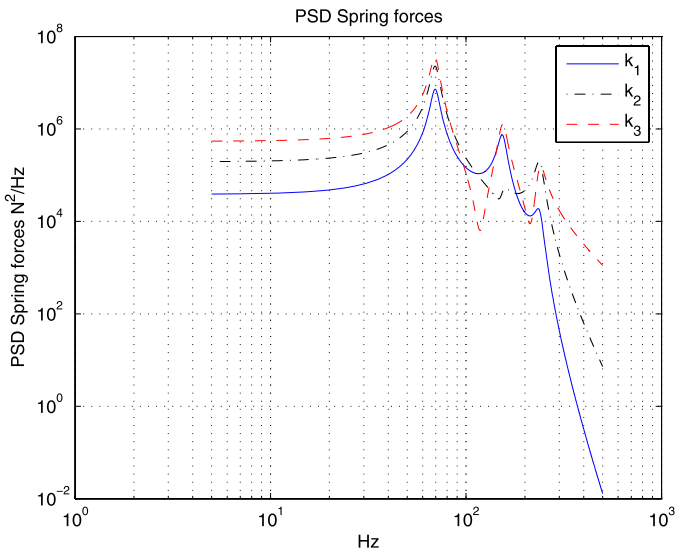


Fig. 2.21. PSD forces in springs

The PSD values of the accelerations and spring forces are shown in Figs. 2.20 and 2.21.

In the previous example we calculated the rms force in the spring k_3 . This can also be done using the concept of modal effective masses [224] and the

relation as derived in Appendix C to obtain the reaction force $F_{base}(\omega)$ as a consequence of the base excitation $\ddot{U}(\omega)$ at a determinate interface.

$$\{F_{base}(\omega)\} = \sum_{k=1}^n [M_{eff,k}(\omega_k)(1 + \omega^2 H_k(\omega))] \{\ddot{U}(\omega)\}. \quad (2.214)$$

The *rms* value of the reaction force $F_{base}(\omega)$ is approximated by

$$F_{rms,base}^2 \approx \sqrt{\sum_{i=1}^n \left[M_{eff,i}^2 \frac{\pi}{2} f_i Q_i W_{\ddot{u}}(f_i) \right]}, \quad (2.215)$$

where n is the number of modes taken into account. This result can be traced in (2.160). This approach is called the *mass participation method* [30]. Equation (2.215) will be proven using (2.193)

$$[M]\{\ddot{z}(t)\} + [K]\{z(t)\} = -[M]\{T\}\ddot{u}(t).$$

The relative displacement vector $\{z(t)\}$ will be depicted on the modal base $[\Phi]$, thus $\{z(t)\} = [\Phi]\{\eta(t)\}$, where $\{\eta(t)\}$ is the vector of generalized coordinates.

The mean square value of the modal reaction force $F_{ik,rms,base}$, $|F_{ik,rms,base}|^2$ is given by

$$\begin{aligned} |F_{ik,rms,base}|^2 &= \{\phi_i\}^T [M]\{T_k\}\{T_k\}^T [M]\{\phi_i\} \ddot{\eta}_{i,rms}^2 \\ &= M_{eff,ik} m_i \ddot{\eta}_{i,rms}^2, \end{aligned} \quad (2.216)$$

or

$$\begin{aligned} |F_{ik,rms,base}|^2 &= M_{eff,ik} m_i \ddot{\eta}_{i,rms}^2 \\ &= M_{eff,ik}^2 \frac{\pi}{2} f_i Q_i W_{\ddot{u}}(f_i), \quad i = 1, 2, \dots, n, \quad k = 1, 2, \dots, 6. \end{aligned} \quad (2.217)$$

The total mean square of the reaction force $F_{k,ms,base}$ is obtained by adding the contribution of the modal mean square reaction forces $|F_{ik,rms,base}|^2$, thus

$$\begin{aligned} F_{k,ms,base} &= \sum_{i=1}^n |F_{ik,rms,base}|^2 \\ &= \sum_{i=1}^n \left[M_{eff,ik}^2 \frac{\pi}{2} f_i Q_i W_{\ddot{u}}(f_i) \right], \quad k = 1, 2, \dots, 6. \end{aligned} \quad (2.218)$$

Finally the *rms* value of the reaction force $F_{k,rms,base}$ is found:

$$F_{k,rms,base} = \sqrt{\sum_{i=1}^n \left[M_{eff,ik}^2 \frac{\pi}{2} f_i Q_i W_{\ddot{u}}(f_i) \right]}, \quad k = 1, 2, \dots, 6. \quad (2.219)$$

Example. The previous example will be continued. The PSD function of the base excitation is constant $W_{\dot{u}} = 0.01 \text{ g}^2/\text{Hz}$. The constant amplification factor is $Q = 10$.

The natural frequencies of the MDOF system as illustrated in Fig. 2.19 are

$$\{f_n\} = \left\{ \begin{array}{l} 69.5745 \\ 153.7834 \\ 238.3038 \end{array} \right\} \text{ Hz},$$

and the associated modal effective masses

$$\{M_{eff}\} = \left\{ \begin{array}{l} 602.2987 \\ 111.7754 \\ 35.9259 \end{array} \right\} \text{ kg}.$$

The mean square value of the reaction force F_{base} is

$$E\{F_{base}^2\} = R_{F_{base}F_{base}}(0) = 4.1523 \times 10^8 \text{ N}^2,$$

and the rms value of F_{base} is

$$F_{base,rms} = \sqrt{E\{F_{base}^2\}} = \sqrt{R_{F_{base}F_{base}}(0)} = 2.0377 \times 10^4 \text{ N}.$$

The approximation of the rms value of the reaction force F_{base} using Miles' equation gives a somewhat lower value (0.2%) as calculated in the previous example ($2.0416 \times 10^4 \text{ N}$), because the term $4\zeta^2$ in $(1+4\zeta^2)$ has been neglected.

Problems

2.18. A cantilevered beam is shown in Fig. 2.22. The length of the beam is l . This cantilevered beam will be idealized by three Bernoulli-Euler beam elements with constant mass and stiffness. The mass and stiffness matrix of the beam element are as given below

$$[M_b] = \frac{mL}{420} \begin{pmatrix} 156 & 22L & 54 & -13L \\ 22L & 4L^2 & 13L & -3L^2 \\ 54 & 13L & 156 & -22L \\ -13L & -3L^2 & -22L & 4L^2 \end{pmatrix},$$

$$[K_b] = \frac{EI}{L^3} \begin{pmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{pmatrix},$$

where L is the length of the beam element, $m = \rho l$ is the mass per unit of length, ρ is the density, E is Young's modulus and I is the second moment of area.

The design parameters have the following values:

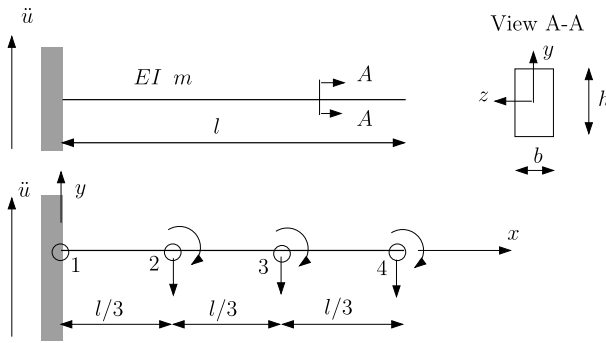


Fig. 2.22. FEM cantilevered beam

Table 2.6. Random libration level

Frequency spectrum (Hz)	$W_{\ddot{u}\ddot{u}}(f)$ g^2/Hz
20–50	5.5 dB/octave
50–800	0.053
800–2000	–5.5 dB/octave
G_{rms}	? g

- $E = 70 \text{ GPa}$,
- $\rho = 2700 \text{ kg/m}^3$,
- $l = 0.3 \text{ m}$,
- $h = 0.01 \text{ m}$,
- $b = 0.006 \text{ m}$,

Calculate the three lowest natural frequencies (Hz) of the cantilevered beam.

The beam is excited at the base by a random enforced acceleration $\ddot{u}(t)$. The random enforced acceleration PSD $W_{\ddot{u}\ddot{u}}(f)$ is specified in Table 2.6. The modal damping ratio for all modes is $\zeta = 0.05$. Perform the following assignments using the mode superposition method:

- Plot PSD of the acceleration responses of nodes 2, 3 and 4 versus frequency (Hz)
- Calculate rms values of the acceleration responses of the nodes 2, 3 and 4.
- Calculate the number of positive zero crossings associated with the accelerations of the nodes 2, 3 and 4.

Repeat the random response analysis applying a general purpose finite element software package. Try to validate your finite element random response analysis by an analytical solution [19].

Answers (partly): MSC.Nastran SOL 111 analysis, frequency range

20–2000 Hz, natural frequencies 91.34, 572.02 and 1605.2 Hz, rms translational (y-dir) acceleration node 4 is 25.1 g and associated number of positive zero crossings node 4 is $\nu_0^+ = 731.6$ Hz.

2.6 Complex Modal Analysis, Lyapunov's Equation

2.6.1 State-Space Equation

Complex modal analysis had been developed by Fang and Wang in [61]. This approach is valid for both the classical and nonclassical damping cases. It is a time domain analysis which is most suitable to calculate correlation functions and covariances of the random responses.

An MDOF linear dynamic system is considered with the following matrix equations of damped motion

$$[M]\{\ddot{x}\} + [C]\{\dot{x}\} + [K]\{x\} = \{f(t)\}, \quad (2.220)$$

where $[M]$, $[C]$ and $[K]$ are $n \times n$ symmetrical positive definite matrices and $\{f(t)\}$ is a stationary **white noise** excitation with following properties

$$E[\{f(t)\}] = \{0\}, \quad E[\{f(t)\}\{f(t+\tau)\}^T] = \frac{1}{2}[W_0]\delta(\tau), \quad (2.221)$$

where $[W_0] = 2[S_0]$ the single-sided white noise autospectrum matrix (PSD) of $\{f(t)\}$ in the cyclic frequency domain. $[S_0]$ is the constant PSD of the double side spectrum in the rad/s domain. The equations of motion (2.220) are transformed into a space-state

$$\{\dot{y}\} = [A]\{y\} + [B]\{f(t)\}, \quad (2.222)$$

where the *state vector* is given by $\{y\}^T = [y_1^T, y_2^T] = [x^T, \dot{x}^T]$, $[A]$ is the time-invariant state matrix, $[B]$ is the time-invariant input matrix and $\{f(t)\}$ the excitation vector. The *state matrix* $[A]$ is defined by

$$[A] = \begin{pmatrix} 0 & I \\ -[M]^{-1}[K] & -[M]^{-1}[C] \end{pmatrix}, \quad (2.223)$$

and the *input matrix* $[B]$ is as follows

$$[B] = \begin{pmatrix} 0 \\ -[M]^{-1} \end{pmatrix}, \quad (2.224)$$

because we can write

$$\begin{aligned} \{\dot{y}_1\} &= \{y_2\}, \\ \{\dot{y}_2\} &= -[M]^{-1}[K]\{y_1\} - [M]^{-1}[C]\{y_2\} + [M]^{-1}\{f(t)\}. \end{aligned} \quad (2.225)$$

The solution of (2.222) is derived in [66]

$$\{y(t)\} = e^{[A](t-\tau)}\{y(\tau)\} + \int_{\tau}^t e^{[A](t-\zeta)}[B]\{f(\zeta)\}d\zeta, \quad (2.226)$$

where $e^{[A](t)} = [I] + [A]t + [A]^2 \frac{t^2}{2} + [A]^3 \frac{t^3}{3!} + \dots$ [10].

The state matrix $[A]$ has $2n$ eigenvalues λ_i and $2n$ corresponding eigenvectors ψ_i . When the damping is below the critical damping, the λ_i and $\{\psi_i\}$ appear in conjugate pairs. The real parts of the eigenvalues are negative, $\Re\{\lambda_i\} < 0$. The complex modal base is given by

$$[\Psi] = [\psi_1, \psi_2, \dots, \psi_{2n}]. \quad (2.227)$$

The state matrix $[A]$ can be expressed in the eigenvalues λ_i and associated eigenvectors ψ_i , if there are no repeating eigenvalues, in the following way [10, 198]

$$[A] = [\Psi][\Lambda][\Psi]^{-1}, \quad (2.228)$$

where $[\Lambda] = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_{2n})$. This equation is known as a “similarity transformation”. It is said that the matrices $[A]$ and $[\Psi]$ are “similar”. If the following base transformation is applied $\{y\} = [\Psi]\{v\}$ and substituting (2.228) in (2.222) the following state-space equation is obtained

$$\{\dot{v}\} = [\Lambda]\{v\} + [\Psi]^{-1}[B]\{f(t)\}. \quad (2.229)$$

The solution for $\{v(t)\}$ is

$$\{v(t)\} = e^{[\Lambda](t-\tau)}\{v(\tau)\} + \int_{\tau}^t e^{[\Lambda](t-\zeta)}[B]\{f(\zeta)\}d\zeta, \quad (2.230)$$

and the stationary solution (when $\tau = 0$ and $t \rightarrow \infty$) becomes

$$\{v(t)\} = \int_0^{\infty} e^{[\Lambda](t-\zeta)}[B]\{f(\zeta)\}d\zeta = \int_0^{\infty} e^{[\Lambda](\zeta)}[B]\{f(t-\zeta)\}d\zeta. \quad (2.231)$$

The correlation matrix $[R_{vv}(\tau)]$ of the state vector $\{v(t)\}$ can be written as

$$[R_{vv}(\tau)] = E[\{v(t)\}\{v^T(t+\tau)\}] = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^{\infty} \{v(t)\}\{v^T(t+\tau)\}dt, \quad (2.232)$$

where $\{v\}^T$ is the transpose of the vector $\{v\}$.¹³ The correlation matrix of the state vector $\{v(t)\}$ can now be obtained by

¹³ The transpose matrix or vector containing complex numbers is formed by interchanging its rows and columns and changing the complex numbers in its conjugate values, $\begin{pmatrix} 2+5j \\ 3-4j \end{pmatrix}^T = [2-5j, 3+4j]$.

$$[R_{vv}(\tau)] = \int_0^\infty \int_0^\infty e^{[A](\zeta)} [\Psi]^{-1} [B] E[\{f(t-\zeta)\}\{f(t-\eta+\tau)\}^T] \\ \times [B]^T [\Psi]^{-T} e^{[A]^T(\eta)} d\zeta d\eta, \quad (2.233)$$

or

$$[R_{vv}(\tau)] = \int_0^\infty \int_0^\infty e^{[A](\zeta)} [\Psi]^{-1} [B] [S_0] \delta(\tau - \eta + \zeta) [B]^T [\Psi]^{-T} e^{[A]^T(\eta)} d\zeta d\eta, \quad (2.234)$$

thus finally

$$[R_{vv}(\tau)] = \int_0^\infty e^{[A](\zeta)} [\Psi]^{-1} [B] [S_0] [B]^T [\Psi]^{-T} e^{[A]^T(\tau+\zeta)} d\zeta \\ = \int_0^\infty e^{[A](\zeta)} [G] e^{[A]^T(\tau+\zeta)} d\zeta. \quad (2.235)$$

The element (i, j) of the correlation matrix $[R_{vv}(\tau)]$ is as follows

$$R_{vv}(\tau)(i, j) = G(i, j) \int_0^\infty e^{\lambda_i \zeta + \bar{\lambda}_j (\zeta + \tau)} d\zeta \\ = \frac{-G(i, j) e^{\bar{\lambda}_j \tau}}{\lambda_i + \bar{\lambda}_j} = R_{vv}(-\tau)(j, i), \quad (2.236)$$

where $\bar{\lambda}$ is the conjugate of λ . The correlation function $R_{vv}(0)(i, j)$ becomes

$$R_{vv}(0)(i, j) = \frac{-G(i, j)}{\lambda_i + \bar{\lambda}_j}. \quad (2.237)$$

The correlation function $R_{yy}(\tau)$ can be calculated as follows

$$[R_{yy}(\tau)] = [\Psi][R_{vv}(\tau)][\Psi]^T. \quad (2.238)$$

Referring to (2.235), the correlation matrix $[R_{yy}(\tau)]$ can be expressed as

$$[R_{yy}(\tau)] = \int_0^\infty e^{[A](\zeta)} [B] [S_0] [B]^T e^{[A]^T(\tau+\zeta)} d\zeta. \quad (2.239)$$

This equation will be used to derive the equation of “The stochastic dual of the direct method of *Lyapunov*”¹⁴ [72]

$$[A][R_{yy}(0)] + [R_{yy}(0)][A]^T = -[B][S_0][B]^T. \quad (2.240)$$

The proof of (2.240) is given in [10]. If the expression

¹⁴ Aleksandr Lyapunov 1857–1918. Sometimes his name is also written as Ljapunov, Liapunov or Ljapunow.

$$[X] = - \int_0^{\infty} e^{[A]t} [C] e^{[B]t} dt, \quad (2.241)$$

exists for all $[C]$, it represents the unique solution of

$$[A][X] + [X][B] = [C]. \quad (2.242)$$

The solution of Lyapunov's equation (2.242) is discussed in [9]. If $[B] = [A]^T$ and the matrix $[C]$ is positive semi definite the matrix $[X]$ is positive-definite when the eigenvalues of matrix $[A]$ are in the negative plane [66].

Example. A SDOF system defined by the following equation of motion is

$$\ddot{x} + 5\dot{x} + 2500x = f(t).$$

The white noise forcing function $f(t)$ had a double sided PSD $S_{ff} = 1 \text{ N}^2/\text{rad/s}$ ($W_{ff} = 2 \text{ N}^2/\text{Hz}$). Calculate the correlation function $R_{xx}(0)$, $R_{x\dot{x}}(0)$, $R_{\dot{x}\dot{x}}(0)$ and $R_{\dot{x}x}(0)$.

The state matrix $[A]$ and the input vector $\{B\}$ are

$$[A] = \begin{pmatrix} 0.0 & 1.0 \\ -2500.0 & -5.0 \end{pmatrix}, \quad \{B\} = \begin{pmatrix} 0.0 \\ 1.0 \end{pmatrix},$$

and the eigenvalues Λ and corresponding eigenvectors become

$$[\Lambda] = \begin{pmatrix} -2.5 + 49.9375j & \\ & -2.5 - 49.9375j \end{pmatrix},$$

and

$$[\Psi] = \begin{pmatrix} -0.0010 - 0.0200j & -0.0010 + 0.0200j \\ 0.9998 & 0.9998 \end{pmatrix}.$$

The undamped natural frequency can be calculated from $\omega_0 = \sqrt{\lambda_1 \lambda_2} = 50 \text{ rad/s}$ and the modal damping ratio $\zeta = -\frac{\lambda_1 + \lambda_2}{2\omega_0} = 0.05$.

The PSD matrix $[G] = [\Psi]^{-1} [B] [S_0] [B]^T [\Psi]^{-T}$ is

$$[G] = \begin{pmatrix} 0.2507 & 0.2495 + 0.0250j \\ 0.2495 - 0.0250j & 0.2507 \end{pmatrix}.$$

The correlation function $[R_{vv}(0)]$ is calculated and becomes

$$\begin{aligned} [R_{vv}(0)] &= \begin{pmatrix} \frac{-G(1,1)}{\lambda_1 + \lambda_1} & \frac{-G(1,2)}{\lambda_1 + \lambda_2} \\ \frac{-G(2,1)}{\lambda_2 + \lambda_1} & \frac{-G(2,2)}{\lambda_2 + \lambda_2} \end{pmatrix} \\ &= \begin{pmatrix} 0.0501 & -0.0001 + 0.0025j \\ -0.0001 - 0.0025j & 0.0501 \end{pmatrix}, \end{aligned}$$

and finally $[R_{yy}(0)]$ is obtained

$$R_{yy}(0) = [\Psi][R_{vv}(0)][\Psi]^T = \begin{pmatrix} 0.00004 & 0 \\ 0 & 0.1 \end{pmatrix}.$$

That means that $\sqrt{\langle x^2 \rangle} = \sqrt{R_{xx}(0)(1,1)} = 0.0063$ m and $\sqrt{\langle \dot{x}^2 \rangle} = \sqrt{R_{\dot{x}\dot{x}}(0)(2,2)} = 0.3162$ m/s. The displacement x and the velocity \dot{x} are not correlated. These results can be checked using (2.116) and (2.118).

Solving (2.240) the correlation matrix $[R_{yy}(0)]$ becomes

$$[R_{yy}(0)] = 10^{-10} \begin{pmatrix} 400\,000 & 0.0 \\ 0.0 & 10^9 \end{pmatrix},$$

which had been expected.

Left Eigenvectors

The standard eigenvalue equation $[A][\Psi] = [\Lambda][\Psi]$ involves the so-called “right eigenvectors” or *right modal matrix*. The “left eigenvectors” or *left modal matrix* is obtained from $[A]^T[\Phi] = [\Lambda][\Phi]$, [234]. $[\Lambda]$ is the diagonal matrix of eigenvalues. The right and left eigenvalue problems have the same eigenvalues. The right and left eigenvectors are mutually orthogonal for distinct eigenvalues

$$\begin{cases} \{\phi_i\}^T \{\psi_j\} = 0, & \lambda_i \neq \lambda_j; \\ \{\phi_i\}^T \{\psi_i\} \neq 0, & \lambda_i = \lambda_j. \end{cases} \quad (2.243)$$

The consequence of (2.243) is that the product of left and right modal matrices is a diagonal matrix

$$[\Phi]^T[\Psi] = [\Upsilon]. \quad (2.244)$$

Equation (2.229) can be altered as follows using $[A] = [\Psi][\Lambda][\Psi]^{-1}$

$$\{\dot{v}\} = [\Lambda]\{v\} + [\Upsilon]^{-1}[\Phi]^T[B]\{f(t)\}, \quad (2.245)$$

which can be solved as showed before. There is no need to calculate $[\Psi]^{-1}$. If (2.244) is scaled such that $[\Upsilon] = [I]$ the left modal matrix

$$[\Phi]^T = [\Psi]^{-1}. \quad (2.246)$$

If the left eigenvectors are introduced there is no need to calculate all eigenvalues, e.g. $\lambda_p, p < 2n$, which depends on the frequency range of interest.

2.6.2 Enforced Acceleration

The damped matrix equation of motion of a linear MDOF system excited at the base by an enforced acceleration is

$$[M]\{\ddot{x}\} + [C]\{\dot{z}\} + [K]\{z\} = \{0\}, \quad (2.247)$$

where the relative displacement vector is $\{z\} = \{x\} - \{u\}$, $\{x\}$ is the absolute displacement vector and u is the displacement at the base associated with the enforced acceleration \ddot{u} . If $\{T\}$ is the rigid body vector in the direction of the enforced acceleration \ddot{u} , (2.247) can be rewritten as

$$[M]\{\ddot{z}\} + [C]\{\dot{z}\} + [K]\{z\} = -[M]\{T\}\ddot{u}. \quad (2.248)$$

The absolute acceleration vector $\{\ddot{x}\}$ can be calculated from (2.247).

The state equation of the linear MDOF dynamic system can now be written as

$$\{\dot{y}\} = [A]\{y\} + \{B\}\ddot{u}, \quad (2.249)$$

$$\{\ddot{x}\} = [\bar{C}]\{y\}, \quad (2.250)$$

where the state matrix $[A]$ is equal to (2.223), the input matrix $[B]$ is given by

$$\{B\} = \begin{Bmatrix} 0 \\ T \end{Bmatrix}, \quad (2.251)$$

and the *output matrix* $[\bar{C}]$ is

$$[\bar{C}] = [-M^{-1}K \quad -M^{-1}C], \quad (2.252)$$

and finally the state vector $\{y\}$ is

$$\{y\} = \begin{Bmatrix} z \\ \dot{z} \end{Bmatrix}. \quad (2.253)$$

The solution for $\{\ddot{x}(t)\}$ is derived in [66]

$$\{\ddot{x}(t)\} = [\bar{C}]e^{[A](t-\tau)}\{y(\tau)\} + \int_{\tau}^t [\bar{C}]e^{[A](t-\zeta)}\{B\}\ddot{u}(\zeta)d\zeta. \quad (2.254)$$

The correlation matrix $[R_{vv}(\tau)]$ becomes

$$\begin{aligned} [R_{vv}(\tau)] &= \int_0^{\infty} e^{[A](\zeta)}[\Psi]^{-1}\{B\}S_{\ddot{u}}\{B\}^T[\Psi]^{-T}e^{[A]^T(\tau+\zeta)}d\zeta \\ &= \int_0^{\infty} e^{[A](\zeta)}[G]e^{[A]^T(\tau+\zeta)}d\zeta. \end{aligned} \quad (2.255)$$

The correlation function $R_{yy}(\tau)$ can be calculated as follows;

$$[R_{yy}(\tau)] = [\Psi][R_{vv}(\tau)][\Psi]^T, \quad (2.256)$$

and the correlation function $R_{\ddot{x}\ddot{x}}(\tau)$ becomes

$$[R_{\ddot{x}\ddot{x}}(\tau)] = [\bar{C}][R_{yy}(\tau)][\bar{C}]^T. \quad (2.257)$$

The correlation matrix $[R_{yy}(\tau)]$ can also be written as

$$\begin{aligned}
 [R_{yy}(\tau)] &= \int_0^\infty e^{[A](\zeta)} \{B\} S_{\ddot{u}} \{B\}^T e^{[A]^T(\tau+\zeta)} d\zeta \\
 &= \int_0^\infty e^{[A](\zeta)} [G] e^{[A]^T(\tau+\zeta)} d\zeta,
 \end{aligned}
 \tag{2.258}$$

thus

$$[A][R_{yy}(0)] + [R_{yy}(0)][A]^T = -[G],
 \tag{2.259}$$

and finally

$$[R_{\ddot{x}\ddot{x}}(0)] = [\bar{C}][R_{yy}(0)][\bar{C}]^T.
 \tag{2.260}$$

Example. A 3 mass-spring-damper dynamic system, as shown in Fig. 2.23, is excited at the base with a white noise random enforced acceleration $W_{\ddot{u}} = 2S_{\ddot{u}} = 0.01 \text{ g}^2/\text{Hz}$. The discrete masses are $m_1 = 200 \text{ kg}$, $m_2 = 250 \text{ kg}$ and $m_3 = 300 \text{ kg}$. The spring stiffness are $k_1 = 10^8 \text{ N/m}$, $k_2 = 2 \times 10^8 \text{ N/m}$ and $k_3 = 3 \times 10^8 \text{ N/m}$. The discrete dampers are defined by $c_1 = 2\zeta\sqrt{k_1m_1}$, $c_2 = 2\zeta\sqrt{k_2m_2}$ and $c_3 = 2\zeta\sqrt{k_3m_3}$ and the damping ratio is $\zeta = 0.05$. Calculate the rms accelerations $\{\ddot{x}\}$ and the rms internal loads.

The damped equations of motion are

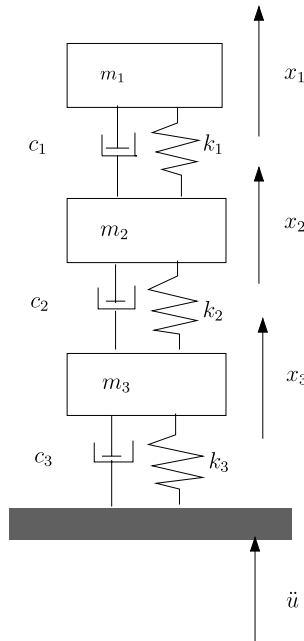


Fig. 2.23. 3 mass-spring-damper system, enforced acceleration

$$\begin{aligned} & \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{Bmatrix} + \begin{bmatrix} c_1 & -c_1 & 0 \\ -c_1 & c_1 + c_2 & -c_2 \\ 0 & -c_2 & c_2 + c_3 \end{bmatrix} \begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{Bmatrix} \\ & + \begin{bmatrix} k_1 & -k_1 & 0 \\ -k_1 & k_1 + k_2 & -k_2 \\ 0 & -k_2 & k_2 + k_3 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}. \end{aligned}$$

The previous equation can be expressed in the relative motion $\{z(t)\} = \{x(t)\} - \{T\}u(t)$, thus

$$\begin{aligned} & \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix} \begin{Bmatrix} \ddot{z}_1 \\ \ddot{z}_2 \\ \ddot{z}_3 \end{Bmatrix} + \begin{bmatrix} c_1 & -c_1 & 0 \\ -c_1 & c_1 + c_2 & -c_2 \\ 0 & -c_2 & c_2 + c_3 \end{bmatrix} \begin{Bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \end{Bmatrix} \\ & + \begin{bmatrix} k_1 & -k_1 & 0 \\ -k_1 & k_1 + k_2 & -k_2 \\ 0 & -k_2 & k_2 + k_3 \end{bmatrix} \begin{Bmatrix} z_1 \\ z_2 \\ z_3 \end{Bmatrix} = - \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix} \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix} \ddot{u}(t), \end{aligned}$$

where

$$\{T\} = \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}.$$

The damping ratios and undamped natural frequencies can be extracted from the eigenvalues $[A]$ of the state matrix $[A]$:

$$\begin{pmatrix} \zeta_1 \\ \zeta_2 \\ \zeta_3 \end{pmatrix} = \begin{pmatrix} 0.025 \\ 0.061 \\ 0.084 \end{pmatrix},$$

and the undamped natural frequencies

$$\begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix} = \begin{pmatrix} 69.58 \\ 153.80 \\ 238.29 \end{pmatrix} \text{ Hz.}$$

The auto correlation matrix $[R_{yy}(0)]$ can be calculated using

$$[A][R_{yy}(0)] + [R_{yy}(0)][A]^T = -[G],$$

and the positive-definite zero-lag correlation function $[R_{\ddot{x}\ddot{x}}(0)]$ of the absolute accelerations is

$$[R_{\ddot{x}\ddot{x}}(0)] = [\bar{C}][R_{yy}(0)][\bar{C}]^T$$

so that

$$[R_{\ddot{x}\ddot{x}}(0)] = \begin{bmatrix} 4334.4 & 2131.6 & 846.6 \\ 2131.6 & 1896.6 & 823.06 \\ 846.6 & 823.06 & 753.76 \end{bmatrix}.$$

The rms accelerations (g) are the square root of the diagonal terms in $[R_{\ddot{x}\ddot{x}}(0)]$, divided by 9.81 m/s^2

$$rms \begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{pmatrix} = \begin{pmatrix} 6.7111 \\ 4.4393 \\ 2.7986 \end{pmatrix} \text{ (g)}.$$

The internal force matrix $D_{\bar{F}}$ is set up as follows

$$[D_{\bar{F}}] = \begin{bmatrix} k_1 & -k_1 & 0 \\ 0 & k_2 & -k_2 \\ 0 & 0 & k_3 \end{bmatrix}.$$

If the output matrix is given by $[\bar{C}] = [D_{\bar{F}}, 0]$, then the positive-definite auto correlation function $[R_{\bar{F}\bar{F}}(0)]$ is defined by

$$[R_{\bar{F}\bar{F}}(0)] = [\bar{C}][R_{yy}(0)][\bar{C}]^T$$

and becomes

$$[R_{\bar{F}\bar{F}}(0)] = 10^8 \begin{bmatrix} 1.72 & 2.79 & 3.30 \\ 2.79 & 5.04 & 6.16 \\ 3.30 & 6.16 & 7.96 \end{bmatrix}.$$

The rms values of the internal forces on the main diagonal are

$$\sqrt{R_{\{\bar{F}_i \bar{F}_i\}}(0)} = \{\bar{F}_{ii,rms}\} = 10^4 \begin{Bmatrix} 1.31 \\ 2.24 \\ 2.82 \end{Bmatrix} \text{ N}.$$

Modal Reduction

If the modal reduction method is applied the relative displacement vector $\{z\}$ (n DOFs) is expressed as follows $\{z\} = [\Phi]\{\eta\}$, where $[\Phi]$ ($n \times m$) is the modal base of the undamped unloaded system in (2.248) and $\{\eta\}$ (m DOFs) the generalized, principal coordinates (also called participation factors). In general, the number of reduced DOFs is much less than the number of original DOFs, $m \ll n$.

When the damping is introduced by the ad-hoc damping coefficient $2\zeta_i\omega_i m_i$ (2.248) can be written as

$$[m_i]\{\ddot{\eta}_i\} + [2\zeta_i\omega_i m_i]\{\dot{\eta}_i\} + [\omega_i^2 m_i]\{\eta_i\} = -[\Phi]^T[M]\{T\}\ddot{u}, \quad (2.261)$$

where the generalized mass $m_i = \{\phi_i\}^T[M]\{\phi_i\}$ and ω_i is the natural frequency associated with mode $\{\phi_i\}$.

The state-space equation of the (modal) reduced linear MDOF dynamic system can now be written as

$$\{\dot{v}\} = [A]\{v\} + \{B\}\ddot{u}, \quad (2.262)$$

$$\{y\} = [\tilde{C}]\{v\}, \quad (2.263)$$

$$\{\ddot{x}\} = [\bar{C}]\{y\}, \quad (2.264)$$

where the state matrix $[A]$ ($2m \times 2m$) is given by

$$[A] = \begin{pmatrix} 0 & I \\ -[\omega_i^2] & -[2\zeta_i\omega_i] \end{pmatrix}, \quad (2.265)$$

and the input matrix $[B]$ is given by

$$\{B\} = \left\{ \begin{array}{c} 0 \\ -[m_i]^{-1}[\Phi]^T[M]\{T\} \end{array} \right\}, \quad (2.266)$$

and the *output matrix* $[\tilde{C}]$ ($2n \times 2m$) is

$$[\tilde{C}] = \begin{bmatrix} \Phi & 0 \\ 0 & \Phi \end{bmatrix}, \quad (2.267)$$

and finally the state vector $\{y\}$ is composed by

$$\{v\} = \left\{ \begin{array}{c} \eta \\ \dot{\eta} \end{array} \right\}, \quad (2.268)$$

and

$$\{y\} = \left\{ \begin{array}{c} z \\ \dot{z} \end{array} \right\}. \quad (2.269)$$

The accelerations $\{\ddot{x}\}$ can be obtained using the output matrix $[\bar{C}]$, derived from (2.247)

$$[\bar{C}] = [-M^{-1}K, -M^{-1}C], \quad (2.270)$$

an ($n \times 2n$) matrix. The damping matrix $[C]$ may be reconstructed as follows [51]

$$[C] = [M] \left[\sum_{i=1}^m 2\zeta_i\omega_i m_i \{\phi_i\} \{\phi_i\}^T \right] [M], \quad (2.271)$$

where $[M]$ is the mass matrix.

The internal stress of forces (generalized forces) $\{\bar{F}\}$ can be directly related to the generalized coordinate $\{\eta\}$. The general forces can be expressed as follows

$$\{\bar{F}\} = [D_F]\{z\} = [D_F][\Phi]\{\eta\} = [\Phi]_\sigma\{\eta\} = [\Phi_\sigma, 0]\{v\}, \quad (2.272)$$

where $[\Phi]_\sigma$ are the so-called *stress modes* or is the stress modal matrix. Equation (2.262) can be extended with

$$\{\bar{F}\} = [\Phi_\sigma, 0]\{v\} = [\hat{C}]\{v\}. \quad (2.273)$$

The auto correlation function of the generalized forces $\{\bar{F}\}$ becomes

$$R_{\bar{F}\bar{F}}(0) = [\hat{C}][R_{vv}(0)][\hat{C}]^T. \quad (2.274)$$

Example. A 3 mass-spring dynamic system, as shown in Fig. 2.19, is excited at the base with a white noise random enforced acceleration $W_{\ddot{u}} = 2S_{\ddot{u}} = 0.01 \text{ g}^2/\text{Hz}$. The discrete masses are $m_1 = 200 \text{ kg}$, $m_2 = 250 \text{ kg}$ and $m_3 = 300 \text{ kg}$. The spring stiffnesses are $k_1 = 10^8 \text{ N/m}$, $k_2 = 2 \times 10^8 \text{ N/m}$ and $k_3 = 3 \times 10^8 \text{ N/m}$. The modal damping ratio for all modes is $\zeta = 0.05$. Calculate the accelerations $\{\ddot{x}\}$ and the internal loads.

The undamped equations of motion are

$$\begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{Bmatrix} + \begin{bmatrix} k_1 & -k_1 & 0 \\ -k_1 & k_1 + k_2 & -k_2 \\ 0 & -k_2 & k_2 + k_3 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix},$$

or previous equation can be expressed in the relative motion $\{z(t)\} = \{x(t)\} - \{T\}u(t)$, thus

$$\begin{aligned} &\begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix} \begin{Bmatrix} \ddot{z}_1 \\ \ddot{z}_2 \\ \ddot{z}_3 \end{Bmatrix} + \begin{bmatrix} k_1 & -k_1 & 0 \\ -k_1 & k_1 + k_2 & -k_2 \\ 0 & -k_2 & k_2 + k_3 \end{bmatrix} \begin{Bmatrix} z_1 \\ z_2 \\ z_3 \end{Bmatrix} \\ &= - \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix} \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix} \ddot{u}(t), \end{aligned}$$

where

$$\{T\} = \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}.$$

The natural frequencies and associated mode shapes of the MDOF system are

$$\{f_n\} = \begin{pmatrix} 69.5745 \\ 153.7834 \\ 238.3038 \end{pmatrix} \text{ Hz}, \quad [\Phi] = \begin{pmatrix} 0.8277 & -0.6490 & 0.1843 \\ 0.5114 & 0.5628 & -0.6421 \\ 0.2310 & 0.5119 & 0.7441 \end{pmatrix}.$$

The responses will be calculated considering the first mode. The rms accelerations and rms spring forces are respectively

$$\begin{aligned} &\text{rms} \begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{pmatrix} = \begin{pmatrix} 4.5440 \\ 2.8073 \\ 1.2683 \end{pmatrix} \text{ g}, \\ &\sqrt{R\{\bar{F}_i \bar{F}_i\}}(0) = \{\bar{F}_{ii,rms}\} = 10^4 \begin{Bmatrix} 0.8915 \\ 1.5800 \\ 1.9533 \end{Bmatrix} \text{ N}. \end{aligned}$$

The responses will be now calculated considering the first and second mode. The rms accelerations and rms spring forces are respectively

$$rms \begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{pmatrix} = \begin{pmatrix} 5.0075 \\ 3.4000 \\ 2.1428 \end{pmatrix} \text{ g,}$$

$$\sqrt{R\{\bar{F}_i \bar{F}_i\}}(0) = \{\bar{F}_{ii,rms}\} = 10^4 \left\{ \begin{array}{l} 0.9825 \\ 1.5809 \\ 2.0334 \end{array} \right\} \text{ N.}$$

The responses will be finally calculated considering all modes. The rms accelerations and rms spring forces are respectively

$$rms \begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{pmatrix} = \begin{pmatrix} 5.0174 \\ 3.6437 \\ 2.7527 \end{pmatrix} \text{ g,}$$

$$\sqrt{R\{\bar{F}_i \bar{F}_i\}}(0) = \{\bar{F}_{ii,rms}\} = 10^4 \left\{ \begin{array}{l} 0.9844 \\ 1.6019 \\ 2.0485 \end{array} \right\} \text{ N.}$$

The state matrix $[A]$ (2.265) of the reduced system with a modal damping ratio $\zeta_i < 1$, and the eigenvalues ω_i^2 of the undamped system can be written as

$$[A] = \begin{pmatrix} 0 & I \\ -[\omega_i^2] & -[2\zeta_i \omega_i] \end{pmatrix}.$$

The left eigenvalue problem of the state matrix is

$$[A][\Psi] = [\Psi][\Lambda], \quad (2.275)$$

where the matrix of the left eigenvectors $[\Psi]$ is

$$[\Psi] = \begin{pmatrix} I & I \\ \Lambda_c & \bar{\Lambda}_c \end{pmatrix}, \quad (2.276)$$

where $\Lambda_c = \Omega[-\zeta + j\sqrt{1-\zeta^2}]$, $\bar{\Lambda}_c = \Omega[-\zeta - j\sqrt{1-\zeta^2}]$ and $[\Omega] = [\omega_i^2]$. The diagonal matrix of the state matrix can be written

$$[\Lambda] = \begin{pmatrix} \Lambda_c & 0 \\ 0 & \bar{\Lambda}_c \end{pmatrix}. \quad (2.277)$$

The inverse of the left eigenvectors $[\Psi]^{-1}$ is computed as follows

$$\begin{aligned} [\Psi]^{-1} &= \begin{pmatrix} I + [\bar{\Lambda}_c - \Lambda_c]^{-1} \Lambda_c & -[\bar{\Lambda}_c - \Lambda_c]^{-1} \\ -[\bar{\Lambda}_c - \Lambda_c]^{-1} \Lambda_c & [\bar{\Lambda}_c - \Lambda_c]^{-1} \end{pmatrix} \\ &= \begin{pmatrix} [\bar{\Lambda}_c - \Lambda_c]^{-1} \Lambda_c & -[\bar{\Lambda}_c - \Lambda_c]^{-1} \\ -[\bar{\Lambda}_c - \Lambda_c]^{-1} \Lambda_c & [\bar{\Lambda}_c - \Lambda_c]^{-1} \end{pmatrix}. \end{aligned} \quad (2.278)$$

By using (2.275) we show that the spectral decomposition of the state matrix $[A]$ is [182]

$$[A] = [\Psi][\Lambda][\Psi]^{-1} = \begin{pmatrix} 0 & I \\ -\bar{\Lambda}_c \Lambda_c & \Lambda_c + \bar{\Lambda}_c \end{pmatrix}. \quad (2.279)$$

2.6.3 Derivation of Miles' Equation

Miles's equation was derived in Sect. 2.4.2, and expressed in (2.145) and (2.146). In this section Miles' equation [127] will be derived with the aid of the direct solution of Lyapunov's equation (F.9), $[A][M] + [M][A]^T = -[C]$, discussed in detail in Appendix F, where the state matrix is denoted by $[A]$, the covariance or correlation matrix by $[M]$; the RHS of the equation, the matrix $[C] = \{B\}S_u\{B\}^T = [G]$, contains information about the white noise spectrum of the enforced acceleration (2.258).

Both the correlation matrix $[M]$ and the matrix $[C]$ are symmetric. This can be easily proven by taking the transpose of (2.258) and assuming the time shift is $\tau = 0$. In general, for an n DOF system, there are $n(2n + 1)$ unknowns components of the M matrix. In fact, considering the steady-state or stationary conditions, the elements

$$m_{i,n+i} = E\{y_i\dot{y}_i\} = E\{\dot{y}_iy_i\} = m_{n+i,i} = 0, \quad i = 1, 2, \dots, n, \quad (2.280)$$

and

$$m_{i,n+j} = E\{y_i\dot{y}_j\} = -E\{\dot{y}_iy_j\} = -m_{n+i,j}, \quad i, j = 1, 2, \dots, n, \quad (2.281)$$

and the number of unknowns elements m_{ij} is reduced from $n(2n + 1)$ to $n(2n + 1) - n = 2n^2$.

To generate the state-space equations, we use (2.119), (2.120), (2.121) and (2.122) are used. The variables are illustrated in Fig. 2.24. The state-space equations are

$$\begin{pmatrix} \dot{z} \\ \ddot{z} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\omega_n^2 & -2\zeta\omega_n \end{pmatrix} \begin{pmatrix} z \\ \dot{z} \end{pmatrix} + \begin{pmatrix} 0 \\ -1 \end{pmatrix} \ddot{u}, \quad (2.282)$$

where $k = \omega_n^2 m$, $c = 2\zeta\sqrt{km}$ and with (2.122) the state equations become

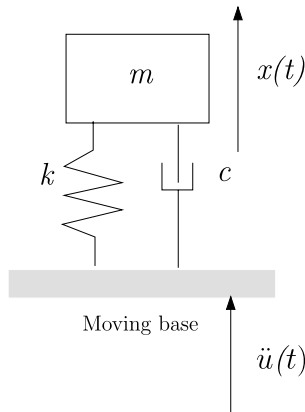


Fig. 2.24. Enforced acceleration of a damped SDOF system

$$\begin{aligned} \{\dot{y}\} &= [A]\{y\} + \{B\}\ddot{u}, \\ \ddot{x} &= [-\omega_n^2, -2\zeta\omega_n] \begin{pmatrix} z \\ \dot{z} \end{pmatrix} = [C]\{y\}. \end{aligned} \quad (2.283)$$

The covariance or correlation matrix $[M]$ is given by (2.280) and (2.281)

$$[M] = \begin{pmatrix} E\{z^2\} & E\{z\dot{z}\} \\ E\{\dot{z}z\} & E\{\dot{z}^2\} \end{pmatrix} = \begin{pmatrix} E\{z^2\} & 0 \\ 0 & E\{\dot{z}^2\} \end{pmatrix}, \quad (2.284)$$

and Lyapunov's equation is

$$[A][M] + [M][A]^T = -\{B\}S_{\ddot{u}}\{B\}^T = -[G] = -\begin{pmatrix} 0 & 0 \\ 0 & S_{\ddot{u}} \end{pmatrix}. \quad (2.285)$$

The direct solution of (2.285) can be written as follows

$$\begin{pmatrix} 1 & -\omega_n^2 \\ 0 & 4\zeta\omega_n \end{pmatrix} \begin{pmatrix} E\{z^2\} \\ E\{\dot{z}^2\} \end{pmatrix} = \begin{pmatrix} 0 \\ S_{\ddot{u}} \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{W_{\ddot{u}}}{2} \end{pmatrix}, \quad (2.286)$$

where $W_{\ddot{u}}$ is the single sided PSD of the white noise enforced acceleration. The solution of (2.286) is

$$E\{z^2\} = R_{zz}(0) = \frac{W_{\ddot{u}}}{8\zeta\omega_n^3} = \frac{W_{\ddot{u}}}{8\zeta(2\pi f_n)^3}, \quad (2.287)$$

and

$$E\{\dot{z}^2\} = R_{\dot{z}\dot{z}}(0) = \frac{W_{\ddot{u}}}{8\zeta\omega_n} = \frac{W_{\ddot{u}}}{8\zeta(2\pi f_n)}. \quad (2.288)$$

The expected value of the acceleration $E\{\ddot{x}^2\}$ can be solved as follows

$$E\{\ddot{x}^2\} = [C][M][C]^T = \omega_n^4 E\{z^2\} + (2\zeta\omega_n)^2 E\{\dot{z}^2\}. \quad (2.289)$$

Finally $E\{\ddot{x}^2\}$ becomes

$$E\{\ddot{x}^2\} = R_{\ddot{x}\ddot{x}}(0) = \frac{\pi}{2}f_n Q W_{\ddot{u}} (1 + 4\zeta^2) \approx \frac{\pi}{2}f_n Q W_{\ddot{u}}, \quad (2.290)$$

where the amplification factor $Q = \frac{1}{2\zeta}$ and $\zeta \ll 1$. Equation (2.290) is equal to (2.145) and (2.146).

2.6.4 Power Transfer Between Two Oscillators

Consider two SDOF systems coupled by a single linear spring k_c . Both systems are loaded by two uncorrelated white noise random forces F_1 and F_2 . The double sided PSD values of the forces are $S_{F_1} = \frac{W_{F_1}}{2}$ and $S_{F_2} = \frac{W_{F_2}}{2}$ (W_{F_1} and W_{F_2} are the single sided PSD functions respectively). We seek the averaged total energies¹⁵ stored in both SDOF systems $\langle E_1 \rangle = m_1 E\{\dot{x}_1^2\}$ and $\langle E_2 \rangle =$

¹⁵ $\langle E \rangle = \frac{1}{2}(mE\{\dot{x}^2\} + kE\{x^2\}) = \frac{1}{2}(mE\{\dot{x}^2\} + m\omega_0^2 E\{x^2\}) = mE\{\dot{x}^2\}$.

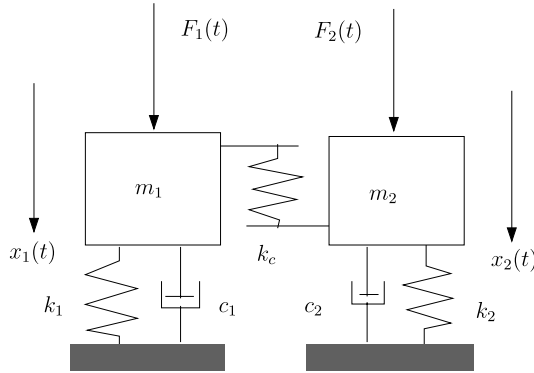


Fig. 2.25. Two coupled oscillators

$m_2 E\{\dot{x}_2^2\}$ and the power transfer between both coupled oscillators Π_{12} . These energies and the power transfer will be calculated using Lyapunov’s equation. The equations of motion of the two SDOF systems, as shown in Fig. 2.25, are

$$m_1 \ddot{x}_1 + c_1 \dot{x}_1 + k_1 x_1 + k_c(x_1 - x_2) = F_1, \tag{2.291}$$

$$m_2 \ddot{x}_2 + c_2 \dot{x}_2 + k_2 x_2 + k_c(x_2 - x_1) = F_2. \tag{2.292}$$

The following parameters are introduced [118]:

- $\omega_1 = \sqrt{\frac{k_1+k_c}{m_1}}, \omega_2 = \sqrt{\frac{k_2+k_c}{m_2}},$
- $\omega_{12}^2 = \frac{k_c}{\sqrt{m_1 m_2}},$
- $\frac{\sqrt{c_1}}{m_1} = \Delta_1, \frac{\sqrt{c_2}}{m_2} = \Delta_2,$
- $\gamma = \sqrt{\frac{m_1}{m_2}}.$

Equations (2.291) and (2.292) can now be written

$$\ddot{x}_1 + \Delta_1 \dot{x}_1 + \omega_1^2 x_1 - \frac{\omega_{12}^2}{\gamma} x_2 = \frac{F_1}{m_1}, \tag{2.293}$$

$$\ddot{x}_2 + \Delta_2 \dot{x}_2 + \omega_2^2 x_2 - \omega_{12}^2 \gamma x_1 = \frac{F_2}{m_2}. \tag{2.294}$$

The power Π_{12} going from system 1 to system 2 is given by [118]

$$\Pi_{12} = -k_c E\{x_2 \dot{x}_1\} = -\omega_{12}^2 \sqrt{m_1 m_2} E\{x_2 \dot{x}_1\}, \tag{2.295}$$

the averaged energy of the uncoupled system 1 is

$$\langle \bar{E}_1 \rangle = m_1 E\{\dot{x}_1^2\} = \frac{W_{F_1}}{4\Delta_1 m_1}, \tag{2.296}$$

and the averaged energy of the uncoupled system 2 is

$$\langle \bar{E}_2 \rangle = m_2 E\{\dot{x}_2^2\} = \frac{W_{F_2}}{4\Delta_2 m_2}. \quad (2.297)$$

Equations (2.293) and (2.294) are converted into a state-space equation

$$\{\dot{y}\} = [A]\{y\} + [B]\{F\},$$

where the state-space is

$$\{y\} = \begin{pmatrix} x_1 \\ x_2 \\ \dot{x}_1 \\ \dot{x}_2 \end{pmatrix}, \quad (2.298)$$

the state matrix $[A]$ is given by

$$[A] = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\omega_1^2 & \frac{\omega_{12}^2}{\gamma} & -\Delta_1 & 0 \\ \omega_{12}^2 \gamma & -\omega_2^2 & 0 & -\Delta_2 \end{pmatrix}, \quad (2.299)$$

the input matrix $[B]$ is

$$[B] = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ \frac{1}{m_1} & 0 \\ 0 & \frac{1}{m_2} \end{pmatrix}, \quad (2.300)$$

and finally the force vector $\{F\}$ can be expressed as

$$\{F\} = \begin{pmatrix} F_1 \\ F_2 \end{pmatrix}. \quad (2.301)$$

The white noise PSD matrix of the applied uncorrelated forces $[S_F]$ becomes

$$[S_F] = \begin{pmatrix} S_{F_1} & 0 \\ 0 & S_{F_2} \end{pmatrix} = \begin{pmatrix} \frac{W_{F_1}}{2} & 0 \\ 0 & \frac{W_{F_2}}{2} \end{pmatrix}. \quad (2.302)$$

The RHS of Lyapunov's equation $[C] = [G]$ (2.235) can be obtained by the following matrix multiplication

$$[C] = [B][S_F][B]^T = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{S_{F_1}}{m_1^2} & 0 \\ 0 & 0 & 0 & \frac{S_{F_2}}{m_2^2} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{W_{F_1}}{2m_1^2} & 0 \\ 0 & 0 & 0 & \frac{W_{F_2}}{2m_2^2} \end{pmatrix}. \quad (2.303)$$

The covariance or correlation matrix $[M]$ is build up and reduced as follows using (2.280) and (2.281)

$$\begin{aligned}
[M] &= \begin{pmatrix} E\{x_1^2\} & E\{x_1x_2\} & E\{x_1\dot{x}_1\} & E\{x_1\dot{x}_2\} \\ E\{x_2x_1\} & E\{x_2^2\} & E\{x_2\dot{x}_1\} & E\{x_2\dot{x}_2\} \\ E\{\dot{x}_1x_1\} & E\{\dot{x}_1x_2\} & E\{\dot{x}_1^2\} & E\{\dot{x}_1\dot{x}_2\} \\ E\{\dot{x}_2x_1\} & E\{\dot{x}_2x_2\} & E\{\dot{x}_2\dot{x}_1\} & E\{\dot{x}_2^2\} \end{pmatrix} \\
&= \begin{pmatrix} E\{x_1^2\} & E\{x_1x_2\} & 0 & E\{x_1\dot{x}_2\} \\ E\{x_1x_2\} & E\{x_2^2\} & -E\{x_1\dot{x}_2\} & 0 \\ 0 & -E\{x_1\dot{x}_2\} & E\{\dot{x}_1^2\} & E\{\dot{x}_1\dot{x}_2\} \\ E\{x_1\dot{x}_2\} & 0 & E\{\dot{x}_1\dot{x}_2\} & E\{\dot{x}_2^2\} \end{pmatrix}. \quad (2.304)
\end{aligned}$$

There are 8 unknowns in (2.304), expressed in the elements $m(i, j)$ of the covariance or correlation matrix $[M]$, $m(1, 1) = E\{x_1^2\}$, $m(1, 2) = E\{x_1x_2\}$, $m(1, 4) = E\{x_1\dot{x}_2\}$, $m(2, 2) = E\{x_2^2\}$, $m(3, 3) = E\{\dot{x}_1^2\}$, $m(3, 4) = E\{\dot{x}_1\dot{x}_2\}$ and $m(4, 4) = E\{\dot{x}_2^2\}$.

The state matrix $[A]$ (2.299) will be now written in a very general form

$$[A] = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ a_1 & b_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \end{pmatrix}. \quad (2.305)$$

Lyapunov equation

$$[A][M] + [M][A]^T = -[C]$$

is transformed into 7 equations with unknowns $m(1, 1)$, $m(1, 2)$, $m(1, 4)$, $m(2, 2)$, $m(3, 3)$, $m(3, 4)$ and $m(4, 4)$ as follows

$$\begin{pmatrix} a_1 & a_2 & a_4 & 0 & 1 & 0 & 0 \\ b_1 & b_2 & b_4 & 0 & 0 & 1 & 0 \\ 0 & a_1 & -a_3 & a_2 & 0 & 1 & 0 \\ 0 & b_1 & -b_3 & b_2 & 0 & 0 & 1 \\ 0 & 0 & -2a_2 & 0 & 2a_3 & 2a_4 & 0 \\ 0 & 0 & a_1 - b_2 & 0 & b_3 & a_3 + b_4 & a_4 \\ 0 & 0 & 2b_1 & 0 & 0 & 2b_3 & 2b_4 \end{pmatrix} \begin{pmatrix} m(1, 1) \\ m(1, 2) \\ m(1, 4) \\ m(2, 2) \\ m(3, 3) \\ m(3, 4) \\ m(4, 4) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -c(3, 3) \\ 0 \\ -c(4, 4) \end{pmatrix}, \quad (2.306)$$

and $E\{x_2\dot{x}_1\} = m(2, 3) = -m(1, 4)$. The independent values of $m_{i,j}$ (2.306) can now be solved. The total energy of coupled SDOF system 1 is

$$\begin{aligned}
\langle E_1 \rangle &= \frac{W_{F_1}}{4\Delta_1 m_1 D} \left[(\omega_1^2 - \omega_2^2)^2 + (\Delta_1 + \Delta_2)(\omega_1^2 \Delta_2^2 + \omega_2^2 \Delta_1^2) + \omega_{12}^4 \left(1 + \frac{\Delta_1}{\Delta_2} \right) \right] \\
&\quad + \frac{W_{F_2}}{4\Delta_2 m_2 D} \left[\omega_{12}^4 \left(1 + \frac{\Delta_2}{\Delta_1} \right) \right], \quad (2.307)
\end{aligned}$$

and the total energy of the coupled SDOF system 2 is

$$\begin{aligned}
\langle E_2 \rangle &= \frac{W_{F_2}}{4\Delta_1 m_2 D} \left[(\omega_1^2 - \omega_2^2)^2 + (\Delta_1 + \Delta_2)(\omega_1^2 \Delta_2^2 + \omega_2^2 \Delta_1^2) + \omega_{12}^4 \left(1 + \frac{\Delta_2}{\Delta_1} \right) \right] \\
&\quad + \frac{W_{F_1}}{4\Delta_2 m_1 D} \left[\omega_{12}^4 \left(1 + \frac{\Delta_1}{\Delta_2} \right) \right], \quad (2.308)
\end{aligned}$$

where the denominator D is given by

$$D = (\omega_1^2 - \omega_2^2)^2 + (\Delta_1 + \Delta_2)(\omega_1^2\Delta_2^2 + \omega_2^2\Delta_1^2) + \omega_{12}^4 \left(2 + \frac{\Delta_1}{\Delta_2} + \frac{\Delta_2}{\Delta_1} \right). \quad (2.309)$$

The power transfer between SDOF 1 and SDOF 2, coupled by the spring k_c , becomes

$$\begin{aligned} \Pi_{12} &= -k_c E\{x_2\dot{x}_1\} = -\omega_{12}^2 \sqrt{m_1 m_2} E\{x_2\dot{x}_1\} \\ &= \frac{\omega_{12}^4 (\Delta_1 + \Delta_2)}{D} \left[\frac{W_{F_1}}{4\Delta_1 m_1} - \frac{W_{F_2}}{4\Delta_2 m_2} \right] \\ &= \beta_{12} [\langle \bar{E}_1 \rangle - \langle \bar{E}_2 \rangle]. \end{aligned} \quad (2.310)$$

The following remarks can be made with respect (2.310):

- The power flow Π_{12} is proportional to the actual uncoupled vibration energies of the system, SDOF 1 and SDOF 2, and the constant of proportionality is β_{12} .
- The *constant of proportionality* β_{12} is positive and symmetric in the system parameters.

2.6.5 Augmented State-Space Equation under Non-White Excitation

White noise can be used to predict the response of a dynamic system in many practical solutions. However, when the PSD does not vary substantially over the frequency range covered by the lowest natural frequencies, a formal extension to the specified band-limited PSD spectrum case is needed. Any PSD spectrum can be obtained as the output of some linear dynamic system exposed to a white noise input. The actual PSD spectrum input can be approximated as accurately as desired. The cost depend on the complexity (i.e. the number of states) of the filter model called the *auxiliary system*.

The spectral density function $S(\omega)$ is approximated by *meromorphic functions*¹⁶ [187]

$$S(\omega) = \frac{b_0 + b_1\omega^2 + \dots + a_m\omega^{2m}}{a_0 + a_1\omega^2 + \dots + a_n\omega^{2n}} = h(j\omega)h(-j\omega) = |h(\omega)|^2, \quad (2.311)$$

where the order m of the numerator does not exceed the order n of the denominator. This follows from the fact that

$$\lim_{\omega \rightarrow \infty} S(\omega) < \infty.$$

All coefficients $b_i, i = 0 \dots m$ and $a_j, j = 0 \dots n$ of the PSD function are real because the system parameters are real. Complex zeros and poles of the transfer function $h(s) = h(j\omega)$ must occur in conjugate pairs.

¹⁶ Rational function with non constant denominator.

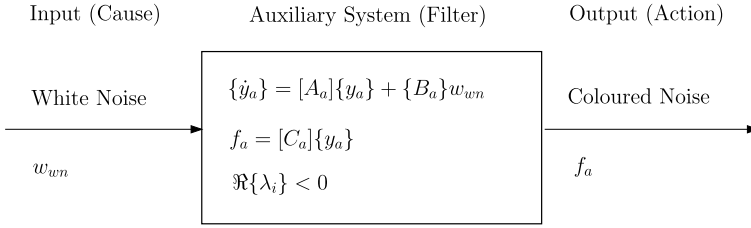


Fig. 2.26. Auxiliary system (shape filter)

We assumed that the properties of the auxiliary system (subscript a) can be found and described by the following state-space equations

$$\{\dot{y}_a\} = [A_a]\{y_a\} + \{B_a\}w_{wn}, \tag{2.312}$$

where w_{wn} is the white noise input of the force and the output f_a of the auxiliary system is the filtered PSD spectrum is given by

$$f_a = [C_a]\{y_a\}. \tag{2.313}$$

The auxiliary system is illustrated in Fig. 2.26. The real parts of the complex eigenvalues λ_i of the state matrix $[A_a]$ will be negative, thus $\Re\{\lambda_i\} < 0$.

The frequency response function $h_a(s)$ of the auxiliary system (Laplace transform) is

$$h(s) = \frac{f_a(s)}{w_{wn}(s)} = [C_a](s[I] - [A_a])^{-1}\{B_a\} = \frac{p(s)}{q(s)}, \tag{2.314}$$

where $p(s)$ and $q(s)$ are real polynomials of the Laplace variable s . If the PSD function of the white noise signal w_{wn} is $S_w = 1$, then the output PSD function of f_a can be obtained as

$$S_{f_a} = h(j\omega)h(-j\omega) = |h(j\omega)|^2. \tag{2.315}$$

When the output f_a of the auxiliary system is injected as the input of the main system

$$\{\dot{y}\} = [A]\{y\} + \{B\}u = [A]\{y\} + \{B\}f_a, \quad \{x\} = [C]\{y\},$$

the following augmented state-space equation will be obtained

$$\{\dot{y}_t\} = [A_t]\{y_t\} + \{B_t\}w_{wn}, \tag{2.316}$$

and

$$\{y\} = [C_t]\{y_t\}, \tag{2.317}$$

where the total number of states is

$$\{y_t\} = \begin{pmatrix} y \\ y_a \end{pmatrix}, \quad (2.318)$$

then the state matrix $[A_t]$ is as follows

$$[A_t] = \begin{pmatrix} A & BC_a \\ 0 & A_a \end{pmatrix}. \quad (2.319)$$

The input matrix $\{B_t\}$ is

$$\{B_t\} = \begin{pmatrix} 0 \\ B_a \end{pmatrix}, \quad (2.320)$$

and the output matrix is

$$[C_t] = [C \quad 0]. \quad (2.321)$$

In [137], random seismic excitation is modelled as a non-white Gaussian process, and the excitation PSD is represented by the *Kanai-Tajimi spectrum*

$$\Phi(\omega) = \frac{1 + 4\zeta_a^2(\frac{\omega}{\omega_a})^2}{[1 - (\frac{\omega}{\omega_a})^2]^2 + 4\zeta_a^2(\frac{\omega}{\omega_a})^2} S_0. \quad (2.322)$$

The Kanai-Tajimi spectrum can be obtained from a second order differential system subjected to white-noise excitation. In this way, the ground acceleration \ddot{x}_a is produced by

$$\ddot{x}_a = -\omega_a^2 z_a - 2\zeta_a \omega_a \dot{z}_a, \quad (2.323)$$

and

$$\ddot{z}_a + 2\zeta_a \omega_a \dot{z}_a + \omega_a^2 z_a = -\ddot{\xi}, \quad (2.324)$$

where $\ddot{\xi}$ is the stationary Gaussian white noise with intensity S_0 . z_a is the response of the filtering system, ω_a and ζ_a are the natural frequency and damping ratio of the filter. Equation (2.322) is illustrated in Fig. 2.27. The state-space representation of the filter is

$$\begin{pmatrix} \dot{z}_a \\ \ddot{z}_a \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -2\zeta_a \omega_a & -\omega_a^2 \end{pmatrix} \begin{pmatrix} z_a \\ \dot{z}_a \end{pmatrix} + \begin{pmatrix} 0 \\ -1 \end{pmatrix} \ddot{\xi}, \quad (2.325)$$

and

$$\ddot{x}_a = \begin{pmatrix} -\omega_a^2 & -2\zeta_a \omega_a \end{pmatrix} \begin{pmatrix} z_a \\ \dot{z}_a \end{pmatrix}. \quad (2.326)$$

Equations (2.325) and (2.326) are similar to (2.312) and (2.313). The Laplace response function $h_a(s)$ (2.314) now becomes

$$h(s) = \frac{\ddot{x}_a(s)}{\ddot{\xi}(s)} = [C_a](s[I] - [A_a])^{-1} \{B_a\} = \frac{\omega_a(\omega_a + 2\zeta_a s)}{s^2 + 2\zeta_a s \omega_a + \omega_a^2}, \quad (2.327)$$

and the frequency response function $h_a(j\omega)$ is

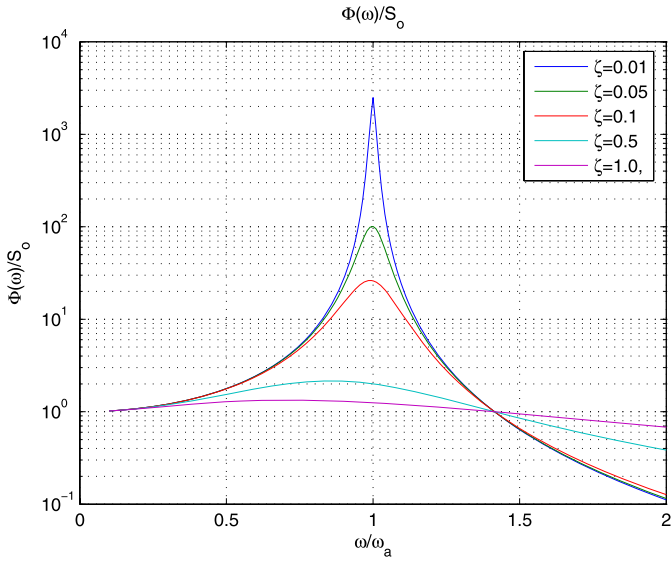


Fig. 2.27. Kanai-Tajimi spectrum

$$h_a(j\omega) = \frac{\omega_a(\omega_a + 2j\zeta_a\omega)}{-\omega^2 + 2j\zeta_a\omega\omega_a + \omega_a^2}. \tag{2.328}$$

That proves that the selected filtering system represents the filter as given in (2.322).

The Laplace transfer function $h(s)$ will be represented in the *canonical form* (simplest) of the state-space.

First Companion Form

If the single input single output transfer function $h(s)$ is given by

$$h(s) = \frac{1}{s^k + a_1s^{k-1} + \dots + a_k}, \tag{2.329}$$

the corresponding state-space representation (companion form) is [66]

$$[A] = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -a_k & -a_{k-1} & -a_{k-2} & \dots & -a_1 \end{pmatrix}, \quad \{B\} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}, \tag{2.330}$$

$$[C] = (1 \ 0 \ 0 \ \dots \ 0),$$

or

$$\begin{aligned}
 [A] &= \begin{pmatrix} -a_1 & -a_2 & \cdots & -a_{k-1} & -a_k \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}, \\
 \{B\} &= \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \\
 [C] &= (0 \ 0 \ 0 \ \cdots \ 1).
 \end{aligned} \tag{2.331}$$

If the single input single output transfer function $h(s)$ is given by

$$h(s) = \frac{b_0 s^m + b_1 s^{m-1} + \cdots + b_m}{s^n + a_1 s^{n-1} + \cdots + a_n}, \quad m \leq n, \tag{2.332}$$

then

$$\lim_{\omega \rightarrow \infty} h(j\omega) < \infty,$$

and the complex zeros and poles of the Laplace transfer function $h(s)$ occur in conjugate pairs.

The output matrix $[C]$ in (2.330) (right-to-left state variable numbering) must be replaced by

$$[C] = [b_k - a_k b_0, b_{k-1} - a_{k-1} b_0, \dots, b_1 - a_1 b_0], \tag{2.333}$$

and the output matrix $[C]$ in (2.331) (left-to-right state variable numbering) must be replaced by

$$[C] = [b_1 - a_1 b_0, b_2 - a_2 b_0, \dots, b_k - a_k b_0]. \tag{2.334}$$

Example. A 2 DOF dynamic system is exposed to white noise enforced acceleration \ddot{u} , with a PSD function $S_{\ddot{u}} = 1$. The Laplace response function between the enforced acceleration and the first DOF acceleration \ddot{x}_1 is calculated. The absolute displacement is $\{x\}$ and the relative displacement vector $\{z\} = \{x\} - \{T\}u$, where the rigid body mode is $\{T\} = [1, 1]^T$. The damping matrix is denoted by $[\bar{C}]$ instead of $[C]$. The output matrix is denoted by $[C]$. The following equations are applicable:

$$[M]\{\ddot{z}\} + [\bar{C}]\{\dot{z}\} + [K]\{z\} = -[M]\{T\}\ddot{u},$$

and

$$\{\ddot{x}\} = -[M]^{-1}[K]\{z\} - [M]^{-1}[\bar{C}]\{\dot{z}\}.$$

The 2 DOF system has the following mass, damping and stiffness characteristics:

$$[M] = \begin{pmatrix} 100 & 0 \\ 0 & 15 \end{pmatrix}, \quad [\bar{C}] = \begin{pmatrix} 75 & -50 \\ -50 & 50 \end{pmatrix},$$

$$[K] = \begin{pmatrix} 76\,915 & -14\,415 \\ -14\,415 & 14\,415 \end{pmatrix}.$$

The state-space matrix $[A]$, the input matrix $\{B\}$ and the output matrix $[C]$ of that system are given by

$$[A] = \begin{pmatrix} O & I \\ -[M]^{-1}[K] & -[M]^{-1}[\bar{C}] \end{pmatrix} \quad \{B\} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix},$$

$$[C] = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} (-[M]^{-1}[K] \quad -[M]^{-1}[\bar{C}]),$$

or

$$\{\dot{z}\} = [A]\{z\} + \{B\}\ddot{u}, \quad \ddot{x}_1 = [C]\{z\} = [C] \begin{pmatrix} z_1 \\ z_2 \\ \dot{z}_1 \\ \dot{z}_2 \end{pmatrix}.$$

We will calculate the Laplace response function $h(s) = \frac{\ddot{x}_1(s)}{\ddot{u}(s)} = \{B\}(s[I] - [A])^{-1}[C]$ of the state-space system.

The Laplace response transfer function is given by

$$h(s) = \frac{b_0s^4 + b_1s^3 + b_2s^2 + b_3s + b_4}{s^4 + a_1s^3 + a_2s^2 + a_3s + a_4},$$

where the coefficients are listed in Table 2.7.

The following companion matrix $[A_c]$ is chosen

Table 2.7. Coefficients

k	a_k	b_k
0	1.0000	0.0000
1	4.0833	0.2500
2	1731.0	625.8333
3	2323.6	2323.6
4	6.0063E5	6.0063E5

Table 2.8. Eigenvalues of state-space matrix $\{A\}$

k	$[A]$	$[A_c]$
1	$-1.8079 + 35.2813j$	$-1.8079 + 35.2813j$
2	$-1.8079 - 35.2813j$	$-1.8079 - 35.2813j$
3	$-0.2337 + 21.9363j$	$-0.2337 + 21.9363j$
4	$-0.2337 - 21.9363j$	$-0.2337 - 21.9363j$

$$\begin{aligned}
 [A_c] &= \begin{pmatrix} -a_1 & -a_2 & -a_3 & -a_4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \\
 &= \begin{pmatrix} -4.0833 & -1731.0 & -2323.6 & -6.0063E5 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix},
 \end{aligned}$$

and the reconstructed input matrix $\{B_c\}$ is as expected (2.333)

$$\{B_c\} = [1, 0, 0, 0]^T,$$

and the reconstructed output matrix $[C_c]$ is calculated to be, with $b_0 = 0$ (2.334)

$$\begin{aligned}
 [C_c] &= [b_1 - a_1 b_0, b_2 - a_2 b_0, b_3 - a_3 b_0, b_4 - a_4 b_0] \\
 &= [0.2500, 625.8333, 2323.6, 6.0063E5].
 \end{aligned}$$

The eigenvalues of the original state-space matrix $[A]$ and the companion matrix $[A_c]$ are listed in Table 2.8.

The damping ratio's and natural frequencies corresponding with the pair of eigenvalues are $\zeta = [0.1024, 0.0213]$ and $\omega_n = [35.3276, 21.9375]$ rad/s respectively.

The companion matrix $[A_c]$ is similar to the original state-space matrix $[A]$ with respect to the Laplace response function $h(s)$ and the eigenvalues $[A]$. The calculated mean square values of \ddot{x}_1 are exactly the same for both the original state-space equations and the reconstructed state-space equations, $E\{\ddot{x}_1^2\} = 351.2603$.

The approximate frequency response function $h_{approx}(s)$ can be expressed in terms of the *elementary factors* by using the *minimum phase approach*:¹⁷

- Constant term (gain) K , $-K = |K|e^{j\pi}$
- Pole at the origin $h(s) = \frac{1}{s}$
- Zero at the origin $h(s) = s$
- Real pole in LHP $h(s) = \frac{\omega_0}{s + \omega_0}$

¹⁷ Poles and zeros are in the left half plane (LHP).

- Real zero in LHP $h(s) = \frac{s+\omega_0}{\omega_0}$
- Two complex poles $h(s) = \frac{\omega_0^2}{s^2+2\zeta s\omega_0+\omega_0^2}$, $|\zeta| < 1$
- Two complex zeros $h(s) = \frac{s^2+2\zeta s\omega_0+\omega_0^2}{\omega_0^2}$, $|\zeta| < 1$,

where ω_0 is the *break frequency* and ζ is the tuning factor. The approximate transfer function $h_{approx}(s)$ can be generally expressed as follows [68]

$$h_{approx}(s) = K \frac{(s + \alpha_1) \prod_i (s^2 + 2\zeta_{2i}s\omega_{2i} + \omega_{2i}^2)}{(s + \beta_3) \prod_j (s^2 + 2\zeta_{4j}s\omega_{4j} + \omega_{4j}^2)}, \quad (2.335)$$

where the constants K , α_1 , β_3 , ζ_{2i} , ζ_{4j} , ω_{2i} and ω_{4j} are all positive.¹⁸ In any case, the approximate PSD $S_{approx}(\omega)$, for a unitary white noise PSD function, is

$$S_{approx}(\omega) = |h_{approx}(j\omega)|^2. \quad (2.336)$$

The magnitude of the approximate PSD function in (2.336) is, in general, expressed in logarithmic form, in a so-called *Bode plot*

$$\text{Magnitude} = 10 \log |h_{approx}(j\omega)|^2 = 20 \log |h_{approx}(j\omega)| \text{ dB}. \quad (2.337)$$

The magnitude Bode plot is accompanied by the Bode phase plot, which is not of interest for the PSD function. However, phase properties of the elementary terms will be discussed too. The following elementary factors are discussed:

- **Constant term**, $h(s) = K$
 - For a constant term, the magnitude plot is a straight line,
 - the phase plot is also a straight line, 0° for a positive constant, $\pm 180^\circ$ for a negative constant.
- **Real pole**, $h(s) = \frac{\omega_0}{\omega_0+s}$
 - For a simple real pole the piecewise linear asymptotic Bode plot for magnitude is at 0 dB until the break frequency ω_0 and then drops at 20 dB per decade (6 dB/oct) (i.e., the slope is -20 dB/decade). An n th order plot has a slope of $-20n$ dB/decade,
 - the phase plot is 0° until one tenth of the break frequency ω_0 and then drops linearly to -90° at ten times the break frequency. An n th order pole drops to $-90n^\circ$.
- **Real zero**, $h(s) = \frac{\omega_0+s}{\omega_0}$
 - For a simple real zero the piecewise linear asymptotic Bode plot for magnitude is at 0 dB until the break frequency ω_0 and then rises at $+20$ dB per decade (6 dB/oct) (i.e., the slope is 20 dB/decade). An n th order plot has a slope of $20n$ dB/decade,
 - the phase plot is 0° until one tenth of the break frequency ω_0 and then rises linearly to 90° at ten times the break frequency. An n th order pole drops to $90n^\circ$.

¹⁸ It is clear that, for a physical realizable system, the order of the denominator polynomial must be greater or equal than that of the nominator.

- **Pole at origin**, $h(s) = \frac{1}{s}$
 - For a simple real pole at the origin draw a straight line with a slope -20 dB per decade and going through 0 dB at 1 rad/s. When n poles exist the phase plot has a slope of $-20n$ dB/decade,
 - the phase plot is -90° . An n th order pole drops to $-90n^\circ$.
- **Zero at origin**, $h(s) = s$
 - For a simple real zero at the origin draw a straight line with a slope $+20$ dB per decade and going through 0 dB at 1 rad/s. When n zeros exist the phase plot has a slope of $+20n$ dB/decade,
 - The phase plot is $+90^\circ$. An n th order zero is at $+90n^\circ$.
- **Complex conjugate pair of poles**, $h(s) = \frac{\omega_0^2}{s^2 + 2\zeta\omega_0s + \omega_0^2}$
 - For the magnitude plot of complex conjugate pair of poles draw a 0 dB at low frequencies, go through a peak of height

$$h(j\omega_r) = -20 \log(2\zeta \sqrt{1 - \zeta^2}) \approx -20 \log(2\zeta)$$

at the resonant frequency $\omega_r \approx \omega_0$ and then drop with a slope -40 dB per decade (-12 dB/oct) (i.e., the slope is -40 dB/decade). The high frequency asymptote goes through the break frequency. Note that the peak exists only for

$$0 < \zeta < \frac{1}{2}\sqrt{2}.$$

- To draw the phase plot, simply follow low frequency asymptote at 0° until

$$\omega = \omega_0 \frac{\log(\frac{2}{\zeta})}{2}$$

then decrease linearly to meet the high frequency asymptote at -180° at

$$\omega = \omega_0 \frac{2}{\log(\frac{2}{\zeta})}.$$

If $\zeta < 0.02$ the approximation can be simply a vertical line at the break frequency.

- **Complex conjugate pair of zeros**, $h(s) = \frac{s^2 + 2\zeta\omega_0s + \omega_0^2}{\omega_0^2}$
 - For the magnitude plot of a complex conjugate pair of zeros draw, a 0 dB at low frequencies, go through a dip of magnitude

$$h(j\omega_r) = 20 \log(2\zeta \sqrt{1 - \zeta^2}) \approx 20 \log(2\zeta)$$

at the resonant frequency $\omega_r \approx \omega_0$ and then rise at 40 dB per decade (12 dB/oct) (i.e., the slope is 40 dB/decade). The high frequency asymptote goes through the break frequency. Note that the peak exists only for

$$0 < \zeta < \frac{1}{2}\sqrt{2}.$$

Table 2.9. PSD spectrum enforced acceleration (Grms = 9.75 g)

Frequency (Hz)	PSD g^2/Hz (single sided)	Transfer function (-)
20–100	6 dB/oct (20 dB/dec)	6 dB/oct
100–400	0.2	$\sqrt{0.2}$
400–2000	-6 dB/oct	-6 dB/oct
Frequency (rad/s)	PSD $g^2/rad/s$ (double sided)	Transfer function (-)
$2\pi 20$ – $2\pi 100$	6 dB/oct (20 dB/dec)	6 dB/oct
$2\pi 100$ – $2\pi 400$	0.1	$\sqrt{0.1}$
$2\pi 400$ – $2\pi 2000$	-6 dB/oct	-6 dB/oct

- To draw the phase plot, simply follow low frequency asymptote at 0° until

$$\omega = \omega_0 \frac{\log(\frac{\xi}{\zeta})}{2}$$

then increase linearly to meet the high frequency asymptote at 180° at

$$\omega = \omega_0 \frac{2}{\log(\frac{\xi}{\zeta})}.$$

Example. The single sided PSD acceleration spectrum $W_a(f)$ is given in Table 2.9 and taken from [68]. This single sided PSD acceleration spectrum $W_a(f)$ will be transformed into a double sided PSD acceleration spectrum $2S_a(\omega) = W_a(f)$, $f \leq \infty$, $|\omega| \leq \infty$. A first examination of the required $h_{approx}(s)$ reveals that the this transfer function may be approximated with a zero at $\omega = 0$ and poles at $\omega = 2\pi 100$ and $\omega = 2\pi 400$. Furthermore the transfer function must decrease at large frequencies at a rate of 20 dB/dec, that is as $\frac{1}{s}$. The resulting transfer function is

$$h_{approx}(s) = K \frac{2\pi 100 \times 2\pi 400s}{(s + 2\pi 100)(s + 2\pi 400)},$$

where the value of the gain K is calculated as to give a value $\sqrt{0.1}$ at a frequency of $\omega = 2\pi\sqrt{100 \times 400}$, resulting in $K = 6.2912 \times 10^{-4}$. The approximate Laplace transfer function becomes

$$h_{approx}(s) = \frac{b_1 s}{s^2 + a_1 s + a_2} = \frac{993.4588s}{s^2 + 3141.5927s + 1579136.7}.$$

The Bode diagrams magnitude and phase are illustrated Fig. 2.28. The corresponding auxiliary state-space system, with state matrix $[A_a]$, input matrix $\{B_a\}$ and output matrix $[C_a]$, is

$$[A_a] = \begin{pmatrix} -3141.5927 & -1579136.7 \\ 1.0 & 0.0 \end{pmatrix}, \quad \{B_a\} = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

$$[C_a] = [993.4588, 0.0].$$

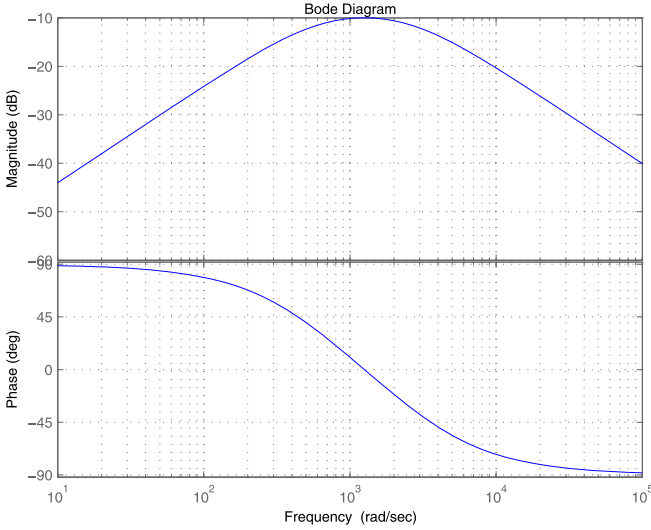


Fig. 2.28. Bode diagram

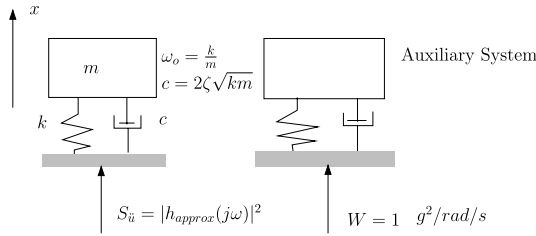


Fig. 2.29. Dynamic system + auxiliary system

The SDOF system together with the auxiliary system to filter the white noise $W = 1 \text{ g}^2/\text{rad/s}$ is shown in Fig. 2.29. The natural frequency of the SDOF system $f_0 = \frac{1}{2\pi}\sqrt{\frac{k}{m}} = 150 \text{ Hz}$. The amplification factor is $Q = 10$, then $\zeta = 0.05$. At first, the $E\{\ddot{x}^2\}$ of the SDOF will be calculated from Miles' equation assuming white noise base excitation (Table 2.9)

$$E\{\ddot{x}^2\} = \frac{\pi}{2}f_0QW_{\ddot{u}}(1 + 4\zeta^2) \approx \frac{\pi}{2} \times 150 \times 10 \times 0.2 = 245.0442 \text{ g}^2.$$

Secondly the mean square acceleration $E\{\ddot{x}^2\}$ of the SDOF will be calculated using the auxiliary system excited by a unit white noise enforced acceleration. The characteristics of the auxiliary system are

$$[A] = \begin{pmatrix} 0 & 1 \\ -\omega_0^2 & -2\zeta\omega_0 \end{pmatrix}, \quad \{B\} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}, \quad [C] = [-\omega_0^2, -2\zeta\omega_0].$$

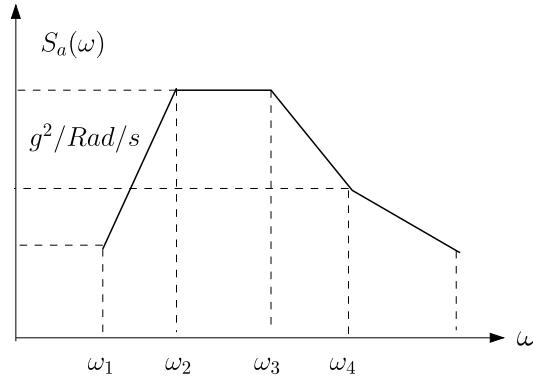


Fig. 2.30. PSD specification

The total state-space system $[A_t]$, $\{B_t\}$ and $[C_t]$, the combination of the auxiliary system and the SDOF, can now be formed.

The mean square acceleration becomes $E\{\ddot{x}^2\} = 211.9628 \text{ g}^2$. As expected, the filtered response is somewhat lower than the white noise response, because the real spectrum is band-limited. The main effort is to set-up the approximate Laplace transfer function $h_{approx}(s)$ such that $|h_{approx}(s)|^2$ will represent the specified PSD acceleration specification.

The following example is again a representation of a standard PSD acceleration specification.

Example. The PSD $S_a(\omega)$ of a random process is illustrated in Fig. 2.30 can be approximated by a transfer function [66].

$$h_{approx}(s) = K \frac{s^2(s^2 + 2\zeta_n s\omega_n + \omega_n^2)}{(s + \alpha)(s^2 + 2\zeta_{d1} s\omega_{d1} + \omega_{d1}^2)(s^2 + 2\zeta_{d2} s\omega_{d2} + \omega_{d2}^2)}, \quad (2.338)$$

where all parameters α , ζ_n , ζ_{d1} , ζ_{d2} , ω_n , ω_{d1} and ω_{d2} are positive. We must vary the parameters to find a good approximated for $h_{approx}(s)$.

Problems

2.19. Consider a two degrees of freedom nonclassically damped linear system

$$[M] = \begin{pmatrix} 100 & 0 \\ 0 & 15 \end{pmatrix} \text{ kg}, \quad [C] = \begin{pmatrix} 75 & -50 \\ -50 & 50 \end{pmatrix} \text{ N s/m} \quad \text{and}$$

$$[K] = \begin{pmatrix} 76915 & -14415 \\ -14415 & 14415 \end{pmatrix} \text{ N/m.}$$

The random load matrix is

$$[S_0] = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ N}^2/\text{Hz}.$$

The state vector $\{y\}$ is given by

$$\{y\} = \begin{Bmatrix} x \\ \dot{x} \end{Bmatrix}.$$

Apply the complex mode approach:

- the eigenvalues λ_i of the state matrix $[A]$,
- the eigenvectors $[\Psi]$,
- the natural frequencies ω_n from conjugate pair of eigenvalues λ_i ,
- the modal damping ratio's ζ_n
- the damped natural frequencies
- the correlation matrix $R_{yy}(0)$,
- the correlation matrix $R_{yy}(\tau)$, $\tau = 0.0-0.60$ s.

This problem is taken from [61].

Answers:

$$\omega_n = \begin{pmatrix} 21.9375 \\ 35.3276 \end{pmatrix} \text{ rad/s}, \quad \zeta_n = \begin{pmatrix} 0.0107 \\ 0.0512 \end{pmatrix}, \quad \omega_d = \begin{pmatrix} 21.9363 \\ 35.2813 \end{pmatrix} \text{ rad/s},$$

$$R_{yy}(0) = 10^{-10} \begin{pmatrix} 4533.57 & 8045.51 & 0.0 & 3110.72 \\ 8045.51 & 19427.876 & 3110.72 & 0.0 \\ 0.0 & 3110.72 & 2325680.3 & 3385340.0 \\ 3110.72 & 0.0 & 3385340.0 & 10948826.0 \end{pmatrix}.$$

2.20. The equation of motion of a SDOF system excited at the base by an enforced acceleration \ddot{u} is given by

$$m\ddot{x} + c\dot{z} + kz = 0,$$

where x is the absolute motion and $z = x - u$ is the motion with respect to the base. The damping ratio is $\zeta = \frac{c}{2\sqrt{km}}$ and the natural frequency is $\omega_0 = \sqrt{\frac{k}{m}}$.

The double sided white noise PSD function of the acceleration is given by $S_{\ddot{u}}$, and the single sided white noise PSD in the cyclic frequency domain is $W_{\ddot{u}} = 2S_{\ddot{u}}$.

Show the following expressions using the state space approach and applying “the stochastic dual of the direct method of Lyapunov”, (2.259) and (2.260) and knowing that $\omega_0 = 2\pi f_0$ and $Q = \frac{1}{2\zeta}$:

- $R_{zz}(0) = \langle z^2 \rangle = \frac{S_{\ddot{u}}}{4\zeta\omega_0^3} = \frac{W_{\ddot{u}}}{8\zeta\omega_0^3}$
- $R_{\dot{z}\dot{z}}(0) = \langle \dot{z}\dot{z} \rangle = R_{zz}(0) = 0$
- $R_{\dot{z}z}(0) = \langle \dot{z}z \rangle = \frac{S_{\ddot{u}}}{4\zeta\omega_0} = \frac{W_{\ddot{u}}}{8\zeta\omega_0}$
- $R_{\ddot{x}\ddot{x}}(0) = \langle \ddot{x}^2 \rangle = \frac{S_{\ddot{u}}\omega_0}{4\zeta} (1 + 4\zeta^2) = \frac{\pi}{2} f_0 Q W_{\ddot{u}} (1 + 4\zeta^2)$

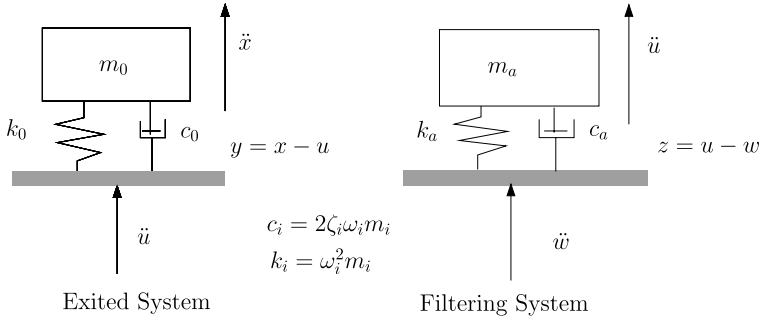


Fig. 2.31. Excited and filtering system

2.21. The original excited SDOF system is indicated by “0” and the filtering SDOF system is indicated by “a”. Both systems are illustrated in Fig. 2.31.

Derive the Laplace response function $h_{\ddot{x}\ddot{w}}(s)$ and the PSD function $S_{\ddot{x}}(\omega)$, using the following steps:

- Derive equation $\ddot{u} = \ddot{u}(z, \dot{z})$
- Derive equation $\ddot{x} = \ddot{x}(y, \dot{y})$
- Derive equation $\ddot{z} = \ddot{z}(z, \dot{z}, \ddot{w})$
- Derive equation $\ddot{y} = \ddot{y}(y, \dot{y}, z, \dot{z})$

Answers:

1. $\ddot{u} = -2\zeta_a\omega_a\dot{z} - \omega_a^2z$
2. $\ddot{x} = -2\zeta_0\omega_0\dot{y} - \omega_0^2y$
3. $\ddot{z} + 2\zeta_a\omega_a\dot{z} + \omega_a^2z = -\ddot{w}$
4. $\ddot{y} + 2\zeta_0\omega_0\dot{z} + \omega_0^2y = 2\zeta_a\omega_a\dot{z} + \omega_a^2z$

- Derive space-state equation $\{\dot{v}\} = [A]\{v\} + \{B\}\ddot{w}$
- Derive output relation $\ddot{x} = [C]\{v\}, \{v\} = [y, \dot{y}, z, \dot{z}]^T$

Answers:

1. $[A] = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -\omega_0^2 & -2\zeta_0\omega_0 & \omega_a^2 & 2\zeta_a\omega_a \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -\omega_a^2 & -2\zeta_a\omega_a \end{pmatrix}$
2. $\{B\} = [0, 0, 0, 1]^T$
3. $[C] = [-\omega_0^2, -2\zeta_0\omega_0, 0, 0]$

- Derive the Laplace response function $h_{\ddot{x}\ddot{w}}(s) = [C](s[I] - [A])^{-1}\{B\}$

Answer:

1. $h_{\ddot{x}\ddot{w}}(s) = \frac{\omega_0\omega_a(\omega_0+2s\zeta_0)(\omega_a+2s\zeta_a)}{(s^2+2s\zeta_0\omega_0+\omega_0^2)(s^2+2s\zeta_a\omega_a+\omega_a^2)}$

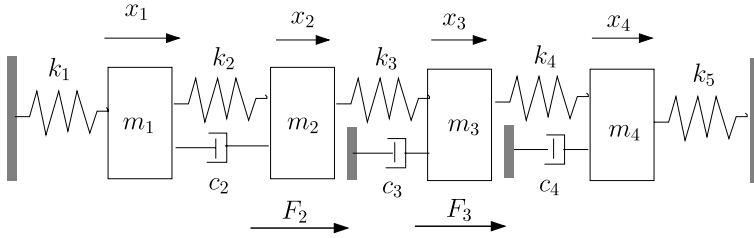


Fig. 2.32. The four degrees of freedom dynamic system

- Derive the expression of the PSD function $S_{\ddot{x}}(\omega)$ assuming a white noise PSD function of the enforced acceleration $S_{\ddot{w}}$

Answer:

$$1. S_{\ddot{x}}(\omega) = S_{\ddot{w}} \frac{\omega_0^2 \omega_a^2 (\omega_0^2 + 4\omega^2 \zeta_0^2) (\omega_a^2 + 4\omega^2 \zeta_a^2)}{[(\omega_0^2 - \omega^2)^2 + 4\omega^2 \zeta_0^2 \omega_0^2][(\omega_a^2 - \omega^2)^2 + 4\omega^2 \zeta_a^2 \omega_a^2]}$$

2.22. A dynamic system consists of four discrete masses m_1, m_2, m_3, m_4 , five spring elements k_1, k_2, k_3, k_4, k_5 , three viscous dampers c_1, c_2, c_3 . The system is loaded by two statistically independent white noise random forces F_2 and F_3 , and the spectral moments are $\langle F_2 \rangle = \langle F_3 \rangle = 0, \langle F_2 F_3 \rangle = 0, \langle F_2 F_2 \rangle = R_{F_2}(\tau) = \frac{W_{F_2}}{2} \delta(\tau)$, and $\langle F_3 F_3 \rangle = R_{F_3}(\tau) = \frac{W_{F_3}}{2} \delta(\tau)$. The dynamic system is shown in Fig. 2.32. The system is taken from [67].

Perform the following activities:

- generate the equations of motion of the dynamic system,
- set up the force matrix to calculate the forces in the springs $\{F_k\} = [S_F]\{x\}$,
- set up the force matrix to calculate the forces in the dampers $\{F_d\} = [S_D]\{\dot{x}\}$,
- set up the space state equations.
- set up the state space output matrix $[C_F]$ for the spring forces.
- set up the state space output matrix $[C_D]$ for the damper forces.
- calculate for $k_1 = k_2 = k_3 = k_4 = k_5 = 10\,000$ N/m, $m_1 = m_2 = m_3 = m_4 = 10$ kg, $c_2 = c_3 = c_4 = 10$ N s/m:
 - damped natural frequencies ω_d rad/s.
 - (undamped) natural frequencies ω_n rad/s.
 - modal damping ratio's ζ .
- If the one-sided white noise power spectral densities of both forces F_2 and F_3 are $W_{F_2} = W_{F_3} = 10\,000$ N²/Hz, calculate
 - The rms spring forces $\{F_k\}_{rms}$.
 - The rms damper forces $\{F_d\}_{rms}$.

Answers:

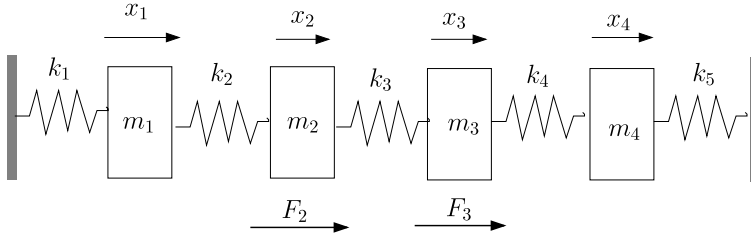


Fig. 2.33. The four degrees of freedom dynamic system

$$\{\omega_d\} = \begin{pmatrix} 19.54 \\ 37.17 \\ 51.17 \\ 60.14 \end{pmatrix} \text{ rad/s}, \quad \{\omega_n\} = \begin{pmatrix} 19.55 \\ 37.17 \\ 51.17 \\ 60.14 \end{pmatrix} \text{ rad/s}, \quad \{\zeta\} = \begin{pmatrix} 0.014 \\ 0.007 \\ 0.014 \\ 0.012 \end{pmatrix},$$

$$\{F_k\}_{rms} = \begin{pmatrix} 1275.3 \\ 998.4 \\ 1002.5 \\ 1003.3 \\ 1275.0 \end{pmatrix} \text{ N} \quad \text{and} \quad \{F_d\}_{rms} = \begin{pmatrix} 43.2 \\ 43.4 \\ 35.4 \end{pmatrix} \text{ N}.$$

2.23. This problem is about the same as stated in problem 2.22, however, no discrete dampers are present. The system is loaded by two statistically independent white noise random forces F_2 and F_3 , and the spectral moments are $\langle F_2 \rangle = \langle F_3 \rangle = 0$, $\langle F_2 F_3 \rangle = 0$, $\langle F_2 F_2 \rangle = R_{F_2}(\tau) = \frac{W_{F_2}}{2} \delta(\tau)$, and $\langle F_3 F_3 \rangle = R_{F_3}(\tau) = \frac{W_{F_3}}{2} \delta(\tau)$. The dynamic system is illustrated in Fig. 2.33. Calculate for $k_1 = k_2 = k_3 = k_4 = k_5 = 10\,000$ N/m, $m_1 = m_2 = m_3 = m_4 = 10$ kg the natural frequencies ω_n rad/s. The one-sided white noise power spectral densities of both forces F_2 and F_3 are $W_{F_2} = W_{F_3} = 10\,000$ N²/Hz. Using the modal reduction method described in Sect. 2.6.2, p. 78, calculate the rms spring forces $\{F_k\}_{rms}$

- the first mode considered,
- the first two mode considered,
- the first three mode considered,
- all modes considered.

The modal damping ratio is $\zeta = 0.015$ for the modes considered. Compare the results with the results of problem 2.22.

Answers:

$$\{\omega_n\} = \begin{pmatrix} 19.54 \\ 37.17 \\ 51.17 \\ 60.17 \end{pmatrix} \text{ rad/s},$$

first mode

$$\{F_k\}_{rms} = \begin{pmatrix} 1056.6 \\ 653.0 \\ 0.0 \\ 653.0 \\ 1056.6 \end{pmatrix} \text{ N,}$$

two first modes

$$\{F_k\}_{rms} = \begin{pmatrix} 1130.7 \\ 670.9 \\ 497.8 \\ 670.9 \\ 1130.7 \end{pmatrix} \text{ N,}$$

three first modes

$$\{F_k\}_{rms} = \begin{pmatrix} 1157.7 \\ 783.2 \\ 497.8 \\ 783.2 \\ 1157.7 \end{pmatrix} \text{ N}$$

and for all mode considered

$$\{F_k\}_{rms} = \begin{pmatrix} 1173.9 \\ 935.5 \\ 806.9 \\ 935.5 \\ 1173.9 \end{pmatrix} \text{ N.}$$

2.24. This problem is taken from [169]. Consider a two degrees of freedom system governed by differential equations with the following matrices

$$\begin{aligned} [M] &= \begin{pmatrix} m & 0 \\ 0 & 2\varepsilon m \end{pmatrix}, & [C] &= \begin{pmatrix} c(1 + \varepsilon_1) & -\varepsilon_1 c \\ -\varepsilon_1 c & 2\varepsilon c \end{pmatrix}, \\ [K] &= \begin{pmatrix} k(1 + \varepsilon_1) & -\varepsilon_1 k \\ -\varepsilon_1 k & 2\varepsilon k \end{pmatrix}, & \{F(t)\} &= \begin{pmatrix} F_1(t) \\ 0 \end{pmatrix}, \end{aligned}$$

where ε_1 and ε_2 are some nonnegative coefficients, $2\varepsilon = \varepsilon_1 + \varepsilon_2$ and $F_1(t)$ is an ideal white noise with intensity $W_0/2$.

Proof that the natural frequencies are:

$$\omega_{1,2}^2 = \frac{k}{2m} [2 + \varepsilon_1 \pm \varepsilon_1 \tau],$$

where $\tau = \sqrt{1 + \frac{2}{\varepsilon}}$. Show that the corresponding mode shapes $[\Phi]$ are given by:

$$[\Phi] = \begin{pmatrix} 2\lambda_1 & 2\lambda_1 \\ \lambda_1(1-\tau) & \lambda_2(1+\tau) \end{pmatrix},$$

where

$$\lambda_1 = \frac{1}{2m}[2 + \varepsilon - \varepsilon\tau],$$

$$\lambda_2 = \frac{1}{2m}[2 + \varepsilon + \varepsilon\tau].$$

And proof that the modal damping ratio's are given by

$$\zeta_{1,2} = \frac{c}{4km}[2 + \varepsilon_1 \pm \varepsilon_1\tau].$$

Calculate, using Lyapunov equation, the mean square values $E(x_1^2)$ and $E(x_2^2)$ and proof the following relations:

$$E(x_1^2) = \frac{W_0}{kc} \{ [2 + \varepsilon_1 + \varepsilon_1\tau]^{-2} [2 + \varepsilon - \varepsilon\tau]^{-2} \\ + [2 + \varepsilon_1 - \varepsilon_1\tau]^{-2} [2 + \varepsilon + \varepsilon\tau]^{-2} \\ - \Psi(4 + 4\varepsilon_1 - 2\varepsilon_1^2\varepsilon_2^2)^{-2} (4 + 2\varepsilon)^{-1} \},$$

$$E(x_2^2) = \frac{W_0}{4kc} \{ [2 + \varepsilon_1 + \varepsilon_1\tau]^{-2} [2 + \varepsilon - \varepsilon\tau]^{-2} [1 - \tau]^2 \\ + [2 + \varepsilon_1 - \varepsilon_1\tau]^{-2} [2 + \varepsilon + \varepsilon\tau]^{-2} [1 + \tau]^2 \\ + \Psi(4 + 4\varepsilon_1 - 2\varepsilon_1^2\varepsilon_2^2)^{-2} (4 + 2\varepsilon)^{-1} \},$$

where

$$\Psi = 2\bar{c} \left[\bar{c} + \frac{\varepsilon_1^2(\varepsilon + 2)}{(2\varepsilon + 2\varepsilon\varepsilon_1 - \varepsilon_1^2)(\varepsilon_1 + 2)} \right]^{-1},$$

and

$$\bar{c} = \frac{c^2}{km}.$$

2.7 Limit Load Factors

2.7.1 Introduction

In spacecraft, the equipment is generally mounted to (sandwich) panels. This is illustrated in Fig. 2.34. The design loads for these equipment have to be established, estimated, before the spacecraft is built. The acoustic pressure loads are transferred into random mechanical vibrations in the panel, exciting the equipment at its base. This acceleration is indicated by \ddot{u} . The equipment panel plus equipment form SDOF system consisting of mass m_1 , spring k_1 and damper c_1 (support structure). A piece of equipment (e.g. box) forms the second SDOF system consisting of mass m_2 , spring k_2 and damper c_2

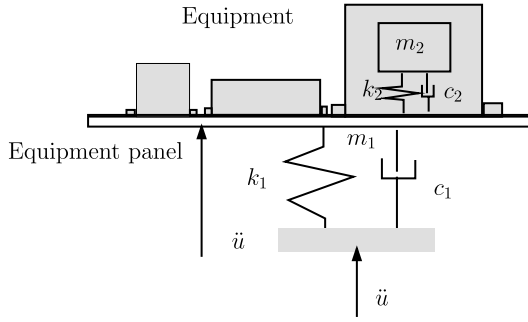


Fig. 2.34. Equipment panel

(component). In general, it is assumed that $m_2 < m_1$. The *limit load factor* (LLF) is defined as the dynamic 3σ rms acceleration response $\langle \ddot{x}_2 \rangle$ of SDOF 2 increased with the *quasi static loads* (QSL). The QSL can be found in the User’s manual of the selected launch vehicle.

$$LLF = 3\langle \ddot{x}_2 \rangle + QSL. \tag{2.339}$$

The QSL are the steady state accelerations increased by very low frequency accelerations (see for details in the Launch Vehicle User’s Manual). The LLF is dependent on the mass m_2 . The estimation of the LLF for the equipment is discussed in the next sections. The theory of the LLFs is based on [28].

2.7.2 Equations of Motion 2 SDOF System

The 2DOF dynamic system is illustrated in Fig. 2.35. This system is excited at the base by an enforced acceleration \ddot{u} . The absolute displacements, velocities and accelerations of the discrete masses are denoted by $x(t)$, $\dot{x}(t)$ and $\ddot{x}(t)$. Further in this section variable t will be skipped. The equations of motion of the 2DOF dynamic system are given by

$$\begin{aligned} & \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{pmatrix} c_1 + c_2 & -c_2 \\ -c_2 & c_2 \end{pmatrix} \begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} + \begin{pmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{pmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} \\ & = - \begin{Bmatrix} c_1 \\ 0 \end{Bmatrix} \dot{u} - \begin{Bmatrix} k_1 \\ 0 \end{Bmatrix} u. \end{aligned} \tag{2.340}$$

If the absolute displacements, velocities and displacements are replaced by the relative ones;

- $y_1 = x_1 - u$
- $y_2 = x_1 - x_2$
- $\ddot{x}_1 = \ddot{y}_1 + \ddot{u}$
- $\ddot{x}_2 = \ddot{y}_2 + \ddot{x}_1 = \ddot{y}_1 + \ddot{y}_2 + \ddot{u}$

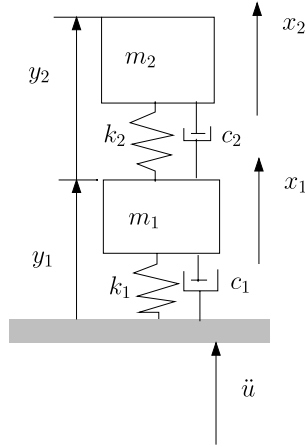


Fig. 2.35. Two SDOF systems

then the following equations of motion are obtained:

$$\begin{pmatrix} m_1 & 0 \\ m_2 & m_2 \end{pmatrix} \begin{Bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{Bmatrix} + \begin{pmatrix} c_1 & -c_2 \\ 0 & c_2 \end{pmatrix} \begin{Bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{Bmatrix} + \begin{pmatrix} k_1 & -k_2 \\ 0 & k_2 \end{pmatrix} \begin{Bmatrix} y_1 \\ y_2 \end{Bmatrix} = - \begin{Bmatrix} m_1 \\ m_2 \end{Bmatrix} \ddot{u}. \tag{2.341}$$

The following notation will be introduced

- $\omega_1^2 = \frac{k_1}{m_1}, \omega_2^2 = \frac{k_2}{m_2}$
- $c_1 = 2\zeta_1\sqrt{k_1m_1}, c_2 = 2\zeta_2\sqrt{k_2m_2}$
- $\frac{c_1}{m_1} = 2\zeta_1\omega_1, \frac{c_2}{m_2} = 2\zeta_2\omega_2$
- $\frac{c_2}{m_1} = 2\zeta_2\omega_2\gamma$
- $\frac{k_2}{m_1} = 2\omega_2\omega_2^2\gamma$ and
- $\gamma = \frac{m_2}{m_1}$

Equation (2.341) can be written as

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{Bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{Bmatrix} + \begin{pmatrix} 2\zeta_1\omega_1 & -2\zeta_2\omega_2\gamma \\ 0 & 2\zeta_2\omega_2 \end{pmatrix} \begin{Bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{Bmatrix} + \begin{pmatrix} \omega_1^2 & -\omega_2^2\gamma \\ 0 & \omega_2^2 \end{pmatrix} \begin{Bmatrix} y_1 \\ y_2 \end{Bmatrix} = - \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \ddot{u}. \tag{2.342}$$

The solution of (2.342) will be in the frequency domain (harmonic solution), then the variables are expressed as follows

- $x_1(t) = X_1(\omega_1)e^{-j\omega t}, x_2(t) = X_2(\omega_1)e^{-j\omega t}$
- $y_1(t) = Y_1(\omega_1)e^{-j\omega t}, y_2(t) = Y_2(\omega_1)e^{-j\omega t}$ and
- $\ddot{u}_1(t) = \ddot{U}_1(\omega_1)e^{-j\omega t},$

then (2.342) becomes

$$\begin{aligned} & \left[-\omega^2 \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} + j\omega \begin{pmatrix} 2\zeta_1\omega_1 & -2\zeta_2\omega_2\gamma \\ 0 & 2\zeta_2\omega_2 \end{pmatrix} + \begin{pmatrix} \omega_1^2 & -\omega_2^2\gamma \\ 0 & \omega_2^2 \end{pmatrix} \right] \begin{Bmatrix} Y_1(\omega) \\ Y_2(\omega) \end{Bmatrix} \\ & = - \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \ddot{U}(\omega). \end{aligned} \quad (2.343)$$

The relative displacements $Y_1(\omega)$ and $Y_2(\omega)$ can be solved from (2.343) using Cramer's rule

$$Y_1(\omega) = \frac{\ddot{U}(\omega)}{D} [\omega^2 - j\omega\zeta_2\omega_2(1 + \gamma) - \omega_2^2(1 + \gamma)], \quad (2.344)$$

and

$$Y_2(\omega) = -\frac{\ddot{U}(\omega)}{D} [j\omega_1\zeta_1\omega_1^2 + \omega_1^2], \quad (2.345)$$

where the determinant is $D = (j\omega)^4 a_4 + (j\omega)^3 a_3 + (j\omega)^2 a_2 + (j\omega) a_1 + a_0$ and the coefficients are

- $a_4 = 1$
- $a_3 = 2\zeta_1\omega_1 + 2\zeta_2\omega_2(1 + \gamma) = \omega_1[2\zeta_1 + 2\zeta_2\phi(1 + \gamma)]$
- $a_2 = \omega_1^2 + \omega_2^2(1 + \gamma) + 4\zeta_1\zeta_2\omega_1\omega_2 = \omega_1^2[1 + \phi^2(1 + \gamma) + 4\zeta_1\zeta_2\phi]$
- $a_1 = 2\zeta_1\omega_1\omega_2^2 + 2\zeta_2\omega_2\omega_1^2 = \omega_1^3[2\zeta_1\phi^2 + 2\zeta_2\phi]$
- $a_0 = \omega_1^2\omega_2^2 = \omega_1^4[\phi^2]$,

where $\phi = \frac{\omega_2}{\omega_1}$.

The absolute accelerations $\ddot{X}_1(\omega)$ and $\ddot{X}_2(\omega)$ can now be found

$$\begin{aligned} \ddot{X}_1(\omega) &= \frac{\ddot{U}(\omega)}{D} [(j\omega)^3 2\zeta_1\omega_1 + (j\omega)^2(\omega_1^2 + 4\zeta_1\zeta_2\omega_1\omega_2) \\ &\quad + j\omega(\zeta_1\omega_1\omega_2^2 + \zeta_2\omega_2\omega_1^2) + \omega_1^2\omega_2^2], \end{aligned} \quad (2.346)$$

and

$$\ddot{X}_2(\omega) = \frac{\ddot{U}(\omega)}{D} [(j\omega)^2 4\zeta_1\zeta_2\omega_1\omega_2 + j\omega(\zeta_1\omega_1\omega_2^2 + \zeta_2\omega_2\omega_1^2) + \omega_1^2\omega_2^2]. \quad (2.347)$$

2.7.3 Frequency Transfer Function

The frequency transfer functions between the absolute accelerations $\ddot{X}_1(\omega)$, $\ddot{X}_2(\omega)$ and the base excitation $\ddot{U}(\omega)$ can now be calculated by $H_{\ddot{X}_1\ddot{U}} = \frac{\ddot{X}_1(\omega)}{\ddot{U}(\omega)}$ and $H_{\ddot{X}_2\ddot{U}} = \frac{\ddot{X}_2(\omega)}{\ddot{U}(\omega)}$ and are expressed in a general form

$$H_{\ddot{X}_1\ddot{U}} = \frac{(j\omega)^3 b_{1,3} + (j\omega)^2 b_{1,2} + (j\omega) b_{1,1} + b_{1,0}}{(j\omega)^4 a_4 + (j\omega)^3 a_3 + (j\omega)^2 a_2 + (j\omega) a_1 + a_0}, \quad (2.348)$$

- $b_{1,3} = 2\zeta_1\omega_1 = \omega_1[2\zeta_1]$
- $b_{1,2} = \omega_1^2 + 4\zeta_1\zeta_2\omega_1\omega_2 = \omega_1^2[1 + 4\zeta_1\zeta_2\phi]$
- $b_{1,1} = \zeta_1\omega_1\omega_2^2 + \zeta_2\omega_2\omega_1^2 = \omega_1^3[\zeta_1\phi^2 + \zeta_2\phi]$
- $b_{1,0} = \omega_1^2\omega_2^2 = \omega_1^4[\phi^2]$,

and

$$H_{\ddot{x}_2\ddot{v}} = \frac{(j\omega)^3 b_{2,3} + (j\omega)^2 b_{2,2} + (j\omega) b_{2,1} + b_{2,0}}{(j\omega)^4 a_4 + (j\omega)^3 a_3 + (j\omega)^2 a_2 + (j\omega) a_1 + a_0}, \quad (2.349)$$

- $b_{2,3} = 0$
- $b_{2,2} = 4\zeta_1\zeta_2\omega_1\omega_2 = \omega_1^2[4\zeta_1\zeta_2\phi]$
- $b_{2,1} = \zeta_1\omega_1\omega_2^2 + \zeta_2\omega_2\omega_1^2 = \omega_1^3[\zeta_1\phi^2 + \zeta_2\phi]$
- $b_{2,0} = \omega_1^2\omega_2^2 = \omega_1^4[\phi^2]$.

2.7.4 Random Responses

In this section we calculate the mean square and the root mean square (rms) of the absolute accelerations $\ddot{x}_1(t)$ and $\ddot{x}_2(t)$ due the random enforced acceleration $\ddot{u}(t)$ with a PSD function $S_{\ddot{u}\ddot{u}}(\omega)$. The mean square values of $\ddot{x}_1(t)$ and $\ddot{x}_2(t)$ are

$$\langle \ddot{x}_1^2 \rangle = \int_{-\infty}^{\infty} |H_{\ddot{x}_1\ddot{v}}|^2 S_{\ddot{u}\ddot{u}}(\omega) d\omega, \quad (2.350)$$

and

$$\langle \ddot{x}_2^2 \rangle = \int_{-\infty}^{\infty} |H_{\ddot{x}_2\ddot{v}}|^2 S_{\ddot{u}\ddot{u}}(\omega) d\omega. \quad (2.351)$$

The PSD function of the enforced acceleration $S_{\ddot{u}\ddot{u}}(\omega)$ is considered to be constant over the frequency domain ω , thus $S_{\ddot{u}}$. The expressions for the mean square values of $\ddot{x}_1(t)$ and $\ddot{x}_2(t)$ can now be written

$$\langle \ddot{x}_1^2 \rangle = S_{\ddot{u}} \int_{-\infty}^{\infty} |H_{\ddot{x}_1\ddot{v}}|^2 d\omega, \quad (2.352)$$

and

$$\langle \ddot{x}_2^2 \rangle = S_{\ddot{u}} \int_{-\infty}^{\infty} |H_{\ddot{x}_2\ddot{v}}|^2 d\omega. \quad (2.353)$$

In Wirsching, Paez and Ortiz [222], Appendix A, a solution of the definite integral of the transfer function $H_{\ddot{x}_1\ddot{v}}$ is provided

$$\begin{aligned} I_{4,i} &= \int_{-\infty}^{\infty} |H_{\ddot{x}_i\ddot{v}}(\omega)|^2 \\ &= \int_{-\infty}^{\infty} \left| \frac{b_{i,0} + (j\omega)b_{i,1} + (j\omega)^2 b_{i,2} + (j\omega)^3 b_{i,3}}{a_0 + (j\omega)a_1 + (j\omega)^2 a_2 + (j\omega)^3 a_3 + (j\omega)^4 a_4} \right|^2 d\omega. \end{aligned} \quad (2.354)$$

The solution of the integral $I_{4,i}$ is given by

$$I_{4,i} = \pi \left[\frac{a_0 b_{i,3}^2 (a_0 a_3 - a_1 a_2) + a_0 a_1 a_4 (2b_{i,1} b_{i,3} - b_{i,2}^2)}{a_0 a_4 (a_0 a_3^2 + a_1^2 a_4 - a_1 a_2 a_3)} \right] + \pi \left[\frac{-a_0 a_3 a_4 (b_{i,1}^2 - 2b_{i,0} b_{i,2}) + a_4 b_{i,0}^2 (a_1 a_4 - a_2 a_3)}{a_0 a_4 (a_0 a_3^2 + a_1^2 a_4 - a_1 a_2 a_3)} \right]. \quad (2.355)$$

2.7.5 Feedback Factor

The mean square values of the accelerations $\ddot{x}_1(t)$ and $\ddot{x}_2(t)$ can be calculated by (2.355). The *feedback factor* Q_{fb} is the ratio of the rms acceleration $\langle \ddot{x}_1 \rangle$ and $\langle \ddot{x}_2 \rangle$, thus

$$Q_{fb} = \sqrt{\frac{\langle \ddot{x}_2^2 \rangle}{\langle \ddot{x}_1^2 \rangle}}. \quad (2.356)$$

The feedback factor Q_{fb} is dependent on the following parameters:

- the damping ratios ζ_1 and ζ_2
- the hard mounted natural frequency ratio (tuning factor) $\phi = \frac{\omega_2}{\omega_1}$
- the mass ratio factor $\gamma = \frac{m_2}{m_1}$

In Fig. 2.36 a parametric study of the feedback factor Q_{fb} is illustrated considering the following values for the parameters:

- the *tuning factor* $\phi = 0.0 \dots$ to $\dots 2.5$,
- the mass ratio $\gamma = 0.001, 0.01, 0.1, 1.0$,
- the damping ratios $\zeta_1 = 0.05$ and $\zeta_2 = 0.05$.

The following conclusions can be drawn:

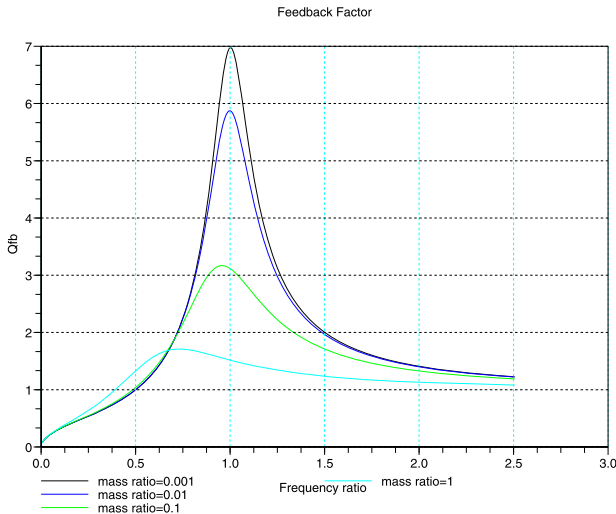


Fig. 2.36. Feedback factor Q_{fb}

1. The feedback factor Q_{fb} has the most influence when $\phi = 1.0$.
2. The ratio $\gamma = \frac{m_2}{m_1}$ strongly influences the feedback factor Q_{fb} when $\phi = 1.0$; it does not influence Q_{fb} when $\phi \ll 1.0$, $\phi \gg 1.0$.
3. When $\phi \gg 1.0$, the two masses m_1 and m_2 tend to have the same acceleration, such that the Q_{fb} value is near unity, regardless of the values of the damping ratios of ζ_1 and ζ_2 .
4. When $\phi \ll 1.0$, Q_{fb} is lower than unity.
5. Due to the dynamic impedance of the hard mounted dynamic system m_1 , c_1 and k_1 , it can be stated that Q_{fb} is always lower than the amplification $Q_2 = \frac{1}{2\zeta_2}$ in the hard mounted condition, $Q_{fb} < Q_2 = \frac{1}{2\zeta_2}$.

Based on the previous results, the feedback factor Q_{fb} will be further investigated for the worst condition, the “tuned case”, $\phi = 1$. The parameters γ and Q_2 will be varied in the following ranges:

- the mass ratio $\gamma = 0.00001 \dots$ to 1
- the amplification factor $Q_2 = \frac{1}{2\zeta_2} = 2.5, 5, 10, 20, 40$
- the damping ration $\zeta_1 = 0.05$

The results of the calculation are shown in Fig. 2.37. From Fig. 2.37 it can be interpreted that:

1. For $\gamma \rightarrow 1$ the $Q_{fb} \rightarrow 1.5$.
2. The feedback factor Q_{fb} is very sensitive to a mass ratio $\gamma = 0.00001 \dots 0.1$: for the lower γ values, the Q_{fb} factor has an asymptotic value depending on the damping ration ζ_2 , but always lower than Q_2 .
3. Vibration loads for mass m_2 can be derived as a tuned case of the 2DOF system, when at least the following parameters are known: $W_{\ddot{u}} = 2S_{\ddot{u}}$, ζ_1 ,

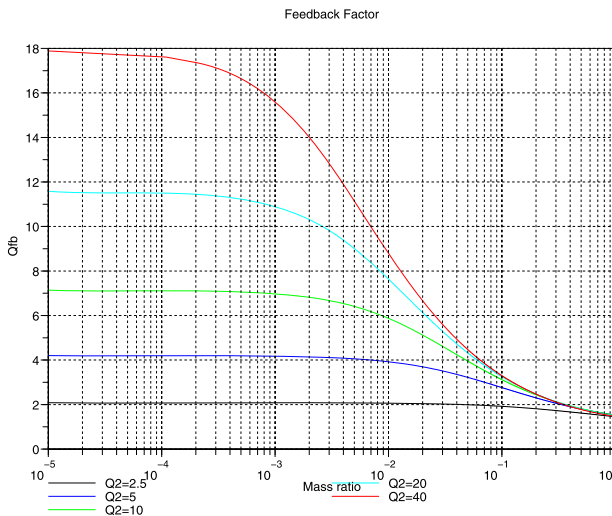


Fig. 2.37. Feedback factor Q_{fb} , $\phi = 1$

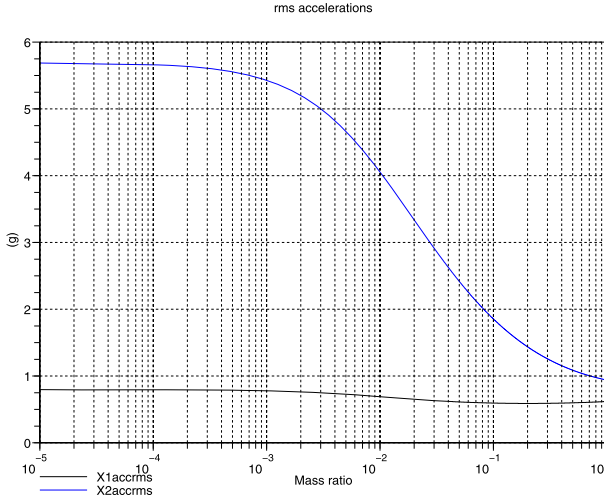


Fig. 2.38. rms accelerations (g) for x_1 and x_2 and $\phi = 1$

ζ_2 , m_1 and m_2 . In general, both damping ratio's ζ_1 , ζ_2 are made equal $\zeta_1 = \zeta_2 = \zeta$.

The rms values of the accelerations $\langle \ddot{x}_1 \rangle$ and $\langle \ddot{x}_2 \rangle$ are calculated using the damping ratio's $\zeta_1 = \zeta_2 = \zeta = 0.05$ and the PSD of the enforced acceleration is $W_{\ddot{u}} = 0.02 \text{ g}^2/\text{Hz}$. The ratio $\phi = 1$, corresponds to the tuned case. The rms accelerations are dependent on the mass ratio γ . The rms accelerations are shown in Fig. 2.38. The rms acceleration response $\langle \ddot{x}_1 \rangle$ is hardly influenced by SDOF system 2 (m_2).

2.7.6 Limit Load Factors

To calculate the limit load factors or the mass acceleration curve 3σ values of the rms acceleration $\langle \ddot{x}_2 \rangle$ shall be considered and the quasi-static loads (QSL) from the launch vehicle User's manual added to that 3σ values. Consider further a mass $m_1 = 500 \text{ kg}$ and the $QSL = 10 \text{ g}$ the LLF is (2.339)

$$LLF = 3\langle \ddot{x}_{2,\phi=1} \rangle + QSL.$$

The limit load factors or the mass acceleration curve is shown in Fig. 2.39.

Another method to calculate the LLF is to calculate the rms acceleration response of $\langle \ddot{x}_1 \rangle$ independent of x_1 . The rms response of the SDOF 1 system can be obtained using Miles' equation

$$\langle \ddot{x}_1 \rangle = \sqrt{\frac{\pi}{2} f_1 Q_1 W_{\ddot{u}}(f_1)}. \tag{2.357}$$

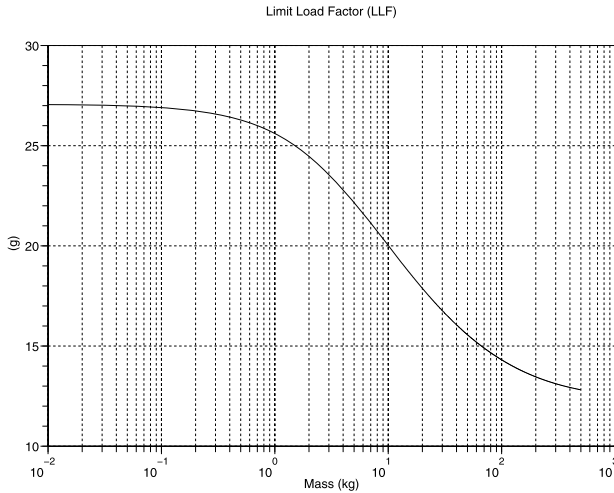


Fig. 2.39. LLF (g) $m_1 = 500$ kg and $\phi = 1$

This approach is regularly used because the definition of the SDOF 1 is not always clear.

The limit load factor definition is now

$$LLF = 3Q_{fb,\phi=1} \sqrt{\frac{\pi}{2} f_1 Q_1 W_{\ddot{u}(f_1)} + QSL}, \tag{2.358}$$

where f_1 is the (assumed or specified) natural frequency (Hz) and Q_1 is the amplification factor (mostly taken as $Q_1 = 10$) and $Q_{fb,\phi=1}$ can be taken from Fig. 2.37. Notice that the latest approach to calculate the LLFs is conservative.

2.8 Force-Limit Prediction Methods

2.8.1 Introduction

In conventional vibration testing of space hardware, the acceleration input to the base (enforced acceleration) of the test item is controlled by the specifications, namely, the envelope of the acceleration peaks of the flight environment. This conventional approach to testing has been known for decades to greatly overtest the test item at its own resonant frequencies [189]. Space structures are normally designed to survive vibration testing as well as the flight environment, thus this overtesting phenomenon normally leads directly in overdesign.

An improved vibration test approach was developed by T.D. Scharton [135, 174, 175]. This approach is called *force-limiting vibration testing* (FLVT). The FLVT measures and limits the reaction forces between the test item and the shaker through real-time notching of the input acceleration.

To estimate the force limits and specification, two mathematical approaches and one empirical method were developed:

- Simple two coupled SDOF method
- Complex two coupled SDOF method
- Semi-empirical method

These mathematical and semi-empirical methods will be discussed in subsequent sections.

2.8.2 Simple Coupled 2DOF System

The source and the load are idealized as lumped masses connected to each other by corresponding springs and dampers. The simple 2DOF source-load system is shown in Fig. 2.40. The damped equations of motion can be derived from Lagrange's equations

$$\begin{aligned} m_1 \ddot{x}_1 + k_1(x_1 - u) - k_2(x_2 - x_1) + c_1(\dot{x}_1 - \dot{u}) - c_2(\dot{x}_2 - \dot{x}_1) &= 0, \\ m_2 \ddot{x}_2 + k_2(x_2 - x_1) + c_2(\dot{x}_2 - \dot{x}_1) &= 0. \end{aligned} \quad (2.359)$$

Equation (2.359) will be transformed into the frequency domain assuming $x_i(t) = X_i(\omega)e^{j\omega t}$ and $u(t) = U(\omega)e^{j\omega t}$, thus

$$\begin{aligned} \begin{pmatrix} -\omega^2 m_1 + j\omega(c_1 + c_2) + k_1 + k_2 & -(j\omega c_2 + k_2) \\ -(j\omega c_2 + k_2) & -\omega^2 m_2 + j\omega c_2 + k_2 \end{pmatrix} \begin{pmatrix} X_1(\omega) \\ X_2(\omega) \end{pmatrix} \\ = - \begin{pmatrix} \frac{k_1}{\omega^2} + j\frac{c_1}{\omega} \\ 0 \end{pmatrix} \ddot{U}(\omega) \end{aligned} \quad (2.360)$$

or when expressed in the accelerations by multiplying (2.360) with $-\omega^2$ will yield

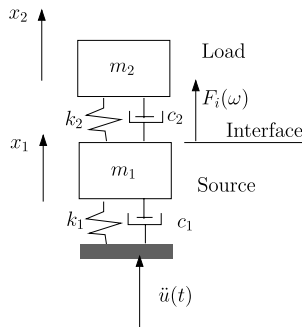


Fig. 2.40. Simple coupled 2DOF Source-Load dynamic system

$$\begin{aligned}
& \begin{pmatrix} -\omega^2 m_1 + j\omega(c_1 + c_2) + k_1 + k_2 & -(j\omega c_2 + k_2) \\ -(j\omega c_2 + k_2) & -\omega^2 m_2 + j\omega c_2 + k_2 \end{pmatrix} \begin{pmatrix} \ddot{X}_1(\omega) \\ \ddot{X}_2(\omega) \end{pmatrix} \\
& = \begin{pmatrix} k_1 + j\omega c_1 \\ 0 \end{pmatrix} \ddot{U}(\omega).
\end{aligned} \tag{2.361}$$

The accelerations $\ddot{X}_1(\omega)$ and $\ddot{X}_2(\omega)$ can be expressed in terms of $\ddot{U}(\omega)$, and can be written as follows

$$\begin{aligned}
\ddot{X}_1(\omega) &= H_{\ddot{X}_1}(\omega) \ddot{U}(\omega), \\
\ddot{X}_2(\omega) &= H_{\ddot{X}_2}(\omega) \ddot{U}(\omega),
\end{aligned} \tag{2.362}$$

where $H_{\ddot{X}_1}(\omega)$ and $H_{\ddot{X}_2}(\omega)$ are the transmissibility functions. Solving (2.361), we find the transmissibility functions

$$\begin{aligned}
H_{\ddot{X}_1}(\omega) &= \frac{[-j\omega^3 2\zeta_1 \omega_1 - \omega^2(\omega_1^2 + 4\zeta_1 \zeta_2 \omega_1 \omega_2) + j\omega(2\zeta_1 \omega_1 \omega_2^2 + 2\zeta_2 \omega_1^2 \omega_2) + \omega_1^2 \omega_2^2]}{D}, \\
H_{\ddot{X}_2}(\omega) &= \frac{[-\omega^2 4\zeta_1 \zeta_2 \omega_1 \omega_2 + j\omega(2\zeta_1 \omega_1 \omega_2^2 + 2\zeta_2 \omega_1^2 \omega_2) + \omega_1^2 \omega_2^2]}{D},
\end{aligned} \tag{2.363}$$

where the determinant D of the matrix of (2.360) is given by

$$\begin{aligned}
D &= m_1 m_2 [\omega^4 - j\omega^3(2\zeta_1 \omega_1 + 2(1 + \mu)\zeta_2 \omega_2) \\
&\quad - \omega^2(\omega_1^2 + (1 + \mu)\omega_2^2 + 4\zeta_1 \zeta_2 \omega_1 \omega_2) \\
&\quad + j\omega(2\zeta_1 \omega_1 \omega_2^2 + 2\zeta_2 \omega_2 \omega_1^2) + \omega_1^2 \omega_2^2],
\end{aligned} \tag{2.364}$$

and

- $\omega_1 = \sqrt{k_1/m_1}$,
- $\omega_2 = \sqrt{k_2/m_2}$,
- $\mu = m_2/m_1$,
- $c_1 = 2\zeta_1 \sqrt{k_1 m_1}$,
- $c_2 = 2\zeta_2 \sqrt{k_2 m_2}$.

The interface load $F_i(\omega)$ (see Fig. 2.40) is given by

$$\begin{aligned}
F_i(\omega) &= k_2(x_2 - x_1) + c_2(\dot{x}_2 - \dot{x}_1) \\
&= k_2 \left(\frac{H_{\ddot{X}_2}(\omega)}{H_{\ddot{X}_1}(\omega)} - 1 \right) X_1(\omega) + c_2 \left(\frac{H_{\ddot{X}_2}(\omega)}{H_{\ddot{X}_1}(\omega)} - 1 \right) \dot{X}_1(\omega) \\
&= -\frac{1}{\omega^2} (\omega_2^2 + 2j\zeta_2 \omega \omega_2) \left(\frac{H_{\ddot{X}_2}(\omega)}{H_{\ddot{X}_1}(\omega)} - 1 \right) \ddot{X}_1(\omega) \\
&= m_2 \ddot{X}_1(\omega) \frac{H_{\ddot{X}_2}(\omega)}{H_{\ddot{X}_1}(\omega)} = M_2(\omega) \ddot{X}_1(\omega),
\end{aligned} \tag{2.365}$$

where $M_2(\omega)$ is the dynamic mass of the “load” and \ddot{x}_1 the acceleration of the “source”.

The PSD of the interface load $W_{F_i F_i}(f)$ can be calculated with the following expression

$$W_{F_i F_i}(f) = |M_2(2\pi f)|^2 W_{x_1 x_1}(f), \tag{2.366}$$

where $W_{x_1 x_1}(f)$ is the PSD of the interface acceleration x_1 dependent on the cyclic frequency f (Hz).

In the case $\omega_1 = \omega_2 = \omega_0$, $m_1 = m_2 = k_1 = k_2 = 1$, $\ddot{U}(\omega) = 1$ and $\zeta_1 = \zeta_2 = 0.01$ the magnitudes of the interface force $F_i(\omega)$, the interface acceleration $\ddot{X}_1(\omega)$ and the dynamic mass of the “load” $M_2(\omega)$ will be calculated and shown in Figs. 2.41, 2.42 and 2.43. This example, shows that both the interface force and interface acceleration peak at the *resonant frequencies* (maximum responses) of the coupled source-load system. The load dynamic mass, however, peaks at the resonant frequency of the load uncoupled from the source, a point corresponding to the *antiresonance frequency* of the coupled system. It is important to note that the load dynamic mass evaluated at the system resonant frequencies can be considerably less than when evaluated at the system antiresonance frequency.

The underlying basis of the *frequency-shift method* is now easily understood. In a conventional vibration test, the high impedance shaker will deliver a peak interface force equal to the product of the acceleration spectral envelope and the peak value of the load dynamic mass; this occurs at the system antiresonance frequency

$$\begin{aligned} |F(\omega)|_{peak} &= |M_2(\omega)|_{peak} |a(\omega)|_{peak}, \quad \text{or} \\ |F(\omega)|_{peak} &= |M_2(\omega)|_{antiresonance} |a(\omega)|_{resonant}, \end{aligned} \tag{2.367}$$

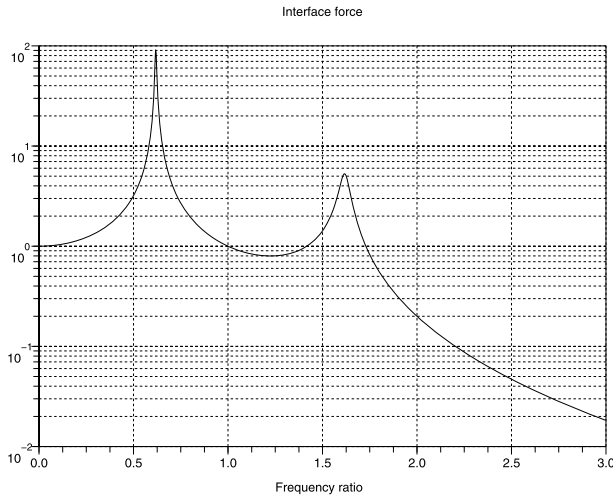


Fig. 2.41. Interface force

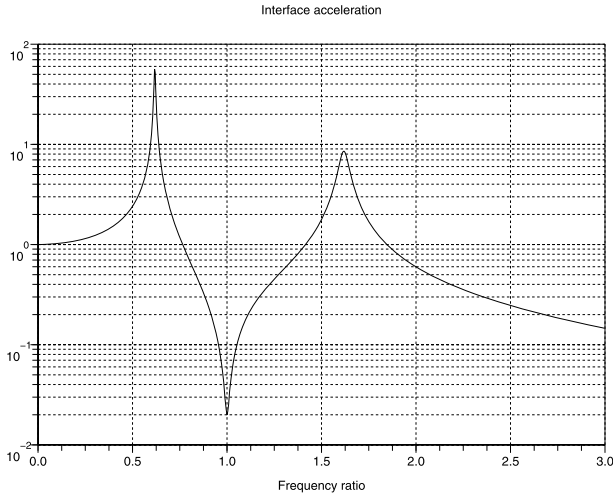


Fig. 2.42. Interface acceleration

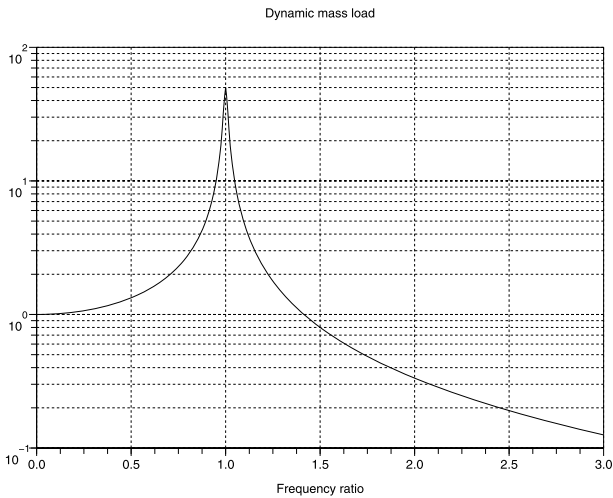


Fig. 2.43. Dynamic mass $M_2(\omega)$

where $a(\omega) = \ddot{x}_1(\omega)$ is the interface acceleration. In contrast, for a force-limited vibration test, the frequency-shift method, the shaker is limited to delivering a peak interface force equal to the product of the acceleration spectral envelope and the frequency-shifted values of the load dynamic mass evaluated at the system resonant frequencies

$$\begin{aligned}
 |\tilde{F}(\omega)|_{peak} &= |M_2(\omega)|_{frequency-shifted} |a(\omega)|_{peak}, \quad \text{or} \\
 |\tilde{F}(\omega)|_{peak} &= |M_2(\omega)|_{resonance} |a(\omega)|_{resonant},
 \end{aligned}
 \tag{2.368}$$

where the tildes denote different values for the peak force. This limiting method can accomplish a significant reduction in the interface force.

Example. Applying (2.367) to the example Figs. 2.41, 2.42 and 2.43, we find yields a peak interface force $|F| = 50 \times 50 = 2500$. Applying (2.368) to the same example yields a peak interface force $|\tilde{F}| = 1.6 \times 50 = 80$, a substantial reduction. This reduction in interface force corresponds to a notch factor at the system resonant frequency of about 31 or 15 dB in the interface acceleration spectrum.

Using the frequency-shift technique we assume that the control acceleration spectrum $|a_c(\omega)|$ properly envelopes the interface accelerations $|a(\omega)|_{peak}$ seen in service. Consequently

$$\begin{aligned} |F_c(\omega)| &= |M_2(\omega)|_{resonance} |a_c(\omega)| \quad \text{or} \\ W_{F_c F_c}(f) &= |M_2(2\pi f)|_{resonance}^2 W_{a_c a_c}(f). \end{aligned} \quad (2.369)$$

Equation (2.366) will be rearranged such that

$$\frac{W_{F_i F_i}(f)}{|M_2(2\pi f)|^2 W_{x_1 x_1}(f)} = \frac{W_{F_i F_i}(f)}{m_2^2 \left| \frac{H_{\ddot{x}_2}(2\pi f)}{H_{\ddot{x}_1}(2\pi f)} \right|^2 W_{x_1 x_1}(f)} = 1, \quad (2.370)$$

or

$$\frac{W_{F_i F_i}(f)}{m_2^2 W_{x_1 x_1}(f)} = \left| \frac{H_{\ddot{x}_1}(2\pi f)}{H_{\ddot{x}_2}(2\pi f)} \right|^2. \quad (2.371)$$

For a lightly damped system, the undamped natural frequencies are about the same as the damped natural frequencies. The undamped natural frequencies will be calculated from (2.359) neglecting the damping. The characteristic determinant can now be written

$$\begin{vmatrix} k_1 + k_2 - \lambda m_1 & -k_2 \\ -k_2 & k_2 - \lambda m_2 \end{vmatrix} = 0, \quad (2.372)$$

so that the characteristic equation is

$$(k_1 + k_2 - \lambda m_1)(k_2 - \lambda m_2) - k_2^2 = 0. \quad (2.373)$$

The roots of (2.373) are undamped eigenvalues $\lambda_{1,2}$. They can be written

$$\lambda_{1,2} = \frac{1}{2} \{ \omega_1^2 (1 + \mu) + \omega_2^2 \} \pm \frac{1}{2} \sqrt{ \{ \omega_1^2 (1 + \mu) + \omega_2^2 \}^2 - 4 \omega_1^4 \omega_2^4 }, \quad (2.374)$$

where $\omega_1^2 = k_1/m_1$, $\omega_2^2 = k_2/m_2$ and the mass ratio is $\mu = m_2/m_1$.

The maximum interface for $F_i(\omega)$ is at the lowest undamped natural frequency $\omega = \sqrt{\lambda_1}$ (see Fig. 2.41) when $\omega_1 = \omega_2 = \omega_0$

$$\lambda_1 = \omega_0^2 \left(1 + \frac{\mu}{2} - \sqrt{ \mu + \frac{\mu^2}{4} } \right). \quad (2.375)$$

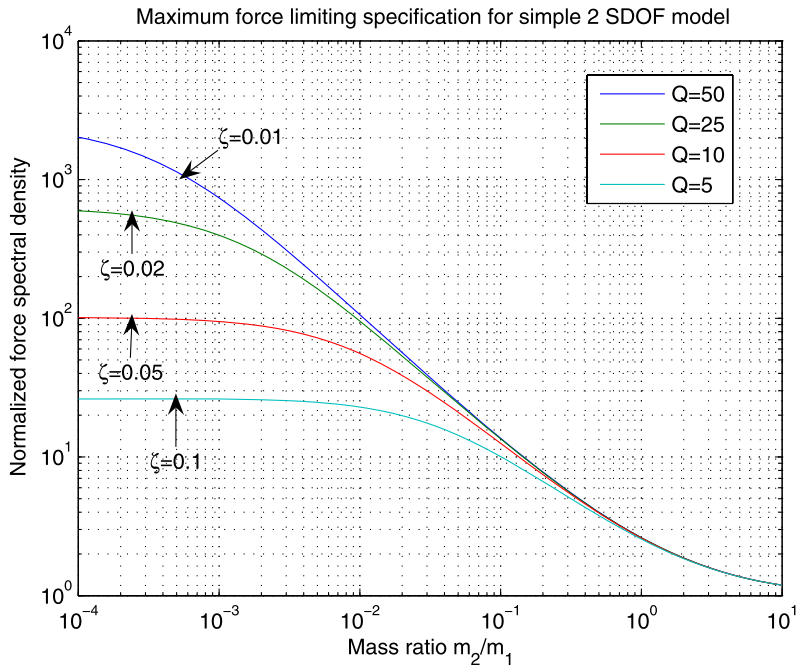


Fig. 2.44. Maximum force limiting specification for 2DOF model

The “load” will then act as a vibration absorber.

We now use (2.371) to find the lowest natural frequency $\sqrt{\lambda_1} = \omega_{\lambda_1} = 2\pi f_{\lambda_1}$ for varying the mass ratio μ and damping ratio $\zeta_1 = \zeta_2 = \zeta$. Thus

$$\frac{W_{F_i F_i}(f_{\lambda_1})}{m_2^2 W_{\ddot{x}_1 \ddot{x}_1}(f_{\lambda_1})} = \left| \frac{H_{\ddot{x}_1}(2\pi f_{\lambda_1})}{H_{\ddot{x}_2}(2\pi f_{\lambda_1})} \right|^2, \quad \mu = 0.0001..10, \tag{2.376}$$

$$\zeta = 0.01, 0.02, 0.05, 0.1.$$

Equation (2.376) is visualized in Fig. 2.44. For small values of μ the normalized force PSD approaches $Q^2 = 1/4\zeta^2$. For very small mass ratios the feedback of the load to the source via the *vibration absorber* mechanism is negligible or the source mass provides a high relative mechanical impedance interface to the load mass. Consequently, the maximum response of the load mass will approach its resonant response, or for random vibration, $W_{FF}(f) = m_2^2 Q^2 W_{\ddot{x}_1 \ddot{x}_1}(f)$. At larger values of μ , force limiting begins to occur because the feedback of the load mass to the source mass via the vibration absorber mechanism becomes greater the source mass provides a lower relative mechanical impedance interface to the load mass.

The simple 2DOF system is a useful initial model for developing force limiting criteria according to the frequency-shift method. Unfortunately, by treating both the source and load as simple SDOF systems, this model is

inherently unable to describe the modal behavior of more realistic distributed mass systems. For a given mode in such an MDOF system, there is modal effective mass that participates in the motion, and a complementary residual mass that does not. This more complex behavior can be imperfectly incorporated into the existing 2DOF model by treating distributed masses as frequency dependent lumped masses and using the smoothed dynamic mass as an estimate of the residual mass. The lumped masses of the simple 2DOF are not an accurate representation of either the modal effective or residual masses for a MDOF system. A more sophisticated model able to account for both resonant and antiresonance structural modes is required to more accurately model a MDOF source-load vibratory system. This model is called the complex 2DOF system.

Distributed structures exhibit several modes of vibration in the frequency band of interest. To build the 2DOF system we have to select the modes for both the source and the load in that frequency band, and that is not always straightforward. The width of these frequency band is traditionally one-third of an octave.

Example. This example is taken from [97]. The KOMPSAT-II earth observation satellite of 800 kg has a Multi-Spectral Camera (MSC) as a main payload (P/L). The MSC has a mass of 76.7 kg. The force limit specification had been established by applying the simple 2DOF model using Fig. 2.44 ($Q = 5$). The lateral modal effective masses for the source m_1 and the load m_2 are calculated by the finite element method. The results of the calculation are presented in Table 2.10.

Table 2.10. Modal effective masses of the satellite bus, and the MSC and force limits

1/3-Oct. frequency band (Hz)	Satellite Modal effective mass (kg) m_1 (kg)	MSC Modal effective mass m_2 (kg)	m_2/m_1	Normalized force spectrum ($Q = 5$)
31.5	0.18			
40	0.62			
50	1.77			
63	338.80			
80	2.29			
100	126.14			
125	120.66	0.48	0.004	25
160	3.98			
200	4.96	1.74	0.35	16
250	2.29	23.03	10.05	1.1
315	1.67	28.02	16.78	1
400	0.42	13.74	32.34	1
500	0.11	0.91	8.45	1.2

2.8.3 Complex Coupled Two-SDOF System

In Fig. 2.45 the two-SDOF system of the so-called source and the so-called load are illustrated in Fig. 2.45. The capital M represents the residual mass, and the small m^* the represents the modal effective mass. The theory behind the modal effective mass and the residual mass is discussed in Appendix D. The dampers and the springs are denoted by c^* and k^* respectively, because they represent mathematical, no physical, values. The following theoretical discussion is based on [48]. The differential equations of motion can be derived using Lagrange’s method. The external force $F_{ext} = 0$ for the time being.

$$\begin{aligned} (M_1 + M_2)\ddot{x} - k_1^*(x_1 - x) - k_2^*(x_2 - x) - c_1^*(\dot{x}_1 - \dot{x}) - c_2^*(\dot{x}_2 - \dot{x}) &= 0, \\ m_1^*\ddot{x}_1 + k_1^*(x_1 - x) + c_1^*(\dot{x}_1 - \dot{x}) &= 0, \\ m_2^*\ddot{x}_2 + k_2^*(x_2 - x) + c_2^*(\dot{x}_2 - \dot{x}) &= 0. \end{aligned} \tag{2.377}$$

The first equation of motion in (2.377) will be decomposed into symmetric SDOF systems:

$$[M_1\ddot{x} - c_1^*(\dot{x}_1 - \dot{x}) - k_1^*(x_1 - x)] + [M_2\ddot{x} - k_2^*(x_2 - x) - c_2^*(\dot{x}_2 - \dot{x})] = 0, \tag{2.378}$$

$$F_{source} + F_{load} = F_{int} - F_{int} = 0,$$

where F_{int} in the interface force between M_1 and M_2 . The interface force can be expressed as follows using (2.377)

$$\begin{aligned} F_{int} &= M_2\ddot{x} - k_2^*(x_2 - x) - c_2^*(\dot{x}_2 - \dot{x}), \\ F_{int} &= M_2\ddot{x} + m_2^*\ddot{x}_2. \end{aligned} \tag{2.379}$$

Using the third Fourier transformed equation of motion in (2.377) we can express the acceleration $\ddot{X}_2(\omega)$ can in terms of $\ddot{X}(\omega)$

$$\ddot{X}_2(\omega) = \left(\frac{\omega_2^2 + 2j\zeta_2\omega\omega_2}{-\omega^2 + \omega_2^2 + 2j\zeta_2\omega\omega_2} \right) \ddot{X}(\omega), \tag{2.380}$$

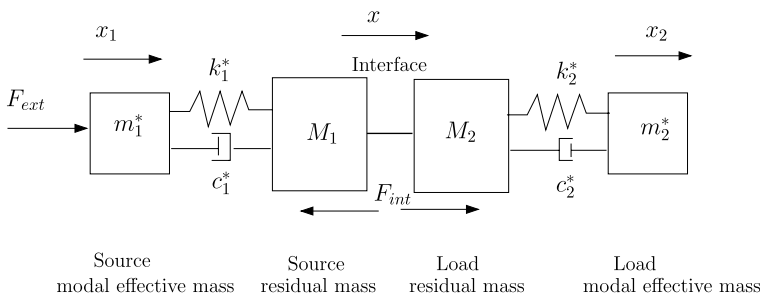


Fig. 2.45. Modal effective and residual mass two-SDOF source-load system

where the natural frequency $\omega_2^2 = k_2/m_2^*$, the viscous damping is $c_2^* = 2j\zeta_2\omega_2$ and the damping ratio is $\zeta_2 = c_2/2\sqrt{k_2^*m_2^*}$. Equation (2.379) gives the interface force in the frequency domain $F_{int}(\omega)$:

$$F_{int}(\omega) = \left(M_2 + m_2^* \frac{\omega_2^2 + 2j\zeta_2\omega\omega_2}{-\omega^2 + \omega_2^2 + 2j\zeta_2\omega\omega_2} \right) \ddot{X}(\omega). \quad (2.381)$$

The quantity between the brackets in (2.381) corresponds to the load dynamic mass $M_2(\omega)$:

$$\begin{aligned} |M_2(\omega)|^2 &= M_2^2 \frac{[(1 - \beta_2^2) + \alpha_2]^2 + \frac{\beta_2^2(1 + \alpha_2)^2}{Q_2^2}}{(1 - \beta_2^2)^2 + \frac{\beta_2^2}{Q_2^2}} \\ &= M_2^2 \frac{[(1 - \beta_2^2) + \alpha_2]^2 + 4\zeta_2^2\beta_2^2(1 + \alpha_2)^2}{(1 - \beta_2^2)^2 + 4\zeta_2^2\beta_2^2}, \end{aligned} \quad (2.382)$$

where $\alpha_1 = \frac{m_1^*}{M_1}$, $\alpha_2 = \frac{m_2^*}{M_2}$, $\beta_1 = \frac{\omega}{\omega_1}$, $\beta_2 = \frac{\omega}{\omega_2}$, $Q_1 = 1/2\zeta_1$ and $Q_2 = 1/2\zeta_2$. Because of the symmetry in (2.378) the expression for the source dynamic mass $M_1(\omega)$ will have exactly the same form as (2.382). Equation (2.382) must be expressed in the systems natural frequencies to determine the general force limits.

The governing equation of the dynamic system illustrated in Fig. 2.45 is

$$\begin{aligned} &\begin{pmatrix} M_1 + M_2 & 0 & 0 \\ 0 & m_1^* & 0 \\ 0 & 0 & m_2^* \end{pmatrix} \begin{pmatrix} \ddot{x} \\ \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} + \begin{pmatrix} c_1^* + c_2^* & -c_1^* & -c_2^* \\ -c_1^* & c_1^* & 0 \\ -c_2^* & 0 & c_2^* \end{pmatrix} \begin{pmatrix} \dot{x} \\ \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} \\ &+ \begin{pmatrix} k_1^* + k_2^* & -k_1^* & -k_2^* \\ -k_1^* & k_1^* & 0 \\ -k_2^* & 0 & k_2^* \end{pmatrix} \begin{pmatrix} x \\ x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \end{aligned} \quad (2.383)$$

The characteristic undamped equation for the eigenvalue problem is

$$\begin{vmatrix} k_1^* + k_2^* - \omega^2(M_1 + M_2) & -k_1^* & -k_2^* \\ -k_1^* & k_1^* - \omega^2 m_1^* & 0 \\ -k_2^* & 0 & k_2^* - \omega^2 m_2^* \end{vmatrix} = 0, \quad (2.384)$$

which yields

$$\begin{aligned} &\omega^2[(M_1 + M_2)(k_1^* - \omega^2 m_1^*)(k_2^* - \omega^2 m_2^*) \\ &+ k_2^* m_2^*(k_1^* - \omega^2 m_1^*) + k_1^* m_1^*(k_2^* - \omega^2 m_2^*)] = 0. \end{aligned} \quad (2.385)$$

Equation (2.385) has one root $\omega = 0$ corresponding to the rigid body motion. The remaining two roots are given by

$$\begin{aligned} &(1 - \Omega^2\beta_2^2)(1 - \beta_2^2) + \alpha_1(1 - \beta_2^2) \\ &+ \mu(1 - \Omega^2\beta_2^2)(1 - \beta_2^2) + \mu\alpha_2(1 - \Omega^2\beta_2^2) = 0, \end{aligned} \quad (2.386)$$

where $\Omega = \omega_2/\omega_1$ and $\mu = M_2/M_1$. Equation (2.386) has two real roots, β_+ and β_- , corresponding to the upper and lower natural frequencies. Ω is called the *tuning parameter*. The natural frequencies β_2^2 can be obtained as follows [175]

$$\beta_2^2 = \frac{1}{2}[-B \pm \sqrt{B^2 - 4C}], \quad (2.387)$$

where

$$B = \frac{-(1 + \mu + \alpha_1)/\Omega^2 - (1 + \mu + \mu\alpha_2)}{1 + \mu},$$

$$C = \frac{(1 + \mu + \alpha_1 + \mu\alpha_2)}{(1 + \mu)\Omega^2}.$$

At this point, the development of force limits for the complex two-SDOF model becomes more involved than for the simple two-SDOF system. It was shown that the interface force and the interface acceleration peak at the same system natural frequencies ($\omega_1 = \omega_2$). This condition does not always hold for the complex two-SDOF system: the maximum interface force may occur at upper natural frequency while the maximum interface acceleration may occur at the lower natural frequency. Consequently, the natural frequencies at which the respective maxima occur, must be identified. The bounding value of the load dynamic mass will then be the ratio of the larger of the two interface forces to the larger of the two interface accelerations. This task is further complicated by the fact that the maximum magnitude of the interface acceleration is dependent upon how the complex two-SDOF system is excited:

1. A constant free acceleration over the system frequency band applied to the source modal effective mass m_1^* .
2. A constant blocked force over the system frequency band applied to the source modal effective mass m_1^* .
3. A constant external force over the system frequency band applied to the source modal effective mass m_1^* .

Scharton [174] chose an external force F_{ext} that is constant over the system's frequency band applied to the source modal effective mass m_1^* . Expressions for the peak interface acceleration will now be derived based in constant source excitation.

Thevenin's theorem (Appendix E) is used to derive an expression relating the free acceleration to the interface acceleration for the source residual mass M_1 . It is convenient to replace the mechanical impedance $Z(\omega)$ by the dynamic mass $M(\omega)$ and the free velocity $v_0(\omega)$ by the free acceleration $A_0(\omega)$. Applying Thevenin's theorem to the residual masses, we find

$$F_0(\omega) = A(\omega)(M_1(\omega) + M_2(\omega)), \quad (2.388)$$

keeping mind that

$$F_0(\omega) = A_0(\omega)M_1(\omega), \quad (2.389)$$

where $A_0(\omega)$ is the free acceleration of the source. The blocked force $F_0(\omega)$ can be eliminated from (2.388) and (2.389):

$$\frac{A(\omega)}{A_0(\omega)} = \frac{M_1(\omega)}{M_1(\omega) + M_2(\omega)}. \quad (2.390)$$

The dynamic mass of (2.381) can be substituted in (2.390), leading to a rather involved algebraic expression for the ratio of the interface to the free acceleration, which will be simply denoted here as

$$\left| \frac{A(\omega)}{A_0(\omega)} \right|^2 = D(\alpha_1, \alpha_2, \mu, \beta_1, \beta_2, \zeta_1, \zeta_2). \quad (2.391)$$

The dynamic masses $M_1(\omega)$ and $M_2(\omega)$ are given by

$$\begin{aligned} M_1(\omega) &= M_1 \left[1 + \alpha_1 \frac{\omega_1^2 + 2j\zeta_1\omega\omega_1}{-\omega^2 + \omega_1^2 + 2j\zeta_1\omega\omega_1} \right] \\ &= M_1 \left[1 + \alpha_1 \frac{1 + 2j\zeta_1\beta_1}{1 - \beta_1^2 + 2j\zeta_1\beta_1} \right], \\ M_2(\omega) &= M_2 \left[1 + \alpha_2 \frac{\omega_2^2 + 2j\zeta_2\omega\omega_2}{-\omega^2 + \omega_2^2 + 2j\zeta_2\omega\omega_2} \right] \\ &= M_2 \left[1 + \alpha_2 \frac{1 + 2j\zeta_2\beta_2}{1 - \beta_2^2 + 2j\zeta_2\beta_2} \right]. \end{aligned} \quad (2.392)$$

The equations of motion (2.377) can be used to derive an expression for the free acceleration of the source residual mass M_1 . For an external force applied at the source modal effective mass m_1^* , (2.377) becomes

$$m_1^* \ddot{x}_1 + c_1(\dot{x}_1 - \dot{x}) + k_1(x_1 - x) = F_{ext}, \quad (2.393)$$

where the parameters are shown in Fig. 2.45.

The free acceleration of the source residual mass will be the motion with the load removed, thus the equation of equilibrium for the residual mass M_1 becomes

$$M_1 \ddot{x} - c_1(\dot{x}_1 - \dot{x}) - k_1(x_1 - x) = 0. \quad (2.394)$$

Adding (2.393) and (2.394) we eliminate the damping and stiffness terms:

$$F_{ext}(\omega) = M_1 \ddot{X}(\omega) + m_1^* \ddot{X}_1(\omega). \quad (2.395)$$

The acceleration $\ddot{X}(\omega)$ is in fact the free acceleration $A_0(\omega)$. The acceleration \ddot{X}_1 can be expressed in terms of \ddot{X} , and (2.395) can be written as

$$F_{ext}(\omega) = \left[m_1^* \frac{-\omega^2 M_1 + 2j\zeta_1\omega_1\omega + \omega_1^2}{2j\zeta_1\omega_1\omega + \omega_1^2} + M_1 \right] \ddot{X}(\omega), \quad (2.396)$$

or

$$\frac{F_{ext}(\omega)}{m_1^*} = \left[\frac{1}{\alpha_1} + \frac{\frac{1-\beta_1^2}{\alpha_1} + 2j\zeta_1\beta_1}{1 + 2j\zeta_1\beta_1} \right] \ddot{X}(\omega). \quad (2.397)$$

Using $\ddot{X}(\omega) = A_0(\omega)$ we find

$$\begin{aligned} \left| \frac{A_0(\omega)}{\frac{F_{ext}(\omega)}{m_1^*}} \right|^2 &= \frac{\beta_1^4(1 + 4\beta_1^2\zeta_1^2)}{[(1 - \beta_1^2)(1 - \frac{\beta_1^2}{\alpha_1}) - 1]^2 + 4\beta_1^6\zeta_1^2(1 + \frac{1}{\alpha_1})^2} \\ &= G(\alpha_1, \beta_1, \zeta_1). \end{aligned} \quad (2.398)$$

The acceleration A_0 can be eliminated from (2.391) and (2.398), thus

$$\left| \frac{A(\omega)}{\frac{F_{ext}(\omega)}{m_1^*}} \right|^2 = D(\alpha_1, \alpha_2, \mu, \beta_1, \beta_2, \zeta_1, \zeta_2)G(\alpha_1, \beta_1, \zeta_1). \quad (2.399)$$

The external force $F_{ext}(\omega)$ will be held constant over the frequency band of interest, thus (2.399) can be evaluated from (2.386) at the upper and lower system natural frequencies β_+ and β_- to yield the ratio of the corresponding system acceleration peaks

$$\left| \frac{A_+}{A_-} \right|^2 = \frac{D(\alpha_1, \alpha_2, \mu, \beta_{1+}, \beta_{2+}, \zeta_1, \zeta_2)G(\alpha_1, \beta_{1+}, \zeta_1)}{D(\alpha_1, \alpha_2, \mu, \beta_{1-}, \beta_{2-}, \zeta_1, \zeta_2)G(\alpha_1, \beta_{1-}, \zeta_1)}. \quad (2.400)$$

The damping ratio of the source and the load are equal, $\zeta_1 = \zeta_2 = \zeta$. β_1 will be eliminated, and $\Omega = \frac{\omega_2}{\omega_1} = \frac{\beta_1}{\beta_2}$. We will write (2.400) as

$$\left| \frac{A_+}{A_-} \right|^2 = H(\alpha_1, \alpha_2, \mu, \Omega, \beta_{2\pm}, \zeta). \quad (2.401)$$

The purpose of deriving the ratio of the resonant interface accelerations is to properly identify the locations of the force and accelerations maxima; the proper “load” dynamic mass will be the ratio of the larger of the two interface forces to the larger of the two interface accelerations. This is illustrated by the following equations

$$\begin{aligned} F_+ &= M_{2+}A_+ = M_{2+}\frac{A_+}{A_-}A_-, \\ F_- &= M_{2-}A_- = M_{2-}\frac{A_-}{A_+}A_+. \end{aligned} \quad (2.402)$$

Table 2.11 summarizes the possible outcomes of this ratio for all combinations of natural frequency locations of the force and the acceleration maxima, where the + and the - denote the higher and lower natural frequencies respectively. The last row in Table 2.11 shows the proper force to acceleration ratio will be the maximum of the load dynamic mass $M_2(\omega)$ evaluated at either the higher acceleration peak or the lower acceleration peak scaled by the ratio

Table 2.11. Possible ratios of the maximum force to maximum acceleration

Position	A_{max}, F_{max}	A_-, F_-	A_-, F_+	A_+, F_-	A_+, F_+
F_{max}		$M_{2-}A_-$	$M_{2+}A_+$	$M_{2-}A_-$	$M_{2+}A_+$
Ratio	$\frac{F_{max}}{A_{max}}$	$\frac{M_{2-}A_-}{A_-} = M_{2-}$	$\frac{M_{2+}A_+}{A_-}$	$\frac{M_{2-}A_-}{A_+}$	$\frac{M_{2+}A_+}{A_+} = M_{2+}$

of the lower to higher acceleration. Symbolically, this logic can be expressed as

```

if  $|A_+| > |A_-|$  then
     $|\bar{M}_2(\beta)| = \max\{|M_2(\beta_+)|, |M_2(\beta_-)|\} \frac{A_-}{A_+}$ 
else if  $|A_+| < |A_-|$  then
     $|\bar{M}_2(\beta)| = \max\{|M_2(\beta_-)|, |M_2(\beta_+)|\} \frac{A_+}{A_-}$ 
end if
    
```

$\bar{M}_2(\beta)$ is the scaled apparent or dynamic mass. It is clear that both β_{\pm} and A_{\pm} depend on the tuning parameter Ω . In the simple two-SDOF, the tuning parameter $\Omega = 1$ always yields the maximum value for the “load” dynamic mass, unfortunately, this is not generally true for the complex coupled two-SDOF system. Consequently, $|\bar{M}_2(\omega)|$, in the algorithm above must be maximized by numerically tuning Ω for all possible combinations of mass, stiffness, and damping for the complex two-SDOF system. This process is summarized as follows

1. Start procedure
2. Pick $\alpha_1 = m_1^*/M_1$, $\alpha_2 = m_2^*/M_2$ and $\mu = M_2/M_1$
3. Solve characteristic equation for $\beta_{2\pm}$ ($\beta_2 = \omega/\omega_2$)
4. For a given Ω , determine $\beta_{1\pm}$ ($\beta_1 = \omega/\omega_1$)
5. Evaluate the dynamic masses $M_1(\beta_{\pm}), M_2(\beta_{\pm})$
6. Evaluate $|A_+/A_-|$,
7. Determine F_{max} from A_{\pm} and scaled $M_2(\beta_{\pm})$
8. if $F_{max,new} < F_{max,old}$ then
 - Pick new Ω , goto 4
 - else
 - Save $F_{max,new}$, goto 2
 - end if
9. End of procedure

The relation between interface force PSD and interface acceleration PSD is given by (2.366). The magnitude squared of the load dynamic mass $M_2(\omega)$ is given by (2.382). We can calculate the normalized force limit PSD $W_{FF}(f)$ normalized by the “load” mass M_2 and acceleration PSD $W_{AA}(f)$:

$$\frac{W_{FF}(f)}{M_2^2 W_{AA}(f)}$$

Analysis results of the maximum normalized interface force PSD are discussed and reported in [175]. The results are shown in Table 2.12. Ω^2 is varied in

Table 2.12. Maximum force spectral density $\frac{W_{FF}(f)}{M_2^2 W_{AA}(f)}$, for $Q = 20$, normalized by load residual mass squared and acceleration spectral density

α_1	α_2	μ								
		0.001	0.003	0.01	0.03	0.1	0.3	1	3	10
8.0	8.0	932	933	936	948	1001	1180	1240	1234	1238
8.0	4.0	233	233	233	235	239	256	294	265	250
8.0	2.0	58	58	58	58	59	60	68	73	68
8.0	1.0	15	15	15	15	15	15	17	23	22
8.0	0.5	4	4	4	4	4	4	4	7	6
8.0	0.25	1	1	1	1	1	1	1	2	5
8.0	0.125	1	1	1	1	1	1	1	1	3
4.0	8.0	871	867	858	849	904	1042	1067	1110	1229
4.0	4.0	218	218	217	216	220	250	254	250	252
4.0	2.0	55	55	55	55	56	61	72	68	67
4.0	1.0	14	14	14	14	14	16	21	23	22
4.0	0.5	3	3	4	4	4	4	6	10	10
4.0	0.25	1	1	1	1	1	1	2	5	5
4.0	0.125	1	1	1	1	1	1	1	3	3
2.0	8.0	1586	1478	1260	1061	990	946	982	1099	1201
2.0	4.0	406	391	355	305	272	259	238	236	254
2.0	2.0	103	101	97	88	79	82	70	63	62
2.0	1.0	26	26	26	25	24	25	25	23	22
2.0	0.5	7	7	7	7	7	9	10	10	10
2.0	0.25	2	2	2	2	2	3	5	5	6
2.0	0.125	1	1	1	1	1	1	3	3	4
1.0	8.0	11041	5731	2714	1486	967	901	984	1095	1181
1.0	4.0	3869	2206	1105	567	332	247	233	238	248
1.0	2.0	1228	826	432	226	125	83	71	66	64
1.0	1.0	359	283	166	100	50	34	26	23	23
1.0	0.5	100	89	63	42	24	15	12	11	11
1.0	0.25	28	27	23	17	11	8	6	6	6
1.0	0.125	8	8	8	7	5	5	4	4	4
0.5	8.0	13889	7720	3501	1726	1023	880	974	1093	1171
0.5	4.0	4516	2895	1417	695	357	247	225	240	244
0.5	2.0	1346	1003	561	283	136	89	70	64	65
0.5	1.0	377	319	211	117	59	39	27	24	22
0.5	0.5	102	95	74	48	27	17	12	11	10
0.5	0.25	28	27	25	19	13	8	7	6	6
0.5	0.125	8	8	8	8	6	5	4	4	4
0.25	8.0	17378	9978	4092	1944	1017	833	936	1092	1166
0.25	4.0	5194	3725	1805	812	380	249	225	241	242
0.25	2.0	1455	1205	741	359	173	93	71	66	65
0.25	1.0	391	354	269	160	74	43	28	23	22
0.25	0.5	103	99	86	63	38	22	14	12	11
0.25	0.25	28	28	27	23	16	10	8	7	7
0.25	0.125	8	8	8	8	7	5	5	4	4

steps of 1/16 from 8/16 to 32/16. In general, the relationship between the normalized force PSD $W_{FF}(f)$ and its controlling input parameters is more complicated for the complex two-SDOF system model than for the simple two-SDOF model. At smaller values of μ , the complex two-SDOF model predicted force limits are larger for $\alpha_1 = \alpha_2 = 1$ and smaller at $\alpha_1 = \alpha_2 = 0.25$ than are their simple two-SDOF counterparts. At larger values for μ , the complex two-SDOF model predicted force limits are larger than their simple two-SDOF model counterparts.

In the limiting case of the load modal effective mass equal to zero, that is $\alpha_2 = 0$, the normalized force PSD reduces to 1.

In practice, the modal effective mass and residual mass for both the source and the load must be known as a function of frequency to implement the above force limits.

Selection Modal Effective Mass and Residual Mass

The ratio between the interface load $F(\omega)$ and the enforced acceleration $\ddot{U}(\omega)$ is apparent mass, and can be described using the asparagus patch model (Appendix D) as follows, [39]

$$\frac{F(\omega)}{\ddot{U}(\omega)} = \underbrace{\sum_{i=1}^{\nu-1} M_{eff,i} H\left(\frac{\omega_i}{\omega}\right)}_{\text{Lower modes}} + \underbrace{M_{eff,\nu} H\left(\frac{\omega_\nu}{\omega}\right)}_{\text{Flexible mode}} + \underbrace{\sum_{i=\nu+1}^{\infty} M_{eff,i} H\left(\frac{\omega_i}{\omega}\right)}_{\text{Higher modes}}, \quad (2.403)$$

where the frequency transfer function $H\left(\frac{\omega_i}{\omega}\right)$ is

$$H\left(\frac{\omega_i}{\omega}\right) = \frac{\left(\frac{\omega_i}{\omega}\right)^2 + 2j\zeta_i\left(\frac{\omega_i}{\omega}\right)}{\left(\frac{\omega_i}{\omega}\right)^2 - 1 + 2j\zeta_i\left(\frac{\omega_i}{\omega}\right)},$$

and $M_{eff,i}$ is the modal effective mass, ω_i is the natural frequency, corresponding with mode i , and ω is the circular frequency.

We can make the following observations from (2.403):

- For lower modes, $\frac{\omega_i}{\omega}$ tends to zero because the excitation frequency is higher than the natural frequencies of the modes. This means that the transfer function H tends to zero, and so does the contribution of the lower modes to the interface force.
- For higher modes, $\frac{\omega_i}{\omega}$ tends to infinity and the transfer function H tends to one. Therefore, the higher modes tend to behave like a rigid body.
- For a mode with a natural frequency near the excitation frequency ($\frac{\omega_i}{\omega} \approx 1$), the contribution depends on the amplification factor $Q = 1/2\zeta$ as the function H tends to $1 - jQ$ (or the modulus is very close to Q).

The complex two-SDOF method used the trends of the transfer function H to derive the approximation. It assumes that the lower modes do not contribute and that the higher modes behave like a rigid body. The approximations of modal effective mass and the residual mass become $m^* = M_{eff,\nu}$ and $M = M_0 - \sum_{i=1}^{\nu} M_{eff,i}$. The approximation is good for low- and high-frequency ratio $\frac{\omega_i}{\omega}$. However, as the frequency ratio approaches unity, the approximation becomes less accurate. That is why when a mode has its natural frequency in the frequency band of interest, no approximation is made and the mode is considered fully.

A step by step procedure is given to determine the interface force specification [39]:

1. Divide the frequency spectrum into frequency bands (usually one-third octave band)
2. Select the next frequency band
3. Create a simple model of the Source
4. Create a simple model of the Load
5. Couple both models
6. Excite the coupled model and get the worst case ratio of the interface force to the interface acceleration
7. Multiply the interface force/acceleration ratio by the specified acceleration, in relevant frequency band, to obtain the interface force specification
8. Are all frequency band covered, if not go to step 2
9. The complete interface force specification has been obtained

Example. For simplicity, the mathematical models of “source” and “load” are similar and illustrated in Fig. 2.46. The complex two-SDOF models will be generated in the octave band. Calculate the bandwidth Δf , the upper band f_{max} and lower band frequency f_{min} . Both the “source” and “load” are fixed in m_1 . The characteristics of both are shown in Table 2.13. The dynamic properties of the “source” and the “load” are given Tables 2.14 and 2.15.

The generated complex two-SDOF models are presented in Table 2.16. In the 63 Hz octave band the modal effective mass of the source is $m_1^* = 0$, however, the residual mass of the source is $M_1 = 285.26$ kg, the same as for the 31.5 Hz frequency band. The source acts like a rigid body. In the 250 Hz octave band the load is partly present, because $m_2^* = 0$ and $M_2 = 50$ kg.

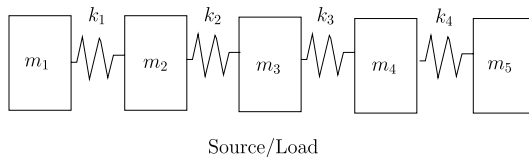


Fig. 2.46. Source/load

Table 2.13. Properties “source” and “load”

Mass (kg)	Source	Load	Stiffness (N/m)	Source	Load
m_1	200	50	k_1	10^8	10^7
m_2	200	50	k_2	10^8	10^7
m_3	200	50	k_3	10^8	10^7
m_4	200	50	k_4	10^8	10^7
m_5	200	50			

Table 2.14. Dynamic properties “source”

Natural frequency	(Hz)	Modal effective mass	(kg)	Residual mass	(kg)	α_1
f_1	39.08	m_1^*	714.74	M_1	285.26	2.51
f_2	112.54	m_1^*	66.67	M_1	218.59	0.31
f_3	172.42	m_1^*	15.65	M_1	202.94	0.07
f_4	211.51	m_1^*	2.94	M_1	200.00	0.01

Table 2.15. Dynamic properties “load”

Natural frequency	(Hz)	Modal effective mass	(kg)	Residual mass	(kg)	α_2
f_1	24.72	m_2^*	178.69	M_2	71.31	2.51
f_2	71.18	m_2^*	16.67	M_2	54.64	0.31
f_3	109.05	m_2^*	3.91	M_2	50.73	0.07
f_4	133.77	m_2^*	0.74	M_2	50.00	0.01

Table 2.16. Complex two-SDOF models

Octave band	Source	Load	α_1	α_2	μ
31.5	f_1	f_1	2.51	2.51	0.25
63		f_2	0.00	0.31	0.19
125	f_2	f_3	0.31	0.07	0.23
	f_2	f_4	0.31	0.01	0.23
	f_3	f_3	0.07	0.07	0.25
	f_3	f_4	0.07	0.01	0.25
250	f_4		0.01	0.00	0.25

2.8.4 Semi-Empirical Method

The semi-empirical force-limit approach is a method to establish force-limits based on the extrapolation of interface force data for similar mounting structures, [135].

$$\begin{aligned}
 W_{FF}(f) &= C^2 M^2 W_{aa}(f) \quad f \leq f_0, \\
 W_{FF}(f) &= C^2 M^2 W_{aa}(f) \left(\frac{f_0}{f}\right)^2 \quad f > f_0,
 \end{aligned}
 \tag{2.404}$$

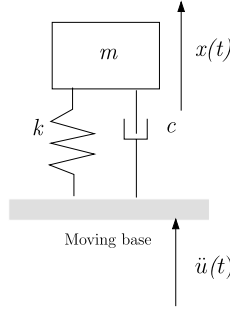


Fig. 2.47. Enforced acceleration of a damped SDOF system

where $W_{FF}(f)$ is the force spectral density, $W_{aa}(f)$ is the acceleration spectral density, M is the total mass of the test item and C is a dimensionless constant which depends on the configuration. f (Hz) is the frequency and f_0 is the frequency of the primary mode with a significant modal effective mass. In [190] some recommendation for C^2 are given, however, based on limited number of flight data. It has been observed that in normal conditions $C^2 = 2$ might be chosen for complete spacecraft or strut mounted heavier equipment. $C^2 = 5$ might be considered for directly mounted lightweight test items.

Example. The base excited SDOF system is shown in Fig. 2.47. The reaction force $F(\omega)$ at the base caused by the enforced acceleration is given by

$$\begin{aligned} F(\omega) &= m\ddot{X}(\omega) = m\left(\frac{\omega_0^2 + 2j\zeta\omega_0\omega}{\omega_0^2 - \omega^2 + 2j\zeta\omega_0\omega}\right)\ddot{U}(\omega) \\ &= m\left(\frac{1 + 2j\zeta\frac{\omega}{\omega_0}}{1 - \frac{\omega^2}{\omega_0^2} + 2j\zeta\frac{\omega}{\omega_0}}\right)\ddot{U}(\omega) \\ &= m\left(\frac{1 + 2j\zeta\frac{f}{f_0}}{1 - \frac{f^2}{f_0^2} + 2j\zeta\frac{f}{f_0}}\right)\ddot{U}(2\pi f), \end{aligned}$$

where $\omega_0 = \sqrt{k/m}$, $f_0 = \frac{1}{2\pi}\sqrt{k/m}$, $\zeta = c/2\sqrt{km}$. The excitation and the responses are in the frequency domain. The PSD of the reaction force $W_{FF}(f)$ will be represented in the cyclic frequency domain and expressed in the properties of the SDOF system and the PSD of the enforced acceleration $W_{\ddot{u}\ddot{u}}(f)$. The normalized force becomes

$$\frac{W_{FF}(f)}{m^2 W_{\ddot{u}\ddot{u}}(f)} = \left| \frac{1 + 2j\zeta\frac{f}{f_0}}{1 - \frac{f^2}{f_0^2} + 2j\zeta\frac{f}{f_0}} \right|^2 = \frac{1 + (2\zeta\frac{f}{f_0})^2}{(1 - \frac{f^2}{f_0^2})^2 + (2\zeta\frac{f}{f_0})^2}.$$

The semi-empirical equations to define the force limiting have already been given by (2.404)

Table 2.17. Random vibration acceleration loads

Frequency (Hz)	Acceleration power spectral density $W_{\ddot{u}\ddot{u}}(f)$ (g^2/Hz)
20–50	3 dB/oct.
50	0.05
300	0.05
300–2000	–3 dB/oct.
Overall	6.48 G_{rms}

$$\frac{W_{FF}(f)}{m^2 W_{\ddot{u}\ddot{u}}(f)} = C^2 \quad f \leq f_0,$$

$$\frac{W_{FF}(f)}{m^2 W_{\ddot{u}\ddot{u}}(f)} = C^2 \left(\frac{f_0}{f}\right)^2 \quad f > f_0.$$

We take the following values for the parameters are assumed:

- $\zeta = 0.05$,
- $f_0 = 150$ Hz,
- $m = 25$ kg,
- $g = 9.81$ m/s²,
- $C^2 = 5$.

The PSD of the enforced acceleration is defined in Table 2.17.

The objective of the calculations is to obtain the notch-curve. The notch-curve will be applied to the PSD of the enforced acceleration $W_{\ddot{u}\ddot{u}}(f)$ in such a way that the specification of the force limiting will be met. In that framework the following calculations and visualizations (Fig. 2.48) are carried out:

- Visualization of the PSD of the enforced acceleration $W_{\ddot{u}\ddot{u}}(f)$.
- Calculation and visualization of the PSD of the force limited interface force. Maximum PSD interface force is 15 036 N²/Hz.
- Calculation and visualization of the PSD of the dynamic interface force. Maximum PSD dynamic interface force is 303 484 N²/Hz.
- Calculation and visualization of the notch curve. Depth of the notch (notch curve) is 0.0495 at $f = 150$ Hz. The minimum PSD of the notched enforced acceleration is $0.0495 \times 0.05 = 0.0000257$ g²/Hz.

Problems

2.25. For a simple 2DOF system (Fig. 2.40), the ratio of the interface force spectrum $W_{FF}(f)$ to the interface acceleration spectrum $W_{aa}(f)$ ($a = \ddot{x}_1$) given by [174]

$$\frac{W_{FF}(f)}{m_2^2 W_{aa}(f)} = \frac{1 + 4\zeta_2^2 \left(\frac{f}{f_2}\right)^2}{\left\{1 - \left(\frac{f}{f_2}\right)^2\right\}^2 + 4\zeta_2^2 \left(\frac{f}{f_2}\right)^2},$$

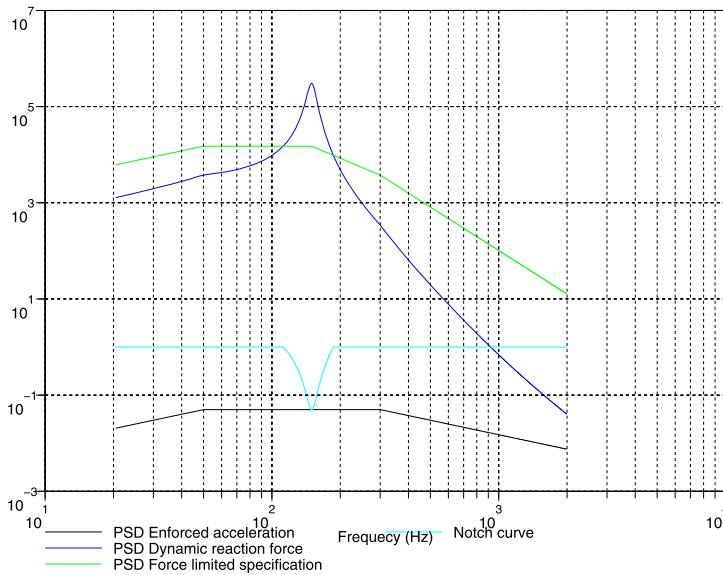


Fig. 2.48. Calculation of notch curve

Table 2.18. Random vibration acceleration loads

Frequency (Hz)	Acceleration spectral density $W_{aa}(f)$ (g^2/Hz)
20	0.0104
	? dB/oct.
50	0.0637
800	0.0637
	? dB/oct.
2000	0.0104
Overall	? G_{rms}

where $\zeta_2 = \frac{c_2}{2\sqrt{k_2 m_2}}$, $f_2 = \frac{1}{2\pi} \sqrt{\frac{k_2}{m_2}}$ and m_2 is the mass of the load. Prove this relation. Hint: think about the expression for the modal effective mass of a base excited SDOF system.

2.26. The qualification random accelerations are specified in Table 2.18. Calculate the slopes between 20–50 Hz, 800–2000 Hz and the Grms value of the spectrum. The test item placed on the shaker table has a mass $M = 50$ kg, and during the low level sine sweep test the first significant resonance frequency measured is $f_0 = 125$ Hz. Establish the force-limit specification $W_{FF}(f)$ (N^2/Hz), between 20–2000 Hz, for that test item, by applying the following relations:

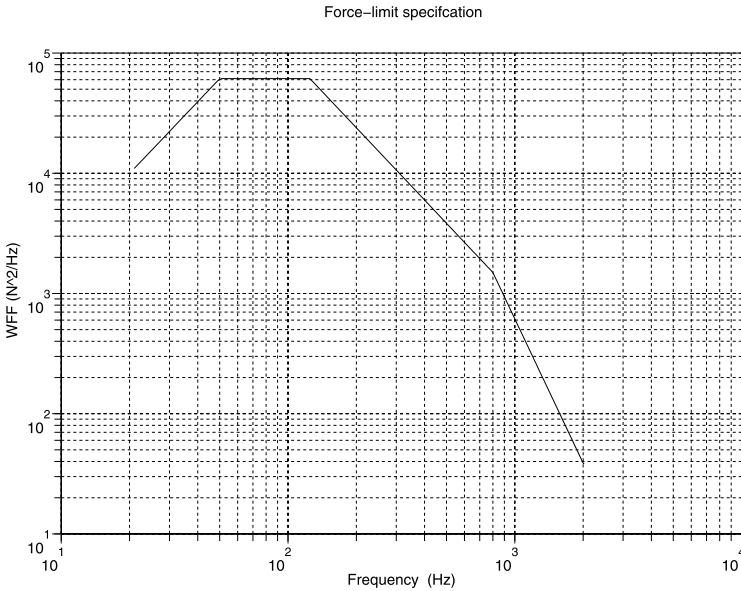


Fig. 2.49. Force limits

$$W_{FF}(f) = C^2 M^2 W_{aa}(f) \quad f \leq f_0,$$

$$W_{FF}(f) = C^2 M^2 W_{aa}(f) \left(\frac{f_0}{f}\right)^2 \quad f > f_0,$$

where $C = 2$ and $W_{aa}(f)$ must be converted into $(\text{m/s}^2)^2/\text{Hz}$ ($g = 9.81 \text{ m/s}^2$).
 Answers: $\pm 5.95 \text{ dB/oct.}$, $8.9 G_{rms}$, Fig. 2.49.

2.27. This problem is based on information given in [97]. The KOMPSAT-II earth observation satellite of 800 kg has a Multi-Spectral Camera (MSC) as a main payload (P/L). The MSC has a mass of 76.7 kg. The force limit specification was established by applying the simple 2DOF model in Fig. 2.44 ($Q = 5$). The lateral modal effective masses for the source m_1 and the load m_2 are calculated by the finite element method. The results of the calculation are presented in Table 2.10. The semi-empirical force-limit method will be applied to establish the force limit specification (2.404). Define the following parameters f_0 and C^2 ($Q = 10$).
 Answers: 250 Hz, 1.1.

2.9 Analysis of Narrow-Band Processes

In this section some interesting properties of narrow-band stationary random processes will be discussed:

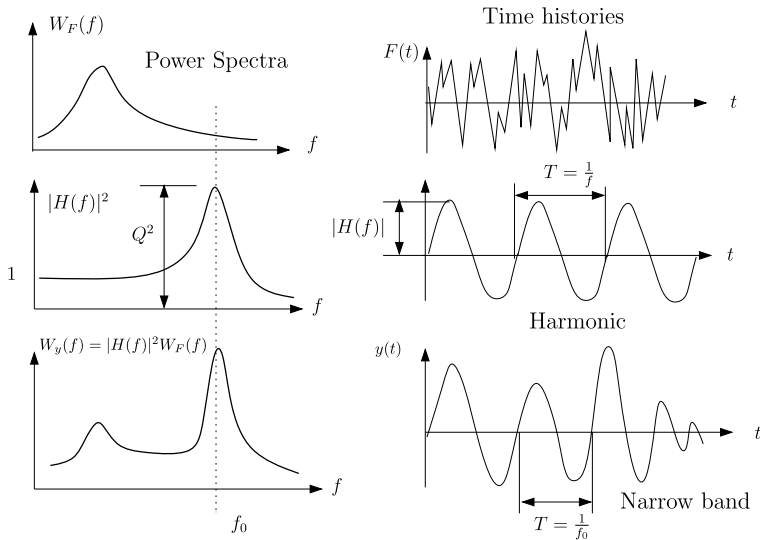


Fig. 2.50. The relationship between spectrum of input and spectrum of output

- Number of crossings per unit of time through a certain level [13]
- Fatigue damage due to random loads.

Besides the rms value of the response of deterministic structures exposed to random forces, these properties are important for further investigation of strength characteristics.

Narrow band responses represent the filtered response of a few modes, sometimes only one dominant mode. This is illustrated in Fig. 2.50. The system shows dominant response characteristics at one natural frequency f_0 , which means the FRF $H(j\omega)$ peaks around that natural frequency. The system acts like a filter and lets pass only responses around the natural frequency f_0 pass; the response is called narrow banded.

2.9.1 Crossings

Consider the event that a stationary process $x(t)$ crosses the level α from below with a certain positive velocity $\dot{x}(t) = v(t)$. This is called a crossing with a positive slope. This is illustrated in Fig. 2.51.

$N_\alpha(\tau)$ is the number of expected crossings for a time period of τ . The random process $x(t)$ is stationary, so the number of expected crossings does not depend on the time at which the process starts. The sum of the number of crossings at level α (with positive and negative velocity) will be a linear function of time, hence

$$N_\alpha(\tau_1 + \tau_2) = N_\alpha(\tau_1) + N_\alpha(\tau_2). \tag{2.405}$$

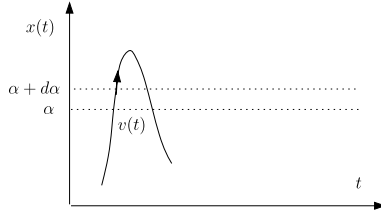


Fig. 2.51. Positive crossings

The number ν_{α}^{+} of positive crossings per unit of time, that the signal $x(t)$ will cross the level α with a positive slope (positive velocity), is defined by

$$N_{\alpha}^{+}(\tau) = \nu_{\alpha}^{+} \tau. \tag{2.406}$$

The joint probability that the values $x(t)$ and $v(t)$ are between certain values, for all times t , is defined as

$$f(\alpha, \beta) d\alpha d\beta = Prob(\alpha \leq x(t) \leq \alpha + d\alpha, \beta \leq v(t) \leq \beta + d\beta). \tag{2.407}$$

The cumulative joint probability is defined by

$$F(\alpha + d\alpha, \beta + d\beta) - F(\alpha, \beta) = \int_{\alpha}^{\alpha+d\alpha} \int_{\beta}^{\beta+d\beta} f(x, \dot{x}) dx d\dot{x}, \tag{2.408}$$

or

$$F(\alpha + d\alpha, \beta + d\beta) - F(\alpha, \beta) = \frac{\partial^2 F(\alpha, \beta)}{\partial \alpha \partial \beta} d\alpha d\beta = f(\alpha, \beta) d\alpha d\beta, \tag{2.409}$$

where $f(\alpha, \beta)$ is the joint probability density function and $F(\alpha, \beta)$ the cumulative joint probability function.

Equations (2.407), (2.408) and (2.409) define the time $d\tau$ the signal $x(t)$ is in the interval $[\alpha, \alpha + d\alpha]$ with a velocity $\dot{x}(t) = v(t) \approx \beta$.

The number ν_{α}^{+} of expected positive crossings per unit of time through the interval $[\alpha, \alpha + d\alpha]$ with a velocity $v(t) = \beta$ is estimated by dividing the amount of time spent inside this interval by the time required to pass this interval. The time $d\tau$ to pass (up or down) the interval $[\alpha, \alpha + d\alpha]$ is

$$d\tau = \frac{d\alpha}{|\beta|}. \tag{2.410}$$

The amount of time the signal $x(t)$ is in the interval $[\alpha, \alpha + d\alpha]$ with a velocity in the interval $[\beta, \beta + d\beta]$ is $f(\alpha, \beta) d\alpha d\beta$. The expected number of crossings (up and down) $\nu_{\alpha}(\beta)$ per unit of time through the level α with the velocity $v(t) = |\beta|$ is

$$\frac{f(\alpha, \beta) d\alpha d\beta}{d\tau} = |\beta| f(\alpha, \beta) d\beta. \tag{2.411}$$

The expected number of crossings (up and down) ν_α per unit of time through level α with all possible velocities is

$$\nu_\alpha = \int_{-\infty}^{\infty} |\beta| f(\alpha, \beta) d\beta. \tag{2.412}$$

If we assume the velocity $v_\alpha(\beta) \geq 0$, then the expected number of positive zero crossings ν_α^+ is (2.412)

$$\nu_\alpha^+ = \frac{1}{2} \int_{-\infty}^{\infty} \beta f(\alpha, \beta) d\beta. \tag{2.413}$$

The Gaussian distribution can be used in (2.407), i.e. the joint probability density function is

$$f(x, \dot{x}) = \frac{1}{2\pi\sigma_x\sigma_{\dot{x}}} e^{(-\frac{x^2}{2\sigma_x^2} - \frac{\dot{x}^2}{2\sigma_{\dot{x}}^2})}. \tag{2.414}$$

Substituting (2.414) in (2.413), and performing the integration, we find

$$\nu_\alpha^+ = \frac{1}{2} \int_{-\infty}^{\infty} \beta \frac{1}{2\pi\sigma_\alpha\sigma_\beta} e^{(-\frac{\alpha^2}{2\sigma_\alpha^2} - \frac{\beta^2}{2\sigma_\beta^2})} d\beta = \frac{\sigma_\beta}{2\pi\sigma_\alpha} e^{-\frac{\alpha^2}{2\sigma_\alpha^2}}, \tag{2.415}$$

with

- $\sigma_x^2 = \sigma_\alpha^2 = E\{x^2(t)\} - (E\{x(t)\})^2 = E\{x^2(t)\} = \int_0^\infty W_x(f)df$
- $\sigma_{\dot{x}}^2 = \sigma_\beta^2 = E\{\dot{x}^2(t)\} - (E\{\dot{x}(t)\})^2 = E\{\dot{x}^2(t)\} = \int_0^\infty (2\pi f)^2 W_x(f)df$

In general, we take zero mean values for both the $x(t)$ and $\dot{x}(t)$. The process $x(t)$ may be any kind of response in the structure due to random loads, i.e. internal forces, stresses, etc.

The number of zero positive crossings (zero up crossings, apparent frequency, characteristic frequency) ν_0^+ can be obtained from (2.415) with $x(t) = 0$ and the spectral moments (2.53)

$$\nu_0^+ = \frac{\sigma_{\dot{x}}}{2\pi\sigma_x} = \sqrt{\frac{m_2}{m_0}} = \sqrt{\frac{\int_0^\infty f^2 W_x(f)df}{\int_0^\infty W_x(f)df}}. \tag{2.416}$$

Equation (2.415) can also be written as

$$\nu_\alpha^+ = \frac{\sigma_{\dot{x}}}{2\pi\sigma_x} e^{-\frac{\alpha^2}{2\sigma_\alpha^2}} = \nu_0^+ e^{-\frac{\alpha^2}{2\sigma_\alpha^2}}. \tag{2.417}$$

The probability density function $f(x) = e^{-\frac{x^2}{2\sigma_\alpha^2}}$ is a *Rayleigh distribution*.

Example. For a SDOF system as shown in Fig. 2.14 the variance of the relative displacement $z(t) = x(t) - u(t)$ given by σ_z and that of the relative velocity $\sigma_{\dot{z}}$ had been obtained in (2.138) and (2.139). The mean square of the relative displacement $z(t)$ is given by

$$E\{z(t)^2\} = \sigma_z^2 = \frac{W_{\ddot{u}}}{8\zeta(2\pi f_n)^3}.$$

The mean square of the relative velocity $\dot{z}(t)$ is given by

$$E\{\dot{z}(t)^2\} = \sigma_{\dot{z}}^2 = \frac{W_{\dot{u}}}{8\zeta(2\pi f_n)}.$$

Again we assume zero average values.

With the aid of (2.416) the number of positive zero crossings per unit of time (frequency) ν_0^+ (Hz) can be obtained:

$$\nu_0^+ = \frac{\sigma_{\dot{x}}}{2\pi\sigma_x} = \frac{1}{2\pi} \sqrt{\frac{\frac{W_{\dot{u}}}{8\zeta(2\pi f_n)}}{\frac{W_{\ddot{u}}}{8\zeta(2\pi f_n)^3}}} = f_n.$$

The number of positive zero crossings ν_0^+ is equal to the natural frequency f_n of the SDOF system, as illustrated in Fig. 2.14.

The following example is illustrated in Fig. 2.19.

Example. Calculate the number of positive zero crossings of the acceleration of the dofs x_1 , x_2 and x_3 and the spring force in the springs k_1 , k_2 and k_3 .

The natural frequencies calculated were

$$\{f_n\} = \left\{ \begin{array}{l} 69.5745 \\ 153.7834 \\ 238.3038 \end{array} \right\} \text{ Hz.}$$

The positive zero crossings $\{\nu_0^+\}$ of the acceleration of the dofs x_1 , x_2 and x_3 become

$$\{\nu_0^+\} = \left\{ \begin{array}{l} 92.2923 \\ 130.5300 \\ 173.9503 \end{array} \right\} \text{ Hz,}$$

and for the positive zero crossings $\{\nu_0^+\}$ of the spring force of the springs k_1 , k_2 and k_3 we get

$$\{\nu_0^+\} = \left\{ \begin{array}{l} 92.4567 \\ 79.3611 \\ 81.3201 \end{array} \right\} \text{ Hz.}$$

The zero positive crossings of the spring forces are of the same order, however, the zero positive crossings of the acceleration show a deviation of a factor for two. The integration to obtain the positive zero crossings is done by the trapezoidal rule with a frequency increment $\Delta f = 0.2$ Hz.

2.9.2 Peaks

Consider a stationary Gaussian process, $x(t)$, having a zero mean $\mu_x = 0$ and a standard deviation σ_x . The p.d.f of its peaks (including both positive and negative maxima) at any time instant is given by [59, 114]

$$f_p(\eta) = \frac{\lambda}{\sigma_x \sqrt{2\pi}} e^{-\frac{\eta^2}{2\lambda^2 \sigma_x^2}} + \frac{\eta \sqrt{1-\lambda^2}}{\sigma_x^2} \Phi\left(\frac{\eta \sqrt{1-\lambda^2}}{\lambda \sigma_x}\right) e^{-\frac{\eta^2}{2\sigma_x^2}}, \quad -\infty < \eta < \infty, \quad (2.418)$$

where

- $\Phi(u) = \text{erf}(u) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^u e^{-\frac{x^2}{2}} dx$. A numerical approximate method to calculate $\text{erf}(u)$ is given in Appendix B
- m_n is the n -th moment of a one-sided PSD function $m_n = \int_0^\infty f^n W_s(f) df$
- $\gamma = \frac{m_2}{m_0 m_4}$
- λ is the spectral width parameter, $\lambda = \sqrt{1-\gamma^2}$

The *cumulative probability function* (c.p.f.) $F_p(\eta)$ of the peaks corresponding to (2.418) can be derived in closed form:

$$F_p(\eta) = \Phi\left(\frac{\eta}{\lambda \sigma_x}\right) - \sqrt{1-\lambda^2} \Phi\left(\frac{\eta \sqrt{1-\lambda^2}}{\lambda \sigma_x}\right) e^{-\frac{\eta^2}{2\sigma_x^2}}. \quad (2.419)$$

For a narrow band random process $\lambda \rightarrow 0$, and the c.p.f. reduces to a Rayleigh density (distribution) function

$$F_p(\eta) = 1 - e^{-\frac{\eta^2}{2\sigma_x^2}}, \quad (2.420)$$

and for a very wide random process $\lambda \rightarrow 1$ the c.d.f. reduces to a Gaussian density function

$$F_p(\eta) = \Phi\left(\frac{\eta}{\sigma_x}\right). \quad (2.421)$$

For a sequence of random variables $\eta_1, \eta_2, \dots, \eta_N$ representing peaks, which are assumed to be uniformly distributed and statistically independent, the c.p.f. of the largest peak becomes

$$F_{p_{max}}(\eta) = P[\max(\eta_1, \eta_2, \dots, \eta_N) \leq \eta] = [F_p(\eta)]^N, \quad (2.422)$$

where \max is added to P to denote that it is the maximum peak in a sequence of N peaks.

The c.p.f. of the maximum of a zero mean stationary Gaussian narrow band random process with a given time interval T is approximated by

$$F_{p_{max}}(\eta) = e^{\{-\nu_0^+ T [e^{-\frac{\eta^2}{2\sigma_x^2}}]\}}. \quad (2.423)$$

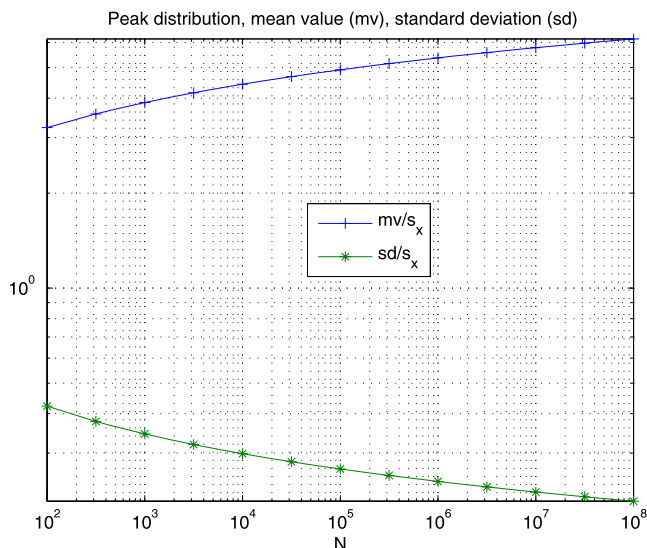


Fig. 2.52. Peak distribution

The corresponding mean value $\bar{\eta}_{max}$ and the standard deviation $\sigma_{\eta_{max}}$ of the peaks were found to be

$$\frac{\bar{\eta}_{max}}{\sigma_x} = \sqrt{2 \ln(\nu_0^+ T)} + \frac{\varepsilon}{\sqrt{2 \ln(\nu_0^+ T)}}, \quad (2.424)$$

and

$$\frac{\sigma_{\eta_{max}}}{\sigma_x} = \frac{\pi}{\sqrt{6}} \frac{1}{\sqrt{2 \ln(\nu_0^+ T)}}, \quad (2.425)$$

where the rate of positive zero crossings is $\nu_0^+ = \sqrt{\frac{m_2}{m_0}}$, and the Euler constant¹⁹ is $\varepsilon = 0.577216$. Equations (2.424) and (2.425) are illustrated in Fig. 2.52. The number of cycles is $N = \nu_0^+ T$.

It is well known that the expected number of positive zero crossings is very close to the expected number of peaks per unit of time for a narrow band process. If the parameter λ of a process become larger and larger, the results of (2.424) and (2.425) become less accurate. The following approximation can be used

$$\frac{\bar{\eta}_{max}}{\sigma_x} = \int_{-\infty}^0 [F_p(\eta)]^N d\eta + \int_0^{\infty} [1 - F_p(\eta)]^N d\eta, \quad (2.426)$$

where $1 - F_p(\eta)$ can be approximated as follows

¹⁹ The Euler constant ε is given by $\varepsilon = \lim_{n \rightarrow \infty} [\sum_{k=1}^n \frac{1}{k} - \ln(n+1)] = 0.577216$, [53].

$$1 - F_p(\eta) \approx \sqrt{1 - \lambda^2} e^{-\frac{\eta^2}{2\sigma_x^2}} + \left[\frac{\sigma_x^3}{\eta^3} e^{-\frac{\eta^2}{2\lambda^2\sigma_x^2}} \right]. \quad (2.427)$$

The mean number of up-crossings η_{mean} is given by [47]

$$\eta_{mean} = \nu_0^+ T \sqrt{1 - \lambda^2}, \quad (2.428)$$

where the spectral width factor is $\lambda = \sqrt{1 - \gamma^2}$, and the irregularity factor is $\gamma = \frac{m_2^2}{m_0 m_4}$, and the spectral moments are $m_i = \int \omega^i S(\omega) d\omega$.

2.9.3 Fatigue Damage due to Random Loads

There are quite a number of failure modes, one of them is the failure of a structure due to fatigue behavior of materials. Fatigue appears when the structure is exposed to oscillating loads (stresses). The material will crack and failure occurs. Fatigue damage is caused by micro plastic deformations (strains) that will damage the structure of the material locally and accumulate to micro cracks and ultimately to failure of the structure.

With the Palmgren-Miner rule, one is able to predict the fatigue life of a structure or part of the structure caused by cumulative damage when the structure is exposed to oscillating loads or stresses.

At a certain stress level s_i (for random vibration the one-sigma value of the stress) one can take the allowable number of oscillations N_i from a so-called s - N curve. In general, the relation between the stress level and the allowable number of oscillations, the s - N , or Wöhler fatigue curve, is

$$N(s)s^b = a, \quad (2.429)$$

where a and b are constants.

In the *model of cumulative damage*, as formulated by Palmgren and Miner, the *Palmgren-Miner* damage function $D(t)$ is

$$D(t) = \sum_{i=1}^{N(t)} \Delta D_i. \quad (2.430)$$

The damage function $D(t)$ is a nondecreasing function of time that starts at zero for a new structure and is normalized to unity when failure occur, the instant of time $t_{failure}$ at which $D(t_{failure}) = 1$.

The Palmgren-Miner rule can be formulated as follows: if the i -th cycle occurs at the stress level s_i at which, in accordance with the s - N curve, N_i causes failure, then the i -th increment of damage is

$$\Delta D_i = \frac{1}{N_i}. \quad (2.431)$$

Grouping the cycles of approximately equal amplitude together, we find a situation in which n_i cycles at the stress level s_i can be identified. Then each

one of the groups i will produce $\frac{n_i}{N_i}$ incremental damage. The failure condition becomes

$$D = \sum_i \frac{n_i}{N_i} = 1. \quad (2.432)$$

Nothing is stated about the sequence of the stress levels. The number of positive crossings at the level $x(t) = \alpha$ is given by (2.417)

$$\nu_\alpha^+ = \frac{\sigma_{\dot{x}}}{2\pi\sigma_x} e^{-\frac{\alpha^2}{2\sigma_x^2}} = \nu_0^+ e^{-\frac{\alpha^2}{2\sigma_x^2}}.$$

With (2.417) it is possible to calculate the number of peaks $n_p(\alpha)$ per unit of time of $x(t)$ in the range $\alpha \leq x(t) \leq \alpha + d\alpha$:

$$n_p(\alpha)d\alpha = \nu_\alpha^+ - \nu_{\alpha+d\alpha}^+ \approx -\frac{d\nu_\alpha^+}{d\alpha}d\alpha, \quad (2.433)$$

thus the Rayleigh distribution of peaks is given by

$$f_p(\alpha) = \frac{n_p(\alpha)}{\nu_0^+} = \frac{\alpha}{\sigma_x^2} e^{-\frac{\alpha^2}{2\sigma_x^2}}, \quad \alpha \geq 0. \quad (2.434)$$

If one mode is dominant in the responses, the number of peaks $n_p(\alpha)$ is about the same as the number of positive zero crossings $n_p(\alpha) \approx \nu_0^+$. The total number of peaks during the time period T is given by $n_p(\alpha)T$. Substitute (2.434) into (2.432) and replace summation by integration. The expectation of the failure $E\{D(T)\}$ becomes

$$E\{D(T)\} = \sum_i \frac{n_i}{N_i} = T \int_0^\infty \frac{n_p(\alpha)}{N(\alpha)} d\alpha. \quad (2.435)$$

Substituting (2.429), we find the number of allowable oscillations at stress level $s = \alpha$ is

$$N(\alpha) = a\alpha^{-b}. \quad (2.436)$$

If both equations (2.434) and (2.436) are substituted into (2.435) we get

$$\begin{aligned} E\{D(T)\} &= T \int_0^\infty \frac{n_p(\alpha)}{N(\alpha)} d\alpha = \frac{\nu_0^+ T}{a\sigma_x^2} \int_0^\infty \alpha^{b+1} e^{-\frac{\alpha^2}{2\sigma_x^2}} d\alpha \\ &= \frac{\nu_0^+ T}{a} (\sqrt{2}\sigma_x)^b \Gamma\left(1 + \frac{b}{2}\right). \end{aligned} \quad (2.437)$$

Equation (2.437) was derived by Miles [127] and is called Miles single degree of freedom formula. The standard deviation of the occurring stress is denoted by $\sigma_x = \sigma_s$.

If failure occurs at time T , so that $E\{D(T)\} = 1$, the time to failure can be calculated:

$$T = \frac{a}{\nu_0^+ (\sqrt{2}\sigma_x)^b \Gamma(1 + \frac{b}{2})}, \quad (2.438)$$

Table 2.19. Some values $\Gamma(x)$ function

x	$\Gamma(x)$	x	$\Gamma(x)$
1.0	1.000	3.5	3.323
1.5	0.886	4.0	6.000
2.0	1.000	4.5	11.632
2.5	1.329	5.0	24.000
3.0	2.000	5.5	52.343

Table 2.20. Value of $\psi_1(b)$ as a function of the fatigue law exponent b [43, 202]

b	1	3	5	7
$\psi_1(b)$	0.0414	0.369	1.280	3.72
b	9	11	13	15
$\psi_1(b)$	10.7	31.5	96.7	308

where the Gamma function $\Gamma(z)$ is defined by [3]

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt.$$

Some values of the Gamma function are given in Table 2.19.

The expression for the standard deviation $\sigma_{D(t)}$ is given in [43, 202]

$$\frac{\sigma_{D(t)}}{E\{D(T)\}} \approx \frac{1}{\sqrt{\nu_0^+ T}} \sqrt{\frac{\psi_1(b)}{\zeta}}, \tag{2.439}$$

where $\psi_1(b)$ is a function of b that is tabulated in Table 2.20, and ζ is the damping ratio of the structure.

The standard deviation σ_T of the fatigue life is derived by [202]

$$\sigma_T = \frac{h_2 \sqrt{h_2^2 + 4h_1}}{6h_1^2}, \tag{2.440}$$

where

$$\begin{aligned} h_1 &= \nu_0^+ \frac{(\sqrt{2}\sigma_x)^b}{a} \Gamma\left(\frac{b}{2} + 1\right), \\ h_2 &= \frac{3(\sqrt{2}\sigma_x)^b}{a} \Gamma\left(\frac{b}{2} + 1\right) \sqrt{\nu_0^+ \frac{\psi_1(b)}{\zeta}}. \end{aligned} \tag{2.441}$$

Example. Given an s - N curve

$$N(s)s^b = a,$$

where

- $a = 1.56 \times 10^{39}$
- $b = 4$.

The natural frequency of a SDOF system is $f_n = 100$ Hz, the mass $m = 100$ kg and the damping ratio is $\zeta = 0.05$. The cross section of the spring is $A = 10^{-4}$ m². In the frequency range from 50–500 Hz the PSD of the base excitation is $W_{\ddot{u}} = 0.1$ g²/Hz. Predict the fatigue life of the spring using the Palmgren-Miner cumulative damage rule. The spring constant k is $k = (2\pi f_n)^2 m = (2\pi 100)^2 \times 100 = 3.948 \times 10^7$ N/m. The rms value of displacement $x(t)$ is about

$$x_{rms} = \sqrt{\frac{W_{\ddot{u}}(f_n)}{8\zeta(2\pi f_n)^2}} = \sqrt{\frac{0.1 \cdot 9.81^2}{8 \times 0.05 \times (2\pi 100)^3}} = 7.958 \times 10^{-3}.$$

The rms stress in the spring is

$$s_{rms} = \frac{kz_{rms}}{A} = \frac{3.948 \times 10^7 \times 2.469 \times 10^{-3}}{10^{-4}} = 3.142 \times 10^8 \text{ Pa.}$$

The number of positive zero crossings $\nu_0^+ = f_n = 100$ Hz. The time to failure T , the fatigue lifetime, is

$$T = \frac{a}{\nu_0^+ (\sqrt{2}\sigma_x)^b \Gamma(1 + \frac{b}{2})} = \frac{1.56 \times 10^{39}}{100 \times (\sqrt{2} \times 3.142 \times 10^8)^4 \Gamma(3)} = 200 \text{ s.}$$

In general, the predicted fatigue life T can be expressed as (2.435)

$$T = \frac{1}{E\{D(T)\} \int_0^\infty \frac{n_p(\alpha)}{N(\alpha)} d\alpha} = \frac{a}{E\{D(T)\} \int_0^\infty n_p(\alpha) \alpha^b d\alpha}. \quad (2.442)$$

In (2.442) we can use a probability density function $n_p(\alpha)$ of peaks, other than Rayleigh one, may be substituted.

Problems

2.28. To survive the launch loads a solar array sandwich panel is supported at 6 hold down points, in particular the steady state accelerations and the acoustic sound pressure. A circular reinforcement (insert) is built in the sandwich panel to transfer the shear loads to the spacecraft at the location of the hold down points. The insert is bonded to an Al-alloy honeycomb core with an allowable shear stress $\tau_{allowable} = 2.3 \times 10^5$ Pa. The measured s - N curve $Ns^b = a$ is $Ns^{5.967} = 1.385 \times 10^{34}$. The predicted average rms shear stress τ_{rms} along the circumference of the insert due to the acoustic sound pressures is $\tau_{rms} = 5.0 \times 10^4$ Pa. The number of associated positive zero crossings is $\nu_0^+ = 130$ Hz.

- Predict the fatigue life of the bonded insert/honeycomb core construction.
- Use the 3σ approach to calculate the margin of safety (MS) against the allowable shear stress of the honeycomb core.

Answers: $T = 617$ s, $MS = 0.53$.

2.29. The output PSD function W_0 of a response is constant in a frequency band $f \in [f_1, f_2]$ and zero otherwise.

- Calculate the number of positive zero crossings ν_0^+ .
- Calculate the number of positive zero crossings ν_0^+ if $f_2 = f_1 + \Delta F$, $\frac{\Delta f}{f_1} \ll 1$.

Answers: $\sqrt{\frac{\frac{1}{3}(f_2^3 - f_1^3)}{f_2 - f_1}}$, f_1 .

2.30. This problem is taken from [14].

Assume a record $x(t)$ from an ergodic random process has a one-sided PSD function given by

$$W_{xx}(f) = \frac{1}{20 + f^2} \quad 5 \leq f \leq 30 \quad \text{otherwise zero.}$$

Define the average number of positive zero crossings ν_0^+ per second in the record $x(t)$.

Answer: $\nu_0^+ = 13.122$ Hz.

2.31. $x(t)$ is a stationary random variable in time with properties shown in Fig. 2.53. Estimate the mean value of peaks η_{max} of $x(t)$ and the standard deviation of peaks σ_{max} of $x(t)$ within a time interval $T = 3600$ s.

Answers: $\eta_{max} = 3.3915$ m, $\sigma_{max} = 0.2301$ m.

2.32. A Sea-launch platform is exposed to a storm with waves of standard deviation, $\sigma = 2.5$ m and an average wave period $\bar{T}_0 = 10$ s. Design a platform height h , so that the deck is flooded only once per $\bar{T}_h = 15$ minutes. The diffraction of the waves a neglected, thus the incoming waves are not affected by the presence of the platform.

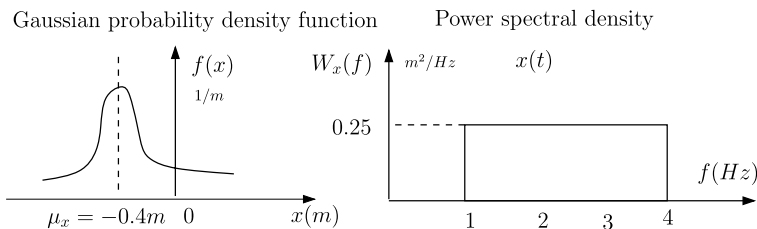


Fig. 2.53. Properties random process $x(t)$

- Calculate the number of positive zero crossings ν_0^+ .
- Calculate the number of positive crossing at h , ν_h^+ .
- Calculate h .

Answers: $\nu_0^+ = \frac{1}{10}$ Hz, $\nu_h^+ = \frac{1}{900}$ Hz, $h = 7.50$ m.

2.33. A 3 mass-spring dynamic system, as shown in Fig. 2.54, is excited at the base with a constant band limited random enforced acceleration $W_{\ddot{u}} = 0.05 \text{ g}^2/\text{Hz}$ in a frequency range $f \in [5, 750]$ Hz. The discrete masses are $m_1 = 150 \text{ kg}$, $m_2 = 200 \text{ kg}$ and $m_3 = 250 \text{ kg}$. The spring stiffnesses of the springs are $k_1 = 10^8 \text{ N/m}$, $k_2 = 2 \times 10^8 \text{ N/m}$ and $k_3 = 3 \times 10^8 \text{ N/m}$. The modal damping ratio for each mode is $\zeta = 0.025$ (the amplification factor, the transmissibility $Q = 20$). Calculate the rms values (diagonal terms) of the accelerations $\{\ddot{x}\}$, the rms values (diagonal terms) of the internal loads and associated positive zero crossings.

Answers with $\delta f = 0.2$ Hz:

$$\{\ddot{x}_{rms}\} = \begin{Bmatrix} 17.3652 \\ 12.3362 \\ 8.9667 \end{Bmatrix} \text{ g.}$$

$$\{F_{ii,rms}\} = 10^4 \begin{Bmatrix} 2.5542 \\ 4.2544 \\ 5.5106 \end{Bmatrix} \text{ N.}$$

$$\{\nu_0^+\} = \begin{Bmatrix} 106.4830 \\ 89.5507 \\ 91.8656 \end{Bmatrix} \text{ Hz.}$$

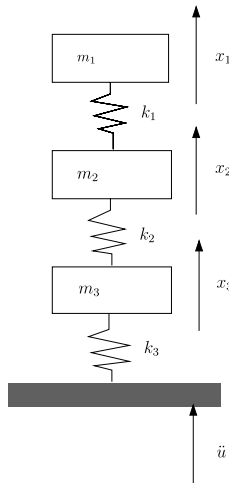


Fig. 2.54. 3 mass-spring system, enforced acceleration

2.34. This problem is taken from [202]. The bending stress response at the critical point of a structural element under random vibration turns out to be a stationary narrow-band process with a resonant frequency of 25 Hz and a rms value $\sigma_S = 5.4 \times 10^3$ psi²⁰ ($\sigma_S = 3.72 \times 10^7$ Pa). This element is made of 75S-T6 Al-alloy whose s - N curve is given by

$$N(s)s^{6.09} = (2 \times 10^5)^{6.09},$$

where $b = 6.09$ and $a = (2 \times 10^5)^{6.09}$. The damping ratio of this element is $\zeta = 0.01$. Determine the mean and standard deviation of fatigue life, T and $\sigma_{D(t)}$ respectively, for this structural element.

Answer: $T = 757.35$ hr, $\sigma_{D(t)} = 1.39$ hr.

2.10 Wide-Band Fatigue Damage

In this section the fatigue life prediction is based on [12, 50, 109]. Dirlik developed an empirical closed form expression for the p.d.f. of rainflow amplitude $f_s(s)$, based on extensive Monte Carlo simulations of the stress amplitude. Dirlik's solutions were successfully verified in theory. Dirlik's damage model for a time period T is as follows

$$E\{D(T)\} = \frac{\nu_p T}{a} \int_0^\infty s^b f_s(s) ds, \tag{2.443}$$

where the s - N curve is given by $Ns^b = a$ and the p.d.f. of the stress amplitudes $f_s(s)$ is given by Dirlik and is approximated as a sum of one Exponential and two Rayleigh densities

$$f_s(s) = \frac{D_1}{Q\sqrt{m_0}} e^{-\frac{s}{Q}} + \frac{D_2 Z}{R^2 \sqrt{m_0}} e^{-\frac{Z^2}{2R^2}} + \frac{D_3 Z}{\sqrt{m_0}} e^{-\frac{Z^2}{2}}, \tag{2.444}$$

where

- m_n is the n -th moment of a one-sided PSD function $m_n = \int_0^\infty f^n W_s(f) df$
- ν_p is the *expected rate of peaks* $\nu_p = \sqrt{\frac{m_4}{m_2}}$
- γ is the *irregularity factor* $\gamma = \frac{\nu_0^+}{\nu_p} = \sqrt{\frac{m_2^2}{m_0 m_4}}$
- λ is the *spectral width parameter*, $\lambda = \sqrt{1 - \gamma^2}$, and is introduced to classify whether the random process is narrow-banded or wide-banded. If $\gamma \rightarrow 1$ the random process is narrow-banded (NB) and if $\gamma \rightarrow 0$ the random process is broad-banded (BB).
- $Z = \frac{s}{\sqrt{m_0}}$
- $X_m = \frac{m_1}{m_0} \sqrt{\frac{m_2}{m_4}}$

²⁰ 1 psi = 6894.75729 Pa.

- $D_1 = \frac{2(X_m - \gamma^2)}{1 + \gamma^2}$
- $R = \frac{\gamma - X_m - D_1^2}{1 - \gamma - D_1 + D_1^2}$
- $D_2 = \frac{1 - \gamma - D_1 + D_1^2}{1 - R}$
- $D_3 = 1 - D_1 - D_2$
- $Q = \frac{1.25(\gamma - D_3 - D_2 \times R)}{D_1}$

Dirlik's method is widely accepted for fatigue damage calculations. The narrow band fatigue damage method (2.437) is conservative. The closed form solution of (Dirlik's p.d.f.) (2.444) is given by [12]

$$E\{D(T)\} = \frac{\nu_p T}{a} m_0^{\frac{b}{2}} \left[D_1 Q^b \Gamma(1 + b) + (\sqrt{2})^b \Gamma\left(1 + \frac{b}{2}\right) (D_2 |R|^b + D_3) \right]. \quad (2.445)$$

The equivalent stress *range* under random loading can be expressed as [114]

$$S_h = \left[\int_0^\infty (2s)^b f_s(s) ds \right]^{\frac{1}{b}}. \quad (2.446)$$

Numerical methods to approximate the equivalent stress range are discussed in [114].

Example. The PSD of a stress response is a constant spectrum $W_\sigma = 100 \text{ MPa}^2/\text{Hz}$ between 20–120 Hz. Outside that frequency range $W_\sigma = 0$. The s - N curve is given by $N(s)s^{5.56} = 1.02 \times 10^{17}$ (s in MPa).

- Calculate $E\{D(T = 1)\}$ and T_{NB} applying (2.437), the narrow-band approach
- Calculate $E\{D(T = 1)\}$ and T_{WB} applying (2.445), the wide-band approach

The spectral moments $m_n = \int_0^\infty f^n W_\sigma(f) df$, $n = 0, 1, 2, 4$ become:

- $m_0 = \int_{20}^{120} W_\sigma(f) df = 10\,000$
- $m_1 = \int_{20}^{120} f W_\sigma(f) df = 700\,000$
- $m_2 = \int_{20}^{120} f^2 W_\sigma(f) df = 5.7333 \times 10^7$
- $m_4 = \int_{20}^{120} f^4 W_\sigma(f) df = 4.9760 \times 10^{11}$

From the spectral moments m_n the positive zero crossings ν_0^+ , the rate of peaks ν_p , the irregularity factor γ and the spectral width parameter can be derived:

- $\nu_0^+ = \sqrt{\frac{m_2}{m_0}} = 75.7188$
- $\nu_p = \sqrt{\frac{m_4}{m_2}} = 93.1615$

- $\gamma = \frac{\nu_0^+}{\nu_p} = \sqrt{\frac{m_2^2}{m_0 m_4}} = 0.8128$
- $\lambda = \sqrt{1 - \gamma^2} = 0.5826$

The constants to be substituted in (2.444) are:

- $X_m = 0.7514$
- $D_1 = 0.1093$
- $D_2 = 0.1997$
- $D_3 = 0.6909$
- $Q = 0.1367$
- $R = 0.5502$

The narrow-band cumulative damage prediction becomes $[E\{D(T = 1)\}] = \frac{\nu_0^+}{a} (\sqrt{2}\sigma_x)^b \Gamma(1 + \frac{b}{2}) = 0.0031$ and the associated fatigue life becomes $T_{NB} = 318.2370$ s. The visualization of Dirlik's p.d.f. $f_s(s)$ is shown in Fig. 2.55. The cumulative probability function $F(\infty) = \int_0^\infty f_s(s) ds = 1$, as expected. The wide-band fatigue life prediction now becomes $E\{D(T = 1)\} = \frac{\nu_p}{a} m_0^{\frac{b}{2}} [D_1 Q^b \Gamma(1 + b) + (\sqrt{2})^b \Gamma(1 + \frac{b}{2}) (D_2 |R|^b + D_3)] = 0.0027$ and the associated fatigue life becomes $T_{BB} = 370.4843$ s. The narrow-band approach is more conservative than Dirlik's wide-band approach.

In [202, 221] a wide band random stresses fatigue damage prediction method had been proposed based on the narrow band random process fatigue damage prediction as calculated by (2.437), denoted by $E\{D_{NB}(T)\}$. The wide band fatigue damage is given by

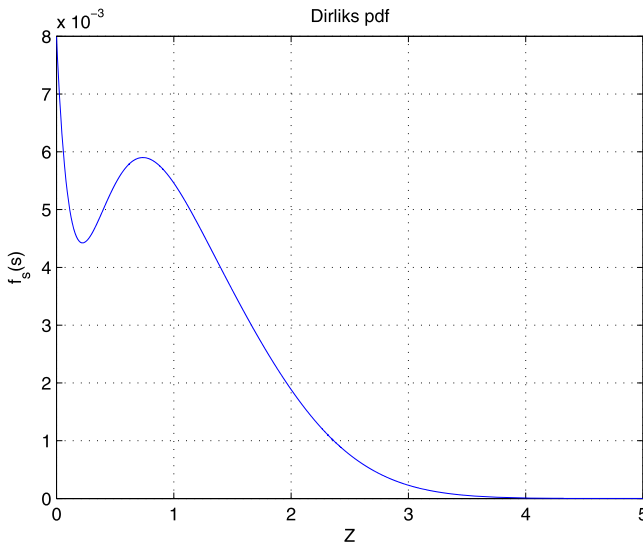


Fig. 2.55. Dirlik's probability density function $f_s(s)$

$$E\{D_{BB}(T)\} = \varphi(\lambda, b)E\{D_{NB}(T)\}, \tag{2.447}$$

where $\varphi(\lambda, b)$ is a modification factor which is a function of λ and b ($N(s)s = a$), λ is the spectral width parameter. The *modification factor* is derived in [221] and is a mix of spectral shapes and the rainflow method of measure the magnitudes

$$\varphi(\lambda, b) = K_1(b) + [1 - K_1(b)](1 - \lambda)^{k_2(b)}, \tag{2.448}$$

where

$$\begin{aligned} K_1(b) &= 0.926 - 0.033b, \\ K_2(b) &= 1.587b - 2.323. \end{aligned} \tag{2.449}$$

The standard deviation of the cumulative damage of the wide band process $\sigma_{BB,D(T)}$ is given by [202]

$$\sigma_{BB,D(T)} = \varphi(\lambda, b)\sigma_{NB,D(T)}, \tag{2.450}$$

where $\sigma_{NB,D(T)}$ can be calculated using (2.439).

For a wide-band process, having a spectral width parameter $\lambda = 1$, the expected maximum peak is given by [232]

$$\frac{\bar{\eta}_{max}}{\sigma_x} = \sqrt{2 \ln(\nu_0^+ T)} + \frac{\varepsilon}{\sqrt{2 \ln(\nu_0^+ T)}} - \frac{\ln \ln(\nu_0^+ T) + \ln(4\pi)}{2\sqrt{2 \ln(\nu_0^+ T)}}. \tag{2.451}$$

This is an extension to (2.424) and the Euler constant is $\varepsilon = 0.577216$.

Problems

2.35. Calculate the spectral width parameter λ for the following two spectra as illustrated in Fig. 2.56

- Spectrum 1 is a unimodal spectrum which approaches the narrow band case when $a \rightarrow b$, $W_0 = 0.04 \text{ g}^2/\text{Hz}$, $a = 50 \text{ Hz}$ and $b = 75 \text{ Hz}$.
- Spectrum 2 is a bimodal spectrum, $W_1 = 0.03 \text{ g}^2/\text{Hz}$, $a = 50 \text{ Hz}$ and $b = 75 \text{ Hz}$, $W_2 = 0.01 \text{ g}^2/\text{Hz}$, $c = 100 \text{ Hz}$ and $d = 200 \text{ Hz}$.

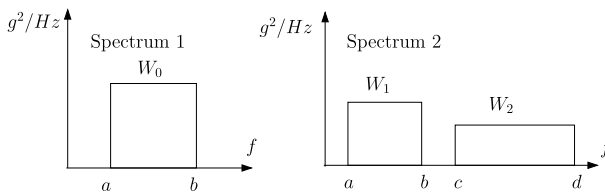


Fig. 2.56. Spectral density models

2.36. The equivalent stress range is defined as $S_h = [\int_0^\infty (2s)^b f_s(s) ds]^{1/b}$. Two peak distributions are defined:

- Rayleigh peak distribution (irregularity factor $\gamma = 1$, spectral band with parameter $\lambda = 0$, NB process) $f_s(s) = \frac{s}{\sigma^2} e^{-\frac{s^2}{2\sigma^2}}, 0 \leq s < \infty$, and
- Gaussian peak distribution (irregularity factor $\gamma = 0$, spectral band with parameter $\lambda = 1$, BB process) $f_s(s) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{s^2}{2\sigma^2}}, -\infty < s < \infty$.

Calculate the equivalent stress range S_h for each peak distribution.

Answers: $\frac{S_h[NB]}{\sigma} = 2\sqrt{2}[\Gamma(1 + \frac{b}{2})]^{1/b}, \frac{S_h[BB]}{\sigma} = 2\sqrt{2}[\frac{1}{2\sqrt{\pi}}\Gamma(1 + \frac{b}{2})]^{1/b}$.

2.37. The equivalent stress amplitude is defined as $\bar{S}_h = [\int_0^\infty (s)^b f_s(s) ds]^{1/b}$ and the s - N curve is given by $N(s)^b = a$. The expected number of peaks per unit of time is ν_p and the duration is T .

Prove that cumulative damage (Palgren-Miner method) $E\{D(T)\}$ becomes $E\{D(T)\} = \frac{T\bar{S}_h\nu_p}{a}$.

2.38. Consider a stationary random process with a one-sided PSD function

$$W(\omega) = \begin{cases} S_0, & \omega_a \leq \omega \leq \omega_b; \\ 0, & \text{elsewhere.} \end{cases}$$

- Define the spectral moments m_0, m_1, m_2 and m_4 .
- Define the bandwidth parameters δ and γ .
- Show that when $(\omega_b - \omega_a) \ll \frac{1}{2}(\omega_b + \omega_a)$, the process is narrow band.

This problem is taken from [203].

2.11 Practical Aspects Enforced Acceleration

In most cases the random mechanical loads for spacecraft and subsystems of spacecraft are specified in a very special manner. The PSD values of the acceleration depend on the frequency (Hz). In general, the frequency range is between 20–2000 Hz. The specification must be accompanied by the G_{rms} value of the random acceleration in the frequency range. An example of a typical acceleration specification is given below.

- 20–150 Hz 6 dB/oct
- 150–700 Hz $W_{\ddot{u}} = 0.04 \text{ g}^2/\text{Hz}$
- 700–2000 Hz –3 dB/oct
- $G_{rms} = 7.3 \text{ g}$.

The graphical representation of the random acceleration specification is shown in Fig. 2.57.

The *octave band* is between f_2 and f_1 defined by

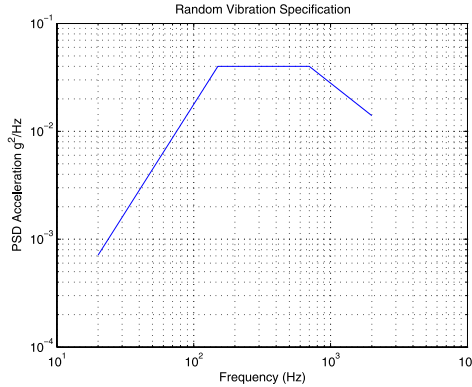


Fig. 2.57. Specification PSD acceleration $G_{rms} = 7.3 \text{ g}$

$$\frac{f_2}{f_1} = 2^1. \tag{2.452}$$

The power of 1 denotes the octave band. The number x is the number of octaves between two frequencies f and the reference frequency f_{ref} can be obtained using

$$\frac{f}{f_{ref}} = 2^x; \tag{2.453}$$

this yields

$$x = \frac{\ln(\frac{f}{f_{ref}})}{\ln 2} = \frac{\log(\frac{f}{f_{ref}})}{\log 2}. \tag{2.454}$$

The relation between the PSD values depends on the number of dBs per octave n (dB/oct) and the number of octaves between two frequencies f and f_{ref} . The relation in dB between $W_{\ddot{u}}(f)$ and $W_{\ddot{u}}(f_{ref})$ is given by

$$10 \log \left\{ \frac{W_{\ddot{u}}(f)}{W_{\ddot{u}}(f_{ref})} \right\} = nx = \frac{n \log(\frac{f}{f_{ref}})}{\log 2}, \tag{2.455}$$

or

$$\left\{ \frac{W_{\ddot{u}}(f)}{W_{\ddot{u}}(f_{ref})} \right\} = \left(\frac{f}{f_{ref}} \right)^{\frac{n}{10 \log 2}} \approx \left(\frac{f}{f_{ref}} \right)^{\frac{n}{3}}. \tag{2.456}$$

If both the frequency f axis and the axis of the PSD function $W(f)$ have a log scale than the angle m (dB/freq) can be obtained by

$$m = \frac{\log W_{\ddot{u}}(f) - \log W_{\ddot{u}}(f_{ref})}{\log f - \log f_{ref}} = \frac{\log \left\{ \frac{W_{\ddot{u}}(f)}{W_{\ddot{u}}(f_{ref})} \right\}}{\log(\frac{f}{f_{ref}})} = \frac{n}{3}. \tag{2.457}$$

Finally the derivation of the following expression is obtained, a relation between the PSD functions and the frequencies

$$W_{\ddot{u}}(f) = W_{\ddot{u}}(f_{ref}) \left(\frac{f}{f_{ref}} \right)^{\frac{n}{3}} = W_{\ddot{u}}(f_{ref}) \left(\frac{f}{f_{ref}} \right)^m. \quad (2.458)$$

The total root mean square (rms) value (magnitude) of $\ddot{u}(t)$ is equal to the square root of the area bounded by the PSD function between frequency limits f_1 and f_2 . This can be written as

$$\ddot{u}_{rms} = \sqrt{E\{\ddot{u}^2(t)\}} = \int_{f_1}^{f_2} W_{\ddot{u}}(f) df. \quad (2.459)$$

The integral (2.459) is in accordance with (2.61). Substituting (2.458) into (2.459) to obtain the following expression:

$$\begin{aligned} \ddot{u}_{rms} &= \sqrt{E\{\ddot{u}^2(t)\}} \\ &= \sqrt{\int_{f_1}^{f_2} W_{\ddot{u}}(f_1) \left(\frac{f}{f_1} \right)^m df} = \sqrt{\frac{W_{\ddot{u}}(f_1) f_1}{m+1} \left[\left(\frac{f_2}{f_1} \right)^{m+1} - 1 \right]}, \quad f_1 < f_2 \\ &= \sqrt{-\int_{f_2}^{f_1} W_{\ddot{u}}(f_2) \left(\frac{f}{f_2} \right)^m df} = \sqrt{\frac{W_{\ddot{u}}(f_2) f_2}{m+1} \left[1 - \left(\frac{f_1}{f_2} \right)^{m+1} \right]}, \quad f_1 < f_2. \end{aligned} \quad (2.460)$$

Example. Use (2.460) to calculate the rms value of $\ddot{u}(t)$ of a typical random specification. Assume that the test specifies the following values:

- $f_1 = 20$ Hz, $f_2 = 150$ Hz
- $W_{\ddot{u}}(f_2) = 0.02$ g²/Hz
- $m = \frac{n}{3} = 2$.

The PSD value at $f_1 = 20$ Hz can be calculated using (2.458)

$$W_{\ddot{u}}(f_1) = W_{\ddot{u}}(f_2) \left(\frac{f_1}{f_2} \right)^m = 0.02 \left(\frac{20}{150} \right)^2 = 3.556 \times 10^{-4} \text{ g}^2/\text{Hz}.$$

The rms value of the base acceleration $\ddot{u}(t)$ becomes

$$\begin{aligned} \ddot{u}_{rms} &= \sqrt{\frac{W_{\ddot{u}}(f_2) f_2}{m+1} \left[1 - \left(\frac{f_1}{f_2} \right)^{m+1} \right]} \\ &= \sqrt{\frac{0.02 \times 150}{2+1} \left[1 - \left(\frac{20}{150} \right)^{2+1} \right]} = 1.000 \text{ g}. \end{aligned}$$

The parameters needed to calculate the G_{rms} value of the random acceleration spectrum are illustrated in Fig. 2.58.

The specification of the PSD (sometimes called *acceleration spectral density* (ASD)) of the enforced acceleration or base excitation can be divided into three regions:

- Spectrum with a positive slope n_1 (rising)
- Flat spectrum (slope is zero)
- Spectrum with negative slope n_2 (falling)

Spectrum with a Positive Slope

Figure 2.58 shows a rising spectrum with a constant slope $n_1 > 0$ between f_1 and f_2 . The constant slope is expressed in decibels per octave. The area A_1 can be calculated as follows

$$A_1 = \left\{ \frac{W(f_2)f_2}{m_1 + 1} \left[1 - \left(\frac{f_1}{f_2} \right)^{m_1+1} \right] \right\}, \quad (2.461)$$

where $m_1 = n_1/3$ and $n_1 > 0$ is the increase of the PSD value in decibels per octave.

Flat Spectrum

For a flat spectrum with a zero slope between f_2 and f_3 with $m_1 = 0$ in (2.461) the area A_2 becomes

$$A_2 = W(f_2)[f_3 - f_2], \quad (2.462)$$

as shown in Fig. 2.58.

Spectrum with a Negative Slope

For a falling spectrum of a constant slope $n_2 < 0$ between f_3 and f_4 the constant slope is expressed in decibels per octave. The area A_3 can be calculated as follows

$$A_3 = \left\{ \frac{W(f_3)f_3}{m_2 + 1} \left[\left(\frac{f_4}{f_3} \right)^{m_2+1} - 1 \right] \right\}, \quad m_2 \neq -1, \quad (2.463)$$

where $m_2 = n_2/3$, and $n_2 (< 0)$ is the decrease of the PSD value in decibels per octave. Equation (2.463) is not applicable if $m_2 = -1$. In that case we have to calculate the value of A_3 when $\lim_{m_2 \rightarrow -1}$. This limit can be found

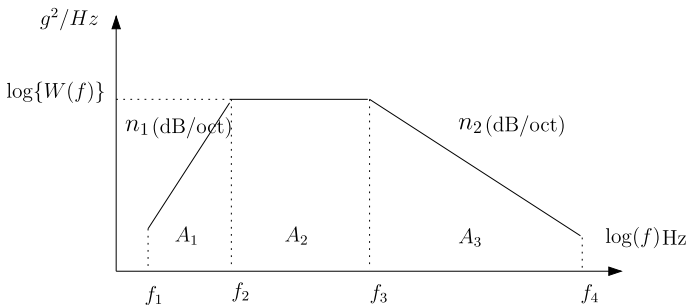


Fig. 2.58. Calculation of G_{rms}

Table 2.21. Power spectral density enforced acceleration

Frequency (Hz)	PSD (g ² /Hz)
10	0.005
100	0.05
500	0.05
2000	0.0125

using L'Hôpital's Rule²¹ [100]. If $u(m_2) = (\frac{f_4}{f_3})^{m_2+1} - 1$ and $v(m_2) = m_2 + 1$, then *L'Hôpital's rule* gives

$$A_3 = W(f_3)f_3 \ln\left(\frac{f_4}{f_3}\right) = 2.30W(f_3)f_3 \log\left(\frac{f_4}{f_3}\right), \quad m_2 = -1. \quad (2.464)$$

The G_{rms} of the enforced random acceleration specification can be obtained, as illustrated in Fig. 2.58, by the following expression

$$G_{rms} = \sqrt{A_1 + A_2 + A_3}. \quad (2.465)$$

Example. Consider the PSD values of a specification of enforced accelerations as shown in Table 2.21. The G_{rms} value of that spectrum is calculated.

The slopes m_1 and m_2 are now calculated

$$m_1 = \frac{\log(\frac{0.05}{0.005})}{\log(\frac{100}{10})} = 1.00, \quad m_2 = \frac{\log(\frac{0.0125}{0.05})}{\log(\frac{2000}{500})} = -1.00.$$

The areas A_1 , A_2 and A_3 become

$$\begin{aligned} A_1 &= \left\{ \frac{W(f_2)f_2}{m_1 + 1} \left[1 - \left(\frac{f_1}{f_2}\right)^{m_1+1} \right] \right\} = 2.48 \text{ g}^2, \\ A_2 &= W(f_2)[f_3 - f_2] = 20.00 \text{ g}^2, \\ A_3 &= 2.30W(f_3)f_3 \log\left(\frac{f_4}{f_3}\right) = 34.62 \text{ g}^2. \end{aligned}$$

The G_{rms} value of the enforced acceleration spectrum as shown in Table 2.21 can now be obtained using (2.465)

$$G_{rms} = \sqrt{A_1 + A_2 + A_3} = 7.56 \text{ g}.$$

²¹ Let $u(a) = v(a) = 0$. If there exists a neighborhood of $x = a$ such that (1) $v(x) \neq 0$, except for $x = a$, and (2) $u'(x)$ and $v'(x)$ exist and do not vanish simultaneously, then

$$\lim_{x \rightarrow a} \frac{u(x)}{v(x)} = \lim_{x \rightarrow a} \frac{u'(x)}{v'(x)}$$

whenever the limit on the right exists.

Problems

2.39. The random vibration levels for a certain component are given in Table 2.22. Calculate the G_{rms} of that specification?

Answer: 14.1 g.

2.40. A PSD spectrum for the enforced acceleration is given in Table 2.23. Calculate the PSD values of the enforced acceleration at 20 and 2000 Hz and calculate G_{rms}

Answers: $W(20) = 0.0833 \text{ g}^2/\text{Hz}$, $W(2000) = 0.0875 \text{ g}^2/\text{Hz}$, $G_{rms} = 18.5 \text{ g}$.

2.41. A spectrum for the random enforced acceleration is given in Table 2.24. Calculate the slopes (dB/oct) and calculate G_{rms}

Answers: 6 dB/oct, -3 dB/oct, $G_{rms} = 14.7 \text{ g}$.

2.42. Both ESA²² and NASA²³ enforced acceleration specifications are specified (see Table 2.25). Investigate the severity of both specifications when exposed to a SDOF system with a mass $M = 5 \text{ kg}$ and a natural frequency $f_n = 140 \text{ Hz}$.

- Calculate for both specifications G_{rms}
- The value of M both specifications have equal G_{rms} .
- The value of M for which both specifications have equal maximum PSD values (g^2/Hz).
- Find the worst case random enforced acceleration specification for the given unit. Explain your choice.

Table 2.22. Power spectral density enforced acceleration

Frequency (Hz)	PSD (g^2/Hz)
20	0.026
20–50	+6 dB/oct
50–800	0.16
800–2000	-6 dB/oct
2000	0.026

Table 2.23. Random vibration test specification

Frequency range (Hz)	PSD (g^2/Hz)	G_{rms} (g)
20	?	
20–60	3 dB/oct	
60–700	0.25	?
700–2000	-3 dB/oct	
2000	?	

²² European Space Agency.

²³ National Aeronautics and Space Administration.

Table 2.24. Random vibration test specification

Frequency range (Hz)	PSD (g^2/Hz)	G_{rms} (g)
20	0.0027	
20–150	? dB/oct	
150–800	0.15	?
800–2000	? dB/oct	
2000	0.06	

Table 2.25. ESA and NASA random enforced acceleration specifications

ESA f (Hz)	ESA PSD (g^2/Hz)	NASA f (Hz)	NASA PSD (g^2/Hz)
20–100	3 dB/oct	20–50	6 dB/oct
100–400	$0.05 \frac{M+20}{M+1}$	50–800	0.16
400–2000	–3 dB/oct	800–2000	–6 dB/oct

- What is the 3σ reaction force for both specifications for the given unit, if $Q = 10$ and $g = 9.81 \text{ m/s}^2$?

Answers:

- ESA $G_{rms} = 31.5 \sqrt{0.05 \frac{M+20}{M+1}}$ g, NASA $G_{rms} = 14.1$ g
- $M = 5.3$ kg
- $M = 7.6$ kg
- ESA
- ESA $F_{reaction} = 3150$ N, NASA $F_{reaction} = 2760$ N

2.43. In a structural mechanical test plan for the CanX-1 satellite (Space Flight Laboratory, SFL, University of Toronto Institute of Aerospace Studies, UTIAS) the following random vibration load were specified (see Table 2.26, log-log scale), however, the G_{rms} was missing. Calculate the G_{rms} .

Answer: 6.8 g.

2.12 3-Sigma Strength Verification

In this section the following items about the 3-sigma approach will be discussed

- Strength verification
- Estimation of payload random vibration load for structure design
- Random vibration reduction on the basis of design loads
- Shock response spectrum (SRS) random enforced acceleration

Table 2.26. Random vibration spectrum

Frequency (Hz)	Power Spectral Density (g^2/Hz)
20–46	0.000312
52–98	0.019656
102–140	0.22776
144–180	0.00156
185–250	0.2808
256–464	0.00156
472–535	0.156
537–866	0.00156
972–1020	0.0702
1030–1940	0.00156
1940–2000	0.00156
Overall G_{rms}	? (g)

2.12.1 Strength Verification

When fatigue and fracture are ignored it is very common to use the 3-sigma (3σ) values of the responses (displacements, acceleration, stresses, forces, etc.) for verification against the more or less static requirements like a maximum excursion at a certain location, maximum design loads (load factors), buckling loads, etc., [222, 132]. However, Gumbel in [76] stated the following about the “Three-sigma fallacy”:

“The founders of the calculus of probabilities were too occupied with the general behavior of statistical masses to be interested in the extremes. The oldest remarks about such values seem to be due to Fourier who stated that, for a normal distribution, the probability to exceed $3\sqrt{2}$ times the standard deviation is about 1 in 50 000, and could therefore be neglected. From this small probability, the erroneous conclusion was drawn that about three times the standard deviation should be considered as the maximum for any statistical variate, for any number of observations. Helmert (1877) stated, on the contrary, that the probability of surpassing a values depends upon the number of observations. The idea that three times the standard deviation should be considered as maximum-irrespective of the number of observations and the distribution-still prevails among most “practical” people. It is even advocated, although in vague form in the classical books by Charlier (1920) and by Yule (1937). However, the fallacy of this “kitchen” rule is obvious. If the initial variate is unlimited, the largest value is unlimited too, and if the *sample size* is increased, the *largest value* encountered will likewise increase. The question is *HOW MUCH?* For very *small* sample sizes, the three sigma condition is too *strong*; for the very large sample sizes, it is too *weak*.”

The calculated rms value is equal to the 1σ value if the average or mean value is $\mu = 0$, then, for the signal $x(t)$ the covariance function $C_{xx}(\tau)$ is equal to the autocorrelation function $R_{xx}(\tau)$, thus

$$C_{xx}(0) = R_{xx}(0) = \sigma_x^2 + \mu_x^2 = \sigma_x^2 = x_{rms}^2.$$

The probability that the Gaussian signal $x(t) \leq 3\sigma_x$, with the normalized variable $z = \frac{x - \mu_x}{\sigma_x}$, is given by

$$Prob\{x(t) \leq 3\sigma\} = \frac{1}{\sqrt{\pi}} \int_{-\infty}^3 e^{-\frac{z^2}{2}} dz = 0.99865, \tag{2.466}$$

thus

$$Prob\{x(t) \geq 3\sigma\} = \frac{1}{\sqrt{\pi}} \int_3^{\infty} e^{-\frac{z^2}{2}} dz = 1 - 0.99865 = 0.00135. \tag{2.467}$$

The probability that the Gaussian signal is $|x(t)| \geq 3\sigma_x$, with the normalized variable $z = \frac{x - \mu_x}{\sigma_x}$, is given by

$$Prob\{|x(t)| \leq 3\sigma\} = \frac{1}{\sqrt{\pi}} \int_{-3}^3 e^{-\frac{z^2}{2}} dz = 0.99730, \tag{2.468}$$

thus

$$Prob\{|x(t)| \geq 3\sigma\} = \frac{2}{\sqrt{\pi}} \int_3^{\infty} e^{-\frac{z^2}{2}} dz = 1 - 0.99730 = 0.00270. \tag{2.469}$$

Example. The strength analysis of a 3σ worst-case design will be traced in this example.

A mass $M = 40$ kg is mounted to the free end of a fixed-free beam. The beam has a bending stiffness EI and a length $L = 1$ m. The fixed-free beam with attached mass is excited at the base with a constant random acceleration with a PSD $W_{\ddot{u}} = 0.40$ g²/Hz (1 g = 9.81 m/s²). The lowest natural frequency of the is $f_n = 25$ Hz. The fixed-free beam with attached mass is illustrated in Fig. 2.59. The cross section of the beam is an I-section with the height of the profile h , a width $b = \frac{h}{2}$ and a constant thickness in web and flanges $\frac{h}{40}$. The material of the beam (Al-alloy) has an allowable yield stress $F_y = 250$ MPa, and Young’s modulus $E = 70$ GPa. The dynamic amplification factor $Q = 10$ ($\zeta = 0.05$). The 3-sigma approach will be applied

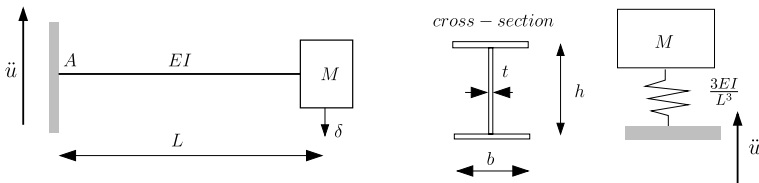


Fig. 2.59. Fixed-free bending beam with tip mass

to verify the strength capability of the fixed beam at A , the point where the maximum bending moment M_{bend} is expected. The fixed-free beam is represented by a SDOF system with mass M , the associated stiffness can be obtained applying a unit load at the tip and using the Myosotis equations [80]. The deflection of the beam due to unit load at the free end is $\delta = \frac{L^3}{3EI}$; the stiffness of the spring is

$$k = \frac{1}{\delta} = \frac{3EI}{L^3}.$$

The second moment of area of the I-section is given by

$$I = \frac{h^3t}{12} + 2(bt) \times \left(\frac{h}{2}\right)^2 = \frac{h^4}{120}.$$

The fundamental natural frequency of the fixed-free beam with attached mass at the free end can be calculated by

$$f_n = \frac{1}{2\pi} \sqrt{\frac{k}{M}} = \frac{1}{2\pi} \sqrt{\frac{3EI}{ML^3}} \text{ Hz.}$$

The required second moment of area I can now be calculated

$$I \geq (2\pi f_n)^2 \frac{ML^3}{3E} = 4.7 \times 10^{-6} \text{ m}^4.$$

This gives a section height $h = 154$ mm, the section width $b = 77$ mm and the thickness of flanges and web is $t = 3.85$ mm. The resistance against bending is

$$W = \frac{2I}{h} = \frac{h^3}{60} = 6.1 \times 10^{-5} \text{ m}^3.$$

We obtain the rms acceleration of the attached mass using Miles' equation

$$a_{rms} = 9.81 \sqrt{\frac{\pi}{2} f_n Q W \ddot{u}} = 122.95 \text{ m/s}^2.$$

The bending moment at A is given by

$$M_{bend} = M a_{rms} L = 4.917 \times 10^3 \text{ Nm.}$$

The bending stress at A at the extreme fibre distance $\frac{h}{2}$ is

$$\sigma_{bend} = \frac{M_{bend}}{W} = 6.0475 \times 10^7 \text{ Pa.}$$

With the 3-sigma approach the margin of safety (MS) of the bending stress σ_{bend} with respect to the allowable yield stress

$$MS = \frac{F_y}{3 \times \sigma_{bend}} - 1 = 0.034.$$

The design is acceptable. The 3-sigma bending stress $3\sigma_{bend}$ is used as “static” stress at point A and is compared with an allowable yield stress F_y . The MS value tells us about the margin which is left with respect to the allowable stress.

2.12.2 Estimation of Payload Random Vibration Load Factors for Structure Design

In their paper [30] Chung et al. discussed six estimation methods to determine 3σ vibration load factors related to random enforced vibration for designing secondary structures:

1. Miles’ equation,
2. Vibration specifications,
3. PSD integration over the full bandwidth, 20 → 2000 Hz,
4. Half power point approach (derived from measured data),
5. Mass participation approach,
6. PSD integration over a reduced bandwidth, 20 → 300 Hz.

Miles’ Equation

This approach assumes the system has a dominant natural frequency (with respect to structural response) and the 3σ peak random vibration load factor \ddot{X} can be obtained using Miles’ equation (2.149)

$$\ddot{X} = 3\sqrt{\frac{\pi}{2} f_i Q_i W_{\ddot{u}}(f_i)}, \quad (2.470)$$

where Q_i is the amplification factor, normally for random response analysis $Q_i = 10$ is taken, f_i (Hz) is the dominant natural frequency and $W_{\ddot{u}}(f_i)$ g^2/Hz is the value of the enforced vibration acceleration PSD at the frequency f_i . The dominant frequency can be determined with the aid of the principle of the modal effective mass (see Appendix D).

Vibration Specifications

This approach uses the G_{rms} value of the design specification directly. The load factor \ddot{X} can be obtained from the following equation

$$\ddot{X} = 3G_{rms}. \quad (2.471)$$

This method is very conservative because it assumes all the energy within the frequency band contributes to the random vibration loads.

PSD integration over the Full Bandwidth

This approach uses the analytical or measured PSD data $W_{resp}(f)$ to derive the random vibration load factor \ddot{X} . It is assumed that all modes up to 2000 Hz contribute to the random vibration loads.

$$\ddot{X} = 3\sqrt{\int_{20}^{2000} W_{resp}(f)df}. \tag{2.472}$$

This method is less conservative than the vibration specification method in predicting the random vibration load factor because it uses the responses (stress, acceleration) of the structure instead of the enforced acceleration to which the structure is exposed. However, it still assumes that the high frequency modes contribute to the structural random vibration load.

Half Power Point Approach

The half power point bandwidth Δf_{HP} is used to calculate the random vibration load factor from test data. The fundamental frequency f_n and the PSD amplitude $W_{resp,max}$ at that fundamental frequency can be easily extracted from the test data. This is illustrated in Fig. 2.60. The random vibration load factor \ddot{X} is

$$\ddot{X} = 3\sqrt{\frac{W_{resp,max}}{2} \Delta f_{HP}} = 3\sqrt{W_{resp,HP} \Delta f_{HP}}. \tag{2.473}$$

Mass Participation Approach

The mass participation approach is based on (2.219), however, the 3σ philosophy will be applied, thus

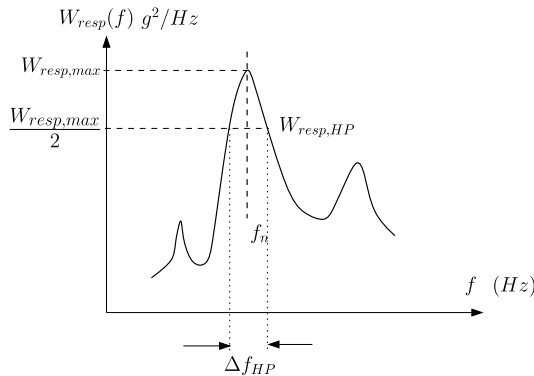


Fig. 2.60. Half Power (HP) approach

$$F_{k,3\sigma,base} = 3 \sqrt{\sum_{i=1}^n \left[M_{eff,ik}^2 \frac{\pi}{2} f_i Q_i W_{\ddot{u},k}(f_i) \right]} = M_{tot,k} \ddot{X}_k, \quad k = 1, 2, \dots, 6, \quad (2.474)$$

where $F_{k,rms,base}$ is the interface force caused by the enforced acceleration \ddot{u}_k , $k = 1, 2, \dots, 6$ with PSD values $W_{\ddot{u},k}(f_i)$, f_i , $i = 1, 2, \dots, n$ are the natural frequencies, $M_{tot,k}$, $k = 1, 2, \dots, 6$ are the total mass or second moments of mass, $M_{eff,ik}$, $i = 1, 2, \dots, n$, $k = 1, 2, \dots, 6$ are the modal effective masses, n is the number of modes considered, Q_i , $i = 1, 2, \dots, n$ are the amplification factors and \ddot{X}_k are the design load factors. It will be assumed that the inertia loads $M_{tot,k} \ddot{X}_k$, $k = 1, 2, \dots, 6$ will lead to the same interface forces. The load factors can now easily be calculated using (2.474):

$$\ddot{X}_k = \frac{1}{M_{tot,k}} \sqrt{\sum_{i=1}^n \left[3M_{eff,ik} \sqrt{\frac{\pi}{2} f_i Q_i W_{\ddot{u},k}(f_i)} \right]^2}, \quad k = 1, 2, \dots, 6. \quad (2.475)$$

The process of estimating the random vibration load factors \ddot{X}_k is as follows:

1. Determine the modal effective masses of each mode within the bandwidth of interest, in general, from the finite element modal analysis.
2. Determine the corresponding random vibration level of each mode from the specification.
3. Compute the peak load associated with the random level obtained in 2 for each mode by using Miles' equation.
4. Multiply the peak load obtained in 3 by the modal effective mass determined in 1 for each mode.
5. The composite random vibration load factor is determined by the root-sum-square (RSS) of the mass weighted peak load of each mode computed in 4, and normalized by the total mass using (2.475).

PSD Integration over a Reduced Bandwidth

This approach uses the analytical or measured PSD data $W_{resp}(f)$ to derive the random vibration load factor \ddot{X} . It is assumed that all modes up to 300 Hz contribute to the random vibration loads. The random vibration load factor is defined as follows

$$\ddot{X} = 3 \sqrt{\int_{20}^{300} W_{resp}(f) df}. \quad (2.476)$$

From several acoustic tests it has been noticed that the accumulated strain generally stabilizes below 300 Hz, while the rms acceleration is still building up

Example. Figure 2.61 shows a simple NDOF dynamic system, to which the 3-sigma approaches is applied. This system is exposed to an enforced random acceleration, for which spectrum is given in Table 2.27 and visualized in Fig. 2.62. The number of discrete masses $m = 1$ kg and the number of spring elements $k = 10^7$ N/m is set to $N = 50$. The total mass is $M_{tot} = 50$ kg. The modal damping ratio $\zeta = 0.05$ is constant for all N modes.

The modal effective mass is plotted against the natural frequency in Fig. 2.63. The five lowest natural frequencies and associated modal effective masses are listed in Table 2.28. The PSD values of the responses, and accelerations (g^2/Hz) of the first discrete mass and N th spring force (N^2/Hz), respectively, are shown in Fig. 2.64. The increase of the rms values vs the frequency are shown in Fig. 2.65. Notice that the convergence of the rms acceleration is slower than the convergence of the spring force.

The rms acceleration per discrete mass ($1 - N$) and the rms force in the spring elements ($1 - N$) are shown in Fig. 2.66. This completes the response

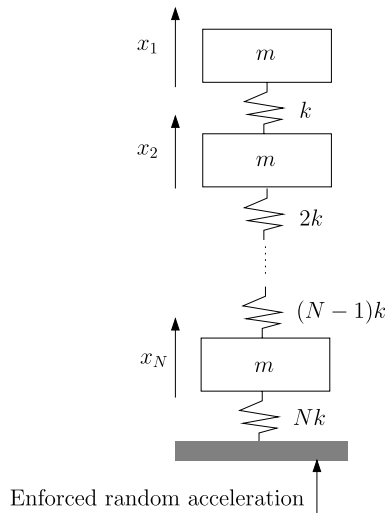


Fig. 2.61. NDOF dynamic system

Table 2.27. Random vibration specification [30]

Frequency (Hz)	PSD g^2/Hz
20	0.018
20–125	1.68 dB/oct
125	0.05
300	0.05
300–2000	–3.12 dB/oct
2000	0.007
Overall	6.33 G_{rms}

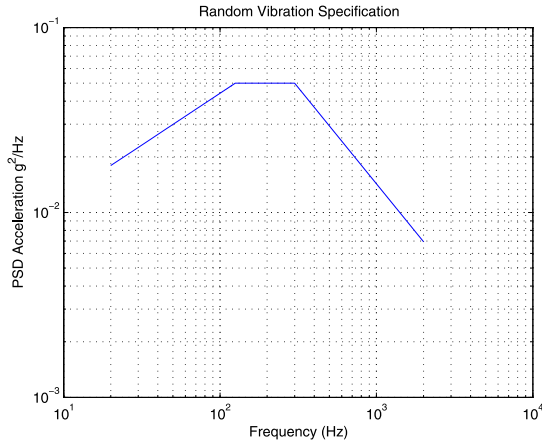


Fig. 2.62. Random vibration specification, Table 2.27

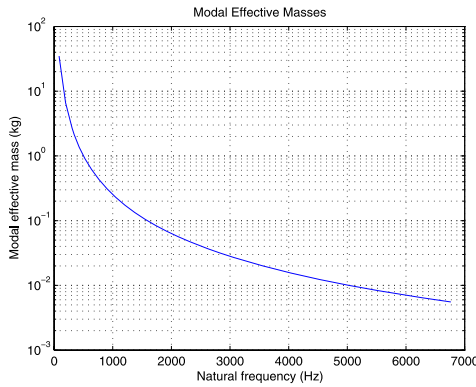


Fig. 2.63. Modal effective mass vs natural frequency (Hz)

Table 2.28. Modal effective mass vs natural frequency

Mode \neq	Natural frequency (Hz)	Modal effective mass (kg)
1	85.1598	34.9278
2	195.4970	6.6277
3	306.5329	2.6958
4	417.7903	1.4512
5	529.2051	0.9045

analysis of the N-SDOF system, as illustrated in Fig. 2.61, excited by enforced random acceleration.

The complete response analysis will be compared to the approximate 3σ calculation of the load factors \ddot{X} . This is shown in Table 2.29.

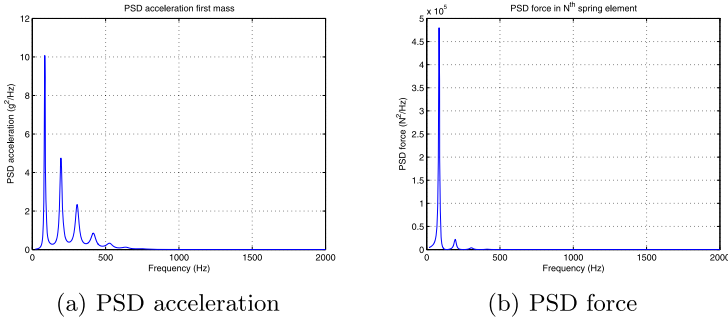


Fig. 2.64. PSD responses

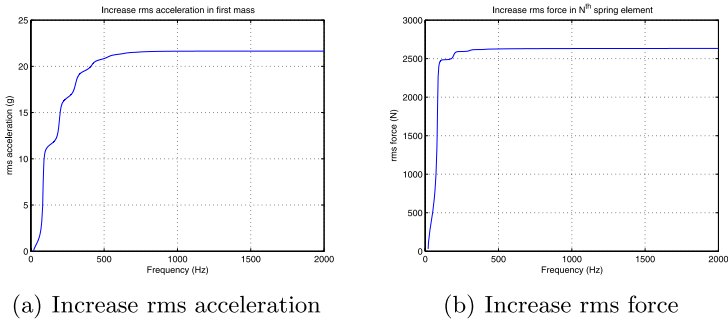


Fig. 2.65. Rms responses

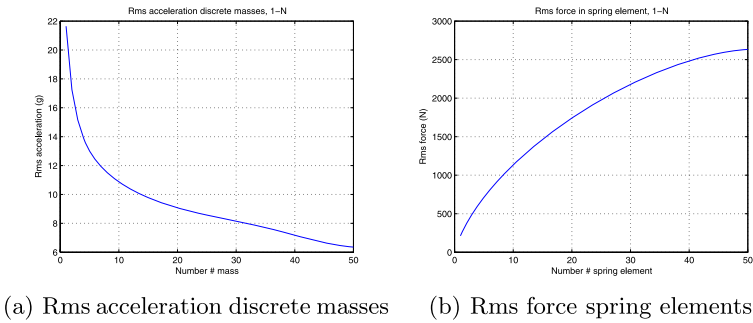


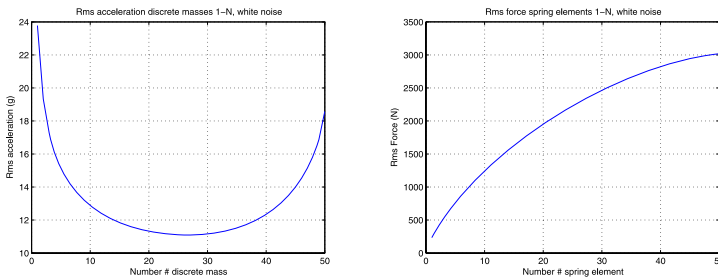
Fig. 2.66. Rms responses, discrete masses, and spring elements

Observations

- Miles’ equation. As expected, Miles’ approach gave a conservative estimation of the random vibration load factors with the assumption of constant damping for all modes and one dominant mode containing most of the energy of the system.

Table 2.29. Approximate 3σ approaches

Method	rms acceleration masses (g)	\ddot{X} (3σ) (g)	rms force Nth spring element (N)
Miles's equation			
1 mode		22.04	
Specification		18.96	
Complete solution (20–2000 Hz) (all modes)	6.35–21.64 Fig. 2.66(a)		2632.2 Fig. 2.66(b)
Half power			
1 mode		19.65	
Mass Participation (all modes) (1 + 2 + 3 modes)		16.45 16.35	2689.4
Complete solution (20–300 Hz) (all modes)	3.59–17.78		2602.1
White noise ($0 \rightarrow \infty$ Hz) Lyapunov equation page 78 ($0.05 \text{ g}^2/\text{Hz}$, all modes)	Fig. 2.67(a)		3017.9 Fig. 2.67(b)



(a) Rms acceleration discrete masses (b) Rms force spring elements

Fig. 2.67. Rms responses discrete masses, spring elements, white noise

- Vibration specification approach. This is the most straightforward approach for estimating the random vibration load factors.
- Half power approach. This is similar to Miles' equation, however, instead of using the spectrum and the amplification factor Q , the half power method uses the response power spectrum density function and the bandwidth corresponding to the half power points to estimate the random vibration load factors. To compute the load factors, apply this approach to every mode below 300 Hz. To obtain the composite random vibration load factors, the root-sum-square can be applied. The random vibration load factors may be weighted by the relative modal effective mass (M_{eff}/M_{tot}) of each mode.
- Mass participation approach. This approach provides a reasonable way to estimate the random vibration load factors if a mathematical (finite

Table 2.30. Derived random vibration load factors for MTSAS [30]

Directions	Miles' equation	Spec	0–2000 Hz bandwidth	0–300 Hz bandwidth	Half power point	Mass participation
X-load factor (g)	29.1	19.0	26.3	14.5	8.5	15.7
Y-load factor (g)	23.0	19.0	42.1	17.7	6.9	14.7
Z-load factor (g)	18.9	19.0	38.8	12.0	3.0	9.4

element) model is available. This method takes into account the dynamic behavior of each mode, and the significance of each mode. To compute the composite random vibration load factors we use the individual modes below 300 Hz.

- Complete solution (20–300 Hz). This approach is a modification of the complete solution over 20–2000 Hz bandwidth for the estimation of reasonable random vibration load factors. The modification is based on the assessments of the strain gauge data measured from several cargo element level acoustic tests. This approach can be employed to either analytical or test derived PSDs. In general, the random vibration load factors estimated from this approach is about 50% of those derived from the full bandwidth approach.
- The white noise (max. PSD) enforced acceleration will of course give envelope response characteristics.

In summary, we recommend the employment of the 300 Hz cut-off bandwidth to estimate the random vibration load factors for structural design. The general procedure is to obtain either analytical or measured PSD functions first, then estimate the random vibration load factors based on integration of the PSD functions using the 300 Hz cut-off frequency.

Reference [30] describes a case study about Module and Truss Structure Attachment System (MTSAS). The random vibration specification levels for MTSAS are given in Table 2.27. The final results of the several approaches and methods are presented in Table 2.30.

2.12.3 Random Vibration Input Reduction

The quasi-static design limit load applied for the design of equipment and instruments is mostly based on experience from previous spacecraft projects and is defined, in general, using the *mass acceleration curve* (MAC). Such a curve can be derived from analytical and flight data, and includes the effects of both transient and mechanically transmitted random vibration [132]. The design loads (factors) are dependent on the mass of the equipment or instrument. The load factors (inertia loads) γ (g) define the interface loads between the equipment, instruments and the spacecraft to which the boxes are mounted. The maximum random interface loads will occur at the resonance frequencies or approximately at the natural frequencies. If the random interface load is higher than the interface loads caused by the design limit load the

input acceleration levels may be reduced (notched) at resonant frequencies. However, it must be pointed out that notching of vibration input must be always negotiated with the customer. The notch philosophy is based on [110].

The notch criteria with regard to the design limit loads is defined as

$$3\sqrt{\int_{f_{min}}^{f_{max}} W_{F_{interface}}(f)df} \leq M_{total}\gamma, \quad (2.477)$$

where $W_{F_{interface}}(f)$ the PSD function of the random interface loads, f_{min} the minimum frequency and f_{max} the maximum frequency of interest in the frequency band, M_{total} the total mass of the equipment or instrument and γ the load factor. Equation (2.477) means that the 3σ ($3 \times rms$) value of the interface force must be less than or equal to the interface load caused by the design loads.

The expected 3σ peak value of the interface force $F_{interface}$ is based on the mass participation method (2.215)

$$F_{interface} = 3\sqrt{\sum_{k=1}^n \left[M_{eff,k}^2 \frac{\pi}{2} f_k Q_k W_{\ddot{u}}(f_k) \right]} \leq M_{total}\gamma, \quad (2.478)$$

where n is the number of modes taken into account. The sum of all modal effective masses $M_{eff,k}$ is equal to the total mass M_{total} of the system. If m ($m < n$) modes are considered, residual mass $M_{residual}$ must be taken into account

$$M_{residual} = M_{total} - \sum_{k=1}^m M_{eff,k}. \quad (2.479)$$

Equation (2.478) must be adapted as follows:

$$F_{interface} = 3\sqrt{\sum_{k=1}^m \left[M_{eff,k}^2 \frac{\pi}{2} f_k Q_k W_{\ddot{u}}^*(f_k) \right] + M_{residual}^2 \int_{f_{min}}^{f_{max}} W_{\ddot{u}}^*(f)df} \leq M_{total}\gamma, \quad (2.480)$$

where $W_{\ddot{u}}^*(f_k)$ and $W_{\ddot{u}}^*(f)$ are the adapted enforced acceleration PSD, $W_{\ddot{u}}(f_k)$ the original PSD function, and f_k are the natural frequencies. The random enforced acceleration spectrum may be reduced (notched) when the 3σ random interface loads are higher than the quasi-static interface loads based on the design loads.

The modal power contribution per mode with natural frequency f_k is related to the original PSD value of the enforced random acceleration $W_{\ddot{u}}(f_k)$. The power is defined by the force $M_{eff,k}\ddot{x}_{rms,k}$ times the velocity $\dot{x}_{rms,k}$. The modal power contribution is given by

$$M_{eff,k} \ddot{x}_{rms,k} \dot{x}_{rms,k} = CW_{\ddot{u}}(f_k), \quad (2.481)$$

where $\ddot{x}_{rms,k}$ is the *rms* acceleration at the interface, $\dot{x}_{rms,k}$ the *rms* velocity at the interface, C is a constant and $W_{\ddot{u}}(f_k)$ the original PSD function of the enforced acceleration at the natural frequency f_k . Rewrite (2.481) as follows

$$M_{eff,k} \frac{\ddot{x}_{rms,k}^2}{2\pi f_k} = CW_{\ddot{u}}(f_k). \quad (2.482)$$

By introducing of Miles' equation for the *rms* acceleration $\ddot{x}(t)$, we obtain the following expression:

$$M_{eff,k} \frac{\frac{\pi}{2} f_k Q_k W_{\ddot{u}}^*(f_k)}{2\pi f_k} = \frac{M_{eff,k}}{4} Q_k W_{\ddot{u}}^*(f_k) = CW_{\ddot{u}}(f_k). \quad (2.483)$$

The constant C is the same for all modes in the frequency range of interest and can be written as

$$\frac{M_{eff,k} Q_k W_{\ddot{u}}^*(f_k)}{W_{\ddot{u}}(f_k)} = 4C = \bar{C} \quad k = 1, 2, \dots, m. \quad (2.484)$$

If all n modes are considered, the residual mass $M_{residual} = 0$ and (2.480) becomes

$$3 \sqrt{\sum_{k=1}^n \left[M_{eff,k}^2 \frac{\pi}{2} f_k Q_k W_{\ddot{u}}^*(f_k) \right]} \leq M_{total} \gamma. \quad (2.485)$$

After substituting (2.484) into (2.485) we obtain the following relation:

$$3 \sqrt{\frac{\pi}{2} \bar{C}} \sqrt{\sum_{k=1}^n [M_{eff,k} f_k W_{\ddot{u}}(f_k)]} \leq M_{total} \gamma. \quad (2.486)$$

Equation (2.486) gives the constant \bar{C} as

$$\bar{C} = \frac{(M\gamma)^2}{\frac{9\pi}{2} \sum_{k=1}^n [M_{eff,k} f_k W_{\ddot{u}}(f_k)]}. \quad (2.487)$$

If the residual mass is not negligible (say $M_{residual} > 0.2M_{total}$) (2.480) must be applied to obtain the reduced PSD of $W_{\ddot{u}}^*(f_k)$ at the natural frequencies f_k . This is an iterative process.

The width of the notch around f_k (if needed) is dependent on the amplification factor Q_k :

$$\Delta f_k = \frac{3f_k}{Q_k} \quad k = 1, 2, \dots, m, \quad (2.488)$$

and the slopes p_k of the reduced PSD function $W_{\ddot{u}}^*$ around f_k are approximately

$$p_k = \mp Q_k \text{ dB/oct.} \quad (2.489)$$

That is three times the half power band width.

The resonant frequencies and associated amplification factors may be extracted from a low level sine sweep test with a low *sweep rate*.

Example. A little unit has a mass $M = 30$ kg. The three resonant frequencies (\approx the natural frequency) with significant modal effective mass are $f_n = 120, 180, 250$ Hz respectively. The associated modal effective masses are $M_{eff} = 20, 8, 2$ kg. The amplification factor for all modes is $Q = 30$. The design load factor $\gamma = 25$ g. The specified PSD of the enforced acceleration is $W_{\ddot{u}} = 0.2$ g²/Hz. Define the notch levels at the resonant frequencies such that no over-testing with respect to the static design load factor γ will occur.

The constant \bar{C} (2.487) becomes

$$\bar{C} = \frac{(M\gamma)^2}{\frac{9\pi}{2} \sum_{k=1}^n [M_{eff,k} f_k W_{\ddot{u}}(f_k)]} = 45.8396.$$

Using (2.484) we can calculate the new notched value of the PSD;

$$W_{\ddot{u}}^*(f_k) = \frac{\bar{C} W_{\ddot{u}}(f_k)}{M_{eff,k} Q_k}, \quad k = 1, 2, 3.$$

Thus the notched values of the PSD can be obtained and are as follows $W_{\ddot{u}}(120) = 0.0153$, $W_{\ddot{u}}(180) = 0.0382$ and $W_{\ddot{u}}(250) = 0.1528$ g²/Hz.

The bandwidths at the resonant frequencies will be in accordance with (2.488):

$$\Delta f = 12, 18, 25 \text{ Hz}$$

respectively.

2.12.4 Acceleration Response Curve

Power Spectral Density input specification $W_{\ddot{u}}(f)$ (g²/Hz) (levels) are sometimes converted to response acceleration plots to be established the input loads to each of the assemblies (subsystems). The conversion from PSD to acceleration response plots is based on the following assumptions:

- Single degree of freedom system
- Amplification factor $Q = 10$ ($\zeta = 0.05$), for all units
- Acceleration is defined by $a(f) = \sqrt{\frac{\pi}{2} f Q W_{\ddot{u}}(f)}$ (g)

In order to apply the acceleration curve, the natural frequencies of each subsystem must be determined. Corresponding accelerations levels are established and applied as a static (inertia) load to determine the stress distribution.

Example. A random enforced acceleration requirement is given in Table 2.31. The corresponding *acceleration response curve* (RAC) is illustrated in Fig. 2.68.

Table 2.31. Component “Random vibration levels”, $M \leq 22.7$ kg [73]

Frequency (Hz)	$W_{\ddot{u}}(f) \text{ g}^2/\text{Hz}$
20–50	6 dB/oct
50	0.16
800	0.16
800–2000	–6.0 dB/oct
Overall	14.12 G_{rms}

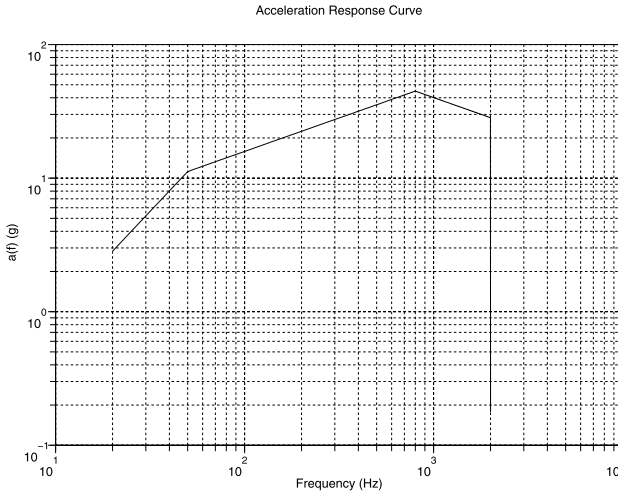


Fig. 2.68. Acceleration response curve

2.12.5 Random Response Spectrum

The relative response $z(t) = x(t) - u(t)$ of an SDOF system exposed to an enforced acceleration $\ddot{u}(t)$, with zero initial conditions, is given by

$$z(t) = - \int_{-\infty}^t \ddot{u}(\tau)h(t - \tau)d\tau, \tag{2.490}$$

where $h(t) = e^{-j\omega t} \frac{\sin \omega_d t}{\omega_d}$ is the damped impulse response function with damping ratio ζ , $\ddot{x}(t) = -2\zeta\omega_0\dot{z}(t) - \omega_0^2z(t)$, is the absolute acceleration, ω_0 the natural frequency of the SDOF system and $\omega_d = \omega_0\sqrt{1 - \zeta^2}$ the damped natural frequency. The maximum absolute responses $|x(t)|$, $|\dot{x}(t)|$ and $|\ddot{x}(t)|$ of the SDOF system at a natural frequency $f_0 = \frac{\omega_0}{2\pi}$ (Hz) are the values of the shock response spectrum (SRS) at the frequency f_0 for the displacements, velocity and acceleration, respectively. Varying the values of the natural frequencies f_0 in a frequency range between a minimum frequency f_{min} and a maximum frequency f_{max} will give the relevant SRS functions of response. In general, within the framework of the design of spacecraft structures, instruments, etc.

the SRS of absolute accelerations is of the most interest, because this type of SRS is specified in the user's manual of the launch vehicle.

The SRS functions describe the maximum responses of the structure, exposed to base excitation. However, the structure has been replaced by a series of SDOF systems with natural frequencies in the frequency range of interest.

The 3σ peak values of the acceleration of an SDOF system, enforced by random base acceleration $W_{\ddot{u}}(f)$ (g^2/Hz) with an amplification factor $Q = \frac{1}{2\zeta}$ and a natural frequency f_0 is given by Miles' equation

$$\ddot{x}_{peak}(f_0) = 3\sqrt{\frac{\pi}{2}f_0QW_{\ddot{u}}(f_0)}, \quad (2.491)$$

with the assumption that the average acceleration is $\mu_{\ddot{x}} = 0$.

Varying the natural frequencies f_0 , and the characteristics of the SDOF system, will give a *random response spectrum* (RRS) function of the 3σ peak accelerations.

Example. With constant random enforced acceleration $W_{\ddot{u}}(f) = 0.1 \text{ g}^2/\text{Hz}$ in a frequency range $20 \leq f \leq 1000 \text{ Hz}$ and a constant amplification factor $Q = 10$, the RRS based on the 3σ approach becomes

$$SRS(f) = 3\sqrt{\frac{\pi}{2}fQW_{\ddot{u}}(f)} = 3.760\sqrt{f} \text{ g} \quad 20 \leq f \leq 1000.$$

2.12.6 Relating Random to Sinusoidal Vibration

This section discusses a method to relate two testing methods, and is based on the equivalent damage theory, for example of electronic boxes [217]. Sinusoidal vibrating testing is mostly done in a single excitation frequency changed with time. The upper f_2 and lower limit f_1 of the frequency, the duration of the sweep, and the amplitude of the sine vibration will affect the severity of the vibration.

During random vibration, multiple random frequency vibrations are generated. The duration of the testing t_{rnd} , the PSD level $W_{\ddot{u}}(f)$ of the enforced acceleration and the upper and lower frequency will determine the severity of the vibration.

The proposed method to relate random and sinusoidal vibration is based on the equal effective damage theory.

A simple supported printed circuit board (PCB) generally has one major resonant frequency. The relation of random to sinusoidal vibration is monitored around the resonant frequency of the PCB.

The calculations are simplified to a SDOF dynamic system

$$m\ddot{z} + c\dot{z} + kz = -m\ddot{u}, \quad (2.492)$$

where m is the discrete mass, k is the stiffness of the spring element, c is the damping constant, \ddot{u} is the enforced acceleration, the relative displacement is $z = x - u$ and x is the absolute displacement.

If PSD of the random enforced acceleration is denoted by $W_{\ddot{u}}(f)$, the rms acceleration \ddot{x}_{rms} (g^2/Hz) is given by Miles equation

$$\ddot{x}_{rms} \approx \ddot{z}_{rms} = \sqrt{\frac{\pi}{2} f_n Q W_{\ddot{u}}(f_n)} \text{ (g)}, \quad (2.493)$$

where the natural frequency is $f_n = \frac{1}{2\pi} \sqrt{\frac{k}{m}}$, the amplification factor is $Q = \frac{1}{2\zeta}$, ζ is the damping ratio and the damping constant $c = 2\zeta \sqrt{km}$.

The acceleration response $\ddot{z}(t)$ in the frequency domain $\ddot{Z}(j\omega)$ of the SDOF system (2.492) caused by the enforced acceleration $\ddot{U}(j\omega)$ is; calculated

$$\ddot{Z}(j\omega) = \frac{-\omega^2 \ddot{U}(j\omega)}{-\omega^2 + \omega_n^2 + 2j\zeta\omega\omega_n}. \quad (2.494)$$

The response at the natural frequency $\omega_n = 2\pi f_n$ is

$$|\ddot{Z}(f_n)| = \frac{\ddot{U}(f_n)}{2\zeta} = Q |\ddot{U}(f_n)|, \quad (2.495)$$

the rms value of $|\ddot{Z}(f_n)|$ is given by

$$|\ddot{Z}(f_n)|_{rms} = \frac{|\ddot{Z}(f_n)|}{\sqrt{2}}, \quad (2.496)$$

and the rms value of the absolute sinusoidal response is

$$|\ddot{X}(f_n)|_{rms} = (Q + 1) |\ddot{Z}(f_n)|_{rms} \approx \frac{Q |\ddot{U}(f_n)|}{\sqrt{2}}. \quad (2.497)$$

An example of a typical random enforced vibration test specification is shown in Table 2.32. Besides the PSD spectrum, the G_{rms} value and the time duration t_{rnd} are mentioned.

If both random and sinusoidal vibration levels generate the same deflections, the equivalence can be established. For comparing the sinusoidal vibration and the random vibration

$$|\ddot{X}(f_n)|_{rms} = \ddot{x}_{rms}, \quad (2.498)$$

Table 2.32. Random vibration test specification

Frequency range (Hz)	PSD (g^2/Hz)	G_{rms} (g)
20–60	6 dB/oct	
60–700	0.2	16.6
700–2000	–3 dB/oct	
Time duration t_{rnd}	120 s	

thus

$$|\ddot{U}(f_n)| = \frac{\ddot{x}_{rms}\sqrt{2}}{Q} = \sqrt{\frac{\pi f_n W_{\ddot{u}}(f_n)}{Q}}. \quad (2.499)$$

Two types of sinusoidal test are proposed to simulate the random vibration environment [224]:

- Sinusoidal dwell test
- Sinusoidal (logarithmic) sweep test

During a sinusoidal dwell test the excitation frequency dwells at the natural frequency f_n . The total number of cycles is

$$N_{dwell} = t_{rnd} * f_n. \quad (2.500)$$

In case of a sinusoidal sweep test the dimension of sweep rate n is Octaves/min. The total time to sweep through a frequency band $f_1 \rightarrow f_2$ is given by [224]

$$t_{sweep} = t_{rnd} = \frac{60}{n \ln 2} \ln\left(\frac{f_2}{f_1}\right). \quad (2.501)$$

The frequencies f_1 and f_2 are selected to be the half power frequencies. The damping ratio ζ can be expressed in terms of the natural frequency f_n and the half power frequencies

$$2\zeta = \frac{f_2 - f_1}{f_n} = \frac{1}{Q}, \quad (2.502)$$

where

$$\begin{aligned} f_1 &= f_n(1 - \zeta) = f_n\left(1 - \frac{1}{2Q}\right), \\ f_2 &= f_n(1 + \zeta) = f_n\left(1 + \frac{1}{2Q}\right). \end{aligned} \quad (2.503)$$

The required sweep rate n can be calculated using (2.501)

$$n = \frac{60}{t_{rnd} \ln 2} \ln\left(\frac{f_2}{f_1}\right) \approx \frac{86.6 \ln(1 + 2\zeta)}{t_{rnd}}. \quad (2.504)$$

The number of cycles is given by

$$N = \frac{60}{n \ln 2} (f_2 - f_1) = \frac{86.6}{n} (f_2 - f_1). \quad (2.505)$$

Example. The specification of the random enforced vibration is given in Table 2.32. The natural frequency of the SDOF system is $f_n = 120$ Hz and the damping ratio $\zeta = 0.05$, which means the amplification factor is $Q = 10$. The rms acceleration of the SDOF system becomes

$$\ddot{x}_{rms} = \sqrt{\frac{\pi}{2} f_n Q W_{\ddot{u}}(f_n)} = \sqrt{\frac{\pi}{2} 120 \times 10 \times 0.2} = 19.4 \text{ (g)},$$

and the associated sinusoidal enforced vibration amplitude yields

$$\ddot{U}(f_n) = \sqrt{\frac{\pi f_n W_{\ddot{u}}(f_n)}{Q}} = 2.7 \text{ (g)}.$$

The number of cycles during the sinusoidal dwell test is

$$N_d = f_n t_{rmd} = 120 \times 120 = 14\,400 \text{ cycles.}$$

The sweep rate n can be calculated by the following expression:

$$n = \frac{60}{t_{rmd} \ln 2} \ln\left(\frac{f_2}{f_1}\right) = \frac{60}{120 \ln 2} \ln\left(\frac{f_2}{f_1}\right) = 0.072 \text{ Oct/min,}$$

where $f_1 = 114$ Hz and $f_2 = 126$ Hz. The number of cycles during the sweep from $f_1 \rightarrow f_2$ can be calculated as follows:

$$N_s = \frac{60}{n \ln 2} (f_2 - f_1) = 14\,382 \text{ cycles.}$$

The number of cycles for both the sinusoidal dwell and sweep test are comparable.

The auto correlation function $R(\tau)$ of the enforced sinusoidal vibration $\ddot{u}(t) = \ddot{U} \sin(\omega t)$ can be obtained using

$$R_{\ddot{u}}(\tau) = \frac{\omega}{2\pi} \int_0^{\frac{2\pi}{\omega}} \ddot{u}(t) \ddot{u}(t + \tau) dt = \frac{\ddot{U}^2}{2} \cos(\omega\tau), \quad (2.506)$$

and the corresponding PSD function

$$S_{\ddot{u}}(\omega) = \int_0^{\frac{2\pi}{\omega}} R_{\ddot{u}}(\tau) \cos(\omega\tau) d\tau = \frac{\ddot{U}^2}{2} \int_0^{\frac{2\pi}{\omega}} \cos^2(\omega\tau) d\tau = \frac{\ddot{U}^2}{2} \frac{\pi}{\omega}. \quad (2.507)$$

In the frequency domain (Hz)

$$W_{\ddot{u}}(f) = 2S_{\ddot{u}}(\omega) = \frac{\ddot{U}^2}{2f}. \quad (2.508)$$

2.12.7 Method for Calculating rms Von Mises Stress

In this section we discuss the calculation of the von Mises stress in a random vibration environment [173]. The transient von Mises stress, for a general tridimensional stress tensor

$$[\sigma(t)] = \begin{pmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{pmatrix}, \quad (2.509)$$

is expressed as

$$\begin{aligned} \sigma_{vM}(t) &= \sqrt{\sigma_{xx}^2 + \sigma_{yy}^2 + \sigma_{zz}^2 - \sigma_{xx}\sigma_{yy} - \sigma_{xx}\sigma_{zz} - \sigma_{yy}\sigma_{zz} + 3(\tau_{xy}^2 + \tau_{xz}^2 + \tau_{yz}^2)}. \end{aligned} \quad (2.510)$$

The square $\sigma_{vM}^2(t)$ can be expressed in a quadratic form on the stress vector $\{\sigma\}$

$$\{\sigma(t)\}^T = [\sigma_{xx} \quad \sigma_{yy} \quad \sigma_{zz} \quad \tau_{xy} \quad \tau_{xz} \quad \tau_{yz}], \quad (2.511)$$

and is given by

$$\sigma_{vM}^2(t) = \{\sigma(t)\}^T [A] \{\sigma(t)\}, \quad (2.512)$$

where $[A]$ is the symmetric, positive-definite matrix

$$[A] = \begin{pmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 \\ -\frac{1}{2} & 1 & -\frac{1}{2} & 0 & 0 & 0 \\ -\frac{1}{2} & -\frac{1}{2} & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 \end{pmatrix}. \quad (2.513)$$

In general, the undamped modal characteristics are the following: natural frequencies, vibration modes, modal effective masses. Associated with the vibration modes are the *stress modes* $[\Phi_\sigma]$. The stress mode reflects the modal stress tensor at a certain location a . The physical stress tensor $[\sigma^a(t)]$ has 6 independent stress components and can be expressed as follows

$$\{\sigma^a(t)\} = \sum_{k=1}^N \eta_k(t) \{\Phi_{\sigma,k}^a\}, \quad (2.514)$$

where $\eta_k(t)$ is the generalized coordinate, $\{\Phi_{\sigma,k}^a\}$ is the column vector of the stress tensor at location a , and N is the number of vibration modes considered.

Substituting (2.514) into (2.512) the square of the von Mises stress at location a becomes

$$\sigma_{vM}^a{}^2(t) = \{\sigma^a(t)\}^T [A] \{\sigma^a(t)\} = \sum_i^N \sum_j^N \eta_i(t) \eta_j(t) \{\Phi_{\sigma,i}^a\}^T [A] \{\Phi_{\sigma,j}^a\}. \quad (2.515)$$

The zero-time lag auto correlation function of the von Mises stress $R_{\sigma_{vM}^a}(0)$ is given by

$$R_{\sigma_{vM}^a}(0) = \sum_i^N \sum_j^N R_{\eta_i \eta_j}(0) \{\Phi_{\sigma,i}^a\}^T [A] \{\Phi_{\sigma,j}^a\}. \quad (2.516)$$

The PSD function $S_{\eta_i \eta_j}(\omega)$ is related to the auto correlation function $R_{\eta_i \eta_j}(\tau)$ by the so-called Wiener-Khinchine relations (page 22)

$$\begin{aligned} [S_{\eta_i \eta_j}(\omega)] &= \int_{-\infty}^{\infty} [R_{\eta_i \eta_j}(\tau)] e^{-j\omega t} dt, \\ [R_{\eta_i \eta_j}(\tau)] &= \frac{1}{2\pi} \int_{-\infty}^{\infty} [S_{\eta_i \eta_j}(\omega)] e^{j\omega t} d\omega. \end{aligned} \quad (2.517)$$

This means that the zero-lag auto correlation matrix $[R_{\eta_i \eta_j}(0)]$ becomes

$$[R_{\eta_i \eta_j}(0)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} [S_{\eta_i \eta_j}(\omega)] d\omega. \quad (2.518)$$

The CSD matrix of the generalized coordinates $[S_{\eta_i \eta_j}(\omega)]$ is related to the input CSD matrix of generalized forces $S_{ff}(\omega)$ (e.g. forces, enforced acceleration) by the frequency response diagonal matrix $[H_{\eta, f}(j\omega)]$ as follows

$$\begin{aligned} [S_{\eta_m \eta_n}(\omega)] &= [H_{\eta, f}(j\omega)] [S_{ff}(\omega)] [H_{\eta, f}^*(-j\omega)] \\ &= [|H_{\eta, f}(j\omega)|^2] [S_{ff, (n)}(\omega)] \\ &= [|H_{\eta, f, (m, n)}(j\omega)|^2] [\Phi]^T [S_{FF, (n)}(\omega)] [\Phi], \end{aligned} \quad (2.519)$$

where m, n are running indices $m, n = 1, 2, \dots, N$, $[\Phi]$ is the of the vibration modes and $[S_{FF, (n)}(\omega)]$ is the PSD/CSD matrix of the physical forces and or enforced excitation.

The mean square of the von Mises stress can be evaluated by

$$\begin{aligned} (\sigma_{vM, rms}^a)^2 &= R_{\sigma_{vM}}^a(0) = \sum_i^N \sum_j^N \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{\eta_i \eta_j}(\omega) d\omega \{\Phi_{\sigma, i}^a\}^T [A] \{\Phi_{\sigma, j}^a\} \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} [H_{\eta, f}(j\omega)] [\Phi]^T [S_{FF}(\omega)] [\Phi] [H_{\eta, f}^*(-j\omega)] d\omega [\Phi_{\sigma}^a]^T [A] [\Phi_{\sigma}^a] \\ &= \frac{1}{\pi} \int_0^{\infty} [H_{\eta, f}(j\omega)] [\Phi]^T [S_{FF}(\omega)] [\Phi] [H_{\eta, f}^*(-j\omega)] d\omega [\Phi_{\sigma}^a]^T [A] [\Phi_{\sigma}^a]. \end{aligned} \quad (2.520)$$

The integral in (2.520) may be approximated by a discrete sum:

$$(\sigma_{vM, rms}^a)^2 = \sum_{k=0}^N \frac{\Delta\omega}{\pi} [H_{\eta, f}(j\omega_k)] [\Phi]^T [S_{FF}(\omega_k)] [\Phi] [H_{\eta, f}^*(-j\omega_k)] [\Phi_{\sigma}^a]^T [A] [\Phi_{\sigma}^a], \quad (2.521)$$

where N is the number of discrete frequency steps, $\Delta\omega = (\omega_{max} - \omega_{min})/N$ and $\omega_k = \omega_{min} + k\Delta\omega$. In the frequency domain (2.521) can be written

$$\begin{aligned} (\sigma_{vM, rms}^a)^2 &= \sum_{k=0}^N \Delta f [H_{\eta, f}(j2\pi f_k)] [\Phi]^T [W_{FF}(f_k)] [\Phi] \\ &\quad \times [H_{\eta, f}^*(-j2\pi f_k)] [\Phi_{\sigma}^a]^T [A] [\Phi_{\sigma}^a], \end{aligned} \quad (2.522)$$

where $\Delta f = (f_{max} - f_{min})/N$ and $f_k = f_{min} + k\Delta f$.

In [153], Preumont presented an alternative for calculating the rms von Mises stress. The alternative definition to (2.512) is

$$\sigma_{vM}^2(t) = \{\sigma(t)\}^T [A] \{\sigma(t)\} = \text{Trace}([A] \{\sigma(t)\} \{\sigma(t)\}^T). \tag{2.523}$$

The location indicator a had been deleted. The auto correlation function $R_{\sigma_{vM}^2}(0)$ can be written

$$\begin{aligned} R_{\sigma_{vM}^2}(0) &= E[\sigma_{vM}^2] = \text{Trace}([A] E[\{\sigma(t)\} \{\sigma(t)\}^T]) \\ &= \int_{-\infty}^{\infty} \text{Trace}([A] S_{\sigma\sigma}(\omega) d\omega) \\ &= \int_{-\infty}^{\infty} S_{\sigma_{vM}^2}(\omega) d\omega. \end{aligned} \tag{2.524}$$

Equation (2.524) is exact and does not involve any assumption. A Gaussian random process of zero mean is defined by the PSD of the equivalent von Mises stress

$$S_{\sigma_{vM}^2}(\omega) = \text{Trace}([A] S_{\sigma\sigma}(\omega)). \tag{2.525}$$

The same procedures can be applied for the Tsai-Hill failure criterion for multi-axial stress states (plane stress) in composite structures. Tsai and Hill have established a suitable fracture criterion based on maximum strain energy. The Tsai-Hill theory is related to strength of the 1-direction (direction of fibers) [93]

$$\sigma_{TH}^2 = \left(\frac{\sigma_{xx}}{X}\right)^2 + \left(\frac{\sigma_{yy}}{Y}\right)^2 - \left(\frac{\sigma_{xx}\sigma_{yy}}{X^2}\right) + \left(\frac{\tau_{xy}}{S}\right)^2 = 1, \tag{2.526}$$

where X is the axial strength, Y is the transverse strength and S the shear strength, of the lamina.

Example. A cantilevered massless beam with a tip mass M is loaded by a random force F with a PSD function $W_{FF}(f)$ in the frequency range f_1 - f_2 . The length of the beam is L and the bending stiffness EI . The cross-section of the beam is H-shaped with a constant thickness $t = h/20$. The width of the flanges and the height of the shear web is h . The cantilevered beam is illustrated in Fig. 2.69.

The deflection $w(x, t)$ will be expressed as an assumed mode $\Phi(x)$ multiplied by the generalized coordinate $\eta(t)$

$$w(x, t) = \Phi(x)\eta(t) = \left[\frac{3}{2} \left(\frac{x}{L}\right)^2 - \frac{1}{2} \left(\frac{x}{L}\right)^3 \right] \eta(t),$$

where $\eta(t)$ has the dimension of (m). Rayleigh's quotient gives the natural frequency of the cantilevered beam with tip mass can be obtained

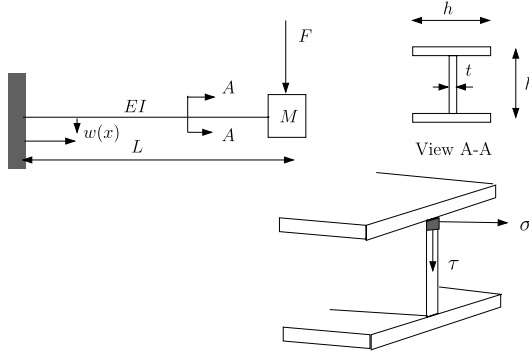


Fig. 2.69. Cantilevered beam

$$R(\Phi) \approx \omega_0^2 = \frac{\frac{1}{2} \int_0^L EI(w'')^2 dx}{\frac{1}{2} M w^2(L)} = \frac{3EI}{ML^3}.$$

The bending moment and the shear force at the root ($x = 0$) are given by

$$M(0) = -EIw''(0) = -\frac{3EI}{L^2}\eta(t),$$

$$D(0) = EIw'''(0) = -\frac{3EI}{L^3}\eta(t).$$

The second moment of area of the H-shaped cross-section is approximated by

$$I = \frac{7}{240}h^4,$$

and the first moment of area S at $h/2$ is

$$S = ht \times \frac{h}{2} = \frac{ht^2}{2}.$$

The modal bending stress σ_b at the root can be calculated as follows

$$\sigma_{mb}(t) = \frac{M(0)\frac{h}{2}}{I} = -\frac{3}{2} \frac{Eh}{L^2}\eta(t),$$

and the modal shear stress at $h/2$ can be calculated in accordance with Jourawki

$$\tau_m(t) = \frac{D(0)S}{It}\eta(t) = -30 \frac{Eh^2}{L^3}\eta(t).$$

The damped equation of motion of the cantilevered beam with tip mass is

$$\ddot{\eta}(t) + 2\zeta\omega_0\dot{\eta}(t) + \omega_0^2\eta(t) = \frac{F(t)}{M},$$

where ζ is the damping ratio. The frequency transfer function $H(j\omega)$ is

$$H(j\omega) = \frac{1}{-\omega^2 + \omega_0^2 + 2j\zeta\omega_0\omega},$$

and

$$H^*(j\omega) = \frac{1}{-\omega^2 + \omega_0^2 - 2j\zeta\omega_0\omega}.$$

The PSD function of the ratio $F(t)/M$ is $W_{FM}(f) = W_{FF}(f)/M^2$ and the constant matrix $[A]$ can be written as

$$[A] = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix},$$

because the von Mises stress for this case is $\sigma_{vM} = \sqrt{\sigma_b^2 + 3\tau^2}$. The modal stress vector is

$$\Phi_\sigma = \begin{pmatrix} -\frac{3}{2} \frac{Eh}{L^2} \\ -30 \frac{Eh^2}{L^3} \end{pmatrix}.$$

The assumed mode at $x = L$ is $\Phi(L) = 1$. Thus (2.522) can be simplified and yields

$$\begin{aligned} & \sigma_{vM,rms}^2 \\ &= \sum_{k=0}^N \Delta f H(j2\pi f_k) \Phi(L) W_{FM}(f_k) \Phi(L) H^*(-j2\pi f_k) \{\Phi_\sigma\}^T [A] \{\Phi_\sigma\} \\ &= \sum_{k=0}^N \Delta f |H(j2\pi f_k)|^2 W_{FM}(f_k) \{\Phi_\sigma\}^T [A] \{\Phi_\sigma\} \\ &\approx \frac{W_{FF}(f_0)}{64\pi^3 \zeta f_0^3 M^2} \{\Phi_\sigma\}^T [A] \{\Phi_\sigma\}, \end{aligned}$$

where $f_k = f_{min} + k \frac{(f_{max} - f_{min})}{N} = f_{min} + k \Delta f$.

The approximation will be proven using Lyapunov's equation as discussed in Sect. 2.6. The SDOF damped equation of motion representing the cantilevered beam with tip mass M and loaded by the force F , will be expressed in terms of the generalized coordinate $\eta(t)$ as

$$\ddot{\eta}(t) + 2\zeta\omega_0\dot{\eta}(t) + \omega_0^2\eta(t) = \frac{F(t)}{M},$$

where ζ is the damping ratio. This equation can be transformed into space-state variables $x_1 = \eta$ and $x_2 = \dot{\eta}(t)$:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\omega_0^2 & -2\zeta\omega_0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{1}{M} \end{pmatrix} F,$$

$$\{\dot{x}\} = [A]\{x\} + \{B\}F.$$

The auto correlation function of the applied force F is

$$R_{FF}(\tau) = \frac{W_{FF}}{2}\delta(\tau),$$

where $W_{FF}(N^2/Hz)$ is the white noise PSD in the frequency domain. The correlation matrix of x_1, x_2 is given by

$$[R_{x_1x_2}(0)] = [R_{x_1x_2}] = \begin{pmatrix} \langle x_1x_1 \rangle & \langle x_1x_2 \rangle \\ \langle x_2x_1 \rangle & \langle x_2x_2 \rangle \end{pmatrix} = \begin{pmatrix} \langle x_1x_1 \rangle & 0 \\ 0 & \langle x_2x_2 \rangle \end{pmatrix},$$

where $\langle x_i x_j \rangle$ are the expected values. The process is assumed stationary, thus $\frac{d}{dt}\langle x_1x_1 \rangle = \langle x_2x_1 \rangle + \langle x_1x_2 \rangle = 0$. This means that $\langle x_2x_1 \rangle = -\langle x_1x_2 \rangle = 0$. Lyapunov equation becomes

$$[A][R_{x_1x_2}] + [R_{x_1x_2}][A]^T = -\{B\} \frac{W_{FF}}{2} \{B\}^T$$

or

$$\begin{aligned} & \begin{bmatrix} 0 & 1 \\ -\omega_0^2 & -2\zeta\omega_0 \end{bmatrix} \begin{bmatrix} \langle x_1x_1 \rangle & 0 \\ 0 & \langle x_2x_2 \rangle \end{bmatrix} + \begin{bmatrix} \langle x_1x_1 \rangle & 0 \\ 0 & \langle x_2x_2 \rangle \end{bmatrix} \begin{bmatrix} 0 & -\omega_0^2 \\ 1 & -2\zeta\omega_0 \end{bmatrix} \\ & = \begin{bmatrix} 0 & 0 \\ 0 & -\frac{W_{FF}}{2M^2} \end{bmatrix}. \end{aligned}$$

The solution of this equation is

$$\begin{aligned} \langle x_1x_1 \rangle &= E\{\eta^2\} = \frac{W_{FF}}{8\zeta\omega_0^3M^2} = \frac{W_{FF}}{64\pi^3\zeta f_0^3M^2}, \\ \langle x_2x_2 \rangle &= E\{\dot{\eta}^2\} = \frac{W_{FF}}{8\zeta\omega_0M^2} = \frac{W_{FF}}{16\pi\zeta f_0M^2}. \end{aligned}$$

The state-space equation can be expanded as follows

$$\begin{aligned} \{\dot{x}\} &= [A]\{x\} + \{B\}F, \\ \{\sigma\} &= [C]\{x\}, \end{aligned}$$

where the output matrix $[C]$ is given by

$$[C] = \begin{bmatrix} \sigma_{mb} & 0 \\ \tau_m & 0 \end{bmatrix},$$

where σ_{mb} is the modal bending stress, τ_m the modal shear stress and

$$\{\sigma\} = \begin{pmatrix} \sigma_b \\ \tau \end{pmatrix}.$$

The expectation of $\{\sigma\}$, $E[\{\sigma\}\{\sigma\}^T]$, can be obtained from

$$E[\{\sigma\}\{\sigma\}^T] = [C][R_{x_1x_2}][C]^T = \langle x_1x_1 \rangle \begin{pmatrix} \sigma_{mb}^2 & \sigma_{mb}\tau_m \\ \sigma_{mb}\tau_m & \tau_m^2 \end{pmatrix}.$$

The mean square of the von Mises stress becomes

$$\sigma_{vM,rms}^2 = \text{Trace}([A]E[\{\sigma\}\{\sigma\}^T]) = \frac{W_{FF}}{64\pi^3\zeta f_0^3 M^2} (\sigma_{mb}^2 + 3\tau_m^2).$$

The following numerical values for the design parameters are taken:

- Young’s modulus $E = 70$ GPa,
- the length of the cantilevered beam $L = 500$ mm,
- the tip mass $M = 100$ kg,
- the width of the flange and the height of the shear web $h = 150$ mm ($t = 7.5$ mm),
- the white noise PSD of the force F is $W_{FF}(f) = 10^6$ N²/Hz,
- the band limited PSD is $W_{FF} = 0, f < f_1, W_{FF} = 10^6$ N²/Hz, $f_1 \leq f \leq f_2, W_{FF} = 0, f > f_2, f_1 = 20$ Hz and $f_2 = 500$ Hz.

The natural frequency $f_0 = 79.3$ Hz and for white noise force excitation the rms value of $\sigma_{vM,rms} = 1.01 \times 10^8$ Pa. For band-limited PSD force excitation, with $N = 1000$ steps, $\Delta f = 0.48$ Hz, the rms value of $\sigma_{vM,rms} = 1.00 \times 10^8$ Pa. The PSD of the σ_{vM}^2 is shown in Fig. 2.70. The PSD of the von Mises is highly peaked around the natural frequency f_0 , therefore the white noise approximation is rather good.

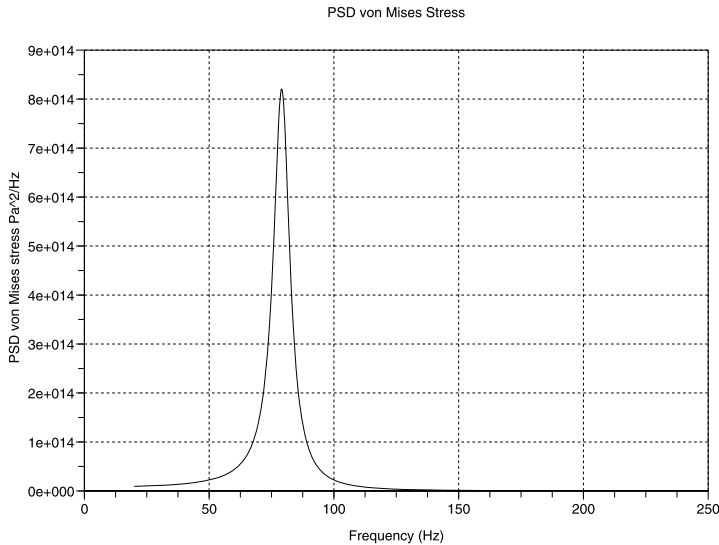


Fig. 2.70. PSD von Mises stress σ_{vM}^2 (Pa²/Hz)

Problems

2.44. A massless bending beam with total length $2L = 1$ m and constant bending stiffness EI N m² is clamped at A and simply supported at C . The beam is deflecting in the x - y plane in the y direction. A mass $M = 50$ kg is attached at the midpoint B . The required minimum natural frequency is $f_n = 50$ Hz. The beam is simultaneously base excited at both ends in the point A and C by a random enforced acceleration $\ddot{u}(t)$ with a constant PSD function $W_{\ddot{u}} = 0.3$ g²/Hz ($g = 9.81$ m/s²) in a frequency range $20 \leq f \leq 250$ Hz. The Young's modulus of the material is $E = 70$ GPa and the allowable yield stress $F_y = 250$ MPa. The bending beam is made of a tube with a diameter $2R$ and a wall thickness $t = \frac{R}{10}$. The fixed hinged beam is illustrated in Fig. 2.71.

1. Apply an unit force $F = 1$ N in the y direction and calculate the associated displacement δ .
2. Calculate the reaction forces R_A , M_A and R_C caused by the unit force $F = 1$ N acting at point B .
3. Calculate the stiffness k at point B in the direction of the force F and the displacement δ .
4. Define the radius R and wall thickness t of the tube ($I = \pi R^3 t$) knowing that the lowest natural frequency $f_n = 50$ Hz.
5. Set up an equivalent sdof system with damping ratio $\zeta = 0.05$.
6. Calculate the $\ddot{\delta}_{rms}$ acceleration of the mass M in the y direction by applying Miles' equation.
7. Calculate the reaction forces R_A , M_A , M_B and R_C caused by the inertia force acting on the mass $F_{inertia} = M\ddot{\delta}_{rms}$.
8. Calculate the bending and shear stress at point A with $\sigma_{bending} = \frac{MR}{I}$, $\tau_{max} = \frac{11}{8} \frac{D}{2\pi Rt}$, and D is the shear force. The maximum bending and shear stress occur at different locations of the cross section!

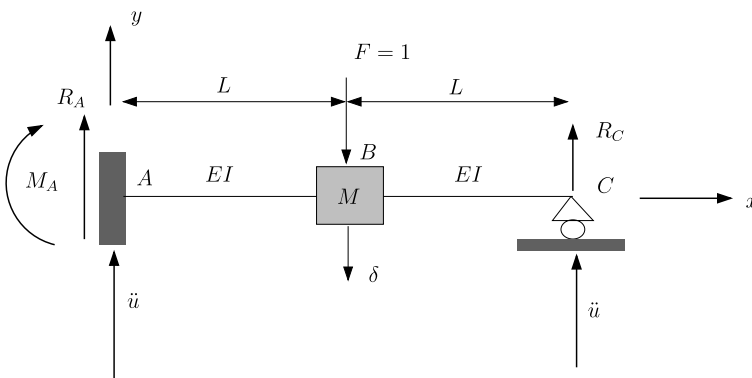


Fig. 2.71. Fixed hinged bending beam

9. Calculate margin of safety against the yielding stress using the ideal stress ($\sigma_{id} = \sqrt{\sigma^2 + 3\tau^2}$, $MS = \frac{F_y}{3\sigma_{id}} - 1$).

Answers: (1) $\delta = \frac{7}{96} \frac{L^3}{EI}$, (2) $k = \frac{96}{7} \frac{EI}{L^3}$, (3) $R_A = \frac{11}{16}$, $R_C = \frac{5}{16}$, $M_A = -\frac{3}{8}L$, $M_B = -\frac{5}{16}L$, (4) $I = 6.426 \times 10^{-7} \text{ m}^4$, $R = 38 \text{ mm}$, $t = 3.8 \text{ mm}$, (6) $\ddot{\delta}_{rms} = 150.583 \text{ m/s}^2$, $F_{inertia} = 7529 \text{ N}$, (7) $R_A = 2353 \text{ N}$, $R_C = 5176 \text{ N}$, $M_A = -1412 \text{ Nm}$, (8) $\sigma_{bending} = 8.309 \times 10^7 \text{ Pa}$, $\tau_{max} = 7.921 \times 10^6 \text{ Pa}$, (9) $MS_{bending} = 0.003$, $MS_{shear} = 5.074$.

2.45. A solar array sandwich panel is supported at 6 hold down points to survive the launch loads, especially the steady state accelerations and the acoustic sound pressures. A circular reinforcement (insert) is built in the sandwich panel to transfer the shear loads to the spacecraft at the location of the hold down points. The insert is bonded to an Al-alloy honeycomb core with an allowable shear stress $\tau_{allowable} = 2.3 \times 10^5 \text{ Pa}$. The measured s - N curve $Ns^b = c$ is $Ns^{5.967} = 1.385 \times 10^{34}$. The predicted average rms shear stress τ_{rms} along the circumference of the insert due to the acoustic sound pressures is $\tau_{rms} = 5.0 \times 10^4 \text{ Pa}$. The associated positive zero crossings are $\nu_0^+ = 130 \text{ Hz}$.

- Predict the fatigue life of the bonded insert/honeycomb core construction.
- Calculate the margin of safety (MS) against the allowable shear stress of the honeycomb core using the 3σ approach.

Answers: $T = 617 \text{ s}$, $MS = 0.53$.

2.46. The piping system of a cooler is mounted on a light weight radiator plate. The predicted radiator random vibration input to the pipes is surprisingly high:

- 20–200 Hz, 3 dB/oct
- 200–300 Hz, 70.0 g^2/Hz
- 300–1000 Hz, -12 dB/oct

Calculate the G_{rms} of the given PSD of the enforced accelerations and generate the $RRS(f)$, $f \in [20, 1000]$, based on the 3σ peak values with $Q = 20$.

Answers: $G_{rms} = 144.0 \text{ g}$, for example $RRS(200) = 1.99 \times 10^3 \text{ g}$.

2.47. Repeat the finite element analysis of the **example** on page 163 with the finite element programme which is frequently used at your school, university, laboratory or company.

2.48. The stress tensor (plane stress) is given by

$$\{\sigma\}^T = [\sigma_{xx} \quad \sigma_{yy} \quad \tau_{xy}].$$

The Tsai-Hill expression is as follows

$$\sigma_{TH}^2 = \left(\frac{\sigma_{xx}}{X}\right)^2 + \left(\frac{\sigma_{yy}}{Y}\right)^2 - \left(\frac{\sigma_{xx}\sigma_{yy}}{X^2}\right) + \left(\frac{\tau_{xy}}{S}\right)^2.$$

Derive the constant matrix $[A]$ which gives

$$\sigma_{TH}^2 = \text{Trace}([A]\{\sigma\}\{\sigma\}^T) = \{\sigma\}^T[A]\{\sigma\}.$$

2.12.8 Random Vibration Component Test Specification

In general, the purpose of performing a random vibration analysis is to create a component test level specification. With a finite element model, a random vibration analysis can be performed to predict acceleration responses from 20–2000 Hz. This response is in turn used as a template to derive a test level specification. NASA GFSC²⁴ proposed the following guideline to create the random vibration specification.

1. From a finite element random vibration response analysis create a log-log plot of the acceleration PSD response (g^2/Hz) versus the frequency (Hz).
2. Add in Flight Limit random vibration level specification for the components to the plot.
3. Enclose response curve inside test spec curve using the following rules:
 - Slopes should be less than 25 dB/oct. or greater than -25 dB/oct (this depends on the performance of the shaker table. Consult the test engineer (test house) for exact slope limits),
 - Frequency bands should be greater than 10 Hz,
 - Don't plot specification below Flight Limit levels,
 - Sharp peaks can be cut off at about 1/2 their height (-3 dB) e.g., a sharp peak of $0.4 g^2/\text{Hz}$ can be cut off at $>0.2 g^2/\text{Hz}$,
 - Drop specification curve into valleys that are large,
 - Try to keep the overall G_{rms} level of the specification to be no more than 1.25 times the overall G_{rms} level of the response curve.

The plot in Fig. 2.72 is an example of a specification to be used for component random vibration testing.

2.13 Random Responses Analysis in the Time Domain

2.13.1 Introduction

In this section the discrete sampled or digitized signals at equally space (equidistant) intervals Δt are discussed. The sampling is generally done by an analog-to digital (A/D) converter. The sampling interval Δt must be in accordance with the maximum frequency f_{max} in the process analyzed. The sampling frequency $f_s = 1/\Delta t$ is chosen according the Nyquist (Shannon-Kotelnikow) criterion

$$f_s = \frac{1}{\Delta t} = 2f_{max} = \frac{\omega_{max}}{\pi}. \quad (2.527)$$

The random process sampling in equidistant intervals Δt is illustrated in Fig. 2.73.

The simulated random time series can eventually being applied for response analysis of non-linear (e.g. geometrical) structures.

²⁴ NASA Goddard Space and Flight Center, Greenbelt, Maryland, USA.

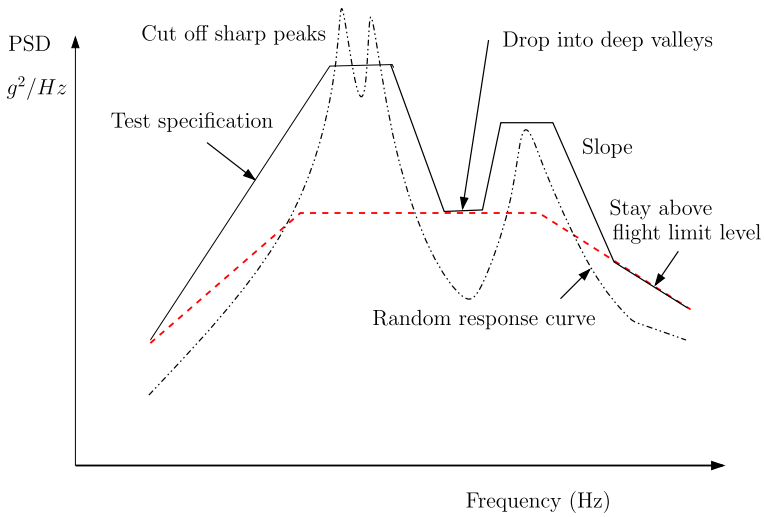


Fig. 2.72. Random vibration specification

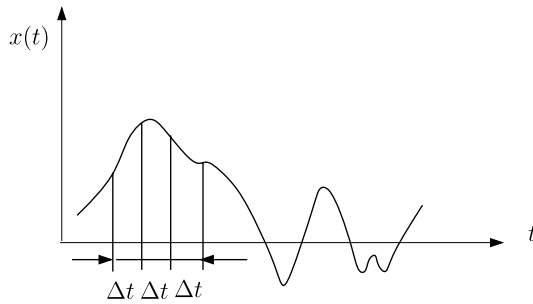


Fig. 2.73. The random process sampling in equidistant intervals Δt

2.13.2 Simulation of the Random Time Series

Details of the simulation of random vibration specification spectra in random time series are discussed in Appendix A.

The simulated time series can be calculated using the following expression:

$$x_k = x(k\Delta t) = \sqrt{2}\Re[FFT(a_n)], \quad k = 0, 1, 2, \dots, 2N - 1, \quad (2.528)$$

where Δt can be calculated (Nyquist criterion) by

$$\Delta t \leq \frac{\pi}{\omega_{max}} = \frac{1}{2f_{max}}. \quad (2.529)$$

The term a_n is

$$\begin{aligned}
 a_n &= [W(f_n)\Delta f]^{\frac{1}{2}}e^{-j\phi_n} \quad 0 \leq n \leq N-1 \\
 &= 0, \quad n \geq N,
 \end{aligned} \tag{2.530}$$

where the angles ϕ_n are uniformly distributed random numbers on the interval $(0-2\pi)$ and $f_n = n\Delta f$ and $\Delta f = f_{max}/N$. f_{max} is the maximum frequency in the power spectrum density $W(f)$, and N is the total number of terms in the frequency range $(0-f_{max})$ Hz.

The PSD function $W(f)$ can be reconstructed using the following expression:

$$\begin{aligned}
 W(n\Delta f) &= W(f_n) = 2 \frac{|IFFT(x_k)|^2}{\Delta f}, \\
 k &= 0, 1, 2, \dots, 2N-1, \quad n = 0, 1, 2, \dots, N-1,
 \end{aligned} \tag{2.531}$$

where the PSD function is in the cyclic frequency domain, and f is the cyclic frequency (Hz).

The estimated mean value of the simulated random time series x_n can be calculated using the following expression

$$x_{mean} = \frac{1}{2N} \sum_{n=0}^{2N-1} x_n, \tag{2.532}$$

and the mean square value x_{rms}^2 can be calculated by

$$x_{rms}^2 = \frac{1}{2N} \sum_{n=0}^{2N-1} x_n^2. \tag{2.533}$$

The variance can be calculated by

$$\begin{aligned}
 Var(x) &= \frac{1}{2N} \sum_{n=0}^{2N-1} (x_n - x_{mean})^2 \\
 &= \frac{1}{2N} \sum_{n=0}^{2N-1} \left[x_n - \frac{1}{2N} \sum_{k=0}^{2N-1} x_k \right]^2,
 \end{aligned} \tag{2.534}$$

and the estimated standard deviation s_x becomes

$$s_x = \sqrt{Var(x)}. \tag{2.535}$$

The autocorrelation function or *mean lagged product* can be calculated by applying the following expression [197]

$$r_m = \frac{1}{2N-m} \sum_{n=0}^{2N-m-1} x_n x_{n+m}, \quad m = 0, 1, 2, \dots, M, \tag{2.536}$$

where, in general, $M < 2N/10$ to prevent too lengthy calculations.

Table 2.33. Component “Random vibration levels”, $M \leq 22.7$ kg [73]

Frequency (Hz)	$W_{\ddot{u}}(f)$ g^2/Hz
20–50	6 dB/oct
50	0.16
800	0.16
800–2000	–6.0 dB/oct
Overall	14.12 G_{rms}

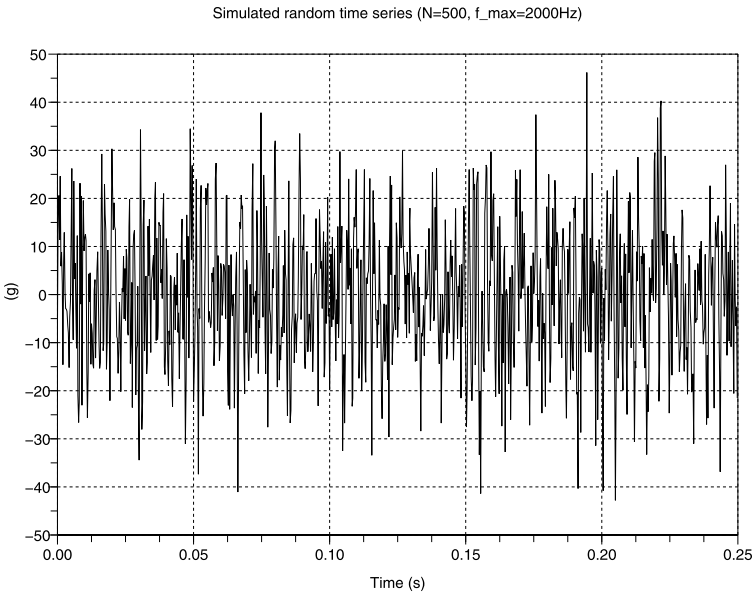


Fig. 2.74. Simulated random time series \ddot{u}_n

Example. A typical random vibration specification for components ($M \leq 22.7$ kg, equipment, boxes, . . .) is given in [73] and shown in Table 2.33. The simulated random time series \ddot{u}_n from Table 2.33 is shown in Fig. 2.74. The number of samples $N = 500$ is taken to get a random time series which is not too dense. The maximum time on the ordinate is $N\Delta t = N/2f_{max} = 0.25$ s and the maximum considered frequency $f_{max} = 2000$ Hz. The reconstructed PSD enforced acceleration $W_{\ddot{u}}(f)$ is shown in Fig. 2.75. This Fig. 2.74 reflects the PSD spectrum as given in Table 2.33. The estimated mean value of the simulated time series \ddot{u}_n is

$$\ddot{u}_{mean} = \frac{1}{2N} \sum_{n=0}^{2N-1} \ddot{u}_n = -0.0011 \text{ g,}$$

and the rms value

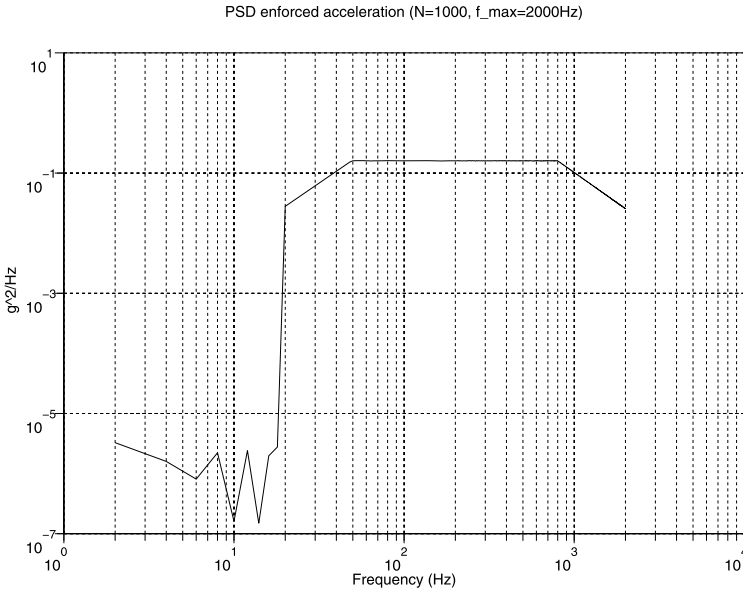


Fig. 2.75. Reconstructed PSD enforced acceleration $W_{\ddot{u}}(f)$

$$\ddot{u}_{rms} = \sqrt{\frac{1}{2N} \sum_{n=0}^{2N-1} \ddot{u}_n^2} = 14.1173 \text{ g},$$

and the estimated standard deviation $s_{\ddot{u}} = 14.1173 \text{ g}$. The auto correlation function is or mean lagged products r_m are illustrated in Fig. 2.76. The auto correlation function shows a white noise characteristic of the simulated random time series.

In the following, a response analysis on a simple structure will be performed. This simple structure may represent a antennae boom. In Fig. 2.77 a cantilevered beam is shown. The beam is excited by a random enforced acceleration $\ddot{u}(t)$. The PSD of the enforced acceleration $W_{\ddot{u}}(f)$ is given in Table 2.33. The random time series is shown in Fig. 2.74. The discrete mass at the tip of the beam is $M = 50 \text{ kg}$. The bending stiffness is given by EI and the mass per unit of length of the beam is $m \text{ kg/m}$. The length of the beam is $L = 0.4 \text{ m}$. The cross-section is a thin-walled square tube $b \times b$ and the wall-thickness is t . The beam is made of Al-alloy, $E = 70 \text{ GPa}$, and the density is $\rho = 2700 \text{ kg/m}^3$. The beam must be designed such that the lowest natural frequency $f_1 = 30 \text{ Hz}$. The assumed mode $\phi(x)$ is the analog of the static deformation corresponding to a unit force at the tip. The assumed mode taken is

$$\phi(x) = \frac{3}{2} \left(\frac{x}{L}\right)^2 - \frac{1}{2} \left(\frac{x}{L}\right)^3. \tag{2.537}$$

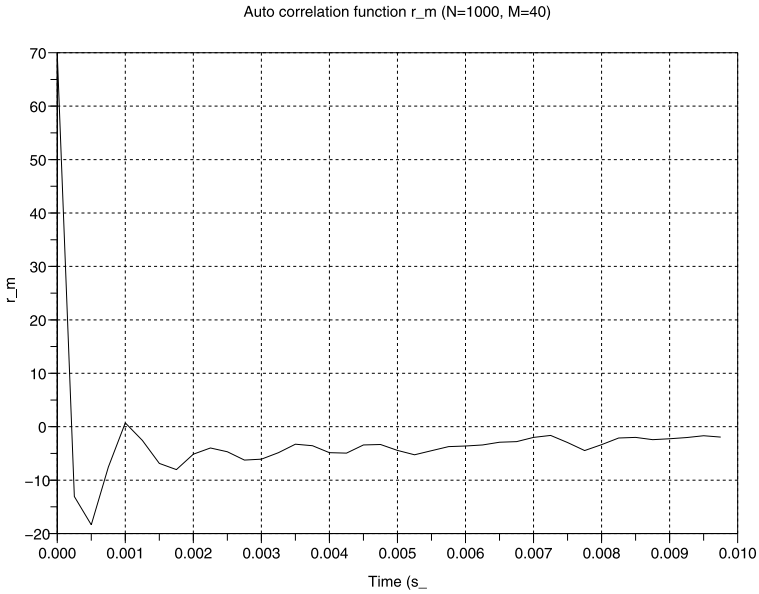


Fig. 2.76. Auto correlation function or mean lagged products

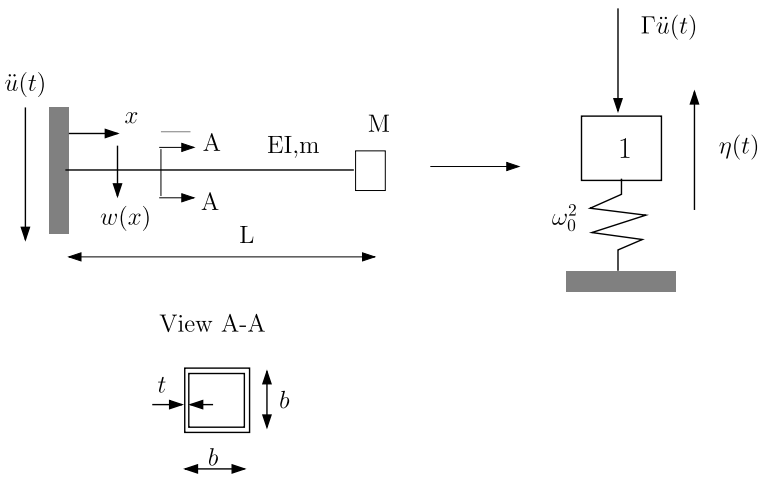


Fig. 2.77. Cantilevered beam

The deflection $w(x, t)$ will now be expressed as follows

$$w(x, t) = \phi(x)\eta(t), \tag{2.538}$$

where $\eta(t)$ is the generalized coordinate. The equation of motion will be derived using the Lagrange equations. First the strain energy U and the kinetic

T have to be obtained

$$U = \frac{EI}{2} \int_0^L \left(\frac{\partial^2 w(x, t)}{\partial x^2} \right)^2 dx, \quad (2.539)$$

$$T = \frac{m}{2} \int_0^L [\dot{w}(x, t) + \dot{u}(t)]^2 dx + \frac{1}{2} M [\dot{w}(L, t) + \dot{u}(t)]^2.$$

The undamped equation of motion is expressed in terms of the generalized coordinate $\eta(t)$, and becomes

$$\ddot{\eta}(t) + \omega_1^2 \eta(t) = -\Gamma \ddot{u}(t), \quad (2.540)$$

where the natural frequency ω_1^2 and the modal participation factor Γ are

$$\omega_1^2 = \frac{420EI}{L^3(33mL + 140M)}, \quad (2.541)$$

$$\Gamma = \frac{35(3mL + 8M)}{2(33mL + 140M)}.$$

The equivalent SDOF system is shown in Fig. 2.77. The damped equation of motion is obtained by adding the modal damping term $2\zeta\omega_1\dot{\eta}(t)$, thus (2.540) becomes

$$\ddot{\eta}(t) + 2\zeta\omega_1\dot{\eta}(t) + \omega_1^2\eta(t) = -\Gamma\ddot{u}(t), \quad (2.542)$$

where ζ is the modal damping ratio. The wall-thickness of the beam cross-section is $t = b/50$. The second moment of area $I = \frac{2}{3}b^3t$, the area is $A = 4bt$ and the bending resistance is $W = \frac{4}{3}b^2t$. The bending moment in the cantilevered beam at $M(x)$, is given by

$$M(x, t) = EIw''(x) = EI\phi''(x)\eta(t). \quad (2.543)$$

Thus at the root of the beam, $x = 0$, $M(0, t) = \phi''(0)\eta(t) = \frac{3EI\eta(t)}{L^2}$. The bending stress $\sigma_b = |\frac{M(x, t)}{W}|$. The lowest natural frequency $f_1 = 30$ Hz can be achieved when $b = 0.08$ m and $t = 0.0016$ m. The mass $m = 1.3824$ kg/m. The lowest natural frequency f_1 is

$$f_1 = \frac{1}{2\pi} \sqrt{\frac{420EI}{L^3(33mL + 140M)}} = 30.091 \text{ Hz},$$

and the modal participation Γ becomes

$$\Gamma = \frac{35(3mL + 8M)}{2(33mL + 140M)} = 1.002.$$

When the beam is taken massless $m = 0$, and $\Gamma = 1$. In [223] the single-step three-stage Newmark algorithm is discussed, and can be written

$$\{X_{n+1}\} = [A]\{X_n\} + \{F_n\}, \quad (2.544)$$

where the vector $\{X_n\}$ is

$$\{X_n\} = \begin{pmatrix} x_n \\ \Delta t v_n \\ \Delta t^2 a_n \end{pmatrix}, \quad (2.545)$$

x_n is the displacement, v_n is the velocity and a_n the acceleration, all at step n . The matrix $[A]$ is the *amplification matrix*. The equation of motion to be solved is

$$m\ddot{x}(t) + c\dot{x}(t) + kx(t) = f(t). \quad (2.546)$$

The amplification matrix $[A] = [A_{ij}]$ is given by

$$\begin{aligned} A_{1,1} &= (m + \gamma\Delta tc)/D, \\ A_{1,2} &= \{m + (\gamma - \beta)\Delta tc\}/D, \\ A_{1,3} &= \{(0.5 - \beta)m + (0.5\gamma - \beta)\Delta tc\}/D, \\ A_{2,1} &= -\gamma\Delta t^2 k/D, \\ A_{2,2} &= \{m + (\beta - \gamma)\Delta t^2 k\}/D, \\ A_{2,3} &= \{(1 - \gamma)m + (\beta - 0.5\gamma)\Delta t^2 k\}/D, \\ A_{3,1} &= -\Delta t^2 k/D, \\ A_{3,2} &= -\Delta t(c + \Delta tk)/D, \\ A_{3,3} &= \{(1 - \gamma)\Delta tc + (0.5 - \beta)\Delta t^2 k\}/D, \\ D &= m + \gamma\Delta tc + \beta\Delta t^2 k. \end{aligned} \quad (2.547)$$

The vector $\{F_n\}$ is

$$\{F_n\}^T = \frac{\Delta t^2}{D} \{\beta f_{n+1}, \gamma f_{n+1}, f_{n+1}\}^T. \quad (2.548)$$

The solution of (2.545) is stable if $2\beta \geq \gamma \geq 0.5$. To solve the transients the following values for the parameters $\beta = 0.25$ and $\gamma = 0.5$ are taken. The initial conditions are $\eta(0) = \dot{\eta}(0) = 0$ and $\ddot{\eta}(0) = -\Gamma\ddot{u}(0)$. The damping ratio is $\zeta = 0.01$. The solution for the generalized coordinate $\eta(t)$, its velocity $\dot{\eta}(t)$ and acceleration $\ddot{\eta}(t)$ are shown in Figs. 2.78, 2.79 and 2.80 respectively. The bending stress $\sigma_b(t)$ at the root of the beam is shown in Fig. 2.81 and the normalized auto-correlation function in Fig. 2.82. It can be concluded that the bending stress $\sigma_b(t)$ is narrow-banded. The auto-correlation function of the bending stress shows a narrow-banded process. The bending acts like a mechanical filter at a natural frequency $f_1 = 30.09$ Hz. This natural frequency is clearly seen in Fig. 2.82.

In the next example the Duffing oscillator will be solved in the time domain. The internal force in the spring of the Duffing oscillator is a combination of linear and cubic restoring forces.

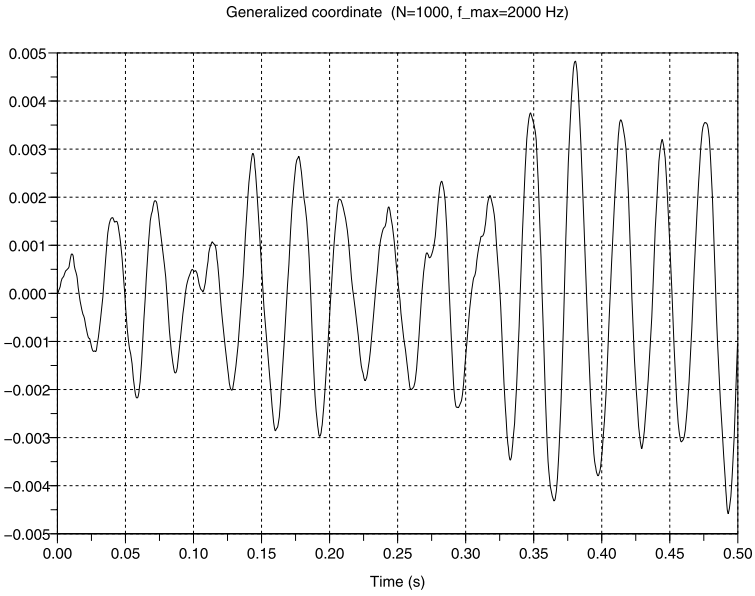


Fig. 2.78. Generalized coordinate $\eta(t)$

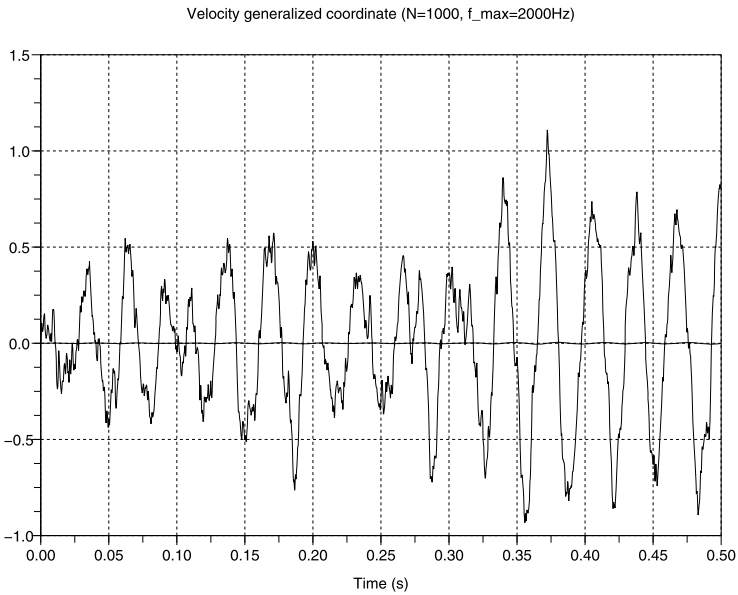


Fig. 2.79. Generalized coordinate $\dot{\eta}(t)$

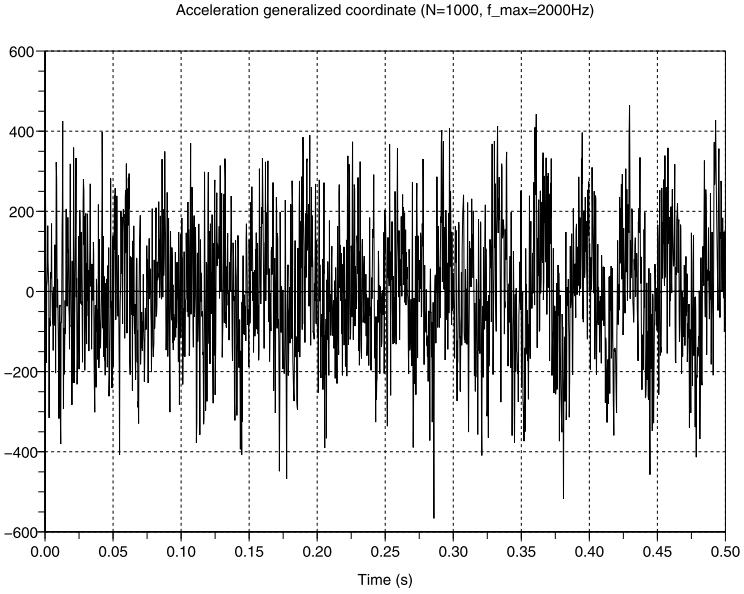


Fig. 2.80. Generalized coordinate $\ddot{\eta}(t)$

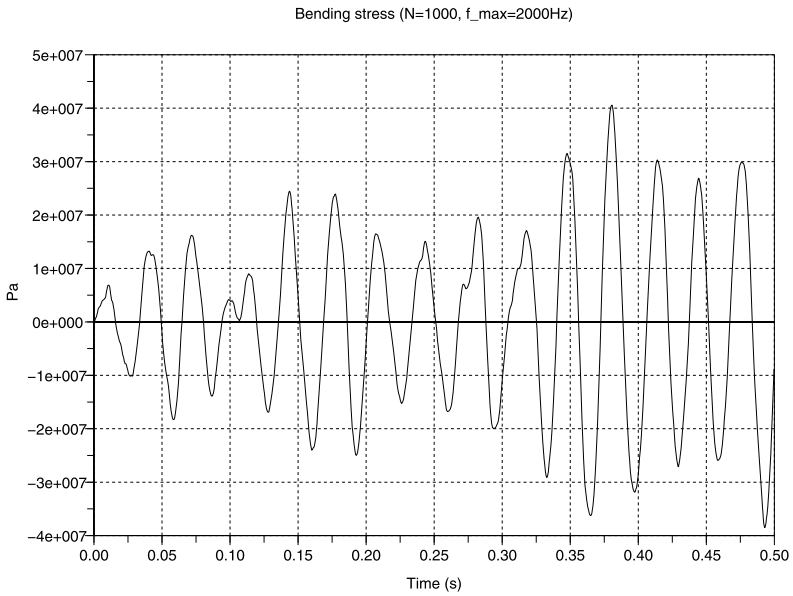


Fig. 2.81. Bending stress $\sigma_b(t)$

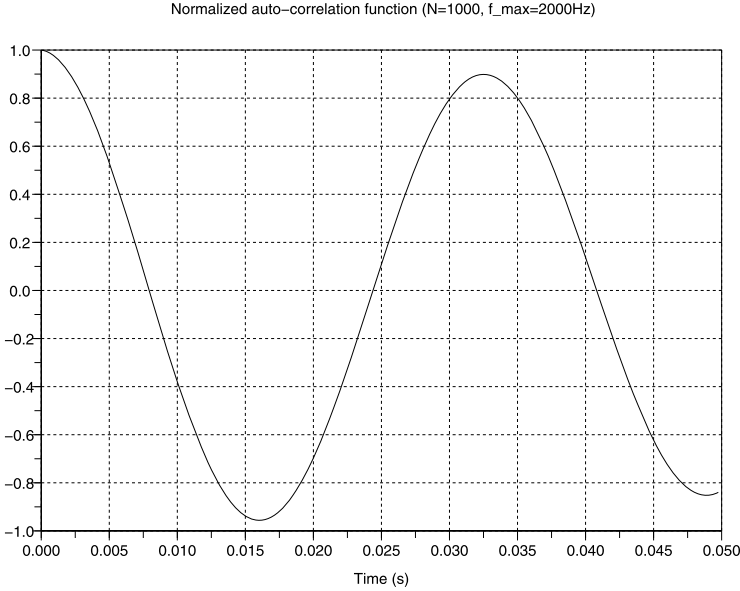


Fig. 2.82. Normalized auto-correlation function

Example. The damped equation of motion of the Duffing oscillator is

$$\ddot{z}(t) + 2\zeta\omega_0\dot{z}(t) + \omega_0^2(z + \varepsilon z^3) = -\ddot{u}(t), \tag{2.549}$$

where $z(t)$ is the relative displacement, $\dot{z}(t)$ is the relative velocity, $\ddot{z}(t)$ is the relative acceleration, all with respect to the base; ζ is the damping ratio, ε is a scaling parameter and $\ddot{u}(t)$ is the random enforced acceleration specified in Table 2.33. This PSD spectrum will be simulated by random time series as shown in Fig. 2.74. The numerical solution of the nonlinear *Duffing oscillator* (2.549) in the time domain is based on the Newmark method and is discussed in detail in [223]. The approximation of the acceleration $\ddot{z}(t)$ at time $(n+1)\Delta t$ is denoted by a_{n+1} , the velocity $\dot{z}(t)$ is denoted by v_{n+1} and the displacement $z(t)$ is denoted by z_{n+1} . The enforced acceleration $\ddot{u}(t)$ is denoted by \ddot{u}_{n+1} .

The acceleration a_{n+1} and velocity v_{n+1} are expressed as

$$\begin{aligned} a_{n+1} &= \frac{1}{\beta\Delta t^2}(z_{n+1} - z_n) - \frac{v_n}{\beta\Delta t} - \left(\frac{1}{2\beta} - 1\right)a_n, \\ v_{n+1} &= \frac{\gamma}{\beta\Delta t}(z_{n+1} - z_n) - \left(\frac{\gamma}{\beta} - 1\right)v_n - \Delta t\left(\frac{\gamma}{2\beta} - 1\right)a_n. \end{aligned} \tag{2.550}$$

If the approximate values of the solution are substituted in (2.549) the following equation is obtained

$$r_{n+1} = a_{n+1} + 2\zeta\omega_0v_{n+1} + \omega_0^2(z_{n+1} + \varepsilon z_{n+1}^3) + \ddot{u}_{n+1} = 0. \tag{2.551}$$

The initial condition can be calculated using r_0 ($t = 0$)

$$r_0 = a_0 + 2\zeta\omega_0 v_0 + \omega_0^2(z_0 + \varepsilon z_0^3) + \ddot{u}_0 = 0.$$

If $v_0 = a_0 = 0$ then

$$a_0 = -2\zeta\omega_0 v_0 - \omega_0^2(z_0 + \varepsilon z_0^3) - \ddot{u}_0 = -\ddot{u}_0. \quad (2.552)$$

The value of r_{n+1} can be approximated by a first order Taylor series expansion;

$$r_{n+1} \approx r_n + \frac{dr_n}{dz_n} \Delta z_n = r_n + K(z_n) \Delta z_n = r_n + \Delta F_n, \quad (2.553)$$

where $K(z_n)$ is the *tangent stiffness* or the Jacobian at z_n . The solution for Δz_n can be obtained by $\Delta z_n = -K(z_n)^{-1}(r_{n+1} - r_n)$. Substituting (2.550) in (2.553) will leads to

$$\begin{aligned} r_{n+1} &= \left(\frac{1}{\beta \Delta t^2} + \frac{2\zeta\omega_0\gamma}{\beta \Delta t} \right) z_{n+1} + \omega_0^2(z_{n+1} + \varepsilon z_{n+1}^3) - F_{int,n} + \ddot{u}_{n+1} \\ &= 0, \end{aligned} \quad (2.554)$$

where the force $F_{int,n}$ is given by

$$\begin{aligned} F_{int,n} &= \left(\frac{1}{\beta \Delta t^2} + \frac{2\zeta\omega_0\gamma}{\beta \Delta t} \right) z_n + \left[\frac{1}{\beta \Delta t} + 2\zeta\omega_0 \left(\frac{\gamma}{\beta} - 1 \right) \right] v_n \\ &\quad + \left[\left(\frac{1}{2\beta} - 1 \right) - 2\zeta\omega_0 \Delta t \left(\frac{\gamma}{2\beta} - 1 \right) \right] a_n. \end{aligned} \quad (2.555)$$

The Jacobian or tangent stiffness $K(z_n) = dr_n/dz_n$ becomes

$$K(z_n) = \left(\frac{1}{\beta \Delta t^2} + \frac{2\zeta\omega_0\gamma}{\beta \Delta t} \right) + \omega_0^2(1 + 3\varepsilon z_n^2). \quad (2.556)$$

The solution procedure discussed in [206] is as follows. The increment Δz is

$$\begin{aligned} r_{n+1} - r_n &= \Delta a_n + 2\zeta\omega_0 \Delta v_n + \omega_0^2(z_{n+1} + \varepsilon z_{n+1}^3) - \omega_0^2(z_n + \varepsilon z_n^3) + \Delta \ddot{u}_n = 0 \\ &\approx \Delta a_n + 2\zeta\omega_0 \Delta v_n + K_t(z_n) \Delta z_n + \Delta \ddot{u}_n = 0, \end{aligned} \quad (2.557)$$

where $\Delta a_n = a_{n+1} - a_n$, $\Delta v_n = v_{n+1} - v_n$, $\Delta \ddot{u}_n = \ddot{u}_{n+1} - \ddot{u}_n$ and the tangent stiffness is $K_t(z_n) = \omega_0^2(1 + 3\varepsilon z_n^2)$. The increment of the acceleration Δa_n and the velocity Δv_n can be derived from (2.550) and are given by

$$\begin{aligned} \Delta a_n &= \frac{1}{\beta \Delta t^2} \left[\Delta z_n - \Delta t v_n - \frac{\Delta t^2}{2} a_n \right], \\ \Delta v_n &= \gamma \Delta t \Delta a_n + \Delta t a_n. \end{aligned} \quad (2.558)$$

Substituting (2.558) in (2.557) will give

$$\begin{aligned}
 K(z_n)\Delta z_n &= \left(\frac{1}{\beta\Delta t} + 2\zeta\omega_0\frac{\gamma}{\beta}\right)v_n + \left[\frac{1}{2\beta} + 2\zeta\omega_0\left(\frac{\gamma\Delta t}{2\beta} - \Delta t\right)\right]a_n - \Delta\ddot{u}_n \\
 &= \Delta F_n.
 \end{aligned}
 \tag{2.559}$$

The next step solutions for the acceleration, velocity and displacement are

$$\begin{aligned}
 a_{n+1} &= a_n + \Delta a_n, \\
 v_{n+1} &= v_n + \Delta v_n, \\
 z_{n+1} &= z_n + \Delta z_n.
 \end{aligned}
 \tag{2.560}$$

The *Newmark-β method* is called an implicit method²⁵ to solve the transient response problem, however, without equilibrium iterations. The error introduced into the approximation can be minimized if a sufficiently small integration step Δt is employed in the analysis. In general, for an conditionally stable Newmark method $\gamma = 1/2$ and $\beta = 1/4$ are taken.

For highly nonlinear cases the implicit numerical scheme can be improved doing equilibrium iterations every time step $n \rightarrow n + 1$. Equation (2.554) will be rewritten as follows

$$\left(\frac{1}{\beta\Delta t^2} + \frac{2\zeta\omega_0\gamma}{\beta\Delta t}\right)z_{n+1} + \omega_0^2(z_{n+1} + \varepsilon z_{n+1}^3) = F_{n+1}.
 \tag{2.561}$$

Within a time step, the following iteration procedure is proposed:

- $\Delta z_n^k = \Delta R_n^k / K_n(z_n)$ (modified Newton-Raphson method, Fig. 2.83),
- $z_{n+1}^k = z_{n+1}^{k-1} + \Delta z_n^k$,

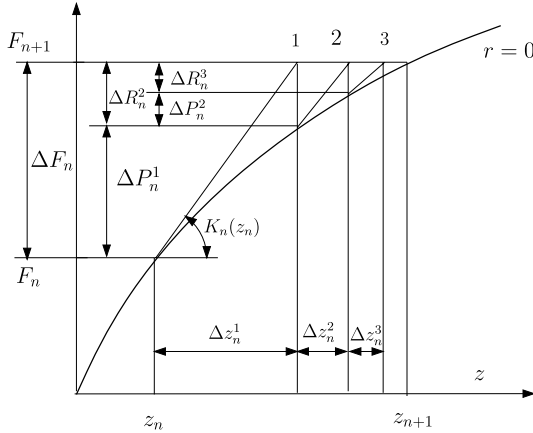


Fig. 2.83. Modified Newton-Raphson iteration [206]

²⁵ Explicit methods calculate the state of a system at a later time from the state of the system at the current time, while an implicit method finds it by solving an equation involving both the current state of the system and the later one.

- $\Delta P_n^k = (\frac{1}{\beta \Delta t^2} + \frac{2\zeta \omega_0 \gamma}{\beta \Delta t}) \Delta z_n^k + \omega_0^2 [\Delta z_n^k + \varepsilon (z_{n+1}^k - z_{n+1}^{k-1})]$,
- $\Delta R_n^{k+1} = \Delta R_n^k - \Delta P_n^k$.

The procedure starts at $k = 1$ with the initial values $z_{n+1}^0 = z_n$ and $\Delta R_n^1 = \Delta F_n$. The iterations are continued until convergence, that is, until the incremental displacement Δz_n^k have become sufficiently small in comparison to the current estimate Δz_n^1 . Thus the total incremental displacement after m iterations is given by

$$\Delta z_n = \sum_{k=1}^m \Delta z_n^k. \tag{2.562}$$

The iteration process may be terminated at iteration m when

$$\frac{\Delta z_n^m}{\Delta z_n^1} \leq \delta, \tag{2.563}$$

where δ is the specified tolerance. The design parameters of Duffing’s equation have the following values:

- The linear natural frequency $\omega_0 = 2\pi f_0$, $f_0 = 30$ Hz.
- The tuning factor $\varepsilon = 0.2$.
- The damping ratio $\zeta = 0.01$.

The displacement $z(t)$ and the non linear restoring force $F_{nl} = \omega_0^2(z + \varepsilon z^3)$ are calculated using the equilibrium iterations. The displacement $z(t)$ is shown in Fig. 2.84 and the nonlinear restoring force $F_{nl}(t)$ in Fig. 2.85.

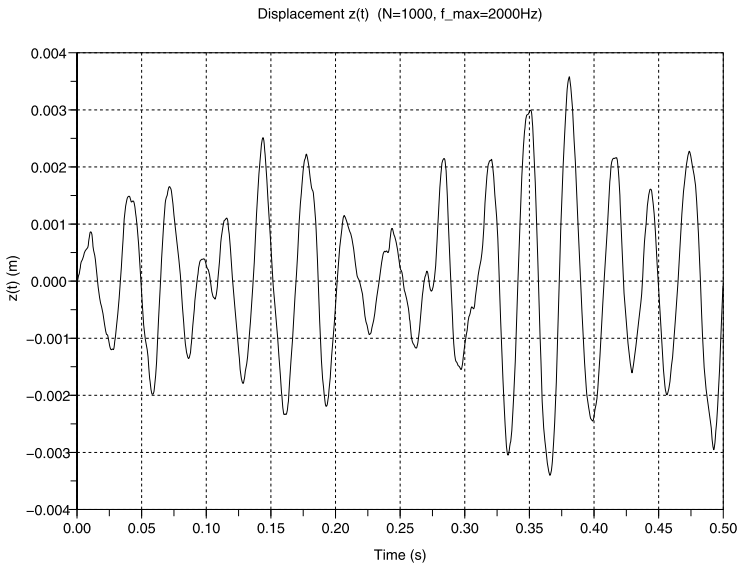


Fig. 2.84. Displacement $z(t)$

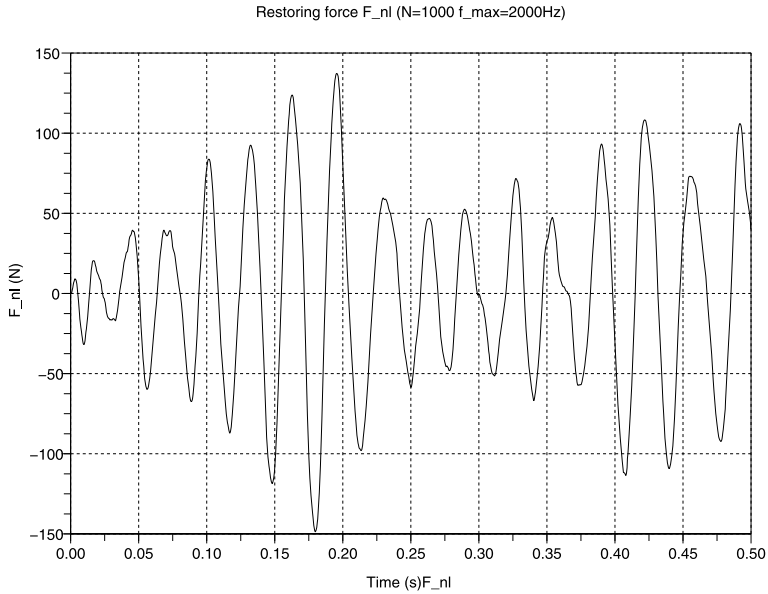


Fig. 2.85. Non linear restoring force $F_{nl}(t)$

Problems

Table 2.34. Power spectrum densities shapes

Spectrum	f_1 (Hz)	f_2 (Hz)	$W(f)$	f_3 (Hz)	f_4 (Hz)	$W(f)$	ν_0^+	ν_p^+
1	5	55	$W_0 f^2$					
2	10	70	W_0					
3	20	100	W_0/f^2					
4	40	100	W_0/f^2					
5	29	35	$W_0 f^2$					
6	9	11	W_0	29	35	W_0		
7	16	40	W_0					
8	9	11	W_0	29	35	$W_0/20$		
9	5	30	W_0/f^2					

2.49. This problem is based on the theory discussed in [128, 177]. The PSD $W(f)$ (1/Hz) in Table 2.34 is non zero between the specified frequency ranges (f_1-f_2) and (f_3-f_4) . Perform the following assignments:

- Calculate for all spectra the constant W_0 such that $\int_0^\infty W(f)df = 1$.

- Calculate for all spectra the number of positive zero crossing $\nu_0^+ = \sqrt{m_2/m_0}$.²⁶
- Calculate for all spectra the average number of maximum peaks $\nu_p^+ = \sqrt{m_4/m_2}$.
- Generate for all spectra the random time series.

²⁶ The spectral moment is $m_n = \int_0^\infty f^n W(f) df$.

Acoustic Random Vibration

Low Frequency Acoustic Loads

3.1 Introduction

Acoustic vibration is the structural response of structures exposed to an acoustic sound field. In this chapter low-frequency acoustic vibrations are discussed using the modal displacement method (MDM). In the higher-frequency bands, statistical energy analysis (SEA) is a good substitute for the classical modal approach.

Lightweight and large antenna structures and solar arrays, of spacecraft (see Fig. 3.1) are very sensitive to acoustic loads during the launch phase. Spacecraft external structures are severely exposed to acoustic loads.

In general, the modal characteristics of the dynamic system are calculated with the aid of the finite element method. The accuracy is determined by the detail of the finite element model and the complexity of the structure. As stated above, the equations of motion will be solved using the classical modal approach and therefore linear structural behavior is assumed.

The structure is deterministic, however, the *acoustic loads* have a random nature.

3.2 Acoustic Loads

Acoustic loads appear as design specifications for spacecraft and spacecraft attachments such as solar arrays and antennae. Acoustic loads are generated during launch, or in acoustic facilities for test purposes. It is very common to specify a reverberant sound field, which means that the intensity of the sound is the same for all directions.

In general, the acoustic loads are described as *sound pressure levels* (SPL) and specified in decibels (dB). The SPL is defined by

$$SPL = 10 \log \left(\frac{p}{p_{ref}} \right)^2, \quad (3.1)$$

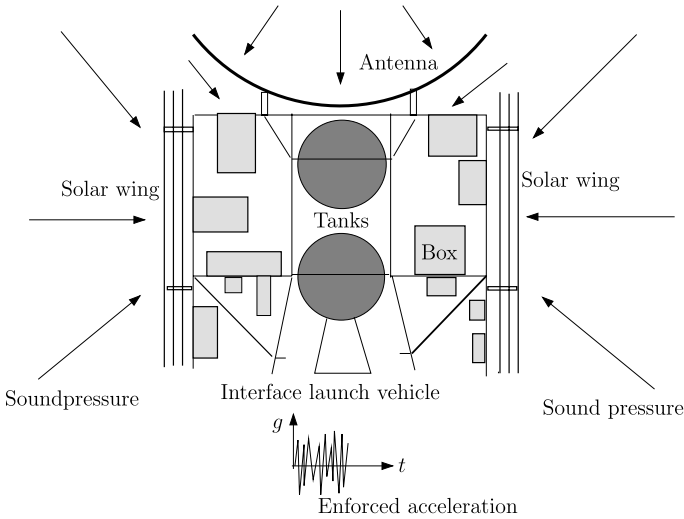


Fig. 3.1. Typical spacecraft

where p is the rms pressure in a certain frequency band with frequency band width Δf , mostly *one octave* or *one third octave band*, and p_{ref} is the *reference pressure* 2×10^{-5} Pa.

The x -th octave band of two sequential frequencies f_1 and f_2 is given by

$$\frac{f_2}{f_1} = 2^x, \tag{3.2}$$

where $x = 1$ for the octave, and $x = \frac{1}{3}$ when the one-third-octave band; then $\frac{f_2}{f_1} = 2^{\frac{1}{3}} = 1.260$.

The center frequency f_c (Hz) is defined by

$$f_c = \sqrt{f_{min} f_{max}}, \tag{3.3}$$

where f_{min} (f_{max}) is the minimum (maximum) frequency (Hz).

The frequency bandwidth Δf (Hz) is given by

$$\Delta f = f_{max} - f_{min}. \tag{3.4}$$

With $\frac{f_{max}}{f_{min}} = 2^x$ the bandwidth Δf can be expressed in terms of the center frequency f_c as follows

$$\Delta f = (2^{\frac{x}{2}} - 2^{-\frac{x}{2}}) f_c. \tag{3.5}$$

When

- $x = 1$, the one octave band width is $\Delta f = 0.7071 f_c$
- $x = \frac{1}{3}$ the one-third octave band width is $\Delta f = 0.2316 f_c$

The PSD of the pressure field $W_p(f_c)$ (Pa²/Hz) in the frequency band with center frequency f_c , bandwidth Δf and rms pressure $p(f_c)$ is defined as

$$W_p(f_c) = \frac{p^2(f_c)}{\Delta f(f_c)}. \tag{3.6}$$

The mean square value p_{rms}^2 of the sound pressure level is given by

$$p_{rms}^2 = \int_0^\infty W(f)df = \sum_{i=1}^k W_i(f_c)\Delta f = \sum_{i=1}^k p_i^2(f_c), \tag{3.7}$$

where k is the number of one octave or one third octave bands.

The *overall sound pressure level* (OASPL) in dB is defined as

$$OASPL = 10 \log \left(\frac{p_{rms}^2}{p_{ref}^2} \right). \tag{3.8}$$

Example. In Table 3.1 the center frequency f_c (Hz) and the corresponding SPL levels (dB) are specified in the octave band. We will calculate the corresponding bandwidths $\Delta f(f_c)$ (Hz), the mean square values $p^2(f_c)$ (Pa²), the PSD values $W_p(f_c)$ (Pa²/Hz) and finally the OASPL.

The following relation determines the conversion of a one-third- octave-band to a one-octave-band;

$$SPL_{1-octave} = 10 \log \left[\sum_{k=1}^3 10^{\frac{SPL_{\frac{1}{3}-octave}}{10}} \right], \tag{3.9}$$

and the following relation determines the conversion of the a one-octave band to the a one-third-octave-band

$$SPL_{\frac{1}{3}-octave} = SPL_{1-octave} + 10 \log \left[\frac{\Delta f_{\frac{1}{3}-octave}}{\Delta f_{1-octave}} \right]. \tag{3.10}$$

Table 3.1. SPL specification

f_c (Hz)	SPL (dB)	Δf_c (Hz)	p^2 (Pa ²)	$W_p(f_c)$ (Pa ² /Hz)
31.5	124	22.27	1004.75	45.11
63	130	44.55	4000.00	89.79
125	135	88.39	12649.11	143.11
250	139	176.78	31773.13	179.74
500	134	353.55	10047.55	28.42
1000	128	707.11	2523.83	3.57
2000	124	1414.21	1004.75	0.71
4000	120	2828.43	400.00	0.14
8000	116	5656.85	159.24	0.03

OASPL 142.01 dB

Table 3.2. Conversion calculation one octave to one third octave band

Octave band (Hz)	$SPL_{1-octave}$ (dB)	$\Delta f_{1-octave}$ (Hz)
125	135	88.4
$\frac{1}{3}$ octave band (Hz)	$\Delta f_{\frac{1}{3}-octave}$ (Hz)	$SPL_{\frac{1}{3}-octave}$ (dB)
100	23.2	129.2
125	28.9	130.1
160	37.1	131.2

Table 3.3. Conversion calculation one third octave to octave band

$\frac{1}{3}$ octave band (Hz)	$SPL_{\frac{1}{3}-octave}$ (dB)	$\frac{SPL_{\frac{1}{3}-octave}}{10}$
100	129.2	12.92
125	130.1	13.01
160	131.2	13.12
Octave band (Hz)	$SPL_{1-octave}$ (dB)	
125	135	

Example. The conversion of the octave band to the one-third-octave band is shown in Table 3.2. The reverse conversion is shown in Table 3.3.

Problems

3.1. Prove the following relations:

- The conversion of the SPL of one third octave band to the one octave band;

$$SPL_{1-octave} = 10 \log \left[\sum_{k=1}^3 10^{\frac{SPL_{\frac{1}{3}-octave}}{10}} \right].$$

- The conversion of the SPL of the one octave band to the one third octave band

$$SPL_{\frac{1}{3}-octave} = SPL_{1-octave} + 10 \log \left[\frac{\Delta f_{\frac{1}{3}-octave}}{\Delta f_{1-octave}} \right].$$

3.2. The following typical specification of the SPL is taken from [133]. Calculate the OASPL of the spectrum given in Table 3.4. Convert the one third octave SPL spectrum given in Table 3.4 to an octave band spectrum and recalculate the OASPL.

Answer: $OASPL = 144.9$ dB.

3.3. The power spectral density in ‘pink noise’ is proportional to the inverse of the frequency f , and is given by $W_p(f) = \frac{\sqrt{2}A^2}{f}$. Show that the rms value

Table 3.4. 1/3 Octave band specification

Center frequency (Hz)	SPL (dB) $p_{ref} = 2 \times 10^{-5}$ Pa
31.5	122.0
40.0	124.0
50.0	126.0
63.0	127.5
80.0	129.5
100.0	130.5
125.0	132.0
160.0	133.0
200.0	133.5
250.0	134.0
315.0	134.5
400.0	134.5
500.0	134.0
630.0	133.5
800.0	133.0
1000.0	132.0
1250.0	131.5
1600.0	130.0
2000.0	129.0
2500.0	128.0
3150.0	126.5
4000.0	125.0
5000.0	124.0
6300.0	122.5
8000.0	121.0
10000.0	120.0

of the sound pressure p in the octave frequency band is independent of the center frequency.

Answer: $p = A$.

3.4. The SPL's given in Table 3.1 are specified in the octave band. Convert the SPL values i the one-third-octave-band and recalculate the OASPL.

3.3 Response Analysis

The undamped equations of motion of a discrete number of coupled mass-spring systems with mass matrix $[M]$, stiffness matrix $[K]$, dynamic force vector $\{F(t)\}$, displacement vector $\{x(t)\}$ and acceleration vector $\{\ddot{x}(t)\}$ can be written as

$$[M]\{\ddot{x}(t)\} + [K]\{x(t)\} = \{F(t)\}. \quad (3.11)$$

The damping will be introduced later.

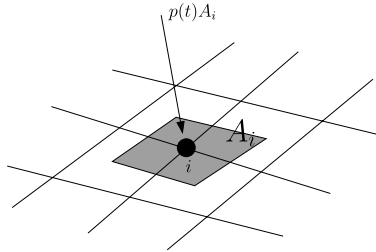


Fig. 3.2. Node i with associated nodal area A_i

In general, the discrete system is a finite element representation of a real structure.

The component $F_{ik}(t)$ of the force vector $\{F(t)\}$ consists of a pressure or pressure difference $p(t)$ applied normal to the area associated with node i . One force applied to the node i is

$$F_{ik}(t) = \int_{A_i} p(t) dA \approx p(t) A_{ik} \quad k = 1, 2, 3, \quad (3.12)$$

where A_{ik} is the area associated with node i in the direction of DOF k . This is illustrated in Fig. 3.2. The pressure $p(t)$ is normal to the area A_i .

The associated nodal area in one finite element can be obtained from [155]

$$\{A_i\} = \int_A \{\Psi_n\}^T dA, \quad i = 1, 2, 3, \quad (3.13)$$

where n is the number of nodes in one finite element and $\{\Psi\}$ the set of shape functions relating the internal displacement field to the nodal displacements. For a diffuse sound field, a correlation exists between the nodal forces. Radiation effects are not taken into account.

Example. A rod with a length L has two nodes. The displacement shape functions $\{\Psi(x)\}$ are

$$\{\Psi(x)\} = \left[1 - \frac{x}{L}, \frac{x}{L} \right].$$

The representative lengths $\{L_i\}$ become

$$\{L_i\} = \int_L \{\Psi\}^T dL = L \int_0^L \left(1 - \frac{x}{L} \right) d\left(\frac{x}{L}\right) = \left(\frac{L}{2} \right).$$

The associated lengths L_i of the two nodes are half of the length L of the rod.

The displacement vector $\{x(t)\}$ will be projected onto the independent mode shapes (eigenvectors), the modal base $[\Phi]$, multiplied by the generalized coordinates $\{\eta(t)\}$

$$\{x(t)\} = [\Phi]\{\eta(t)\}. \quad (3.14)$$

The orthogonality properties of the mode shapes with respect to the mass matrix $[M]$ and the stiffness matrix $[K]$ yields in diagonal matrices

$$[\Phi]^T[M][\Phi] = [\delta_{ij}m_i], \quad [\Phi]^T[K][\Phi] = [\delta_{ij}m_i\omega_i^2], \quad (3.15)$$

where δ_{ij} is the Kronecker Delta function, m_i are the generalized masses and ω_i^2 the eigenvalues of the dynamic system.

The coupled equations of motion are decoupled and expressed in terms of the generalized coordinates $\eta_i(t)$. We will now introduce the modal damping ratio ζ_i and the modal viscous damping term $2\zeta_i\omega_i\dot{\eta}_i(t)$. The decoupled damped equations of motion become

$$\ddot{\eta}_i(t) + 2\zeta_i\omega_i\dot{\eta}_i(t) + \omega_i^2\eta_i(t) = \frac{\{\phi\}^T\{F(t)\}}{m_i} = f_i(t), \quad i = 1, 2, \dots, n, \quad (3.16)$$

where n is the number of generalized coordinates.

The solution in the time domain with zero initial conditions (displacement and velocity) is

$$\eta_i(t) = \int_{-\infty}^{\infty} e^{-\zeta_i\omega_i\tau} \frac{\sin(\omega_d t)}{\omega_d} f_i(t - \tau) d\tau = \int_{-\infty}^{\infty} h_i(\tau) f_i(t - \tau) d\tau, \quad (3.17)$$

where $h_i(\tau)$ is the impulse response function and the damped natural (circular) frequency ω_d is defined as

$$\omega_d = \omega_i \sqrt{1 - \zeta_i^2}. \quad (3.18)$$

In the frequency domain, with $\Pi_i(\omega) = \int_{-\infty}^{\infty} \eta_i(t) e^{-j\omega t} dt$

$$\Pi_i(\omega) = H_i(\omega) f_i(\omega), \quad (3.19)$$

where $H_i(\omega) = \frac{1}{\omega_i^2 - \omega^2 + 2j\zeta_i\omega_i\omega}$ is the frequency response function and $f_i(\omega) = \int_{-\infty}^{\infty} f_i(t) e^{-j\omega t} dt$ is the Fourier transform of the force function $f_i(t)$.

The cross correlation function $R_{\eta_i\eta_j}(\tau)$ of the generalized coordinates $\eta_i(t)$ and $\eta_j(t)$ for an ergodic stationary process is given by

$$R_{\eta_i\eta_j}(\tau) = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T \eta_i(t) \eta_j(t - \tau) dt. \quad (3.20)$$

The relation between the cross PSD function $S_{\eta_i\eta_j}(\omega)$ and the cross correlation function $R_{\eta_i\eta_j}(\tau)$ is given by the Wiener-Khintchine relationships, namely

$$S_{\eta_i\eta_j}(\omega) = \int_{-\infty}^{\infty} R_{\eta_i\eta_j}(\tau) e^{-j\omega\tau} d\tau \quad i, j = 1, 2, \dots, n, \quad (3.21)$$

where n is the number of modes considered and

$$R_{\eta_i \eta_j}(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{\eta_i \eta_j}(\omega) e^{j\omega\tau} d\omega \quad i, j = 1, 2, \dots, n. \quad (3.22)$$

The matrix of cross correlation functions $[R_{f_i f_j}(\tau)]$ for an ergodic stationary forces $f_i(t)$ and $f_j(t)$ is given by

$$[R_{f_i f_j}(\tau)] = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T [f_i(t) f_j(t - \tau)] dt \quad i, j = 1, 2, \dots, n. \quad (3.23)$$

Equation (3.23) can be rewritten using (3.16) in matrix form

$$[R_{f_i f_j}(\tau)] = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T \frac{\{\phi_i\}^T \{F_k(t)\} \{F_l(t - \tau)\}^T \{\phi_j\}}{m_i m_j} dt$$

$$i, j = 1, 2, \dots, n, \quad k, l = 1, 2, \dots, m, \quad (3.24)$$

or

$$[R_{f_i f_j}(\tau)] = \frac{1}{m_i m_j} \{\phi_i\}^T [A] \left[\lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T \{p_k(t)\} \{p_l(t - \tau)\}^T dt \right] [A] \{\phi_j\}$$

$$i, j = 1, 2, \dots, n, \quad k, l = 1, 2, \dots, m \quad (3.25)$$

where $[A]$ is the diagonal matrix of areas associated with the m DOFs of the finite element model as illustrated in Fig. 3.2.

Finally, the cross correlation function $R_{f_i f_j}$ can be related to the cross correlation matrix of pressures all over the surface of the structure

$$[R_{f_i f_j}(\tau)] = \frac{1}{m_i m_j} \{\phi_i\}^T [A] [R_{p_k p_l}(\tau)] [A] \{\phi_j\}$$

$$i, j = 1, 2, \dots, n, \quad k, l = 1, 2, \dots, m, \quad (3.26)$$

where the matrix $[R_{p_k p_l}(\tau)]$ is the matrix of correlation functions of pressures over the surface of the structure, k and l denote the sequential numbers of DOFs and n the number of modes that are considered.

The matrix of cross PSD functions becomes

$$[S_{f_i f_j}(\omega)] = \frac{1}{m_i m_j} \{\phi_i\}^T [A] [S_{p_k p_l}(\omega)] [A] \{\phi_j\}$$

$$i, j = 1, 2, \dots, n, \quad k, l = 1, 2, \dots, m, \quad (3.27)$$

where the matrix $[S_{p_k p_l}(\omega)]$ is the matrix of PSD functions of pressures over the surface of the structure.

The matrix of cross PSD functions $[S_{p_k p_l}(\omega)]$ of the pressures all over the surface and related to the DOFs $i = 1, 2, \dots, m$ is as follows

$$[S_{p_k p_l}(\omega)] = \begin{pmatrix} S_{p_1 p_1}(\omega) & S_{p_1 p_2}(\omega) & \cdots & S_{p_1 p_m}(\omega) \\ S_{p_2 p_1}(\omega) & S_{p_2 p_2}(\omega) & \cdots & S_{p_2 p_m}(\omega) \\ \vdots & \vdots & \ddots & \vdots \\ S_{p_m p_1}(\omega) & S_{p_m p_2}(\omega) & \cdots & S_{p_m p_m}(\omega) \end{pmatrix}. \quad (3.28)$$

Equation 3.21 can be written as

$$S_{\eta_i \eta_j}(\omega) = \int_{-\infty}^{\infty} \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T \left(\int_{-\infty}^{\infty} h_i(\alpha) f_i(t - \alpha) d\alpha \right) \times \left(\int_{-\infty}^{\infty} h_j(\beta - \tau) f_j(t - (\beta - \tau)) d\beta \right) e^{-j\omega\tau} d\tau dt, \quad (3.29)$$

and

$$S_{\eta_i \eta_j}(\omega) = \int_{-\infty}^{\infty} R_{f_i f_j}(\tau - \alpha + \beta) e^{-j\omega(\tau - \alpha + \beta)} d\tau \times \left(\int_{-\infty}^{\infty} h_i(\alpha) e^{-j\omega\alpha} d\alpha \right) \left(\int_{-\infty}^{\infty} h_j(\beta) e^{j\omega\beta} d\beta \right), \quad (3.30)$$

where $H_i(\omega) = \int_{-\infty}^{\infty} h_i(\tau) e^{-j\omega\tau} d\tau$. The PSD function matrix for the generalized coordinates $[S_{\eta_i \eta_j}(\omega)]$ can be obtained by the following matrix equation

$$[S_{\eta_i \eta_j}(\omega)] = \frac{H_i(\omega) \{\phi_i\}^T [A] [S_{p_k p_l}(\omega)] [A] \{\phi_j\} H_j^*(\omega)}{m_i m_j \omega_i^2 \omega_j^2} \\ i, j = 1, 2, \dots, n, \quad k, l = 1, 2, \dots, m, \quad (3.31)$$

where $H_i(\omega) = \frac{\omega_i^2}{\omega_i^2 - \omega^2 + 2j\zeta_i \omega_i \omega}$ the frequency response function and $H_j^*(\omega) = \frac{\omega_j^2}{\omega_j^2 - \omega^2 - 2j\zeta_j \omega_j \omega}$ called the conjugate frequency response function with $H_i(\omega) H_i^*(\omega) = |H_i(\omega)|^2$. The matrix of the cross spectral density functions of the generalized coordinates η_i and η_j is

$$[S_{\eta_i \eta_j}(\omega)] = \left[\frac{H_i(\omega)}{m_i \omega_i^2} \right] [\Phi]^T [A] [S_{p_k p_l}(\omega)] [A] [\Phi] \left[\frac{H_j^*(\omega)}{m_j \omega_j^2} \right], \quad (3.32)$$

where $[\Phi]$ is the modal base.

In general, (3.28) is expressed as follows

$$[S_{p_k p_l}(\omega)] = S_p(\omega) \begin{pmatrix} C_{p_1 p_1}(\omega) & C_{p_1 p_2}(\omega) & \cdots & C_{p_1 p_m}(\omega) \\ C_{p_2 p_1}(\omega) & C_{p_2 p_2}(\omega) & \cdots & C_{p_2 p_m}(\omega) \\ \vdots & \vdots & \ddots & \vdots \\ C_{p_m p_1}(\omega) & C_{p_m p_2}(\omega) & \cdots & C_{p_m p_m}(\omega) \end{pmatrix}, \quad (3.33)$$

where $S_p(\omega)$ is the reference PSD function of the applied pressures. This reference PSD of the pressure is, in general, related to the sound pressure levels (SPL) of the sound field exposed to the surface of the structure. $C_{p_i p_j}(\omega)$ is the correlation (coherence) function between pressures at the nodes i and j .

Some typical pressure fields can be described [220]:

1. If the dimension of the surface is less than a quarter of the wave length λ the correlation function $C_{p_i p_j}(\omega) = 1.0$. The wave length is $\lambda = \frac{2\pi}{k}$ with the wave number $k = \frac{\omega}{c}$ and c is the speed of sound. At room temperature the speed of sound under 1 Bar is $c = 340$ m/s.

2. If the sound pressure field is completely random (rain on the roof) the off-diagonal terms of (3.33) are zero and the diagonal terms $C_{p_i p_i}(\omega) = 1.0$. The correlation matrix $[C_{p_i p_j}(\omega)] = [I]$.
3. A three dimensional wave field with uniform intensity $I = \frac{p^2}{\rho c}$ in all directions is commonly called a reverberant sound field, with ρ the density of the air, $\rho \approx 1.2 \text{ kg/m}^3$. The coherence function $C_{p_i p_j}(\omega) = \frac{\sin(k|\bar{r}_i - \bar{r}_j|)}{(k|\bar{r}_i - \bar{x}_j|)}$ and $|\bar{r}_i - \bar{r}_j|$ the distance between node i and j . The correlation function $\frac{\sin(k|\bar{r}_i - \bar{r}_j|)}{(k|\bar{r}_i - \bar{x}_j|)}$ in the x - y plane is often replaced by $\frac{\sin(k|\bar{r}_i - \bar{r}_j|)}{(k|\bar{r}_i - \bar{x}_j|)} \approx \frac{\sin(k|\Delta x|)}{(k|\Delta x|)} \frac{\sin(k|\Delta y|)}{(k|\Delta y|)}$.

Equation (3.31) can be written as

$$[S_{\eta_i \eta_j}(\omega)] = S_p(\omega) \frac{H_i(\omega) J_{ij}^2 H_j^*(\omega)}{m_i m_j \omega_i^2 \omega_j^2} \quad (3.34)$$

$$i, j = 1, 2, \dots, n, \quad k, l = 1, 2, \dots, m,$$

where J_{ij}^2 is called the *joint acceptance*

$$J_{ij}^2 = \{\phi_i\}^T [A] [C_{p_k p_l}(\omega)] [A] \{\phi_j\}. \quad (3.35)$$

Thus (3.32) can be rewritten as

$$[S_{\eta_i \eta_j}(\omega)] = S_p(\omega) \left[\frac{H_i(\omega)}{m_i \omega_i^2} \right] [J_{ij}^2] \left[\frac{H_j^*(\omega)}{m_j \omega_j^2} \right], \quad (3.36)$$

with the matrix of joint acceptances $[J_{ij}^2]$

$$[J_{ij}^2] = [\Phi]^T [A] [C_{p_k p_l}(\omega)] [A] [\Phi]. \quad (3.37)$$

Example. The joint acceptance function will be calculated at a frequency $f = 54.113 \text{ Hz}$ and a speed of sound $c = 340 \text{ m/s}^2$. The wave number becomes $k = 2\pi/c = 1$. The mode shape $\phi(x, y) = \sin \frac{\pi x}{2a} \sin \frac{\pi y}{2b}$, the first mode shape of a simply supported rectangular plate with length $a = 1$ and width $b = 1$. The joint acceptance functions J_{ij}^2 are calculated using integrals instead of matrix multiplications:

1. Rigid body mode and a plane wave (complete correlation $C_{p_i p_j} = 1$)
2. Elastic mode $\phi(x, y)$ and a plane wave (complete correlation $C_{p_i p_j} = 1$)
3. Rigid body mode and a reverberant sound field with

$$C_{p_i p_j} = \frac{\sin(k\Delta x) \sin(k\Delta y)}{k\Delta x k\Delta y}$$

4. Elastic mode $\phi(x, y)$ and a reverberant sound field with

$$C_{p_i p_j} = \frac{\sin(k\Delta x) \sin(k\Delta y)}{k\Delta x k\Delta y}$$

The solution of the first joint acceptance function J_{00}^2 (index 0 stands for rigid body mode) is

$$J_{00}^2 = \int_{a=-1}^{a=1} \int_{b=-1}^{b=1} \int_{a=-1}^{a=1} \int_{b=-1}^{b=1} dx dy du dv = 16$$

The second joint acceptance function J_{11}^2 (index 1 stands for the first elastic mode) is

$$J_{11}^2 = \int_{a=-1}^{a=1} \int_{b=-1}^{b=1} \int_{a=-1}^{a=1} \int_{b=-1}^{b=1} \sin \frac{\pi x}{2a} \sin \frac{\pi y}{2b} \sin \frac{\pi u}{2a} \sin \frac{\pi v}{2b} dx dy du dv = 2.628.$$

This result is conform [55]. The third joint acceptance function J_{00}^2 becomes

$$J_{00}^2 = \int_{a=-1}^{a=1} \int_{b=-1}^{b=1} \int_{a=-1}^{a=1} \int_{b=-1}^{b=1} \frac{\sin |x-y|}{|x-y|} \frac{\sin |u-v|}{|u-v|} dx dy du dv = 12.883,$$

and the fourth joint acceptance function J_{11}^2 becomes

$$J_{11}^2 = \int_{a=-1}^{a=1} \int_{b=-1}^{b=1} \int_{a=-1}^{a=1} \int_{b=-1}^{b=1} \frac{\sin |x-y|}{|x-y|} \frac{\sin |u-v|}{|u-v|} \\ \times \sin \frac{\pi x}{2a} \sin \frac{\pi y}{2b} \sin \frac{\pi u}{2a} \sin \frac{\pi v}{2b} dx dy du dv = 2.321.$$

The joint acceptance function J_{ij}^2 is a measure for the coupling with the sound field.

The matrix of the cross PSD $[S_{x_i x_j}(\omega)]$ of the physical displacements $\{x\}$ can be obtained by

$$[S_{x_i x_j}(\omega)] = [\Phi][S_{\eta_i \eta_j}(\omega)][\Phi]^T, \quad (3.38)$$

the matrix of cross PSD $[S_{\dot{x}_i \dot{x}_j}(\omega)]$ of the velocities $\{\dot{x}\}$ becomes

$$[S_{\dot{x}_i \dot{x}_j}(\omega)] = \omega^2 [\Phi][S_{\eta_i \eta_j}(\omega)][\Phi]^T, \quad (3.39)$$

and the matrix of cross PSD $[S_{\ddot{x}_i \ddot{x}_j}(\omega)]$ of the accelerations $\{\ddot{x}\}$

$$[S_{\ddot{x}_i \ddot{x}_j}(\omega)] = \omega^4 [\Phi][S_{\eta_i \eta_j}(\omega)][\Phi]^T. \quad (3.40)$$

The matrix of cross PSD of the stresses $S_{\sigma_i \sigma_j}(\omega)$ of the internal stresses σ can be expressed as follows

$$[S_{\sigma_i \sigma_j}(\omega)] = [\Phi_\sigma][S_{\eta_i \eta_j}(\omega)][\Phi_\sigma]^T, \quad (3.41)$$

where $[\Phi_\sigma]$ is the matrix of stress modes.

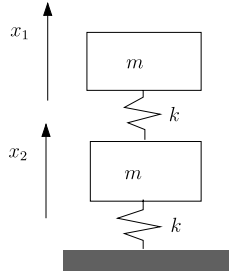


Fig. 3.3. 2 mass-spring system

Example. A two mass-spring system is illustrated in Fig. 3.3. The stress modes $[\Phi_\sigma]$ will be calculated. The stress modes are defined as follows

$$[\Phi_\sigma] = [D_\sigma][\Phi],$$

where $[D_\sigma]$ is the stress matrix (the stress matrix times the displacement vector $\{x\}$ will yield the static stresses)

$$[D_\sigma] = \frac{k}{A} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix},$$

with A the cross section of the springs.

The mode shapes are

$$[\Phi] = \frac{1}{\sqrt{m}} \begin{pmatrix} -0.8507 & -0.5257 \\ -0.5257 & 0.8507 \end{pmatrix},$$

with the generalized masses $[\Phi]^T [M][\Phi] = [I]$. The stress modes now become

$$[\Phi_\sigma] = [D_\sigma][\Phi] = \frac{k}{A\sqrt{m}} \begin{pmatrix} -0.3249 & -1.3764 \\ -0.5257 & 0.8507 \end{pmatrix}.$$

The PSD is symmetric with respect to $\omega = 0$ and, if the circular frequency ω is replaced by the number of cycles per second f , the PSD function $S(\omega)$ can be replaced by

$$W(f) = 2S(\omega). \tag{3.42}$$

In all equations the PSD function $S(\omega)$ can be replaced by $W(f)$ and ω by $\omega = 2\pi f$.

Example. The purpose of the example is the show the procedure to calculate the response of a simply supported beam caused by an acoustic field.

A simple supported beam with length $4L$ is mathematically represented by a finite element model consisting of 5 nodes and 4 bar elements. The beam has a width b . The mass per unit of length of the beam is m . The finite element model is illustrated in Fig. 3.4. The total area of the beam is $4bL$. The natural frequencies of the simply supported beam with mode shapes $\phi(x) = \sin \frac{n\pi x}{4L}$, $n = 1, 2, \dots$ are given by

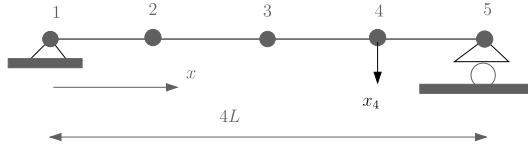


Fig. 3.4. Finite element model simply supported beam

$$f_n = \frac{(n\pi)^2}{2\pi} \sqrt{\frac{EI}{m(4L)^4}}, \quad n = 1, 2, \dots$$

The damping ratio for all modes is $\zeta = 0.05$. The beam is exposed to a random pressure field with the coherence matrix $[C_{p_i p_j}(f)]$.

The modal matrix $[\Phi]$ of the first two modes, the lumped mass matrix $[M]$ and the diagonal matrix of areas $[A]$ are

$$[\Phi] = \begin{bmatrix} 0 & 0 \\ \frac{1}{2}\sqrt{2} & 1 \\ 1 & 0 \\ -\frac{1}{2}\sqrt{2} & -1 \\ 0 & 0 \end{bmatrix}, \quad [M] = mL \begin{bmatrix} \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} \end{bmatrix},$$

$$[A] = Lb \begin{bmatrix} \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} \end{bmatrix}.$$

The generalized mass matrix becomes

$$[\Phi]^T [M] [\Phi] = 2mL \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

The following coherence matrices are considered:

1. Rain on the roof sound field

$$[C_{p_i p_j}(f)] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

2. Plane wave sound field

$$[C_{p_i p_j}(f)] = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

3. Reverberant sound field

$$[C_{p_i p_j}(f)] = \begin{bmatrix} 1 & \theta_1 & \theta_2 & \theta_3 & \theta_4 \\ \theta_1 & 1 & \theta_1 & \theta_2 & \theta_3 \\ \theta_2 & \theta_1 & 1 & \theta_1 & \theta_2 \\ \theta_3 & \theta_2 & \theta_1 & 1 & \theta_1 \\ \theta_4 & \theta_3 & \theta_2 & \theta_1 & 1 \end{bmatrix},$$

where $\theta_n = \sin(nkl)/nkl$, the wave number is $k = \frac{2\pi f}{c}$, and the speed of sound is $c \approx 340$ m/s. Assume $kL = 1$ then

$$[C_{p_i p_j}(f)] = \begin{bmatrix} 1 & 0.8415 & 0.4546 & 0.0470 & -0.1892 \\ 0.8415 & 1 & 0.8415 & 0.4546 & 0.0470 \\ 0.4546 & 0.8415 & 1 & 0.8415 & 0.4546 \\ 0.0470 & 0.4546 & 0.8415 & 1 & 0.8415 \\ -0.1892 & 0.0470 & 0.4546 & 0.8415 & 1 \end{bmatrix}.$$

The response of the beam exposed to a rain on the roof type of sound field will be worked out. The joint acceptance matrix can be obtained using (3.37)

$$[J_{ij}^2] = [\Phi]^T [A] [C_{p_k p_l}(\omega)] [A] [\Phi] = 2(Lb)^2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

The PSD of the two generalized coordinates, associated with the two modes, can be now calculated using (3.36)

$$\begin{aligned} [W_{\eta_i \eta_j}(f)] &= W_p(f) \left[\frac{H_i(f)}{m_i (2\pi f_i)^2} \right] [J_{ij}^2] \left[\frac{H_j^*(\omega)}{m_j (2\pi f_j)^2} \right] \\ &= \frac{W_p(f)b^2}{2m^2(2\pi)^4} \begin{bmatrix} \frac{|H_1(f)|^2}{f_1^4} & 0 \\ 0 & \frac{|H_2(f)|^2}{f_2^4} \end{bmatrix} = \frac{W_p(f)b^2}{2m^2(2\pi)^4} \begin{bmatrix} h_1 & 0 \\ 0 & h_2 \end{bmatrix}, \end{aligned}$$

where $|H_i(f)|^2 = \frac{1}{[1 - (\frac{f}{f_i})^2]^2 + [2\zeta_i (\frac{f}{f_i})]^2}$, ζ_i is the modal damping ratio and f_i is the natural frequency for $i = 1, 2$. The PSD of physical displacements $\{x\}$ can be calculated with the aid of (3.38)

$$\begin{aligned} [W_{x_i x_j}(f)] &= [\Phi] [W_{\eta_i \eta_j}(f)] [\Phi]^T \\ &= \frac{W_p(f)b^2}{2m^2(2\pi)^4} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0.5h_1 + h_2 & 0.7071h_1 & 0.5h_1 - h_2 & 0 \\ 0 & 0.7071h_1 & h_1 & 0.7071h_1 & 0 \\ 0 & 0.5h_1 - h_2 & 0.7071h_1 & 0.5h_1 + h_2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \end{aligned}$$

After the calculations of the PSD of the displacements $\{x\}$, the accelerations, forces, stresses, etc. and associated rms values can be numerically obtained. The lowest two natural frequencies of the simply supported beam are

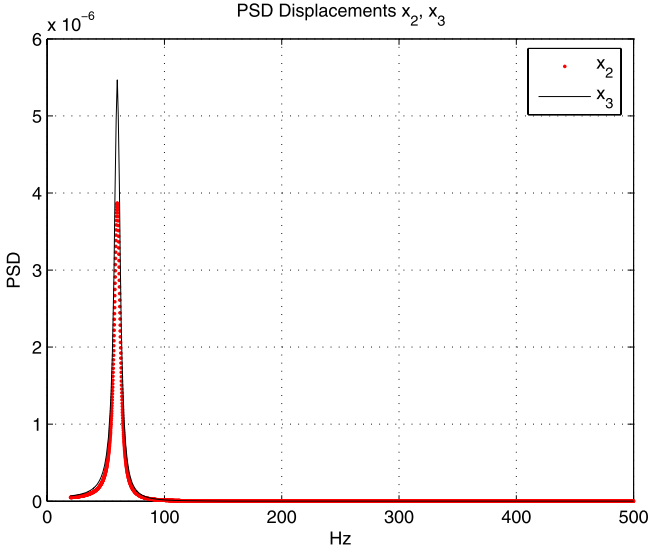


Fig. 3.5. PSD displacements

$f_1 = 60$ Hz, and $f_2 = \frac{(2\pi)^2}{2\pi} f_1 = 377.0$ Hz. The modal damping ratio is constant for both modes $\zeta = 0.05$. The PSD of the pressure field W_p is constant in a frequency band $f = 20\text{--}500$ Hz and zero otherwise. The PSD of the pressure is normalized, e.g. $\frac{W_p b^2}{2m^2(2\pi)^4} = 1$. The PSD of the normalized displacements and accelerations and associated positive zero crossings of the nodes 2 and 3 (4 is the same as 2) are calculated. The responses are shown in Figs. 3.5 and 3.6. The rms values of the normalized displacements and positive zero crossings are

$$\begin{Bmatrix} x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 0.0060 \\ 0.0071 \end{Bmatrix}, \quad \begin{Bmatrix} \nu_{0,2}^+ \\ \nu_{0,3}^+ \end{Bmatrix}_{rms} = \begin{Bmatrix} 68.4615 \\ 60.4381 \end{Bmatrix},$$

and the rms values of the normalized accelerations and positive zero crossings are

$$\begin{Bmatrix} \ddot{x}_2 \\ \ddot{x}_3 \end{Bmatrix} = \begin{Bmatrix} 3.1084 \times 10^3 \\ 1.2504 \times 10^3 \end{Bmatrix}, \quad \begin{Bmatrix} \nu_{0,2}^+ \\ \nu_{0,3}^+ \end{Bmatrix} = \begin{Bmatrix} 370.1744 \\ 184.8545 \end{Bmatrix}.$$

The integration had been done with a frequency increment $\Delta f = 0.1$ Hz.

Problems

3.5. Calculate the modal bending stress at point A near the fixation of the bending beam with bending stiffness EI , length L and a bending resistance W .

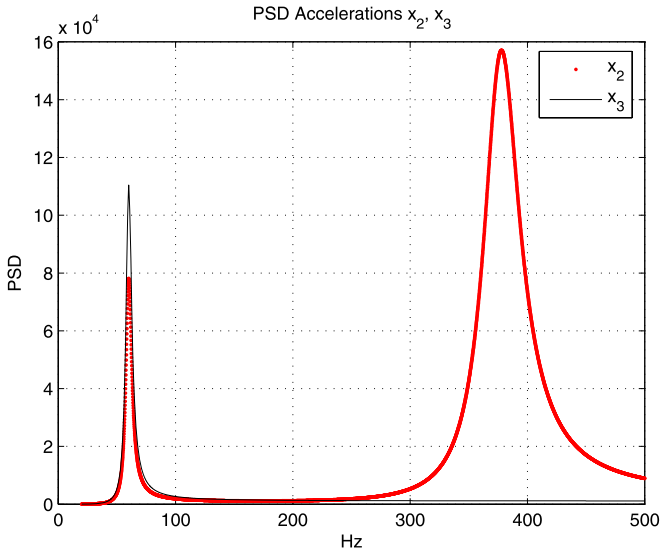


Fig. 3.6. PSD accelerations

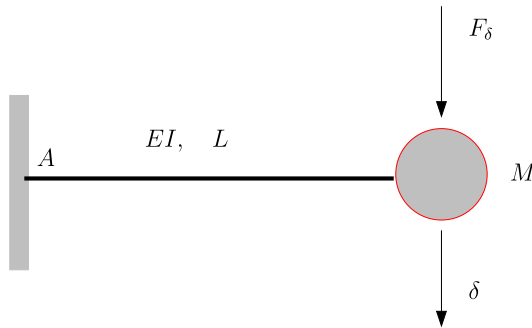


Fig. 3.7. Calculation modal stress

At the free end of the beam a mass M has been connected. The bending beam is massless. The bending beam is illustrated in Fig. 3.7.

- Calculate the natural frequency.
- Calculate the mode shape ϕ at the location of the mass M . The generalized mass $M\phi^2 = 1$.
- Calculate the force $F_{\delta=1}$.
- Calculate the bending stress σ_b at A due to $F_{\delta=1}$.
- Calculate the modal bending stress σ_{bm} at A .

Answers: $f_n = \frac{1}{2\pi} \sqrt{\frac{3EI}{ML^3}}$, $\phi = \frac{1}{\sqrt{M}}$, $F_{\delta=1} = \frac{3EI}{L^3}$, $\sigma_b = \frac{3EI}{WL^2}$, $\sigma_{bm} = \frac{3EI}{\sqrt{MWL^2}}$.

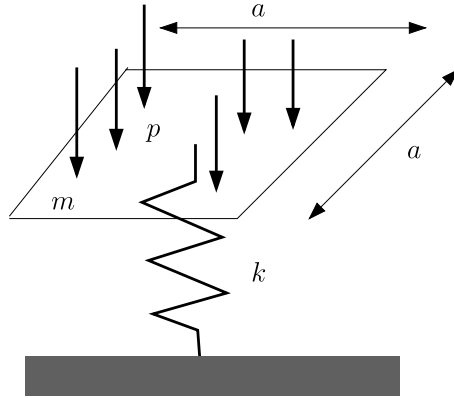


Fig. 3.8. Acoustic load applied to supported rigid plate

3.6. A rigid square plate with length and width a has a mass m . This system is illustrated in Fig. 3.8. The plate is supported in the center by a spring with spring stiffness k and cross section A . The amplification factor $Q = \frac{1}{2\zeta}$, with ζ the damping ratio. The PSD of the pressure $W_p(f) = W_p$. The wavelength is $\lambda > a$. Perform the following operations:

- Calculate the natural frequency f_0 of the system.
- Define the normalized mode ϕ with $\phi^2 m = 1$ and $\delta = \phi \eta$.
- Define the stress mode ϕ_σ .
- Calculate the joint acceptance J_{00}^2 .
- Calculate the PSD response $W_{\eta\eta}(f_0)$ of the modal participation η .
- Calculate the PSD response $W_{\delta\delta}(f_0)$ of the displacement δ of the mass m at the natural frequency f_0 .
- Calculate the PSD stress $W_{\sigma\sigma}(f_0)$ in the spring at the natural frequency f_0 .

Answers: $f_0 = \frac{1}{2\pi} \sqrt{\frac{k}{m}}$, $\phi = \frac{1}{\sqrt{m}}$, $\phi_\sigma = \frac{k}{A} \frac{1}{\sqrt{m}}$, $J_{00}^2 = a^4$, $W_{\eta\eta}(f_0) = \frac{Q^2}{(2\pi f_0)^4} \frac{a^4}{m} W_p$, $W_{\delta\delta}(f_0) = \frac{Q^2}{(2\pi f_0)^4} \frac{a^4}{m^2} W_p$, $W_{\sigma\sigma}(f_0) = \left(\frac{k}{A}\right)^2 \frac{Q^2}{(2\pi f_0)^4} \frac{a^4}{m^2} W_p$.

3.7. Repeat the example given on page 214 of the response analysis of a simply supported beam (Fig. 3.4), however, now exposed to an acoustic field in the form of a plane wave. All the terms in the coherence matrix $[C_{p_i p_j}(f)]$ are equal to one. All other parameters are the same.

- Calculate the joint acceptance matrix $[J_{ij}^2]$.
- Calculate the matrix of response of the two generalized coordinates $[W_{\eta_i \eta_j}(f)]$.
- Calculate the matrix of physical displacements $[W_{x_i x_j}(f)]$.
- Calculate numerically with $\Delta f = 0.2$ Hz the normalized diagonal terms of the matrix of PSD values $[W_{x_i x_j}(f)]$, x_2 and x_3 , and associated numbers of upward crossings ν_0^+ .

- Calculate numerically with $\Delta f = 0.2$ Hz the normalized diagonal terms of the matrix of PSD values $[W_{\ddot{x}_i \ddot{x}_j}(f)] = (2\pi f)^4 [W_{x_i x_j}(f)]$, \ddot{x}_2 and \ddot{x}_3 , and associated numbers of upward crossings ν_0^+ .

3.8. The wavelength of acoustic waves propagating in air and flexural waves propagating in large bending plates are as follows [18]

$$\lambda_{acoustic} = \frac{c}{f},$$

$$\lambda_{plate} = \left(\frac{Et^3}{12m(1-\nu^2)} \right)^{\frac{1}{4}} \left(\frac{2\pi}{f} \right)^{\frac{1}{2}},$$

where c is the speed of sound $c = 340$ m/s², m the mass per unit of area $m = 2$ kg/m², t the thickness of the plate $t = 2$ mm, E is the Young's modulus $E = 70$ GPa, the Poisson's ratio $\nu = 0.3$, and f is the frequency in Hz.

- At which frequency (critical or coincidence frequency) do the two waves coincide?
- For lower frequencies, is the flexural wavelength greater than or less than the acoustic wavelength?
- Evaluate the numerical values of the coincidence frequency and the associated wavelength.
- The compressional wave velocity is given by $c_p = [\frac{Et}{m(1-\nu^2)}]^{0.5}$. Express the coincidence frequency $f_{coincidence}$ in c_p .

Answers: $f_{coincidence} = \frac{c^2}{2\pi\sqrt{\frac{Et^3}{12m(1-\nu^2)}}}$, $f < f_{coincidence} \rightarrow \frac{\lambda_{plate}}{\lambda_{acoustic}} < 1$, $f > f_{coincidence} \rightarrow \frac{\lambda_{plate}}{\lambda_{acoustic}} > 1$, $f_{coincidence} = 3633$ Hz, $\lambda_{plate} = \lambda_{acoustic} = 0.094$ m,
 $f_{coincidence} = \frac{\sqrt{12}c^2}{2\pi t\sqrt{c_p}}$.

3.4 Modal Damping

Table 3.5 shows the NASA recommended modal damping ratio profile for vibroacoustic analysis. The modal damping is expressed as percent of the critical damping.

Table 3.5. Recommended modal damping ratio [31]

Frequency (Hz)	Modal damping ratio ζ
0.0–10.0	0.01
10.0–35.0	0.02
75.0	0.03
130.0	0.04
200.0	0.05

3.5 Simplified Acoustic Response Analysis

3.5.1 Introduction

In the early stage, structures must be dimensioned before more detailed strength and stiffness analyzes can be made. For that simple analysis technique may applied very fruitful. A few of this methods will be discussed and illustrated with examples.

3.5.2 Acoustic Loads Transformed into Mechanical Random Vibrations

Large (sandwich) panels, with all kinds of instruments, electronic boxes, etc. mounted onto them, are fairly sensitive to acoustic loads. The acoustic loads are transformed into mechanical random vibrations by the panel. These mechanical random vibrations vibrate in tune with the mounted instruments, electronic boxes, etc. In this section we determine the applied mechanical vibrations due to the acoustic loads.

The SPL is expressed with respect to a reference pressure $p_{ref} = 2 \times 10^{-5}$ Pa and is given by

$$SPL = 10 \log \left(\frac{p^2}{p_{ref}^2} \right).$$

The SPL are usually specified in the 1/1 octave band or 1/3 octave band. The center frequency f_c (Hz) can be associated with the bandwidth Δf :

- 1/1 octave band $\Delta f = 0.7071f_c$
- 1/3 octave band $\Delta f = 0.2316f_c$

The PSD function $W_p(f)$ (Pa²/Hz), where p is the rms sound pressure for a specific center frequency f_c with bandwidth Δf can be obtained as follows

$$W_p(f) = \frac{p^2}{\Delta f}. \quad (3.43)$$

The power spectral density of the acceleration of the box, due to the acoustic pressure (Fig. 3.9), is determined as follows [191]

$$W_a(f) = \beta^2 Q^2 \left(\frac{A}{gM} \right)^2 W_p(f), \quad (3.44)$$

where $W_a(f)$ is the PSD of the accelerations (g²/Hz) of the box and the supporting structures, $\beta = 2.5$ is the effectiveness *vibro acoustic factor*, $Q = 4.5$ is the amplification factor, A is the area of the side of the box (m²) mounted to the panel, M is the total mass of the box and the supporting structure (kg) and g is the gravitational acceleration (9.81 m/s²).

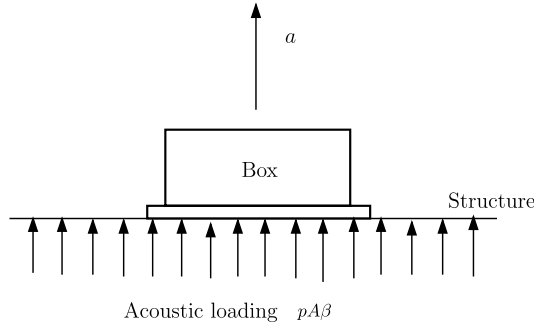


Fig. 3.9. Model for evaluation of the effects of acoustic pressure

Table 3.6. Calculation of wave length λ , $c = 340$ m/s

Center frequency f_c	Wave length $\lambda = \frac{c}{f}$ (m)
31.5	10.79
63	5.40
125	2.72
250	1.36
500	0.68
1000	0.34

If the wave length $\lambda = \frac{c}{f} \geq \sqrt{A}$, where \sqrt{A} is the characteristic length or width of the panel, the sound wave may be considered as a plane wave. The speed of sound is $c = 340$ m/s. The wave length λ in the one octave band is illustrated in Table 3.6.

The expression (3.44) can be easily derived from the undamped equation of motion for an isotropic plate with a deflection w , bending stiffness D , mass per unit of area m and exposed to a uniform pressure p

$$D \left(\frac{\partial^4 w}{\partial x^4} + 2 \frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} \right) + m \frac{\partial^2 w}{\partial t^2} = p, \tag{3.45}$$

where x and y are rectangular coordinates.

The deflection w can be expressed

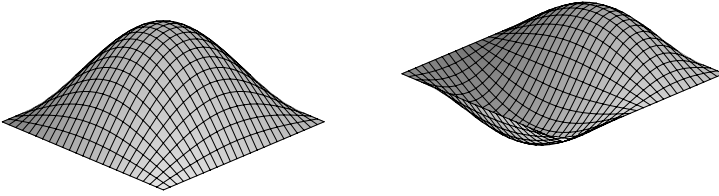
$$w(x, y) = \sum_{i=1}^n \sum_{j=1}^n \eta_{ij}(t) \phi_{ij}(x, y), \tag{3.46}$$

where $\phi_{ij}(x, y)$ is the ij th mode shape and $\eta_{ij}(t)$ the ij th generalized coordinate. The mode shapes are orthogonal, so

$$m \int_0^b \int_0^a \phi_{ij}(x, y) \phi_{kl}(x, y) dx dy = m_{gij} \delta_{ik} \delta_{jl}, \tag{3.47}$$

where a and b are the length and width of the rectangular bending plate.

The mode shapes can be approximated, e.g.


 (a) First mode $i = j = 1$

 (b) Second mode $i = 2, j = 1$
Fig. 3.10. Vibration modes simply supported plate $\phi_{ij}(x, y) = (\sin \frac{i\pi x}{a} \sin \frac{j\pi y}{b})$

- simply supported plate, $\phi_{ij}(x, y) = (\sin \frac{i\pi x}{a} \sin \frac{j\pi y}{b})$, $i, j = 1, 2, \dots, n$ and
- clamped supported plate, $\phi_{ij}(x, y) = \sin \frac{i\pi x}{a} (1 - \cos \frac{2\pi x}{a}) \sin \frac{j\pi y}{b} (1 - \cos \frac{2\pi y}{b})$, $i, j = 1, 2, \dots, n$.

The natural frequency of the supported plate is ω_{ij} , $i, j = 1, 2, \dots, n$. The uncoupled damped equations of motions for η_{ij} can now be obtained when the orthogonality relations for the mode shapes ϕ_{ij} and the modal damping ratio ζ_{ij} are introduced

$$\dot{\eta}_{ij} + 2\zeta_{ij}\omega_{ij}\dot{\eta}_{ij} + \omega_{ij}^2\eta_{ij} = \frac{p \int_0^b \int_0^a \phi_{ij} dx dy}{m \int_0^b \int_0^a \phi_{ij}^2 dx dy}, \quad i, j = 1, 2, \dots, n. \quad (3.48)$$

The equations of motion (3.48) are Fourier transformed in the frequency domain. With $\eta_{ij} = \Pi_{ij}(\omega)e^{j\omega t}$ and $p = P(\omega)e^{j\omega t}$ the equations for Π_{ij} become

$$\begin{aligned} \Pi_{ij} &= \left(\frac{1}{\omega_{ij}^2 - \omega^2 + 2j\zeta_{ij}\omega_{ij}\omega} \right) \frac{P \int_0^b \int_0^a \phi_{ij} dx dy}{m \int_0^b \int_0^a \phi_{ij}^2 dx dy} \\ &= H_{ij}(\omega) \frac{P \int_0^b \int_0^a \phi_{ij} dx dy}{m \int_0^b \int_0^a \phi_{ij}^2 dx dy}, \end{aligned} \quad (3.49)$$

and for the acceleration of the generalized coordinates $\ddot{\Pi}_{ij} = -\omega^2 \Pi_{ij}$, thus

$$\begin{aligned} \ddot{\Pi}_{ij} &= \left(\frac{-\omega^2}{\omega_{ij}^2 - \omega^2 + 2j\zeta_{ij}\omega_{ij}\omega} \right) \frac{P \int_0^b \int_0^a \phi_{ij} dx dy}{m \int_0^b \int_0^a \phi_{ij}^2 dx dy} \\ &= -\omega^2 H_{ij}(\omega) \frac{P \int_0^b \int_0^a \phi_{ij} dx dy}{m \int_0^b \int_0^a \phi_{ij}^2 dx dy}. \end{aligned} \quad (3.50)$$

The PSD function of the displacement and the acceleration for the generalized coordinates can now be expressed by

$$S_{\eta_{ij}}(\omega) = |H_{ij}(\omega)|^2 \frac{S_p(\omega) (\int_0^b \int_0^a \phi_{ij} dx dy)^2}{(m \int_0^b \int_0^a \phi_{ij}^2 dx dy)^2}, \quad (3.51)$$

and

$$S_{\ddot{\eta}_{ij}}(\omega) = \omega^4 |H_{ij}(\omega)|^2 \frac{S_p(\omega) \left(\int_0^b \int_0^a \phi_{ij} dx dy \right)^2}{\left(m \int_0^b \int_0^a \phi_{ij}^2 dx dy \right)^2}. \quad (3.52)$$

The rms value of the displacement of the generalized coordinate $\langle \eta_{ij}^2 \rangle$ is given by

$$\langle \eta_{ij}^2 \rangle = \frac{1}{4\zeta_{ij}^2 \omega_{ij}^2} \frac{S_p(\omega_{ij}) \left(\int_0^b \int_0^a \phi_{ij} dx dy \right)^2}{\left(m \int_0^b \int_0^a \phi_{ij}^2 dx dy \right)^2}. \quad (3.53)$$

The rms of the acceleration of the generalized coordinate $\langle \ddot{\eta}_{ij}^2 \rangle$ does not exist. The PSD function of the acceleration of the deflection $\ddot{w}(x, y)$ is

$$S_{\ddot{w}}(\omega) = \sum_{i=1}^n \sum_{j=1}^n S_{\ddot{\eta}_{ij}}(\omega) \phi_{ij}^2(x, y), \quad (3.54)$$

or

$$W_{\ddot{w}}(f) = \sum_{i=1}^n \sum_{j=1}^n W_{\ddot{\eta}_{ij}}(f) \phi_{ij}^2(x, y), \quad (3.55)$$

where $W_{\ddot{w}}(f) = 2S_{\ddot{w}}(\omega)$. To keep it simple, only the first mode ϕ_{11} will be considered, and the contribution of all other modes neglected. The integrals for the simply supported plate are calculated:

$$\int_0^b \int_0^a \left(\sin \frac{\pi x}{a} \sin \frac{\pi y}{b} \right) dx dy = \frac{4ab}{(\pi)^2}, \quad (3.56)$$

$$\int_0^b \int_0^a \left(\sin \frac{\pi x}{a} \sin \frac{\pi y}{b} \right)^2 dx dy = \frac{ab}{4}. \quad (3.57)$$

The expressions for the clamped plate modes are too lengthy and are not given here, but may be obtained numerically or by using special purpose software (e.g. Maple®).

The rms value of the deflection of the first generalized coordinate η_{11} is

$$\langle \eta_{11}^2 \rangle = \frac{1}{8\zeta_{11}^2 (2\pi f_{11})^3} \left(\frac{16}{\pi^2} \right)^2 W_p(f_{11}) \left(\frac{A}{M} \right)^2, \quad (3.58)$$

where the area of the plate is $A = ab$ and the total mass is $M = mab$.

The PSD function of the acceleration of the first generalized coordinate η_{11} at the first natural frequency f_{11} , associated with simply supported plate mode ϕ_{11} , is

$$W_{\ddot{\eta}_{11}}(f_{11}) = \frac{1}{(2\zeta_{11})^2} \left(\frac{16}{\pi^2} \right)^2 W_p(f_{11}) \left(\frac{A}{M} \right)^2, \quad (3.59)$$

and the PSD function of the acceleration of the deflection $w(x, y)$ at the first natural frequency f_{11} , neglecting all other modes, can now be calculated, and becomes

$$W_{\ddot{w}}(x, y, f_{11}) = \frac{1}{(2\zeta_{11})^2} \left(\frac{16}{\pi^2} \right)^2 W_p(f_{11}) \left(\frac{A}{M} \right)^2 \left(\sin \frac{\pi x}{a} \sin \frac{\pi y}{b} \right)^2. \quad (3.60)$$

Equation (3.60) is very similar to (3.44). If $\zeta_{11} = 5\% = \frac{1}{Q_{11}}$ then $(Q_{11}^2\beta^2) = 16.211^2 = 262.8$, which is somewhat higher than that in (3.44).

3.5.3 The Stress in an Acoustically Loaded Panel

Based on the NASA guidelines, [134], the following steps can be followed for establishing acoustic loads of the planar components:

1. Estimate the panel fundamental natural frequency f_n .
2. Convert the sound pressure levels to sound pressure PSD (3.44).
3. Based on a damping ratio ζ , determine the amplification factor Q for the panel.
4. Use Miles' equation to determine the rms equivalent static pressure p_{rms} using $p_{rms} = \sqrt{\frac{\pi}{2} f_n Q W_p(f_n)}$.
5. The equivalent static pressure is given by the 3σ value $p_{peak} = 3p_{rms}$. This assumes the panel fundamental mode is the same as the deflection of the panel exposed to a uniform static pressure and the spatial correlation of the acoustic pressure field over the panel is uniform too.

This approach is conservative.

Example. The maximum bending stress σ_b at the center of a circular simply supported plate with a radius $R = 0.5$ m, a thickness $t = 2$ mm, a Poisson's ratio $\nu = 0.33$ and an uniform pressure p is given by [207]:

$$\sigma_b = \frac{3(3 + \nu)pR^2}{8t^2}.$$

Assume a fundamental frequency $f_n = 100$ Hz a $SPL = 135$ dB at a center frequency $f_c = 125$ Hz in 1/1 octave band. The reference pressure $p_{ref} = 2.0 \times 10^{-5}$ Pa. The amplification factor $Q = 10$. The bandwidth Δf at a center frequency f_c is given by $\Delta f = \frac{f_c}{\sqrt{2}} = 88.35$ Hz. The rms pressure $p_{rms}^2 = p_{ref}^2 10^{\frac{SPL}{10}} = 1.265 \times 10^4$ Pa². The PSD value of the pressure at the center frequency $f_c = 125$ Hz becomes $W_p(125) = \frac{p_{rms}^2}{\Delta f} = 88.4$ Pa²/Hz. The 3σ or the peak value of the pressure $P_{peak} = 3\sqrt{\frac{\pi}{2} f_n Q W_p(125)} = 1422$ Pa.

The bending stress σ_b at the center of the circular plate is the same in all directions:

$$\sigma_b = \frac{3(3 + \nu)p_{peak}R^2}{8t^2} = \frac{3(3 + 0.33) \times 1422 \times 0.5^2}{8 \times 0.002^2} = 1.11 \times 10^8 \text{ Pa.}$$

In [17, 18] Blevins discussed a method to calculate rms stresses in plates exposed to random pressures.

The modal pressure \bar{p} (inertia load) is defined as follows

$$\bar{p} = (2\pi f_0)^2 m |\delta|, \quad (3.61)$$

where m (kg/m²) is the mass per unit area of the plate, f_0 (Hz) is the fundamental natural frequency of the plate and $|\delta|$ is the maximum modal displacement of $\phi(x, y)$ which is the mode shape or vibration mode. The rms stress $\sigma_{rms}(x, y)$ at a certain point (x, y) in the plate can be determined from the following equation

$$\sigma_{rms}(x, y) = \frac{\sigma_{\bar{p}}(x, y)}{\bar{p}} \sqrt{\frac{\pi}{2} f_0 Q W_p(f_0)} \quad (3.62)$$

where Q is the amplification factor, $W_p(f_0)$ is the power spectral density of the pressure, and $\sigma_{\bar{p}}(x, y)$ is the stress at the location (x, y) corresponding to the modal pressure \bar{p} .

The calculated bending stress in plates exposed to a random acoustic pressure calculated either with the approximate method proposed in [134] or that proposed by Blevins [17, 18] are identical. The Blevins approximate method is discussed in more detail in Appendix H.

3.5.4 Acoustic and Random Vibration Test Tailoring

When developing a qualification test program for spacecraft, it is necessary to determine whether there should be an acoustic or random vibration test for each component (instrument, box, ...). Several criteria must be considered when making such a decision.

First, frequency sensitivity of the component must be established. While many subsystems are not sensitive to frequencies above 2000 Hz, those that do respond above this threshold cannot be tested by random vibration, but must instead be tested with acoustic noise. In these cases, decisions must be made regarding precisely how to conduct the acoustic test. The problem is easy for high area-to-mass components, which can be simply suspended with bungee cords within a reverberant acoustic chamber and directly “hit” with the acoustic source.

Low area-to-mass components are somewhat more problematic, as the vibration energy input should come through the base or mounting structure in the same manner as random vibration. This is because, when incident acoustic energy excites a spacecraft structure, high frequency random vibration is passed through the structure to mounted components.

To simulate this in an acoustic test, the component should be mounted on a plate. The area distribution and dynamic properties of the plate must be similar to the spacecraft structure, so that a “flight-like” vibration environment is imparted to the unit under test. The plate should then be excited by acoustic energy, which will be passed along to the component in the form of high frequency random vibration.

Below 2000 Hz, low area/mass components can be tested with random vibration only. The random vibration input specification should be a combination of acoustic and random response of the spacecraft structure at the component interface.

The “break even” area/mass ratio, representing the regime in which both acoustic and random vibration tests induce equal stochastic acceleration responses, can be calculated analytically for a SDOF system. A schematic of an example component is shown in Fig. 3.11. The example component is a SDOF system having a natural frequency f_0 (Hz), a mass m and a modal damping ζ . From the two simple SDOF systems the “break even” area/mass ration can be calculated. The area/mass ratio A/m will be derived by applying Lyapunov’s equation. The damped equations of motion of both SDOF systems (left and right Fig. 3.11) can be written as follows

$$\begin{aligned} \ddot{z}(t) + 2\zeta\omega_0\dot{z}(t) + \omega_0^2z(t) &= -\ddot{u}, \\ \ddot{x}(t) + 2\zeta\omega_0\dot{x}(t) + \omega_0^2z(x) &= \frac{Ap(t)}{m} = \frac{F(t)}{m}, \end{aligned} \tag{3.63}$$

where the relative motion of the left SDOF system is $z = x - u$, the enforced acceleration is $\ddot{u}(t)$, A is the surface area of the right SDOF and $p(t)$ the plane wave pressure field exposed to the surface area of the SDOF systems. The natural frequency of both SDOF system is $\omega_0^2 = k/m$, the damping is $c/m = 2\zeta\omega_0$ and the damping ratio is ζ ($c = 2\zeta\sqrt{km}$). The enforced acceleration and the exposed pressure have a random nature with PSD’s $W_{\ddot{u}\ddot{u}}(f)$ and $W_{pp}(f)$ respectively. The PSD of the equivalent force $F(t)$ is given by $W_{FF}(f)$. The left SDOF in Fig. 3.11 will be denoted by “Base Excited SDOF System” and the right SDOF by “Pressure Loaded SDOF System”.

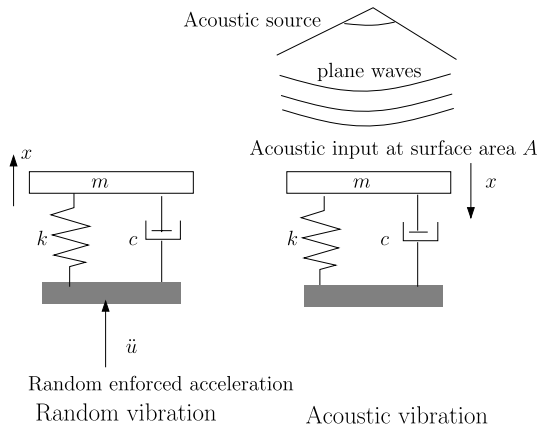


Fig. 3.11. Simple component under acoustic and random vibration [63, 64]

The random responses for z and x for both equations of motion (3.63) will be found applying Lyapunov's equation, as discussed in Sect. 2.6. The same solution procedure is followed on page 172.

Base Excited SDOF System

The SDOF damped equation of motion with the enforced acceleration $\ddot{u}(t)$ will be expressed in terms of the relative displacement $z(t) = x(t) - u(t)$

$$\ddot{z}(t) + 2\zeta\omega_0\dot{z}(t) + \omega_0^2z(t) = -\ddot{u}. \tag{3.64}$$

Equation (3.64) will be transformed into space-state variables $y_1 = z(t)$ and $y_2 = \dot{z}(t)$. The state-space equation is

$$\begin{aligned} \begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} &= \begin{pmatrix} 0 & 1 \\ -\omega_0^2 & -2\zeta\omega_0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} 0 \\ -1 \end{pmatrix} \ddot{u}, \\ \{ \dot{y} \} &= [A]\{y\} + \{B\}\ddot{u}. \end{aligned}$$

The spring force F_k and the force in the damper F_c are considered to be the output variables

$$\{F_0\} = \begin{pmatrix} F_k \\ F_c \end{pmatrix} = \begin{pmatrix} k & 0 \\ 0 & c \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} \omega_0^2m & 0 \\ 0 & 2\zeta\omega_0m \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = [C]\{y\}. \tag{3.65}$$

The auto correlation function of the enforced acceleration \ddot{u} is given by

$$R_{\ddot{u}\ddot{u}}(\tau) = \frac{W_{\ddot{u}\ddot{u}}}{2}\delta(\tau), \tag{3.66}$$

where $W_{\ddot{u}\ddot{u}}(\text{g}^2/\text{Hz})$ is the white noise PSD in the cyclic frequency domain. The correlation matrix of y_1, y_2 is given by

$$[R_{y_1y_2}(0)] = [R_{y_1y_2}] = \begin{pmatrix} \langle y_1y_1 \rangle & \langle y_1y_2 \rangle \\ \langle y_2y_1 \rangle & \langle y_2y_2 \rangle \end{pmatrix} = \begin{pmatrix} \langle y_1y_1 \rangle & 0 \\ 0 & \langle y_2y_2 \rangle \end{pmatrix}, \tag{3.67}$$

where $\langle y_iy_j \rangle$ are the expected values. The process is assumed to be stationary, thus $\frac{d}{dt}\langle y_1y_1 \rangle = \langle y_2y_1 \rangle + \langle y_1y_2 \rangle = 0$. This means that $\langle y_2y_1 \rangle = -\langle y_1y_2 \rangle = 0$.

The Lyapunov equation becomes

$$\begin{aligned} [A][R_{y_1y_2}] + [R_{y_1y_2}][A]^T &= -\{B\}\frac{W_{\ddot{u}\ddot{u}}}{2}\{B\}^T, \\ \begin{bmatrix} 0 & 1 \\ -\omega_0^2 & -2\zeta\omega_0 \end{bmatrix} \begin{bmatrix} \langle y_1y_1 \rangle & 0 \\ 0 & \langle y_2y_2 \rangle \end{bmatrix} &+ \begin{bmatrix} \langle y_1y_1 \rangle & 0 \\ 0 & \langle y_2y_2 \rangle \end{bmatrix} \begin{bmatrix} 0 & -\omega_0^2 \\ 1 & -2\zeta\omega_0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 \\ 0 & -\frac{W_{\ddot{u}\ddot{u}}}{2} \end{bmatrix}. \end{aligned} \tag{3.68}$$

The solution of (3.68) is

$$\begin{aligned}\langle y_1 y_1 \rangle &= E\{z^2\} = \frac{W_{\ddot{u}\ddot{u}}}{8\zeta\omega_0^3} = \frac{W_{\ddot{u}\ddot{u}}}{64\pi^3\zeta f_0^3}, \\ \langle y_2 y_2 \rangle &= E\{\dot{z}^2\} = \frac{W_{\ddot{u}\ddot{u}}}{8\zeta\omega_0} = \frac{W_{\ddot{u}\ddot{u}}}{16\pi\zeta f_0}.\end{aligned}\quad (3.69)$$

The expected value of the spring force and the damper force $\{F_0\}$, $E[\{F_0\}\{F_0\}^T]$, can be obtained from

$$\begin{aligned}E[\{F_0\}\{F_0\}^T] &= [C][R_{y_1 y_2}][C]^T = \begin{pmatrix} E[F_k^2] & 0 \\ 0 & E[F_c^2] \end{pmatrix} = \begin{pmatrix} \langle F_k F_k \rangle & 0 \\ 0 & \langle F_c F_c \rangle \end{pmatrix} \\ &= \begin{pmatrix} \omega_0^4 m^2 \langle y_1 y_1 \rangle & 0 \\ 0 & 4\zeta^2 \omega_0^2 m^2 \langle y_2 y_2 \rangle \end{pmatrix} \\ &= \frac{\pi}{2} f_0 Q m^2 W_{\ddot{u}\ddot{u}} \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{Q^2} \end{pmatrix},\end{aligned}\quad (3.70)$$

where $Q = 1/2\zeta$.

The rms values of the spring force and the damper force are respectively

$$\begin{aligned}F_{k,rms} &= m \sqrt{\frac{\pi}{2} f_0 Q W_{\ddot{u}\ddot{u}}}, \\ F_{c,rms} &= \frac{m}{Q} \sqrt{\frac{\pi}{2} f_0 Q W_{\ddot{u}\ddot{u}}}.\end{aligned}\quad (3.71)$$

The rms damper force is very small compared to the rms spring force.

Pressure Loaded SDOF System

The SDOF damped equation of motion representing the pressure loaded SDOF and loaded by the force $F = Ap(t)$, is expressed in terms of the absolute displacement $x(t)$ is

$$\ddot{x}(t) + 2\zeta\omega_0\dot{x}(t) + \omega_0^2 x(t) = \frac{F(t)}{m}.\quad (3.72)$$

Equation (3.72) will be transformed into space-state variables $y_1 = x(t)$ and $y_2 = \dot{x}(t)$:

$$\begin{aligned}\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} &= \begin{pmatrix} 0 & 1 \\ -\omega_0^2 & -2\zeta\omega_0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{1}{m} \end{pmatrix} F, \\ \{ \dot{y} \} &= [A]\{y\} + \{B\}F.\end{aligned}\quad (3.73)$$

The spring force F_k and the force in the damper F_c are considered to be the output variables

$$\{F_0\} = \begin{pmatrix} F_k \\ F_c \end{pmatrix} = \begin{pmatrix} k & 0 \\ 0 & c \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} \omega_0^2 m & 0 \\ 0 & 2\zeta\omega_0 m \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = [C]\{y\}.\quad (3.74)$$

The auto correlation function of the applied force F is

$$R_{FF}(\tau) = \frac{W_{FF}}{2} \delta(\tau), \quad (3.75)$$

where W_{FF} (N^2/Hz) is the white noise PSD in the cyclic frequency domain. The correlation matrix of y_1, y_2 is given by

$$[R_{y_1 y_2}(0)] = [R_{y_1 y_2}] = \begin{pmatrix} \langle y_1 y_1 \rangle & \langle y_1 y_2 \rangle \\ \langle y_2 y_1 \rangle & \langle y_2 y_2 \rangle \end{pmatrix} = \begin{pmatrix} \langle y_1 y_1 \rangle & 0 \\ 0 & \langle y_2 y_2 \rangle \end{pmatrix}, \quad (3.76)$$

where $\langle y_i y_j \rangle$ are the expected values. The process is assumed to be stationary, thus $\frac{d}{dt} \langle y_1 y_1 \rangle = \langle y_2 y_1 \rangle + \langle y_1 y_2 \rangle = 0$. This means that $\langle y_2 y_1 \rangle = -\langle y_1 y_2 \rangle = 0$.

The Lyapunov equation becomes

$$\begin{aligned} [A][R_{y_1 y_2}] + [R_{y_1 y_2}][A]^T &= -\{B\} \frac{W_{FF}}{2} \{B\}^T, \\ \begin{bmatrix} 0 & 1 \\ -\omega_0^2 & -2\zeta\omega_0 \end{bmatrix} \begin{bmatrix} \langle y_1 y_1 \rangle & 0 \\ 0 & \langle y_2 y_2 \rangle \end{bmatrix} &+ \begin{bmatrix} \langle y_1 y_1 \rangle & 0 \\ 0 & \langle y_2 y_2 \rangle \end{bmatrix} \begin{bmatrix} 0 & -\omega_0^2 \\ 1 & -2\zeta\omega_0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 \\ 0 & -\frac{W_{FF}}{2m^2} \end{bmatrix}. \end{aligned} \quad (3.77)$$

The solution of (3.77) is

$$\begin{aligned} \langle y_1 y_1 \rangle &= E\{x^2\} = \frac{W_{FF}}{8\zeta\omega_0^3 m^2} = \frac{W_{FF}}{64\pi^3 \zeta f_0^3 m^2} = \frac{A^2 W_{pp}}{64\pi^3 \zeta f_0^3 m^2}, \\ \langle y_2 y_2 \rangle &= E\{\dot{x}^2\} = \frac{W_{FF}}{8\zeta\omega_0 m^2} = \frac{W_{FF}}{16\pi\zeta f_0 m^2} = \frac{A^2 W_{pp}}{16\pi\zeta f_0 m^2}. \end{aligned} \quad (3.78)$$

The expected value of the spring force and the damper force $\{F_0\}$, $E[\{F_0\}\{F_0\}^T]$, can be obtained from

$$\begin{aligned} E[\{F_0\}\{F_0\}^T] &= [C][R_{y_1 y_2}][C]^T = \begin{pmatrix} E[F_k^2] & 0 \\ 0 & E[F_c^2] \end{pmatrix} = \begin{pmatrix} \langle F_k F_k \rangle & 0 \\ 0 & \langle F_c F_c \rangle \end{pmatrix} \\ &= \begin{pmatrix} \omega_0^4 m^2 \langle y_1 y_1 \rangle & 0 \\ 0 & 4\zeta^2 \omega_0^2 m^2 \langle y_2 y_2 \rangle \end{pmatrix} \\ &= \frac{\pi}{2} f_0 Q A^2 W_{pp} \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{Q^2} \end{pmatrix}, \end{aligned} \quad (3.79)$$

where $Q = 1/2\zeta$.

The rms values of the spring force and the damper force are respectively

$$\begin{aligned} F_{k,rms} &= A \sqrt{\frac{\pi}{2} f_0 Q W_{pp}}, \\ F_{c,rms} &= \frac{A}{Q} \sqrt{\frac{\pi}{2} f_0 Q W_{pp}}. \end{aligned} \quad (3.80)$$

The rms damper force is very small compared to the rms spring force.

Synthesis

The internal spring forces F_s in both SDOF systems will be equated to obtain the “break even” area/mass ratio (A/m). The area/mass ratio can be calculated using (3.71) and (3.80)

$$\frac{A}{m} = \sqrt{\frac{W_{\ddot{u}u}}{W_{pp}}} \approx \sqrt{\frac{W_{\ddot{u}u}(f_0)}{W_{pp}(f_0)}}. \tag{3.81}$$

To calculate a possible “break even” a the following SDOF system will be considered:

- The discrete mass is $m = 50$ kg.
- The natural frequency $f_0 = 125$ Hz.

In the ECSS standard “Testing” [56] qualification levels for the random enforced vibration and acoustic SPL’s are recommended. The random vibration specification is shown in Table 3.7 and the SPL in Table 3.8. The PSD of the pressure field at 125 Hz is

$$W_{pp} = \frac{(2.0 \times 10^{-5})^2 10^{\frac{SPL(f)}{10}}}{\Delta f} = \frac{(2.0 \times 10^{-5})^2 10^{\frac{139}{10}}}{0.7071 \times 125} = 332.88 \frac{\text{Pa}^2}{\text{Hz}}.$$

The PSD of the random enforced acceleration is

$$W_{\ddot{u}u} = 0.12 \frac{m + 20}{m + 1} = 0.12 \frac{50 + 20}{50 + 1} = 0.16 \frac{\text{g}^2}{\text{Hz}} = 15.55 \frac{(\frac{\text{m}}{\text{s}^2})^2}{\text{Hz}}.$$

Table 3.7. Random Vibration test levels

Frequency (Hz)	levels $W_{\ddot{u}u}$ (g^2/Hz)
20–100	+3 dB/octave
100–300	$0.12 \times (m + 20)/(m + 1)$
300–2000	–5 dB/octave

Table 3.8. Acoustic qualification test levels

Center frequency (Hz)	SPL (dB) 0 dB = 2×10^{-5} (Pa)
31.5	130
63	135.5
125	139
250	143
500	138
1000	132
2000	128
4000	124
8000	120
OASPL	147

The area/mass ratio becomes

$$\frac{A}{m} = \sqrt{\frac{W_{\ddot{u}i}(f_0)}{W_{pp}(f_0)}} = \sqrt{\frac{15.55}{332.88}} = 0.216 \frac{\text{m}^2}{\text{kg}}.$$

In [63, 64] a typical “break even” of $A/m = 0.215 \frac{\text{m}^2}{\text{kg}}$ is given.

The area/mass ratio of a test object is important in the selection of a vibration test method. Acoustic testing is more effective for high area/mass components, while random vibration is more effective for low area/mass components.

Problems

3.9. Consider a simply supported plate made of an Al-alloy with the following properties; the Young’s modulus $E = 70. \times 10^9$ Pa, the Poisson’s ratio $\nu = 0.3$, the density of the material $\rho = 2700$ kg/m³, the thickness of the plate $t = 1.25$ mm, the length of the plate is $b = 1$ m, the width is $a = 0.5$ m. The plate is exposed to a random pressure field, uniform over the surface, with an overall sound pressure $OASPL = 150$ dB. The random pressure field has a constant power spectral density over the frequency range from 20–1020 Hz. The modal damping ratio $\zeta = 0.05$. The maximum bending stress at the center of the plate, exposed to a uniform pressure p , is given by $\sigma_b = \frac{0.1017pa^2}{t^2}$ [207].

- Calculate the PSD W_p of the pressure field.
- What is the first (fundamental) natural frequency f_0 (Hz) with the mode shape $\phi(x, y) = \sin \frac{\pi x}{a} \sin \frac{\pi y}{b}$?
- What is the maximum deflection at the center of the plate.
- Calculate the modal pressure \bar{p} .
- What is the maximum bending stress at the center of the plate exposed to the modal pressure \bar{p} ?
- Calculate the peak stress $p_p = \sqrt{\frac{\pi}{2} f_0 Q W_p}$.
- What is the maximum rms bending stress at center of the plate applying the Blevins approximation method?

Answers: 400 Pa²/Hz, $f_0 = 60.5$ Hz, $\delta = 1$, $\bar{p} = 1.951 \times 10^6$ Pa, 1.191×10^{10} Pa, $p_p = 616.6$ Pa, 3.76×10^6 Pa.

3.10. An equipment mounting panel (1×1 m²) is made of a sandwich construction with Al-alloy face sheets with a thickness $t = 0.3$ mm and the core height $h = 25$ mm. The Young’s modulus of the face sheets is $= 70$ GPa. The mass per unit of area of the sandwich panel is $m = 3.7$ kg/m². A number of boxes are randomly placed on the panel and the total mass of the boxes is $M = 75$ kg. Calculate the lowest natural frequency of the simply supported sandwich panel inclusive the total mass of the boxes. The panel is exposed to

Table 3.9. SPL specification

Octave band (Hz)	SPL (dB), $0 \text{ dB} = 2 \times 10^{-5} W_p(f) \text{ Pa}^2$	
32.5	128	?
63	134	?
125	139	?
250	143	?
500	138	?
1000	132	?
2000	128	?
OASPL	?	?

sound pressure field as specified in Table 3.9. Calculate the OASPL and the PSD function $W_p(f)$.

Specify the mechanical random acceleration test environment $W_a(f) \text{ g}^2/\text{Hz}$ in the octave band to be applied by an electro-dynamic shaker system testing one of the boxes. Visualize the acceleration specification in a x - y plot.

3.6 Fluid Structure Interaction

3.6.1 Introduction

In Sect. 3.3 no dynamic interaction between vibrating structure and acoustic loads had been considered. The dynamic response of the structure did not influence the acoustic pressure field.

In this section the dynamic interaction between the vibrating structure and the acoustic field will be discussed. This fluid structure interaction is illustrated in Fig. 3.12.

The following topics will be discussed:

- the derivation of the wave equation
- the principle of fluid structure interaction
- structural response analysis

Examples will be worked out to illustrate the theory.

3.6.2 Wave Equation

At first the static position of the air pressure will be defined. That means the absence of sound pressures. The physical state is given by:

- pressure P_0 (Pa) and
- density ρ_0 (kg/m^3).

The physical state in the presence of sound is given by:

- total pressure $P_0 + p(x, y, z, t)$ (Pa),

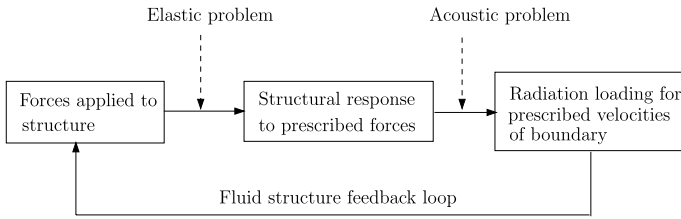


Fig. 3.12. Dynamic interaction of an elastic structure and an ambient fluid [94]

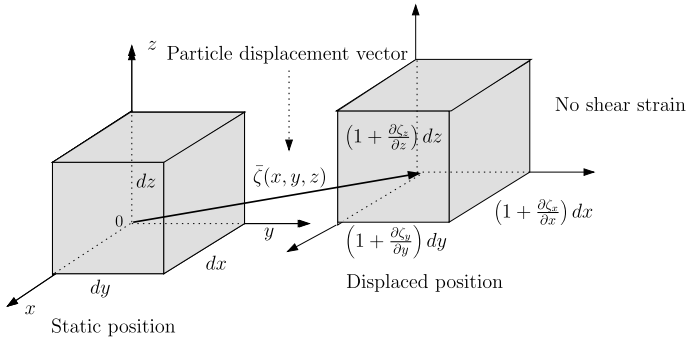


Fig. 3.13. Deformed fluid particle

- density $\rho_0 + \rho(x, y, z, t)$ (kg/m²) and
- displacement $\bar{X}_0 + \bar{\xi}(x, y, z, t)$ (m),

where $p(x, y, z, t)$ is the sound pressure and $\bar{\xi}(x, y, z, t)$ is the displacement of the particle.

The deformation of the particle is shown in Fig. 3.13. The shear deformation is not considered thus the particle remains cubic after deformation. The change in volume is as follows

$$\begin{aligned}
 V + dV &= \left(1 + \frac{\partial \xi_x}{\partial x}\right) \left(1 + \frac{\partial \xi_y}{\partial y}\right) \left(1 + \frac{\partial \xi_z}{\partial z}\right) dx dy dz \\
 &\approx \left(1 + \frac{\partial \xi_x}{\partial x} + \frac{\partial \xi_y}{\partial y} + \frac{\partial \xi_z}{\partial z}\right) dx dy dz,
 \end{aligned}
 \tag{3.82}$$

thus the ratio dV/V becomes

$$\frac{dV}{V} = \left(\frac{\partial \xi_x}{\partial x} + \frac{\partial \xi_y}{\partial y} + \frac{\partial \xi_z}{\partial z}\right) = \text{div } \bar{\xi} = \nabla \bar{\xi} = \xi_{i,i},
 \tag{3.83}$$

where $i = x, y, z$.

The mass balance at constant temperature (Boyle’s Law) is

$$\rho_0 V = (\rho_0 + \rho)(V + dV) = \rho_0 V + \rho V + \rho_0 dV + \rho dV,
 \tag{3.84}$$

which yields neglecting the second order term ρdV

$$\frac{dV}{V} = \frac{-\rho}{\rho_0}. \quad (3.85)$$

Combining (3.83) and (3.85) will result in the following relation

$$\rho = -\rho_0 \left(\frac{\partial \xi_x}{\partial x} + \frac{\partial \xi_y}{\partial y} + \frac{\partial \xi_z}{\partial z} \right) = -\rho_0 \bar{\xi}_{i,i} = -\rho_0 \operatorname{div}(\bar{\xi}). \quad (3.86)$$

The sound pressure increase p can be calculated using the *Hooke's law* related to the *compressibility* of the fluid (air) by the following expression

$$p = -B \frac{dV}{V} = B \frac{\rho}{\rho_0} = c^2 \rho, \quad (3.87)$$

where c is the speed of sound (m/s), B is the *bulk modulus* of the fluid. The minus sign has been applied because a decrease in the volume V will give an increase in the sound pressure p . For an adiabatic¹ gas applies that

$$PV^\gamma = \text{constant}, \quad (3.88)$$

where the total pressure is $P = P_0 + p$ and $\gamma = \frac{c_p}{c_v}$ is ratio of the *specific heats* at constant pressure and volume. This ratio is $\gamma = 1.66$ for an ideal gas and is $\gamma = 1.4$ for air. Differentiation (3.88) will give the following expression

$$dPV^\gamma + \gamma PV^{\gamma-1} dV \approx dPV^\gamma + \gamma P_0 V^{\gamma-1} dV, \quad (3.89)$$

thus yielding

$$dP = p = -\gamma P_0 \frac{dV}{V} = -B \frac{dV}{V}. \quad (3.90)$$

The definition of the *speed of sound* in air c (m/s) is defined by

$$c = \sqrt{\frac{B}{\rho_0}} = \sqrt{\frac{\gamma P_0}{\rho_0}} = \sqrt{\frac{1.4 \times 10^5}{1.2}} = 342 \text{ m/s}, \quad (3.91)$$

where the density of the air $\rho_0 = 1.2 \text{ kg/m}^3$ at sea level and at 20°C.

The acceleration of the mass particle $A\rho_0 dx$ can be obtained from *d'Alembert's* equation of equilibrium, shown in Fig. 3.14

$$A(P_0 + p) - A\left(P_0 + p + \frac{dp}{dx}\right) dx - A\rho_0 dx \ddot{u} = 0, \quad (3.92)$$

or in the form of *Newton's law*

$$\frac{dp}{dx} = -\rho_0 \ddot{u}. \quad (3.93)$$

¹ An adiabatic process is one in which no heat is gained or lost by the system.

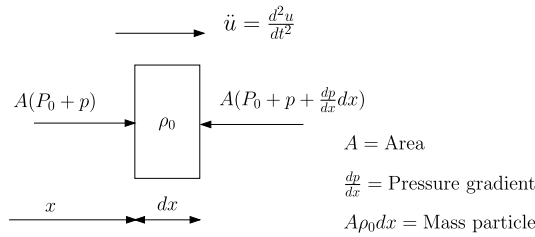


Fig. 3.14. Acceleration of particle

A generalization of (3.93) will give the expression

$$\begin{aligned} \frac{dp}{dx} \mathbf{e}_1 + \frac{dp}{dy} \mathbf{e}_2 + \frac{dp}{dz} \mathbf{e}_3 &= -\rho_0(\ddot{u} + \ddot{v} + \ddot{w}), \\ \frac{dp}{dx_i} \mathbf{e}_i &= -\rho_0 \ddot{\xi}_{j,j}, \\ \text{grad}(p) &= -\rho_0 \ddot{\xi}. \end{aligned} \tag{3.94}$$

Following physical relations were derived:

- $\text{grad}(p) = -\rho_0 \ddot{\xi},$
- $p = -B \frac{dV}{V} = B \frac{\rho}{\rho_0} = c^2 \rho,$
- $\rho = -\rho_0 \text{div}(\bar{\xi}).$

Combining these equations we derive the wave equation

$$\text{div}(\text{grad}(p)) = \frac{1}{c^2} \ddot{p}, \tag{3.95}$$

or

$$\nabla^2 p = \frac{\partial^2 p}{dx^2} + \frac{\partial^2 p}{dy^2} + \frac{\partial^2 p}{dz^2} = \frac{1}{c^2} \ddot{p}. \tag{3.96}$$

The Fourier transform in the frequency domain of (3.96) is called the *Helmholtz* equation

$$\frac{\partial^2 p}{dx^2} + \frac{\partial^2 p}{dy^2} + \frac{\partial^2 p}{dz^2} + k^2 p = 0, \tag{3.97}$$

where $k = \omega/c$ is called the *wave number*.

In the x -direction the Helmholtz equation can be written

$$\frac{\partial^2 p}{dx^2} + k^2 p = 0. \tag{3.98}$$

For x -variation, the solution of (3.95) has the form

$$p(x, t) = \hat{p}(x)g(t). \tag{3.99}$$

Then

$$\hat{p}''(x)g(t) = \frac{1}{c^2}\hat{p}(x)\ddot{g}(t), \tag{3.100}$$

$$\frac{\hat{p}''(x)}{\hat{p}(x)} = \frac{1}{c^2}\frac{\ddot{g}(t)}{g(t)} = \text{constant} = -k^2, \tag{3.101}$$

$$\begin{aligned} \hat{p}(x) &= Ae^{jkx} + Be^{-jkx}, \\ g(t) &= Ce^{jkt} + Be^{-jkt}. \end{aligned} \tag{3.102}$$

The solution of $\hat{p}(x)$ can be expressed as follows

$$\hat{p}(x) = Ae^{jkx} + Be^{-jkx} = C \sin(kx) + D \cos(kx). \tag{3.103}$$

If the sound wave $\hat{p}(x)$ travels one period, one *wavelength* further, than $k\lambda = 2\pi$. The wave length becomes

$$\lambda = \frac{2\pi}{k} = \frac{2\pi c}{\omega} = \frac{c}{f}, \tag{3.104}$$

where f is the cyclic frequency (Hz).

Example. An infinite rigid plate moves sinusoidally (harmonically) with a velocity amplitude $v_w = \hat{v}_w e^{j\omega t}$. This is illustrated in Fig. 3.15. Calculate the pressures $\hat{p}(x)$, $\hat{p}(0)$ and the radiated energy $\Pi_{rad} = \frac{1}{2}\Re(\hat{F}\hat{v}_w)$ per unit of area. Only the sound wave travelling in the positive x -direction is considered. At the plate $x = 0$ the pressure gradient $\frac{d\hat{p}}{dx} = -\omega\rho_0\hat{v}_w$, so that $\hat{p}(x) = Be^{-jkx}$. Hence, at $x = 0$

$$\left. \frac{d\hat{p}}{dx} \right|_{x=0} = \{-Bjk e^{jkx}\}|_{x=0} = -Bjk = -j\omega\rho_0\hat{v}_w.$$

It follows that the constant is

$$B = \frac{\omega}{k}\rho_0\hat{v}_w = \rho_0 c \hat{v}_w.$$

The sound pressure $p(x, t)$ can now be written as

$$p(x, t) = \rho_0 c \hat{v}_w e^{j\omega t - jkx}.$$

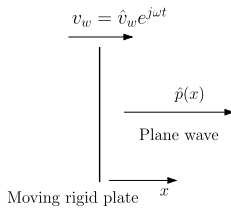


Fig. 3.15. Sound waves caused by a moving rigid plate

If x and t are allowed to vary simultaneously, the pressure will remain constant if $\frac{d\hat{p}}{dt} = \frac{d(kx)}{dt} - \omega = 0$, or $\frac{dx}{dt} = \frac{\omega}{k} = c$. The sound pressure $\hat{p}(0)$ at $x = 0$ becomes $\hat{p}(0) = \rho_0 c \dot{U}_w$, and the absolute value of the impedance (ratio force/velocity) is $|\frac{\hat{p}(0)}{\hat{v}_w}| = \rho_0 c$, the so-called *specific acoustic impedance*. In general, the impedance $\frac{\hat{p}(0)}{\hat{v}_w} = \rho_0 c e^{j\alpha}$.

The radiated energy \dot{H}_{rad} per unit of area at $x = 0$ can be calculated as follows

$$\dot{H}_{rad} = \frac{1}{2} \Re\{\hat{p}(0)\hat{v}_w^*\} = \frac{1}{2} \rho_0 c \hat{v}_w \hat{v}_w^* = \frac{1}{2} \rho_0 c \hat{v}_w^2,$$

where $\frac{1}{2} \Re\{\hat{p}(0)\hat{v}_w^*\} = \lim_{2T \rightarrow \infty} \int_{-T}^T p(0, t) \dot{u}_w(t) dt = \langle p(0, t) \dot{v}_w(t) \rangle$, and $\dot{u}_w(t) = v_w e^{j\omega t}$.

Spherical Radiator

In a spherical coordinate system with the radius r as the only variable, the Helmholtz equation (3.97) can be written as

$$\frac{d^2 \hat{p}}{dr^2} + \frac{2}{r} \frac{d\hat{p}}{dr} + k^2 \hat{p} = 0. \tag{3.105}$$

Equation (3.105) describes the radiation of a sound wave from a spherical source (radiator) as shown in Fig. 3.16. The general solution of the pressure wave $\hat{p}(r)$ for an outgoing sound wave is

$$\hat{p}(r) = \frac{B e^{-jkr}}{r}, \tag{3.106}$$

where B is a constant of integration. Using the boundary condition

$$\frac{d\hat{p}}{dr} = -A \left[\left(\frac{1}{r^2} + \frac{jk}{r} \right) e^{-jkr} \right]_{r=a} = -j\omega \rho_0 \hat{v}_a, \tag{3.107}$$

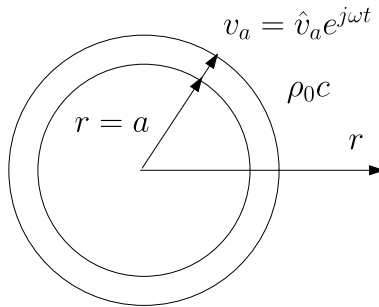


Fig. 3.16. Spherical radiator

the general solution can be obtained, where \hat{v}_a is the pulsation velocity of the spherical radiator of the surface at $r = a$. After introducing the boundary condition (3.107), the solution (3.106) becomes

$$\hat{p}(r) = \hat{v}_a \frac{j\omega\rho_a}{1 + jka} \frac{a^2}{r} e^{-jk(r-a)}, \quad r \geq a. \quad (3.108)$$

Equation (3.108) shows that the sound pressure $\hat{p}(r) \sim \frac{1}{r}$, which is called the $\frac{1}{r}$ -law of spherical sources. The (point) *source strength* of the spherical source is defined by

$$\hat{Q}_s = \hat{v}_a A_{\text{sphere}, r=a} = \hat{v}_a 4\pi a^2, \quad (3.109)$$

thus (3.108) can be rewritten as follows

$$\hat{p}(r) = \frac{\hat{Q}_s}{4\pi} \frac{j\omega\rho_0}{1 + jka} \frac{1}{r} e^{-jk(r-a)}, \quad r \geq a. \quad (3.110)$$

For $a \ll r$ and $ka = \frac{2\pi a}{\lambda} \ll 1$ (3.110) becomes

$$\hat{p}(r) = j\omega\rho_0 \frac{\hat{Q}_s}{4\pi r} e^{-jkr}. \quad (3.111)$$

Figure 3.17 shows half a radiating sphere, connected to a rigid wall (baffle). The source strength of the half radiating sphere, with radius $r = a$ and velocity $\hat{v}(a)$ is given by

$$\hat{Q}_s = 2\pi a^2 \hat{v}_a, \quad (3.112)$$

thus the radiated sound pressure expressed in (3.111) is modified, and the radiated sound pressure $\hat{p}(r)$ from the half spherical radiator becomes

$$\hat{p}(r) = j\omega\rho_0 \frac{\hat{Q}_s}{2\pi r} e^{-jkr}. \quad (3.113)$$

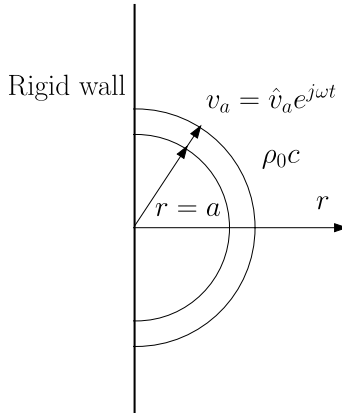


Fig. 3.17. Spherical radiator in rigid wall

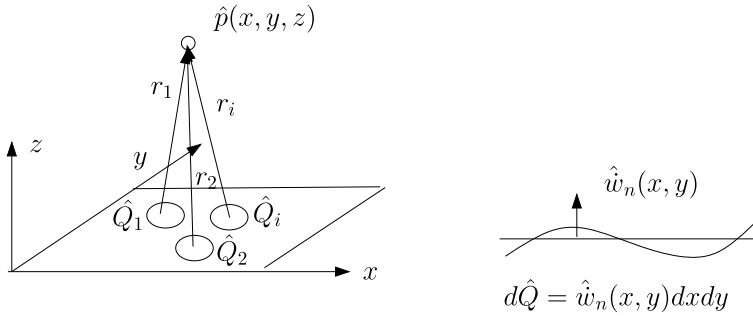


Fig. 3.18. Plane radiator as an array of point sources

Figure 3.18 shows a number of one-sided point sources in the z -direction. The total sound pressure $\hat{p}(x, y, z)$, the sum of the sound pressures $\hat{p}_i(r_i)$ radiated by the individual point sources with source strength \hat{Q}_i , becomes

$$\hat{p}(x, y, z) = \frac{j\omega\rho_0}{2\pi} \sum_i \frac{\hat{Q}_i}{r_i} e^{-jkr_i}. \tag{3.114}$$

For a vibrating plate, the infinitesimal source is given by

$$d\hat{Q} = \hat{w}_n(x, y) dx dy = \hat{w}_n(x, y) dA, \tag{3.115}$$

where $\dot{w}(x, y) = \hat{w}_n(x, y) = \hat{w}_n(x, y)e^{j\omega t}$ is the velocity normal to the xy -plane. Equation (3.114) can be rewritten as *Rayleigh's*² *equation*

$$\hat{p}(r) = \frac{j\omega\rho_0}{2\pi} \iint_A \frac{\hat{w}_n(x, y)}{r} e^{-jkr} dA. \tag{3.116}$$

3.6.3 Pressure Structure Interaction

The actual sound pressure $p = \hat{p}e^{j\omega t}$ existing at any point in the fluid (air) is the sum of two components [184]

$$\hat{p} = \hat{p}_{bl} + \hat{p}_{rad}, \tag{3.117}$$

where the pressure p_{bl} is defined as the pressure that would be observed with the sound sources turned on but with the flexible structure (transponder, [184]) replaced by a rigid body (i.e. its motion is blocked), the pressure \hat{p}_{rad} is defined as the radiated pressure that would be observed with the sound sources turned off but the flexible structure vibrating.

² Lord Rayleigh 1842–1919.

3.6.4 Structural Responses

The matrix equation of motion for a linear structural system can be written as follows

$$[M]\{\ddot{x}\} + [C]\{\dot{x}\} + [K]\{x\} = \left\{ - \int_{A_i} p dA_i \right\} = \{F\}, \quad (3.118)$$

where the force vector $\{F\}$ represents the equivalent pressure loads at the nodes. The pressure p is normal to the surface of the structure and directed into the structure. The node sequence number is indicated by the index i . The area A_i represents the node equivalent area.

The physical displacement vector $\{x\}$ will be projected onto the vibration modes multiplied by the generalized coordinates (weight factors)

$$\{x\} = [\Phi]\{\eta(t)\}, \quad (3.119)$$

where $[\Phi]$ is the modal matrix of undamped vibration modes and $\{\eta\}$ is the vector of generalized coordinates.

The coupled equations of motion (3.118) can be uncoupled using the orthogonality properties of the vibration (normal) modes

$$m_k \ddot{\eta}_k + 2\zeta_k \omega_k m_k \dot{\eta}_k + \omega_k^2 m_k \eta_k = \{\phi_k\}^T \{F\}, \quad k = 1, 2, \dots, M, \quad (3.120)$$

where M is the number of kept vibration modes, the generalized mass is $m_k = \{\phi_k\}^T [M] \{\phi_k\}$, the generalized stiffness $k_k = \{\phi_k\}^T [K] \{\phi_k\} = \omega_k^2 m_k$ and the generalized damping $c_k = \{\phi_k\}^T [C] \{\phi_k\} = 2\zeta_k \omega_k m_k$. In general, the generalized damping or modal damping will be introduced later on after the undamped equations of motion are uncoupled. The damping matrix $[C]$ is normally not known. Finally the uncoupled damped equations of motion can be expressed as follows

$$\ddot{\eta}_k + 2\zeta_k \omega_k \dot{\eta}_k + \omega_k^2 \eta_k = \frac{\{\phi_k\}^T \{F\}}{m_k} = \frac{\{\phi_k\}^T \left\{ - \int_{A_i} p dA_i \right\}}{m_k},$$

$$k = 1, 2, \dots, M, \quad i = 1, 2, \dots, N, \quad (3.121)$$

or substituting (3.117) in (3.121) we obtain the following equations are obtained

$$\ddot{\eta}_k + 2\zeta_k \omega_k \dot{\eta}_k + \omega_k^2 \eta_k = \frac{\{\phi_k\}^T \left\{ - \int_{A_i} (p_{bl} + p_{rad}) dA_i \right\}}{m_k},$$

$$k = 1, 2, \dots, M, \quad i = 1, 2, \dots, N. \quad (3.122)$$

The radiation pressure \hat{p}_{rad} can be calculated with (3.116), however, the normal velocity \hat{w}_n will be projected on to the modal base $[\Phi]$ weighted by the generalized coordinates $\{\eta\}$ as follows

$$\{\hat{w}_{i,n}\} = [\Phi_n] \{\dot{\eta}\}. \quad (3.123)$$

The radiated pressure \hat{p}_{rad} becomes

$$\begin{aligned} \hat{p}_{rad}(x, y, z) &= -\frac{\omega^2 \rho_0}{2\pi} \sum_i^N A_i \hat{w}_n(x_i, y_i, z_i) \frac{e^{-jkr_i}}{r_i} \\ &= -\frac{\omega^2 \rho_0}{2\pi} \sum_i^N A_i [\phi_i] \{\hat{\eta}\} \frac{e^{-jkr_i}}{r_i}, \end{aligned} \tag{3.124}$$

where $\eta(t) = \hat{\eta}e^{j\omega t}$, r_i the distance between the node and the position (x, y, z) and $[\phi_i]$ are the modal properties in relation with the normal displacement $w_{i,n}$ of the node i on the surface of the vibrating structure. The equivalent nodal load vector $\{-\int_{A_i} p_{rad} dA\}$ can now be obtained by the following expression

$$\begin{aligned} \left\{ -\int_{A_i} p_{rad} dA \right\} &= \frac{\omega^2 \rho_0}{2\pi} \sum_i A_i \left(\sum_n A_n [\phi_n] \{\hat{\eta}\} \frac{e^{-jkr_n}}{r_n} \right) \\ &= \frac{\omega^2 \rho_0}{2\pi} \sum_i A_i \left(\sum_n A_n [\phi_n] \{\hat{\eta}\} \frac{\{\cos kr_i - j \sin kr_n\}}{r_n} \right), \\ i, n &= 1, 2, \dots, N, \end{aligned} \tag{3.125}$$

where all nodes i and n are located on the surface of the vibrating structure, as is the point (x, y, z) where the radiated pressure is measured. In fact this location is a node.

The distance between nodes i and k , $\Delta r_{ik} = |r_i - r_k|$. This is illustrated in Fig. 3.19. Equation (3.125) is now completely written in vector and matrix notation

$$\begin{aligned} \left\{ \int_{A_i} p_{rad} dA \right\} &= -\frac{\omega^2 \rho_0}{2\pi} [A_i] \left[\frac{\{\cos k \Delta r_{in} - j \sin k \Delta r_{in}\}}{\Delta r_{in}} \right] [A_n] [\Phi] \{\eta\} \\ &= -\frac{\omega^2 \rho_0}{2\pi} [A] \left[\frac{\{\cos k \Delta r_{in} - j \sin k \Delta r_{in}\}}{\Delta r_{in}} \right] [A] [\Phi] \{\eta\}, \\ i, n &= 1, 2, \dots, N. \end{aligned} \tag{3.126}$$

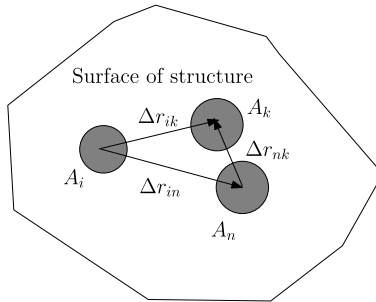


Fig. 3.19. Surface nodes

The generalized force associated with vibration mode ψ_l becomes

$$\begin{aligned}
 & \{\phi_l\}^T \left\{ \int_{A_i} p_{rad} dA \right\} \\
 &= -\frac{\omega^2 \rho_0}{2\pi} \{\phi_l\}^T \sum_n \left([A_i] \left[\frac{\{\cos k \Delta r_{in} - j \sin k \Delta r_{in}\}}{\Delta r_{in}} \right] [A_n] \{\phi_m\} \{\eta\} \right) \\
 &= j\omega \sum_l (R_{lm} + j\omega M_{lm}) \eta_m, \\
 & i, n = 1, 2, \dots, N, \quad l, m = 1, 2, \dots, M,
 \end{aligned} \tag{3.127}$$

where R_{kl} is called the *modal resistance* and M_{kl} the *modal inertia* and are given by

$$\begin{aligned}
 R_{lm} &= \frac{\omega \rho_0}{2\pi} \{\phi_l\}^T [A] \left[\frac{\sin k \Delta r_{in}}{\Delta r_{in}} \right] [A] \{\phi_m\}, \quad i, n = 1, 2, \dots, N, \\
 M_{lm} &= \frac{\rho_0}{2\pi} \{\phi_l\}^T [A] \left[\frac{\cos k \Delta r_{in}}{\Delta r_{in}} \right] [A] \{\phi_m\}, \quad l, m = 1, 2, \dots, M,
 \end{aligned} \tag{3.128}$$

and where N is the number of nodes and M the number of modes.

In case $i = n$, thus $\Delta r_{in} = 0$, the following expressions can be derived using L'Hôpital's rule

$$\begin{aligned}
 \lim_{\Delta r_{in} \rightarrow 0} \frac{\sin k \Delta r_{in}}{\Delta r_{in}} &= k, \\
 \lim_{\Delta r_{in} \rightarrow 0} \frac{\cos k \Delta r_{in}}{\Delta r_{in}} &= 0.
 \end{aligned} \tag{3.129}$$

Equation (3.122) is the basis for further evaluation the equations of motion.

$$\ddot{\eta}_k + 2\zeta_k \omega_k \dot{\eta}_k + \omega_k^2 \eta_k = \frac{\{\phi_k\}^T \left\{ -\int_{A_i} (p_{bl} + p_{rad}) dA_i \right\}}{m_k}, \quad k = 1, 2, \dots, M,$$

and can be written as follows, where $\eta(t)$ is the Fourier transform $\eta(t) = \hat{\eta}(\omega) e^{j\omega t}$

$$\begin{aligned}
 & \left[\omega_k^2 - \omega^2 \left(1 + \frac{M_{kk}}{m_k} \right) + j\omega \left(2\zeta_k \omega_k + \frac{R_{kk}}{m_k} \right) \right] \hat{\eta}_k(\omega) \\
 &+ \sum_{j=1, j \neq k}^N \left[\omega^2 \frac{M_{jk}}{m_k} + j\omega \frac{R_{jk}}{m_k} \right] \hat{\eta}_j(\omega) = -\frac{1}{m_k} \{\phi_k\}^T \left\{ \int_{A_i} p_{bl} dA_i \right\}, \\
 & k = 1, 2, \dots, M.
 \end{aligned} \tag{3.130}$$

The coupling terms in (3.130) will be neglected assuming $M_{kl} \ll M_{kk}$ and $R_{kl} \ll R_{kk}$, hence

$$\begin{aligned}
 & \left[\omega_k^2 - \omega^2 \left(1 + \frac{M_{kk}}{m_k} \right) + j\omega \left(2\zeta_k \omega_k + \frac{R_{kk}}{m_k} \right) \right] \hat{\eta}_k(\omega) \\
 &= -\frac{1}{m_k} \{\phi_k\}^T \left\{ \int_{A_i} p_{bl} dA_i \right\} = -\frac{1}{m_k} \{\phi_k\}^T [A_i] \{p_{bl,i}\}, \quad k = 1, 2, \dots, N.
 \end{aligned} \tag{3.131}$$

The joint acceptance function J_{lm} had been defined in (3.37), page 212. For a diffuse sound field, the joint acceptance is

$$J_{lm} = \{\phi_l\}^T [A_i] \left[\frac{\sin k\Delta_{in}}{k\Delta_{in}} \right] [A_n] \{\phi_m\}, \quad i, n = 1, 2, \dots, N. \quad (3.132)$$

The modal inertia M_{lm} and the modal resistance R_{lm} can be related to the joint acceptance J_{ml} . The following expressions can be derived

$$\begin{aligned} M_{lm} &= \frac{\rho_0 k}{2\pi} J_{lm}, \\ R_{lm} &= \frac{\omega \rho_0 k}{2\pi} J_{lm} = \frac{\omega^2 \rho_0}{2\pi c} J_{lm}. \end{aligned} \quad (3.133)$$

The cross PSD of the generalized coordinates $\hat{\eta}_i$, $S_{\hat{\eta}_i \hat{\eta}_j}(\omega)$ can be calculated as follows

$$S_{\hat{\eta}_i \hat{\eta}_j}(\omega) = \frac{H_i(\omega) H_j^*(\omega) J_{ij}}{\omega_i^2 \omega_j^2 m_i m_j} S_p(\omega), \quad (3.134)$$

where $S_p(\omega)$ is the PSD of the blocked pressure field. The FRF $H_i(\omega)$ is given by

$$H_i(\omega) = \frac{\omega_i^2}{1 - \frac{\omega^2}{\omega_i^2} \left(1 + \frac{M_{ii}}{m_i}\right) + \frac{j\omega}{\omega_i^2} (2\omega_i \zeta_i + \frac{R_{ii}}{m_i})}. \quad (3.135)$$

The PSD of the physical displacement $\{x\}$ can be calculated

$$S_{\hat{x}_i \hat{x}_j}(\omega) = \{\phi_i\} S_{\hat{\eta}_i \hat{\eta}_j}(\omega) \{\phi_j\}^T, \quad (3.136)$$

where the displacement vector $\{x(t)\} = \{\hat{x}(t)\} e^{j\omega t}$. The matrix of the cross PSD $[S_{\hat{x}_i \hat{x}_j}(\omega)]$ of the physical displacements $\{\hat{x}\}$ can be obtained by (3.38)

$$[S_{\hat{x}_i \hat{x}_j}(\omega)] = [\Phi] [S_{\hat{\eta}_i \hat{\eta}_j}(\omega)] [\Phi]^T,$$

the matrix of cross PSD $[S_{\dot{x}_i \dot{x}_j}(\omega)]$ of the velocities $\{\dot{x}\}$ can be calculated by (3.39)

$$[S_{\dot{x}_i \dot{x}_j}(\omega)] = \omega^2 [\Phi] [S_{\hat{\eta}_i \hat{\eta}_j}(\omega)] [\Phi]^T,$$

and the matrix of cross PSD $[S_{\ddot{x}_i \ddot{x}_j}(\omega)]$ of the accelerations $\{\ddot{x}\}$ is in accordance with (3.40)

$$[S_{\ddot{x}_i \ddot{x}_j}(\omega)] = \omega^4 [\Phi] [S_{\hat{\eta}_i \hat{\eta}_j}(\omega)] [\Phi]^T.$$

The matrix of cross PSD of the stresses $S_{\hat{\sigma}_i \hat{\sigma}_j}(\omega)$ of the internal stress field $\tilde{\sigma}$ can be expressed as follows (3.41)

$$[S_{\hat{\sigma}_i \hat{\sigma}_j}(\omega)] = [\Phi_\sigma] [S_{\hat{\eta}_i \hat{\eta}_j}(\omega)] [\Phi_\sigma]^T,$$

where $[\Phi_\sigma]$ is the matrix of stress modes.

The single sided PSD functions given in the cyclic frequency domain are in general (2.61)

$$W(f) = 2S(\omega). \tag{3.137}$$

The relation between the blocked pressure and the incidence (free-free) pressure, specified by the usual SPL levels (dB) (see (3.1)) is roughly

$$W_{pbl}(f) = W_p(f)\langle\Gamma^2\rangle, \tag{3.138}$$

where

- $\langle\Gamma^2\rangle = 1, f \leq \frac{c}{2a},$
- $\langle\Gamma^2\rangle = 2, f > \frac{c}{2a},$

where a is the characteristic dimension of the plate and shell structure, e.g. $a \approx \sqrt{A_{plate}}$. A more detailed discussion can be read in [184, 185].

Example. A simply supported bending beam has bending stiffness $EI(\text{Nm}^2)$, and m (kg/m) mass per unit of length of the beam. The width of the beam is b (m), and L (m) is the length of the beam. Only the first vibration mode $\phi(x) = \sin \frac{\pi x}{L}$ is taken into account to calculate the response characteristics of the beam when it is exposed to a reverberant sound field with PSD $W_p(f)$ (Pa^2/Hz).

The deflection $w(x, t)$ of the beam is given by $w(x, t) = \phi(x)\eta(t)$. The natural frequency f_1 associated with the vibration mode $\phi(x)$ can be calculated using Rayleigh’s quotient $R(\phi)$

$$\omega_1 = 2\pi f_1 = \sqrt{R(\phi)} = \sqrt{\frac{0.5 \int_0^L EI \left(\frac{d^2\phi}{dx^2}\right)^2 dx}{0.5 \int_0^L m\phi^2 dx}} = \pi^2 \sqrt{\frac{EI}{mL^4}}.$$

The generalized mass m_1 becomes

$$m_1 = \int_0^L m \sin^2 \frac{\pi x}{L} dx = \frac{1}{2}mL.$$

The joint acceptance function for an unidirectional reverberant sound field can be calculated as follows

$$\begin{aligned} J_{11}(\omega) &= b^2 \int_0^L \int_0^L \phi(x)\phi(y) \frac{\sin k(x-y)}{k(x-y)} dx dy \\ &= b^2 \int_0^L \int_0^L \sin \frac{\pi x}{L} \sin \frac{\pi y}{L} \frac{\sin k(x-y)}{k(x-y)} dx dy, \end{aligned}$$

where the wave number $k = \omega/c$.

The modal inertia M_{11} and the modal resistance R_{11} can be obtained by the following expression

$$M_{11} = \frac{\rho_0 b^2}{2\pi} \int_0^L \int_0^L \sin \frac{\pi x}{L} \sin \frac{\pi y}{L} \frac{\cos k(x-y)}{(x-y)} dx dy,$$

$$R_{11} = \frac{\omega \rho_0 b^2}{2\pi} \int_0^L \int_0^L \sin \frac{\pi x}{L} \sin \frac{\pi y}{L} \frac{\sin k(x-y)}{(x-y)} dx dy.$$

The equation of motion for the generalized coordinated $\eta_1(t) = \hat{\eta}_1(\omega)e^{j\omega t}$ yields

$$[-\omega^2(m_1 + M_{11}) + j\omega(2\zeta_1\omega_1 m_1 + R_{11})]\hat{\eta}_1 = -b \int_0^L \sin \frac{\pi x}{L} \hat{p}_{bl} dx,$$

where the blocked pressure is $p_{bl}(t) = \hat{p}_{bl}(t)e^{j\omega t}$. The PSD of the reverberant blocked pressure is given by

$$S_{bl}(x, \omega) = S_{bl}(\omega) \frac{\sin k(\Delta x)}{k\Delta x} = \frac{W_{bl}(f)}{2} \frac{\sin k(\Delta x)}{k\Delta x} = \frac{W_p(f)}{2} \frac{\sin k(\Delta x)}{k\Delta x}.$$

The PSD function $S_{\eta_1}(\omega)$ is defined by $S_{\eta_1}(\omega) = \lim_{T \rightarrow \infty} \frac{|\hat{\eta}_1|^2}{2T}$ and the PSD function of the blocked pressure $S_{p_{bl}}(\omega)$ is defined by $S_{p_{bl}}(\omega) = \lim_{T \rightarrow \infty} \frac{|p_{bl}|^2}{2T}$. The PSD function the PSD function $S_{\eta_1}(\omega)$ can be expressed as follows

$$\begin{aligned} S_{\eta_1}(\omega) &= \frac{|H_1(\omega)|^2}{m_1^2 \omega_1^4} b^2 \int_0^L \sin^2 \frac{\pi x}{L} S_{p_{bl}}(x, \omega) dx \\ &= \frac{|H_1(\omega)|^2}{m_1^2 \omega_1^4} b^2 S_{p_{bl}}(\omega) \int_0^L \int_0^L \sin \frac{\pi x}{L} \sin \frac{\pi y}{L} \frac{\sin k(x-y)}{k(x-y)} dx dy \\ &= \frac{|H_1(\omega)|^2 J_{11}(\omega)}{m_1^2 \omega_1^4} S_{p_{bl}}(\omega), \end{aligned}$$

where the FRF $H_1(\omega)$ is given by

$$H_1(\omega) = \frac{1}{1 - \frac{\omega^2}{\omega_1^2} (1 + \frac{M_{11}}{m_1}) + \frac{j\omega}{\omega_1^2} (2\zeta_1\omega_1 + \frac{R_{11}}{m_1})}.$$

The bending beam has a length $L = 1.0$ m and the mass per unit of length $m = 1.866$ kg/m. The lowest natural frequency is $f_1 = 50$ Hz and the width of the beam is $b = 0.5$ m. The speed of sound in air is $c = 342$ m/s (the wave number is $k = \omega_1/c = 0.919$). The modal damping ratio $\zeta = 0.02$.

The joint acceptance J_{11} for the unidirectional diffuse sound field can now be calculated.

$$J_{11}(\omega_1) = b^2 \int_0^L \int_0^L \sin \frac{\pi x}{L} \sin \frac{\pi y}{L} \frac{\sin k(x-y)}{k(x-y)} dx dy = 0.1.$$

The modal added mass

$$M_{11}(\omega_1) = \frac{\rho_0 k}{2\pi} J_{11}(\omega_1) = 0.018,$$

and the modal resistance

$$R_{11}(\omega_1) = \frac{\omega \rho_0 k}{2\pi} J_{11}(\omega_1) = 5.511.$$

The mass ratio

$$\frac{M_{11}}{m_1} = 0.019,$$

and the damping ratio

$$\frac{R_{11}}{2m_1\omega_1\zeta} = 0.47,$$

mean that the added mass is relatively low with respect to the generalized mass and the modal resistance is 47% of the modal damping.

Statistical Energy Analysis

Statistical Energy Analysis

4.1 Introduction

Statistical energy analysis (SEA) originated in the aerospace industry in the early 1960s. Today, SEA is applied to a large variety of products, from cars and trucks to aircraft, spacecraft, electronic equipment, buildings, consumer products and more. SEA is based on the principle of energy conservation. All the energy input to a system, through mechanical or acoustic excitation, must leave the system through structural damping or acoustic radiation. The method is fast and is applicable over a wide frequency range. SEA is very good for problems that combine many different sources of excitation, whether mechanical or acoustic. In SEA the structural vibrational behavior of elements (subsystems) is described statistically. For high-frequencies a deterministic modal description of the dynamic behavior of structures is not very useful. The modes (oscillators) are grouped statistically and the energy transfer from one group of modes to another group of modes is statistically proportional to the difference in the subsystem total energies. Readers who are interested in a more detailed description of the SEA method are encouraged to read the following interesting literature [116, 214, 229]. A very clear discussion and explanation of the SEA can be found in [138]. SEA is attractive for high-frequency regions where a deterministic analysis of all resonant modes of vibration is not practical.

In [120] Richard Lyon explained his motivation for calculating the power flow between modes of vibration excited by random noise sources.

4.2 Some Basics about Averaged Quantities

The average power of the product of the displacement $x_1(t)$ and $x_2(t)$ is defined as [185, 96]

$$\langle x_1 x_2 \rangle = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T x_1(t) x_2(t) dt = \frac{1}{2} \Re[X_1(\omega) X_2^*(\omega)], \quad (4.1)$$

where the star * denote the *complex conjugate*. The displacement $x_{1,2}(t)$ can be written, using Euler' identity [53], as follows

$$x_{1,2}(t) = \Re[X_{1,2}(\omega)e^{j\omega t}] = \Re(X_{1,2}) \cos(\omega t) - \Im(X_{1,2}) \sin(\omega t). \quad (4.2)$$

The average power $\langle x_1 x_2 \rangle$ of $x_1(t)$ and $x_2(t)$ now becomes

$$\begin{aligned} \langle x_1 x_2 \rangle &= [\Re\{X_1(\omega)\}\Re\{X_2(\omega)\}]\langle \cos^2(\omega t) \rangle \\ &\quad + [\Im\{X_1(\omega)\}\Im\{X_2(\omega)\}]\langle \sin^2(\omega t) \rangle \\ &\quad - \Re\{X_1(\omega)\}\Im\{X_2(\omega)\}\langle \sin(\omega t) \cos(\omega t) \rangle \\ &\quad - \Re\{X_2(\omega)\}\Im\{X_1(\omega)\}\langle \sin(\omega t) \cos(\omega t) \rangle. \end{aligned} \quad (4.3)$$

The time average values of $\langle \sin^2(\omega t) \rangle$, $\langle \cos^2(\omega t) \rangle$ and $\langle \sin(\omega t) \cos(\omega t) \rangle$ are

$$\langle \sin^2(\omega t) \rangle = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \sin^2(\omega t) dt = \frac{\omega}{2\pi} \int_0^{\frac{2\pi}{\omega}} \sin^2(\omega t) dt = \frac{1}{2}, \quad (4.4)$$

$$\langle \cos^2(\omega t) \rangle = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \cos^2(\omega t) dt = \frac{\omega}{2\pi} \int_0^{\frac{2\pi}{\omega}} \cos^2(\omega t) dt = \frac{1}{2}, \quad (4.5)$$

and

$$\begin{aligned} \langle \sin(\omega t) \cos(\omega t) \rangle &= \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \sin(\omega t) \cos(\omega t) dt \\ &= \frac{\omega}{2\pi} \int_0^{\frac{2\pi}{\omega}} \sin(\omega t) \cos(\omega t) dt = 0. \end{aligned} \quad (4.6)$$

Equation (4.3) can be rewritten as follows

$$\begin{aligned} \langle x_1 x_2 \rangle &= \frac{1}{2} [\{\Re\{X_1(\omega)\}\Re\{X_2(\omega)\} + \Im\{X_1(\omega)\}\Im\{X_2(\omega)\}\}] \\ &= \frac{1}{2} \Re\{X_1(\omega)X_2^*(\omega)\}. \end{aligned} \quad (4.7)$$

As a consequence of (4.7) the average of $x^2(t)$, namely $\langle x^2 \rangle$, is

$$\langle x^2 \rangle = \frac{1}{2} \Re[X(\omega)X^*(\omega)] = \frac{1}{2} |X(\omega)|^2. \quad (4.8)$$

With the aid of (4.7) we can write the following combinations of time average values $\langle x_1 \dot{x}_2 \rangle$, $\langle x_2 \dot{x}_1 \rangle$, $\langle \dot{x}_1 \dot{x}_2 \rangle$ and $\langle \ddot{x}_2 \dot{x}_1 \rangle$:

$$\begin{aligned} \langle x_1 \dot{x}_2 \rangle &= \frac{1}{2} \Re\{X_1(\omega)\dot{X}_2^*(\omega)\} = \frac{1}{2} \Re\{j\omega X_1(\omega)X_2^*(\omega)\} \\ &= \frac{\omega}{2} [\Re\{X_1(\omega)\}\Im\{X_2(\omega)\} - \Re\{X_2(\omega)\}\Im\{X_1(\omega)\}], \end{aligned} \quad (4.9)$$

and

$$\begin{aligned}\langle x_2 \dot{x}_1 \rangle &= \frac{1}{2} \Re \{ X_2(\omega) \dot{X}_1^*(\omega) \} = \frac{1}{2} \Re \{ j\omega X_2(\omega) X_1^*(\omega) \} \\ &= \frac{\omega}{2} [\Re \{ X_2(\omega) \} \Im \{ X_1(\omega) \} - \Re \{ X_1(\omega) \} \Im \{ X_2(\omega) \}].\end{aligned}\quad (4.10)$$

From (4.9) and (4.10) we can conclude that $\langle x_1 \dot{x}_2 \rangle$ and $\langle x_2 \dot{x}_1 \rangle$ are skew-symmetric, thus

$$\langle x_1 \dot{x}_2 \rangle = -\langle x_2 \dot{x}_1 \rangle. \quad (4.11)$$

If $x_1 = x_2 = x$, (4.11) will give

$$\langle x \dot{x} \rangle = -\langle x \dot{x} \rangle = 0. \quad (4.12)$$

It can be proven that the average value of $\langle \ddot{x}_1 \dot{x}_2 \rangle$ can be expressed in terms of $\langle x_1 \dot{x}_2 \rangle$

$$\langle \ddot{x}_1 \dot{x}_2 \rangle = -\omega^2 \langle x_1 \dot{x}_2 \rangle, \quad (4.13)$$

and therefore $\langle \dot{x}_1 \dot{x}_2 \rangle$ is skew-symmetric too

$$\langle \dot{x}_1 \dot{x}_2 \rangle = -\langle \dot{x}_2 \dot{x}_1 \rangle, \quad (4.14)$$

from which, it follows that

$$\langle \ddot{x} \dot{x} \rangle = -\langle \dot{x} \ddot{x} \rangle = 0. \quad (4.15)$$

The equation of motion of a SDOF dynamic system with a discrete mass m , a spring stiffness k and damper c and external force $F(t)$ is given by

$$m\ddot{x} + c\dot{x} + kx = F(t). \quad (4.16)$$

The average input power Π_{in} is expressed by

$$\Pi_{in} = \langle F \dot{x} \rangle = \frac{1}{2} \Re \{ F(\omega) \dot{X}(\omega) \}, \quad (4.17)$$

where $x(t) = X(\omega)e^{j\omega t}$ and $F(t) = F(\omega)e^{j\omega t}$. If the averages are applied to (4.16) the following relation is obtained

$$m\langle \ddot{x} \dot{x} \rangle + c\langle \dot{x}^2 \rangle + k\langle x \dot{x} \rangle = \langle F(t) \dot{x} \rangle = \Pi_{in}. \quad (4.18)$$

In accordance with (4.12) and (4.15) the average values of $\langle \ddot{x} \dot{x} \rangle = \langle x \dot{x} \rangle = 0$ and therefore (4.18) becomes

$$\Pi_{in} = c\langle \dot{x}^2 \rangle = \Pi_{diss}, \quad (4.19)$$

where Π_{diss} is the dissipated energy in the damper c . The interpretation of (4.19), is that the average power Π_{in} introduced in the SDOF system is equal to the power Π_{diss} dissipated by the damper. The dissipated power Π_{diss} can be rewritten and expressed in terms of the mass m , modal damping ratio ζ , the natural frequency ω_n and $\dot{X}(\omega)$

$$\Pi_{diss} = c\langle \dot{x}^2 \rangle = 2\zeta\sqrt{km}\langle \dot{x}^2 \rangle = m\zeta\omega_n|\dot{X}(\omega)|^2, \quad (4.20)$$

where $2\sqrt{km}$ is the critical damping. The *loss factor* η is the ratio of the dissipated energy per cycle divided by the peak strain energy per cycle. Normally the loss factor $\eta = 2\zeta$. Thus the *average dissipated power* can be rewritten as

$$\Pi_{diss} = m\eta\omega_n\langle \dot{x}^2 \rangle = \frac{1}{2}m\eta\omega_n|\dot{X}(\omega)|^2. \quad (4.21)$$

The *mobility function* $Y(\omega)$ is the ratio of the velocity $\dot{X}(\omega)$ and the applied force $F(\omega)$

$$Y(\omega) = \frac{\dot{X}(\omega)}{F(\omega)}, \quad (4.22)$$

where $\Re\{Y(\omega)\}$ is called the *conductance* and $\Im\{Y(\omega)\}$ the *susceptance*. For a SDOF system, the mobility function $Y(\omega)$ is

$$Y(\omega) = \frac{j\omega}{m\{\omega_n^2 - \omega^2 + j\eta\omega\omega_n\}}. \quad (4.23)$$

The average of $\dot{x}^2(t)$, $\langle \dot{x}^2 \rangle$, can be expressed as follows

$$\begin{aligned} \langle \dot{x}^2 \rangle &= \frac{1}{2}|\dot{X}^2(\omega)| = \frac{1}{2}\{Y(\omega)F(\omega)F^*(\omega)Y^*(\omega)\} \\ &= |Y^2(\omega)|\frac{1}{2}\{F(\omega)F^*(\omega)\} = |Y^2(\omega)|\langle F^2 \rangle. \end{aligned} \quad (4.24)$$

The *average power input* $\Pi_{in} = \langle F\dot{x} \rangle$ can now be expressed as follows

$$\begin{aligned} \Pi_{in} = \langle F\dot{x} \rangle &= \frac{1}{2}\Re\{F(\omega)\dot{X}^*(\omega)\} = \frac{1}{2}|F(\omega)|^2\Re\{Y^*(\omega)\} \\ &= \frac{1}{2}m\eta\omega_n|F(\omega)|^2|Y(\omega)|^2 = \frac{1}{2}m\eta\omega_n|X(\omega)|^2. \end{aligned} \quad (4.25)$$

Example. A SDOF system has a natural frequency $\omega_n = 2\pi f_n = 208.2$ rad/s. The discrete mass is $m = 150$ kg. A steady-state oscillation force with an amplitude $F = 100$ N produces a steady-state velocity $v = \dot{X} = 0.2$ m/s. Estimate the loss factor η .

The average input power is given by $\Pi_{in} = \frac{1}{2}F\dot{X}$ and the average dissipated power is $\Pi_{diss} = \frac{1}{2}m\eta\omega_n\dot{X}^2$. These are equal $\Pi_{in} = \Pi_{diss}$. The loss factor η can be calculated from

$$\eta = \frac{F}{m\omega_n\dot{X}} = \frac{100}{150 \times 208.2 \times 0.2} = 0.0016.$$

For a stationary and ergodic random process, the average power of the velocity $\dot{x}(t)$ can be expressed as

$$\langle \dot{x}^2 \rangle = \int_0^\infty W_{\dot{x}\dot{x}}(f)df, \quad (4.26)$$

and the average power of $F(t)$ is given by

$$\langle F^2 \rangle = \int_0^\infty W_{FF}(f)df, \tag{4.27}$$

where $W_{\dot{x}\dot{x}}(f)$ is the PSD of the velocity $\dot{x}(t)$, and $W_{FF}(f)$ is the PSD of the external force $F(t)$.

In a narrow frequency band with the center frequency f and bandwidth Δf , the average power of the velocity $\dot{x}(t)$ and external force $F(t)$ can be obtained using (4.26) and (4.27):

$$\langle \dot{x}^2 \rangle_{\Delta f} = W_{\dot{x}\dot{x}}(f)\Delta f, \tag{4.28}$$

and

$$\langle F^2 \rangle_{\Delta f} = W_{FF}(f)\Delta f. \tag{4.29}$$

The PSD of the velocity $\dot{x}(t)$ can be expressed in terms of the PSD of the external force $F(t)$ using (4.24), thus

$$\begin{aligned} W_{\dot{x}\dot{x}}(f) &= |Y(f)|^2 W_{FF}(f), \\ S_{\dot{x}\dot{x}}(\omega) &= |Y(\omega)|^2 S_{FF}(\omega), \end{aligned} \tag{4.30}$$

where $W(f) = 2S(\omega)$. $W(f)$ is the single-sided PSD function dependent upon the frequency (Hz, cps) and $S(\omega)$ is the double-sided PSD function in the frequency domain (rad/s).

The average power or mean square of $\dot{x}(t)$ can now be obtained

$$\begin{aligned} \langle \dot{x}^2 \rangle &= \int_0^\infty W_{\dot{x}\dot{x}}(f)df = \int_0^\infty |Y(f)|^2 W_{FF}(f)df \\ &= \frac{1}{4\pi} \int_{-\infty}^\infty |Y(\omega)|^2 S_{FF}(\omega)d\omega, \end{aligned} \tag{4.31}$$

where $Y(\omega)$ is already given in (4.23)

The integral $\frac{1}{2\pi} \int_{-\infty}^\infty |Y(\omega)|^2 d\omega$ has a closed form solution (see Appendix B) and with

- $a_0 = m$
- $a_1 = m\eta\omega_n$
- $a_2 = m\omega_n^2$
- $b_0 = -1$
- $b_1 = 0$

it is

$$\frac{1}{2\pi} \int_{-\infty}^\infty |Y(\omega)|^2 d\omega = \frac{\frac{a_0 b_1}{a_2} - b_0}{2a_0 a_1} = \frac{1}{2m^2 \eta \omega_n}. \tag{4.32}$$

The integral (4.31) can be further approximated because $|Y(\omega_n)|^2$ is very peaked and the PSD $W_{FF}(f) = 2S_{FF}(\omega)$ is smooth and constant near ω_n or f_n , therefore (4.31) may be written as follows

$$\begin{aligned}\langle \dot{x}^2 \rangle &= \int_0^\infty W_{\dot{x}\dot{x}}(f)df = \frac{1}{2\pi} \int_0^\infty |Y(\omega)|^2 S_{FF}(\omega) d\omega \\ &\approx \frac{W_{FF}(f_n)}{4\pi} \int_{-\infty}^\infty |Y(\omega)|^2 d\omega = \frac{W_{FF}(f_n)}{4m^2\eta\omega_n} = \frac{W_{FF}(f_n)}{8m^2\pi\eta f_n}.\end{aligned}\quad (4.33)$$

The total average energy $\langle E \rangle$ is the average of the sum of the kinetic energy T and the strain energy U

$$\begin{aligned}\langle E \rangle &= \langle T + U \rangle = \left\langle \frac{1}{2}m\dot{x}^2(t) + \frac{1}{2}kx^2(t) \right\rangle \\ &= \frac{1}{2} \left(m\langle \dot{x}^2 \rangle + \frac{k}{\omega_n^2} \langle \dot{x}^2 \rangle \right) = m\langle \dot{x}^2 \rangle,\end{aligned}\quad (4.34)$$

where $\langle \dot{x}^2 \rangle = \omega_n^2 \langle x^2 \rangle$ and $k = m\omega_n^2$. With (4.33) the average energy $\langle E \rangle$ of the SDOF system with mass m and average power of the velocity $\langle \dot{x}^2 \rangle$ is, becomes

$$\langle E \rangle = m\langle \dot{x}^2 \rangle = \frac{W_{FF}(f_n)}{8m\pi\eta f_n} = \frac{W_{FF}(f_n)}{4m\eta\omega_n}.\quad (4.35)$$

The average dissipated energy Π_{diss} can now expressed in terms of the average energy $\langle E \rangle$ as follows:

$$\Pi_{diss} = m\eta\omega_n \langle \dot{x}^2 \rangle = \eta\omega_n \langle E \rangle.\quad (4.36)$$

Furthermore the average power $\langle \dot{x}^2 \rangle$ of the acceleration can be derived with the knowledge that $\langle \ddot{x}^2 \rangle = \omega_n^2 \langle \dot{x}^2 \rangle$

$$\langle \ddot{x}^2 \rangle = \frac{\omega_n W_{FF}(f_n)}{4m^2\eta} = \frac{1}{m^2} \frac{\pi}{2} f_n Q W_{FF}(f_n),\quad (4.37)$$

which is Miles' equation, where the amplification factor is $Q = \frac{1}{\eta} = \frac{1}{2\zeta}$.

The average input power Π_{in} of the external force $F(t)$ is equal to the average dissipated power Π_{diss} :

$$\Pi_{in} = m\eta\omega_n \langle \dot{x}^2 \rangle = \eta\omega_n \langle E \rangle = \frac{W_{FF}(f_n)}{4m}.\quad (4.38)$$

4.3 Two Coupled Oscillators

Consider a simple two-SDOF system as shown in Fig. 4.1. The coupling element between the two SDOFs is a linear spring and is non dissipative. The quantities of interest in this section are the average energies of each oscillator and the average energy flow between them.

The 2DOF systems, as shown Fig. 4.1 are coupled with a linear spring k_{12} . The equations of motion of the two coupled oscillators are

$$m_1\ddot{x}_1 + c_1\dot{x}_1 + k_1x_1 + k_{12}(x_1 - x_2) = F_1,\quad (4.39)$$

$$m_2\ddot{x}_2 + c_2\dot{x}_2 + k_2x_2 + k_{12}(x_2 - x_1) = F_2,\quad (4.40)$$

where

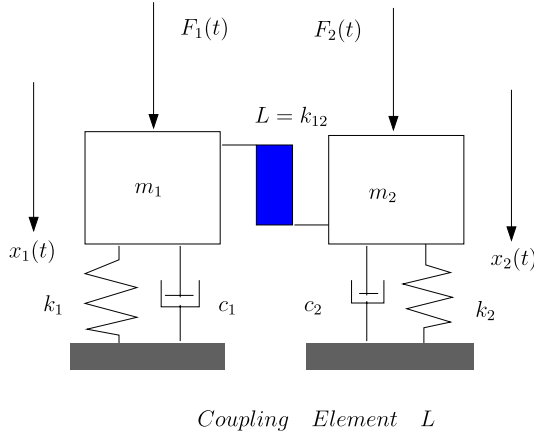


Fig. 4.1. Two coupled oscillators [214]

- $\omega_1 = \sqrt{\frac{k_1}{m_1}}, \omega_2 = \sqrt{\frac{k_2}{m_2}}, \omega_{12} = \sqrt{\frac{k_{12}}{m_1 m_2}}$
- $c_1 = 2\zeta_1 \sqrt{k_1 m_1}, c_2 = 2\zeta_2 \sqrt{k_2 m_2}$
- $\frac{c_1}{m_1} = 2\zeta_1 \omega_1 = \eta_1 \omega_1, \frac{c_2}{m_2} = 2\zeta_2 \omega_2 = \eta_2 \omega_2.$

The *mass* and *gyroscopic coupling* in the *coupling element* L is not considered because these coupling elements are discussed in detail in [116]. Adding mass and gyroscopic coupling complicates the analysis unnecessarily. Dividing the equations motion of the coupled system by m_1 and m_2 respectively we find

$$\ddot{x}_1 + 2\zeta_1 \omega_1 \dot{x}_1 + \omega_1^2 x_1 + m_2 \omega_{12}^2 (x_1 - x_2) = \frac{F_1}{m_1}, \quad (4.41)$$

$$\ddot{x}_2 + 2\zeta_2 \omega_2 \dot{x}_2 + \omega_2^2 x_2 + m_1 \omega_{12}^2 (x_2 - x_1) = \frac{F_2}{m_2}. \quad (4.42)$$

In this mathematical model of the two coupled oscillators the applied forces F_1 and F_2 are not correlated and therefore $\langle F_1 F_2 \rangle = 0$.

The average power supplied by the force F_1 is $\langle F_1 \dot{x}_1 \rangle$ and the average power supplied by F_2 is $\langle F_2 \dot{x}_2 \rangle$. The modified equations of motion (4.41) become

$$\begin{aligned} \langle \ddot{x}_1 \dot{x}_1 \rangle + 2\zeta_1 \omega_1 \langle \dot{x}_1^2 \rangle + \omega_1^2 \langle x_1 \dot{x}_1 \rangle \\ + m_2 \omega_{12}^2 \langle (x_1 - x_2) \dot{x}_1 \rangle = \frac{\langle F_1 \dot{x}_1 \rangle}{m_1}, \end{aligned} \quad (4.43)$$

and

$$\begin{aligned} \langle \ddot{x}_2 \dot{x}_2 \rangle + 2\zeta_2 \omega_2 \langle \dot{x}_2^2 \rangle + \omega_2^2 \langle x_2 \dot{x}_2 \rangle \\ + m_1 \omega_{12}^2 \langle (x_2 - x_1) \dot{x}_2 \rangle = \frac{\langle F_2 \dot{x}_2 \rangle}{m_2}. \end{aligned} \quad (4.44)$$

The averages $\langle \ddot{x}_i \dot{x}_i \rangle = 0$ and $\langle x_i \dot{x}_i \rangle = 0$ and $\langle x_i \dot{x}_j \rangle = -\langle x_j \dot{x}_i \rangle$ and therefore

$$\langle F_1 \dot{x}_1 \rangle + \langle F_2 \dot{x}_2 \rangle = 2m_1 \zeta_1 \omega_1 \langle \dot{x}_1^2 \rangle + 2m_2 \zeta_2 \omega_2 \langle \dot{x}_2^2 \rangle, \quad (4.45)$$

which means that the average powers supplied to the two coupled SDOFs is dissipated by the dampers and no energy is dissipated in the coupling element L . This coupling element is called conservative.

The average power flow from oscillator 1 to oscillator 2 is defined as

$$\begin{aligned} \Pi_{12} &= -\langle k_{12} x_2 \dot{x}_1 \rangle = -m_1 m_2 \omega_{12}^2 \langle x_2 \dot{x}_1 \rangle \\ &= -\frac{1}{2} m_1 m_2 \omega_{12}^2 \Re\{X_2 \dot{X}_1^*\}. \end{aligned} \quad (4.46)$$

The negative sign introduced to be consistent with average damping energy because the power flow from system 1 to system 2 is extracted from the energy of system 1. The power flow Π_{12} will be evaluated and expressed in terms of the PSD functions of the applied forces $W_{F_1 F_1}(f)$ and $W_{F_2 F_2}(f)$. We will show that the average power flow Π_{12} can be written as

$$\Pi_{12} = \beta_{12} [\langle E_1 \rangle - \langle E_2 \rangle], \quad (4.47)$$

where β_{12} is the *coupling factor* and $\langle E_i \rangle$ is taken from (4.35). The coupling factor β_{12} will now be calculated.

The displacement x_1 and x_2 are harmonic and can be written

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} X_1(\omega) \\ X_2(\omega) \end{pmatrix} e^{j\omega t} = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} e^{j\omega t}. \quad (4.48)$$

The introduction of (4.48) into the equations of motion of the coupled SDOF systems in (4.41) leads to

$$\begin{pmatrix} -\omega^2 + 2j\zeta_1\omega\omega_1 + \omega_1^2 + m_2\omega_{12}^2 & -m_2\omega_{12}^2 \\ -m_1\omega_{12}^2 & -\omega^2 + 2j\zeta_2\omega\omega_2 + \omega_2^2 + m_1\omega_{12}^2 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} \frac{F_1}{m_1} \\ \frac{F_2}{m_2} \end{pmatrix}. \quad (4.49)$$

The unknown displacement X_1 and X_2 will be expressed in terms of F_1 and F_2 .

The determinant $D(\omega)$ of the matrix in (4.49) is

$$\begin{aligned} D(\omega) &= a_0 \omega^4 - j a_1 \omega^3 - a_2 \omega^2 + j a_3 \omega + a_4 \\ &= a_0 (j\omega)^4 + a_1 (j\omega)^3 + a_2 (j\omega)^2 + a_3 j\omega + a_4, \end{aligned} \quad (4.50)$$

where

- $a_0 = 1$
- $a_1 = 2(\zeta_1 \omega_1 + \zeta_2 \omega_2)$
- $a_2 = \{\omega_1^2 + \omega_2^2 + \omega_{12}^2(m_1 + m_2) + 4\zeta_1 \zeta_2 \omega_1 \omega_2\}$

- $a_3 = 2(\zeta_1\omega_1\omega_2^2 + \zeta_2\omega_2\omega_1^2 + \zeta_1m_1\omega_{12}^2\omega_1 + \zeta_2m_2\omega_2\omega_{12}^2)$
- $a_4 = (\omega_1^2\omega_2^2 + m_1\omega_{12}^2\omega_1^2 + m_2\omega_{12}^2\omega_2^2)$.

Use Cramer's rule to solve for X_1 and X_2 :

$$X_1 = \frac{1}{m_1m_2D(\omega)} \{ \{m_2F_1(-\omega^2 + 2j\zeta_2\omega\omega_2 + \omega_2^2 + m_1\omega_{12}^2)\} + m_1m_2\omega_{12}^2F_2 \}, \quad (4.51)$$

and

$$X_2 = \frac{1}{m_1m_2D(\omega)} \{ \{m_1F_2(-\omega^2 + 2j\zeta_1\omega\omega_1 + \omega_1^2 + m_2\omega_{12}^2)\} + m_1m_2\omega_{12}^2F_1 \}. \quad (4.52)$$

The displacements X_1 and X_2 in (4.51) and (4.52) can be expressed in a more straight forward manner:

$$X_1(\omega) = H_{11}(\omega)F_1(\omega) + H_{12}(\omega)F_2(\omega), \quad (4.53)$$

$$X_2(\omega) = H_{21}(\omega)F_1(\omega) + H_{22}(\omega)F_2(\omega), \quad (4.54)$$

where $H_{ij}(\omega)$ are the *receptances*

- $H_{11}(\omega) = \frac{1}{m_1D(\omega)}(-\omega^2 + 2j\zeta_2\omega\omega_2 + \omega_2^2 + m_1\omega_{12}^2)$
- $H_{12}(\omega) = H_{21}(\omega) = \frac{\omega_{12}^2}{D(\omega)}$
- $H_{22}(\omega) = \frac{1}{m_2D(\omega)}(-\omega^2 + 2j\zeta_1\omega\omega_1 + \omega_1^2 + m_2\omega_{12}^2)$.

The velocities \dot{X}_1 and \dot{X}_2 , in the frequency domain, are also expressed in terms of the applied forces F_1 and F_2 and the mobility functions $Y_{ij} = j\omega H_{ij}$

$$\dot{X}_1(\omega) = Y_{11}(\omega)F_1(\omega) + Y_{12}(\omega)F_2(\omega), \quad (4.55)$$

$$\dot{X}_2(\omega) = Y_{21}(\omega)F_1(\omega) + Y_{22}(\omega)F_2(\omega). \quad (4.56)$$

The average power flow from SDOF 1 to SDOF 2 is calculated using (4.46)

$$\Pi_{12} = -m_1m_2\omega_{12}^2\langle x_2\dot{x}_1 \rangle = -\frac{1}{2}m_1m_2\omega_{12}^2\Re\{X_2(\omega)\dot{X}_1^*(\omega)\}.$$

The average power flow Π_{12} when the expressions for $\dot{X}_1(\omega)$ and $X_2(\omega)$ are substituted in (4.46) becomes

$$\begin{aligned} \langle x_2\dot{x}_1 \rangle &= \frac{1}{2}\Re\{X_2\dot{X}_1^*\} \\ &= \frac{1}{2}\Re\{H_{21}Y_{11}^*|F_1|^2 + H_{22}Y_{12}^*|F_2|^2 + H_{21}Y_{12}^*F_1F_2^* + H_{22}Y_{11}^*F_2F_1^*\}. \end{aligned} \quad (4.57)$$

The external forces F_1 and F_2 are not correlated, hence $\langle F_1F_2 \rangle = \frac{1}{2}\Re\{F_1F_2^*\} = \frac{1}{2}\Re\{F_2F_1^*\} = 0$ and therefore (4.57) can be simplified as follows

$$\begin{aligned}\langle x_2 \dot{x}_1 \rangle &= \frac{1}{2} \Re \{ X_2 \dot{X}_1^* \} \\ &= \frac{1}{2} \Re \{ H_{21} Y_{11}^* |F_1|^2 + H_{22} Y_{12}^* |F_2|^2 \}.\end{aligned}\quad (4.58)$$

Equation (4.58) will be elaborated term by term

$$H_{21} Y_{11}^* = \frac{\omega_{12}^2}{m_1 |D(\omega)|^2} \{ j\omega^3 - 2\zeta_2 \omega_2 \omega^2 - j\omega(\omega_2^2 + m_1 \omega_{12}^2) \}, \quad (4.59)$$

$$H_{22} Y_{12}^* = \frac{\omega_{12}^2}{m_2 |D(\omega)|^2} \{ j\omega^3 + 2\zeta_1 \omega_1 \omega^2 - j\omega(\omega_1^2 + m_2 \omega_{12}^2) \}. \quad (4.60)$$

The average of $\langle x_2 \dot{x}_1 \rangle = \frac{1}{2} \Re \{ X_2 \dot{X}_1 \}$ will be further worked out and the following expressions obtained

$$\langle x_2 \dot{x}_1 \rangle = \frac{\omega^2 \omega_{12}^2}{m_2 |D(\omega)|^2} \{ \zeta_1 \omega_1 \} |F_2|^2 - \frac{\omega^2 \omega_{12}^2}{m_1 |D(\omega)|^2} \{ \zeta_2 \omega_2 \} |F_1|^2, \quad (4.61)$$

$$\langle x_2 \dot{x}_1 \rangle = \frac{\omega_{12}^2}{m_1 m_2 |D(\omega)|^2} \{ m_1 \omega^2 \zeta_1 \omega_1 |F_2|^2 - m_2 \omega^2 \zeta_2 \omega_2 |F_1|^2 \}, \quad (4.62)$$

$$\langle x_2 \dot{x}_1 \rangle = -\zeta_1 \zeta_2 \omega_1 \omega_2 \omega_{12}^2 \left\{ \frac{\omega^2 |F_1|^2}{|D(\omega)|^2 m_1 \zeta_1 \omega_1} - \frac{\omega^2 |F_2|^2}{|D(\omega)|^2 m_2 \zeta_2 \omega_2} \right\}. \quad (4.63)$$

The average power flow Π_{12} now becomes

$$\Pi_{12} = m_1 m_2 \zeta_1 \zeta_2 \omega_1 \omega_2 \omega_{12}^4 \left\{ \frac{\omega^2 |F_1|^2}{|D(\omega)|^2 m_1 \zeta_1 \omega_1} - \frac{\omega^2 |F_2|^2}{|D(\omega)|^2 m_2 \zeta_2 \omega_2} \right\}. \quad (4.64)$$

The loss factor $\eta = 2\zeta$ will be introduced in (4.64), thus

$$\Pi_{12} = \frac{m_1 m_2 \eta_1 \eta_2 \omega_1 \omega_2 \omega_{12}^4}{2} \left\{ \frac{\omega^2 |F_1|^2}{|D(\omega)|^2 m_1 \eta_1 \omega_1} - \frac{\omega^2 |F_2|^2}{|D(\omega)|^2 m_2 \eta_2 \omega_2} \right\}. \quad (4.65)$$

The time average $\langle x_2 \dot{x}_1 \rangle$ can also be written as

$$\langle x_2 \dot{x}_1 \rangle = \int_0^\infty W_{x_2 \dot{x}_1}(f) df = \frac{1}{2\pi} \int_{-\infty}^\infty S_{x_2 \dot{x}_1}(\omega) d\omega. \quad (4.66)$$

The PSD functions $W_{F_1 F_1}(f)$ and $W_{F_2 F_2}(f)$ of the external forces $F_1(\omega)$ and $F_2(\omega)$ are assumed to be smooth and constant functions in the neighborhood of the natural frequencies ω_1 and ω_2 respectively. The integral $\int_0^\infty W_{FF}(f) df = \frac{1}{2\pi} \int_{-\infty}^\infty S_{FF}(\omega) d\omega$, where $W_{FF}(f) = 2S_{FF}(\omega)$. Equation (4.65) in conjunction with (4.66) can be rewritten and the average cross power Π_{12} expressed in terms of the PSD functions of the external forces:

$$\Pi_{12} = -\frac{m_1 m_2 \eta_1 \eta_2 \omega_1 \omega_2 \omega_{12}^4}{2\pi} \int_{-\infty}^\infty \frac{(j\omega)^2 d\omega}{|D(\omega)|^2} \left[\frac{W_{F_1 F_1}}{4m_1 \eta_1 \omega_1} - \frac{W_{F_2 F_2}}{4m_2 \eta_2 \omega_2} \right]. \quad (4.67)$$

From (4.67) we can conclude that the cross power flow Π_{12} is proportional to the difference of the total energies $\langle E_i \rangle$ of the SDOF systems 1 and 2 respectively, which accords with (4.47),

$$\Pi_{12} = \beta_{12}[\langle E_1 \rangle - \langle E_2 \rangle].$$

It is easy to prove that (4.47) can be reversed and the cross power Π_{21} is

$$\Pi_{21} = \beta_{21}[\langle E_2 \rangle - \langle E_1 \rangle], \quad (4.68)$$

where $\beta_{12} = \beta_{21}$.

If two conservatively coupled oscillators are identical and excited by independent random forces, the spectrum of power flow between them is proportional to the difference in the spectra of their energies. For oscillators that are not identical, the total energy flow will be proportional to the difference in energies, provided the excitation spectra are relatively flat near the resonance frequencies [8].

What is left now is the calculation of the coupling factor β_{12} :

$$\beta_{12} = -\frac{m_1 m_2 \eta_1 \eta_2 \omega_1 \omega_2 \omega_{12}^4}{2\pi} \int_{-\infty}^{\infty} \frac{(j\omega)^2 d\omega}{|D(\omega)|^2}. \quad (4.69)$$

The (standard) integral is (see Appendix B)

$$\begin{aligned} & \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{(j\omega)^2 d\omega}{|D(\omega)|^2} \\ &= \frac{b_0(-a_1 a_4 + a_2 a_3) - a_0 a_3 b_1 + a_0 a_1 b_2 + \frac{a_0 b_3}{a_4}(a_0 a_3 - a_1 a_2)}{2a_0(a_0 a_3^2 + a_1^2 a_4 - a_1 a_2 a_3)}, \end{aligned}$$

where

- $a_0 = 1$
- $a_1 = \eta_1 \omega_1 + \eta_2 \omega_2$
- $a_2 = \{\omega_1^2 + \omega_2^2 + \omega_{12}^2(m_1 + m_2) + \eta_1 \eta_2 \omega_1 \omega_2\}$
- $a_3 = \eta_1 \omega_1 \omega_2^2 + \eta_2 \omega_2 \omega_1^2 + \eta_1 m_1 \omega_1 \omega_{12}^2 + \eta_2 m_2 \omega_2 \omega_{12}^2$
- $a_4 = \omega_1^2 \omega_2^2 + m_1 \omega_1^2 \omega_{12}^2 + m_2 \omega_2^2 \omega_{12}^2$,

and

- $b_0 = 0$
- $b_1 = 0$
- $b_2 = 1$
- $b_3 = 0$,

thus

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{(j\omega)^2 d\omega}{|D(\omega)|^2} = \frac{a_1 b_2}{2(a_0 a_3^2 + a_1^2 a_4 - a_1 a_2 a_3)}. \quad (4.70)$$

Finally after many manipulations the coupling factor β_{12} becomes

$$\beta_{12} = \frac{\omega_{12}^4 m_1 m_2 (\eta_1 \omega_1 + \eta_2 \omega_2)}{D}, \quad (4.71)$$

where the denominator D is as follows

$$\begin{aligned} D = & [(\omega_1^2 + \omega_{12}^2 m_2) - (\omega_2^2 + \omega_{12}^2 m_1)]^2 \\ & + m_1 m_2 \omega_{12}^4 \left(2 + \frac{\eta_1 \omega_1}{\eta_2 \omega_2} + \frac{\eta_2 \omega_2}{\eta_1 \omega_1} \right) \\ & + \eta_1 \eta_2 \omega_1 \omega_2 \left[\omega_1^2 + \omega_2^2 + \omega_1 \omega_2 \left(\frac{\eta_1}{\eta_2} + \frac{\eta_2}{\eta_1} \right) \right] \\ & + \eta_1 \eta_2 \omega_1 \omega_2 \left[m_1 \omega_{12}^2 \left(1 + \frac{\eta_1 \omega_1}{\eta_2 \omega_2} \right) + m_2 \omega_{12}^2 \left(1 + \frac{\eta_2 \omega_2}{\eta_1 \omega_1} \right) \right]. \end{aligned} \quad (4.72)$$

The coupling factor β_{12} is positive definite and symmetric with respect to both SDOF systems.

If SDOF 2 is blocked, i.e. $\langle E_2 \rangle = 0$, then $\bar{\Pi}_{12} = \beta_{12} \langle E_1 \rangle$. That means the spring k_{12} in the coupling element is grounded. Like to the average dissipated damping energy $\Pi_{diss} = \omega \eta \langle E \rangle$, the power transferred from SDOF 1 to SDOF 2 blocking SDOF 2 can be expressed as

$$\bar{\Pi}_{12} = \beta_{12} \langle E_1 \rangle = \eta_{12} \omega_1 \langle E_1 \rangle. \quad (4.73)$$

If SDOF 1 is blocked the power transferred from SDOF 2 to SDOF 1 can be written as in (4.73):

$$\bar{\Pi}_{21} = \beta_{21} \langle E_2 \rangle = \eta_{21} \omega_2 \langle E_2 \rangle. \quad (4.74)$$

The average power Π_{12} transferred from SDOF 1 to SDOF 2 becomes

$$\Pi_{12} = \bar{\Pi}_{12} - \bar{\Pi}_{21} = \eta_{12} \omega_1 \langle E_1 \rangle - \eta_{21} \omega_2 \langle E_2 \rangle, \quad (4.75)$$

where η_{ij} are called the *coupling loss factors*, ω_i are the circular frequencies and $\langle E_i \rangle = m_i \langle x_i^2 \rangle$ the *total energies*.

From the symmetry of the coupling factor $\beta_{12} = \beta_{21}$ the reciprocity relation of the coupling loss factors can be derived

$$\eta_{12} \omega_1 = \eta_{21} \omega_2. \quad (4.76)$$

Figure 4.2 shows two (water) subsystems (tanks), system 1 and 2. Both tanks contain water (energy) $\langle E_1 \rangle$ and $\langle E_2 \rangle$. Water is supplied to tank 1 $\Pi_{1,in}$ and water leaves the tanks directly as $\Pi_{1,diss}$ and $\Pi_{2,diss}$. Water streams from tank 1 to tank 2, as Π_{12} . The power balance equations for the tanks are

$$\Pi_{1,in} = \Pi_{1,diss} + \Pi_{12}, \quad (4.77)$$

$$\Pi_{12} = \Pi_{2,diss}, \quad (4.78)$$

$$\Pi_{1,in} = \eta_1 \omega_1 \langle E_1 \rangle + \eta_{12} \omega_1 \langle E_1 \rangle - \eta_{21} \omega_2 \langle E_2 \rangle, \quad (4.79)$$

$$\eta_{12} \omega_1 \langle E_1 \rangle - \eta_{21} \omega_2 \langle E_2 \rangle = \eta_2 \omega_2 \langle E_2 \rangle. \quad (4.80)$$

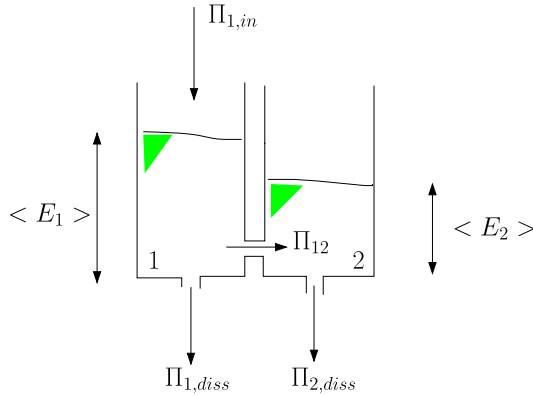


Fig. 4.2. Power flow analogy (Courtesy R.G. Dejong, Calvin College, USA)

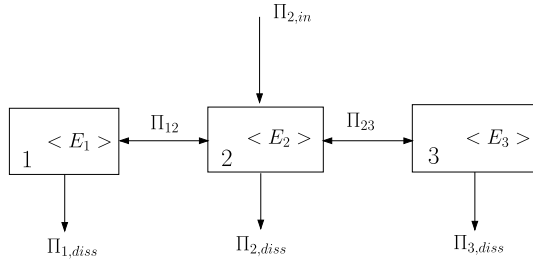


Fig. 4.3. Three coupled subsystems

Equations (4.79) and (4.80) written in matrix notation

$$\begin{bmatrix} (\eta_{12} + \eta_1)\omega_1 & -\eta_{21}\omega_2 \\ -\eta_{12}\omega_1 & (\eta_{21} + \eta_2)\omega_1 \end{bmatrix} \begin{Bmatrix} \langle E_1 \rangle \\ \langle E_2 \rangle \end{Bmatrix} = \begin{Bmatrix} \Pi_{1,in} \\ 0 \end{Bmatrix}. \tag{4.81}$$

The second equation in (4.81) gives the ratio between the energies:

$$\frac{\langle E_2 \rangle}{\langle E_1 \rangle} = \frac{\eta_{21}}{\eta_{21} + \eta_2}, \tag{4.82}$$

and from both equations (4.81) the following known result is obtained

$$\Pi_{1,in} = \Pi_{1,diss} + \Pi_{2,diss} = \eta_1\omega_1\langle E_1 \rangle + \eta_2\omega_2\langle E_2 \rangle. \tag{4.83}$$

The average power supplied is dissipated.

Example. Figure 4.3 shows a system, which consists of three coupled subsystems. Only subsystem 2 receives an external power $\Pi_{2,in}$. Set-up the power balance equations for the subsystems.

Subsystem 1: $\eta_1\omega_1\langle E_1 \rangle + \eta_{12}\omega_1\langle E_1 \rangle - \eta_{21}\omega_2\langle E_2 \rangle = 0$

$$\text{Subsystem 2: } \eta_2\omega_2\langle E_2 \rangle + \eta_{21}\omega_2\langle E_2 \rangle + \eta_{23}\omega_2\langle E_2 \rangle - \eta_{12}\omega_1\langle E_1 \rangle - \eta_{32}\omega_3\langle E_3 \rangle = \Pi_{2,in}$$

$$\text{Subsystem 3: } \eta_3\omega_3\langle E_3 \rangle + \eta_{32}\omega_3\langle E_3 \rangle - \eta_{23}\omega_2\langle E_2 \rangle = 0$$

In matrix notation

$$\begin{bmatrix} (\eta_1 + \eta_{12})\omega_1 & -\eta_{21}\omega_2 & 0 \\ -\eta_{12}\omega_1 & (\eta_2 + \eta_{21} + \eta_{23})\omega_2 & -\eta_{32}\omega_3 \\ 0 & -\eta_{23}\omega_2 & (\eta_3 + \eta_{32})\omega_3 \end{bmatrix} \begin{Bmatrix} \langle E_1 \rangle \\ \langle E_2 \rangle \\ \langle E_3 \rangle \end{Bmatrix} = \begin{Bmatrix} 0 \\ \Pi_{2,in} \\ 0 \end{Bmatrix}.$$

It can be seen that the matrix is symmetric. The diagonal terms represent the outgoing power (the sum of the loss factor and coupling loss factors is called the *total loss factor*) and the off diagonal terms represent the incoming power internal to the system. The right hand vector represents the external power supplied to the system. The total vibrational energies $\langle E_i \rangle$ must be found.

4.4 Multimode Systems

Two linear multimode elastic structural systems, subsystem 1 and subsystem 2, are coupled via their common junction, as shown in Fig. 4.4. In a frequency band $\Delta\omega$, each subsystem has a number of active modes $N_1(\omega)$ and $N_2(\omega)$. The term *modal density* n (number of modes per unit of frequency, (modes/rad/s) is now introduced. The number of modes in the frequency band $\Delta\omega$ can be written as

$$\begin{aligned} N_1(\Delta\omega) &= n_1(\omega)\Delta\omega, \\ N_2(\Delta\omega) &= n_2(\omega)\Delta\omega. \end{aligned} \tag{4.84}$$

The modes $N_1(\Delta\omega)$ for subsystem 1 are denoted by α , $1 \leq \alpha \leq N_1(\Delta\omega)$, and the modes $N_2(\Delta\omega)$ for subsystem 2 are denoted by σ , $1 \leq \sigma \leq N_2(\Delta\omega)$. There are N_1N_2 interacting modal pairs α and σ , as shown in Fig. 4.5. This modal pair will be considered as two coupled oscillators (SDOF). The power flow Π_{12} between subsystem 1 and subsystem 2 will be derived from the following rules:

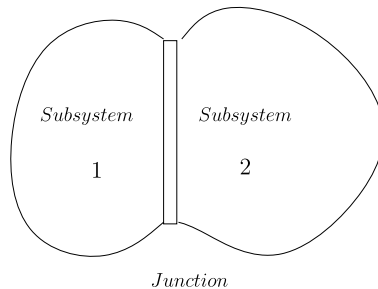


Fig. 4.4. Two coupled subsystems

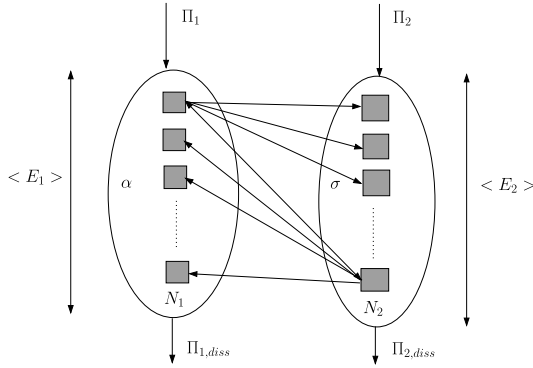


Fig. 4.5. Power flow in multimode systems

- Each mode of each subsystem has a (circular) natural frequency uniformly distributed in the frequency range $\Delta\omega$ (rad/s)
- Each mode in a subsystem has equal time average energy $\langle E_\alpha \rangle$ or $\langle E_\sigma \rangle$, hence $\langle E_1 \rangle = N_1 \langle E_\alpha \rangle$ and $\langle E_2 \rangle = N_2 \langle E_\sigma \rangle$
- Each mode in a subsystem has an equal loss factor η_α or η_σ ; all modes have equal band width $\eta\omega$
- The response is determined by resonant modes, with the responses of non resonant modes being neglected

The average inter modal power flow $\Pi_{\alpha\sigma}$ between mode α and mode σ , in a frequency band $\Delta\omega$, is

$$\Pi_{\alpha\sigma} = \beta_{\alpha\sigma} [\langle E_{\alpha,\Delta\omega} \rangle - \langle E_{\sigma,\Delta\omega} \rangle]. \quad (4.85)$$

The total power flow $\Pi_{1\sigma}$ from all modes of subsystem 1 to mode σ in the frequency band $\delta\omega$ is given by

$$\Pi_{1\sigma} = \beta_{\alpha\sigma} N_1 [\langle E_{\alpha,\Delta\omega} \rangle - \langle E_{\sigma,\Delta\omega} \rangle]. \quad (4.86)$$

The total power flow Π_{12} between subsystem 1 and subsystem 2 in terms of the frequency band $\Delta\omega$ now becomes

$$\Pi_{12} = \beta_{\alpha\sigma} N_1 N_2 [\langle E_{\alpha,\Delta\omega} \rangle - \langle E_{\sigma,\Delta\omega} \rangle]. \quad (4.87)$$

The *total power flow* Π_{12} can be expressed in subsystem total energies $\langle E_1 \rangle$ and $\langle E_2 \rangle$, in the frequency band $\Delta\omega$, using (4.87)

$$\Pi_{12} = \beta_{\alpha\sigma} N_1 N_2 \left[\frac{\langle E_1 \rangle}{N_1} - \frac{\langle E_2 \rangle}{N_2} \right]. \quad (4.88)$$

The *coupling loss factor* η_{12} is defined, when subsystem 2 is blocked, as

$$\bar{\Pi}_{12} = \beta_{\alpha\sigma} N_1 N_2 \frac{\langle E_1 \rangle}{N_1} = \beta_{\alpha\sigma} N_2 \langle E_1 \rangle = \omega \eta_{12} \langle E_1 \rangle, \quad (4.89)$$

and the coupling loss factor η_{21} is defined, when subsystem 1 is blocked, as

$$\bar{\Pi}_{21} = \beta_{\alpha\sigma} N_1 N_2 \frac{\langle E_2 \rangle}{N_2} = \beta_{\alpha\sigma} N_1 \langle E_2 \rangle = \omega \eta_{21} \langle E_2 \rangle. \quad (4.90)$$

The total power flow Π_{12} can now be written as follows

$$\Pi_{12} = \bar{\Pi}_{12} - \bar{\Pi}_{21} = \omega \eta_{12} \langle E_1 \rangle - \omega \eta_{21} \langle E_2 \rangle. \quad (4.91)$$

The *reciprocity law* (also called *consistency*) for the coupling loss factors η_{12} and η_{21} can be easily derived, because

$$\eta_{12} N_2 = \eta_{21} N_1 = \frac{\beta_{\alpha\sigma} N_1 N_2}{\omega}. \quad (4.92)$$

The dissipated power $\Pi_{i,diss}$ per subsystem i is defined as follows

$$\Pi_{i,diss} = \omega \eta_i \langle E_i \rangle, \quad (4.93)$$

where η_i is the *dissipated loss factor* of subsystem i , and ω the center frequency of the frequency band with bandwidth $\Delta\omega$. The center frequencies may be defined in the octave or one-third octave bands.

The modal density n_i of subsystem i is the average number of modes N_i in the frequency band $\Delta\omega$, and is defined as in (4.84):

$$n_i = \frac{N_i}{\Delta\omega}. \quad (4.94)$$

The reciprocity law (consistency) of the coupling loss factors (4.92) can be written (no summation)

$$\eta_{ij} n_i = \eta_{ji} n_j. \quad (4.95)$$

Substituting (4.95) into (4.91), we obtain the following expression for the power exchange Π_{12} :

$$\Pi_{12} = \omega \eta_{12} \left[\langle E_1 \rangle - \frac{n_1}{n_2} \langle E_2 \rangle \right], \quad (4.96)$$

or

$$\Pi_{12} = \omega \eta_{12} n_1 \left[\frac{\langle E_1 \rangle}{n_1} - \frac{\langle E_2 \rangle}{n_2} \right]. \quad (4.97)$$

Figure 4.6 shows two coupled multi-mode subsystems. The power balance equations are

$$\Pi_{in,1} = \Pi_{12} + \Pi_{diss,1}, \quad (4.98)$$

$$\Pi_{in,2} = \Pi_{21} + \Pi_{diss,2}. \quad (4.99)$$

When rewritten using the modal densities of both subsystems 1 and 2 these equations become

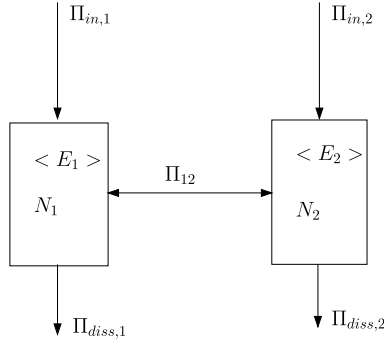


Fig. 4.6. Two interacting subsystems

$$\Pi_{in,1} = \omega \eta_{12} n_1 \left[\frac{\langle E_1 \rangle}{n_1} - \frac{\langle E_2 \rangle}{n_2} \right] + \omega \eta_1 \langle E_1 \rangle, \quad (4.100)$$

$$\Pi_{in,2} = \omega \eta_{21} n_2 \left[\frac{\langle E_2 \rangle}{n_2} - \frac{\langle E_1 \rangle}{n_1} \right] + \omega \eta_2 \langle E_2 \rangle. \quad (4.101)$$

In matrix notation these can be written as

$$\begin{pmatrix} \eta_1 + \eta_{12} & -\frac{\eta_{12}n_1}{n_2} \\ -\frac{\eta_{21}n_2}{n_1} & \eta_2 + \eta_{21} \end{pmatrix} \begin{Bmatrix} \langle E_1 \rangle \\ \langle E_2 \rangle \end{Bmatrix} = \frac{1}{\omega} \begin{Bmatrix} \Pi_{in,1} \\ \Pi_{in,2} \end{Bmatrix}. \quad (4.102)$$

Using the consistency of the modal densities (4.95), we can write (4.102) as follows

$$\begin{pmatrix} \eta_1 + \eta_{12} & -\eta_{21} \\ -\eta_{12} & \eta_2 + \eta_{21} \end{pmatrix} \begin{Bmatrix} \langle E_1 \rangle \\ \langle E_2 \rangle \end{Bmatrix} = \frac{1}{\omega} \begin{Bmatrix} \Pi_{in,1} \\ \Pi_{in,2} \end{Bmatrix}. \quad (4.103)$$

The *loss factor matrix* in (4.103) is not symmetric. The matrix can be made symmetrical multiplying and dividing with the appropriate model densities

$$\begin{pmatrix} (\eta_1 + \eta_{12})n_1 & -\eta_{12}n_1 \\ -\eta_{21}n_2 & (\eta_2 + \eta_{21})n_2 \end{pmatrix} \begin{Bmatrix} \frac{\langle E_1 \rangle}{n_1} \\ \frac{\langle E_2 \rangle}{n_2} \end{Bmatrix} = \frac{1}{\omega} \begin{Bmatrix} \Pi_{in,1} \\ \Pi_{in,2} \end{Bmatrix}. \quad (4.104)$$

The power flow equations for a structure made up of N subsystems is expressed in the following matrix form:

$$\begin{bmatrix} (\eta_1 + \sum_{i \neq 1}^N \eta_{1i})n_1 & -\eta_{12}n_1 & \cdots & -\eta_{1N}n_1 \\ -\eta_{21}n_2 & (\eta_2 + \sum_{i \neq 2}^N \eta_{2i})n_2 & \cdots & -\eta_{2N}n_2 \\ \vdots & \vdots & \ddots & \vdots \\ -\eta_{N1}n_N & \cdots & \cdots & (\eta_N + \sum_{i \neq N}^N \eta_{Ni})n_N \end{bmatrix} \times \begin{Bmatrix} \frac{\langle E_1 \rangle}{n_1} \\ \frac{\langle E_2 \rangle}{n_2} \\ \vdots \\ \frac{\langle E_N \rangle}{n_N} \end{Bmatrix} = \frac{1}{\omega} \begin{Bmatrix} \Pi_{in,1} \\ \Pi_{in,2} \\ \vdots \\ \Pi_{in,N} \end{Bmatrix}. \quad (4.105)$$

The following example is taken from [139] problem 6.1.

Example. As a first approximation, a spacecraft structure can be modelled as a flat aluminium platform coupled to a large aluminium cylinder, as illustrated in Fig. 4.7. The density of Al-alloy $\rho = 2700 \text{ kg/m}^3$. The Al-alloy plate is 5 mm thick and its area is 3.53 m^2 . The cylinder is 2 m long, has a mean diameter of 1.5 m, and a 3 mm wall thickness. The following information is available about the structure in the range 500 Hz octave band: the platform is driven directly and the cylinder is driven only via the coupling joints; the internal loss factor of the platform (subsystem 1) $\eta_1 = 4.4 \times 10^{-3}$, the internal loss factor of the cylinder (subsystem 2), $\eta_2 = 2.4 \times 10^{-3}$; the platform rms vibrational velocity is $\langle v_1 \rangle = 27.2 \text{ mm/s}$; and the cylinder rms velocity $\langle v_2 \rangle = 13.2 \text{ mm/s}$. Estimate the coupling loss factors η_{12} and η_{21} , and the input power $\Pi_{in,1}$.

The energies for the subsystems 1 and 2, the platform and cylinder respectively, are given in Table 4.1.

The total input power is dissipated in both subsystems 1 and 2, thus

$$\Pi_{in,1} = \omega \eta_1 \langle E_1 \rangle + \omega \eta_1 \langle E_1 \rangle = 1.3060 \text{ Nm/s},$$

and the power balance equations (4.104) of the platform and the cylinder are

$$\begin{pmatrix} (\eta_1 + \eta_{12})n_1 & -\eta_{12}n_1 \\ -\eta_{21}n_2 & (\eta_2 + \eta_{21})n_2 \end{pmatrix} \begin{Bmatrix} \langle \frac{E_1}{n_1} \rangle \\ \langle \frac{E_2}{n_2} \rangle \end{Bmatrix} = \frac{1}{\omega} \begin{Bmatrix} \Pi_{in,1} \\ 0 \end{Bmatrix}.$$

The second equation gives

$$\frac{\langle E_1 \rangle}{\langle E_2 \rangle} = \frac{\eta_{21}}{\eta_{21} + \eta_2} = 0.1525,$$

and with the power balance equation for the platform

$$(\eta_{12} + \eta_1) \langle E_1 \rangle - \eta_{21} \langle E_2 \rangle = \frac{\Pi_{in,1}}{\omega}.$$

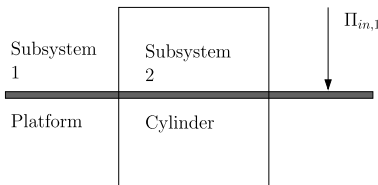


Fig. 4.7. Simple spacecraft structure

Table 4.1. Spacecraft properties

Structural part	Mass (kg)	Velocity (rms) (m/s)	Energy (Nm, J)
Platform	$M_1 = 117.8935$	$\langle v_1 \rangle = 27.2 \times 10^{-3}$	$\langle E_1 \rangle = 8.7222 \times 10^{-2}$
Cylinder	$M_2 = 76.3407$	$\langle v_2 \rangle = 13.2 \times 10^{-3}$	$\langle E_2 \rangle = 1.3302 \times 10^{-2}$

The coupling loss factors can now be calculated from the last two equations, $\eta_{12} = 4.26 \times 10^{-4}$ and $\eta_{21} = 3.92 \times 10^{-4}$.

4.5 SEA Parameters

To generate the SEA power balance equations, we need the following parameters:

- Dissipation loss factors η_i
- Coupling Loss factors η_{ij}
- Modal densities n_i
- Subsystem energies $\langle E_i \rangle$
- Source Power inputs Π_{in}

4.5.1 Dissipation Loss Factor

For one system the power balance equation tells us that the input power $\Pi_{in,s}$ is dissipated by the system, $\Pi_{diss,s} = \omega \eta_s \langle E_s \rangle$. That means that the dissipated (*intrinsic*) loss factor η_s for that system is

$$\eta_s = \frac{\Pi_{in,s}}{\omega \langle E_s \rangle}, \quad (4.106)$$

where $\langle E_s \rangle$ is the total energy.

The power input $\Pi_{in,i}$ of one SDOF i due to an applied force $F_i(\omega)$ has already been derived in (4.38), but is rewritten as follows

$$\Pi_{in,i} = \frac{\langle F_i^2 \rangle}{4 \Delta f_i m_i}, \quad (4.107)$$

where $\langle F_i^2 \rangle = W_f \Delta f_i$ is the mean square applied load, and m_i is the mass of SDOF i . Δf_i is the bandwidth.

For a multi mode system there are N_s modes (oscillators) in the frequency band Δf_i , equally mass and force distributed in Δf_i and the forces F_i are like rain on the roof and are spatially not correlated (spatially delta-correlated), thus $\langle F_i F_j \rangle = \delta_{ij} F_i^2$. For a multi mode system s the total input power $\Pi_{in,s}$ is written with the use of (4.107)

$$\Pi_{in,s} = \sum_{i=1}^{N_s} \Pi_{in,i} = \frac{N_s \langle F_i^2 \rangle}{4 \Delta f_i m_i} = \frac{\langle F_s^2 \rangle N_s}{4 \Delta f_i M_s} = \frac{n_s(f) \langle F_s^2 \rangle}{4 M_s}, \quad (4.108)$$

where the system has a total mass $M_s = N_s m_i$, $\langle F_s^2 \rangle = N_s \langle F_i^2 \rangle$ and a modal density $n_s(f)$ ($n_s(f) = 2\pi n_s(\omega)$). The total input power $\Pi_{s,in}$ can be expressed in terms of the mean square of the applied force F_s multiplied by

the real part of the mobility of the system at the location, where the force is applied

$$\Pi_{in,s} = \langle F_s^2 \rangle \Re\{Y^*(\omega)\} = \frac{n_s(f) \langle F_s^2 \rangle}{4M_s}. \quad (4.109)$$

From (4.109) it follows that the modal density $n_s(f)$ of the system can be expressed as follows

$$n_s(f) = 4M_s \Re\{Y^*(\omega)\}. \quad (4.110)$$

The modal density $n_s(f)$ of the system can be averaged over the frequency band $\Delta\omega = \omega_2 - \omega_1$ with a center frequency ω it can be calculated using the following expression

$$n_s(f) = \frac{4M_s}{\Delta\omega} \int_{\omega_1}^{\omega_2} \Re\{Y^*(\omega)\} d\omega = \frac{4M_s}{\Delta\omega} \int_{\omega_1}^{\omega_2} \Re\left\{\frac{1}{Z^*(\omega)}\right\} d\omega, \quad (4.111)$$

where $Z^*(\omega)$ is the impedance function.

Example. For a thin bending plate with bending stiffness D , density ρ and plate thickness t , and area A_p , the real part of the driving point mobility function can be approximated by that of an infinite thin bending plate $Y_\infty = \frac{1}{8\sqrt{D\rho t}}$. The modal density $n_p(f)$ of the infinite bending plate is

$$n_p(f) = 4M_p \Re\{Y^*(\omega)\} = \frac{A_p}{2} \sqrt{\frac{\rho t}{D}},$$

or

$$n_p(\omega) = \frac{A_p}{4\pi} \sqrt{\frac{\rho t}{D}}.$$

The total energy $\langle E_s \rangle$ of the system can be written as

$$\langle E_s \rangle = M_s \langle v_s^2 \rangle, \quad (4.112)$$

where $\langle v_s^2 \rangle$ is the spatial average of the mean square velocity squared at a center circular frequency ω .

The modal density $n(f) = \frac{n(\omega)}{2\pi}$ and (4.106) can now be written as follows

$$\eta_s = \frac{n_s(f) \langle F_s^2 \rangle}{4\pi\omega M_s^2 \langle v_s^2 \rangle} = \frac{n_s(f) \langle F_s^2 \rangle}{8\pi f M_s^2 \langle v_s^2 \rangle}. \quad (4.113)$$

Equation (4.113) is given in [163].

Equation (4.113) can be expressed in terms of the input power $\Pi_{in,s}$:

$$\eta_s = \frac{\Pi_{in,s}}{2\pi f M_s \langle v_s^2 \rangle} = \frac{\Pi_{in,s} 2\pi f}{M_s \langle a_s^2 \rangle}, \quad (4.114)$$

where $\langle a_s^2 \rangle$ is the spatially average acceleration. In [158], a band average loss factor is defined

$$\eta_s = \frac{1}{f_2 - f_1} \int_{f_1}^{f_2} \frac{\Pi_{in,s} 2\pi f}{M_s \langle a_s^2 \rangle} df. \quad (4.115)$$

In the following sections some typical values for the dissipated loss factor associated with structural elements will be given. Still it is considered essential to determine the dissipation loss factor of the system experimentally.

The dissipation loss factor η is often expressed in dB, denoted by $\bar{\eta}$, using [40]

$$\bar{\eta} = 10 \log \left(\frac{\eta}{10^{-12}} \right) \text{ dB}. \quad (4.116)$$

Plates

The dissipation **loss** factor η for a plate as reported in the literature [55, 163] can be obtained from the following expression

$$\eta = \begin{cases} 0.050, & \text{up to } f = 80 \text{ Hz;} \\ \frac{1.8}{f^{0.87}}, & 80 < f < f_{pivot} \text{ Hz;} \\ 0.002, & \text{above } f_{pivot} \text{ Hz,} \end{cases} \quad (4.117)$$

where the *pivot frequency* $f_{pivot} = 2500 \text{ Hz}$.

Sandwich Panels

For structures in sandwich design without equipment the dissipation loss factor can be obtained from [212]

$$\eta = \frac{0.3}{f^{0.63}}. \quad (4.118)$$

Here η decreases as frequency increases.

The built-up (sandwich) panel structure dissipation loss factor is estimated to be

$$\eta = \begin{cases} 0.050, & f \leq f_{pivot} \text{ Hz;} \\ 0.050 \left(\frac{f_{pivot}}{f} \right)^{0.5}, & f > f_{pivot} \text{ Hz,} \end{cases} \quad (4.119)$$

where f_{pivot} is the pivot frequency. In [37] the pivot frequency is $f_{pivot} = 500 \text{ Hz}$.

In [37] the attached resonant equipment has an estimated average loss factor to be $\eta = 0.06$. The loss factor is twice the (critical) damping ratio $\eta = 2\zeta$.

The *Jet Propulsion Laboratory* (JPL) used the following expressions for the loss factor in the vibro acoustic high frequency SEA response analysis of stowed solar arrays (solar panels) of their spacecraft Magellan, Mars Observer and TOPEX:

$$\eta = \begin{cases} 0.050, & f \leq f_{pivot} \text{ Hz;} \\ 0.050 \left(\frac{f_{pivot}}{f} \right), & f > f_{pivot} \text{ Hz,} \end{cases} \quad (4.120)$$

where the pivot frequency is $f_{pivot} = 250 \text{ Hz}$.

Table 4.2. Structural properties and dimensions spacecraft solar panels

Structural parameters	Magellan	Mars Observer	TOPEX
Length (m)	2.520	2.268	3.287
Width (m)	2.494	1.829	1.925
Face sheet thickness (mm)	0.381	0.254	0.305
Core thickness (mm)	12.7	25.4	34.29
Mass per unit of area (kg/m ²)	4.66	3.72	5.42
Face sheet material	Aluminum	Kevlar	Aluminum
Core material	Aluminum	Aluminum	Aluminum

The general structural properties of the JPL spacecraft solar panels are given in Table 4.2.

Cylinders

The loss factor of cylinders can vary from 0.002–0.03 for unstiffened cylinders and from 0.001–0.03 for stiffened cylinders. Above 3000 Hz, the loss factor is almost constant, 0.004 and 0.006 for unstiffened and stiffened cylinders, respectively. Several tests [157, 158] gave the average loss factor of a uniform open-ended cylinder as 0.00219. Radiation damping was neglected.

Acoustic Room

In acoustics, damping is measured from reverberation time T_R , the time required for the vibration energy to decrease by a factor 10^{-6} ; see [118]. Thus

$$e^{-2\pi f \eta T_R} = 10^{-6}, \quad (4.121)$$

which gives

$$\eta(f) = \frac{2.2}{f T_R} = \frac{13.816}{\omega T_R}, \quad (4.122)$$

where f is the center frequency.

4.5.2 Coupling Loss Factor

This section summarizes the coupling loss factors associated with some of the more common coupling joints, [24].

There are two approaches for deriving coupling loss factors for structures:

- The modal approach. In the modal approach, the couplings between individual modes are computed and an average taken over the modes in each frequency band, e.g. [42].
- The wave approach. In this approach, the coupling loss factor is related to the *power transmissibility* for semi-infinite structures, which is often easier to estimate than the average of couplings between modes of finite structures.

The power transmitted from the first to the second structure through the junction is then the energy lost by the first structure via the coupling. The coupling loss factor η_{12} has been defined as the energy lost per radian of motion relative to the total energy of the subsystem $\eta_{12}\omega\langle E_1 \rangle$, and the source of energy loss is the transmission through the *junction* at the boundary of the first subsystem. The coupling loss factor η_{ij} is related to the *power transmission coefficient* τ_{ij} as follows

$$\tau_{ij} = \frac{\Pi_{trans}}{\Pi_{inc}} = \frac{\omega\eta_{ij}\langle E_i \rangle}{c_g\langle E_i \rangle/l_f} = \frac{\omega\eta_{ij}l_f}{c_g}, \quad (4.123)$$

where Π_{trans} is the power transmitted through the junction, Π_{inc} is the power incident on the junction, l_f is the mean free path length between the incidents on the junction, and c_g is the *group velocity* as discussed in Appendix K. For a beam, the mean free length is the length of the source beam $l_f = L$, and for a plate $l_f = \frac{\pi A}{L_{ij}}$, where L_{ij} is the length of the junction. The power per length normal to the boundary $\Pi_{inc} = \frac{c_g\langle E_i \rangle}{A_i\pi}$ for a reverberant field [96].

L-beam Structures Coupling Loss Factors

Theoretical estimates to determine the coupling loss factors η_{ij} for beams assembled at a right angle, so-called L-beam structures, are available in [2]. They are given as functions of the *transmission coefficients* τ_{ij} between the two subsystems. The coupling loss factors η_{ij} may be calculated using the following expression

$$\eta_{ij} = \frac{c_{bi}\tau_{ij}}{\omega L_i}, \quad (4.124)$$

where $c_{bi} = \sqrt{\omega}^4 \sqrt{\frac{E_i I_i}{\rho_i A_i}}$ is the sound speed of the flexural waves, E_i is Young's modulus, I_i is the second moment of area, ρ_i is the density, A_i is the cross-section, L_i is the length of beam i and τ_{ij} is the transmission coefficient across the joint relating the incident waves in subsystem i to be transmitted in subsystem j . The transmission coefficient for each wave type may be calculated as follows

$$\tau_{bb} = \frac{2\beta^2 + 1}{9\beta^2 + 6\beta + 2}, \quad (4.125)$$

$$\tau_{bl} = \tau_{lb} = \frac{8\beta^2 + 5\beta}{9\beta^2 + 6\beta + 2}, \quad (4.126)$$

$$\tau_{ll} = \frac{\beta^2}{9\beta^2 + 6\beta + 2}, \quad (4.127)$$

where $\beta = \frac{c_{bi}}{c_{li}}$, with the longitudinal wave speed $c_{li} = \sqrt{\frac{E_i}{\rho_i}}$ and the index b stands for bending and the index l for longitudinal respectively.

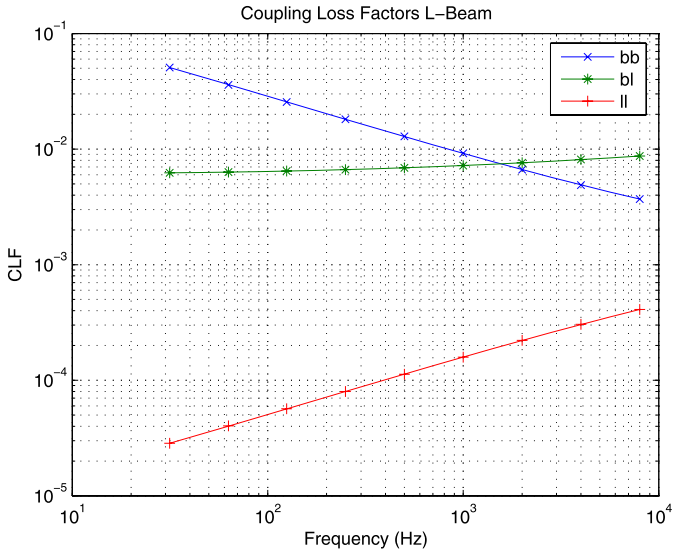


Fig. 4.8. Coupling loss factors L-beam structure

Example. A L-beam structure consists of two beams with the same properties: a length $L = 1$ m, a cross-section $A = b \times h = 0.03 \times 0.05$ m², a second moment of area $I = \frac{bh^3}{12} = 1.25 \times 10^{-7}$ m⁴. The beams are made of an Al-alloy with $E = 70$ GPa and a density $\rho = 2700$ kg/m³. Calculate the coupling loss factors η_{ij} for “bb”, “bl” and “ll” respectively in the octave band $f = 31.5\text{--}8000$ Hz. The coupling loss factors are given in Fig. 4.8.

Plate-to-Plate Coupling Loss Factors

The most commonly encountered structure-to-structure coupling is a line junction between two plate structures. The coupling loss factor for a line junction is given in terms of the wave transmission coefficient for the line junction [15]. The coupling loss factor of a line junction from plate 1 to plate 2 is given by

$$\eta_{12} = \frac{2c_B L \tau_{12}}{\pi \omega A_{p,1}} = \frac{2}{\pi} \frac{L}{k_1 A_{p,1}} \tau_{12}, \tag{4.128}$$

where the following properties of plate 1 are of importance:

$$c_B = \sqrt{\omega} \left\{ \frac{D}{\rho t} \right\}^{0.25} = \sqrt{t c_l \omega} \sqrt{\frac{1}{12}}$$

is the *bending wave (phase) velocity* (or phase velocity), t is the thickness of the plate, D is the bending stiffness of the plate, ρ is the density of plate,

L is the length of the junction of the two plates, ω is the radian frequency, $A_{p,1}$ is the area of plate 1 and τ_{12} the *wave transmission coefficient*. The speed of sound in plane of the plate 1 $c_l = \sqrt{\frac{E}{\rho(1-\nu^2)}}$, the wave number is $k_1 = \sqrt{\omega} \sqrt[4]{\frac{\rho t}{D}}$ and ν is Poisson's ratio. The transmission coefficient τ_{12} for *random incidence* vibrational energy of two L-shape plates can be calculated using the approximate formula, [15],

$$\tau_{12} = \tau_{12}(0) \frac{2.754X}{(1 + 3.24X)}, \quad (4.129)$$

where $X = \frac{t_1}{t_2}$. The *normal transmission coefficient* $\tau_{12}(0)$ for thin plates may be calculated as follows

$$\tau_{12}(0) = 2[\psi^{0.5} + \psi^{-0.5}]^{-2}, \quad (4.130)$$

where

$$\psi = \frac{\rho_1 c_{l,1}^{\frac{3}{2}} t_1^{\frac{5}{2}}}{\rho_2 c_{l,2}^{\frac{3}{2}} t_2^{\frac{5}{2}}}. \quad (4.131)$$

When both plates are made of the same sheet material, (4.130) becomes

$$\tau_{12}(0) = 2[X^{\frac{5}{4}} + X^{-\frac{5}{4}}]^{-2}. \quad (4.132)$$

Example. Two identical Al-alloy square plates are firmly connected in a L-shape form (right angle). The Young's modulus is $E = 70$ GPa, the Poisson's ratio is $\nu = 0.3$, the density is $\rho = 2700$ kg/m³, the length is equal to the width $a = b = 1$ m and the thickness of the plates is $t = 5$ mm. Calculate the coupling loss factor $\eta_{12}(f)$ in the octave band with center frequencies in the range $f = 31.5$ –8000 Hz.

The speed of the flexural waves $c_b = 622.2972$ m/s, the longitudinal speed in the plates is $c_l = 5091.8$ m/s. The normal transmission coefficient $\tau_{12}(0) = 0.5$. The coupling loss factors $\eta_{12}(f)$ are shown in Fig. 4.9.

Acoustic Radiation

General

The radiated power is given by

$$P_{rad} = R_{rad} \langle v_p^2 \rangle = \rho c A_p \langle v_p^2 \rangle \sigma_{rad} = \eta_{sa} \omega M_p \langle v_p^2 \rangle, \quad (4.133)$$

where η_{sa} is the coupling loss factor from the structure to the acoustic room, R_{rad} is the radiation resistance, ρc is the *acoustic impedance*, A_p is the area of the panel, ρ is the density of surrounding fluid (air), c the speed of sound

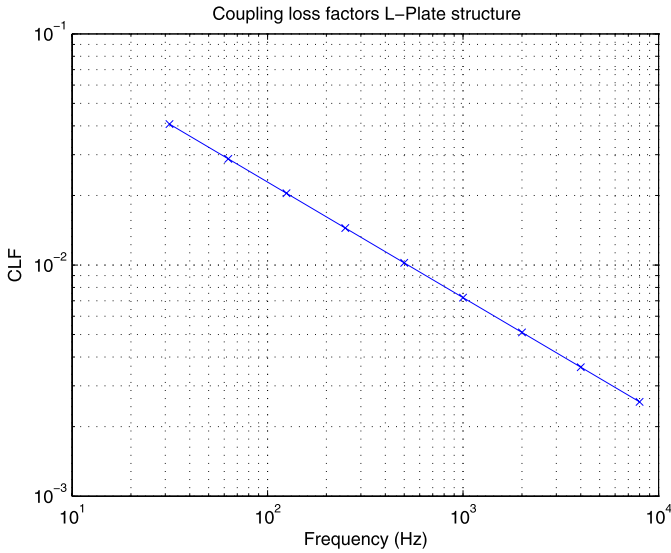


Fig. 4.9. Coupling loss factors L-shape plates

in the surrounding fluid, σ_{rad} is the *radiation efficiency*, M_p the total mass of the panel and ω the radian frequency. The coupling loss factor of a panel, for a structure acoustic volume coupling, can now be obtained

$$\eta_{sa} = \frac{R_{rad}}{M_p \omega} = \frac{A_p \rho c \sigma_{rad}}{M_p \omega}. \tag{4.134}$$

Using the reciprocity law for coupling loss factors, we can obtain the following expression for the coupling factor from the acoustic space to the structure:

$$\eta_{as} = \frac{R_{rad}}{M_p \omega} \frac{n_s}{n_{as}} = \frac{A_p \rho c \sigma_{rad}}{M_p \omega} \frac{n_s}{n_{as}}, \tag{4.135}$$

where n_s is the modal density of the panel and n_{as} is the modal density of the *acoustic space*. The dimension of the radiation resistance R_{rad} is $W/(m/s)^2$ ($W = N\ m/s$).

Plates

The panel radiation of a plate to half space is given by [46, 121, 161]

$$R_{rad} = A_p \rho c \begin{cases} \frac{\lambda_c \lambda_a}{A_p} 2 \frac{f}{f_{cr}} g_1 \left(\frac{f}{f_{cr}} \right) + \frac{P \lambda_c}{A_p} g_2 \left(\frac{f}{f_{cr}} \right), & f < f_{cr}, ka, kb > 2; \\ \frac{4}{\pi^4} \frac{P \lambda_c}{A_p} \left(\frac{f}{f_{cr}} \right)^{\frac{1}{2}}, & f < f_{cr}, ka, kb < 2; \\ \left(\frac{a}{\lambda_c} \right)^{\frac{1}{2}} + \left(\frac{b}{\lambda_c} \right)^{\frac{1}{2}}, & f = f_{cr}; \\ \left(1 - \frac{f_{cr}}{f} \right)^{-\frac{1}{2}}, & f > f_{cr}, \end{cases} \tag{4.136}$$

where a and b are the length and width of the panel and further

- A_p the panel surface area of one side of the panel
- c the speed of sound
- c_b the wave speed of bending in the panel
- f the frequency
- f_{cr} the critical frequency of the panel
- $k = \frac{\omega}{c}$ is the wave number
- P the perimeter of the panel
- $\lambda_a = \frac{c}{f}$ the acoustic wavelength
- $\lambda_c = \frac{c_b}{f_{cr}}$ the critical wavelength of the panel at the critical frequency f_{cr}
- ρ is the density of the air (fluid)

and $g_1(\frac{f}{f_{cr}})$ and $g_2(\frac{f}{f_{cr}})$ are defined as follows

$$g_1\left(\frac{f}{f_{cr}}\right) = \begin{cases} \left(\frac{4}{\pi^4}\right) \frac{1-2\alpha^2}{\alpha\sqrt{(1-\alpha^2)}}, & f < \frac{1}{2}f_{cr} \\ 0, & f > \frac{1}{2}f_{cr} \end{cases},$$

$$g_2\left(\frac{f}{f_{cr}}\right) = \frac{1}{(2\pi)^2} \frac{\{(1-\alpha^2)\ln[\frac{1+\alpha}{1-\alpha}] + 2\alpha\}}{\sqrt{(1-\alpha^2)^3}},$$

where $\alpha = \sqrt{(\frac{f}{f_{cr}})}$.

In the book of Craik [40] Leppington has been cited for expressions of the radiation efficiency σ

$$\sigma = \begin{cases} \frac{Pc}{4\pi^2\sqrt{ff_{cr}A_p\sqrt{\alpha^2-1}}}\left[\ln\left\{\frac{1+\alpha}{1-\alpha}\right\} + \frac{2\alpha}{\alpha^2-1}\right], & f < f_{cr}; \\ a\sqrt{\frac{2\pi f}{c}}(0.5 - 0.15\frac{a}{b}), & f = f_{cr}, a < b; \\ (1 - \frac{f_{cr}}{f})^{-\frac{1}{2}}, & f > f_{cr}. \end{cases} \quad (4.137)$$

With respect to (4.136) $g_1 = 0$ and the radiation efficiency had been corrected at $f = f_{cr}$.

Example. A sandwich panel has dimensions $a = 2.5$ m and $b = 2.5$ m. The critical frequency is $f_{cr} = 500$ Hz. Calculate the radiation resistance in the octave band, 32.5–8000 Hz, using Leppington's equations. The speed of sound in air is $c = 340$ m/s. The radiation efficiency is illustrated in Fig. 4.10.

The subcritical baffled plate radiation efficiency $\sigma_{baf} = \sigma_{corner} + \sigma_{edge}$ when $f < f_{cr}$ in which σ_{corner} and σ_{edge} are the modal average radiation efficiency for the so-called *corner and edge modes* [141]. The σ_{corner} is

$$\sigma_{corner} = \frac{8}{\pi^4} \left(\frac{\lambda_c^2}{A_p}\right) \begin{cases} \left(\frac{4}{\pi^4}\right) \frac{1-2\alpha^2}{\alpha\sqrt{(1-\alpha^2)}}, & f < \frac{1}{2}f_{cr} \\ 0, & f > \frac{1}{2}f_{cr} \end{cases}, \quad (4.138)$$

and σ_{edge} is given by

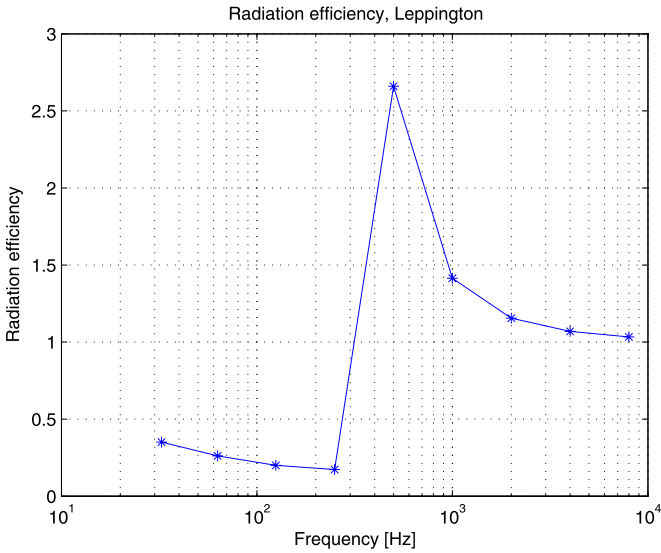


Fig. 4.10. Radiation efficiency

$$\sigma_{edge} = \frac{1}{4\pi^2} \left(\frac{P\lambda_c}{A_p} \right) \frac{\{(1 - \alpha^2) \ln[\frac{1+\alpha}{1-\alpha}] + 2\alpha\}}{\sqrt{(1 - \alpha^2)^3}}. \tag{4.139}$$

The subcritical unbaffled plate radiation efficiency is given by [141]

$$\sigma_{unb} = F_{plate}(F_{corner}\sigma_{corner} + F_{edge}\sigma_{edge}), \quad f < f_{cr}. \tag{4.140}$$

Here F_{plate} is a plate correction that accounts for the effect of inertial flows that surround the plate at low frequencies where the acoustic wavelength exceeds the dimensions of the plate. The local corrections F_{corner} and F_{edge} account for the effect on radiation from corner and edge modes at higher frequencies due to localized inertial flows near the plate perimeter. The correction factors are

$$F_{plate} = \frac{53f^4 \frac{A_p^2}{c^4}}{1 + 53f^4 \frac{A_p^2}{c^4}}, \tag{4.141}$$

$$F_{corner} = \frac{1}{2} \left[\frac{13 \frac{f}{f_{cr}}}{1 + 13 \frac{f}{f_{cr}}} \right], \tag{4.142}$$

and

$$F_{edge} = \frac{1}{2} \left[\frac{49 \frac{f}{f_{cr}}}{1 + 49 \frac{f}{f_{cr}}} \right]. \tag{4.143}$$

Conlon in his paper [37] presented a very simple equation for the radiation efficiency σ_{rad} of a baffled complex sandwich panel, loaded and unloaded

$$\sigma_{rad} = \begin{cases} 0.47\left(\frac{k_a}{k_p}\right)^{2.24}, & k_a < 1.5k_p; \\ 1.0, & k_a > 1.5k_p, \end{cases} \quad (4.144)$$

where k_a is the acoustic wave number and k_p is the unloaded panel wave number.

Stiffened Panels

The radiation resistance of a stiffened panel is discussed in [60, 121] and is given by

$$R_{rad} = R_{rad,panel} + R_{rad,rib}, \quad (4.145)$$

where $R_{rad,panel}$ is the radiation resistance of the panel as discussed in the previous section and $R_{rad,rib}$ is the radiation resistance per unit length of the rib (stiffener) and is given by

$$R_{rad,rib} = \rho c \lambda_c g_3 \left(\frac{f}{f_{cr}} \right). \quad (4.146)$$

The function g_3 is given by a complicated expression which includes an elliptic integral and is given in [121].

$$\begin{aligned} g_3 = & \left(\frac{1}{2\pi} \right)^2 \left\{ (2 - \alpha^2) \ln \left[\frac{1 + \alpha}{1 - \alpha} \right] - \frac{1}{3} \alpha^3 (1 + 2\alpha^2) \right\} \\ & \times (1 - \alpha^2)^{-\frac{3}{2}} + \left[2\sqrt{2}(1 + \alpha^2) F \left(\frac{1}{\sqrt{2}}, \phi \right) + \sqrt{2}(2 + \alpha^2) E \left(\frac{1}{\sqrt{2}}, \phi \right) \right] \\ & \times \left(1 - \frac{1}{2} \alpha^4 \right) (1 + \alpha^2)^{-\frac{3}{2}}, \end{aligned} \quad (4.147)$$

where $\alpha = \frac{f}{f_{cr}}$ and $\phi = \cos^{-1} \left[\frac{1 - \alpha^2}{1 + \alpha^2} \right]^{\frac{1}{2}}$ and F and E are Legendre's elliptic integrals of the first and second kind, respectively. The elliptic integral of the first type is [90]

$$F(k, \phi) = \int_0^\phi \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}}, \quad (4.148)$$

and the elliptic integral of the second type is given by

$$E(k, \phi) = \int_0^\phi \sqrt{1 - k^2 \sin^2 \theta} d\theta, \quad (4.149)$$

with $0 < k^2 < 1$. Quite often in numeric tables the modulus k is replaced by $\sin \theta$.

The approximation of g_3 is as follows [60]

$$10 \log g_3 = \begin{cases} a + b \times \left(\frac{f}{f_{cr}} \right), & 0.01 \leq \frac{f}{f_{cr}} < 0.3; \\ c + d \times \ln \left(\frac{f}{f_{cr}} \right), & 0.3 \leq \frac{f}{f_{cr}} < 0.9, \end{cases} \quad (4.150)$$

where $a = -6.15$, $b = 2.33$, $c = 2.91$ and $d = 11.87$.

Unstiffened Cylinder

In [205] the average radiation efficiency of modes in frequency bands containing some acoustically fast (AF) modes of an unstiffened cylinder is derived. The average radiation efficiency σ_{rad} is

$$\sigma_{rad} = \frac{\nu_0^{3/2} f_R/f_c}{2B(F - 1/F)} \left(1 - \nu_0 \sqrt{1 - \nu_0^2 (f_R/f_c)^2} \right) \times \left[\frac{1}{\sqrt{1/F - \nu_0}} - \frac{1}{\sqrt{F - \nu_0}} \right] \sqrt{12(1 - \nu^2)}, \quad (4.151)$$

where $\nu_0 = \frac{f}{f_R}$, f is the center frequency, the ring frequency is $f_R = \frac{1}{2\pi R} \sqrt{\frac{E}{\rho_m}}$, R is the radius of the cylinder, E is Young’s modulus, ρ_m is density of the applied material, ν is Poisson’s ratio and the critical frequency is $f_c = \frac{c^2}{2\pi t} \sqrt{\frac{12\rho_m(1-\nu^2)}{E}}$. The frequency bandwidth factor F is

- the upper frequency limit $f_u = fF$ and
- the lower frequency limit $f_l = f/F$

The frequency bandwidth factor is $F = 1.122$ for the one-third octave band and $F = 1.414$ for the octave band. The modal density function B is given in Table 4.3. Equation (4.151) is always valid when:

1. $\nu_0 < 1/F$ and $\nu_0 < 0.65 \log(3f_c/f_R)$
2. $f_R/f_c > 1.5$ and $\nu_0 < f_c/f_R$.

For small values of ν_0 and f_R/f_c (4.151) simplifies to

$$\sigma_{rad} = \frac{\nu_0^{3/2} f_R/f_c}{2B} \left[\frac{F^{1/2} - (1/F)^{1/2}}{F - 1/F} \right] \sqrt{12(1 - \nu^2)}, \quad (4.152)$$

and if only one-third octave band or narrower bands are considered

$$\left[\frac{F^{1/2} - (1/F)^{1/2}}{F - 1/F} \right] \simeq \frac{1}{2},$$

we obtain

$$\sigma_{rad} \simeq \frac{\nu_0^{3/2} f_R/f_c}{4B} \sqrt{12(1 - \nu^2)}. \quad (4.153)$$

Table 4.3. Factor B (modal density) unstiffened cylinder

Factor B	Applicable frequency range ν_0
$B = 2.5\sqrt{\nu_0}$	$\nu_0 \leq 0.48$
$B = 3.6\nu_0$	$0.48 < \nu_0 \leq 0.83$
$B = 2 + \frac{0.23}{F - \frac{1}{F}} \left[F \cos\left(\frac{1.745}{F^2 \nu_0^2}\right) - \frac{1}{F} \cos\left(\frac{1.745 F^2}{\nu_0^2}\right) \right]$	$\nu_0 > 0.83$

In [205] it is discussed that the internal radiation may taken equal to the external radiation which had been derived in this section.

Some cylinder characteristic parameters for aluminium cylinders are:

- $\nu = 0.33$
- $E = 70 \text{ GPa}$
- $\rho = 2700 \text{ kg/m}^3$
- $f_R \approx 810/R(\text{m}) \text{ Hz}$
- $f_R/f_c \approx 68.6(t/R)$
- $f_c \approx 11.8/t(\text{m}) \text{ Hz}$

with the associated parameters for air at normal temperature and pressure

- $c = 340 \text{ m/s}$
- $\rho \approx 1.2 \text{ kg/m}^3$
- $\rho c \approx 408 \text{ kg s/m}^2$

Critical Frequency

The derivation of the *critical frequency* f_{cr} is discussed in detail in [159]. When an infinite plate is excited, the frequency at which the speed of the free bending wave becomes equal to the speed of the acoustic wave in air is called the critical frequency. The derivation of the critical frequency for a isotropic thin plate will be summarized. The critical frequencies for thick isotropic thick plates, thin and thick composite panels are given Table 4.4, where

- $\alpha = \frac{D_{12}+2D_{66}}{D}$, see Appendix J
- c speed of sound in the fluid (i.e. air)
- $\bar{D} = D_{11} = D_{22}$ is the bending stiffness in x - and y -direction respectively for a balanced lay-up of layers see Appendix J
- $D = \frac{Et(h+t)^2}{2(1-\nu^2)}$ for a sandwich panel with isotropic face sheets
- $D = \frac{Et^3}{12(1-\nu^2)}$ for a isotropic plate
- E is Young's modulus
- G is the shear modulus (isotropic assumed)

Table 4.4. Critical frequencies [159]

Type of panels	Critical Frequency ω_{cr}	Critical Frequency f_{cr} (Hz)
Thin isotropic plate	$\omega_{cr} = c^2 \sqrt{\frac{m}{D}}$	$f_{cr} = \frac{c^2}{2\pi} \sqrt{\frac{m}{D}}$
Thick isotropic plate	$\omega_{cr} = \frac{c^2 \sqrt{\frac{m}{D}}}{\sqrt{1 - \frac{c^2 m}{S}}}$	$f_{cr} = \frac{c^2 \sqrt{\frac{m}{D}}}{2\pi \sqrt{1 - \frac{c^2 m}{S}}}$
Thin composite panel	$\omega_{cr} = \frac{c^2 \sqrt{\frac{m}{D}}}{\sqrt{\frac{3+\alpha}{4}}}$	$f_{cr} = \frac{c^2 \sqrt{\frac{m}{D}}}{2\pi \sqrt{\frac{3+\alpha}{4}}}$
Thick composite panel	$\omega_{cr} = \frac{c^2 \sqrt{\frac{m}{D}}}{\sqrt{\frac{3+\alpha}{4} - \frac{c^2 m}{S}}}$	$f_{cr} = \frac{c^2 \sqrt{\frac{m}{D}}}{2\pi \sqrt{\frac{3+\alpha}{4} - \frac{c^2 m}{S}}}$

- h is the core height
- m is the mass per unit of area
- $S = Gh$ is the shear stiffness for a thick plate
- $S = Gh(1 + \frac{t}{h})^2$ is the shear stiffness for a sandwich panel
- t is thickness of plate or the face sheet thickness (both face sheets equal thickness)
- $\frac{c^2 m}{S} \geq \frac{3+\alpha}{4}$ for a composite sandwich panel
- $\frac{c^2 m}{S} \geq 1$ for a thick isotropic plate sandwich panel

The free vibration equation of motion for a thin isotropic plate is given by

$$\nabla^4 w + \frac{m}{D} \frac{\partial^2 w}{\partial t^2} = 0, \tag{4.154}$$

where the plate is assumed to be in the x - y plane and $\nabla^4 = \frac{\partial^4}{\partial x^4} + 2\frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4}$. The solution for the infinite plate may be written

$$w = e^{j(\omega t - k_x x - k_y y)}, \tag{4.155}$$

where k_x and k_y are the wave number components. They are related by the expression $k_x^2 + k_y^2 = k^2$, where k is the wave number. The wave number can be written as $k = \frac{\omega}{c}$. For a thin plate the wave number components can be expressed as follows by substituting (4.155) in (4.154)

$$k_x^4 + 2k_x^2 k_y^2 + k_y^4 = k^4 = \frac{m\omega^2}{D}. \tag{4.156}$$

The speed of the bending wave (phase velocity) $c_b = \frac{\omega}{k}$ becomes

$$c_b^4 = \frac{\omega^2 D}{m}. \tag{4.157}$$

At the critical frequency $c_b = c$. The critical frequency ω_{cr} for the thin isotropic plate is given by

$$\omega_{cr}^2 = \frac{c^4 m}{D}, \tag{4.158}$$

or

$$f_{cr} = \frac{c^2}{2\pi} \sqrt{\frac{m}{D}}. \tag{4.159}$$

Example. A sandwich panel with dimensions $a = 2.5$ m and $b = 2.25$ m has a mass per unit of area $m = 2.5$ kg/m². The core height $h = 20$ mm and both face sheets have a thickness $t = 0.2$ mm. The flexural bending stiffness of the composite sandwich panel is $\bar{D} = D_{11} = D_{22} = 5000$ Nm, $D_{12} = 70$ Nm and $D_{66} = 165$ Nm. The shear modulus of the core is $G = 8.2 \times 10^7$ N/m². The shear stiffness is $S = Gh(1 + \frac{t}{h})^2 = 16.7 \times 10^5$ N/m. The speed of sound in air $c = 340$ m/s. Calculate the critical frequency f_{cr} for

- the composite sandwich panel
- the sandwich panel in case the face sheets are made of an isotropic Al-alloy with $E = 70$ GPa and $\nu = 0.3$.

For the composite sandwich panel the critical frequency is given by

$$f_{cr} = \frac{c^2 \sqrt{\frac{m}{D}}}{2\pi \sqrt{\frac{3+\alpha}{4} - \frac{c^2 m}{S}}} = 532.3 \text{ Hz},$$

where $\alpha = \frac{D_{12}+2D_{66}}{D} = 0.08$ and for the sandwich panel with isotropic face sheets the critical frequency is

$$f_{cr} = \frac{c^2 \sqrt{\frac{m}{D}}}{2\pi \sqrt{1 - \frac{c^2 m}{S}}} = 570.9 \text{ Hz},$$

where $D = \frac{Et(h+t)^2}{2(1-\nu^2)} = 3139 \text{ Nm}$.

4.5.3 Modal Densities

General

The modal density $n(\omega)$ is the number of modes per radian frequency (rad/s) and the modal density $n(f)$ is the number of modes per cyclic frequency (Hz). The relation between $n(\omega)$ and $n(f)$ is given by

$$n(\omega) = \frac{N}{\Delta\omega} = \frac{N}{2\pi\Delta f} = \frac{n(f)}{2\pi}, \quad (4.160)$$

where N is the *mode count* in the frequency band $\Delta\omega$ and Δf .

A compendium of modal densities for typical structural elements of engineering importance is given in [79].

Composite Structure

A composite structure consists of a number of structural elements for which the modal density $n_i(\omega)$ is known. In [79] it is postulated that the modal density of the composite structure is the sum of the modal densities of its components. The modal density of the composite consisting of m basic structural elements at a center frequency ω is given by

$$n(\omega) = \sum_{j=1}^m n_j(\omega). \quad (4.161)$$

General Structure

The expression for the modal density $n(f)$ of a general structure is given in [33] as

$$n(f) = \frac{1}{(f_2 - f_1)} \int_{f_1}^{f_2} 4M\Re(Y)df, \quad (4.162)$$

where $\Re(Y)$ is the real part of the point mobility, and M is the mass of the structure. The point mobility is the ratio between the velocity response of the structure measured at the excitation location and the force input to the structure.

Bending Beam

The natural frequencies of a simply supported bending beam are given by

$$\omega_p = \left(\frac{p\pi}{L}\right)^2 \sqrt{\frac{EI}{m}}, \quad p = 1, 2, \dots, n, \quad (4.163)$$

with associated vibration modes $\phi_p(x) = \sin \frac{p\pi x}{L}$. Here EI is the bending stiffness, m is the mass per unit of length, L is the length of the beam. Equation (4.163) can be rewritten such that mode count $N(\omega)$, is dependent on the maximum frequency ω :

$$N(\omega) = \frac{L}{\pi} \sqrt{\omega}^4 \sqrt{\frac{m}{EI}} = \frac{1}{\Delta k} \sqrt{\omega}^4 \sqrt{\frac{m}{EI}} = \frac{1}{\Delta k} k_b. \quad (4.164)$$

Here $\Delta k = \frac{L}{\pi}$, and $k_b = \sqrt{\omega}^4 \sqrt{\frac{m}{EI}}$ is the wave number of the bending beam. The equation of motion of the bending beam is $EI \frac{\partial^4 w}{\partial x^4} + m \frac{\partial^2 w}{\partial t^2} = 0$. Substituting the solution $w(x, t) = e^{j(\omega t - k_b x)}$ we can obtain the expression for the wave number k_b . The modal density of the bending beam $n(\omega)$ now becomes

$$n(\omega) = \frac{dN(\omega)}{d\omega} = \frac{1}{\Delta k} \frac{dk_b}{d\omega} = \frac{L}{2\pi} \frac{1}{\sqrt{\omega}} \sqrt{\frac{m}{EI}}. \quad (4.165)$$

The modal density $n(f) = 2\pi n(\omega)$ can be easily obtained

$$n(f) = L \frac{1}{\sqrt{2\pi f}} \sqrt{\frac{m}{EI}}, \quad (4.166)$$

where f in (4.166) is the cyclic frequency (Hz). The modal $n(f)$ is inversely proportional to the square root of the frequency.

The speed of sound (phase velocity) in the bending beam is defined by [52] $c_p = \frac{\omega}{k_b} = \sqrt{\omega}^4 \sqrt{\frac{EI}{m}}$ and the *group speed* is given by $c_g = \frac{d\omega}{dk_b} = 2c_p$.

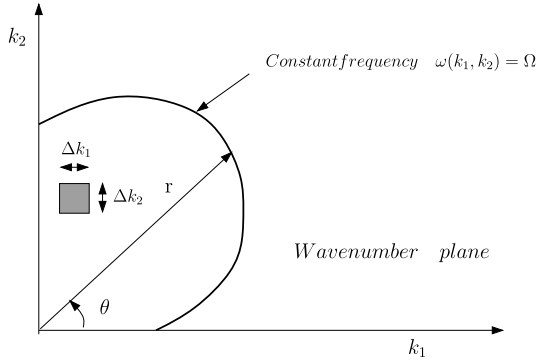


Fig. 4.11. The wave number plane [159]

Plates

After [21, 219] the number of modes $N(\Omega)$ can be calculated as follows assuming that the boundary conditions do not much effect the higher frequency range (asymptotic solution)

$$N(\Omega) \approx \frac{1}{\Delta k_1 \Delta k_2} \iint_S dk_1 dk_2, \tag{4.167}$$

where S is the region in the first quadrant of the k_1 and k_2 plane bounded by the curve $\omega(k_1, k_2) = \Omega$. This is illustrated in Fig. 4.11. The wave numbers k_1 and k_2 can be estimated by

$$k_1 = \frac{m_1 \pi}{a}, \quad k_2 = \frac{m_2 \pi}{b}, \quad m_1, m_2 = 1, 2, \dots, \tag{4.168}$$

where a and b are the sides of the rectangular area, and Δk_1 and Δk_2 are

$$\Delta k_1 = \frac{\pi}{a}, \quad \Delta k_2 = \frac{\pi}{b}. \tag{4.169}$$

For a simply supported plate the natural frequency $\omega(k_1, k_2)$ is given by

$$\omega^2(k_1, k_2) = \frac{D}{m} (k_1^2 + k_2^2)^2 = \frac{D}{m} r^4,$$

where D is the bending stiffness, m is the mass per unit of area and $r^2 = k_1^2 + k_2^2$. If $\frac{k_2}{k_1} = \tan \theta$, (4.167) can be written as follows

$$N(\Omega) \approx \frac{ab}{\pi^2} \int_0^{\frac{\pi}{2}} \int_0^r \eta d\eta d\theta = \frac{ab}{2\pi^2} \int_0^{\frac{\pi}{2}} r^2 d\theta, \tag{4.170}$$

where r is the maximum value for r for any given θ . The modal density $n(\omega)$ can derived using (4.170)

$$n(\omega) = \frac{N(\omega)}{d\omega} = \frac{ab}{2\pi^2} \int_0^{\frac{\pi}{2}} \frac{d(r^2)}{d\omega} d\theta. \tag{4.171}$$

Example. The natural frequency of a simply supported plate is given by

$$\omega^2(k_1, k_2) = \frac{D}{m}(k_1^2 + k_2^2)^2 = \frac{D}{m}r^4.$$

The radius r will be expressed in terms of the natural frequency ω as follows

$$r^2 = \omega \left(\frac{m}{D} \right)^{\frac{1}{2}}.$$

The derivative of r^2 with respect to ω becomes

$$\frac{d(r^2)}{d\omega} = \left(\frac{m}{D} \right)^{\frac{1}{2}}.$$

The modal density $n(\omega)$ can be obtained using (4.171)

$$n(\omega) = \frac{ab}{2\pi^2} \int_0^{\frac{\pi}{2}} \frac{d(r^2)}{d\omega} d\theta = \frac{ab}{2\pi^2} \left(\frac{m}{D} \right)^{\frac{1}{2}} \int_0^{\frac{\pi}{2}} d\theta = \frac{ab}{4\pi} \left(\frac{m}{D} \right)^{\frac{1}{2}};$$

the modal density $n(f)$ is

$$n(f) = \frac{ab}{2} \left(\frac{m}{D} \right)^{\frac{1}{2}}.$$

In the following example the modal density $n(f)$ for a plate will be illustrated numerically.

Example. A plate is made of an Al-alloy having dimensions of $a \times b = 2.19 \times 1.22$ m and thickness $t = 4.95$ mm. The Young's modulus of the material is $E = 72$ GPa, the Poisson's ratio $\nu = 0.3$ and the density is $\rho = 2800$ kg/m³. The modal density $n(f)$ is calculated using equation

$$n(f) = \frac{A}{2} \left(\frac{\rho t}{D} \right)^{\frac{1}{2}} = 0.176 \text{ modes/Hz},$$

where the area $A = a \times b$ and the bending stiffness (flexural rigidity) of the plate is given by $D = \frac{Et^3}{12(1-\nu^2)}$. The modal density of an isotropic plate is also called *Courant's density* [57].

The medium surrounding the plate is air with a speed of sound $c = 346$ m/s. The critical frequency of the plate f_{cr} is calculated using the equation

$$f_{cr} = \frac{c^2}{2\pi} \left(\frac{\rho t}{D} \right)^{\frac{1}{2}} = 2508 \text{ Hz}.$$

In a paper of Xie et al. [230], the modal density of rectangular plates is presented accounting for the boundary conditions. For a simply supported plate the modal density is given by

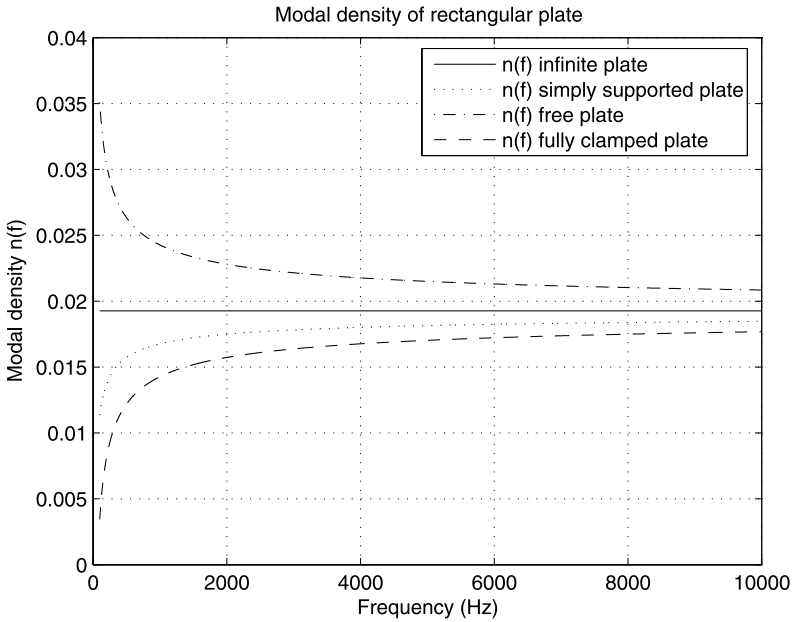


Fig. 4.12. Modal density rectangular plate

$$n(\omega) = \frac{A}{2\pi} \sqrt{\frac{m}{D}} - \frac{1}{4} \sqrt[4]{\frac{m}{D}} \frac{a+b}{\pi\sqrt{\omega}}, \tag{4.172}$$

the modal density of a free plate is given by

$$n(\omega) = \frac{A}{2\pi} \sqrt{\frac{m}{D}} + \frac{1}{2} \sqrt[4]{\frac{m}{D}} \frac{a+b}{\pi\sqrt{\omega}}, \tag{4.173}$$

and the modal density of a fully clamped plate is

$$n(\omega) = \frac{A}{2\pi} \sqrt{\frac{m}{D}} - \frac{1}{2} \sqrt[4]{\frac{m}{D}} \frac{a+b}{\pi\sqrt{\omega}}, \tag{4.174}$$

where $A = ab$, a and b are the length and the width of the plate, respectively. Remember that the modal density in the cyclic frequency domain is $n(f) = 2\pi n(\omega)$.

Example. The modal density for an Al-alloy rectangular plate, with a length $a = 0.4$ m, width $b = 0.3$ m and a thickness $t = 2$ mm with different boundary conditions is calculated. Young’s modulus is $E = 70$ GPa, Poisson’s ratio is $\nu = 0.33$ and the density is $\rho = 2700$ kg/m³. In Fig. 4.12 modal densities are shown.

Sandwich Panels

The modal densities of sandwich panels with identical isotropic face sheets and isotropic core are given in Table 4.5. The variables in the expressions for the modal densities are explained later:

Table 4.5. Modal density sandwich panels (isotropic face sheets and core)

Reference	Modal density n modes/Hz
[37]	$n(f) = \frac{A_p \omega}{c_{b,eff}^2} \left(1 - \frac{[\frac{1}{c_b^3} + \frac{1}{c_s^3}]^{-\frac{4}{3}}}{2c_{b,eff} c_b^3}\right)$
[159]	$n(f) = \frac{\pi A_p m_p f}{S} \left\{1 + \left(m_p^2 \omega^4 + \frac{4m_p \omega^2 S^2}{D}\right)^{-\frac{1}{2}} (m_p \omega^2 + \frac{2S^2}{D})\right\}$
[33]	$n(f) = \frac{\pi A_p m_p f}{gB} \times \left\{1 + [m_p^2 \omega^4 + 4m_p \omega^2 g^2 B(1 - \nu^2)]^{-\frac{1}{2}} [m_p \omega^2 + 2g^2 B(1 - \nu^2)]\right\}$

- A_p is the panel surface area
- $B = \frac{E(h+t_f)^2 t_f}{2} \approx \frac{Eh^2 t_f}{2}$
- c_b is the bending wave speed $c_b = \sqrt{\omega} [\frac{D}{m_p}]^{\frac{1}{4}}$
- c_s is the shear wave speed $c_s = [\frac{Gh}{m_p}]^{\frac{1}{2}}$
- $c_{b,eff}$ is the effective bending wave speed $c_b = [\frac{1}{c_b^3} + \frac{1}{c_s^3}]^{-\frac{1}{3}}$
- D is the section bending stiffness $D = \frac{Et_f(h+t_f)^2}{2(1-\nu^2)} \approx \frac{Eh^2 t_f}{2}$
- f is the cyclic frequency (Hz)
- $g = \frac{G}{h} [\frac{2}{Et_f}]$
- G is the shear modulus $G = \sqrt{G_L G_W}$
- h is the core height (thickness)
- m_p is the build-up panel mass/area
- $n(f)$ is the modal density in the cyclic frequency domain
- $n(\omega)$ is the modal density in the radian frequency domain $n(\omega) = \frac{n(f)}{2\pi}$
- S is the shear stiffness $S = Gh(1 + \frac{t_f}{h})^2 \approx Gh$
- t_f is the face sheet thickness
- ν is the Poisson's ratio of the face sheet material
- ω is the radian frequency (rad/s) $\omega = 2\pi f$

Example. A square sandwich panel with a surface area $A_p = 2.5 \times 2.5 \text{ m}^2$ has identical face sheets with a thickness $t_f = 0.3 \text{ mm}$, the core height is $h = 20 \text{ mm}$ and the mass per unit of area is $m_p = 2.5 \text{ kg/m}^2$. The face sheets are made of an Al-alloy with a Young's modulus $E = 70 \text{ GPa}$, and the Poisson's ratio is $\nu = 0.3$. The shear modulus of the core $G = 100 \text{ MPa}$. Calculate the modal density $n(f)$ modes/Hz using the equations written in Table 4.5 from [33, 37, 159]. The results of the modal density calculations, in the octave band $f = 31.5\text{--}8000 \text{ Hz}$, for all methods are about the same, and shown in Fig. 4.13.

The modal density for a composite sandwich panel had been derived in [159] and is recapitulated in Appendix J.

Unstiffened Cylinders

The ratio $\nu_0 = \frac{\omega}{\omega_r}$ is the ratio between the center frequency ω and the ring frequency $\omega_r = \frac{1}{R} (\frac{E}{\rho})^{\frac{1}{2}}$, where R is the radius of the cylinder, E is Young's

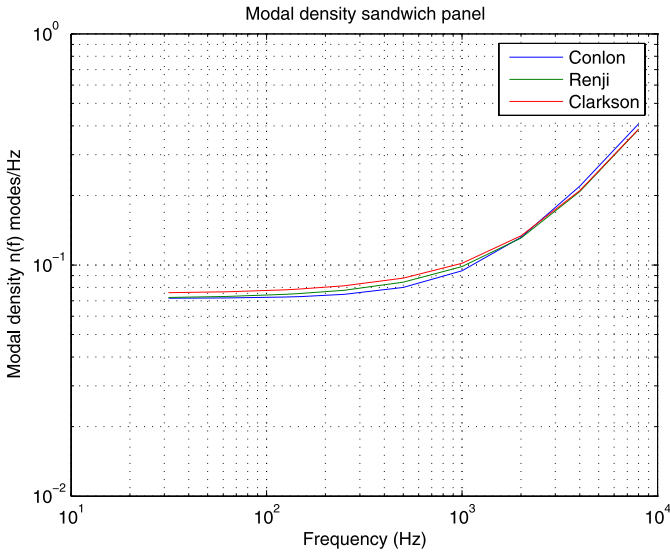


Fig. 4.13. Modal density sandwich panel [33, 37, 159]

Table 4.6. Factor B (modal density) unstiffened cylinder

Factor B	Applicable frequency range ν_0
$B = 2.5\sqrt{\nu_0}$	$\nu_0 \leq 0.48$
$B = 3.6\nu_0$	$0.48 < \nu_0 \leq 0.83$
$B = 2 + \frac{0.23}{F - \frac{1}{F}} [F \cos(\frac{1.745}{F^2 \nu_0^2}) - \frac{1}{F} \cos(\frac{1.745 F^2}{\nu_0^2})]$	$\nu_0 > 0.83$

modulus, and ρ the density of the structural material. The modal density of an unstiffened cylinder is taken from [55, 205] and can be calculated using a factor B

$$B = \frac{\pi n(f) t f_r}{2L} = \frac{n(f) \pi t C_l}{2A}, \quad n(f) = \frac{2BL}{\pi t f_r} = \frac{4AB}{\pi t C_l}, \quad (4.175)$$

where t is the wall thickness, L is the length of the cylinder, $C_l = \sqrt{\frac{E}{\rho}}$ is the longitudinal wave speed, A is the surface area of the cylinder and $n(f)$ is the modal density of the unstiffened cylinder. In Table 4.6 the approximate factors B are given in the following references [34, 55, 205]. The parameters used are listed below:

- $F = 2^{\frac{x}{6}}$
- $x = 1$ for the octave band width with $f_{max} = fF$ and $f_{min} = \frac{f}{F}$ becomes $F = 2^{\frac{1}{2}} = 1.414$ and
- $x = \frac{1}{3}$ for the third octave band width $f_{max} = fF$ and $f_{min} = \frac{f}{F}$ becomes $F = 2^{\frac{1}{6}} = 1.122$.

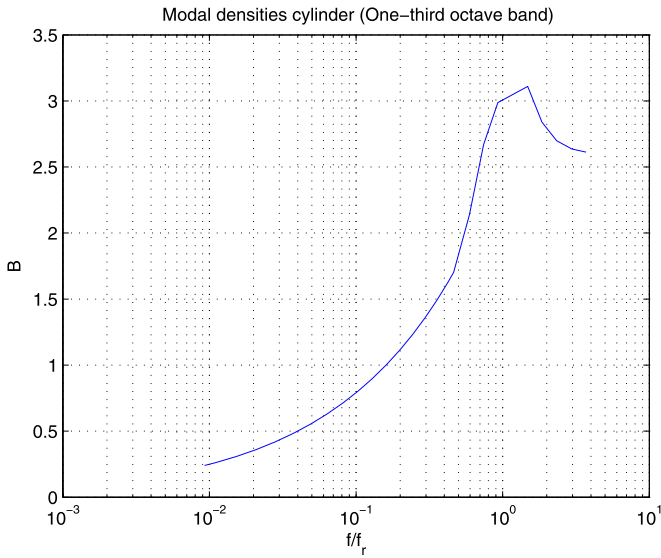


Fig. 4.14. Factor B calculated in the one-third octave band

The B factor for the one-third octave band is shown in Fig. 4.14.

Acoustic Chamber

The total number of modes (eigenfrequencies) from 0 to an upper limit frequency f in a rectangular room is given by [104]

$$N(f) = \frac{4\pi}{3}V\left(\frac{f}{c}\right)^3 + \frac{\pi}{4}A\left(\frac{f}{c}\right)^2 + \frac{L}{8}\left(\frac{f}{c}\right), \quad (4.176)$$

where $V = L_x L_y L_z$ is the real geometrical volume of the room, $A = 2(L_x L_y + L_x L_z + L_y L_z)$ is the area of all walls and $L = 4(L_x + L_y + L_z)$ is the sum of all edge lengths. The modal density of the acoustic room can now be easily derived from (4.176)

$$n(f) = \frac{dN(f)}{df} = \frac{4\pi}{c}V\left(\frac{f}{c}\right)^2 + \frac{\pi}{2c}A\left(\frac{f}{c}\right) + \frac{L}{8c}. \quad (4.177)$$

Modal Overlap

The modal overlap M_{jk} of two modes is the ratio of the bandwidth $\Delta = \eta\omega$ and the modal spacing $|\omega_j - \omega_k|$

$$M_{jk} = \frac{\Delta}{|\omega_j - \omega_k|}. \quad (4.178)$$

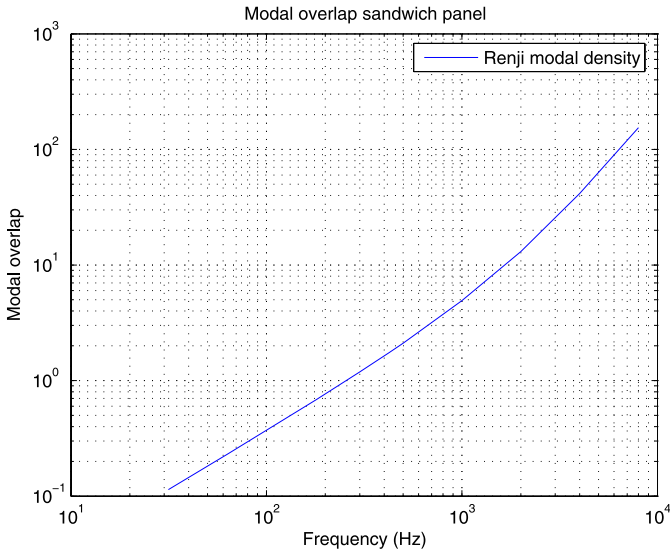


Fig. 4.15. Modal overlap of sandwich panel

The modal overlap of a system as a whole is given by

$$M = n(\omega)\Delta = n(\omega)\eta\omega = n(f)\eta f, \quad (4.179)$$

where η is the loss factor, ω (f) is the center frequency and $n(f)$ is the modal density.

For $M < 1$ deterministic methods can be used, e.g., *the finite element or boundary element methods*, but if $M > 1$ then statistical methods are more suitable.

Example. From the previous example of the modal densities for a sandwich panel with identical isotropic face sheets and core, the modal overlap will be calculated assuming $\eta = 0.05$ in the octave band $f = 31.5\text{--}8000$ Hz. The modal overlap, using the description of [159], is shown in Fig. 4.15. The modal overlap $M = 1$ at $f \approx 250$ Hz.

4.5.4 Subsystem Energies

Mechanical Subsystem

The total energy of a beam, plate or shell subsystem can be expressed in terms of the total mass M and the spatial average of the velocity response $\langle v^2 \rangle$

$$M\langle v^2 \rangle = \int_A m|v(x)|^2 dx, \quad (4.180)$$

where m is the mass per unit of length or area, $v(x)$ is the spatial velocity and x denotes the spatial location, one or two dimensional.

The velocity response $v(x)$ normal to the neutral line or plane can be expressed in a modal expansion of modes $\phi_n(x)$

$$v(x) = \sum_n v_n \phi_n(x), \tag{4.181}$$

where the mode functions are orthogonal, i.e.

$$\int_A m \phi_i(x) \phi_j(x) dx = \begin{cases} \Lambda_i, & i = j, \\ 0, & i \neq j, \end{cases} \tag{4.182}$$

where Λ_i is the generalized mass (*normalization constant*) for the mode function i .

For a single localized source, $p(x)$ is modelled by

$$p(x) = F_0 \delta(x - x_0), \tag{4.183}$$

where $\delta(x - x_0)$ is the Dirac delta function. The *modal participation* (generalized coordinate) $v_n(x)$ becomes

$$v_n(\omega) = \frac{j\omega F_0 \phi_n(x_0)}{[\omega_n^2 - \omega^2 + j\eta_n \omega \omega_n]}. \tag{4.184}$$

The average velocity v^2 can be evaluated using (4.180), thus

$$v^2 = \frac{1}{M} \sum_n \frac{\omega^2 F_0^2 \phi_n^2(x_0)}{[(\omega_n^2 - \omega^2)^2 + \eta_n^2 \omega^2 \omega_n^2] \Lambda_n}, \tag{4.185}$$

where the mode function originality was applied.

Uniformly distributed point sources (“rain on the roof”) are spatially averaged over the area A .

$$\langle v^2 \rangle = \frac{1}{AmM} \int_A \sum_n \frac{\omega^2 F_0^2 m \phi_n^2(x_0) dx_0}{[(\omega_n^2 - \omega^2)^2 + \eta_n^2 \omega^2 \omega_n^2] \Lambda_n}, \tag{4.186}$$

assuming a uniformly distributed mass per unit of length or area m . Again the mode function orthogonality property, we can express (4.186) as follows:

$$\langle v^2 \rangle = \frac{\omega^2 F_0^2}{M^2} \sum_n \frac{1}{[(\omega_n^2 - \omega^2)^2 + \eta_n^2 \omega^2 \omega_n^2]}. \tag{4.187}$$

Equation (4.187) shows that, for independent localized forces (or random broadband excitation), the input energy to all modes is uniform and indicates that the response of a subsystem to random broadband excitation is dominant at natural frequencies ω_n .

Averaging over the frequency band $\Delta\omega = \omega_1 \rightarrow \omega_2$ which contain $N = n_2 - n_1$ modes, we can find the following expression for (4.186):

$$\langle v^2 \rangle_{\Delta\omega} = \frac{F_{\Delta\omega}^2}{M^2} \sum_{n=n_1}^{n_2} \frac{1}{\Delta\omega} \int_{\Delta\omega} \frac{\omega^2 d\omega}{[(\omega_n^2 - \omega^2)^2 + \eta_n^2 \omega^2 \omega_n^2]}, \quad (4.188)$$

where $F_{\Delta\omega}^2$ is the mean square of the forces within the frequency band $\Delta\omega$. If $\omega_n \approx \omega$, the range of natural frequencies is small compared to the center frequency ω , and the loss factor η_n is small so that $\eta_n \omega_n \ll \omega$. Equation (4.188) becomes (see also Appendix I)

$$\langle v^2 \rangle_{\Delta\omega} = \frac{F_{\Delta\omega}^2}{\Delta\omega M^2} \sum_{n=n_1}^{n_2} \frac{\pi}{2\eta_n \omega_n}. \quad (4.189)$$

Assuming a constant loss factor η in the frequency band ω and $\omega_n \approx \omega$ for all modes n , we can evaluate (4.189) as follows:

$$\langle v^2 \rangle_{\Delta\omega} = \frac{F_{\Delta\omega}^2}{M^2} \frac{\pi}{2\eta\omega} \frac{N}{\Delta\omega} = \frac{F_{\Delta\omega}^2}{M^2} \frac{n(f)}{8\pi\eta f}. \quad (4.190)$$

The derivation of (4.190) is discussed in [24] and is in accordance with [102] where the *asymptotic modal analysis* (AMA) had been applied.

For one resonant mode $\phi_n(x)$ (SDOF system) the average energy is given by (4.35)

$$M \langle v^2 \rangle = \frac{F_{\Delta\omega}^2}{M} \frac{1}{8\pi\eta f_n}.$$

Acoustic Room

The temporally averaged energy of an acoustic volume (room) is known through its relationship to the mean square spatial average pressure

$$\langle E_{av} \rangle = \frac{\langle p^2 \rangle}{\rho c^2} V = \frac{M \langle p^2 \rangle}{Z_0^2}, \quad (4.191)$$

where $\frac{p^2}{\rho c^2}$ is called the *acoustic potential energy density* [185], $\langle p^2 \rangle$ is the mean square sound pressure averaged in space and time, which can be calculated with $\langle p^2(f) \rangle = p_{ref}^2 10^{\frac{SPL(f)}{10}}$, and $p_{ref} = 2.0 \times 10^{-5}$ Pa. $Z_0 = \rho c$ is the *specific acoustic impedance* or *characteristic impedance*¹ and $M = V\rho$ is the total mass: $SPL(f)$ is the sound pressure level, $SPL(f) = 10 \log(\frac{p^2}{p_{ref}^2})$ dB, V is the volume of the acoustic room, ρ is the density of the fluid in the acoustic room (air $\rho \approx 1.2$ kg/m³) and c is the ambient speed of sound in the fluid (air $c \approx 340$ m/s). The energy of an acoustic room can also be derived as follows.

¹ For air at normal condition $\rho c = 414$ kg/(m² s).

The energy of a subsystem is given by $\langle E \rangle = M\langle v^2 \rangle$. The relation between the pressure variation p and the velocity v in the fluid is given by $p = \rho cv$. Thus

$$\langle E_{av} \rangle = M\langle v^2 \rangle = V\rho \frac{\langle p^2 \rangle}{(\rho c)^2} = V \frac{\langle p^2 \rangle}{\rho c^2}. \quad (4.192)$$

What is the dimension of $\langle E_{av} \rangle$?

Example. Calculate the PSD of the pressure and the average energy in an acoustic room with a volume $V = 1600 \text{ m}^3$. The sound pressure level in the room is $SPL = 140 \text{ dB}$ ($p_{ref} = 2 \times 10^{-5} \text{ Pa}$) at the center frequency $f_c = 250 \text{ Hz}$ in the octave band. The speed of sound in the air is $c = 340 \text{ m/s}$, and the density of air $\rho = 1.2 \text{ kg/m}^3$.

The mean square of the pressure $\langle p^2 \rangle = (2 \times 10^{-5})^2 \times 10^{\frac{SPL}{10}} = 4.0 \times 10^4 \text{ Pa}^2$. The PSD of the pressure $W_p = \frac{\langle p^2 \rangle}{0.707f_c} = 226.274 \text{ Pa}^2/\text{Hz}$. The average energy in the acoustic room $\langle E_{av} \rangle = \frac{\langle p^2 \rangle}{\rho c^2} V = 461.361 \text{ Nm(W)}$.

4.5.5 Source Power Inputs

The average input power Π_{in} of a mechanical load to a subsystem is given by

$$\Pi_{in} = \frac{1}{2} F_{\Delta f}^2 \Re\{Y(\omega)\} = \frac{1}{2} F_{\Delta f}^2 \frac{n(f)}{4M}, \quad (4.193)$$

where $F_{\Delta f}^2$ is the mean square force in the frequency band Δf , $n(f)$ is the modal density and M is to the total mass of the subsystem. The spatial average of the mobility function $\Re\{Y(\omega)\}$ is derived in Appendix I.

Some examples of source power inputs $\Pi_{in,s}$ of some infinite mechanical systems are given in Table 4.7.

The symbols used in Table 4.7 are given below:

A	Cross-section area of beam
D	Bending stiffness of plate
E	Young's modulus
F	Force
G	Shear modulus
I	Second moment of area of beam
J	Torsion constant of beam
j	$\sqrt{-1}$
M	Bending moment
T	Torque
t	Thickness of plate
V	Velocity
$\dot{\theta}$	Angular velocity
ρ	Density

Table 4.7. Properties of infinite systems

System	Driving point mobility Y_∞	Power flow into system $\Pi_{in,s}$
Beam longitudinal wave motion; force excitation	$\frac{V}{F} = \frac{1}{2A\sqrt{E\rho}}$	$\frac{\langle F^2 \rangle}{4A\sqrt{E\rho}}$
Beam torsional wave motion; torque excitation	$\frac{\dot{\theta}}{T} = \frac{1}{2\sqrt{GJ_p}}$	$\frac{\langle T^2 \rangle}{4\sqrt{GJ_p}}$
Beam flexural wave motion; force excitation	$\frac{v}{F} = \frac{j-1}{4A\rho\sqrt{\omega}} \left(\frac{A\rho}{EI}\right)^{\frac{1}{4}}$	$\frac{\langle F^2 \rangle}{4A\rho\sqrt{\omega}} \left(\frac{A\rho}{EI}\right)^{\frac{1}{4}}$
Beam flexural wave motion; bending moment excitation	$\frac{v}{M} = \frac{(1+j)\sqrt{\omega}}{4EI} \left(\frac{EI}{A\rho}\right)^{\frac{1}{4}}$	$\frac{\langle M^2 \rangle\sqrt{\omega}}{8EI} \left(\frac{EI}{A\rho}\right)^{\frac{1}{4}}$
Plate flexural wave motion; force excitation	$\frac{v}{F} = \frac{1}{8\sqrt{D\rho t}}$	$\frac{\langle F^2 \rangle}{16\sqrt{D\rho t}}$

4.5.6 Stresses and Strains

The *strain energy* per unit of volume U^* in an elastic body can be expressed in the principal stresses and strains:

$$U^* = \frac{1}{2}[\sigma_1\varepsilon_1 + \sigma_2\varepsilon_2 + \sigma_3\varepsilon_3]. \tag{4.194}$$

For thin plate and shell structures the stress perpendicular to the neutral plane is $\sigma_3 = 0$, hence U^* becomes

$$U^* = \frac{1}{2}[\sigma_1\varepsilon_1 + \sigma_2\varepsilon_2] \approx [\sigma\varepsilon]. \tag{4.195}$$

For isotropic materials, U^* can be expressed as follows

$$U^* = \frac{\sigma^2}{E}, \tag{4.196}$$

where E is the Young’s modulus.

In pure bending, the stress can be expressed in terms of the stress in the extreme fiber $z = e$ (see Fig. 4.16)

$$\sigma(z) = \frac{2z\sigma_{max}}{t}. \tag{4.197}$$

The strain energy per unit of area U is now

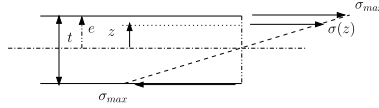


Fig. 4.16. Plate, beam

$$U = \int_{-\frac{l}{2}}^{\frac{l}{2}} U^* dz = \frac{1}{E} \int_{-\frac{l}{2}}^{\frac{l}{2}} \sigma(z)^2 dz = \frac{4\sigma_{max}^2}{Et^2} \int_{-\frac{l}{2}}^{\frac{l}{2}} (z)^2 dz = \frac{t\sigma_{max}^2}{3E}. \quad (4.198)$$

The average strain energy U is equal to the average kinetic energy per unit of area of the plate, thus

$$\frac{t\langle\sigma_{max}^2\rangle}{3E} = \frac{M_p\langle v^2\rangle}{A_p}. \quad (4.199)$$

For σ_{max} the following relation can be obtained

$$\langle\sigma_{max}^2\rangle = \frac{3EM_p\langle v^2\rangle}{tA_p}. \quad (4.200)$$

This is in accordance with [176].

For a sandwich plate with a face sheet thickness t_f , (4.198) becomes

$$U = \int_{-\frac{b}{2}}^{\frac{b}{2}+t_f} U^* dz = \frac{1}{E} \int_{-\frac{b}{2}}^{\frac{b}{2}+t_f} \sigma(z)^2 dz = \frac{2t_f\sigma_{max}^2}{E}, \quad (4.201)$$

thus (4.200) changes into

$$\langle\sigma_{max}^2\rangle = \frac{EM_p\langle v^2\rangle}{2t_fA_p}. \quad (4.202)$$

We can now estimate the average stresses $\langle\sigma_{max}^2\rangle$ from the average energy of the plate or sandwich panel.

The average value for the maximum strain $\langle\varepsilon_{max}^2\rangle$ can be derived from (4.200) and (4.202) by dividing both equations by E^2 .

Example. A sandwich panel, with CFRP face sheets $t = 0.2$ mm, Young’s modulus $E = 100$ GPa, an area $A_p = 6.25$ m² and a mass $m = 2.5$ kg/m², is exposed to an acoustic excitation in the octave band at 250 Hz, and responds with an average velocity $\langle v \rangle = 0.05$ m/s. Calculate the average stress $\langle\sigma_{max}\rangle$ and average strain $\langle\varepsilon_{max}\rangle$. The average stress $\langle\sigma_{max}\rangle$ is

$$\langle\sigma_{max}\rangle = \sqrt{\frac{EM_p\langle v^2\rangle}{2t_fA_p}} = 0.624 \times 10^6 \text{ Pa},$$

and the average strain $\langle\varepsilon_{max}\rangle$ is

$$\langle\varepsilon_{max}\rangle = \frac{\langle\sigma_{max}\rangle}{E} = 0.624 \times 10^{-5} \text{ m/m}.$$

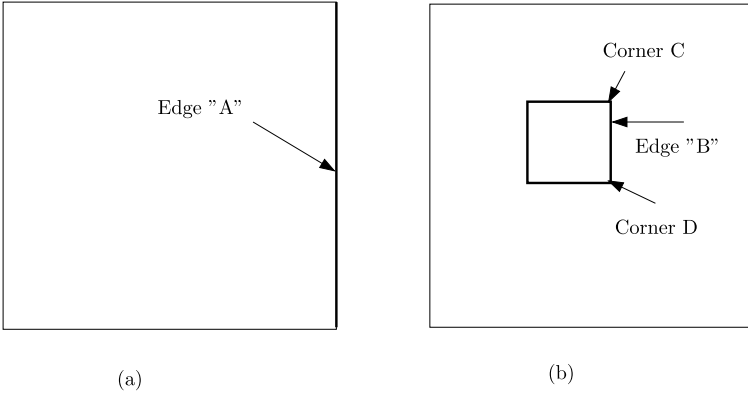


Fig. 4.17. (a) Full plate with clamped or free edges; (b) Plate with clamped square hole

In [176], Shankar mentioned the “*stress doubling rule*”. For the prediction of spatial variation of stresses, consistent with the simplicity of the SEA approach, a doubling of the mean square stress for every constrained boundary is recommended. For example, at a rigidly clamped boundary (having both rotations and translations constrained) the mean square peak stress must be four times the spatially averaged stress, and at a clamped corner (the junction of two edges) there is a further doubling, i.e. a total factor of 8. The respective stress concentrations are given by the square roots, i.e. 2 and 2.8. The edges and corners are illustrated in Fig. 4.17.

4.5.7 Non-resonant Response

The theory discussed in this section comes from papers [46, 162]. For a limp panel the non-resonant response is given by

$$\langle v^2 \rangle = \frac{2\langle p^2 \rangle}{m\omega^2}, \quad (4.203)$$

where $\langle v^2 \rangle$ is the mean square of the average velocity, $\langle p^2 \rangle$ is the mean square acoustic pressure, and m the mass per unit of area.

Consider two reverberant rooms separated by a panel. The transmission room is denoted by system 1, the receiving room by system 3 and the panel by system 2. This is illustrated in Fig. 4.18. The power flow balance equations for the three systems are:

$$\begin{aligned} \Pi_1 = \omega\eta_1\langle E_1 \rangle + \omega\eta_{12}n_1 \left(\frac{\langle E_1 \rangle}{n_1} - \frac{\langle E_2 \rangle}{n_2} \right) \\ + \omega\eta_{13}n_1 \left(\frac{\langle E_1 \rangle}{n_1} - \frac{\langle E_3 \rangle}{n_3} \right), \end{aligned} \quad (4.204)$$

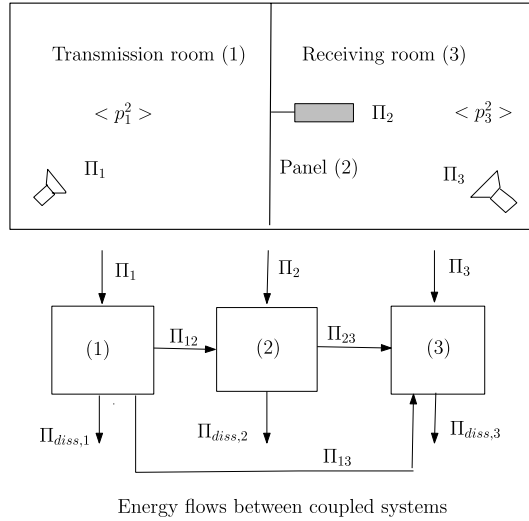


Fig. 4.18. Transmission suite

$$\begin{aligned} \Pi_2 = \omega \eta_2 \langle E_2 \rangle + \omega \eta_{21} n_2 \left(\frac{\langle E_2 \rangle}{n_2} - \frac{\langle E_1 \rangle}{n_1} \right) \\ + \omega \eta_{23} n_2 \left(\frac{\langle E_2 \rangle}{n_2} - \frac{\langle E_3 \rangle}{n_3} \right), \end{aligned} \tag{4.205}$$

$$\begin{aligned} \Pi_3 = \omega \eta_3 \langle E_3 \rangle + \omega \eta_{31} n_3 \left(\frac{\langle E_3 \rangle}{n_3} - \frac{\langle E_1 \rangle}{n_1} \right) \\ + \omega \eta_{32} n_3 \left(\frac{\langle E_3 \rangle}{n_3} - \frac{\langle E_2 \rangle}{n_2} \right). \end{aligned} \tag{4.206}$$

In [162] the following expression for the indirect CLF η_{13} , for non-resonant response, is given

$$\eta_{13} = \frac{\tau A c}{4 \omega V_1}, \tag{4.207}$$

where *sound power transmission coefficient* τ is given by

$$\tau = \frac{8 \rho^2 c^2}{m^2 \omega^2}, \tag{4.208}$$

and A is the area of the panel, c is the speed of sound in air and ρ is the density of the air. The coupling loss factor η_{31} is given by

$$\eta_{31} = \frac{2 \rho c}{m \omega}. \tag{4.209}$$

4.6 Applications

4.6.1 Panel in an Acoustic Field

For a panel with a given radiation efficiency σ_{rad} and dissipation loss factor η_p , the ratio of the panel average energy $\langle E_p \rangle$ to the acoustic average field energy $\langle E_f \rangle$ is proportional to the ratio of the respective modal densities

$$\frac{\langle E_p \rangle}{\langle E_f \rangle} = \frac{n_p}{n_f} \frac{\eta_{rad}}{\eta_{rad} + \eta_p}. \quad (4.210)$$

Here η_{rad} is the radiation coupling loss factor, n_p is the modal density of the panel and n_f is the modal density of the acoustic field.

Equation (4.210) states that a decrease of the panel modal density implies a decrease in its response. The modal density of a panel is inversely proportional to the stiffness of the panel because $n_p(f) \propto \sqrt{\frac{m}{D}}$. That means that an increase in the panel natural frequency f_n , which is proportional to the stiffness $f_n \propto \sqrt{\frac{D}{m}}$, will cause a lower modal density and an increase in the distance between consecutive modes.

Decreasing the radiation coupling loss factor η_{rad} and increasing the dissipation loss factor η_p of the panel will lead to lower average structural responses of that panel. If the internal loss factor of the panel structure is small compared to the radiation coupling loss factor, $\eta_p \ll \eta_{rad}$, the value of η_{rad} is of no importance anymore and the ratio of average energies is proportional to the ratio of modal densities, thus

$$\frac{\langle E_p \rangle}{\langle E_a \rangle} \approx \frac{n_p}{n_f}.$$

4.6.2 Sandwich Panels

In this vibro-acoustic application we discuss the response characteristics of a sandwich panel in a spacecraft structure, carrying equipment, and exposed to sound pressures.

If the spacecraft is placed in the acoustic room or under the fairing of a launch vehicle the sandwich panels (side panels) full with equipment and instruments will be exposed to sound pressure. The response PSD of the acceleration is used to specify the random vibration (test) environment for the equipment and instruments as illustrated in Fig. 4.19.

The analysis procedure is taken from [168]. There it is shown in that the power flow between sandwich panels is negligible when a typical honeycomb sandwich panels are exposed to vibro-acoustic reverberant pressure loads. This is shown in Fig. 4.20. The sandwich panel will be considered unconnected from adjacent structures.

The acoustic input power to sandwich panel or panel element i is $\Pi_{in,i}$ and equals the dissipated power of that sandwich panel:

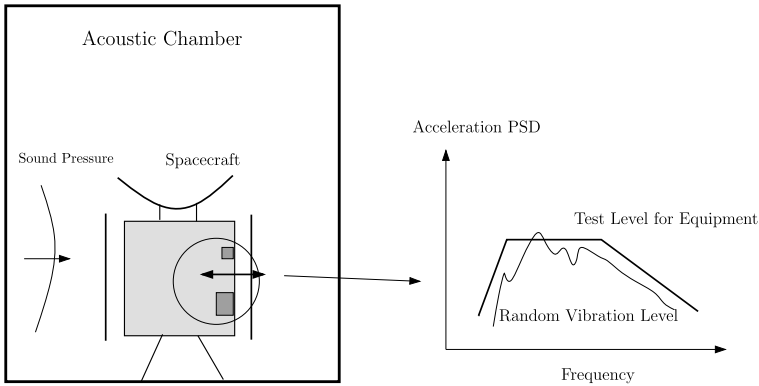


Fig. 4.19. Typical random response of spacecraft panel

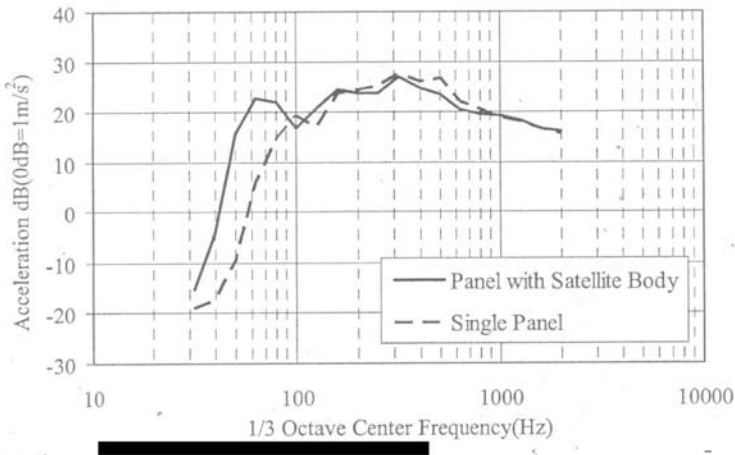


Fig. 4.20. Comparison of the vibration response of a panel either separated from or to the spacecraft [4]

$$H_{in,i} = \eta_i \omega \langle E_i \rangle = \eta_i \omega M_i \langle v_i^2 \rangle = \eta_i \omega M_i \frac{\langle a_i^2 \rangle}{(2\pi f)^2}, \quad (4.211)$$

where $\langle E_i \rangle$ is the energy of element i , ω is the center frequency of either an octave or third octave band, η_i is the dissipation loss factor, M_i is the total mass of the bare sandwich panel without any equipment mounted to it, $\langle v_i^2 \rangle$ is the spatial average mean square velocity, and $\langle a_i^2 \rangle$ the spatial average mean square acceleration of the panel.

For sandwich panels with an honeycomb core, the following frequency dependent dissipation loss factor η_i is recommended in [168]

$$\eta_i = f^{-0.7}, \quad (4.212)$$

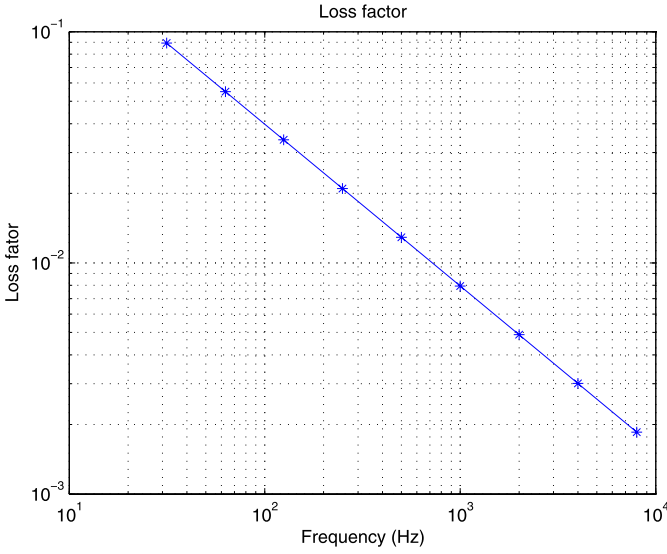


Fig. 4.21. Loss factor $\eta = f^{-0.7}$

where f is the center frequency (Hz). The values of the loss factor are shown in Fig. 4.21.

The acoustic input power is given by

$$\Pi_{in,i} = \frac{c^2 \langle p^2 \rangle \sigma_i A_i n_i(f)}{4\pi f^2 M_i}, \quad (4.213)$$

where c is the speed of sound, $\langle p^2 \rangle$ is the mean square sound pressure, σ_i is the radiation efficiency, $n_i(f)$ is the modal density, f is the center frequency (Hz) and M_i is the mass of the bare sandwich panel i .

The NASA Lewis empirical radiation efficiency σ for a flat baffled panel is given by [36]

$$\sigma = \begin{cases} \frac{f^2}{f_{cr}^2}, & f < f_{cr}; \\ 1, & f \geq f_{cr}, \end{cases} \quad (4.214)$$

where f is the center frequency and f_{cr} is the critical frequency of the panel.

The modal density of a sandwich panel with isotropic face sheets is taken from [159] and is expressed as follows.

$$n(f) = \frac{\pi A_p m_p f}{S} \left\{ 1 + \left(m_p^2 \omega^4 + \frac{4m_p \omega^2 S^2}{D} \right)^{-\frac{1}{2}} \left(m_p \omega^2 + \frac{2S^2}{D} \right) \right\}, \quad (4.215)$$

where the parameters can be found in Sect. 4.5. Note that m_p is the surface mass of the bare sandwich panel. If the material properties of the face

sheets are not isotropic, equivalent isotropic material properties can be applied. The modal density $n(f)$ for composite honeycomb sandwich panels with orthotropic properties is discussed in [159].

The critical frequency $\omega_{cr} = 2\pi f_{cr}$ is that frequency at which the sandwich panel has the same wave speed (speed of sound) as the surrounding medium, in general, air with a speed of sound $c \approx 340$ m/s. The critical frequency of a sandwich panel with isotropic face sheets is given by [160]

$$\omega_{cr}^2 = \frac{\frac{c^4 m_p}{D}}{\left[1 - \frac{c^2 m_p}{S}\right]}, \quad (4.216)$$

where the parameters can be found in Sect. 4.5. The critical frequency ω_{cr} does not exist if $\frac{c^2 m_p}{S} \geq 1$.

The spatial average mean square acceleration $\langle a_i^2(f) \rangle$ in the band with center frequency f can now be obtained using (4.211):

$$\langle a_i^2(f) \rangle = \frac{c^2 \langle p^2 \rangle \sigma_i A_i n_i(f)}{2f M_i^2 \eta_i}. \quad (4.217)$$

The mass loaded sandwich panel response (instrumentation and equipment mounted to the panel) can be obtained by two methods discussed in [168].

- **The mass ratio method** to calculate the mean square spatial averaged acceleration response $\langle a_b^2(f) \rangle$ is based on the total mass of the mounted equipment and instruments M_b with respect to the total panel mass $M_i + M_b$:

$$\langle a_b^2(f) \rangle = \langle a_i^2(f) \rangle \frac{M_i}{M_i + M_b}. \quad (4.218)$$

- **The mass area density ratio method** to calculate the mean square spatial averaged acceleration response $\langle a_b^2(f) \rangle$ is based on distributed masses of the box $m_b = \frac{M_b}{A_b}$ with respect to the total distributed mass per unit of area $m_p + m_b$, where A_b is total area of the foot prints of the equipment or instruments.

$$\langle a_b^2(f) \rangle = \langle a_i^2(f) \rangle \frac{m_p}{m_p + m_b}. \quad (4.219)$$

The use of the mass ratio method leads to a more conservative acceleration PSD compared with the mass density ratio method. PSD values obtained, using the mass density ratio method agree well with test results.

In general, the equipment's random acceleration vibration test specification is a PSD function $W_a(f)$ (g^2/Hz). The spatial average mean square of the acceleration $\langle a^2(f) \rangle$ in the frequency band with center frequency f is calculated using the expression

$$\langle a^2(f) \rangle = W_a(f) \Delta f,$$

where $W_a(f)$ is the constant acceleration PSD value in the frequency band f with associated band width Δf . In [168] it is proposed to calculate peak

Table 4.8. Specified SPL, OASPL = ??? dB

One octave frequency band (Hz)	SPL (dB) 0 dB = 2.0×10^{-5} Pa
31.5	124
63	130
125	135
250	139
500	134
1000	128
2000	124
4000	120
8000	116

PSD values $W_{a,p}(f)$ from the constant $W_a(f)$ in the frequency band f as follows

$$W_{a,p}(f) = \frac{\langle a^2(f) \rangle}{\Delta f} \frac{1}{\eta f n(f)} = W_a(f) \frac{1}{\eta f n(f)}, \quad (4.220)$$

where ηf is the half power band width, and $\eta f n(f)$ is the modal overlap.

Example. A sandwich panel with isotropic face sheets has the following geometry and material properties; the length is $a = 1.82$ m, the width $b = 0.91$ m and the total height $h = 25$ mm. The two identical isotropic face sheets have a thickness $t = 0.3$ mm. The face sheet are made of an Al-alloy with a Young's modulus $E = 71.6$ GPa and the Poisson's ratio is $\nu = 0.3$. The isotropic shear modulus of the honeycomb core is $G = 134$ MPa. The mass per unit of area of the sandwich panel is $m_p = 4.5$ kg/m². The sandwich panel is excited by a diffuse sound field as given in Table 4.8. The mean square pressures can be calculated by $\langle p^2 \rangle = p_{ref}^2 10^{\frac{SPL}{10}}$ and $p_{ref} = 2^{-5}$ Pa. The speed of sound in air is $c = 340$ m/s.

Calculate the OASPL of the SPL levels given in Table 4.8. The answer to the previous question is OASPL = 142 dB.

The total mass of the equipment and instrumentation is $M_b = 30$ kg, and the total area of the footprint is 40% of sandwich panel area $A_i = ab$ m². The mathematical model for the loss factor is $\eta_i = f^{-0.7}$, and the radiation efficiency will be approximated using $\sigma_i = 1$ if $f \geq f_{cr}$ and $\sigma_i = \frac{f^2}{f_{cr}^2}$ if $f < f_{cr}$. Calculate the acceleration responses $W_a(f)$ and $W_{a,p}(f)$ for the bare panel and the acceleration responses $W_a(f)$ and $W_{a,p}(f)$ for the mass loaded panel when this panel is exposed to acoustic pressures.

To calculate the radiation efficiency, we need the critical frequency f_{cr} . The critical frequency of the bare sandwich panel is $f_{cr} = 487.5$ Hz.

The PSD accelerations $W_a(f)$ and peaked PSD accelerations $W_{a,p}(f)$ of the bare sandwich panel are shown in Fig. 4.22. The PSD accelerations $W_a(f)$ and peaked PSD accelerations $W_{a,p}(f)$ of the mass loaded honeycomb sandwich panel are shown in Fig. 4.23.

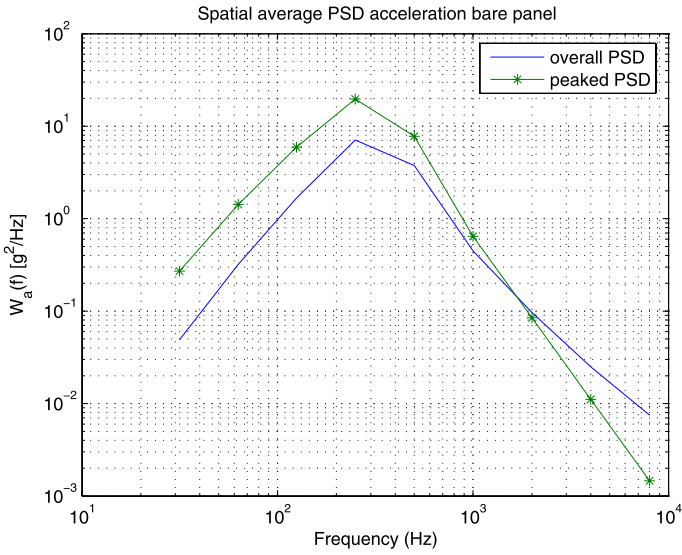


Fig. 4.22. PSD accelerations bare sandwich panel

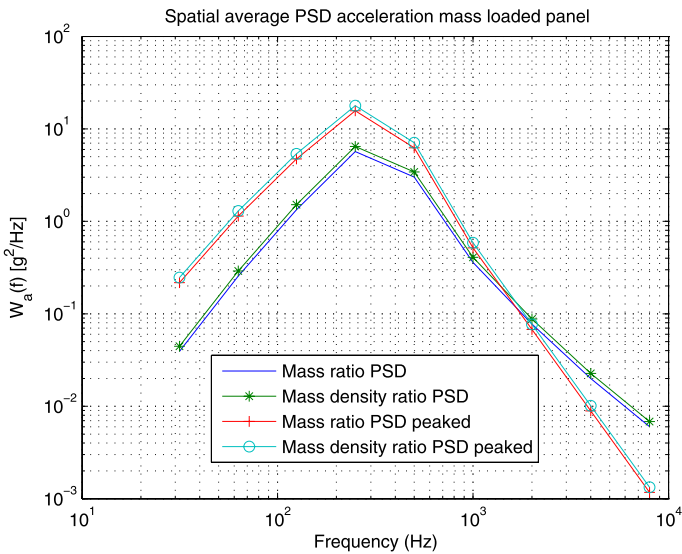


Fig. 4.23. PSD accelerations mass loaded sandwich panel

4.7 Test-Based SEA Equations

Statistical Energy Analysis is used to predict wide-band noise and vibration. That prediction may rely on parameters derived from theory or from test. The latest is called test-based SEA [32].

The power flow equations for a structure made up of N subsystems are given in (4.105).

For a system composed of N subsystems, N^2 unknowns must be estimated: N damping loss factors and $N(N - 1)$ coupling loss factors.

Getting enough test data to generate N^2 equations is achieved by applying input power on one subsystem only and measuring the response of all subsystems to that specific excitation. This generates N equations and therefore this operation must be repeated for all subsystems in order to generate N^2 equations. If $N = 2$ (4.103) can be rearranged. Equation (4.103) is given by

$$\begin{pmatrix} \eta_1 + \eta_{12} & -\eta_{21} \\ -\eta_{12} & \eta_2 + \eta_{21} \end{pmatrix} \begin{Bmatrix} \langle E_1 \rangle \\ \langle E_2 \rangle \end{Bmatrix} = \frac{1}{\omega} \begin{Bmatrix} \Pi_{in,1} \\ \Pi_{in,2} \end{Bmatrix}.$$

The space averaged energy of subsystem i is denoted by $\langle E_{ij} \rangle$ when subsystem j is excited. Equation (4.103) can be written as

$$\omega \begin{pmatrix} \langle E_{11} \rangle & \langle E_{11} \rangle & -\langle E_{21} \rangle & 0 \\ 0 & \langle E_{11} \rangle & -\langle E_{21} \rangle & -\langle E_{21} \rangle \\ -\langle E_{12} \rangle & -\langle E_{12} \rangle & \langle E_{22} \rangle & 0 \\ 0 & -\langle E_{12} \rangle & \langle E_{22} \rangle & \langle E_{22} \rangle \end{pmatrix} \begin{Bmatrix} \eta_1 \\ \eta_{12} \\ \eta_{21} \\ \eta_2 \end{Bmatrix} = \begin{Bmatrix} \Pi_{in,1} \\ 0 \\ 0 \\ \Pi_{in,2} \end{Bmatrix}. \quad (4.221)$$

In [122] it is shown that the condition number of the energy matrix in (4.221) is even acceptable in case the coupling between subsystems become stronger (see also problem 4.17), however, the *energy matrix* (4.221) may be ill-conditioned and errors in the calculation get larger with an increasing number of subsystems [71]. Therefore it is proposed to break down the $N \times N$ matrix into N sets of $(N - 1) \times (N - 1)$ matrices giving the coupling loss factors and N equations giving the damping loss factors. For the coupling loss factors related to subsystem i the following expression is used [71], with $j = 1, 2, \dots, N$

$$\begin{Bmatrix} \eta_{1i} \\ \vdots \\ \eta_{Ni} \end{Bmatrix}_{j \neq i} = \frac{\Pi_{in,i}}{\omega \langle E_{ii} \rangle} \begin{pmatrix} (\frac{\langle E_{11} \rangle}{\langle E_{i1} \rangle} - \frac{\langle E_{1i} \rangle}{\langle E_{ii} \rangle}) & \dots & (\frac{\langle E_{N1} \rangle}{\langle E_{i1} \rangle} - \frac{\langle E_{Ni} \rangle}{\langle E_{ii} \rangle}) \\ \vdots & \ddots & \vdots \\ (\frac{\langle E_{1N} \rangle}{\langle E_{iN} \rangle} - \frac{\langle E_{1i} \rangle}{\langle E_{ii} \rangle}) & \dots & (\frac{\langle E_{NN} \rangle}{\langle E_{iN} \rangle} - \frac{\langle E_{Ni} \rangle}{\langle E_{ii} \rangle}) \end{pmatrix}_{j \neq i}^{-1} \begin{Bmatrix} 1 \\ \vdots \\ 1 \end{Bmatrix}. \quad (4.222)$$

This matrix is usually well conditioned since the diagonal terms are large.

The damping loss factor of subsystem i is then obtained as [71]

$$\eta_i = \frac{\frac{\Pi_{in,i}}{\omega} - [\sum_{j=1}^N (\langle E_{ji} \rangle \eta_{ji} - \langle E_{ii} \rangle \eta_{ij})]_{j \neq i}}{\langle E_{ii} \rangle}, \quad (4.223)$$

or directly using

$$\begin{Bmatrix} \eta_1 \\ \vdots \\ \eta_N \end{Bmatrix} = \frac{1}{\omega} \begin{pmatrix} \langle E_{11} \rangle & \dots & \langle E_{1N} \rangle \\ \vdots & \ddots & \vdots \\ \langle E_{N1} \rangle & \dots & \langle E_{NN} \rangle \end{pmatrix} \begin{Bmatrix} \Pi_{in,1} \\ \vdots \\ \Pi_{in,N} \end{Bmatrix}. \tag{4.224}$$

Under certain assumptions the problem can be greatly simplified

$$\eta_{ij} \approx \frac{1}{\omega} \left\{ \frac{\langle E_{ij} \rangle}{\langle E_{ii} \rangle} \right\} \left\{ \frac{\Pi_{in,j}}{\langle E_{jj} \rangle} \right\}. \tag{4.225}$$

The following assumptions must be considered:

1. The modal energy $\frac{\langle E \rangle}{N}$ (N is the number of modes per band) of a directly driven subsystem is greater than that of a subsystem connected to it. This assumption is not valid for strong coupled subsystems, where $\frac{\langle E_{ij} \rangle}{N_j} \cong \frac{\langle E_{ii} \rangle}{N_i}$.
2. The expression calculates the coupling loss factor between two subsystems from measurements on these subsystems only. The underlying assumption is that most of the power flows directly between the two subsystems, not through a third one.

In [156] Radcliffe proposed the following expression to identify the damping loss factors from measurements of the subsystem energies $\langle E_i \rangle$, $i = 1, 2 \dots N$

$$\eta_i = \frac{1}{n_i e_i} \begin{cases} \sum_{k=1}^{i-1} n_k \eta_{kj} (e_k - e_i), \\ + \sum_{k>i}^N n_k \eta_{ik} (e_k - e_i), & i = 2 \dots N - 1; \\ \sum_{k=1}^{N-1} n_k \eta_{ki} (e_k - e_i), & i = N, \end{cases} \tag{4.226}$$

where $e_i = \frac{\langle E_i \rangle}{n_i}$ and n_i is the modal density of subsystem i .

It is assumed that the responses from all subsystems in the SEA model are measurable at each band center frequency of interest, and that all coupling loss factors and modal densities are known at these frequencies. For convenience it is assumed that power is input to subsystem 1.

The internal loss factor for the powered subsystem with unknown input power cannot be identified from measured subsystem energies and other methods must be used to estimate it.

In acoustics the damping is measured from the *reverberation time* T_R [118], the time required for the vibration energy to decrease by a factor 10^{-6} . Thus

$$e^{-2\pi f \eta T_R} = 10^{-6}, \tag{4.227}$$

which leads to

$$\eta(f) = \frac{2.2}{f T_R}, \tag{4.228}$$

where f is the center frequency.

In [156], Radcliffe proposed a least square error fit to a reverberation time \bar{T}_R from the identified damping loss factor $\eta(f_i)$ as follows

$$\bar{T}_R = \frac{2.2 \sum_{i=1}^M \frac{1}{f_i^2}}{\sum_{i=1}^M \frac{\eta(f_i)}{f_i}}, \quad (4.229)$$

where M is the number of center frequencies in the frequency band. Equation (4.229) can be obtained by applying the *method of Least Squares* [226] to the series of equation, $T_{R,i} = 2.2 \frac{1/f_i^2}{\eta_i/f_i}$.

The new computed damping loss factor $\bar{\eta}_i$ using the fitted reverberation time \bar{T}_R is

$$\bar{\eta}(f_i) = \frac{2.2}{f_i \bar{T}_R}. \quad (4.230)$$

Example. This simple example is illustrated in Fig. 4.24 and is taken from [156]. An SEA model consists of two identical cubic spaces, separated by an Al-alloy panel. Acoustic powers are input to Space 1 and the working medium is air. Space 1 is designated as subsystem 1, the panel as subsystem 2 and space 2 as subsystem 3. The properties of all subsystems are given in Table 4.9.

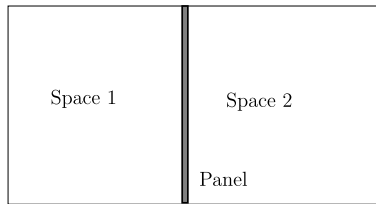


Fig. 4.24. Space panel space

Table 4.9. Geometry, material properties, assumed measurements at 5000 Hz

Panel	length x width	$0.5 \times 0.5 \text{ m}^2$
	Thickness t	0.007 m
	Young's modulus E	70 GPa
	Poisson's ratio ν	0.3
	Density ρ	2700 kg/m ³
	Velocity v	0.000885 m/s
Air	Density ρ	1.244 kg/m ³
	Speed of sound c	344 m/s ²
Space 1	Volume V	$0.5 \times 0.5 \times 0.5 \text{ m}^3$
	Pressure p	39 Pa
Space 2	Volume V	$0.5 \times 0.5 \times 0.5 \text{ m}^3$
	Pressure p	19 Pa

Properties of Subsystem 2 (Panel)

The bending stiffness of the panel is given by

$$D = \frac{Et^3}{12(1-\nu^2)}.$$

The modal density of the panel is constant over the frequency band is given by

$$n_2 = \frac{A_p}{2} \sqrt{\frac{\rho t}{D}} = 0.0116 \text{ modes/Hz.}$$

The energy E_2 can be calculated as follows:

$$E_2 = M\langle v^2 \rangle = 3.7007 \times 10^{-6} \text{ Joules,}$$

and the modal energy e_2 is

$$e_2 = \frac{E_2}{n_2} = 3.1932 \times 10^{-4} \text{ Joules/mode.}$$

The critical frequency f_{cr} of the panel is

$$f_{cr} = \frac{c^2}{2\pi} \sqrt{\frac{m}{D}} = 1746 \text{ Hz.}$$

Subsystems 1 and 3, Space 1 and Space 2

The modal densities of subsystem 1 and 3 are

$$n_1 = n_3 = \frac{4\pi f^2 V}{c^3} + \frac{\pi f A_s}{2c^2} + \frac{L_s}{8c} = 1.0664 \text{ modes/Hz.}$$

The energies of the subsystem 1 and 3, $\langle E_1 \rangle$ and $\langle E_3 \rangle$, are

$$\langle E_1 \rangle = \frac{V p_1^2}{\rho c^2} = 0.0013 \text{ Joules,}$$

and

$$\langle E_3 \rangle = \frac{V p_2^2}{\rho c^2} = 3.1991 \times 10^{-4} \text{ Joules.}$$

The modal energies are

$$e_1 = \frac{\langle E_1 \rangle}{n_1} = 0.0012 \text{ Joules/mode,}$$

and

$$e_3 = \frac{\langle E_3 \rangle}{n_3} = 2.9998 \times 10^{-4} \text{ Joules/mode.}$$

Coupling Loss Factors

Expressions for the coupling loss factors η_{21} and η_{23} are taken from [156]

$$\eta_{21} = \eta_{23} = \frac{\rho c A_p}{2\pi f M_p} \begin{cases} \frac{2\lambda_{cr} P}{\pi^2 A_p} \sin^{-1}\left(\frac{f}{f_{cr}}\right)\beta, & f < f_{cr}; \\ (1 - \frac{f_{cr}}{f})^{-\frac{1}{2}}, & f > f_{cr}, \end{cases}$$

where ρ is the fluid density, c is the speed of sound in the fluid, A_p is the area of the panel, P is the perimeter of the panel, M_p is the total mass of the panel, f_{cr} is the critical frequency of the panel, $\lambda_{cr} = \frac{c}{f_{cr}}$ is the wavelength at f_{cr} and β is

$$\beta = \begin{cases} 1, & \text{for simply supported edge supports;} \\ 2, & \text{for clamped edge supports;} \\ \sqrt{2}, & \text{for typical mounting conditions.} \end{cases}$$

The coupling loss factors $\eta_{21} = \eta_{23}$ can be calculated ($f > f_{cr}$), $f = 5000$ Hz

$$\eta_{21} = \eta_{23} = \frac{\rho c A_p}{2\pi f M_p} \left(1 - \frac{f_{cr}}{f}\right)^{-\frac{1}{2}} = 8.9342 \times 10^{-4}.$$

The coupling loss factor η_{12} and η_{32} can now be computed:

$$\eta_{12} = \frac{n_2}{n_1} \eta_{21} = 9.7092 \times 10^{-6},$$

and

$$\eta_{32} = \frac{n_2}{n_3} \eta_{23} = 9.7092 \times 10^{-6}.$$

The coupling loss factor for the flanking path between subsystems 1 and 3 is assumed constant for this example, i.e. $\eta_{13} = 0.0001$, thus $\Pi_{13} = \omega \eta_{13} n_1 \left(\frac{\langle E_1 \rangle}{n_1} - \frac{\langle E_3 \rangle}{n_3}\right)$.

Loss Factor

The damping loss factor of subsystem 1 cannot be independently computed because the measured energy in that element is considered as input.

The damping loss factors of subsystem 2 and 3 will be computed using (4.226)

$$\eta_2 = 2.4 \times 10^{-3},$$

and

$$\eta_3 = 3.0434 \times 10^{-4}.$$

Experimental Determination of the Radiation Efficiency

Let us assume that the structure, the cylinder is system 1, and the surrounding acoustic volume, the reverberant chamber is system 2. The power balance

equations of the two systems are, [157]

$$\begin{aligned} \Pi_1 &= \omega(\eta_1 + \eta_{12})\langle E_1 \rangle - \omega\eta_{21}\langle E_2 \rangle, \\ \Pi_2 &= -\omega\eta_{12}\langle E_1 \rangle + \omega(\eta_2 + \eta_{21})\langle E_2 \rangle, \end{aligned} \quad (4.231)$$

where η_1 and η_2 are the loss factors (dissipation loss factor, DLF) of the cylinder and acoustic volume, η_{12} is the coupling loss factor (CLF) between the cylinder and the acoustic volume, $\langle E_1 \rangle$ and $\langle E_2 \rangle$ are the energies stored in the cylinder and acoustic volume, and Π_1 and Π_2 are the power inputs to these systems. The CLF η_{21} between the acoustic volume and the cylinder can be written as

$$\eta_{21} = \eta_{12} \frac{n_1}{n_2}, \quad (4.232)$$

where n_1 and n_2 are the modal densities of the cylinder and acoustic volume, respectively. Only the cylinder will be mechanically excited, thus the second part of (4.231) can be written

$$\eta_{12} = \frac{n_2 \eta_2 \langle E_2 \rangle}{\langle E_1 \rangle n_2 - \langle E_2 \rangle n_1}, \quad (4.233)$$

and the radiation efficiency σ_{rad} is calculated using

$$\eta_{12} = \frac{\sigma_{rad} \rho c}{2\pi f \mu_s}, \quad (4.234)$$

where ρ is the density of air, c the speed of sound in air and μ_s the surface mass density of the structure.

The modal density n_2 of the acoustic volume is given by

$$n_2 = \frac{4\pi f^2 V_2}{c^3} + \frac{\pi f A_2}{c^2} + \frac{P_2}{c}, \quad (4.235)$$

where P_2 , A_2 , V_2 are the total edge length, surface area and volume of the reverberation chamber, respectively. It can be seen that the modal density of the acoustic volume increases rapidly with frequency, and hence $n_1 \ll n_2$ for the frequency $f > 100$ Hz. Hence (4.233) can be simplified to

$$\eta_{12} = \frac{n_2 \langle E_2 \rangle}{\langle E_1 \rangle}. \quad (4.236)$$

The DLF of the chamber η_2 is given by

$$\eta_2 = \frac{13.816}{2\pi f T_r}, \quad (4.237)$$

where T_r is the reverberation time of the chamber.

The energy of the cylinder $\langle E_1 \rangle$ is given by

$$\langle E_1 \rangle = M \langle v_1^2 \rangle, \quad (4.238)$$

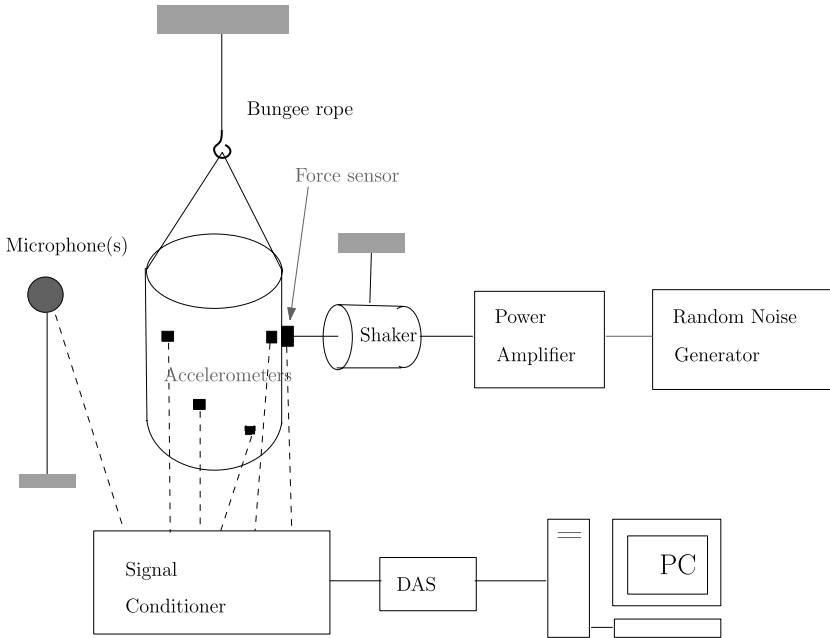


Fig. 4.25. Experimental setup

where $\langle v_1^2 \rangle$ is the space averaged square of the velocity and M is the total mass of the cylinder. The energy of the acoustic volume is given by

$$\langle E_2 \rangle = \frac{\langle p_2^2 \rangle V_2}{\rho c^2}, \quad (4.239)$$

where $\langle p_2^2 \rangle$ is the space average square of the sound pressure in the acoustic chamber.

To measure the radiation efficiency of the cylinder, it is suspended using bungee ropes in a reverberant chamber, as shown in Fig. 4.25, and excited laterally using an electromagnetic shaker on the outer surface of the cylinder. A small force transducer is kept in between the shaker and the cylinder to measure the input excitation level. A miniature accelerometer with built-in preamplifier is placed on the inner surface of the cylinder exactly at the location of excitation to measure the point mobility. A few number of miniature accelerometers are mounted on the cylinder at random locations. Four microphones are suspended at different locations to measure the sound pressure level (SPL) in the reverberation chamber. The cylinder is excited at a limited number of randomly selected locations by a random noise generator in the frequency range of interest. The signals from the force transducer, accelerometers and microphones are recorded on a PC or laptop using a multichannel data acquisition system. The data is analyzed for point mobility, average cylinder

velocity and average sound pressure level in the chamber for each excitation location. Then the spatial average values of these parameters at all excitation locations are considered for calculation e.g. the radiation efficiency.

4.8 SEA Strategy

The SEA prediction procedure can be divided into the four following steps:

1. Modelling of the dynamic system into subsystems and junctions (coupling)
2. Determination of SEA parameters for the SEA model
3. Calculation of subsystem energies
4. Calculation of average response levels for the systems

Example. Calculate the responses of a plate placed in a reverberant chamber with the aid of the SEA.

- Two subsystems:
 1. Subsystem 1 is the acoustic chamber
 2. Subsystem 2 is the plate or panel placed (hung) in the acoustic chamber
- Coupling
 1. Radiation
- Space-time average energies
 1. Acoustic chamber $\langle E_1 \rangle = \frac{V \langle p^2 \rangle}{\rho c^2}$
 2. Plate/panel $\langle E_2 \rangle = \frac{M \langle a^2 \rangle}{\omega^2}$
- Modal densities
 1. Acoustic chamber $n_1 = \frac{V \omega^2}{c^3}$
 2. Plate/panel $n_2 = \frac{A}{2} \sqrt{\left(\frac{m}{D}\right)}$
- Loss factors
 1. Acoustic chamber η_1 not considered
 2. Plate/panel η_2
 3. One sided radiation loss factor η_{21}
- Energy balance
 1. $\frac{\langle E_2 \rangle}{\langle E_1 \rangle} = \frac{n_2}{n_1} \left[\frac{2\eta_{21}}{\eta_2 + 2\eta_{21}} \right]$
- Result
 1. $\frac{\langle a^2 \rangle}{\langle p^2 \rangle} = \frac{c}{2\rho\sqrt{mD}} \left[\frac{2\eta_{21}}{\eta_2 + 2\eta_{21}} \right]$

where

- A is the surface area of the plate/panel
- a is the acceleration of the plate/panel
- c is the speed of sound in air
- D is the bending stiffness of the plate/panel
- ρ is the density of the air
- p is the pressure in the acoustic chamber
- V is the volume of the acoustic chamber
- ω is the radian frequency

Problems

4.1. A spring-mounted rigid body with a $m = 100$ kg mass can be modelled as an oscillator with a spring stiffness $k = 6.25106$ N/m. A steady-state applied force of $F = 75$ N produces a velocity of $v = 0.15$ m/s. Estimate the damping ratio ζ , the loss factor η and the amplification factor Q (quality factor).

Answers: $\zeta = 0.0125$, $\eta = 0.025$ and $Q = 40$.

4.2. The dynamic response of a SDOF system

$$\ddot{x} + 2\zeta\omega_0\dot{x} + \omega_0^2x = 0,$$

with, certain initial conditions, is given by

$$x(t) = Ae^{-\omega_0\zeta t} \sin(\omega_0\sqrt{1-\zeta^2}t),$$

where $x(t)$ is the decaying oscillating displacement, ζ is the damping ratio and ω_0 is the natural frequency. The mass of the SDOF system is m and stiffness is $k = m\omega_0^2$. Prove that the average total energy

$$\langle E \rangle = \langle PE \rangle + \langle KE \rangle = \frac{1}{2}kA^2e^{-2\omega_0\zeta t}.$$

4.3. Prove that $\langle \dot{x}^2 \rangle = \omega_n^2 \langle x^2 \rangle$.

Hint: Use $\langle x^2 \rangle = \frac{W_{FF}(f_n)}{4\pi} \int_{-\infty}^{\infty} |H(\omega)|^2 d\omega$ and $Y(\omega) = j\omega H(\omega)$ and compare that with (4.33).

4.4. Prove that $\langle \ddot{x}^2 \rangle = \omega_n^2 \langle \dot{x}^2 \rangle$.

Hint: Use $\langle \ddot{x}^2 \rangle = \frac{W_{FF}(f_n)}{4\pi} \int_{-\infty}^{\infty} \omega^2 |H(\omega)|^2 d\omega$ and compare that with (4.33).

4.5. The power flow between to SDOF system is given by $\Pi_{12} = \beta_{12}[\langle E_1 \rangle - \langle E_2 \rangle]$. Prove that $\beta_{12} = \beta_{21}$.

Hint: Use (4.46).

4.6. A SEA model consists of two subsystems and is illustrated in Fig. 4.26. The power transfer Π_{12} is given by

$$\Pi_{12} = \omega\eta_{12}\langle E_1 \rangle - \omega\eta_{21}\langle E_2 \rangle.$$

Show that the power ratio $\frac{\Pi_{12}}{\Pi_{in,1}}$ is

$$\frac{\Pi_{12}}{\Pi_{in,1}} = \frac{\eta_{12}\eta_2}{\eta_{12}\eta_{12}\eta_2 + \eta_{21}\eta_1 + \eta_1\eta_2}.$$

4.7. Set up the power balance equations for the system with for subsystems as shown in Fig. 4.27. The coupling loss factors η_{ij} and the loss factors are η_i , with $i, j = 1, 2, 3, 4$.

Answers: $[\omega\eta_{ij}]\{\langle E_i \rangle\} = \{\Pi_{i,in}\}$, $\eta_{ii} = \eta_i + \sum_{j,j \neq i} \eta_{ij}$.

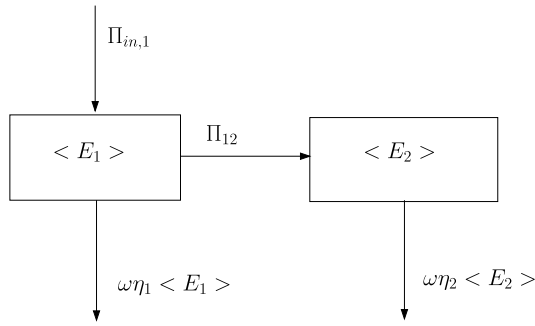


Fig. 4.26. SEA model

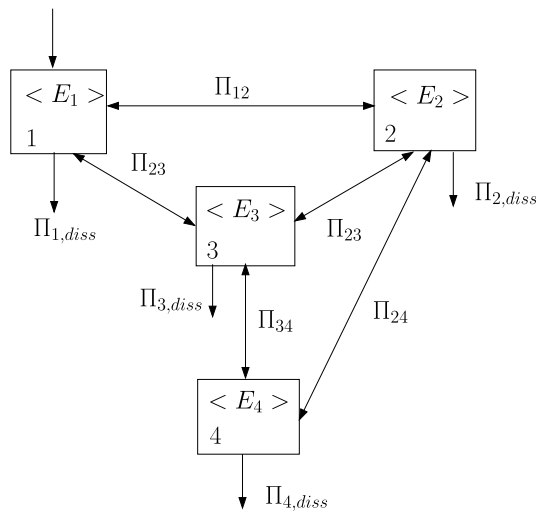


Fig. 4.27. Four coupled subsystems

4.8. The dimensions of the Large European Acoustic Facility (LEAF) are $9.0 \times 11.0 \times 16.4$ (width, length, height) m^3 and filled with air. Calculate the modal densities of the LEAF in the one octave frequency band from 31.5–8000 Hz. The speed of sound in air is $c = 340$ m/s.

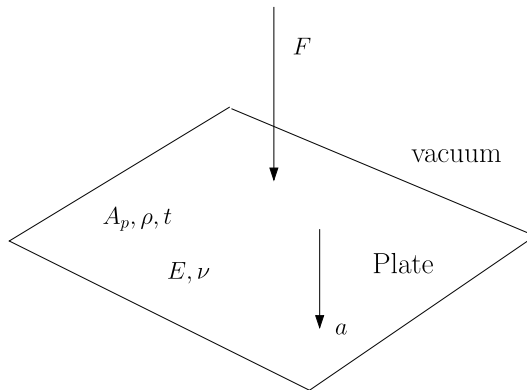
Answers: $V = 1624 \text{ m}^3$, $A = 854 \text{ m}^2$, $P = 145.6 \text{ m}$, e.g. $n(250) = 16.70$ modes/Hz.

4.9. The dimensions of the Large European Acoustic Facility (LEAF) are $9.0 \times 11.0 \times 16.4$ (width, length, height) m^3 and filled with air. Calculate the OASPL and the average energies $\langle E_{LEAF} \rangle$ of the LEAF in the one octave frequency band from 31.5–8000 Hz. The speed of sound in air is $c = 340$ m/s and the density of air is $\rho = 1.2 \text{ kg/m}^3$. The SPLs are given in Table 4.10.

Answers: e.g. $OASPL = 154.0 \text{ dB}$, $\langle E_{LEAF}(250) \rangle = 1.592 \times 10^6 \text{ Nm}$.

Table 4.10. SPL in LEAF

One octave frequency band (Hz)	SPL (dB) $0 \text{ dB} = 2.0 \times 10^{-5} \text{ Pa}$
31.5	136
63	141
125	147
250	150
500	147
1000	144
2000	137
4000	131
8000	125

**Fig. 4.28.** Plate excited by random force

4.10. A square plate (in vacuum) made of Al-alloy with the following properties; the Young's modulus is $E = 70 \text{ GPa}$, the Poisson's ratio is $\nu = 0.3$ and the density is $\rho = 2800 \text{ kg/m}^2$ is excited by a random force F . The PSD of the force F is constant over the octave band $f_i = 31.5\text{--}1000 \text{ Hz}$, $i = 1, 2, \dots, 6$ and is given by $W_F = 100 \text{ N}^2/\text{Hz}$. The total area of the plate is $A_p = 4 \text{ m}^2$ and the thickness is $t = 5 \text{ mm}$. The dissipated loss factor is constant $\eta = 0.02$. The plate is illustrated in Fig. 4.28.

- Calculate $\langle F_i^2 \rangle$
- Calculate the overall rms value F_{rms}
- Derive the expression for the mean square of the acceleration $\langle a_i^2 \rangle$ in the center frequency $\langle f_i \rangle$
- Derive the expression for the overall rms value of the acceleration $\langle a \rangle$
- Calculate the mean square acceleration $\langle a_i^2 \rangle$ in the octave band in the center frequency $\langle f_i \rangle$ $1 = 1, 2, \dots, 6$
- Calculate the rms value of the acceleration $\langle a \rangle$ over the complete band

Answers: $\langle F_i^2 \rangle = W_F \Delta f_i$, 373.09 N, $\langle a_i^2 \rangle = \frac{(2\pi f_i) \langle F_i^2 \rangle n(f_i)}{4\eta M_p^2}$, $\langle a \rangle = \sqrt{\sum_{i=1}^6 \langle a_i^2 \rangle} = 52.7942 \text{ m/s}^2$.

4.11. A rectangular steel plate has been supported horizontally. The physical properties of the plate are: length $l = 0.9 \text{ m}$, width $b = 0.6 \text{ m}$, thickness $t = 0.00115 \text{ m}$, density $\rho = 6956 \text{ kg/m}^3$, Young's modulus $E = 210.0 \text{ GPa}$, shear modulus $G = 80.77 \text{ GPa}$ and the Poisson's ratio $\nu = 0.3$. The shear rigidity is defined by $S = Gt$. The speed of sound in air $c = 340 \text{ m/s}$.

- Calculate mass of plate M_p kg.
- Calculate modal density $n_p(f)$ modes/Hz.
- Calculate critical frequency of plate f_{crit} Hz.
- Calculate critical frequency of plate $f_{crit,S}$ Hz, taking into account the shear rigidity $S \text{ N/m}$ and use following expression $f_{crit,S} = \sqrt{f_{crit}^2 [1 - \frac{c^2 \rho t}{S}]}$ Hz [160].

Answers: $M_p = 4.32 \text{ kg}$, $n_p(f) = 0.141 \text{ modes/Hz}$, $f_{crit} = 9622 \text{ Hz}$ and $f_{crit,S} = 9574 \text{ Hz}$.

4.12. Calculate the modal densities $n(f)$ and $n(\omega)$ of a bending beam with bending stiffness EI , length L , cross section A , and density ρ , using the expression $\Re\{Y(\omega)\} = \frac{n(f)}{4M}$. The complex mobility function $Y(\omega)$ is given by

$$Y(\omega) = \frac{j-1}{4A\rho\sqrt{\omega}} \sqrt[4]{\frac{A\rho}{EI}}$$

and the total mass is $M = A\rho L$.

Answers: $n(f) = \frac{L}{\sqrt{2\pi f}} \sqrt[4]{\frac{A\rho}{EI}}$, $n(\omega) = \frac{L}{2\pi\sqrt{\omega}} \sqrt[4]{\frac{A\rho}{EI}}$.

4.13. Calculate the modal densities $n(f)$ of a unstiffened cylinder with the following geometrical and material properties; the radius is $R = 0.302 \text{ m}$, the length $L = 0.75 \text{ m}$, the wall thickness $t = 0.00146 \text{ m}$, the total mass is $M = 5.55 \text{ kg}$, the Young's modulus $E = 70 \times 10^9 \text{ N/m}^2$ and the Poisson's ratio $\nu = 0.3$. This problem is taken from [34].

- Calculate the ring frequency f_r Hz.
- Calculate the wave speed C_l m/s.
- Calculate the modal density $n(f)$ in the octave band. Create a plot $\frac{f}{f_r}$ versus $n(f)$.
- Calculate the modal density $n(f)$ in the one-third octave band. Create a plot $\frac{f}{f_r}$ versus $n(f)$.

Answers: $f_r = 2698 \text{ Hz}$, $C_l = 5119 \text{ m/s}$, $n(f)$ in the octave band is given in Fig. 4.29.

4.14. This problem is taken from [159]. A sandwich panel has dimensions $a = 2.15 \text{ m}$ and $b = 1.80 \text{ m}$. The total mass is $M = 13.8 \text{ kg}$. The core height is $h = 18 \text{ mm}$, the face sheet thickness $t = 0.2 \text{ mm}$. The face sheet is a laminate of two layers (0/90) of CFRP. Each CFRP layer has the following properties:

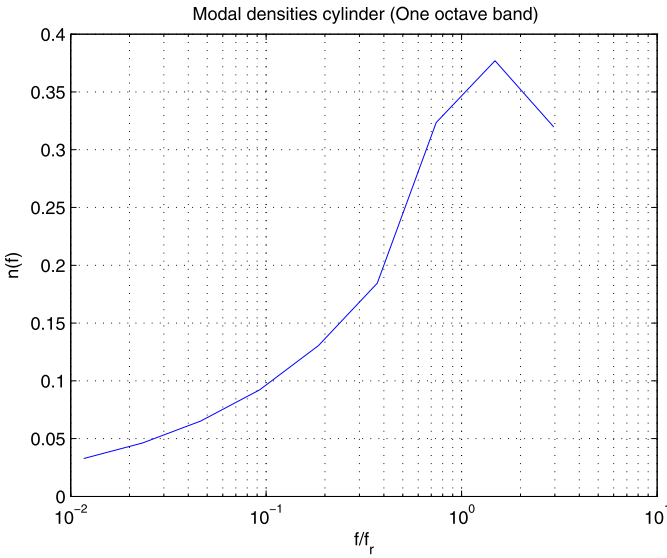


Fig. 4.29. Modal densities problem 4.13

- Young’s modulus along the fiber direction $E = 30 \times 10^{10}$ Pa
- Young’s modulus along the transverse direction $E = 0.607 \times 10^{10}$ Pa
- Poisson’s ratio $\nu = 0.346$
- Shear modulus $G = 0.50 \times 10^{10}$ Pa

The shear modulus of the honeycomb core is 6.32×10^7 , 10.53×10^7 Pa.

Calculate the $[D]$ matrix, assuming that the layers of both face sheets are balanced, and calculate the shear rigidity S . Further calculate the modal density $n(f)$ of the sandwich panel in the octave band (32.5–8000 Hz).

Answers: $D_{11} = 5135.3$ Nm, $D_{22} = 5028.0$ Nm, $D_{12} = 69.7$ Nm, $D_{66} = 165.6$ Nm, $S = 15.0$ N/m, $n(32.5) = 0.061$, $n(8000) = 0.634$.

4.15. Consider a panel kept in an acoustic chamber and excited mechanically. For SEA modelling, the acoustic chamber field is taken as subsystem 1 and the panel is taken as subsystem 2. The energy of acoustic chamber, subsystem 1, is given by $\langle E_1 \rangle = \frac{p^2 V}{\rho c^2}$ and the energy of the panel, subsystem 2, is given by $\langle E_2 \rangle = M \langle v^2 \rangle$, where p is the rms pressure field, V is the volume of the acoustic chamber, ρ is the density of the air in the chamber, c is the speed of sound in the air, M is the total mass of the panel and $\langle v \rangle$ is the velocity of the panel averaged of the surface.

Prove the relation

$$\langle E_1 \rangle = \frac{\eta_{21}}{\eta_1 + \eta_{12}} \langle E_2 \rangle$$

where η_{21} and η_{12} are the coupling loss factors and η_1 is the loss factor of the acoustic chamber given by $\eta_1 = \frac{S \bar{\alpha}}{8\pi f V}$, where S is the surface of the acoustic

Table 4.11. Identified damping loss factors

Third octave band f_i (Hz)	$\eta(f_i)$
500	0.010
630	0.030
800	0.010
1000	0.008
1250	0.006
1600	0.010
2000	0.008
2500	0.006
3150	0.004
4000	0.003
5000	0.001

chamber and $\bar{\alpha}$ is the sound power absorption coefficient and f is the central cyclic frequency.

Prove that the radiation resistance R_{rad} of the panel can be expressed by [161]

$$R_{rad} = \frac{p^2 S \bar{\alpha}}{[4\rho c \langle v^2 \rangle - \frac{n_2 c^2 p^2}{\pi f^2 M}]}$$

using the reciprocity relation $\eta_{12} = \eta_{21} \frac{n_2}{n_1}$. The modal density of the acoustic chamber is given by $n_1 = \frac{4\pi f^2 V}{c^3}$ and n_2 is the modal density of the panel. What is the dimension of R_{rad} ? [$\frac{W}{(m/s)^2}$].

4.16. The identified damping loss factor $\eta(f_i)$ in the third octave frequency band are given in Table 4.11. Compute the Least Squared error fit of the reverberation time \bar{T}_R (4.229) and calculate the damping loss factors $\bar{\eta}(f_i)$ in the third octave frequency band.

Answers: $\bar{T}_R = 0.25$ s, see Fig. 4.30.

4.17. Consider a three subsystems model which is subjected sequentially to power $\Pi_{in,1}$, $\Pi_{in,2}$ and $\Pi_{in,3}$ respectively. The loss factors of each of the subsystems are η_1 , η_2 and η_3 , the coupling loss factors between the subsystems are η_{12} , η_{21} , η_{13} , η_{31} , η_{23} and η_{32} . Subsystem 1 is first subjected to an input power $\Pi_{in,1}$ and the energies of all subsystems are measured and are respectively $\langle E_{11} \rangle$, $\langle E_{21} \rangle$ and $\langle E_{31} \rangle$. The first index denotes the subsystem number and the second index the subsystem subjected to power input. Similarly, the subsystem 2 will be injected by the power $\Pi_{in,2}$ and the measured energies are respectively $\langle E_{12} \rangle$, $\langle E_{22} \rangle$ and $\langle E_{32} \rangle$. Finally subsystem 3 will be powered by $\Pi_{in,3}$ and the measured subsystem energies are respectively $\langle E_{13} \rangle$, $\langle E_{23} \rangle$ and $\langle E_{33} \rangle$.

Generate the 9×9 energy matrix [$\langle E_{ij} \rangle$], $i, j = 1, 2, 3$. The sequence of loss and coupling loss factors is η_1 , η_2 and η_3 , η_{12} , η_{21} , η_{13} , η_{31} , η_{23} and η_{32} .

The subsystem energy are related as follows $\langle E_{ij} \rangle = \varepsilon \langle E_{ij} \rangle$, $\varepsilon = 0.05, 0.10, 0.30, 0.50, 0.80$.

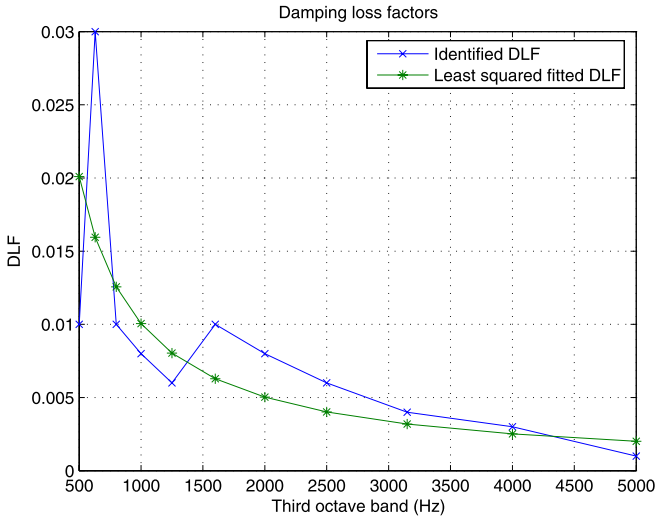


Fig. 4.30. Damping loss factors

Table 4.12. Condition number of the energy matrix

No.	Energy ratio ε	Condition number
1	0.05	5.3826
2	0.10	5.8944
3	0.30	9.5020
4	0.50	16.5558
5	0.80	54.4306

Perform the following numerical experiments:

1. Subsystem 1 is driven first and its energy is assumed to be $\langle E_{11} \rangle = 1.0$. The energy of subsystems 2 and 3 are $\langle E_{21} \rangle = \varepsilon \langle E_{11} \rangle$ and $\langle E_{31} \rangle = \varepsilon \langle E_{11} \rangle$.
2. Subsystem 2 is driven first and its energy is assumed to be $\langle E_{22} \rangle = 1.0$. The energy of subsystems 1 and 3 are $\langle E_{12} \rangle = \varepsilon \langle E_{22} \rangle$ and $\langle E_{32} \rangle = \varepsilon \langle E_{22} \rangle$.
3. Subsystem 3 is driven first and its energy is assumed to be $\langle E_{33} \rangle = 1.0$. The energy of subsystems 1 and 2 are $\langle E_{13} \rangle = \varepsilon \langle E_{33} \rangle$ and $\langle E_{23} \rangle = \varepsilon \langle E_{33} \rangle$.

The above energy ratio ε corresponds to weak coupling to strong coupling.

Compute the condition number of the energy matrix (the ratio biggest eigenvalue to lowest eigenvalue).

Answer: See Table 4.12.

This problem is taken from [122].

4.18. There are two sinusoidal waves:

1. The first wave has a wave number $k_1 = 2$ rad/m and an angular frequency $\omega_1 = 2$ rad/s.

2. The second wave has a wave number $k_2 = 2.2$ rad/s and an angular frequency $\omega_2 = 8$ rad/s.

Calculate the phase velocities v_1 and v_2 .

The third sinusoidal wave is the summation of the waves 1 and 2. Calculate the wave number k_3 , the angular frequency ω_3 of the carrier wave and Δk_3 and $\Delta\omega_3$ of the modulation impressed on the carrier.

Calculate the phase velocity v_3 of the carrier wave and the group velocity v_{g3} .

Answers: $v_1 = 1$ m/s, $v_2 = 3.636$ m/s, $k_3 = 2.1$ rad/m, $\omega_3 = 5$ rad/s, $\Delta k_3 = 0,1$ rad/m, $\Delta\omega_3 = 3$ rad/s, $v_3 = 2.381$ m/s and $v_{g3} = 30$ m/s.

- 4.19. Calculate the group velocity c_g of an infinite isotropic bending plate described by (4.154), (4.155) and (4.156).

Answer: $c_g = \frac{3}{2}\sqrt{\omega^4 \frac{D}{\bar{m}}}$.

- 4.20. The natural frequencies of a simply supported, rectangular plate can be computed by

$$\omega_{m,n} = \pi^2 \sqrt{\frac{D}{\bar{m}}} \left[\left(\frac{m\pi}{L_x} \right)^2 + \left(\frac{n\pi}{L_y} \right)^2 \right],$$

where $D = \frac{Et^3}{12(1-\nu^2)}$ is the bending stiffness, $\bar{m} = \rho t$ is the mass per unit of area, E is Young's modulus, ν is Poisson's ratio, ρ is the density of the plate material, t is the thickness of the plate, m is the mode number in the x direction, n the mode number in y direction, L_x the length of the x direction and L_y the width of the plate in y direction. The wave number in x direction is $k_x = \frac{m\pi}{L_x}$ and the wave number in y direction is $k_y = \frac{n\pi}{L_y}$.

Derive the group speed c_{gx} in x direction and the group speed c_{gy} in y direction.

Show numerically that the group speed in the x direction can be approximated by $c_{gx} = \frac{\partial\omega}{\partial k_x} |_{\omega_{m,n}} = \frac{\omega_{m,n} - \omega_{m+1,n}}{k_m - k_{m+1}}$ and the group speed in the y direction can be approximated by $c_{gy} = \frac{\partial\omega}{\partial k_y} |_{\omega_{m,n}} = \frac{\omega_{m,n} - \omega_{m,n+1}}{k_n - k_{n+1}}$.

The properties of the plate are given in Table 4.13. Plot the group speed c_{gx} and c_{gy} as a function of the frequency f (frequency range 0–5000 Hz).

Table 4.13. Properties of single plate

Parameter	Unit	Dimension
L_x	m	1
L_y	m	0.7
t	m	0.003
E	Pa	200×10^{11}
ν		0.3
ρ	kg/m ²	7800

Fokker-Planck-Kolmogorov Method
or Diffusion Equation Method

Fokker-Planck-Kolmogorov Method or Diffusion Equation Method

5.1 Introduction

In the previous parts, random vibration of linear dynamic systems are discussed. In general, spacecraft structures are considered to behave linearly and the modal characteristics play an important role in the design and verification process. In this part, random vibrations of nonlinear dynamic systems where modal properties do not exist will be briefly discussed. Many solution methods exist to solve nonlinear systems exposed to random loads. In [123] an overview of several solution methods is given. In this chapter one particular method based on the Fokker-Planck-Kolmogorov (FPK) equation is applied to predict the response characteristics. Wang and Uhlenbeck's paper entitled: "On the Theory of the Brownian Motion II", published in [218] was the starting point to study the derivation and applications of the FPK equation. This paper was originally published in *Reviews of Modern Physics* in 1945. In [26] Caughey repeated and extended the derivation of the FPK equation for a Markoff vector process and showed the steady state solution of the FPK equation for the nonlinear Duffing equation with a Gaussian white noise load. Risken's book [165], is completely dedicated to the FPK equation, methods of solution and applications, however, it has little to do with the mechanical engineering problems. Ibrahim [84] describes in his book the parametric random vibrations and Roberts and Spanos [166] discuss in their book the solution of random vibrations of nonlinear dynamic systems using statistical linearization. To's book, [210], is more or less an union of the topics described in [84, 166].

In this chapter, the derivation of the FPK equation, in conjunction with the Markoff process, will be discussed and, Miles' equation is obtained as example of a linear SDOF system, [127], by solving the stationary probability density function, and the related first and second moments of the random response.

In general, no closed form solution of the conditional or transition and joint probability density function from the FPK equation can be obtained

for nonlinear random vibration problems and therefore the finite element and finite difference methods are discussed to numerically solve the FPK equation. In a paper of Kumar and Narayanan, [103], the numerical solution of the FPK equation by the finite element method, and the finite difference method, for nonlinear systems is summarized and examples presented. The numerical solution of the FPK equation demands significant computer resources to calculate both the stationary and transient (non-stationary) probability density functions.

The FPK equation is strongly related to the Itô and Stratonovich *stochastic differential equation* (SDE). The difference between both SDEs will be explained.

This chapter includes many examples and problems.

5.2 Probability Theory

The following set of probability density functions completely defines a random function [26, 218]:

- $f_1(x, t)dx$ is the probability of finding x in the range x to $x + dx$ at time t .
- $f_2(x_1, t_1, x_2, t_2)dx_1dx_2$ is the probability of finding x in the range x_1 to $x_1 + dx_1$ at time t_1 and x_2 in the range x_2 to $x_2 + dx_2$ at time t_2 .
- $f_3(x_1, t_1, x_2, t_2, x_3, t_3)dx_1dx_2dx_3$ is the probability of finding x in the range x_1 to $x_1 + dx_1$ at time t_1 , in the range x_2 to $x_2 + dx_2$ at time t_2 and in the range x_3 to $x_3 + dx_3$ at time t_3 .

The higher probability density functions $f_n(x_1, t_1, \dots, x_n, t_n)$ are defined in a similar manner, however, each f_n must satisfy the following criteria:

- $f_n \geq 0$
- f_n is symmetric in the pairs $x_1, t_1, x_2, t_2, \dots, x_n, t_n$
- $f_k = \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} f_n dx_{k+1} \dots dx_n$.

The last equation determines a *marginal probability*. The probability function f_n can be used as a means classifying a purely random function, which means that the value of x at some time t_1 does not depend upon, or is not correlated with, the value of x at any other time t_2 . The probability $f_1(x, t)dx$ completely describes the function in this case, since the higher probability density function f_n are found from the following equation

$$f_n(x_1, t_1, x_2, t_2, \dots, x_n, t_n) = \prod_{i=1}^n f_1(x_i, t_i). \quad (5.1)$$

It becomes more complicated when the probability density f_2 completely describes the random functions. This is a so-called **Markoff process**.¹ To

¹ Also called a Markov process or a Markovian process [149].

define a *Markoff process* more precisely the *conditional probability* will be introduced. The conditional (transition) probability density function is defined by $f_{c_2}(x_1, t_1|x_2, t_2)dx_2$ as the probability that, for a given $x = x_1$ at $t = t_1$, the value of x is in the range from x_2 to $x_2 + dx_2$ at a time t_2 later.

$$f_2(x_1, t_1, x_2, t_2) = f_1(x_1, t_1)f_{c_2}(x_1|x_2, t_2 - t_1). \quad (5.2)$$

The function f_{c_2} must satisfy the following conditions:

- $f_{c_2}(x_1, t_1|x_2, t_2) \geq 0$;
- $\int_{-\infty}^{\infty} f_{c_2}(x_1, t_1|x_2, t_2)dx_2 = 1$;
- $f_1(x_2, t_2) = \int_{-\infty}^{\infty} f_1(x_1, t_1)f_{c_2}(x_1|x_2, t_2 - t_1)dx_1$.

For *Brownian motion*

$$\lim_{t \rightarrow \infty} f_{c_2}(x_1|x_2, t) = f_1(x_2). \quad (5.3)$$

5.3 Markoff Process

A Markoff process is defined such that the conditional probability that x lies in the interval, from x_1 to $x_1 + dx_1$ at t_1 , from x_2 to $x_2 + dx_2$ at t_2 , ... from x_{n-1} to $x_{n-1} + dx_{n-1}$ at t_{n-1} depends only on the values of x at t_n and t_{n-1} . Thus for a Markoff process

$$f_{c_n}(x_1, t_1, x_2, t_2, \dots, x_{n-1}, t_{n-1}|x_n, t_n) = f_{c_2}(x_{n-1}, t_{n-1}|x_n, t_n). \quad (5.4)$$

It is possible to derive f_3, f_4, \dots from f_2 and (5.2), e.g.

$$\begin{aligned} f_3(x_1, t_1, x_2, t_2, x_3, t_3) &= f_2(x_1, t_1, x_2, t_2)f_{c_2}(x_2, t_2|x_3, t_3) \\ &= \frac{f_2(x_1, t_1, x_2, t_2)f_2(x_2, t_2, x_3, t_3)}{f_1(x_2, t_2)}, \end{aligned} \quad (5.5)$$

and

$$\begin{aligned} f_4(x_1, t_1, x_2, t_2, x_3, t_3, x_4, t_4) &= f_2(x_1, t_1, x_2, t_2, x_3, t_3)f_{c_2}(x_3, t_3|x_4, t_4) \\ &= \frac{f_2(x_1, t_1, x_2, t_2)f_2(x_2, t_2, x_3, t_3)}{f_1(x_2, t_2)} \\ &\quad \times \frac{f_2(x_3, t_3, x_4, t_4)}{f_1(x_3, t_3)}, \end{aligned} \quad (5.6)$$

using (5.2) and (5.5). The conditional probability density function must also satisfy the Smoluchowski or the Chapman-Kolmogorov equation, which will be discussed in the next section.

5.4 Smoluchowki or Chapman-Kolmogorov Equation

The Smoluchowki or Chapman-Kolmogorov equation [165, 218] is

$$f_{c_2}(x_1, t_1|x_2, t_2) = \int_{-\infty}^{\infty} f_{c_2}(x_1, t_1|x, t)f_{c_2}(x, t|x_2, t_2 - t)dx. \tag{5.7}$$

If $t_1 = 0$, the *time homogeneity* equation (5.7) can be written

$$f_{c_2}(x_1|x_2, t_2) = \int_{-\infty}^{\infty} f_{c_2}(x_1|x, \tau)f_{c_2}(x, \tau|x_2, t_2 - \tau)dx, \tag{5.8}$$

as illustrated in Fig. 5.1.

The joint probability density function f_2 is obtained from f_3 by integrating over one coordinate:

$$f_2(x_1, t_1, x_2, t_2) = \int_{-\infty}^{\infty} f_3(x_1, t_1, x, t, x_2, t_2)dx \tag{5.9}$$

The joint probability function f_2 is written as follows:

$$f_2(x_1, t_1, x_2, t_2) = f_1(x_1, t_1)f_{c_2}(x_1, t_1|x_2, t_2), \tag{5.10}$$

thus

$$f_1(x_1, t_1)f_{c_2}(x_1, t_1|x_2, t_2) = \int_{-\infty}^{\infty} f_1(x_1, t_1)f_{c_2}(x_1, t_1|x, t)f_{c_2}(x, t|x_2, t_2 - t)dx. \tag{5.11}$$

Because $f_1(x_1, t_1)$ is arbitrary, the Smoluchowski or Chapman-Kolmogorov equation is obtained:

$$f_{c_2}(x_1, t_1|x_2, t_2) = \int_{-\infty}^{\infty} f_{c_2}(x_1, t_1|x, t)f_{c_2}(x, t|x_2, t_2 - t)dx. \tag{5.12}$$

Equation (5.12) is interpreted as follows. The transition probability density from x_1 at time t_1 to x_2 at time t_2 is the same as the transition probability

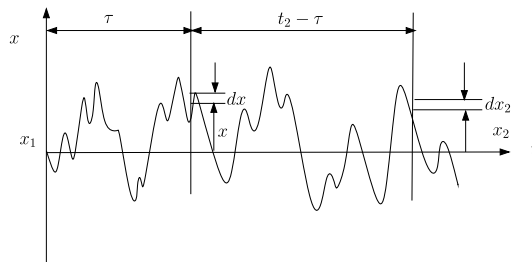


Fig. 5.1. Illustration of the Smoluchowski or Chapman-Kolmogorov equation

density from x_1 at time t_1 to x at time t times the transition probability density from x at time t to x_2 at time t_2 for all possible x and $t_1 < t < t_2$. Equation (5.12) may also be expressed in terms of the probability function P :

$$P(x_1, t_1 | x_2, t_2) = \int_{-\infty}^{\infty} P(x_1, t_1 | x, t) P(x, t | x_2, t_2 - t) dx. \tag{5.13}$$

For two random variables y and z , (5.12) can be written

$$f_{c_2}(y_1 z_1, t_1 | y_2 z_2, t_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{c_2}(y_1 z_1, t_1 | yz, t) f_{c_2}(yz, t | y_2 z_2, t_2 - t) dy dz, \tag{5.14}$$

and for an N -dimensional state-space $\{x\}$ (5.12) and (5.14) becomes

$$\begin{aligned} f_{c_2}(\{x_1\}, t_1 | \{x_2\}, t_2) \\ = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \prod_{i=1}^N f_{c_2}(\{x_1\}, t_1 | \{x\}, t) f_{c_2}(\{x\}, t | \{x_2\}, t_2 - t) dx_i. \end{aligned} \tag{5.15}$$

5.5 Derivation of the Fokker-Planck-Kolmogorov Equation

The derivation of the Fokker²-Planck-Kolmogorov (FPK) equation is based on the work of Wang and Uhlenbeck presented in [26, 218]. From the elementary theory of probability in random processes, the n th order moment of a random process x is given by

$$E\{x^n\} = \langle x^n \rangle = \int_{-\infty}^{\infty} x^n f(x) dx. \tag{5.16}$$

For a Markoff process, the transitional behavior is of concern, in particular the change of various moments of the process with time. A random process has a certain value x_1 at time t and which changes by a small amount to the a value x at time $t + dt$ as illustrated in Fig. 5.2. The moments of change in the process in time dt , analogous to (5.16) are defined and assumed to exist for one-dimensional state-space as follows

$$a_n(x, dt) = \int_{-\infty}^{\infty} (x_2 - x)^n f_{c_2}(x | x_2, dt) dx_2, \quad n = 1, 2, \dots \tag{5.17}$$

The rate of changes of the first and second moments of the increments in x_2 are

$$\begin{cases} A(x) = \lim_{dt \rightarrow 0} \frac{d}{dt} [a_1(x, dt)], & n = 1; \\ B(x) = \lim_{dt \rightarrow 0} \frac{d}{dt} [a_2(x, dt)], & n = 2. \end{cases} \tag{5.18}$$

² Adriaan Daniël Fokker 1887–1972.

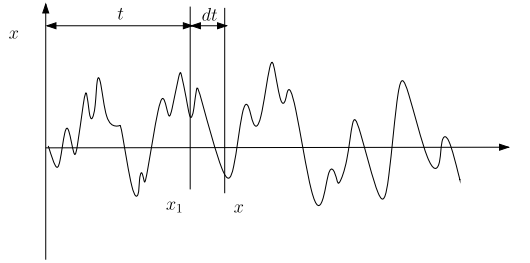


Fig. 5.2. Illustration Smoluchowski or Chapman-Kolmogorov equation

Only the first and second moments are assumed non zero in the limit as $dt \rightarrow 0$.

The Smoluchowski or Chapman-Kolmogorov equation of the Markoff process in one-dimensional form is written

$$f_{c_2}(x_1|x_2, t + dt) = \int_{-\infty}^{\infty} f_{c_2}(x_1|x, t) f_{c_2}(x, t|x_2, dt) dx. \tag{5.19}$$

Consider an infinitely differentiable and arbitrary function $R(x)$ such that

$$\lim_{x \rightarrow \pm\infty} \frac{d^n R(x)}{dx^n} = 0, \quad \text{for any } n > 0. \tag{5.20}$$

For example $R(x)$ could be a decaying exponential function. Multiplying (5.19) by $R(x)$ and integrating over all phase space x yields

$$\begin{aligned} & \int_{-\infty}^{\infty} R(x_2) f_{c_2}(x_1|x_2, t + dt) dx_2 \\ &= \int_{-\infty}^{\infty} R(x_2) dx_2 \int_{-\infty}^{\infty} f_{c_2}(x_1|x, t) f_{c_2}(x, t|x_2, dt) dx. \end{aligned} \tag{5.21}$$

To describe the variation of $f_{c_2}(x_1|x_2, t+dt)$ with time, we write $\frac{\partial f_{c_2}(x_1|x_2, t)}{\partial t} = \lim_{dt \rightarrow 0} [f_{c_2}(x_1|x_2, t + dt) - f_{c_2}(x_1|x_2, t)]/dt$. Thus

$$\begin{aligned} & \lim_{dt \rightarrow 0} \frac{d}{dt} \int_{-\infty}^{\infty} R(x_2) f_{c_2}(x_1|x_2, t + dt) dx_2 \\ &= \lim_{dt \rightarrow 0} \frac{d}{dt} \int_{-\infty}^{\infty} R(x_2) dx_2 \int_{-\infty}^{\infty} f_{c_2}(x_1|x, t) f_{c_2}(x, t|x_2, dt) dx. \end{aligned} \tag{5.22}$$

Since x_2 and x are close together, $R(x_2)$ can be developed in a Taylor series expansion about the point x

$$R(x_2) = R(x) + (x_2 - x)R'(x) + \frac{1}{2}(x_2 - x)^2 R''(x) + \dots, \tag{5.23}$$

where higher order terms are ignored. Equation (5.23) is now inserted in (5.22)

$$\begin{aligned} & \lim_{dt \rightarrow 0} \frac{d}{dt} \int_{-\infty}^{\infty} R(x_2) f_{c_2}(x_1|x_2, t + dt) dx_2 \\ &= \lim_{dt \rightarrow 0} \frac{d}{dt} \int_{-\infty}^{\infty} f_{c_2}(x_1|x, t) \\ & \quad \times \int_{-\infty}^{\infty} \left[R(x) + (x_2 - x)R'(x) + \frac{1}{2}(x_2 - x)^2 R''(x) \right] f_{c_2}(x, t|x_2, dt) dx dx_2. \end{aligned} \tag{5.24}$$

If $dt \rightarrow 0$ and $x_2 \rightarrow x$ then (5.24) becomes after substituting (5.18)

$$\begin{aligned} & \int_{-\infty}^{\infty} R(x) \frac{\partial f_{c_2}(x_1|x, t)}{\partial t} dx \\ &= \lim_{dt \rightarrow 0} \frac{d}{dt} \int_{-\infty}^{\infty} R(x) [f_{c_2}(x_1|x, t + dt) - f_{c_2}(x_1|x, t)] dx \\ &= \int_{-\infty}^{\infty} f_{c_2}(x_1|x, t) \left[A(x)R'(x) + \frac{1}{2}B(x)R''(x) \right] dx. \end{aligned} \tag{5.25}$$

The conditional probability density function $f_{c_2}(x_1|x, t)$ will be abbreviated by f_{c_2} . Integrating (5.25) twice by parts yields

$$\begin{aligned} \int_{-\infty}^{\infty} R(x) \frac{\partial f_{c_2}}{\partial t} dx &= f_{c_2} A(x)R(x)|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} R(x) \frac{\partial}{\partial x} [A(x)f_{c_2}] dx \\ & \quad + \frac{1}{2} f_{c_2} R'(x)B(x)|_{-\infty}^{\infty} - R(x) \frac{\partial}{\partial x} [B(x)f_{c_2}]|_{-\infty}^{\infty} \\ & \quad + \int_{-\infty}^{\infty} R(x) \frac{\partial^2}{\partial x^2} [B(x)f_{c_2}] dx. \end{aligned} \tag{5.26}$$

Recalling (5.20) and rearranging terms in (5.26), we find that

$$\int_{-\infty}^{\infty} R(x) \left\{ \frac{\partial f_{c_2}}{\partial x} + \frac{\partial}{\partial x} [A(x)f_{c_2}] - \frac{1}{2} \frac{\partial^2}{\partial x^2} [B(x)f_{c_2}] \right\} dx = 0. \tag{5.27}$$

Let $R(x)$ be an arbitrary function, (5.27) must hold for any $R(x)$, thus the expression between the braces must be zero. The FPK equation or Kolmogorov forward equation, for a one-dimensional case, becomes

$$\frac{\partial f_{c_2}}{\partial t} = - \frac{\partial}{\partial x} [A(x)f_{c_2}] + \frac{1}{2} \frac{\partial^2}{\partial x^2} [B(x)f_{c_2}]. \tag{5.28}$$

The term containing $A(x)$ is known as the *drift* term with deterministic behavior of the system, while the term $B(x)$ is known as the *diffusion* term due to the stochastic nature of the excitation or load.

The Pawula theorem states [164, 165] that the generalized Fokker-Planck equation with finite derivatives greater than two leads to a contradiction to the positivity of the distribution function.

If $x \rightarrow x_1$ and $t \rightarrow 0$ the conditional probability density function f_{c_2} approaches unity. For the value if x_1 in (5.28) as $t \rightarrow 0$

$$\lim_{t \rightarrow 0} f_{c_2}(x_1|x, t) \rightarrow \delta(x - x_1), \quad (5.29)$$

where $\delta(x - x_1)$ is the Dirac delta function. Equation (5.29) implies that no transition from one state to another can take place in zero time. The solution of (5.28) must satisfy the initial conditions (5.29) and boundary conditions

$$\lim_{|x| \rightarrow \pm\infty} f_{c_2}(x_1|x, t) \rightarrow 0, \quad (5.30)$$

For an N -dimensional state-space system the FPK equation is a parabolic partial differential equation³ (PDE) [26], which can be written as follows

$$\frac{\partial f_{c_2}}{\partial t} = - \sum_{i=1}^N \frac{\partial}{\partial x_i} [A_i(x) f_{c_2}] + \frac{1}{2} \sum_{i=1}^N \sum_{j=1}^N \frac{\partial^2}{\partial x_i \partial x_j} [B_{ij}(x) f_{c_2}]. \quad (5.31)$$

The required solution of (5.31) is the positive solution satisfying the initial conditions

$$\lim_{t \rightarrow 0} f_{c_2}(\{x_1\}|\{x\}, t) \rightarrow \prod_{i=1}^N \delta(x_i - x_{1,i}), \quad (5.32)$$

where $x_{1,i}$ is the initial value of x_i . Thus, the conditional probability density function approaches unity when the time approaches zero (for $t \geq 0$). The boundary conditions have to fulfill

$$\lim_{|x_i| \rightarrow \pm\infty} f_{c_2}(\{x_1\}|\{x\}, t) \rightarrow 0, \quad i = 1, 2, \dots, N. \quad (5.33)$$

Stationary Solution

If the conditional probability density function $f_{c_2}(x_1|x, t)$ is independent of the time, the joint probability function will become a stationary probability density function, thus

$$f_{c_2}(x_1|x, t) = f(x), \quad (5.34)$$

and

$$\frac{\partial f_{c_2}}{\partial t} = 0. \quad (5.35)$$

The stationary solution is often called the steady-state solution.

³ A parabolic partial differential equation is a type of second-order partial differential equation, describing a wide family of problem in science including heat diffusion and stock option pricing. These problems, also known as evolution problems, describe physical or mathematical systems with a time variable, and which behave essentially like heat diffusing through a medium like a metal plate.

The N -dimensional FPK equation (5.31) can be written

$$\frac{1}{2} \sum_{i=1}^N \sum_{j=1}^N \frac{\partial^2}{\partial x_i \partial x_j} [B_{ij}(x)f(x)] - \sum_{i=1}^N \frac{\partial}{\partial x_i} [A_i(x)f(x)] = 0. \quad (5.36)$$

In [27] solution techniques are provided for solving the steady-state response of systems exposed to stochastic excitation applying by the FPK equation.

Example. The following example of a SDOF nonlinear system is taken from [26, 98]. The system is defined by

$$\ddot{x} + 2\zeta\omega_n\dot{x} + F(x) = \xi(t), \quad (5.37)$$

where $\xi(t)$ is a stationary Gaussian exciting force per unit of mass having a white noise spectrum, and the non-linearity is present only in the spring stiffness component, ζ is the damping ratio and ω_n is natural frequency of the linearized spring stiffness (e.g. $F(x) = \frac{k_1x}{m} + \frac{k_2x^2}{m} + \dots = \omega_n^2x + \dots$, m is the mass of the system). If the nonlinear oscillator is fixed, then $x(t)$ is the absolute displacement response of the mass of the oscillator, and $\xi(t)$ is the ratio of the exciting force to the mass. On the other hand, if the nonlinear oscillator is attached to an oscillating base (e.g. enforced acceleration), then $x(t)$ is the displacement response of the mass relative to the base and $\xi(t)$ is the negative of the enforced random acceleration at the base. Equation (5.37) will be written in the state-space form, writing $y_1 = x$ and $y_2 = \dot{x}$

$$\begin{cases} \dot{y}_1 = y_2, \\ \dot{y}_2 = -2\zeta\omega_n y_2 - F(y_1) + \xi(t). \end{cases} \quad (5.38)$$

The state-space equation is two dimensional corresponding to the dimensions of y_1 and y_2 . Deriving the FPK equation (5.31), $N=2$ and $i, j = 1, 2$. The first step is evaluating the average values of the first and second order moments for $N = 2$

$$\begin{cases} a_i(z, dt) = \int_{-\infty}^{\infty} (y_i - z_i) f_{c_2}(y|z, dt) dz_1 dz_2, \\ b_{ij}(z, dt) = \int_{-\infty}^{\infty} (y_i - z_i)(y_j - z_j) f_{c_2}(y|z, dt) dz_1 dz_2, \quad i, j = 1, 2. \end{cases} \quad (5.39)$$

$$\begin{cases} A_1(x) = \lim_{dt \rightarrow 0} \frac{d}{dt} [a_1(x, dt)], \quad i = 1, 2; \\ B_{ij}(x) = \lim_{dt \rightarrow 0} \frac{d}{dt} [b_{ij}(x, dt)], \quad i, j = 1, 2. \end{cases} \quad (5.40)$$

In the limit as $dt \rightarrow 0$ and $y_i \rightarrow x_i$ $f_{c_2} \rightarrow 0$. Noting that $y_i - z_i = dy_i$, then

$$\begin{cases} a_1 = \langle dy_1 \rangle \\ a_2 = \langle dy_2 \rangle \\ b_{11} = \langle dy_1^2 \rangle \\ b_{12} = b_{21} = \langle dy_1 dy_2 \rangle \\ b_{22} = \langle dy_2^2 \rangle, \end{cases} \quad (5.41)$$

thus

$$\begin{cases} A_1 = \lim_{dt \rightarrow 0} \frac{\langle dy_1 \rangle}{dt} = y_2 = \dot{x}, \\ A_2 = \lim_{dt \rightarrow 0} \frac{\langle dy_2 \rangle}{dt} = -2\zeta\omega_n\dot{x} - F(x), \\ B_{11} = \lim_{dt \rightarrow 0} \frac{\langle dy_1^2 \rangle}{dt} \rightarrow 0, \\ B_{12} = B_{21} = \lim_{dt \rightarrow 0} \frac{\langle dy_1 dy_2 \rangle}{dt} = 0, \\ B_{22} = \lim_{dt \rightarrow 0} \frac{\langle dy_2^2 \rangle}{dt} = (2\zeta\omega_n)^2 \dot{x}^2 dt + S_f = S_f. \end{cases} \quad (5.42)$$

The following remarks must be made. The term $B_{11} \rightarrow 0$ because it is of second order. The term B_{11} is evaluated using (5.38). To determine the averaged value of A_2 and B_{22} , $\langle dy_2 \rangle$ will be integrated over a short time dt

$$\langle dy_2 \rangle = -2\zeta\omega_n \langle \dot{x} \rangle dt - \langle F(x) \rangle dt + \int_t^{t+dt} \langle f(\varsigma) \rangle d\varsigma, \quad (5.43)$$

however, since $\langle \xi(t) \rangle = 0$, $\langle dy_2 \rangle = -2\zeta\omega_n \dot{x} dt - F(x) dt$ and $A_2 = -2\zeta\omega_n \dot{x} - F(x)$. To determine B_{22} , the term $\langle dy_2^2 \rangle$ must be evaluated.

$$\begin{aligned} \langle dy_2^2 \rangle &= \left\langle \zeta\omega_n \dot{x} dt - F(x) dt + \int_t^{t+dt} \xi(\varsigma) d\varsigma \right\rangle \\ &\times \left\langle \zeta\omega_n \dot{x} dt - F(x) dt + \int_t^{t+dt} \xi(\eta) d\eta \right\rangle. \end{aligned} \quad (5.44)$$

Now $F(y_1)$ is the nonlinear force/unit of mass expressed in the characteristic form

$$F(y_1) = k_1 y_1 + k_2 y_1^2 + k_3 y_1^3 + \dots \quad (5.45)$$

For a small nonlinearity in stiffness it is assumed that the response to a stationary random excitation is also stationary. In that case the correlation $\langle \dot{x} \sum x^n \rangle = 0$.

Furthermore, for stationary white noise force $\xi(t)$, the autocorrelation function of the process is given by

$$\langle \xi(t)\xi(t + \tau) \rangle = R_f(\tau) = S_f \delta(\tau), \quad (5.46)$$

where S_f is the constant PSD function for the white noise force, and $\delta(\tau)$ is the delta function. The autocorrelation function becomes

$$\begin{cases} R_f(\tau) = 0, & \tau \neq 0; \\ R_f(0) = \langle \xi(t)^2 \rangle = S_f, & \tau = 0. \end{cases} \quad (5.47)$$

The second moment B_{22} can now be obtained:

$$B_{22} = \lim_{dt \rightarrow 0} \frac{\langle dy_2^2 \rangle}{dt} = (2\zeta\omega_n)^2 y_2^2 dt + S_f = S_f. \quad (5.48)$$

Equation (5.48) states that the time rate of change of the mean square acceleration is constant. Substituting the first moments A_i and second moments B_{ij} , $i, j = 1, 2$, we find that (5.31) becomes

$$\frac{\partial f_{c_2}}{\partial t} = -\frac{\partial}{\partial y_1}[A_1 f_{c_2}] - \frac{\partial}{\partial y_2}[A_2 f_{c_2}] + \frac{1}{2} \frac{\partial^2}{\partial y_2^2}[B_{22} f_{c_2}], \quad (5.49)$$

or

$$-\frac{\partial}{\partial y_1}[y_2 f_{c_2}] + \frac{\partial}{\partial y_2}[(2\zeta\omega_n y_2 + F(y_1))f_{c_2}] + \frac{S_f}{2} \frac{\partial^2 f_{c_2}}{\partial y_2^2} = 0. \quad (5.50)$$

This is the stationary form of Kramer's equation and will be solved in accordance with [26]. Equation (5.50) can be factorized as follows

$$\left[2\zeta\omega_n \frac{\partial}{\partial y_2} - \frac{\partial f_{c_2}}{\partial y_1}\right] \left[f_{c_2} y_2 + \frac{S_f}{4\zeta\omega_n} \frac{\partial}{\partial y_2}\right] + \frac{\partial}{\partial y_2} \left[(f_{c_2} F(y_1)) + \frac{S_f}{4\zeta\omega_n} \frac{\partial f_{c_2}}{\partial y_1}\right] = 0. \quad (5.51)$$

We see that $f_{c_2} = f_2(y_1, y_2) = f(y_1)f(y_2) = f(\dot{x})f(x)$ is a function of the displacement and velocity. One way to solve (5.51) is to take

$$f(y_2)y_2 + \frac{S_f}{4\zeta\omega_n} \frac{\partial f(y_2)}{\partial y_2} = 0, \quad (5.52)$$

and

$$f(y_1)F(y_1) + \frac{S_f}{4\zeta\omega_n} \frac{\partial f_1}{\partial y_1} = 0. \quad (5.53)$$

The solutions of (5.52) and (5.53) are respectively

$$f(y_1) = A e^{-\frac{4\zeta\omega_n}{S_f} \int_0^{y_1} F(\eta) d\eta}, \quad (5.54)$$

and

$$f(y_2) = B e^{-\frac{4\zeta\omega_n}{S_f} \frac{y_2^2}{2}}. \quad (5.55)$$

The probability density function $f_2(y_1, y_2)$ becomes

$$f_2(y_1, y_2) = f(y_1)f(y_2) = C e^{-\frac{4\zeta\omega_n}{S_f} [\frac{y_2^2}{2} + \int_0^{y_1} F(\eta) d\eta]}. \quad (5.56)$$

Equation (5.56) can also be written

$$f_2(x, \dot{x}) = C e^{-\frac{4\zeta\omega_n H}{S_f}}, \quad (5.57)$$

where $H = \frac{y_2^2}{2} + \int_0^{y_1} F(\eta) d\eta$, the total energy of the system per unit of mass. As already noticed, the probability density function of the displacement and the velocity are statistically independent. The probability density function $f(x)$ is non Gaussian if the nonlinear spring stiffness is present.

The spring stiffness per unit of mass is given by

$$F(\eta) = \sum_{i=1}^n k_i x^i, \quad (5.58)$$

representing a hardening spring, then (5.57) yields

$$f(x) = Ae^{-\frac{4\zeta\omega_n}{S_f} \sum_{i=1}^n \frac{k_i x^{i+1}}{i+1}}. \quad (5.59)$$

The hardening spring stiffness reduces the large displacement peaks in a random process, thereby reducing the rms response. The presence of such nonlinearities in structures subject to random excitation is beneficial and there is a case for introducing them at the design stage wherever possible.

The probability density function of the displacement of a SDOF linear

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2x = \frac{f(t)}{m}$$

system can be written as

$$f(x) = Ae^{-\frac{4\zeta\omega_n m^2}{S_f} \frac{\omega_n^2 x^2}{2}}, \quad (5.60)$$

and the probability function of the velocity is

$$f(\dot{x}) = Be^{-\frac{4\zeta\omega_n m^2}{S_f} \frac{\dot{x}^2}{2}}, \quad (5.61)$$

The probability density functions $f(x)$ and $f(\dot{x})$ are symmetric with respect to $x = 0$ and $\dot{x} = 0$, because the average values of x and \dot{x} are equal to zero ($\langle x \rangle = \int_{-\infty}^{\infty} x f(x) dx = 0$ and $\langle \dot{x} \rangle = \int_{-\infty}^{\infty} \dot{x} f(\dot{x}) d\dot{x} = 0$), thus

$$\int_{-\infty}^{\infty} f(x) dx = 2 \int_0^{\infty} f(x) dx = 1, \quad \int_{-\infty}^{\infty} f(\dot{x}) dx = 2 \int_0^{\infty} f(\dot{x}) d\dot{x} = 1, \quad (5.62)$$

Thus the constants A and B can now be calculated using (see also Appendix B)

$$\int_0^{\infty} e^{-ax^2} dx = \frac{1}{2} \sqrt{\frac{\pi}{a}}. \quad (5.63)$$

Thus both constants A and B become

$$A = \frac{\sqrt{\frac{2\zeta\omega_n^3 m^2}{S_f}}}{\sqrt{\pi}}, \quad B = \frac{\sqrt{\frac{2\zeta\omega_n m^2}{S_f}}}{\sqrt{\pi}}. \quad (5.64)$$

The variance of x and \dot{x} can now be obtained using the following standard integral

$$\int_0^{\infty} x^2 e^{-ax^2} dx = \frac{1}{4a} \sqrt{\frac{\pi}{a}}. \quad (5.65)$$

Thus

$$\text{Var}\{x\} = E\{x^2\} = \langle x^2 \rangle = \sigma_x^2 = 2 \int_0^\infty x^2 f(x) dx = \frac{S_f}{4\zeta\omega_n^3 m^2}, \quad (5.66)$$

and

$$\text{Var}\{\dot{x}\} = E\{\dot{x}^2\} = \langle \dot{x}^2 \rangle = \sigma_{\dot{x}}^2 = 2 \int_0^\infty \dot{x}^2 f(\dot{x}) d\dot{x} = \frac{S_f}{4\zeta\omega_n m^2}. \quad (5.67)$$

Equations (5.66) and (5.67) were derived in Sect. 2.4.1 by using the spectral approach.

5.5.1 Calculation of FPK Equation Coefficients

The coefficients A , and B , (5.18) in (5.28), can be calculated for several typical cases of space-state differential equations given by [149]

$$\dot{x}_i = \psi_i[x_1, x_2, \dots, x_n; \xi_1(t), \xi_2(t), \dots, \xi_n(t)], \quad i = 1, 2, \dots, n, \quad (5.68)$$

where ψ_i are linear or nonlinear functions of the state variables x_i , $i = 1, 2, \dots, n$, and the white noise random stationary functions ξ_i , $i = 1, 2, \dots, n$. The auto and cross correlation of $\xi_i(t)$ and $\xi_j(t)$ is of the δ type with a PSD S_{ij} , so that

$$R_{\xi_i \xi_j}(\tau) = S_{ij} \delta(\tau), \quad i, j = 1, 2, \dots, n. \quad (5.69)$$

The output from the system (5.68) will be a Markoff process.

The following state-space system of equations will be examined

$$\dot{x}_i = \psi_i(x_1, x_2, \dots, x_n) + \xi_i(t), \quad i = 1, 2, \dots, n, \quad (5.70)$$

The external forces $\xi_i(t)$ are stationary, statistically independent white noise with the following characteristics

$$\begin{aligned} E\{\xi_i(t)\} &= \langle \xi_i(t) \rangle = 0, \\ E\{\xi_i(t)\xi_j(t+\tau)\} &= R_{\xi_i \xi_j} = S_j \delta_{ij} \delta(\tau), \quad i, j = 1, 2, \dots, n, \end{aligned} \quad (5.71)$$

where S_j are constant spectral densities of the j th process, and δ_{ij} is the Kronecker delta.

Equation (5.70) denotes the Markoff process $[X_1(t), X_2(t), \dots, X_n(t)]$ for its probability density, provided $x_i(t)$, $i = 1, 2, \dots, n$ for $t \leq t_0$ depends only on the value $x_i(t_0)$ since

$$x_i(t) = x_i(t_0) + \int_{t_0}^t \psi_i(x_1, x_2, \dots, x_n) d\tau + \int_{t_0}^t \xi_i(\tau) d\tau \quad (5.72)$$

does not depend on time up to t_0 . From (5.70) the increments can be calculated:

$$\Delta_i = x_i(t+\delta t) - x_i(t) = \psi_i[x_1(t), x_2(t), \dots, x_n(t)] \Delta t + \int_t^{t+\Delta t} \xi_i(\tau) d\tau. \quad (5.73)$$

When (5.73) is substituted in (5.17), the term a_i becomes

$$a_i = \psi_i[x_1(t), x_2(t), \dots, x_n(t)]\Delta t + \int_t^{t+\Delta t} \langle \xi_i(\tau) \rangle d\tau; \tag{5.74}$$

equation (5.18) gives the coefficients A_i :

$$A_i = \psi_i[x_1(t), x_2(t), \dots, x_n(t)]. \tag{5.75}$$

The same procedure will be followed to obtain the coefficients B_{ij} . Equation (5.73) yields

$$\begin{aligned} \Delta x_i \Delta x_j &= \left[\psi_i \Delta t + \int_t^{t+\Delta t} \xi_i(\tau_1) d\tau_1 \right] \left[\psi_j \Delta t + \int_t^{t+\Delta t} \xi_j(\tau_2) d\tau_2 \right] \\ &= \psi_i \psi_j (\Delta t)^2 + \psi_i \Delta t \int_t^{t+\Delta t} \xi_i(\tau_1) d\tau_1 + \psi_j \Delta t \int_t^{t+\Delta t} \xi_j(\tau_1) d\tau_2 \\ &\quad + \int_t^{t+\Delta t} \int_t^{t+\Delta t} \psi_i(\tau_1) \psi_j(\tau_2) d\tau_1 d\tau_2. \end{aligned} \tag{5.76}$$

The terms b_{ij} are given by (5.17)

$$\begin{aligned} b_{ij} &= \langle \Delta x_i \Delta x_j \rangle \\ &= \psi_i \psi_j (\Delta t)^2 + \int_t^{t+\Delta t} \int_t^{t+\Delta t} \langle \psi_i(\tau_1) \psi_j(\tau_2) \rangle d\tau_1 d\tau_2 \\ &= \psi_i \psi_j (\Delta t)^2 + S_i \delta_{ij} \Delta t, \end{aligned} \tag{5.77}$$

and, applying (5.18), we can obtain the coefficients B_{ij} knowing $\Delta t \rightarrow 0$

$$B_{ij} = S_i \delta_{ij}. \tag{5.78}$$

If the loads $w_i(t)$ are statistically dependent so that

$$\langle \xi_i \xi_j \rangle = R_{\xi_i \xi_j}(\tau) = S_{ij} \delta(\tau), \tag{5.79}$$

then the coefficient B_{ij} becomes

$$B_{ij} = S_{ij}, \quad i, j = 1, 2, \dots, n. \tag{5.80}$$

Example. A nonlinear Duffing’s oscillator is described by the following equation of motion

$$\ddot{x} + 2\zeta\omega_0\dot{x} + \omega_0^2(x^2 + \varepsilon x^3) = \xi(t), \tag{5.81}$$

where $2\zeta\omega_0$ is the damping ratio, ω_0 the natural frequency, ε is the ratio of nonlinearity and $\xi(t)$ is a Gaussian white noise excitation with the statistical properties

$$E\{\xi\} = \langle \xi(t) \rangle = 0, \quad E\{\xi(t)\xi(t + \tau)\} = R_{\xi\xi}(\tau) = 2D\delta(\tau).$$

Letting $z_1 = x$ and $z_2 = \dot{x}$, we can write (5.81) the space-state form

$$\begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \end{pmatrix} = \begin{pmatrix} z_2 \\ -2\zeta\omega_0 z_2 - \omega_0^2(z_1 + \varepsilon z_1^3) \end{pmatrix} + \begin{pmatrix} 0 \\ \xi(t) \end{pmatrix}. \quad (5.82)$$

The coefficients of drift and diffusion are

$$\begin{aligned} A_1 &= z_2, \\ A_2 &= -2\zeta\omega_0 z_2 - \omega_0^2(z_1 + \varepsilon z_1^3), \\ B_{11} &= 0, \\ B_{22} &= 2D. \end{aligned}$$

Substituting the coefficients in (5.31) we find the FPK equation becomes

$$\frac{\partial f_{c2}}{\partial t} = -z_2 \frac{\partial f_{c2}}{\partial z_1} + 2\zeta\omega_0 f_{c2} + [2\zeta\omega_0 z_2 + \omega_0^2(z_1 + \varepsilon z_1^3)] \frac{\partial f_{c2}}{\partial z_2} + D \frac{\partial^2 f_{c2}}{\partial z_2^2} \quad (5.83)$$

where f_{c2} is the abbreviation of the conditional density function $f_{c2}(\{z_0\}|\{z\}, t)$.

If the space-state system of equations has a more general form

$$\dot{x}_i = \psi_i(x_1, x_2, \dots, x_n) + \sum_{k=1}^n \varphi_{ik}(x_1, x_2, \dots, x_n) \xi_k(t), \quad i = 1, 2, \dots, n, \quad (5.84)$$

where $\xi_k(t)$ also satisfy (5.71). The coefficients of FPK equation are now written, the drift terms A_i

$$A_i(x_1, \dots, x_n) = \psi_i(x_1, \dots, x_n) + \frac{1}{2} \sum_{k=1}^n \sum_{l=1}^n S_i \frac{\partial \varphi_{il}}{\partial x_k} \varphi_{kl} \quad i = 1, 2, \dots, n, \quad (5.85)$$

and the diffusion coefficients B_{ij} for the space-state system (5.84)

$$B_{ij}(x_1, \dots, x_n) = \sum_{k=1}^n S_k \varphi_{ik} \varphi_{kj} \quad i = 1, 2, \dots, n, \quad (5.86)$$

or, in the more general case

$$B_{ij}(x_1, \dots, x_n) = \sum_{k=1}^n \sum_{m=1}^n S_{km} \varphi_{ik} \varphi_{mj} \quad i = 1, 2, \dots, n. \quad (5.87)$$

If the functions ψ_i in (5.84) depend on time, the diffusion coefficient B_{ij} remain unchanged, however, the drift terms $A_i[x_1, x_2, \dots, x_n, t] = \psi_i$, $i = 1, 2, \dots, n$ are functions of time.

Example. This example is taken from [178]. The governing equation of a nonlinear oscillator under one random external excitation and two random parametric excitations is given by

$$\ddot{x} + 2\alpha\dot{x}[1 + \eta(t)] + \beta_1\dot{x}\left[x^2 + \frac{\dot{x}^2}{\Omega^2}\right] + \Omega^2x[1 + \xi(t)] = \zeta(t), \tag{5.88}$$

where α, β_1, Ω are constants and $\eta(t), \xi(t), \zeta(t)$ are independent zero mean physical Gaussian white noise processes with covariances

$$\begin{aligned} E\{\eta(t)\eta(s)\} &= D_\eta\delta(t - s), \\ E\{\xi(t)\xi(s)\} &= D_\xi\delta(t - s), \\ E\{\zeta(t)\zeta(s)\} &= D_\zeta\delta(t - s). \end{aligned}$$

Letting $z_1 = x$ and $z_2 = \dot{x}$, we can write (5.88) in state-space form

$$\begin{aligned} \begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \end{pmatrix} &= \begin{pmatrix} z_2 \\ -2\alpha z_2 - \beta_1 z_2 \left[z_1^2 + \frac{z_2^2}{\Omega^2} \right] - \Omega^2 z_1 \end{pmatrix} \\ &+ \begin{pmatrix} 0 & 0 & 0 \\ -2\alpha z_2 & -\Omega^2 z_1 & -1 \end{pmatrix} \begin{pmatrix} \eta \\ \xi \\ \zeta \end{pmatrix}. \end{aligned} \tag{5.89}$$

Equations (5.85) and (5.86) give the coefficients

$$\begin{aligned} A_1 &= z_2, \\ A_2 &= -2\alpha z_2 - \beta_1 z_2 \left[z_1^2 + \frac{z_2^2}{\Omega^2} \right] - \Omega^2 z_1, \\ B_{11} &= B_{12} = B_{21} = 0, \\ B_{22} &= 4\alpha^2 z_2^2 D_\eta + \Omega^4 z_1^2 D_\xi + D_\zeta. \end{aligned}$$

Substituting the coefficients in (5.31), we find the FPK equation

$$\begin{aligned} \frac{\partial f_{c2}}{\partial t} &= -z_2 \frac{\partial f_{c2}}{\partial z_1} + \Omega^2 z_1 \frac{\partial f_{c2}}{\partial z_2} \\ &+ \frac{\partial}{\partial z_2} \left\{ \left[(2\alpha - 2\alpha^2 D_\eta) z_2 + \beta_1 z_2 \left(z_1^2 + \frac{z_2^2}{\Omega^2} \right) \right] f_{c2} \right\} \\ &+ \frac{1}{2} \frac{\partial^2}{\partial z_2^2} [(4\alpha^2 z_2^2 D_\eta + \Omega^4 z_1^2 D_\xi + D_\zeta) f_{c2}], \end{aligned} \tag{5.90}$$

where f_{c2} is an abbreviation for the conditional density function $f_{c2}(\{z_0\}|\{z\}, t)$.

5.5.2 Exact Stationary Response Solutions of Nonlinear Dynamic Systems

The theory described in this section is based on a paper written by Wang et al. [215].

Consider the following general nonlinear SDOF system

$$\ddot{x}(t) + g[\dot{x}(t), x(t)] = \xi(t), \quad (5.91)$$

where $g[\dot{x}(t), x(t)]$ is a mass normalized nonlinear restoring force, and $\xi(t)$ is a mass normalized zero-mean Gaussian white noise with a delta-type correlation function $E\{\xi(t)\xi(t+\tau)\} = 2D\delta(\tau)$. By the introduction of the state variables $y_1 = x(t)$ and $y_2 = \dot{x}(t)$ the space-state representation of (5.91) becomes

$$\begin{aligned} \dot{y}_1 &= y_2, \\ \dot{y}_2 &= -g(y_1, y_2) + \xi(t). \end{aligned} \quad (5.92)$$

The stationary probability density function $f(y_1, y_2)$ of the system response is governed by the reduced (stationary) FPK equation

$$-y_1 \frac{\partial f}{\partial y_1} + \frac{\partial [g(y_1, y_2)f]}{\partial y_2} + D \frac{\partial^2 f}{\partial y_2^2} = 0. \quad (5.93)$$

Assume a solution for the probability density function as follows

$$f(y_1, y_2) = C \exp\left[-\frac{p(y_1, y_2)}{D}\right], \quad (5.94)$$

where C is the normalization constant ($\int_0^\infty \int_0^\infty f(y_1, y_2) dy_1 dy_2 = 1$) and $p(y_1, y_2)$ is an arbitrary nonlinear function. The probability density function (5.94) must be nonnegative and normalized for $p(y_1, y_2)$ in order to achieve a valid probability density.

Substituting (5.94) in (5.93), and deleting arguments, we find

$$f \left[y_1 \frac{\partial p}{\partial y_1} \frac{1}{D} - g \frac{\partial p}{\partial y_2} \frac{1}{D} + \left(\frac{\partial p}{\partial y_2} \right)^2 \frac{1}{D} + \frac{\partial g}{\partial y_2} - \frac{\partial^2 p}{\partial y_2^2} \right] = 0. \quad (5.95)$$

Since $f(y_1, y_2) \neq 0$, (5.95) implies

$$\frac{\partial g}{\partial y_2} - g \frac{\partial p}{\partial y_2} \frac{1}{D} + y_1 \frac{\partial p}{\partial y_1} \frac{1}{D} + \left(\frac{\partial p}{\partial y_2} \right)^2 \frac{1}{D} - \frac{\partial^2 p}{\partial y_2^2} = 0. \quad (5.96)$$

Equation (5.96) express $\partial g/\partial y_2$ as a implicit function of $p(y_1, y_2)$ with the solution

$$\begin{aligned} g(y_1, y_2) &= g_1(y_1) e^{\left[\frac{p(y_1, y_2)}{D}\right]} + \frac{\partial p(y_1, y_2)}{\partial y_2} \\ &\quad - \frac{1}{D} e^{\left[\frac{p(y_1, y_2)}{D}\right]} \int_{y_1}^{y_2} y_2 \frac{\partial p(y_1, y_2)}{\partial y_1} e^{\left[-\frac{p(y_1, y_2)}{D}\right]} dy_2, \end{aligned} \quad (5.97)$$

where

$$g_1(y_1) = \left[g(y_1, 0) - \frac{p(y_1, 0)}{D y_2} \right] e^{-\frac{p(y_1, 0)}{D}}. \quad (5.98)$$

The known function $g_1(y_1)$ in (5.98) may be arbitrarily selected under the name $p(y_1, y_2)$. We have demonstrated that $g(y_1, y_2)$ and $g(y_1, y_2) + g_1(y_1)e^{\left[\frac{p(y_1, y_2)}{D}\right]}$ fulfill (5.96) and possess the same probability density $f(y_1, y_2)$. Namely

$$\ddot{x} + g(\dot{x}, x) = \xi(t),$$

and

$$\ddot{x} + g(\dot{x}, x) + g_1(x)e^{\left[\frac{p(\dot{x}, x)}{D}\right]} = \xi(t),$$

have the same stationary probability density function

$$f(\dot{x}, x) = C \exp\left[-\frac{p(\dot{x}, x)}{D}\right].$$

Example. The function $p(y_1, y_2)$ is given by

$$p(y_1, y_2) = ay_1^n + by_1^m y_2^2. \quad (5.99)$$

Substituting (5.99) in (5.97) yields

$$\begin{aligned} g(y_1, y_2) = & \left[g_1(y_1)e^{\frac{ay_1^n}{D}} - \frac{1}{2b^2 y_1^{2m}} (any_1^{n+m-1} + Dmby_1^{m-1}) \right] e^{\frac{by_1^m y_2^2}{D}} \\ & + 2by_1^m y_2 + \frac{my_2^2}{2y_1} + \frac{an}{2b} y_1^{n-m-1} + \frac{Dm}{2by_1^{m+1}}, \end{aligned} \quad (5.100)$$

and with a choice of $g_1(y_1)$, such that the function $p(y_1, y_2)$ can be simplified to

$$g(y_1, y_2) = 2by_1^m y_2 + \frac{my_2^2}{2y_1} + \frac{an}{2b} y_1^{n-m-1} + \frac{Dm}{2by_1^{m+1}}. \quad (5.101)$$

Finally the nonlinear SDOF system is

$$\ddot{x} + 2bx^m \dot{x} + \frac{m\dot{x}^2}{2x} + \frac{an}{2b} x^{n-m-1} + \frac{Dm}{2bx^{m+1}} = \xi(t) \quad (x \neq 0). \quad (5.102)$$

Thus according to (5.94), the exact stationary joint probability density function $f(x, \dot{x})$ of the nonlinear SDOF system as defined by (5.99) is given by

$$f(x, \dot{x}) = C \exp\left[-\frac{ax^n + bx^m \dot{x}^2}{D}\right]. \quad (5.103)$$

This probability density function satisfies (5.96) in two regions $(-\infty, 0)$ and $(0, +\infty)$.

If the function $p(y_1, y_2)$ satisfies the following condition

$$\frac{\partial^2 p}{\partial y_2^2} - \frac{\partial g}{\partial y_2} = 0, \quad (5.104)$$

equation (5.96) becomes

$$-g \frac{\partial p}{\partial y_2} + y_1 \frac{\partial p}{\partial y_1} + \left(\frac{\partial p}{\partial y_2} \right)^2 = 0, \tag{5.105}$$

and $p(y_1, y_2)$ can satisfy (5.96).

Integrating (5.104), we can obtain the following expression with integration constants

$$\frac{\partial p(y_1, y_2)}{\partial y_2} = g(y_1, y_2) - g(y_1, 0) + \frac{\partial p(y_1, 0)}{\partial y_2}. \tag{5.106}$$

Further integration of (5.106) and the introduction of an appropriate integration constant yields

$$p(y_1, y_2) = \int_0^{y_2} g(y_1, y_2) dy_2 - y_2 g(y_1, 0) + y_2 \frac{\partial p(y_1, 0)}{\partial y_2} + p(y_1, 0). \tag{5.107}$$

Differentiation of (5.107) with respect to y_1 gives

$$\frac{\partial p(y_1, y_2)}{\partial y_1} = \int_0^{y_2} \frac{g(y_1, y_2)}{\partial y_1} dy_2 - y_2 \frac{\partial g(y_1, 0)}{\partial y_1} + y_2 \frac{\partial^2 p(y_1, 0)}{\partial y_2 \partial y_1} + \frac{\partial p(y_1, 0)}{\partial y_1}. \tag{5.108}$$

The results of (5.106) and (5.107) substituted in (5.105) will give

$$y_2 \left[\int_0^{y_2} \frac{g(y_1, y_2)}{\partial y_1} dy_2 - y_2 \frac{\partial g(y_1, 0)}{\partial y_1} + y_2 \frac{\partial^2 p(y_1, 0)}{\partial y_2 \partial y_1} + \frac{\partial p(y_1, 0)}{\partial y_1} \right] - \left[g(y_1, 0) - \frac{\partial p(y_1, 0)}{\partial y_2} \right] \left[g(y_1, y_2) - g(y_1, 0) + \frac{\partial p(y_1, 0)}{\partial y_2} \right] = 0, \tag{5.109}$$

and if

$$\frac{\partial p(y_1, 0)}{\partial y_2} = 0,$$

equation (5.109) can be expressed

$$\begin{aligned} \frac{\partial p(y_1, 0)}{\partial y_1} &= \frac{1}{y_2} g(y_1, 0) [g(y_1, y_2) - g_1(y_1, 0)] \\ &+ y_2 \frac{\partial g(y_1, 0)}{\partial y_1} - \int_0^{y_2} \frac{g(y_1, y_2)}{\partial y_1} dy_2. \end{aligned} \tag{5.110}$$

When the right-hand side of (5.110) is only a function of the function y_1 , then the following equation can also be established by integrating (5.110):

$$\begin{aligned} p(y_1, 0) &= \int_0^{y_2} \frac{\partial p(y_1, 0)}{\partial y_1} dy_1 \\ &= \frac{1}{y_2} \int_0^{y_1} g(y_1, 0) [g(y_1, y_2) - g_1(y_1, 0)] dy_1 + y_2 [g(y_1, 0) - g(0, 0)] \\ &\quad - \int_0^{y_2} [g(y_1, y_2) - g(0, y_2)] dy_2. \end{aligned} \tag{5.111}$$

Substituting (5.111) in (5.107) yields

$$p(y_1, y_2) = \int_0^{y_2} g(y_1, y_2) dy_2 + \frac{1}{y_2} \int_0^{y_1} g(y_1, 0)[g(y_1, y_2) - g_1(y_1, 0)] dy_1 - \int_0^{y_2} [g(y_1, y_2) - g(0, y_2)] dy_2 - y_2 g(0, 0). \quad (5.112)$$

Example. The function $g(y_1, y_2)$ is given by $g(y_1, y_2) = \beta y_2 + g_1(y_1)$. The first check is to find out if the right-hand side of (5.110) is a function of y_1 only. The right-hand side of (5.110) becomes

$$\frac{\partial p(y_1, 0)}{\partial y_1} = g_1(y_1) \beta y_2 \frac{1}{y_2} + y_2 \frac{dg_1(y_1)}{dy_1} - \int_0^{y_2} \frac{dg_1(y_1)}{dy_1} dy_2 = \beta g_1(y_1), \quad (5.113)$$

which is only a function of y_1 .

Equation (5.111) will give an expression for the function $p(y_1, y_2)$, namely

$$p(y_1, y_2) = \frac{\beta}{2} y_2^2 + \beta \int_0^{y_1} g_1(y_1) dy_1. \quad (5.114)$$

It is easily to verify that (5.104) is satisfied.

When

$$g(y_1, y_2) = \beta y_2 + g(y_1) + g_2(y_1) e^{\frac{y_2^2 \beta^2}{2D}},$$

the exact probability density is also the same function as in (5.114).

5.5.3 Stationary Solution FPK Equation of Conservative Systems

In [69] for a number of SDOF and MDOF conservative dynamic systems exposed to random excitation the stationary solution of the FPK equation had been obtained. The following conservative dynamic systems are discussed:

- Second-order system (SDOF)
- High-order system (MDOF)

Second-Order System (SDOF)

The nonlinear SDOF dynamic system is given by

$$m\ddot{x} + c\dot{x} + \frac{\partial U}{\partial x} = \xi(t), \quad (5.115)$$

where m is the discrete mass, c the viscous damping, $\frac{\partial U}{\partial x}$ represents a conservative elastic restoring force and $\xi(t)$ is the Gaussian white noise excitation with zero mean and $\langle \xi(t)\xi(t+\tau) \rangle = 2D\delta(\tau)$. The state-space equations can be written, with $y = m\dot{x}$

$$\begin{aligned}\dot{x} &= \frac{y}{m}, \\ \dot{y} &= -\frac{\partial U}{\partial x} - \frac{c}{m}y + \xi(t).\end{aligned}\tag{5.116}$$

Retaining the symbol H for the sum of the potential and kinetic energy, so that

$$H = U(x) + \frac{y^2}{2m},\tag{5.117}$$

and (5.116) can be written as Itô SDEs

$$\begin{aligned}\Delta x &= \frac{\partial H}{\partial y} \Delta t, \\ \Delta y &= \left[-\frac{\partial H}{\partial x} - c \frac{\partial H}{\partial y} \right] \Delta t + \sqrt{2D} \Delta W(t),\end{aligned}\tag{5.118}$$

where $W(t)$ is the standard Wiener process. The stationary FPK equation for the system (5.118) becomes

$$\left[-\frac{\partial}{\partial x} \left(\frac{\partial H}{\partial y} f \right) + \frac{\partial}{\partial y} \left(\frac{\partial H}{\partial x} f \right) \right] + \frac{\partial}{\partial y} \left(c \frac{\partial H}{\partial y} f \right) + D \frac{\partial^2 f}{\partial y^2} = 0.\tag{5.119}$$

The terms in the square brackets are the terms which remain if the damping and white noise are removed from the system, i.e. if the system is conservative. Find a function $f(H)$ which makes the remaining two terms in (5.119) cancel and a solution of (5.119) is found. Substitute in (5.119) the function

$$f(x, y) = f(H(x, y)),\tag{5.120}$$

which becomes

$$\frac{\partial}{\partial y} \left(c \frac{\partial H}{\partial y} f(H) \right) + D \frac{\partial^2 f(H)}{\partial y^2} = 0.\tag{5.121}$$

Integration of (5.121) with respect to y yields

$$c \frac{\partial H}{\partial y} f(H) + D \frac{\partial f(H)}{\partial y} = L(x),\tag{5.122}$$

where $L(x)$ is an arbitrary function.

In view $\lim_{y \rightarrow \infty} f(H) = 0$ and $\lim_{y \rightarrow \infty} \frac{\partial f(H)}{\partial y} = 0$ than the arbitrary function becomes

$$L(x) = 0,\tag{5.123}$$

and (5.122) simplifies to

$$c \frac{\partial H}{\partial y} f(H) + D \frac{\partial f(H)}{\partial H} \frac{\partial H}{\partial y} = 0,\tag{5.124}$$

Assume that $\partial H / \partial y$ is not identically zero, otherwise H is independent of y and hence so is f , then

$$\frac{df}{dH} + \frac{c}{D}f = 0. \tag{5.125}$$

The general solution of (5.125) is

$$f(H) = Ce^{[-\frac{c}{D}H]}, \tag{5.126}$$

where C is a constant, which is to be chosen to normalize the solution.

The result is that for the second-order system

$$\begin{aligned} \Delta x &= \frac{\partial H}{\partial y} \Delta t, \\ \Delta y &= \left[-\frac{\partial H}{\partial x} - c\frac{\partial H}{\partial y} \right] \Delta t + \sqrt{2D} \Delta W(t), \end{aligned} \tag{5.127}$$

with $D = \text{constant}$ and $C = \text{constant}$, thus the steady-state probability function $f(x, y)$ is

$$f(x, y) = Ce^{[-\frac{c}{D}H(x,y)]}, \tag{5.128}$$

Example. The nonlinear term in (5.115) will be denoted by $g(x)$, thus

$$g(x) = \frac{\partial U}{\partial x}. \tag{5.129}$$

Then from (5.117)

$$H = \int_0^x g(\theta)d\theta + g(0) + \frac{y^2}{2m}. \tag{5.130}$$

Thus for the dynamic system

$$m\ddot{x} + c\dot{x} + g(x) = \xi(t), \tag{5.131}$$

where $\langle \xi(t)\xi(t+\tau) \rangle = 2D\delta(\tau)$, results (5.128) for the steady-state probability density function

$$f(x, y) = Ce^{[-\frac{c}{D}(\int_0^x g(\theta)d\theta + \frac{y^2}{2m})]}, \tag{5.132}$$

where $y = m\dot{x}$ and C is the normalizing constant. This solution is well known.

5.5.4 High-Order System (MDOF)

Following MDOF dynamic system is illustrated in Fig. 5.3. The discrete masses $m_i, 1 = 1, 2, \dots, n$ are connected by linear or nonlinear springs (nonlinear restoring forces). The discrete mass m_n is connected to the ground with a nonlinear spring and in parallel a viscous damping characterized by c_n . The total strain energy stored in the springs is denoted by $U(x_i)$. The equations of motion of the MDOF dynamic system are given by:

$$\begin{aligned} m_i\ddot{x}_i + \frac{\partial U}{\partial x_i} &= 0, \quad i = 1, 2, \dots, n-1, \\ m_n\ddot{x}_n + \frac{\partial U}{\partial x_n} + c_n\dot{x}_n &= \xi_n(t), \end{aligned} \tag{5.133}$$

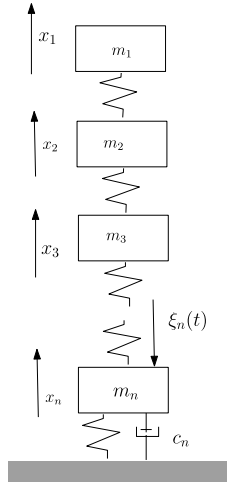


Fig. 5.3. MDOF dynamic system

where $\xi(t)$ is a Gaussian white noise force with zero-mean and $E\{\xi_n(t)\xi_n(t + \tau)\} = \langle \xi_n(t)\xi_n(t + \tau) \rangle = 2D\delta(\tau)$. To set up the state-space equations the state-space variables $\dot{x}_i = y_i/m_i$, $i = 1, 2, \dots, n$ is introduced, thus the equations of motion (5.133) can be transferred into the space-state form

$$\begin{aligned} \dot{x}_i &= \frac{y_i}{m_i} = \frac{\partial H}{\partial y_i}, \quad i = 1, 2, \dots, n, \\ \dot{y}_i &= -\frac{\partial H}{\partial x_i}, \quad i = 1, 2, \dots, n - 1, \\ \dot{y}_n &= -\frac{\partial H}{\partial x_n} + c_n \frac{\partial H}{\partial y_n} + \xi_n(t), \end{aligned} \tag{5.134}$$

where the total energy function $H(x_i, y_i)$ is given by

$$H(x_i, y_i) = U(x_i) + \frac{1}{2} \sum_{k=1}^n \frac{y_k^2}{m_k}. \tag{5.135}$$

The stationary FPK equation for the joint probability density function $f(x_i, y_i) = f(H)$ becomes

$$\left[\sum_{i=1}^n \left\{ -\frac{\partial}{\partial x_i} \left(\frac{\partial H}{\partial y_i} f \right) + \frac{\partial}{\partial y_i} \left(\frac{\partial H}{\partial x_i} f \right) \right\} \right] + \frac{\partial}{\partial y_n} \left(c_n \frac{\partial H}{\partial y_n} f \right) + D \frac{\partial^2 f}{\partial y_n^2} = 0. \tag{5.136}$$

The terms in between the square brackets cancel out if f is a function of H , thus

$$\frac{\partial}{\partial y_n} \left(c_n \frac{\partial H}{\partial y_n} f(H) \right) + D \frac{\partial^2 f(H)}{\partial y_n^2} = 0. \tag{5.137}$$

A similar solution as for (5.121) is found

$$f(x_1 \dots, x_n, y_1 \dots, y_n) = Ae^{[-\frac{c_n}{D} H(x_1 \dots, x_n, y_1 \dots, y_n)]}, \tag{5.138}$$

where A is the normalization constant.

5.6 Itô-Stratonovich Dilemma in Stochastic Processes

White noise is the formal time derivative of the *standard Wiener process*, $W(t)$, $t \geq 0$. This stochastic process is also discussed in Appendix N. The Wiener process and white noise are related with each other in the following way:

$$\begin{aligned} &(W(t_2) - W(t_1)) \perp (W(t_3) - W(t_2)), \quad t_1 < t_2 < t_3, \\ &E\{W(t) - W(\tau)\} = 0, \\ &R_{WW}(t, \tau) = 2D\min(t, \tau), \\ &\xi(t) = \frac{W(t + \Delta) - W(t)}{\Delta}, \\ &R_{\xi\xi}(t - \tau) = 2D\delta|t - \tau|, \end{aligned} \tag{5.139}$$

where the symbol \perp means perpendicular to. For a standard Wiener process, $2D = 1$. The last property is the cause of the dilemma, because it causes the inclusion of a second order term in the noise in equations that contain first order terms of time.

Consider the following one dimensional noise driven dynamical system, represented by the *Langevin equation*

$$\dot{x} = f(x, t) + g(x, t)\xi(t). \tag{5.140}$$

This equation can be written as a stochastic differential equation (SDE), and *Itô differential equation* becomes

$$\Delta x = f(x, t)\Delta t + g(x, t)\Delta W, \tag{5.141}$$

where $\Delta W = \xi\Delta t$ is the increment of the Wiener process. Equation (5.141) represents a Markoff process x . The SDE form of system dynamics is considered to be a more accurate description of the stochastic process, because of the difficulties in taking time derivatives of the Wiener process $W(t)$. Now consider the integral form of the SDE (5.141)

$$x(t) = x(t_0) + \int_{t_0}^t f(x, t)dt + \int_{t_0}^t g(x, t)dW. \tag{5.142}$$

The problem of differing interpretations comes from the second integral (which is a stochastic integral) in (5.142). It is not possible to write the standard Riemann sum form for this integral, nor the Riemann-Stieltjes' form, since the process $W(t)$ is not of bounded variation. Then, depending upon the way the equations is discretized (forward/central) we can get two interpretations of this integral, known as the Itô and Stratonovich interpretations respectively. Consider the stochastic integral of the function $h(t)$

$$\Xi = \int_0^t h(W(t), t)dW(t). \tag{5.143}$$

then, the Itô and Stratonovich interpretation of Ξ are given as (see also Appendix N)

$$\Xi_{Ito} = \lim_{\Delta \rightarrow 0} \sum_{i=0}^{N-1} h(W(t_i), t_i)[W(t_{i+1}) - W(t_i)], \tag{5.144}$$

and

$$\Xi_{Strat} = \lim_{\Delta \rightarrow 0} \sum_{i=0}^{N-1} h\left(\frac{W(t_i) + W(t_{i+1})}{2}, \frac{t_i + t_{i+1}}{2}\right)[W(t_{i+1}) - W(t_i)], \tag{5.145}$$

where $\Delta = \max(t_{i+1} - t_i)$ and $0 < t_0 < t_1 < \dots < t_N = t$. There is no reason to prefer one discretization scheme over the other, and both are equally valid. From (5.144) and (5.145) it follows directly that if $h(W(t), t) = h(t)$, $h(t)$ is independent of the noise term, and the two interpretations match. However, when this is not the case, two different results are obtained. To illustrate this, take $h(t) = W(t)$. The stochastic integral (5.143) will be evaluated in the Itô and Stratonovich senses. The Itô interpretation becomes

$$\begin{aligned} E\{\Xi_{Ito}\} &= E\left\{\sum_{i=0}^{N-1} W(t_i)[W(t_{i+1}) - W(t_i)]\right\} \\ &= \sum_{i=0}^{N-1} E\{W(t_i)W(t_{i+1}) - W(t_i)W(t_i)\} \\ &= \sum_{i=0}^{N-1} (2Dt_i - 2Dt_i) \\ &= 0. \end{aligned} \tag{5.146}$$

On the other hand, with the Stratonovich interpretation, the stochastic integral becomes

$$\begin{aligned}
 E\{\Xi_{Stat}\} &= E\left\{\sum_{i=0}^{N-1}\left[\frac{W(t_i)+W(t_{i+1})}{2}\right][W(t_{i+1})-W(t_i)]\right\} \\
 &= \frac{1}{2}\sum_{i=0}^{N-1}E\{W(t_i)W(t_{i+1})+W(t_{i+1})W(t_{i+1})\} \\
 &\quad -E\{W(t_i)W(t_i)-W(t_i)W(t_{i+1})\} \\
 &= \frac{1}{2}\sum_{i=0}^{N-1}(2Dt_i+2Dt_{i+1}-2Dt_i-2Dt_i) \\
 &= D\sum_{t=0}^{N-1}(t_{i+1}-t_i) \\
 &= Dt.
 \end{aligned}
 \tag{5.147}$$

Clearly, the results are different. Later it will be shown how these different interpretations lead to different results for the FPK diffusion equation.

The FPK equation will now be derived from the stochastic differential equation (SDE)

$$\dot{x} = f(x, t) + g(x, t)\xi(t), \tag{5.148}$$

where x is a Markoff process and is completely characterized by its transition density function $f_{c2}(y, \tau|x, t)$. A function $R(x) \geq 0$ will be considered which is a twice differentiable function and

$$\lim_{|x|\rightarrow\infty}R(x) = 0, \quad \lim_{|x|\rightarrow\infty}R'(x) = 0, \quad \lim_{|x|\rightarrow\infty}R''(x) = 0. \tag{5.149}$$

Further it is assumed that f_{c2} can be expanded in a power series about t

$$\begin{aligned}
 &\int_{-\infty}^{\infty}[f_{c2}(y, \tau|x, t+dt) - f_{c2}(y, \tau|x, t)]R(x)dx \\
 &= \int_{-\infty}^{\infty}\left[\frac{\partial f_{c2}(y, \tau|x, t)}{\partial t}\Delta t + O(\Delta t^2)\right]R(x)dx.
 \end{aligned}
 \tag{5.150}$$

And according to the Chapman-Kolmogorov equation

$$f_{c2}(y, \tau|x, t+dt) = \int_{-\infty}^{\infty}f_{c2}(y, \tau|z, t)f_{c2}(z, t|x, t+dt)dz, \tag{5.151}$$

equation (5.150) becomes

$$\begin{aligned}
 &\int_{-\infty}^{\infty}\left[\frac{\partial f_{c2}(y, \tau|x, t)}{\partial t}\Delta t\right]R(x)dx \\
 &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f_{c2}(y, \tau|x, t)f_{c2}(x, t|z, t+dt)R(z)dzdx \\
 &\quad - \int_{-\infty}^{\infty}f_{c2}(y, \tau|x, t)R(x)dx \\
 &= \int_{-\infty}^{\infty}f_{c2}(y, \tau|x, t)\left[\int_{-\infty}^{\infty}f_{c2}(x, t|z, t+dt)R(z)dz - R(x)\right]dx.
 \end{aligned}
 \tag{5.152}$$

The function $R(z)$ is expanded about x :

$$R(z) = R(x) + R'(x)(z - x) + \frac{R''(x)}{2}(z - x)^2 + O(|z - x|^3). \quad (5.153)$$

Substituting (5.153) in (5.152) yields

$$\begin{aligned} & \int_{-\infty}^{\infty} \left[\frac{\partial f_{c2}(y, \tau|x, t)}{\partial t} dt \right] R(x) dx \\ &= \int_{-\infty}^{\infty} f_{c2}(y, \tau|x, t) \left[\int_{-\infty}^{\infty} f_{c2}(x, t|z, t + dt) R(z) dz - R(x) \right] dx \\ &= \int_{-\infty}^{\infty} f_{c2}(y, \tau|x, t) \left[R(x) \int_{-\infty}^{\infty} f_{c2}(x, t|z, t + dt) dz \right. \\ &\quad + R'(x) \int_{-\infty}^{\infty} (z - x) f_{c2}(x, t|z, t + dt) dz \\ &\quad + R''(x) \int_{-\infty}^{\infty} (z - x)^2 f_{c2}(x, t|z, t + dt) dz \\ &\quad \left. - \frac{R''(x)}{2} \int_{-\infty}^{\infty} O(z - x)^3 f_{c2}(x, t|z, t + dt) dz - R(x) \right] dx \\ &= \int_{-\infty}^{\infty} f_{c2}(y, \tau|x, t) \left[R'(x) \int_{-\infty}^{\infty} (z - x) f_{c2}(x, t|z, t + dt) dz \right. \\ &\quad + R'(x) \int_{-\infty}^{\infty} (z - x) f_{c2}(x, t|z, t + dt) dz \\ &\quad + R''(x) \int_{-\infty}^{\infty} (z - x)^2 f_{c2}(x, t|z, t + dt) dz \\ &\quad \left. - \frac{R''(x)}{2} \int_{-\infty}^{\infty} O(z - x)^3 f_{c2}(x, t|z, t + dt) dz \right] dx. \end{aligned} \quad (5.154)$$

Now, looking at the integrals, we find

$$\begin{aligned} \int_{-\infty}^{\infty} (z - x)^n f_{c2}(x, t|z, t + dt) dz &= \int_{-\infty}^{\infty} (z - x)^n f_{c2}(x, t|z - x, t + dt) dz \\ &= E\{(z - x)^n, x, t\}. \end{aligned} \quad (5.155)$$

Equation (5.154) becomes

$$\begin{aligned} & \int_{-\infty}^{\infty} \left[\frac{\partial f_{c2}(y, \tau|x, t)}{\partial t} dt \right] R(x) dx \\ &\approx \int_{-\infty}^{\infty} f_{c2}(y, \tau|x, t) \left[R'(x) E\{(z - x), x, t\} + \frac{R''(x)}{2} E\{(z - x)^2, x, t\} \right] dx \\ &= \int_{-\infty}^{\infty} f_{c2}(y, \tau|x, t) \left[R'(x) E\{(\Delta x), x, t\} + \frac{R''(x)}{2} E\{\Delta x^2, x, t\} \right] dx, \end{aligned} \quad (5.156)$$

where third order terms are neglected.

Itô and Stratonovich Approach for Integration

To calculate the moments of (5.156), (5.141) can be written

$$E\{(\Delta x, x, t) = E\{f(x, t)\Delta t + g(x, t)dW(t)\}. \quad (5.157)$$

Two possible integration schemes for integration of a discretized variable can be applied, the Itô and Stratonovich approaches.

Itô Approach

In the Itô approach the integration scheme used is forward integration, that is

$$\int_{x_1}^{x_2} \Phi(x)dx = \sum_{i=0}^{N-1} [\Phi(x_i)\{x(t_{i+1}) - x(t_i)\}]. \quad (5.158)$$

The first moment $E\{(z - x), x, t\}$ becomes

$$\begin{aligned} E\{(z - x), x, t\} &= E\{f(x, t)\Delta t + g(x, t)\Delta W(t)\} \\ &= E\{f(x, t)\Delta t\} + E\{g(x, t)\Delta W(t)\} \\ &= f(x, t)\Delta t + E\{g(x, t)\}E\{\Delta W(t)\} \\ &= f(x, t)\Delta t + E\{g(x, t)\} \times 0 \\ &= f(x, t)\Delta t, \end{aligned} \quad (5.159)$$

and the second moment $E\{(z - x)^2, x, t\}$ is

$$\begin{aligned} E\{(z - x)^2, x, t\} &= E\{[f(x, t)\Delta t + g(x, t)\Delta W(t)]^2\} \\ &= E\{[f(x, t)\Delta t]^2\} + E\{[g(x, t)\Delta W(t)]^2\} \\ &\quad + 2E\{f(x, t)\Delta t g(x, t)\Delta W(t)\} \\ &= O(\Delta t)^2 + g(x, t)^2 E\{\Delta W(t)^2\} + 2f(x, t)g(x, t)E\{\Delta W(t)\} \\ &= g(x, t)^2 2D\Delta t. \end{aligned} \quad (5.160)$$

Higher order moments $E\{(z - x)^n, x, t\}$, $n > 2$ have terms in $O(\Delta t)^2$.

Stratonovich Approach

In the Stratonovich approach, the integration scheme used is the middle point integration, that is

$$\int_{x_1}^{x_2} \Phi(x)dx = \sum_{i=0}^{N-1} \Phi\left(\frac{x(t_i) + x(t_{i+1})}{2}\right)\{x(t_{i+1}) - x(t_i)\}. \quad (5.161)$$

The first moment $E\{(z - x), x, t\}$ becomes

$$E\{(z - x), x, t\} = E\left\{f\left(\frac{x(t + \Delta t) + x(t)}{2}\right)\Delta t + g\left(\frac{x(t + \Delta t) + x(t)}{2}\right)\Delta W(t)\right\}. \quad (5.162)$$

In this case $f\left(\frac{x(t + \Delta t) + x(t)}{2}\right)$ is not a deterministic function, and $g\left(\frac{x(t + \Delta t) + x(t)}{2}\right)$ is not independent of $\Delta W(t)$. The functions, f and g can be expanded about $x(t)$ as follows

$$\begin{aligned} f\left(\frac{x(t + \Delta t) + x(t)}{2}\right) &= f(x(t)) + \frac{\Delta x}{2}f'(x(t)) + \frac{1}{2}\frac{\Delta x^2}{4}f''(x(t)) + \dots, \\ g\left(\frac{x(t + \Delta t) + x(t)}{2}\right) &= g(x(t)) + \frac{\Delta x}{2}g'(x(t)) + \frac{1}{2}\frac{\Delta x^2}{4}g''(x(t)) + \dots, \end{aligned} \quad (5.163)$$

$$\begin{aligned} E\{(z - x), x, t\} &= E\{f(x, t)\Delta t\} + E\left\{\frac{\Delta x}{2}f'(x(t))\Delta t\right\} + E\left\{\frac{1}{2}\frac{\Delta x^2}{4}f''(x(t))\Delta t\right\} \\ &+ E\{g(x, t)\Delta W(t)\} + E\left\{\frac{\Delta x}{2}g'(x(t))\Delta W(t)\right\} \\ &+ E\left\{\frac{1}{2}\frac{\Delta x^2}{4}g''(x(t))\Delta W(t)\right\} + \dots. \end{aligned} \quad (5.164)$$

Using (5.141) and (5.163) in (5.164) will give

$$\begin{aligned} E\{(z - x), x, t\} &= E\{f\Delta t\} + E\left\{f'\left(\frac{[f + \frac{\Delta x}{2}f']\Delta t + [g + \frac{\Delta x}{2}g']\Delta W(t)}{2}\right)\Delta t\right\} \\ &+ E\left\{\frac{1}{2}f''\left(\frac{[f + \frac{\Delta x}{2}f']\Delta t + [g + \frac{\Delta x}{4}g'](\Delta W)^2}{2}\right)\Delta t\right\} \\ &+ E\{g\Delta W\} + E\left\{g'\left(\frac{[f + \frac{\Delta x}{2}f']\Delta t + [g + \frac{\Delta x}{2}g']\Delta W}{2}\right)\Delta W\right\} \\ &+ E\left\{\frac{1}{2}g''\left(\frac{[f + \frac{\Delta x}{2}f']\Delta t + [g + \frac{\Delta x}{4}g'](\Delta W)^2}{2}\right)\Delta W\right\} + \dots. \end{aligned} \quad (5.165)$$

The term with $E\{\Delta W\} = 0$ vanish and the terms $E\{\Delta W\}^2\Delta t = 2D\Delta t^2$ and higher order are $O(\Delta t)^2$. Thus only terms remaining are

$$\begin{aligned} E\{(z - x), x, t\} &= E\{f\Delta t\} + E\left\{\frac{1}{2}g'g\Delta W^2\right\} + O(\Delta t^2) \\ &= f(x, t)\Delta t + \frac{1}{2}g'(x, t)g(x, t)2D\Delta t, \end{aligned} \quad (5.166)$$

and the second moment $E\{(z - x)^2, x, t\}$

$$\begin{aligned}
 & E\{(z-x)^2, x, t\} \\
 &= E\left\{\left[f\left(\frac{x(t+\Delta t)+x(t)}{2}\right)\Delta t + g\left(\frac{x(t+\Delta t)+x(t)}{2}\right)\Delta W(t)\right]^2\right\} \\
 &= E\left\{\left[f\left(\frac{x(t+\Delta t)+x(t)}{2}\right)\Delta t\right]^2\right\} \\
 &\quad + 2E\left\{\left[f\left(\frac{x(t+\Delta t)+x(t)}{2}\right)g\left(\frac{x(t+\Delta t)+x(t)}{2}\right)\right]\Delta t\Delta W\right\} \\
 &\quad + E\left\{\left[g\left(\frac{x(t+\Delta t)+x(t)}{2}\right)\Delta W\right]^2\right\}. \tag{5.167}
 \end{aligned}$$

Using (5.141) and (5.163) in (5.167) will give

$$\begin{aligned}
 & E\{(z-x)^2, x, t\} \\
 &= E\{f^2\Delta t^2\} + E\left\{f'^2\left(\frac{[f + \frac{\Delta x}{2}f']\Delta t}{2}\right)^2\Delta t\right\} \\
 &\quad + E\left\{\frac{1}{2}f''^2\left(\frac{\{[f + \frac{\Delta x}{2}f']\Delta t\}^2}{4}\right)\Delta t^2\right\} + 2E\{fg\Delta t\Delta W\} \\
 &\quad + 2E\left\{fg'\frac{\Delta x}{2}\Delta t\Delta W\right\} + 2E\left\{f'g\frac{\Delta x}{2}\Delta t\Delta W\right\} + E\{g^2\Delta W^2\} \\
 &\quad + E\left\{g'^2\left(\frac{[f + \frac{\Delta x}{2}f']\Delta t}{2}\right)^2\Delta W\right\} \\
 &\quad + E\left\{\frac{1}{4}g''^2\left(\frac{\{[f + \frac{\Delta x}{2}f']\Delta t\}^2}{4}\right)\Delta W^2\right\} + \dots \tag{5.168}
 \end{aligned}$$

And in this case the only terms remaining with order $O(\Delta t)^2$ are

$$E\{(z-x)^2, x, t\} = E\{g^2\Delta W^2\} = g(x, t)^2 2D\Delta t. \tag{5.169}$$

For this term, the result of the integration is the same as found using Itô integration scheme.

Fokker-Planck-Kolmogorov Equation

Use the results of the previous sections and substitute the derived moment terms in the **Itô sense** in (5.156):

$$\begin{aligned}
 & \int_{-\infty}^{\infty} \left[\frac{\partial f_{c2}(y, \tau|x, t)}{\partial t}\Delta t\right] R(x)dx \\
 &= \int_{-\infty}^{\infty} f_{c2}(y, \tau|x, t) \left[R'(x)f(x, t)\Delta t + \frac{R''(x)}{2}g(x, t)^2 2D\Delta t\right] dx. \tag{5.170}
 \end{aligned}$$

Dividing all terms in (5.170) by Δt , and making $\Delta t \rightarrow 0$ will yield

$$\begin{aligned} & \int_{-\infty}^{\infty} \left[\frac{\partial f_{c2}(y, \tau|x, t)}{\partial t} \right] R(x) dx \\ &= \int_{-\infty}^{\infty} f_{c2}(y, \tau|x, t) \left[R'(x)f(x, t) + \frac{R''(x)}{2}g(x, t)^2 2D \right] dx. \end{aligned} \quad (5.171)$$

Integrating the right hand side of (5.171) by parts (twice the second term and applying (5.149) we find

$$\begin{aligned} & \int_{-\infty}^{\infty} \left[\frac{\partial f_{c2}(y, \tau|x, t)}{\partial t} \right] R(x) dx \\ &= - \int_{-\infty}^{\infty} R(x) \frac{\partial}{\partial x} [f_{c2}(y, \tau|x, t)f(x, t)] \\ &+ \frac{R(x)}{2} \frac{\partial^2}{\partial x^2} [f_{c2}(y, \tau|x, t)g(x, t)^2] 2D dx, \end{aligned} \quad (5.172)$$

then

$$\begin{aligned} & \int_{-\infty}^{\infty} \left[\frac{\partial f_{c2}(y, \tau|x, t)}{\partial t} \right] R(x) dx \\ &= - \int_{-\infty}^{\infty} \left\{ \frac{\partial}{\partial x} [f_{c2}(y, \tau|x, t)f(x, t)] \right. \\ &\quad \left. + D \frac{\partial^2}{\partial x^2} [f_{c2}(y, \tau|x, t)g(x, t)^2] \right\} R(x) dx. \end{aligned} \quad (5.173)$$

Since this has to hold for every $R(x)$ the FPK equation becomes

$$\begin{aligned} \frac{\partial f_{c2}(y, \tau|x, t)}{\partial t} &= - \frac{\partial}{\partial x} [f_{c2}(y, \tau|x, t)f(x, t)] \\ &+ D \frac{\partial^2}{\partial x^2} [f_{c2}(y, \tau|x, t)g(x, t)^2]. \end{aligned} \quad (5.174)$$

Now use the results of the previous sections and substitute the derived moment terms in the **Stratonovich sense** in (5.156):

$$\begin{aligned} & \int_{-\infty}^{\infty} \left[\frac{\partial f_{c2}(y, \tau|x, t)}{\partial t} \Delta t \right] R(x) dx \\ &= \int_{-\infty}^{\infty} f_{c2}(y, \tau|x, t) \left[R'(x)(f(x, t) + g'(x, t)g(x, t)D) \Delta t \right. \\ &\quad \left. + \frac{R''(x)}{2}g(x, t)^2 2D \Delta t \right] dx. \end{aligned} \quad (5.175)$$

Dividing all terms of (5.175) and making $\Delta t \rightarrow 0$ we find

$$\begin{aligned} & \int_{-\infty}^{\infty} \left[\frac{\partial f_{c2}(y, \tau|x, t)}{\partial t} \right] R(x) dx \\ &= \int_{-\infty}^{\infty} f_{c2}(y, \tau|x, t) \left[R'(x)(f(x, t) + g'(x, t)g(x, t)D) \right. \\ & \quad \left. + \frac{R''(x)}{2}g(x, t)^2 2D \right] dx. \end{aligned} \tag{5.176}$$

Integrating the right hand side of (5.171) by parts (twice the second term) and applying (5.149) then

$$\begin{aligned} & \int_{-\infty}^{\infty} \left[\frac{\partial f_{c2}(y, \tau|x, t)}{\partial t} \right] R(x) dx \\ &= - \int_{-\infty}^{\infty} R(x) \frac{\partial}{\partial x} \left[f_{c2}(y, \tau|x, t) \left\{ f(x, t) + \frac{1}{2}g'(x, t)g(x, t)2D \right\} \right] \\ & \quad + \frac{R(x)}{2} \frac{\partial^2}{\partial x^2} [f_{c2}(y, \tau|x, t)g(x, t)^2] 2D dx, \end{aligned} \tag{5.177}$$

Since this has to hold for every $R(x)$ the FPK equation becomes

$$\begin{aligned} \frac{\partial f_{c2}(y, \tau|x, t)}{\partial t} &= - \frac{\partial}{\partial x} \left[f_{c2}(y, \tau|x, t) \left\{ f(x, t) + \frac{1}{2}g'(x, t)g(x, t)2D \right\} \right] \\ & \quad + D \frac{\partial^2}{\partial x^2} [f_{c2}(y, \tau|x, t)g(x, t)^2]. \end{aligned} \tag{5.178}$$

Rearranging (5.178) will yield

$$\begin{aligned} \frac{\partial f_{c2}(y, \tau|x, t)}{\partial t} &= - \frac{\partial}{\partial x} [f_{c2}(y, \tau|x, t)f(x, t)] \\ & \quad + D \frac{\partial}{\partial x} \left[g(x, t) \frac{\partial}{\partial x} \{f_{c2}(y, \tau|x, t)g(x, t)\} \right]. \end{aligned} \tag{5.179}$$

Notice the difference between (5.174) and (5.179) achieved with two equally valid schemes of integration.

The most important usages of both approaches are:

- Stratonovich in physics and engineering
- Itô in mathematical analysis, financial mathematics

In [152] a number of Itô and Stratonovich SDEs are given and repeated in Table 5.1.

MDOF Stochastic Differential Equations

Consider the following stochastic differential equation written in the *Stratonovich form* [200]

$$\{ \dot{y} \} = \mathbf{f}_S(\{y\}, t) + \mathbf{g}(\{y\}, t)\{\xi(t)\}, \tag{5.180}$$

Table 5.1. Itô and Stratonovich SDEs

SDE	Definition
Itô	$\Delta X = -aX \Delta t + \sigma \Delta W$
Stratonovich	$\Delta X = -aX \Delta t + \sigma \Delta W$
Itô	$\Delta X = (aX + b)\Delta t + \sigma \Delta W$
Stratonovich	$\Delta X = (aX + b)\Delta t + \sigma \Delta W$
Itô	$\Delta X = (aX - \sigma^2/2)\Delta t + \sigma \Delta W$
Stratonovich	$\Delta X = (aX - \sigma^2/2)\Delta t + \sigma \Delta W$
Itô	$\Delta X = aX \Delta t + bX \Delta W$
Stratonovich	$\Delta X = (aX - \{1/2\}b^2 X \Delta t + bX \Delta W$
Itô	$\Delta X = (aX + b)\Delta t + (bX + d)\Delta W$
Stratonovich	$\Delta X = [(a - \{1/2\}b)X + c - \{1/2\}bd]\Delta t + (bX + d)\Delta W$
Itô	$\Delta X = [\{1/2\}a(a - 1)X^{1-2/a}]\Delta t + aX^{1-1/a} \Delta W$
Stratonovich	$\Delta X = aX^{1-1/a} \Delta W$
Itô	$\Delta X = [\{1/2\}a^2 X]\Delta t + a\sqrt{1 - X^2} \Delta W$
Stratonovich	$\Delta X = a\sqrt{1 - X^2} \Delta W$
Itô	$\Delta X = [a^2 X(1 + X^2)]\Delta t + a(1 + X^2)\Delta W$
Stratonovich	$\Delta X = a(1 + X^2)\Delta W$
Itô	$\Delta X = a(b - X)\Delta t + \sigma\sqrt{X} \Delta W$
Stratonovich	$\Delta X = [a(b - X) - \sigma^2/4]\Delta t + \sigma\sqrt{X} \Delta W$

where $\{y\}$ is the $2n$ -vector of state variables; $\mathbf{f}_S(\{y\}, t)$ and $\mathbf{g}(\{y\}, t)$ are, respectively, and n -vector and $n \times j$ matrix, whose elements are deterministic functions depending nonlinearly on $\{y\}$. $\{\xi(t)\}$ is the j -vector of stationary, zero-mean white noise processes. As already mentioned the Stratonovich approach is quite often applied in engineering problems. Equation (5.180) can be rewritten in the following *Itô standard form* [200]

$$\{\Delta y\} = \mathbf{f}_I(\{y\}, t)\Delta t + \mathbf{g}(\{y\}, t)\{\Delta W(t)\}, \tag{5.181}$$

where $\mathbf{f}_I(\{y\}, t)$ is the vector of *drift coefficients*, which accounts for the presence of the Wong-Zakai or Stratonovich correction term. $\{\Delta W(t)\}$ is the j -vector of the Wiener processes, whose increments satisfy the following relations

$$\begin{aligned} E\{\Delta W(t)\} &= \{0\}, \\ E\{\Delta W(t_1)\Delta W(t_2)\} &= [2D]\delta(t_1 - t_2)\Delta t_1\Delta t_2 = [2D]\Delta t, \end{aligned} \tag{5.182}$$

In Appendix N it is stated that the Wiener process, denoted by $W(t)$, is also called Brownian motion, and denoted by $B(t)$. In this section the notation $W(t)$ will be used to indicate a Wiener process. A Wiener process is called a standard (or unit) Wiener process if $2D = 1$. Equation (5.181) can be written

$$\{\Delta y\} = \left[\mathbf{f}_S(\{y\}, t) + \frac{1}{2} \sigma(\{y\}, t) \frac{\partial \sigma(\{y\}, t)}{\partial y} \right] \Delta t + \mathbf{g}(\{y\}, t) \{\Delta W(t)\}, \quad (5.183)$$

where the second term inside the brackets on the right hand side of (5.183) is the Wong-Zakai or Stratonovich correction term. Partial differentiation, $\frac{\sigma(\{y\}, t)}{\partial y}$ should be used in accordance with the rules of matrix operations. One can write

$$\begin{aligned} \Delta y_i &= \left[f_{S,i} + \frac{1}{2} \sum_{k=1}^{2n} \sum_{j=1}^m \sigma_{kj} \frac{\sigma_{ij}}{\partial y_k} \right] \Delta t + \sum_{j=1}^m g_{ij} \Delta W_j \\ &= f_{I,i} \Delta t + \sum_{j=1}^m g_{ij} \Delta W_j, \end{aligned} \quad (5.184)$$

where m is the number of stochastic excitations ΔW_j and $\sigma_{ij} = g_{il} \sqrt{2D_{lj}}$, $l = 1, 2, \dots, m$.

Example. This example is taken from [210]. An SDOF system is disturbed by parametric (multiplicative) and external (additive) Gaussian white noise excitations. The equation of motion of the SDOF system is

$$\ddot{x} + \alpha[1 + \xi(t)]\dot{x} + \beta[1 + \xi(t)]x = \xi(t), \quad (5.185)$$

where x is the stochastic displacement, α, β are constants and $\xi(t)$ is the Gaussian white noise excitation. The second order ordinary differential equation is transformed into two first order differential equations, the so-called two space-state, with $y_1 = x$ and $\dot{y}_1 = x$

$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\beta[1 + \xi(t)] & -\alpha[1 + \xi(t)] \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \xi(t) \\ \xi(t) \end{pmatrix}. \quad (5.186)$$

Equation (5.186) can be rewritten as an Stratonovich stochastic differential equation

$$\begin{pmatrix} \Delta y_1 \\ \Delta y_2 \end{pmatrix} = \begin{pmatrix} y_2 \\ -\beta y_1 - \alpha y_2 \end{pmatrix} \Delta t + \begin{pmatrix} 0 & 0 \\ 0 & 1 - \beta y_1 - \alpha y_2 \end{pmatrix} \begin{pmatrix} \Delta W \\ \Delta W \end{pmatrix} \quad (5.187)$$

or

$$\{\Delta y\} = \mathbf{f}_S(\{y\}, t) \Delta t + \mathbf{g}(\{y\}, t) \Delta W, \quad (5.188)$$

where $\Delta W = \xi(t) \Delta t$ is the Wiener process. In accordance with (5.184), the Wong-Zakai or Stratonovich correction terms can be calculated

$$\begin{aligned} \Delta y_i &= \left[f_{S,i} + \frac{1}{2} \sum_{k=1}^2 \sum_{j=1}^2 \sigma_{kj} \frac{\sigma_{ij}}{\partial y_k} \right] \Delta t + \sum_{j=1}^2 g_{ij} \Delta W_j \\ &= f_{I,i} \Delta t + \sum_{j=1}^2 g_{ij} \Delta W_j, \quad i = 1, 2, \end{aligned} \quad (5.189)$$

where $[\sigma] = [g]\sqrt{2D}$ is given by

$$[\sigma] = \begin{pmatrix} 0 & 0 \\ 0 & (1 - \beta y_1 - \alpha y_2)\sqrt{S_w} \end{pmatrix}, \tag{5.190}$$

and the autocorrelation function $R_w(\tau)$ is given by

$$R_w(\tau) = E\{W_i(t)W_j(t + \tau)\} = 2D\delta(\tau) = S_w\delta(\tau).$$

Thus the Itô form of the SDE becomes

$$\begin{aligned} \{\Delta y\} &= \{f_I\}\Delta t + [G]\{\Delta W\} \\ &= \begin{pmatrix} f_{S,1} \\ f_{S,2} - \frac{1}{2}\alpha S_w(1 - \beta y_1 - \beta y_2) \end{pmatrix} \Delta t \\ &\quad + \begin{pmatrix} 0 & 0 \\ 0 & (1 - \beta y_1 - \beta y_2) \end{pmatrix} \begin{pmatrix} \Delta W \\ \Delta W \end{pmatrix} \\ &= \begin{pmatrix} y_2 \\ -\beta y_1 - \alpha y_2 - \frac{1}{2}\alpha S_w(1 - \beta y_1 - \beta y_2) \end{pmatrix} \Delta t \\ &\quad + \begin{pmatrix} 0 & 0 \\ 0 & (1 - \beta y_1 - \beta y_2) \end{pmatrix} \begin{pmatrix} \Delta W \\ \Delta W \end{pmatrix}. \end{aligned} \tag{5.191}$$

The associated FPK equation is

$$\frac{\partial f_{c_2}}{\partial t} = - \sum_{i=1}^2 \frac{\partial [f_{I,i} f_{c_2}]}{\partial y_i} + \sum_{i=1}^2 \sum_{j=1}^2 \frac{\partial^2 [(GDG^T)_{ij} f_{c_2}]}{\partial y_i \partial y_j}, \tag{5.192}$$

where the conditional probability density function is $f_{c_2} = f_{c_2}(\{y_0\}, t_0 | \{y\}, t)$ and $f(\{y\}, t) = \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} f_{c_2}(\{y_0\}, t_0 | \{y\}, t) f(\{y_0\}, t_0) d\{y_0\}$.

5.7 Behavior of Linear Systems with Random Parametric Excitation

In this section, we discuss the behavior of linear systems with parametric parameters varying as white noise, and is based on a paper of Gray [75]. The general associated FPK equation is given and moment equations are derived. In the previous section an example of a parametric system was given. As a starting point Gray considered an $(n + 1)$ th order linear differential equation given by

$$\frac{d^{n+1}x}{dt^{n+1}} + \sum_{k=0}^n [a_k + \xi_k(t)] \frac{d^k x}{dt^k} = \xi_{n+1}(t) + f(t), \tag{5.193}$$

where the a_k are constants, $f(t)$ is a known signal possessing a PSD spectrum, and the $\xi_k(t)$ are the stationary random variables that are ergodic, Gaussian and white. The mean or expectation of $\xi_k(t)$ is

$$\begin{aligned}
 E\{\xi_k(t)\} &= \langle \xi_k(t) \rangle = 0, \\
 E\{\xi_k(t)\xi_i(t + \tau)\} &= \langle \xi_k(t)\xi_i(t + \tau) \rangle = 2D_{ki}\delta(\tau).
 \end{aligned}
 \tag{5.194}$$

The process in (5.193) defines a continuous Markoff process of $n + 1$ dimensions. Due to the presence of $f(t)$ in the equation, this process may not be stationary, however, it still possesses a transition probability. Let the variables y_0, y_1, \dots, y_n be defined by

$$y_k = \frac{d^k x}{dt^k}. \tag{5.195}$$

Then, if $f(\{y\}, t) = f$ is the joint probability density function in the variable y_k for $k = 0, 1, \dots, n$, f will satisfy the FPK equation given by

$$\begin{aligned}
 \frac{\partial f}{\partial t} &= - \sum_{k=0}^{n-1} \frac{\partial(y_{k+1}f)}{\partial y_k} - \frac{\partial}{\partial y_n} \left\{ \left[f(t) - D_{n+1,n} - \sum_{k=0}^n (a_k - D_{nk})y_k \right] f \right\} \\
 &+ \frac{\partial^2}{\partial y_n^2} \left\{ \left[\sum_{k=0}^n \sum_{i=0}^n D_{ki}y_k y_i - 2 \sum_{k=0}^n D_{n+1,k}y_k + D_{n+1,n+1} \right] f \right\}.
 \end{aligned}
 \tag{5.196}$$

Example. Consider the following equation of motion:

$$\frac{d^2x(t)}{dt^2} + [2\omega_0\zeta + \xi_1(t)]\frac{dx(t)}{dt} + [\omega_0^2 + \xi_0(t)]x = \xi_2(t).$$

The following variable transformations are made in comparison with (5.193)

- $a_0 = \omega_0^2,$
- $a_1 = 2\omega_0\zeta,$
- $E\{\xi_k(t)\} = 0, k = 0, 1, 2,$
- $E\{\xi_i(t)\xi_j(t + \tau)\} = 2D_{ij}\delta_{ij}\delta(\tau) \quad i, j = 0, 1, 2,$
- $f(t) = 0,$
- $y_0 = x,$
- $y_1 = \frac{dx}{dt},$ and
- $n = 1.$

Equation (5.196) becomes

$$\begin{aligned}
 \frac{\partial f}{\partial t} &= - \sum_{k=0}^0 \frac{\partial(y_{k+1}f)}{\partial y_k} - \frac{\partial}{\partial y_1} \left\{ \left[D_{21}\delta_{21} - \sum_{k=0}^1 (a_k - D_{1k}\delta_{1k})y_k \right] f \right\} \\
 &+ \frac{\partial^2}{\partial y_1^2} \left\{ \left[\sum_{k=0}^1 \sum_{i=0}^1 D_{ki}\delta_{ki}y_k y_i - 2 \sum_{k=0}^1 D_{2k}\delta_{2k}y_k + D_{22}\delta_{22} \right] f \right\},
 \end{aligned}$$

or

$$\begin{aligned}
 \frac{\partial f}{\partial t} &= - \frac{\partial(y_1f)}{\partial y_0} + \frac{\partial}{\partial y_1} \left\{ \left[\omega_0^2 y_0 + (2\zeta\omega_0 - D_{11})y_1 \right] f \right\} \\
 &+ \frac{\partial^2}{\partial y_1^2} \{ [D_{00}y_0^2 + D_{11}y_1^2 + D_{22}] f \}.
 \end{aligned}$$

5.7.1 Moments and Autocorrelation

Let M be a well-behaved function of the functions y_k , so that the stochastic mean or expectation of M exists and is defined by the integral

$$E\{M\} = \langle M \rangle = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} M f dy_0 dy_1 \cdots dy_n. \quad (5.197)$$

If the FPK equation (5.196) is multiplied by M , integrated over all y_k , then integrated by parts, it yields the result

$$\begin{aligned} \frac{d\langle M \rangle}{dt} &= \sum_{k=0}^{n-1} \left\langle y_{k+1} \frac{\partial M}{\partial y_k} \right\rangle + [f(t) - D_{n+1,n}] \left\langle \frac{\partial M}{\partial y_n} \right\rangle \\ &\quad - \sum_{k=0}^n (a_k - D_{nk}) \left\langle y_k \frac{\partial M}{\partial y_n} \right\rangle + \sum_{k=0}^n \sum_{i=0}^n D_{ki} \left\langle y_k y_i \frac{\partial^2 M}{\partial y_n^2} \right\rangle \\ &\quad - 2 \sum_{k=0}^n D_{n+1,k} \left\langle y_k \frac{\partial^2 M}{\partial y_n^2} \right\rangle + D_{n+1,n+1} \left\langle \frac{\partial^2 M}{\partial y_n^2} \right\rangle. \end{aligned} \quad (5.198)$$

This equation (5.198) can now be used for the finding the means moments and the autocorrelation function.

Mean

By setting $M = y_k$ in (5.198), we can obtain the following result:

$$\begin{aligned} \frac{d\langle y_k \rangle}{dt} &= \langle y_{k+1} \rangle, \quad k = 0, 1, 2, \dots, n-1, \\ \frac{d\langle y_n \rangle}{dt} &= - \sum_{k=0}^n (a_k - D_{nk}) \langle y_k \rangle + f(t) - D_{n+1,n}. \end{aligned} \quad (5.199)$$

Combining these equations and using the definition $y_k = \frac{d^k x}{dt^k}$, we obtain the equation for the mean of x , given by

$$\frac{d^{n+1} \langle x \rangle}{dt^{n+1}} + \sum_{k=0}^n (a_k - D_{nk}) \frac{d^k \langle x \rangle}{dt^k} = f(t) - D_{n+1,n}. \quad (5.200)$$

Example. Consider a dynamic system defined by the differential equation

$$\frac{d^2 x}{dt^2} + [2\zeta\omega_0 + \xi_1(t)] \frac{dx}{dt} + [\omega_0^2 + \xi_0(t)]x = \xi_2(t),$$

where $\xi_0(t)$, $\xi_1(t)$ and $\xi_2(t)$ are zero-mean Gaussian white noises and

$$\langle \xi_i(t) \xi_j(t + \tau) \rangle = 2D_{ij} \delta(\tau), \quad i, j = 0, 1, 2.$$

This is identical to the general case with $n+1 = 2$, $a_1 = 2\zeta\omega_0$, $a_0 = \omega_0^2$ and $f(t) = 0$. The mean will satisfy the differential equation

$$\frac{d^2 \langle x \rangle}{dt^2} + (2\zeta\omega_0 - D_{11}) \frac{d\langle x \rangle}{dt} + (\omega_0^2 - D_{10}) \langle x \rangle = -D_{21}.$$

Autocorrelation Function

In an analogous manner used for finding the equation for the mean, the equation for the autocorrelation $E\{x(t)x(s)\} = \langle x(t)x(s) \rangle$ is given by

$$\begin{aligned} \frac{d^{n+1}\langle x(t)x(s) \rangle}{dt^{n+1}} + \sum_{k=0}^n (a_k - D_{nk}) \frac{d^k \langle x(t)x(s) \rangle}{dt^k} \\ = [f(t) - D_{n+1,n}] \langle x(s) \rangle, \end{aligned} \tag{5.201}$$

which will be valid for $t \geq s$.

Second Moments and the Variance

We obtain by using (5.198), with M defined by the relation $M = y_k y_i - \langle y_k \rangle \langle y_i \rangle$, the differential equations for C_{ki} , where

$$C_{ik} = C_{ik} = \langle y_k y_i \rangle - \langle y_k \rangle \langle y_i \rangle. \tag{5.202}$$

This leads to the following set of equations:

$$\begin{aligned} \frac{dC_{ik}}{dt} &= C_{i,k+1} + C_{k,i+1}, \quad i, k = 0, 1, \dots, n-1, \quad i \neq n, \quad k \neq n, \\ \frac{dC_{in}}{dt} &= C_{n,i+1} - \sum_{k=0}^n (a_k - D_{nk}) C_{ik}, \quad i = 0, 1, \dots, n-1, \quad i \neq n, \\ \frac{1}{2} \frac{dC_{nn}}{dt} &= - \sum_{k=0}^n (a_k - D_{nk}) C_{nk} + \sum_{k=0}^n \sum_{i=0}^n D_{ik} C_{ik} \\ &\quad + \sum_{k=0}^n \sum_{i=0}^n D_{ik} \langle y_k \rangle - 2 \sum_{k=0}^n D_{n+1,k} \langle y_k \rangle + D_{n+1,n+1}. \end{aligned} \tag{5.203}$$

The first equation of (5.203) represents $\frac{1}{2}n(n+1)$ equations (since $C_{ki} = C_{ik}$). The second equation of (5.203) represent n equations, and added to the third equation of (5.203), gives a total of $\frac{1}{2}(n+1)(n+2)$ equations for the same number of unknowns $C_{ik} = C_{ki}$. In principle, once the means are found, the second moments found also.

Example. Consider a dynamic system defined by the differential equation

$$\frac{d^2x}{dt^2} + [2\zeta\omega_0 + \xi_1(t)] \frac{dx}{dt} + [\omega_0^2 + \xi_0(t)]x = \xi_2(t),$$

where $\xi_0(t)$, $\xi_1(t)$ and $\xi_2(t)$ are zero-mean Gaussian white noises and

$$\langle \xi_i(t)\xi_j(t+\tau) \rangle = 2D_{ij}\delta(\tau), \quad i, j, = 0, 1, 2.$$

This is identical to the general case with $n+1 = 2$, $a_1 = 2\zeta\omega_0$, $a_0 = \omega_0^2$ and $f(t) = 0$. The mean will satisfy the differential equation

$$\frac{d^2\langle x \rangle}{dt^2} + (2\zeta\omega_0 - D_{11})\frac{d\langle x \rangle}{dt} + (\omega_0^2 - D_{10})\langle x \rangle = -D_{21}.$$

The mean will be stable if and only if $2\zeta\omega_0 > D_{11}$ and $\omega_0^2 > D_{10}$. The equations for the covariance function C_{ij} are given by

$$\begin{aligned}\frac{dC_{00}}{dt} &= 2C_{01}, \\ \frac{dC_{01}}{dt} &= C_{11} - (\omega_0^2 - D_{10})C_{00} - (2\zeta\omega_0 - D_{11})C_{01}, \\ \frac{1}{2}\frac{dC_{11}}{dt} &= -(\omega_0^2 - 3D_{10})C_{01} - (2\zeta\omega_0 - 2D_{11})C_{11} + D_{00}C_{00} \\ &\quad + D_{00}\langle y_0 \rangle^2 + 2D_{10}\langle y_0 \rangle\langle y_1 \rangle + D_{11}\langle y_1 \rangle^2 \\ &\quad - 2D_{20}\langle y_0 \rangle - 2D_{11}\langle y_1 \rangle + D_{22}.\end{aligned}$$

Consider the following simplified linear equation of motion:

$$\frac{d^2x}{dt^2} + 2\zeta\omega_0\frac{dx}{dt} + \omega_0^2x = \xi_2(t),$$

then for a stationary process the following equations are obtained:

$$\begin{aligned}2C_{01} &= 0, \\ C_{11} - \omega_0^2C_{00} - 2\zeta\omega_0C_{01} &= 0, \\ -\omega_0^2C_{01} - 2\zeta\omega_0C_{11} + D_{22} &= 0.\end{aligned}$$

Solving the set of algebraic equations we find the covariances become

$$\langle \dot{x}x \rangle = C_{01} = C_{10} = 0, \quad \langle \dot{x}\dot{x} \rangle = C_{11} = \frac{D_{22}}{2\zeta\omega_0}, \quad \langle xx \rangle = C_{00} = \frac{D_{22}}{2\zeta\omega_0^3},$$

because the mean values are $\langle x \rangle = \langle \dot{x} \rangle = 0$.

The space state of the previous equation of motion is given by

$$\begin{aligned}\begin{pmatrix} \dot{y}_0 \\ \dot{y}_1 \end{pmatrix} &= \begin{pmatrix} 0 & 1 \\ -\omega_0^2 & -2\zeta\omega_0 \end{pmatrix} \begin{pmatrix} y_0 \\ y_1 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \xi_2(t) \\ &= [A] \begin{pmatrix} y_0 \\ y_1 \end{pmatrix} + \{G\}\xi(t).\end{aligned}$$

The stationary correlation matrix is $[R] = [C] + [\langle y_i \rangle \langle y_j \rangle]$, $i, j = 0, 1$ and can be obtained by solving the Lyapunov equation (see Sect. 2.6)

$$[A][R] + R[A]^T = -\{G\}2D_{22}[G].$$

As before, we find

$$[R] = [C] = D_{22} \begin{pmatrix} \frac{1}{2\zeta\omega_0^2} & 0 \\ 0 & \frac{1}{2\zeta\omega_0} \end{pmatrix}.$$

5.8 Generation FPK Equation for MDOF Systems

The equations of motion for a lumped-parameter systems with MDOF can be, in general, written as follows [210]:

$$\ddot{x}_i + h_i(x_1, x_2, \dots, x_n, \dot{x}_1, \dot{x}_2, \dots, \dot{x}_n) = f_{ir}^p, \tag{5.204}$$

$$f_{ir}^p = f_{ir}(x_1, x_2, \dots, x_n, \dot{x}_1, \dot{x}_2, \dots, \dot{x}_n)\xi_r(t), \tag{5.205}$$

and with $\{y_1\} = [x_1, x_2, \dots]$ and $\{y_2\} = [\dot{x}_1, \dot{x}_2, \dots, \dot{x}_n]$ the equations can be written in matrix form

$$\{\ddot{y}\} + h(\{y_1\}, \{y_2\}) = \{f^p\}_{n \times 1}, \tag{5.206}$$

and

$$\{f^p\}_{n \times 1} = [f]_{n \times m} \{\xi(t)\}_{m \times 1}. \tag{5.207}$$

With

$$\{y\} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix},$$

the set of equations can be expressed in a state space

$$\{\dot{y}\} = \begin{pmatrix} \{y_2\} \\ -h(\{y_1\}, \{y_2\}) \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & [f] \end{pmatrix} \begin{pmatrix} 0 \\ \{\xi\} \end{pmatrix}, \tag{5.208}$$

where $\{\xi\}$ is the vector of delta correlated white noise processes.

The corresponding *Itô equation* can be written

$$\begin{aligned} \{\Delta y\} &= \begin{pmatrix} \{y_2\} \\ -h(\{y_1\}, \{y_2\}) + \kappa \end{pmatrix} \Delta t + \begin{pmatrix} 0 & 0 \\ 0 & [f] \end{pmatrix} \begin{pmatrix} \{\Delta W\} \\ \{\Delta W\} \end{pmatrix} \\ &= \{m\} \Delta t + [G] \Delta W, \end{aligned} \tag{5.209}$$

where κ is the Wong-Zakai or Stratonovich correction term, $\{\Delta W\} = \{\xi\} \Delta t$ which is a Brownian motion or *Wiener process*, and

$$E[\{W(t)\}\{W(t + \tau)\}] = [2D]\delta(\tau), \tag{5.210}$$

The FPK equation can be easily derived from (5.209) and (5.210)

$$\frac{\partial f_{c_2}}{\partial t} = - \sum_{i=1}^{2n} \frac{\partial [m_i f_{c_2}]}{\partial y_i} + \sum_{i=1}^{2n} \sum_{j=1}^{2n} \frac{\partial^2 [(GDG^T)_{ij} f_{c_2}]}{\partial y_i \partial y_j}. \tag{5.211}$$

where the conditional probability density function is given by

$$f_{c_2} = f_{c_2}(\{y_1(0)\}, \{y_2(0)\} | \{y_1(t)\}, \{y_2(t)\}).$$

In terms of the first and second moments A_i and B_{ij} the FPK equation (5.211) can be written as follows

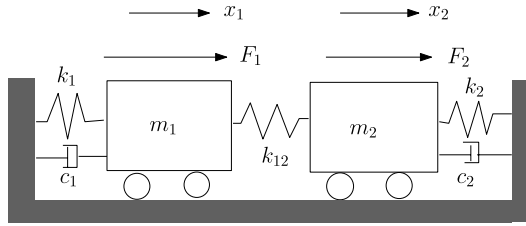


Fig. 5.4. Two degrees of freedom system (four state-space system)

$$\frac{\partial f_{c_2}}{\partial t} = - \sum_{i=1}^{2n} \frac{\partial [A_i f_{c_2}]}{\partial y_i} + \frac{1}{2} \sum_{i=1}^{2n} \sum_{j=1}^{2n} \frac{\partial^2 [B_{ij} f_{c_2}]}{\partial y_i \partial y_j}, \tag{5.212}$$

where

$$A_i = m_i, \quad B_{ij} = 2(GDG^T)_{ij}. \tag{5.213}$$

Example. A MDOF linear dynamic system is illustrated in Fig. 5.4. The state-space variables of that dynamic system are given by

$$\{y\} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} \tag{5.214}$$

The state matrix $[A]$ of the dynamic system showed now becomes

$$[A] = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{-(k_1+k_2)}{m_1} & \frac{k_{12}}{m_1} & \frac{-c_1}{m_1} & 0 \\ \frac{k_{12}}{m_2} & \frac{-(k_1+k_2)}{m_2} & 0 & \frac{-c_2}{m_2} \end{pmatrix}, \tag{5.215}$$

and the input state matrix denoted by B is

$$[G] = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ \frac{1}{m_1} & 0 \\ 0 & \frac{1}{m_2} \end{pmatrix} \tag{5.216}$$

The forces are denoted by $\{\xi\}$

$$\{\xi\} = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}, \tag{5.217}$$

where $\{\xi(t)\}$ is a Gaussian and white noise process, with the following statistical properties

$$E\{\xi(t)\} = \{0\},$$

$$E\{\xi(t)\xi(t + \tau)\} = [2D]\delta(\tau) = \begin{pmatrix} S_{\xi_1} & 0 \\ 0 & S_{\xi_2} \end{pmatrix} \delta(\tau),$$

where S_{ξ_i} are the constant PSD functions. The following notations are now introduced and substituted in (5.215)

- $\Omega_1^2 = \frac{(k_1+k_2)}{m_1}$
- $\Omega_2^2 = \frac{(k_1+k_2)}{m_2}$
- $\omega_1^2 = \frac{k_2}{m_1}$
- $\omega_2^2 = \frac{k_2}{m_2}$
- $\omega_{11}^2 = \frac{k_{12}}{m_1}$
- $\omega_{22}^2 = \frac{k_{12}}{m_2}$
- $c_1 = 2\zeta_1\sqrt{k_1m_1}$
- $c_2 = 2\zeta_2\sqrt{k_2m_2}$
- $2\zeta_1\omega_1 = \frac{c_1}{m_1}$
- $2\zeta_2\omega_2 = \frac{c_2}{m_1}$

thus (5.215) becomes

$$[A] = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\Omega_1^2 & \omega_{11}^2 & -2\zeta_1\omega_1 & 0 \\ \omega_{22}^2 & -\Omega_2^2 & 0 & -2\zeta_2\omega_2 \end{pmatrix} \tag{5.218}$$

The Itô and Stratonovich stochastic differential equations can be generated

$$\{ \Delta y \} = \{ f \} \Delta t + [g] \{ \Delta W \} = \{ m \} \Delta t + [g] \{ \Delta W \}, \tag{5.219}$$

where $\{ f \} = \{ m \} = [A] \{ y \}$, because the matrix $[g] = [G]$ is not dependent on the state variables $\{ y \}$, but

$$[G] = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{m_1} & 0 \\ 0 & 0 & 0 & \frac{1}{m_1} \end{pmatrix},$$

and

$$[G][D][G]^T = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{S_{\xi_1}}{2m_1^2} & 0 \\ 0 & 0 & 0 & \frac{S_{\xi_2}}{2m_1^2} \end{pmatrix}.$$

The FPK equation for this linear system, using (5.192), with $f_{c_2} = f_{c_2}(\{y_0\}|\{y\}, t)$ is given in the following expression

$$\begin{aligned}
 \frac{\partial f_{c_2}}{\partial t} &= \frac{S_{\xi_1}}{2m_1^2} \frac{\partial^2 f_{c_2}}{\partial y_3^2} + \frac{S_{\xi_2}}{2m_2^2} \frac{\partial^2 f_{c_2}}{\partial y_4^2} - y_3 \frac{\partial f_{c_2}}{\partial y_1} - y_4 \frac{\partial f_{c_2}}{\partial y_2} \\
 &\quad - \frac{\partial}{\partial y_3} [(-\Omega_1^2 y_1 + \omega_{11}^2 y_2 - 2\zeta_1 \omega_1 y_3) f_{c_2}] \\
 &\quad - \frac{\partial}{\partial y_4} [(\omega_{22}^2 y_1 - \Omega_1^2 y_2 - 2\zeta_2 \omega_2 y_4) f_{c_2}]. \tag{5.220}
 \end{aligned}$$

Problems

5.1. When x is a Markoff process prove the following equation

$$f_3(x_1, t_1, x_2, t_2, x_3, t_3) = \frac{f_2(x_1, t_1, x_2, t_2) f_2(x_2, t_2, x_3, t_3)}{f_1(x_2, t_2)}.$$

5.2. Derive Miles' equation of a SDOF linear system with enforced acceleration \ddot{u} at the base using the FPK equation. The equation governing the relative motion z is given by

$$\ddot{z} + 2\zeta\omega_n \dot{z} + \omega_n^2 z = -\ddot{u}.$$

Prove that the expected value of the square of the absolute acceleration \ddot{x}^2 is

$$E\{\ddot{x}^2\} = (-2\zeta\omega_n)^2 E\{\dot{z}^2\} + (-\omega_n^2)^2 E\{z^2\}.$$

The white noise PSD function is $S_{\ddot{u}} = \frac{W_{\ddot{u}}}{2}$, $\omega_n = 2\pi f_n$ and $Q = \frac{1}{2\zeta}$.

Answer: $E\{\ddot{x}\} = \frac{\pi}{2} f_n Q W_{\ddot{u}} (1 + 4\zeta^2)$.

5.3. This problem is taken from [101]; parametric excitation is combined with nonparametric excitation. The equation of motion of a SDOF system is given by

$$\ddot{x} + \omega_0 [2\zeta + \xi_2(t)] \dot{x} + \omega_0^2 [1 + \xi_1(t)] x = \xi_0(t).$$

Here $\xi_0(t)$, $\xi_1(t)$ and $\xi_2(t)$ are white noise processes with spectral densities

$$[2D] = \begin{pmatrix} S_{00} & 0 & 0 \\ 0 & S_{11} & S_{12} \\ 0 & S_{21} & S_{22} \end{pmatrix}.$$

- Set up the nonlinear Stratonovich equation $\{dy\} = \{f\}dt + [G]\{dW\}$ with $y_1 = x$ and $y_2 = \dot{x}$
- Derive Itô's equation $\{dy\} = \{m\}dt + [G]\{dW\}$ with $y_1 = x$ and $y_2 = \dot{x}$
- Derive from Itô's equation the FPK equation for $f_{c_2}(\{y_0\}|\{y\}, t)$
- Solve the stationary FPK equation for $f_1(\{y\})$

5.4. This problem is taken from [92]. A hardening Duffing oscillator subjected to additive and multiplicative (parametric) white noise is given by

$$\ddot{x} + 2\zeta\dot{x} + [1 - \xi_1(t)]x + \varepsilon x^3 = \xi_2(t),$$

where the noise terms are independent, and

$$R_w(\tau) = E[\{\xi(t)\}\{\xi(t + \tau)\}^T] = [2D]\delta(\tau) = \begin{pmatrix} S_1 & 0 \\ 0 & S_2 \end{pmatrix}.$$

Generate the corresponding FPK equation.

Answer:

$$\frac{\partial f_{c_2}}{\partial t} = \frac{\partial[(2\zeta y_2 + y_1 + \varepsilon y_1^3)]f_{c_2}}{\partial y_2} - \frac{\partial[y_2 f_{c_2}]}{\partial y_1} + \frac{\partial^2[(S_1 y_2^2 + S_2)f_{c_2}]}{2\partial y_2^2}.$$

5.5. This problem is taken from [183]. Solve the following parabolic partial differential equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2},$$

with the finite difference approximation. Perform the following assignments:

- Set up an explicit finite difference scheme
 Answer: $u_{i,k+1} = ru_{i-1,k} + (1 - 2r)u_{i,k} + ru_{i+1,k}$, $r = \Delta t/\Delta x^2$, $x = x_{min} + i\Delta x$, $t = k\Delta t$, $i, k = 0, 1, 2, \dots$
- Solve the partial differential equation, using the finite difference explicit scheme, with given initial and boundary conditions for $t = 0.02$:
 - Initial condition $t = 0$ $u = 2x$ for $0 \leq x \leq 0.5$, $u = 2(1 - x)$ for $0.5 < x \leq 1.0$.
 - Boundary conditions $u = 0$ at $x = 0$ and $x = 1$ for all t .
 - $\Delta x = 0.1$ and $\Delta t = 0.001$
 Answer: $t = 0.02$, $(x, u) = (0.0, 0.0), (0.1, 0.1939), (0.2, 0.3781), (0.3, 0.5373), (0.4, 0.6486), (0.5, 0.6891), (0.6, 0.6486), (0.7, 0.5373), (0.8, 0.3781), (0.9, 0.1939), (1.0, 0.0)$.

5.6. A linear mass spring system is coupled with a Duffing-like oscillator, expressed by the following set of equations of motion

$$\begin{cases} M\ddot{x} + \lambda_1\dot{x} + k_1x + \gamma(x - y) = f_1(t), \\ m\ddot{y} + \lambda_s\dot{y} + Cy^3 + \gamma(y - x) = f_2(t), \end{cases} \tag{5.221}$$

where $f_1(t)$ and $f_2(t)$ are two white non-correlated noises whose diffusion coefficients are respectively W_{01} and W_{02} :

$$\begin{cases} \langle f_1(t) \rangle = 0, \\ \langle f_2(t) \rangle = 0, \\ \langle f_i(t_k)f_j(t_l) \rangle = \frac{W_{0i}}{2}\delta_{ij}\delta(t_k - t_l). \end{cases} \tag{5.222}$$

Write down Itô equations (4 DOFs), expressed in terms of $y_1 = x$, $y_2 = \dot{x}$, $y_3 = y$ and $y_4 = \dot{y}$.

Answer:

$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dot{y}_3 \\ \dot{y}_3 \end{pmatrix} = \begin{pmatrix} y_2 \\ -\frac{1}{M}[\lambda_1 y_2 + k_1 y_1 + \gamma(y_1 - y_3)] \\ y_4 \\ -\frac{1}{m}[\lambda_s y_4 + C y_3^3 + \gamma(y_3 - y_1)] \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{f_1(t)}{M} \\ 0 \\ \frac{f_2(t)}{m} \end{pmatrix},$$

or

$$\begin{pmatrix} \Delta y_1 \\ \Delta y_2 \\ \Delta y_3 \\ \Delta y_3 \end{pmatrix} = \begin{pmatrix} y_2 \\ -\frac{1}{M}[\lambda_1 y_2 + k_1 y_1 + \gamma(y_1 - y_3)] \\ y_4 \\ -\frac{1}{m}[\lambda_s y_4 + C y_3^3 + \gamma(y_3 - y_1)] \end{pmatrix} \Delta t + \begin{pmatrix} 0 \\ \frac{\sqrt{W_{01}}}{\sqrt{2M}} \Delta W_1 \\ 0 \\ \frac{\sqrt{W_{02}}}{\sqrt{2m}} \Delta W_2 \end{pmatrix},$$

where W_i are standard Wiener processes with $dW(t) = W(t+dt) - W(t)$ and $E\{\Delta W(t)\Delta W(t)\} = \Delta t$.

The transition probability density function $f_{4c}(\mathbf{y}_0|\mathbf{y}, t) = f$. Derive the Fokker-Planck equation expressed in terms of f .

Answer:

$$\begin{aligned} \frac{\partial f}{\partial t} = & -y_2 \frac{\partial f}{\partial y_1} + \frac{\partial}{\partial y_2} \frac{1}{M} [\{\lambda_1 y_2 + k_1 y_1 + \gamma(y_1 - y_3)\}f] + \frac{W_{01}}{4M^2} \frac{\partial^2 f}{\partial y_2^2} \\ & - y_4 \frac{\partial f}{\partial y_4} \frac{1}{m} [\{\lambda_s y_4 + C y_3^3 + \gamma(y_3 - y_1)\}f] + \frac{W_{02}}{4m^2} \frac{\partial^2 f}{\partial y_4^2}. \end{aligned}$$

5.7. This problem is taken from [6, 7, 78]. Consider the stochastic ordinary differential equation

$$\Delta x = (x - x^3)\Delta t + \sigma \Delta W,$$

where $W(t)$ is the standard Wiener process. Write down the FPK equation and solve the stationary FPK equation for the probability density function $f(x) = f$. Calculate the first and second moment $\langle x \rangle$, $\langle x^2 \rangle$ as function of $\sigma = 0.2 \dots 1.0$, respectively.

Answers: $\frac{\partial f}{\partial t} = -\frac{\partial}{\partial x}[(x - x^3)f] + \frac{\sigma^2}{2} \frac{\partial^2 f}{\partial x^2}$, $f(x) = C e^{(x^2 - 0.5x^4)/\sigma^2}$, e.g. $\sigma = 0.2$, $C = 0.7312 \times 10^{-5}$, $\langle x^2 \rangle = 0.9785$.

5.8. This problem is taken from [6]. The following set of equations is given

$$\begin{aligned} \dot{x} = & \mu_o x + \mu_r(t)x + [1 - (x^2 + y^2)](x^2 + y^2)x - \omega y - b(x^2 + y^2)y + \xi_x(t), \\ \dot{y} = & \mu_o y + \mu_r(t)y + [1 - (x^2 + y^2)](x^2 + y^2)y + \omega x + b(x^2 + y^2)x + \xi_y(t), \end{aligned}$$

where the additive noise terms are considered uncorrelated with $\mu_r(t)$. The correlation functions for $\mu_r(t)$, $\xi_x(t)$ and $\xi_y(t)$ are given as

$$\begin{aligned} R_{\mu\mu} = & \langle \mu(t)\mu(t + \tau) \rangle = 2D_m \delta(\tau), \\ R_{xx} = & \langle \xi_x(t)\xi_x(t + \tau) \rangle = 2D \delta(\tau), \\ R_{yy} = & \langle \xi_y(t)\xi_y(t + \tau) \rangle = 2D \delta(\tau), \end{aligned}$$

where D_m and D are the spectral densities. Expressing the white noise processes as formal derivatives of the Wiener processes W_m, W_x and W_y , set-up the Itô-type of SDEs taking into account the Wong-Zakai correction. Write down the FPK equation when $f(x, y) = f$ is the joint probability density function. Answers:

$$\begin{aligned} \Delta x &= \{\mu_o x + [1 - (x^2 + y^2)](x^2 + y^2)x - \omega y - b(x^2 + y^2)y + D_m x\} \Delta t \\ &\quad + \sqrt{2D} \Delta W_x(t) + \sqrt{2D_m x} \Delta W_m(t), \\ \Delta y &= \{\mu_o y + [1 - (x^2 + y^2)](x^2 + y^2)y + \omega x + b(x^2 + y^2)x + D_m y\} \Delta t \\ &\quad + \sqrt{2D} \Delta W_y(t) + \sqrt{2D_m y} \Delta W_m(t), \\ \frac{\partial f}{\partial t} &= -\frac{\partial f}{\partial x} [\{\mu_o x + \mu_r(t)x + [1 - (x^2 + y^2)](x^2 + y^2)x - \omega y - b(x^2 + y^2)y \\ &\quad + \sqrt{2D_m x}\} f] \\ &\quad - \frac{\partial f}{\partial y} [\{\mu_o y + \mu_r(t)y + [1 - (x^2 + y^2)](x^2 + y^2)y + \omega x - b(x^2 + y^2)x \\ &\quad + \sqrt{2D_m y}\} f] \\ &\quad + \frac{\partial^2 f}{\partial x^2} [(D + D_m x^2) f] + 2 \frac{\partial^2 f}{\partial x \partial y} [(D_m xy) f] + \frac{\partial^2 f}{\partial y^2} [(D + D_m y^2) f]. \end{aligned}$$

5.9. A nonlinear vibration is described by the following equation of motion (Duffing's equation)

$$\ddot{x}(t) + 2\zeta\omega_0\dot{x}(t) + \omega_0^2x(t) + k_3x^3(t) = \xi(t),$$

where $E\{\xi(t)\xi(t + \tau)\} = 2D\delta(\tau)$.

The following assignments are required:

- Derive the FPK equation, for the non stationary and stationary problem
- Derive the joint probability function $f_2(x, \dot{x})$ for the stationary problem
- Derive the number of up-crossings at level a , given that $\nu^+(a) = \nu_a^+ = \int_0^\infty \dot{x} f_2(a, \dot{x}) d\dot{x}$.
- Try to derive an approximation of $\nu^+(a)$ for a large $|x|$, by using (5.338)

Answers: $f_2(x, \dot{x}) = C e^{-\frac{1}{2\sigma^2}(\frac{\dot{x}^2}{2} + \frac{\omega_0^2 x^2}{2} + \frac{k_3 x^4}{4})}$, $\sigma^2 = \frac{D}{4\zeta\omega_0}$, $\nu^+(a) = \frac{1}{\sigma\sqrt{2\pi}} \times e^{-\frac{1}{2\sigma^2}[\int_0^a (\omega_0^2 x + k_3 x^3) dx]} / \int_{-\infty}^\infty e^{-\frac{1}{2\sigma^2}(\frac{\omega_0^2 x^2}{2} + \frac{k_3 x^4}{4})} dx$.

5.10. Let a vector process $\{z(t)\} = [z_1(t), z_2(t)]^T$ be defined by $z_1(t) = x(t)$ and $z_2(t) = \dot{x}(t)$. we can write Itô SDE

$$\Delta\{z(t)\} = \{f(\{z(t)\})\} \Delta t + \{g\} \Delta W(t),$$

with

$$\{f\{z(t)\}\} = \begin{bmatrix} z_2(t) \\ -2\zeta z_1^2(t) - 1)z_2(t) - z_1(t) \end{bmatrix},$$

and

$$\{g\} = \begin{pmatrix} 0 \\ \sqrt{2D} \end{pmatrix}.$$

The Wiener process is given by $\Delta W(t) = \xi(t)\Delta t$, where $\xi(t)$ is white noise with the properties $E\{\xi(t)\} = 0$ and $E\{\xi(t)\xi(t + \tau)\} = 2D\delta(\tau)$. Write down the original equation of motion (van der Pol). The transition probability density function is given by $f_{c2}(\{z_0\}, 0, \{z\}, t) = f$. What is the corresponding FPK equation and what are the initial and boundary conditions for f ?

Answers: $\ddot{x}(t) + 2\zeta[x(t)^2 - 1]\dot{x}(t) + x(t) = \xi(t)$, $\frac{\partial f}{\partial t} = 2\zeta(z_1^2 - 1)f - z_2\frac{\partial f}{\partial z_1} + [2\zeta(z_1^2 - 1)z_2 + z_1]\frac{\partial f}{\partial z_2} + D\frac{\partial^2 f}{\partial z_1^2}$.

5.11. The nonlinear SDOF dynamic system (Duffing) is excited by a filtered white noise, [87]. The governing equations are

$$\begin{aligned} \ddot{x} + 2\zeta\omega_0\dot{x} + \alpha\omega_0^2x^3 &= f(t), \\ \ddot{f} + 2\gamma\lambda\dot{f} + \lambda^2f &= \xi(t), \end{aligned}$$

where $\xi(t)$ is a white noise with zero mean and autocorrelation

$$E\{\xi(t)\xi(t + \tau)\} = \langle \xi(t)\xi(t + \tau) \rangle = 2D\delta(\tau).$$

We can control the input bandwidth γ and center frequency λ .

Write down the FPK equation when the following state variables are introduced; $z_1 = x$, $z_2 = \dot{x}$, $z_3 = f$ and $z_4 = \dot{f}$. First set up the system of Itô stochastic differential equations.

5.12. This problem is based on the paper of Dunne and Gandbari [54]. A general SDOF system is given by

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \alpha_1\dot{x}x^2 + \alpha_2\dot{x}|x| + \omega_n^2z + k_3x^3 = \xi(t),$$

where α_1 and α_2 are constants, ζ is the damping ratio, ω_n is the natural frequency of the linear dynamic system, k_3 is a constant and $\xi(t)$ is Gaussian white noise with zero mean and $E\{\xi(t)\xi(t + \tau)\} = \langle \xi(t)\xi(t + \tau) \rangle = 2D\delta(\tau)$. Write the two-state ($z_1 = x$, $z_2 = \dot{x}$) equations described by the Itô SDE, the corresponding stationary FPK equation for the joint probability density function $f(z_1, z_2) = f$ and solve the stationary FPK equation with $\alpha_1 = \alpha_2 = 0$. Generate an expression for the mean up-crossings $\nu^+(a)$ of level a , $\nu^+(a) = \int_0^\infty z_2 f(a, z_2) dz_2$. The following parameters are to be used; $\zeta = 0.0138$, $\alpha_1 = \alpha_2 = 0$, $\omega_n = 144.34$, $k_3 = 3021$ and $\sqrt{2D} = 200$. Calculate $\nu^+(a)$.

Answers:

$$\begin{pmatrix} \Delta z_1 \\ \Delta z_2 \end{pmatrix} = \begin{bmatrix} -2\zeta\omega_n z_1 - \alpha_1 z_2 z_1^2 - \alpha_2 z_2 |z_2| - \omega_n^2 z_1 - k_3 z_1^3 \\ z_1 \end{bmatrix} \Delta t + \begin{bmatrix} 0 \\ \sqrt{2D} \end{bmatrix} \Delta W(t),$$

$$-\frac{\partial}{\partial z_1}[z_2 f] + \frac{\partial}{\partial z_2}[2\zeta\omega_n z_1 + \alpha_1 z_2 z_1^2 + \alpha_2 z_2 |z_2| + \omega_n^2 z_1 + k_3 z_1^3] + D \frac{\partial^2 f}{\partial z_2^2} = 0,$$

$$f(z_1, z_2) = C \exp\left\{-\frac{2\zeta\omega_n}{D}\left[\frac{z_2^2}{2} + \frac{\omega_n^2 z_1^2}{2} + \frac{k_3 z_1^4}{4}\right]\right\},$$

where C is normalization constant,

$$\nu^+(a) = \frac{C}{2} \sqrt{\frac{\pi D}{\zeta\omega_n}} \exp\left[-\frac{2\zeta\omega_n}{D} \int_0^a (\omega_n^2 z_1 + k_3 z_1^2) dz_1\right].$$

5.13. This problem is based on an contribution of Scheurkogel and Elishakoff in the proceedings [233], entitled: “An Exact Solution of the Fokker-Planck Equation for Nonlinear Random Vibration of a Two-Degrees-of-Freedom System”.

Consider the system of random differential equations

$$m_1 \ddot{x}_1 + a_{11} \dot{x}_1 + a_{12} \dot{x}_2 + \frac{\partial U}{\partial x_1} = \xi_1(t),$$

$$m_2 \ddot{x}_2 + a_{21} \dot{x}_1 + a_{22} \dot{x}_2 + \frac{\partial U}{\partial x_2} = \xi_2(t),$$

where $\xi_1(t)$ and $\xi_2(t)$ are stationary white Gaussian processes with zero mean and auto and cross correlation function

$$E\{\xi_i(t)\xi_j(t + \tau)\} = \langle \xi_i(t)\xi_j(t + \tau) \rangle = 2D_{ij}\delta(\tau), \quad i, j = 1, 2.$$

The two differential equations describe a coupled two-degrees-of-freedom system with linear damping and nonlinear stiffness, the latter derivable from potential function $U(x_1, x_2)$

$$U(x_1, x_2) = \frac{k}{2}(x_1^2 + x_2^2) + \frac{\alpha}{2}(x_1 - x_2)^2 + \frac{\beta}{4}(x_1 - x_2)^4,$$

where k, α are positive and β is a small and positive nonlinearity parameter.

The spectral density matrix $2D_{ij}$ is proportional to the damping matrix

$$2D_{ij} = \frac{a_{ij} + a_{ji}}{\lambda},$$

where λ is a constant.

Prove that the stationary solution of the FPK equation is given by

$$f(x_1, x_2, \dot{x}_1, \dot{x}_2) = \gamma e^{-\frac{\lambda}{2}[m_1 \dot{x}_1^2 + m_2 \dot{x}_2^2 + 2U(x_1, x_2)]},$$

where γ follows from the normalization condition

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x_1, x_2, \dot{x}_1, \dot{x}_2) dx_1 dx_2 d\dot{x}_1 d\dot{x}_2 = 1.$$

Introduce two new variables

$$u = x_1 + x_2, \quad v = x_1 - x_2.$$

and prove that the following marginal densities are given by

$$\begin{aligned} f(\dot{x}_1) &= \sqrt{\frac{\lambda m_1}{2\pi}} e^{-\frac{\lambda m_1 \dot{x}_1^2}{2}}, \\ f(\dot{x}_2) &= \sqrt{\frac{\lambda m_2}{2\pi}} e^{-\frac{\lambda m_2 \dot{x}_2^2}{2}}, \\ f(u) &= \frac{\lambda k}{4\pi} e^{-\frac{\lambda k u^2}{4}}, \\ f(v) &= C e^{-[\frac{\lambda \beta v^4}{4} + \frac{\lambda(k+2\alpha)v^2}{4}]}, \end{aligned}$$

where C is the normalization constant. Show that

$$f(x_1, x_2, \dot{x}_1, \dot{x}_2) = f(\dot{x}_1, \dot{x}_2, u, v) = f(\dot{x}_1) f(\dot{x}_2) f(u) f(v).$$

Further prove that

$$\begin{aligned} E\{\dot{x}_1\} = E\{\dot{x}_2\} = E\{x_1\} = E\{x_2\} = E\{u\} = E\{v\} = 0, \\ E\{\dot{x}_1^2\} = \frac{1}{\lambda m_1}, \quad E\{\dot{x}_2^2\} = \frac{1}{\lambda m_2}, \quad E\{u^2\} = \frac{2}{\lambda k}, \quad E\{x_1^2\} = E\{x_2^2\}. \end{aligned}$$

and prove, knowing that $E\{x_1^2\} = E\{x_2^2\} = \frac{E\{u^2\} + E\{v^2\}}{4}$, and $E\{x_1 x_2\} = \frac{E\{u^2\} - E\{v^2\}}{4}$, the following relations

$$\begin{aligned} E\{v^2\} &= \frac{2}{\lambda(k+2\alpha)} \left(1 - \frac{12\beta}{\lambda(k+2\alpha)^2} \right) + O(\beta^2), \quad \beta \rightarrow 0, \\ E\{x_1^2\} = E\{x_2^2\} &= \frac{1}{2\lambda k} + \frac{2}{\lambda(k+2\alpha)} \left(1 - \frac{12\beta}{\lambda(k+2\alpha)^2} \right) + O(\beta^2), \quad \beta \rightarrow 0, \\ E\{x_1 x_2\} &= \frac{1}{2\lambda k} - \frac{2}{\lambda(k+2\alpha)} \left(1 - \frac{12\beta}{\lambda(k+2\alpha)^2} \right) + O(\beta^2), \quad \beta \rightarrow 0. \end{aligned}$$

hints:

$$\begin{aligned} \int_{-\infty}^{\infty} x^{2n} e^{-px^2} dx &= \frac{\Gamma(n+1/2)}{p^{n+1/2}}, \\ E\{v^{2m}\} &= \left[\frac{\lambda}{4}(k+2\alpha) \right]^{-m} \sigma_m \left(\frac{4\beta}{\lambda(k+2\alpha)^2} \right) \\ \sigma_m(z) &= (2m)! \left[\frac{1 - m(m+2)z}{m!4^m} \right] + O(z^2), \quad z \downarrow 0. \end{aligned}$$

5.14. This problem is inspired on a paper of Dimentberg [49]. Consider the following second-order system

$$\ddot{x} + 2\alpha\dot{x}[1 + \eta(t)] + \beta_1\dot{x}(x^2 + \dot{x}^2/\Omega^2) + \Omega^2x[1 + \xi(t)] = \zeta(t), \quad \beta_1 \geq 0,$$

where $\eta(t)$, $\xi(t)$ and $\zeta(t)$ are independent zero-mean Gaussian white noise processes or white noise in the Stratonovich sense with intensities D_η , D_ξ and D_ζ respectively.

Write down the stationary joint of probability function $f = f(x, v)$ of the coordinates $x(t)$ and velocity $v(t) = \dot{x}$.

The intensities D_ξ and D_η satisfy the following condition

$$\Omega^2 D_\xi = 4\alpha^2 D_\eta.$$

Prove that the following solution of f

$$f(x, v) = \frac{C e^{-\beta(x^2+v^2/\Omega^2)}}{(\aleph + x^2 + v^2/\Omega^2)^{\delta-\aleph\beta}},$$

where

$$\aleph = D_\zeta/D_\xi\Omega^4, \quad \delta = 2\alpha/D_\xi\Omega^2, \quad \beta = \beta_1/D_\xi\Omega^2,$$

and

$$C^{-1} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, v) dx dv.$$

For the further calculations assume that $\beta_1 = 0$. Calculate the normalization constant C , $f(x) = \int_{-\infty}^{\infty} f(x, v) dv$, $\langle x^{2n} \rangle$, $\langle x^2 v^2 \rangle$ and the positive crossings at level a , $\nu^+(a) = \int_0^\infty v f(x, v) dv$.

Answers:

$$\begin{aligned} v \frac{\partial f}{\partial x} &= \Omega^2 x \frac{\partial f}{\partial v} + \frac{\partial}{\partial v} \{ [(2\alpha - 2\alpha^2 D_\eta)v + \beta_1 v (x^2 + v^2/\Omega^2)] f \} \\ &\quad + \frac{1}{2} \frac{\partial^2}{\partial v^2} [(4\alpha^2 v^2 D_\eta + \Omega^4 x^2 D_\xi + D_\zeta) f], \\ C &= (\pi\Omega)^{-1} (\delta - 1) \aleph^{\delta-1}, \\ f(x) &= \frac{\aleph^{\delta-1} \Gamma(\delta - 1/2)}{\sqrt{\pi} \Gamma(\delta - 1)} (\aleph + x^2)^{-(\delta-1/2)}, \\ \langle x^{2n} \rangle &= \frac{\aleph^n \Gamma(n+1) \Gamma(\delta - n - 1)}{\sqrt{\pi} \Gamma(\delta - 1)}, \\ \langle x^2 v^2 \rangle &= \frac{\aleph^2 \Omega^2}{4(\delta - 2)(\delta - 3)}, \\ \nu^+(a) &= \frac{\Omega}{2\pi} \frac{1}{(1 + a^2/\aleph)^{\delta-1}}. \end{aligned}$$

5.15. This problem is taken from [216]. Consider the following non-linear oscillator

$$\ddot{x} + \frac{\beta\dot{x}}{1 + \dot{x}^2/2} + \frac{1 + \dot{x}^2/2}{1 + x^2/2} = \xi(t),$$

where the white noise $\xi(t)$ has a zero-mean and a second moment $\langle \xi(t)\xi(t + \tau) \rangle = 2D\delta(\tau)$. Show that the stationary joint probability function $f(x, \dot{x})$ is given by

$$f(x, \dot{x}) = \ln[(1 + x^2/2)^{\beta-D}(1 + \dot{x}^2/2)^\beta],$$

where $\beta > 2D$.

5.9 Numerical Solution of the FPK Equation

5.9.1 Solution of the FPK Equation by the Finite Element Method

Many papers have been published about the numerical solution of the FPK equation using the finite element method. In 1985 Langley published a paper [107] using the finite element method to analyze the statistics of nonlinear random vibration, but only for stationary problems. In 1991 Langtangen [108], discussed the application of the finite element method for stationary structural reliability problems. In 1991 Spencer and Bergman published a paper [194] about the numerical solution of the transient FPK equation for nonlinear stochastic systems. In [170] the finite element method was applied to the transient Duffing oscillator. In 2006 [103] both the finite element method and the finite difference method were applied to solve two-dimensional (two state variables, SDOF system) nonlinear systems and a four-dimensional linear system consisting of two coupled mass-spring systems. In [178] Shiau introduced adaptive finite element meshes to solve stationary stochastic systems, and [54] gives overview of some contributors of the development and use of numerical method for solving the FPK equation. The following methods are discussed:

- *Finite Element method*
- *Finite Difference method*

The Finite Element Method (FEM) which uses simple piece-wise shape functions defined over a finite, rather than an infinite region, thus allowing greater flexibility in satisfying complicated boundary conditions.

The Finite Difference method is amongst others the earliest used to solve parabolic partial differential equations, but they have not widely applied to the FPK equation.

5.9.2 General Finite Element Approach

The derivation of the finite element matrices will be in accordance with the theory proposed by Langley [107]. We use Langley's notation. The FPK equation can be applied to any dynamical system whose equations of motion can be written in the form

$$\{\dot{z}\} = \{g(z)\} + [A]\{w\}, \quad (5.223)$$

where $\{z\}$ represents displacements and velocities of the system, $[A]$ is a square matrix and $\{w\}$ is a vector of uncorrelated Gaussian white noise processes, each having a spectral density of unity. $\{g(x)\}$ is a general vector function of the variables $\{z\}$. The vector $\{z\}$ is a Markov process and the corresponding FPK equation is

$$\frac{\partial f}{\partial t} = - \sum_{i=1}^n \frac{\partial f}{\partial z_i} (g_i(z) f) + \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n \frac{\partial^2}{\partial z_i \partial z_j} [B_{ij} f], \tag{5.224}$$

where $f_{2c}(\{z_0\}, 0|\{z\}, t) = f$ is the transition or conditional probability density function and matrix $[B] = [A][A]^T$.

Example. The equation of motion of a Duffing oscillator excited by a Gaussian white noise random force $f(t)$ is given by

$$\ddot{x} + 2\zeta\omega_0\dot{x} + \omega_0^2(1 + \epsilon x^2)x = f(t).$$

The state equation can now be written by putting $z_1 = x$ and $z_2 = \dot{x}$

$$\begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \end{pmatrix} = \begin{pmatrix} z_2 \\ -2\zeta\omega_0 z_2 - \omega_0^2(1 + \epsilon z_1^2)z_1 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & \sqrt{2D} \end{pmatrix} \begin{pmatrix} 0 \\ \xi(t) \end{pmatrix},$$

where $\langle f(t) \rangle = 0$, $\langle \xi(t) \rangle = 0$, $\langle f(t)f(t+\tau) \rangle = 2D\delta(\tau)$ and $\langle \xi(t)\xi(t+\tau) \rangle = \delta(\tau)$.

The stochastic differential equations (SDEs) can now be written

$$\begin{pmatrix} \Delta z_1 \\ \Delta z_2 \end{pmatrix} = \begin{pmatrix} z_2 \\ -2\zeta\omega_0 z_2 - \omega_0^2\{1 + \epsilon z_1^2\}z_1 \end{pmatrix} \Delta t + \begin{pmatrix} 0 & 0 \\ 0 & \sqrt{2D} \end{pmatrix} \begin{pmatrix} 0 \\ \Delta W(t) \end{pmatrix},$$

where $\Delta W(t) = \xi(t)\Delta t$.

The SDEs lead to the following FPK equation for the transition probability density function $f_{2c}(\{z_0\}, 0|\{z\}, t) = f$.

$$\frac{\partial f}{\partial t} = - \frac{\partial f}{\partial z_1} (z_2 f) + \frac{\partial f}{\partial z_2} [\{+2\zeta\omega_0 z_2 + \omega_0^2(1 + \epsilon z_1^2)z_1\} f] + D \frac{\partial^2 f}{\partial z_2^2}.$$

Equation (5.224) can be rewritten as

$$\frac{\partial f}{\partial t} = L_1[f] + L_2[f], \tag{5.225}$$

where the differential operators L_1 and L_2 are given by

$$L_1 = - \sum_{i=1}^n \frac{\partial}{\partial z_i} [g_i(z)], \tag{5.226}$$

$$L_2 = \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n \frac{\partial^2}{\partial z_i \partial z_j} [B_{ij}]. \tag{5.227}$$

For any weighting or test function $\phi(z)$, the *weak form* of (5.225) (see Appendix M) implies that

$$\int_R \phi(z) \frac{\partial f}{\partial t} d\tau - \int_R \phi(z) L_1[f] d\tau - \int_R \phi(z) L_2[f] d\tau = 0, \tag{5.228}$$

where \int_R represents the integration over the whole domain and $d\tau = dz_1 dz_2 \cdots dz_n$. The second and third term in (5.228) will be integrated by parts, thus

$$\begin{aligned} \int_R \phi(z) L_1[f] d\tau &= - \sum_{i=1}^n \int_R \phi(z) \frac{\partial}{\partial z_i} [g_i(z) f] d\tau \\ &= - \sum_{i=1}^n \int_{\hat{R}} [\phi(z) g_i(z) f] \Big|_{z_{i1}}^{z_{i2}} d\hat{\tau} \\ &\quad + \sum_{i=1}^n \int_R g_i(z) f \frac{\partial}{\partial z_i} [\phi(z)] d\tau, \end{aligned} \tag{5.229}$$

and

$$\begin{aligned} \int_R \phi(z) L_2[f] d\tau &= \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n \int_R \phi(z) \frac{\partial^2}{\partial z_i \partial z_j} [B_{ij} f] d\tau \\ &= \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n \int_{\hat{R}} \left[\phi(z) \frac{\partial}{\partial z_j} B_{ij} f \right] \Big|_{z_{i1}}^{z_{i2}} d\hat{\tau} \\ &\quad - \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n \int_{\hat{R}} \int_R \frac{\partial}{\partial z_i} [\phi(z)] \frac{\partial}{\partial z_j} [B_{ij} f] d\tau, \end{aligned} \tag{5.230}$$

where \hat{R} and $d\hat{\tau}$ refer to the reduced region in which z_i does not appear, and z_{i2} and z_{i1} are the upper and lower limits of z_i respectively. In many problems the region R will represent an infinite domain in which $f(z)$ tends to zero as z_i becomes large. In such cases $f(z)$ and its derivatives will become zero at infinity, thus (5.228) becomes

$$\begin{aligned} \int_R \phi(z) \frac{\partial f}{\partial t} d\tau - \sum_{i=1}^n \int_R g_i(z) f \frac{\partial}{\partial z_i} [\phi(z)] d\tau \\ + \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n \int_R \frac{\partial}{\partial z_i} [\phi(z)] \frac{\partial}{\partial z_j} [B_{ij} f] d\tau = 0. \end{aligned} \tag{5.231}$$

Since (5.231) involves only the first derivatives of $f(z)$ it is sufficient for the shape functions to have C_0 continuity.

In his paper [107] Langley discussed three possible methods of approximating the solution of transition probability density function $f(z)$ by applying the finite element method:

1. The interior region of the domain is divided into finite elements and surrounded by “infinite elements” which extend to infinity.
2. The interior region of the domain is divided into finite elements, while the outer region is modelled using boundary elements or series solutions.
3. The interior region, which is considered to be sufficiently large, is divided into finite elements while the outer region is neglected.

Approach 3 will be adopted because $f(z)$ tends to zero in the outer regions of the domain. With this approach, the question arises: “How large an interior region and the density of finite elements must be considered?” Convergence studies are of great importance.

The interior region is divided into a number of n -dimensional finite elements (n is number of state variables), which have straight parallel sides, e.g.

- $n = 1$ line elements,
- $n = 2$ rectangular elements,
- $n = 3$ cuboid elements.

To solve (5.231) by the weighted residual method (WRM) (Galerkin), it is assumed that f can be represented as a sum of *shape functions* $H_i(z)$. The shape functions are chosen such that the value of f_e within the finite element can be expressed in terms of the nodal values

$$f_e = \sum_{i=1}^m H_i(z) f_i, \quad (5.232)$$

where f_i is the value of f at node i , $H_i(z)$ is the corresponding shape function and m is the number of nodes in one finite element. The shape function $H_i(z)$ is chosen to give unity at node i and zero at all other nodes. It is convenient to describe the shape functions in global coordinates because g_i and B_{ij} are in general functions of z .

For the 1-dimensional problem, the shape functions $\bar{H}_{k,i}(z_k)$, $i = 1, 2, k = 1, 2, \dots, n$ are [178]

$$\begin{aligned} \bar{H}_{k,i}(z_k) &= \bar{H}_{k,1}(z_k) = \frac{-z_k + z_{k,j}}{z_{k,j} - z_{k,i}}, \\ \bar{H}_{k,j}(z_k) &= \bar{H}_{k,2}(z_k) = \frac{-z_k + z_{k,i}}{z_{k,i} - z_{k,j}}, \end{aligned} \quad (5.233)$$

where i, j represents two nodes and k is the number of state variables, e.g. $z_1 = x$ and $z_2 = \dot{x}$. In general, the shape function $H_i(z_k)$ can be written to be

$$H_i(\{z\}) = \prod_{k=1}^n \bar{H}_{k,i}(z_k). \quad (5.234)$$

The shape functions $H_i(z)$, $z = z_1$, in the 1-D element with two nodes $i = 1$ and $j = i = 2$ can be written

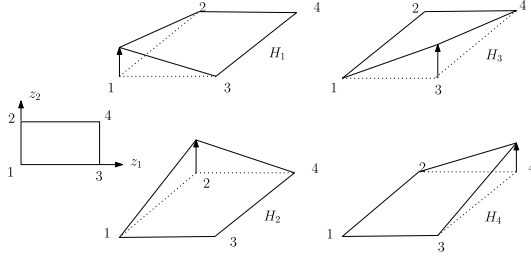


Fig. 5.5. Shape functions for a two-dimensional finite element

$$\begin{aligned}
 H_1(z) &= \prod_{k=1}^{n=1} \bar{H}_{k,1}(z_k) = \frac{-z + z_2}{z_2 - z_1}, \\
 H_2(z) &= \prod_{k=1}^{n=1} \bar{H}_{k,2}(z_k) = \frac{-z + z_1}{z_2 - z_1},
 \end{aligned} \tag{5.235}$$

For two variables, $n = 2$, the finite element to be used is rectangular with four nodes, $m = 4$. The four shape functions H_i are illustrated in Fig. 5.5. The shape functions $H_i(\{z\})$, $\{z\} = [z_1, z_2]$, in the 2-D rectangular element with four nodes $i = 1, 2, 3, 4$ can be written

$$\begin{aligned}
 H_1(\{z\}) &= \prod_{k=1}^{n=2} \bar{H}_{k,1}(z_k) = \left(\frac{-z_1 + z_{1,3}}{z_{1,3} - z_{1,1}} \right) \left(\frac{-z_2 + z_{2,2}}{z_{2,2} - z_{2,1}} \right), \\
 H_2(\{z\}) &= \prod_{k=1}^{n=2} \bar{H}_{k,2}(z_k) = \left(\frac{-z_1 + z_{1,4}}{z_{1,4} - z_{1,2}} \right) \left(\frac{-z_2 + z_{2,1}}{z_{2,1} - z_{2,2}} \right), \\
 H_3(\{z\}) &= \prod_{k=1}^{n=2} \bar{H}_{k,3}(z_k) = \left(\frac{-z_1 + z_{1,1}}{z_{1,1} - z_{1,3}} \right) \left(\frac{-z_2 + z_{2,4}}{z_{2,4} - z_{2,3}} \right), \\
 H_4(\{z\}) &= \prod_{k=1}^{n=2} \bar{H}_{k,4}(z_k) = \left(\frac{-z_1 + z_{1,2}}{z_{1,2} - z_{1,4}} \right) \left(\frac{-z_2 + z_{2,3}}{z_{2,3} - z_{2,4}} \right), \\
 z_{1,4} &= z_{1,3}, \quad z_{2,4} = z_{2,2}.
 \end{aligned} \tag{5.236}$$

Example. This example is taken from [103]. For the two-dimensional problem, the shape functions for a bilinear 4-node element (Fig. 5.6) are given by

$$\begin{aligned}
 H_1(r, s) &= \frac{(r - r_2)(s - s_4)}{(r_1 - r_2)(s_1 - s_4)}, & H_2(r, s) &= \frac{(r - r_1)(s - s_3)}{(r_2 - r_1)(s_2 - s_3)}, \\
 H_3(r, s) &= \frac{(r - r_4)(s - s_2)}{(r_3 - r_4)(s_3 - s_2)}, & H_4(r, s) &= \frac{(r - r_3)(s - s_1)}{(r_4 - r_3)(s_4 - s_1)}.
 \end{aligned} \tag{5.237}$$

In general, both g_i and B_{ij} in general, are functions of z_i so that it is convenient to describe the shape functions in a global coordinate system [178].

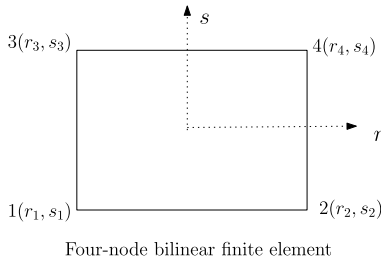


Fig. 5.6. Bilinear 4-nod element

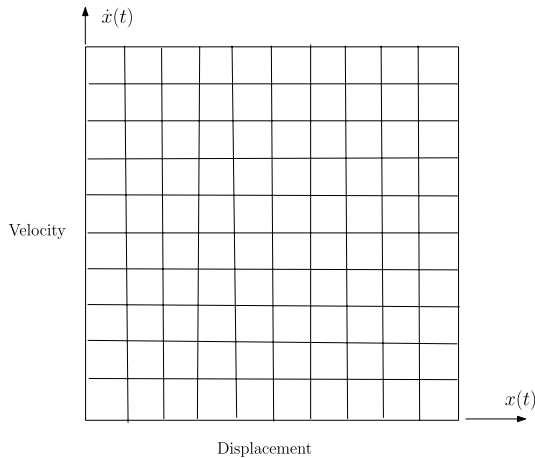


Fig. 5.7. Finite element mesh for two-dimensional system

A finite element mesh used for a two-dimensional (state-variables) dynamic system is illustrated in Fig. 5.7.

Replacing the integral (5.231) over the region with the sum of the integrals over each finite element, we find

$$\begin{aligned}
 & \sum_e \left\{ \int_e \phi(z) \frac{\partial f_e}{\partial t} d\tau - \sum_{i=1}^n \int_e g_i(z) f_e \frac{\partial}{\partial z_i} [\phi(z)] d\tau \right. \\
 & \left. + \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n \int_e \frac{\partial}{\partial z_i} [\phi(z)] \frac{\partial}{\partial z_j} [B_{ij} f_e] d\tau \right\} = 0, \quad (5.238)
 \end{aligned}$$

where \sum_e represents the sum over all finite elements, \int_e represents the integral over an element, and f_e the transition probability density function within an element. $\phi(z)$ is now chosen to be zero except over those elements which contain a particular node, say global node q . Over these finite elements, the weighting or test function $\phi(z)$ is chosen to be one of the shape functions

corresponding to this node. For one finite element with m nodes and n state variables, (5.238) can be rewritten as follows

$$\sum_e \left\{ \sum_{s=1}^m \left[\int_e \{H_i\} \frac{\partial(H_s f_s)}{\partial t} d\tau - \sum_{i=1}^n \int_e g_i(z) (H_s f_s) \frac{\partial}{\partial z_i} [\{H_i\}] d\tau \right. \right. \\ \left. \left. + \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n \int_e \frac{\partial}{\partial z_i} [\{H_i\}] \frac{\partial}{\partial z_j} [B_{ij} H_s f_s] d\tau \right] \right\} = 0, \quad (5.239)$$

or

$$\sum_e \left\{ \int_e \{H_i\} [H_i] d\tau \{f_i\} - \sum_{i=1}^n \int_e g_i(z) \{H_i\} \frac{\partial}{\partial z_i} [[H_i]] d\tau \{f_i\} \right. \\ \left. + \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n \int_e \frac{\partial}{\partial z_i} [\{H_i\}] \frac{\partial}{\partial z_j} [B_{ij} [H_i]] d\tau \{f_i\} \right\} = 0, \quad (5.240)$$

or

$$\sum_e \{ [m_e] \{ \dot{f}_i \} + [k_e] \{ f_i \} \} = 0, \quad (5.241)$$

where $\{ \dot{f}_i \}$ are the time derivatives and the $\{ f_i \}$ are nodal values at m nodal values, $[m_e]$ the so-called mass matrix and $[k_e]$ the so-called stiffness matrix given by

$$[m_e] = \int_e \{H_i\} [H_i] d\tau, \quad (5.242)$$

and

$$[k_e] = - \sum_{i=1}^n \int_e g_i(z) \{H_i\} \frac{\partial}{\partial z_i} [H_i] d\tau \\ + \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n \int_e \frac{\partial}{\partial z_i} \{H_i\} \frac{\partial}{\partial z_j} [B_{ij} [H_i]] d\tau. \quad (5.243)$$

All the element matrices should be assembled in the overall “mass” matrix $[M]$ and “stiffness” matrix $[K]$ and the finite element representation of the FPK equation can be expressed by

$$[M] \{ \dot{f} \} + [K] \{ f \} = \{ 0 \}, \quad (5.244)$$

where $\{ f(\{z\}, t) \}$ is the vector of nodal transition probability density functions. The finite difference method (5.244) can be written as follows

$$[M] \left[\frac{f(t + \Delta t) - f(t)}{\Delta t} \right] + [K] \{ f(t) \} = \{ 0 \}. \quad (5.245)$$

Rearrangement of the vector terms using the θ method will give a recurrence relation

$$[M - (1 - \theta)\Delta tK]\{f(t + \Delta t)\} = [M + \theta\Delta tK]\{f(t)\}, \quad (5.246)$$

where $0 \leq \theta \leq 1$ and Δt is the time step. The Crank-Nicholson scheme is applied with $\theta = 0.5$ for stability:

$$\left[M - \frac{\Delta t}{2}K\right]\{f(t + \Delta t)\} = \left[M + \frac{\Delta t}{2}K\right]\{f(t)\}. \quad (5.247)$$

The initial condition of (5.247) is

$$\lim_{t \rightarrow 0} f_{2c}(\{z_0\}, 0|\{z\}, t) = \delta(\{z\} - \{z_0\}), \quad (5.248)$$

and the normalization condition is

$$\int_R f_{2c}(\{z_0\}, 0|\{z\}, t)d\tau = 1. \quad (5.249)$$

The theorem of total probability can then be applied, yielding

$$f_1(\{z\}, t) = \int_R f_{2c}(\{z_0\}, 0|\{z\}, t)f_1(\{z_0\})d\tau. \quad (5.250)$$

Equation (5.247) can now be written as follows, [194]

$$\left[M - \frac{\Delta t}{2}K\right]\{f_1(t + \Delta t)\} = \left[M + \frac{\Delta t}{2}K\right]\{f_1(t)\}, \quad (5.251)$$

subject to the initial condition

$$f_1(0) = f_1(\{z_0\}). \quad (5.252)$$

Solution 1-D Problem

In [85], the diffusion of bistable potential was investigated with the aid of the FPK equation. The bistable potential is represented by the nonlinear Langevin equation (NLE)

$$\frac{d}{dt}x = \gamma x - gx^3 + \eta(t), \quad (5.253)$$

where x is the driven random variable, γ is a positive friction coefficient, and g is the nonlinearity parameter. The driving random force $\eta(t)$ is to be a Gaussian white noise obeying

$$\{\eta(t)\} = \langle \eta(t) \rangle = 0, \quad E\{\eta(t + \tau)\eta(t)\} = \langle \eta(t + \tau)\eta(t) \rangle = 2\varepsilon\delta(\tau),$$

where ε is the diffusion coefficient. The equivalent FPK equation for the probability density (distribution) function $f(x, t) = f$ is

$$\frac{\partial f}{\partial t} = -\frac{\partial}{\partial x}[(\gamma x - gx^3)f] + \varepsilon \frac{\partial^2 f}{\partial x^2}. \quad (5.254)$$

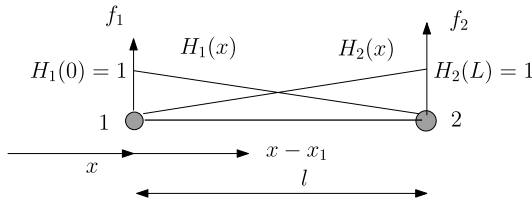


Fig. 5.8. Illustration of linear 1-D element

The initial probability density function $f(x, 0) = f(x)$ is a Gaussian distribution

$$f(x) = \frac{1}{\sigma_0 \sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma_0^2}}. \tag{5.255}$$

The stationary solution of (5.254) is given by [35]

$$f(x) = A e^{\frac{2\gamma x^2}{\varepsilon} - \frac{g x^3}{\varepsilon}}, \tag{5.256}$$

where A is an constant coefficient, derived from $\int_{-\infty}^{\infty} f(x) dx = 1$.

Equation (5.254) will be solved by the finite element method using linear 1-D elements as shown in Fig. 5.8. Equation (5.254) is transformed into the weak form

$$\int_a^b \left[\phi \frac{\partial f}{\partial t} + \frac{\partial}{\partial x} [(\gamma x - g x^3) f] + \varepsilon \frac{\partial \phi}{\partial x} \frac{\partial f}{\partial x} \right] dx = \varepsilon \phi \frac{\partial f}{\partial x} \Big|_a^b. \tag{5.257}$$

The natural boundary conditions $\frac{\partial f}{\partial x} \Big|_a = \frac{\partial f}{\partial x} \Big|_b = 0$ apply. We also have $\int_{-\infty}^{\infty} f(x, t) = 1$ or $\int_a^b f(x, t) = 1$.

For one linear 1-D element (5.257) can be written

$$\int_{x_1}^{x_2} \left[\phi \frac{\partial f}{\partial t} + \phi \frac{\partial}{\partial x} [(\gamma x - g x^3) f] + \varepsilon \frac{\partial \phi}{\partial x} \frac{\partial f}{\partial x} \right] dx = 0, \tag{5.258}$$

where x_1 and x_2 are the coordinates of the nodes 1 and 2, respectively (Fig. 5.8). The “mass” matrix and “stiffness” matrix for the two-node linear 1-D element will be derived from (5.258).

The probability density function $f(x)$ will be approximated by two shape functions $H_1(x) = 1 - \frac{(x-x_1)}{(x_2-x_1)} = \frac{-x+x_2}{x_2-x_1}$ and $H_2(x) = \frac{(x-x_1)}{(x_2-x_1)} = \frac{-x+x_1}{x_1-x_2}$ in the following way

$$f(x) = H_1(x) f_1 + H_2(x) f_2, \tag{5.259}$$

where f_1 and f_2 are the unknown nodal values of the probability density function at node 1 and node 2 respectively (see Fig. 5.8). Both shape functions $H_i(x)$ apply in $x_1 \leq x \leq x_2$.

The two weighting functions are in accordance with Galerkin’s method, thus

$$\phi_1(x) = H_1(x), \quad \phi_2(x) = H_2(x). \quad (5.260)$$

The approximate function for $f(x)$ in (5.259) and the weighting functions in (5.260) are substituted in (5.258) and further evaluated to get the “mass” and “stiffness” matrices for the 1-D linear finite element. The weak form (5.258) will be evaluated term by term.

$$\begin{aligned} \int_{x_1}^{x_2} \phi \frac{\partial f}{\partial t} dx &= - \int_{x_1}^{x_2} \begin{pmatrix} H_1(x) \\ H_2(x) \end{pmatrix} (H_1(x) \ H_2(x)) \begin{pmatrix} \dot{f}_1 \\ \dot{f}_2 \end{pmatrix} dx \\ &= [m_e] \begin{pmatrix} \dot{f}_1 \\ \dot{f}_2 \end{pmatrix}, \end{aligned} \quad (5.261)$$

where $[m_e]$ is the element “mass” matrix. Now the “stiffness” matrices $[k_{e,i}]$, $i = 1, \dots, 3$ will be derived.

$$\begin{aligned} \int_{x_1}^{x_2} \gamma \phi \frac{\partial [x f]}{\partial x} dx &= \gamma \int_{x_1}^{x_2} \begin{pmatrix} H_1(x) \\ H_2(x) \end{pmatrix} \frac{\partial}{\partial x} [x (H_1(x) \ H_2(x))] \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} dx \\ &= \gamma [k_{1,e}] \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \end{aligned} \quad (5.262)$$

$$\begin{aligned} \int_{x_1}^{x_2} g \phi \frac{\partial [x^3 f]}{\partial x} dx &= g \int_{x_1}^{x_2} \begin{pmatrix} H_1(x) \\ H_2(x) \end{pmatrix} \frac{\partial}{\partial x} [x^3 (H_1(x) \ H_2(x))] \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} dx \\ &= g [k_{2,e}] \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \end{aligned} \quad (5.263)$$

and

$$\begin{aligned} \int_{x_1}^{x_2} \varepsilon \frac{\partial \phi}{\partial x} \frac{\partial f}{\partial x} dx &= \varepsilon \int_{x_1}^{x_2} \frac{\partial}{\partial x} \begin{pmatrix} H_1(x) \\ H_2(x) \end{pmatrix} \frac{\partial}{\partial x} (H_1(x) \ H_2(x)) \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} dx \\ &= \varepsilon [k_{3,e}] \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}. \end{aligned} \quad (5.264)$$

Further evaluation of the integrals in (5.261), (5.262), (5.263) and (5.264) will lead to the “mass” and “stiffness” matrices. The length of the 1-D linear element is $l_i = x_2 - x_1$. The matrices entries are:

$$\begin{aligned} m_e(1,1) &= \frac{x_2}{3} - \frac{x_1}{3}, \\ m_e(1,2) &= \frac{x_2}{6} - \frac{x_1}{6}, \\ m_e(2,1) &= \frac{x_2}{6} - \frac{x_1}{6}, \\ m_e(2,2) &= \frac{x_2}{3} - \frac{x_1}{3}, \end{aligned} \quad (5.265)$$

$$\begin{aligned}
 k_{1,e}(1,1) &= \frac{x_1}{x_2 - x_1}, \\
 k_{1,e}(1,2) &= -\frac{x_2}{x_2 - x_1}, \\
 k_{1,e}(2,1) &= -\frac{x_1}{x_2 - x_1}, \\
 k_{1,e}(2,2) &= \frac{x_2}{x_2 - x_1},
 \end{aligned} \tag{5.266}$$

$$\begin{aligned}
 k_{2,e}(1,1) &= \frac{x_1^3}{x_2 - x_1}, \\
 k_{2,e}(1,2) &= -\frac{x_2^3}{x_2 - x_1}, \\
 k_{2,e}(2,1) &= -\frac{x_1^3}{x_2 - x_1}, \\
 k_{2,e}(2,2) &= \frac{x_2^3}{x_2 - x_1},
 \end{aligned} \tag{5.267}$$

and

$$\begin{aligned}
 k_{3,e}(1,1) &= \frac{1}{x_2 - x_1}, \\
 k_{3,e}(1,2) &= -\frac{1}{x_2 - x_1}, \\
 k_{3,e}(2,1) &= -\frac{1}{x_2 - x_1}, \\
 k_{3,e}(2,2) &= \frac{1}{x_2 - x_1}.
 \end{aligned} \tag{5.268}$$

The total “stiffness” matrix $[k_e]$ is

$$[k_e] = \gamma[k_{1,e}] + g[k_{2,e}] + \varepsilon[k_{3,e}]. \tag{5.269}$$

All the element matrices should be assembled into the overall “mass” matrix $[M]$ and “stiffness” matrix $[K]$; is the finite element representation of the FPK equation can be expressed by

$$[M]\{\dot{f}(t)\} + [K]\{f(t)\} = \{0\}, \tag{5.270}$$

where $\{f\}$ is the vector of nodal probability density functions. Applying the finite difference method, we can write (5.270) as follows:

$$[M] \left[\frac{f(t + \Delta t) - f(t)}{\Delta t} \right] + [K]\{f(t)\} = \{0\}. \tag{5.271}$$

Rearrangement of the vector terms using the θ method will give the recurrence relation

$$[M - (1 - \theta)\Delta tK]\{f(x, t + \Delta t)\} = [M + \theta\Delta tK]\{f(x, t)\}, \quad (5.272)$$

where $0 \leq \theta \leq 1$ and Δt is the time step. The Crank-Nicholson scheme is applied with $\theta = 0.5$ for stability:

$$\left[M - \frac{\Delta t}{2}K\right]\{f(x, t + \Delta t)\} = \left[M + \frac{\Delta t}{2}K\right]\{f(x, t)\}. \quad (5.273)$$

The probability density function $f(x, t)$ has to fulfill the normalization rule

$$\begin{aligned} \int_{-\infty}^{\infty} f(x, t)dx &= \int_a^b f(x, t)dx = \sum_{i=1}^N \int_{x_{i,1}}^{x_{i,2}} f(x, t)dx \\ &= \sum_{i=1}^N \frac{1}{2}(x_{i,2} - x_{i,1})(f_{i,1} + f_{i,2}) \\ &= \sum_{i=1}^N \frac{1}{2}l_i(f_{i,1} + f_{i,2}) = 1, \end{aligned} \quad (5.274)$$

where N is the number of linear finite elements, l_i is the length of the element and $f_{i,1}$ and $f_{i,2}$ are the values of the probability density function at the nodes 1 and 2 (see Fig. 5.8).

The first moment becomes

$$\begin{aligned} \int_{-\infty}^{\infty} xf(x, t)dx &= \sum_{i=1}^N \int_{x_{i,1}}^{x_{i,2}} xf(x, t)dx \\ &= \sum_{i=1}^N \frac{1}{3l_i}(f_{i,2} - f_{i,1})(x_{i,2}^3 - x_{i,1}^3) \\ &\quad + \frac{1}{2}\left[\left(1 + \frac{x_{i,1}}{l_i}\right)f_{i,1} - \frac{x_{i,1}f_{i,2}}{l_i}\right](x_{i,2}^2 - x_{i,1}^2), \end{aligned} \quad (5.275)$$

and the second moment is given by

$$\begin{aligned} \int_{-\infty}^{\infty} x^2f(x, t)dx &= \sum_{i=1}^N \int_{x_{i,1}}^{x_{i,2}} x^2f(x, t)dx \\ &= \sum_{i=1}^N \frac{1}{4l_i}(f_{i,2} - f_{i,1})(x_{i,2}^4 - x_{i,1}^4) \\ &\quad + \frac{1}{3}\left[\left(1 + \frac{x_{i,1}}{l_i}\right)f_{i,1} - \frac{x_{i,1}f_{i,2}}{l_i}\right](x_{i,2}^3 - x_{i,1}^3). \end{aligned} \quad (5.276)$$

Problems

5.16. This problem is based on an example given in [58]. Consider a stochastic dynamic system

$$\begin{pmatrix} \Delta x_1 \\ \Delta x_2 \end{pmatrix} = \begin{pmatrix} a_{11} & 0 \\ 0 & a_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \Delta t + \begin{pmatrix} c_{11} & 0 \\ 0 & c_{22} \end{pmatrix} \begin{pmatrix} \Delta W_1 \\ \Delta W_2 \end{pmatrix},$$

with $[x_1(0), x_2(0)]$ being Gaussian with mean $[m_1, m_2]$, and covariance diagonal matrix

$$\begin{pmatrix} \sigma_{01}^2 & 0 \\ 0 & \sigma_{02}^2 \end{pmatrix},$$

and $\Delta W_1, \Delta W_2$ are the standard Wiener processes.

The FPK equation is

$$\frac{\partial f}{\partial t} = -(a_{11} + a_{22})f - a_{11}x_1 \frac{\partial f}{\partial x_1} - a_{22}x_2 \frac{\partial f}{\partial x_2} + \frac{1}{2} \left(c_{11}^2 \frac{\partial^2 f}{\partial x_1^2} + c_{22}^2 \frac{\partial^2 f}{\partial x_2^2} \right),$$

with the probability density function $f(x, t)$ at $t = 0$ is given by

$$f(x, 0) = \frac{1}{2\pi\sigma_{01}\sigma_{02}} e^{-\frac{1}{2} \left[\left(\frac{x_1(0)-m_1}{\sigma_{01}} \right)^2 + \left(\frac{x_2(0)-m_2}{\sigma_{02}} \right)^2 \right]},$$

The system is linear and its exact solution is known to be Gaussian and is given by

$$f(x, t) = \frac{1}{2\pi\sigma_1\sigma_2} e^{-\frac{1}{2} \left[\left(\frac{x_1 - \hat{x}_1}{\sigma_1} \right)^2 + \left(\frac{x_2 - \hat{x}_2}{\sigma_2} \right)^2 \right]},$$

where

$$\begin{aligned} \sigma_1^2(t) &= \left(\sigma_{01}^2 + \frac{c_{11}^2}{2a_{11}} \right) e^{2a_{11}t} - \frac{c_{11}^2}{2a_{11}}, \\ \sigma_2^2(t) &= \left(\sigma_{02}^2 + \frac{c_{22}^2}{2a_{22}} \right) e^{2a_{22}t} - \frac{c_{22}^2}{2a_{22}}, \end{aligned}$$

and

$$\begin{aligned} \hat{x}_1(t) &= m_1 e^{a_{11}t}, \\ \hat{x}_2(t) &= m_2 e^{a_{22}t}. \end{aligned}$$

The following numerical values for the parameters are taken:

$$\begin{aligned} a_{11} &= -0.5, & a_{22} &= -1, & m_1 &= m_2 = 0.5, \\ \sigma_{01} &= \sigma_{02} = 1, & c_{11} &= c_{22} = 1. \end{aligned}$$

The domain $\Omega(x_1, x_2)$ is given by

$$\Omega = \{m_1 - 3\sigma_{01}, m_1 + 3\sigma_{01}\}, \{m_2 - 3\sigma_{02}, m_2 + 3\sigma_{02}\}.$$

Write in MATLAB® or SCILAB® or Octave® or other package a finite element program to solve the FPK equation of the given dynamic system. Apply a mesh of 60×60 divisions of linear quadrilateral elements and a time step $\delta t = 0.1$ s, for $t = 0.1-8.0$ s.

5.9.3 Solution of the FPK Equation by the Finite Difference Method

The numerical solution the FPK equation by the Finite Difference Method (FDM) will be illustrated by working out an example is taken from [194]. Consider the SDOF system with a non-linear spring $g(x)$, given as

$$\ddot{x} + 2\zeta\omega_0\dot{x} + g(x) = \xi(t), \tag{5.277}$$

where $\xi(t)$ is Gaussian white noise with a autocorrelation function $E\{\xi(t)\xi(t+\tau)\} = 2D\delta(\tau)$. Equation (5.277) will be transformed into state-space variables $x_1 = x$ and $x_2 = \dot{x}$ as follows:

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= -2\zeta\omega_0x_2 - g(x_1) + \xi(t). \end{aligned} \tag{5.278}$$

The corresponding FPK equation is given by

$$\frac{\partial f}{\partial t} = -\frac{\partial(x_2f)}{\partial x_1} + \frac{\partial[\{2\zeta\omega_0x_2 + g(x_1)\}f]}{\partial x_2} + D\frac{\partial^2 f}{\partial x_2^2}, \tag{5.279}$$

where $f = f_{c2}$ is the transition or conditional probability density function. The following initial and boundary conditions will be applied:

$$\lim_{t \rightarrow 0} f_{c2}(\mathbf{x}_0|\mathbf{x}, t) = \delta(\mathbf{x} - \mathbf{x}_0), \tag{5.280}$$

and

$$\lim_{|x_1, x_2| \rightarrow \infty} f_{c2}(\mathbf{x}_0|\mathbf{x}, t) = 0. \tag{5.281}$$

The normalization condition is given by

$$\int_{R^2} f_{c2}(\mathbf{x}_0|\mathbf{x}, t) d\mathbf{x} = 1, \tag{5.282}$$

where $\mathbf{x} = \{x\} = [x_1, x_2]^T$. The joint probability function $f_2(\mathbf{x}, t)$ can be obtained as follows

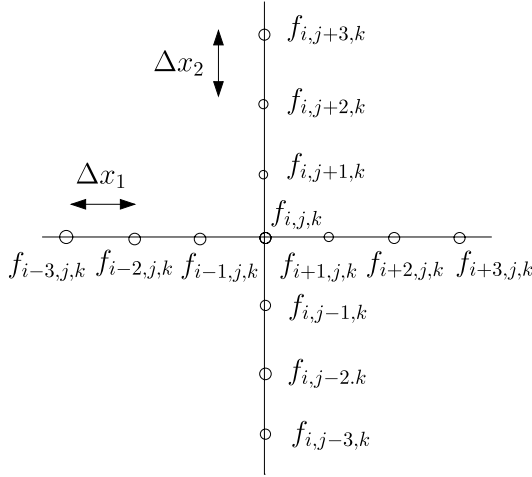
$$f_2(\mathbf{x}, t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{c2}(\mathbf{x}_0|\mathbf{x}, t) f_1(\mathbf{x}_0) d\mathbf{x}_0, \tag{5.283}$$

so that

$$\lim_{t \rightarrow 0} f_2(\mathbf{x}, t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \lim_{t \rightarrow 0} f_{c2}(\mathbf{x}_0|\mathbf{x}, t) f_1(\mathbf{x}_0) d\mathbf{x}_0 = f_1(\mathbf{x}_0). \tag{5.284}$$

The probability density function is given by

$$f_2(x_1, x_2, t) = \int_{-\infty}^{\infty} f_{c2}(x_{1,0}, x_{2,0}|x_1, x_2, t) f_2(x_1, x_2, 0) dx_1 dx_2.$$



$$f_{i,j,k} = f(x_{1,min} + i\Delta x_1, x_{2,min} + j\Delta x_2, k\Delta t)$$

Fig. 5.9. Finite difference discretization

The initial condition is

$$f_2(x_1, x_2, 0) = \frac{1}{\sigma_1\sqrt{2\pi}} e^{-\frac{(x_1-\mu_1)^2}{2\sigma_1^2}} \frac{1}{\sigma_2\sqrt{2\pi}} e^{-\frac{(x_2-\mu_2)^2}{2\sigma_2^2}}.$$

The FPK equation (5.279), when $f = f_2$ and $g(x) = \omega_0^2(x + \varepsilon x^3)$, becomes

$$\frac{\partial f}{\partial t} = -x_2 \frac{\partial f}{\partial x_1} + 2\zeta\omega_0 f + \{2\zeta\omega_0 x_2 + \omega_0^2(x_1 + \varepsilon x_1^3)\} \frac{\partial f}{\partial x_2} + D \frac{\partial^2 f}{\partial x_2^2}, \quad (5.285)$$

Let $f_{i,j,k}$ denote the probability density function at discrete locations relative to a candidate point as shown in Fig. 5.9 and is given by

$$f_{i,j,k} = f(x_{1,min} + i\Delta x_1, x_{2,min} + j\Delta x_2, k\Delta t), \quad i, j, k = 0, 1, 2, \dots \quad (5.286)$$

The probability density function at $f_{i,j+1,k}$ and $f_{i,j-1,k}$ may be expressed in a Taylor series [92] (limited to two terms for simplicity)

$$\begin{aligned} f_{i,j+1,k} &= f_{i,j,k} + f'_{i,j,k} \Delta x_2 + \frac{1}{2} f''_{i,j,k} \Delta x_2^2 + O(\Delta x_2^3), \\ f_{i,j-1,k} &= f_{i,j,k} - f'_{i,j,k} \Delta x_2 + \frac{1}{2} f''_{i,j,k} \Delta x_2^2 + O(\Delta x_2^3), \\ f_{i,j,k+1} &= f_{i,j,k} + f'_{i,j,k} \Delta t + O(\Delta t^2), \end{aligned} \quad (5.287)$$

Solving for the derivative terms at the candidate point $f_{i,j,k}$ gives

$$\begin{aligned}
 \frac{\partial f(x_1, x_2, t)}{\partial x_1} &= \frac{f_{i+1,j,k} - f_{i-1,j,k}}{2\Delta x_1} + O(\Delta x_1^2), \\
 \frac{\partial f(x_1, x_2, t)}{\partial x_2} &= \frac{f_{i,j+1,k} - f_{i,j-1,k}}{2\Delta x_2} + O(\Delta x_2^2), \\
 \frac{\partial^2 f(x_1, x_2, t)}{\partial x_1^2} &= \frac{f_{i+1,j,k} - 2f_{i,j,k} + f_{i-1,j,k}}{\Delta x_1^2} + O(\Delta x_1^2), \\
 \frac{\partial^2 f(x_1, x_2, t)}{\partial x_2^2} &= \frac{f_{i,j+1,k} - 2f_{i,j,k} + f_{i,j-1,k}}{\Delta x_2^2} + O(\Delta x_1^2), \\
 \frac{\partial f(x_1, x_2, t)}{\partial t} &= \frac{f_{i,j,k+1} - f_{i,j,k}}{\Delta t} + O(\Delta t).
 \end{aligned}
 \tag{5.288}$$

By substituting (5.288) into (5.285), we may formulate one equation for the nodal probability density function (TPDF):

$$\begin{aligned}
 \frac{f_{i,j,k+1} - f_{i,j,k}}{\Delta t} &= -x_2 \left\{ \frac{f_{i+1,j,k} - f_{i-1,j,k}}{2\Delta x_1} \right\} + 2\zeta\omega_0 f_{i,j,k} \\
 &\quad + (2\zeta\omega_0 x_2 + \omega_0^2 x_1 + \varepsilon x_1^3) \left\{ \frac{f_{i,j+1,k} - f_{i,j-1,k}}{2\Delta x_2} \right\} \\
 &\quad + D \left\{ \frac{f_{i,j+1,k} - 2f_{i,j,k} + f_{i,j-1,k}}{\Delta x_2^2} \right\} \\
 &= F_{i,j,k}.
 \end{aligned}
 \tag{5.289}$$

The operator $F_{i,j,k}$ is given by

$$\begin{aligned}
 F_{i,j,k} &= \left\{ 2\zeta\omega_0 - \frac{2D}{\Delta x_2^2} \right\} f_{i,j,k} \\
 &\quad - \frac{x_2}{2\Delta x_1} f_{i+1,j,k} + \frac{x_2}{2\Delta x_1} f_{i-1,j,k} \\
 &\quad + \left\{ \frac{[2\zeta\omega_0 x_2 + \omega_0^2(x_1 + \varepsilon x_1^3)]}{2\Delta x_2} + \frac{D}{\Delta x_2^2} \right\} f_{i,j+1,k} \\
 &\quad \times \left\{ -\frac{[2\zeta\omega_0 x_2 + \omega_0^2(x_1 + \varepsilon x_1^3)]}{2\Delta x_2} + \frac{D}{\Delta x_2^2} \right\} f_{i,j-1,k}.
 \end{aligned}
 \tag{5.290}$$

Equation (5.290) multiplied by Δt can be written as follows:

$$\begin{aligned}
 F_{i,j,k}\Delta t &= \alpha_{i,j,k} f_{i,j,k} + \alpha_{i+1,j,k} f_{i+1,j,k} + \alpha_{i-1,j,k} f_{i-1,j,k} \\
 &\quad + \alpha_{i,j+1,k} f_{i,j+1,k} + \alpha_{i,j-1,k} f_{i,j-1,k},
 \end{aligned}
 \tag{5.291}$$

where:

- $\alpha_{i,j,k} = 2\zeta\omega_0\Delta t - 2D\frac{\Delta t}{\Delta x_2^2}$
- $\alpha_{i+1,j,k} = -x_2\frac{\Delta t}{2\Delta x_1}$
- $\alpha_{i-1,j,k} = x_2\frac{\Delta t}{2\Delta x_1}$
- $\alpha_{i,j+1,k} = [2\zeta\omega_0 x_2 + \omega_0^2(x_1 + \varepsilon x_1^3)]\frac{\Delta t}{2\Delta x_2} + D\frac{\Delta t}{\Delta x_2^2}$

- $\alpha_{i,j-1,k} = -[2\zeta\omega_0x_2 + \omega_0^2(x_1 + \varepsilon x_1^3)]\frac{\Delta t}{2\Delta x_2} + D\frac{\Delta t}{\Delta x_2^2}$
- $x_1 = x_{1,min} + i\Delta x_1, i = 0, 1, 2, \dots$
- $x_2 = x_{2,min} + j\Delta x_2, j = 0, 1, 2, \dots$

A more general finite difference approximation for (5.289) is the weighted average approximation [183]

$$\frac{f_{i,j,k+1} - f_{i,j,k}}{\Delta t} = \theta F_{i,j,k+1} + (1 - \theta)F_{i,j,k}, \tag{5.292}$$

where, in practice $0 \leq \theta \leq 1$. $\theta = 0$ will give the explicit scheme, $\theta = 0.5$ is the Crank-Nicolson scheme and $\theta = 1$ is the fully implicit backward time-difference method. The equations are unconditionally stable and convergent for $\frac{1}{2} \leq \theta \leq 1$.

Equation (5.292) can be transformed into an explicit finite difference iteration scheme using $\theta = 0$

$$f_{i,j,k+1} = (1 + \alpha_{i,j,k})f_{i,j,k} + \alpha_{i+1,j,k}f_{i+1,j,k} + \alpha_{i-1,j,k}f_{i-1,j,k} + \alpha_{i,j+1,k}f_{i,j+1,k} + \alpha_{i,j-1,k}f_{i,j-1,k}. \tag{5.293}$$

Equation (5.293) is illustrated in Fig. 5.10.

Equation (5.292) will be rewritten to bring the terms “ $k + 1$ ” to the LHS, and the terms “ k ” all at the RHS of the equation

$$\begin{aligned} &\beta_{i,j,k+1}f_{i,j,k+1} + \beta_{i+1,j,k+1}f_{i+1,j,k+1} + \beta_{i-1,j,k+1}f_{i-1,j,k+1} \\ &\quad + \beta_{i,j+1,k+1}f_{i,j+1,k+1} + \beta_{i,j-1,k+1}f_{i,j-1,k+1} \\ &= \beta_{i,j,k}f_{i,j,k} + \beta_{i+1,j,k}f_{i+1,j,k} + \beta_{i-1,j,k}f_{i-1,j,k} \\ &\quad + \beta_{i,j+1,k}f_{i,j+1,k} + \beta_{i,j-1,k}f_{i,j-1,k}, \quad i, j, k = 0, 1, 2, \dots \end{aligned} \tag{5.294}$$

where

- $\beta_{i,j,k+1} = 1 - \theta\alpha_{i,j,k+1}$
- $\beta_{i+1,j,k+1} = -\theta\alpha_{i+1,j,k+1}$

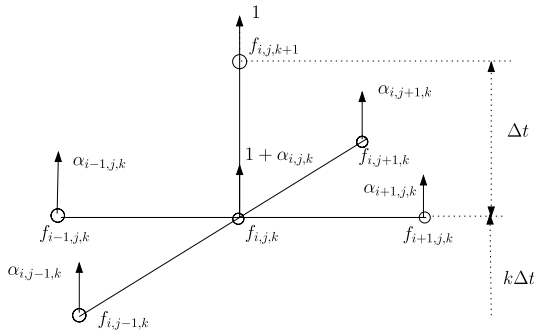


Fig. 5.10. Finite difference explicit scheme

- $\beta_{i-1,j,k+1} = -\theta\alpha_{i-1,j,k+1}$
- $\beta_{i,j+1,k+1} = -\theta\alpha_{i,j+1,k+1}$
- $\beta_{i,j-1,k+1} = -\theta\alpha_{i,j-1,k+1}$
- $\beta_{i,j,k} = 1 + (1 - \theta)\alpha_{i,j,k}$
- $\beta_{i+1,j,k} = (1 - \theta)\alpha_{i+1,j,k}$
- $\beta_{i-1,j,k} = (1 - \theta)\alpha_{i-1,j,k}$
- $\beta_{i,j+1,k} = (1 - \theta)\alpha_{i,j+1,k}$
- $\beta_{i,j-1,k} = (1 - \theta)\alpha_{i,j-1,k}$

5.9.4 Boundary Conditions Numerical Solution FPK Equation

Consider the SDE describing a time-dependent discrete SDOF dynamic system subjected to external (additive) Gaussian white noise given by

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{bmatrix} g_1(x_1, x_2) \\ g_2(x_1, x_2) \end{bmatrix} + \begin{bmatrix} 0 \\ \xi(t) \end{bmatrix}, \tag{5.295}$$

where the white noise excitation is fully defined by its first two moments

$$E\{\xi(t)\} = 0, \quad E\{\xi(t), \xi(t + \tau)\} = 2D\delta(\tau). \tag{5.296}$$

The discussion is partly based on [194]. The transition probability density function $f_{c2}(x_1, x_2, 0|x_1, x_2, t) = f_{c2}$ gives the probability, being differential elements $(x_1, x_2, x_1 + dx_1, x_2 + dx_2)$ in the phase plane at time t , having started from $(x_1(0), x_2(0))$ at time $t = 0$, satisfying the FPK equation

$$\frac{\partial f_{c2}}{\partial t} = -\frac{\partial[g_1(x_1, x_2)f_{c2}]}{\partial x_1} - \frac{\partial[g_2(x_1, x_2)f_{c2}]}{\partial x_2} + D\frac{\partial^2 f_{c2}}{\partial x_2^2}, \tag{5.297}$$

with the initial condition

$$\lim_{t \rightarrow 0} f_{c2}(x_1, x_2, 0|x_1, x_2, t) = \delta\{x_1(t) - x_1(0)\}\delta\{x_2(t) - x_2(0)\}, \tag{5.298}$$

and the normalization condition

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{c2}(x_1, x_2, 0|x_1, x_2, t) dx_1 dx_2 = 1, \tag{5.299}$$

together with

$$\lim_{|x_1|, |x_2| \rightarrow \infty} f_{c2}(x_1, x_2, 0|x_1, x_2, t) = 0. \tag{5.300}$$

The theorem of the total probability can be obtained by using the following equation:

$$f(x_1, x_2, t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f\{x_1(0), x_2(0)\} f_{c2}(x_1, x_2, 0|x_1, x_2, t) dx_1(0) dx_2(0). \tag{5.301}$$

The stationary probability density function $f(x_1, x_2)$ is

$$f(x_1, x_2) = \lim_{t \rightarrow \infty} f_{c2}(x_1, x_2, t). \quad (5.302)$$

When the finite element method is applied, all element matrices should be assembled in the overall “mass” matrix $[M]$ and “stiffness” matrix $[K]$ and the finite element representation of the FPK equation can be expressed

$$[M]\{\dot{f}\} + [K]\{f\} = \{0\}, \quad (5.303)$$

where $\{f(x_i, t)\}$ is the vector of nodal probability density functions, with the initial conditions $f(x_1, x_2, 0)$.

Example. Random initial conditions can be prescribed, for example as a binormal probability density function

$$f(x_1, x_2, 0) = \phi\left(\frac{x_{1,0} - \mu_{1,0}}{\sigma_{1,0}}\right)\phi\left(\frac{x_{2,0} - \mu_{2,0}}{\sigma_{2,0}}\right),$$

where $\phi()$ is the standard normal density function, $\mu_{i,0}$ is the expected values of the initial condition of x_i and $\sigma_{i,0}$ is the corresponding standard deviation.

The boundary condition for the stationary solution of (5.303) is discussed in [103]. The system of n homogeneous equations is given by

$$[K]\{f\} = \{0\}, \quad (5.304)$$

where n also is the number of nodes in the computational mesh. The matrix $[K]$ has $n \times n$ and vector $\{f\}$ has n elements. This system admits both the trivial solution $\{f\} = \{0\}$ and a nontrivial solution through enforcement of the normalization condition.

The nontrivial solution is obtained by first fixing the probability density function at the node corresponding to the origin as unity $f_c = 1$. This degree of freedom f_c becomes constrained and is treated as an additional boundary condition. The matrix $[K]$ will be partitioned into set u , the unknowns, and set c

$$[K]\{f\} = \begin{bmatrix} K_{uu} & K_{uc} \\ K_{cu} & K_{cc} \end{bmatrix} \begin{pmatrix} f_u \\ f_c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad (5.305)$$

and (5.305) can be rewritten using the first set of equations

$$[K_{uu}]\{f_u\} = -[K_{uc}]f_c = -[K_{uc}], \quad (5.306)$$

where the matrix $[K_{uu}]$ is obtained by removing the c th row and column from the matrix $[K]$ and we obtain matrix $[K_{uc}]$ from the c th column after removing the c th component. The solution of the stationary FPK equation is obtained by enforcing the normalization condition given in (5.299).

An alternative method to solve the homogeneous set of (5.304) with the constraining condition $f_c = 1$ is by the introduction of the Lagrange's multiplier λ connected to the constraining condition as follows

$$\begin{bmatrix} K_{uu} & K_{uc} & 0 \\ K_{cu} & K_{cc} & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{pmatrix} f_u \\ f_c \\ \lambda \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \tag{5.307}$$

The sets u and c are extended by the Lagrange multiplier λ , thus the total number of degrees of freedom (unknowns) becomes $n + 1$. Afterwards the normalization condition shall be applied.

In [108] a generalized Lagrange multiplier method is introduced and the normalization condition (5.299) is now written as

$$[c]\{f\} = 1, \tag{5.308}$$

where $[c]$ is vector of weighting elements. The system of $n + 1$ coupled equations (5.307) can now being written as

$$\begin{bmatrix} K & c^T \\ c & 0 \end{bmatrix} \begin{pmatrix} f \\ \lambda \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{5.309}$$

We eliminate of $\{f\}$ from (5.309), and obtain the following two expressions

$$\begin{aligned} \{\lambda\} &= [c^T K^{-1} c]^{-1}, \\ \{f\} &= \lambda [K]^{-1} [c], \end{aligned} \tag{5.310}$$

or

$$\{f\} = \frac{1}{[c^T K^{-1} c]} [K]^{-1} [c]. \tag{5.311}$$

The computational algorithm becomes

1. Solve $[K]\{f\} = \{c\}$.
2. Compute $\psi = [c]\{f\}$.
3. Set $\{f\} = \{f\}/\psi$.

Most of the computational effort spent, is solving the system $[K]\{f\} = \{c\}$, where $[K]$ is nonsymmetric.

Problems

5.17. This problem concerns the diffusion in a bistable potential [85]. The diffusion of in a bistable potential is represented by the nonlinear Langevin equation (NLE)

$$\dot{x} = \gamma x - gx^3 + \xi(t),$$

where $x(t)$ is the driven random variable, $\gamma = 1$ is the positive friction coefficient, $g = 1$ the parameter of nonlinearity. The driven random force $\xi(t)$ is assumed to be a Gaussian white noise obeying

$$\langle \xi(t) \rangle = 0, \quad \langle \xi(t)\xi(t + \tau) \rangle = 2\varepsilon\delta(\tau),$$

where $\varepsilon = 0.5 \times 10^{-6}$ is the diffusion coefficient.

The transition density function $f_{c1}(x_0|x, t) = f_{c1}$ will be calculated by using the weighted average finite difference method, with $\theta = 1$ (fully implicit backward time difference method and unconditional stable). The discrete transition density function in the x, t grid is given by

$$f_{c1}(x_{min} + (i - 1)\Delta x, (k - 1)\Delta t), \quad i, k = 1, 2, \dots,$$

where $x_{min} = -1, x_{max} = 1$, representing zero transition density function $\lim_{|x| \rightarrow \infty} f_{c2}(x_0|x, t) = 0$, and $t_{max} = 10$ s.

The probability density function $f_1(x, t)$ can be obtained from the following equation

$$f_1(x, t) = \int_{-\infty}^{\infty} f_{c1}(x_0|x, t)f_1(x_0)dx_0,$$

and the mean value of the variable $x(t)$ can be obtained from

$$\langle x \rangle = \int_{-\infty}^{\infty} xf_1(x, t)dx.$$

Carry out the following assignments:

- Show that the FPK equation of the NLE is given by

$$\frac{\partial f_{c1}}{\partial t} = -\frac{\partial}{\partial x}[(\gamma x - gx^3)f_{c1}] + \varepsilon \frac{\partial^2 f_{c1}}{\partial x^2}$$

and that also

$$\frac{\partial f_1}{\partial t} = -\frac{\partial}{\partial x}[(\gamma x - gx^3)f_1] + \varepsilon \frac{\partial^2 f_1}{\partial x^2},$$

where $\lim_{t \rightarrow 0} f_{c2}(x_0|x, t) = \delta(x - x_0)$ thus $f_1(x, 0) = f_1(x_0)$.

- The implicit backward time difference scheme, with $f_{i,k} = f_1(x_{min} + (i - 1)\Delta x, (k - 1)\Delta t)$, $i, k = 1, 2, \dots$ is given by

$$a_1 f_{i-1,k+1} + a_2 f_{i,k+1} + a_3 f_{i+1,k+1} + f_{i,k} = 0,$$

Derive the coefficients a_1, a_2 and a_3 .

- $a_1 = [\frac{\gamma x - gx^3}{2\Delta x} + \frac{\varepsilon}{\Delta x^2}]\Delta t$
- $a_2 = [-\gamma + 3gx^2 - \frac{2\varepsilon}{\Delta x^2}]\Delta t - 1$
- $a_3 = [-\frac{\gamma x - gx^3}{2\Delta x} + \frac{\varepsilon}{\Delta x^2}]\Delta t,$
- $x = x_i.$

- Solve the time difference scheme

$$a_1 f_{i-1,k+1} + a_2 f_{i,k+1} + a_3 f_{i+1,k+1} = -f_{i,k},$$

- for $t = 0 \cdots t_{max}$, $t_{max} = 4, 5, 6$ and 10 s,
 - $\Delta x = 0.005$ and $\Delta t = 0.001$, $x_{min} = -1$ and $x_{max} = 1$,
 - with initial condition $f_1(x_0) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x_0-\mu)^2}{2\sigma^2}}$, $\mu = 0$ and $\sigma^2 = 1 \times 10^{-6}$,
 - and boundary conditions $f(x_{min}, t) = 0$ and $f(x_{max}, t) = 0$.
- Calculate $\langle x(t) \rangle$ for $t_{max} = 10, 15$ s using

$$f_1(x, t) = \int_{-\infty}^{\infty} f_{c1}(x_0|x, t) f_1(x_0) dx_0,$$

and

$$\langle x(t) \rangle = \int_{-\infty}^{\infty} x f_1(x, t) dx.$$

5.10 Applications

5.10.1 Vibrating Thin Plates Exposed to Acoustic Loads

Introduction

In this section we discuss the nonlinear dynamic behavior of a membrane type structure, exposed to random acoustic loads. The deformation of the pre-stress membrane will be represented using one assumed mode. The resulting nonlinear SDOF equation of motion becomes a Duffing's equation.

Vibration Theory Thin Plates

The partial differential equation of motion for a thin pre-stressed plate is given by [77]

$$\begin{aligned} m \frac{\partial^2 w(x, y, t)}{\partial t^2} + D \nabla^2 \nabla^2 w(x, y, t) \\ = p(x, y, t) + N_x \frac{\partial^2 w(x, y, t)}{\partial x^2} + N_{xy} \frac{\partial^2 w(x, y, t)}{\partial x \partial y} \\ + N_y \frac{\partial^2 w(x, y, t)}{\partial y^2}, \end{aligned} \tag{5.312}$$

where

- $w(x, y, t)$ is the deflection of the thin plate
- $D = \frac{Et^3}{12(1-\nu^2)}$ is the bending stiffness
- E is Young's modulus
- t is thickness of plate

- ν is Poisson's ratio
- $\nabla^2 \nabla^2 w(x, y, t) = \left(\frac{\partial^2 w(x, y, t)}{\partial x^2} + \frac{\partial^2 w(x, y, t)}{\partial y^2} \right)^2$
- m mass per unit of area
- $p(x, y, t)$ is the random pressure field
- N_x is the normal pre-stress in x -direction
- N_y is the normal pre-stress in y -direction
- N_{xy} is the shear pre-stress in x -direction

If the deflection $w(x, y, t)$ is a function of x, t only, than

$$w(x, y, t) = w(x, t). \quad (5.313)$$

Equation (5.312) becomes the partial differential equation of motion of a bending beam

$$m \frac{\partial^2 w(x, t)}{\partial t^2} + D \frac{\partial^4 w(x, y, t)}{\partial x^4} = p(x, y, t) + N_x \frac{\partial^2 w(x, y, t)}{\partial x^2}, \quad (5.314)$$

where m is now defined to be the mass per unit of length. Due to the deflection $w(x, t)$ of the thin plate, a pre-stress will be built up [150]

$$\bar{N}_x = \frac{1}{2L} \frac{Et}{2L} \int_0^L \left\{ \frac{\partial w(x, t)}{\partial x} \right\}^2 dx, \quad (5.315)$$

where $2L$ is the length of the plate. Substituting (5.315) in (5.314) yields

$$m \frac{\partial^2 w(x, t)}{\partial t^2} + D \frac{\partial^4 w(x, y, t)}{\partial x^4} = p(x, y, t) + (N_x + \bar{N}_x) \frac{\partial^2 w(x, y, t)}{\partial x^2}. \quad (5.316)$$

The deflection $w(x, t)$ will be projected on one assumed mode $\phi(x, t)$ as follows

$$w(x, t) = \eta(t) \phi(x) = \eta(t) \sin\left(\frac{\pi x}{2L}\right), \quad (5.317)$$

where $\eta(t)$ is the generalized coordinate. The sinusoidal function reflects more or less the motion of a very thin (hinged-hinged) plate.

The following partial derivatives can be obtained:

$$\begin{aligned} \frac{\partial w(x, t)}{\partial x} &= \left(\frac{\pi}{2L}\right) \cos\left(\frac{\pi x}{2L}\right), \\ \frac{\partial^2 w(x, t)}{\partial x^2} &= -\left(\frac{\pi}{2L}\right)^2 \sin\left(\frac{\pi x}{2L}\right), \\ \frac{\partial^3 w(x, t)}{\partial x^3} &= -\left(\frac{\pi}{2L}\right)^3 \cos\left(\frac{\pi x}{2L}\right), \\ \frac{\partial^4 w(x, t)}{\partial x^4} &= \left(\frac{\pi}{2L}\right)^4 \sin\left(\frac{\pi x}{2L}\right), \end{aligned} \quad (5.318)$$

and the following integrals can be established

$$\begin{aligned} \int_0^{2L} \frac{\partial^2 w(x, t)}{\partial x^2} dx &= \left(\frac{\pi}{2L}\right)^2 \left\{ \frac{x}{2} + \frac{L}{2\pi} \sin\left(\frac{\pi x}{2L}\right) \right\}_0^{2L} = \left(\frac{\pi}{2L}\right)^2 L, \\ \int_0^{2L} \left\{ \sin\left(\frac{\pi x}{2L}\right) \right\}^2 dx &= \left\{ \frac{x}{2} - \frac{L}{2\pi} \sin\left(\frac{\pi x}{2L}\right) \right\}_0^{2L} = L, \\ \int_0^{2L} \left\{ \sin\left(\frac{\pi x}{2L}\right) \right\} dx &= -\frac{2L}{\pi} \cos\left(\frac{\pi x}{2L}\right)_0^{2L} = \frac{4L}{\pi}. \end{aligned} \quad (5.319)$$

The assumed mode (5.317) will be substituted in (5.316), (5.318) yields the following ordinary differential equation of motion:

$$\begin{aligned} m\ddot{\eta}(t) \sin\left(\frac{\pi x}{2L}\right) + \eta(t) D \left(\frac{\pi}{2L}\right)^4 \sin\left(\frac{\pi x}{2L}\right) \\ = p(x, t) - \eta(t)(N_x + \bar{N}_x) \left(\frac{\pi}{2L}\right)^2 \sin\left(\frac{\pi x}{2L}\right). \end{aligned} \quad (5.320)$$

Multiplying (5.320) with $\sin\left(\frac{\pi x}{2L}\right)$ and integrating \int_0^{2L} all terms in (5.320) will yield

$$\begin{aligned} m\ddot{\eta}(t) \int_0^{2L} \sin^2\left(\frac{\pi x}{2L}\right) dx + D \left(\frac{\pi}{2L}\right)^4 \eta(t) \int_0^{2L} \sin^2\left(\frac{\pi x}{2L}\right) dx \\ = p(x, t) \int_0^{2L} \sin\left(\frac{\pi x}{2L}\right) dx \\ - \eta(t)(N_x + \bar{N}_x) \left(\frac{\pi}{2L}\right)^2 \int_0^{2L} \sin^2\left(\frac{\pi x}{2L}\right) dx. \end{aligned} \quad (5.321)$$

Using the results of (5.319) and dividing by L , (5.321) becomes

$$m\ddot{\eta}(t) + D \left(\frac{\pi}{2L}\right)^4 \eta(t) = \frac{4}{\pi} p(x, t) - (N_x + \bar{N}_x) \left(\frac{\pi}{2L}\right)^2 \eta(t). \quad (5.322)$$

Introducing (5.315) into (5.322) and using (5.319), we find in the following nonlinear equation of motion

$$\begin{aligned} m\ddot{\eta}(t) + D \left(\frac{\pi}{2L}\right)^4 \eta(t) \\ = \frac{4}{\pi} p(x, t) - \left[N_x + \frac{Et}{4} \left(\frac{\pi}{2L}\right)^2 \eta^2(t) \right] \left(\frac{\pi}{2L}\right)^2 \eta(t), \end{aligned} \quad (5.323)$$

however, after further evaluation, the nonlinear equation of motion becomes

$$\begin{aligned} m\ddot{\eta}(t) + \left[\frac{Et^3}{12(1-\nu^2)} \left(\frac{\pi}{2L}\right)^4 + N_x \left(\frac{\pi}{2L}\right)^2 \right] \eta(t) + \frac{Et}{4} \left(\frac{\pi}{2L}\right)^4 \eta^3(t) \\ = \frac{4}{\pi} p(x, t). \end{aligned} \quad (5.324)$$

Damping will be introduced in (5.324), (5.324) will give Duffing's oscillator

$$\ddot{\eta}(t) + 2\zeta\omega_0\dot{\eta}(t) + \omega_0^2\{\eta(t) + \varepsilon\eta^3(t)\} = \frac{4}{\pi} \frac{p(x, t)}{m} = w(t), \quad (5.325)$$

where

- $\omega_0 = \sqrt{k/m}$,
- $k = \frac{Et^3}{12(1-\nu^2)} \left(\frac{\pi}{2L}\right)^4 + N_x \left(\frac{\pi}{2L}\right)^2$ and
- $\varepsilon = \frac{Et}{4k} \left(\frac{\pi}{2L}\right)^4$.
- $E\{w(t)w(t+\tau)\} = 2D\delta(\tau)$.

The uniform membrane stress σ_x is given by

$$\sigma_x = \frac{N_x}{t} + \frac{\bar{N}_x}{t} = \frac{N_x}{t} + \frac{E}{4} \left(\frac{\pi}{2L}\right)^4 \eta^2(t). \quad (5.326)$$

Random Pressure Load

The random white noise acoustic pressure $p(t)$ has a constant double-sided PSD function in the radian frequency domain $S_p(\omega) = S_p$ or can be expressed as a single-sided constant PSD function in the cyclic frequency domain $W_p(f) = W_p = 2S_p$. The amplitude of the correlation function $2D$ can be written

$$2D = \left(\frac{\pi}{4}\right)^2 \frac{S_p}{m^2} = \left(\frac{\pi}{4}\right)^2 \frac{W_p}{2m^2}. \quad (5.327)$$

Equation (5.325) has the general form

$$\ddot{x} + g(\dot{x}, x) = \ddot{x} + \beta\dot{x} + g_1(x) = w(t), \quad (5.328)$$

where $g_1(x) = \omega_0^2(x + \varepsilon x^3)$ and $\beta = 2\zeta\omega_0$. Introduce the state-variables $y_1 = x = \eta$ and $y_2 = \dot{x} = \dot{\eta}$. The solution of the joint probability density function $f(y_1, y_2)$ of the FPK equation of (5.328) is

$$f(y_1, y_2) = C e^{-\frac{p(y_1, y_2)}{D}}, \quad (5.329)$$

where

$$p(y_1, y_2) = \frac{\beta}{2} y_2^2 + \beta \int_0^{y_1} g_1(\zeta) d\zeta.$$

Finally the function $p(y_1, y_2)$ can be written as

$$p(y_1, y_2) = \zeta\omega_0 y_2^2 + \zeta\omega_0^3 \left(y_1^2 + \frac{\varepsilon y_1^4}{2} \right). \quad (5.330)$$

The joint probability density function $f(y_1, y_2)$ can be expressed as follows

$$f(y_1, y_2) = f_1(y_1)f_2(y_2), \quad (5.331)$$

where

$$f_1(y_1) = C_1 e^{-\frac{\zeta\omega_0^3}{D^2}(y_1^2 + \frac{\varepsilon y_1^4}{2})}, \tag{5.332}$$

$$f_2(y_2) = C_2 e^{-\frac{\zeta\omega_0}{D^2}y_2^2}. \tag{5.333}$$

The constants C_1 and C_2 can be derived from the relations $\int_{-\infty}^{\infty} f_i(y_i) dy_i = 1$, $i = 1, 2$. To obtain the constants C_1 and C_2 reference is made to Appendix B.

The following relations are defined

$$\sigma_v^2 = \frac{(\frac{\pi}{4})^2 S_p}{4\zeta\omega_0 m^2}, \tag{5.334}$$

and

$$\sigma_d^2 = \frac{(\frac{\pi}{4})^2 S_p}{4\zeta\omega_0^3 m^2}. \tag{5.335}$$

Equations (5.332) and (5.333) become

$$f_1(y_1) = \frac{\sqrt{2\varepsilon}}{e^{\frac{1}{8\sigma_d^2\varepsilon}} K_{\frac{1}{4}}(\frac{1}{8\sigma_d^2\varepsilon})} e^{-\frac{1}{2\sigma_d^2}(y_1^2 + \frac{\varepsilon y_1^4}{2})}, \tag{5.336}$$

$$f_2(y_2) = \frac{1}{\sigma_v\sqrt{2\pi}} e^{-\frac{1}{2\sigma_v^2}y_2^2}, \tag{5.337}$$

where $K_\nu(z)$ is the modified Bessel function of the second kind of order ν [125]. The function $K_\nu(z)$ is defined as

$$K_\nu(z) = \frac{\frac{1}{2}\pi\{I_{-\nu}(z) - I_\nu(z)\}}{\sin \nu z},$$

and

$$I_\nu(z) = \sum_{r=0}^{\infty} \frac{(\frac{1}{2}z)^{\nu+2r}}{r!\Gamma(\nu+r+1)},$$

where $I_\nu(z)$ is a modified Bessel function of the first kind and order ν .

When $|z|$ is large enough and $-\pi < \text{phase}(z) < \pi$

$$K_\nu(z) = \sqrt{\frac{\pi}{2z}} e^{-z} \left[1 + \frac{4\nu^2 - 1^2}{1!8z} + \frac{(4\nu^2 - 1^2)(4\nu^2 - 3^2)}{2!(8z)^2} + \dots \right. \\ \left. + \frac{(4\nu^2 - 1^2) \dots (4\nu^2 - \{2r - 3\}^2)}{(r-1)!(8z)^{r-1}} + \dots \right],$$

then

$$e^z K_\nu(z) \approx \sqrt{\frac{\pi}{2z}} \left(1 + \frac{4\nu^2 - 1^2}{1!8z} + \dots \right). \tag{5.338}$$

The frequency of positive zero-crossings ν_0^+ (cycles) is given by [106]

$$\nu_0^+ = f_1(y_1 = 0) \int_0^\infty y_2 f_2(y_2) dy_2. \quad (5.339)$$

Applying (5.338), we can approximate the number of positive zero crossing ν_0^+

$$\nu_0^+ \approx \frac{1}{2\pi} \frac{\sigma_v}{\sigma_d} \left(1 + \frac{3}{4} \sigma_d^2 \varepsilon \right). \quad (5.340)$$

The expected value of η , $E(\eta^2)$, can be calculated with the following approximation

$$E(\eta^2) = \langle \eta^2 \rangle = \int_{-\infty}^\infty x^2 f_1(x) dx \approx \sigma_d^2 \left(1 + \frac{3}{4} \sigma_d^2 \varepsilon \right). \quad (5.341)$$

A SDOF Vibro-Impact System

A SDOF vibration system is considered [91]. An elastic amplitude constraint is placed under the mass as shown in Fig. 5.11. When the system is vibrating, the mass m can move with limited amplitude as in a simple linear system with a damper c and spring stiffness k , but for a finite amplitude, the elastic constraint will make its contribution. For simplicity, the constraint is modeled by a non-linear spring $\eta g(x)$ according to the Hertz contact law and the inertial effect is neglected. The system is excited by a stationary random force $\xi(t)$ with a zero mean and correlation $\langle \xi(t)\xi(t+\tau) \rangle = 2m^2 D \delta(\tau)$. The equation of motion is given by

$$m\ddot{x} + c\dot{x} + kx + \eta g(x) = \xi(t), \quad (5.342)$$

where

$$g(x) = \begin{cases} x^{\frac{3}{2}} & x \geq 0 \\ 0 & x < 0 \end{cases}, \quad (5.343)$$

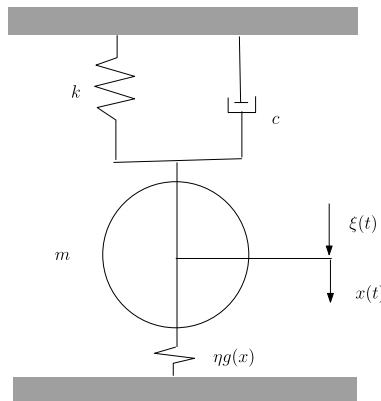


Fig. 5.11. SDOF vibro-impact system [91]

and η is the contact stiffness, which is a function of the elastic properties of the two contacting bodies.

The joint probability density function $f_2(x, \dot{x}) = f$, with $\dot{x} = v$, $c/m = 2\zeta\omega_0$ and $k/m = \omega_0^2$, satisfies the time-independent FPK equation

$$v \frac{\partial f}{\partial x} = 2\zeta\omega_0 f + \left[2\zeta\omega_0 + \omega_0^2 x + \frac{\eta}{m} g(x) \right] \frac{\partial f}{\partial v} + D \frac{\partial^2 f}{\partial v^2}. \tag{5.344}$$

The resulting joint probability density function $f_2(x, v)$ is

$$f_2(x, v) = C e^{-\frac{v^2}{2\sigma_0^2\omega_0^2} - \frac{x^2}{2\sigma_0^2} - \frac{\eta}{m\sigma_0^2\omega_0^2} \int_0^x g(u) du}, \tag{5.345}$$

where σ_0 is given by

$$\sigma_0^2 = \frac{D}{2\zeta\omega_0^3}, \tag{5.346}$$

and the normalization constant can be determined by the normalization condition

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_2(x, v) dx dv = 1. \tag{5.347}$$

The normalization constant becomes

$$C^{-1} = \sqrt{2\pi}\sigma_0^2\omega_0 \left[\sqrt{\frac{\pi}{2}} + \frac{2}{5} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{1}{2^n} \psi^{-(4n+2)/5} \Gamma\left(\frac{4n+2}{5}\right) \right], \tag{5.348}$$

where the non-dimensional contact stiffness, or non-linearity parameter ψ is given by

$$\psi = \frac{2\eta\sqrt{\sigma_0}}{5m\omega_0^2}. \tag{5.349}$$

The exact stationary solution of the FPK equation for the vibro-impact problem is then

$$f_2(x, v) = \begin{cases} C e^{-\frac{v^2}{2\sigma_0^2\omega_0^2} - \frac{x^2}{2\sigma_0^2} - \psi \frac{x^{5/2}}{\sigma_0^{5/2}}}, & x \geq 0; \\ C e^{-\frac{v^2}{2\sigma_0^2\omega_0^2} - \frac{x^2}{2\sigma_0^2}}, & x < 0. \end{cases} \tag{5.350}$$

From preceding analysis, the separated first order probability density functions for the displacement x and the velocity v are

$$f(x) = \begin{cases} C\sqrt{2\pi}\sigma_0\omega_0 e^{-\frac{x^2}{2\sigma_0^2} - \psi \frac{x^{5/2}}{\sigma_0^{5/2}}}, & x \geq 0; \\ C\sqrt{2\pi}\sigma_0\omega_0 e^{-\frac{x^2}{2\sigma_0^2}}, & x < 0, \end{cases} \tag{5.351}$$

and

$$f(v) = \frac{1}{\sqrt{2\pi}\sigma_0\omega_0} e^{-\frac{v^2}{2\sigma_0^2\omega_0^2}}. \tag{5.352}$$

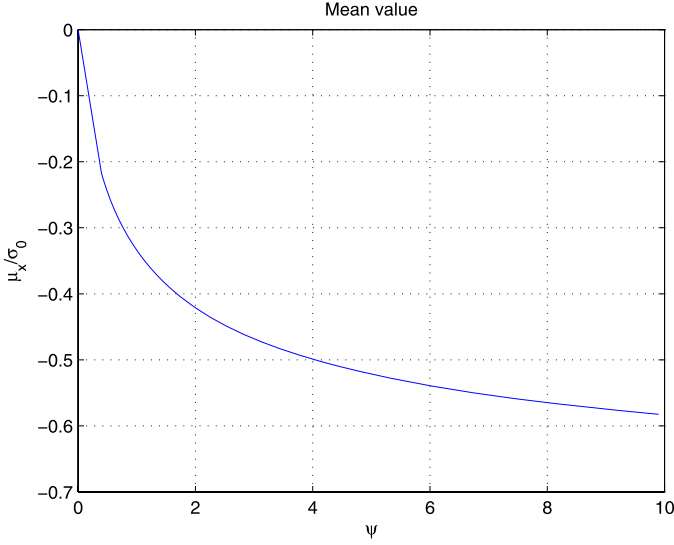


Fig. 5.12. Normalized expected number of positive zero-crossing f_0^+ of stationary response

The mean value of the displacement x , μ_x , is as follows

$$\mu_x = \sigma_0 \frac{-1 + \frac{2}{5} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{1}{2^n} \psi^{-(4n+4)/5} \Gamma(\frac{4n+4}{5})}{\sqrt{\frac{\pi}{2}} + \frac{2}{5} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{1}{2^n} \psi^{-(4n+2)/5} \Gamma(\frac{4n+2}{5})}. \tag{5.353}$$

When the stiffness of the constraint is reduced to zero, i.e. the system is linear, it can be proved that μ_x equals to zero. The normalized mean value μ_x versus ψ is plotted in Fig. 5.12. When the stiffness becomes very large the mean value approaches a finite value

$$\lim_{\psi \rightarrow \infty} \mu_x = -\sqrt{\frac{2}{\pi}} \sigma_0. \tag{5.354}$$

The variance of the displacement x , σ_x^2 can also be found exactly

$$\sigma_x^2 = \sigma_0^2 \frac{\sqrt{\frac{\pi}{2}} + \frac{2}{5} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{1}{2^n} \psi^{-(4n+6)/5} \Gamma(\frac{4n+6}{5})}{\sqrt{\frac{\pi}{2}} + \frac{2}{5} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{1}{2^n} \psi^{-(4n+2)/5} \Gamma(\frac{4n+2}{5})} - \mu_x^2. \tag{5.355}$$

If ψ approaches to zero, the variance $\sigma_x \rightarrow \sigma_0$. The normalized variance versus ψ is plotted in Fig. 5.13. When ψ approaches infinity, σ_x^2 approaches a finite value

$$\lim_{\psi \rightarrow \infty} \sigma_x^2 = \sigma_0^2 \left(1 - \frac{2}{\pi}\right). \tag{5.356}$$

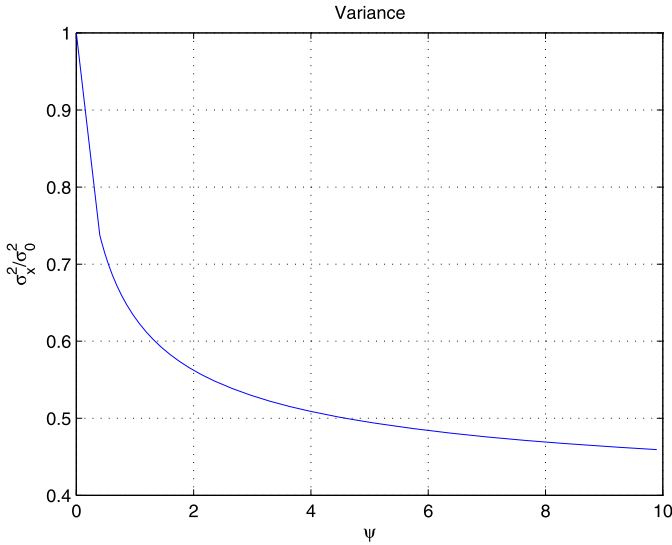


Fig. 5.13. Normalized expected number of positive zero-crossing f_0^+ of stationary response

Let ν_0^+ be the expected number of positive zero-crossings per unit (expected frequency) of time

$$\nu_0^+ = \int_0^\infty v f_2(0, v) dv. \tag{5.357}$$

Then

$$\nu_0^+ = \frac{\omega_0}{2\pi} \frac{\sqrt{2\pi}}{\sqrt{\frac{\pi}{2}} + \frac{2}{5} \sum_{n=0}^\infty \frac{(-1)^n}{n!} \frac{1}{2^n} \psi^{-(4n+2)/5} \Gamma(\frac{4n+2}{5})}. \tag{5.358}$$

When ψ approaches zero, it can be proved that $\nu_0^+ \rightarrow \frac{\omega_0}{2\pi}$. The normalized positive zero-crossings per unit of time is defined by $f_0^+ = 2\pi\nu_0^+/\omega_0$. The normalized expected frequency f_0^+ versus ψ is plotted in Fig. 5.14. When ψ approaches zero, it can be proved that the normalized expected frequency f_0^+ approaches 1. As ψ increases, f_0^+ also increases since the system is now becoming more rigid. The limiting normalized expected frequency is found to be 2 simply by taking the limiting value of (5.358). The positive crossings at a level $x = a$ can be obtained as follows

$$\nu_a^+ = \frac{\omega_0}{2\pi} \frac{\sqrt{2\pi} e^{-\frac{a^2}{2\sigma_0^2}}}{\sqrt{\frac{\pi}{2}} + \frac{2}{5} \sum_{n=0}^\infty \frac{(-1)^n}{n!} \frac{1}{2^n} \psi^{-(4n+2)/5} \Gamma(\frac{4n+2}{5})}. \tag{5.359}$$

An increase in the spectral density $2D$ of the excitation corresponds to an increase of the input energy, i.e. amplitude in some sense, and the non-linearity parameter ψ will increase.

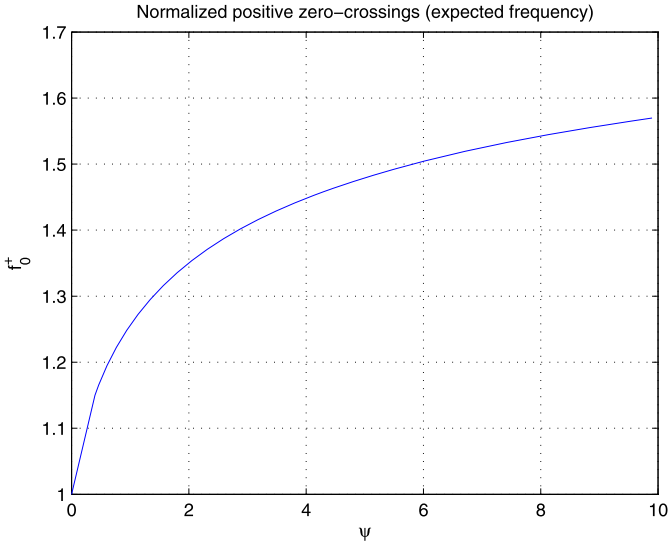


Fig. 5.14. Normalized expected number of positive zero-crossing f_0^+ of stationary response

The impact oscillator can also be viewed as a limiting case of a vibration system with bilinear spring, a spring with different stiffnesses for positive and negative displacements. If one of the stiffnesses approaches infinity, the system becomes an impact oscillator.

A

Simulation of the Random Time Series

The simulation of random time series is discussed in [25, 129, 170, 179, 180]. In this appendix we discuss the efficiency of fast computation of the random time series. The analysis is based on [128, 129]. If $\Phi_x^1(\omega)$ is the one-sided PSD of the desired signal $x(t)$, then $x(t)$ may be approximated by

$$x(t) = \sqrt{2} \sum_{n=0}^{N-1} [\Phi_x^1(\omega_n) \Delta\omega]^{\frac{1}{2}} \cos(\omega_n t - \phi_n), \quad (\text{A.1})$$

where ϕ_n are uniformly distributed random numbers on the interval $(0 - 2\pi)$ and $\omega_n = n\Delta\omega$ and $\Delta\omega = \omega_{max}/N$. ω_{max} is the maximum frequency in the power spectrum $\Phi_x^1(\omega)$, and N is the total number of terms in the summation.

A considerable improvement in the computational effort can be obtained by recasting (A.1) to allow the use of the Fast Fourier Transform (FTT). To accomplish this, (A.1) may be written as

$$x(t) = \Re \left[\sqrt{2} \sum_{n=0}^{N-1} [\Phi_x^1(\omega_n) \Delta\omega]^{\frac{1}{2}} e^{j(\omega_n t - \phi_n)} \right]. \quad (\text{A.2})$$

If the simulated time series $x(t)$, is needed only at discrete values of time t , let $x_n = x(t_k) = x(k\Delta t)$, where the time duration between the equally spaced samples is Δt . Evaluation of (A.2) at time $t = t_k$ gives

$$x(t_k) = x(k\Delta t) = \Re \left[\sqrt{2} \sum_{n=0}^{N-1} [\Phi_x^1(\omega_n) \Delta\omega]^{\frac{1}{2}} e^{j(\omega_n k\Delta t - \phi_n)} \right]. \quad (\text{A.3})$$

To satisfy the Nyquist sampling criterion, the time series, $x(t)$, must be sampled at a high enough rate to obtain two samples during one period of the highest frequency component ω_{max} of interest in the original PSD $\Phi_1(\omega)$. Hence, Δt is chosen to be

$$\Delta t \leq \frac{\pi}{\omega_{max}}. \quad (\text{A.4})$$

The term $\omega_n k \Delta t$ in (A.3) can be rewritten as follows

$$\omega_n k \Delta t = n \Delta \omega k \Delta t = n \frac{\omega_{max}}{N} k \frac{\pi}{\omega_{max}} = \frac{nk2\pi}{2N}. \tag{A.5}$$

Thus (A.3) can be expressed in the following form

$$x(k \Delta t) = \Re \left[\sqrt{2} \sum_{n=0}^{N-1} [\Phi_x^1(\omega_n) \Delta \omega]^{\frac{1}{2}} e^{-j\phi_n} e^{\frac{jnk2\pi}{2N}} \right]. \tag{A.6}$$

Equation (A.6) may be evaluated using the FFT algorithm. given a discrete sequence a_n , the FFT provides an efficient means of computing A_k , where

$$A_k = \sum_{n=0}^{N-1} a_n e^{\frac{jnk2\pi}{2N}}, \quad k = 0, 1, 2, \dots, N - 1. \tag{A.7}$$

Equation (A.6) may be evaluated using the FFT by defining a sequence

$$\begin{aligned} a_n &= [\Phi_x^1(\omega) \Delta \omega]^{\frac{1}{2}} e^{-j\phi_n} \quad n \leq N - 1, \\ &= 0, \quad n \geq N. \end{aligned} \tag{A.8}$$

Equation (A.6) may then be written as

$$x_k = x(k \Delta t) = \Re \left[\sqrt{2} \sum_{n=0}^{2N-1} a_n e^{\frac{jnk2\pi}{2N}} \right] \quad k = 0, 1, 2, \dots, 2N - 1. \tag{A.9}$$

In finding the real part of (A.9) we may use the complex conjugate¹ of the right side of (A.9) to give

$$x_k = x(k \Delta t) = \Re \left[\sqrt{2} \sum_{n=0}^{2N-1} a_n e^{\frac{-jnk2\pi}{2N}} \right] \quad k = 0, 1, 2, \dots, 2N - 1. \tag{A.10}$$

This is equivalent to

$$x_k = \sqrt{2} \Re[FFT(a_n)]. \tag{A.11}$$

Note that the length of the sequence a_n is $2N$.

The PSD function $\phi_x^1(\omega)$ can be reconstructed using the following expression

$$\phi_{x,n}^1 = 2 \frac{|IFFT(x_k)|^2}{\Delta \omega} \quad k = 0, 1, 2, \dots, 2N - 1, \quad n = 0, 1, 2, \dots, N - 1, \tag{A.12}$$

where *IFFT* is the inverse Fast Fourier Transform.

¹ $\cos u = \Re[e^{ju}] = \Re[e^{-ju}]$.

Table A.1. Random vibration specification [30]

Frequency (Hz)	$W(f)$ g^2/Hz
20	0.018
20–125	1.68 dB/oct
125	0.05
300	0.05
300–2000	−3.12 dB/oct
2000	0.007
Overall	6.33 G_{rms}

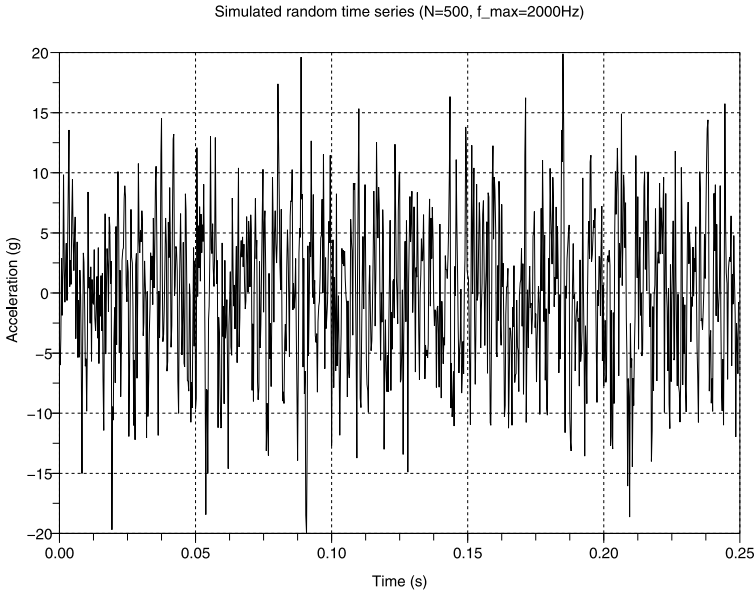


Fig. A.1. Simulated random time series from Table A.1

If the product $\phi_x^1(\omega)\Delta\omega$ is replaced by $W_x^1(f)\Delta f$, (A.12) can be written as follows

$$W_{x,n}^1 = 2 \frac{|IFFT(x_k)|^2}{\Delta f} \quad k = 0, 1, 2, \dots, 2N - 1, \quad n = 0, 1, 2, \dots, N - 1, \quad (\text{A.13})$$

where $W(f)$ is the PSD function in the cyclic frequency domain and f is the cyclic frequency (Hz).

Example. The enforced random acceleration spectrum is specified in Table A.1. This random acceleration spectrum will be simulated by a random time series. The number of terms in the time series is $N = 500$, and the maximum frequency of interest is $f_{max} = 2000$ Hz. The simulated random time

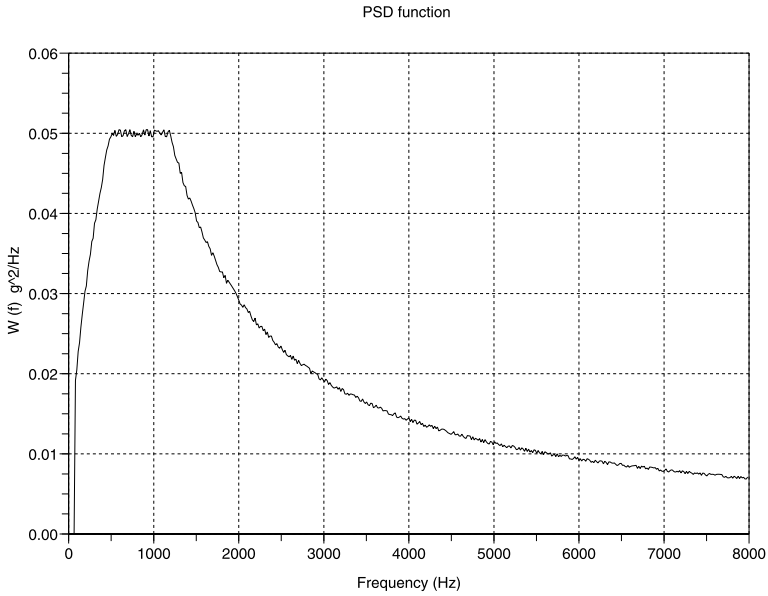


Fig. A.2. Reconstructed PSD $W(f)$ from Fig. A.1

series is shown in Fig. A.1. The reconstructed PSD function $W(f)$ from the random time series in Fig. A.1 is shown in Fig. A.2.

Problems

A.1. This problem is taken from [180]. The PSD function is given by

$$W_0(f) = W_0 \frac{\alpha}{\alpha^2 + f^2}, \quad 0 \leq f \leq f_u,$$

where

$$W_0 = \frac{1}{[\arctan \frac{f_u}{\alpha}]},$$

where $f_u = 40$ Hz and $\alpha = 4$ Hz.

Carry out the following assignments:

- Simulate the random time series
- Reconstruct the original PSD function from the random time series.

B

Tables of Integrals

The solution of the type of integral

$$I_n = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{G_n(j\omega)}{H_n(j\omega)H_n(-j\omega)} d\omega, \quad (\text{B.1})$$

with the *Hurwitz polynomial*

$$H_n(j\omega) = a_0(j\omega)^n + a_1(j\omega)^{n-1} + \dots + a_n \quad (\text{B.2})$$

and

$$G_n(j\omega) = b_0(j\omega)^{2(n-1)} + b_1(j\omega)^{2(n-2)} + \dots + b_{n-1} \quad (\text{B.3})$$

is given by [43, 74, 171]:

- $I_1 = \frac{b_0}{2a_0a_1}$
- $I_2 = \frac{\frac{a_0b_1}{a_2} - b_0}{2a_0a_1}$
- $I_3 = \frac{b_0a_2 + \frac{a_0a_1b_2}{a_3} - a_0b_1}{2a_0(a_1a_2 - a_0a_3)}$
- $I_4 = \frac{b_0(-a_1a_4 + a_2a_3) - a_0a_3b_1 + a_0a_1b_2 + \frac{a_0b_3}{a_4}(a_0a_3 - a_1a_2)}{2a_0(a_0a_3^2 + a_1^2a_4 - a_1a_2a_3)}$

The solutions of (B.1) will be used to solve integrals of this type in the course of the book.

The integral

$$\int_{-\infty}^{\infty} \frac{1}{|H_g(j\omega)|^2} d\omega, \quad (\text{B.4})$$

with the frequency response function

$$H_g(\omega) = -\omega^2 + \omega_0^2(1 + jg), \quad (\text{B.5})$$

cannot be evaluated using (B.1) if $n = 2$, I_2 is not defined because $a_1 = 0$. The integral (B.4) can be obtained using the residue theorem [195].

Table B.1. Accuracy of (B.7)

g	$\frac{(1+g^2)^{0.25}}{\cos(0.5g)}$
0.01	1.000
0.05	1.001
0.10	1.004
0.50	1.091

$$\int_{-\infty}^{\infty} \frac{1}{|H_g(j\omega)|^2} d\omega = \frac{\pi \cos(0.5g)}{2g\omega_0^3(1+g^2)^{0.25}}. \tag{B.6}$$

When the term $g \ll 1$ (structural damping coefficient) then

$$\int_{-\infty}^{\infty} \frac{1}{|H_g(j\omega)|^2} d\omega \approx \frac{\pi}{2g\omega_0^3}. \tag{B.7}$$

The accuracy of (B.7) is illustrated in Table B.1.

Some Definite Integrals

The following definite integral is of interest [17] for the calculation of the mean square response:

$$\begin{aligned} I_0 &= \int_{\omega_1}^{\omega_2} \frac{d\omega}{(1-\omega^2)^2 + (2\zeta\omega)^2} \\ &= \frac{1}{8\sqrt{1-\zeta^2}} \ln \left\{ \frac{[\omega_2^2 + 2\omega_2\sqrt{1-\zeta^2} + 1][\omega_1^2 - 2\omega_1\sqrt{1-\zeta^2} + 1]}{[\omega_2^2 - 2\omega_2\sqrt{1-\zeta^2} + 1][\omega_1^2 + 2\omega_1\sqrt{1-\zeta^2} + 1]} \right\} \\ &\quad + \frac{1}{4\zeta} \left[\arctan \frac{2\zeta\omega_2}{1-\omega_2^2} - \arctan \frac{2\zeta\omega_1}{1-\omega_1^2} \right]. \end{aligned} \tag{B.8}$$

Error Function erf(x)

A numerical approximate procedure for calculating the erf(x) is discussed in [113, 114]. The error function erf(x) is given by $\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_{-\infty}^x e^{-y^2} dy$. The numerical approximation is as follows

$$\text{erf}(x) = \begin{cases} \frac{2}{\sqrt{\pi}} e^{-x^2} \sum_{k=0}^9 \frac{2^k x^{2k+1}}{(2k+1)!!}, & 0 \leq x \leq 2; \\ 1 - e^{-x^2} \frac{1}{\sqrt{\pi x}} (1 - \frac{1}{2x^2}), & 2 < x < \infty. \end{cases} \tag{B.9}$$

where (n)!! is the *double factorial* defined as follows

$$(n)!! = \begin{cases} 1, & n = 0 \text{ or } n = 1; \\ n(n-2)!!, & n \geq 2. \end{cases} \tag{B.10}$$

Some identities of double factorials are

$$(2n)!! = 2^n n!, \tag{B.11}$$

and

$$(2n + 1)!! = \frac{(2n + 1)!}{2^n n!}. \tag{B.12}$$

The maximum absolute error of (B.9) is 0.0047% and the maximum relative error is 0.470% [113], which is satisfactory for engineering purposes.

Another approximation of the error function is discussed in [231] and given by

$$\operatorname{erf}(x) = \begin{cases} 0.30121 + 0.49161x^2 + 0.91811x^3 - 2.35341x^4 \\ \quad - 3.33071x^5 + 15.65241x^6 - 10.7846x^7, & 0.13 \leq x < 0.96; \\ 1, & 0.96 \leq x. \end{cases} \tag{B.13}$$

In [192] more details about calculating random vibration integrals are discussed.

B.1 Approximations of Inverse Transform of the Standard Normal

If $X = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-y^2} dy$ is expressed by $X = F(x)$ then the inverse value $x = F^{-1}(X)$ for the standard normal can be approximated as follows [20]

$$x = z + \frac{p_0 + p_1 z^1 + p_2 z^2 + p_3 z^3 + p_4 z^4}{q_0 + q_1 z^1 + q_2 z^2 + q_3 z^3 + q_4 z^4}, \tag{B.14}$$

where $z = \sqrt{-\log([1 - X]^2)}$, and

- $p_0 = -0.322232431088$
- $p_1 = -1$
- $p_2 = -0.342242088547$
- $p_3 = -0.0204231210245$
- $p_4 = -0.0000453642210148$
- $q_0 = 0.099348462606$
- $q_1 = 0.588581570495$
- $q_2 = 0.531103462366$
- $q_3 = 0.10353775285$
- $q_4 = 0.0038560700634$

This approximation has a relative accuracy of about six decimal digits and is valid for $0.5 < X < 1$. The symmetry of the normal allows us to extend it to $0.0 < X < 0.5$ by the transformation $X = 1 - X$ and $x = -x$.

B.2 Integrals of Probability Density Functions

Solutions of the joint probability functions from the FPK equation are in general exponential expression. Integral for of these exponential functions are

given.

$$\int_{-\infty}^{\infty} e^{-ax^2} dx = 2 \int_0^{\infty} e^{-ax^2} dx = \sqrt{\frac{\pi}{a}}, \tag{B.15}$$

$$\int_{-\infty}^{\infty} x e^{-ax^2} dx = 2 \int_0^{\infty} x e^{-ax^2} dx = \frac{1}{a}, \tag{B.16}$$

$$\int_{-\infty}^{\infty} x^2 e^{-ax^2} dx = 2 \int_0^{\infty} x^2 e^{-ax^2} dx = \frac{1}{2a} \sqrt{\frac{\pi}{a}}, \tag{B.17}$$

$$\int_{-\infty}^{\infty} x^4 e^{-ax^2} dx = 2 \int_0^{\infty} x^4 e^{-ax^2} dx = \frac{3}{4a^2} \sqrt{\frac{\pi}{a}}, \tag{B.18}$$

$$\int_{-\infty}^{\infty} x^n e^{-ax^2} dx = 2 \int_0^{\infty} x^n e^{-ax^2} dx = a^{-\frac{1}{2}(n+1)} \Gamma\left\{\frac{1}{2}(n+1)\right\}. \tag{B.19}$$

When z is real number and n is an integer, than

$$\Gamma(1) = 1, \quad \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}, \quad \Gamma(z+1) = z\Gamma(z), \quad \Gamma(n) = n! \tag{B.20}$$

$$\int_{-\infty}^{\infty} e^{-a(x^2+bx^4)} dx = 2 \int_0^{\infty} e^{-a(x^2+bx^4)} dx = \frac{\sqrt[4]{\frac{a}{b}} e^{\frac{a}{8b}} K_{\frac{1}{4}}\left(\frac{a}{8b}\right)}{2\sqrt[4]{ab}}, \tag{B.21}$$

where $K_\nu(z)$ is a Bessel function of the second kind and ν order [125].

$$\begin{aligned} \int_{-\infty}^{\infty} x e^{-a(x^2+bx^4)} dx &= 2 \int_0^{\infty} x e^{-a(x^2+bx^4)} dx \\ &= \frac{\sqrt{\pi} e^{\frac{a}{4b}}}{2\sqrt{ab}} \left[1 - \operatorname{erf}\left(\sqrt{\frac{a}{4b}}\right) \right]. \end{aligned} \tag{B.22}$$

$$\begin{aligned} \int_{-\infty}^{\infty} x^2 e^{-a(x^2+bx^4)} dx &= 2 \int_0^{\infty} x^2 e^{-a(x^2+bx^4)} dx \\ &= \frac{\sqrt[4]{\frac{a}{b}} e^{\frac{a}{8b}}}{8b\sqrt[4]{ab}} \left[K_{\frac{3}{4}}\left(\frac{a}{8b}\right) - K_{\frac{1}{4}}\left(\frac{a}{8b}\right) \right]. \end{aligned} \tag{B.23}$$

$$\begin{aligned} \int_{-\infty}^{\infty} x^4 e^{-a(x^2+bx^4)} dx &= 2 \int_0^{\infty} x^4 e^{-a(x^2+bx^4)} dx \\ &= \frac{\sqrt[4]{\frac{a}{b}} e^{\frac{a}{8b}}}{16b^2\sqrt[4]{ab}} \left[K_{\frac{3}{4}}\left(\frac{a}{8b}\right) - K_{\frac{1}{4}}\left(\frac{a}{8b}\right) - \frac{2b}{a} K_{\frac{1}{4}}\left(\frac{a}{8b}\right) \right]. \end{aligned} \tag{B.24}$$

C

Some Fourier Transforms

The following Fourier transforms are to be solved

$$h(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} H(j\omega) e^{j\omega t} d\omega, \quad (\text{C.1})$$

and

$$R(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} |H(j\omega)|^2 e^{j\omega t} d\omega, \quad (\text{C.2})$$

with

$$H(j\omega) = \frac{1}{\omega_n^2 - \omega^2 + 2j\zeta\omega\omega_n} \quad (\text{C.3})$$

The integrals (C.1) and (C.2) will be evaluated by using the theory of complex functions [195]. Apply the residue theorem to evaluate the integrals. Let $f(z)$ be single-valued and analytic inside and on a simple closed curve C except at the singularities a, b, c, \dots inside C which have residues $a_{-1}, b_{-1}, c_{-1}, \dots$. Then the *residue theorem* states that

$$\oint_C f(z) dz = 2\pi j (a_{-1} + b_{-1} + c_{-1} + \dots), \quad (\text{C.4})$$

i.e. the integral of $f(z)$ around C is $2\pi j$ times the sum of the residues of $f(z)$ at the singularities enclosed by C .

To obtain the residue of a function $f(z)$ at $z = a$ the following equation with a pole of order k can be applied.

$$a_{-1} = \lim_{z \rightarrow a} \frac{1}{(k-1)!} \frac{d^{k-1}}{dz^{k-1}} \{(z-a)^k f(z)\}. \quad (\text{C.5})$$

If $k = 1$ (simple pole) the result of the residue is

$$a_{-1} = \lim_{z \rightarrow a} (z-a) f(z). \quad (\text{C.6})$$

The poles of (C.3) are

$$\begin{aligned}\omega_1 &= \omega_n(j\zeta - \sqrt{1 - \zeta^2}), \\ \omega_2 &= \omega_n(j\zeta + \sqrt{1 - \zeta^2}).\end{aligned}\tag{C.7}$$

Calculate two residues associated with the poles ω_1 and ω_2 . The first residue a_{-1} can be calculated as follows

$$a_{-1} = \lim_{\omega \rightarrow \omega_1} (\omega - \omega_1) \frac{e^{j\omega t}}{(\omega - \omega_1)(\omega - \omega_2)} = \frac{e^{j\omega_n(j\zeta - \sqrt{1 - \zeta^2})t}}{-2\omega_n\sqrt{1 - \zeta^2}},\tag{C.8}$$

and the second residue b_{-1} becomes

$$b_{-1} = \lim_{\omega \rightarrow \omega_2} (\omega - \omega_2) \frac{e^{j\omega t}}{(\omega - \omega_1)(\omega - \omega_2)} = \frac{e^{j\omega_n(j\zeta + \sqrt{1 - \zeta^2})t}}{2\omega_n\sqrt{1 - \zeta^2}}.\tag{C.9}$$

Solve the definite integral of the type $\int_{-\infty}^{\infty} f(x)dx$. The contour integral (C.4) can be written as follows

$$\oint_C f(z)dz = \lim_{R \rightarrow \infty} \left[\int_{-R}^R f(x)dx + \int_{\Gamma} f(z)dz \right],\tag{C.10}$$

where $z = x + jy$, Γ the semi-circle with radius R . If $|f(z)| \leq \frac{M}{R^k}$ for $z = Re^{j\theta}$ where $k > 1$ and M is a constant then it can be proved that $\lim_{R \rightarrow \infty} \int_{\Gamma} f(z)dz = 0$ [195].

The integral (C.1) can be solved

$$h(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{j\omega t}}{\omega_n^2 - \omega^2 + 2j\zeta\omega\omega_n} d\omega = j(a_{-1} + b_{-1}),\tag{C.11}$$

The sum of the residues $a_{-1} + b_{-1}$ becomes

$$\begin{aligned}a_{-1} + b_{-1} &= \frac{e^{-\omega_n\zeta t}}{2\omega_n\sqrt{1 - \zeta^2}} [-e^{-j\omega_n\sqrt{1 - \zeta^2}t} + e^{j\omega_n\sqrt{1 - \zeta^2}t}] \\ &= \frac{e^{-\omega_n\zeta t}}{\omega_n\sqrt{1 - \zeta^2}} j \sin(\omega_n\sqrt{1 - \zeta^2}t).\end{aligned}\tag{C.12}$$

Integral (C.11) becomes finally

$$\begin{aligned}h(t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{j\omega t}}{\omega_n^2 - \omega^2 + 2j\zeta\omega\omega_n} d\omega \\ &= -\frac{e^{-\omega_n\zeta t}}{\omega_n\sqrt{1 - \zeta^2}} \sin(\omega_n\sqrt{1 - \zeta^2}t).\end{aligned}\tag{C.13}$$

The result of integral (C.13) was expected because it is the impulse response function of a damped sdof system.

Now the integral (C.2) is solved. The FRF $|H(\omega)|^2$ can be written as

$$|H(\omega)|^2 = H(\omega)H(-\omega), \tag{C.14}$$

and the four poles of $|H(\omega)|^2$ are:

$$\begin{aligned} \omega_1 &= \omega_n(j\zeta - \sqrt{1 - \zeta^2}), \\ \omega_2 &= \omega_n(j\zeta + \sqrt{1 - \zeta^2}), \\ \omega_3 &= \omega_n(-j\zeta - \sqrt{1 - \zeta^2}), \\ \omega_4 &= \omega_n(-j\zeta + \sqrt{1 - \zeta^2}). \end{aligned} \tag{C.15}$$

The contour integral $\oint_C f(z)dz$ is only applicable for the positive side of the imaginary axis and therefore the solution of (C.2) will be

$$R(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} |H(\omega)|^2 e^{j\omega t} d\omega = j(a_1 + b_1). \tag{C.16}$$

The residues a_{-1} and b_{-1} can be calculated as follows

$$a_{-1} = \lim_{\omega \rightarrow \omega_1} (\omega - \omega_1) \frac{e^{j\omega t}}{(\omega - \omega_1)(\omega - \omega_2)(\omega - \omega_3)(\omega - \omega_4)}, \tag{C.17}$$

and

$$b_{-1} = \lim_{\omega \rightarrow \omega_2} (\omega - \omega_2) \frac{e^{j\omega t}}{(\omega - \omega_1)(\omega - \omega_2)(\omega - \omega_3)(\omega - \omega_4)}. \tag{C.18}$$

Substituting the poles ω_1 and ω_2 of (C.15) in (C.17) and (C.18) the residues become

$$a_{-1} = \frac{e^{j\omega_n[j\zeta - \sqrt{1 - \zeta^2}]t}}{(-2\omega_n\sqrt{1 - \zeta^2})(2j\omega_n\zeta)(2\omega_n[j\zeta - \sqrt{1 - \zeta^2}])}, \tag{C.19}$$

and

$$b_{-1} = \frac{e^{j\omega_n[j\zeta + \sqrt{1 - \zeta^2}]t}}{(-2\omega_n\sqrt{1 - \zeta^2})(2j\omega_n\zeta)(2\omega_n[j\zeta + \sqrt{1 - \zeta^2}])}. \tag{C.20}$$

The sum of the residues $a_{-1} + b_{-1}$ can now be obtained

$$\begin{aligned} a_{-1} + b_{-1} &= \frac{e^{-\omega_n\zeta t}}{(2\omega_n\sqrt{1 - \zeta^2})(2j\omega_n\zeta)(2\omega_n^2)} \\ &\quad \times [2\omega_n\sqrt{1 - \zeta^2} \cos(\omega_n\sqrt{1 - \zeta^2}t) \\ &\quad + 2\omega_n\zeta \sin(\omega_n\sqrt{1 - \zeta^2}t)]. \end{aligned} \tag{C.21}$$

The final solution of (C.2) becomes

$$R(t) = j(a_{-1} + b_{-1}), \tag{C.22}$$

and substituting (C.21) into (C.22) the following expression can be obtained

$$R(t) = \frac{e^{-\omega_n\zeta t}}{4\omega_n^3\zeta} \left[\cos(\omega_n\sqrt{1 - \zeta^2}t) + \frac{\zeta}{1 - \zeta^2} \sin(\omega_n\sqrt{1 - \zeta^2}t) \right], \quad t \geq 0. \tag{C.23}$$

Problems

C.1. Find the Fourier transform $F(\omega)$ of the function

$$f(t) = \frac{1}{b^2 + t^2},$$

using contour integration.

Answer: $F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t} dt = \frac{\pi}{b} e^{b|\omega|}$.

D

Modal Effective Mass

A flexible linear structure is idealized as a finite element system with N degrees of freedom (DOF). The undamped equations of motion of the MDOF dynamic system are

$$[M]\{\ddot{x}(t)\} + [K]\{x(t)\} = \{F(t)\}, \quad (\text{D.1})$$

where

- $[M]$ the mass matrix
- $[K]$ the stiffness matrix
- $\{\ddot{x}(t)\}$ the acceleration vector
- $\{x(t)\}$ the displacement vector and
- $\{F(t)\}$ the external force vector.

The interface DOFS are denoted by $\{x_r(t)\}$ (r-set) and the internal DOFS by $\{x_i(t)\}$ (i-set). The interface DOFS describe the six motions as a rigid body: three translations and three rotations. The six rigid body motions may be introduced in one node with six DOFS. The interface DOFS constitute a determinate interface. Partitioning the equations of motion (D.1) into the r-set and i-set we find in the following equations

$$\begin{bmatrix} M_{rr} & M_{ri} \\ M_{ir} & M_{ii} \end{bmatrix} \begin{Bmatrix} \ddot{x}_r(t) \\ \ddot{x}_i(t) \end{Bmatrix} + \begin{bmatrix} K_{rr} & K_{ri} \\ K_{ir} & K_{ii} \end{bmatrix} \begin{Bmatrix} x_r(t) \\ x_i(t) \end{Bmatrix} = \begin{Bmatrix} F_r(t) \\ F_i(t) \end{Bmatrix}. \quad (\text{D.2})$$

The physical displacements are expressed as a superposition of rigid body modes $[\Phi_r]$ and elastic modes $[\Phi_e]$ multiplied by the associated generalized coordinates $x_r(t)$ (r-set) and $\eta_n(t)$ (n-set) respectively

$$\{x(t)\} = [\Phi_r]\{x_r(t)\} + [\Phi_e]\{\eta_n(t)\} = [\Psi] \begin{Bmatrix} x_r(t) \\ \eta_n(t) \end{Bmatrix}. \quad (\text{D.3})$$

We can obtain the matrix of rigid body motions Φ_r from the stiffness matrix (D.2), ignoring the inertia terms and $F_i(t)$, from the second equation

$$[K_{ri}]\{x_r\} + [K_{ii}]\{x_i\} = 0. \quad (\text{D.4})$$

From (D.4) we can derive a relation between the internal and external DOFs if there is no load applied to the internal DOFs, hence $\{F_i\} = \{0\}$

$$\{x_i\} = -[K_{ii}]^{-1}[K_{ri}]\{x_r\} = -[K_{ii}]^{-1}[K_{ri}][I] = -[K_{ii}]^{-1}[K_{ri}] = [\phi_r]. \tag{D.5}$$

The complete rigid body modes $[\Phi_r]$ become

$$[\Phi_r] = \begin{bmatrix} I \\ \phi_r \end{bmatrix}. \tag{D.6}$$

If the r-set DOFs are constrained $\{x_r\} = \{0\}$ the elastic vibration modes can be obtained with respect to the external DOFs from the eigenvalue problem

$$(-\omega_k^2[M_{ii}] + [K_{ii}])\{\phi_k\} = \{0\}, \tag{D.7}$$

where ω_k^2 is the k -th eigenvalue with the associated eigenvector $\{\phi_k\}$.

The matrix of the elastic vibration modes $[\Phi_e]$ is

$$[\Phi_e] = \begin{bmatrix} 0 \\ \Phi_i \end{bmatrix} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ \phi_{i1} & \phi_{i2} & \cdots & \phi_{in} \end{bmatrix} \tag{D.8}$$

The introduction of the modal matrix $[\Psi]$ of (D.3) into (D.2) leads to the following equations of motion expressed in terms of the external and generalized DOFs, $\{x_r(t)\}$ and $\{\eta_n(t)\}$

$$\begin{bmatrix} M_o & L^T \\ L & [m] \end{bmatrix} \begin{Bmatrix} \ddot{x}_r(t) \\ \ddot{\eta}_n(t) \end{Bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & [k] \end{bmatrix} \begin{Bmatrix} x_r(t) \\ \eta_n(t) \end{Bmatrix} = \begin{Bmatrix} F_r(t) \\ 0 \end{Bmatrix} \tag{D.9}$$

where

- $[M_o] = [\Phi_r]^T[M][\Phi_r]$ the rigid body mass matrix with respect to the r-set
- $[K][\Phi_r] = [0]$ the rigid body force vector
- $[L]^T = [\Phi_r]^T[M][\Phi_e]$ the modal participation factors
- $[m] = [\phi_i]^T[M][\phi_i]$ the generalized masses
- $[k] = [\phi_i]^T[K][\phi_i]$ the generalized stiffness.
- $\{F_r(t)\} = [\Phi_r]^T\{F(t)\}$

Introduce the modal damping ratio in the uncoupled equations of motion of the generalized coordinated $\eta_k(t)$ then the second part of equations of (D.9) will become

$$\ddot{\eta}_k(t) + 2\zeta_k\omega_k\dot{\eta}_k(t) + \omega_k^2\eta_k(t) = \frac{-\{L_k\}\{\ddot{x}_r(t)\}}{m_k}, \quad k = 1, 2, \dots, n. \tag{D.10}$$

The number of modes $\{\phi_k\}$, $k = 1, 2, \dots, n$ is in general much less than the number if internal dofs i . The Fourier transform of (D.10), using $\eta(t) = \Pi(\omega)e^{j\omega t}$, can be written as

$$(-\omega^2 + 2j\zeta_k\omega_k + \omega_k^2)\Pi_k(\omega) = \frac{-\{L_k\}\{\ddot{X}_r(\omega)\}}{m_k}, \quad k = 1, 2, \dots, n, \tag{D.11}$$

and if $\Pi_k(\omega)$ is made explicit then

$$\Pi_k(\omega) = H_k(\omega) \frac{-\{L_k\}\{\ddot{X}_r(\omega)\}}{m_k}, \quad k = 1, 2, \dots, n. \quad (\text{D.12})$$

where $H_k(\omega) = (-\omega^2 + 2j\zeta_k\omega_k + \omega_k^2)^{-1}$. Use the following expression $\ddot{I}_k = -\omega^2 \Pi_k$.

The first part of equation in (D.9), an equation applicable for the external dofs $x_r(t)$ is

$$[M_0]\{\ddot{x}_r\} + [L]^T\{\ddot{\eta}(t)\} = \{F_r(t)\}. \quad (\text{D.13})$$

If (D.11) is substituted in the Fourier transform of (D.13) with $\ddot{I}_k = -\omega^2 \Pi_k$ then

$$\left([M_0] + \omega^2 \sum_{k=1}^n \frac{H_k(\omega)\{L_k\}^T\{L_k\}}{m_k} \right) \ddot{X}_r(\omega) = F_r(\omega), \quad (\text{D.14})$$

or

$$\left([M_0] + \omega^2 \sum_{k=1}^n H_k(\omega)[M_{eff,k}] \right) \ddot{X}_r(\omega) = F_r(\omega), \quad (\text{D.15})$$

where the 6×6 modal effective mass $M_{eff,k}$ is defined as

$$[M_{eff,k}] = \frac{\{L_k\}^T\{L_k\}}{m_k}. \quad (\text{D.16})$$

The mass matrix $[M_0]$ can be expanded as the sum of the modal effective masses $M_{eff,k}$

$$\begin{aligned} [M_0] &= \sum_{k=1}^i [M_{eff,k}] \\ &= [\Phi_r]^T [M] [\Phi_e] ([\Phi_e]^T [M] [\Phi_e])^{-1} [\Phi_e]^T [M] [\Phi_r] \\ &= [\Phi_r]^T [M] [\Phi_r]. \end{aligned} \quad (\text{D.17})$$

The inverse of the modal matrix $[\Phi_e]$ exists, if all modes $\{\phi_k\}$, $k = 1, 2, \dots, i$ are calculated!

The modal effective mass matrix $[M_{eff,k}]$ of mode k is the fraction of the total static mass $[M_0]$ that can be attributed to this mode.

The residual mass matrix $[M_{res}]$ is defined by

$$[M_{res}] = [M_0] - \sum_{k=1}^n [M_{eff,k}], \quad (\text{D.18})$$

where n is the number of kept modes $n \ll i$, where i is the total number of degrees of freedom.

Finally (D.15) can now be written

$$\sum_{k=1}^n \{ [M_{eff,k}] (1 + \omega^2 H_k(\omega)) \} \{ \ddot{X}_r(\omega) \} = \{ F_r(\omega) \}, \quad (D.19)$$

Equation (D.19) shows the relation between the reaction forces $\{F_r(\omega)\}$ at the location of the determinate supported structure excited by enforced acceleration $\{\ddot{X}_r(\omega)\}$. This equation will be used in combination with Miles' equation to calculate the rms reaction forces of an elastic structure exposed to random dynamic enforced acceleration. The term between the curly brackets is called the dynamic mass matrix

$$[M_d(\omega)] = \sum_{k=1}^n [M_{eff,k}] (1 + \omega^2 H_k(\omega)). \quad (D.20)$$

Using the concept of modal effective mass we can decompose a MDOF structure into a series of SDOF systems. The discrete mass reflects the modal effective mass $m_{eff,n}$ with corresponding angular natural frequency ω_n , spring constant $k_n = \omega_n^2 m_{eff,n}$, modal damping ratio ζ_n and the damping constant $c_n = 2\zeta_n \omega_n m_{eff,n}$. The representing series of SDOF systems is called the asparagus model of the structure [39]. This is illustrated in Fig. D.1.

More about the modal effective masses can be read in [151, 172].

Example. This example is taken from [86]. Consider the two degrees of freedom dynamic system shown on Fig. D.2. The parameters have the following values, the discrete masses are $m_1 = 2$ kg and $m_2 = 1$ kg, the spring constants are $k_1 = 1000$ N/m, $k_2 = 2000$ N/m and $k_3 = 3000$ N/m. Calculate the modal effective masses.

The homogeneous equations of motion are

$$\begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} \begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} + \begin{pmatrix} k_1 + k_3 & -k_3 \\ -k_3 & k_2 + k_3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The eigenvalue problem will yield the natural frequencies (Hz), $f_1 = 4.78$ Hz and $f_2 = 12.43$ Hz. The corresponding vibration modes $[\Phi_e]$ are

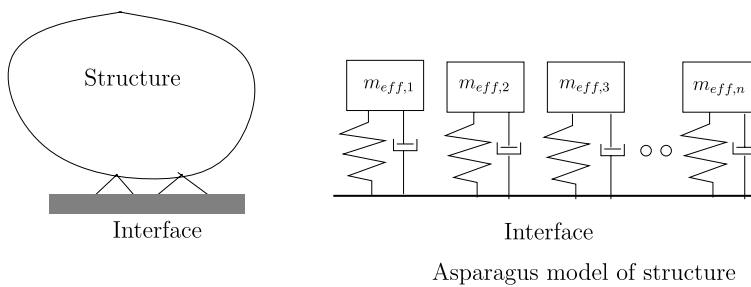


Fig. D.1. Asparagus patch model [39]

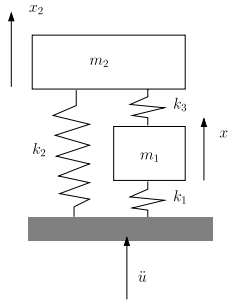


Fig. D.2. Two degree of freedom system

$$[\Phi_e] = \begin{pmatrix} 0.6280 & -0.3251 \\ 0.4597 & 0.8881 \end{pmatrix}.$$

The generalized (diagonal) mass matrix is expressed by

$$[mg] = [\Phi_e]^T [M] [\Phi_e] = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

where $[M]$ is the mass matrix. The vibration modes have been scaled so that the generalized mass matrix becomes the identity matrix.

The rigid body vector $\{\Phi_r\}$ can be easily generated

$$\{\Phi_r\} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

The modal participation factors are given by

$$L_k = \{\phi_{e,k}\} [M] \{\Phi_r\}, \quad k = 1, 2.$$

thus $L_1 = 1.7157$ and $L_2 = -0.2379$. A modal participation factor represents the coupling of the vibration mode with the motion as a rigid body.

The modal effective mass is given by

$$m_{eff,k} = \frac{L_k^2}{mg(k,k)} \quad k = 1, 2.$$

The results are presented in Table D.1. The sum of the modal effective masses equals the total system mass.

The absolute value of the dynamic mass $|md|$ is presented in Fig. D.3. Notice that, at the first natural frequency, the dynamic mass is very high.

Problems

D.1. Calculate modal effective masses for the dynamic system shown in Fig. D.4 shall be calculated. The discrete mass parameter m (kg) and spring

Table D.1. Modal effective and residual mass

Mode	Natural frequency (Hz)	Modal effective mass (kg)	Residual mass (kg)
1	4.78	2.944	0.056
2	12.43	0.056	0
Total mass		3	

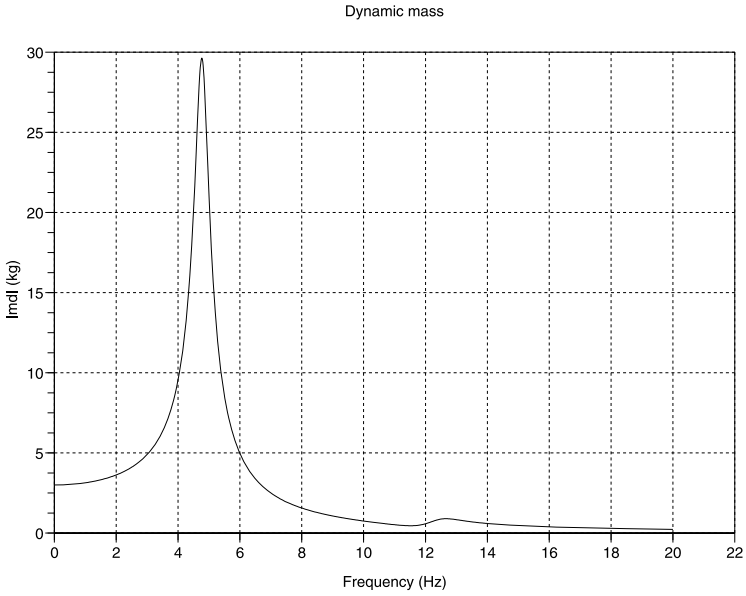


Fig. D.3. Dynamic mass, modal damping ratio $\zeta = 0.05$

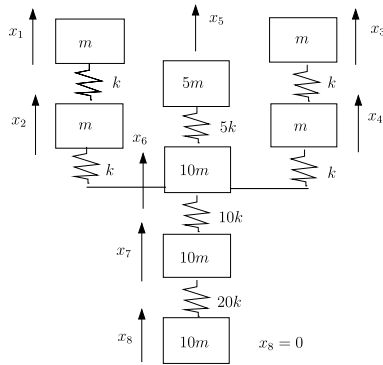


Fig. D.4. Dynamic system 8 DOFs

Table D.2. Calculation of the modal effective mass

Mode shape ≠	Natural frequency (Hz)	Modal Participation L_k	Generalized mass m_k	Modal effective mass m_{eff} (kg)
1	24.5422	5.5874	1.5746	19.8271
2	31.1052	0.0000	1.0000	0.0000
3	36.6716	2.7421	1.9255	3.9048
4	64.4657	3.7104	5.0429	2.7300
5	81.4344	0.0000	1.0000	0.0000
6	82.0637	0.7400	1.0989	0.4983
7	95.9164	3.8552	7.2863	2.0398
Total mass (without $m_8 = 10$)				29.0000

stiffness parameters k (N/m) are $m = 1$ and $k = 100\,000$ respectively. The constrained degree of freedom is $x_8 = 0$. The answers are shown in Table D.2.

D.2. During a sine qualification vibration test of a complete spacecraft, with a total mass $M = 6100$ kg, the constant enforced acceleration is $\ddot{U}(f) = 12.5$ m/s² in the frequency range between 20–100 Hz. The most significant vibration mode (mode 1) in the axial direction has the following properties:

- The natural frequency is $f_1 = 36$ Hz.
- The modal effective mass is $m_{eff,1} = 2000$ kg.
- The generalized mass is $m_1 = \{\Phi_1\}^T [M] \{\Phi_1\} = 1$. $[M]$ is the mass matrix and $\{\Phi_1\}$ is the vibration mode.
- Modal damping ratio $\zeta_1 = 0.02$
- Maximum modal displacement $\{\Phi_1\}$ is 0.05.

Perform the following assignments:

- Calculate the modal participation factor L_1 .
- Set up the two equations of motion expressed in terms of the enforced acceleration $\ddot{u}(t) = \ddot{U}(\omega)e^{j\omega t}$ and the generalized coordinate $\eta_1(t) = \ddot{\Pi}(\omega)e^{j\omega t}$.
- Calculate the acceleration of the generalized coordinate $\ddot{\Pi}(2\pi 36)$.
- Calculate maximum physical acceleration $a_{max,1}$.
- Calculate dynamic mass $|m_d(2\pi 36)|$.
- Calculate reaction force $|F_r(2\pi 36)| = |m_d((2\pi 36)) * \ddot{U}(2\pi 36)|$ or (D.19).

Answers (partly): $L_1 = 44.7214$ kg, $\ddot{\Pi}(2\pi 36) = -1.3975 \times 10^4 j$, $|a_{max,1}| = 6.9877 \times 10^2$ m/s².

E

Thevenin's and Norton's Theorems for Equivalent Linear Mechanical Systems

The theory discussed in this appendix is based on [48, 193]. In Fig. E.1 the configuration of a load on its mount (source or foundation) can be idealized as a simple 2DOF harmonic oscillator. Here m_1 and m_2 are the source and load masses, respectively, and Z_1 and Z_2 are the source and load impedances, respectively. When using a mechanical impedance formulation, it is convenient to analyze the idealized 2DOF system using Thevenin's and Norton's theorems about equivalent linear mechanical systems.

Thevenin's theorem states: *At a given frequency, any linear system of loads and sources may be reduced to a simpler system consisting of a single constant-force source in parallel with a single impedance connected to the load.* Rephrased, Thevenin's theorem stated that insofar as the load is concerned, the source can be considered as an equivalent force driving an equivalent source impedance in parallel with a single impedance of the load, (see Fig. E.2). The equivalent force is given by the force at the interface when the source is driving an infinite load (blocked force). The source equivalent impedance is the source impedance measured with the forces removed.

Norton's theorem states: *At a given frequency, any linear system of loads and sources may be reduced to a simpler system consisting of a single*

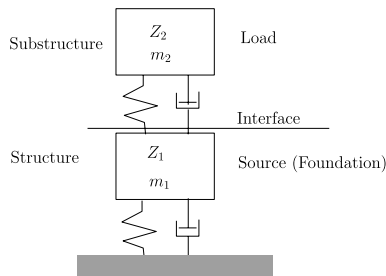


Fig. E.1. Idealized 2DOF system

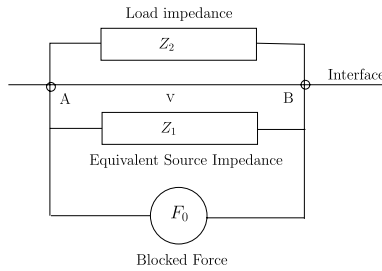


Fig. E.2. Thevenin's equivalent system

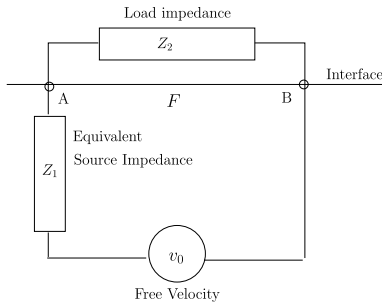


Fig. E.3. Norton's equivalent system

Table E.1. Impedance definitions

Name	Symbol	Definition	Alternative names
Dynamic stiffness	K	F/x	Dynamic modulus
Mechanical impedance	Z	F/v	
Dynamic mass	m	F/a	Apparent mass
Flexibility	α	x/F	Receptance compliance
Mobility	Y	v/F	Admittance
Accelerance	A	a/F	Inertance

constant-velocity source in series with a single impedance connected to the load.

Rephrased, Norton's theorem states that insofar as the load is concerned, the source can be considered as an equivalent velocity (motion) driving an equivalent source impedance in series with the load (see Fig. E.3). The equivalent velocity-source is the interface velocity when the load is removed (in the *free velocity*). The source equivalent impedance is the source impedance measured with the motions removed.

Investigations in vibration mechanics are often described in terms of mechanical impedances. In Table E.1 the definitions of impedances are summarized.

The 2DOF system illustrated in Fig. E.1 is transformed into a equivalent mechanical system using Thevenin's theorem. All parameters are defined in the frequency domain. This equivalent mechanical system is shown in Fig. E.2. From this equivalent system it follows that

$$F_0 = v(Z_1 + Z_2), \quad (\text{E.1})$$

where F_0 is the blocked force, v is the interface velocity with the load attached, and Z_1 and Z_2 are the source and load impedances, respectively. The associated acceleration is given by $a = j\omega v$. Furthermore, if no load is attached to the interface, the free velocity v_0 and the associated free acceleration $a_0 = j\omega v_0$, are related to the blocked force by

$$F_0 = Z_1 v_0 \quad (\text{E.2})$$

All parameters are in the frequency domain. Thus the following relation can be written

$$\frac{a}{v} = \frac{a_0}{v_0} = j\omega. \quad (\text{E.3})$$

Combining (E.1) and (E.2) yields the following important relation

$$a = \frac{a_0}{1 + \frac{Z_2}{Z_1}}, \quad (\text{E.4})$$

where a is the interface acceleration and a_0 is the free acceleration (acceleration resulting from no restraining forces at the interface). Equation (E.4) shows that the interface acceleration a is a function of the source and load impedances, Z_1 and Z_2 , respectively.

During a vibration test on a shaker the shaker provides an input acceleration that is largely independent of the dynamic characteristics of the source and the load, an input corresponding to the free acceleration to the free acceleration a_0 . That means that the impedance of the source $Z_1 \rightarrow \infty$. The shaker acts as an infinite impedance source.

The 2DOF system (Fig. E.1) can be reduced to Norton's equivalent mechanical system (Fig. E.3). Applying Norton's theorem yield

$$v_0 = F \left(\frac{1}{Z_1} + \frac{1}{Z_2} \right), \quad (\text{E.5})$$

where v_0 is the free velocity and F is the interface force.

From the previous equations (E.1), (E.3) and (E.5) the impedances can be eliminated and the following relation can be derived

$$\frac{v}{v_0} + \frac{F}{F_0} = 1. \quad (\text{E.6})$$

The ratio of the interface to free velocity can be replaced by the corresponding ratio interface to free acceleration in (E.6) to derive

$$\frac{a}{a_0} + \frac{F}{F_0} = 1, \quad (\text{E.7})$$

where a and F are the interface acceleration and force, respectively, and a_0 is the free acceleration and F_0 the blocked force.

Equation (E.7) provides the basis for exact dual control of the source load vibratory system shown in Fig. E.1. It appears that dual-control vibration test, which involves limiting force levels as well as acceleration levels at the input to the test item, is a viable solution for improve otest conditions on a variety of hardware as applied in the force-limiting vibration testing.

Problems

E.1. A SDOF system has a discrete mass m , a natural frequency ω_0 and a damping ratio ζ . The SDOF is loaded by a harmonic force $F(\omega)$. The harmonic displacement of the mass is denoted by $X(\omega)$. Derive the impedance $Z(\omega)$ of the SDOF system.

Answer: $Z(\omega) = -j(-\omega^2 + 2j\zeta\omega_0\omega + \omega_0^2)/m\omega$.

E.2. Derive the following relation, [29]

$$W_{FF}(f) = \left| \frac{M_1(f)M_2(f)}{M_1(f) + M_2(f)} \right|^2 W_{a_0a_0}(f),$$

where $M_1(f)$ and $M_2(f)$ are the apparent masses of source and load, respectively, $W_{FF}(f)$ is the interface force spectrum and $W_{a_0a_0}(f)$ is the acceleration spectrum at the interface of the unloaded structures (free acceleration, the acceleration resulting from no restraining forces in the interface).

F

Lyapunov's Equation

Let $[A]$ be an $2n \times 2n$ matrix of real or complex terms a_{ij} and suppose that all eigenvalues of $[A]$ lie in the left half-plane. $f(t)$ is an $2n$ -component column-vector with the following property

$$E[\{f(t)\}\{f(t-\tau)\}^T] = [E f_i(t) f_j(t-\tau)^T] = [C]\delta(\tau), \quad (\text{F.1})$$

where $\delta(\tau)$ is the *delta-function* and the Hermitian matrix $[C]$ is positive semi-definite.

$x(t)$ is a stationary stochastic process defined by

$$\{\dot{x}(t)\} = [A]\{x(t)\} + \{f(t)\}. \quad (\text{F.2})$$

Then the expected value of $x(t)$ is [65]

$$E[\{x(t)\}\{x(t-\tau)\}^T] = e^{[A]\tau}[M], \quad (\text{F.3})$$

where the $2n \times 2n$ covariance matrix $[M]$ is uniquely determined by the system of $(2n)^2$ unknowns

$$[A][M] + [M][A]^T = -[C]. \quad (\text{F.4})$$

The steady-state solution of (F.2) is given by

$$\{x(t)\} = \int_{-\infty}^t e^{[A](t-\lambda)} \{f(\lambda)\} d\lambda = \int_0^{\infty} e^{[A]\lambda} \{f(t-\lambda)\} d\lambda. \quad (\text{F.5})$$

Therefore,

$$\begin{aligned} E[\{x(t)\}\{x(t-\tau)\}^T] &= E \left[\int_0^{\infty} \int_0^{\infty} e^{[A](t-\lambda)} \{f(t-\lambda)\} \{f(t-\tau-\mu)\}^T e^{[A]^T \mu} d\lambda d\mu \right] \\ &= \int_0^{\infty} \int_0^{\infty} e^{[A]\lambda} [C] \delta(\tau + \mu - \lambda) e^{[A]^T \mu} d\lambda d\mu \\ &= \int_0^{\infty} e^{[A](\mu+\tau)} [C] e^{[A]^T \mu} d\mu = [M] e^{[A]\tau}, \end{aligned} \quad (\text{F.6})$$

where

$$[M] = E\{\{x(t)\}\{x(t)\}^T\} = \int_0^\infty e^{[A]\mu}[C]e^{[A]^T\mu}d\mu. \tag{F.7}$$

The integrand of (F.7) is called $[Z(\mu)] = e^{[A]\mu}[C]e^{[A]^T\mu}$. Differentiation gives

$$\frac{d[Z(\mu)]}{d\mu} = [A][Z(\mu)] + [Z(\mu)][A]^T. \tag{F.8}$$

Since the matrix $[A]$ has all its eigenvalues in the left half-plane, the matrix $[Z(\mu)]$ tends exponentially to 0 as $\mu \rightarrow \infty$. Therefore, integration of (F.8) from 0 to ∞ will obtain

$$[A][M] + [M][A]^T = -[C]. \tag{F.9}$$

This is the classical Lyapunov equation of stability-theory. The covariance matrix $[M]$ is unique for any fixed $[C]$ [65].

For convenience (F.2) is written as

$$\{\dot{x}(t)\} = [A]\{x(t)\} + \{f(t)\} = [A]\{x(t)\} + \{B\}w(t), \tag{F.10}$$

where the input vector $\{B\}$ is a $2n \times 1$ vector, and $w(t)$ is a scalar white noise with $E\{w^2(t)\} = S_w$. The Lyapunov equation (F.9) can be written as [72]

$$[A][M] + [M][A]^T = -\{B\}S_w\{B\}^T = -S_w\{B\}\{B\}^T. \tag{F.11}$$

The equation (F.11) will be transformed to a "phase canonical form" by the transformation

$$\{x(t)\} = [K]\{s(t)\}, \tag{F.12}$$

where $[K]$ is the state transformation matrix; (F.10) becomes

$$\{\dot{s}(t)\} = [\Phi]\{s(t)\} + \{G\}w(t), \tag{F.13}$$

with

$$\bullet \quad [\Phi] = [K]^{-1}[A][K] = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \ddots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ -d_{2n} & -d_{2n-1} & -d_{2n-2} & \cdots & -d_2 & -d_1 \end{pmatrix} \text{ and}$$

$$\bullet \quad \{G\} = [K]^{-1}\{B\} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}.$$

The elements d_1, d_2, \dots, d_{2n} are the coefficients of the *characteristic polynomial* of $[A]$

$$\chi(\lambda) = \lambda^{2n} + d_1\lambda^{2n-1} + d_2\lambda^{2n-2} + \dots + d_{2n-1}\lambda + d_{2n}, \quad (\text{F.14})$$

and the matrix $[\Phi]$ is recognized as the *companion matrix*.

The state variables of the realization (F.13) are called the “phase variables”.

The steady-state solution of (F.11) is given by (F.5)

$$\{x(t)\} = \int_0^\infty e^{[A](t-\tau)}\{B\}w(t)d\tau. \quad (\text{F.15})$$

Now, because a convergent infinite series of the $2n \times 2n$ matrix $[A]$ can be expressed as an unique polynomial of degree $2n - 1$ ($2n$ is the degree of the minimum polynomial $\chi(\lambda) = 0$ of $[A]$), hence

$$e^{[A](t-\tau)} = \sum_{i=0}^{2n-1} \alpha_i(t - \tau)[A]^i. \quad (\text{F.16})$$

Substituting (F.16) into (F.15) will result in

$$\begin{aligned} \{x(t)\} &= \int_0^\infty e^{[A](t-\tau)}\{B\}w(t)d\tau \\ &= \sum_{i=0}^{2n-1} [A]^i\{B\} \int_0^\infty \alpha_i(t - \tau)w(\tau)d\tau \\ &= \sum_{i=0}^{2n-1} [A]^i\{B\}\beta_i(t) = [C_{A,B}]\{\beta\}, \end{aligned} \quad (\text{F.17})$$

where the *observability matrix* $[C_{A,B}]$ is

$$[C_{A,B}] = [[A]^{2n-1}\{B\}, [A]^{2n-2}\{B\}, \dots, [A]\{B\}, \{B\}]. \quad (\text{F.18})$$

An analogous solution can be obtained for (F.13). The steady-state solution of (F.13) is given by (F.5)

$$\{s(t)\} = \int_0^\infty e^{[\Phi](t-\tau)}\{G\}w(t)d\tau. \quad (\text{F.19})$$

The $2n \times 2n$ matrix $[\Phi]$ can be expressed as an unique polynomial of degree $2n - 1$ ($2n$ is the degree of the minimum polynomial $\chi(\lambda) = 0$ of $[\Phi]$), hence

$$e^{[\Phi](t-\tau)} = \sum_{i=0}^{2n-1} \alpha_i(t - \tau)[\Phi]^i. \quad (\text{F.20})$$

Substituting (F.20) into (F.19) will result in

$$\begin{aligned}
 \{s(t)\} &= \int_0^\infty e^{[\Phi](t-\tau)}\{G\}w(t)d\tau \\
 &= \sum_{i=0}^{2n-1} [\Phi]^i\{G\} \int_0^\infty \alpha_i(t-\tau)w(\tau)d\tau \\
 &= \sum_{i=0}^{2n-1} [\Phi]^i\{G\}\beta_i(t) = [C_{\Phi,G}]\{\beta\},
 \end{aligned}
 \tag{F.21}$$

where the matrix $[C_{\Phi,G}]$ is

$$[C_{\Phi,G}] = [[\Phi]^{2n-1}\{G\}, [\Phi]^{2n-2}\{G\}, \dots, [\Phi]\{G\}, \{G\}].
 \tag{F.22}$$

The matrix $[C_{\Phi,G}]$ is called the *controllability matrix*.

The phase canonical transformation (F.12) is

$$\{x(t)\} = [C_{A,B}]\{\beta\} = [K]\{s(t)\} = [K][C_{\Phi,G}]\{\beta\},
 \tag{F.23}$$

which means that

$$\{[C_{A,B}] - [K][C_{\Phi,G}]\}\{\beta\} = \{0\},
 \tag{F.24}$$

thus

$$[K] = [C_{A,B}][C_{\Phi,G}]^{-1}.
 \tag{F.25}$$

The inverse of the controllability matrix $[C_{\Phi,G}]^{-1}$ can be verified to be [72]

$$[C_{\Phi,G}]^{-1} = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ d_1 & 1 & \cdots & 0 & 0 \\ d_2 & d_1 & \ddots & 0 & 0 \\ \vdots & \vdots & \cdots & 1 & 0 \\ d_{2n-1} & d_{2n-2} & \cdots & d_1 & 1 \end{pmatrix}.
 \tag{F.26}$$

Writing the observability matrix $[\bar{C}_{A,B}]$ and the controllability matrix $[C_{\Phi,G}]$ in an opposite sequence will have no influence on the state transformation matrix $[K]$. The inverse of the controllability matrix $[C_{\Phi,G}]^{-1}$ can be verified to be

$$[\bar{C}_{\Phi,G}]^{-1} = \begin{pmatrix} d_1 & d_2 & \cdots & d_{2n-1} & 1 \\ d_2 & d_3 & \cdots & 1 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ d_{2n-1} & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{pmatrix}.
 \tag{F.27}$$

The covariance matrix $[M]$ can now calculated using the following expression

$$[M] = [K][Q][K]^T,
 \tag{F.28}$$

where the covariance matrix $[Q]$ is obtained from

$$[\Phi][Q] + [Q][\Phi]^T = -S_w\{G\}\{G\}^T.
 \tag{F.29}$$

Example. Two coupled oscillators illustrate the details of the computation. The system is

$$[A] = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 1 & -0.02 & 0.02 \\ 1 & -2 & 0.02 & -0.04 \end{pmatrix}, \quad \{B\} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}.$$

Calculate the state transformation matrix $[K]$ to obtain the second realization $[\Phi]$ and $\{G\}$.

The coefficients d_i of the characteristic polynomial $\chi(\lambda)$ are

$$d_1 = 0.06, \quad d_2 = 3.0004, \quad d_3 = 0.04, \quad d_4 = 1.0$$

The observability matrix $[C_{A,B}]$ is

$$[C_{A,B}] = \begin{pmatrix} -0.9992 & -0.02 & 1.0 & 0.0 \\ 0.9988 & 0.02 & 0.0 & 0.0 \\ 0.07996 & -0.9992 & -0.02 & 1.0 \\ -0.119936 & 0.9988 & 0.02 & 0.0 \end{pmatrix},$$

and the controllability matrix $[C_{\Phi,G}]$ is

$$[C_{\Phi,G}] = \begin{pmatrix} 1.0 & 0.0 & 0.0 & 0.0 \\ -0.06 & 1.0 & 0.0 & 0.0 \\ -2.9968 & -0.06 & 1.0 & 0.0 \\ 0.319832 & -2.9968 & -0.06 & 1.0 \end{pmatrix},$$

and the inverse of the controllability matrix $[C_{\Phi,G}]^{-1}$ is

$$[C_{\Phi,G}]^{-1} = \begin{pmatrix} 1.0 & 0.0 & 0.0 & 0.0 \\ 0.06 & 1.0 & 0.0 & 0.0 \\ 3.0004 & 0.06 & 1.0 & 0.0 \\ 0.04 & 3.0004 & 0.06 & 1.0 \end{pmatrix}.$$

Finally the state transformation matrix $[K] = [C_{A,B}][C_{\Phi,G}]^{-1}$ can be obtained:

$$[K] = \begin{pmatrix} 2.0 & 0.04 & 1.0 & 0.0 \\ 1.0 & 0.02 & 0.0 & 0.0 \\ 0.0 & 2.0 & 0.04 & 1.0 \\ 0.0 & 1.0 & 0.04 & 0.0 \end{pmatrix}.$$

Lyapunov's equation

$$[\Phi][Q] + [Q][\Phi]^T = -\{G\}\{G\}^T = -C, \tag{F.30}$$

can be solved more explicitly [65], where the RHS of (F.30) is given by

$$[C] = \begin{pmatrix} 0 & \cdots & 0 & 0 \\ 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 1 \end{pmatrix}, \tag{F.31}$$

and the transformed state matrix $[\Phi]$ and the transformed input vector $\{G\}$ are given in (F.13). The covariance matrix is $[Q] = [E\{s_i s_j\}]$. The expected values are expressed in terms the matrix $[Q]$, $i, j = 1, 2, \dots, 2n$, [65]

$$E\{s_i s_j\} = \begin{cases} 0, & i + j \text{ odd}; \\ (-1)^{\frac{(j-i)}{2}} q_{\frac{i+j}{2}}, & i + j \text{ even}, \end{cases} \tag{F.32}$$

where the numbers q_1, q_2, \dots, q_{2n} are uniquely determined by the $2n$ linear equations

$$\begin{cases} (-1)^k \sum_{\frac{k}{2} \leq s \leq \frac{2n+k}{2}} (-1)^s d_{2n-2s+k} q_{s+1} = 0, & k = 0, 1, \dots, 2n - 2; \\ (-1)^k \sum_{\frac{k}{2} \leq s \leq \frac{2n+k}{2}} (-1)^s d_{2n-2s+k} q_{s+1} = \frac{1}{2}, & k = 2n - 1, \end{cases} \tag{F.33}$$

where $d_0 = 1$.

Example. For $2n = 6$ equations (F.32) and (F.33) states that

$$[E\{s_i s_j\}] = \begin{pmatrix} q_1 & 0 & -q_2 & 0 & q_3 & 0 \\ 0 & q_2 & 0 & -q_3 & 0 & q_4 \\ -q_2 & 0 & q_3 & 0 & -q_4 & 0 \\ 0 & -q_3 & 0 & q_4 & 0 & -q_5 \\ q_3 & 0 & -q_4 & 0 & q_5 & 0 \\ 0 & q_4 & 0 & -q_5 & 0 & q_6 \end{pmatrix},$$

where q_1, q_2, \dots, q_{2n} are determined by the linear equations

$$\begin{pmatrix} d_6 & -d_4 & d_2 & 1 & 0 & 0 \\ 0 & d_5 & -d_3 & d_1 & 0 & 0 \\ 0 & -d_4 & d_4 & -d_2 & 1 & 0 \\ 0 & 0 & -d_5 & d_3 & -d_1 & 0 \\ 0 & 0 & d_6 & -d_4 & d_2 & -1 \\ 0 & 0 & 0 & d_5 & -d_3 & d_1 \end{pmatrix} = \begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \frac{1}{2} \end{pmatrix}.$$

G

Correlation Coefficients

The 3D wave equation is derived in [185] and is given by

$$\nabla^2 p - \frac{\ddot{p}}{c^2} = 0, \quad (\text{G.1})$$

where p is the fluctuation in pressure or sound pressure, the Laplacian $\nabla^2 = \sum_j \frac{\partial^2}{\partial^2 x_j}$, $j = 1, 2, 3$ and $c = \sqrt{\gamma \frac{P_0}{\rho_0}} \approx 340$ m/s the speed of sound with $P_0 = 10^5$ Pa the ambient pressure, $\rho_0 = 1.21$ kg/m³ and $\gamma = 1.4$ is the ratio of the specific heats of the gas.

Using the Fourier transform we can transform the sound pressure into the frequency domain, $p(t) = p(\omega)e^{j\omega t}$ and the Helmholtz equation [185] will be obtained

$$\nabla^2 p(x, y, z, \omega) + k^2 p(x, y, z, \omega) = 0, \quad (\text{G.2})$$

where $k = \frac{\omega}{c}$ is the wave number. The Helmholtz equation in for example only in the x -direction can be written as follows

$$\frac{d^2 p(x, \omega)}{dx^2} + k^2 p(x, \omega) = 0, \quad (\text{G.3})$$

with the general solution

$$p(x, \omega) = Ae^{-jkx} + Be^{jkx}, \quad (\text{G.4})$$

where A and B are arbitrary constants and can be solved using the boundary conditions. The term $Ae^{j(\omega t - kx)}$ is a wave travelling in the positive x -direction and the term $Be^{j(\omega t + kx)}$ is a wave travelling in $-x$ -direction. The waves are periodic and the pressures repeat after $\omega t - kx = 2n\pi$ and $\omega t + kx = 2n\pi$, respectively, where n is an integer. At a given time, the pressure repeats after a distance of $\lambda = 2\pi/k$. The quantity λ is called the wave-length.

We define a vectorial wave number $\{k\} = k\{l\}$ propagating in the positive direction of $\{k\}$ with modulus k . We define also a position vector $\{r\}$.

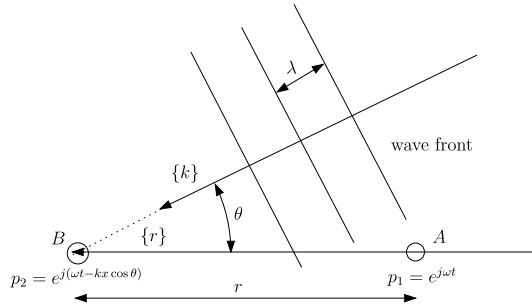


Fig. G.1. Correlation in a plane wave

Any sound wave travelling in the direction of the vector $\{k\}$ gives a sound pressure at the position vector $\{r\}$

$$p(r, t) = Ae^{j(\omega t - \{k\}^T \{r\})}, \tag{G.5}$$

where $\{k\}^T \{r\} = k_1x + k_2y + k_3z$, and $k^2 = k_1^2 + k_2^2 + k_3^2$.

Suppose a plane wave of wavelength λ passes from the point A to B , and θ let be the angle between the vector $\{k\}$ of the wave fronts and the vector $\{r\}$ (the line AB with a length r). This is illustrated in Fig. G.1.

The inner product $\{k\}^T \{r\} = kr \cos \theta$. Then the pressure at point B is given by $p_2 = e^{j(\omega t - kr \cos \theta)}$.

We define the correlation coefficient between the sound pressure at two points A and B in a sound field as [38]

$$R = \frac{\langle p_1 p_2 \rangle_{av}}{[\langle p_1^2 \rangle_{av} \langle p_2^2 \rangle_{av}]^{\frac{1}{2}}}, \tag{G.6}$$

where $p_1(t)$ and $p_2(t)$ are the respective instantaneous sound pressures at time t at the two points, and the angular brackets denote long time average. That means that

$$\langle p_1 p_2 \rangle_{av} = \frac{1}{T} \int_0^T p_1(t) p_2(t) dt. \tag{G.7}$$

Cancelling out the common factor $\frac{1}{T}$ we obtain the correlation coefficient

$$R = \frac{\int_0^T p_1(t) p_2(t) dt}{[\int_0^T p_1^2(t) dt \int_0^T p_2^2(t) dt]^{\frac{1}{2}}}. \tag{G.8}$$

Following the Cauchy-Schwarz inequality [100]

$$\left| \int_0^T p_1(t) p_2(t) dt \right|^2 \leq \left| \int_0^T p_1^2(t) dt \right| \left| \int_0^T p_2^2(t) dt \right|. \tag{G.9}$$

We see that the correlation coefficient is $|R|^2 \leq 1$; that means $-1 \leq R \leq 1$.

In the following calculation of the correlation coefficients we will make use of the relations given below [100]

$$\sin(\alpha \pm \beta) = \sin \alpha \cos \beta \pm \cos \alpha \sin \beta, \quad (\text{G.10})$$

$$\cos(\alpha \pm \beta) = \cos \alpha \cos \beta \mp \sin \alpha \sin \beta, \quad (\text{G.11})$$

$$\int_0^x \sin^2 ay dy = \frac{x}{2} - \frac{\sin 2ax}{4a}, \quad (\text{G.12})$$

$$\int_0^x \cos^2 ay dy = \frac{x}{2} + \frac{\sin 2ax}{4a}. \quad (\text{G.13})$$

If $p_1(t)$ and $p_2(t)$ are both sinusoidal of the same frequency, but differ in phase by ϕ , so that $p_1(t) = \cos \omega t$ and $p_2(t) = \cos(\omega t - \phi)$, then the correlation coefficient $R = \cos \phi$. The integration is done over the period $T = \frac{2\pi}{\omega}$.

The correlation coefficient R between the point A and B in Fig. G.1 becomes $R = \cos(kr \cos \theta)$. The real parts of the pressures p_1 and p_2 respectively in the points A and B are given by

$$p_1(t) = \cos(\omega t), \quad (\text{G.14})$$

$$p_2(t) = \cos(\omega t - kr \cos \theta). \quad (\text{G.15})$$

To obtain the correlation coefficient R the following integrals has to be evaluated with $T = \frac{2\pi}{\omega}$:

$$\int_0^{\frac{2\pi}{\omega}} p_1(t)p_2(t)dt = \frac{\pi}{\omega} \cos(kr \cos \theta), \quad (\text{G.16})$$

$$\int_0^{\frac{2\pi}{\omega}} p_1^2(t)dt = \frac{\pi}{\omega}, \quad (\text{G.17})$$

$$\int_0^{\frac{2\pi}{\omega}} p_2^2(t)dt = \frac{\pi}{\omega}. \quad (\text{G.18})$$

The value of \bar{R} for a random sound field can now be obtained from the definition of such field as having equal weights to all directions of the incident sound: a the correlation coefficient \bar{R} is the average for all directions. This average \bar{R} is

$$\begin{aligned} \bar{R} &= \frac{1}{2\pi} \int_0^\pi \int_0^{2\pi} \cos(kr \cos \theta) \sin \theta d\varphi d\theta \\ &= \frac{\sin(kr)}{kr}, \end{aligned} \quad (\text{G.19})$$

which is the cross-correlation coefficient for the sound pressure at two points, distance r apart, in a random sound field of wave number k .

The derivation of the correlation coefficient \bar{R} can also be found in Henrich Kuttruff's book [104] about room acoustics.

H

Approximations for the Joint Acceptance and Acoustic Response Analysis

H.1 Theory

The theory discussed in this appendix is based on [17, 18]. The method is applied to calculate structural response of sandwich panels of the Herschel spacecraft service module (SVM) exposed to acoustic pressures [211].

A linear elastic plate or shell is exposed to acoustic pressures over large portions of its surface. The undamped equation of motion of the continuous structure is

$$m \frac{\partial^2 w}{\partial t^2} + L(w) = p, \quad (\text{H.1})$$

where $w(x, y, z, t)$ is the displacement field, x, y, z are the spatial coordinates, $p(x, y, z, t)$ is the pressure field on the exposed surface and t is the time, $m(x, y, z)$ is the mass per unit of area and $L()$ is a linear operator representing the load-deflection relationship of the structure.

Example. The linear operator for the out-of-plane bending of a thin flat plate is given for $w(x, y, t)$

$$L(w) = \frac{Et^3}{12(1-\nu^2)} \left(\frac{\partial^4 w}{\partial x^4} + 2 \frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} \right),$$

where E is the Young's modulus, t is the plate thickness, ν is Poisson's ratio, and x, y are the rectangular coordinates.

If the operator $L(w)$ is self-adjoint, then the eigenvalue problem associated with (H.1) can be written as follows

$$\omega_i^2 m \tilde{w}_i - L(\tilde{w}_i) = 0, \quad (\text{H.2})$$

where ω_i^2 is the eigenvalue (ω_i is the natural frequency) for the i th vibration mode (eigenfunction) \tilde{w}_i . The vibration modes are orthogonal over the structure

$$\int_A \tilde{w}_i m \tilde{w}_j ds = 0, \quad i \neq j, \quad \int_A \tilde{w}_i L(\tilde{w}_j) ds = 0, \quad i \neq j, \quad (\text{H.3})$$

assuming no repeated eigenvalues, thus $\omega_i^2 \neq \omega_j^2$. A is the domain of the structure and ds is an element of that domain.

The physical displacement function $w(w, y, z, t)$ will be expressed in terms of time dependent generalized coordinates $\eta_i(t)$ and spatial linear independent eigenfunctions $\tilde{w}_i(x, y, z)$ as follows

$$w(x, y, x, t) = \sum_{i=1}^N \eta_i(t) \tilde{w}_i(x, y, z). \quad (\text{H.4})$$

Substitute the series (H.4) in (H.1), apply the orthogonality relations of (H.3) and (H.2), and subsequently introduce the ad hoc (uncoupled) modal viscous damping. The equation of motion of the continuous structural system can be expressed in terms the generalized coordinates $\eta_i(t)$, to give the following uncoupled equations of motion;

$$\ddot{\eta}_i + 2\zeta_i \omega_i \dot{\eta}_i + \omega_i^2 \eta_i = \frac{\int_A \tilde{w}_i p ds}{\int_A \tilde{w}_i^2 m ds}, \quad i = 1, 2, \dots, N. \quad (\text{H.5})$$

Those are the well known N equations of motion of a viscously damped SDOF systems.

It is reasonable to decompose the spatial physical pressure field $p(x, y, z, t)$ in the same manner as the displacement field, thus

$$p(x, y, x, t) = \sum_{i=1}^N \psi_i(t) \tilde{p}_i(x, y, z). \quad (\text{H.6})$$

In the following analysis we consider the modal structural response η_i in the vicinity of the natural frequency ω_i . It is reasonable to consider each element \tilde{p}_i as representing the behavior of the plate in the neighborhood of the i th vibration mode \tilde{w}_i . Then equations (H.5) become

$$\ddot{\eta}_i + 2\zeta_i \omega_i \dot{\eta}_i + \omega_i^2 \eta_i = \frac{\int_A \tilde{w}_i \psi_i \tilde{p}_i ds}{\int_A \tilde{w}_i^2 m ds} = J_i \psi_i, \quad i = 1, 2, \dots, N, \quad (\text{H.7})$$

where J_i is the *modal joint acceptance* given by

$$J_i = \frac{\int_A \tilde{w}_i \tilde{p}_i ds}{\int_A \tilde{w}_i^2 m ds}, \quad i = 1, 2, \dots, N. \quad (\text{H.8})$$

The following approximations for the modal joint acceptance J_i is proposed by Blevins in [17]. They give approximations for the shape of the surface pressure distributions:

1. $\tilde{p}_i(x, y, z) = 1$. The surface pressure distribution can be approximated as a constant pressure over the surface. This approximation is very simple. It can be expected to be applicable when the acoustic half wavelength exceeds the lateral dimensions of the structure, as is often the case for fundamental modes of plate structures. This method cannot be applied for anti-symmetric modes.
2. $\tilde{p}_i(x, y, z) = \text{Sign}[\tilde{w}_i(x, y, z)]$. The surface pressure distribution can be approximated by $\text{Sign}(\tilde{w}_i)$, a function whose value is either +1 and -1 corresponding to the sign of \tilde{w}_i . The disadvantage is that the square wave form of $\text{Sign}(\tilde{w}_i)$ is not a realistic representation of a pressure field, and numerical integration is required to determine the joint acceptance.
3. $\tilde{p}_i(x, y, z) = [\sin kx, \cos kx]$. The surface pressure distribution can be simulated by a sinusoidal dependence of a travelling wave. This more complex representation requires numerical evaluation of the joint acceptance.
4. $\tilde{p}_i(x, y, z) = m(x, y, z)\tilde{w}_i(x, y, z)$. The surface pressure distribution can be approximated by the mass-weighted structural mode shape. For the fundamental mode of a panel, the estimate is similar to the assumption of uniform pressure. The advantage of the approach is that it tends to produce a conservative estimate for each mode is exactly the modal joint acceptance $J_i = 1$. The validity of the approximation depends on the degree of which the sound field conforms to the vibration mode.

The mass-weighted mode shape will be used for the shape of the pressure field on the surface of the structure, thus

$$\tilde{p}_i = m\tilde{w}_i, \quad (\text{H.9})$$

and the associated joint acceptance (H.8)

$$J_i = 1. \quad (\text{H.10})$$

The modal approximation of the generalized response (H.7) can now be written as

$$\ddot{\eta}_i + 2\zeta_i\omega_i\dot{\eta}_i + \omega_i^2\eta_i = \psi_i, \quad i = 1, 2, \dots, N, \quad (\text{H.11})$$

For a stationary random pressure loading, the single-sided pressure PSD function is $W_p(f)$ (Pa^2/Hz), thus

$$\psi_{rms}^2 = \int_{f_1}^{f_2} W_p(f)df, \quad (\text{H.12})$$

where ψ_{rms} is the root mean square pressure over the frequency band f_1 to f_2 . As the integration limits are spread to encompass the full frequency range $f_1 = 0$ to $f_2 \rightarrow \infty$ the mean square of the generalized coordinate η_i is given by

$$\eta_{i,rms}^2 = \frac{W_p(f_i)}{8\zeta_i(2\pi f_i)^3}, \quad i = 1, 2, \dots, N. \quad (\text{H.13})$$

The dynamic response of plates and shells to surface pressures is assembled on a mode by mode basis, using the single mode relation

$$\frac{w_i}{\tilde{w}_i} = \eta_i, \quad (\text{H.14})$$

where w_i is the physical deflection of the i th mode (H.4). The procedure for calculating the displacement for any mode of the plate or shell exposed to the acoustic loading can be carried out in four steps [17]:

1. Identify the rms acoustic pressure applied to the surface of the structure in the frequency range of interest. In general, the sound pressure level (SPL) are expressed in dB. The rms of the pressure can be calculated using

$$p_{rms}^2 = p_{ref}^2 10^{\frac{SPL}{10}},$$

where the reference pressure is $p_{ref} = 2 \times 10^{-5}$ Pa. The single-sided PSD of the pressure is

$$W_p(f) = \frac{p_{rms}^2}{\Delta f}$$

(Pa²/Hz), where Δf (Hz) is the bandwidth in e.g. the octave or one-third octave band. For the octave band, the band width is $\Delta f_1 = 1.414f_c$; for the one-third octave band, the band width is $\Delta f_{\frac{1}{3}} = 0.232f_c$ and f_c (Hz) is the center frequency.

2. Perform the modal analysis to calculate the natural frequencies $f_i = \frac{\omega_i}{2\pi}$, vibration modes \tilde{w}_i , and associated modal stresses $\tilde{\sigma}_i$ in the frequency range of interest. Estimate a modal damping ratio ζ_i .
3. Select a characteristic point on the structure which will be matched to the acoustic pressure. The applied pressure field is approximated by the mass-weighted mode shape $\tilde{p}_i = m\tilde{w}_i$. Ordinarily this reference point is a point of maximum response in the mode of interest. Calculate a characteristic modal pressure

$$\tilde{p}_i = m(x_r, y_r, z_r) |\tilde{w}_i(x_r, y_r, z_r)|,$$

where x_r , y_r and z_r are the coordinates of the reference point and $m(x_r, y_r, z_r)$ is the mass per unit of area of the thin-walled structure at the reference point.

4. The response to the acoustic pressure field is scaled relative to the modal response using the solution of (H.13). If the pressure field has a random broadband spectrum $W_p(f)$, then

$$\frac{w_{i,rms}(x_r, y_r, z_r, t)}{\tilde{w}_i(x_r, y_r, z_r)} = \frac{\sigma_{i,rms}(x_r, y_r, z_r, t)}{\tilde{\sigma}_i(x_r, y_r, z_r)} = \frac{1}{\tilde{p}_i} \sqrt{\frac{W_p(f_i)}{8\zeta_i(2\pi f_i)^3}}. \quad (\text{H.15})$$

This is a very useful equation for estimating the structural response of plate and shell structures exposed to random pressure fields.

Example. Consider an Al-alloy simply supported rectangular thin plate $a = 210$ mm by $b = 140$ mm and thick $t = 1.2$ mm. The Young's modulus is $E = 70$ GPa, the density is $\rho = 2770$ kg/m³ and the modal damping ratio is $\zeta = 0.017$. The SPL is 145.7 dB ($p_{ref} = 2.0 \times 10^{-5}$ Pa) in the one-third octave band about the center frequency $f_c = 200$ Hz ($f_u = 224$ Hz, $f_l = 178$ Hz).

The PSD of the pressure can be calculated

$$W_p = \frac{p_{rms}^2}{0.232f_c} = 3202.9 \text{ Pa}^2/\text{Hz}.$$

The natural frequencies (Hz) of the simply supported plate are given by

$$f_{ij} = \frac{\pi}{2} \sqrt{\frac{D}{m}} \left[\left(\frac{i}{a} \right)^2 + \left(\frac{j}{b} \right)^2 \right], \quad i, j = 1, 2, \dots,$$

and the corresponding vibration modes are

$$\tilde{w}_{ij}(x, y) = \sin \frac{i\pi x}{a} \sin \frac{j\pi y}{b}, \quad i, j = 1, 2, \dots,$$

where the bending stiffness of the plate is $D = \frac{Eh^3}{12(1-\nu^2)}$ (Nm²) and $m = \rho h$ is the mass per unit of area (kg/m²).

The corresponding modal stresses are

$$\tilde{\sigma}_x = \frac{6\pi^2 D(j^2 b^2 + \nu j^2 a^2) \sin \frac{i\pi x}{a} \sin \frac{j\pi y}{b}}{a^2 b^2 h^2}, \quad i, j = 1, 2, \dots,$$

and

$$\tilde{\sigma}_y = \frac{6\pi^2 D(j^2 a^2 + \nu i^2 b^2) \sin \frac{i\pi x}{a} \sin \frac{j\pi y}{b}}{a^2 b^2 h^2}, \quad i, j = 1, 2, \dots$$

The mass-weighted pressure field is

$$\tilde{p} = m \sin \frac{i\pi x}{a} \sin \frac{j\pi y}{b}.$$

For $i = 1$, $j = 1$, $x = \frac{a}{2}$, $y = \frac{b}{2}$ and $\nu = 0.33$ the following results can be calculated:

- $f_{11} = 213.6$ Hz
- $\tilde{\sigma}_x = 1.8380 \times 10^{10}$ Pa
- $\tilde{\sigma}_y = 2.7215 \times 10^{10}$ Pa
- $\tilde{p} = 3.3240$ kg/m

The stress at the center of the simply supported plate can now being calculated using

$$\sigma_{x,rms} = \frac{\tilde{\sigma}_x}{\tilde{p}} \sqrt{\frac{W_p}{8\zeta(2\pi f_{11})^3}} = 1.7265 \times 10^7 \text{ Pa},$$

and

$$\sigma_{y,rms} = \frac{\tilde{\sigma}_y}{\tilde{p}} \sqrt{\frac{W_p}{8\zeta(2\pi f_{11})^3}} = 2.5564 \times 10^7 \text{ Pa.}$$

The rms acceleration at the center of the simply supported plate is

$$a_{rms} = \frac{(2\pi f_{11})^2 \tilde{w}(0,0)}{\tilde{p}} \sqrt{\frac{W_p}{8\zeta(2\pi f_{11})^3}} = \frac{\tilde{w}(0,0)}{\tilde{p}} \sqrt{\frac{\pi f_{11} W_p}{4\zeta}} = 172.4 \text{ g.}$$

This is a rather high acceleration, however, the SPL is very severe.

The bending structural wave length λ_b of uniform panels is given by

$$\lambda_b = \sqrt[4]{\frac{Et^3}{12m}} \sqrt{\frac{2\pi}{f}}, \quad (\text{H.16})$$

the shear structural wave length of the uniform panel is

$$\lambda_s = \frac{1}{f} \sqrt{\frac{Et}{2m(1+\nu)}}, \quad (\text{H.17})$$

and the acoustic wave length is

$$\lambda_a = \frac{f}{c} = \frac{2\pi}{k}. \quad (\text{H.18})$$

The coincidence frequency is the frequency at which the acoustic wave length is equal to the bending structural (or shear structural) wave length, $\lambda_a = \lambda_b$, thus

$$f_{coincidence} = \frac{c^2}{2\pi t} \sqrt{\frac{12m}{Et}}. \quad (\text{H.19})$$

Example. An Al-alloy uniform plate is thick $t = 1.25$ mm, the density is $\rho = 2800$ kg/m², the Young's modulus is $E = 70$ GPa and the Poisson's ratio $\nu = 0.33$. The speed of sound in air is $c = 340$ m/s.

The bending structural wave length at $f = 200$ Hz is $\lambda_b = 0.2381$ m, the shear structural wave length at $f = 200$ Hz is $\lambda_s = 17.6777$ m and the coincidence frequency for bending of the uniform plate is $f_{coincidence} = 10197$ Hz. The wave length at $f_{coincidence}$ is $\lambda = 0.0333$ m

Below the coincidence frequency the wave length of the sound field exceeds the wave length of the bending waves of the plate.

Blevins proposed an improvement of the approximation of structural response of plates and shells exposed to an acoustic pressure field by using one-dimensional sinusoidal acoustic and vibration modes, thus for the pressure field

$$\tilde{p}(x) = m \sin(kx + \phi), \quad (\text{H.20})$$

and for the vibration mode

$$\tilde{w}(x) = \sin\left(\frac{i\pi x}{L}\right), \quad (\text{H.21})$$

where ϕ is the phase angle, k the acoustical wave number $k = \frac{2\pi}{\lambda_a}$, L is the characteristic length of the panel (length or width) and the *modal index* $i = \frac{2L}{\lambda_b}$ is called the number of half waves.

The corresponding joint acceptance $J_x(\phi)$ can be calculated and becomes

$$\begin{aligned} J_x(\phi) &= \frac{\int_L \tilde{w}(x)\tilde{p}(x)dx}{\int_L m\tilde{w}(x)^2dx} \\ &= \frac{\sin(kL - i\pi + \phi) - \sin\phi}{kL - i\pi} - \frac{\sin(kL + i\pi + \phi) - \sin\phi}{kL + i\pi}. \end{aligned} \quad (\text{H.22})$$

A sound pressure field is the sum of standing and travelling waves. A simple travelling wave in x -direction can be written as follows

$$P(x, t) = P_0 \sin(kx - \omega t) = P_0 \sin kx \cos \omega t - P_0 \cos kx \sin \omega t, \quad (\text{H.23})$$

where the pressure field is decomposed into two standing waves. This is the solution of the one dimensional wave equation (see Appendix G (G.4)).

The travelling wave is thus the sum of two components, 90 degrees out of phase; the response is also the sum of two phase-shifted components. The magnitude of the response can be calculated by using the magnitude of the joint acceptance as the vector sum of two components

$$J_{x,wave} = \sqrt{J_x^2(\phi = 0) + J_x^2(\phi = \pi/2)}. \quad (\text{H.24})$$

In general, the acoustic and structural waves will travel in both orthogonal directions for plates and shells. The 2-D joint acceptance can be approximated by

$$J_{2D} = J_{x,wave} J_{y,wave}. \quad (\text{H.25})$$

Blevins proved that if the acoustic wavelength is equal to the structural wavelength $\lambda_a = \lambda_b$, the joint acceptance $J_{2D} = 1$. If there is a mismatch between the acoustic and structural wave length by 20% or more, the joint acceptance decays rapidly for the higher modes (large vales of i).

If the acoustic wavelength of a panel exceeds the structural wavelength in a fundamental mode, the maximum value of the two-dimensional joint acceptance is $16/\pi^2 = 1.621$. This can be easily proved, see [211]. The spatial pressure field is assumed as follow:

$$\tilde{p}_1(x, y) = m, \quad (\text{H.26})$$

and the mode shapes in the x - and y -direction are respectively

$$\tilde{w}_1(x) = \sin \frac{\pi x}{a}, \quad \tilde{w}_1(y) = \sin \frac{\pi y}{b}, \quad (\text{H.27})$$

where a is the length and b is the width of the panel. The two-dimensional joint acceptance J_{11} can be written

$$\begin{aligned}
 J_{11} &= \frac{\int \int_A \tilde{p}_1 \tilde{w}_1(x) \tilde{w}_1(y) dA}{\int \int_A m \tilde{w}_1^2(x) \tilde{w}_1^2(y) dA} = \frac{\int \int_A \tilde{w}_1(x) \tilde{w}_1(y) dA}{\int \int_A \tilde{w}_1^2(x) \tilde{w}_1^2(y) dA} \\
 &= \frac{\int_A \tilde{w}_1(x) dx \int_A \tilde{w}_1(y) dy}{\int_A \tilde{w}_1^2(x) dx \int_A \tilde{w}_1^2(y) dy} = \frac{\frac{2a}{\pi} \times \frac{2b}{\pi}}{\frac{a}{2} \times \frac{b}{2}} = \frac{16}{\pi^2}. \tag{H.28}
 \end{aligned}$$

Equation (H.15) can be corrected by introducing the two-dimensional joint acceptance factor J_{2D} as follows

$$\frac{w_{i,rms}(x_r, y_r, z_r, t)}{\tilde{w}_i(x_r, y_r, z_r)} = \frac{\sigma_{i,rms}(x_r, y_r, z_r, t)}{\tilde{\sigma}_i(x_r, y_r, z_r)} = \frac{J_{2D}}{\tilde{p}_i} \sqrt{\frac{W_p(f_i)}{8\zeta_i(2\pi f_i)^3}}. \tag{H.29}$$

An approximate analytical method is discussed for the dynamic response of panels to surface sound pressures. The method is most applicable to relatively uniform flat and curved panels which respond out of plane to the pressures. The method requires the natural frequencies and mode shapes of the panels. The method also requires that the relationship between modal deformation and modal stress be known if stress predictions are wanted. Damping and the PSD of the applied pressure are also required inputs.

The method does not require exact knowledge of the distribution of the applied surface pressures. Corrections are then applied to allow for the influence of the acoustic wavelength relative to the wavelength of the panel modes. The method can be applied for both fundamental and higher modes. This Appendix is based on a paper of Blevins [17].

Problems

H.1. Consider a thin simply supported rectangular plate of length a , width b , thickness t and a constant mass per unit of area m , which is exposed to a sound pressure field. The Young’s modulus is E and the Poisson’s ratio is ν . The vibration modes $\tilde{w}_{ij}(x, y)$, the natural frequencies f_{ij} (Hz) and the modal stress $\tilde{\sigma}_{ij}(x, y)$ distribution are given by

- $f_{ij} = \frac{\omega_{ij}}{2\pi} = \frac{\pi}{2} \left[\frac{i^2}{a^2} + \frac{j^2}{b^2} \right] \sqrt{\frac{Et^3}{12m(1-\nu^2)}} \text{ (Hz)}, i, j = 1, 2 \dots$
- $\tilde{w}_{ij}(x, y) = \sin \frac{i\pi x}{a} \sin \frac{j\pi y}{b}, i, j = 1, 2 \dots$
- $\tilde{\sigma}_{ij}(x, y) = \frac{Et}{2(1-\nu^2)} \left[\frac{(i\pi)^2}{a^2} + \nu \frac{(j\pi)^2}{b^2} \right] \sin \frac{i\pi x}{a} \sin \frac{j\pi y}{b}, i, j = 1, 2 \dots$

Evaluate the modal joint acceptance J_{ij} for the given modes and the various approximations of the pressure field distribution $\tilde{p}_{ij}(x, y)$:

1. $\tilde{p}_{ij}(x, y, z) = m$, constant
2. $\tilde{p}_{ij}(x, y, z) = m \text{ Sign}[\tilde{w}_{ij}(x, y, z)]$, square wave

3. $\tilde{p}_{ij}(x, y, z) = m \sin kx$, sinusoidal wave
4. $\tilde{p}_{ij}(x, y, z) = m \cos kx$, cosine wave
5. $\tilde{p}_{ij}(x, y, z) = m\tilde{w}_{ij}(x, y, z)$, mass-weighted mode shape

Answers:

1. $J_{ij} = \frac{4[-\cos i\pi][1-\cos j\pi]}{\pi^2 i j}$.
2. $J_{ij} = \frac{16}{\pi^2}$
3. $J_{ij} = \frac{2}{j\pi} [1 - \cos j\pi] \left[\frac{\sin(ka-i\pi)}{ka-i\pi} - \frac{\sin(ka+i\pi)}{ka+i\pi} \right]$
4. $J_{ij} = \frac{2}{j\pi} [1 - \cos j\pi] \left[\frac{1-\cos(ka+i\pi)}{ka+i\pi} - \frac{1-\cos(ka-i\pi)}{ka-i\pi} \right]$
5. $J_{ij} = 1$

Calculate the natural frequencies f_{ij} , modal joint acceptances J_{ij} and modal stresses $\tilde{\sigma}_{ij}(x, y)$ in the middle of the plate ($x, y = \frac{a}{2}, \frac{b}{2}$) for $i, j = 1, 2, 3$ for a simply supported rectangular Al-alloy plate, $a = 200$ mm, $b = 125$ mm, $t = 1.25$ mm, the density is $\rho = 2700$ kg/m³, the Young's modulus is $E = 70$ GPa and the Poisson's ratio is $\nu = 0.33$. The speed of sound in air is $c = 340$ m/s and the wave number is given by $k = \frac{\omega_{ij}}{c}$.

Answers: $f_{11} = 272.1$ Hz, wave number $k = 5.03$, wave length $\lambda = \frac{c}{f_{11}} = 1.25$ m, the modal stress $\tilde{\sigma}_x = 1.8380 \times 10^{10}$ Pa, $\tilde{\sigma}_y = 2.7215 \times 10^{10}$ Pa, and the joint acceptances become:

1. $J_{11} = \frac{8}{\pi^2}$
2. $J_{11} = \frac{16}{\pi^2}$
3. $J_{11} = 0.3814$
4. $J_{11} = 0.6934$
5. $J_{11} = 1$

H.2. The deflection of an simply supported circular plate uniformly loaded by unit pressure is given by [207]

$$w(r) = \frac{(R^2 - r^2)}{64D(1 + \nu)} \left(\frac{5 + \nu}{1 + \nu} R^2 - r^2 \right),$$

and the maximum corresponding stresses (radial, tangential) at the center of the plate are

$$\sigma_r = \sigma_t = \frac{3(3 + \nu)R^2}{8t^2},$$

and the mass per unit of area is m , where the bending stiffness is $D = \frac{Et^3}{12(1-\nu^2)}$, R is the radius of the plate, E is Young's modulus, ν is the Poisson's ratio and t the constant thickness of the plate. The deflection $w(r)$ of plate loaded by an unit static pressure is considered to be the vibration mode $\tilde{w}(r)$.

Calculate the natural frequency of the simply supported plate with the aid of Rayleigh's quotient $R(w) \approx \omega_w^2 = \frac{U}{T^*}$, inserting $\nu = 0.33$, where

- the strain energy is $U = \frac{1}{2} \int D \left[\left(\frac{d^2 w}{dr^2} + \frac{1}{r} \frac{dw}{dr} \right)^2 - \frac{2(1-\nu)}{r} \frac{dw}{dr} \frac{d^2 w}{dr^2} \right] r dr d\theta$,

- the kinetic energy is $T^* = \frac{1}{2} \int \int mw^2 r dr d\theta$.

Answer: $f_w = \frac{\omega_w}{2\pi} = 0.2429 \sqrt{\frac{Et^3}{mR^4}}$ (Hz). This answer had been obtained using the software programme MAPLE[®], V9.5.

The modal damping ratio is ζ and the constant single-sided PSD of the pressure is given W_p (Pa²/Hz).

Calculate the rms stresses $\sigma_{r,rms}$ and $\sigma_{t,rms}$ at the center of the simply supported circular plate applying Blevins approximation method assuming a mass-weighted pressure field distribution and an associated joint acceptance $J = 1$.

Answers:

- $\tilde{\sigma}(0)_{r,t} = \frac{3(3+\nu)R^2}{8t^2} = 1.2488 \frac{R^2}{t^2}$
- $\tilde{p} = m|\tilde{w}(0)| = m|w(0)| = 0.6696 \frac{mR^4}{Et^3}$
- $f_w = 0.2429 \sqrt{\frac{Et^3}{mR^4}}$
- $\sigma(0)_{r,t,rms} = \frac{\tilde{\sigma}(0)_{r,t}}{\tilde{p}} \sqrt{\frac{W_p}{8\zeta(2\pi f_w)^3}}$

H.3. The mode shape of a one-dimensional structure is given by

$$\tilde{w}(x) = \sin\left(\frac{i\pi x}{L}\right),$$

and the one-dimensional pressure field is defined as

$$\tilde{p}(x) = m \sin(kx + \phi),$$

where ϕ is the phase angle, k the acoustical wave number $k = \frac{2\pi}{\lambda_a}$, L is the length and the modal index is $i = \frac{2L}{\lambda_b}$ or also called the number of half waves.

The corresponding joint acceptance $J_x(\phi)$ becomes

$$J_x(\phi) = \frac{\sin(kL - i\pi + \phi) - \sin \phi}{kL - i\pi} - \frac{\sin(kL + i\pi + \phi) - \sin \phi}{kL + i\pi}.$$

The joint acceptance is the vector sum of two components, thus

$$J_{x,wave} = \sqrt{J_x^2(\phi = 0) + J_x^2(\phi = \pi/2)}.$$

Calculate $J_{x,wave}$ for $\frac{\lambda_b}{\lambda_a} = 1$ for $i = 1, 2, \dots$.

Answer: $J_{x,wave} = 1.3066$ for all i .

Simplification of Conductance

An elastic 2-D body is excited by a force $F(\omega)$ at (x_0, y_0) and the velocity $v(\omega)$ is observed at location (x, y) . Generally the mobility function $Y(x, y, \omega)$ can be written as follows

$$Y(x, y, \omega) = \frac{j\omega}{M_p} \sum_r \frac{\phi_r(x, y)\phi_r(x_0, y_0)}{[(\omega_r^2 - \omega^2) + j\eta_r\omega_r\omega]}, \quad (\text{I.1})$$

where $\phi_r(x, y)$ is a vibration mode (mode shape) normalized for a uniform mass per unit of area m_p and the total mass $M_p = A_p \times m_p$ such that

$$\frac{1}{A_p m_p} \int_A m_p \phi_i(x, y)\phi_j(x, y)dA = \frac{1}{A_p} \int_A \phi_i(x, y)\phi_j(x, y)dA = \delta_{ij}, \quad (\text{I.2})$$

and ω_r is the radian natural frequency and η_r the loss factor associated with mode r . The point mobility at location (x_0, y_0) can be easily derived from (I.1) and is as follows

$$Y(x_0, y_0, \omega) = \frac{j\omega}{M_p} \sum_r \frac{\phi_r^2(x_0, y_0)}{[(\omega_r^2 - \omega^2) + j\eta_r\omega_r\omega]}. \quad (\text{I.3})$$

The real part of the mobility $\Re\{Y(x_0, y_0, \omega)\}$ can be written

$$\Re\{Y(x_0, y_0, \omega)\} = \frac{1}{M_p} \sum_r \frac{\eta_r \omega_r \omega^2 \phi_r^2(x_0, y_0)}{[(\omega_r^2 - \omega^2)^2 + (\eta_r \omega_r \omega)^2]}. \quad (\text{I.4})$$

The average of the real part of the mobility $\Re\{Y(x_0, y_0, \omega)\}$ can be calculated over all driving points (x_0, y_0) on the plate

$$\langle \Re\{Y(x_0, y_0, \omega)\} \rangle_A = \frac{1}{M_p} \frac{1}{A_p} \int_A \sum_r \frac{\eta_r \omega_r \omega^2 \phi_r^2(x_0, y_0)}{[(\omega_r^2 - \omega^2)^2 + (\eta_r \omega_r \omega)^2]} dA, \quad (\text{I.5})$$

with $\frac{1}{A_p} \int_A \phi_r^2(x_0, y_0) dA = 1$ the average value $\langle \Re\{Y(x_0, y_0, \omega)\} \rangle_A$ becomes

$$\langle \Re\{Y(x_0, y_0, \omega)\} \rangle_A = \frac{1}{M_p} \sum_r \frac{\eta_r \omega_r \omega^2}{[(\omega_r^2 - \omega^2)^2 + (\eta_r \omega_r \omega)^2]}. \quad (\text{I.6})$$

Equation (I.6) describes the average conductance of a structure over all the driving points (x_0, y_0) . An average over the frequency band $\Delta\omega$ will also be taken, thus

$$\langle \Re\{Y(x_0, y_0, \omega)\} \rangle_{A\Delta\omega} = \frac{1}{M_p} \frac{1}{\Delta\omega} \int_{\Delta\omega} \sum_r \frac{\eta_r \omega_r \omega^2}{[(\omega_r^2 - \omega^2)^2 + (\eta_r \omega_r \omega)^2]} d\omega. \quad (\text{I.7})$$

The integral

$$\int_{\Delta\omega} \frac{\omega^2}{[(\omega_r^2 - \omega^2)^2 + (\eta_r \omega_r \omega)^2]} d\omega = \frac{\pi}{2\eta_r \omega_r}, \quad (\text{I.8})$$

because (see Appendix B)

$$\int_{-\infty}^{\infty} \frac{\omega^2}{[(\omega_r^2 - \omega^2)^2 + (\eta_r \omega_r \omega)^2]} d\omega = \frac{\pi}{\eta_r \omega_r}. \quad (\text{I.9})$$

The evaluation of (I.7) will result in

$$\langle \Re\{Y(x_0, y_0, \omega)\} \rangle_{A\Delta\omega} = \frac{1}{M_p} \frac{\pi N(\Delta\omega)}{2\Delta\omega} = \frac{n(\omega)\pi}{2M_p}, \quad (\text{I.10})$$

where N is the number of modes in the frequency band width $\Delta\omega$ and when introducing for the modal density $n(f) = \frac{n(\omega)}{2\pi}$, the following known equation will be obtained and (I.10) becomes

$$\langle \Re\{Y(x_0, y_0, \omega)\} \rangle_{A\Delta\omega} = \frac{n(f)}{4M_p}. \quad (\text{I.11})$$

This means that if the average conductance of all driving points as well as over the frequency band $\Delta\omega = 2\pi\Delta f$ is measured the modal density $n(f)$ can be computed directly using the total mass M_p of the test structure under investigation. The natural frequencies should be lie close to each other in the frequency band.

Modal Density of Composite Sandwich Panels

This appendix J is based on an article of Renji [159].

J.1 Equation of Motion

The vertical equilibrium of an element in a rectangular plate is given by [213]

$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} - p = 0, \quad (\text{J.1})$$

where Q_x is the shear force in a plane x - z , Q_y is the shear force in the y - z plane and p is the uniform pressure load. The equilibrium of the moment about the y -axis is

$$\frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} - Q_x = 0, \quad (\text{J.2})$$

and the equilibrium of the moment about the x -axis is

$$\frac{\partial M_y}{\partial y} + \frac{\partial M_{yx}}{\partial x} - Q_y = 0, \quad (\text{J.3})$$

where M_{xx} , M_{xy} , M_{yx} and M_{yy} are the moment resultants. From the equations (J.2), (J.3) and (J.1) the following equilibrium equation can be derived

$$\frac{\partial^2 M_x}{\partial x^2} + \frac{\partial M_{xy}}{\partial x \partial y} + \frac{\partial M_{yx}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} - p = 0. \quad (\text{J.4})$$

The twisting moments M_{xy} and M_{yx} are equal because

$$M_{xy} = \int_{-\frac{t}{2}}^{\frac{t}{2}} \tau_{xy} z dz = \int_{-\frac{t}{2}}^{\frac{t}{2}} \tau_{yx} z dz = M_{yx}, \quad (\text{J.5})$$

thus (J.4) can be written as follows

$$\frac{\partial^2 M_x}{\partial x^2} + 2\frac{\partial M_{xy}}{\partial x\partial y} + \frac{\partial^2 M_y}{\partial y^2} - p = 0. \tag{J.6}$$

The relation between the moment resultants and the curvatures for a symmetric laminate (balanced lay-up)¹ is [93]

$$\begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \begin{bmatrix} D_{11} & D_{12} & 0 \\ D_{21} & D_{22} & 0 \\ 0 & 0 & D_{66} \end{bmatrix} = \begin{Bmatrix} \frac{\partial\phi_x}{\partial x} \\ \frac{\partial\phi_y}{\partial y} \\ \frac{\partial\phi_x}{\partial y} + \frac{\partial\phi_y}{\partial x} \end{Bmatrix}, \tag{J.7}$$

where ϕ_x and ϕ_y are the rotations of the transverse plane due to bending. To include the effect of transverse shear the Mindlin’s theory is used. The average shear angle is taken to be the rotation of the transverse plane. For a sandwich panel, this theory gives accurate results. Hence, if γ_x and γ_y are the average rotations of the transverse plane due to shear, $\phi_x = \frac{\partial w}{\partial x} - \gamma_x$ and $\phi_y = \frac{\partial w}{\partial y} - \gamma_y$. Equation (J.6) will now rewritten and becomes

$$\begin{aligned} D_{11}\frac{\partial^4 w}{\partial x^4} + 2(D_{12} + 2D_{66})\frac{\partial^4 w}{\partial x^2\partial y^2} + D_{22}\frac{\partial^4 w}{\partial y^4} \\ = D_{11}\frac{\partial^3\gamma_x}{\partial x^3} + (D_{12} + 2D_{66})\frac{\partial^2}{\partial x\partial y}\left(\frac{\partial\gamma_x}{\partial x} - \partial\gamma_y\right) + D_{22}\frac{\partial^3\gamma_y}{\partial y^3} + p. \end{aligned} \tag{J.8}$$

The *shear rigidities* are defined by

$$S_x = G_x h \left(1 + \frac{t}{h}\right)^2, \quad S_y = G_y h \left(1 + \frac{t}{h}\right)^2, \tag{J.9}$$

where G_x and G_y are the shear moduli, h the height of the core and t is the thickness of the face sheet (equal face sheets). In general the “isotropic” shear modulus $G = \sqrt{G_x G_y}$ is taken, $S = S_x = S_y = Gh(1 + \frac{t}{h})^2$.

The average shear angles can be expressed in terms of the shear force divided by the shear rigidity

$$\gamma_x = \frac{Q_x}{S_x}, \quad \gamma_y = \frac{Q_y}{S_y}. \tag{J.10}$$

When (J.10) is substituted in (J.1) the following expression is obtained

$$S\left(\frac{\partial\gamma_x}{\partial x} + \frac{\partial\gamma_y}{\partial x}\right) + p = 0, \tag{J.11}$$

or

$$S\left(\frac{\partial^3\gamma_x}{\partial x^3} + \frac{\partial^3\gamma_y}{\partial x^2\partial y}\right) + \frac{\partial^2 p}{\partial x^2} = 0, \tag{J.12}$$

¹ For an isotropic plate with thickness t , Young’s modulus E and the Poisson’s ratio ν the bending stiffness matrix becomes $D_{11} = D_{22} = D$, $D_{12} = D_{21} = \nu D$, $D_{66} = \frac{1}{2}D(1 - \nu)$ and $D = \frac{Et^3}{12(1-\nu^2)}$.

and

$$S \left(\frac{\partial^3 \gamma_y}{\partial y^3} + \frac{\partial^3 \gamma_x}{\partial x \partial y^2} \right) + \frac{\partial^2 p}{\partial y^2} = 0. \quad (\text{J.13})$$

When equations (J.12) and (J.13) are used, (J.8) becomes now

$$\begin{aligned} & D_{11} \frac{\partial^4 w}{\partial x^4} + 2(D_{12} + 2D_{66}) \frac{\partial^4 w}{\partial x^2 \partial y^2} + D_{22} \frac{\partial^4 w}{\partial y^4} \\ &= -\frac{1}{S} \left(D_{11} \frac{\partial^2 p}{\partial x^2} + D_{22} \frac{\partial^2 p}{\partial y^2} \right) + p \\ &+ (D_{12} + 2D_{66} - D_{11}) \frac{\partial^3 \gamma_y}{\partial x^2 \partial y} + (D_{12} + 2D_{66} - D_{22}) \frac{\partial^3 \gamma_x}{\partial x \partial y^2}. \end{aligned} \quad (\text{J.14})$$

Terms with the third order of derivatives of the shear angles are now neglected and the pressure is replaced by the inertia force $p = -m \frac{\partial^2 w}{\partial t^2}$, thus the simplified equation (J.14) becomes [159]

$$\begin{aligned} & D_{11} \frac{\partial^4 w}{\partial x^4} + 2(D_{12} + 2D_{66}) \frac{\partial^4 w}{\partial x^2 \partial y^2} + D_{22} \frac{\partial^4 w}{\partial y^4} \\ &= -\frac{m}{S} \frac{\partial^2}{\partial t^2} \left(D_{11} \frac{\partial^2 w}{\partial x^2} + D_{22} \frac{\partial^2 w}{\partial y^2} \right) - m \frac{\partial^2 w}{\partial t^2}, \end{aligned} \quad (\text{J.15})$$

where m the mass of the panel per unit of area. Equation (J.15) reflects the equation of motion of a balanced composite sandwich plate, however, some shear effects are neglected because D_{11} and D_{22} are much larger than D_{12} and D_{66} .

J.2 Modal Density

The area below a constant ω curve in the wave number plane is a measure of the number of modes $N(\omega)$, [21, 219]. For a plate with dimensions a and b the mode count is given by

$$N(\omega) = \frac{ab}{2\pi^2} \int_0^{\frac{\pi}{2}} r^2 d\theta, \quad (\text{J.16})$$

where for each value of θ , r is the maximum distance to the curve with a constant ω (see Fig. 4.11). The modal density $n(\omega)$ can be derived from (J.16) and becomes

$$n(\omega) = \frac{dN(\omega)}{d\omega} = \frac{ab}{2\pi^2} \int_0^{\frac{\pi}{2}} \frac{d(r^2)}{d\omega} d\theta \quad (\text{J.17})$$

and r^2 can be obtained from (J.15).

The solution of a finite plate is assumed to be

$$W(x, y, \omega) = C e^{j(\omega t - k_x x - k_y y)}, \quad (\text{J.18})$$

where C is a constant and k_x and k_y are the wave numbers in x and y direction respectively. When (J.18) is substituted into (J.15) the undamped equation of motion of the plate will be transformed into the frequency and wave number domain. And when the deflected W is deleted the following expression is obtained

$$D_{11}k_x^4 + 2(D_{12} + 2D_{66})k_x^2k_y^2 + D_{22}k_y^4 - \frac{\omega^2 m}{S}(D_{11}k_x^2 + D_{22}k_y^2) = m\omega^2. \quad (\text{J.19})$$

The following wave number space $\bar{k}_x = \sqrt[4]{D_{11}}k_x$ and $\bar{k}_y = \sqrt[4]{D_{22}}k_y$ will be constructed. The radius r is related to the wave numbers as follows $r \cos \theta = \bar{k}_x$ and $r \sin \theta = \bar{k}_y$ or $r^2 = \bar{k}_x^2 + \bar{k}_y^2$. Then the mode count become

$$N(\omega) = \frac{ab}{2\pi^2 \sqrt[4]{D_{11}D_{22}}} \int_0^{\frac{\pi}{2}} r^2 d\theta, \quad (\text{J.20})$$

where r^2 can be obtained from (J.19) by using the relationships $r \cos \theta = \sqrt[4]{D_{11}}k_x$ and $r \sin \theta = \sqrt[4]{D_{22}}k_y$. Hence

$$r^4 \left(\sin^4 \theta + \cos^4 \theta + \frac{D_{12} + 2D_{66}}{2\sqrt{D_{11}D_{22}}} \sin^2 2\theta \right) - r^2 \frac{m\omega^2}{S} (\sqrt{D_{11}} \cos^2 \theta + \sqrt{D_{22}} \sin^2 \theta) - m\omega^2 = 0, \quad (\text{J.21})$$

and with

- $\gamma_1^2 = \frac{1}{2} \left\{ 1 - \frac{D_{12} + 2D_{66}}{\sqrt{D_{11}D_{22}}} \right\}$
- $f_1(\theta) = 1 - \gamma_1^2 \sin^2 2\theta$
- $f_2(\theta) = \sqrt[4]{\frac{D_{11}}{D_{22}}} \cos^2 \theta + \sqrt[4]{\frac{D_{22}}{D_{11}}} \sin^2 \theta,$

equation (J.21) can be written as follows

$$f_1(\theta)r^4 - \frac{m\omega^2}{S} \sqrt[4]{D_{11}D_{22}} f_2(\theta)r^2 - m\omega^2 = 0. \quad (\text{J.22})$$

The solution of (J.22) can be easily extracted

$$r_{1,2}^2 = \frac{1}{2f_1(\theta)} \times \left[\frac{m\omega^2}{S} \sqrt[4]{D_{11}D_{22}} f_2(\theta) \pm \sqrt{\left(\frac{m\omega^2}{S} \sqrt[4]{D_{11}D_{22}} f_2(\theta) \right)^2 + 4f_1(\theta)m\omega^2} \right]. \quad (\text{J.23})$$

The function $f_1(\theta) > 0$ thus

$$\sqrt{\left(\frac{m\omega^2}{S} \sqrt[4]{D_{11}D_{22}f_2(\theta)}\right)^2 + 4f_1(\theta)m\omega^2} > \frac{m\omega^2}{S} \sqrt[4]{D_{11}D_{22}f_2(\theta)}, \quad (\text{J.24})$$

hence the positive root $r^2 = r_1^2$ is only applicable. The mode count can be calculated by substituting (J.23) into (J.20)

$$N(\omega) = \frac{abm\omega^2}{4\pi^2 S} \int_0^{\frac{\pi}{2}} \left\{ \frac{f_2(\theta)}{f_1(\theta)} + \frac{1}{f_1(\theta)} \sqrt{\left[f_2^2(\theta) + \frac{4S^2 f_1(\theta)}{m\omega^2 \sqrt{D_{11}D_{22}}} \right]} \right\} d\theta. \quad (\text{J.25})$$

To calculate the modal density $\frac{dr^2}{d\omega}$ (J.17) needs to be evaluated

$$\begin{aligned} \frac{dr^2}{d\omega} = & \frac{1}{2f_1(\theta)} \frac{2m\omega}{S} \sqrt[4]{D_{11}D_{22}} \left[f_2(\theta) \right. \\ & \left. + \frac{1}{2} \frac{2m\omega^2 f_2^2(\theta) + \frac{4f_1(\theta)S^2}{\sqrt{D_{11}D_{22}}}}{\sqrt{m^2\omega^4 f_2^2(\theta) + \frac{4m\omega^2 f_1(\theta)S^2}{\sqrt{D_{11}D_{22}}}}} \right]. \end{aligned} \quad (\text{J.26})$$

The modal density $n(\omega)$ can be derived from (J.20):

$$n(\omega) = \frac{ab}{2\pi^2 \sqrt[4]{D_{11}D_{22}}} \int_0^{\frac{\pi}{2}} \frac{dr^2}{d\theta} d\theta. \quad (\text{J.27})$$

Equation (J.26) will be substituted in (J.27) and the expression for the modal density $n(\omega)$ becomes

$$n(\omega) = \frac{abm\omega}{2\pi^2 S} \int_0^{\frac{\pi}{2}} \left[\frac{f_2(\theta)}{f_1(\theta)} + \frac{1}{f_1(\theta)} \frac{m\omega^2 f_2^2(\theta) + \frac{2f_1(\theta)S^2}{\sqrt{D_{11}D_{22}}}}{\sqrt{m^2\omega^4 f_2^2(\theta) + \frac{4m\omega^2 f_1(\theta)S^2}{\sqrt{D_{11}D_{22}}}}} \right] d\theta, \quad (\text{J.28})$$

or

$$n(f) = \frac{2abmf}{S} \int_0^{\frac{\pi}{2}} \left[\frac{f_2(\theta)}{f_1(\theta)} + \frac{1}{f_1(\theta)} \frac{m(2\pi f)^2 f_2^2(\theta) + \frac{2f_1(\theta)S^2}{\sqrt{D_{11}D_{22}}}}{\sqrt{m^2(2\pi f)^4 f_2^2(\theta) + \frac{4m(2\pi f)^2 f_1(\theta)S^2}{\sqrt{D_{11}D_{22}}}}} \right] d\theta. \quad (\text{J.29})$$

If no shear is considered (J.22) can be simplified assuming $S \rightarrow \infty$

$$f_1(\theta)r^4 - m\omega^2 = 0. \quad (\text{J.30})$$

The root r^2 of interest is

$$r^2 = \frac{1}{f_1(\theta)} \sqrt{4m\omega^2 f_1(\theta)}. \quad (\text{J.31})$$

The derivative $\frac{dr^2}{d\omega}$ can be easily obtained

$$\frac{dr^2}{d\omega} = \frac{1}{f_1(\theta)} \frac{4m\omega f_1(\theta)}{\sqrt{4m\omega^2 f_1(\theta)}}. \quad (\text{J.32})$$

We can calculate the modal density $n(\omega)$ by substituting (J.32) into (J.17)

$$\begin{aligned}
 n(\omega) &= \frac{ab}{2\pi^2 \sqrt[4]{D_{11}D_{22}}} \int_0^{\frac{\pi}{2}} \frac{1}{f_1(\theta)} \frac{2\sqrt{m}f_1(\theta)}{\sqrt{f_1(\theta)}} d\theta \\
 &= \frac{ab}{2\pi^2} \sqrt{\frac{m}{D_{11}}} \sqrt[4]{\frac{D_{11}}{D_{22}}} \int_0^{\frac{\pi}{2}} \frac{d(2\theta)}{\sqrt{f_1(\theta)}} \\
 &= \frac{ab}{2\pi^2} \sqrt{\frac{m}{D_{11}}} \sqrt[4]{\frac{D_{11}}{D_{22}}} F\left(\gamma_1, \frac{\pi}{4}\right), \tag{J.33}
 \end{aligned}$$

where $F(\gamma_1, \frac{\pi}{4})$ is the elliptic integral of the first kind, with $\gamma_1^2 = \frac{1}{2}\{1 - \frac{D_{12}+2D_{66}}{\sqrt{D_{11}D_{22}}}\}$, $f_1(\theta) = 1 - \gamma_1^2 \sin^2 2\theta$, is defined by [90]

$$F(k, \phi) = \int_0^\phi \frac{dx}{\sqrt{1 - k^2 \sin^2 x}}. \tag{J.34}$$

The modal density $n(f)$ becomes

$$n(f) = \frac{ab}{\pi} \sqrt{\frac{m}{D_{11}}} \sqrt[4]{\frac{D_{11}}{D_{22}}} F\left(\gamma_1, \frac{\pi}{4}\right). \tag{J.35}$$

In [22] the following approximate expression for the modal density of an orthotropic plate is given

$$n(f) = \frac{\sqrt{m}A}{4} \left(\frac{1}{\sqrt{D_{11}}} + \frac{1}{\sqrt{D_{22}}} \right), \tag{J.36}$$

where A is the area of the plate.

Example. A sandwich panel has dimensions $a = 2.5$ m and $b = 2.0$ m. The total mass is $M = 15$ kg. The flexural and shear rigidities of the sandwich panel are $D_{11} = D_{22} = 5000$ Nm, $D_{12} = 70$ Nm, $D_{66} = 165$ Nm and $S = 15$ N/m. Calculate the modal densities of the sandwich panel in the octave band (32.5–8000 Hz). Calculate the modal densities of the same plate, however, $S \rightarrow \infty$, again in the octave band. The calculated modal densities are plotted in Fig. J.1.

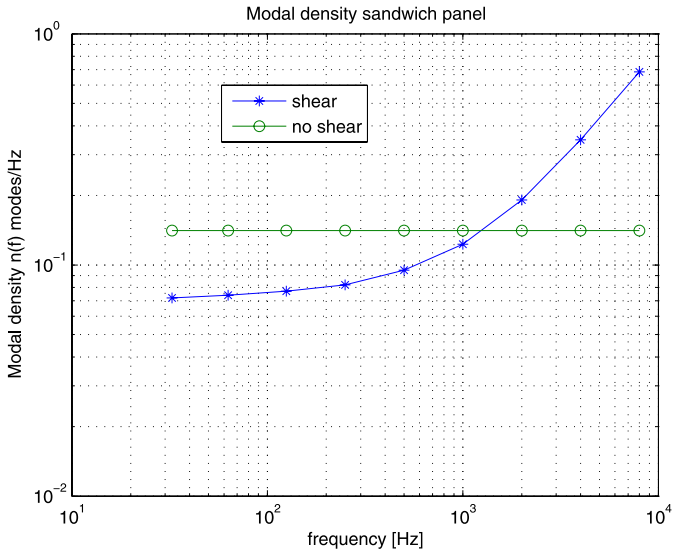


Fig. J.1. Modal densities sandwich panel

K

Wave Propagation and Group Velocity

The theory about phase velocity and group velocity is based on the book of Brillouin [23] and [124].

The usual phase velocity c_p of waves is defined as the phase difference between the vibrations observed at two different locations in a free plane wave. In a wave

$$\psi = A \cos(\omega t - kx) = A \cos \left[\omega \left(t - \frac{x}{c_p} \right) \right], \quad (\text{K.1})$$

where the phase velocity c_p is

$$c_p = \frac{\omega}{k}. \quad (\text{K.2})$$

Another velocity can be defined considering the propagation of a train of waves with a changing amplitude. This is called the modulation impressed on a carrier. The modulation results in building up some “groups” (wave-packets) of large amplitude which moves along with the group velocity c_g . A simple combination of groups obtains when two waves

$$\omega_1 = \omega + \Delta\omega, \quad k_1 = k + \Delta k, \quad (\text{K.3})$$

and

$$\omega_2 = \omega - \Delta\omega, \quad k_2 = k - \Delta k, \quad (\text{K.4})$$

are superimposed resulting in

$$\begin{aligned} \psi &= A \cos(\omega_1 t - k_1 x) + A \cos(\omega_2 t - k_2 x) \\ &= 2A \cos(\omega t - kx) \cos(\Delta\omega t - \Delta kx). \end{aligned} \quad (\text{K.5})$$

The wave in (K.5) represents a carrier with frequency ω and a modulation with frequency $\Delta\omega$. This wave may be described as a succession of moving beats (or groups, or wave-packets). The carrier’s phase velocity is $c_p = \frac{\omega}{k}$ while the group velocity c_g is given by

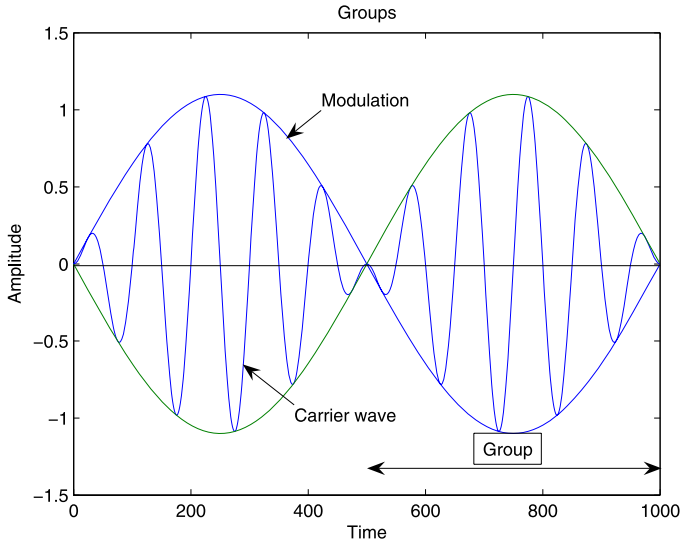


Fig. K.1. Series of wavelets

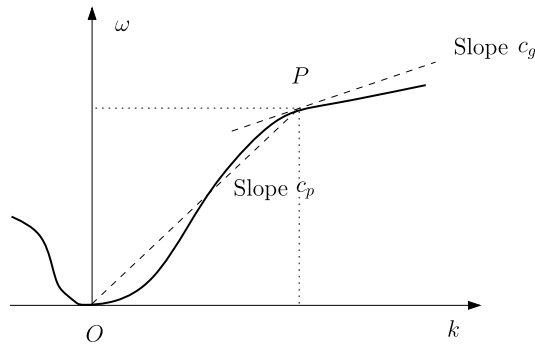


Fig. K.2. k - ω diagram [23]

$$c_g = \frac{\Delta\omega}{\Delta k} = \frac{\partial\omega}{\partial k}, \quad \Delta k \rightarrow 0. \tag{K.6}$$

The situation is represented in Fig. K.1 where is shown the series of wavelets (ω, k) with variable amplitude ($\Delta\omega, \Delta k$). A very useful graphical representation is created when the frequency ω is plotted against the wave number k as shown Fig. K.2. The slope of the chord OP gives the phase velocity c_p , while the slope of the tangent at point P yields the group velocity c_g .

The frequency ω can be expressed in terms of the wave number k and the phase velocity c_p

$$\omega = kc_p, \tag{K.7}$$

hence

$$c_g = c_p + k \frac{\partial c_p}{\partial k} = c_p - \lambda \frac{\partial c_p}{\partial \lambda}, \quad (\text{K.8})$$

where the wave length $\lambda = k2\pi$. A medium exhibiting a phase velocity is called a dispersive medium.

Example. The homogeneous undamped equation of motion of a bending beam is given by

$$EI \frac{\partial^4 w}{\partial x^4} + m \frac{\partial^2 w}{\partial t^2} = 0, \quad (\text{K.9})$$

where EI is the bending stiffness, w is the deflection and m is the mass per unit of length.

The assumed solution for the deflection is

$$w(x, t) = Ae^{-j(\omega t - kx)}, \quad (\text{K.10})$$

and will be substituted in the undamped equation of motion. This equation becomes

$$(EI k^4 - \omega^2 m) Ae^{-j(\omega t - kx)} = 0.$$

The wave number k becomes

$$k = \sqrt{\omega} \sqrt[4]{\frac{m}{EI}}. \quad (\text{K.11})$$

The phase speed c_p in the bending beam can now calculated

$$c_p = \frac{\omega}{k} = \sqrt{\omega} \sqrt[4]{\frac{EI}{m}}, \quad (\text{K.12})$$

and the groups velocity becomes

$$c_g = \frac{\partial \omega}{\partial k} = 2c_p. \quad (\text{K.13})$$

Summary of Wave Relations [52]

The following is a collection of terms often used in wave analysis:

ω angular frequency (rad/s)

f cyclic frequency $f = \frac{\omega}{2\pi}$ (Hz)

T period $T = \frac{1}{f} = \frac{2\pi}{\omega}$ (s)

k wave number $k = \frac{2\pi}{\lambda} = \frac{\omega}{c_p}$ (1/m)

λ wave length $\lambda = \frac{2\pi c_p}{\omega} = \frac{2\pi}{k}$ (m)

ϕ phase of wave $\phi = kx - \omega t = \frac{\omega c_p}{c_p}(x - c_p t) = \frac{2\pi}{\lambda}(x - c_p t)$ (rad)

c_p phase velocity $c_p = \frac{\omega}{k} = \frac{\omega \lambda}{2\pi}$ (m/s)

c_g group velocity $c_g = \frac{d\omega}{dk}$ (m/s)

The vibration of a bending beam is considered to show the relation between power and energy [96].

The undamped equation of motion of a bending beam is given by (K.9) and the assumed solution of the infinite long beam is given by (K.10)

$$w(x, t) = W e^{-j(kx - \omega t)}.$$

The velocity $\dot{w}(x, t)$ is

$$\dot{w}(x, t) = \frac{\partial w(x, t)}{\partial t} = j\omega W e^{-j(kx - \omega t)}, \quad (\text{K.14})$$

and the angular velocity $\frac{\dot{w}(x, t)}{\partial x}$ becomes

$$\frac{\dot{w}(x, t)}{\partial x} = k\omega W e^{-j(kx - \omega t)}. \quad (\text{K.15})$$

The bending moment $M(x)$ in the bending beam is defined as

$$M(x) = -EI \frac{\partial^2 w(x, t)}{\partial x^2} = k^2 EI W e^{-j(kx - \omega t)}. \quad (\text{K.16})$$

The shear force $D(x)$ becomes

$$D(x) = -\frac{\partial M(x)}{\partial x} = -jk^3 EI W e^{-j(kx - \omega t)}. \quad (\text{K.17})$$

The average power Π applied to the beam

$$\Pi = \frac{1}{2} \Re\{D(x, \omega) \dot{w}^*(x, \omega)\} + \frac{1}{2} \Re\left\{M(x, \omega) \frac{\dot{w}(x, \omega)}{\partial x}\right\}, \quad (\text{K.18})$$

or

$$\Pi = \omega k^3 EI W^2. \quad (\text{K.19})$$

The power input Π is independent of x and t .

The total energy E_{tot} in the beam per unit of length is given by

$$E_{tot} = \frac{1}{2} m \dot{w}^2(x, t) + \frac{1}{2} EI \left(\frac{\partial^2 w(x, t)}{\partial x^2} \right)^2, \quad (\text{K.20})$$

or

$$\begin{aligned} E_{tot} &= \frac{1}{2} m \frac{1}{2} \Re\{\dot{w}(x, \omega) \dot{w}^*(x, \omega)\} \\ &\quad + \frac{1}{2} EI \frac{1}{2} \Re\left\{ \frac{\partial^2 w(x, \omega)}{\partial x^2} \frac{\partial^2 w^*(x, \omega)}{\partial x^2} \right\}. \end{aligned} \quad (\text{K.21})$$

The total energy per unit of length of the beam E_{tot} becomes

$$E_{tot} = \frac{1}{4} m \omega^2 W^2 + \frac{1}{4} EI k^4 W^2. \quad (\text{K.22})$$

The wave number k is given by (K.11)

$$k = \sqrt{\omega} \sqrt[4]{\frac{m}{EI}},$$

thus the total energy E_{tot} can now be written as follows

$$E_{tot} = \frac{1}{2}EI k^4 W^2. \tag{K.23}$$

Rewriting (K.23) the following expression is obtained

$$E_{tot} = \frac{k}{2\omega} (EI\omega k^3 W^2) = \frac{1}{2c_p} (\omega EI k^3 W^2) = \frac{1}{c_g} \Pi, \tag{K.24}$$

or

$$\Pi = c_g E_{tot}. \tag{K.25}$$

The power Π flows at group velocity.

The term energy density Ξ will now be introduced, energy per unit of length, area, or volume and the energy density is the sum of the kinetic and strain energy and can be written as

$$\Xi = \rho \left(\frac{\partial w}{\partial t} \right)^2 = \rho \omega^2 W^2, \tag{K.26}$$

where ρ is the density. The power intensity $\$$ is the power per unit width or area as the waves propagate. The power intensity can be expressed as follows

$$\$ = c_g \Xi. \tag{K.27}$$

The undamped equation of motion of a general continuous structure (beam, plate, etc.) can be written as follows

$$m(\bar{x}) \frac{\partial^2 w(\bar{x}, t)}{\partial t^2} + \Lambda(\bar{x}) w(\bar{x}, t) = 0, \tag{K.28}$$

where \bar{x} is the position vector, $w(\bar{x}, t)$ the deflection, $m(\bar{x})$ is the mass per unit of length, area or volume and $\Lambda(\bar{x})$ a differential operator. Substituting (K.10) in (K.28) the following expression will be obtained

$$-m(\bar{x})\omega^2 W + \Lambda(-jk)W = 0, \tag{K.29}$$

thus the frequency

$$\omega = \sqrt{\frac{\Lambda(-jk)}{m(\bar{x})}}. \tag{K.30}$$

The group velocity c_g can be expressed into the differential operator Λ

$$c_g = \frac{d\omega}{dk} = \frac{1}{2} \left[\frac{1}{m(\bar{x})} \Lambda(-jk) \right]^{-\frac{1}{2}} \frac{-j}{m(\bar{x})} \dot{\Lambda}(-jk), \tag{K.31}$$

where $\dot{\Lambda}(-jk)$ is derivative op the differential operator with respect to $-jk$. Substituting (K.30) into (K.31) will give

$$c_g = \frac{-j}{2\omega m(\bar{x})} \dot{\Lambda}(-jk). \tag{K.32}$$

Example. For a bending beam $\Lambda(\bar{x}) = EI \frac{\partial^4}{\partial x^4}$, $\Lambda(-jk) = EI(-jk)^4$, $\dot{\Lambda}(-jk) = 4EI(-jk)^3$ and $m(\bar{x}) = m$. The frequency is $\omega = \sqrt{\frac{EI k^4}{m}}$ and the group velocity becomes $c_g = \frac{-j}{2\omega m(\bar{x})} \dot{\Lambda}(jk) = 2\sqrt{\omega} \sqrt[4]{\frac{EI}{m}}$.

Relationship Modal Energy and Energy Striking the Boundary

The vibration field that results from the repeated reflections of the outgoing waves from the boundaries of a plate is called a reverberant field [96]. It is comprised of modal response. Let’s find the relationship between the modal energy and the energy striking the boundary of a plate. Suppose the mean square velocity of the reverberant field is $\langle v^2 \rangle$. The intensity \$ over the differential angle $d\theta$ is

$$d\$ = \rho c_g W_v(f) \frac{d\theta}{2\pi}, \tag{K.33}$$

where ρ is the density of the plate, c_g is the group velocity and $W_v(f)$ is the PSD of the velocity. The normal component of the intensity $d\$_n$ into the boundary is illustrated in Fig. K.3 and is given by

$$d\$_n = d\$ \cos \theta. \tag{K.34}$$

The total power per length $\$ _n$ normal to the boundary is obtained integrating between $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2} \leq \theta$:

$$\$ _n = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \rho c_g W_v(f) \cos \theta \frac{d\theta}{2\pi} = \frac{\rho c_g W_v(f)}{\pi}. \tag{K.35}$$

The modal energy e_{modal} is given by

$$e_{modal} = \frac{M \langle v^2 \rangle}{n(f) \Delta f} = \frac{M W_v(f)}{n(f)}, \tag{K.36}$$

where M is the total mass, $\langle v^2 \rangle$ is the average mean square velocity, $n(f)$ is the modal density and Δf the bandwidth. Introduction of expressions for the modal density $n(f)$, the group velocity c_g and bending waves k of the plate in (K.35) and (K.36) one can show that the bending and modes of the plate are related to each other as follows

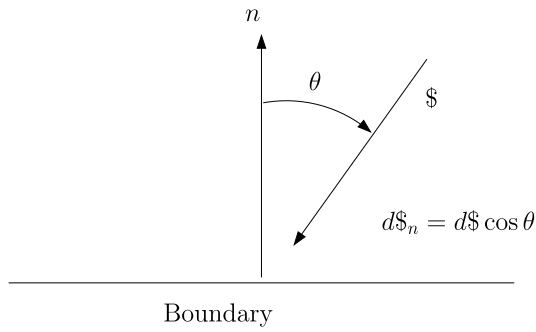


Fig. K.3. Intensity into boundary

$$\$ _n = \frac{k}{\pi} e_{modal}. \tag{K.37}$$

This expression will be applied deriving the coupling loss factor of two or more connected plates.

L

Finite Difference Approximations for Various Orders

By using higher-order Taylor series expansion and more discretization points, high-order accuracy finite difference schemes are derived by Wojtkiewicz in [227].

L.1 Introduction

Let

$$f(x + i\Delta x, y + j\Delta y) = f_{i,j}, \quad (\text{L.1})$$

$$\sum^m = f_{i+m,j} + f_{i-m,j}, \quad (\text{L.2})$$

$$\sum_k = f_{i,j+k} + f_{i,j-k}, \quad (\text{L.3})$$

$$\delta_m^x = f_{i+m,j} - f_{i-m,j}, \quad (\text{L.4})$$

and

$$\delta_k^y = f_{i,j+k} - f_{i,j-k}. \quad (\text{L.5})$$

L.2 Second Order Approximation

$$\frac{\partial^2 f_{i,j}}{\partial x^2} = \frac{\sum^1 - \sum^0}{\Delta x^2}, \quad (\text{L.6})$$

$$\frac{\partial^2 f_{i,j}}{\partial y^2} = \frac{\sum_1 - \sum_0}{\Delta y^2}, \quad (\text{L.7})$$

$$\frac{\partial f_{i,j}}{\partial x} = \frac{\delta_1^x - \delta_0^x}{2\Delta x}, \quad (\text{L.8})$$

$$\frac{\partial f_{i,j}}{\partial y} = \frac{\delta_1^y - \delta_0^y}{2\Delta y}. \quad (\text{L.9})$$

L.3 Fourth Order Approximation

$$\frac{\partial^2 f_{i,j}}{\partial x^2} = \frac{16 \sum^1 - \sum^2 - 15 \sum^0}{12 \Delta x^2}, \tag{L.10}$$

$$\frac{\partial^2 f_{i,j}}{\partial y^2} = \frac{16 \sum_1 - \sum_2 - 15 \sum_0}{12 \Delta y^2}, \tag{L.11}$$

$$\frac{\partial f_{i,j}}{\partial x} = \frac{8\delta_1^x - \delta_2^x}{12 \Delta x}, \tag{L.12}$$

$$\frac{\partial f_{i,j}}{\partial y} = \frac{8\delta_1^y - \delta_2^y}{12 \Delta y}. \tag{L.13}$$

L.4 Sixth Order Approximation

$$\frac{\partial^2 f_{i,j}}{\partial x^2} = \frac{270 \sum^1 - 27 \sum^2 + 2 \sum^3 - 245 \sum^0}{180 \Delta x^2}, \tag{L.14}$$

$$\frac{\partial^2 f_{i,j}}{\partial y^2} = \frac{270 \sum_1 - 27 \sum_2 + 2 \sum_3 - 245 \sum_0}{180 \Delta y^2}, \tag{L.15}$$

$$\frac{\partial f_{i,j}}{\partial x} = \frac{45\delta_1^x - 9\delta_2^x + \delta_3^x}{60 \Delta x}, \tag{L.16}$$

$$\frac{\partial f_{i,j}}{\partial y} = \frac{45\delta_1^y - 9\delta_2^y + \delta_3^y}{60 \Delta y}. \tag{L.17}$$

L.5 Eighth Order Approximation

$$\frac{\partial^2 f_{i,j}}{\partial x^2} = \frac{8064 \sum^1 - 1008 \sum^2 + 128 \sum^3 - 9 \sum^4 - 7175 \sum^0}{5040 \Delta x^2}, \tag{L.18}$$

$$\frac{\partial^2 f_{i,j}}{\partial y^2} = \frac{8064 \sum_1 - 1008 \sum_2 + 128 \sum_3 - 9 \sum_4 - 7175 \sum_0}{5040 \Delta y^2}, \tag{L.19}$$

$$\frac{\partial f_{i,j}}{\partial x} = \frac{672\delta_1^x - 168\delta_2^x + 32\delta_3^x - 3\delta_4^x}{840 \Delta x}, \tag{L.20}$$

$$\frac{\partial f_{i,j}}{\partial y} = \frac{672\delta_1^y - 168\delta_2^y + 32\delta_3^y - 3\delta_4^y}{840 \Delta y}. \tag{L.21}$$

L.5.1 Tenth Order Approximation

$$\frac{\partial^2 f_{i,j}}{\partial x^2} = \frac{42\,000 \sum^1 - 6\,000 \sum^2 + 1\,000 \sum^3 - 125 \sum^4 + 8 \sum^5 - 36\,883 \sum^0}{25\,200 \Delta x^2}, \tag{L.22}$$

$$\frac{\partial^2 f_{i,j}}{\partial y^2} = \frac{42\,000 \sum_1 - 6000 \sum_2 + 1000 \sum_3 - 125 \sum_4 + 8 \sum_5 - 36\,883 \sum_0}{25200 \Delta y^2}, \quad (\text{L.23})$$

$$\frac{\partial f_{i,j}}{\partial x} = \frac{2100\delta_1^x - 600\delta_2^x + 150\delta_3^x - 25\delta_4^x + 2\delta_5^x}{2520 \Delta x}, \quad (\text{L.24})$$

$$\frac{\partial f_{i,j}}{\partial y} = \frac{2100\delta_1^y - 600\delta_2^y + 150\delta_3^y - 25\delta_4^y + 2\delta_5^y}{2520 \Delta y}. \quad (\text{L.25})$$

L.5.2 Stability

The simple convection-diffusion equation will be considered to discuss the stability of the explicit two-level finite difference scheme. The simple FKP equation is denoted by

$$\frac{\partial f}{\partial t} = -u \frac{\partial f}{\partial x} + D \frac{\partial^2 f}{\partial x^2}. \quad (\text{L.26})$$

The corresponding finite difference scheme is

$$\frac{f_{j,k+1} - f_{j,k}}{\Delta t} = -u \frac{f_{j+1,k} - f_{j-1,k}}{2\Delta x} + D \frac{f_{j+1,k} - 2f_{j,k} + f_{j-1,k}}{\Delta x^2}, \quad (\text{L.27})$$

or

$$f_{j,k+1} = f_{j,k} - \beta \frac{f_{j+1,k} - f_{j-1,k}}{2\Delta x} + \alpha \frac{f_{j+1,k} - 2f_{j,k} + f_{j-1,k}}{\Delta x^2}, \quad (\text{L.28})$$

where $k = t$, $k + 1 = t + \Delta t$, $j = x$, $j + 1 = x + \Delta x$, $j - 1 = x - \Delta x$ and further $\beta = \frac{u\Delta t}{2\Delta x}$ is the Courant-Friedrichs-Lewy (CFL) number and the coefficient $\alpha = \frac{D\Delta t}{\Delta x^2}$, both are typical measures for the advection (bringing) and diffusion velocities relative to the characteristic propagation $\frac{\Delta x}{\Delta t}$.

It can be proven that the explicit two-dimensional finite difference scheme is numerically stable when, [201],

$$\alpha = \frac{D\Delta t}{\Delta x^2} < \frac{1}{2}, \quad \beta = \frac{u\Delta t}{2\Delta x} < 1, \quad (\text{L.29})$$

or

$$\Delta t < \min\left(\frac{2\Delta x}{u}, \frac{\Delta x^2}{2D}\right). \quad (\text{L.30})$$

Jain in [88] derived the following stability criteria

$$\frac{u\Delta t}{\Delta x} < 1, \quad \frac{u\Delta t}{D} \leq 2. \quad (\text{L.31})$$

Problems

L.1. The following nonlinear problem is given by the Duffing’s equation

$$\ddot{x} + 2\zeta\dot{x} + \gamma x + \varepsilon x^3 = w(t),$$

where the white noise excitation is fully defined by the first and second moments

$$E\{w(t)\} = 0, \quad E\{w(t)w(t + \tau)\} = 2D\delta(\tau).$$

The state-space variables are defined by

$$x_1 = x, \quad x_2 = \dot{x}.$$

The space state equation is as follows

$$\{\dot{\mathbf{x}}\} = [m(\mathbf{x})] + \{G(\mathbf{x})\}w(t),$$

where $[\mathbf{x}] = [x_1, x_2]$.

Generate the drift matrix $[m(\mathbf{x})]$ and the diffusion coefficients $\{G(\mathbf{x})\}$.

Answers: $[m(\mathbf{x})] = \begin{pmatrix} x_2 \\ -2\zeta x_2 - \gamma x_1 - \varepsilon x_1^3 \end{pmatrix}$, $\{G(\mathbf{x})\} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

Derive the associated FPK equation and initial, boundary and normalization conditions.

Answers: $\frac{\partial f_{c2}}{\partial t} = \frac{\partial^2}{\partial x_2^2}[Df_{c2}] - \frac{\partial}{\partial x_1}[x_2f_{c2}] + \frac{\partial}{\partial x_2}[\{2\zeta x_2 + \gamma x_1 + \varepsilon x_1^3\}f_{c2}]$,
 $\lim_{t \rightarrow 0} f_{c2}(\mathbf{x}_0|\mathbf{x}, t) = \delta(\mathbf{x}_0 - \mathbf{x})$, $\lim_{|x_1|, |x_2| \rightarrow \infty} f_{c2}(\mathbf{x}_0|\mathbf{x}, t) = 0$,
 $\int \int_{-\infty}^{\infty} f_{c2}(\mathbf{x}_0|\mathbf{x}, t) d\mathbf{x} = 1$, $f_2(\mathbf{x}, t) = \int \int_{-\infty}^{\infty} f_{c2}(\mathbf{x}_0|\mathbf{x}, t) f_2(\mathbf{x}_0) d\mathbf{x}_0$ and
 $\lim_{t \rightarrow 0} f_2(\mathbf{x}, t) = \int \int_{-\infty}^{\infty} f_{c2}(\mathbf{x}_0|\mathbf{x}, t) f_2(\mathbf{x}_0) d\mathbf{x}_0 = f_2(\mathbf{x}_0)$.

Solve the probability density function $f_2(\mathbf{x}, t)$ of the parabolic partial differential FPK equation, given the following parameters and initial and boundary conditions:

- $x_{min} = -10, x_{max} = 10, \dot{x}_{min} = -15, \dot{x}_{max} = 15$
- $t_{max} = 1, 4, 8, 10$ s
- $\zeta = 0.2, \gamma = 1, \varepsilon = 0.1$
- $D = 2\zeta = 0.4$
- Initial condition $t = 0$ is an Gaussian binormal distribution, $\mu_x = 0, \mu_{\dot{x}} = 10, \sigma_{0,x}^2 = \sigma_{0,\dot{x}}^2 = 1/2$

Solve the stationary solution of $f_2(\mathbf{x})$ from the FPK equation, $\frac{\partial f}{\partial t} = 0$:

- Analytically. Stationary solution of the FPK equation of the following equation of motion $\ddot{x} + \beta\dot{x} + g(x) = w(t)$, with $E\{w(t)\} = 0, E\{w(t)w(t + \tau)\} = 2D\delta(\tau)$, is given by $f_2(x, \dot{x}) = \frac{1}{g} \sqrt{\frac{2\pi D}{\beta}} e^{-\frac{\beta}{2D}[\frac{1}{2}\dot{x}^2 + g(x)x]}$, [27].
- Numerically with the initial condition $f_2(x = 0, \dot{x} = 0) = f_2(x_1 = 0, x_2 = 0) = 1$, applying the second order, fourth order and sixth order finite difference approximation.

M

The Galerkin Method

Galerkin¹ developed in 1915 an series expansion to approximate the solution of differential equations [186]. A brief description of the method is as follows.

The differential equation of the problem is written in the form

$$L\{u(x, t)\} = 0, \quad (\text{M.1})$$

where L is the differential operator and u represents the physical quantities, e.g. displacement, velocity, acceleration. The governing partial differential equations with the boundary conditions is called the “strong form” of the problem.

The approximate solution is expressed in series as follows

$$u = \sum_{i=1}^N \eta_i(t) \phi_i(x), \quad (\text{M.2})$$

where $\phi_i(x)$ represents a sequence of kinematically admissible functions only dependent on the space coordinate x , and η_i are the generalized coordinates only dependent on the time t . Galerkin’s method is based on the fact that some measure of the error in $L\{u(x, t)\} = 0$ is minimized for any fixed value of N satisfying simultaneously, for $i = 1..N$ the conditions

$$\int_V L\{u(x, t)\} \phi_i dV = \int_V L \left\{ \sum_{j=1}^N \eta_j(t) \phi_j(x) \right\} \phi_i dV = 0, \quad i = 1, 2, \dots, N. \quad (\text{M.3})$$

The problem is now reformulated in the “weak form”.

Example. Given a simply supported beam with constant properties. The bending stiffness is EI , the mass per unit length is m and dynamic load is $q(t)$ and the length is L . The undamped partial differential equation of motion is given by

¹ Boris G. Galerkin 1871–1945.

$$EI \frac{\partial^4 w}{\partial x^4} + m \frac{\partial^2 w}{\partial t^2} = q(t),$$

where w is the deflection of the beam.

The deflection $w(x, t)$ will be expressed as follows

$$w(x, t) = \eta(t)\phi(x) = \eta(t) \sin\left(\frac{\pi x}{L}\right).$$

In accordance with Galerkin's method the following integral equation should be solved

$$\int_0^L \left[EI \frac{\partial^4 w}{\partial x^4} + m \frac{\partial^2 w}{\partial t^2} - q(t) \right] \phi dx = 0, \tag{M.4}$$

or

$$\int_0^L \left[EI \eta(t) \frac{\partial^4 \sin\left(\frac{\pi x}{L}\right)}{\partial x^4} + m \sin\left(\frac{\pi x}{L}\right) \frac{\partial^2 \eta(t)}{\partial t^2} - q(t) \right] \sin\left(\frac{\pi x}{L}\right) dx = 0,$$

or

$$\int_0^L \left[EI \left(\frac{\pi}{L}\right)^4 \eta(t) \sin^2\left(\frac{\pi x}{L}\right) + m \sin^2\left(\frac{\pi x}{L}\right) \ddot{\eta}(t) - q(t) \sin\left(\frac{\pi x}{L}\right) \right] dx = 0,$$

or

$$EI \left(\frac{\pi}{L}\right)^4 \eta(t) \frac{L}{2} + m \frac{L}{2} \ddot{\eta}(t) - \frac{2L}{\pi} q(t) = 0.$$

Finally the approximate equation of motion expressed in the generalized coordinate $\eta(t)$ becomes

$$EI \left(\frac{\pi}{L}\right)^4 \eta(t) + m \ddot{\eta}(t) = \frac{4}{\pi} q(t),$$

or

$$\ddot{\eta}(t) + \frac{EI}{m} \left(\frac{\pi}{L}\right)^4 \eta(t) = \frac{4}{\pi} \frac{q(t)}{m}.$$

The natural frequency ω can now be obtained

$$\omega = \left(\frac{\pi}{L}\right)^2 \sqrt{\frac{EI}{m}},$$

and the generalized force is

$$f_g(t) = \frac{4}{\pi} \frac{q(t)}{m}.$$

Thus,

$$\ddot{\eta}(t) + \omega^2 \eta(t) = f_g(t).$$

This is the well known equation of motion for an undamped SDOF system.

The first term of the LHS of (M.4) will be integrated by parts twice² (assuming EI constant)

$$\begin{aligned} \int_0^L \frac{\partial^4 w}{\partial x^4} \phi dx &= \left(\phi \frac{\partial^3 w}{\partial x^3} \right) \Big|_0^L - \int_0^L \frac{\partial^3 w}{\partial x^3} \frac{d\phi}{dx} dx \\ &= \left(\phi \frac{\partial^3 w}{\partial x^3} \right) \Big|_0^L - \left[\left(\frac{d\phi}{dx} \frac{\partial^2 w}{\partial x^2} \right) \Big|_0^L - \int_0^L \frac{\partial^2 w}{\partial x^2} \frac{d^2 \phi}{dx^2} dx \right]. \end{aligned} \quad (M.5)$$

The test function ϕ shall obey the boundary conditions, thus:

- $\phi(0) = 0$ at the fixation of the beam,
- $\frac{d\phi}{dx} \Big|_0 = 0$ at the fixation of the beam,
- $\frac{\partial^3 w}{\partial x^3} \Big|_L = 0$, no shear force at the tip of the beam,
- $\frac{\partial^2 w}{\partial x^2} \Big|_L = 0$, no bending moment at the tip of the beam.

Equation (M.4) can be expressed as follows

$$\begin{aligned} \int_0^L \left[EI \frac{\partial^4 w}{\partial x^4} + m \frac{\partial^2 w}{\partial t^2} - q(t) \right] \phi dx \\ = \int_0^L \left[EI \frac{\partial^2 w}{\partial x^2} \frac{d^2 \phi}{dx^2} + m \frac{\partial^2 w}{\partial t^2} \phi - q(t) \phi \right] dx = 0. \end{aligned} \quad (M.6)$$

The approximate functions for $w(x)$ in (M.6) may be of lower order than in (M.4), C_0 continuous instead of C_1 continuous.

The following example concerns the solution of ordinary differential equation using the finite element method. The “stiffness” and the “load vector” will be derived. Galerkin’s method will be applied to obtain the finite element stiffness matrix and external load vector.

Example. This example is taken from [105]. The ordinary differential equation to be solved is

$$a \frac{d^2 u}{dx^2} + b \frac{du}{dx} + cu = f(x), \quad 0 < x < L, \quad (M.7)$$

with boundary conditions

$$u(0) = U(L) = 0.$$

² Integration by parts is defined as:

$$\int_a^b f(x)g'(x)dx = f(x)g(x) \Big|_a^b - \int_a^b g(x)f'(x)dx.$$

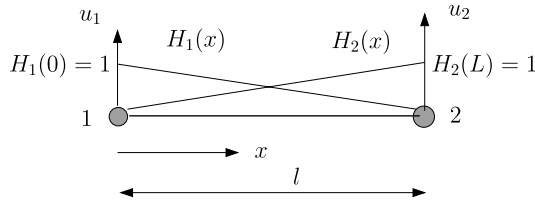


Fig. M.1. Linear finite element

The weak formulation of the equation is

$$\int_0^L \left[-a \frac{du}{dx} \frac{dw}{dx} + bw \frac{du}{dx} + cwu \right] dx = \int_0^L wf(x) dx - \left[aw \frac{du}{dx} \right]_0^L.$$

The natural boundary conditions at $x = 0$ and $x = L$ shall be $\frac{du}{dx}|_0 = \frac{du}{dx}|_L = 0$, and $w(x)$ is the test function.

Within one linear finite element as illustrated in Fig. M.1 the shape $H_i(x)$ and test functions w are the same. Thus for one element the weak formulation becomes

$$\int_0^{l_i} \left[-a \frac{du}{dx} \frac{dw}{dx} + bw \frac{du}{dx} + cwu \right] dx = \int_0^{l_i} wf(x) dx.$$

The shape functions of internal the linear finite element are defined as follows (see also Fig. M.1):

- $H_1(x) = (1 - x/l_i),$
- $H_2(x) = x/l_i.$

The internal unknown $u(x)$ can be expressed into the shape functions $H_1(x)$ and $H_2(x)$ in combination with the nodal properties $u(0) = u_1$ and $u(l_1) = u_2$, hence

$$u(x) = H_1(x)u_1 + H_2(x)u_2 = [H_1(x), H_2(x)] \begin{pmatrix} u_1 \\ u_2 \end{pmatrix},$$

and

$$u'(x) = \frac{du}{dx} = [H_1'(x), H_2'(x)] \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \left[-\frac{1}{l_i}, \frac{1}{l_i} \right] \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}.$$

The test or weighting functions w are defined by Galerkin's method

$$\begin{pmatrix} w_1(x) \\ w_2(x) \end{pmatrix} = \begin{pmatrix} H_1(x) \\ H_2(x) \end{pmatrix},$$

and

$$\begin{pmatrix} \frac{dw_1}{dx} \\ \frac{dw_2}{dx} \end{pmatrix} = \begin{pmatrix} H_1'(x) \\ H_2'(x) \end{pmatrix} = \begin{pmatrix} -\frac{1}{l_i} \\ \frac{1}{l_i} \end{pmatrix}.$$

The stiffness matrix $[K_i]$ if the i th element can be obtained using the weak form and the selected shape and test functions for $u(x)$ and $w(x)$

$$[K_i] = \int_0^{l_i} \left[-a \begin{pmatrix} H'_1 \\ H'_2 \end{pmatrix} [H'_1, H'_2] + b \begin{pmatrix} H_1 \\ H_2 \end{pmatrix} [H'_1, H'_2] + c \begin{pmatrix} H_1 \\ H_2 \end{pmatrix} [H_1, H_2] \right] dx.$$

Evaluation of the integral gives

$$[K_i] = -\frac{a}{l_1} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} + \frac{b}{2} \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} + \frac{cl_i}{6} \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}.$$

The element generalized load vector

$$\{F_i\} = \int_0^{l_i} \begin{pmatrix} H_1 \\ H_2 \end{pmatrix} f(x) dx.$$

For a constant uniform load $f(x) = f_0$ the element load vector becomes

$$\{F_i\} = \frac{f_0 l_i}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

The differential equation (M.7) and associated boundary conditions will be solved assigning the following values for the parameters:

- $a = 1,$
- $b = -3,$
- $c = 2,$
- $f_0 = 1,$
- $L = 1,$
- $u(0) = 0,$
- $u(1) = 0,$
- Number of linear elements $N = 1000.$

The results of the finite element analysis are shown in Fig. M.2 and correlate very good with the numerical solution using Maple®.

Problems

M.1. A beam has a constant thickness t but a width which varies linearly from b_0 at the root to $b_0/2$ at the tip as shown in Fig. M.3. The density of the material is ρ and the Young's modulus is E . The flexural vibration $w(x, t)$ of the beam is approximated by the following expression

$$w(x, t) = \eta(t) \sin\left(\frac{\pi x}{2L}\right).$$

Applying the Galerkin's method yields the undamped equation of motion expressed in the generalized coordinate $\eta(t)$. Estimate the fundamental natural frequency.

Answers: $m_0 \ddot{\eta} + 6.0881 EI_0 \eta = 0, \omega^2 = 6.0881 \frac{EI_0}{m_0 L^4}, f = 0.3927 \sqrt{\frac{EI_0}{m_0 L^4}}$ (Hz),
 $m_0 = b_0 t \rho, EI_0 = b_0 t^3 / 12.$

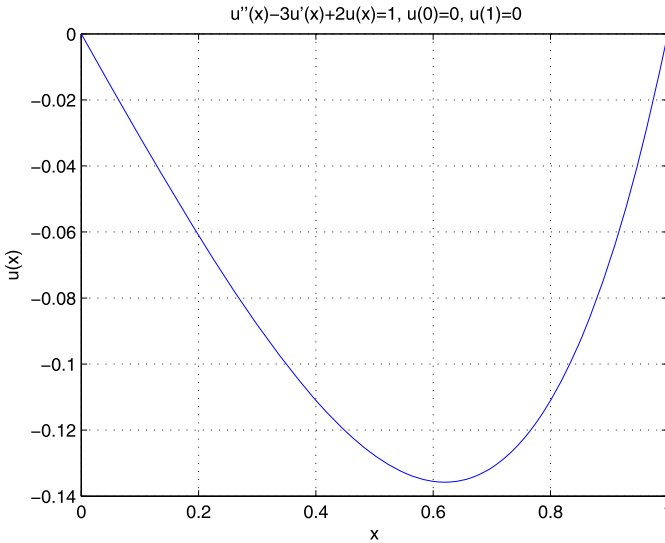


Fig. M.2. Solution $\frac{d^2u}{dx^2} - 3\frac{du}{dx} + 2u = f(x)$, $0 < x < 1$, $u(0) = u(1) = 0$

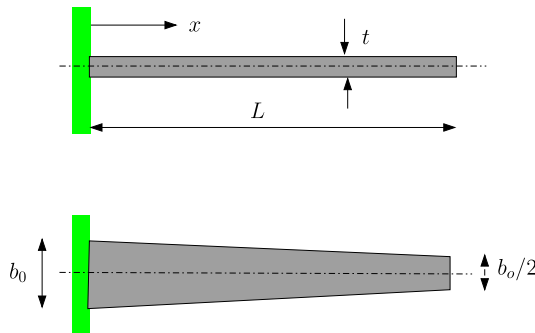


Fig. M.3. Tapered beam

M.2. Derive the stiffness matrix, mass matrix and force vector for a beam element with two nodes; 1 and 2. In node 1 there are two unknown degrees of freedom; the deflection w_1 and the rotation ϕ_1 . At node 2 there are two degrees of freedom too; the deflection w_2 and the rotation ϕ_2 . The bending stiffness EI is constant over the beam element with length L . The constant mass per unit of length is given by m and the constant load per unit of length is q . The beam element is illustrated in Fig. M.4.

The following weak form of the equation of motion for the beam shall be used to derive the mass and stiffness matrices and the force vector

$$\int_0^L \left[EI \frac{\partial^2 w}{\partial x^2} \frac{d^2 \phi}{dx^2} + m \frac{\partial^2 w}{\partial t^2} \phi - q(t) \phi \right] dx = 0.$$

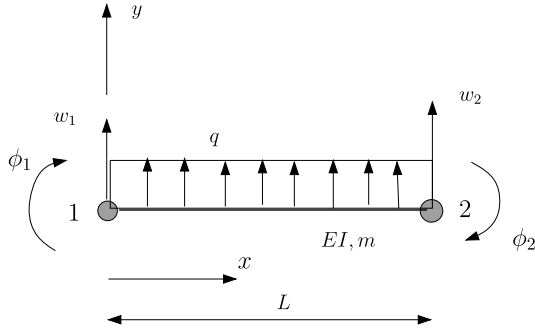


Fig. M.4. Beam element

The approximate deflection shape $w(x)$ is given by

$$w(x) = a_0 + a_1x + a_2x^2 + a_3x^3$$

and shall be expressed in the unknown degrees of freedom w_1, ϕ_1, w_2 and ϕ_2 as follows

$$w(x) = H_1(x)w_1 + H_2(x)\phi_1 + H_3(x)w_2 + H_4(x)\phi_2.$$

Define the shape functions $H_1(x), H_2(x), H_3(x),$ and $H_4(x)$ and perform the following assignments:

- Derive the stiffness matrix $[k]$ from $\int_0^L EI \frac{\partial^2 w}{\partial x^2} \frac{d^2 \phi}{dx^2} dx$ with the aid of Galerkin's method.
- Derive the mass matrix $[m]$ from $\int_0^L m \frac{\partial^2 w}{\partial t^2} \phi dx$ with the aid of Galerkin's method.
- Derive the force vector $[f]$ from $\int_0^L q(t)\phi dx$ with the aid of Galerkin's method.
- The lowest natural frequency of a fixed-free beam is given by $f_n = 0.560 \sqrt{\frac{EI}{mL^4}}$ (Hz) (verify). Calculate the lowest natural frequency f_1 (Hz) with a finite element model (FEM) consisting of 10 beam elements and 11 nodes. The DOF's in node 1 are constrained, $w_1 = \phi_1 = 0$ (hint: set $EI = m = L = 1$).

Answers:

- $H_1(x) = 1 - 3(\frac{x}{L})^2 + 2(\frac{x}{L})^3,$
- $H_2(x) = x - 2(\frac{x^2}{L}) + 2(\frac{x^3}{L^2}),$
- $H_3(x) = 3(\frac{x}{L})^2 - 2(\frac{x}{L})^3,$
- $H_4(x) = -(\frac{x^2}{L}) + (\frac{x^3}{L^2}),$
- $[k] = \frac{EI}{L^3} \begin{pmatrix} 12 & 6L & -12 & 6L \\ & 4L^2 & -6L & 2L^2 \\ & & 12 & -6L \\ \text{sym.} & & & 4L^2 \end{pmatrix},$

- $[m] = mL \begin{pmatrix} \frac{13}{35} & \frac{11L}{210} & \frac{9}{70} & -\frac{13L}{420} \\ & \frac{L^2}{105} & \frac{13L}{420} & \frac{L^2}{140} \\ & & \frac{13}{35} & -\frac{11L}{210} \\ \text{sym.} & & & \frac{L^2}{105} \end{pmatrix},$

- $\{f\} = qL \begin{pmatrix} \frac{1}{2} \\ \frac{L}{12} \\ \frac{1}{2} \\ -\frac{L}{12} \end{pmatrix}.$

- Finite element analysis $f_1 = 0.559\sqrt{\frac{EI}{mL^4}}$ Hz.

M.3. The axial deformation $u(x)$ of a bar subjected to an uniform load $q(x)$ (1-D Poisson equation) is given by

$$EA \frac{d^2u}{dx^2} = q,$$

where E is Young’s modulus and A the constant cross-section of the bar. The following boundary conditions apply

$$\begin{cases} u(0) = 0, \\ EA \frac{du}{dx} \Big|_L = 0. \end{cases}$$

The weak form of the problem is expressed as

$$\int_0^L \left[EA \frac{d^2u}{dx^2} - q \right] v dx = 0,$$

where v is the test function and obeys the boundary conditions.

Derive from the weak form the following equation

$$EA \int_0^L \frac{du}{dx} \frac{dv}{dx} dx = - \int_0^L qv dx.$$

M.4. Solve the following ordinary differential equation

$$x^2 \frac{d^2u}{dx^2} - 3x \frac{du}{dx} - 4u = x^2, \quad 10 < x < 20, \tag{M.8}$$

with the boundary conditions $u(10) = 0$ and $u(20) = 100$, with the aid of the finite element method using linear elements as illustrated in Fig. M.1, [105]. Perform the assignments:

1. Set up the weak form of (M.8)
2. Derive the stiffness matrix for the linear element based on the weak form and using Galerkin’s method

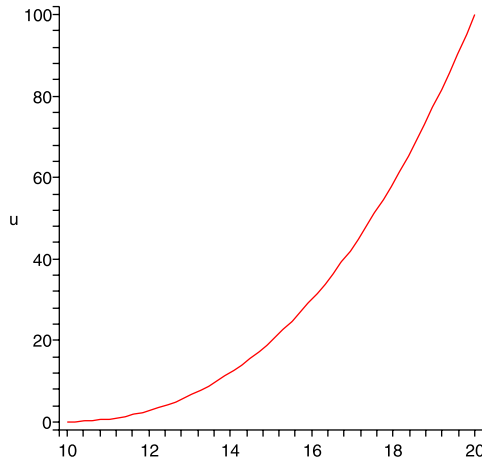


Fig. M.5. Solution $x^2 \frac{d^2u}{dx^2} - 3x \frac{du}{dx} - 4u = x^2$, $10 < x < 20$, $u(10) = 0$ and $u(20) = 100$

3. Derive the force vector of the linear element based on the weak form and using Galerkin’s method
4. Solve numerically (M.8) and associated boundary conditions using 10, 50 and 100 elements.

Answers:

$$\int_{10}^{20} \left(x^2 \frac{dw}{dx} \frac{du}{dx} + 4xw \frac{du}{dx} + 4wu \right) dx = - \int_{10}^{20} wx^2 dx + \left[x^2 w \frac{du}{dx} \right]_{10}^{20},$$

$$[K_e] = \frac{1}{l_i^2} \begin{pmatrix} 4x_1x_2^2 - 6x_1^2x_2 - x_2^3 + 3x_1^3 & 2x_1x_2^2 - x_2^3 - x_1^3 \\ -2x_2x_1^2 + x_2^3 + x_1^3 & 6x_1x_2^2 - 4x_1^2x_2 - 3x_2^3 + x_1^3 \end{pmatrix},$$

$$\{F_e\} = \frac{1}{12l_i} \begin{pmatrix} -4x_2x_1^3 + x_2^4 + 3x_1^4 \\ -4x_1x_2^3 + 3x_2^4 + x_1^4 \end{pmatrix},$$

where l_i is the length of the linear element and x_1 and x_2 are the coordinates of respectively nodes 1 and 2 (Fig. M.1). The solution of (M.8) for $x = 15$ is $u = 19.7816$. In Fig. M.5 the numerical solution, using Maple[®], is shown.

Wiener Process, Stochastic Integrals, Stochastic Differential Equations

N.1 Wiener Process

A *Wiener process* (notation $W(t)$, $t \geq 0$) is named in the honor of Norbert Wiener.¹ Another name is the *Brownian motion* (notation $B(t)$, $t \geq 0$). The Wiener process is a Gaussian process and is completely described by its expectation and correlation function and is probably the simplest stochastic process and can be taken the prototype of all stochastic processes. The standard Wiener process has the following main properties [81]:

1. $W(0) = 0$
2. Paths (trajectories are continuous functions of $t \in [0, \infty)$)
3. Expectation $E\{W(t)\} = 0$
4. Correlation (Covariance) function $E\{W(t)W(s)\} = \min(t, s)$
5. For any t_1, t_2, \dots, t_b the random vector $[W(t_1), W(t_2), \dots, W(t_n)]$ is Gaussian
6. For any s, t

$$\begin{aligned} E\{W^2(t)\} &= t, \\ E\{W(t) - W(s)\} &= 0, \\ E\{[W(t) - W(s)]^2\} &= |t - s|. \end{aligned}$$

7. Increments of Wiener process on non overlapping intervals are independent, i.e. $W(t_2) - W(s_2)$, $W(t_1) - W(s_1)$ are independent
8. Paths of Wiener process are not differentiable functions

The properties 1–5 are nothing but the definition of the Wiener process. Property 6 is implied in 3 and 4. Property 4 provides the orthogonality of increments, that is for $s_1 < s_2 < s_3 < s_4$

$$E\{[W(s_4) - W(s_3)][W(s_2) - W(s_1)]\} = (s_2 - s_1) - (s_2 - s_1) = 0.$$

¹ Norbert Wiener 1894–1964.

To verify the validity of 8, with $h > 0$ let define

$$\Delta(h) = \frac{W(s+h) - W(s)}{h}$$

and show that $\lim_{h \rightarrow 0} \Delta(h)$ does not exist. Assume that this limit exists then the limit for the Fourier transform $\lim_{h \rightarrow 0} E\{e^{j\lambda\Delta(h)}\}$ exists and is a continuous function of λ . Hence, since the random variable $\Delta(h)$ is non zero mean Gaussian with the variance $E\{\frac{[W(s+h)-W(s)]^2}{h^2}\} = \frac{1}{h}$, thus

$$E\{e^{j\lambda\Delta(h)}\} = e^{-\frac{\lambda^2}{2h}} \rightarrow_{h \rightarrow 0} = \begin{cases} 0 & \lambda = 0 \\ 1 & \lambda \neq 0 \end{cases} := U(\lambda).$$

Since the Heaviside step function $U(\lambda)$ is discontinuous the assumed differentiability is not valid.

Let $X(t)$ be a standard Wiener process, from property 6 the probability density function $f_t(x)$ can be derived

$$f_t(x) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}, \quad t > 0. \tag{N.1}$$

For the differential $dW(t)$ apply the following properties [204]:

1. $dW(t) = \xi_t dt$, where ξ_t is white noise
2. Expectation $E\{dW(t)\} = 0$
3. Correlation (Covariance) function $E\{dW(t)dW(s)\} = 0$
4. $E\{dW^2(t)\} = dt$

The practical importance of the Wiener process is in the relation property 1.

Simulation of White Noise

The simulation of the white noise is described in [204]. In the case of simulation of stochastic processes an important question is the reproducibility of the results. For this purpose, random numbers are computed using deterministic algorithms. The numerical simulation of stochastic process requires a random number generator which produces uniformly distributed numbers independent from each other using certain initial values from the interval $(0, 1]$. There are several ways to produce such random numbers from a uniformly distributed sequence.

Let R_n be a sequence of numbers with standard normal distribution from the interval $[0, 1]$

$$\begin{aligned} E\{R_n\} &= 0, \\ E\{R_n^2\} &= 1. \end{aligned} \tag{N.2}$$

Thus, the equation

$$\Delta W = R_n \sqrt{\Delta t}, \tag{N.3}$$

fulfills the requirements of the Wiener process

$$E\{\Delta W\} = E\{R_n \sqrt{\Delta t}\} = E\{R_n\}E\{\sqrt{\Delta t}\} = E\{R_n\}\sqrt{\Delta t} = 0, \tag{N.4}$$

and

$$E\{\Delta W^2\} = E\{[R_n \sqrt{\Delta t}]^2\} = E\{R_n^2\}E\{\sqrt{\Delta t}^2\} = E\{R_n^2\}\Delta t = \Delta t. \tag{N.5}$$

Such a sequence of R_n that satisfies (N.2), can be constructed from two uniformly distributed, statistically independent sequences U_n and V_n using the *Box-Müller transformation*. These sequences can be obtained by applying a linear generator with arbitrary initial values $0 < U_0, V_0 \leq 1$ ($U_0 \neq V_0$). The algorithm reads

$$U_0 = 0.3, \quad V_0 = 0.7, \quad a = 9821.0, \quad b = 0.211322, \tag{N.6}$$

$$U_n = \text{frac}(aU_{n-1} + b), \tag{N.7}$$

$$V_n = \text{frac}(aV_{n-1} + b), \tag{N.8}$$

where the function frac^2 limits the number to the interval $[0, 1]$ by cutting the integer digits off. Finally, the Box-Müller transformation, which yields the sequences R_{n_1} and R_{n_2} with standard normal distribution from the evenly distributed sequences U_n and V_n

$$\begin{aligned} R_{n_1} &= \sqrt{-2 \ln U_n} \sin(2\pi V_n), \\ R_{n_2} &= \sqrt{-2 \ln U_n} \cos(2\pi V_n). \end{aligned} \tag{N.9}$$

Example. Generate a Wiener process within a time interval $[0, 1]$ with a number of simulations $n = 10\,000$ ($\Delta t = 1/n$) using the Box-Müller transformation to simulate a white noise, normal Gaussian $N(0, 1)$ distributed. The distribution of the simulation of the Gaussian process is shown in Fig. N.1. The simulation of the Wiener process is illustrated in Fig. N.2.

More examples are given in [82].

N.2 Itô versus Stratonovich

In this section a not to formal discussion on the differences between Itô and Stratonovich interpretation of stochastic differential equation (SDE). The discussion is taken from [148].

The interpretation question arises when dealing with the multiplicative stochastic (parametric excitation) equation

² $\text{frac}(x) = x - \text{fix}(x)$.

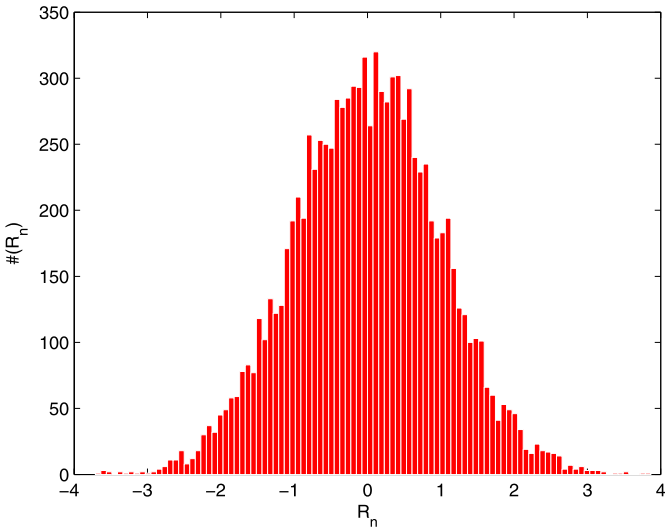


Fig. N.1. Distribution of R_n (10^4 iterations)

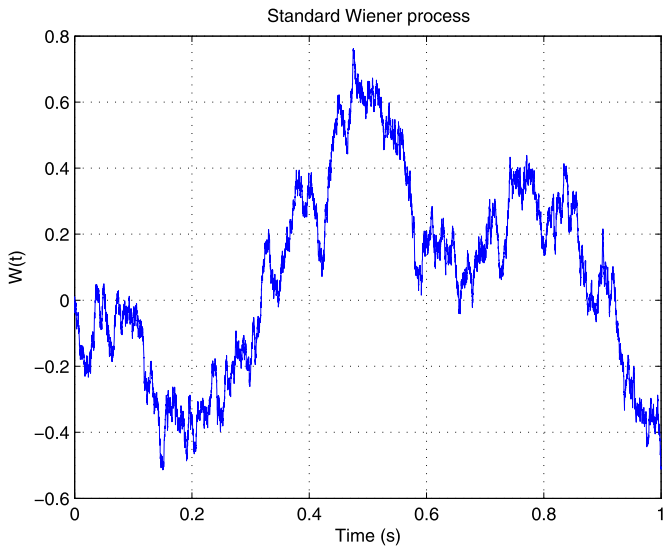


Fig. N.2. Simulated standard Wiener process

$$\dot{X}(t) = f(X(t)) + g(X(t))\xi(t), \tag{N.10}$$

where f and g are given functions, and $\xi(t)$ is a Gaussian white noise, that is, Gaussian and stationary random process with zero mean and delta correlated. Alternatively, (N.10) can be written in terms of the Wiener process $W(t)$ as

$$\Delta X(t) = f(X(t))\Delta t + g(X(t))\Delta W(t), \tag{N.11}$$

where $\Delta W(t) = \xi(t)\Delta t$. When g depends on $X(t)$, (N.10) and (N.11) have no meaning, unless an interpretation of the multiplicative term $g(X(t))\xi(t)$ is provided. These different interpretations of the multiplicative term must be given because, due to extreme randomness of the white noise, it is not clear what X should be used even during an infinitesimal time step Δt . According to Itô, that value of $X(t)$ is one before the beginning of the time step, i.e. $X = X(t)$, whereas Stratonovich uses the value of X at the middle of the time step: $X = X(t + \Delta t/2) = X(t) + \Delta X(t)/2$.

Before proceeding further with the consequences of the above discussion a more precise meaning of the differential of random processes driven by Gaussian white noise and its implications will be given. The differential of any random process $X(t)$ is defined by

$$\Delta X(t) = X(t + \Delta t) - X(t). \tag{N.12}$$

It can be proven that $\Delta X(t) = O(\sqrt{\Delta t})$.

The differential of the product of two random processes will be investigated since the differential adopts a different expression depending on the interpretation (Itô or Stratonovich) chosen. In accordance with (N.12) the following expression becomes

$$\Delta(XY) = [(X + \Delta X)(Y + \Delta Y)] - XY. \tag{N.13}$$

This expression can be rewritten in many different ways. One possibility is

$$\Delta(XY) = \left(X + \frac{\Delta X}{2}\right)dY + \left(Y + \frac{\Delta Y}{2}\right)dX, \tag{N.14}$$

but it is also allowed to write the product as

$$\Delta(XY) = X\Delta Y + Y\Delta X + \Delta X\Delta Y. \tag{N.15}$$

Therefore, (N.14) reads in the Stratonovich interpretation when

$$\Delta(XY) = X_S\Delta Y + Y_S\Delta X, \tag{N.16}$$

where

$$X_S(t) \equiv X(t + \Delta t/2) = X(t) + \Delta X(t)/2, \tag{N.17}$$

and the same for $Y_S(t)$. The differential of the product follows the Itô interpretation when

$$\Delta(XY) = X_I\Delta Y + Y_I\Delta X + \Delta X\Delta Y, \tag{N.18}$$

where

$$X_I(t) \equiv X(t), \tag{N.19}$$

and $Y_I(t) \equiv Y(t)$. Note that (N.16) formally agrees with the rules of calculus while (N.18) does not. Previous equations (N.16) and (N.18) can be generalized to the product of two function $U(X)$ and $V(X)$, of the random process $X = X(t)$. Thus

$$\Delta(UV) = U(X_S)\Delta V(X) + V(X_S)\Delta U(X), \tag{N.20}$$

where X_S is given by (N.17) and $\Delta V(X) = V(X + \Delta X) - V(X)$ with an analogous expression for $\Delta U(X)$. Within Itô convention it becomes

$$\Delta(UV) = U(X_I)\Delta V(X) + V(X_I)\Delta U(X) + \Delta U(X)\Delta V(X). \tag{N.21}$$

The expected value of the multiplicative term in (N.10), $g(X)\xi(t)$, depends on the interpretation given. In the Itô interpretation it is clear that $E\{g(X)\xi(t)\} = 0$, because the value X and the value $g(X)$ anticipates the jump in the noise. In other words, $g(X)$ is independent of $\xi(t)$. On the other hand, it can be proved that within the Stratonovich framework the average of the multiplicative term reads $g(X)g'(X)/2$ where the prime denotes the derivative. The zero value of $E\{g(X)\xi(t)\}$ makes Itô very appealing because then the deterministic equation for the mean value of X only depends on the drift term $f(X)$. In this sense, note that any multiplicative stochastic differential equation has different expression for the function $f(X)$ and $g(X)$ depending the interpretation is chosen. In the Stratonovich framework, the SDE of type (N.11) can be written

$$\Delta X = f^{(S)}(X_S)\Delta t + g^{(S)}(X_S)\Delta W(t), \tag{N.22}$$

where $X_S = X + dX/2$. In the Itô sense the SDE becomes

$$\Delta X = f^{(I)}(X_I)dt + g^{(I)}(X_I)\Delta W(t), \tag{N.23}$$

where $X_I = X$. The function $f^{(S)}$ and $f^{(I)}$ are not only evaluated at different values of X but are also different functions depending in the interpretation given. The same applies for $g^{(S)}$ and $g^{(I)}$. From (N.17), (N.22) and (N.23) it can be shown, keeping terms up to order Δt , the relation between $f^{(S)}$ and $f^{(I)}$ is

$$f^{(I)}(X) = f^{(S)}(X) - \frac{1}{2}g^{(S)}(X)\frac{\partial g^{(S)}(X)}{\partial X}, \tag{N.24}$$

while the multiplicative functions $g^{(S)}$ and $g^{(I)}$ are equal

$$g^{(I)}(X) = g^{(S)}(X). \tag{N.25}$$

Conversely, it is possible to pass from the Stratonovich SDE to an equivalent Itô SDE. The difference between both interpretations only effects the drift term given by function f while the function g remains unaffected. For an additive SDE, i.e. when g is independent of X , the interpretation question is irrelevant.

A crucial difference between Itô and Stratonovich interpretations appears when a change of variables is performed on the original equation. Then it can be proved that, Stratonovich convention, the standard rules of calculus hold, but new rules appear when the equation is understood in Itô sense. From the point of view of this property, the Stratonovich criterion seems to be convenient. For the sake of completeness, the rules to change of variables will be repeated in each interpretation. Let $h(X, t)$ be an arbitrary function of X and t . In the Itô sense, the differential of $h(X, t)$ reads

$$\Delta h = \frac{\partial h(X, t)}{\partial X} \Delta X + \left[\frac{\partial h(X, t)}{\partial t} + \frac{1}{2} g^2(X, t) \frac{\partial^2 h(X, t)}{\partial X^2} \right] \Delta t, \quad (\text{N.26})$$

whereas in the Stratonovich sense, the usual expression will be obtained

$$\Delta h = \frac{\partial h(X_S, t)}{\partial X_s} \Delta X + \frac{\partial h(X_S, t)}{\partial t} \Delta t, \quad (\text{N.27})$$

where

$$\frac{\partial h(X_S, t)}{\partial X_s} = \frac{\partial h(X, t)}{\partial X} \Big|_{X=X_S}, \quad (\text{N.28})$$

and X_S is given by (N.17). Equation (N.26) is known as the Itô's lemma and it is extensively used.

If the SDE for X is given by

$$\Delta X = f(X) \Delta t + g(X) \Delta W,$$

the Itô SDE for $h(X, t)$ can be written as follows

$$\Delta h = L^0 h(X) \delta t + L^1 h(X) dW, \quad (\text{N.29})$$

where the operators L^0 and L^1 are given by

$$\begin{aligned} L^0 &= \frac{\partial}{\partial t} + f(X) \frac{\partial}{\partial x} + \frac{1}{2} g^2(X) \frac{\partial^2}{\partial x^2}, \\ L^1 &= g(X) \frac{\partial}{\partial x}. \end{aligned} \quad (\text{N.30})$$

Example. The following SDE will be considered to derive Itô formula

$$\Delta X = (\alpha - X) dt + \beta \sqrt{X} \Delta W.$$

The change in variable is given by

$$V(X) = \sqrt{X}.$$

Itô's formula can be written as

$$\Delta V = L^0 V(X) \Delta t + L^1 V(X) \Delta W,$$

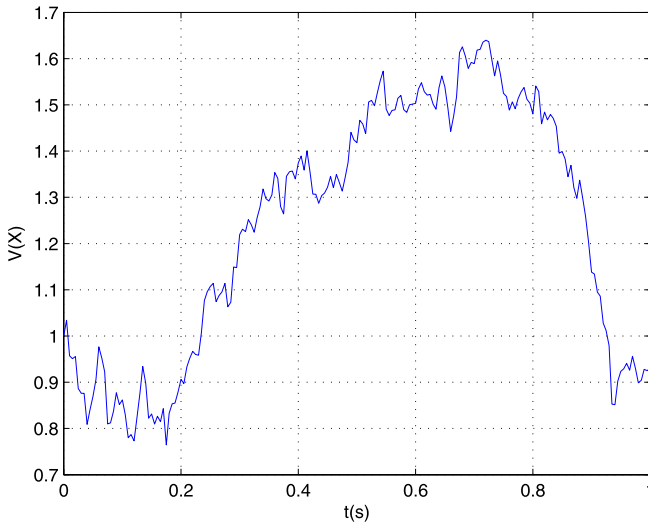


Fig. N.3. $V(X)$

and with

$$\frac{\partial V(X)}{\partial t} = 0, \quad \frac{\partial V(X)}{\partial x} = \frac{1}{2\sqrt{X}}, \quad \frac{\partial^2 V(X)}{\partial x^2} = -\frac{1}{4X\sqrt{X}},$$

then

$$\Delta V = \left(\frac{4\alpha - b^2}{8V} - \frac{V}{2} \right) \Delta t + \frac{\beta}{2} \Delta W.$$

If $\alpha = 2$, $\beta = 1$, the total time $T = 1\text{s}$ and the number of steps $N = 200$. The increment of the Wiener process is $\Delta W = \xi\sqrt{\Delta t}$, where $\xi \sim N(0, 1)$. The graphical representation of $V(X)$ is given in Fig. N.3.

N.3 Stochastic Integrals

The theory about the stochastic integrals is taken from [82] will be discussed in a simple engineering way. A more mathematical derivation of the stochastic integrals (Itô and Stratonovich) can be read in [200].

Given a suitable function $h(t)$, the integral

$$\int_0^T h(t) dt$$

may be approximated by the *Riemann sum*

$$\sum_{j=0}^{N-1} h(t_j)(t_{j+1} - t_j), \tag{N.31}$$

where the discrete points are defined by $t_j = j\Delta t$, $j = 0, 1, \dots, N - 1$. The integral may be defined by taking $\Delta \rightarrow 0$ in (N.31).

In a similar way the following sum may be considered

$$\sum_{j=0}^{N-1} h(t_j)[W(t_{j+1}) - W(t_j)], \tag{N.32}$$

which, is similar with (N.31), may be regarded as an approximation to a *stochastic integral* with respect to the Brownian motion

$$\int_0^T h(t)dW(t).$$

Equation (N.32) is known as the *Itô integral*.

An alternative way to (N.31) is given by

$$\sum_{j=0}^{N-1} h\left(\frac{t_j + t_{j+1}}{2}\right)(t_{j+1} - t_j), \tag{N.33}$$

which is also a Riemann sum approximation to the integral $\int_0^T h(t)dt$. The corresponding alternative of (N.32) is

$$\sum_{j=0}^{N-1} h\left(\frac{t_j + t_{j+1}}{2}\right)[W(t_{j+1}) - W(t_j)], \tag{N.34}$$

which is indicated by the *Stratonovich integral*.

It is possible to evaluate exactly the stochastic integrals that are approximated when the deterministic function $h(t)$ will be replaced by the Wiener process $W(t)$. The Itô version is the limiting case of

$$\begin{aligned} & \sum_{j=0}^{N-1} W(t_j)[W(t_{j+1}) - W(t_j)] \\ &= \frac{1}{2} \sum_{j=0}^{N-1} [W(t_{j+1})^2 - W(t_j)^2 - \{W(t_{j+1}) - W(t_j)\}^2] \\ &= \frac{1}{2} \left[W(T)^2 - W(0)^2 - \sum_{j=0}^{N-1} \{W(t_{j+1}) - W(t_j)\}^2 \right] \\ &= \frac{1}{2} W(T)^2 - \frac{1}{2} T. \end{aligned} \tag{N.35}$$

The Itô integral can be written to be

$$\int_0^T W(t)dW(t) = \frac{1}{2}W(T)^2 - \frac{1}{2}T. \tag{N.36}$$

The Stratonovich integral is the limiting case of

$$\begin{aligned} & \sum_{j=0}^{N-1} \frac{1}{2}[W(t_j) + W(t_{j+1})][W(t_{j+1}) - W(t_j)] \\ &= \frac{1}{2} \sum_{j=0}^{N-1} [W(t_{j+1})^2 - W(t_j)^2] \\ &= \frac{1}{2} [W(T)^2 - W(0)^2] \\ &= \frac{1}{2} W(T)^2. \end{aligned} \tag{N.37}$$

Thus in place of (N.36) the Stratonovich integral is given by

$$\int_0^T W(t)dW(t) = \frac{1}{2}W(T)^2. \tag{N.38}$$

It can be shown that forming $\frac{1}{2}[W(t_j) + W(t_{j+1})]$ and adding an independent $\delta Z_j, N(0, \Delta/4)$, gives an value for $W(\frac{1}{2}\{t_j + t_{j+1}\})$.

Itô and Stratonovich integrals both have their uses in mathematical modeling. A simple transformation converts Itô to Stratonovich. This is explained in more mathematical detail in [200].

Applying Itô formula it can be verified that [70]

$$\begin{aligned} \int_a^b W^n(t)dW(t) &= \frac{1}{n+1}[W^{n+1}(b) - W^{n+1}(a)] \\ &\quad - \frac{n}{2} \int_a^b W^{n+1}(t)dW(t). \end{aligned} \tag{N.39}$$

N.4 Correlation Function

The derivation of the correlation function R is taken from an introductory lecture about stochastic differential equation given by Chris Williams, Institute for Adaptive and Neural Computations, School of Informatics, University of Edinburgh, UK. Given the set of SDE's

$$\Delta\{x(t)\} = -[A]\{x(t)\}\Delta t + [G]\{\Delta W(t)\}. \tag{N.40}$$

The solution is

$$\{x(t)\} = e^{-At}\{x(0)\} + \int_0^t e^{-A(t-s)}[G]\{dW(s)\}. \quad (\text{N.41})$$

The Wiener have with zero means and expectation given

$$E[\{W(t)\}\{W(s)\}] = \langle\{W(t)\}\{W(s)\}\rangle = [2D]\min(t, s), \quad (\text{N.42})$$

and the expectation of the standard Wiener process $\Delta W(t)$ is

$$E[\{\Delta W(t)\}\{\Delta W(t)\}] = dt. \quad (\text{N.43})$$

For the stationary solution the $\{x(0)\}$ dependency will be removed

$$\begin{aligned} E[\{x(t)\}\{x(s)\}] &= R_{xx}(t-s) = R_{xx}(\tau) \\ &= \int_0^{\min(t,s)} e^{-A(t-s)}[G][2D][G]^T e^{-A^T(t-s)} ds. \end{aligned} \quad (\text{N.44})$$

For autocorrelation function $[R_{xx}(\tau)]$ stationary processes and zero lag $\tau = 0$

$$R_{xx}(0) = \int_0^\infty e^{-A(t-s)}[G][2D][G]^T e^{-A^T(t-s)} ds. \quad (\text{N.45})$$

Equation (N.45) is the solution of the following Lyapunov equation [10, 66]

$$[A][R_{xx}(0)] + [R_{xx}(0)][A]^T = -[G][2D][G]^T. \quad (\text{N.46})$$

Problems

N.1. Describe in detail the Polar-Marsaglia algorithm for generating random variates, X , from a standard normal density $X \sim N(0, 1)$.

N.2. Prove that for a standard Wiener process the following relation applies [99]

$$E\{|W(t) - W(s)|^4\} = 3|t - s|^2, \quad \text{for all } s, t \geq 0.$$

N.3. The Wiener process is defined such that $W(0) = 0$, $E\{W^2(t_1)\} = t_1$ and $E\{W^2(t_0)\} = t_0$. Prove that the expected value of the Itô stochastic integral

$$E\left\{\int_{t_0}^{t_1} W(t)dW(t) = 0\right\}.$$

This problem is taken from [130].

N.4. Let X be the solution of the Itô SDE

$$\Delta X = \frac{b^2}{2}X \Delta t + bX dW.$$

Derive that the same process X is also the solution of the Stratonovich equation

$$\Delta X = bX dW.$$

N.5. The formal scalar linear SDE is given by

$$\Delta X = A(t)X\Delta t + B(t)X\Delta W.$$

Prove that the solution of the SDE is in case of the

- Stratonovich equation

$$X = C e^{(\int_{t_0}^t A(s)ds + \int_{t_0}^t B(s)dW)},$$

- Itô equation

$$X = C e^{(\int_{t_0}^t [A(s) - B(s)^2/2]ds + \int_{t_0}^t B(s)dW)}.$$

N.6. Prove that

$$[R_{xx}(t - s)] = e^{-A(t-s)}[R_{xx}(0)] \quad \text{for } t > s,$$

and

$$[R_{xx}(t - s)] = [R_{xx}(s - t)]^T.$$

N.7. This problem is taken from [228]. The equations of motion for the four-dimensional linear system are

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{pmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -(k_1 + k_2) & -c_1 & k_2 & 0 \\ 0 & 0 & 0 & 1 \\ k_2 & 0 & -(k_2 + k_3) & -c_2 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{pmatrix} \xi_1(t) \\ \xi_2(t) \end{pmatrix},$$

where $\xi_1(t)$ and $\xi_2(t)$ are Gaussian white noise stochastic processes, such that

$$E\{\xi_1(t)\} = 0, \quad E\{\xi_2(t)\} = 0, \quad E\{\xi_i(t)\xi_j(t + \tau)\} = 2D_{ij}\delta_{ij}\delta(\tau).$$

The system parameters are: $k_1 = k_2 = k_3 = 1$, $c_1 = c_2 = 0.4$, and $D_{11} = D_{22} = 0.2$.

Reconstruct the MDOF dynamic system and calculate the correlation matrix $[R_{xx}]$ and covariance matrix $[C_{xx}]$.

Answer:

$$[R_{xx}] = [C_{xx}] = \begin{bmatrix} 0.3333 & 0 & 0.1667 & 0 \\ 0 & 0.5 & 0 & 0 \\ 0.1667 & 0 & 0.3333 & 0 \\ 0 & 0 & 0 & 0.5 \end{bmatrix}.$$

O

Glossary

Adjoint operator—An operator is self-adjoint if $L^\dagger(w_1, w_2) = \int [w_2 L(w_1) - w_1 L(w_2)] dx = 0$ applies.

Autocorrelation function—The autocorrelation function of a signal is the average of the product of the value of the signal at time t with the value at time $t + \tau$.

Apparent Mass—Complex frequency response function which is the ratio of force to acceleration.

Brownian motion—Is the seemingly random movement of particles suspended in a liquid or gas or the mathematical model used to describe such random movements, often called a particle theory (named after the Scottish botanist Robert Brown).

Coupling loss factor—The coupling loss factor (CLF) gives the loss rate when power transmits from one subsystem to another.

Convolution—The process in which one function $h(\lambda)$ is convolved or folded back along another function $f(t)$. If the result of the convolution is $c(t)$ we define the process as follows

$$c(t) = \int_{-\infty}^{\infty} h(\lambda) f(t - \lambda) d\lambda.$$

Critical Frequency—When an infinite plate is excited, the frequency at which the speed of the free bending wave becomes equal to the speed of the acoustic wave in air is called the critical frequency.

Diffuse Sound Field—A diffuse sound field consists of an infinite number of statistically uncorrelated plane progressive waves. A sound field in which the time average of the mean square sound pressure is everywhere the same and the flow of acoustic energy in all directions is equally probable.

Ergodic Process—The mean and the variance of a stationary ergodic process can be computed using temporal averaging instead of ensemble averaging

$$E\{x(t)\} = \langle x(t) \rangle = \mu_x = \frac{1}{T} \int_0^T x(t) dt.$$

Excitation—Excitation is an external force (or other input) applied to a system causing it to respond.

Fokker-Planck-Kolmogorov equation—Represents the evolution in time and space of the probability density function of the states of a stochastic dynamical system.

Force limited vibration test—Any vibration test in which the force between the test item and shaker is measured and controlled.

Frequency—The frequency of a periodic function is the reciprocal of the period $\frac{1}{T}$. The unit of frequency is Hertz (Hz).

Frequency Response Function—The frequency response function for a linear system is the Fourier transform of the unit impulse function $h(\tau)$ which describes this system. In equation form

$$H(\omega) = \int_{-\infty}^{\infty} h(\tau)e^{-j\omega\tau} d\tau.$$

Gaussian Random Vibration—Gaussian random vibration is vibration which has instantaneous magnitudes that occur according to the Gaussian distribution.

Group velocity—The group velocity c_g is the velocity of energy propagation.

Linear System—A system is linear when the principle of linear superposition holds: if the output of $x_1(t)$ for the input $F_1(t)$ and $x_2(t)$ for an input $F_2(t)$ then in the input $F_1(t) + F_2(t)$ would for a linear system yield $x_1(t) + x_2(t)$ as an output.

Loss factor—The loss factor of a subsystem represents the loss percentage when the input power to the subsystem from an external excitation source is converted to the dynamical energy of the subsystem. (Also called damping loss factor, intrinsic loss factor.)

Markov (Markoff) process—A simple stochastic process in which the distribution of future states depends only on the present state and not on how it arrived in the present state.

Mechanical Impedance—Is a measure of how much a structure resists motion when subjected to a given force. It relates forces with velocities acting on a mechanical system.

Modal Effective Mass—Masses in the model consisting of SDOF systems connected in parallel to a common base, so as to represent the apparent mass of a base-driven dynamic system. the sum of the modal effective masses equals the total mass.

Modal Energy—The average energy per mode in the subsystem.

Modal Overlap—The modal overlap is the ratio of the average damping bandwidth for an individual mode to the average spacing between resonance frequencies.

Mode count—The mode count represents the number of resonance modes

available in the band of interest for the subsystem to receive and store dynamical energy.

Narrow Band—A process whose spectral density function is narrow, encompassing a small finite set of frequencies which are adjacent.

Norton's equivalent system—At a given, any linear system of loads and sources may be reduced to a simpler system consisting of a single constant velocity source in series with a single impedance connected to the load.

Notching—Reduction of acceleration input spectrum in narrow frequency bands, usually where items have resonances.

Parametric Vibration—Parametric vibration refers to the oscillatory motion that occurs in a structure or mechanical system as a result of time-dependent variation of such parameters as inertia, damping, or stiffness.

Power Spectral density—Measures how a signal is distributed in the frequency domain.

Probability distribution function—The probability density function $F(x)$ defines the probability that $x(t) \leq x$ at any value of t . In terms of the **probability density function** $f(x)$,

$$F(x) = \int_{-\infty}^x f(u)du,$$

where u is a dummy variable of integration.

Radiation efficiency—Radiation efficiency σ is defined as the proportionality between radiated sound power Π and the square of the surface normal velocity $\langle v^2 \rangle$ averaged over time and radiating surface A .

Random Process (Stochastic Process)—A random process is a set (ensemble) of time functions that can be characterized through statistical properties.

Random Vibration—Random vibration is vibration whose instantaneous magnitude is not specified for any given instant of time. Random Vibration may be broad-band, covering a wide and continuous range of frequencies, narrow-band, covering a relatively narrow range of frequencies.

Resonant Frequency—In physics, resonance is the tendency of a system to oscillate at maximum amplitude at certain frequencies, known as the system's resonant (or resonance) frequencies.

Response—The response of a system is the motion (or other output) resulting from an excitation.

Single Degree of Freedom System (SDOF)—Vibration model with one mass attached to a base with a spring and optional a damper.

Standard Deviation σ —The standard deviation is equal to the root mean square (rms) value of the deviation of a function (or a set of numbers) from the mean value. In vibration theory, the mean value is zero, therefore, the standard deviation is equal to the rms value.

Stationary process—A stationary process is a collection of time-history records having statistical properties that are invariant with respect to translations in time.

Statistical Energy Analysis—The Statistical Energy Analysis (SEA) is a technique to analyze and predict the vibro-acoustic response of a complex system by calculating the energy flow between subsystems.

Strict Stationarity—All statistical properties are invariant with time.

Subsystem—A subsystem can be seen as a part or physical element of the structure (“the system”) that is to be analyzed. Subsystems are structural or acoustical entities that have modes which are similar in nature and have equal modal energies.

Thevenin’s equivalent system—At a given, any linear system of loads and sources may be reduced to a simpler system consisting of a single constant force source in parallel with a single impedance connected to the load.

Total Energy—The total time averaged vibrational energy stored in the resonant modes of the subsystem.

Transition probability density function—Transition probability density function $f_{c2}(x_0|x, t)$ gives the probability of being in a differential element $(x, x + dx)$ of the phase plane at time t , having started at x_0 at time zero, and satisfies the Fokker-Planck equation.

Variance σ^2 —The variance is equal to the square of the standard deviation, where the mean value is zero, it is the mean square value of a variable which represents the magnitude of a vibration.

Wavelength—The wavelength is the distance between repeating units of a wave pattern.

Wave Number—The wave number is the number of wavelengths in a distance 2π

$$k = \frac{2\pi}{\lambda}.$$

Weak form—The weak form is a variational statement of the problem in which the residual is integrated against a test function. This has the effect of relaxing the problem; instead of finding an exact solution everywhere, the solution is found satisfying the strong form on average over the domain. This is illustrated by the following example:

Strong form $\frac{d^2u}{dx^2} = p_o,$

Residual form $R = \frac{d^2u}{dx^2} - p_o = 0,$

Weak form $\int_0^L Rvdxdx = 0, v$ is test function.

Weak Stationarity—The mean and the autocorrelation are invariant with time.

Wide Band—A process whose spectral density function is wide, encompassing a large finite set of frequencies which are adjacent.

Wiener Process—A Wiener process $W(t)$ starting at the origin at time zero is a Gaussian stochastic process with independent increments which are temporally homogeneous.

White Noise—A noise with a δ correlation is called a white noise, because the spectral distribution is independent of the frequency.

White Random Vibration—White random vibration has a constant acceleration spectral density over the frequency spectrum of interest. It is a form of white noise.

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