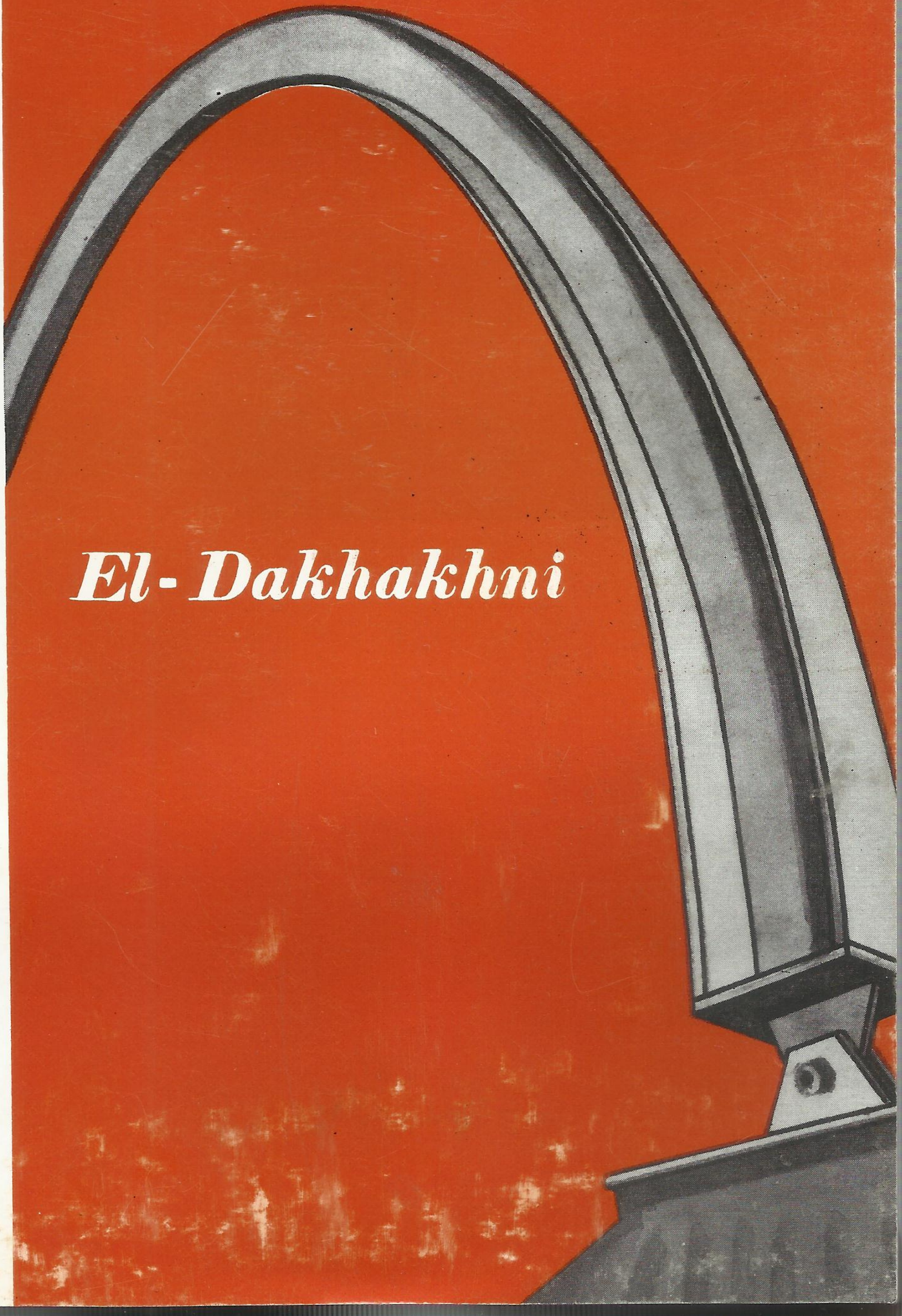
# Theory of Structures

*Part 1*



***El-Dakhakhni***

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# THEORY OF STRUCTURES

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**Part I**


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# PREFACE TO SIXTH EDITION

As a result of a new outlook among instructors and students, and a growing realization of the benefits that may be gained from a suitable text book, this edition has been prepared.

New material has been added particularly in chapters dealing with stresses. In chapter 8 on normal stresses, the general equation of stress and the core theory have been introduced. In chapter 9 on shear stresses, torsion of non-circular solid sections and thin-walled open and closed sections has been included. Further, additional problems to be worked out are presented at the end of each chapter so as to give the instructor a wider choice in the selection of the problems the student may solve.

On the preparation of this edition, no effort was spared to meet most of the advices and suggestions offered by my colleagues and students alike. The many critical questions that my former students have asked, while seeking to clarify the basic principles in their own minds, have been of immeasurable help in this respect.

The publication of Part 2 of this book will follow this edition. While Part 1 is mainly concerned with the analysis of statically determinate structures, Part 2 will deal with deformations and the analysis of statically indeterminate structures.

It was hoped to present Part 2 earlier but this was postponed owing to a full-time engagement of the author on a major research programme conducted at Lehigh University, U.S.A., which lasted almost two years.

Finally, the author wishes to extend his gratitude to Architect, S. El-Komey and Mr. A. Rasmey who were responsible for the design of the new cover of this book.

Cairo, December 1983

**W.M. El-Dakhakhni**

## PREFACE TO FIRST EDITION

Dictating notes to students consumes a great deal of effort on both the instructor and the students sides, and wastes a large part of the lecture's time. Should this effort and time be saved, they could be directed towards better understanding of the principles taught and attempting to cover more material.

Although several good English-written books are available on the subject, they do not serve the purpose because in these the ft-1b system is adopted. Moreover, the subjects taught in our universities under Theory of Structures may be partly found in books on Theory of Structures and partly in books on Strength of Materials.

This book is designed to serve the need of our engineering students and to introduce the Theory of Structures for those among them who are approaching the subject for the first time.

The content covers the course taught to engineering students in their first year and a significant part of their second year. It is hoped that, by the aid of this book, the latter part may be relegated to the first year course so as to give room to modern and more advanced material to be taught in subsequent courses.

The book contains a large number of fully worked out examples. A large proportion of these are based on questions set in examination papers of both Egyptian and foreign universities. Also, the book contains a large number of examples to be worked out that appear at the end of each chapter. These are presented in the same sequence as the various sections within the chapter and are generally arranged in order of difficulty. In many cases the data given are so chosen for Arithmetical simplicity.

The author is thankful to his colleagues in the Civil Engineering Dept., Assiut University for many fruitful discussions. Mrs. Gamal El-Dakhakhni for typing the manuscript, and the Staff of Dar Al-Maaref for their

wonderful co-operation without which the production of this book in this form would not have been possible.

At last, and by no means the least, the author wishes to express his gratitude to his family particularly his parents, to whom this book is dedicated, fore their continuous encouragement.

**W.M. El-Dakhakhni**

Cairo, September 1968.

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CHAPTER 1

# PRINCIPLES OF PLANE STATICS

## 1.1 Introduction

Statics is that branch of Mechanics which deals with the determination of forces keeping a rigid body in a state of equilibrium. There are two methods of solving problems of this nature; graphical and analytical. The selection of the method depends on the type of the problem and on personal preference of the student. Among the problems that are solved more advantageously by graphical methods are the determination of the member forces and the resultant deflection of various truss joints. The author finds that not only is a knowledge of graphical methods helpful in the solution of probfems such as these, but also that it emphasizes the analytical solution and helps the student to visualize the physical meaning of the problem at hand.

In this book, therefore, graphic statics will be dealt with in details and discussion of its application to various problems will be made in the appropriate place.

## 1.2 Definitions

Rigid body : Structures and structural members are referred to as rigid bodies. In the exact sense, a rigid body is one whose dimensions never change under any applied forces. It must be remembered that there is no perfect rigid body as structures are made of materials that deform slightly under the application of loads.

Force : A force is any action that changes the state of body from rest to motion or vice versa. It is completely specified by : (1) magnitude in terms of a chosen unit; kilogramme (kg) or ton (t), (2) point of application, (3) direction. The last two particulars may be replaced by the line of action and sense of direction (arrow head). For example, a force of 3 t. acting at point a and in a direction making an angle of 30° with the x-axis could be equally specified by saying 3 t. acting along line ab from a to b. A force is a vector quantity as it has both magnitude and direction. Hence it may be represented graphically by a line drawn through the point of application and having a length equal to the magnitude of the force to a suitable scale. The slope of this line indicates the direction of the force and an arrow head the sense in which the force acts. Referring to Fig. 1.1, the force P is completely defined by the vector ab; given as P units of length, acting through point a in a direction making an angle $\theta$ with the x-axis from a to b.

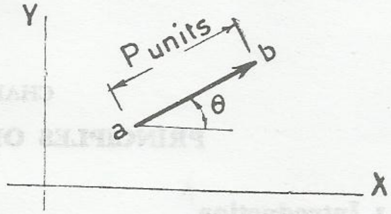


Fig. 1.1

As far as the equilibrium condition of a rigid body is concerned the effect of a force may be considered the same at any point along its line of action. For example, the reactions $R_a$ and $R_b$ of the simple truss shown in Fig. 1.2 are the same whether the load acts at c or d. Only the internal forces in the truss would change. With P applied at c, member cd has zero force, and if P is applied at d member cd will have a force equal to P.

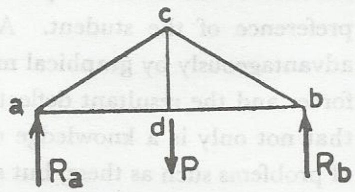


Fig. 1.2

Moment : Moment of a force is its ability to make a body turn. Referring to Fig. 1.3, the moment of a force about any point o in its plane is given by the product of the force and the perpendicular distance from o to its line of action, i.e. moment of force P about o; $M_o = Pr$. The units of moment are distance times force, i.e. cm.kg. or m.t.

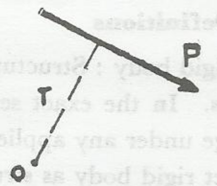


Fig. 1.3

Concurrent forces : A concurrent force system is shown in Fig. 1.4 a. It consists of a number of forces whose lines of action meet in a common point.

Parallel forces : A parallel force system consists of a number of forces whose lines of action are all parallel as shown in Fig. 1.4 b.

Nonconcurrent forces : A system of nonconcurrent forces is shown in Fig. 1.4 c. It consists of a number of forces in various directions and their lines of action do not meet in a common point.

Couple : Any two forces equal in magnitude, opposite in direction and not having a common line of action form a couple. This system of

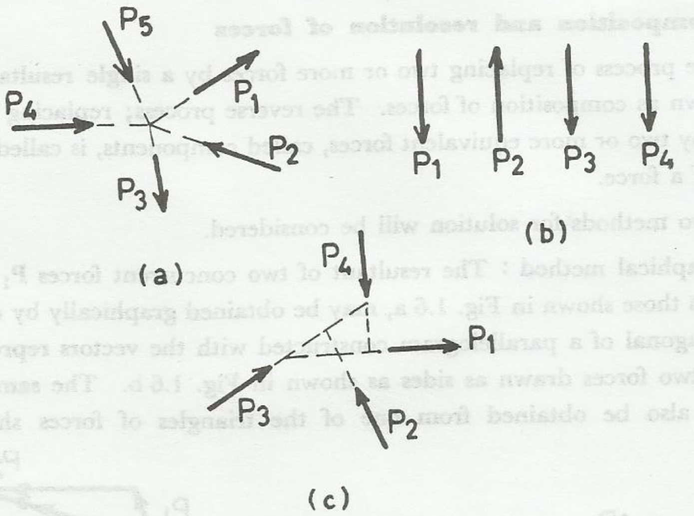



Fig. 1.4

forces has a purely rotational effect on any rigid body. Referring to Fig. 1.5 and taking moments about point m then, moment of couple = $P \times 0 + P \times mn = Pr$. The same result can be obtained by taking moments about n, o, or any other point in the plane of the forces. Thus, it can be deduced that the moment of a couple about any point in its plane is constant and is equal to the product of one of the forces and the perpendicular distance between the lines of action of the two forces.

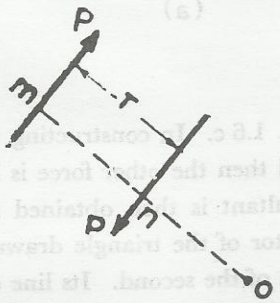



Fig. 1.5

Resultant : The resultant of a system of forces acting on a rigid body is a single force or couple, which has the same effect on the equilibrium of the body under consideration.

Equilibrant : A force which holds a system of forces in equilibrium is called the equilibrant of these forces.

Equilibrium : A body which is initially at rest and remains so when acted upon by a system of forces is said to be in a state of static equilibrium. For a rigid body to be in equilibrium, it is necessary that the resultant is neither a force nor a couple otherwise it will tend to cause the body to translate or rotate.

**1.3 Composition and resolution of forces**

The process of replacing two or more forces by a single resultant force is known as composition of forces. The reverse process; replacing a single force by two or more equivalent forces, called components, is called resolution of a force.

Two methods for solution will be considered.

Graphical method : The resultant of two concurrent forces $P_1$ and $P_2$ such as those shown in Fig. 1.6 a, may be obtained graphically by drawing the diagonal of a parallelogram constructed with the vectors representing these two forces drawn as sides as shown in Fig. 1.6 b. The same result could also be obtained from one of the triangles of forces shown in

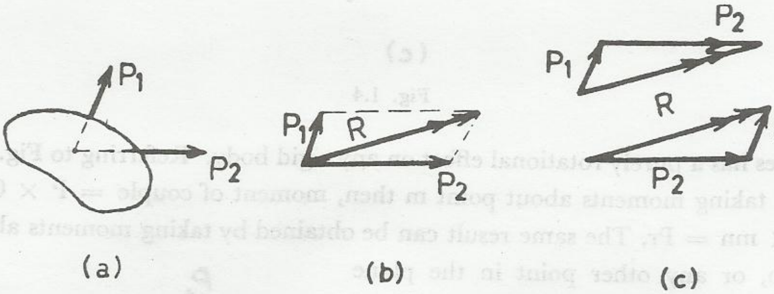



Fig. 1.6

Fig. 1.6 c. In constructing these triangles, either force may be drawn first and then the other force is laid out from the end of the first vector. The resultant is then obtained in magnitude and direction from the closing vector of the triangle drawn from the beginning of the first vector to the end of the second. Its line of action must pass through the point of intersection of $P_1$ and $P_2$ otherwise it would not have the same effect as the two forces.

If a force R is required to be resolved into two components $P_1$ and $P_2$ any two of the four quantities defining the two components; magnitude and direction of each, must be given and the other two are found from the force triangle.

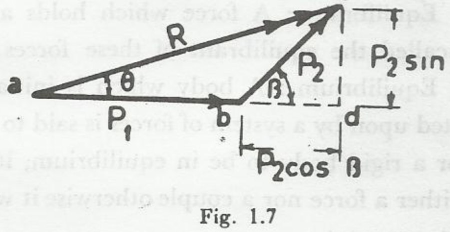



Fig. 1.7

Analytical method : Referring to the triangle of forces shown in Fig. 1.7,

$$ac = \sqrt{\overline{ad}^2 + \overline{dc}^2}$$

$$R = \sqrt{(P_1 + P_2 \cos\beta)^2 + (P_2 \sin \beta)^2} \quad \ldots 1.1$$

$$\tan \theta = \frac{cd}{ad} = \frac{P_2 \sin \beta}{P_1 + P_2 \cos \beta} \quad \ldots 1.2$$

Equations 1.1 and 1.2 give the magnitude and direction of the resultant R.

The reverse process, that of replacing a force by two components is possible provided that two of the unknowns defining the two components are given. These may be the directions of the two components, their magnitudes, or the magnitude and direction of one. It is customary, however, to resolve a force along two directions perpendicular to each other, taken for convenience as the x and y directions. In Fig. 1.8, X and Y are the rectangular components of R.

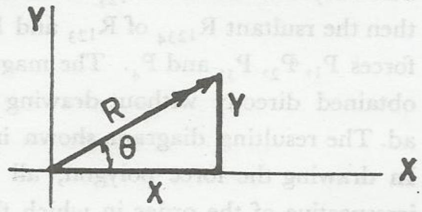


Fig. 1.8

$$X = R \cos \theta \quad \ldots 1.3$$

$$Y = R \sin \theta \quad \ldots 1.4$$

Given the rectangular components, the magnitude and direction of the resultant are given by :

$$R = \sqrt{X^2 + Y^2} \quad \ldots 1.5$$

$$\tan \theta = \frac{Y}{X} \quad \ldots 1.6$$

## 1.4 Resultant of a system of concurrent forces

Graphical method : Consider a system of concurrent forces, $P_1$, $P_2$, $P_3$, and $P_4$, as shown in Fig. 1.9 a. This figure, which shows the forces in their relative positions is called a space diagram. Referring to Fig. 1.9 b, the resultant $R_{12}$ of the two forces $P_1$ and $P_2$ is obtained in magnitude and direction from the triangle of forces abc and its line of action is parallel to

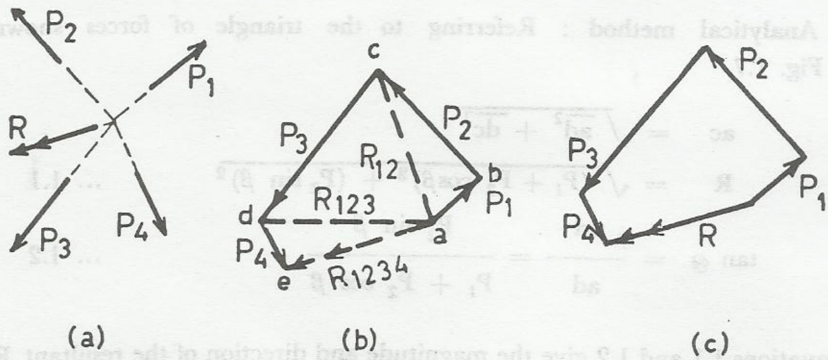



Fig. 1.9

vector ac and through the common point of intersection of the forces. Similarly. the resultant $R_{123}$ of the forces $R_{12}$ and $P_3$ may be obtained, and then the rsultant $R_{1234}$ of $R_{123}$ and $P_4$ which is the resultant of all the four forces $P_1$, $P_2$, $P_3$, and $P_4$. The magnitude and direction could have been obtained directly without drawing the intermediate dashed lines, ac and ad. The resulting diagram, shown in Fig. 1.9 c, is called the *force polygon.* In drawing the force polygon, all the forces are arranged in one sense, irrespective of the order in which they are drawn, and then the magnitude and direction of the resultant are given by the closing vector of the polygon drawn in the opposite sense. Its line of action is obviously parallel to vector ae and passes through the common point of intersection.

Analytical method : The first step in the analytical solution is to choose two suitable rectangular axes, usually taken as the x and y axes, and then resolve all the forces along these axes not forgetting that components acting in opposite directions have opposite signs.

Referring to Fig. 1.10,

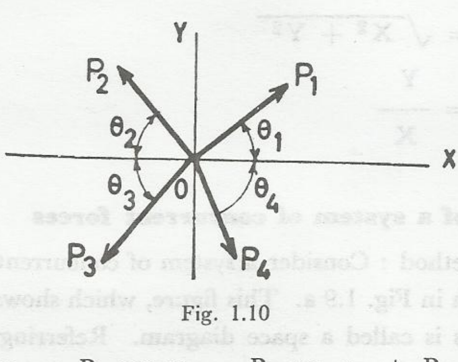



Fig. 1.10

$$X = P_1 \cos \theta_1 - P_2 \cos \theta_2 - P_3 \cos \theta_3 + P_4 \cos \theta_4$$

$$Y = P_1 \sin \theta_1 + P_2 \sin \theta_2 - P_3 \sin \theta_3 - P_4 \sin \theta_4$$

The resultant acts though o and its magnitude and direction are given by :

$$R = \sqrt{X^2 + Y^2}$$

$$\tan \Theta = \frac{Y}{X}$$

**Example 1.1** Determine graphically and analytically the resultant of the concurrent forces shown in Fig. 1.11 a.

Graphical solution : Starting from point a, Fig. 1.11 b, draw vectors representing forces 2, 3, 5 and 4 t. The resultant is given by R (vector ab) which scales 1.8 t and acts through point o in the direction shown in the force polygon, Fig. 1.11 b.

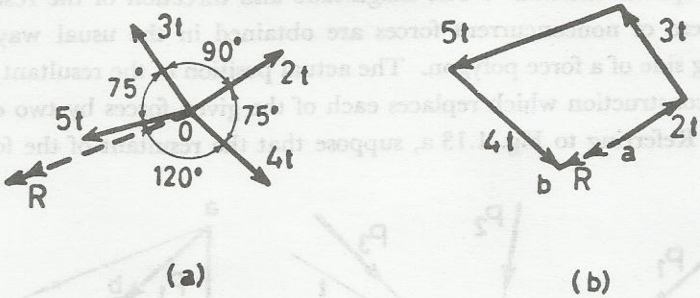



Fig. 1.11

Analytical solution : Choosing the rectangular axes as the x and y axes, and resolving along these directions then referring to Fig. 1.12,

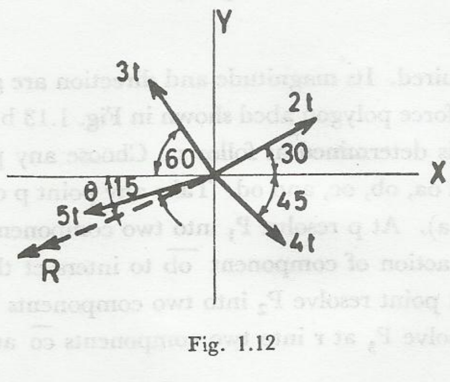



Fig. 1.12

$$X = 2\cos 30 - 3\cos 60 - 5\cos 15 + 4\cos 45$$
$$= 2 \times 0.866 - 3 \times 0.5 - 5 \times 0.966 + 4 \times 0.707$$
$$= 1.73 - 1.50 - 4.83 + 2.83 = 1.77\ t \leftarrow$$
$$Y = 2\sin 30 + 3\sin 60 - 5\sin 15 - 4\sin 45$$
$$= 2 \times 0.5 + 3 \times 0.866 - 5 \times 0.286 - 4 \times 0.707$$
$$= 1.0 + 2.60 - 1.43 - 2.83 = 0.66\ t \downarrow$$
$$R = \sqrt{1.77^2 + 0.66^2} = 1.84\ t \leftarrow$$
$$\tan \Theta = \frac{0.66}{1.77} = 0.373, \text{ i.e. } \Theta = 20^\circ 27'$$

### 1.5 Resultant of a system of nonconcurrent forces

Graphical method : The magnitude and direction of the resultant of a system of nonconcurrent forces are obtained in the usual way by the closing side of a force polygon. The actual position of the resultant is found by a construction which replaces each of the given forces by two components. Referring to Fig. 1.13 a, suppose that the resultant of the forces $P_1$,

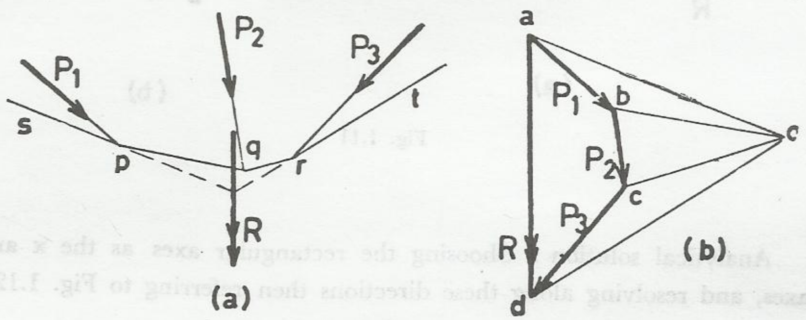



Fig. 1.13

$P_2$ and $P_3$ is required. Its magnitude and direction are given by the closing vector ad in the force polygon abcd shown in Fig. 1.13 b. The line of action of the resultant is determined as follows : Choose any point o in the force polygon and join oa, ob, oc, and od. Take any point p on the line of action of $P_1$, (Fig. 1.13 a). At p resolve $P_1$ into two components $\overline{ao}$ and $\overline{ob}$. Extend the line of action of component $\overline{ob}$ to intersect the line of action of $P_2$, at q. At this point resolve $P_2$ into two components $\overline{bo}$ and $\overline{oc}$. In the same manner resolve $P_3$ at r into two components $\overline{co}$ and $\overline{od}$. Thus, the original system of forces is replaced by six components; two pairs of which $\overline{ob}$ & $\overline{bo}$ and $\overline{oc}$ & $\overline{co}$ being equal and opposite, cancel each other. The resultant of the six components, and therefore of the original force system, is the resultant of the two remaining components $\overline{ao}$ and $\overline{od}$ and acts through the point of intersection of their lines of action.

The diagram constructed from the force polygon is called the *polar diagram*, o is the *pole*, and each of the lines oa, ob, and od, a *ray*. The diagram constructed from the lines spqrt on the space diagram is called the *link* or *funicular polygon* and each of the lines sp, pq, qr and rt, a *link*. It is important to remember that a link is drawn between the lines of action of two forces which are adjacent to each other and also parallel to the ray directed through the intersection of these two forces in the force polygon.

Analytical method : In order that the resultant of a system of noncon-current forces may have the same effect as the given system, its components along two perpendicular axes must be equal to the algibraic sum of the components of the given forces along the same axes and also it must have the same rotational effect, i.e. same moment about any point in the plane of the given force system. Referring to Fig. 1.14, if R is the resultant and X and Y are its components along the x and y axes respectively then,

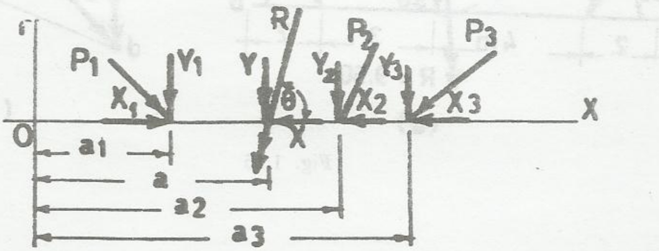



Fig. 1.14

$$X = \Sigma X = X_2 + X_3 - X_1 \leftarrow$$

$$Y = \Sigma Y = Y_1 + Y_2 + Y_3 \downarrow$$

$$R = \sqrt{X^2 + Y^2}$$

$$\tan \theta = \frac{Y}{X}$$

The last two equations give the magnitude and direction of the resultant. Its position is determined from the condition that the moment of the resultant about any point in the plane must be equal to the algibraic sum of the moments of the given system of forces. Hence, choosing any point o and taking moments of the given forces, or for convenience in this case of the components of the given forces, about this point then,

$$Y\ a = Y_1\ a_1 + Y_2\ a_2 + Y_3\ a_3$$

$$\text{or}\quad a = \frac{Y_1\ a_1 + Y_2\ a_2 + Y_3\ a_3}{\Sigma Y} \qquad \ldots\ 1.7$$

i.e. the resultant passes through a point distance a; given by equation 1.7, from the chosen point o.

**Example 1.2** Determine graphically and analytically the resultant of the forces acting on beam ab shown in Fig. 1.15 a.

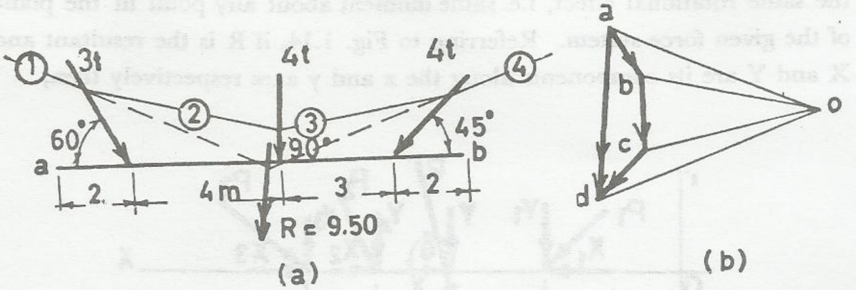



Fig. 1.15

Graphical solution : Draw the force polygon (Fig. 1.15 b) to a convenient scale. The resultant is given by ad which scales 9.5 t. Choose pole o and join oa, ob, oc and od. Draw the link polygon (Fig. 1.15 a) with links 1, 2, 3 and 4 parallel to their corresponding rays oa, ob, oc and od. Extend links 1 and 4 to meet in a point. From this point, draw a parallel to ad to get the line of action of the resultant.

Analytical solution : Choose the x and y axes as ab and the normal to ab through a.

$$X = 3 \cos 60 + 4 \cos 90 - 4 \cos 45$$

$$= 3 \times 0.5 - 4 \times 0.707 = -1.33, \text{ i.e. } 1.33\ t \leftarrow$$

$$Y = 3 \sin 60 + 4 \sin 90 + 4 \sin 45$$
$$= 3 \times 0.866 + 4 + 4 \times 0.707 = 9.43 \text{ t} \downarrow$$
$$R = \sqrt{1.33^2 + 9.43^2} = 9.5 \text{ t} \swarrow$$
$$\tan \theta = \frac{9.43}{1.33} = 7.0902 \quad \text{i.e. } \theta = 81^0\, 58'$$

Taking moments about a and assuming that the resultant R cuts ab at a distance x from a then,

$$9.43 \times x = 3 \times 0.866 \times 2 + 4 \times 6 + 4 \times 0.707 \times 9$$
$$x = \frac{54.65}{9.43} = 5.8 \text{ m.}$$

## 1.6 Conditions of equilibrium for concurrent forces

Referring to Fig. 1.9 a which is reproduced in Fig. 1.16a, if a force $P_5$ equal and opposite to the resultant R were added to the given force system, the resultant would be zero and therefore the force polygon must close as shown in Fig. 1.16 b. This indicates that the algebraic sums of the components along any chosen rectangular axes are zero.

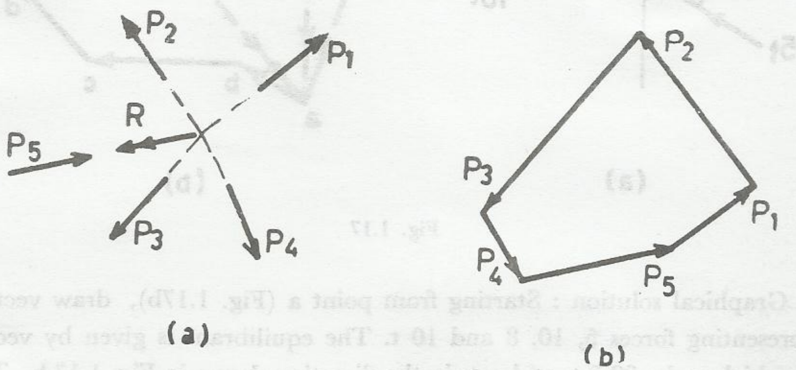



Fig. 1.16

Hence, if several concurrent forces are known to be in equilibrium, the following conditions must be satisfied.

Graphical : Closed force polygon.

Analytical : $\Sigma X = 0$ ... 1.8

and $\Sigma Y = 0$ ... 1.9

From the analytical conditions, it is seen that there are only two equations of equilibrium. Hence, the maximum number of unknowns that could be determined is two. These may be :

(a) A single force in magnitude and direction.

(b) The magnitudes of two forces of known directions

(c) The magnitude of one force and the direction of another.

**Example 1.3** For the system of concurrent forces shown in Fig. 1.17 determine graphically and analytically.

(a) The single force required to keep them in equilibrium.

(b) The two balancing forces along xx and yy.

(c) The two balancing forces; one of magnitude 18t. and the other acts along xx.

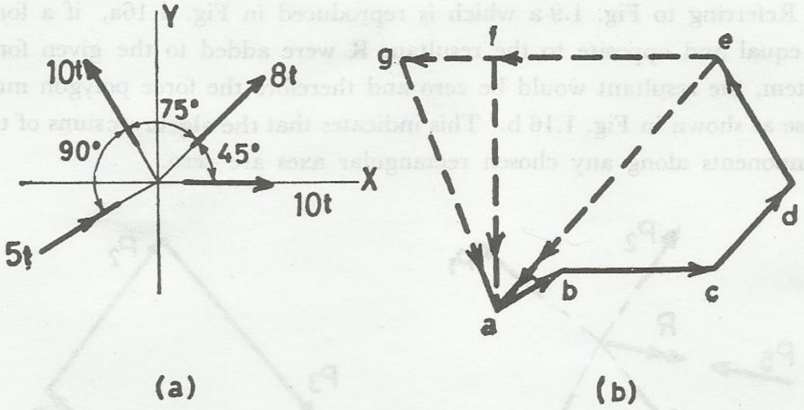



Fig. 1.17

Graphical solution : Starting from point a (Fig. 1.17b), draw vectors representing forces 5, 10, 8 and 10 t. The equilibrant is given by vector $\overline{ea}$, which scales 22.5 t. and acts in the direction shown in Fig. 1.17 b. The balancing forces along xx and yy are given by vectors $\overline{ef}$ and $\overline{fa}$. These forces scale 15 t. and 17 t. respectively and act in the directions shown in Fig. 1.17 b. The two required forces in (c) are given by vectors $\overline{ga}$ and $\overline{eg}$ and these forces are 18 t. and 21 t. respectively.

Analytical solution : Assuming that P is the equilibrant required in (a), X and Y are its components along the x and y axes, and resolving along these two directions,

$$\Sigma X = 0 = X + 10 + 8 \cos 45 - 10 \cos 60 + 5 \cos 30$$

$$X = -10 - 8 \times 0.707 + 10 \times 0.5 - 5 \times 0.866 = -14.99 \text{ t},$$

i.e. 14.99 t ←

$$\Sigma Y = 0 = Y + 8 \sin 45 + 10 \sin 60 + 5 \sin 30$$

$$Y = -8 \times 0.707 - 10 \times 0.866 - 5 \times 0.5 = -16.57 \text{ t.}$$

i.e. 16.57 ↓

$$P = \sqrt{14.99^2 + 16.57^2} = 22.5 \text{ t} \quad \swarrow$$

$$\tan \Theta = \frac{16.57}{14.99} = 1.1054, \qquad \text{i.e. } \Theta = 46^0\ 55'$$

The two balancing forces required in (b) are :

X = 14.99 t ← and Y = 16.57 t ↓

The two balancing forces required in (c) are found as follows : Assuming that the force of 18 t. makes an angle $\Theta$ with the x-axis and that the magnitude of the other along the positive direction of the x-axis is F then by resolving along the x and y axes and equating to zero, two equations are obtained from which the two unknowns, $\Theta$ and F, are determined. Thus :

$$\Sigma X = 0 = F + 10 + 8 \cos 45 - 10 \cos 60 + 5 \cos 30 + 18 \cos \Theta$$

$$\Sigma Y = 0 = 8 \sin 45 + 10 \sin 60 + 5 \sin 30 + 18 \sin \Theta$$

From the second equation, $\sin \Theta = -0.934$, i.e. $\Theta = -69^0\ 10'$

Substituting the value of $\Theta$ in the first equation,

$$F = -10 - 8 \times 0.707 + 10 \times 0.5 - 5 \times 0.866 - 18 \times 0.345$$

$$= -10 - 5.66 + 5 - 4.33 - 6.2 = -21.19, \text{ i.e. } 21.19 \text{ t} \leftarrow$$

(Note that the negative sign for F means that its direction is opposite to that assumed).

### 1.7 Conditions of equilibrium for nonconcurrent forces

Referring to Fig. 1.13 a which is reproduced in Fig. 1.18 a, if a force $P_4$ equal and opposite to the resultant R were added to the given system, and if in addition it were collinear with the resultant then a system in equilibrium is obtained. For this system the force polygon is closed indicating

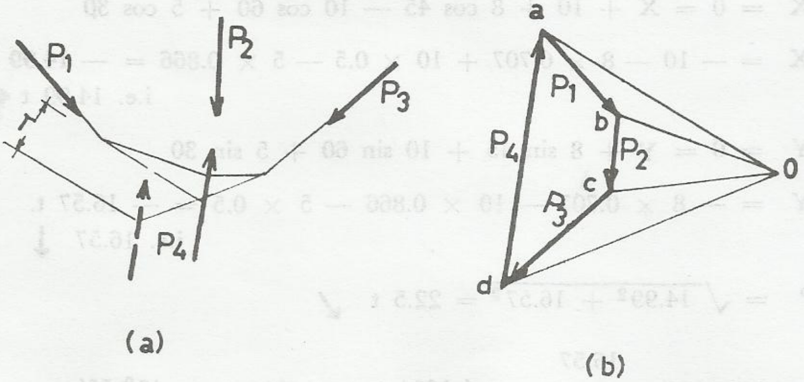



Fig. 1.18

that the algebraic sums of the components along the chosen rectangular axes are zero. Also, when the link polygon is drawn, it will be found that it closes on itself. From the link polygon it is seen that the given force system is replaced by four pairs of forces, each of which consists of two equal, opposite and collinear forces. Therefore the original system is in equilibrium.

Suppose that $P_4$ were not collinear with R but acts parallel to it along the dashed-line shown in Fig. 1.18a. The force polygon will still close but from the link polygon it is seen that the given force system is replaced by three equal and opposite pairs of forces in equilibrium, and a pair of forces equal, opposite and at a distance r apart and thus forms a couple of moment equal to oa × r clockwise (oa being measured to force scale and r to linear scale). Therefore, the given system of forces is not in equilibrium.

Hence, if several nonconcurrent forces are known to be in equilibrium, the following conditions must be satisfied.

Graphical : Closed force polygon.

Closed link polygon.

If the force polygon closes but the link polygon does not, the given system of forces will be equivalent to a couple.

Analytical : $\Sigma X = 0$ ... 1.10

$\Sigma Y = 0$ ... 1.11

$\Sigma M_o = 0$ ... 1.12

where o is any point in the plane of the forces.

From the analytical conditions, it is seen that there are three equations. Hence the maximum number of unknowns that could be determined is three. These may be :

(a) Three forces known in direction but of unknown magnitudes.

(b) The magnitude only of one force and the magnitude and direction of another force.

It should be remembered that a parallel force system is a special case of nonconcurrent forces. In this case, however, the analytical condition $\Sigma X = 0$ is initially satisfied. Hence, the maximum number of unknowns that can be determined for a system of parallel forces in equilibrium is two only.

## 1.8 Three-force theorem

*"Three nonparallel forces in equilibrium must intersect in a common point. 'The magnitudes and directions of these forcese are given by the sides of a triangle of forces with the arrow heads in one continuous sense".*

This is obvious when one notices that any two of the three forces can be replaced by a resultant force through their point of intersection and that in order that the third force may be in equilibrium with the resultant, their lines of action must be coinsident.

This theorem is commonly used in the solution of statical problems. Its importance is demonstrated in the solution of the problems given in the following section.

## 1.9 Treatment of some problems in statics

(1) Determination of two parallel equilibrants of a system of parallel forces.

Graphical method : Consider the system of parallel forces $P_1$, $P_2$ and $P_3$ shown in Fig. 1.19 a, and suppose it is required to find the two parallel forces through a and b that would keep the system in equilibrium. If R is the resultant of the two parallel equilibrants required then, for equilibrium conditions, it must be equal and opposite to the resultant of the given force system. Hence the two required equilibrants must be parallel to the given force system. Draw the part in the force polygon corresponding

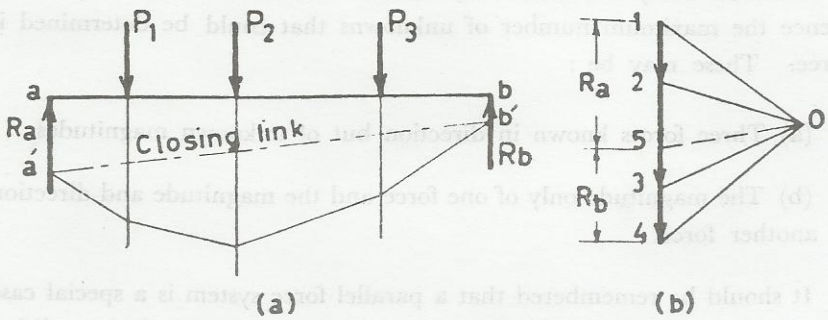



Fig. 1.19

to $P_1$, $P_2$ and $P_3$. Choose a pole 0 and join 01, 02, 03 and 04. Draw a link parallel to ray 01 between $P_1$ and the known line of action of $R_a$. Similarly, draw successively the links between $P_1$ & $P_2$, and $P_2$ & $P_3$, and then between $P_3$ and the line of action of $R_b$. Since $R_a$ and $R_b$ are both vertical, point 5 in the force polygon (Fig. 1.19 b) must lie on the line 1234. Also, since $P_1$, $P_2$, $P_3$, $R_a$ and $R_b$ are known to be in equilibrium, the link polygon must close, which condition is satisfied by drawing the line a'b' which is called the *closing link*. This link lying between $R_a$ and $R_b$ in the space diagram must have a corresponding parallel ray in the polar diagram. Hence, 5 is the point of intersection between ray 05 drawn parallel to the closing link and line 1234. Vectors $\overline{51}$ and $\overline{45}$, therefore, represent $R_a$ and $R_b$ respectively.

Analytical method : Referring to Fig. 1.20 and taking moments about a,

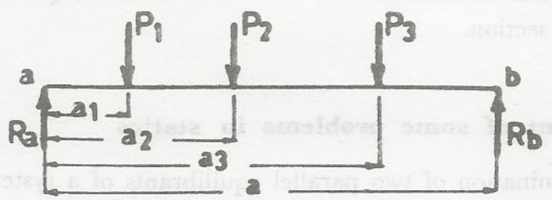



Fig. 1.20

$$\Sigma M_a = 0 = R_b\, a - P_1\, a_1 - P_2\, a_2 - P_3\, a_3$$

$$R_b = \frac{P_1\, a_1 + P_2\, a_2 + P_3\, a_3}{a} \uparrow$$

$$\Sigma Y = 0 = R_a - P_1 - P_2 - P_3 + R_b$$

$$R_a = P_1 + P_2 + P_3 - R_b \uparrow$$

Note that a check can be applied to the values thus obtained from the $\Sigma M_b = 0$ condition.

**Example 1.4** Fig. 1.21 shows a system of parallel forces. Find analytically and graphically the equilibrants A and B passing through a and b if A and B are parallel.

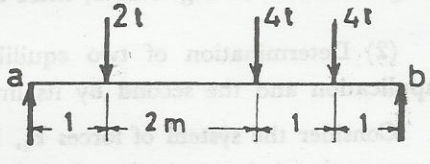



Fig. 1.21

Analytical solution :

Since A and B are parallel and since the equation $\Sigma X = 0$ must be satisfied, it follows that both A and B must be parallel to the given force system, i.e. vertical.

$$\Sigma M_a = 0 = B \times 5 - 4 \times 4 - 4 \times 3 - 2 \times 1$$

$$B = \frac{16 + 12 + 2}{5} = 6 \text{ t.} \uparrow$$

$$\Sigma M_b = 0 = A \times 5 - 2 \times 4 - 4 \times 2 - 4 \times 1$$

$$A = \frac{8 + 8 + 4}{5} = 4 \text{ t.} \uparrow$$

As a check the condition $\Sigma Y = 0$ must be satisfied. Hence,

$$\Sigma Y = 4 - 2 - 4 - 4 + 6 = 0$$

Graphical solution :

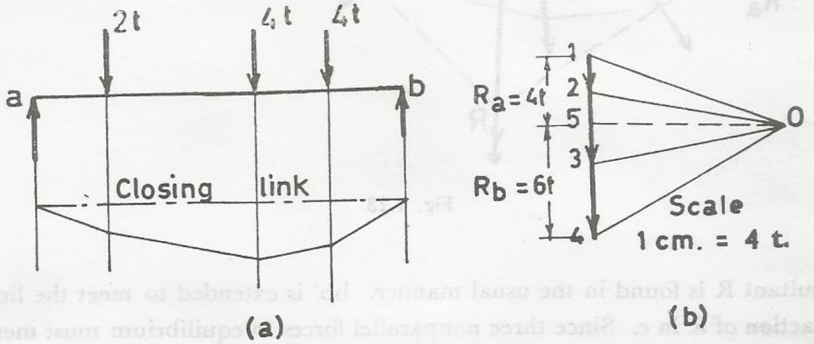



Fig. 1.22

Draw the force polygon for the given forces to a suitable scale. Choose any pole 0, join 01, 02, 03 and 04 and draw the corresponding link polygon.

From 0 draw a parallel to the closing link to intersect line 1234 in 5. Then vector $\overline{51}$ represents equilibrant A and vector $\overline{45}$ equilibrant B. From the diagram shown in Fig. 1.22 b, these scale 4 t. and 6 t. respectively.

(2) Determination of two equilibrants; one given by its point of application and the second by its line of action.

Consider the system of forces $P_1$, $P_2$ and $P_3$ shown in Fig. 1.23a and suppose that it is required to find the two forces that keep them in equilibrium. In this case, it is known that the left equilibrant passes through the given point a and that the right equilibrant acts along the given line bb. There are two graphical solutions to this problem :

(a) By use of the three-force theorem.

Referring to Fig. 1.23, the force and link polygons are drawn and the

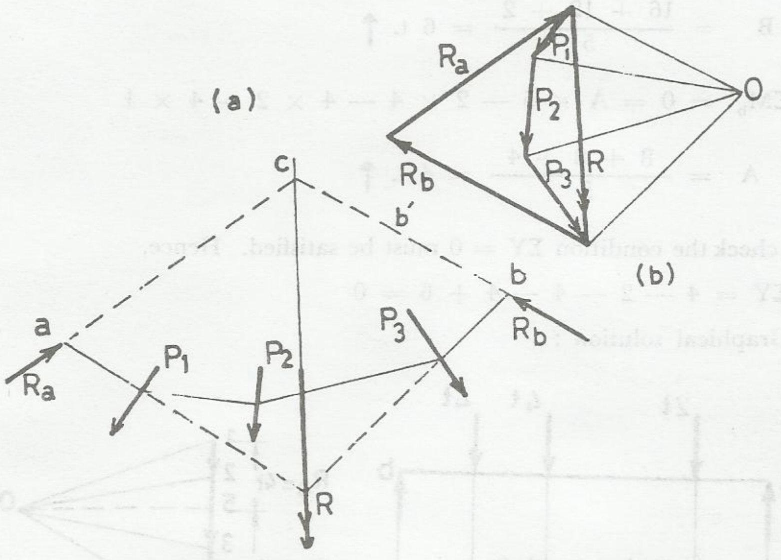



Fig. 1.23

resultant R is found in the usual manner. bb′ is extended to meet the line of action of R in c. Since three nonparallel forces in equilibrium must meet in a point, the equilibrant through a must also pass through c. Knowing the directions of $R_a$ and $R_b$, their magnitudes are obtained from the force polygon (Fig. 1.23 b) by resolving R along ca and cb.

(b) By use of the link polygon.

Draw the part of the force polygon corresponding to the given forces $P_1$, $P_2$ and $P_3$. Choose a pole 0 and draw rays to points 1, 2, 3 and 4.

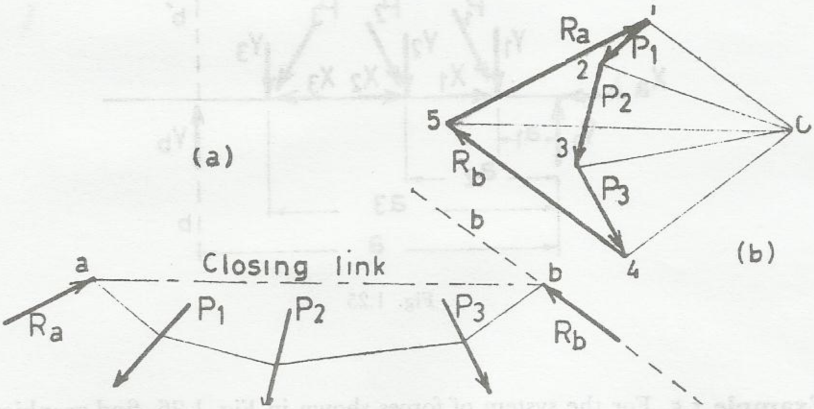



Fig. 1.24

Through the given point a, draw a link parallel to ray 01 between $P_1$ and the unknown line of action of $R_a$. Similarly, draw successively the links between $P_1$ & $P_2$, and $P_2$ & $P_3$ and then between $P_3$ and the known line of action of $R_b$. This last link intersects the given line of action of $R_b$ at b. For equilibrium, the link polygon must close and ab will be the closing link. Draw the corresponding ray in the force polygon (Fig. 1.24 b) parallel to the closing link. This ray must pass through the intersection of the vectors representing $R_a$ and $R_b$. Now this point of intersection lies on the ray parallel to the closing link and also on the line drawn from point 4 in the force polygon parallel to bb and is thus determined. Vectors 51 and 45, therefore, give the magnitudes and directions of $R_a$ and $R_b$ respectively.

Analytical method :

Choose any two rectangular axes, for convenience taken as bb and the normal to it through the given point a, and resolve all the forces along these chosen axes. Then applying the three equations of equilibrium, and refer ring to Fig. 1.25,

$$\Sigma X = 0 \ , \ X_a = X_1 + X_2 - X_3 \rightarrow$$

$$\Sigma M_a = 0 \ , \ Y_b = \frac{Y_1 a_1 + Y_2 a_2 + Y_3 a_3}{a} \uparrow$$

$$\Sigma Y = 0 \ , \ Y_a = Y_1 + Y_2 + Y_3 - Y_b \uparrow$$

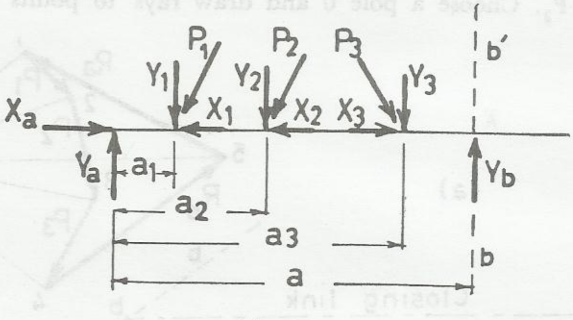



Fig. 1.25

**Example 1.5** For the system of forces shown in Fig. 1.26, find graphically and analytically the equilibrants A and B passing through a and b if B is vertical.

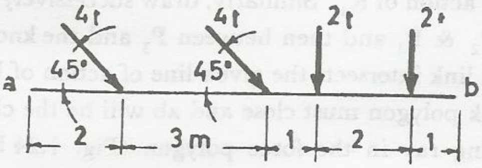



Fig. 1.26

Graphical solution :

The force and link polygons are drawn and the resultant R is found as shown in Fig. 1.27. Extend the line of action of R to meet the vertical through b in point c. Join c to a. ca and cb are the lines of action of the equilibrants A and B respectively. R is resolved along these two directions in the force polygon and the forces are scaled. These are found to be : A = 7.1 t. and B = 5.3 t.

Analytical solution :

$$\Sigma M_a = 0 = 9\,B - 2 \times 8 - 2 \times 6 - \frac{4}{\sqrt{2}} \times 5 - \frac{4}{\sqrt{2}} \times 2$$

$$B = \frac{1}{9}\,(16 + 12 + 14.2 + 5.68) = 5.3\text{t.} \uparrow$$

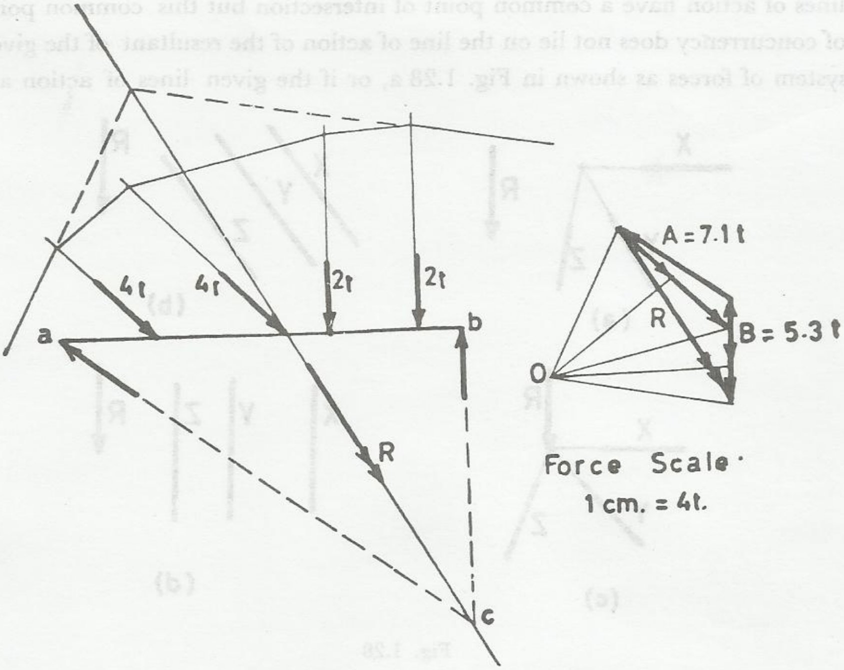



Fig. 1.27

$$\Sigma X = 0 = X_a \quad \frac{4}{\sqrt{2}} \quad \frac{4}{\sqrt{2}}$$

$$X_a = 5.68 \text{ t.} \leftarrow$$

$$\Sigma Y = 0 = Y_a - \frac{4}{\sqrt{2}} - \frac{4}{\sqrt{2}} - 2 \quad 2 + 5.3$$

$$Y_a = 4.38 \text{ t.} \uparrow$$

$$A = \sqrt{5.68^2 + 4.38^2} = 7.16 \text{ t.}$$

and makes an angle $\Theta$ with the horizontal,

$$\tan \Theta = \frac{4.38}{5.68} = 0.772$$

$$\Theta = 37^0 \ 40'$$

(3) Determination of three equilibrants given by their lines of action. A determinate solution to this problem is only possible provided that

the given lines of action are neither parallel nor concurrent. If the given lines of action have a common point of intersection but this common point of concurrency does not lie on the line of action of the resultant of the given system of forces as shown in Fig. 1.28 a, or if the given lines of action are

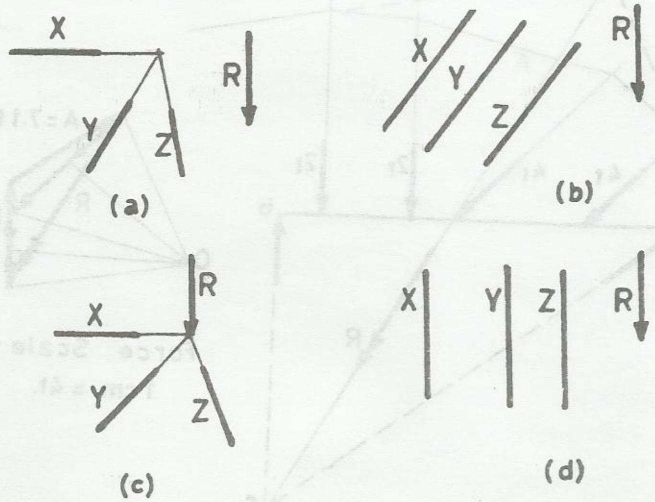



Fig. 1.28

parallel but not parallel to the resultant as shown in Fig. 1.28 b, a solution is not possible. This is obvious as in both cases the unknown equilibrants may be replaced by a single force which is impossible to be collinear with R and therefore can never be in equilibrium. If, however, the given lines of action and the resultant of the given system of forces have a common point of concurrency as shown in Fig. 1.28 c, or if the lines of action are parallel and also parallel to the resultant as shown in Fig. 1.28 d, a solution is possible but not determinate. This can be verified from the analytical conditions; two equations and three unknowns.

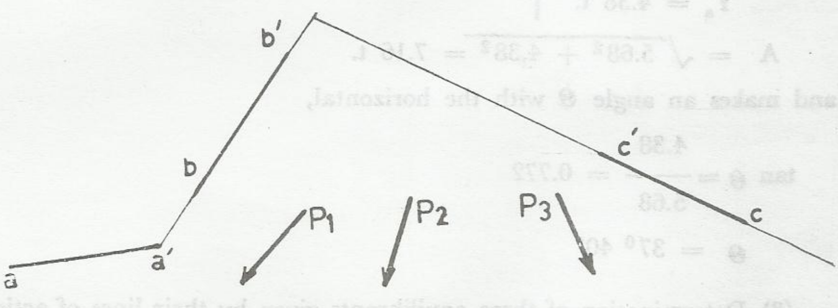



Fig. 1.29

Consider the system of forces $P_1$, $P_2$ and $P_3$ shown in Fig. 1.29, and suppose that it is required to find three forces acting along the given lines aa′, bb′ and cc′ to keep the given system of forces in equilibrium.

This problem can always be reduced to the previous one. Referring to Fig. 1.30a, assume that R is the resultant of the given system which can be found in the manner described in section 1.5. Combine any two of the unknown forces (in this case X and Y are combined) into a single force P. Applying the three-force theorem to R, P and Z, P must pass through the point of intersection of the lines of action of R and Z. From the force polygon (Fig. 1.30 b), knowing their directions, the magnitudes of P and Z are found. Resolving P along aa′ and bb′ (Fig. 1.29) the two left forces required, X and Y, are obtained.

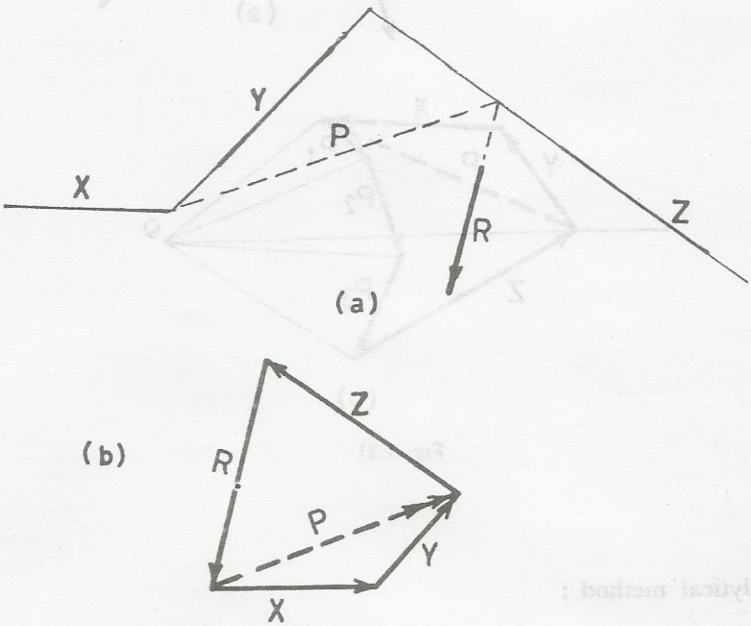



Fig. 1.30

This problem may equally be solved by use of the link polygon. Referring to Fig. 1.31a, replace any two of the required equilibrants by a single force (in this case X and Y are replaced by P which must pass through the point of intersection of their lines of action d). Proceed to find the two equilibrants Z and P, the first being given by its line of action cc′ and the second by its point of application d, as in the previous case. Once P is found, it will be a simple matter to resolve it along aa′ and bb′ to obtain X and Y as shown in Fig. 1.31 b.

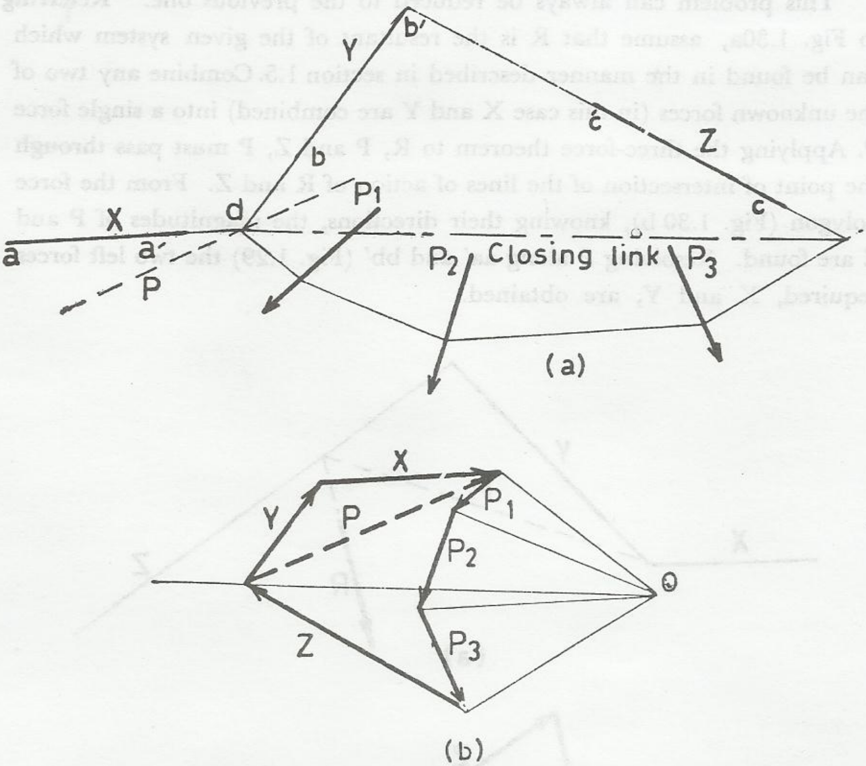



Fig. 1.31

Analytical method :

By applying the three conditions of equilibrium, $\Sigma X = 0$, $\Sigma Y = 0$ and $\Sigma M = 0$, three equations are obtained from which the unknown magnitudes of X, Y and Z are determined. Although this solution is possible yet it is inefficient as it involves the solution of three simultaneous equations. By ingenuity in applying the equations of equilibrium the solution can be simplified. For instance, by taking moments about point a (Fig. 1.32), the only unknown entering the equation will be Z and a direct solution for it will be possible. Similarly X and Y can be obtained directly by taking oments about points b and c respectively.

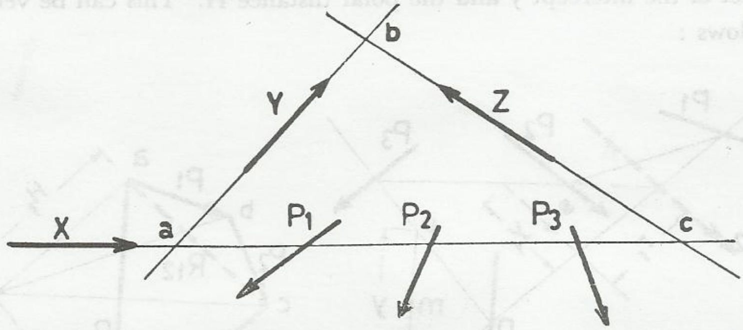



Fig. 1.32

To generalize, the three conventional equations of equilibrium, $\Sigma X = 0$, $\Sigma Y = 0$, and $\Sigma M = 0$ can always be replaced by the moment equations $\Sigma M_a = 0$, $\Sigma M_b = 0$, $\Sigma M_c = 0$, where points a, b, and c are three points which do not lie on one straight line. This may be proved as follows : If a system of forces satisfies any one of the three moment equations, say $\Sigma M_a = 0$, then the resultant cannot be a couple but may be a force acting through point a. If the system also satisfies the equation $\Sigma M_b = 0$ then the possibility the resultant being a force acting along ab still exists. If in addition, the system satisfies the equation $\Sigma M_c = 0$, where c does not lie on the straight line passing through a and b, the possibility of a resultant force is also eliminated. Since the resultant is neither a couple nor a force, the system must therefore be in equilibrium. By similar reasoning it can be shown that the conventional equations of equilibrium may be replaced by the three independent equations $\Sigma M_a = 0$, $\Sigma M_b = 0$, and $\Sigma X = 0$, where $\Sigma X$ is the algibraic sum of the force components in any direction other than perpendicular to ab.

## 1.10 Further applications of the link polygons

(1) Determination of moment of a system of forces.

Consider the force system $P_1$, $P_2$ and $P_3$ shown in Fig. 1.33a, and suppose it is required to calculate the moment of these forces about point m. Construct the force polygon for the given system (Fig. 1.33 b). Choose a pole 0, and draw the corresponding link polygon. Draw through point m a line parallel to the direction of the resultant as determined from the force polygon and measure, to the linear scale, the intercept y of this line between the first and last links. Also measure, to the force scale, the polar distance H which is the perpendicular distance from the pole 0 to the resultant vector in the force polygon. The moment is then given by the product of the intercept y and the polar distance H. This can be verified as follows :

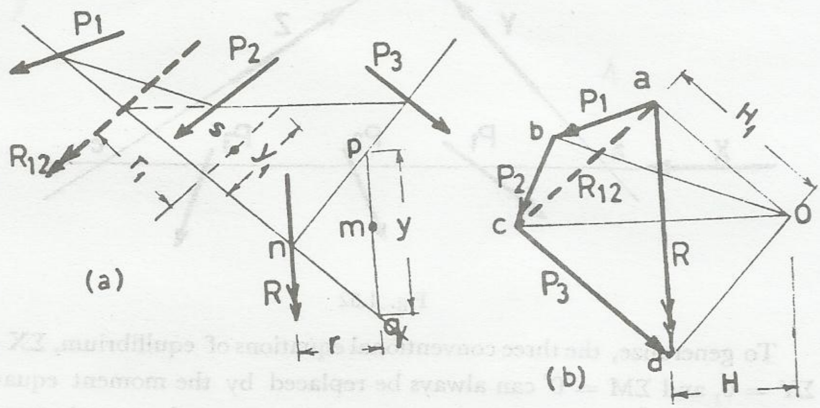



Fig. 1.33

The sum of the moments of the given forces is equal to the moment of their resultant about point m. Then,

$$M_m = R\ r$$

Triangles aod and npq are similar, and if H is drawn perpendicular to ad then,

$$\frac{R}{H} = \frac{y}{r}$$

$$\text{or } H\ y = R\ r = M_m$$

The method just illustrated is general in the sense that it can be used to find the moment of a part of the given system of forces. Suppose, for instance, that it is required to compute the moments of $P_1$ and $P_2$ about a point s (Fig. 1.33 a). This is equal to the moment of $R_{12}$ represented by vector $\overline{ac}$ in the force polygon and whose line of action is located by the intersection of the appropriate links of the link polygon. Hence, the moment is equal to the product of the resultant $R_{12}$ and its arm $r_1$ which is again equal to the product of the polar distance $H_1$ that is the perpendicular distance from the pole 0 to $R_{12}$ and the intercept $y_1$ of the line parallel to $R_{12}$ drawn through s between the links locating the resultant $R_{12}$.

This concept is very useful in the graphical determination of the bending moment diagrams which will be described later in section 3.9.

(2) Line of pressure.

There is a principle associated with link polygons which is often used. If the pole is chosen so as to coincide with the initial point of the first force in a force polygon, a special link polygon is developed. This link polygon is known as the *line of pressure* or as it is sometimes called the linear arch, and has the useful characteristic that each link represents the resultant of the preceding forces. This is obvious when one considers the basic principles involved in the construction of a link polygon.

(3) Link polygons drawn through a given point.

Suppose it is required to draw a link polygon passing through a given point m (Fig. 1.34 a). This may be done by drawing between forces $P_2$ and $P_3$ any link that passes through point m. Corresponding to this link there is a ray in the force polygon (Fig. 1.34 b) that must be parallel to it and also passing through the intersection of the vectors representing $P_2$ and $P_3$ in the force polygon. The pole 0 may now be chosen as any point on this ray and the remainder of the link polygon completed as shown.

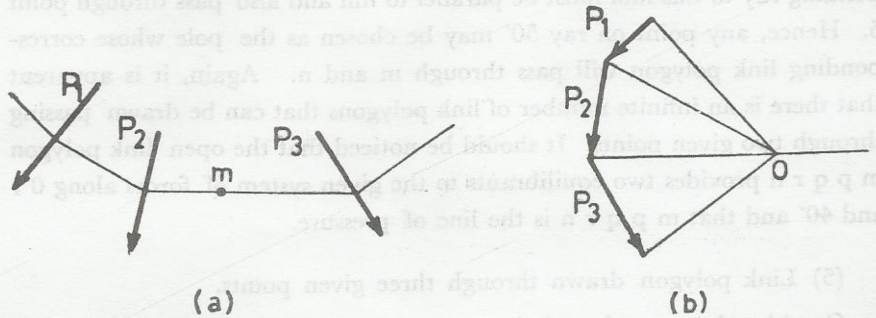



Fig. 1.34

Obviously this ray is the locus to the pole whose corresponding link polygons pass through m, and hence there is an infinite number of link polygons satisfying this condition.

(4) Link polygons drawn through two given points.

Sometimes it becomes necessary to pass a link polygon through two particular points such as m and n (Fig. 1.35 a). The procedure is as follows:

Suppose temporarily that it is required to find two equilibrants for the given system of forces; the first given by its point of application m and the

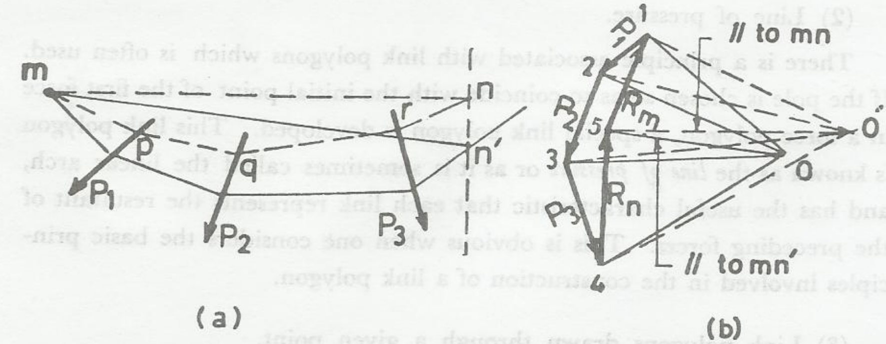



Fig. 1.35

second by its line of action nn′ which is assumed vertical for convenience. Passing the first link in the link polygon corresponding to any pole 0 through point m and proceeding as described in section 1.9, point 5 in the force polygon defining $R_m$ and $R_n$ is located (Fig. 1.35b). Had the pole 0 been chosen as desired, the resulting link polygon would have passed through m and n and the closing link would have been mn. The corresponding ray to this link must be parallel to mn and also pass through point 5. Hence, any point on ray 50′ may be chosen as the pole whose corresponding link polygon will pass through m and n. Again, it is apparent that there is an infinite number of link polygons that can be drawn passing through two given points. It should be noticed that the open link polygon m p q r n provides two equilibrants to the given system of forces along 0′1 and 40′ and that m p q r n is the line of pressure.

(5) Link polygon drawn through three given points.

Consider the case where it is necessary to pass a link polygon through three given points m, n and p as shown in Fig. 1.36 a. As before assume temporarily that it is required to find two equilibrants to the given system of forces; the first being given by its point of application m and the other by its line of action which is taken vertical through n for convenience. Continue as before and locate point 5 in the force polygon (Fig. 1.36 b). As mentioned in the preceding paragraph, line 55′ which is the parallel to mn through point 5, is the locus to the pole whose corresponding link polygons pass through m and n, Now consider the forces lying between m and p and again assume temporarily that it is required to find their vertical equilibrant through p and that through m. In this manner point 6 in the force polygon is located. Similarly line 66′, the parallel to mp through 6, is the locus to the pole whose corresponding link polygons pass through points m and p. Hence, for the link polygon to pass through all the given three points, the pole 0′ must be the intersection of the two locii 55′ and 66′. In this case, only one pole can be located and hence only one link polygon may be passed through three specified points. It should be noticed that the open link polygon through m, p and n is the line of pressure and that it provides two equilibrants at m and n along 0′1 and 40′.

Further, a link polygon cannot be made to passs through more than three specified points.

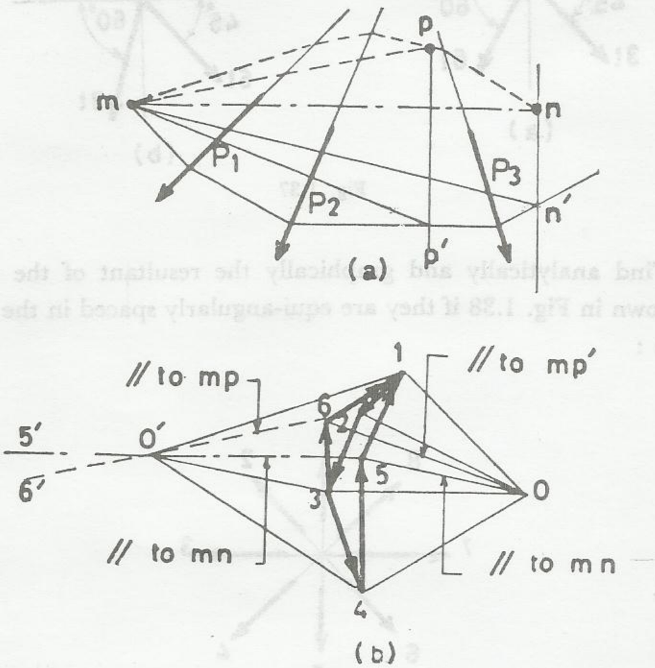



Fig. 1.36

## EXAMPLES TO BE WORKED OUT

(1) Find analytically and graphically the resultant of each of the two systems of forces shown in Figs. 1.37 a and b.

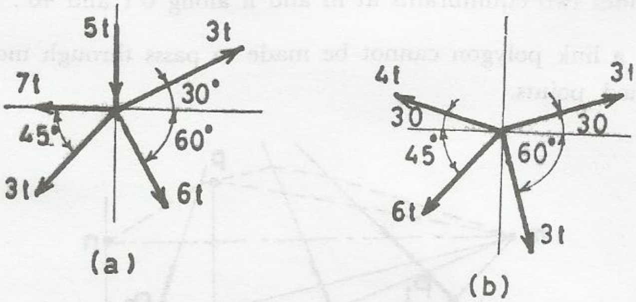



Fig. 1.37

(2) Find analytically and graphically the resultant of the system of forces shown in Fig. 1.38 if they are equi-angularly spaced in the following two case :

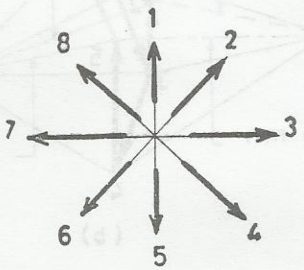



Fig. 1.38

| Force number | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | |
|---|---|---|---|---|---|---|---|---|---|
| Case 1 | 10 | 9 | 8 | 7 | 5 | 4 | 3 | 2 | kgs. |
| Case 2 | 10 | 4 | 2 | 2 — | 4 | 0 | 5 | 2 | kgs. |

(3) Figs. 1.39 a and b show two joints in a truss. Determine analytically and graphically the forces in members A and B.

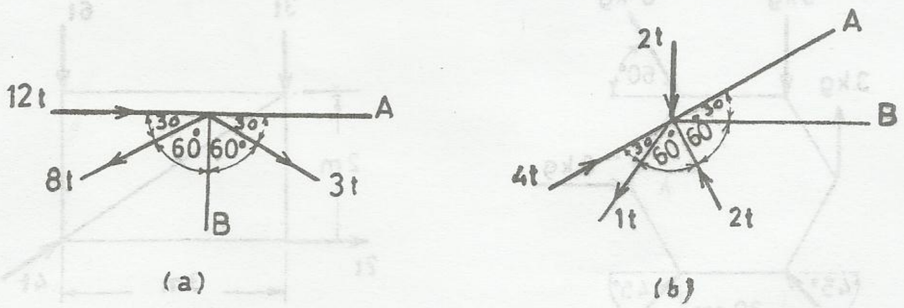



Fig. 1.39

(4) Figs. 1.40 a and b show two systems of forces in equilibrium. Find the values of P and $\Theta$ in each case.

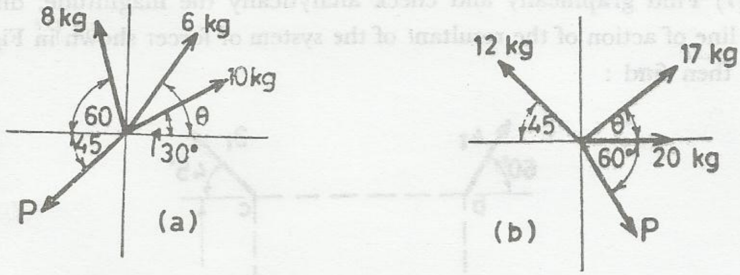



Fig. 1.40

(5) Determine graphically and analytically the pulls on the pullies a and b of the arrangement shown in Fig. 1.41.

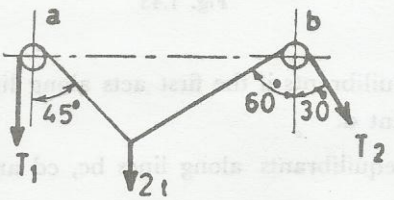



Fig. 1.41

(6) Find graphically and verify analytically the magnitude, direction and line of action of the resultant of the system of forces shown in Figs. 1.42 a and b.

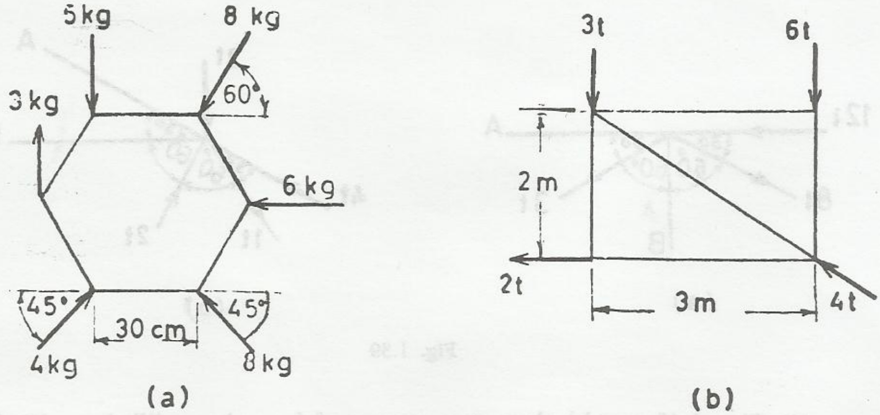



Fig. 1.42

(7) Find graphically and check analytically the magnitude, direction and line of action of the resultant of the system of forces shown in Fig. 1.43 and then find :

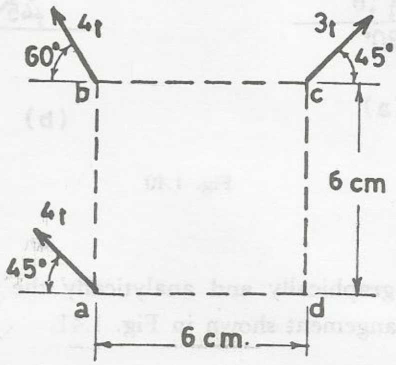



Fig. 1.43

(a) the two equilibrants if the first acts along line ab and the second passes through point d.

(b) the three equilibrants along lines bc, cd and ad.

(8) Find analytically and graphically using the three-force theorem the equilibrants A and B, through points a and b, to the system of forces shown in Figs. 1.44 a and b if :

in Fig. (a) B is vertical

in Fig. (b) B is horizontal.

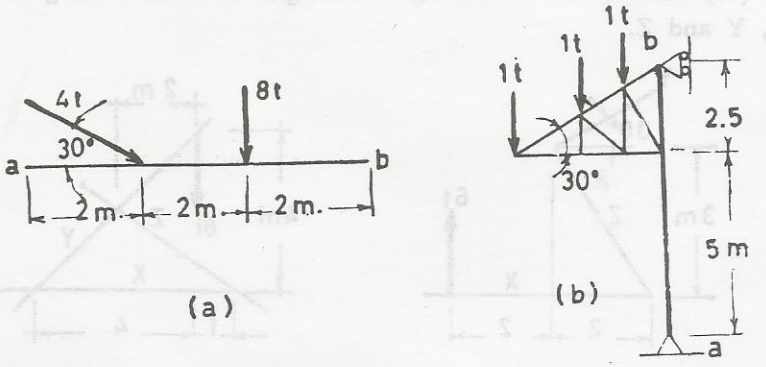



Fig .1.44

(9) Using the three-force theorem, find the reactions of the structure shown in Fig. 1.45 if the reaction at b is vertical.

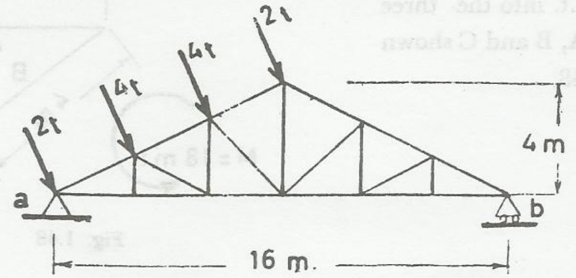



Fig. 1.45

(10) A horizontal rigid bar is acted upon by the forces shown in Fig. 1.46. Determine graphically the two equilibrants A and B passing through points a and b and satisfying the following conditions :

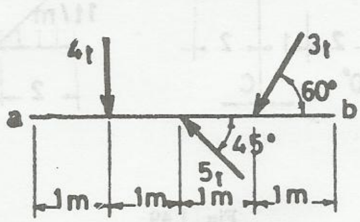



Fig. 1.46

(a) A is vertical.

(b) B is vertical.

(c) A and B are parallel.

(d) A and B are equal.

(11) Resolve the two forces given in Figs. 1.47 a and b along directions X, Y and Z.

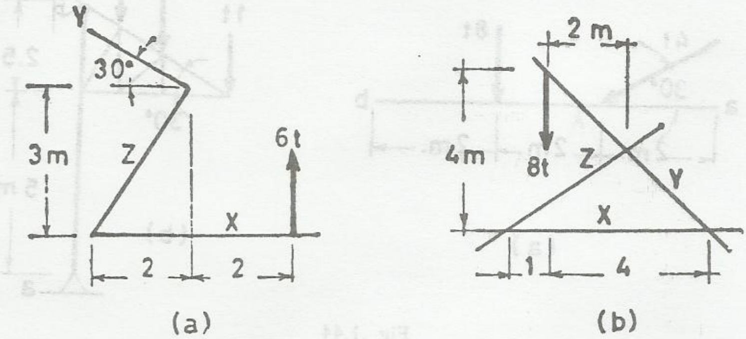



Fig. 1 47

(12) Resolve the couple M = 18 m.t. into the three directions A, B and C shown in Fig. 1.48.

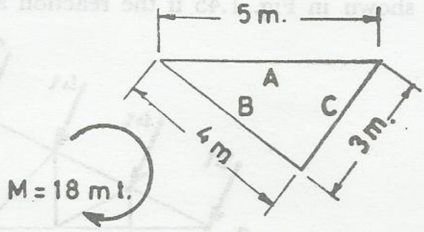



Fig. 1.48

(13) Find analytically and graphically the equilibrants A, B and C fo the two systems of forces shown in Figs. 1.49 a and b.

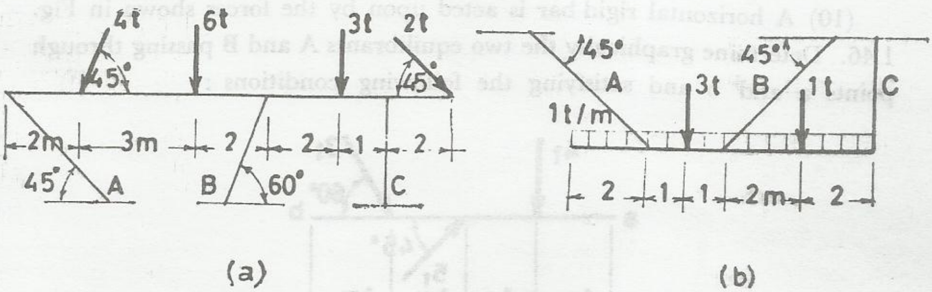



Fig. 1.49

(14) Two posts are rigidly fixed at the base and acted upon by forces A, B and C. If due to these forces the reactions at the base consist of a force and a moment as shown in Figs. 1.50 a and b, determine analytically and graphically the forces A, B and C.

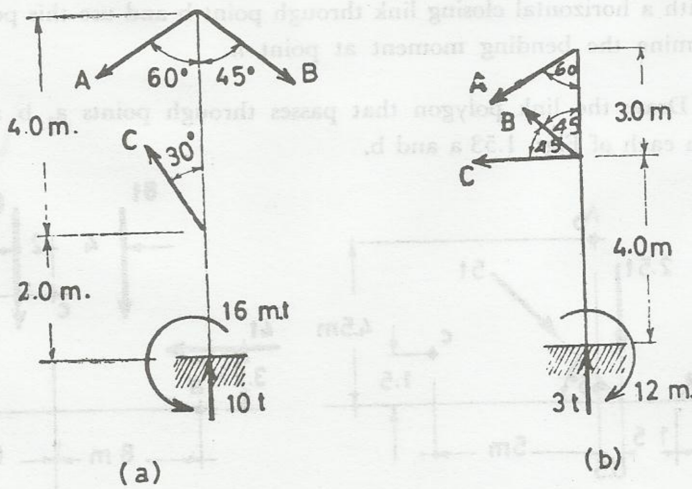



Fig. 1.50

(15) The system of parallel forces shown in Fig. 1.51 is applied to a string fixed at a and b. If the maximum force carried by the string should not exceed 5 kg., what is the least possible length of the sting between a and b ?

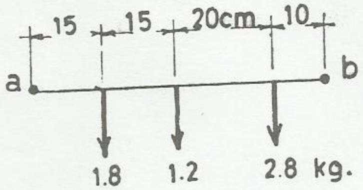



Fig. 1.51

(16) Find graphically the two parallel equilibrants through points a and b of the forces shown in Fig. 1.52. Draw a link polygon for the set of

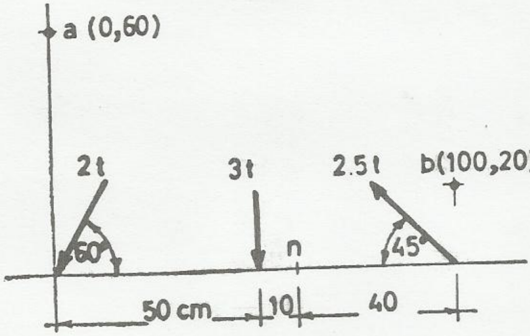



Fig. 1.52

forces with a horizontal closing link through point b and use this polygon to determine the bending moment at point n.

(17) Draw the link polygon that passes through points a, b and c shown in each of Figs. 1.53 a and b.

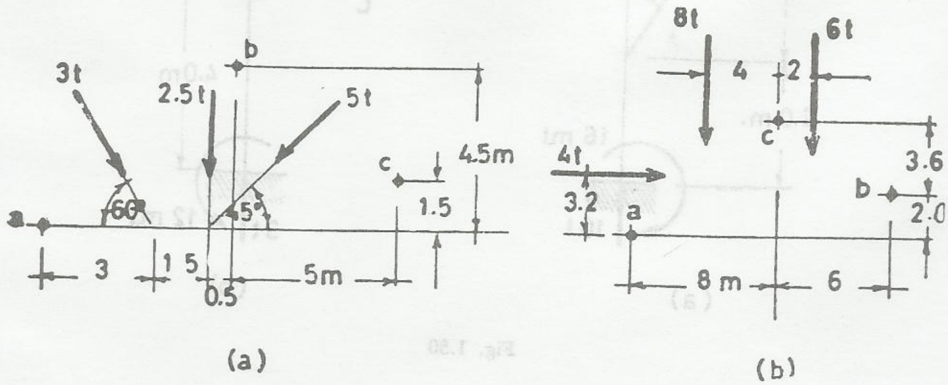



Fig. 1.53

CHAPTER 2

# LOADS AND REACTIONS

## 2.1 Loads

Forces acting on a structure are called *loads.* It is not intended to discuss load computation as this is dealt with in most design books. Hence, throughout this book loads will be assumed as given and it is only for convenience that they are classified as :

(1) Dead and live loads : *Dead loads* consist mainly of the own weight of the structure. If the dimensions of the structure are known, dead loads can be computed provided that the unit weights of the materials of construction are known. In practical design, the dimensions of structures are not known and therefore they have to be assumed guided by data concerning the dead weight of other similar existing structures or by some imperical formula. *Live loads* are loads that vary in position such as the weight of pedestrians, locomotives and vehicles. On dealing with such loads attention must be given to their placing on the structure so that the load function considered; reaction, shearing force, bending moment, deflection, etc., may have its maximum possible value.

(2) Concentrated and distributed loads : A *concentrated load* is a load which is assumed to be acting on a point such as the wheel loads of a train or a crane. Strictly speaking there is no concentrated loads as these would cause infinite stresses. *Distributed loads* are loads distributed over a certain length or area of the structure. It may be *uniformly distributed* such as some goods of the same kind piled up to the same height, or *uniformly varying* such as hydrostatic or earth pressures, or *nonuniformly distributed* like wind forces acting on a wing of an aeroplane.

## 2.2. Reactions

Most structures are restrained against free motion by means of *supports* that connect them to some stationary body. The resistances offered by the supports to counteract the action of the applied loads are called *reactions.* These reactions are in effect the equilibrants to the loads acting on the structure.

There are three main types of supports. Each type provides certain kinds of restraints.

(1) Roller support is usually represented by rollers as shown in Fig. 2.1 a. Rollers permit translation on the surface along which they roll as well as rotation. They provide a single reaction component normal to the surface on which they roll. On applying the equations of equilibrium rollers provide one unknown even if the rollers are inclined as in this case the ratio between the two rectangular components of the reaction is known.

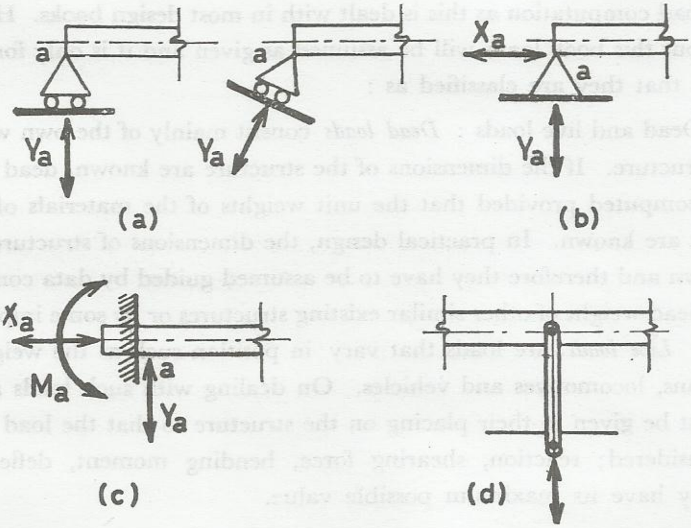



Fig. 2.1

(2) Hinge support is usually represented by a detail as shown in Fig. .2 b. It permits rotation only and is capable of providing a single force in any direction. Thus when the equations of equilibrium are applied, a hinge support provides two unknowns; magnitude and direction of the reaction. These are usually substituted for by the magnitudes of its horizontal and vertical components.

(3) Fixed support is usually represented by a detail as shown in Fig. 2.1 c. It neither permits translation in any direction nor does it allow rotation. It is capable of providing a single force in any direction at any location. Hence when the equations of equilibrium are applied, a fixed support provides three unknowns; magnitude, direction and line of action of the reaction. These are usually substituted for by two rectangular components and a moment.

Another type of support that may be used is shown in Fig. 2.1 d, and is called link support or pendulum. It has the same action as the roller support, i.e. it provides one reaction component along the link.

To summerize, in genereal a roller support at point a provides one reaction component $Y_a$, a hinge support provides two components $X_a$ and $Y_a$ while a fixed support provides three reaction components $X_a$, $Y_a$ and $M_a$. It must be remembered that one or more of these reaction components may be zero for structures under particular cases of loading. Further, it is to be known that a reaction component may act in either sense, i.e. the vertical component Y may act upwards or downwards, the horizontal component X may act to the right or to the left and finally the moment M may act either clockwise or anticlockwise as shown in Fig. 2.1.

### 2.3 Calculation of reactions

In general a structure is acted upon by a system of nonconcurrent forces consisting of the known applied loads and the unknown reactions, which as mentioned before are the equilibrants to the applied loads. Hence, if the structure is to remain in equilibrium the three equations of equilibrium, $\Sigma X = 0$, $\Sigma Y = 0$ and $\Sigma M = 0$ must be fulfilled simultaneously by the loads and reactions. The application of these equations to several problems is illustrated below.

**Example 2.1** Find the reactions for beam ab loaded as shown in Fig. 2.2 a.

Solution : The centre line of the beam, with the unknown reaction components and all the given loads, are drawn in Fig. 2.2 b. At a, two

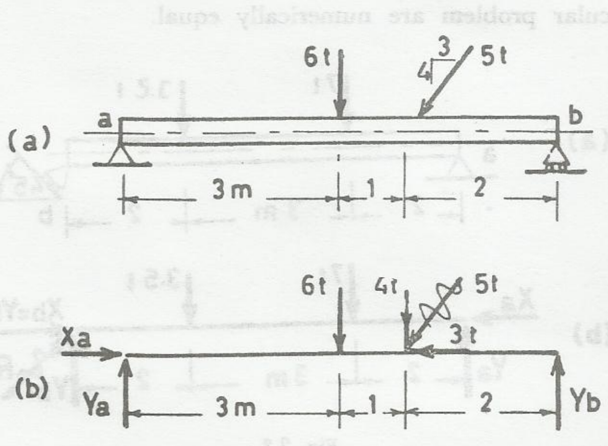



Fig. 2.2

reaction components $X_a$ and $Y_a$ may exist as the end is hinged. At b, there cannot be but one vertical reaction component $Y_b$ as the end is on a roller. For convenience the inclined force is replaced with its horizontal and vertical components as shown. Applying the equations of equilibrium,

$$\Sigma X = 0 = X_a \quad 3$$

$$X_a = 3 \text{ t.} \rightarrow$$

$$\Sigma M_a = 0 = Y_b \times 6 - 4 \times 4 - 6 \times 3$$

$$Y_b = \frac{16 + 18}{6} = 5.67 \text{ t.} \uparrow$$

$$\Sigma M_b = 0 = Y_a \times 6 - 6 \times 3 - 4 \times 2$$

$$Y_a = \frac{18 + 8}{6} = 4.33 \text{ t.} \uparrow$$

Check : $\Sigma Y = 0$

$$5.67 - 6 - 4 + 4.33 = 0$$

**Example 2.2** Find the reactions for beam ab loaded as shown in Fig. 2.3.

Solution : As in the preceding example, the centre line of the beam with the unknown reaction components and all the given loads, are drawn in Fig. 2.3 b. At a, two reaction components $X_a$ and $Y_a$ may exist as the end is hinged. At b, the reaction $R_b$ acts normal to the supporting plane as the end is on a roller. For convenience in applying the equations of equilibrium, $R_b$ is replaced with its two components $X_b$ and $Y_b$ which in this particular problem are numerically equal.

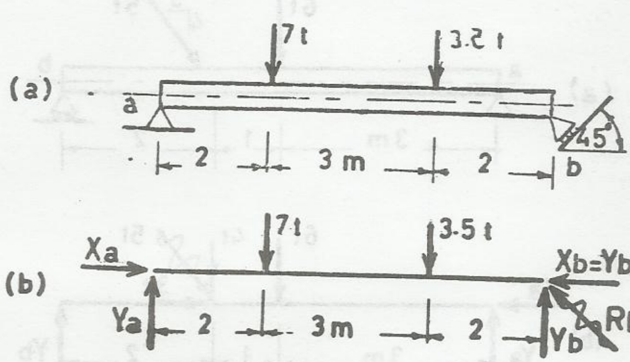



Fig. 2 3

$$\Sigma M_a = 0 = Y_b \times 7 - 3.5 \times 5 - 7 \times 2$$

$$Y_b = \frac{17.5 + 14}{7} = 4.5 \text{ t.} \uparrow$$

Since the slope of the supporting plane at b is 1 : 1,

$$X_b = 4.5 \text{ t} \leftarrow$$

$$\Sigma M_b = 0 = Y_a \times 7 - 7 \times 5 - 3.5 \times 2$$

$$Y_a = \frac{35 + 7}{7} = 6 \text{ t.} \uparrow$$

$$\Sigma X = 0 = X_a - 4.5$$

$$X_a = 4.5 \text{ t.} \rightarrow$$

Check : $\Sigma Y = 0$

$$7 + 3.5 - 6 - 4.5 = 0$$

**Example 2.3** Determine the reactions for the overhanging beam shown in Fig. 2.4

Solution : For calculating the reactions, the distributed load is replaced with an equivalent concentrated load. This load is equal to the sum of the distributed load on the beam. As the intensity of the distributed load varies abruptly at b the portion of the load between a and b is replaced with one concentrated load = 1.5 × 5 = 7.5 t. acting at the centroid of the distributed load of this portion, i.e. at 2.5 m. from a. Similarly, the rest of the distributed load is replaced with a concentrated load = 2 × 2 = 4 t. at 1 m. from b. The rest of the procedures are similar to those

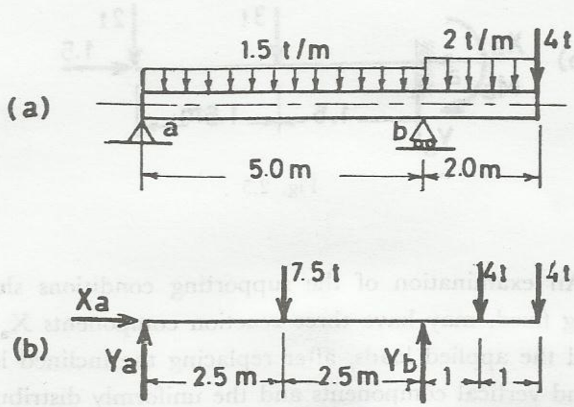



Fig. 2.4

in the preceding examples. Referring to Fig. 2.4 b and applying the three equations of equilibrium,

$$\Sigma X = 0 = X_a$$

$$X_a = 0$$

$$\Sigma M_a = 0 = 4 \times 7 + 4 \times 6 + 7.5 \times 2.5 - Y_b \times 5$$

$$Y_b = \frac{28 + 24 + 18.75}{5} = 14.15 \text{ t.} \uparrow$$

$$\Sigma M_b = 0 = 4 \times 2 + 4 \times 1 - 7.5 \times 2.5 + Y_a \times 5$$

$$Y_a = \frac{18.75 - 8 - 4}{5} = 1.35 \text{ t.} \uparrow$$

Check : $\Sigma Y = 0$

$$1.35 - 7.5 + 14.15 - 4 - 4 = 0$$

**Example 2.4** Determine the reactions for the cantilever ab loaded as shown in Fig. 2.5 a.

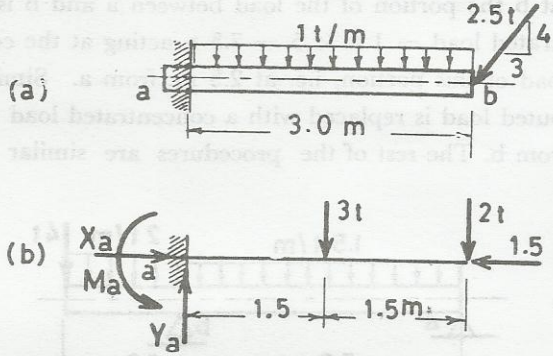


Fig. 2.5

Solution : An examination of the supporting conditions shows that support a, being fixed, may have three reaction components $X_a$, $Y_a$ and $M_a$. These and the applied loads, after replacing the inclined load with its horizontal and vertical components and the uniformly distributed load with its equivalent concentrated load, are shown in Fig. 2.5 b.

$$\Sigma X = 0 = X_a - 1.5$$

$$X_a = 1.5 \text{ t.} \rightarrow$$

$$\Sigma Y = 0 = Y_a - 3 - 2$$

$$Y_a = 5 \text{ t.} \uparrow$$

$$\Sigma M_a = 0 = 3 \times 1.5 + 2 \times 3 - M_a$$

$$M_a = 10.5 \text{ m.t. (anticlockwise)}$$

Check : Taking moments about any point, say b then,

$$\Sigma M_b = 0$$

$$3 \times 1.5 + 10.5 - 5 \times 3 = 0$$

**Example 2.5** Determine the reactions for beam abcd under the given loads if it is supported by three link members arranged as shown in Fig. 2.6 a.

Solution : An inspection of the supporting conditions of the given structure shows that there are three reaction components $R_b$, $R_c$ and $R_d$ of fixed directions along the link members bb′, cc′ and dd′ respectively. These together with the applied loads are shown in Fig. 2.6 b.

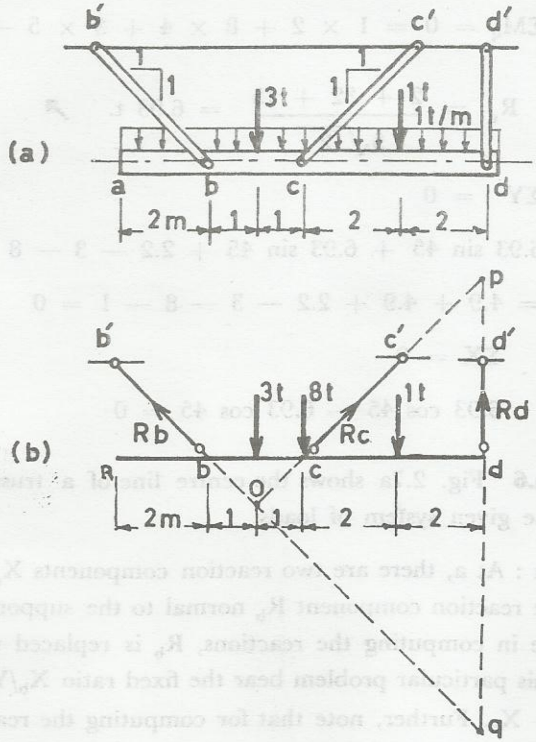



Fig. 2.6

Applying the three conventional conditions of equilibrium; $\Sigma X = 0$, $\Sigma Y = 0$ and $\Sigma M = 0$, three simultaneous equations are obtained, which when solved give the three unknown magnitudes of $R_b$ $R_c$ and $R_d$. Although this solution is quite permissible, yet it is not very clever as it involves the solution of three simultaneous equations. A direct solution may be carried out as follows :

By taking moments about the point of intersection of the lines of action of $R_b$ and $R_c$ the only unknown which enters the equation will be $R_d$. Thus,

$$\Sigma M_o = 0 = 3 \times 0 + 8 \times 1 + 1 \times 3 - R_d \times 5$$

$$R_d = \frac{8 + 3}{5} = 2.2 \text{ t.} \uparrow$$

Similarly, $\Sigma M_p = 0 = 1 \times 2 + 8 \times 4 + 3 \times 5 - R_b \times 5\sqrt{2}$

$$R_b = \frac{2 + 32 + 15}{5\sqrt{2}} = 6.93 \text{ t.} \nwarrow$$

$$\Sigma M_q = 0 = 1 \times 2 + 8 \times 4 + 3 \times 5 - R_c \times 5\sqrt{2}$$

$$R_c = \frac{2 + 32 + 15}{5\sqrt{2}} = 6.93 \text{ t.} \nearrow$$

Check : $\Sigma Y = 0$

$$6.93 \sin 45 + 6.93 \sin 45 + 2.2 - 3 - 8 - 1$$

$$= 4.9 + 4.9 + 2.2 - 3 - 8 - 1 = 0$$

$$\Sigma X = 0$$

$$6.93 \cos 45 - 6.93 \cos 45 = 0$$

**Example 2.6** Fig. 2.7a shows the centre line of a truss. Find its reactions for the given system of loads.

Solution : At a, there are two reaction components $X_a$ and $Y_a$. At b, there is one reaction component $R_b$ normal to the supporting plane. For convenience in computing the reactions, $R_b$ is replaced with $X_b$ and $Y_b$ which in this particular problem bear the fixed ratio $X_b/Y_b = \tan 30$, i.e. $Y_b = \sqrt{3}\ X_b$. Further, note that for computing the reactions the shape of the structure is not important. Any shape supported in the same manner and acted upon by the same loads will have the same reactions as those of the given structure. To emphasize this point, consider the arbitrary shape shown in Fig. 2.7 b which is supported in the same manner and subjected to the same system of loads as the original structure. Applying the equations of equilibrium,

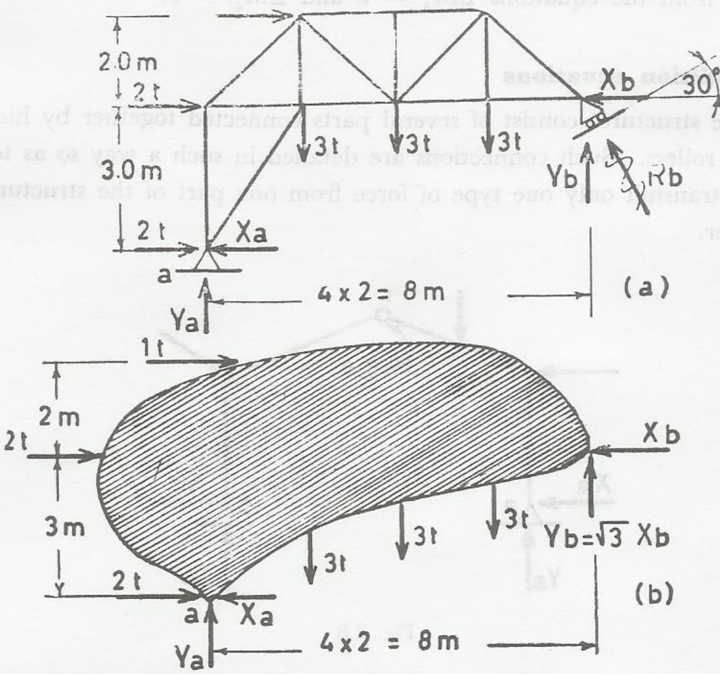



Fig. 2.7

$$\Sigma M_a = 0 = 2 \times 3 + 1 \times 5 + 3 \times 2 + 3 \times 4 + 3 \times 6 - Y_b \times 8 - 3\, Y_b / \sqrt{3}$$

$$Y_b = \frac{6 + 5 + 6 + 12 + 18}{8 + \sqrt{3}} = 4.83 \text{ t.} \uparrow$$

$$X_b = Y_b / \sqrt{3} = 4.83 / \sqrt{3} = 2.8 \text{ t.} \leftarrow$$

$$\Sigma X = 0 = 1 + 2 + 2 - 2.8 - X_a$$

$$X_a = 2.2 \text{ t.} \leftarrow$$

$$\Sigma M_b = 0 = Y_a \times 8 - 2 \times 3 + 2.2 \times 3 + 1 \times 2 - 3 \times 6 - 3 \times 4 - 3 \times 2$$

$$Y_a = \frac{6 - 6.6 - 2 + 18 + 12 + 6}{8} = 4.17 \text{ t.} \uparrow$$

Check: $\Sigma Y = 0$

$$4.17 - 3 - 3 - 3 + 4.83 = 0$$

Had the supports a and b been on the same level. The solution for the reactions would have been much easier, for in that case a direct solution for the vertical components of the reactions could have been obtained directly from the equations $\Sigma M_a = 0$ and $\Sigma M_b = 0$.

## 2.4 Condition equations

Some structures consist of several parts connected together by hinges, links or rollers. Such connections are detailed in such a way so as to be able to transmit only one type of force from one part of the structure to the other.

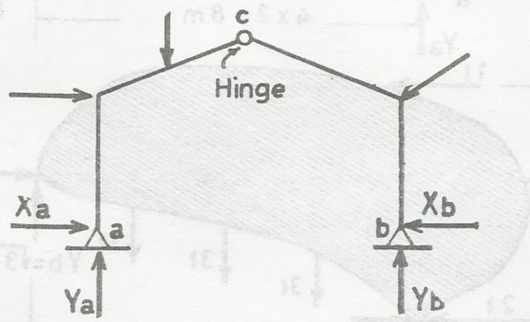



Fig. 2.8

Consider, for example, the structure shown in Fig. 2.8. It is supported by hinge supports at a and b and is provided with an intermediate hinge at c. Since a hinge is free to rotate, it cannot transmit moment from part ac to part bc or vice versa. Therefore, the algebraic sum of the moments about the hinge c of all the forces, including the reactions, acting on either part ac or part bc must be zero. In an equation form this is expressed as :

$$\sum_a^c M_c = 0 \quad \text{or} \quad \sum_b^c M_c = 0$$

Either of these equations together with the three equations of equilibrium are sufficient for the determination of the four reaction components $X_a$, $Y_a$, $X_b$ and $Y_b$.

Further, consider the structure shown in Fig. 2.9. It is fixed at a, supported on a hinge at b, and has a roller inserted at c. Since the roller cannot

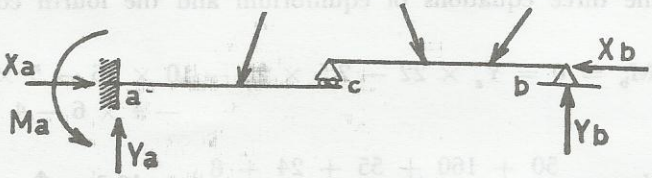



Fig. 2.9

transmit but a vertical force from one part to the other, and that it is free to translate horizontally and also to rotate, it follows that :

(1) The sum algebraic of the horizontal components of the forces acting on either part to the right or to the left of c must be equal to zero.

(2) The sum algebraic of the moments about c of the forces acting on either side of c must be zero.

In an equation form these conditions are expressed as :

(1) $\sum_{a}^{c} X = 0$ or $\sum_{b}^{c} X = 0$

(2) $\sum_{a}^{c} M_c =$ or $\sum_{b}^{c} M_c = 0$

Either equations in (1) and one of the two equations in (2) together with the three equations of equilibrium applied to the structure as a whole are sufficient to determine the five reaction components $X_a$, $Y_a$, $M_a$, $X_b$, and $Y_b$.

**Example 2.7** Fig. 2.10 shows a three-hinged polygonal arch. Find its reactions under the given system of loads.

Solution : There are four reaction components; $X_a$, $Y_a$, $X_b$ and $Y_b$.

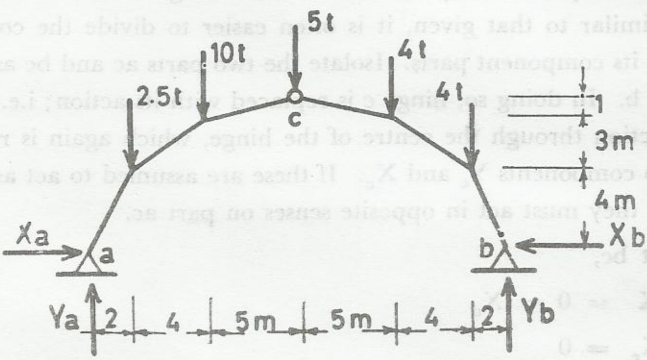



Fig. 2.10

Applying the three equations of equilibrium and the fourth condition equation,

$$\Sigma M_b = 0 = Y_a \times 22 - 2.5 \times 20 - 10 \times 16 - 5 \times 11 - 4 \times 6 - 4 \times 2$$

$$Y_a = \frac{50 + 160 + 55 + 24 + 8}{22} = 13.5 \text{ t.} \uparrow$$

$$\Sigma M_a = 0 = Y_b \times 22 - 2.5 \times 2 - 10 \times 6 - 5 \times 11 - 4 \times 16 - 4 \times 20$$

$$Y_b = \frac{5 + 60 + 55 + 64 + 80}{22} = 12 \text{ t.} \uparrow$$

$$\sum_b^c M_c = 0 = 12 \times 11 - 4 \times 9 - 4 \times 5 - X_b \times 8$$

$$X_b = \frac{132 - 36 - 20}{8} = 9.5 \text{ t.} \leftarrow$$

$$\Sigma X = 0 = X_a - 9.5$$

$$X_a = 9.5\text{t} \rightarrow$$

Check : $\Sigma Y = 0$

$$13.5 - 2.5 - 10 - 5 - 4 - 4 + 12 = 0$$

**Example 2.8** Determine the reactions of the structure shown in Fig. 2.11 a.

**Solution** : There are four reaction components $X_a$, $Y_a$, $M_a$ and $Y_b$. These could be obtained from the three conditions of equilibrium and the fourth condition provided by the intermediate hinge at c. However, for structures similar to that given, it is often easier to divide the composed structure to its component parts. Isolate the two parts ac and bc as shown in Fig. 2.11 b. In doing so, hinge c is replaced with its action; i.e. a force in any direction through the centre of the hinge, which again is replaced with its two components $Y_c$ and $X_c$. If these are assumed to act as shown on part bc, they must act in opposite senses on part ac.

For part bc,

$$\Sigma X = 0 = X_c$$

$$X_c = 0$$

$$\Sigma M_c = 0 = Y_b \times 4 - 4 \times 2 - 12 \times 3$$

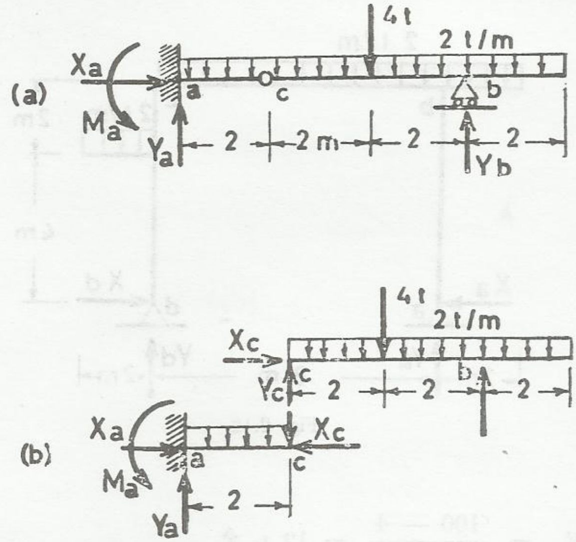



Fig. 2.11

$$Y_b = \frac{8 + 36}{4} = 11 \text{ t.} \uparrow$$

$$\Sigma Y = 0 = 4 + 6 \times 2 - 11 - Y_c$$

$$Y_c = 4 + 12 - 11 = 5 \text{ t.} \uparrow$$

For part ac,

$$\Sigma X = 0 = X_a$$

$$X_a = 0$$

$$\Sigma Y = 0 = Y_a - 2 \times 2 - 5$$

$$Y_a = 4 + 5 = 9 \text{ t.} \uparrow$$

$$\Sigma M_a = 0 = 2 \times 2 \times 1 + 5 \times 2 - M_a$$

$$M_a = 4 + 10 = 14 \text{ m.t. (anticlockwise)}$$

Check : $\Sigma M_b = 0$

$$2 \times 8 \times 2 + 4 \times 2 + 14 - 9 \times 6 = 0$$

**Example 2.9** The three-hinged frame abcd has two hinged supports at a and d, and an intermediate hinge at c as shown in Fig. 2.12. Determine its reactions under the given system of loads.

Solution : There are four reaction components $X_a$, $Y_a$, $X_d$ and $Y_d$.

$$\Sigma M_d = 0 = Y_a \times 8 - 2 \times 10 \times 5 + 2 \times 2 \times 1$$

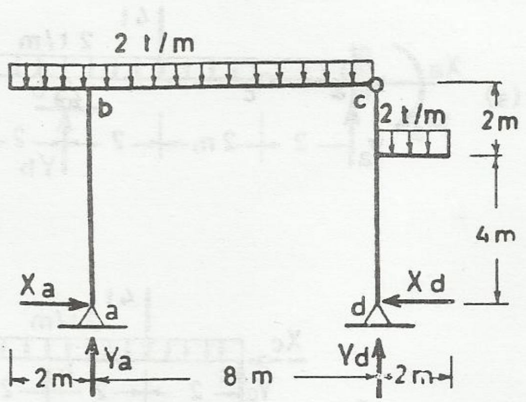



Fig. 2.12

$$Y_a = \frac{100 - 4}{8} = 12 \text{ t.} \uparrow$$

$$\Sigma M_a = 0 = Y_d \times 8 - 2 \times 10 \times 3 - 2 \times 2 \times 9$$

$$Y_d = \frac{60 + 36}{8} = 12 \text{ t.} \uparrow$$

$$\sum_{d}^{c} M_c = 0 = X_d \times 6 + 2 \times 2 \times 1$$

$$X_d = -\frac{4}{6} = -0.67 \text{ t., i.e. } 0.67 \text{ t} \rightarrow$$

$$\Sigma X = 0 = X_a + 0.67$$

$$X_a = -0.67 \text{ t. , i.e. } 0.67 \text{ t.} \leftarrow$$

Check : A partial check can be obtained from $\Sigma Y = 0$

$$2 \times 10 + 2 \times 2 - 12 - 12 = 0$$

This check does not include the horizontal components $X_a$ and $X_d$.

$$\sum_{a}^{c} M_c = 0 = X_a \times 6 + 2 \times 10 \times 5 - 12 \times 8$$

$$X_a = \frac{-100 + 96}{6} = -0.67 \text{ t., i.e. } 0.67 \text{ t.} \leftarrow$$

which is the value obtained before from the independent equation $\Sigma X = 0$.

**Example 2.10** The frame abcd is fixed at a and hinged at d, and has two intermediate hinges at b and c as shown in Fig. 2.13. Determine its reactions under the given vertical and horizontal loads.

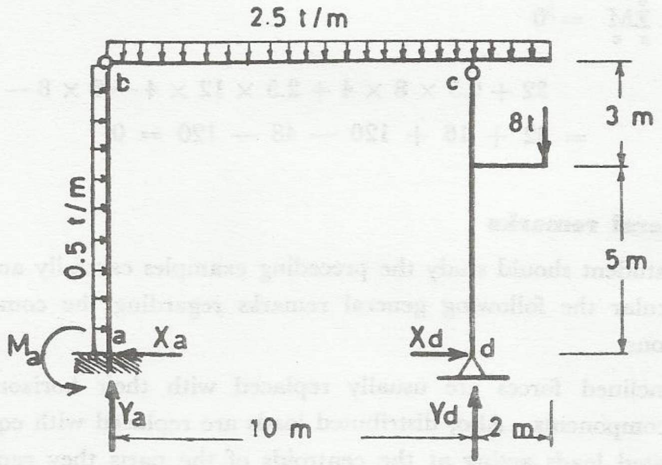



Fig. 2.13

Solution : There are five reaction components; three at the fixed support; $X_a$, $Y_a$ and $M_a$, and two at the hinged support; $X_d$ and $Y_d$. Applying the three equations of equilibrium and the two condition equations provided by the hinges at b and c,

$$\sum_{d}^{c} M = 0 = X_d \times 8 - 8 \times 2$$

$$X_d = \frac{16}{8} = 2 \text{ t.} \rightarrow$$

$$\sum_{d}^{b} M_b = 0 = Y_d \times 10 + 2 \times 8 - 12 \times 2.5 \times 6 - 8 \times 12$$

$$Y_d = \frac{180 + 96 - 16}{10} = 26 \text{ t.} \uparrow$$

$$\Sigma Y = 0 = Y_a + 26 - 2.5 \times 12 - 8$$

$$Y_a = 30 + 8 - 26 = 12 \text{ t.} \uparrow$$

$$\Sigma X = 0 = X_a - 0.5 \times 8 - 2$$

$$X_a = 4 + 2 = 6 \text{ t.} \leftarrow$$

$$\Sigma M_b = 0 = M_a + 0.5 \times 8 \times 4 - 6 \times 8$$

$$M_a = 48 - 16 = 32 \text{ m.t. (anticlockwise)}$$

Check : $\overset{c}{\underset{a\ \ c}{\Sigma M}} = 0$

$$32 + 0.5 \times 8 \times 4 + 2.5 \times 12 \times 4 - 6 \times 8 - 12 \times 10$$

$$= 32 + 16 + 120 - 48 - 120 = 0$$

## 2.5 General remarks

The student should study the preceding examples carefully and notice in particular the following general remarks regarding the computation of reactions.

(1) Inclined forces are usually replaced with their horizontal and vertical components. Also, distributed loads are replaced with equivalent concentrated loads acting at the centroids of the parts they replace.

(2) For direct computation of the reactions of some structures, the conventional equations of equilibrium; $\Sigma X = 0$, $\Sigma Y = 0$ and $\Sigma M = 0$ may be replaced with any other three convenient forms. (See section 1.9).

(3) If the answer of a reaction component comes out positive it acts in the assumed direction. If, however, it comes out negative the component acts in a direction opposite to that assumed.

(4) In problems including condition equations where it is sometimes convenient to break up the given structure into separate component parts, the interacting forces between adjacent parts may be assumed to act in either sense, for vertical components upwards or downwards, and for horizontal components to the right or to the left, but they must act in opposite senses on two adjacent parts.

(5) A check on the results is always desirable. Note that some of the checks given in the preceding examples are only partial checks. Also, the student is strongly advised to clearly indicate his results by underlining them and showing the units and directions of the forces.

## 2.6 Stability and determinancy

There are two main types of statical stability and determinancy of structures; external concerned with the reactions and internal concerned with the internal forces and moments. This discussion will be confined to external stability and determinancy.

A *stable structure* is one which can support a general system of loads elastically and immediately on its application.

Since there are three possible movements of a plane structure under a general case of loading; translation in the horizontal and vertical directions and rotation, no less than three reaction components can make a structure stable. However, three or even more reaction components are not always sufficient to make a structure stable. For example, when the lines of action of the reaction components are concurrent as shown in Fig. 2.14 a, or parallel as shown in Fig. 2.14 b, the structure is unstable as in the first case the structure is not completely restrained and will tend to rotate about the the point of concurrency 0, and in the second case it has no restraint to prevent any tendency of the structure to move normal to the direction of the links.

A *determinate structure* is defined as one for which the reactions can be determined by the application of the equations of equilibrium in addition to condition equations, if any, introduced by special constructional details.

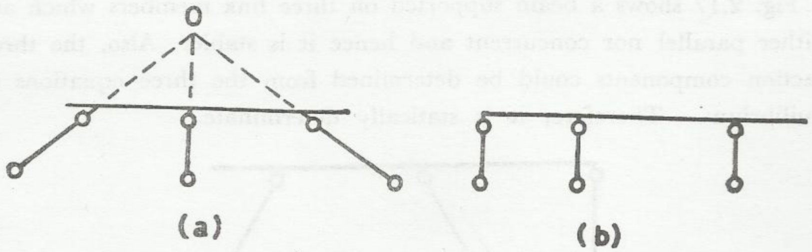



Fig. 2.14

If the number of the available equations is fewer than the number of the unknown reaction components, the structure is said to be ***statically indeterminate***. The *degree of indeterminancy* corresponds to the number reaction components in excess to the available number of equations. If, on the other hand, the number of the available equations is more than the number of the reaction components, the structure is ***unstable***.

**2.7 Classified examples**

The beam shown in Fig. 2.15 is supported on two rollers. The number of reaction components is one less than the three required for stability and hence the beam is unstable.

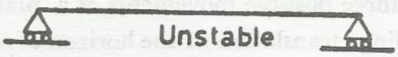



Fig. 2 15

Fig. 2.16 shows a truss supported on a hinge at a and a roller at b. It has three reaction components which could be determined from the three equations of equilibrium. Hence, the truss is staticaly determinate.

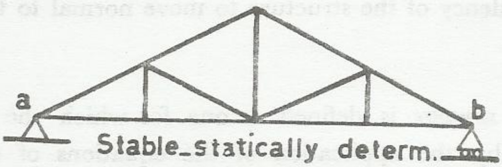



Fig. 2.16

Fig. 2.17 shows a beam supported on three link members which are neither parallel nor concurrent and hence it is stable. Also, the three reaction components could be determined from the three equations of equilibrium. Therefore, it is statically determinate.

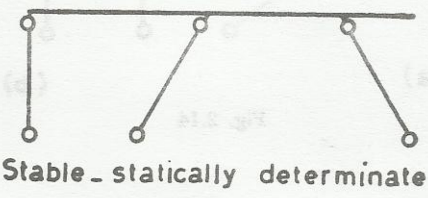



Fig. 2.17

Fig. 2.18 shows a three-hinged arch. It has four reaction components which could be determined from the three equations of equilibrium and a fourth condition provided by the central hinge. Hence, the arch is statically determinate.

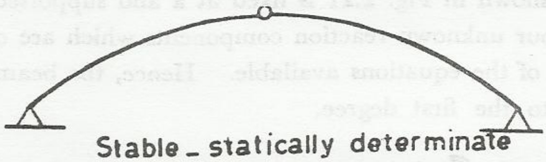



Fig. 2.18

Fig. 2.19 shows a frame which is fixed at b and supported on a hinge at a. It has five reaction components which could be determined from the three equations of equilibrium and the two condition equations provided by the two intermediate hinges at c and d. Hence, the frame is statically determinate.

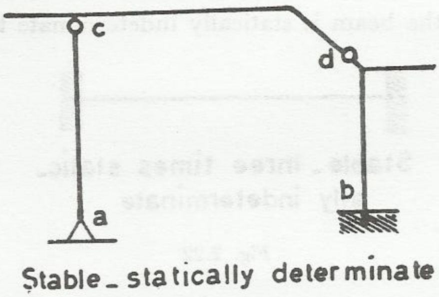



Fig. 2.19

The frame shown in Fig. 2.20 is fixed at a and b and is provided with three intermediate hinges at c, d and e. It has six reaction components which could be determined from the three equations of equilibrium in addition to the condition equations provided by the three intermediate hinges. The given structure is therefore statically determinate.

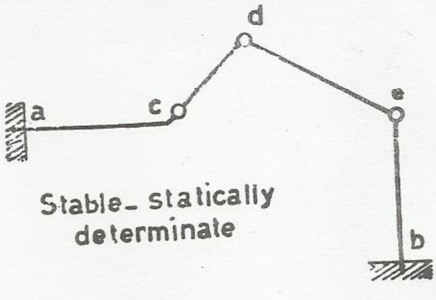



Fig. 2.20

The beam shown in Fig. 2.21 is fixed at a and supported on a roller at b. It has four unknown reaction components which are one in excess to the number of the equations available. Hence, the beam is statically indeterminate to the first degree.


Stable – once statically indeterminate

Fig. 2.21

The beam shown in Fig. 2.22 is fixed at both ends. It has six unknown reaction components which are three in excess to the number of equations available. Hence, the beam is statically indeterminate to the third degree.

Stable – three times statically indeterminate

Fig. 2.22

## EXAMPLES TO BE WORKED OUT

(1) — (20) Determine the reactions of the structures shown in Figs. 2.23 - 2.42.

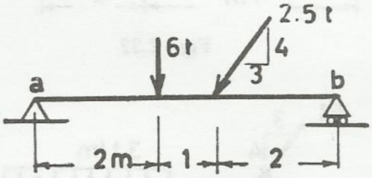



Fig. 2.23

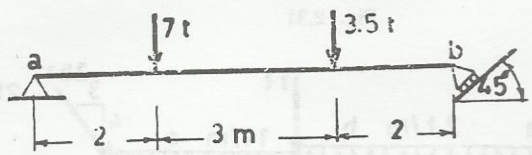



Fig. 2.24

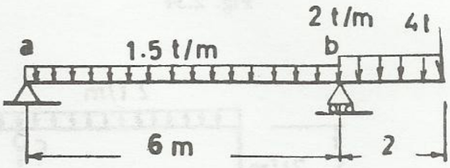



Fig. 2.25

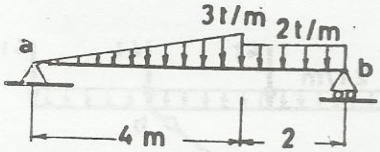



Fig. 2.26

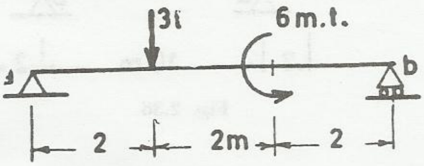



Fig. 2.27

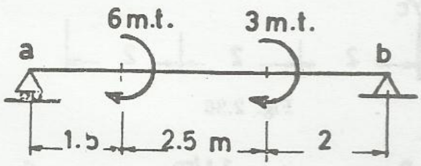



Fig. 2.28

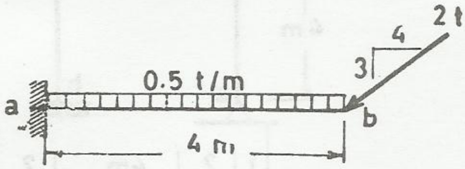



Fig. 2.29

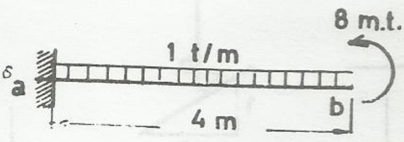



Fig. 2.30

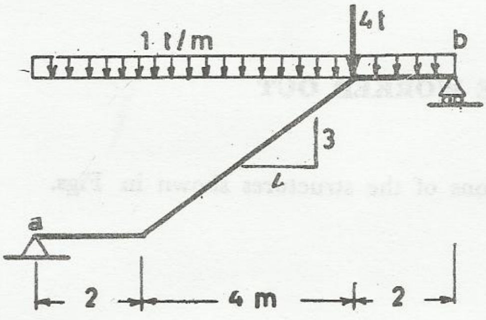



Fig. 2.31

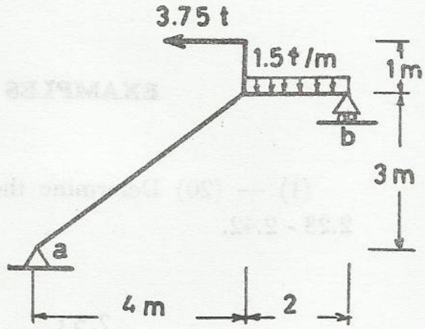



Fig. 2.32

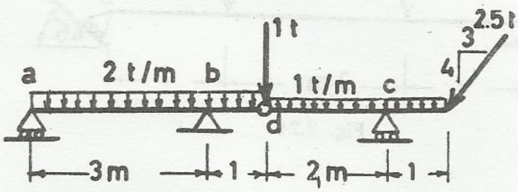



Fig. 2.33

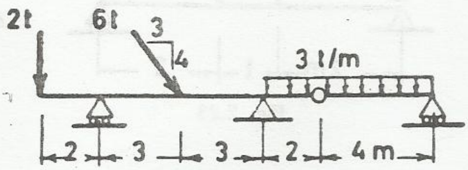



Fig. 2.34

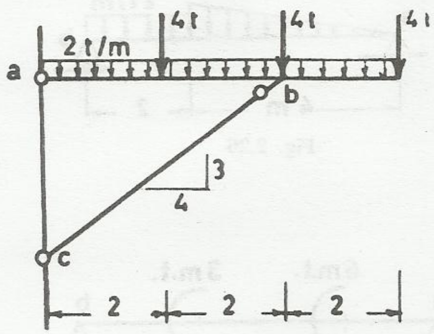



Fig. 2.35

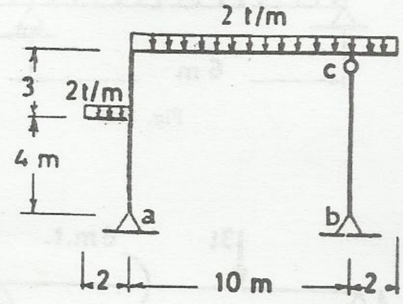



Fig. 2.36

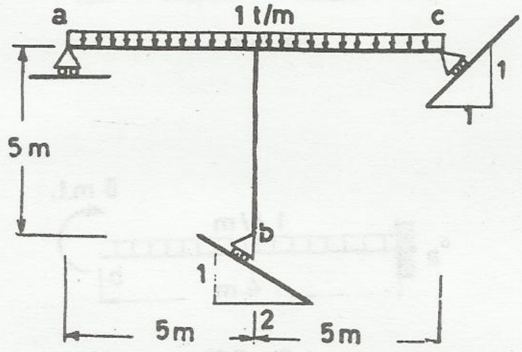



Fig. 2.37

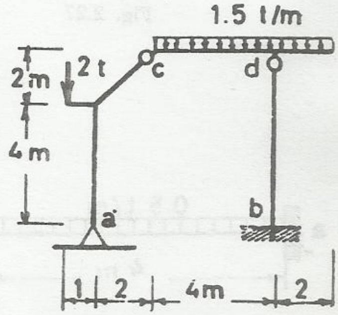



Fig. 2.38

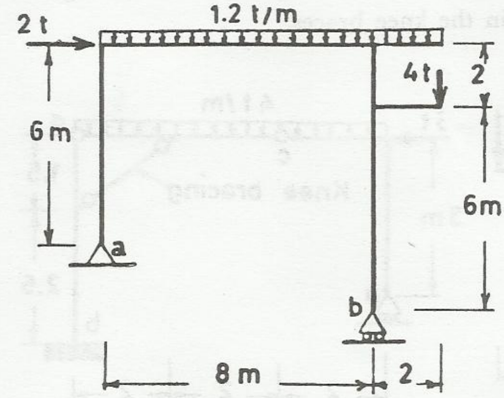



Fig. 2.39

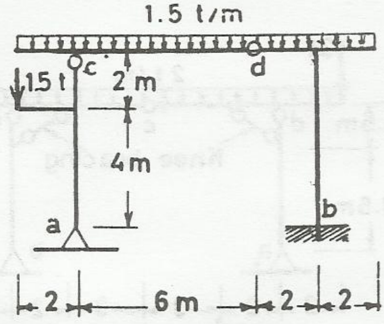



Fig. 2.40

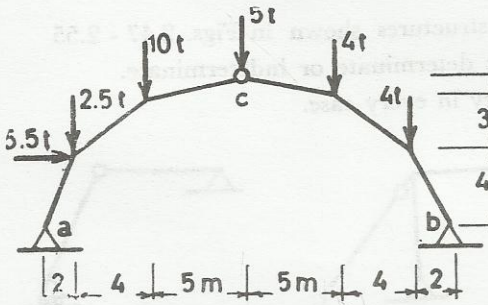



Fig. 2.41

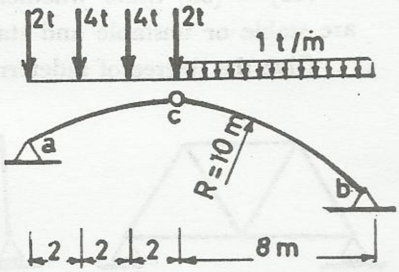



Fig. 2.42

(21), (22) Determine the reactions of the two frames shown in Figs. 2.43 and 2.44 and also the forces in the link members de.

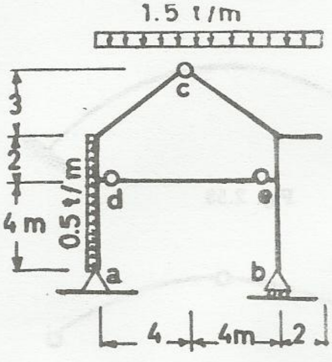



Fig. 2.43

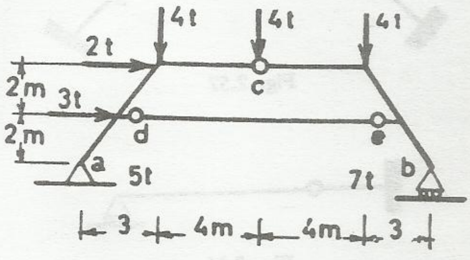



Fig. 2.44

(23), (24) Determine the reactions of the two frames shown in Figs. 2.45 and 2.46 and find the forces in the knee braces.

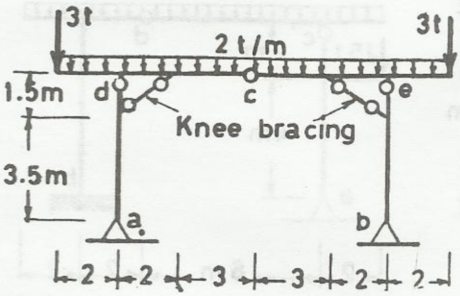



Fig. 2.45

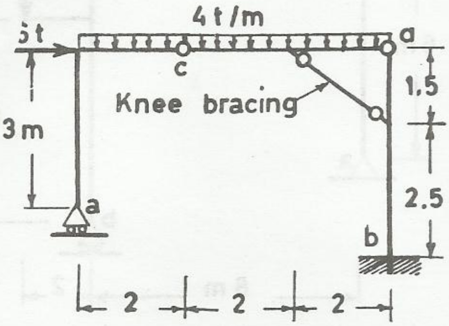



Fig. 2.46

(25) — (33) State whether the structures shown in Figs. 2.47 - 2.55 are stable or unstable and statically determinate or indeterminate. Give the degree of indeterminancy in every case.

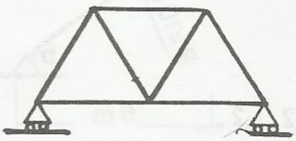


Fig. 2.47

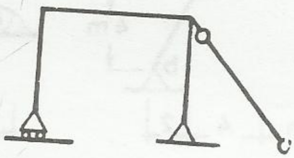


Fig. 2.48

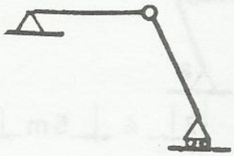


Fig. 2.49

Fig. 2.50

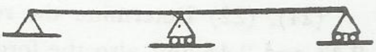


Fig. 2.51

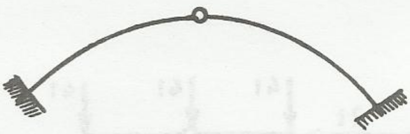


Fig. 2.52

Fig. 2.53

Fig. 2.54

Fig. 2.55

CHAPTER 3

# THRUST, SHEARING FORCE AND BENDING MOMENT

## 3.1 Introduction

The main aim of structural analysis is to find out whether a structure can carry safely all the loads that may act on it during its life time. This involves a comparison of the greatest values of the internal forces produced by the applied loads and the resistances of the structural element under consideration according to its dimensions and material of construction. In this chapter various internal forces will be studied. At this elementary stage, however, discussion will be limited to straight members, in which the axis joining the centroids of successive cross-sections is a straight line, subjected to loads lying in a single plane which also contains an axis of symmetry of every cross-section. Under these two conditions, members will bend in the plane of loading without twisting.

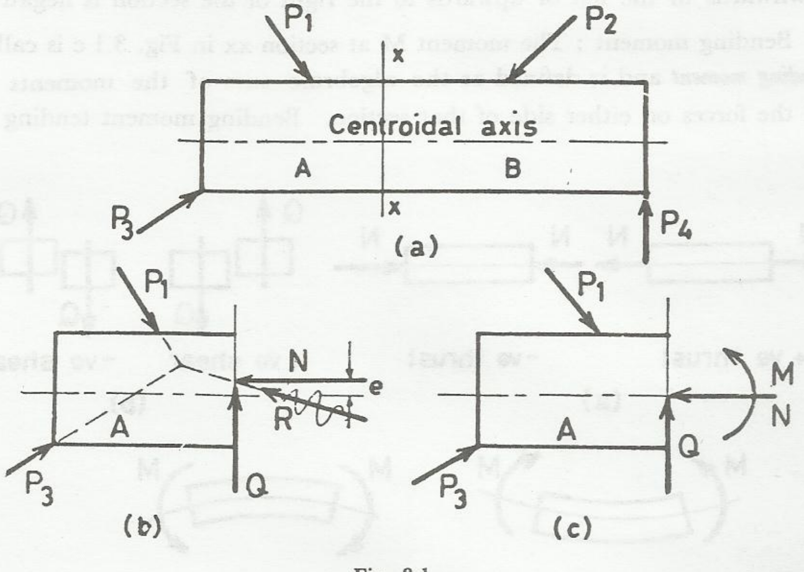



Fig. 3.1

Consider a structural member and let it be in equilibrium under the system of loads shown in Fig. 3.1 a. Let xx be any cross-section which cuts it into two separate parts A and B. If the member as a whole is in equilibrium then every part of it must be in equilibrium. Part A, for instance, must be in equilibrium under forces $P_1$ , $P_3$ and a force R which is the action of part B on part A (Fig. 3.1 b). In general, R could be resolved to a tangential force Q and an eccentric normal force N which is equivalent to a central normal force N and a moment $M = N \times e$. The various internal forces at section xx are shown in Fig. 3.1 c.

**3.2 Definitions**

**Normal force : The force N acting along the axis of the member in Fig. 3.1 c is called *normal force* or *thrust*. Thrust at a section is defined as the algebraic sum of the components along the axis of the member of all the forces on either side of that section. When the axial force tends to pull two parts of a member apart, thrust is termed positive and when it tends to press them together it is negative.**

Shearing force : The force Q acting tangential to section xx in Fig. 3.1c **is called *shearing force*. Shearing force at a section is defined as the algebraic** sum of the components perpendicular to the axis of the member on either side of that section. Shearing force acting upwards to the left or downwards to the right of the section is termed positive while shearing force acting downwards to the left or upwards to the right of the section is negative.

Bending moment : The moment M at section xx in Fig. 3.1 c is called ***bending moment*** and is defined as the algebraic sum of the moments of all the forces on either side of that section. Bending moment tending to

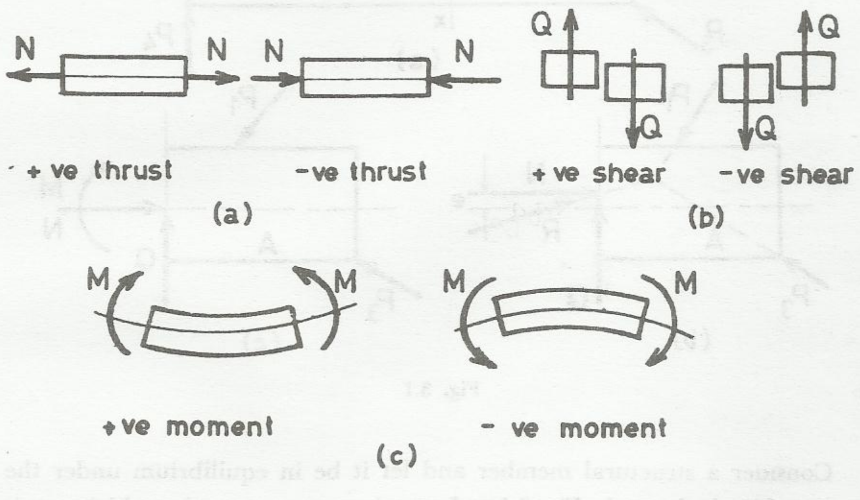



Fig. 3.2

produce tension in the lower fibres of a structural member and compression in the upper fibres is termed positive and vice versa.

Sign conventions for thrust, shearing force and bending moment are shown in Figs. 3.2 a, b and c respectively.

Beam : One of the most common structural members is the beam. It is a member which is subjected to bending by loads generally oblique to its longitudinal axis. Several examples are shown in Figs. 3.3 a-h. The

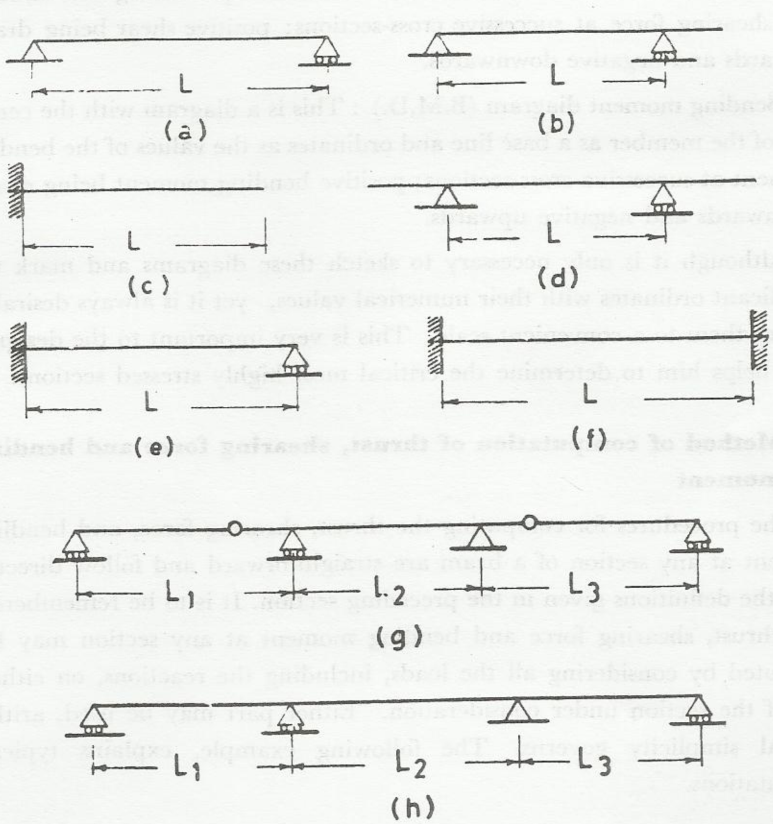



Fig. 3.3

beam shown in Fig. 3.3a is called *simple beam*, that in Fig. 3.3 b is said to have an overhang or an *overhanging beam*, that in Fig. 3.3 c is a *cantitever*, that in Fig. 3.3 d is a double overhanging beam, that in Fig. 3.3 e is a *restrained beam*, that in Fig. 3.3 f is a *fixed or fixed-ended beam*, that in Fig. 3.3 g. is termed a *cantilever beam*, while that in Fig. 3.3 h is a *continuous beam*. For all beams the distance L between supports is called a *span.* In continuous or cantilever beams there are several spans which are not necessarily of the same length.

Thrust or normal force diagram (N.F.D.): This is a diagram with the centre line of the member as base line and ordinates representing the values of the thrust at successive cross-sections; compression and tension being drawn on opposite sides of the base line.

Shearing force diagram (S.F.D.) : This is a diagram with the centre line of the member as a base line and ordinates representing the values of the shearing force at successive cross-sections; positive shear being drawn upwards and negative downwards.

Bending moment diagram (B.M.D.) : This is a diagram with the centre line of the member as a base line and ordinates as the values of the bending moment at successive cross-sections; positive bending moment being drawn downwards and negative upwards.

Although it is only necessary to sketch these diagrams and mark the significant ordinates with their numerical values, yet it is always desirable to plot them to a convenient scale. This is very important to the designer as it helps him to determine the critical most highly stressed sections.

## 3·3 Method of computation of thrust, shearing force and bending moment

The procedures for computing the thrust, shearing force, and bending moment at any section of a beam are straightforward and follow directly from the definitions given in the preceding section. It is to be remembered that thrust, shearing force and bending moment at any section may be computed by considering all the loads, including the reactions, on either side of the section under consideration. Either part may be used, arithmetical simplicity governs. The following example, explains typical computations.

**Example 3·1** Calculate the thrust, shearing force and bending moment at sections c and d of the beam shown in Fig. 3.4.

Solution : The first step is finding the unknown reactions. These are found by the methods described in chapter 2 and are shown, with the applied loads after being resolved into vertical and horizontal components, in Fig. 3.4 b.

Section c is investigated first. As mentioned before, the part of the

**beam on either side of the section may be used but it is obvious in this case that the forces on the part to the left of the section are simpler.**

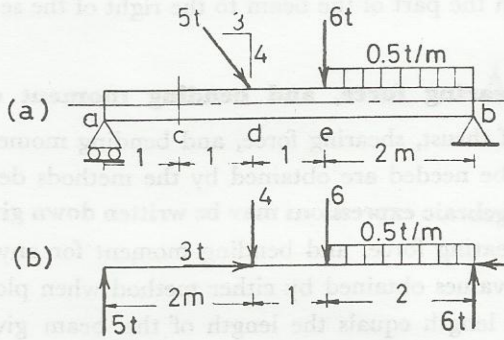



Fig. 3.4

By definition,

$N_c = 0$

$Q_c = 5$ t., upwards to the left of the section, i.e. + 5 t

$M_c = 5 \times 1 = 5$ m.t., causing tension in the lower fibers, i.e. + 5 m.t.

**To demonstrate the advantage of using the left part instead of the right** one, computations are re-carried out using the right part.

$N_c = -3 + 3 = 0$

$Q_c = 4 + 6 + 0.5 \times 2 - 6 = 5$ t., downwards to the right of the section, i.e. + 5t.

$M_c = 6 \times 4 - 0.5 \times 2 \times 3 - 6 \times 2 - 4 \times 1 = 5$ m.t., causing tension in the lower fibers, i.e. + 5 m.t.

These are the same results which have been more readily obtained before.

Consider now section d. At a section just to the left of point d,
$N = 0$, $Q = +5$ t., and $M = 5 \times 2 = +10$ m.t.,
while just to its right and again computing the straining actions from the part of the beam to the left of the section.

$N = -3$ t., $Q = +5 - 4 = +1$ t., and $M = 5 \times 2 = +10$ m.t.

This indicates the importance of determining the thrust and shearing force on either side of a concentrated load. The bending moment in both cases is the same.

(As an exercise, the student may compute N. Q and M at section d from the loads on the part of the beam to the right of the section).

### 3·4 Thrust, shearing force, and bending moment diagrams

The values of thrust, shearing force, and bending moment at as many sections as may be needed are obtained by the methods described above. **Alternatively, algebraic expressions mav be written down giving the values** of the thrust, shearing force, and bending moment for any section along the beam. The values obtained by either method when plotted against a base line whose length equals the length of the beam give the thrust, shearing force, and bending moment diagrams. Some typical examples of N.F., S.F., and B.M.Ds. follow.

**Example 3.2** Draw the N.F., S.F., and B.M.Ds. for the beam and loading given in example 3.1.

Solution : The loads and reactions are reproduced in Fig. 3.5 a. The

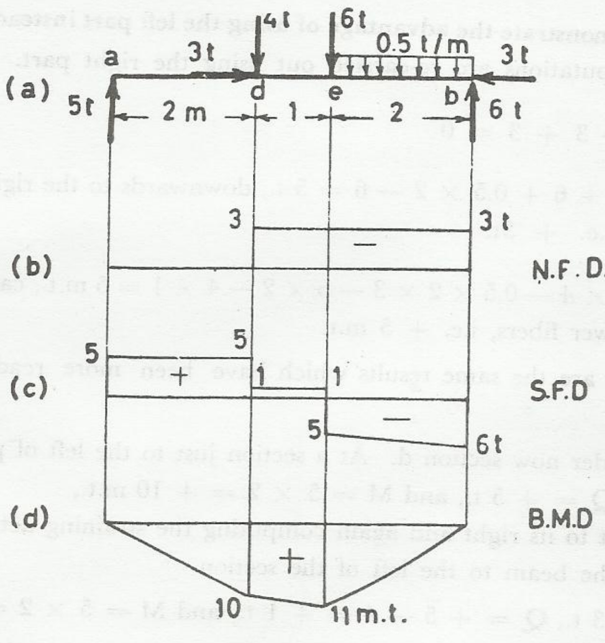


Fig. 3.5

thrust on a section just to the right of point a is zero and remains so until just to the left of point d. Just to the right of d, the thrust is — 3 t. which means that the N.F.D. rises abruptly from zero to 3 at this point, and since no additional horizontal loads are applied between point d and b, the thrust remains constant throughout this part of the beam as shown in Fig. 3.5 b. The shearing force on a section just to the right of a is + 5 t., and therefore the S.F.D. rises abruptly from zero to + 5 at this point, and since no additional vertical loads are applied between a and d, the shear remains constant throughout this part of the beam. Just to the right of point d, the 4 t. concentrated load has caused the shear to be reduced from + 5 to + 1. Between d and e the shear remains + 1 t. and the S.F.D. follows a horizontal line. Just to the right of point e the 6 t. concentrated load has caused the shear to be further reduced to — 5 t. In part eb, the shear on any section at a distance x to the right of point e is $Q = -5 - 0.5\,x$, which indicates that the S.F.D.in this part is a straight line increasing numerically from an ordinate of 5 t. at point e to 6 t. at point b. This last value may be more simply obtained considering a section just to the left of point b and considering the forces to its right which in this particular case is 6 t. upwards, i.e. $Q = -6$ t. The S.F.D. is shown in Fig. 3.5 c. The bending moment at a section distance x from point a in part ad is $M = 5\,x$. Therefore, the B.M.D. starts from zero at point a and increases linearly to an ordinate of 10 m.t. at point d. In part de, the bending moment at any section distance x from point d is $M = 5\,(2 + x) - 4\,x = 10 + x$, or more simply $M = 10 + x$ where 10 is the moment at section d and x is the moment of the resultant vertical force to the left of d. which is the shearing force at d. Hence, the B.M.D. in this part is a straight line increasing from an ordinate of 10 at d to 11 at e. Finally, in the part eb the bending moment at any section distance x to the right of e is $M = 11 - 5x - x^2/4$. Therefore, the B.M.D. starts at 11 at point e and decreases along a curve to zero at point b. This last part of the beam may be investigated more simply by considering a section at a distance x from point b, then the bending moment at any section in this part is $M = 6\,x - x^2/4$ which means that the B.M.D. is zero at b and increases along a curve to an ordinate of 11 at e which is the same result obtained before. The B.M.D. is shown in Fig. 3.5 d.

**Example 3.3** Construct the N.F., S.F. and B.M.Ds. for the overhanging beam loaded as shown in Fig. 3.6 a.

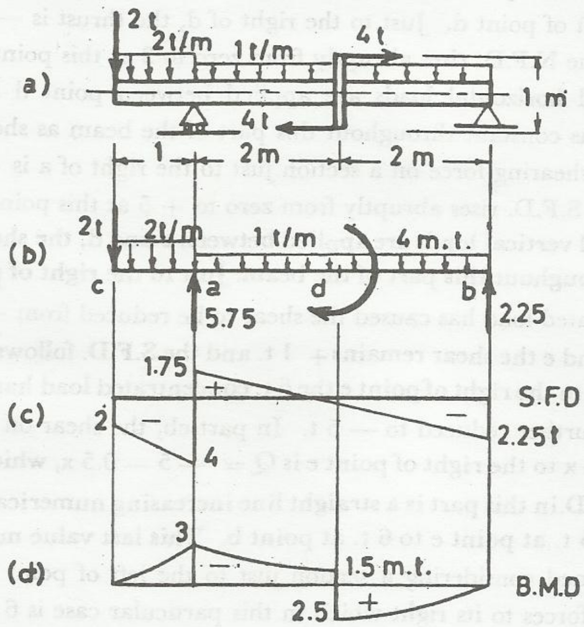



Fig. 3.6

Solution : Reactions,

$$\Sigma M_a = 0 = 4 + 1 \times 4 \times 2 - 2 \times 1 \times 0.5 - 2 \times 1 - Y_b \times 4$$

$$Y_b = \frac{4 + 8 - 1 - 2}{4} = 2.25 \text{ t.} \uparrow$$

$$\Sigma M_b = 0 = 2 \times 5 + 2 \times 1 \times 4.5 + 1 \times 4 \times 2 - 4 - Y_a \times 4$$

$$Y_a = \frac{10 + 9 + 8 - 4}{4} = 5.75 \text{ t.} \uparrow$$

$$\Sigma X = 0 = X_b \quad , \quad X_b = 0$$

Check, $\Sigma Y = 0$

$$2 + 2 \times 1 + 1 \times 4 - 5.75 - 2.25 = 0$$

Straining actions : Since all the loads are normal to the axis of the beam, there is no thrust at any section of the beam. Starting from the left end, just to the right of point c the shearing force is — 2 t. In the part ac, the shear on any section at a distance x from point c is $Q = -2 - 2x$, which indicates that the S.F.D. in this part is a straight line decreasing from a value of — 2 at point c to — 4 just to the left of point a, but just to its right the shear is increased by the value of the reaction to + 1.75 t., i.e. at point a there is a sudden change in the S.F.D. from — 4 to + 1.75. In the same manner the remainder of the S.F.D. shown in Fig. 3.6 c may be easily verified. The bending moment at a section distance x from point c in part ac is $M = -(2x + x^2)$. Therefore, the B.M.D. starts at zero at point c and varies along a curve to an ordinate of — 3 m.t. at point a. Similarly, in part ad. the B.M.D. is a curve increasing to an ordinate of — 1.5 m.t. at a section just to the left of point d. However, just to the right of point d the B.M.D. has increased by 4 m.t. to + 2.5 m.t. Therefore, at the point of application of a concentrated moment, there is a sudden change in the B.M.D. similar to that in the S.F.D. due to concentrated loads. Finally, the B.M.D. in the part db is a curve decreasing from an ordinate of + 2.5 at d to zero at b. The B.M.D. is shown in Fig. 3.6 d.

### 3·5 Relationships between load, shearing force and bending moment

Although the methods discussed in the preceding section for finding the N.F., S.F., and B.M.Ds. are most important and fundamental, yet in cases where a beam is subjected to loads perpendicular to its axis, as the majority of the beams are, the construction of the S.F. and B.M.Ds. may be facilitated by knowing certain relationships that exist between load, shear, and bending moment.

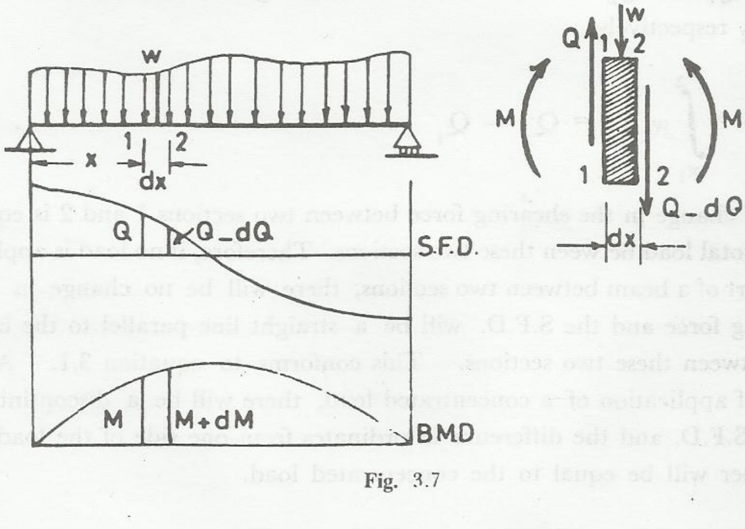



Fig. 3.7

Consider a simple beam subjected to a load of varying intensity and assume that the S.F. and B.M.Ds. are as shown in Fig. 3.7. Consider now the equilibrium of a small element of the beam of legnth dx over which the intensity of the load may be assumed uniform and equal to w as indicated.

$$\Sigma Y = 0 = Q - wdx - (Q - dQ)$$

$$w = \frac{dQ}{dx} \qquad \ldots 3.1$$

i.e. the slope of the S.F.D. at any section is measured by the value of the load intenisty at this section. Therefore, if a uniformly distributed load is applied to a part of a beam, w will be constant and hence the S.F.D. will have a constant slope ; i.e. will be a straight line in this part. If no load is applied to a beam between two points, the slope of the S.F.D. will be zero; i.e. it will be a straight line parallel to the base line between these two points. If a concentrated load is applied at a point of a beam, w, which is the load intensity, will be infinite and hence the slope of the S.F.D. will be infinite or vertical at this point. Generally, if the applied load is distributed but its intensity varies from point to point, the S.F.D. will be a curve whose slope changes continuously to correspond.

Integrating both sides of equation 3.1,

$$\int_{x_1}^{x_2} w\,dx = \int_{Q_1}^{Q_2} dQ$$

where $Q_1$ and $Q_2$ are the values of the shearing force at $x = x_1$ and $x = x_2$ respectively.

$$\left(\int_{x_1}^{x_2} w\,dx = Q_2 - Q_1\right) \qquad \ldots 3.2$$

i.e. the change in the shearing force between two sections 1 and 2 is equal to the total load between these two sections. Therefore, if no load is applied to a part of a beam between two sections, there will be no change in the shearing force and the S.F.D. will be a straight line parallel to the base line between these two sections. This conforms to equation 3.1. At a point of application of a concentrated load, there will be a discontinuity in the S.F.D. and the difference in ordinates from one side of the load to the other will be equal to the concentrated load.

Taking moments about face 2.2 (Fig. 3.7),

$$\Sigma M = 0 = Qdx + M - wdx\frac{dx}{2} - (M + dM)$$

Neglecting infinitismals of second order,

$$Q = \frac{dM}{dx} \qquad \ldots 3.3$$

i.e. the slope of the B.M.D. at any section is measured by the ordinate of the shearing force at this section. Therefore, if the shear is constant in a part of a beam, the B.M.D. will be a straight line in this part. However, if the shear varies in any manner the B.M.D. will be a curve. At the point of application of a concentrated load, there is an abrupt change in the ordinate of the S.F.D. and therefore an abrupt change in the slope of the B.M.D. at such a point. Further, the ordinate of the B.M.D. is a maximum or a minimum at the point of zero shear. According to the sign conventions used, and starting from the left end, then if at any point the ordinate of the S.F.D. changes from positive to negative then the bending moment is a maximum (maximum positive) at this point. If, on the other hand, the ordinate of the S.F.D. at a point changes from negative to positive, the bending moment is a minimum (maximum negative) at this point.

Integrating both sides of equation 3.3,

$$\int_{x_1}^{x_2} Q\,dx = \int_{M_1}^{M_2} dM$$

where $M_1$ and $M_2$ are the values of the bending moment at $x = x_1$ and $x = x_2$ respectively.

$$\int_{x_1}^{x_2} Q\,dx = M_2 - M_1 \qquad \ldots 3.4$$

i.e. the change in the bending moment between two sections 1 and 2 is equal to the area of the S.F.D. between these two sections.

### 3.6 Standard cases of S.F. and B.M.Ds.

By making use of the relationships given in the preceding section, the forms of the S.F. and B.M.Ds. may be sketched and by computing numerical values of shearing forces and bending moments at the points where the shapes of the diagrams change or at sections where the maximum or minimum occur, full knowledge of these diagrams is obtained.

Several standard cases are considered in the following.

(a) Simple beam with a non-central vertical load.

From statics, $Y_a = \dfrac{Pb}{L} \uparrow$ and $Y_b = \dfrac{Pa}{L} \uparrow$

S.F.D. : Between a and c the load is zero and therefore the shear is constant and equal to $+$ Pb/L. Similarly, between c and b, the load being zero, the shear is constant and equal to — Pa/L.

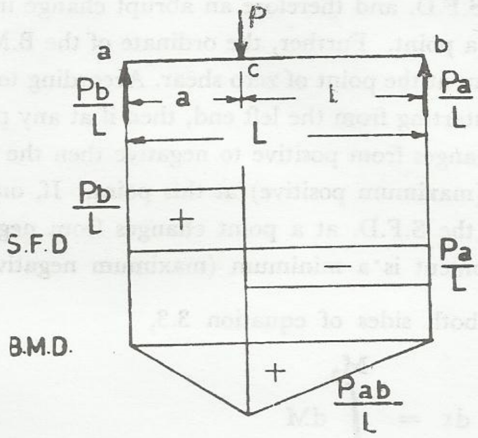



Fig. 3.8

B.M.D. : Between a and c the shear is constant and therefore the bending moment varies linearly from zero at a to $+$ Pab/L at c. Similarly, in part bc the moment varies linearly from zero at b to $+$ Pab/L at c. (Note that the maximum moment occurs at c where the shear changes from positive to negative).

Hence the S.F. and B.M.Ds. are as shown in Fig. 3.8.

(b) Simple beam with a central vertical load.

This is a special case of the previous one. The S.F. and B.M.Ds. are as shown in Fig. 3.9.

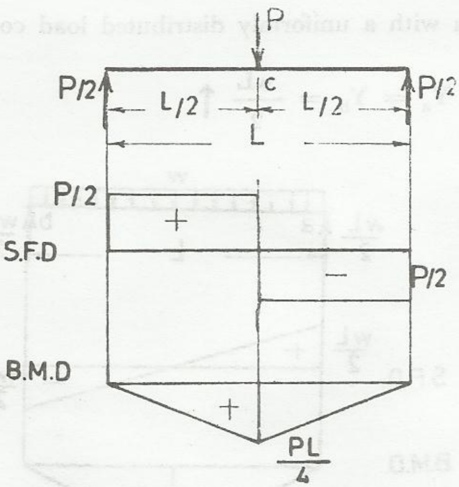



Fig. 3.9

(c) Simple beam with two equidistant equal loads.

From statics, $Y_a = Y_b = P \uparrow$

S.F.D. : Between a and c the load is zero and therefore the shear is constant and equal to + P. Between c and d the load is zero and the shear is equal to P — P = 0. Similarly, between b and d the load being zero the shear is constant and equal to — P.

B.M.D. : Between a and c the shear is constant and therefore the bending moment varies linearly from zero at a to Pa at c. Between c and d the shear is zero and therefore the bending moment is constant and equal to Pa. Similarly in part bd the moment varies linearly from + Pa at d to zero at b. The S.F. and B.M.Ds. are shown in Fig. 3.10.

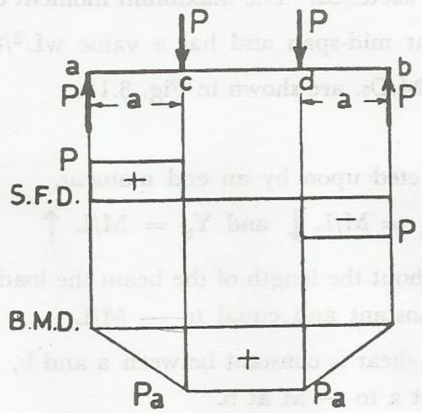



Fig. 3.10

(d) Simple beam with a uniformly distributed load covering the span.

From statics, $Y_a = Y_b = \frac{wL}{2} \uparrow$

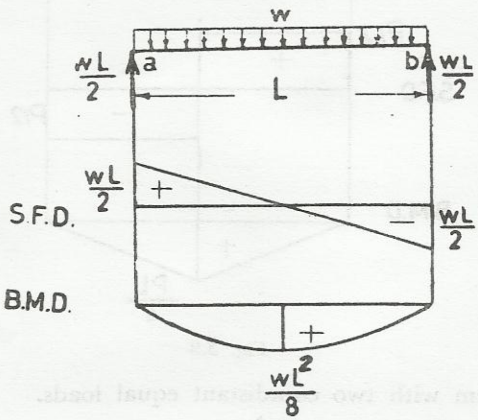



Fig. 3.11

S.F.D. : Between a and b, w is constant and therefore the shear varies linearly from + wL/2 at a to — wL/2 at b.

B.M.D. : For any section at a distance x from a,

$$M = \frac{wLx}{2} - \frac{wx^2}{2}$$

i.e. the B.M.D. varies parabolically along the span. The form of the B.M.D. can now be sketched. The maximum moment occurs at the point of zero shear, i.e. at mid-span and has a value $wL^2/8$.

The S.F. and B.M.Ds. are shown in Fig. 3.11.

(e) Simple beam acted upon by an end moment.

From statics, $Y_a = M/L \downarrow$ and $Y_b = M/L \uparrow$

S.F.D. : Throughout the length of the beam the load is zero and therefore the shear is constant and equal to — M/L.

B.M.D. : As the shear is constant between a and b, the moment varies linearly from zero at a to — M at b.

The S.F. and B.M.Ds. are shown in Fig. 3.12.

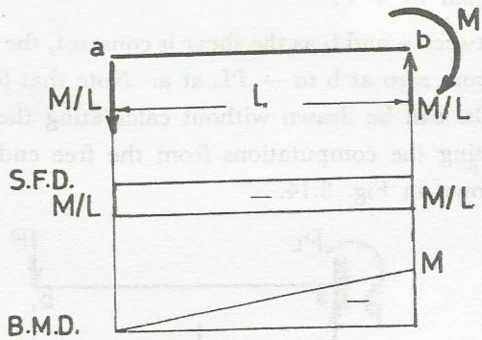



Fig. 3.12

(f) Simple beam with an intermediate concentrated moment.

From statics, $Y_a = M/L \downarrow$ and $Y_b = M/L \uparrow$

S.F.D. : Throughout the length of the beam the load is zero and therefore the shear is constant and equal to — M/L .

B.M.D. :Between a and c the shear is constant and therefore the bending moment varies linearly from zero at a to — Ma/L just to the left of c. Just to the right of c the moment is increased by M to + Mb/L, and again varies linearly to zero at b.

The S.F. and B.M.Ds. are shown in Fig. 3.13.

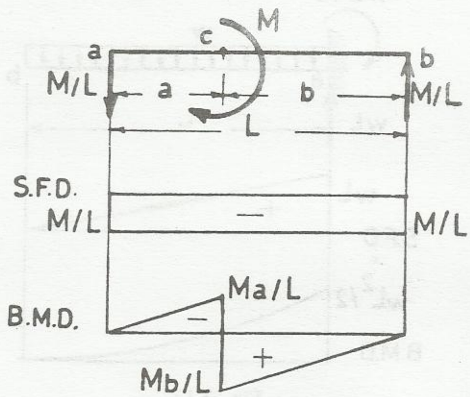



Fig. 3.13

(g) Cantilever with a concentrated end load.

From statics, $Y_a = P \uparrow$ and $M_a = PL$ (anticlockwise)

S.F.D. : Between a and b the load is zero and therefore the shear is constant and equal to + P.

B.M.D. : Between a and b as the shear is constant, the bending moment varies linearly from zero at b to — PL at a. Note that for cantilevers the S.F. and B.M.Ds. can be drawn without calculating the reactions. This is done by starting the computations from the free end. The S.F. and B.M.Ds. are shown in Fig. 3.14.

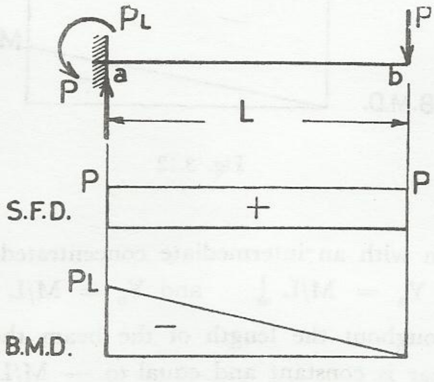



Fig. 3.14

(h) Cantilever with a uniformly distributed load.

From statics, $Y_a = wL \uparrow$ and $M_a = wL^2/2$ (anticlockwise)

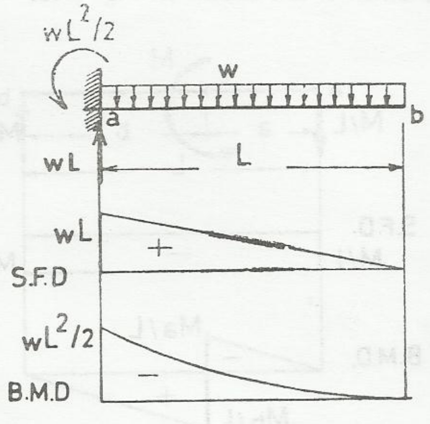



Fig. 3.15

S.F.D. : Theroughout the length of the beam the load is constant and therefore the shear varies linearly from zero at b to + wL at a.

B.M.D. : As the shear varies linearly along the beam the bending moment varies parabolically from zero at b to — $wL^2/2$ at a. Note again that both the S.F. and B.M.Ds. can be drawn without calculating the reactions. The S.F. and B.M.Ds. are shown in Fig. 3.15.

### 3·7 Principle of superposition

**Very often, the loading on a beam consists of a combination of two or more of the standard cases given in the preceding section. The resulting S.F. and B.M.Ds. for any of these loading combination can be readily determined by the *principle of superposition*, which may be stated as follows :**

***"The effect of several loads acting simultaneously on an elastic body is the same as the algebraic sum of the effects of these loads when each load acts seperately".***

This principle is the base to the elastic methods of structural analysis and is applicable provided that :

(1) The body is elastic, i.e. it regains its original shape on the removal of the applied loads.

(2) Stresses are proportional to strains, i.e. the material of the body obeys Hook's law.

(3) The geometry of the structure does not change during the application of loads; elastic strains being neglected.

The principle of superposition may best be explained by the following examples.

**Example 3·4** Consider the case of a simple beam loaded as shown in Fig. 3.16 a. Taking the concentrated load and the uniformly distributed load as separate systems, the resulting S.F. and B.M.Ds. for each individual case are shown in Figs. 3.16d,e,l &m, and the combination in Figs. 3.16f & n or, as shown on the centre line as base line in Figs. 3.16 g. & o respectively. By drawing the S.F. and B.M.Ds. for individual cases to the same scale the ordinates for the final S.F. and B.M.Ds. in Figs. 3.16 g & o can be scaled by dividers. Sometimes it is useful to draw the B.M.D. in parts as shown in Fig. 3.16 n. This type of diagram is of particular value in computing deflections of beams.

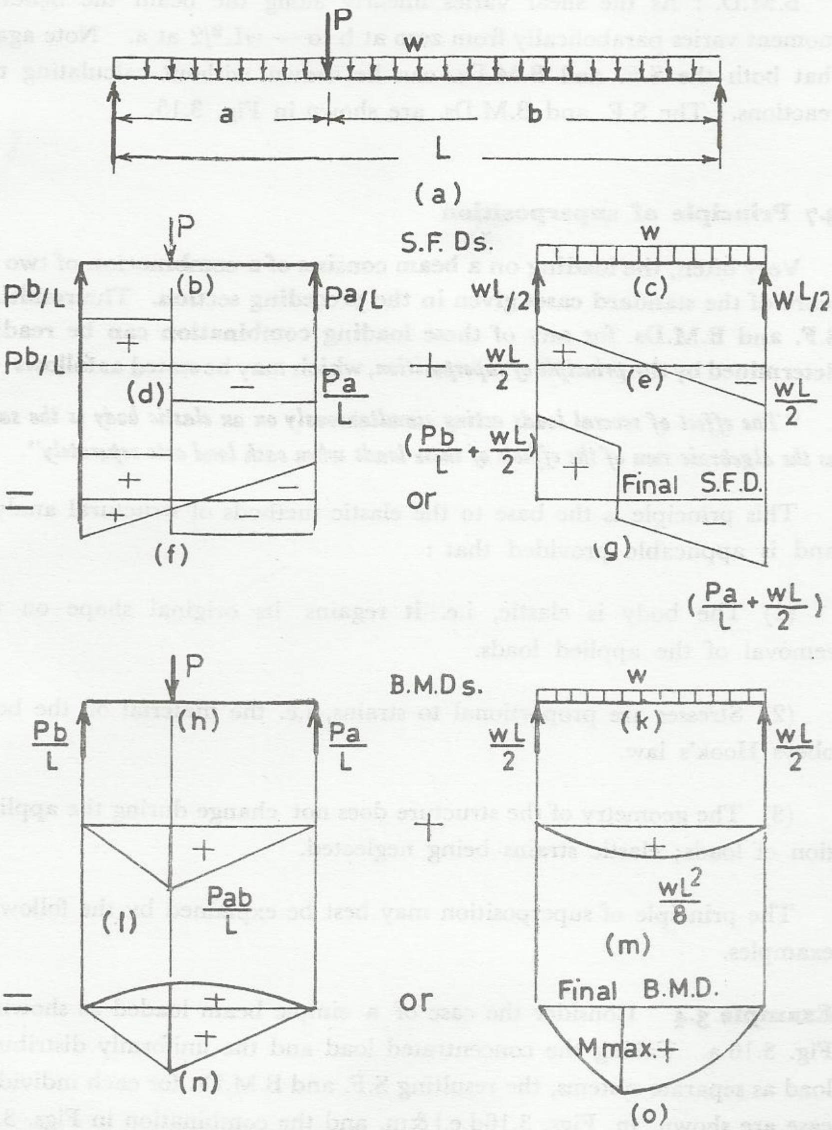



Fig. 3.16

**Example 3.5** Consider the case of a simple beam loaded as shown in Fig. 3.17 a. In this case the separate systems are shown in Figs. 3.17 b&c. The separate S.F. and B.M.Ds. are shown in Figs. 3.17 d, e, l & m, and the combined S.F. and B.M.Ds. are shown in Figs. 3.17 g and o respectively.

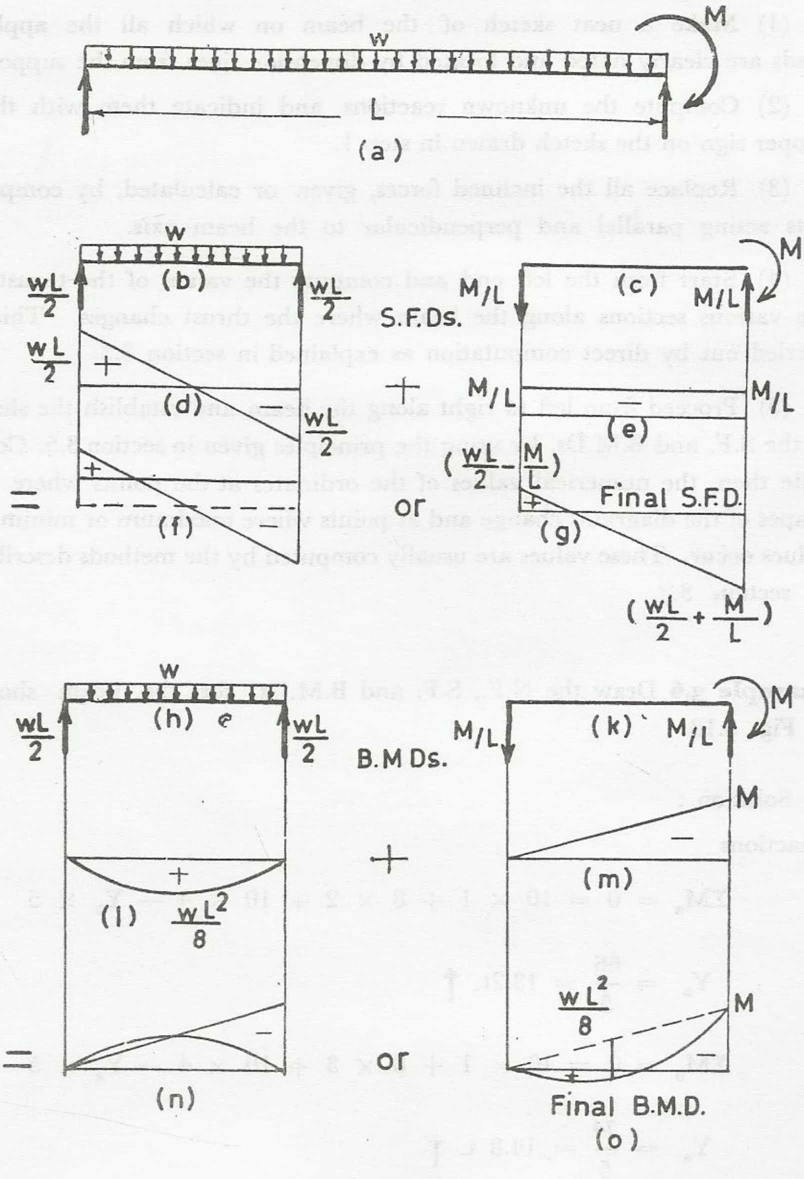



Fig. 3.17

### 3.8 Illustrative examples

The following examples will illustrate the construction of the N.F., S.F. and B.M.Ds. for statically determinate beams utilizing the principles discussed in the preceding sections in this chapter. For further emphasis, the steps used in such problems are summerized.

(1) Make a neat sketch of the beam on which all the applied loads are clearly noted and located by dimension lines from the supports.

(2) Compute the unknown reactions and indicate them with their proper sign on the sketch drawn in step 1.

(3) Replace all the inclined forces, given or calculated, by components acting parallel and perpendicular to the beam axis.

(4) Start from the left end and compute the values of the thrust at the various sections along the beam where the thrust changes. This is carried out by direct computation as explained in section 3.3.

(5) Proceed from left to right along the beam and establish the shape of the S.F. and B.M.Ds. by using the principles given in section 3.5. Compute then, the numerical values of the ordinates at the points where the shapes of the diagrams change and at points where maximum or minimum values occur. These values are usually computed by the methods described in section 3.3.

**Example 3.6** Draw the N.F., S.F. and B.M.Ds. for the beam shown in Fig. 3.18.

Solution :

Reactions

$$\Sigma M_a = 0 = 10 \times 1 + 8 \times 2 + 10 \times 4 - Y_b \times 5$$

$$Y_b = \frac{66}{5} = 13.2\text{t.} \uparrow$$

$$\Sigma M_b = 0 = 10 \times 1 + 8 \times 3 + 10 \times 4 - Y_a \times 5$$

$$Y_a = \frac{74}{5} = 14.8 \text{ t.} \uparrow$$

$$\Sigma X = 0 = X_a - 6$$

$$X_a = 6 \text{ t.} \rightarrow$$

N.F.D

N (in ad) = — 6 t.

N (in db) = 0

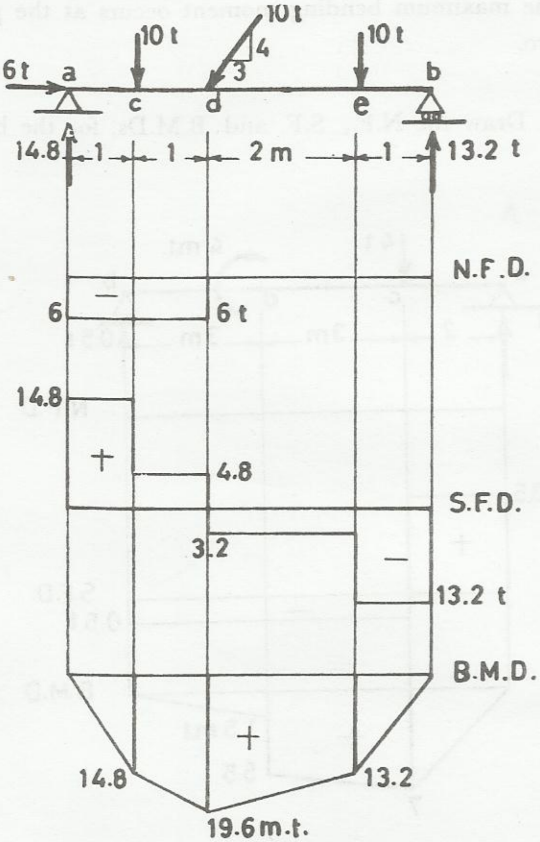



Fig. 3.18

S.F.D.

$$Q \text{ (in ac)} = +\ 14.8 \text{ t.}$$

$$Q \text{ (in cd)} = 14.8 - 10 = +\ 4.8 \text{ t.}$$

$$Q \text{ (in de)} = 4.8 - 8 = -\ 3.2 \text{ t.}$$

$$Q \text{ (in eb)} = -\ 3.2 - 10 = -\ 13.2 \text{ t.}$$

B.M.D.

$$M_a = 0$$

$$M_c = 14.8 \times 1 = +\ 14.8 \text{ m.t.}$$

$$M_d = 14.8 \times 2 - 10 \times 1 = +\ 19.6 \text{ m.t.}$$

$$M_e = 14.8 \times 4 - 10 \times 3 - 8 \times 2 = +\ 13.2 \text{ m.t.}$$

$$\text{or } M_e = 13.2 \times 1 = +\ 13.2 \text{ m.t.}$$

$$M_b = 0$$

Note that the maximum bending moment occurs at the point where the shear is zero.

**Example 3·7** Draw the N.F., S.F. and B.M.Ds. for the beam shown in Fig. 3.19.

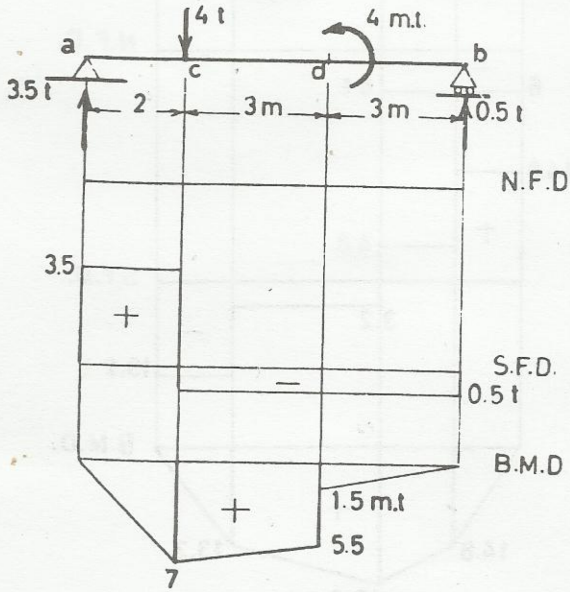



Fig. 3.19

Solution :

Reactions

$$\Sigma M_a = 0 = 4 \times 2 - 4 - Y_b \times 8$$

$$Y_b = \frac{4}{8} = 0.5 \text{ t. } \uparrow$$

$$\Sigma Y = 0 = 4 - 0.5 - Y_a$$

$$Y_a = 3.5 \text{ t. } \uparrow$$

N.F.D.

Since no axial loads are applied to the beam, the N.F.D. is zero

S.F.D.

$$Q \text{ (in ac)} = +\ 3.5 \text{ t.}$$

$$Q \text{ (in cb)} = +\ 3.5 - 4 = -\ 0.5 \text{ t.}$$

B.M.D.

$M_c = 3.5 \times 2 = +7$ m.t.

$M_d$ (left) $= 3.5 \times 5 - 4 \times 3 = +5.5$ m.t.

Since a moment of 4 m.t. acts at point d, the bending moment changes suddenlty and its value just to the right of d will be :

$M_d$ (right) $= +5.5 - 4 = +1.5$ m.t.

or $M_d$ (right) $= 0.5 \times 3 = +1.5$ m.t.

$M_b = 0$

Note that the slope of the B.M.D. along cd is the same as that along db as the S.F. is constant in both parts.

**Example 3.8** Draw the N.F., S.F. and B.M.Ds. for the beam shown in Fig. 3.20. Also calculate the position and value of the maximum positive bending moment.

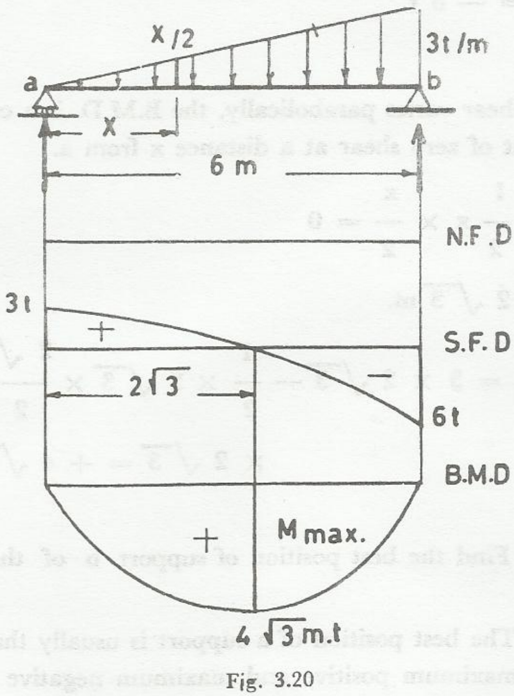



Fig. 3.20

**Solution :**

**Reactions**

$$\Sigma M_a = 0 = \frac{3\times 6}{2} \times \frac{2}{3} . \times 6 - Y_b \times 6$$

$$Y_b = \frac{36}{6} = 6 \text{ t.} \uparrow$$

$$\Sigma Y = 0 = \frac{3 \times 6}{2} - 6 - Y_a$$

$$Y_a = 9 - 6 = 3 \text{ t.} \uparrow$$

**N.F.D.**

As in the previous example since all the loads are perpendicular to the axis of the beam, the N.F.D. is zero.

**S.F.D.**

As the load varies linearly, the S.F.D. varies parabolically.

$$Q_a = + 3 \text{ t.}$$
$$Q_b = - 6 \text{ t.}$$

**B.M.D.**

Since the shear varies parabolically, the B.M.D. is a cubic curve. Assume the point of zero shear at a distance x from a.

$$3 - \frac{1}{2} x \times \frac{x}{2} = 0$$

$$x = 2\sqrt{3} \text{ m.}$$

$$M_{max} = 3 \times 2\sqrt{3} - \frac{1}{2} \times 2\sqrt{3} \times \frac{2\sqrt{3}}{2} \times \frac{1}{3} \times 2\sqrt{3} = + 4\sqrt{3} \text{ m.t.}$$

**Example 3.9** Find the best position of support b of the beam shown in Fig. 3.21.

**Solution :** The best position of a support is usually that which corresponds to the maximum positive and maximum negative moments being equal. Assume that the required position of support b is at a distance **x** from end c.

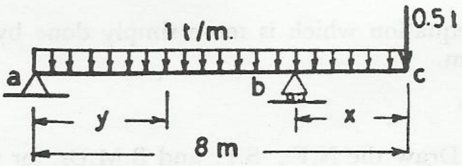



Fig. 3.21

Reactions

$$\Sigma M_a = 0 = 1 \times 8 \times 4 + 0.5 \times 8 - Y_b\,(8 - x)$$

$$Y_b = \frac{36}{8 - x}$$

$$\Sigma M_b = 0 = 0.5\,x - 8\,(4 - x) + (8 - x)\,Y_a$$

$$Y_a = \frac{32 - 8.5\,x}{8 - x}$$

$$\text{Check : } \Sigma Y = \frac{36 + 32 - 8.5\,x}{8 - x} - 8 \times 1 - 0.5 = 0$$

Maximum negative moment occurs at support b.

$$M_{max} \text{ negative} = 0.5\,x + \frac{x^2}{2}$$

Maximum positive moment occurs at point of zero shear. Assume this point to be at a distance y from support a.

$$\frac{32 - 8.5\,x}{8 - x} - 1 \times y = 0$$

$$y = \frac{32 - 8.5\,x}{8 - x}$$

$$M_{max} \text{ positive} = \left(\frac{32 - 8.5\,x}{8 - x}\right)^2 - \frac{1}{2}\left(\frac{32 - 8.5}{8 - x}\right)^2$$

$$= \frac{1}{2}\left(\frac{32 - 8.5\,x}{8 - x}\right)^2$$

Equating the maximum positive to the maximum negative moments,

$$x + x^2 = \left(\frac{32 - 8.5\,x}{8 - x}\right)^2$$

Solving this equation which is more simply done by trial,
$x \doteqdot 2$ m.

**Example 3.10** Draw the N.F., S.F. and B.M.Ds. for the inclined beam shown in Fig. 3.22.

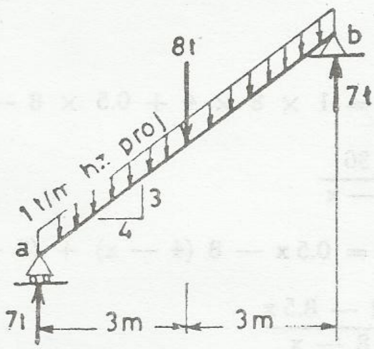



Fig. 3.22

Solution :

Reactions

Since $R_a$ is vertical, $R_b$ must also be vertical.

From symmetry, $R_a = R_b = \dfrac{1 \times 6 + 8}{2} = 7$ t. ↑

Resolve all the loads along and perpendicular to the axis of the beam. The components of all the loads are shown in Fig. 3.23 a. From these components the N.F., S.F. and B.M.Ds. are drawn in the usual manner and are shown in Figs. 3.23 b, c & d respectively. Note that the B.M.D. in this case is the same as for a horizontal beam having a span equal to the horizontal projection of the given beam and acted upon by the same vertical loads. To generalize, when inclined beams are subjected to concentrated vertical loads or distributed vertical load per unit length of the horizontal projection, the B.M.D. is identical to that of a horizontal beam of a span length equal to the horizontal projection of the inclined one and acted upon by the same vertical loads.

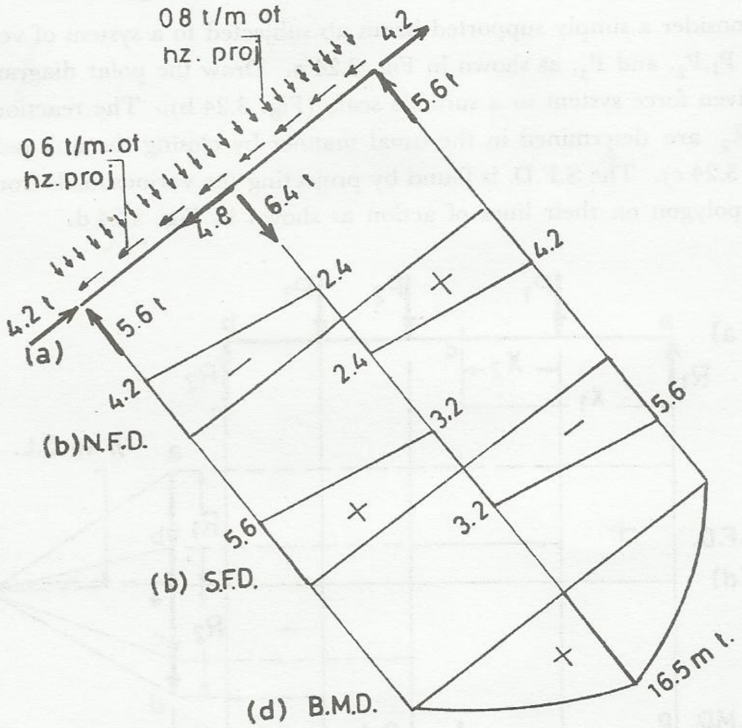



Fig. 3.23

All the beams considered till now have been statically determinate. For statically indeterminate beams, once the *redundants* are determined they may be treated as external loads and the procedures for drawing the N.F., S.F. and B.M.Ds. are basically the same as for statically determinate beams.

### 3.9 Graphical method for determining the N.F., S.F. and B.M.Ds.

Thrust, shear and bending moment can be determined graphically by means of the line of pressure. This method is suitable for curved beams or arches and will be discussed later (section 4.5).

**In those cases, however, where a straight beam is subjected to loads normal to its axis, as the majority of the beams are, the thrust is zero and**

**the construction of the S.F. and B.M.D. is more easily carried out by the link polygon method which is described in the following.**

Consider a simply supported beam ab subjected to a system of vertical loads $P_1, P_2$, and $P_3$, as shown in Fig. 3.24 a. Draw the polar diagram for the given force system to a suitable scale (Fig. 3.24 b). The reactions $R_1$ and $R_2$ are determined in the usual manner by closing the link polygon (Fig. 3.24 c). The S.F.D. is found by projecting the various loads from the force polygon on their lines of action as shown in Fig. 3.24 d.

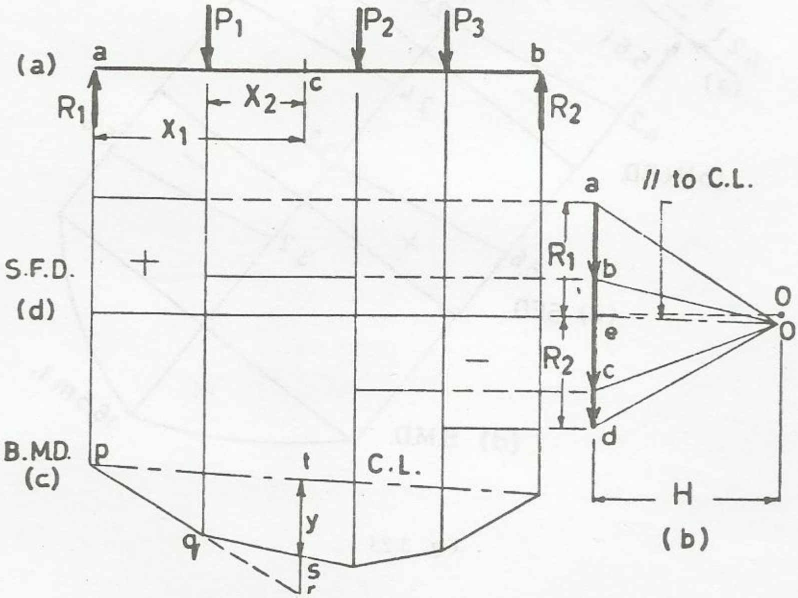



Fig. 3.24

Consider any section as c in the beam.

Now triangles oae and prt are similar.

$$\frac{rt\ (\text{distance})}{x_1\ (\text{distance})} = \frac{ae\ \text{force})}{H\ (\text{force})} = \frac{R_1\ (\text{force})}{H\ (\text{force})}$$

$$R_1\ (\text{force}).\ x_1\ (\text{distance}) = H\ (\text{force}).\ rt\ (\text{distance})$$

$$R_1\ x_1 = H\ rt \qquad \ldots\ (a)$$

also triangles oab and qrs are similar.

$$\frac{sr\ (\text{distance})}{x_2\ (\text{distance})} = \frac{ab\ (\text{force})}{H\ (\text{force})} = \frac{P_1\ (\text{force})}{H\ (\text{force})}$$

$$P_1 \text{ (force)}.\ x_2 \text{ (distance)} = H \text{ (force)}.\ sr \text{ (distance)}$$

$$P_1\, x_2 = H\, sr \qquad \text{... (b)}$$

but $$M_c = R_1\, x_1 - P_1\, x_2 \qquad \text{... (c)}$$

substituting from (a) and (b) into (c),

$$M_c = H \text{ (force)} \times (rt - sr) \text{ (distance)}$$

$$= H\, y$$

i.e. the closed link polygon represents the B.M.D. and the bending moment at any section is the polar distance H, measured to the force scale times the intercept y of the link polygon measured to the linear scale. Usually it is convenient to have a B.M.D. drawn on a horizontal base. This is done by drawing the link polygon that corresponds to the new pole o′ on the horizontal line through e.

The procedures for drawing the S.F. and B.M.Ds. for the general case of a nonuniformly distributed load are mainly similar to those outlined above. Referring to Fig. 3.25, the load diagram is divided into vertical strips and

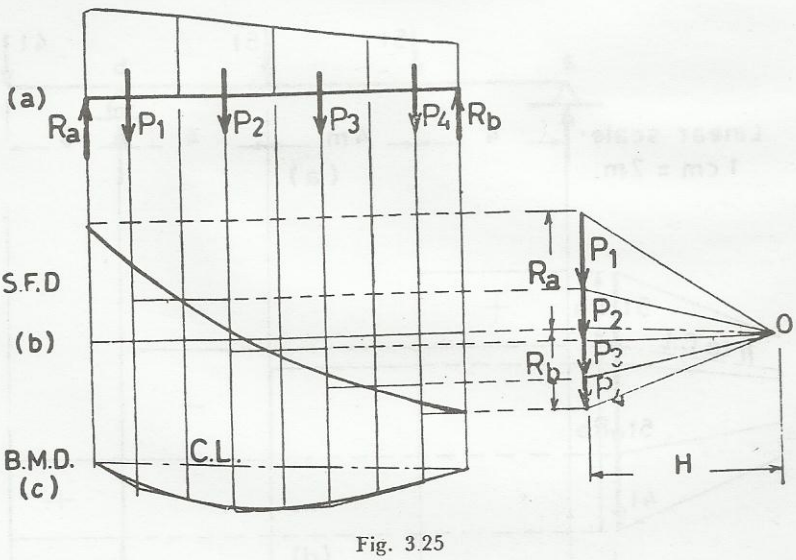



Fig. 3.25

the area of each strip is replaced by an equivalent concentrated load acting at the controid of the area. The S.F. and B.M.Ds. are then drawn in the usual manner. The diagrams thus drawn, however, will have the exact values at the ends of the strips because for these points the shearing force and bending moment will be the same whether the loads on their right or left are concentrated or distributed. The final S.F.D. will be a curve joining the ends of strips with equal give- and-take areas. Also the final B.M D. will be a curve touching the link polygon at the ends of strips as shown in Figs. 3.25 b and c. It should be remembered that these diagrams are approximately correct and the shorter the length of the strip the more accurate will the result be. Further, points of particular interest or points of application of concentrated loads should be taken as ends of strips.

**Example 3.11** Find graphically the S.F. and B.M.Ds. for the overhanging beam shown in Fig. 3.26 a.

**Solution : The polar diagram (Fig. 3.26 b) is drawn to scale 1 cm = 4 t. For convenience the polar distance H is taken 16 t. The link polygon** is constructed in the usual manner and the closing link is drawn as shown in Fig. 3.26 c. The reactions are determined by drawing a ray parallel to the closing link. The S.F.D. is constructed by projecting the forces on their lines of action and is shown in Fig. 3.26 d. Fig. 3.26 c is the required B.M.D to scale 1 cm. = 32 m.t. in which 32 is the product of the polar distance

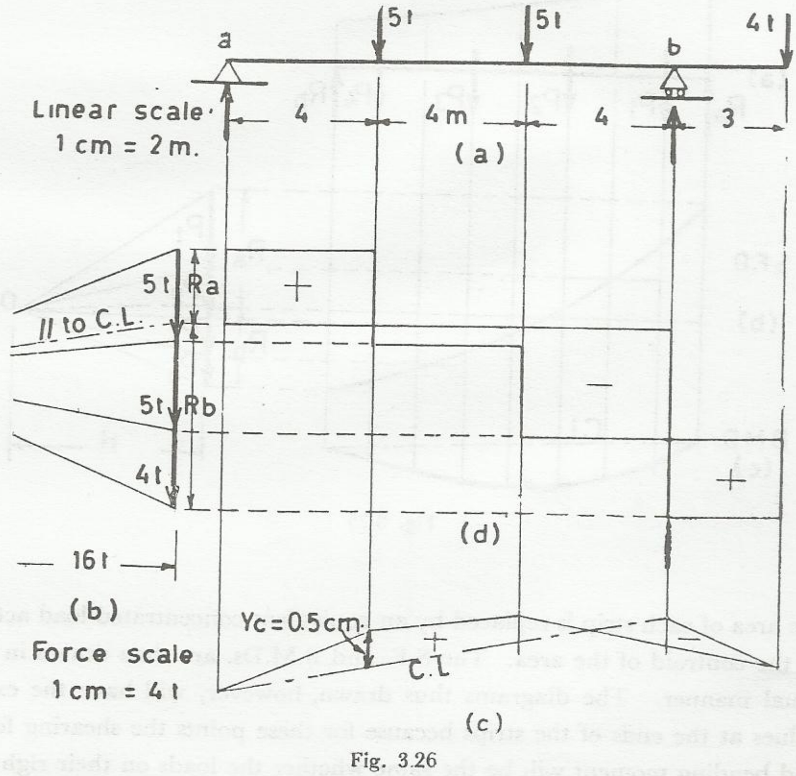



Fig. 3.26

(16 t.) and the linear scale (1 cm. = 2 m.). For example, the bending moment at point c is measured to be $y_c \times 32 = 16$ m.t. The signs may be found by inspection.

**Example 3.12** Construct the S.F. and B.M.Ds. for the cantilever shown in Fig. 3.27 a.

Solution : Starting at the free end, the force polygon in Fig. 3.27 b is drawn. The pole 0 is taken at H = 10 t. horizontally to the left of the starting point of the force polygon. This is done for convenience so as to obtain a B.M.D. with a horizontal base. The link polygon is drawn as shown

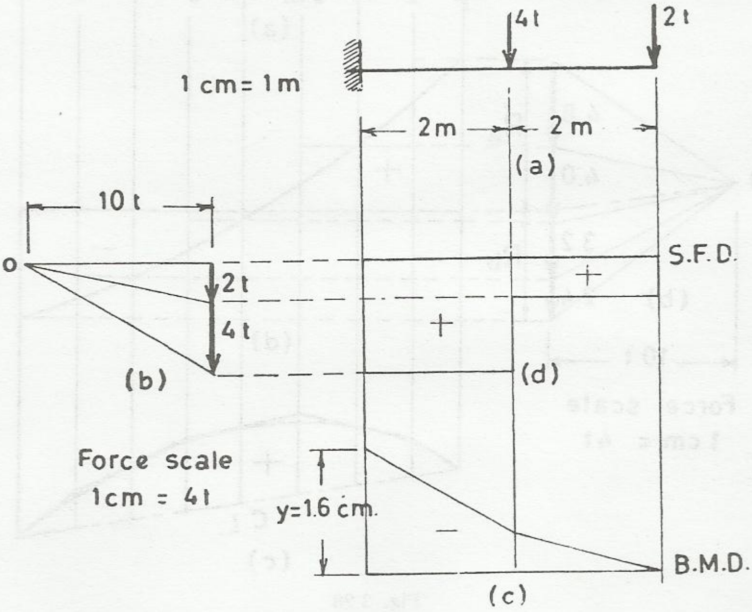



Fig. 3.27

in Fig. 3.27 c. This is the required B.M.D. with scale 1 cm. = 10 m.t., in which 10 is the product of the polar distance (10 t.) and the linear scale (1 cm. = 1 m.). With this scale the bending moment at the fixed end is measured to be :

$$M = 1.6 \times 10 = 16 \text{ m.t.}$$

The S.F.D. is obtained in the usual way by projecting the forces on their corresponding lines of action and is shown in Fig. 3.27 d.

**Example 3.13** Construct the S.F. and B.M.Ds. for the simple beam shown in Fig. 3.28 a.

Solution : The distributed load is replaced by a series of concentrated loads. This is done by dividing the load diagram into strips. In this case four strips, each 2 m. long, are used.

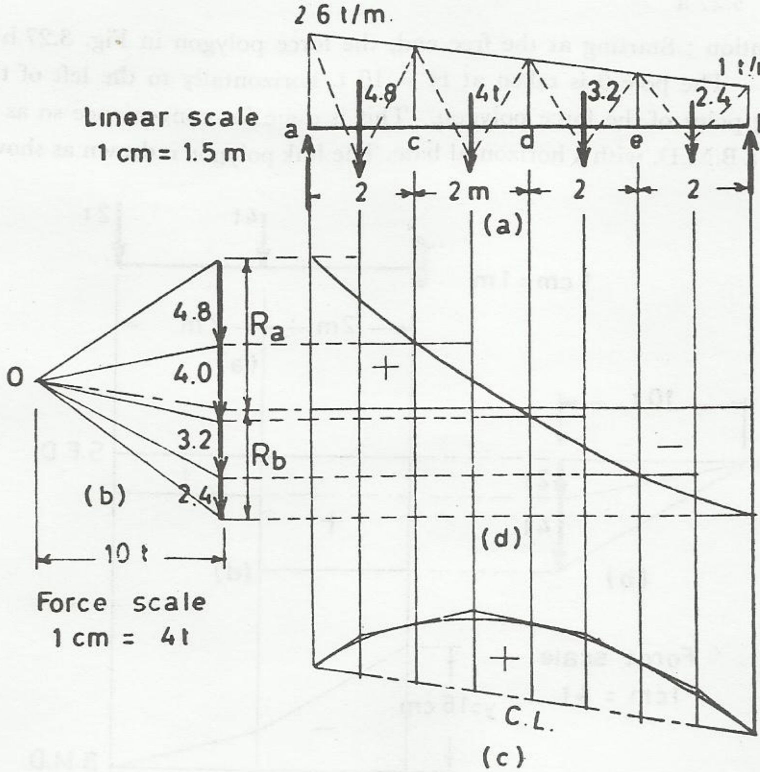



Fig. 3.28

The equivalent concentrated loads are given by :

$$P_1 = \frac{2.6 + 2.2}{2} \times 2 = 4.8 \text{ t.}$$

$$P_2 = \frac{2.2 + 1.8}{2} \times 2 = 4.0 \text{ t.}$$

$$P_3 = \frac{1.8 + 1.4}{2} \times 2 = 3.2 \text{ t.}$$

$$P_4 = \frac{1.4 + 1}{2} \times 2 = 2.4 \text{ t}$$

The centroids of the trapizoidal strips are found by the simple graphical method illustrated in Fig. 3.28 a. **The polar diagram** is drawn and the polar distance is taken 10 t. (Fig. 3.28 b). The link polygon (Fig. 3.28 c) is then drawn and the closing link is found. A ray parallel to this closing link will determine the reactions. The link polygon represents the B.M.D. and the S.F.D. is found in the usual manner by projecting the forces on their corresponding lines of action. It is to be noted that the ordinates of these diagrams are exact only at points a, b, c, d and e. The values at these points are joined by a curve as explained in section 3.9. This is done and the final S.F. and B.M.Ds. are shown in Fig. 3.28 c and d.

## EXAMPLES TO BE WORKED OUT

(1) - (8) Draw the N.F., S.F. and B.M.Ds. for the statically determinate beams shown in Figs. 3.29-3.36.

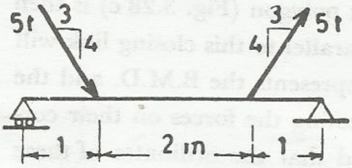


Fig. 3.29

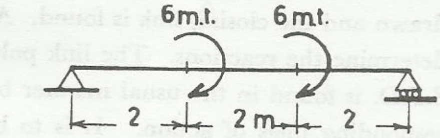


Fig. 3.30

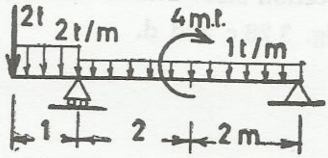


Fig. 3.31

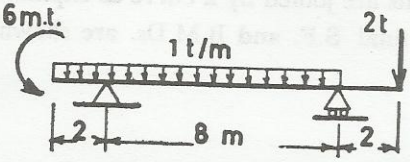


Fig. 3.32

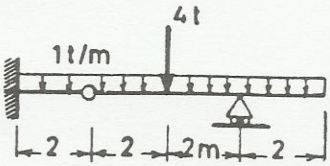


Fig. 3.33

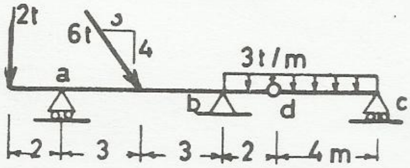


Fig. 3.34

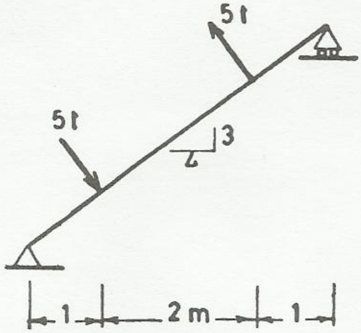


Fig. 3.35

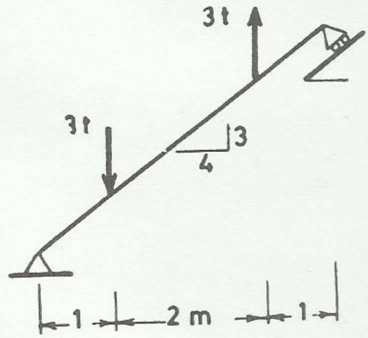


Fig. 3.36

(9) , (10) Draw the N.F., S.F. and B.M.Ds. of the two beams shown in Figs. 3.37 and 3.38. Calculate the position and value of the maximum positive bending moment.

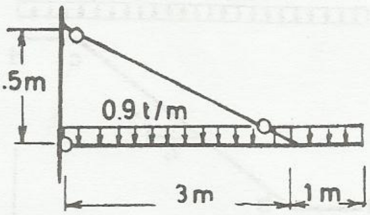



Fig. 3.37

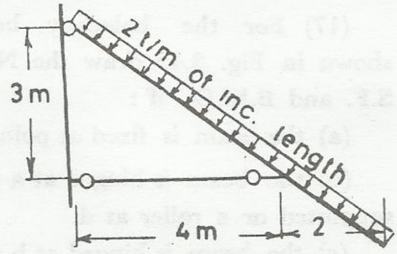



Fig. 3.38

(11) - (16) Draw the N.F., S.F. and B.M.Ds. for the beams shown in Fig. 3.39 - 3.44.

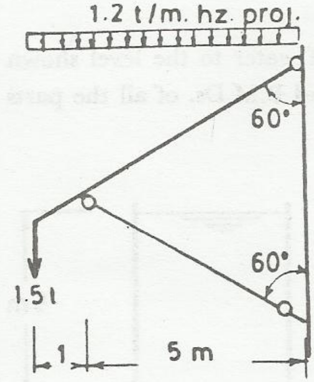



Fig. 3.39

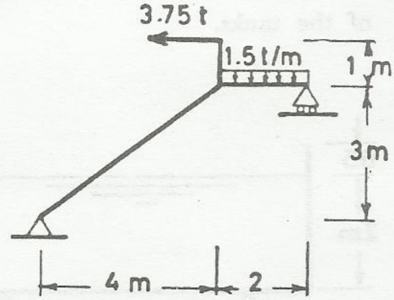



Fig. 3.40

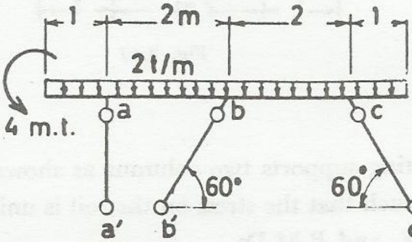



Fig. 3.41

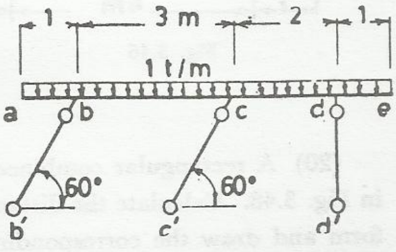



Fig. 3.42

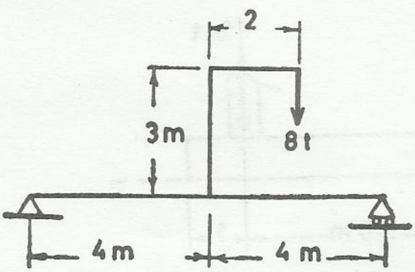



Fig. 3.43

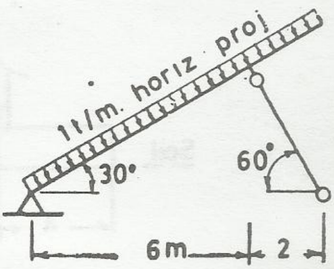



Fig. 3.44

(17) For the balacony beam shown in Fig. 3.45 draw the N.F., S.F. and B.M.Ds. if :

(a) the beam is fixed at point d.

(b) the beam is hinged at a and supported or a roller at d.

(c) the beam is hinged at b and supported on a roller at c.

Compare the values of the maximum moment in each case.

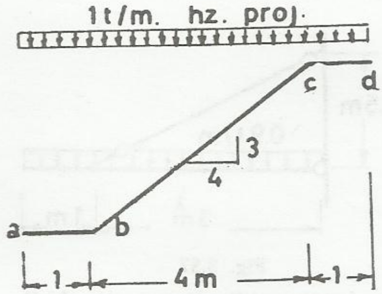



Fig. 3.45

(18), (19) Long open water tanks are full of water to the level shown in Figs. 3.46 and 3.47. Draw the N.F., S.F. and B.M.Ds. of all the parts of the tanks.

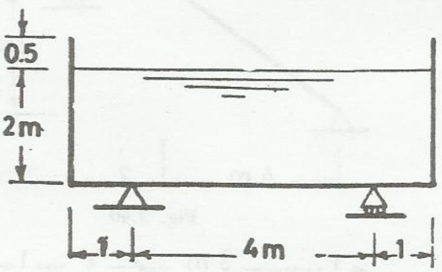



Fig. 3.46

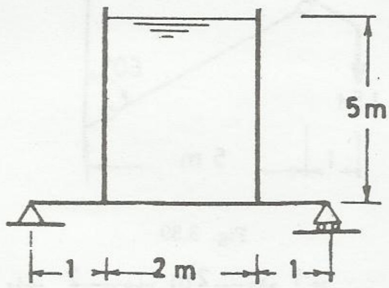



Fig. 3.47

(20) A rectangular combined footing supports two columns as shown in Fig. 3.48. Calculate the distance x such that the stress on the soil is uniform and draw the corresponding S.F. and B.M.Ds.

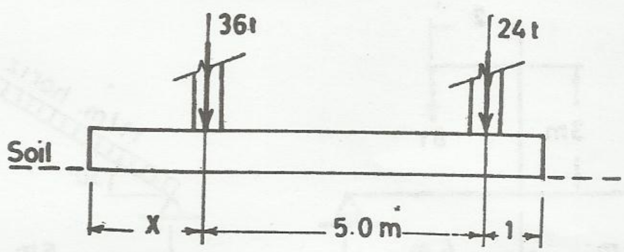



Fig. 3.48

(21) The beam given in Fig. 3.49 is provided with an intermediate hinge at point c. Find the best position of the hinge and draw the corresponding S.F. and B.M.Ds.

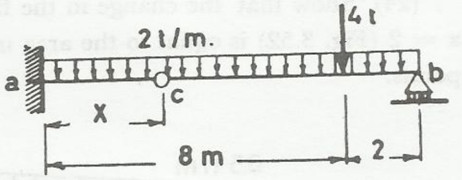



Fig. 3.49

(22) A pair of lock gates are strengthened by two beams ac and bc. If the load on the beams is as shown in Fig. 3.50, draw the N.F., S.F. and B.M.Ds. for one beam.

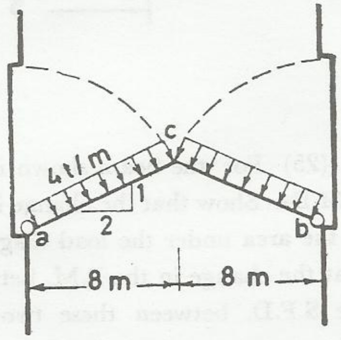



Fig. 3.50

(23) A timber coffer dam is made of planking supported by vertical piles. The piles are fixed at the bed level and supported by struts at the water level as shown in Fig. 3.51. If the piles are spaced 0.70 m. and the struts support 2/7 of the total water pressure, what will be the S.F. and B.M. in the pile at the bed level ?

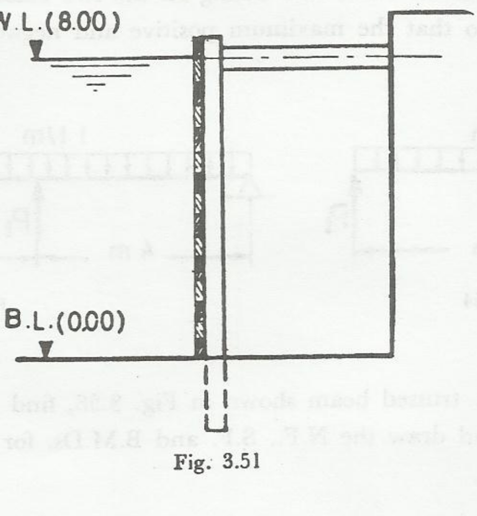



Fig. 3.51

(24) Show that the change in the B.M. between the points x = 1 and x = 2 (Fig. 3.52) is equal to the area under the S.F.D. between these two points.

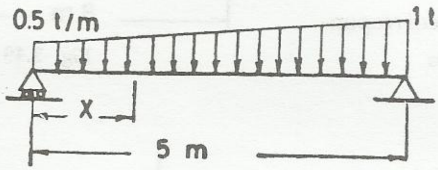



Fig. 3.52

(25) For the beam shown in Fig. 3.53., find graphically the S.F. and B.M.Ds. Show that the change in the shear between points b and c is equal to the area under the load diagram between these two points. Also show that the change in the B.M. between points a and c equals the area under the S.F.D. between these two points.

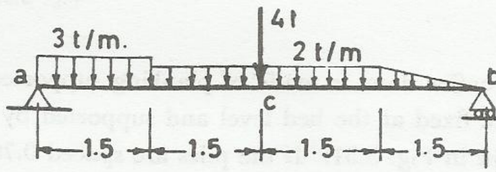



Fig. 3.53

(26),(27) Find the value of P acting on the two beams shown in Figs. 3.54 and 3.55 so that the maximum positive and negative moments are equal.

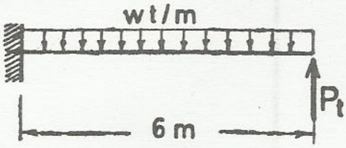



Fig. 3.54

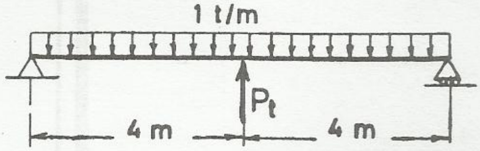



Fig. 3.55

(28) For the trussed beam shown in Fig. 3.56, find the forces in the link members and draw the N.F., S.F. and B.M.Ds. for the beam.

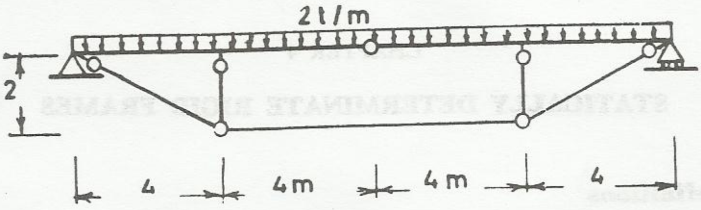



Fig. 3.56

(29) - (31) For the statically indeterminate beams shown in Figs. 3.57, 3.58 and 3.59 draw the S.F. and B.M.Ds. if :

in Fig. 3.57, $F_{de} = +4$ t.

in Fig. 3.58, $M_b = -wL^2/8$

in Fig. 3.59, $M_a = -13.2$ m.t. and $Y_c = 9$ t. ↑

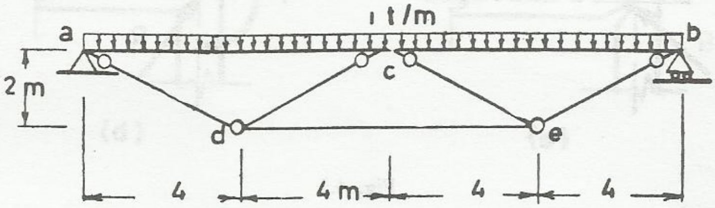



Fig. 3.57

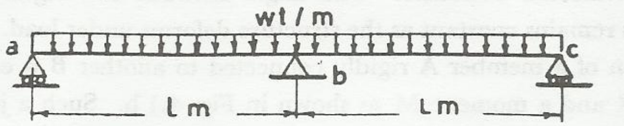



Fig. 3.58

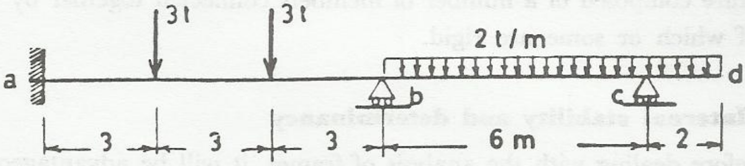



Fig. 3.59

CHAPTER 4

# STATICALLY DETERMINATE RIGID FRAMES

## 4.1 Definitions

Connections : There are two main types of structural connections :

(1) Hinged or pin-connection allows relative rotation between the ends of the connected members and hence no moment can be transmitted from one member to the other. Referring to Fig. 4.1 a, the action of member A on member B cannot be but a single force R acting through the centre of the hinge. Such a joint may be formed by a proper pin or by riveting.

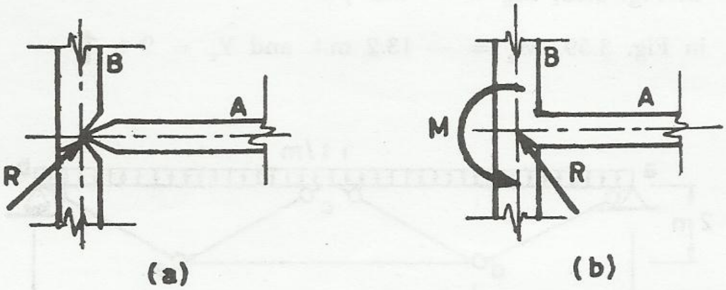



Fig. 4.1

(2) Rigid connection does not allow relative rotation between the ends of the connected members. The angle between two rigidly connected members remains constant as the structure deforms under load. In general, the action of a member A rigidly connected to another B is equivalent to a force R and a moment M as shown in Fig. 4.1 b. Such a joint may be formed by riveting, welding or by monolithic casting.

Rigid frame : A rigid frame, or as often briefly called *"Frame"*, is a structure composed of a number of members connected together by joints all of which or some are rigid.

## 4.2 Internal stability and determinancy

Before dealing with the analysis of frames, it will be advantageous to discuss the criteria of stability and determinancy. External stability and determinancy having been considered in section 2.6, the following discussion will be limited mainly to combined and internal stability and determinancy.

Consider a frame and let it have m members, j joints and also let there be r external reaction components. In general, a cross-section in a member has three straining actions; thrust, shearing force and bending moment, which when determined corresponding values at any other section along the same member may be computed. Thus, the number of independent unknowns is equal to $3m + r$. On the other hand, if the frame is in equilibrium, every joint in it is in equlibrium, and since a rigid joint is generally subject to a system of forces equivalent to a force and a couple the number of the available equations of equilibrium is equal to $3j$; three equations for each joint.

Sometimes frames consist of several parts connected together by hinges or links. If s condition equations (See section 2.4) are introduced due to these connections, the total number of equations available for the solution of the unknowns will be $3j + s$.

A frame may be classified as unstable, statically determinate or statically indeterminate, or as sometimes called redundant, by comparing the number of unknowns with the available number of independent equations. Thus,

If $3j + s > 3m + r$, the frame is unstable,

$3j + s = 3m + r$, the frame is statically determinate,

$3j + s < 3m + r$, the frame is statically indeterminate,

and if n is the *degree of indeterminancy* (or *redundancy*) then,

$$n = 3m + r - (3j + s)$$

**Example 4.1** Classify the frames shown in Figs. 4.2 a-k as being unstable, statically determinate or statically indeterminate. Indicate the degree of indeterminancy in each case.

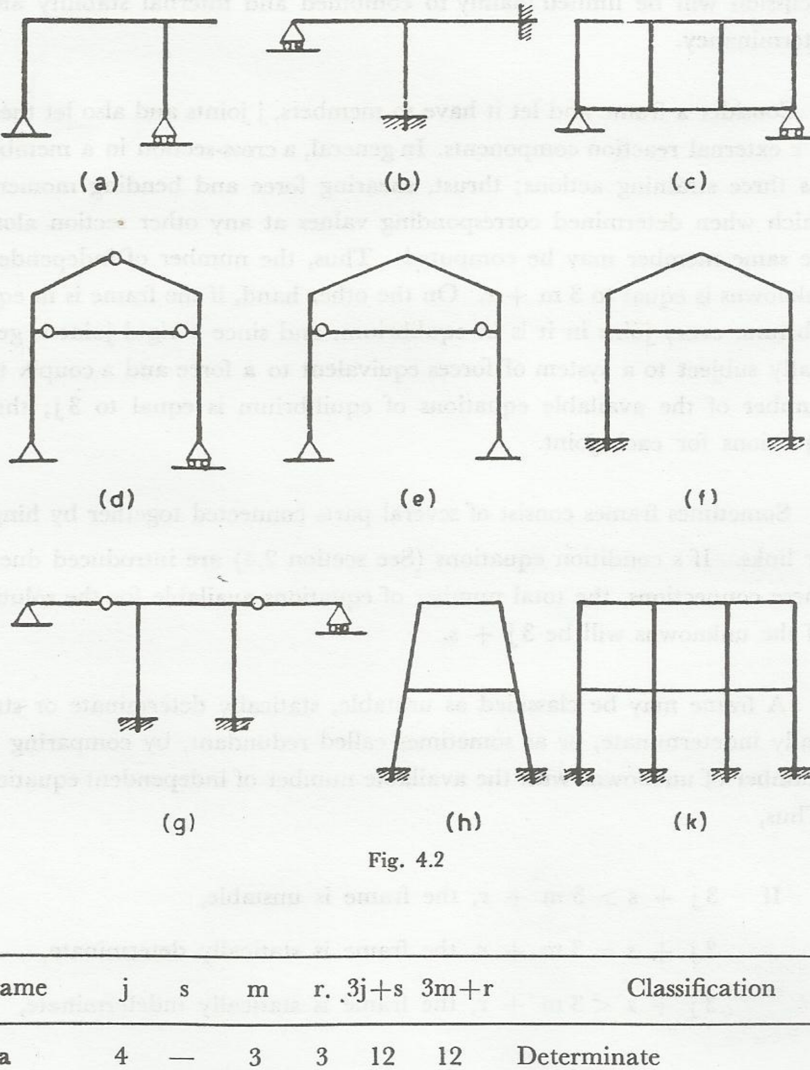



Fig. 4.2

| Frame | j | s | m | r. | 3j+s | 3m+r | Classification |
|---|---|---|---|---|---|---|---|
| a | 4 | — | 3 | 3 | 12 | 12 | Determinate |
| b | 4 | — | 3 | 7 | 12 | 16 | Indeterminate- 4th degree |
| c | 8 | — | 10 | 3 | 24 | 33 | Indeterminate- 9th degree |
| d | 7 | 3 | 7 | 3 | 24 | 24 | Determinate |
| e | 7 | 2 | 7 | 4 | 23 | 25 | Indeterminate- 2nd degree |
| f | 7 | — | 7 | 6 | 21 | 27 | Indeterminate- 6th degree |
| g | 8 | 2 | 7 | 9 | 26 | 30 | Indeterminate- 4th degree |
| h | 6 | — | 6 | 6 | 18 | 24 | Indeterminate- 6th degree |
| k | 12 | — | 14 | 12 | 36 | 54 | Indeterminate-18th degree |

It should be noted that n denotes the combined degree of indeterminancy, or redundancy, with respect to both the reactions and the internal straining actions. The degree of external indeterminancy, i.e. with respect to reactions only may be found in the manner discussed previously in section 2.6. The degree of internal indeterminancy is obviously the difference between the combined and external degree of indeterminancy. Thus,

$$n_i = n_c - n_e$$

Further, the comparison between the available number of equations and the number of unknowns, though necessary, is not always sufficient to decide whether a frame is stable or not. If $3j + s$ is greater than $3m + r$, then this comparison is sufficient for deciding that the frame is unstable. If, however, $3j + s$ is equal to or less than $3m + r$ it does not automatically mean that the frame is stable. To explain this statement, consider the frame shown in Fig. 4.3 a.

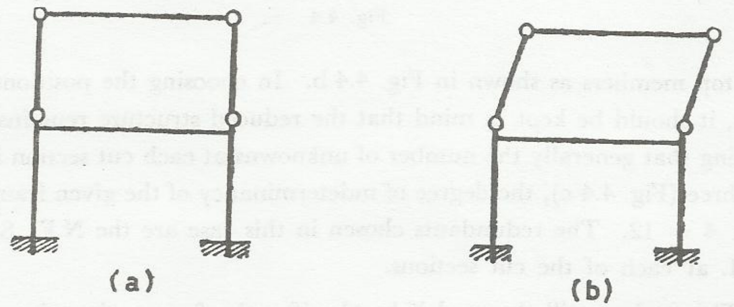



Fig. 4.3

$$j = 6 \qquad m = 6$$

$$s = 4 \qquad r = 6$$

$$3j + s = 22 < 3m + r = 24$$

A comparison between the number of the available equations and the number of independent unknowns will show that the frame is statically indeterminate or redundant to the second degree. However, it is obvious that there is nothing to prevent it from collapsing in the manner shown in Fig. 4.3 b, and is therefore classified as unstable.

This example shows that blind comparison between the number of the available equations and the number of independent unknowns does not necessarily yield the right answer. It is not argued that the method is sometimes useful but it is advised that when adopted, it should be used with care.

The most basic method to determine the degree of indeterminancy of an indeterminate frame is to remove as many supports and or to cut as many members to reduce the frame to a statically determinate structure. The degree of indeterminancy, or redundancy, corresponds then tc the number of the removed supports and or cut members. The removed restraints are called *redundants*. To explain this statement, consider the highly indeterminate frame, known as the Virendeel girder, shown in Fig. 4.4 a. It may be reduced to a statically determinate frame by cutting

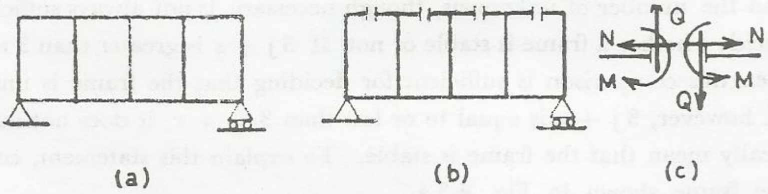



Fig. 4.4

the top members as shown in Fig. 4.4 b. In choosing the positions of the cuts, it should be kept in mind that the reduced structure remains stable. Noting that generally the number of unknowns at each cut section is equal to three (Fig. 4.4 c), the degree of indeterminancy of the given frame, $n = 3 \times 4 = 12$. The redundants chosen in this case are the N.F., S.F. and B.M. at each of the cut sections.

The student will do good if he classifies the frames given in example 4.1 according to this method.

### 4.3 Method of analysis

The first step in the analysis of frames is the computation of the external reaction components. The beginner may then isolate the various members as free bodies and draw N.F., S.F. and B.M.Ds. for each individual member utilizing the relationships between load, shearing force and bending moment developed in section 3.5. and bearing in mind the following important basic principle :

*"If the frame as a whole is in equilibrium, then every part af it, whether a member or a joint, is in equilibrium under the loads acting on it together with the action of the other members connected to it".*

Later when the student becomes familiar with the method of analysis, he will find that direct drawing of N.F., S.F. and B.M.Ds. of the whole frame is a simple matter.

The sign conventions used in regard to frames are similar to those used for beams and mentioned in section 3.2. Some difficulty may arise regarding the signs of B.M.Ds. To avoid confusion, various members may be considered as beams being looked at from the inside of the frame. The student is advised, however, to accustom himself to drawing the B.M.D. on the tension side of the various members of the frame. The parts of the B.M.D. lying inside the frame are positive and those outside the frame are negative. The determination of the tension side of a member is of great importance in the design of structures. For reinforced concrete structures the tension side defines the side where the main reinforcement should be placed and in metallic structures the compression side generally calls for additional investigation with regard to the stability of the member considered.

As no new theory is involved, the analysis may best be followed by working out some typical examples.

**Example 4.2** Draw free-body, thrust, shearing force and bending moment diagrams for all the members of the frame shown in Fig. 4.5 a.

Solution : The three unknown reaction components at the fixed support are found by the application of the three equations of equilibrium.

$$\Sigma X = 0 = X_a$$

$$X_a = 0$$

$$\Sigma Y = 0 = Y_a - 1 - 2 - 2 - 2 - 1$$

$$Y_a = 8 \text{ t.} \uparrow$$

$$\Sigma M_a = 0 = M_a + 1 \times 2 - 2 \times 2 - 2 \times 4 - 1 \times 6$$

$$M_a = 16 \text{ m.t. (anticlockwise)}$$

Member ab

This member is in equilibrium under the forces and moments shown in Fig. 4.5 b. The force 8 t. acting downwards and the clockwise moment of 16 m.t. acting at b, represent the action of members bc and cd on member ab which are found by considering the equilibrium of the latter as a free-body. The corresponding N.F., S.F. and B.M.Ds. are shown in Fig. 4.5 b.

Member bc

The moment and forces maintaining member bc in equilibrium

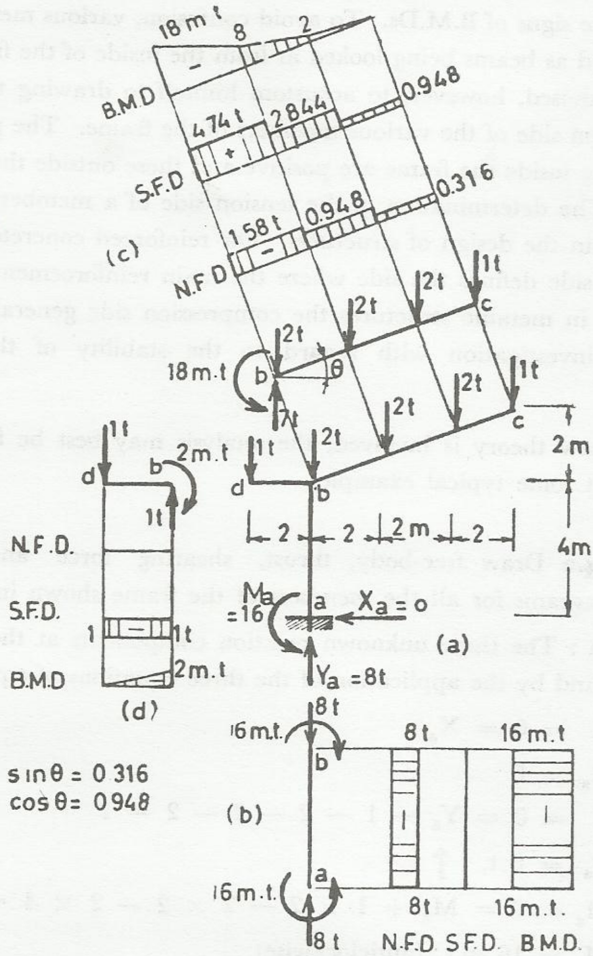


Fig. 4.5

together with the corresponding N.F., S.F. and B.M.Ds. are shown in Fig. 4.5 c. The N.F. and S.F.Ds. are obtained in the usual manner by resolving the forces along and normal to the axis of the member.

Member bd

The applied force and the actions of the rest of the frame on member bd together with the N.F., S.F. and B.M.Ds. are shown in Fig. 4.5 d.

**Example 4·3** Draw the free-body, thrust, shearing force and bending moment diagrams of all the members and the joints of the frame shown in Fig. 4.6 a.

Solution : $\Sigma X = 0 = 1 - X_a$

$X_a = 1$ t. →

$\Sigma M_b = 0 = 1 \times 2.5 + 2.5 \times 2 - Y_a \times 5$

$Y_a = 1.5$ t. ↑

$\Sigma Y = 0 = 2.5 - 1.5 - Y_b$

$Y_b = 1$ t. ↑

The free-body, N.F., S.F. and B.M.Ds. of members ac, cd and db are shown in succession in Figs. 4.6 b, c and d. Free-body diagrams for joints c and d are shown in Figs. 4.6 e and f respectively.

**Example 4·4** Fig. 4.7 a shows a frame in a shed. Draw the free-body, N.F., S.F. and B.M.Ds. for member ab.

Solution : Member bc is a link member, i.e. subjected to axial load $R_c$ only.

$\Sigma M_a = 0 = 0.8\ R_c \times 7.5 \times 0.75 - 4.2 \times 2 - 2.25 \times 4$

$R_c = 4$ t. ↙

$X_c = 0.6 \times 4 = 2.4$ t. ←

$Y_e = 0.8 \times 4 = 3.2$ t. ↓

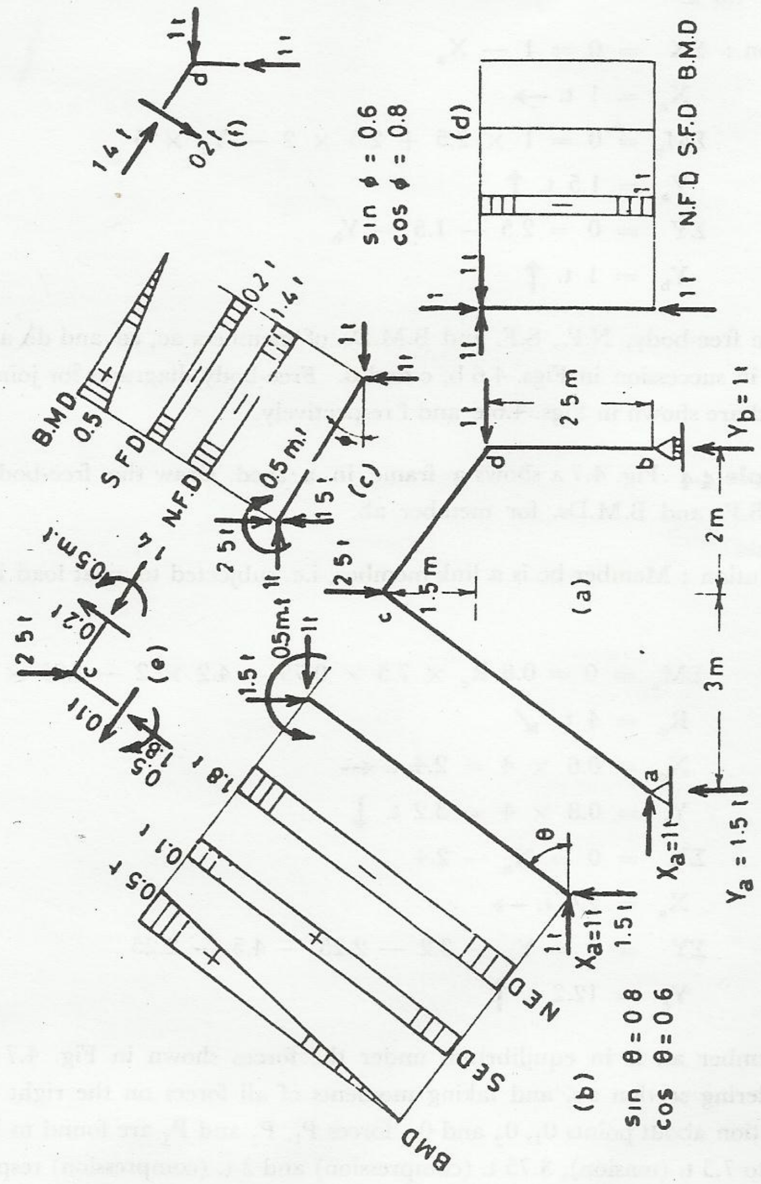
$\Sigma X = 0 = X_a - 2.4$

$X_a = 2.4$ t. →

$\Sigma Y = 0 = Y_a - 3.2 - 2.25 - 4.5 - 2.25$

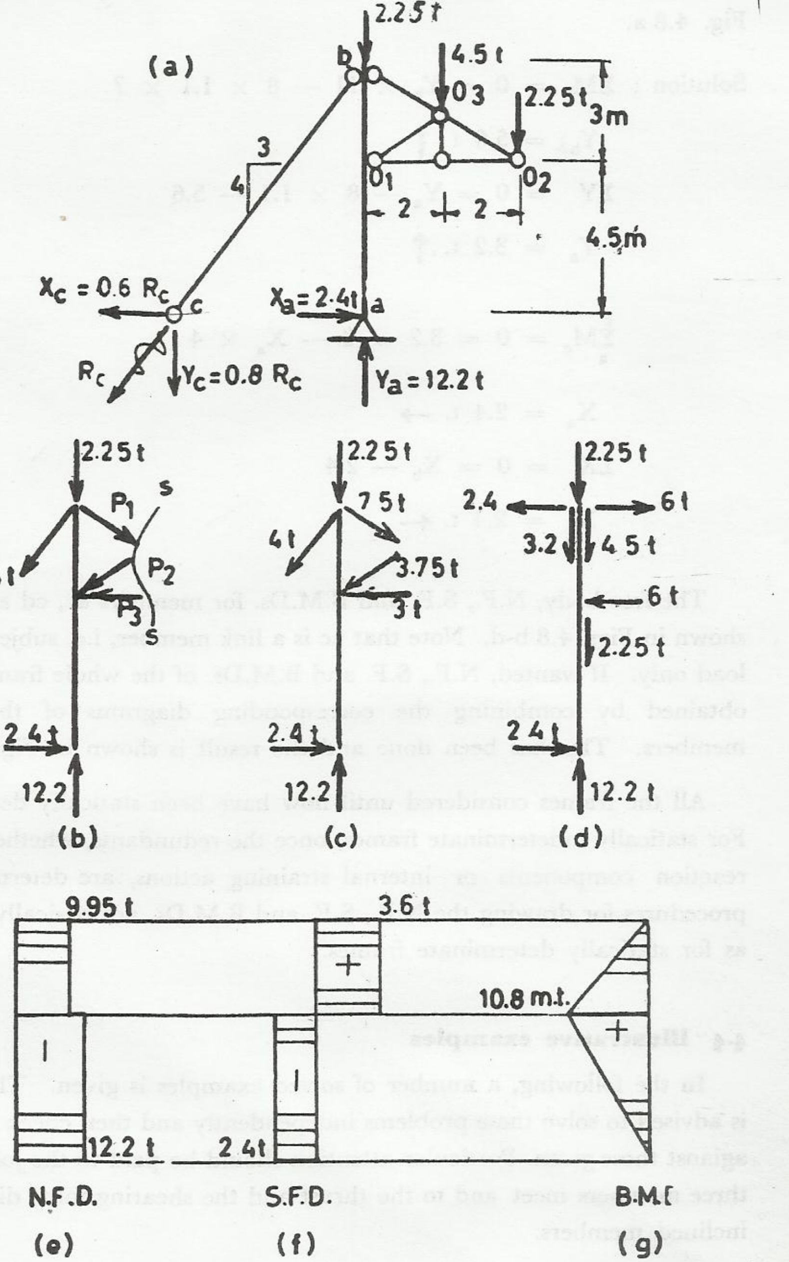
$Y_a = 12.2$ t. ↑

Member ab is in equilibrium under the forces shown in Fig. 4.7 b. Considering section s-s, and taking moments of all forces on the right of the section about points $0_1$, $0_2$ and $0_3$, forces $P_1$, $P_2$ and $P_3$ are found to be equal to 7.5 t. (tension), 3.75 t. (compression) and 3 t. (compression) respectively. Hence, member ab as a free-body and the forces maintaining it in equilibrium are as shown in Fig. 4.7 c. The components of these forces along and normal to the axis of the member are shown in Fig. 4.7 d. The N.F., S.F. and B.M.Ds. are given in Figs. 4.7 e, f and g respectively.



sin φ = 0.6
cos φ = 0.8
N.F.D S.F.D B.M.D
(d)
(f)
(e)
(c)
(a)
(b)
sin θ = 0.8
cos θ = 0.6
B.M.D
S.F.D
N.F.D
BMD
SFD
NFD
2.5 m
1.5 m
2 m
3 m
Yb = 1t
Xa=1t
Ya = 1.5 t


Fig. 4.6



(a)
2.25 t
b
4.5 t
0 3
2.25 t
3m
3
4
0 1
0 2
2
2
4.5 m
$X_c = 0.6\ R_c$
c
$X_a = 2.4$ t
a
$R_c$
$Y_c = 0.8\ R_c$
$Y_a = 12.2$ t
2.25 t
s
$P_1$
4 t
$P_2$
$P_3$
s
2.4 t
12.2
(b)
2.25 t
7.5 t
4 t
3.75 t
3 t
2.4 t
12.2
(c)
2.25 t
2.4
6 t
3.2
4.5 t
6 t
2.25 t
2.4 t
12.2 t
(d)
9.95 t
3.6 t
+
10.8 m.t.
+
12.2 t
2.4 t
N.F.D.
S.F.D.
B.M.
(e)
(f)
(g)


Fig. 4.7

**Example 4.5** Draw the free-body, thrust, shearing force and bending moment diagrams for all the members of the three-hinged frame shown in Fig. 4.8 a.

Solution : $\Sigma M_a = 0 = Y_b \times 11 - 8 \times 1.1 \times 7$

$$Y_b = 5.6 \text{ t} \uparrow$$

$$\Sigma Y = 0 = Y_a - 8 \times 1.1 - 5.6$$

$$Y_a = 3.2 \text{ t.} \uparrow$$

$$\overset{c}{\underset{a}{\Sigma}} M_c = 0 = 3.2 \times 3 - X_a \times 4$$

$$X_a = 2.4 \text{ t.} \rightarrow$$

$$\Sigma X = 0 = X_b - 2.4$$

$$X_b = 2.4 \text{ t.} \leftarrow$$

The free-body, N.F., S.F. and B.M.Ds. for members ac, cd and db are shown in Figs. 4.8 b-d. Note that ac is a link member, i.e. subject to axial load only. If wanted, N.F., S.F. and B.M.Ds. of the whole frame may be obtained by combining the corresponding diagrams of the various members. This has been done and the result is shown in Figs. 4.8 e- g.

All the frames considered until now have been statically determinate. For statically indeterminate frames, once the redundants, whether external reaction components or internal straining actions, are determined, the procedures for drawing the N.F., S.F. and B.M.Ds. are basically the same as for statically determinate frames.

## 4.4 Illustrative examples

In the following, a number of solved examples is given. The student is advised to solve these problems independently and then check his results agianst those given. Particular attention should be paid to the joints where three members meet and to the thrust and the shearing force diagrams of inclined members.

**Examples 4.6 - 4.13** Draw the N.F., S.F. and B.M.Ds. of the statically determinate frames shown in Figs. 4.9 - 4.15.

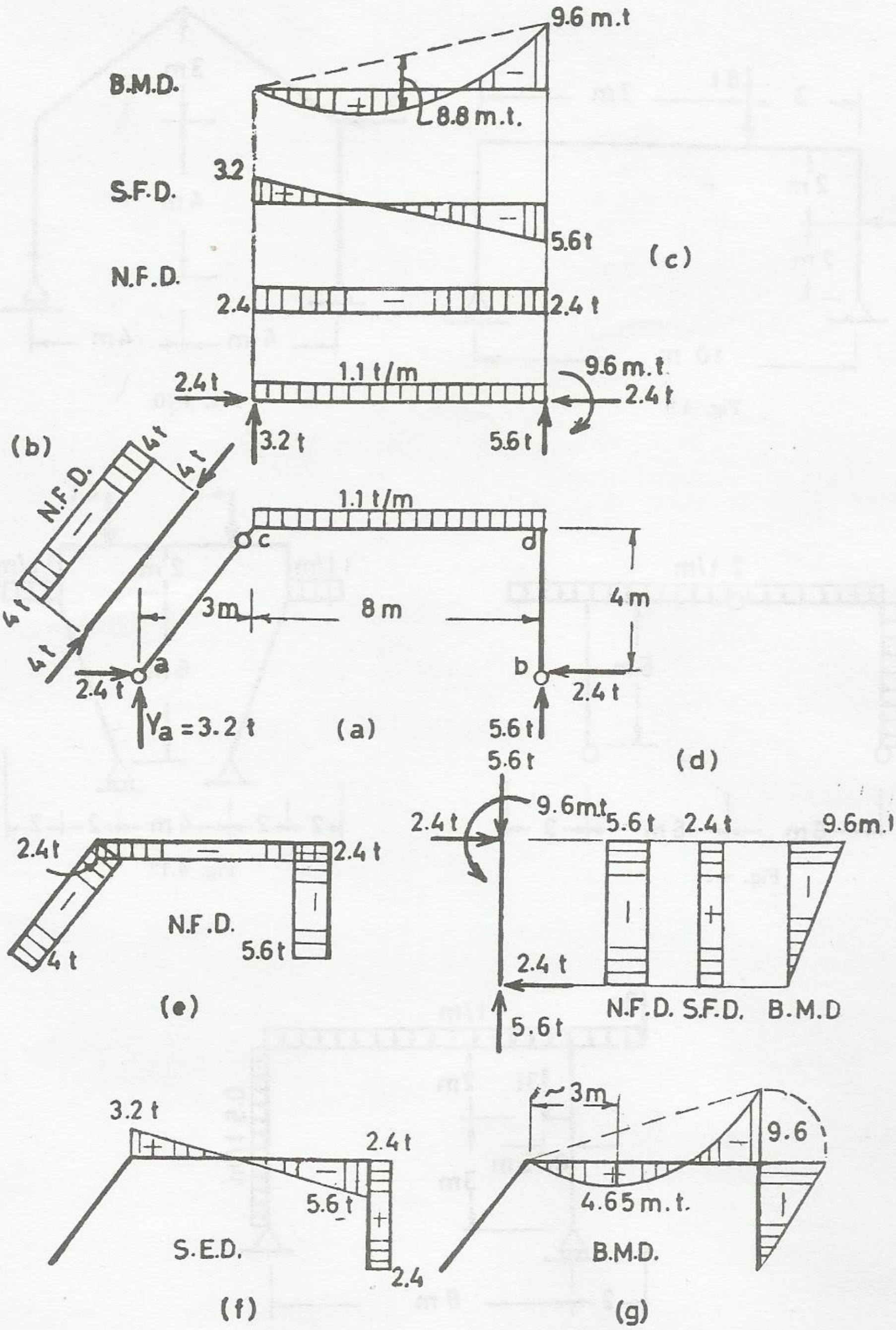

9.6 m.t
B.M.D.
8.8 m.t.
3.2
S.F.D.
5.6 t
(c)
N.F.D.
2.4
2.4 t
2.4 t
1.1 t/m
9.6 m.t
2.4 t
3.2 t
5.6 t
(b)
N.F.D.
4 t
1.1 t/m
c
d
4m
3m
8 m
a
b
2.4 t
Ya = 3.2 t
(a)
2.4 t
5.6 t
5.6 t
(d)
9.6m.t
2.4 t
5.6 t
2.4 t
9.6m.t
2.4t
2.4 t
N.F.D.
4 t
5.6t
2.4 t
N.F.D. S.F.D. B.M.D
5.6t
(e)
3m
3.2 t
2.4t
9.6
5.6 t
4.65 m.t.
S.E.D.
B.M.D.
2.4
(f)
(g)

Fig. 4.8

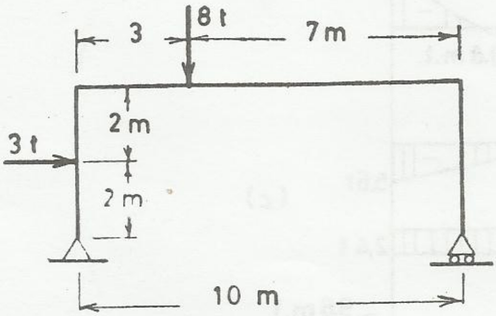



Fig. 4.9

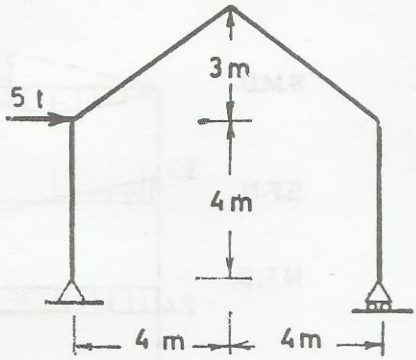



Fig. 4.10

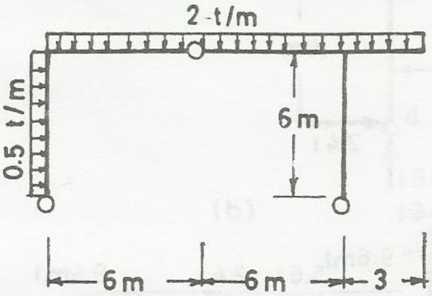



Fig. 4.11

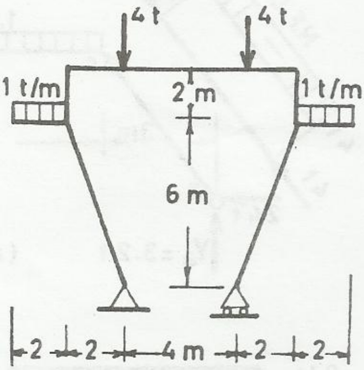



Fig. 4.12

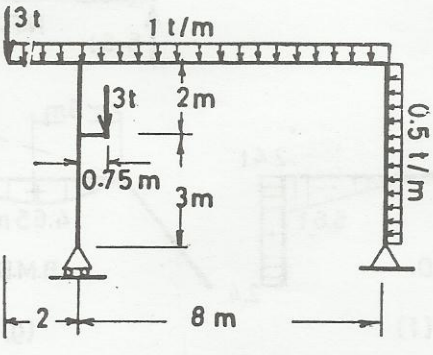



Fig. 4.13

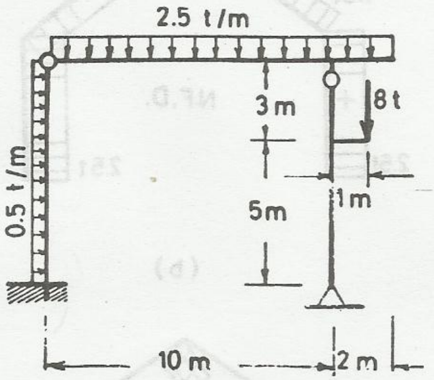



Fig. 4.14

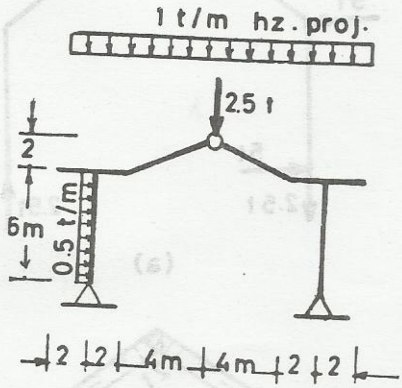



Fig. 4.15

Solution : The solution of problems 4.6-4.13 is given in Figs. 4.16-4.22. Figs. (a) show the external reactions, Figs. (b) the thrust diagrams, Figs. (c) the shearing force diagrams and Figs. (d) the bending moment diagrams.

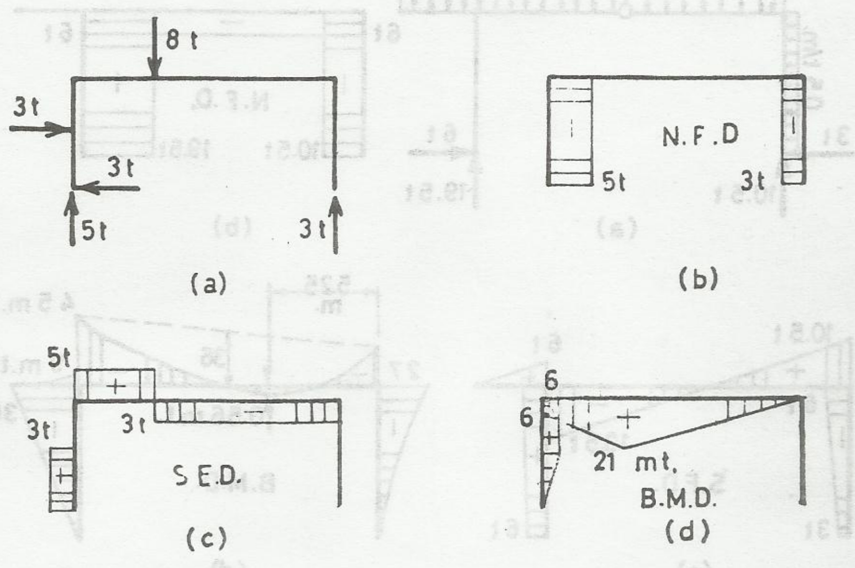



Fig. 4.16

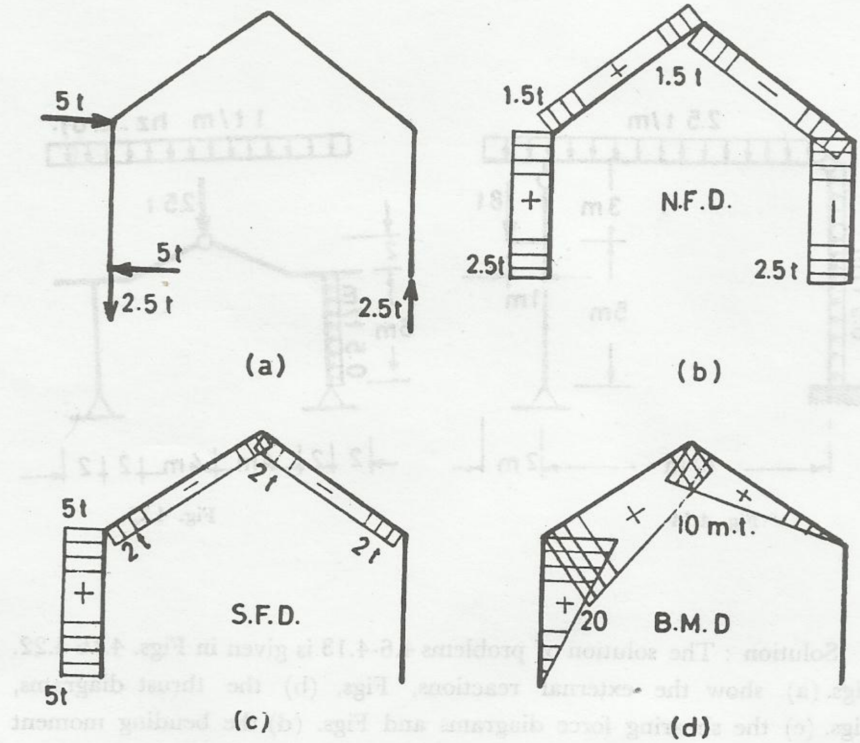


Fig. 4.17

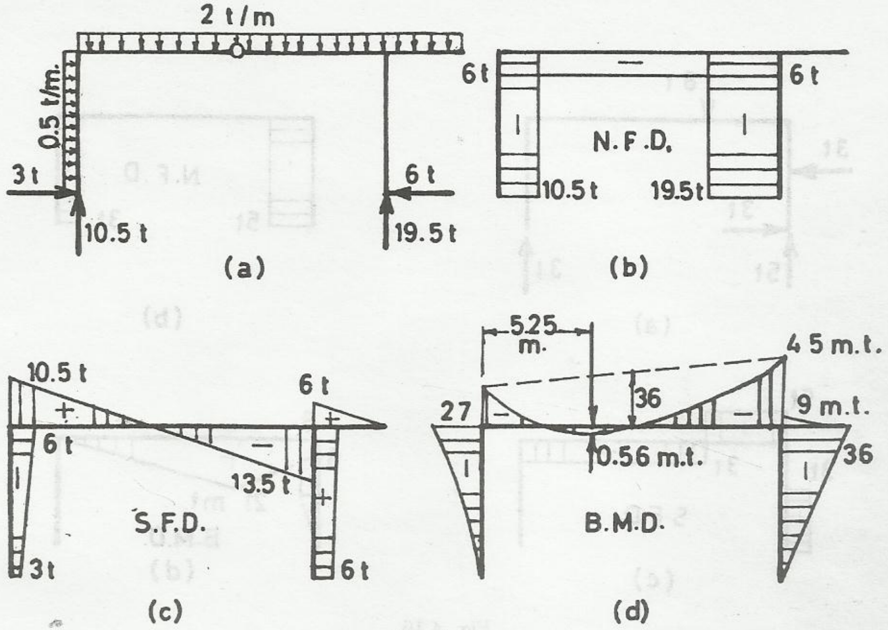


Fig. 4.18

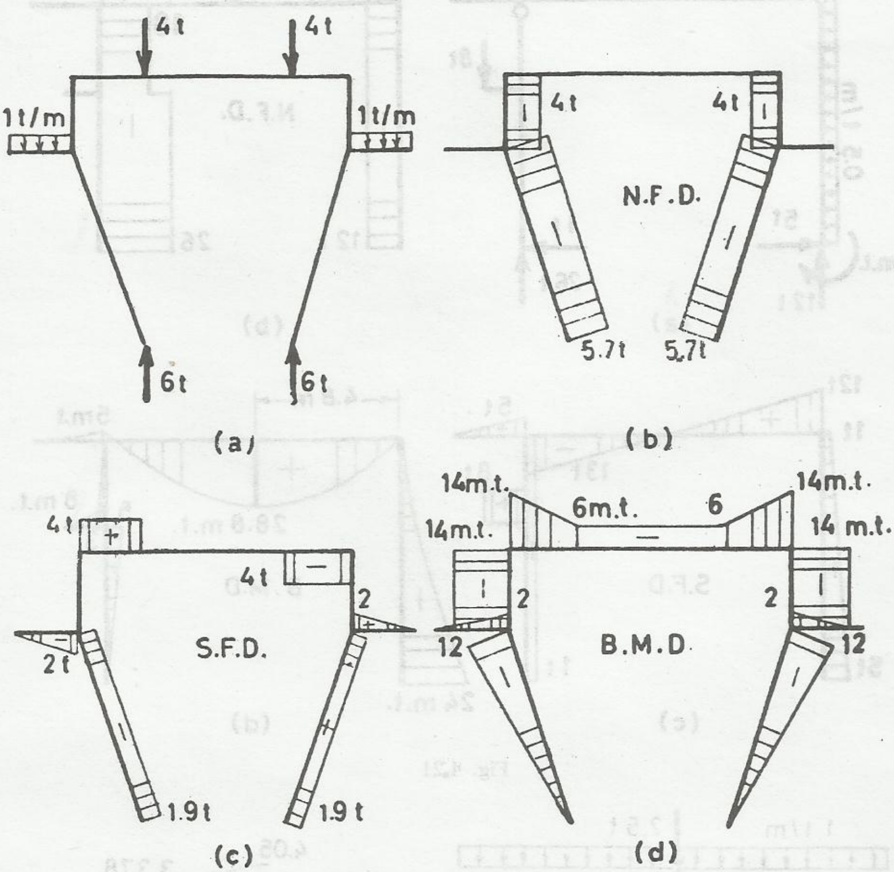



Fig. 4.19

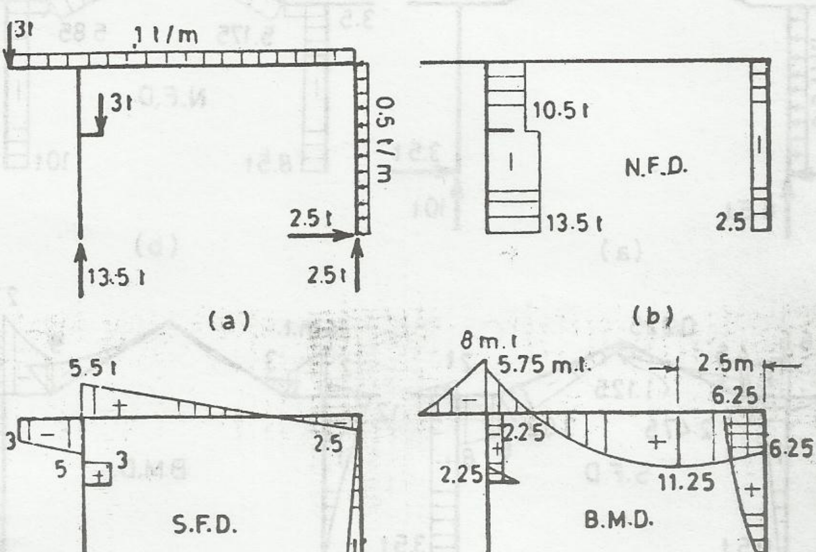



Fig. 4.20

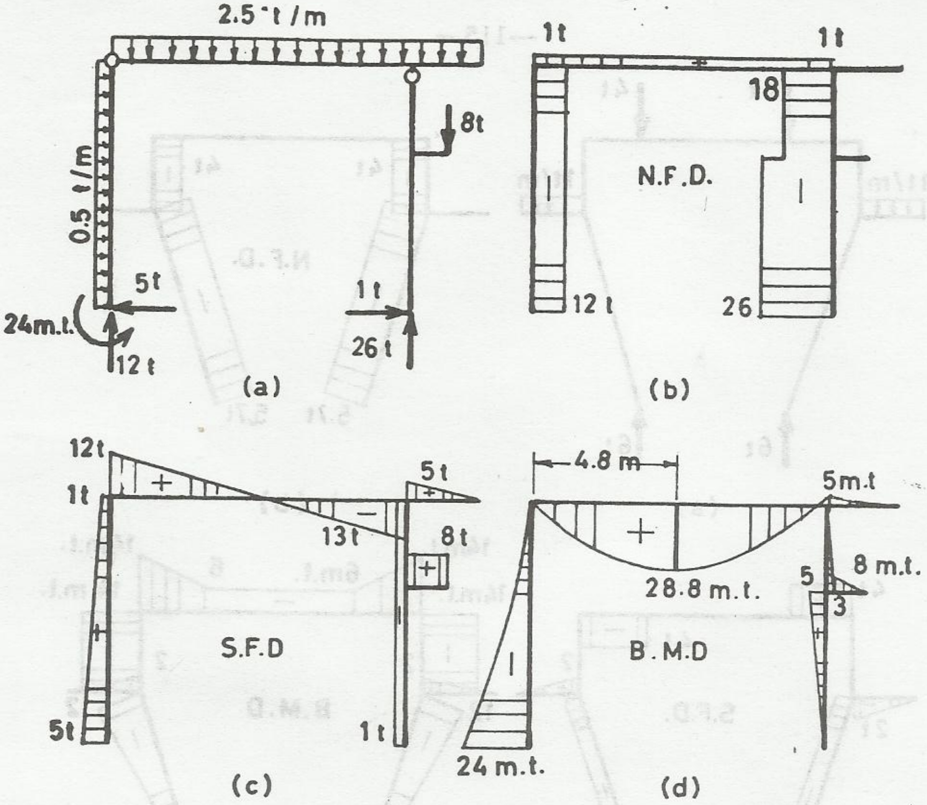



Fig. 4.21

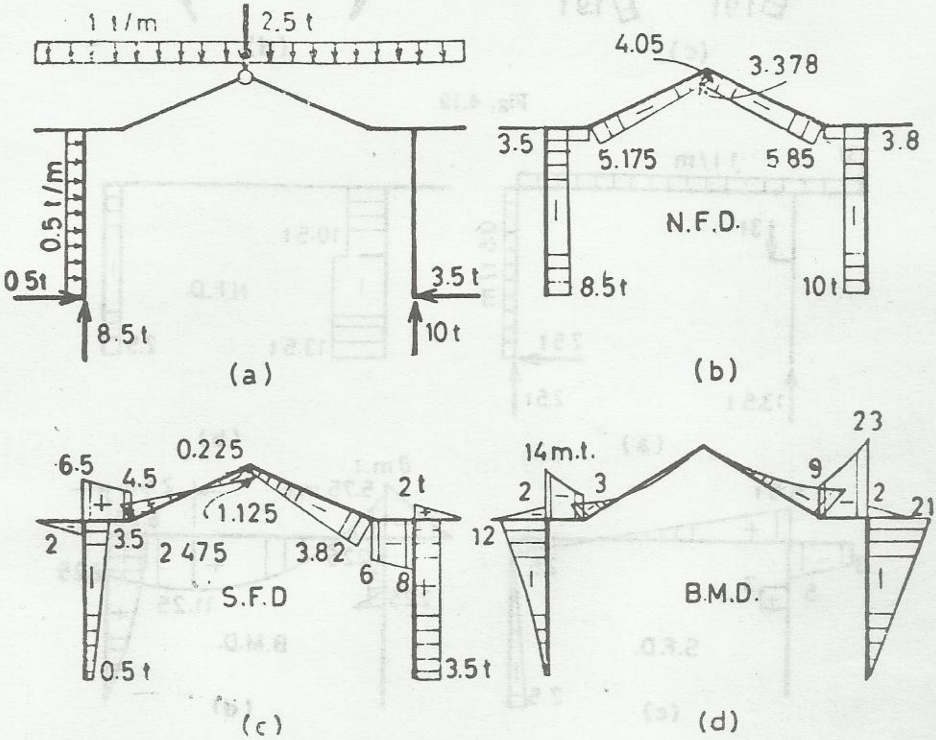



Fig. 4.22

### 4·5 Three-hinged arches

An arch is a structure which developes horizontal reaction components as well as vertical components even when it is subjected to vertical loading only. The main advantage of the arch is that the horizontal reaction components produce moments that counteract those due to the vertical components.

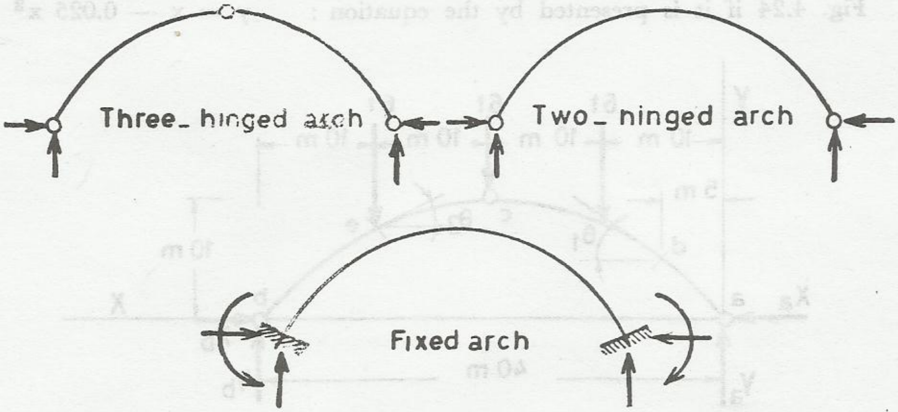



Fig. 4.23

Fig. 4.23 shows three different types of arches; the three-hinged, the two-hinged and the hingeless or fixed arches.

Among these three types, the three-hinged arch is the only statically determinate one and hence is the only one considered here. The three-hinged arch can be analysed by either analytical or graphical methods.

Analytical solution :

The four reaction components of the arch can be calculated, in the manner described in section 2.4, from the three equations of equilibrium and the fourth condition equation that the bending moment at the intermediate hinge is zero. Once the reactions are determined, the bending moment at any section along the arch can be determined in the usual manner by calculating the moments of all the forces either to the right or the left of the section. Determining the thrust and shearing force, however, requires more work as the resultant of all the forces to the left or the right of the section must be resolved into components along the tangent to the arch and the normal to it at the section considered. If the shape of the arch is represented by an equation, then the slope of the tangent can be determined from the first derivative of the equation. As most arches are parabolic or circular, this presents no difficulty. The method is illustrated by a numerical example.

**Example 4·14** Determine the thrusts, shearing forces and bending moments at points d and e of the three-hinged parabolic arch shown in Fig. 4.24 if it is presented by the equation : $y = x - 0.025\, x^2$

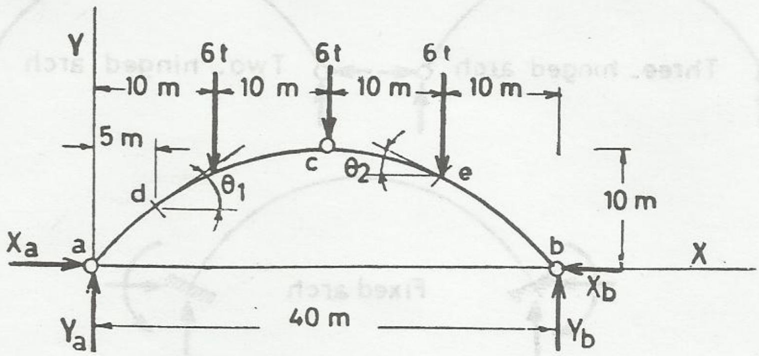



Fig. 4.24

Solution :

$$\Sigma M_a = 0 = Y_b \times 40 - 6\,(10 + 20 + 30)$$

$$Y_b = 9 \text{ t.} \uparrow$$

$$\Sigma Y = 0 = Y_a - 3 \times 6 + 9$$

$$Y_a = 9 \text{ t.} \uparrow$$

$$\sum_{a}^{c} M_c = 0 = 9 \times 20 - 10 \times 6 - 10\, X_a$$

$$X_a = 12 \text{ t.} \rightarrow$$

$$\Sigma X = 0 = 12 - X_b$$

$$X_b = 12 \text{ t.} \leftarrow$$

If a section is taken through point d, the free-body diagram of the part of the arch on the left hand side of the section will be as shown in either Fig. 4.25 a or 4.25 b.

At $x = 5$, $y = y_d = 5 - 0.025 \times 5^2 = 4.375$ m.

$$\tan \Theta_1 = \left(\frac{dy}{dx}\right)_{x=5} = 1 - 0.05 \times 5 = 0.75$$

$$\sin \Theta_1 = 0.6 \quad \text{and} \quad \cos \Theta_1 = 0.8$$

Since the slope of the arch changes from point to point, it is easier to calculate the reactions X, Y and M acting on the free body. Thus, referring to Fig. 4.25 a,

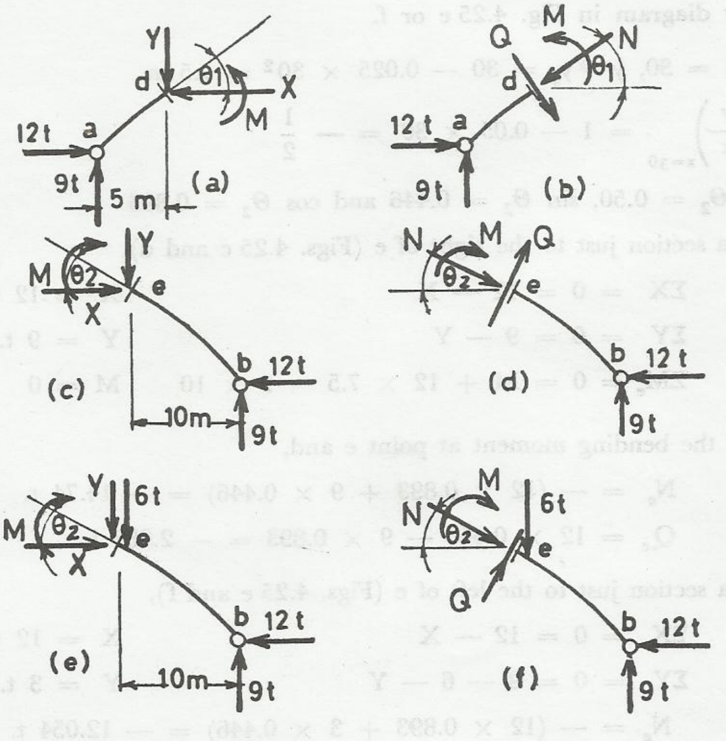



Fig. 4.25

$$\Sigma X = 0 = 12 - X \qquad X = 12 \text{ t.} \leftarrow$$

$$\Sigma Y = 0 = 9 - Y \qquad Y = 9 \text{ t.} \downarrow$$

$$\Sigma M_d = 0 = 12 \times 4.375 - 9 \times 5 + M$$

$$M = -7.5 \text{ m.t., i.e. } M = 7.5 \text{ m.t. (clockwise)}$$

M is the bending moment at section d. The thrust and shearing force can be easily expressed in terms of X and Y. Thus, referring to Fig. 4.25 b,

$$N_d = -(Y \sin \Theta_1 + X \cos \Theta_1)$$

$$= -(9 \times 0.6 + 12 \times 0.8) = -15 \text{ t.}$$

$$Q_d = Y \cos \Theta_1 - X \sin \Theta_1$$

$$= 9 \times 0.8 - 12 \times 0.6 = 0$$

Similarly, when a section is taken through point e, the part of the arch on the right hand side may be considered as a free-body. However, since this section coinsides witlr a concentrated load, two sections; just to the right and just to the left of the load must be considered. The former is shown in the free-body diagram in Figs. 4.25 c or d, and the latter in the free-body diagram in Fig. 4.25 e or f.

At $x = 30$, $y = y_e = 30 - 0.025 \times 30^2 = 7.5$ m.

$$\left(\frac{dy}{dx}\right)_{x=30} = 1 - 0.05 \times 30 = -\frac{1}{2}$$

$\tan \Theta_2 = 0.50$, $\sin \Theta_2 = 0.446$ and $\cos \Theta_2 = 0.893$

For a section just to the right of e (Figs. 4.25 c and d),

$$\Sigma X = 0 = 12 - X \qquad X = 12 \text{ t.} \rightarrow$$

$$\Sigma Y = 0 = 9 - Y \qquad Y = 9 \text{ t.} \downarrow$$

$$\Sigma M_e = 0 = M + 12 \times 7.5 - 9 \times 10 \qquad M = 0$$

M is the bending moment at point e and,

$$N_e = -(12 \times 0.893 + 9 \times 0.446) = -14.74 \text{ t.}$$

$$Q_e = 12 \times 0.446 - 9 \times 0.893 = -2.685 \text{ t.}$$

For a section just to the left of e (Figs. 4.25 e and f),

$$\Sigma X = 0 = 12 - X \qquad X = 12 \text{ t.} \rightarrow$$

$$\Sigma Y = 0 = 9 - 6 - Y \qquad Y = 3 \text{ t.} \downarrow$$

$$N_e = -(12 \times 0.893 + 3 \times 0.446) = -12.054 \text{ t.}$$

$$Q_e = 12 \times 0.446 - 3 \times 0.893 = 2.673 \text{ t.}$$

Graphical solution :

Many engineers prefer the graphical method to the analytical method of solution not only because it is quicker but also as it gives the values of the reactions of the arch, thrusts, shearing forces and bending moments at various points along its axis in one operation.

The determination of the reactions is based on the fact that the bending moment at the intermediate hinge is zero. Consider for example the three-hinged arch shown in Fig. 4.26 a. According to the principle of superposition, the final reactions to the applied loads is equal to the sum of the reactions to two load systems corresponding to the loads on either side of the intermediate hinge acting separately as shown in Figs. 4.26 b and c. It is easier to find the reactions for each of cases (b) and (c) as the reaction on the unloaded side must pass through the intermediate hinge if the bending moment there is to be zero. The problem thus reduces to finding two equilibrants to a system of forces, one given by its point of application and

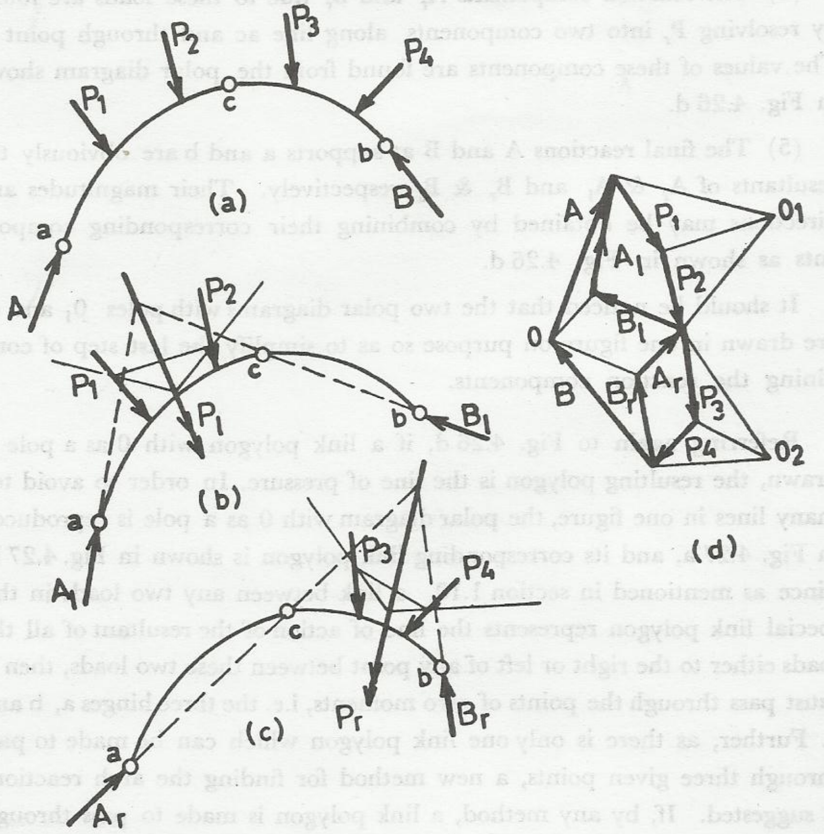



Fig. 4.26

the other by its line of action. These may be easily found by the methods described in section 1.9.

With reference to Fig. 4.26, the method of determining the reactions is summerized in the following :

(1) The resultant $P_1$ of the loads shown in Fig. 4.26 b is determined by means of a polar diagram with pole $0_1$ and its corresponding link polygon.

(2) The reaction components $A_1$ and $B_1$ due to these loads are found by resolving $P_1$ along line bc and through point a. Once their directions are known, the magnitudes of these components are found from the polar diagram shown in Fig. 4.26 d.

(3) **Similarly,** the resultant $P_r$ of the loads shown in Fig. 4.26 c is found by means of a polar diagram with pole $0_2$ and its corresponding link polygon.

(4) The reaction components $A_r$ and $B_r$ due to these loads are found by resolving $P_r$ into two components along line ac and through point b. The values of these components are found from the polar diagram shown in Fig. 4.26 d.

(5) The final reactions A and B at supports a and b are obviously the resultants of $A_r$ & $A_l$ and $B_r$ & $B_l$ respectively. Their magnitudes and directions may be obtained by combining their corresponding components as shown in Fig. 4.26 d.

It should be noticed that the two polar diagrams with poles $0_1$ and $0_2$ are drawn in one figure on purpose so as to simplify the last step of combining the reaction components.

Referring again to Fig. 4.26 d, if a link polygon with 0 as a pole is drawn, the resulting polygon is the line of pressure. In order to avoid too many lines in one figure, the polar diagram with 0 as a pole is reproduced in Fig. 4.27 a, and its corresponding link polygon is shown in Fig. 4.27 b. Since as mentioned in section 1.10, a link between any two loads in this special link polygon represents the line of action of the resultant of all the loads either to the right or left of any point between these two loads, then it must pass through the points of zero moments, i.e. the three hinges a, b and c. Further, as there is only one link polygon which can be made to pass through three given points, a new method for finding the arch reactions is suggested. If, by any method, a link polygon is made to pass through

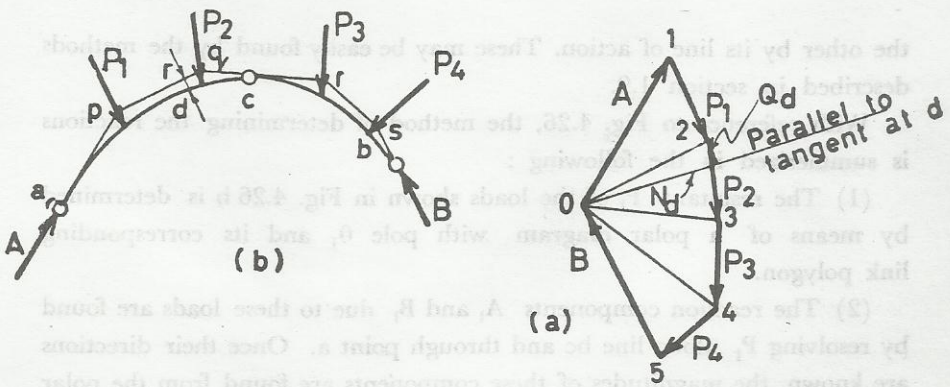



Fig. 4.27

the three hinges then the magnitudes of the reactions are simply obtained by measuring the first and last rays in the corresponding polar diagram and their lines of action coinside with the first and last links in the line of pressure.

Once the line of pressure is drawn, the thrust and shearing force at a point between any two loads are readily obtainable by resolving the corresponding link along and perpendicular to the tangent to the arch at that point, and the bending moment is the product of the force represented by the link in the panel where the point lies, and the perpendicular distance from the point to the link. For example, the thrust and shearing force at point d (Fig. 4.27) is found by resolving the force represented by link pq along the tangent and the normal to the arch at d. This is done on the polar diagram as shown in Fig. 4.27 a. Also, the bending moment at point d is the product of force $\overline{02}$ in the polar diagram, by the distance r which is the perpendicular from point d to link pq.

A special case of practical importance is the three-hinged arch subjected to vertical loads. The thrust and shearing force at any point along the arch axis may be found in the manner described above. The determination of the bending moment, however, is further simplified. This is explained with reference to Fig. 4.28. The bending moment at point d on the arch is given by :

$$M_d = \overline{02}\ r = \overline{02}\cos\Theta\ \frac{r}{\cos\Theta} = H \times y$$

i.e. the area between the arch axis and the line of pressure represents the bending moment diagram. When at any point, the vertical ordinate y

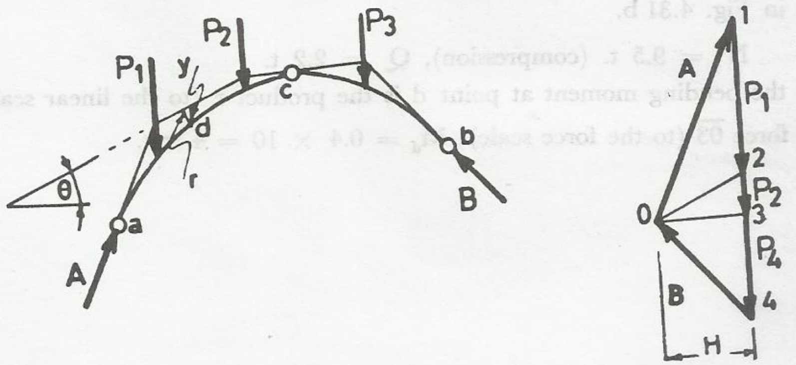



Fig. 4.28

of this area is multiplied by the polar distance H, the product gives the bending moment at this point.

**Example 4.15** Find graphically the reactions, thrust, shearing force and bending moment at point d of the three-hinged circular arch shown in Fig. 4.29.

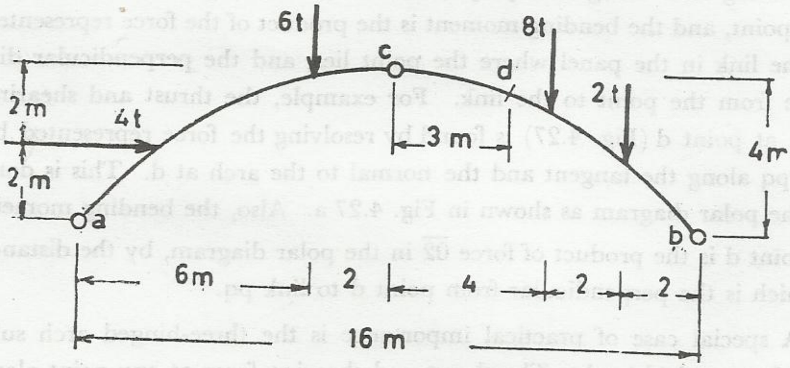



Fig. 4.29

Solution :

Following the procedure described in this section the determination of the reactions is illustrated in Figs. 4.30 a, b and c. From the force polygon in Fig. 4.30 c, the reactions A and B scale 8 t. and 14.5 t. respectively and act in the directions indicated.

The link polygon with 0 as a pole is shown in Fig. 4.31 a.

The thrust and shearing force at point d are scaled from the force polygon in Fig. 4.31 b.

$N = 9.5$ t. (compression), $Q = 2.2$ t.

and the bending moment at point d is the product r (to the linear scale and force $\overline{03}$ (to the force scale); $M_d = 0.4 \times 10 = 4$ m.t.

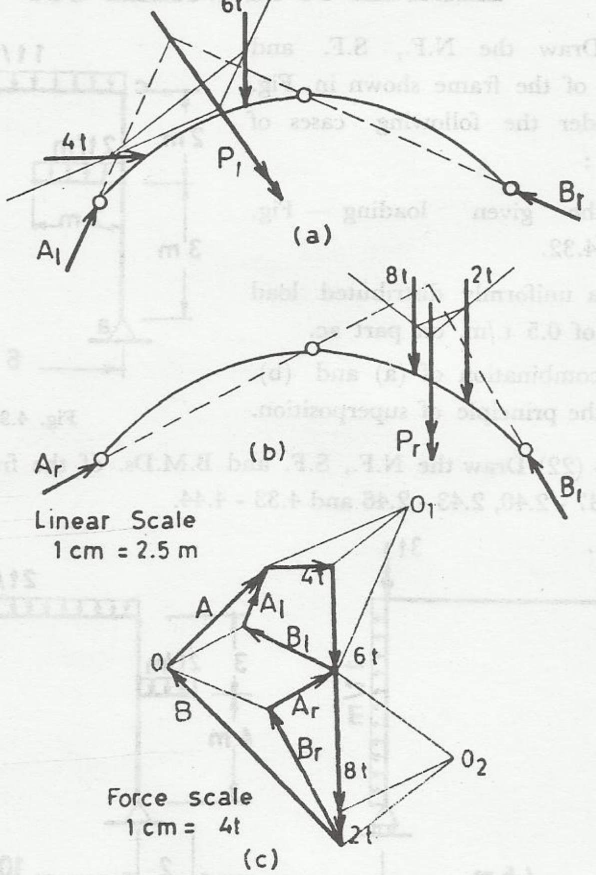



Fig. 4.30

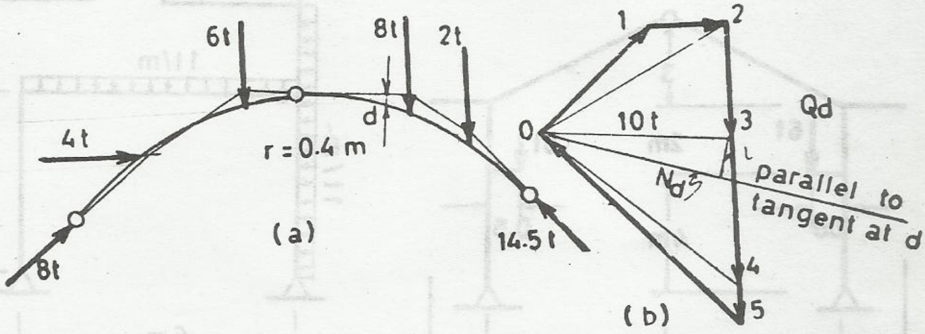



Fig. 4.31

## EXAMPLES TO BE WORKED OUT

(1) Draw the N.F., S.F. and B.M.Ds. of the frame shown in Fig. 4.32 under the following cases of loading :

(a) the given loading — Fig. 4.32.

(b) a uniformly distributed load of 0.5 t./m. on part ac.

(c) combination of (a) and (b).

Verify the principle of superposition.

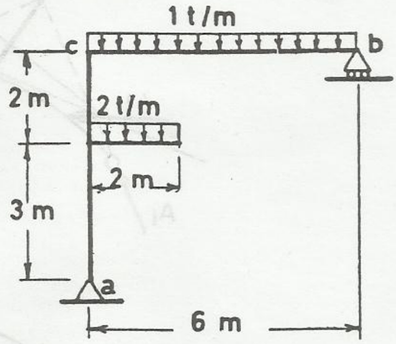



Fig. 4.32

(2) - (22) Draw the N.F., S.F. and B.M.Ds. of the frames shown in Figs. 2.37 - 2.40, 2.43 - 2.46 and 4.33 - 4.44.

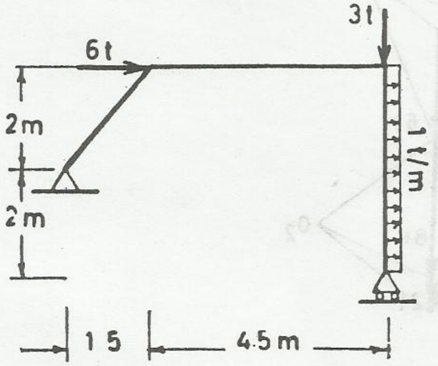



Fig. 4.33

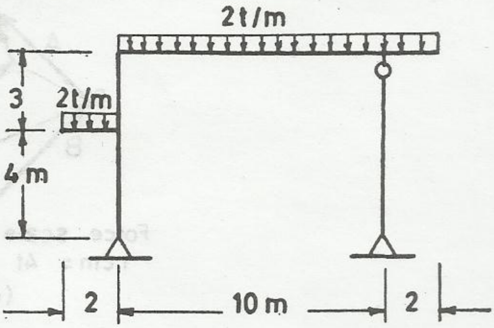



Fig. 4.34

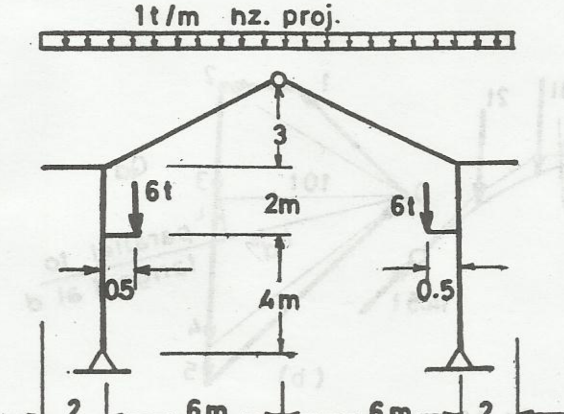



Fig. 4.35

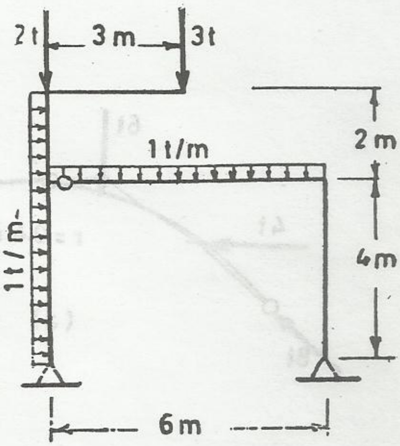



Fig. 4.36

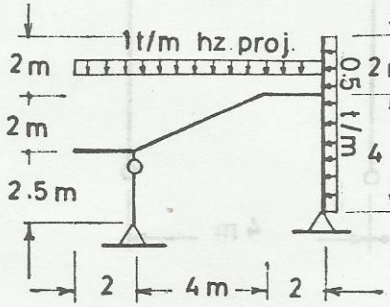



Fig. 4.37

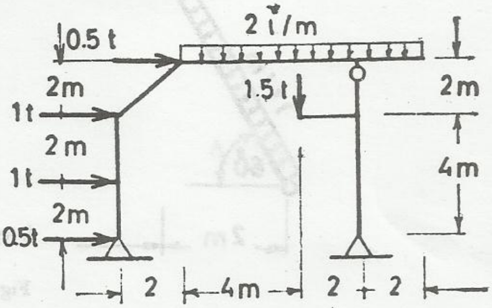



Fig. 4.38

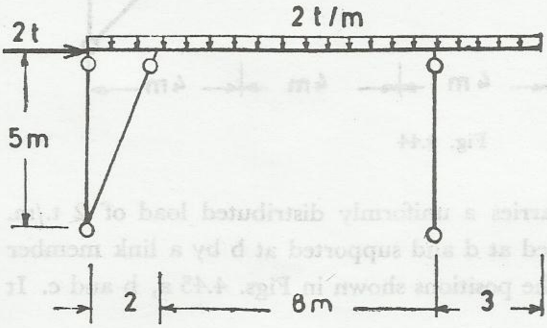



Fig. 4.39

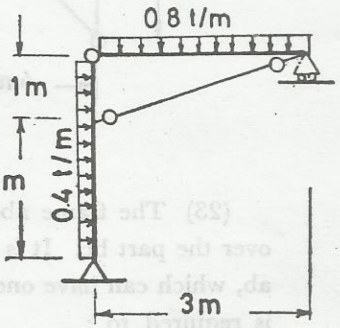



Fig. 4.40

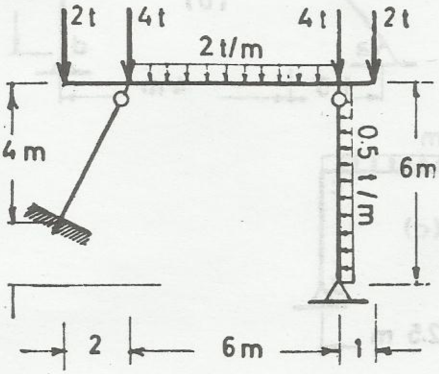



Fig. 4.41

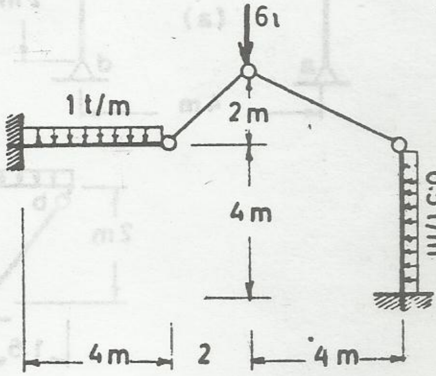



Fig. 4.42

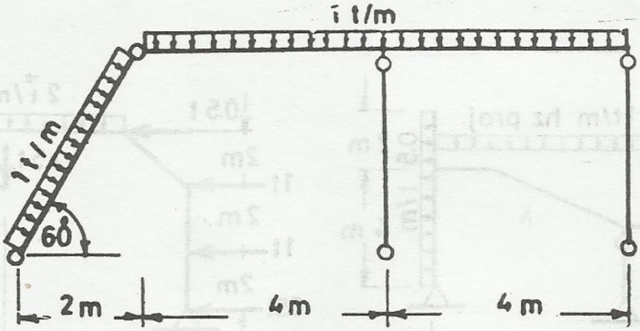



Fig. 4.43

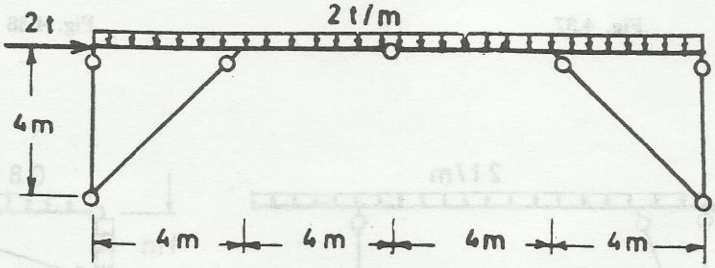



Fig. 4.44

(23) The frame abcd carries a uniformly distributed load of 2 t./m. over the part bc. It is hinged at d and supported at b by a link member ab, which can have one of the positions shown in Figs. 4.45 a, b and c. It is required to :

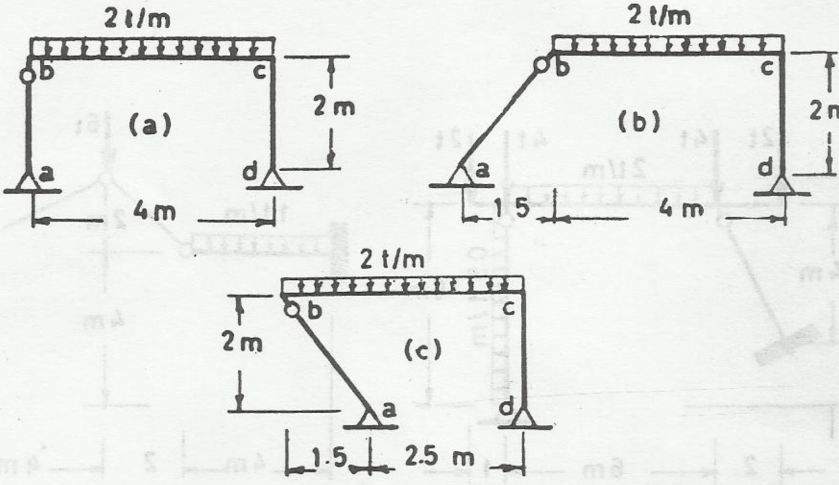



Fig. 4.45

(a) draw the N.F., S.F. and B.MDs. for each case.

(b) determine the position and value of the maximum positive moment.

(c) study the three systems from the statical point of view.

(24) - (27) Draw the N.F., S.F. and B.M.Ds. of the statically indeterminate frames shown in Figs. 4.46 - 4.49 if :

in Fig. 4.46, $X_a = 1.2$ t. (to the left)

in Fig. 4.47, $M_a = 6$ m.t. (anticlockwise)

in Fig. 4.48, $M_a = 26$ m.t. (clockwise) and

$M_b = 30$ m.t. (anticlockwise)

in Fig. 4.49, $N_e = +0.25$ t., $Q_e = 0$ and $M_e = +3.5$ m.t.

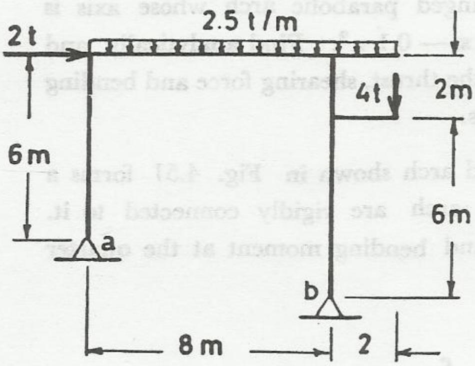



Fig. 4.46

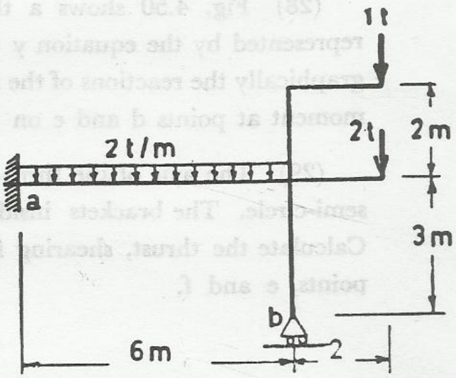



Fig. 4.47

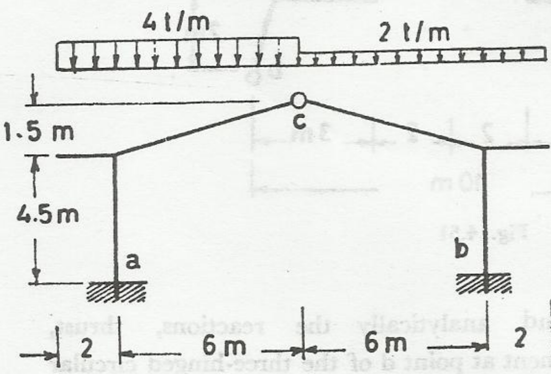



Fig. 4.48

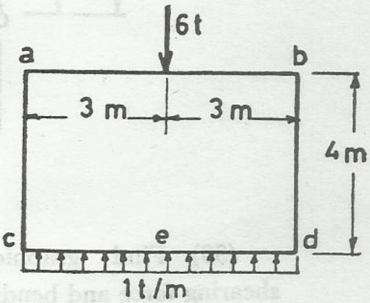



Fig. 4.49

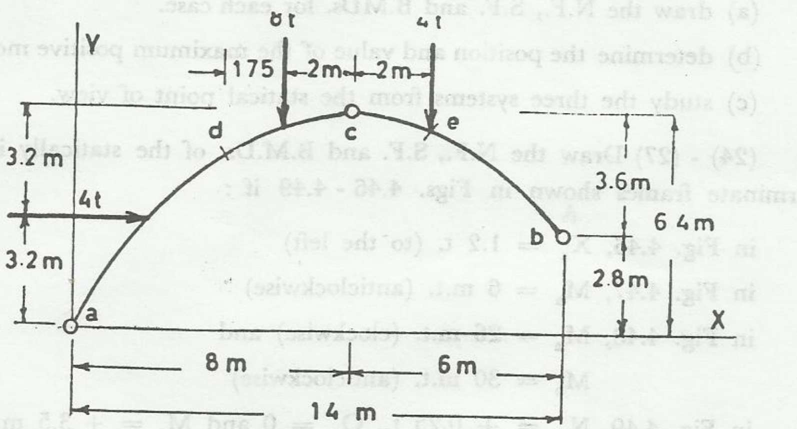



Fig. 4.50

(28) Fig. 4.50 shows a three-hinged parabolic arch whose axis is represented by the equation $y = 1.6\,x - 0.1\,x^2$. Find analytically and graphically the reactions of the arch, the thrust, shearing force and bending moment at points d and e on its axis.

(29) The axis of the three-hinged arch shown in Fig. 4.51 forms a semi-circle. The brackets inside the arch are rigidly connected to it. Calculate the thrust, shearing force and bending moment at the quarter points, e and f.

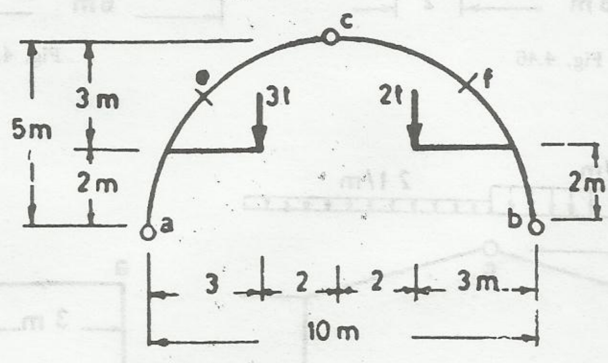



Fig. 4.51

(30) Find graphically and analytically the reactions, thrust, shearing force and bending moment at point d of the three-hinged circular arch shown in Fig. 4.52.

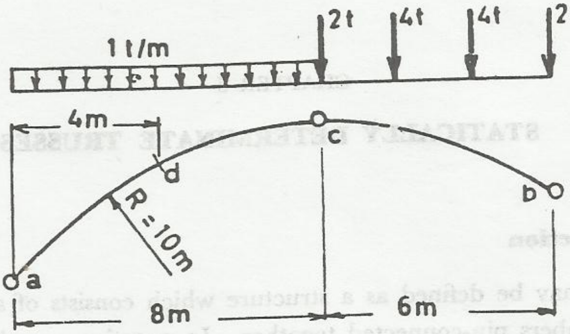



Fig. 4.52

(31) For the three-hinged parabolic arch shown in Fig. 4.53, draw the bending moment diagram under a uniformly distributed load of 2 t./m. of horizontal projection on : (a) part ac, (b) part bc, (c) the whole span.

Explain why the bending moment diagram for the case of loading in (c) zero.

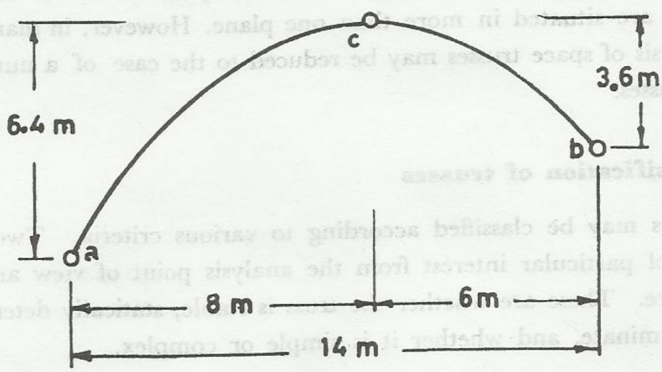



Fig. 4 53

CHAPTER 5

# STATICALLY DETERMINATE TRUSSES

## 5·1 Introduction

A truss may be defined as a structure which consists of a number of straight members pin-connected together. In practice, members are bolted, riveted or welded at their ends. Nevertheless, they are assumed to be pin-connected in the analysis because it has been found that such an analysis gives a good estimate of the forces in the members. Provided the external loads are applied at the pin joints, all the truss members will be link members subjected to either axial tension or compression. Members carrying tension are usually called *ties* and those carrying compression are termed *struts.*

Trusses considered in this chapter are plane and co-planer with the applied loads. Apart from this type, there are space trusses in which the members are situated in more than one plane. However, in many cases, the analysis of space trusses may be reduced to the case of a number of plane trusses.

## 5·2 Classification of trusses

Trusses may be classified according to various criteria. Two classifications of particular interest from the analysis point of view are considered here. These are whether the truss is stable, statically determinate or indeterminate, and whether it is simple or complex.

## 5·3 Stability and determinancy

Before dealing with the criteria of stability and determinancy of trusses, the student is advised to recall the relevant discussions in sections 2.6. and 4.2.

Consider a truss and let it have m members, j joints and r external reaction components. By definition, the truss members carry only axial loads. Thus, the number of unknowns is equal to m + r; m member forces and r reaction components. On the other hand, since each joint is subjected to a system of concurrent forces in equilibrium, the number of the available equations is equal to 2 j; two equations for each joint.

A truss may be classified as unstable, statically determinate or statically indeterminate by comparing the number of unknowns with the number of equations available.

If $2j > m + r$, the truss is unstable.

$2j = m + r$, the truss is statically determinate.

$2j < m + r$, the truss is statically indeterminate.

It should be emphasized that the abovementioned comparison is not always sufficient to decide whether a truss is stable or not.

If $2j > m + r$, this comparison is sufficient to indicate that the truss is unstable. If, however, $2j \leq m + r$, it does not automatically mean that the truss is stable. To explain this statement, consider the two trusses shown in Figs. 5.1 a and b.

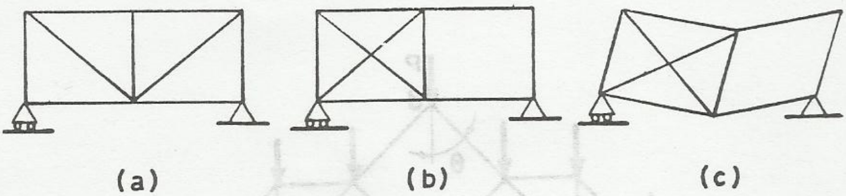



Fig. 5.1

Both trusses satisfy the relationship $2j = m + r$. However, while the truss shown in Fig. 5.1 a is stable and statically determinate, that shown in Fig. 5.1 b is unstable since it can distorte as shown in Fig. 5.1 c without offering any resistance to a general case of loading.

This example shows that blind comparison between the number of the available equations and the number of unknowns does not necessarily yield the right answer.

It may be noticed that no mention, as yet, has been made to the criteria of internal and external indeterminancy. This has been done on purpose as the computation of the external reaction components is sometimes related to the disposition of the truss members and it is difficult, therefore, to distinguish between internal and external indeterminancy. The following examples may illustrate this point.

Fig. 5.2. shows a truss resting on two hinges at a and b which provide 4 > 3 reaction components but is still statically determinate as an additional condition is furnished by equating the moments of all the forces either

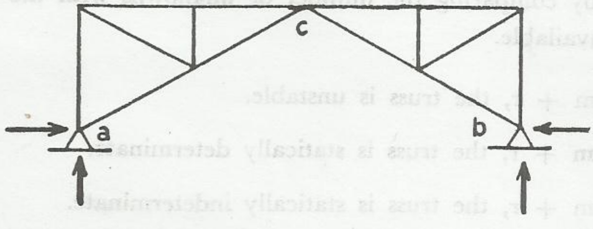



Fig. 5.2

to the right or to the left of hinge c to zero as in the case of three-hinged frames and arches. It may be argued that, by definition, all the joints of the truss are hinges so why joint c in particular has been chosen to provide the required condition. The answer is that joint c differs from all the others in that it divides the structure into two entirely separate parts.

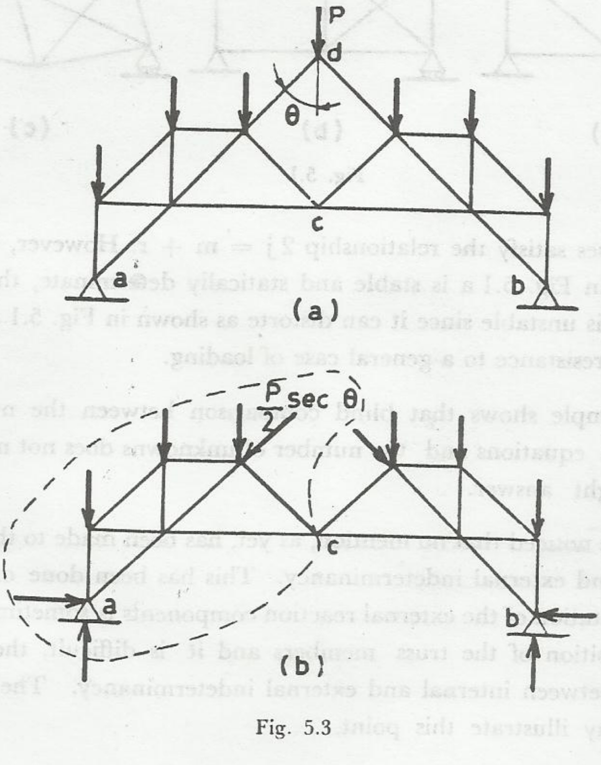



Fig. 5.3

Fig. 5.3 a shows another two-hinged truss with an intermediate hinge which is not so obvious. The forces in the two members meeting at d are found by considering the equilibrium of the joint. The members may then be replaced by the forces they carry as shown in Fig. 5.3 b. A fourth condition is thus established by equating the moments of all the forces either to the right or the left of the hinge c to zero. This condition together with the three conditions of equilibrium are sufficient to determine the four reaction components at the supports.

Fig. 5.4 a shows a truss resting on two rollers and a hinged support which provide 4>3 reaction components but is still statically determinate. The reactions may be found with reference to Fig. 5.4 b as follows :

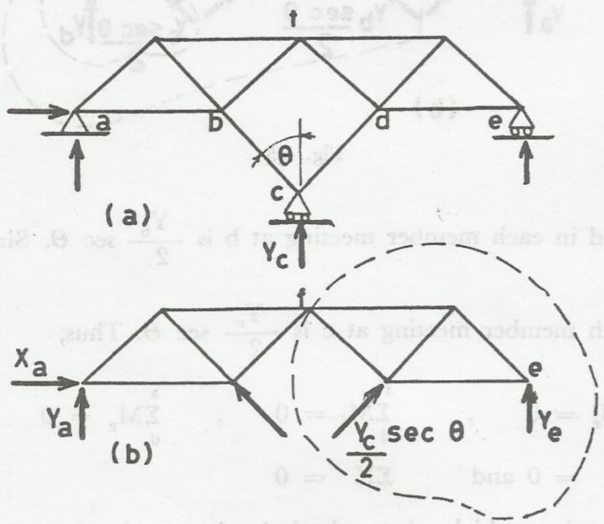



Fig. 5.4

(1) Members cb and cd are replaced by the forces they carry; each force being equal to $\frac{Y_c}{2}$ sec $\Theta$

(2) $\Sigma M_f = 0$ and $\Sigma M_e = 0$, are two equations which when solved give the values of $Y_c$ and $Y_e$.

(3) $X_a$ and $Y_a$ are found from the conditions $\Sigma X = 0$ and $\Sigma Y = 0$ applied to the whole structure.

Fig. 5.5a shows a truss resting on a hinged support at a and three rollers at b, c and d, which provide 5 > 3 reaction components but is still statically determinate. The five reaction components may be found with reference to Fig. 5.5 b as follows :

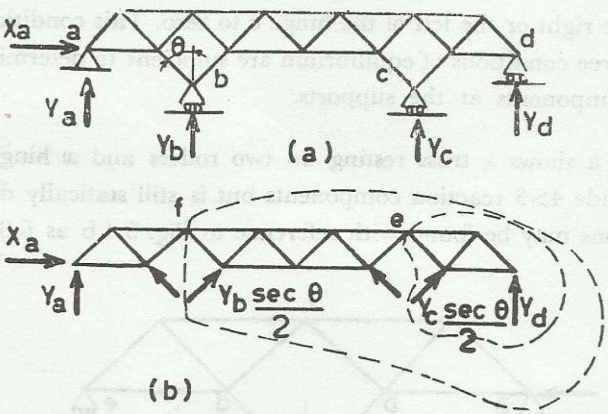



Fig. 5.5

The load in each member meeting at b is $\frac{Y_b}{2}$ sec $\Theta$. Similarly, the load in each member meeting at c is $\frac{Y_c}{2}$ sec $\Theta$. Thus,

$$\sum_{d}^{e} M_e = 0 \quad , \quad \sum_{d}^{f} M_f = 0 \quad , \quad \sum_{d}^{a} M_a = 0$$

$$\Sigma Y = 0 \text{ and } \quad \Sigma X = 0$$

are five equations which when solved simultaneously give the five reaction components at the supports.

## 5·4. Simple and complex trusses

Any truss supported in a statically determinate manner and developed by successive addition of joints to a basic triangle such that each new joint is connected to two existing ones by two members not in alignment is called a simple truss. Whenever the composition of a truss does not follow this scheme, it is called a complex truss. Simple trusses allow a much easier solution than complex ones. When the complexity consists of a few modifications from a simple truss, complex trusses can also be easily analysed. Fig. 5.6 shows two examples of such trusses.

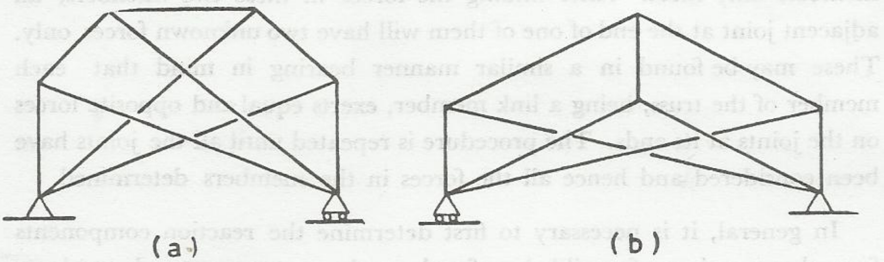



Fig. 5.6

### 5·5 Methods of analysis

Considering first simple trusses, forces in the members can be determined by either analytical, graphical or a combination of both methods. In any method, the following guiding principles should be noticed.

(1) The truss as a whole is a rigid body kept in equilibrium by the applied loads and the reaction components at the supports. These forces generally form a system of non-concurrent forces for which three equations of equilibrium may be written.

(2) Any part of the truss is a rigid body kept in equilibrium by the external forces including the pre-calculated reactions together with the reactions acting upon it by the rest of the structure.

(3) Any joint in a truss is kept in equilibrium by the effect of the external forces and the forces in the members meeting at this joint. These forces form a system of concurrent forces for which two equations of equilibrium may be written.

(4) Each of the truss members is a link member which may be replaced by equal and opposite forces at its ends.

### 5·6 Application of the method of joints

In this method the equilibrium of each of the truss joints is considered separately. When a number of members meet at a joint, all but two of the forces they carry are known, the other two may be found from the application of the two equations available for a set of concurrent forces in equilibrium; $\Sigma X = 0$ and $\Sigma Y = 0$. Since all the forces in the truss members are unknown, the analysis should start with a joint where two members only meet. After finding the forces in these two members, an adjacent joint at the end of one of them will have two unknown forces only. These may be found in a similar manner bearing in mind that each member of the truss, being a link member, exerts equal and opposite forces on the joints at its ends. The procedure is repeated until all the joints have been considered and hence all the forces in the members determined.

In general, it is necessary to first determine the reaction components from the equations of equilibrium for the entire structure in order to have only two unknown forces at a joint where the analysis could be started.

The analysis of a truss by the method of joints will be illustrated by a numerical example.

**Example 5.1** Find the forces in all the members of the truss shown in Fig. 5.7 using the method of joints.

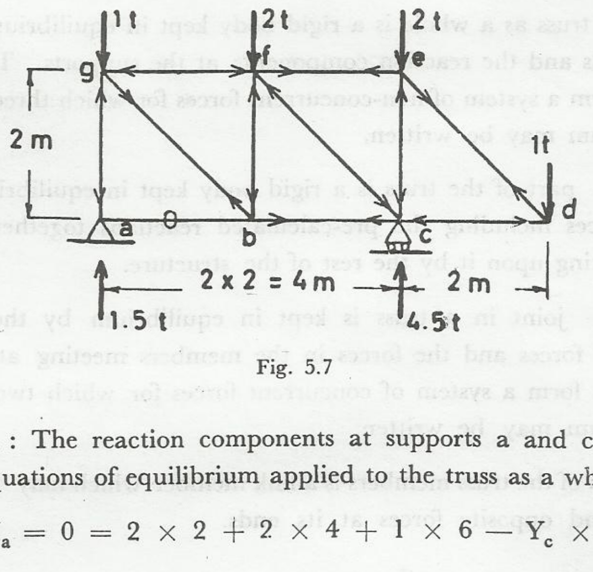



Fig. 5.7

Solution : The reaction components at supports a and c are found from the equations of equilibrium applied to the truss as a whole. Thus,

$$\Sigma M_a = 0 = 2 \times 2 + 2 \times 4 + 1 \times 6 - Y_c \times 4$$

$$Y_c = \frac{4 + 8 + 6}{4} = 4.5 \text{ t.} \uparrow$$

$$\Sigma M_c = 0 = Y_a \times 4 + 1 \times 2 - 2 \times 2 - 1 \times 4$$

$$Y_a = \frac{4 + 4 - 2}{4} = 1.5 \text{ t.} \uparrow$$

$$\Sigma X = 0 = X_a$$

In considering the equilibrium of individual joints, it is preferable to assume that all the members are in tension and thus the forces are directed away from the joints. In such a case, a negative result means that the member is in compression and the force is directed towards the joint under consideration.

The analysis could be started at either joint a or joint d as they have two unknown forces only.

Considering the equilibrium of joint a,

$$\Sigma X = 0 = F_{ab}\ ,\ F_{ab} = 0$$

$$\Sigma Y = 0 = 1.5 + F_{ag}\ ,\ F_{ag} = -1.5 \text{ t.}$$

Considering the equilibrium of joint g.

$$\Sigma Y = 0 = 1.5 - 1 - F_{gb} \cos 45,\ F_{gb} = 0.5\sqrt{2} \text{ t.}$$

$$\Sigma X = 0 = F_{gf} + 0.5\sqrt{2} \cos 45,\ F_{gf} = -0.5 \text{ t.}$$

Considering the equilibrium of joint b.

$$\Sigma X = 0 = F_{bc} - 0.5\sqrt{2} \cos 45,\ F_{bc} = 0.5 \text{ t.}$$

$$\Sigma Y = 0 = F_{bf} + 0.5\sqrt{2} \cos 45,\ F_{bf} = -0.5 \text{ t.}$$

Considering the equilibrium of joint f,

$$\Sigma Y = 0 = 0.5 - 2 - F_{fc} \cos 45,\ F_{fc} = -1.5\sqrt{2} \text{ t.}$$

$$\Sigma X = 0 = 0.5 + F_{fe} - 1.5\sqrt{2} \cos 45,\ F_{fe} = 1 \text{ t.}$$

Considering the equilibrium of joint e,

$$\Sigma X = 0 = 1 - F_{ed} \cos 45,\ F_{ed} = \sqrt{2} \text{ t.}$$

$$\Sigma Y = 0 = 2 + \sqrt{2} \cos 45 + F_{ec},\ F_{ec} = -3 \text{ t.}$$

Considering the equilibrium of joint c,

$$\Sigma X = 0 = 0.5 - 1.5\sqrt{2} \cos 45 - F_{cd}\ ,\ F_{cd} = -1 \text{ t.}$$

Check at joint d,

$$\Sigma X = 1 - \sqrt{2} \cos 45 = 0$$

$$\Sigma Y = 1 - \sqrt{2} \cos 45 = 0$$

Positive F's correspond to tension and negative F's to compression.

### 5·7 Zero members

In the preceding example, member ab has been found to carry a zero force. Such members are called *zero members.* As far as the analysis is concerned zero members may be omitted altogether without affecting the forces in the rest of the members. If this is the case, students often enquire, why such members are provided in the first place. The answer is that a zero member under a certain case of loading may carry a force under another case of loading. For instance if the truss shown in Fig. 5.7 is subjected to a horizontal force, member ab will not remain a zero member but will carry a force equal to the horizontal reaction component at a. Also, members carrying zero calculated force are provided for design purposes such as to avoid buckling of long compression members or sagging of long horizontal tension members.

Usually, it simplifies the problem a great deal to detect the zero members in a truss and omitt them before carrying out a complete analysis.

Referring to Figs. 5.8 a and b, the following two rules may help to spot the zero members.

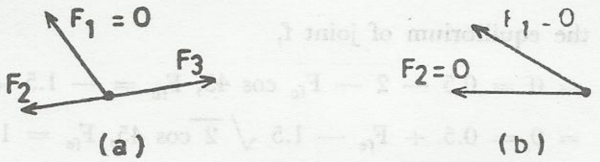



Fig. 5.8

(1) If a joint is acted upon by only three member forces $F_1$, $F_2$ and $F_3$, two of which, say $F_2$ and $F_3$, have the same line of action, then the remaining force $F_1$ must be zero.

(2) If a joint is acted upon by only two member forces $F_1$ and $F_2$ which do not have the same line of action then both forces must be zero.

These two rules follow from the consideration of equilibrium of the joint or by resolving the forces along members 2 and 3 in case (a) and along either member 1 or 2 in case (b).

The simplification resulting from first determining the zero members is illustrated in the following example.

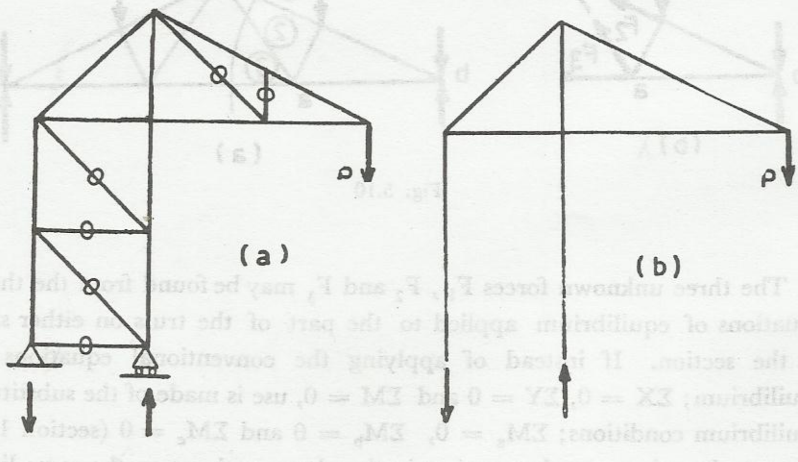



Fig. 5.9

By rule number (1) given in this section, the truss given in Fig. 5.9 a may be shown to have six zero members; marked 0 in the figure. Thus, regarding the analysis, the given truss may be replaced by that shown in Fig. 5.9 b which has seven unknown member forces only against the seventeen forces of the original truss.

## 5.8 Application of the method of sections

It is often desirable to find the force in a single member of a truss without analysing the entire structure. An analysis by the method of sections generally gives this force by a single operation without the necessity of finding the forces in the other members as in the case of the method of joints.

Instead of considering the equilibrium of a joint, a section is taken through the truss dividing it into two entirely separate parts, and the equillibrium of either part is considered. Generally, the section is chosen so that it cuts three members including the one whose force is to be determined. This force is then simply obtained by taking moments about the point of intersection of the other two members.

If the forces in any or all of members 1, 2 and 3 of the truss shown in Fig. 5.10 a are desired, the section will be as shown.

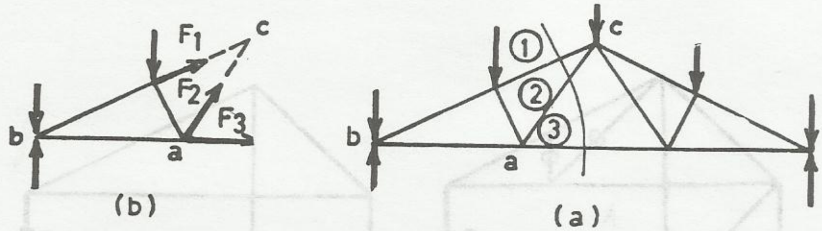



Fig. 5.10

The three unknown forces $F_1$, $F_2$ and $F_3$ may be found from the three equations of equilibrium applied to the part of the truss on either side of the section. If instead of applying the conventional equations of equilibrium; $\Sigma X = 0, \Sigma Y = 0$ and $\Sigma M = 0$, use is made of the substitute equilibrium conditions; $\Sigma M_a = 0$, $\Sigma M_b = 0$ and $\Sigma M_c = 0$ (section 1.9) where a, b and c are three points in the plane and not on the same line. The forces are then obtained directly. Thus, with reference to Fig. 5.10 b.

$\Sigma M_a = 0$ gives the value of $F_1$,

$\Sigma M_b = 0$ gives the value of $F_2$,

and $\Sigma M_c = 0$ gives the value of $F_3$.

Although it is preferable to take the section across three members, it is possible sometimes to take it across four or more members provided that all but the one whose force is to be determined intersect. This force is then found by taking moments about the common point of intersection of the other members.

Figs. 5-11-5.13 show examples of such sections with the corresponding centre of moments for the determination of forces $F_1$ in the members marked (1).

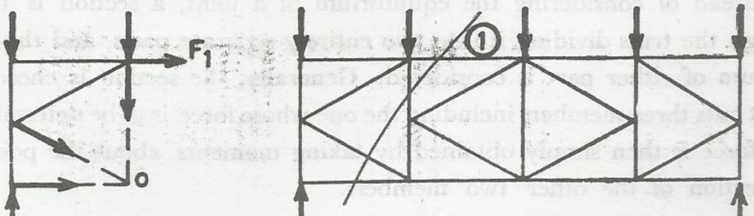



Fig. 5.11

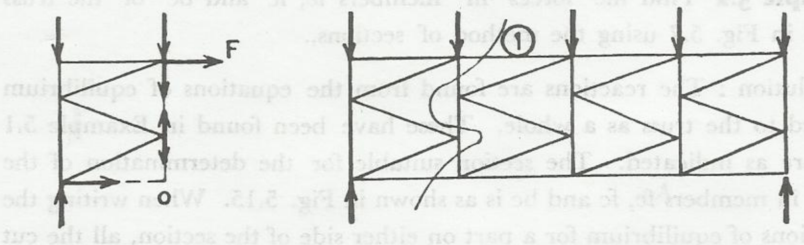



Fig. 5.12

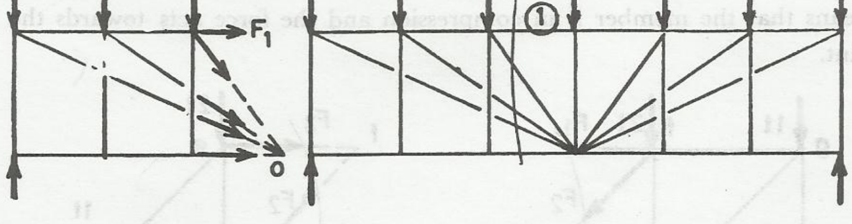



Fig. 5.13

Sometimes the moments equation does not give directly the required force. Such cases are usually encountered when the forces in the diagonals of trusses with parallel chords are desired.

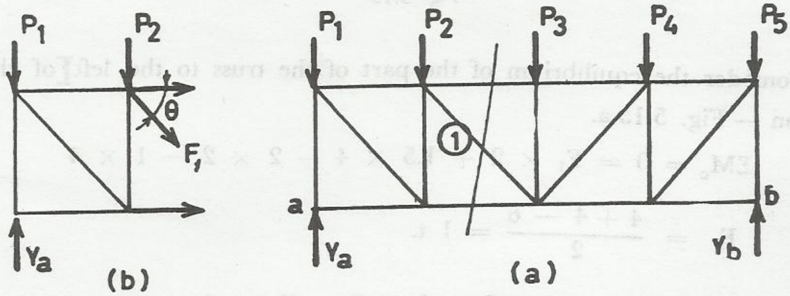



Fig. 5.14

For example the force $F_1$ in member (1) of the truss shown in Fig. 5.14 a cannot be determined by the application of the moments equation as the other two members do not intersect. This force, however, can be obtained directly by applying the equilibrium condition $\Sigma Y = 0$. Thus, with reference to Fig. 5.14 b, $\Sigma Y = 0 = Y_a - P_1 - P_2 - F_1 \sin \theta$, gives the value of $F_1$.

The analysis of a truss by the method of sections will be illustrated by a numerical example.

**Example 5.2** Find the forces in members fe, fc and bc of the truss shown in Fig. 5.7 using the method of sections,.

Solution : The reactions are found from the equations of equilibrium applied to the truss as a whole. These have been found in Example 5.1 and are as indicated. The section suitable for the determination of the forces in members fe, fc and bc is as shown in Fig. 5.15. When writing the equations of equilibrium for a part on either side of the section, all the cut members are assumed in tension and the forces are directed away from the joints. Thus, whenever a negative sign for the force is obtained it means that the member is in compression and the force acts towards the joint.

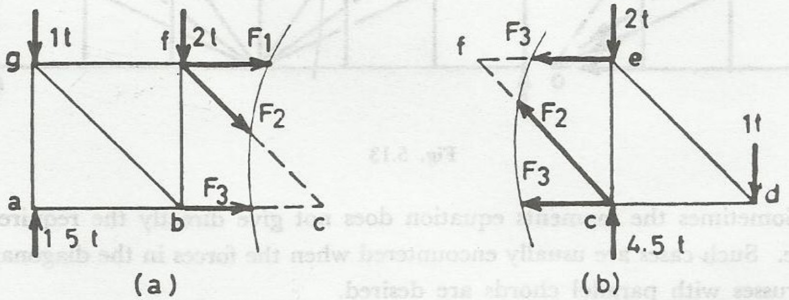



Fig. 5.15

Consider the equilibrium of the part of the truss to the left of the section — Fig. 5.15 a.

$$\Sigma M_c = 0 = F_1 \times 2 + 1.5 \times 4 - 2 \times 2 - 1 \times 4$$

$$F_1 = \frac{4 + 4 - 6}{2} = 1 \text{ t.}$$

$$\Sigma M_f = 0 = 1.5 \times 2 - 1 \times 2 - F_3 \times 2$$

$$F_3 = \frac{3 - 2}{2} = \frac{1}{2} \text{ t.}$$

$$\Sigma Y = 0 = 1.5 - 1 - 2 - F_2 \times \frac{1}{\sqrt{2}}$$

$$F_2 = -1.5 \sqrt{2} \text{ t.}$$

Positive F's correspond to tension and negative F's to compression. These values check with those obtained in Example 5.1 by the method of joints. The part of the truss to the right of the section might have equally been used. Referring to Fig. 5.15 b, the equations of equilibrium would be as follows :

$$\Sigma M_c = 0 = 1 \times 2 - F_1 \times 2$$

$$\Sigma M_f = 0 = 1 \times 4 + 2 \times 2 - 4.5 \times 2 + F_3 \times 2$$

$$\Sigma Y = 0 = 4.5 - 2 - 1 + F_2 \times \frac{1}{\sqrt{2}}$$

These equations give results identical to those obtained previously.

### 5·9 Application of the method of force coefficients

Consider member ab of a truss which is connected to other members at a and b as shown in Fig. 5.16.

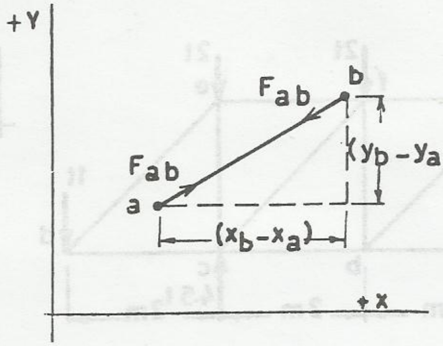



Fig. 5.16

Let $L_{ab}$ and $F_{ab}$ be the length and the force in member ab. Now imagine the forces at all joints to be resolved in the same directions x and y. The components of $F_{ab}$ at a, parallel to the x and y axes are then,

$$\frac{x_b - x_a}{L_{ab}} \quad \text{and} \quad F_{ab} \frac{y_b - y_a}{L_{ab}}$$

These components may be written as :

$$(x_b - x_a) \frac{F_{ab}}{L_{ab}} \quad \text{and} \quad (y_b - y_a) \frac{F_{ab}}{L_{ab}}$$

The ratio $\frac{F_{ab}}{L_{ab}}$, which is common to both components, is called the *force* *coefficient* of member ab and is denoted by $f_{ab}$. Thus,

$$f_{ab} = \frac{F_{ab}}{L_{ab}} \qquad \text{and} \qquad F_{ab} = f_{ab}\, L_{ab}$$

When the resolution of forces at a joint is made, the forces in the members are always assumed to be tensile and the terms, $(x_a - x_b)$, $(y_a - y_b)$, etc.; in the equations are considered positive or negative according to whether they tend to move the joint in the positive or negative directions of the x and y-axes.

As an example on the application of this method, the truss shown in Fig. 5.7 and already solved by the method of joints, will be analysed.

**Example 5·3** Find the forces in all the members of the truss shown in Fig. 5.7 which is reproduced in Fig. 5.17 using the method of force coefficients,

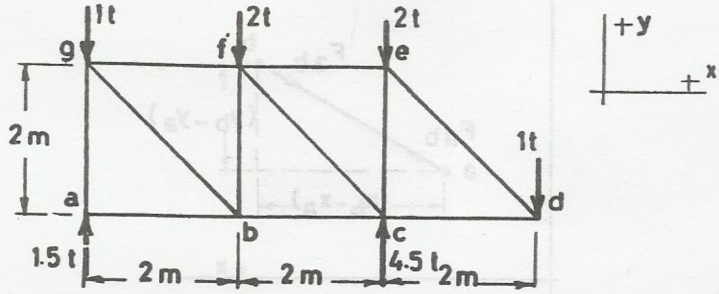



Fig. 5.17

Solution : Taking the co-ordinate axes positive in the directions shown and starting from joint a the equilibrium equations are formed as follows : Considering first the equation for the x-direction,

$$2\, f_{ab} = 0 \qquad f_{ab} = 0$$

and $\quad F_{ab} = f_{ab}\, L_{ab} = 0$

Considering now the equation for the y-direction,

$$1.5 + 2\, f_{ag} = 0 \qquad f_{ag} = -\,{}^{3}/_{4}$$

and $\quad F_{ag} = f_{ag}\, L_{ag} = -\,{}^{3}/_{4} \times 2 = -\,1.5$ t.

Following the same procedure at every joint, the necessary equations are formed and are best set out in a tabulated form as in Table 5.1.

| Joint | Equations | Member | f | Length | Force |
|---|---|---|---|---|---|
| a | $2 f_{ab} = 0$ | ab | $0$ | $2$ | $0$ |
| | $2 f_{ag} + 1.5 = 0$ | ag | $-3/4$ | $2$ | $-1/3$ |
| g | $2 f_{gf} + 2 f_{gb} = 0$ | gf | $-1/4$ | $2$ | $-1/2$ |
| | $-2 f_{gb} - 2 f_{ga} - 1 = 0$ | gb | $1/4$ | $2\sqrt{2}$ | $\sqrt{2}/2$ |
| b | $2 f_{bc} - 2 f_{gb} - 2 f_{ab} = 0$ | bc | $1/4$ | $2$ | $1/2$ |
| | $2 f_{bf} + 2 f_{gb} = 0$ | bf | $-1/4$ | $2$ | $-1/2$ |
| f | $2 f_{fe} + 2 f_{fc} - 2 f_{gf} = 0$ | fe | $1/2$ | $2$ | $1$ |
| | $-2 f_{bf} - 2 f_{fc} - 2 = 0$ | fc | $-3/4$ | $2\sqrt{2}$ | $3\sqrt{2}/2$ |
| e | $2 f_{ed} - 2 f_{fe} = 0$ | ed | $1/2$ | $2\sqrt{2}$ | $\sqrt{2}$ |
| | $-2 f_{ed} - 2 f_{ec} - 2 = 0$ | ec | $-3/2$ | $2$ | $-3$ |
| c | $2 f_{cd} - 2 f_{cb} - 2 f_{cf} = 0$ | cd | $-1/2$ | $2$ | $-1$ |
| | $2 f_{ec} + 2 f_{ef} + 4.5 = 0$ | ce | $-1/2$ | $2$ | $-3$ |

Table 5.1

## 5.10 Graphical method — stress diagram

In the analysis of trusses by the method of joints, two unknown forces were obtained from the two equations of equilibrium; $\Sigma X = 0$ and $\Sigma Y = 0$, applied to each joint. As has been shown in chapter 1, it is also possible to find two unknown forces at each joint graphically by the use of the force polygon for the forces meeting at the joint. The joints must be considered in the same sequence as used in the method of joints, i.e. the analysis could only be started at a joint where no more than two unknown forces exist. Consider for example the simple truss shown in Fig. 5.18. The known forces are the applied load at the apex and the two reactions at the supports.

In the graphical method, it is convenient to use *Bow's notation* to describe the forces. All spaces between the forces; applied loads, pre-calculated reactions and unknown member forces, are lettered and each force is designated by the two letters corresponding to the spaces on its sides. Thus referring to Fig. 5.18 and reading clockwise round the joints, the load

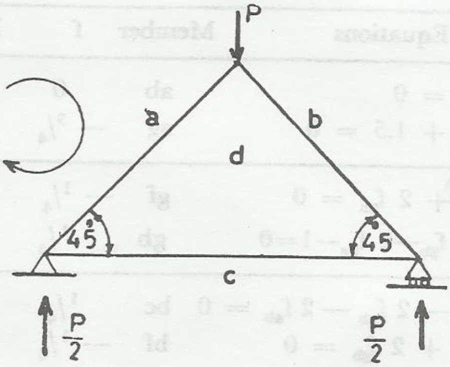



Fig. 5.18

P at the apex is called load ab, the reaction at the left support is load ca, and the reaction at the right support is load bc. Similarly, the force in the horizontal member is force dc or force cd according to which end of the member is being considered.

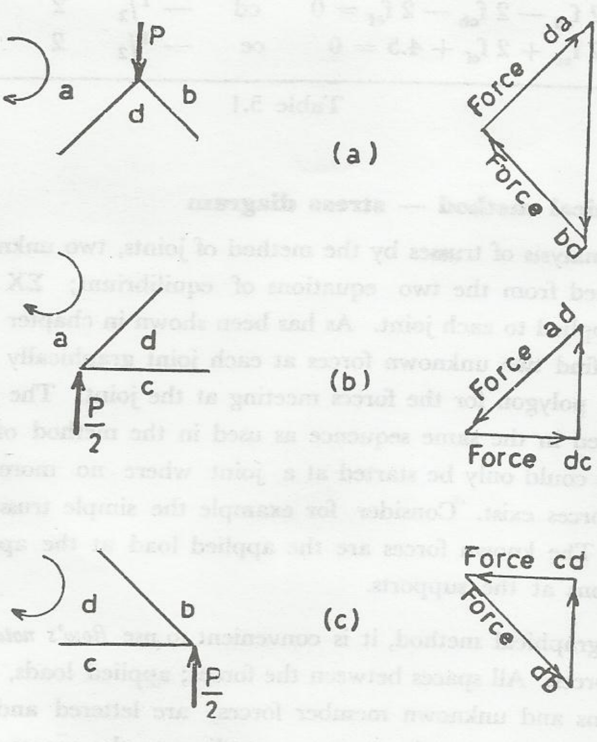



Fig. 5.19

The forces in the inclined members can be obtained by drawing a triangle of forces for the forces meeting at the apex as shown in Fig. 5.19a. The force in the horizontal member can be obtained either by drawing a triagnle of forces for the forces meeting at the left or the right supports — Figs. 5.19 b and c.

The triangles of forces drawn for individual joints and shown in Figs. 5.19 a, b and c can be combined to form a single diagram called a *stress diagram* — Fig. 5.20.

The directions of the arrows in the truss diagram, which indicate the action of the member forces on the joints, are obtained by considering each joint to be the centre of a clock and the letters are read clockwise round it. Thus, having drawn the stress diagram consider, say, the joint at the left support. Reading clockwise round this joint, the inclined member is ad. On the stress diagram, the direction from a to d is downward to the left. Therefore, the arrow is placed in the truss diagram thus ↙ near the joint. The horizontal member is dc, and d to c on the stress diagram is a direction left to right so the arrow is placed thus → near the joint The directions of the arrows in the other members are found in a similar manner.

The principles described in the previous simple example can be applied to simple trusses having any number of members.

The general procedures of analysis are as follows :

(1) The external reactions are determined. This can be done graphically but it is usually more convenient to use the analytical method. For cantilever trusses, this step may be done without.

(2) Using Bow's notation, a force polvgon, for the external loads including the reactions, is drawn.

(3) Starting with a terminal joint, where no more than two unknown member forces meet, the force polygon for this joint is drawn. This is done on the force diagram drawn in step (2).

4 — Going round the joints in a clockwise direction, the joint forces are marked by arrows on the truss diagram, close to the joint.

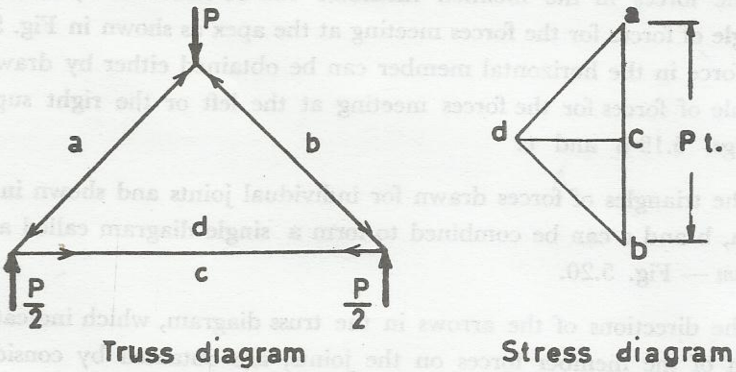



Fig. 5.20

(5) These arrows are transferred to the other end of the member in a reversed direction.

(6) A joint at the end of one of the members whose force has been determined in step (3), and where no more than two other members meet is considered next.

(7) The procedures are continue l until all the joints have been considered. The result is a complete stress diagram from which the magnitudes of the forces can be scaled, and a truss diagram with arrows at both ends of all the members. A member with arrows thus ←→ is in compression and that with arrows thus →← is in tension.

As an example to the application of these procedures, the truss in Fig. 5.7, previously analysed by analytical met ods, will be analysed graphically.

Solution : The notations used are as shown in Fig. 5.2 la. The stress diagram drawn to scale 1 cm. = 1 t., is shown in Fig. 5.21 b. The final result is given in Table 5.2.. Positive signs indicate tension and negative signs indicate compression.

| Member | bg | cg | dh | hg | hj | jb | ek | kj | kl | ia | if |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Force | 0 | -1.5 | -0.5 | +0.7 | -0.5 | +0.5 | +1.0 | -2.1 | -3.0 | -1.0 | +1.4 |

Table 5.2

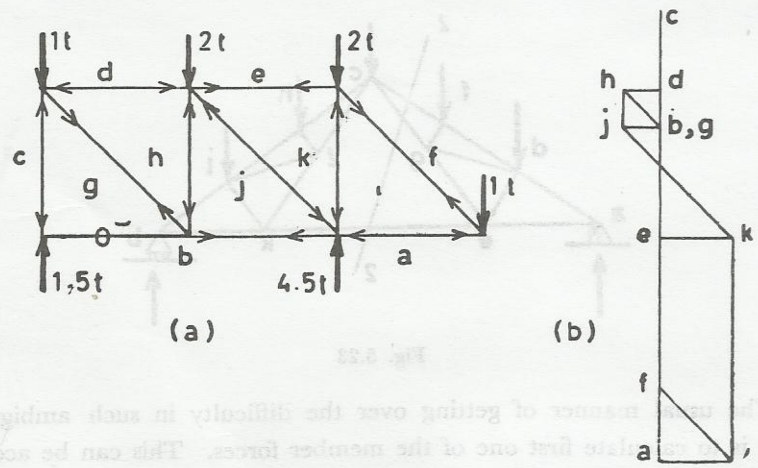



Fig. 5.21

## 5.11 Ambiguous trusses

Sometimes difficulty arises in analysing trusses by the method of joints alone or in drawing the stress diagram either because all the joints have more than two members in which case the stress diagram cannot be directly started, or it can be started but not continued for a similar reason as that mentioned above.

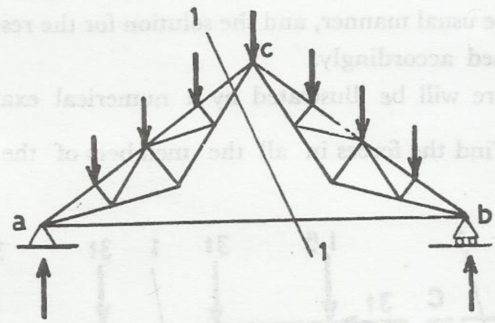



Fig. 5.22

In the truss shown in Fig. 5.22, for example, a direct solution cannot be started since all the joints of the truss have more than two members.

In the truss shown in Fig. 5.23, the solution may be started by considering the equilibrium of joint a after the components of the reactions have been determined. The next step will lead to either joint d or joint e, at which three unknown member forces meet. If the analysis is started at joint b, similar situation will develop at joints i and k.

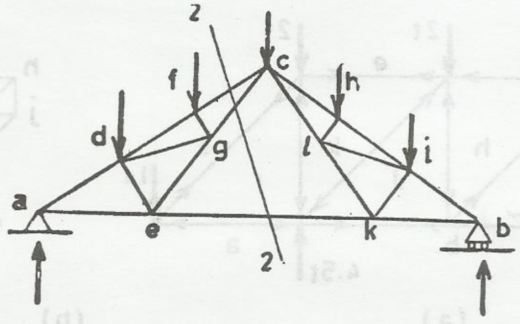



Fig. 5.23

The usual manner of getting over the difficulty in such ambiguous cases is to calculate first one of the member forces. This can be accomplished by the method of sections.

By passing section 1 - 1 through the truss shown in Fig. 5.22 as indicated the force in the tie ab can be calculated by taking moments of all the forces to the left or to the right of the section about joint c. As a result, the analysis can be started and, in this case, completed.

Similarly, by passing section 2 - 2 through the truss shown in Fig. 5.23 as indicated the force in member ek can be calculated by taking moments of all the forces to the left or to the right of the section about joint c. As a result, joint e will have only two unknown forces which can be obtained graphically in the usual manner, and the solution for the rest of the member forces is continued accordingly.

The procedure will be illustrated by a numerical example.

**Example 5.5** Find the forces in all the members of the truss shown in Fig. 5.24 b.

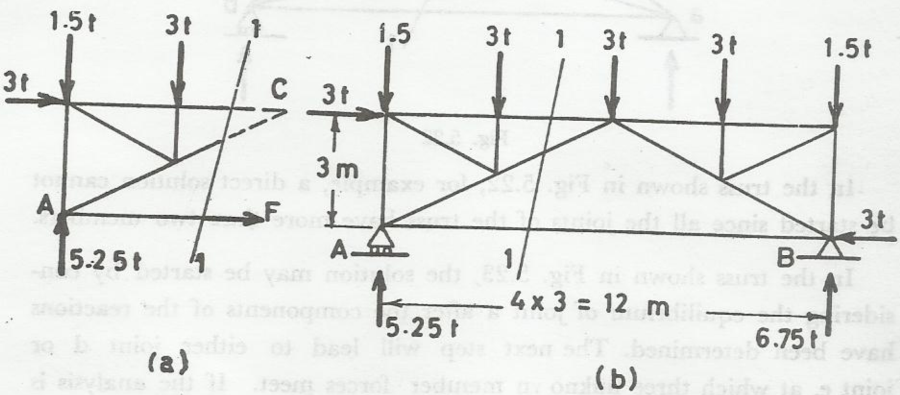



Fig. 5.24

Solution : The reactions are found in the usual manner by considering the equilibrium of the truss as a whole. Thus,

$$\Sigma X = 0 = 3 - X_B$$

$$X_B = 3 \text{ t.} \leftarrow$$

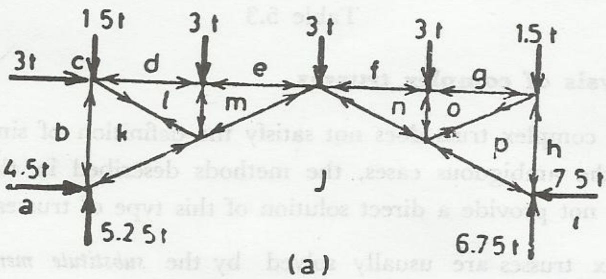
$$\Sigma M_A = 0 = 3 \times 3 + 3\ (3 + 6 + 9) + 1.5 \times 12 - Y_B \times 12$$

$$Y_B = 6.75 \text{ t.} \uparrow$$

$$\Sigma Y = 0 = 1.5 + 3 \times 3 + 1.5 - 6.75 - Y_A$$

$$Y_A = 5.25 \text{ t.} \uparrow$$

Considering the equilibrium of the part to the left of section 1-1 shown in Fig. 5.24 a, the force in member AB is found as follows :



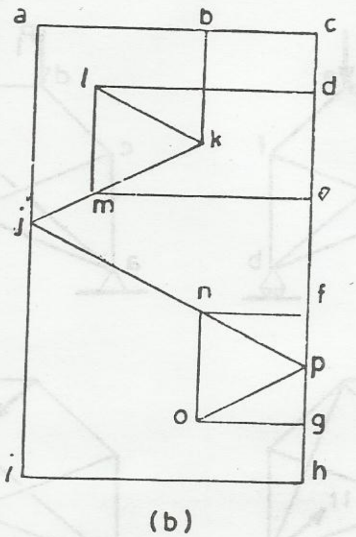




Fig. 5.25

$$\Sigma M_c = 0 = 5.25 \times 6 - 1.5 \times 6 - 3 \times 3 - F \times 3$$

$$F = +4.5 \text{ t.}$$

Now member AB can be replacced by two equal tensile forces at joints A and B, and the stress diagram may be drawn starting from either joint A or joint B.

The stress diasram drawn to scale 1 cm. = 2 t. is shown in Fig. 5.25 b, and the notations used are shown in Fig. 5.25 a. he final result is given in Table 5.3.

| Member | bk | kj | dl | lk | em | ml | mj | fn | nj | go | on | op | pj | ph |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Force | -3 | -5 | -6 | 3.5 | -6 | -3 | -1.7 | -3 | -5 | -3 | -3 | 3.3 | -8.3 | 3.1 |

Table 5.3

### 5.12 Analysis of complex trusses

Since a complex truss does not satisfy the definition of simple trusses including the ambiguous cases, the methods described in the previous sections do not provide a direct solution of this type of trusses.

Complex trusses are usually solved by the *substitute member method* which is sometimes referred to as Henneberg's method. By replacing one

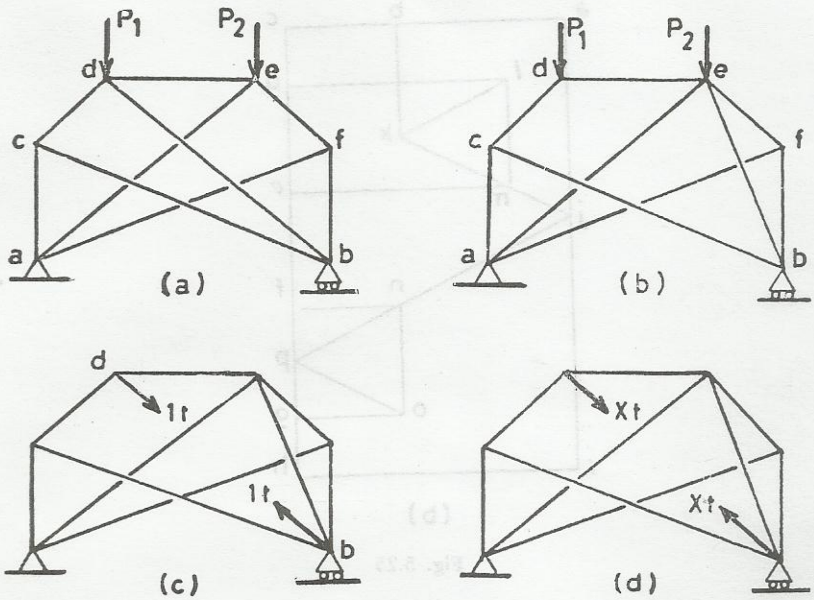



Fig. 5.26

or more members in a complex truss, it is possible to reduce it to a simple truss that can be solved directly by the methods described earlier for the analysis of simple trusses. Then, a correction may be applied later to eliminate the effect of the substitute member or members.

The complex truss shown in Fig. 5.26 a, for example, may be reduced to the simple truss shown in Fig. 5.26 b by replacing member bd by member eb.

Let the forces in the members of the original truss be denoted by F, and the forces in the members of the reduced simple truss by $F_o$. If the external loads are now removed and two equal and opposite unit forces are applied at the joints at the ends of the removed member; joints d and b as shown in Fig. 5.26 c, another set of member forces $F_1$ will be obtained. If further the unit opposite forces at joints d and b are replaced by forces of magnitude X as shown in Fig. 5.26 d, the set of forces in the reduced simple truss will be $F_1$ X. The unknown factor X, which is the force in the removed member db is found from the condition that the final force in the substitute member eb must be zero. Thus,

$$F_{eb} = F_{o\ eb} + F_{1\ eb}\, X = 0$$

$$\text{from which, } X = -\left(\frac{F_o}{F_1}\right)_{eb}$$

Once the force in member db is found, it may be treated as an external load applied at joints d and b and the analysis completed by the methods used earlier for simple trusses. Alternatively the forces in every member of the original truss may be obtained by superimposing the forces in Figs. 5.26 b and d, or by applying the following relation for individual members,

$$F = F_o + F_1 X$$

In some cases, it is necessary to introduce more than one substitute member in order to reduce a complex truss to a simple one. Since, however, the basic idea has been explained and complex trusses are seldom encountered in practice, no further space will be spared for a more comprehensive study. For further discussion on the subject the student is referred to "Timoshenko and Young, Theory of structures".

The method will be illustrated by a numerical example.

**Example 5.6** Determine the forces in all the members of the complex truss shown in Fig. 5 27 a, using Henneberg's method.

Solution : If member ef is temp rarily removed and a substitute member bf is added as shown in Fig. 5.27 b, the details of the calculations will be as pres nted in Table 5.4.

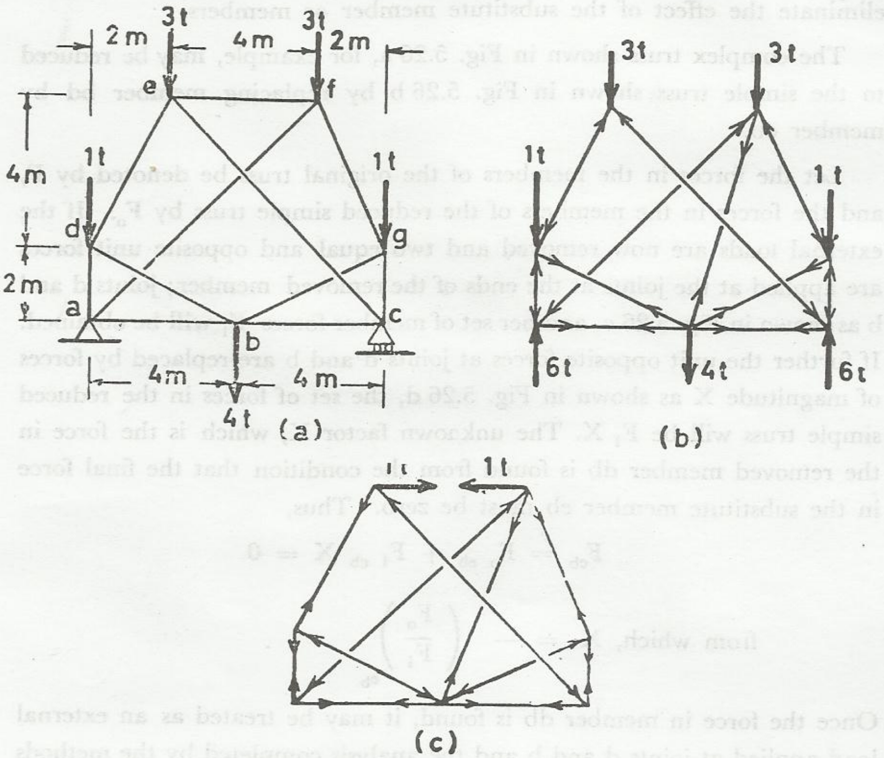



Fig. 5.27

| Member | $F_o$ | $F_1$ | $F = F_o + F_1 X$ |
|---|---|---|---|
| ab | $+5/2$ | $+5/6$ | $-5$ |
| bc | $+1$ | $+2/3$ | $-5$ |
| ad | $-7/2$ | $+5/6$ | $-11$ |
| bd | $+\sqrt{5}/2$ | $\sqrt{5}/6$ | $+2\sqrt{5}$ |
| cg | $5$ | $+2/3$ | $11$ |
| bg | $+4\sqrt{5}/5$ | $-2\sqrt{5}/15$ | $+2\sqrt{5}$ |
| de | $-\sqrt{5}$ | $+\sqrt{5}/3$ | $-4\sqrt{5}$ |
| fg | $-8\sqrt{5}/5$ | $+4\sqrt{5}/15$ | $-4\sqrt{5}$ |
| ce | $-\sqrt{2}$ | $-2\sqrt{2}/3$ | $+5\sqrt{2}$ |
| af | $-5\sqrt{2}/2$ | $-5\sqrt{2}/6$ | $+5\sqrt{2}$ |
| ef | $0$ | $+1$ | $-9$ |
| bf | $+9\sqrt{10}/10$ | $+\sqrt{10}/10$ | $.0$ |

Table 5.4

The factor X has been obtained from the relation,

$$X = -\left(\frac{F_0}{F_1}\right)_{bf} = -\frac{9\sqrt{10}}{10} \times \frac{10}{\sqrt{10}} = -9$$

## 5·13 Illustrative examples

The following examples illustrate the application of the various methods to the analysis of several typ:s of statically determinate trusses. The student is advised to solve the given problems independently and then check his results against those given. Further, he will do good to solve the problems by different methods and compare their relative ease when applied to individual trusses.

**Examples 5·7-5·14** Calculate the forces in all the members of the trusses shown in Figs. 5.28-5.35 by the method of joints.

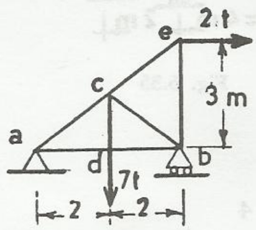



Fig. 5.28

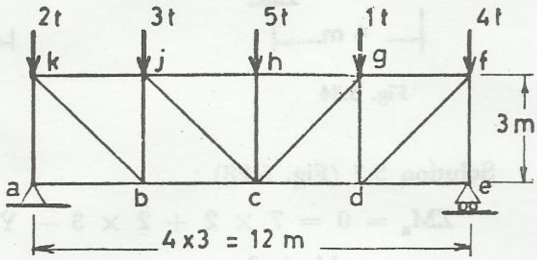



Fig. 5.29

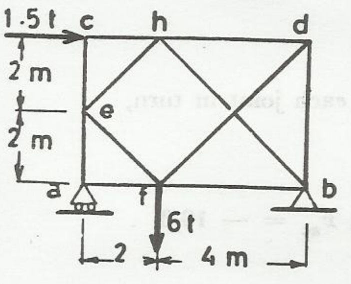



Fig. 5.30

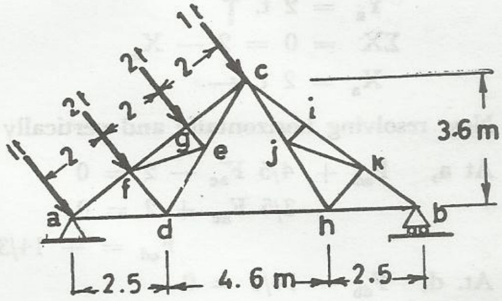



Fig. 5.31

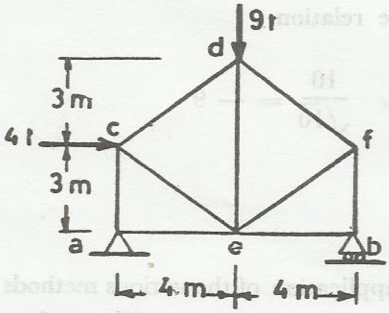



Fig. 5.32

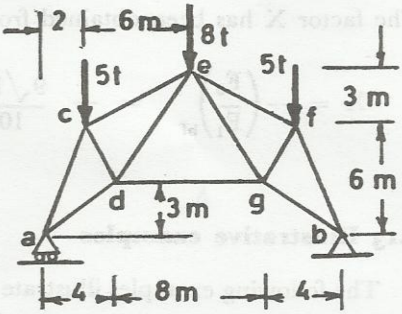



Fig. 5.33

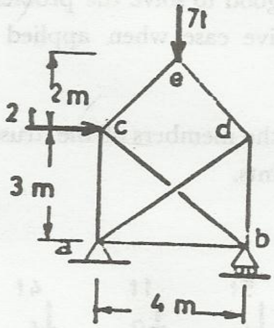



Fig. 5.34

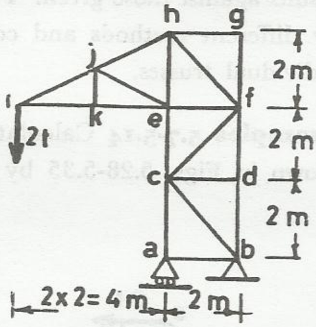



Fig. 5.35

**Solution 5.7 (Fig. 5.28) :**

$$\Sigma M_a = 0 = 7 \times 2 + 2 \times 3 - Y_b \times 4$$

$$Y_b = \frac{14 + 6}{4} = 5 \text{ t.} \uparrow$$

$$\Sigma Y = 0 = 7 - 5 - Y_a$$

$$Y_a = 2 \text{ t.} \uparrow$$

$$\Sigma X = 0 = 2 - X_a$$

$$X_a = 2 \text{ t.} \leftarrow$$

Now resolving horizontally and vertically at each joint in turn,

At a, $F_{ad} + 4/5\ F_{ac} - 2 = 0$

$3/5\ F_{ac} + 2 = 0$

$$F_{ad} = +\ 14/3 \text{ t.}, \quad F_{ac} = -\ 10/3 \ .$$

At. d, $F_{db} - 14/3 = 0$

$F_{dc} - 7 = 0$

$$F_{db} = +\ 4/3 \text{ t.}, \quad F_{dc} = +\ 7 \ .$$

At c, $F_{ce} \times 4/5 + F_{cb} \times 4/5 + 10/3 \times 4/5 = 0$

$F_{ce} \times 3/5 - F_{cb} \times 3/5 + 10/3 \times 3/5 - 7 = 0$

$F_{ce} = + 5/2$ t., $F_{cb} = - 35/6$ t.

At e, $5/2 \times 3/5 + F_{eb} = 0$

$F_{eb} = - 3/2$ t.

At b, this last joint provides a check on the results.

$\Sigma X = 14/3 - 35/6 \times 4/5 = 0$

$\Sigma = 5 - 35/6 \times 3/5 - 3/2 = 0$

Solution 5.8 (Fig. 5.29) :

$\Sigma M_a = 0 = 3 \times 3 + 5 \times 6 + 1 \times 9 + 4 \times 12 - Y_e \times 12$

$$Y_e = \frac{9 + 30 + 9 + 43}{12} = 8 \text{ t.} \uparrow$$

$\Sigma Y = 0 = 2 + 3 + 5 + 1 + 4 - 8 - Y_a$

$Y_a = 7$ t. $\uparrow$

At a, $F_{ak} + 7 = 0$

$F_{ab} = 0$

$I_{ab} = 0$ , $F_{ak} = - 7$ t.

At k, $F_{kb} \times \frac{1}{\sqrt{2}} + F_{kj} = 0$

$F_{kb} \times \frac{1}{\sqrt{2}} + 2 - 7 = 0$

$F_{kb} = + 5\sqrt{2}$ t. , $F_{kj} = - 5$ t.

At b, $F_{bc} - 5\sqrt{2} \times \frac{1}{\sqrt{2}} - 0 = 0$

$F_{bj} + 5\sqrt{2} \times \frac{1}{\sqrt{2}} = 0$

$F_{bc} = + 5$ t. , $F_{bj} = - 5$ t.

At j, $F_{jh} + F_{jc} \times \frac{1}{\sqrt{2}} + 5 = 0$

$F_{jc} \times \frac{1}{\sqrt{2}} - 5 + 3 = 0$

$F_{jc} = + 2\sqrt{2}$ t. , $F_{jh} = - 7$ t.

At h, $F_{hg} + 7 = 0$

$F_{hc} + 5 = 0$

$F_{hg} = - 7$ t. , $F_{hc} = -$ t

At c, $F_{cd} + F_{cg} \times \frac{1}{\sqrt{2}} - 5 - 2\sqrt{2} \times \frac{1}{\sqrt{2}} = 0$

$$F_{cg} \times \frac{1}{\sqrt{2}} - 5 + 2\sqrt{2} \times \frac{1}{\sqrt{2}} = 0$$

$$F_{cg} = +3\sqrt{2} \text{ t.}, \quad F_{cd} = +4 \text{ t.}$$

At g, $F_{gf} + 7 - 3\sqrt{2} \times \frac{1}{\sqrt{2}} = 0$

$$F_{gd} + 3\sqrt{2} \times \frac{1}{\sqrt{2}} + 1 = 0$$

$$F_{gf} = -4 \text{ t.}, \quad F_{gd} = -4 \text{ t.}$$

At d, $F_{df} \times \frac{1}{\sqrt{2}} + F_{de} - 4 = 0$

$$F_{df} \times \frac{1}{\sqrt{2}} - 4 = 0$$

$$F_{df} = +4\sqrt{2} \text{ t.}, \quad F_{de} = 0$$

At f, $F_{df} \times \frac{1}{\sqrt{2}} - 4 = 0$

$$F_{df} \times \frac{1}{\sqrt{2}} + F_{fe} + 4 = 0$$

$$F_{fe} = -8 \text{ t.}$$

At e, t is last joint provides a check on the results,

$\Sigma X = 0$

$\Sigma Y = 8 - 8 = 0$

S lution 5.9 (Fig. 5.30) :

$\Sigma M_a = 0 = 1.5 \times 4 + 6 \times 2 - Y_b \times 6$

$$Y_b = \frac{6 + 12}{6} = 3 \text{ t.} \uparrow$$

$\Sigma Y = 0 = 6 - 3 - Y_a$

$Y_a = 3 \text{ t.} \uparrow$

$\Sigma X = 0 = 1.5 - X_b$

$X_b = 1.5 \text{ t.} \leftarrow$

At a $F_{af} = 0$

$F_{ae} + 3 = 0$

$$F_{af} = 0, \quad F_{ae} = -3\text{t.}$$

At c, $F_{ch} + 1.5 = 0$

$F_{ce} = 0$

$$F_{ce} = 0 \;,\; F_{ch} = -1.5 \text{ t.}$$

At e, $F_{eh} \times \frac{1}{\sqrt{2}} + F_{ef} \times \frac{1}{\sqrt{2}} = 0$

$$F_{eh} \times \frac{1}{\sqrt{2}} - F_{ef} \times \frac{1}{\sqrt{2}} + 3 = 0$$

$$F_{eh} = -1.5\sqrt{2} \text{ t.} \;,\; F_{ef} = +1.5\sqrt{2}\text{t.}$$

At h, $F_{hd} + F_{hb} \times \frac{1}{\sqrt{2}} + 1.5 + 1.5\sqrt{2} \times \frac{1}{\sqrt{2}} = 0$

$$F_{hb} \times \frac{1}{\sqrt{2}} - 1.5\sqrt{2} \times \frac{1}{\sqrt{2}} = 0$$

$$F_{hb} = +1.5\sqrt{2} \text{ t.} \quad F_{hd} = -4.5 \text{ t.}$$

At d, $F_{df} \times \frac{1}{\sqrt{2}} - 4.5 = 0$

$$F_{df} \times \frac{1}{\sqrt{2}} + F_{db} = 0$$

$$F_{df} = +4.5\sqrt{2} \text{ t.} \;,\; F_{db} = -4.5 \text{ t.}$$

At f, $F_{fb} + 4.5\sqrt{2} \times \frac{1}{\sqrt{2}} - 1.5\sqrt{2} \times \frac{1}{\sqrt{2}} + 0 = 0$

$$F_{df} \times \frac{1}{\sqrt{2}} + F_{ef} \times \frac{1}{\sqrt{2}} - 6 = 0$$

$$F_{fb} = -3 \text{ t.}$$

At b, this last joint provides a check on the results,

$$\Sigma X = 3 - 1.5\sqrt{2} \times \frac{1}{\sqrt{2}} - 1.5 = 0$$

$$\Sigma Y = 3 - 4.5 + 1.5\sqrt{2} \times \frac{1}{\sqrt{2}} = 0$$

Solution 5.10 (Fig. 5.31) :

Resolving the resultant of the inclined forces horizontally and vertically then,

$$\Sigma M_a = 0 = 4.8 \times 2.4 + 3.6 \times 1.8 - Y_b \times 9.6$$

$Y_b = 1.875$ t. ↑

$\Sigma Y = 0 = 4.8 - 1.875 - Y_a$

$Y_a = 2.925$ t. ↑

$\Sigma X = 0 = X_a - 3.6$

$X_a = 3.6$ t. ←

Generally, difficulty arises in such a truss at either joints d or f if the analysis is started at joint a, or at joints h or k if the analysis is started at joint b, due to the presence of three unknown member forces. However, by inspection, it could be seen that members ij, jk, kh, hj and jc have zero forces for the given case of loading.

Consequently, no difficulty will arise in this particular case if the analysis is started at joint b.

At b, $F_{bc} \times 4/5 + F_{bd} = 0$

$F_{bc} \times 3/5 + 1.875 = 0$

$F_{bc} = -3.12$ t. , $F_{bd} = +2.5$ t.

At a, $F_{ad} + F_{af} \times 4/5 + 1 \times 3/5 - 3.6 = 0$

$F_{af} \times 3/5 - 1 \times 4/5 + 2.925 = 0$

$F_{af} = -3.54$ t. , $F_{ad} = +5.83$ t.

At d, $F_{de} \times 0.54 - F_{df} \times 0.6 + 2.5 - 5.83 = 0$

$F_{de} \times 0.84 + F_{df} \times 0.8 = 0$

$F_{de} = +2.85$ t. , $F_{df} = -3$ t.

At f, resolving along member ac and normal to it,

$F_{fg} + F_{fe} \times 0.935 + 3.54 = 0$

$F_{fe} \times 0.35 + 2 - 3 = 0$

$F_{fe} = +2.86$ t. , $F_{fg} = -6.8$ t.

At g. resolving along member ac and normal to it,

$F_{gc} + 6.8 = 0$

$F_{ge} + 2 = 0$

$F_{ge} = -2$ t. , $F_{gc} = -6.8$ t.

At e. $F_{ec} \times 0.54 - 2.85 \times 0.54 - 2.86 \times 0.935 + 2 \times 0.6 = 0$

$F_{ec} = +5.6$ t.

Solution 5.11 (Fig. 5.32) :

$\Sigma M_a = 0 = 9 \times 4 + 4 \times 3 - Y_b \times 8$

$Y_b = 6$ t. ↑

$\Sigma Y = 0 = 9 - 6 - Y_a$

$Y_a = 3$ t. ↑

$\Sigma X = 0 = 4 - X_a$

$X_a = 4$ t. ←

At a, $F_{ae} - 4 = 0$

$F_{ac} + 3 = 0$

$$F_{ae} = 4 \text{ t.}, \; F_{ac} = -3$$

At c, $F_{cd} \times 4/5 + F_{cd} \times 4/5 + 4 = 0$

$F_{cd} \times 3/5 - F_{ce} \times 3/5 + 3 = 0$

$$F_{cd} = -5 \text{ t.}, \; F_{ce} = 0$$

At b, $F_{be} = 0$

$F_{bf} + 6 = 0$

$$F_{be} = 0, \; F_{bf} = -6 \text{ t.}$$

At f, $F_{fd} \times 4/5 + F_{fe} \times 4/5 = 0$

$F_{fd} \times 3/5 - F_{fe} \times 3/5 + 6 = 0$

$$F_{fd} = -5 \text{ t.}, \; F_{fe} = +5 \text{ t.}$$

At e, $F_{ef} \times 4/5 - 4 = 0$

$F_{ef} \times 3/5 + F_{ed} = 0$

$$F_{ed} = -3 \text{ t.}$$

At d, this last joint provides a check on the results,

$$\Sigma X = 5 \times 4/5 - 5 \times 4/5 = 0$$

$$\Sigma Y = 3 + 5 \times 3/5 + 5 \times 3/5 - 9 = 0$$

Solution 5.12 (Fig. 5.33) :

From symmetry, $Y_a = Y_b = \dfrac{5 + 8 + 5}{2} = 9 \text{ t.} \uparrow$

$$X_b = 0$$

At a, $F_{ad} \times 4/5 + F_{ac} \times \dfrac{1}{\sqrt{10}} = 0$

$$F_{ad} \times 3/5 + F_{ac} \times \frac{3}{\sqrt{10}} + 9 = 0$$

$$F_{ad} = +5 \text{ t.}, \; F_{ac} = -4\sqrt{10} \text{ t.}$$

At c, $F_{ce} \times \dfrac{2}{\sqrt{5}} + F_{cd} \times \dfrac{2}{\sqrt{13}} + 4\sqrt{10} \times \dfrac{1}{\sqrt{10}} = 0$

$$F_{ce} \times \frac{1}{\sqrt{5}} - F_{cd} \times \frac{3}{\sqrt{13}} - 5 + 4\sqrt{10} \times \frac{3}{\sqrt{10}} = 0$$

$$F_{cd} = +5\sqrt{13}/4 \text{ t.}, \; F_{ce} = -13\sqrt{5}/4 \text{ t.}$$

At d, $F_{de} \times \frac{2}{\sqrt{13}} + F_{dg} - 5 \times 4/5 - 5\frac{\sqrt{13}}{4} \times \frac{2}{\sqrt{13}} = 0$

$F_{de} \times \frac{3}{\sqrt{13}} + 5\frac{\sqrt{13}}{4} \times \frac{3}{\sqrt{13}} - 5 \times 3/5 = 0$

$F_{de} = -\sqrt{13}/4$ t. , $F_{dg} = +7$ t.

Since the truss and the loads are symmetrical there is no need to consider further joints.

Solution 5.13 (Fig. 5.34) :

$\Sigma M_a = 0 = 2 \times 3 + 7 \times 2 - Y_b \times 4$

$Y_b = \frac{6 + 14}{4} = 5$ t. ↑

$\Sigma Y = 0 = 7 - 5 - Y_a$

$Y_a = 2$ t. ↑

$\Sigma X = 0 = 2 - X_a$

$X_a = 2$ t. ←

At e $F_{ed} \times \frac{1}{\sqrt{2}} - F_{ec} \times \frac{1}{\sqrt{2}} = 0$

$F_{ed} \times \frac{1}{\sqrt{2}} + F_{ec} \times \frac{1}{\sqrt{2}} + 7 = 0$

$F_{ed} = -7\sqrt{2}/2$ t. , $F_{ec} = -7\sqrt{2}/2$ t.

At c, $F_{cb} \times 4/5 - \frac{7\sqrt{2}}{2} \times \frac{1}{\sqrt{2}} + 2 = 0$

$F_{cb} \times 3/5 + F_{ca} + \frac{7\sqrt{2}}{2} \times \frac{1}{\sqrt{2}} = 0$

$F_{cb} = +15/8$ t. , $F_{ca} = -37/8$ t.

At d, $F_{ad} \times 4/5 - \frac{7\sqrt{2}}{2} \times \frac{1}{\sqrt{2}} = 0$

$F_{ad} \times 3/5 + \frac{7\sqrt{2}}{2} \times \frac{1}{\sqrt{2}} + F_{db} = 0$

$F_{ad} = +35/8$ t. , $F_{db} = -49/8$ t.

At a, $F_{ab} + 35/8 \times 4/5 - 2 = 0$

$$F_{ab} = -3/2 \text{ t.}$$

At b, this last joint provides a check on the results,

$$\Sigma X = 15/8 \times 4/5 - 3/2 = 0$$

$$\Sigma Y = 5 + 15/8 \times 3/5 - 49/8 = 0$$

Solution 5.14 (Fig. 5.35) :

$$\Sigma M_a = 0 = 2 \times 4 + Y_b \times 2$$

$$Y_b = -8/2 = 4 \text{ t.} \downarrow$$

$$\Sigma Y = 0 = 2 + 4 - Y_a$$

$$Y_a = 6 \text{ t.} \uparrow$$

$$\Sigma X = 0 = X_a, \quad X_a = 0$$

By inspection, members ab, bc, cd, cf, fg. gh, kj and je have zero forces.

At i, $F_{ie} + F_{ih} \times \frac{2}{\sqrt{5}} = 0$

$$F_{ih} \times \frac{1}{\sqrt{5}} - 2 = 0$$

$$F_{ih} = +2\sqrt{5} \text{ t.}, \quad F_{ie} = -4 \text{ t.}$$

At h, $F_{hf} \times \frac{1}{\sqrt{2}} - 2\sqrt{5} \times \frac{2}{\sqrt{5}} = 0$

$$F_{hf} \times \frac{1}{\sqrt{2}} + F_{he} + 2\sqrt{5} \times \frac{1}{\sqrt{5}} = 0$$

$$F_{hf} = +4\sqrt{2} \text{ t.}, \quad F_{he} = -6 \text{ t.}$$

At f, $F_{fe} + 4\sqrt{2} \times \frac{1}{\sqrt{2}} = 0$

$$F_{fb} - 4\sqrt{2} \times \frac{1}{\sqrt{2}} = 0$$

$$F_{fe} = -4 \text{ t.}, \quad F_{fb} = +4 \text{ t.}$$

At e, $F_{ae} + 6 = 0$

$$F_{ae} = -6 \text{ t.}$$

It is to be noted that in calculating the member forces, no use has been made of the calculated reactions. This is generally true for cantilever trusses if the analysis is started from the free end.

**Examples 5.15 - 5.20** Find the forces in the marked members of the trusses shown in Figs. 5.36-5.41 using the method of sections.

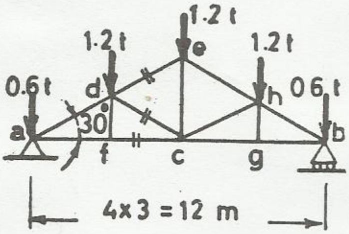



Fig. 5.36

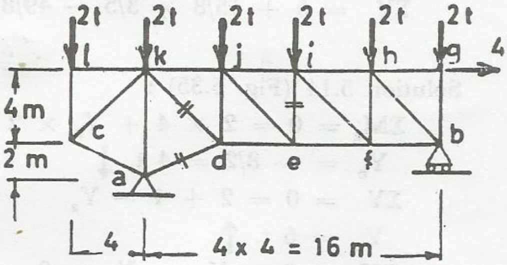



Fig. 5.37

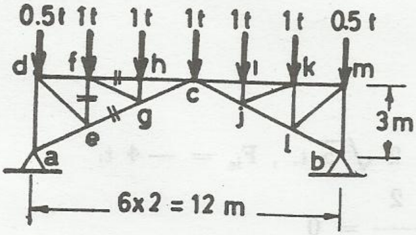



Fig. 5.38

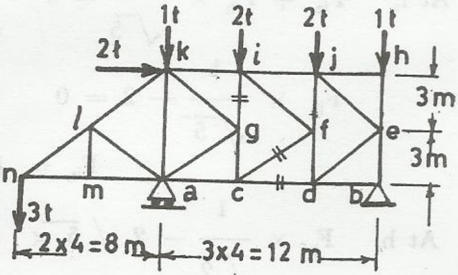



Fig. 5.39

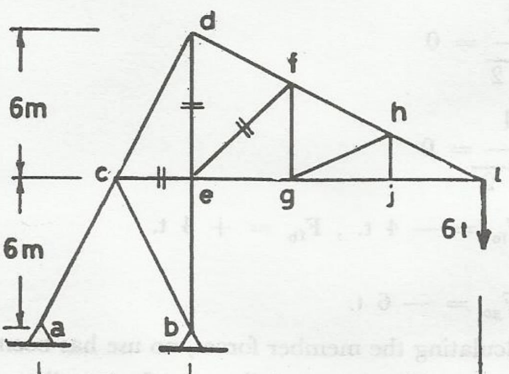



Fig. 5.40

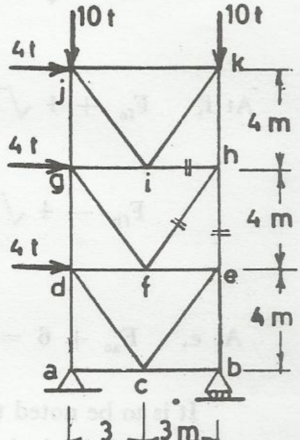



Fig. 5.41

Solution 5.15 (Fig. 5.36) :

$\Sigma M_a = 0 = 1.2 \times 3 + 1.2 \times 6 + 1.2 \times 9 + 0.6 \times 12 - Y_b \times 12$

$$Y_b = \frac{3.6 + 7.2 + 10.8 + 7.2}{12} = 2.4 \text{ t.} \uparrow$$

$\Sigma Y = 0 = 0.6 + 1.2 + 1.2 + 0.6 - 2.4 - Y_a$

$Y_a = 2.4 \text{ t.} \uparrow$

$\Sigma X = 0 = X_a \qquad X_a = 0$

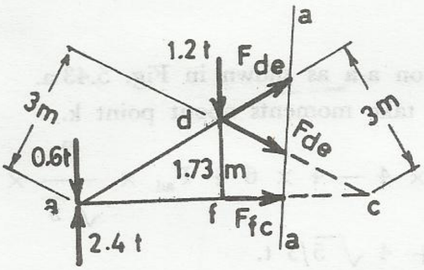



Fig. 5.42

Consider section a-a as shown in Fig. 5.42

For member de, take moments about point c.

$\Sigma M_c = 0 = (2.4 - 0.6) \times 6 - 1.2 \times 3 + F_{de} \times 3$

$F_{de} = -2.4 \text{ t.}$

For member fc, take moments about point d.

$\Sigma M_d = 0 = (2.4 - 0.6) \times 3 + F_{fc} \times 1.73$

$F_{fc} = +3.12 \text{ t.}$

For member dc take moments about point a.

$\Sigma M_a = 0 = 1.2 \times 3 + F_{dc} \times 3$

$F_{dc} = -1.2 \text{ t.}$

Solution 5.16 (Fig. 5.37) :

$\Sigma M_a = 0 = 4 \times 6 + 2\,(16 + 12 + 8 + 4 - 4) - Y_b \times 16$

$Y_b = 96/16 = 6 \text{ t.} \uparrow$

$\Sigma Y = 0 = 6 \times 2 - 6 - Y_a$

$Y_a = 6 \text{ t} \uparrow$

$\Sigma X = 0 = 4 - X_a$

$X_a = 4 \text{ t.} \leftarrow$

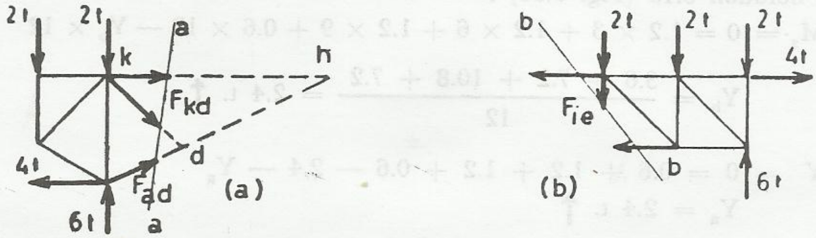



Fig. 5.43

Consider section a-a as shown in Fig. 5.43 a.
For member ad, take moments about point k.

$$\Sigma M_k = 0 = 2 \times 4 - 4 \times 6 + F_{ad} \times \frac{2}{\sqrt{5}} \times 6$$

$$F_{ad} = + 4\sqrt{5}/3 \text{ t.}$$

For member kd, take moments about point h.

$$\Sigma M_h = 0 = 6 \times 12 + 4 \times 6 - 2 \times 16 - 2 \times 12 - F_{kd} \times \frac{1}{\sqrt{2}} \times 12$$

$$F_{kd} = + 10\sqrt{2}/3 \text{ t.}$$

For member ie, consider section b-b shown in Fig. 5.43 b.

$$\Sigma Y = 0 = 6 - 2 - 2 - 2 - F_{ie}$$

$$F_{ie} = 0$$

Solution 5.17 (Fig. 5.38) :

$$\Sigma M_a = 0 = 0.5 \times 12 + 1\,(10 + 8 + 6 + 4 + 2) - Y_b \times 12$$

$$Y_b = \frac{6 + 30}{12} = 3 \text{ t.} \uparrow$$

$$\Sigma Y = 0 = 6 - 3 = 3 \text{ t.} \uparrow$$

$$\sum_a^c M_c = 0 = 3 \times 6 - 0.5 \times 6 - 1 \times 4 - 1 \times 2 - X_a \times 3$$

$$X_a = \frac{18 - 3 - 4 - 2}{3} = 3 \text{ t.} \rightarrow$$

$$\Sigma X = 0 = 3 - X_b$$

$$X_b = 3 \text{ t.} \leftarrow$$

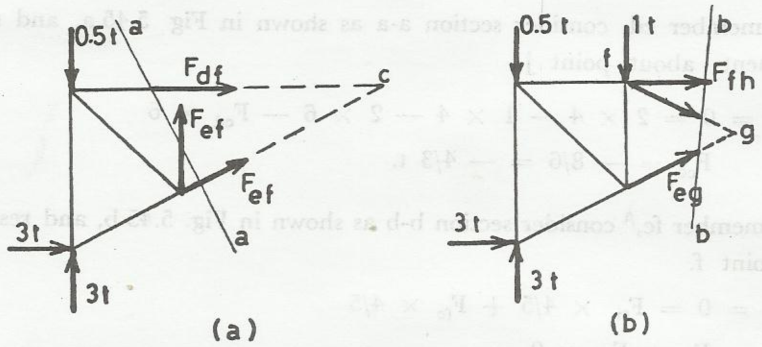



Fig. 5.44

Consider section a-a as shown in Fig. 5.44 a.

For member ef, take moments about point c.

$\Sigma M_c = 0 = 3 \times 6 - 0.5 \times 6 - 3 \times 3 + F_{ef} \times 4$

$F_{ef} = -1.5$ t.

Consider section b-b as shown in Fig. 5.44 b.

For member fh, take moments about point g.

$\Sigma M_g = 0 = 3 \times 4 - 0.5 \times 4 - 3 \times 2 - 1 \times 2 + F_{fh} \times 1$

$F_{fh} = -2$ t.

For member eg, take moments about point f.

$\Sigma M_f = 0 = 3 \times 2 - 0.5 \times 2 - 3 \times 3 - F_{eg} \times \frac{2}{\sqrt{5}} \times 2$

$F_{eg} = -\sqrt{5}$ t.

Solution 5.18 (Fig. 5.39) :

$\Sigma M_a = 0 = 1 \times 12 + 2 \times 8 + 2 \times 4 + 2 \times 6 - 3 \times 8 - Y_b \times 12$

$Y_b = 24/12 = 2$ t. ↑

$\Sigma Y = 0 = 3 + 1 + 2 + 2 + 1 - 2 - Y_a$

$Y_a = 7$ t. ↑

$\Sigma X = 0 = 2 - X_b$ $\qquad X_b = 2$ t. ←

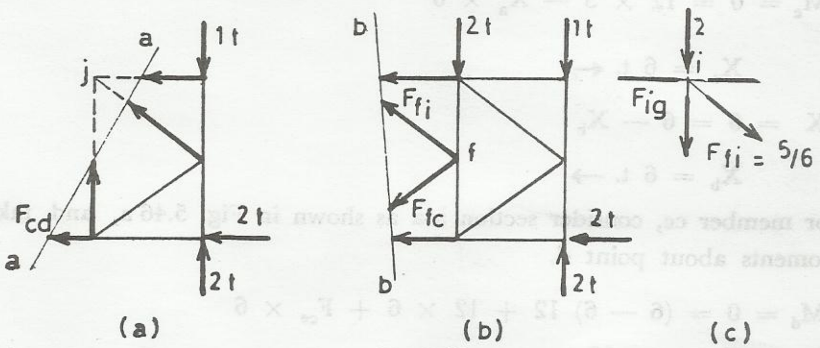



Fig. 5.45

For member cd, consider section a-a as shown in Fig. 5.45 a, and take moments about point j.

$$\Sigma M = 0 = 2 \times 4 - 1 \times 4 - 2 \times 6 - F_{cd} \times 6$$

$$F_{cd} = -8/6 = -4/3 \text{ t.}$$

For member fc, consider section b-b as shown in Fig. 5.45 b, and resolve at joint f.

$$\Sigma X = 0 = F_{fi} \times 4/5 + F_{fc} \times 4/5$$

$$F_{fi} + F_{fc} = 0$$

By considering the equilibrium of the part of the truss to the right of section b-b,

$$\Sigma Y = 0 = 2 - 1 - 2 + F_{fi} \times 3/5 - F_{fc} \times 3/5$$

$$3/5\, F_{fi} - 3/5\, F_{fc} - 1 = 0$$

From these two equations,

$$F_{fi} = +5/6 \text{ t.} \quad \text{and} \quad F_{fc} = -5/6 \text{ t.}$$

For member ig, consider the equilibrium of joint i shown in Fig. 5.45 c.

$$\Sigma Y = 0 = 2 + 5/6 \times 3/5 + F_{ig}$$

$$F_{ig} = -2.5 \text{ t.}$$

**Solution 5.19 (Fig. 5.40) :**

$$\Sigma M_a = 0 = 6 \times 18 - Y_b \times 6$$

$$Y_b = 18 \text{ t.} \uparrow$$

$$\Sigma Y = 0 = 6 - 18 - Y_a$$

$$Y_a = 12 \text{ t.} \downarrow$$

$$\underset{a}{\Sigma M_c} = 0 = 12 \times 3 - X_a \times 6$$

$$X_a = 6 \text{ t.} \leftarrow$$

$$\Sigma X = 0 = 6 - X_b$$

$$X_b = 6 \text{ t.} \rightarrow$$

For member ce, consider section a-a as shown in Fig. 5.46 a, and take moments about point d.

$$\Sigma M_d = 0 = (6 - 6)\, 12 + 12 \times 6 + F_{ce} \times 6$$

$$F_{ce} = -12 \text{ t}$$

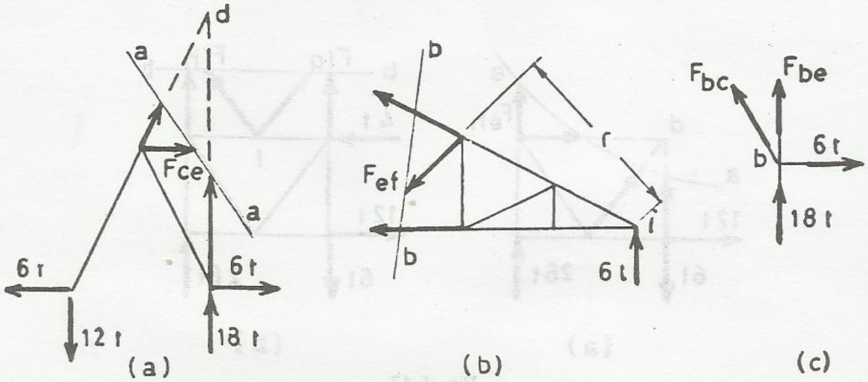



Fig. 5.46

For member ef, consider section b-b as shown in Fig. 5.46 b and take moments about point i.

$$\Sigma M_i = 0 = F_{ef} \times r$$

$$F_{ef} = 0$$

Member ed cannot be obtained directly. It may be found by first calculating the force in member be and then resolving vertically at joint e.

With reference to Fig. 5.46 c,

$$\Sigma X = 0 = 6 - F_{bc} \times \frac{1}{\sqrt{5}}$$

$$\Sigma Y = 0 = 18 + F_{bc} \times \frac{2}{\sqrt{5}} + F_{be}$$

$$F_{be} = -30 \text{ t.}$$

By considering the equilibrium of point e and noting that $F_{ef} = 0$,

$$F_{ed} = F_{be} = -30 \text{ t.}$$

Solution 5.20 (Fig. 5.41) :

$$\Sigma M_a = 0 = 4\,(4 + 8 + 12) + 10 \times 6 - Y_b \times 6$$

$$Y_b = 26 \text{ t.} \uparrow$$

$$\Sigma Y = 0 = 10 + 10 - 26 - Y_a$$

$$Y_a = 6 \text{ t.} \downarrow$$

$$\Sigma X = 0 = 4 + 4 + 4 - X_a$$

$$X_a = 12 \text{ t.} \leftarrow$$

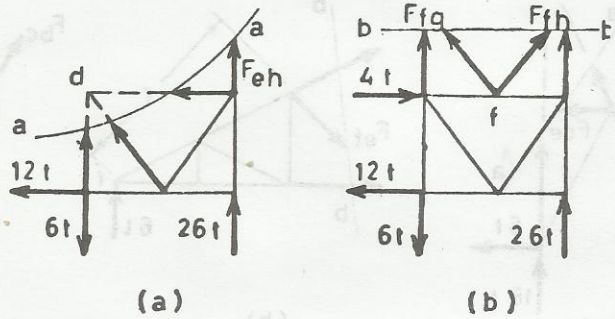



Fig. 5.47

For member eh, consider section a-a as shown in Fig. 5.47 a and take moments about point d.

$$\Sigma M_d = 0 = 26 \times 6 - 12 \times 4 + F_{eh} \times 6$$

$$F_{eh} = -18 \text{ t.}$$

Consider section b-b as shown in Fig. 5.47 b. Resolving vertically at joint f,

$$\Sigma Y = 0 = F_{fg} \times 4/5 + F_{fh} \times 4/5$$

Considering the equilibrium of the part of the truss below section b-b,

$$\Sigma X = 0 = F_{fg} \times 3/5 - F_{fh} \times 3/5 - 4 + 12$$

From these two equations,

$$F_{fh} = +20/3 \text{ t.}$$

The force in member ih may be obtained by resolving the forces at joint h. Thus,

$$\Sigma X = 0 = F_{ih} + 20/3 \times 3/5$$

$$F_{ih} = -4 \text{ t.}$$

**Examples 5.21-5.23** Calculate the forces in all the members of the trusses shown in Figs. 5.30, 5.33 and 5.34 by the method of force coefficients.

Solation : The solutiom of problms 5.21-5.23 are presented in Tables 5.5-5.7 respectively.

Solution 5.21 (Fig. 5.30) :

| Joint | Equations | Member | f | Length | Force |
|---|---|---|---|---|---|
| a | $2 f_{af} = 0$ | af | 0 | 2 | 0 |
| | $2 f_{ae} + 3 = 0$ | ae | $-3/2$ | 2 | $-3$ |
| c | $2 f_{ch} + 1.5 = 0$ | ch | $-3/4$ | 2 | $-3/2$ |
| | $2 f_{ce} = 0$ | ce | 0 | 2 | 0 |
| e | $2 f_{eh} + 2 f_{ef} = 0$ | eh | $-3/4$ | $2\sqrt{2}$ | $-3\sqrt{2}/2$ |
| | $2 f_{eh} - 2 f_{ef} - 2 f_{ea} = 0$ | ef | $3/4$ | $2\sqrt{2}$ | $3\sqrt{2}/2$ |
| h | $4 f_{hd} + 4 f_{hb} - 2 f_{he} - 2 f_{hc} = 0$ | hd | $-9/8$ | 4 | $-9/2$ |
| | $4 f_{hb} + 2 f_{he} = 0$ | hb | $3/8$ | $4\sqrt{2}$ | $3\sqrt{2}/2$ |
| f | $4 f_{fb} + 4 f_{fd} - 2 f_{fe} - 2 f_{fa} = 0$ | fd | $9/8$ | $4\sqrt{2}$ | $9\sqrt{2}/2$ |
| | $-4 f_{fd} - 2 f_{fe} + 6 = 0$ | fb | $-3/4$ | 4 | $-3$ |
| b | $4 f_{bh} + 4 f_{fb} + 1.5 = 0$ | bh | $3/8$ | $4\sqrt{2}$ | $3\sqrt{2}/2$ |
| | $4 f_{bh} + 4 f_{bd} + 3 = 0$ | bd | $-9/8$ | 4 | $-9/2$ |
| d | $4 f_{hd} + 4 f_{fd} = 0$ | | | | |
| | $4 f_{bd} + 4 f_{fd} = 0$ | | | | |

Table 5.5

Note that the last two equations provide a check on the results since they are satisfied when the values of the force coefficients already found are substituted in them.

Solution 5.22 (Fig. 5.33) :

| Joint | Equations | Member | f | Length | Force |
|---|---|---|---|---|---|
| a | $4 f_{ad} + 2 f_{ac} = 0$ | ad | 1 | 5 | 5 |
| | $3 f_{ad} + 6 f_{ac} + 9 = 0$ | ac | $-2$ | $2\sqrt{10}$ | $-4\sqrt{10}$ |
| c | $6 f_{ce} + 2 f_{cd} - 2 f_{ac} = 0$ | ce | $-13/12$ | $3\sqrt{5}$ | $-13\sqrt{5}/4$ |
| | $3 f_{ce} - 3 f_{cd} - 6 f_{ac} - 5 = 0$ | cd | $5/4$ | $\sqrt{13}$ | $5\sqrt{13}/4$ |
| d | $8 f_{dg} + 4 f_{de} - 2 f_{dc} - 4 f_{da} = 0$ | dg | $7/8$ | 8 | 7 |
| | $6 f_{de} + 3 f_{dc} - 3 f_{da} = 0$ | de | $-1/8$ | $2\sqrt{13}$ | $-\sqrt{13}/4$ |

Table 5.6

Solution 5.23 (Fig. 5.34):

| Joint | Equations | Member | f | Length | Force |
|---|---|---|---|---|---|
| e | $2\ f_{ed} - 2\ f_{ec} = 0$ | ed | $-\ 7/4$ | $2\ \sqrt{2}$ | $-\ 7\sqrt{2}/2$ |
| | $2\ f_{ed} + 2\ f_{ec} + 7 = 0$ | ec | $-\ 7/4$ | $2\ \sqrt{2}$ | $-\ 7\sqrt{2}/2$ |
| d | $2\ f_{ed} + 4\ f_{ad} = 0$ | ad | $7/8$ | $5$ | $35/8$ |
| | $3\ f_{ed} - 3\ f_{ad} - 3\ f_{db} = 0$ | bd | $-\ 49/24$ | $3$ | $49/8$ |
| b | $3\ f_{db} + 3\ f_{bc} + 5 = 0$ | bc | $3/8$ | $5$ | $15/8$ |
| | $4\ f_{bc} + 4\ f_{ba} = 0$ | ba | $-\ 3/8$ | $4$ | $-\ 3/2$ |
| a | $4\ f_{ba} + 4\ f_{ad} - 2 = 0$ | ad | $7/8$ | $5$ | $35/8$ |
| | $3\ f_{ad} + 3\ f_{ac} + 2 = 0$ | ac | $-\ 37/24$ | $3$ | $-\ 37/8$ |
| c | $2\ f_{ce} + 4\ f_{bc} + 2 = 0$ | | | | |
| | $-\ 2\ f_{ce} + 3\ f_{bc} + 3\ f_{ac} = 0$ | | | | |

Table 5.7

**Examples 5·24-5·30** Find graphically the forces in all the members of the trusses shown in Figs. 5.37 and 5.48-5.53.

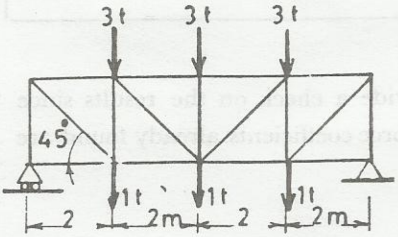



Fig. 5.48

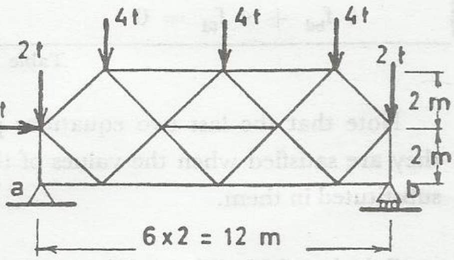



Fig. 5.49

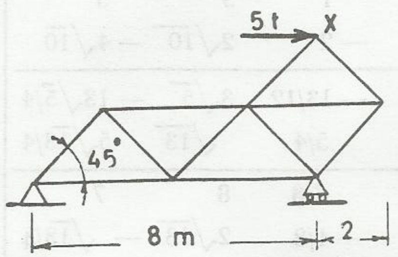



Fig. 5.50

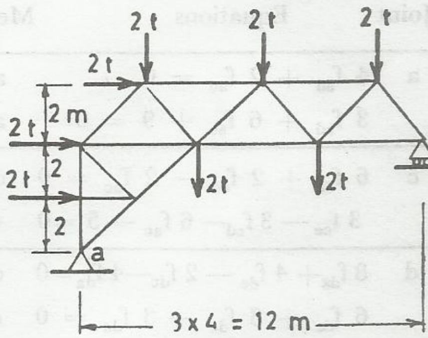



Fig. 5.51

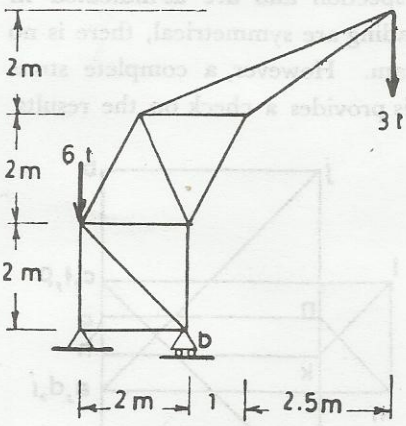


Fig. 5.52

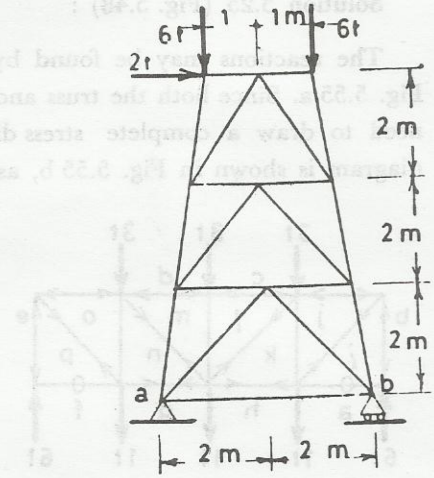


Fig. 5.53

Solution 5.24 (Fig. 5.37) :

Having found the reactions (Example 5.16), and lettering the forces as shown in Fig. 5.54 a, the stress diagram can be drawn as shown in Fig. 5.54 b. The magnitude of the forces can be scaled from the stress diagram, while their type can be noted from the arrows marked on Fig. 5.54 a.

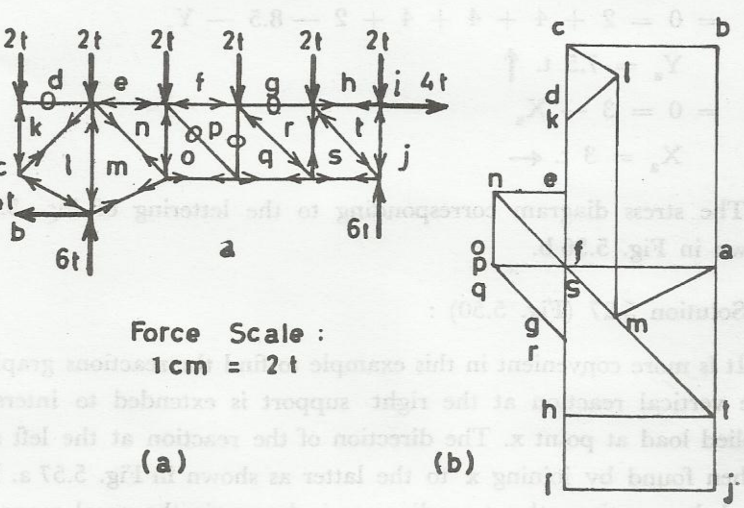


Fig. 5.54

Solution 5.25 (Fig. 5.48) :

The reactions may be found by inspection and are as indicated in Fig. 5.55 a. Since both the truss and loading are symmetrical, there is no need to draw a complete stress diagram. However, a complete stress diagram is shown in Fig. 5.55 b, as this provides a check on the results.

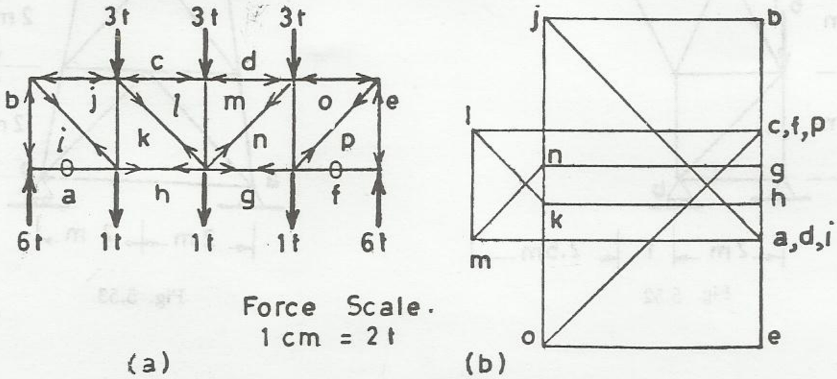



Fig. 5.55

Solution 5.26 (Fig. 5.49) :

$\Sigma M_a = 0 = 2 \times 12 + 4 \times 10 + 4 \times 6 + 4 \times 2 + 3 \times 2 - Y_b \times 12$

$Y_b = 8.5$ t. ↑

$\Sigma Y = 0 = 2 + 4 + 4 + 4 + 2 - 8.5 - Y_a$

$Y_a = 7.5$ t. ↑

$\Sigma X = 0 = 3 - X_a$

$X_a = 3$ t. ←

The stress diagram corresponding to the lettering of Fig. 5.56 a is shown in Fig. 5.56 b.

Solution 5.27 (Fig. 5.50) :

It is more convenient in this example to find the reactions graphically. The vertical reaction at the right support is extended to intersect the applied load at point x. The direction of the reaction at the left support is then found by joining x to the latter as shown in Fig. 5.57 a. Having found the reactions, the stress diagram is drawn in the usual manner and is as shown in Fig. 5.57 b.

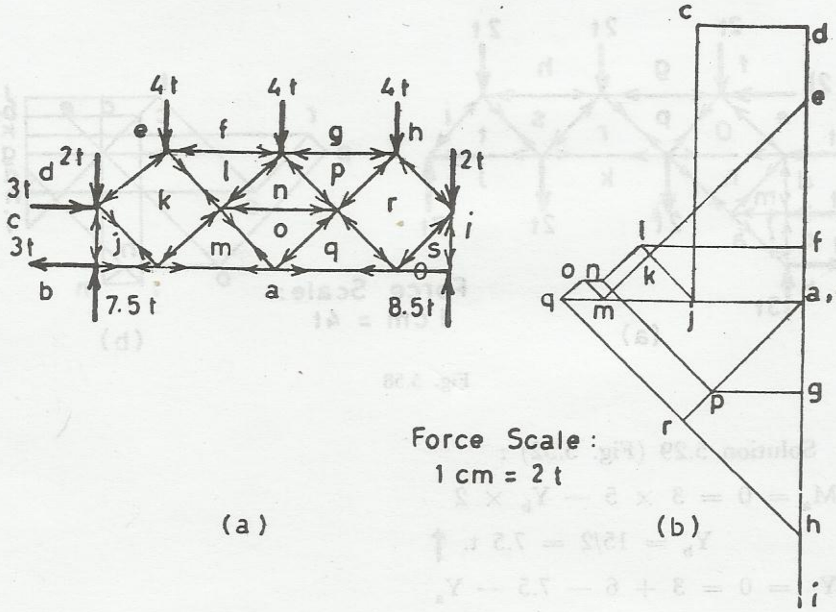



Fig. 5.56

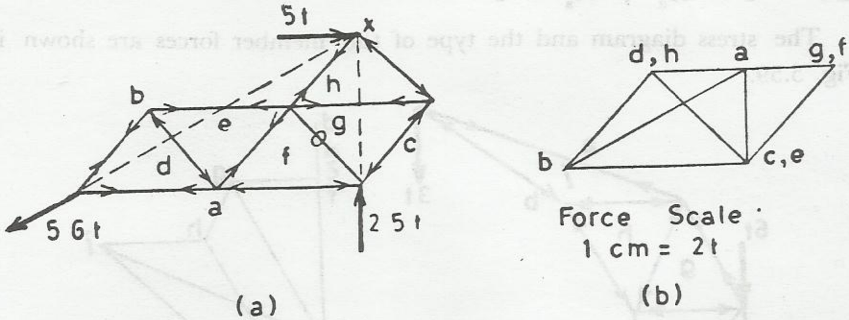



Fig. 5.57

Solution 5.28 (Fig. 5.51) :

$$\Sigma M_a = 0 = 2\,(10 + 8 + 6 + 4 + 2 + 6 + 4 + 2) - Y_b \times 12$$

$$Y_b = 7 \text{ t.} \uparrow$$

$$\Sigma Y = 0 = 2 \times 5 - 7 - Y_a$$

$$Y_a = 3 \text{ t.} \uparrow$$

$$\Sigma X = 0 = 2 \times 3 - X_a$$

$$X_a = 6 \text{ t.} \leftarrow$$

Having found the reactions, and lettering the forces as shown in Fig. 5.58 a, the stress diagram can be drawn as shown in Fig. 5.58 b.

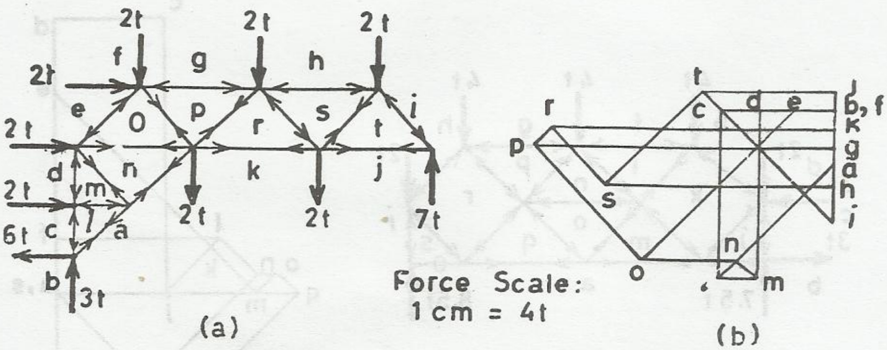



Fig. 5.58

Solution 5.29 (Fig. 5.52) :

$$\Sigma M_a = 0 = 3 \times 5 - Y_b \times 2$$

$$Y_b = 15/2 = 7.5 \text{ t.} \uparrow$$

$$\Sigma Y = 0 = 3 + 6 - 7.5 - Y_a$$

$$Y_a = 1.5 \text{ t.} \uparrow$$

$$\Sigma X = 0 = X_a \ , \ X_a = 0$$

The stress diagram and the type of the member forces are shown in Fig. 5.59.

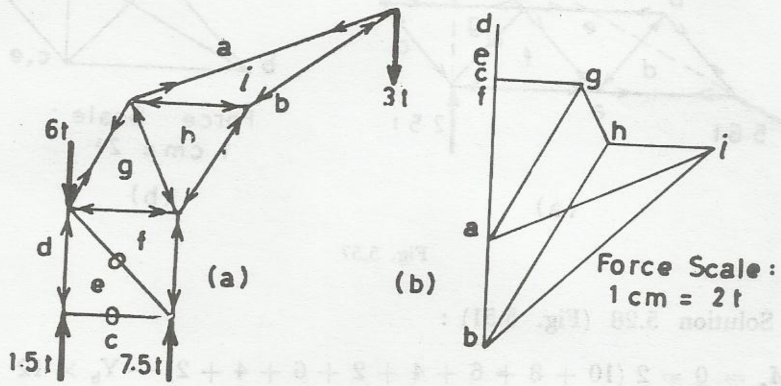



Fig. 5.59

It should be noted that, as for all cantilever trusses, the stress diagram could be drawn without calculating the reactions.

Solution 5.30 (Fig. 5.53) :

$$\Sigma M_a = 0 = 2 \times 6 + 6 \times 1 + 6 \times 3 - Y_b \times 4$$

$$Y_b = 36/4 = 9 \text{ t.} \uparrow$$

$$\Sigma Y = 0 = 6 + 6 - 9 - Y_a$$

$$Y_a = 3 \text{ t.} \uparrow$$

$$\Sigma X = 0 = 2 - X_a$$

$$X_a = 2 \text{ t.} \leftarrow$$

The stress diagram and the type of the member forces are shown in **Fig.** 5.60. Again, it should be noted that the stress diagram in this case could be drawn without calculating the reactions.

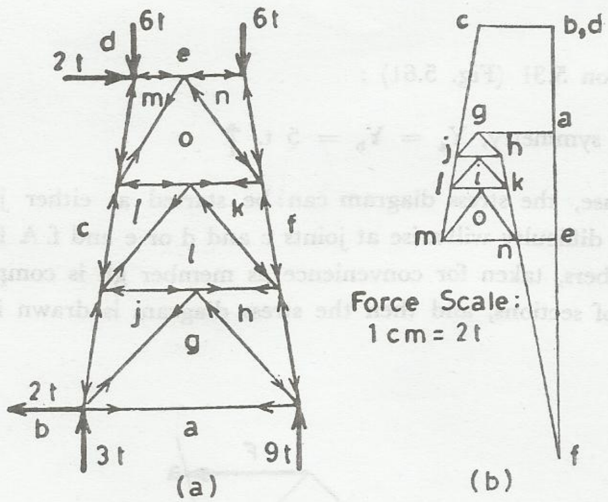



Fig. 5.60

**Example 5·31 - 5·34** Draw the stress diagrams for the trusses shown in Figs. 5.61 - 5.63 and Fig. 5.31.

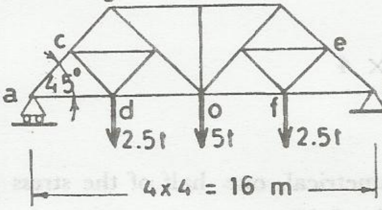



Fig. 5.61

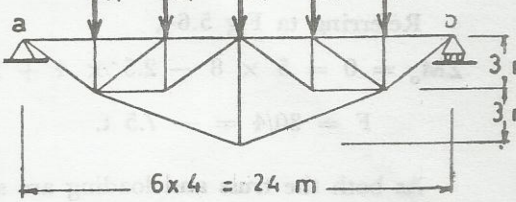



Fig. 5.62

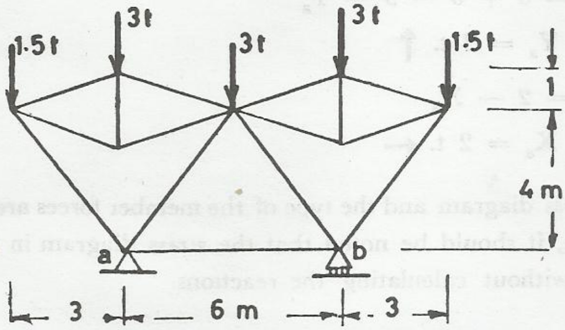



Fig. 5.63

Solution 5.31 (Fig. 5.61) :

From symmetry, $Y_a = Y_b = 5$ t. ↑

In this case, the stress diagram can be started at either joints a or b. However difficulty will arise at joints c and d or e and f. A force in one of the members, taken for convenience as member gh is computed by the method of sections, and then the stress diagram is drawn in the usual manner.

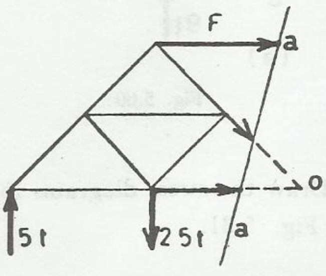



Fig. 5.64

Referring ta Fig 5.64,

$\Sigma M_o = 0 = 5 \times 8 - 2.5 \times 4 + F \times 4$

$F = 30/4 = -7.5$ t.

As both the truss and loading are symmetrical, one half of the stress diagram is enough. This is shown in Fig. 5.65.

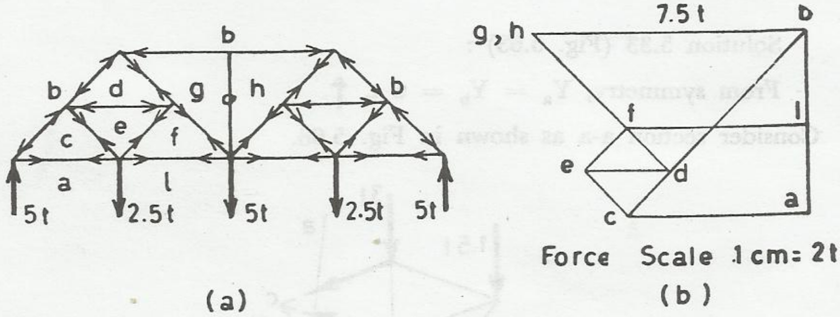



Fig. 5.65

Solution 5.32 (Fig. 5.62) :

From symmetry, $Y_a = Y_b = 5$ t. ↑

Consider section a-a as shown in Fig. 5.66.

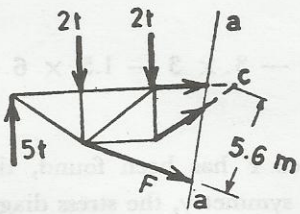



Fig. 5.66

$$\Sigma M_c = 0 = 5 \times 12 - 2\ (8 + 4) - F \times 5.6$$

$$F = 36/5.6 = 6.42 \text{ t.}$$

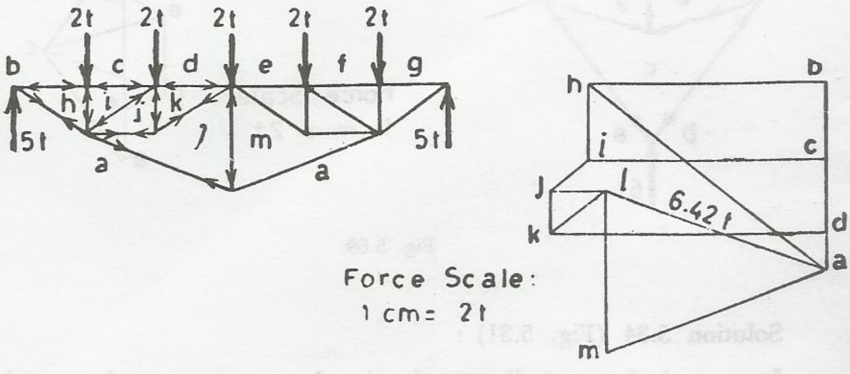



Fig. 5.67

The stress diagram for half the truss and the type of member forces are shown in Fig. 5.67.

Solution 5.33 (Fig. 5.63) :

From symmetry, $Y_a = Y_b = 6$ t. ↑

Consider section a-a as shown in Fig. 5.68.

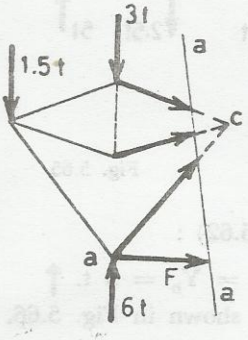



Fig. 5.68

$$\Sigma M_c = 0 = 6 \times 3 - 3 \times 3 - 1.5 \times 6 - F \times 4$$
$$F = 0$$

Now that the force F has been found, the stress diagram can be started at joint a. For symmetry, the stress diagram for half the truss only is given. This is shown in Fig. 5.69.

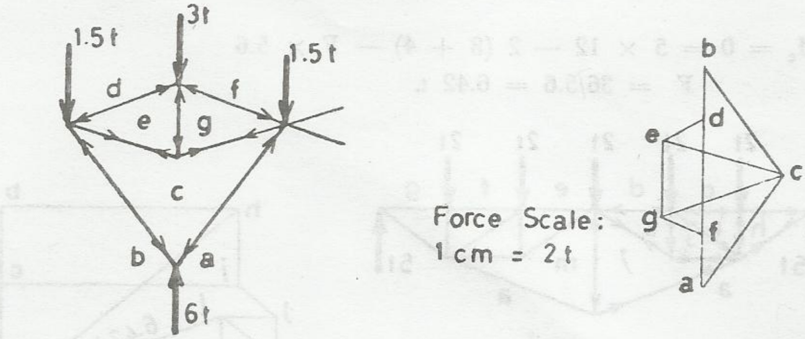



Fig. 5.69

Solution 5.34 (Fig. 5.31) :

In general, the stress diagram for the given truss cannot be completed without calculating one of the member forces. However, for the particular case of loading considered it could be seen that members ij, ik, kh, hi and jc have zero forces. Thus, no difficulty in drawing the stress diagram will arise if it is started at joint b.

The reactions having already been calculated, the stress diagram will be as shown in Fig. 5.70.

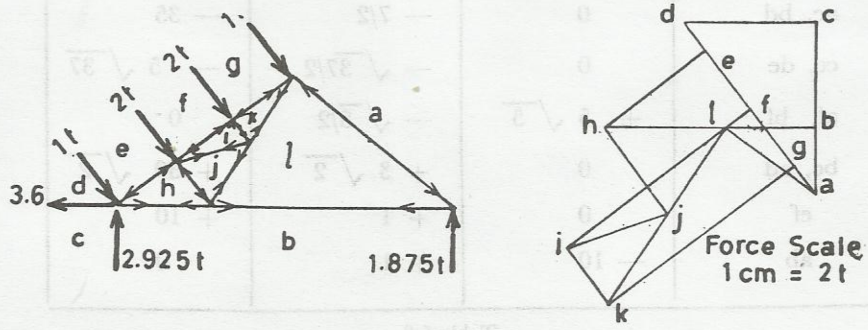



Fig. 5.70

**Example 5·35** Calculate the forces in all the members of the complex truss shown in Fig. 5.71.

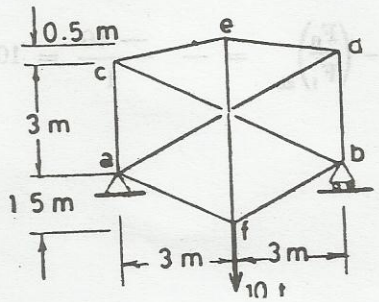



Fig. 5.71

Solution 5.35 :

Member ef is removed and a substitute member ab is added as shown in Fig. 5.72. The details of the calculations are listed in Table 5.8

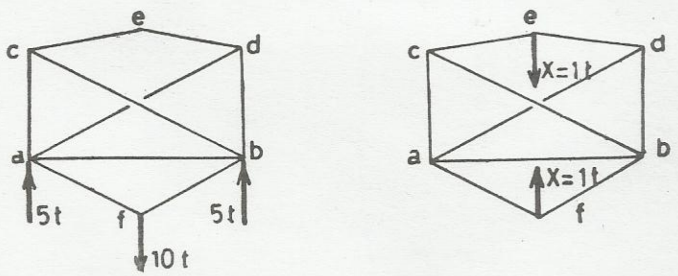



Fig. 5.72

| Member | $F_0$ | $F_1$ | $F$ |
|---|---|---|---|
| ac, bd | 0 | $-\ 7/2$ | $-\ 35$ |
| ce, de | 0 | $-\sqrt{37}/2$ | $-\ 5\sqrt{37}$ |
| af, bf | $+\ 5\sqrt{5}$ | $-\sqrt{5}/2$ | 0 |
| bc, ad | 0 | $+\ 3\sqrt{2}$ | $+\ 30\sqrt{2}$ |
| ef | 0 | $+\ 1$ | $+\ 10$ |
| ab | $-\ 10$ | $+\ 1$ | 0 |

Table 5.8

The factor X is found from the condition that the final force in the substitute member ab should be zero. Thus,

$$X = -\left(\frac{F_0}{F_1}\right)_{ab} = -\ \frac{-\ 10}{1} = 10$$

## EXAMPLES TO BE WORKED OUT

(1) - (21) Find graphically the forces in all the members of the trusses shown in Figs. 5.73-5.93.

Check analytically the forces in the marked members.

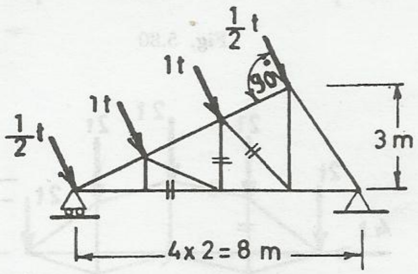



Fig. 5.73

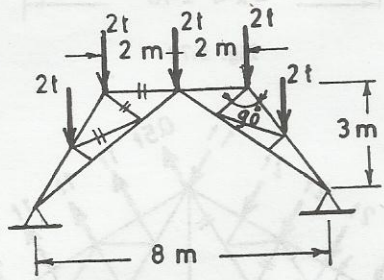



Fig. 5.74

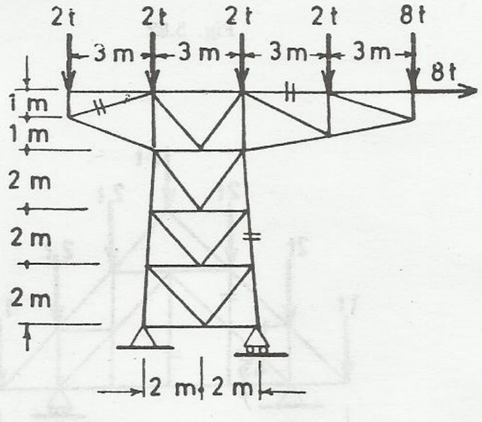



Fig. 5.75

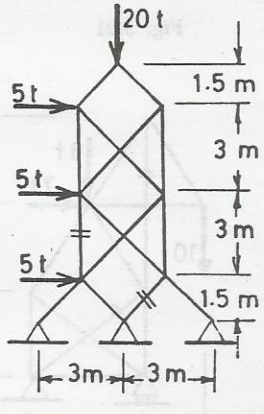



Fig. 5.76

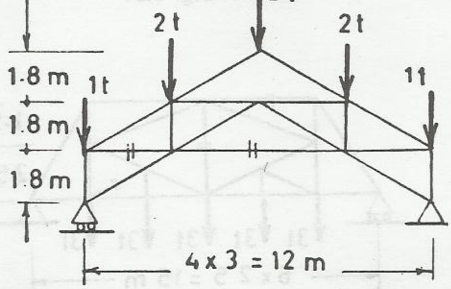



Fig. 5.77

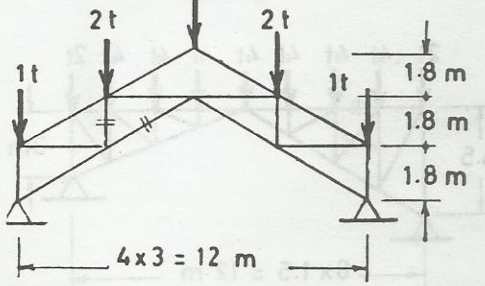



Fig. 5.78

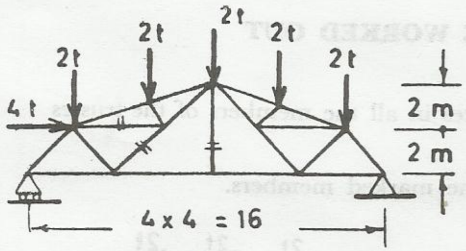


Fig. 5.79

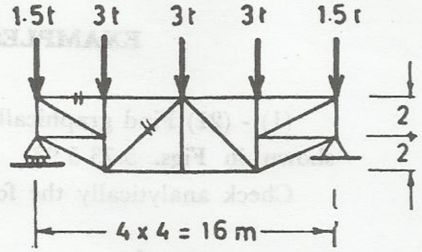


Fig. 5.80

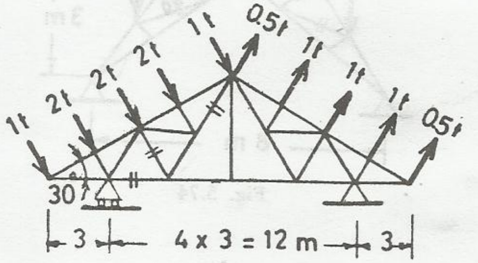


Fig. 5.81

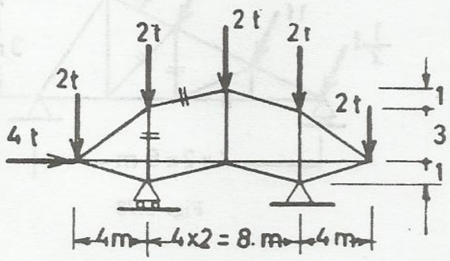


Fig. 5.82

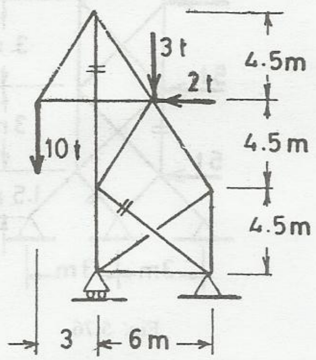


Fig. 5.83

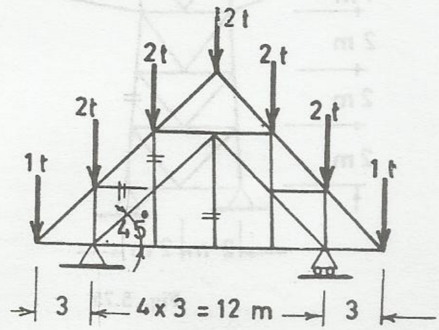


Fig. 5.84

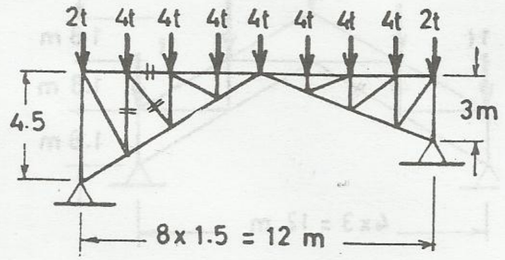


Fig. 5.85

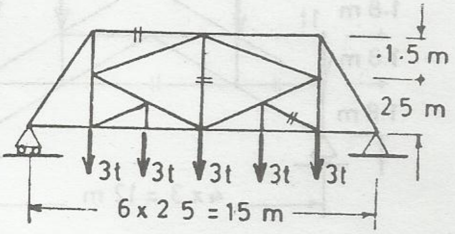


Fig. 5.86

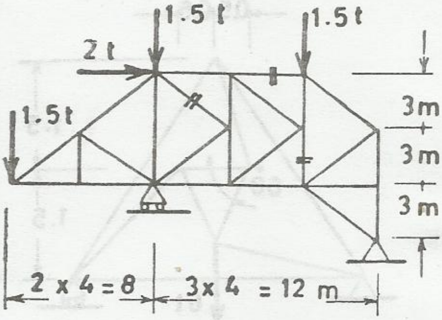



Fig. 5.87

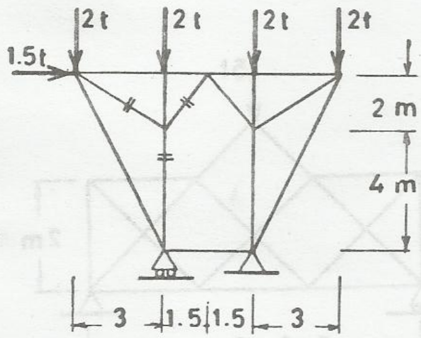



Fig. 5.88

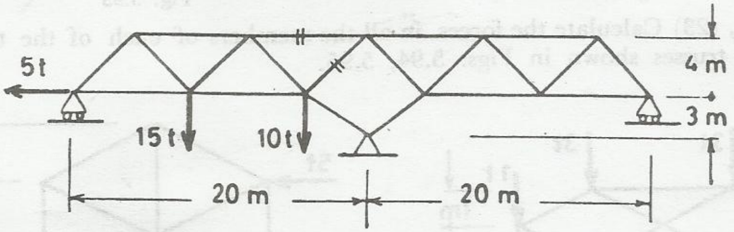



Fig. 5.89

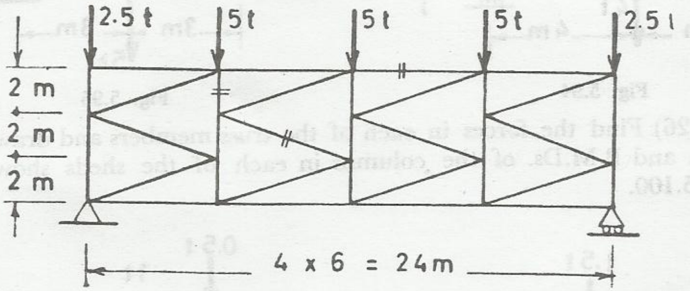



Fig. 5.90

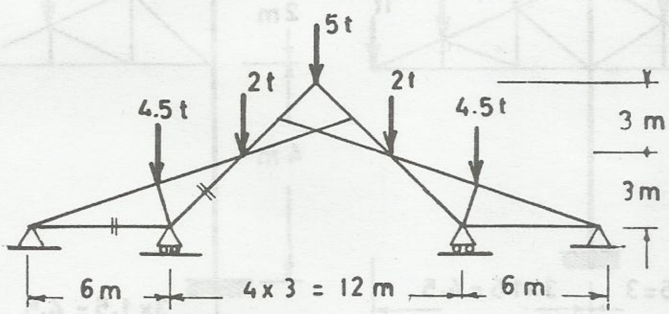



Fig. 5.91

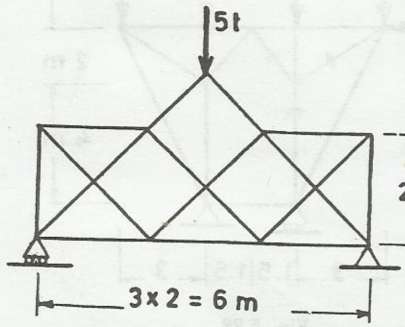



Fig. 5.92

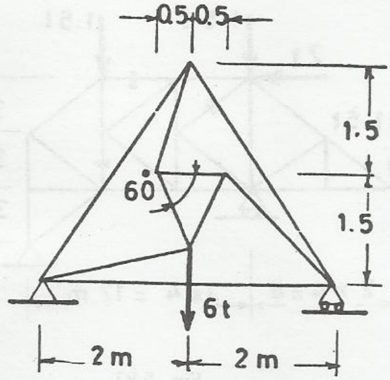



Fig. 5.93

(22) , (23) Calculate the forces in all the members of each of the two complex trusses shown in Figs. 5.94, 5.95.

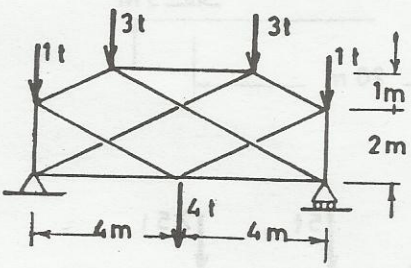



Fig. 5.94

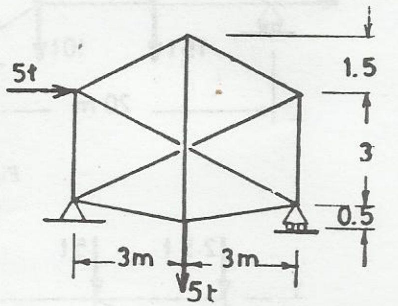



Fig. 5.95

(24) , (26) Find the forces in each of the truss members and draw the N.F., S.F. and B.M.Ds. of the columns in each of the sheds shown in Figs. 5.96-5.100.

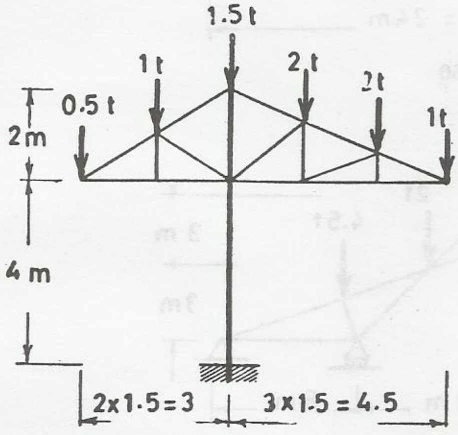



Fig. 5.96

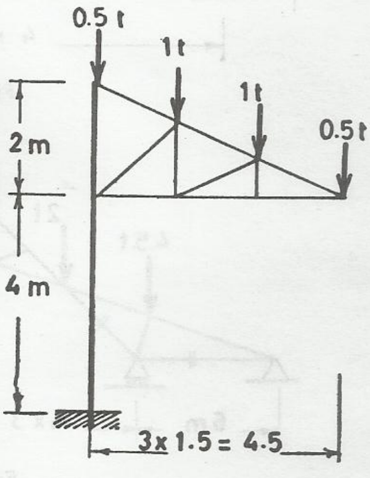



Fig. 5.97

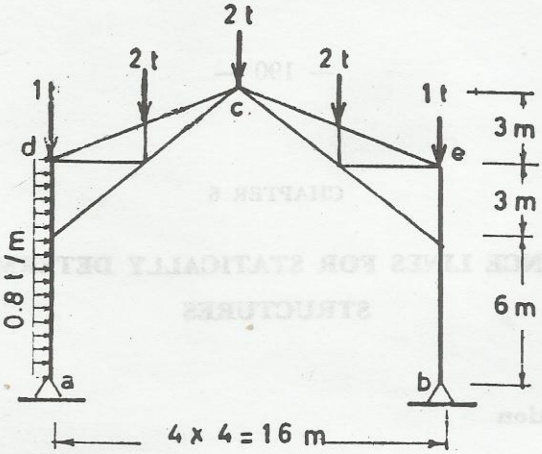



Fig. 5.98

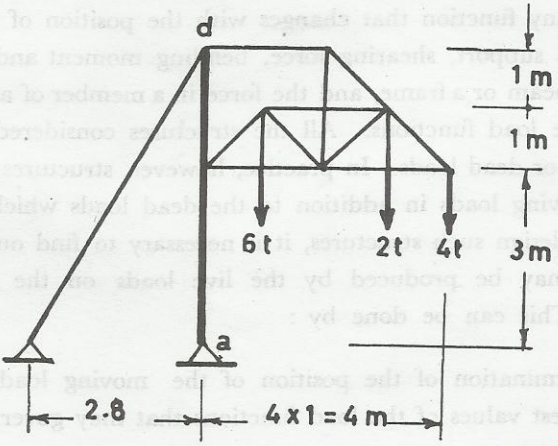



Fig. 5.99

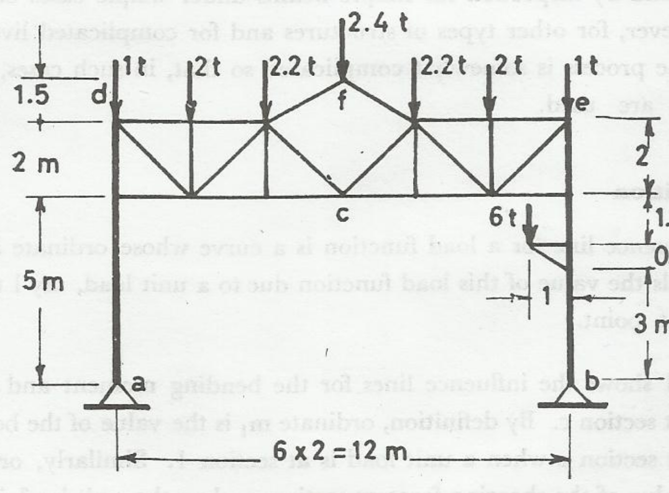



Fig. 5.100

CHAPTER 6

# INFLUENCE LINES FOR STATICALLY DETERMINATE STRUCTURES

## 6.1 Introduction

In chapters 2 - 5, it has been shown how to find the values of various *load functions* in statically determinate structures. By a load function, it is meant any function that changes with the position of the load. A reaction at a support, shearing force, bending moment and thrust at a section of a beam or a frame, and the force in a member of a truss are examples of the load functions. All the structures considered were subjected to fixed or dead loads. In practice, however, structures may be subjected to moving loads in addition to the dead loads which are always acting. To design such structures, it is necessary to find out the biggest effect that may be produced by the live loads on the various load functions. This can be done by :

(1) Determination of the position of the moving loads which will give the largest values of the load functions that may govern the design.

(2) Calculation of the values of the load functions required. These may be found by inspection for simple beams under simple cases of loading. However, for other types of structures and for complicated live load systems, the process is somewhat complicated so that, in such cases, influence lines are used.

## 6.2 Definition

An influence line for a load function is a curve whose ordinate at any point equals the value of this load function due to a unit load, say 1 t., acting at that point.

Fig. 6.1 shows the influence lines for the bending moment and shearing force at section c. By definition, ordinate $m_1$ is the value of the bending moment at section c when a unit load is at section 1. Similarly, ordinate $q_1$ is the value of the shearing force at section c when the unit load is at 1.

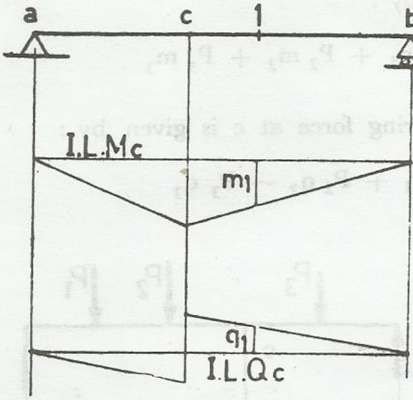



Fig. 6.1

It is essential to understand the difference between the infiuence lines for shear or moment at a section in a beam and the shearing force and bending moment diagrams for the beam. A shear, or moment, diagram is drawn for a condition in which all the loads are fixed in position on the structure, and it gives the values of the shear, or moment, at all the sections of the structure for the particular loading position considered. On the other hand, an influence line for shear, or moment, at a section is drawn when the loading consists of a unit load moving across the structure, and gives the value of the shear, or moment, at one particular point for various positions of the unit load on the structure.

**6.3 Properties of the influence lines**

Since the ordinate of an influence line at a section equals the value of a particular function for which the influence line is constructed when a unit load is placed at that section, the following general rules hold :

(1) The value of a function due to a single concentrated load equals the product of the load and the ordinate of the influence line of that function at the point of application of the load. Further, the total value of a function due to a number of concentrated loads is the sum of the products of each load and the ordinate of the influence line at the position of the load. This follows from the principle of superposition.

Thus, referring to Fig. 6.2, the moment at c due to the shown load system is given by :

$$M_c = P_1 m_1 + P_2 m_2 + P_3 m_3$$

Similarly, the shearing force at c is given by :

$$Q_c = P_1 q_1 + P_2 q_2 - P_3 q_3$$

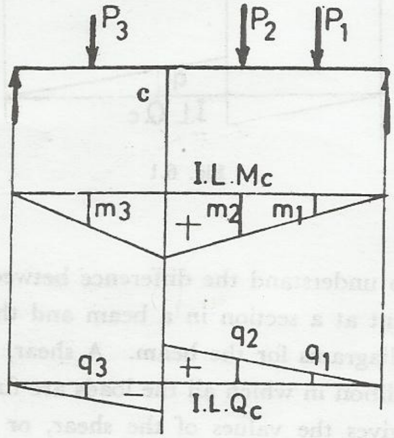



Fig. 6.2

(2) In order to obtain the maximum value of a function, due to a single concentrated load, the load should be placed at the section where the ordinate of the influence line for that function is a maximum. It is obvious that if the maximum positive value of a function is required, the load should be placed at the point where the ordinate of the influence line has its maximum positive value. On the other hand, if the maximum negative value is required, the load should be placed at the point where the ordinate of the influence line has its maximum negative value.

Thus, referring to Fig. 6.3, the moment at point c due to a single concentrated load P is a maximum when P is at c. Similarly, the shearing force at c has its maximum positive value when the load P is just to the right of c, and its maximum negative value when P is just to the left of c.

(3) The value of a load function due to uniformly distributed load is equal to the product of the load intensity and the net area, under the

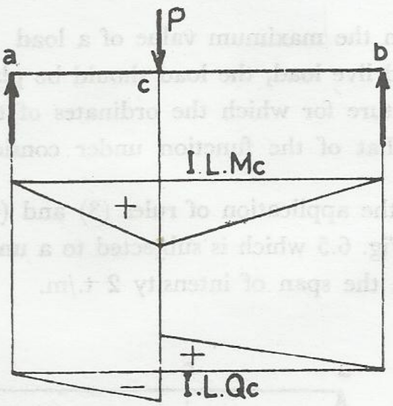



Fig. 6.3

influence line of the considered function, that corresponds to the loaded part.

This may be easily proved with reference to Fig. 6.4.

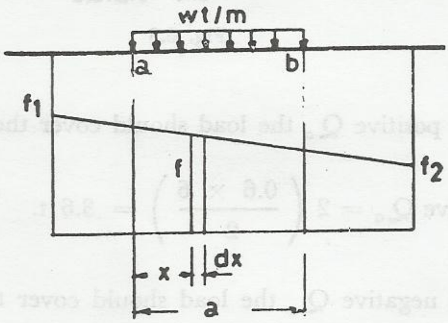



Fig. 6.4

Let $f_1$ $f_2$ be the influence line of some load function F for a part of a structure loaded along part ab as shown. The part of the uniform load applied in distance dx may be considered as a concentrated load of magnitude wdx, and from rule (1), the value of the function due to this elemental concentrated load is given by :

$$dF = wdx\ f$$

The total value of the function F due to the shown load is then obtained by integrating dF between x = 0 and x = a.

Thus, $$F = \int_o^a wdx\ f = w \int_o^a f\ dx$$

$$= w \times \text{area under the influence line in part ab.}$$

(4) To obtain the maximum value of a load function due to a uniformly distributed live load, the load should be placed so as to cover the part of the structure for which the ordinates of the influence line have similar sign as that of the function under consideration.

To illustrate the application of rules (3) and (4), consider the simple beam shown in Fig. 6.5 which is subjected to a uniformly distributed live load longer than the span of intensity 2 t./m.

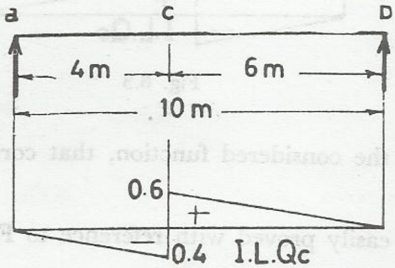



Fig. 6.5

For maximum positive $Q_c$ the load should cover the part bc. Then,

$$\text{max. positive } Q_c = 2\left(\frac{0.6 \times 6}{2}\right) = 3.6 \text{ t.}$$

For maximum negative $Q_c$ the load should cover the part ac. Then,

$$\text{max. negative } Q_c = 2\left(\frac{-0.4 \times 4}{2}\right) = -1.6 \text{ t.}$$

If the load covers the entire length of the beam, the shear at c will be given by :

$$Q_c = 2\left(\frac{0.6 \times 6}{2} - \frac{0.4 \times 4}{2}\right) = 2 \text{ t.}$$

## 6.4 Extreme values

On designing any part of a structure, it is necessary to determine the greatest possible values; maximum positive and maximum negative, of various load functions due to combinations of the dead load which is always acting, and the live loads placed in such a way so as to produce maximum effects. The values of the load functions resulting from these combinations are called the extreme values of these functions. The extreme values of a load function, therefore, define upper and lower limits of the range within which the load function considered may vary.

Referring to the simple beam shown in Fig. 6.5, if it carries a uniformly distributed dead load of 1 t./m in addition to a uniformly distributed live load of 2 t./m, the extreme values of the shear at section c will be found as follows :

$$Q_c \text{ (due to D.L.)} = 1 \left( \frac{0.6 \times 6}{2} - \frac{0.4 \times 4}{2} \right) = 1 \text{ t.}$$

$$\text{max. positive } Q_c \text{ (due to L.L.)} = 2 \left( \frac{0.6 \times 6}{2} \right) = 3.6 \text{ t.}$$

$$\text{max. negative } Q_c \text{ (due to L.L.)} = 2 \left( \frac{-0.4 \times 4}{2} \right) = -1.6 \text{ t.}$$

The extreme values for the shearing force at c are thus given by :

max. positive $Q_c = 1 + 3.6 = 4.6$ t.

max. negative $Q_c = 1 - 1.6 = -0.6$ t.

## 6.5 Construction of influence lines for simply supported beams

Consider a unit load crossing a simply supported beam as shown in Fig. 6.6 a. For the load at a distance x from a, the reactions are given by :

$$Y_a = \frac{(L - x)}{L} \quad \text{and} \quad Y_b = \frac{x}{L}$$

It is noted that the variation of both $Y_a$ and $Y_b$ is linear with x. At $x = 0$, $Y_a = 1$ and at $x = L$, $Y_a = 0$. Similarly, $Y_b = 1$ at $x = L$ and $Y_b = 0$ at $x = 0$. The influence lines for the reactions at a and b are therefore as shown in Figs. 6.6 b and c. The positive sign indicates that the reactions are upward.

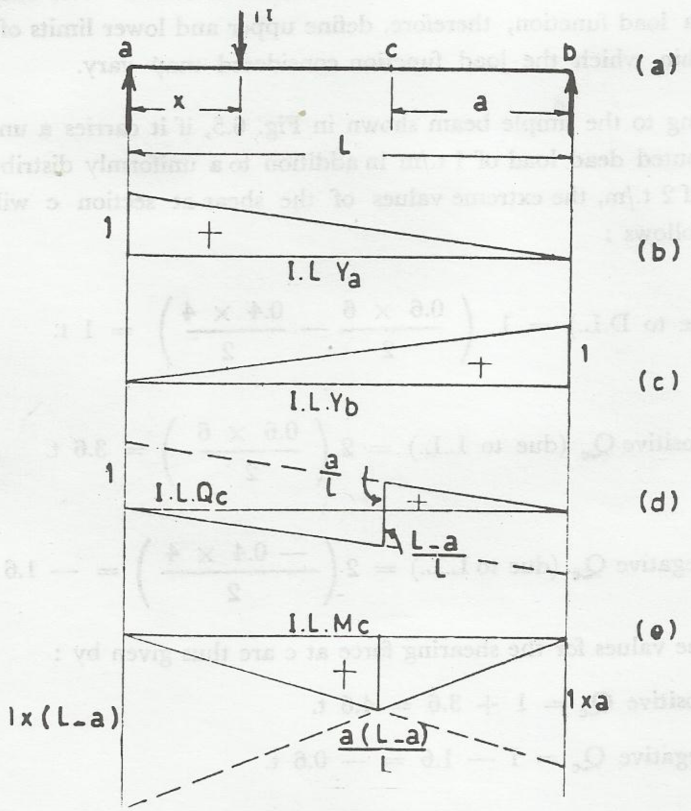



Fig. 6.6

To construct the influence line for the shearing force at any given section, consider section c at distance a from the right suport as shown in Fig. 6.6a As the load moves from a to c, i.e. to the left of c, the shearing force at c is negative and is equal to the reaction at b, i.e. $Q_c = -x/L$ As the load crosses c, the shearing force is no longer equal to the reaction at b but to $(1 - Y_b)$, or more conveniently, to $Y_a$. Thus as the unit load moves from c to b, i.e. to the right of c, the shearing force at c is positive and is equal to the reaction at a, i.e. $Q_c = +(L - x)/L$ The influence line for the shearing force at c is thus as shown in Fig. 6.6. d. The critical ordinates at c are readily obtainable from similar triangles.

To construct the influence line for the bending moment at any given section, consider section c shown in Fig. 6.6 a. As the load moves to the right of c the bending moment at c is positive and equal to $Y_a$ (L — a), or $M_c = \frac{(L - x)}{L} (L - a)$. As the load moves to the left of c, the bending moment at c is positive and equal to $Y_b \times a$, or $M_c = \frac{x}{L} \times a$.

The influence line for the bending moment at c is thus as shown in Fig. 6.6 e. The maximum ordinate at c can be calculated either from similar triangles or from the first principles due to the action of a load of 1 t. acting at c, $M_c = \frac{a (L - a)}{L}$

It should be noticed that the influence lines of all the load functions considered are linear functions of the distance x. This is characteristic of all statically determinate structures. Therefore, in order to construct the influence line for any load function of any statically determinate structure only main ordinates are calculated and then connected by straight lines to form the required influence line. For instance, for the influence line for the reaction $Y_a$, the unit load is first placed at a and then at b. When the load is in the former position, $Y_a = 1$ and when it is in the latter position, $Y_a = 0$. After obtaining these main ordinates, a straight line is drawn between them to form the influence line shown in Fig. 6.6 b. Further, it is often more convenient in computing the ordinates of the influence lines to work from the forces on the side of the section which is away from the unit moving load.

## 6.6 Maximum S.F. and B.M. at a given section in a simple beam

When a simple beam is subjected to a single concentrated load or a uniformly distributed live load longer than the span, the position of the load to give maximum value of shearing force and bending moment at a given section can be easily found with the aid of influence lines by the application of the rules given in section 6.3. There are another two important cases of loading that need to be considered. These are :

(1) **Uniformly distributed load shorter than the span.**

From the influence line for the shearing force at section c shown in Fig. 6.7 b, it can be seen that the maximum positive shearing force at c occurs when the head of the load is at c and the load covers the part of the span to the right of the section as shown in Fig. 6.7 c, and the maximum negative value occurs when the end of the load is at c and the load covers the part of the span to the left of the section as shown in Fig. 6.7 d.

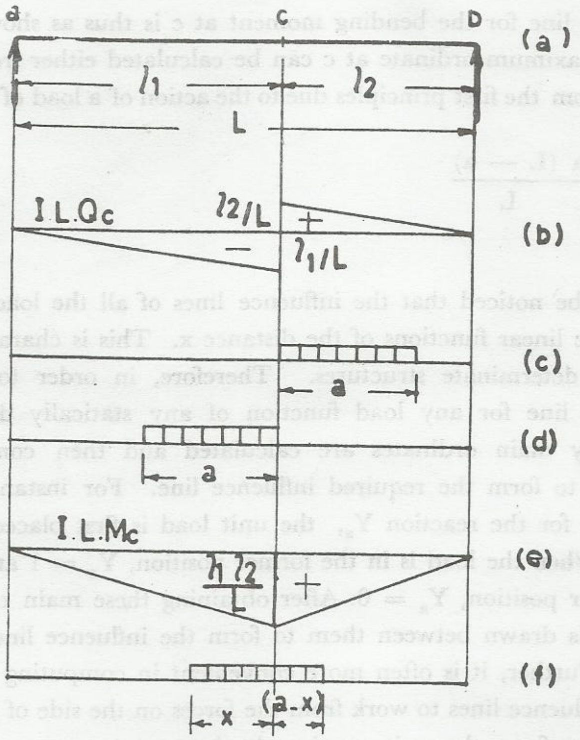



Fig. 6.7

From consideration of the influence line for the bending moment at c shown in Fig. 6.7 e, it is obvious that the load should be placed as shown in Fig. 6.7 f covering a part of the span on either side of c. It can be proved that for maximum moment at section c the section should divide the load in the same ratio as it divides the span. Thus referring to Fig. 6.7 f, the maximum moment occurs at c when:

$$\frac{x}{l_1} = \frac{a - x}{l_2}$$

(2) Series of concentrated loads at fixed distances apart.

When a series of concentrated loads crosses a span the maximum positive shearing force at a given section c will generally occur when the load at the head of the series is just to the right of c and the rest of the loads are all to its right as shown in Fig. 6.8 c. Similarly, the maximum negative shearing force will occur when the end load is at c and the rest of the loads are all to the left of the section as shown in Fig. 6.8 d.

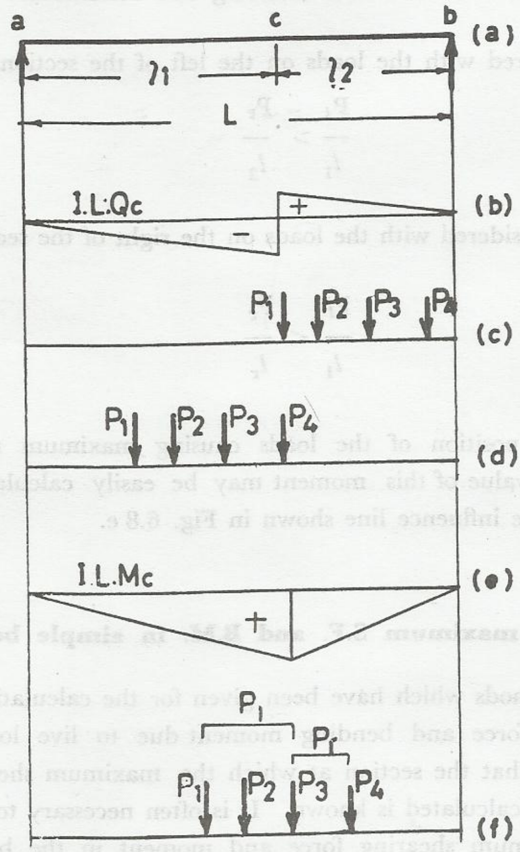



Fig. 6.8

While this may be generally true, it is possible sometimes, particularly when the first load or loads are light compared to the loads that follow, to have maximum values when the first load or loads have actually passed the section. In such cases, the values of the shearing force for the various possible load positions may be calculated and the biggest value is the maximum required.

Referring to Fig. 6.8 f, the maximum bending moment occurs at c when a load P is at c. This is because the bending moment diagram for a series of concentrated loads consists of a number of straight lines intersecting at the positions of the loads. It can be proved that the load P gives the required position of the loads on the beam to cause maximum bending moment at c if it satisfies the following two conditions :

if P is considered with the loads on the left of the section,

$$\frac{P_l}{l_1} > \frac{P_r}{l_2}$$

and if it is considered with the loads on the right of the section,

$$\frac{P_l}{l_1} < \frac{P_r}{l_r}$$

Once the position of the loads causing maximum moment at c is known, the value of this moment may be easily calculated from the ordinates of the influence line shown in Fig. 6.8 e.

### 6.7 Absolute maximum S.F. and B.M. in simple beams

In the methods which have been given for the calculation of maximum shearing force and bending moment due to live load systems it was assumed that the section at which the maximum shearing force or moment to be calculated is known. It is often necessary to calculate the absolute maximum shearing force and moment in the beam. By the absolute maximum shearing force and moment, it is meant the biggest value that can occur at any section of the beam under a given live load system.

The absolute maximum positive shearing force in a simple beam occurs at the left support, while the absolute maximum negative shearing force occurs at the right support.

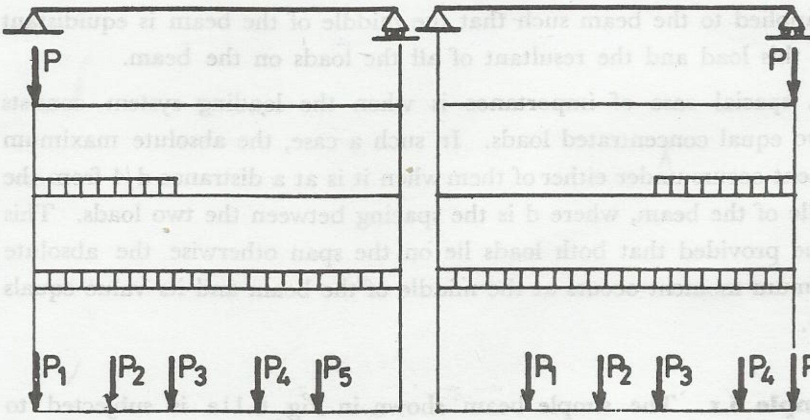



Fig. 6.9 Fig. 6.10

Fig. 6.9 shows the load positions for various live load systems; single concentrated load, uniformly distributed load shorter than the span, uniformly distributed load longer than the span and a series of concentrated loads, to produce absolute maximum positive shearing force. Fig. 6.10 shows the load positions to cause absolute maximum negative shearing force. These load positions correspond also to the maximum upward reactions at the supports.

The absolute maximum moment occurs at mid span under either a single concentrated load or a uniformly distributed load. Under a series of concentrated loads, the section where absolute maximum moment occurs is not so obvious.

As mentioned before, since the bending moment diagram for a series of concentrated loads consists of a number of straight lines intersecting at the positions of the loads, then the absolute maximum moment occurs under one of the loads. Before calculating the absolute maximum moment, two questions must be answered; under which load does the absolute maximum moment occur ? and what is the position of this load on the span to produce absolute maximum moment ?

The answer to the first question is determined by trial, but the second is found by the following rule :

The absolute maximum moment occurs under one of the loads which are applied to the beam such that the middle of the beam is equidistant from this load and the resultant of all the loads on the beam.

A special case of importance is when the loading system consists of two equal concentrated loads. In such a case, the absolute maximum moment occurs under either of them when it is at a distrance d/4 from the middle of the beam, where d is the spacing between the two loads. This is true provided that both loads lie on the span otherwise the absolute maximum moment occurs at the middle of the beam and its value equals PL/4.

**Example 6.1** The simple beam shown in Fig. 6.11a is subjected to a uniformly distributed live load of intensity 2 t./m. In addition, the beam carries a uniformly distributed dead load of intensity 1 t./m. Determine for section c :

(1) the shearing force and bending moment due to dead load.

(2) the maximum positive and maximum negative shearing force. and also the maximum moment due to live load.

(3) the extreme values for the shearing force and moment.

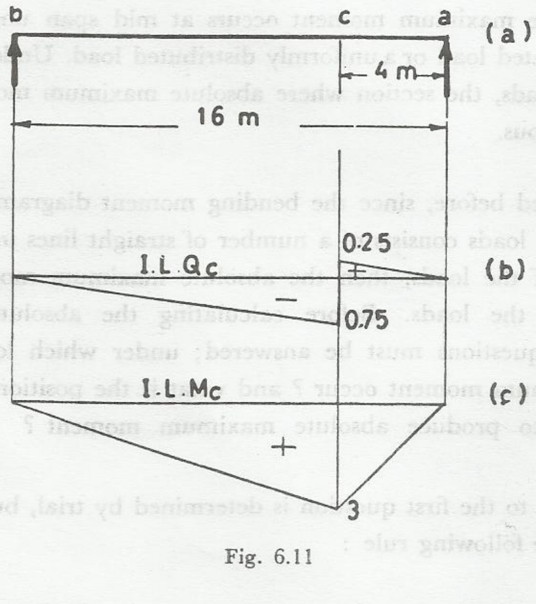



Fig. 6.11

Solution : The influence lines for the shear and moment at c are shown in Figs. 6.11 b and c.

$$Q_c \text{ (due to D.L.)} = 1 \left( \frac{0.25 \times 4}{2} - \frac{0.75 \times 12}{2} \right) = -4 \text{ t.}$$

$$M_c \text{ (due to D.L.)} = 1 \times \frac{16 \times 3}{2} = 24 \text{ m.t.}$$

Maximum positive $Q_c$ due to L.L. occurs with the load covering part ac and is equal to :

$$2 \times \frac{0.25 \times 4}{2} = 1 \text{ t.}$$

Maximum negative $Q_c$ due to L.L. occurs with the load covering part bc and is equal to :

$$-2 \times \frac{0.75 \times 12}{2} = -9 \text{ t.}$$

Maximum positive $M_c$ occurs with the load covering all the span and is equal to :

$$2 \times \frac{16 \times 3}{2} = 48 \text{ m.t.}$$

Extreme values for $Q_c$ :

$$-4 + 1 = -3 \text{ t.}$$

$$-4 - 9 = -13 \text{ t.}$$

Extreme values for $M_c$ :

24 m.t.

24 + 48 = 72 m.t.

**Example 6.2** A simply supported beam is subjected to the load system shown in Fig. 6.12 a. Using the influence lines, find the positions of the loads to cause maximum positive and maximum negative shearing force and maximum moments at points c and d. Calculate these values.

Solution : The influence lines for the shear and moment at points c and d are shown in Figs. 6.12 b - e.

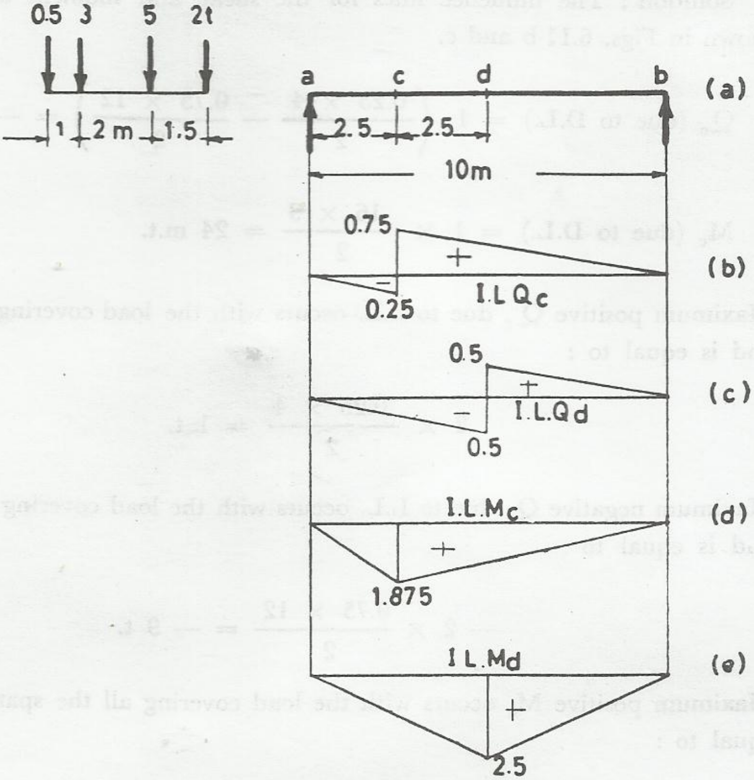



Fig. 6.12

Maximum positive $Q_c$ may occur with the first load just to the right of c, but since the first load is light compared to the loads that follow, there is a possibility that the maximum positive shear may occur with the second load (3 t.) just to the right of c. In such a case $Q_c$ is calculated for both positions and the bigger value is considered.

For 0.5 t. at c,

$$Q_c = 0.5 \times 0.75 + 3 \times 0.65 + 5 \times 0.45 + 2 \times 0.3 = 5.175 \text{ t.}$$

For 3 t. at c,

$$Q_c = 3 \times 0.75 + 5 \times 0.55 + 2 \times 0.4 - 0.5 \times 0.15 = 5.725 \text{ t.}$$

Thus, the maximum positive $Q_c$ occurs with 3 t. just to the right of c and is equal to 5.725 t.

Maximum negative $Q_c$ occurs with 2 t. just to the left of c.

Maximum negative $Q_c = -0.25 \times 2 - 0.1 \times 5 = -1.0$ t.

It should be noted that with the 2 t. load just to the left of c, both the 3 t. and 0.5 t. loads are off the span.

For maximum positive $Q_d$ there are two possible positions of the load; 0.5 t. load just to the right of d and 3 t. load just to the right of d. It will be found by trial that the latter position gives maximum positive $Q_d$ and its value is given by :

$$\text{Maximum positive } Q_d = 0.5 \times 3 + 0.3 \times 5 + 0.15 \times 2 - 0.4 \times 0.5 = 3.1 \text{ t.}$$

Maximum negative $Q_d$ occurs with 2 t. just to the left of d.

$$\text{Maximum negative } Q_d = -0.5 \times 2 - 0.35 \times 5 - 0.15 \times 3 - 0.05 \times 0.5 = -3.225 \text{ t.}$$

For maximum positive $M_c$, apply the rule given in section 6.6. With the 3 t. load at c it will be found that :

$$3.5/2.5 > 7/7.5 \quad \text{and} \quad 0.5/2.5 < 10/7.5$$

Thus, the maximum moment occurs with the load of 3 t. at c and its value is found from the influence line in Fig. 6.12 d.

$$\text{Maximum positive } M_c = 0.5 \times 1.125 + 3 \times 1.875 + 5 \times 1.375 + 2 \times 1 = 15.06 \text{ m.t.}$$

For maximum positive $M_d$, it will be found, with the load of 5 t. at d, that :

$$8.5/5 > 2/5 \quad \text{and} \quad 3.5/5 < 7/5$$

Thus, the maximum moment occurs with the load of 5 t. at d and its value is found from the influence line in Fig. 6.12 e.

$$\text{Maximum positive } M_d = 0.5 \times 1 + 3 \times 1.5 + 5 \times 2.5 + 2 \times 1.75 = 21 \text{ m.t.}$$

**Example 6.3** The simple beam shown in Fig. 6.13 a is subjected to the given load system. Determine the position of the loads causing absolute maximum moment in the beam and then using the influence lines calculate its value. Find also the equivalent uniformly distributed live load longer than the span which will cause the same absolute maximum bending moment.

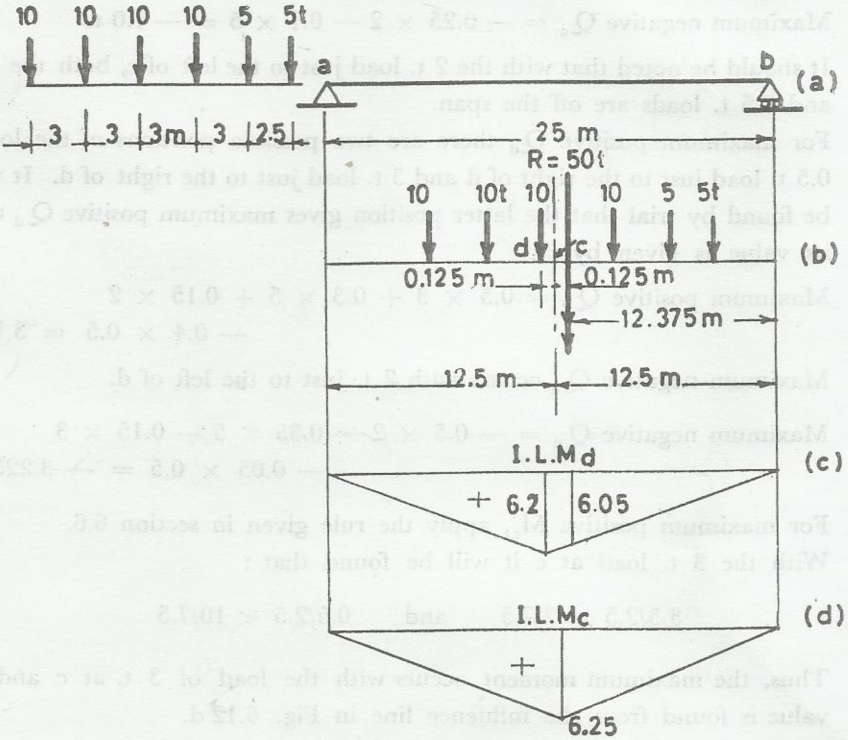



Fig. 6.13

Solution :

Applying the rule given in section 6.7, the absolute maximum moment occurs at d with the position of the load as shown in Fig. 6.13 b. The influence line for the moment at d is shown in Fig. 6.13 c, from which,

Absolute maximum $M_d = 50 \times 6.05 = 302.5$ m.t.

Under a uniformly distributed load, the absolute maximum moment occurs at c. Let the intensity of the equivalent uniformly distributed live load be w, and using the influence line in Fig. 6.13 d then,

$$\frac{25 \times 6.25}{2} \times w = 302.5$$

$$w = 3.872 \text{ t./m.}$$

## 6.8 Construction of influence lines for overhanging beams

Consider the beam shown in Fig. 6.14 a. The influence lines for the reactions will first be constructed.

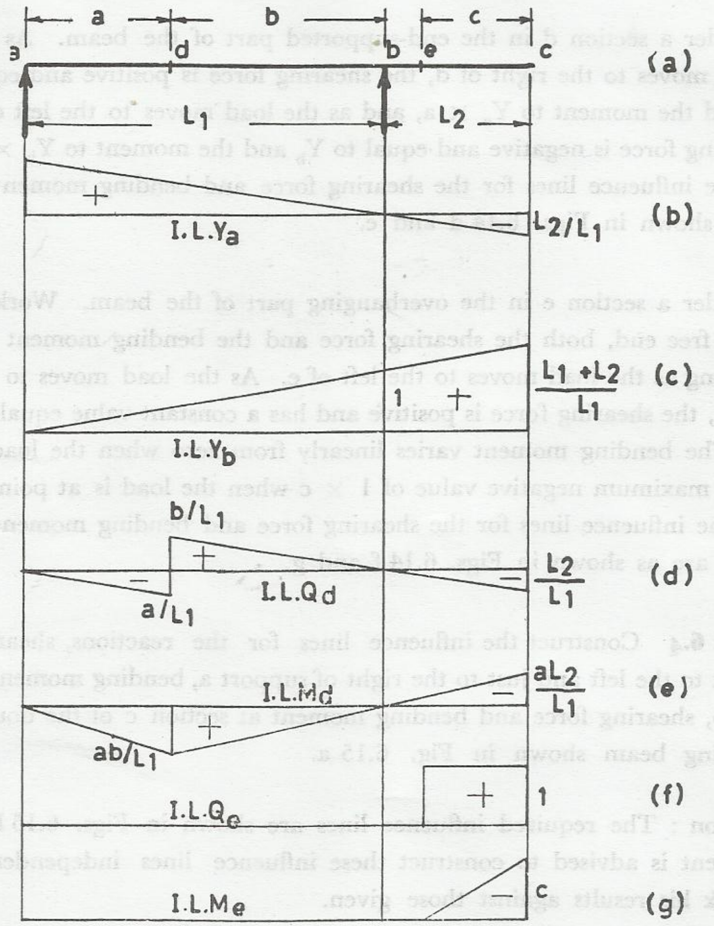



Fig. 6.14

As a unit load moves from a to b, the reaction at a is as for a simple beam. When the load passes b, the reaction at a is negative, i.e. downward. When the load is at c, $Y_a = -1 \times L_2 / L_1$. Hence, the influence line for the reaction at a is as shown in Fig. 6.14 b. Similarly, as a unit load moves from a to b, the reaction at b is as for a simple beam. When the load passes b, the reaction is still upward. When the load is at c, $Y_b = 1 \times \frac{L_1 + L_2}{L_1}$. Hence, the influence line for $Y_b$ is as shown in Fig. 6.14 c.

Consider a section d in the end-supported part of the beam. As the unit load moves to the right of d, the shearing force is positive and equal to $Y_a$, and the moment to $Y_a \times a$, and as the load moves to the left of d the shearing force is negative and equal to $Y_b$ and the moment to $Y_b \times b$. Hence the influence lines for the shearing force and bending moment at d are as shown in Figs. 6.14 d and e.

Consider a section e in the overhanging part of the beam. Working from the free end, both the shearing force and the bending moment are zero as long as the load moves to the left of e. As the load moves to the right of e, the shearing force is positive and has a constant value equals to unity. The bending moment varies linearly from zero when the load is at e to a maximum negative value of $1 \times c$ when the load is at point c. Hence, the influence lines for the shearing force and bending moment at section e are as shown in Figs. 6.14 f and g.

**Example 6.4** Construct the influence lines for the reactions, shearing forces just to the left and just to the right of support a, bending moment at support b, shearing force and bending moment at section c of the double overhanging beam shown in Fig. 6.15 a.

Solution : The required influence lines are shown in Figs. 6.15 b-h. The student is advised to construct these influence lines independently and check his results against those given.

## 6.9 Construction of influence lines for cantilever beams

The procedures adopted for constructing the influence lines for various load functions in simple and overhanging beams can be easily applied to cantilever beams. A unit load may be applied at a number of points along the beam where a sudden change of the load function under consideration is expected and the ordinates calculated at these points are then connected by straight lines to orm the required influence line. It should

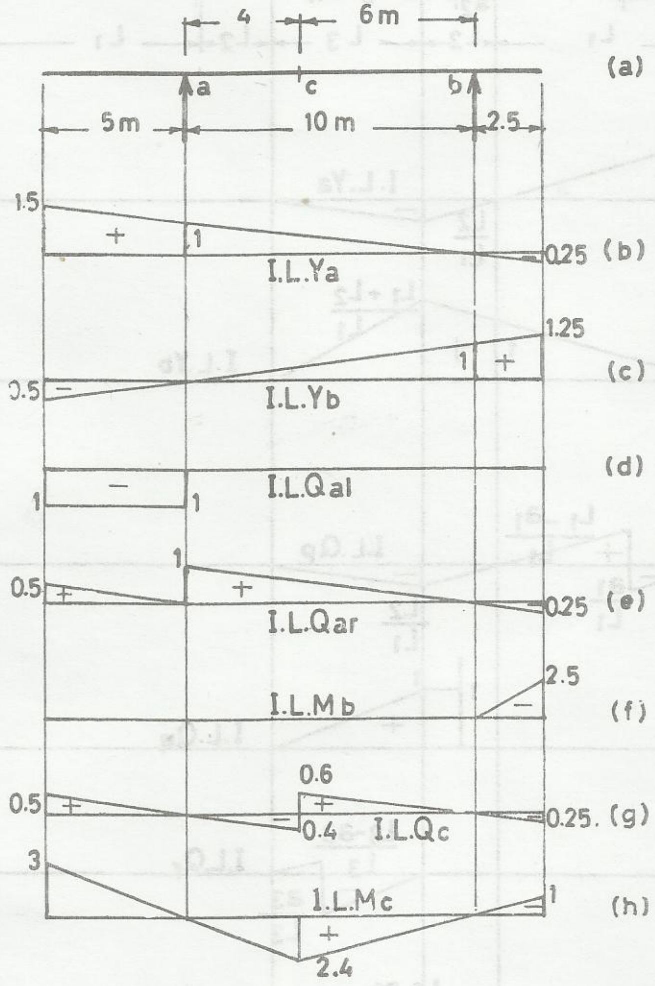
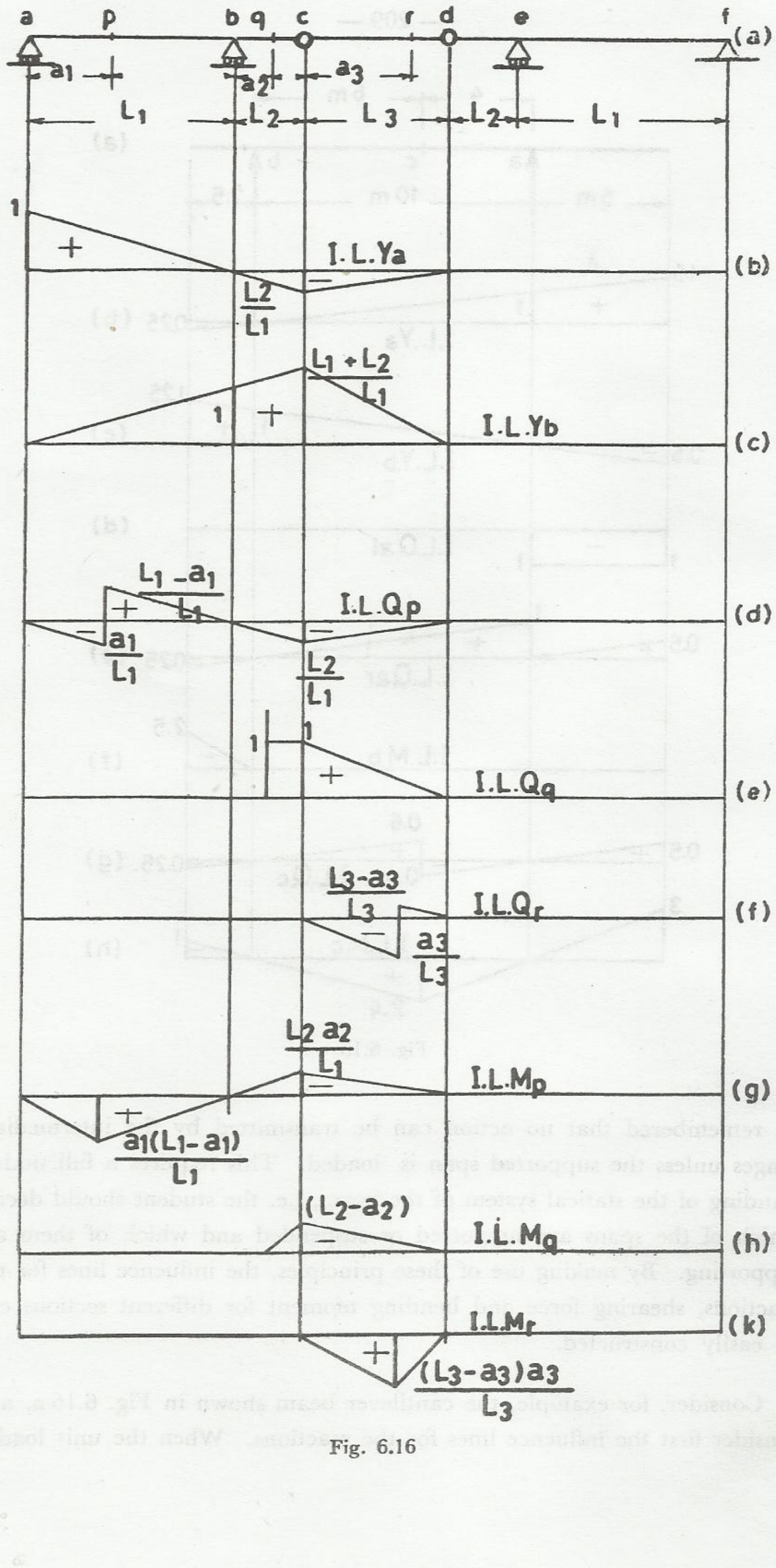



Fig. 6.15

be remembered that no action can be transmitted by the intermediate hinges unless the supported span is loaded. This requires a full understanding of the statical system of the beam, i.e. the student should decide which of the spans are supported or suspended and which of them are supporting. By making use of these principles, the influence lines for the reactions, shearing force and bending moment for different sections can be easily constructed.

Consider, for example, the cantilever beam shown in Fig. 6.16 a, and consider first the influence lines for the reactions. When the unit load is



a
p
b
q
c
r
d
e
f
(a)
a1
a2
a3
L1
L2
L3
L2
L1
1
+
I.L.Ya
(b)
−
L2/L1
L1 + L2/L1
1
+
I.L.Yb
(c)
L1 − a1/L1
+
I.L.Qp
(d)
−
a1/L1
−
L2/L1
1
1
+
I.L.Qq
(e)
L3-a3/L3
I.L.Qr
(f)
−
a3/L3
L2 a2/L1
−
I.L.Mp
(g)
+
a1(L1- a1)/L1
(L2-a2)
−
I.L.Mq
(h)
I.L.Mr
(k)
+
(L3-a3)a3/L3


Fig. 6.16

at a, $Y_a = 1$ while $Y_b = 0$. When the unit load is at b, $Y_b = 1$ while $Y_a = 0$. If the unit load is at c, the reactions at a and b are respectively :

$$Y_a = -\frac{L_2}{L_1} \quad \text{and} \quad Y_b = \frac{L_1 + L_2}{L_1}$$

When the unit load is at d, both $Y_a$ and $Y_b$ are zero, and as long as the unit load moves outside part ad of the beam the reactions $Y_a$ and $Y_b$ will remain zero. Hence, the influence lines for $Y_a$ and $Y_b$ are as shown in Figs. 6.16 b and c.

Next consider the influence lines for the shearing forces at points p (in the end-supported span), q (in the overhang) and r (in the supported or suspended span).

As the unit load moves to the right of point p, the shearing force at p is positive and is equal to $Y_a$, and as the load moves to the left of p, the shearing force is negative and is equal to $Y_b$. Hence the influence line for the shearing force at p is as shown in Fig. 6.16 d.

As the unit load moves from a to q, $Q_q = 0$ as for a section in the overhang of the overhanging beam considered in section 6.8. As the load moves between q and c, the shearing force is positive and has a constant value of unity. When the unit load is at d, $Q_q = 0$ as the load will be carried only by the right supporting span. When the load moves from d to f, it will have no effect on the shear at q. Thus, the influence line for the shearing force at q is as shown in Fig. 6.16 e.

The influence line for the shearing force at section r is the same as that of a simple span cd. As long as the load is outside span cd, it will have no effect on the shearing force at r. Hence, the influence line for the shearing force at r is as shown in Fig. 6.16 f.

Next consider the influence lines for the bending moments at points p, q and r.

As the unit load moves to the right of point p, the bending moment at p is equal to $Y_a a_1$. When the unit load is at a or any section along part df, the bending moment at p is zero. Hence, the influence line for the bending moment at p is as shown in Fig. 6.16 g.

As the unit load moves from a to q, the bending moment at q is zero, as is the case in the overhanging beam. When the unit load is at c, the bending moment at q is equal to $(L_2 - a_2)$, and when it is at d or beyond, the bending moment is zero. The influence line for the bending moment at q is thus as shown in Fig. 6.16 h.

The influence line for the bending moment at r is as for a simple span cd and is as shown in Fig. 6.16 k.

**Example 6.5** For the cantilever beam shown in Fig. 6.17 a, construct the influence lines for the reactions, the shearing force and bending moment at sections p, q and r.

Using these diagrams, calculate the extreme values of the shearing force and bending moment at section r due to a dead load of 2 t./m. and a live load of any length of 4 t./m.

Solution :

The required influence lines are shown in Figs. 6.17 b-1.

$$Q_r \text{ (due to D.L.)} = 2\left(\frac{0.25 \times 6}{2} + \frac{0.25 \times 2}{2} - \frac{0.75 \times 6}{2} - \frac{0.25 \times 2}{2}\right) = -3 \text{ t.}$$

$$\text{max. positive } Q_r \text{ (due to L.L.)} = 4\left(\frac{0.25 \times 6}{2} + \frac{0.25 \times 2}{2}\right) = 4 \text{ t.}$$

$$\text{max. negative } Q_r \text{ (due to L.L.)} = 4\left(\frac{-0.75 \times 6}{2} - \frac{0.25 \times 2}{2}\right) = -10 \text{ t.}$$

The extreme values for $Q_r$ are : — 13 t. and 1 t.

From the influence line for $M_r$ (Fig. 6.17*l*)

$$M_r \text{ (due to D.L.)} = 2\left(\frac{1.5 \times 8}{2} - \frac{0.5 \times 6}{2} - \frac{1.5 \times 2}{2}\right) = 6 \text{m.t.}$$

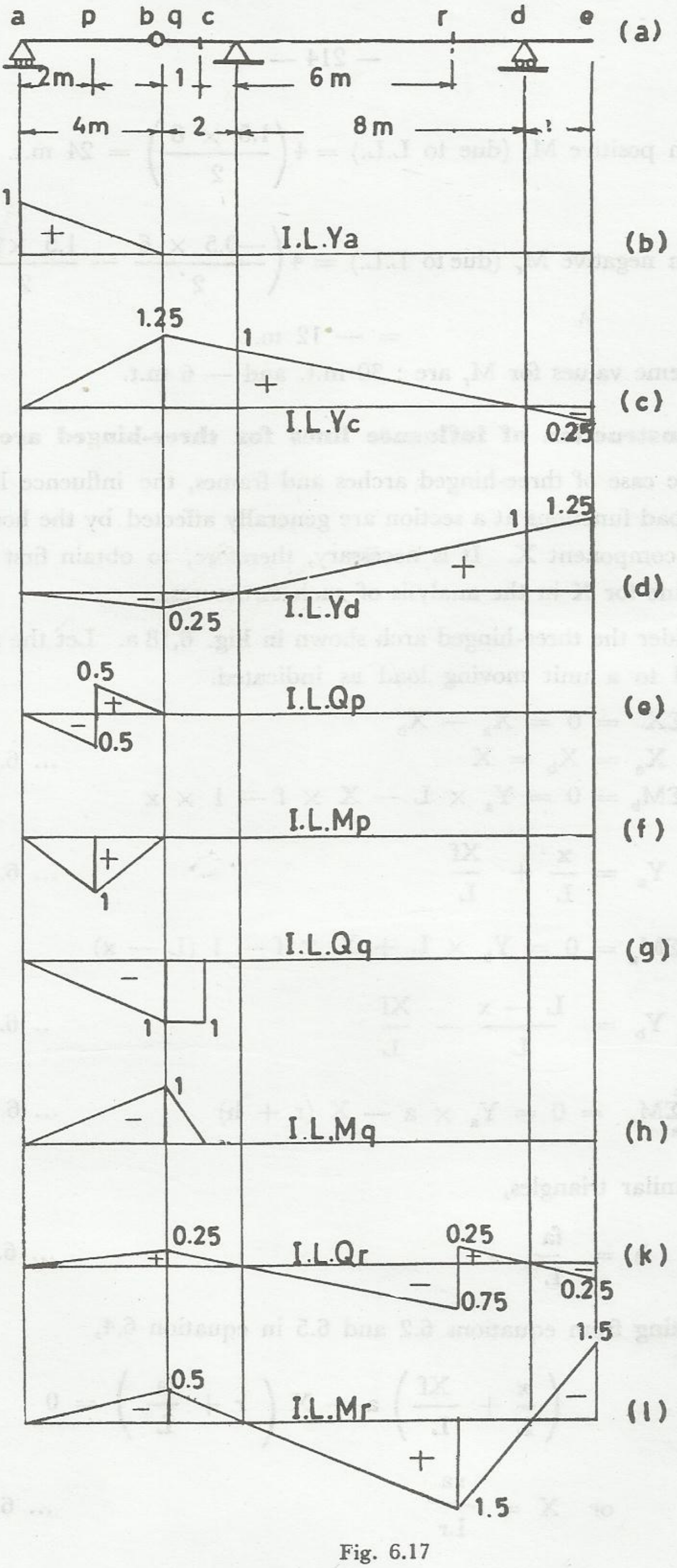

a p b q c r d e
(a)
2m
1
6 m
4m
2
8 m
1
I.L.Ya
(b)
1.25
1
I.L.Yc
0.25
(c)
1.25
1
0.25
I.L.Yd
(d)
0.5
0.5
I.L.Qp
(e)
I.L.Mp
1
(f)
I.L.Qq
1
1
(g)
1
I.L.Mq
(h)
0.25
I.L.Qr
0.25
0.75
0.25
(k)
0.5
I.L.Mr
1.5
1.5
(l)

Fig. 6.17

maximum positive $M_r$ (due to L.L.) $= 4\left(\frac{1.5 \times 8}{2}\right) = 24$ m.t.

maximum negative $M_r$ (due to L.L.) $= 4\left(\frac{-0.5 \times 6}{2} - \frac{1.5 \times 2}{2}\right)$

$= -12$ m.t.

The extreme values for $M_r$ are : 30 m.t. and $-$ 6 m.t.

### 6.10 Construction of influence lines for three-hinged arches

In the case of three-hinged arches and frames, the influence lines for various load functions at a section are generally affected by the horizontal reaction component X. It is necessary, therefore, to obtain first the influence line for X in the analysis of such structures.

Consider the three-hinged arch shown in Fig. 6,18 a. Let the arch be subjected to a unit moving load as indicated.

$$\Sigma X = 0 = X_a - X_b$$

$$X_a = X_b = X \qquad \text{... 6.1}$$

$$\Sigma M_b = 0 = Y_a \times L - X \times f - 1 \times x$$

$$Y_a = \frac{x}{L} + \frac{Xf}{L} \qquad \text{... 6.2}$$

$$\Sigma M_a = 0 = Y_b \times L + X \times f - 1\,(L - x)$$

$$Y_b = \frac{L - x}{L} - \frac{Xf}{L} \qquad \text{.. 6.3}$$

$$\sum_a^c M_c = 0 = Y_a \times a - X\,(r + h) \qquad \text{... 6.4}$$

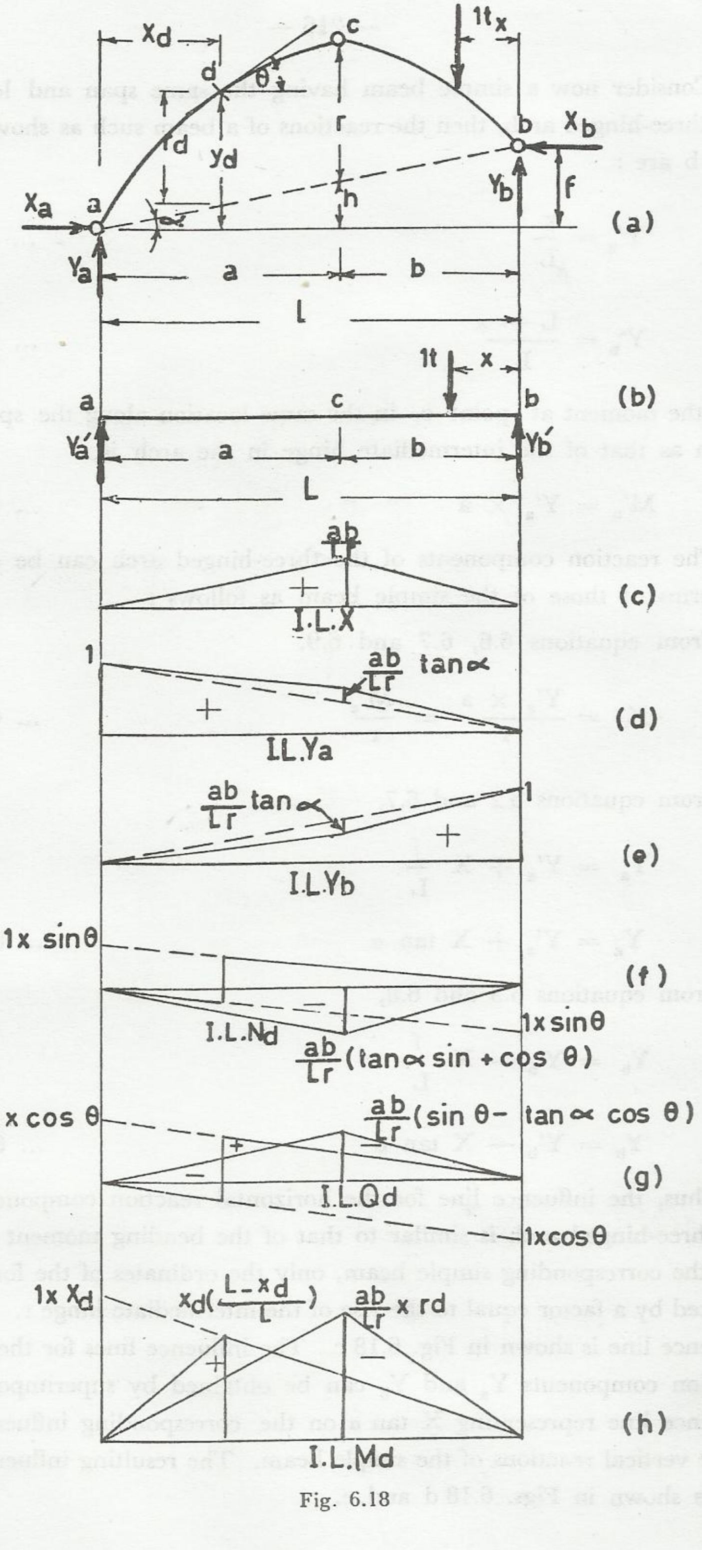
From similar triangles,

$$h = \frac{fa}{L} \qquad \text{... 6.5}$$

Substituting from equations 6.2 and 6.5 in equation 6.4,

$$\left(\frac{x}{L} + \frac{Xf}{L}\right) a - X\left(r + \frac{fa}{L}\right) = 0$$

$$\text{or} \quad X = \frac{xa}{L.r} \qquad \text{... 6.6}$$



$x_d$
c
$1t_x$
d
θ
r
$x_b$
b
$r_d$
$y_d$
$Y_b$
f
$X_a$
a
h
α
(a)
$Y_a$
a
b
l
1t
x
a
c
b
(b)
$Y_a'$
a
b
$Y_b'$
L
$\frac{ab}{Lr}$
+
(c)
I.L.X
1
$\frac{ab}{Lr}$ tan α
+
(d)
I.L.Ya
$\frac{ab}{Lr}$ tan α
1
+
(e)
I.L.Yb
1x sinθ
(f)
−
I.L.Nd
1x sinθ
$\frac{ab}{Lr}$(tan α sin + cos θ)
1x cos θ
$\frac{ab}{Lr}$(sin θ − tan α cos θ)
+
−
−
(g)
I.L.Qd
1xcosθ
1x $X_d$
$x_d(\frac{L-x_d}{L})$
$\frac{ab}{Lr}$ x rd
−
+
(h)
I.L.Md


Fig. 6.18

Consider now a simple beam having the same span and loading as the three-hinged arch, then the reactions of a beam such as shown in Fig. 6.18 b are :

$$Y'_a = \frac{x}{L} \qquad \text{... 6.7}$$

$$Y'_b = \frac{L - x}{L} \qquad \text{... 6.8}$$

and the moment at point c, in the same location along the span of the beam as that of the intermediate hinge in the arch is,

$$M'_c = Y'_a \times a \qquad \text{... 6.9}$$

The reaction components of the three-hinged arch can be expressed in terms of those of the simple beam as follows :

From equations 6.6, 6.7 and 6.9.

$$X = \frac{Y'_a \times a}{r} = \frac{M'_c}{r} \qquad \text{... 6.10}$$

From equations 6.2 and 6.7,

$$Y_a = Y'_a + X \frac{f}{L}$$

or

$$Y_a = Y'_a + X \tan \alpha \qquad \text{... 6.11}$$

From equations 6.3 and 6.8,

$$Y_b = Y'_b - X \frac{f}{L}$$

or

$$Y_b = Y'_b - X \tan \alpha \qquad \text{... 6.12}$$

Thus, the influence line for the horizontal reaction component X of the three-hinged arch is similar to that of the bending moment at point c of the corresponding simple beam, only the ordinates of the former are reduced by a factor equal to the rise of the intermediate hinge r. Such an influence line is shown in Fig. 6.18 c. The influence lines for the vertical reaction components $Y_a$ and $Y_b$ can be obtained by superimposing the influence line representing X tan $\alpha$ on the corresponding influence lines of the vertical reactions of the simple beam. The resulting influence lines are as shown in Figs. 6.18 d and e.

Once the influence lines for the reaction components are constructed, the influence lines for the thrust, shearing force and bending moment at a given section d can be constructed in the usual way by moving a unit load once to the right and once to the left of the section and in every case the function under consideration is calculated in terms of the reaction components on the side away from the unit load. Thus for the unit load to the right of point d,

$$N_d = -(Y_a \sin \Theta + X \cos \Theta)$$
$$Q_d = Y_a \cos \Theta - X \sin \Theta$$
$$M_d = Y_a x_d - X y_d$$

Expressing these functions in terms of the reaction components of the corresponding simple beam,

$$N_d = -Y'_a \sin \Theta - X (\tan \alpha \sin \Theta + \cos \Theta)$$
$$Q_d = Y'_a \cos \Theta - X (\sin \Theta - \tan \alpha \cos \Theta)$$
$$M_d = Y'_a x_d - X r_d$$

Similarly, for the unit load to the left of d,

$$N_d = Y_b \sin \Theta - X \cos \Theta$$
$$Q_d = -(Y_b \cos \Theta + X \sin \Theta)$$
$$M_d = Y_b (L - x_d) - X (y_d - f)$$

or,

$$N_d = Y'_b \sin \Theta - X (\tan \alpha \sin \Theta + \cos \Theta)$$
$$Q_d = -Y'_b \cos \Theta - X (\sin \Theta - \tan \alpha \cos \Theta)$$
$$M_d = Y'_b (L - x_d) - X r_d$$

Hence, the influence lines for the thrust, shearing force, and bending moment at point d are as shown in Figs. 6.18 f, g and h respectively.

It should be noticed that for a given point on the arch the angle $\Theta$ can be calculated from the shape of the arch.

A special case of importance is the three-hinged arch whose supports are on the same level. In such a case, the general solution given above holds and is further simplified due to the fact that the vertical reaction components of the arch are the same as those of a simple beam. This can be found from first principles or by substituting $\alpha = 0$ in equations 6.11 and 6.12. The influence line for the horizontal reaction component X has a maximum ordinate at the position of the intermediate hinge which may, again, be obtained by dividing the bending moment in the simple beam having the same span at the position of the intermediate hinge by the rise of the arch.

**Example 6.6** The three-hinged parabolic arch in Fig. 6.19 a can be expressed by the equation $y = 1.6\,x - 0.1\,x^2$ with the co-ordinate axes as shown. Construct the influence lines for the reactions, thrust and bending moment at point d of the arch.

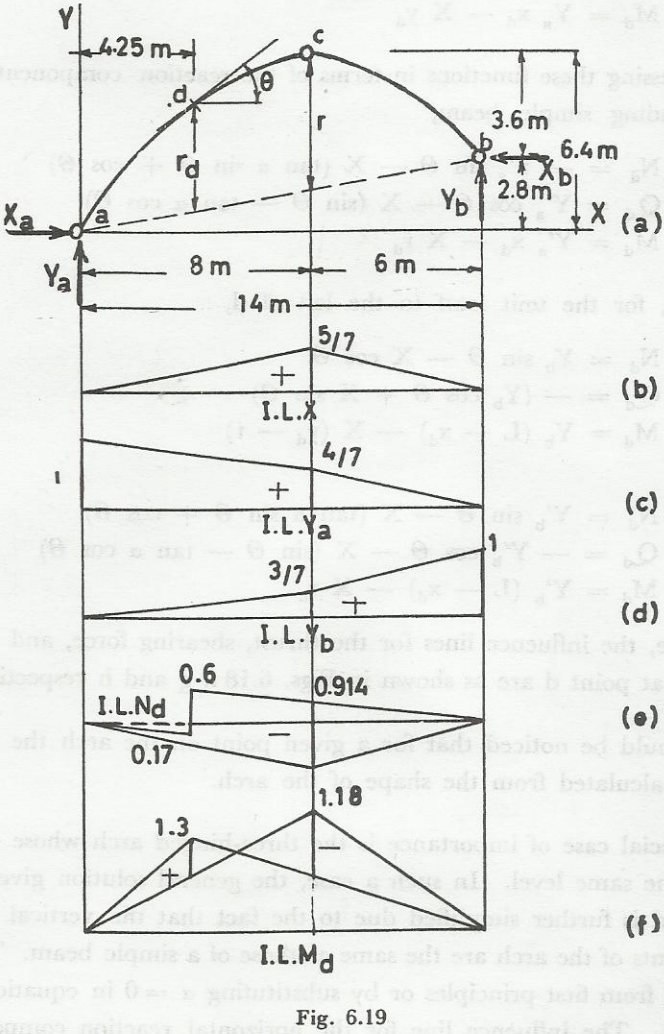



Fig. 6.19

$$\tan \alpha = \frac{2.8}{14} = 0.2$$

$$\tan \Theta = \left(\frac{dy}{dx}\right)_{x = 4.25} = 1.6 - 0.2\,x = 0.75$$

from which $\sin \Theta = 0.6$ and $\cos \Theta = 0.8$

at $x = 4.25$, $y = 1.6 \times 4.25 - 0.1 \times 4.25^2 = 5$ m.

$$r_d = 5 - \frac{4.25}{14} \times 2.8 = 4.15 \text{ m.}$$

Thus the influence lines for the reactions, the thrust and bending moment at section d can be constructed and are as shown in Figs. 6.19 b-f.

**Example 6.7** The three-hinged parabolic arch shown in Fig. 6.20 a is subjected to a uniformly distributed load of 1 t./m. of horizontal projection and a live load consisting of a uniformly distributed load of 2 t./m. of horizontal projection and a concentrated moving load of 10 t.

Construct the influence lines for the reactions, the thrust and bending moment at section d. Calculate the extreme values of the thrust and bending moment at section d,

Solution :

Taking the origin at a, the equation of the arch can be expressed as :

$$y = x - x^2 / 40$$

$$\tan \Theta = \left(\frac{dy}{dx}\right)_{x = 10} = 1 - \frac{x}{20} = \frac{1}{2}$$

from which $\sin \Theta = \frac{1}{\sqrt{5}} = 0.446$ and $\cos \Theta = \frac{2}{\sqrt{5}} = 0.893$

at $x = 10$, $y = 10 - \frac{100}{40} = 7.5$ m.

Thus, the influence lines for the reactions and the thrust and bending moment at section d can be constructed and are as shown in Figs. 6.20 b-f.

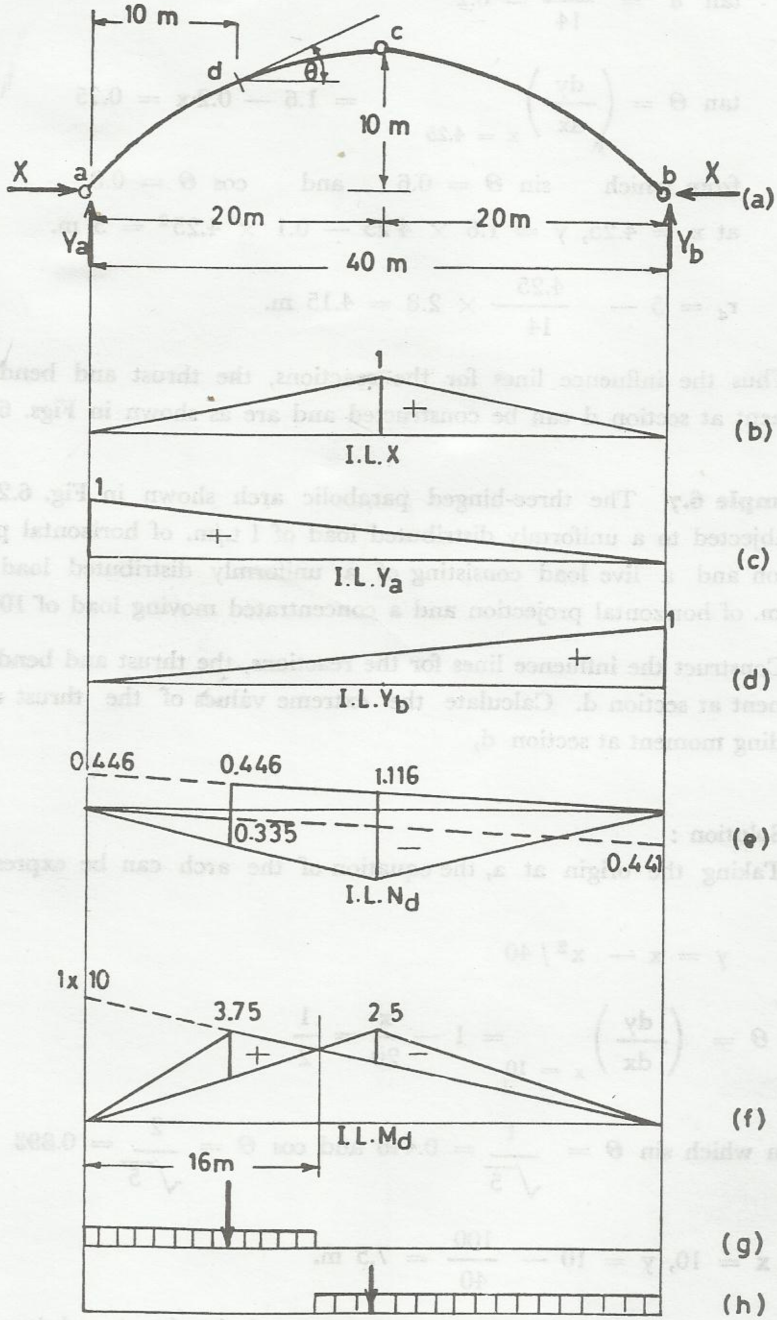

10 m
c
d
θ
10 m
X
a
b
X
(a)
20m
20m
Ya
Yb
40 m
1
+
(b)
I.L.X
1
+
(c)
I.L.Ya
1
+
(d)
I.L.Yb
0.446
0.446
1.116
0.335
−
(e)
0.441
I.L.Nd
1x 10
3.75
2.5
+
−
(f)
I.L.Md
16m
(g)
(h)

Fig. 6.20

$$N_d \text{ (due to D.L.)} = -1 \left( \frac{0.335 \times 10}{2} + \frac{0.781 + 1.116}{2} \times 10 + \frac{1.116 \times 20}{2} \right) = -22.32 \text{ t.}$$

The maximum thrust at d occurs with the uniformly distributed load covering all the span with the concentrated load at c. Thus, from the influence line in Fig. 6.20 e.

$$N_d \text{ (due to L.L.)} = -22.32 \times 2 - 1.116 \times 10 = -55.8 \text{ t.}$$

The extreme values for $N_d$ are :— 22.32t and — 78.12t

$$M_d \text{ (due to D.L.)} = 1 \left( \frac{3.75 \times 16}{2} - \frac{2.5 \times 24}{2} \right) = 0$$

The load position for maximum positive $M_d$ is shown in Fig. 6.20 g. Thus, from the influence line in Fig. 6.20 f.

$$\text{maximum positive } M_d \text{ (due to L.L.)} = 2 \times \frac{3.75 \times 16}{2} + 10 \times 3.75 = 97.5 \text{ m.t.}$$

The position of the load for maximum negative $M_d$ is shown in Fig. 6.20h.

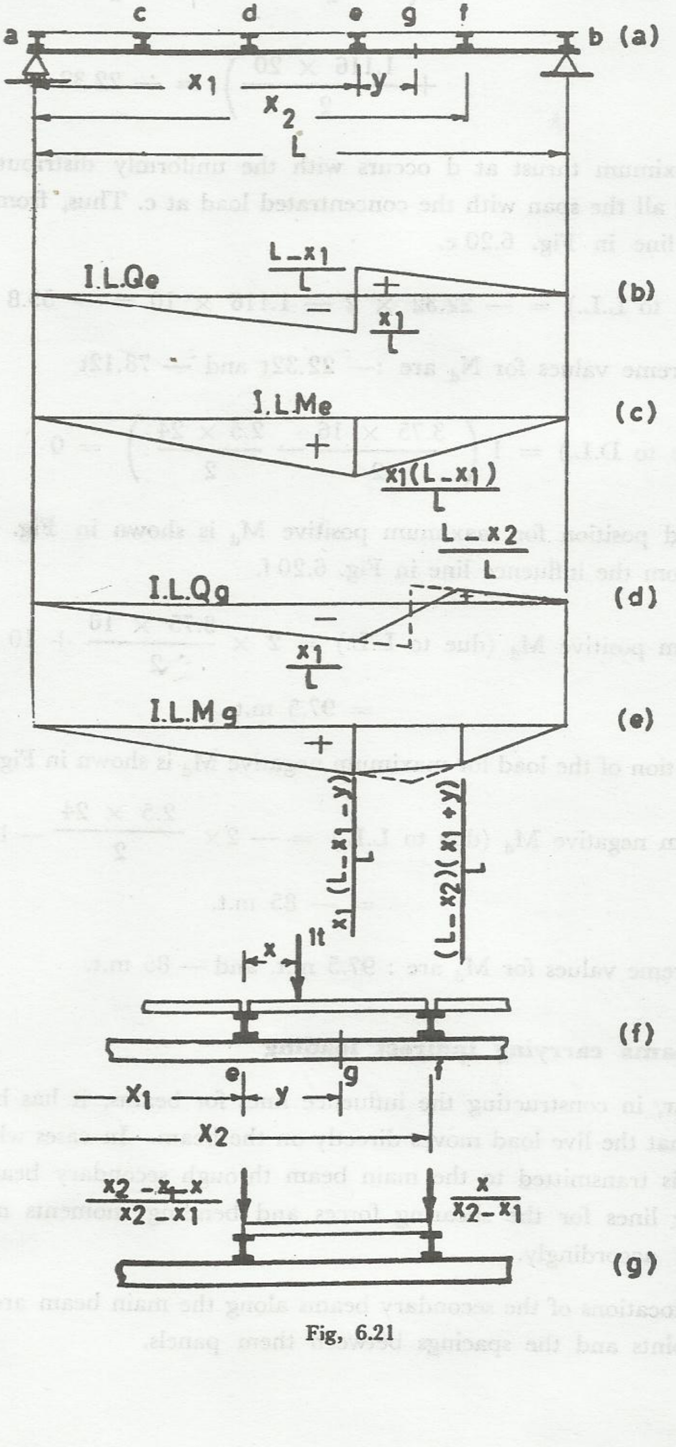
$$\text{maximum negative } M_d \text{ (due to L.L.)} = -2 \times \frac{2.5 \times 24}{2} - 10 \times 2.5 = -85 \text{ m.t.}$$

The extreme values for $M_d$ are : 97.5 m.t. and — 85 m.t.

### 6.11. Beams carrying indirect loading

So far, in constructing the influence lines for beams, it has been assumed that the live load moves directly on the beam. In cases where the loading is transmitted to the main beam through secondary beams, the influence lines for the shearing forces and bending moments must be modified accordingly.

The locations of the secondary beams along the main beam are called panel points and the spacings between them panels.



a
c
d
e
g
f
b
(a)
x1
y
x2
L
I.L.Qe
L - x1 / L
x1 / L
(b)
I.L.Me
x1(L - x1) / L
(c)
I.L.Qg
L - x2 / L
x1 / L
(d)
I.L.Mg
x1(L - x1 - y) / L
(L - x2)(x1 + y) / L
(e)
x
1t
(f)
e
g
f
X1
y
X2
x2 - x1 - x / x2 - x1
x / x2 - x1
(g)


Fig, 6.21

Consider, for example, the beam shown in Fig. 6.21 a. The load is transmitted to the beam at the panel points a, b, c, d, e and f.

Obviously, the influence lines for the reactions of the beam do not change whether the load is transmitted to it directly or indirectly. Also, the influence lines for the shearing force and bending moment at point e, which is a panel point, are not affected by the indirect loading as shown in Figs. 6.21 b and c.

Figs. 6.21 d and e show the influence lines for the shearing force and bending moment at point g within panel ef. For direct loading to the beam, the resulting influence lines are as shown by the dashed lines, and for indirect loading, the influence lines are shown by the continuous lines in the corresponding figures. Note that the variation in the shear and moment within panel ef is linear. This may be proved with reference to Figs. 6.21 f and g as follows :

Consider a unit load moving across panel ef and at a distance x from point e.

Referring to Fig. 6.21g, the shearing force at point g is given dy the relationship :

$$Q_g = \frac{L - x_1 - x}{L} - \frac{x_2 - x_1 - x}{x_2 - x_1}$$

From this relationship it can be seen that the variation of the shearing force at point g in panel ef is linear.

When $x = 0$, $Q_g = - x_1 / L$

$$x = x_2 - x_1,\ Q_g = \frac{L - x_2}{L}$$

Hence the influence line for the shearing force at point g is as shown in Fig. 6.21 d.

The bending moment at point g as the load moves from e to f is given by the relationship :

$$M_g = Y_a (x_1 + y) - \frac{x_2 - x_1 - x}{x_2 - x_1} y$$

which becomes

$$M_g = \frac{L - x_1 - x}{L} (x_1 + y) - \frac{x_2 - x_1 - x}{x_2 - x_1} y$$

Hence the variation of bending moment at point g in panel ef is linear.

When $x = 0$, $M_g = \dfrac{x_1}{L}(L - x_1 - y)$

$$x = x_2 - x_1,\ M_g = \frac{L - x_2}{L}(x_1 + y)$$

Hence the influence line for the bending moment at point g is as shown in Fig. 6.21 e.

From the influence lines for the shearing force and bending moment at point g, it should be noted that the simplest method of constructing such diagrams is to draw them as if the load was carried directly by the beam and then join the points on the influence lines where the verticals from the panel points e and f meet the corresponding influence lines.

In the case of three-hinged arches carrying indirect loading, the usual procedures for the construction of influence lines may be applied with only slight modifications similar to those mentioned above.

**Example 6·8** The simple beam shown in Fig. 6.22 a supports secondary beams at 4 m. spacing. Construct the influence lines for the shearing force and bending moment at section c.

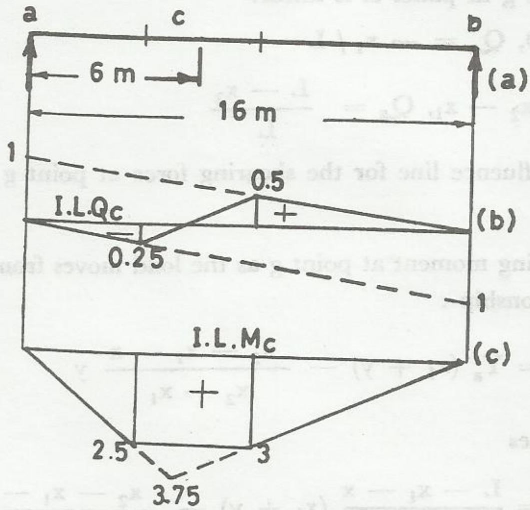



Fig. 6.22

Solution :

The influence lines for the shearing force and bending moment at point c are shown in Figs. 6.22 a and b respectively. The critical ordinates are calculated from similar triangles.

## 6.12 Construction of influence lines for trusses

The same general procedures as those used for constructing the influence lines for beams can be applied to trusses. Before constructing the influence lines for the forces in the members, it is necessary to determine first the most suitable method for calculating the member force. The influence line for a member force may be constructed directly by resolution at a joint, or by taking a section and considering the shear in a panel or the moment at a point as a unit load is placed at each of the panel points of the loaded chord of the truss. In general, calculations are preferably carried out for the part of the truss on the side of the section which is away from the unit load. Sometimes it is necessary to construct first the influence lines for forces in members other than that considered and use the results in constructing the influence line actually required.

The methods of constructing the influence lines for the forces in truss members will be illustrated by a consideration of a number of examples.

**Example 6.9** Construct the influence lines for the forces in members cc′, c′d, d′d, c′d′ and de of the truss shown in Fig. 6.23 a.

Solution :

For member cc′, the force can be directly determined from the consideration of equilibrium of joint c. When the unit load is at c, the force in cc′ is tensile and equal to unity. When the unit load is at any other joint, the force in the member is zero. Thus, the influence line for the force in member cc′ is as shown in Fig. 6.23 b.

For member c′d, the force can be calculated by considering the shear in panel cd. With a unit load to the right of section a-a, the tension in c′d equals $Y_a$ multiplied by 5/4, and hence is directly proportional to $Y_a$. Since $Y_a$ varies linearly, the influence line is a straight line varying from zero at b to $5/4 \times 18/27 = 5/6$ at d. With the unit load to the left of the section, the compression in c′d equals $Y_b$ multiplied by 5/4 and since $Y_b$

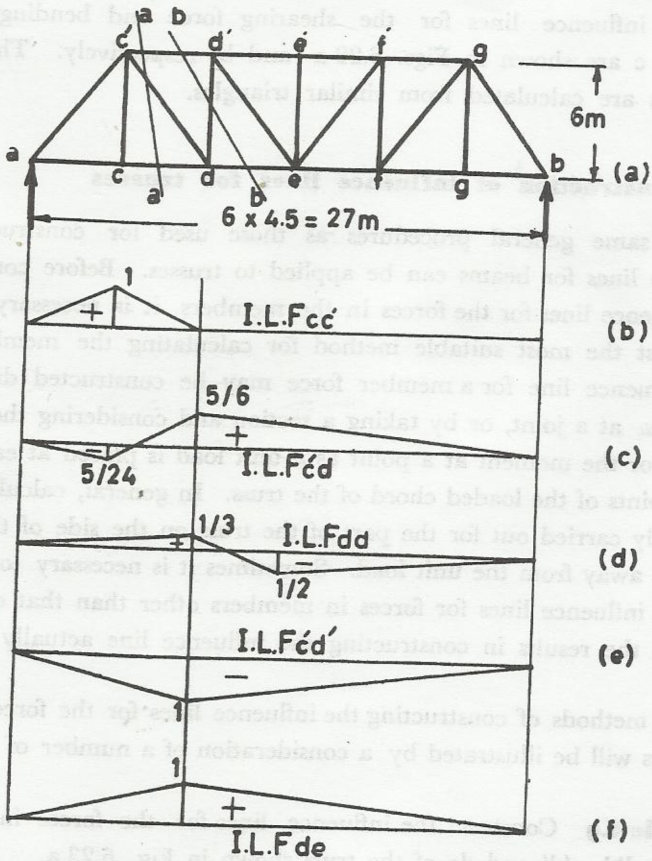



Fig. 6.23

varies linearly, the influence line is a straight line varying from zero at a to — 5/4 × 4.5/27 = — 5/24 at c. The influence line for the force in member c′d is thus as shown in Fig. 6.23 c.

The force in member dd′ can be calculated in a similar way as member c′d. With a unit load to the right of section b-b, the compression in the member is equal to $Y_a$. Hence the influence line is a straight line varying from zero at b to — 1/2 at e. When the unit load is to the left of the section, the tension in the member is equal to $Y_b$. Hence the influence line is a straight line varying from zero at a to 9/27 = 1/3 at d. Between panel points d and e, the influence line is a straight line (section 6.11). The influence line is shown in Fig. 6.23 d.

For member c′d′, take moments about point d of the forces acting on one side of section a-a. With the unit load to the left of the section, the compression in c′d′ equals to $Y_b$ multiplied by 18 and divided by the truss height of 6 m., and hence the influence line is proportional to $Y_b$ and varies linearly from zero at a to $-3/2 \times 18/27 = -1$ at d. With the unit load to the right of the section, the compression in c′d′ equals to $Y_a$ multiplied by 9 and divided by 6. Hence, the influence line is a straight line varying from $-3/2 \times 18/27 = -1$ at d to zero at b. This influence line is shown in Fig. 6.23 e.

In a similar manner, the force in member de is obtained by taking moments about point d′. The resulting influence line is shown in Fig. 6.23 f.

It should be noticed that both influence lines in Figs. 6.23 e and f have maximum ordinates at d which is the centre of moments. This is always the case and this result can be used to simplify the construction of such influence lines.

**Example 6.10** Construct the influence lines for the forces in members cc″, c′d′, c′c″, c″d, c″d′, ef and ef″ of the k-truss shown in Fig. 6.24 a.

Solution :

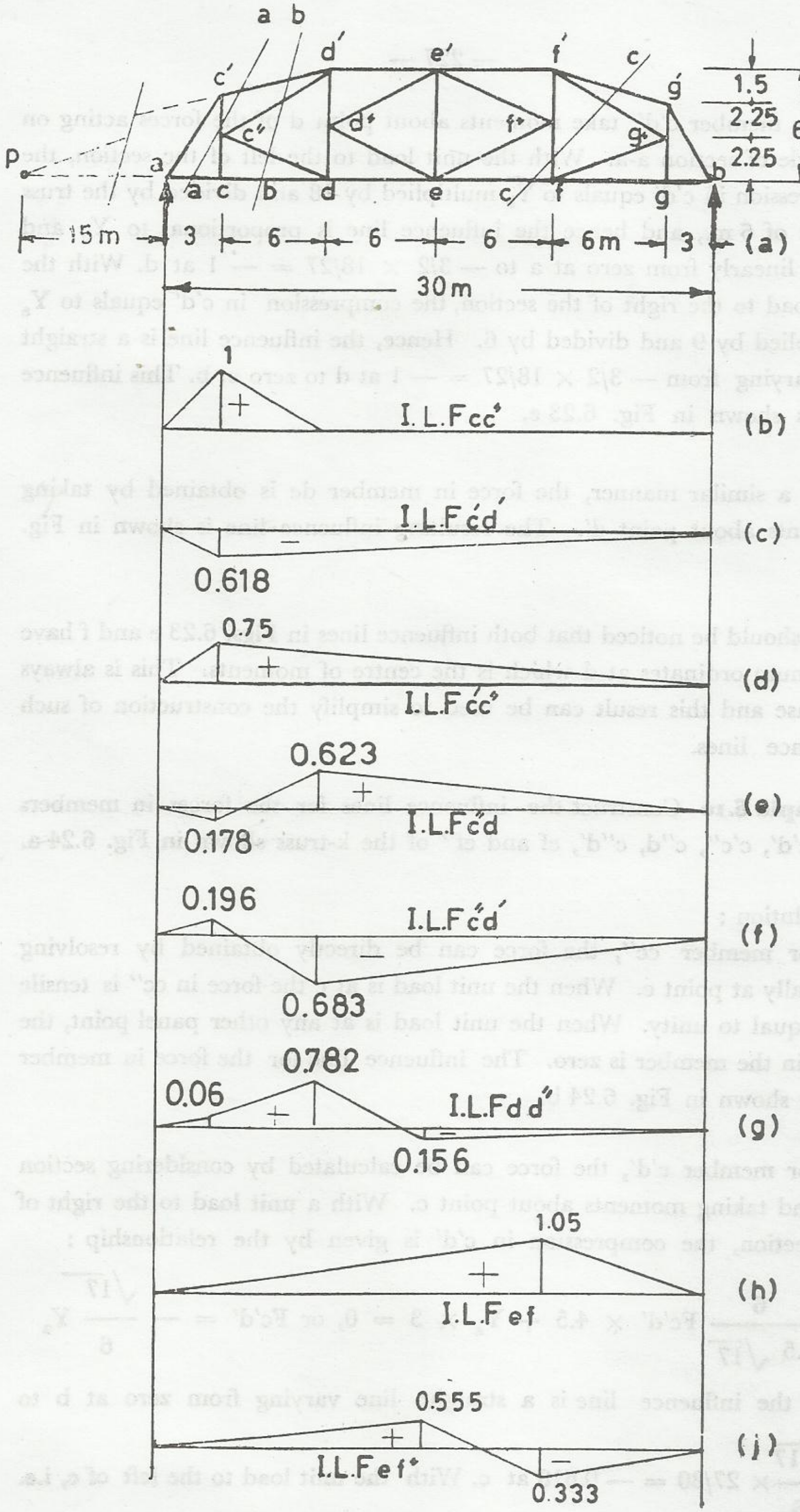
For member cc″, the force can be directly obtained by resolving vertically at point c. When the unit load is at c the force in cc″ is tensile and equal to unity. When the unit load is at any other panel point, the force in the member is zero. The influence line for the force in member cc″ is shown in Fig. 6.24 b.

For member c′d′, the force can be calculated by considering section a-a and taking moments about point c. With a unit load to the right of the section, the compression in c′d′ is given by the relationship :

$$\frac{6}{1.5\sqrt{17}} F_{c'd'} \times 4.5 + Y_a \times 3 = 0, \text{ or } F_{c'd'} = -\frac{\sqrt{17}}{6} Y_a$$

Thus the influence line is a straight line varying from zero at b to

$-\frac{\sqrt{17}}{6} \times 27/30 = -0.618$ at c. With the unit load to the left of c, i.e.



a b
d′
e′
f′
c
c′
g′
1.5
2.25
2.25
6m
c″
d*
f*
g*
p
a
a
c
b
d
e
c
f
g
b
15m
3
6
6
6
6m
3
(a)
30m
1
+
I.L.Fcc″
(b)
I.L.F′c′d
−
(c)
0.618
0.75
+
(d)
I.L.F′c′c″
0.623
+
(e)
I.L.F″c″d
0.178
0.196
I.L.F″c″d′
(f)
0.683
0.782
0.06
+
I.L.Fdd″
−
(g)
0.156
1.05
+
(h)
I.L.Fef
0.555
+
−
(i)
I.L.Fef*
0.333


Fig. 6.24

at a it is obvious that the force is zero. The influence line for the force in member c'd' is shown in Fig. 6.24 c.

For member c'c'', the force can be calculated by considering section a-a and taking moments about the point of intersection of the extensions of the top and bottom chords; point p. When the unit load is to the right of the section, the tension in c'c'' is given by the relationship :

$$Y_a \times 15 - Fc'c'' \times 18 = 0, \text{ or } Fc'c'' = 5/6\ Y_a$$

Thus the influence line is a straight line, varying from zero at b to 5/6 × 27/30 = 0.75 at c. With the unit load at panel point a, it is obvious that the force is zero. Thus, the influence line for the force in member c'c'' is as shown in Fig. 6.24 d.

For the forces in members c''d and c''d', the relationship between the two forces is determined first by resolving horizontally at joint c''. Thus.

$$\frac{8}{\sqrt{89}} Fc''d' + \frac{8}{\sqrt{73}} Fc''d = 0$$

$$\text{or } Fc''d' = -1.1\ Fc''d$$

The value of either force can be calculated by considering section b-b and taking moments about point p. The horizontal components of the two forces will cancel each other. As the unit load moves to the right of the section,

$$Y_a \times 15 + \left( \frac{5}{\sqrt{89}} Fc''d' - \frac{3}{\sqrt{73}} Fc''d \right) 18 = 0$$

Substituting $Fc''d' = -1.1\ Fc''d$

$$Fc''d = 0.89\ Y_a$$

$$Fc''d' = -0.975\ Y_a$$

Thus the influence lines are straight lines varying from zero at b to 0.89 × 21/30 = 0.623 for Fc''d, and —0.975 × 21/30 = — 0.683 for Fc''d' at d. As the unit load moves to the left of the section,

$$Y_b \times 45 + \left( \frac{3}{\sqrt{73}} Fc''d - \frac{5}{\sqrt{89}} Fc''d' \right) 27 = 0$$

Substituting $Fc''d' = -1.1\ Fc''d$

$$Fc''d = -1.78\ Y_b$$

$$Fc''d' = +1.96\ Y_b$$

Thus the influence lines are straight lines varying from zero at a to — 1.78 × 3/30= — 0.178 for Fc″d, and 1.96 × 3/30 = 0.196 for Fc″d′ at c. Between panel points c and d the variation in the influence lines is linear. The influence lines for the forces in c″d and c″d′ are shown in Figs. 6.4 e and f respectively.

For member dd″, the force can best be found by vertical resolution at joint d. Thus .

Fdd″= — 0.35 Fc″d ± the effect of unit load that may come between c and e.

With the unit load at e, the force in dd″ is,

$$-0.35 \times 0.623 \times 15/21 = -0.156$$

With the unit load at d, the force in dd″ is,

$$1 - 0.35 \times 0.632 = 0.782$$

With the unit load at c, the force in dd″ is,

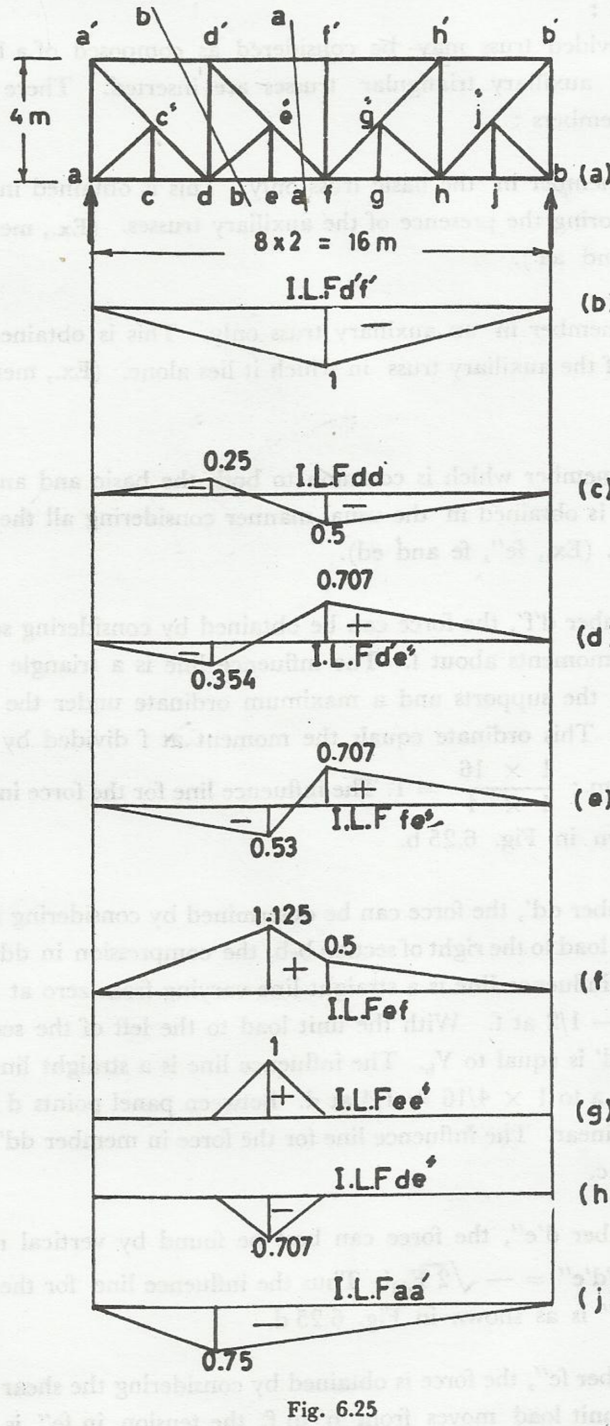
$$-0.35 \times -0.178 = 0.06$$

The influence line for the force in member dd″ is as shown in Fig. 6.24 g.

For member ef, the force can be found by considering section c-c and taking moments about point f′. When the unit load is to the right of the section, the tension in ef equals $Y_a \times 21/6$. The influence line varies linearly from zero at b to 9/30 × 21/6 = 1.05 at f. With the unit load to the left of the section, the tension in ef is equal to $Y_b \times 9/6$. The influence line for the force in ef is shown in Fig. 6.24 h.

For member ef″. the force can be obtained by considering the shear in panel ef. The resulting influence line is shown in Fig. 6.24 j.

**Example 6.11** Construct the influence lines for the forces in members d′f′, dd′, d′e″, fe″, ef, ee″ and aa′ of the subdivided truss shown in Fig. 6.25 a if ab is the loaded chord.



a'
b
d'
a
f'
h'
b'
4 m
c'
e'
g'
j'
a
b (a)
c
d
b
e
a
f
g
h
j
8 x 2 = 16 m
I.L.F d'f'
(b)
1
0.25
I.L.F dd'
(c)
0.5
0.707
(d)
I.L.F d'e''
0.354
0.707
I.L.F fe''
(e)
0.53
1.125
0.5
I.L.F ef
(f)
1
I.L.F ee'
(g)
I.L.F de'
(h)
0.707
I.L.F aa'
(j)
0.75


Fig. 6.25

Solution :

A subdivided truss may be considered as composed of a basic truss into which auxiliary triangular trusses are inserted. There are three types of members :

(1) A member in the basic truss only. This is obtained in the usual manner ignoring the presence of the auxiliary trusses. (Ex., members d'f' dd', d'e'' and aa').

(2) A member in an auxiliary truss only. This is obtained by consideration of the auxiliary truss in which it lies alone. (Ex., members ee'' and de'').

(3) A member which is common to both the basic and an auxiliary truss. This is obtained in the usual manner considering all the members of the truss. (Ex., fe'', fe and ed).

For member d'f', the force can be obtained by considering section a-a and taking moments about f. The influence line is a triangle with zero ordinates at the supports and a maximum ordinate under the centre of moments f. This ordinate equals the moment at f divided by the truss height of 4 m.; $\frac{1 \times 16}{4 \times 4} = 1$. The influence line for the force in member d'f' is shown in Fig. 6.25 b.

For member dd', the force can be determined by considering the shear. With a unit load to the right of section b-b, the compression in dd' is equal to $Y_a$. The influence line is a straight line varying from zero at b to $-1 \times 8/16 = -1/2$ at f. With the unit load to the left of the section, the tension in dd' is equal to $Y_b$. The influence line is a straight line varying from zero at a to $1 \times 4/16 = 1/4$ at d. Between panel points d and f the variation is linear. The influence line for the force in member dd' is shown in Fig. 6.25 c.

For member d'e'', the force can best be found by vertical resolution at joint d', $F_{d'e''} = -\sqrt{2}\, F_{dd'}$. Thus the influence line for the force in member d'e'' is as shown in Fig. 6.25 d.

For member fe'', the force is obtained by considering the shear in panel ef As the unit load moves from b to f, the tension in fe'' is equal to

$\sqrt{2}\,Y_a$ and as the unit load moves from a to e, the compression in fe″ is equal to $\sqrt{2}\,Y_b$. Thus, the influence line for the force in member fe″ is as shown in Fig. 6.25 e.

The influence line for the force in member ef (or de as they are identical by consideration of the equilibrium at joint e) can be obtained by considering section a-a and taking moments about d′. As the unit load moves from b to f, the tension in ef is equal to $Y_a \times 4/4 = Y_a$, and as the unit load moves from a to e, the tension is equal to $Y_b \times 12/4 = 3\,Y_b$. Thus, the influence line varies linearly from zero at b to $1 \times 8/16 = 1/2$ at f, and from zero at a to $3 \times 6/16 = 1.125$ at e. Between e and f the variation is linear. The influence line for the force in member ef, or de, is shown in Fig. 6.25 f.

For member ee″ in the auxiliary truss, the force can be obtained directly by vertical resolution at joint e. The influence line is shown in Fig. 6.25 g.

For member de″ in the auxiliary truss, the force can be found by resolution at joint e″; $F_{de''} = - F_{ee''} / \sqrt{2}$. The influence line is thus as shown in Fig. 6.25 h.

For member aa′ in the basic truss, as the unit load moves across the span, the force is equal to $Y_a$. Only when the load is at a it is directly taken by the support and the force is zero. The influence line is shown in Fig. 6.25 j.

## 6.13 Alternative method for the construction of influence lines

In this method, an imaginary deformation in the beam section or the truss member under consideration is introduced. The method is based on Müller-Berslau principle which indicates that the shape of an influence line of a load function is identical to the deformed form of the structure when it is deformed by an action similar to that of the load function considered.

The method will be illustrated by considering the simple beam shown in Fig. 6.26 a. If the influence line for the reaction at support a is required, the left end of the beam is lifted a unit distance. Since the beam remains straight, its deflected shape and hence the shape of the influence line is a triangle as shown in Fig. 6.26 b. The shape of the influence line for the bending moment at point c is the deflected shape of the beam formed by introducing an imaginary hinge at c to allow the two parts ac and

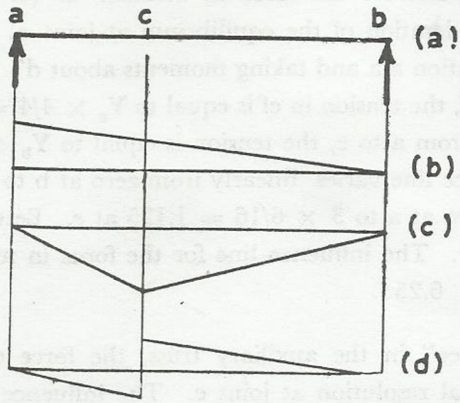



Fig. 6.26

bc to rotate with respect to each other. Since both ends of the beam are free, parts ac and bc would rotate, thus deflecting point c downward. This results in the shape of the influence line shown in Fig. 6.26 c. The shape of the influence line for the shearing force at c is found by cutting the beam at c and subjecting the ends to a shearing force. The deflected shape would then be two triangles as shown in Fig. 6.26 d.

Once the shape of the influence line is sketched, the ordinates at the critical sections are calculated to completely define the influence line. In addition, this method provides an easy and quick way for determining the parts of a structure to be loaded in order to obtain the maximum value of the load function for which the influence line is sketched. This method is seldom used in connection with statically determinate structures, but it is of importance in constructing the influence lines for statically indeterminate structures.

## 6.14 Illustrative examples

As an illustration to the construction of influence lines for various types of structures, a number of examples will be considered. In each example, the influence lines for various load functions are constructed. The student is advised to solve these problems independently using the principles presented in this chapter and then check his results against those given. He should also remember that an influence line is not complete unless the signs and also the ordinates at all critical sections are indicated.

**Example 6.12** For the double overhanging beam shown in Fig. 6.27, construct the influence lines for the reactions, shearing forces at a section just to the right and another just to the left of support b, and bending moments at points a and c.

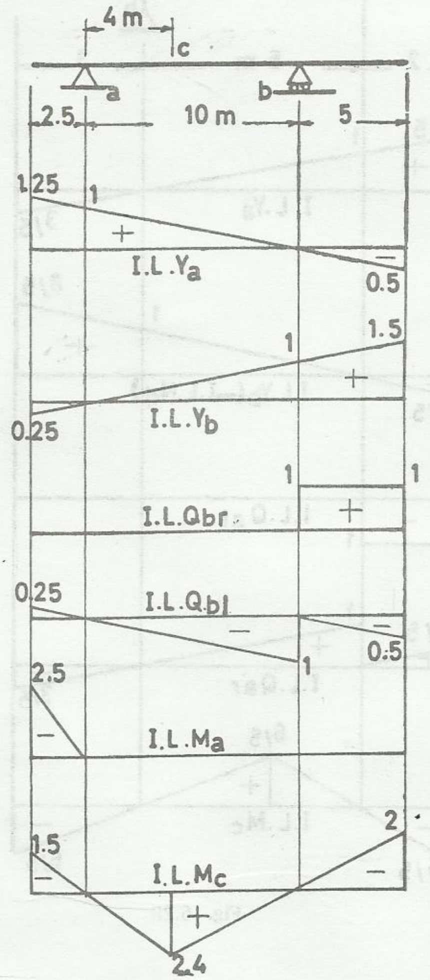



Fig. 6.27

**Example 5.13** For the frame shown in Fig. 6.28, construct the influence lines for the reactions, shearing forces at a section just to the right and another just to the left of support a, thrust at d and bending moment at c.

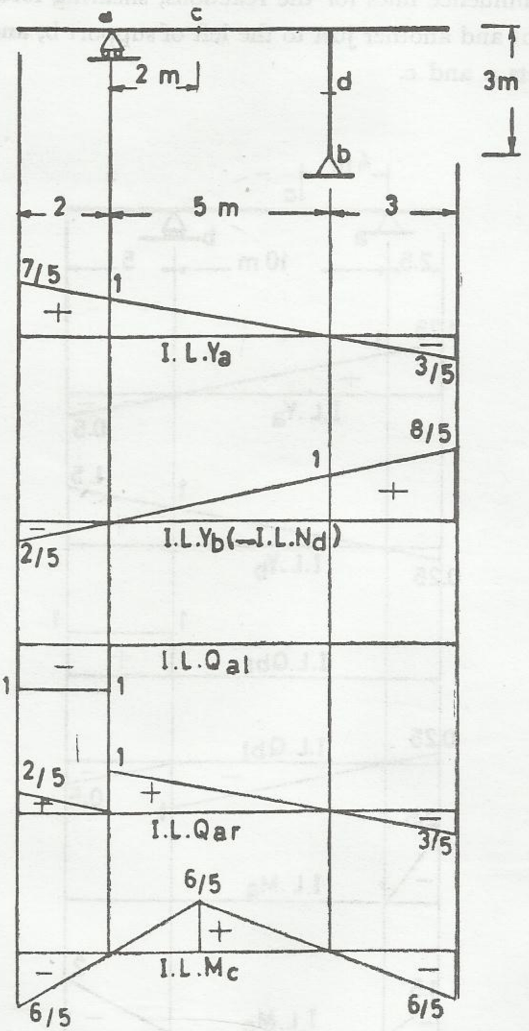



Fig. 6.28

**Example 6.14** If a a load moves from c to d on the frame shown in Fig. 6.29, construct the influence lines for the reactions, the shearing forces and bending moments at sections e and f.

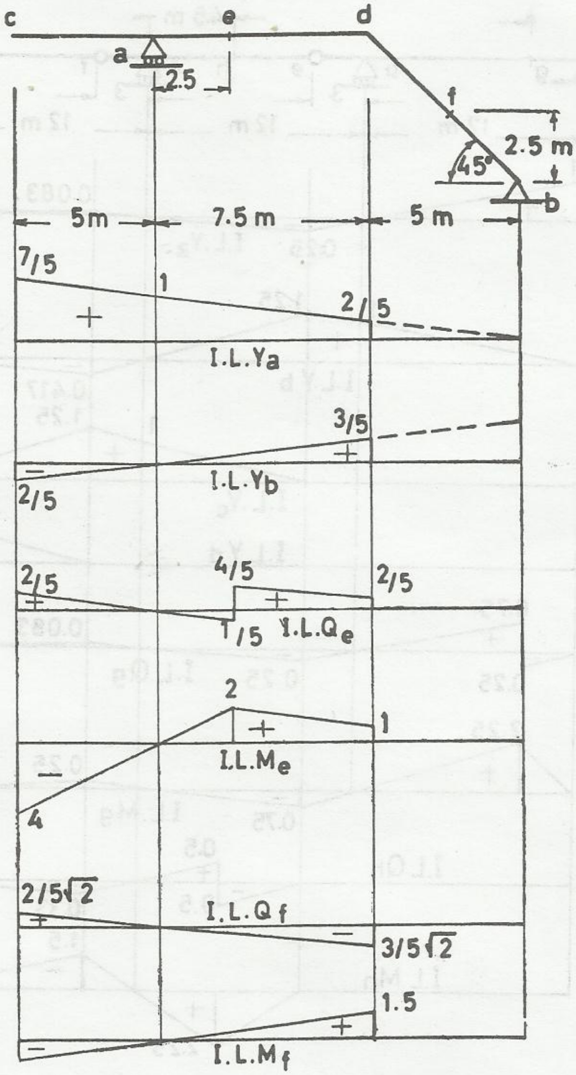



Fig. 6.29

**Example 6.15** For the eantilever beam shown in Fig. 6.30, construct the influenee lines for the reactions at the supports, the shearing forces and bending moments at sections g and h.

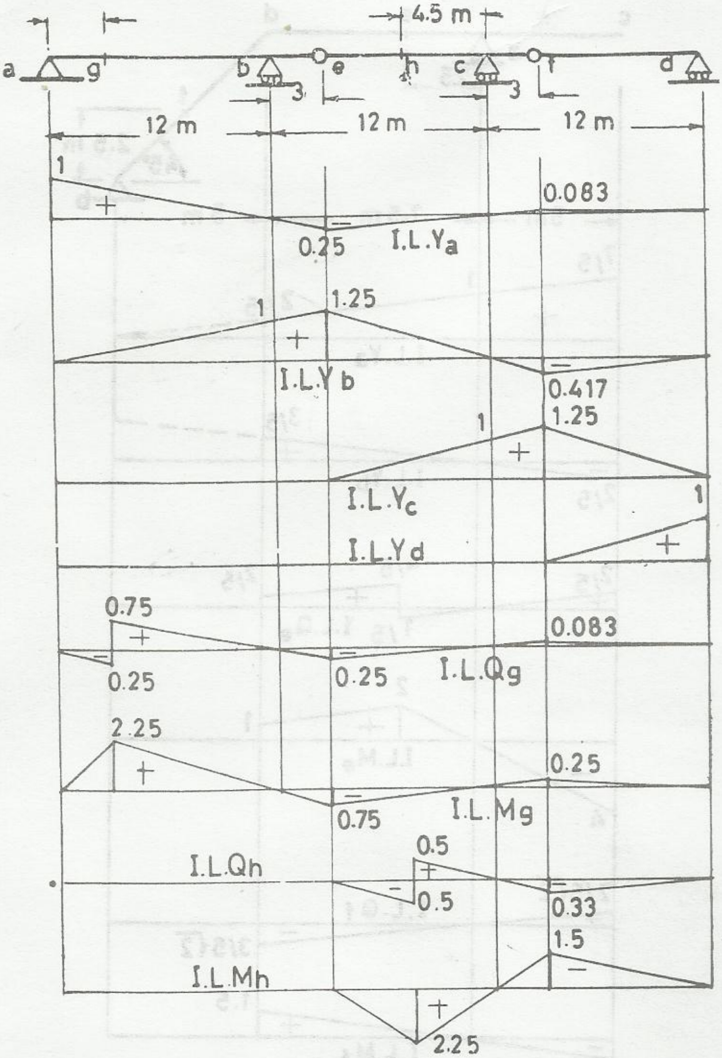



Fig. 6.30

**Example 6.16** For the three-hinged circular arch shown in Fig. 6.31, construct the influence lines for the reaction components, thrust and moment at section d.

From the properties of the circle,

$y_d$ = 4.74 m., sin $\Theta$ = 0.4 and cos $\Theta$ = 0.916

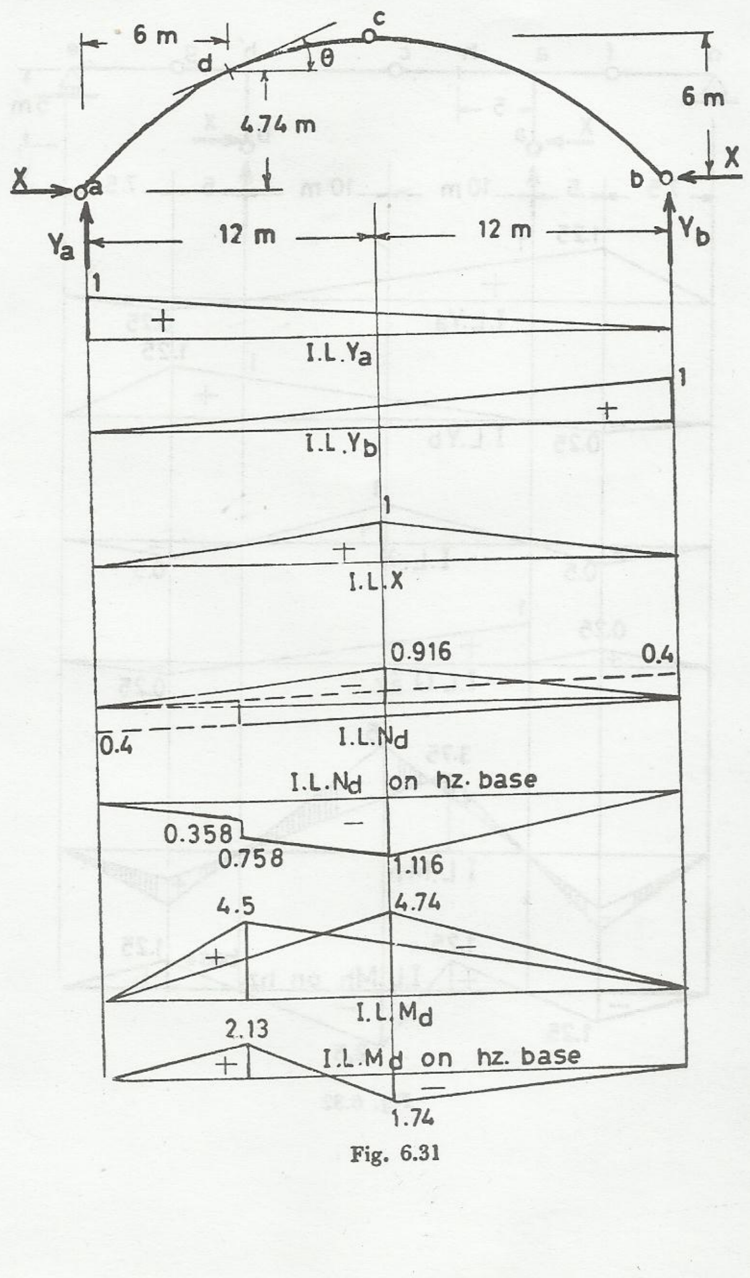



Fig. 6.31

**Example 6.18** Construct the influence lines for the forces in members fk, ek, bc, hl, ci, mn and id of the truss shown in Fig. 6.33.

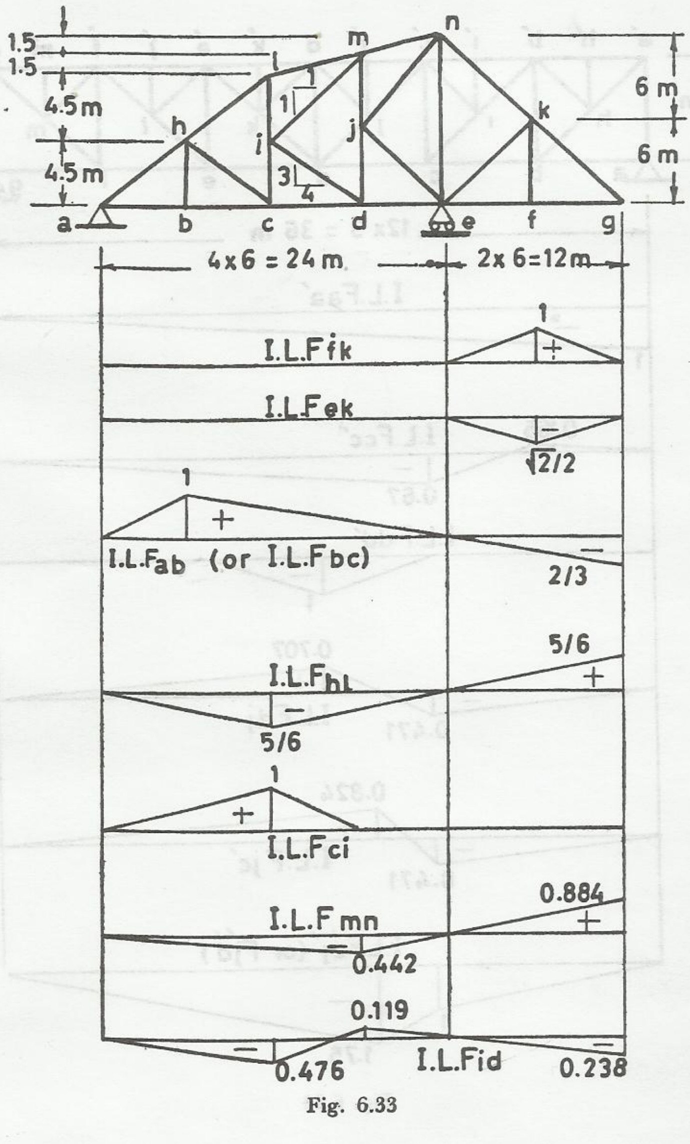



Fig. 6.33

**Example 6.19** For the subdivided truss shown in Fig. 6.34 and loaded on the top chord, construct the influence lines for the forces in members aa′, cc′, dd′, dj, jc′ and c′j′.

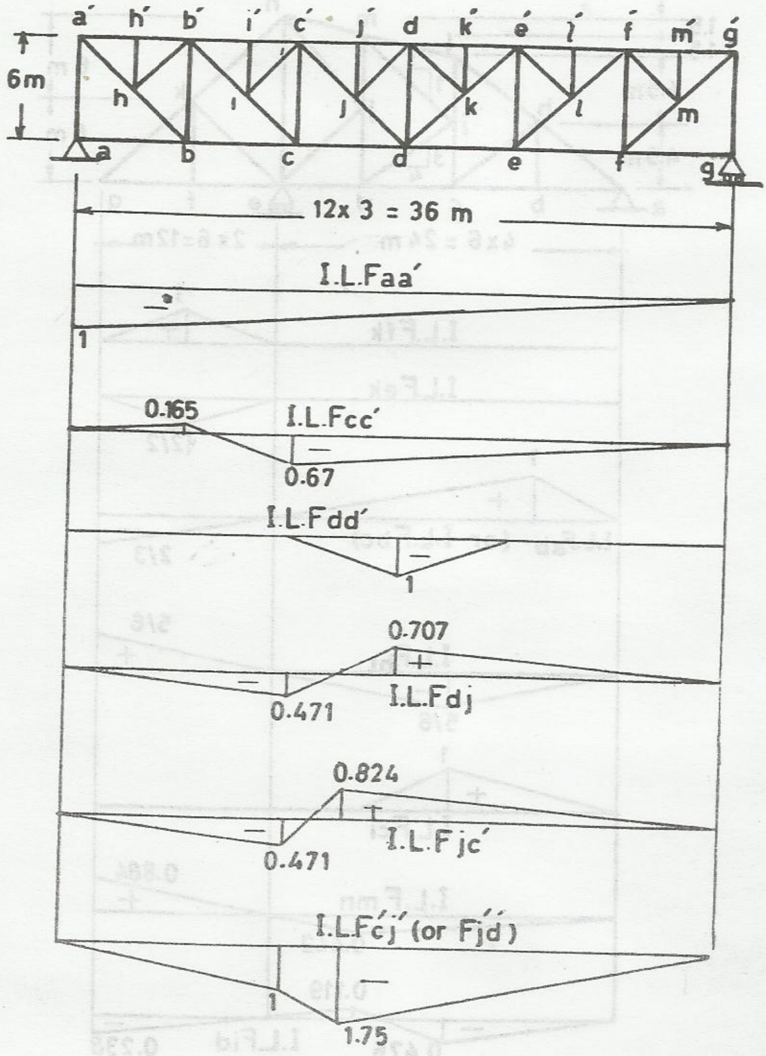



Fig. 6.34

**Example 6.20** For the three-hinged truss shown in Fig. 6.35, construct the influence lines for the reactions and forces in members ea′, fa′, and fg.

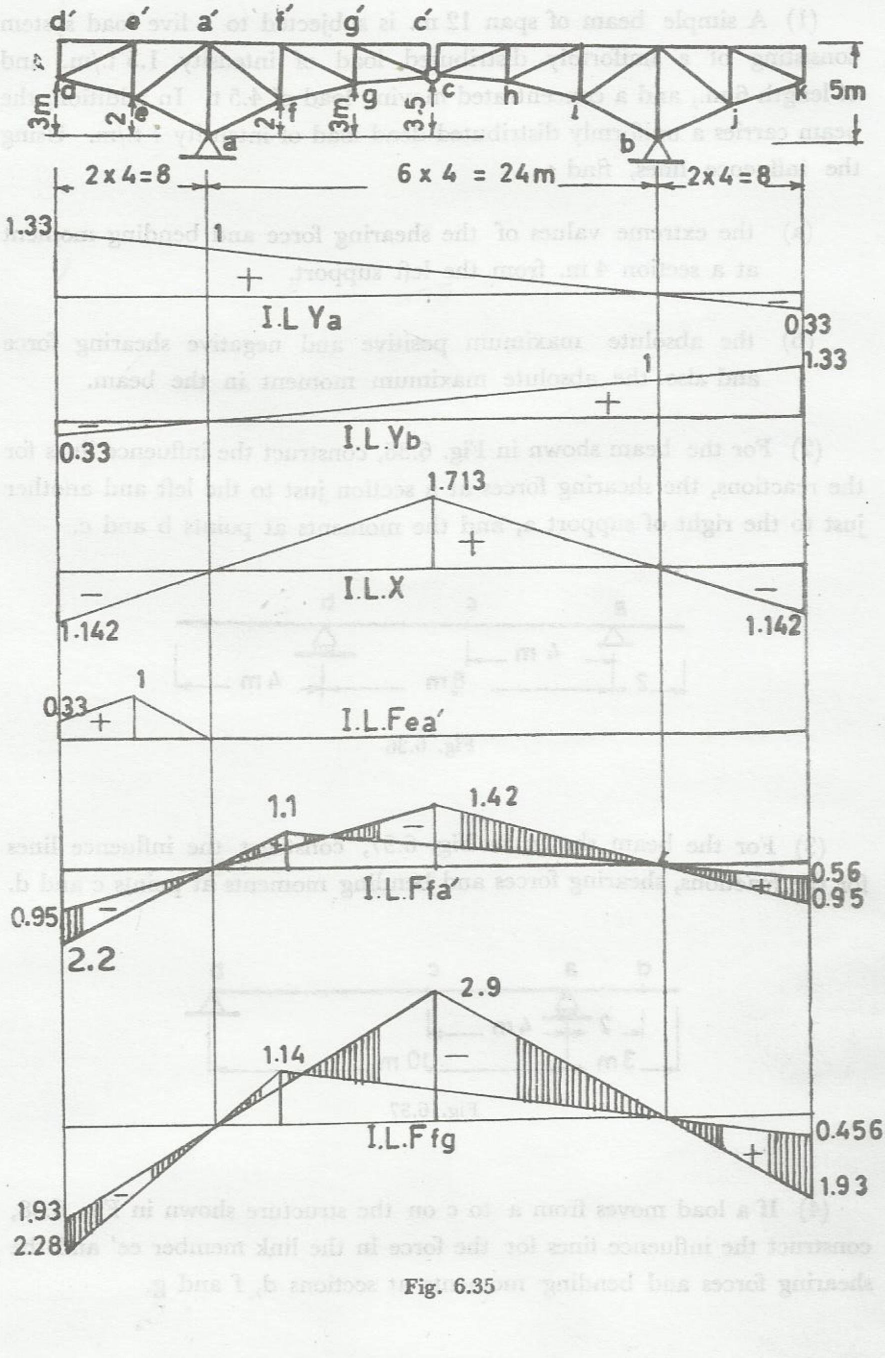



**Fig. 6.35**

## EXAMPLES TO BE WORKED OUT

(1) A simple beam of span 12 m. is subjected to a live load system consisting of a uniformly distributed load of intensity 1.5 t./m. and of length 6 m., and a concentrated moving load of 4.5 t. In addition, the beam carries a uniformly distributed dead load of intensity 1 t./m. Using the influence lines, find :

(a) the extreme values of the shearing force and bending moment at a section 4 m. from the left support.

(b) the absolute maximum positive and negative shearing force and also the absolute maximum moment in the beam.

(2) For the beam shown in Fig. 6.36, construct the influence lines for the reactions, the shearing forces at a section just to the left and another just to the right of support a, and the moments at points b and c.

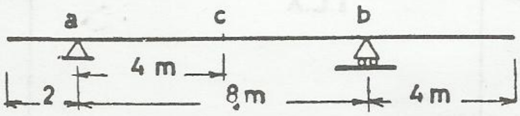



Fig. 6.36

(3) For the beam shown in Fig. 6.37, construct the influence lines for the reactions, shearing forces and bending moments at points c and d.

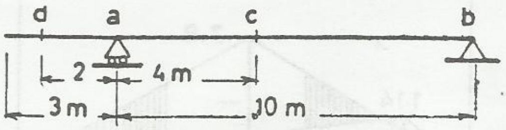



Fig. 6.37

(4) If a load moves from a to c on the structure shown in Fig. 6.38, construct the influence lines for the force in the link member ee′ and the shearing forces and bending moments at sections d, f and g.

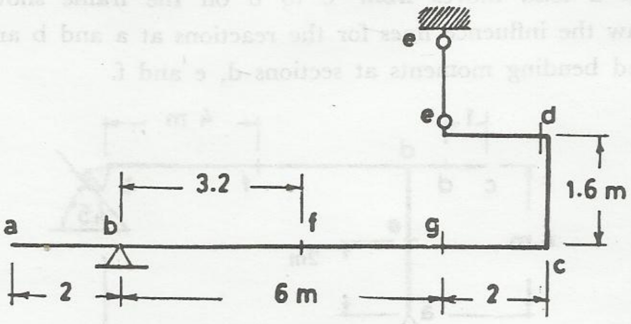



Fig. 6.38

(5) If a load moves from a to b on the structure shown in Fig. 6.39, draw the influence lines for the reactions and the shearing forces and the bending moments at sections e, f and g.

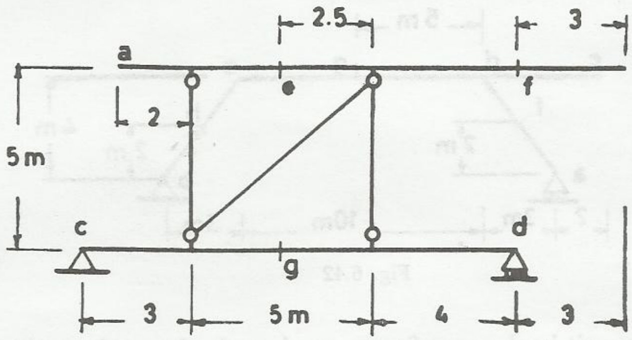



Fig. 6.39

(6) For the frame shown in Fig. 6.40. construct the influence lines for the reactions, the shearing forces at a section just to the left and another just to the right of support b and the moment at c.

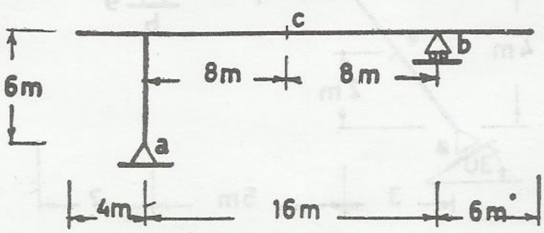



Fig. 6.40

(7) If a load moves from c to b on the frame shown in Fig. 6.41, draw the influence lines for the reactions at a and b and shearing forces and bending moments at sections d, e and f.

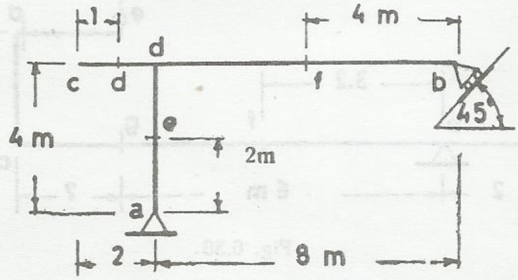



Fig. 6.41

(8) If a load moves from c to e on the frame shown in Fig. 6.42, construct the influence lines for the reactions, shearing forces and bending moments at sections f, g and h.

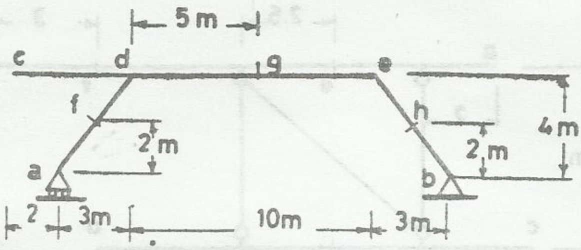



Fig. 6.42

(9) If a unit load moves from c to d on the frame shown in Fig. 6.43, construct the influence lines for the reaction components at a and b and the thrusts, shearing forces and bending moments at sections e, f and g.

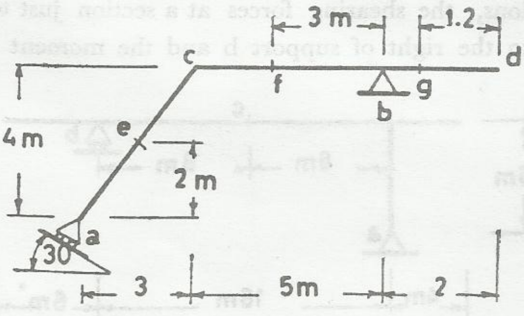



Fig. 6.43

(10) If a load moves from e to d on the structure shown in Fig. 6.44, draw the influence lines for the force in the link member bc and the bending moment at e in member ac.

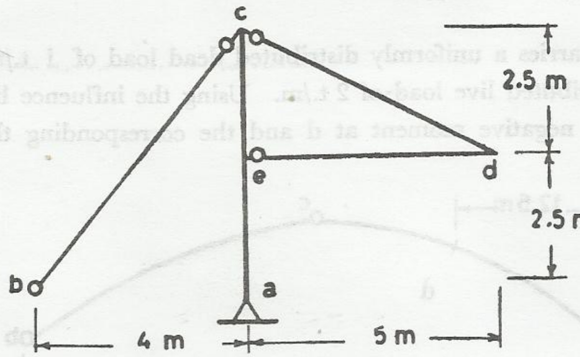



Fig. 6.44

(11) For the beam shown in Fig. 6.45, construct the influence lines for the reactions at a and b, shearing forces and bending moments at sections g and h. Find the extreme values of the shearing force and bending moment at h if the beam is subjected to a dead load of 0.75 t./m. and a live load of 1.5 t./m.

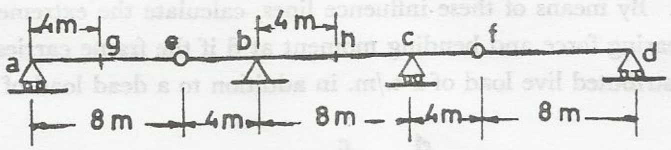



Fig. 6.45

(12) For the cantilever beam shown in Fig. 6.46, calculate the extreme values of the shearing force and bending moment at section e due to a uniformly distributed live load of 1.8 t./m. plus a moving concentrated load of 6 t., and a uniformly distributed dead load of 1.2 t./m.

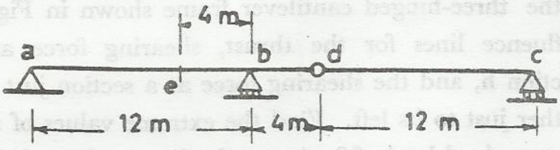



Fig. 6.46

(13) Taking the origin at a, the three-hinged parabolic arch shown in Fig. 6.47 is represented by the equation :

$$y = 0.8\,x - 0.016\,x^2$$

The arch carries a uniformly distributed dead load of 1 t./m. and a uniformly distributed live load of 2 t./m. Using the influence lines, find the maximum negative moment at d and the corresponding thrust.

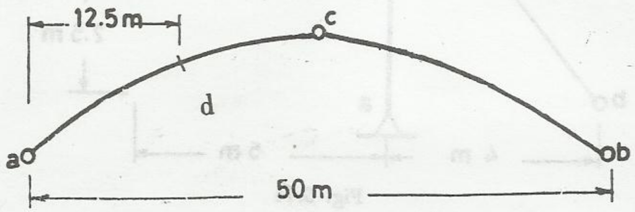



Fig. 6.47

What would be these values if the load is transmitted to the arch through secondary beams spaced 5 m. apart ?

(14) For the three-hinged frame shown in Fig. 6.48, construct the influence lines for the reactions, shearing force and bending moment at section d. By means of these influence lines, calculate the extreme values of the shearing force and bending moment at d if the frame carries a uniformly distributed live load of 2 t./m. in addition to a dead load of 1 t./m.

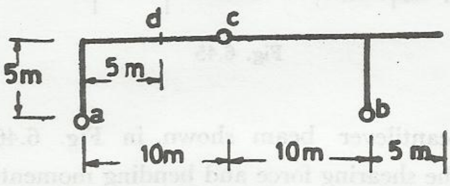



Fig. 6.48

(15) For the three-hinged cantilever frame shown in Fig. 6.49, construct the influence lines for the thrust, shearing force and bending moment at section h, and the shearing force at a section just to the right of b and another just to its left. Find the extreme values of the shearing force at h due to a dead load of 2 t./m. and a live load consisting of a uniformly distributed load of 2 t./m. plus a moving concentrated load of 6 t.

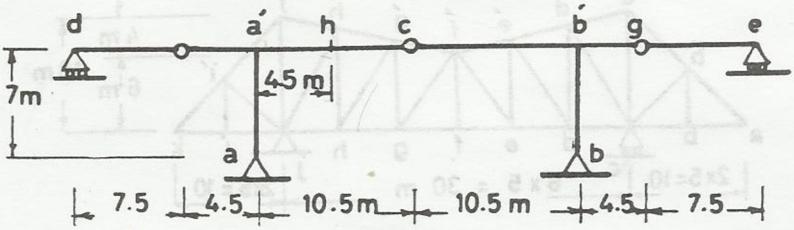



Fig. 6.49

(16) For the structure shown in Fig. 6.50, draw the influence lines for the thrusts, shearing forces and bending moments at sections g and h.

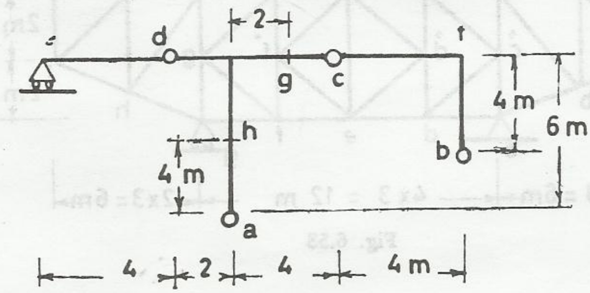



Fig. 6.50

(17) Construct the influence lines for the forces in members bc, cc', de' d'e', fe' and ee' of the truss shown in Fig. 6.51.

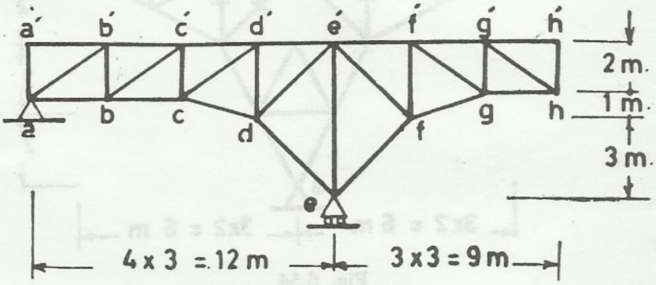



Fig. 6.51

(18) Construct the influence lines for the forces in members b'c', ec', ef, ef', e'f' and ff' of the truss shown in Fig. 6.52.

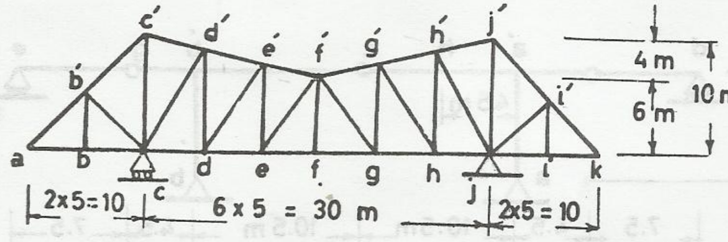



Fig. 6.52

(19) Construct the influence lines for the forces in members bc′, d′d″, e′d″ and ee′ of the truss shown in Fig. 6.53.

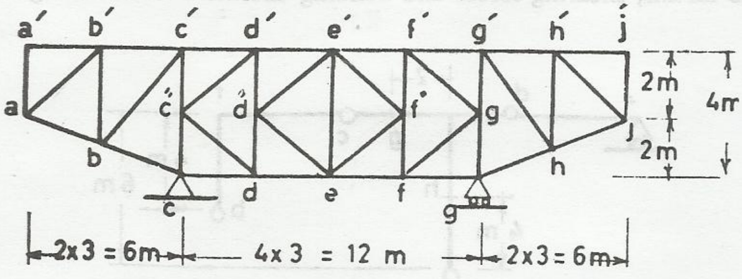



Fig. 6.53

(20) If a load moves from b to c on the truss shown in Fig. 6.54, construct the influence lines for the reactions and the forces in the marked members.

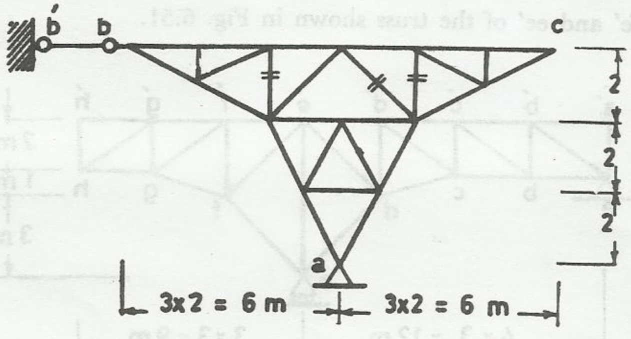



Fig. 6.54

(21) If a load moves on the bottom chord ab of the truss shown in Fig. 6.55, construct the influence lines for the reaction components at a, the force in the link member cc′ and the forces in the marked members.

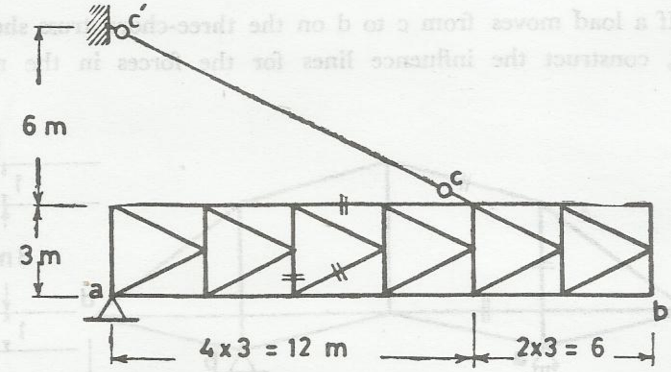



Fig. 6.55

(22) Construct the influence lines for the forces in members aa', k'b', bc, ff', ll' and gg' of the truss shown in Fig. 6.56.

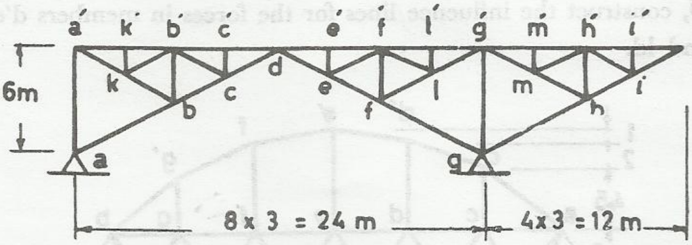



Fig. 6.56

(23) Fig. 6.57 shows a truss supported on two hinges at a and b. If a load moves from c to d, construct the influence lines for the reactions and the forces in members gf, fe and eg.

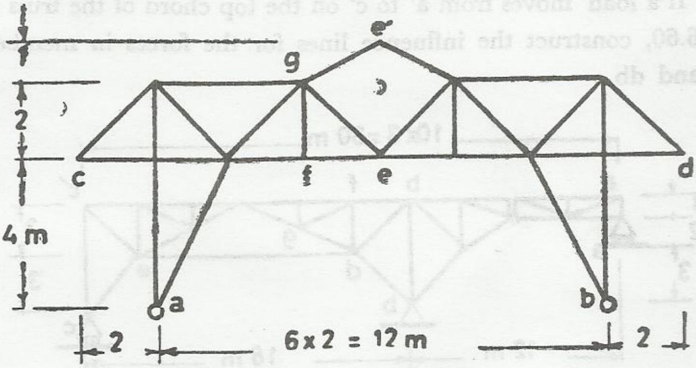



Fig. 6.57

(24) If a load moves from c to d on the three-chord truss shown in Fig. 6.58, construct the influence lines for the forces in the marked members.

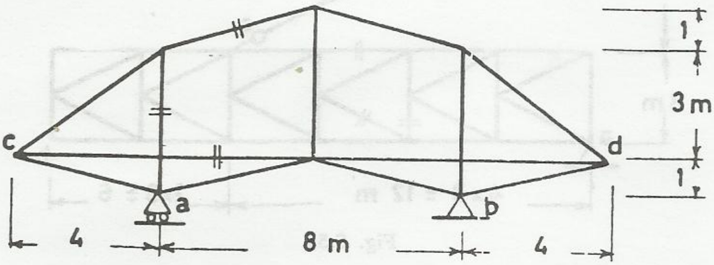



Fig. 6.58

(25) If a load moves from a to b on the three-chord truss shown in Fig. 6.59, construct the influence lines for the forces in members d'e', ee', de, kl and ld.

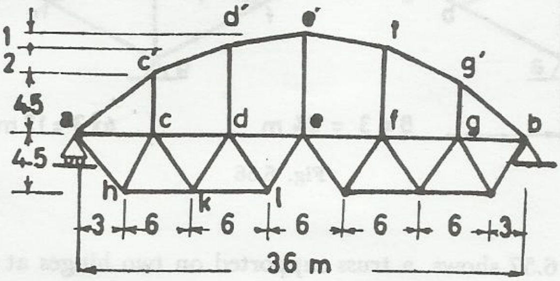



Fig. 6.59

(26) If a load moves from a' to c' on the top chord of the truss shown in Fig. 6.60, construct the influence lines for the forces in members de, dg, bd and db.

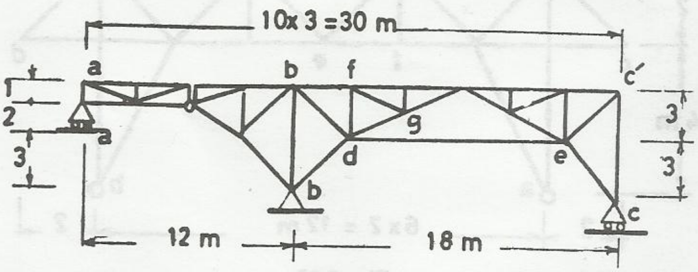



Fig. 6.60

(27) Construct the influence lines of the forces in members bc, cb and b′c′ of the three-hinged truss shown in Fig. 6.61. If the truss is subjected to a uniformly distributed dead load of 1 t./m. and a live load consisting of a uniformly distributed load of 1.5 t./m. and a concentrated moving load of 10 t., determine the extreme values of the forces in the three members.

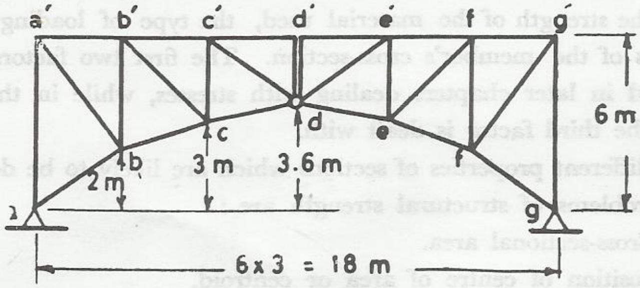



Fig. 6.61

CHAPTER 7

# PROPERTIES OF PLANE AREAS

## 7.1 Introduction

The carrying capacity of a structural member depends on three main factors; the strength of the material used, the type of loading and the properties of the member's cross-section. The first two factors will be considered in later chapters dealing with stresses, while in the present chapter the third factor is dealt with.

The different properties of sections which are likely to be determined in the problems of structural strength are :

(1) Cross-sectional area.
(2) Position of centre of area or centroid.
(3) Moment of inertia or second moment of area.
(4) Polar moment of inertia.
(5) Radius of gyration.
(6) Product of inertia.
(7) Principal axes of inertia.

It should be remembered that for a particular problem it may not be necessary to calculate but a few of these properties.

The cross-sectional area, A, needs no description and may be easily calculated for most structural shapes.

## 7.2 Centre of area or centroid

This is defined as the point 0 in the plane at which the area may be assumed to be concentrated to cause the same moment about an axis in the plane as the distributed area. Thus, referring to Fig. 7.1, the co-ordinates of the centroid $(\bar{x}, \bar{y})$ with respect to the rectangular axes y and x are given by :

$$\bar{x} = \frac{\int x dA}{A} \qquad \text{... 7.1 a}$$

$$\bar{y} = \frac{\int y dA}{A} \qquad \text{... 7.1 b}$$

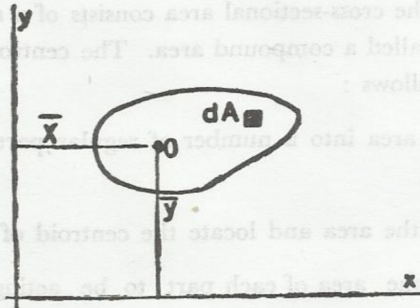



Fig. 7.1

where A is the total area, and $\int ydA$ and $\int xdA$ are the first moments of area or the statical moments about the x and y axes respectively. These are commonly denoted by $S_y$ and $S_x$. Thus,

$$S_x = \int ydA \qquad \text{... 7.2 a}$$

$$S_y = \int xdA \qquad \text{... 7.2 b}$$

It follows from the definition that if an area has an axis of symmetry, the centroid lies on it and if, further, it has two or more axes of symmetry, it is at their point of intersection. This result helps in detecting the centroids of many areas as may be seen from Fig. 7.2.

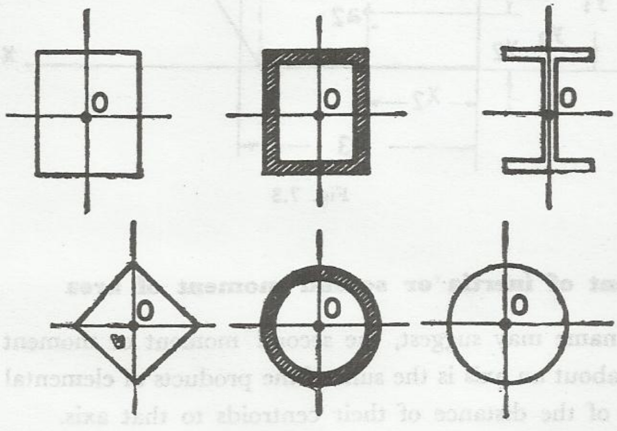



Fig. 7.2

The centroids of regular areas such as the triangle, the semi-circle, the parabola, etc. are best found by calculus and the student is advised to memorize them. These are given in Appendix 1.

In many cases, the cross-sectional area consists of a number of regular parts and is hence called a compound area. The centroid of a compound area is found as follows :

(a) Divide the area into a number of regular parts whose centroids are known.

(b) Determine the area and locate the centroid of each part.

(c) Assuming the area of each part to be acting at its centroid, take moments about two convenient rectangular axes. The centroid of the compound area shown in Fig. 7.3 has thus the co-ordinates $\bar{x},\bar{y}$ where $\bar{x}$ and $\bar{y}$ are given by :

$$\bar{x} = \frac{a_1 x_1 + a_2 x_2 + a_3 x_3}{A}$$

$$\bar{y} = \frac{a_1 y_1 + a_2 y_2 + a_3 y_3}{A}$$

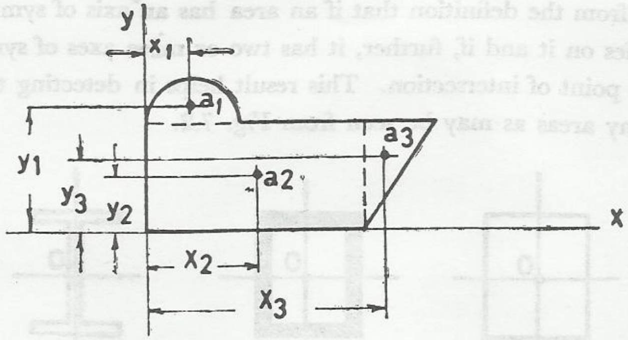



Fig. 7.3

### 7·3 Moment of inertia or second moment of area

As the name may suggest, the second moment or moment of inertia of an area about an axis is the sum of the products of elemental areas and the square of the distance of their centroids to that axis.

Thus, referring to Fig. 7.4, the moment of inertia which is usually denoted by I may be expressed mathematically as :

$$I_x = \int y^2 dA \qquad \ldots\ 7.3$$

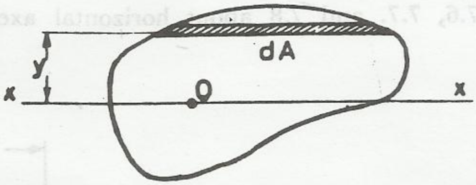



Fig. 7.4

It follows from the definition that the moment of inertia is always positive and its unit is length to the fourth power; $cm^4$ or $m.^4$. The moments of inertia of the common shapes given in Appendix 1 should be memorized as they are frequently used.

### 7.4 Theorem of parallel axes

The moment of inertia about any axis is equal to the moment of inertia about a parallel axis through the centroid plus the product of the total area and the square of the distance between the two axes. Referring to Fig. 7.5, this could be expressed mathematically as :

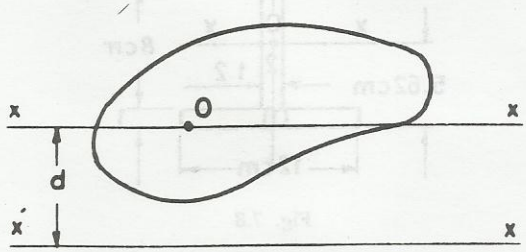



Fig. 7.5

$$I'_x = I_x + Ad^2 \qquad \dots 7.4$$

In transferring a moment of inertia between two axes, neither of which is through the centroid, it is necessary first to find the moment of inertia about the centroidal axis then transfer it to the required axis by using equation 7.4 twice. It is seen that I about a centroidal axis is smaller than that about any other parallel axis.

**Examples 7.1-7.3** Determine the moments of inertia of the areas shown in Figs. 7.6, 7.7. and 7.8 about horizontal axes through their centroids.

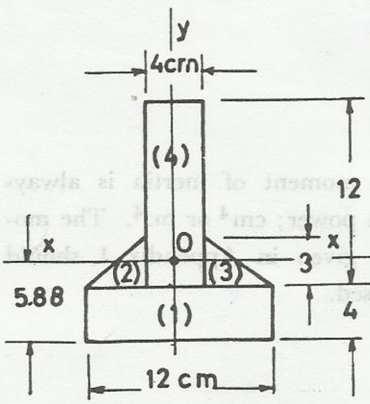



Fig. 7.6

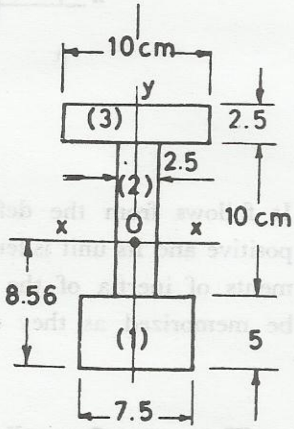



Fig. 7.7

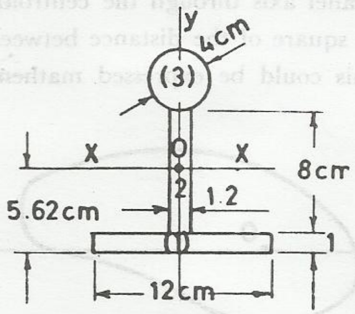



Fig. 7.8

Solution to example 7.1 :

Referring to Fig. 7.6, since the area is symmetrical about the y-axis, the centroid must lie on it, and it remains to calculate its distance from a convenient axis such as that at the lower edge of the section to completely locate the centroid. In order to do so, the area is divided into triangles and rectangles. For convenience, the computations are given in a tabulated form as shown in Table 7.1. The areas of individual parts are tabulated in column (2), the co-ordinates of the centroids of the elements are tabulated in column (3), and the first moments of area, Ay, are tabulated in column (4). The centroid of the total area is obtained by dividing the sum of the terms in column (4) by the sum of the terms in column (2). Thus,

$$\bar{y} = \frac{636}{108} = 5.88 \text{ cm.}$$

The moment of inertia of the total area about the x-axis will be obtained as the sum of the moments of inertia of the various elements about this axis; the moment of inertia of each element being given by equation 7.4. The terms $I_x$ and $Ad^2$ for all the elements are given in columns (5) and (6) respectively. The moment of inertia of the entire area about the x-axis is equal to the sum of all the terms in columns (5) and (6). Thus,

$$I_x = 646 + 1553 = 2199 \text{ cm}^4$$

| Element | A cm². | y cm. | Ay cm.³ | I cm⁴. | Ad² cm.⁴ |
|---|---|---|---|---|---|
| 1 | 48 | 2 | 96 | 64 | 728 |
| 2 | 6 | 5 | 30 | 3 | 5 |
| 3 | 6 | 5 | 30 | 3 | 5 |
| 4 | 48 | 10 | 480 | 576 | 815 |
| Total | 108 | | 636 | 646 | 1553 |

Table 7.1

Solution to Examples 7.2 & 7.3 :

The calculations for the other two areas in Figs. 7.7. and 7.8 are summerized in Tables 7.2 and 7.3 respectively.

| Element | A cm.² | y cm. | Ay cm.³ | I cm.⁴ | Ad² cm.⁴ |
|---|---|---|---|---|---|
| 1 | 37.5 | 2.5 | 94 | 78 | 1380 |
| 2 | 25 | 10 | 250 | 208 | 52 |
| 3 | 25 | 16.25 | 406 | 13 | 1482 |
| Total | 87.5 | | 750 | 299 | 2914 |

Table 7.2

$$\bar{y} = \frac{750}{87.5} = 8.56 \text{ cm.}$$

$$I_x = 299 + 2914 = 3213 \text{ cm.}^4$$

| Element | A $cm.^2$ | y cm. | Ay $cm.^3$ | I $cm.^4$ | $Ad^2$ $cm.^4$ |
|---|---|---|---|---|---|
| 1 | 12 | 0.5 | 6 | 1 | 315 |
| 2 | 9.6 | 5 | 48 | 51.2 | 3.7 |
| 3 | 12.6 | 11 | 138.6 | 12.6 | 364 |
| Total | 34.2 | | 192.6 | 64.8 | 682.7 |

Table 7.3

$$\bar{\bar{y}} = \frac{192.6}{34.2} = 5.62 \text{ cm.}$$

$$I_x = 64.8 + 682.7 = 7.47.5 \text{ cm.}^4$$

It should be noticed that if the area has more than one axis of symmetry, the calculations are simpler as the position of the centroid is known to be at the point of intersection of these axes. Also, if the moment of inertia of one of the compound areas about a vertical centroidal axis is required, calculations may still be conducted in the manner outlined above.

The following example will illustrate these points.

**Example 7.4** Find the moments of inertia for the section shown in Fig. 7.9 about the horizontal and vertical axes through the centroid.

Solution : The centroid is known to be at the point of intersection of the two axes of symmetry. The momenc of inertia of the whole area about the x-axis will be obtained as the sum of the moments of inertia of a rectangle about its centroidal axis and the moments of inertia of four triangles about an axis through their base. Thus,

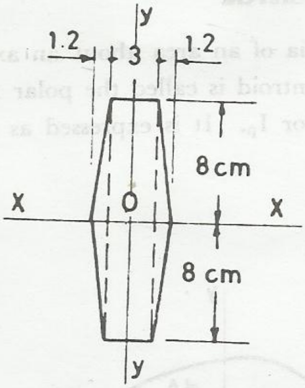



Fig. 7 9

$$I_x = \frac{3 \times 16^3}{12} + 4\left(\frac{1.2 \times 8^3}{12}\right) = 1228.8 \text{ cm.}^4$$

The moment of inertia of the whole area about the y-axis will be obtained as the sum of the moment of inertia of a rectangle about its centroidal axis and the moments of inertia of two triangles about an axis parallel to their centroidal axes and at a distance d,

$$d = 1.5 + 1.2/3 = 1.9 \text{ cm.}$$

Thus, $I_y = \frac{16 \times 3^3}{12} + 2\left(\frac{16 \times 1.2^3}{36} + \frac{16 \times 1.2 \times 1.9^2}{2}\right)$

$$= 106.4 \text{ cm.}^4$$

It should be noticed that a beam with a cross-section such as that shown in Fig. 7.9 may be used with the longer dimension either vertical or horizontal, and it will offer more resistance to bending when placed in the former position. This is because the moment of inertia about the x-axis is larger than that about the y-axis as indicated by the above calculations.

The student is advised to solve the previous examples independently and check his results against those given. Further, he can attempt to calculate the moments of inertia about the y-axis.

### 7.5 Polar moment of inertia

The moment of inertia of an area about an axis perpendicular to its plane and through its centroid is called the polar moment of inertia and is usually denoted by J or $I_p$. It is expressed as :

$$J = \int r^2 dA \qquad \text{... 7.5}$$

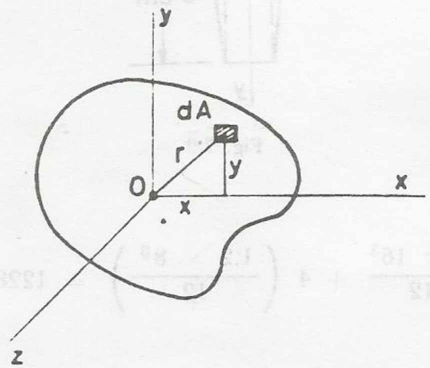



Fig. 7.10

Referring to Fig. 7.10, $r^2 = x^2 + y^2$ and from equation 7.5.

$$J = \int (x^2 + y^2)\, dA = I_x + I_y \qquad \text{...7.6}$$

Equation 7.6 shows that the sum of the moments of inertia about any two rectangular centroidal axes is constant. It follows that if the moment of inertia about one of these axes is a maximum, the moment about the other must be a minimum. The polar moment of inertia of a circular section is frequently used in problems dealing with torsion of shafts with circular cross-section (chapter 9). For this section, it is known from symmetry that $I_x = I_y$. Hence from equation 7.5.

$$J = \frac{\pi r^4}{4} + \frac{\pi r^4}{4} = \frac{\pi r^4}{2}$$

### 7.6 Radius of gyration

The radius of gyration of a section is the distance from the inertia axis that the entire area may be assumed to be concentrated in order to give the same moments of inertia. Thus by definition, the radius of gyration which is usually denoted by i may be expressed as :

$$I = Ai^2 \quad \text{... 7.7}$$

$$\text{or} \quad i = \sqrt{\frac{I}{A}} \quad \text{... 7.8}$$

It is seen that the point where the area is assumed to be concentrated is not the same as the centroid. It is also different for each inertia axis chosen.

$$i_x = \sqrt{\frac{I_x}{A}} \quad \text{... 7.9 a}$$

$$i_y = \sqrt{\frac{I_y}{A}} \quad \text{... 7.9 b}$$

This area property is of particular importance in regard to problems dealing with buckling of columns.

### 7·7 Product of inertia

The product of inertia about two rectangular axes x and y is defined as the sum of the products of elemental areas and the co-ordinates of their centroids to the reference axes. Thus, referring to Fig. 7.11, the product of inertia which is usually denoted by $I_{xy}$ may be expressed mathematically as :

$$I_{xy} = \int xydA \quad \text{... 7.10}$$

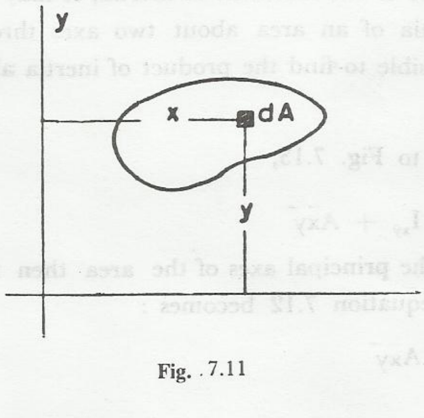



Fig. 7.11

It is evaluated by methods similar to those used in evaluating the moments of inertia. When both the x and y co-ordinates have similar signs, either positive or negative, the product of inertia is positive, but when the two co-ordinates have different signs, it is negative. If the area is symmetrical with respect to one of the axes as shown in Fig. 7.12, each elemental area dA on one side of the axis of symmetry will have a

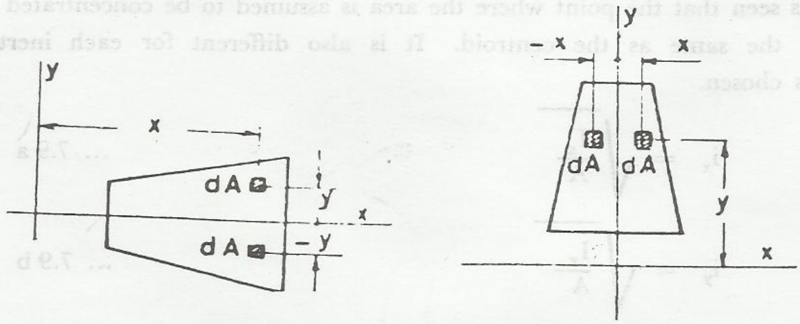



Fig. 7.12

corresponding area on the other side which while having one similar co-ordinate. the second is of opposite sign. It is obvious therefore that the sum of the products of inertia of the two elements will be zero, and consequently the product of inertia of the whole area will also be zero. Thus, it may be stated that when either one of the two centroidal axes is an axis of symmetry, the product of inertia is zero.

$$I_{xy} = 0 \qquad \dots 7.11$$

Such two axes are called principal axes of inertia.

As in the case of the moments of inertia, it may be shown that if the product of inertia of an area about two axes through the centroid is known, it is possible to find the product of inertia about any other set of parallel axes.

Thus, referring to Fig. 7.13,

$$I_{x'y'} = I_{xy} + A\bar{x}\bar{y} \qquad \dots 7.12$$

If x and y are the principal axes of the area then from equation 7.11 $I_{xy} = 0$, and equation 7.12 becomes :

$$I_{x'y'} = A\bar{x}\bar{y} \qquad \dots 7.13$$

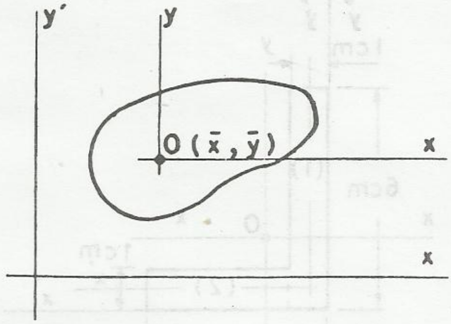



Fig. 7.13

For an area composed of symmetrical elements, the product of inertia of the entire area is obtained as the sum of the values found by using equation 7.13 for each element. The following examples will illustrate this point.

**Example 7.5** Find the product of inertia $I_{xy}$ for the Z-section shown in Fig. 7.14.

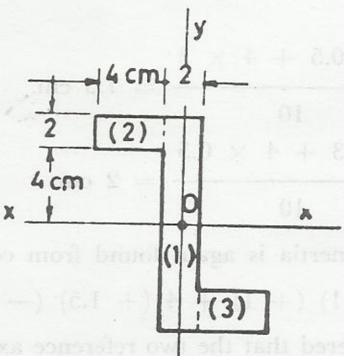



Fig. 7.14

Solution :

The total area is divided into three rectangular elements (1), (2) and (3). Rectangle (1) is symmetrical about both x and y axes, hence its product of inertia is zero. Rectangles (2) and (3) are symmetrical about axes through their centroids, therefore their product of inertia may be found from equation 7.13.

for rectangle (2), $I_{xy} = (4 \times 2)\ (-3)\ (+5) = -120$ cm.$^4$

for rectangle (3), $I_{xy} = (4 \times 2)\ (+3)\ (-5) = -120$ cm.$^4$

for the whole section, $I_{xy} = -120 - 120 = -240$ cm.$^4$

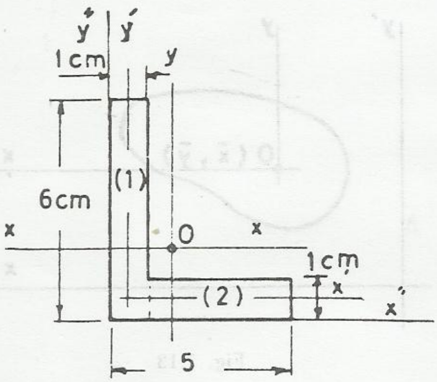



Fig. 7.15

**Example 7.6** Find the product of inertia about horizontal and vertical axes through the centroid of the unequal angle shown in Fig. 7.15.

Solution : The total area is divided into two rectangular elements (1) and (2) as shown. The co-ordinates of the centroid with respect to two axes x'' and y'' chosen as the outer edges of the angle are obtained as follows :

$$x_1 = \frac{6 \times 0.5 + 4 \times 3}{10} = 1.5 \text{ cm.}$$

$$y_1 = \frac{6 \times 3 + 4 \times 0.5}{10} = 2 \text{ cm.}$$

and the product of inertia is again found from equation 7.13

$$I_{xy} = 6\,(-1)\,(+1) + 4\,(+1.5)\,(-1.5) = -15 \text{ cm.}^4$$

It should be remembered that the two reference axes x'' and y'' are arbi trarily chosen. If the reference axes were chosen as x' and y' through the centroids of the two rectangles (1) and (2), another shorter solution may be worked out.

Since rectangle (1) is symmetrical about the y' axis and rectangle (2) is symmetrical about the x' axis then $I_{x'y'} = 0$. The co-ordinates of the centroid with respect to these two axes are :

$$x_2 = \frac{4 \times 2.5}{10} = 1 \text{ cm.}$$

$$y_2 = \frac{6 \times 2.5}{10} = 1.5 \text{ cm.}$$

and the product of inertia is readily obtained from equation 7.11.

$$I_{x'y'} = 0 = I_{xy} + 10 \times 1 \times 1.5$$

$$I_{xy} = -15 \text{ cm.}^4$$

which is identical to the value obtained before.

## 7.8 Moments and product of inertia about inclined axes

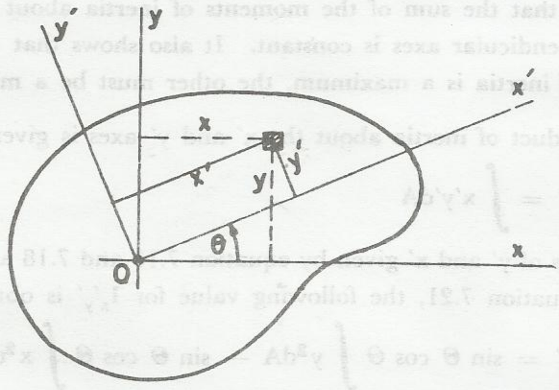



Fig. 7.16

The moments of intertia of an area about inclined axes may be obtained from the properties of the area with respect to the horizontal and vertical axes.

Referring to Fig. 7.16,

$$I_x' = \int y'^2 dA \qquad \dots 7.14$$

$$y' = y \cos\Theta - x \sin\Theta \qquad \dots 7.15$$

If this value of y' is substituted in equation 7.14, the following value for $I_x'$ is obtained,

$$I_x' = \cos^2\Theta \int y^2 dA + \sin^2\Theta \int x^2 dA - 2 \sin\Theta \cos\Theta \int xy dA \qquad \dots 7.16$$

The first and second integrals of equation 7.16 represent the moments of inertia of the area about the x and y axes respectively. The last integral represents the product of inertia $I_{xy}$. Equation 7.16 may thus be written as :

$$I_x' = I_x \cos^2\Theta + I_y \sin^2\Theta - I_{xy} \sin 2\Theta \qquad \dots \mathbf{7.17a}$$

Using the trigonometric relation,

$$x' = x \cos\Theta + y \sin\Theta \qquad \dots 7.18$$

a similar expression may be derived for the moment of inertia about the y′-axis.

$$I_{y}' = I_x \sin^2\Theta + I_y \cos^2\Theta + I_{xy} \sin 2\Theta \quad \ldots 7.19\text{ a}$$

If equations 7.17 and 7.19 are added, the following relationship is obtained.

$$I_x' + I_y' = I_x + I_y \quad \ldots 7.20$$

This shows that the sum of the moments of inertia about any two centroidal perpendicular axes is constant. It also shows that if one of the moments of inertia is a maximum, the other must be a minimum.

The product of inertia about the x′ and y′ axes is given by :

$$I_{x'y'} = \int x'y'dA \quad \ldots 7.21$$

If the values of y′ and x′ given by equation 7.15 and 7.18 are then substituted in equation 7.21, the following value for $I_{x'y'}$ is obtained.

$$I_{x'y'} = \sin\Theta \cos\Theta \int y^2dA - \sin\Theta \cos\Theta \int x^2dA + \cos^2\Theta \int xydA - \sin^2\Theta \int xydA$$

$$I_{x'y'} = (I_x - I_y) \sin\Theta \cos\Theta + I_{xy} (\cos^2\Theta - \sin^2\Theta) \ldots 7.22\text{ a}$$

If the relationships

$$\cos^2\Theta = \frac{1}{2} + \frac{1}{2}\cos 2\Theta,\ \sin^2\Theta = \frac{1}{2} - \frac{1}{2}\cos 2\Theta, \text{ and } \sin 2\Theta = 2 \sin\Theta \cos\Theta$$

ore substituted in equations 7.17a, 7.19a and 7.22a, the result will by :

$$I_x' = \frac{I_x + I_y}{2} + \frac{I_x - I_y}{2}\cos 2\Theta - I_{xy} \sin 2\Theta \ldots \quad 7.17\text{ b}$$

$$I_y' = \frac{I_x + I_y}{2} - \frac{I_x - I_y}{2}\cos 2\Theta + I_{xy} \sin 2\Theta \quad \ldots 7.19\text{ b}$$

$$I_{x'y'} = \frac{I_x - I_y}{2}\sin 2\Theta + I_{xy} \cos 2\Theta \quad \ldots 7.22\text{ b}$$

## 7.9 Principal axes of inertia

The principal axes of inertia of a plane area, which are usually referred to as axes u and v, may be defined as the two perpendicular axes passing through the centroid of the area such that the moment of inertia about one is a maximum and about the other a minimum, or alternatively, the two axes the product of inertia about which is zero.

As may be seen from equations 7.17 and 7.19, the moment of inertia of an area about an inclined axis is a function of the angle $\Theta$. The angle $\Theta$ at which the moment of inertia is a maximum or a minimum is obtained by differentiating equation 7.17 with respect to $\Theta$ and equating the derivative to zero. Thus,

$$\frac{dI_x}{d\Theta} = - I_x \sin 2\Theta + I_y \sin 2\Theta - 2 I_{xy} \cos 2\Theta = 0$$

or, $$\tan 2\Theta = \frac{- 2 I_{xy}}{I_x - I_y} \qquad \ldots 7.23$$

Since there are two angles under $360^0$ which have the same tangent, equ-equation 7.23 defines two values for the angle $2\Theta$ which are at $180^o$ The two corresponding values of $\Theta$ will be at $90^0$. By definition, the two perpendicular axes defined by equation 7.23 are the principal axes.

By the alternative definition, if $\Theta$ is to define the principal axes, $I_{x'y'} = 0$. Thus, **from equation 7.22 b,**

$$\tan 2\Theta = \frac{- 2 I_{xy}}{I_x - I_y}$$

which is the same result obtained in equation 7.23. The moments of inertia about the principal axes $I_u$ and $I_v$ may be obtained by substituting the values of $\Theta$ obtained from equation 7.23 into equation 7.17**b.** Thus,

$$I_u = \frac{I_x + I_y}{2} + \sqrt{\left(\frac{I_x - I_y}{2}\right)^2 + I^2_{xy}} \qquad \ldots 7.24$$

$$I_v = \frac{I_x + I_y}{2} - \sqrt{\left(\frac{I_x - I_y}{2}\right)^2 + I^2_{xy}} \qquad \ldots 7.25$$

where $I_u$ and $I_v$ are the maximum and minimum values of the moments of inertia respectively.

## 7.10 Semi-graphical treatment — Mohr's circle of inertia

The expressions for the moments and products of inertia about inclined axes given by equations 7.17 and 7.22 are difficult to remember. It is convenient, therefore, to use a graphical solution which is easy to remember. A careful study of equations $7.17_a$ and $7.22_a$ shows that they represent a circle written in a parametric form. That they represent a circle is made clearer by first re-writing them as :

$$I_{x\theta} - \frac{I_x + I_y}{2} = \frac{I_x - I_y}{2} \cos 2\Theta - I_{xy} \sin 2\Theta$$

$$I_{xy\theta} = \frac{I_x - I_y}{2} \sin 2\Theta + I_{xy} \cos 2\Theta$$

This is done by making use of the trigonometric relationships,

$$\sin^2 \Theta = 1/2 - 1/2 \cos 2\Theta$$

$$\cos^2 \Theta = 1/2 + 1/2 \cos 2\Theta$$

Eliminating the parameter $\Theta$ by squaring and adding,

$$\left( I_{x\theta} - \frac{I_x + I_y}{2} \right)^2 + I^2_{xy\theta} = \left( \frac{I_x - I_y}{2} \right)^2 + I^2_{xy} \quad \ldots \quad 7.26$$

However, in every problem $I_x$, $I_y$ and $I_{xy}$ are constants while $I_{x\theta}$ and $I_{xy\theta}$ are the variables. Hence equation 7.26 may be written in a simplified form,

$$(I_{x\theta} - a)^2 + I^2_{xy\theta} = b^2 \quad \ldots \quad 7.27$$

where $a = \dfrac{I_x + I_y}{2}$ ... 7.28 a

$b = \sqrt{\left( \dfrac{I_x - I_y}{2} \right)^2 + I^2_{xy}}$ ... 7.28 b

Equation 7.27 is the familiar expression known in analytical geometry; $(x-a)^2 + y^2 = b^2$ of a circle of radius b and centre at (a, 0). Hence if a circle satisfying this equation is plotted, the co-ordinates of a point (x, y) on this circle correspond to $I_{x\theta}$ and $I_{xy\theta}$ for a particular inclination $\Theta$ with respect to the reference axes x and y. The x co-ordinate represents the moment of inertia while the y co-ordinate represents the product of inertia. The circle so constructed is called ***Mohr's circle* of inertia.**

There are several methods of plotting the circle defined by equation 7.26. It may be constructed by locating the centre at (a, 0) and using the radius b given by equation 7.28 b, but this is not the best procedure for the purpose at hand. The moment of inertia and product of inertia about two centroidal rectangular axes x and y are usually known. These two values, $I_x$ and $I_{xy}$, define one point on the circle. This Knowledge, together with the fact that the centre of circle is located on the abscessa at $(I_x + I_y)/2$ is sufficient to plot the circle. The procedures are outlined below with reference to Fig. 7.17.

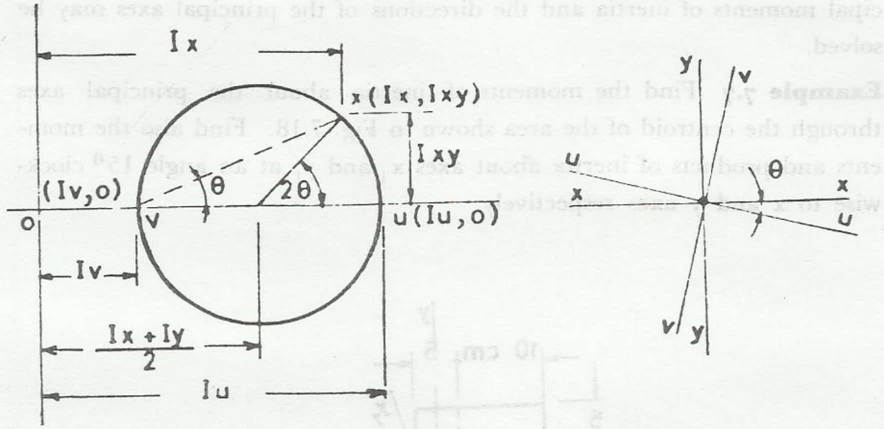



Fig. 7.17

(1) Set up a rectangular co-ordinate system of axes where the horizontal axis is the moment of inertia and the vertical axis is the product of inertia axis. Directions of positive axes are taken, as usual, to the right and upward.

(2) Locate the centre of the circle, which is on the horizontal axis at a distance of $(I_x+I_y)/2$ from the origin o

(3) Locate the point x of co-ordinates $I_x$ and $I_{xy}$ with respect to the origin; $I_{xy}$ measured upwards if positive and downwards if negative.

(4) Connect the centre of the circle found in (2) with the point located in (3) and determine this distance which is the radius of the circle.

(5) Draw a circle with the radius found in (4). The two points of intersection with the horizontal axis give the values of the two principal moments of inertia $I_u$ and $I_v$.

(6) The directions of the u-axis makes an angle $\Theta$, found from the geometry of the circle, with the x-axis. A clockwise rotation of the axis to be found corresponds to a clockwise rotation round the circle. The v-axis is obtained in a similar manner and makes an angle $\Theta$ with the y-axis.

By following the above procedures in constructing Mohr's circle of inertia, the complete problem of determining the magnitude of the principal moments of inertia and the directions of the principal axes may be solved.

**Example 7.7** Find the moments of inertia about the principal axes through the centroid of the area shown in Fig. 7.18. Find also the moments and products of inertia about axes $x_1$ and $y_1$ at an angle $15^0$ clockwise to x and y axes respectively.

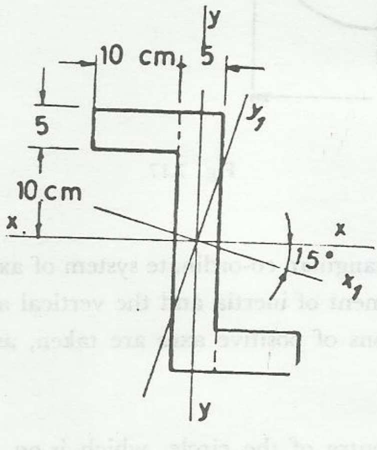



Fig. 7.18

Solution :

The moments and product of inertia about x and y axes are obtained as follows :

$$I_x = \frac{5 \times 30^3}{12} + 2\left(\frac{10 \times 5^3}{12} + 10 \times 5 \times 12.5^2\right) = 27098 \text{ cm.}^4$$

$$I_y = \frac{30 \times 5^3}{12} + 2\left(\frac{5 \times 10^3}{12} + 10 \times 5 \times 7.5^2\right) = 6774 \text{ cm.}^4$$

$$I_{xy} = (5 \times 10)(-7.5)(12.5) + (5 \times 10)(7.5)(-12.5) = -9360 \text{ cm.}^4$$

Mohr's circle may now be plotted from these three values. The centre of the circle has the co-ordinates $\left( \frac{I_x + I_y}{2}, 0 \right)$ or (16936,0), while point x has the co-ordinates $(I_x, I_{xy})$ or (27098, — 9360) as shown in Fig. 7.19 a. Therefore,

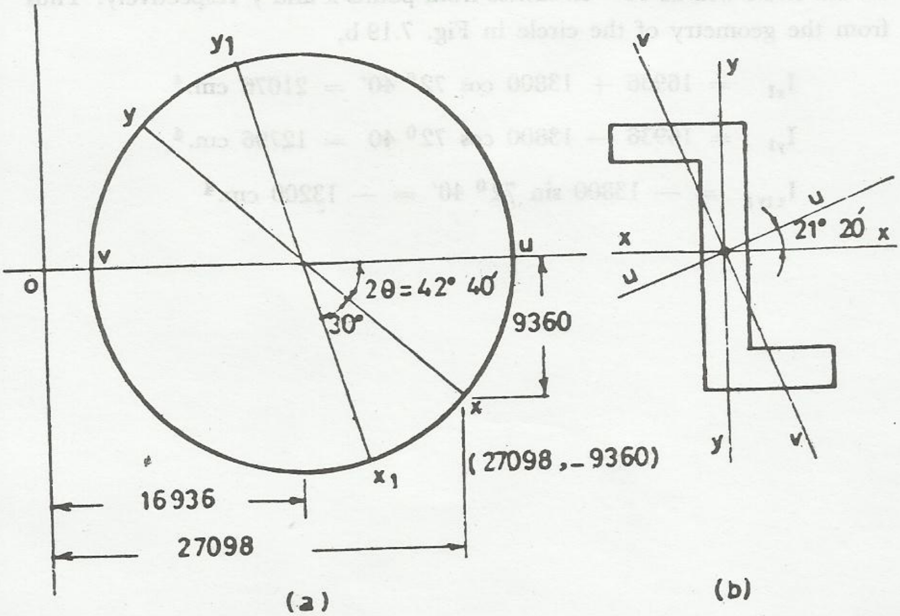



Fig. 7.19

$$\text{Radius} = \sqrt{9360^2 + 10162^2} = 13800$$

$$\tan 2\theta = \frac{9360}{10162} = 0.922$$

$$2\theta = 42^0\ 40'$$

The principal moments of inertia are equal to the distance from the origin to the centre of the circle plus or minus its radius.

$$I_u = 16936 + 13800 = 30736 \text{ cm.}^4$$

$$I_v = 16936 - 13800 = 3136 \text{ cm.}^4$$

Since point u on the circle is anticlockwise from point x, the u-axis is anticlockwise from the x-axis at an angle,

$$\theta = \frac{42^0\ 40'}{2} = 21^0\ 20'$$

The v-axis is perpendicular to the u-axis through the centroid as shown in Fig. 7.19 b.

The moments and product if inertia about the $x_1$ and $y_1$ axes are obtained from the co-ordinates of points $x_1$ and $y_1$ on the circle. Since $x_1$ and $y_1$ axes are 15° clockwise from x and y axes, the points $x_1$ and $y_1$ on the circle will be $30^0$ clockwise from points x and y respectively. Thus from the geometry of the circle in Fig. 7.19 b,

$$I_{x1} = 16936 + 13800 \cos 72^0\, 40' = 21076 \text{ cm.}^4$$

$$I_{y1} = 16936 - 13800 \cos 72^0\, 40 = 12796 \text{ cm.}^4$$

$$I_{x1y1} = -\, 13800 \sin 72^0\, 40' = -\, 13200 \text{ cm.}^4$$

## EXAMPLES TO BE WORKED OUT

(1) - (9) Calculate the moments of inertia about horizontal and vertical axes through the centroids of the plane areas shown in Figs. 7.20-7.28

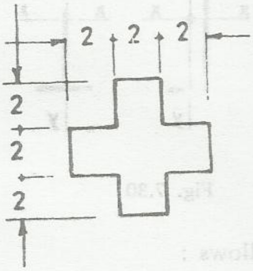



Fig. 7.20

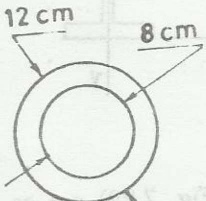



Fig. 7.21

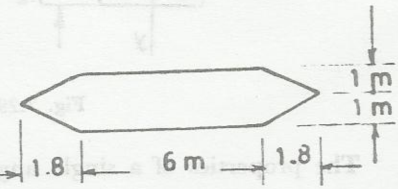



Fig. 7.22

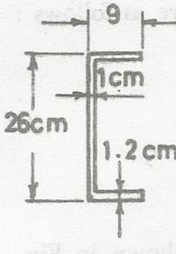



Fig. 7.23

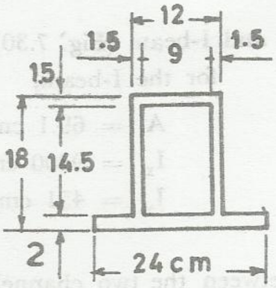



Fig. 7.24

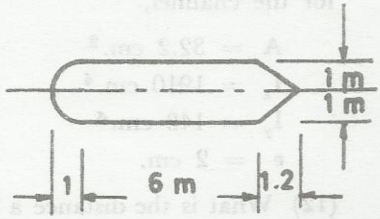



Fig. 7.25

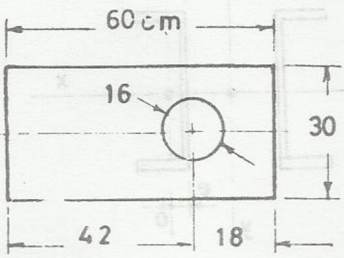



Fig 7..26

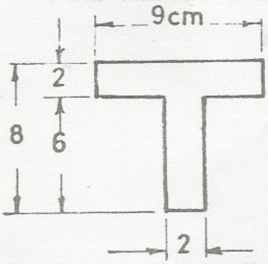



Fig. 7.27

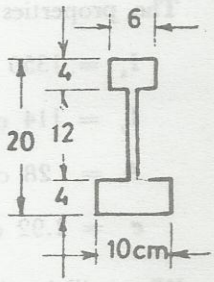



Fig. 7.28

(10), (11) Calculate the values of $I_x$ and $I_y$ of the compound sections shown in Figs. 7.29 and 7.30.

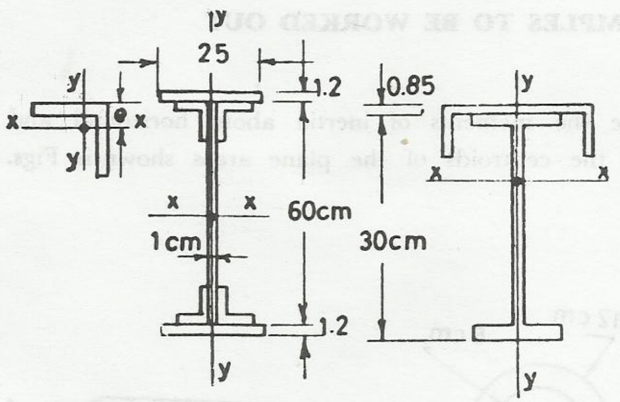



Fig. 7.29

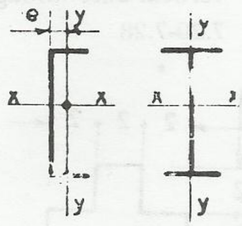



Fig. 7.30

The properties of a single angle (Fig. 7.29) are as follows :

$I_x = I_y = 207$ cm.[4]

$A = 22.7$ cm.[2]

$e = 2.9$ cm.

The properties of the channel and I-beam (Fig. 7.30) are as follows :

for the channel,

$A = 32.2$ cm.[2]

$I_x = 1910$ cm.[4]

$I_y = 148$ cm.[4]

$e = 2$ cm.

for the I-beam,

$A = 69.1$ cm.[2]

$I_x = 9300$ cm.[4]

$I_y = 451$ cm.[4]

(12) What is the distance a between the two channels shown in Fig. 7.31 so as to produce equal moments of inertia about the x and y axes ?

The properties of one channel are :

$I_x = 1350$ cm.[4]

$I_y = 114$ cm.[4]

$A = 28$ cm.[2]

$e = 1.92$ cm.

What will be the radius of gyration ?

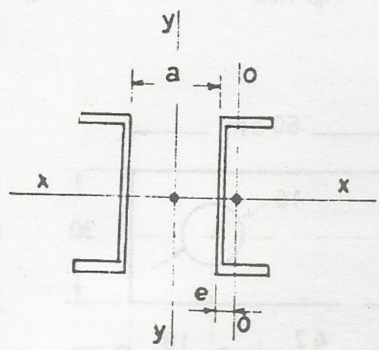



Fig. 7.31

(13)- (20) Find the moments and product of inertia about horizontal and vertical axes through the centroids of each of the areas given in Figs. 7.32 to 7.39. Then find, either analytically or semi-graphically using Mohr's circle of inertia, the principal moments of inertia and the directions of the principal axes showing them on sketches of the areas in each case.

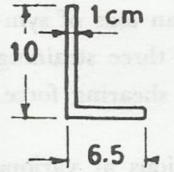



Fig. 7.32

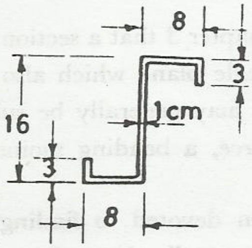



Fig. 7.33

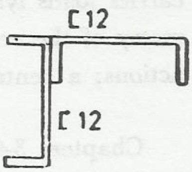



Fig. 7.34

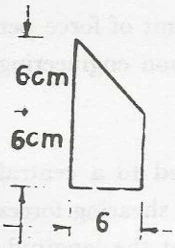



Fig. 7.35

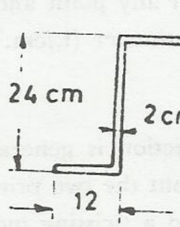



Fig. 7.36

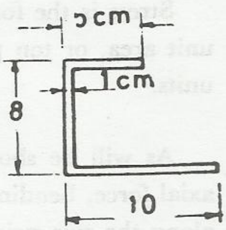



Fig. 7.37

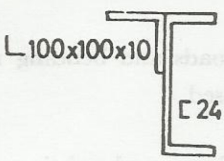



Fig. 7.38

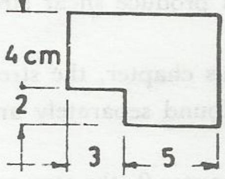



Fig. 7.39

# CHAPTER 8
# NORMAL STRESSES

## 8.1 Introduction

It has been shown in chapter 3 that a section in a straight beam which carries loads lying in a single plane which also contains an axis of symmetry of the cross-sections may generally be subjected to three straining actions; a central axial force, a bending moment and a shearing force.

Chapters 3-6 have been devoted to finding these actions at various parts of different types of statically determinate structures. After finding the values of the straining actions at the sections of a structural member, it is necessary to find the resulting stresses in order to design the member.

Stress is the force intensity at any point and has the unit of force per unit area, or ton per square centimeter ($t./cm.^2$) in common engineering units.

As will be shown later, a section is generally subjected to a central axial force, bending moment about the two principal axes, shearing forces along the two principal axes and a twisting moment about the longitudinal axis of the beam.

The axial force and bending moments produce normal stresses, i.e. perpendicular to the section, while the shearing forces and the twisting moments produce shear stresses, i.e. in the plane of the section.

In this chapter, the stresses due to axial loads and bending moments will be found separately and then superimposed.

In chapter 9, the stresses due to shearing forces and twisting moments and their combined effect will be found.

Finally, in chapter 10, the combined effect of normal and shear stresses; principal stresses will be considered.

**8.2 Stresses due to central normal force**

Consider a member having a cross-sectional area A and subjected to an axial load N as shown in Fig. 8.1. At any cross-section such as x-x, a

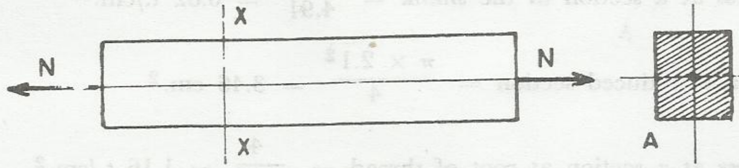



Fig. 8.1

number of forces acting on elemental unit areas are developed. By definition, these forces are the stresses, and for equilibrium, the resultant of all the stresses must coincide with N, i.e. it must pass through the centroid. Hence, the stress must be uniformly distributed over the cross-section and its value is given by :

$$f = \pm\ N/A$$

The positive sign is used when N is tensile while the negative sign is used when N is compressive.

The stress distribution across the section may be represented diagramatically as shown in Fig. 8.2 c.

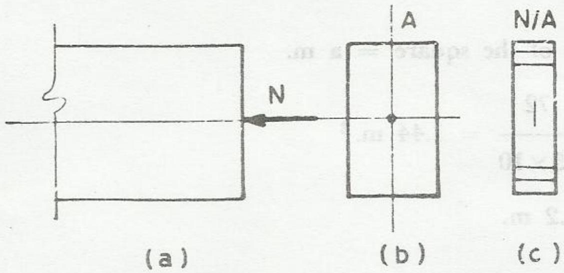



Fig. 8.2

**Example 8.1** A bolt 25 mm. diameter carries an axial tension of 4 t. Calculate the tensile stress at a section in the shank and another at the root of the thread where the diameter is reduced to 21 mm.

Solution :

Area of bolt at a section in the shank $= \frac{\pi \times 2.5^2}{4} = 4.91$ cm.$^2$

Stress at a section in the shank $= \frac{4}{4.91} = 0.82$ t./cm.$^2$

Area of reduced section $= \frac{\pi \times 2.1^2}{4} = 3.46$ cm.$^2$

Stress at a section at root of thread $= \frac{4}{3.46} = 1.16$ t./cm.$^2$

**Example 8.2** A short hollow steel tube 10 cm. external diameter and 7.5 cm. internal diameter is subjected to an axial compression of 37 t. Calculate the stress in the tube.

Solution :

$$A = \pi (10^2 - 7.5^2)/4 = 34.4 \text{ cm.}^2$$

$$f = \frac{-37}{34.4} = -1.08 \text{ t./cm.}^2$$

**Example 8.3** Calculate the cross-sectional dimensions of a square brick pier to support an axial load of 72 t. including its own weight if the allowable compressive stress for brickwork is 5 kg/cm.$^2$

Solution :

Let the side of the square = a m.

$$a^2 = \frac{72}{5 \times 10} = 1.44 \text{ m.}^2$$

$$a = 1.2 \text{ m.}$$

## 8.3 Stresses due to bending moment

In this theory for the determination of bending stresses, it will be assumed that :

(1) The member is subjected to pure bending, i.e. no shearing force or thrust act.

(2) Plane of loading passes through the centroid of the section and contains ι principal axis.

(3) The material is not stressed beyond its proportional limit.

(4) The modulus of elasticity or Young's modulus of the material is the same for tension and compression.

(5) Plane sections before bending remain plane after bending.

Fig. 8.3 shows an originally straight beam acted upon by two end couples of moment M. Under pure bending, the beam will be bent in a circular arc of radius R. If the beam is considered as composed of longitudinal fibers, the fibers in the upper part of the beam will be contracted and those in the lower part will be extended. The fiber on some intermediate plane, called the *neutral plane*, will be neither contracted nor extended. The intersection of the neutral plane and the plane of the cross-section is a line called the *neutral axis* of the cross-section; NA.

Consider two sections a-a and b-b at a distance dx apart. The strain at a fiber distance y from the neutral axis is given by :

$$\text{strain} = \frac{\text{extension}}{\text{original length}}$$

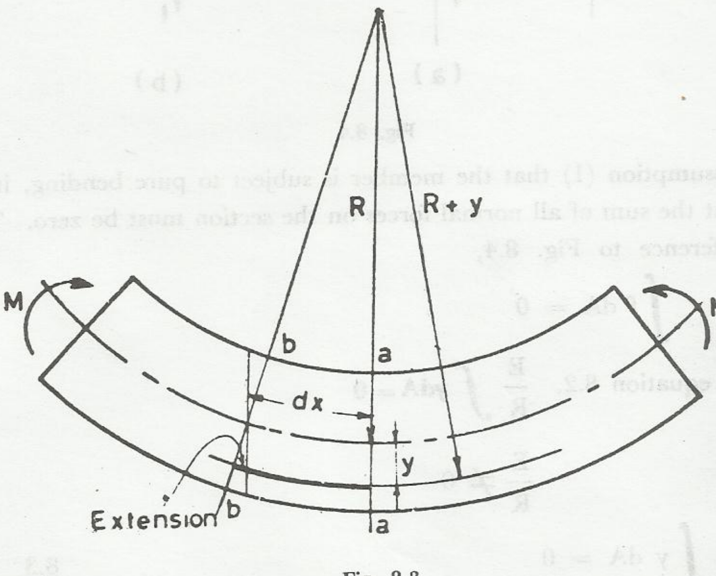



Fig. 8.3

From similar triangles,

$$\text{extension} = \frac{ydx}{R}$$

$$\text{strain} = \frac{y}{R}$$

and from assumption (3) that the material is not stressed beyond its proportional limit, the stress is given by :

$$f = \frac{Ey}{R} \qquad \text{... 8.2}$$

i.e. the stress due to bending moment is proportional to the distance from the neutral axis. Such stress distribution is shown in Fig. 8.4 b.

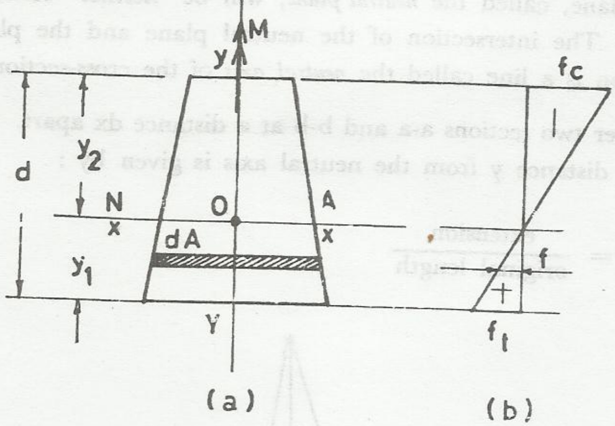



Fig. 8.4

From assumption (1) that the member is subject to pure bending, it follows that the sum of all normal forces on the section must be zero. Thus with reference to Fig. 8.4,

$$\int f\,dA = 0$$

or from equation 8.2, $\dfrac{E}{R}\displaystyle\int ydA = 0$

but $\dfrac{E}{R} \neq 0$

then, $$\int y\,dA = 0 \qquad 8.3$$

Since the first moment of area about an axis is equal to zero only if this axis passes through the centroid, it follows that the neutral axis must pass through the centroid of the cross-section.

For equilibrium, the sum of the moments of all the normal forces on the cross-section about the x-axis must be equal to the applied moment M. Thus,

$$\int f \, dA \, y = M$$

or from equation 8.2, $\frac{E}{R} \int y^2 dA = M$

but $\int y^2 dA$ is the second moment of area $I_x$. Thus,

$$\frac{EI_x}{R} = M \qquad \dots 8.4$$

This equation shows that there is a linear relationship between the applied moment M and the curvature of the beam; $K = 1/R$. The constant $EI_x$ in this linear relationship is called the *bending rigidity* of the beam. It is the product of the modulus of elasticity and the moment of inertia of the section about the axis of bending.

Substituting from equation 8.2 into 8.4,

$$f = \frac{M\,y}{I_x} \qquad \dots 8.5$$

This formula gives the bending stress in the section at any fiber distance y from the neutral axis. It also shows that the stress varies linearly from zero on the fiber at the centroid to maximum values at the fibers furthermost from the centroid. If the distances to these fibres are $y_1$ and $y_2$ (Fig. 8.4 a), then the maximum stresses are given by :

$$f_t = \frac{My_1}{I_x} \text{ and } f_c = \frac{My_2}{I_x} \qquad \dots 8.6$$

If the section is symmetrical about the x-axis,

$$y_1 = y_2 = d/2$$

and

$$f_t = f_c = \frac{Md}{2I_x} \qquad \dots 8.7$$

In most practical problems, the maximum stresses given by equation 8.7 are the quantities required. It is noted that both the moment of inertia $I_x$ and the depth of the section d are constants for a given section. Hence, $\frac{2I_x}{d}$ is constant. Further, since this ratio depends only on the dimensions of the section, it can be considered as one of its properties. This ratio is called the *elastic section modulus* and is denoted by Z. Thus, equations 8.7 may be expressed as :

$$f_t = f_c = \frac{M}{Z} \qquad \text{... 8.8}$$

In the case of rolled steel sections common in structural work, the values of Z are normally taken directly from tables similar to those given in Appendix 2. For examination purposes, however, the student may be expected to calculate his own value of Z.

It should be noted that for sections which are not symmetrical about the x-axis, there will be two section moduli,

$$Z_1 = \frac{I_x}{y_1} \quad \text{and} \quad Z_2 = \frac{I_x}{y_2} \qquad \text{... 8.9}$$

Equation 8.8 needs further consideration. The maximum moment that the section can carry which is called the *moment of resistance* of the section, $M_r$, is given by :

$$M_r = f\,Z \qquad \text{... 8.10}$$

where f is the maximum allowable stress of the material used. Formula 8.10 shows that the efficient sections for resisting bending moment should have as large values of Z as possible for a given amount of material. This is accomplished by locating as much of the material as far as possible from the neutral axis.

For equilibrium, there is no moment for all the normal forces on the section about the y-axis. Thus,

$$\int f\,dA\,x = 0$$

or from equation 8.2, $\frac{E}{R} \int xydA = 0$

but $\frac{E}{R} \neq 0$

then, $\int x\ y\ dA = 0$ ... 8.11

but $\int$ xydA is the product of inertia of the section about the x and y axes, and it is known that it is equal to zero only if x and y axes are the principal axes of inertia. Hence, the bending stresses given by equation 8.5 are correct only when the bending occurs about a principal axis of inertia.

The main formulae arising from the analysis in this section are obtained by combining equations 8.2 and 8.4 and may be stated as :

$$\frac{E}{R} = \frac{M}{I} = \frac{f}{y} \qquad \text{... 8.12}$$

It remains to mention that throughout the given analysis, it has been assumed that the beam is subject to pure bending, i.e. no shearing force is present. However, it may be assumed that the bending stresses on a section are not affected by the shearing force on that section and that the given formulae hold with sufficient accuracy.

**Example 8.4** A simply supported beam has a span of 6 m. and a symmetrical cross-section whose I = 9800 $cm.^4$ and depth = 30 cm. Find the maximum uniformly distributed load it can support if the stress is not to exceed 1.2 $t./cm.^2$

Solution :

Let the required uniformly distributed load = w t./m.

$$M_{max} = \frac{wL^2}{8} = \frac{36\ w}{8} = 4.5\ w\ \text{m.t.}$$

$$M_r = Z\ f = \frac{9300 \times 1.2}{15 \times 100} = 7.84\ \text{m.t.}$$

Equating the maximum applied moment to the moment of resistance of the section,

$$4.5\,w = 7.84$$

$$\text{or} \quad w = \frac{7.84}{4.5} = 1.74 \text{ t./m.}$$

**Example 8.5** Find the diameter of the drum round which a strip of steel 0.2 cm. thick may be bent without the stress exceeding 2.2 t./cm$^2$. E (for steel) = 2200 t./cm.$^2$

Solution :

From equation 8.12, $R = \frac{y\,E}{f}$

$$y = \frac{0.2}{2} = 0.1 \text{ cm.}$$

$$E = 2200 \text{ t./cm.}^2$$

$$f = 2.2 \text{ t./cm.}^2$$

$$\text{Diameter} = 2R = 2 \times \frac{0.1 \times 2200}{2.2} = 200 \text{ cm.} = 2 \text{ m.}$$

**Example 8.6** A simply supported beam has a span of 4 m. and carries a uniformly distributed load of 1.5 t./m. If the beam has an I-section of the following properties :

area of cross-section = 35 cm.$^2$

moment of inertia about the axis of bending = 2500 cm.$^4$

height of section = 20 cm.,

calculate the maximum normal stress. If instead, two I-sections on top of each other and fastened together to form one section were used, what would be the maximum concentrated load P that may act at the middle of the beam without the stress exceeding that obtained for the single I-section ?

Solution :

$$M_{max} \text{ (under uniformly distributed load)} = \frac{1.5 \times 4^2}{8} = 3 \text{ m.t.}$$

$$f_{max} = \pm \frac{M\,y}{I} = \pm \frac{300 \times 20}{2 \times 2500} = \pm 1.2 \text{ t./cm.}^2$$

I (fore the compound section) $= 2\,(2500+35\times10^2) = 12000$ cm.$^4$

$M_{max}$ (under central concentrated load) $= \dfrac{P \times 4}{4} = P$ m.t.

$$f = 1.2 = \frac{P \times 100 \times 20}{12000}$$

$$P = 7.2 \text{ t.}$$

**Example 8.7** A steel tube 6 cm. and 4 cm. external and internal diameters respectively is used as a simply supported beam spaning 4 m. and loaded with a concentrated load at mid-span. It is found that the maximum load it can thus carry is 0.238 t. If four of these tubes are placed parallel to one another and fastened together to form a single beam, the centres of the tubes forming a square 6 × 6 cm., find the maximum central load which this beam can carry if the maximum stress is not to exceed that of the single tube.

Solution :

$M_{max}$ (for the single tube) $= \dfrac{0.238 \times 4}{4} = 0.238$ m.t.

I (for single tube) $= \pi(D_e^{\,4} - D_i^{\,4})/64 = 51$ cm.$^4$

$$f = \frac{M \times r}{I} = \frac{0.238 \times 100 \times 3}{51} = 1.4 \text{ t./cm.}^2$$

area of single tube $= \pi\,(6^2 - 4^2)/4 = 15.7$ cm.$^2$
I (for compound beam) $= 4\,(51 + 15.7 \times 3^2) = 770$ cm.$^4$
Let the maximum load the compound beam can carry = P t.

$$M_{max} = \frac{PL}{4} = P \text{ m.t.}$$

$$f = 1.4 = \frac{P \times 100 \times 6}{770}$$

$$P = 1.8 \text{ t.}$$

**Example 8.8** A four wheel trolly weighs 17.6 t. when loaded. If the bearings are 20 cm. off the centre-lines of the rails what size of round steel axle will be required to resist the resulting moment ? — allowable stress in bending = 0.8 t./cm$^2$.

Solution :

Each axle carries two concentrated loads P t., each acts at 20 cm. from the support.

$$P = \frac{17.6}{4} = 4.4 \text{ t.}$$

$$M_{max} = 4.4 \times 0.2 = 0.88 \text{ m.t.}$$

$$Z \text{ (for circular section)} = \frac{\pi r^3}{4}$$

$$f = \frac{M}{Z}, \quad \text{or} \quad 0.8 = \frac{88 \times 4}{\pi r^3}$$

$$r = \sqrt[3]{\frac{88 \times 4}{0.8\,\pi}} = \sqrt[3]{140} = 5.2 \text{ cm.}$$

The minimum section required for the axle of the trolly is 10.4 cm. diameter.

**Example 8.9** Determine the allowable bending moment of the cast iron T-section shown in Fig. 8.5. What will be this moment if the section is turned upside down ? For cast iron, $f_c = 0.8$ t./cm.$^2$ and $f_t = 0.4$ t./cm.$^2$

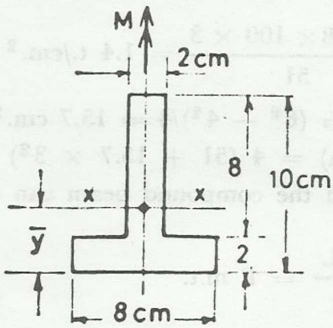



Fig. 8.5

Solution :

$$\bar{y} = \frac{16 \times 1 + 16 \times 6}{16 + 16} = 3.5 \text{ cm.}$$

$$I_x = \frac{8 \times 2^3}{12} + 16 \times 2.5^2 + \frac{2 \times 8^3}{12} + 16 \times 2.5^2 = 290.7 \text{ cm.}^4$$

$$Z_1 \text{ (tension side)} = \frac{290.7}{3.5} = 83 \text{ cm.}^3$$

$$Z_2 \text{ (compression side)} = \frac{290.7}{6.5} = 44.6 \text{ cm.}^3$$

$$M_r = f_t Z_1 = 0.4 \times 83 = 0.332 \text{ m.t.}$$

$$M_r = f_c Z_2 = 0.8 \times 44.6 = 0.3568 \text{ m.t.}$$

The allowable moment of the section = 0.332 m.t.

For the turned section.

$$Z_1 \text{ (tension side)} = \frac{290.7}{6.5} = 44.6 \text{ cm.}^3$$

$$Z_2 \text{ (compression side)} = \frac{290.7}{3.5} = 83 \text{ cm.}^3$$

$$M_r = f_t Z_1 = 0.4 \times 44.6 = 0.1784 \text{ m.t.}$$

$$M_r = f_c Z_2 = 0.8 \times 83 = 0.664 \text{ m.t.}$$

The allowable moment of the section in this case = 0.1784 m.t. It is obvious, therefore, that the section in the first position is more efficient in resisting the moment.

## 8.4 Double bending

In the analysis given in the previous section, it has been assumed that bending occurs about a principal axis. In some practical cases, bending occurs about any axis of the cross-section. Such sections are said to be under double, oblique or unsymmetrical bending. The resulting stresses are best found by resolving the moment, or more conveniently the loads causing this moment, into two components along the principal axes and then superimposing the stresses found separately from the stress formula 8.5.

Consider a cross-section as shown in Fig. 8.6 a, in which the bending moment occurs in a plane making an anlge $\Theta$ with the y-axis. This moment may be resolved into the two components shown in Fig. 8.6 b. The component of the moment which acts about the x-axis is $M_x = M \cos\Theta$, and that about the y-axis is $M_y = M \sin \Theta$. These moment components may be used separately in the usual bending formula and the resulting normal stresses are found by superposition. Thus, the stress at any point (x, y) on the section is given by :

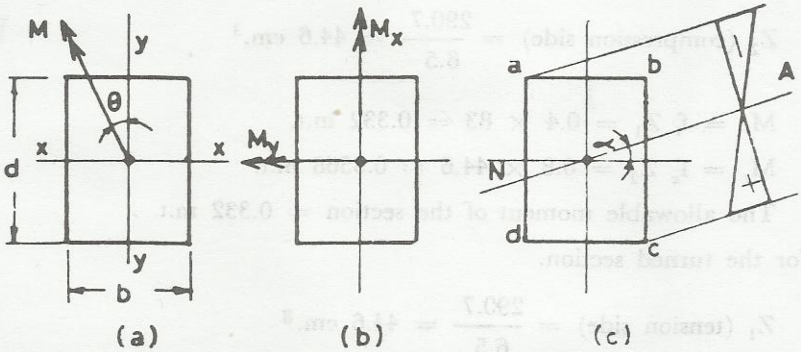



Fig. 8.6

$$f = \pm \frac{M_x\, y}{I_x} \pm \frac{M_y\, x}{I_y} \qquad \text{... 8.13}$$

where positive and negative signs correspond to tensile and compressive stresses respectively.

The stress given by equation 8.13 is zero for points on the line :

$$\pm \frac{M_x\, y}{I_x} \pm \frac{M_y\, x}{I_y} = 0 \qquad \text{... 8.14}$$

This is the equation of the neutral axis of the section. It is seen that it passes through the centroid and has an inclination given by :

$$\tan \alpha = \mp \frac{M_y}{I_y} \Big/ \frac{M_x}{I_x} \qquad \text{... 8.15}$$

To avoid any confusion that may result from the sign conventions, the signs in equation 8.15 are discarded and the neutral axis is plotted so as to lie in the two quadrants not occupied by the moment vector.

The neutral axis together with the stress distribution are shown in Fig. 8.6 c. The maximum stress occurs at the fibres farthermost from the neutral axis. Corner a has a maximum compressive stress of magnitude :

$$f_a = -\frac{M_x\, d}{2I_x} - \frac{M_y\, b}{2I_y}$$

while corner c has a maximum tensile stress of magnitude :

$$f_t = \frac{M_x\, d}{2I_x} + \frac{M_y\, b}{2I_y}$$

**Example 8.10** A 15 × 10 cm. wooden purlin spans between two sloping rafters as shown in Fig. 8.7. The purlin has a span of 4 m. and carries a uniformly distributed load w = 0.13 t./m., from the roof. Locate the neutral axis and plot the normal stress distribution at the section of maximum moment.

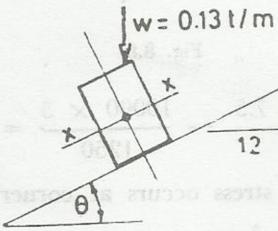



Fig. 8.7

Solution :

$\sin \Theta = 5/13$ and $\cos \Theta = 12/13$

$$M_x = 12/13 \times \frac{0.13 \times 4^2}{8} = 0.24 \text{ m.t.}$$

$$M_y = 5/13 \times \frac{0.13 \times 4^2}{8} = 0.10 \text{ m.t.}$$

$$I_x = \frac{10 \times 15^3}{12} = 2810 \text{ cm.}^4$$

$$I_y = \frac{15 \times 10^3}{12} = 1250 \text{ cm.}^4$$

$$\tan a = \frac{0.1}{1250} \times \frac{2810}{0.24} = 0.938$$

$$a = 43^0\ 10'$$

Fig. 8.8. shows the location of the neutral axis and the normal stress distribution. The maximum compressive stress occurs at corner a,

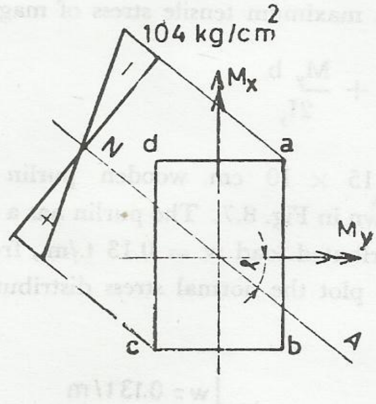



Fig. 8.8

$$f_a = -\frac{24000 \times 7.5}{2810} - \frac{10000 \times 5}{1250} = -104 \text{ kg/cm.}^2$$

and the maximum tensile stress occurs at corner c,

$$f_c = +104 \text{ kg/cm.}^2$$

**Example 8.11** An unequal angle 65 × 130 × 10 has the following properties :

$I_u = 340$ cm.$^4$ $\qquad I_v = 35$ cm.$^4$ $\qquad \alpha = 14^0\ 30$

If the section is subjected to a moment M = 0.25 m.t. about the x-axis, calculate the normal stress at point a on the section located as shown in Fig. 8.9 with respect to the u and v axes.

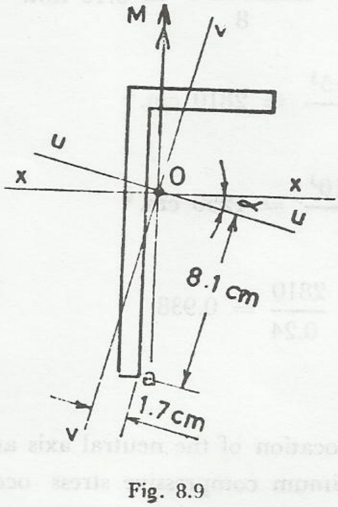



Fig. 8.9

Solution :

The bending formula cannot be used directly as the moment is not applied about one of the principal axes. For this reason the moment is resolved into two components about the u and v axes.

$$M_u = M \cos \alpha = 0.25 \times 0.9682 = 0.242 \text{ m.t.}$$

$$M_v = M \sin \alpha = 0.25 \times 0.2504 = 0.063 \text{ m.t.}$$

$$f_a = \frac{24.2 \times 8.1}{340} + \frac{6.3 \times 1.7}{35} = 0.58 + 0.31 = 0.89 \text{ t./cm.}^2$$

**Example 8.12** Fig. 8.10 a shows an elevation and a plan of a beam under a set of vertical and horizontal loads. If the cross-section is an I-beam of the given properties, find the maximum normal stresses and plot the corresponding distribution.

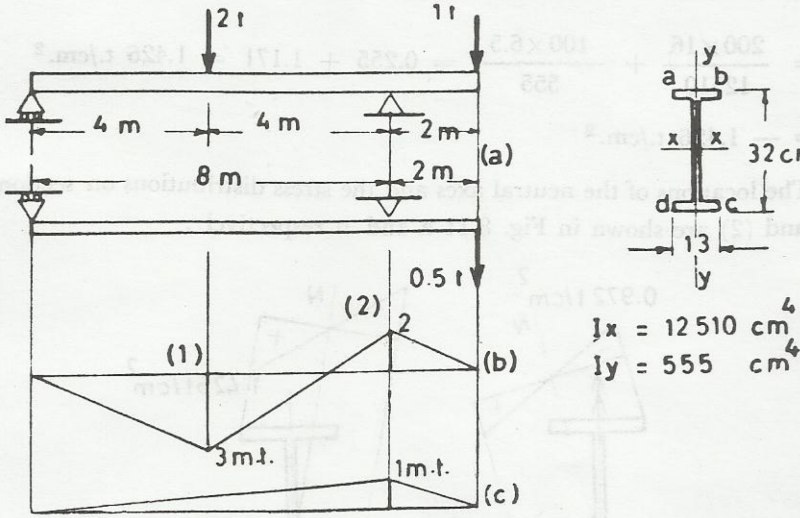



Fig. 8.10

Solution :

Figs. 8.10 b and c show the B.M.Ds. about the x and y axes respectively, drawn on the tension side. Maximum normal stress may occur at either one of the two most stressed sections; section (1) and section (2).

For section (1), $M_x = 3$ m.t. and $M_y = 0.5$ m.t.

$$\tan \alpha = \frac{0.5 \times 12510}{3 \times 555} = 3.75$$

$$\alpha = 75^0$$

maximum normal stress occurs at points a and c on the section.

$$f_a = -\frac{300 \times 16}{12510} - \frac{50 \times 6{,}5}{555} = -0.384 - 0.588 = -0.972 \text{ t./cm.}^2$$

$$f_c = +0.972 \text{ t./cm.}^2$$

For section (2), $M_x = 2$ m.t. and $M_y = 1$ m.t.

$$\tan \alpha = \frac{1 \times 12510}{2 \times 555} = 11.25$$

$$\alpha = 84^0\ 54$$

maximum normal stress occurs at points b and d on the section.

$$f_b = \frac{200 \times 16}{12510} + \frac{100 \times 6.5}{555} = 0.255 + 1.171 = 1.426 \text{ t./cm.}^2$$

$$f_d = -1.426 \text{ t./cm.}^2$$

The locations of the neutral axes and the stress distributions on sections (1) and (2) are shown in Fig. 8.11 a and b respectivel/.

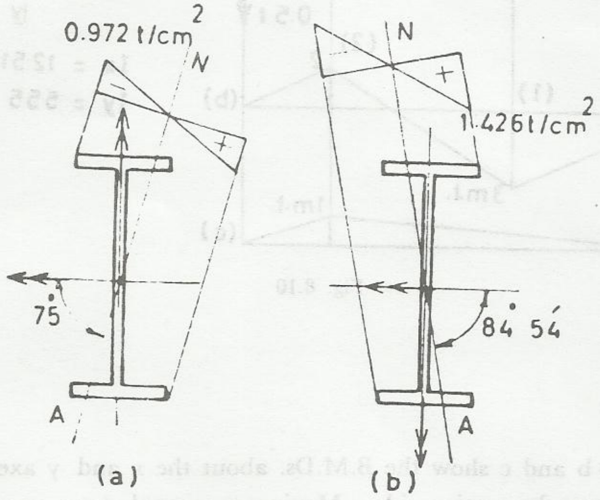



Fig. 8.11

**8.5 Combined axial force and bending moment stresses**

In many practical problems, cross-sections are subjected to the combined effect of bending moment and axial force, tension or compression. In the latter case, however, it is assumed throughout the following discussion that the compression does not lead to buckling of the member upon which it acts. This is dealt with elsewhere.

Consider a member of rectangular cross-section carrying an axial compressive force N together with a bending moment $M_x$ as shown in Fig. 8.12 a.

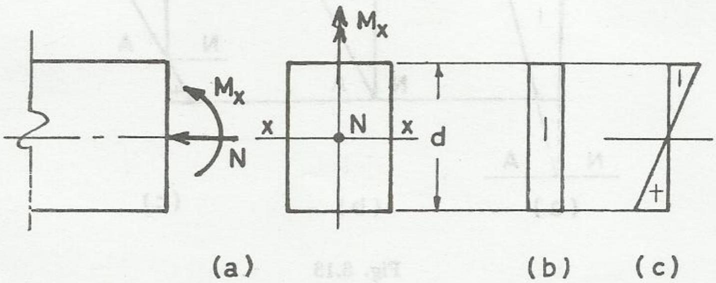



Fig. 8.12

If N acts alone, the normal stress over the section is given by :

$$f = -N/A$$

and the stress distribution over the section is as shown in Fig. 8.12 b. If the moment acts alone, the normal stress is given by equation 8.5,

$$f = \pm \frac{M_x\, y}{I_x}$$

and the stress distribution over the section is as shown in Fig. 8.12 c. From the principle of superposition, the stress due to the combined effect of thrust and moment is given by :

$$f = -\frac{N}{A} \pm \frac{M_x\, y}{I_x} \qquad \text{... 8.16}$$

Examination of this equation shows that the greatest compressive stress occurs at the extreme upper fiber and has the value :

$$f = -\frac{N}{A} - \frac{M\,d}{2\,I_x}$$

The stress at the extreme lower fiber is given by :

$$f = -\frac{N}{A} + \frac{M\,d}{2I_x}$$

Whether this stress is tensile or compressive depends on the relative values of the two stress components; N/A and $\frac{M\,d}{2I_x}$ The three possible stress distributions are shown in Figs. 8.13 a-c.

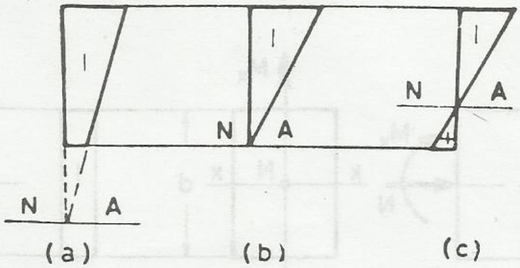



Fig. 8.13

**Example 8.13** A masonary chimney of 1 m. and 0.7 m. external and internal radii is subjected to a wind pressure of 100 kg/m.$^2$ of projected area. Determine the maximum height to which the chimney may be built without having any tensile stress at the base section or the maximum compressive stress exceeding 4 kg/m.$^2$, if the weight of masonary is 2 t./m.$^3$

Solution :

$A = \pi(r_e^2 - r_i^2) \quad = \pi(1 - 0.7^2) \quad = 1.6$ m.$^2$

$I = \pi(r_e^4 - r_i^4)/4 = \pi(1 - 0.7^4)/4 = 0.6$ m.$^4$

Let the height of the chimney = h m.

$N = 1.6 \times h \times 2 \times 1000 = 3200\,h$ kg.

$M = 100 \times 2\,h \times h/2 = 100\,h^2$ m.kg.

$$f_{1/2} = -\frac{N}{A} \pm \frac{M\,y}{I}$$

for no tension,

$$f_1 = 0 = -\frac{3200\,h}{1.6} + \frac{100\,h^2 \times 1}{0.6}$$

$h = 12$ m.

for compressive stresses not to exceed 4 kg./m.,[2]

$$f_2 = -4 = \frac{3200\,h}{1.6} - \frac{100\,h^2 \times 1}{0.6}$$

$h = 10.6$ m.

The maximum height of the chimney required = 10.6 m.

**Example 8.14** Find the maximum stresses on section a-a of the clamp shown in Fig. 8.14 when the force provided by the screw is 125 kg. Plot the stress distribution across the section.

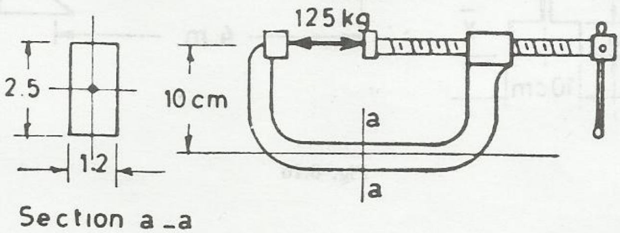



Fig. 8.14

Solution :

$A = 1.2 \times 2.5 = 3$ cm.$^2$

$$I_x = \frac{1.2 \times 2.5^3}{12} = 1.56 \text{ cm.}^4$$

$N = +125$ kg.

$M_x = 125 \times 10 = 1250$ cm. kg.

$$f_1 = \frac{125}{3} + \frac{1250 \times 2.5}{2 \times 1.56} = 1041.7 \text{ kg./cm.}^2$$

$$f_2 = \frac{125}{3} - \frac{1250 \times 2.5}{2 \times 1.56} = -958.3 \text{ kg./cm.}^2$$

The stress distribution is thus as shown in Fig. 8.15

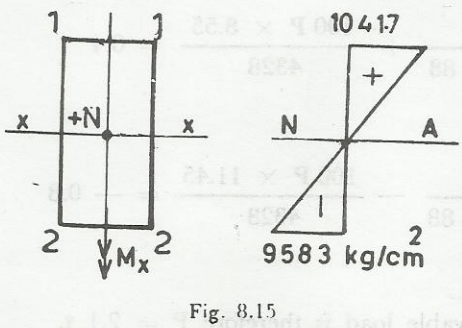



Fig. 8.15

**Example 8.15** Fig. 8.16 shows a cast iron beam having a cross-section of the given dimensions. If the allowable stresses for cast iron are 0.4 t./cm.$^2$ and 0.8 t./cm.$^2$ for tension and compression respectively, find the maximum allowable value of the central concentrated load P.

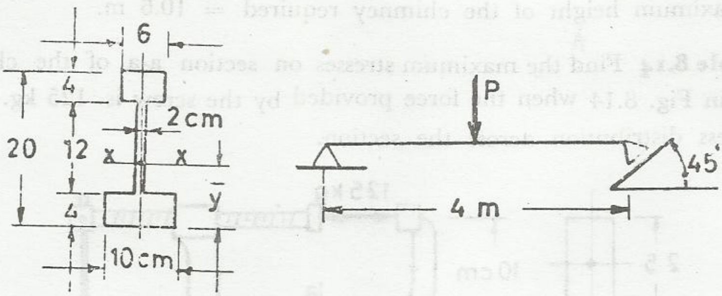



Fig. 8.16

**Solution :**

$$A = 6 \times 4 + 2 \times 12 + 4 \times 10 = 88 \text{ cm.}^2$$

$$\bar{y} = \frac{40 \times 2 + 24 \times 10 + 24 \times 18}{88} = 8.55 \text{ cm.}$$

$$I_x = \left(\frac{6\times4^3}{12} + 24 \times 9.45^2\right) + \left(\frac{2\times12^3}{12} + 24 \times 1.45^2\right) + \left(\frac{10\times4^3}{12} + 40 \times 6.55^2\right) = 4328 \text{ cm.}^4$$

The maximum moment occurs at mid-span where,

$M = PL/4 = P$ m.t.

$N = -P/2$ t.

$$f_t = -\frac{P}{2 \times 88} + \frac{100 P \times 8.55}{4328} = 0.4$$

$P = 2.1$ t.

$$f_c = -\frac{P}{2 \times 88} - \frac{100 P \times 11.45}{4328} = -0.8$$

$P = 3$ t.

The maximum allowable load is therefore $P = 2.1$ t.

## 8.6 Eccentric thrust

The analysis given in section 8.5 can be used to find the stresses due to eccentric thrust.

Consider a post subjected to an eccentric compressive force lying on the x-axis at a distance e from the y-axis as shown in Fig. 8.17 a. This eccentric force is equivalent to a central thrust N and a bending moment $M_y = N \times e$ about the y-axis as shown in Fig. 8.17 b.

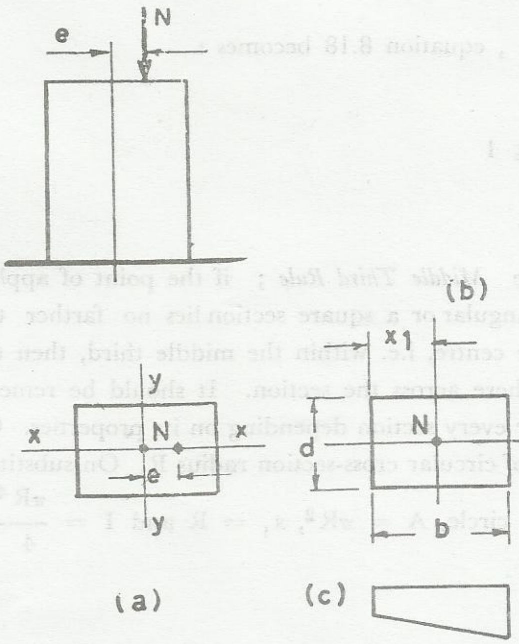



Fig. 8.17

From equation 8.16, the normal stress at any fiber at a distance x from the y-axis is given by :

$$f = \frac{N}{A} \pm \frac{Ne\ x}{I_y}$$

The maximum normal stresses occur at the extreme outer fibers and from Fig. 8.17 c, which shows a possible stress distribution, tensile stresses are most likely to occur in the fiber at the left edge. Its value is given by :

$$f = -\frac{N}{A}\left(1 - \frac{x\ eA}{I_y}\right) \qquad \ldots\ 8.17$$

The tensile stress is of particular interest as some materials such as brickwork and plain concrete are not allowed to carry tension. Thus, from equation 8.17, for no tension to occur in the section,

$$\frac{Ae\ x_1}{I_y} \leq 1 \qquad \ldots 8.18$$

On substituting for the properties of the rectangle, $A = bd$, $x_1 = \frac{b}{2}$

and $I_y = \frac{db^3}{12}$, equation 8.18 becomes :

$$\frac{6e}{b} \leq 1 \qquad \ldots 8.19$$

$$\text{or } e \leq b/6$$

This is called the *Middle Third Rule* ; if the point of application of a thrust on a rectangular or a square section lies no farther than b/6 on either side of the centre, i.e. within the middle third, then there will be no tension anywhere across the section. It should be remembered that there is a rule for every section depending on its properties. Consider for example, a post of circular cross-section radius R. On substituting for the properties of the circle, $A = \pi R^2$, $x_1 = R$ and $I = \frac{\pi R^4}{4}$ in equation 8.18, it becomes :

$$e \leq R/4 \qquad \ldots 8.20$$

i.e. if the point of application of a thrust on a circular section lies no farther than R/4 from its centre, i.e. within the middle quarter, then the stresses everywhere across the section will be compressive. This is the *Middle Fourth Rule* for circular sections.

In the previous two examples the stress considered was that within the material, i.e. in the post itself. Sometimes, however, it is the stress between the faces of two materials resting on each other that need to be considered.

Consider a footing resting on the soil at plane a-a and let the thrust N cut the base at a distance e from its centre as shown in Fig. 8-18 a. If N lies within the middle third of the base width b the stress distribution

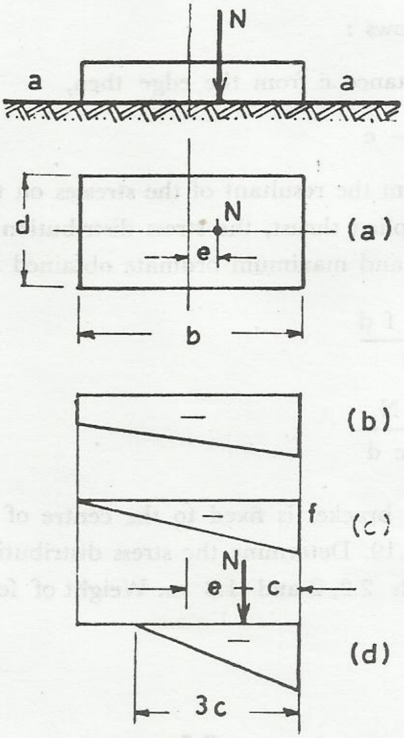



Fig. 8.18

will be as shown in Fig. 8.18 b and the stress values will be given by equation 8.17. In the limiting case when the thrust lies on the middle third point, the stress distribution is triangular as shown in Fig. 8.18 c. The maximum stress may again be obtained from equation 8.17, or more easily, from equilibrium consideration. Thus,

$$N = \frac{f\ b\ d}{2}$$

or $\quad f = 2\,N/A$

which is equal to double the average stress: N/A.

If N cuts the base outside the middle third but still within the section, then according to formula 8.17, tensile stresses are expected to be produced between the two materials at the contact plane. But since the soil cannot develop such a stress, i.e. it cannot prevent any tendency of the base to lift above the contact plane, formula 8.17 cannot be used to determine the stress distribution in this case. Instead, it is found from equilibrium as follows :

Let N lie at a distance c from the edge then,

$$c = b/2 - e \qquad \text{... 8.21}$$

since for equilibrium the resultant of the stresses on the base must be co-linear with the applied thrust, the stress distribution is a triangle with a base length of 3 c and maximum ordinate obtained from the condition :

$$N = \frac{3\,c\,f\,d}{2}$$

$$\text{or} \quad f = \frac{2\,N}{3\,c\,d} \qquad \text{... 8.22}$$

**Example 8.16** A bracket is fixed to the centre of a square foundation as shown in Fig. 8.19. Determine the stress distribution on the soil if the base side b equals 2.2, 2 and 1.8 m. Weight of foundation = 2 t/m³.

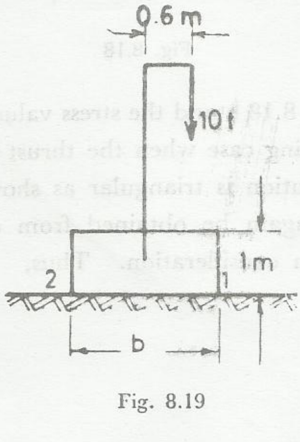



Fig. 8.19

Solution :

For b = 2.2 m., $A = 2.2^2 = 4.84 \text{ m.}^2$ and $I = 2.2^4/12 = 1.94 \text{ m.}^4$

weight of foundation = 4.84 × 1 × 2 = 9.68 t.

N = — 10 — 9.68 = — 19.68 t.

M = 10 × 0.6 = 6 m.t.

$$e = \frac{6}{19.68} = 0.304 \text{ m., } b/6 = \frac{2.2}{6} = 0.366 \text{ m.}$$

N lies within the middle third.

$$f_1 = -\frac{19.68}{4.84} - \frac{6 \times 1.1}{1.94} = -4.06 - 3.39 = -7.45 \text{ t./m.}^2$$

$$f_2 = -4.06 + 3.39 = -0.65 \text{ t./m.}^2$$

For $b = 2$ m., $A = 4$ m.$^2$ and $I = 2^4/12 = 1.33$ m.$^4$

weight of foundation $= 4 \times 1 \times 2 = 8$ t.

$N = -10 - 8 = -18$ t.

$e = 6/18 = 0.33$ m., $b/6 = 2/6 = 0.33$ m.

N lies on the middle third point.

$$f_1 = -2\,N/A = \frac{-2 \times 18}{4} = -9 \text{ t./m.}^2$$

$$f_2 = 0$$

For $b = 1.8$ m., $A = 1.8^2 = 3.24$ m.$^2$

weight of foundation $= 3.24 \times 1 \times 2 = 6.48$ t.

$N = -10 - 6.48 = -16.48$ t.

$$e = \frac{6}{16.48} = 0.364 \text{ m., } b/6 = \frac{1.8}{6} = 0.3 \text{ m.}$$

N lies outside the middle third and the stress is found from equation 8.22.

$c = 0.9 - 0.364 = 0.536$ m.

$$f = \frac{-2 \times 16.48}{3 \times 0.536 \times 1.8} = -11.4 \text{ t./m.}^2$$

and the stress is equal to zero at a distance $= 3 \times 0.536 = 1.608$ m. from edge 1.

The stress distributions corresponding to the three cases are shown in Figs. 8.20 a, b and c.

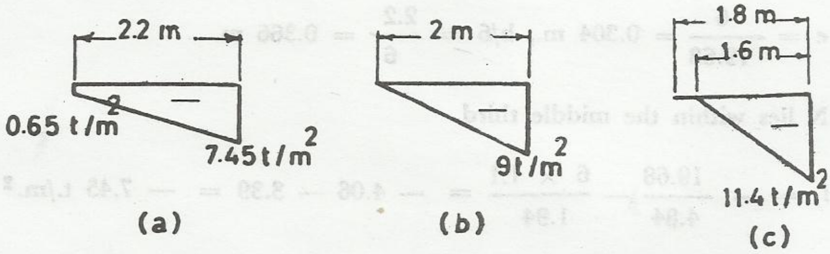



Fig. 8.20

## 8.7 Eccentricity about both axes

Consider the general case of a section acted upon by a thrust which is eccentric to both axes as shown in Fig. 8.21 a.

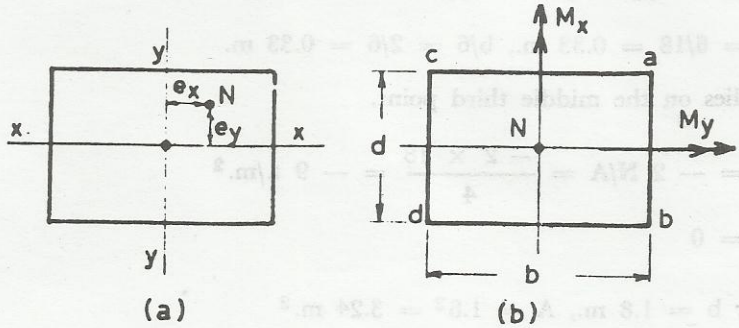



Fig. 8.21

The eccentric force N is equivalent to a central force N, a bending moment about the x-axis; $M_x = Ne_y$ and a bending moment about the y-axis; $M_y = Ne_x$. By applying the usual axial and bending stress formulae and superimposing the results, the final normal stress at a point (x, y) is obtained.

$$f = \pm \frac{N}{A} \pm \frac{M_x\, y}{I_x} \pm \frac{M_y\, x}{I_y} \qquad .. \; 8.23$$

As mentioned before, the positive sign is used when the stress is tensile while the negative sign is used when the stress is compressive. The sign of the final stress at any point is found by inspection. Referring to Fig. 8.21 b, for example, it is seen that corner a has the maximum compressive stress,

$$f_a = -\frac{N}{A} - \frac{M_x\, d}{2I_x} - \frac{M_y\, b}{2I_y}$$

and corner d has the maximum tensile stress,

$$f_d = -\frac{N}{A} + \frac{M_x\, d}{2I_x} + \frac{M_y\, b}{2I_y}$$

The stresses at corners b and c are given by :

$$f_b = -\frac{N}{A} + \frac{M_x\, d}{2I_x} - \frac{M_y\, b}{2I_y}$$

$$f_c = -\frac{N}{A} - \frac{M_x\, d}{2I_x} + \frac{M_y\, b}{2I_y}$$

It should be emphasized here that the moment vectors act along the principal axes of inertia. For any unsymmetrical section, equation 8.23 becomes :

$$f = \pm \frac{N}{A} \pm \frac{M_u}{I_u} v \pm \frac{M_v}{I_v} u \qquad \text{... 8.24}$$

where $I_u$ and $I_v$ are the principal moments of inertia and u and v are the co-ordinates with respect to the principal axes u and v

The equation of the neutral axis is obtained by setting the stress given by equation 8.23 to zero. Thus,

$$\frac{N}{A} + \frac{M_x\, y}{I_x} + \frac{M_y\, x}{I_y} = 0 \qquad \text{... 8.25}$$

But $\quad M_x = Ne_y \quad , \quad M_y = Ne_x$

$\qquad I_x = Ai_x^{\,2} \quad$ and $\quad I_y = Ai_y^{\,2}$

On substituting these values in equation 8.25 it becomes,

$$\frac{e_x}{i^2_y}\, x + \frac{e_y}{i^2_x}\, y + 1 = 0 \qquad \text{... 8.26}$$

Equations 8.25 or 8.26 represent the neutral axis. The respective intercepts of the neutral axis with the x and y axes are given by :

$$x_1 = -\frac{i_y^{\,2}}{e_x} \quad \text{and} \quad y_1 = -\frac{i_x^{\,2}}{e_y} \qquad \text{... 8.27}$$

Hence the neutral axis may be represented by a line NA as shown in Figs. 8.22.

The following remarks regarding the neutral axis are of importance :

(1) The neutral axis lies in the quadrant opposite to that in which N acts.

(2) The position of the neutral axis does not depend on the magnitude of N but on its position.

(3) On plotting the neutral axis, two cases may arise :

(a) The axis cuts the section as shown in Fig. 8.22 a. In this case the stresses across the section will be of two different types.

(b) The axis does not cut the section as shown in Fig. 8.22 b. In this case all the stresses across the section will be of one type only.

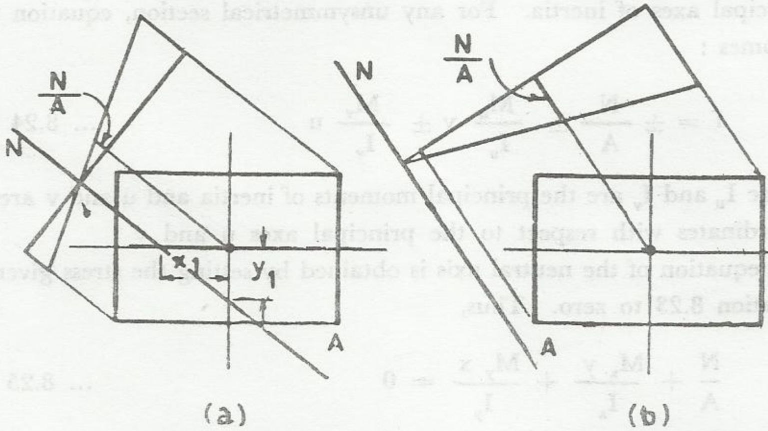



Fig. 8.22

(4) The points of maximum stresses are those farthermost from the neutral axis. The stress on a fiber through the centroid and parallel to the neutral axis is equal to N/A.

This last remark offers a quick and simple method for the determination of the stress distribution. Once the neutral axis is located by means of equations 8.27, it will be an easy matter to plot the stress distribution across the section knowing that it varies linearly and that it has a zero ordinate at the neutral axis and an ordinate equal to N/A at the centroid.

(5) If the point of application of a normal force N moves along a vector such as on in Fig. 8.23, the neutral axis moves parallel to itself. This is proved in the following :

Equation 8.26 expresses the conditions that the normal stress along a certain line, known as the neutral axis, is zero. Let the normal force move along vector on which makes an angle $\Theta$ with the x-axis.

$$e_x = e \cos \Theta \qquad \text{and} \qquad e_y = e \sin \Theta$$

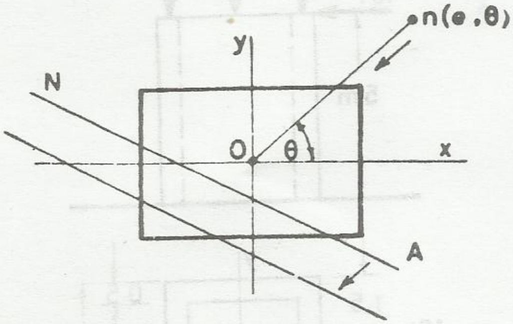



Fig. 8:23

Substituting these values in equation 8.26,

$$\frac{e \cos \Theta}{i^2_y} x + \frac{e \sin \Theta}{i^2_x} y + 1 = 0$$

the slope of the axis NA is given by :

$$\text{slope} = - \frac{e \cos \Theta}{i_y^2} \times \frac{i_x^2}{e \sin \theta} = - \frac{i^2_x}{i_y^2} \cot \Theta = \text{constant}$$

i.e. all the neutral axes corresponding to a normal force moving along a vector of inclination $\Theta$ are parallel.

Note that as the normal force approaches the centroid the neutral axis moves away from the section. In the extreme case when the normal force is central, $e_x = e_y = 0$ and the neutral axis lies at infinity. In the other extreme case when the normal force is at infinity, the section is subject to bending only and the neutral axis passes through the centroid.

**Example 8.17** A masonary pier of hollow rectangular section carries beside its own weight the set of loads shown in Fig. 8.24. Find the normal stress distribution at the base section of the pier if it weighs 2 t./m.$^3$

Solution :

Weight of pier $= 2 \times 5 \; (4 \times 3 - 3 \times 2) = 60$ t.

N $= 60 + 40 + 10 + 40 = 150$ t. ↓

$M_x = 40 \times 1.25 = 50$ m.t. ↓

$M_y = 10 \times 5 + 40 \times 1.75 - 10 \times 1.75 = 102.5$ m.t. →→

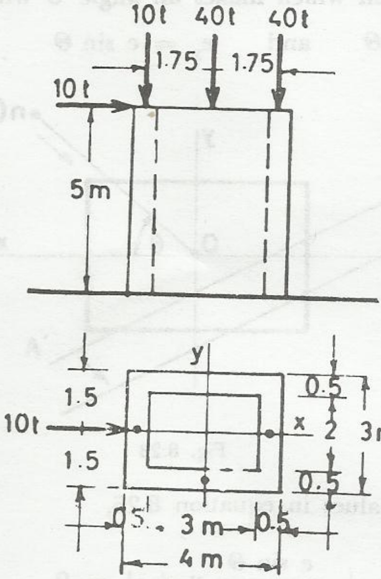



Fig. 8.24

$$A = 4 \times 3 - 3 \times 2 = 6 \text{ m.}^2$$

$$I_x = \frac{4 \times 3^3}{12} - \frac{3 \times 2^3}{12} = 7 \text{ m}^4.$$

$$I_y = \frac{3 \times 4^3}{12} - \frac{2 \times 3^3}{12} = 11.5 \text{ m}^4.$$

The position of the neutral axis is found from equations 8.27.

$$e_x = \frac{M_y}{N} = \frac{102.5}{150} \qquad e_y = \frac{M_x}{N} = \frac{50}{150}$$

$$i_y^2 = \frac{I_y}{A} = \frac{11.5}{6} \qquad i^2_x = \frac{I_x}{A} = \frac{7}{6}$$

$$x_1 = -\frac{11.5}{6} \times \frac{150}{102.5} = -2.85 \text{ m.}$$

$$y_1 = -\frac{7}{6} \times \frac{150}{50} = -3.5 \text{ m.}$$

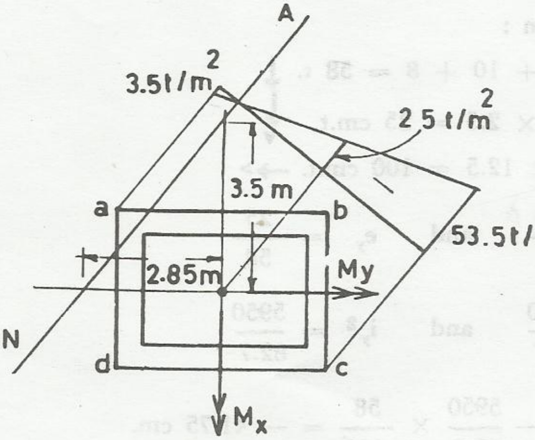



Fig. 8.25

The neutral axis is thus located as shown in Fig. 8.25. It is obvious that the maximum stresses occur at corners a and c.

$$f_a = -\frac{150}{6} + \frac{50 \times 1.5}{7} + \frac{102.5 \times 2}{11.5} = 3.5 \text{ t./m}^2.$$

$$f_c = -\frac{150}{6} - \frac{50 \times 1.5}{7} - \frac{102.5 \times 2}{11.5} = -53.5 \text{ t./m}^2.$$

**Example 8.18** A short column carries three loads as shown in Fig. 8.26. Locate the neutral axis and plot the normal stress distribution across the section if it has the following properties :

$A = 82.7$ cm²., $I_y = 5950$ cm⁴. and $I_x = 2140$ cm⁴.

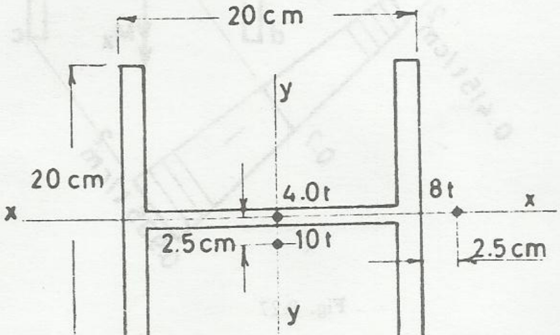



Fig. 8 26

Solution :

$N = 40 + 10 + 8 = 58$ t. $\downarrow$

$M_x = 10 \times 2.5 = 25$ cm.t. $\downarrow$

$M_y = 8 \times 12.5 = 100$ cm.t. $\rightarrow$

$e_x = \dfrac{100}{58}$ and $e_y = \dfrac{25}{58}$

$i_x^2 = \dfrac{2140}{82.7}$ and $i_y^2 = \dfrac{5950}{82.7}$

$$x_1 = -\frac{5950}{82.7} \times \frac{58}{100} = -41.75 \text{ cm.}$$

$$y_1 = -\frac{2140}{82.7} \times \frac{58}{25} = -60 \text{ cm.}$$

The neutral axis is located as shown in Fig. 8.27.

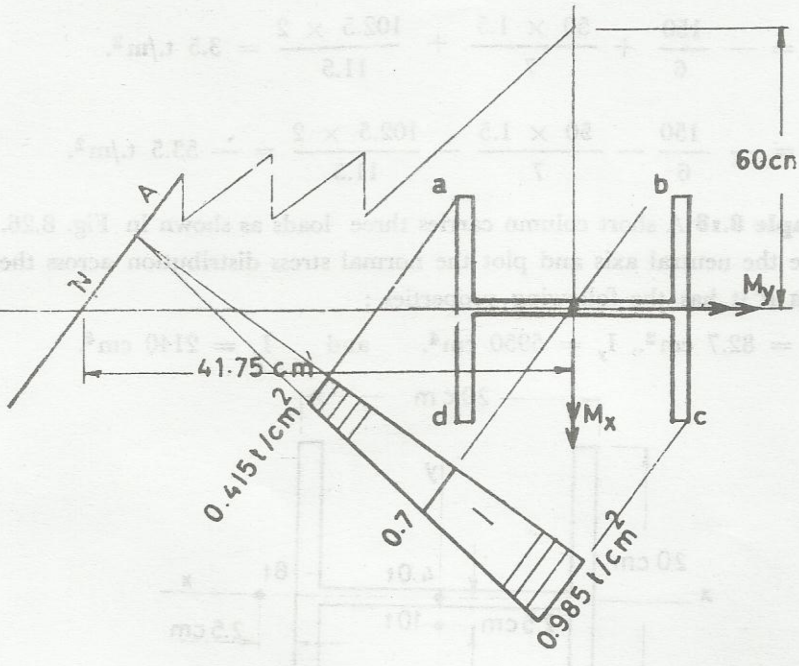



Fig. 8.27

$$f_a = -\frac{58}{82.7} + \frac{25 \times 10}{2149} + \frac{100 \times 10}{5950} = -0.415 \text{ t./cm}^2.$$

$$f_d = -\frac{58}{82.7} - \frac{25 \times 10}{2140} - \frac{100 \times 10}{5950} = -0.985 \text{ t./cm}^2.$$

**8.8 Core of plane areas.**

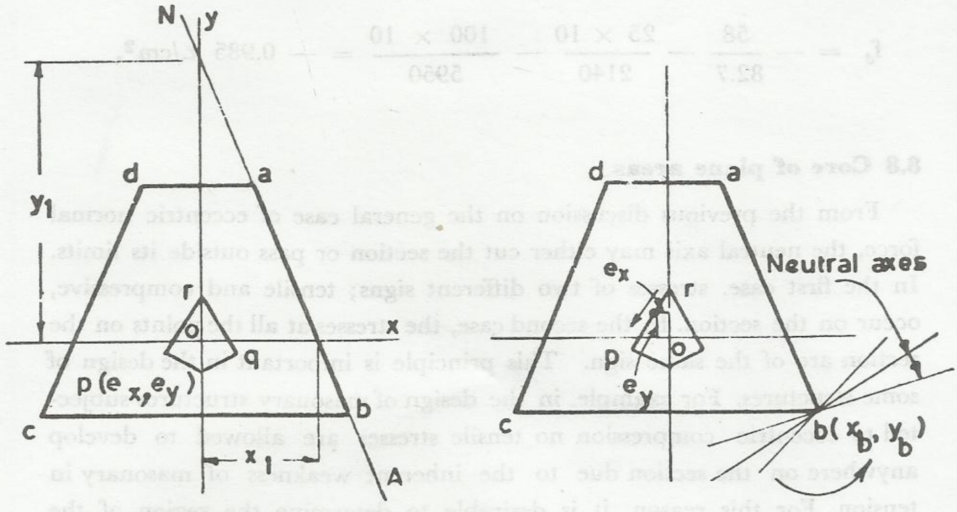
From the previous discussion on the general case of eccentric normal force, the neutral axis may either cut the section or pass outside its limits. In the first case, stresses of two different signs; tensile and compressive, occur on the section. In the second case, the stresses at all the points on the section are of the same sign. This principle is important in the design of some structures. For example, in the design of masonary structures subjected to eccentric compression no tensile stresses are allowed to develop anywhere on the section due to the inherent weakness of masonary in tension. For this reason, it is desirable to determine the region of the section within which the eccentric compressive force may act without producing tensile stresses on the section.

In the region near the centroid of every plane area there exists an area where if the point of application of the normal force lies within its boundary, the stresses allover the section will be of the same sign. This area is called the *core* of the section. In the limiting case, when the point of application of the force is on the boundary of the core, the neutral axis is tangential to the section. It follows that the core is the locus of the point of application of the normal force N for which the neutral axis is tangential to the section. Therefore, in order to determine a point on the core of a section, it is necessary to draw a neutral axis touching the perimeter of the section under consideration and locate the corresponding point of application of the normal force. This point obviously lies on the core of the section. This procedure is repeated until a number of points sufficient for plotting the core is obtained. This may seem rather tedious but normally only few points are needed to plot the core. The following remarks are helpful in this respect.

(1) A symmetrical section has a symmetrical core; the axis of symmetry being the same for both figures.

(2) For every corner on the perimeter of the section, there is a corresponding straight line on the core.

(3) For every straight line on the perimeter of the section, there is a corresponding corner in the core.




(a) Fig. 8.28 (b)

These remarks are explained with reference to Figs. 8.28 a and b. Referring to Fig. 8.28 a, the co-ordinates $(e_x, e_y)$ of point p on the core corresponding to the position of the neutral axis NA as coinsiding with line ab on the perimeter of the area is found from equations 8.27,

$$e_x = -\frac{i_y^2}{x_1} \quad \text{and} \quad e_y = -\frac{i_x^2}{y_1}$$

The co-ordinates of point r which corresponds to the NA coinsiding with line cb is found in a similar way. Only this point lies on the y-axis as the intercept $x_1$ of the tangent cb with the x-axis is infinite and hence $e_x = 0$.

Equation 8.26 expresses the condition that the normal stress at a certain point on the section is zero. Referring to Fig. 8.28 b, if the stress at point b is required to be zero, equation 8.26 gives :

$$\frac{x_b}{i_y^2}\, e_x + \frac{y_b}{i_x^2}\, e_y + 1 = 0$$

Consider now the co-ordinates that define the point of application of the normal force N, $e_x$ and $e_y$ as variables. It is obvious that the locus of N corresponding to all neutral axes through point b is a straight line. Note that as N moves from point r to point p the neutral axis rotates anticlockwise from position cb to ba.

The core of some common types of sections is shown in Fig. 8.29.

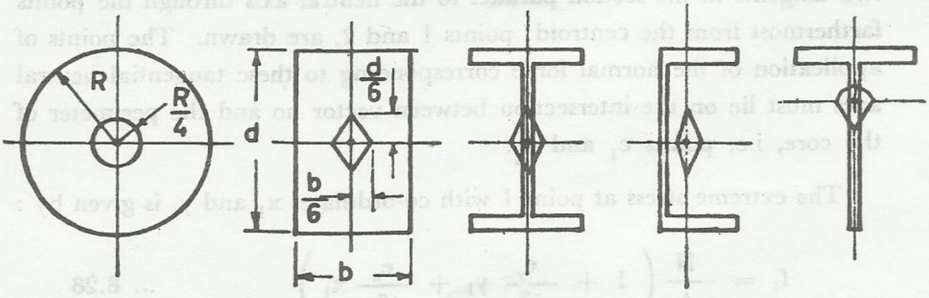



Fig. 8.29

The core has further use in that it provides a convenient method of checking the values of the extreme normal stresses on the section. This will be considered in the next section.

## 8.9 Extreme stresses by means of core

Consider the section shown in Fig. 8.30 and let it be subject to an eccentric normal force N at point n at a distance e from the centroid o and on vector no which makes an angle $\theta$ with the x-axis.

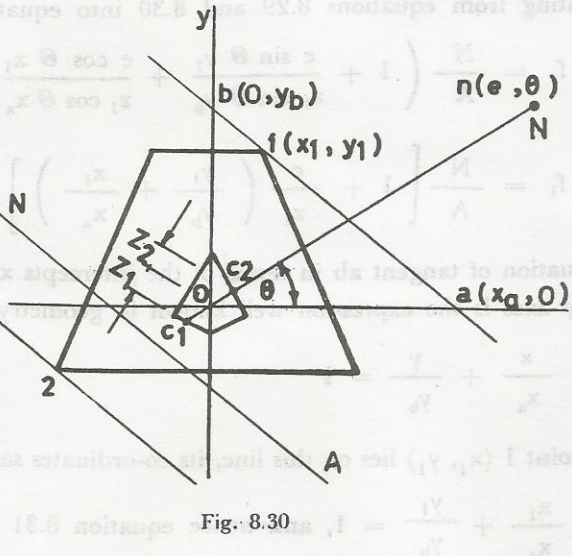



Fig. 8.30

Let NA be the neutral axis corresponding to the normal force at point n. As shown before, the neutral axes corresponding to various positions of N on vector no are all parallel. To get the points of extreme stresses, two tangents to the section parallel to the neutral axis through the points farthermost from the centroid; points 1 and 2, are drawn. The points of application of the normal force corresponding to these tangential neutral axes must lie on the intersection between vector no and the perimeter of the core, i.e. points $c_1$ and $c_2$.

The extreme stress at point 1 with co-ordinates $x_1$ and $y_1$ is given by :

$$f_1 = \frac{N}{A}\left(1 + \frac{e_y}{i^2_x} y_1 + \frac{e_x}{i^2_y} x_1\right). \quad \text{... 8.28}$$

$$e_y = e \sin \Theta \quad \text{and} \quad e_x = e \cos \Theta \quad \text{... 8.29}$$

Also from equation 8.27,

$$i^2_x = \text{constant} = e_y \, y_b$$

$$\text{and } i^2_y = \text{constant} = e_x \, x_a$$

$$\text{Let } z_1 = oc_1 \quad \text{and} \quad z_2 = oc_2$$

Then for the particular position of the normal force at point $c_1$,

$$i_x^2 = z_1 \sin \Theta \, y_b$$

$$i_y^2 = z_1 \cos \Theta \, x_a \quad \text{... 8.30}$$

Substituting from equations 8.29 and 8.30 into equation 8.28,

$$f_1 = \frac{N}{A}\left(1 + \frac{e \sin \Theta \, y_1}{z_1 \sin \Theta \, y_b} + \frac{e \cos \Theta \, x_1}{z_1 \cos \Theta \, x_a}\right)$$

$$f_1 = \frac{N}{A}\left[1 + \frac{e}{z_1}\left(\frac{y_1}{y_b} + \frac{x_1}{x_a}\right)\right] \quad \text{... 8.31}$$

The equation of tangent ab in terms of the intercepts $x_a$ and $y_b$ with the x and y axes is the expression well known in geometry,

$$\frac{x}{x_a} + \frac{y}{y_b} = 1$$

Since point 1 $(x_1, y_1)$ lies on this line, its co-ordinates satisfy this equation.

Thus, $\dfrac{x_1}{x_a} + \dfrac{y_1}{y_b} = 1$, and hence equation 8.31 reduces to :

$$f_1 = \frac{N}{A}\left(1 + \frac{e}{z_1}\right)$$

or $$f_1 = \frac{N(z_1 + e)}{Az_1} \qquad \ldots 8.32$$

Similarly, it may be shown that the maximum normal stress at point 2 is given by :

$$f_2 = \frac{N(z_2 - e)}{Az_2} \qquad \ldots 8.33$$

A study of equations 8.32 and 8.33 will show that for maximum fiber stress to occur in the section, $z_1$ and $z_2$ must be a minimum. In other words, the load should be placed such that the load vector is normal to the boundary of the core.

The quantities $N(z_1 + e)$ and $N(z_2 - e)$ in equations 8.32 and 8.33 represent the moment of the eccentric normal force N about the core points $c_1$ and $c_2$. Also, in accordance with equation 8.8, the quantities $Az_1$ and $Az_2$ are referred to as the bending section modulii for the extreme points 1 and 2.

Thus the extreme stress on the section is given by :

$$f = \frac{\text{moment of N about appropriate core point}}{\text{appropriate bending section modulus}}$$

The procedures for determining the core and its use in finding the extreme normal stresses are illustrated in the following examples.

**Example 8.19** Fig. 8.31 a shows the straining actions on the base section of a masonary pier. Determine the maximum normal stress on the section using the core method.

$A = 15.6\ m^2$ $i^2_x = 0.29\ m^2$ and $i^2_y = 5.3\ m^2$

Solution :

Before finding the maximum normal stresses, the core of the section and the location of the normal force N have to be determined.
The core is detemrined by applying equations 8.27,

$$x_1 = -\frac{i^2_y}{e_x}, \quad y_1 = -\frac{i^2_x}{e_y}$$

For N.A. to coinside with side 2-3,

$$x_1 = \infty \quad , \quad y_1 = 1\ m.$$

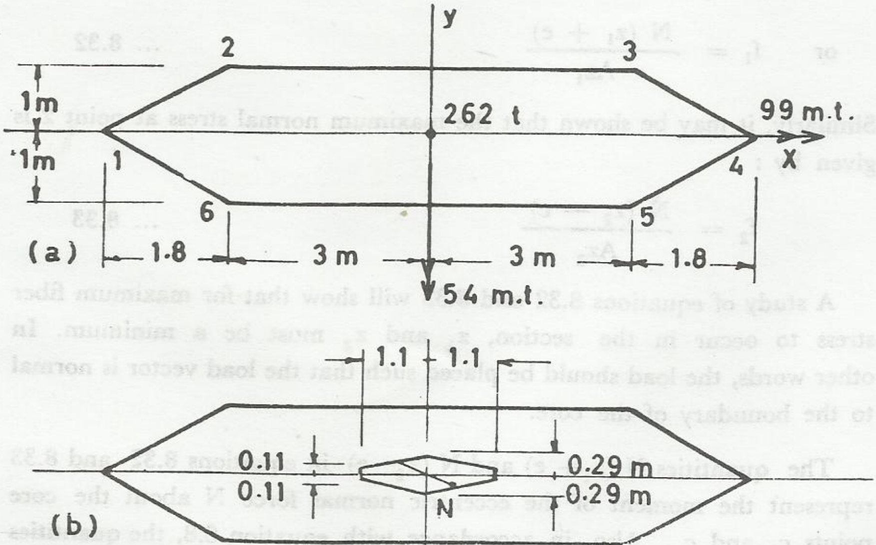



Fig. 8.31

$$e_x = x_c = 0 \quad , \quad e_y = y_c = \frac{0.29}{1} = 0.29 \text{ m.}$$

For N.A. to coinside with side 3-4,

$$x_1 = 4.8 \text{ m.} \quad , \quad y_1 = 2.67 \text{ m.}$$

$$x_c = -\frac{5.3}{4.8} = -1.1 \text{ m.} \quad , \quad y_c = -\frac{0.29}{2.67} = -0.11 \text{ m.}$$

$x_c$ and $y_c$ represent the x and y co-ordinates of the core points.

The remainder of the core may be found from symmetry and is as shown in Fig. 8.31 b.

The location of N is found in the usual way from the values of the straining actions,

$$e_x = \frac{M_y}{N} = \frac{99}{262} = 0.37 \text{ m.}$$

$$e_y = \frac{M_x}{N} = -\frac{54}{262} = -0.21 \text{ m.}$$

Thus N lies as indicated in Fig. 8.31 b, and it is seen that it lies within the core. The stresses allover the section are therefore expected to be of the same sign.

From the figure,

$z_1 = z_2 = 0.47$ m. and $e = 0.42$ m.

$$f_5 = f_{max} = -\frac{262\ (0.47 + 0.42)}{15.6 \times 0.47} = -32 \quad t./m.^2$$

$$f_2 = f_{min} = -\frac{262\ (0.47 - 0.42)}{15.6 \times 0.47} = -1.8 \ t./m.^2$$

**Example 8.20** Check the maximum normal stresses in example 8.17 by the core method.

Solution :

The cross-section and the straining actions are reproduced in Fig. 8.32 a.

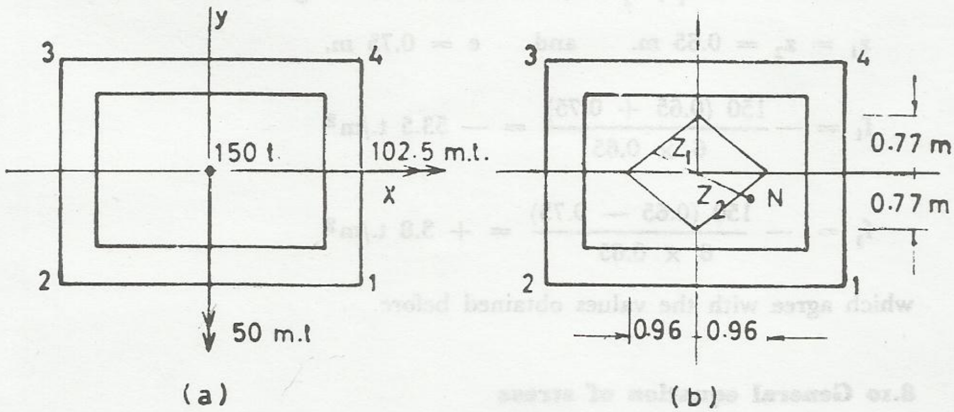



Fig. 8.32

The properties of the section have been found and are as follows :

$A = 6\ m^2$ , $I_x = 7\ m^4$ and $I_y = 11.5\ m^4$

Determination of the core :

For N.A. to coinside with side 1-2,

$x_1 = \infty$ , $y_1 = -1.5$ m.

$$x_c = 0 \ , \ y_c = +\frac{7}{6} \times \frac{2}{3} = 0.77 \text{ m}$$

For N.A. to coinside with side 2-3,

$x_1 = -2$ , $y_1 = \infty$

$$x_c = \frac{11.5}{6} \times \frac{1}{2} = 0.96 \text{ m.} \quad , \quad y_c = 0$$

The rest of the core is found from symmetry.

Location of N :

$$e_x = \frac{102.5}{150} = 0.68 \text{ m.}$$

$$e_y = -\frac{50}{150} = -0.33 \text{ m.}$$

The core of the section and the position of N are shown in Fig. 8.32 b. The values of $z_1$ , $z_2$ and e are scaled from Fig. 8.32 b.

$z_1 = z_2 = 0.65$ m. and $e = 0.75$ m.

$$f_1 = -\frac{150\ (0.65 + 0.75)}{6 \times 0.65} = -53.5 \text{ t./m}^2$$

$$f_3 = -\frac{150\ (0.65 - 0.75)}{6 \times 0.65} = +3.8 \text{ t./m}^2$$

which agree with the values obtained before.

### 8.10 General equation of stress

Equation 8.23 of the normal stresses on a section subject to the general case of an axial load eccentric to both of the centroidal axes cannot be applied unless the two reference axes chosen are the principal axes of inertia.

In the case of unsymmetrical sections where the principal axes of inertia are not obvious, the solution becomes too involved and resort is preferably made to the general equation of stress which is developed in the following.

Fig. 8.33 shows a section subject to a normal force N which is eccentric with respect to two rectangular centroidal axes x-x and y-y. From assumption (5) in section 8.3 that plane sections before bending remain plane after bending, the stresses on the section vary directly with the x and y co-ordinates of the various points. Therefore the normal stress on the section is represented by a plane. Mathematically, the normal stress f at any point (x , y) may be expressed by :

$$f = ax + by + c \qquad \dots 8.34$$

For equilibrium, the sum of all the normal forces on the cross-section must be equal to the applied normal force and the sum of the moments of all the normal forces on the section about the x and y axes must be equal to the applied moments about the respective axes.

Mathematically these conditions are expressed as :

$$N = \int f\, dA \qquad \dots 8.35$$

$$Ne_y = \int f\, dA\, y = M_x \qquad \dots 8.36$$

$$Ne_x = \int f\, dA\, x = M_y \qquad \dots 8.37$$

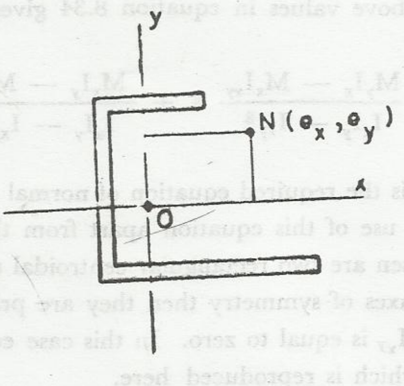



Fig. 8.33

Substituting the value of f from equation 8.34 into equations 8.35, 8.36 and 8.37 gives :

$$N = a\int x\, dA + b\int y\, dA + c\int dA \qquad \dots 8.38$$

$$M_x = a\int x\, y\, dA + b\int y^2\, dA + c\int y\, dA \qquad \dots 8.39$$

$$M_y = a\int x^2\, dA + b\int x\, y\, dA + c\int x\, dA \qquad \dots 8.40$$

Since the two reference axes are taken through the centroid, the statical moments of area; $\int x\, dA$ and $\int y\, dA$ are both equal to zero, and the remainder of the integrals are the well known expressions for the geometric properties of the section.

Equations 8.38-8.40 thus reduce to :

$$N = cA \qquad \dots 8.41$$

$$M_x = aI_{xy} + bI_x \qquad \dots 8.42$$

$$M_y = aI_y + bI_{xy} \qquad \dots 8.43$$

In any problem the section is given so that its properties A, $I_x$, $I_y$ and $I_{xy}$ are known. Also the strainging actions N, $M_x$ and $M_y$ are known. Solving equations 8.41, 8.42 and 8.43 simultaneously gives the following values for the constants a, b and c.

$$c = \frac{N}{A} \quad , \quad a = \frac{M_yI_x - M_xI_{xy}}{I_xI_y - I_{xy}^2} \quad , \quad b = \frac{M_xI_y - M_yI_{xy}}{I_xI_y - I_{xy}^2}$$

Substituting the above values in equation 8.34 gives :

$$f = \frac{N}{A} + \frac{M_yI_x - M_xI_{xy}}{I_xI_y - I_{xy}^2}\, x + \frac{M_xI_y - M_yI_{xy}}{I_xI_y - I_{xy}^2}\, y \qquad \dots 8.44$$

Equation 8.44 is the required equation of normal stress. There are no restrictions on the use of this equation apart from the fact that the two reference axes chosen are two rectangular centroidal axes. If one or both of these axes are axes of symmetry then they are principal axes and the product of inertia $I_{xy}$ is equal to zero. In this case equation 8.44 reduces to equation 8.23 which is reproduced here.

$$f = \pm \frac{N}{A} \pm \frac{M_x y}{I_x} \pm \frac{M_y x}{I_y} \qquad \dots 8.45$$

## 8.11 Sign conventions

On applying equation 8.45 the proper signs for the various terms are easily found by inspection. This is not the case however with regard to the general equation of stress, 8.44. Specified sign conventions must be adopted. If the positive directions of the co-ordinate axes are taken as usual to the right and upwards, the corresponding sign conventions for the moments are as indicated in Fig. 8.34, and as previously considered, N is positive when tensile and negative wh n co npressive.

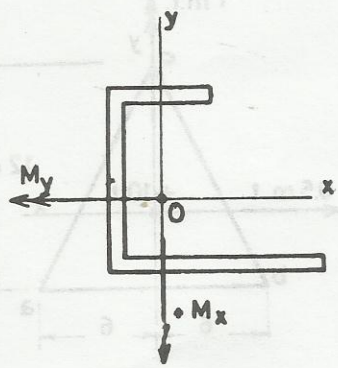



Fig. 8.34

As long as the stresses at one or several points in a section are required, this method provides the best approach to the problem especially when the section is unsymmetrical. If however the maximum normal stress or the normal stress distribution on the section are required reference must be made to the neutral axis.

### 8.12 The neutral axis

As mentioned before, the neutral axis N.A. is that straight line along which the normal stresses are zero. The equation of the neutral axis is therefore readily obtained by setting the right hand side of equation 8.44 to zero.

$$\frac{N}{A} + \left( \frac{M_y I_x - M_x I_{yx}}{I_x I_y - I_{xy}^{\ 2}} \right) x + \left( \frac{M_x I_y - M_y I_{xy}}{I_x I_y - I_{xy}^{\ 2}} \right) y = 0 \quad \ldots 8.46$$

The x-intercept of the neutral axis is found by setting y to zero, and the y-intercept is found by setting x to zero in equation 8 46.

The procedures for applying the general equation of stress and for locating the neutral axis and the points of maximum stress are illustrated in the following examples.

**Example 8.21** Find the equation of the normal stress on the cross-section of Fig. 8.35 due to the shown straining actions, then calculate the stresses at each of the three corners.

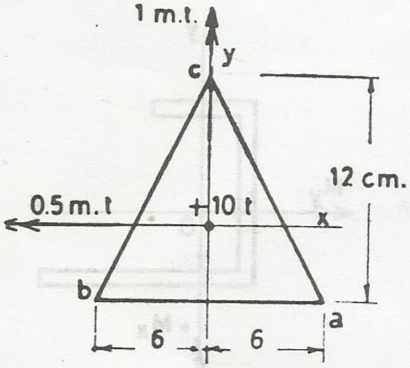



Fig. 8.35

Solution:

The reference axes are chosen as the horizontal and vertical axes through the centroid. Since the y axis is an axis of symmetry, x and y are the principal axes of inertia and $I_{xy} = 0$.

$$A = \frac{12 \times 12}{2} = 72 \text{ cm}^2$$

$$I_x = \frac{12 \times 12^3}{36} = 576 \text{ cm}^4$$

$$I_y = \frac{12 \times 12^3}{48} = 432 \text{ cm}^4$$

According to the sign conventions used,

$N = +10$ t., $M_x = -100$ cm.t. and $M_y = +50$ cm.t.

The normal stress equation is thus given by :

$$f = +\frac{10}{12} + \frac{50 \times 576}{576 \times 432} x + \frac{-100 \times 432}{576 \times 432} y$$

$$f = 0.139 + 0.116\, x - 0.176\, y$$

At corner a, $x = 6$ , $y = -4$

$$f = 0.139 + 0.116 \times 6 + 0.176 \times 4 = 1.539 \text{ t./cm}^2$$

At corner b, $x = -6$ , $y = -4$

$$f = 0.139 - 0.116 \times 6 + 0.176 \times 4 = 0.147 \text{ t./cm}^2$$

At corner c, $x = 0$ , $y = 8$

$$f = 0.139 - 0.176 \times 8 = -1.269 \text{ t./cm}^2$$

**Example 8.22** Plot the normal stress distribution on the section shown in Fig. 8.36 if it is subjected to a bending moment $M_x = 0.5$ m.t. as indicated.

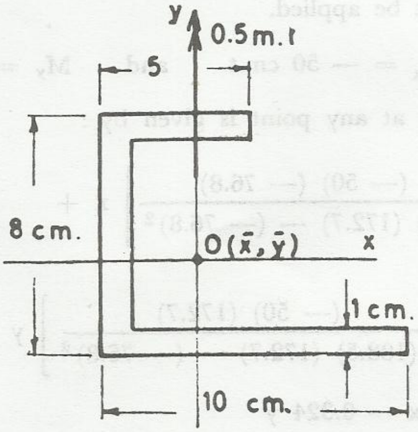



**Fig. 8.36**

Solution :

The centroid is determined first. Choosing two temporary axes as the left and lower edges of the section, the co-ordinates of 0 are determined as follows.

$$A = 5 \times 1 + 6 \times 1 + 10 \times 1 = 21 \text{ cm}^2.$$

$$\bar{y} = \frac{5 \times 7.5 + 6 \times 4 + 10 \times 0.5}{21} = 3.17 \text{ cm.}$$

$$\bar{x} = \frac{5 \times 2.5 + 6 \times 0.5 + 10 \times 5}{21} = 3.12 \text{ cm.}$$

Referring to the centroidal axes x and y,

$$I_x = \frac{5 \times 1^3}{12} + 5 \times 4.33^2 + \frac{1 \times 6^3}{12} + 6 \times 0.83^2 + \frac{10 \times 1^3}{12} + 10 \times 2.67^2 = 188.5 \text{ cm}^4.$$

$$I_y = \frac{1 \times 5^3}{12} + 5 \times 0.62^2 + \frac{6 \times 1^3}{12} + 6 \times 2.62^2 + \frac{1 \times 10^3}{12} + 10 \times 1.88^2 = 172.7 \text{ cm.}^4$$

$I_{xy} = 5\ (-0.62)\ (4.33) + 6\ (-2.62)\ (0.83) + 10\ (1.88)\ (-2.67)$
$= -\ 76.8\ cm.^4$

Since the principal axes of inertia are not being used, the general equation of stress must be applied.

$N = 0, \quad M_x = -\ 50\ cm.t. \quad \text{and} \quad M_y = 0$

The normal stress at any point is given by :

$$f = \left[ \frac{-(-50)\ (-76.8)}{(188.5)\ (172.7) - (-76.8)^2} \right] x + \left[ \frac{(-50)\ (172.7)}{(188.5)\ (172.7) - (-76.8)^2} \right] y$$

$f = -\ 0.114\ x - 0.324\ y$

The equation of the neutral axis is given by :

$0.114\ x + 0.324\ y = 0$

This is an equation of a straight line through the centroid which may be represented by first finding its slope; $\tan \alpha = \frac{-0.144}{0.324} = -\frac{4}{9}$ as shown in Fig. 8.37.

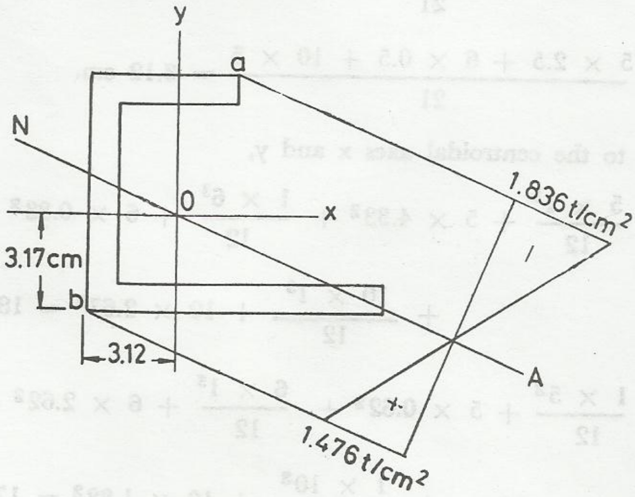



Fig. 8.37

The maximum stresses occur at point a (1.88, 4.83) and point b (— 3.12, — 3.17). Substituting the co-ordinates of these points in the normal stress equation,

$$f_a = -0.144\,(1.88) - 0.324\,(4.83) = -1.836 \text{ t./cm}^2$$

$$f_b = -0.144\,(-3.12) - 0.324\,(-3.17) = 1.476 \text{ t./cm.}^2$$

The normal stress distribution is shown in Fig. 8.37.

**Example 8.23** Find the maximum normal stresses on the cross-section shown in Fig. 8.38 due to a single compressive force of 10 t. at point a of the section. Plot the normal stress distribution.

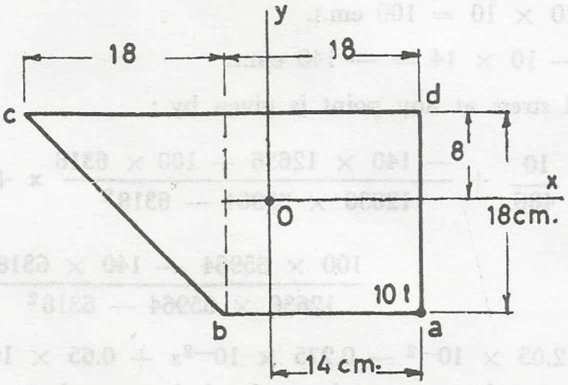



Fig. 8.38

Solution :

The centroid is determined first. Choosing two temporary axes as the upper and right edges of the section, the co-ordinates of 0 are calculated as fellows :

$$A = 18 \times 18 + \frac{18 \times 18}{2} = 486 \text{ cm.}^2$$

$$\bar{x} = \frac{18 \times 18 \times 9 + 18 \times 9\,(18 + 6)}{486} = 14 \text{ cm.}$$

$$\bar{y} = \frac{18 \times 18 \times 9 + 18 \times 9 \times 6}{486} = 8 \text{ cm.}$$

Referring to the centroidal axes x and y,

$$I_x = \frac{18 \times 18^3}{12} + 18 \times 18 \times 1^2 + \frac{18 \times 18^3}{36} + 18 \times 9 \times 2^2$$

$$= 12636 \text{ cm.}^4$$

$$I_y = \frac{18 \times 18^3}{12} + 18 \times 18 \times 5^2 + \frac{18 \times 18^3}{36} + 18 \times 9 \times 10^2$$

$$= 35964 \text{ cm.}^4$$

$$I_{xy} = 18 \times 18 \, (5) \, (-1) - \frac{1}{72} \times 18^2 \times 18^2 + 18 \times 9 \, (-10) \, (2)$$

$$= - 6318 \text{ cm.}^4$$

Since the principal exes of inertia are not being used, the general equation of stress must be applied.

$N = - 10$ t.

$M_x = 10 \times 10 = 100$ cm.t.

$M_y = - 10 \times 14 = - 140$ cm.t.

The normal stress at any point is given by :

$$f = - \frac{10}{486} + \frac{- 140 \times 12636 + 100 \times 6318}{12636 \times 35964 - 6318^2} x + \frac{100 \times 35964 - 140 \times 6318}{12636 \times 35964 - 6318^2} y$$

$$f = - 2.05 \times 10^{-2} - 0.275 \times 10^{-2} x + 0.65 \times 10^{-2} y$$

Since the maximum normal stress and the normal stress distribution are required, the neutral axis has to be found. Its equation is given by :

$$2.05 + 0.275 \, x - 0.65 \, y = 0 \qquad \text{(i)}$$

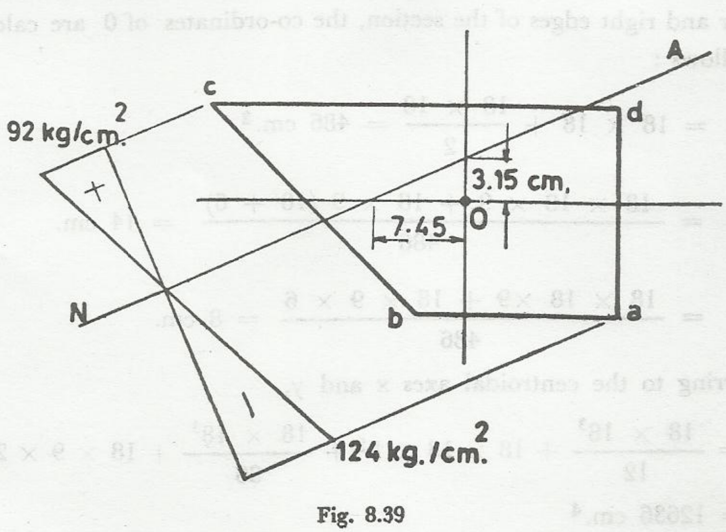



Fig. 8.39

The neutral axis is plotted in the known manner by finding the x and y intercepts and is as shown in Fig. 8.39.

The maximum normal stresses occur at the points which are farthermost from the neutral axis, or in this particular case at points a (14, —10) and c (—22, 8).

Substituting the co-ordinates of these points in the normal stress equation,

$$f_a = [-2.05 - 0.275\ (14) + 0.65\ (-10)]\ 10^{-2} = -0.124\ t/cm^2$$

$$f_c = [-2.05 - 0.275\ (-22) + 0.65\ (8)]\ 10^{-2} = +0.092\ t/cm.^2$$

The normal stress distribution is shown in Fig. 8.39.

**Example 8.24** Find the location of a single normal force on the cross-section shown in Fig. 8.38 such that the neutral axis may coinside with side ad.

Solution :

Let a normal force N act on the given section with eccentricities $e_x$ and $e_y$ with respect to the centroidal axes x and y then, $M_x = Ne_y$ and $M_y = Ne_x$

and the general equation of stress becomes :

$$f = N\left(\frac{1}{A} + \frac{e_xI_x - e_yI_{xy}}{I_xI_y - I_{xy}{}^2}\,x + \frac{e_yI_y - e_xI_{xy}}{I_xI_y - I_{xy}{}^2}\,y\right)$$

At the neutral axis the quantity between brackets must be zero.

$$\frac{1}{A} + \frac{e_xI_x - e_yI_{xy}}{I_xI_y - I_{xy}{}^2}\,x + \frac{e_yI_y - e_yI_{xy}}{I_xI_y - I_{xy}{}^2}\,y = 0 \qquad \text{(ii)}$$

In order the neutral axis may coinside with side ad, its equation must be represented by:

$$x - 14 = 0 \qquad \text{(iii)}$$

Equations (ii) and (iii) are identical. Therefore, the ratios between the coefficients of similar terms in both equations must be equal.

$$\frac{1}{A} \Big/ -14 = \frac{e_x I_x - e_y I_{xy}}{I_x I_y - I_{xy}^2} \Big/ 1 = \frac{e_y I_y - e_x I_{xy}}{I_x I_y - I_{xy}^2} \Big/ 0 \qquad \text{(iv)}$$

Using the properties of the cross-section found previously in example 8.23 and solving the two equations in (iv),

$e_x = -6.4$ cm. and $e_y = 3.2$ cm.

It should be remembered that (— 6.4, 3.2) define the co-ordinates of a point on the core of the section.

**Example 8.25** For the cross-section shown in Fig. 8.38, plot the line along which a force must act so that zero stress may develop at point a.

Solution :

Let the section be subjected to an axial force at location ($e_x$, $e_y$). The stress at any point on the section is thus given by :

$$f = N \left[ \frac{1}{A} + \frac{e_x I_x - e_y I_{xy}}{I_x I_y - I_{xy}^2} x + \frac{e_y I_y - e_x I_{x,y}}{I_x I_y - I_{xy}^2} y \right]$$

At point a, $x = 14$, $y = -10$ and $f = 0$. Substituting these values and the geometric properties of the cross-section already found in example 8.23,

$$2\,05 + 0.275 e_x - 0.65 e_y = 0 \qquad \text{(v)}$$

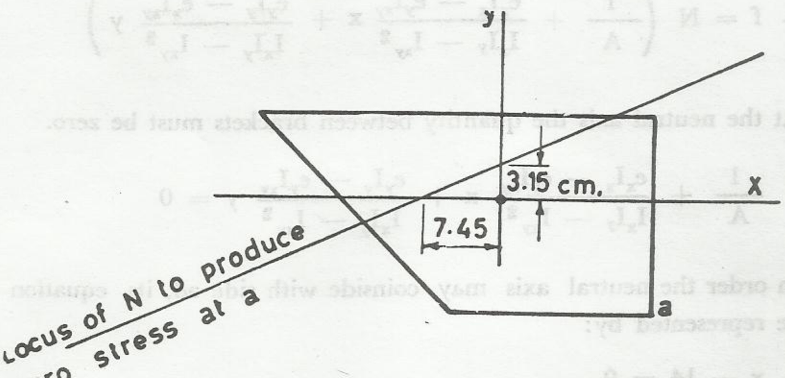



Fig. 8.40

## EXAMPLES TO BE WORKED OUT

(1) A simple beam of span 4 m. carries a uniformly distributed load of 0.25 t./m. The beam is of rectangular section 25 cm. deep. How wide should it be if the stress must not exceed 0.2 $t/cm^2$ ?

(2) A T-section of uniform thickness 1 cm. has a flange breadth of 12 cm. and an overall depth of 12 cm. Calculate the moments of resistance of the section about the principal axes if the stresses are limited to 1.2 $t./cm.^2$

(3) The front axle of a lorry carries the loads shown in Fig. 8.41. Check the normal stresses in the axle if it has an I-section of the given dimensions.

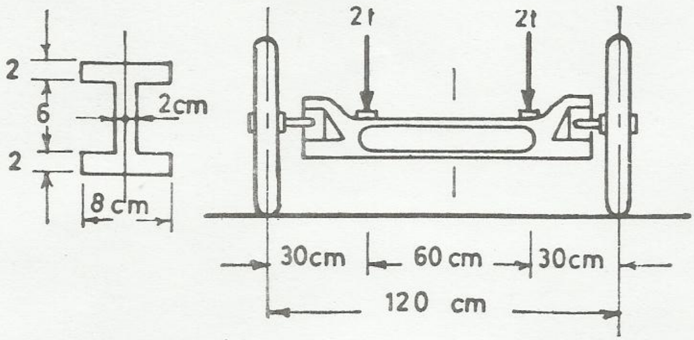



Fig. 8.41

(4) A simply supported beam of span 4 m. has a cross-section 20 cm. deep and moment of inertia about the axis of bending of 2140 $cm.^4$ What central load can the beam carry if the bending stresses are not to exceed 1.2 $t./cm.^2$ ? For the same beam, what total uniformly distributed load it may carry ?

(5) A steel plate 0.6 cm. thick and 10 cm. wide is to be bent to form an arc of a circle. What is the maximum bending stress in the plate when bent to a 2.5 m. radius ?

(6) In an experiment in the laboratory to determine the modulus of elasticity of steel E, a steel rod 6 cm. wide and 1 cm. deep was supported

This equation represents a straight line along which the normal force must act in order to produce zero stress at point a. It is plotted in the usual way by finding the intercepts $e_x$ and $e_y$ on the x and y axes.

$$e_x = \frac{-2.05}{0.275} = -7.45 \text{ cm.} \quad \text{and} \quad e_y = \frac{2.05}{0.65} = 3.15 \text{ cm.}$$

The result is shown in Fig. 8.40.

It is of interest to notice that equation (v) is identical to equation (i) which represents the N.A. due to a normal force acting at point a. This is always true in the sense that the N.A. due to a normal force acting at some point is the locus of the normal force to produce zero stress at the particular point considered.

## EXAMPLES TO BE WORKED OUT

(1) A simple beam of span 4 m. carries a uniformly distributed load of 0.25 t./m. The beam is of rectangular section 25 cm. deep. How wide should it be if the stress must not exceed 0.2 t/cm$^2$ ?

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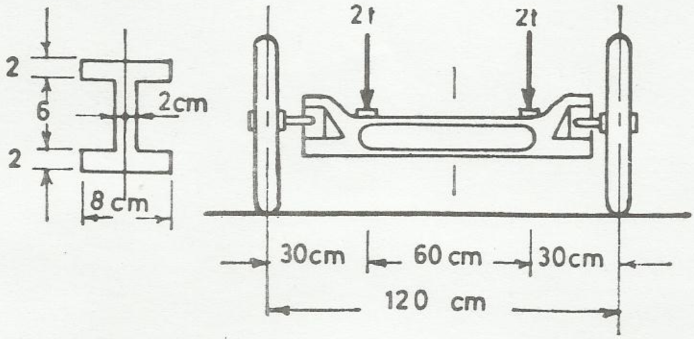



Fig. 8.41

(4) A simply supported beam of span 4 m. has a cross-section 20 cm. deep and moment of inertia about the axis of bending of 2140 cm.$^4$ What central load can the beam carry if the bending stresses are not to exceed 1.2 t./cm.$^2$ ? For the same beam, what total uniformly distributed load it may carry ?

(5) A steel plate 0.6 cm. thick and 10 cm. wide is to be bent to form an arc of a circle. What is the maximum bending stress in the plate when bent to a 2.5 m. radius ?

(6) In an experiment in the laboratory to determine the modulus of elasticity of steel E, a steel rod 6 cm. wide and 1 cm. deep was supported

on a span of 1 m. and subjected to a constant B.M. = 1000 cm. kg. The central deflection was measured by a dial gauge and was found to be equal to 1.25 cm. Calculate the value of E.

(7) A steel channel as shown in Fig. 8.42 carries water across a span of 6 m. and is simply supported at the ends. If the water weighs 1 t./m.$^3$, calculate the maximum normal stress for the middle cross-section of the channel. Neglect own weight of channel.

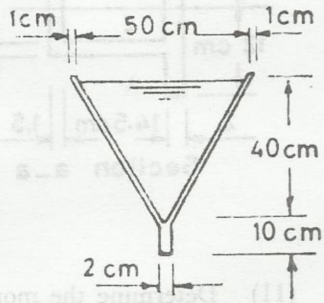



Fig. 8.42

(8) Fig. 8.43 shows a cast iron section. If the allowable stresses for cast iron are 0.4 t./cm.$^2$ and 0.8 t./cm.$^2$ in tension and compression respectively, what is the safe moment for the section ? and what safe uniformly distributed load will a simply supported beam carry over a span of 4.8 m. if it has the given cross-section ?

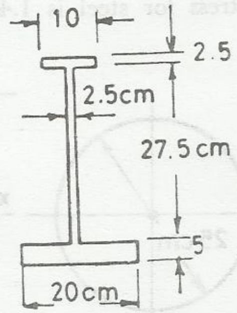



Fig. 8.43

(9) Find the maximum normal stresses on section a-a of the machine bracket shown in Fig. 8.44 if it has the given cross-section.

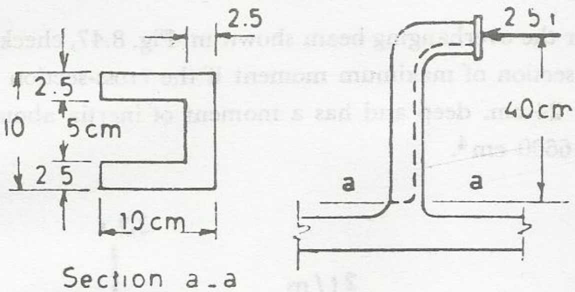



Fig. 8.44

(10) Determine the maximum value of the load P that may be applied to the machine bracket shown in Fig. 8.45 so that the extreme stresses do not exceed 0.4 t./cm.$^2$ in tension and 0.8 t./cm.$^2$ in compression.

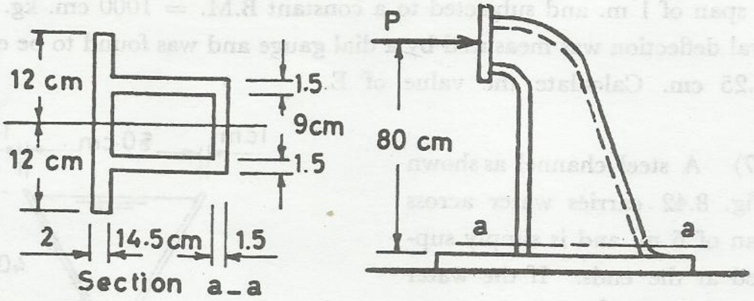



Fig. 8.45

(11) Determine the moments of resistance of the steel sections shown in Figs. 8.46 a-d, when bent about the two principal axes and find the ratio between the two moments for each section. Allowable bending stress for steel is 1.4 t./cm².

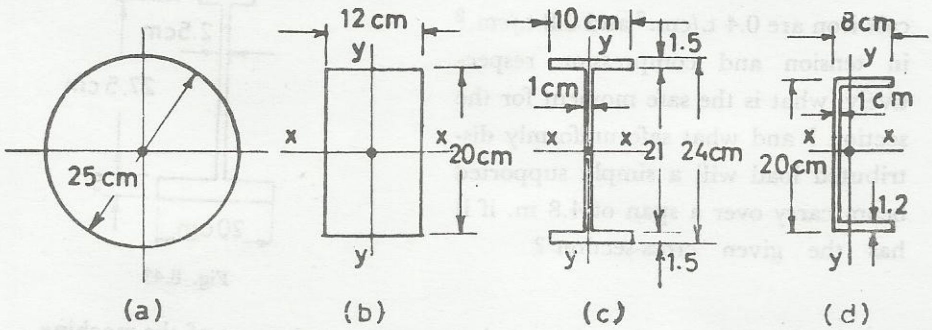



Fig. 8.46

(12) For the overhanging beam shown in Fig. 8.47, check the normal stress at the section of maximum moment if the cross-section of the beam is an I-beam 24 cm. deep and has a moment of inertia about the axis of bending of 16690 cm⁴.

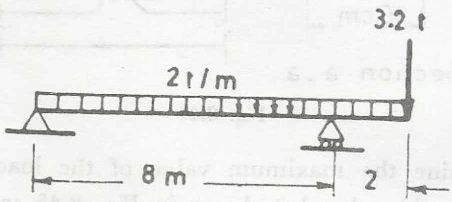



Fig. 8.47

(13) A simple beam of span L m., carries two concentrated loads at quarter points each of 4 t. Find the maximum value of L if the beam has the steel cross-section shown in Fig. 8.48 a. If the beam is strengthend by a cover plate as shown in Fig. 8.48 b, what will be the new fiber stresses ? Allowable stress for steel equals 1. 2 t./cm$^2$.

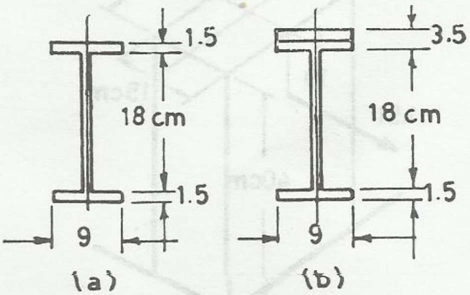



Fig. 8.48

(14) A simply supported beam carries two concentrated loads as shown in Fig. 8.49. The section of the beam is the given channel with the following properties (obtained from tables) :

$h = 30$ cm., $b = 10$ cm., $e = 2.7$ cm.,

$I_x = 8030$ cm.$^4$ and $I_y = 495$ cm.$^4$

The channel may take either position (1) or (2). Calculate the maximum value of P in both cases and hence deduce which of the two positions is better if the allowable normal stress is 1.2 t./cm.$^2$

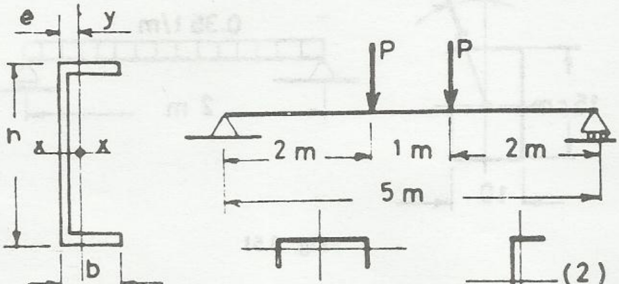



Fig. 8.49

(15) A timber post of the given dimensions is loaded as shown in Fig. 8.50. Neglecting the own weight of the post, calculate the maximum

normal stress on the base section and plot the normal stress distribution.

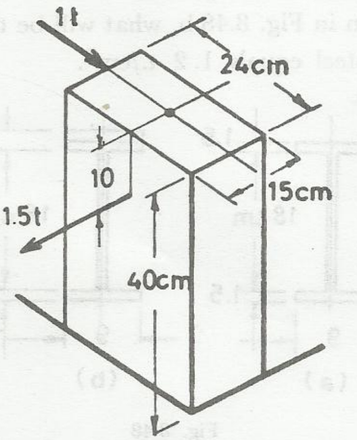



Fig. 8.50

(16) The 10 × 15 cm. wooden beam shown in Fig. 8.51 is used to support a uniformly distributed load of 0.35 t./m. on a simple span of 2 m. The applied load acts in a plane making an angle $30^0$ with the vertical as shown. Calculate the maximum bending stresses at mid span and plot the normal stress distribution on the section.

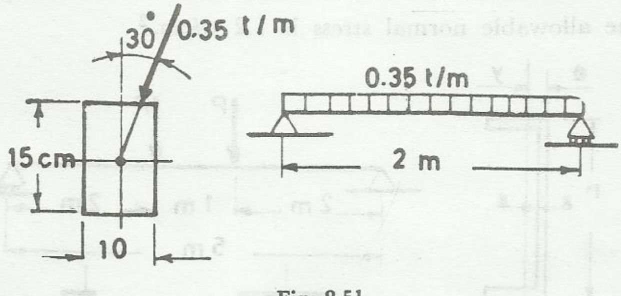



Fig. 8.51

(17) A crane girder whose cross-section consists of a plate 36 × 1.4 cm. and two channels for each of which A = 58.8 cm.$^2$, $I_x$ = 8030 cm$^4$. $I_y$ = 495 cm$^4$., carries the vertical and horizontal loads shown in Fig. 8.52. Plot the stress distribution at the section of maximum moment indicating the values of maximum compressive and tensile stresses.

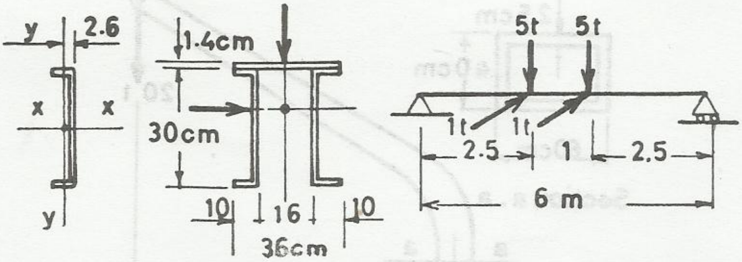



Fig. 8.52

(18) Fig. 8.53 shows a channel purlin on a rafter sloping 5 : 12. The purlin has a span of 4 m. and carries a vertical uniformly distributed load from the roof of 0.52 t./m. Plot the stress distribution at the section of maximum moment indicating the values of maximum tensile and compressive stresses. For channel, $I_x = 4820$ cm$^4$., $I_y = 317$ cm$^4$.

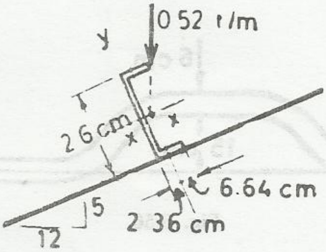



Fig. 8.53

(19) Find the maximum value of P the cantilever shown in Fig. 8.54 may carry without the stress exceeding 1.2 t./cm$^2$.

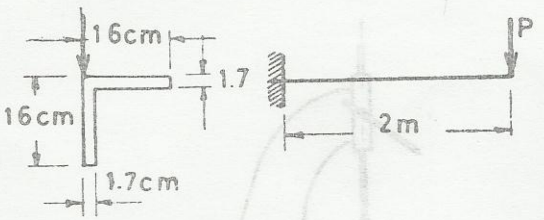



Fig. 8.54

(20) A cantilever crane supports a load of 20 t. as shown in Fig. 8.55. Check the normal stress on section a-a if the cross-section is a hollow rectangular section of the given dimensions.

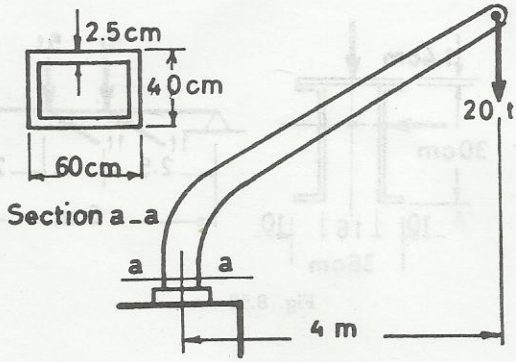



Fig. 8.55

(21) A link in a machine has to be bent as shown in Fig. 8.56 for the sake of clearance. Calculate the maximum tensile and compressive stresses in the link if it is subjected to a tensile force of 1.5 t., and has a 6 × 4 cm. rectangular cross-section. Plot the normal stress distribution across the section.

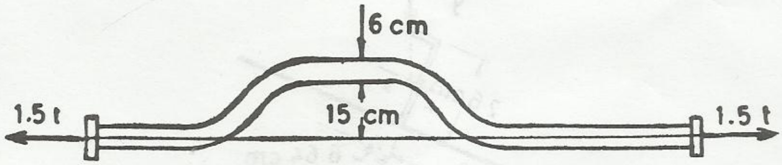



Fig. 8.56

(22) Fig. 8.57 shows a cast iron stand of a drilling machine. Find the greatest value of the drilling force P without the tensile stress exceeding 0.4 t./cm.$^2$ if the stand has a hollow rectangular cross-section. The outside dimensions are 16 cm. and 12 cm. and the material is 2 cm. thick all round.

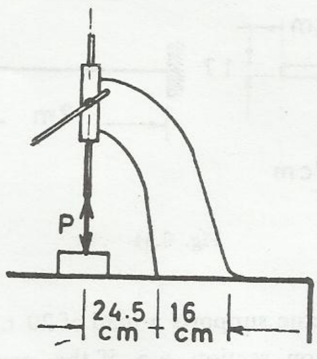



Fig. 8.57

(23) Fig. 8.58 shows a wooden cantilever of rectangular cross-section 18 × 12 cm. Calculate the maximum normal stresses at the sections of maximum positive and maximum negative moments. Plot the normal stress distribution in both cases.

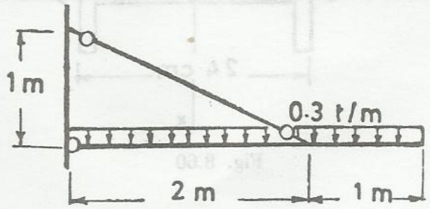



Fig. 8.58

(24) A top chord member in a steel truss is subjected to an axial compression of 10 t. and also a bending moment from a concentrated load of 1.2 t. as shown in Fig. 8.59. The member is composed of two angles 10 × 10 × 1.2 cm., the properties for each of which are :

$A = 22.7$ cm.$^2$ and $I_x = 207$ cm.$^4$

Calculate the maximum normal stresses in the member.

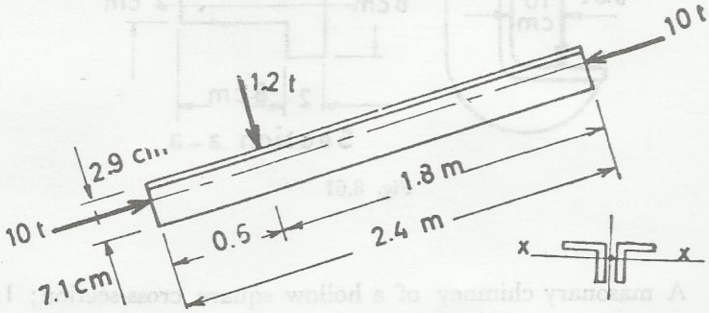



Fig. 8.59

(25) A short steel column consists of a standard I-beam number 24 and carries the loads shown in Fig. 8.60. Calculate the extreme normal stresses and plot the normal stress distribution across the section. The properties of the I-beam are :

$A = 46.1$ cm.$^2$ and $Z_x = 354$ cm$^3$.

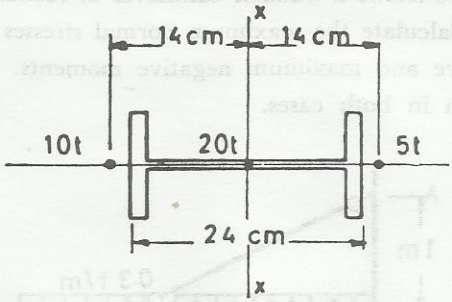



Fig. 8.60

(26) A cast iron bracket carries a load of 0.8 t. as shown in Fig. 8.61. Calculate the values of the extreme normal stresses on section a-a if it is a T-section of the given dimensions.

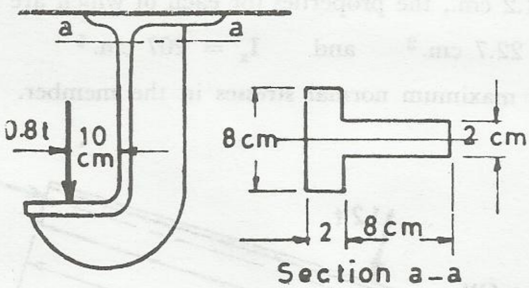



Fig. 8.61

(27) A masonary chimney of a hollow square cross-section; 1.8 m. and 1.2 m. outside and inside dimensions respectively, is subjected to a horizontal wind pressure of 0.1 t./m². across one of the sides. If the masonary weighs 2 t./m³., find the normal stress distribution at the base section of the chimney if it is 8 m. high. What would the maximum height be if no tensile stresses are allowed ?

(28) A bracket is fixed to the centre of a square foundation as shown in Fig. 8.62. Determine the pressure distribution on the soil if the base side b is equal to 3.5,3 and 2.5m. and the weight of foundations to 2 t./m³.

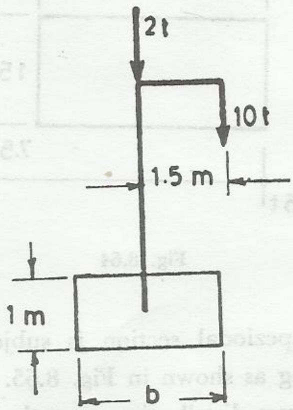



Fig. 8.62

(29) For the frame shown in Fig. 8.63 draw the N.F., S.F. and B.M.Ds. Also, plot the normal stress distribution on the soil under the footings if they are square of side length = 2m. Weight of footing = 2.2 t./m$^3$.

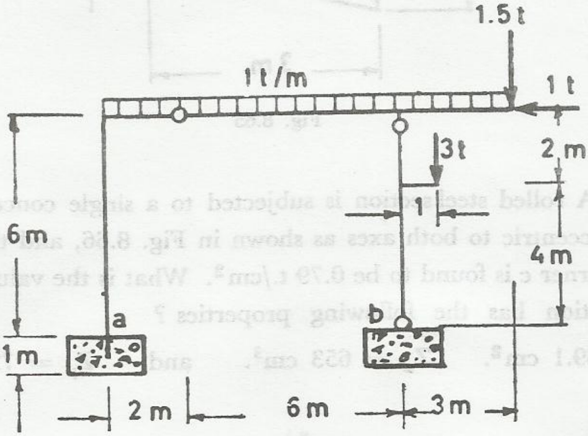



Fig. 8.63

(30) Find the maximum normal stresses over the rectangular section shown in Fig. 8.64 if it is subjected to a compressive force of 6.5 t. located as shown. Plot the stress distribution.

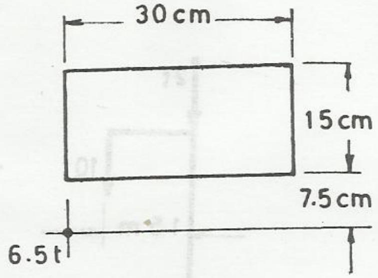



Fig. 8.64

(31) The given trapeziodal section is subjected to three eccentric compressive forces acting as shown in Fig. 8.65. Locate the neutral axis and plot the normal stress distribution over the section.

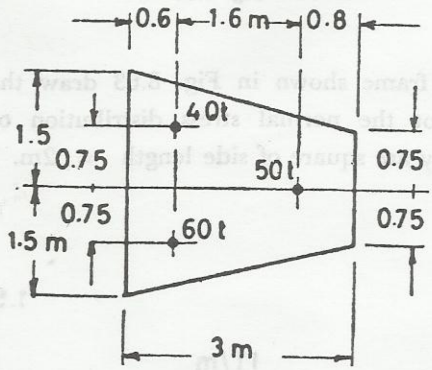



Fig. 8.65

(32) A rolled steel section is subjected to a single concentrated load which is eccentric to both axes as shown in Fig. 8.66, and the maximum stress at corner c is found to be 0.79 t./cm². What is the value of this load if the section has the following properties ?

$A = 69.1 \text{ cm}^2$. $Z_x = 653 \text{ cm}^3$. and $Z_y = 72.2 \text{ cm}^3$.

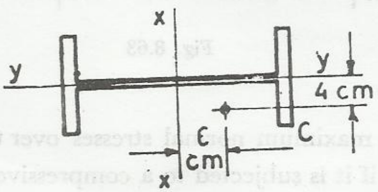



Fig 8.66

(33) - (35) Masonary piers have cross-sections as shown in Figs. 8.67-8.69, and carry the given set of vertical and horizontal loads beside their own weights. If the masonary weighs 2 t./m³., find the normal stress distributions on the sections at the base of the piers and locate the points of maximum stresses.

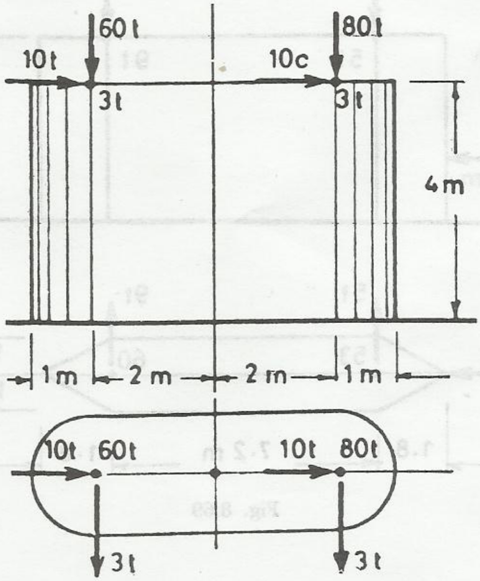



Fig. 8.67

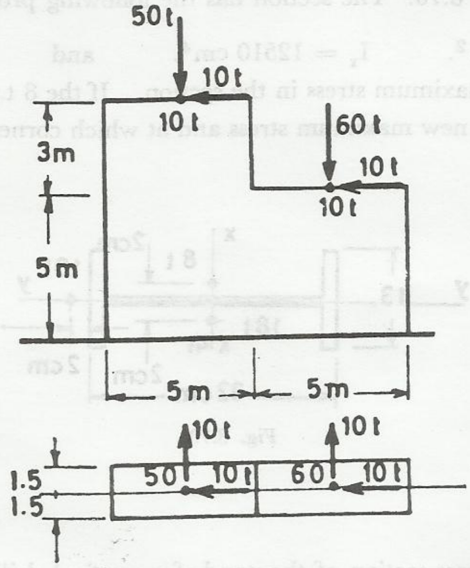



Fig. 8.68

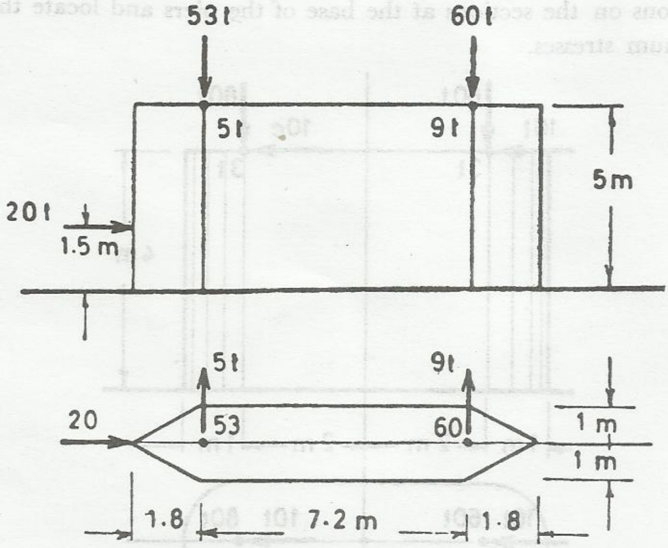



Fig. 8.69

(36) A rolled steel section carries a central load and three other loads as shown in Fig. 8.70. The section has the following properties.

$A = 77.8\ cm^2$. $I_x = 12510\ cm^4$. and $I_y = 555\ cm^4$.
Determine the maximum stress in the section. If the 8 t. load is removed, what will be the new maximum stress and at which corner will it occur ?

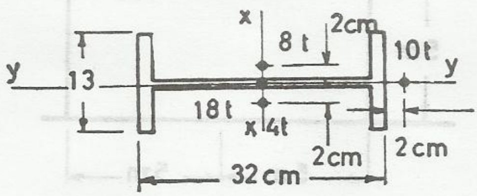



Fig. 8.70

(37) The cross-section of the stand of a vertical drilling machine has the form shown in Fig. 8.71. The line of action of thrust of the drill passes

through N which moves on an arc of a circle radius 36 cm as indicated. Find the position of N to produce maximum stress and calculate the value of N if the tensile stress is limited to 0.4 t./cm². Check the result using the core method.

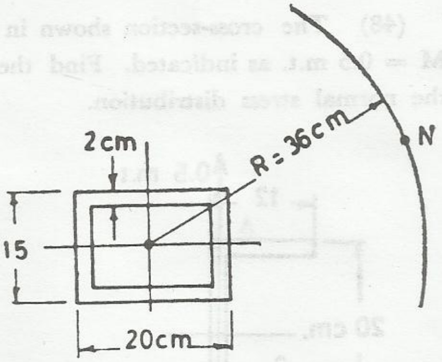


Fig. 8.71

(38) The section shown in Fig. 8.72 is subjected to a normal tensile force P at the given location. Using the core method, determine the value of P so that the maximum normal stress may not exceed ± 100 kg/cm².

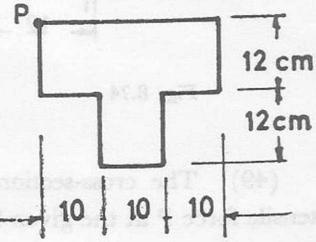


Fig. 8.72

(39) - (45) Using the core method, check the maximum normal stress in problems number (20), (21), (22), (26), (31), (35) and (36).

(46) A cross-sectional area having the shape of an angle as shown in Fig. 8.73 is acted on by a single compressive force of 8 t. Using centroidal axes which are parallel to the sides of the angle, determine the maximum normal stresses and plot the normal stress distribution.

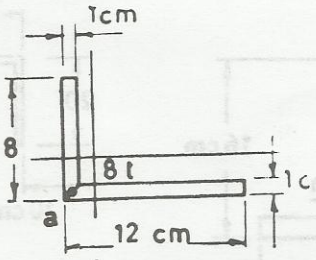


Fig. 8.73

(47) Given the area in Fig. 8.73, plot the line along which a normal force must be placed to produce zero stress at corner a.

(48) The cross-section shown in Fig. 8.74 is acted on by a moment M = 0.5 m.t. as indicated. Find the maximum normal stresses and plot the normal stress distribution.

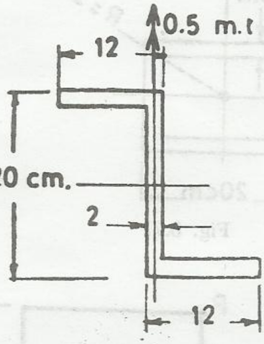



Fig. 8.74

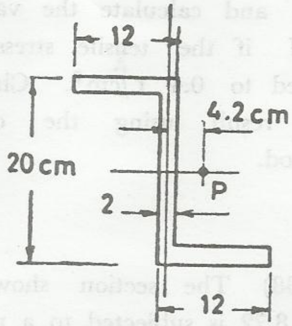



Fig. 8.75

(49) The cross-section shown in Fig. 8.75 is acted on by a single tensile force P at the given location. Find the maximum value of P so that the maximum normal stress may not exceed 1.2 t/cm$^2$. Find the location of P so that the neutral axis may coinside with upper edge of the section.

(50), (51) The cross-sections shown in Figs. 8.76 and 8.77 are subjected to a compressive normal force N at the indicated location. Find the value of N in each case so that the maximum compressive stress may not exceed 1 t./cm$^2$. and the maximum tensile stress 1.4 t./cm$^2$. Plot the normal stress distribution in each case.

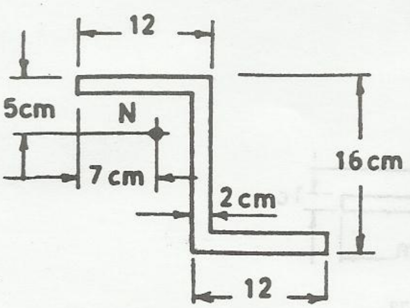



Fig. 8.76

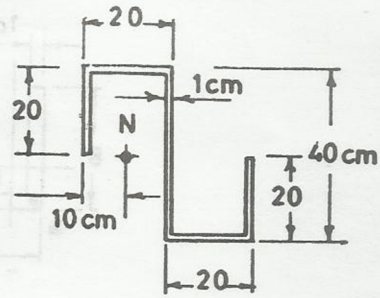



Fig. 8.77

(52) Solve problem (19) using the general equation of stress.

CHAPTER 9

# SHEAR STRESSES

## 9.1 Introduction

In chapter 8, a study of the normal stresses has been made. There is another type of stress of no less importance. This is the shear stress produced, as mentioned previously, by shearing forces and twisting moments.

In this chapter, shear stresses produced by direct shear, shear stress distribution across sections of members resisting varying moments along their lengths, and shear stresses due to twisting moments will be considered. Finally, the combined effect of shearing forces and twisting moments will be dealt with.

## 9.2 Direct shear stress

Consider a block which is glued to a table and let it be subjected to a force Q as shown in Fig. 9.1. If the block is assumed to be divided by

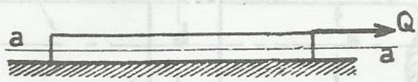



Fig 9.1

a horizontal section a-a, the upper part of the block will tend to slide over the lower part and stresses in the plane of the section, i.e. parallel to the applied load, will be developed resisting the tendency of the two parts to slide on each other. These stresses are called shear stresses. They arise in many practical problems, civil and mechanical alike.

Consider, for example, two plates held together by one rivet and carrying a force Q as shown in Fig. 9.2 a. Provided there is no friction between the two surfaces of contact, the area that resists the applied load is the cross-sectional area of the rivet. If the rivet is imagined to be divided to two parts by plane a-a, then the upper part will tend to slide over the lower one and shear stresses are set up on the section of the rivet.

This joint is called a lap joint and the connecting rivet is said to be in *single shear*. Another type of joints frequently used in practice is shown

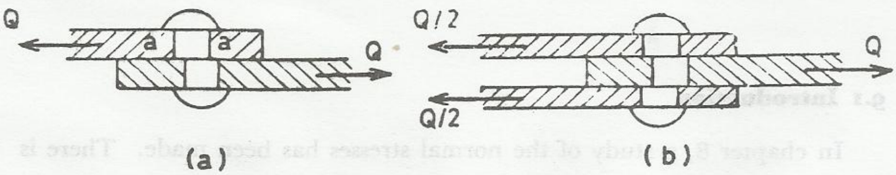



Fig. 9.2

in Fig. 9.2 b. In this joint, two cross-sectional areas of the same rivet resist the applied load, and thus the rivet is said to be in *double shear*.

As a second example, consider the case of a circular shaft with a collar held in a bearing as shown in Fig. 9.3. Under the action of a force Q there is, firstly, a tendency for the shaft to push through the collar thus inducing shear stresses on a cylinderical surface of perimeter $\pi a$ and height c.

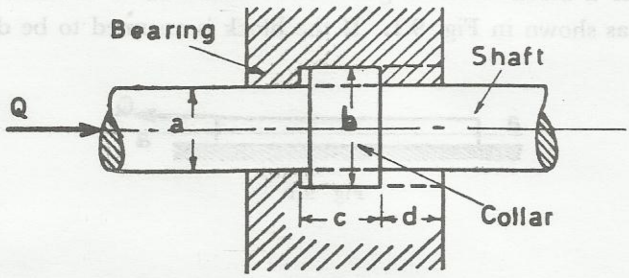



Fig. 9.3

Secondly, there is a tendency for the collar to push through the bearing so that shear stresses are set up on a cylinderical surface of perimeter $\pi b$ and height d.

The shear stress on any surface is defined as the force divided by the area resisting the tendency of sliding that results from the applied force. Thus, referring again to Fig. 9.1, and assuming that the shear stresses are uniformly distributed over the plane a-a, then for equilibrium $qA = Q$, from which,

$$q_{av} = \frac{Q}{A} \qquad \ldots 9.1$$

where Q is the applied shearing force, A the sheared area and $q_{av}$ the average shear stress.

In the first example of the riveted joints, the average shear stress developed in the single shear rivet (Fig. 9.2 a) is given by :

$$q_{av} = \frac{4Q}{\pi d^2}$$

and in the double shear rivet (Fig. 9.2 b),

$$q_{av} = \frac{2Q}{\pi d^2}$$

where d is the diameter of the rivet.

In the second example (Fig. 9.3), the average shear stress developed between the shaft and the collar is :

$$q_{av} = \frac{Q}{\pi ac}$$

and that between the collar and the bearing is :

$$q_{av} = \frac{Q}{\pi bd}$$

Thus, if $\frac{Q}{\pi ac} > \frac{Q}{\pi bd}$ or bd > ac, shear failure is more likely to occur between the shaft and the collar, and vice versa, if ac > bd, failure is more likely to occur between the collar and the bearing.

It should be mentioned that the shear stress given by equation 9.1 is only approximately true. This is mainly due to the nonuniform distribution of stress and in some cases, such as those of the riveted or bolted joints, due to the complex nature in which the applied shearing force is distributed among the various rivets or bolts of the joint and also due to the presence of friction.

**Example 9.1** Fig. 9.4 shows a lever keyed to a shaft 4 cm. diameter. The key is 1 cm. wide and 5 cm. long. Find the maximum load P that can be applied to the lever without the shear stress in the key exceeding 1000 kg./cm².

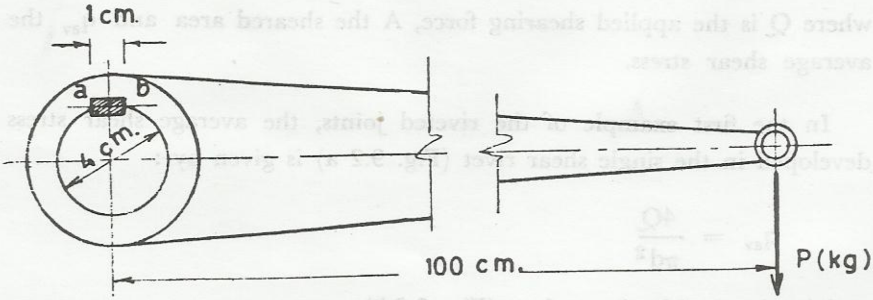



Fig. 9.4

Solution :

The torque produced = 100 P kg.cm. If the key shears, it will do so on plane ab. Let the shearing force on this plane be Q. Then for equilibrium,

$$2Q = 100\ P$$

$$Q\ (\text{allowable}) = 1 \times 5 \times 1000 = 5000\ \text{kg.}$$

$$P\ (\text{allowable}) = \frac{2 \times 5000}{100} = 100\ \text{kg.}$$

**Example 9.2** Two steel rods are connected together by the joint shown in Fig. 9.5. Show which part of the joint is likely to shear first if the pin has a rectangular cross-section 8 × 1.5 cm. If the allowable shear stress is 1 t./cm²., what will be the maximum allowable value of the pull P ?

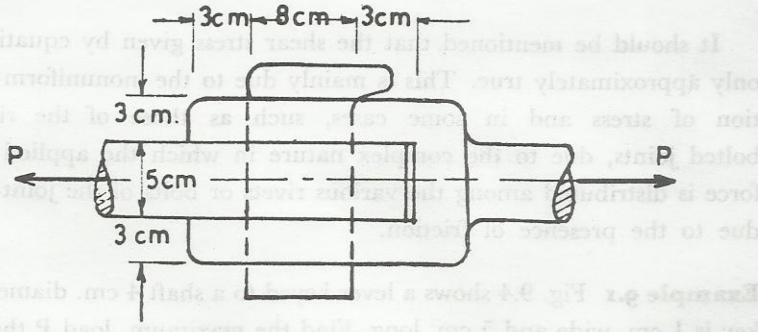



Fig. 9.5

Solution : There are three possible modes of failure :

(a) Failure of the rod

Area resisting shear = $2 \times 5 \times 3 = 30$ cm$^2$.

(b) Failure of the pin.

Area resisting shear = $2 \times 8 \times 1.5 = 24$ cm$^2$.

(c) Failure of the head.

Area resisting shear = $4 \times 3 \times 3 = 36$ cm$^2$.

Since the various components of the joint have the same shear resistance, failure is likely to occur in the component which offers the least area, i.e. the pin.

$$P \text{ (allowable)} = qA = 1 \times 24 = 24 \text{ t.}$$

**Example 9.3** Determine the safe load P that can be applied to the riveted connection shown in Fig. 9.6 if the allowable shear stress in the rivets is 0.96 t./cm$^2$., and the allowable stresses for the plates are 0.8 t./cm$^2$ and 1.2 t./cm$^2$ for shear and tension respectively. Diameter of rivet = 2 cm.

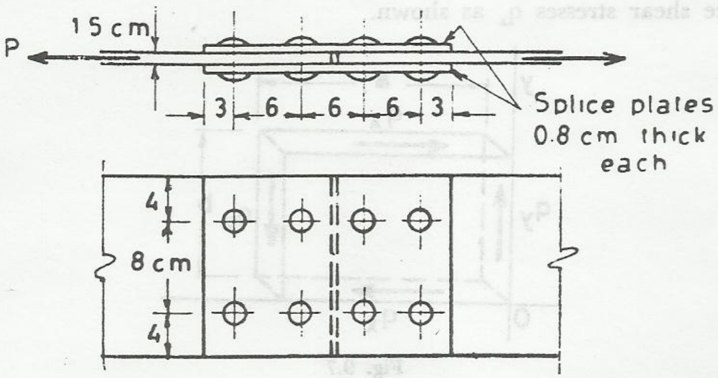



Fig. 9.6

Solution : There are three possible modes of failure of the joint.

(a) Shear of rivets : The area resisting the shear is twice the cross-sectional area of the rivets. Thus,

$$\text{Area resisting shear} = 2 \times 4 \times \frac{\pi \times 2^2}{4} = 8\,\pi \text{ cm}^2.$$

Allowable load carried by rivets = $8\,\pi \times 0.96 = 24$ t.

(b) Shear of plates : It is obvious that if shear failure occurs in the plates, it will do so in the main plate as its thickness is less than the combined thicknesses of the splice plates.

Area resisting shear $= 4 \times 9 \times 1.5 = 54$ cm$^2$.

Allowable load carried by plates $= 54 \times 0.8 = 43.2$ t.

(c) Tensile failure of the plates : It is obvious that if tension failure occurs it will do so across the rivet line where the area resisting tension is a minimum.

Net area $= 1.5\ (16 - 2 \times 2) = 18$ cm$^2$.

Allowable load in tension $= 18 \times 1.2 = 21.6$ t.

P (allowable) $= 21.6$ t.

**9·3 Compl**k **mentary shear**

Consider an element of unit thickness as shown in Fig. 9.7, and let it be subjected to a vertical shearing force which produces vertical shear stress $q_y$ on the vertical faces. For equilibrium, two equal and opposite forces must act on the horizontal faces of the element. These forces produce shear stresses $q_x$ as shown.

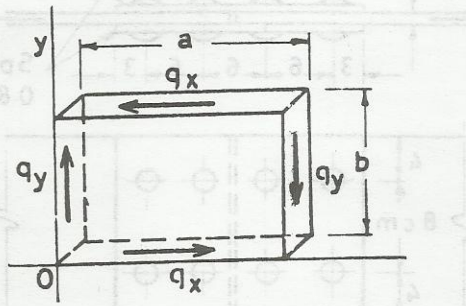



Fig. 9.7

Thus, shear stresses in vertical planes are always accompanied by shear stresses in horizontal planes perpendicular to the applied shearing force. Such stress is called complementary shear stress.

Referring again to Fig. 9.7, and taking moments of the forces on the faces about 0 then,

$$\Sigma M_o = (q_y \times a \times 1)\ b - (q_x \times b \times 1)\ a$$

$$q_x = q_y \qquad \dots 9.2$$

This shows that the complementary shear stress is equal to the applied shear stress. The directions of the two shear stresses must be either both towards, or both away from, the line of intersection of the two planes in which they act.

### 9.4 Shear stress formula for beams

Consider an element of a beam contained between two cross-sections a-a and b-b, at a small distance dx apart and let M & Q and (M + dM) & (Q — dQ) be the values of the bending moments and shearing forces across the two sections as shown in Fig. 9.8 a. Due to the bending moment, the normal stress on face a-a is given by :

$$f = \frac{M\ y}{I_x}$$

while the corresponding value on face b-b is :

$$f + df = \frac{(M + dM)\ y}{I_x}$$

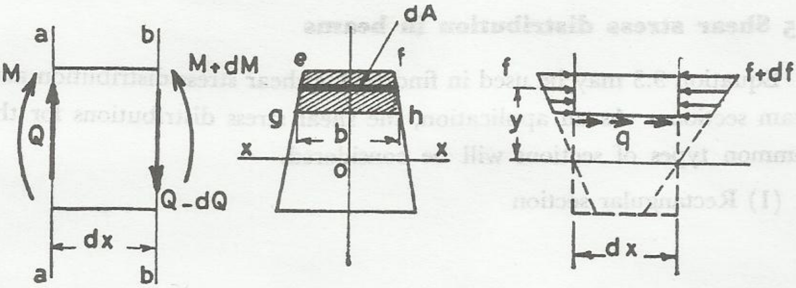



Fig. 9.8

Consider now the part of the element contained between the upper extreme fiber and that at some distance from the centroidal axis. Referring to Figs. 9.8 b and c, and assuming that the shear stress q is uniformly distributed on the horizontal plane of area b dx then for equilibrium,

$$q\ b\ dx = \int_{\substack{area\\efgh}} (f + df)\ dA - \int_{\substack{area\\efgh}} f dA = \frac{dM}{I_x} \int_{\substack{area\\efgh}} y dA$$

but $\int_{\substack{\text{area}\\ \text{efgh}}} y\, dA$ is the statical moment of area efgh, S, about the centroidal axis,

and from equation 3.3, $\dfrac{dM}{dx} = Q$ Thus,

$$q = \frac{Q\,S}{I_x\,b} \qquad \dots\ 9.3$$

Since the shear stress is complementary (section 9.3), it follows that the vertical shear stress on the cross-section is also given by equation 9.3. It gives the shear stress at any fiber at a distance y from the centroidal axis. Q is the shearing force on the section, I is the moment of inertia of the whole crosss-sectional area about the axis of bending, S is the statical moment of area of the part contained between the extreme fiber and that at a distance y from the centroidal axis, and b is the width of the beam at the fiber at which the shear stress is being calculated. At a given section, both Q and I are constants, but the shear stresses at various fibers have different values as the values of S and b for such fibers differ.

## 9·5 Shear stress distribution in beams

Equation 9.3 may be used in finding the shear stress distribution across beam sections. As an application, the shear stress distributions for three common types of sections will be considered.

(1) Rectangular section

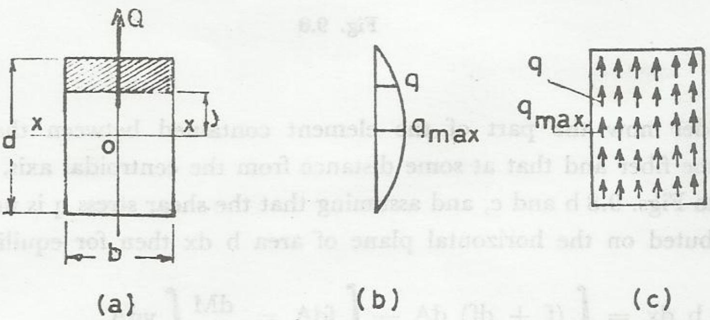



Fig. 9.9

Referring to Fig. 9.9 a and by applying equation 9.3, the shear stress at a fiber distance y from the x-axis is found.

The statical moment of the area contained between the upper extreme fiber and that at a distance y from the x-axis about the x-axis is given by :

$$S = b\left(\frac{d}{2} - y\right)\left(\frac{d}{4} + \frac{y}{2}\right) = \frac{bd^2}{8}\left(1 - \frac{4y^2}{d^2}\right)$$

The moment of inertia of the rectangular area is given by :

$$I_x = \frac{bd^3}{12}$$

Substituting these values in equation 9.3 and simplifying noting that the cross-sectional area A = bd then,

$$q = \frac{3Q}{2A}\left(1 - \frac{4y^2}{d^2}\right) \qquad \ldots 9.4$$

Equation 9.4 shows that in a beam of rectangular cross-section the shear stresses vary parabolically. The maximum value occurs when y = 0, i.e. at the centroidal axis.

$$q_{max} = \frac{3Q}{2A} \qquad \ldots 9.5$$

This equation shows that the maximum shear stress in a rectangular section is one and half times the average shear stress given by equation 9.1; $q_{av} = Q/A$ Since beams of rectangular cross-section are frequently used in practice, and design is usually governed by the maximum stress values, the student is advised to memorize equation 9.5 rather than the more complicated one in 9.4.

At $y = \pm d/2$, i.e. at the extreme upper and lower edges of the section, the shear stress is zero. The values of the shear stress at various levels of the beam cross-section may be represented diagramatically by the parabola shown in Fig. 9.9 b.

In order to satisfy the equilibrium condition, $\Sigma Y = 0$, at a section, the sum of the products of all the vertical shear stresses and their respective elemental areas must be equal to the applied shearing force Q.

$$\int q\,dA = Q$$

Using the expression in equation 9.4 and noting that $dA = b dy$.

$$\int q\,dA = \int_{-d/2}^{+d/2} \frac{3Q}{2A}\left(1 - \frac{4y^2}{d^2}\right) b\,dy = Q$$

This does not only verify equation 9.4, but also shows that the direction of the shear stresses on a section is the same as that of the shearing force. The direction of the shear stresses corresponding to that of the shearing force Q in Fig. 9.9 a is shown on the cross-section in Fig. 9.9 c.

(2) Circular section

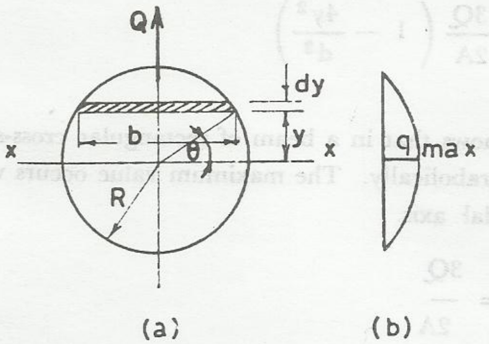



Fig. 9.10

Referring to Fig. 9.10, the statical moment of the area contained between the extreme upper fiber and that at a distance y from the centroidal axis is found by integration as follows :

$$S = \int_{y}^{R} b\,y\,dy$$

$$b = 2R\cos\Theta,$$

$$y = R\sin\Theta,$$

$$dy = R\cos\Theta\,d\Theta$$

$$S = 2R^3 \int_{\Theta}^{\pi/2} \cos^2\Theta \sin\Theta \, d\Theta = 2/3 \; R^3 \cos^3\Theta$$

$$I_x = \pi R^4/4$$

Substituting these values in equation 9.3, and remembering that the cross-sectional area in this case is $A = \pi R^2$ then,

$$q = \frac{4Q}{3A} \cos^2\Theta = \frac{4Q}{3A} (1 - \sin^2\Theta),$$

$$q = \frac{4Q}{3A}\left(1 - \frac{y^2}{R^2}\right) \quad \ldots 9.6$$

Equation 9.6 shows that in a beam of circular cross-section, the shear stresses vary parabolically. The maximum value occurs when $y = 0$, i.e. at the centroidal axis.

$$q_{max} = \frac{4Q}{3A} \quad \ldots 9.7$$

This equation shows that maximum shear stress in a circular section is one third more than the average shear stress; $q_{av} = Q/A$. The shear stress distribution is represented by the parabola shown in Fig. 9.10 b.

(3) I-section

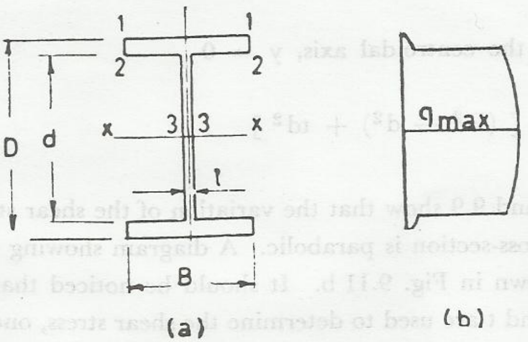



Fig. 9.11

Let the dimensions of the I-section be as shown in Fig. 9.11 a. Consider a fiber in the flange at a distance y from the x-axis.

$$q = \frac{Q}{IB}\left[ B\left(\frac{D}{2} - y\right)\left(y + \frac{D}{4} - \frac{y}{2}\right)\right]$$

$$= \frac{QD^2}{8I}\left(1 - \frac{4y^2}{D^2}\right) \qquad \ldots 9.8$$

For the top fiber, $y = D/2$, $q_1 = 0$

For a fiber at level 2-2, $y = d/2$.

$$q_2 = \frac{Q}{8I}(D^2 - d^2)$$

Consider a fiber in the web at a distance y from the x-axis.

$$q = \frac{Q}{It}\left[ B\left(\frac{D - d}{2}\right)\left(\frac{D + d}{4}\right) + t\left(\frac{d}{2} - y\right)\left(\frac{d}{4} + \frac{y}{2}\right)\right]$$

$$= \frac{Q}{It}\left[\frac{B}{8}(D^2 - d^2) + \frac{t}{2}\left(\frac{d^2}{4} - y^2\right)\right] \qquad \ldots 9.9$$

For a fiber at level 2-2, $y = d/2$,

$$q_2 = \frac{QB}{8It}(D^2 - d^2)$$

For a fiber at the centroidal axis, $y = 0$

$$q_3 = \frac{QB}{8It}\left[(D^2 - d^2) + td^2\right]$$

Equations 9.8 and 9.9 show that the variation of the shear stress along the depth of the cross-section is parabolic. A diagram showing this stress distribution is shown in Fig. 9.11 b. It should be noticed that at level 2-2, two widths B and t are used to determine the shear stress, one in the flange and the other in the web. This explains the abrupt increase in the shear stress at this level. Again it is noticed that the maximum shear stress occurs on the fiber at the centroidal axis.

The total shearing force taken by the web can be found by integrating the shear stress as given by equation 9.9. on the whole area of the web.

$$Q_w = \int_{-d/2}^{+d/2} q\, t\, dy$$

$$= \frac{Q}{I}\left[\frac{Bd}{8}(D^2 - d^2) + \frac{t\, d^3}{12}\right] \qquad \text{... 9.10}$$

For an I-section,

$$I = 2B\left(\frac{D}{2} - \frac{d}{2}\right)\left(\frac{D}{4} + \frac{d}{4}\right)^2 + \frac{t\, d^3}{12}$$

$$= \frac{B}{8}(D^2 - d^2)\left(\frac{D}{2} + \frac{d}{2}\right) + \frac{t\, d^3}{12}$$

For a deep I-section girder, $\left(\frac{D}{2} + \frac{d}{2}\right) \doteqdot d$

$$I = \frac{Bd}{8}(D^2 - d^2) + \frac{t\, d^3}{12}$$

Substituting this value in equation 9.10,

$$Q_w \doteqdot Q \qquad \text{... 9.11}$$

Equation 9.11 shows that for deep I-section girders, the web carries nearly all the shearing force. This is an assumption which is usually made in the design of deep plate girders.

In the last three examples, the maximum shear stress was found to be at the centroidal axis. This is generally the case if the sides of the cross-sectional area are parallel. Then, S becomes the only variable in the shear stress formula and it is a maximum at the centroidal axis. If the sides of the cross-sectional area are not parallel, q becomes a function of both S and t and the maximum shear stress may not occur at the centroidal axis. For example, the maximum shear stress occurs at the middle of the height of a symmetrical triangular cross-section and not at one third from the base as one might be led to think.

**Example 9.4** The beam shown in Fig. 9.12 has a channel cross-section of the given dimensions. Calculate the maximum shear stress and plot the shear stress distribution at the section of maximum shearing force.

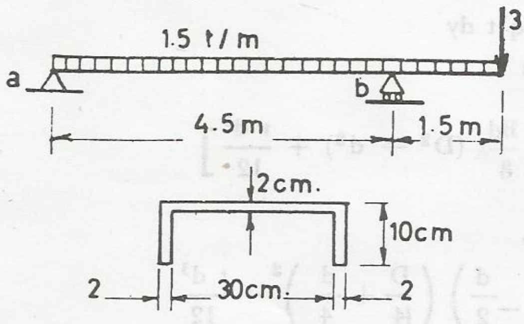



Fig. 9.12

Solution : The maximum shearing force occurs at a section just to the right of support b. Its value is given by :

$$Q_{max} = 1.5 \times 1.5 + 3 = 5.25 \text{ t.}$$

Assuming that the centroid lies on an axis at a distance $\bar{y}$ from the upper edge,

$$\bar{y} = \frac{2 \times 10 \times 2 \times 5 + 30 \times 2 \times 1}{2 \times 10 \times 2 + 30 \times 2} = \frac{260}{100} = 2.6 \text{ cm.}$$

$$I_x = 2\left(\frac{2 \times 10^3}{12} + 20 \times 2.4^2\right) + \frac{30 \times 2^3}{12} + 60 \times 1.6^2$$

$$= 737.34 \text{ cm}^4$$

The maximum shear stress occurs at the centroidal x-axis.

$$q_{max} = \frac{5.25 \; (34 \times 2 \times 1.6 + 2 \times 2 \times 0.6 \times 0.3)}{737.34 \times 4} = 0.195 \text{ t./cm}^2.$$

The shear stresses at the various levels indicated in Fig. 9.13 a are calculated below.

$$q_1 = 0$$

$$q_2 = \frac{5.25 \; (34 \times 2 \times 1.6)}{737.34 \times 34} = 0.023 \text{ t./cm}^2.$$

$$q_3 = \frac{5.25 \; (34 \times 2 \times 1.6)}{737.34 \times 4} = 0.194 \text{ t./cm}^2.$$

$$q_5 = 0$$

The shear stress distribution is shown in Fig. 9.13 b.

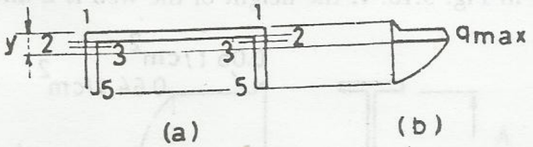



Fig. 9.13

**Example 9.5** Calculate the maximum shearing force which the I-section shown in Fig. 9.14 can carry if the permissible shear stress is 0.9 t./cm$^2$. Compute the percentage shearing force carried by the web.

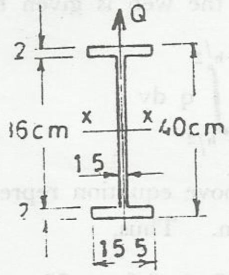



Fig. 9.14

Solution : The maximum shear stress occurs at the centroidal axis. The moment of inertia of the whole section about this axis, and the statical moment of the area between the upper edge and the x-axis about the x-axis must be found first in order to apply equation 9.3.

$$I_x = \frac{15.5 \times 40^3}{12} - 2\left(\frac{7 \times 36^3}{12}\right) = 28394 \text{ cm}^4$$

$$S = 15.5 \times 2 \times 19 + 1.5 \times 18 \times 9 = 832 \text{ cm}^3.$$

$$q' = 0.9 = \frac{Q_{max} \times 832}{28394 \times 1.5}$$

$$Q_{max} = \frac{0.9 \times 28394 \times 1.5}{832} = 46.1 \text{ t.}$$

In order to calculate the percentage shearing force carried by the web, the shear stress distribution on the section has to be found first. This has been done, in the usual way, by the application of equation 9.3 and the result is shown in Fig. 9.15. If the height of the web is h and its thickness

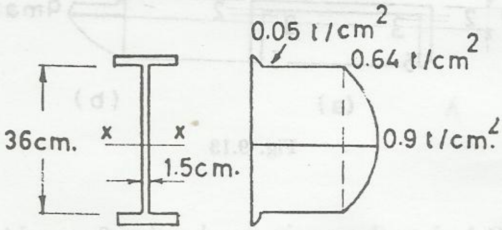



Fig. 9.15

is t and the shear stress at a fiber distance y from the x-axis is q, then the shearing force carried by the web is given by :

$$Q_w = \int_{-h/2}^{+h/2} q \; t \; dy = t \int_{-h/2}^{+h/2} q \; dy$$

but the integral in the above equation represents the area of the shear stress distribution diagram. Thus,

$$Q_w = 1.5 \; (0.64 \times 36 + 2/3 \times 36 \times 0.26) = 43.9 \text{ t.}$$

percentage shearing force carried by the web

$$= \frac{43.9}{46.1} \times 100 = 95.4\%$$

### 9.6 Shear flow

The shearing force per unit length of a longitudinal section of a beam is called the shear flow. Since the force is measured in kg. or ton, the shear flow denoted here by $\tau$ has the units of kg./cm. or t./cm. The formula for the shear flow in beams may be easily obtained by modifying the shear stress formula. Since, as mentioned above, the shear flow on a fiber is the shearing force per unit length of the beam, it needs only to multiply the shear stress q by the width of the beam b at the fiber considered. Thus,

$$\tau = q \; b \qquad \text{... 9.12}$$

or $$\tau = \frac{Q \; S}{I} \qquad \text{... 9.13}$$

where Q, S and I have the same meanings as those in the shear stress formula.

Equation 9.13 is very useful in determining the necessary connections between the various component parts of a compound beam. This will be illustrated by the following examples.

**Example 9.6** Two wooden planks form a T-section of a beam as shown in Fig. 9.16. If the beam is subjected to a constant shear of 350 kg, find the necessary spacing of the nails between the two planks to make the beam act as one unit, assuming that the allowable shearing force per nail is 60 kg.

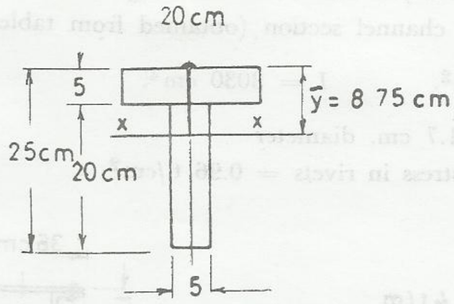



Fig. 9.16

Solution : The shear flow in the plane of contact of the two planks may be determined from the application of equation 9:13. For this reason, the centroidal x-axis, the moment of inertia about this axis and the statical moment of the area of the upper plank about the x-axis must be found first.

Assuming that the x-axis to be at a distance $\bar{y}$ from the upper edge of the section, then,

$$\bar{y} = \frac{20 \times 5 \times 2.5 + 20 \times 5 \times 15}{20 \times 5 \times 2} = \frac{1750}{200} = 8.75 \text{ cm.}$$

$$I_x = \frac{20 \times 5^3}{12} + 20 \times 5 \times 6.25^2 + \frac{5 \times 20^3}{12} + 5 \times 20 \times 6.25^2 = 11058 \text{ cm}^4$$

$$S = 20 \times 5 \times 6.25 = 625 \text{ cm}^3.$$

$$\tau = \frac{Q\,S}{I} = \frac{350 \times 625}{11058} = 20 \text{ kg./cm.}$$

This means that a force of 20 kg. must be transferred from one plank to the other in every centimeter of the beam's length. Thus, if s is the spacing of the nails then,

$$60 = 20\ s$$

$$\text{or} \quad s = \frac{60}{20} = 3 \text{ cm.}$$

**Example 9.7** A simply supported beam of span 6 m. carries a uniformly distributed load of 4 t./m. The cross-section of the beam is built up from two channels and a cover plate as shown in Fig. 9.17. Specify the spacing for the rivets necessary to fasten this beam together for the following data. Properties of the channel section (obtained from tables) are :

$$A = 58.8 \text{ cm}^2. \qquad I = 8030 \text{ cm}^4.$$

Rivets used are 1.7 cm. diameter
Allowable shear stress in rivets = 0.96 t./cm$^2$.

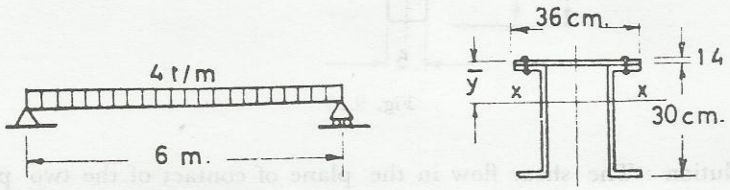



Fig. 9.17

Solution : This problem may be solved by applying equation 9.13 to determine the shear flow in the plane between the cover plate and the channels. For this reason, the centroidal x-axis and the moment of inertia of the whole section about this axis must be found. Also, the statical moment of the cover plate about the x-axis should be calculated.

Let the x-axis be at a distance $\bar{y}$ from the top edge of the section.

$$\bar{y} = \frac{1.4 \times 36 \times 0.7 + 2 \times 58.8 (1.4 + 15)}{1.4 \times 36 + 2 \times 58.8} = \frac{1965}{168} = 11.7 \text{ cm.}$$

$$I_x = \frac{36 \times 1.4^3}{12} + 36 \times 1.4 \times 11^2 + 2(8030 + 58.8 \times 4.7^2) = 24760 \text{ cm}^4.$$

$$S = 1.4 \times 36 \times 11 = 555 \text{ cm}^3.$$

The largest shear flow occurs near the supports where the maximum shearing force $Q_{max}$ acts.

$$Q_{max} = \frac{4 \times 6}{2} = 12 \text{ t.}$$

$$\tau_{max} = \frac{Q\,S}{I} = \frac{12 \times 555}{24760} = 0.27 \text{ t./cm.}$$

This shearing force per unit length of the beam is resisted by two rows of rivets, and for the data given, each rivet can resist a force of $\frac{\pi \times 1.7^2}{4} \times 0.96 = 2.17$ t. Hence, if s is the spacing of rivets, or the pitch as it is sometimes called, then,

$$\frac{0.27\ s}{2} = 2.17$$

$$s = \frac{2.17 \times 2}{0.27} = 16 \text{ cm.}$$

This pitch of rivets applies at a section where the shearing force Q = 12 t. Similar calculations for sections subjected to smaller shearing forces may be carried out leading to longer spacing and hence smaller number of rivets required for the beam. However, most specifications specify a maximum pitch for reasons to be found in the design books. According to the E.S.S. (Egyptian Standard Specifications); and for the case considered, $s_{max} = 12$ d, where d is the diameter of the rivet used, i.e. $12 \times 1.7 = 20.4$ cm. This pitch may be adopted for the part of the beam where the shearing force does not exceed a certain value found from the relationship :

$$\frac{Q \times 555 \times 20.4}{24760 \times 2} = 2.17$$

$$Q = \frac{2.17 \times 24760 \times 2}{555 \times 20.4} = 9.5 \text{ t.}$$

Referring to Fig. 9.18, which shows the shearing force diagram, this shearing force occurs at a distance x = 0.625 m. from either supports.

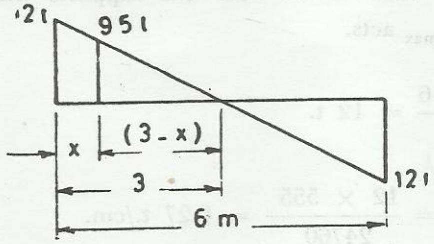



Fig. 9.18

Thus, for economy, it may be specified to use 1.7 cm. rivets at 16 cm spacing for a distance 64 cm. adjacent to the supports and 20.4 cm. for the remaining length of the beam.

## 9.7 Shear centre

The shear centre of a cross-section is the point in the plane of the section through which the shearing force must act if torsion is not to occur. Obviously, this is the point through which the resultant of the shear stresses on the section acts. In sections 9.5 and 9.6, the shear stress and shear flow in a beam having an axis of symmetry which lies in the plane of loading have been discussed. Under these conditions, there can be no torsion of the beam as the resultant of the shear stresses on the section coincides with the applied shearing force, and the shear centre lies on the axis of symmetry. In general, the resultant of the shear stresses on a section does not lie in the plane of loading. This causes torsion of the beams; i.e. the cross-sections of the beam rotate about its longitudinal axis. To avoid this twisting action, it is necessary that the shearing force be applied through the shear centre. In the following discussion, the location of the shear centre will be limited to thin-walled sections having one axis of symmetry. This is because solid sections twist too little, in virtue of their high torsional rigidity, to justify the complicated procedure necessary to determine their shear centre. The usual method in this case is to find the resultant of the shear stresses acting on the section and then find the location of the external shearing force to keep this resultant in equilibrium.

Consider for example the channel section shown in Fig. 9.19 a. Let the external shearing force Q be applied at point c on the x-axis such that no twisting of the section occurs, and let c be at a distance e from the centre of the web.

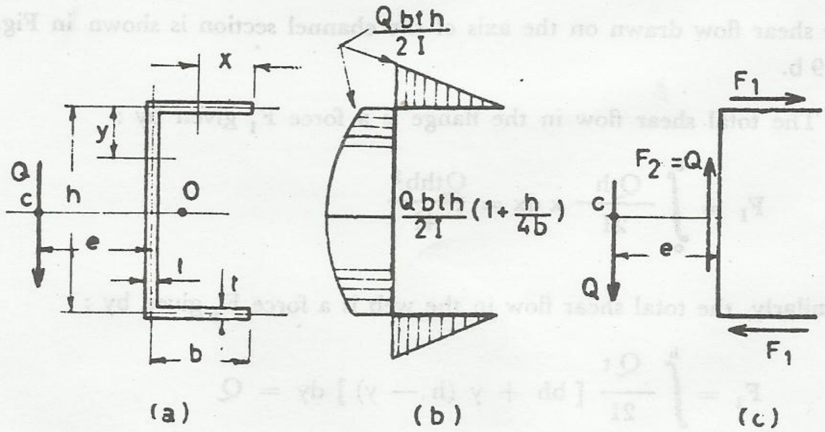



Fig. 9.19

Applying equation 9.13, at a distance x from the free edge of the flange, the shear flow is given by :

$$\tau = \frac{Q\,x\,t\,h}{2I}$$

Thus, the shear flow varies linearly along the flange from zero at the free edge to a value at the junction between the web and the flange given by :

$$\tau = \frac{Q\,b\,t\,h}{2I}$$

Also, at a distance y along the web from the junction between the web and the flange,

$$\tau = \frac{Q}{I}\left[\frac{b\,t\,h}{2} + yt\left(\frac{h}{2} - \frac{y}{2}\right)\right]$$

$$= \frac{Qt}{2I}\left[bh + y\,(h - y)\right]$$

Thus, the shear flow along the web varies parabolically from a value of Qbth/2I at the junction between the web and the flange to a value of

$\frac{Qbth}{2I}\left(1+\frac{h}{4b}\right)$ at the mid-height of the web. The variation of the shear flow drawn on the axis of the channel section is shown in Fig. 9.19 b.

The total shear flow in the flange is a force $F_1$ given by :

$$F_1 = \int_o^b \frac{Qth}{2I} x\,dx = \frac{Qthb^2}{4I}$$

Similarly, the total shear flow in the web is a force $F_2$ given by :

$$F_2 = \int_o^h \frac{Qt}{2I}\left[bh + y(h - y)\right] dy = Q$$

These shearing forces, which are shown in Fig. 9.19 c, indicate that the resultant of the shear stresses on the section is equivalent to a vertical force, $F_2 = Q$ and a couple, $F_1h = Qth^2b^2/4I$

An expression for the distance e locating the plane in which the external shearing force Q must be applied to produce no torsion may now be obtained.

$$Qe = F_1h$$

or $$e = \frac{th^2b^2}{4I} \qquad \dots 9.14$$

Equation 9.14 shows that the location of the shear centre is independent of the applied shearing force, and that it is a function of the dimensions of the section only. Therefore, e is a property of the cross-sectional area and may be calculated for any shape of a thin-walled section from a knowledge of its dimensions.

An analysis similar to that of the channel section leads to the conclusion that the shear centre for an angle, symmetrical or unsymmetrical, is at the point of intersection of the centre lines of the two legs as shown in Figs. 9.20 a and b. For a T-section, the shear centre lies at the point of intersection of centre lines of the web and the flange as shown in Fig. 9.20 c. The location of the shear centre has wide applications in

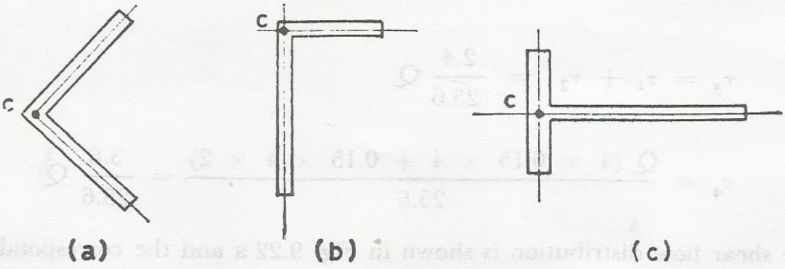



Fig. 9.20

aircraft structures and in the design of modern thin-walled cold drawn sections.

**Example 9.8** Locate the shear centre for a beam having the cross-sectional dimensions shown in Fig. 9.21.

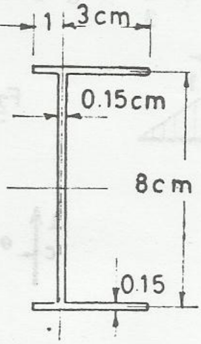



Fig. 9.21

Solution : The flanges, being thin, the moment of inertia about their centoridal axis may be neglected. Thus,

$$I = \frac{0.15 \times 8^3}{12} + 2 \times 4 \times 0.15 \times 4^2 = 25.6 \text{ cm}^4.$$

The values of the shear flow at the location shown in Fig. 9.22 a are calculated below :

$$\tau_1 = \frac{Q\,(3 \times 0.15 \times 4)}{25.6} = \frac{1.8}{25.6}\,Q$$

$$\tau_2 = \frac{Q\,(1 \times 0.15 \times 4)}{25.6} = \frac{0.6}{25.6}\,Q$$

$$\tau_3 = \tau_1 + \tau_2 = \frac{2.4}{25.6} Q$$

$$\tau_4 = \frac{Q\,(4 \times 0.15 \times 4 + 0.15 \times 4 \times 2)}{25.6} = \frac{3.6}{25.6} Q$$

The shear flow distribution is shown in Fig. 9.22 a and the corresponding shearing forces are shown in Fig. 9.22 b where,

$$F_1 = \frac{1.8Q}{25.6} \times \frac{3}{2} = \frac{2.7}{25.6} Q$$

$$F_2 = \frac{0.6Q}{25.6} \times \frac{1}{2} = \frac{0.3}{25.6} Q$$

$$F_3 = \frac{2.4Q}{25.6} \times 8 + \frac{2}{3} \times 8 \times \frac{1.2Q}{25.6} = \frac{25.6Q}{25.6} = Q$$

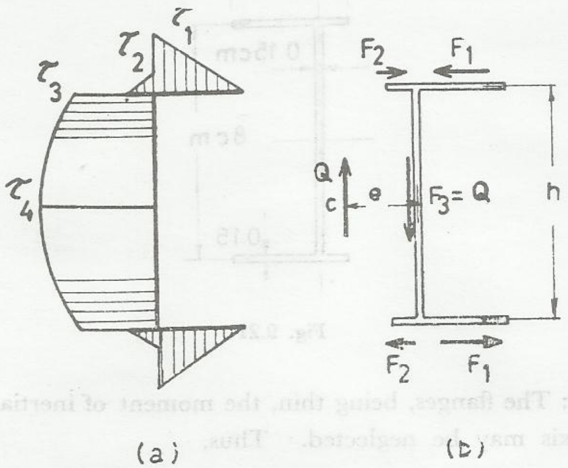



Fig. 9.22

Thus, referring to Fig. 9.22 b,

$$Q\,e = (F_1 - F_2)\,h$$

$$e = \frac{2.4Q \times 8}{25.6Q} = 0.75 \text{ cm.}$$

**Example 9·9** Determine the location of the shear centre for the cross-section shown in Fig. 9.23. Assume that the cross-sectional area of the plate is neglegible compared with the areas A of the flanges.

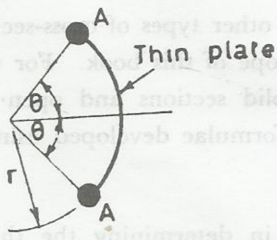



Fig. 9.23

Solution : The shear flow at any point along the plate will be in the direction of the tangent at this point, and its magnitude is given by equation 9.13.

Neglecting the area of the plate as assumed,

$$I = 2A\ (r \sin\Theta)^2$$

$$S = A\ r \sin\Theta$$

Thus, the shear flow along the plate has a constant value of :

$$\tau = \frac{Q \times A\ r \sin\Theta}{2A\ (r \sin\Theta)^2} = \frac{Q}{2r \sin\Theta}$$

The moment of the resulting shearing forces about the centre is equal to :

shear flow $\times$ length of plate $\times$ radius of arc

$$= \frac{Q}{2r \sin\Theta} \times 2r\Theta \times r = \frac{Q\Theta r}{\sin\Theta}$$

Assuming that the shear centre is at a distance e from the centre,

$$Qe = \frac{Q\Theta r}{\sin\Theta}$$

$$e = \frac{\Theta r}{\sin\Theta}$$

## 9.8 Torsion

Twisting moment or torque, which is usually denoted by T or $M_t$ is a moment causing the sections of the member upon which it acts to rotate about its longitudinal axis. In the analysis that follows, shear stresses due to torque will be determined. Only, members of solid and hollow circular sections and tubular cross-sections of any shape will be considered in details as the analysis of other types of cross-sections is generally complicated and beyond the scope of this book. For the sake of completeness, however, non-circular solid sections and open thin-walled sections are dealt with in brief and formulae developed using advanced methods are given.

The basic approach in determining the thrust, shearing force and bending moment at a section of a member is also followed in finding the torque. A cross-section along the member is taken, and the equilibrium of a part of the member on either side of the section is considered. For finding the torque at that section only one equation; $\Sigma M_z = 0$, is applied, z being the longitudinal axis of the member.

For example, consider a shaft which is acted upon by three torques $T_1$, $T_2$ and $T_3$ as shown in Fig. 9.24 a, and assume that the torque at section a-a is required.

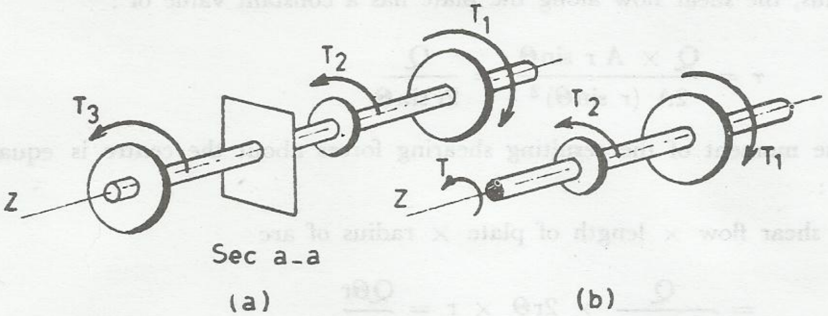



Fig. 9.24

First of all the three torques must be in equilibrium; i.e., $T_1 = T_2 + T_3$. Next, by passing a cross-section a-a at the location required, and considering the equilibrium of a part on either side of that section, the required torque is found. Thus, with reference to Fig. 9.24 b,

$$\Sigma M_z = 0 = T_1 - T_2 - T$$

$$T = T_1 - T_2$$

**Example 9.10** A cantilever consists of three members AB, BC and CD rigidly connected to each other. BC and CD lie in a horizontal plane and at a right angle to each other, and member AB is in the vertical plane. Calculate the torques at sections a and b located as shown in Fig. 9.25.

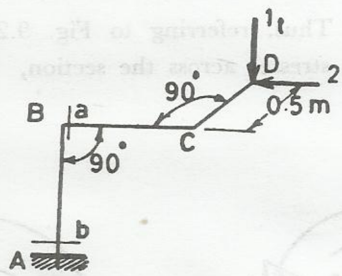



Fig. 9.25

Solution :

Section a in member BC.

Taking the z-axis along member BC,

$$\Sigma M_z = 0 = 1 \times 0.5 - T_a$$

$$T_a = 0.5 \text{ m.t. (anticlockwise)}$$

Section b in member AB.

Taking the z-axis along member AB,

$$\Sigma M_z = 0 = 2 \times 0.5 - T_b$$

$$T_b = 1 \text{ m.t. (clockwise)}$$

## 9.9 The torsion formula

In this theory for the determination of shear stresses due to torque developed in solid, hollow and tubular circular cross-sections, it will be assumed that :

( 1 ) The member is subjected to torsion only, i.e. no shearing force acts.

( 2 ) Plane cross-sections remain plane after torques are applied.

( 3 ) Radii of the sections remain straight after torques are applied. This means that the shear strain varies linearly from the centre to the outer surface of the shaft.

( 4 ) Hook's law applies, i.e. shear stresses are proportional to shear strains.

On the basis of the above assumptions the torsion formula will be developed.

From assumptions (3) and (4), the shear stress distribution across the section varies linearly from zero at the centre to a maximum at the extreme outer fiber. Thus, referring to Fig. 9.26 a which shows the variation of the shear stresses across the section,

$$\frac{q_r}{r} = \frac{q_{max}}{R} \qquad \text{... 9.15}$$

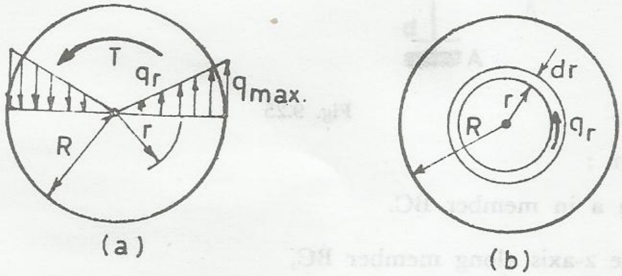



Fig. 9.26

From assumption (1), since the section is subjected to torque only, the resulting shear stresses may now be determined by equating the applied torque to the internal twisting moment of resistance. Referring to Fig. 9.26 b,

$$T = \int_0^R 2\pi\, r\, dr \times q_r \times r$$

Substituting from equation 9.15 for $q_r$,

$$T = \frac{q_{max}}{R} \int_0^R 2\pi\, r^3\, dr$$

$$\int_0^R 2\pi\, r^3\, dr = \frac{\pi R^4}{2} \qquad \text{... 9.16}$$

This is the polar moment of inertia usually denoted by J. Thus,

$$T = \frac{q_{max}\, J}{R}$$

or $$q_{max} = \frac{T\,R}{J} \qquad \text{... 9.17}$$

Consider next the strains caused by these shear stresses. A solid circular shaft of radius R, length L and subjected to end torques T is shown in Fig. 9.27.

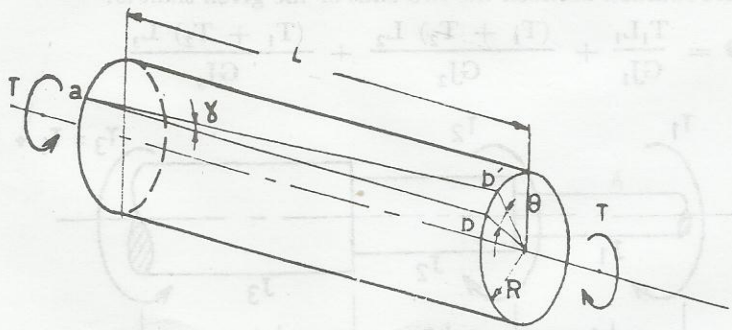



Fig. 9.27

Due to torsion, a line such as ab on the surface of the shaft will assume the new position ab'. From assumption (4) that shear stresses are proportional to shear strains, the shear strain $\gamma$ is given by :

$$\gamma = \frac{q}{G} \qquad \ldots 9.18$$

where G is the shear modulus.

But if $\Theta$ is the angle of twist of the near end of the shaft with respect to its far end, then from geometry,

$$bb' = \gamma L = R\Theta$$

$$\text{or} \quad \Theta = \frac{\gamma L}{R} \qquad \ldots 9.19$$

Substituting, from equations 9.17 and 9.18 into equation 9.19, the rate of change of twist, or the angle of twist per unit length of the shaft is given by :

$$\frac{\Theta}{L} = \frac{q_{max}}{GR} = \frac{T}{GJ} \qquad \ldots 9.20$$

Thus, in order to find the relative rotation between two cross-sections along a shaft subjected to uniform torque and at a distance L apart, it is only necessary to multiply either quantity in equation 9.20 by L. A study of equation 9.20 shows that this is only true in a uniform length of the shaft i.e. of constant J, subjected to a constant torque. In general, the applied torque as well as the diameter of the shaft may vary along its length. In such a case, the relative rotations along the various lengths of the shaft must be summed. Thus, referring to Fig. 9.28, the relative rotation between the two ends of the given shaft is:

$$\Theta = \frac{T_1L_1}{GJ_1} + \frac{(T_1 + T_2)\ L_2}{GJ_2} + \frac{(T_1 + T_2)\ L_3}{GJ_3}$$

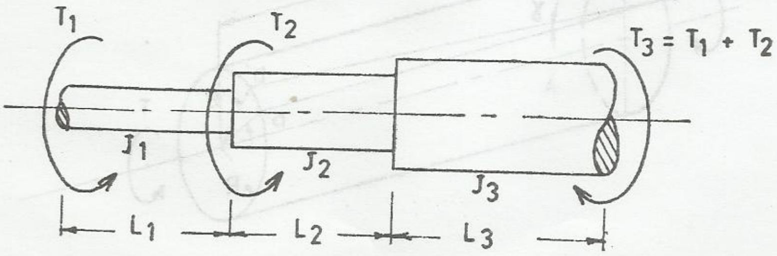



Fig. 9.28

Combining equations 9.15 and 9.20,

$$\frac{\Theta G}{L} = \frac{T}{J} = \frac{q_r}{r} = \frac{q_{max}}{R} \qquad \ldots\ 9.21$$

This is the general torque expression that relates the angle of twist $\Theta$ and the shear stress q with the applied torque T, in terms of the dimensions of the shaft and the property of its material.

Further, it should be noted that the direction of the shear stress due to torque at a point along any radius of the cross-section, is normal to the radius and in the plane of the section — Fig. 9.26 a.

The general torque expression in equation 9.21, equally applies to shafts with hollow and tubular circular cross-sections since the assumptions used in its derivation hold also to these two cases. However, it is necessary to modify J accordingly.

For a hollow circular section, the limits of integration in equation 9.16 extend, instead from zero to R, from $R_2$ to $R_1$ where $R_2$ and $R_1$ are the internal and external radii respectively. Hence for a hollow circular section.

$$J = \frac{\pi}{2}\ (R_1{}^4 - R_2{}^4) \qquad \ldots\ 9.22$$

Equation 9.22 may be re-written as :

$$J = \frac{\pi}{2}\ (R_1 - R_2)\ (R_1 + R_2)\ (R_1{}^2 + R_2{}^2)$$

For a thin circular tube, $R_1 - R_2 = t$. and $R_1 \doteqdot R_2 = R$

Hence for a circular tubular section,

$$J \doteqdot 2\pi R^3\, t \qquad \ldots\ 9.23$$

where R is the mean radius of the tube and t its thickness.

**Example 9.11** A solid circular shaft 20 cm. diameter is to be replaced by a hollow one; the ratio between its external and internal diameters being 2 : 1. Find the size of this hollow shaft if the maximum shear stress is to be the same for both shafts. What percentage economy in weight will this change cause ?

Solution :

Let r = the internal radius of the hollow shaft.

Then, 2r = the external radius of the hollow shaft.

$J$ (for the hollow shaft) $= \pi\ (16r^4 - r^4)/2 = 7.5\ \pi r^4\ cm^4.$

$J$ (for the solid shaft) $= \pi \times 10^4/2 = 5000\pi\ cm^4.$

If T is the applied torque then,

$$q_{max} \text{ (for the hollow shaft) } = \frac{T \times 2r}{7.5\ \pi r^4}$$

$$q_{max} \text{ (for the solid shaft) } = \frac{T \times 10}{5000\pi}$$

Equating the maximum stresses in the two shafts,

$r = 5.1$ cm.

i.e. the hollow shaft must have 10.2 cm. internal diameter and 20.4 cm. external diameter.

$$\text{Percentage saving in weight} = \frac{100\pi - \pi\ (10.2^2 - 5.1^2)}{100\pi} \times 100 = 22\%$$

**Example 9.12** Find the maximum shear stress in a hollow shaft 40 cm. and 20 cm. external and internal diameters when subjected to a torque of 45 m.t. If G = 700 t./cm²., what is the angle of twist in a length of 10 m. ? What diameter would be required for a solid shaft to have a similar angle of twist under the same torque and in the same length ?

Solution :

$$J = \pi\ (20^4 - 10^4)/2 = 75000\pi\ cm^4.$$

$$q_{max} = \frac{TR}{J} = \frac{4500 \times 20}{75000\pi} = 0.38\ t./cm^2.$$

$$\Theta = \frac{TL}{GJ} = \frac{4500 \times 1000}{700 \times 75000\pi} = 0.028 \text{ radians}$$

$$= 0.028 \times 180/\pi = 1.56^0 = 1^0\ 30'\ 36''$$

Let the diameter of the solid shaft be D cm.,

$$J = \pi D^4/32 \text{ cm}^4.$$

$$\Theta = \frac{TL}{G \times 75000\pi} = \frac{32TL}{G \times \pi D^4}$$

$$D = 39.4 \text{ cm.}$$

**Example 9.13** Experiments in the laboratory were carried out to determine the values of the shear modules G for a number of materials. During one of these experiments on a tube 0.3 cm. thick and 5 cm. mean radius, the relative angle of twist in a length of 100 cm. was measured to be 0.008 radians at a torque of 15 cm.t. Calculate the value of G.

Solution : From equation 9.21,

$$G = \frac{T\,L}{\Theta\,J}$$

$$J \text{ (for tube)} = 2\pi R^3 t = 2\pi \times 125 \times 0.3 = 75\pi \text{ cm}^4.$$

$$G = \frac{15 \times 100}{0.008 \times 75\pi} = 795 \text{ t./cm}^2.$$

**Example 9.14** Determine the maximum shear stress in the compound shaft shown in Fig. 9.29 if part ab has a solid circular section 10 cm. diameter and part bc a hollow circular section 20 and 10 cm. external and internal diameters. If G = 800 t./cm²., calculate the rotation of the free end of the shaft.

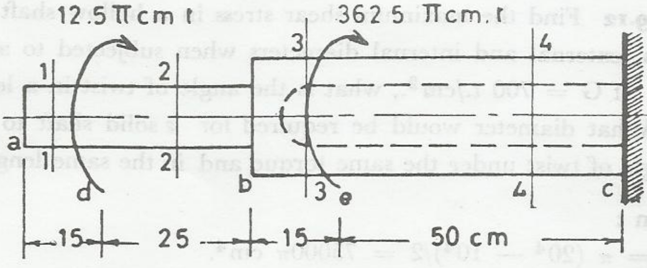



Fig. 9.29

Solution :

$$J_{ab} = \frac{\pi \times 10^4}{32} = 312.5\pi \text{ cm}^4.$$

$$J_{bc} = \pi\,(20^4 - 10^4)/32 = 4700\pi \text{ cm}^4.$$

By taking sections 1-1, 2-2, 3-3 and 4-4, and considering each time the part of the shaft to the left of the section, the torques at various parts are found to be :

$T_{ad} = 0$ , $T_{db} = 12.5\pi$ cm.t.

$T_{be} = 12.5\pi$ cm.t and $T_{ec} = 12.5\pi + 362.5\pi = 375.5\pi$ cm.t.

By inspection, the maximum shear stress may occur either between db or ec.

$$q_{max} \text{ (in part db)} = \frac{12.5\pi \times 5}{312.5\pi} = 0.2 \text{ t./cm}^2.$$

$$q_{max} \text{ (in part ec)} = \frac{375.5\pi \times 10}{4700\pi} = 0.8 \text{ t./cm}^2.$$

The maximum shear stress occurs in the part of the shaft between e and c.

To find the rotation at the free end, lengths of the shaft having constant T and J are considered separately and the final result is obtained by summing the various relative rotations. Thus,

$$\Theta = \Theta_{ad} + \Theta_{db} + \Theta_{be} + \Theta_{ec}$$

$$= \frac{1}{800}\left(0 + \frac{12.5\pi \times 25}{312.5\pi} + \frac{12.5\pi \times 15}{4700\pi} + \frac{375.5\pi \times 50}{4700\pi}\right)$$

$$= \frac{1}{800}\left(0 + 1 + 0.08 + 4\right) = \frac{5.08}{800} \times \frac{180}{\pi} = 0^0 21' \, 42''.$$

## 9.10 Power transmitting shafts

Rotating shafts transmitting power are widely used. It will be advantageous, therefore, to establish a relation between the transmitted horsepower HP and the torque T acting on the shaft.

By definition, one HP does work of 75 m.kg. per second, or $75 \times 100 \times 60 = 450000$ cm.kg. per minute. Also, from the principles of Dynamics, work is equal to torque multiplied by the angle, measured in radians, through which the torque rotates per unit of time. For a shaft rotating at n r.p.m., the angle rotated is $2\pi n$ radians per minute. Thus if a shaft transmits a torque T, measured in cm.kg., it will do work of $2\pi n$ T cm.kg. per minute. Equating this to the HP supplied,

$$450000 \text{ HP} = 2\pi n T$$

$$\text{or} \quad T = \frac{450000 \text{ HP}}{2\pi n} \qquad \text{... 9.24 a}$$

where T is the torque in cm.kg and n the number of rotations of the shaft per minute.

If T is to be expressed in m.t., equation 9.24 a is re-written as :

$$T = \frac{4.5 \text{ HP}}{2\pi n} \qquad \text{... 9.24 b.}$$

**Example 9.15** Find the diameter of a solid circular shaft for a 200 HP motor operating at 150 r.p.m. if the shear stress is limited to 1000 kg./cm². What will be the diameter of the shaft if the motor operates at 1500 r.p.m. ?

Solution:

Let $T_1$ and $T_2$ be the torques transmitted by the low-speed and high-speed shafts respectively, and $D_1$ and $D_2$ their corresponding diameters. From equation 9.24 a,

$$T_1 = \frac{450000 \times 200}{2\pi \times 150} = 95500 \text{ cm.kg.}$$

$$T_2 = \frac{450000 \times 200}{2\pi \times 1500} = 9550 \text{ cm.kg.}$$

$$q_{max} = 1000 = \frac{95500 \times 16}{\pi D_1^3}$$

$$D_1^3 = \frac{95.5 \times 16}{\pi} \quad \text{or } D_1 = 7.86 \text{ cm.}$$

$$\text{Similarly,} \quad q_{max} = 1000 = \frac{9550 \times 16}{\pi D_2^3}$$

$$D_2^3 = \frac{9.55 \times 16}{\pi} \quad \text{or } D_2 = 3.64 \text{ cm.}$$

This demonstrates the advantage of using high-speed machines. Appreciable economy in the size of a shaft results from increasing its rotating speed. As may be seen from Example 9.11, further economy of less significance may be achieved by making use of hollow sections.

## 9.11 Torsion of non-circular solid sections

In contrast with circular sections, the analysis of non-circular solid sections subject to torsion is mathematically complicated. This is because the assumptions made in the derivation of the torsion formula for circular sections are no longer valid. Plane sections before torque do not remain plane after torque and stresses at various points of a section are not proportional to their distance from the centre of rotation.

The following two remarks are helpful in visualizing the nature of stress distribution on solid cross-sections of non-circular shape.

(1) The shear stresses at points on the perimeter of a section act in a tangential direction to it.

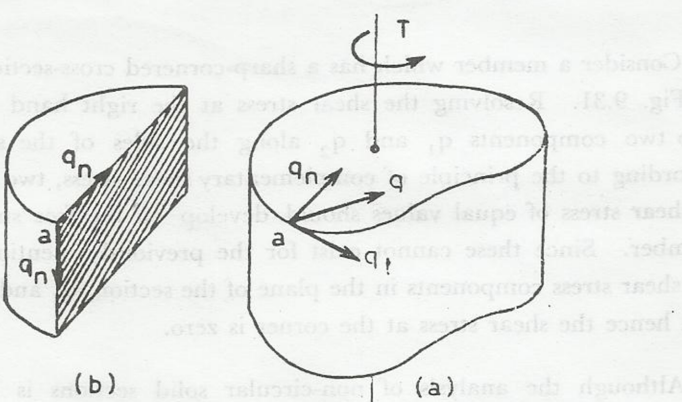



**Fig. 9.30**

This may be proved with reference to Fig. 9.30. Suppose the shear stress q at point a on the perimeter of the given section act at an angle to the tangent then this stress could be resolved into two components in the plane of the section; a tangential component $q_t$ and another normal to it $q_n$. If, however, $q_n$ were to exist, another complementary shear stress of equal value and in a direction perpendicular to it must act on the outer surface of the member as shown in Fig. 9.30 b. Since the outer surface is free of load, the shear stress is zero and hence $q_n$ cannot exist. Thus the shear stress at a point on the perimeter of a section acts in a direction tangential to the perimeter of the section.

(2) The shear stress at corners of non-circular sections must be zero.

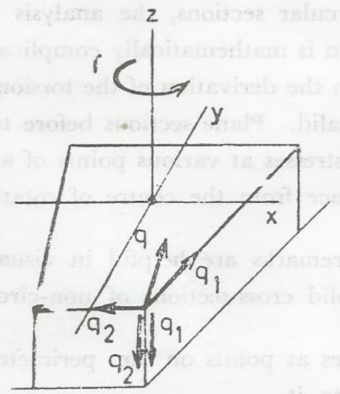



Fig. 9.31

Consider a member which has a sharp-cornered cross-section as shown in Fig. 9.31. Resolving the shear stress at the right hand near corner into two components $q_1$ and $q_2$ along the sides of the section then according to the principle of complementary shear stress, two components of shear stress of equal values should develop on the free surface of the member. Since these cannot exist for the previously mentioned reason, the shear stress components in the plane of the section, $q_1$ and $q_2$ are zero and hence the shear stress at the corner is zero.

Although the analysis of non-circular solid sections is beyond the scope of this book, yet for the sake of completeness formulae developed for common types of sections are given.

(1) **Rectangular section**

$$q_{max} = \frac{T}{\alpha d b^2} \quad \dots 9.25$$

$$\Theta = \frac{TL}{\beta d b^3 G} \quad \dots 9.26$$

where d and b are the long and short sides of the section respectively, and $\alpha$ and $\beta$ are factors depending on the d/b ratio.

Values of $\alpha$ and $\beta$ for the full range of d/b ratios are reproduced from Timoshenko's Strength of Materials, and are given in the following table:

| d/b | 1 | 1.5 | 1.75 | 2 | 2.5 | 3 | 4 | 6 | 8 | 10 | ∞ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $\alpha$ | 0.208 | 0.231 | 0.239 | 0.246 | 0.258 | 0.267 | 0.282 | 0.299 | 0.307 | 0.313 | 0.33 |
| $\beta$ | 0.141 | 0.196 | 0.214 | 0.229 | 0.249 | 0.263 | 0.281 | 0.299 | 0.307 | 0.313 | 0.33 |

Two special cases of the rectangular section deserve particular attention. These are:

(a) **Square section**; d = b and d/b = 1

$$q_{max} = \frac{T}{0.208d^3} \qquad \dots 9.27$$

$$\Theta = \frac{TL}{0.141d^4G} \qquad \dots 9.28$$

(b) **Flat strip**; the thickness b is very small with respect to d and b/d = ∞

$$q_{max} = \frac{T}{0.33db^2} \qquad \dots 9.29$$

$$\Theta = \frac{TL}{0.33db^3G} \qquad \dots 9.30$$

(2) **Elleptic section**

$$q = \frac{2\,T}{\pi\, ab^2} \qquad \dots 9.31$$

$$\Theta = \frac{TL}{\left(\dfrac{\pi a^3b^3}{a^2+b^2}\right)G} \qquad \dots 9.32$$

where a and b are the lengths of the semi-major and semi-minor axes respectively.

**(3) Equi-lateral triangular section**

$$q_{max} = \frac{20\ T}{a^3} \qquad \text{... 9.33}$$

$$\theta = \frac{46.2\ TL}{a^4\ G} \qquad \text{... 9.34}$$

where a is a side length of the triangle.

It is impotant to note that on both rectangular and elleptic sections the maximum shear stress does not occur at the fiber farthermost from the centre as in the case of circular sections but at the ends of the shortest axes of symmetry, and in the equi-lateral triangular sections at the points nearest to the centre on the extreme outer fiber.

Examination of formulae 9.25-9.34 shows that the maximum shear stress and the angle of twist may be expressed in terms of factors depending on the shape and dimensions of the cross-sectional area. Thus, generalizing,

$$q_{max} = \frac{T}{Z_p} \qquad \text{... 9.35}$$

$$\theta = \frac{TL}{GJ} \qquad \text{... 9.36}$$

where $Z_p$ is the torsion section modulus, and J another torison constant. Attention should be drawn to the fact that J is just a constant for the section. For circular cross-sections, J is equal to the polar moment of inertia but this is not generally true. For instance the torsion constant J for a flat strip is equal to $d\,b^3/3$ while its polar moment of inertia is nearly equal to $b\,d^3/12$

## 9.12 Torsion of open thin-walled sections

A thin-walled section is that whose thickness is very small compared to its other dimensions. Thin-walled sections may be divided into two groups; open and closed sections. Since their analysis is different, open and closed thin-walled sections will be considered separately.

An open thin-walled section of any shape may be treated as a flat strip. Thus, referring to Fig. 9.32 a which shows a number of open thin-walled sections of uniform thickness t and using equations 9.29 and 9.30 for flat strips,

$$q_{max} = \frac{T}{0.33\ t^2 s} \quad \text{... 9.37}$$

$$\Theta = \frac{TL}{0.33 G t^3 s} \quad \text{... 9.38}$$

where s is the extended length of the section.

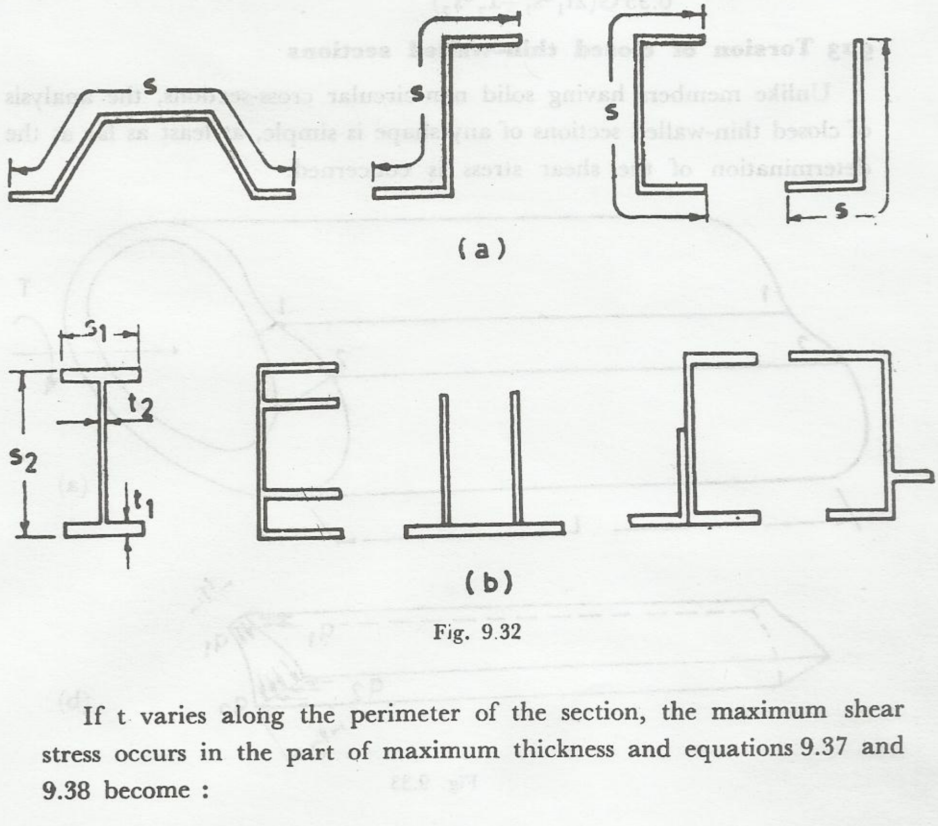



Fig. 9.32

If t varies along the perimeter of the section, the maximum shear stress occurs in the part of maximum thickness and equations 9.37 and 9.38 become :

$$q_{max} = \frac{T\ t_{max}}{0.33\ \Sigma t^3 s} \quad \text{... 9.39}$$

$$\Theta = \frac{TL}{0.33\ G\ \Sigma t^3 s} \quad \text{... 9.40}$$

where the summation $\Sigma t^3 s$ is carried out for lengths of the perimeter having constant thickness.

If the section is compound as shown in Fig. 9.32 b, and cannot be extended into a single flat strip, equations 9.39 and 9.40 still hold. In this case however, s is equal to the sum of separate lengths. For example for the I-section shown in Fig. 9.32 b, $t_1 > t_2$ the maximum shear stress occurs in the flange and its value is given by :

$$q_{max} = \frac{Tt_1}{0.33\,(2t_1{}^3 s_1 + t_2{}^3 s_2)}$$

$$\text{and } \Theta = \frac{TL}{0.33\,G(2t_1{}^3 s_1 + t_2{}^3 s_2)}$$

## 9.13 Torsion of closed thin-walled sections

Unlike members having solid non-circular cross-sections, the analysis of closed thin-walled sections of any shape is simple, at least as far as the determination of the shear stress is concerned.

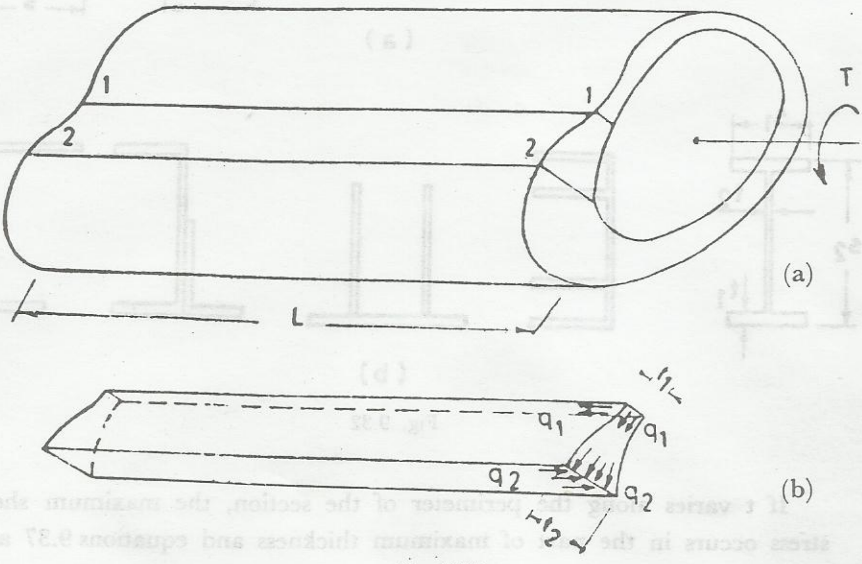


Fig. 9.33

Consider a tube which has a closed thin-walled cross-section with varying thickness as shown in Fig. 9.33, and consider the equilibrium of an element cut off by two longitudinal sections 1-1 and 2-2.

If the shear stresses in the plane of the cross-section at points 1 and 2 are $q_1$ and $q_2$ respectively then according to the principle of complementary shear stress equal shear stresses must exist on the longitudinal planes as shown in Fig. 9.33 b.

Consider now the forces in the longitudinal direction of the element then for equilibrium,

$$q_1 t_1 L = q_2 t_2 L$$

$$\tau_1 = \tau_2 = \tau \qquad \dots 9.41$$

Equation 9.41' shows that the shear flow $\tau$ is constant along the perimeter of a closed thin-walled section. It also shows that, in contrast with thin-walled open sections, the maximum shear stress occurs at the location of least thickness.

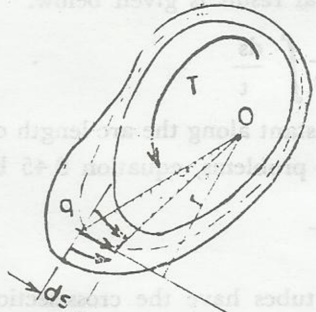



Fig. 9.34

Consider next a cross-section of the tube as shown in Fig. 9.34. The shear stress q may be obtained by equating the applied torque to the internal twisting moment about any point 0 Thus,

$$T = \int q \ t \ ds \ r$$

But qt is equal to $\tau$ which is constant along the perimeter of the section. Hence,

$$T = qt \int ds \ r \qquad \dots 9.42$$

The product ds r is equal to twice the area of the triangle shown hatched in Fig. 9.34 and the integral of this product for the whole arc length of the perimeter is equal to double the area enclosed within the centre line of the section. Denoting this area by $\bar{A}$ to distinguish it from the cross-sectional area A, equation 9.42 reduces to :

$$T = 2 \ q \ t \ \bar{A}$$

$$q = \frac{T}{2 \ t \ \bar{A}} \qquad \dots 9.43$$

As mentioned before, the maximum shear stress occurs at the location of least thickness. Thus,

$$q_{max} = \frac{T}{2\ \overline{A}\ t_{min}} \qquad \dots 9.44$$

It remains to determine the angle of twist $\Theta$. This cannot be found from simple considerations as in the case of circular sections due to the nonvalidity of the assumptions made in that case and resort has to be made to the strain energy method. Since, however, the student has not yet been introduced to this method, no attempt is made here to derive the expression for $\Theta$ and the final result is given below.

$$\Theta = \frac{TL}{4\ G\ \overline{A}^2} \int \frac{ds}{t} \qquad \dots 9.45$$

If the thickness t is constant along the arc length of the perimeter as is the case in most practical problems, equation 9.45 becomes :

$$\Theta = \frac{TLs}{4G\overline{A}^2t} \qquad \dots 9.46$$

**Example 9.16** Two tubes have the cross-sections shown in Fig. 9.35. Calculate the shear stress in each if they are subjected to a torque T = 0.1 m.t. What will be the angle of twist in a length of 1.6 m. of each of the two tubes. G = 800 t./cm².

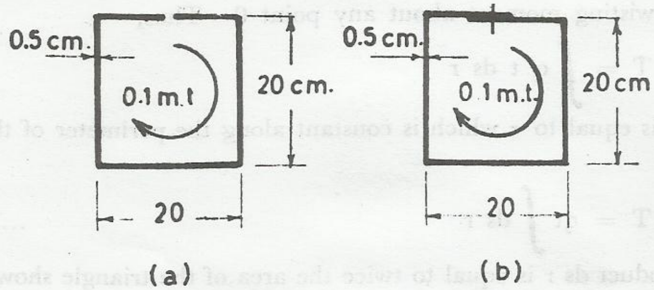



Fig. 9.35

Solution :

For the open section in Fig. 9.35 a,

$$q = \frac{T}{0.33\ t^2 s} = \frac{10000}{0.33 \times 0.25 \times 80} = 1500\ kg./cm^2$$

$$\Theta = \frac{TL}{0.33\ Gt^3 s} = \frac{10 \times 160}{0.33 \times 800 \times 0.125 \times 80} = 0.6 \text{ rad.}$$

For the closed section in Fig. 9.35 b,

$$\bar{A} = 19.5 \times 19.5 = 380 \text{ cm}^2.$$

$$q = \frac{T}{2\bar{A}t} = \frac{10000}{2 \times 380 \times 0.5} = 26.3 \text{ kg./cm}^2.$$

$$\Theta = \frac{TLs}{4G\bar{A}^2 t} = \frac{10 \times 160 \times 80}{4 \times 800 \times 380^2 \times 0.5} = 0.00055 \text{ rad.}$$

This example shows the high torsional strength and stiffness of closed sections in comparison with open sections.

**Example 9.17** If the lap joint in the tube shown in Fig. 9.35 b is riveted, what will be the spacing of these rivets along the length of the tube to ensure it act as a closed section ? Shear resistance of rivet = 125 kg.

Solution :

$$\tau = \frac{T}{2\ \bar{A}} = \frac{10000}{2 \times 380} = 13.2 \text{ kg./cm.}$$

Let the spacing between the rivets = s cm.

$$\tau s = 125$$

$$s = \frac{125}{13.2} = 9.5 \text{ cm.}$$

**Example 9.18** A member has a channel section as shown in Fig. 9.36 a is subjected to a torque of 11.8 cm.t. If two channels of the same size were welded together to form one closed section as shown in Fig. 9.36 b, calculate the torque that can be carried by this section if the allowable shear stress for both sections is the same.

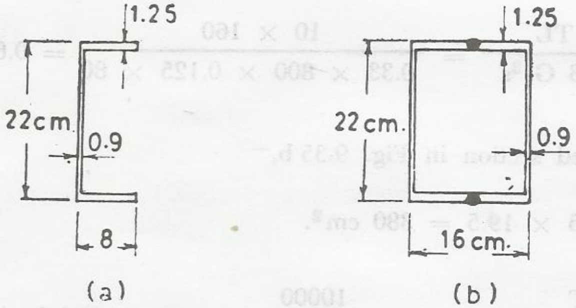



Fig. 9.36

Solution :

$$q_a = \frac{T\, t_{max}}{0.33\ \Sigma t^3 s} = \frac{11.8 \times 1.25}{0.33\ (22 \times 0.9^3 + 2 \times 7.1 \times 1.25^3)} = 1\ \text{t./cm}^2.$$

$$q_b = \frac{T}{2\ \bar{A}\ t_{min}}$$

$$1 = \frac{T}{2 \times 0.9\ (20.75\ \times\ 15.1)}$$

$$T = 564\ \text{cm.t.}$$

A comparison between the torques that can be carried by the two sections demonstrates the high torsional strength of closed thin-walled sections in comparison with open ones.

**Example 9.19** Determine the shear stresses in the cross-section shown in Fig. 9.37 if it is subjected to a torque of 1 m.t. Calculate the angle of twist of the tube in a length of 2 m. $G = 800$ t./cm$^2$.

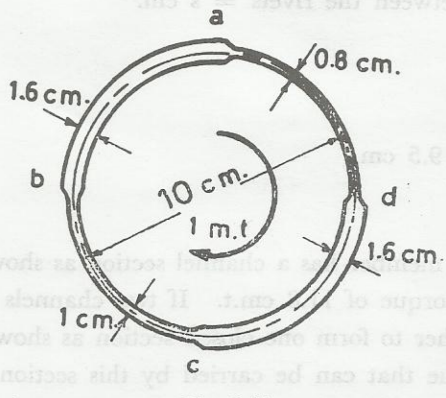



Fig 9.37

Solution : $\bar{A} = \frac{\pi \times 10^2}{4} = 78.8 \text{ cm}^2.$

$$\tau = \frac{T}{2\,\bar{A}} = \frac{100}{2 \times 78.8} = 0.64 \text{ t./cm.}$$

$$q_{ad} = \frac{0.64}{0.8} = 0.8 \text{ t./cm}^2.$$

$$q_{dc} = q_{ab} = \frac{0.64}{1.6} = 0.4 \text{ t./cm}^2.$$

$$q_{bc} = \frac{0.64}{1} = 0.64 \text{ t./cm}^2.$$

To calculate the angle of twist, the $\int \frac{ds}{t}$ has to be evaluated

$$\int \frac{ds}{t} = \frac{10\,\pi}{4}\left(\frac{1}{1} + \frac{2}{1.6} + \frac{1}{0.8}\right) = 27.5$$

$$\Theta = \frac{TL}{4\bar{A}^2 G}\int \frac{ds}{t} = \frac{100 \times 200 \times 27.5}{4 \times 78.8^2 \times 800} = 0.028 \text{ rad.}$$

## 9.14 Combined shearing force and torsion stresses

In some problems, members are subjected to the combined effect of shearing force and torque. The shear stresses caused by each of these were considered earlier in this chapter. The final stresses may be found by superposition as in the case of combined axial force and bending stresses discussed earlier in section 8.5. However, while normal stresses act only normal to the cross-section, either toward or away from it, shear stresses acting in the plane of the cross-section may have any direction. This causes some difficulty, and therefore in this section attention will be directed to cases where the shear stresses due to shearing force and torque have the same line of action and hence may be easily superimposed.

**Example 9.20** Find the maximum shear stress at section a-a of the structure shown in Fig. 9.38 if it has a circular cross-section 10 cm. diameter.

Solution: The straining actions causing shear stresses at section a-a are:

$Q = 3$ t. ↓

$T = 1 \times 1 - \times 1 \times 0.5 = 0.5$ m.t. (clockwise).

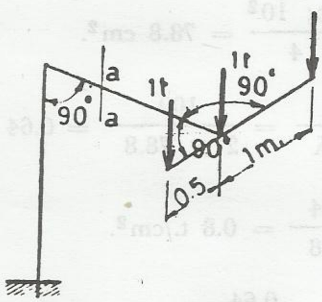



Fig. 9.38

Due to the torque T, the shear stresses vary linearly from zero at the centre to a maximum value, $q_{max} = TR/J$, at the extreme outer fiber and have directions tangential to the circumference. Their directions at points 1-4 are shown in Fig. 9.39 a.

Due to the shearing force Q, the shear stresses vary parabolically from zero at points 2 and 4 to a maximum value, 4Q/3A, at fiber 1-3. Their directions are parallel to the applied shearing force as shown in Fig. 9.39 b.

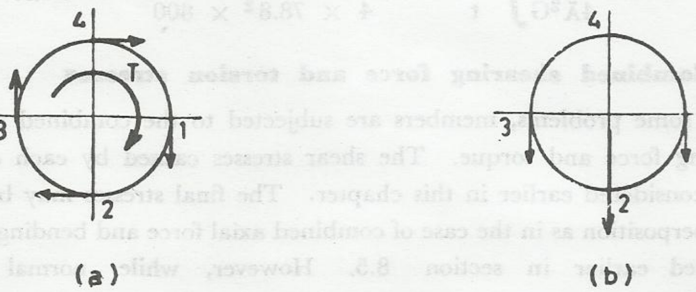



Fig. 9.39

By inspection, point 1 on the cross-section has the maximum shear stress since the two stress components due to T and Q act in the same direction.

$$q_{max} = \frac{50 \times 5}{0.5\,\pi \times 5^4} + \frac{4 \times 3}{3 \times \pi \times 5^2} = 0.305 \text{ t./cm}^2.$$

## EXAMPLES TO BE WORKED OUT

( 1 ) Three pieces of wood having 15 × 15 cm. square sections are glued together as shown in Fig. 9.40. If the arrangement carries a load of 12 t. what will be the average shear stress in the joints ?

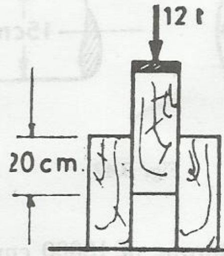



Fig. 9.40

( 2 ) Rivet holes 2.4 cm. diameter are puched in a metal sheet 0.8 cm. thick. If the shearing strength of the sheet is 2.1 t./cm$^2$., what will be the compressive stress in the punch at the time of punching ?

( 3 ) A lever is secured to a shaft by a pin as shown in Fig. 9.41. The shaft is 5 cm. diameter and the cross-sectional area of the pin is 2 cm$^2$. What torque can be applied to the lever without causing the shear stress in the pin to exceed 0.8 t./cm$^2$. ?

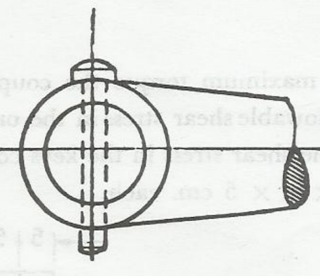


Fig. 9.41

( 4 ) A copper disc 10 cm. diameter and 0.2 cm. thick is fitted in the casing of an air compressor so as to function as a safty valve and blow in case of serious increase of the pressure of the compressed air. Calculate at what pressure the disc will blow out assuming that failure occurs by shear round the edges of the disc and that the shearing strength of copper is 1.25 t./cm$^2$.

( 5 ) Estimate the maximum torque the coupling shown in Fig. 9.42 can carry if the shear stress in the bolts is not to exceed 0.8 t./cm$^2$. The bolts are 2 cm. diameter.

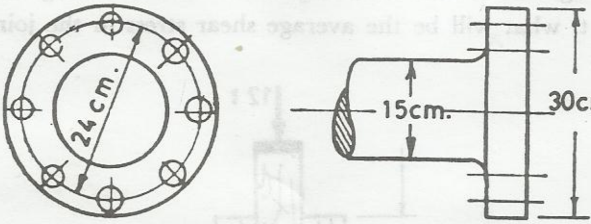



Fig. 9.42

( 6 ) A gear transmitting torque of 18000 cm.kg. to a 6 cm. diameter shaft is keyed to it as shown in Fig. 9.43. The key is 1.5 × 1.5 cm. in section and 5 cm. long. Determine the shear stress in the key.

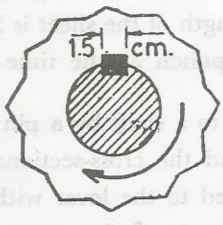



Fig. 9.43

( 7 ) Estimate the maximum torque the coupling shown in Fig. 9.44 can transmit if the allowable shear stress in the one centimeter bolts is 0.8 t./cm$^2$ Check then the shear stress in the keys connecting the shaft to the flange if they are 1 × 1 × 5 cm. each.

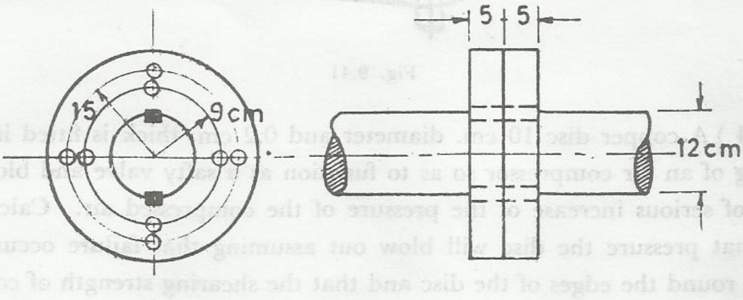



Fig. 9.44

(8) The beam shown in Fig. 9.45 has a T-section of the given dimensions. Calculate the maximum shear stress and plot the shear stress distribution at the section of maximum shearing force.

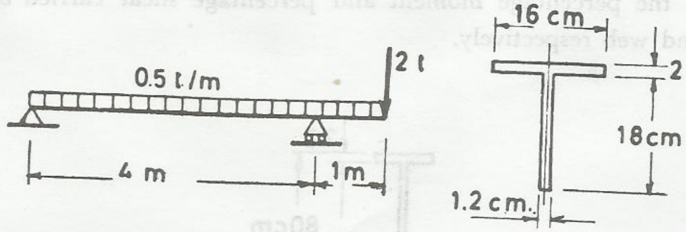



Fig. 9.45

(9) - (14) — For the cross-sectional areas shown in Figs. 9.46 a - f, plot the shear stress distributions indicating the value and position of the maximum shear stress in every case

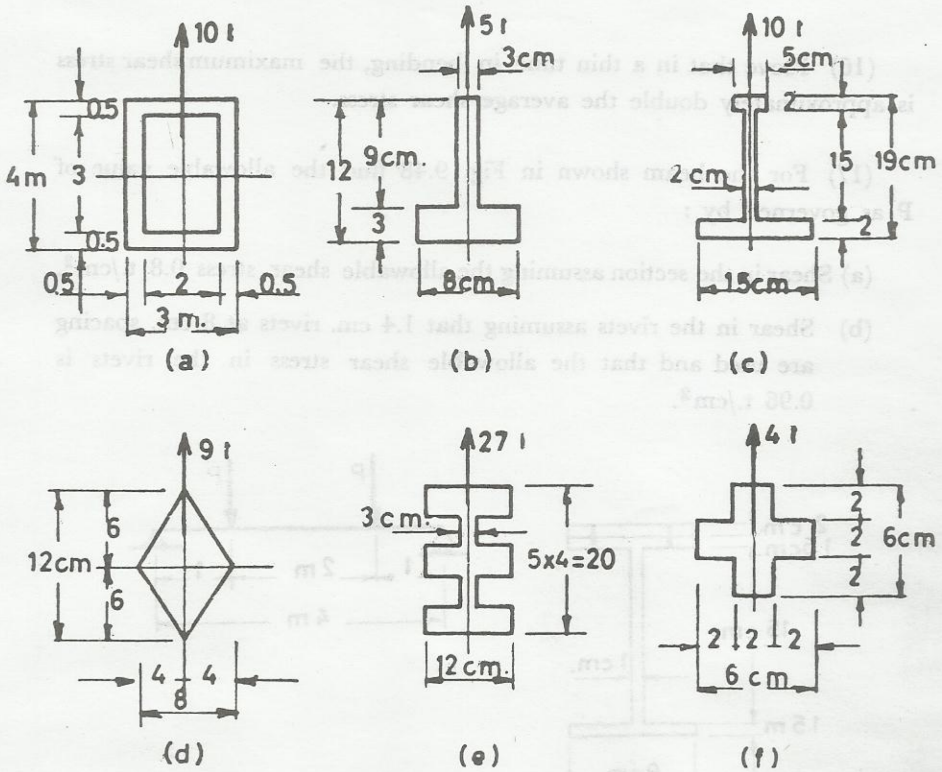



Fig. 9.46

(15) A girder has a cross-section of the dimensions shown in Fig. 9.47. At a certain section, it carries a moment of 60 m.t. and a shearing force of 40 t. Plot the normal and shear stress distributions on the section. Calculate the percentage moment and percentage shear carried by the flanges and web respectively.

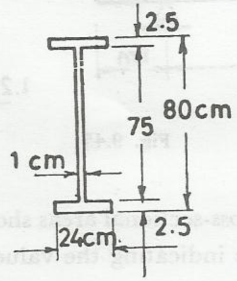



Fig. 9 47

(16) Prove that in a thin tube in bending, the maximum shear stress is approximately double the average shear stress.

(17) For the beam shown in Fig. 9.48 find the allowable value of P as governed by :

(a) Shear in the section assuming the allowable shear stress 0.8 t./cm$^2$.

(b) Shear in the rivets assuming that 1.4 cm. rivets at 8 cm. spacing are used and that the allowable shear stress in the rivets is 0.96 t./cm$^2$.

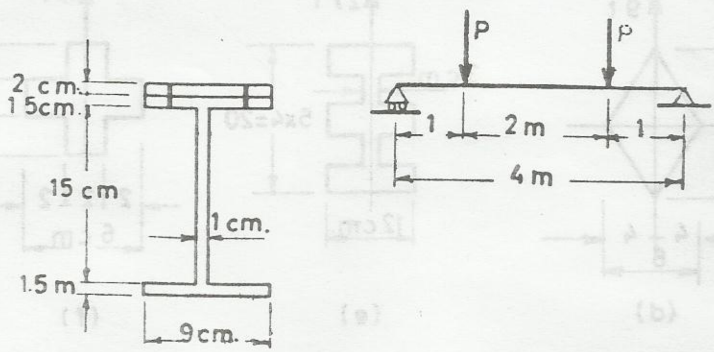



Fig. 9.48

(18) A plate girder has a span of 12 m. and carries a uniformly distributed load of 8 t./m. If the cross-section of the girder is as shown in Fig. 9.49, calculate the maximum shear stresses in rivets a and b if they are 2.3 cm. diameter and spaced at 10 cm. pitch.

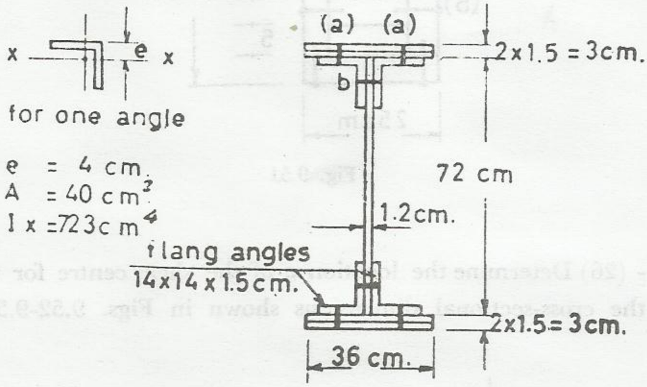



Fig. 9.49

(19) A simply supported beam carries a set of loads as shown in Fig. 9.50. The beam has a cross-section of two I-sections fastened together, as shown in the figure, by 1.7 cm. diameter rivets. If the allowable shear stress in the rivets is 0.8 t./cm$^2$., specify the proper pitch. Assume that for one I-section,

$$A = 35.5 \text{ cm}^2. \quad \text{and} \quad I = 2142 \text{ cm}^4.$$

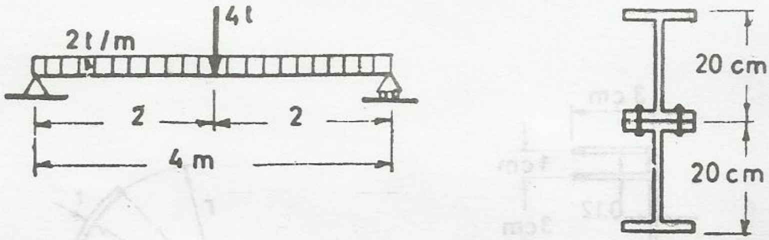



Fig. 9.50

(20) A wooden beam has the cross-section shown in Fig. 9.51. It is built up of planks 5 cm. thick each, nailed together with nails which have shear resistance of 42 kg. per nail. If the beam is subjected to a constant shear of 1000 kg., what should be the spacings of nails (a) and (b) ?

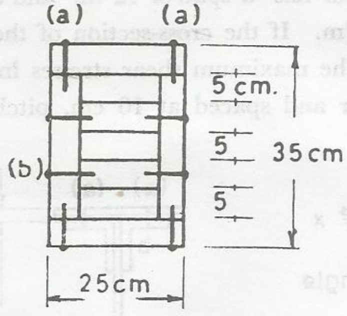



Fig. 9.51

(21) - (26) Determine the location e of the shear centre for the beams having the cross-sectional dimensions shown in Figs. 9.52-9.57.

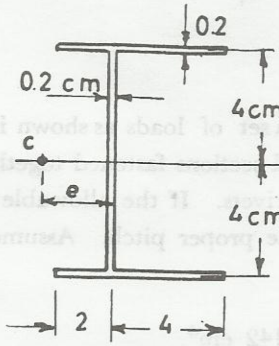



Fig. 9.52

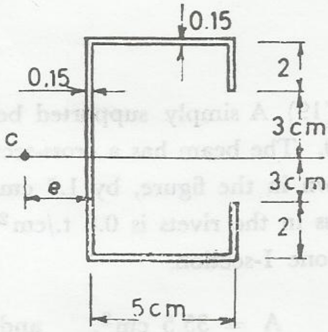



Fig. 9.53

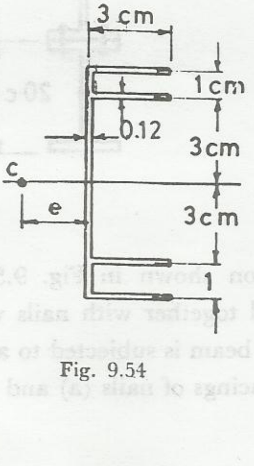



Fig. 9.54

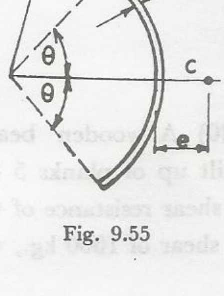



Fig. 9.55

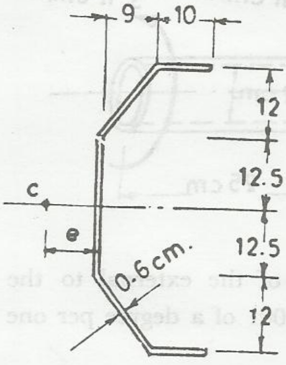



Fig. 9.56

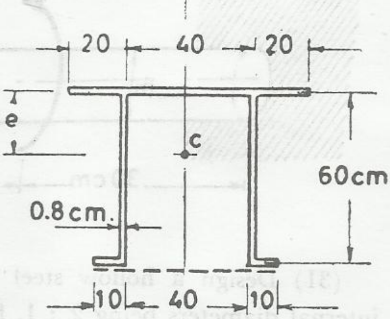



Fig. 9.57

(27) Determine the maximum shear stress in the stepped shaft with diameters 2, 6, 4 and 3 cm. starting from the left end, and subjected to the torques shown in Fig. 9.58. If G = 800 t./cm$^2$., what is the relative angle of twist between its two ends ?

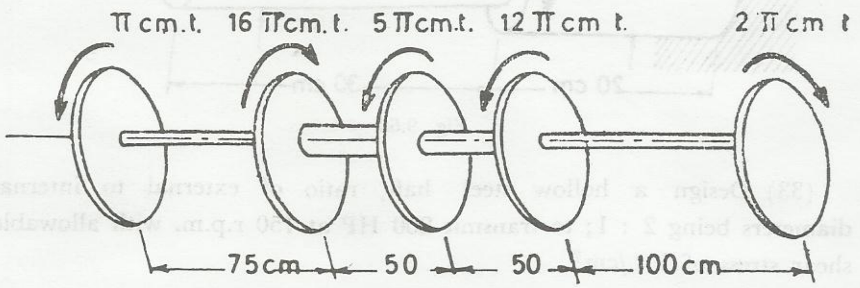



Fig. 9.58

(28) A circular tube 0.318 cm. thick and 8 cm. mean diameter transmits a torque T. If the allowable shear stress is 1 t./cm$^2$., what torque can it transmit ? For this tube what will be the relative angle of twist in a length of 80 cm. if G = 800 t./cm$^2$. ?

(29) What must be the length of 0.5 cm. diameter aluminum wire so that it could be twisted through one complete revolution without exceeding a shear stress of 0.25 t./cm$^2$. ? G = 250 t./cm$^2$.

(30) Determine the maximum shear stress in the shaft shown in Fig. 9.59. If G = 800 t./cm$^2$., what will be the angle of twist at the free end ?

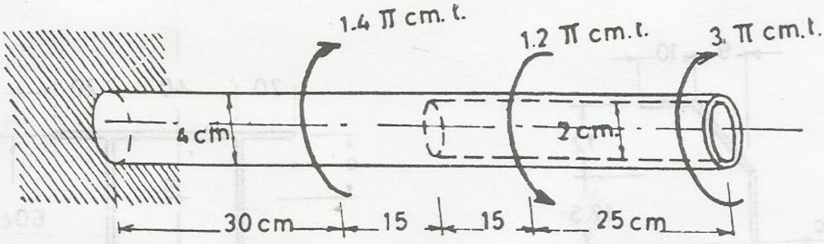



Fig. 9.59

(31) Design a hollow steel shaft, the ratio of the external to the internal diameters being 2 : 1, for a stifness of 0.001 of a degree per one cm.kg. torque. G for steel = 800 t./cm$^2$.

(32) The stepped shaft shown in Fig. 9.60 carries the given torques. What is the total angle of twist at the free end if the maximum shear stress is limited to 1 t./cm$^2$. ? G = 800 t./cm$^2$.

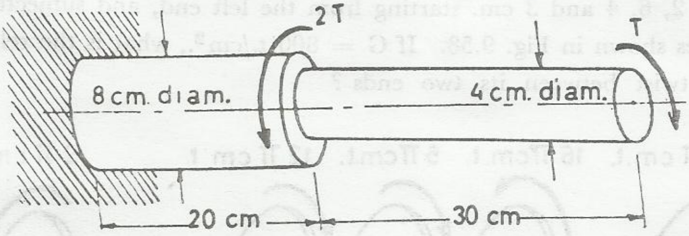



Fig. 9.60

(33) Design a hollow steel haft, ratio of external to internal diameters being 2 : 1; to transmit 200 HP at 150 r.p.m. with allowable shear stress of 1 t./cm$^2$.

(34) The shaft shown in Fig. 9.61 carries a torque of 8. m.t. at the given location. Calculate the values of the resisting torques at the fixed ends.

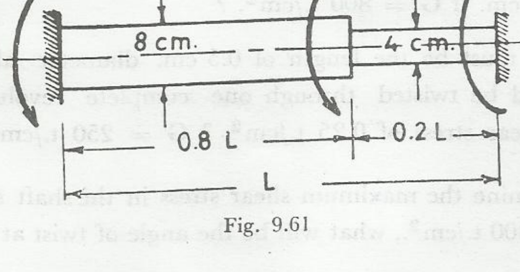



Fig. 9.61

(35) Compare the torsional moment of resistance and the angle of twist of the following steel sections using a maximum shear stress of 1 $t./cm^2$. and a length of 3 m.

(a) A solid circular section 10 cm. diameter.

(b) A hollow circular section 10 cm. and 8 cm. outside and inside diameters.

(c) A solid square section side length 10 cm. ($\alpha = 0.208$, $\beta = 0.141$).

(d) A solid rectangular section $5 \times 20$ cm. ($\alpha = 0.282$, $\beta = 0.281$).

(e) An eleptical section 5 cm. and 20 cm. minor and major diameters. G (for steel) = 800 $t./cm^2$.

(36) Compare the shear stress and angle of twist for a thin tube of equilateral triangle cross-section, side length 12 cm. and wall thickness 0.4 cm. to a similar tube having a longitudinal slot if both tubes are subjected to the same torque.

(37) A tube with a hollow square cross-section measuring 30 cm. between the centre lines of the walls is subjected to a torque T = 1.5 m.t. If the thicknesses of the four sides are 0.6,0.8, 1.2 and 1.6 cm., what are the shear stresses in the walls of the tube ? Calculate the angle of twist in a 3 m. length of the tube. G = 800 $t./cm^2$.

(38) Determine the shear stress at points a, b, c and d at the fixed end of the cantilever shown in Fig. 9.62, if it has a circular cross-section 10 cm. diameter.

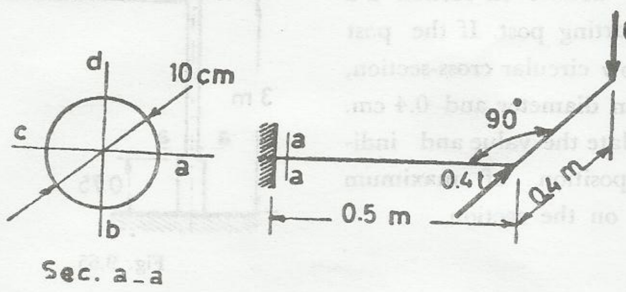



Fig. 9.62

(39) A concrete girder has a hollow rectangular section 80 cm. × 40 cm. outside dimensions and a uniform wall thickness of 10 cm. Find the shear stress distributions due to a vertical shearing force of 40 t. and a torque of 12 m.t.

(40) Fig. 9.63 shows a riveted bracket. If the area of the rivet is 5 $cm^2$., calculate the shear stress in rivets 1 and 2.

(41) Fig. 9.64 shows a hand-rail connected to a bracket by three 17 mm. rivets. Calculate the shear stress in each of the three rivets.

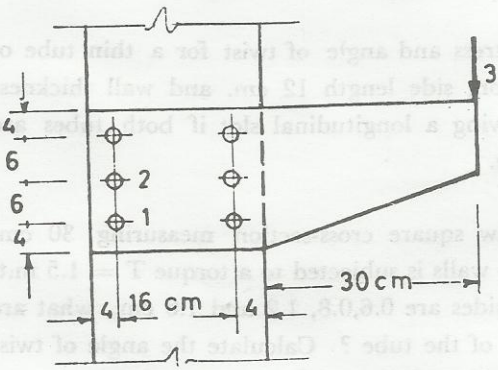



Fig. 9.63

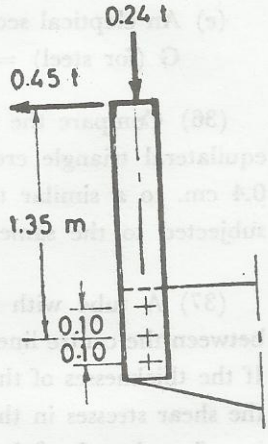



Fig. 9.64

(42) Fig. 9.65 shows an advertising board. If the wind pressure on the board is 200 kg./$m^2$. and it's own weight is 0.5 t., indicate the straining actions on section a-a of the supporting post. If the post has a hollow circular cross-section, 20 om. mean diameter and 0.4 cm. thick, calculate the value and indicate the position of maximum shear stress on the section.

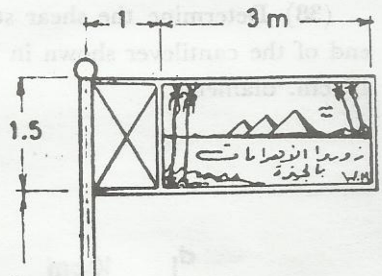



Fig. 9.65

CHAPTER 10

# PRINCIPAL STRESSES

## 10.1 Introduction

In the preceding two chapters, formulae to determine separately the normal and shear stresses have been developed. Combining normal stresses due to N, $M_x$ and $M_y$ on one hand, and shear stresses due $Q_x$, $Q_y$ and T on the other hand has also been discussed. In this chapter, equations to transform the normal and shear stresses at a certain point, given by the established formulae, into equivalent stress systems will be developed. Then, the maximum and minimum values of the normal stresses and their directions will be determined. Also, the maximum shear stress will be found. At this elementary stage, however, discussion will be limited to two-dimensional stress systems in which all the stress components lie in one plane. Apart from such systems, there are three-dimensional stress systems in which the stress components lie in more than one plane. In practice, most members are usually subjected to two-dimensional stress system. Thus, although one type of the stress systems is considered, the majority of the normal applications fall within its scope.

For example, an element of a beam as that shown in Fig. 10-1 a, will be subjected to a normal stress due to the bending moment at the section, $f = -My/I$, and a shear stress due to the shearing force at the section, $q = QS/Ib$, both being in the same plane. It is a normal practice to indicate the various stress components on an element as shown in Fig. 10.1 b.

As a second example, consider the shaft shown in Fig. 10.2 a, which is subjected to a torque T and an axial tension N.

An element on the surface of the shaft given in Fig. 10.2 a will be

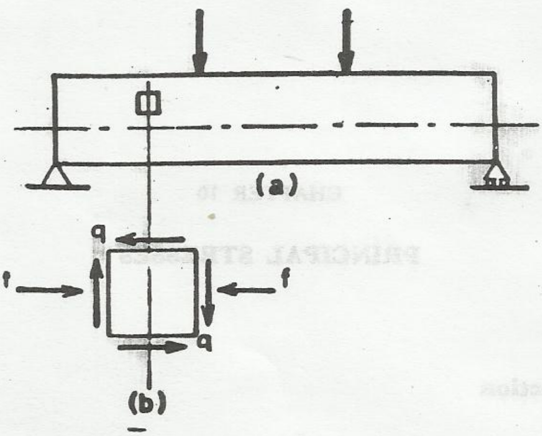



Fig. 10 1

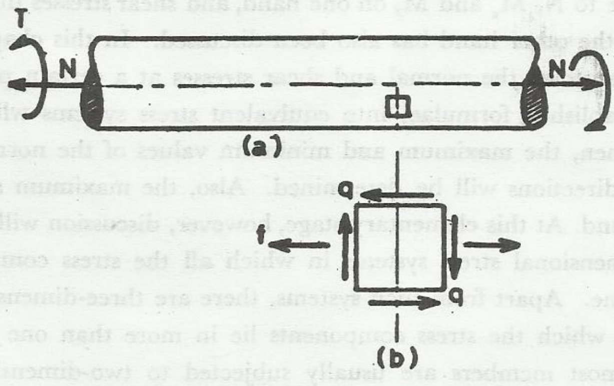



Fig. 10 2

**subjected to a shear stress, q = TR/J and a normal stress, f = N/A These stress components, which again lie in one plane, are shown on an element in Fig. 10.2 b.**

**The normal stress f and the shear stress q mentioned in the previous two examples are not the greatest values that may occur at the locations of the elements considered. Since the design of a member is usually governed by the maximum stress that may occur in the member, it is essential to establish means by which these maximum stresses may be determined.**

**Example 10.1** **Indicate on small elements the stress components at locations 1-5 on the simple beam shown in Fig. 10.3.**

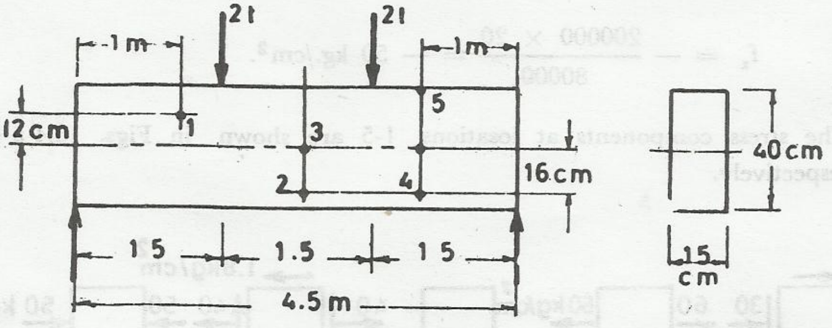



Fig. 10.3

Solution :

$$I_x = \frac{15 \times 40^3}{12} = 80000 \text{ cm}^4.$$

For location 1, M = 2 m.t. and Q = + 2 t

$$f_x = \frac{My}{I} = -\frac{200000 \times 12}{80000} = -30 \text{ kg./cm}^2.$$

$$q = \frac{QS}{Ib} = \frac{2000\ (15 \times 8 \times 16)}{80000 \times 15} = +3.2 \text{ kg./cm}^2.$$

For location 2, M = 3 m.t. and Q = 0

$$f_x = \frac{300000 \times 16}{80000} = +60 \text{ kg./cm}^2.$$

$$q = 0$$

For location 3, since it lies on the neutral axis, the normal stress $f_x$ is zero. Also, since the shearing force at the cross-section on which the considered location lies is zero, the shear stress is also zero.

For location 4, M = 2 m.t. and Q = — 2 t.

$$f_x = \frac{200000 \times 16}{80000} = +40 \text{ kg./cm}^2.$$

$$q = \frac{-2000\ (4 \times 15 \times 1.8)}{80000 \times 15} = -1.8 \text{ kg./cm}^2.$$

For location 5, since it lies at the extreme upper fiber, the shear stress is zero and the normal stress is a maximum and given by :

$$f_x = -\frac{200000 \times 20}{80000} = -50 \text{ kg./cm}^2.$$

The stress components at losations 1-5 are shown in Figs. 10.4 a-e respectively.

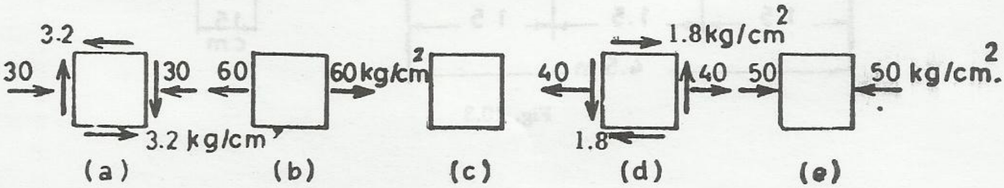



Fig. 10.4

## 10.2 General two-dimensional stress system

In a two-dimensional stress system, the stresses at any point in a member act in the same plane. In general, a point is subjected to a stress system consisiting of two normal stress components $f_x$ and $f_y$ in the x-and y directions respectively, and shear stresses $q_x$ and $q_y$. These being equal (section 9.3), are denoted by q in the analysis that follows.

The various stress components are indicated, in their assumed positive directions, in Fig. 10.5.

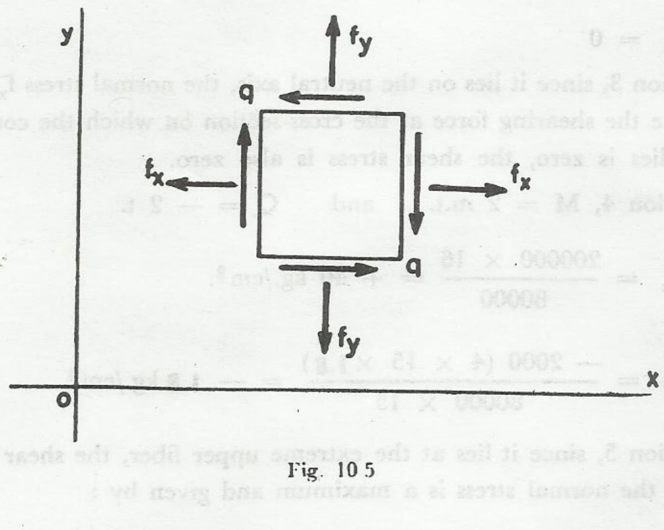



Fig. 10 5

**10.3 Stresses on an inclined plane**

Consider the stresses acting on an inclined plane ab of the element shown in Fig. 10.6 a. The inclined plane makes an angle $\Theta$ with the y-axis and cuts off a triangular element as shown in Fig. 10.6 b. If the area of plane ab is dA, then the areas of the faces ac and bc are dA cos $\Theta$ and dA sin $\Theta$ respectively.

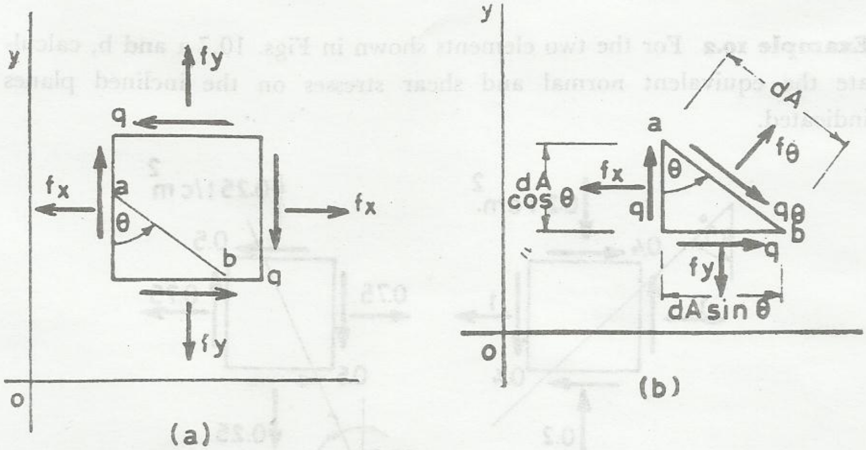



Fig. 10.6

Multiplying the stresses shown in Fig. 10.6 b by their respective areas, and then by considering the equilibrium of the resulting forces, the values of the normal stress $f_\theta$ and the shear stress $q_\theta$ on the inclined plane are obtained.

Resolving the forces along the x-axis,

$$f_\theta \, dA \cos\Theta - f_x \, dA \cos\Theta + q dA \sin\Theta + q_\theta \, dA \sin\Theta = 0 \quad \ldots \; 10.1$$

Resolving the forces along the y-axis,

$$f_\theta \, dA \sin\Theta - f_y \, dA \sin\Theta + q dA \cos\Theta - q_\theta \, dA \cos\Theta = 0 \quad \ldots \; 10.2$$

Eliminating $q_\theta$ between equations 10.1 and 10.2,

$$f_\theta = \tfrac{1}{2}(f_x + f_y) + \tfrac{1}{2}(f_x - f_y)\cos 2\Theta - q \sin 2\Theta \quad \ldots \; 10.3$$

Eliminating $f_\theta$ between equations 10.1 and 10.2,

$$q_\theta = \tfrac{1}{2}(f_x - f_y)\sin 2\Theta + q \cos 2\Theta \quad \ldots \; 10.4$$

Equations 10.3 and 10.4 are the general expressions for the equivalent normal and shear stresses on a plane making an angle $\Theta$ with the y-direction and caused by a known system of stresses; $f_x$, $f_y$ and q. It is important to note the signs of all the quantities in these equations. Tensile normal stresses are considered positive. Positive shear stresses correspond to shear acting upward on the left face and downward on the right face of the element. The angle $\Theta$ refers to the y-axis and is considered positive when measured in anticlockwise direction. The positive signs correspond to the directions indicated in Fig. 10.5.

**Example 10.2** For the two elements shown in Figs. 10.7 a and b, calculate the equivalent normal and shear stresses on the inclined planes indicated.

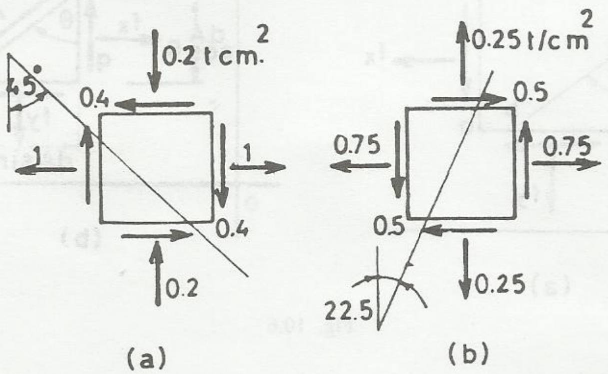



Fig. 10.7

For the element in Fig. 10.7 a,

$f_x = +1$, $f_y = -0.2$, $q = +0.4$ and $\Theta = +45^0$

From equations 10.3 and 10.4,

$$f\theta = \frac{1 - 0.2}{2} + \frac{1 + 0.2}{2} \cos 90 - 0.4 \sin 90$$

$$= 0.4 + 0 - 0.4 = 0$$

$$q\theta = \frac{1 + 0.2}{2} \sin 90 + 0.4 \cos 90$$

$$= 0.6 + 0 = 0.6 \text{ t./cm}^2.$$

For the element in Fig. 10.7 b,

$f_x = 0.75$, $f_y = 0.25$, $q = -0.5$ and $\Theta = -22.5^0 = +157.5^\circ$

$$f_\theta = \frac{0.75 + 0.25}{2} + \frac{0.75 - 0.25}{2} \cos 315 + 0.5 \sin 315$$

$$= 0.5 + 0.25 \times 0.707 - 0.5 \times 0.707 = 0.323 \text{ t./cm}^2.$$

$$q\theta = \frac{0.75 - 0.25}{2} \sin 315 - 0.3 \cos 315$$

$$= -0.25 \times 0.707 - 0.5 \times 0.707 = -0.53 \text{ t./cm}^2.$$

The stress components on the inclined planes in Fig. 10.7 a and b, are shown in Figs. 10.8 a and b respectively

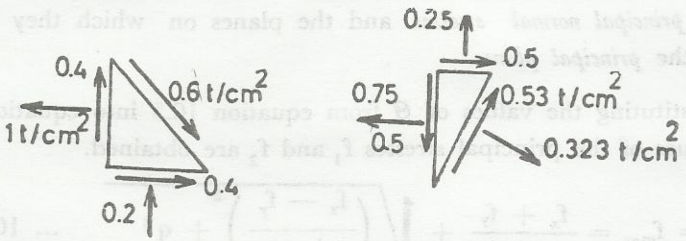



Fig. 10.8

## 10.4 Principal normal stresses

Usually, the greatest possible normal stress value, as given by equation 10.3, and the direction of the plane on which it acts are the quantitites required. The direction of this plane is found first.

In order to find the planes of maximum or minimum normal stresses, equation 10.3 is differentiated with respect to $\Theta$ and the derivative equated to zero. Thus,

$$\frac{df}{d\Theta} = 0 = -(f_x - f_y) \sin 2\Theta - 2q \cos 2\Theta$$

Hence, $$\tan 2\Theta = \frac{-2q}{f_x - f_y} \qquad \ldots 10.5$$

Equation 10.5 defines two values of $2\Theta$ differing by $180^0$, and hence two values of $\Theta$ separated by $90^0$. The normal stress corresponding to one of these values is a maximum and to the other a minimum. To determine which one of these angles is associated with the maximum or minimum stress, each of the two values obtained from equation 10.5 is substituted separately in equation 10.3, and the values of stresses are then compared.

Considering next equation 10.4, the shear stress q vanishes at an angle $\theta$ where,

$$\tan 2\theta = \frac{-2q}{f_x - f_y} \qquad \ldots 10.6$$

which is identical to equation 10.5, defining the directions of the planes of the maximum and minimum normal stresses. Thus, in a two-dimensional stress system, there are two planes separated by $90^0$ on which the shear stress is zero, and on one of which the normal stress is a maximum and on the other the normal stress is a minimum. These stresses are called *principal normal stresses* and the planes on which they act are called the *principal planes.*

Substituting the values of $\theta$ from equation 10.5 into equation 10.3, the values of the principal stresses $f_1$ and $f_2$ are obtained.

$$f_1 = f_{max} = \frac{f_x + f_y}{2} + \sqrt{\left(\frac{f_x - f_y}{2}\right)^2 + q^2} \qquad \ldots 10.7$$

$$f_2 = f_{min} = \frac{f_x + f_y}{2} - \sqrt{\left(\frac{f_x - f_y}{2}\right)^2 + q^2} \qquad \ldots 10.8$$

Alternatively, the values of the principal stresses can be calculated directly, without first finding the directions of the principal planes, by considering the equilibrium of an element as shown in Fig. 10.9. It is noted that the shear stress on the inclined plane is zero. This follows from the definition of the principal planes mentioned above.

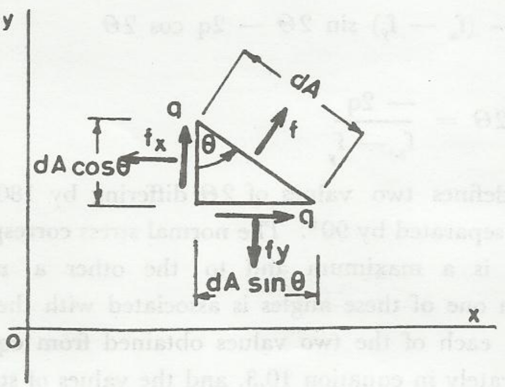



Fig. 10.9

$$\Sigma X = 0 = f\, dA \cos\Theta + q\, dA \sin\Theta - f_x\, dA \cos\Theta$$

$$f - f_x = - q \tan\Theta \qquad \dots 10.9$$

$$\Sigma Y = 0 = f\, dA \sin\Theta + q\, dA \cos\Theta - f_y\, dA \sin\Theta$$

$$f - f_y = - q \cot\Theta \qquad \dots 10.10$$

Eliminating $\Theta$ between equations 10.9 and 10.10,

$$(f - f_x)(f - f_y) = q^2$$

This is a quadratic equation in f, the roots to which are :

$$f_1 = \frac{f_x + f_y}{2} + \sqrt{\left(\frac{f_x - f_y}{2}\right)^2 + q^2}$$

$$\text{and } f_2 = \frac{f_x + f_y}{2} - \sqrt{\left(\frac{f_x - f_y}{2}\right)^2 + q^2}$$

which are the same results obtained before.

**10.5 Maximum shear stress**

To find the planes of maximum or minimum shear stresses, equation 10.4 is differentiated with respect to $\Theta$ and the derivative equated to zero.

$$\frac{dq}{d\Theta} = 0 = (f_x - f_y) \cos 2\Theta - 2q \sin 2\Theta$$

$$\text{Hence,} \quad \cot 2\Theta = \frac{2q}{f_x - f_y} \qquad \dots 10.11$$

A comparison between equation 10.11 and equation 10.6, shows that the maximum and minimum shear stresses occur on planes inclined at an angle of $45^0$ to the principal planes. Substituting the values of $\Theta$ from equation 10.11 into equation 10.4, the value of the maximum shear stress is obtained. Thus,

$$q_{max} = \sqrt{\left(\frac{f_x - f_y}{2}\right)^2 + q^2} \qquad \dots 10.12$$

From equations 10.7 and 10.8,

$$\sqrt{\left(\frac{f_x - f_y}{2}\right)^2 + q^2} = \tfrac{1}{2}(f_1 - f_2) \qquad \dots 10.13$$

$$\text{Hence,} \quad q_{max} = \tfrac{1}{2}(f_1 - f_2) \qquad \dots 10.14$$

which shows that the numerical value of the maximum shear stress at a point is equal to one half the difference between the principal normal stresses at that point.

### 10.6 Semi-graphical treatment — Mohr's circle of stress

The relationship between the normal and shear stresses on inclined planes is similar to the relationship between the moments and products of inertia about inclined axes.

If the values of $f_\theta$ and $q_\theta$, as defined by equations 10.3 and 10.4, are plotted for different values of $\Theta$, all the points will lie on a circle. To prove this, equations 10.3 and 10.4 are re-written in the following form:

$$f_\theta - \frac{f_x + f_y}{2} = \frac{f_x - f_y}{2} \cos 2\Theta - q \sin 2\Theta$$

$$q_\theta = \frac{f_x - f_y}{2} \sin 2\Theta + q \cos 2\Theta$$

Eliminating the parameter $\Theta$ between these two equations by squaring and adding,

$$\left( f_\theta - \frac{f_x + f_y}{2} \right)^2 + q_\theta^2 = \left( \frac{f_x - f_y}{2} \right)^2 + q^2 \qquad \cdots 10.15$$

However, in every problem $f_x$, $f_y$ and $q$ are constant quantities while $f_\theta$ and $q_\theta$ are the variables. Hence equation 10.15 may be written in a simplified form,

$$(f_\theta - a)^2 + q_\theta^2 = b^2 \qquad \cdots 10.16$$

where, $$a = \frac{f_x + f_y}{2} \qquad \cdots 10.17$$

and $$b = \sqrt{\left( \frac{f_x - f_y}{2} \right)^2 + q^2} \qquad \cdots 10.18$$

Equation 10.16 is the familiar expression known in analytical geometry; $(x - a)^2 + y^2 = b^2$, of a circle of radius b and centre at (a, o). Hence, if a circle satisfying this equation is plotted, the co-ordinates of a point (x, y) on this circle correspond to $f_\theta$ and $q_\theta$ for a particular orientation $\Theta$ with respect to the planes on which the given stress components act. The x co-ordinate represents the normal stresses while the y co-ordinate represents the shear stresses. The circle so constructed is called *Mohr's circle of stress.*

There are several methods of plotting the circle defined by equation 10.15. It may be constructed by locating the centre at (a, o) and using the radius given by equation 10.18, but this is not the best procedure for the purpose at hand. The normal and shear stresses acting on two perpendicular planes are usually known. The normal stress on a plane and its associated shear stress define a point on the circle. This, together with the knowledge that the centre of the circle is located on the abscessa at $\frac{1}{2}(f_x + f_y)$, is sufficient to plot the circle.

The procedures are outlined below with reference to Fig. 10.10.

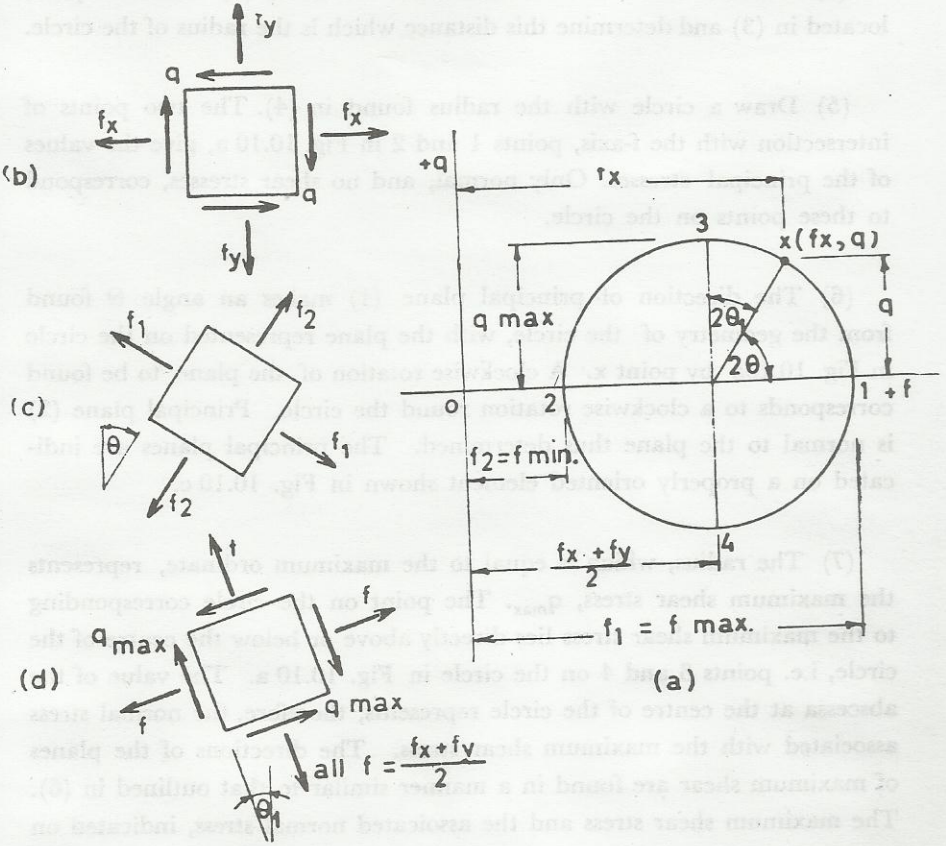


Fig. 10·10

(1) Set up a rectangular co-ordinate system of axes where the horizontal and vertical axes represent the normal and shear stresses axes respectively. Positive directions are taken, as usual, to the right and upward.

(2) Locate the centre of the circle which is on the horizontal axis at a distance $\frac{1}{2}(f_x + f_y)$ from the origin 0.

(3) Locate the point x of co-ordinates $f_x$ and q. f is measured to the right of the origin if tensile and to its left if compressive. Also, q is measured upwards when it follows the positive direction assumed throughout this text; i.e. upwards on the left face of the element and downwards on the right face, and is measured downwards when it is negative.

(4) Connect the centre of the circle found in (2) with the point located in (3) and determine this distance which is the radius of the circle.

(5) Draw a circle with the radius found in (4). The two points of intersection with the f-axis, points 1 and 2 in Fig. 10.10 a, give the values of the principal stresses. Only normal, and no shear stresses, correspond to these points on the circle.

(6) The direction of principal plane (1) makes an angle $\Theta$ found from the geometry of the circle, with the plane represented on the circle in Fig. 10.10a by point x. A clockwise rotation of the plane to be found corresponds to a clockwise rotation round the circle. Principal plane (2) is normal to the plane thus determined. The principal planes are indicated on a properly oriented element shown in Fig. 10.10 c.

(7) The radius, which is equal to the maximum ordinate, represents the maximum shear stress, $q_{max}$. The point on the circle corresponding to the maximum shear stress lies directly above or below the centre of the circle, i.e. points 3 and 4 on the circle in Fig. 10.10 a. The value of the abscessa at the centre of the circle represents, therefore, the normal stress associated with the maximum shear stress. The directions of the planes of maximum shear are found in a manner similar to that outlined in (6). The maximum shear stress and the assoicated normal stress, indicated on a properly oriented element, are shown in Fig. 10.10 d.

**Example 10.3** Using Mohr's circle of stress, re-work example 10.2

Solution :

From the data given in Fig. 10.7 a, Mohr's circle of stress can be drawn in the manner described in section 10.6 and is as shown in Fig. 10.11; centre at (0.4, 0) and radius $= \sqrt{0.6^2 + 0.4^2} = \sqrt{0.52} = 0.72$. Point $x_1$ on the circle is located by taking an angle $2 \times 45 = 90^0$ counter-clockwise from x.

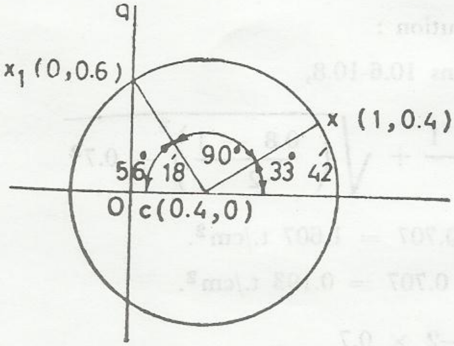



Fig. 10.11

From the geometry of the circle,

$f = 0.4 - 0.72 \cos 56^0\ 18' = 0.4 - 0.72 \times 0.5548 = 0$

$q = 0.72 \sin 56^0\ 18' = 0.72 \times 0.8319 = 0.6$ t./cm².

From the data given in Fig. 10.7 b, Monr's circle can be drawn and is as shown in Fig. 10.12; centre at (0.5, 0) and radius. $R = \sqrt{0.75^2 + 0.5^2} = 0.558$. Point $x_1$ on the circle is located at an angle of $2 \times 22.5 = 45^0$ clockwise from point x.

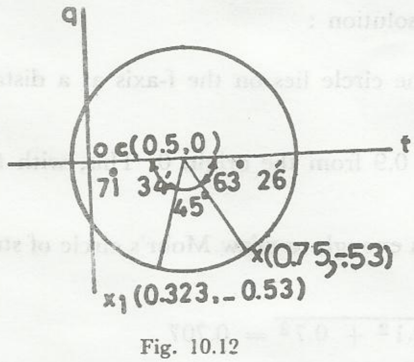



Fig. 10.12

$f = 0.5 - 0.558 \cos 71^0 34' = 0.5 - 0.558 \times 0.3184 = 0.323$ t./cm$^2$.

$q = 0.558 \sin 71^0 34' = 0.558 \times 0.9487 = -0.53$ t./cm$^2$.

**Example 10.4** At some point in a structural member, the stresses are given by : $f_x = 0.8$ t./cm$^2$., $f_y = 1$ t./cm$^2$. and $q = 0.7$ t./cm$^2$. Following the sign conventions in Fig. 10.5, find analytically and semi-graphically using Mohr's circle of stress the values and directions of the principal stresses. What is the value of the maximum shear stress and the normal stress associated with it ?

Analytical solution :

From equations 10.6-10.8,

$$f_1 = \frac{0.8 + 1}{2} + \sqrt{\left(\frac{0.8 - 1}{2}\right)^2 + 0.7^2}$$

$f_1 = 0.9 + 0.707 = 1.607$ t./cm$^2$.

$f_2 = 0.9 - 0.707 = 0.193$ t./cm$^2$.

$$\tan 2\Theta = \frac{-2 \times 0.7}{0.8 - 1} = 7$$

$2\Theta = 81^0 54'$ or $\Theta = 40^0 57'$

From equation 10.14,

$$q_{max} = \frac{1.607 - 0.193}{2} = 0.707 \text{ t./cm}^2.$$

The normal stress associated with the maximum shear stress is given by :

$$f = \frac{f_x + f_y}{2} = \frac{0.8 + 1}{2} = 0.9 \text{ t./cm}^2.$$

Semi-graphical solution :

The centre of the circle lies on the f-axis at a distance $\frac{f_x + f_y}{2}$

$= \frac{0.8 + 1}{2} = 0.9$ from the origin 0. This, with the co-ordinates of point x (0.8, 0.7), is enough to draw Mohr's circle of stress which is shown in Fig. 10.13.

Radius $= \sqrt{0.1^2 + 0.7^2} = 0.707$

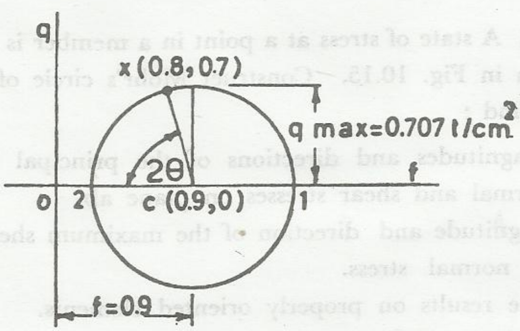



Fig. 10.13

From the geometry of the circle,

$$f_1 = 0.9 + 0.707 = +1.607 \text{ t./cm}^2.$$

$$f_2 = 0.9 - 0.707 = +0.193 \text{ t./cm}^2.$$

$$\tan 2\Theta = \frac{0.7}{0.1} = 7$$

$$2\Theta = 81^0\,54' \quad \text{or} \quad \Theta = 40^0\,57'$$

The principal stresses indicated on a properly oriented element are shown in Fig. 10.14 a.

$q_{max}$ = radius = 0.707 t./cm².

and the associated normal stress equals to the abscessa of the centre of the circle; f = 0.9 t. cm².

These stresses indicated on a properly oriented element are shown in Fig. 10.14 b.

As a check, the angle between the planes of principal normal stresses and those of maximum shear is $40^0\,57' + 4^0\,3' = 45^0$ as it shoud be.

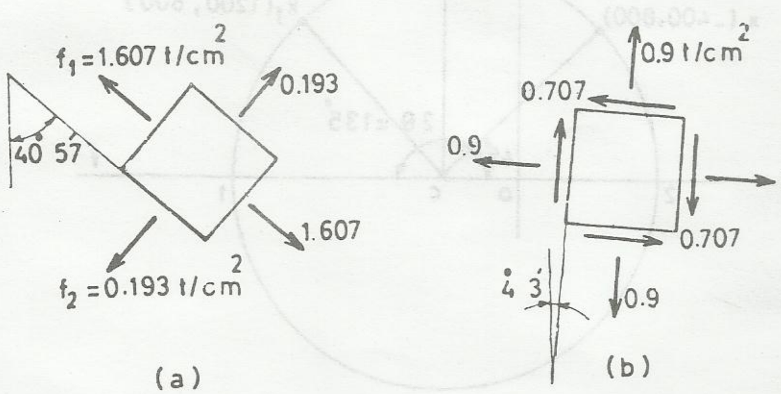



Fig. 10.14

**Example 10.5** A state of stress at a point in a member is defined by the element shown in Fig. 10.15. Construct Mohr's circle of stress for this element and find :

(1) the magnitudes and directions of the principal stresses.

(2) the normal and shear stresses on plane ab.

(3) the magnitude and direction of the maximum shear stress, and the associated normal stress.

Indicate the results on properly oriented elements.

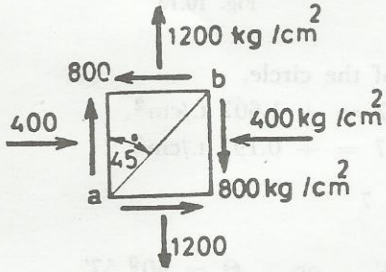



Fig. 10.15

Solution :

Mohr's circle of stress is plotted from the three given stress components; $f_x = -400$, $f_y = 1200$ and $q = 800$ The centre of the circle lies on the f-axis at a distance $\frac{1}{2}(f_x + f_y) = \frac{1}{2}(-400 + 1200) = 400$, from the origin 0. Point x on the circle has the co-ordinates $(f_x, q)$ or

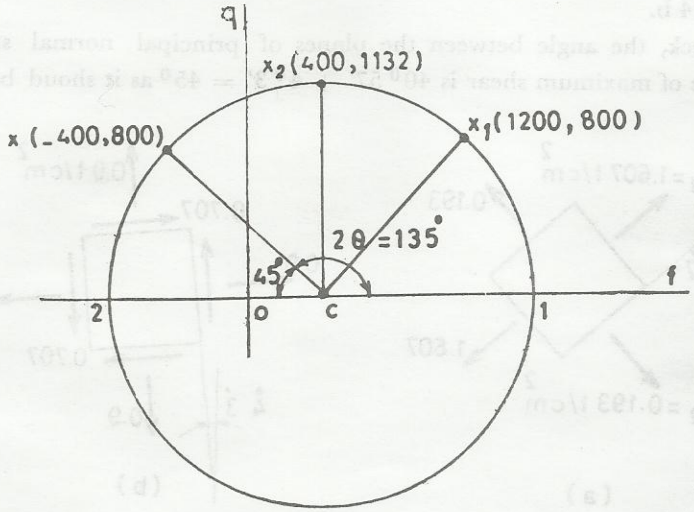



Fig. 10.16

(— 400,800) as shown in Fig. 10.16. Therefore, the radius of the circle is given by :

$$R = \sqrt{800^2 + 800^2} = 800\sqrt{2} = 1132$$

$$\text{and } 2\Theta = 135^0$$

The principal stresses are eoual to the distance from the origin 0 to the centre of the circle c plus or minus the radius. Thus,

$$f_1 = 400 + 1132 = 1532 \text{ kg./cm}^2.$$

$$f_2 = 400 - 1132 = 732 \text{ kg./cm}^2.$$

Since point 1 on the circle is clockwise from point x, the plane on which $f_1$ acts is clockwise from the plane on which the stress — 400 kg./cm². acts and at an angle $\Theta = 135/2 = 67^0\ 30'$. The direction of $f_2$ is of course normal to that of $f_1$. The principal stresses and their directions are shown in Fig. 10.17 a.

The normal and shear stresses on plane ab are obtained from the co-ordinates of point $x_1$ on the circle. Since plane ab is $45^0$ clockwise from the plane on which the stress — 400 kg./cm². acts and which is represented by point x, point $x_1$ will be $90^0$ cloockwise from x. Thus, from the geometry of the circle in Fig. 10.16, the normal and shear stresses on plane ab are given by :

$$f = 400 + 800\sqrt{2}\cos 45 = 1200 \text{ kg./cm}^2.$$

$$q = 800\sqrt{2}\sin 45 = 800 \text{ kg./cm}^2.$$

The stresses on plane ab and the associated stresses on a plane normal to it are shown in Fig. 10.17 b.

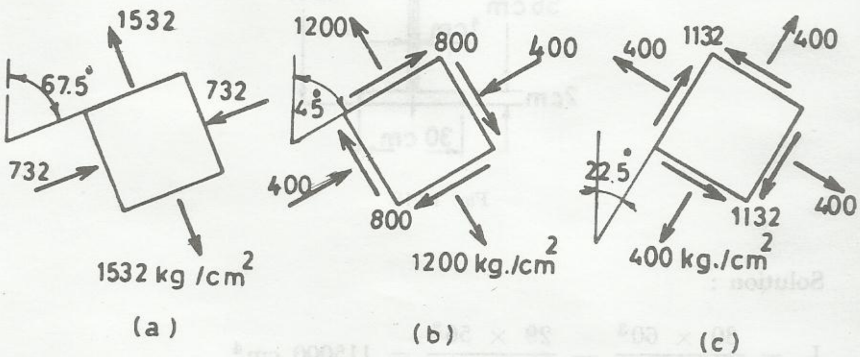



Fig. 10.17

Reference to Fig. 10.16 shows that point $x_2$ on the circle represents the plane on which the maximum positive shear stress occurs. Then, since point $x_2$ on the circle is $45^0$ clockwise from point x, the plane of the maximum shear stress makes an angle of $22^0$ 30′ clockwise from the plane on which the stress of — 400 kg./cm². acts. The magnitude of the maximum shear stress is equal to the radius of the circle. Thus,

$$q_{max} = 1132 \text{ kg./cm}^2.$$

The abscessa of the centre of the circle gives the normal stress associated with this shear stress; f = 400 kg./cm². These stresses are shown on the element in Fig. 10.17 c. As a check, it will be noted that the angle between the principal planes and the planes of maximum shear stresses is $45^0$ as it should be.

**Example 10.6** The cross-section of a beam is an I-section of the dimensions shown in Fig. 10.18. At a certain section along the beam, the bending moment is 45 m.t. and the shearing force is 42 t. For a point at the junction between the web and the top flange, find :

(1) the normal stress.

(2) the shear stress.

(3) the principal normal stresses.

(4) the maximum shear stress.

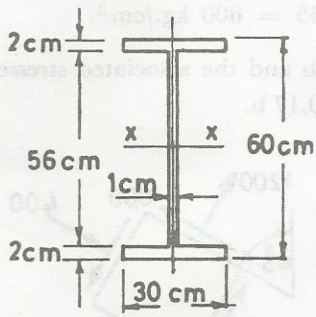



Fig. 10.18

Solution :

$$I_x = \frac{30 \times 60^3}{12} - \frac{29 \times 56^3}{12} = 115000 \text{ cm}^4.$$

$$f = \frac{4500 \times 28}{115000} = -1.1 \text{ t./cm}^2.$$

$$q = \frac{42\ (30 \times 2 \times 29)}{115000 \times 1} = 0.63 \text{ t./cm}^2.$$

The principal stresses are then,

$$f_1 = -0.55 + \sqrt{0.55^2 + 0.68^2}$$
$$= -0.55 + 0.83 = +0.28 \text{ t./cm}^2.$$
$$f_2 = -0.55 - 0.83 = -1.38 \text{ t./cm}^2.$$

The principal maximum shear is given by :

$$q = \frac{0.28 + 1.38}{2} = 0.83 \text{ t./cm}^2.$$

The maximum normal stress due to bending moment alone occurs at the top fiber,

$$f_{max} = -1.1 \times \frac{30}{28} = -1.18 \text{ t./cm}^2.$$

Thus the greatest principal normal stress $f_2$ is 17% greater than $f_{max}$. The maximum shear stress due to shearing force alone occurs at the middle fiber of the section,

$$q_{max} = \frac{42\ (30 \times 2 \times 29 + 28 \times 1 \times 14)}{115000 \times 1} = 0.78 \text{ t./cm}^2.$$

Thus the principal shear stress q is about 6% greater than $q_{max}$.

**Example 10.7** Fig. 10.19 shows a shaft of hollow circular section, 16 cm. and 12 cm. external and internal diameters. The shaft is subjected to an axial compressive force of 40 t. and a torque of 4 m.t. as shown. Find the magnitudes and directions of the principal normal stresses at any point on the surface of the shaft. What is the percentage increase in the shear stress due to the axial force ?

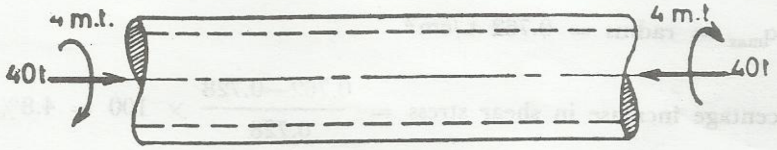



Fig. 10.19

Solution :

$A = \pi (8^2 - 6^2) = 88 \text{ cm}^2.$

$J = \pi (8^4 - 6^4)/2 = 4400 \text{ cm}^4.$

$f = -40/88 = -0.455 \text{ t./cm}^2$

$$q = \frac{-400 \times 8}{4400} = -0.728 \text{ t./cm}^2.$$

These stress components are indicated on the element in Fig. 10.20 a. The corresponding Mohr's circle of stress is shown in Fig. 10.20 b; centre at (— 0.227,0) and radius of 0.762.

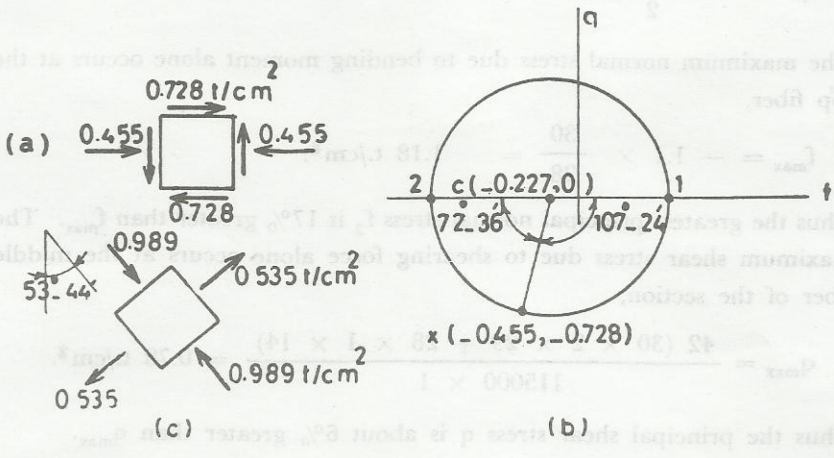



Fig. 10.20

The principal stresses are shown on a properly oriented element in Fig. 10.20 c.

From the geometry of the circle, the maximum principal shear stress is given by :

$q_{max} = \text{radius} = 0.762 \text{ t./cm}^2.$

$$\text{percentage increase in shear stress} = \frac{0.762-0.728}{0.728} \times 100 = 4.8\%$$

## EXAMPLES TO BE WORKED OUT

(1) Indicate on small elements the stress components at the locations on the structures shown in Figs. 10.21-10.24.

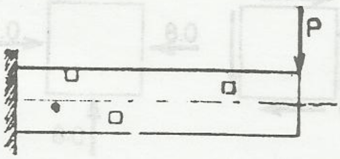



Fig. 10.21

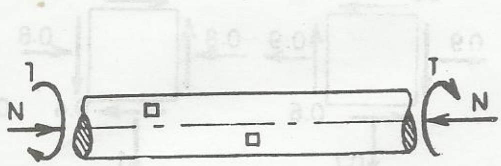



Fig. 10.22

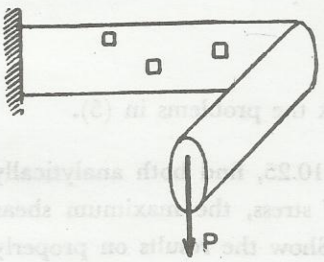



Fig 10.23

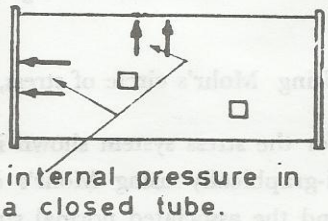



Fig. 10.24

(2) For the elements shown in Fig. 10.25, find from the first principles the equivalent normal and shear stresses acting on the indicated inclined planes.

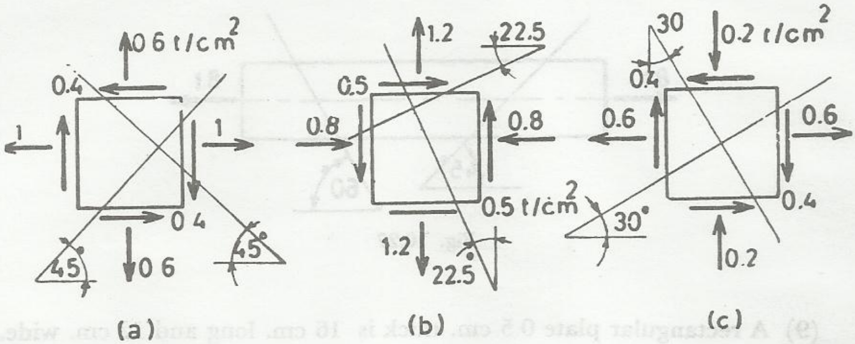



Fig. 10.25

(3) Using equations 10.3 and 10.4, re-work the problems in (2).

(4) Using Mohr's circle of stress, re-work the problem in (2).

(5) For the stress systems shown in Fig. 10.26, calculate the principal normal stresses and show their directions on properly oriented elements. Calculate in each case the value of the maximum shear stress.

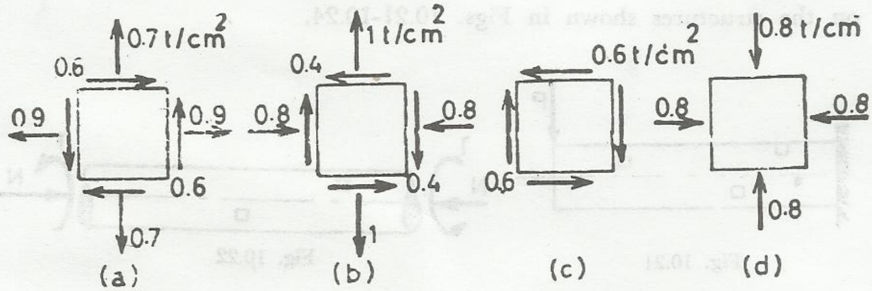



Fig. 10.26

(6) Using Mohr's circle of stress, re-work the problems in (5).

(7) For the stress system shown in Fig. 10.25, find both analytically and semi-graphically using Mohr's circle of stress, the maximum shear stresses and the associated normal stresses. Show the results on properly oriented elements.

(8) A steel bar of 8 cm². cross-sectional area is subjected to an axial tension as shown in Fig. 10.27. Calculate the normal and shear stresses on the two inclined planes indicated.

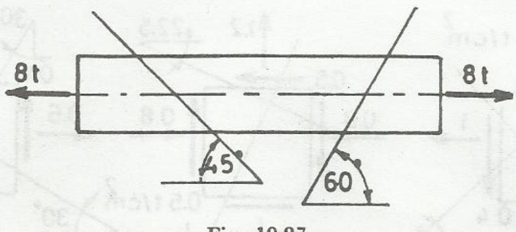



Fig. 10.27

(9) A rectangular plate 0.5 cm. thick is 16 cm. long and 12 cm. wide. A tensile force of 6 t., uniformly distributed along the longer side of the plate, is applied to the plate as shown in Fig. 10.28. Find the normal and shear stresses on planes across the diagonals. What total force on the other two sides of the plate will create a state of pure shear stress in the plate ?

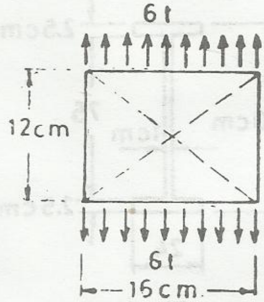



Fig. 10.28

(10) A rivet 5 $cm^2$. cross-sectional area is subjected to a shearing force of 4 t. and tension of 3 t. Determine the magnitudes of the greatest tensile and shear stresses in the rivet.

(11) A solid circular shaft 8 cm. diameter is subjected to a torque of 800 cm.t. and an axial tension of 12.5 t. as shown in Fig. 10.29. Find the magnitude and direction of the principal stresses on the surface of the shaft. Indicate the result on a properly oriented element.

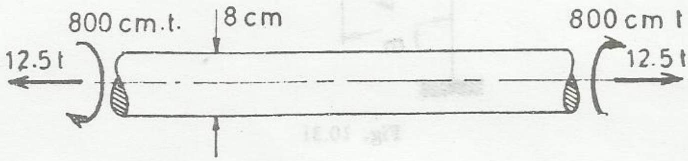



Fig. 10.29

(12) A hollow circular shaft, 12 cm. and 8 cm. external and internal diameters is subjected to a torque of 2 m.t. and a thrust of 20 t. What is the shear stress due to the torque alone and the percentage increase when the thrust is considered ?

(13) A steel tube 8 cm. internal diameter and 0.4 cm. thick has closed ends and is subjected to an intrenal pressure of 100 kg./$cm^2$. Calculate the magnitude of maximum shear stress and indicate its direction on the surface of the tube.

(14) A girder has an I-section of the dimensions shown in Fig. 10.30. At a certain section it carries a positive moment of 60 m.t. and a shearing force of 40 t. For a point on this section at the junction between the web and the top flange, determine the normal stress and the percentage increase when shear is considered.

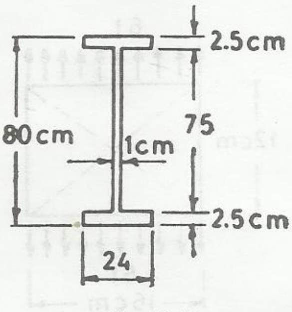



Fig. 10.30

(15) The cantilever shown in Fig. 10.31 has a circular cross-section 20 cm. diameter. Find the maximum tensile normal stress at section a-a due to the bending moment alone and the percentage increase when the torque is considered.

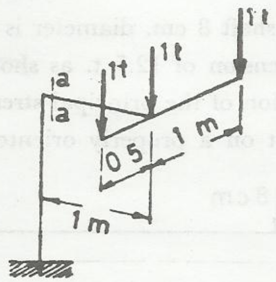



Fig. 10.31

(16) A cylindrical pressure vessel 50 cm. diameter with walls 1 cm. thick is subjected to an internal pressure of 40 kg./cm². If the plates are butt-welded as shown in Fig. 10.32, determine the normal and shearing forces that have to be carried per 1 cm. length of the weld.

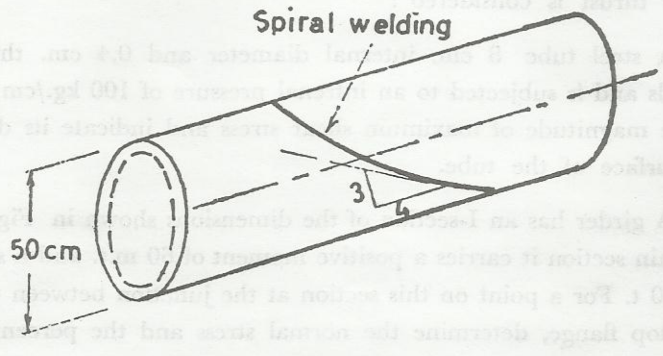



Fig. 10.32

(17) A cylindrical pressure vessel 160 cm. diameter and 1 cm. thick is subjected to a uniform internal fluid pressure of 40 kg./cm$^2$. and supported as shown in Fig. 10.33. If the weight of the vessel and the contained fluid is 1.5 t./m., find the magnitudes of the principal stresses at the extreme upper and lower fibers of the middle section.

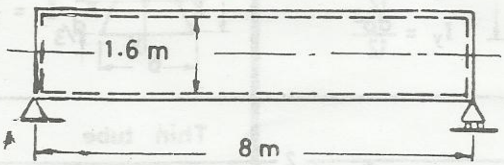



Fig. 10.33

(18) Fig. 10.34 shows an advertising post. If the wind pressure on the board is 100 kg./m$^2$. and its own weight is 0.25 t., indicate the straining actions on section a-a of the supporting post. If the post has a hollow square section of mean side length 25 cm. and uniform thickness of 0.5 cm., calculate for four points diametrically opposite, the normal stresses, shear stresses and the principal normal stresses. Indicate your results on properly oriented elements.

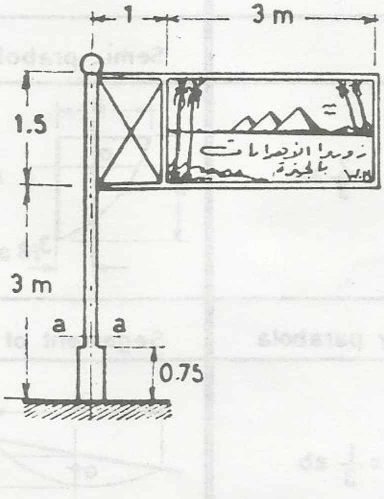
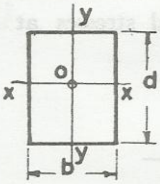
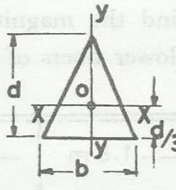
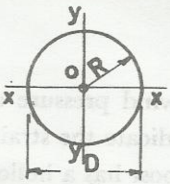
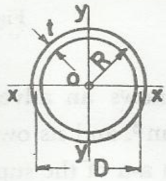
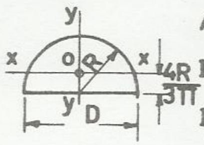
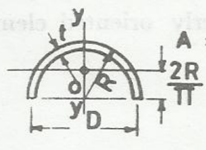
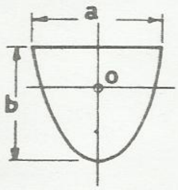
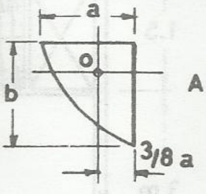
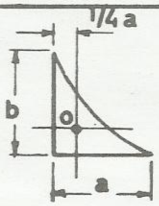
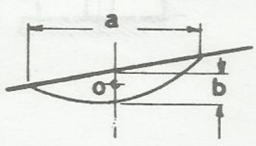



Fig. 10.34

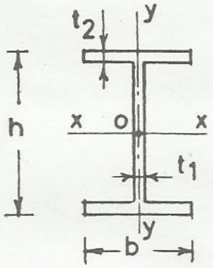
## Appendix 1

## GEOMETRICAL PROPERTIES OF PCANE SECTIONS

| Rectangle | Triangle |
|---|---|
| <br>$A = bd$<br>$I_x = \frac{bd^3}{12}$<br>$I_y = \frac{db^3}{12}$ | <br>$A = \frac{1}{2}bd$<br>$I_x = \frac{bd^3}{36}$<br>$I_y = \frac{db^3}{48}$ |
| **Circle** | **Thin tube** |
| <br>$A = \pi R^2 = \frac{\pi D^2}{4}$<br>$I_x = \frac{\pi R^4}{4} = \frac{\pi D^4}{64}$<br>$I_y = I_x$<br>$J = \frac{\pi R^4}{2} = \frac{\pi D^4}{32}$ | <br>$A = 2\pi Rt = \pi Dt$<br>$I_x = \pi R^3 t = \frac{\pi D^3 t}{8}$<br>$I_y = I_x$<br>$J = 2\pi R^3 t = \frac{\pi D^3 t}{4}$ |
| **Semi_circle** | **Half thin tube** |
| <br>$A = \frac{\pi R^2}{2} = \frac{\pi D^2}{8}$<br>$I_x = 0.11 R^4$<br>$I_y = \frac{\pi R^4}{8}$ | <br>$A = \pi Rt$ |
| **Parabola** | **Semi_prabola** |
| <br>$A = \frac{2}{3}ab$ | <br>$A = \frac{2}{3}ab$ |
| **Complementary parabola** | **Segement of a parabola** |
| <br>$A = \frac{1}{3}ab$ | <br>$A = \frac{2}{3}ab$ |

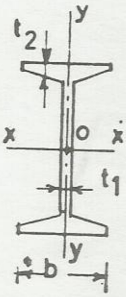
# Appendix 2

# PROPERTIES OF ROLLED STEEL SECTIONS



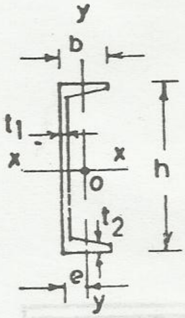

## BROAD FLANGE I-BEAMS

| Size | Dimensions h mm. | b mm. | $t_1$ mm. | $t_2$ mm. | Area $cm^2$ | Weight Kg/m. | x-axis $I_x$ $cm^4$ | $Z_x$ $cm^3$ | $i_x$ cm. | y-axis $I_y$ $cm^4$ | $Z_y$ $cm^3$ | $i_y$ cm. |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 10 | 100 | 100 | 6 | 10 | 26.1 | 20.5 | 447 | 89.3 | 4.14 | 167 | 33.4 | 2.53 |
| 12 | 120 | 120 | 7 | 11 | 34.3 | 26.9 | 864 | 144.0 | 5.02 | 317 | 53.0 | 3.04 |
| 14 | 140 | 140 | 8 | 12 | 44.1 | 34.6 | 1520 | 217.0 | 5.87 | 550 | 79.0 | 3.53 |
| 16 | 160 | 160 | 9 | 14 | 58.4 | 45.8 | 2630 | 329.0 | 6.72 | 958 | 120.0 | 4.05 |
| 18 | 180 | 180 | 9 | 14 | 65.3 | 51.6 | 3840 | 426.0 | 7.63 | 1360 | 151.0 | 4.55 |
| 20 | 200 | 200 | 10 | 16 | 82.7 | 64.9 | 5950 | 595.0 | 8.48 | 2140 | 214.0 | 5.08 |
| 22 | 220 | 220 | 10 | 16 | 91.1 | 71.5 | 8050 | 732.0 | 9.37 | 2840 | 258.0 | 5.59 |
| 24 | 240 | 240 | 11 | 18 | 111.0 | 87.4 | 11690 | 974.0 | 10.30 | 4150 | 346.0 | 6.11 |
| 26 | 260 | 260 | 11 | 18 | 121.0 | 94.8 | 15050 | 1160.0 | 11.20 | 5280 | 406.0 | 6.61 |
| 28 | 280 | 280 | 12 | 20 | 144.0 | 113.0 | 20720 | 1480.0 | 12.00 | 7320 | 523.0 | 7.14 |
| 30 | 300 | 300 | 12 | 20 | 154.0 | 121.0 | 25760 | 1720.0 | 12.90 | 9010 | 600.0 | 7.65 |
| 32 | 320 | 320 | 13 | 22 | 171.0 | 135.0 | 32250 | 2020.0 | 13.70 | 9910 | 661.0 | 7.60 |
| 34 | 340 | 300 | 13 | 22 | 174.0 | 137.0 | 36940 | 2170.0 | 14.50 | 9910 | 661.0 | 7.55 |
| 36 | 360 | 300 | 14 | 24 | 192.0 | 150.0 | 45120 | 2510.0 | 15.30 | 10810 | 721.0 | 7.51 |
| 38 | 380 | 300 | 14 | 24 | 194.0 | 153.0 | 50950 | 2680.0 | 16.20 | 10810 | 721.0 | 7.46 |
| 40 | 400 | 300 | 14 | 26 | 209.0 | 164.0 | 60640 | 3030.0 | 17.00 | 11710 | 781.0 | 7.49 |
| 42.5 | 425 | 300 | 14 | 26 | 212.0 | 166.0 | 69480 | 3270.0 | 18.10 | 11710 | 781.0 | 7.43 |
| 45 | 450 | 300 | 15 | 28 | 232.0 | 182.0 | 84220 | 3740.0 | 19.00 | 12620 | 841.0 | 7.38 |
| 47.5 | 475 | 300 | 15 | 28 | 235.0 | 185.0 | 95120 | 4010.0 | 20.10 | 12620 | 841.0 | 7.32 |
| 50 | 500 | 300 | 16 | 30 | 255.0 | 200.0 | 113200 | 4530.0 | 21.00 | 13530 | 902.0 | 7.28 |
| 55 | 550 | 300 | 16 | 30 | 263.0 | 207.0 | 140300 | 5100.0 | 23.10 | 13530 | 902.0 | 7.17 |
| 60 | 600 | 300 | 17 | 32 | 289.0 | 227.0 | 180800 | 6030.0 | 25.00 | 14440 | 962.0 | 7.07 |
| 65 | 650 | 300 | 17 | 32 | 297.0 | 234.0 | 216800 | 6670.0 | 27.00 | 14440 | 962.0 | 6.97 |
| 70 | 700 | 300 | 18 | 34 | 324.0 | 254.0 | 270300 | 7720.0 | 28.90 | 15350 | 1020.0 | 6.88 |
| 75 | 750 | 300 | 18 | 34 | 333.0 | 261.0 | 316300 | 8430.0 | 30.80 | 15350 | 1020.0 | 6.79 |
| 80 | 800 | 300 | 18 | 34 | 342.0 | 268.0 | 366400 | 9160.0 | 32.70 | 15350 | 1020.0 | 6.70 |
| 85 | 850 | 300 | 19 | 36 | 372.0 | 292.0 | 443900 | 10440.0 | 34.60 | 16270 | 1080.0 | 6.61 |
| 90 | 900 | 300 | 19 | 36 | 381.0 | 299.0 | 506000 | 11250.0 | 36.40 | 16270 | 1080.0 | 6.53 |
| 95 | 950 | 300 | 19 | 36 | 391.0 | 307.0 | 573000 | 12060.0 | 38.30 | 16270 | 1080.0 | 6.45 |
| 100 | 1000 | 300 | 19 | 36 | 400.0 | 314.0 | 644700 | 12900.0 | 40.10 | 16280 | 1080.0 | 6.37 |



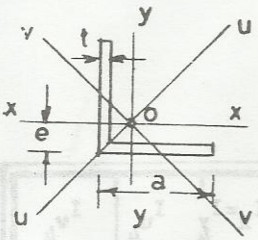

## STANDARD I-BEAMS

| Size | Dimensions h mm. | b mm. | $t_1$ mm. | $t_2$ mm. | Area $cm^2$ | Weight Kg/m. | x-axis $I_x$ $cm^4$ | $Z_x$ $cm^3$ | $i_x$ cm. | y-axis $I_y$ $cm^4$ | $Z_y$ $cm^3$ | $i_y$ cm. |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 12 | 120 | 58 | 5.1 | 7.7 | 14.2 | 11.2 | 328 | 54.7 | 4.81 | 21.5 | 7.41 | 1.23 |
| 14 | 140 | 66 | 5.7 | 8.6 | 18.3 | 14.4 | 573 | 81.9 | 5.61 | 35.2 | 10.70 | 1.40 |
| 16 | 160 | 74 | 6.3 | 9.5 | 22.8 | 17.9 | 935 | 117.0 | 6.40 | 54.2 | 14.80 | 1.55 |
| 18 | 180 | 82 | 6.9 | 10.4 | 27.9 | 21.9 | 1450 | 161.0 | 7.20 | 81.3 | 19.80 | 1.71 |
| 20 | 200 | 90 | 7.5 | 11.3 | 33.5 | 26.3 | 2140 | 214.0 | 8.00 | 117.0 | 26.00 | 1.87 |
| 22 | 220 | 98 | 8.1 | 12.2 | 39.6 | 31.1 | 3060 | 278.0 | 8.80 | 162.0 | 33.10 | 2.02 |
| 24 | 240 | 106 | 8.7 | 13.1 | 46.1 | 36.2 | 4250 | 354.0 | 9.95 | 221.0 | 41.70 | 2.20 |
| 26 | 260 | 113 | 9.4 | 14.1 | 53.4 | 41.9 | 5740 | 442.0 | 10.40 | 288.0 | 51.00 | 2.32 |
| 28 | 280 | 119 | 10.1 | 15.2 | 61.1 | 48.0 | 7590 | 542.0 | 11.10 | 364.0 | 61.20 | 2.45 |
| 30 | 300 | 125 | 10.8 | 16.2 | 69.1 | 54.2 | 9800 | 653.0 | 11.90 | 451.0 | 72.20 | 2.56 |
| 32 | 320 | 131 | 11.5 | 17.3 | 77.8 | 61.1 | 12510 | 782.0 | 12.70 | 555.0 | 84.70 | 2.67 |
| 34 | 340 | 137 | 12.2 | 18.3 | 86.8 | 68.1 | 15700 | 923.0 | 13.50 | 674.0 | 98.40 | 2.80 |
| 36 | 360 | 143 | 13.0 | 19.5 | 97.1 | 76.2 | 19610 | 1090.0 | 14.20 | 818.0 | 114.0 | 2.90 |
| 38 | 380 | 149 | 13.7 | 20.5 | 107.0 | 84.0 | 24010 | 1260.0 | 15.00 | 975.0 | 131.0 | 3.02 |
| 40 | 400 | 155 | 14.4 | 21.6 | 118.0 | 92.6 | 29210 | 1460.0 | 15.70 | 1160.0 | 149.0 | 3.15 |
| 42.5 | 425 | 163 | 15.3 | 23.0 | 132.0 | 104.0 | 36970 | 1740.0 | 16.70 | 1440.0 | 176.0 | 3.30 |
| 45 | 450 | 170 | 16.2 | 24.3 | 147.0 | 115.0 | 45850 | 2040.0 | 17.70 | 1730.0 | 203.0 | 3.43 |
| 47.5 | 475 | 178 | 17.1 | 25.6 | 163.0 | 128.0 | 56480 | 2380.0 | 18.60 | 2090.0 | 235.0 | 3.60 |
| 50 | 500 | 185 | 18.0 | 27.0 | 180.0 | 141.0 | 68740 | 2750.0 | 19.60 | 2480.0 | 268.0 | 3.72 |
| 55 | 550 | 200 | 19.0 | 30.0 | 213.0 | 167.0 | 99180 | 3610.0 | 21.60 | 3490.0 | 349.0 | 4.02 |
| 60 | 600 | 215 | 21.6 | 32.4 | 254.0 | 199.0 | 139000 | 4630.0 | 23.40 | 4670.0 | 434.0 | 4.30 |




## CHANNELS

| Size | Dimensions h mm. | b mm. | $t_1$ mm. | $t_2$ mm. | Area $cm^2$ | Weight Kg/m. | e mm. | x-axis $I_x$ $cm^4$ | $Z_x$ $cm^3$ | $i_x$ cm. | y-axis $I_y$ $cm^4$ | $Z_y$ $cm^3$ | $i_y$ cm. |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 8 | 80 | 45 | 6 | 8 | 11.0 | 8.64 | 14.5 | 106 | 26.5 | 3.10 | 19.4 | 6.36 | 1.33 |
| 10 | 100 | 50 | 6 | 8.5 | 13.5 | 10.60 | 15.5 | 206 | 41.2 | 3.90 | 29.3 | 8.49 | 1.44 |
| 12 | 120 | 55 | 7 | 9 | 17.0 | 13.40 | 16.0 | 364 | 60.7 | 4.63 | 43.2 | 11.1 | 1.59 |
| 14 | 140 | 60 | 7 | 10 | 20.4 | 16.00 | 17.5 | 605 | 86.4 | 5.44 | 62.7 | 14.8 | 1.75 |
| 16 | 160 | 65 | 75 | 10.5 | 24.0 | 18.80 | 18.4 | 925 | 116.0 | 6.20 | 85.3 | 18.3 | 1.89 |
| 18 | 180 | 70 | 8 | 11 | 28.0 | 22.00 | 19.2 | 1350 | 150.0 | 6.94 | 114.0 | 22.4 | 2.02 |
| 20 | 200 | 75 | 85 | 11.5 | 32.2 | 25.30 | 20.1 | 1910 | 191.0 | 7.70 | 148.0 | 27.0 | 2.14 |
| 22 | 220 | 80 | 9 | 12.5 | 37.4 | 29.40 | 21.4 | 2690 | 245.0 | 8.49 | 197.0 | 33.6 | 2.29 |
| 24 | 240 | 85 | 95 | 13 | 42.3 | 33.20 | 22.3 | 3600 | 300.0 | 9.22 | 248.0 | 39.6 | 2.42 |
| 26 | 260 | 90 | 10 | 14 | 48.3 | 37.90 | 23.6 | 4820 | 371.0 | 10.00 | 317.0 | 47.7 | 2.56 |
| 28 | 280 | 95 | 10 | 15 | 53.3 | 41.80 | 25.3 | 6280 | 448.0 | 10.81 | 399.0 | 57.2 | 2.74 |
| 30 | 300 | 100 | 10 | 16 | 58.8 | 46.20 | 27.0 | 8030 | 535.0 | 11.70 | 495.0 | 67.8 | 2.90 |
| 32 | 320 | 100 | 14 | 17.5 | 75.8 | 59.50 | 26.0 | 10870 | 679.0 | 12.00 | 597.0 | 80.6 | 2.81 |
| 35 | 350 | 100 | 14 | 16 | 77.3 | 60.60 | 24.0 | 12840 | 734.0 | 12.80 | 570.0 | 75.0 | 2.71 |
| 38 | 380 | 102 | 134 | 16 | 79.7 | 62.60 | 23.5 | 15730 | 826.0 | 14.08 | 613.0 | 78.4 | 2.78 |
| 40 | 400 | 110 | 14 | 18 | 91.5 | 71.80 | 26.5 | 20350 | 1020.0 | 14.90 | 846.0 | 102.0 | 3.04 |




## EQUAL ANGLES

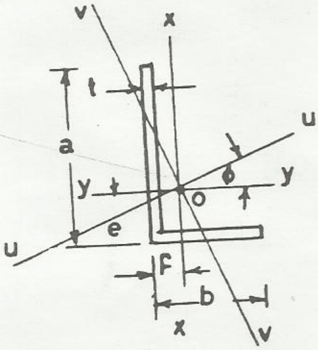
| Size | Dimns. a mm. | Dimns. t mm. | Area cm.$^2$ | Weight Kg/m. | e mm. | $I_x=I_y$ cm.$^4$ | $I_u$ cm.$^4$ | $I_v$ cm.$^4$ |
|---|---|---|---|---|---|---|---|---|
| 45x45x5 | 45 | 5 | 4.30 | 3.38 | 12.8 | 7.33 | 12.4 | 3.25 |
| 45x45x7 | 45 | 7 | 5.86 | 4.60 | 13.6 | 10.4 | 16.4 | 4.39 |
| 50x50x5 | 50 | 5 | 4.80 | 3.77 | 14.4 | 11.0 | 17.4 | 4.59 |
| 50x50x7 | 50 | 7 | 6.56 | 5.15 | 14.9 | 14.6 | 23.1 | 6.02 |
| 50x50x9 | 50 | 9 | 8.24 | 6.47 | 15.6 | 17.9 | 28.1 | 7.67 |
| 55x55x6 | 55 | 6 | 6.31 | 4.95 | 15.6 | 17.3 | 27.4 | 7.24 |
| 55x55x8 | 55 | 8 | 8.23 | 6.46 | 16.4 | 22.1 | 34.8 | 9.35 |
| 55x55x10 | 55 | 10 | 10.10 | 7.90 | 17.2 | 26.3 | 41.4 | 11.30 |
| 60x60x6 | 60 | 6 | 6.91 | 5.42 | 16.9 | 22.8 | 36.1 | 9.43 |
| 60x60x8 | 60 | 8 | 9.03 | 7.09 | 17.7 | 29.1 | 46.1 | 12.10 |
| 60x60x10 | 60 | 10 | 11.10 | 8.69 | 18.5 | 34.9 | 55.1 | 14.60 |
| 65x65x7 | 65 | 7 | 8.70 | 6.83 | 18.5 | 33.4 | 53.0 | 13.80 |
| 65x65x9 | 65 | 9 | 11.00 | 8.62 | 19.3 | 41.3 | 65.4 | 17.20 |
| 65x65x11 | 65 | 11 | 13.20 | 10.30 | 20.0 | 48.8 | 76.8 | 20.70 |
| 70x70x7 | 70 | 7 | 9.4 | 7.38 | 19.7 | 42.4 | 67.1 | 17.60 |
| 70x70x9 | 70 | 9 | 11.9 | 9.34 | 20.5 | 52.6 | 83.1 | 22.00 |
| 70x70x11 | 70 | 11 | 14.3 | 11.20 | 21.3 | 61.8 | 97.6 | 26.00 |
| 75x75x8 | 75 | 8 | 11.5 | 9.03 | 21.3 | 58.9 | 93.3 | 24.40 |
| 75x75x10 | 75 | 10 | 14.1 | 11.10 | 22.1 | 71.4 | 113.0 | 29.80 |
| 75x75x12 | 75 | 12 | 16.7 | 13.10 | 22.9 | 82.4 | 130.0 | 34.70 |
| 80x80x8 | 80 | 8 | 12.3 | 9.66 | 22.6 | 72.3 | 115.0 | 29.60 |
| 80x80x10 | 80 | 10 | 15.1 | 11.90 | 23.4 | 87.5 | 139.0 | 35.90 |
| 80x80x12 | 80 | 12 | 17.9 | 14.10 | 24.1 | 102.0 | 161.0 | 43.00 |
| 90x90x9 | 90 | 9 | 15.5 | 12.20 | 25.4 | 116.0 | 184.0 | 47.80 |
| 90x90x11 | 90 | 11 | 18.7 | 14.70 | 26.2 | 138.0 | 218.0 | 57.10 |
| 90x90x13 | 90 | 13 | 21.8 | 17.10 | 27.0 | 158.0 | 250.0 | 65.90 |

NOTE: Approximate radii of gyration are:
$i_x = 0.30\ a$ $\qquad$ $i_v = 0.20\ a$

**EQUAL ANGLES (Cont.)**

| Size | Dimns. a mm. | Dimns. t mm. | Area cm.$^2$ | Weight Kg/m. | e mm. | $I_x=I_y$ cm.$^4$ | $I_u$ cm.$^4$ | $I_v$ cm.$^4$ |
|---|---|---|---|---|---|---|---|---|
| 100x100x10 | 100 | 10 | 19.2 | 15.1 | 28.2 | 177 | 280 | 73.3 |
| 100x100x12 | 100 | 12 | 22.7 | 17.8 | 29.0 | 207 | 328 | 86.2 |
| 100x100x14 | 100 | 14 | 26.2 | 20 6 | 29.8 | 235 | 372 | 98.3 |
| 110x110x10 | 110 | 10 | 21.2 | 16.6 | 30.7 | 239 | 379 | 98.6 |
| 110x110x12 | 110 | 12 | 25.1 | 19.7 | 31.5 | 280 | 444 | 116.0 |
| 110x110x14 | 110 | 14 | 29.0 | 22.8 | 32.1 | 319 | 505 | 133.0 |
| 120x120x11 | 120 | 11 | 25.4 | 19.9 | 33.6 | 341 | 541 | 140.0 |
| 120x120x13 | 120 | 13 | 29.7 | 23.3 | 34.4 | 394 | 625 | 162.0 |
| 120x120x15 | 120 | 15 | 33.9 | 26.6 | 35.1 | 446 | 705 | 186.0 |
| 130x130x12 | 130 | 12 | 30.0 | 23.6 | 36.4 | 472 | 750 | 194.0 |
| 130x130x14 | 130 | 14 | 34.7 | 27.2 | 37.2 | 540 | 857 | 243.0 |
| 130x130x16 | 130 | 16 | 39.3 | 30.9 | 38.0 | 605 | 959 | 251.0 |
| 140x140x13 | 140 | 13 | 35.0 | 27.5 | 39.2 | 638 | 1010 | 262.0 |
| 140x140x15 | 140 | 15 | 40.0 | 31.4 | 40.0 | 723 | 1150 | 298.0 |
| 140x140x17 | 140 | 17 | 45.0 | 35.3 | 40.8 | 805 | 1280 | 334.0 |
| 150x150x14 | 150 | 14 | 40.3 | 31.6 | 42.1 | 845 | 1340 | 347.0 |
| 150x150x16 | 150 | 16 | 45.7 | 35.9 | 42.9 | 949 | 1510 | 391.0 |
| 150x150x18 | 150 | 18 | 51.0 | 40.1 | 43.6 | 1050 | 1670 | 438.0 |
| 160x160x15 | 160 | 15 | 46.1 | 36.2 | 44.9 | 1100 | 1750 | 453.0 |
| 160x160x17 | 160 | 17 | 51.8 | 40.7 | 45.7 | 1230 | 1950 | 506.0 |
| 160x160x19 | 160 | 19 | 57.5 | 45.1 | 46.5 | 1350 | 2140 | 558.0 |
| 180x180x16 | 180 | 16 | 55.4 | 43.5 | 50.2 | 1680 | 2690 | 679.0 |
| 180x180x18 | 180 | 18 | 61.9 | 48.6 | 51.0 | 1870 | 2930 | 757.0 |
| 180x180x20 | 180 | 20 | 68.4 | 53.7 | 51.8 | 2040 | 3260 | 830.0 |
| 200x200x16 | 200 | 16 | 61.8 | 48.5 | 55.2 | 2340 | 3740 | 943.0 |
| 200x200x18 | 200 | 18 | 69.1 | 54.3 | 56.0 | 2600 | 4150 | 1050.0 |
| 200x200x20 | 200 | 20 | 76.4 | 59.9 | 56.8 | 2850 | 4540 | 1160.0 |

NOTE: Approximate radii of gyration are:
$i_x$=0.30 a $i_v$=0.20 a




## UNEQUAL ANGLES

| Size | Dimns. mm. | | | Area | Weight | Dist. of G mm. | | tan Ø | Moments of inertia cm.$^4$ | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | b | a | t | cm.$^2$ | Kg/m. | f | e | | $I_y$ | $I_x$ | $I_u$ | $I_v$ |
| 30x45x5 | 30 | 45 | 5 | 3.53 | 2.77 | 7.3 | 15.2 | .430 | 7.0 | 2.5 | 8.0 | 1.44 |
| 40x60x5 | 40 | 60 | 5 | 4.79 | 3.76 | 9.7 | 19.6 | .437 | 17.2 | 6.1 | 19.8 | 3.50 |
| 40x60x7 | 40 | 60 | 7 | 6.55 | 5.14 | 10.5 | 20.4 | .429 | 23.0 | 8.1 | 26.3 | 4.73 |
| 50x75x7 | 50 | 75 | 7 | 8.33 | 6.50 | 12.4 | 24.7 | .430 | 40.3 | 16.4 | 53.1 | 9.58 |
| 50x75x9 | 50 | 75 | 9 | 10.50 | 8.20 | 13.2 | 25.6 | .427 | 57.2 | 20.1 | 65.4 | 11.90 |
| 65x100x9 | 65 | 100 | 9 | 14.2 | 11.10 | 15.9 | 33.2 | .415 | 140.0 | 46.6 | 160.0 | 26.80 |
| 65x100x11 | 65 | 100 | 11 | 17.1 | 13.4 | 16.7 | 34.0 | .410 | 167.0 | 55.3 | 189.0 | 32.90 |
| 80x120x10 | 80 | 120 | 10 | 19.1 | 15.0 | 19.5 | 39.2 | .438 | 276.0 | 98.1 | 318.0 | 56.10 |
| 80x120x12 | 80 | 120 | 12 | 22.7 | 17.8 | 20.3 | 40.0 | .433 | 323.0 | 114.0 | 371.0 | 66.10 |
| 100x150x12 | D0 | 150 | 12 | 28.7 | 22.6 | 24.2 | 48.9 | .439 | 650.0 | 232.0 | 749.0 | 132.0 |
| 100x150x14 | D0 | 150 | 14 | 33.2 | 26.1 | 25.0 | 49.7 | .435 | 744.0 | 264.0 | 856.0 | 152.0 |
| 30x60x5 | 30 | 60 | 5 | 4.29 | 3.37 | 6.8 | 21.5 | .256 | 15.6 | 2.6 | 16.5 | 1.69 |
| 30x60x7 | 30 | 60 | 7 | 5.85 | 4.59 | 7.6 | 22.4 | .248 | 20.7 | 3.41 | 21.8 | 2.28 |
| 40x80x6 | 40 | 80 | 6 | 6.89 | 5.31 | 8.8 | 28.5 | .259 | 44.9 | 7.59 | 47.6 | 4.99 |
| 40x80x8 | 40 | 80 | 8 | 9.01 | 7.07 | 9.5 | 29.4 | .253 | 57.6 | 9.68 | 60.9 | 6.41 |
| 50x100x8 | 50 | 100 | 8 | 11.50 | 8.99 | 11.3 | 35.9 | .259 | 116.0 | 19.50 | 123 | 12.60 |
| 50x100x10 | 50 | 100 | 10 | 14.1 | 11.10 | 12.0 | 36.7 | .252 | 141.0 | 23.40 | 149 | 15.50 |
| 65x130x10 | 65 | 130 | 10 | 18.6 | 14.60 | 14.5 | 46.5 | .259 | 321.0 | 54.20 | 340 | 35.00 |
| 65x130x12 | 65 | 130 | 12 | 22.1 | 17.3 | 15.3 | 47.4 | .255 | 376.0 | 63.00 | 397 | 41.20 |
| 80x160x12 | 80 | 160 | 12 | 27.5 | 21.6 | 17.7 | 57.2 | .259 | 720.0 | 122.00 | 763 | 78.90 |

NOTE: Approximate radii of gyration are:
For b:a=1:1.5 angles, $i_y$=0.31 a and $i_x$=0.28 b
For b:a=1:2 angles, $i_y$=0.31 a and $i_x$=0.26 b

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