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ELASTIC SOLUTIONS FOR SOIL AND ROCK MECHANICS

by
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and
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ELASTIC SOLUTIONS FOR SOIL AND ROCK MECHANICS

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T. William Lambe
Robert V. Whitman

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PREFACE TO REPRINTED EDITION

The original edition of this book has been out of print for several years, but there have been many requests for it to be reprinted. The original publishers, John Wiley and Son Inc, New York, have been gracious in re-assigning copyright to the surviving author, and hence, the book is now being reprinted through the Centre for Geotechnical Research at the University of Sydney.

This reprinted edition contains a significant number of corrections which were brought to the author's attention by a number of users, in particular, Dr John T Christian, Dr Peter T Brown, Professor M R Madhav, Mr J M Shen, Sir Alan Muir Wood, Dr K J Cheverton, Professor Michael J Pender, Dr I D Moore, Associate Professor J C Small and Mr M A Adler. I am very grateful to these persons for their interest in bringing the errors to my attention. I am also grateful for the encouragement of my colleagues within the Centre for Geotechnical Research at the University of Sydney to prepare the corrected edition, and to Ms Monica Martin, who undertook the typing of the corrections and Miss Kim Pham for correcting the figures.

Harry G Poulos
August, 1991

PREFACE

The authors have attempted to assemble as comprehensive a collection as possible of graphs, tables and explicit solution of problems in elasticity relevant to soil and rock mechanics. Many of these solutions are well known and widely used in geotechnical practice, and are available in standard references. However, new solutions of relevance appear at frequent intervals and in diverse publications, and it is difficult for the practising engineer to locate, or even to know of the existence of, a solution which may be of interest. The large majority of solutions are for an isotropic homogeneous mass, but some important solutions are also included for cross-anisotropic and non-homogeneous elastic materials. Because of the vast literature in the theory of elasticity and the need to keep the book to a reasonable size, coverage of solutions in this book is by no means exhaustive, and solutions which may be considered of relevance by some people will doubtless have been omitted. In a number of instances, a reference is given even though no solution is reproduced in the book.

It has not been found practicable to maintain a uniform notation throughout the book; where there appeared to be valid reasons for doing so, the original author's notation has been adhered to, but particularly in the more basic material, a common notation has been used. However, a uniform sign convention has been used in that the following are considered as positive: compressive stress, reduction in length or volume, and displacement in the positive co-ordinate direction.

The authors have not attempted the immense task of a full check of all the solutions they have reproduced, but a more limited check has been carried out by testing solutions for self-consistency and consistency with other solutions and this has uncovered a number of errors in the original solutions which have been corrected. However, it is probable that some further errors will have escaped the authors' notice and any information on such errors will be gratefully received by them.

The book is divided into essentially four parts:

(a) an introductory summary of the basic equations and relationships in elastic theory (Chapter 1) and then basic solutions for problems involving concentrated loads on elastic media (Chapter 2);

(b) solutions for loading of simple geometrical areas, both uniformly loaded and rigid (Chapters 3 to 9);

(c) solutions of a more complicated nature having relevance to practical soil mechanics, rock mechanics and foundation problems (Chapters 10 to 15);

(d) appendices containing complete solutions for various cases of surface loading on an anisotropic or isotropic elastic half space.

As a reference for students, research workers and practising engineers, this book may be used in a number of ways:

(a) as an immediate source of solutions for use in solving geotechnical problems;

(b) as a source of basic solutions from which more complicated solutions may be evaluated by the user;

(c) as a source of reference solutions against which numerical computer solutions (e.g. from the finite element method) may be checked.

Grateful achnowledgement is given to the great number of persons and institutions, too numerous to list individually, who have given permission for their solutions to be reproduced. Special thanks are due to Dr. T. William Lambe; Edmund K. Turner Professor of Civil Engineering at M.I.T., for his original encouragement of the preparation of the book; Dr. J.P. Giroud of the University of Grenoble, France, for his generous permission to reproduce many of his results, both published and unpublished; Drs. J.R. Booker and P.T. Brown of the University of Sydney for their comments and advice, and Dr. C. M. Gerrard and Mrs. W.J. Harrison for permission to reproduce their papers in full as Appendices A and B. Finally, the authors are greatly indebted to Mrs. M. Brown, who cheerfully and patiently carried out the major task of typing the manuscript, and to Mr. R. Brew, Mrs. H. Papallo and Miss A. Chittendon, who undertook the onerous task of preparing the diagrams.

H. G. Poulos
E. H. Davis
June 1973

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Chapter 1
FUNDAMENTAL DEFINITIONS AND RELATIONSHIPS

1.1 Analysis of Stress

1.1.1 BASIC DEFINITIONS AND SIGN CONVENTION

Since it is often convenient in soil mechanics to consider compressive stresses as positive, this conventional will be adopted here. The normal and shear stresses acting on an element are shown in Fig.1.1, the stresses all being of positive sign.

The normal stresses $\sigma_x, \sigma_y, \sigma_z$ are positive when directed into the surface.

The notation for the shear stress τ_{ij} is as follows:

τ_{ij} is the shear stress acting in the j direction on a plane normal to the i axis.

The sign convention for shear stress is as follows:

The shear stress is *positive* when directed in a *negative* Cartesian direction while acting on a plane whose outward normal points in a *positive* direction, or, when directed in a *positive* Cartesian direction while acting on a plane whose outward normal points in a *negative* Cartesian direction.

Equilibrium requires that

$$\tau_{xy} = \tau_{yx}$$
$$\tau_{yz} = \tau_{zy}$$
$$\tau_{zx} = \tau_{xz}$$

For the definition of stresses in other coordinate systems, see Section 1.3.

1.1.2 STRESS COMPONENTS ON ANY PLANE

Referring to Fig.1.2, the stress components p_{nx}, p_{ny}, p_{nz} on any plane with a directed normal n can be expressed in terms of the stresses in the x, y and z coordinates as

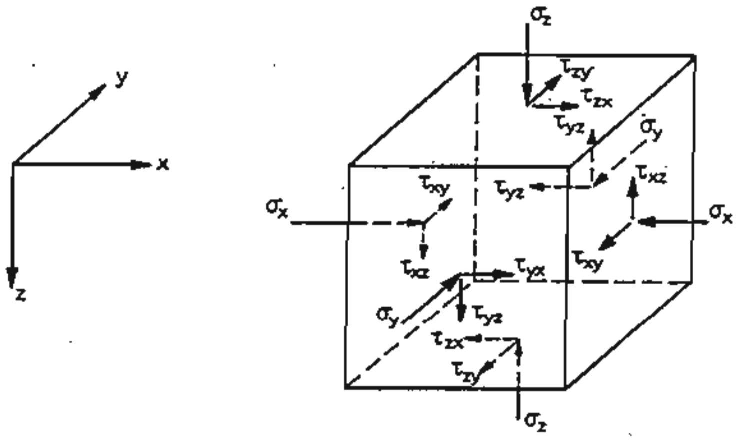


FIG.1.1

$$\begin{vmatrix} p_{nx} \\ p_{ny} \\ p_{nz} \end{vmatrix} = \begin{vmatrix} \sigma_x & \tau_{yx} & \tau_{zx} \\ \tau_{xy} & \sigma_y & \tau_{zy} \\ \tau_{xz} & \tau_{yz} & \sigma_z \end{vmatrix} \begin{vmatrix} cos(n,x) \\ cos(n,y) \\ cos(n,z) \end{vmatrix} \quad \ldots (1.1)$$

where $cos(n,x)$ is the cosine of the angle between the n and x directions, and similarly for $cos(n,y)$ and $cos(n,z)$.

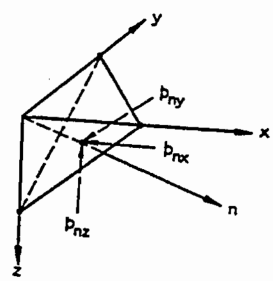


FIG.1.2

1.1.3 TRANSFORMATION OF AXES

If a new set of orthogonal axes x', y', z' are chosen, the stress components in this coordinate system are related to the stress components in the original x, y, z system as follows:

$$S_1 = A\ S\ A^T \quad \ldots (1.2)$$

where S_1 is the stress matrix with respect to the $x'y'z'$ axes,

S is the stress matrix with respect to the $x\ y\ z$ axes,

A is the direction cosine matrix, i.e.,

$$A = \begin{vmatrix} cos(x',x) & cos(x',y) & cos(x',z) \\ cos(y',x) & cos(y',y) & cos(y',z) \\ cos(z',x) & cos(z',y) & cos(z',z) \end{vmatrix}$$

A^T is the transpose of A.

1.1.4 PRINCIPAL STRESSES

It is possible to show that there is one set of axes with respect to which all shear stresses are zero and the normal stresses have their extreme values. The three mutually perpendicular planes where this condition exists are called the *principal planes*, and the normal stresses acting on these planes are the principal stresses.

The principal stresses, σ_1, σ_2 and σ_3 (the maximum, intermediate and minimum stresses respectively) may be found as the roots of the equation

$$\sigma_i^3 - J_1\sigma_i^2 + J_2\sigma_i - J_3 = 0 \quad \ldots (1.3)$$

where $J_1 = \sigma_x+\sigma_y+\sigma_z = \Theta$ *(bulk stress)* ... (1.4a)

$$J_2 = \sigma_x\sigma_y+\sigma_y\sigma_z+\sigma_z\sigma_x-\tau_{xy}^2-\tau_{yz}^2-\tau_{zx}^2 \quad \ldots (1.4b)$$

$$J_3 = \sigma_x\sigma_y\sigma_z-\sigma_x\tau_{yz}^2-\sigma_y\tau_{zx}^2-\sigma_z\tau_{xy}^2 + 2\tau_{xy}\tau_{yz}\tau_{zx} \quad \ldots (1.4c)$$

J_1 (or Θ), J_2, J_3 are often known as the first, second and third stress invariants, as they remain constant, independent of the coordinate system.

In terms of the principal stresses,

$$J_1 = \Theta = \sigma_1 + \sigma_2 + \sigma_3 \quad \ldots (1.5a)$$

$$J_2 = \sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1 \quad \ldots (1.5b)$$

$$J_3 = \sigma_1\sigma_2\sigma_3 \quad \ldots (1.5c)$$

The directions of the normals to the principal planes are given by

$$cos(n_i,x) = \frac{A_i}{\sqrt{A_i^2+B_i^2+C_i^2}} \quad \ldots (1.6a)$$

$$cos(n_i,y) = \frac{B_i}{\sqrt{A_i^2+B_i^2+C_i^2}} \quad \ldots (1.6b)$$

$$cos(n_i,z) = \frac{C_i}{\sqrt{A_i^2+B_i^2+C_i^2}} \quad \ldots (1.6c)$$

where $A_i = (\sigma_y-\sigma_i)(\sigma_z-\sigma_i)-\tau_{zy}\tau_{yz}$

$B_i = \tau_{zy}\tau_{xz} - \tau_{xy}(\sigma_z-\sigma_i)$

$C_i = \tau_{xy}\tau_{yz} - \tau_{xz}(\sigma_y-\sigma_i)$

and σ_i are the principal stresses $(i = 1,2,3)$.

1.1.5 MAXIMUM SHEAR STRESS

The maximum shear stress occurs on a plane whose normal makes an angle of 45° with the σ_1 and σ_3 directions.

The maximum shear stress, τ_{max} at a point is given by

$$\tau_{max} = \tfrac{1}{2}(\sigma_1-\sigma_3) \quad \ldots (1.7)$$

1.1.6 OCTAHEDRAL STRESSES

The octahedral normal stress σ_{oct} and the octahedral shear stress τ_{oct} at a point are the stresses acting on the eight planes of an imaginary octahedron surrounding the point, the normals to the faces of the octahedron having direction cosines of $\pm 1/\sqrt{3}$ with the direction of the principal stresses.

The magnitudes of the octahedral stresses are

$$\sigma_{oct} = \frac{\sigma_1+\sigma_2+\sigma_3}{3} = \frac{\Theta}{3} = \frac{\sigma_x+\sigma_y+\sigma_z}{3} \quad \ldots (1.8)$$

$$\tau_{oct} = \frac{1}{3}\left((\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2\right)^{\frac{1}{2}}$$

$$= \frac{1}{3}\left((\sigma_x-\sigma_y)^2+(\sigma_y-\sigma_z)^2+(\sigma_z-\sigma_x)^2 + 6(\tau_{xy}^2+\tau_{xz}^2+\tau_{yz}^2)\right)^{\frac{1}{2}} \quad \ldots (1.9)$$

1.1.7 TWO-DIMENSIONAL STRESS SYSTEMS

Many situations in soil mechanics can be treated as two-dimensional problems in which only the stresses in a single plane need be considered. The most important case is that of *plane strain*, in which the strain (see Section 1.2) in one of the coordinate directions (usually the y direction here) is zero. Another class of problems are those involving *plane stress* conditions, in which the stress in one of the coordinate directions (usually y here) is zero.

In two-dimensional stress situations, the stress relationships are considerably simplified in relation to the general three-dimensional case. Referring to Fig.1.3, the stresses on a plane making an angle θ with the z direction are

$$\sigma_\theta = \tfrac{1}{2}(\sigma_x+\sigma_z)+\tfrac{1}{2}(\sigma_x-\sigma_z)\cos 2\theta+\tau_{xz}\sin 2\theta \quad \ldots (1.10)$$

$$\tau_\theta = \tau_{xz}\cos 2\theta-\tfrac{1}{2}(\sigma_x-\sigma_z)\sin 2\theta \quad \ldots (1.11)$$

The principal stresses are given by

$$\left.\begin{matrix}\sigma_1\\ \sigma_3\end{matrix}\right\} = \tfrac{1}{2}(\sigma_x+\sigma_z) \pm\tfrac{1}{2}\left((\sigma_x-\sigma_z)^2+4\tau_{xz}^2\right)^{\frac{1}{2}} \quad \ldots (1.12)$$

The principal planes are inclined at an angle

$$\theta_1 = \tfrac{1}{2}\tan^{-1}\frac{2\tau_{xz}}{\sigma_x-\sigma_z} \quad \ldots (1.13)$$

and

$\theta_1+90°$ to the z axis.

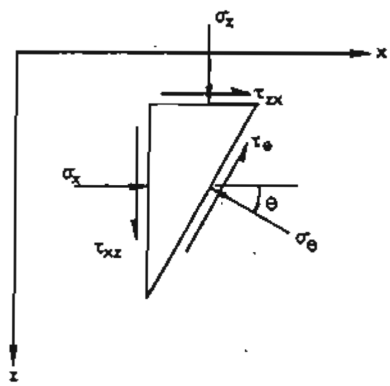


FIG.1.3

The maximum shear stress occurs on planes inclined at 45° to the principal planes and is of magnitude

$$\tau_{max} = \tfrac{1}{2}\left((\sigma_x-\sigma_z)^2+4\tau_{xz}^2\right)^{\frac{1}{2}} \quad \ldots (1.14)$$

(It should be noted that the sign of this maximum shear stress is opposite on the two planes, in order to conform to the sign convention given in Section 1.1.1).

1.1.8 MOHR'S CIRCLE OF STRESS

A geometrical solution for stresses in any direction is provided by Mohr's circle, shown in Fig.1.4 for a two-dimensional stress system. The circle is drawn in relation to a set of orthogonal axes, one for normal stress (σ) and the other for shear stress (τ). The scale of these two axes must be equal.

If the principal stresses σ_1, σ_3 are known, the circle can be drawn with the centre at $\sigma=\tfrac{1}{2}(\sigma_1+\sigma_3)$ and of radius $(\sigma_1-\sigma_3)/2$.

If the normal and shear stresses are known, the circle can be drawn with the centre at $\sigma=\tfrac{1}{2}(\sigma_x+\sigma_y)$ and passing through the points (σ_x, τ_{xz}) and $(\sigma_z, -\tau_{xz})$.

The radius of the circle thus constructed is equal to the maximum shear stress τ_{max} (see Equation 1.14).

The angle $2\theta_1$ is twice the angle between the x-z coordinate axes and the axes corresponding to the directions of principal stress (the 1-3 axes in Fig.1.4). The direction of rotation of the radius from its original constructed position to where the circle intersects the normal stress axis is in the same angular sense as the direction of rotation of the axes for the x-z axes to become the principal 3-1 axes.

The stresses in any other directions x', z' may similarly be determined by drawing a diameter, through the centre of the circle, at an angle $2\theta'$ to the diameter describing the stress conditions on the

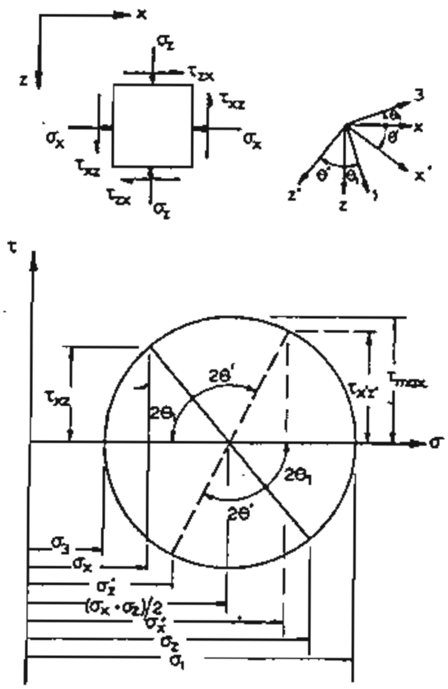


FIG.1.4 Mohr circle of stress.

x-z axes, where θ' is the angle between the x-z axes and the x'-z' axes (see Fig.1.4).

It should be noted that shear stresses are considered positive if they tend to produce a clockwise rotation about a point, outside the element, at the plane on which they act (Fig.1.5). This convention is consistent with that previously developed for three-dimensional conditions.

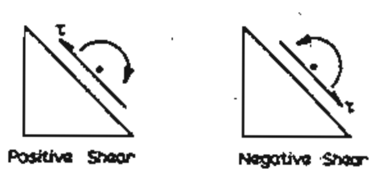


FIG.1.5

1.1.9 POLE CONSTRUCTION

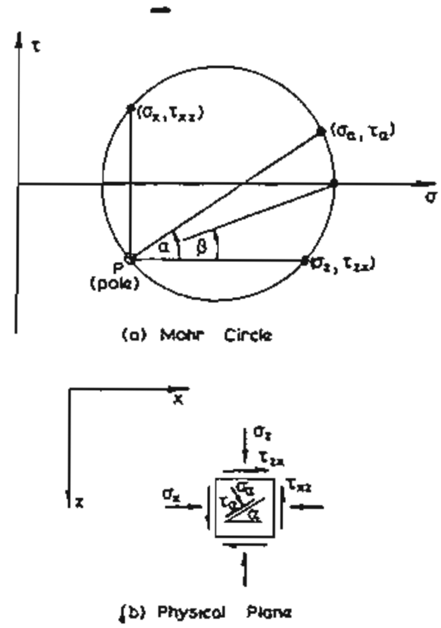


FIG.1.6 Pole construction.

The pole construction is a useful way of linking the stresses at a point in the physical plane (Fig. 1.6b) to the Mohr circle diagram for the stresses (Fig.1.6a). The pole, P, is the point on the circle such that the normal and shear stresses on any plane α (perpendicular to the physical plane) are given by the intersection with the Mohr circle of a line through P parallel with the plane α. For example the stresses on vertical and horizontal planes are as indicated in Fig.1.6a and the major principal plane is inclined at the angle β above the horizontal.

1.2 Analysis of Strain

1.2.1 BASIC DEFINITIONS

Considering first the case of two-dimensional strain (Fig.1.7), the normal strains ε_x and ε_z are defined as

$$\varepsilon_x = -\frac{\partial \rho_x}{\partial x} \qquad \text{... (1.15a)}$$

$$\varepsilon_z = -\frac{\partial \rho_z}{\partial z} \qquad \ldots (1.15b)$$

where ρ_x, ρ_z are the displacements in the x and z directions. A positive normal strain corresponds to a decrease in length.

The shear strain γ_{xz} is the angular change in a right angle in a material and is related to the displacements ρ_x and ρ_z as

$$\gamma_{xy} = -\frac{\partial \rho_x}{\partial z} - \frac{\partial \rho_z}{\partial x} \qquad \ldots (1.16)$$

A positive shear strain represents an increase in the right angle and a negative shear strain represents a decrease in the right angle.

Considering the xy and yz planes similarly, the six strain components are related to the displacements ρ_x, ρ_y, ρ_z in the x, y and z directions as

$$\varepsilon_x = -\frac{\partial \rho_x}{\partial x} \qquad \gamma_{xy} = -\frac{\partial \rho_x}{\partial y} - \frac{\partial \rho_y}{\partial x} \qquad \ldots (1.17a)$$

$$\varepsilon_y = -\frac{\partial \rho_y}{\partial y} \qquad \gamma_{yz} = -\frac{\partial \rho_y}{\partial z} - \frac{\partial \rho_z}{\partial y} \qquad \ldots (1.17b)$$

$$\varepsilon_z = -\frac{\partial \rho_z}{\partial z} \qquad \gamma_{zx} = -\frac{\partial \rho_z}{\partial x} - \frac{\partial \rho_x}{\partial z} \qquad \ldots (1.17c)$$

As for shear stresses $\gamma_{ij} = \gamma_{ji}$

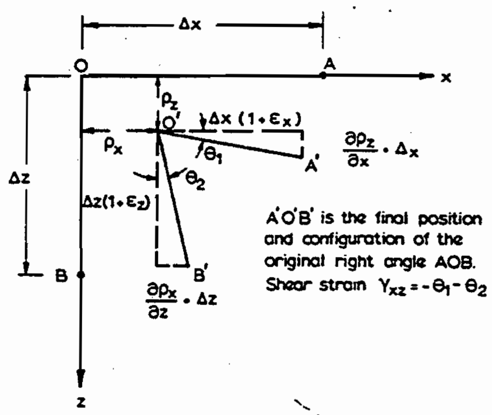


FIG.1.7

1.2.2 STRAIN IN A PLANE

Considering again a two-dimensional strain situation, the normal strain ε_θ in a plane inclined at θ to the x axis is

$$\varepsilon_\theta = \frac{\varepsilon_x+\varepsilon_z}{2} + \frac{\varepsilon_x-\varepsilon_z}{2}\cos 2\theta + \frac{\gamma_{xz}}{2}\sin 2\theta \qquad \ldots (1.18)$$

and the shear strain is

$$\gamma_\theta = \gamma_{xz}\cos 2\theta - (\varepsilon_x-\varepsilon_z)\sin 2\theta \qquad \ldots (1.19)$$

(Note that the above expressions correspond to those for the normal and shear stresses (Section 1.1), except for a factor of ½ in the last term).

1.2.3 TRANSFORMATION OF AXES

If a new set of orthogonal axes x', y', z' are chosen, the strain components in this coordinate system are related to the strain components in the original x, y, z system as

$$D_1 = A\,D\,A^T \qquad \ldots (1.20)$$

where D is the strain matrix in the x, y, z system,

i.e.,

$$D = \begin{vmatrix} \varepsilon_x & \tfrac{1}{2}\gamma_{xy} & \tfrac{1}{2}\gamma_{xz} \\ \tfrac{1}{2}\gamma_{yx} & \varepsilon_y & \tfrac{1}{2}\gamma_{yz} \\ \tfrac{1}{2}\gamma_{zx} & \tfrac{1}{2}\gamma_{zy} & \varepsilon_z \end{vmatrix} \qquad \ldots (1.21)$$

D_1 is the strain matrix in the x', y', z' system.

A is the direction cosine matrix defined in Section 1.1.3.

A^T is the transpose of A.

In matrix operations, it is convenient to use the double suffix notation and to define $\tfrac{1}{2}\gamma_{ij}$ as ε_{ij}. The strain matrix is then

$$D = \begin{vmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{yz} & \varepsilon_{zz} \end{vmatrix} \qquad \ldots (1.22)$$

1.2.4 PRINCIPAL STRAINS

Analogous to the principal planes of stress, there are three principal planes of strain. The shear strains in these planes are zero and the normal strains are the principal strains. The major and minor principal strains are respectively, the greatest and least normal strains at the point. For an isotropic elastic material, the principal planes of strain can be shown to coincide with the principal planes of stress.

The principal strains are determined, in a similar manner to principal stresses, as the roots of the equation

$$\varepsilon_i^3 - I_1\varepsilon_i^2 + I_2\varepsilon_i - I_3 = 0 \qquad \ldots (1.23)$$

$$\text{where } I_1 = \varepsilon_x + \varepsilon_y + \varepsilon_z \qquad \ldots (1.24a)$$

$$I_2 = \varepsilon_x\varepsilon_y + \varepsilon_y\varepsilon_z + \varepsilon_z\varepsilon_x - \frac{\gamma_{xy}^2}{4} - \frac{\gamma_{yz}^2}{4} - \frac{\gamma_{zx}^2}{4} \qquad \ldots (1.24b)$$

$$I_3 = \varepsilon_x\varepsilon_y\varepsilon_z - \frac{\varepsilon_x\gamma_{yz}^2}{4} - \frac{\varepsilon_y\gamma_{zx}^2}{4} - \frac{\varepsilon_z\gamma_{xy}^2}{4} + \frac{\gamma_{xy}\,\gamma_{xz}\,\gamma_{yz}}{4} \qquad \ldots (1.24c)$$

I_1, I_2, I_3 are the strain invariants, analogous to the stress invariants.

In two-dimensional systems, the principal strains ε_1, ε_3 are as follows:

$$\left.\begin{matrix}\varepsilon_1\\ \varepsilon_3\end{matrix}\right\} = \frac{\varepsilon_x+\varepsilon_z}{2} \pm\tfrac{1}{2}\sqrt{(\varepsilon_x-\varepsilon_z)^2+\gamma_{xz}^2} \qquad \ldots (1.25)$$

and the principal planes are inclined at an angle θ_1 to the x and z axes, where

$$\theta_1 = \tfrac{1}{2}\tan^{-1}\frac{\gamma_{xz}}{\varepsilon_x-\varepsilon_z} \qquad \ldots (1.26)$$

1.2.5 MAXIMUM SHEAR STRAIN

$$\gamma_{max} = \varepsilon_1 - \varepsilon_3 \qquad \ldots (1.27)$$

where ε_1 = maximum principal normal strain,

ε_3 = minimum principal normal strain.

γ_{max} occurs on a plane whose normal makes an angle of $45°$ with the ε_1 and ε_3 directions.

1.2.6 MOHR'S CIRCLE OF STRAIN

A geometrical solution for strains in any direction is provided by Mohr's circle of strain (Fig.1.8). The only difference between the circle of strain and the circle of stress is that, in the circle of strain, the ordinate represents only one-half the shear strain (i.e. the ordinate axis is $\gamma/2$). As in Fig.1.4, the axes *1-3* represent the principal axes, x-z the horizontal and vertical space axes and x'-z' the axes in direction at an angle θ' to the x-z axes.

The diameter of the circle is equal to the maximum shear strain

$$\gamma_{max} = \sqrt{(\varepsilon_x-\varepsilon_z)^2 + \gamma_{xz}^2}$$

The pole construction as described for the Mohr circle of stress may be adapted for the Mohr circle of strain.

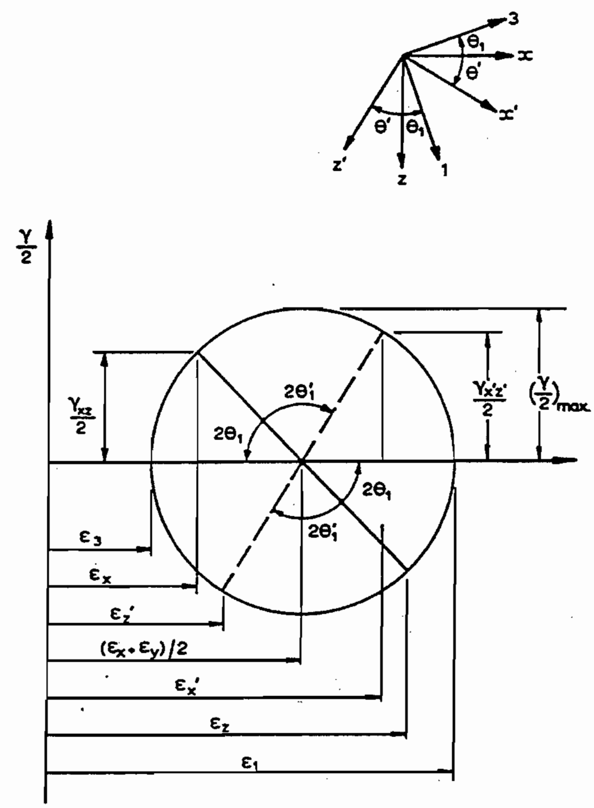


FIG.1.8 Mohr circle of strain.

1.3 Equilibrium Equations

1.3.1 CARTESIAN COORDINATES

By considering the equilibrium of the element shown in Fig.1.1 in the Cartesian coordinate system, the following equilibrium equations are obtained:

$$\frac{\partial\sigma_x}{\partial x} + \frac{\partial\tau_{yx}}{\partial y} + \frac{\partial\tau_{xz}}{\partial z} - X = 0 \qquad \ldots (1.28a)$$

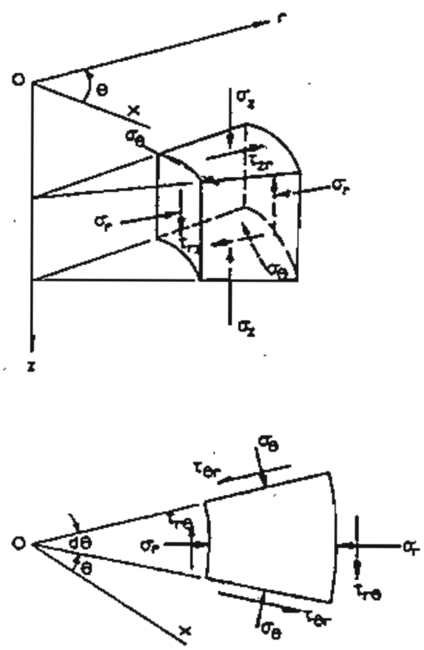
$$\frac{\partial\tau_{xy}}{\partial x} + \frac{\partial\sigma_y}{\partial y} + \frac{\partial\tau_{yz}}{\partial z} - Y = 0 \qquad \ldots (1.28b)$$

$$\frac{\partial\tau_{xz}}{\partial x}+\frac{\partial\tau_{yz}}{\partial y}+\frac{\partial\sigma_z}{\partial z}-Z=0$$

where X, Y, Z are the body forces, per unit volume, in the x, y and z directions.

With an ordinary gravity field and the z direction vertically downwards, X and Y are zero and Z is the unit weight, γ, of the material.

1.3.2 CYLINDRICAL COORDINATES



Plan of element in rθ plane.

FIG.1.9

Considering the equilibrium of the element in the cylindrical (r,z,θ) coordinate system shown in Fig. 1.9, the equilibrium equations are (neglecting body forces)

$$\frac{\partial\sigma_r}{\partial r}+\frac{1}{r}\frac{\partial\tau_{r\theta}}{\partial\theta}+\frac{\partial\tau_{zr}}{\partial z}+\frac{\sigma_r-\sigma_\theta}{r}=0 \qquad \ldots (1.29a)$$

$$\frac{\partial\tau_{r\theta}}{\partial r}+\frac{1}{r}\frac{\partial\sigma_\theta}{\partial\theta}+\frac{\partial\tau_{\theta z}}{\partial z}+\frac{2\tau_{r\theta}}{r}=0 \qquad \ldots (1.29b)$$

$$\frac{\partial\tau_{zr}}{\partial r}+\frac{1}{r}\frac{\partial\tau_{\theta z}}{\partial\theta}+\frac{\partial\sigma_z}{\partial z}+\frac{\tau_{zr}}{r}=0 \qquad \ldots (1.29c)$$

With axial symmetry, these become

$$\frac{\partial\sigma_r}{\partial r}+\frac{\partial\tau_{zr}}{\partial z}+\frac{\sigma_r-\sigma_\theta}{r}=0 \qquad \ldots (1.30a)$$

$$\frac{\partial\tau_{zr}}{\partial r}+\frac{\partial\sigma_z}{\partial z}+\frac{\tau_{zr}}{r}=0 \qquad \ldots (1.30b)$$

1.3.3. SPHERICAL COORDINATES (Fig.1.10)

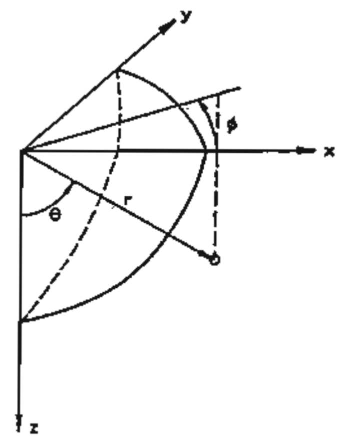


FIG.1.10

$$\frac{\partial\sigma_r}{\partial r}+\frac{1}{r}\frac{\partial\tau_{r\theta}}{\partial\theta}+\frac{1}{r\sin\theta}\frac{\partial\tau_{r\phi}}{\partial\phi}+\frac{2\sigma_r-\sigma_\theta-\sigma_\phi+\tau_{r\theta}\cot\theta}{r}=0 \qquad \ldots (1.31a)$$

$$\frac{\partial\tau_{r\theta}}{\partial r}+\frac{1}{r}\frac{\partial\sigma_\theta}{\partial\theta}+\frac{1}{r\sin\theta}\frac{\partial\tau_{\theta\phi}}{\partial\phi}+\frac{3\tau_{r\theta}+(\sigma_\theta-\sigma_\phi)\cot\theta}{r}=0 \qquad \ldots (1.31b)$$

$$\frac{\partial\tau_{r\phi}}{\partial r}+\frac{1}{r}\frac{\partial\tau_{\theta\phi}}{\partial\theta}+\frac{1}{r\sin\theta}\frac{\partial\sigma_\phi}{\partial\phi}+\frac{3\tau_{r\phi}+2\tau_{\theta\phi}\cot\theta}{r}=0 \qquad \ldots (1.31c)$$

For complete spherical symmetry these become

$$\frac{\partial\sigma_r}{\partial r}+\frac{2(\sigma_r-\sigma_\theta)}{r}=0 \qquad \ldots (1.32)$$

1.4 Strain-Displacement and Compatibility Equations

1.4.1 CARTESIAN COORDINATES

The strain - displacement relationships are given in equation (1.17). Since six strain components are derived from only three displacements, the strains are not independent of each other. Six further relationships, known as the compatibility equations, can be derived. These are as follows:

$$\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y} \quad \ldots (1.33a)$$

$$\frac{\partial^2 \varepsilon_y}{\partial z^2} + \frac{\partial^2 \varepsilon_z}{\partial y^2} = \frac{\partial^2 \gamma_{yz}}{\partial y \partial z} \quad \ldots (1.33b)$$

$$\frac{\partial^2 \varepsilon_z}{\partial x^2} + \frac{\partial^2 \varepsilon_x}{\partial z^2} = \frac{\partial^2 \gamma_{zx}}{\partial z \partial x} \quad \ldots (1.33c)$$

$$2\left(\frac{\partial^2 \varepsilon_x}{\partial y \partial z}\right) = \frac{\partial}{\partial x}\left(-\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{zx}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z}\right) \quad \ldots (1.33d)$$

$$2\left(\frac{\partial^2 \varepsilon_y}{\partial z \partial x}\right) = \frac{\partial}{\partial y}\left(\frac{\partial \gamma_{yz}}{\partial x} - \frac{\partial \gamma_{zx}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z}\right) \quad \ldots (1.33e)$$

$$2\left(\frac{\partial^2 \varepsilon_z}{\partial x \partial y}\right) = \frac{\partial}{\partial z}\left(\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{zx}}{\partial y} - \frac{\partial \gamma_{xy}}{\partial z}\right) \quad \ldots (1.33f)$$

1.4.2 CYLINDRICAL COORDINATES

The strain - displacement equations are

$$\varepsilon_r = -\frac{\partial \rho_r}{\partial r} \qquad \gamma_{r\theta} = -\frac{1}{r}\frac{\partial \rho_r}{\partial \theta} - \frac{\partial \rho_\theta}{\partial r} + \frac{\rho_\theta}{r} \quad \ldots (1.34a)$$

$$\varepsilon_\theta = -\frac{\rho_r}{r} - \frac{1}{r}\frac{\partial \rho_\theta}{\partial \theta} \qquad \gamma_{\theta z} = -\frac{\partial \rho_\theta}{\partial z} - \frac{1}{r}\frac{\partial \rho_z}{\partial \theta} \quad \ldots (1.34b)$$

$$\varepsilon_z = -\frac{\partial \rho_z}{\partial z} \qquad \gamma_{zr} = -\frac{\partial \rho_z}{\partial r} - \frac{\partial \rho_r}{\partial z} \quad \ldots (1.34c)$$

The corresponding compatability equations are quoted by L'ure (1964).

1.4.3. SPHERICAL COORDINATES

The strain - displacement equations are:

$$\varepsilon_r = -\frac{\partial \rho_r}{\partial r} \qquad \gamma_{r\theta} = -\frac{1}{r}\frac{\partial \rho_r}{\partial \theta} - \frac{\partial \rho_\theta}{\partial r} + \frac{\rho_\theta}{r} \quad \ldots (1.35a)$$

$$\varepsilon_\theta = -\frac{\rho_r}{r} - \frac{1}{r}\frac{\partial \rho_\theta}{\partial \theta} \qquad \gamma_{\theta\phi} = -\frac{1}{r\sin\theta}\frac{\partial \rho_\theta}{\partial \phi} - \frac{1}{r}\frac{\partial \rho_\phi}{\partial \theta} + \frac{\rho_\phi \cot\theta}{r} \quad \ldots (1.35b)$$

$$\varepsilon_\phi = -\frac{\rho_r}{r} - \frac{\rho_\theta}{r}\cot\theta - \frac{1}{r\sin\theta}\frac{\partial \rho_\phi}{\partial \phi} \qquad \gamma_{\phi r} = -\frac{\partial \rho_\phi}{\partial r} + \frac{\rho_\phi}{r} - \frac{1}{r\sin\theta}\frac{\partial \rho_r}{\partial \phi} \quad \ldots (1.35c)$$

The compatibility equations, for the case of axial symmetry, are quoted by L'ure (1964).

1.5 Stress-Strain Relationships

1.5.1 LINEAR HOMOGENEOUS ISOTROPIC MATERIAL

Strains in terms of stress:

$$\varepsilon_x = \frac{1}{E}[\sigma_x - \nu(\sigma_y + \sigma_z)] \quad \ldots (1.36a)$$

$$\varepsilon_y = \frac{1}{E}[\sigma_y - \nu(\sigma_x + \sigma_z)] \quad \ldots (1.36b)$$

$$\varepsilon_z = \frac{1}{E}[\sigma_z - \nu(\sigma_x + \sigma_y)] \quad \ldots (1.36c)$$

$$\gamma_{xy} = \frac{1}{G}\tau_{xy} \quad \ldots (1.36d)$$

$$\gamma_{yz} = \frac{1}{G}\tau_{yz} \quad \ldots (1.36e)$$

$$\gamma_{zx} = \frac{1}{G}\tau_{zx} \quad \ldots (1.36f)$$

where E = Young's modulus
ν = Poisson's ratio
G = shear modulus

$$= \frac{E}{2(1+\nu)} \quad \ldots (1.37)$$

Also, volume strain

$$\varepsilon_v = \frac{(1-2\nu)}{E}\Theta = \frac{\Theta}{3K} \quad \ldots (1.37)$$

where $\varepsilon_v = \varepsilon_x + \varepsilon_y + \varepsilon_z$
$\Theta = \sigma_x + \sigma_y + \sigma_z$ = bulk stress

K = bulk modulus.

Stresses in terms of strains:

$$\sigma_x = \lambda\varepsilon_v + 2G\varepsilon_x \quad \text{... (1.38a)}$$

$$\sigma_y = \lambda\varepsilon_v + 2G\varepsilon_y \quad \text{... (1.38b)}$$

$$\sigma_z = \lambda\varepsilon_v + 2G\varepsilon_z \quad \text{... (1.38c)}$$

$$\tau_{xz} = G\gamma_{xz} \quad \text{etc.} \quad \text{... (1.38d)}$$

where λ, G are Lame's parameters

$$\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)} \quad \text{... (1.39a)}$$

$$G = \frac{E}{2(1+\nu)} = \text{shear modulus} \quad \text{... (1.39b)}$$

(G is also often denoted as μ).

For the special case of plane stress e.g. in the x-z plane, $\sigma_y = 0$ in the above equations.

For the special case of plane strain in the x-z plane, $\varepsilon_y = 0$ and hence

$$\sigma_y = \nu(\sigma_x + \sigma_z) \quad \text{... (1.40)}$$

Equations (1.36) then reduce to

$$\varepsilon_x = \frac{(1+\nu)}{E}[\sigma_x(1-\nu) - \nu\sigma_z] \quad \text{... (1.41a)}$$

$$\varepsilon_y = 0 \quad \text{... (1.41b)}$$

$$\varepsilon_z = \frac{(1+\nu)}{E}[\sigma_z(1-\nu) - \nu\sigma_x] \quad \text{... (1.41c)}$$

$$\gamma_{xz} = \frac{1}{G}\tau_{xz} \quad \text{... (1.41d)}$$

Solutions for a plane strain problem can be used for the corresponding plane stress problem provided that the following equivalent values of E and ν are used in the plane strain problem:

$$E_e = \frac{(1+2\nu)E}{(1+\nu)^2} \quad \text{... (1.42a)}$$

$$\nu_e = \frac{\nu}{1+\nu} \quad \text{... (1.42b)}$$

Conversely, to use solutions for a plane stress problem for the corresponding plane strain problem, the equivalent moduli are

$$E_T = \frac{E}{1-\nu^2} \quad \text{... (1.43a)}$$

$$\nu_T = \frac{\nu}{1-\nu} \quad \text{... (1.43b)}$$

Plane stress solutions which do not involve the elastic parameters are therefore identical with the corresponding plane strain solutions e.g. stresses within a semi-infinite plate and stresses due to line loading on a semi-infinite mass.

Summary of Relationships Between Elastic Parameters

$$G \text{ (or } \mu) = \frac{E}{2(1+\nu)} \quad \text{... (1.44)}$$

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)} \quad \text{... (1.45)}$$

$$K = \frac{E}{3(1-2\nu)} = \frac{2(1+\nu)G}{3(1-2\nu)} \quad \text{... (1.46)}$$

$$E = \frac{9KG}{3K+G} \quad \text{... (1.47)}$$

$$\nu = \frac{(3K-2G)}{2(3K+G)} \quad \text{... (1.48)}$$

$$\frac{\lambda}{G} = \frac{2\nu}{1-2\nu} \quad \text{... (1.49)}$$

Constrained modulus ($1/m_v$ in Soil Mechanics)

$$= \frac{(1-\nu)E}{(1+\nu)(1-2\nu)} \quad \text{... (1.50)}$$

1.5.2 CROSS ANISOTROPIC MATERIAL

Stresses in terms of strains:

Cartesian coordinates:

$$\sigma_x = a\varepsilon_x + b\varepsilon_y + c\varepsilon_z \quad \text{... (1.51a)}$$

$$\sigma_y = b\varepsilon_x + a\varepsilon_y + c\varepsilon_z \quad \text{... (1.51b)}$$

$$\sigma_z = c\varepsilon_x + c\varepsilon_y + d\varepsilon_z \quad \text{... (1.51c)}$$

$$\tau_{xz} = f\varepsilon_{xz} \quad \text{... (1.51d)}$$

$$\tau_{zy} = f\varepsilon_{zy} \quad \text{... (1.51e)}$$

$$\tau_{xy} = (a-b)\varepsilon_{xy} \quad \text{... (1.51f)}$$

Cylindrical coordinates:

$$\sigma_r = a\varepsilon_r + b\varepsilon_\theta + c\varepsilon_z \quad \text{... (1.52a)}$$

$$\sigma_\theta = b\varepsilon_r + a\varepsilon_\theta + c\varepsilon_z \quad \text{... (1.52b)}$$

$$\sigma_z = c\varepsilon_r + c\varepsilon_\theta + d\varepsilon_z \quad \text{... (1.52c)}$$

$$\tau_{rz} = f\varepsilon_{rz} \quad \text{... (1.52d)}$$

$$\tau_{\theta z} = f\varepsilon_{\theta z} \quad \text{... (1.52e)}$$

$$\tau_{r\theta} = (a-b)\varepsilon_{r\theta} \quad \text{... (1.52f)}$$

where
$$a = \frac{E_h(1-\nu_{hv}\nu_{vh})}{(1+\nu_h)(1-\nu_h-2\nu_{hv}\nu_{vh})} \quad \ldots (1.53a)$$

$$b = \frac{E_h(\nu_h+\nu_{hv}\nu_{vh})}{(1+\nu_h)(1-\nu_h-2\nu_{hv}\nu_{vh})} \quad \ldots (1.53b)$$

$$c = \frac{E_h\,\nu_{vh}}{1-\nu_h-2\nu_{hv}\nu_{vh}} \quad \ldots (1.53c)$$

$$d = \frac{E_v(1-\nu_h)}{1-\nu_h-2\nu_{hv}\nu_{vh}} \quad \ldots (1.53d)$$

and E_h = modulus of elasticity in the horizontal direction

E_v = modulus of elasticity in the vertical direction

ν_h = Poisson's ratio for effect of horizontal stress on complementary horizontal strain

ν_{hv} = Poisson's ratio for effect of horizontal stress on vertical strain

ν_{vh} = Poisson's ratio for effect of vertical stress on horizontal strain.

It can be shown that

$$\frac{E_h}{E_v} = \frac{\nu_{hv}}{\nu_{vh}} \quad \ldots (1.54)$$

The elastic constant f is a shear modulus and cannot be expressed in terms of the Young's moduli or Poisson's ratios. f is often denoted as G_v.

Strains in terms of stresses:

$$\varepsilon_x = \frac{\sigma_x}{E_h} - \frac{\nu_h\sigma_y}{E_h} - \frac{\nu_{vh}\sigma_z}{E_v} \quad \ldots (1.55a)$$

$$\varepsilon_y = -\frac{\nu_h\sigma_x}{E_h} + \frac{\sigma_y}{E_h} - \frac{\nu_{vh}\sigma_z}{E_v} \quad \ldots (1.55b)$$

$$\varepsilon_z = -\frac{\nu_{hv}\sigma_x}{E_h} - \frac{\nu_{hv}\sigma_y}{E_h} + \frac{\sigma_z}{E_v} \quad \ldots (1.55c)$$

$$\varepsilon_{xz} = \frac{\tau_{xz}}{f} = \frac{\tau_{xz}}{G_v} \quad \ldots (1.55d)$$

$$\varepsilon_{yz} = \frac{\tau_{yz}}{f} = \frac{\tau_{yz}}{G_v} \quad \ldots (1.55e)$$

$$\varepsilon_{xy} = \frac{\tau_{xy}}{(a-b)} = \frac{(1+\nu_h)\tau_{xy}}{E_h} = \frac{\tau_{xy}}{G_h} \quad \ldots (1.55f)$$

In some works (e.g. Urena et al, 1966) ν_{hv} is denoted merely as ν_v and the use of ν_{vh} is avoided by using equation (1.54).

The fact that the strain energy must be positive imposes restrictions on the values of the elastic parameters. For a cross-anisotropic material with a vertical axis of elastic symmetry, Hearmon (1961) gives these restrictions as

$$a > 0 \quad \ldots (1.56a)$$

$$d > 0 \quad \ldots (1.56b)$$

$$f > 0 \quad \ldots (1.56c)$$

$$a^2 > b^2 \quad \ldots (1.56d)$$

$$(a+b)d > 2c^2 \quad \ldots (1.56e)$$

$$ad > c^2 \quad \ldots (1.56f)$$

In terms of the Poisson's ratios, these restrictions impose the limits

$$1 - \nu_h - 2\nu_{hv}\,\nu_{vh} > 0 \quad \ldots (1.57a)$$

$$1 - \nu_h > 0 \quad \ldots (1.57b)$$

$$1 + \nu_h > 0 \quad \ldots (1.57c)$$

1.6 Differential Equations of Isotropic Elasticity

1.6.1 EQUATIONS IN TERMS OF STRESSES

Cartesian Coordinates

$$\nabla^2\sigma_x + \frac{1}{1+\nu}\frac{\partial^2\Theta}{\partial x^2} = \frac{\nu}{1-\nu}\left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} + \frac{\partial Z}{\partial z}\right) + 2\frac{\partial X}{\partial x} \quad \ldots (1.58a)$$

$$\nabla^2\sigma_y + \frac{1}{1+\nu}\frac{\partial^2\Theta}{\partial y^2} = \frac{\nu}{1-\nu}\left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} + \frac{\partial Z}{\partial z}\right) + 2\frac{\partial Y}{\partial y} \quad \ldots (1.58b)$$

$$\nabla^2\sigma_z + \frac{1}{1+\nu}\frac{\partial^2\Theta}{\partial z^2} = \frac{\nu}{1-\nu}\left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} + \frac{\partial Z}{\partial z}\right) + 2\frac{\partial Z}{\partial z} \quad \ldots (1.58c)$$

$$\nabla^2\tau_{yz} + \frac{1}{1+\nu}\frac{\partial^2\Theta}{\partial y\partial z} = \frac{\partial Y}{\partial z} + \frac{\partial Z}{\partial y} \quad \ldots (1.58d)$$

$$\nabla^2\tau_{zx} + \frac{1}{1+\nu}\frac{\partial^2\Theta}{\partial z\partial x} = \frac{\partial Z}{\partial x} + \frac{\partial X}{\partial z} \quad \ldots (1.58e)$$

$$\nabla^2\tau_{xy} + \frac{1}{1+\nu}\frac{\partial^2\Theta}{\partial x\partial y} = \frac{\partial X}{\partial y} + \frac{\partial Y}{\partial x} \quad \ldots (1.58f)$$

where $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$

$\Theta = \sigma_x + \sigma_y + \sigma_z$

For constant or zero body forces, the first three equations of (1.58) reduce to the Laplace equation

$$\nabla^2\Theta = 0 \quad \ldots (1.59)$$

For the special case of *plane stress*, the equations are the equilibrium equations

$$\frac{\partial\sigma_x}{\partial x} + \frac{\partial\tau_{xz}}{\partial z} - X = 0 \quad \ldots (1.60a)$$

$$\frac{\partial\sigma_z}{\partial z} + \frac{\partial\tau_{xz}}{\partial x} - Z = 0 \quad \ldots (1.60b)$$

and

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\right)(\sigma_x + \sigma_z) = (1+\nu)\left(\frac{\partial X}{\partial x} + \frac{\partial Z}{\partial z}\right) \quad \ldots (1.60c)$$

For *plane strain*, the first two of the above three equations are again applicable. The third equation is

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\right)(\sigma_x + \sigma_z) = \frac{1}{1-\nu}\left(\frac{\partial X}{\partial x} + \frac{\partial Z}{\partial z}\right) \quad \ldots (1.61)$$

If body forces are constant, the equations for plane stress and plane strain conditions are identical.

Cylindrical Coordinates

With zero or constant body forces:

$$\nabla^2\sigma_z + \frac{1}{1+\nu}\frac{\partial^2\Theta}{\partial z^2} = 0 \quad \ldots (1.62a)$$

$$\nabla^2\sigma_r + \frac{2}{r^2}(\sigma_\theta - \sigma_r) - \frac{4}{r^2}\frac{\partial\tau_{r\theta}}{\partial\theta} + \frac{1}{1+\nu}\frac{\partial^2\Theta}{\partial r^2} = 0 \quad \ldots (1.62b)$$

$$\nabla^2\sigma_\theta - \frac{2}{r^2}(\sigma_\theta - \sigma_r) + \frac{4}{r^2}\frac{\partial\tau_{r\theta}}{\partial\theta} + \frac{1}{1+\nu}\left(\frac{1}{r}\frac{\partial\Theta}{\partial r} + \frac{1}{r^2}\frac{\partial^2\Theta}{\partial\theta^2}\right) = 0 \quad \ldots (1.62c)$$

$$\nabla^2\tau_{r\theta} - \frac{2}{r^2}\frac{\partial}{\partial\theta}(\sigma_\theta - \sigma_r) - \frac{4}{r^2}\tau_{r\theta} + \frac{1}{1+\nu}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\Theta}{\partial\theta}\right) = 0 \quad \ldots (1.62d)$$

$$\nabla^2\tau_{rz} - \frac{\tau_{rz}}{r^2} - \frac{2}{r^2}\frac{\partial\tau_{\theta z}}{\partial\theta} + \frac{1}{1+\nu}\frac{\partial^2\Theta}{\partial r\partial z} = 0 \quad \ldots (1.62e)$$

$$\nabla^2\tau_{z\theta} - \frac{\tau_{z\theta}}{r^2} + \frac{2}{r}\frac{\partial\tau_{rz}}{\partial\theta} + \frac{1}{1+\nu}\frac{1}{r}\frac{\partial^2\Theta}{\partial\theta\partial z} = 0 \quad \ldots (1.62f)$$

For the general case of non-constant body forces, the corresponding equations, in tensor form, are given by L'ure (1964).

1.6.2 EQUATIONS IN TERMS OF STRESS FUNCTION ϕ

Cartesian Coordinates

$$\nabla^4\phi = 0 = \nabla^2\nabla^2\phi \quad \ldots (1.63)$$

where $\nabla^2 \equiv \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)$

For plane stress or plane strain,

$$\frac{\partial^4\phi}{\partial x^4} + \frac{2\partial^4\phi}{\partial x^2\partial z^2} + \frac{\partial^4\phi}{\partial z^4} = 0 \quad \ldots (1.64)$$

and the stresses are related to ϕ as follows:

$$\sigma_z = \frac{\partial^2\phi}{\partial z^2} \quad \ldots (1.65a)$$

$$\sigma_x = \frac{\partial^2\phi}{\partial x^2} \quad \ldots (1.65b)$$

$$\tau_{xz} = \frac{-\partial^2\phi}{\partial x\partial z} \quad \ldots (1.65c)$$

Cylindrical Coordinates

For axial symmetry,

$$\nabla^4\phi = 0 = \nabla^2\nabla^2\phi \quad \ldots (1.66)$$

where $\nabla^2 \equiv \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}\right)$

The stresses are related to ϕ as

$$\sigma_r = \frac{\partial}{\partial z}\left(\nu\nabla^2\phi - \frac{\partial^2\phi}{\partial r^2}\right) \quad \ldots (1.67a)$$

$$\sigma_\theta = \frac{\partial}{\partial z}\left(\nu\nabla^2\phi - \frac{1}{r}\frac{\partial\phi}{\partial r}\right) \quad \ldots (1.67b)$$

$$\sigma_z = \frac{\partial}{\partial z}\left[(2-\nu)\nabla^2\phi - \frac{\partial^2\phi}{\partial z^2}\right] \quad \ldots (1.67c)$$

$$\tau_{rz} = \frac{\partial}{\partial r}\left[(1-\nu)\nabla^2\phi - \frac{\partial^2\phi}{\partial z^2} \right] \quad \ldots (1.67d)$$

1.6.3 EQUATIONS IN TERMS OF DISPLACEMENTS

Cartesian Coordinates

$$(\lambda+G)\frac{\partial\varepsilon_v}{\partial x} + G\nabla^2\rho_x - X = 0 \quad \ldots (1.68a)$$

$$(\lambda+G)\frac{\partial\varepsilon_v}{\partial y} + G\nabla^2\rho_y - Y = 0 \quad \ldots (1.68b)$$

$$(\lambda+G)\frac{\partial\varepsilon_v}{\partial z} + G\nabla^2\rho_z - Z = 0 \quad \ldots (1.68c)$$

where λ, G are Lame's parameters

ε_v = volume strain

= $\varepsilon_x + \varepsilon_y + \varepsilon_z$

Cylindrical Coordinates

For axial symmetry,

$$(\lambda+2G)\left(\frac{\partial^2\rho_r}{\partial r^2} + \frac{1}{r}\frac{\partial\rho_r}{\partial r} - \frac{\rho_r}{r^2}\right) + G\frac{\partial^2\rho_r}{\partial z^2} + (\lambda+G)\frac{\partial^2\rho_z}{\partial r\partial z} = R \quad \ldots (1.69a)$$

$$(\lambda+2G)\frac{\partial^2\rho_z}{\partial z^2} + \frac{G}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\rho_z}{\partial r}\right) + \frac{\rho_r}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\rho_r}{\partial z}\right) + \frac{\lambda}{r}\frac{\partial}{\partial z}\left(\frac{\partial}{\partial r}(\rho_r\, r)\right) = Z \quad \ldots (1.69b)$$

where R, Z are the body forces in the r and z directions.

On the Z axis $(r=0)$ the relevant equation is

$$(\lambda+2G)\frac{\partial^2\rho_z}{\partial z^2} + 2G\frac{\partial^2\rho_z}{\partial r^2} + 2(\lambda+G)\frac{\partial^2\rho_r}{\partial r\partial z} = Z \quad \ldots (1.69c)$$

1.7 Convenient Methods of Considering Loaded Areas

1.7.1 SUPERPOSITION OF RECTANGLES

If the loaded area can be approximated by a rectangle, or by a series of rectangles, and appropriate influence factors for stress or displacement beneath the corner of a rectangle are available, the stress or displacement at any point may be determined by superposition of rectangles.

For the simple case of a single rectangle, the stress beneath an interior point O_I (see Fig.1.11) may simply be calculated as

$$\sigma = p(K_1 + K_2 + K_3 + K_4) \quad \ldots (1.70)$$

where K_1, K_2, K_3, K_4 are the appropriate influence factors for areas 1,2,3 and 4, for the appropriate geometry of each rectangle.

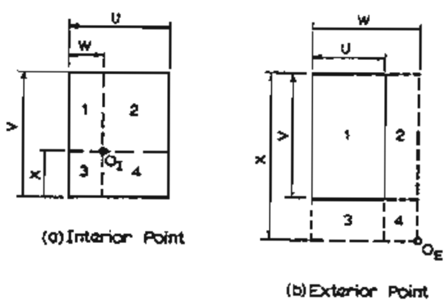


FIG.1.11

For an exterior point O_E (Fig.1.11)

$$\sigma = p(K_{1+2+3+4} - K_{2+4} - K_{3+4} + K_4) \quad \ldots (1.71)$$

For computer calculations, the superposition principle can be stated as (see Fig.1.11):

$$\sigma = J(W,X) - J(W-U,X) - J(W,X-V) + J(W-U,X-V) \quad \ldots (1.72)$$

where $J(m,n)$ = sign $(mn)\sigma(|m|,|n|)$

sign (mn) = 1 when $mn > 0$; -1 when $mn < 0$

$\sigma(|m|,|n|)$ = stress beneath corner of a rectangle $m \times n$.

Displacements are calculated similarly.

For horizontal and shear stresses, care must be taken to take account of the sign of K for each rectangle.

1.7.2 NEWMARK'S METHOD

This method was developed by Newmark (1935) and is a graphical method involving the use of an influence chart, examples of which are shown in Figs.3.68-3.78. A drawing is made of the loaded area to a scale which is marked on the chart, and this drawing is so placed on the chart that the origin of the chart coincides with the point at or beneath which the stress or displacement is required.

The number of blocks covered by the loaded area is then counted and multiplied by an appropriate factor (shown on the chart) and the applied loading to give the required stress or displacement.

When the area is not uniformly loaded, the charts can still be used by considering the non-uniform loading to be made up of several sets of uniformly loaded areas.

In using the charts, parts of blocks may be estimated with sufficient accuracy for practical purposes. In general, the loaded area will be drawn on tracing paper and laid upon the chart.

Several "Newmark Charts" for stresses and displacements in a semi-infinite mass are given in Section 3.6. For a finite layer, Burmister (1956) has prepared charts, but the use of these charts is more complicated as they must be used in conjunction with a table of influence values (see Section 5.4.1).

1.7.3 SECTOR METHOD

This method has been described by Poulos (1967a). For any particular problem, a set of curves relating the stress or displacement influence factor beneath the apex of a uniformly loaded sector to the sector radius may be obtained by integration of the appropriate point load influence factors over a sector. Such sets of curves are referred to as "sector curves", and typical examples are given in Sections 3.6.2 and 5.4.2.

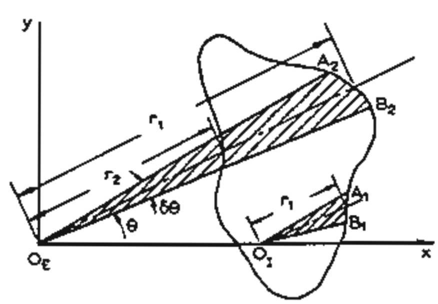


FIG.1.12 Division of loaded area into sectors.

In order to use the sector curves for calculation of the required influence factors for a loaded area of any shape, a scale diagram of the area is drawn, and a number of relatively small-angled sectors are drawn to cut the loaded area, each sector emanating from an apex which lies on the same vertical line as the point at which the influence factor is required. The point on the surface of the elastic solid through which this vertical line passes will be termed the "surface origin". In Fig.1.12, typical sectors $O_E A_2 B_2$ and $O_I A_1 B_1$ are shown in plan for surface origins O_E outside and O_I inside the loaded area.

For invariant stresses such as the bulk stress θ, and for the stress σ_z and the displacement ρ_z, the influence of the typical sector $O_E A_2 B_2$ for the external surface origin O_E is

$$(I_{s_1} - I_{s_2}).\delta\theta$$

where I_{s_1} is the sector influence value at the required depth for a mean sector radius r_1, and similarly for I_{s_2}.

The influence factor at O_E for the whole loaded area is

$$I = \sum (I_{s_1} - I_{s_2}).\delta\theta \quad \ldots (1.73)$$

For the surface origin O_I within the loaded area, the influence factor for the whole loaded area is

$$I = \sum I_{s_1}.\delta\theta \quad \ldots (1.74)$$

When evaluating the influence factor for a stress or displacement which is in a direction other than the z direction, the summation of sector influence factors must be vectorial. To calculate the horizontal stress σ_x in the x direction at O_E, both the tangential and radial stress influence factors for each sector are required, the influence value for the whole loaded area being given by

$$I_{\sigma_x} = \sum\{(\sigma_r I_{s_1} - \sigma_r I_{s_2}).\delta\theta.\cos^2\theta + (\sigma_\theta I_{s_1} - \sigma_\theta I_{s_2})\delta\theta.\sin^2\theta\} \quad \ldots (1.75)$$

where $I\sigma_x$ is the influence factor for due to the loaded area,

$\sigma_r I_{s_1}$, $\sigma_r I_{s_2}$ are the sector influence factors for the radial stress, for sector radii of r_1 and r_2 respectively.

$\sigma_\theta I_{s_1}$, $\sigma_\theta I_{s_2}$ are the sector influence factors for the tangential stress, for sector radii of r_1, r_2.

The influence factors for horizontal stress σ_y in the y direction may be obtained similarly,

$$I_{\sigma_y} = \sum\{(\sigma_r I_{s_1} - \sigma_r I_{s_2}).\delta\theta.\sin^2\theta + (\sigma_\theta I_{s_1} - \sigma_\theta I_{s_2})\delta\theta.\cos^2\theta\} \quad \ldots (1.76)$$

In the same manner, it may be shown that the influence factors for the three shear stresses in the Cartesian coordinate system are as follows:

$$I_{\tau_{yx}} = I_{\tau_{xy}} = \sum\{(\sigma_r I_{s_1} - \sigma_r I_{s_2} - \sigma_\theta I_{s_1} + \sigma_\theta I_{s_2}).\delta\theta.\sin\theta.\cos\theta\} \quad \ldots (1.77a)$$

$$I_{\tau_{zx}} = I_{\tau_{xz}} = \sum\{(\tau_{rz}I_{s_1} - \tau_{rz}I_{s_2}).\cos\theta.\delta\theta \qquad \text{... (1.77b)}$$

$$I_{\tau_{zy}} = I_{\tau_{yz}} = \sum(\tau_{rz}I_{s_1} - \tau_{rz}I_{s_2}).\sin\theta.\delta\theta \qquad \text{... (1.77c)}$$

where $\tau_{rz}I_{s_1}$, $\tau_{rz}I_{s_2}$ are the sector influence factors for radial shear stress, for sector radii of r_1, r_2.

For the displacements ρ_z and ρ_y in the x and y directions for the surface origin O_E, the influence factors are

$$I_{\rho_x} = \sum(\rho_r I_{s_1} - \rho_r I_{s_2}).\delta\theta.\cos\theta \qquad \text{... (1.78a)}$$

$$I_{\rho_y} = \sum(\rho_r I_{s_1} - \rho_r I_{s_2}).\delta\theta.\sin\theta \qquad \text{... (1.78b)}$$

where $I\rho_z$ and $I\rho_y$ are influence factors for the displacements in the x and y directions due to the whole loaded area,

$\rho_r I_{s_1}$, $\rho_r I_{s_2}$ are the sector influence factors for radial displacement, for sector radii of r_1, r_2.

Having found the influence factor I for the whole area, the stresses and displacements for O_E and O due to uniform loading are given in all cases by

$$\sigma = \frac{p}{2\pi} . I \qquad \text{... (1.79)}$$

$$\text{and, } \rho = \frac{pX}{2\pi E} . I \qquad \text{... (1.80)}$$

The accuracy of the influence factors calculated by the sector method increases with the number of sectors used, and the more irregular the shape of the loaded area, the greater is the desirable number of sectors. For the calculation of influence factors for stresses and displacements which are neither invariant nor in the z direction, the sector angle $\delta\theta$ must be small in order to preserve the accuracy of both the magnitude and direction of the calculated influence factor.

Stresses and Displacements Beneath the Centre of a Uniformly Loaded Circle

Beneath the centre of a uniformly loaded circle, the expressions for stresses and displacements reduce to very simple forms.

For σ_z, ρ_z and the invariant stresses,

$$I = 2\pi I_{sa} \qquad \text{... (1.81)}$$

where I_{sa} is the sector influence factor for a sector radius equal to the radius of the circle.

For the horizontal stresses σ_x and σ_y,

$$I_{\sigma_x} = I_{\sigma_y} = \pi(\sigma_r I_{sa} + \sigma_\theta I_{sa}) \qquad \text{... (1.82)}$$

where $\sigma_r I_{sa}$, $\sigma_\theta I_{sa}$ are sector influence factors for σ_θ and σ_r for a sector radius equal to the radius of the circle.

The influence factors for all shear stresses and for the horizontal displacements ρ_x and ρ_y are zero in this case.

1.8 Superposition of Solutions for Various Loadings

Solutions are usually only available for relatively simple types of loading. If the loading pattern is complicated, superposition of solutions for simple loadings may frequently be employed. Examples of the decomposition of complicated loadings into simpler loads have been given by Giroud (1968) in terms of four simple loading types, uniform vertical load, linearly varying vertical load, uniform horizontal load and linearly varying horizontal load. Fig.1.13 shows the examples given by Giroud, the signs in each case referring to the signs of the four simple loading types.

The foregoing is exact for generalized linear loading. The approach can be extended approximately to completely general non-linear loading by division of this loading into a series of general linear loadings.

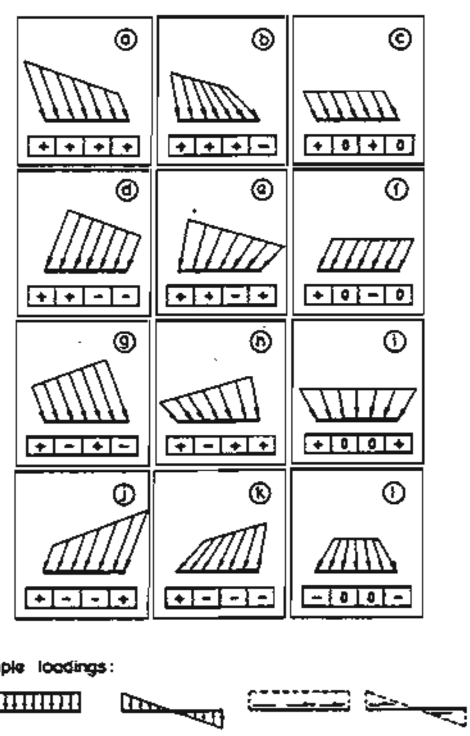


FIG.1.13 Decomposition of loadings (Giroud,1968).

1.9 Equations of Simple Bending Theory

1.9.1 HORIZONTAL BEAM

$$EI\frac{d^2\rho}{dx^2} = -M \qquad \ldots (1.83)$$

where EI = flexural rigidity
ρ = deflection (positive downwards)
x = distance along beam
M = bending moment ("sagging" moments positive, "hogging" moments negative)

Slope
$$\theta = \frac{d\rho}{dx} \qquad \ldots (1.84)$$

Shear force
$$V = -\frac{dM}{dx} \qquad \ldots (1.85a)$$

$$= EI\frac{d^3\rho}{dx^3} \quad \text{(for constant } EI) \qquad \ldots (1.85b)$$

Load per unit length
$$p = -\frac{d^2M}{dx^2} \qquad \ldots (1.86a)$$

$$= EI\frac{d^4\rho}{dx^4} \quad \text{(for constant } EI) \qquad \ldots (1.86b)$$

1.9.2 CIRCULAR PLATE

For axially-symmetrical loading,

$$\frac{d^4\rho}{dr^4} + \frac{2}{r}\frac{d^3\rho}{dr^3} - \frac{1}{r^2}\frac{d^2\rho}{dr^2} + \frac{1}{r^3}\frac{d\rho}{dr} = \frac{q}{D} \qquad \ldots (1.87)$$

where ρ = deflection (positive downwards)
r = radial distance from centre load intensity
q = load intensity
D = flexural rigidity of plate
$= \dfrac{Et^3}{12(1-\nu^2)}$
E = Young's modulus of plate
ν = Poisson's ratio of plate
t = plate thickness

The bending moments M_r and M_θ per unit length in the radial and tangential directions are given by

$$M_r = -D\left(\frac{d^2\phi}{dr^2} + \frac{\nu}{r}\frac{d\rho}{dr}\right) \qquad \ldots (1.88a)$$

and
$$M_\theta = -D\left(\frac{d\rho}{dr} + \nu\frac{d^2\rho}{dr^2}\right) \qquad \ldots (1.88b)$$

1.9.3 RECTANGULAR PLATE

$$\frac{\partial^4\rho}{\partial x^4} + 2\frac{\partial^4\rho}{\partial x^2\partial y^2} + \frac{\partial^4\rho}{\partial y^4} = \frac{q}{D} \qquad \ldots (1.89a)$$

i.e.,
$$\nabla^4\rho = q/D \qquad \ldots (1.89b)$$

where q = intensity of load
D = flexural rigidity of plate as before.

The moments per unit length, M_x and M_y, in the x and y directions are

$$M_x = -D\left(\frac{\partial^2\rho}{\partial x^2} + \nu\frac{\partial^2\rho}{\partial y^2}\right) \qquad \ldots (1.90a)$$

$$M_y = -D\left(\frac{\partial^2\rho}{\partial y^2} + \nu\frac{\partial^2\rho}{\partial x^2}\right) \qquad \ldots (1.90b)$$

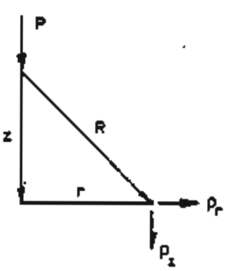
Chapter 2
BASIC SOLUTIONS FOR CONCENTRATED LOADING

2.1 Point Loading

2.1.1 KELVIN PROBLEM

Point load acting within an infinite elastic mass (Fig.2.1).

FIG.2.1



$$\sigma_z = \frac{P}{8\pi(1-\nu)} \left[\frac{3z^3}{R^5} + \frac{(1-2\nu)z}{R^3} \right] \quad \ldots (2.1a)$$

$$\sigma_r = \frac{P}{8\pi(1-\nu)} \frac{z}{R^3} \left[\frac{3r^2}{R^2} - (1-2\nu) \right] \quad \ldots (2.1b)$$

$$\sigma_\theta = -\frac{P(1-2\nu)}{8\pi(1-\nu)} \frac{z}{R^3} \quad \ldots (2.1c)$$

$$\theta = \frac{P}{8\pi(1-\nu)} \cdot \frac{2(1+\nu)z}{R^3} \quad \ldots (2.1d)$$

$$\tau_{rz} = \frac{P}{8\pi(1-\nu)} \frac{r}{R^3} \left[\frac{3z^2}{R^2} + (1-2\nu) \right] \quad \ldots (2.1e)$$

$$\rho_z = \frac{P(1+\nu)}{8\pi(1-\nu)ER} \left[3 - 4\nu + \frac{z^2}{R^2} \right] \quad \ldots (2.1f)$$

$$\rho_r = -\frac{P(1+\nu)}{8\pi(1-\nu)E} \cdot \frac{rz}{R^3} \quad \ldots (2.1g)$$

2.1.2 BOUSSINESQ PROBLEM

Point load acting on the surface of a semi-infinite mass (Fig.2.2).

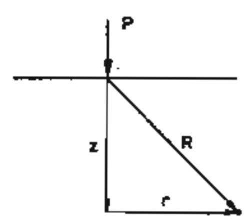


FIG.2.2

$$\sigma_z = \frac{3Pz^3}{2\pi R^5} \quad \ldots (2.2a)$$

$$\sigma_r = -\frac{P}{2\pi R^2} \left[\frac{-3r^2 z}{R^3} + \frac{(1-2\nu)R}{R+z} \right] \quad \ldots (2.2b)$$

$$\sigma_\theta = -\frac{(1-2\nu)P}{2\pi R^2} \left[\frac{z}{R} - \frac{R}{R+z} \right] \quad \ldots (2.2c)$$

$$\theta = \frac{(1+\nu)\, Pz}{\pi R^3} \quad \ldots (2.2d)$$

$$\tau_{rz} = \frac{3Prz^2}{2\pi R^5} \quad \ldots (2.2e)$$

$$\rho_z = \frac{P(1+\nu)}{2\pi ER} \left[2(1-\nu) + \frac{z^2}{R^2} \right] \quad \ldots (2.2f)$$

$$\rho_r = \frac{P(1+\nu)}{2\pi ER} \left[\frac{rz}{R^2} - \frac{(1-2\nu)r}{R+z} \right] \quad \ldots (2.2g)$$

2.1.3 CERUTTI'S PROBLEM

Horizontal point load acting along the surface of a semi-infinite mass (Fig.2.3).

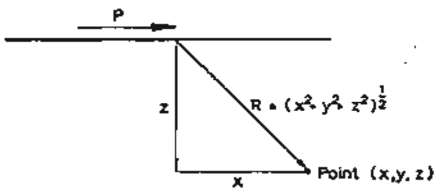


FIG. 2.3

$$\sigma_z = \frac{3Pxz^2}{2\pi R^5} \qquad \ldots (2.3a)$$

$$\sigma_x = \frac{-Px}{2\pi R^3}\left[\frac{-3x^2}{R^2} + \frac{1-2\nu}{(R+z)^2}\left(R^2 - y^2 - \frac{2Ry^2}{R+z}\right)\right] \qquad \ldots (2.3b)$$

$$\sigma_y = \frac{-Px}{2\pi R^3}\left[\frac{-3y^2}{R^2} + \frac{1-2\nu}{(R+z)^2}\left(3R^2 - x^2 - \frac{2Rx^2}{R+z}\right)\right] \qquad \ldots (2.3c)$$

$$\theta = \frac{(1+\nu)Px}{\pi R^3} \qquad \ldots (2.3d)$$

$$\tau_{xy} = \frac{-Py}{2\pi R^3}\left[-\frac{3x^2}{R^2} + \frac{(1-2\nu)}{(R+z)^2}\left(-R^2 + x^2 + \frac{2Rx^2}{R+z}\right)\right] \qquad \ldots (2.3e)$$

$$\tau_{yz} = \frac{3Pxyz}{2\pi R^5} \qquad \ldots (2.3f)$$

$$\tau_{zx} = \frac{3Px^2z}{2\pi R^5} \qquad \ldots (2.3g)$$

$$\rho_z = \frac{P(1+\nu)}{2\pi ER}\left[\frac{xz}{R^2} + \frac{(1-2\nu)x}{R+z}\right] \qquad \ldots (2.3h)$$

$$\rho_x = \frac{P(1+\nu)}{2\pi ER}\left[1 + \frac{x^2}{R^2} + (1-2\nu)\left(\frac{R}{R+z} - \frac{x^2}{(R+z)^2}\right)\right] \qquad \ldots (2.3i)$$

$$\rho_y = \frac{P(1+\nu)}{2\pi ER}\left[\frac{xy}{R^2} - \frac{(1-2\nu)xy}{(R+z)^2}\right] \qquad \ldots (2.3j)$$

2.1.4 MINDLIN'S PROBLEM NO.1

Vertical point load P acting beneath the surface of a semi-infinite mass. (Mindlin, 1936). (Fig.2.4).

$$\sigma_x = \frac{-P}{8\pi(1-\nu)}\left[\frac{(1-2\nu)(z-c)}{R_1^3} - \frac{3x^2(z-c)}{R_1^5} + \frac{(1-2\nu)[3(z-c)-4\nu(z+c)]}{R_2^3} - \frac{3(3-4\nu)x^2(z-c)-6c(z+c)[(1-2\nu)z-2\nu c]}{R_2^5} - \frac{30cx^2z(z+c)}{R_2^7} - \frac{4(1-\nu)(1-2\nu)}{R_2(R_2+z+c)} \times \left(1 - \frac{x^2}{R_2(R_2+z+c)} - \frac{x^2}{R_2^2}\right)\right] \qquad \ldots (2.4a)$$

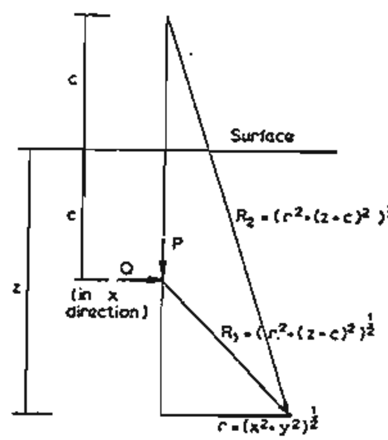


FIG. 2.4

$$\sigma_y = \frac{-P}{8\pi(1-\nu)}\left[\frac{(1-2\nu)(z-c)}{R_1^3} - \frac{3y^2(z-c)}{R_1^5} + \frac{(1-2\nu)[3(z-c)-4\nu(z+c)]}{R_2^3} - \frac{3(3-4\nu)y^2(z-c)-6c(z+c)[(1-2\nu)z-2\nu c]}{R_2^5} - \frac{30cy^2z(z+c)}{R_2^7} - \frac{4(1-\nu)(1-2\nu)}{R_2(R_2+z+c)} \times \left(1 - \frac{y^2}{R_2(R_2+z+c)} - \frac{y^2}{R_2^2}\right)\right] \qquad \ldots (2.4b)$$

$$\sigma_z = \frac{-P}{8\pi(1-\nu)}\left[-\frac{(1-2\nu)(z-c)}{R_1^3} + \frac{(1-2\nu)(z-c)}{R_2^3} - \frac{3(z-c)^3}{R_1^5} - \frac{3(3-4\nu)z(z+c)^2-3c(z+c)(5z-c)}{R_2^5} - \frac{30cz(z+c)^3}{R_2^7}\right] \qquad \ldots (2.4c)$$

$$\tau_{yz} = \frac{-Py}{8\pi(1-\nu)}\left[-\frac{(1-2\nu)}{R_1^3}+\frac{1-2\nu}{R_2^3}-\frac{3(z-c)^2}{R_1^5} - \frac{3(3-4\nu)z(z+c)-3c(3z+c)}{R_2^5}-\frac{30cz(z+c)^2}{R_2^7}\right] \quad \ldots (2.4d)$$

$$\tau_{zx} = \frac{-Px}{8\pi(1-\nu)}\left[-\frac{1-2\nu}{R_1^3}+\frac{1-2\nu}{R_2^3}-\frac{3(z-c)^2}{R_1^5} - \frac{3(3-4\nu)z(z+c)-3c(3z+c)}{R_2^5}-\frac{30cz(z+c)^2}{R_2^7}\right] \quad \ldots (2.4e)$$

$$\tau_{xy} = \frac{-Pxy}{8\pi(1-\nu)}\left[-\frac{3(z-c)}{R_1^5}-\frac{3(3-4\nu)(z-c)}{R_2^5} + \frac{4(1-\nu)(1-2\nu)}{R_2^2(R_2+z+c)}\left(\frac{1}{R_2+z+c}+\frac{1}{R_2}\right) - \frac{30cz(z+c)}{R_2^7}\right] \quad \ldots (2.4f)$$

$$\sigma_r = \frac{-P}{8\pi(1-\nu)}\left[\frac{(1-2\nu)(z-c)}{R_1^3}-\frac{(1-2\nu)(z+7c)}{R_2^3} + \frac{4(1-\nu)(1-2\nu)}{R_2(R_2+z+c)}-\frac{3r^2(z-c)}{R_1^5} + \frac{6c(1-2\nu)(z+c)^2-6c^2(z+c)-3(3-4\nu)r^2(z-c)}{R_2^5} - \frac{30cr^2z(z+c)}{R_2^7}\right] \quad \ldots (2.4g)$$

$$\sigma_\theta = \frac{-P(1-2\nu)}{8\pi(1-\nu)}\left[\frac{z-c}{R_1^3}+\frac{(3-4\nu)(z+c)-6c}{R_2^3} - \frac{4(1-\nu)}{R_2(R_2+z+c)}+\frac{6c(z+c)^2}{R_2^5}-\frac{6c^2(z+c)}{(1-2\nu)R_2^5}\right] \quad \ldots (2.4h)$$

$$\tau_{rz} = \frac{-Pr}{8\pi(1-\nu)}\left[-\frac{1-2\nu}{R_1^3}+\frac{1-2\nu}{R_2^3}-\frac{3(z-c)^2}{R_1^5} - \frac{3(3-4\nu)z(z+c)-3c(3z+c)}{R_2^5}-\frac{30cz(z+c)^2}{R_2^7}\right] \quad \ldots (2.4i)$$

$$\rho_r = \frac{Pr}{16\pi G(1-\nu)}\left[\frac{z-c}{R_1^3}+\frac{(3-4\nu)(z-c)}{R_2^3} - \frac{4(1-\nu)(1-2\nu)}{R_2(R_2+z+c)}+\frac{6cz(z+c)}{R_2^5}\right] \quad \ldots (2.4j)$$

$$\rho_z = \frac{P}{16\pi G(1-\nu)}\left[\frac{3-4\nu}{R_1}+\frac{8(1-\nu)^2-(3-4\nu)}{R_2} + \frac{(z-c)^2}{R_1^3}+\frac{(3-4\nu)(z+c)^2-2cz}{R_2^3}+\frac{6cz(z+c)^2}{R_2^5}\right] \quad \ldots (2.4k)$$

Influence factors for σ_z, and σ_r and σ_θ on the axis have been tabulated by Geddes (1966).

2.1.5 MINDLIN'S PROBLEM NO.2.

Horizontal point load Q acting beneath the surface of a semi-infinite mass. (Mindlin, 1936). (Fig.2.4).

$$\sigma_x = \frac{-Qx}{8\pi(1-\nu)}\left[-\frac{(1-2\nu)}{R_1^3}+\frac{(1-2\nu)(5-4\nu)}{R_2^3}-\frac{3x^2}{R_1^5} - \frac{3(3-4\nu)x^2}{R_2^5}-\frac{4(1-\nu)(1-2\nu)}{R_2(R_2+z+c)^2} \times \left(3-\frac{x^2(3R_2+z+c)}{R_2^2(R_2+z+c)}\right) + \frac{6c}{R_2^5}\left(3c-(3-2\nu)(z+c)+\frac{5x^2z}{R_2^2}\right)\right] \quad \ldots (2.5a)$$

$$\sigma_y = \frac{-Qx}{8\pi(1-\nu)}\left[\frac{(1-2\nu)}{R_1^3}+\frac{(1-2\nu)(3-4\nu)}{R_2^3}-\frac{3y^2}{R_1^5} - \frac{3(3-4\nu)y^2}{R_2^5}-\frac{4(1-\nu)(1-2\nu)}{R_2(R_2+z+c)^2} \times \left(1-\frac{y^2(3R_2+z+c)}{R_2^2(R_2+z+c)}\right)\right.$$

(Continued)

$$+ \frac{6c}{R_2^5}\left(c - (1-2\nu)(z+c) + \frac{5y^2 z}{R_2^2}\right)\Bigg] \qquad \dots (2.5b)$$

$$\sigma_z = \frac{-Qx}{8\pi(1-\nu)}\Bigg[\frac{(1-2\nu)}{R_1^3} - \frac{(1-2\nu)}{R_2^3} - \frac{3(z-c)^2}{R_1^5}$$

$$- \frac{3(3-4\nu)(z+c)^2}{R_2^5}$$

$$+ \frac{6c}{R_2^5}\left(c + (1-2\nu)(z+c) + \frac{5z(z+c)^2}{R_2^2}\right)\Bigg] \qquad \dots (2.5c)$$

$$\tau_{yz} = \frac{-Qxy}{8\pi(1-\nu)}\Bigg[-\frac{3(z-c)}{R_1^5} - \frac{3(3-4\nu)(z+c)}{R_2^5}$$

$$+ \frac{6c}{R_2^5}\left(1 - 2\nu + \frac{5z(z+c)}{R_2^2}\right)\Bigg] \qquad \dots (2.5d)$$

$$\tau_{zx} = \frac{-Q}{8\pi(1-\nu)}\Bigg[-\frac{(1-2\nu)(z-c)}{R_1^3} + \frac{(1-2\nu)(z-c)}{R_2^3}$$

$$- \frac{3x^2(z-c)}{R_1^5} - \frac{3(3-4\nu)x^2(z+c)}{R_2^5}$$

$$- \frac{6c}{R_2^5}\left(z(z+c) - (1-2\nu)x^2 - \frac{5x^2 z(z+c)}{R_2^2}\right)\Bigg] \qquad \dots (2.5e)$$

$$\tau_{xy} = \frac{-Qy}{8\pi(1-\nu)}\Bigg[-\frac{(1-2\nu)}{R_1^3} + \frac{(1-2\nu)}{R_2^3} - \frac{3x^2}{R_1^5}$$

$$- \frac{3(3-4\nu)x^2}{R_2^5} - \frac{4(1-\nu)(1-2\nu)}{R_2(R_2+z+c)^2} \times$$

$$\times\left(1 - \frac{x^2(3R_2+z+c)}{R_2^2(R_2+z+c)}\right) - \frac{6cz}{R_2^5}\left(1 - \frac{5x^2}{R_2^2}\right)\Bigg] \qquad \dots (2.5f)$$

$$\rho_x = \frac{Q}{16\pi G(1-\nu)}\Bigg[\frac{(3-4\nu)}{R_1} + \frac{1}{R_2} + \frac{x^2}{R_1^3} + \frac{(3-4\nu)x^2}{R_2^3}$$

$$+ \frac{2cz}{R_2^3}\left(1 - \frac{3x^2}{R_2^2}\right) + \frac{4(1-\nu)(1-2\nu)}{R_2+z+c} \times$$

$$\times\left(1 - \frac{x^2}{R_2(R_2+z+c)}\right)\Bigg] \qquad \dots (2.5g)$$

$$\rho_y = \frac{Qxy}{16\pi G(1-\nu)}\Bigg[\frac{1}{R_1^3} + \frac{(3-4\nu)}{R_2^3} - \frac{6cz}{R_2^5}$$

$$- \frac{4(1-\nu)(1-2\nu)}{R_2(R_2+z+c)^2}\Bigg] \qquad \dots (2.5h)$$

$$\rho_z = \frac{Qx}{16\pi G(1-\nu)}\Bigg[\frac{z-c}{R_1^3} + \frac{(3-4\nu)(z-c)}{R_2^3}$$

$$- \frac{6cz(z+c)}{R_2^5} + \frac{4(1-\nu)(1-2\nu)}{R_2(R_2+z+c)}\Bigg] \qquad \dots (2.5i)$$

2.1.6 POINT LOAD ON FINITE LAYER

Vertical point load acting at the surface of a layer underlain by a rough rigid base (Fig.2.5). This problem has been studied in detail by Burmister (1943,1945).

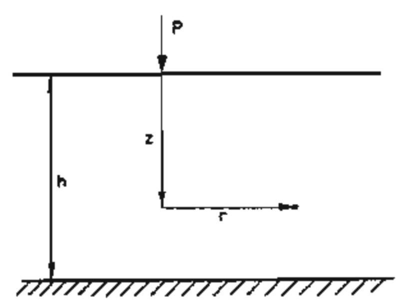


FIG. 2.5

Numerical values for the stresses and displacements in this problem have been tabulated by Poulos (1967b), and are given in TABLES 2.1 to 2.7.

2.1.7 OTHER SOLUTIONS

(a) *Point Loads Inside an Infinite Two Layer System* - Solutions for stresses and displacements have been given by Plevako (1969) for both vertical and horizontal point loading.

(b) *Point Load Within an Elastic Layer* - A formal solution for stresses and displacements has been given by Shekter and Prikhodchenko (1964), but no numerical values are evaluated.

TABLE 2.1

INFLUENCE FACTORS I_{σ_z} FOR VERTICAL STRESS σ_z

POINT LOAD

$$\sigma_z = I_{\sigma_z} P/2\pi h^2$$

r/h \ z/h	I_{σ_z} for Poisson's Ratio ν = 0.0									
	1.0	0.9	0.8	0.7	0.6	0.5	0.4	0.3	0.2	0.1
0.0	4.422	4.993	5.779	7.005	9.017	12.50	19.10	33.35	75.11	300.0
0.1	4.308	4.844	5.571	6.680	8.448	11.37	16.45	25.82	43.04	53.07
0.2	3.989	4.431	5.005	5.826	7.022	8.738	11.05	13.49	13.36	5.400
0.3	3.523	3.842	4.225	4.713	5.316	5.970	6.426	6.067	4.028	0.980
0.4	2.982	3.178	3.386	3.601	3.782	3.827	3.554	2.741	1.419	0.280
0.5	2.432	2.528	2.606	2.642	2.595	2.397	1.937	1.323	0.594	0.112
0.6	1.923	1.947	1.943	1.886	1.747	1.499	1.132	0.690	0.288	0.057
0.7	1.461	1.460	1.412	1.319	1.165	0.944	0.669	0.384	0.155	0.035
0.8	1.114	1.070	1.006	0.910	0.770	0.597	0.404	0.223	0.089	0.023
0.9	0.821	0.768	0.702	0.615	0.504	0.376	0.245	0.131	0.052	0.015
1.0	0.593	0.541	0.481	0.408	0.324	0.233	0.146	0.075	0.029	0.010
1.25	0.238	0.201	0.164	0.127	0.090	0.056	0.028	0.010	0.001	-0.001
1.5	0.074	0.055	0.036	0.020	0.006	-0.006	-0.012	-0.013	-0.010	-0.007
1.75	0.007	-0.003	-0.011	-0.018	-0.021	-0.023	-0.022	-0.018	-0.014	-0.010
2.0	-0.017	-0.020	-0.023	-0.025	-0.025	-0.023	-0.020	-0.017	-0.013	-0.010
2.5	-0.019	-0.018	-0.018	-0.016	-0.015	-0.013	-0.011	-0.009	-0.007	-0.006
3.0	-0.010	-0.009	-0.008	-0.007	-0.006	-0.005	-0.004	-0.003	-0.002	-0.001
3.5	-0.004	-0.003	-0.002	-0.002	-0.001	-0.000	0.000	0.001	0.001	0.001
4.0	-0.000	0.000	0.000	0.001	0.001	0.001	0.002	0.002	0.002	0.002
6.0	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.001
8.0	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
10.0	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000

r/h \ z/h	I_{σ_z} for Poisson's Ratio ν = 0.2									
	1.0	0.9	0.8	0.7	0.6	0.5	0.4	0.3	0.2	0.1
0.0	4.358	5.028	5.853	7.089	9.098	12.57	19.15	33.59	75.13	300.0
0.1	4.243	4.874	5.640	6.761	8.525	11.44	16.51	25.86	43.06	53.08
0.2	3.919	4.448	5.060	5.894	7.089	8.797	11.10	13.52	13.38	5.415
0.3	3.447	3.840	4.259	4.763	5.369	6.018	6.466	6.098	4.050	0.994
0.4	2.902	3.157	3.399	3.630	3.817	3.862	3.585	2.766	1.437	0.293
0.5	2.352	2.490	2.597	2.651	2.613	2.418	1.998	1.341	0.069	0.124
0.6	1.844	1.898	1.918	1.878	1.750	1.507	1.143	0.701	0.298	0.067
0.7	1.407	1.405	1.376	1.298	1.155	0.941	0.671	0.389	0.161	0.041
0.8	1.048	1.015	0.964	0.879	0.752	0.586	0.399	0.222	0.091	0.026
0.9	0.764	0.716	0.659	0.581	0.480	0.360	0.235	0.126	0.051	0.016
1.0	0.543	0.494	0.439	0.374	0.297	0.214	0.133	0.068	0.025	0.008
1.25	0.213	0.172	0.135	0.099	0.066	0.036	0.013	-0.001	-0.007	-0.006
1.5	0.067	0.042	0.021	0.004	-0.010	-0.018	-0.022	-0.021	-0.017	-0.012
1.75	0.010	-0.003	-0.014	-0.022	-0.026	-0.028	-0.026	-0.022	-0.017	-0.012
2.0	-0.006	-0.014	-0.019	-0.022	-0.024	-0.023	-0.020	-0.017	-0.013	-0.010
2.5	-0.008	-0.009	-0.009	-0.010	-0.009	-0.009	-0.007	-0.006	-0.005	-0.004
3.0	-0.002	-0.002	-0.002	-0.001	-0.001	-0.001	-0.001	-0.000	0.000	0.001
3.5	0.001	0.001	0.001	0.002	0.002	0.002	0.002	0.002	0.002	0.002
4.0	0.002	0.002	0.002	0.002	0.002	0.002	0.002	0.002	0.002	0.002
6.0	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
8.0	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000
10.0	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000

r/h \ z/h	I_{σ_z} for Poisson's Ratio ν = 0.4									
	1.0	0.9	0.8	0.7	0.6	0.5	0.4	0.3	0.2	0.1
0.0	4.434	5.259	6.113	7.330	9.297	12.73	19.25	33.64	75.15	300.0
0.1	4.329	5.091	5.887	6.990	8.717	11.58	16.60	25.92	43.08	53.07
0.2	3.980	4.626	5.270	6.093	7.257	8.926	11.18	13.57	13.40	5.404
0.3	3.472	3.963	4.416	4.917	5.502	6.121	6.536	6.135	4.058	0.981
0.4	2.890	3.221	3.496	3.734	3.910	3.936	3.634	2.791	1.440	0.279
0.5	2.306	2.500	2.639	2.705	2.665	2.461	2.026	1.353	0.605	0.108
0.6	1.775	1.865	1.912	1.888	1.767	1.523	1.153	0.702	0.290	0.051
0.7	1.323	1.343	1.336	1.276	1.143	0.935	0.666	0.381	0.150	0.027
0.8	0.959	0.936	0.902	0.833	0.720	0.563	0.382	0.209	0.079	0.014
0.9	0.671	0.632	0.587	0.523	0.436	0.327	0.212	0.109	0.039	0.007
1.0	0.466	0.412	0.365	0.312	0.248	0.177	0.107	0.050	0.015	0.002
1.25	0.163	0.116	0.080	0.050	0.025	0.007	-0.007	-0.012	-0.010	-0.003
1.5	0.043	0.016	-0.007	-0.022	-0.031	-0.004	-0.031	-0.023	-0.013	-0.004
1.75	0.009	-0.007	-0.020	-0.028	-0.031	-0.030	-0.025	-0.017	-0.009	-0.002
2.0	0.003	-0.006	-0.013	-0.017	-0.018	-0.017	-0.014	-0.009	-0.005	0.001
2.5	0.007	0.005	0.003	0.001	0.001	-0.000	0.000	0.001	0.001	0.002
3.0	0.008	0.007	0.006	0.005	0.005	0.004	0.003	0.003	0.003	0.002
3.5	0.006	0.006	0.005	0.004	0.004	0.003	0.003	0.002	0.002	0.002
4.0	0.004	0.003	0.003	0.003	0.002	0.002	0.001	0.001	0.001	0.001
6.0	0.000	0.000	0.000	0.000	0.000	-0.000	-0.000	-0.000	-0.000	-0.000
8.0	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000
10.0	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000

r/h \ z/h	I_{σ_z} for Poisson's Ratio ν = 0.5									
	0.1	0.9	0.8	0.7	0.6	0.5	0.4	0.3	0.2	0.1
0.0	4.695	5.603	6.457	7.631	9.542	12.91	19.39	33.75	75.26	300.0
0.1	4.556	5.417	6.215	7.279	8.951	11.76	16.73	26.01	43.17	53.17
0.2	4.171	4.905	5.554	6.345	7.463	9.080	11.29	13.64	13.44	5.438
0.3	3.612	4.175	4.638	5.118	5.666	6.241	6.610	6.167	4.053	0.947
0.4	2.974	3.359	3.649	3.876	4.027	4.019	3.679	2.795	1.403	0.205
0.5	2.341	2.571	2.726	2.791	2.738	2.512	2.049	1.344	0.561	0.032
0.6	1.770	1.882	1.945	1.927	1.801	1.549	2.162	0.692	0.257	-0.001
0.7	1.291	1.321	1.329	1.279	1.152	0.943	0.669	0.378	0.139	0.010
0.8	0.911	0.891	0.869	0.813	0.710	0.561	0.385	0.215	0.089	0.029
0.9	0.623	0.576	0.540	0.490	0.415	0.319	0.216	0.124	0.064	0.042
1.0	0.412	0.355	0.314	0.272	0.223	0.166	0.111	0.069	0.047	0.046
1.25	0.123	0.072	0.037	0.015	0.000	-0.007	-0.007	0.000	0.014	0.029
1.5	0.023	-0.011	-0.034	-0.048	-0.052	-0.048	-0.039	-0.025	-0.010	0.003
1.75	-0.003	-0.023	-0.037	-0.046	-0.048	-0.046	-0.039	-0.030	-0.021	-0.013
2.0	-0.005	-0.016	-0.024	-0.030	-0.032	-0.033	-0.030	-0.026	-0.021	-0.018
2.5	0.004	-0.012	-0.002	-0.005	-0.007	-0.008	-0.009	-0.010	-0.010	-0.011
3.0	0.008	0.007	0.005	-0.003	0.002	0.001	-0.001	-0.002	-0.003	-0.003
3.5	0.008	0.007	0.006	0.005	0.004	0.003	0.002	0.001	0.001	0.000
4.0	0.006	0.006	0.005	0.004	0.003	0.003	0.002	0.002	0.002	0.002
6.0	0.002	0.002	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.001
8.0	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
10.0	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000

TABLE 2.2

INFLUENCE FACTORS I_{σ_r} FOR RADIAL STRESS σ_r

POINT LOAD

$$\theta_r = I_{\sigma_r} \; P/2\pi h^2$$

I_{σ_r} for Poisson's Ratio $\nu = 0.0$

r/h \ z/h	1.0	0.9	0.8	0.7	0.6	0.5	0.4	0.3	0.2	0.1
0.0	0	-0.444	-0.795	-1.150	-1.598	-2.273	-3.460	-5.960	-12.99	-50.61
0.1	0	-0.431	-0.718	-1.014	-1.350	-1.784	-2.308	-2.686	-0.311	23.14
0.2	0	-0.301	-0.516	-0.673	-0.767	-0.729	0.273	1.325	5.468	7.069
0.3	0	-0.167	-0.263	-0.275	-0.161	0.173	0.937	2.277	3.474	0.401
0.4	0	-0.038	-0.030	0.056	0.264	0.636	1.212	1.776	1.509	-1.197
0.5	0	0.062	0.140	0.269	0.475	0.750	1.043	1.099	0.438	-1.473
0.6	0	0.125	0.238	0.370	0.529	0.683	0.765	0.589	-0.077	-1.395
0.7	0	0.155	0.278	0.391	0.494	0.553	0.514	0.261	-0.307	-1.228
0.8	0	0.159	0.276	0.364	0.421	0.418	0.321	0.065	-0.394	-1.050
0.9	0	0.148	0.250	0.315	0.339	0.302	0.185	-0.048	-0.409	-0.885
1.0	0	0.128	0.213	0.258	0.261	0.209	0.091	-0.108	-0.390	-0.740
1.25	0	0.075	0.119	0.133	0.115	0.064	-0.024	-0.146	-0.296	-0.463
1.5	0	0.036	0.054	0.055	0.036	0.001	-0.054	-0.124	-0.204	-0.285
1.75	0	0.015	0.019	0.041	-0.001	-0.023	-0.055	-0.093	-0.135	-0.175
2.0	0	0.004	0.003	-0.003	-0.015	-0.028	-0.046	-0.066	-0.086	-0.104
2.5	0	-0.003	-0.006	-0.010	-0.015	-0.021	-0.026	-0.032	-0.036	-0.039
3.0	0	-0.003	-0.005	-0.007	-0.010	-0.012	-0.014	-0.015	-0.015	-0.015
3.5	0	-0.002	-0.003	-0.004	-0.005	-0.006	-0.007	-0.007	-0.007	-0.006
4.0	0	-0.001	-0.002	-0.002	-0.003	-0.003	-0.003	-0.003	-0.003	-0.003
6.0	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
8.0	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
10.0	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000

I_{σ_r} for Poisson's Ratio $\nu = 0.2$

r/h \ z/h	1.0	0.9	0.8	0.7	0.6	0.5	0.4	0.3	0.2	0.1
0.0	1.090	0.407	-0.065	-0.452	-0.845	-1.347	-2.143	-3.725	-8.033	-30.71
0.1	1.061	0.431	0.002	-0.327	-0.613	-0.878	-1.054	-0.627	3.865	34.75
0.2	0.980	0.488	0.172	-0.156	-0.677	0.112	0.843	3.011	8.351	12.50
0.3	0.863	0.552	0.376	0.338	0.486	0.926	1.884	3.586	5.412	3.347
0.4	0.726	0.595	0.547	0.615	0.845	1.296	2.000	2.785	2.855	0.614
0.5	0.589	0.604	0.649	0.768	0.990	1.320	1.695	1.885	1.413	-0.259
0.6	0.462	0.576	0.678	0.808	0.980	1.173	1.306	1.213	0.657	-0.528
0.7	0.352	0.522	0.650	0.769	0.885	0.972	0.966	0.766	0.264	-0.578
0.8	0.263	0.452	0.586	0.686	0.756	0.775	0.701	0.479	0.063	-0.544
0.9	0.192	0.378	0.504	0.586	0.624	0.605	0.505	0.297	-0.036	-0.479
1.0	0.137	0.307	0.420	0.465	0.503	0.466	0.363	0.182	-0.080	-0.292
1.25	0.054	0.282	0.266	0.291	0.263	0.226	0.176	0.087	-0.089	-0.181
1.5	0.017	0.080	0.121	0.140	0.136	0.112	0.067	0.008	-0.062	-0.134
1.75	0.008	0.053	0.067	0.074	0.069	0.051	0.031	-0.002	-0.038	-0.069
2.0	0.021	0.011	0.021	0.025	0.025	0.019	0.023	-0.002	-0.018	-0.031
2.5	-0.002	0.000	0.002	0.003	0.003	0.002	-0.002	-0.001	-0.003	-0.003
3.0	-0.001	-0.001	-0.000	-0.000	-0.000	-0.000	0.000	0.001	0.001	0.002
3.5	-0.000	-0.000	-0.000	-0.000	-0.000	0.000	0.000	0.001	0.001	0.002
4.0	0.000	0.000	0.000	0.000	0.000	0.001	0.001	0.001	0.001	0.001
6.0	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.001
8.0	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
10.0	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000

I_{σ_r} for Poisson's Ratio $\nu = 0.4$

r/h \ z/h	1.0	0.9	0.8	0.7	0.6	0.5	0.4	0.3	0.2	0.1
0.0	2.969	1.785	1.024	0.492	0.073	-0.318	-0.775	-1.487	-3.121	-10.92
0.1	2.816	1.777	1.071	0.601	0.288	0.125	0.257	1.442	8.002	46.27
0.2	2.653	1.753	1.192	0.875	0.792	1.049	2.014	4.709	11.20	17.83
0.3	2.315	1.695	1.320	1.172	1.283	1.774	2.888	4.912	7.322	6.214
0.4	1.926	1.590	1.397	1.377	1.573	2.048	2.845	3.814	4.180	2.359
0.5	1.537	1.441	1.411	1.449	1.641	1.980	2.405	2.696	2.377	0.905
0.6	1.183	1.261	1.314	1.404	1.554	1.747	1.906	1.866	1.390	0.306
0.7	0.882	1.066	1.182	1.282	1.384	1.469	1.475	1.303	0.844	0.056
0.8	0.639	0.876	1.022	1.121	1.186	1.204	1.136	0.929	0.537	-0.037
0.9	0.451	0.701	0.856	0.949	0.990	0.973	0.878	0.679	0.362	-0.058
1.0	0.311	0.549	0.700	0.784	0.812	0.781	0.684	0.511	0.262	-0.045
1.25	0.108	0.278	0.391	0.453	0.470	0.445	0.381	0.282	0.158	0.023
1.5	0.030	0.131	0.204	0.247	0.263	0.254	0.224	0.179	0.125	0.073
1.75	0.006	0.061	0.104	0.133	0.147	0.148	0.140	0.126	0.107	0.092
2.0	0.002	0.031	0.055	0.073	0.085	0.091	0.092	0.091	0.089	0.090
2.5	0.003	0.013	0.021	0.029	0.036	0.041	0.047	0.052	0.057	0.062
3.0	0.006	0.009	0.013	0.016	0.020	0.024	0.027	0.031	0.034	0.037
3.5	0.004	0.006	0.008	0.010	0.008	0.014	0.016	0.018	0.020	0.021
4.0	0.003	0.004	0.005	0.007	0.008	0.009	0.010	0.011	0.011	0.012
6.0	0.001	0.001	0.001	0.001	0.002	0.002	0.002	0.002	0.002	0.002
8.0	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
10.0	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000

I_{σ_r} for Poisson's Ratio $\nu = 0.5$

r/h \ z/h	1.0	0.9	0.8	0.7	0.6	0.5	0.4	0.3	0.2	0.1
0.0	4.695	2.981	1.902	1.184	0.672	0.274	-0.067	-0.393	-0.733	-1.101
0.1	4.556	2.947	1.937	1.288	0.884	0.711	0.947	2.464	10.02	51.95
0.2	4.171	2.846	2.018	1.537	1.366	1.606	2.641	5.556	12.58	20.42
0.3	3.612	2.672	2.081	1.793	1.824	2.289	3.437	5.581	8.242	7.561
0.4	2.974	2.429	2.077	1.946	2.071	2.515	3.321	4.344	4.817	3.146
0.5	2.341	2.133	1.983	1.957	2.092	2.397	2.815	3.122	2.841	1.412
0.6	1.770	1.812	1.801	1.847	1.953	2.115	2.259	2.216	1.737	0.743
0.7	1.291	1.492	1.589	1.660	1.732	1.791	1.780	1.597	1.134	0.547
0.8	0.911	1.195	1.349	1.437	1.485	1.484	1.400	1.182	0.785	0.412
0.9	0.623	0.934	1.113	1.210	1.244	1.216	1.110	0.903	0.585	0.320
1.0	0.412	0.725	0.899	0.997	1.026	0.991	0.888	0.712	0.467	0.256
1.25	0.123	0.344	0.492	0.578	0.608	0.593	0.535	0.444	0.331	0.173
1.5	0.023	0.156	0.258	0.313	0.356	0.355	0.342	0.310	0.269	0.135
1.75	-0.003	0.104	0.139	0.203	0.227	0.234	0.240	0.238	0.216	0.102
2.0	-0.003	0.039	0.078	0.110	0.134	0.152	0.163	0.171	0.176	0.082
2.5	0.004	0.019	0.035	0.049	0.063	0.076	0.087	0.098	0.107	0.041
3.0	0.008	0.015	0.023	0.037	0.038	0.045	0.053	0.059	0.065	0.019
3.5	0.008	0.012	0.016	0.021	0.025	0.029	0.034	0.037	0.040	0.008
4.0	0.006	0.009	0.012	0.015	0.017	0.020	0.022	0.024	0.026	0.028
6.0	0.001	0.002	0.003	0.004	0.004	0.004	0.005	0.005	0.005	0.005
8.0	0.000	0.000	0.000	0.000	0.000	0.001	0.001	0.001	0.001	0.001
10.0	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000

TABLE 2.3

INFLUENCE FACTORS I_{σ_θ} FOR TANGENTIAL STRESS σ_θ

POINT LOAD

$$\sigma_\theta = I_{\sigma_\theta} P/2\pi h^2$$

z/h \ r/h	I_{σ_θ} for Poisson's Ratio ν = 0.0									
r/h	1.0	0.9	0.8	0.7	0.6	0.5	0.4	0.3	0.2	0.1
0.0	0	-0.444	-0.795	-1.150	-1.598	-2.273	-3.460	-5.960	-12.99	-30.61
0.1	0	-0.431	-0.769	-1.104	-1.513	-2.101	-3.054	-4.758	-7.822	-6.674
0.2	0	-0.394	-0.697	-0.970	-1.293	-1.681	-2.161	-2.600	-2.002	4.276
0.3	0	-0.341	-0.595	-0.810	-1.014	-1.200	-1.299	-1.064	0.206	3.849
0.4	0	-0.281	-0.482	-0.633	-0.743	-0.788	-0.691	-0.279	0.757	2.738
0.5	0	-0.222	-0.373	-0.472	-0.517	-0.486	-0.324	0.062	0.788	1.913
0.6	0	-0.169	-0.279	-0.339	-0.346	-0.283	-0.119	0.190	0.681	1.353
0.7	0	-0.124	-0.201	-0.235	-0.223	-0.154	-0.011	0.221	0.534	0.972
0.8	0	-0.089	-0.141	-0.158	-0.137	-0.073	0.042	0.213	0.440	0.708
0.9	0	-0.062	-0.096	-0.103	-0.080	-0.025	0.065	0.190	0.346	0.522
1.0	0	-0.042	-0.064	-0.064	-0.042	0.003	0.072	0.162	0.271	0.388
1.25	0	-0.014	-0.018	-0.013	0.003	0.028	0.062	0.103	0.148	0.192
1.5	0	-0.002	-0.001	0.006	0.014	0.027	0.044	0.063	0.081	0.099
1.75	0	0.001	0.004	0.008	0.014	0.021	0.029	0.038	0.046	0.052
2.0	0	0.002	0.005	0.008	0.011	0.015	0.019	0.023	0.026	0.028
2.5	0	0.001	0.003	0.004	0.005	0.007	0.008	0.008	0.009	0.009
3.0	0	0.001	0.001	0.002	0.002	0.003	0.003	0.003	0.003	0.003
3.5	0	0.000	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.001
4.0	0	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
6.0	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000
8.0	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000
10.0	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000

z/h \ r/h	I_{σ_θ} for Poisson's Ratio ν = 0.2									
r/h	1.0	0.9	0.8	0.7	0.6	0.5	0.4	0.3	0.2	0.1
0.0	1.089	0.407	-0.065	-0.452	-0.845	-1.347	-2.143	-3.725	-8.033	-30.71
0.1	1.060	0.400	-0.057	-0.429	-0.797	-1.245	-1.900	-3.003	-4.929	-4.342
0.2	0.979	0.378	-0.039	-0.368	-0.672	-0.996	-1.365	-1.705	-1.430	2.237
0.3	0.861	0.344	-0.014	-0.287	-0.516	-0.712	-0.847	-0.779	-0.097	1.996
0.4	0.725	0.302	0.009	-0.207	-0.367	-0.471	-0.482	-0.303	0.246	1.349
0.5	0.587	0.257	0.025	-0.138	-0.243	-0.296	-0.261	-0.090	0.279	0.877
0.6	0.460	0.211	0.035	-0.086	-0.159	-0.181	-0.136	-0.006	0.231	0.566
0.7	0.351	0.169	0.038	-0.050	-0.100	-0.108	-0.070	0.023	0.171	0.363
0.8	0.261	0.131	0.036	-0.027	-0.061	-0.064	-0.035	0.027	0.119	0.231
0.9	0.190	0.099	0.031	-0.014	-0.037	-0.039	-0.018	0.023	0.079	0.144
1.0	0.136	0.073	0.025	-0.007	-0.023	-0.024	-0.010	0.016	0.050	0.087
1.25	0.053	0.030	0.011	-0.002	-0.008	-0.010	-0.006	0.002	0.011	0.018
1.5	0.016	0.009	0.002	-0.003	-0.005	-0.006	-0.005	-0.004	-0.002	-0.002
1.75	0.002	0.002	-0.002	-0.003	-0.004	-0.005	-0.005	-0.005	-0.005	-0.006
2.0	-0.002	-0.002	-0.003	-0.003	-0.003	-0.004	-0.004	-0.004	-0.004	-0.005
2.5	-0.002	-0.002	-0.001	-0.001	-0.001	-0.001	-0.001	-0.001	-0.001	-0.001
3.0	-0.000	-0.000	-0.000	-0.000	0.000	0.000	0.000	0.000	0.000	0.001
3.5	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.001
4.0	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.001
6.0	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
8.0	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000
10.0	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000

z/h \ r/h	I_{σ_θ} for Poisson's Ratio ν = 0.4									
r/h	1.0	0.9	0.8	0.7	0.6	0.5	0.4	0.3	0.2	0.1
0.0	2.970	1.785	1.024	0.492	0.073	-0.318	-0.775	-1.487	-3.121	-10.91
0.1	2.886	1.742	1.004	0.487	0.082	-0.287	-0.695	-1.245	-2.082	-2.115
0.2	2.653	1.621	0.945	0.470	0.103	-0.214	-0.518	-0.807	-0.903	0.099
0.3	2.315	1.440	0.855	0.440	0.123	-0.134	-0.348	-0.489	-0.439	0.052
0.4	1.926	1.227	0.745	0.397	0.134	-0.072	-0.228	-0.319	-0.298	-0.120
0.5	1.538	1.006	0.625	0.345	0.131	-0.033	-0.156	-0.234	-0.256	-0.226
0.6	1.183	0.796	0.507	0.288	0.118	-0.014	-0.116	-0.189	-0.238	-0.273
0.7	0.882	0.610	0.398	0.231	0.099	-0.007	-0.093	-0.162	-0.221	-0.283
0.8	0.640	0.454	0.303	0.179	0.077	-0.007	-0.079	-0.142	-0.202	-0.269
0.9	0.452	0.329	0.223	0.133	0.056	-0.010	-0.069	-0.124	-0.180	-0.244
1.0	0.311	0.232	0.160	0.096	0.038	-0.014	-0.061	-0.107	-0.156	-0.212
1.25	0.108	0.086	0.061	0.035	0.009	-0.017	-0.043	-0.070	-0.099	-0.132
1.5	0.030	0.026	0.018	0.009	-0.002	-0.014	-0.026	-0.039	-0.054	-0.070
1.75	0.006	0.006	0.005	0.002	-0.002	-0.007	-0.012	-0.018	-0.024	-0.031
2.0	0.002	0.003	0.003	0.002	0.001	-0.001	-0.003	-0.005	-0.007	-0.009
2.5	0.005	0.005	0.006	0.006	0.006	0.006	0.006	0.006	0.007	0.007
3.0	0.006	0.006	0.006	0.006	0.007	0.007	0.007	0.007	0.008	0.008
3.5	0.004	0.004	0.005	0.005	0.005	0.005	0.006	0.006	0.006	0.006
4.0	0.003	0.003	0.003	0.003	0.004	0.004	0.004	0.004	0.004	0.004
6.0	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
8.0	-0.000	-0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
10.0	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000

z/h \ r/h	I_{σ_θ} for Poisson's Ratio ν = 0.5									
r/h	1.0	0.9	0.8	0.7	0.6	0.5	0.4	0.3	0.2	0.1
0.0	4.695	2.981	1.902	1.184	0.672	0.274	-0.067	-0.393	-0.733	-1.101
0.1	4.557	2.905	1.859	1.159	0.658	0.267	-0.069	-0.391	-0.729	-1.100
0.2	4.171	2.690	1.736	1.088	0.618	0.248	-0.074	-0.385	-0.716	-1.089
0.3	3.613	2.373	1.552	0.980	0.557	0.218	-0.080	-0.373	-0.688	-1.056
0.4	2.975	2.001	1.330	0.849	0.482	0.183	-0.086	-0.353	-0.644	-0.991
0.5	2.341	1.618	1.096	0.707	0.401	0.145	-0.090	-0.326	-0.586	-0.896
0.6	1.770	1.259	0.871	0.568	0.321	0.108	-0.092	-0.294	-0.517	-0.780
0.7	1.291	0.946	0.668	0.440	0.247	0.074	-0.091	-0.259	-0.443	-0.658
0.8	0.912	0.688	0.496	0.330	0.182	0.043	-0.087	-0.223	-0.371	-0.540
0.9	0.623	0.485	0.356	0.238	0.128	0.023	-0.082	-0.189	-0.305	-0.434
1.0	0.413	0.331	0.247	0.165	0.085	0.005	-0.075	-0.158	-0.247	-0.343
1.25	0.124	0.107	0.083	0.053	0.019	-0.017	-0.056	-0.096	-0.138	-0.181
1.5	0.023	0.023	0.017	0.008	-0.005	-0.020	-0.037	-0.054	-0.072	-0.089
1.75	-0.003	-0.001	-0.002	-0.005	-0.009	-0.014	-0.021	-0.027	-0.034	-0.040
2.0	-0.004	-0.003	-0.003	-0.004	-0.005	-0.007	-0.009	-0.011	-0.012	-0.013
2.5	0.005	0.005	0.004	0.004	0.004	0.005	0.005	0.005	0.006	0.007
3.0	0.009	0.008	0.008	0.008	0.008	0.008	0.009	0.009	0.010	0.011
3.5	0.008	0.008	0.008	0.008	0.008	0.008	0.009	0.009	0.009	0.010
4.0	0.007	0.007	0.007	0.007	0.007	0.007	0.007	0.007	0.008	0.008
6.0	0.002	0.002	0.002	0.002	0.002	0.002	0.002	0.002	0.002	0.002
8.0	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.001
10.0	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000

TABLE 2.4

INFLUENCE FACTORS I_θ FOR BULK STRESS Θ

POINT LOAD

$$\Theta = I_\theta P/2\pi h^2$$

z/h \ r/h	I_θ for Poisson's Ratio ν = 0.0									
r/h	1.0	0.9	0.8	0.7	0.6	0.5	0.4	0.3	0.2	0.1
0.0	4.422	4.105	4.188	4.705	5.821	7.958	12.18	21.63	49.12	198.8
0.1	4.308	4.008	4.084	4.563	5.586	7.495	11.09	18.38	34.90	69.34
0.2	3.989	3.736	3.793	4.174	4.963	6.337	8.614	12.21	16.82	16.74
0.3	3.523	3.333	3.368	3.628	4.142	4.951	6.063	7.280	7.708	5.229
0.4	2.982	2.859	2.874	3.024	3.303	3.683	4.075	4.238	3.685	1.822
0.5	2.432	2.368	2.372	2.440	2.552	2.667	2.696	2.484	1.820	0.552
0.6	1.923	1.903	1.902	1.917	1.930	1.904	1.778	1.469	0.891	0.015
0.7	1.481	1.490	1.489	1.475	1.436	1.347	1.171	0.867	0.402	-0.221
0.8	1.114	1.140	1.140	1.140	1.054	0.945	0.767	0.500	0.135	-0.319
0.9	0.821	0.854	0.856	0.827	0.762	0.655	0.494	0.273	-0.011	-0.348
1.0	0.593	0.627	0.630	0.602	0.542	0.446	0.309	0.130	-0.090	-0.342
1.25	0.238	0.262	0.266	0.247	0.208	0.147	0.066	-0.033	-0.147	-0.271
1.5	0.074	0.088	0.089	0.079	0.056	0.022	-0.022	-0.074	-0.132	-0.192
1.75	0.007	0.013	0.013	0.006	-0.006	-0.024	-0.047	-0.073	-0.101	-0.130
2.0	-0.017	-0.015	-0.016	-0.020	-0.027	-0.036	-0.047	-0.060	-0.073	-0.086
2.5	-0.018	-0.019	-0.021	-0.022	-0.025	-0.027	-0.029	-0.032	-0.034	-0.036
3.0	-0.010	-0.011	-0.011	-0.012	-0.013	-0.014	-0.014	-0.014	-0.014	-0.014
3.5	-0.003	-0.004	-0.005	-0.005	-0.005	-0.005	-0.005	-0.005	-0.005	-0.004
4.0	-0.001	-0.001	-0.001	-0.001	-0.001	-0.001	-0.001	-0.001	-0.001	-0.001
6.0	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.001
8.0	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
10.0	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000

z/h \ r/h	I_θ for Poisson's Ratio ν = 0.2									
r/h	1.0	0.9	0.8	0.7	0.6	0.5	0.4	0.3	0.2	0.1
0.0	6.537	5.834	5.724	6.184	7.405	9.875	14.86	26.13	59.06	238.6
0.1	6.364	5.705	5.584	6.005	7.115	9.315	13.55	22.23	42.00	83.49
0.2	5.879	5.314	5.193	5.511	6.350	7.914	10.58	14.83	20.30	20.15
0.3	5.171	4.736	4.621	4.814	5.337	6.231	7.502	8.905	9.365	6.337
0.4	4.353	4.055	3.954	4.039	4.295	4.687	5.102	5.249	4.538	2.256
0.5	3.527	3.351	3.271	3.281	3.356	3.442	3.342	3.136	2.300	0.742
0.6	2.766	2.685	2.630	2.599	2.570	2.500	2.313	1.909	1.185	0.103
0.7	2.110	2.096	2.064	2.017	1.940	1.805	1.567	1.177	0.597	-0.174
0.8	1.572	1.598	1.585	1.537	1.447	1.297	1.065	0.728	0.274	-0.287
0.9	1.146	1.193	1.194	1.153	1.067	0.927	0.723	0.445	0.094	0.320
1.0	0.818	0.874	0.884	0.852	0.777	0.656	0.487	0.265	-0.005	-0.197
1.25	0.319	0.483	0.412	0.388	0.320	0.253	0.184	0.188	-0.085	-0.169
1.5	0.100	0.131	0.144	0.141	0.122	0.087	0.040	-0.017	-0.081	-0.148
1.75	0.020	0.050	0.051	0.049	0.038	0.016	0.001	-0.029	-0.059	-0.087
2.0	-0.011	-0.003	-0.001	-0.000	-0.002	-0.007	-0.001	-0.022	-0.035	-0.046
2.5	-0.011	-0.011	-0.009	-0.008	-0.008	-0.008	-0.008	-0.008	-0.008	-0.008
3.0	-0.002	-0.002	-0.002	-0.002	-0.001	-0.001	-0.000	0.001	0.002	0.003
3.5	0.002	0.002	0.002	0.002	0.002	0.002	0.003	0.003	0.004	0.005
4.0	0.003	0.003	0.003	0.003	0.003	0.003	0.003	0.003	0.003	0.004
6.0	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.002	0.001
8.0	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000
10.0	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000

z/h \ r/h	I_θ for Poisson's Ratio ν = 0.4									
r/h	1.0	0.9	0.8	0.7	0.6	0.5	0.4	0.3	0.2	0.1
0.0	10.39	8.826	8.158	8.309	9.438	12.08	17.70	30.66	68.90	278.2
0.1	10.10	8.610	7.961	8.077	9.086	11.42	16.17	26.11	49.00	97.23
0.2	9.286	8.000	7.406	7.438	8.153	9.761	12.68	17.47	23.69	23.34
0.3	8.102	7.098	6.592	6.529	6.909	7.761	9.075	10.56	10.94	7.248
0.4	6.742	6.038	5.638	5.507	5.617	5.911	6.251	6.286	5.321	2.518
0.5	5.381	4.947	4.675	4.499	4.438	4.407	4.275	3.816	2.726	0.788
0.6	4.142	3.921	3.733	3.580	3.438	3.256	2.943	2.379	1.442	0.084
0.7	3.088	3.019	2.916	2.788	2.626	2.396	2.048	1.522	0.773	-0.200
0.8	2.238	2.266	2.226	2.133	1.983	1.760	1.439	0.995	0.414	-0.292
0.9	1.580	1.662	1.166	1.605	1.482	1.290	1.021	0.665	0.221	-0.295
1.0	1.087	1.194	1.225	1.191	1.098	0.945	0.730	0.454	0.121	-0.255
1.25	0.379	0.479	0.531	0.538	0.504	0.434	0.330	0.120	0.049	-0.112
1.5	0.106	0.173	0.215	0.234	0.230	0.207	0.170	0.116	0.058	-0.001
1.75	0.022	0.061	0.089	0.106	0.113	0.112	0.103	0.089	0.073	0.059
2.0	0.007	0.028	0.045	0.059	0.068	0.073	0.076	0.077	0.078	0.081
2.5	0.017	0.023	0.030	0.036	0.042	0.048	0.053	0.059	0.065	0.071
3.0	0.019	0.022	0.025	0.028	0.031	0.034	0.038	0.041	0.044	0.047
3.5	0.015	0.016	0.018	0.020	0.021	0.023	0.025	0.026	0.028	0.029
4.0	0.009	0.010	0.011	0.012	0.013	0.014	0.015	0.016	0.017	0.017
6.0	0.001	0.001	0.001	0.001	0.002	0.002	0.002	0.002	0.002	0.002
8.0	-0.000	-0.000	-0.000	-0.000	-0.000	0.000	0.000	0.000	0.000	0.000
10.0	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000

z/h \ r/h	I_θ for Poisson's Ratio ν = 0.5									
r/h	1.0	0.9	0.8	0.7	0.6	0.5	0.4	0.3	0.2	0.1
0.0	14.08	11.56	10.26	9.999	10.88	13.46	19.26	32.97	73.79	298.0
0.1	13.67	11.27	10.01	9.726	10.49	12.74	17.61	28.09	53.45	104.0
0.2	12.52	10.44	9.308	8.970	9.447	10.93	13.86	18.81	25.30	24.77
0.3	10.84	9.219	8.271	7.891	8.047	8.748	9.968	11.38	11.61	7.452
0.4	8.923	7.789	7.056	6.670	6.581	6.716	6.914	6.786	5.576	2.360
0.5	7.022	6.322	5.805	5.455	5.231	5.053	4.773	4.140	2.816	0.548
0.6	5.310	4.953	4.627	4.342	4.077	3.771	3.329	2.614	1.487	-0.038
0.7	3.873	3.759	3.587	3.379	3.131	2.808	2.358	1.716	0.829	-0.101
0.8	2.734	2.776	2.714	2.580	2.376	2.091	1.698	1.173	0.503	-0.099
0.9	1.869	1.995	2.010	1.937	1.787	1.558	1.243	0.837	0.343	-0.072
1.0	1.237	1.401	1.460	1.434	1.333	1.163	0.925	0.623	0.267	-0.040
1.25	0.370	0.523	0.612	0.645	0.627	0.568	0.476	0.348	0.208	0.021
1.5	0.068	0.169	0.217	0.273	0.305	0.287	0.267	0.231	0.187	0.048
1.75	-0.010	0.072	0.108	0.153	0.170	0.174	0.182	0.180	0.162	0.049
2.0	-0.013	0.021	0.051	0.076	0.097	0.113	0.125	0.135	0.143	0.051
2.5	0.013	0.025	0.037	0.049	0.061	0.072	0.083	0.093	0.103	0.038
3.0	0.024	0.030	0.036	0.042	0.048	0.054	0.061	0.067	0.073	0.026
3.5	0.024	0.027	0.030	0.034	0.037	0.041	0.044	0.047	0.050	0.018
4.0	0.019	0.021	0.023	0.025	0.028	0.030	0.032	0.033	0.035	0.012
6.0	0.005	0.006	0.006	0.007	0.007	0.007	0.008	0.008	0.008	0.008
8.0	0.001	0.001	0.001	0.001	0.001	0.002	0.002	0.002	0.002	0.002
10.0	-0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000

TABLE 2.5

INFLUENCE FACTORS $I_{\tau_{rz}}$ FOR SHEAR STRESS τ_{rz}

POINT LOAD

$$\tau_{rz} = I_{\tau_{rz}} \; P/2\pi h^2$$

$I_{\tau_{rz}}$ for Poisson's Ratio ν = 0.0

r/h \ z/h	1.0	0.9	0.8	0.7	0.6	0.5	0.4	0.3	0.2	0.1
0	0	0	0	0	0	0	0	0	0	0
0.1	0.258	0.312	0.462	0.731	1.204	2.093	3.957	8.480	21.42	53.01
0.2	0.473	0.564	0.814	1.246	1.957	3.154	5.230	8.750	13.18	10.69
0.3	0.616	0.719	0.999	1.456	2.138	3.116	4.416	5.735	5.791	2.781
0.4	0.678	0.774	1.025	1.408	1.916	2.517	3.080	3.262	2.540	0.928
0.5	0.671	0.747	0.939	1.211	1.527	1.820	1.966	1.785	1.164	0.347
0.6	0.614	0.668	0.796	0.963	1.128	1.232	1.199	0.962	0.541	0.124
0.7	0.529	0.564	0.638	0.725	0.791	0.797	0.706	0.505	0.294	0.027
0.8	0.435	0.455	0.490	0.523	0.532	0.493	0.398	0.249	0.086	-0.016
0.9	0.344	0.353	0.363	0.365	0.345	0.294	0.209	0.104	0.008	-0.035
1.0	0.262	0.265	0.261	0.246	0.214	0.163	0.095	0.023	-0.031	-0.041
1.25	0.112	0.111	0.098	0.075	0.045	0.011	-0.022	-0.046	-0.052	-0.032
1.5	0.032	0.032	0.023	0.009	-0.009	-0.026	-0.038	-0.044	-0.036	-0.014
1.75	-0.004	-0.002	-0.006	-0.012	-0.020	-0.026	-0.029	-0.026	-0.017	0.001
2.0	-0.016	-0.013	-0.013	-0.014	-0.016	-0.017	-0.015	-0.010	-0.002	0.011
2.5	-0.014	-0.011	-0.008	-0.006	-0.003	-0.000	0.004	0.009	0.015	0.021
3.0	-0.007	-0.004	-0.001	0.002	0.005	0.008	0.012	0.015	0.019	0.023
3.5	-0.002	0.001	0.003	0.006	0.008	0.011	0.013	0.017	0.019	0.022
4.0	0.001	0.003	0.005	0.007	0.009	0.011	0.014	0.016	0.018	0.020
6.0	0.001	0.003	0.004	0.005	0.006	0.007	0.008	0.009	0.010	0.011
8.0	0.001	0.001	0.002	0.002	0.003	0.004	0.004	0.005	0.005	0.006
10.0	0.000	0.001	0.001	0.001	0.002	0.002	0.002	0.003	0.003	0.003

$I_{\tau_{rz}}$ for Poisson's Ratio ν = 0.2

r/h \ z/h	1.0	0.9	0.8	0.7	0.6	0.5	0.4	0.3	0.2	0.1
0	0	0	0	0	0	0	0	0	0	0
0.1	0.330	0.342	0.472	0.731	1.199	2.086	3.949	8.472	21.42	53.00
0.2	0.607	0.623	0.835	1.248	1.949	3.141	5.215	8.734	13.16	10.68
0.3	0.797	0.804	1.032	1.462	2.129	3.099	4.395	5.713	5.771	2.765
0.4	0.889	0.880	1.071	1.420	1.909	2.499	3.056	3.237	2.516	0.909
0.5	0.896	0.868	0.996	1.231	1.524	1.804	1.943	1.758	1.138	0.326
0.6	0.839	0.798	0.864	0.992	1.131	1.220	1.178	0.937	0.515	0.102
0.7	0.746	0.697	0.713	0.761	0.801	0.790	0.689	0.482	0.215	0.005
0.8	0.636	0.586	0.570	0.566	0.550	0.495	0.386	0.229	0.064	-0.038
0.9	0.526	0.480	0.446	0.413	0.368	0.299	0.202	0.088	-0.012	-0.056
1.0	0.424	0.386	0.343	0.297	0.242	0.173	0.093	0.012	-0.048	-0.061
1.25	0.229	0.206	0.170	0.127	0.079	0.031	-0.015	-0.049	-0.063	-0.049
1.5	0.114	0.103	0.082	0.054	0.024	-0.005	-0.029	-0.043	-0.045	-0.031
1.75	0.054	0.050	0.039	0.024	0.007	-0.008	-0.020	-0.027	-0.026	-0.017
2.0	0.025	0.024	0.019	0.012	0.004	-0.004	-0.010	-0.013	-0.012	-0.008
2.5	0.007	0.007	0.007	0.006	0.004	0.003	0.001	0.000	-0.000	-0.000
3.0	0.004	0.005	0.005	0.005	0.005	0.004	0.004	0.003	0.002	0.001
3.5	0.004	0.004	0.004	0.004	0.004	0.004	0.003	0.003	0.002	0.001
4.0	0.004	0.004	0.004	0.003	0.003	0.003	0.003	0.002	0.002	0.001
6.0	0.002	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.000	0.000
8.0	0.001	0.001	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
10.0	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000

$I_{\tau_{rz}}$ for Poisson's Ratio ν = 0.4

r/h \ z/h	1.0	0.9	0.8	0.7	0.6	0.5	0.4	0.3	0.2	0.1
0.0	0	0	0	0	0	0	0	0	0	0
0.1	0.436	0.375	0.470	0.713	1.175	2.060	3.923	8.448	21.40	52.99
0.2	0.806	0.687	0.834	1.216	1.904	3.091	5.165	8.689	13.12	10.65
0.3	1.064	0.896	1.037	1.423	2.069	3.032	4.327	5.649	5.717	2.727
0.4	1.195	0.995	1.085	1.381	1.843	2.421	2.976	3.161	2.450	0.862
0.5	1.214	1.001	1.024	1.198	1.459	1.723	1.856	1.675	1.065	0.273
0.6	1.149	0.941	0.904	0.969	1.073	1.142	1.091	0.851	0.439	0.045
0.7	1.033	0.843	0.765	0.752	0.753	0.720	0.607	0.391	0.139	-0.052
0.8	0.893	0.730	0.631	0.570	0.514	0.435	0.313	0.152	-0.008	-0.094
0.9	0.750	0.616	0.512	0.428	0.346	0.252	0.140	0.020	-0.078	-0.109
1.0	0.616	0.510	0.412	0.322	0.232	0.139	0.042	-0.047	-0.107	-0.110
1.25	0.352	0.301	0.235	0.164	0.091	0.022	-0.038	-0.082	-0.102	-0.088
1.5	0.191	0.170	0.135	0.091	0.045	0.002	-0.034	-0.059	-0.068	-0.060
1.75	0.102	0.096	0.079	0.055	0.030	0.004	-0.017	-0.032	-0.040	-0.040
2.0	0.056	0.055	0.048	0.036	0.022	0.008	-0.005	-0.016	-0.023	-0.028
2.5	0.020	0.020	0.019	0.016	0.012	0.006	0.001	-0.006	-0.012	-0.019
3.0	0.009	0.008	0.007	0.006	0.004	0.001	-0.003	-0.007	-0.011	-0.016
3.5	0.003	0.002	0.001	0.000	-0.001	-0.003	-0.006	-0.008	-0.011	-0.015
4.0	-0.000	-0.001	-0.002	-0.003	-0.004	-0.006	-0.007	-0.009	-0.011	-0.013
6.0	-0.004	-0.004	-0.005	-0.005	-0.005	-0.006	-0.006	-0.007	-0.007	-0.008
8.0	-0.003	-0.004	-0.004	-0.004	-0.004	-0.004	-0.004	-0.004	-0.005	-0.005
10.0	-0.002	-0.003	-0.003	-0.003	-0.003	-0.003	-0.003	-0.003	-0.003	-0.003

$I_{\tau_{rz}}$ for Poisson's Ratio ν = 0.5

r/h \ z/h	1.0	0.9	0.8	0.7	0.6	0.5	0.4	0.3	0.2	0.1
0.0	0	0	0	0	0	0	0	0	0	0
0.1	0.530	0.395	0.459	0.690	1.148	2.035	3.905	8.440	21.40	53.01
0.2	0.980	0.728	0.815	1.173	1.853	3.043	5.128	8.668	13.13	10.67
0.3	1.294	0.957	1.017	1.367	2.001	2.965	4.270	5.610	5.702	2.739
0.4	1.456	1.074	1.071	1.322	1.766	2.340	2.901	3.101	2.412	0.851
0.5	1.481	1.094	1.020	1.144	1.380	1.636	1.771	1.599	1.007	0.241
0.6	1.406	1.044	0.915	0.927	1.002	1.057	1.004	0.769	0.372	0.007
0.7	1.268	0.952	0.791	0.726	0.696	0.646	0.527	0.322	0.075	-0.087
0.8	1.102	0.841	0.671	0.562	0.476	0.378	0.249	0.089	-0.059	-0.118
0.9	0.932	0.725	0.565	0.438	0.328	0.216	0.096	-0.023	-0.110	-0.119
1.0	0.773	0.616	0.474	0.348	0.234	0.124	0.020	-0.068	-0.119	-0.105
1.25	0.463	0.394	0.309	0.219	0.132	0.053	-0.013	-0.058	-0.075	-0.057
1.5	0.274	0.251	0.209	0.158	0.105	0.056	0.015	-0.012	-0.024	-0.020
1.75	0.169	0.164	0.146	0.120	0.091	0.063	0.038	0.019	0.006	0.001
2.0	0.111	0.113	0.106	0.094	0.078	0.062	0.046	0.032	0.020	0.009
2.5	0.059	0.061	0.060	0.057	0.053	0.046	0.039	0.030	0.021	0.011
3.0	0.037	0.038	0.037	0.036	0.033	0.030	0.026	0.021	0.015	0.008
3.5	0.025	0.025	0.024	0.023	0.021	0.019	0.016	0.013	0.009	0.005
4.0	0.017	0.017	0.016	0.015	0.014	0.012	0.010	0.008	0.006	0.003
6.0	0.004	0.004	0.004	0.003	0.003	0.003	0.002	0.002	0.001	0.001
8.0	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.000	0.000	0.000
10.0	0.001	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000

TABLE 2.6
INFLUENCE VALUES FOR SURFACE DISPLACEMENTS
POINT LOAD
$\rho = I_\rho P/2\pi hE$

VERTICAL DISPLACEMENT ρ_z

(Taylor, 1962)

r/h \ ν	0	0.2	0.4	0.5
0.05	37.580	35.921	31.052	27.351
0.1	17.586	16.728	14.260	13.360
0.2	7.624	7.162	5.897	4.914
0.3	4.327	4.016	3.154	2.480
0.4	2.720	2.478	1.827	1.320
0.5	1.792	1.599	1.092	0.699
0.6	1.212	1.048	0.635	0.290
0.7	0.823	0.690	0.352	0.051
0.8	0.560	0.450	0.168	-0.079
0.9	0.373	0.286	0.053	-0.160
1.0	0.250	0.182	-0.011	-0.183
1.25	0.080	0.031	-0.085	-0.194
1.5	0.013	-0.002	-0.077	-0.156
1.75	-0.007	-0.007	-0.048	-0.123
2.0	-0.012	-0.011	-0.039	-0.083
2.5	-0.004	-0.017	-0.025	-0.036
3.0	-0.003	0.001	-0.008	-0.025
3.5	-0.003	0.000	-0.004	-0.018
4.0	-0.001	0.000	-0.003	-0.012
6.0	-0.000	-0.000	-0.001	-0.002
8.0	-0.000	-0.000	-0.000	-0.000
10.0	-0.000	-0.000	-0.000	-0.000

TABLE 2.7
INFLUENCE VALUES FOR SURFACE DISPLACEMENTS
POINT LOAD
$\rho = I_\rho P/2\pi hE$

RADIAL DISPLACEMENT ρ_r

(Taylor, 1962)

r/h \ ν	0	0.2	0.4	0.5
0.05	19.959	14.362	5.559	-0.041
0.1	9.948	7.124	2.723	-0.078
0.2	4.896	3.455	1.250	-0.156
0.3	3.183	2.184	0.716	-0.225
0.4	2.308	1.523	0.426	-0.288
0.5	1.773	1.064	0.232	-0.326
0.6	1.277	0.824	0.102	-0.376
0.7	1.000	0.620	0.008	-0.405
0.8	0.789	0.465	-0.063	-0.420
0.9	0.627	0.349	-0.111	-0.421
1.0	0.499	0.259	-0.141	-0.417
1.25	0.292	0.150	-0.175	-0.380
1.5	0.167	0.048	-0.163	-0.315
1.75	0.097	0.012	-0.134	-0.250
2.0	0.060	0.002	-0.109	-0.195
2.5	0.027	0.003	-0.070	-0.118
3.0	0.010	-0.002	-0.038	-0.072
3.5	0.003	-0.008	-0.022	-0.046
4.0	0.002	-0.000	-0.014	-0.029
6.0	0.000	-0.000	-0.002	-0.002
8.0	0.000	-0.000	-0.000	-0.001
10.0	0.000	-0.000	-0.000	-0.000

2.2 Line Loading

In sub-sections 2.2.1 to 2.2.7, only solutions for stress are presented. Displacements due to line loading on or in a semi-infinite mass are only meaningful if evaluated as the displacement of one point relative to another point, both points being located neither at the origin of loading nor at infinity.

2.2.1 INFINITE LINE LOAD ACTING WITHIN AN INFINITE SOLID (Integrated Kelvin problem). (Fig.2.6).

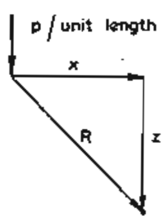


FIG. 2.6

$$\sigma_z = \frac{p}{2\pi(1-\nu)} \frac{z}{R^2} \left[\frac{(3-2\nu)}{2} - \frac{x^2}{R^2} \right] \quad \ldots (2.6a)$$

$$\sigma_x = \frac{p}{2\pi(1-\nu)} \frac{z}{R^2} \left[- \frac{(1-2\nu)}{2} + \frac{x^2}{R^2} \right] \quad \ldots (2.6b)$$

$$\sigma_y = \frac{p}{2\pi} \frac{\nu}{(1-\nu)} \frac{z}{R^2} \quad \ldots (2.6c)$$

$$\tau_{xz} = \frac{p}{2\pi(1-\nu)} \frac{x}{R^2} \left[\frac{(1-2\nu)}{2} + \frac{z^2}{R^2} \right] \quad \ldots (2.6d)$$

2.2.2 INFINITE VERTICAL LINE LOAD ON THE SURFACE OF A SEMI-INFINITE MASS (Integrated Boussinesq problem)(Fig.2.7).

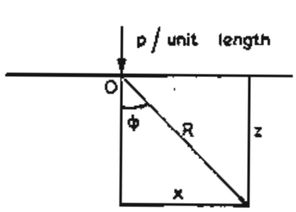


FIG. 2.7

$$\sigma_x = \frac{2p}{\pi} \frac{x^2 z}{R^4} \quad \ldots (2.7a)$$

$$\sigma_z = \frac{2p}{\pi} \frac{z^3}{R^4} \quad \ldots (2.7b)$$

$$\sigma_y = \frac{2p\nu}{\pi} \frac{z}{R^2} \quad \ldots (2.7c)$$

$$\tau_{xz} = \frac{2p}{\pi} \frac{xz^2}{R^4} \quad \ldots (2.7d)$$

Principal Stresses:

$$\sigma_1 = \sigma_R = \frac{2p}{\pi} \frac{z}{R^2} \quad \ldots (2.8a)$$

$$\sigma_2 = \sigma_\psi = 0 \quad \ldots (2.8b)$$

$$\tau_{max} = \frac{p}{\pi} \frac{z}{R^2} \quad \ldots (2.8c)$$

Loci of constant σ_1, σ_2 and τ_{max}
are circles tangent to x-axis at 0.

Trajectories of σ_1
are radial lines through 0.

Trajectories of σ_2
are a family of semi-circles, centres at 0.

Trajectories of τ_{max}
are two orthogonal families of equi-angular spirals intersecting the radial lines at ±45°.

For the case of a vertical line load of finite length, influence values for σ_z are tabulated by Lysmer and Duncan (1969).

2.2.3 HORIZONTAL LINE LOAD ACTING ON SURFACE OF SEMI-INFINITE MASS (Integrated Cerruti Problem)(Fig.2.8).

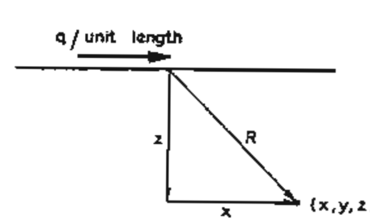


FIG. 2.8

$$\sigma_z = \frac{2qxz^2}{\pi R^4} \qquad \ldots (2.9a)$$

$$\sigma_x = \frac{2qx^3}{\pi R^4} \qquad \ldots (2.9b)$$

$$\sigma_y = \frac{2qxz^2}{\pi R^4} \qquad \ldots (2.9c)$$

$$\tau_{xz} = \frac{2qx^2z}{\pi R^4} \qquad \ldots (2.9d)$$

2.2.4 MELAN'S PROBLEM I

Vertical line loading p/unit length beneath the surface of a semi-infinite mass (Fig.2.9).

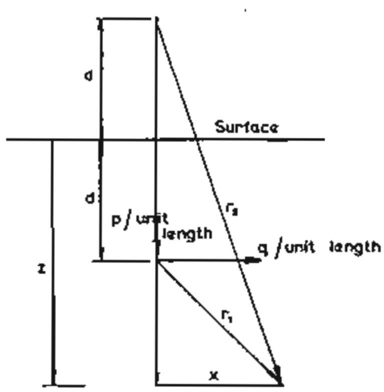


FIG. 2.9

$$\sigma_z = \frac{p}{\pi}\left[\frac{m+1}{2m}\left\{\frac{(z-d)^3}{r_1^4} + \frac{(z+d)\,[(z+d)^2 + 2dz]}{r_2^4} - \frac{8dz(d+z)x^2}{r_2^6}\right\} + \frac{m-1}{4m}\left(\frac{z-d}{r_1^2} + \frac{3z+d}{r_2^2} - \frac{4zx^2}{r_2^4}\right)\right] \qquad \ldots (2.10a)$$

$$\sigma_x = \frac{p}{\pi}\left[\frac{m+1}{2m}\left\{\frac{(z-d)x^2}{r_1^4} + \frac{(z+d)(x^2+2d^2) - 2dx^2}{r_2^4} + \frac{8dz(d+z)x^2}{r_2^6}\right\} + \frac{m-1}{4m}\left(-\frac{z-d}{r_1^2} + \frac{z+3d}{r_2^2} + \frac{4zx^2}{r_2^4}\right)\right] \qquad \ldots (2.10b)$$

$$\tau_{xz} = \frac{px}{\pi}\left[\frac{m+1}{2m}\left\{\frac{(z-d)^2}{r_1^4} + \frac{z^2-2dz-d^2}{r_2^4} + \frac{8dz(d+z)^2}{r_2^6}\right\} + \frac{m-1}{4m}\left(\frac{1}{r_1^2} - \frac{1}{r_2^2} + \frac{4z(d+z)}{r_2^4}\right)\right] \qquad \ldots (2.10c)$$

where $m = \frac{1-\nu}{\nu}$.

2.2.5 MELAN'S PROBLEM II

Horizontal line load q/unit length acting beneath the surface of a semi-infinite mass (Fig.2.9).

$$\sigma_z = \frac{qx}{\pi}\left[\frac{m+1}{2m}\left\{\frac{(z-d)^2}{r_1^4} - \frac{d^2-z^2+6dz}{r_2^4} + \frac{8dzx^2}{r_2^6}\right\} - \frac{m-1}{4m}\left\{\frac{1}{r_1^2} - \frac{1}{r_2^2} - \frac{4z(d+z)}{r_2^4}\right\}\right] \qquad \ldots (2.11a)$$

$$\sigma_x = \frac{qx}{\pi}\left[\frac{m+1}{2m}\left\{\frac{x^2}{r_1^4} + \frac{x^2-4dz-2d^2}{r_2^4} + \frac{8dz(d+z)^2}{r_2^6}\right\} + \frac{m-1}{4m}\left(\frac{1}{r_1^2} + \frac{3}{r_2^2} - \frac{4z(d+z)}{r_2^4}\right)\right] \qquad \ldots (2.11b)$$

$$\tau_{xz} = \frac{q}{\pi}\left[\frac{m+1}{2m}\left\{\frac{(z-d)x^2}{r_1^4} + \frac{(2dz+x^2)(d+z)}{r_2^4} - \frac{8dz(d+z)x^2}{r_2^6}\right\} + \frac{m-1}{4m}\left(\frac{z-d}{r_1^2} + \frac{3z+d}{r_2^2} - \frac{4z(d+z)^2}{r_2^4}\right)\right] \qquad \ldots (2.11c)$$

2.2.6 VERTICAL LINE LOADING ON QUARTER-SPACE (Figure 2.10).

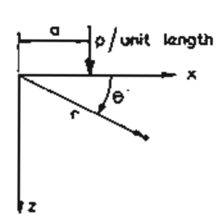


FIG. 2.10

Solutions for the stresses within the quarter-space have been given by Shepherd (1935). Hetenyi (1960) has obtained solutions for the stresses on the boundaries due to both a vertical and a horizontal line load.

Values of stresses obtained by Shepherd are given in Table 2.8. Polar coordinates are used as the problem originally considered was that of an infinite sector.

TABLE 2.8

STRESSES IN A QUARTER-SPACE (After Shepherd, 1935)

Stress	r/a θ°	0.358	0.504	0.710	1.0	1.409	1.985	2.796
	0	0	0	0	∞	0	0	0
	15	0.02	0.29	2.63	14.83	1.30	0.09	-0.06
	30	0.13	1.12	4.46	6.59	2.30	0.27	0.01
	45	0.29	1.36	3.03	3.36	1.52	0.34	0.04
$\frac{2\pi a}{p}\sigma_\theta$	60	0.19	0.84	1.56	1.53	0.80	0.21	0.01
	75	0.10	0.23	0.41	0.36	0.22	0.05	-0.06
	90	0	0	0	0	0	0	0
	0	-0.88	-1.26	-1.61	∞	-1.96	-1.90	-1.70
	15	0.04	0.75	1.37	-1.06	3.03	0.63	-0.31
	30	0.22	0.31	-0.42	-0.32	1.93	1.46	0.69
	45	0.04	0.03	-0.24	0.48	1.74	1.85	1.25
$\frac{2\pi a}{p}\sigma_r$	60	0.02	0.09	0.46	1.16	2.09	2.18	1.89
	75	-0.16	0.35	0.45	1.56	2.55	2.79	2.47
	90	-0.64	-0.48	0.23	1.45	2.60	3.13	2.99
	0	0	0	0	0	0	0	0
	15	-0.09	0.26	2.33	-2.49	-2.95	-0.77	-0.35
	30	0.05	0.48	0.70	-2.88	-3.31	-1.45	-0.69
	45	0.02	0.01	-0.90	-2.46	-2.74	-1.59	-0.84
$\frac{2\pi a}{p}\tau_{r\theta}$	60	0.01	-0.38	-1.20	-1.80	-2.01	-1.35	-0.75
	75	-0.01	-0.28	-0.79	-1.19	-1.15	-0.81	-0.43
	90	0	0	0	0	0	0	0

2.2.7 LINE LOADING AT THE APEX OF AN INFINITE WEDGE (Fig.2.11)

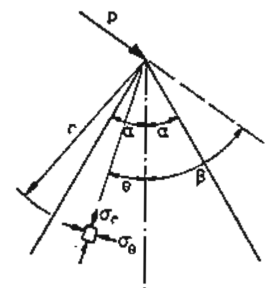


FIG. 2.11

Michell (1900) gives the following solutions:

$$\sigma_r = \frac{2p}{r}\left[\frac{\cos\beta\ \cos\theta}{2\alpha + \sin 2\alpha} + \frac{\sin\beta\ \sin\theta}{2\alpha - \sin 2\alpha}\right] \quad \text{... (2.12a)}$$

$$\sigma_\theta = 0 \quad \text{... (2.12b)}$$

$$\tau_{r\theta} = 0 \quad \text{... (2.12c)}$$

2.2.8 VERTICAL LINE LOAD ON FINITE LAYER (Fig.2.12)

Values of stresses and displacements are given in TABLES 2.9 to 2.13 (Poulos, 1966).

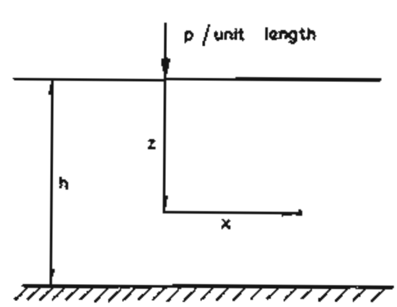


FIG. 2.12

TABLE 2.9

INFLUENCE VALUES I_{σ_z} FOR VERTICAL STRESS σ_z

LINE LOAD

$$\sigma_z = \frac{P}{\pi h} I_{\sigma_z}$$

ν=0	ν=0.2
ν=0.4	ν=0.5

x/h \ z/h	1.0		0.9		0.8		0.7		0.6		0.4		0.2	
0	2.634	2.566	2.787	2.759	2.980	2.975	3.249	3.256	3.641	3.653	5.157	5.127	9.891	9.899
	2.539	2.580	2.787	2.865	3.025	3.113	3.312	3.391	3.704	3.772	5.201	5.234	9.905	9.911
0.1	2.573	2.503	2.713	2.682	2.885	2.877	3.118	3.122	3.443	3.452	4.516	4.505	5.946	5.952
	2.471	2.508	2.702	2.774	2.920	3.001	3.171	3.243	3.498	3.562	4.585	4.586	5.957	5.960
0.2	2.400	2.331	2.505	2.471	2.627	2.614	2.774	2.773	2.948	2.954	3.251	3.217	2.341	2.347
	2.291	2.312	2.477	2.528	2.642	2.703	2.810	2.867	2.990	3.040	3.283	3.305	2.351	2.343
0.3	2.144	2.075	2.203	2.162	2.261	2.240	2.311	2.302	2.335	2.333	2.099	2.081	0.918	0.922
	2.021	2.023	2.145	2.170	2.245	2.279	2.318	2.349	2.352	2.383	2.117	2.128	0.923	0.907
0.4	1.840	1.774	1.855	1.810	1.857	1.828	1.830	1.812	1.751	1.741	1.301	1.257	0.407	0.409
	1.711	1.689	1.773	1.766	1.810	1.815	1.808	1.815	1.744	1.755	1.307	1.307	0.409	0.389
0.5	1.525	1.462	1.504	1.455	1.465	1.429	1.391	1.365	1.265	1.247	0.803	0.774	0.205	0.204
	1.389	1.350	1.397	1.365	1.387	1.365	1.338	1.320	1.231	1.223	0.792	0.786	0.201	0.187
0.6	1.223	1.168	1.179	1.130	1.117	1.077	1.024	0.993	0.889	0.867	0.497	0.443	0.110	0.107
	1.092	1.039	1.061	1.008	1.021	0.978	0.951	0.916	0.837	0.814	0.476	0.466	0.105	0.099
0.7	0.954	0.906	0.898	0.850	0.827	0.785	0.733	0.698	0.611	0.584	0.308	0.270	0.062	0.056
	0.830	0.771	0.774	0.711	0.718	0.661	0.644	0.594	0.543	0.509	0.276	0.264	0.053	0.058
0.8	0.721	0.683	0.661	0.619	0.592	0.553	0.508	0.473	0.408	0.379	0.185	0.124	0.032	0.024
	0.615	0.553	0.547	0.478	0.485	0.421	0.416	0.360	0.335	0.294	0.150	0.137	0.023	0.035
0.9	0.536	0.507	0.479	0.443	0.417	0.380	0.345	0.311	0.267	0.237	0.108	0.066	0.015	0.006
	0.447	0.390	0.375	0.309	0.314	0.251	0.254	0.195	0.192	0.149	0.070	0.057	0.005	0.020
1.0	0.357	0.338	0.306	0.279	0.254	0.224	0.199	0.169	0.144	0.116	0.045	-0.020	-0.000	-0.011
	0.294	0.241	0.224	0.162	0.167	0.106	0.117	0.062	0.075	0.032	0.010	-0.005	-0.007	0.005
1.25	0.121	0.123	0.091	0.082	0.062	0.047	0.035	0.018	0.013	-0.005	-0.016	-0.057	-0.014	-0.024
	0.111	0.079	0.059	0.020	0.020	-0.023	-0.010	-0.055	-0.028	-0.064	-0.037	-0.057	-0.015	-0.021
1.5	0.010	0.027	-0.005	0.003	-0.018	-0.016	-0.028	-0.031	-0.033	-0.040	-0.035	-0.097	-0.019	-0.026
	0.043	0.026	0.011	-0.013	-0.014	-0.042	-0.031	-0.064	-0.040	-0.070	-0.035	-0.061	-0.011	-0.034
1.75	-0.030	-0.006	-0.036	-0.019	-0.040	-0.029	-0.042	-0.036	-0.039	-0.038	-0.032	-0.064	-0.017	-0.019
	0.024	0.018	0.005	-0.006	-0.010	-0.027	-0.021	-0.046	-0.026	-0.049	-0.021	-0.047	-0.005	-0.034
2.0	-0.042	-0.014	-0.042	-0.019	-0.041	-0.022	-0.039	-0.026	-0.035	-0.026	-0.025	-0.131	-0.013	-0.011
	0.025	0.024	0.015	0.010	0.006	-0.003	-0.001	-0.014	-0.005	-0.020	-0.005	-0.026	0.002	-0.025
2.5	-0.025	-0.003	-0.022	-0.004	-0.019	-0.004	-0.016	-0.004	-0.013	-0.004	-0.006	-0.042	0.000	0.002
	0.027	0.034	0.022	0.027	0.018	0.020	0.014	0.001	0.011	0.009	0.007	-0.000	0.006	-0.006
3.0	-0.010	0.004	-0.007	0.004	-0.005	0.004	-0.003	0.004	-0.000	0.004	0.003	0.005	0.006	0.006
	0.022	0.032	0.019	0.028	0.017	0.023	0.014	0.015	0.012	0.015	0.008	0.008	0.005	0.003
4.0	0.006	0.005	0.006	0.005	0.007	0.004	0.007	0.005	0.008	0.005	0.009	0.005	0.009	0.004
	0.008	0.019	0.006	0.017	0.005	0.015	0.004	0.013	0.003	0.011	0.001	0.008	0.000	0.007
6.0	0.005	0.000	0.005	0.000	0.005	0.000	0.005	0.000	0.005	0.000	0.005	0.000	0.005	0.000
	0.000	0.006	-0.000	0.005	-0.000	0.004	-0.001	0.004	-0.001	0.004	-0.001	0.003	-0.001	0.002
8.0	0.002	-0.000	0.002	-0.000	0.002	-0.001	0.002	-0.001	0.002	-0.001	0.002	-0.001	0.002	-0.001
	-0.001	0.001	-0.001	0.000	-0.001	0.000	-0.001	0.000	-0.001	0.000	-0.001	-0.000	-0.001	-0.001

TABLE 2.10

INFLUENCE VALUES I_θ FOR BULK STRESS θ

LINE LOAD

$$\theta = \frac{P}{\pi h} I_\theta$$

$\nu=0$	$\nu=0.2$
$\nu=0.4$	$\nu=0.5$

x/h \ z/h	1.0		0.9		0.8		0.7		0.6		0.4		0.2	
0	2.634	3.789	2.546	3.567	2.569	3.508	2.706	3.611	2.986	3.902	4.292	5.419	8.885	10.92
	5.923	7.739	5.419	6.933	5.145	6.433	5.120	6.217	5.322	6.295	6.846	7.709	13.02	14.12
0.1	2.573	3.707	2.492	3.499	2.512	3.441	2.637	3.530	2.887	3.786	3.975	5.042	6.647	8.230
	5.766	7.524	5.294	6.761	5.012	6.291	4.997	6.075	5.162	6.121	6.390	7.217	9.880	10.76
0.2	2.400	3.436	2.338	3.267	2.352	3.214	2.444	3.272	2.620	3.445	3.253	4.165	3.759	4.764
	5.346	6.936	4.960	6.318	4.726	5.903	4.673	5.683	4.755	5.651	5.371	6.116	5.848	6.432
0.3	2.144	3.063	2.107	2.947	2.115	2.900	2.166	2.920	2.256	2.997	2.458	3.206	1.961	2.605
	4.716	6.068	4.447	5.632	4.216	5.315	4.188	5.106	4.176	4.997	4.226	4.875	3.326	3.737
0.4	1.840	2.559	1.829	2.538	1.832	2.504	1.847	2.492	1.862	2.490	1.770	2.363	0.970	1.413
	3.992	5.068	3.847	4.840	3.718	4.620	3.635	4.434	3.553	4.272	3.239	3.803	1.952	2.275
0.5	1.525	2.142	1.536	2.129	1.536	2.107	1.524	2.075	1.489	2.021	1.239	1.716	0.423	0.754
	3.240	4.050	3.206	3.992	3.112	3.882	3.057	3.739	2.937	3.565	2.452	2.947	1.183	1.476
0.6	1.223	1.685	1.249	1.709	1.249	1.701	1.221	1.662	1.157	1.583	0.844	1.220	0.107	0.370
	2.548	3.117	2.599	3.203	2.568	3.175	2.517	3.079	2.390	2.919	1.868	2.308	0.754	1.045
0.7	0.954	1.302	0.989	1.349	0.989	1.352	0.955	1.313	0.879	1.228	0.559	0.867	-0.070	0.153
	1.937	2.314	2.045	2.484	2.037	2.533	2.026	2.486	1.910	2.362	1.423	1.820	0.499	0.808
0.8	0.721	0.949	0.759	1.008	0.760	1.019	0.724	0.987	0.647	0.909	0.350	0.590	-0.175	0.020
	1.436	1.661	1.575	1.890	1.612	1.979	1.608	1.973	1.514	1.887	1.100	1.462	0.360	0.679
0.9	0.537	0.696	0.574	0.756	0.575	0.773	0.541	0.747	0.470	0.679	0.207	0.409	-0.224	-0.044
	1.044	1.171	1.195	1.409	1.230	1.533	1.263	1.560	1.194	1.514	0.858	1.191	0.279	0.610
1.0	0.357	0.394	0.390	0.451	0.392	0.472	0.362	0.457	0.300	0.409	0.080	0.213	-0.258	-0.098
	0.686	0.724	0.837	0.972	0.901	1.105	0.936	1.160	0.895	1.150	0.651	0.956	0.234	0.565
1.25	0.121	0.079	0.142	0.112	0.142	0.128	0.122	0.127	0.082	0.109	-0.052	0.029	-0.242	-0.090
	0.258	0.238	0.373	0.417	0.411	0.547	0.484	0.619	0.483	0.666	0.389	0.634	0.212	0.514
1.5	0.010	-0.014	0.020	0.002	0.018	0.011	0.004	0.013	-0.020	0.009	-0.097	-0.017	-0.200	-0.056
	0.101	0.079	0.176	0.229	0.214	0.299	0.268	0.353	0.285	0.405	0.272	0.475	0.222	0.465
1.75	-0.030	-0.021	-0.027	-0.015	-0.030	-0.011	-0.039	-0.009	-0.053	-0.009	-0.095	-0.014	-0.148	-0.020
	0.056	0.055	0.102	0.141	0.142	0.200	0.170	0.252	0.191	0.299	0.211	0.376	0.217	0.419
2.0	-0.042	-0.022	-0.043	-0.018	-0.047	-0.015	-0.052	-0.013	-0.060	-0.012	-0.081	-0.013	-0.105	-0.013
	0.058	0.072	0.084	0.118	0.110	0.163	0.130	0.205	0.149	0.243	0.179	0.311	0.206	0.368
2.5	-0.025	-0.005	-0.027	-0.004	-0.030	-0.003	-0.033	-0.001	-0.036	-0.000	-0.041	0.002	-0.045	0.005
	0.062	0.102	0.073	0.124	0.086	0.147	0.097	0.170	0.108	0.193	0.131	0.238	0.152	0.281
3.0	-0.010	0.005	-0.012	0.005	-0.013	0.005	-0.015	0.006	-0.016	0.007	-0.017	0.009	-0.019	0.013
	0.052	0.097	0.058	0.111	0.064	0.125	0.073	0.139	0.079	0.154	0.093	0.182	0.105	0.208
4.0	0.006	0.007	0.005	0.007	0.005	0.007	0.004	0.007	0.004	0.007	0.004	0.007	-0.000	0.008
	0.018	0.058	0.021	0.064	0.026	0.070	0.027	0.075	0.029	0.081	0.034	0.091	0.037	0.101
6.0	0.005	0.001	0.005	0.001	0.005	0.001	0.005	0.001	0.005	0.001	0.005	0.001	0.003	0.000
	0.000	0.017	0.001	0.018	0.001	0.020	0.002	0.021	0.003	0.022	0.004	0.025	0.004	0.027
8.0	0.002	-0.001	0.002	-0.001	0.002	-0.001	0.002	-0.001	0.002	-0.001	0.002	-0.001	0.002	-0.001
	-0.002	0.002	-0.002	0.002	-0.002	0.003	-0.001	0.003	-0.001	0.003	-0.001	0.004	-0.001	0.004

TABLE 2.11

INFLUENCE VALUES $I_{\tau_{xz}}$ FOR HORIZONTAL SHEAR STRESS τ_{xz}

LINE LOAD

$$\tau_{xz} = \frac{p}{\pi h} I_{\tau_{xz}}$$

$\nu=0$	$\nu=0.2$
$\nu=0.4$	$\nu=0.5$

x/h \ z/h	1.0		0.9		0.8		0.7		0.6		0.4		0.2	
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0.1	0.143	0.195	0.164	0.193	0.219	0.234	0.309	0.315	0.447	0.448	1.031	1.026	2.914	2.906
	0.266	0.330	0.223	0.250	0.242	0.248	0.310	0.307	0.436	0.427	1.008	0.999	2.888	2.888
0.2	0.265	0.364	0.300	0.357	0.391	0.422	0.536	0.550	0.744	0.747	1.441	1.432	2.230	2.215
	0.496	0.617	0.416	0.468	0.439	0.453	0.542	0.537	0.725	0.710	1.396	1.380	2.181	2.178
0.3	0.348	0.485	0.388	0.470	0.490	0.536	0.643	0.665	0.843	0.849	1.322	1.309	1.224	1.203
	0.666	0.830	0.554	0.630	0.563	0.588	0.657	0.655	0.820	0.803	1.260	1.238	1.155	1.147
0.4	0.391	0.554	0.427	0.529	0.517	0.578	0.645	0.676	0.792	0.802	1.008	0.994	0.638	0.612
	0.766	0.960	0.635	0.732	0.616	0.655	0.671	0.677	0.771	0.757	0.936	0.911	0.553	0.541
0.5	0.391	0.572	0.421	0.539	0.489	0.561	0.576	0.615	0.660	0.676	0.694	0.680	0.325	0.295
	0.799	1.007	0.660	0.775	0.610	0.666	0.617	0.634	0.646	0.640	0.617	0.594	0.227	0.215
0.6	0.367	0.552	0.385	0.514	0.427	0.510	0.476	0.524	0.510	0.532	0.448	0.435	0.155	0.122
	0.780	0.993	0.644	0.774	0.571	0.643	0.534	0.567	0.508	0.516	0.372	0.358	0.050	0.045
0.7	0.319	0.506	0.332	0.465	0.353	0.442	0.370	0.424	0.370	0.397	0.270	0.258	0.059	0.025
	0.724	0.933	0.600	0.740	0.511	0.599	0.444	0.494	0.380	0.404	0.198	0.198	0.048	-0.042
0.8	0.268	0.446	0.271	0.407	0.273	0.372	0.275	0.335	0.256	0.290	0.150	0.141	0.008	-0.027
	0.649	0.849	0.540	0.688	0.447	0.550	0.363	0.432	0.280	0.324	0.086	0.105	-0.100	-0.077
0.9	0.212	0.383	0.215	0.348	0.210	0.307	0.197	0.261	0.169	0.205	0.072	0.066	-0.018	-0.054
	0.567	0.757	0.475	0.627	0.385	0.500	0.295	0.380	0.205	0.267	0.018	0.055	-0.124	-0.084
1.0	0.152	0.303	0.148	0.274	0.139	0.234	0.119	0.185	0.090	0.129	0.014	0.010	-0.030	-0.066
	0.460	0.636	0.392	0.544	0.312	0.438	0.226	0.329	0.139	0.222	-0.028	0.034	-0.120	-0.070
1.25	0.055	0.177	0.054	0.161	0.046	0.131	0.030	0.092	0.010	0.049	-0.026	-0.028	-0.020	-0.059
	0.285	0.440	0.252	0.401	0.199	0.338	0.136	0.263	0.069	0.186	-0.050	0.049	-0.112	-0.022
1.5	0.012	0.098	0.004	0.090	0.003	0.073	-0.004	0.048	-0.012	0.021	-0.020	-0.025	0.006	-0.038
	0.170	0.311	0.157	0.300	0.127	0.268	0.087	0.223	0.043	0.174	-0.038	0.080	-0.085	0.020
1.75	-0.014	0.053	-0.015	0.051	-0.011	0.042	-0.010	0.028	-0.009	0.013	-0.000	-0.013	0.031	-0.019
	0.099	0.228	0.094	0.228	0.079	0.213	0.055	0.189	0.028	0.159	-0.025	0.095	-0.064	0.041
2.0	0.011	0.029	-0.018	0.029	-0.010	0.026	-0.004	0.020	0.003	0.013	0.021	0.000	0.052	-0.005
	0.054	0.172	0.053	0.175	0.045	0.169	0.033	0.156	0.017	0.139	-0.018	0.096	-0.051	0.050
2.5	0.002	0.017	-0.005	0.018	0.005	0.018	0.015	0.016	0.025	0.014	0.048	0.010	0.073	0.005
	0.016	0.116	0.015	0.118	0.011	0.114	0.005	0.109	-0.003	0.100	-0.023	0.075	-0.048	0.042
3.0	-0.004	0.015	0.005	0.015	0.015	0.015	0.026	0.015	0.036	0.014	0.057	0.011	0.079	0.007
	-0.000	0.087	-0.003	0.087	-0.006	0.084	-0.011	0.079	-0.016	0.073	-0.031	0.056	-0.051	0.031
4.0	0.007	0.012	0.015	0.012	0.023	0.011	0.030	0.010	0.038	0.009	0.054	0.007	0.070	0.004
	-0.020	0.045	-0.022	0.043	-0.025	0.040	-0.028	0.037	-0.021	0.034	-0.039	0.025	-0.049	0.014
6.0	0.007	0.007	0.012	0.006	0.017	0.005	0.022	0.005	0.027	0.004	0.037	0.003	0.047	0.001
	-0.026	0.015	-0.028	0.014	-0.029	0.013	-0.030	0.012	-0.032	0.010	-0.035	0.007	-0.038	0.004
8.0	0.004	0.004	0.007	0.003	0.011	0.003	0.014	0.003	0.017	0.002	0.024	0.001	0.030	0.001
	-0.028	0.006	-0.024	0.005	-0.025	0.005	-0.026	0.004	-0.026	0.004	-0.028	0.003	-0.030	0.001

TABLE 2.12

INFLUENCE VALUES $I_{\rho x}$ FOR SURFACE DISPLACEMENTS

LINE LOAD

$\rho_x = \frac{p}{\pi E} I_{\rho x}$

HORIZONTAL DISPLACEMENT ρ_x

x/h \ ν	0	0.2	0.4	0.5
0.1	1.366	0.952	0.316	-0.094
0.2	1.246	0.834	0.217	-0.184
0.3	1.130	0.727	0.127	-0.269
0.4	1.019	0.613	0.042	-0.342
0.5	0.872	0.518	-0.033	-0.407
0.6	0.749	0.436	-0.097	-0.460
0.7	0.646	0.353	-0.151	-0.499
0.8	0.556	0.288	-0.194	-0.524
0.9	0.476	0.227	-0.223	-0.538
1.0	0.390	0.180	-0.247	-0.537
1.25	0.257	0.084	-0.261	-0.509
1.5	0.167	0.026	-0.241	-0.448
1.75	0.111	0.005	-0.211	-0.384
2.0	0.074	0.001	-0.175	-0.308
2.5	0.034	-0.002	-0.114	-0.210
3.0	0.012	-0.015	-0.067	-0.135
4.0	0.002	-0.002	-0.026	-0.053
6.0	0.000	0.000	-0.005	-0.007
8.0	0.000	0.000	-0.000	-0.000

TABLE 2.13

INFLUENCE VALUES $I_{\rho z}$ FOR SURFACE DISPLACEMENTS

LINE LOAD

$\rho_z = \frac{p}{\pi E} I_{\rho z}$

VERTICAL DISPLACEMENT ρ_z

x/h \ ν	0	0.2	0.4	0.5
0.1	3.756	3.466	2.633	1.926
0.2	2.461	2.222	1.558	0.973
0.3	1.730	1.533	0.964	0.458
0.4	1.244	1.069	0.583	0.132
0.5	0.896	0.749	0.324	-0.079
0.6	0.643	0.511	0.145	-0.217
0.7	0.453	0.347	0.045	-0.299
0.8	0.313	0.218	-0.057	-0.344
0.9	0.212	0.141	-0.101	-0.358
1.0	0.126	0.059	-0.139	-0.359
1.25	0.023	-0.006	-0.146	-0.315
1.5	-0.012	-0.024	-0.112	-0.254
1.75	-0.023	-0.023	-0.086	-0.198
2.0	-0.017	-0.015	-0.071	-0.134
2.5	-0.010	-0.009	-0.034	-0.085
3.0	-0.008	-0.001	-0.014	-0.057
4.0	-0.002	-0.000	-0.007	-0.025
6.0	-0.000	-0.000	-0.002	-0.004
8.0	-0.000	-0.000	-0.000	-0.000

2.3 Line Loading—Axial Symmetry

2.3.1 UNIFORM VERTICAL RING LOADING ON SURFACE OF SEMI-INFINITE MASS (Fig. 2.13)

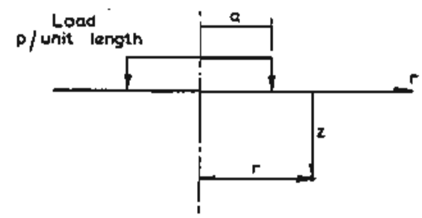


FIG. 2.13

On the axis (r=0),

$$\sigma_z = \frac{3pz^3a}{(a^2+z^2)^{5/2}} \qquad \dots (2.13a)$$

$$\sigma_r = \sigma_\theta = \frac{pza}{2(a^2+z^2)^{5/2}}\left[2(1+\nu)\,(a^2+z^2) - 3z^2\right] \qquad \dots (2.13b)$$

$$\Theta = \frac{2p(1+\nu)za}{(a^2+z^2)^{3/2}} \qquad \dots (2.13c)$$

$$\tau_{rz} = 0 \qquad \dots (2.13d)$$

$$\rho_z = \frac{p(1+\nu)a}{E(a^2+z^2)^{\frac{1}{2}}}\left[2(1-\nu) + \frac{z^2}{(a^2+z^2)}\right] \qquad \dots (2.13e)$$

$$\rho_r = 0 \qquad \dots (2.13f)$$

2.3.2 UNIFORM VERTICAL SUBSURFACE LINE LOAD (Fig. 2.14)

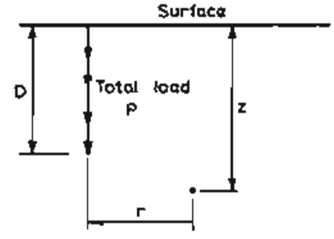


FIG. 2.14

Geddes (1966) evaluated the following expressions for the stresses from Mindlin's equations:

$$K_{zz} = \sigma_z \frac{D^2}{P} = \frac{-1}{8\pi(1-\nu)}\Bigg[-\frac{2(2-\nu)}{A} + \frac{2(2-\nu) + 2(1-2\nu)\frac{m}{n}\left(\frac{m}{n}+\frac{1}{n}\right)}{B} - \frac{(1-2\nu)2\left(\frac{m}{n}\right)^2}{F} + \frac{n^2}{A^3} + \frac{4m^2-4(1+\nu)\left(\frac{m}{n}\right)^2 m^2}{F^3}$$
$$+ \frac{4m(1+\nu)(m+1)\left(\frac{m}{n}+\frac{1}{n}\right)^2 - (4m^2+n^2)}{B^3} + \frac{6m^2\left(\frac{m^4-n^4}{n^2}\right)}{F^5} + \frac{6m\left(mn^2 - \frac{1}{n^2}[m+1]^5\right)}{B^5}\Bigg] \quad \ldots (2.14a)$$

$$K_{rr} = \sigma_r \frac{D^2}{P} = \frac{-1}{8\pi(1-\nu)}\Bigg[\frac{(1-2\nu)}{A} + \frac{(7-2\nu)-12(1-\nu)\frac{m}{n}\left(\frac{m}{n}+\frac{1}{n}\right)}{B} - \left\{\frac{4(2-\nu)-12(1-\nu)\left(\frac{m}{n}\right)^2}{F}\right\} - \frac{n^2}{A^3}$$
$$+ \frac{4n^2-2m^2+2(1+2\nu)\left(\frac{m}{n}\right)^2 m^2}{F^3} - \left\{\frac{3n^2-2m^2+2(1+2\nu)\frac{m}{n}(m+1)^2\left(\frac{m}{n}+\frac{1}{n}\right)}{B^3}\right\}$$
$$+ \frac{6\left[n^2m^2-m^4\left(\frac{m}{n}\right)^2\right]}{F^5} + \frac{6\left[\frac{m}{n}(m+1)^4\left(\frac{m}{n}+\frac{1}{n}\right) - m^2n^2\right]}{B^5} + 4(1-\nu)(1-2\nu)\left\{\frac{1}{F+m} - \frac{1}{B+m+1}\right\}\Bigg] \quad \ldots (2.14b)$$

$$K_{\theta\theta} = \sigma_\theta \frac{D^2}{P} = \frac{-1}{8\pi(1-\nu)}\Bigg[\frac{(1-2\nu)}{A} + \frac{6-(1-2\nu)(3-4\nu)+6(1-2\nu)\frac{m}{n}\left(\frac{m}{n}+\frac{1}{n}\right)}{B} + \frac{2(1-2\nu)^2-6(1-2\nu)\left(\frac{m}{n}\right)^2-6}{F}$$
$$+ \frac{2m^2-4\nu n^2+2(1+2\nu)\frac{m}{n}(m+1)^2\left(\frac{m}{n}+\frac{1}{n}\right)}{B^3} + \frac{4\nu n^2-2m^2-2(1+2\nu)m^2\left(\frac{m}{n}\right)^2}{F^3} - 4(1-\nu)(1-2\nu)\left\{\frac{1}{F+m} - \frac{1}{B+m+1}\right\}\Bigg] \quad \ldots (2.14c)$$

$$K_{rz} = \tau_{rz} \frac{D^2}{P} = \frac{-n}{8\pi(1-\nu)}\Bigg[\frac{(1-2\nu)\frac{1}{n}\left(\frac{m}{n}-\frac{1}{n}\right)}{A} + \frac{(m-1)\left(\frac{m}{n}-\frac{1}{n}\right)^2}{A^3} + \frac{(1-2\nu)\frac{1}{n}\left(\frac{m}{n}+\frac{1}{n}\right) - \frac{6}{n}\left(\frac{m}{n}\right)^2\left(\frac{m}{n}+\frac{1}{n}\right)}{B}$$
$$+ \frac{12m-4\nu m+(m+1)^3\left[\left(\frac{1}{n}\right)^2+12\frac{m^2}{n^4}\right]}{B^3} - \frac{6\left(\frac{m}{n}\right)^2(m+1)^3\left(\frac{m}{n}+\frac{1}{n}\right)^2+6mn^2}{B^5} + \frac{6\left(\frac{m}{n}\right)^3\left(\frac{1}{n}\right) - 2(1-2\nu)\frac{m}{n^2}}{F}$$
$$+ \frac{4\nu m-12m-m^3\left[\frac{2}{n^2}+12\frac{m^2}{n^4}\right]}{F^3} + \frac{6mn^2+6m^3\left(\frac{m}{n}\right)^4}{F^5}\Bigg] \quad \ldots (2.14d)$$

On the axis with $n=0$ and $m>1.0$,

$$K_{zz} = \frac{-1}{8\pi(1-\nu)}\left[-\frac{4(1-\nu)}{m} - \frac{2(2-\nu)}{(m-1)} + \frac{2(2-\nu)}{(m+1)} + \frac{4m(2-\nu)}{(m+1)^2} - \frac{4m^2}{(m+1)^3}\right] \quad \ldots (2.14e)$$

$$K_{rr} = K_{\theta\theta} = \frac{-1}{8\pi(1-\nu)}\left[-\frac{2+2\nu(1-2\nu)}{m} + \frac{(1-2\nu)}{(m-1)} + \frac{6-(1-2\nu)^2}{(m+1)} - \frac{6m}{(m+1)^2} + \frac{2m^2}{(m+1)^3}\right] \quad \ldots (2.14f)$$

$$K_{rz} = 0 \quad \ldots (2.14g)$$

where $n = r/D$, $m = z/D$

$F^2 = n^2 + m^2$

$A^2 = [n^2 + (m-1)^2]$

$B^2 = [n^2 + (m+1)^2]$.

K_{zz} throughout the mass, and K_{rr} and $K_{\theta\theta}$ on the axis, are tabulated by Geddes (1966).

2.3.3 LINEARLY VARYING SUBSURFACE LINE LOAD (Fig. 2.15)

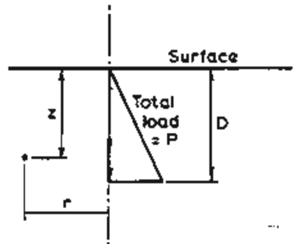


FIG. 2.15

The following expressions for stress have been obtained by Geddes (1966):

$$K_{zz} = \sigma_z \frac{D^2}{P} = \frac{-1}{4\pi(1-\nu)} \Bigg[\frac{-2(2-\nu)}{A} + \frac{2(2-\nu)(4m+1)-2(1-2\nu)(\frac{m}{n})^2(m+1)}{B} + \frac{2(1-2\nu)\frac{m^3}{n^2}-8(2-\nu)m}{F} + \frac{mn^2+(m-1)^3}{A^3}$$

$$+ \frac{4\nu n^2m+4m^3-15n^2m-2(5+2\nu)(\frac{m}{n})^2(m+1)^3+(m+1)^3}{B^3} + \frac{2(7-2\nu)mn^2-6m^3+2(5+2\nu)(\frac{m}{n})^2m^3}{F^3}$$

$$+ \frac{6mn^2(n^2-m^2)+12(\frac{m}{n})^2(m+1)^5}{B^5} - \frac{12(\frac{m}{n})^2m^5+6mn^2(n^2-m^2)}{F^5} - 2(2-\nu)\log_e\left(\frac{A+m-1}{F+m}\cdot\frac{B+m+1}{F+m}\right)\Bigg] \quad \ldots (2.15a)$$

$$K_{rr} = \sigma_r \frac{D^2}{P} = \frac{-1}{4\pi(1-\nu)} \Bigg[\frac{(1-2\nu)}{A} + \frac{(7-2\nu)-12m+12(1-\nu)(\frac{m}{n})^2(m+1)}{B} + \frac{12m-12(1-\nu)\frac{m^3}{n^2}}{F} - \frac{(m-1)^3+mn^2}{A^3}$$

$$+ \frac{3(m+1)^3-2m^3+(21-4\nu)mn^2+2(5+2\nu)(\frac{m}{n})^2(m+1)^3}{B^3} - \frac{2(5+2\nu)\frac{m^5}{n^2}+4(5-\nu)mn^2}{F^3}$$

$$+ \frac{6mn^2(m^2-n^2)-12(\frac{m}{n})^2(m+1)^5}{B^5} - \frac{6mn^2(m^2-n^2)-12\frac{m^7}{n^2}}{F^5} + (1-2\nu)\log_e\frac{A+m-1}{F+m}$$

$$+ \{(1-2\nu)^2-6\}\log\left(\frac{B+m+1}{F+m}\right) + 2(1-\nu)(1-2\nu)\left\{\frac{m-1}{B+m+1}-\frac{m}{F+m}\right\}\Bigg] \quad \ldots (2.15b)$$

$$K_{\theta\theta} = \sigma_\theta \frac{D^2}{P} = \frac{-1}{4\pi(1-\nu)} \Bigg[\frac{(1-2\nu)}{A} - \frac{(1-2\nu)(3-4\nu)+6(1-2\nu)(\frac{m}{n})^2(m+1)+6(2m-1)}{B} + \frac{6(1-2\nu)\frac{m^3}{n^2}+12m}{F}$$

$$- (1-2\nu)\left\{\frac{2(m+1)^3+4mn^2-2(\frac{m}{n})^2(m+1)^3}{B^3}\right\} + \frac{2(m+1)^3+6mn^2-2m^3-6(\frac{m}{n})^2(m+1)^3}{B^3}$$

$$+ \frac{(2m^3+4mn^2-2\frac{m^5}{n^2})(1-2\nu)}{F^3} - \frac{6mn^2-6\frac{m^5}{n^2}}{F^3} + (1-2\nu)\log_e\left(\frac{A+m-1}{F+m}\right) + \{(1-2\nu)^2-6\}$$

$$\cdot \log_e\left(\frac{B+m+1}{F+m}\right) - 2(1-\nu)(1-2\nu)\left\{\frac{m-1}{B+m+1}-\frac{m}{F+m}\right\}\Bigg] \quad \ldots (2.15c)$$

and,

$$K_{rz} = \tau_{rz}\frac{D^2}{P} = \frac{-n}{4\pi(1-\nu)} \Bigg[\frac{2(2-\nu)+(1-2\nu)\frac{m}{n^2}(m-1)}{A} - \frac{6(\frac{m}{n})^4}{F} - \frac{2(2-\nu)+(1-2\nu)\frac{m}{n^2}(m+1)-6\frac{m^3}{n^4}(m+1)}{B} + \frac{\frac{m}{n^2}(m-1)^3-n^2}{A^3}$$

$$- \frac{(3-4\nu)\left[\frac{m}{n^2}(m+1)^3+m^2\right]+17m^2-n^2+12\,\frac{m^3}{n^4}(m+1)^3}{B^3}$$

$$+ \frac{2(1-2\nu)\,\frac{m^4}{n^2}+4(5-\nu)m^2+12\,\frac{m^6}{n^4}}{F^3}$$

$$+ \frac{6\,\frac{m^3}{n^4}\,(m+1)^5-6\,\frac{m}{n^2}(m+1)^5+12m^2n^2}{B^5}$$

$$- \left.\frac{6\,\frac{m^8}{n^4}-6\,\frac{m^6}{n^2}+12m^2n^2}{F^5}\right]$$

... (2.15d)

On the loading axis with $m>1.0$

$$K_{zz} = \frac{-1}{4\pi(1-\nu)}\left[2-\frac{2(2-\nu)m}{(m-1)}+\frac{6(2-\nu)m}{(m+1)}-\frac{2(7-2\nu)m^2}{(m+1)^2}\right.$$

$$\left.+\frac{4m^3}{(m+1)^3}-2(2-\nu)\log_e\left\{\frac{m^2-1}{m^2}\right\}\right]$$

... (2.15e)

$$K_{rr} = K_{\theta\theta} = \frac{1}{4\pi(1-\nu)}\left[11-2(1-2\nu)(1-\nu)+(1-2\nu)\right.$$

$$.\ \log_e\left\{\frac{m-1}{m}\right\}+(1-2\nu)^2\log_e\left\{\frac{m+1}{m}\right\}-6\log_e\left\{\frac{m+1}{m}\right\}$$

$$+(1-2\nu)\,\frac{m}{(m-1)}+\left[(1-2\nu)^2-18\right]\,\frac{m}{(m+1)}$$

$$\left.+\frac{9m^2}{(m+1)^2}-\frac{2m^3}{(m+1)^3}\right]$$ ... (2.15f)

$$K_{rz} = 0$$ ... (2.15g)

where $n = r/D$, $m = z/D$
$F^2 = n^2 + m^2$
$A^2 = [n^2 + (m-1)^2]$
$B^2 = [n^2 + (m+1)^2]$

Values of K_{zz} throughout the mass, and K_{rr} and $K_{\theta\theta}$ on the axis, are tabulated by Geddes (1966).

Chapter 3
DISTRIBUTED LOADS ON THE SURFACE OF A SEMI-INFINITE MASS

3.1 Loading on an Infinite Strip

3.1.1 UNIFORM VERTICAL LOADING (Fig.3.1)

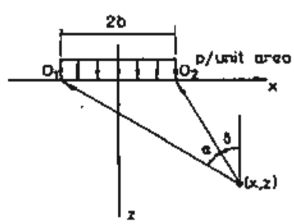


FIG.3.1

$$\sigma_z = \frac{p}{\pi}\left[\alpha + \sin\alpha\,\cos(\alpha+2\delta)\right] \qquad \ldots (3.1a)$$

$$\sigma_x = \frac{p}{\pi}\left[\alpha - \sin\alpha\,\cos(\alpha+2\delta)\right] \qquad \ldots (3.1b)$$

$$\sigma_y = \frac{2p}{\pi}\nu\alpha \qquad \ldots (3.1c)$$

$$\tau_{xz} = \frac{p}{\pi}\sin\alpha\,\sin(\alpha+2\delta) \qquad \ldots (3.1d)$$

$$\sigma_1 = \frac{p}{\pi}\left[\alpha + \sin\alpha\right] \qquad \ldots (3.1e)$$

$$\sigma_3 = \frac{p}{\pi}\left[\alpha - \sin\alpha\right] \qquad \ldots (3.1f)$$

$$\tau_{max} = \frac{p}{\pi}\sin\alpha \qquad \ldots (3.1g)$$

Loci of constant σ_1 and σ_2 are circles passing through O_1 and O_2.

Loci of constant τ_{max} are circles passing through O_1 and O_2.

Trajectories of σ_1 are a family of confocal hyperbolas, foci at O_1 and O_2. These curves bisect the angle, α, at all points.

Trajectories of σ_2 are a family of confocal ellipses, foci at O_1 and O_2.

Trajectories of τ_{max} are two orthogonal families: equiangular spirals intersecting the ellipses under 45°.

Maximum $\tau_{max} = p/\pi$, occurs at all points of the semi-circle through O_1 and O_2.

Maximum $\sigma_1 = p$, occurs at points $(x,0)$, $-b<x<b$.

Minimum $\sigma_3 = 0$, " " " $(x,0)$, $-b>x>b$.

Values of σ_x, σ_z, τ_{xz}, σ_1 σ_3 and τ_{max} are tabulated in Table 3.1, and contours of σ_z and τ_{max} are given in Fig.3.2 (Jurgenson, 1934).

As for line loading, displacements due to strip loading on or in a semi-infinite mass are only meaningful if evaluated as the displacement of one point relative to another point, neither point being located at infinity. The vertical displacement at the surface, relative to the centre of the strip, is given by

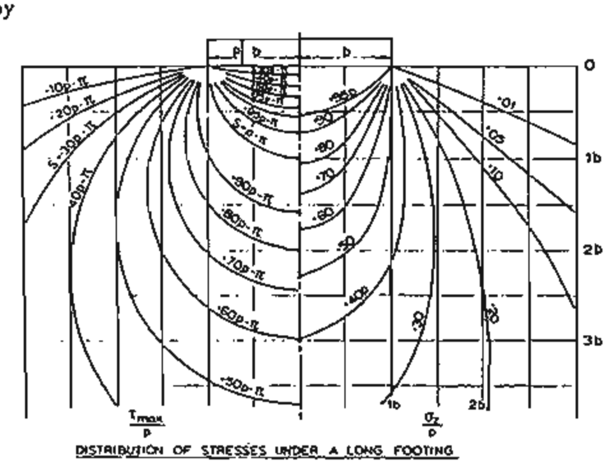


FIG.3.2 Stresses beneath a strip (Jurgenson,1934).

$$\rho_z(x,0) - \rho_z(0,0) = \frac{2p(1-\nu^2)}{\pi E}\{(x-b)\ln|x-b| - (x+b)\ln|x+b| + 2b\ln b\} \quad \ldots (3.2)$$

(See plot in Fig.9.2a, Chapter 9)

TABLE 3.1

STRESSES BENEATH A UNIFORMLY LOADED STRIP (Jurgenson, 1934)

x/b	z/b	σ_z/p	σ_x/p	τ_{zx}/p	β	τ_{max}/p	σ_1/p	σ_3/p
0	0	1.0000	1.0000	0	0	0	1.0000	1.0000
	.5	.9594	.4498	0	0	.2548	.9594	.4498
	1	.8183	.1817	0	0	.3183	.8183	.1817
	1.5	.6678	.0803	0	0	.2937	.6678	.0803
	2	.5508	.0410	0	0	.2546	.5508	.0410
	2.5	.4617	.0228	0	0	.2195	.4617	.0228
	3	.3954	.0138	0	0	.1908	.3954	.0138
	3.5	.3457	.0091	0	0	.1683	.3457	.0091
	4	.3050	.0061	0	0	.1499	.3050	.0061
0.5	0	1.0000	1.0000	0	0	0	1.0000	1.0000
	.25	.9787	.6214	.0522	8°35'	.1871	.9871	.6129
	.5	.9028	.3920	.1274	13°17'	.2848	.9323	.3629
	1	.7352	.1863	.1590	14°52'	.3158	.7763	.1446
	1.5	.6078	.0994	.1275	13°18'	.2847	.6370	.0677
	2	.5107	.0542	.0959	11°25'	.2470	.5298	.0357
	2.5	.4372	.0334	.0721	9°49'	.2143	.4693	.0206
1	.25	.4996	.4208	.3134	41°25'	.3158	.7760	.1444
	.5	.4969	.3472	.2996	37°59'	.3088	.7308	.1133
	1	.4797	.2250	.2546	31°43'	.2847	.6371	.0677
	1.5	.4480	.1424	.2037	26°34'	.2546	.5498	.0406
	2	.4095	.0908	.1592	22°30'	.2251	.4751	.0249
	2.5	.3701	.0595	.1243	19°20'	.1989	.4137	.0159
1.5	.25	.0177	.2079	.0606	73°47'	.1128	.2281	.0025
	.5	.0892	.2850	.1466	61°50'	.1765	.3636	.0106
	1	.2137	.2488	.2101	47°23'	.2115	.4428	.0198
	1.5	.2704	.1807	.2022	38°44'	.2071	.4327	.0184
	2	.2876	.1268	.1754	32°41'	.1929	.4007	.0143
	2.5	.2851	.0892	.1469	28°09'	.1765	.3637	.0106
2	.25	.0027	.0987	.0164	80°35'	.0507	.1014	.0002
	.5	.0194	.1714	.0552	71°59'	.0940	.1893	.0014
	1	.0776	.2021	.1305	58°17'	.1424	.2834	.0052
	1.5	.1458	.1847	.1568	48°32'	.1578	.3232	.0074
	2	.1847	.1456	.1567	41°27'	.1579	.3232	.0073
	2.5	.2045	.1256	.1442	36°02'	.1515	.3094	.0064
2.5	.5	.0068	.1104	.0254	76°43'	.0569	.1141	.0003
	1	.0357	.1615	.0739	65°12'	.0970	.1957	.0016
	1.5	.0771	.1645	.1096	55°52'	.1180	.2388	.0029
	2	.1139	.1447	.1258	48°32'	.1265	.2556	.0036
	2.5	.1409	.1205	.1266	42°45'	.1269	.2575	.0036
3	.5	.0026	.0741	.0137	79°25'	.0379	.0758	.0001
	1	.0171	.1221	.0449	69°42'	.0690	.1384	.0005
	1.5	.0427	.1388	.0757	61°15'	.0895	.1803	.0012
	2	.0705	.1341	.0954	54°12'	.1006	.2029	.0018
	2.5	.0952	.1196	.1036	48°20'	.1054	.2128	.0020
	3	.1139	.1019	.1057	43°22'	.1058	.2137	.0020

β is angle between direction of σ_1 and the vertical.

3.1.2 UNIFORM HORIZONTAL LOADING (Fig.3.3).

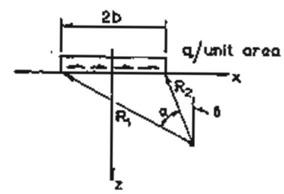


FIG.3.3

$$\sigma_x = \frac{q}{\pi}\left[\log_e \frac{R_1^2}{R_2^2} - \sin\alpha \cdot \sin(\alpha+2\delta)\right] \quad \ldots (3.3a)$$

$$\sigma_z = \frac{q}{\pi}\left[\sin\alpha \ \sin(\alpha+2\delta)\right] \quad \ldots (3.3b)$$

$$\tau_{xz} = \frac{q}{\pi}\left[\alpha - \sin\alpha \ \cos(\alpha+2\delta)\right] \quad \ldots (3.3c)$$

Values of σ_x/q have been tabulated and are given in Table 3.2. It should be noted that σ_z/q values are equal to the corresponding values of τ_{xz}/p for uniform vertical loading.

TABLE 3.2

VALUES OF σ_x/q FOR UNIFORM HORIZONTAL LOADING OVER STRIP

(Scott, 1963)

z/b \ x/b	0	0.5	1.0	2.0	4.0
0.0	0.00	0.70	-	0.60	0.33
0.2	0.00	0.62	1.16	0.68	0.32
0.5	0.00	0.39	0.60	0.57	0.32
1.0	0.00	0.13	0.26	0.39	0.28
1.5	0.00	0.06	0.12	0.24	0.25
2.0	0.00	0.03	0.06	0.15	0.20
3.0	0.00	0.01	0.02	0.06	0.14
5.0	0.00	0.00	0.00	0.01	0.05

The expression for the horizontal displacement of a point on the surface, relative to the centre of the strip, $\rho_x(x,0)-\rho_x(0,0)$, is identical with the expression for the relative vertical displacement due to uniform vertical load in equation (3.2).

3.1.3 VERTICAL LOADING INCREASING LINEARLY (Fig.3.4)

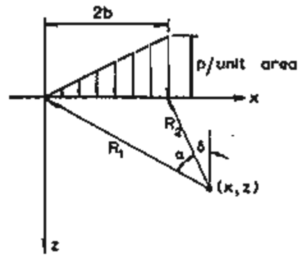


FIG.3.4

$$\sigma_z = \frac{p}{2\pi}\left[\frac{x}{b}\alpha - \sin 2\delta\right] \qquad \ldots (3.4a)$$

$$\sigma_x = \frac{p}{2\pi}\left[\frac{x}{b}\alpha - \frac{z}{b}\log_e\frac{R_1^2}{R_2^2} + \sin 2\delta\right] \qquad \ldots (3.4b)$$

$$\tau_{xz} = \frac{p}{2\pi}\left[1 + \cos 2\delta - \frac{z\alpha}{b}\right] \qquad \ldots (3.4c)$$

Values of σ_z/p have been tabulated by Scott (1963) and are given in Table 3.3.

TABLE 3.3

VALUES OF σ_z/p FOR LINEARLY INCREASING LOAD ON STRIP

(Scott, 1963)

z/b \ x/b	-3.0	-2.0	-1.0	0	1.0	2.0	3.0	5.0
0.0	0.00	0.00	0.00	0.00	0.50	0.50	0.00	0.00
0.5	0.00	0.00	0.00	0.08	0.48	0.42	0.02	0.00
1.0	0.00	0.00	0.02	0.13	0.41	0.35	0.06	0.00
2.0	0.01	0.03	0.06	0.16	0.28	0.25	0.13	0.01
3.0	0.02	0.05	0.10	0.15	0.20	0.19	0.12	0.04
4.0	0.03	0.06	0.09	0.13	0.16	0.15	0.11	0.05

The vertical displacement of a point on the surface, relative to the value at $x=0$, is given by

$$\rho_z(x,0)-\rho_z(0,0) = \frac{p(1-\nu^2)}{\pi b E}\{2b^2 \ln 2b - \frac{x^2}{2}\ln x + (\frac{x^2}{2} - 2b^2)\ln|2b-x| + bx\} \qquad \ldots (3.5)$$

3.1.4 HORIZONTAL LOADING LINEARLY INCREASING (Max. loading = q, Fig.3.4)

$$\sigma_x = \frac{q}{2\pi}\left\{\frac{3z\alpha}{b} + \frac{x}{b}\log_e\frac{R_1^2}{R_2^2} - \cos 2\delta - 5\right] \qquad \ldots (3.6a)$$

$$\sigma_z = \frac{q}{2\pi}\left[1 + \cos 2\delta - \frac{z}{b}\alpha\right] \qquad \ldots (3.6b)$$

$$\tau_{xz} = \frac{q}{2\pi}\left[\frac{x}{b}\cdot\alpha - \frac{z}{b}\log_e\frac{R_1^2}{R_2^2} + \sin 2\delta\right] \qquad \ldots (3.6c)$$

The expression for the horizontal displacement of a point on the surface, relative to $x=0$, is identical with the expression for relative vertical movement due to vertical loading (equation 3.5).

3.1.5 SYMMETRICAL VERTICAL TRIANGULAR LOADING (Fig.3.5, Gray, 1936)

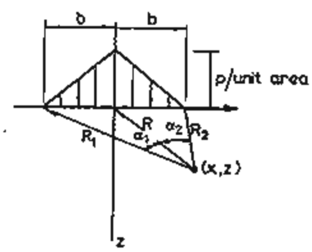


FIG.3.5

$$\sigma_z = \frac{p}{\pi}\left[(\alpha_1+\alpha_2) + \frac{x}{b}(\alpha_1-\alpha_2)\right] \qquad \ldots (3.7a)$$

$$\sigma_x = \frac{p}{\pi}\left[(\alpha_1+\alpha_2) + \frac{x}{b}(\alpha_1-\alpha_2) - 2\frac{z}{b}\log_e\frac{R_1R_2}{R_0^2}\right] \qquad \ldots (3.7b)$$

$$\tau_{xz} = -\frac{pz}{\pi b}(\alpha_1-\alpha_2) \qquad \ldots (3.7c)$$

$$\sigma_1 = \frac{p}{\pi}\left[(\alpha_1+\alpha_2) + \frac{x}{b}(\alpha_1-\alpha_2) - \frac{z}{b}\log_e\frac{R_1R_2}{R_0^2}\right] + \frac{pz}{\pi b}\left(\log_e^2\frac{R_1R_2}{R_0^2} + (\alpha_1-\alpha_2)^2\right)^{\frac{1}{2}} \qquad \ldots (3.7d)$$

$$\sigma_3 = \frac{p}{\pi}\left[(\alpha_1+\alpha_2) + \frac{x}{b}(\alpha_1-\alpha_2) - \frac{z}{b}\log_e\frac{R_1R_2}{R_0^2}\right] - \frac{pz}{\pi b}\left(\log_e^2\frac{R_1R_2}{R_0^2} + (\alpha_1-\alpha_2)^2\right)^{\frac{1}{2}} \qquad \ldots (3.7e)$$

$$\tau_{max} = \frac{pz}{\pi b} \left(\log_e^2 \frac{R_1 R_2}{R_0^2} + (\alpha_1 - \alpha_2)^2 \right)^{\frac{1}{2}} \quad \ldots (3.7e)$$

Stresses have been tabulated by Jurgenson (1934) and are given in TABLE 3.4.

Contours of σ_z and τ_{max} are shown in Fig.3.6 (Jurgenson, 1934).

TABLE 3.4

STRESSES BENEATH A STRIP WITH SYMMETRICAL TRIANGULAR LOADING

(Jurgenson, 1934)

x/b	z/b	σ_z/p	σ_x/p	τ_{zx}/p	β	τ_{max}/p	σ_1/p	σ_3/p
0	0	1.0000	1.0000	0	0	0	1.0000	1.0000
	.25	.8440	.3931	0	0	.2255	.8440	.3931
	.5	.7048	1925	0	0	.2562	.7048	.1925
	.75	.5904	.1025	0	0	.2440	.5904	.1025
	1.0	.5000	.0588	0	0	.2206	.5000	.0588
	1.25	.4296	.0359	0	0	.1969	.4296	.0359
	1.5	.3744	.0234	0	0	.1755	.3744	.0234
	1.75	.3305	.0158	0	0	.1574	.3305	.0158
	2.0	.2952	.0111	0	0	.1421	.2952	.0111
	2.5	.2422	.0062	0	0	0	.2422	.0062
.25	0	.7500	.7500	0	-	0	.7500	.7500
	.25	.7196	.3874	.1151	17°22'	.2021	.7556	.3514
	.5	.6344	.2026	.1146	13°59'	.2444	.6629	.1741
	.75	.5462	.1138	.0951	11°53'	.2361	.5661	.0939
	1.0	.4711	.0681	.0756	10°17'	.2152	.4848	.0544
	1.25	.4101	.0432	.0597	9°01'	.1930	.4197	.0337
0.5	0	.5000	.5000	0	-	0	.5000	.5000
	.25	.4949	.3357	.1525	31°13'	.1720	.5873	.2433
	.5	.4714	.2152	.1762	27°00'	.2178	.5611	.1255
	.75	.4350	.1385	.1570	23°19'	.2160	.5028	.0708
	1.0	.3955	.0913	.1299	20°15'	.2000	.4434	.0434
	1.25	.3577	.0617	.1055	17°45'	.1817	.3914	.0280
	1.5	.3238	.0433	.0858	15°44'	.1644	.3480	.0192
	2.0	.2682	.0229	.0582	12°42'	.1358	.2814	.0098
	2.5	.2266	.0130	.0415	10°37'	.1146	.2344	.0052
0.75	0	.2500	.2500	0	-	0	.2500	.2500
	.25	.2620	.2620	.1476	45°00'	.1476	.4096	.1144
	.5	.2875	.2162	.1810	39°26'	.1845	.4364	.0674
	.75	.3000	.1611	.1735	34°05'	.1869	.4175	.0437
	1.0	.2980	.1167	.1528	29°39'	.1777	.3851	.0297
	1.25	.2869	.0847	.1309	26°10'	.1654	.3512	.0204
1.0	0	0	0	0	-	0	0	0
	.25	.0766	.1956	.0959	60°54'	.1129	.2490	.0232
	.5	.1393	.2005	.1414	51°06'	.1447	.3146	.0252
	.75	.1813	.1693	.1534	43°53'	.1535	.3288	.0218

β is angle between vertical and direction σ_1.

TABLE 3.4 (Cont.)

x/b	z/b	σ_z/p	σ_x/p	τ_{zx}/p	β	τ_{max}/p	σ_1/p	σ_3/p
1.0	1.0	.2048	.1338	.1476	38°14'	.1518	.3211	.0175
	1.25	.2148	.1033	.1343	33°43'	.1454	.3045	.0137
	1.5	.2159	.0794	.1189	30°04'	.1371	.2848	.0106
	1.75	.2048	.0471	.0903	29°25'	.1199	.2459	.0061
	2.5	.1874	.0298	.0685	20°30'	.1044	.2130	.0042
1.25	.25	.0155	.1278	.0399	72°18'	.0689	.1406	.0028
	.5	.0580	.1668	.0899	60°35'	.1051	.2175	.0073
	.75	.1002	.1599	.1169	52°10'	.1207	.2508	.0094
	1.0	.1319	.1379	.1256	45°41'	.1256	.2605	.0093
	1.25	.1526	.1137	.1232	40°30'	.1247	.2579	.0085
1.5	.25	.0046	.0840	.0186	77°27'	.0439	.0882	.0004
	.5	.0250	.1294	.0540	67°01'	.0751	.1523	.0021
	.75	.0545	.1396	.0828	58°37'	.0934	.1905	.0037
	1.0	.0824	.1315	.0992	51°58'	.1022	.2092	.0048
	1.25	.1049	.1156	.1053	46°28'	.1054	.2157	.0049
	1.5	.1211	.0981	.1044	41°51'	.1050	.2146	.0046
	2.0	.1375	.0681	.0929	34°45'	.0992	.2020	.0036
	2.5	.1403	.0470	.0781	29°33'	.0910	.1847	.0027
2.0	.5	.0064	.7773	.0222	74°02'	.0420	.0000	.0841
	1.0	.0332	.1041	.0572	60°54'	.0673	.1360	.0014
	1.5	.0636	.0955	.0763	50°55'	.0780	.1576	.0016
	2.0	.0862	.0761	.0791	43°10'	.0793	.0019	.1605
	2.5	.0985	.0581	.0741	37°23'	.0768	.0015	.1551

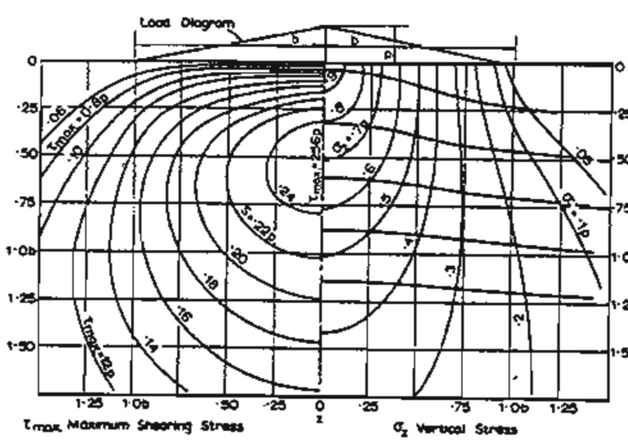


FIG.3.6 Stresses beneath strip with triangular loading (Jurgenson, 1934).

3.1.6 ASYMMETRICAL VERTICAL TRIANGULAR LOADING (Fig.3.7, Gray, 1936)

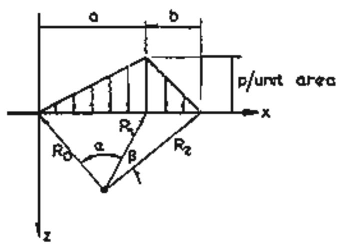


FIG.3.7

$$\sigma_z = \frac{p}{\pi}\left[\frac{x\alpha}{a} + \frac{a+b-x}{b}\beta\right] \qquad \text{... (3.8a)}$$

$$\sigma_x = \frac{p}{\pi}\left[\frac{x\alpha}{a} + \frac{a+b-x}{b}\beta + \frac{2z}{a}\log_e\frac{R_1}{R_0} + \frac{2z}{b}\log_e\frac{R_1}{R_2}\right] \qquad \text{... (3.8b)}$$

$$\tau_{xz} = \frac{pz}{\pi}\left[\frac{\alpha}{a} - \frac{\beta}{b}\right] \qquad \text{... (3.8c)}$$

$$\sigma_1 = \frac{p}{\pi}\left[\frac{x}{a}\alpha + \frac{a+b-x}{b}\beta + \frac{z}{a}\log_e\frac{R_1}{R_0} + \frac{z}{b}\log_e\frac{R_1}{R_2}\right] + \frac{pz}{\pi}\left(\left(\frac{1}{a}\log_e\frac{R_1}{R_0} + \frac{1}{b}\log_e\frac{R_1}{R_2}\right)^2 + \left(\frac{\alpha}{a} - \frac{\beta}{b}\right)^2\right)^{\frac{1}{2}} \qquad \text{... (3.8d)}$$

$$\sigma_3 = \frac{p}{\pi}\left[\frac{x\alpha}{a} + \frac{a+b-x}{b}\beta + \frac{z}{a}\log_e\frac{R_1}{R_0} + \frac{z}{b}\log_e\frac{R_1}{R_2}\right] - \frac{pz}{\pi}\left(\left(\frac{1}{a}\log_e\frac{R_1}{R_0} + \frac{1}{b}\log_e\frac{R_1}{R_2}\right)^2 + \left(\frac{\alpha}{a} - \frac{\beta}{b}\right)^2\right)^{\frac{1}{2}} \qquad \text{... (3.8e)}$$

$$\tau_{max} = \frac{pz}{\pi}\left(\left(\frac{1}{a}\log_e\frac{R_1}{R_0} + \frac{1}{b}\log_e\frac{R_1}{R_2}\right)^2 + \left(\frac{\alpha}{a} - \frac{\beta}{b}\right)^2\right)^{\frac{1}{2}} \qquad \text{... (3.8f)}$$

3.1.7 VERTICAL "EMBANKMENT" LOADING (Fig.3.8, Gray, 1936)

$$\sigma_z = \frac{p}{\pi}\left[\beta + \frac{x\alpha}{a} - \frac{z}{R_2^2}(x-b)\right] \qquad \text{... (3.9a)}$$

$$\sigma_x = \frac{p}{\pi}\left[\beta + \frac{x\alpha}{a} + \frac{z}{R_2^2}(x-b) + \frac{2z}{a}\log_e\frac{R_1}{R_0}\right] \qquad \text{... (3.9b)}$$

$$\tau_{xz} = -\frac{p}{\pi}\left[\frac{z\alpha}{a} - \frac{z^2}{R_2^2}\right] \qquad \text{... (3.9c)}$$

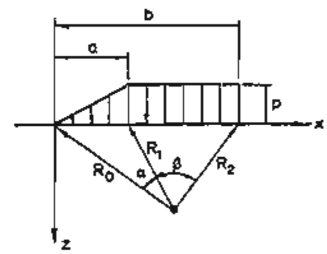


FIG.3.8

$$\left.\begin{matrix}\sigma_1\\ \sigma_3\end{matrix}\right\} = \frac{p}{\pi}\left[\beta + \frac{x\alpha}{a} + \frac{z}{a}\log_e\frac{R_1}{R_2} \pm \frac{z}{a}\left(\left\{\frac{a}{R_2^2}(x-b) + \log_e\frac{R_1}{R_0}\right\}^2 + \left(\alpha - \frac{az}{R_2^2}\right)^2\right)^{\frac{1}{2}}\right] \qquad \text{... (3.9d)}$$

Influence factors for σ_z beneath the edge of the loading have been published by Osterberg (1957) and are shown in Fig.3.9.

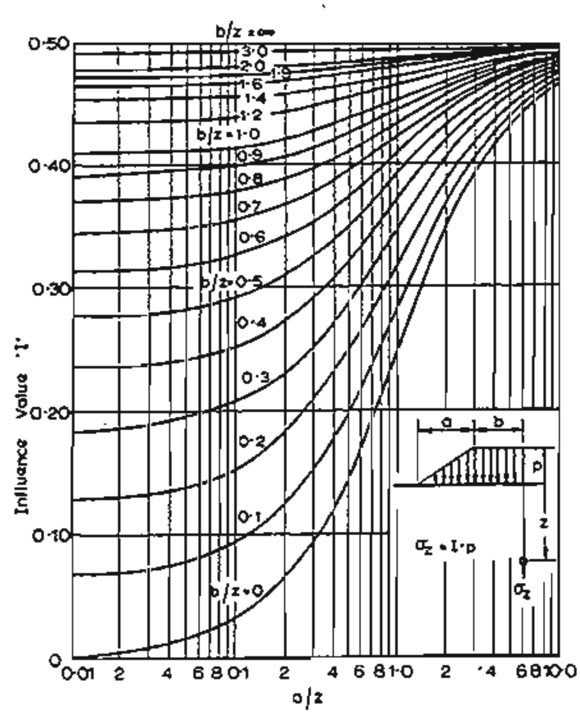


FIG.3.9 Influence chart for vertical stress due to embankment loading (Osterberg, 1957).

3.2 Loading over Half the Infinite Surface

3.2.1 UNIFORM VERTICAL LOADING (Fig.3.10, Gray, 1936).

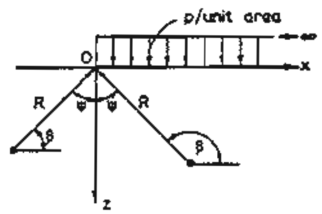


FIG.3.10

$$\sigma_z = \frac{p}{\pi}\left(\beta + \frac{xz}{R^2}\right) \quad \ldots (3.10a)$$

$$\sigma_x = \frac{p}{\pi}\left(\beta - \frac{xz}{R^2}\right) \quad \ldots (3.10b)$$

$$\tau_{xz} = -\frac{p}{\pi}\sin^2\beta \quad \ldots (3.10c)$$

$$\sigma_1 = \frac{p}{\pi}(\beta + \sin\beta) \quad \ldots (3.10d)$$

$$\sigma_3 = \frac{p}{\pi}(\beta - \sin\beta) \quad \ldots (3.10e)$$

$$\tau_{max} = \frac{p}{\pi}\sin\beta \quad \ldots (3.10f)$$

Loci of constant σ_1 and σ_3 are radial lines, OR.

Loci of constant τ_{max} are radial lines, OR.

Trajectories of σ_1 are a family of confocal parabolas*. Focus at 0. Horizontal axis.

Trajectories of σ_3 are a family of confocal parabolas*. Focus at 0. Horizontal axis.

The two families are orthogonal.

Trajectories of τ_{max} are two orthogonal families of parabolas. Focus at 0. Vertical axis.

Maximum $\tau_{max} = p/\pi$, occurs at points $(0,z)$, $z>0$.

Maximum $\sigma_1 = p$, " " " $(x,0)$, $x>0$.

Minimum $\sigma_3 = 0$, " " " $(x,0)$, $x<0$.

*These curves bisect the angle, β, at all points.

3.2.2 LINEARLY INCREASING VERTICAL LOADING (Fig.3.11, Gray, 1936).

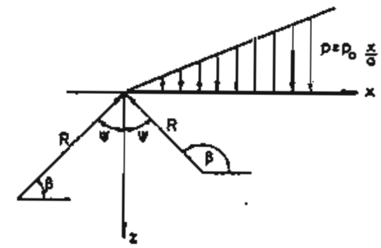


FIG.3.11

$$\sigma_z = \frac{p_o}{\pi a}[x\beta + z] \quad \ldots (3.11a)$$

$$\sigma_x = \frac{p_o}{\pi a}[x\beta - z - 2z\,\log_e R] \quad \ldots (3.11b)$$

$$\tau_{xz} = \frac{p_o}{\pi a}z\,\beta \quad \ldots (3.11c)$$

$$\left.\begin{matrix}\sigma_1\\ \sigma_3\end{matrix}\right\} = \frac{p_o}{\pi a}(x\beta - z\,\log_e R) \pm \frac{p_o z}{\pi a}\left[(1+\log_e R)^2 + \beta^2\right]^{\frac{1}{2}} \quad \ldots (3.11d)$$

$$\tau_{max} = \frac{p_o z}{\pi a}\left\{(1+\log_e R)^2 + \beta^2\right\}^{\frac{1}{2}} \quad \ldots (3.11e)$$

3.2.3 VERTICAL "EMBANKMENT" LOADING (Fig.3.12, Gray, 1936).

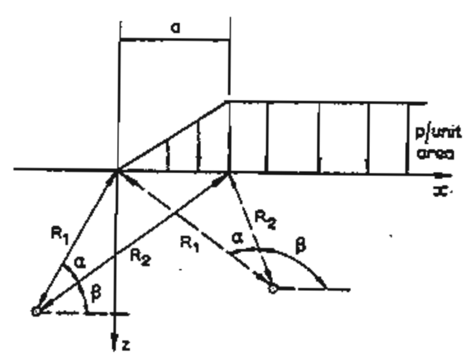


FIG.3.12

$$\sigma_z = \frac{p}{\pi a}[a\beta + x\alpha] \quad \ldots (3.12a)$$

$$\sigma_x = \frac{p}{\pi a}\left[a\beta + x\alpha + 2z\,\log_e\frac{R_2}{R_1}\right] \quad \ldots (3.12b)$$

$$\tau_{xz} = -\frac{p}{\pi a} z \alpha \quad \ldots (3.12c)$$

$$\left.\begin{matrix}\sigma_1 \\ \sigma_3\end{matrix}\right\} = \frac{p}{\pi a} \left[\left(a\beta + x\alpha + z \log_e \frac{R_2}{R_1}\right) \pm z \left(\log_e^2 \frac{R_2}{R_1} + \alpha^2\right)^{\frac{1}{2}} \right] \quad \ldots (3.12d)$$

$$\tau_{max} = \frac{pz}{\pi a} \left(\log_e^2 \frac{R_2}{R_1} + \alpha^2\right)^{\frac{1}{2}} \quad \ldots (3.12e)$$

Values of stresses, presented originally by Jurgenson (1934), are tabulated in TABLE 3.5.

TABLE 3.5

STRESSES DUE TO SEMI-INFINITE EMBANKMENT LOADING (Jurgenson, 1934)

x/a	z/a	σ_z/p	σ_x/p	τ_{zx}/p	90°-β	τ_{max}/p	σ_1/p	σ_3/p
0	0	0	0	0	-	0	0	0
	.25	.0780	.3034	.1055	21°33'	.1544	.3451	.0363
	.5	.1476	.4038	.1762	27°00'	.2178	.4935	.0579
	.75	.2048	.4487	.2214	30°35'	.2528	.5796	.0740
	1.0	.2500	.4706	.2500	33°06'	.2732	.6335	.0871
	1.25	.2852	.4821	.2685	34°56'	.2860	.6697	.0977
	1.5	.3129	.4883	.2807	36°20'	.2941	.6947	.1065
	2.0	.3524	.4946	.2952	38°14'	.3036	.7271	.1199
	2.5	.3789	.4968	.3028	39°30'	.3085	.7464	.1294
0.25	0	.2500	.2500	0	-	0	.2500	.2500
	.25	.2643	.3924	.1619	34°13'	.1741	.5025	.1543
	.5	.3023	.4544	.2302	35°52'	.2424	.6208	.1360
	.75	.3381	.4784	.2643	37°34'	.2734	.6817	.1349
	1.0	.3659	.4885	.2828	38°53'	.2894	.7166	.1378
	1.25	.3867	.4935	.2936	39°51'	.2984	.7385	.1417
-0.25	0	0	0	0	-	0	0	0
	.25	.0161	.2201	.0468	12°19'	.1123	.2304	.0058
	.5	.0633	.3430	.1157	19°48'	.1816	.3848	.0216
	.75	.1156	.4077	.1692	24°36'	.2235	.4852	.0382
	1.0	.1630	.4431	.2072	27°59'	.2501	.5532	.0530
	1.25	.2032	.4634	.2339	30°28'	.2677	.6010	.0656
0.5	0	.5000	.5000	0	-	0	.5000	.5000
	.25	.5000	.5000	.1762	45°00'	.1762	.6762	.3238
	.5	.5000	.5000	.2500	45°00'	.2500	.7500	.2500
	.75	.5000	.5000	.2807	45°00'	.2807	.7807	.2193
	1.0	.5000	.5000	.2952	45°00'	.2952	.7952	.2048
	1.25	.5000	.5000	.3028	45°00'	.3028	.8028	.1972
	1.5	.5000	.5000	.3072	45°00'	.3072	.8072	.1928
	2.0	.5000	.5000	.3119	45°00'	.3119	.8119	.1881
	2.5	.5000	.5000	.3144	45°00'	.3144	.8144	.1856
-0.5	0	0	0	0	-	0	0	0
	.25	.0051	.1642	.0237	8°19'	.0829	.1676	.0018
	.5	.0286	.2847	.0738	14°59'	.1478	.3045	.0089
	.75	.0650	.3614	.1239	19°57'	.1932	.4064	.0200
	1.0	.1045	.4086	.1653	23°42'	.2246	.4812	.0320
	1.25	.1422	.4381	.1972	26°41'	.2465	.5367	.0437
	1.5	.1762	.4566	.2214	28°50'	.2621	.5785	.0543
	2.0	.2317	.4771	.2537	32°06'	.2818	.6362	.0726
	2.5	.2734	.4869	.2729	34°19'	.2930	.6732	.0872

TABLE 3.5 Continued

x/a	z/a	σ_z/p	σ_x/p	τ_{zx}/p	90°-β	τ_{max}/p	σ_1/p	σ_3/p
0.75	0	.7500	.7500	0	-	0	.7500	.7500
	.25	.7357	.6076	.1619	55°47'	.1741	.8458	.4976
	.5	.6978	.5457	.2302	54°08'	.2424	.8642	.3794
	.75	.6619	.5217	.2643	52°26'	.2734	.8652	.3184
	1.0	.6341	.5114	.2828	51°07'	.2894	.8622	.2834
	1.25	.6133	.5065	.2936	50°09'	.2984	.8583	.2615
-0.75	0	0	0	0	0	0	0	0
	.25	.0023	.1302	.0143	6°18'	.0656	.1319	.0007
	.5	.0147	.2382	.0493	11°54'	.1222	.2487	.0043
1.0	0	1.0000	1.0000	0	90°00'	0	1.0000	1.0000
	.25	.9220	.6973	.1055	68°24'	.1541	.9638	.6556
	.5	.8524	.5962	.1762	63°01'	.2178	.9421	.5065
	.75	.7951	.5512	.2214	59°26'	.2528	.9260	.4204
	1.0	.7500	.5294	.2500	56°54'	.2732	.9129	.3665
	1.25	.7148	.5178	.2685	55°05'	.2860	.9023	.3303
	1.5	.6871	.5117	.2807	53°41'	.2941	.8935	.3053
	2.0	.6476	.5054	.2952	51°47'	.3036	.8801	.2729
	2.5	.6211	.5032	.3028	50°31'	.3085	.8707	.2537
-1.0	0	0	0	0	0	0	0	0
	.5	.0083	.2031	.0348	9°50'	.1034	.2091	.0023
	1.0	.0452	.3369	.1024	17°33'	.1782	.3693	.0129
	1.5	.0967	.4092	.1621	23°02'	.2252	.4782	.0278
	2.0	.1475	.4465	.2049	26°57'	.2536	.5506	.0434
	2.5	.1915	.4674	.2343	29°45'	.2719	.6014	.0576
1.25	0	1.0000	1.0000	0	90°00'	0	1.0000	1.0000
	.25	.9803	.7763	.0461	77°51'	.1120	.9903	.7663
	.5	.9367	.6570	.1157	70°13'	.1816	.9785	.6153
	.75	.8844	.5923	.1692	65°24'	.2235	.9619	.5149
	1.0	.8370	.5569	.2072	62°02'	.2501	.9471	.4469
	1.5	.7632	.5247	.2528	57°38'	.2795	.9235	.3645
	2.0	.7123	.5124	.2763	54°57'	.2938	.9062	.3186
	2.5	.6765	.5069	.2896	53°10'	.3018	.8935	.2899
1.5	0	1.0000	1.0000	0	90°00'	0	1.0000	1.0000
	.25	.9949	.8357	.0237	81°42'	.0829	.9982	.8234
	.5	.9714	.7152	.0738	75°02'	.1478	.9911	.6955
	.75	.9350	.6385	.1239	70°03'	.1932	.9800	.5936
	1.0	.8955	.5913	.1653	66°19'	.2246	.9680	.5188
	1.5	.8238	.5433	.2214	61°11'	.2621	.9457	.4215
	2.0	.7682	.5229	.2537	57°54'	.2818	.9274	.3638
	2.5	.7266	.5130	.2729	55°41'	.2930	.9128	.3268
2.0	0	1.0000	1.0000	0	90°00'	0	1.0000	1.0000
	.25	.9989	.8919	.0096	84°55'	.0543	.9997	.8911
	.5	.9916	.7968	.0348	80°10'	.1034	.9976	.7908
	1.0	.9548	.6631	.1024	72°28'	.1781	.9871	.6309
	1.5	.9032	.5908	.1621	66°59'	.2252	.9722	.5218
	2.0	.8524	.5534	.2049	63°04'	.2537	.9566	.4492
	2.5	.8085	.5325	.2343	60°15'	.2719	.9424	.3986

3.3 Loading on a Circular Area

3.3.1 UNIFORM VERTICAL LOADING (Fig.3.13)

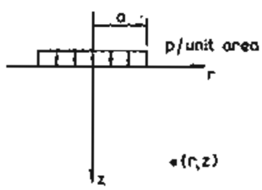


FIG.3.13

On the axis *(r=o)*,

$$\sigma_z = p\left[1 - \left\{\frac{1}{1+(a/z)^2}\right\}^{3/2}\right] \quad \ldots (3.13a)$$

$$\sigma_r = \sigma_\theta = \frac{p}{2}\left[(1+2\nu) - \frac{2(1+\nu)z}{(a^2+z^2)^{\frac{1}{2}}} + \frac{z^3}{(a^2+z^2)^{3/2}}\right] \quad \ldots (3.13b)$$

$$\Theta = 2(1+\nu)p\left[1 - \frac{z}{(a^2+z^2)^{\frac{1}{2}}}\right] \quad \ldots (3.13c)$$

$$\rho_z = \frac{2pa(1-\nu^2)}{E}\left(\sqrt{1+(z/a)^2} - z/a\right) \cdot \left[1 + \frac{z/a}{2(1-\nu)\sqrt{1+(z/a)^2}}\right] \quad \ldots (3.13d)$$

General expressions and tabulated values for all stresses, strains and displacements are given in Appendix B, as a special case of the values for an anisotropic material.

Values of σ_z, σ_r, σ_θ, τ_{rz} and ρ_z for $\nu=0.5$ have been presented in graphical form by Foster and Ahlvin (1954) and are reproduced in Figures 3.14 to 3.18.

A complete tabulation of stresses, strains and deflections for all values of ν has been presented by by Ahlvin and Ulery (1962) and is given in Tables 3.6 to 3.13. The key to the use of these Tables is shown below.

Bulk stress

$$\Theta = 2p(1+\nu)A = \sigma_z+\sigma_r+\sigma_\theta = e\,\frac{E}{1-2\nu} \quad \ldots (3.14a)$$

Vertical stress

$$\sigma = p[A + B] \quad \ldots (3.14b)$$

Radial horizontal stress

$$\sigma_r = p[2\nu A + C + (1-2\nu)F] \quad \ldots (3.14c)$$

Tangential horizontal stress

$$\sigma_\theta = p[2\nu A - D + (1-2\nu)E_1] \quad \ldots (3.14d)$$

Vertical-radial shear stress

$$\tau_{rz} = \tau_{zr} = pG_1 \quad \ldots (3.14e)$$

Bulk strain

$$e = p\,\frac{2(1+\nu)}{E}(1-2\nu)A = \varepsilon_z+\varepsilon_r+\varepsilon_\theta = \Theta\,\frac{1-2\nu}{E} \quad \ldots (3.14f)$$

Vertical strain

$$\varepsilon_z = p\,\frac{1+\nu}{E}[(1-2\nu)A + B] \quad \ldots (3.14g)$$

Radial horizontal strain

$$\varepsilon_r = p\,\frac{1+\nu}{E}\,[(1-2\nu)F + C] \quad \ldots (3.14h)$$

Tangential horizontal strain

$$\varepsilon_\theta = p\,\frac{1+\nu}{E}\,[(1-2\nu)E_1 - D] \quad \ldots (3.14i)$$

Vertical-radial shear strain

$$\gamma_{rz} = \gamma_{zr} = p\,\frac{2(1+\nu)}{E}\,G_1 = \frac{2(1+\nu)}{E}\,\tau_{rz} \quad \ldots (3.14j)$$

Vertical deflection

$$\rho_z = p\,\frac{1+\nu}{E}\,a\left[\frac{z}{a}A + (1-\nu)H\right] \quad \ldots (3.14k)$$

Radial horizontal deflection

$$\rho_r = p\,\frac{1+\nu}{E}\,(-r)\;[(1-2\nu)E_1 - D] = -r\;\varepsilon_\theta \quad \ldots (3.14l)$$

Tangential horizontal deflection

$$\rho_\theta = 0 \quad \ldots (3.14m)$$

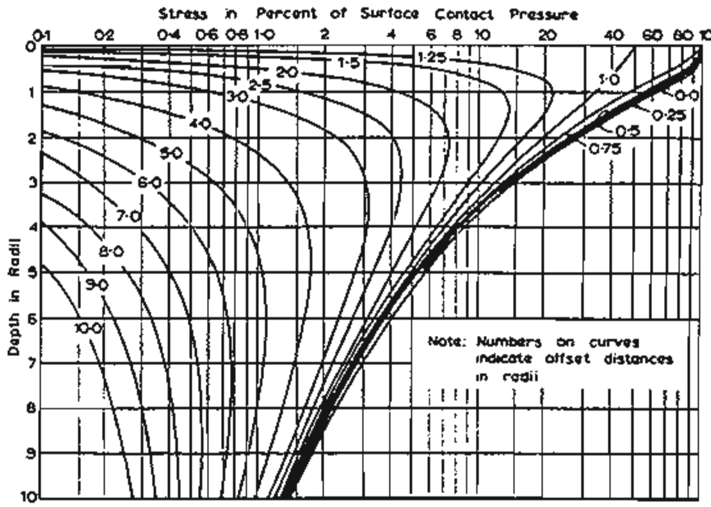


FIG.3.14 Vertical stress σ_z beneath uniform circular load (Foster and Ahlvin, 1954).

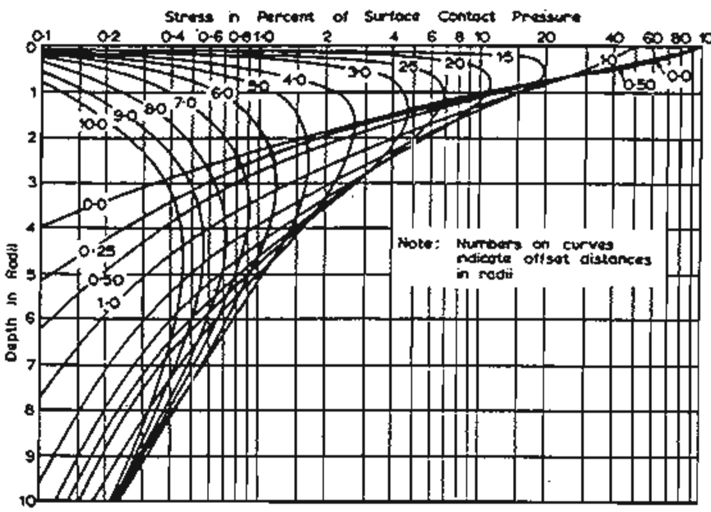


FIG.3.15 Horizontal stress σ_r beneath uniform circular load (ν=0.5) (Foster and Ahlvin,1954).

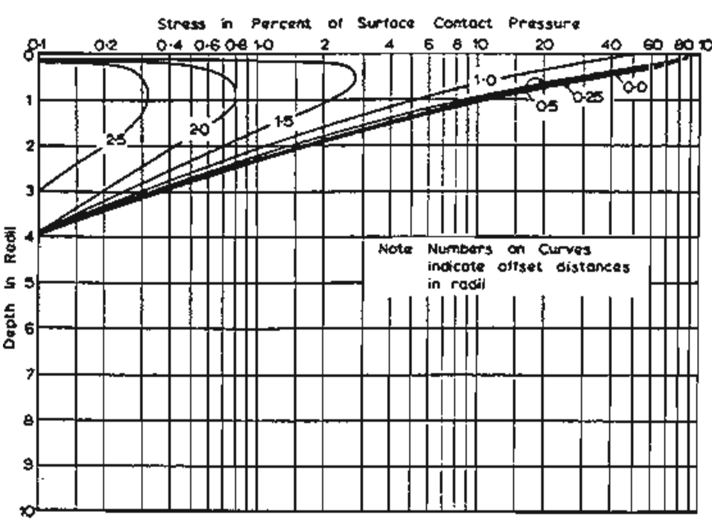


FIG.3.16 Horizontal stress σ_θ beneath uniform circular load (ν=0.5) (Foster and Ahlvin,1954).

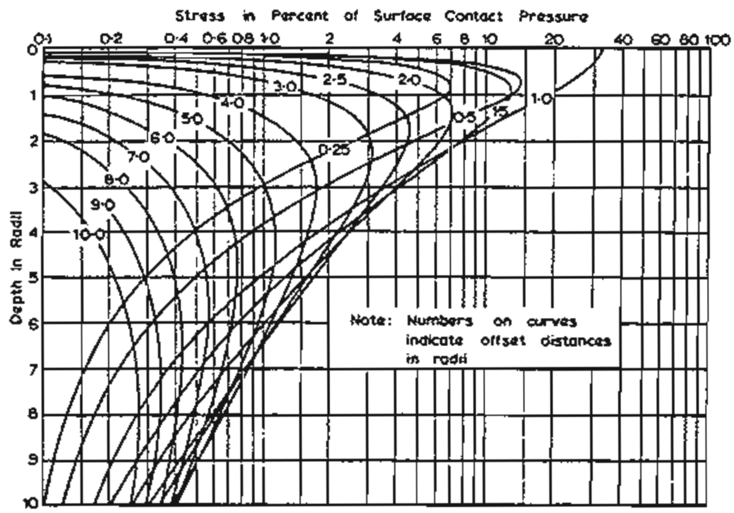


FIG.3.17 Shear stress τ_{rz} beneath uniform circular load (Foster and Ahlvin, 1954).

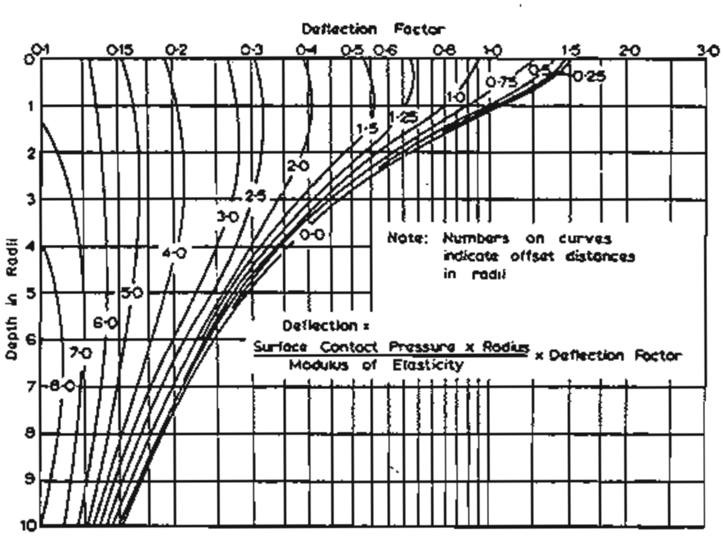


FIG.3.18 Vertical deflection ρ_z beneath uniform circular load (Foster and Ahlvin, 1954).

TABLE 3.6

FUNCTION "A"

(Ahlvin and Ulery, 1962)

z/a \ r/a	0	0.2	0.4	0.6	0.8	1	1.2	1.5	2	3	4	5	6	7	8	10	12	14
0	1.0	1.0	1.0	1.0	1.0	.5	0	0	0	0	0	0	0	0	0	0	0	
0.1	.90050	.89748	.88679	.86126	.78797	.43015	.09645	.02787	.00856	.00211	.00084	.00042						
0.2	.80388	.79824	.77884	.73483	.63014	.38269	.15433	.05251	.01680	.00419	.00167	.00083	.00048	.00030	.00020			
0.3	.71265	.70518	.68316	.62690	.52081	.34375	.17964	.07199	.02440	.00622	.00250							
0.4	.62861	.62015	.59241	.53767	.44329	.31048	.18709	.08593	.03118									
0.5	.55279	.54403	.51622	.46448	.38390	.28156	.18556	.09499	.03701	.01013	.00407	.00209	.00118	.00071	.00053	.00025	.00014	.00009
0.6	.48550	.47691	.45078	.40427	.33676	.25588	.17952	.10010										
0.7	.42654	.41874	.39491	.35428	.29833	.21727	.17124	.10228	.04558									
0.8	.37531	.36832	.34729	.31243	.26581	.21297	.16206	.10236										
0.9	.33104	.32492	.30669	.27707	.23832	.19488	.15253	.10094										
1	.29289	.28763	.27005	.24697	.21468	.17868	.14329	.09849	.05185	.01742	.00761	.00393	.00226	.00143	.00097	.00050	.00029	.00018
1.2	.23178	.22795	.21662	.19890	.17626	.15101	.12570	.09192	.05260	.01935	.00871	.00459	.00269	.00171	.00115			
1.5	.16795	.16552	.15877	.14804	.13436	.11892	.10296	.08048	.05116	.02142	.01013	.00548	.00325	.00210	.00141	.00073	.00043	.00027
2	.10557	.10453	.10140	.09647	.09011	.08269	.07471	.06275	.04496	.02221	.01160	.00659	.00399	.00264	.00180	.00094	.00056	.00036
2.5	.07152	.07098	.06947	.06698	.06373	.05974	.05555	.04880	.03787	.02143	.01221	.00732	.00463	.00308	.00214	.00115	.00068	.00043
3	.05152	.05101	.05022	.04886	.04707	.04487	.04241	.03839	.03150	.01980	.01220	.00770	.00505	.00346	.00242	.00132	.00079	.00051
4	.02986	.02976	.02907	.02802	.02832	.02749	.02651	.02490	.02193	.01592	.01109	.00768	.00536	.00384	.00282	.00160	.00099	.00065
5	.01942	.01938				.01835			.01573	.01249	.00949	.00708	.00527	.00394	.00298	.00179	.00113	.00075
6	.01361					.01307			.01168	.00983	.00795	.00628	.00492	.00384	.00299	.00188	.00124	.00084
7	.01005					.00976			.00894	.00784	.00661	.00548	.00445	.00360	.00291	.00193	.00130	.00091
8	.00772					.00755			.00703	.00635	.00554	.00472	.00398	.00332	.00276	.00189	.00134	.00094
9	.00612					.00600			.00566	.00520	.00466	.00409	.00353	.00301	.00256	.00184	.00133	.00096
10								.00477	.00465	.00438	.00397	.00352	.00326	.00273	.00241			

TABLE 3.7

FUNCTION "B"

(Ahlvin and Ulery, 1962)

z/a \ r/a	0	0.2	0.4	0.6	0.8	1	1.2	1.5	2	3	4	5	6	7	8	10	12	14
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0				
0.1	.09852	.10140	.11138	.13424	.18796	.05388	-.07899	-.02672	-.00845	-.00210	-.00084	-.00042						
0.2	.18857	.19306	.20772	.23524	.25983	.08513	-.07759	-.04448	-.01593	-.00412	-.00166	-.00083	-.00024	-.00015	-.00010			
0.3	.26362	.26787	.28018	.29483	.27257	.20757	-.04316	-.04999	-.02166	-.00599	-.00245							
0.4	.32016	.32259	.32748	.32273	.26925	.12404	-.00766	-.04535	-.02522									
0.5	.35777	.35752	.35323	.33106	.26236	.13591	.02165	-.03455	-.02651	-.00991	-.00388	-.00199	-.00116	-.00073	-.00049	-.00025	-.00014	-.00009
0.6	.37831	.37531	.36308	.32822	.25411	.14440	.04457	-.02101										
0.7	.38487	.37962	.36072	.31929	.24638	.14986	.06209	-.00702	-.02329									
0.8	.38091	.37408	.35133	.30699	.23779	.15292	.07530	.00614										
0.9	.36962	.36275	.33734	.29299	.22891	.15404	.08507	.01795										
1	.35355	.34553	.32075	.27819	.21978	.15355	.09210	.02814	-.01005	-.01115	-.00608	-.00344	-.00210	-.00135	-.00092	-.00048	-.00028	-.00018
1.2	.31485	.30730	.28481	.24836	.20113	.14915	.10002	.04378	.00023	-.00995	-.00632	-.00378	-.00236	-.00156	-.00107			
1.5	.25602	.25025	.23338	.20694	.17368	.13732	.10193	.05745	.01385	-.00669	-.00600	-.00401	-.00265	-.00181	-.00126	-.00068	-.00040	-.00026
2	.17889	.18144	.16644	.15198	.13375	.11331	.09254	.06371	.02836	.00028	-.00410	-.00371	-.00278	-.00202	-.00148	-.00084	-.00050	-.00033
2.5	.12807	.12633	.12126	.11327	.10298	.09130	.07869	.06022	.03429	.00661	-.00130	-.00271	-.00250	-.00201	-.00156	-.00094	-.00059	-.00039
3	.09487	.09394	.09099	.08635	.08033	.07325	.06551	.05354	.03511	.01112	.00157	-.00134	-.00192	-.00179	-.00151	-.00099	-.00065	-.00046
4	.05707	.05666	.05562	.05383	.05145	.04773	.04532	.03995	.03066	.01515	.00595	.00155	-.00029	-.00094	-.00109	-.00094	-.00068	-.00050
5	.03772	.03760				.03384			.02474	.01522	.00810	.00371	.00132	.00013	-.00043	-.00070	-.00061	-.00049
6	.02666					.02468			.01968	.01380	.00867	.00496	.00254	.00110	.00028	-.00037	-.00047	-.00045
7	.01980					.01868			.01577	.01204	.00842	.00547	.00332	.00185	.00093	-.00002	-.00029	-.00037
8	.01526					.01459			.01279	.01034	.00779	.00554	.00372	.00236	.00141	.00035	-.00008	-.00025
9	.01212					.01170			.01054	.00888	.00705	.00533	.00386	.00265	.00178	.00066	.00012	-.00012
10								.00924	.00879	.00764	.00631	.00501	.00382	.00281	.00199			

TABLE 3.8

FUNCTION "C"

(Ahlvin and Ulery, 1962)

r/a z/a	0	0.2	0.4	0.6	0.8	1	1.2	1.5	2	3	4	5	6	7	8	10	12	14
0	0	0	0	0	0	0												
0.1	-.04926	-.05142	-.05903	-.07708	-.12108	.02247	.12007	.04475	.01536	.00403	.00164	.00082						
0.2	-.09429	-.09755	-.10872	-.12977	-.14552	.02419	.14896	.07892	.02951	.00796	.00325	.00164	.00094	.00059	.00039			
0.3	-.13181	-.13484	-.14415	-.15023	-.12990	.01988	.13394	.09816	.04148	.01169	.00483							
0.4	-.16008	-.16188	-.16519	-.15985	-.11168	.01292	.11014	.10422	.05067									
0.5	-.17889	-.17835	-.17497	-.15625	-.09833	.00483	.08730	.10125	.05690	.01824	.00778	.00399	.00231	.00146	.00098	.00050	.00029	.00018
0.6	-.18915	-.18664	-.17336	-.14934	-.08967	-.00304	.06731	.09313										
0.7	-.19244	-.18831	-.17393	-.14147	-.08409	-.01061	.05028	.08253	.06129									
0.8	-.19046	-.18481	-.16784	-.13393	-.08066	-.01744	.03582	.07114										
0.9	-.18481	-.17841	-.16024	-.12664	-.07828	-.02337	.02359	.05993										
1	-.17678	-.17050	-.15188	-.11995	-.07634	-.02843	.01331	.04939	.05429	.02726	.01333	.00726	.00433	.00278	.00188	.00098	.00057	.00036
1.2	-.15742	-.15117	-.13467	-.10763	-.07289	-.03575	-.00245	.03107	.04522	.02791	.01467	.00824	.00501	.00324	.00221			
1.5	-.12801	-.12277	-.11101	-.09145	-.06711	-.04124	-.01702	.01088	.03154	.02652	.01570	.00933	.00585	.00386	.00266	.00141	.00083	.00039
2	-.08944	-.08491	-.07976	-.06925	-.05560	-.04144	-.02687	-.00782	.01267	.02070	.01527	.01013	.00671	.00462	.00327	.00179	.00107	.00069
2.5	-.06403	-.06068	-.05839	-.05259	-.04522	-.03605	-.02800	-.01536	.00103	.01384	.01314	.00987	.00707	.00506	.00369	.00209	.00128	.00083
3	-.04744	-.04560	-.04339	-.04089	-.03642	-.03130	-.02587	-.01748	-.00528	.00792	.01030	.00888	.00689	.00520	.00392	.00232	.00145	.00096
4	-.02854	-.02737	-.02562	-.02585	-.02421	-.02112	-.01964	-.01586	-.00956	.00038	.00492	.00602	.00562	.00476	.00389	.00254	.00168	.00115
5	-.01886	-.01810				-.01568			-.00939	-.00293	-.00128	.00329	.00391	.00380	.00341	.00250	.00177	.00127
6	-.01333					-.01118			-.00819	-.00405	-.00079	.00129	.00234	.00272	.00272	.00227	.00173	.00130
7	-.00990					-.00902			-.00678	-.00417	-.00180	-.00004	.00113	.00174	.00200	.00193	.00161	.00128
8	-.00763					-.00699			-.00552	-.00393	-.00225	-.00077	.00029	.00096	.00134	.00157	.00143	.00120
9	-.00607					-.00423			-.00452	-.00353	-.00235	-.00118	-.00027	.00037	.00082	.00124	.00122	.00110
10								-.00381	-.00373	-.00314	-.00233	-.00137	-.00063	.00030	.00040			

TABLE 3.9

FUNCTION "D"

(Ahlvin and Ulery, 1962)

r/a z/a	0	0.2	0.4	0.6	0.8	1	1.2	1.5	2	3	4	5	6	7	8	10	12	14
0	0	0	0	0	0	0	0	0	0	0	0	0						
0.1	.04926	.04998	.05235	.05716	.06687	.07635	.04108	.01803	.00691	.00193	.00080	.00041						
0.2	.09429	.09552	.09900	.10546	.11431	.10932	.07139	.03444	.01359	.00384	.00159	.00081	.00047	.00029	.00020			
0.3	.13181	.13305	.14051	.14062	.14267	.12745	.09078	.04817	.01982	.00927	.00238							
0.4	.16008	.16070	.16229	.16288	.15756	.13696	.10248	.05887	.02545									
0.5	.17889	.17917	.17826	.17481	.16403	.14074	.10894	.06670	.03039	.00921	.00390	.00200	.00116	.00073	.00049	.00025	.00015	.00009
0.6	.18915	.18867	.18573	.17887	.16489	.14137	.11186	.07212										
0.7	.19244	.19132	.18679	.17782	.16229	.13926	.11237	.07551	.03801									
0.8	.19046	.18927	.18348	.17306	.15714	.13548	.11115	.07728										
0.9	.18481	.18349	.17709	.16635	.15063	.13067	.10866	.07788										
1	.17678	.17503	.16886	.15824	.14344	.12513	.10540	.07753	.04456	.01611	.00725	.00382	.00224	.00142	.00096	.00050	.00029	.00018
1.2	.15742	.15618	.15014	.14073	.12823	.11340	.09757	.07484	.04575	.01796	.00835	.00446	.00264	.00169	.00114			
1.5	.12801	.12754	.12237	.11549	.10657	.09608	.08491	.06833	.04539	.01983	.00970	.00532	.00320	.00205	.00140	.00073	.00043	.00027
2	.08944	.09080	.08668	.08273	.07814	.07187	.06566	.05589	.04103	.02098	.01117	.00643	.00398	.00260	.00179	.00095	.00056	.00036
2.5	.06403	.06565	.06284	.06068	.05777	.05525	.05069	.04486	.03532	.02045	.01183	.00717	.00457	.00306	.00213	.00115	.00068	.00044
3	.04744	.04834	.04760	.04548	.04391	.04195	.03963	.03606	.02983	.01904	.01187	.00755	.00497	.00341	.00242	.00133	.00080	.00052
4	.02854	.02928	.02996	.02798	.02724	.02661	.02568	.02408	.02110	.01552	.01087	.00757	.00533	.00382	.00280	.00160	.00100	.00065
5	.01886	.01950				.01816			.01535	.01230	.00939	.00700	.00523	.00392	.00299	.00180	.00114	.00077
6	.01333					.01351			.01149	.00976	.00788	.00625	.00488	.00381	.00301	.00190	.00124	.00086
7	.00990					.00966			.00899	.00787	.00662	.00542	.00445	.00360	.00292	.00192	.00130	.00092
8	.00763					.00759			.00727	.00641	.00554	.00477	.00402	.00332	.00275	.00192	.00131	.00096
9	.00607					.00746			.00601	.00533	.00470	.00415	.00358	.00303	.00260	.00187	.00133	.00099
10								.00542	.00506	.00450	.00398	.00364	.00319	.00278	.00239			

TABLE 3.10

FUNCTION "E_1"

(Ahlvin and Ulery, 1962)

z/a \ r/a	0	0.2	0.4	0.6	0.8	1	1.2	1.5	2	3	4	5	6	7	8	10	12	14
0	.5	.5	.5	.5	.5	.5	.34722	.22222	.12500	.05556	.03125	.02000	.01389	.01020	.00781	.00500	.00347	.00255
0.1	.45025	.44949	.44698	.44173	.43008	.39198	.30445	.20399	.11806	.05362	.03045	.01959						
0.2	.40194	.40043	.39591	.38660	.36798	.32802	.26598	.18633	.11121	.05170	.02965	.01919	.01342	.00991	.00762			
0.3	.35633	.35428	.33809	.33674	.31578	.28003	.23311	.16967	.10450	.04979	.02886							
0.4	.31431	.31214	.30541	.29298	.27243	.24200	.20526	.15428	.09801									
0.5	.27639	.27407	.26732	.25511	.23639	.21119	.18168	.14028	.09180	.04608	.02727	.01800	.01272	.00946	.00734	.00475	.00332	.00246
0.6	.24275	.24247	.23411	.22289	.20634	.18520	.16155	.12759										
0.7	.21327	.21112	.20535	.19525	.18093	.16356	.14421	.11620	.08027									
0.8	.18765	.18550	.18049	.17190	.15977	.14523	.12928	.10602										
0.9	.16552	.16337	.15921	.15179	.14168	.12954	.11634	.09686										
1	.14645	.14483	.14610	.13472	.12618	.11611	.10510	.08865	.06552	.03736	.02352	.01602	.01157	.00874	.00683	.00450	.00318	.00237
1.2	.11589	.11435	.11201	.10741	.10140	.09431	.08657	.07476	.05728	.03425	.02208	.01527	.01113	.00847	.00664			
1.5	.08398	.08356	.08159	.07885	.07517	.07088	.06621	.05871	.04703	.03003	.02008	.01419	.01049	.00806	.00636	.00425	.00304	.00228
2	.05279	.05105	.05146	.05034	.04850	.04675	.04442	.04078	.03454	.02410	.01706	.01248	.00943	.00738	.00590	.00401	.00290	.00219
2.5	.03576	.03426	.03489	.03435	.03360	.03211	.03150	.02953	.02599	.01945	.01447	.01096	.00850	.00674	.00546	.00378	.00276	.00210
3	.02566	.02519	.02470	.02491	.02444	.02389	.02330	.02216	.02007	.01585	.01230	.00962	.00763	.00617	.00505	.00355	.00263	.00201
4	.01493	.01452	.01495	.01526	.01446	.01418	.01395	.01356	.01281	.01084	.00900	.00742	.00612	.00511	.00431	.00313	.00237	.00185
5	.00971	.00927				.00929			.00873	.00774	.00673	.00579	.00495	.00425	.00364	.00275	.00213	.00168
6	.00680					.00632			.00629	.00574	.00517	.00457	.00404	.00354	.00309	.00241	.00192	.00154
7	.00503					.00493			.00466	.00438	.00404	.00370	.00330	.00296	.00264	.00213	.00172	.00140
8	.00386					.00377			.00354	.00344	.00325	.00297	.00273	.00250	.00228	.00185	.00155	.00127
9	.00306					.00227			.00275	.00273	.00264	.00246	.00229	.00212	.00194	.00163	.00139	.00116
10								.00210	.00220	.00225	.00221	.00203	.00200	.00181	.00171			

TABLE 3.11

FUNCTION "F"

(Ahlvin and Ulery, 1962)

z/a \ r/a	0	0.2	0.4	0.6	0.8	1	1.2	1.5	2	3	4	5	6	7	8	10	12	14
0	.5	.5	.5	.5	.5	0	-.34722	-.22222	-.12500	-.05556	-.03125	-.02000	-.01389	-.01020	-.00781	-.00500	-.00347	-.00255
0.1	.45025	.44794	.43981	.41954	.35789	.03817	-.20800	-.17612	-.10950	-.05151	-.02961	-.01917						
0.2	.40194	.39781	.38294	.34823	.26215	.05466	-.11165	-.13381	-.09441	-.04750	-.02798	-.01835	-.01295	-.00961	-.00742			
0.3	.35633	.35094	.34508	.29016	.20503	.06372	-.05346	-.09768	-.08010	-.04356	-.02636							
0.4	.31431	.30801	.28681	.24469	.17086	.06848	-.01818	-.06835	-.06684									
0.5	.27639	.26997	.24890	.20937	.14752	.07037	.00388	-.04529	-.05479	-.03595	-.02320	-.01590	-.01154	-.00875	-.00681	-.00450	-.00318	-.00237
0.6	.24275	.23444	.21667	.18138	.13042	.07068	.01797	-.02749										
0.7	.21327	.20762	.18956	.15903	.11740	.06963	.02704	-.01392	-.03469									
0.8	.18765	.18287	.16679	.14053	.10604	.06774	.03277	-.00365										
0.9	.16552	.16158	.14747	.12528	.09664	.06533	.03619	.00408										
1	.14645	.14280	.12395	.11225	.08850	.06256	.03819	.00984	-.01367	-.01994	-.01591	-.01209	-.00931	-.00731	-.00587	-.00400	-.00289	-.00219
1.2	.11589	.11360	.10460	.09449	.07486	.05670	.03913	.01716	-.00452	-.01491	-.01337	-.01068	-.00844	-.00676	-.00550			
1.5	.08398	.08196	.07719	.06918	.05919	.04804	.03686	.02177	.00413	-.00879	-.00995	-.00870	-.00723	-.00596	-.00495	-.00353	-.00261	-.00201
2	.05279	.05348	.04994	.04614	.04162	.03593	.03029	.02197	.01043	-.00189	-.00546	-.00589	-.00544	-.00474	-.00410	-.00307	-.00233	-.00183
2.5	.03576	.03673	.03459	.03263	.03014	.02762	.02406	.01927	.01188	.00198	-.00226	-.00364	-.00386	-.00366	-.00332	-.00263	-.00208	-.00166
3	.02566	.02586	.02255	.02395	.02263	.02097	.01911	.01623	.01144	.00396	-.00010	-.00192	-.00258	-.00271	-.00263	-.00223	-.00183	-.00150
4	.01493	.01536	.01412	.01259	.01386	.01331	.01256	.01134	.00912	.00508	.00209	.00026	-.00076	-.00127	-.00148	-.00153	-.00137	-.00120
5	.00971	.01011				.00905			.00700	.00475	.00277	.00129	.00031	-.00030	-.00066	-.00096	-.00099	-.00093
6	.00680					.00675			.00538	.00409	.00278	.00170	.00088	.00030	-.00010	-.00053	-.00066	-.00070
7	.00503					.00483			.00428	.00346	.00258	.00178	.00114	.00064	.00027	-.00020	-.00041	-.00049
8	.00386					.00380			.00350	.00291	.00229	.00174	.00125	.00082	.00048	.00003	-.00020	-.00033
9	.00306					.00374			.00291	.00247	.00203	.00163	.00124	.00089	.00062	.00020	-.00005	-.00029
10								.00267	.00246	.00213	.00176	.00149	.00126	.00092	.00070			

TABLE 3.12

FUNCTION "G_1" (Ahlvin and Ulery, 1962)

z/a \ r/a	0	0.2	0.4	0.6	0.8	1	1.2	1.5	2	3	4	5	6	7	8	10	12	14
0	0	0	0	0	0	.31831	0	0	0	0	0	0	0	0	0	0	0	0
0.1	0	.00315	.00802	.01951	.06682	.31405	.05555	.00865	.00159	.00023	.00007	.00003						
0.2	0	.01163	.02877	.06441	.16214	.30474	.13592	.03060	.00614	.00091	.00026	.00010	.00005	.00003	.00002			
0.3	0	.02301	.05475	.11072	.21465	.29228	.18216	.05747	.01302	.00201	.00059							
0.4	0	.03460	.07883	.14477	.23442	.27779	.20195	.08233	.02138									
0.5	0	.04429	.09618	.16426	.23652	.26216	.20731	.10185	.03033	.00528	.00158	.00063	.00030	.00016	.00009	.00004	.00002	.00001
0.6	0	.04966	.10729	.17192	.22949	.24574	.20496	.11541										
0.7	0	.05484	.11256	.17126	.21772	.22924	.19840	.12373	.04718									
0.8	0	.05590	.11225	.16534	.20381	.21295	.18953	.12855										
0.9	0	.05496	.10856	.15628	.18904	.29712	.17945	.28881										
1	0	.05266	.10274	.14566	.17419	.18198	.16884	.12745	.06434	.01646	.00555	.00233	.00113	.00062	.00036	.00015	.00007	.00004
1.2	0	.04585	.08831	.12323	.14615	.15408	.14755	.12038	.06967	.02077	.00743	.00320	.00159	.00087	.00051			
1.5	0	.03483	.06688	.09293	.11071	.11904	.11830	.10477	.07075	.02599	.01021	.00460	.00235	.00130	.00078	.00033	.00016	.00009
2	0	.02102	.04069	.05721	.06948	.07738	.08067	.07804	.06275	.03062	.01409	.00692	.00369	.00212	.00129	.00055	.00027	.00015
2.5	0	.01293	.02534	.03611	.04484	.05119	.05509	.05668	.05117	.03099	.01650	.00886	.00499	.00296	.00185	.00082	.00041	.00023
3	0	.00840	.01638	.02376	.02994	.03485	.03843	.04124	.04039	.02886	.01745	.01022	.00610	.00376	.00241	.00110	.00057	.00032
4	0	.00382	.00772	.01149	.01480	.01764	.02004	.02271	.02475	.02215	.01639	.01118	.00745	.00499	.00340	.00167	.00090	.00052
5	0	.00214				.00992		.01343	.01551	.01601	.01364	.01105	.00782	.00560	.00404	.00216	.00122	.00073
6	0					.00602		.00845	.01014	.01148	.01082	.00917	.00733	.00567	.00432	.00243	.00150	.00092
7	0					.00396			.00687	.00830	.00842	.00770	.00656	.00539	.00432	.00272	.00171	.00110
8	0					.00270			.00481	.00612	.00656	.00631	.00568	.00492	.00413	.00278	.00185	.00124
9	0					.00177			.00347	.00459	.00513	.00515	.00485	.00438	.00381	.00274	.00192	.00133
10	0							.00199	.00258	.00351	.00407	.00420	.00411	.00382	.00346			

TABLE 3.13

FUNCTION "H" (Ahlvin and Ulery, 1962)

z/a \ r/a	0	0.2	0.4	0.6	0.8	1	1.2	1.5	2	3	4	5	6	7	8	10	12	14
0	2.0	1.97987	1.91751	1.80575	1.62553	1.27319	.93676	.71185	.51671	.33815	.25200	.20045	.16626	.14315	.12576	.09918	.08346	.07023
0.1	1.80998	1.79018	1.72886	1.61961	1.44711	1.18107	.92670	.70888	.51627	.33794	.25184	.20081						
0.2	1.63961	1.62068	1.56242	1.46001	1.30614	1.09996	.90098	.70074	.51382	.33726	.25162	.20072	.16688	.14288	.12512			
0.3	1.48806	1.47044	1.40979	1.32442	1.19210	1.02740	.86726	.68823	.50966	.33638	.25124							
0.4	1.35407	1.33802	1.28963	1.20822	1.09555	.96202	.83042	.67238	.50412									
0.5	1.23607	1.22176	1.17894	1.10830	1.01312	.90298	.79308	.65429	.49728	.33293	.24996	.19982	.16668	.14273	.12493	.09996	.08295	.07123
0.6	1.13238	1.11998	1.08350	1.02154	.94120	.84917	.75653	.63469										
0.7	1.04131	1.03057	.99794	.91049	.87742	.80030	.72143	.61442	.48061									
0.8	.96125	.95175	.92386	.87928	.82136	.75571	.68809	.59398										
0.9	.89072	.88251	.85856	.82616	.77950	.71495	.65677	.57361										
1	.82843	.85005	.80465	.76809	.72587	.67769	.62701	.55364	.45122	.31877	.24386	.19673	.16516	.24182	.12394	.09952	.08292	.07104
1.2	.72410	.71882	.70370	.67937	.64814	.61187	.57329	.51552	.43013	.31162	.24070	.19520	.16369	.14099	.12350			
1.5	.60555	.60233	.57246	.57633	.55559	.53138	.50496	.46379	.39872	.29945	.23495	.19053	.16199	.14058	.12281	.09876	.08270	.07064
2	.47214	.47022	.44512	.45656	.44502	.43202	.41702	.39242	.35054	.27740	.22418	.18618	.15846	.13762	.12124	.09792	.08196	.07026
2.5	.38518	.38403	.38098	.37608	.36940	.36155	.35243	.33698	.30913	.25550	.21208	.17898	.15395	.13463	.11928	.09700	.08115	.06980
3	.32457	.32403	.32184	.31887	.31464	.30969	.30381	.29364	.27453	.23487	.19977	.17154	.14919	.13119	.11694	.09558	.08061	.06897
4	.24620	.24588	.24820	.25128	.24168	.23932	.23668	.23164	.22188	.19908	.17640	.15596	.13864	.12396	.11172	.09300	.07864	.06848
5	.19805	.19785				.19455			.18450	.17080	.15575	.14130	.12785	.11615	.10585	.08915	.07675	.06695
6	.16554					.16326			.15750	.14868	.13842	.12792	.11778	.10836	.09990	.08562	.07452	.06522
7	.14217					.14077			.13699	.13097	.12404	.11620	.10843	.10101	.09387	.08197	.07210	.06377
8	.12448					.12352			.12112	.11680	.11176	.10600	.09976	.09400	.08848	.07800	.06928	.06200
9	.11079					.10989			.10854	.10548	.10161	.09702	.09234	.08784	.08298	.07407	.06678	.05976
10								.09900	.09820	.09510	.09290	.08980	.08500	.08180	.07710			

3.3.2 CONICAL LOADING

Distributions of σ_z down the axis (Harr and Lovell, 1963) and σ_r along the axis (Schiffman, 1963) are shown in Figs. 3.19 and 3.20.

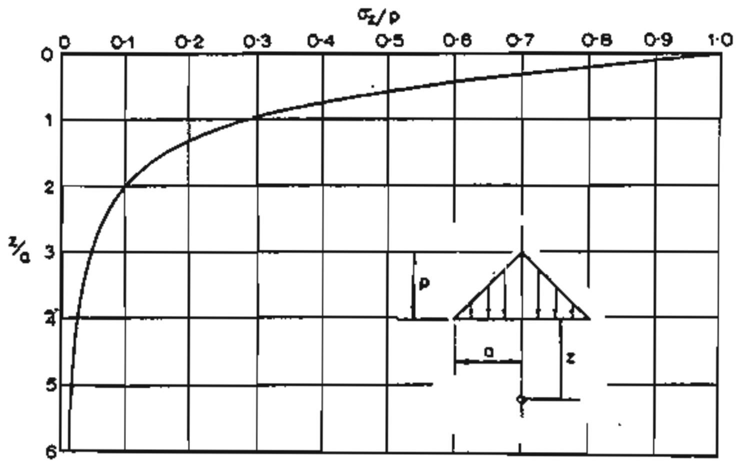


FIG.3.19 Vertical stress σ_z on axis due to conical loading (Harr and Lovell, 1963).

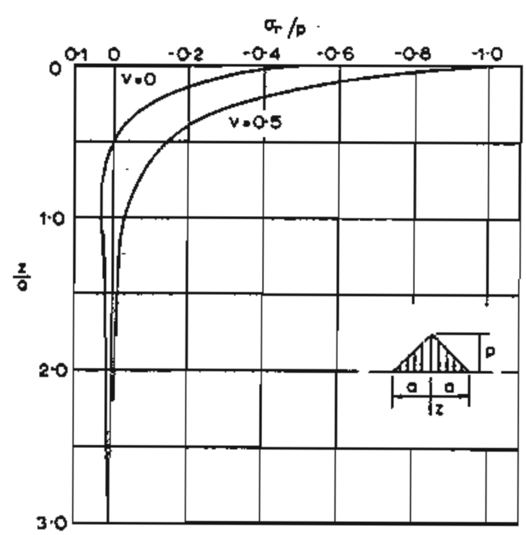


FIG.3.20 Radial stress σ_r on axis due to conical loading (Schiffman, 1963).

3.3.3 UNIFORM HORIZONTAL LOADING

As pointed out by Barber (1966), the vertical stress due to a uniform horizontal load is the same as the shear stress due to a uniform vertical load. Thus, these stresses may be evaluated from the values in Section 3.3.1.

Vertical and horizontal normal stresses presented by Barber (1963) are reproduced in Figs. 3.21 and 3.22.

Charts for determining the normal stresses at any point due to horizontal loading on any shape of loaded area are given in Section 3.6.

General expressions and tabulated values for all stresses, strains and displacements are given in Appendix B.

3.3.4 INWARD SHEAR LOADINGS

For uniform inward shear on a circle, values of σ_z presented by Barber (1963) are shown in Fig.3.23. General expressions and tabulated values for all stresses, strains and displacements are given in Appendix B.

For non-uniform inward shear on a circle, values of σ_z presented by Barber (1963) are shown in Figure 3.24.

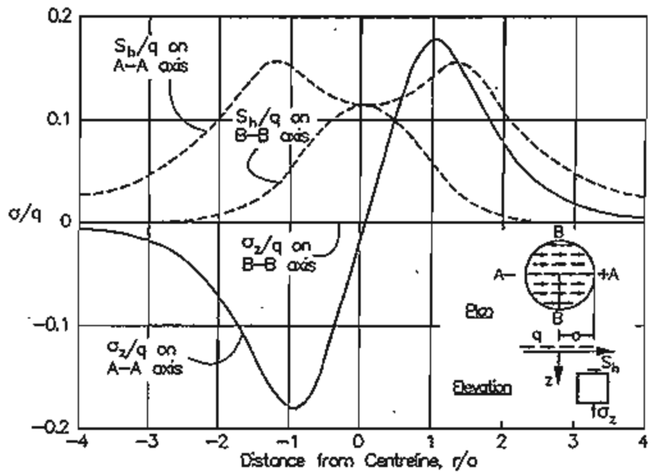


FIG.3.21 Stresses due to horizontal loading on circle. z/a=1, ν=0.5. (Barber, 1963).

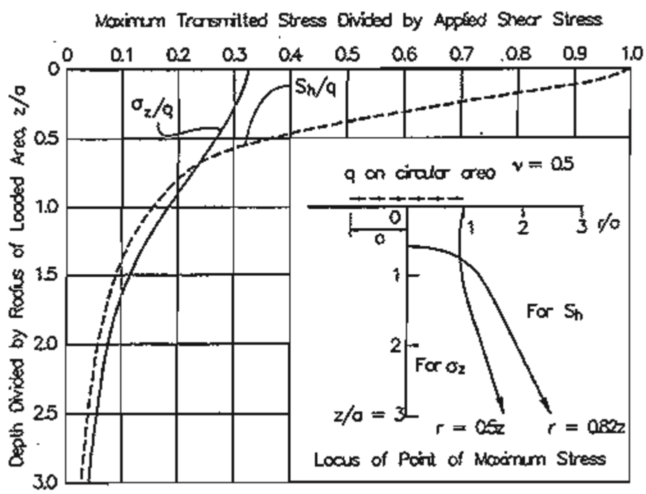


FIG.3.22 Maximum stress at different depths due to horizontal loading on circle (Barber, 1963).

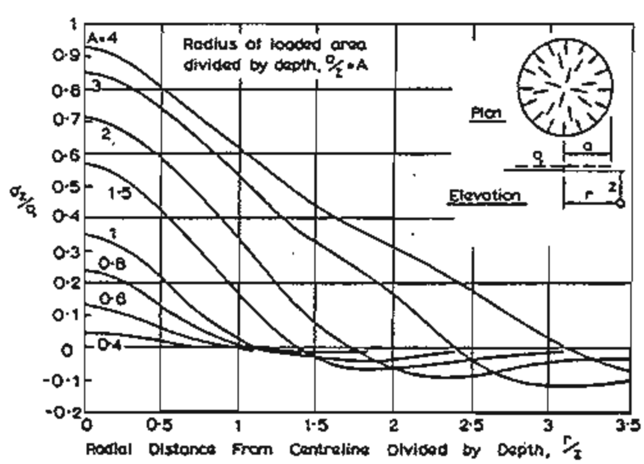


FIG.3.23 Vertical stress σ_z due to uniform inward shear on circle (Barber, 1963).

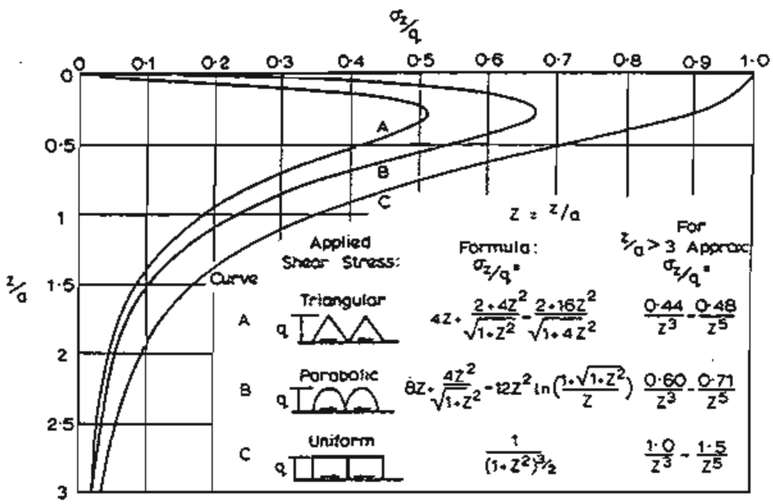


FIG.3.24 Vertical stress σ_z on axis due to various shear loads on circle (Barber, 1963).

3.3.5 UNIFORM VERTICAL LOADING ON A PERFECTLY ROUGH CIRCULAR AREA (Fig.3.13)

This problem was considered by Schiffman (1968), who obtained general solutions for stresses and displacements.

Along the axis, the stresses are as follows:

$$\sigma_z = p[1 - \{\frac{1}{1+(\frac{a}{z})^2}\}^{3/2} - n\frac{z}{a} K_o^2(0,\frac{z}{a})] \quad \ldots (3.15a)$$

$$\sigma_\theta = \sigma_r = p\,[\frac{(1+2\nu)}{2} - \frac{2(1+\nu)z}{(a^2+z^2)^{1/2}} + \frac{z^3}{(a^2+z^2)^{3/2}} - n\{(1+\nu)K_o^1(0,\frac{z}{a}) - \frac{z}{2a}K_o^2(0,\frac{z}{a})\}] \quad \ldots (3.15b)$$

$$\tau_{rz} = 0 \quad \ldots (3.15c)$$

where $K_o^1(0,z/a) = \sqrt{2/\pi}[\tan^{-1}(a/z) - \frac{z/a}{(1+(\frac{z}{a})^2)}]$

$K_o^2(0,z/a) = \sqrt{8/\pi}\,/\,(1+(z/a)^2)^2$

The distributions of σ_r and σ_z along the axis are shown in Figs. 3.25a and 3.25b for $\nu=0$ and 0.5. Also shown are the corresponding distributions for a frictionless or smooth surface. For $\nu=0.5$, friction has no influence on the stresses.

Along the surface ($z=0$), the shear stress is

$$\tau_{rz} = \frac{(1-2\nu)\,p\,r}{(1-\nu)\pi\sqrt{a^2-r^2}} \qquad (r/a < 1) \quad \ldots (3.16a)$$

$$\tau_{rz} = 0 \qquad (r/a > 1) \quad \ldots (3.16b)$$

The radial distribution of τ_{rz} beneath the surface of the circle is shown in Fig.3.26 for various values of ν.

The surface displacements are as follows:

$$\rho_z = \frac{pa(1+\nu)}{E}\{4(1-\nu)E(r/a) - \frac{(1-2\nu)^2}{1-\nu}\cdot\sqrt{1-(\frac{r}{a})^2}\} \qquad (r/a < 1) \quad \ldots (3.17a)$$

$$\rho_z = \frac{4p(1-\nu^2)r}{\pi E}[E(\frac{a}{r}) - \frac{(r^2-a^2)}{r^2}K(\frac{a}{r})] \qquad (r/a > 1) \quad \ldots (3.17b)$$

$$\rho_r = 0 \qquad (r/a < 1) \quad \ldots (3.17c)$$

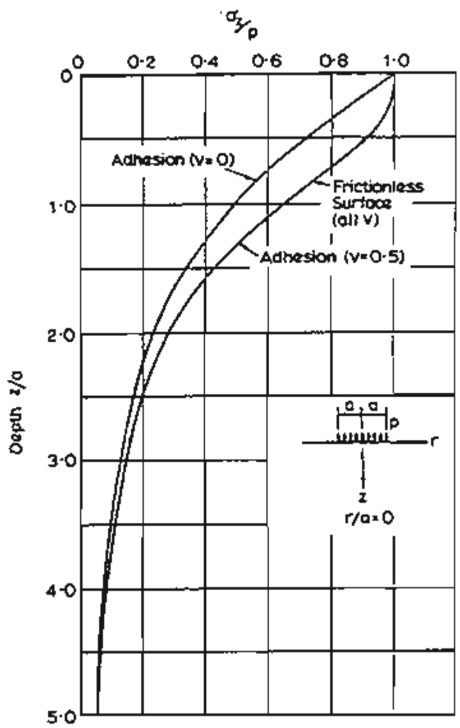
$$\rho_r = \frac{(1+\nu)(1-2\nu)pa}{E\sqrt{\pi}}[\frac{r}{a}\sin^{-1}(\frac{a}{r}) + (\frac{1+\sqrt{r^2-a^2}}{r})] \qquad (r/a > 1) \quad \ldots (3.17d)$$

where $K(k)$ is the complete elliptic integral of the first kind,

$E(k)$ is the complete elliptic integral of the second kind.

Surface displacement profiles are shown in Fig. 3.27 for both a frictionless and fully rough circular areas. For $\nu=0.5$, friction has no influence on displacement.

The influence of ν on the central vertical surface displacement is shown in Fig.3.28.



(a) Vertical stress σ_z

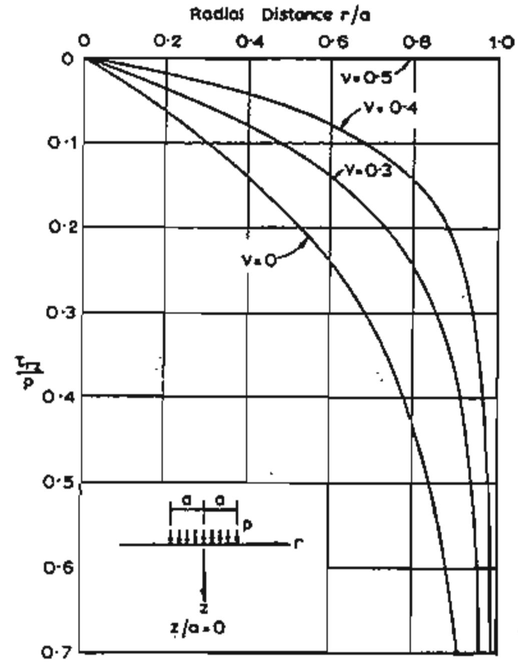
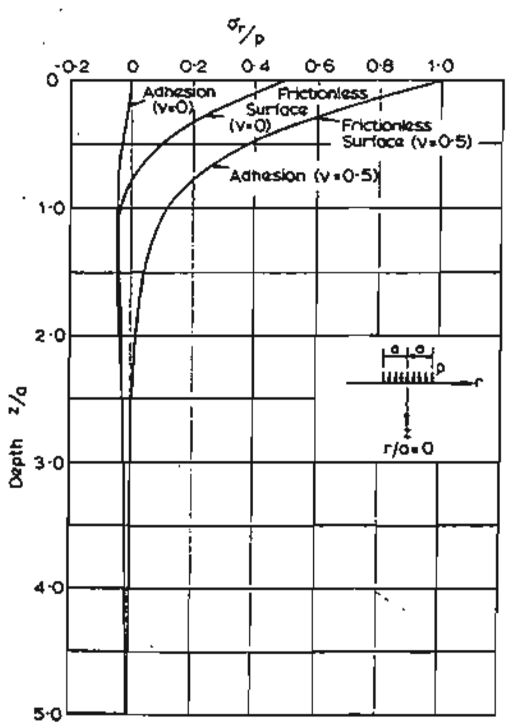


FIG.3.26 Distribution of shear stress τ_{rz} along surface (Schiffman, 1968).



(b) Radial stress σ_r

FIG.3.25 Distribution of stress on axis of circle. (Schiffman, 1968).

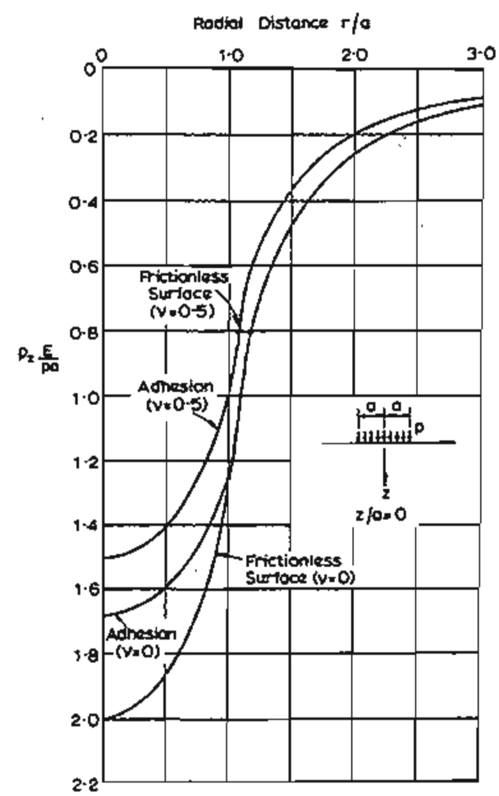


FIG.3.27 Vertical displacement profile along surface. (Schiffman, 1968).

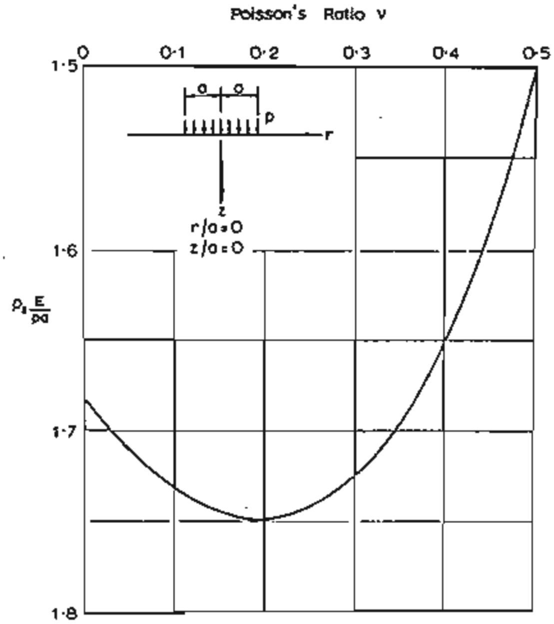


FIG.3.28 Influence of ν on ρ_z at surface on axis. (Schiffman, 1968).

3.3.6 OTHER TYPES OF LOADING

(i) Parabolic loading - see Harr and Lovell (1963) and Schiffman (1963).

(ii) Linearly varying vertical stress - see Appendix B.

(iii) Linearly varying torsional stress - see Appendix B.

3.4 Loading on a Rectangular Area

3.4.1 UNIFORM VERTICAL LOADING

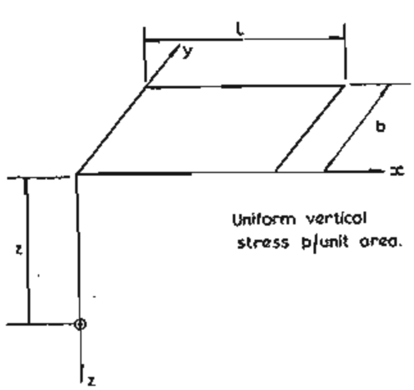


FIG. 3.29

Beneath the corner of the rectangle (see Fig. 3.29), Holl (1940) gives the following expressions for stresses for $\nu = 0.5$:

$$\sigma_z = \frac{p}{2\pi}\left[\tan^{-1}\frac{\ell b}{zR_3} + \frac{\ell bz}{R_3}\left(\frac{1}{R_1^2}+\frac{1}{R_2^2}\right)\right] \quad \text{... (3.18a)}$$

$$\sigma_x = \frac{p}{2\pi}\left[\tan^{-1}\frac{\ell b}{zR_3} - \frac{\ell bz}{R_1^2R_3}\right] \quad \text{... (3.18b)}$$

$$\sigma_y = \frac{p}{2\pi}\left[\tan^{-1}\frac{\ell b}{zR_3} - \frac{\ell bz}{R_2^2R_3}\right] \quad \text{... (3.18c)}$$

$$\tau_{xz} = \frac{p}{2\pi}\left[\frac{b}{R_2} - \frac{z^2 b}{R_1^2R_3}\right] \quad \text{... (3.18d)}$$

$$\tau_{yz} = \frac{p}{2\pi}\left[\frac{\ell}{R_1} - \frac{z^2 \ell}{R_2^2R_3}\right] \quad \text{... (3.18e)}$$

$$\tau_{xy} = \frac{p}{2\pi}\left[1 + \frac{z}{R_3} - z\left(\frac{1}{R_1}+\frac{1}{R_2}\right)\right] \quad \text{... (3.18f)}$$

where $R_1 = (\ell^2+z^2)^{1/2}$

$R_2 = (b^2+z^2)^{1/2}$

$R_3 = (\ell^2+b^2+z^2)^{1/2}$

Influence factors for the normal stresses have been presented by Giroud (1970). These stresses are expressed as follows:

Under the corners:

$$\sigma_z = p\,K_0 \quad \text{... (3.19a)}$$

$$\sigma_x = p\,[K_2 - (1-2\nu)K_2^1] \quad \text{... (3.19b)}$$

$$\sigma_y = p\,[L_2 - (1-2\nu)L_2^1] \quad \text{... (3.19c)}$$

The influence factors K_0($\equiv I$ in Fig.3.30), K_2, K_2^1, L_2, L_2^1 are reproduced in Tables 3.14 to 3.18.

Influence factors for σ_z beneath the corner are shown in Fig.3.30(Fadum, 1948). For points other than the corner, the principle of superposition may be employed.

Beneath the corner of a rectangle (see Fig.3.29), Harr (1966) quotes the following expression for vertical displacement at depth z:

$$\rho_z = \frac{pb}{E}(1-\nu^2)\left(A - \frac{1-2\nu}{1-\nu}B\right) \quad \text{... (3.20)}$$

where $$A = \frac{1}{2\pi}\left(\ell n\,\frac{\sqrt{1+m_1^2+n_1^2}+m_1}{\sqrt{1+m_1^2+n_1^2}-m_1} + m_1\,\ell n\,\frac{\sqrt{1+m_1^2+n_1^2}+1}{\sqrt{1+m_1^2+n_1^2}-1}\right)$$

TABLE 3.14

VALUES OF K_o

(Giroud, 1970)

z/ℓ \ b/ℓ	0	0.1	0.2	1/3	0.4	0.5	2/3	1	1.5	2	2.5	3	5	10	∞
0	0.000	0.250	0.250	0.250	0.250	0.250	0.250	0.250	0.250	0.250	0.250	0.250	0.250	0.250	0.250
0.2	0.000	0.137	0.204	0.234	0.240	0.244	0.247	0.249	0.249	0.249	0.249	0.249	0.249	0.249	0.249
0.4	0.000	0.076	0.136	0.187	0.202	0.218	0.231	0.240	0.243	0.244	0.244	0.244	0.244	0.244	0.244
0.5	0.000	0.061	0.113	0.164	0.181	0.200	0.218	0.232	0.238	0.239	0.240	0.240	0.240	0.240	0.240
0.6	0.000	0.051	0.096	0.143	0.161	0.182	0.204	0.223	0.231	0.233	0.234	0.234	0.234	0.234	0.234
0.8	0.000	0.037	0.071	0.111	0.127	0.148	0.173	0.200	0.214	0.218	0.219	0.220	0.220	0.220	0.220
1	0.000	0.028	0.055	0.087	0.101	0.120	0.145	0.175	0.194	0.200	0.202	0.203	0.204	0.205	0.205
1.2	0.000	0.022	0.043	0.069	0.081	0.098	0.121	0.152	0.173	0.182	0.185	0.187	0.189	0.189	0.189
1.4	0.000	0.018	0.035	0.056	0.066	0.080	0.101	0.131	0.154	0.164	0.169	0.171	0.174	0.174	0.174
1.5	0.000	0.016	0.031	0.051	0.060	0.073	0.092	0.121	0.145	0.156	0.161	0.164	0.166	0.167	0.167
1.6	0.000	0.014	0.028	0.046	0.055	0.067	0.085	0.112	0.136	0.148	0.154	0.157	0.160	0.160	0.160
1.8	0.000	0.012	0.024	0.039	0.046	0.056	0.072	0.097	0.121	0.133	0.140	0.143	0.147	0.148	0.148
2	0.000	0.010	0.020	0.033	0.039	0.048	0.061	0.084	0.107	0.120	0.127	0.131	0.136	0.137	0.137
2.5	0.000	0.007	0.013	0.022	0.027	0.033	0.043	0.060	0.080	0.093	0.101	0.106	0.113	0.115	0.115
3	0.000	0.005	0.010	0.016	0.019	0.024	0.031	0.045	0.061	0.073	0.081	0.087	0.096	0.099	0.099
4	0.000	0.003	0.006	0.009	0.011	0.014	0.019	0.027	0.038	0.048	0.055	0.060	0.071	0.076	0.076
5	0.000	0.002	0.004	0.006	0.007	0.009	0.012	0.018	0.026	0.033	0.039	0.043	0.055	0.061	0.062
10	0.000	0.000	0.001	0.002	0.002	0.002	0.003	0.005	0.007	0.009	0.011	0.013	0.020	0.028	0.032
15	0.000	0.000	0.000	0.001	0.001	0.001	0.001	0.002	0.003	0.004	0.005	0.006	0.010	0.016	0.021
20	0.000	0.000	0.000	0.000	0.000	0.001	0.001	0.001	0.002	0.002	0.003	0.004	0.006	0.010	0.016
50	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.001	0.001	0.002	0.006

TABLE 3.15

VALUES OF K_2

(Giroud, 1970)

z/ℓ \ b/ℓ	0	0.1	0.2	1/3	0.4	0.5	2/3	1	1.5	2	2.5	3	5	10	∞
0	0.000	0.250	0.250	0.250	0.250	0.250	0.250	0.250	0.250	0.250	0.250	0.250	0.250	0.250	0.250
0.2	0.000	0.069	0.116	0.149	0.159	0.169	0.177	0.184	0.187	0.188	0.188	0.188	0.188	0.188	0.188
0.4	0.000	0.031	0.058	0.085	0.095	0.106	0.118	0.128	0.133	0.134	0.134	0.134	0.135	0.135	0.135
0.5	0.000	0.022	0.043	0.064	0.073	0.083	0.094	0.105	0.110	0.112	0.112	0.112	0.113	0.113	0.113
0.6	0.000	0.017	0.032	0.049	0.056	0.065	0.075	0.086	0.091	0.093	0.093	0.094	0.094	0.094	0.094
0.8	0.000	0.009	0.018	0.029	0.034	0.040	0.047	0.057	0.062	0.064	0.064	0.065	0.065	0.065	0.065
1	0.000	0.006	0.011	0.018	0.021	0.025	0.030	0.037	0.042	0.044	0.045	0.045	0.045	0.045	0.045
1.2	0.000	0.003	0.007	0.011	0.013	0.016	0.020	0.025	0.029	0.031	0.032	0.032	0.032	0.032	0.032
1.4	0.000	0.002	0.004	0.007	0.008	0.010	0.013	0.017	0.020	0.022	0.023	0.023	0.023	0.023	0.023
1.5	0.000	0.002	0.004	0.006	0.007	0.008	0.011	0.014	0.017	0.019	0.019	0.020	0.020	0.020	0.020
1.6	0.000	0.001	0.003	0.005	0.006	0.007	0.009	0.012	0.015	0.016	0.017	0.017	0.017	0.017	0.017
1.8	0.000	0.001	0.002	0.003	0.004	0.005	0.006	0.008	0.011	0.012	0.012	0.013	0.013	0.013	0.013
2	0.000	0.001	0.001	0.002	0.003	0.003	0.004	0.006	0.008	0.009	0.009	0.010	0.010	0.010	0.010
2.5	0.000	0.000	0.001	0.001	0.001	0.002	0.002	0.003	0.004	0.005	0.005	0.005	0.006	0.006	0.006
3	0.000	0.000	0.000	0.001	0.001	0.001	0.001	0.002	0.002	0.003	0.003	0.003	0.003	0.003	0.003
4	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.001	0.001	0.001	0.001	0.001	0.001	0.002	0.002
5	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.001	0.001	0.001	0.001	0.001
10	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
15	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
20	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
50	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000

TABLE 3.16

VALUES OF K_2'

(Giroud, 1970)

z/ℓ \ b/ℓ	0	0.1	0.2	1/3	0.4	0.5	2/3	1	1.5	2	2.5	3	5	10	∞
0	0.000	0.234	0.219	0.199	0.189	0.176	0.156	0.125	0.094	0.074	0.061	0.051	0.031	0.016	0.000
0.2	0.000	0.059	0.097	0.118	0.121	0.122	0.118	0.103	0.082	0.067	0.056	0.048	0.030	0.016	0.000
0.4	0.000	0.026	0.048	0.069	0.075	0.082	0.086	0.083	0.071	0.060	0.051	0.045	0.029	0.015	0.000
0.5	0.000	0.019	0.036	0.054	0.060	0.067	0.073	0.074	0.066	0.056	0.049	0.043	0.028	0.015	0.000
0.6	0.000	0.015	0.028	0.043	0.049	0.056	0.062	0.066	0.061	0.053	0.047	0.041	0.028	0.015	0.000
0.8	0.000	0.009	0.018	0.029	0.033	0.039	0.046	0.052	0.052	0.047	0.043	0.038	0.026	0.015	0.000
1	0.000	0.007	0.013	0.021	0.024	0.029	0.035	0.042	0.044	0.042	0.039	0.035	0.025	0.014	0.000
1.2	0.000	0.005	0.009	0.015	0.018	0.022	0.027	0.034	0.037	0.037	0.035	0.032	0.024	0.014	0.000
1.4	0.000	0.004	0.007	0.012	0.014	0.017	0.021	0.027	0.032	0.033	0.032	0.030	0.023	0.014	0.000
1.5	0.000	0.003	0.006	0.010	0.012	0.015	0.019	0.025	0.029	0.031	0.030	0.029	0.022	0.014	0.000
1.6	0.000	0.003	0.006	0.009	0.011	0.013	0.017	0.023	0.027	0.029	0.028	0.027	0.022	0.013	0.000
1.8	0.000	0.002	0.005	0.007	0.009	0.011	0.014	0.019	0.024	0.025	0.026	0.025	0.021	0.013	0.000
2	0.000	0.002	0.004	0.006	0.007	0.009	0.012	0.016	0.020	0.023	0.023	0.023	0.020	0.013	0.000
2.5	0.000	0.001	0.002	0.004	0.005	0.006	0.008	0.011	0.015	0.017	0.018	0.019	0.017	0.012	0.000
3	0.000	0.001	0.002	0.003	0.003	0.004	0.006	0.008	0.011	0.013	0.015	0.015	0.015	0.011	0.000
4	0.000	0.000	0.001	0.002	0.002	0.002	0.003	0.005	0.007	0.008	0.010	0.011	0.012	0.010	0.000
5	0.000	0.000	0.001	0.001	0.001	0.002	0.002	0.003	0.004	0.006	0.007	0.008	0.009	0.009	0.000
10	0.000	0.000	0.000	0.000	0.000	0.000	0.001	0.001	0.001	0.002	0.002	0.002	0.003	0.005	0.000
15	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.001	0.001	0.001	0.001	0.002	0.003	0.000
20	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.001	0.001	0.002	0.000
50	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000

TABLE 3.17

VALUES OF L_2

(Giroud, 1970)

z/ℓ \ b/ℓ	0	0.1	0.2	1/3	0.4	0.5	2/3	1	1.5	2	2.5	3	5	10	∞
0	0.000	0.250	0.250	0.250	0.250	0.250	0.250	0.250	0.250	0.250	0.250	0.250	0.250	0.250	0.250
0.2	0.000	0.010	0.045	0.094	0.112	0.134	0.158	0.184	0.201	0.208	0.211	0.214	0.217	0.218	0.219
0.4	0.000	0.002	0.010	0.032	0.045	0.064	0.091	0.128	0.156	0.169	0.176	0.179	0.186	0.188	0.189
0.5	0.000	0.001	0.006	0.020	0.029	0.044	0.068	0.105	0.136	0.151	0.159	0.164	0.172	0.175	0.176
0.6	0.000	0.000	0.003	0.013	0.019	0.031	0.051	0.086	0.118	0.134	0.144	0.149	0.158	0.163	0.164
0.8	0.000	0.000	0.001	0.006	0.009	0.016	0.029	0.057	0.087	0.106	0.117	0.124	0.135	0.141	0.143
1	0.000	0.000	0.001	0.003	0.005	0.009	0.017	0.037	0.064	0.083	0.095	0.103	0.116	0.123	0.125
1.2	0.000	0.000	0.000	0.002	0.003	0.005	0.011	0.025	0.047	0.065	0.077	0.085	0.100	0.108	0.111
1.4	0.000	0.000	0.000	0.001	0.002	0.003	0.007	0.017	0.035	0.051	0.062	0.071	0.087	0.095	0.099
1.5	0.000	0.000	0.000	0.001	0.001	0.003	0.005	0.014	0.030	0.045	0.056	0.064	0.081	0.090	0.094
1.6	0.000	0.000	0.000	0.001	0.001	0.002	0.004	0.012	0.026	0.040	0.051	0.059	0.076	0.085	0.089
1.8	0.000	0.000	0.000	0.000	0.001	0.001	0.003	0.008	0.020	0.031	0.041	0.049	0.066	0.077	0.081
2	0.000	0.000	0.000	0.000	0.001	0.001	0.002	0.006	0.015	0.025	0.034	0.041	0.058	0.069	0.074
2.5	0.000	0.000	0.000	0.000	0.000	0.000	0.001	0.003	0.008	0.014	0.021	0.027	0.043	0.055	0.061
3	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.002	0.004	0.008	0.013	0.018	0.032	0.045	0.051
4	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.001	0.002	0.003	0.006	0.008	0.018	0.031	0.039
5	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.001	0.002	0.003	0.004	0.011	0.022	0.031
10	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.001	0.006	0.016
15	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.002	0.011
20	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.001	0.008
50	0.000	0.000	0.000	0.030	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.003

TABLE 3.18

VALUES OF L_2'

(Giroud, 1970)

z/ℓ \ b/ℓ	0	0.1	0.2	1/3	0.4	0.5	2/3	1	1.5	2	2.5	3	5	10	∞
0	0.000	0.016	0.031	0.051	0.061	0.074	0.094	0.125	0.156	0.176	0.189	0.199	0.219	0.234	0.250
0.2	0.000	0.013	0.025	0.041	0.049	0.060	0.076	0.103	0.130	0.148	0.160	0.169	0.188	0.203	0.219
0.4	0.000	0.010	0.020	0.032	0.039	0.047	0.061	0.083	0.106	0.122	0.133	0.141	0.159	0.174	0.189
0.5	0.000	0.009	0.017	0.029	0.034	0.042	0.054	0.074	0.096	0.111	0.121	0.129	0.146	0.161	0.176
0.6	0.000	0.008	0.015	0.025	0.030	0.037	0.048	0.066	0.086	0.100	0.110	0.118	0.134	0.149	0.164
0.8	0.000	0.006	0.012	0.020	0.023	0.029	0.037	0.052	0.069	0.082	0.091	0.098	0.114	0.127	0.143
1	0.000	0.005	0.009	0.015	0.018	0.023	0.029	0.042	0.056	0.067	0.075	0.082	0.097	0.110	0.125
1.2	0.000	0.004	0.007	0.012	0.015	0.018	0.024	0.034	0.046	0.056	0.063	0.069	0.083	0.096	0.111
1.4	0.000	0.003	0.006	0.010	0.012	0.015	0.019	0.027	0.038	0.046	0.053	0.058	0.072	0.084	0.099
1.5	0.000	0.003	0.005	0.009	0.011	0.013	0.017	0.025	0.035	0.043	0.049	0.054	0.067	0.079	0.094
1.6	0.000	0.002	0.005	0.008	0.010	0.012	0.016	0.023	0.032	0.039	0.045	0.050	0.062	0.074	0.089
1.8	0.000	0.002	0.004	0.007	0.008	0.010	0.013	0.019	0.027	0.033	0.039	0.043	0.055	0.066	0.081
2	0.000	0.002	0.003	0.006	0.007	0.008	0.011	0.016	0.023	0.029	0.033	0.038	0.048	0.059	0.074
2.5	0.000	0.001	0.002	0.004	0.005	0.006	0.007	0.011	0.016	0.020	0.024	0.027	0.036	0.047	0.061
3	0.000	0.001	0.002	0.003	0.003	0.004	0.005	0.008	0.012	0.015	0.018	0.021	0.028	0.038	0.051
4	0.000	0.000	0.001	0.002	0.002	0.002	0.003	0.005	0.007	0.009	0.011	0.013	0.018	0.026	0.039
5	0.000	0.000	0.001	0.001	0.001	0.002	0.002	0.003	0.005	0.006	0.007	0.009	0.013	0.019	0.031
10	0.000	0.000	0.000	0.000	0.000	0.000	0.001	0.001	0.001	0.002	0.002	0.002	0.004	0.007	0.016
15	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.001	0.001	0.001	0.001	0.002	0.003	0.011
20	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.001	0.001	0.002	0.008
50	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.003

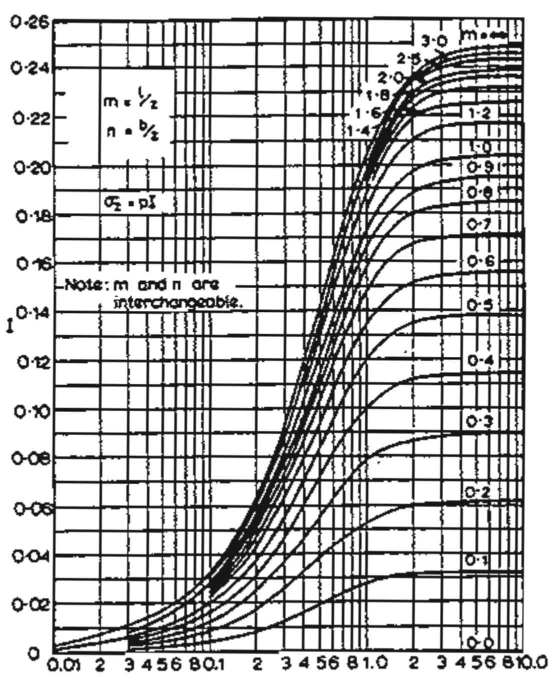


FIG.3.30 Vertical stress beneath corner of uniformly loaded rectangle. (Fadum, 1948).

$$B = \frac{n_1}{2\pi} \tan^{-1} \frac{m_1}{n_1 \sqrt{1 + m_1^2 + n_1^2}}$$

$$m_1 = \ell/b$$

$$n_1 = z/b$$

Explicit expressions and influence factors for the vertical displacement at the surface ($z=0$) have been evaluated for four points beneath the rectangle, and for the mean displacement ρ_m, by Giroud (1968). These influence factors are shown in Fig.3.31 and are tabulated in Table 3.19.

For all points,

$$\rho_z = \frac{(1-\nu^2)}{E} p\, b\, I \qquad \ldots (3.21)$$

where b is the length of the shorter side
M is the centre of the longer side
N the centre of the shorter side
C the corner
O the centre of the rectangle
m the mean

TABLE 3.19

INFLUENCE FACTORS FOR VERTICAL SURFACE DISPLACEMENT BENEATH RECTANGLE (Giroud, 1968)

$\alpha = \frac{\ell}{b}$	I_C	I_M	I_N	I_O	I_m	$\alpha = \frac{\ell}{b}$	I_C	I_M	I_N	I_O	I_m
1	0.561	0.766	0.766	1.122	0.946	15	1.401	2.362	1.621	2.802	2.498
1.1	0.588	0.810	0.795	1.176	0.992	20	1.493	2.544	1.713	2.985	2.677
1.2	0.613	0.852	0.822	1.226	1.035	25	1.564	2.686	1.784	3.127	2.817
1.3	0.636	0.892	0.847	1.273	1.075	30	1.622	2.802	1.842	3.243	2.932
1.4	0.658	0.930	0.870	1.317	1.112	40	1.713	2.985	1.934	3.426	3.113
1.5	0.679	0.966	0.892	1.358	1.148	50	1.784	3.127	2.005	3.568	3.254
1.6	0.698	1.000	0.912	1.396	1.181	60	1.842	3.243	2.063	3.684	3.370
1.7	0.716	1.033	0.931	1.433	1.213	70	1.891	3.341	2.112	3.783	3.467
1.8	0.734	1.064	0.949	1.467	1.244	80	1.934	3.426	2.154	3.868	3.552
1.9	0.750	1.094	0.966	1.500	1.273	90	1.971	3.501	2.192	3.943	3.627
2	0.766	1.122	0.982	1.532	1.300	100	2.005	3.568	2.225	4.010	3.693
2.2	0.795	1.176	1.012	1.590	1.353	200	2.225	4.010	2.446	4.451	4.134
2.4	0.822	1.226	1.039	1.644	1.401	300	2.355	4.268	2.575	4.709	4.391
2.5	0.835	1.250	1.052	1.669	1.424	400	2.446	4.451	2.667	4.892	4.574
3	0.892	1.358	1.110	1.783	1.527	500	2.517	4.593	2.738	5.034	4.717
3.5	0.940	1.450	1.159	1.880	1.616	600	2.575	4.709	2.796	5.150	4.833
4	0.982	1.532	1.201	1.964	1.694	700	2.624	4.807	2.845	5.248	4.931
4.5	1.019	1.604	1.239	2.038	1.763	800	2.667	4.892	2.887	5.333	5.015
5	1.052	1.669	1.272	2.105	1.826	900	2.704	4.967	2.925	5.408	5.092
6	1.110	1.783	1.330	2.220	1.935	10^3	2.738	5.034	2.958	5.476	5.158
7	1.159	1.880	1.379	2.318	2.028	10^4	3.471	6.500	3.691	6.941	6.623
8	1.201	1.964	1.422	2.403	2.110	10^5	4.204	7.966	4.424	8.407	8.089
9	1.239	2.038	1.459	2.477	2.182	10^6	4.937	9.432	5.157	9.874	9.555
10	1.272	2.105	1.493	2.544	2.246	∞	∞	∞	∞	∞	∞

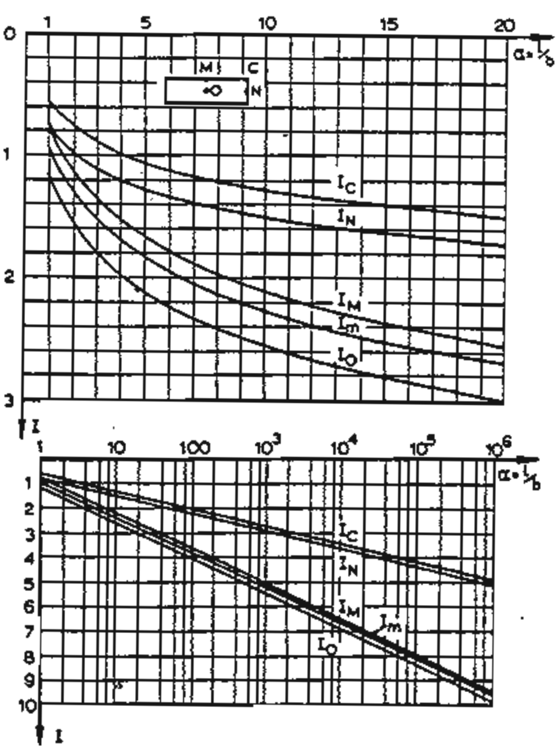


FIG.3.31 Influence factors for vertical surface displacement beneath rectangle. (Giroud, 1968).

For a point on the centre-line of the rectangle, distance x from the centre (Fig.3.32), Giroud (1969) gives the following expression for the horizontal surface displacement ρ_x:

$$\rho_x = \frac{(1+\nu)(1-2\nu)}{2\pi E} p\ell \left\{ \frac{b}{2\ell} \ln \frac{4(\ell-x)^2+b^2}{4x^2+b^2} + 2\left(1-\frac{x}{\ell}\right)\arctan\frac{b}{2(\ell-x)} - 2\frac{x}{\ell}\arctan\frac{b}{2x} \right\} \quad \ldots (3.21)$$

From this expression, the solution for ρ_x beneath the corner of a rectangle of proportions $2\frac{\ell}{b}$ may be obtained by taking half the value of ρ_x obtained from equation (3.21) when $x=\ell$.

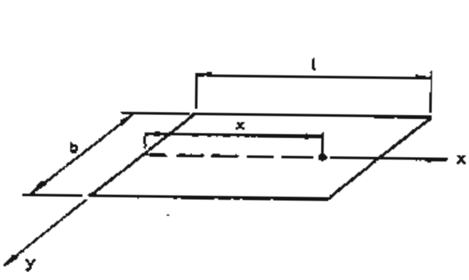


FIG.3.32

3.4.2 LINEARLY VARYING VERTICAL LOADING (Fig. 3.33)

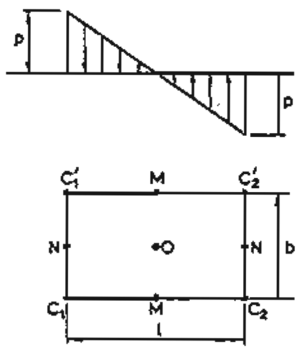


FIG. 3.33

The normal stresses are expressed as follows by Giroud (1970):

Under the corners:

$$\sigma_z = \varepsilon\, p\, M_0 \qquad \text{... (3.22a)}$$

$$\sigma_x = \varepsilon\, p[\, M_2 - (1-2\nu)M_2^1] \qquad \text{... (3.22b)}$$

$$\sigma_y = \varepsilon\, q[\, N_2 - (1-2\nu)N_2^1] \qquad \text{... (3.22c)}$$

Under the centre:

$$\sigma_z = 0 \qquad \text{... (3.23a)}$$

$$\sigma_x = 0 \qquad \text{... (3.23b)}$$

$$\sigma_y = 0 \qquad \text{... (3.23c)}$$

where $\varepsilon = +1$ at C_1 and C_1^1, and,
-1 at C_2 and C_2^1,
$M_0, M_2, M_2^1, N_2, N_2^1$ are influence factors which are given in Tables 3.21 to 3.25.

Influence factors for the vertical surface displacement beneath various points have been obtained by Giroud (1968), and are tabulated in Table 3.20. Explicit expressions for the displacements are given by Giroud.

At the corners

$$\rho_z = \frac{(1-\nu^2)p\ell}{E} I_c \qquad \text{if } b \geq \ell \qquad \text{... (3.24a)}$$

or

$$\frac{(1-\nu^2)pb}{E} I_c' \qquad \text{if } \ell \geq b \qquad \text{... (3.24b)}$$

where $$I_c = \frac{\alpha}{\pi}\left[\alpha - \sqrt{1+\alpha^2} + \ln \frac{1+\sqrt{1+\alpha^2}}{\alpha}\right]$$

and $\alpha = b/\ell$ ... (3.24c)

$$I_c' = \frac{1}{\pi}\left[\frac{1}{\alpha} + \frac{\sqrt{1+\alpha^2}}{\alpha} + \ln(\alpha + \sqrt{1+\alpha^2})\right]$$

and $\alpha = \ell/b$ ... (3.24d)

At N,

$$\rho_z = \frac{(1-\nu^2)p\ell}{E} I_N \qquad \text{if } b \geq \ell \qquad \text{... (3.24e)}$$

or

$$\frac{(1-\nu^2)pb}{E} I_N' \qquad \text{if } \ell \geq b \qquad \text{... (3.24f)}$$

$$\rho_z = 0 \qquad \text{at points } M \text{ and } O \qquad \text{... (3.24g)}$$

TABLE 3.20

INFLUENCE FACTORS FOR VERTICAL SURFACE DISPLACEMENT DUE TO LINEARLY VARYING COMPRESSION TO TENSION LOADING

(Giroud, 1968)

$b \geq \ell$			$\ell \geq b$					
b/ℓ	I_C	I_N	ℓ/b	I_C'	I_N'	ℓ/b	I_C'	I_N'
1	0.149	0.263	1	0.149	0.263	15	0.785	0.995
1.1	0.150	0.269	1.1	0.162	0.282	20	0.872	1.084
1.2	0.151	0.274	1.2	0.174	0.300	25	0.940	1.154
1.3	0.152	0.279	1.3	0.187	0.317	30	0.995	1.211
1.4	0.153	0.282	1.4	0.198	0.334	40	1 084	1.301
1.5	0.154	0.286	1.5	0.210	0.349	50	1.154	1.371
1.6	0.154	0.289	1.6	0.221	0.364	60	1.211	1.429
1.7	0.155	0.291	1.7	0.232	0.379	70	1.259	1.478
1.8	0.155	0.294	1.8	0.243	0.392	80	1.301	1.520
1.9	0.156	0.296	1.9	0.253	0.406	90	1.338	1.557
2	0.156	0.297	2	0.263	0.418	100	1.371	1.590
2.2	0.157	0.300	2.2	0.282	0.442	200	1.590	1.810
2.4	0.157	0.303	2.4	0.300	0.465	300	1.719	1.939
2.5	0.157	0.304	2.5	0.309	0.475	400	1.810	2.031
3	0.158	0.308	3	0.349	0.524	500	1.881	2.101
3.5	0.158	0.310	3.5	0.386	0.566	600	1.939	2.159
4	0.158	0.312	4	0.418	0.603	700	1.988	2.208
4.5	0.159	0.313	4.5	0.448	0.636	800	2.031	2.251
5	0.159	0.314	5	0.475	0.666	900	2.068	2.288
6	0.159	0.315	6	0.524	0.719	10^3	2.101	2.322
7	0.159	0.316	7	0.566	0.765	10^4	2.834	3.055
8	0.159	0.317	8	0.603	0.804	10^5	3.567	3.788
10	0.159	0.317	9	0.636	0.840	10^6	4.300	4.521
∞	0.159	0.318	10	0.666	0.872	∞	∞	∞

TABLE 3.21

VALUES OF M_0

(Giroud, 1970)

z/ℓ \ b/ℓ	0	0.1	0.2	1/3	0.4	0.5	2/3	1	1.5	2	2.5	3	5	10	∞
0	0.000	0.250	0.250	0.250	0.250	0.250	0.250	0.250	0.250	0.250	0.250	0.250	0.250	0.250	0.250
0.2	0.000	0.109	0.160	0.180	0.183	0.186	0.187	0.188	0.188	0.188	0.188	0.188	0.188	0.188	0.188
0.4	0.000	0.047	0.083	0.111	0.118	0.126	0.131	0.134	0.134	0.135	0.135	0.135	0.135	0.135	0.135
0.5	0.000	0.033	0.060	0.085	0.092	0.100	0.107	0.111	0.112	0.113	0.113	0.113	0.113	0.113	0.113
0.6	0.000	0.023	0.044	0.065	0.072	0.079	0.087	0.092	0.093	0.094	0.094	0.094	0.094	0.094	0.094
0.8	0.000	0.013	0.025	0.038	0.043	0.049	0.056	0.062	0.064	0.065	0.065	0.065	0.065	0.065	0.065
1	0.000	0.007	0.015	0.023	0.026	0.031	0.036	0.042	0.045	0.045	0.045	0.045	0.045	0.045	0.045
1.2	0.000	0.005	0.009	0.014	0.017	0.020	0.024	0.029	0.031	0.032	0.032	0.032	0.032	0.032	0.032
1.4	0.000	0.003	0.006	0.009	0.011	0.013	0.016	0.020	0.022	0.023	0.023	0.023	0.023	0.023	0.023
1.5	0.000	0.002	0.005	0.007	0.009	0.011	0.013	0.017	0.019	0.020	0.020	0.020	0.020	0.020	0.020
1.6	0.000	0.002	0.004	0.006	0.007	0.009	0.011	0.014	0.016	0.017	0.017	0.017	0.017	0.017	0.017
1.8	0.000	0.001	0.003	0.004	0.005	0.006	0.008	0.010	0.012	0.013	0.013	0.013	0.013	0.013	0.013
2	0.000	0.001	0.002	0.003	0.003	0.004	0.005	0.007	0.009	0.010	0.010	0.010	0.010	0.010	0.010
2.5	0.000	0.000	0.001	0.001	0.002	0.002	0.003	0.004	0.005	0.005	0.005	0.006	0.006	0.006	0.006
3	0.000	0.000	0.000	0.001	0.001	0.001	0.001	0.002	0.003	0.003	0.003	0.003	0.003	0.003	0.003
4	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.001	0.001	0.001	0.001	0.001	0.002	0.002	0.002
5	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.001	0.001	0.001	0.001	0.001	0.001
10	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
15	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
20	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
50	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000

TABLE 3.22

VALUES OF M_z

(Giroud, 1970)

z/ℓ \ b/ℓ	0	0.1	0.2	1/3	0.4	0.5	2/3	1	1.5	2	2.5	3	5	10	∞
0	0.000	0.250	0.250	0.250	0.250	0.250	0.250	0.250	0.250	0.250	0.250	0.250	0.250	0.250	0.250
0.2	0.000	0.026	0.040	0.046	0.046	0.046	0.045	0.043	0.042	0.042	0.042	0.042	0.042	0.042	0.042
0.4	0.000	0.002	0.003	0.002	0.001	-0.000	-0.002	-0.005	-0.007	-0.008	-0.008	-0.008	-0.008	-0.008	-0.008
0.5	0.000	-0.001	-0.002	-0.005	-0.006	-0.008	-0.010	-0.013	-0.015	-0.016	-0.016	-0.016	-0.016	-0.016	-0.016
0.6	0.000	-0.002	-0.004	-0.007	-0.009	-0.010	-0.013	-0.016	-0.018	-0.019	-0.019	-0.020	-0.020	-0.020	-0.020
0.8	0.000	-0.002	-0.005	-0.007	-0.009	-0.010	-0.013	-0.016	-0.018	-0.019	-0.019	-0.019	-0.019	-0.019	-0.019
1	0.000	-0.002	-0.003	-0.006	-0.007	-0.008	-0.010	-0.013	-0.015	-0.015	-0.016	-0.016	-0.016	-0.016	-0.016
1.2	0.000	-0.001	-0.002	-0.004	-0.005	-0.006	-0.007	-0.009	-0.011	-0.012	-0.012	-0.012	-0.013	-0.013	-0.013
1.4	0.000	-0.001	-0.002	-0.003	-0.003	-0.004	-0.005	-0.007	-0.008	-0.009	-0.009	-0.009	-0.010	-0.010	-0.010
1.5	0.000	-0.001	-0.001	-0.002	-0.003	-0.003	-0.004	-0.006	-0.007	-0.008	-0.008	-0.008	-0.008	-0.009	-0.009
1.6	0.000	-0.001	-0.001	-0.002	-0.002	-0.003	-0.004	-0.005	-0.006	-0.007	-0.007	-0.007	-0.007	-0.008	-0.008
1.8	0.000	-0.000	-0.001	-0.001	-0.002	-0.002	-0.003	-0.004	-0.005	-0.005	-0.005	-0.006	-0.006	-0.006	-0.006
2	0.000	-0.000	-0.001	-0.001	-0.001	-0.002	-0.002	-0.003	-0.003	-0.004	-0.004	-0.004	-0.005	-0.005	-0.00
2.5	0.000	-0.000	-0.000	-0.000	-0.001	-0.001	-0.00	-0.001	-0.002	-0.002	-0.002	-0.002	-0.003	-0.003	-0.00
3	0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.001	-0.001	-0.001	-0.001	-0.001	-0.001	-0.002	-0.002	-0.002
4	0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.001	-0.001	-0.001	-0.001	-0.001
5	0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000
10	0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000
15	0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000
20	0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000
50	0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000

TABLE 3.23

VALUES OF M'_2

(Giroud, 1970)

z/ℓ \ b/ℓ	0	0.1	0.2	1/3	0.4	0.5	2/3	1	1.5	2	2.5	3	5	10	∞
0	0.000	0.161	0.115	0.077	0.063	0.048	0.031	0.015	0.006	0.003	0.002	0.001	0.000	0.000	0.000
0.2	0.000	0.026	0.037	0.036	0.033	0.028	0.021	0.011	0.005	0.002	0.001	0.001	0.000	0.000	0.000
0.4	0.000	0.007	0.012	0.015	0.015	0.015	0.012	0.008	0.004	0.002	0.001	0.001	0.000	0.000	0.000
0.5	0.000	0.004	0.008	0.010	0.010	0.010	0.009	0.006	0.003	0.002	0.001	0.001	0.000	0.000	0.000
0.6	0.000	0.003	0.005	0.007	0.007	0.008	0.007	0.005	0.003	0.002	0.001	0.001	0.000	0.000	0.000
0.8	0.000	0.001	0.002	0.003	0.004	0.004	0.004	0.003	0.002	0.001	0.001	0.001	0.000	0.000	0.000
1	0.000	0.001	0.001	0.002	0.002	0.002	0.002	0.002	0.002	0.001	0.001	0.000	0.000	0.000	0.000
1.2	0.000	0.000	0.001	0.001	0.001	0.001	0.002	0.002	0.001	0.001	0.001	0.000	0.000	0.000	0.000
1.4	0.000	0.000	0.000	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.000	0.000	0.000	0.000
1.5	0.000	0.000	0.000	0.000	0.001	0.001	0.001	0.001	0.001	0.001	0.000	0.000	0.000	0.000	0.000
1.6	0.000	0.000	0.000	0.000	0.000	0.001	0.001	0.001	0.001	0.001	0.000	0.000	0.000	0.000	0.000
1.8	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.001	0.001	0.000	0.000	0.000	0.000	0.000	0.000
2	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
2.5	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
3	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
4	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
5	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
10	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
15	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
20	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
50	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000

TABLE 3.24

VALUES OF N_2

(Giroud, 1972)

z/ℓ \ b/ℓ	0	0.1	0.2	1/3	0.4	0.5	2/3	1	1.5	2	2.5	3	5	10	∞
0	0.000	0.250	0.250	0.250	0.250	0.250	0.250	0.250	0.250	0.250	0.250	0.250	0.250	0.250	0.250
0.2	0.000	0.008	0.034	0.067	0.078	0.090	0.101	0.110	0.114	0.114	0.115	0.115	0.115	0.115	0.115
0.4	0.000	0.001	0.006	0.018	0.024	0.033	0.044	0.056	0.061	0.062	0.063	0.063	0.063	0.063	0.063
0.5	0.000	0.000	0.003	0.010	0.014	0.020	0.029	0.040	0.045	0.047	0.048	0.048	0.048	0.048	0.048
0.6	0.000	0.000	0.002	0.006	0.008	0.013	0.019	0.028	0.034	0.036	0.036	0.037	0.037	0.037	0.037
0.8	0.000	0.000	0.000	0.002	0.003	0.005	0.009	0.015	0.019	0.021	0.022	0.022	0.023	0.023	0.023
1	0.000	0.000	0.000	0.001	0.001	0.002	0.004	0.008	0.011	0.013	0.014	0.014	0.015	0.015	0.015
1.2	0.000	0.000	0.000	0.000	0.001	0.001	0.002	0.004	0.007	0.008	0.009	0.009	0.010	0.010	0.010
1.4	0.000	0.000	0.000	0.000	0.000	0.000	0.001	0.002	0.004	0.005	0.006	0.006	0.007	0.007	0.007
1.5	0.000	0.000	0.000	0.000	0.000	0.000	0.001	0.002	0.003	0.004	0.005	0.005	0.006	0.006	0.006
1.6	0.000	0.000	0.000	0.000	0.000	0.000	0.001	0.001	0.003	0.004	0.004	0.004	0.005	0.005	0.005
1.8	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.001	0.002	0.002	0.003	0.003	0.004	0.004	0.004
2	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.001	0.002	0.002	0.002	0.003	0.003	0.003
2.5	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.001	0.001	0.001	0.001	0.001	0.002
3	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.001	0.001	0.001	0.001
4	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
5	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
10	0.000	0.000	0.000	0.000	-0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
15	0.000	0.000	0.000	0.000	0.000	-0.000	0.000	-0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
20	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
50	0.000	0.000	0.000	0.000	0.000	0.000	-0.000	0.000	0.000	-0.000	0.000	0.000	0.000	0.000	0.000

TABLE 3.25

VALUES OF N_2'

(Giroud, 1970)

z/ℓ \ b/ℓ	0	0.1	0.2	1/3	0.4	0.5	2/3	1	1.5	2	2.5	3	5	10	∞
0	0.000	0.039	0.135	0.173	0.187	0.202	0.219	0.235	0.244	0.247	0.248	0.249	0.250	0.250	0.250
0.2	0.000	0.022	0.042	0.062	0.070	0.079	0.091	0.103	0.110	0.112	0.113	0.114	0.115	0.115	0.115
0.4	0.000	0.009	0.018	0.028	0.033	0.038	0.045	0.054	0.059	0.061	0.062	0.063	0.063	0.063	0.063
0.5	0.000	0.006	0.013	0.020	0.023	0.027	0.033	0.040	0.044	0.046	0.047	0.047	0.048	0.048	0.048
0.6	0.000	0.005	0.009	0.014	0.017	0.020	0.024	0.029	0.033	0.035	0.036	0.036	0.037	0.037	0.037
0.8	0.000	0.002	0.005	0.008	0.009	0.011	0.013	0.017	0.020	0.021	0.022	0.022	0.023	0.023	0.023
1	0.000	0.001	0.003	0.004	0.005	0.006	0.008	0.010	0.012	0.013	0.014	0.014	0.015	0.015	0.015
1.2	0.000	0.001	0.002	0.003	0.003	0.004	0.005	0.006	0.008	0.009	0.009	0.009	0.010	0.010	0.010
1.4	0.000	0.000	0.001	0.002	0.002	0.002	0.003	0.004	0.005	0.006	0.006	0.006	0.007	0.007	0.007
1.5	0.000	0.000	0.001	0.001	0.002	0.002	0.002	0.003	0.004	0.005	0.005	0.005	0.006	0.006	0.006
1.6	0.000	0.000	0.001	0.001	0.001	0.001	0.002	0.003	0.003	0.004	0.004	0.004	0.005	0.005	0.005
1.8	0.000	0.000	0.000	0.001	0.001	0.001	0.001	0.002	0.002	0.003	0.003	0.003	0.004	0.004	0.004
2	0.000	0.000	0.000	0.000	0.001	0.001	0.001	0.001	0.002	0.002	0.002	0.002	0.003	0.003	0.003
2.5	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.002
3	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.001	0.001	0.001	0.001	0.001	0.001
4	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
5	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
10	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
15	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	-0.000
20	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	-0.000
50	0.000	0.000	0.000	0.000	0.000	0.000	-0.000	0.000	0.000	-0.000	0.000	0.000	0.000	0.000	-0.000

For a point at a distance x from the edge $C_1^1NC_1$ on the axis NON (Fig.3.33), Giroud (1969) gives the following solution for the horizontal displacement ρ_x:

$$\rho_x = \frac{(1+\nu)(1-2\nu)}{2\pi E}\,\ell\Big[\frac{b}{2\ell}\Big(1-\frac{2x}{\ell}\Big)\ln\frac{4(\ell-x)^2+b^2}{4x^2+b^2} - \frac{2x}{\ell}\Big(1-\frac{x}{\ell}\Big)\Big(\arctan\frac{b}{2x}+\arctan\frac{b}{2(\ell-x)}\Big) + \frac{b^2}{2\ell^2}\Big(\arctan\frac{2x}{b}+\arctan\frac{2(\ell-x)}{b}\Big) - \frac{b}{\ell}\Big] \quad \ldots (3.25)$$

As in Section 3.4.1, the solution for the corner of a rectangle of proportions $2\ell/b$ may be obtained by taking half the value of ρ_x for $x=\ell$.

The results in this section can be combined with those in Section 3.4.1 to give the results for a trapezoidal distribution of loading. For the particular case when the loading varies linearly across the rectangle to zero at one edge, expressions and graphs for the vertical stress are given by Gray (1943,1948), expressions for the horizontal and shear stresses by Ambraseys (1960) and graphs and expressions for vertical displacement by Stamatopoulos (1959).

3.4.3 VERTICAL EMBANKMENT LOADING (Fig.3.34)

Vertical surface displacements ρ_z have been evaluated for several points by Giroud (1968b).

Influence factors K_O, K_c etc for the seven points marked in Fig.3.34 are shown in Figs. 3.35 to 3.41.

In all cases

$$\rho_z = \frac{1-\nu^2}{E}\,p\,a\,K \quad \ldots (3.26)$$

Expressions for the influence factors K_O(centre) and K_c(corner) are as follows:

$$K_O = \frac{1}{2\pi}\Big[\frac{1}{\beta}\ln\big(\alpha+\sqrt{1+\alpha^2}\big) + \frac{\alpha^2}{\beta}\ln\frac{1+\sqrt{1+\alpha^2}}{\alpha} - \frac{(1-2\beta)^2}{\beta}\ln\frac{\alpha-2\beta+\sqrt{(1-2\beta)^2+(\alpha-2\beta)^2}}{1-2\beta} - \frac{(\alpha-2\beta)^2}{\beta}\ln\frac{1-2\beta+\sqrt{(1-2\beta)^2+(\alpha-2\beta)^2}}{\alpha-2\beta}$$

$$- \frac{(\alpha-1)^2}{\sqrt{2}\,\beta} \ln \frac{\sqrt{2(1+\alpha^2)}+\alpha+1}{\sqrt{2(1-2\beta)^2+2(\alpha-2\beta)^2-4\beta+\alpha+1}} \Bigg] \qquad \ldots (3.27a)$$

$$K_C = \frac{1}{2\pi}\Bigg[\frac{1}{\beta}\ln\left(\alpha+\sqrt{1+\alpha^2}\right) + \frac{\alpha^2}{\beta}\ln\frac{1+\sqrt{1+\alpha^2}}{\alpha}$$

$$+\beta \ln \frac{\beta-\alpha+\sqrt{\beta^2+(\alpha-\beta)^2}}{1-\beta+\sqrt{\beta^2+(1-\beta)^2}}$$

$$-\frac{1}{\sqrt{2}\beta}\ln\frac{\sqrt{2\beta^2+2(1-\beta)^2}+2\beta-1}{\sqrt{2}-1}$$

$$-\frac{\alpha^2}{\sqrt{2}\beta}\ln\frac{\sqrt{2\beta^2+2(\alpha-\beta)^2}+2\beta-\alpha}{\alpha(\sqrt{2}-1)}$$

$$-\frac{(\alpha-1)^2}{\sqrt{2}\beta}\ln\frac{\sqrt{2(1-\beta)^2+2(\alpha-\beta)^2}+2\beta-\alpha-1}{\sqrt{2(1+\alpha^2)}-(1+\alpha)}$$

$$-\frac{(1-\beta)^2}{\beta}\ln\frac{\sqrt{\beta^2+(1-\beta)^2}-\beta}{\sqrt{(1-\beta)^2+(\alpha-\beta)^2}+\beta-\alpha}$$

$$-\frac{(\alpha-\beta)^2}{\beta}\ln\frac{\sqrt{\beta^2+(\alpha-\beta)^2}-\beta}{\sqrt{(1-\beta)^2+(\alpha-\beta)^2}+\beta-1}$$

$$+2\beta\ln(1+\sqrt{2})\Bigg] \qquad \ldots (3.27b)$$

For the point A,

$$K_A = \frac{1}{2\pi}\Bigg[\frac{1}{2\beta}\ln\left(2\alpha+\sqrt{1+4\alpha^2}\right) + \frac{2\alpha^2}{\beta}\ln\frac{1+\sqrt{1+4\alpha^2}}{2\alpha}$$

$$-\frac{1}{2\sqrt{2}\beta}\ln\frac{\sqrt{2(1-2\beta)^2+8\beta^2}+4\beta-1}{\sqrt{2}-1}$$

$$-\frac{(2\alpha-1)^2}{2\sqrt{2}\beta}\ln\frac{\sqrt{2(1-2\beta)^2+8(\alpha-\beta)^2}+4\beta-2\alpha-1}{\sqrt{2(1+4\alpha^2)}-(1+2\alpha)}$$

$$-\frac{(1-2\beta)^2}{2\beta}\ln\frac{\sqrt{4\beta^2+(1-2\beta)^2}-2\beta}{\sqrt{(1-2\beta)^2+4(\alpha-\beta)^2}+2(\beta-\alpha)}$$

$$-\frac{2(\alpha-\beta)^2}{\beta}\ln\frac{2(\alpha-\beta)}{\sqrt{(1-2\beta)^2+4(\alpha-\beta)^2}+2\beta-1}$$

$$-2\beta\ln\frac{2\beta}{\sqrt{(1-2\beta)^2+4\beta^2}+2\beta-1}\Bigg] \qquad \ldots (3.27c)$$

In the above expressions,

$\alpha = b/a$

$\beta = c/a$.

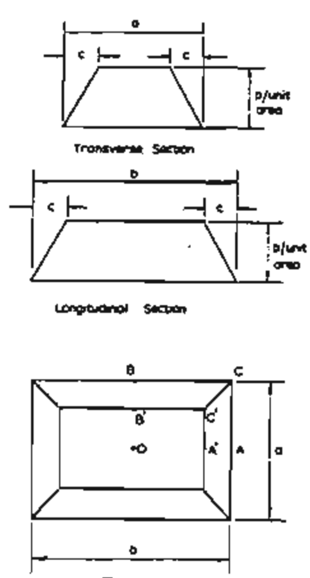


FIG.3.34

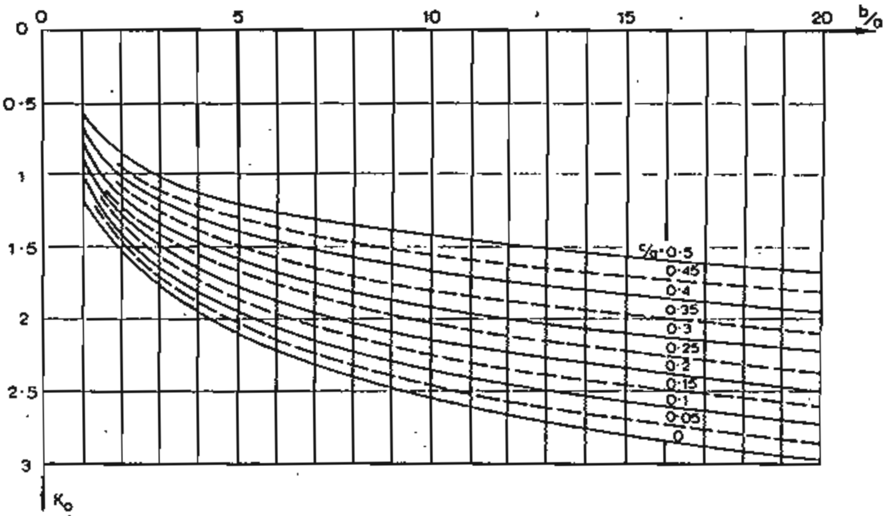


FIG.3.35 Displacement Influence Factors K_o. (Giroud, 1968).

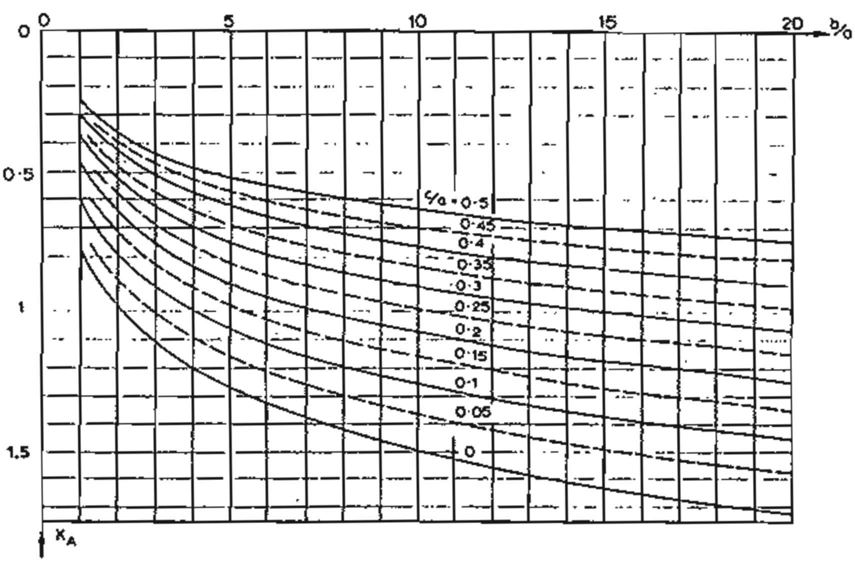


FIG.3.36 Displacement Influence Factors K_A. (Giroud, 1968).

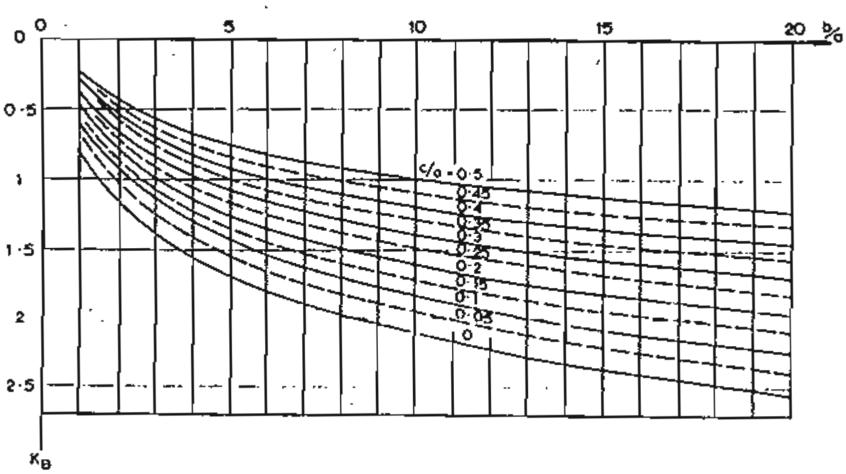


FIG.3.37 Displacement Influence Factors K_B. (Giroud, 1968).

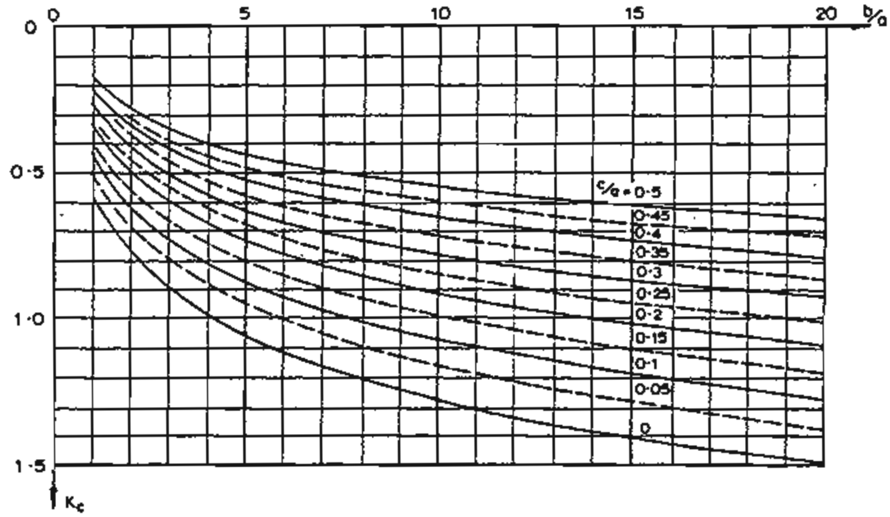


FIG.3.38 Displacement Influence Factors K_c. (Giroud, 1968).

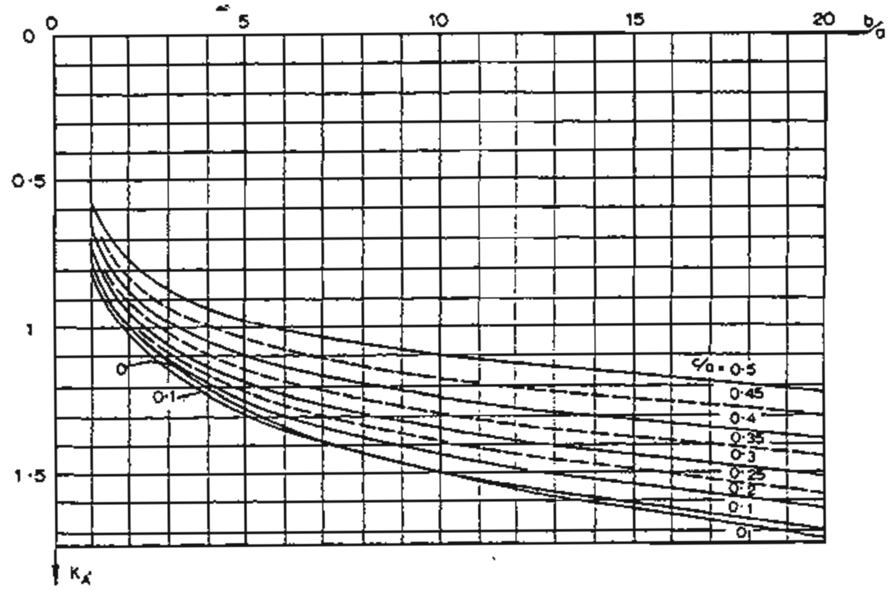


FIG.3.39 Displacement Influence Factors K_A'. (Giroud, 1968).

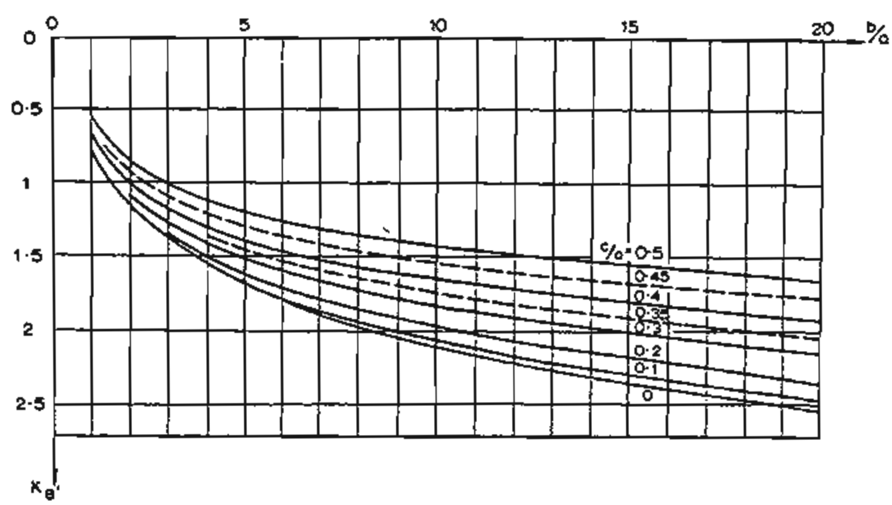


FIG.3.40 Displacement Influence Factors K_B'. (Giroud, 1968).

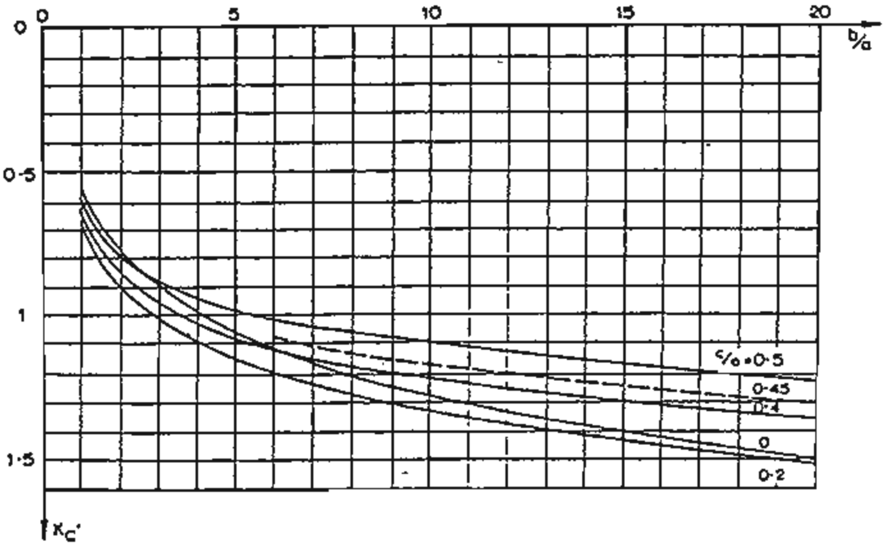


FIG.3.41 Displacement Influence Factors K_c'. (Giroud, 1968).

3.4.4 UNIFORM HORIZONTAL LOADING (Fig.3.42)

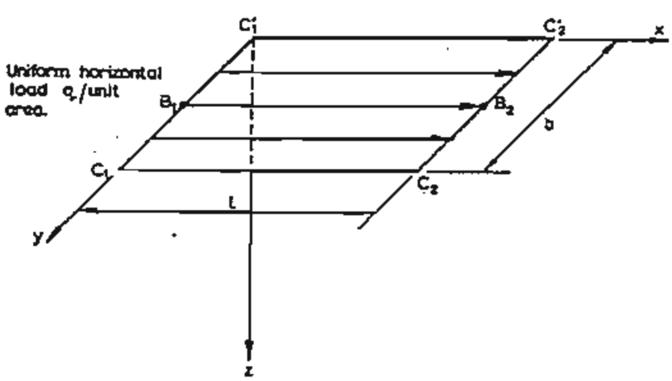


FIG.3.42

Holl (1940) gives the following solutions for the stresses beneath the corners C_2 and C_1^1 of the rectangle:

$$\sigma_z = \frac{q}{2\pi}\left[\frac{b}{R_2} - \frac{z^2 b}{R_1^2 R_3}\right] \quad \ldots (3.28a)$$

$$\sigma_x = \frac{q}{\pi}\left[\ell n \frac{R_1(b+R_2)}{z(b+R_3)} - \frac{\ell^2 b}{2R_1^2 R_3}\right] \quad \ldots (3.28b)$$

$$\sigma_y = \frac{q}{2\pi}\left[\ell n \frac{R_1(b+R_2)}{z(b+R_3)} - b\left(\frac{1}{R_2} - \frac{1}{R_3}\right)\right] \quad \ldots (3.28c)$$

$$\tau_{xz} = \frac{q}{2\pi}\left[\tan^{-1}\frac{\ell b}{zR_3} - \frac{\ell bz}{R_1^2 R_3}\right] \quad \ldots (3.28d)$$

$$\tau_{yz} = \frac{q}{2\pi}\left[1 + \frac{z}{R_3} - z\left(\frac{1}{R_1} + \frac{1}{R_2}\right)\right] \quad \ldots (3.28e)$$

$$\tau_{xy} = \frac{q}{2\pi}\left[\ell n \frac{(R_1+\ell)(R_3-\ell)}{zR_2} + \ell\left(\frac{1}{R_3} - \frac{1}{R_1}\right)\right] \quad \ldots (3.28f)$$

where $R_1 = (\ell^2+z^2)^{1/2}$

$R_2 = (b^2+z^2)^{1/2}$

$R_3 = (\ell^2+b^2+z^2)^{1/2}$

It should be noted that the values of τ_{xz}, τ_{yz} and σ_z for uniform horizontal loading correspond to the values of σ_x, τ_{xy} and τ_{xz} for uniform vertical loading (from the reciprocal theorem).

The principle of superposition may be applied to determine the stresses at points not beneath the corner of the rectangle.

Influence factors for the normal stresses have been obtained by Giroud (1970). The stresses are expressed as follows:

Under the corners,

$$\sigma_z = \varepsilon\, q\, K_1 \quad \ldots (3.29a)$$

$$\sigma_x = \varepsilon\, q\, [K_3 - (1-2\nu)K_3^1] \quad \ldots (3.29b)$$

$$\sigma_y = \varepsilon\, q\, [K_5 - (1-2\nu)K_5^1] \quad \ldots (3.29c)$$

where $\epsilon = +1$ at C_2 and C_2^1 (see Fig.3.42)
and $ -1$ at C_1 and C_1^1.

K_1, K_3, K_3^1, K_5, K_5^1 are influence factors which are given in Tables 3.26 to 3.30.

Under the centre,

$\sigma_z = 0$

$\sigma_x = 0$

$\sigma_y = 0$.

TABLE 3.26

VALUES OF K_1 (Giroud, 1970)

z/ℓ \ b/ℓ	0	0.1	0.2	1/3	0.4	0.5	2/3	1	1.5	2	2.5	3	5	10	∞
0	0.000	0.159	0.159	0.159	0.159	0.159	0.159	0.159	0.159	0.159	0.159	0.159	0.159	0.159	0.159
0.2	0.000	0.071	0.111	0.135	0.140	0.145	0.149	0.152	0.153	0.153	0.153	0.153	0.153	0.153	0.153
0.4	0.000	0.037	0.067	0.095	0.105	0.115	0.125	0.133	0.136	0.137	0.137	0.137	0.137	0.137	0.137
0.5	0.000	0.028	0.054	0.079	0.089	0.100	0.111	0.121	0.125	0.127	0.127	0.127	0.127	0.127	0.127
0.6	0.000	0.023	0.043	0.066	0.075	0.085	0.097	0.109	0.115	0.116	0.117	0.117	0.117	0.117	0.117
0.8	0.000	0.015	0.029	0.046	0.053	0.062	0.073	0.086	0.093	0.095	0.096	0.097	0.097	0.097	0.097
1	0.000	0.010	0.020	0.032	0.037	0.045	0.054	0.067	0.075	0.077	0.079	0.079	0.079	0.080	0.080
1.2	0.000	0.007	0.014	0.023	0.027	0.033	0.040	0.051	0.059	0.062	0.064	0.064	0.065	0.065	0.065
1.4	0.000	0.005	0.010	0.017	0.020	0.024	0.030	0.040	0.047	0.050	0.052	0.053	0.054	0.054	0.054
1.5	0.000	0.004	0.009	0.014	0.017	0.021	0.026	0.035	0.042	0.045	0.047	0.048	0.049	0.049	0.049
1.6	0.000	0.004	0.008	0.013	0.015	0.018	0.023	0.031	0.038	0.041	0.043	0.044	0.045	0.045	0.045
1.8	0.000	0.003	0.006	0.010	0.011	0.014	0.018	0.024	0.030	0.034	0.035	0.036	0.037	0.038	0.038
2	0.000	0.002	0.004	0.007	0.009	0.011	0.014	0.019	0.025	0.028	0.029	0.030	0.032	0.032	0.032
2.5	0.000	0.001	0.003	0.004	0.005	0.006	0.008	0.011	0.015	0.018	0.019	0.020	0.022	0.022	0.022
3	0.000	0.001	0.002	0.003	0.003	0.004	0.005	0.007	0.010	0.012	0.013	0.014	0.015	0.016	0.016
4	0.000	0.000	0.001	0.001	0.001	0.002	0.002	0.003	0.005	0.006	0.007	0.007	0.009	0.009	0.009
5	0.000	0.000	0.000	0.001	0.001	0.001	0.001	0.002	0.003	0.003	0.004	0.004	0.005	0.006	0.006
10	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.001	0.001	0.001	0.001	0.002
15	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.001	0.001
20	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
50	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000

TABLE 3.27

VALUES OF K_3 (Giroud, 1970)

z/ℓ \ b/ℓ	0	0.1	0.2	1/3	0.4	0.5	2/3	1	1.5	2	2.5	3	5	10	∞
0	0.000	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞
0.2	0.000	0.107	0.189	0.259	0.282	0.307	0.332	0.353	0.362	0.364	0.365	0.365	0.365	0.366	0.366
0.4	0.000	0.037	0.069	0.104	0.117	0.133	0.150	0.167	0.175	0.177	0.177	0.178	0.178	0.178	0.178
0.5	0.000	0.023	0.045	0.069	0.079	0.091	0.104	0.118	0.125	0.127	0.128	0.129	0.129	0.129	0.129
0.6	0.000	0.016	0.030	0.047	0.054	0.063	0.074	0.085	0.091	0.093	0.094	0.094	0.094	0.095	0.095
0.8	0.000	0.007	0.014	0.023	0.026	0.031	0.038	0.045	0.050	0.052	0.052	0.052	0.053	0.053	0.053
1	0.000	0.004	0.007	0.012	0.014	0.016	0.020	0.025	0.028	0.030	0.030	0.030	0.031	0.031	0.031
1.2	0.000	0.002	0.004	0.006	0.007	0.009	0.011	0.014	0.017	0.018	0.018	0.018	0.019	0.019	0.019
1.4	0.000	0.001	0.002	0.004	0.004	0.005	0.006	0.009	0.010	0.011	0.011	0.012	0.012	0.012	0.012
1.5	0.000	0.001	0.002	0.003	0.003	0.004	0.005	0.007	0.008	0.009	0.009	0.009	0.010	0.010	0.010
1.6	0.000	0.001	0.001	0.002	0.003	0.003	0.004	0.005	0.006	0.007	0.007	0.008	0.008	0.008	0.008
1.8	0.000	0.000	0.001	0.001	0.002	0.002	0.002	0.003	0.004	0.005	0.005	0.005	0.005	0.005	0.005
2	0.000	0.000	0.001	0.001	0.001	0.001	0.002	0.002	0.003	0.003	0.003	0.004	0.004	0.004	0.004
2.5	0.000	0.000	0.000	0.000	0.000	0.000	0.001	0.001	0.001	0.001	0.001	0.002	0.002	0.002	0.002
3	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.001	0.001	0.001	0.001	0.001	0.001	0.001
4	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
5	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
10	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
15	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
20	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
50	0.000	-0.000	-0.000	-0.000	-0.000	-0.000	0.000	-0.000	-0.000	0.000	0.000	-0.000	0.000	0.000	0.000

TABLE 3.28

VALUES OF K_3^1

(Giroud, 1970)

z/ℓ \ b/ℓ	0	0.1	0.2	1/3	0.4	0.5	2/3	1	1.5	2	2.5	3	5	10	∞
0	0.000	0.143	0.128	0.109	0.100	0.088	0.071	0.047	0.027	0.017	0.011	0.008	0.003	0.001	0.000
0.2	0.000	0.025	0.040	0.048	0.049	0.048	0.044	0.033	0.021	0.014	0.010	0.007	0.003	0.001	0.000
0.4	0.000	0.009	0.016	0.023	0.025	0.026	0.026	0.023	0.016	0.011	0.008	0.006	0.003	0.001	0.000
0.5	0.000	0.006	0.011	0.016	0.018	0.020	0.021	0.019	0.014	0.010	0.008	0.006	0.003	0.001	0.000
0.6	0.000	0.004	0.008	0.012	0.013	0.015	0.016	0.016	0.012	0.009	0.007	0.005	0.002	0.001	0.000
0.8	0.000	0.002	0.004	0.007	0.008	0.009	0.010	0.011	0.009	0.007	0.006	0.005	0.002	0.001	0.000
1	0.000	0.001	0.003	0.004	0.005	0.006	0.007	0.008	0.007	0.006	0.005	0.004	0.002	0.001	0.000
1.2	0.000	0.001	0.002	0.003	0.003	0.004	0.005	0.006	0.006	0.005	0.004	0.004	0.002	0.001	0.000
1.4	0.000	0.001	0.001	0.002	0.002	0.003	0.003	0.004	0.004	0.004	0.004	0.003	0.002	0.001	0.000
1.5	0.000	0.000	0.001	0.002	0.002	0.002	0.003	0.004	0.004	0.004	0.003	0.003	0.002	0.001	0.000
1.6	0.000	0.000	0.001	0.001	0.002	0.002	0.002	0.003	0.003	0.003	0.003	0.003	0.002	0.001	0.000
1.8	0.000	0.000	0.001	0.001	0.001	0.001	0.002	0.002	0.003	0.003	0.003	0.002	0.001	0.001	0.000
2	0.000	0.000	0.000	0.001	0.001	0.001	0.001	0.002	0.002	0.002	0.002	0.002	0.001	0.001	0.000
2.5	0.000	0.000	0.000	0.000	0.000	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.000	0.000
3	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.001	0.001	0.001	0.001	0.001	0.001	0.000	0.000
4	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.001	0.001	0.001	0.000	0.000
5	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
10	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
15	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
20	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
50	0.000	-0.000	0.000	0.000	-0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000

TABLE 3.29

VALUES OF K_5

(Giroud, 1970)

z/ℓ \ b/ℓ	0	0.1	0.2	1/3	0.4	0.5	2/3	1	1.5	2	2.5	3	5	10	∞
0	0.000	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞
0.2	0.000	0.005	0.027	0.066	0.085	0.109	0.142	0.185	0.218	0.234	0.242	0.247	0.255	0.258	0.259
0.4	0.000	0.001	0.005	0.017	0.025	0.038	0.059	0.091	0.118	0.133	0.141	0.145	0.153	0.156	0.158
0.5	0.000	0.000	0.003	0.010	0.015	0.024	0.039	0.065	0.090	0.104	0.111	0.116	0.123	0.127	0.128
0.6	0.000	0.000	0.002	0.006	0.009	0.015	0.026	0.048	0.070	0.082	0.089	0.094	0.101	0.105	0.106
0.8	0.000	0.000	0.001	0.002	0.004	0.007	0.013	0.026	0.042	0.053	0.059	0.063	0.070	0.074	0.075
1	0.000	0.000	0.000	0.001	0.002	0.003	0.007	0.015	0.027	0.035	0.040	0.044	0.051	0.054	0.055
1.2	0.000	0.000	0.000	0.001	0.001	0.002	0.004	0.009	0.017	0.024	0.028	0.032	0.038	0.041	0.042
1.4	0.000	0.000	0.000	0.000	0.001	0.001	0.002	0.005	0.011	0.016	0.020	0.023	0.029	0.032	0.033
1.5	0.000	0.000	0.000	0.000	0.000	0.001	0.002	0.004	0.009	0.013	0.017	0.020	0.025	0.028	0.029
1.6	0.000	0.000	0.000	0.000	0.000	0.001	0.001	0.003	0.007	0.011	0.015	0.017	0.022	0.025	0.026
1.8	0.000	0.000	0.000	0.000	0.000	0.000	0.001	0.002	0.005	0.008	0.011	0.013	0.017	0.020	0.021
2	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.001	0.003	0.006	0.008	0.010	0.014	0.017	0.018
2.5	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.001	0.001	0.003	0.004	0.005	0.008	0.011	0.012
3	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.001	0.001	0.002	0.003	0.005	0.007	0.008
4	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.001	0.001	0.002	0.004	0.005
5	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.001	0.002	0.003
10	0.000	-0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.001
15	0.000	-0.000	-0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
20	0.000	-0.000	-0.000	-0.000	-0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
50	0.000	-0.000	-0.000	-0.000	-0.000	-0.000	0.000	-0.000	-0.000	0.000	0.000	0.000	0.000	0.000	0.000

TABLE 3.30

VALUES OF K_5^1

(Giroud, 1970)

z/ℓ \ b/ℓ	0	0.1	0.2	1/3	0.4	0.5	2/3	1	1.5	2	2.5	3	5	10	∞
0	0.000	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞
0.2	0.000	0.036	0.069	0.105	0.120	0.139	0.164	0.197	0.223	0.236	0.244	0.248	0.255	0.258	0.259
0.4	0.000	0.016	0.031	0.049	0.058	0.069	0.085	0.107	0.127	0.137	0.143	0.147	0.153	0.157	0.158
0.5	0.000	0.011	0.023	0.036	0.043	0.051	0.064	0.083	0.100	0.109	0.115	0.118	0.124	0.127	0.128
0.6	0.000	0.009	0.017	0.028	0.032	0.039	0.050	0.065	0.080	0.088	0.093	0.096	0.102	0.105	0.106
0.8	0.000	0.005	0.010	0.017	0.020	0.024	0.031	0.042	0.053	0.059	0.063	0.066	0.071	0.074	0.075
1	0.000	0.003	0.007	0.011	0.013	0.016	0.020	0.028	0.036	0.041	0.045	0.047	0.052	0.054	0.055
1.2	0.000	0.002	0.004	0.007	0.009	0.011	0.014	0.019	0.025	0.030	0.033	0.035	0.039	0.041	0.042
1.4	0.000	0.002	0.003	0.005	0.006	0.007	0.010	0.014	0.018	0.022	0.024	0.026	0.030	0.032	0.033
1.5	0.000	0.001	0.003	0.004	0.005	0.006	0.008	0.012	0.016	0.019	0.021	0.023	0.026	0.028	0.029
1.6	0.000	0.001	0.002	0.004	0.004	0.005	0.007	0.010	0.014	0.016	0.019	0.020	0.023	0.025	0.026
1.8	0.000	0.001	0.002	0.003	0.003	0.004	0.005	0.008	0.010	0.013	0.014	0.016	0.019	0.020	0.021
2	0.000	0.001	0.001	0.002	0.002	0.003	0.004	0.006	0.008	0.010	0.011	0.012	0.015	0.017	0.018
2.5	0.000	0.000	0.001	0.001	0.001	0.002	0.002	0.003	0.005	0.006	0.007	0.007	0.009	0.011	0.012
3	0.000	0.000	0.000	0.001	0.001	0.001	0.001	0.002	0.003	0.004	0.004	0.005	0.006	0.008	0.008
4	0.000	0.000	0.000	0.000	0.000	0.000	0.001	0.001	0.001	0.002	0.002	0.002	0.003	0.004	0.005
5	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.001	0.001	0.001	0.001	0.002	0.003	0.003
10	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.001
15	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
20	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
50	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000

Explicit expressions and influence factors for vertical displacement ρ_z beneath the points C_1, C_2, B_1 and B_2 of the rectangle have been evaluated by Giroud (1968) and are given in Table 3.31 (refer Fig. 3.42 for definition of ℓ and b).

At $\left.\begin{matrix} C_2 \\ C_1 \end{matrix}\right\}$ $\quad \rho_z = \dfrac{\pm(1+\nu)(1-2\nu)q\, b\, I_C}{E}$ if $\ell \geq b$ ... (3.30a)

or $\dfrac{\pm(1+\nu)(1-2\nu)q\, \ell\, I_C^1}{E}$ if $b \geq \ell$ ... (3.30b)

where $I_C = \dfrac{1}{2\pi}\left[\arctan\alpha + \alpha \ln \dfrac{\sqrt{1+\alpha^2}}{\alpha}\right]$ ... (3.30c)

and $\alpha = b/\ell$

$I_C^1 = \dfrac{\alpha}{2\pi}\left[\arctan\dfrac{1}{\alpha} + \dfrac{1}{\alpha}\ln\sqrt{1+\alpha^2}\right]$ ... (3.30d)

and $\alpha = \ell/b$.

At $\left.\begin{matrix} B_2 \\ B_1 \end{matrix}\right\}$ $\quad \rho_z = \dfrac{\pm(1+\nu)(1-2\nu)q\, b\, I_B}{E}$ if $\ell \geq b$ ... (3.31a)

or $\dfrac{\pm(1+\nu)(1-2\nu)q\, \ell\, I_B^1}{E}$ $\quad b \geq \ell$ ... (3.31b)

Giroud (1969a) gives the following expression for the horizontal displacement ρ_x of a point on the centre-line B_1B_2 (Fig.3.42), distance x from B_1:

$$\rho_x = \frac{(1+\nu)}{\pi E} q\ell \left[2(1-\nu)\left\{\left(1-\frac{x}{\ell}\right)\ln\frac{b+\sqrt{4(\ell-x)^2+b^2}}{2(\ell-x)} + \frac{x}{\ell}\ln\frac{b+\sqrt{4x^2+b^2}}{2|x|}\right\} + \frac{b}{\ell}\ln\frac{2(\ell-x)+\sqrt{4(\ell-x)^2+b^2}}{-2x+\sqrt{4x^2+b^2}}\right] \quad \text{... (3.32)}$$

For the corners C_1 and C_2, Giroud (1969b) gives

$$\rho_x = \frac{(1+\nu)}{\pi E} q\ell \left[(1-\nu)\ln\frac{b+\sqrt{\ell^2+b^2}}{\ell} + \frac{b}{\ell}\ln\frac{\ell+\sqrt{\ell^2+b^2}}{b}\right] \quad \text{... (3.33a)}$$

$$\rho_y = \frac{\pm\nu(1+\nu)}{\pi E} q(\ell+b-\sqrt{\ell^2+b^2}) \quad \text{... (3.33b)}$$

(+ for C_1^1 and C_2
− for C_1 and C_2^1).

TABLE 3.31

INFLUENCE FACTORS FOR VERTICAL SURFACE DISPLACEMENT DUE TO UNIFORM HORIZONTAL LOADING

(Giroud, 1968)

$b \geq \ell$			$\ell \geq b$					
b/ℓ	I_C'	I_B'	ℓ/b	I_C	I_B	ℓ/b	I_C	I_B
1	0.180	0.276	1	0.180	0.276	15	0.590	0.701
1.1	0.185	0.288	1.1	0.192	0.290	20	0.636	0.746
1.2	0.190	0.299	1.2	0.204	0.303	25	0.671	0.782
1.3	0.194	0.309	1.3	0.214	0.315	30	0.701	0.811
1.4	0.197	0.318	1.4	0.225	0.326	40	0.746	0.857
1.5	0.200	0.327	1.5	0.234	0.337	50	0.782	0.892
1.6	0.203	0.335	1.6	0.243	0.347	60	0.811	0.921
1.7	0.206	0.342	1.7	0.252	0.356	70	0.835	0.946
1.8	0.208	0.348	1.8	0.260	0.365	80	0.857	0.967
1.9	0.210	0.355	1.9	0.268	0.373	90	0.875	0.986
2	0.212	0.360	2	0.276	0.381	100	0.892	1.002
2.2	9.215	0.371	2.2	0.290	0.396	200	1.002	1.113
2.5	0.219	0.384	2.4	0.303	0.410	300	1.067	1.177
3	0.224	0.401	2.5	0.309	0.416	400	1.113	1.223
3.5	0.228	0.413	3	0.337	0.445	500	1.148	1.259
4	0.230	0.423	3.5	0.361	0.469	600	1.177	1.288
4.5	0.232	0.431	4	0.381	0.491	700	1.202	1.312
5	0.234	0.438	4.5	0.400	0.509	800	1.223	1.333
7	0.239	0.455	5	0.416	0.526	900	1.242	1.352
10	0.242	0.468	6	0.445	0.555	10^3	1.259	1.369
15	0.245	0.479	7	0.469	0.579	10^4	1.625	1.735
20	0.246	0.484	8	0.491	0.601	10^5	1.991	2.102
40	0.248	0.492	9	0.509	0.619	10^6	2.358	2.469
∞	0.250	0.500	10	0.526	0.638	∞	∞	∞

3.4.5 LINEARLY VARYING HORIZONTAL LOADING (Fig.3.43)

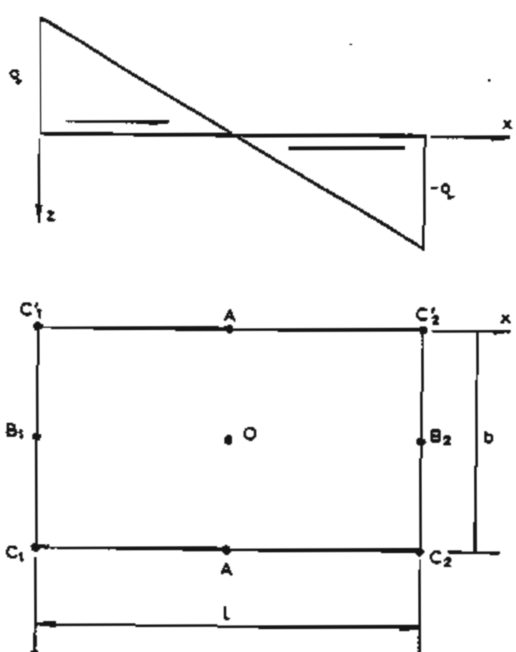


FIG.3.43

Influence factors for the normal stresses have been obtained by Giroud (1970). The stresses are expressed as follows:

Under the corners:

$$\sigma_z = -q\,M_1 \qquad \ldots (3.34a)$$

$$\sigma_x = -q[M_3 - (1-2\nu)M_3^1] \qquad \ldots (3.34b)$$

$$\sigma_y = -q[M_5 - (1-2\nu)M_5^1] \qquad \ldots (3.34c)$$

Under the centre,

$$\sigma_z = 2q(K_1 - M_1) \qquad \ldots (3.35a)$$

$$\sigma_x = 2q[K_3 - M_3 - (1-2\nu)(K_3^1 - M_3^1)] \qquad \ldots (3.35b)$$

$$\sigma_y = 2q[K_5 - M_5 - (1-2\nu)(K_5^1 - M_5^1)] \qquad \ldots (3.35c)$$

The influence factors $M_1, M_3, M_3^1, M_5, M_5^1$ are given in Tables 3.32 to 3.36. $K_1, K_3, K_3^1, K_5, K_5^1$ are given in Tables 3.26 to 3.30.

Explicit expressions and influence factors for vertical displacement ρ_z beneath the points $O, A, C_1, C_1^1, B_1, C_2, C_2^1, B_2$ of the rectangle have been evaluated by Giroud (1968) and are shown in Table 3.37. Influence factors for the mean settlement ρ_m are also given.

At C_2, C_2^1, C_1 & C_1^1

$$\rho_z = \frac{-(1+\nu)(1-2\nu)q\,\ell\,I_C}{E} \quad \text{if } b \geq \ell \qquad \ldots (3.36a)$$

$$\text{or} \quad \frac{-(1+\nu)(1-2\nu)q\,b\,I_C^1}{E} \quad \text{if } \ell \geq b \qquad \ldots (3.36b)$$

and similarly for points B_2 and B_1,

$$\text{where} \quad I_C = \frac{\alpha}{2\pi}\left[-1 + \alpha \arctan\frac{1}{\alpha} + \ell n\,\frac{\sqrt{1+\alpha^2}}{\alpha}\right] \qquad \ldots (3.36c)$$

and $\alpha = b/\ell$

$$I_C^1 = \frac{1}{2\pi}\left[-1 + \frac{1}{\alpha}\arctan\alpha + \ell n\sqrt{1+\alpha^2}\right] \qquad \ldots (3.36d)$$

and $\alpha = \ell/b$

$$\text{At } O, \; \rho_z = \frac{(1+\nu)(1-2\nu)q\,\ell\,I_O}{E} \quad \text{if } b \geq \ell \qquad \ldots (3.37a)$$

$$\text{or} \quad \frac{(1+\nu)(1-2\nu)qb\,I_O^1}{E} \quad \text{if } \ell \geq b \qquad \ldots (3.37b)$$

and similarly for point A and the mean settlement ρ_m.

Giroud (1969b) gives the following expressions for the horizontal displacements at the corners of the rectangle:

$$\rho_x = \frac{\pm(1+\nu)}{\pi E} q \left[b \, \ell n \frac{\ell + \sqrt{\ell^2 + b^2}}{b} - (1+\nu) \frac{b}{\ell} (\sqrt{\ell^2 + b^2} - b) \right] \quad \ldots (3.38a)$$

(+ for C_1 and C_1^1, - for C_2 and C_2^1)

$$\rho_y = \frac{\pm\nu(1+\nu)}{\pi E} q \left[b - \frac{b^2}{\ell} \ell n \frac{\ell + \sqrt{\ell^2 + b^2}}{b} \right] \quad \ldots (3.38b)$$

(+ for C_1^1 and C_2^1, - for C_1 and C_2).

TABLE 3.32

VALUES OF M_1

(Giroud, 1970)

b/ℓ z/ℓ	0	0.1	0.2	1/3	0.4	0.5	2/3	1	1.5	2	2.5	3	5	10	∞
0	0.000	0.159	0.159	0.159	0.159	0.159	0.159	0.159	0.159	0.159	0.159	0.159	0.159	0.159	0.159
0.2	0.000	0.043	0.065	0.075	0.076	0.078	0.078	0.078	0.078	0.078	0.078	0.078	0.078	0.078	0.078
0.4	0.000	0.012	0.021	0.027	0.029	0.030	0.031	0.030	0.030	0.030	0.030	0.030	0.030	0.030	0.030
0.5	0.000	0.006	0.011	0.015	0.016	0.017	0.017	0.016	0.015	0.015	0.015	0.015	0.015	0.015	0.015
0.6	0.000	0.003	0.005	0.007	0.007	0.008	0.007	0.006	0.005	0.005	0.005	0.005	0.005	0.004	0.004
0.8	0.000	-0.000	-0.000	-0.001	-0.001	-0.002	-0.003	-0.004	-0.006	-0.006	-0.007	-0.007	-0.007	-0.007	-0.007
1	0.000	-0.001	-0.002	-0.003	-0.004	-0.005	-0.006	-0.008	-0.010	-0.011	-0.011	-0.011	-0.011	-0.011	-0.011
1.2	0.000	-0.001	-0.002	-0.004	-0.004	-0.005	-0.007	-0.009	-0.011	-0.011	-0.012	-0.012	-0.012	-0.012	-0.012
1.4	0.000	-0.001	-0.002	-0.003	-0.004	-0.005	-0.006	-0.008	-0.010	-0.011	-0.011	-0.012	-0.012	-0.012	-0.012
1.5	0.000	-0.001	-0.002	-0.003	-0.004	-0.004	-0.006	-0.008	-0.009	-0.010	-0.011	-0.011	-0.011	-0.011	-0.011
1.6	0.000	-0.001	-0.002	-0.003	-0.003	-0.004	-0.005	-0.007	-0.009	-0.010	-0.010	-0.011	-0.011	-0.011	-0.011
1.8	0.000	-0.001	-0.001	-0.002	-0.003	-0.003	-0.004	-0.006	-0.008	-0.009	-0.009	-0.009	-0.010	-0.010	-0.010
2	0.000	-0.001	-0.001	-0.002	-0.002	-0.003	-0.004	-0.005	-0.007	-0.007	-0.008	-0.008	-0.009	-0.009	-0.009
2.5	0.000	-0.000	-0.001	-0.001	-0.001	-0.002	-0.002	-0.003	-0.004	-0.005	-0.006	-0.006	-0.006	-0.006	-0.006
3	0.000	-0.000	-0.000	-0.001	-0.001	-0.001	-0.001	-0.002	-0.003	-0.004	-0.004	-0.004	-0.005	-0.005	-0.005
4	0.000	-0.000	-0.000	-0.000	-0.000	-0.001	-0.001	-0.001	-0.001	-0.002	-0.002	-0.002	-0.003	-0.003	-0.003
5	0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.001	-0.001	-0.001	-0.001	-0.001	-0.002	-0.002	-0.002
10	0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.001
15	0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000
20	0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000
50	0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000

TABLE 3.33

VALUES OF M_3

(Giroud, 1970)

b/ℓ z/ℓ	0	0.1	0.2	1/3	0.4	0.5	2/3	1	1.5	2	2.5	3	5	10	∞
0	0.000	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞
0.2	0.000	0.013	0.017	0.012	0.008	0.002	-0.006	-0.015	-0.019	-0.020	-0.021	-0.021	-0.021	-0.021	-0.021
0.4	0.000	-0.006	-0.012	-0.020	-0.024	-0.028	-0.035	-0.042	-0.046	-0.047	-0.047	-0.048	-0.048	-0.048	-0.048
0.5	0.000	-0.006	-0.012	-0.019	-0.022	-0.026	-0.031	-0.037	-0.041	-0.042	-0.042	-0.043	-0.043	-0.043	-0.043
0.6	0.000	-0.005	-0.010	-0.015	-0.018	-0.021	-0.026	-0.031	-0.034	-0.035	-0.036	-0.036	-0.036	-0.036	-0.036
0.8	0.000	-0.003	-0.006	-0.009	-0.011	-0.013	-0.016	-0.020	-0.022	-0.023	-0.023	-0.023	-0.024	-0.024	-0.023
1	0.000	-0.002	-0.003	-0.005	-0.006	-0.008	-0.009	-0.012	-0.014	-0.014	-0.015	-0.015	-0.015	-0.015	-0.015
1.2	0.000	-0.001	-0.002	-0.003	-0.004	-0.004	-0.006	-0.007	-0.009	-0.009	-0.009	-0.010	-0.010	-0.010	-0.010
1.4	0.000	-0.001	-0.001	-0.002	-0.002	-0.003	-0.003	-0.004	-0.005	-0.006	-0.006	-0.006	-0.006	-0.006	-0.006
1.5	0.000	-0.000	-0.001	-0.001	-0.002	-0.002	-0.003	-0.004	-0.004	-0.005	-0.005	-0.005	-0.005	-0.005	-0.005
1.6	0.000	-0.000	-0.001	-0.001	-0.001	-0.002	-0.002	-0.003	-0.004	-0.004	-0.004	-0.004	-0.004	-0.004	-0.004
1.8	0.000	-0.000	-0.000	-0.001	-0.001	-0.001	-0.001	-0.002	-0.002	-0.003	-0.003	-0.003	-0.003	-0.003	-0.003
2	0.000	-0.000	-0.000	-0.000	-0.001	-0.001	-0.001	-0.001	-0.002	-0.002	-0.002	-0.002	-0.002	-0.002	-0.002
2.5	0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.001	-0.001	-0.001	-0.001	-0.001	-0.001	-0.001
3	0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000
4	0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000
5	0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000
10	0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	0.000
15	0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	0.000
20	0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	0.000
50	0.000	-0.000	-0.000	-0.000	-0.000	0.000	0.000	0.000	-0.000	0.000	-0.000	0.000	0.000	0.000	0.000

TABLE 3.34

VALUES OF M_3^1

(Giroud, 1970)

b/ℓ z/ℓ	0	0.1	0.2	1/3	0.4	0.5	2/3	1	1.5	2	2.5	3	5	10	∞
0	0.000	0.080	0.043	0.017	0.009	0.001	-0.006	-0.009	-0.007	-0.005	-0.003	-0.003	-0.001	-0.000	0.000
0.2	0.000	0.004	0.005	0.001	-0.001	-0.003	-0.006	-0.007	-0.006	-0.004	-0.003	-0.002	-0.001	-0.000	0.000
0.4	0.000	-0.000	-0.001	-0.002	-0.003	-0.003	-0.004	-0.005	-0.004	-0.003	-0.003	-0.002	-0.001	-0.000	0.000
0.5	0.000	-0.001	-0.001	-0.002	-0.003	-0.003	-0.004	-0.004	-0.004	-0.003	-0.002	-0.002	-0.001	-0.000	0.000
0.6	0.000	-0.001	-0.001	-0.002	-0.002	-0.003	-0.003	-0.004	-0.003	-0.003	-0.002	-0.002	-0.001	-0.000	0.000
0.8	0.000	-0.000	-0.001	-0.001	-0.002	-0.002	-0.002	-0.003	-0.003	-0.002	-0.002	-0.001	-0.001	-0.000	0.000
1	0.000	-0.000	-0.001	-0.001	-0.001	-0.001	-0.002	-0.002	-0.002	-0.002	-0.002	-0.001	-0.001	-0.000	0.000
1.2	0.000	-0.000	-0.000	-0.001	-0.001	-0.001	-0.001	-0.002	-0.002	-0.002	-0.001	-0.001	-0.001	-0.000	0.000
1.4	0.000	-0.000	-0.000	-0.001	-0.001	-0.001	-0.001	-0.001	-0.001	-0.001	-0.001	-0.001	-0.001	-0.000	0.000
1.5	0.000	-0.000	-0.000	-0.000	-0.001	-0.001	-0.001	-0.001	-0.001	-0.001	-0.001	-0.001	-0.001	-0.000	0.000
1.6	0.000	-0.000	-0.000	-0.000	-0.000	-0.001	-0.001	-0.001	-0.001	-0.001	-0.001	-0.001	-0.001	-0.000	0.000
1.8	0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.001	-0.001	-0.001	-0.001	-0.001	-0.001	-0.000	-0.000	0.000
2	0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.001	-0.001	-0.001	-0.001	-0.001	-0.000	-0.000	0.000
2.5	0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	0.000
3	0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	0.000
4	0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	0.000
5	0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	0.000
10	0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	0.000
15	0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	0.000
20	0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	0.000
50	0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	0.000

TABLE 3.35

VALUES OF M_5

(Giroud, 1970)

b/ℓ z/ℓ	0	0.1	0.2	1/3	0.4	0.5	2/3	1	1.5	2	2.5	3	5	10	∞
0	0.000	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞
0.2	0.000	0.003	0.015	0.031	0.036	0.042	0.046	0.045	0.040	0.036	0.034	0.032	0.030	0.029	0.028
0.4	0.000	0.000	0.001	0.004	0.006	0.007	0.008	0.006	0.002	-0.002	-0.004	-0.005	-0.008	-0.009	-0.009
0.5	0.000	0.000	0.001	0.002	0.002	0.003	0.003	0.001	-0.004	-0.007	-0.009	-0.010	-0.013	-0.014	-0.014
0.6	0.000	0.000	0.000	0.001	0.001	0.001	0.000	-0.002	-0.006	-0.009	-0.011	-0.012	-0.014	-0.015	-0.016
0.8	0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.001	-0.003	-0.006	-0.009	-0.010	-0.012	-0.014	-0.015	-0.015
1	0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.001	-0.002	-0.005	-0.007	-0.009	-0.010	-0.012	-0.013	-0.013
1.2	0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.001	-0.002	-0.004	-0.005	-0.007	-0.008	-0.010	-0.011	-0.011
1.4	0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.001	-0.003	-0.004	-0.005	-0.006	-0.008	-0.009	-0.009
1.5	0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.001	-0.002	-0.003	-0.004	-0.005	-0.007	-0.008	-0.008
1.6	0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.001	-0.002	-0.003	-0.004	-0.005	-0.006	-0.007	-0.008
1.8	0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.001	-0.001	-0.002	-0.003	-0.004	-0.005	-0.006	-0.006
2	0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.001	-0.002	-0.002	-0.003	-0.004	-0.005	-0.005
2.5	0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.001	-0.001	-0.002	-0.003	-0.003	-0.004
3	0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.001	-0.001	-0.002	-0.002	-0.003
4	0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.001	-0.001	-0.002
5	0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.001	-0.001
10	0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000
15	0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000
20	0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000
50	0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	0.000	-0.000	-0.000	-0.000	-0.000	-0.000

TABLE 3.36

VALUES OF M_5^1

(Giroud, 1970)

b/ℓ z/ℓ	0	0.1	0.2	1/3	0.4	0.5	2/3	1	1.5	2	2.5	3	5	10	∞
0	0.000	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞
0.2	0.000	0.016	0.028	0.038	0.041	0.044	0.045	0.043	0.038	0.035	0.033	0.032	0.030	0.029	0.028
0.4	0.000	0.002	0.004	0.006	0.006	0.006	0.006	0.003	-0.000	-0.003	-0.005	-0.006	-0.008	-0.009	-0.009
0.5	0.000	0.001	0.001	0.001	0.001	0.001	0.000	-0.002	-0.006	-0.008	-0.010	-0.011	-0.013	-0.014	-0.014
0.6	0.000	-0.000	-0.000	-0.001	-0.001	-0.002	-0.003	-0.005	-0.008	-0.010	-0.012	-0.013	-0.014	-0.015	-0.016
0.8	0.000	-0.001	-0.001	-0.002	-0.002	-0.003	-0.004	-0.006	-0.009	-0.010	-0.012	-0.012	-0.014	-0.015	-0.015
1	0.000	-0.001	-0.001	-0.002	-0.002	-0.003	-0.004	-0.005	-0.007	-0.009	-0.010	-0.011	-0.012	-0.013	-0.013
1.2	0.000	-0.000	-0.001	-0.002	-0.002	-0.002	-0.003	-0.004	-0.006	-0.007	-0.008	-0.009	-0.010	-0.011	-0.011
1.4	0.000	-0.000	-0.001	-0.001	-0.001	-0.002	-0.002	-0.003	-0.005	-0.006	-0.006	-0.007	-0.008	-0.009	-0.009
1.5	0.000	-0.000	-0.001	-0.001	-0.001	-0.002	-0.002	-0.003	-0.004	-0.005	-0.006	-0.006	-0.007	-0.008	-0.008
1.6	0.000	-0.000	-0.001	-0.001	-0.001	-0.001	-0.002	-0.003	-0.004	-0.004	-0.005	-0.006	-0.007	-0.007	-0.008
1.8	0.000	-0.000	-0.000	-0.001	-0.001	-0.001	-0.001	-0.002	-0.003	-0.004	-0.004	-0.005	-0.005	-0.006	-0.006
2	0.000	-0.000	-0.000	-0.001	-0.001	-0.001	-0.001	-0.002	-0.002	-0.003	-0.003	-0.004	-0.004	-0.005	-0.005
2.5	0.000	-0.000	-0.000	-0.000	-0.000	-0.001	-0.001	-0.001	-0.001	-0.002	-0.002	-0.002	-0.003	-0.003	-0.004
3	0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.001	-0.001	-0.001	-0.001	-0.002	-0.002	-0.002	-0.003
4	0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.001	-0.001	-0.001	-0.001	-0.001	-0.002
5	0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.001	-0.001	-0.001
10	0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000
15	0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000
20	0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000
50	0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000

TABLE 3.37

INFLUENCE FACTORS FOR VERTICAL SURFACE DISPLACEMENTS DUE TO LINEARLY VARYING HORIZONTAL LOADING

(Giroud, 1968)

ℓ ≥ b				ℓ ≥ b						b ≥ ℓ					
ℓ/b	I_A'	I_O'	I_m'	ℓ/b	I_C'	I_B'	ℓ/b	I_C'	I_B'	b/ℓ	I_C	I_B	I_A	I_O	I_m
1	0.100	0.159	0.080	1	0.021	0.057	15	0.288	0.390	1	0.021	0.057	0.100	0.159	0.080
1.1	0.107	0.168	0.083	1.1	0.024	0.064	20	0.330	0.434	1.1	0.020	0.056	0.102	0.166	0.083
1.2	0.114	0.176	0.087	1.2	0.028	0.071	25	0.363	0.468	1.2	0.019	0.054	0.104	0.171	0.087
1.3	0.121	0.183	0.090	1.3	0.032	0.078	30	0.390	0.497	1.3	0.018	0.052	0.105	0.176	0.090
1.4	0.127	0.189	0.093	1.4	0.035	0.084	40	0.434	0.541	1.4	0.017	0.051	0.107	0.181	0.093
1.5	0.133	0.195	0.096	1.5	0.039	0.090	50	0.468	0.576	1.5	0.016	0.049	0.108	0.185	0.095
1.6	0.139	0.201	0.098	1.6	0.043	0.096	60	0.497	0.605	1.6	0.015	0.048	0.109	0.188	0.098
1.7	0.144	0.206	0.100	1.7	0.046	0.102	70	0.521	0.629	1.7	0.014	0.046	0.110	0.191	0.100
1.8	0.150	0.210	0.102	1.8	0.050	0.108	80	0.541	0.650	1.8	0.014	0.045	0.110	0.194	0.102
1.9	0.154	0.215	0.104	1.9	0.053	0.114	90	0.560	0.669	1.9	0.013	0.043	0.111	0.197	0.104
2	0.159	0.219	0.106	2	0.057	0.119	100	0.576	0.685	2	0.012	0.042	0.112	0.199	0.106
2.2	0.168	0.226	0.110	2.2	0.064	0.129	200	0.685	0.795	2.2	0.011	0.039	0.113	0.204	0.109
2.5	0.179	0.235	0.114	2.4	0.071	0.139	300	0.749	0.859	2.5	0.010	0.036	0.114	0.209	0.113
3	0.195	0.247	0.119	2.5	0.074	0.144	400	0.795	0.905	3	0.009	0.031	0.116	0.215	0.119
3.5	0.208	0.255	0.124	3	0.090	0.165	500	0.830	0.940	3.5	0.007	0.028	0.117	0.220	0.124
4	0.219	0.262	0.127	3.5	0.105	0.185	600	0.859	0.969	4	0.007	0.025	0.118	0.224	0.127
5	0.235	0.273	0.133	4	0.119	0.202	700	0.884	0.994	5	0.005	0.020	0.120	0.229	0.133
7	0.255	0.285	0.139	4.5	0.132	0.217	800	0.905	1.015	7	0.004	0.015	0.121	0.235	0.140
10	0.273	0.294	0.145	5	0.144	0.232	900	0.924	1.034	10	0.003	0.010	0.122	0.239	0.146
15	0.287	0.302	0.149	6	0.165	0.257	10^3	0.940	1.051	15	0.002	0.007	0.123	0.243	0.151
20	0.294	0.306	0.151	7	0.185	0.278	10^4	1.307	1.417	20	0.001	0.005	0.124	0.245	0.155
50	0.308	0.313	0.156	8	0.202	0.297	10^5	1.673	1.784	50	0.001	0.002	0.124	0.248	0.161
100	0.313	0.316	0.158	9	0.217	0.315	10^6	2.039	2.150	100	0.000	0.001	0.125	0.249	0.163
∞	0.318	0.318	0.159	10	0.232	0.330	∞	∞	∞	∞	0	0	0.125	0.250	0.167

3.5 Loading on an Elliptical Area

3.5.1 UNIFORM VERTICAL LOADING

Stresses and displacements at the surface and on the axis of the ellipse have been obtained by Deresiewicz (1960) (Fig.3.44).

Expressions are derived for the stresses and displacements on the axis within the mass, and on the surface.

The variation of maximum shear stress with depth for various e values is shown in Fig.3.45. Stress distributions along the axis for four values of e are given in Fig.3.46. In all cases, $\nu=0.3$, and e is defined as $e=(1-a^2/b^2)^{\frac{1}{2}}$.

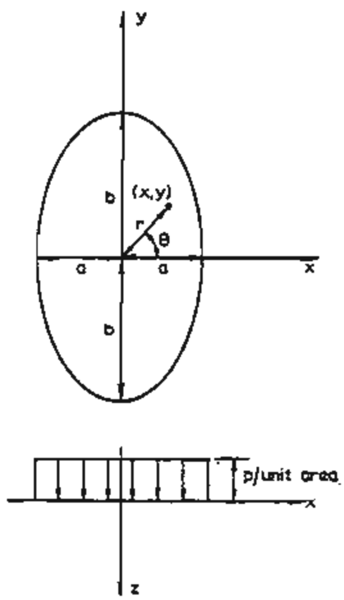


FIG.3.44

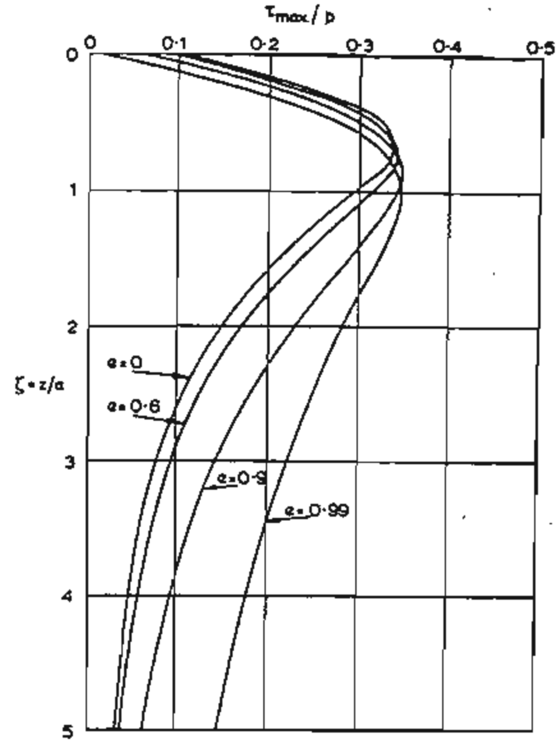


FIG.3.45 Maximum shear stress down axis of ellipse. (Deresiewicz, 1960).

Values of the horizontal stresses on the axis are tabulated in Table 3.38.

TABLE 3.38

HORIZONTAL STRESSES ON AXIS OF ELLIPSE ($\nu=0.3$) (Deresiewicz, 1960)

e	0		0.3		0.6		0.9		0.99	
$\xi=z/a$	σ_x/p	σ_y/p	σ_x/p	σ_y/p	σ_x/p	σ_y/p	σ_x/p	σ_y/p	σ_x/p	σ_y/p
0	0.8000	0.8000	0.8047	0.7953	0.8222	0.7778	0.8786	0.7214	0.9636	0.6364
0.05	0.7351	0.7351	0.7404	0.7330	0.7585	0.7244	0.8157	0.6875	0.8987	0.6158
0.1	0.6711	0.6711	0.6765	0.6716	0.6954	0.6716	0.7535	0.6538	0.8343	0.5954
0.2	0.5488	0.5488	0.5542	0.5542	0.5744	0.5694	0.6338	0.5875	0.7100	0.5551
0.4	0.3428	0.3428	0.3488	0.3531	0.3681	0.3894	0.4236	0.4647	0.5003	0.4849
0.7	0.1488	0.1488	0.1524	0.1599	0.1668	0.1999	0.2094	0.3137	0.2587	0.3839
1.0	0.0575	0.0575	0.0600	0.0648	0.0673	0.0949	0.0928	0.2060	0.1235	0.3105
1.5	0.0064	0.0064	0.0067	0.0090	0.0086	0.0229	0.0134	0.0997	0.0179	0.2270
2	-0.0050	-0.0050	-0.0051	-0.0037	-0.0055	0.0017	-0.0050	0.0457	-0.0277	0.1721
3	-0.0064	-0.0064	-0.0062	-0.0067	-0.0078	-0.0054	-0.0115	0.0080	-0.0604	0.1095
4	-0.0046	-0.0046	-0.0030	-0.0064	-0.0058	-0.0046	-0.0092	-0.0011	-0.0723	0.0761
5	-0.0033	-0.0033	-0.0019	-0.0050	-0.0038	-0.0040	-0.0069	-0.0030	-0.0783	0.0566
10	-0.0009	-0.0009	-0.0015	-0.0001	-0.0010	-0.0013	-0.0021	-0.0018	-0.0898	0.0252

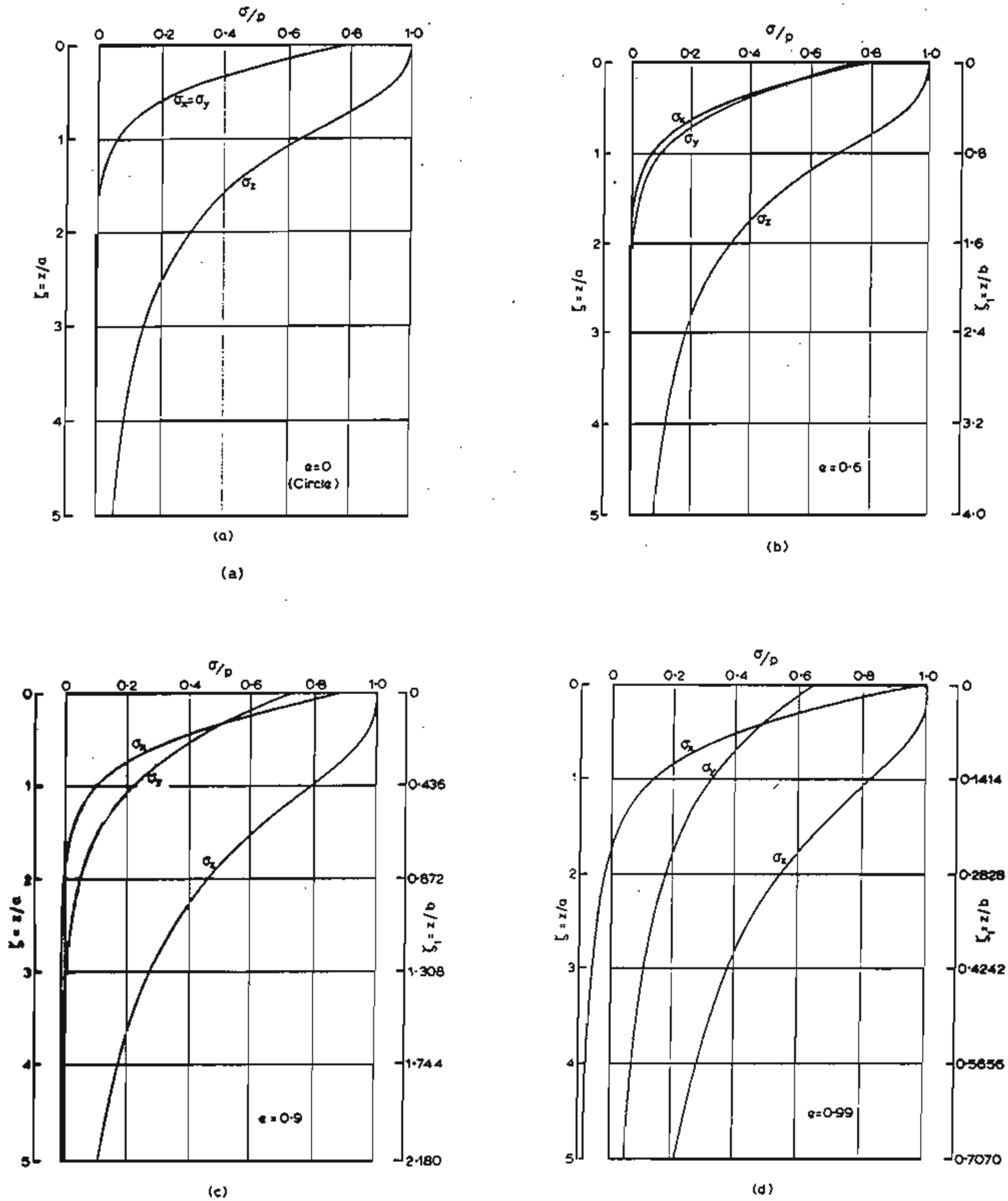


FIG.3.46 Variation of normal stresses with depth on axis of ellipse. ν=0.3. (Deresiewicz, 1960).

On the axis the displacements are given by

$$\rho_z = \frac{4(1-\nu^2)pa}{\pi E} K(e) \qquad \ldots (3.39a)$$

$$\rho_x = \rho_y = 0 \qquad \ldots (3.39b)$$

where $K(e)$ = complete elliptic integral of the first kind.

Relative vertical displacements ρ/ρ_o on the axis of the ellipse are shown in Fig.3.47 for $\nu=0.3$.

The variation of ρ/ρ_o along the boundary of the ellipse with position is given in Fig.3.48, while Fig.3.49 shows the variation of the displacements at the extremity of the major axis (ρ_M) and the minor axis (ρ_m). ρ_z is expressed in all cases as a ratio of the surface displacement ρ_o.

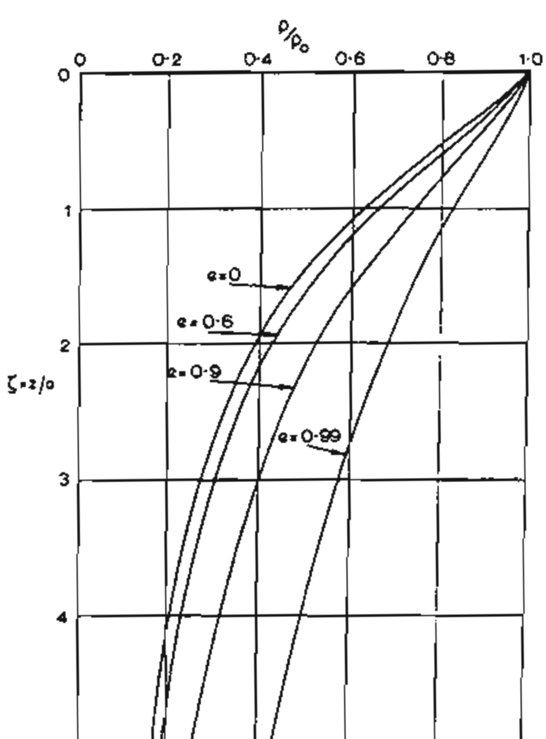


FIG.3.47 Vertical displacement on axis as ratio of surface value ρ_o. (Deresiewicz, 1960).

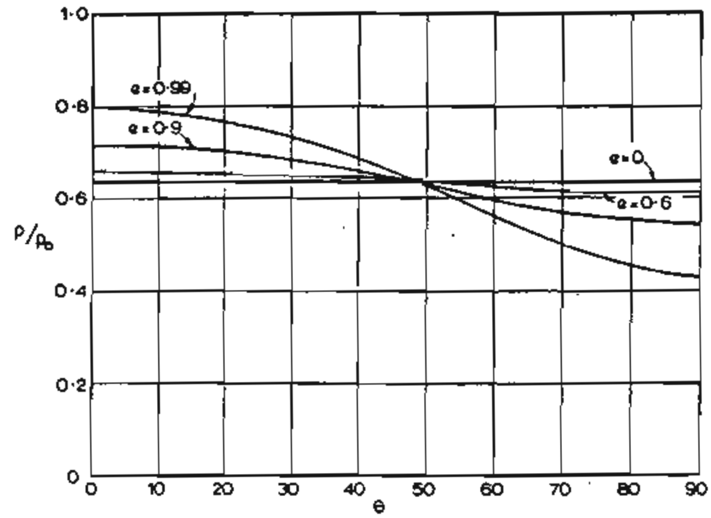


FIG.3.48 Variation of boundary surface displacement ρ to centre value ρ_o with position along boundary. (Deresiewicz, 1960).

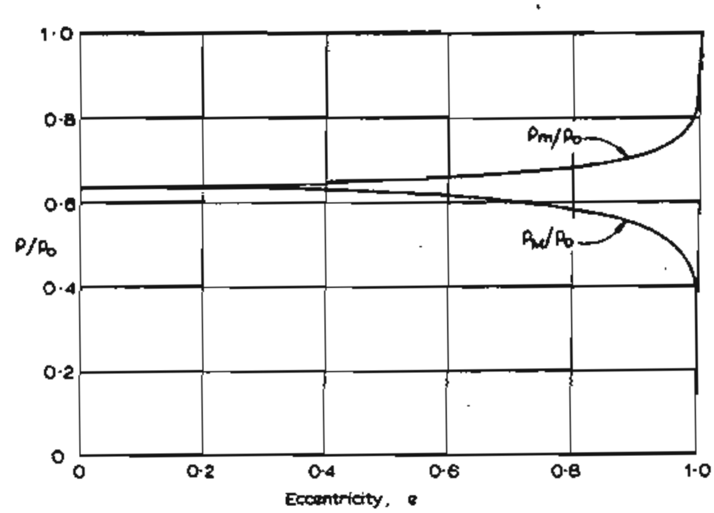


FIG.3.49 Variation of ratio of displacement at extremity of major axis, ρ_M, and minor axis, ρ_m, to that at centre, ρ_o. (Deresiewicz, 1960).

3.5.2 VERTICAL SEMI-ELLIPSOIDAL LOADING

This type of loading has been used to simulate wheel loading on road pavements. Vertical stresses and vertical displacements within the mass have been evaluated by Sanborn and Yoder (1967).

3.6 Loading over Any Area

3.6.1 "NEWMARK CHARTS"

The basis for, and use of, "Newmark Charts", is described in 1.7.2. Charts for vertical stress σ_z, horizontal stress, bulk stress θ and shear stresses τ_{xz} and τ_{xy} (all as a function of the applied stress), originally presented by Newmark (1942), are reproduced in Figs. 3.50 to 3.54. Fig. 3.55 gives correction factors for τ_{xy} when Poisson's ratio is different from 0.5, while Fig. 3.56 gives part of the correction factor for σ_x. When $\nu \neq 0.5$, σ_x is given by the value of σ_x for $\nu = 0.5$, plus $(1-2\nu)/6$ times the value of θ for $\nu = 0.5$ (Fig. 3.52) plus $(1-2\nu)$ times the quantity obtained from Fig. 3.56.

Similar charts for vertical displacement ρ_z on the surface and below the surface were obtained by Newmark (1947) and are shown in Figs.3.57 and 3.58. A chart for correcting the vertical subsurface displacements in Fig.3.58, which are for $\nu = 0.5$, for other values of ν, is given in Fig.3.59. Figs. 3.50 to 3.59 are for vertical surface loading.

Charts for the horizontal normal stress due to an applied surface horizontal shear loading have been prepared by Barber (1965) and are given in Figs.3.60 to 3.63. Stresses parallel to, and perpendicular to, the applied loading are considered for both $\nu = 0.5$ and $\nu = 0$. As pointed out by Barber (1966), the vertical stress due to shear loading is, by the reciprocal theorem, identical to the shear stress due to a vertical load and may thus be determined from Fig.3.53.

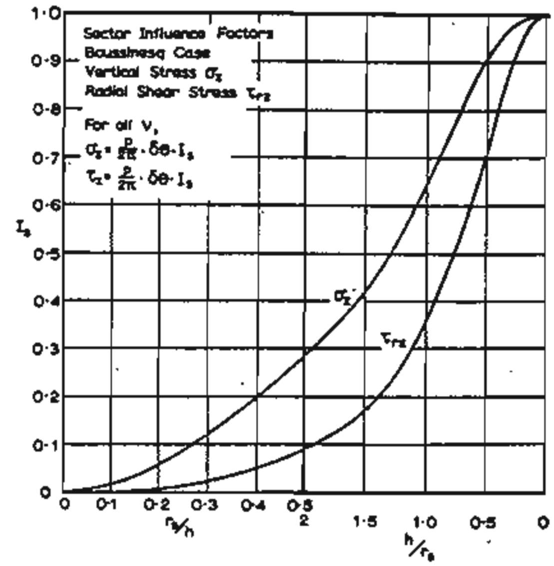


FIG.3.64 Sector Influence Values for σ_z and τ_{rz}.

3.6.2 SECTOR CURVES

The sector method and the use of sector curves have been described in 1.7.3. Sector curves for the normal and shear stresses due to vertical loading, obtained by Poulos (1967a), are shown in Figs.3.64 and 3.65. For the vertical and radial displacements ρ_z and ρ_r, plots of the curves are unnecessary, as the sector curves have the following simple explicit form:

$$\rho_z = \frac{p}{\pi}.\delta\theta.\frac{(1-\nu^2)}{E}\;r_s \qquad \ldots (3.39a)$$

$$\rho_r = \frac{p}{2\pi}.\delta\theta.\frac{(1+\nu)(1-2\nu)}{E}\;r_s \qquad \ldots (3.39b)$$

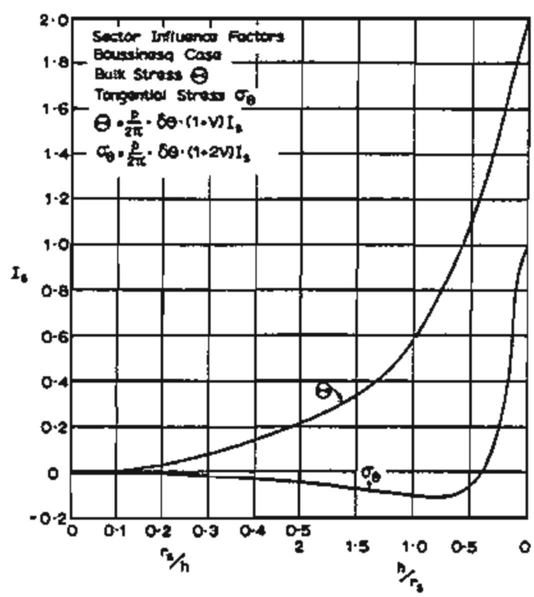


FIG.3.65 Sector Influence values for θ and σ_θ.

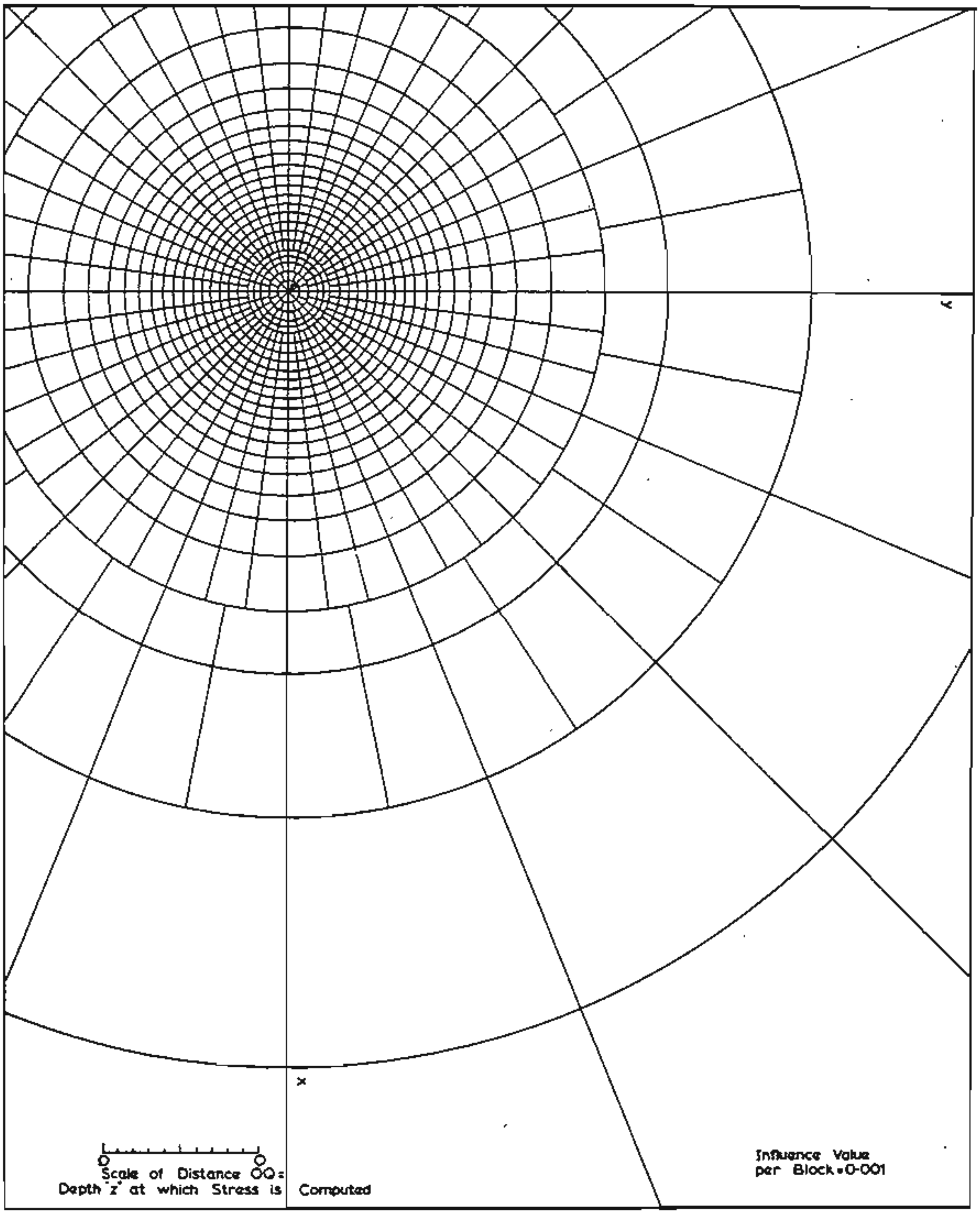


FIG.3.50 Influence chart for vertical stress σ_z (Newmark, 1942)
(All values of ν)
σ_z = .001Np
where N=no.of blocks.

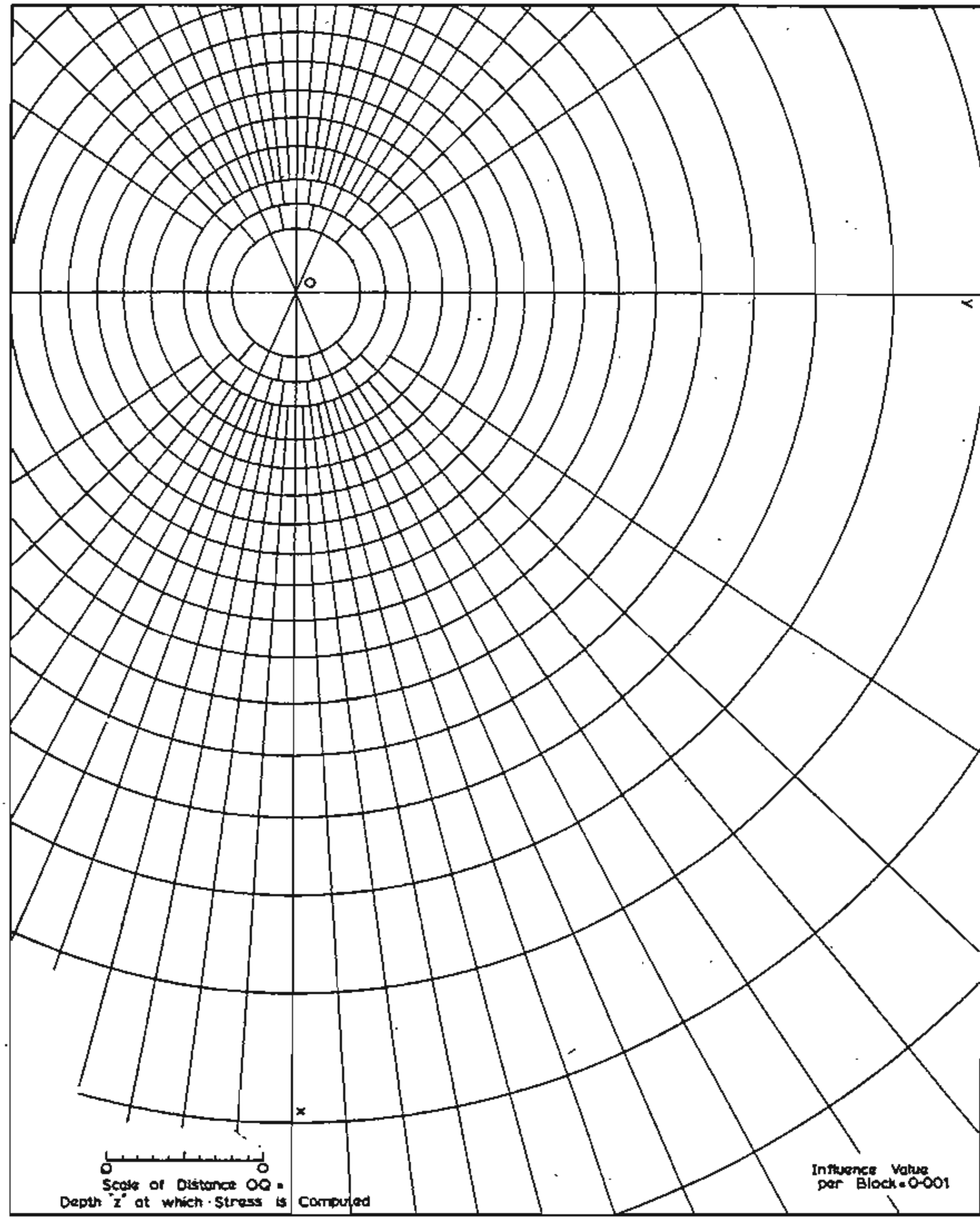


FIG.3.51 Influence chart for horizontal stress σ_x (Newmark, 1942).
$\nu = 0.5$
$\sigma_x = .001Np$
where N=no. of blocks.

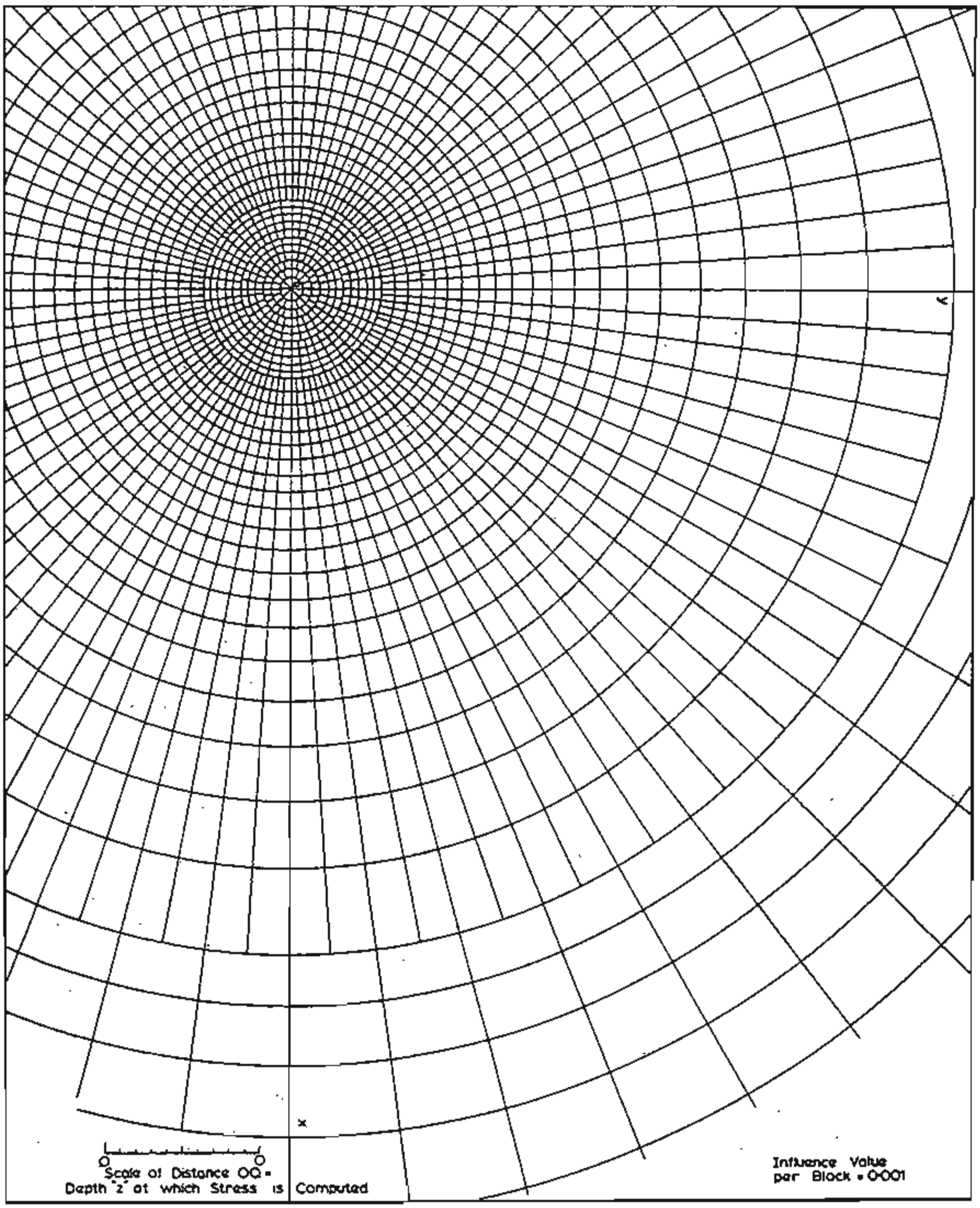


FIG.3.52 Influence chart for bulk stress θ (Newmark, 1942)
(for all values of ν)

$$\theta = \frac{2(1+\nu)}{3} \; .001Np$$

where N=no.of blocks.

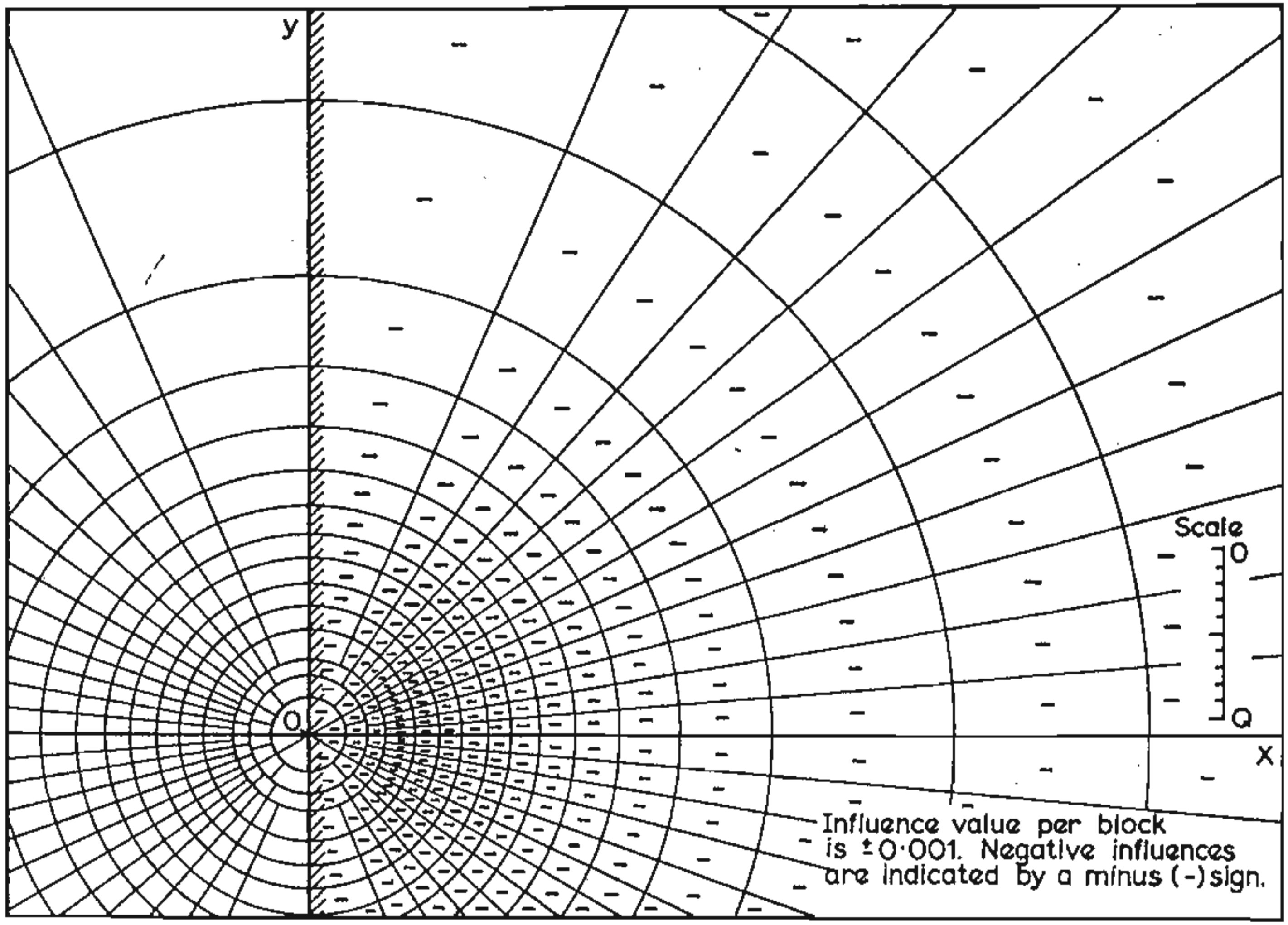


FIG.3.53 Influence chart for shear stress τ_{xz}. (Newmark, 1942).
(for all values of ν)
τ_{xz} = .001N p
where N=no.of blocks.

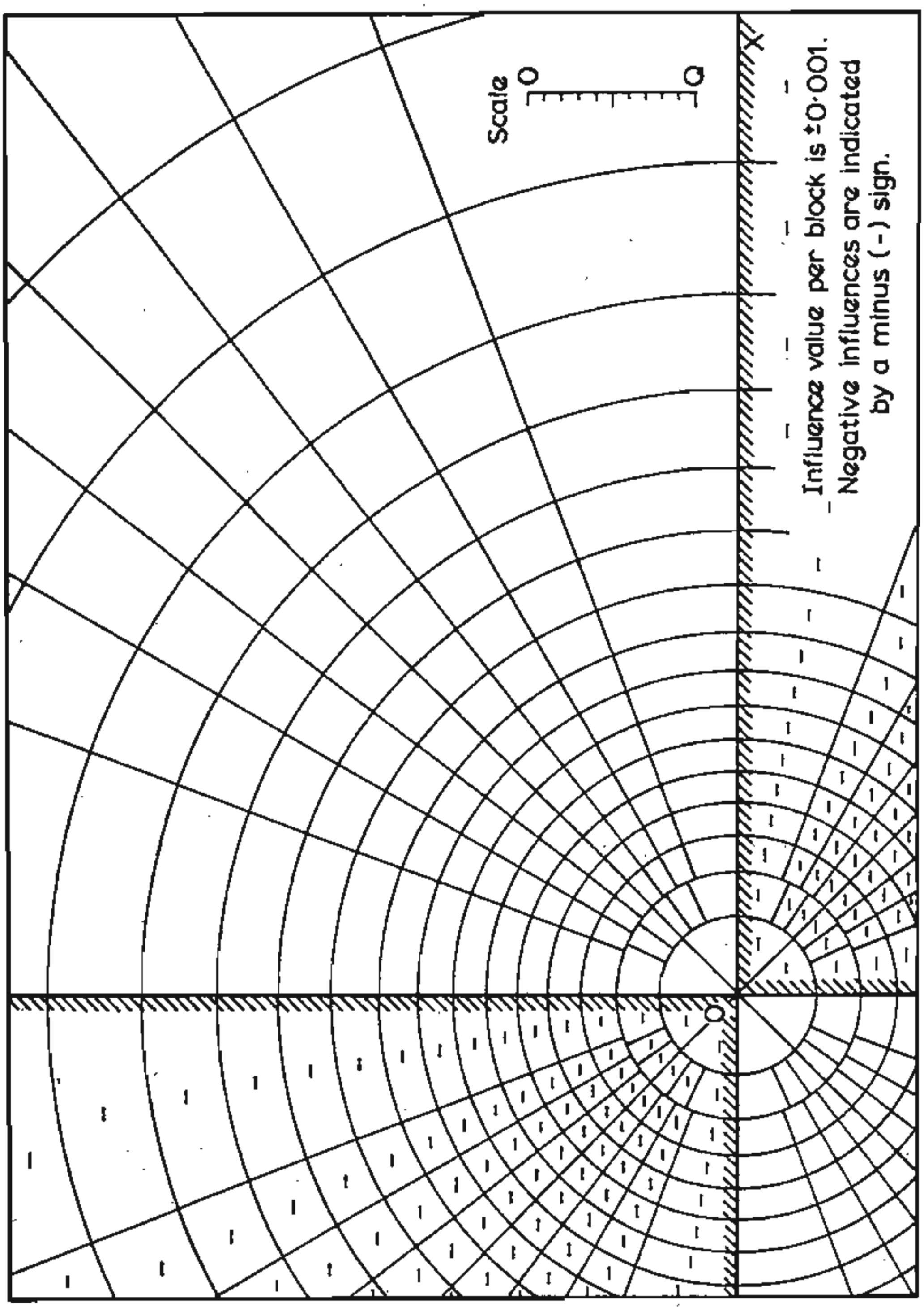


FIG.3.54 Influence chart for shear stress τ_{xy} for ν=0.5 (Newmark, 1942).

τ_{xy} = .001N p

where N=no.of blocks.

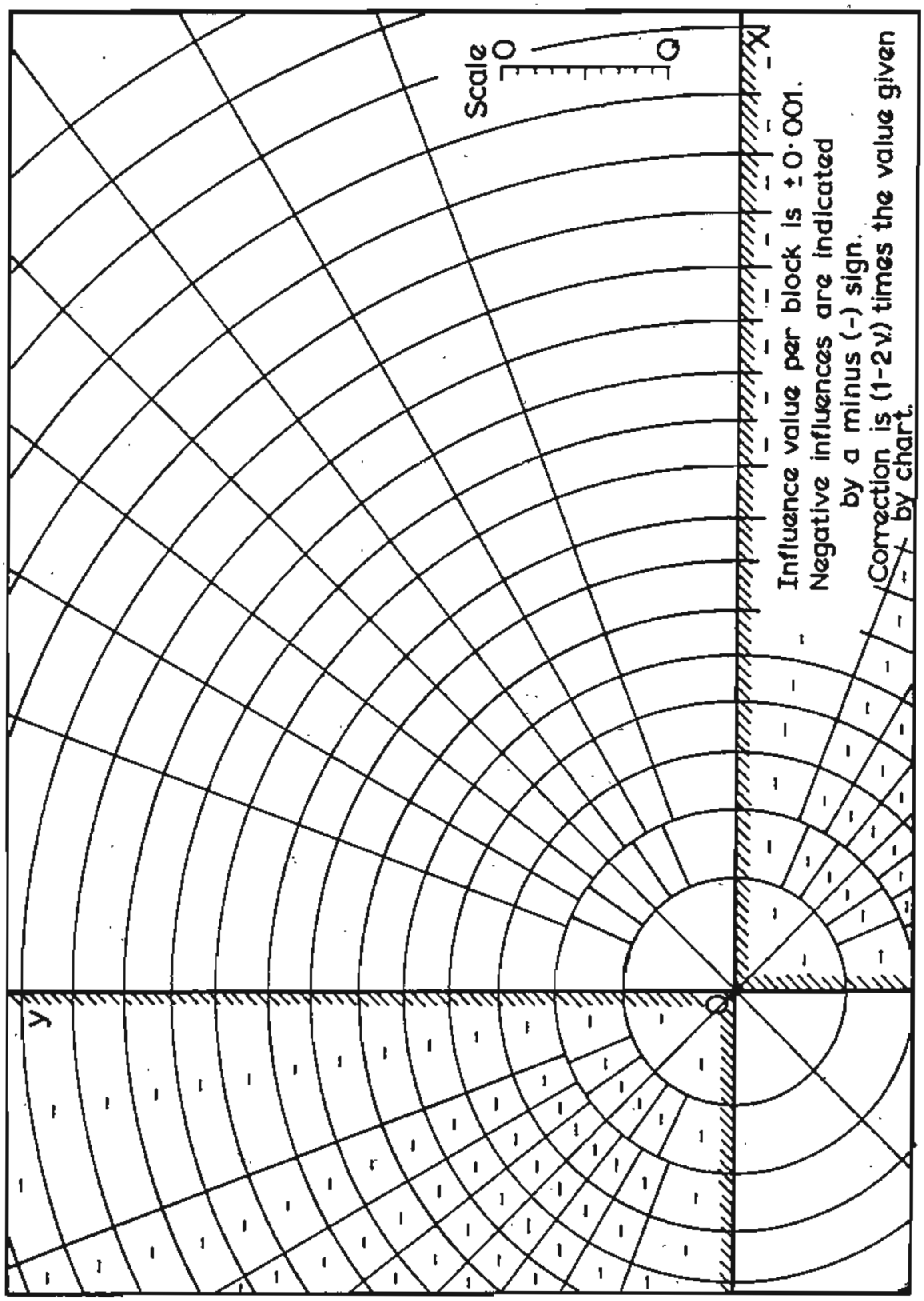


FIG.3.55 Influence chart for correction to τ_{xy} when ν=0.5. (Newmark, 1942).

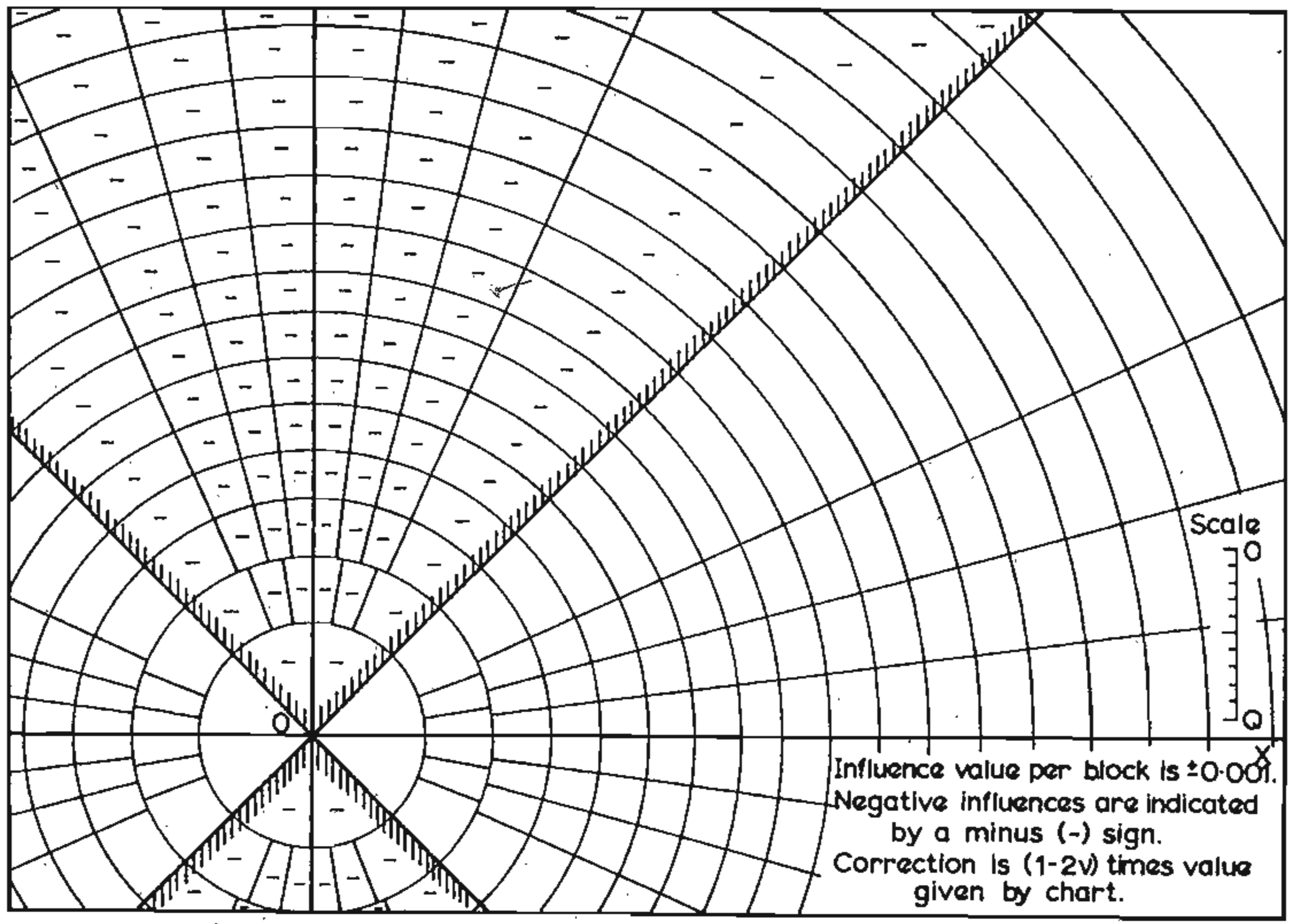


FIG.3.56 Influence chart for part of correction to σ_x for $\nu \neq 0.5$. (Newmark,1942).

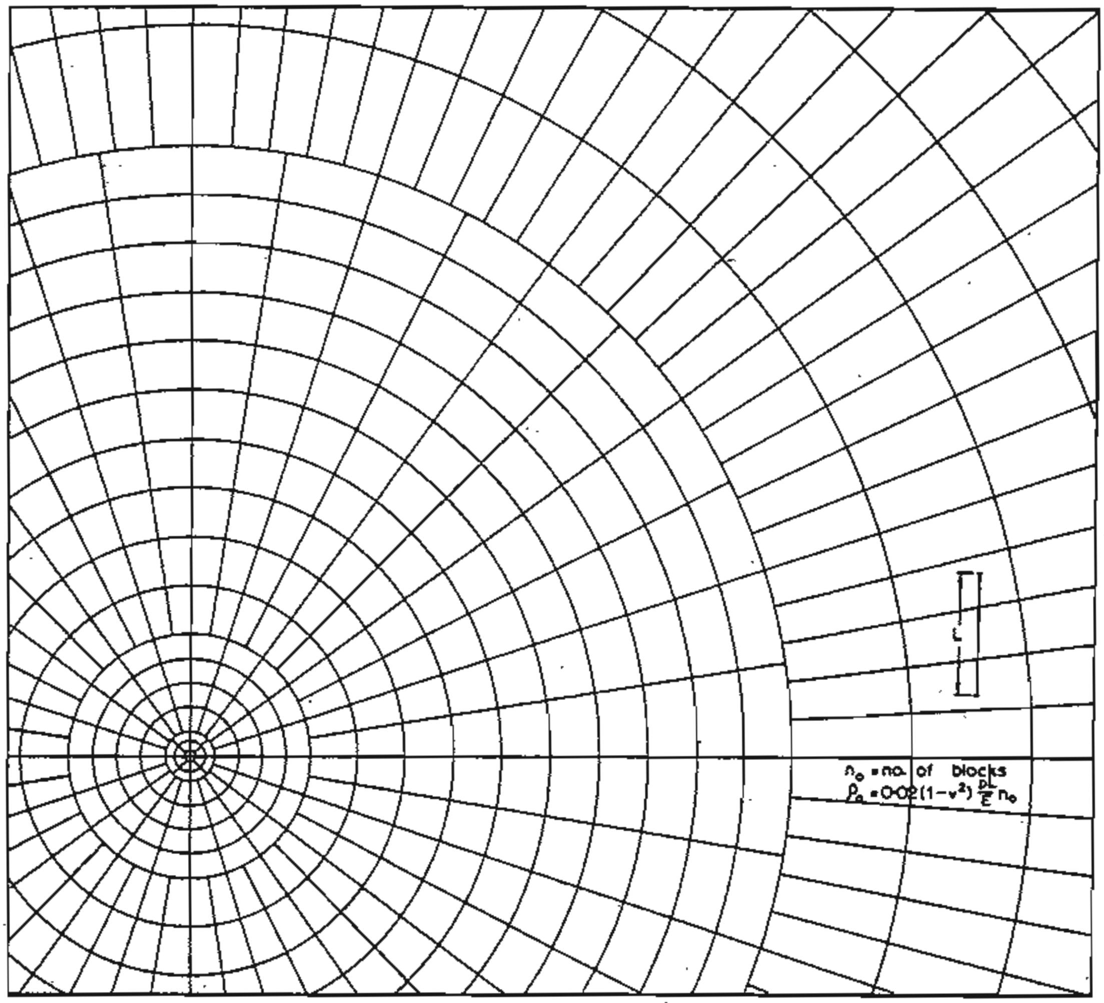


FIG.3.57 Influence chart for vertical displacement at surface. (Newmark, 1947).

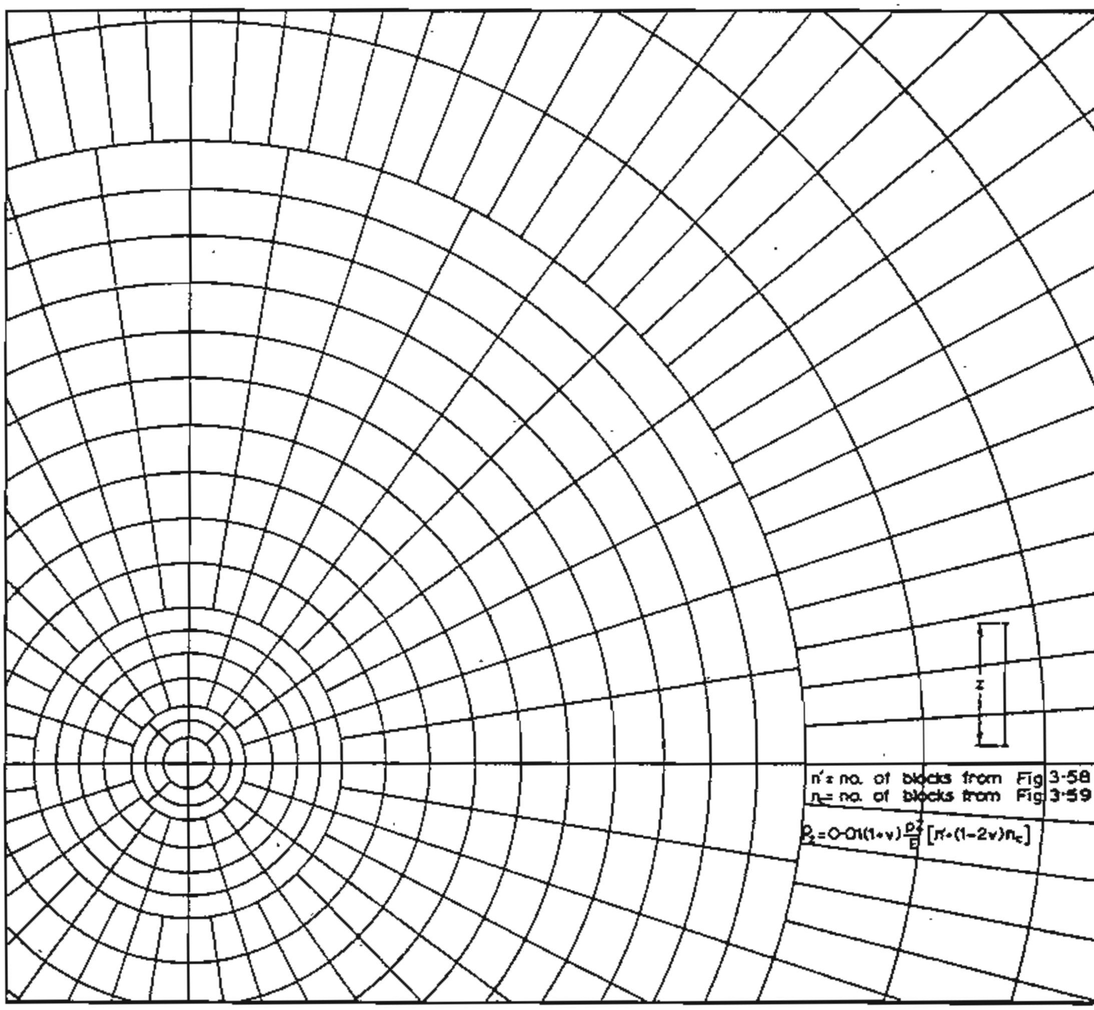


FIG.3.58 Influence chart for vertical displacement at depth z below surface. ν=0.5. (Newmark, 1947).

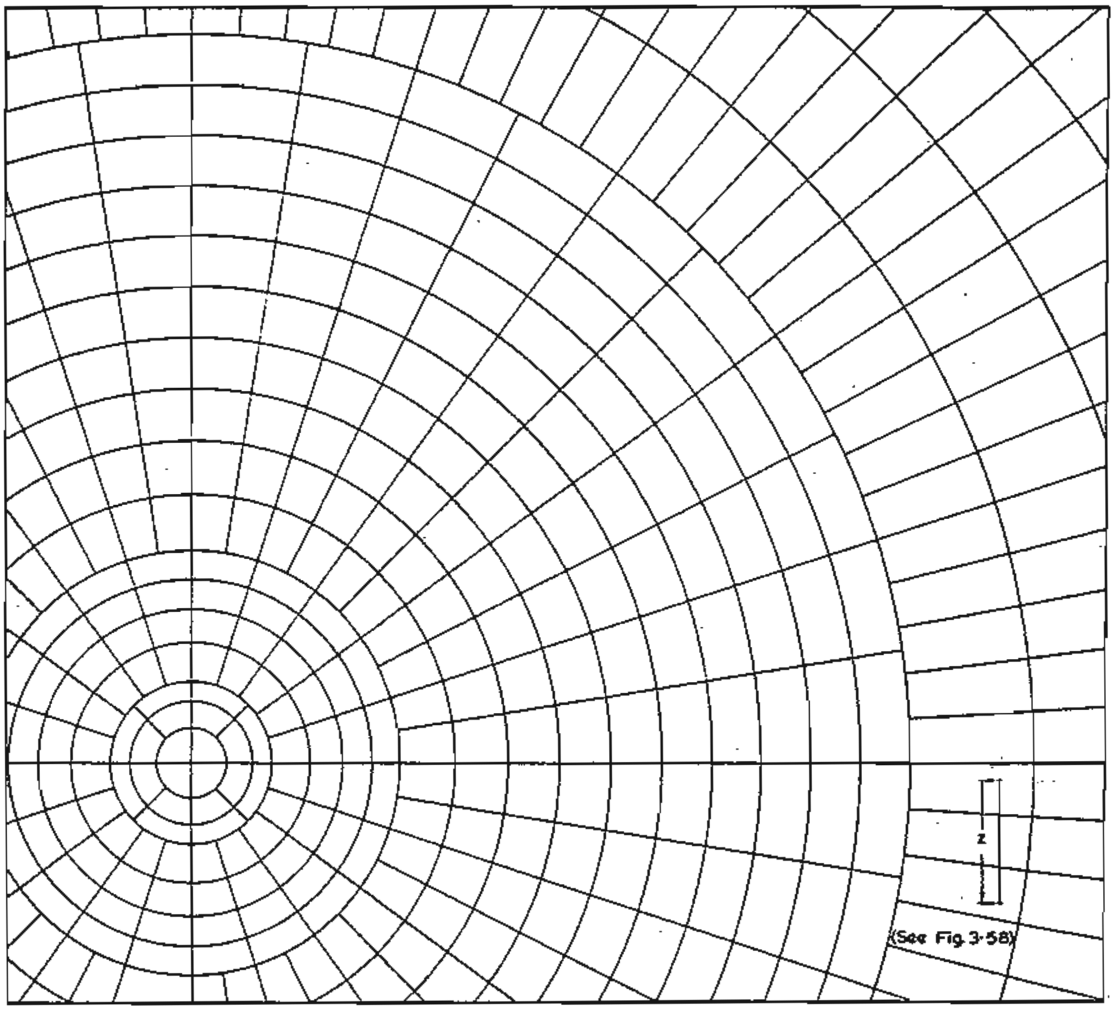


FIG.3.59 Influence chart for Poisson's ratio correction for vertical displacement at depth z below surface. (Newmark, 1947).

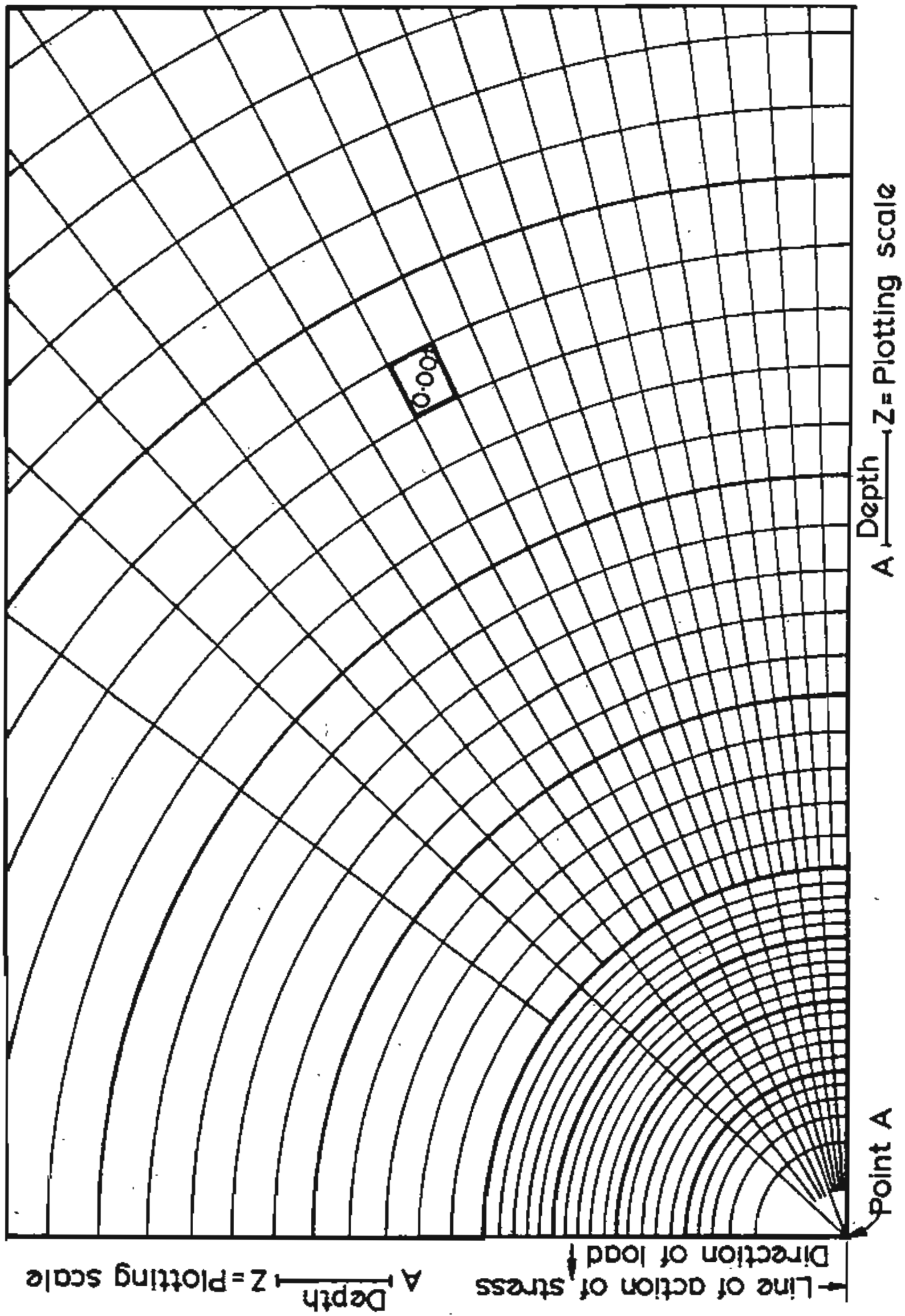


FIG.3.60 Influence chart for horizontal stress, parallel to an applied shear load.
ν=0.5. (Barber, 1964).
σ_h = .001Nq
where N=no.of blocks.

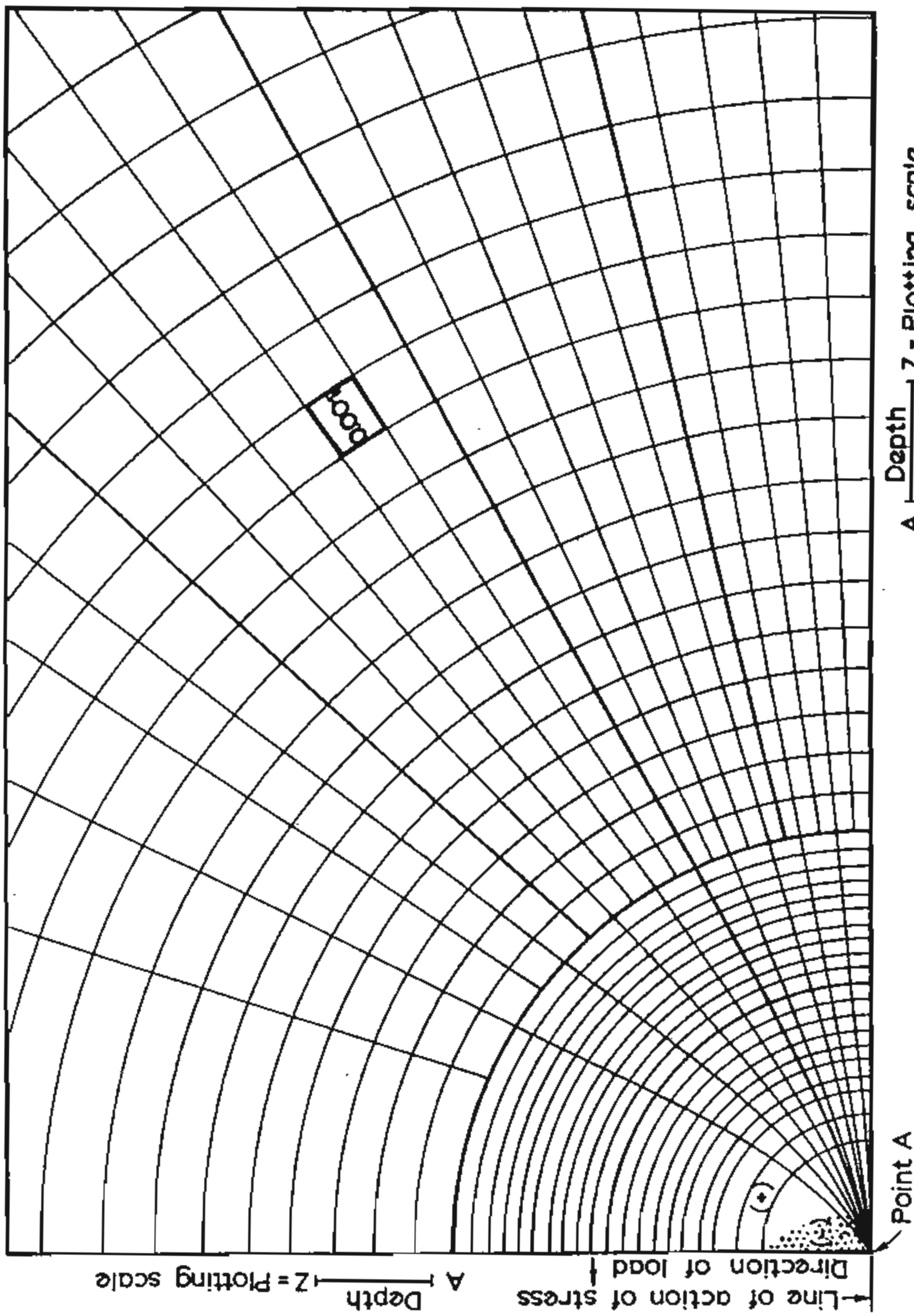


FIG.3.61 Influence chart for horizontal stress, parallel to an applied shear load. ν=0. (Barber, 1965).

σ_h = .001N q

where N=no.of blocks.

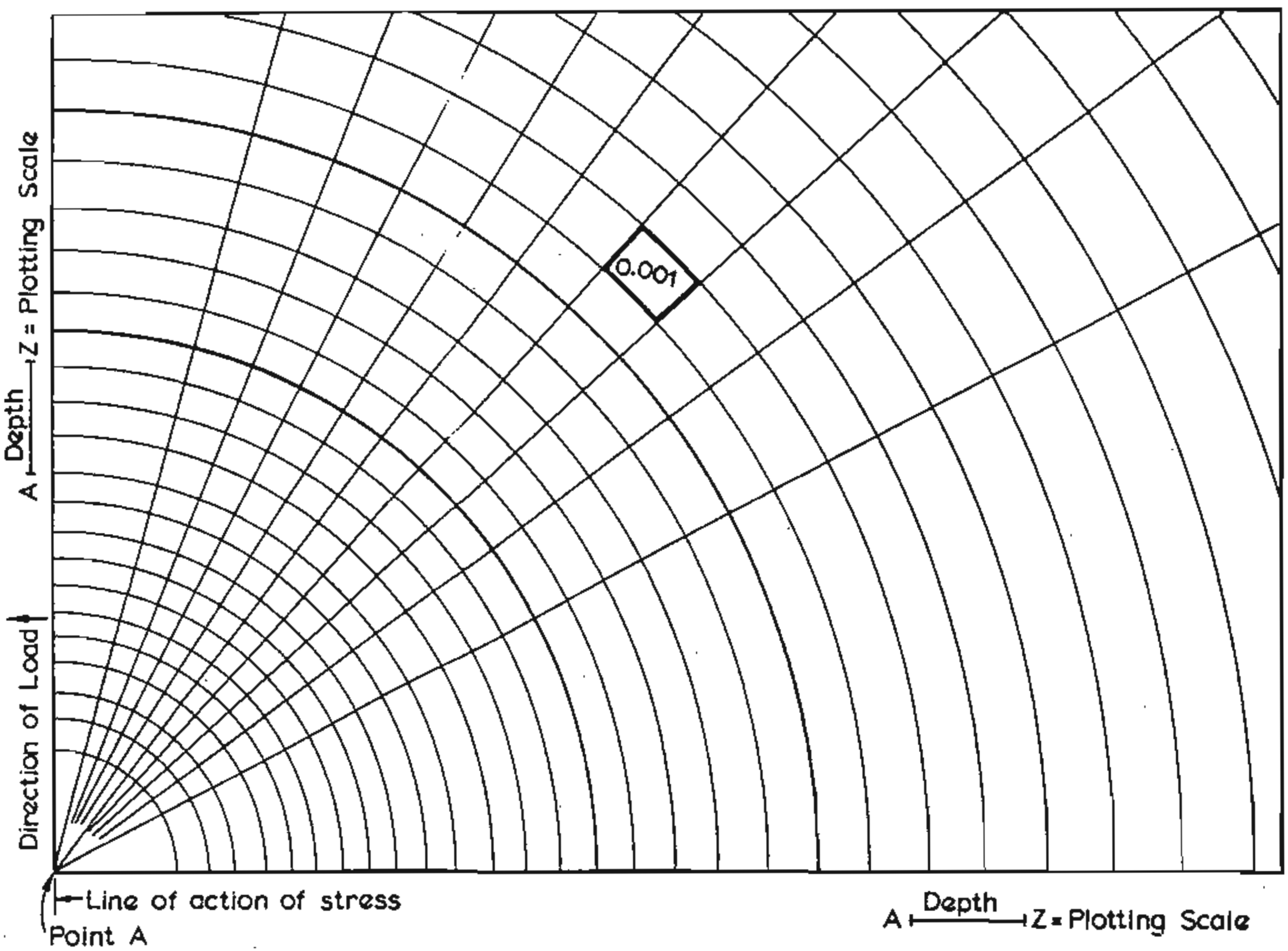


FIG.3.62 Influence chart for horizontal stress, perpendicular to an applied shear load. ν=0.5. (Barber, 1965).
σ_h = .001Nq
where N=no.of blocks.

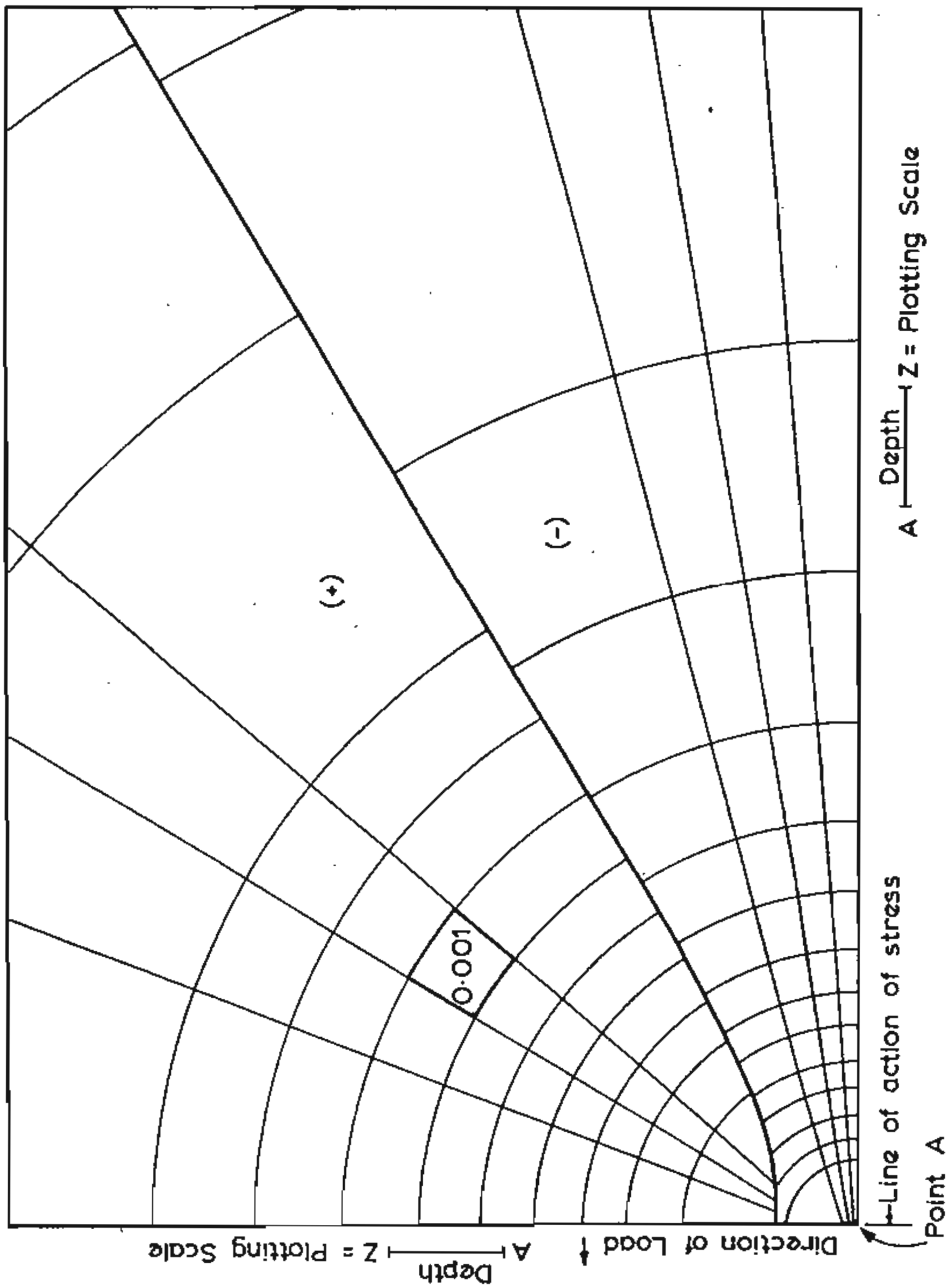


FIG.3.63 Influence chart for horizontal stress, perpendicular to an applied shear load. ν=0. (Barber, 1965).

σ_h = .001Nq
where N=no.of blocks.

Chapter 4
DISTRIBUTED LOADING BENEATH THE SURFACE OF A SEMI-INFINITE MASS

4.1 Vertical Loading on a Horizontal Area

4.1.1 RECTANGULAR AREA

The Mindlin point load equation (Section 2.1.4) for vertical stress σ_z has been integrated over a rectangular area by Skopek (1961). The following expression has been obtained for σ_z beneath the corner of a rectangle $a \times b$ (Fig.4.1):

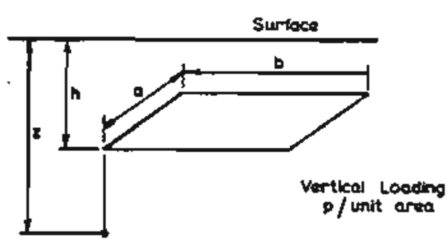


FIG.4.1

$$\sigma_z = \frac{p}{4\pi(1-\nu)}\Bigg[(1-\nu)\left(\arctan\frac{ab}{(z-h)R_1} + \arctan\frac{ab}{(z+h)R_2}\right) + \frac{(z-h)aR_1}{2br_1^2} - \frac{a(z-h)^3}{2br_3^2\,R_1}$$

$$+ \frac{[(3-4\nu)z(z+h)-h(5z-h)\}aR_2}{2(z+h)br_2^2}$$

$$- \frac{[(3-4\nu)z(z+h)^2-h(z+h)(5z-h)]a}{2br_4^2\,R_2}$$

$$+ \frac{2hz(z+h)aR_2^3}{b^3r_2^4} + \frac{3hzaR_2r_5^2}{(z+h)b^3r_2^2} - \frac{hz(z+h)^3a}{br_4^4\,R_2}$$

$$\cdot\left(\frac{2b^2-(z+h)^2}{b^2} - \frac{a^2}{R_2^2}\right)\Bigg] \qquad \ldots (4.1)$$

where $R_1^2 = a^2+b^2+(z-h)^2$ $\quad R_2^2 = a^2+b^2+(z+h)^2$

$r_1^2 = a^2+(z-h)^2$ $\quad r_2^2 = a^2+(z+h)^2$

$r_3^2 = b^2+(z-h)^2$ $\quad r_4^2 = b^2+(z+h)^2$

$r_5^2 = b^2-(z+h)^2$.

The stresses at other points within the mass may be obtained by use of the principle of superposition.

Influence factors for the vertical displacement of the corner of a rectangle are shown in Fig.4.2 (Groth and Chapman, 1969). The displacement is given by

$$\rho = \frac{q\,a\,I}{E} \qquad \ldots (4.2)$$

where a is the shorter side of the rectangle.

The influence factor I is given by

$$I = K_0\Bigg[K_1\left\{\beta\,\ln\left(\frac{1+\sqrt{1+\beta^2}}{\beta}\right) + \ln\left(\beta+\sqrt{1+\beta^2}\right)\right\}$$

$$+ K_2\left\{\ln\left(\frac{\beta+t}{\sqrt{1+4\alpha^2\beta^2}}\right) + \beta\,\ln\left(\frac{1+t}{\beta s}\right)\right.$$

$$\left. - 2\alpha\beta\tan^{-1}\left(\frac{1}{2\alpha\beta}\right) + 4\alpha\beta\tan^{-1}\left(\frac{(1-s)(\beta s-t)}{2\alpha}\right)\right\}$$

$$+ 2\alpha\beta K_1\tan^{-1}\left(\frac{1}{2\alpha t}\right) + \frac{8\alpha^4\beta t}{s^2(1+4\alpha^2t^2)}$$

$$\cdot\left\{2 + \frac{1}{4\alpha^2} - \frac{1}{t^2}\right\}\Bigg] \qquad \ldots (4.2a)$$

where $K_0 = \dfrac{1+\nu}{8\pi(1-\nu)}$

$K_1 = 3-4\nu$

$K_2 = 5-12\nu+8\nu^2$

$\alpha = h/b$

$\beta = b/a$

$$s = \sqrt{1+4\alpha^2}$$
$$t = \sqrt{1+\beta^2(1+4\alpha^2)}$$

Stresses and displacements beneath a rigid rectangle embedded in a semi-infinite mass are given in Section 7.9.

For the limiting case of a uniformly loaded strip ($b/a=\infty$), Skopek gives the following expression for the vertical stress on the central axis of the strip:

$$\sigma_z = \frac{2}{\pi}\Big[\arctan\frac{a}{z-h} + \arctan\frac{a}{z+h} + \frac{a(z-h)}{2[a^2+(z-h)^2](1-\nu)} + \frac{a[(3-4\nu)z+h]}{2[a^2+(z+h)^2](1-\nu)} + \frac{2hz(z+h)a}{[a^2+(z+h)^2]^2(1-\nu)}\Big] \quad \ldots (4.1a)$$

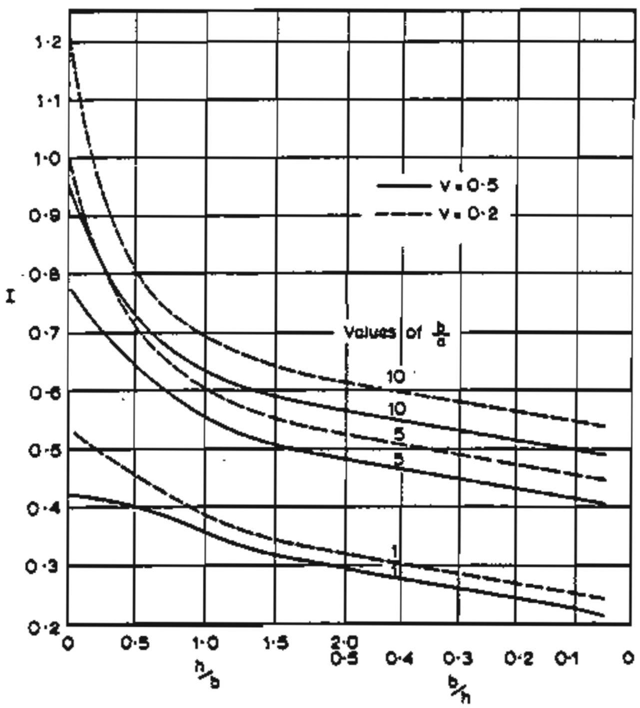


FIG.4.2 Vertical displacement factor for corner of embedded rectangle.

Fox (1948b) has obtained solutions for the relationship between the *mean* vertical displacement ρ_m of a rectangle beneath the surface to the mean displacement ρ_{mo} of a similar rectangle situated at the surface. ρ_m/ρ_{mo} is plotted against $h/\sqrt{ab}$ and $\sqrt{ab}/h$ in Fig.4.3 for various values of b/a and for $\nu=0.5$.

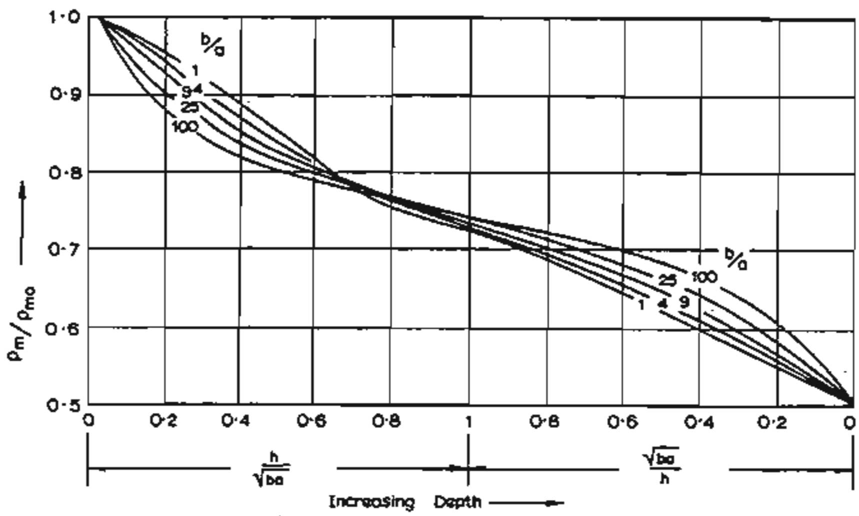


FIG.4.3 Ratio of mean displacement of rectangle at depth h to that of rectangle at surface. $\nu=0.5$. (Fox, 1948).

4.1.2 CIRCULAR AREA (Fig.4.4)

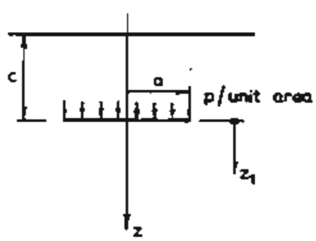


FIG.4.4

Nishida (1966) has derived the following expressions for vertical stress σ_z and vertical displacement ρ_z:

(a) *Beneath the centre*

$$\sigma_{zo} = \frac{p}{4(1-\nu)}\left[(1-2\nu)(z-c)\left\{\frac{1}{\sqrt{a^2+(z+c)^2}} - \frac{1}{\sqrt{a^2+(z-c)^2}} + \frac{1}{z-c} - \frac{1}{z+c}\right\}\right.$$

$$- \left\{\left(\frac{z+c}{\sqrt{a^2+(z+c)^2}}\right)^3 - 1\right\} \cdot \frac{3(3-4\nu)z(z+c)^2-3c(z+c)(5z-c)}{3(z+c)^3}$$

$$+ 1 - \left(\frac{z-c}{\sqrt{a^2+(z-c)^2}}\right)^3 + \frac{6cz}{(z+c)^2} \cdot \left.\left\{1-\left(\frac{z+c}{\sqrt{a^2+(z-c)^2}}\right)^5\right\}\right] \qquad \ldots (4.3)$$

$$\rho_{zo} = \frac{p(1+\nu)}{4E(1-\nu)}\left[(3-4\nu)\left\{\sqrt{a^2+(z-c)^2}-(z-c)\right\}\right.$$

$$+ (5-12\nu+8\nu^2)\left\{\sqrt{a^2+(z+c)^2}-(z+c)\right\} + (z-c)$$

$$- \frac{(z-c)^2}{\sqrt{a^2+(z-c)^2}} + \frac{(3-4\nu)(z+c)^2-2cz}{z+c}$$

$$- \frac{2cz(z+c)^2}{\left(\sqrt{a^2+(z+c)^2}\right)^3} + \frac{2cz}{(z+c)}$$

$$\left. - \frac{(3-4\nu)(z+c)^2-2cz}{\sqrt{a^2+(z+c)^2}}\right] \qquad \ldots (4.4)$$

i.e., $$\rho_{zo} = \frac{pa\,I_o}{E} \qquad \ldots (4.4a)$$

(b) *Beneath the edge* $(r=a)$

See Nishida (1966) for explicit expressions.

ρ_z beneath the edge is given by

$$\rho_{ze} = \frac{pa\, I_e}{E} \qquad \ldots (4.5)$$

Values of σ_z/p beneath the centre and edge of the circle given by Nishida are tabulated in Table 4.1. Influence factors I_o and I_e for the vertical displacement beneath the centre and edge of the circle are tabulated in Table 4.2.

TABLE 4.1

VERTICAL STRESS σ_z BENEATH CIRCULAR AREA (Nishida, 1966)

		σ_{zo}/p (centre)					σ_{ze}/p (edge)				
ν	z_1/a \ c/a	0	1	2	3	∞	0	1	2	3	∞
0.00	0	1.00	0.70	0.56	0.54	0.50	0.50	0.33	0.30	0.28	0.25
	1	0.64	0.35	0.30	0.27	0.25	0.34	0.21	0.18	0.17	0.13
	2	0.28	0.17	0.13	0.12	0.10	0.20	0.12	0.10	0.09	0.07
	4	0.09	0.06	0.05	0.04	0.03	0.12	0.05	0.04	0.03	0.01
0.25	0	1.00	0.71	0.57	0.53	0.50	0.50	0.38	0.31	0.28	0.25
	1	0.64	0.46	0.39	0.29	0.26	0.34	0.24	0.18	0.15	0.13
	2	0.28	0.18	0.15	0.13	0.11	0.20	0.13	0.11	0.09	0.08
	4	0.09	0.07	0.06	0.04	0.03	0.12	0.06	0.05	0.03	0.02
0.50	0	1.00	0.75	0.58	0.54	0.50	0.50	0.40	0.32	0.28	0.25
	1	0.64	0.45	0.38	0.35	0.34	0.34	0.29	0.21	0.19	0.16
	2	0.28	0.22	0.18	0.15	0.14	0.20	0.17	0.13	0.11	0.10
	4	0.09	0.08	0.07	0.04	0.04	0.12	0.07	0.06	0.05	0.04

The centre displacement may also be obtained from

$$\rho_{zo} = F_R \cdot (\rho_z)_s \qquad \ldots (4.6)$$

where $(\rho_z)_s$ is the surface displacement given in Section 3.3.1,

and F_R is a reduction factor, plotted in Fig.4.7. For this case, $r_s = a$.

TABLE 4.2

INFLUENCE FACTORS FOR VERTICAL DISPLACEMENT OF CIRCLE (Nishida, 1966)

	I_o (centre)			I_e (edge)		
c/a \ ν	0.50	0.25	0.00	0.50	0.25	0.00
0.00	1.500	1.875	2.000	0.955	1.194	1.273
3.50	0.908	0.995	0.909			
5.00	0.862	0.947	0.862	0.586	0.640	0.585
100	0.750	0.833	0.750	0.478	0.530	0.478
1000	0.750	0.833	0.750	0.478	0.530	0.478

$$\rho_z = \frac{pa}{E} \cdot I$$

4.1.3 GENERAL AREAS

Sector curves for bulk stress Θ are shown in Figs. 4.5 and 4.6 for $\nu=0$ and 0.5 (Poulos, 1967a).

Sector curves for the ratio F_R of the vertical displacement of a sector at depth d below the surface to the vertical displacement of the same sector situated at the surface, are shown in Fig.4.7 (Poulos, 1967a).

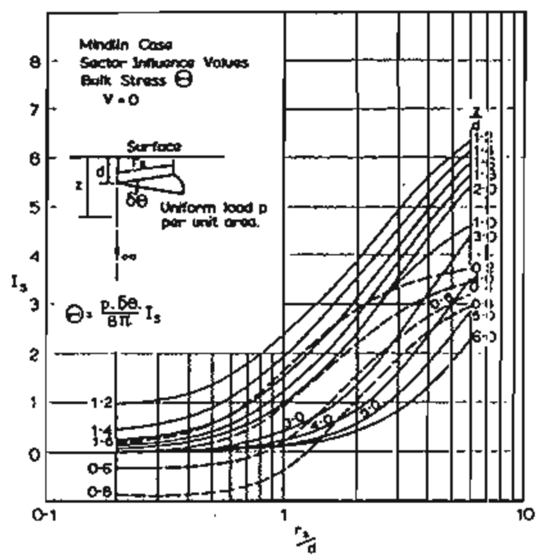


FIG.4.5 Sector curves for bulk stress Θ. $\nu=0$.

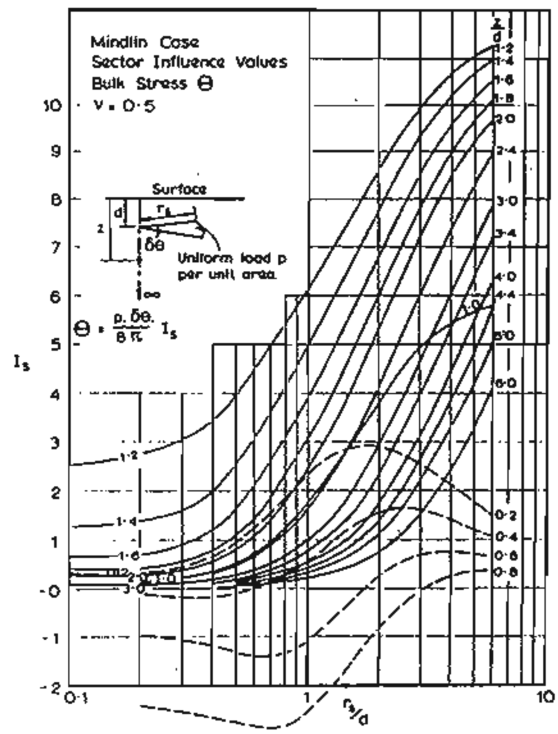


FIG.4.6 Sector curves for bulk stress Θ. $\nu=0.5$.

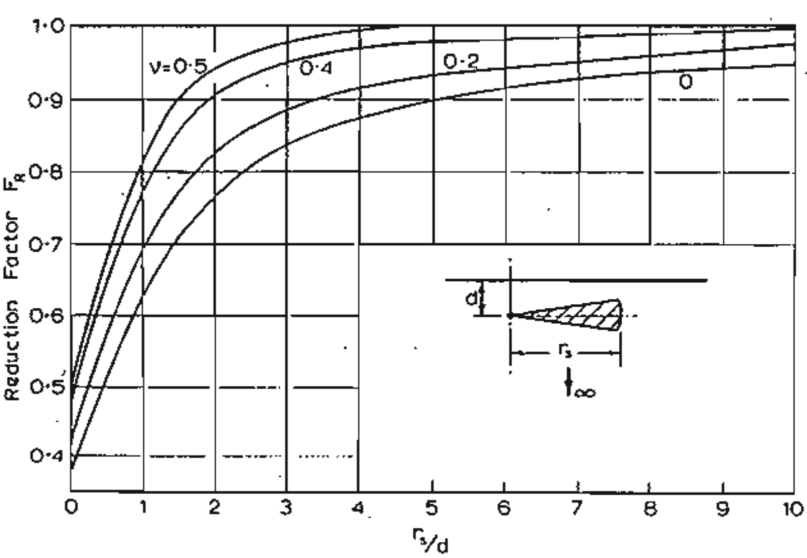


FIG.4.7 Ratio F_R of displacement at apex of sector at depth d to that of sector at surface.

4.2 Horizontal Loading on a Vertical Rectangle

The horizontal displacement ρ_x at the upper and lower corner of a rectangular area (Fig.4.8) has been obtained by Douglas and Davis (1964).

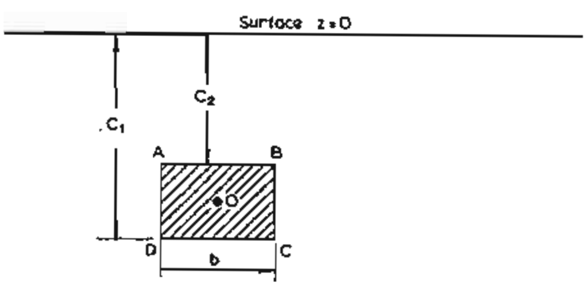


FIG.4.8

At the upper corners A and B, for a uniform horizontal pressure p,

$$\rho_x = \frac{pb}{32\pi G(1-\nu)}\{(3-4\nu)F_1+F_4+4(1-2\nu)(1-\nu)F_5\} \quad \ldots (4.7)$$

At the lower corners D and C,

$$\rho_x = \frac{pb}{32\pi G(1-\nu)}\{(3-4\nu)F_1+F_2+4(1-2\nu)(1-\nu)F_3\} \quad \ldots (4.8)$$

where $K_1 = \dfrac{2c_1}{b}$ $\quad K_2 = \dfrac{2c_2}{b}$.

$$F_1 = -(K_1-K_2)\ \ln\left(\frac{(K_1-K_2)}{2+\sqrt{4+(K_1-K_2)^2}}\right) - 2\ \ln\left(\frac{2}{(K_1-K_2)+\sqrt{4+(K_1-K_2)^2}}\right)$$

$$F_2 = 2\ \ln\left(\frac{2(K_1+\sqrt{1+K_1^2})}{(K_1+K_2)+\sqrt{4+(K_1+K_2)^2}}\right) + (K_1-K_2) \times \ln\left(\frac{2+\sqrt{4+(K_1+K_2)^2}}{(K_1+K_2)}\right) - K_1^2\left(\frac{\sqrt{4+(K_1+K_2)^2}}{(K_1+K_2)} - \frac{\sqrt{1+K_1^2}}{K_1}\right)$$

$$F_3 = -2K_1\ \ln\left(\frac{K_1}{1+\sqrt{1+K_1^2}}\right) + (K_1+K_2) \times \ln\left(\frac{(K_1+K_2)}{2+\sqrt{4+(K_1+K_2)^2}}\right) - \ln\left(\frac{(K_1+K_2)+\sqrt{4+(K_1+K_2)^2}}{2(K_1+\sqrt{1+K_1^2})}\right) + \frac{(K_1+K_2)}{4} \times [\sqrt{4+(K_1+K_2)^2} - (K_1+K_2)] - K_1(\sqrt{1+K_1^2} - K_1)$$

$$F_4 = -2\ \ln\left(\frac{2(K_2+\sqrt{1+K_2^2})}{(K_1+K_2)+\sqrt{4+(K_1+K_2)^2}}\right) + (K_1-K_2)\ \ln\left(\frac{2+\sqrt{4+(K_1+K_2)^2}}{(K_1+K_2)}\right) + K_2^2\left(\frac{\sqrt{4+(K_1+K_2)^2}}{(K_1+K_2)} - \frac{\sqrt{1+K_2^2}}{K_2}\right)$$

$$F_5 = 2K_2\ \ln\left(\frac{K_2}{1+\sqrt{1+K_2^2}}\right) - (K_1+K_2) \times \ln\left(\frac{K_1+K_2}{2+\sqrt{4+(K_1+K_2)^2}}\right) + \ln\left(\frac{(K_1+K_2)+\sqrt{4+(K_1+K_2)^2}}{2(K_2+\sqrt{1+K_2^2})}\right) - \left(\frac{K_1+K_2}{4}\right) \times [\sqrt{4+(K_1+K_2)^2} - (K_1+K_2)] - K_2(K_2-\sqrt{1+K_2^2})$$

For the displacement at other points in the same plane, the principle of superposition may be employed.

Values of F_1 to F_5 are plotted in Figures 4.9 to 4.11.

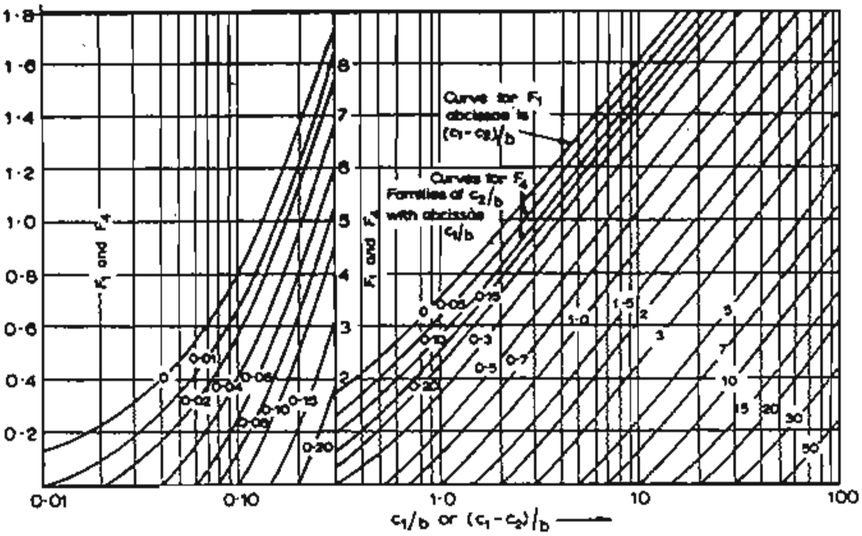


FIG.4.9 Factors F_1 and F_4 (Douglas and Davis, 1964).

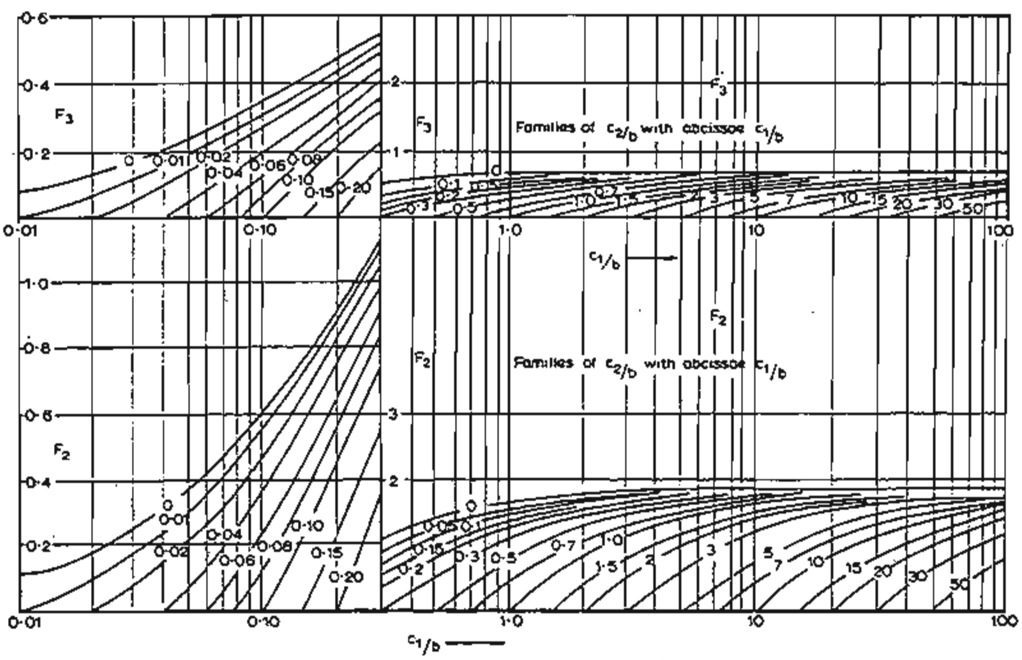


FIG.4.10 Factors F_2 and F_3 (Douglas and Davis, 1964).

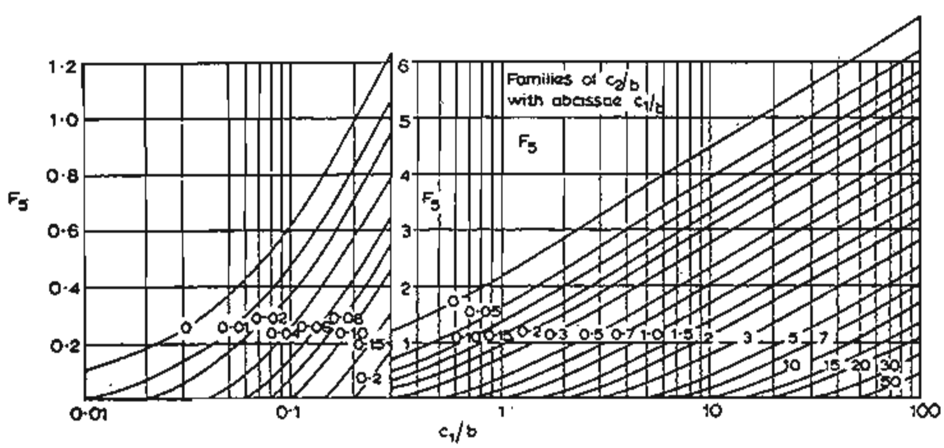


FIG.4.11 Factor F_5 (Douglas and Davis, 1964).

4.3 Rectangles Subjected to Shear Loading

Three cases have been considered by Groth and Chapman (1969), as shown in Fig.4.12, of the corner displacements in the direction of uniform loading applied to subsurface rectangles. Influence factors for the displacements are shown in Figs. 4.13 to 4.17. In all cases, the displacement is expressed as

$$\rho = \frac{q\,a\,I}{E} \qquad \ldots (4.9)$$

Case 1

Top Corner A:

$$I = K_0\Big\{(K_1+1)\,\ell n\big(\beta+\sqrt{1+\beta^2}\big) + \beta K_1\,\ell n\Big(\frac{1+\sqrt{1+\beta^2}}{\beta}\Big)$$

$$+ (K_1+K_2)\,\ell n\Big(\frac{s+\beta(1+2\alpha)}{t+2\alpha\beta}\Big)$$

$$+ \beta(1+2\alpha)K_2\,\ell n\Big(\frac{s+1}{\beta(1+2\alpha)}\Big)$$

$$- 2\alpha\beta K_2\,\ell n\Big(\frac{t+1}{2\alpha\beta}\Big) + \alpha\beta\,\ell n\Big(\frac{t+1}{t-1}\Big)$$

$$- \alpha\beta\,\ell n\Big(\frac{s+1}{s-1}\Big) + 2\alpha\beta\Big\{\frac{1+2\alpha^2\beta^2}{t}$$

$$- \frac{1+\alpha\beta^2(1+2\alpha)}{s}\Big\} + \alpha\Big\{\frac{2\alpha\beta s}{1+2\alpha} - \beta t\Big\}\Big\}$$

where
$$\alpha = \frac{h}{b}$$
$$\beta = \frac{b}{a}$$
$$s = \sqrt{1+\beta^2(1+2\alpha)^2}$$
$$t = \sqrt{1+4\alpha^2\beta^2}$$
$$K_0 = \frac{1+\nu}{8\pi(1-\nu)}$$
$$K_1 = 3-4\nu$$

Bottom Corner B:

$$I = K_0\Big\{\beta K_1\,\ell n\Big(\frac{1+\sqrt{1+\beta^2}}{\beta}\Big) - (K_1+1)\,\ell n\big(\sqrt{1+\beta^2}-\beta\big)$$

$$+(K_1+K_2)\,\ell n\Big(\frac{2\beta(1+\alpha)+s}{\beta(1+2\alpha)+t}\Big)$$

$$+ 2K_2\beta(1+\alpha)\,\ell n\Big(\frac{1+s}{2\beta(1+\alpha)}\Big)$$

$$- K_2\beta(1+2\alpha)\,\ell n\Big(\frac{1+t}{\beta(1+2\alpha)}\Big)$$

$$+ \beta(1+\alpha)\Big[\,\ell n\Big(\frac{t+1}{t-1}\Big) - \ell n\Big(\frac{s+1}{s-1}\Big) + s$$

$$- \frac{2(1+\alpha)t}{1+2\alpha} - \frac{2}{s}\big(1+2\beta^2(1+\alpha)^2\big)$$

$$+ \frac{2}{t}\big(1+\beta^2(1+\alpha)(1+2\alpha)\big)\Big]\Big\}$$

where α, β, K_0, K_1 as above
$$s = \sqrt{1+4\beta^2(1+\alpha)^2}$$
$$t = \sqrt{1+\beta^2(1+2\alpha)^2}$$

Case 2

Top Corner A:

$$I = K_0\Big\{(K_1+1)\Big[\beta\,\ell n\Big(\frac{1+\sqrt{1+\beta^2}}{\beta}\Big)$$

$$+ \beta(1+2\alpha)\,\ell n\Big(\frac{s+1}{\beta(1+2\alpha)}\Big) - 2\alpha\beta\,\ell n\Big(\frac{t+1}{2\alpha\beta}\Big)\Big]$$

$$+ \ell n\Big(\frac{s+\beta(1+2\alpha)}{t+2\alpha\beta}\Big) + K_1\,\ell n\big(\beta+\sqrt{1+\beta^2}\big)$$

$$- c\Big(\frac{1+\alpha\beta^2(1+2\alpha)}{s} - \frac{1+2\alpha^2\beta^2}{t}\Big)$$

$$+ 2(1-\nu)(1-2\nu)\big[\beta(1+2\alpha)\big(s-\beta(1+2\alpha)\big)$$

$$- 2\alpha\beta(t-2\alpha\beta) + \ell n\Big(\frac{s+\beta(1+2\alpha)}{t+2\alpha\beta}\Big)\big]\Big\}$$

where α, β, K_0, K_1 as above
$$s = \sqrt{1+\beta^2(1+2\alpha)^2}$$
$$t = \sqrt{1+4\alpha^2\beta^2}$$
$$c = 2\alpha\beta$$

Bottom Corner B:

$$I = K_0\Big\{\beta(K_1+1)\Big[\,\ell n\Big(\frac{1+\sqrt{1+\beta^2}}{\beta}\Big)$$

$$+ 2(1+\alpha)\,\ell n\Big(\frac{s+1}{2\beta(1+\alpha)}\Big)$$

$$- (1+2\alpha)\,\ell n\Big(\frac{t+1}{\beta(1+2\alpha)}\Big)\Big]$$

$$- K_1\,\ell n\big(\sqrt{1+\beta^2}-\beta\big) + \ell n\Big(\frac{s+2\beta(1+\alpha)}{t+\beta(1+2\alpha)}\Big)$$

$$-2\beta(1+\alpha)\Big(\frac{1}{s} - \frac{1}{t}\Big) - 2\beta^2(1+\alpha)\,.$$

$$\Big(\frac{2\beta(1+\alpha)}{s} - \frac{\beta(1+2\alpha)}{t}\Big)$$

$$+ 2(1-\nu)(1-2\nu)\big[2\beta(1+\alpha)s - \beta(1+2\alpha)t$$

$$- 4\beta^2(1+\alpha)^2 + \beta^2(1+2\alpha)^2$$

$$+ \ln\left(\frac{s+2\beta(1+\alpha)}{t+\beta(1+2\alpha)}\right)]]$$

where α, β, K_0, K_1 as above

$$s = \sqrt{1+4\beta^2(1+\alpha)^2}$$

$$t = \sqrt{1+\beta^2(1+2\alpha)^2}$$

where α, β, K_0, K_1 as above

$$K_2 = 4(1-\nu)(1-2\nu)$$

$$s = \sqrt{1+\beta^2(1+4\alpha^2)}$$

$$t = \sqrt{1+4\alpha^2\beta^2}$$

$$u = \beta\sqrt{1+4\alpha^2}$$

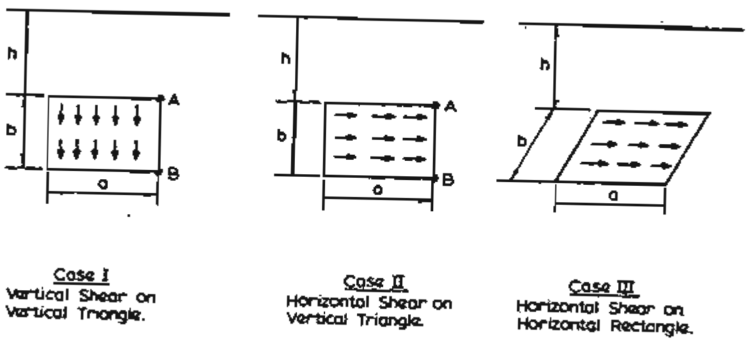


FIG.4.12 Loading cases for horizontal loading on a rectangle.

Case 3

$$I = K_0\{K_1[\ln(\beta+\sqrt{1+\beta^2}) + \tfrac{1}{2}\beta\ln\left(\frac{s+1}{s-1}\right)]$$

$$+ \beta(K_1+1)\ln\left(\frac{1+\sqrt{1+\beta^2}}{\beta}\right)$$

$$+ (K_2+1)\ln\left(\frac{\beta+s}{t}\right) + \beta\ln\left(\frac{1+s}{u}\right)$$

$$+ 2\alpha\beta[2\tan^{-1}\left(\frac{\beta(t-1)}{2\alpha\beta(t+s)}\right) - \tan^{-1}\frac{1}{2\alpha}$$

$$+ \frac{\alpha(s^2-1)}{s(1+4\alpha^2s^2)} - K_1\tan^{-1}\left(\frac{1}{2\alpha s}\right)$$

$$+ K_2\left(2\tan^{-1}\left(\frac{u-\beta}{2\alpha\beta(u+s)}\right) - \tan^{-1}\left(\frac{1}{2\alpha\beta}\right)\right.$$

$$\left. + 2\tan^{-1}\left(\frac{u-2\alpha\beta}{\beta(u+s)}\right)\right)]\}$$

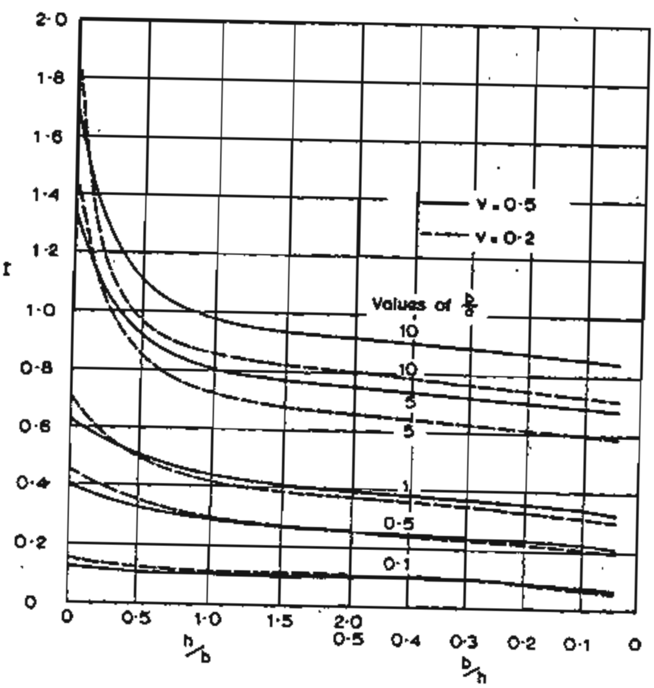


FIG.4.13 Vertical displacement factor for upper corner A. Case I.

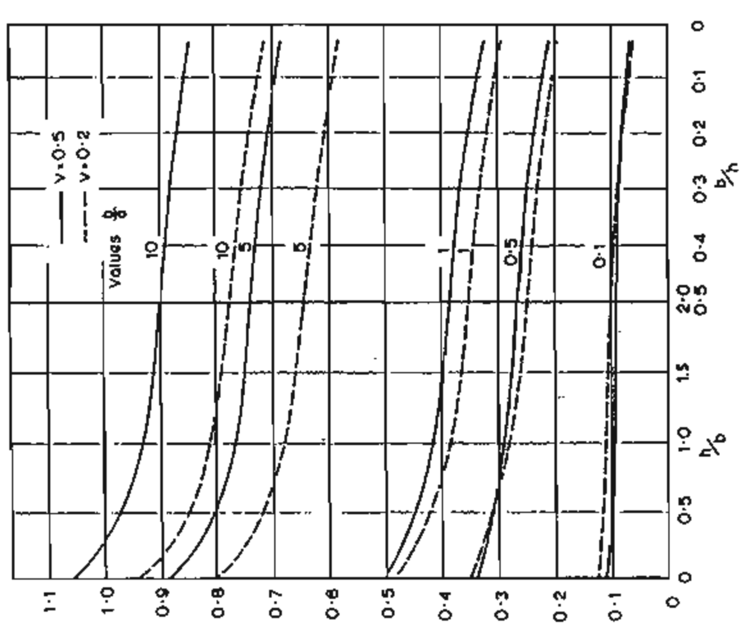


FIG.4.14 Vertical displacement factor for lower corner B. Case I.

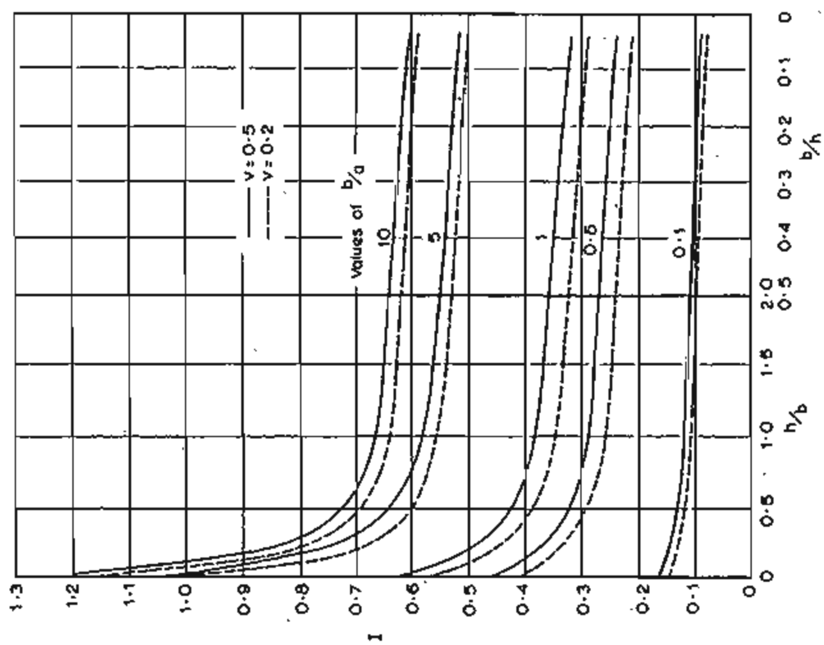


FIG.4.15 Horizontal displacement factor for upper corner A. Case II.

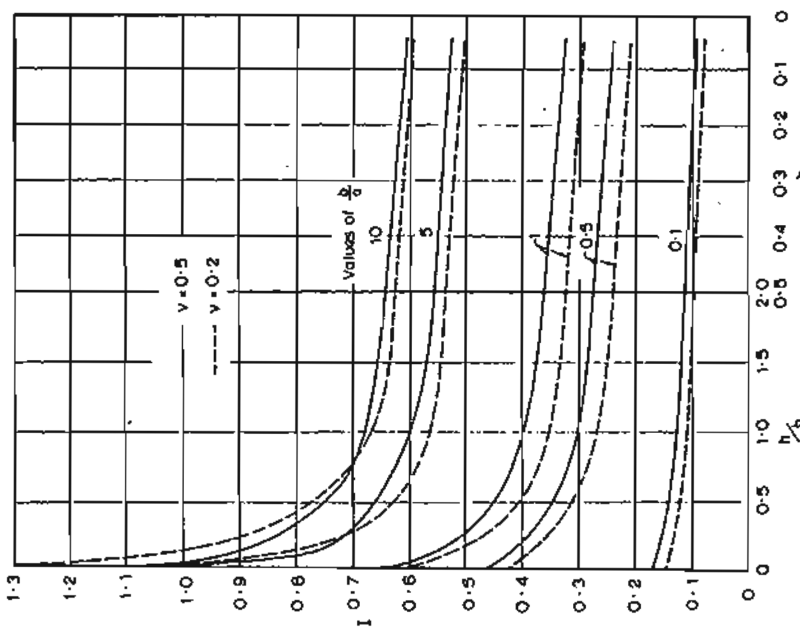


FIG.4.17 Horizontal displacement factor for corners. Case III.

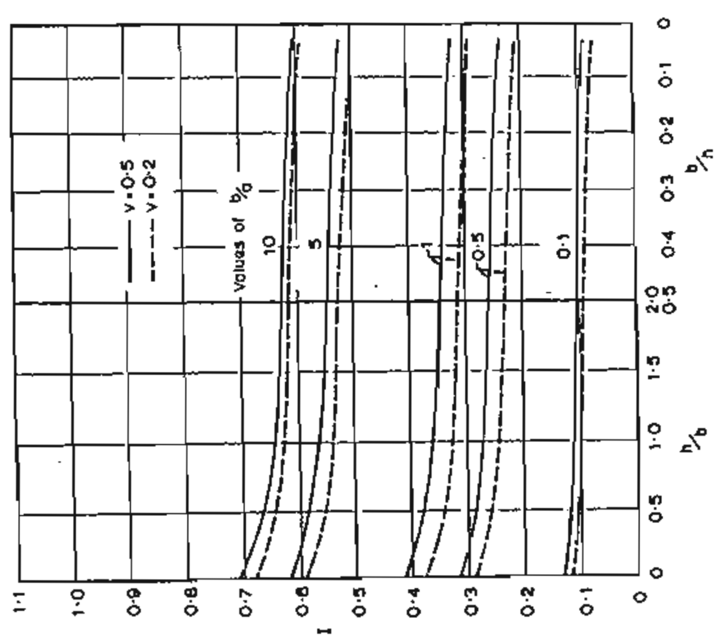


FIG.4.16 Horizontal displacement factor for lower corner B. Case II.

Chapter 5
SURFACE LOADING OF A FINITE LAYER UNDERLAIN BY A RIGID BASE

5.1 Loading on an Infinite Strip

5.1.1 ~~UNIFORM~~ VERTICAL LOADING (Fig.5.1)

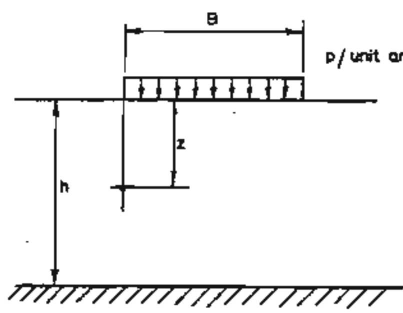


FIG.5.1

Influence factors for the vertical displacement ρ_z and the horizontal displacement ρ_x beneath the edge of the strip, obtained by Poulos (1967b), are shown in Figs. 5.2 and 5.3.

Influence factors for the vertical stress σ_z, bulk stress Θ and shear stress τ_{xz} beneath the edge are shown in Figs. 5.4 and 5.5, for four values of ν. The interface between the layer and the base is rough ("adhesive").

The horizontal stresses σ_x and σ_y may be evaluated as follows:

$$\sigma_x = \frac{\Theta}{1+\nu} - \sigma_z \qquad \ldots (5.1)$$

$$\sigma_y = \nu(\sigma_x + \sigma_z) \qquad \ldots (5.2)$$

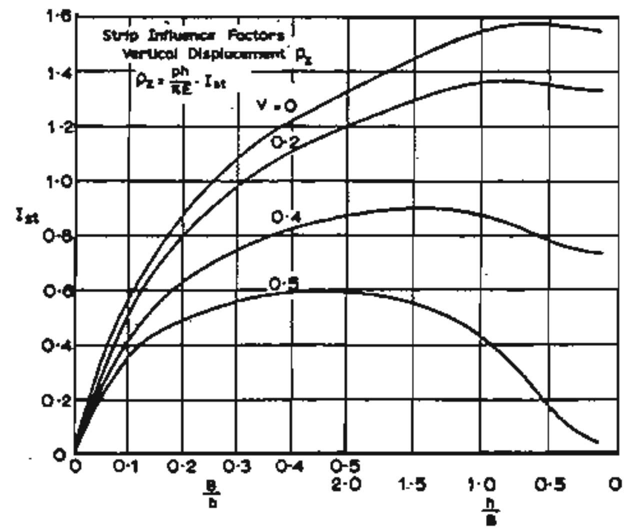


FIG.5.2 Strip curves for ρ_z.

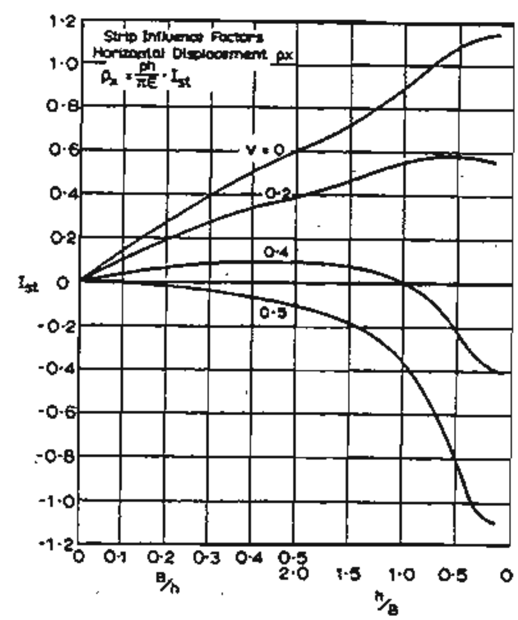


FIG.5.3 Strip curves for ρ_x.

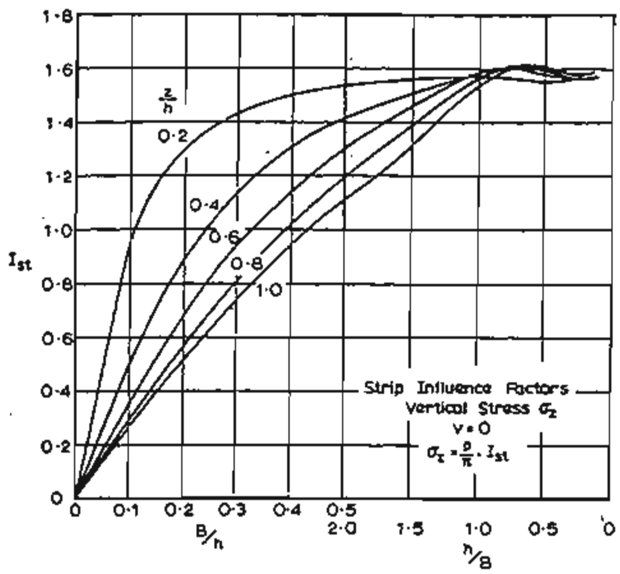


FIG.5.4 Strip curves for σ_z. ν=0.

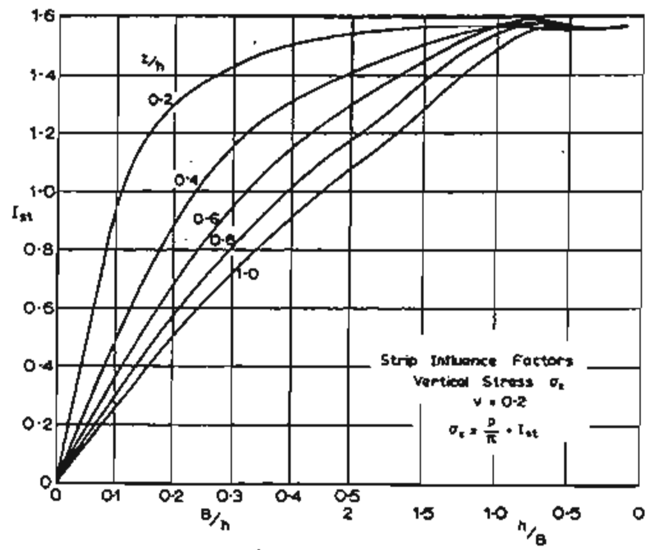


FIG.5.5 Strip curves for σ_z. ν=0.2.

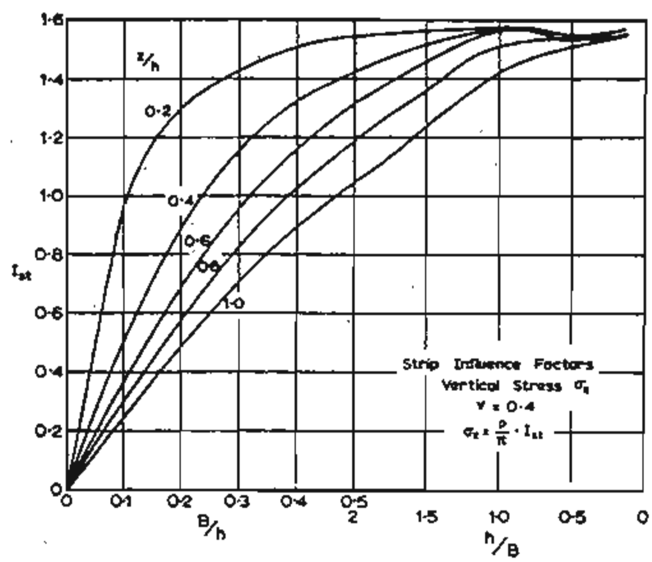


FIG.5.6 Strip curves for σ_z. ν=0.4.

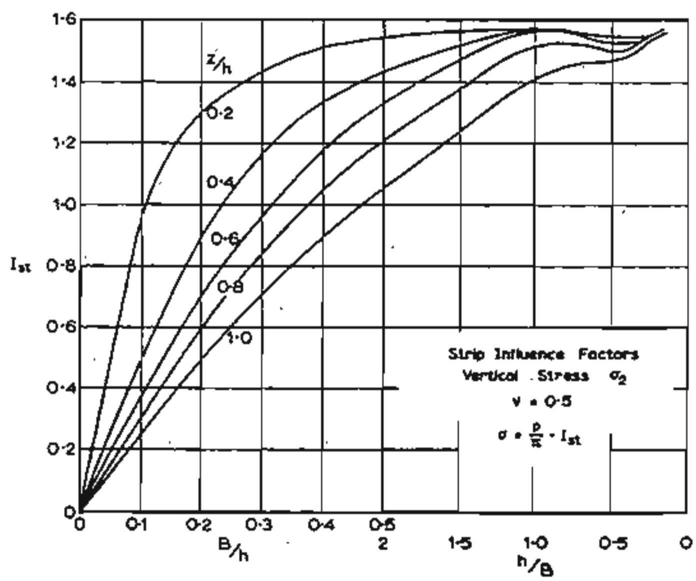


FIG.5.7 Strip curves for σ_z. ν=0.5.

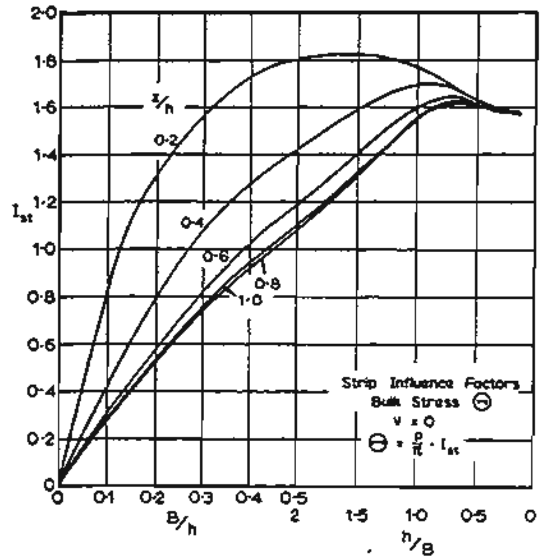


FIG.5.8 Strip curves for θ. ν=0.

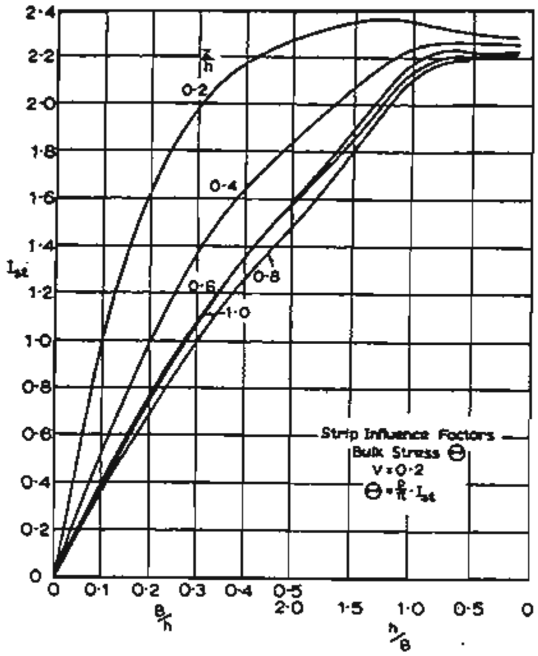


FIG.5.9 Strip curves for θ. ν=0.2

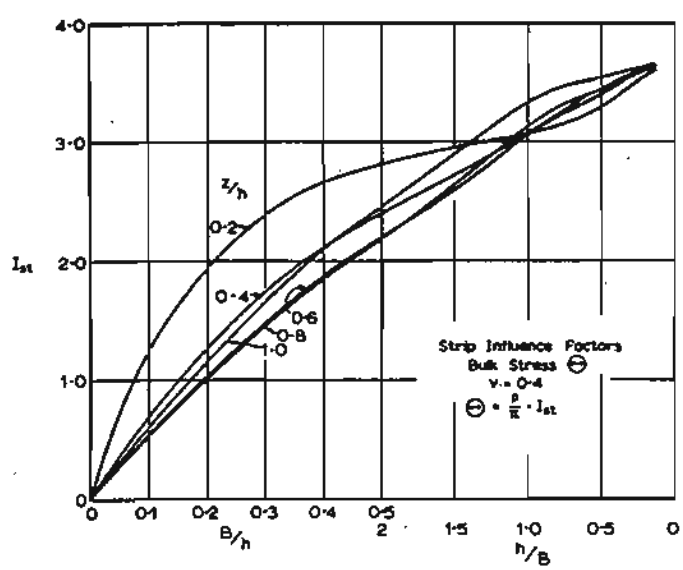


FIG.5.10 Strip curves for θ. ν=0.4.

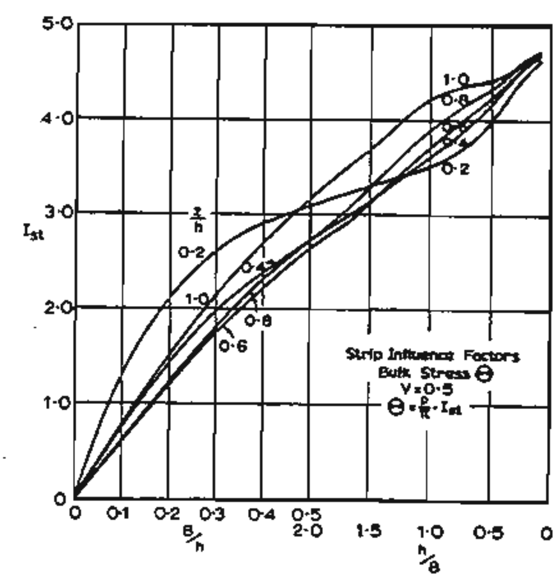


FIG.5.11 Strip curves for θ. ν=0.5.

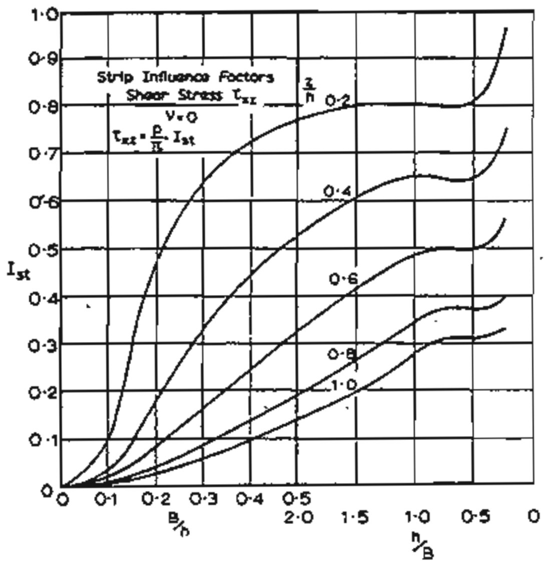


FIG.5.12 Strip curves for τ_{xz}. ν=0.

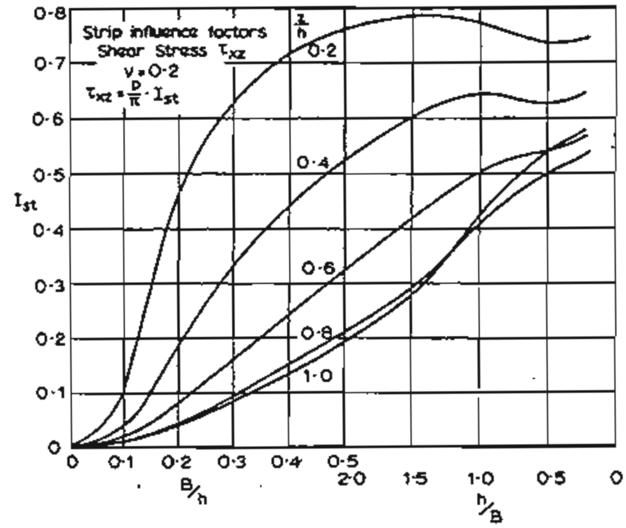


FIG.5.13 Strip curves for τ_{xz}. ν=0.2.

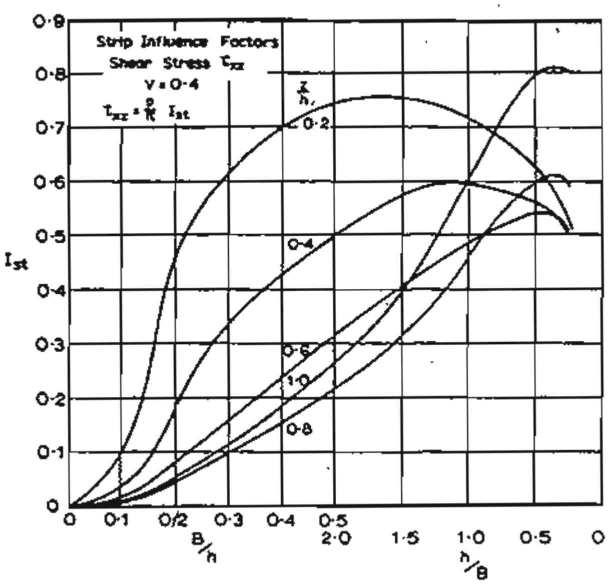


FIG.5.14 Strip curves for τ_{xz}. ν=0.4.

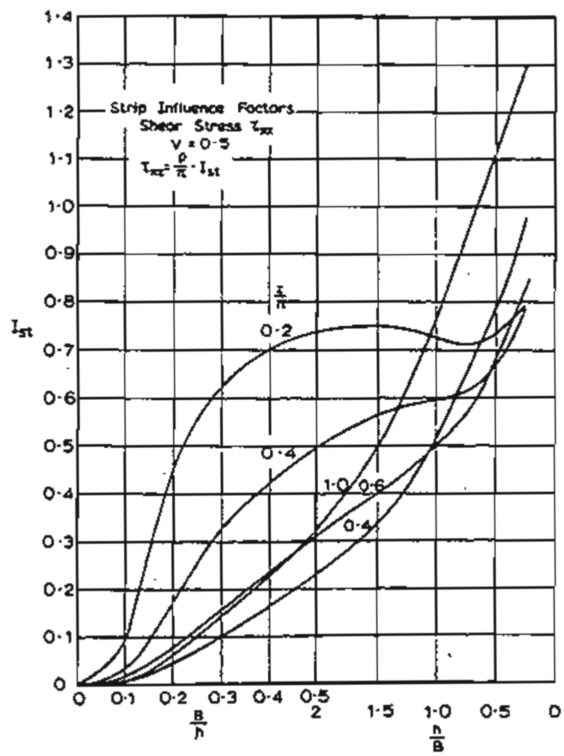


FIG.5.15 Strip curves for τ_{xz}. ν=0.5.

Ueshita and Meyerhof (1968) have also obtained influence factors for ρ_z beneath the edge of the strip, considering both a rough rigid base (adhesive interface) and a smooth rigid base (smooth interface). These influence factors, reproduced in Fig. 5.16, show that the effect of the interface is considerable for $\nu=0.5$ but almost negligible for $\nu=0$.

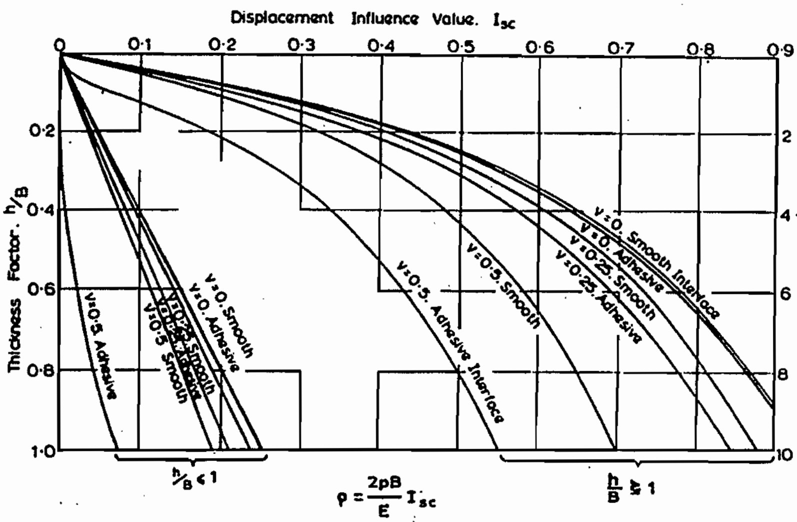


FIG.5.16 Displacement factors for edge of strip (Ueshita and Meyerhof, 1968).

5.1.2 TRIANGULAR VERTICAL LOADING (Fig.5.17)

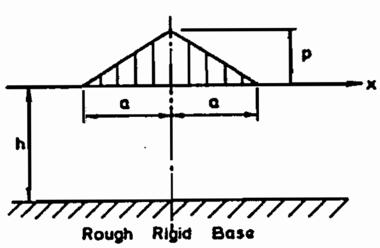


FIG.5.17

Solutions for the stresses within the layer have been obtained by Giroud and Watissee (1972). Some solutions for σ_z, σ_x and τ_{xz} beneath the centre and edge of the loading are shown in Figs.5.18 to 5.20 for $\nu=0.3$.

Giroud (1970) expresses the vertical surface displacement as

$$\rho_z = \frac{pa}{E} \cdot r_H \qquad \ldots (5.3)$$

Values of r_H are plotted in Figs. 5.21 to 5.25 for five values of ν.

The solutions for triangular loading may be superposed to determine solutions for "embankment" or trapezoidal loading.

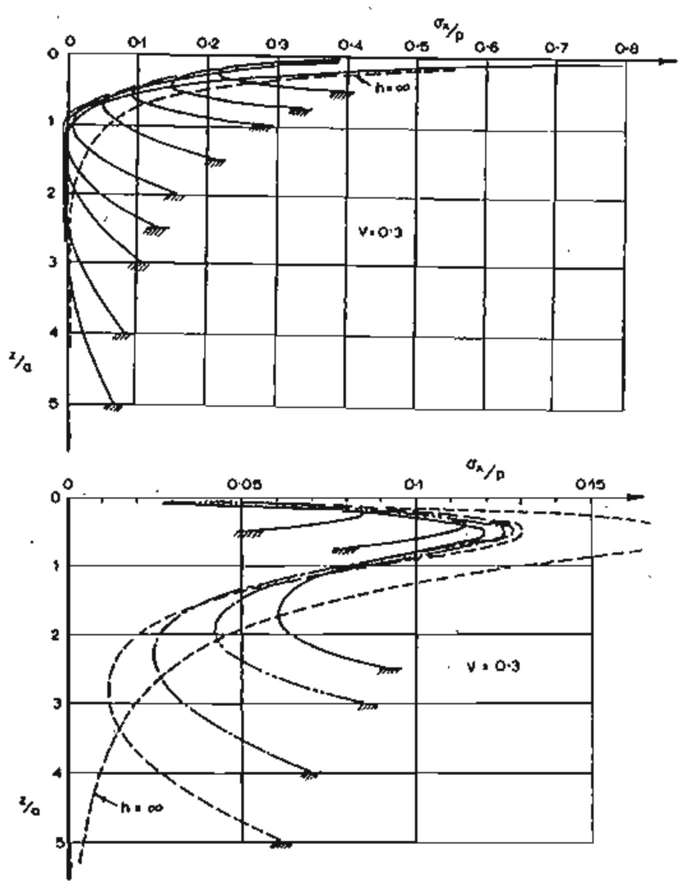


FIG.5.18 Horizontal stress σ_x
(a) beneath centre
(b) beneath edge
(Giroud and Watissee, 1972).

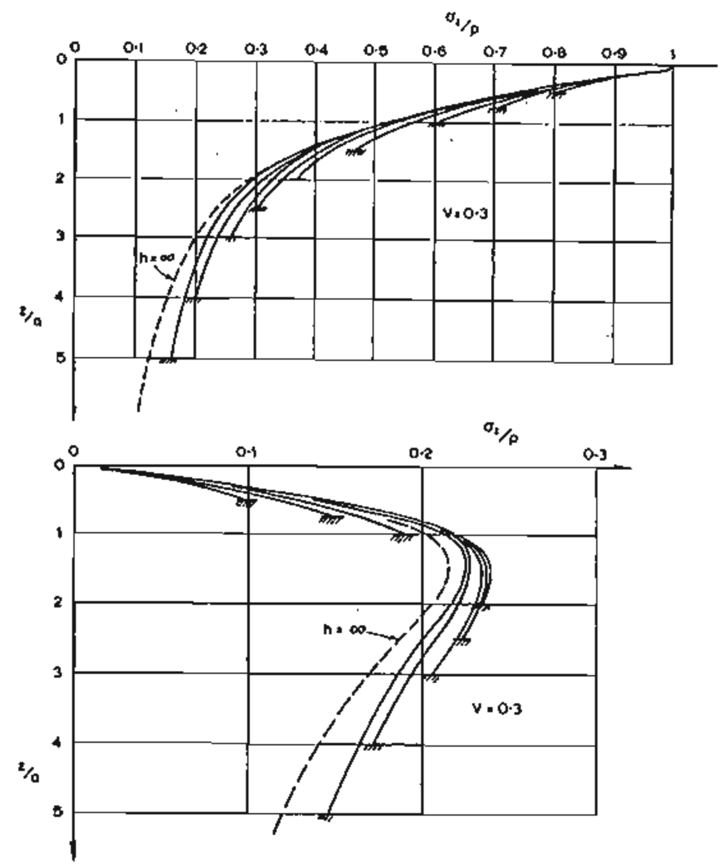


FIG.5.19 Vertical stress σ_z
(a) beneath centre
(b) beneath edge
(Giroud and Watissee, 1972).

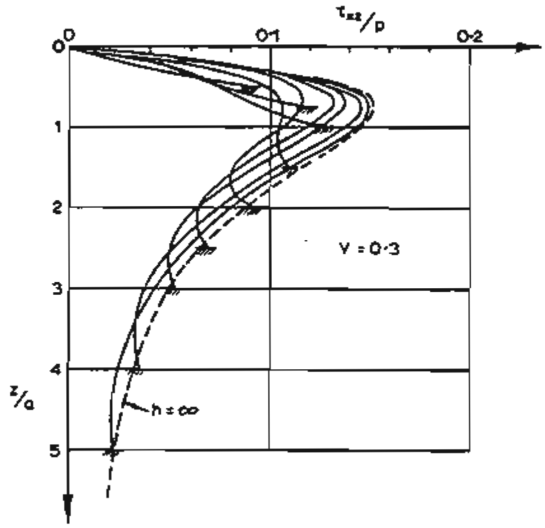


FIG.5.20 Shear stress τ_{xz} beneath edge (Giroud and Watissee, 1972).

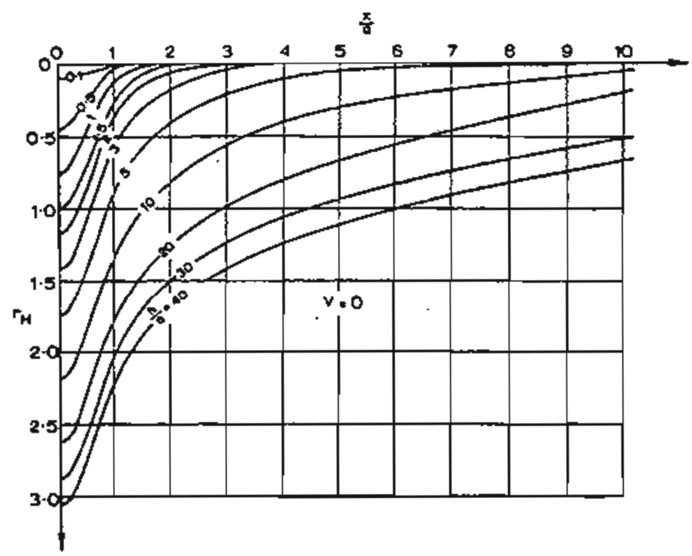


FIG.5.21 Displacement factor r_H. ν=0. Giroud, 1970).

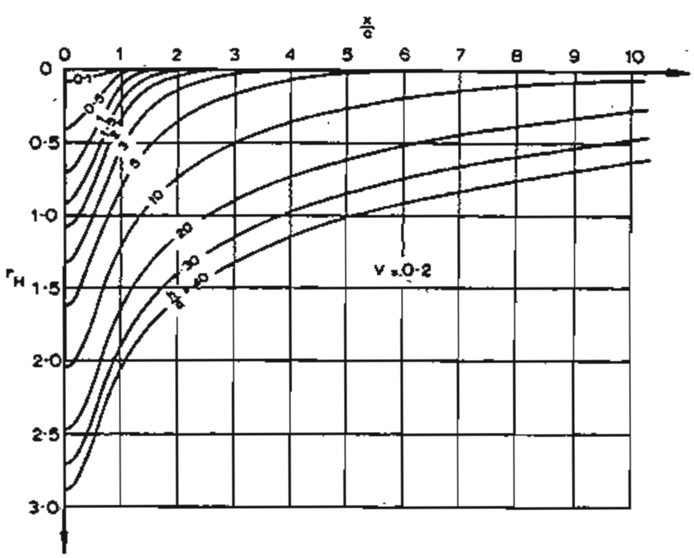


FIG.5.22 Displacement factor r_H. ν=0.2. (Giroud, 1970).

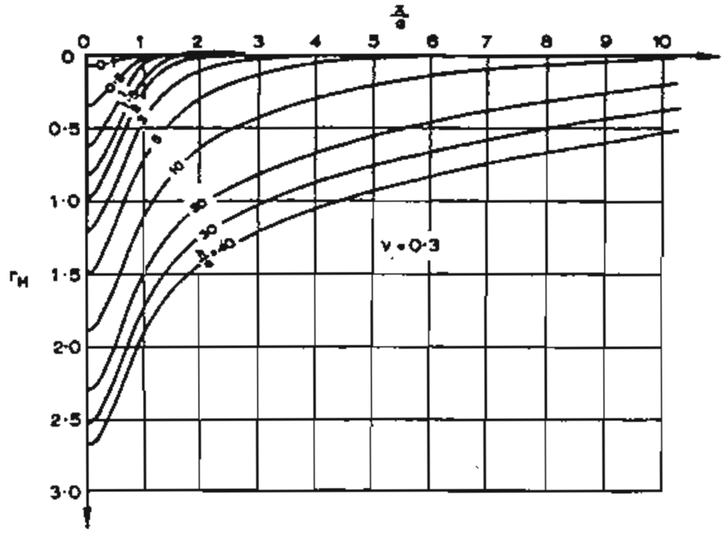


FIG.5.23 Displacement factor r_H. ν=0.3. (Giroud, 1970).

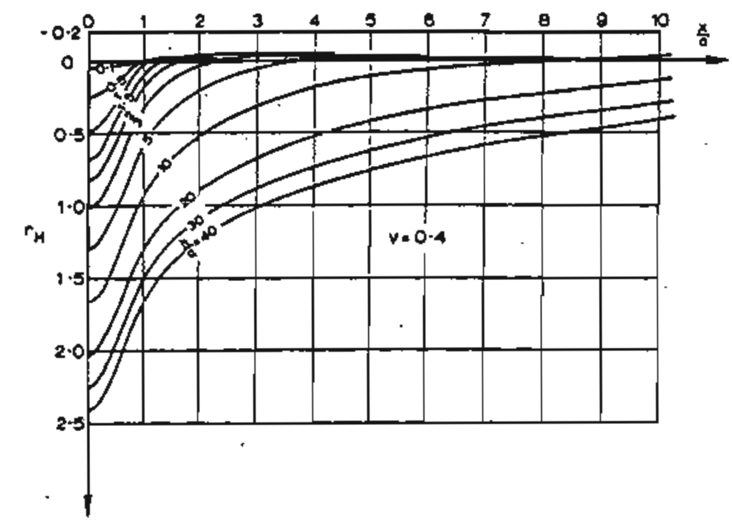


FIG.5.24 Displacement factor r_H. ν=0.4. (Giroud, 1970).

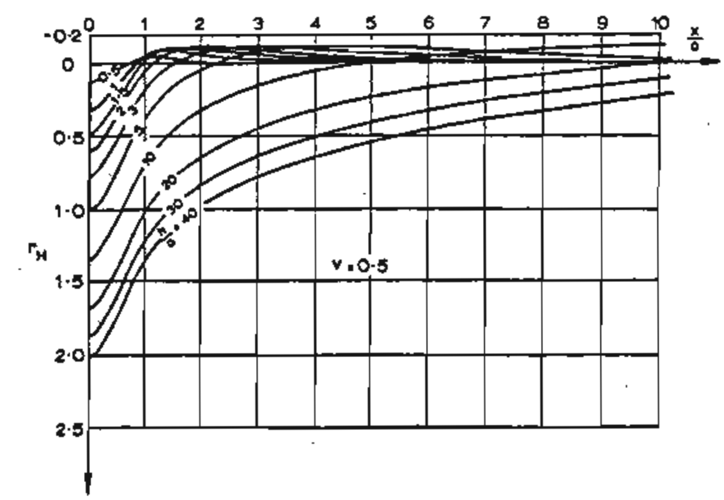


FIG.5.25 Displacement factor r_H. ν=0.5. (Giroud, 1970).

5.2 Loading on a Circular Area (Fig. 5.26)

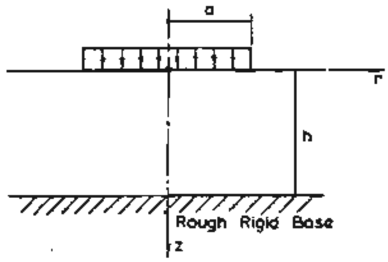


FIG.5.26

For four values of h/a and three values of ν *(0.15, 0.30* and *0.45)* Milovic (1970) has tabulated solutions for the stresses and displacements beneath the the centre *(r/a = 0)*, *(r/a = 0.5)*, and the edge *(r/a = 1)* for the case of uniform vertical loading. Solutions for σ_z and σ_r beneath the centre and edge, and τ_{rz} beneath the edge, are shown in Figs. 5.27 to 5.31 for ν = *0.30.*

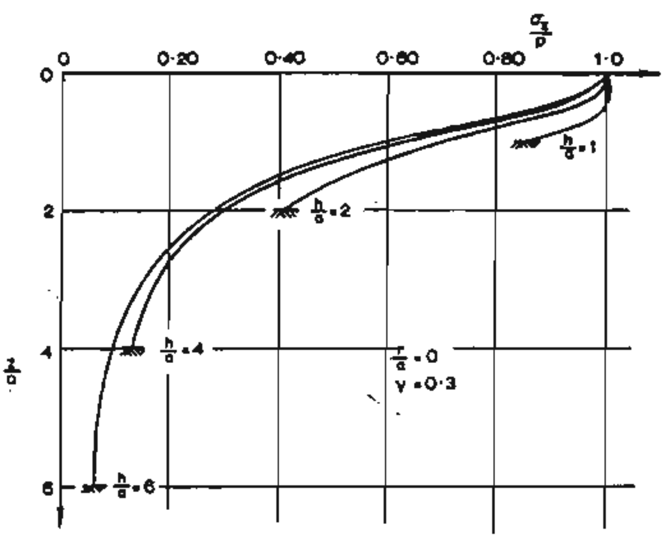


FIG.5.27 σ_z beneath centre (Milovic, 1970).

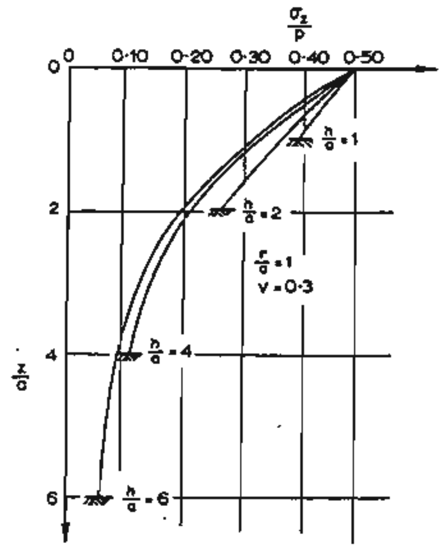


FIG.5.28 σ_z beneath edge (Milovic, 1970).

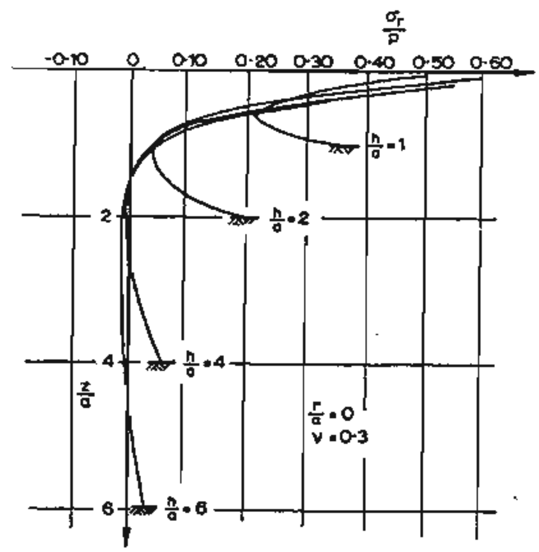


FIG.5.29 σ_r beneath centre (Milovic, 1970).

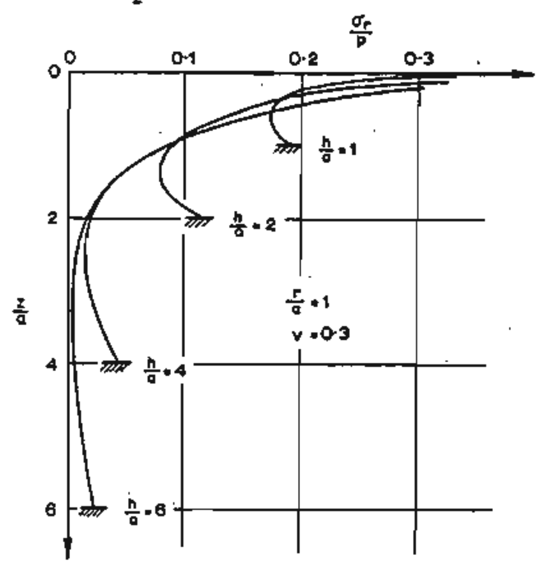


FIG.5.30 σ_r beneath edge (Milovic, 1970).

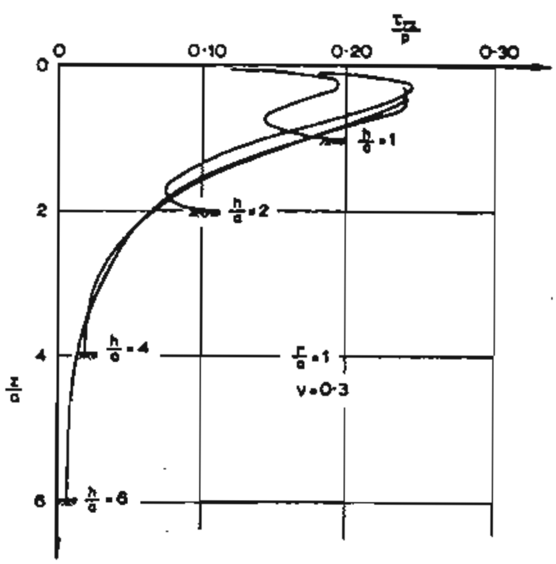


FIG.5.31 τ_{rz} beneath edge (Milovic, 1970).

TABLE 5.1

INFLUENCE FACTORS FOR VERTICAL DISPLACEMENT AT EDGE OF CIRCULAR AREA (Egorov, 1958)

h/a	I_e SMOOTH INTERFACE (All ν)	ADHESIVE INTERFACE (ν=0.3 only)
0.2	0.005	0.04
0.5	0.12	0.10
1	0.23	0.20
2	0.38	0.34
3	0.45	0.42
5	0.52	0.50
7	0.56	0.54
10	0.58	0.57
∞	0.64	0.64

Edge displacement $\rho_e = \dfrac{2pa(1-\nu^2)I_e}{E}$

For other cases, stresses and displacements beneath the centre of a uniformly loaded circular area may conveniently be obtained from the sector curves described in Section 5.4 and shown in Figs. 5.52 to 5.73. The use of the sector method is described in Section 1.7.3.

Influence factors for the vertical displacement ρ_z beneath the centre of the circle have been obtained by Ueshita and Meyerhof (1968) for both a rough underlying base (adhesive interface) and a smooth interface. These factors are shown in Fig.5.32.

Approximate values of the influence factor for vertical displacement at the edge of the circular area have been quoted by Harr (1966) and are shown in Table 5.1 for both an "adhesive interface" and a "smooth interface".

For various points within the loaded area, Milovic (1970) has obtained solution for the vertical surface displacement, and these are shown in Table 5.2. These results are for an adhesive interface.

TABLE 5.2

INFLUENCE FACTORS I_ρ FOR VERTICAL SURFACE DISPLACEMENT WITHIN CIRCULAR AREA

(Rough Rigid Base) (After Milovic, 1970)

ν	h/a \ r/a	0	.2	.4	.6	.8	1.0
0.15	1	0.464	0.458	0.441	0.408	0.348	0.208
	2	0.684	0.674	0.645	0.593	0.509	0.348
	4	0.811	0.800	0.768	0.710	0.619	0.463
	6	0.839	0.827	0.794	0.736	0.646	0.501
0.30	1	0.397	0.392	0.379	0.351	0.301	0.173
	2	0.613	0.604	0.578	0.531	0.456	0.305
	4	0.740	0.732	0.703	0.651	0.568	0.420
	6	0.770	0.762	0.733	0.681	0.597	0.458
0.45	1	0.278	0.276	0.267	0.250	0.213	0.109
	2	0.489	0.482	0.461	0.422	0.361	0.229
	4	0.612	0.608	0.585	0.541	0.472	0.340
	6	0.637	0.635	0.612	0.568	0.499	0.374

$$\rho_z = \frac{2pa\, I_\rho}{E}$$

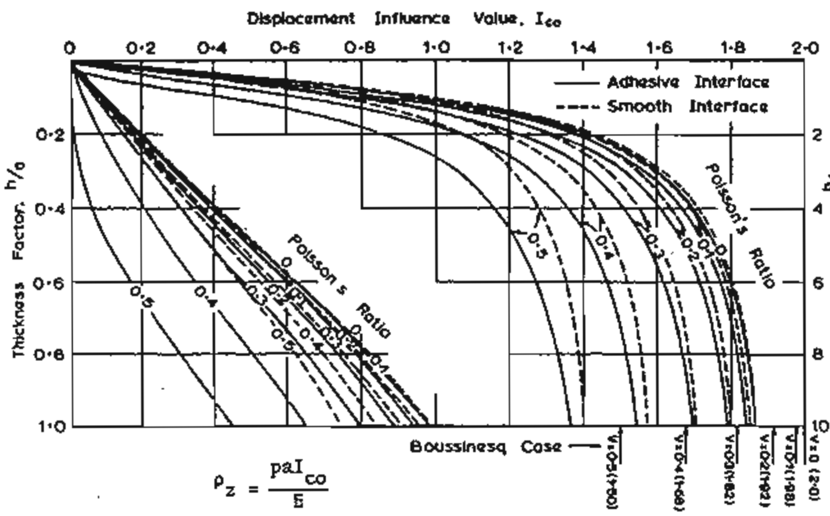


FIG.5.32 Displacement factor for centre of circle (Ueshita and Meyerhof, 1968).

5.3 Loading on a Rectangular Area

5.3.1 ROUGH RIGID BASE (Fig.5.33)

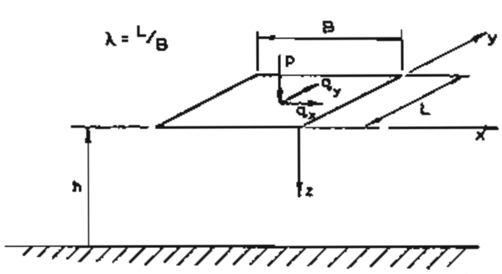


FIG.5.33

For uniform vertical loading p/unit area, a smooth rectangular area and a rough (adhesive) interface, between the layer and the base, Burmister (1956) has evaluated the vertical stress σ_z beneath the corner of the rectangle at various depths in the layer for $\nu=0.4$. These results are shown in Figures 5.34 to 5.38. The value of ν has little influence on these vertical stresses, expecially near the top of the layer.

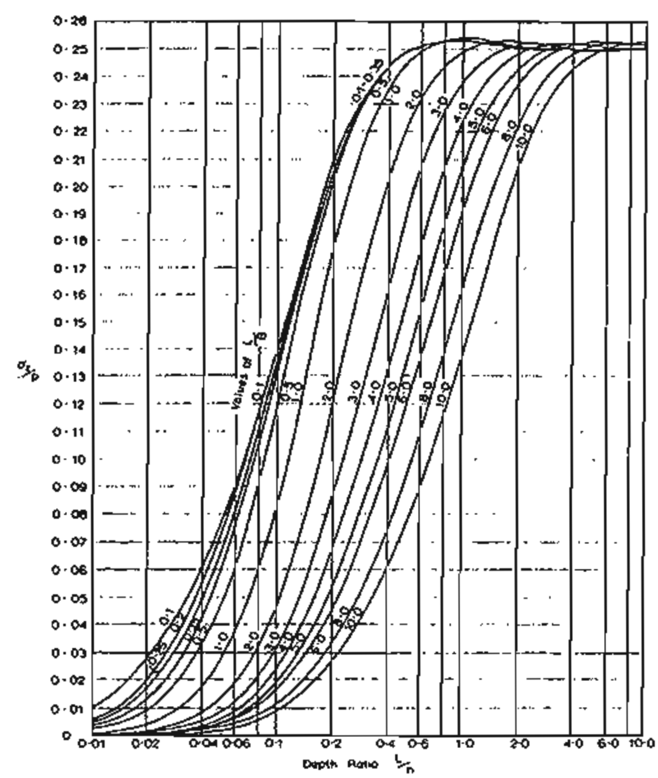


FIG.5.34 Burmister layer theory. σ_z beneath corner at z=0.2h. ν=0.4. (Burmister, 1956).

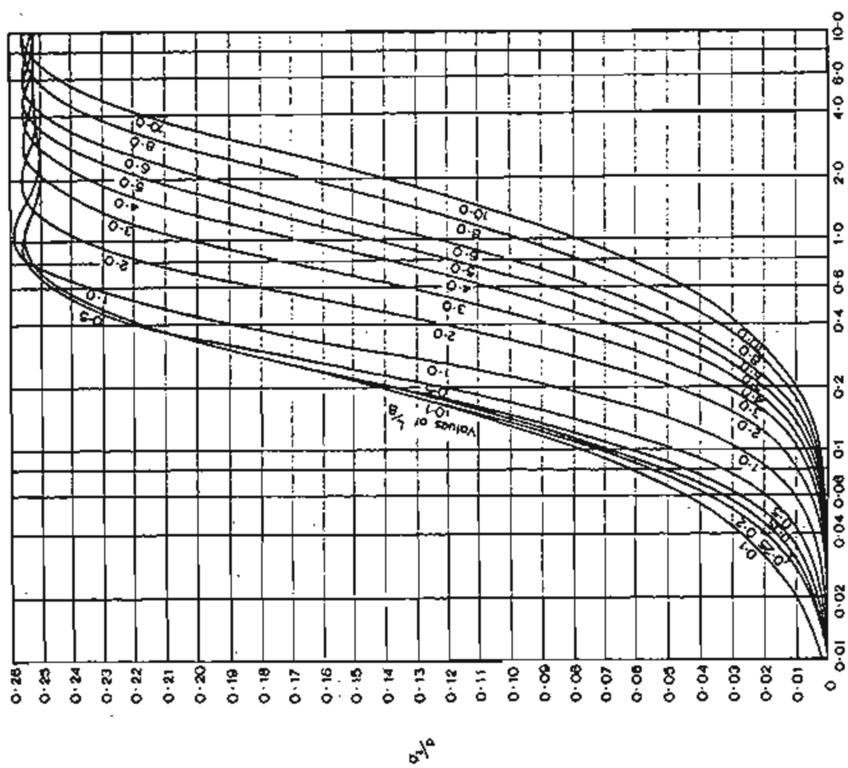


FIG.5.35 Burmister layer theory. σ_z beneath corner at z=0.4h. ν=0.4. (Burmister, 1956).

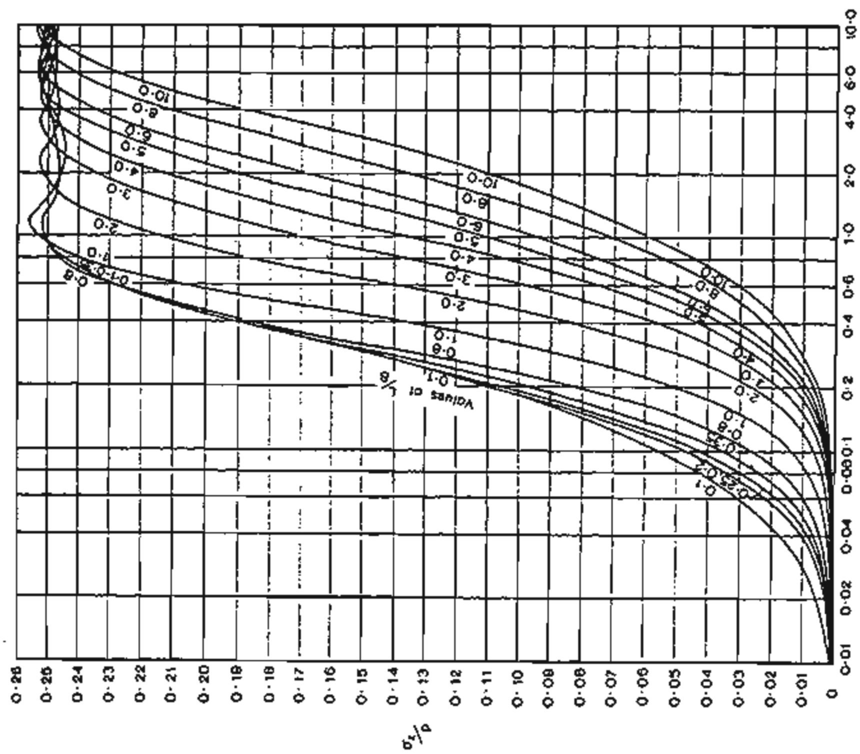


FIG.5.36 Burmister layer theory. σ_z beneath corner at z=0.6h. ν=0.4. (Burmister, 1956).

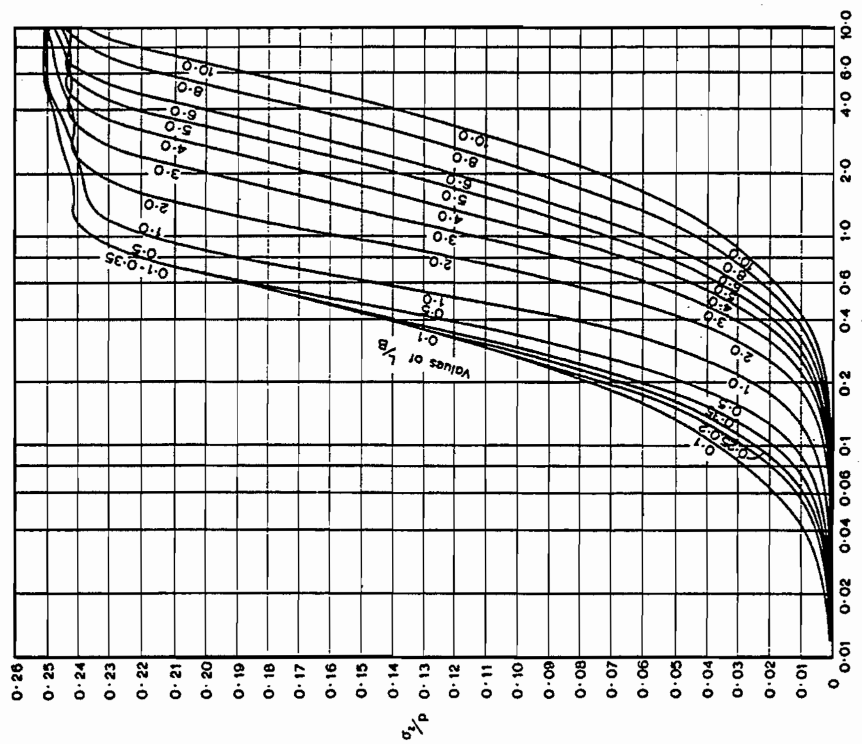


FIG.5.38 Burmister layer theory. σ_z beneath corner at z=1.0h. ν=0.4. (Burmister 1956).

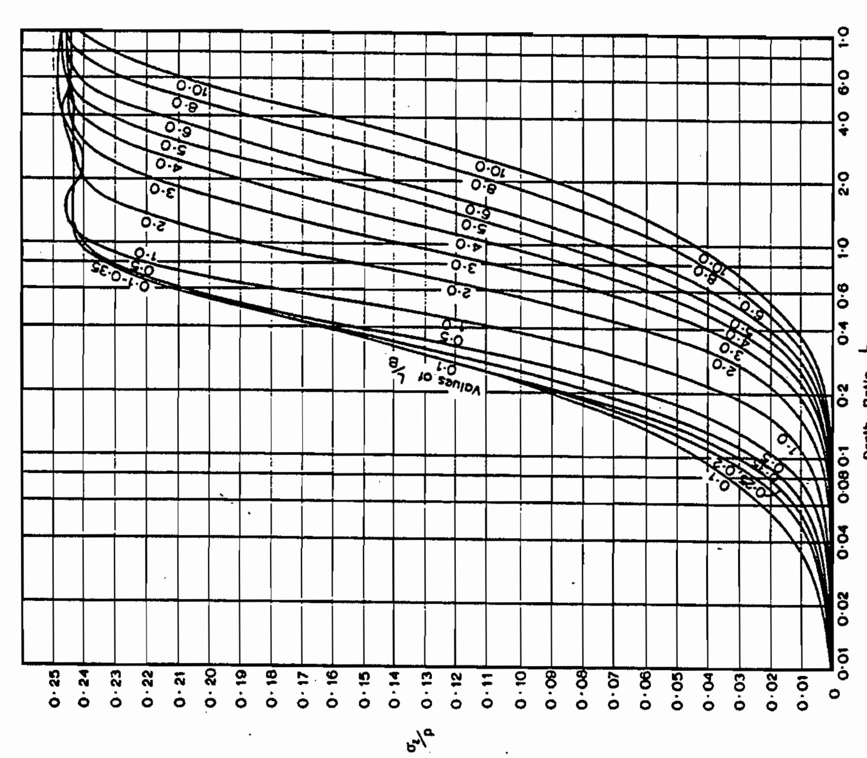


FIG.5.37 Burmister layer theory. σ_z beneath corner at z=0.8h. ν=0.4. (Burmister, 1956).

Influence factors for the vertical displacement of the corner of a rectangle loaded with a uniform vertical stress p per unit area have been presented by Ueshita and Meyerhof (1968) for a rough rigid base (adhesive interface). These influence factors I_{rc} are shown in Figures 5.39 to 5.44 for 6 values of ν. The actual displacement is given by

$$\rho_z = \frac{p\,B\,I_{rc}}{E} \qquad \ldots (5.4)$$

Influence factors for both vertical and horizontal surface displacements at the corner of a rectangle have been presented by Davis and Taylor (1962) for a rough rigid underlying base. Both uniform vertical loading q_z per unit area and horizontal loadings q_x and q_y per unit area, have been considered.

Referring to the key diagram in Figure 5.45(a), for corner 1, i.e. the corner contained in the positive xy quadrant, displacements are expressed as

$$\rho_{ij} = q_j \frac{h}{E}(1+\nu)I_{ij} \qquad \ldots (5.5)$$

where $I_{ij} = {}_0m_{ij} + {}_1m_{ij}\nu + {}_2m_{ij}\nu^2$

i, j are any of x, y, z.

For example, for vertical displacement due to uniform horizontal loading in the y direction,

$$\rho_{zy} = q_y \frac{h}{E}(1+\nu)I_{zy} \qquad \ldots (5.6)$$

The solutions for m_{xx}, m_{yy} and m_{xy} are approximate only as they have been obtained by use of the Steinbrenner approximation (see Section 6.4.1).

The influence factors ${}_0m_{ij}$, ${}_1m_{ij}$, ${}_2m_{ij}$ for corner 1 are plotted in Figures 5.45 to 5.48. It should also be noted that, because of the approximate treatment of m_{xx}, m_{yy} and m_{xy}, ${}_2m_{xy}={}_2m_{yx}=0$, and ${}_2m_{xx}={}_2m_{yy}=0$.

For corner 1, the total displacement in direction i due to combined loading is given by

$$\rho_i(1) = \rho_{ix}(1) + \rho_{iy}(1) + \rho_{iz}(1)$$

$$= \frac{h(1+\nu)}{E}\{q_x I_{ix} + q_y I_{iy} + q_z I_{iz}\} \qquad \ldots (5.7)$$

For the other three corners the displacements can be obtained from the displacement components $\rho_{ij}(1)$ as follows:

$$\rho_x \left|\begin{array}{c} 2 \\ 3 \\ 4 \end{array}\right| = \begin{array}{c} + \\ + \\ + \end{array} \rho_{xx}(1) \begin{array}{c} - \\ + \\ - \end{array} \rho_{xy}(1) \begin{array}{c} + \\ - \\ - \end{array} \rho_{xz}(1) \qquad \ldots (5.8a)$$

$$\rho_y \left|\begin{array}{c} 2 \\ 3 \\ 4 \end{array}\right| = \begin{array}{c} - \\ + \\ - \end{array} \rho_{yx}(1) \begin{array}{c} + \\ + \\ + \end{array} \rho_{yy}(1) \begin{array}{c} - \\ - \\ + \end{array} \rho_{yz}(1) \qquad \ldots (5.8b)$$

$$\rho_z \left|\begin{array}{c} 2 \\ 3 \\ 4 \end{array}\right| = \begin{array}{c} + \\ - \\ - \end{array} \rho_{zx}(1) \begin{array}{c} - \\ - \\ + \end{array} \rho_{zy}(1) \begin{array}{c} + \\ + \\ + \end{array} \rho_{zz}(1) \qquad \ldots (5.8c)$$

Another series of solutions have been presented by Milovic and Tournier (1971) who have evaluated stresses and displacements beneath the centre and corner of the rectangle. The stresses evaluated are defined as follows:

(a) Uniform Vertical Loading p

$$\sigma_z = p.I_{zp} \qquad \ldots (5.9a)$$

$$\sigma_x = p.I_{xp} \qquad \ldots (5.9b)$$

$$\tau_{xz} = p.I_{xzp} \qquad \ldots (5.9c)$$

(b) Uniform Horizontal Loading q_x

$$\sigma_z = q_x.I_{zq} \qquad \ldots (5.10a)$$

$$\sigma_x = q_x.I_{xq} \qquad \ldots (5.10b)$$

$$\tau_{xz} = q_x.I_{xzq} \qquad \ldots (5.10c)$$

For the centre of the rectangle, the stress influence values I_{zp} and I_{xp} for vertical loading and the influence values I_{xzq} for horizontal loading are given in Tables 5.3 to 5.5; for the centre, I_{xzp}, I_{zq} and I_{xq} are all zero. For the corner of the rectangle, the stress influence values I_{zp}, I_{xp} and I_{xzp} for vertical loading are given in Tables 5.6 to 5.8 and I_{zq}, I_{xq} and I_{xzq} for horizontal loading are given in Tables 5.9 to 5.11.

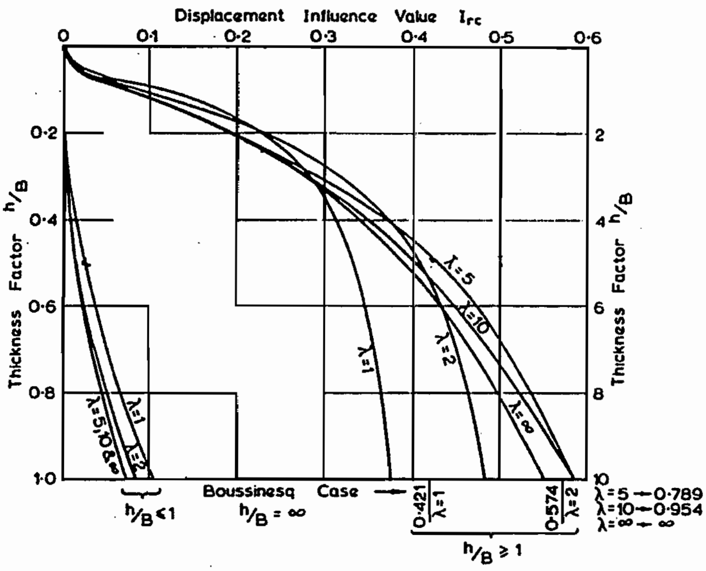


FIG.5.39 Displacement factors for corner of rectangle. ν=0.5. (Ueshita and Meyerhof,1968).

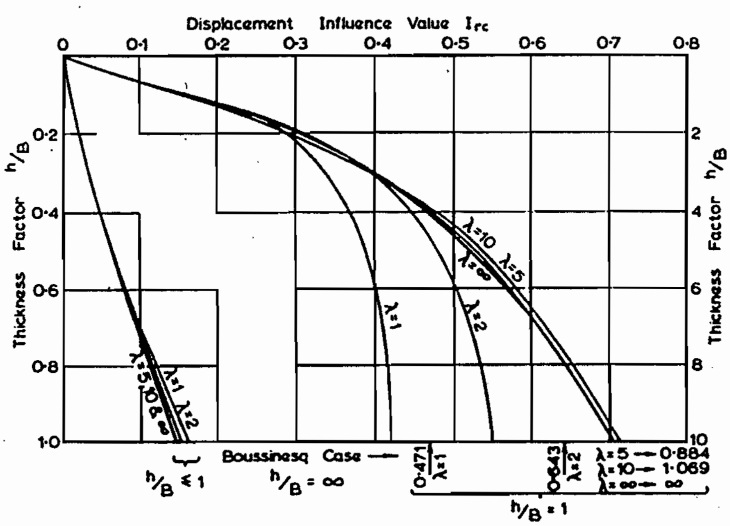


FIG.5.40 Displacement factors for corner of rectangle. ν=0.4.(Ueshita and Meyerhof,1968).

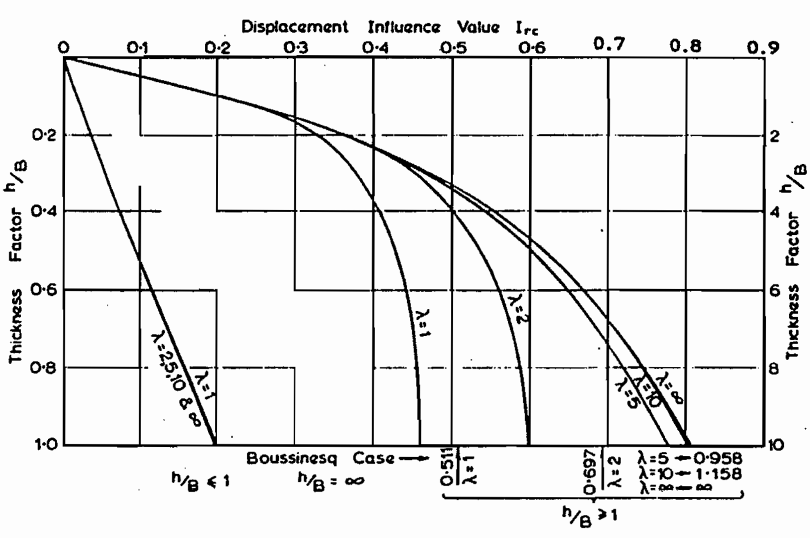


FIG.5.41 Displacement factors for corner of rectangle. ν=0.3. (Ueshita and Meyerhof,1968).

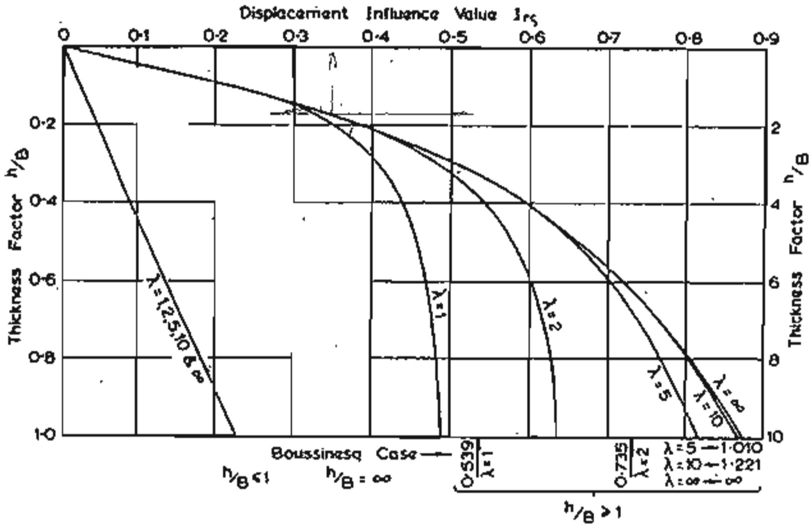


FIG.5.42 Displacement factors for corner of rectangle. ν=0.2. (Ueshita and Meyerhof,1968).

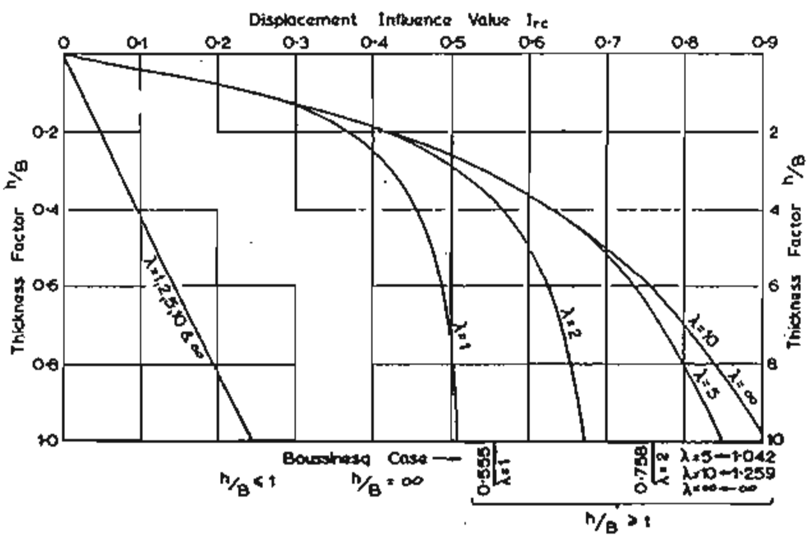


FIG.5.43 Displacement factors for corner of rectangle. ν=0.1. (Ueshita and Meyerhof,1968).

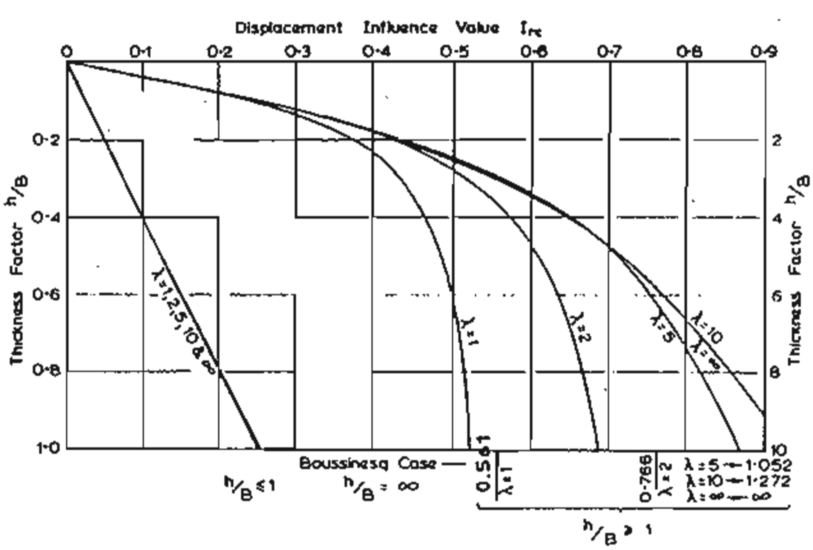
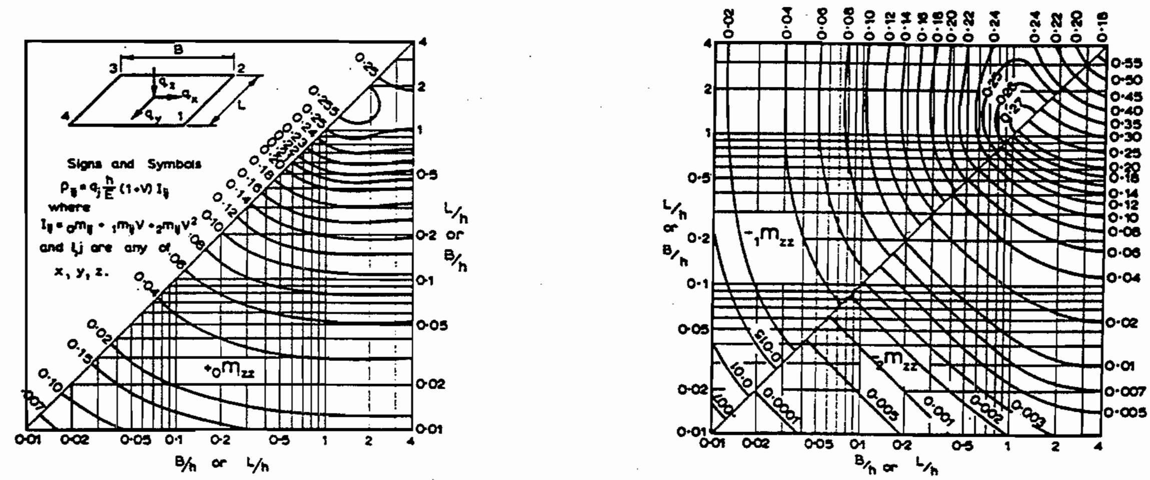
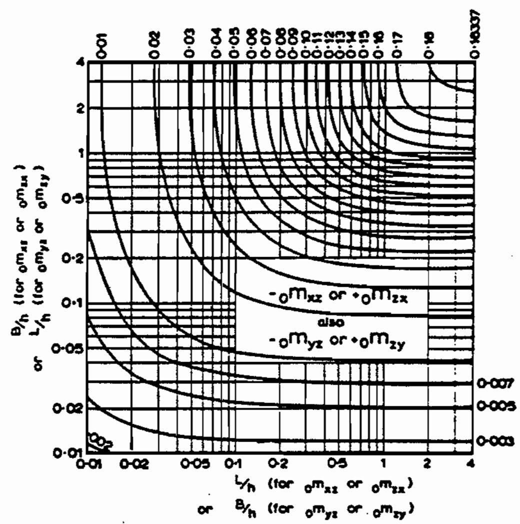


FIG.5.44 Displacement factors for corner of rectangle. ν=0. (Ueshita and Meyerhof,1968).

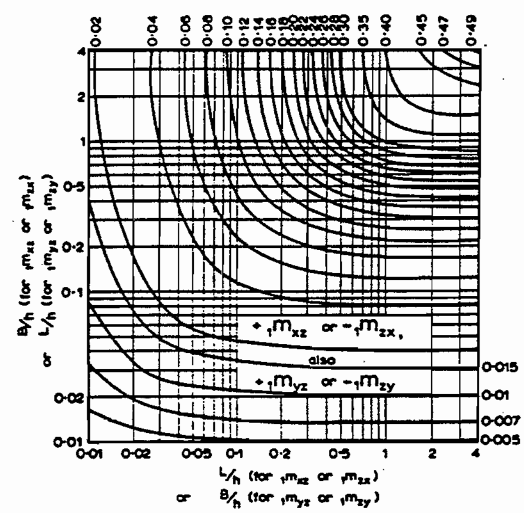


(a) (b)

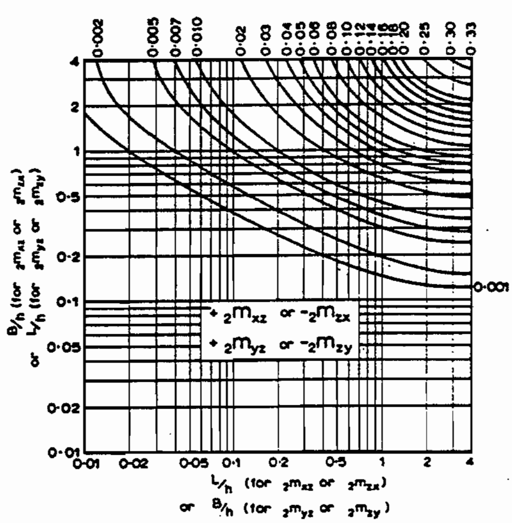
FIG.5.45 Rectangle displacement factors (Davis and Taylor, 1962).



(a)



(b)



(c)

FIG.5.46 Rectangle displacement factors (Davis and Taylor, 1962).

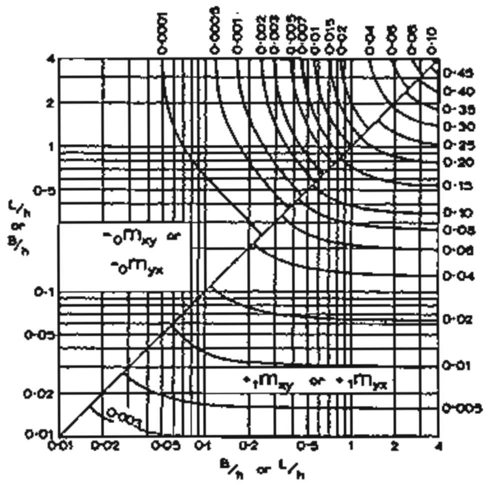


FIG.5.47 Rectangle displacement factors (Note that ${}_2m_{xy} = 0$) (Davis and Taylor, 1962).

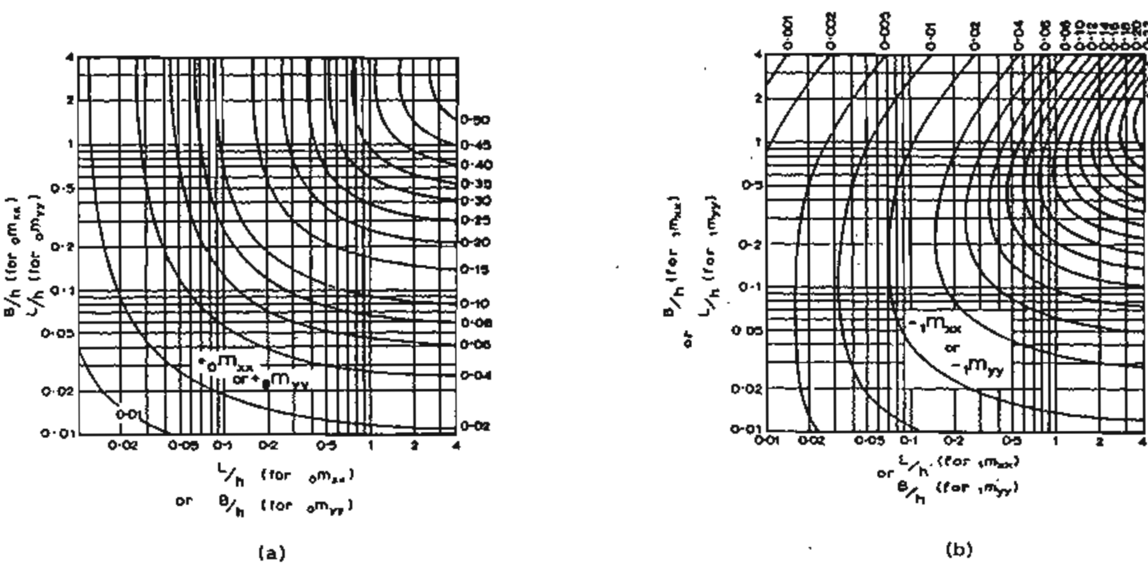


FIG.5.48 Rectangle displacement factors (Davis and Taylor, 1962).

TABLE 5.3

INFLUENCE VALUES I_{zp}, I_{xp} AND I_{xzq}
(Milovic and Tournier, 1971)

$(L/B) = 1.00$ Centre

h/B	z/B	$\nu = 0.15$			$\nu = 0.30$			$\nu = 0.45$		
		I_{zp}	I_{xp}	I_{xzq}	I_{zp}	I_{xp}	I_{xzq}	I_{zp}	I_{xp}	I_{xzq}
1.0	0.00	1.000	0.441	-	1.000	0.549	-	1.000	0.638	-
	0.10	0.974	0.310	0.738	0.974	0.425	0.739	0.974	0.520	0.740
	0.20	0.941	0.214	0.544	0.943	0.308	0.545	0.947	0.397	0.548
	0.40	0.837	0.040	0.276	0.842	0.113	0.275	0.855	0.192	0.274
	0.60	0.682	0.005	0.167	0.690	0.062	0.164	0.712	0.149	0.164
	0.80	0.563	0.013	0.131	0.570	0.093	0.122	0.595	0.211	0.114
	1.00	0.473	0.083	0.117	0.468	0.201	0.096	0.478	0.391	0.067
2.0	0.00	1.000	0.527	-	1.000	0.649	-	1.000	0.766	-
	0.10	0.970	0.385	0.726	0.970	0.500	0.726	0.970	0.590	0.726
	0.20	0.931	0.261	0.521	0.931	0.355	0.522	0.931	0.446	0.522
	0.40	0.802	0.061	0.236	0.802	0.122	0.236	0.804	0.181	0.237
	0.80	0.462	-0.027	0.062	0.464	0.003	0.062	0.469	0.034	0.063
	1.20	0.282	-0.023	0.033	0.286	-0.005	0.033	0.294	0.027	0.033
	1.60	0.200	-0.005	0.030	0.204	0.020	0.028	0.215	0.057	0.025
	2.00	0.157	0.027	0.028	0.155	0.067	-	0.161	0.132	0.011
3.0	0.00	1.000	0.545	-	1.000	0.672	-	1.000	0.795	-
	0.10	0.970	0.400	0.724	0.970	0.502	0.724	0.970	0.635	0.724
	0.20	0.930	0.275	0.518	0.930	0.370	0.519	0.930	0.464	0.519
	0.40	0.799	0.070	0.230	0.799	0.131	0.230	0.799	0.191	0.231
	0.80	0.452	-0.024	0.052	0.453	0.003	0.052	0.454	0.030	0.053
	1.20	0.263	-0.025	0.020	0.264	-0.009	0.020	0.266	0.007	0.021
	1.60	0.170	-0.018	0.013	0.172	-0.006	0.013	0.175	0.007	0.014
	2.00	0.122	-0.011	0.012	0.124	0.000	0.012	0.129	0.013	0.012
	2.50	0.091	-0.001	0.013	0.093	0.011	0.011	0.099	0.029	0.010
	3.00	0.073	0.013	0.012	0.073	0.031	0.009	0.0076	0.063	0.004
5.0	0.00	1.000	0.555	-	1.000	0.684	-	1.000	0.811	-
	0.10	0.970	0.432	0.723	0.970	0.521	0.723	0.970	0.625	0.723
	0.20	0.930	0.282	0.517	0.930	0.380	0.517	0.930	0.476	0.517
	0.40	0.798	0.076	0.228	0.798	0.138	0.228	0.798	0.200	0.223
	0.80	0.450	-0.020	0.048	0.450	0.007	0.049	0.450	0.034	0.049
	1.20	0.258	-0.023	0.015	0.258	-0.009	0.015	0.258	0.006	0.015
	1.60	0.162	-0.017	0.007	0.162	-0.009	0.007	0.163	0.000	0.007
	2.00	0.110	-0.013	0.005	0.111	-0.007	0.005	0.112	0.000	0.005
	2.50	0.075	-0.008	0.004	0.075	-0.004	0.004	0.077	0.001	0.004
	3.00	0.055	-0.006	0.004	0.056	-0.002	0.004	0.057	0.003	0.004
	3.50	0.043	-0.004	0.004	0.044	0.000	0.004	0.046	0.006	0.004
	4.00	0.036	-0.002	0.005	0.037	0.003	0.004	0.039	0.009	0.004
	4.50	0.031	0.001	0.005	0.032	0.006	0.004	0.034	0.015	0.004
	5.00	0.027	0.005	0.005	0.027	0.012	0.003	0.029	0.023	0.002

TABLE 5.4

INFLUENCE VALUES I_{zp}, I_{xp} AND I_{xzq}
(Milovic and Tournier, 1971)

$(L/B) = 2.00$ Centre

h/B	z/B	ν = 0.15 I_{zp}	ν = 0.15 I_{xp}	ν = 0.15 I_{xzq}	ν = 0.30 I_{zp}	ν = 0.30 I_{xp}	ν = 0.30 I_{xzq}	ν = 0.45 I_{zp}	ν = 0.45 I_{xp}	ν = 0.45 I_{xzq}
	0.00	1.000	0.511	-	1.000	0.571	-	1.000	0.607	-
	0.10	0.992	0.364	0.775	0.992	0.445	0.775	0.992	0.501	0.777
	0.20	0.976	0.245	0.589	0.977	0.314	0.589	0.981	0.377	0.593
1.0	0.40	0.919	0.047	0.337	0.924	0.122	0.334	0.936	0.207	0.337
	0.60	0.821	-0.006	0.233	0.827	0.080	0.226	0.847	0.193	0.224
	0.80	0.732	0.020	0.198	0.734	0.130	0.181	0.754	0.291	0.165
	1.00	0.651	0.115	0.181	0.638	0.273	0.145	0.639	0.523	0.099
	0.00	1.000	0.656	-	1.000	0.731	-	1.000	0.796	-
	0.10	0.985	0.502	0.754	0.985	0.545	0.755	0.985	0.620	0.764
	0.20	0.963	0.325	0.552	0.963	0.390	0.552	0.964	0.449	0.554
	0.40	0.877	0.084	0.271	0.878	0.138	0.272	0.880	0.189	0.274
2.0	0.80	0.615	-0.039	0.090	0.619	0.001	0.091	0.627	0.044	0.093
	1.20	0.436	-0.036	0.058	0.441	0.000	0.057	0.455	0.046	0.057
	1.60	0.334	-0.007	0.054	0.340	0.036	0.050	0.356	0.100	0.046
	2.00	0.271	0.048	0.052	0.269	0.115	0.038	0.277	0.227	0.020
	0.00	1.000	0.691	-	1.000	0.772	-	1.000	0.848	-
	0.10	0.982	0.508	0.751	0.982	0.580	0.752	0.982	0.675	0.752
	0.20	0.961	0.350	0.546	0.962	0.418	0.546	0.962	0.482	0.547
	0.40	0.872	0.102	0.261	0.872	0.156	0.261	0.873	0.207	0.263
	0.80	0.598	-0.032	0.073	0.599	0.002	0.073	0.602	0.036	0.074
3.0	1.20	0.403	-0.040	0.033	0.405	-0.016	0.033	0.409	0.010	0.035
	1.60	0.286	-0.030	0.024	0.289	-0.001	0.024	0.295	0.012	0.025
	2.00	0.217	-0.019	0.023	0.220	0.000	0.022	0.229	0.025	0.023
	2.50	0.168	-0.002	0.024	0.171	0.020	0.022	0.180	0.055	0.020
	3.00	0.137	0.024	0.024	0.137	0.059	0.017	0.142	0.116	0.008
	0.00	1.000	0.710	-	1.000	0.795	-	1.000	0.879	-
	0.10	0.981	0.525	0.750	0.981	0.605	0.750	0.981	0.702	0.750
	0.20	0.961	0.366	0.544	0.961	0.436	0.544	0.961	0.505	0.544
	0.40	0.870	0.114	0.257	0.870	0.170	0.257	0.870	0.224	0.258
	0.80	0.594	-0.024	0.066	0.594	0.009	0.066	0.594	0.043	0.066
	1.20	0.393	-0.036	0.024	0.394	-0.014	0.024	0.395	0.007	0.025
5.0	1.60	0.270	-0.030	0.012	0.271	-0.015	0.013	0.272	0.000	0.013
	2.00	0.195	-0.023	0.009	0.195	-0.012	0.009	0.197	0.000	0.010
	2.50	0.138	-0.017	0.008	0.139	-0.008	0.008	0.141	0.002	0.008
	3.00	0.104	-0.001	0.008	0.105	-0.004	0.008	0.108	0.006	0.008
	3.50	0.083	-0.007	0.008	0.085	0.000	0.008	0.088	0.011	0.008
	4.00	0.070	-0.003	0.009	0.071	0.005	0.008	0.075	0.018	0.008
	4.50	0.060	0.002	0.009	0.062	0.012	0.008	0.065	0.029	0.007
	5.00	0.053	0.009	0.009	0.053	0.023	0.007	0.056	0.045	0.004

TABLE 5.5

INFLUENCE VALUES I_{zp}, I_{xp} AND I_{xzq}

$(L/B) = 5.00$

(Milovic and Tournier, 1971)
Centre

h/B	z/B	$\nu = 0.15$			$\nu = 0.30$			$\nu = 0.45$		
		I_{zp}	I_{xp}	I_{xzq}	I_{zp}	I_{xp}	I_{xzq}	I_{zp}	I_{xp}	I_{xzq}
	0.00	1.000	0.525	-	1.000	0.566	-	1.000	0.592	-
	0.10	0.996	0.365	0.796	0.996	0.425	0.796	0.996	0.462	0.797
	0.20	0.980	0.252	0.610	0.981	0.303	0.608	0.983	0.358	0.609
1.0	0.40	0.920	0.050	0.367	0.922	0.117	0.360	0.930	0.200	0.358
	0.60	0.830	0.005	0.271	0.832	0.081	0.257	0.843	0.199	0.246
	0.80	0.753	0.022	0.239	0.751	0.138	0.212	0.760	0.307	0.183
	1.00	0.688	0.121	0.221	0.672	0.288	0.171	0.665	0.544	0.108
	0.00	1.000	0.715	-	1.000	0.745	-	1.000	0.760	-
	0.10	0.990	0.505	0.772	0.990	0.565	0.772	0.990	0.590	0.773
	0.20	0.971	0.362	0.565	0.971	0.390	0.566	0.972	0.410	0.568
	0.40	0.889	0.104	0.290	0.890	0.135	0.290	0.893	0.163	0.293
2.0	0.80	0.667	-0.040	0.119	0.670	0.004	0.118	0.677	0.039	0.120
	1.20	0.524	-0.043	0.092	0.528	0.002	0.088	0.539	0.063	0.088
	1.60	0.441	-0.008	0.092	0.443	0.054	0.082	0.455	0.144	0.072
	2.00	0.385	0.068	0.088	0.377	0.162	0.064	0.379	0.310	0.032
	0.00	1.000	0.778	-	1.000	0.812	-	1.000	0.836	-
	0.10	0.990	0.541	0.767	0.990	0.610	0.767	0.990	0.649	0.768
	0.20	0.969	0.408	0.556	0.969	0.436	0.556	0.970	0.458	0.557
	0.40	0.884	0.137	0.272	0.884	0.165	0.273	0.885	0.188	0.274
	0.80	0.649	-0.027	0.089	0.650	0.000	0.089	0.653	0.026	0.091
3.0	1.20	0.489	-0.049	0.052	0.492	-0.022	0.052	0.498	0.008	0.054
	1.60	0.391	-0.043	0.045	0.395	-0.015	0.044	0.404	0.020	0.046
	2.00	0.329	-0.029	0.046	0.333	0.003	0.044	0.344	0.046	0.043
	2.50	0.278	-0.002	0.049	0.281	0.038	0.043	0.292	0.099	0.038
	3.00	0.241	0.042	0.047	0.238	0.102	0.034	0.242	0.198	0.016
	0.00	1.000	0.819	-	1.000	0.859	-	1.000	0.895	-
	0.10	0.990	0.603	0.765	0.990	0.642	0.765	0.990	0.703	0.765
	0.20	0.969	0.442	0.511	0.969	0.473	0.551	0.969	0.503	0.552
	0.40	0.881	0.164	0.264	0.881	0.194	0.264	0.882	0.221	0.265
	0.80	0.641	-0.010	0.075	0.641	0.015	0.075	0.642	0.039	0.076
	1.20	0.474	-0.040	0.033	0.475	-0.018	0.033	0.476	0.003	0.034
5.0	1.60	0.367	-0.041	0.021	0.368	-0.023	0.021	0.370	-0.003	0.023
	2.00	0.294	-0.037	0.018	0.296	-0.020	0.018	0.299	-0.001	0.019
	2.50	0.233	-0.029	0.017	0.235	-0.014	0.017	0.239	0.005	0.018
	3.00	0.191	-0.022	0.018	0.193	-0.007	0.018	0.199	0.013	0.019
	3.50	0.162	-0.014	0.019	0.165	0.002	0.019	0.172	0.024	0.019
	4.00	0.141	-0.006	0.021	0.144	0.012	0.019	0.152	0.040	0.018
	4.50	0.126	0.005	0.022	0.128	0.027	0.019	0.135	0.062	0.016
	5.00	0.113	0.020	0.021	0.113	0.048	0.015	0.117	0.096	0.008

TABLE 5.6

INFLUENCE VALUES I_{zp}, I_{xp} AND I_{xzp}
(Milovic and Tournier, 1971)

$(L/B) = 1.00$ Corner

h/B	z/B	$\nu = 0.15$			$\nu = 0.30$			$\nu = 0.45$		
		I_{zp}	I_{xp}	I_{xzp}	I_{zp}	I_{xp}	I_{xzp}	I_{zp}	I_{xp}	I_{xzp}
1.0	0.00	0.250	0.083	0.000	0.250	0.112	0.000	0.250	0.134	0.000
	0.20	0.250	0.061	0.121	0.250	0.921	0.119	0.250	0.122	0.114
	0.40	0.250	0.039	0.105	0.250	0.072	0.103	0.250	0.110	0.098
	0.60	0.250	0.028	0.079	0.250	0.064	0.079	0.250	0.112	0.075
	0.80	0.241	0.028	0.059	0.238	0.071	0.064	0.239	0.133	0.070
	1.00	0.227	0.040	0.056	0.220	0.094	0.073	0.215	0.176	0.096
2.0	0.00	0.250	0.131	0.000	0.250	0.163	0.000	0.250	0.190	0.000
	0.20	0.250	0.089	0.136	0.250	0.117	0.136	0.250	0.142	0.135
	0.40	0.243	0.052	0.128	0.244	0.076	0.127	0.245	0.098	0.126
	0.80	0.210	0.010	0.080	0.211	0.028	0.079	0.214	0.048	0.077
	1.20	0.170	0.001	0.044	0.172	0.016	0.043	0.178	0.037	0.042
	1.60	0.141	0.003	0.024	0.142	0.023	0.025	0.149	0.053	0.025
	2.00	0.118	0.021	0.019	0.117	0.050	0.024	0.120	0.098	0.030
3.0	0.00	0.250	0.146	0.000	0.250	0.181	0.000	0.250	0.213	0.000
	0.20	0.249	0.100	0.138	0.249	0.129	0.138	0.249	0.156	0.137
	0.40	0.241	0.060	0.131	0.241	0.083	0.131	0.242	0.106	0.130
	0.80	0.203	-0.013	0.084	0.203	0.028	0.084	0.204	0.044	0.084
	1.20	0.157	-0.003	0.049	0.158	0.008	0.049	0.160	0.020	0.048
	1.60	0.121	-0.007	0.028	0.122	0.003	0.028	0.125	0.014	0.027
	2.00	0.096	-0.005	0.017	0.098	0.004	0.016	0.102	0.016	0.016
	2.50	0.077	0.001	0.009	0.078	0.012	0.009	0.083	0.028	0.010
	3.00	0.064	0.011	0.007	0.064	0.027	0.009	0.066	0.054	0.011
5.0	0.00	0.250	0.156	0.000	0.250	0.192	0.000	0.250	0.227	0.000
	0.20	0.249	0.107	0.138	0.249	0.137	0.138	0.249	0.167	0.138
	0.40	0.241	0.066	0.132	0.241	0.090	0.132	0.240	0.113	0.131
	0.80	0.200	0.016	0.086	0.200	0.032	0.086	0.201	0.047	0.086
	1.20	0.152	-0.001	0.051	0.153	0.009	0.051	0.153	0.019	0.051
	1.60	0.114	-0.006	0.030	0.114	0.001	0.030	0.115	0.008	0.030
	2.00	0.086	-0.007	0.019	0.087	-0.002	0.019	0.087	0.004	0.018
	2.50	0.063	-0.006	0.011	0.064	-0.002	0.011	0.065	0.003	0.011
	3.00	0.049	-0.005	0.007	0.050	-0.001	0.006	0.051	0.004	0.006
	3.50	0.040	-0.003	0.004	0.040	0.001	0.004	0.042	0.006	0.004
	4.00	0.034	-0.001	0.003	0.034	0.003	0.003	0.036	0.009	0.003
	4.50	0.029	0.001	0.002	0.030	0.006	0.002	0.032	0.014	0.002
	5.00	0.026	0.005	0.002	0.026	0.011	0.002	0.027	0.022	0.003

TABLE 5.7

INFLUENCE VALUES I_{zp}, I_{xp} AND I_{xzp} (Milovic and Tournier, 1971)

$(L/B) = 2.00$ Corner

h/B	z/B	ν = 0.15 I_{zp}	ν = 0.15 I_{xp}	ν = 0.15 I_{xzp}	ν = 0.30 I_{zp}	ν = 0.30 I_{xp}	ν = 0.30 I_{xzp}	ν = 0.45 I_{zp}	ν = 0.45 I_{xp}	ν = 0.45 I_{xzp}
	0.00	0.250	0.085	0.000	0.250	0.108	0.000	0.250	0.129	0.000
	0.20	0.250	0.062	0.119	0.250	0.089	0.118	0.250	0.118	0.113
1.0	0.40	0.250	0.041	0.104	0.250	0.072	0.103	0.250	0.112	0.097
	0.60	0.250	0.029	0.079	0.250	0.067	0.081	0.250	0.118	0.078
	0.80	0.248	0.030	0.061	0.244	0.076	0.069	0.240	0.141	0.075
	1.00	0.241	0.042	0.061	0.232	0.099	0.080	0.223	0.183	0.107
	0.00	0.250	0.147	0.000	0.250	0.164	0.000	0.250	0.176	0.000
	0.20	0.250	0.100	0.136	0.250	0.116	0.136	0.250	0.129	0.135
	0.40	0.248	0.059	0.130	0.249	0.076	0.129	0.250	0.092	0.127
2.0	0.80	0.230	0.012	0.086	0.231	0.030	0.085	0.234	0.051	0.082
	1.20	0.205	-0.002	0.051	0.207	0.020	0.051	0.212	0.048	0.049
	1.60	0.183	0.005	0.031	0.183	0.033	0.032	0.188	0.072	0.033
	2.00	0.163	0.029	0.026	0.160	0.068	0.033	0.160	0.131	0.042
	0.00	0.250	0.172	0.000	0.250	0.192	0.000	0.250	0.207	0.000
	0.20	0.250	0.118	0.139	0.250	0.135	0.139	0.250	0.149	0.139
	0.40	0.246	0.072	0.134	0.246	0.088	0.134	0.246	0.102	0.133
	0.80	0.222	0.017	0.093	0.222	0.031	0.092	0.224	0.045	0.091
3.0	1.20	0.190	-0.004	0.059	0.191	0.009	0.058	0.194	0.024	0.057
	1.60	0.162	-0.009	0.037	0.163	0.004	0.036	0.167	0.020	0.035
	2.00	0.139	-0.007	0.023	0.141	0.007	0.023	0.146	0.026	0.022
	2.50	0.119	0.001	0.014	0.120	0.019	0.014	0.125	0.046	0.015
	3.00	0.103	0.018	0.012	0.102	0.044	0.014	0.104	0.085	0.019
	0.00	0.250	0.189	0.000	0.250	0.211	0.000	0.250	0.232	0.000
	0.20	0.250	0.131	0.140	0.250	0.150	0.140	0.250	0.168	0.140
	0.40	0.245	0.083	0.135	0.245	0.100	0.135	0.245	0.116	0.135
	0.80	0.218	0.023	0.095	0.219	0.037	0.095	0.219	0.050	0.095
	1.20	0.183	-0.001	0.062	0.184	0.010	0.062	0.184	0.021	0.061
	1.60	0.151	-0.009	0.040	0.151	0.000	0.040	0.152	0.010	0.040
5.0	2.00	0.124	-0.011	0.027	0.125	-0.003	0.027	0.126	0.006	0.026
	2.50	0.099	-0.010	0.017	0.100	-0.003	0.016	0.102	0.005	0.016
	3.00	0.081	-0.008	0.011	0.082	-0.001	0.011	0.085	0.007	0.010
	3.50	0.068	-0.005	0.007	0.069	0.002	0.007	0.073	0.011	0.007
	4.00	0.059	-0.002	0.005	0.060	0.006	0.005	0.064	0.017	0.005
	4.50	0.053	0.002	0.004	0.053	0.011	0.004	0.057	0.026	0.004
	5.00	0.047	0.008	0.003	0.047	0.020	0.004	0.049	0.040	0.005

TABLE 5.8

INFLUENCE VALUES I_{zp}, I_{xp} AND I_{xzp}
(Milovic and Tournier, 1971)

$(L/B) = 5.00$ Corner

h/B	z/B	$\nu = 0.15$			$\nu = 0.30$			$\nu = 0.45$		
		I_{zp}	I_{xp}	I_{xzp}	I_{zp}	I_{xp}	I_{xzp}	I_{zp}	I_{xp}	I_{xzp}
1.0	0.00	0.250	0.082	0.000	0.250	0.103	0.000	0.250	0.124	0.000
	0.20	0.250	0.062	0.118	0.250	0.088	0.117	0.250	0.120	0.113
	0.40	0.250	0.041	0.103	0.250	0.072	0.103	0.250	0.114	0.098
	0.60	0.250	0.029	0.079	0.250	0.067	0.081	0.250	0.120	0.080
	0.80	0.247	0.030	0.061	0.244	0.076	0.069	0.242	0.143	0.077
	1.00	0.239	0.042	0.060	0.233	0.100	0.081	0.226	0.185	0.109
2.0	0.00	0.250	0.146	0.000	0.250	0.154	0.000	0.250	0.158	0.000
	0.20	0.250	0.101	0.135	0.250	0.112	0.134	0.250	0.121	0.133
	0.40	0.247	0.061	0.120	0.247	0.073	0.138	0.248	0.087	0.126
	0.80	0.230	0.012	0.086	0.230	0.029	0.085	0.232	0.050	0.082
	1.20	0.207	-0.001	0.052	0.208	0.020	0.052	0.211	0.050	0.050
	1.60	0.188	0.006	0.032	0.188	0.035	0.034	0.190	0.077	0.036
	2.00	0.172	0.030	0.028	0.168	0.072	0.035	0.166	0.136	0.046
3.0	0.00	0.250	0.177	0.000	0.250	0.182	0.000	0.250	0.184	0.000
	0.20	0.249	0.124	0.138	0.249	0.131	0.137	0.249	0.136	0.137
	0.40	0.245	0.077	0.133	0.246	0.086	0.133	0.246	0.093	0.132
	0.80	0.224	0.019	0.093	0.225	0.029	0.092	0.223	0.040	0.091
	1.20	0.197	-0.004	0.060	0.197	0.008	0.059	0.199	0.022	0.058
	1.60	0.173	-0.010	0.038	0.174	0.004	0.038	0.176	0.022	0.037
	2.00	0.155	-0.008	0.025	0.156	0.009	0.025	0.159	0.032	0.025
	2.50	0.139	0.002	0.016	0.138	0.024	0.017	0.141	0.056	0.018
	3.00	0.126	0.022	0.014	0.123	0.053	0.018	0.123	0.100	0.023
5.0	0.00	0.250	0.203	0.000	0.250	0.209	0.000	0.250	0.214	0.000
	0.20	0.249	0.146	0.139	0.249	0.153	0.139	0.249	0.158	0.139
	0.40	0.245	0.095	0.135	0.245	0.102	0.135	0.245	0.109	0.135
	0.80	0.221	0.031	0.096	0.221	0.038	0.096	0.222	0.045	0.096
	1.20	0.191	0.002	0.064	0.191	0.010	0.064	0.192	0.018	0.063
	1.60	0.164	-0.009	0.043	0.164	-0.001	0.043	0.166	0.009	0.042
	2.00	0.142	-0.012	0.030	0.143	-0.004	0.030	0.145	0.006	0.029
	2.50	0.122	-0.012	0.020	0.123	-0.004	0.019	0.125	0.008	0.019
	3.00	0.107	-0.010	0.013	0.108	-0.001	0.013	0.111	0.012	0.013
	3.50	0.096	-0.007	0.009	0.097	0.004	0.009	0.101	0.019	0.009
	4.00	0.088	-0.002	0.007	0.089	0.010	0.007	0.092	0.028	0.007
	4.50	0.081	0.004	0.005	0.082	0.019	0.006	0.085	0.042	0.007
	5.00	0.075	0.013	0.005	0.075	0.032	0.007	0.076	0.062	0.008

TABLE 5.9

INFLUENCE VALUES I_{zq} I_{xq} AND I_{xzq}
(Milovic and Tournier, 1971)

(L/B) = 1.00 Corner

h/B	z/B	ν = 0.15 I_{zq}	I_{wq}	I_{xzq}	ν = 0.30 I_{zq}	I_{xq}	I_{xzq}	ν = 0.45 I_{zq}	I_{xq}	I_{xzq}
	0.10	0.157	0.496	0.223	0.157	0.507	0.223	0.157	0.515	0.096
	0.20	0.152	0.303	0.200	0.151	0.313	0.200	0.151	0.321	0.104
1.0	0.40	0.142	0.126	0.157	0.143	0.136	0.155	0.146	0.146	0.076
	0.60	0.130	0.053	0.126	0.131	0.064	0.123	0.135	0.077	0.051
	0.80	0.118	0.022	0.108	0.118	0.038	0.102	0.123	0.059	0.037
	1.00	0.108	0.019	0.100	0.105	0.045	0.090	0.105	0.086	0.044
	0.10	0.155	0.527	0.217	0.155	0.537	0.217	0.155	0.547	0.217
	0.20	0.150	0.328	0.188	0.150	0.337	0.189	0.150	0.346	0.189
	0.40	0.133	0.146	0.135	0.133	0.153	0.135	0.133	0.159	0.136
2.0	0.80	0.089	0.033	0.069	0.089	0.037	0.069	0.090	0.041	0.069
	1.20	0.057	0.007	0.042	0.058	0.010	0.041	0.060	0.013	0.041
	1.60	0.040	0.001	0.033	0.041	0.005	0.031	0.044	0.010	0.028
	2.00	0.031	0.006	0.029	0.031	0.013	0.024	0.032	0.027	0.017
	0.10	0.155	0.531	0.216	0.155	0.543	0.216	0.155	0.553	0.216
	0.20	0.150	0.332	0.186	0.150	0.342	0.186	0.150	0.351	0.186
	0.40	0.132	0.150	0.131	0.132	0.156	0.130	0.132	0.163	0.131
	0.80	0.087	0.036	0.061	0.087	0.039	0.061	0.087	0.042	0.062
3.0	1.20	0.052	0.009	0.031	0.052	0.011	0.031	0.053	0.013	0.032
	1.60	0.032	0.002	0.019	0.033	0.003	0.019	0.033	0.004	0.020
	2.00	0.022	0.000	0.015	0.022	0.001	0.015	0.023	0.002	0.015
	2.50	0.015	0.000	0.014	0.015	0.001	0.013	0.017	0.003	0.011
	3.00	0.012	0.002	0.013	0.012	0.005	0.010	0.012	0.010	0.006
	0.10	0.155	0.533	0.215	0.155	0.545	0.215	0.155	0.556	0.215
	0.20	0.150	0.334	0.185	0.150	0.344	0.185	0.150	0.353	0.185
	0.40	0.132	0.151	0.129	0.132	0.158	0.129	0.132	0.164	0.129
	0.80	0.086	0.037	0.058	0.086	0.040	0.058	0.086	0.044	0.058
	1.20	0.051	0.010	0.027	0.051	0.012	0.027	0.051	0.013	0.027
	1.60	0.031	0.003	0.014	0.031	0.004	0.014	0.031	0.005	0.014
5.0	2.00	0.019	0.000	0.009	0.019	0.001	0.009	0.020	0.002	0.009
	2.50	0.012	0.000	0.006	0.012	0.000	0.006	0.012	0.001	0.006
	3.00	0.008	0.000	0.005	0.008	0.000	0.005	0.008	0.000	0.005
	3.50	0.005	0.000	0.005	0.006	0.000	0.005	0.006	0.000	0.005
	4.00	0.004	0.000	0.005	0.004	0.000	0.005	0.005	0.001	0.004
	4.50	0.003	0.000	0.005	0.003	0.000	0.004	0.004	0.001	0.004
	5.00	0.003	0.000	0.005	0.003	0.001	0.004	0.003	0.002	0.002

TABLE 5.10

INFLUENCE VALUES I_{zq}, I_{xq} AND I_{xzq}
(Milovic and Tournier, 1971)

(L/B) = 2.00 Corner

h/B	z/B	ν = 0.15 I_{zq}	ν = 0.15 I_{xq}	ν = 0.15 I_{xzq}	ν = 0.30 I_{zq}	ν = 0.30 I_{xq}	ν = 0.30 I_{xzq}	ν = 0.45 I_{zq}	ν = 0.45 I_{xq}	ν = 0.45 I_{xzq}
	0.10	0.158	0.509	0.230	0.158	0.515	0.229	0.158	0.517	0.229
	0.20	0.155	0.312	0.210	0.155	0.318	0.208	0.155	0.323	0.207
1.0	0.40	0.147	0.132	0.172	0.148	0.140	0.168	0.150	0.149	0.165
	0.60	0.139	0.056	0.145	0.139	0.068	0.139	0.142	0.082	0.132
	0.80	0.131	0.024	0.128	0.130	0.042	0.119	0.133	0.066	0.018
	1.00	0.125	0.022	0.121	0.120	0.051	0.106	0.118	0.096	0.089
	0.10	0.156	0.552	0.222	0.156	0.556	0.224	0.156	0.558	0.223
	0.20	0.151	0.348	0.195	0.151	0.352	0.195	0.151	0.354	0.196
	0.40	0.137	0.160	0.145	0.137	0.163	0.145	0.138	0.165	0.146
2.0	0.80	0.100	0.039	0.084	0.100	0.042	0.083	0.102	0.045	0.084
	1.20	0.071	0.008	0.058	0.072	0.011	0.057	0.074	0.016	0.056
	1.60	0.055	0.001	0.049	0.056	0.007	0.045	0.059	0.015	0.041
	2.00	0.045	0.008	0.045	0.045	0.019	0.036	0.046	0.037	0.025
	0.10	0.156	0.560	0.220	0.156	0.564	0.220	0.156	0.568	0.221
	0.20	0.151	0.355	0.191	0.151	0.359	0.191	0.151	0.362	0.192
	0.40	0.136	0.166	0.138	0.136	0.169	0.139	0.136	0.172	0.139
	0.80	0.096	0.043	0.072	0.096	0.046	0.072	0.097	0.048	0.072
3.0	1.20	0.064	0.012	0.042	0.064	0.013	0.042	0.065	0.015	0.042
	1.60	0.044	0.002	0.030	0.044	0.004	0.029	0.045	0.006	0.030
	2.00	0.032	0.000	0.025	0.032	0.001	0.024	0.034	0.003	0.023
	2.50	0.024	0.000	0.023	0.024	0.002	0.021	0.026	0.006	0.019
	3.00	0.019	0.003	0.022	0.019	0.008	0.017	0.020	0.016	0.010
	0.10	0.156	0.563	0.219	0.156	0.568	0.219	0.156	0.572	0.219
	0.20	0.151	0.358	0.189	0.151	0.362	0.189	0.151	0.366	0.189
	0.40	0.136	0.168	0.135	0.136	0.172	0.135	0.136	0.175	0.135
	0.80	0.096	0.045	0.066	0.096	0.048	0.066	0.096	0.050	0.066
	1.20	0.063	0.014	0.034	0.063	0.015	0.034	0.063	0.016	0.035
	1.60	0.041	0.004	0.020	0.041	0.005	0.020	0.042	0.006	0.021
5.0	2.00	0.028	0.001	0.014	0.028	0.002	0.013	0.028	0.002	0.014
	2.50	0.018	0.000	0.010	0.018	0.000	0.010	0.019	0.001	0.011
	3.00	0.013	-0.001	0.009	0.013	0.000	0.009	0.013	0.000	0.009
	3.50	0.009	-0.001	0.009	0.010	0.000	0.009	0.010	0.000	0.009
	4.00	0.007	0.000	0.009	0.008	0.000	0.009	0.008	0.001	0.008
	4.50	0.006	0.000	0.009	0.006	0.001	0.008	0.007	0.002	0.007
	5.00	0.005	0.001	0.009	0.005	0.002	0.007	0.006	0.005	0.004

TABLE 5.11

INFLUENCE VALUES I_{zq}, I_{xq} AND I_{xzq} (Milović and Tournier, 1971)

$(L/B) = 5.00$ Corner

h/B	z/B	ν = 0.15			ν = 0.30			ν = 0.45		
		I_{zq}	I_{xq}	I_{xzq}	I_{zq}	I_{xq}	I_{xzq}	I_{zq}	I_{xq}	I_{xzq}
	0.10	0.158	0.510	0.231	0.158	0.514	0.230	0.158	0.516	0.229
	0.20	0.155	0.312	0.211	0.155	0.318	0.209	0.155	0.322	0.207
1.0	0.40	0.148	0.132	0.174	0.148	0.140	0.169	0.150	0.148	0.164
	0.60	0.140	0.056	0.147	0.140	0.067	0.140	0.143	0.082	0.131
	0.80	0.133	0.024	0.131	0.132	0.042	0.120	0.135	0.067	0.106
	1.00	0.127	0.022	0.123	0.122	0.052	0.107	0.120	0.098	0.087
	0.10	0.156	0.555	0.224	0.156	0.556	0.224	0.156	0.555	0.224
	0.20	0.152	0.351	0.197	0.152	0.352	0.197	0.152	0.351	0.198
	0.40	0.138	0.162	0.150	0.138	0.163	0.149	0.138	0.164	0.150
2.0	0.80	0.102	0.039	0.091	0.102	0.042	0.090	0.103	0.044	0.089
	1.20	0.074	0.008	0.068	0.075	0.011	0.064	0.077	0.016	0.061
	1.60	0.059	0.001	0.060	0.060	0.007	0.053	0.063	0.016	0.046
	2.00	0.051	0.009	0.055	0.050	0.021	0.043	0.051	0.041	0.027
	0.10	0.156	0.567	0.221	0.156	0.567	0.222	0.156	0.567	0.222
	0.20	0.151	0.316	0.193	0.151	0.361	0.193	0.151	0.361	0.193
	0.40	0.137	0.170	0.142	0.137	0.171	0.142	0.137	0.171	0.143
	0.80	0.098	0.046	0.078	0.098	0.046	0.078	0.099	0.047	0.079
3.0	1.20	0.067	0.013	0.050	0.068	0.014	0.050	0.068	0.015	0.050
	1.60	0.048	0.002	0.039	0.049	0.004	0.038	0.050	0.005	0.038
	2.00	0.037	-0.001	0.036	0.038	0.001	0.034	0.039	0.004	0.032
	2.50	0.029	0.000	0.035	0.030	0.002	0.030	0.031	0.007	0.026
	3.00	0.025	0.004	0.033	0.025	0.011	0.025	0.025	0.021	0.014
	0.10	0.156	0.572	0.220	0.156	0.573	0.220	0.156	0.573	0.221
	0.20	0.151	0.366	0.190	0.151	0.367	0.191	0.151	0.367	0.191
	0.40	0.136	0.175	0.137	0.136	0.175	0.137	0.136	0.175	0.138
	0.80	0.097	0.049	0.069	0.097	0.050	0.070	0.097	0.050	0.070
	1.20	0.065	0.016	0.039	0.065	0.016	0.039	0.066	0.016	0.040
	1.60	0.045	0.005	0.025	0.045	0.005	0.026	0.045	0.006	0.027
5.0	2.00	0.032	0.001	0.020	0.032	0.002	0.020	0.033	0.002	0.021
	2.50	0.023	0.000	0.017	0.023	0.000	0.017	0.023	0.000	0.018
	3.00	0.017	-0.001	0.017	0.017	0.000	0.016	0.018	0.000	0.017
	3.50	0.014	-0.001	0.017	0.014	0.000	0.016	0.014	0.001	0.016
	4.00	0.011	-0.009	0.017	0.012	0.000	0.016	0.012	0.002	0.015
	4.50	0.010	0.000	0.017	0.010	0.001	0.015	0.011	0.004	0.013
	5.00	0.009	0.002	0.017	0.009	0.004	0.013	0.009	0.007	0.008

5.3.2 SMOOTH RIGID BASE

The corresponding case for a smooth rigid base has been considered by Sovinc (1961). A layer of finite lateral extent rather than one of infinite extent is considered by Sovinc, but the calculated values of stress and displacement are for a layer of sufficiently large lateral dimensions that they may be considered as applying to a layer of infinite lateral extent.

The distribution of vertical stress σ_z beneath the centre of the rectangle is shown in Figure 5.49 for various layer depths.

Values of σ_z/p at various points beneath the rectangle, abstracted from values tabulated by Sovinc (1961), are shown in Table 5.12.

Influence factors for vertical displacement ρ_z at various points on the rectangle are shown in Figure 5.50. In all cases, $\nu=0.5$.

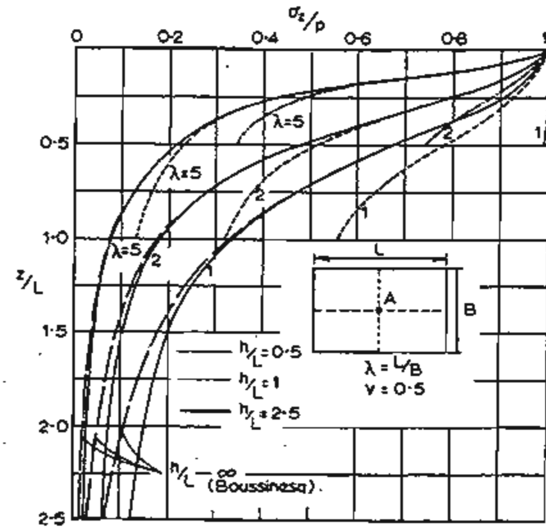


FIG.5.49 Vertical stress beneath centre of rectangle (Sovinc, 1961).

TABLE 5.12

VERTICAL STRESS BENEATH A RECTANGLE ON A FINITE LAYER WITH SMOOTH BASE (Sovinc, 1961)

Values of σ_z/p

L/B	h/B	Centre		Centre of short edge		Centre of long edge		Corner	
		z/h=0.5	1	z/h=0.5	1	z/h=0.5	1	z/h=0.5	1
1	1	.7835	.5616	.4571	.3835	.4571	.3835	-	-
	2.5	.2498	.1217	.1938	.1105	.1938	.1105	-	-
	5	.0744	.0325	.0695	.0317	.0695	.0317	.0616	.0282
2	1	.8808	.7410	.4412	.3842	.5359	.5110	.2678	.2636
	2	.5097	.3219	.2923	.2197	.3881	.2804	.2256	.1921
	5	.1333	.0622	.1050	.0564	.1242	.0606	.0988	.0552
5	2.5	.4847	.3459	.2032	.1675	.3930	.3199	.1826	.1541
	5	.2294	.1346	.1239	.0894	.2094	.1316	.1205	.0875
	12.5	.0526	.0246	.0406	.0214	-	-	-	-

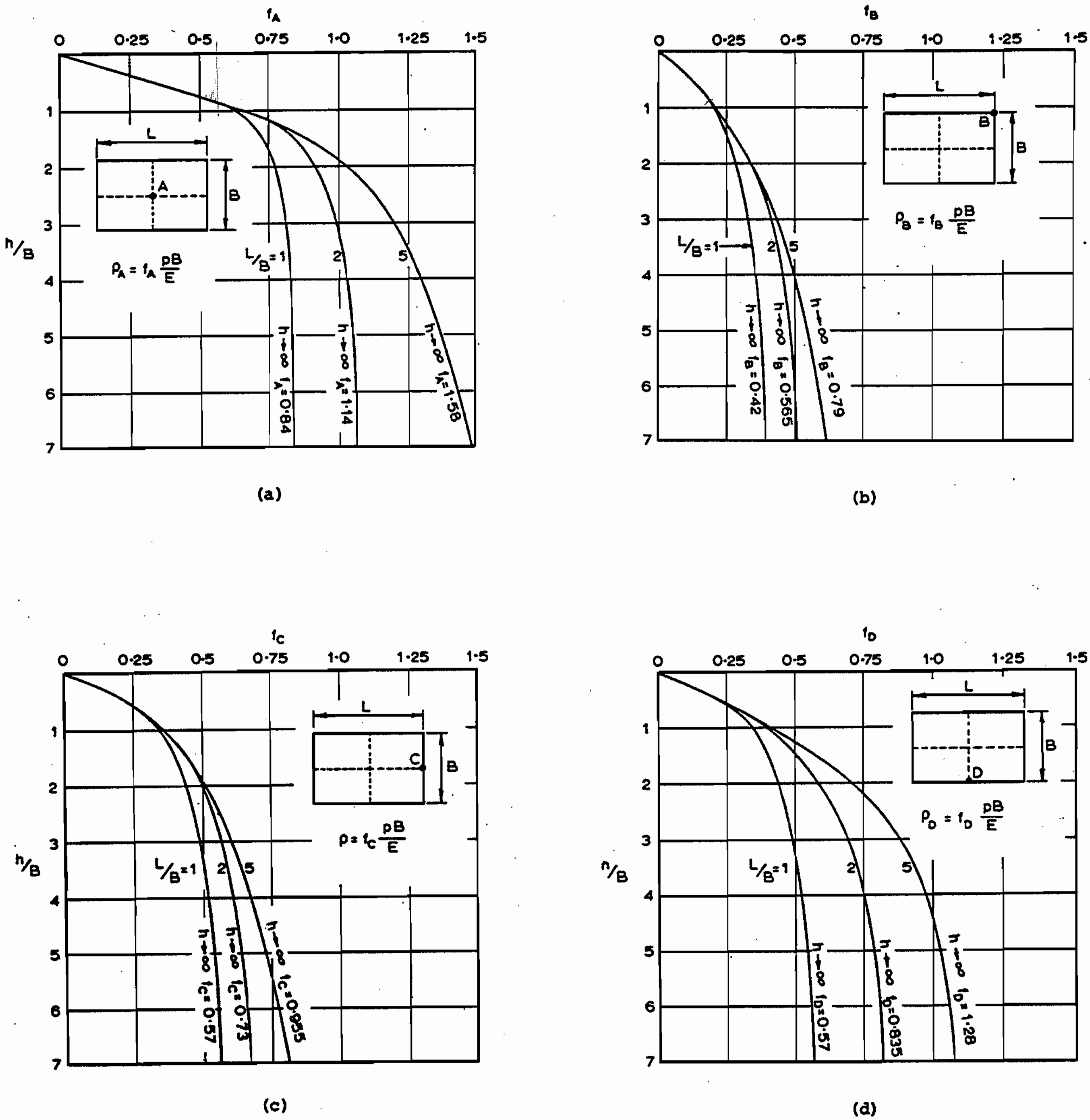


FIG.5.50 Vertical surface displacement at various points on rectangle. ν=0.5. (Sovinc, 1961).

5.4 Vertical Loading over Any Area

5.4.1 "NEWMARK CHARTS"

Burmister (1956) has presented charts for the horizontal stresses σ_x and σ_y, the shear stress τ_{xy} and the shear stress τ_{zx}. These charts are used in conjunction with tables giving influence factors for various depths in the layer. Values of ν of *0.2* and *0.4* are considered.

5.4.2 SECTOR CURVES

Sector curves for the normal and shear stresses, and the vertical and horizontal surface displacements, have been presented by Poulos (1967b) and are reproduced in Figures 5.51 to 5.72. The use of the sector method is described in Section 1.7.3.

In all cases, the stress beneath the sector is given by

$$\sigma = \frac{p.\delta\theta}{2\pi} \cdot I_s \qquad \ldots (5.11)$$

where $\delta\theta$ is the sector angle

and the displacement by

$$\rho = \frac{ph.\delta\theta}{2\pi E} \cdot I_s \qquad \ldots (5.12)$$

5.4.3 ANALYTICAL EXPRESSIONS FOR GENERAL PLANE STRAIN CASES

For any generalized surface loading, Holl (1939) gives expressions for the stresses at the rough rigid base, the maximum shear stress on the axis, and the vertical surface displacement.

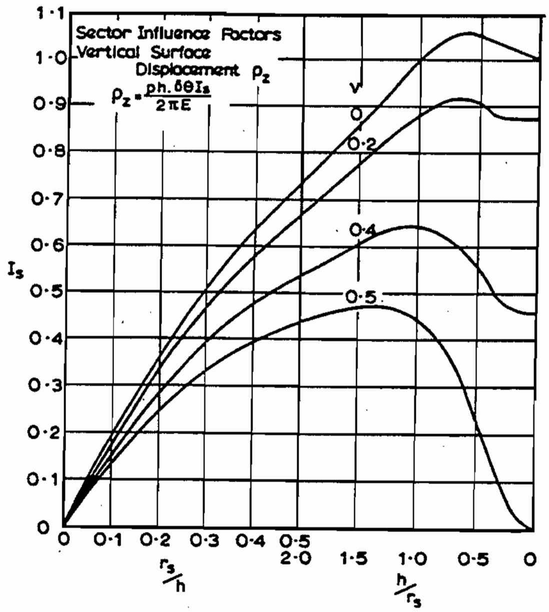


FIG. 5.51 Sector curves for ρ_z.

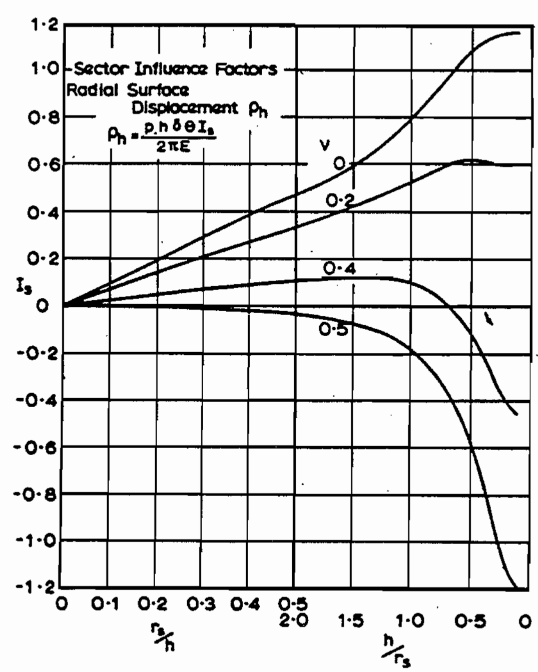


FIG.5.52 Sector curves for ρ_h.

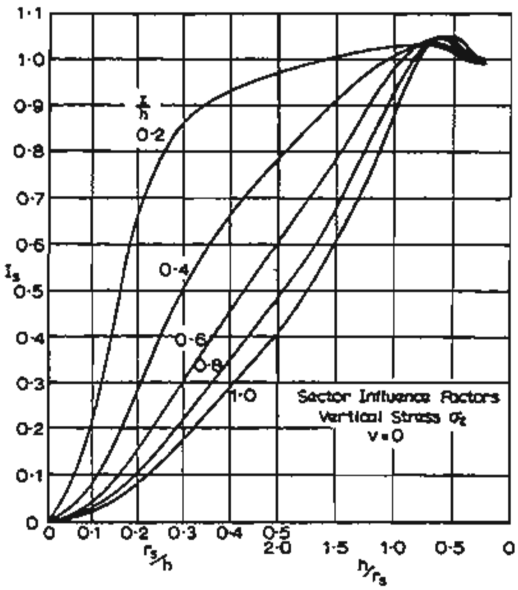


FIG.5.53 Sector curves for σ_z. ν=0.

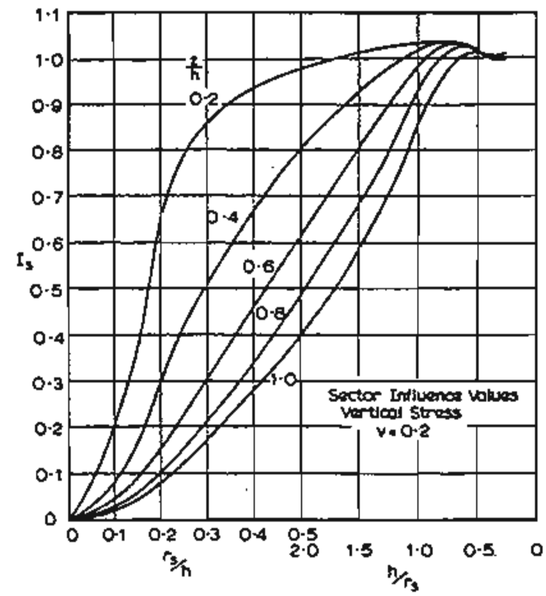


FIG.5.54 Sector curves for σ_z. ν=0.2.

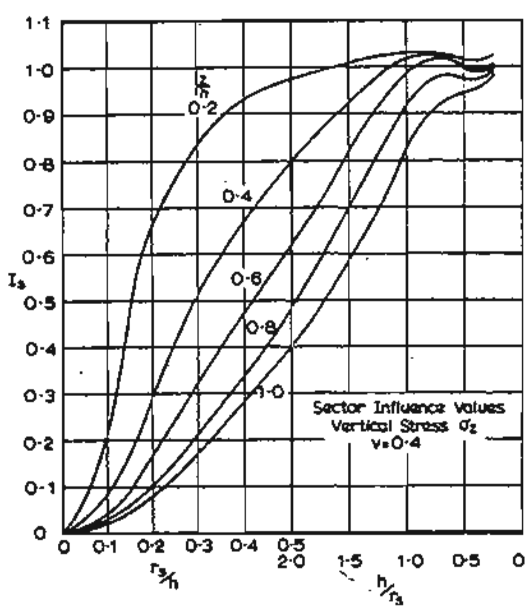


FIG.5.55 Sector curves for σ_z. ν=0.4.

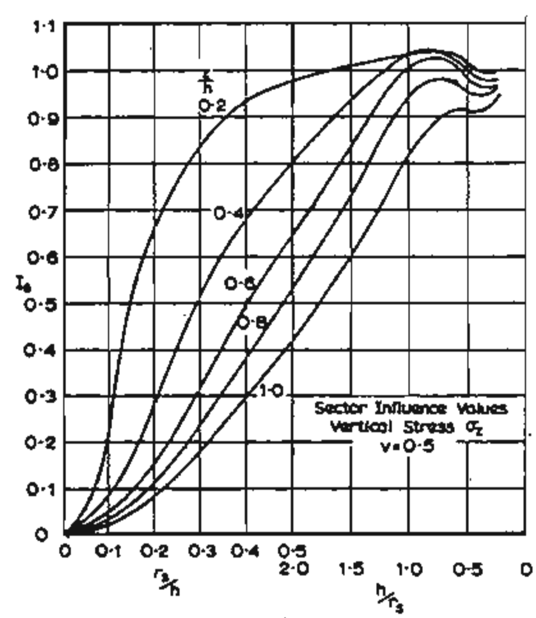


FIG.5.56 Sector curves for σ_z. ν=0.5.

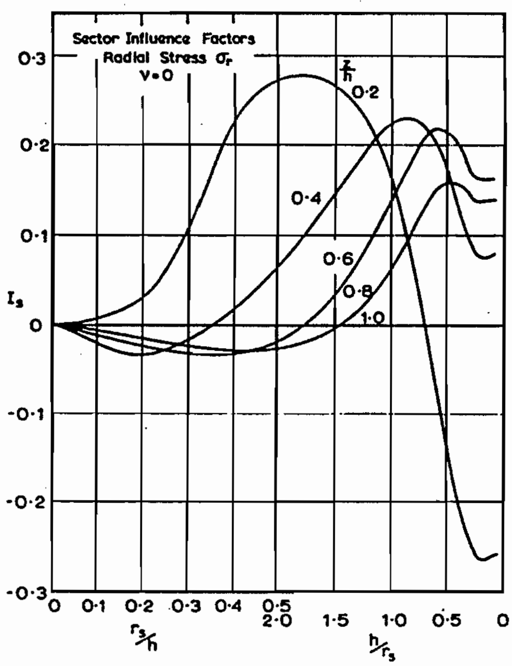


FIG.5.57 Sector curves for σ_r. ν=0.

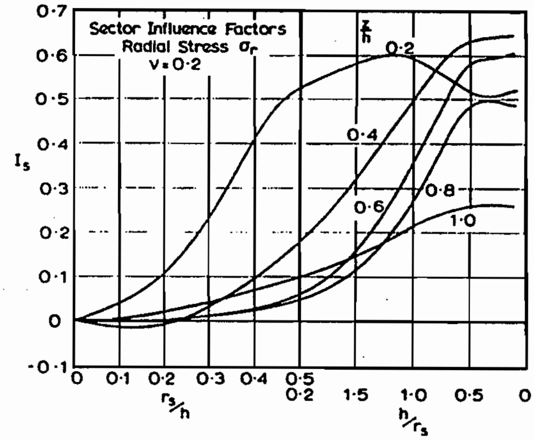


FIG.5.58 Sector curves for σ_r. ν=0.2.

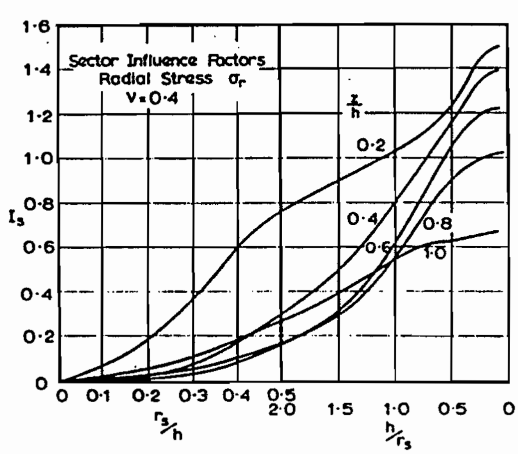


FIG.5.59 Sector curves for σ_r. ν=0.4.

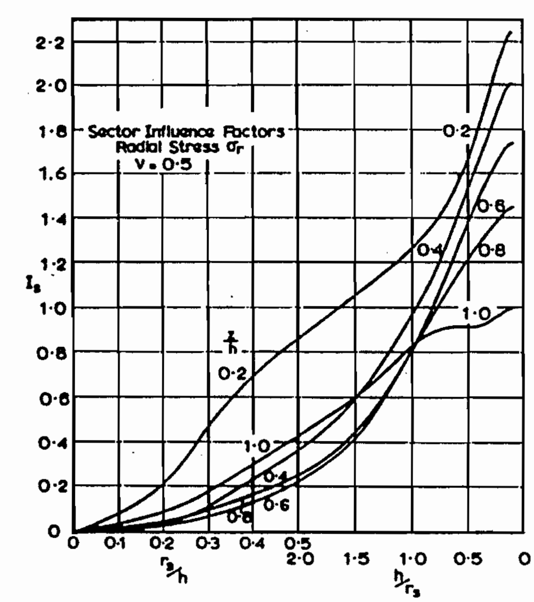


FIG.5.60 Sector curves for σ_r. ν=0.5.

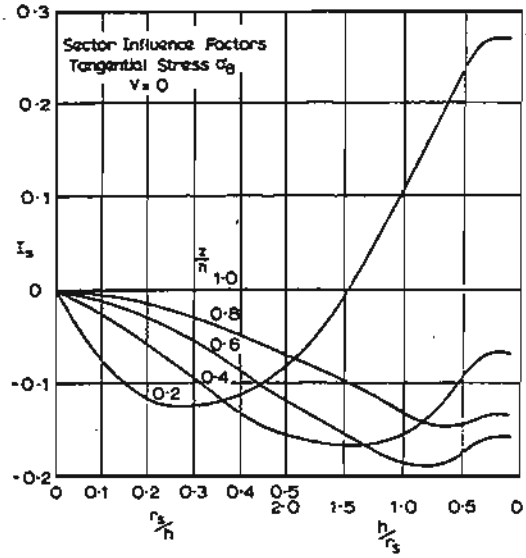


FIG.5.61 Sector curves for σ_θ. ν=0.

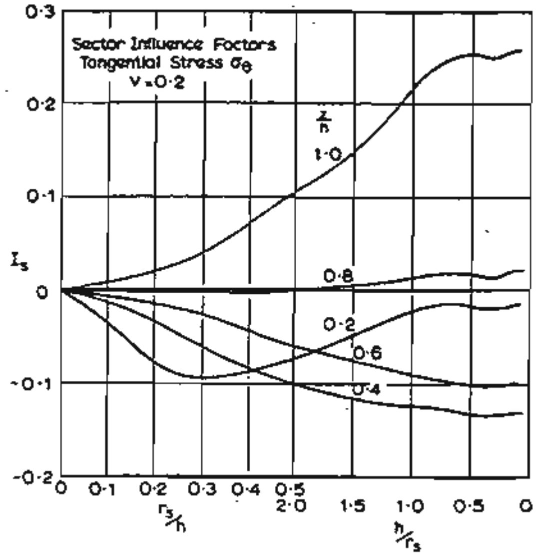


FIG.5.62 Sector curves for σ_θ. ν=0.2.

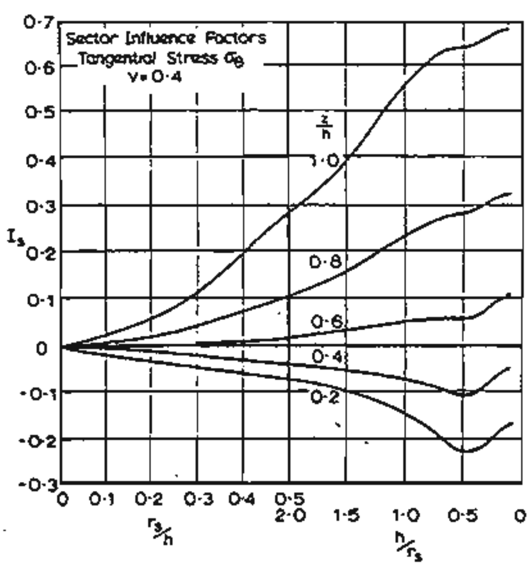


FIG.5.63 Sector curves for σ_θ. ν=0.4.

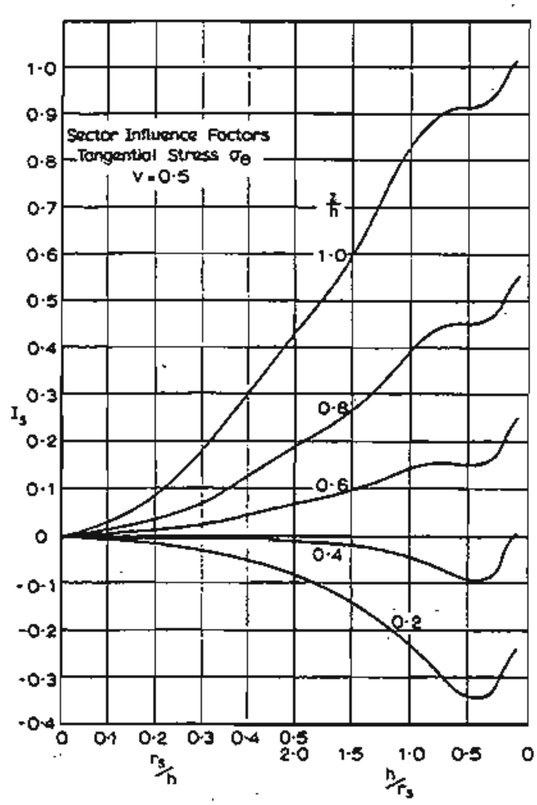


FIG.5.64 Sector curves for σ_θ. ν=0.5

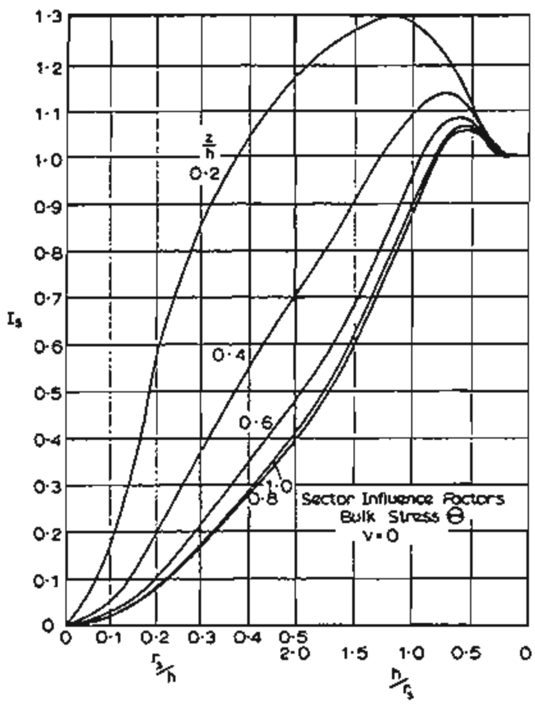


FIG.5.65 Sector curves for θ. ν=0.

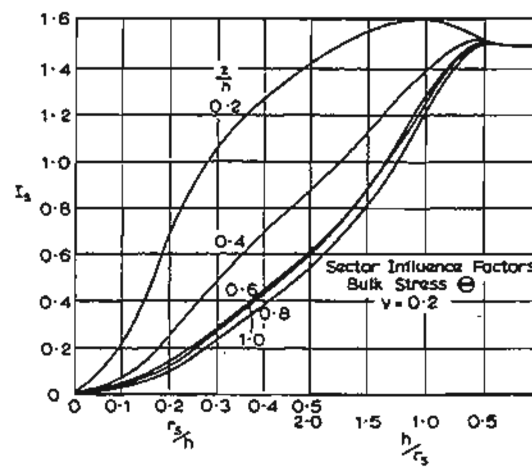


FIG.5.66 Sector curves for θ. ν=0.2.

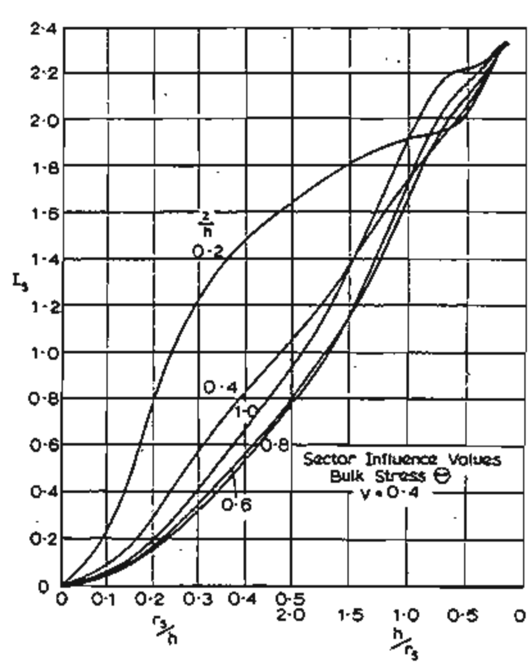


FIG.5.67 Sector curves for θ. ν=0.4.

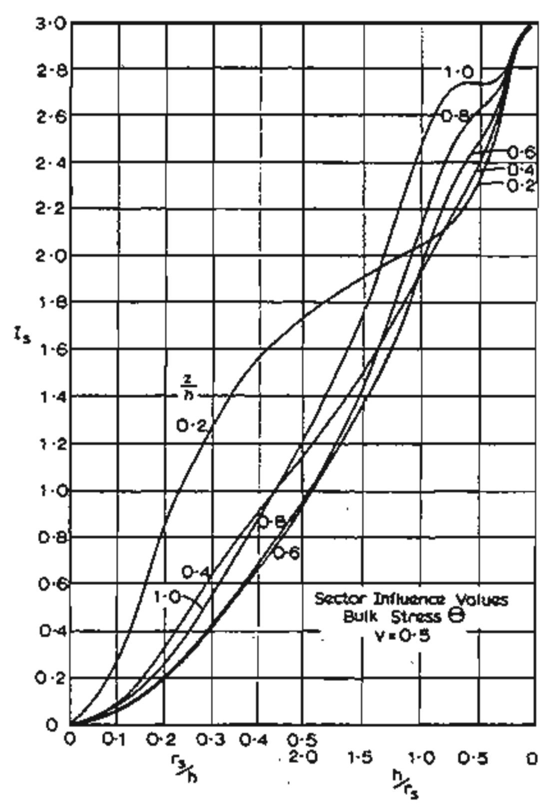


FIG.5.68 Sector curves for θ. ν=0.5.

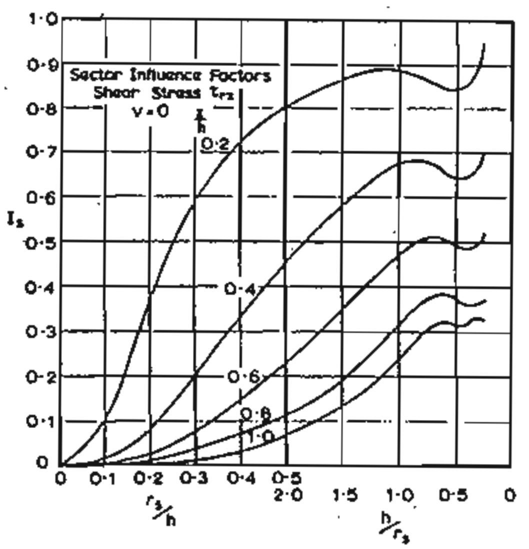


FIG.5.69 Sector curves for τ_{rz}. ν=0.

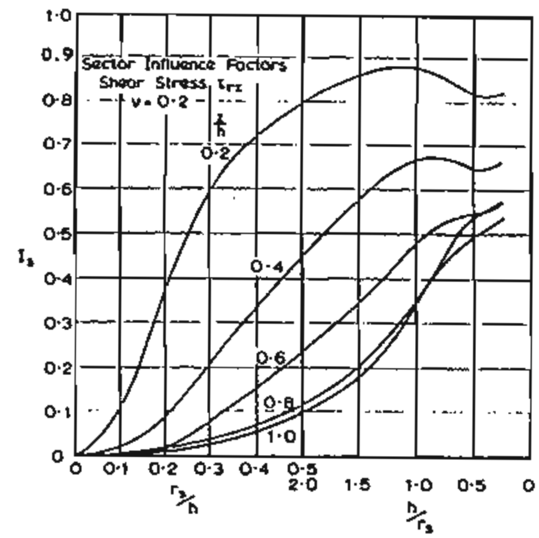


FIG.5.70 Sector curves for τ_{rz}. ν=0.2.

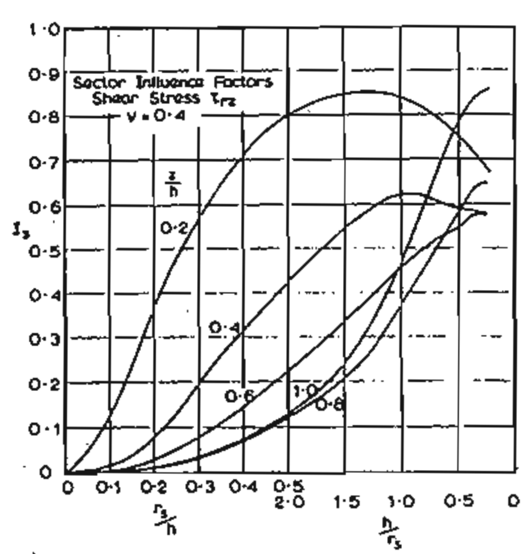


FIG.5.71 Sector curves for τ_{rz}. ν=0.4.

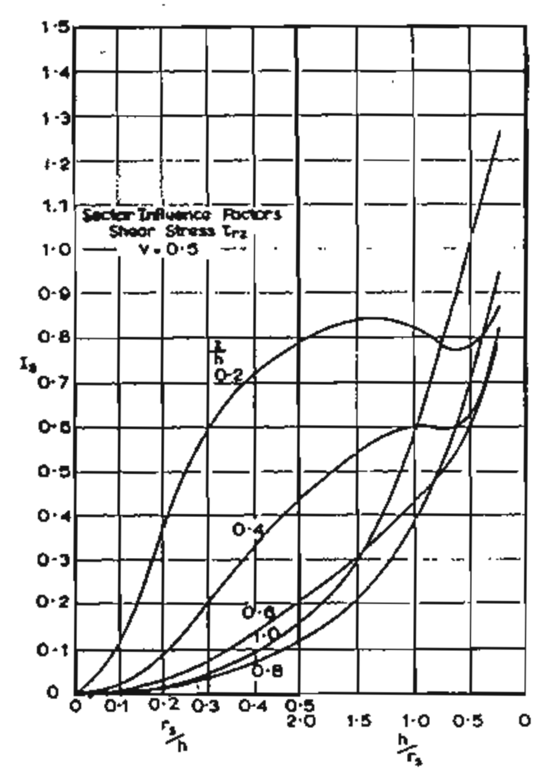


FIG.5.72 Sector curves for τ_{rz}. ν=0.5.

Chapter 6
SURFACE LOADING OF MULTI-LAYER SYSTEMS

6.1 Two-Layer Systems

Unless otherwise stated, the results given in this chapter are for adhesive interfaces between layers.

6.1.1 UNIFORM VERTICAL LOADING ON A CIRCULAR AREA (Fig.6.1)

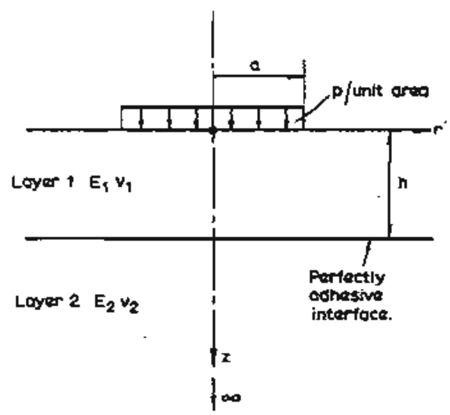


FIG. 6.1

Contours of vertical stress σ_z in a two-layer system, with $E_1/E_2=10$, $\nu_1=\nu_2=0.5$ and $h/a=1$, have been obtained by Fox (1948a) and are shown in Fig. 6.2. These contours are compared with those for the Boussinesq case ($E_1/E_2=1$).

The influence of the ratios E_1/E_2 and h/a on the vertical stress on the interface, obtained by Fox (1948a), are shown in Fig.6.3.

The variation of vertical stress σ_z on the axis at the interface, with E_1/E_2 and r/h, is shown in Fig.6.4.

Values of σ_z and $(\sigma_z-\sigma_r)$ on the axis (Fox, 1948a) are tabulated in Table 6.1. It should be noted that all the above stresses are for a perfectly rough interface between the layers.

Corresponding values of σ_z and $(\sigma_z-\sigma_r)$ for a perfectly smooth interface between the layers are tabulated in Table 6.2 (Fox, 1948a).

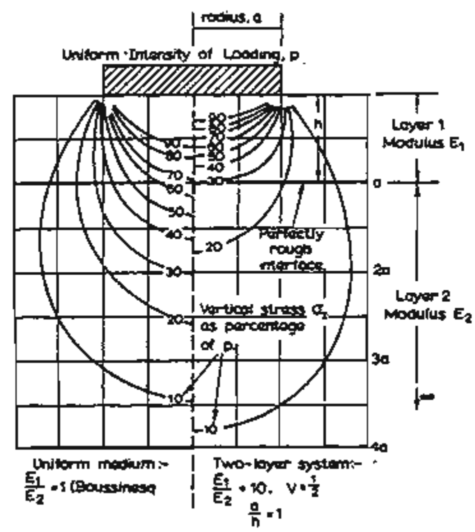


FIG.6.2 Vertical stress in uniform mass and two-layer system (Fox, 1948a).

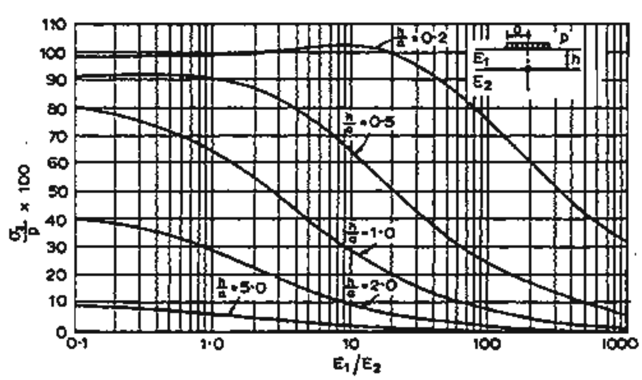


FIG.6.3 Vertical interface stress on axis (Fox, 1948a).

The complete distribution of stress within the two layer system, obtained by Fox (1948a), is shown in Figs. 6.5(a) to (c) for the case $a/h=1$ and for three values of E_1/E_2.

It should be noted that in Figs. 6.3 to 6.5, and in Tables 6.1 and 6.2, $\nu_1=\nu_2=0.5$.

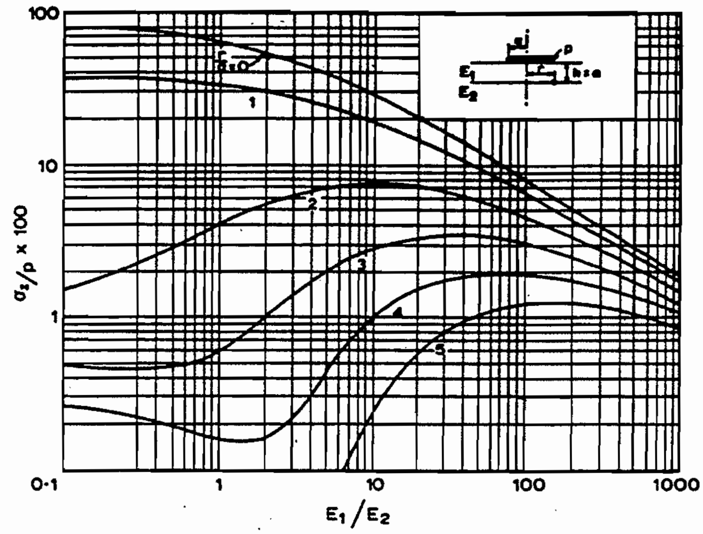
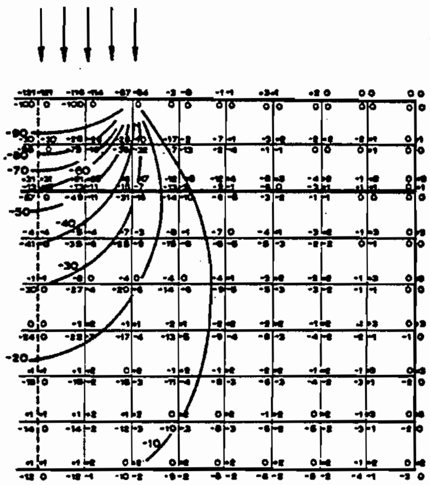


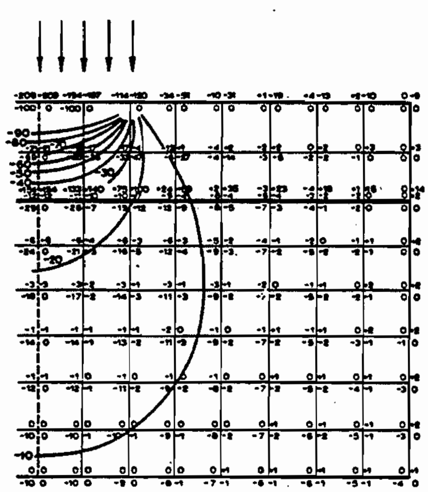
FIG.6.4 Vertical stress along interface for h/a = 1 (Fox, 1948a).



$\frac{a}{h}=1$ $E_1/E_2=2$

$\nu_1=\nu_2=0{\cdot}5$

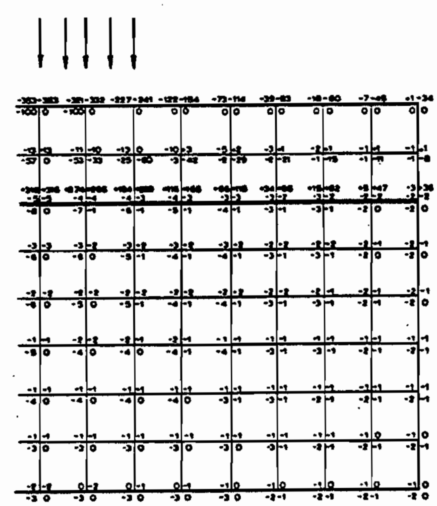
(a)



$\frac{a}{h}=1$ $E_1/E_2=10$

$\nu_1=\nu_2=0{\cdot}5$

(b)



$\frac{a}{h}=1$ $E_1/E_2=100$

$\nu_1=\nu_2=0{\cdot}5$

(c)

FIG.6.5 Stresses in two-layer system (Fox, 1948a)

σ_r	σ_θ
σ_z	τ_{rz}

Note: tension positive

TABLE 6.1

PERFECTLY ROUGH INTERFACE

AXIAL STRESSES σ_z AND $\sigma_z-\sigma_r$ AS PERCENTAGE OF APPLIED LOADING, IN THE LOWER LAYER OF A TWO-LAYER SYSTEM (Fox, 1948a)

	σ_z				$\sigma_z-\sigma_r$			
E_1/E_2	1	10	100	1000	1	10	100	1000
Depth below interface				$a/h = \frac{1}{2}$				
0	28.4	10.1	2.38	0.51	26.8	7.6	1.26	0.160
h	8.7	4.70	1.58	0.42	8.6	3.93	1.02	0.185
2h	4.03	2.78	1.17	0.35	4.00	2.48	0.85	0.195
3h	2.30	1.84	0.91	0.31	2.29	1.70	0.71	0.190
4h	1.48	1.29	0.74	0.28	1.48	1.23	0.61	0.185
				$a/h = 1$				
0	64.6	29.2	8.1	1.85	53.0	18.8	3.6	0.54
h	28.4	16.8	6.0	1.62	26.8	13.5	3.8	0.71
2h	14.5	10.5	4.6	1.43	14.1	9.2	3.3	0.79
3h	8.7	7.0	3.6	1.24	8.6	6.4	2.8	0.76
4h	5.7	5.0	2.9	1.10	5.6	4.7	2.4	0.73
				$a/h = 2$				
0	91.1	64.4	24.6	7.10	53.7	30.4	8.4	1.50
h	64.6	48.0	20.5	6.06	53.0	34.6	11.8	2.52
2h	42.4	34.0	16.5	5.42	38.4	28.2	11.4	2.90
3h	28.4	24.4	13.3	4.80	26.8	21.5	10.2	2.92
4h	20.0	18.1	10.8	4.28	19.2	16.6	8.8	2.82

TABLE 6.2

PERFECTLY SMOOTH INTERFACE

AXIAL STRESSES σ_z AND $\sigma_z-\sigma_r$ AS PERCENTAGE OF APPLIED LOADING, IN THE LOWER LAYER OF A TWO-LAYER SYSTEM (Fox, 1948a)

	σ_z				$\sigma_z-\sigma_r$			
E_1/E_2	1	10	100	1000	1	10	100	1000
Depth below interface				$a/h = \frac{1}{2}$				
0	31.0	10.5	2.41	0.51	0.00	0.00	0.00	0.00
½h					14.5	4.49		
h	14.1	6.3	1.83	0.45	11.5	4.32	0.96	0.16
2h	6.4	3.67	1.36	0.38	5.9	3.03	0.91	0.18
3h	3.46	2.35	1.05	0.33	3.32	2.08	0.79	0.19
4h	2.12	1.61	0.83	0.29	2.07	1.37	0.66	0.18
				$a/h = 1$				
0	72.2	30.5	8.2	1.90	0.00	0.00	0.00	0.00
½h					34.2	12.7		
h	43.7	21.7	6.8	1.72	33.1	14.2	3.41	0.59
2h	22.5	13.6	5.25	1.51	20.2	11.0	3.47	0.74
3h	12.8	8.9	4.09	1.33	12.1	7.8	3.05	0.77
4h	8.1	6.2	3.26	1.17	7.8	5.7	2.61	0.75
				$a/h = 2$				
0	102.5	67.7	24.9	6.7	0.00	0.00	0.00	0.00
½h					37.8	23.1		
h	86.9	57.6	22.5	6.3	52.6	32.0	9.9	1.96
2h	59.6	42.1	18.6	5.7	48.3	31.7	11.6	2.68

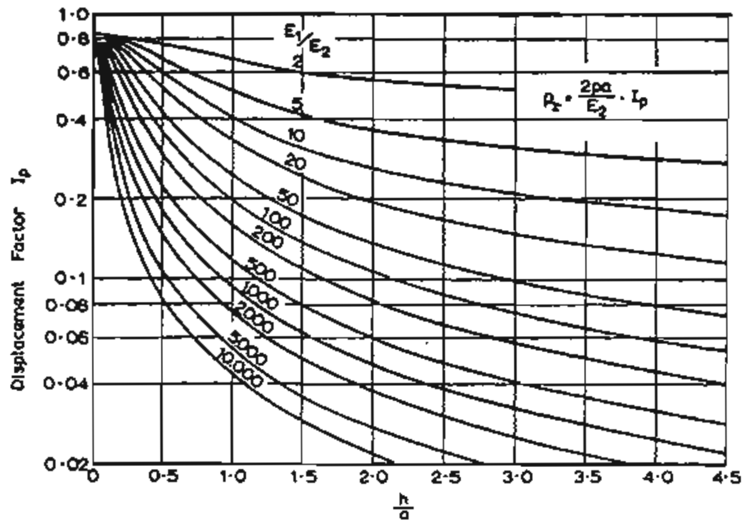


FIG.6.6 Burmister layered system theory. Vertical displacement at centre of circle. ν_1=0.2, ν_2=0.4. (Burmister, 1962).

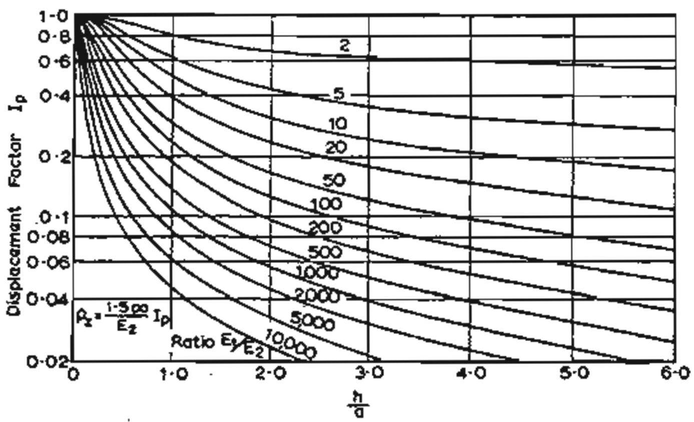


FIG.6.7 Burmister layered system theory. Vertical displacement at centre of circle. ν_1=ν_2=0.5. (Burmister, 1945).

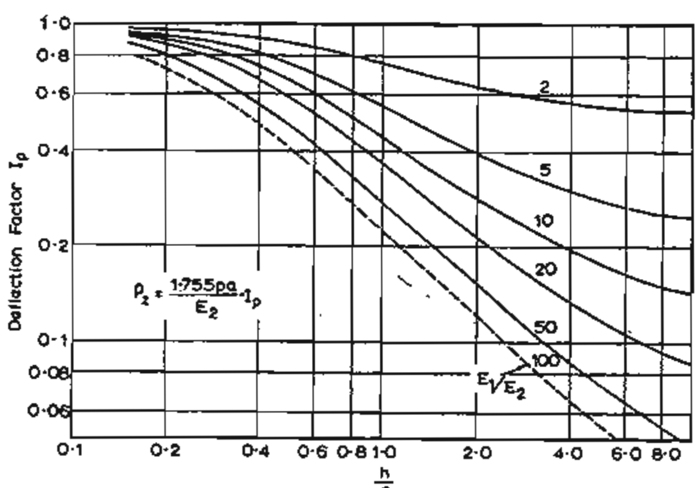


FIG.6.8 Vertical displacement at centre of circle. ν_1=ν_2=0.35. (Thenn de Barros, 1966).

Influence factors for the vertical surface displacement at the centre of the circular area are shown in Fig.6.6 for the case $\nu_1=0.2$, $\nu_2=0.4$ (Burmister, 1962), in Fig.6.7 for the case $\nu_1=\nu_2=0.5$ (Burmister, 1945) and in Fig.6.8 for $\nu_1=\nu_2=0.35$ (Thenn de Barros, 1966). An alternative interpretation of the results in Fig.6.7 has been given by Ueshita and Meyerhof (1967), who plot an equivalent value of Young's modulus (E_e) which may be used with the displacement influence factor for the centre of a circular area on a semi-infinite mass (see Section 3.3.1). The variation of E_e/E_2 with h/a and E_1/E_2 is shown in Fig.6.9.

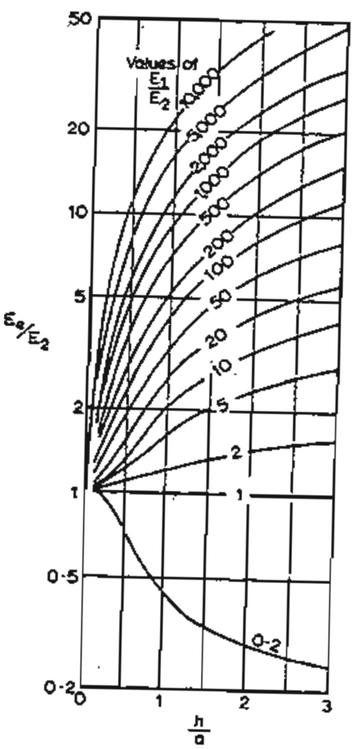


FIG.6.9 Equivalent modulus E_e of two-layer system. $\nu_1=\nu_2=0.5$. (Ueshita and Meyerhof, 1967).

$$\rho_z = \frac{1.5pa}{E_e}$$

A comprehensive series of solutions for the stresses, strains and displacements at selected points within a two-layer system has been obtained by Gerrard (1969). The following combinations of variables have been considered:

E_1/E_2	2, 5, 10
h/a	0.5, 1, 2, 4
ν_1	0.2, 0.35, 0.5
ν_2	0.2, 0.35, 0.5

6.1.2 UNIFORM HORIZONTAL LOADING ON A CIRCULAR AREA (Fig. 6.10).

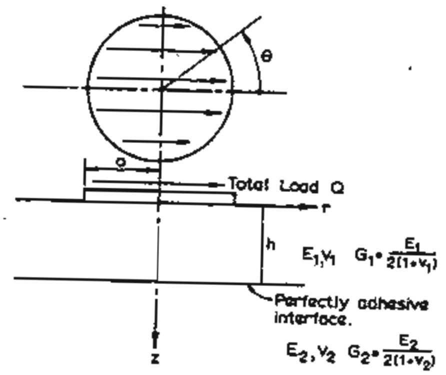


FIG.6.10

This problem has been considered by Westmann (1963). A polar diagram showing contours of interface normal stress are given in Fig. 6.11 for $E_2/E_1=0.5$, $\nu_1=\nu_2=0.5$ and $h/a=1$. Influence factors for the vertical stress σ_z at the interface are given in Figs. 6.12(a) and 6.12(b) for two ratios of G_2/G_1 and for $\nu_1=\nu_2=0.5$, and for various h/a values.

Influence factors for the shearing stresses τ_{zr} and $\tau_{z\theta}$ are given in Figs. 6.13 and 6.14.

Influence factors for the vertical surface displacement ρ_z are given in Fig. 6.15 and for the horizontal tangential displacement ρ_θ in Fig. 6.16.

A comprehensive series of solutions for the stresses, strains and displacements at selected points within a two-layer system have been obtained by Gerrard (1969), for the same combinations of variables as outlined in Section 6.1.1. As well as uniform horizontal loading, the case of a linearly increasing inward shear load, from zero at the centre to a maximum at the edge, has been evaluated.

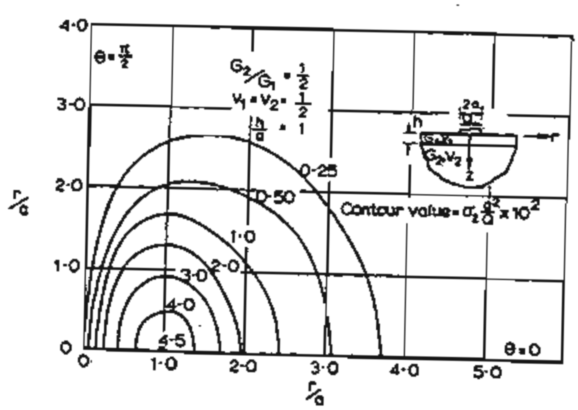


FIG.6.11 Contours of interface vertical stress (Westmann, 1963).

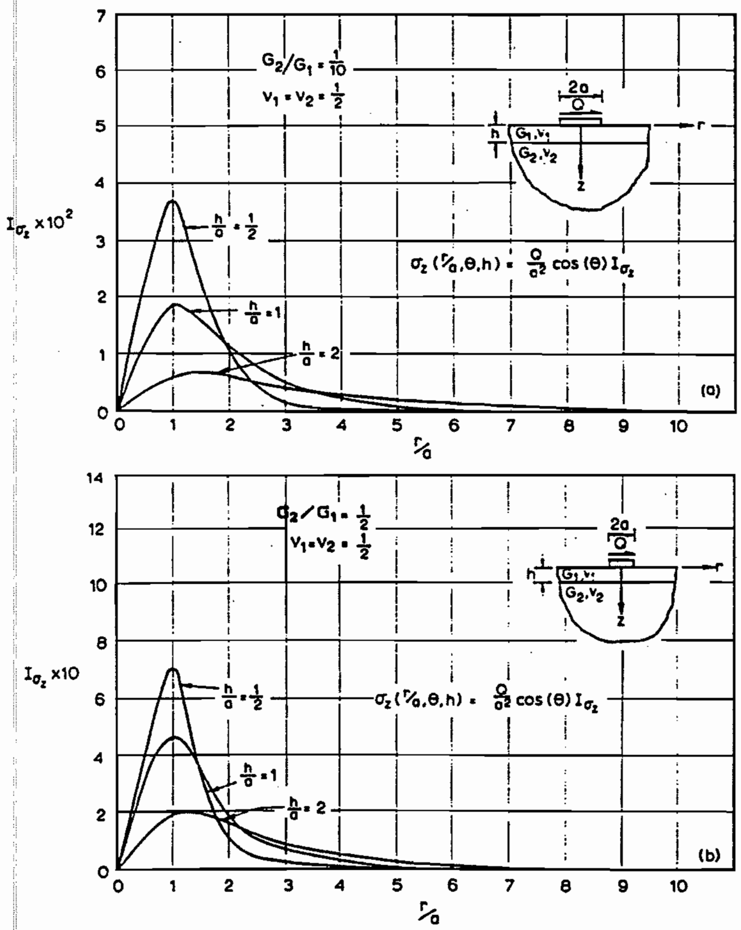


FIG.6.12 Influence factors for interface vertical stress (Westmann, 1963).

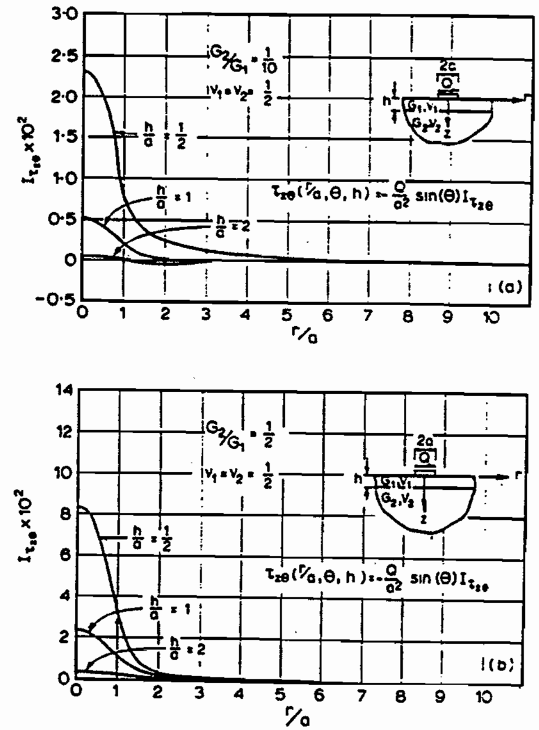


FIG.6.13 Influence factors for interface shear stress $\tau_{z\theta}$ (Westmann, 1963).

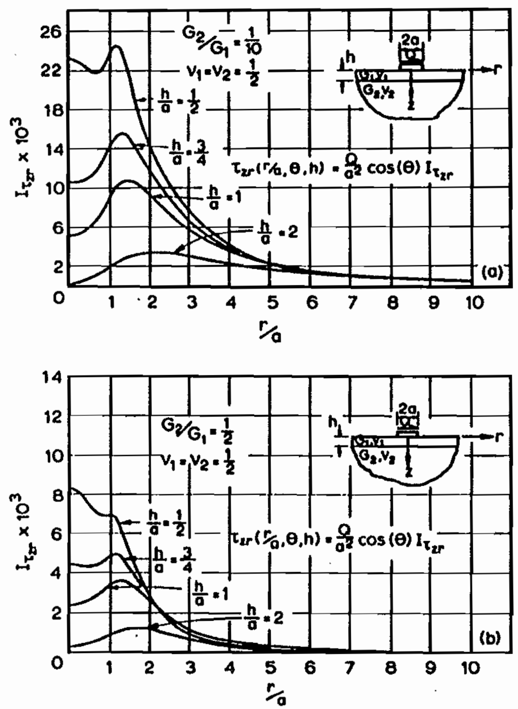


FIG.6.14 Influence factors for interface shear stress τ_{zr} (Westmann, 1963).

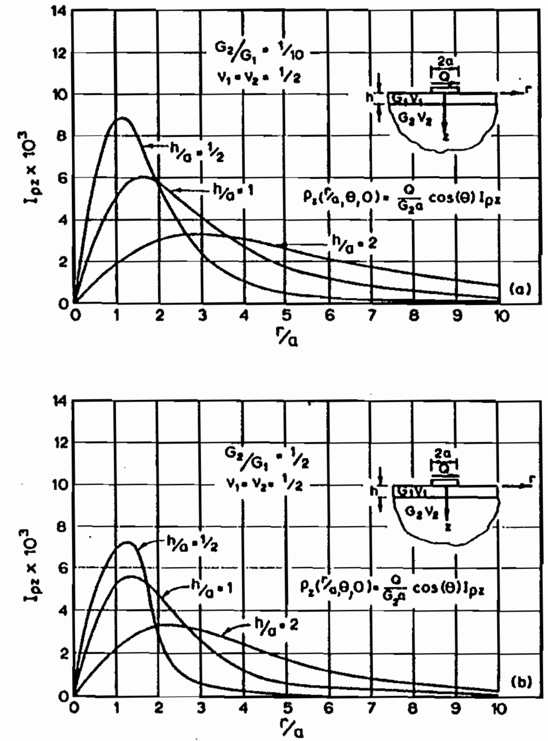


FIG.6.15 Influence factors for surface vertical displacement (Westmann, 1963).

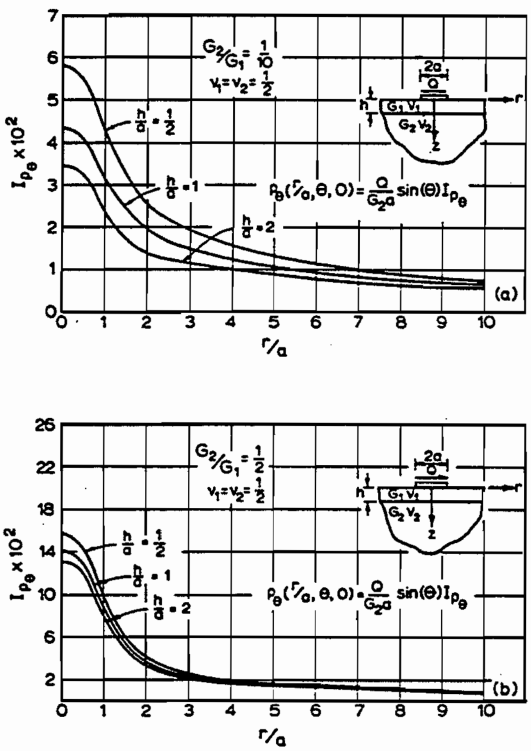


FIG.6.16 Influence factors for surface tangential displacement (Westmann, 1963).

6.2 Three-Layer Systems

6.2.1 UNIFORM VERTICAL LOADING ON A CIRCULAR AREA (Fig.6.17)

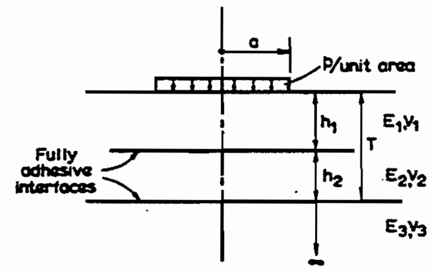


FIG.6.17

Stresses

An extensive tabulation of the stresses at the layer interfaces, on the axis of the circle, has been presented by Jones (1962). Peattie (1962) has presented these stresses graphically together with factors for the horizontal tensile strain at the first interface. Mitchell and Shen (1967) extended Peattie's graphs to include the horizontal stress at the first interface in layer 2 and the horizontal tensile and compressive strains at the lower interface.

An extensive tabulation of the radial stresses within the uppermost layer is given by Kirk (1966) for a wide range of elastic modulus ratios and for various values of ν in the layers.

The tables of stresses presented by Jones (1962) are reproduced in Tables 6.3 to 6.30. Again $\nu_1=\nu_2=\nu_3=0.5$. In these tables,

$$a_1 = a/h_2$$
$$H = h_1/h_2$$
$$k_1 = E_1/E_2$$
$$k_2 = E_2/E_3$$

The organization of these tables is as follows:

Stresses for all values of a_1 appear in a block, each table containing four blocks in ascending order of k_2.

The tables are for ascending values of k_1, and each set of four tables are for ascending values of H.

In all cases, the stresses are expressed as a fraction of the applied stress p.

The first three columns of stresses in the tables refer to the upper interface; the last three to the lower interface.

Compressive stresses are positive.

TABLE 6.3

(Jones, 1962)

$H = 0.125$
$k_1 = 0.2$

	a_1	σ_{z_1}	$\sigma_{z_1}-\sigma_{r_1}$	$\sigma_{z_1}-\sigma_{r_2}$	σ_{z_2}	$\sigma_{z_2}-\sigma_{r_2}$	$\sigma_{z_2}-\sigma_{r_3}$
$k_2 = 0.2$	0.1	0.66045	0.12438	0.62188	0.01557	0.00332	0.01659
	0.2	0.90249	0.13546	0.67728	0.06027	0.01278	0.06391
	0.4	0.95295	0.10428	0.52141	0.21282	0.04430	0.22150
	0.8	0.99520	0.09011	0.45053	0.56395	0.10975	0.54877
	1.6	1.00064	0.08777	0.43884	0.86258	0.13755	0.68777
	3.2	0.99970	0.04129	0.20643	0.94143	0.10147	0.50736
$k_2 = 2.0$	0.1	0.66048	0.12285	0.61424	0.00892	0.01693	0.00846
	0.2	0.90157	0.12916	0.64582	0.03480	0.06558	0.03279
	0.4	0.95120	0.08115	0.40576	0.12656	0.23257	0.11629
	0.8	0.99235	0.01823	0.09113	0.37307	0.62863	0.31432
	1.6	0.99918	-0.04136	-0.20680	0.74038	0.98754	0.49377
	3.2	1.00032	-0.03804	-0.19075	0.97137	0.82102	0.41051
$k_2 = 20.0$	0.1	0.66235	0.12032	0.60161	0.00256	0.03667	0.00183
	0.2	0.90415	0.11787	0.58933	0.01011	0.14336	0.00717
	0.4	0.95135	0.03474	0.17370	0.03838	0.52691	0.02635
	0.8	0.98778	-0.14872	-0.74358	0.13049	1.61727	0.08086
	1.6	0.99407	-0.50533	-2.52650	0.36442	3.58944	0.17947
	3.2	0.99821	-0.80990	-4.05023	0.76669	5.15409	0.25770
$k_2 = 200.0$	0.1	0.66266	0.11720	0.58599	0.00057	0.05413	0.00027
	0.2	0.90370	0.10495	0.52477	0.00226	0.21314	0.00107
	0.4	0.94719	-0.01709	-0.08543	0.00881	0.80400	0.00402
	0.8	0.99105	-0.34427	-1.72134	0.03259	2.67934	0.01340
	1.6	0.99146	-1.21129	-6.05643	0.11034	7.35978	0.03680
	3.2	0.99332	-2.89282	-14.46408	0.32659	16.22830	0.08114

TABLE 6.4

(Jones, 1962)

$H = 0.125$
$k_1 = 2.0$

	a_1	σ_{z_1}	$\sigma_{z_1}-\sigma_{r_1}$	$\sigma_{z_1}-\sigma_{r_2}$	σ_{z_2}	$\sigma_{z_2}-\sigma_{r_2}$	$\sigma_{z_2}-\sigma_{r_3}$
$k_2 = 0.2$	0.1	0.43055	0.71614	0.35807	0.01682	0.00350	0.01750
	0.2	0.78688	1.01561	0.50780	0.06511	0.01348	0.06741
	0.4	0.98760	0.83924	0.41962	0.23005	0.04669	0.23346
	0.8	1.01028	0.63961	0.31981	0.60886	0.11484	0.57418
	1.6	1.00647	0.65723	0.32862	0.90959	0.13726	0.68630
	3.2	0.99822	0.38165	0.19093	0.94322	0.09467	0.47335
$k_2 = 2.0$	0.1	0.42950	0.70622	0.35303	0.00896	0.01716	0.00858
	0.2	0.78424	0.97956	0.48989	0.03493	0.06647	0.03324
	0.4	0.98044	0.70970	0.35488	0.12667	0.23531	0.11766
	0.8	0.99434	0.22319	0.11164	0.36932	0.63003	0.31501
	1.6	0.99364	-0.19982	-0.09995	0.72113	0.97707	0.48853
	3.2	0.99922	-0.28916	-0.14461	0.96148	0.84030	0.42015
$k_2 = 20.0$	0.1	0.43022	0.69332	0.34662	0.00228	0.03467	0.00173
	0.2	0.78414	0.92086	0.46048	0.00899	0.13541	0.00677
	0.4	0.97493	0.46583	0.23297	0.03392	0.49523	0.02476
	0.8	0.97806	-0.66535	-0.33270	0.11350	1.49612	0.07481
	1.6	0.96921	-2.82859	-1.41430	0.31263	3.28512	0.16426
	3.2	0.98591	-5.27906	-2.63954	0.68433	5.05952	0.25298
$k_2 = 200.0$	0.1	0.42925	0.67488	0.33744	0.00046	0.04848	0.00024
	0.2	0.78267	0.85397	0.42698	0.00183	0.19043	0.00095
	0.4	0.97369	0.21165	0.10582	0.00711	0.71221	0.00356
	0.8	0.97295	-1.65954	-0.82977	0.02597	2.32652	0.01163
	1.6	0.95546	-6.47707	-3.23855	0.08700	6.26638	0.03133
	3.2	0.96377	-16.67376	-8.33691	0.26292	14.25621	0.07128

TABLE 6.5

(Jones, 1962) $H = 0.125$ $k_1 = 20.0$

	a_1	σ_{z1}	$\sigma_{z1}-\sigma_{r1}$	$\sigma_{z1}-\sigma_{r2}$	σ_{z2}	$\sigma_{z2}-\sigma_{r2}$	$\sigma_{z2}-\sigma_{r3}$
$k_2 = 0.2$	0.1	0.14648	1.80805	0.09040	0.01645	0.00322	0.01611
	0.2	0.39260	3.75440	0.18772	0.06407	0.01249	0.06244
	0.4	0.80302	5.11847	0.25592	0.23135	0.04421	0.22105
	0.8	1.06594	3.38600	0.16930	0.64741	0.11468	0.57342
	1.6	1.02942	1.81603	0.09080	1.00911	0.13687	0.68436
	3.2	0.99817	1.75101	0.08756	0.97317	0.07578	0.37890
$k_2 = 2.0$	0.1	0.14529	1.81178	0.09059	0.00810	0.01542	0.00771
	0.2	0.38799	3.76886	0.18844	0.03170	0.06003	0.03002
	0.4	0.78651	5.16717	0.25836	0.11650	0.21640	0.10820
	0.8	1.02218	3.43631	0.17182	0.34941	0.60493	0.30247
	1.6	0.99060	1.15211	0.05761	0.69014	0.97146	0.48573
	3.2	0.99893	-0.06894	-0.00345	0.93487	0.88358	0.44179
$k_2 = 20.0$	0.1	0.14447	1.80664	0.09033	0.00182	0.02985	0.00149
	0.2	0.38469	3.74573	0.18729	0.00716	0.11697	0.00585
	0.4	0.77394	5.05489	0.25274	0.02710	0.43263	0.02163
	0.8	0.98610	2.92533	0.14627	0.09061	1.33736	0.06687
	1.6	0.93712	-1.27093	-0.06355	0.24528	2.99215	0.14961
	3.2	0.96330	-7.35384	-0.36761	0.55490	5.06489	0.25324
$k_2 = 200.0$	0.1	0.14422	1.78941	0.08947	0.00033	0.04010	0.00020
	0.2	0.38388	3.68097	0.18405	0.00131	0.15781	0.00079
	0.4	0.77131	4.80711	0.24036	0.00505	0.59391	0.00297
	0.8	0.97701	1.90825	0.09541	0.01830	1.95709	0.00979
	1.6	0.91645	-5.28803	-0.26440	0.06007	5.25110	0.02626
	3.2	0.92662	-21.52546	-1.07627	0.18395	12.45058	0.06225

TABLE 6.6

(Jones, 1962) $H = 0.125$ $k_1 = 200.0$

	a_1	σ_{z1}	$\sigma_{z1}-\sigma_{r1}$	$\sigma_{z1}-\sigma_{r2}$	σ_{z2}	$\sigma_{z2}-\sigma_{r2}$	$\sigma_{z2}-\sigma_{r3}$
$k_2 = 0.2$	0.1	0.03694	2.87564	0.01438	0.01137	0.00201	0.01005
	0.2	0.12327	7.44285	0.03721	0.04473	0.00788	0.03940
	0.4	0.36329	15.41021	0.07705	0.16785	0.02913	0.14566
	0.8	0.82050	19.70261	0.09851	0.53144	0.08714	0.43568
	1.6	1.12440	7.02380	0.03512	1.03707	0.13705	0.68524
	3.2	0.99506	2.35459	0.01177	1.00400	0.06594	0.32971
$k_2 = 2.0$	0.1	0.03481	3.02259	0.01511	0.00549	0.00969	0.00485
	0.2	0.11491	8.02452	0.04012	0.02167	0.03812	0.01906
	0.4	0.33218	17.64175	0.08821	0.08229	0.14286	0.07143
	0.8	0.72695	27.27701	0.13639	0.27307	0.45208	0.22604
	1.6	1.00203	23.38638	0.11693	0.63916	0.90861	0.45430
	3.2	1.00828	11.87014	0.05935	0.92560	0.91469	0.45735
$k_2 = 20.0$	0.1	0.03336	3.17763	0.01589	0.00128	0.01980	0.00099
	0.2	0.10928	8.66097	0.04330	0.00509	0.07827	0.00391
	0.4	0.31094	20.12259	0.10061	0.01972	0.29887	0.01494
	0.8	0.65934	36.29943	0.18150	0.07045	1.01694	0.05085
	1.6	0.87931	49.40857	0.24704	0.20963	2.64313	0.13216
	3.2	0.93309	57.84369	0.28923	0.49938	4.89895	0.24495
$k_2 = 200.0$	0.1	0.03307	3.26987	0.01635	0.00025	0.02809	0.00014
	0.2	0.10810	9.02669	0.04513	0.00098	0.11136	0.00056
	0.4	0.30639	21.56482	0.10782	0.00386	0.43035	0.00215
	0.8	0.64383	41.89878	0.20949	0.01455	1.53070	0.00765
	1.6	0.84110	69.63157	0.34816	0.05011	4.56707	0.02284
	3.2	0.86807	120.95981	0.60481	0.15719	11.42045	0.05710

TABLE 6.7
(Jones, 1962)

$H = 0.25$
$k_1 = 0.2$

	a_1	σ_{z_1}	$\sigma_{z_1}-\sigma_{r_1}$	$\sigma_{z_1}-\sigma_{r_2}$	σ_{z_2}	$\sigma_{z_2}-\sigma_{r_2}$	$\sigma_{z_2}-\sigma_{r_3}$
$k_2 = 0.2$	0.1	0.27115	0.05598	0.27990	0.01259	0.00274	0.01370
	0.2	0.66109	0.12628	0.63138	0.04892	0.01060	0.05302
	0.4	0.90404	0.14219	0.71096	0.17538	0.03744	0.18722
	0.8	0.95659	0.12300	0.61499	0.48699	0.09839	0.49196
	1.6	0.99703	0.10534	0.52669	0.81249	0.13917	0.69586
	3.2	0.99927	0.05063	0.25317	0.92951	0.11114	0.55569
$k_2 = 2.0$	0.1	0.27103	0.05477	0.27385	0.00739	0.01409	0.00704
	0.2	0.66010	0.12136	0.60681	0.02893	0.05484	0.02742
	0.4	0.90120	0.12390	0.61949	0.10664	0.19780	0.09890
	0.8	0.94928	0.06482	0.32410	0.32617	0.56039	0.28019
	1.6	0.99029	-0.00519	-0.02594	0.69047	0.96216	0.48108
	3.2	1.00000	-0.02216	-0.11080	0.95608	0.87221	0.43610
$k_2 = 20.0$	0.1	0.26945	0.05192	0.25960	0.00222	0.03116	0.00156
	0.2	0.66161	0.11209	0.56046	0.00877	0.12227	0.00611
	0.4	0.90102	0.08622	0.43111	0.03354	0.45504	0.02275
	0.8	0.94012	-0.07351	-0.36756	0.11658	1.44285	0.07214
	1.6	0.97277	-0.40234	-2.01169	0.33692	3.37001	0.16850
	3.2	0.99075	-0.71901	-3.59542	0.73532	5.10060	0.25503
$k_2 = 200.0$	0.1	0.27072	0.04956	0.24778	0.00051	0.04704	0.00024
	0.2	0.65909	0.10066	0.50330	0.00202	0.18557	0.00093
	0.4	0.89724	0.04248	0.21242	0.00791	0.70524	0.00353
	0.8	0.93596	-0.24071	-1.20357	0.02961	2.40585	0.01203
	1.6	0.96370	-1.00743	-5.03714	0.10193	6.82481	0.03412
	3.2	0.97335	-2.54264	-12.71320	0.30707	15.45931	0.07730

TABLE 6.8
(Jones, 1962)

$H = 0.25$
$k_1 = 2.0$

	a_1	σ_{z_1}	$\sigma_{z_1}-\sigma_{r_1}$	$\sigma_{z_1}-\sigma_{r_2}$	σ_{z_2}	$\sigma_{z_2}-\sigma_{r_2}$	$\sigma_{z_2}-\sigma_{r_3}$
$k_2 = 0.2$	0.1	0.15577	0.28658	0.14329	0.01348	0.00277	0.01384
	0.2	0.43310	0.72176	0.36088	0.05259	0.01075	0.05377
	0.4	0.79551	1.03476	0.51738	0.19094	0.03842	0.19211
	0.8	1.00871	0.88833	0.44416	0.54570	0.10337	0.51687
	1.6	1.02425	0.66438	0.33219	0.90563	0.14102	0.70510
	3.2	0.99617	0.41539	0.20773	0.93918	0.09804	0.49020
$k_2 = 2.0$	0.1	0.15524	0.28362	0.14181	0.00710	0.01353	0.00677
	0.2	0.42809	0.70225	0.35112	0.02783	0.05278	0.02639
	0.4	0.77939	0.96634	0.48317	0.10306	0.19178	0.09589
	0.8	0.96703	0.66885	0.33442	0.31771	0.55211	0.27605
	1.6	0.98156	0.17331	0.08665	0.66753	0.95080	0.47540
	3.2	0.99840	-0.05691	-0.02846	0.93798	0.89390	0.44695
$k_2 = 20.0$	0.1	0.15436	0.27580	0.13790	0.00179	0.02728	0.00136
	0.2	0.42462	0.67115	0.33557	0.00706	0.10710	0.00536
	0.4	0.76647	0.84462	0.42231	0.02697	0.39919	0.01996
	0.8	0.92757	0.21951	0.10976	0.09285	1.26565	0.06328
	1.6	0.91393	-1.22411	-0.61205	0.26454	2.94860	0.14743
	3.2	0.95243	-3.04320	-1.52160	0.60754	4.89878	0.24494
$k_2 = 200.0$	0.1	0.15414	0.26776	0.13388	0.00036	0.03814	0.00019
	0.2	0.42365	0.63873	0.31937	0.00143	0.15040	0.00075
	0.4	0.76296	0.71620	0.35810	0.00557	0.57046	0.00285
	0.8	0.91600	-0.28250	-0.14125	0.02064	1.92636	0.00963
	1.6	0.88406	-3.09856	-1.54928	0.07014	5.35936	0.02680
	3.2	0.89712	-9.18214	-4.59107	0.21692	12.64318	0.06322

TABLE 6.9

(Jones, 1962) $H = 0.25$ $k_1 = 20.0$

	a_1	σ_{z_1}	$\sigma_{z_1}-\sigma_{r_1}$	$\sigma_{z_1}-\sigma_{r_2}$	σ_{z_2}	$\sigma_{z_2}-\sigma_{r_2}$	$\sigma_{z_2}-\sigma_{r_3}$
	0.1	0.04596	0.61450	0.03072	0.01107	0.00202	0.01011
	0.2	0.15126	1.76675	0.08834	0.04357	0.00793	0.03964
$k_2 = 0.2$	0.4	0.41030	3.59650	0.17983	0.16337	0.02931	0.14653
	0.8	0.85464	4.58845	0.22942	0.51644	0.08771	0.43854
	1.6	1.12013	2.31165	0.11558	1.01061	0.14039	0.70194
	3.2	0.99676	1.24415	0.06221	0.99168	0.07587	0.37934
	0.1	0.04381	0.63215	0.03162	0.00530	0.00962	0.00481
	0.2	0.14282	1.83766	0.09188	0.02091	0.03781	0.01891
$k_2 = 2.0$	0.4	0.37882	3.86779	0.19339	0.07933	0.14159	0.07079
	0.8	0.75904	5.50796	0.27540	0.26278	0.44710	0.22355
	1.6	0.98743	4.24281	0.21213	0.61673	0.90115	0.45058
	3.2	1.00064	1.97494	0.09876	0.91258	0.93254	0.46627
	0.1	0.04236	0.65003	0.03250	0.00123	0.01930	0.00096
	0.2	0.13708	1.90693	0.09535	0.00488	0.07623	0.00381
$k_2 = 20.0$	0.4	0.35716	4.13976	0.20699	0.01888	0.29072	0.01454
	0.8	0.68947	6.48948	0.32447	0.06741	0.98565	0.04928
	1.6	0.85490	6.95639	0.34782	0.20115	2.55231	0.12762
	3.2	0.90325	6.05854	0.30293	0.48647	4.76234	0.23812
	0.1	0.04204	0.65732	0.03287	0.00024	0.02711	0.00014
	0.2	0.13584	1.93764	0.09688	0.00095	0.10741	0.00054
$k_2 = 200.0$	0.4	0.35237	4.26004	0.21300	0.00372	0.41459	0.00207
	0.8	0.67286	6.94871	0.34743	0.01399	1.46947	0.00735
	1.6	0.81223	8.55770	0.42789	0.04830	4.36521	0.02183
	3.2	0.82390	10.63614	0.53181	0.15278	10.93570	0.05468

TABLE 6.10

(Jones, 1962) $H = 0.25$ $k_1 = 200.0$

	a_1	σ_{z_1}	$\sigma_{z_1}-\sigma_{r_1}$	$\sigma_{z_1}-\sigma_{r_2}$	σ_{z_2}	$\sigma_{z_2}-\sigma_{r_2}$	$\sigma_{z_2}-\sigma_{r_3}$
	0.1	0.01139	0.86644	0.00433	0.00589	0.00090	0.00451
	0.2	0.04180	2.71354	0.01357	0.02334	0.00357	0.01784
$k_2 = 0.2$	0.4	0.14196	6.83021	0.03415	0.09024	0.01365	0.06824
	0.8	0.42603	13.19664	0.06598	0.31785	0.04624	0.23118
	1.6	0.94520	13.79134	0.06896	0.83371	0.10591	0.52955
	3.2	1.10738	2.72901	0.01365	1.10259	0.08608	0.43037
	0.1	0.00909	0.96553	0.00483	0.00259	0.00407	0.00203
	0.2	0.03269	3.10763	0.01554	0.01027	0.01611	0.00806
$k_2 = 2.0$	0.4	0.10684	8.37852	0.04189	0.04000	0.06221	0.03110
	0.8	0.30477	18.95534	0.09478	0.14513	0.21860	0.10930
	1.6	0.66786	31.18909	0.15595	0.42940	0.58553	0.29277
	3.2	0.98447	28.98500	0.14493	0.84545	0.89191	0.44595
	0.1	0.00776	1.08738	0.00544	0.00065	0.00861	0.00043
	0.2	0.02741	3.59448	0.01797	0.00257	0.03421	0.00171
$k_2 = 20.0$	0.4	0.08634	10.30923	0.05155	0.01014	0.13365	0.00668
	0.8	0.23137	26.41442	0.13207	0.03844	0.49135	0.02457
	1.6	0.46835	57.46409	0.28732	0.13148	1.53833	0.07692
	3.2	0.71083	99.29034	0.49645	0.37342	3.60964	0.18048
	0.1	0.00744	1.19099	0.00596	0.00014	0.01311	0.00007
	0.2	0.02616	4.00968	0.02005	0.00056	0.05223	0.00026
$k_2 = 200.0$	0.4	0.08141	11.96405	0.05982	0.00224	0.20551	0.00103
	0.8	0.21293	32.97364	0.16487	0.00871	0.77584	0.00388
	1.6	0.40876	82.77997	0.41390	0.03234	2.63962	0.01320
	3.2	0.56613	189.37439	0.94687	0.11049	7.60287	0.03801

TABLE 6.11

(Jones, 1962)

$H = 0.5$
$k_1 = 0.2$

	a_1	σ_{z_1}	$\sigma_{z_1}-\sigma_{r_1}$	$\sigma_{z_1}-\sigma_{r_2}$	σ_{z_2}	$\sigma_{z_2}-\sigma_{r_2}$	$\sigma_{z_2}-\sigma_{r_3}$
$k_2 = 0.2$	0.1	0.07943	0.01705	0.08527	0.00914	0.00206	0.01030
	0.2	0.27189	0.05724	0.28621	0.03577	0.00804	0.04020
	0.4	0.66375	0.13089	0.65444	0.13135	0.02924	0.14622
	0.8	0.91143	0.15514	0.77571	0.38994	0.08369	0.41843
	1.6	0.96334	0.13250	0.66248	0.72106	0.13729	0.68647
	3.2	0.99310	0.06976	0.34879	0.89599	0.12674	0.63371
$k_2 = 2.0$	0.1	0.07906	0.01617	0.08085	0.00557	0.01074	0.00537
	0.2	0.27046	0.05375	0.26877	0.02190	0.04206	0.02103
	0.4	0.65847	0.11770	0.58848	0.08222	0.15534	0.07767
	0.8	0.89579	0.11252	0.56258	0.26429	0.47045	0.23523
	1.6	0.94217	0.04897	0.24486	0.60357	0.90072	0.45036
	3.2	0.99189	0.01380	0.06900	0.91215	0.94385	0.47192
$k_2 = 20.0$	0.1	0.07862	0.01439	0.07196	0.00175	0.02415	0.00121
	0.2	0.26873	0.04669	0.23345	0.00692	0.09519	0.00476
	0.4	0.65188	0.09018	0.45089	0.02676	0.36008	0.01800
	0.8	0.87401	0.01260	0.06347	0.09552	1.19151	0.05958
	1.6	0.89568	-0.24336	-1.21680	0.28721	2.95409	0.14770
	3.2	0.95392	-0.53220	-2.66100	0.66445	4.86789	0.24339
$k_2 = 200.0$	0.1	0.07820	0.01243	0.06213	0.00041	0.03682	0.00018
	0.2	0.26803	0.03912	0.19558	0.00163	0.14576	0.00073
	0.4	0.64904	0.06006	0.30029	0.00643	0.56051	0.00280
	0.8	0.86406	-0.10447	-0.52234	0.02436	1.96771	0.00984
	1.6	0.86677	-0.67154	-3.35768	0.08540	5.77669	0.02888
	3.2	0.89703	-1.86126	-9.30628	0.26467	13.63423	0.06817

TABLE 6.12

(Jones, 1962)

$H = 0.5$
$k_1 = 2.0$

	a_1	σ_{z_1}	$\sigma_{z_1}-\sigma_{r_1}$	$\sigma_{z_1}-\sigma_{r_2}$	σ_{z_2}	$\sigma_{z_2}-\sigma_{r_2}$	$\sigma_{z_2}-\sigma_{r_3}$
$k_2 = 0.2$	0.1	0.04496	0.08398	0.04199	0.00903	0.00181	0.00906
	0.2	0.15978	0.28904	0.14452	0.03551	0.00711	0.03554
	0.4	0.44523	0.72313	0.36156	0.13314	0.02634	0.13172
	0.8	0.83298	1.03603	0.51802	0.42199	0.07992	0.39962
	1.6	1.05462	0.83475	0.41737	0.85529	0.13973	0.69863
	3.2	0.99967	0.45119	0.22560	0.94506	0.10667	0.53336
$k_2 = 2.0$	0.1	0.04330	0.08250	0.04125	0.00465	0.00878	0.00439
	0.2	0.15325	0.28318	0.14159	0.01836	0.03454	0.01727
	0.4	0.42077	0.70119	0.35060	0.06974	0.12954	0.06477
	0.8	0.75683	0.96681	0.48341	0.23256	0.41187	0.20594
	1.6	0.93447	0.70726	0.35363	0.56298	0.85930	0.42965
	3.2	0.98801	0.33878	0.16939	0.88655	0.96353	0.48176
$k_2 = 20.0$	0.1	0.04193	0.08044	0.04022	0.00117	0.01778	0.00089
	0.2	0.14808	0.27574	0.13787	0.00464	0.07027	0.00351
	0.4	0.40086	0.67174	0.33587	0.01799	0.26817	0.01341
	0.8	0.69098	0.86191	0.43095	0.06476	0.91168	0.04558
	1.6	0.79338	0.39588	0.19794	0.19803	2.38377	0.11919
	3.2	0.85940	-0.41078	-0.20539	0.49238	4.47022	0.22351
$k_2 = 200.0$	0.1	0.04160	0.07864	0.03932	0.00024	0.02515	0.00013
	0.2	0.14676	0.26853	0.13426	0.00095	0.09968	0.00050
	0.4	0.39570	0.64303	0.32152	0.00374	0.38497	0.00192
	0.8	0.67257	0.74947	0.37474	0.01416	1.36766	0.00684
	1.6	0.74106	-0.02761	-0.01381	0.04972	4.08937	0.02045
	3.2	0.75176	-1.88545	-0.94273	0.15960	10.25631	0.05128

TABLE 6.13
(Jones, 1962)

$H = 0.5$
$k_1 = 20.0$

	a_1	σ_{z_1}	$\sigma_{z_1}-\sigma_{r_1}$	$\sigma_{z_1}-\sigma_{r_2}$	σ_{z_2}	$\sigma_{z_2}-\sigma_{r_2}$	$\sigma_{z_2}-\sigma_{r_3}$
$k_2 = 0.2$	0.1	0.01351	0.16526	0.00826	0.00596	0.00098	0.00488
	0.2	0.05079	0.58918	0.02946	0.02361	0.00386	0.01929
	0.4	0.16972	1.66749	0.08337	0.09110	0.01474	0.07369
	0.8	0.47191	3.23121	0.16156	0.31904	0.04967	0.24834
	1.6	0.97452	3.54853	0.17743	0.82609	0.11279	0.56395
	3.2	1.09911	1.27334	0.06367	1.08304	0.09527	0.47637
$k_2 = 2.0$	0.1	0.01122	0.17997	0.00900	0.00259	0.00440	0.00220
	0.2	0.04172	0.64779	0.03239	0.01028	0.01744	0.00872
	0.4	0.13480	1.89817	0.09491	0.03998	0.06722	0.03361
	0.8	0.35175	4.09592	0.20480	0.14419	0.23476	0.11738
	1.6	0.70221	6.22002	0.31100	0.42106	0.62046	0.31023
	3.2	0.97420	5.41828	0.27091	0.82256	0.93831	0.46916
$k_2 = 20.0$	0.1	0.00990	0.19872	0.00994	0.00063	0.00911	0.00046
	0.2	0.03648	0.72264	0.03613	0.00251	0.03620	0.00181
	0.4	0.11448	2.19520	0.10976	0.00988	0.14116	0.00706
	0.8	0.27934	5.24726	0.26236	0.03731	0.51585	0.02579
	1.6	0.50790	10.30212	0.51511	0.12654	1.59341	0.07967
	3.2	0.70903	16.38520	0.81926	0.35807	3.69109	0.18455
$k_2 = 200.0$	0.1	0.00960	0.21440	0.01072	0.00013	0.01355	0.00007
	0.2	0.03526	0.78493	0.03925	0.00054	0.05395	0.00027
	0.4	0.10970	2.44430	0.12221	0.00214	0.21195	0.00106
	0.8	0.26149	6.23424	0.31172	0.00831	0.79588	0.00398
	1.6	0.45078	14.11490	0.70574	0.03070	2.67578	0.01338
	3.2	0.57074	29.95815	1.49791	0.10470	7.61457	0.03807

TABLE 6.14
(Jones, 1962)

$H = 0.5$
$k_1 = 200.0$

	a_1	σ_{z_1}	$\sigma_{z_1}-\sigma_{r_1}$	$\sigma_{z_1}-\sigma_{r_2}$	σ_{z_2}	$\sigma_{z_2}-\sigma_{r_2}$	$\sigma_{z_2}-\sigma_{r_3}$
$k_2 = 0.2$	0.1	0.00363	0.22388	0.00112	0.00256	0.00033	0.00163
	0.2	0.01414	0.81903	0.00410	0.01021	0.00130	0.00648
	0.4	0.05256	2.52558	0.01263	0.04014	0.00506	0.02529
	0.8	0.18107	6.11429	0.03057	0.15048	0.01844	0.09221
	1.6	0.53465	10.82705	0.05414	0.48201	0.05399	0.26993
	3.2	1.04537	9.34212	0.04671	1.00671	0.08624	0.43121
$k_2 = 2.0$	0.1	0.00215	0.26620	0.00133	0.00094	0.00128	0.00064
	0.2	0.00826	0.98772	0.00494	0.00373	0.00509	0.00254
	0.4	0.02946	3.19580	0.01598	0.01474	0.01996	0.00998
	0.8	0.09508	8.71973	0.04360	0.05622	0.07434	0.03717
	1.6	0.27135	20.15765	0.10079	0.19358	0.23838	0.11919
	3.2	0.62399	34.25229	0.17126	0.52912	0.54931	0.27466
$k_2 = 20.0$	0.1	0.00149	0.31847	0.00159	0.00023	0.00257	0.00013
	0.2	0.00564	1.19598	0.00598	0.00094	0.01025	0.00051
	0.4	0.01911	4.02732	0.02014	0.00372	0.04047	0.00202
	0.8	0.05574	12.00885	0.06004	0.01453	0.15452	0.00773
	1.6	0.13946	32.77028	0.16385	0.05399	0.53836	0.02692
	3.2	0.30247	77.62943	0.38815	0.18091	1.56409	0.07820
$k_2 = 200.0$	0.1	0.00133	0.37065	0.00185	0.00005	0.00387	0.00002
	0.2	0.00498	1.40493	0.00702	0.00022	0.01544	0.00008
	0.4	0.01649	4.86215	0.02431	0.00086	0.06118	0.00031
	0.8	0.04553	15.33902	0.07670	0.00340	0.23698	0.00118
	1.6	0.10209	45.93954	0.22970	0.01315	0.86345	0.00432
	3.2	0.18358	128.13051	0.64065	0.04854	2.80877	0.01404

TABLE 6.15
(Jones, 1962)

$H = 1.0$
$k_1 = 0.2$

	a_1	σ_{z_1}	$\sigma_{z_1}-\sigma_{r_1}$	$\sigma_{z_1}-\sigma_{r_2}$	σ_{z_2}	$\sigma_{z_2}-\sigma_{r_2}$	$\sigma_{z_2}-\sigma_{r_3}$
$k_2 = 0.2$	0.1	0.02090	0.00464	0.02320	0.00541	0.00128	0.00638
	0.2	0.08023	0.01773	0.08865	0.02138	0.00503	0.02515
	0.4	0.27493	0.05976	0.29878	0.08125	0.01903	0.09516
	0.8	0.67330	0.13818	0.69092	0.26887	0.06192	0.30960
	1.6	0.92595	0.15978	0.79888	0.60229	0.13002	0.65010
	3.2	0.95852	0.09722	0.48612	0.82194	0.14348	0.71742
$k_2 = 2.0$	0.1	0.02045	0.00410	0.02052	0.00356	0.00687	0.00343
	0.2	0.07845	0.01561	0.07805	0.01410	0.02713	0.01357
	0.4	0.26816	0.05166	0.25828	0.05427	0.10351	0.05175
	0.8	0.65090	0.11111	0.55555	0.18842	0.34703	0.17351
	1.6	0.88171	0.10364	0.51819	0.48957	0.79986	0.39993
	3.2	0.94153	0.06967	0.34835	0.81663	0.99757	0.49879
$k_2 = 20.0$	0.1	0.01981	0.00306	0.01529	0.00118	0.01591	0.00080
	0.2	0.07587	0.01145	0.05726	0.00471	0.06310	0.00316
	0.4	0.25817	0.03540	0.17702	0.01846	0.24396	0.01220
	0.8	0.61544	0.05163	0.25817	0.06839	0.86114	0.04306
	1.6	0.78884	-0.07218	-0.36091	0.21770	2.36054	0.11803
	3.2	0.82936	-0.25569	-1.27847	0.53612	4.28169	0.21408
$k_2 = 200.0$	0.1	0.01952	0.00214	0.01068	0.00028	0.02412	0.00012
	0.2	0.07473	0.00777	0.03883	0.00110	0.09587	0.00048
	0.4	0.25368	0.02076	0.10382	0.00436	0.37417	0.00187
	0.8	0.59853	-0.00538	-0.02690	0.01679	1.36930	0.00685
	1.6	0.73387	-0.28050	-1.40250	0.06020	4.23805	0.02119
	3.2	0.70248	-0.90965	-4.54826	0.19189	10.36507	0.05183

TABLE 6.16
(Jones, 1962

$H = 1.0$
$k_1 = 2.0$

	a_1	σ_{z_1}	$\sigma_{z_1}-\sigma_{r_1}$	$\sigma_{z_1}-\sigma_{r_2}$	σ_{z_2}	$\sigma_{z_2}-\sigma_{r_2}$	$\sigma_{z_2}-\sigma_{r_3}$
$k_2 = 0.2$	0.1	0.01241	0.02186	0.01093	0.00490	0.00096	0.00478
	0.2	0.04816	0.08396	0.04198	0.01943	0.00378	0.01890
	0.4	0.17203	0.28866	0.14433	0.07496	0.01448	0.07241
	0.8	0.48612	0.71684	0.35842	0.26193	0.04924	0.24620
	1.6	0.91312	0.97206	0.48603	0.67611	0.11558	0.57790
	3.2	1.04671	0.60091	0.30046	0.95985	0.12527	0.62637
$k_2 = 2.0$	0.1	0.01083	0.02179	0.01090	0.00241	0.00453	0.00227
	0.2	0.04176	0.08337	0.04169	0.00958	0.01797	0.00899
	0.4	0.14665	0.28491	0.14246	0.03724	0.06934	0.03467
	0.8	0.39942	0.71341	0.35670	0.13401	0.24250	0.12125
	1.6	0.71032	1.02680	0.51340	0.38690	0.63631	0.31815
	3.2	0.92112	0.90482	0.45241	0.75805	0.97509	0.48754
$k_2 = 20.0$	0.1	0.00963	0.02249	0.01124	0.00061	0.00920	0.00046
	0.2	0.03697	0.08618	0.04309	0.00241	0.03654	0.00183
	0.4	0.12805	0.29640	0.14820	0.00950	0.14241	0.00712
	0.8	0.33263	0.76292	0.38146	0.03578	0.51815	0.02591
	1.6	0.52721	1.25168	0.62584	0.12007	1.56503	0.07825
	3.2	0.65530	1.70723	0.85361	0.33669	3.51128	0.17556
$k_2 = 200.0$	0.1	0.00925	0.02339	0.01170	0.00013	0.01319	0.00007
	0.2	0.03561	0.09018	0.04509	0.00051	0.05252	0.00026
	0.4	0.12348	0.31470	0.15735	0.00202	0.20609	0.00103
	0.8	0.31422	0.83274	0.41637	0.00783	0.76955	0.00385
	1.6	0.46897	1.53521	0.76760	0.02874	2.53100	0.01265
	3.2	0.51161	2.76420	1.38210	0.09751	6.99283	0.03496

TABLE 6.17
(Jones, 1962)

$H = 1.0$
$k_1 = 20.0$

	a_1	σ_{z_1}	$\sigma_{z_1}-\sigma_{r_1}$	$\sigma_{z_1}-\sigma_{r_2}$	σ_{z_2}	$\sigma_{z_2}-\sigma_{r_2}$	$\sigma_{z_2}-\sigma_{r_3}$
$k_2 = 0.2$	0.1	0.00417	0.04050	0.00202	0.00271	0.00039	0.00195
	0.2	0.01641	0.15675	0.00784	0.01080	0.00155	0.00777
	0.4	0.06210	0.55548	0.02777	0.04241	0.00606	0.03028
	0.8	0.21057	1.53667	0.07683	0.15808	0.02198	0.10991
	1.6	0.58218	2.77359	0.13868	0.49705	0.06327	0.31635
	3.2	1.06296	2.55195	0.12760	1.00217	0.09906	0.49525
$k_2 = 2.0$	0.1	0.00263	0.04751	0.00238	0.00100	0.00160	0.00080
	0.2	0.01029	0.18481	0.00924	0.00397	0.00637	0.00319
	0.4	0.03810	0.66727	0.03336	0.01565	0.02498	0.01249
	0.8	0.12173	1.97428	0.09871	0.05938	0.09268	0.04634
	1.6	0.31575	4.37407	0.21870	0.20098	0.29253	0.14626
	3.2	0.66041	6.97695	0.34885	0.53398	0.65446	0.32723
$k_2 = 20.0$	0.1	0.00193	0.05737	0.00287	0.00024	0.00322	0.00016
	0.2	0.00751	0.22418	0.01121	0.00098	0.01283	0.00064
	0.4	0.02713	0.82430	0.04121	0.00387	0.05063	0.00253
	0.8	0.08027	2.59672	0.12984	0.01507	0.19267	0.00963
	1.6	0.17961	6.77014	0.33851	0.05549	0.66326	0.03316
	3.2	0.34355	15.23252	0.76163	0.18344	1.88634	0.09432
$k_2 = 200.0$	0.1	0.00176	0.06733	0.00337	0.00006	0.00478	0.00002
	0.2	0.00683	0.26401	0.01320	0.00022	0.01908	0.00010
	0.4	0.02443	0.98346	0.04917	0.00088	0.07557	0.00038
	0.8	0.06983	3.23164	0.16158	0.00348	0.29194	0.00146
	1.6	0.14191	9.28148	0.46407	0.01339	1.05385	0.00527
	3.2	0.22655	24.85236	1.24262	0.04911	3.37605	0.01688

TABLE 6.18
(Jones, 1962)

$H = 1.0$
$k_1 = 200.0$

	a_1	σ_{z_1}	$\sigma_{z_1}-\sigma_{r_1}$	$\sigma_{z_1}-\sigma_{r_2}$	σ_{z_2}	$\sigma_{z_2}-\sigma_{r_2}$	$\sigma_{z_2}-\sigma_{r_3}$
$k_2 = 0.2$	0.1	0.00117	0.05507	0.00028	0.00097	0.00010	0.00051
	0.2	0.00464	0.21467	0.00107	0.00388	0.00041	0.00203
	0.4	0.01814	0.78191	0.00391	0.01538	0.00160	0.00801
	0.8	0.06766	2.38055	0.01190	0.05952	0.00607	0.03037
	1.6	0.22994	5.57945	0.02790	0.21214	0.02028	0.10140
	3.2	0.62710	9.29529	0.04648	0.60056	0.04847	0.24236
$k_2 = 2.0$	0.1	0.00049	0.06883	0.00034	0.00029	0.00035	0.00017
	0.2	0.00195	0.26966	0.00135	0.00116	0.00138	0.00069
	0.4	0.00746	1.00131	0.00501	0.00460	0.00545	0.00273
	0.8	0.02647	3.24971	0.01625	0.01797	0.02092	0.01046
	1.6	0.08556	8.92442	0.04462	0.06671	0.07335	0.03668
	3.2	0.25186	20.83387	0.10417	0.22047	0.21288	0.10644
$k_2 = 20.0$	0.1	0.00027	0.08469	0.00042	0.00007	0.00062	0.00003
	0.2	0.00104	0.33312	0.00167	0.00028	0.00248	0.00012
	0.4	0.00384	1.25495	0.00627	0.00110	0.00985	0.00049
	0.8	0.01236	4.26100	0.02130	0.00436	0.03825	0.00191
	1.6	0.03379	12.91809	0.06459	0.01683	0.13989	0.00699
	3.2	0.08859	36.04291	0.18021	0.06167	0.45544	0.02277
$k_2 = 200.0$	0.1	0.00021	0.10075	0.00050	0.00002	0.00087	0.00000
	0.2	0.00082	0.39741	0.00199	0.00006	0.00347	0.00002
	0.4	0.00298	1.51234	0.00756	0.00025	0.01381	0.00007
	0.8	0.00893	5.28939	0.02645	0.00100	0.05403	0.00027
	1.6	0.02065	17.01872	0.08509	0.00392	0.20250	0.00101
	3.2	0.04154	52.23615	0.26118	0.01505	0.70098	0.00350

TABLE 6.19
(Jones, 1962)

$H = 2.0$
$k_1 = 0.2$

	a_1	σ_{z_1}	$\sigma_{z_1}-\sigma_{r_1}$	$\sigma_{z_1}-\sigma_{r_2}$	σ_{z_2}	$\sigma_{z_2}-\sigma_{r_2}$	$\sigma_{z_2}-\sigma_{r_3}$
$k_2 = 0.2$	0.1	0.00540	0.00121	0.00604	0.00242	0.00060	0.00302
	0.2	0.02138	0.00477	0.02386	0.00964	0.00240	0.01202
	0.4	0.08209	0.01821	0.09106	0.03770	0.00939	0.04695
	0.8	0.28150	0.06106	0.30531	0.13832	0.03422	0.17112
	1.6	0.68908	0.13660	0.68299	0.40830	0.09826	0.49131
	3.2	0.93103	0.12899	0.64493	0.73496	0.15705	0.78523
$k_2 = 2.0$	0.1	0.00502	0.00098	0.00494	0.00180	0.00339	0.00170
	0.2	0.01986	0.00389	0.01953	0.00716	0.01350	0.00675
	0.4	0.07630	0.01485	0.07449	0.02815	0.05288	0.02644
	0.8	0.26196	0.04977	0.24875	0.10523	0.19467	0.09733
	1.6	0.63535	0.10924	0.54641	0.33075	0.57811	0.28905
	3.2	0.87025	0.12296	0.61462	0.68388	1.00199	0.50100
$k_2 = 20.0$	0.1	0.00444	0.00056	0.00282	0.00065	0.00825	0.00041
	0.2	0.01756	0.00221	0.01105	0.00260	0.03286	0.00164
	0.4	0.06706	0.00819	0.04097	0.01030	0.12933	0.00647
	0.8	0.22561	0.02431	0.12153	0.03956	0.48595	0.02430
	1.6	0.51929	0.03070	0.15352	0.13743	1.55804	0.07790
	3.2	0.65700	-0.00926	-0.04632	0.37409	3.39883	0.16994
$k_2 = 200.0$	0.1	0.00414	0.00032	0.00160	0.00015	0.01234	0.00006
	0.2	0.01635	0.00124	0.00621	0.00058	0.04922	0.00025
	0.4	0.06231	0.00436	0.02180	0.00231	0.19450	0.00097
	0.8	0.20757	0.00955	0.04774	0.00905	0.74256	0.00371
	1.6	0.45550	-0.02172	-0.10861	0.03363	2.52847	0.01264
	3.2	0.48642	-0.15589	-0.77944	0.11105	6.60835	0.03349

TABLE 6.20
(Jones, 1962)

$H = 2.0$
$k_1 = 2.0$

	a_1	σ_{z_1}	$\sigma_{z_1}-\sigma_{r_1}$	$\sigma_{z_1}-\sigma_{r_2}$	σ_{z_2}	$\sigma_{z_2}-\sigma_{r_2}$	$\sigma_{z_2}-\sigma_{r_3}$
$k_2 = 0.2$	0.1	0.00356	0.00545	0.00272	0.00216	0.00041	0.00203
	0.2	0.01415	0.02155	0.01078	0.00861	0.00162	0.00809
	0.4	0.05493	0.08266	0.04133	0.03386	0.00634	0.03172
	0.8	0.19661	0.28226	0.14113	0.12702	0.02349	0.11744
	1.6	0.55306	0.67844	0.33922	0.40376	0.07109	0.35545
	3.2	0.96647	0.79393	0.39696	0.83197	0.12583	0.62913
$k_2 = 2.0$	0.1	0.00250	0.00555	0.00278	0.00100	0.00188	0.00094
	0.2	0.00991	0.02199	0.01099	0.00397	0.00750	0.00375
	0.4	0.03832	0.08465	0.04231	0.01569	0.02950	0.01475
	0.8	0.13516	0.29365	0.14683	0.05974	0.11080	0.05540
	1.6	0.36644	0.75087	0.37542	0.20145	0.35515	0.17757
	3.2	0.67384	1.17294	0.58647	0.51156	0.77434	0.38717
$k_2 = 20.0$	0.1	0.00181	0.00652	0.00326	0.00025	0.00378	0.00019
	0.2	0.00716	0.02586	0.01293	0.00099	0.01507	0.00075
	0.4	0.02746	0.10017	0.05007	0.00394	0.05958	0.00298
	0.8	0.09396	0.35641	0.17821	0.01535	0.22795	0.01140
	1.6	0.23065	1.00785	0.50392	0.05599	0.78347	0.03917
	3.2	0.37001	2.16033	1.08017	0.17843	2.13215	0.10661
$k_2 = 200.0$	0.1	0.00164	0.00778	0.00389	0.00005	0.00542	0.00003
	0.2	0.00647	0.03090	0.01544	0.00021	0.02163	0.00011
	0.4	0.02470	0.12030	0.06014	0.00085	0.08578	0.00043
	0.8	0.08326	0.43693	0.21847	0.00335	0.33214	0.00166
	1.6	0.19224	1.32870	0.66434	0.01283	1.19190	0.00596
	3.2	0.25526	3.40664	1.70332	0.04612	3.67558	0.01838

TABLE 6.21

(Jones, 1962)

$H = 2.0$
$k_1 = 20.0$

	a_1	σ_{z_1}	$\sigma_{z_1}-\sigma_{r_1}$	$\sigma_{z_1}-\sigma_{r_2}$	σ_{z_2}	$\sigma_{z_2}-\sigma_{r_2}$	$\sigma_{z_2}-\sigma_{r_3}$
$k_2 = 0.2$	0.1	0.00134	0.00968	0.00048	0.00108	0.00014	0.00068
	0.2	0.00533	0.03839	0.00192	0.00429	0.00055	0.00273
	0.4	0.02100	0.14845	0.00741	0.01702	0.00216	0.01078
	0.8	0.07950	0.52414	0.02621	0.06576	0.00820	0.04101
	1.6	0.26613	1.41720	0.07085	0.23186	0.02740	0.13698
	3.2	0.67882	2.38258	0.11913	0.63006	0.06384	0.31919
$k_2 = 2.0$	0.1	0.00059	0.01219	0.00061	0.00033	0.00051	0.00025
	0.2	0.00235	0.04843	0.00242	0.00130	0.00203	0.00101
	0.4	0.00922	0.18857	0.00943	0.00518	0.00803	0.00401
	0.8	0.03412	0.68382	0.03419	0.02023	0.03093	0.01547
	1.6	0.10918	2.04134	0.10207	0.07444	0.10864	0.05432
	3.2	0.29183	4.60426	0.23021	0.23852	0.30709	0.15354
$k_2 = 20.0$	0.1	0.00033	0.01568	0.00078	0.00008	0.00094	0.00005
	0.2	0.00130	0.06236	0.00312	0.00031	0.00374	0.00019
	0.4	0.00503	0.24425	0.01221	0.00123	0.01486	0.00074
	0.8	0.01782	0.90594	0.04530	0.00485	0.05789	0.00289
	1.6	0.05012	2.91994	0.14600	0.01862	0.21190	0.01060
	3.2	0.11331	7.95104	0.39755	0.06728	0.67732	0.03387
$k_2 = 200.0$	0.1	0.00027	0.01927	0.00096	0.00002	0.00131	0.00001
	0.2	0.00106	0.07675	0.00384	0.00007	0.00524	0.00003
	0.4	0.00406	0.30182	0.01509	0.00028	0 02085	0.00010
	0.8	0.01397	1.13555	0.05678	0.00110	0.08180	0.00041
	1.6	0.03538	3.83254	0.19163	0.00431	0.30676	0.00153
	3.2	0.06182	11.55403	0.57770	0.01644	1.04794	0.00524

TABLE 6.22

(Jones, 1962)

$H = 2.0$
$k_1 = 200.0$

	a_1	σ_{z_1}	$\sigma_{z_1}-\sigma_{r_1}$	$\sigma_{z_1}-\sigma_{r_2}$	σ_{z_2}	$\sigma_{z_2}-\sigma_{r_2}$	$\sigma_{z_2}-\sigma_{r_3}$
$k_2 = 0.2$	0.1	0.00036	0.01350	0.00007	0.00033	0.00003	0.00015
	0.2	0.00144	0.05366	0.00027	0.00130	0.00012	0.00058
	0.4	0.00572	0.20911	0.00105	0.00518	0.00046	0.00232
	0.8	0.02231	0.76035	0.00380	0.02038	0.00180	0.00901
	1.6	0.08215	2.29642	0.01148	0.07675	0.00649	0.03244
	3.2	0.26576	5.28589	0.02643	0.25484	0.01912	0.09562
$k_2 = 2.0$	0.1	0.00011	0.01737	0.00009	0.00008	0.00009	0.00004
	0.2	0.00045	0.06913	0.00035	0.00033	0.00036	0.00018
	0.4	0.00179	0.27103	0.00136	0.00131	0.00142	0.00071
	0.8	0.00685	1.00808	0.00504	0.00520	0.00553	0.00277
	1.6	0.02441	3.27590	0.01638	0.02003	0.02043	0.01021
	3.2	0.08061	9.02195	0.04511	0.07248	0.06638	0.03319
$k_2 = 20.0$	0.1	0.00005	0.02160	0.00011	0.00002	0.00014	0.00001
	0.2	0.00018	0.08604	0.00043	0.00007	0.00058	0.00003
	0.4	0.00071	0.33866	0.00169	0.00030	0.00229	0.00011
	0.8	0.00261	1.27835	0.00639	0.00119	0.00901	0.00045
	1.6	0.00819	4.35311	0.02177	0.00467	0.03390	0.00170
	3.2	0.02341	13.26873	0.06634	0.01784	0.11666	0.00583
$k_2 = 200.0$	0.1	0.00003	0.02587	0.00013	0.00000	0.00019	0.00000
	0.2	0.00012	0.10310	0.00052	0100002	0100075	0.00000
	0.4	0.00047	0.40676	0.00203	0.00007	0.00300	0.00002
	0.8	0.00165	1.54951	0.00775	0.00026	0.01183	0.00006
	1.6	0.00445	5.43705	0.02719	0.00104	0.04515	0.00023
	3.2	0.00929	17.58810	0.08794	0.00409	0.16107	0.00081

TABLE 6.23
(Jones, 1962)

$H = 4.0$
$k_1 = 0.2$

	a_1	σ_{z_1}	$\sigma_{z_1}-\sigma_{r_1}$	$\sigma_{z_1}-\sigma_{r_2}$	σ_{z_2}	$\sigma_{z_2}-\sigma_{r_2}$	$\sigma_{z_2}-\sigma_{r_3}$
$k_2 = 0.2$	0.1	0.00139	0.00028	0.00141	0.00086	0.00023	0.00114
	0.2	0.00555	0.00112	0.00562	0.00345	0.00091	0.00454
	0.4	0.02198	0.00444	0.02220	0.01371	0.00360	0.01801
	0.8	0.08435	0.01686	0.08428	0.05323	0.01394	0.06968
	1.6	0.28870	0.05529	0.27647	0.19003	0.04909	0.24545
	3.2	0.70074	0.11356	0.56778	0.51882	0.12670	0.63352
$k_2 = 2.0$	0.1	0.00123	0.00026	0.00131	0.00071	0.00130	0.00065
	0.2	0.00491	0.00104	0.00521	0.00283	0.00518	0.00259
	0.4	0.01942	0.00412	0.02059	0.01126	0.02057	0.01028
	0.8	0.07447	0.01574	0.07869	0.04388	0.07977	0.03989
	1.6	0.25449	0.05311	0.26554	0.15904	0.28357	0.14178
	3.2	0.62074	0.12524	0.62622	0.45455	0.75651	0.37825
$k_2 = 20.0$	0.1	0.00087	0.00018	0.00090	0.00028	0.00325	0.00016
	0.2	0.00346	0.00072	0.00358	0.00111	0.01298	0.00065
	0.4	0.01367	0.00283	0.01417	0.00443	0.05159	0.00258
	0.8	0.05207	0.01089	0.05444	0.01741	0.20134	0.01007
	1.6	0.17367	0.03790	0.18949	0.06525	0.73322	0.03666
	3.2	0.39955	0.10841	0.54203	0.20965	2.13666	0.10683
$k_2 = 200.0$	0.1	0.00069	0.00019	0.00097	0.00006	0.00487	0.00002
	0.2	0.00274	0.00078	0.00389	0.00024	0.01947	0.00010
	0.4	0.01079	0.00309	0.01544	0.00095	0.07752	0.00039
	0.8	0.04074	0.01199	0.05995	0.00378	0.30432	0.00152
	1.6	0.13117	0.04352	0.21758	0.01456	1.13373	0.00567
	3.2	0.26403	0.14445	0.72224	0.05161	3.59608	0.01798

TABLE 6.24
(Jones, 1962)

$H = 4.0$
$k_1 = 2.0$

	a_1	σ_{z_1}	$\sigma_{z_1}-\sigma_{r_1}$	$\sigma_{z_1}-\sigma_{r_2}$	σ_{z_2}	$\sigma_{z_2}-\sigma_{r_2}$	$\sigma_{z_2}-\sigma_{r_3}$
$k_2 = 0.2$	0.1	0.00103	0.00128	0.00064	0.00078	0.00014	0.00071
	0.2	0.00411	0.00511	0.00256	0.00312	0.00057	0.00284
	0.4	0.01631	0.02022	0.01011	0.01241	0.00226	0.01129
	0.8	0.06319	0.07722	0.03861	0.04842	0.00877	0.04384
	1.6	0.22413	0.25955	0.12977	0.17617	0.03133	0.15666
	3.2	0.60654	0.58704	0.29352	0.50917	0.08500	0.42501
$k_2 = 2.0$	0.1	0.00057	0.00147	0.00074	0.00034	0.00065	0.00032
	0.2	0.00228	0.00587	0.00293	0.00137	0.00260	0.00130
	0.4	0.00905	0.02324	0.01162	0.00544	0.01032	0.00516
	0.8	0.03500	0.08957	0.04479	0.02135	0.04031	0.02015
	1.6	0.12354	0.31215	0.15608	0.07972	0.14735	0.07368
	3.2	0.34121	0.81908	0.40954	0.25441	0.43632	0.21816
$k_2 = 20.0$	0.1	0.00030	0.00201	0.00101	0.00008	0.00128	0.00006
	0.2	0.00119	0.00803	0.00402	0.00034	0.00510	0.00026
	0.4	0.00469	0.03191	0.01596	0.00134	0.02032	0.00102
	0.8	0.01790	0.12427	0.06213	0.00532	0.07991	0.00400
	1.6	0.06045	0.45100	0.22550	0.02049	0.29991	0.01500
	3.2	0.14979	1.36427	0.68214	0.07294	0.97701	0.04885
$k_2 = 200.0$	0.1	0.00023	0.00263	0.00131	0.00002	0.00180	0.00001
	0.2	0.00091	0.01050	0.00525	0.00007	0.00720	0.00004
	0.4	0.00360	0.04179	0.02090	0.00029	0.02870	0.00014
	0.8	0.01360	0.16380	0.08190	0.00115	0.11334	0.00057
	1.6	0.04409	0.60898	0.30449	0.00451	0.43251	0.00216
	3.2	0.09323	1.98899	0.99449	0.01705	1.49306	0.00747

TABLE 6.25 (Jones, 1962) $H = 4.0$ $k_1 = 20.0$

	a_1	σ_{z_1}	$\sigma_{z_1}-\sigma_{r_1}$	$\sigma_{z_1}-\sigma_{r_2}$	σ_{z_2}	$\sigma_{z_2}-\sigma_{r_2}$	$\sigma_{z_2}-\sigma_{r_3}$
$k_2 = 0.2$	0.1	0.00042	0.00233	0.00012	0.00037	0.00004	0.00021
	0.2	0.00166	0.00932	0.00047	0.00148	0.00017	0.00085
	0.4	0.00663	0.03692	0.00185	0.00588	0.00068	0.00340
	0.8	0.02603	0.14242	0.00712	0.02319	0.00266	0.01331
	1.6	0.09718	0.49826	0.02491	0.08758	0.00983	0.04914
	3.2	0.31040	1.31627	0.06581	0.28747	0.02990	0.14951
$k_2 = 2.0$	0.1	0.00013	0.00312	0.00016	0.00010	0.00015	0.00007
	0.2	0.00054	0.01245	0.00062	0.00039	0.00059	0.00029
	0.4	0.00214	0.04944	0.00247	0.00154	0.00235	0.00117
	0.8	0.00837	0.19247	0.00962	0.00610	0.00924	0.00462
	1.6	0.03109	0.69749	0.03487	0.02358	0.03488	0.01744
	3.2	0.10140	2.09049	0.10452	0.08444	0.11553	0.05776
$k_2 = 20.0$	0.1	0.00005	0.00413	0.00021	0.00002	0.00025	0.00001
	0.2	0.00021	0.01651	0.00083	0.00009	0.00099	0.00005
	0.4	0.00083	0.06569	0.00328	0.00035	0.00396	0.00020
	0.8	0.00321	0.25739	0.01287	0.00138	0.01565	0.00078
	1.6	0.01130	0.05622	0.04781	0.00542	0.05993	0.00300
	3.2	0.03258	3.10980	0.15549	0.02061	0.20906	0.01045
$k_2 = 200.0$	0.1	0.00003	0.00515	0.00026	0.00000	0.00033	0.00000
	0.2	0.00014	0.02056	0.00103	0.00002	0.00131	0.00001
	0.4	0.00054	0.08191	0.00410	0.00008	0.00524	0.00003
	0.8	0.00206	0.32231	0.01612	0.00030	0.02077	0.00010
	1.6	0.00683	1.21587	0.06079	0.00120	0.08034	0.00040
	3.2	0.01590	4.14395	0.20720	0.00468	0.28961	0.00145

TABLE 6.26 (Jones, 1962) $H = 4.0$ $k_1 = 200.0$

	a_1	σ_{z_1}	$\sigma_{z_1}-\sigma_{r_1}$	$\sigma_{z_1}-\sigma_{r_2}$	σ_{z_2}	$\sigma_{z_2}-\sigma_{r_2}$	$\sigma_{z_2}-\sigma_{r_3}$
$k_2 = 0.2$	0.1	0.00010	0.00334	0.00002	0.00010	0.00001	0.00004
	0.2	0.00042	0.01333	0.00007	0.00039	0.00003	0.00016
	0.4	0.00167	0.05295	0.00026	0.00157	0.00013	0.00065
	0.8	0.00663	0.20621	0.00103	0.00625	0.00051	0.00256
	1.6	0.02562	0.74824	0.00374	0.02427	0.00195	0.00975
	3.2	0.09166	2.25046	0.01125	0.08799	0.00660	0.03298
$k_2 = 2.0$	0.1	0.00003	0.00437	0.00002	0.00002	0.00002	0.00001
	0.2	0.00011	0.01746	0.00009	0.00009	0.00009	0.00005
	0.4	0.00042	0.06947	0.00035	0.00036	0.00036	0.00018
	0.8	0.00168	0.27221	0.00136	0.00142	0.00144	0.00072
	1.6	0.00646	1.01140	0.00506	0.00560	0.00553	0.00277
	3.2	0.02332	3.28913	0.01645	0.02126	0.01951	0.00975
$k_2 = 20.0$	0.1	0.00001	0.00545	0.00003	0.00000	0.00003	0.00000
	0.2	0.00003	0.02178	0.00011	0.00002	0.00014	0.00001
	0.4	0.00013	0.08673	0.00043	0.00008	0.00054	0.00003
	0.8	0.00050	0.34131	0.00171	0.00031	0.00215	0.00011
	1.6	0.00186	1.28773	0.00644	0.00124	0.00833	0.00042
	3.2	0.00612	4.38974	0.02195	0.00483	0.03010	0.00150
$k_2 = 200.0$	0.1	0.00000	0.00652	0.00003	0.00000	0.00004	0.00000
	0.2	0.00002	0.02606	0.00013	0.00000	0.00017	0.00000
	0.4	0.00007	0.10389	0.00052	0.00002	0.00068	0.00000
	0.8	0.00025	0.40997	0.00205	0.00007	0.00269	0.00001
	1.6	0.00086	1.56284	0.00781	0.00027	0.01049	0.00005
	3.2	0.00225	5.48870	0.02744	0.00107	0.03866	0.00019

TABLE 6.27
(Jones, 1962)

$H = 8.0$
$k_1 = 0.2$

	a_1	σ_{z_1}	$\sigma_{z_1}-\sigma_{r_1}$	$\sigma_{z_1}-\sigma_{r_2}$	σ_{z_2}	$\sigma_{z_2}-\sigma_{r_2}$	$\sigma_{z_2}-\sigma_{r_3}$
$k_2 = 0.2$	0.1	0.00035	0.00006	0.00028	0.00027	0.00007	0.00036
	0.2	0.00142	0.00023	0.00113	0.00108	0.00028	0.00142
	0.4	0.00566	0.00090	0.00449	0.00432	0.00113	0.00567
	0.8	0.02240	0.00354	0.01769	0.01711	0.00449	0.02246
	1.6	0.08589	0.01335	0.06673	0.06610	0.01725	0.08624
	3.2	0.29318	0.04270	0.21350	0.23182	0.05907	0.29533
$k_2 = 2.0$	0.1	0.00030	0.00008	0.00038	0.00023	0.00041	0.00021
	0.2	0.00120	0.00030	0.00152	0.00091	0.00165	0.00083
	0.4	0.00479	0.00121	0.00606	0.00364	0.00660	0.00330
	0.8	0.01894	0.00480	0.02399	0.01446	0.02616	0.01308
	1.6	0.07271	0.01841	0.09206	0.05601	0.10080	0.05040
	3.2	0.24933	0.06307	0.31534	0.19828	0.35008	0.17504
$k_2 = 20.0$	0.1	0.00016	0.00010	0.00049	0.00009	0.00105	0.00005
	0.2	0.00065	0.00040	0.00198	0.00037	0.00421	0.00021
	0.4	0.00260	0.00158	0.00790	0.00149	0.01679	0.00084
	0.8	0.01026	0.00629	0.03143	0.00594	0.06664	0.00333
	1.6	0.03926	0.02463	0.12314	0.02320	0.25871	0.01294
	3.2	0.13335	0.09123	0.45615	0.08510	0.92478	0.04624
$k_2 = 200.0$	0.1	0.00009	0.00015	0.00074	0.00002	0.00162	0.00001
	0.2	0.00036	0.00059	0.00294	0.00008	0.00648	0.00003
	0.4	0.00145	0.00235	0.01176	0.00032	0.02587	0.00013
	0.8	0.00573	0.00938	0.04690	0.00127	0.10287	0.00051
	1.6	0.02160	0.03710	0.18549	0.00503	0.40238	0.00201
	3.2	0.06938	0.14226	0.71130	0.01912	1.48097	0.00740

TABLE 6.28
(Jones, 1962)

$H = 8.0$
$k_1 = 2.0$

	a_1	σ_{z_1}	$\sigma_{z_1}-\sigma_{r_1}$	$\sigma_{z_1}-\sigma_{r_2}$	σ_{z_2}	$\sigma_{z_2}-\sigma_{r_2}$	$\sigma_{z_2}-\sigma_{r_3}$
$k_2 = 0.2$	0.1	0.00028	0.00028	0.00014	0.00024	0.00004	0.00022
	0.2	0.00113	0.00111	0.00056	0.00096	0.00017	0.00087
	0.4	0.00451	0.00444	0.00222	0.00384	0.00069	0.00347
	0.8	0.01786	0.01752	0.00876	0.01522	0.00275	0.01373
	1.6	0.06895	0.06662	0.03331	0.05900	0.01060	0.05298
	3.2	0.24127	0.22014	0.11007	0.20949	0.03693	0.18466
$k_2 = 2.0$	0.1	0.00013	0.00039	0.00020	0.00010	0.00020	0.00010
	0.2	0.00053	0.00157	0.00079	0.00041	0.00078	0.00039
	0.4	0.00213	0.00628	0.00314	0.00164	0.00311	0.00156
	0.8	0.00844	0.02487	0.01244	0.00653	0.01237	0.00618
	1.6	0.03269	0.09597	0.04798	0.02556	0.04802	0.02401
	3.2	0.11640	0.33606	0.16803	0.09405	0.17188	0.08594
$k_2 = 20.0$	0.1	0.00005	0.00061	0.00030	0.00002	0.00037	0.00002
	0.2	0.00019	0.00242	0.00121	0.00010	0.00149	0.00007
	0.4	0.00076	0.00967	0.00484	0.00040	0.00596	0.00030
	0.8	0.00300	0.03845	0.01922	0.00159	0.02369	0.00118
	1.6	0.01154	0.15010	0.07505	0.00630	0.09274	0.00464
	3.2	0.04003	0.54942	0.27471	0.02409	0.34233	0.01712
$k_2 = 200.0$	0.1	0.00003	0.00082	0.00041	0.00001	0.00052	0.00000
	0.2	0.00011	0.00328	0.00164	0.00002	0.00206	0.00001
	0.4	0.00042	0.01310	0.00655	0.00008	0.00825	0.00004
	0.8	0.00167	0.05216	0.02608	0.00034	0.03287	0.00016
	1.6	0.00629	0.20491	0.10245	0.00135	0.12933	0.00065
	3.2	0.02020	0.76769	0.38384	0.00527	0.48719	0.00244

TABLE 6.29
(Jones, 1962)

$H = 8.0$
$k_1 = 20.0$

	a_1	σ_{z_1}	$\sigma_{z_1}-\sigma_{r_1}$	$\sigma_{z_1}-\sigma_{r_2}$	σ_{z_2}	$\sigma_{z_2}-\sigma_{r_2}$	$\sigma_{z_2}-\sigma_{r_3}$
	0.1	0.00012	0.00056	0.00003	0.00011	0.00001	0.00006
	0.2	0.00047	0.00223	0.00011	0.00044	0.00005	0.00025
$k_2 = 0.2$	0.4	0.00190	0.00889	0.00044	0.00176	0.00020	0.00099
	0.8	0.00754	0.03522	0.00176	0.00701	0.00079	0.00393
	1.6	0.02947	0.13569	0.00678	0.02746	0.00306	0.01528
	3.2	0.10817	0.47240	0.02362	0.10145	0.01105	0.05524
	0.1	0.00003	0.00079	0.00004	0.00003	0.00004	0.00002
	0.2	0.00013	0.00316	0.00016	0.00011	0.00016	0.00008
$k_2 = 2.0$	0.4	0.00050	0.01260	0.00063	0.00043	0.00064	0.00032
	0.8	0.00200	0.05007	0.00250	0.00170	0.00253	0.00127
	1.6	0.00786	0.19496	0.00975	0.00673	0.00993	0.00496
	3.2	0.02944	0.70709	0.03535	0.02579	0.03678	0.01839
	0.1	0.00001	0.00106	0.00005	0.00001	0.00006	0.00000
	0.2	0.00004	0.00425	0.00021	0.00002	0.00025	0.00001
$k_2 = 20.0$	0.4	0.00014	0.01696	0.00085	0.00009	0.00100	0.00005
	0.8	0.00056	0.06751	0.00338	0.00037	0.00398	0.00020
	1.6	0.00217	0.26466	0.01323	0.00147	0.01565	0.00078
	3.2	0.00791	0.98450	0.04922	0.00576	0.05892	0.00295
	0.1	0.00000	0.00133	0.00007	0.00000	0.00008	0.00000
	0.2	0.00002	0.00531	0.00027	0.00000	0.00032	0100000
$k_2 = 200.0$	0.4	0.00006	0.02122	0.00106	0.00002	0.00128	0.00001
	0.8	0.00025	0.08453	0.00423	0.00008	0.00509	0.00003
	1.6	0.00096	0.33268	0.01663	0.00032	0.02009	0.00010
	3.2	0.00319	1.25614	0.06281	0.00125	0.07660	0.00038

TABLE 6.30
(Jones, 1962)

$H = 8.0$
$k_1 = 200.0$

	a_1	σ_{z_1}	$\sigma_{z_1}-\sigma_{r_1}$	$\sigma_{z_1}-\sigma_{r_2}$	σ_{z_2}	$\sigma_{z_2}-\sigma_{r_2}$	$\sigma_{z_2}-\sigma_{r_3}$
	0.1	0.00003	0.00083	0.00000	0.00003	0.00000	0.00001
	0.2	0.00011	0.00330	0.00002	0.00011	0.00001	0.00005
$k_2 = 0.2$	0.4	0.00046	0.01320	0.00007	0.00044	0.00004	0.00018
	0.8	0.00182	0.05242	0.00026	0.00175	0.00014	0.00072
	1.6	0.00720	0.20411	0.00102	0.00693	0.00056	0.00282
	3.2	0.02751	0.74013	0.00370	0.02656	0.00212	0.01058
	0.1	0.00001	0.00109	0.00001	0.00001	0.00001	0.00000
	0.2	0.00003	0.00438	0.00002	0.00002	0.00002	0.00001
$k_2 = 2.0$	0.4	0.00010	0.01748	0.00009	0.00009	0.00009	0.00005
	0.8	0.00041	0.06956	0.00035	0.00038	0.00037	0.00018
	1.6	0.00162	0.27262	0.00136	0.00149	0.00145	0.00072
	3.2	0.00625	1.01322	0.00507	0.00584	0.00547	0.00273
	0.1	0.00000	0.00136	0.00001	0.00000	0.00001	0.00000
	0.2	0.00001	0.00546	0.00003	0.00001	0.00003	0.00000
$k_2 = 20.0$	0.4	0.00002	0.02181	0.00011	0.00002	0.00013	0.00001
	0.8	0.00010	0.08687	0.00043	0.00008	0.00052	0.00003
	1.6	0.00039	0.34202	0.00171	0.00032	0.00204	0.00010
	3.2	0.00149	1.29190	0.00646	0.00127	0.00777	0.00039
	0.1	0.00000	0.00163	0.00001	0.00000	0.00001	0.00000
	0.2	0.00000	0.00654	0.00003	0.00000	0.00004	0.00000
$k_2 = 200.0$	0.4	0.00001	0.02613	0.00013	0.00000	0.00016	0.00000
	0.8	0.00003	0.10417	0.00052	0.00002	0.00063	0.00000
	1.6	0.00013	0.41121	0.00206	0.00007	0.00249	0.00001
	3.2	0.00047	1.56843	0.00784	0.00027	0.00957	0.00005

Displacements

Ueshita and Meyerhof (1967) have evaluated the vertical surface displacement at the centre and edge of the circle. Influence factors have been computed for the following parameters:

$E_1/E_2 = 2, 10, 100$

$E_2/E_3 = 2, 10, 100$

$\nu_1 = \nu_2 = \nu_3 = 0.5$

$T/a = 0.5, 1, 2, 4$

$h_1/T = 0, 0.2, 0.4, 0.6, 0.8, 1.0$

The symbols are defined in Fig.6.17. The influence factors are shown in Fig.6.18. The centre displacement is given by

$$\rho_c = \frac{1.5pa}{E_3} F_{co} \qquad \ldots (6.1)$$

The edge displacement

$$\rho_e = \frac{1.5pa}{E_3} F_{ce} \qquad \ldots (6.2)$$

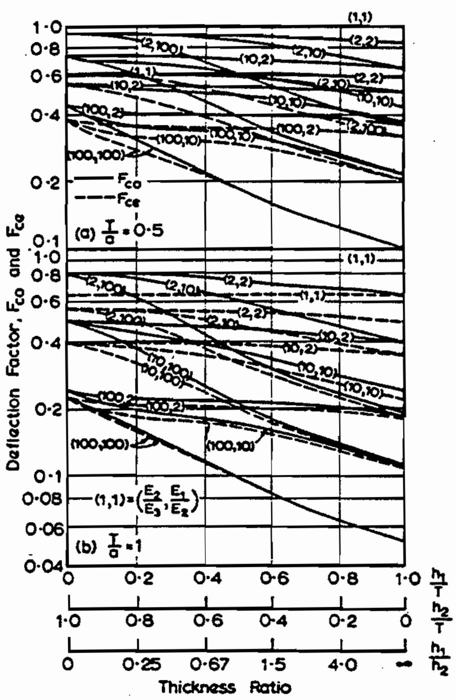


FIG.6.18 Deflection factors F_{co} and F_{ce} for a three-layer system (Ueshita and Meyerhof,1967).

Further influence factors for vertical displacement at the centre of the circle have been presented by Thenn de Barros (1966). The following parameters have been considered:

$n_1 = E_1/E_2 = 2, 5, 10, 20, 50$

$n_2 = E_2/E_3 = 2, 5, 10$

$\nu_1 = \nu_2 = \nu_3 = 0.35$

Influence factors F are tabulated in Tables 6.31 to 6.35.

The actual centre displacement of the circle is given by

$$\rho_c = \frac{1.755pa}{E_3} F \qquad \ldots (6.3)$$

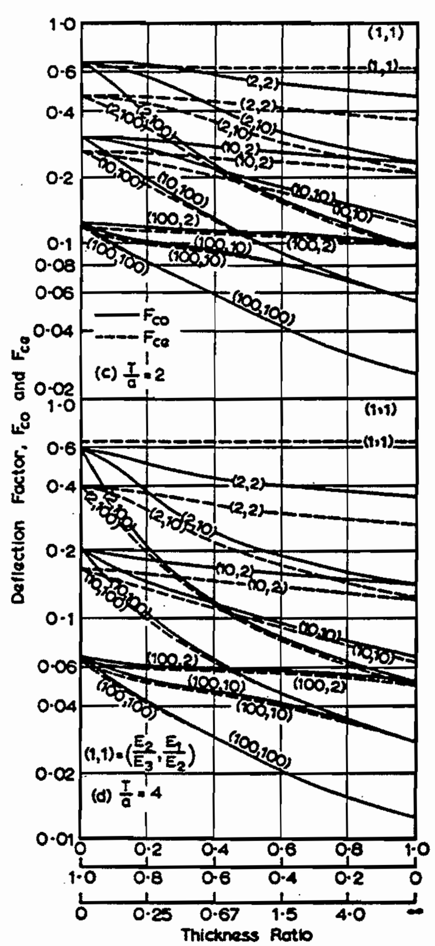


TABLE 6.31

DISPLACEMENT FACTORS F

$n_1 = 2$ (After Thenn de Barros, 1966)

n_2	h_2/a \ h_1/a	0.156	0.312	0.625	1.25	2.5	5
	0.312	0.858	0.789	0.662	0.510	0.394	-
	0.625	0.772	0.717	0.616	0.489	0.387	0.323
2	1.25	0.669	0.633	0.560	0.460	0.375	0.319
	2.5	-	0.564	0.508	0.428	0.360	0.314
	5	-	-	0.470	0.400	0.343	0.306
	0.312	0.747	0.651	0.505	0.352	0.240	-
	0.625	0.601	0.537	0.438	0.324	0.231	0.171
5	1.25	0.449	0.416	0.359	0.284	0.215	0.166
	2.5	-	0.320	0.287	0.240	0.194	0.158
	5	-	-	0.235	0.202	0.172	0.148
	0.312	0.664	0.559	0.415	0.274	0.174	-
	0.625	0.500	0.436	0.346	0.246	0.165	0.112
10	1.25	0.344	0.315	0.267	0.207	0.150	0.108
	2.5	-	0.221	0.198	0.165	0.130	0.100
	5	-	-	0.149	0.128	0.108	0.0903

TABLE 6.32

DISPLACEMENT FACTORS F

$n_1 = 5$ (After Thenn de Barros, 1966)

n_2	h_2/a \ h_1/a	0.156	0.312	0.625	1.25	2.5	5
	0.312	0.830	0.733	0.556	0.372	0.246	-
	0.625	0.745	0.669	0.522	0.359	0.242	0.174
2	1.25	0.648	0.593	0.476	0.339	0.235	0.172
	2.5	-	0.528	0.430	0.313	0.224	0.168
	5	-	-	0.394	0.289	0.211	0.163
	0.312	0.713	0.598	0.427	0.268	0.162	-
	0.625	0.572	0.497	0.378	0.250	0.157	0.101
5	1.25	0.429	0.387	0.312	0.223	0.148	0.0989
	2.5	-	0.299	0.249	0.188	0.134	0.0944
	5	-	-	0.202	0.155	0.116	0.0871
	0.312	0.626	0.508	0.350	0.211	0.122	-
	0.625	0.469	0.401	0.301	0.195	0.118	0.0712
10	1.25	0.325	0.291	0.236	0.168	0.109	0.0691
	2.5	-	0.206	0.174	0.134	0.0954	0.0649
	5	-	-	0.130	0.102	0.0779	0.0579

TABLE 6.33

DISPLACEMENT FACTORS F

$n_1 = 10$ (After Thenn de Barros, 1966)

n_2	h_2/a \ h_1/a	0.156	0.312	0.625	1.25	2.5	5
	0.312	0.811	0.687	0.483	0.298	0.181	-
	0.625	0.729	0.630	0.457	0.290	0.178	0.116
2	1.25	0.634	0.561	0.420	0.275	0.174	0.115
	2.5	-	0.499	0.379	0.255	0.166	0.112
	5	-	-	0.346	0.234	0.156	0.109
	0.312	0.689	0.553	0.369	0.217	0.123	-
	0.625	0.553	0.467	0.334	0.207	0.121	0.0720
5	1.25	0.416	0.367	0.281	0.187	0.115	0.0706
	2.5	-	0.284	0.225	0.160	0.105	0.0679
	5	-	-	0.181	0.130	0.0912	0.0629
	0.312	0.601	0.467	0.302	0.172	0.0946	-
	0.625	0.405	0.376	0.267	0.163	0.0924	0.0525
10	1.25	0.312	0.275	0.214	0.144	0.0876	0.0514
	2.5	-	0.195	0.159	0.117	0.0782	0.0489
	5	-	-	0.117	0.0885	0.0642	0.0442

TABLE 6.34

DISPLACEMENT FACTORS F

$n_1 = 20$ (After Thenn de Barros, 1966)

n_2	h_2/a \ h_1/a	0.156	0.312	0.625	1.25	2.5	5
	0.312	0.789	0.629	0.411	0.239	0.136	-
	0.625	0.711	0.583	0.394	0.234	0.135	0.0809
2	1.25	0.621	0.523	0.365	0.224	0.132	0.0802
	2.5	-	0.465	0.331	0.209	0.127	0.0788
	5	-	-	0.300	0.191	0.120	0.0764
	0.312	0.662	0.501	0.313	0.175	0.0953	-
	0.625	0.535	0.433	0.290	0.169	0.0939	0.0528
5	1.25	0.404	0.345	0.250	0.157	0.0909	0.0521
	2.5	-	0.267	0.202	0.136	0.0846	0.0506
	5	-	-	0.161	0.112	0.0742	0.0475
	0.312	0.573	0.419	0.254	0.139	0.0740	-
	0.625	0.434	0.349	0.232	0.134	0.0729	0.0396
10	1.25	0.301	0.260	0.193	0.123	0.0703	0.0390
	2.5	-	0.184	0.145	0.102	0.0645	0.0377
	5	-	-	0.106	0.0779	0.0542	0.0348

TABLE 6.35

DISPLACEMENT FACTORS F

$n_1 = 50$ (After Themn de Barros, 1966)

n_2	h_2/a \ h_1/a	0.156	0.312	0.625	1.25	2.5	5
	0.312	0.744	0.538	0.324	0.178	0.0960	-
	0.625	0.677	0.508	0.314	0.175	0.0953	0.0532
2	1.25	0.594	0.461	0.297	0.170	0.0940	0.0528
	2.5	-	0.411	0.271	0.161	0.0915	0.0522
	5	-	-	0.245	0.148	0.0868	0.0509
	0.312	0.612	0.421	0.244	0.131	0.0687	-
	0.625	0.507	0.378	0.233	0.128	0.0681	0.0364
5	1.25	0.387	0.311	0.209	0.122	0.0667	0.0361
	2.5	-	0.241	0.173	0.110	0.0637	0.0354
	5	-	-	0.137	0.0915	0.0575	0.0338
	0.312	0.524	0.348	0.197	0.104	0.0539	-
	0.625	0.410	0.305	0.187	0.102	0.0534	0.0280
10	1.25	0.288	0.237	0.164	0.0966	0.0523	0.0277
	2.5	-	0.169	0.128	0.0847	0.0496	0.0272
	5	-	-	0.0926	0.0664	0.0436	0.0258

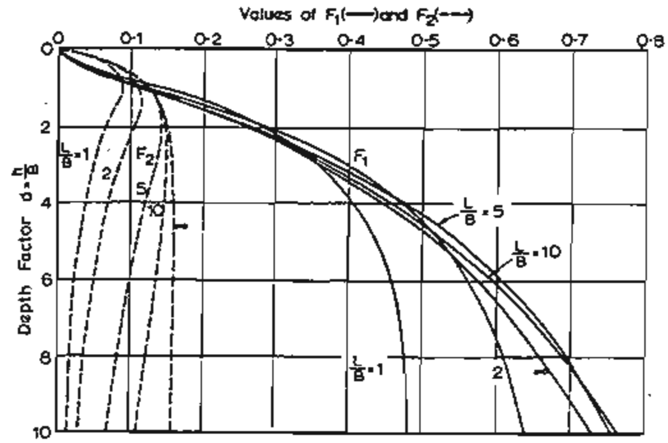


FIG.6.19 Factors F_1 and F_2 for Steinbrenner approximate method of calculating vertical displacement (Steinbrenner, 1934).

6.3 Four-Layer Systems

Some solutions for a four-layer system, subjected to uniform vertical loading over a circular area, have been obtained by Verstraeten (1967). Contours of the major, intermediate and minor principal stresses and the maximum shear stress for a particular geometry and various modular ratios have been given. A brief examination of the influence of uniform horizontal loading on the radial stress has also been made.

6.4 Approximate Solutions for Multi-Layer Systems

6.4.1 STEINBRENNER'S METHOD

This method, first suggested by Steinbrenner (1934), enables the vertical surface displacement of a loaded area to be estimated on the assumption that the stress distribution within the layered system is identical with the Boussinesq distribution for a homogeneous semi-infinite mass. It was originally applied to the problem of a single layer underlain by a rough rigid base, and for the case of a uniformly loaded rectangular area, the resulting approximate influence factors for vertical surface displacement at the corner of the rectangle on a layer of depth h are shown in Fig.6.19. The displacement at the corner is given as

$$\rho_z = \frac{pB}{E} I_\rho \quad \ldots (6.4)$$

where $I_\rho = (1-\nu^2)F_1 + (1-\nu-2\nu^2)F_2$

B = shorter side of rectangle.

Steinbrenner's method for a single layer may be extended to any number of layers as follows:

for the vertical surface displacement of a rectangle on n layers:

$$\rho_z = pB\left\{\sum_{i=1}^{n-1} \frac{(I_{\rho i+1} - I_{\rho i})}{E_i} + \frac{I_{\rho n}}{E_n}\right\} \quad \ldots (6.5)$$

where E_i, ν_i are the elastic parameters of layer i

$I_{\rho i}$ is the vertical displacement influence factor (Fig. 6.19) corresponding to a depth factor h_i/B, where h_i is the depth below the ground surface of the top of layer i.

The method can be adapted to the calculation of displacements in directions other than the vertical and for loadings other than vertical.

Investigations of the accuracy of Steinbrenner's method have been made by Davis and Taylor (1962) and Poulos (1967b), who concluded that, for a single layer underlain by a rough rigid base, the vertical displacement is underestimated by about 10% except for very thin layers for $\nu=0.5$, when this method overestimates the vertical displacement considerably.

Davis and Taylor (1962) have also shown that Steinbrenner's method cannot accurately be applied to the calculation of the horizontal surface displacement of a layer due to vertical loading or the vertical displacement of a layer due to horizontal surface loading. For horizontal displacement due to horizontal loading the method gives reasonably accurate solutions.

Steinbrenner's approximation is most satisfactory for layered systems in which the modulus increases rather than decreases with depth.

6.4.2 PALMER AND BARBER'S METHOD

For a general two-layer system, Palmer and Barber (1940) assume that the upper layer, of thickness h, modulus E_1 and $\nu=\nu_1$, may be replaced by an equivalent thickness h_e of lower layer material (modulus E_2, $\nu=\nu_2$) as follows:

$$h_e = h \left\{\frac{E_1(1-\nu_2^2)}{E_2(1-\nu_1^2)}\right\}^{1/3} \qquad \ldots (6.6)$$

The vertical surface displacement is then obtained by adding the vertical displacement of the equivalent layer between $z=0$ and $z=h_e$ to the vertical displacement at a depth h_e in a semi-infinite mass.

For example, for the case of a circular footing (radius a) on a two-layer system having $\nu_1=\nu_2=0.5$, the vertical displacement ρ_1 at depth h_e is

$$\rho_1 = \frac{1.5\, pa^2}{E_2\left[a^2+ h^2\left(\frac{E_1}{E_2}\right)^{2/3}\right]^{\frac{1}{2}}} \qquad \ldots (6.7)$$

The displacement within the upper layer, ρ_2 is

$$\rho_2 = \frac{E_2}{E_1}\left\{\frac{1.5\, pa}{E_2} - \rho_1\right\} \qquad \ldots (6.8)$$

The vertical surface displacement $\rho_z=\rho_1+\rho_2$

$$\rho_z = \frac{1.5\, pa}{E_2}\left[\frac{a}{\left[a^2+h^2\left(\frac{E_1}{E_2}\right)^{2/3}\right]^{\frac{1}{2}}}\left(1 - \frac{E_2}{E_1}\right) + \frac{E_2}{E_1}\right] \qquad \ldots (6.9)$$

Comparisons between the above approximate solution and the correct solution given by Burmister (see Section 6.1) show excellent agreement.

Palmer and Barber's method can be extended to multi-layer systems by repeated replacement of overlying layers by equivalent thicknesses of the lowermost material.

6.4.3 ODEMARK'S METHOD

This method, developed by Odemark (1949), may be used to estimate the vertical surface displacement of a three-layer elastic system in which $\nu_1=\nu_2=\nu_3=0.5$.

For the centre of a uniformly loaded circle, the displacement ρ_z is (refer to Fig.6.17)

$$\rho_z = \frac{2(1-\nu_3^2)}{E_3}\, pa\, F_{co} \qquad \ldots (6.10)$$

where $F_{co} = F_{co_3} + F_{co_2} + F_{co_1}$

$$F_{co3} = \frac{1}{\left[1+ N_3^2\left(\frac{T}{a}\right)^2\right]^{\frac{1}{2}}}$$

$$F_{co2} = \frac{E_3}{E_2}\left\{\frac{1}{\left[1+N_2^2\left(\frac{T}{a}\right)^2\right]^{\frac{1}{2}}} - \frac{1}{\left[1+N_2'^2\left(\frac{T}{a}\right)^2\right]^{\frac{1}{2}}}\right\}$$

$$F_{co1} = \frac{E_3}{E_1}\left\{1 - \frac{1}{\left[1+N_1^2\left(\frac{T}{a}\right)^2\right]^{\frac{1}{2}}}\right\}$$

$$N_1 = 0.9\,\frac{h_1}{T}$$

$$N_2 = 0.9\,\frac{h_1}{T}\left(\frac{E_1}{E_3}\frac{E_3}{E_m}\right)^{1/3}$$

$$N_2{}^1 = 0.9\,\frac{h_1}{T}\left(\frac{E_1}{E_3}\frac{E_3}{E_m}\right)^{1/3} + 0.9\,\frac{h_2}{T}$$

$$N_3 = 0.9\,\frac{h_1}{T}\left(\frac{E_1}{E_3}\frac{E_3}{E_m}\right)^{1/3} + 0.9\,\frac{h_2}{T}\left(\frac{E_2}{E_3}\right)^{1/3}$$

$$\frac{E_3}{E_m} = \left[\frac{1 + N_2^2\left(\frac{T}{a}\right)^2}{1 + N_3^2\left(\frac{T}{a}\right)^2}\right]^{\frac{1}{2}} + \frac{E_3}{E_2}\left\{1 - \left[\frac{1 + N_2^2\left(\frac{T}{a}\right)^2}{1 + N_2'^2\left(\frac{T}{a}\right)^2}\right]^{\frac{1}{2}}\right\}$$

and E_m = an equivalent modulus of elasticity of the composite underlayer.

The value of $\frac{E_3}{E_m}$ is determined by solving the last equation iteratively. F_{co} may then be determined.

Ueshita and Meyerhof (1967) show that Odemark's method is in good agreement with rigorous analysis if $E_3<E_2<E_1$. However, if E increases with depth, Odermark's method is unsatisfactory. Steinbrenner's method is preferable under these circumstances.

6.4.4 UESHITA AND MEYERHOF'S METHOD

This method, proposed by Ueshita and Meyerhof (1968), may be used to calculate the vertical surface displacement of a uniformly loaded area on a two-layer system in which the lower layer is stiffer than the upper layer.

For a uniformly loaded circle, the centre displacement, ρ_{zc}, and the edge displacement, ρ_{ze}, are given by

$$\rho_{zc} = \frac{2(1-\nu_1^2)}{E_1} pa\, F_{co}' \qquad \ldots (6.11)$$

$$\rho_{ze} = \frac{2(1-\nu_1^2)}{E_1} pa\, F_{ce}' \qquad \ldots (6.12)$$

where $F_{co}' = \frac{1}{2(1-\nu_1^2)} I_{co} + \frac{E_1}{E_2}\left(1 - \frac{I_{co}}{2(1-\nu_1^2)}\right)$

$$F_{ce}' = \frac{1}{2(1-\nu_1^2)} I_{ce} + \frac{E_1}{E_2}\left(0.637 - \frac{I_{ce}}{2(1-\nu_1^2)}\right)$$

I_{co}, I_{ce} are displacement influence factors for the centre and edge respectively of a uniformly loaded circle on a *rough rigid base* (see Section 5.2).

Excellent agreement was found between approximate values of displacement, calculated from the above equations, and accurate solutions obtained by Ueshita and Meyerhof.

6.4.5 VESIC'S METHOD

Vesic (1963) has prepared charts similar to those of Steinbrenner (Fig.6.19) for both a uniform circular load and a rigid circular plate. These charts are shown in Fig.6.20 and are based on the Boussinesq stress distribution for a uniform semi-infinite mass. A convenient form of the equation for vertical surface displacement is:

$$\rho_z = p\, 2a \sum \frac{(1-\nu_n^2)}{E_n} (I_n - I_{n-1}) \qquad \ldots (6.13)$$

where ρ_z = vertical displacement of the centre of the uniformly loaded area

p = uniformly distributed load

a = radius of circular loaded area

ν_n = Poisson's ratio of layer

I_n = displacement factor of the n^{th} layer, corresponding to z/a

z = depth from surface to base of layer n.

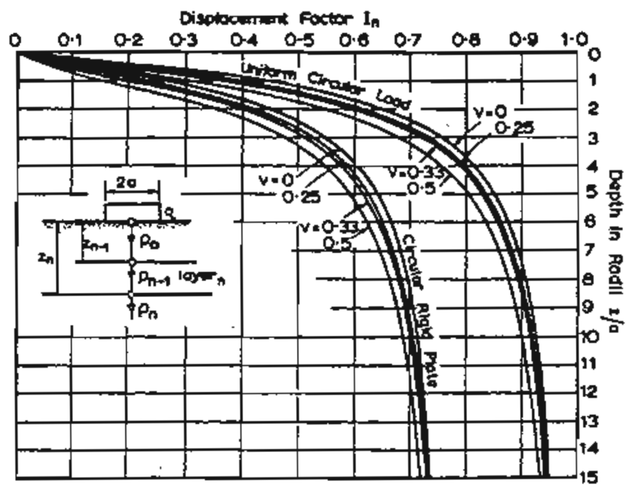


FIG.6.20 Approximate displacement factors for layered systems (Vesic, 1963).

6.4.6 APPROXIMATIONS FOR EQUIVALENT MODULUS OF UPPER LAYERS

Ueshita and Meyerhof (1967) -

For a general two-layer system, the equivalent modulus of the system, which is now considered to be a uniform semi-infinite mass, is plotted in Fig.6.9.

Thenn de Barros (1966) -

For a three-layer system, the upper two layers are replaced by a single layer, of thickness h_1+h_2, and having an equivalent modulus

$$E_e = \left\{ \frac{h_1 \sqrt[3]{E_1} + h_2 \sqrt[3]{E_2}}{h_1 + h_2} \right\}^3 \qquad \ldots (6.14)$$

The three-layer system is thus reduced to a two-layer system, whence the displacement may be determined either from the solutions in Section 6.1, or from one of the approximate methods described in this section.

There seems no reason why this approach should not be extended to more than three layers.

Chapter 7
RIGID LOADED AREAS

7.1 Infinite Strip on Semi-Infinite Mass

7.1.1 SYMMETRICAL VERTICAL LOADING OF SMOOTH STRIP (Fig.7.1)

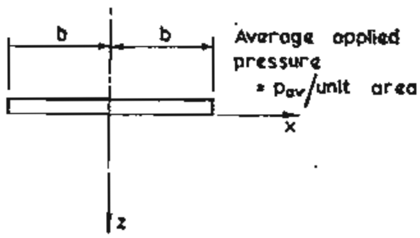


FIG.7.1

Borowicka (1939) obtained the following solution for contact pressure at the base, when the base is smooth:

$$\sigma_z = \frac{2}{\pi} \frac{p_{av}}{\sqrt{1 - (\frac{x}{b})^2}} \qquad \ldots (7.1)$$

7.1.2 MOMENT LOADING OF SMOOTH STRIP (Fig.7.2)

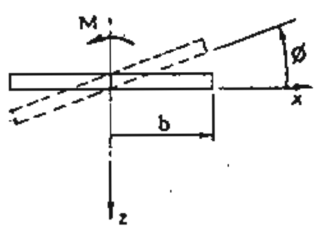


FIG.7.2

Muskhelishvili (1963) derives the following solutions:

$$\text{Rotation } \phi = \frac{(\chi+1)M}{2\pi\mu b^2} \qquad \ldots (7.2)$$

$$\text{Contact pressure } \sigma_z = -\frac{2x}{\pi}\frac{M}{b^3}\frac{1}{\sqrt{1-(\frac{x}{b})^2}} \qquad \ldots (7.3)$$

where $\chi = \dfrac{\lambda+3\mu}{\lambda+\mu}$

λ, μ are Lame's parameters.

7.1.3 SYMMETRICAL VERTICAL LOADING OF ROUGH STRIP (See Fig.7.1).

Muskhelishvili (1963) derives the following solutions for the distribution of vertical and shear stress at the base:

$$\sigma_z = \frac{2p_{av}}{\pi\sqrt{1-(\frac{x}{b})^2}}\frac{1+\chi}{\sqrt{\chi}}\cos\left[\frac{\ln\chi}{2\pi}\ln\frac{1+\frac{x}{b}}{1-\frac{x}{b}}\right] \qquad \ldots (7.4)$$

$$\tau_{xz} = \frac{2p_{av}}{\pi\sqrt{1-(\frac{x}{b})^2}}\frac{1+\chi}{\sqrt{\chi}}\sin\left[\frac{\ln\chi}{2\pi}\ln\frac{1+\frac{x}{b}}{1-\frac{x}{b}}\right] \qquad \ldots (7.5)$$

where $\chi = \dfrac{\lambda+3\mu}{\lambda+\mu}$

λ, μ are Lame's parameters.

It will be observed that when $\nu=0.5$ and $\chi=1$, the above expressions for σ_z and τ_{xz} reduce to those for a smooth strip (Section 7.1.1), i.e. roughness has no effect.

7.1.4 MOMENT LOADING OF ROUGH STRIP (See Fig.7.2)

Muskhelishvili (1963) derives the following solution for base rotation :

$$\phi = \frac{\chi+1}{4\pi\mu(1+4\beta^2)} \frac{M}{b^2} \quad \ldots (7.6)$$

where $\chi = \dfrac{\lambda+3\mu}{\lambda+\mu}$

λ, μ are Lame's parameters.

$$\beta = \frac{\ln\chi}{2\pi}$$

7.2 Circle on Semi-Infinite Mass

7.2.1 SYMMETRICAL VERTICAL LOADING

This case is a particular case of the ellipse on a semi-infinite mass considered in Section 7.4. The following special results are quoted:

Vertical contact pressure distribution within the circle:

$$\sigma_z = \frac{p_{av}}{2(1-\frac{r^2}{a^2})^{1/2}} \quad \ldots (7.7)$$

where r = radial distance from centre

a = radius of circle

p_{av} = average applied pressure $= P/\pi a^2$

P = total load.

Vertical surface displacement of circle:

$$\rho_z = \frac{\pi}{2}(1-\nu^2)\frac{p_{av}a}{E} \quad \ldots (7.8)$$

Other stresses and displacements may be obtained from Section 7.4 for an ellipse with zero eccentricity $(e=0)$.

Sneddon (1946) obtained the following solution for vertical stress σ_z in the mass:

$$\sigma_z = \frac{P_{av}}{2}\left[R^{-1/2}\sin\tfrac{1}{2}\psi + \frac{z}{a}R^{-3/2}\left\{\frac{z}{a}\sin(\tfrac{3}{2}\psi) - \cos(\tfrac{3}{2}\psi)\right\}\right] \quad \ldots (7.9)$$

where $R^2 = \left\{(\frac{r}{a})^2+(\frac{z}{a})^2-1\right\}^2+4(\frac{z}{a})^2$

$$\psi = \tan^{-1}\left[\frac{2\frac{z}{a}}{(\frac{r}{a})^2+(\frac{z}{a})^2-1}\right]$$

The distribution of σ_z beneath the circle, obtained by Muki (1961), is shown in Fig.7.3.

The complete distribution of stress, strain and displacement may also be obtained from Appendix B.

On the axis at depth z,

$$\sigma_z = \frac{P_{av}}{2}\left[\frac{1+3(\frac{z}{a})^2}{\{1+(\frac{z}{a})^2\}^2}\right] \quad \ldots (7.10)$$

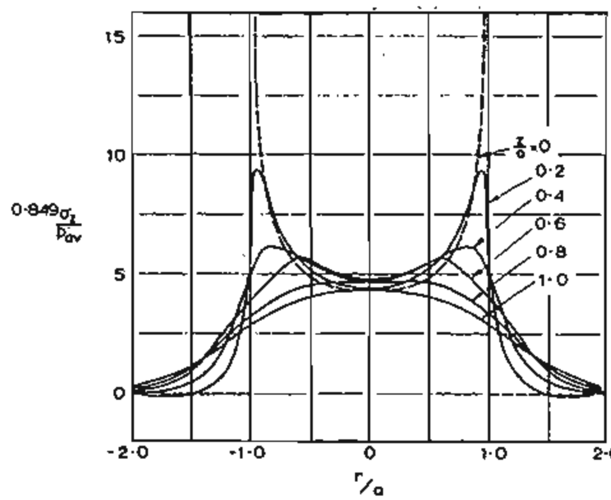


FIG.7.3 Vertical stress distribution beneath rigid circle (Muki, 1961).

7.2.2 HORIZONTAL LOADING

Muki (1961) derives general solutions for stresses and displacements for this problem.

The horizontal displacement ρ_x for an applied average stress p_x in the x direction is given by Bycroft (1956):

$$\rho_x = \frac{(7-8\nu)(1+\nu)p_x\pi a}{16(1-\nu)E} \quad \ldots (7.11)$$

The complete distribution of stress, strain and displacement may also be obtained from Appendix B.

7.2.3 MOMENT LOADING (Fig.7.4)

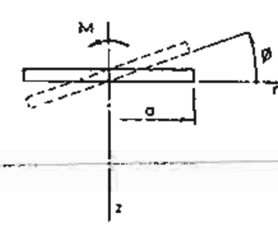


FIG.7.4

Borowicka (1943) obtained the following solution for the rotation ϕ:

$$\phi = \frac{3M(1-\nu^2)}{4Ea^3} \qquad \ldots (7.12)$$

For the particular case of the contact pressure beneath the base with coordinates (r, 0, θ),

$$\sigma_z = \frac{3M(\frac{r}{a})}{2a^3\pi} \frac{\cos\theta}{\sqrt{1-(\frac{r}{a})^2}} \qquad (0 \le \frac{r}{a} \le 1) \qquad \ldots (7.13)$$

The complete distribution of stress, strain and displacement may be obtained from the results in Appendix B.

7.2.4 TORSION LOADING (Fig.7.5)

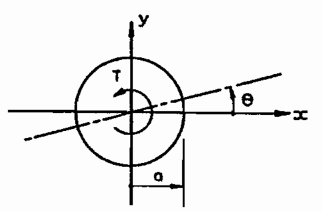


FIG.7.5

Reissner and Sagoci (1944) give the following solution for angular rotation θ:

$$\theta = \frac{3T(1+\nu)}{8Ea^3} \qquad \ldots (7.14)$$

The complete distribution of stress, strain and displacement may be obtained from the solutions given in Appendix B.

7.3 Circular Ring on Semi-Infinite Mass

This problem has been considered by Egorov (1965). (1965).

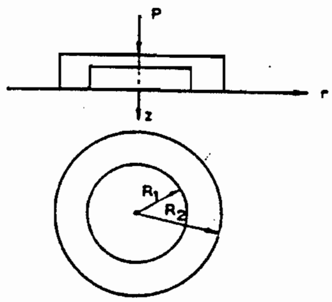


FIG.7.6

The contact pressure distribution is expressed as follows:

$$p = \frac{p_{av}(1-n^2)\sqrt{(\frac{r}{R_2})^2-m^2}}{2\sqrt{1-m^2}\,E_o(k)\sqrt{\{(\frac{r}{R_2})^2-n^2\}\{1-(\frac{r}{R_2})^2\}}} \qquad \ldots (7.15)$$

where $n = \frac{R_1}{R_2}$

$$p_{av} = \frac{P}{\pi R_2^2(1-n^2)}$$

$$E_o(k) = \int_0^{\pi/2} \sqrt{(1-k^2\sin^2\theta)}\,d\theta$$

= complete elliptic integral of second kind

$$k = \left(\frac{1-n^2}{1-m^2}\right)^{\frac{1}{2}}$$

$m = 0.8n$ for $0 \le n \le 0.9$, increasing to 1 for $n=1$ (note that $n=0$ is the case of a circle).

Contact pressure distributions for various values of n are shown in Fig.7.7.

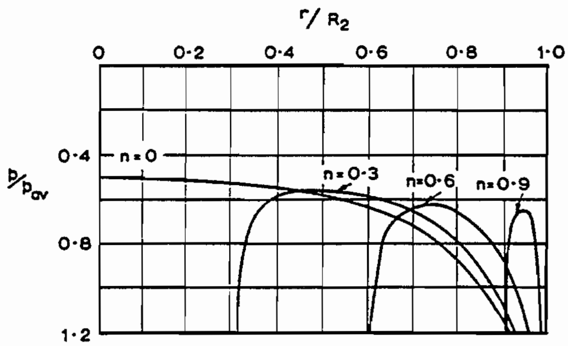


FIG.7.7 Contact pressure beneath rigid circular ring (Egorov, 1965).

It has been found that for $n \geq 0.9$, a rigid ring behaves essentially as a rigid strip foundation (see Section 7.1).

The vertical displacement of a rigid ring may be expressed as

$$\rho_z = \frac{P(1-\nu^2)}{ER_2}\,\omega(n) \qquad \ldots (7.16)$$

Values of $\omega(n)$ are tabulated in Table 7.1.

TABLE 7.1
VALUES OF $\omega(n)$ FOR RIGID RING
(Egorov, 1965)

n	0	0.2	0.4	0.6	0.8	0.9	0.95
$\omega(n)$	0.50	0.50	0.51	0.52	0.57	0.60	0.65

7.4 Rectangle on Semi-Infinite Mass

In all cases below, the rectangle is smooth.

7.4.1 SYMMETRICAL VERTICAL LOADING.

The following approximate solution for the vertical displacement ρ_z is quoted by Whitman and Richart (1967):

$$\rho_z = \frac{P(1-\nu^2)}{\beta_z\sqrt{BL}\,E} \qquad \ldots (7.17)$$

where P = total vertical load
B,L = rectangle dimensions
β_z = factor dependent on L/B and plotted in Fig.7.8.

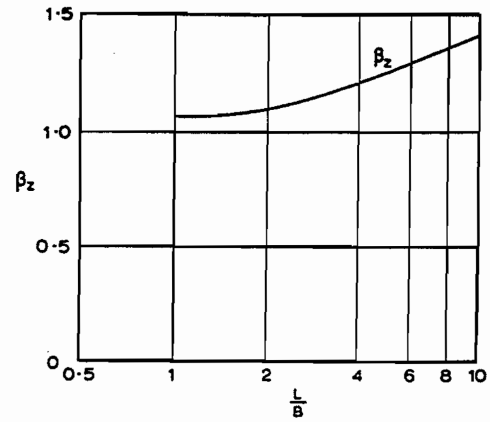


FIG.7.8 Coefficient β_z for rigid rectangle (Whitman and Richart, 1967).

7.4.2 HORIZONTAL LOADING

Barkan (1962) gives the following approximate solution for horizontal displacement ρ_h due to a load Q in the direction of B:

$$\rho_h = \frac{Q(1-\nu^2)}{\beta_x\sqrt{BL}\,E} \qquad \ldots (7.18)$$

β_x is a factor dependent on L/B and ν and is tabulated in Table 7.2.

TABLE 7.2
VALUES OF β_x (Barkan, 1962)

ν	L/B						
	0.5	1	1.5	2	3	5	10
0.1	1.040	1.000	1.010	1.020	1.050	1.150	1.250
0.2	0.990	0.938	0.942	0.945	0.975	1.050	1.160
0.3	0.926	0.868	0.864	0.870	0.906	0.950	1.040
0.4	0.844	0.792	0.770	0.784	0.806	0.850	0.940
0.5	0.770	0.704	0.692	0.686	0.700	0.732	0.840

7.4.3 MOMENT LOADING

Lee (1962) gives the following approximate solution for rotation ϕ of the base of the rectangle due to moment M applied in the direction of B:

$$\phi = \frac{M(1-\nu^2)}{B^2LE}\,I_\theta \qquad \ldots (7.19)$$

I_θ is a function of L/B and is given in Table 7.3.

Solutions quoted by Whitman and Richart (1967) for $L/B<1$ are also given.

TABLE 7.3
VALUES OF I_θ
(Lee, 1963; Whitman and Richart, 1967)

L/B	0.1	0.2	0.5	1	1.5	2	∞
I_θ	1.59	2.29	3.33	3.7	4.12	4.38	5.1

7.5 Ellipse on Semi-Infinite Mass

7.5.1 SYMMETRICAL VERTICAL LOADING (Fig.7.9)

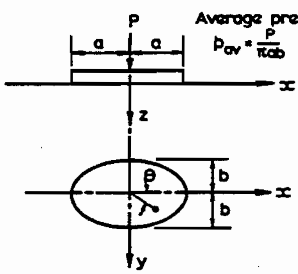


FIG.7.9

This problem has been considered by Schiffman and Aggarwala (1961).

The contact pressure distribution is given by

$$\sigma_z = \frac{p_{av}}{2} \frac{1}{(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2})^{\frac{1}{2}}} \qquad \ldots (7.20)$$

This distribution is similar to that of a rigid circular footing, except that points of equal contact pressure form an elliptical curve similar to the bounding contour of the elliptical footing.

On the axis at depth z below the surface, the normal stresses are as follows:

$$\sigma_z = \frac{p_{av}}{2} \left\{ \frac{(1+\zeta^2)(1+k^2\zeta^2)+\zeta^2(1+k^2\zeta^2)+k\zeta^2\ (1+\zeta^2)}{[(1+k^2\zeta^2)(1+\zeta^2)]^{3/2}} \right\} \qquad \ldots (7.21)$$

$$\sigma_x = \frac{p_{av}}{2e^2} \left\{ \frac{e^2 - (1-2\nu)(1+k^2\zeta^2)^2}{k(1+k^2\zeta^2)[(1+k^2\zeta^2)(1+\zeta^2)]^{\frac{1}{2}}} + (1-2\nu) \right\} \qquad \ldots (7.22)$$

$$\sigma_y = \frac{p_{av}k}{2e^2} \left\{ \frac{e^2 + (1-2\nu)k^2(1+\zeta^2)^2}{k(1+\zeta^2)[(1+k^2\zeta^2)(1+\zeta^2)]^{\frac{1}{2}}} - (1-2\nu) \right\} \qquad \ldots (7.23)$$

where $\zeta = \frac{z}{b}$

$k^2 = \frac{b^2}{a^2}$

$e^2 = 1 - \frac{b^2}{a^2} = 1 - k^2$

For $e=0$, the ellipse reduces to a circle.

The distribution of these normal stresses along the axis is plotted in Figs. 7.10 to 12 for various values of e and for four values of ν. It should be noted that σ_z is independent of ν. For the limiting case $e=0$, the solutions are identical with those for a rigid circle (Section 7.2).

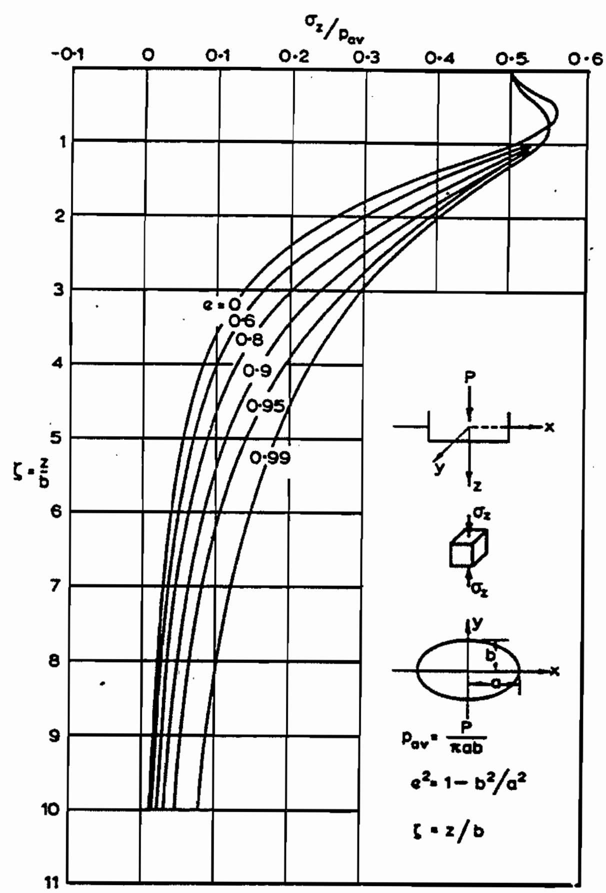
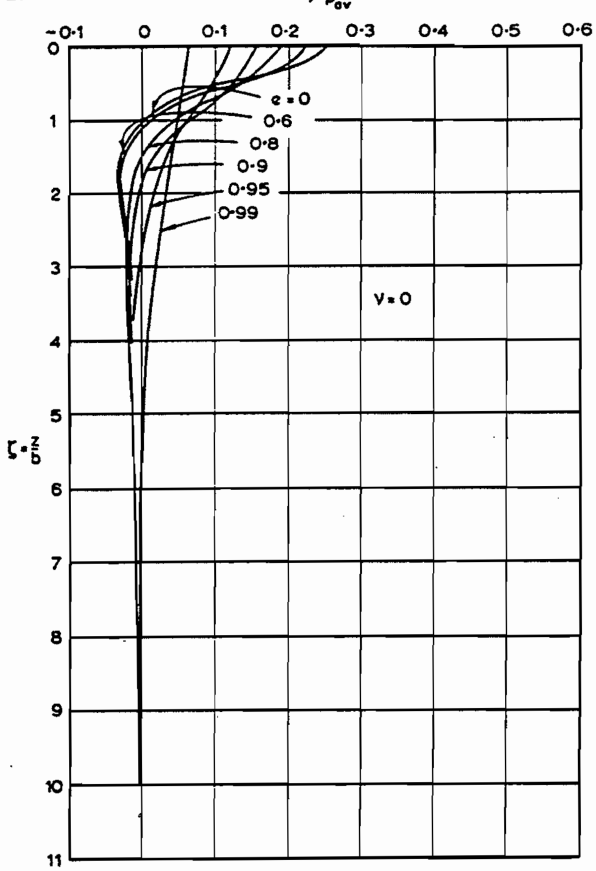
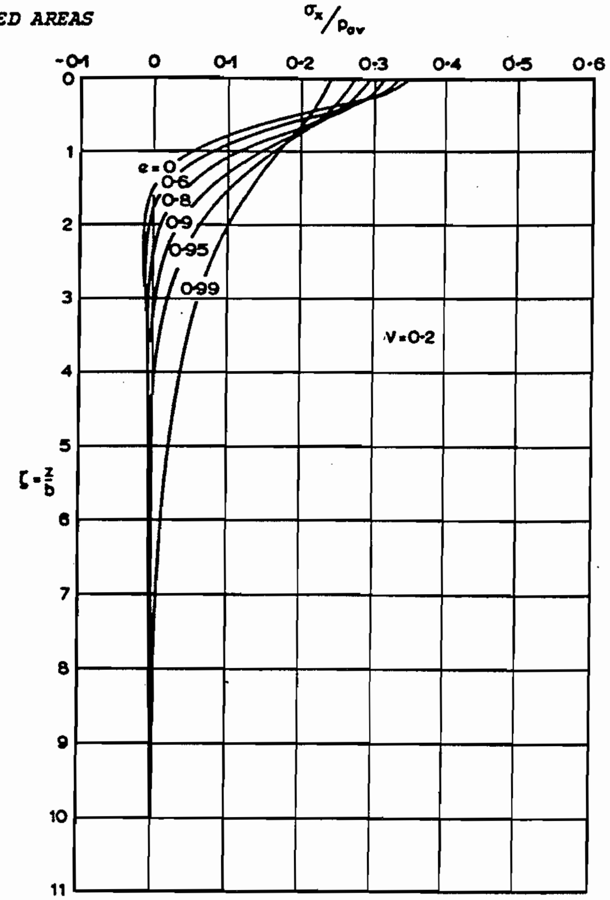


FIG.7.10 σ_z along axis of rigid ellipse (Schiffman and Aggarwala, 1961).

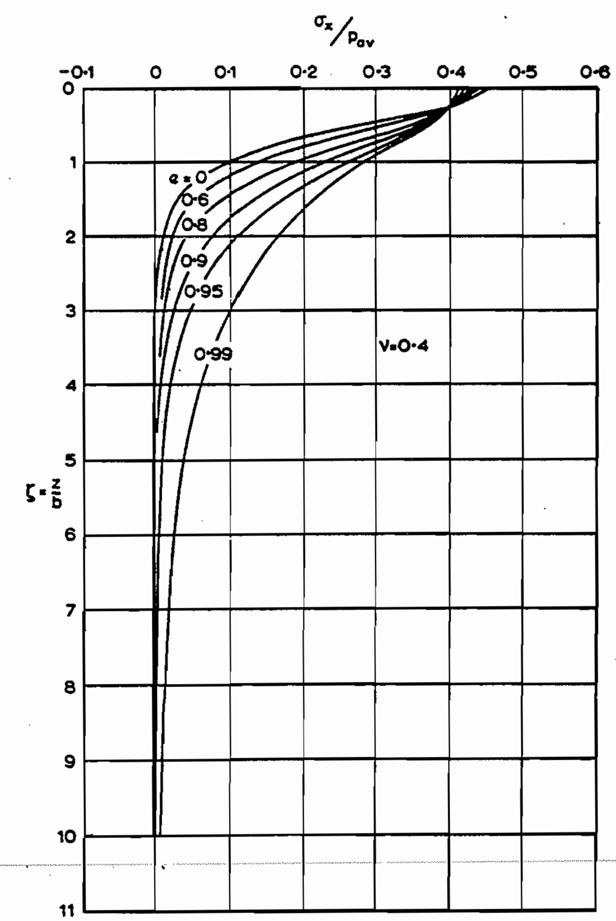
Displacements are not evaluated by Schiffman and Aggarwala, but general expressions are given. Way (1940) gives explicit expressions for the horizontal surface displacements inside and outside the ellipse.

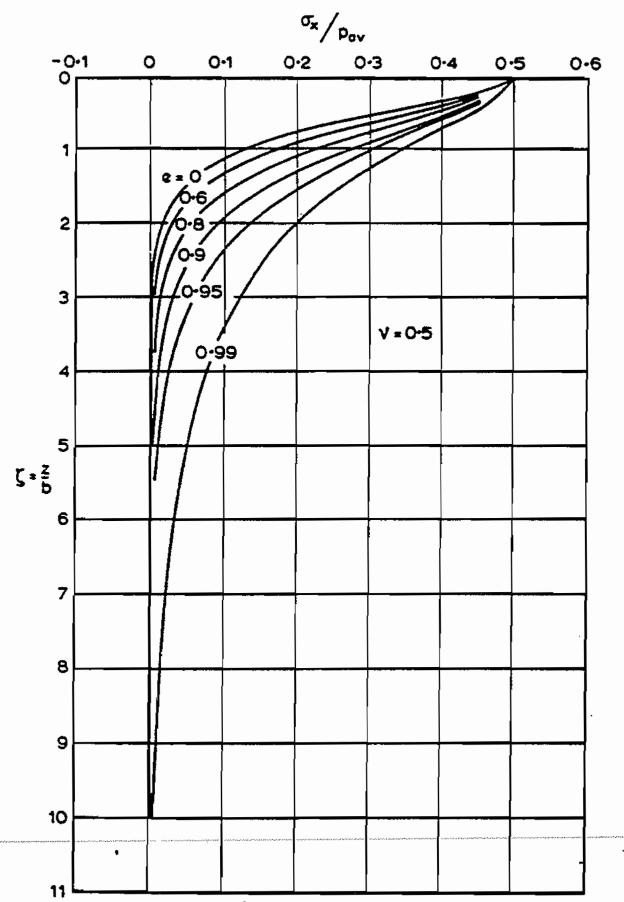

σx/pav
-0·1 0 0·1 0·2 0·3 0·4 0·5 0·6
0 1 2 3 4 5 6 7 8 9 10 11
ζ = z/b
e = 0
0·6
0·8
0·9
0·95
0·99
ν = 0

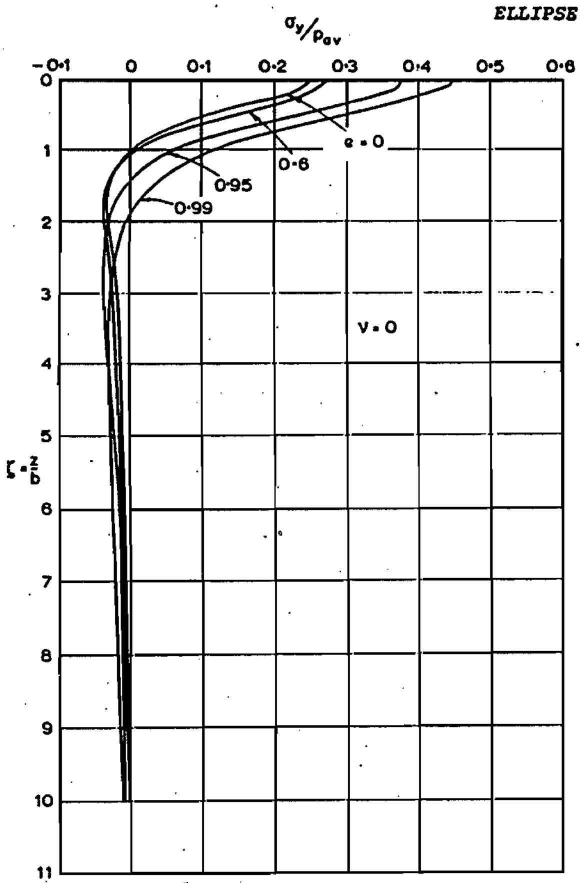
(a)

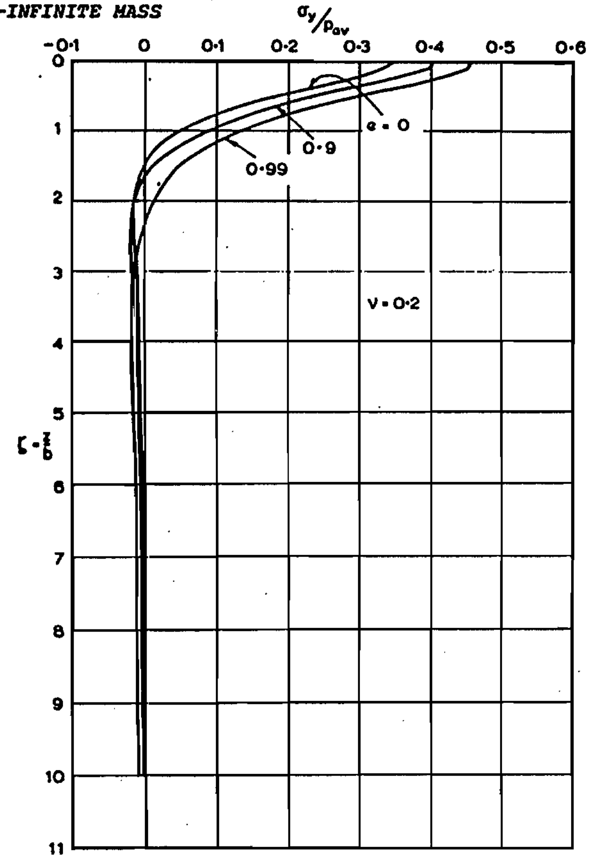

σx/pav
-0·1 0 0·1 0·2 0·3 0·4 0·5 0·6
0 1 2 3 4 5 6 7 8 9 10 11
ζ = z/b
e = 0
0·6
0·8
0·9
0·95
0·99
ν = 0·2

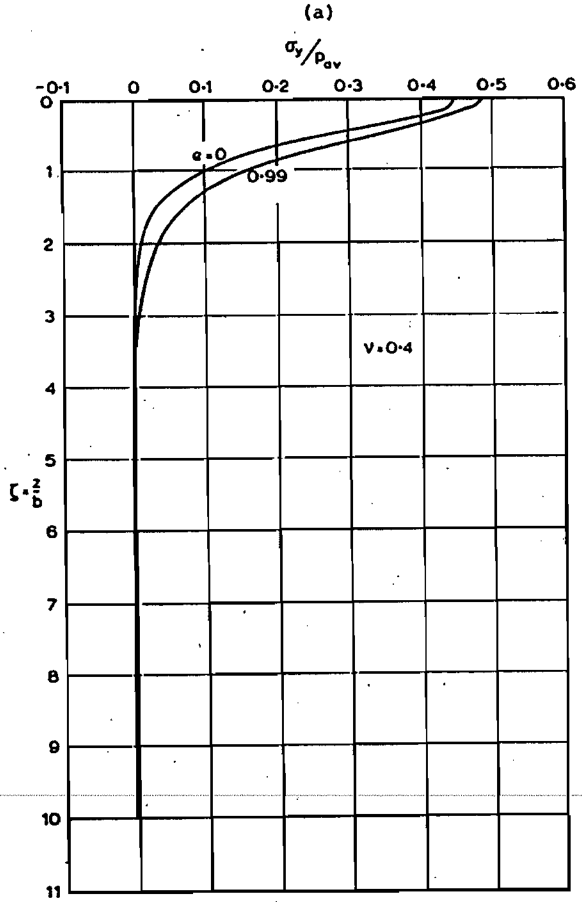
(b)

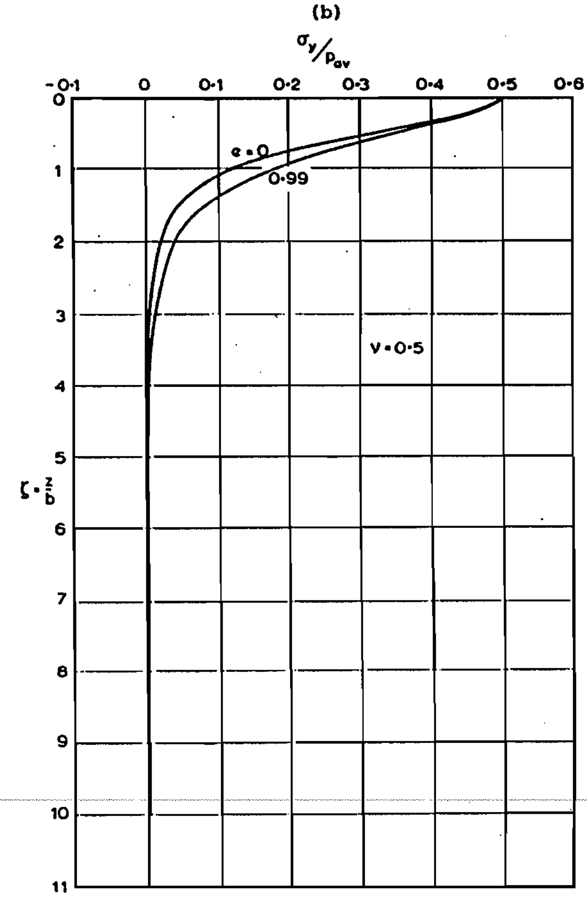

σx/pav
-0·1 0 0·1 0·2 0·3 0·4 0·5 0·6
0 1 2 3 4 5 6 7 8 9 10 11
ζ = z/b
e = 0
0·6
0·8
0·9
0·95
0·99
ν = 0·4


σx/pav
-0·1 0 0·1 0·2 0·3 0·4 0·5 0·6
0 1 2 3 4 5 6 7 8 9 10 11
ζ = z/b
e = 0
0·6
0·8
0·9
0·95
0·99
ν = 0·5


σy/pav
-0·1 0 0·1 0·2 0·3 0·4 0·5 0·6
0 1 2 3 4 5 6 7 8 9 10 11
ζ = z/b
e = 0
0·6
0·95
0·99
v = 0
(a)


σy/pav
-0·1 0 0·1 0·2 0·3 0·4 0·5 0·6
0 1 2 3 4 5 6 7 8 9 10 11
ζ = z/b
e = 0
0·9
0·99
v = 0·2
(b)


σy/pav
-0·1 0 0·1 0·2 0·3 0·4 0·5 0·6
0 1 2 3 4 5 6 7 8 9 10 11
ζ = z/b
e = 0
0·99
v = 0·4
(c)


σy/pav
-0·1 0 0·1 0·2 0·3 0·4 0·5 0·6
0 1 2 3 4 5 6 7 8 9 10 11
ζ = z/b
e = 0
0·99
v = 0·5
(d)

7.6 Infinite Strip on Finite Layer

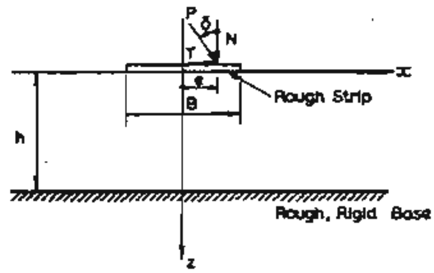


FIG.7.13

The problem of a rough strip, on a finite layer underlain by a rough rigid base, and subjected to inclined eccentric loading, has been studied by Milovic et al (1970) using a finite element analysis.

Contact Stresses

The contact stresses beneath the strip are expressed as

$$\sigma_z = \frac{P}{B} K_z \qquad \dots (7.24a)$$

$$\sigma_x = \frac{P}{B} K_x \qquad \dots (7.24b)$$

$$\tau_{xz} = \frac{P}{B} K_{xz} \qquad \dots (7.24c)$$

where

$$K_z = K_{zN}\cos\delta + K_{zT}\sin\delta + \frac{e}{B} K_{zM}\cos\delta \qquad \dots (7.25a)$$

$$K_x = K_{xN}\cos\delta + K_{xT}\sin\delta + \frac{e}{B} K_{xM}\cos\delta \qquad \dots (7.25b)$$

$$K_{xz} = K_{xzN}\cos\delta + K_{xzT}\sin\delta + \frac{e}{B} K_{xzM}\cos\delta \qquad \dots (7.25c)$$

and

K_{zN}, K_{xN}, K_{xzN} are influence factors for σ_z, σ_x and τ_{xz}, due to the vertical component N of load P,

K_{zT}, K_{xT}, K_{xzT} are influence factors due to the tangential component T,

K_{zM}, K_{xM}, K_{xzM} are influence factors due to moment $M=Ne$.

For $h/B=1, 2$ and 3 and three values of ν, the values of these influence factors are given in Tables 7.4 to 7.6.

Stresses on Axis

The stresses σ_z^a, σ_x^a and τ_{xz}^a on the axis are expressed as

$$\sigma_z^a = \frac{P}{B} K_z^a \qquad \dots (7.26a)$$

$$\sigma_x^a = \frac{P}{B} K_x^a \qquad \dots (7.26b)$$

$$\tau_{xz}^a = \frac{P}{B} K_{xz}^a \qquad \dots (7.26c)$$

where

$$K_z^a = K_{zN}^a \cos\delta \qquad \dots (7.27a)$$

$$K_x^a = K_{xN}^a \cos\delta \qquad \dots (7.27b)$$

$$K_{xz}^a = K_{xzT}^a \sin\delta + \frac{e}{B} K_{xzM} \cos\delta \qquad \dots (7.27c)$$

Values of these influence factors are shown in Tables 7.7 to 7.9.

TABLE 7.4

COEFFICIENTS K FOR $\frac{h}{B} = 1$

(Milovic et al, 1970)

			K_z			K_x			K_{xz}	
	$\frac{x}{B}$	K_{zN}	K_{zT}	K_{zM}	K_{xN}	K_{xT}	K_{xM}	K_{xzN}	K_{xzT}	K_{xzM}
	-0.45	1.3435	-0.5272	-6.5658	0.0357	-0.4321	0.1351	-0.1841	1.2563	-0.8578
	-0.35	1.0234	-0.1788	-3.6458	0.0262	-0.1802	-0.0060	-0.0735	1.0568	-1.3841
	-0.25	0.9162	-0.0820	-2.2486	0.0172	-0.0869	-0.0168	-0.0363	0.9420	-1.5228
	-0.15	0.8682	-0.0368	-1.2488	0.0123	-0.0417	-0.0104	-0.0172	0.8847	-1.5831
	-0.05	0.8489	-0.0106	-0.4024	0.0101	-0.0126	-0.0027	-0.0051	0.8600	-1.6065
ν=0.005	0.05	0.8489	0.0106	0.4024	0.0101	0.0126	0.0027	0.0051	0.8600	-1.6065
	0.15	0.8682	0.0368	1.2488	0.0123	0.0417	0.0104	0.0172	0.8847	-1.5831
	0.25	0.9162	0.0820	2.2486	0.0172	0.0869	0.0168	0.0363	0.9420	-1.5228
	0.35	1.0234	0.1788	3.6458	0.0262	0.1802	0.0060	0.0735	1.0568	-1.3841
	0.45	1.3435	0.5272	6.5658	0.0357	0.4321	-0.1351	0.1841	1.2563	-0.8578
	-0.45	1.4035	-0.3529	-6.7504	0.4360	-0.3914	-1.8784	0.0377	1.2349	-0.8397
	-0.35	1.0010	-0.0417	-3.4908	0.3745	-0.2194	-1.2026	0.0476	1.0600	-0.9327
	-0.25	0.9011	-0.0108	-2.1655	0.3567	-0.1113	-0.8131	0.0309	0.9459	-0.8730
	-0.15	0.8557	0.0023	-1.1983	0.3439	-0.0569	-0.4620	0.0184	0.8910	-0.8511
	-0.05	0.8388	0.0017	-0.3864	0.3387	-0.0175	-0.1506	0.0061	0.8673	-0.8398
ν=0.300	0.05	0.8388	-0.0017	0.3864	0.3387	0.0175	0.1506	-0.0061	0.8673	-0.8398
	0.15	0.8557	-0.0023	1.1983	0.3439	0.0569	0.4620	-0.0184	0.8910	-0.8511
	0.25	0.9011	0.0108	2.1655	0.3567	0.1113	0.8131	-0.0309	0.9459	-0.8730
	0.35	1.0010	0.0417	3.4908	0.3745	0.2194	1.2026	-0.0476	1.0600	-0.9327
	0.45	1.4035	0.3529	6.7504	0.4360	0.3914	1.8784	-0.0377	1.2349	-0.8397
	-0.45	1.5016	-0.2349	-7.1178	0.8455	-0.3369	-3.8359	0.2007	1.1714	-0.7384
	-0.35	0.9178	0.1995	-3.0815	0.6547	-0.3986	-1.9297	0.1527	1.0629	-0.5468
	-0.25	0.8982	0.1193	-2.1259	0.6255	-0.1705	-1.4746	0.0886	0.9556	-0.3142
	-0.15	0.8443	0.0659	-1.1185	0.6238	-0.1127	-0.7898	0.0526	0.9159	-0.2414
	-0.05	0.8384	0.0149	-0.3801	0.6231	-0.0295	-0.2743	0.0164	0.8948	-0.1982
ν=0.450	0.05	0.8384	-0.0149	0.3801	0.6231	0.0295	0.2743	-0.0164	0.8948	-0.1982
	0.15	0.8443	-0.0659	1.1185	0.6238	0.1127	0.7898	-0.0526	0.9159	-0.2414
	0.25	0.8982	-0.1193	2.1259	0.6255	0.1705	1.4746	-0.0886	0.9556	-0.3142
	0.35	0.9178	-0.1995	3.0815	0.6547	0.3986	1.9297	-0.1527	1.0629	-0.5468
	0.45	1.5016	0.2349	7.1178	0.8455	0.3369	3.8359	-0.2007	1.1714	-0.7384

TABLE 7.5

COEFFICIENTS K FOR $\frac{h}{B} = 2$

(Milovic et al, 1970)

		K_z			K_x			K_{xz}		
	$\frac{x}{B}$	K_{zN}	K_{zT}	K_{zM}	K_{xN}	K_{xT}	K_{xM}	K_{xzN}	K_{xzT}	K_{xzM}
ν=0.005	-0.45	1.4239	-0.7483	-6.7512	0.1146	-0.4253	0.0510	-0.3470	1.3617	-0.7322
	-0.35	1.0265	-0.2419	-3.5061	0.0541	-0.1382	-0.0888	-0.1556	1.0542	-1.3336
	-0.25	0.8991	-0.1244	-2.1507	0.0316	-0.0640	-0.0484	-0.0841	0.9145	-1.5062
	-0.15	0.8379	-0.0608	-1.1867	0.0226	-0.0304	-0.0243	-0.0430	0.8487	-1.5864
	-0.05	0.8123	-0.0185	-0.3813	0.0193	-0.0091	-0.0074	-0.0133	0.8206	-1.6184
	0.05	0.8123	0.0185	0.3813	0.0193	0.0091	0.0074	0.0133	0.8206	-1.6184
	0.15	0.8379	0.0608	1.1867	0.0226	0.0304	0.0243	0.0430	0.8487	-1.5864
	0.25	0.8991	0.1244	2.1507	0.0316	0.0640	0.0484	0.0841	0.9145	-1.5062
	0.35	1.0265	0.2419	3.5061	0.0541	0.1382	0.0888	0.1556	1.0542	-1.3336
	0.45	1.4239	0.7483	6.7512	0.1146	0.4253	-0.0510	0.3470	1.3617	-0.7322
ν=0.300	-0.45	1.5314	-0.5708	-6.9558	0.5532	-0.3380	-2.2374	-0.0315	1.3639	-0.8941
	-0.35	1.0029	-0.1052	-3.3210	0.4239	-0.1511	-1.3476	-0.0118	1.0593	-0.9192
	-0.25	0.8732	-0.0592	-2.0582	0.3736	-0.0653	-0.8618	-0.0071	0.9125	-0.9116
	-0.15	0.8090	-0.0244	-1.1271	0.3463	-0.0328	-0.4765	-0.0029	0.8464	-0.9223
	-0.05	0.7834	-0.0074	-0.3623	0.3352	-0.0098	-0.1538	-0.0008	0.8177	-0.9260
	0.05	0.7834	0.0074	0.3623	0.3352	0.0098	0.1538	0.0008	0.8177	-0.9260
	0.15	0.8090	0.0244	1.1271	0.3463	0.0328	0.4765	0.0029	0.8464	-0.9223
	0.25	0.8732	0.0592	2.0582	0.3736	0.0653	0.8618	0.0071	0.9125	-0.9116
	0.35	1.0029	0.1052	3.3210	0.4239	0.1511	1.3476	0.0118	1.0593	-0.9192
	0.45	1.5314	0.5708	6.9558	0.5532	0.3380	2.2374	0.0315	1.3639	-0.8941
ν=0.450	-0.45	1.6792	-0.5789	-7.3255	1.0095	-0.1877	-4.3289	0.1999	1.3350	-0.9264
	-0.35	0.9214	0.1194	-2.9452	0.6800	-0.3207	-2.1602	0.1028	1.0818	-0.5444
	-0.25	0.8606	0.0257	-2.0055	0.6645	-0.0879	-1.5451	0.0530	0.9330	-0.3927
	-0.15	0.7782	0.0254	-1.0353	0.6090	-0.0734	-0.8089	0.0302	0.8750	-0.3594
	-0.05	0.7607	0.0017	-0.3515	0.5985	-0.0141	-0.2773	0.0090	0.8460	-0.3393
	0.05	0.7607	-0.0017	0.3515	0.5985	0.0141	0.2773	-0.0090	0.8460	-0.3393
	0.15	0.7782	-0.0254	1.0353	0.6090	0.0734	0.8089	-0.0302	0.8750	-0.3594
	0.25	0.8606	-0.0257	2.0055	0.6645	0.0879	1.5451	-0.0530	0.9330	-0.3927
	0.35	0.9214	-0.1194	2.9452	0.6800	0.3207	2.1602	-0.1028	1.0818	-0.5444
	0.45	1.6792	0.5789	7.3225	1.0095	0.1877	4.3289	-0.1999	1.3350	-0.9264

TABLE 7.6

COEFFICIENTS K FOR $\frac{h}{B} = 3$

(Milovic et al, 1970)

	$\frac{x}{B}$	K_z			K_x			K_{xz}		
		K_{zN}	K_{zT}	K_{zM}	K_{xN}	K_{xT}	K_{xM}	K_{xzN}	K_{xzT}	K_{xzM}
ν=0.005	-0.45	1.3803	-0.7680	-6.5858	0.1231	-0.4642	0.0610	-0.3501	1.2635	-0.8359
	-0.35	1.0382	-0.3111	-3.6383	0.0714	-0.1952	-0.0447	-0.1753	1.0633	-1.4231
	-0.25	0.9102	-0.1640	-2.2313	0.0468	-0.0954	-0.0390	-0.0991	0.9415	-1.5959
	-0.15	0.8480	-0.0827	-1.2339	0.0353	-0.0462	-0.0223	-0.0519	0.8793	-1.6750
	-0.05	0.8224	-0.0255	-0.3966	0.0308	-0.0140	-0.0072	-0.0163	0.8522	-1.7081
	0.05	0.8224	0.0255	0.3966	0.0308	0.0140	0.0072	0.0163	0.8522	-1.7081
	0.15	0.8486	0.0827	1.2339	0.0353	0.0462	0.0223	0.0519	0.8793	-1.6756
	0.25	0.9102	0.1640	2.2313	0.0468	0.0954	0.0390	0.0991	0.9415	-1.5959
	0.35	1.0382	0.3111	3.6383	0.0714	0.1952	0.0447	0.1753	1.0633	-0.4231
	0.45	1.3803	0.7680	6.5858	0.1231	0.4642	-0.0610	0.3501	1.2635	-0.8359
ν=0.300	-0.45	1.4974	-0.5944	-6.8039	0.5253	-0.3641	-1.9598	-0.0679	1.2624	-0.9291
	-0.35	1.0223	-0.1690	-3.4673	0.4187	-0.1987	-1.2399	-0.0233	1.0708	-1.0719
	-0.25	0.8815	-0.0930	-2.1247	0.3758	-0.0943	-0.8292	-0.0156	0.9404	-1.0451
	-0.15	0.8132	-0.0441	-1.1652	0.3499	-0.0467	-0.4679	-0.0081	0.8771	-1.0445
	-0.05	0.7854	-0.0135	-0.3740	0.3387	-0.0141	-0.1519	-0.0026	0.8491	-1.0438
	0.05	0.7854	0.0135	0.3740	0.3387	0.0141	0.1519	0.0026	0.8491	-1.0438
	0.15	0.8132	0.0441	1.1652	0.3499	0.0467	0.4679	0.0081	0.8771	-1.0445
	0.25	0.8815	0.0930	2.1247	0.3758	0.0943	0.8292	0.0156	0.9404	-1.0451
	0.35	1.0223	0.1690	3.4673	0.4187	0.1987	1.2399	0.0233	1.0708	-1.0719
	0.45	1.4974	0.5944	6.8039	0.5253	0.3641	1.9598	0.0679	1.2624	-0.9291
ν=0.450	-0.45	1.6568	-0.5677	-7.2166	0.9636	-0.2111	-3.9620	0.1266	1.2195	-0.8907
	-0.35	0.9450	0.0544	-3.0324	0.6555	-0.3436	-1.9615	0.0920	1.0805	-0.7488
	-0.25	0.8649	0.0490	-2.0553	0.6494	-0.1152	-1.4760	0.0438	0.9448	-0.5554
	-0.15	0.7773	0.0064	-1.0615	0.5978	-0.0827	-0.7812	0.0252	0.8920	-0.5115
	-0.05	0.7556	0.0063	-0.3592	0.5872	-0.0182	-0.2702	0.0072	0.8636	-0.4831
	0.05	0.7556	-0.0063	0.3592	0.5872	0.0182	0.2702	-0.0072	0.8636	-0.4831
	0.15	0.7773	-0.0064	1.0615	0.5978	0.0827	0.7812	-0.0252	0.8920	-0.5115
	0.25	0.8649	-0.0490	2.0553	0.6494	0.1152	1.4760	-0.0438	0.9448	-0.5554
	0.35	0.9450	-0.0544	3.0324	0.6555	0.3430	1.9615	-0.0920	1.0805	-0.7488
	0.45	1.6568	0.5677	7.2166	0.9636	0.2111	3.9620	-0.1266	1.2195	-0.8907

TABLE 7.7
COEFFICIENTS K^a FOR $\frac{h}{B}=1$ (Milovic et al, 1970)

	$\frac{z}{B}$	K^a_{zN}	K^a_{xN}	K^a_{xzT}	K^a_{xzM}
ν=0.005	0.05	0.8489	0.0101	0.8600	-1.6065
	0.15	0.8396	0.0107	0.8113	-1.6103
	0.25	0.8322	-0.0065	0.7296	-1.5571
	0.35	0.8225	-0.0307	0.6390	-1.4322
	0.45	0.8085	-0.0518	0.5578	-1.2608
	0.55	0.7903	-0.0645	0.4938	-1.0770
	0.65	0.7699	-0.0670	0.4479	-0.9059
	0.75	0.7491	-0.0593	0.4175	-0.7622
	0.85	0.7287	-0.0414	0.3993	-0.6557
	0.95	0.7089	-0.0133	0.3906	-0.5963
ν=0.300	0.05	0.8388	0.3387	0.8673	-0.8398
	0.15	0.8528	0.2919	0.8051	-1.1035
	0.25	0.8654	0.2331	0.7084	-1.2304
	0.35	0.8684	0.1794	0.6047	-1.2154
	0.45	0.8600	0.1419	0.5139	-1.1073
	0.55	0.8421	0.1266	0.4423	-0.9639
	0.65	0.8184	0.1316	0.3891	-0.8268
	0.75	0.7917	0.1564	0.3497	-0.7232
	0.85	0.7626	0.2010	0.3193	-0.6738
	0.95	0.7304	0.2673	0.2933	-0.7026
ν=0.450	0.05	0.8384	0.6231	0.8948	-0.1982
	0.15	0.8703	0.5189	0.8032	-0.6603
	0.25	0.9001	0.4205	0.6849	-0.9208
	0.35	0.9097	0.3397	0.5680	-0.9867
	0.45	0.9052	0.2927	0.4702	-0.9293
	0.55	0.8864	0.2775	0.3936	-0.8204
	0.65	0.8606	0.2944	0.3348	-0.7174
	0.75	0.8299	0.3407	0.2856	-0.6571
	0.85	0.7945	0.4179	0.2385	-0.6713
	0.95	0.7518	0.5302	0.1858	-0.7999

TABLE 7.8
COEFFICIENTS K^a FOR $\frac{h}{B}=2$ (Milovic et al, 1970)

	$\frac{z}{B}$	K^a_{zN}	K^a_{xN}	K^a_{xzT}	K^a_{xzM}
ν=0.005	0.03	0.8123	0.0193	0.8206	-1.3336
	0.10	0.7952	0.0421	0.7963	-1.3936
	0.17	0.7811	0.0503	0.7503	-1.2549
	0.23	0.7692	0.0457	0.6892	-1.0799
	0.30	0.7580	0.0322	0.6211	-0.9403
	0.37	0.7459	0.0143	0.5529	-0.8379
	0.43	0.7319	-0.0045	0.4896	-0.7617
	0.50	0.7158	-0.0222	0.4335	-0.7018
	0.57	0.6982	-0.0375	0.3853	-0.6520
	0.63	0.6794	-0.0503	0.3448	-0.6084
	0.73	0.6503	-0.0635	0.2965	-0.5505
	0.87	0.6129	-0.0748	0.2502	-0.4815
	1.00	0.5782	-0.0795	0.2193	-0.4205
	1.13	0.5474	-0.0793	0.1991	-0.3671
	1.27	0.5205	-0.0754	0.1862	-0.3211
	1.40	0.4974	-0.0685	0.1784	-0.2823
	1.53	0.4776	-0.0587	0.1740	-0.2505
	1.67	0.4606	-0.0460	0.1719	-0.2256
	1.80	0.4458	-0.0298	0.1712	-0.2082
	1.93	0.4324	-0.0093	0.1712	-0.1989
ν=0.300	0.03	0.7834	0.3352	0.8177	-0.9260
	0.10	0.7823	0.3236	0.7901	-1.1267
	0.17	0.7893	0.2949	0.7376	-1.2856
	0.23	0.7944	0.2545	0.6688	-1.3805
	0.30	0.7964	0.2093	0.5935	-1.4008
	0.37	0.7932	0.1652	0.5197	-1.3803
	0.43	0.7843	0.1259	0.4527	-1.3138
	0.50	0.7707	0.0930	0.3947	-1.2226
	0.57	0.7534	0.0665	0.3460	-1.1280
	0.63	0.7338	0.0460	0.3059	-1.0293
	0.73	0.7022	0.0263	0.2590	-0.8848
	0.87	0.6606	0.0113	0.2150	-0.7214
	1.00	0.6216	0.0073	0.1860	-0.3445
	1.13	0.5867	0.0113	0.1668	-0.4797
	1.27	0.5561	0.0215	0.1539	-0.3959
	1.40	0.5294	0.0371	0.1447	-0.3318
	1.53	0.5058	0.0580	0.1375	-0.2853
	1.67	0.4845	0.0848	0.1308	-0.2552
	1.80	0.4644	0.1188	0.1233	-0.2425
	1.93	0.4439	0.1618	0.1134	-0.2500
ν=0.450	0.03	0.7607	0.5985	0.8460	-0.3393
	0.10	0.7741	0.5489	0.8047	-0.6830
	0.17	0.7962	0.4860	0.7385	-0.9609
	0.23	0.8123	0.4116	0.6565	-1.1449
	0.30	0.8239	0.3394	0.5712	-1.2366
	0.37	0.8252	0.2730	0.4900	-1.2494
	0.43	0.8182	0.2177	0.4191	-1.2093
	0.50	0.8043	0.1736	0.3596	-1.1388
	0.57	0.7860	0.1397	0.3112	-1.0533
	0.63	0.7649	0.1145	0.2723	-0.9634
	0.73	0.7311	0.0919	0.2280	-0.8228
	0.87	0.6867	0.0765	0.1878	-0.6679
	1.00	0.6454	0.0752	0.1618	-0.5386
	1.13	0.6068	0.0843	0.1445	-0.4353
	1.27	0.5769	0.1017	0.1320	-0.3559
	1.40	0.5490	0.1266	0.1216	-0.2979
	1.53	0.5241	0.1594	0.1112	-0.2600
	1.67	0.5008	0.2011	0.0988	-0.2427
	1.80	0.4775	0.2540	0.0823	-0.2486
	1.93	0.4521	0.3211	0.0593	-0.2835

TABLE 7.9 COEFFICIENTS K^{α} FOR $\frac{h}{B} = 3$ (Milovic et al, 1970)

	$\frac{z}{B}$	K^{α}_{zN}	K^{α}_{xN}	K^{α}_{xzT}	K^{α}_{xzM}
ν=0.005	0.05	0.8224	0.0308	0.8522	-1.7081
	0.15	0.7927	0.0657	0.7974	-1.7265
	0.25	0.7693	0.0680	0.7029	-1.6960
	0.35	0.7483	0.0491	0.5937	-1.5951
	0.45	0.7251	0.0222	0.4905	-1.4445
	0.55	0.6981	-0.0038	0.4032	-1.2757
	0.65	0.6683	-0.0254	0.3338	-1.1117
	0.75	0.6374	-0.0418	0.2801	-0.9635
	0.85	0.6066	-0.0537	0.2391	-0.8348
	0.95	0.5771	-0.0620	0.2079	-0.7253
	1.10	0.5360	-0.0687	0.1742	-0.5923
	1.30	0.4890	-0.0727	0.1455	-0.4586
	1.50	0.4498	-0.0721	0.1287	-0.3619
	1.70	0.4176	-0.0688	0.1190	-0.2912
	1.90	0.3913	-0.0634	0.1138	-0.2386
	2.10	0.3698	-0.0564	0.1114	-0.1991
	2.30	0.3522	-0.0478	0.1108	-0.1695
	2.50	0.3376	-0.0372	0.1113	-0.1478
	2.70	0.3252	-0.0241	0.1121	-0.1322
	2.90	0.3142	-0.0076	0.1128	-0.1255
ν=0.300	0.05	0.7854	0.3387	0.8491	-1.0438
	0.15	0.7849	0.3186	0.7888	-1.3201
	0.25	0.7885	0.2651	0.6838	-1.4685
	0.35	0.7861	0.1992	0.5650	-1.4738
	0.45	0.7725	0.1377	0.4556	-1.3790
	0.55	0.7485	0.0881	0.3657	-1.2371
	0.65	0.7178	0.0513	0.2963	-1.0855
	0.75	0.6842	0.0253	0.2440	-0.9429
	0.85	0.6501	0.0077	0.2050	-0.8167
	0.95	0.6171	-0.0039	0.1760	-0.7087
	1.10	0.5714	-0.0130	0.1454	-0.5710
	1.30	0.5192	-0.0170	0.1199	-0.4375
	1.50	0.4761	-0.0141	0.1051	-0.3401
	1.70	0.4407	-0.0068	0.0962	-0.2689
	1.90	0.4118	0.0038	0.0907	-0.2164
	2.10	0.3879	0.0174	0.0871	-0.1780
	2.30	0.3678	0.0344	0.0840	-0.1506
	2.50	0.3504	0.0554	0.0806	-0.1329
	2.70	0.3344	0.0815	0.0758	-0.1247
	2.90	0.3185	0.1145	0.0685	-0.1273
ν=0.450	0.05	0.7556	0.5872	0.8636	-0.5312
	0.15	0.7744	0.5135	0.7843	-0.9543
	0.25	0.7994	0.4132	0.6627	-1.2385
	0.35	0.8072	0.3074	0.5323	-1.3257
	0.45	0.7994	0.2195	0.4179	-1.2779
	0.55	0.7752	0.1519	0.3268	-1.1605
	0.65	0.7426	0.1049	0.2591	-1.0232
	0.75	0.7061	0.0729	0.2096	-0.8898
	0.85	0.6691	0.0522	0.1738	-0.7705
	0.95	0.6335	0.0395	0.1478	-0.6686
	1.10	0.5851	0.0297	0.1210	-0.5287
	1.30	0.5303	0.0267	0.1004	-0.4023
	1.50	0.4855	0.0325	0.0881	-0.3086
	1.70	0.4493	0.0439	0.0807	-0.2402
	1.90	0.4200	0.0599	0.0754	-0.1906
	2.10	0.3958	0.0802	0.0705	-0.1557
	2.30	0.3753	0.1053	0.0647	-0.1330
	2.50	0.3570	0.1366	0.0568	-0.1217
	2.70	0.3392	0.1758	0.0453	-0.1227
	2.90	0.3201	0.2256	0.0282	-0.1386

Displacements and Rotation

The vertical and horizontal displacements ρ_z and ρ_x and the rotation ω are given as

$$\rho_z = \frac{P}{E} w_0 \qquad \dots (7.28a)$$

where $w_0 = w_{0N} \cos \delta \qquad \dots (7.28b)$

$$\rho_x = \frac{P}{E} u_0 \qquad \dots (7.29a)$$

where $u_0 = u_{0T} \sin \delta$

$$\tan \omega = \tan \omega_0 \frac{P}{BE} \qquad \dots (7.30a)$$

where $\tan \omega_0 = 2\omega_{cM} \frac{e}{B} \cos \delta \qquad \dots (7.30b)$

Values of u_{0T}, w_{0N} and ω_{cM} are given in Table 7.10.

TABLE 7.10 COEFFICIENTS u_{0T}, w_{0N} AND ω_{cM} (Milovic et al, 1970)

	$\frac{H}{B}$	u_{0T}	w_{0N}	ω_{cM}
ν=0.005	1.0	1.2346	0.7900	2.8185
	2.0	1.6164	1.1959	2.9272
	3.0	1.9778	1.5015	3.1253
ν=0.300	1.0	1.4607	0.6684	2.7698
	2.0	1.8527	1.0685	3.0127
	3.0	2.2334	1.3523	3.1900
ν=0.450	1.0	1.4908	0.4170	2.2435
	2.0	1.8888	0.7618	2.6090
	3.0	2.2246	0.9940	2.7486

7.7 Circle on Finite Layer

7.7.1 SYMMETRICAL VERTICAL LOADING (Fig.7.14)

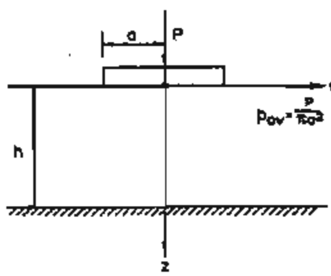


FIG.7.14

Solutions for the contact pressure distribution for various values of h/a, obtained by Poulos (1968a), are shown in Fig.7.15 for $\nu=0.4$. The influence of ν is shown in Fig.7.16.

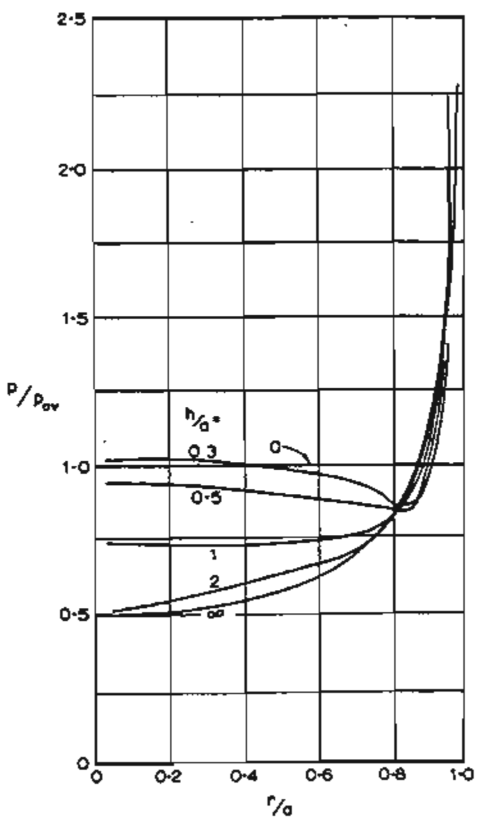


FIG.7.15 Effect of layer depth on contact pressure ($\nu=0.4$).

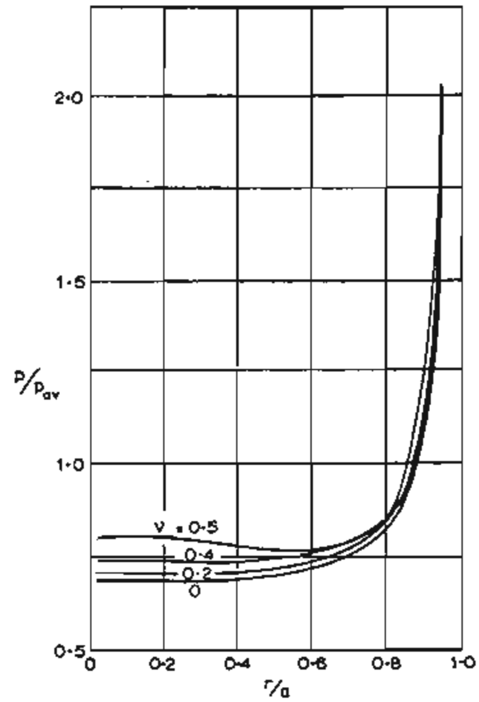


FIG.7.16 Effect of ν on contact pressure ($h/a=1$).

Comparisons between the distribution of stress along the axis of a rigid and a uniformly loaded circle are shown in Fig.7.17 for σ_z and in Fig.7.18 for bulk stress θ.

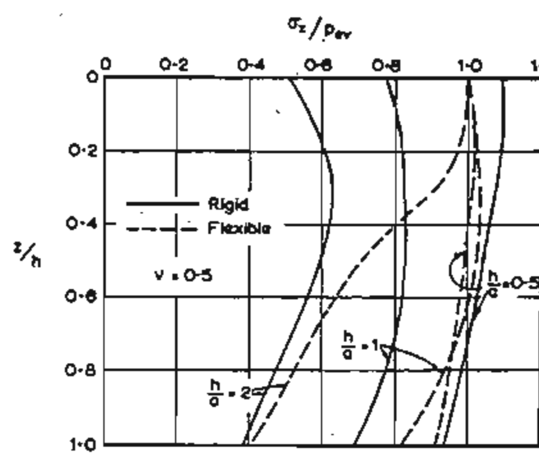


FIG.7.17 Vertical stress along axis.

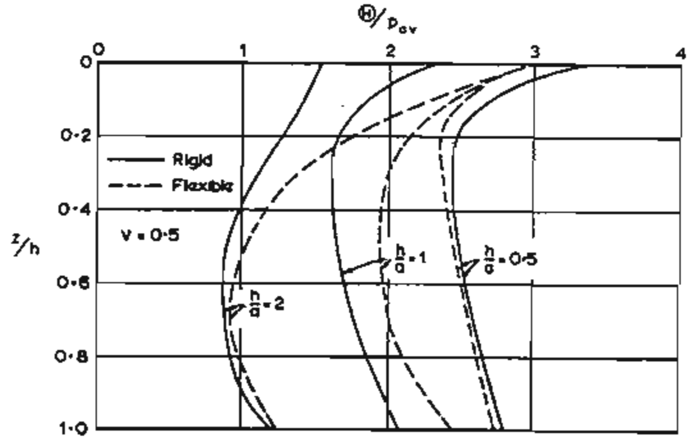


FIG.7.18 Bulk stress along axis.

Influence factors for the vertical displacement obtained by Poulos (1968a), are given in Fig.7.19.

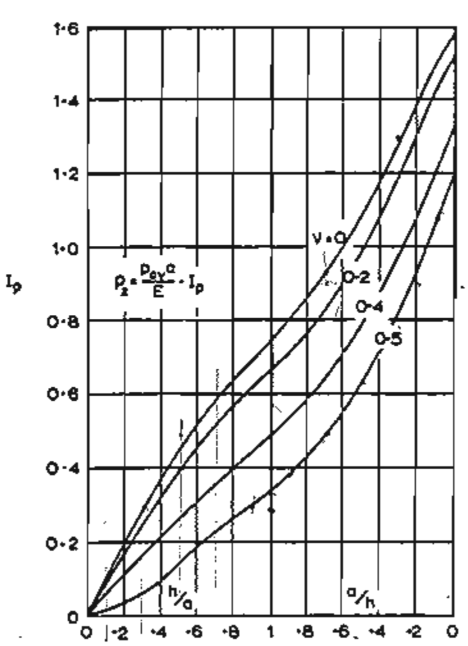


FIG.7.19 Influence factors for vertical displacement of rigid circle.

7.7.2 MOMENT LOADING (Fig.7.20)

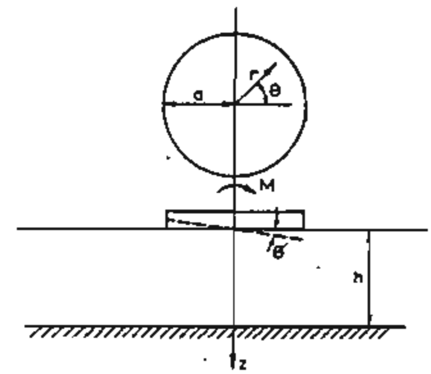


FIG.7.20

Yegorov and Nichiporovich (1961) give the following approximate solutions for rotation ψ and contact pressure distribution p:

$$\psi = \frac{(1-\nu^2)M}{4a^3BE} \qquad \ldots (7.31)$$

$$p = \frac{C_1(\frac{r}{a})+C_3(\frac{r}{a})^3}{B\sqrt{1-(\frac{r}{a})^2}} \cdot \frac{M}{\pi a^3}\cos\theta \qquad \ldots (7.31)$$

where $B = \frac{1}{3}a_1 + \frac{1}{5}a_3$

$C_1 = \frac{1}{2}(a_1 - a_3)$

$C_3 = a_3$

and a_1 and a_3 are given in Table 7.11.

TABLE 7.11

FACTORS FOR RIGID CIRCLE SUBJECTED TO MOMENT (Yegorov and Nichiporovich, 1961)

$\frac{h}{a}$	a_1	a_3
0.25	4.23	-2.33
0.5	2.14	-0.70
1.0	1.25	-0.10
1.5	1.10	-0.03
2.0	1.04	0
3.0	1.01	0
≥ 5.0	1.00	0

7.8 Rectangle on Finite Layer

7.8.1 UNIFORM VERTICAL LOADING

Solutions for the vertical displacement of a rigid rectangle on a finite layer with a smooth frictionless interface at the base, and for $\nu=0.5$, have been obtained by Sovinc (1969) and are shown in Fig.7.21.

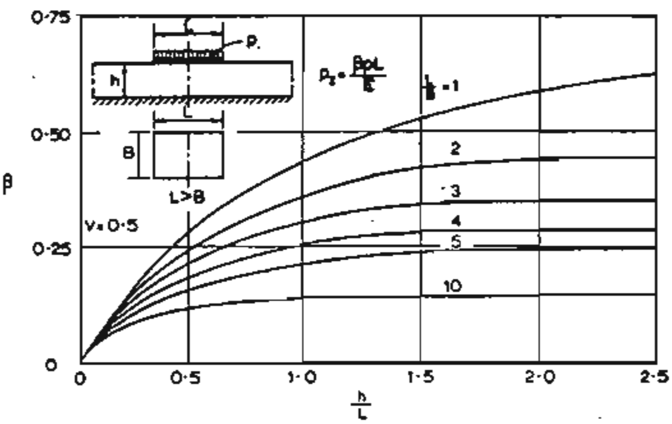


FIG.7.21 Vertical displacement of rigid rectangle (Sovinc, 1969).

7.8.2 MOMENT LOADING

Solutions obtained by Sovinc (1969) for the rotation θ of a rigid rectangle, for $\nu=0.5$, are shown in Fig.7.22.

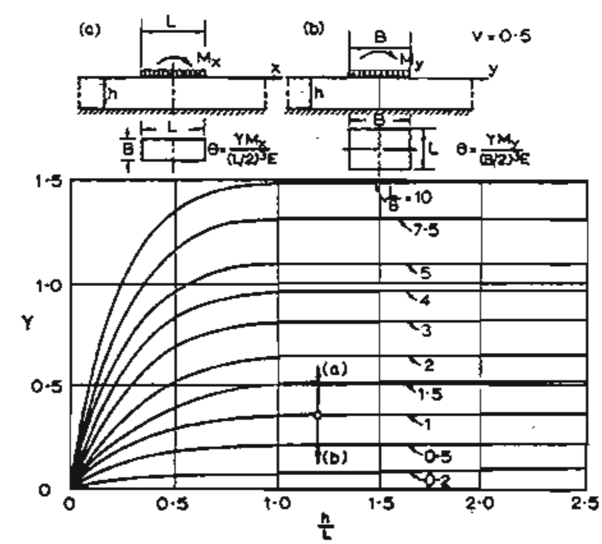


FIG.7.22 Rotation of rigid rectangle (Sovinc, 1969).

7.9 Rigid Areas Embedded Within a Semi-Infinite Mass

7.9.1 VERTICALLY LOADED CIRCLE

For a rigid circle of radius a embedded a distance c below the surface of a semi-infinite mass, Butterfield and Banerjee (1971) have presented solutions for the vertical stress σ_z below the centre and edge and the vertical displacement ρ_z. Solutions for σ_z are shown in Figs.7.23(a) and (b) for $\nu=0.5$. In these figures, z is the distance below the circle. σ_z is not greatly affected by ν, decreasing slightly as ν decreases. Solutions for ρ_z are shown in Fig.7.24.

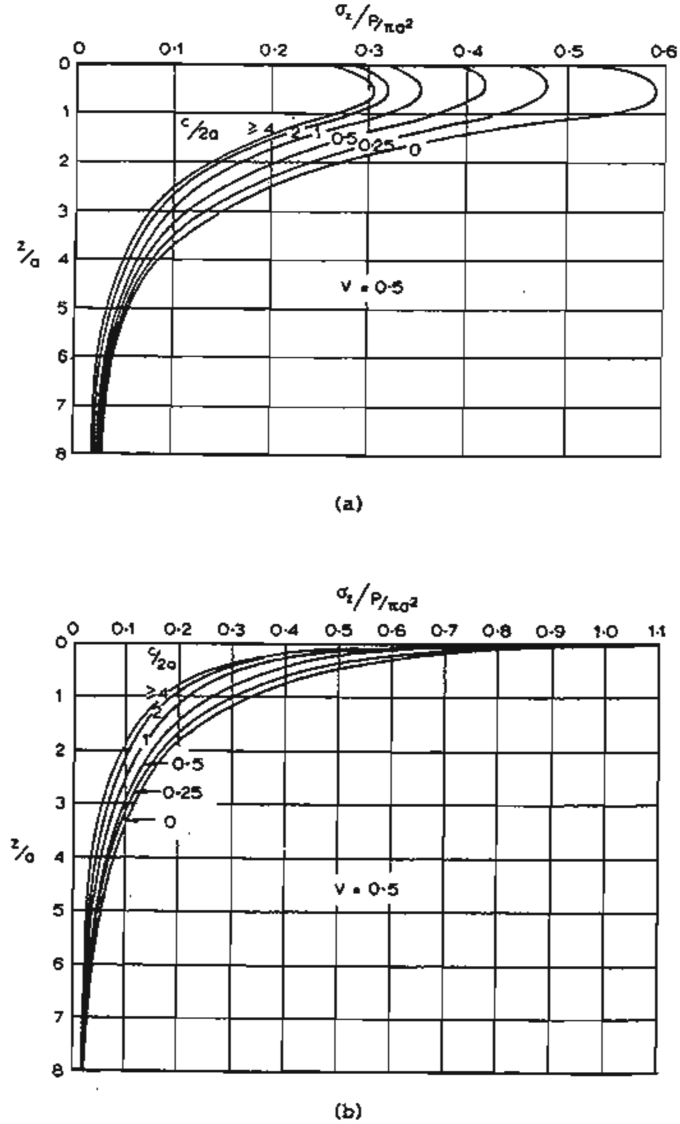


FIG.7.23 Vertical stress beneath embedded rigid circle
(a) beneath centre
(b) beneath edge
(Butterfield and Banerjee, 1971).

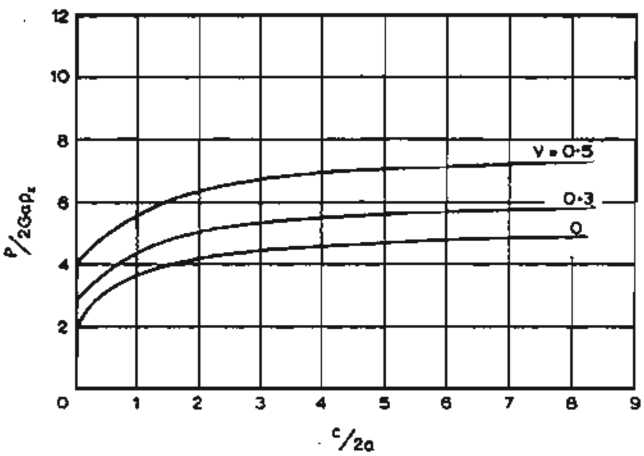
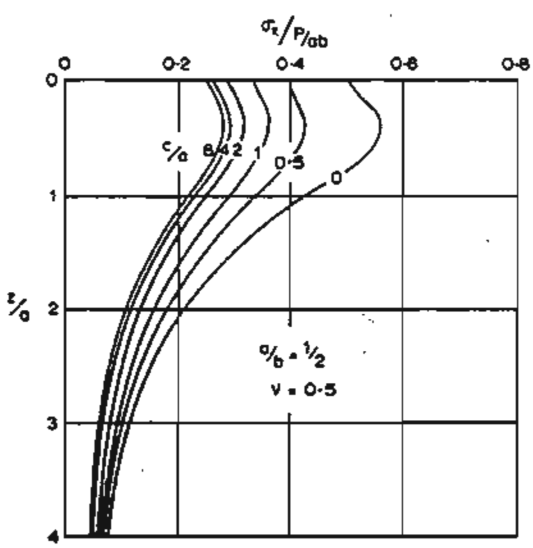


FIG.7.24 Vertical displacement of embedded rigid circle (Butterfield and Banerjee, 1971).

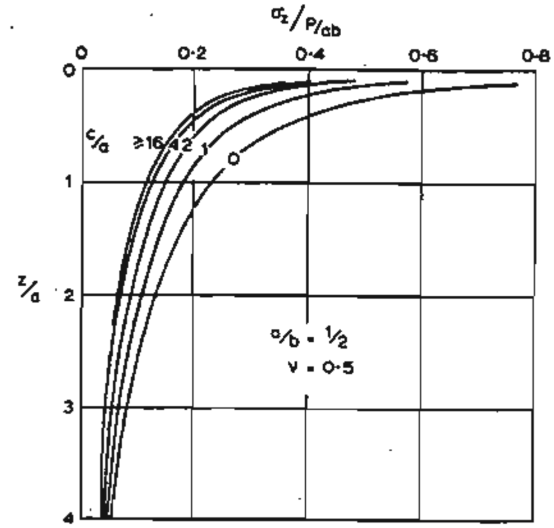
7.9.2 VERTICALLY LOADED RECTANGLE

Butterfield and Banerjee (1971) have given solutions for σ_z and ρ_z beneath the centre and corner of a rigid rectangle, $a \times b$, embedded c below the surface. For values of a/b of $1/2$ and $1/6$, the centre and corner is given in Figs.7.25 (a) & (b) and 7.26 (a) & (b). z is the distance below the rectangle. Butterfield and Banerjee give further solutions for $a/b = 1$, $1/3$ and $1/4$.

Solutions for vertical displacement ρ_z of the rectangle are shown in Fig.7.27. It should be noted that the decrease in ρ_z due to embeddment could also be estimated by applying the results of Fox (1948)(see Fig.4.3) to the surface displacement of a surface rectangle (Fig.7.21).

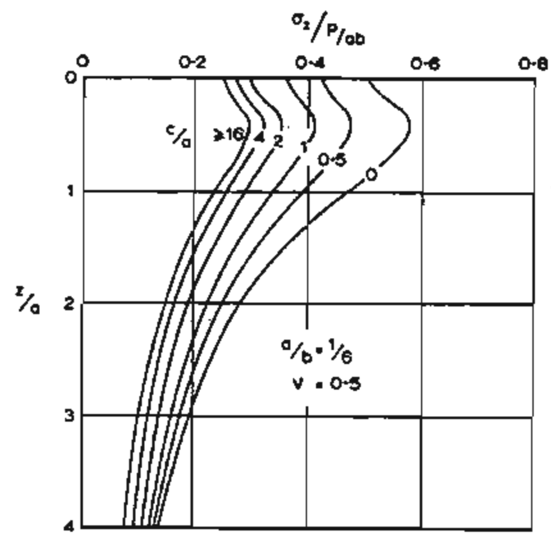


(a) Beneath centre

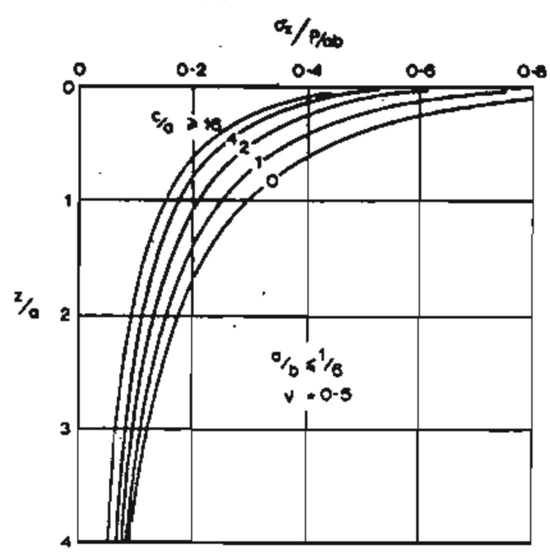


(b) Beneath corner

FIG.7.25 Vertical stress beneath embedded rigid rectangle (Butterfield and Banerjee, 1971).



(a) Beneath centre



(b) Beneath corner

FIG.7.26 Vertical stress beneath embedded rigid rectangle (Butterfield and Banerjee, 1971).

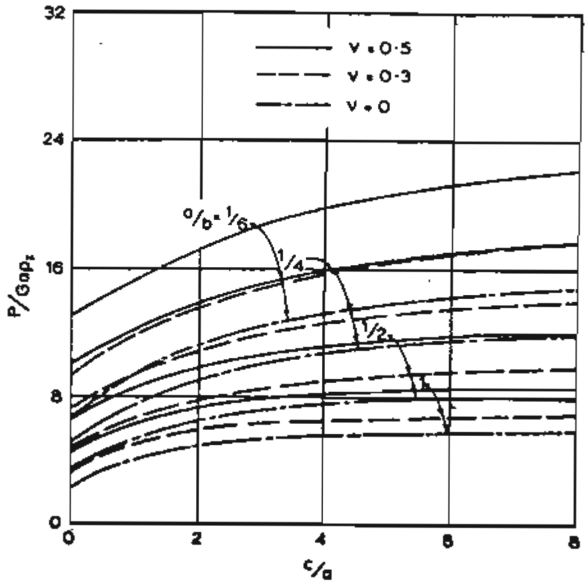


FIG.7.27 Vertical displacement of embedded rigid rectangle (Butterfield and Banerjee, 1971).

7.10 Approximations for Vertical Displacement of Rigid Areas

It is well known (Fox,1948b) that the vertical displacement of a vertically loaded rigid area may be approximated by the mean vertical displacement of a uniformly loaded flexible area of the same shape. The following approximations are quoted by Davis and Taylor (1962) for vertically loaded areas:

Circle and Strip:

$$\rho_{rigid} = \tfrac{1}{2}\left[\rho_{centre} + \rho_{edge}\right]_{flexible} \qquad \ldots (7.33)$$

Rectangle:

$$\rho_{rigid} = \frac{1}{3}\left[2\rho_{centre} + \rho_{corner}\right]_{flexible} \qquad \ldots (7.34)$$

Chapter 8
STRESSES AND DISPLACEMENTS IN CROSS-ANISOTROPIC MEDIA

8.1 Concentrated Loading on a Semi-Infinite Mass

8.1.1 VERTICAL SURFACE POINT LOAD (Fig.2.2)

This problem has been solved by Koning (1957) and de Urena et al (1966). The notation of the latter paper will be used here, the stress-strain relationships being expressed as follows (the notation of Section 1.5 is shown in brackets below):

$$\sigma_x = (\lambda+\mu)\varepsilon_x + \mu\varepsilon_y + \omega\varepsilon_z \qquad \text{... (8.1a)}$$

$$\sigma_y = \mu\varepsilon_x + (\lambda+\mu)\varepsilon_y + \omega\varepsilon_z \qquad \text{... (8.1b)}$$

$$\sigma_z = \omega\varepsilon_x + \omega\varepsilon_y + \rho\varepsilon_z \qquad \text{... (8.1c)}$$

$$\tau_{yz} = \theta\varepsilon_{yz} \qquad \text{... (8.1d)}$$

$$\tau_{xz} = \theta\varepsilon_{xz} \qquad \text{... (8.1e)}$$

$$\tau_{xy} = \lambda\varepsilon_{xy} \qquad \text{... (8.1f)}$$

where $\lambda = \dfrac{E_h}{1+\nu_h}$ (= $a-b$ in Section 1.5)

$$\lambda+\mu = \frac{E_h\left(1-\nu_{hv}^2\frac{E_v}{E_h}\right)}{(1+\nu_h)\left(1-\nu_h-2\nu_{hv}^2\frac{E_v}{E_h}\right)} \qquad (=a)$$

$$\omega = \frac{E_v\nu_{hv}}{1-\nu_h-2\nu_{hv}^2\frac{E_v}{E_h}} \qquad (=c)$$

$$\rho = \frac{E_v(1-\nu_h)}{1-\nu_h-2\nu_{hv}^2\frac{E_v}{E_h}} \qquad (=d)$$

$$\mu = \frac{E_h\left(\nu_h+\nu_{hv}^2\frac{E_v}{E_h}\right)}{(1+\nu_h)\left(1-\nu_h-2\nu_{hv}^2\frac{E_v}{E_h}\right)} \qquad (=b)$$

$$\theta = G_v \qquad (=f)$$

The various elastic parameters are defined in Section 1.5.

The solutions are as follows:-

$$\sigma_z = \frac{\omega a+\rho eB}{(r^2+z^2B)^{3/2}}z + \frac{\omega b+\rho fC}{(r^2+z^2C)^{3/2}}z \qquad \text{... (8.2a)}$$

$$\sigma_r = \frac{(\lambda+\mu)a+\omega eB}{(r^2+z^2B)^{3/2}}z + \frac{(\lambda+\mu)b+\omega fC}{(r^2+z^2C)^{3/2}}z + \frac{\lambda az}{r^2(r^2+z^2B)^{1/2}} + \frac{\lambda bz}{r^2(r^2+z^2C)^{1/2}} \qquad \text{... (8.2b)}$$

$$\sigma_\theta = -\frac{\lambda az}{r^2(r^2+z^2B)^{1/2}} - \frac{\lambda bz}{r^2(r^2+z^2C)^{1/2}} + \frac{\mu a+\omega eB}{(r^2+z^2B)^{3/2}}z + \frac{\mu b+\omega fC}{(r^2+z^2C)^{3/2}}z \qquad \text{... (8.2c)}$$

$$\tau_{rz} = \frac{\frac{\theta}{2}(a-e)r}{(r^2+z^2B)^{3/2}} + \frac{\frac{\theta}{2}(b-f)r}{(r^2+z^2C)^{3/2}} \qquad \text{... (8.2d)}$$

$$\rho_z = \frac{e}{(r^2+z^2B)^{1/2}} + \frac{f}{(r^2+z^2C)^{1/2}} \qquad \text{... (8.2e)}$$

$$\rho_r = \frac{az}{r(r^2+z^2B)^{1/2}} + \frac{bz}{r(r^2+z^2C)^{1/2}} \qquad \text{... (8.2f)}$$

where
$$B = \frac{(\lambda+\mu)\rho-\omega(\omega+\theta)}{\rho\theta} + \sqrt{\frac{[(\lambda+\mu)\rho-\omega(\omega+\theta)]^2}{\rho^2\theta^2} - \frac{\lambda+\mu}{\rho}}$$

$$C = \frac{(\lambda+\mu)\rho-\omega(\omega+\theta)}{\rho\theta} - \sqrt{\frac{[(\lambda+\mu)\rho-\omega(\omega+\theta)]^2}{\rho^2\theta^2} - \frac{\lambda+\mu}{\rho}}$$

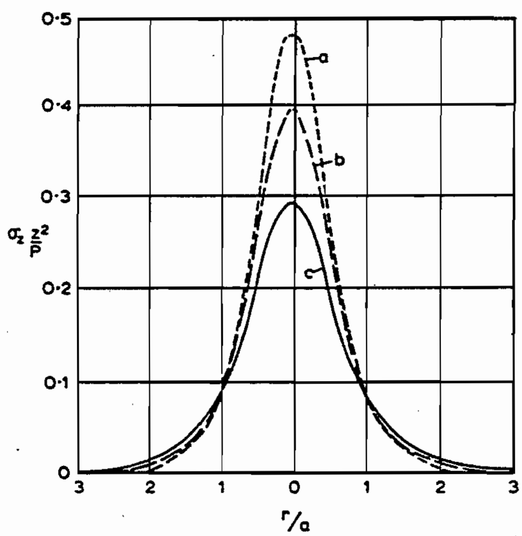
$$a = \frac{P}{4\pi}\frac{(\theta-2\rho B)}{(2\omega+\theta)}\frac{\sqrt{B}(\theta-2\rho C)}{\theta\rho(B-C)}$$

$$b = -\frac{P}{4\pi}\frac{(\theta-2\rho B)}{(2\omega+\theta)}\frac{\sqrt{C}(\theta-2\rho C)}{\theta\rho(B-C)}$$

$$e = \frac{P}{4\pi}\frac{\sqrt{B}(\theta-2\rho C)}{\theta\rho(B-C)}$$

$$f = -\frac{P}{4\pi}\frac{\sqrt{C}(\theta-2\rho B)}{\theta\rho(B-C)}$$

An indication of the influence of anistropy on the stress distribution is given in Fig.8.1 (Koning, 1957) which shows the distribution of σ_z for the isotropic case and two anisotropic cases.



Curve	$\frac{E_h}{E_v}$	ν_{hv}	ν_h	$\frac{\theta}{E_v}$
a (isotropic)	1	½	½	0.67
b	2	¾	⅛	0.89
c	3	¾	⅛	1.28

FIG.8.1 Influence of anisotropy on vertical stress due to point load (Koning, 1957).

Barden (1963) derived solutions to this problem based on earlier solutions by Michell (1900). However, as pointed out by Dooley (1964), Barden made the implicit assumption that the shear modulus for a pair of axes inclined at 45° to the xz-axes is the same as that in the direction of the x- and z-axes. The validity of Barden's solutions is therefore limited.

8.1.2 VERTICAL SURFACE LINE LOAD (Fig.2.7)

de Urena et al (1966) give the following solutions:

$$\sigma_z = \frac{p}{\pi}\frac{\sqrt{BC}}{\sqrt{B}-\sqrt{C}}\, z\left[\frac{1}{x^2+z^2B} - \frac{1}{x^2+z^2C}\right] \quad \ldots (8.3a)$$

$$\sigma_x = \frac{p}{\pi}\frac{\sqrt{BC}}{\sqrt{B}-\sqrt{C}}\, z\left[-\frac{B}{x^2+z^2B} - \frac{C}{x^2+z^2C}\right] \quad \ldots (8.3b)$$

$$\tau_{xz} = \frac{p}{\pi}\frac{\sqrt{BC}}{\sqrt{B}-\sqrt{C}}\, x\left[-\frac{1}{x^2+z^2B} + \frac{1}{x^2+z^2C}\right] \quad \ldots (8.3c)$$

where B, C are defined in Section 8.1.1.

8.1.3 TANGENTIAL SURFACE LINE LOAD (Fig.2.9)

de Urena et al (1966) give the following solutions:

$$\sigma_z = 2Rx\left[-\frac{1}{x^2+z^2C} + \frac{1}{x^2+z^2B}\right] \quad \ldots (8.4a)$$

$$\sigma_x = 2Rx\left[\frac{C}{x^2+z^2C} - \frac{B}{x^2+z^2B}\right] \quad \ldots (8.4b)$$

$$\tau_{xz} = 2Rz\left[-\frac{C}{x^2+z^2C} + \frac{B}{x^2+z^2B}\right] \quad \ldots (8.4c)$$

where $R = \frac{q}{2\pi}\frac{2\rho C-\theta}{(2\omega+\theta)(\sqrt{B}-\sqrt{C})}$

and B and C are defined in Section 8.1.1.

8.2 Strip on Semi-Infinite Mass

Gerrard and Harrison (1970a) have given complete solutions for the stress, displacement and strain distributions within the mass. This paper is reprinted in full as Appendix A. It contains solutions for the following cases:

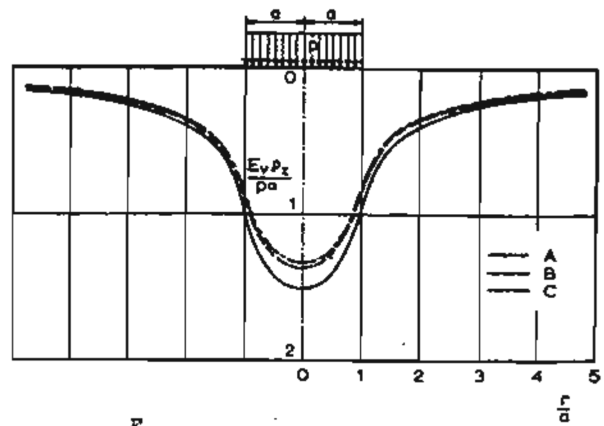
(a) uniform vertical pressure
(b) uniform vertical displacement
(c) linear vertical pressure
(d) linear vertical displacement
(e) uniform lateral shear stress
(f) uniform lateral shear displacement
(g) linear lateral shear stress
(h) linear lateral shear displacement.

8.3 Circle on Semi-Infinite Mass

Gerrard and Harrison (1970b) have given complete solutions for the stress, displacement and strain distributions within the mass. This paper is reprinted in full as Appendix B. It contains solutions for the following cases:

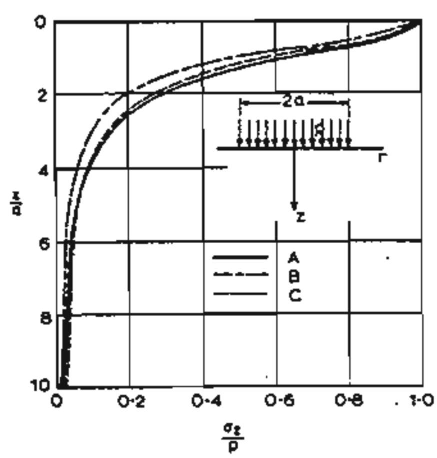
(a) uniform vertical pressure
(b) uniform vertical displacement
(c) linear vertical pressure
(d) linear vertical displacement
(e) linear radial shear stress
(f) linear radial shear displacement
(g) linear torsional shear stress
(h) linear torsional shear displacement
(i) uniform unidirectional shear stress
(j) uniform unidirectional shear displacement.

Some indication of the effects of anisotropy on the distribution of vertical stress and vertical displacement beneath a uniformly loaded circle is given by the results of Koning (1960), reproduced as Figs. 8.2 to 8.4. Similar results for a rigid circle are shown in Figs. 8.5 to 8.7. It should be noted that the distribution of vertical contact stress in the latter case is unaffected by the anisotropy.



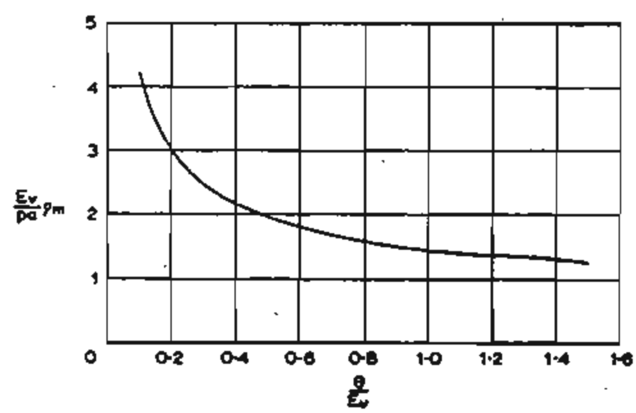
	$\frac{E_h}{E_v}$	ν_{vh}	ν_{hv}	ν_h	$\frac{\theta}{E_v}$	
A	1	1/2	1/2	1/2	0.67	(Boussinesq)
B	2	3/8	3/4	1/8	0.89	
C	4	3/16	3/4	1/8	1.28	

FIG.8.3 Distributions of vertical displacement of surface (Koning, 1960).



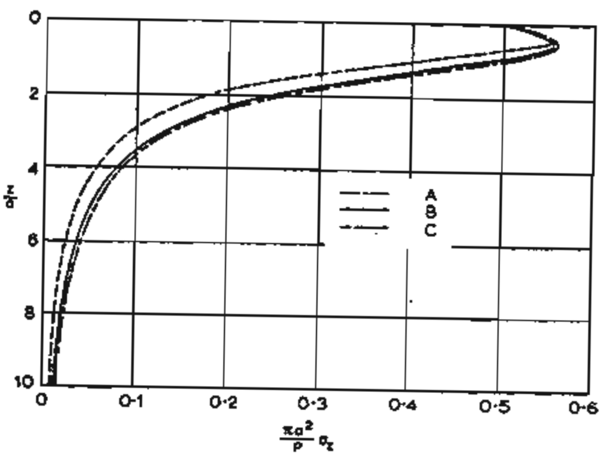
	$\frac{E_h}{E_v}$	ν_{vh}	ν_{hv}	ν_h	$\frac{\theta}{E_v}$	
A	1	1/2	1/2	1/2	0.67	(Boussinesq)
B	2	3/8	3/4	1/8	0.89	
C	4	3/16	3/4	1/8	1.28	

FIG.8.2 Vertical stress on axis of circle (Koning, 1960).



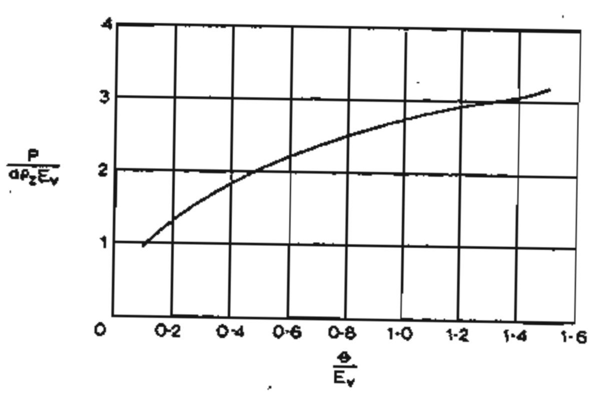
$\frac{E_h}{E_v} = 4, \; \nu_{vh} = 3/16 \quad \nu_{hv} = 3/4, \quad \nu_h = 1/8$

FIG.8.4 Mean displacement of circle (Koning, 1960).



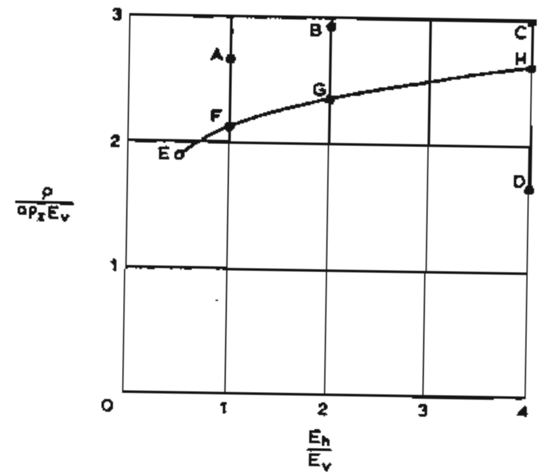
$\frac{E_h}{E_v}$		ν_{vh}	ν_{hv}	ν_h	$\frac{\theta}{E_v}$	
A	1	1/2	1/2	1/2	0.67	(Boussinesq)
B	2	3/8	3/4	1/8	0.89	
C	4	3/16	3/4	1/8	1.28	

FIG.8.5 Vertical stress on axis beneath rigid circle (Koning, 1960).



$\frac{E_h}{E_v} = 4$, $\nu_{vh} = 3/16$, $\nu_{hv} = 3/4$, $\nu_h = 1/8$

FIG.8.6 Vertical displacement of rigid circle (Koning, 1960).



	$\frac{E_h}{E_v}$	ν_{vh}	ν_{hv}	ν_h	$\frac{\theta}{E_v}$	
A	1	1/2	1/2	1/2	0.67	(Boussinesq)
B	2	3/8	3/4	1/8	0.89	
C	4	3/16	3/4	1/8	1.28	
D	4	3/16	3/4	1/8	0.32	
E	1/2	1/6	1/12	3/4	0.60	
F	1	1/4	1/4	1/4	0.80	
G	2	1/6	1/3	1/3	0.90	
H	4	1/12	1/3	1/3	1.20	

FIG.8.7 Vertical displacement of rigid circle (Koning, 1960).

8.4 Loading on Multi-Layer Systems

For general loading of an infinite strip or a circle on the surface of a half space consisting of any number of anisotropic layers, mathematical solutions, (without numerical evaluation), for stresses and displacements, are given by Gerrard and Harrison (1971).

8.4.1 VERTICAL UNIFORM LOADING ON STRIP

This problem has been considered by Gerrard (1967) who obtained a limited number of solutions indicating the effect of anisotropy on the stress distribution beneath the strip. The cases considered by Gerrard are summarised in Table 8.1. All cases involve one or two layers overlying a rough rigid base at a total depth of 6b (see Fig.8.8).

The complete pattern of stresses for Cases 2 and 3 are shown in Figs.8.9 and 8.10, indicating the effect of inserting a softer overlying layer in Case 3.

The distributions of normal stress along the axis are shown in Figs.8.11(a) to (c).

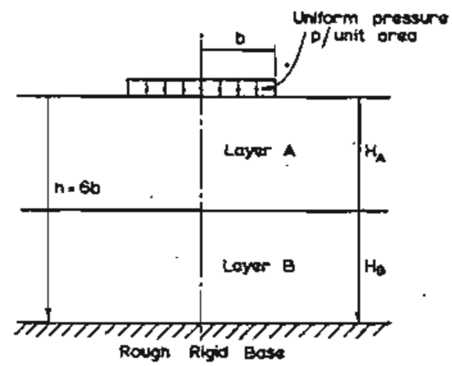


FIG.8.8

TABLE 8.1
CASES CONSIDERED BY GERRARD (1967)

PROBLEM NUMBER	GEOMETRY	LOADING	NO. OF LAYERS	$\frac{H_A}{b}$	$\frac{H_B}{b}$	$\frac{(E_v)_A}{(E_v)_B}$	Layer A					Layer B				
							$\frac{E_h}{E_v}$	$\frac{f}{E_v}$	ν_h	ν_{hv}	ν_{vh}	$\frac{E_h}{E_v}$	$\frac{f}{E_v}$	ν_h	ν_{hv}	ν_{vh}
1	Plane Strain	Uniform Normal Stress (p)	1	6	-	-	1.5	0.9	0.25	0.30	0.20	-	-	-	-	-
2	"	"	1	6	-	-	3.0	1.0	0.1	0.9	0.30	-	-	-	-	-
3	"	"	2	2	4	1/4	1.5	0.9	0.25	0.30	0.20	3.0	1.0	0.1	0.9	0.3
4	"	"	2	2	2	4	3.0	1.0	0.1	0.9	0.3	1.5	0.9	0.25	0.30	0.20
5	"	"	1	6	-	-	1.0	0.7	0.43 (Isotropic)	0.43	0.43	-	-	-	-	-

Each cell: σ_z / σ_x (upper line); τ_{xz} / σ_y (lower line).

z/a \ x/a	0	0.5	1	1.5	2	3	
0	100.0 / 130.8 0.0 / 103.1		50.0 / 56.9 0.0 / 50.7	0.0 / -16.8 0.0 / -1.7	0.0 / -15.9 0.0 / -1.6	0.0 / -13.6 0.0 / -1.4	Surface
.5	93.6 / 37.5 0.0 / 88.0	87.1 / 32.7 16.4 / 81.6	49.5 / 31.6 33.8 / 47.8				
1	76.8 / 7.0 0.0 / 69.8		47.1 / 15.1 26.4 / 43.9	23.6 / 20.2 23.2 / 23.2	10.9 / 18.8 15.2 / 11.7	2.9 / 10.8 5.3 / 3.7	
2	51.2 / -2.3 0.0 / 45.8		39.5 / 3.1 14.2 / 35.9		20.2 / 10.5 14.6 / 19.2	9.1 / 11.9 9.0 / 9.4	
3	38.3 / -0.5 0.0 / 34.4		33.0 / 2.1 7.7 / 29.9		22.3 / 6.8 10.0 / 20.8	13.2 / 9.8 8.2 / 12.8	
4	31.6 / 3.6 0.0 / 28.8		28.7 / 4.8 4.4 / 26.3		22.0 / 7.4 6.6 / 20.5	14.9 / 9.5 6.5 / 14.3	
5	27.7 / 10.5 0.0 / 25.9		25.7 / 10.8 3.0 / 24.2		20.9 / 11.3 5.0 / 19.9	15.1 / 11.3 5.8 / 14.7	
6	24.3 / 24.3 0.0 / 24.3		22.9 / 22.9 3.8 / 22.9		19.2 / 19.2 6.6 / 19.2	14.6 / 14.6 8.0 / 14.6	
	Axis						

σ_z | σ_x
τ_{xz} | σ_y

figures given are percentages of the loading stress, p

Fig. 8.9 Complete Stress Distribution for Case 2 (Gerrard, 1967).

z/a \ x/a	0	0.5	1	1.5	2	3	
0	100.0 / 84.3 0.0 / 51.1		50.0 / 29.2 0.0 / 22.3	0.0 / -26.0 0.0 / -6.5	0.0 / -19.5 0.0 / -4.9	0.0 / -9.3 0.0 / -2.3	Surface
.5	96.1 / 24.7 0.0 / 35.0	89.6 / 19.8 12.7 / 31.8	50.3 / 18.7 28.5 / 19.7				
1	83.0 / 4.2 0.0 / 26.0		49.6 / 11.3 21.1 / 17.7	23.2 / 15.7 16.5 / 10.9	8.9 / 13.6 8.5 / 6.1	0.7 / 5.8 1.1 / 1.7	
2	17.9 61.4 / 13.0 0.0 / 22.7 56.5		13.7 44.6 / 14.3 14.8 / 16.6 41.6		6.8 18.1 / 13.4 13.2 / 7.2 17.7	2.5 5.2 / 8.7 6.2 / 2.2 5.6	Interface
3	45.6 / 1.5 0.0 / 41.2		37.7 / 4.5 9.9 / 34.4		22.4 / 9.5 11.8 / 21.1	10.7 / 11.5 8.2 / 10.8	
4	36.6 / 3.5 0.0 / 33.3		32.4 / 5.3 5.8 / 29.7		23.0 / 8.7 8.1 / 21.5	13.7 / 11.0 7.1 / 13.4	
5	31.5 / 10.8 0.0 / 29.5		28.8 / 11.3 3.7 / 27.0		22.1 / 12.2 6.0 / 21.1	14.6 / 12.3 6.5 / 14.4	
6	27.4 / 27.4 0.0 / 27.4		25.4 / 25.4 4.8 / 25.4		20.4 / 20.4 8.0 / 20.4	14.5 / 14.5 9.1 / 14.5	Lower Boundary
	Axis						

Legend:

In general

σ_z | σ_x
τ_{xy} | σ_y

At an interface

σ_{xA}
σ_z | σ_{xB}
τ_{xz} | σ_{yA}
σ_{yB}

figures given are percentages of the loading stress, p

Fig. 8.10 Complete Stress Distribution for Case 3 (Gerrard, 1967).

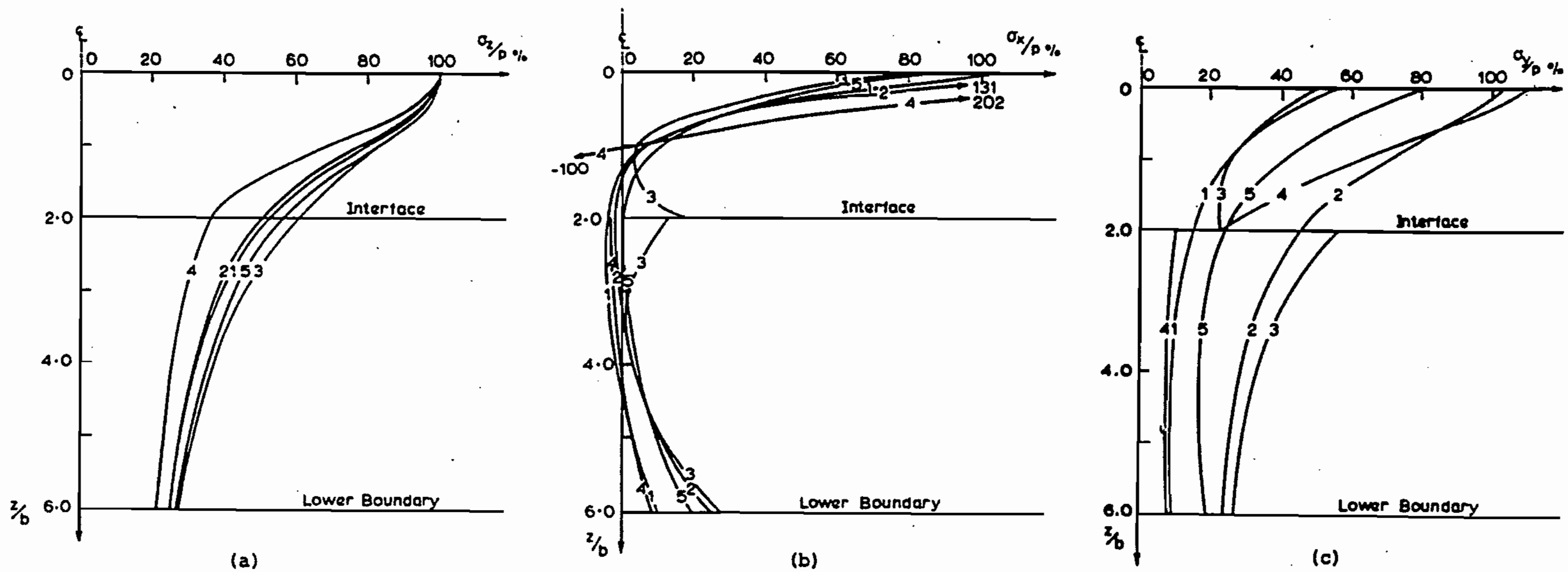


FIG.8.11 Stresses on axis beneath uniformly loaded strip on anisotropic layered system (Gerrard, 1967).

8.4.2 LOADING ON CIRCULAR AREA

Gerrard (1967) has considered a limited number of cases involving uniform vertical stress and linearly varying inward shear stress on the circular area. The cases considered are summarised in Table 8.2. All cases involve a single layer of depth $1.5r_0$ (r_0=circle radius) overlying another layer (generally softer) of infinite depth.

The vertical and radial surface displacements are shown in Fig.8.12. The distributions of normal stress down the axis are shown in Fig.8.13.

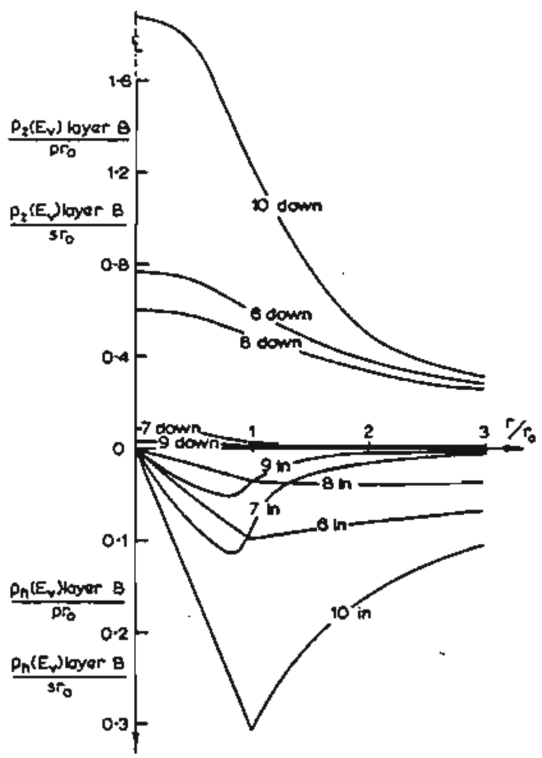


FIG.8.12 Surface deformation profiles for circle on layered anisotropic system (Gerrard, 1967).

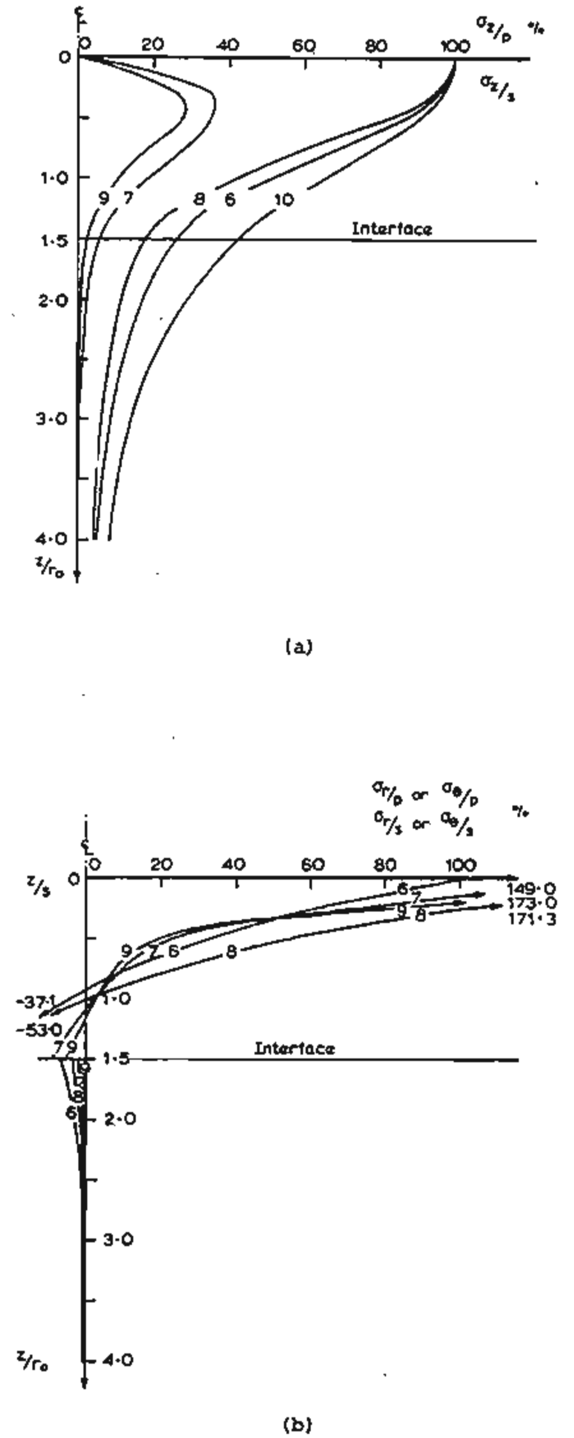


FIG.8.13 Stresses on axis of circle on layered anisotropic system (Gerrard, 1967).

TABLE 8.2
CASES CONSIDERED BY GERRARD (1967)

PROBLEM NUMBER	GEOMETRY	LOADING	NO. OF LAYERS	H_A	H_B	$\frac{(E_v)A}{(E_v)A}$	Layer A					Layer B				
							$\frac{E_h}{E_v}$	$\frac{f}{E_v}$	ν_h	ν_{hv}	ν_{vh}	$\frac{E_h}{E_v}$	$\frac{f}{E_v}$	ν_h	ν_{hv}	ν_{vh}
6	Axisym-metric	Uniform Normal Stress (p)	2	$1.5r_0$	∞	5	1.0	0.8 (Isotropic)	0.25	0.25	0.25	2.0	0.9	0.25	0.35	0.175
7	"	Linear Shear Stress $(s\frac{r}{r_0})$	2	$1.5r_0$	∞	5	1.0	0.8 (isotropic)	0.25	0.25	0.25	"	"	"	"	"
8	"	Uniform Normal Stress (p)	2	$1.5r_0$	∞	5	3.0	1.0	0.1	0.9	0.3	"	"	"	"	"
9	"	Linear Shear Stress $(s\frac{r}{r_0})$	2	$1.5r_0$	∞	5	3.0	1.0	0.1	0.9	0.3	"	"	"	"	"
10.	"	Uniform Normal Stress (p)	2	$1.5r_0$	∞	1	1.0	0.8 (isotropic)	0.25	0.25	0.25	1.0	0.8 (isotropic)	0.25	0.25	0.25

8.5 Particular Cases of Anisotropy

8.5.1 REPEATED LAYER SYSTEMS

It can be shown that a mass of material consisting of an alternating system of individually homogeneous layers is equivalent to one homogeneous mass of cross-anisotropic material, provided the thickness of the group of layers forming the repeated system is small compared with the governing dimensions of the problem being considered. The anisotropic elastic constants of the equivalent material in terms of the elastic constants and thicknesses of the individual layers are given by Salamon (1968). Further treatment is given by Wardle and Gerrard (1972).

For the case of an isotropic two-layer repeated system, with a layer 1 of thickness h_1 and moduli E_1, ν_1 and a layer 2 of thickness h_2 and moduli E_2, ν_2, the equivalent cross-anisotropic properties are as follows:

$$\nu_h = \left[\frac{(1-t)E_1\nu_1}{1-\nu_1^2} + \frac{tE_2\nu_2}{1-\nu_2^2}\right] \Big/ \left[\frac{(1-t)E_1}{1-\nu_1^2} + \frac{tE_2}{1-\nu_2^2}\right] \qquad \ldots (8.5a)$$

$$\nu_{hv} = (1-\nu_h)\left[\frac{(1-t)\nu_1}{1-\nu_1} + \frac{t\nu_2}{1-\nu_2}\right] \qquad \ldots (8.5b)$$

$$E_h = (1-\nu_h^2)\left[\frac{(1-t)E_1}{1-\nu_1^2} + \frac{tE_2}{1-\nu_2^2}\right] \qquad \ldots (8.5c)$$

$$\frac{1}{E_v} = \left[\frac{(1-t)}{E_1}\left(1 - \frac{2\nu_1^2}{1-\nu_1}\right) + \frac{t}{E_2}\left(1 - \frac{2\nu_2^2}{1-\nu_2}\right) + \frac{2\nu_{hv}^2}{(1-\nu_h)E_h}\right] \qquad \ldots (8.5d)$$

$$G_h = \frac{E_h}{2(1+\nu_h)} \qquad \ldots (8.5e)$$

$$\frac{1}{G_v} = \frac{2(1-t)(1+\nu_1)}{E_1} + \frac{2t(1+\nu_2)}{E_2} \qquad \ldots (8.5f)$$

where $t = \frac{h_2}{h_1+h_2}$

8.5.2 REINFORCED MATERIAL

A special case of the repeated layer system is that of only two repeated layers, one of which is very thin but has a high Young's modulus so that it forms a reinforcing sheet. In terms of equations (8.5), if layer 2 is the reinforcing layer, the special case is obtained by putting $t \to 0$ and $E_2 \to \infty$, and letting $tE_2/E_1 = K$, which is in general a finite quantity (Harrison and Gerrard, 1972). The cross-anisotropic material equivalent to the material with the reinforcing layers has only four independent elastic constants instead of the usual five.

8.5.3 THE WESTERGAARD MATERIAL

Westergaard (1938) considered the particular case in which the semi-infinite mass is assumed to be homogeneous but reinforced internally such that no horizontal displacements can occur. Harrison and Gerrard (1972) show that this material is equivalent to a reinforced material having K (Section 8.5.2) equal to infinity. The Westergaard material is equivalent to a cross-anisotropic material with only three independent constants. The following solutions apply for σ_z:

(i) Surface point load P (See Fig.2.2).

$$\sigma_z = \frac{P\eta}{2\pi} \frac{z}{(r^2+\eta^2 z^2)^{3/2}} \qquad \ldots (8.6)$$

(ii) Surface line load p/unit length, and of length y. For a point in plane of end of load,

$$\sigma_z = \frac{p\eta z}{2\pi} \left[\frac{y}{(x^2+\eta^2 z^2)} \frac{1}{(x^2+y^2+\eta^2 z^2)^{\frac{1}{2}}} \right] \qquad \ldots (8.7)$$

where $\eta = \sqrt{\dfrac{1-2\nu}{2(1-\nu)}}$

and $\nu = \nu_1$, Poisson's ratio of the material between the reinforcement.

Influence factors for σ_z due to point loading have been plotted by Harr (1966) while influence factors for σ_z due to line loading of finite length have been plotted by Fadum (1948).

Influence factors for the vertical stress σ_z beneath the corner of a uniformly loaded rectangle (Fadum, 1948) are shown in Fig.8.14 for the case of $\nu = 0$.

It should be noted that since lateral strains and displacements are zero, the stresses on a horizontal plane are functions of vertical displacement only, viz.

$$\sigma_z = \frac{2(1-\nu)G}{(1-2\nu)} \frac{\partial \rho_z}{\partial z} \qquad \ldots (8.8a)$$

$$\tau_{xz} = G\left(\frac{\partial \rho_z}{\partial x}\right) \qquad \ldots (8.8b)$$

$$\tau_{zy} = G\left(\frac{\partial \rho_z}{\partial y}\right) \qquad \ldots (8.8c)$$

where G is the shear modulus.

Thus, vertical displacement ρ_z may be obtained by direct integration of σ_z.

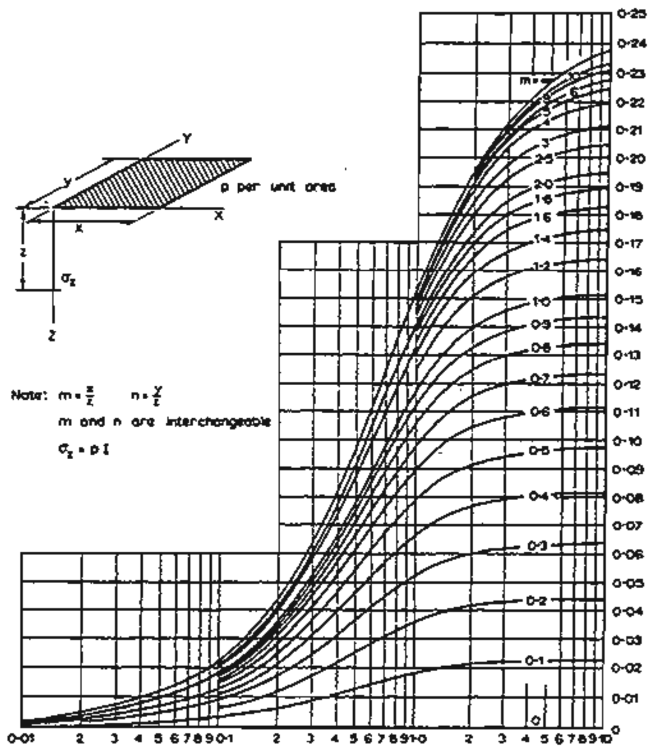


FIG.8.14 Vertical stress beneath corner of rectangle on Westergaard material (Fadum, 1948).

Chapter 9
STRESSES AND DISPLACEMENTS IN A NON-HOMOGENEOUS ELASTIC MASS

9.1 Semi-Infinite Mass with Linear Variation of Modulus

9.1.1 UNIFORM STRIP LOADING

This problem has been solved by Gibson (1967) who considered an elastic half-space where Poisson's ratio remains constant but the shear modulus G increases linearly with depth as follows (Fig. 9.1):

$$G(z) = G(0) + mz \qquad \ldots (9.1)$$

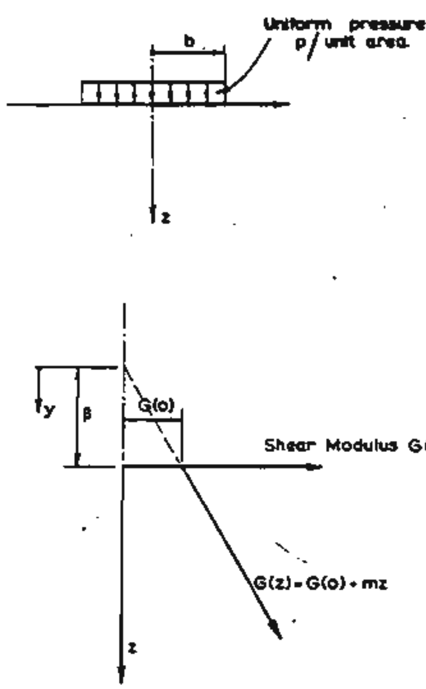


FIG.9.1

Two limiting cases may be recovered from the general expressions for $\nu=0.5$:

(i) $m=0$ $(\beta=\infty)$ i.e. constant $G=G(0)$ with depth. This is the classical homogeneous case.

(ii) $G(0)=0$ $(\beta=0)$. This is the case of a linear increase in G with depth, starting from zero at the surface.

Considering the vertical displacement ρ_z for each case, it is found that for case (i), the actual displacement is infinite (see Section 3.1), but the difference between the displacement of a point and the central surface displacement is finite, and equal to

$$\rho_z(0,0)-\rho_z(x,z) = \frac{pb}{2\pi G(0)}\left\{\tfrac{1}{2}\left(1+\frac{x}{b}\right)\ln\left[\frac{z^2+(b+x)^2}{b^2}\right] + \tfrac{1}{2}\left(1-\frac{x}{b}\right)\ln\left[\frac{z^2+(b-x)^2}{b^2}\right] + \frac{z}{b}\tan^{-1}\left(\frac{b+x}{z}\right) + \frac{z}{b}\tan^{-1}\left(\frac{b-x}{z}\right)\right\} \qquad \ldots (9.2)$$

Values of $[\rho_z(0,0)-\rho_z(x,z)]$ are plotted in Fig.9.2(a).

For case (ii), the vertical displacement is now finite, and given by

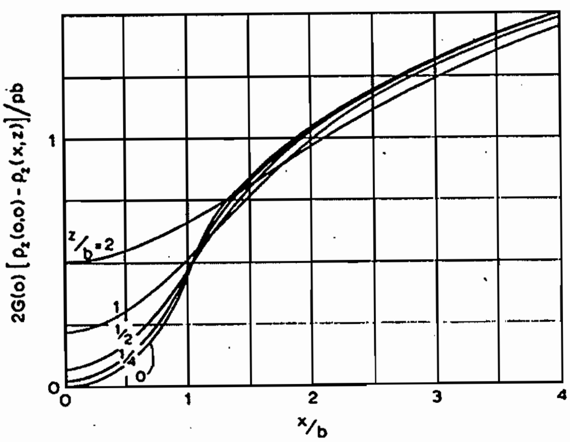
$$\rho_z = \frac{p}{2\pi m}\left[\tan^{-1}\left(\frac{b+x}{z}\right) + \tan^{-1}\left(\frac{b-x}{z}\right)\right] \qquad \ldots (9.3)$$

Values of ρ_z for this case are plotted in Fig. 9.2(b).

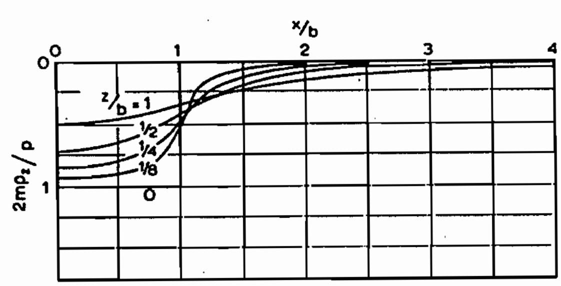
It is noteworthy that the stresses are *identical* for both cases (i) and (ii) i.e. non-homogeneity has no influence on the stress distribution. This result suggests that stresses for finite values of β may not differ appreciably from the values for the limiting cases, provided $\nu=0.5$.

An important conclusion reached from a study of case (ii) is that a material whose modulus varies linearly with depth from zero at the surface behaves as a "Winkler" material i.e. the vertical displacement at any point on the surface is directly proportional to the intensity of vertical stress at that point. This may clearly be seen from the curve for $z/b=0$ in Fig.9.2(b). The coefficient of subgrade reaction k_s in the Winkler material is related to m as $k_s=2m$.

It should be noted that the above conclusion is exact only for $\nu=0.5$.



(a) Relative displacements of strip on uniform mass. ν-0.5. G=G(0).



(b) Displacement of strip on mass with linearly increasing G. ν=0.5. G(0)=0.

FIG.9.2 (Gibson, 1967)

9.1.2 UNIFORM LOADING OVER CIRCULAR AREA

Profiles of vertical surface displacement in terms of the value at the centre have been obtained by Brown and Gibson (1972) for three values of ν and are shown in Figs. 9.3 to 9.5. In these figures, r is the radial distance from the centre, a is the radius and β is as defined in Fig. 9.1.

The variation of central surface vertical displacement $\rho_z(r=0)$ with β and ν is shown in Fig. 9.6.

The conclusion regarding the identical behaviour of an incompressible $(\nu=0.5)$ mass whose modulus varies linearly with depth from zero at the surface, and a Winkler medium, remains valid for this type of loading and indeed, for any type of surface loading.

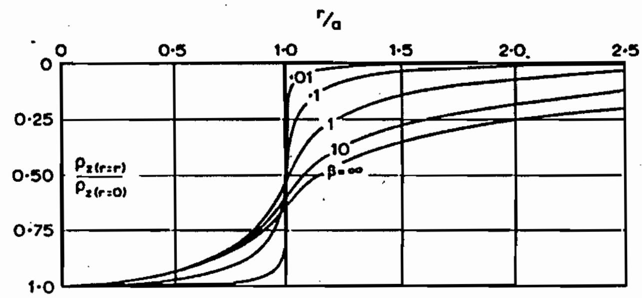


FIG.9.3 Surface displacement profiles for ν=0.5 (Brown and Gibson, 1972).

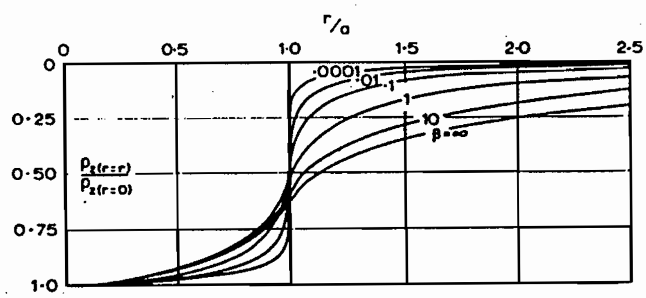


FIG.9.4 Surface displacement profiles for ν=1/3 (Brown and Gibson, 1972).

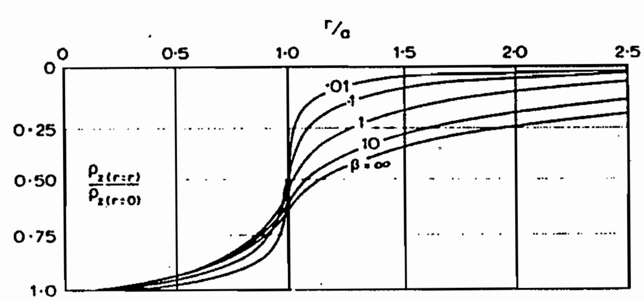


FIG.9.5 Surface displacement profiles for ν=0 (Brown and Gibson, 1972).

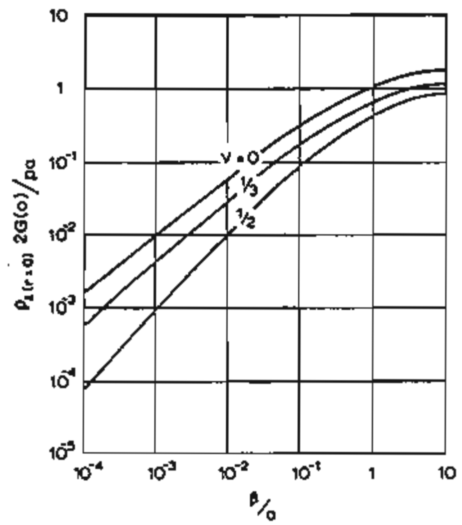


FIG.9.6 Central surface displacement of circular area. (Brown and Gibson, 1972).

p = applied pressure
a = radius

9.2 Generalized Boussinesq Theory for Non-Homogeneous Semi-Infinite Mass

9.2.1 VERTICAL POINT LOADING ON SURFACE

Holl (1940) developed a general form of Boussinesq's classical equations, based on earlier solutions of Griffith (1929) and Frohlich (1933). The solutions for a vertical point loading (Fig. 2.2) are:

$$\sigma_z = \frac{nPz^n}{2\pi R^{n+2}} \qquad \dots (9.4a)$$

$$\sigma_r = \frac{nPz^{n-2}r^2}{2\pi R^{n+2}} \qquad \dots (9.4b)$$

$$\sigma_\theta = 0. \qquad \dots (9.4c)$$

$$\tau_{rz} = \frac{nPz^{n-1}r}{2\pi R^{n+2}} \qquad \dots (9.4d)$$

The above solutions are valid for $n>2$ and satisfy equilibrium and compatibility requirements for the following restricted class of material:

$$E = E_o z^\lambda \qquad \dots (9.5)$$

where E_o = modulus at unit depth

$$\lambda = n-3 = \frac{1}{\nu} - 2.$$

When $n=3$, $\lambda=0$ and $\nu=0.5$, the above solutions reduce to the classical Boussinesq solutions for $\nu=0.5$. When $n=4$, $\lambda=1$ and $\nu=1/3$, the modulus E varies linearly with depth. Thus, this case corresponds to that considered by Gibson (1967) (Section 9.1) except that Gibson considered $\nu=0.5$.

Provided the above restrictions on ν and modulus variation are observed the generalized Boussinesq solution may be used to study the stress distribution in a non-homogeneous mass for all types of surface loading.

9.2.2 HORIZONTAL POINT LOADING ON SURFACE

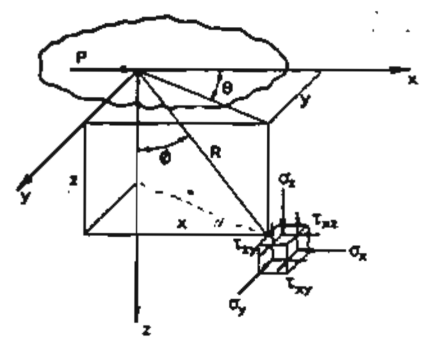


FIG.9.7

Holl (1940) gives the following solutions (refer Fig. 9.7):

$$\sigma_z = \frac{n(n-2)P}{2\pi z^2} \cos^{n+1}\phi \sin\phi \cos\theta \qquad \dots (9.6a)$$

$$\sigma_x = \frac{n(n-2)P}{2\pi z^2} \cos^{n-1}\phi \sin^3\phi \cos^3\theta \qquad \dots (9.6b)$$

$$\sigma_y = \frac{n(n-2)P}{2\pi z^2} \cos^{n-1}\phi \sin^3\phi \cos\theta \sin^2\theta \quad . (9.6c)$$

$$\tau_{yz} = \frac{n(n-2)P}{2\pi z^2} \cos^n\phi \sin^2\phi \sin\theta \cos\theta \qquad \dots (9.6d)$$

$$\tau_{xy} = \frac{n(n-2)P}{2\pi z^2} \cos^n\phi \sin^2\phi \cos^2\theta \qquad \dots (9.6e)$$

$$\tau_{xz} = \frac{n(n-2)P}{2\pi z^2} \cos^{n-1}\phi \sin^3\phi \sin\theta \cos^2\theta \quad . (9.6f)$$

Note that the same restrictions on the relationship between modulus variation and ν apply in this case as in the case of a vertical point load (section 9.2.1).

9.2.3 LINE LOADING ON SURFACE (Fig. 9.8)

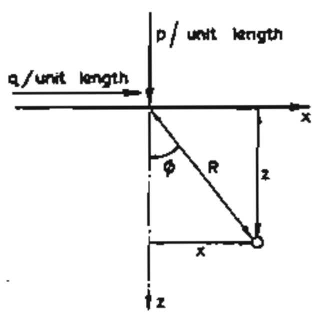


FIG.9.8

(a) Vertical Loading

$$\sigma_z = \frac{p}{z} K \cos^{n+1}\phi \quad \ldots (9.7a)$$

$$\sigma_x = \frac{p}{z} K \cos^{n-1}\phi \sin^2\phi \quad \ldots (9.7b)$$

$$\tau_{xz} = \frac{p}{z} K \cos^{n}\phi \sin\phi \quad \ldots (9.7c)$$

Values of K are tabulated in Table 9.1.

(b) Horizontal Loading

$$\sigma_z = (n-2)K \frac{q}{z} \cos^{n}\phi \sin\phi \quad \ldots (9.8a)$$

$$\sigma_x = (n-2)K \frac{q}{z} \cos^{n-2}\phi \sin^3\phi \quad \ldots (9.8b)$$

$$\tau_{xz} = (n-2)K \frac{q}{z} \cos^{n-1}\phi \sin^2\phi \quad \ldots (9.8c)$$

Values of K are tabulated in Table 9.1 (same as for vertical loading).

TABLE 9.1

K versus n

(Holl, 1940)

n	3	4	5	6	7	8
K	$\frac{2}{\pi}$	$\frac{3}{4}$	$\frac{8}{3\pi}$	$\frac{15}{16}$	$\frac{16}{5\pi}$	$\frac{35}{32}$

9.2.4 UNIFORM VERTICAL LOADING OVER CIRCULAR AREA

Beneath the centre of a circle of radius a, loaded with a uniform surface stress p per unit area,

$$\sigma_z = p[1 - \cos^{n}\phi_0] \quad \ldots (9.9a)$$

$$\sigma_r = \frac{p}{2}\left[\frac{n}{n-2}(1 - \cos^{n-2}\phi_0) - (1 - \cos^{n}\phi_0)\right] \quad \ldots (9.9b)$$

where $\phi_0 = \tan^{-1}\frac{a}{z}$

Values of σ_z are tabulated in Table 9.2 for $n=4$ and 5. Holl (1940) also gives expressions for σ_z and σ_r due to parabolic vertical loading.

TABLE 9.2

GENERALISED BOUSSINESQ PROBLEM

VALUES OF σ_z/p BENEATH CENTRE

Uniform Vertical Loading

(After Harr, 1966)

n	$\frac{z}{a}$	Circle Radius a	Rectangle $m=1$	$m=2$	$m=3$	$m=10$	Infinite Strip $m=\infty$
	0	1	1	1	1	1	1
	0.25	0.996	0.942	0.950	0.957	0.970	0.984
4	0.5	0.960	0.792	0.824	0.852	0.876	0.884
	1	0.750	0.426	0.547	0.581	0.603	0.625
	1.5	0.518	0.255	0.372	0.404	0.437	0.457
	2	0.360	0.142	0.230	0.281	0.316	0.357
	3	0.109	0.074	0.135	0.175	0.214	0.245
	5	0.077	0.025	0.048	0.063	0.125	0.150
	0	1	1	1	1	1	1
	0.25	0.998	0.984	0.987	0.988	0.991	0.993
	0.5	0.982	0.859	0.830	0.902	0.922	0.925
5	1	0.817	0.508	0.625	0.657	0.676	0.686
	1.5	0.591	0.298	0.404	0.443	0.480	0.510
	2	0.429	0.164	0.249	0.294	0.333	0.411
	3	0.229	0.089	0.150	0.192	0.243	0.282
	5	0.096	0.032	0.062	0.088	0.141	0.162

9.2.5 UNIFORM VERTICAL LOADING OVER RECTANGULAR AREA (Fig. 9.9)

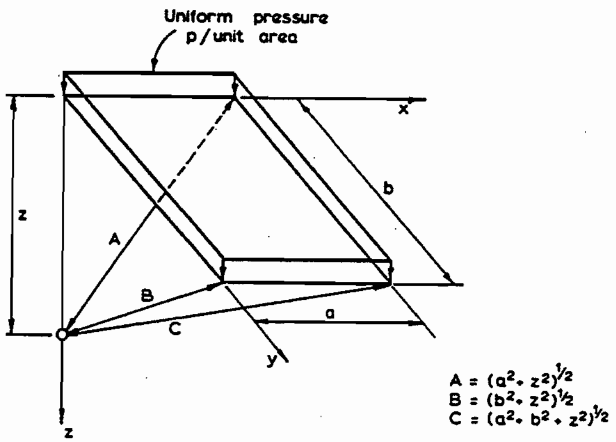


FIG.9.9

Beneath the corner of the rectangle, for $n=4$,

$$\sigma_z = \frac{p}{2\pi}\left[\frac{a}{A}\left(1+\frac{z^2}{2A^2}\right)\arctan\frac{b}{A} + \frac{b}{B}\left(1+\frac{z^2}{2B^2}\right)\arctan\frac{a}{B} + \frac{abz^2}{2C^2}\left(\frac{1}{A^2}+\frac{1}{B^2}\right)\right] \quad \text{... (9.10a)}$$

$$\sigma_x = \frac{p}{4\pi}\left[\frac{a^3}{A^3}\arctan\frac{b}{A} + \frac{b}{B}\arctan\frac{a}{B} - \frac{abz^2}{A^2C^2}\right] \quad \text{... (9.10b)}$$

$$\tau_{xz} = \frac{p}{4\pi}\left[\arctan\frac{b}{z} - \frac{z^3}{A^3}\arctan\frac{b}{A} + bz\left(\frac{1}{B^2}-\frac{z^2}{A^2C^2}\right)\right] \quad \text{... (9.10c)}$$

$$\tau_{xy} = \frac{p}{4\pi}\left[\frac{(a^2+b^2)^2}{z^2C^2} - \frac{1}{z^2}\left(\frac{a^4}{A^2}+\frac{b^4}{B^2}\right)\right] \quad \text{... (9.10d)}$$

Holl (1940) also quotes expressions for n values of 3 (homogeneous mass, see Chapter 3), $5,6,7$ and 8.

Values of σ_z for $n=4$ and 5, given by Harr (1966), are reproduced in Table 9.2 for various values of $m=b/a$.

9.2.6 UNIFORM HORIZONTAL LOADING OVER RECTANGULAR AREA (Fig. 9.10)

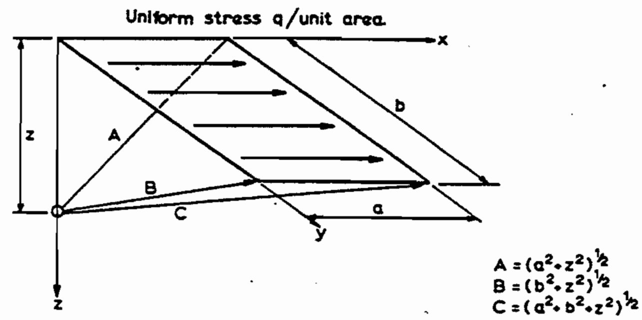


FIG.9.10

For $n=4$,

$$\sigma_x = \frac{q}{\pi}\left[\arctan\frac{b}{z} - \frac{z}{A}\left(1+\frac{a^2}{2A^2}\right)\arctan\frac{b}{A} - \frac{a^2bz}{2A^2C^2}\right] \quad \text{... (9.11a)}$$

$$\sigma_y = \frac{q}{2\pi}\left[\arctan\frac{b}{z} - \frac{z}{A}\arctan\frac{b}{A} - bz\left(\frac{1}{B^2}-\frac{1}{C^2}\right)\right] \quad \text{... (9.11b)}$$

$$\tau_{xy} = \frac{q}{2\pi}\left[\arctan\frac{a}{z} - \frac{z}{B}\arctan\frac{a}{B} + \frac{a}{z}\left(\frac{a^2}{A^2}-\frac{a^2+b^2}{C^2}\right)\right] \quad \text{... (9.11c)}$$

Holl (1940) also quotes solutions for $n=3$ (homogenous case, see Chapter 3) and $n=5$.

The values of τ_{xz}, τ_{yz} and σ_z for a horizontal loading correspond to the values of σ_x, τ_{xy} and τ_{xz} for a vertical loading, multiplied by the factor $(n-2)$.

9.3 Finite Layer with Linear Variation of Modulus (Fig. 9.11)

This problem has been considered by Gibson, Brown and Andrews (1971). Profiles of vertical surface displacement due to uniform strip loading are shown in Fig. 9.12 and due to uniform circular loading in Fig. 9.13. In both figures, $G(0)=0$ and $\nu=0.5$. In this case the vertical displacement of the loaded area is strictly uniform only when h/b or $h/a=\infty$; as the layer thickness decreases, the non-uniformity of settlement, inside and outside the loaded area, increases.

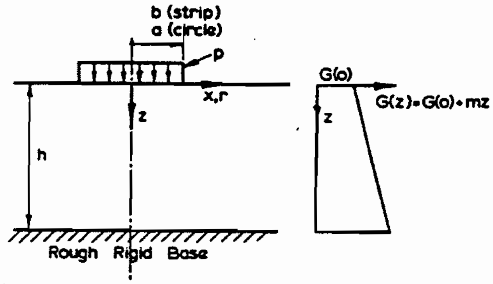


FIG.9.11

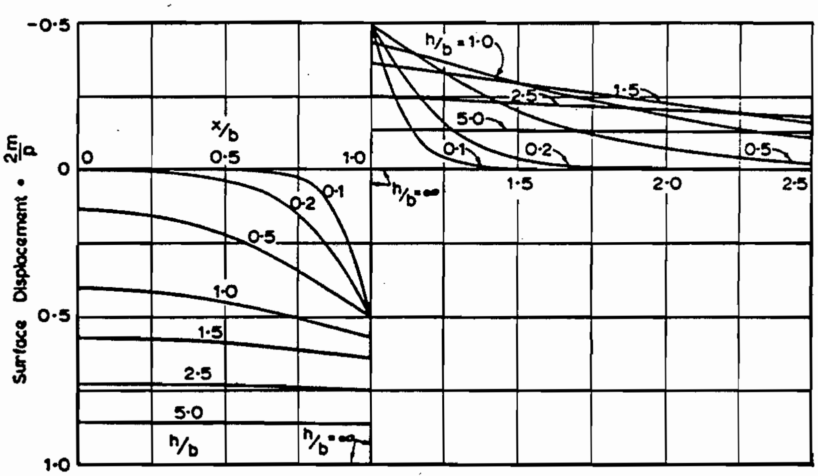


FIG.9.12 Surface displacement profiles due to uniform strip loading (Gibson et at, 1971).

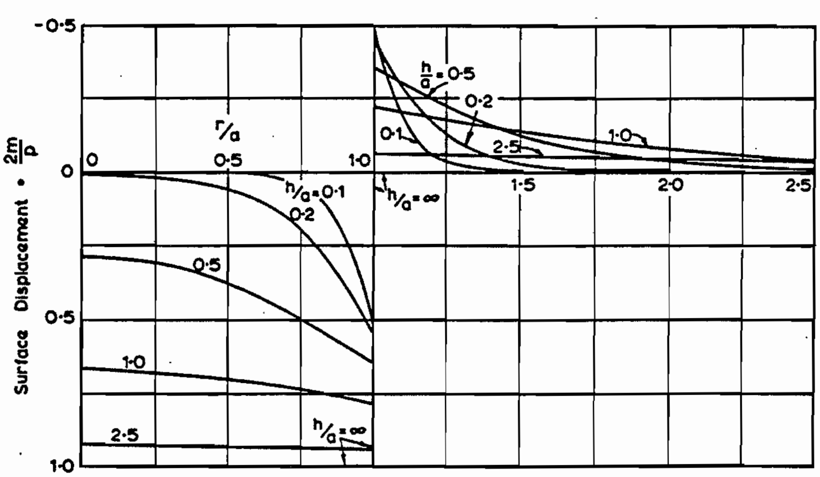


FIG.9.13 Surface displacement profiles due to uniform circular loading (Gibson et al, 1971).

Chapter 10
STRESSES AND DISPLACEMENTS IN EMBANKMENTS AND SLOPES

10.1 Embankment on Rough Rigid Base (Fig. 10.1)

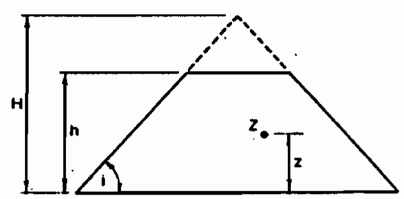


FIG.10.1

This problem has been studied by Clough and Woodward (1967) and Poulos, Booker and Ring (1972). It is important to distinguish between the classical definition of displacement, which is referred to a fixed datum, and assumes the embankment to be created instantaneously, and the displacement of a point in the embankment which would be observed during incremental construction of the embankment. These displacements are termed the "single-lift" and "observed" displacements respectively. Referring to Fig.10.1, for a point Z, height z above the base of the embankment of height h, the "single-lift" displacement is denoted as $\rho(z,h)$ and the "observed" displacement as $v(z,h)$. These displacements are related as follows:

$$v(z,h) = \rho(z,h) - \rho(z,z) \qquad \ldots (10.1)$$

where $\rho(z,z)$ is the single-lift displacement of the point Z when the top of the embankment is at the level of Z, i.e. z above the base.

Thus the observed displacements may be calculated from the "single-lift" displacements for embankment heights of h and z and therefore in this section, all solutions for displacements refer to the single-lift values. The stresses are unaffected by incremental construction.

Stress and displacement contours for a 30° embankment having $\nu=0.3$ are shown in Figs.10.2 to 10.6. Five embankment heights are considered.

Influence factors for the vertical displacement on the axis of the embankment are given in Figs.10.7 to 10.12, for three embankment slopes (20°,30° and 40°) and two Poisson's ratios (0.3 and 0.48). The relationship between displacement influence factor I and relative height of the point above the base, z/H, are plotted for five values of relative embankment height h/H, where H is the maximum possible height of the embankment. The actual displacement is

$$\rho(\tfrac{z}{H}, \tfrac{h}{H}) = I(\tfrac{z}{H}, \tfrac{h}{H})\gamma \frac{H^2}{E} \qquad \ldots (10.2)$$

where γ = unit weight of embankment material.

The observed settlement $v(z/H,h/H)$ may be calculated from equation (10.1).

For a particular embankment (Fig.10.13), Clough and Woodward (1967) give contours of stress, displacement and strain factors. These are shown in Figs. 10.14 to 10.16. The effect of incremental construction on displacements is shown in Fig.10.15; with incremental construction, the "observed" displacements are zero at the top of the embankment, whereas the "single-lift" value of vertical displacement is a maximum at the top.

Clough and Woodward also investigated the effects of departures in ν and side slope from those of their "standard case". The results of this investigation are presented as multipliers for the stresses, strains and displacements for the standard case, and are shown in Fig.10.17.

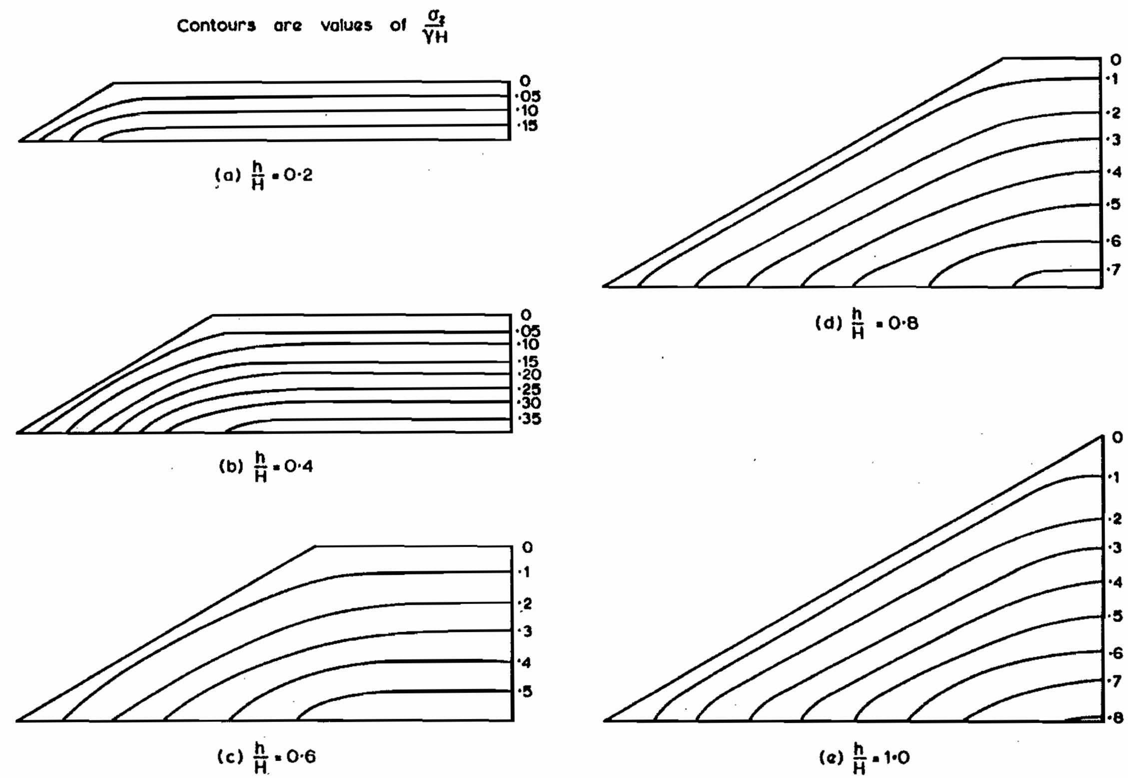


FIG.10.2 Contours of vertical stress for various embankment heights. 30^{o} slope, ν=0.3 (Poulos et al, 1972).

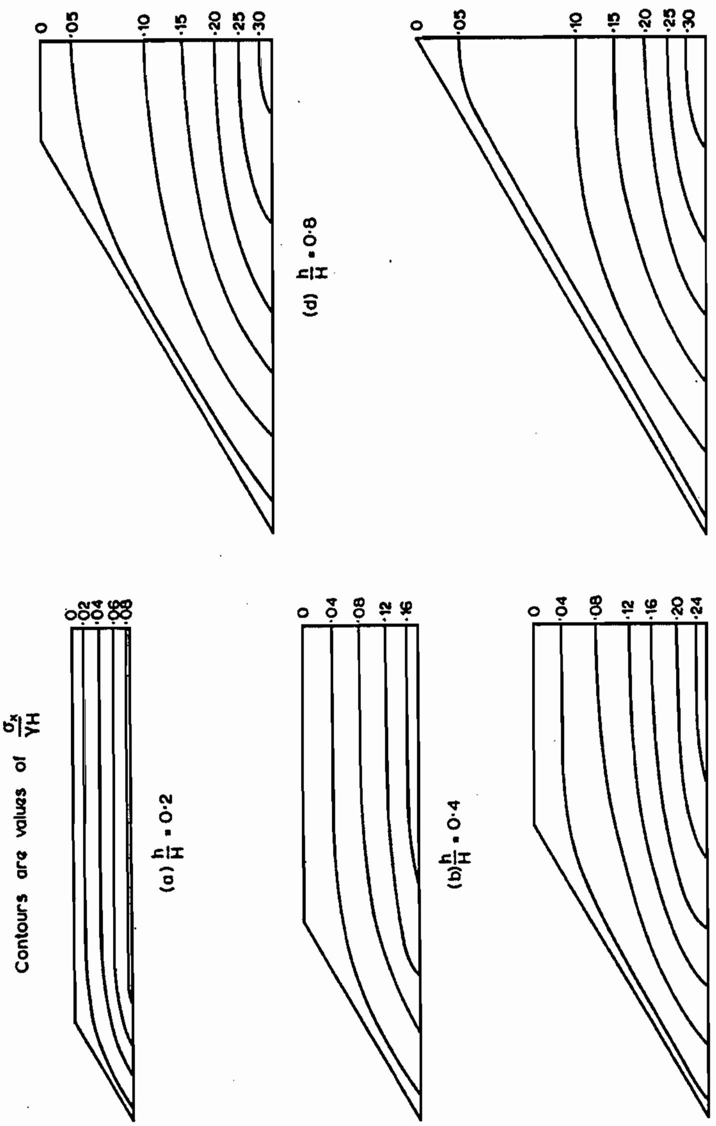


FIG.10.3 Contours of horizontal stress for various embankment heights. 30° slope, ν=0.3 (Poulos et al, 1972).

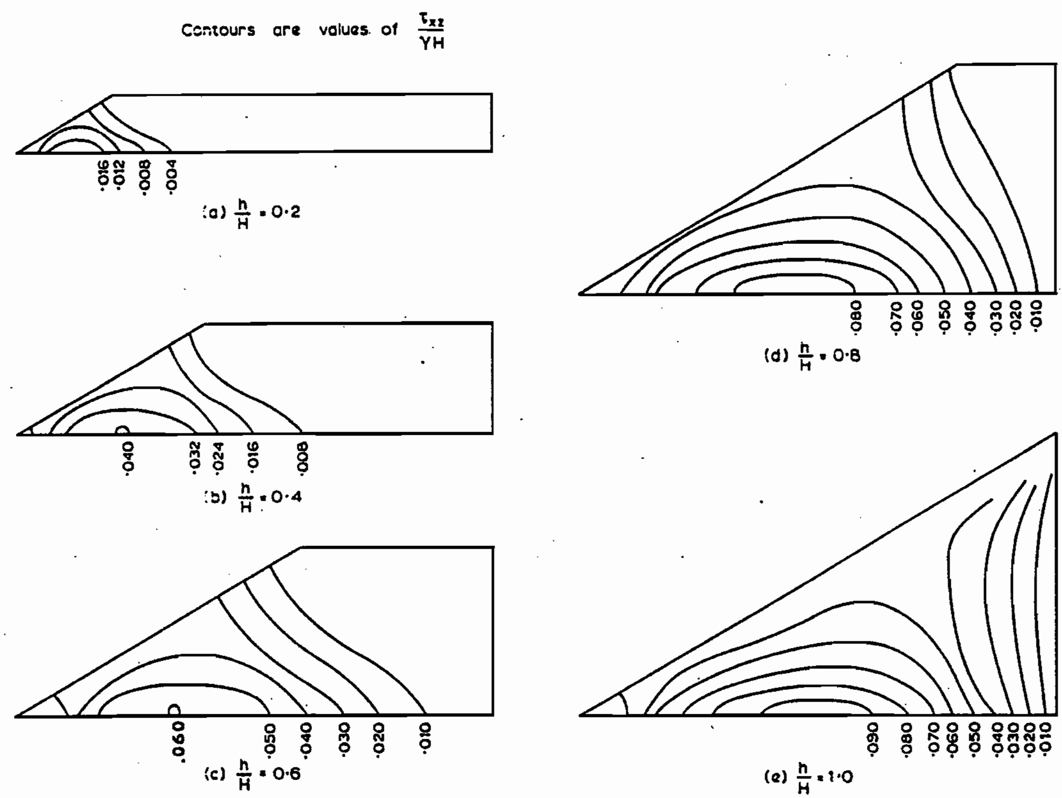


FIG.10.4 Contours of shear stress for various embankment heights. 30° slope, ν=0.3 (Poulos et al, 1972).

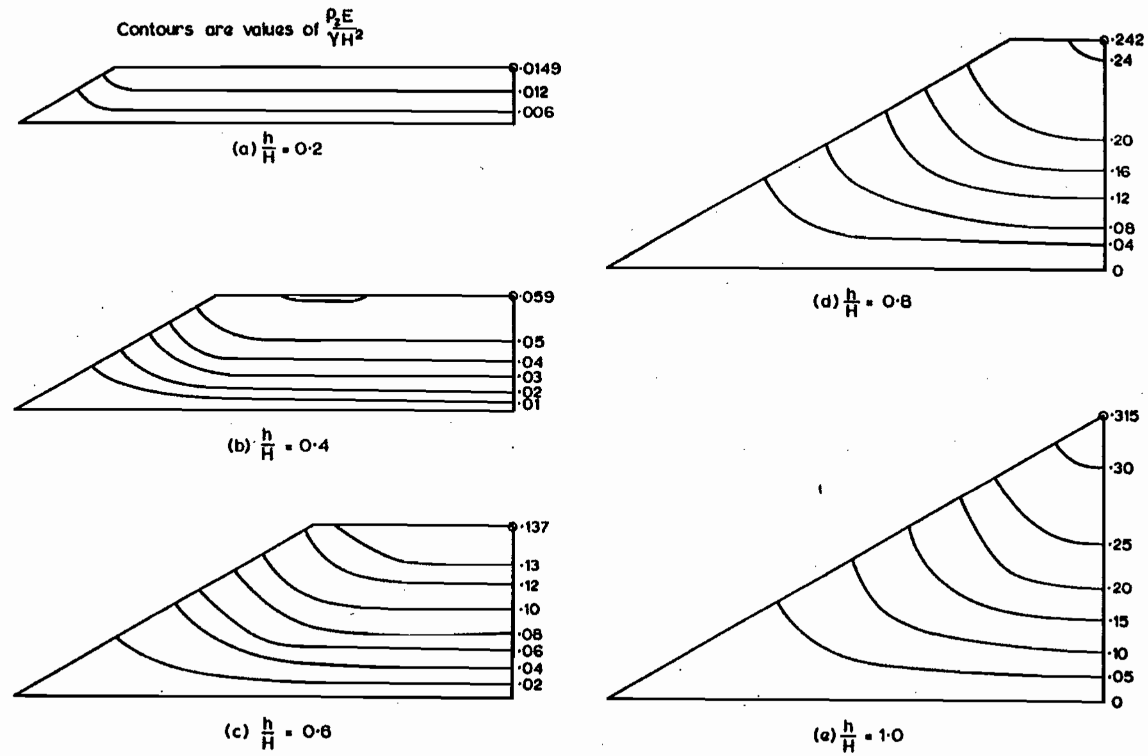


FIG.10.5 Contours of vertical displacement for various embankment heights. 30° slope, ν=0.3 (Poulos et al, 1972).

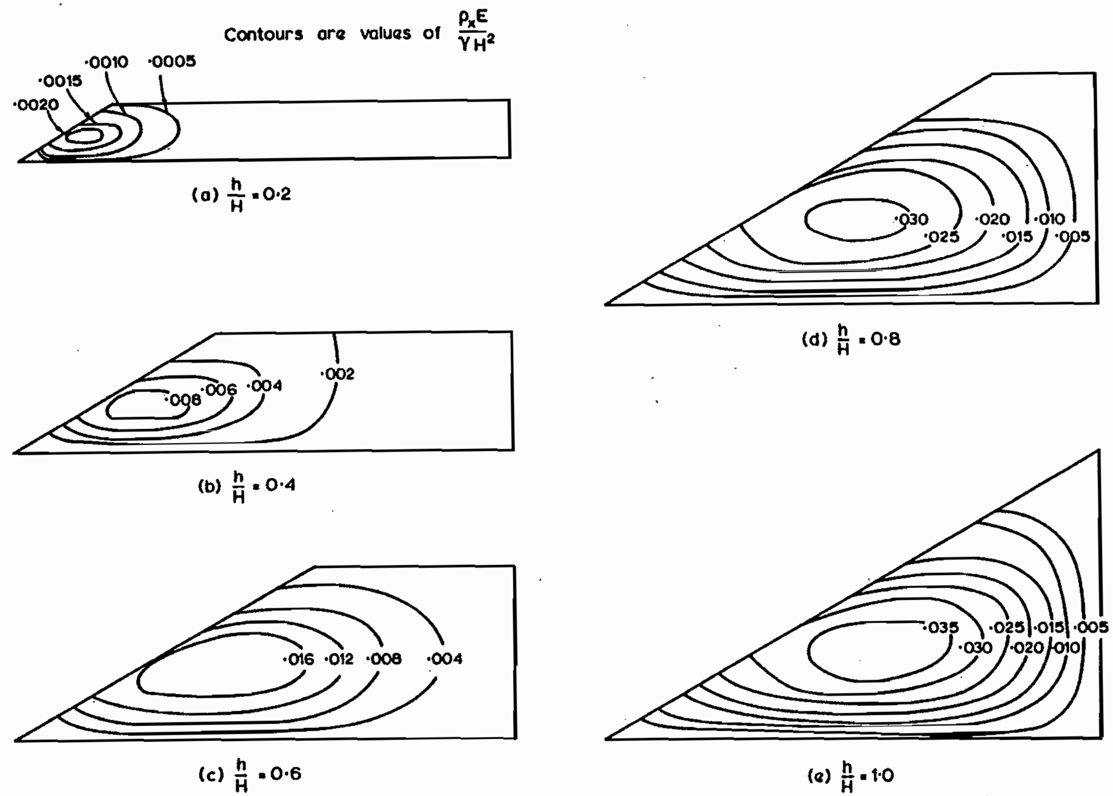


FIG.10.6 Contours of horizontal displacement for various embankment heights. 30° slope, ν=0.3 (Poulos et al, 1972).

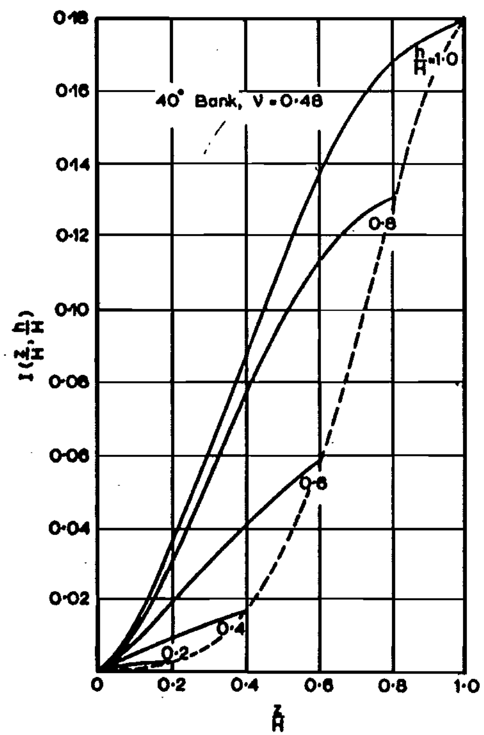


FIG.10.7 Embankment vertical displacement factors. 40° slope, ν=0.48.

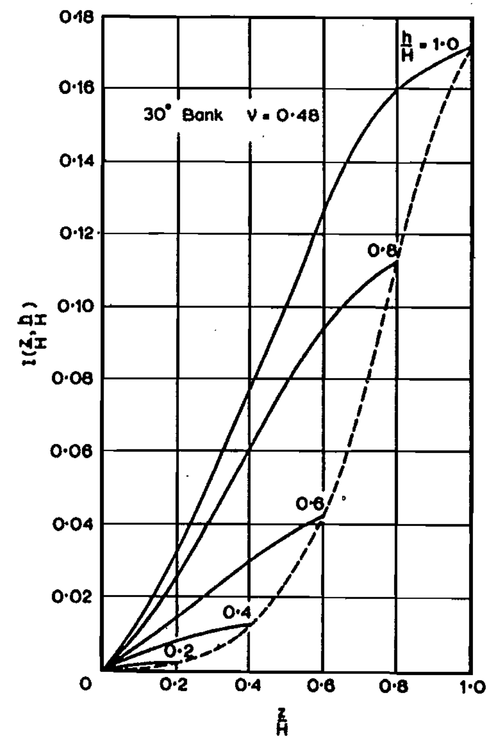


FIG.10.8 Embankment vertical displacement factors. 30° slope, ν=0.48.

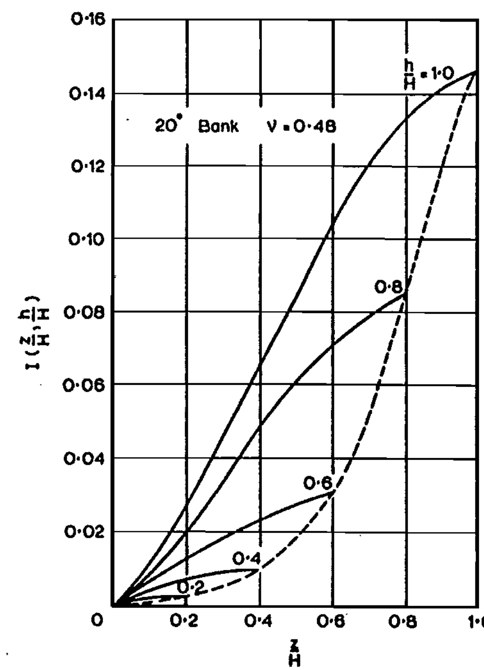


FIG.10.9 Embankment vertical displacement factors. 20° slope, ν=0.48.

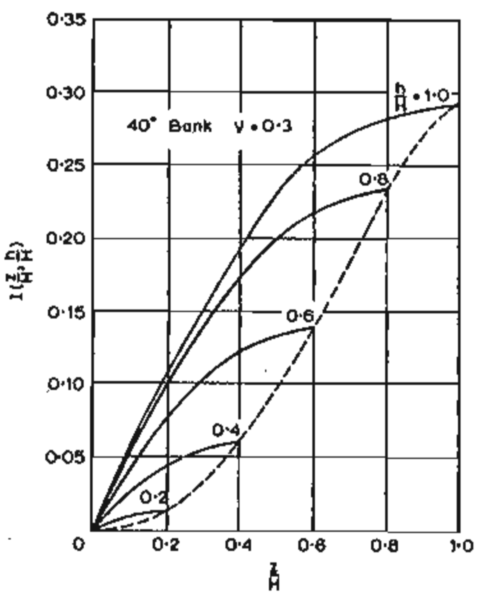


FIG.10.10 Embankment vertical displacement factors. 40° slope, ν=0.3.

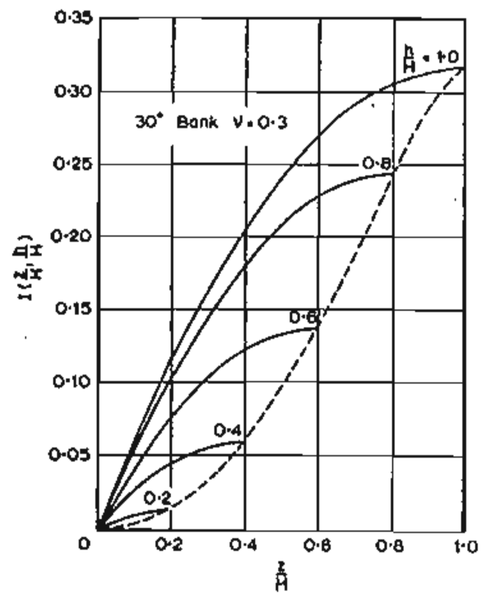


FIG.10.11 Embankment vertical displacement factors. 30° slope, ν=0.3.

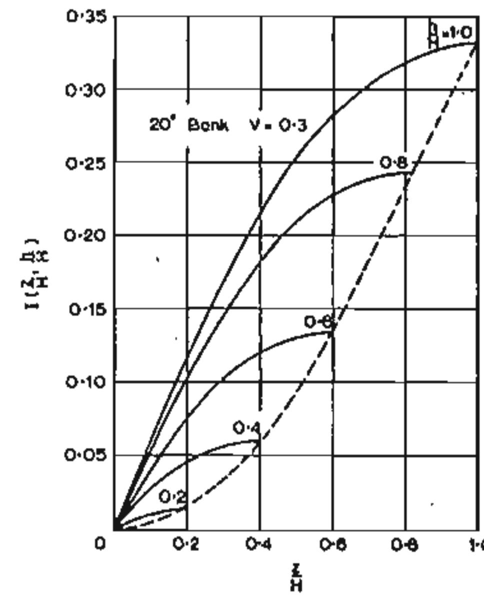


FIG.10.12 Embankment vertical displacement factors. 20° slope. ν=0.3.

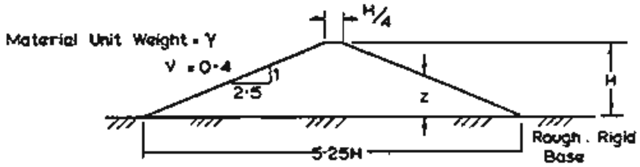


FIG.10.13 Standard embankment (Clough and Woodward, 1967).

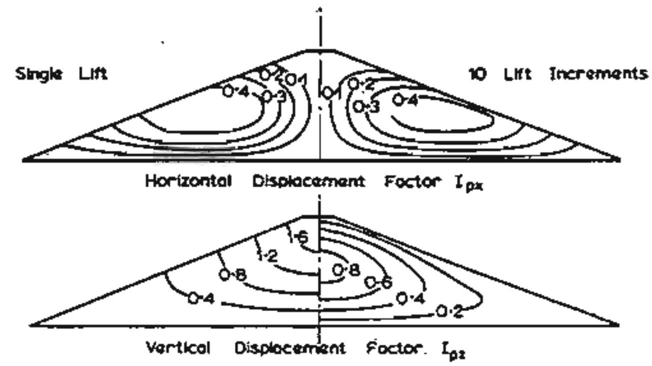
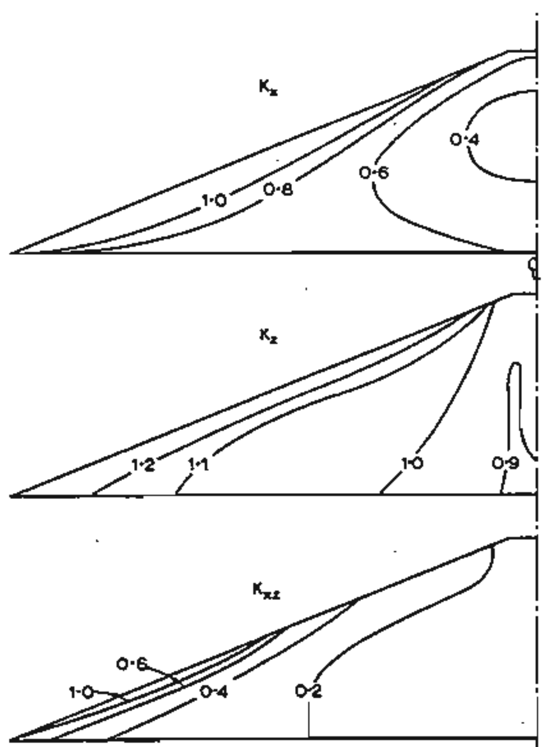


FIG.10.15 Displacement factor contours for standard embankment (Clough and Woodward, 1967)

$$v_x \text{ or } \rho_x = 14.82 \frac{\gamma H}{E} I_{\rho x}$$

$$v_z \text{ or } \rho_z = 14.82 \frac{\gamma H}{E} I_{\rho z}$$



$$\sigma_x = \gamma z K_x$$

$$\sigma_z = \gamma z K_z$$

$$\tau_{xz} = \gamma z K_{xz}$$

FIG.10.14 Stress factor contours for standard embankment (Clough and Woodward, 1967).

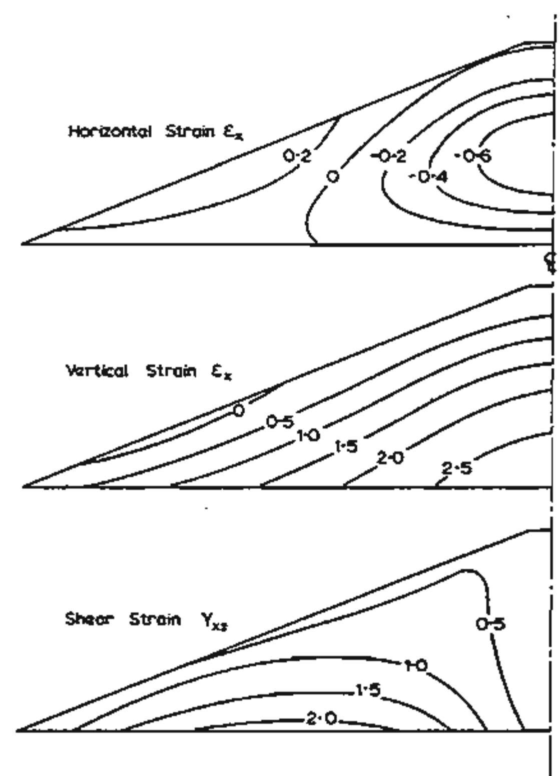


FIG.10.16 Strain contours (in percent) for standard embankment (Clough and Woodward, 1967).

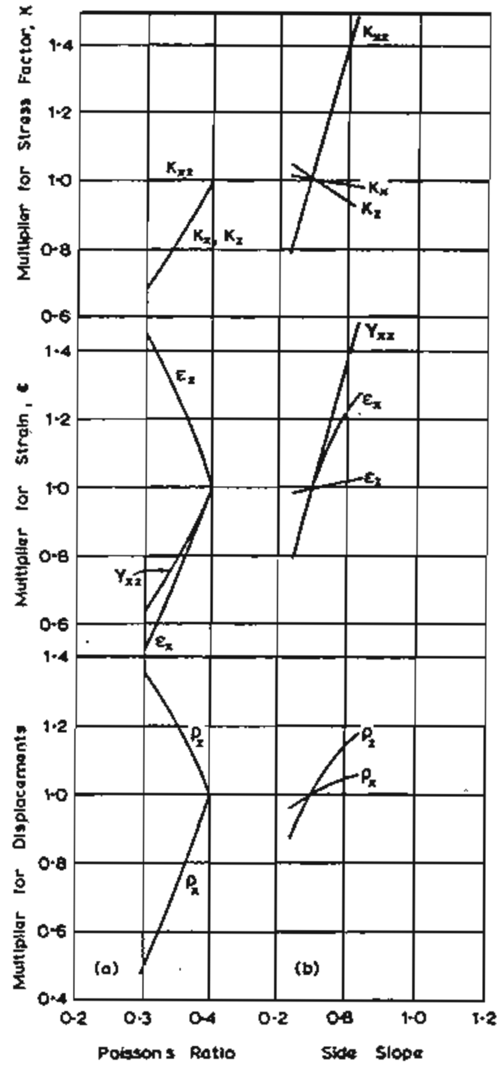


FIG.10.17 Multiplying factors for departures from standard embankment (Clough and Woodward, 1967).

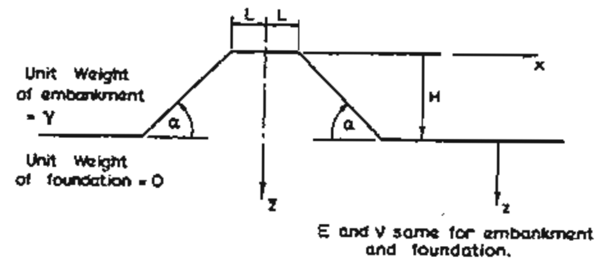


FIG.10.18

10.2 Embankment on Elastic Foundation

10.2.1 SEMI-INFINITE FOUNDATION (Fig.10.18)

This problem has been considered by Perloff et al (1967), who have compared the stresses within the system, due to the weight of the embankment, with those obtained by assuming the embankment to apply a purely vertical stress, proportional to the embankment height, to the surface of the foundation. Stresses from this latter procedure, termed the "normal loading approximation", have been discussed in Chapter 3. Verruijt (1969) has observed that there is some inaccuracy in the results of Perloff et al because of the incorrect implicit assumption in their analysis that the biharmonic equation is invariant for conformal transformations.

Contours for the vertical stress σ_z, horizontal stress σ_x, shear stress τ_{xz} and maximum shear stress τ_{max} within the embankment and foundation are shown in Figs.10.19 to 10.22 for the case $\alpha=45°$, $L/H=3$. These figures show that the "normal loading approximation" may considerably overestimate all the stresses and that consequently, displacements will generally also be overestimated.

Figs.10.19 to 10.22 are all for $\nu=0.3$, except that Fig.10.21 shows some contours for τ_{xz} for $\nu=0.5$ also. The vertical stress σ_z is however virtually independent of ν. An example of the influence of ν on σ_x is shown in Fig.10.23, where the distributions of σ_x with depth beneath the centre and the edge of the embankment, for both $\nu=0.3$ and 0.5. The effect of ν is most pronounced for small values of L/H.

Distributions of σ_z and τ_{xz} along the base of the embankment, for various values of L/H and slope angle α, are shown in Figs. 10.24 and 10.25.

Distributions of σ_z, σ_x and τ_{xz} along selected vertical sections have also been obtained for various slope angles and values of L/H, and for $\nu=0.3$. These distributions are shown in Figs.10.26 to 10.30 (σ_z), Figs.10.31 to 10.35 (σ_x) and Fig.10.36 (τ_{xz}).

It should be emphasized that in all the above solutions, the foundation is assumed weightless and only the effect of the embankment is considered.

The stresses in vicinity of a cutting of trapezoidal shape can also be obtained from the above solutions by considering the material above the level of the base of the cutting to be a pair of wide embankments.

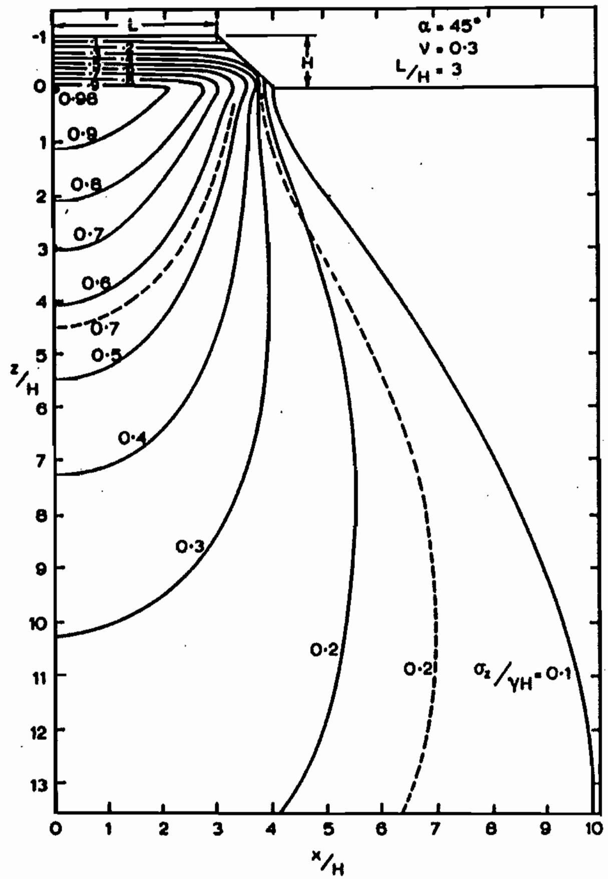


FIG.10.19 Contours of vertical stress σ_z (Perloff et al, 1967). Dashed lines show "normal loading" approximation.

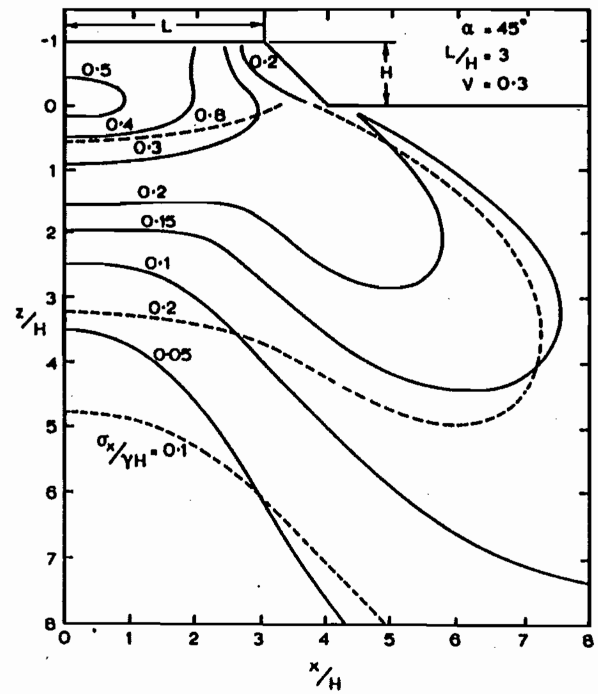


FIG.10.20 Contours of horizontal stress σ_x (Perloff et al, 1967). Dashed lines show "normal loading" approximation.

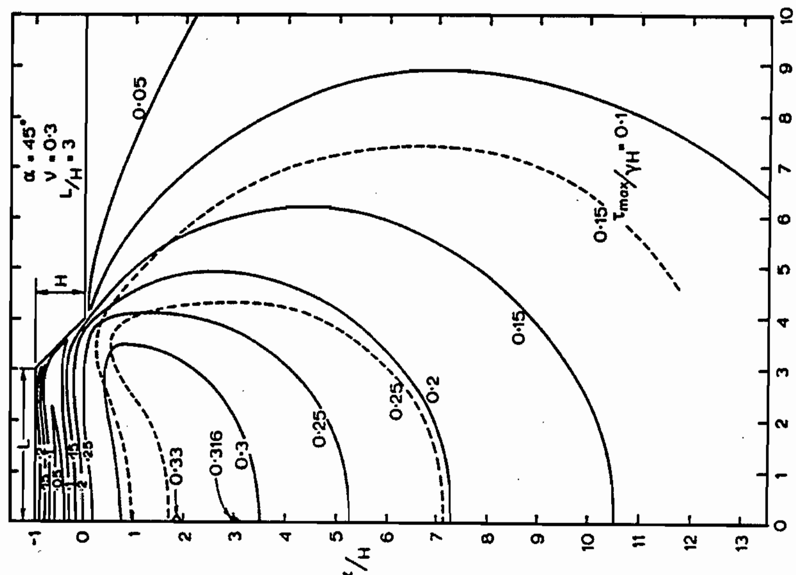


FIG.10.22 Contours of maximum shear τ_{max} (Perloff et al, 1967). Dashed lines show "normal loading" approximation.

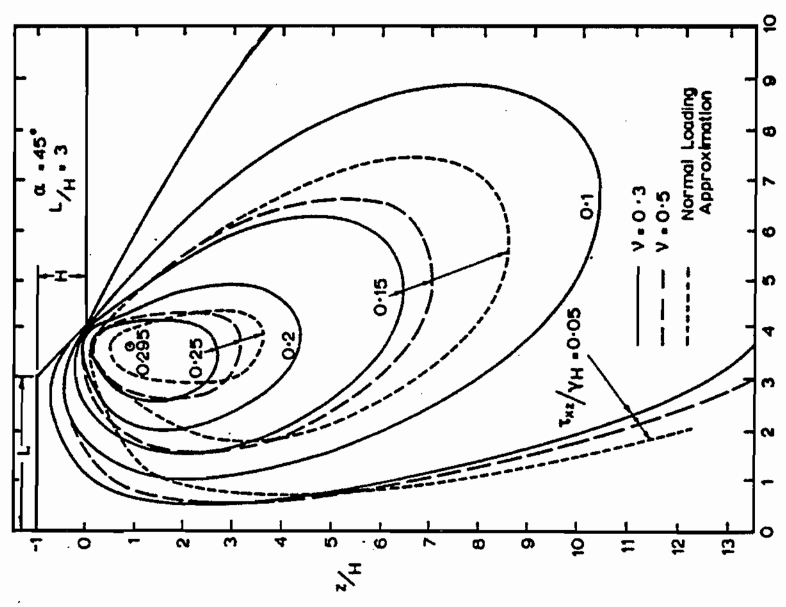
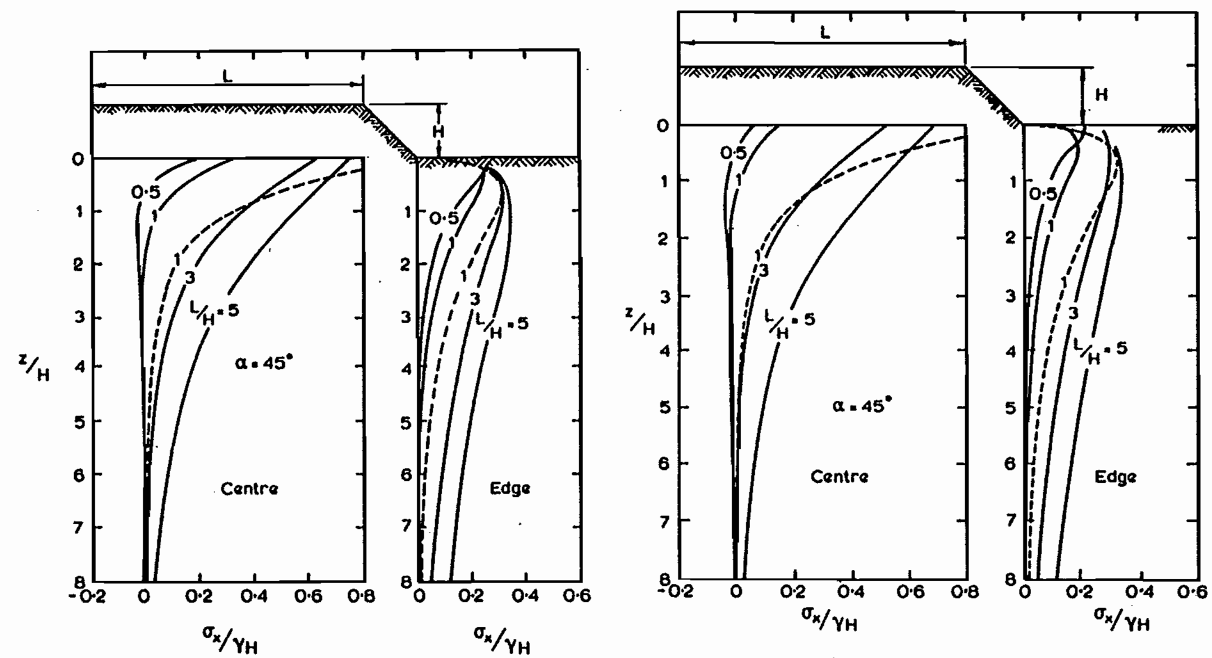
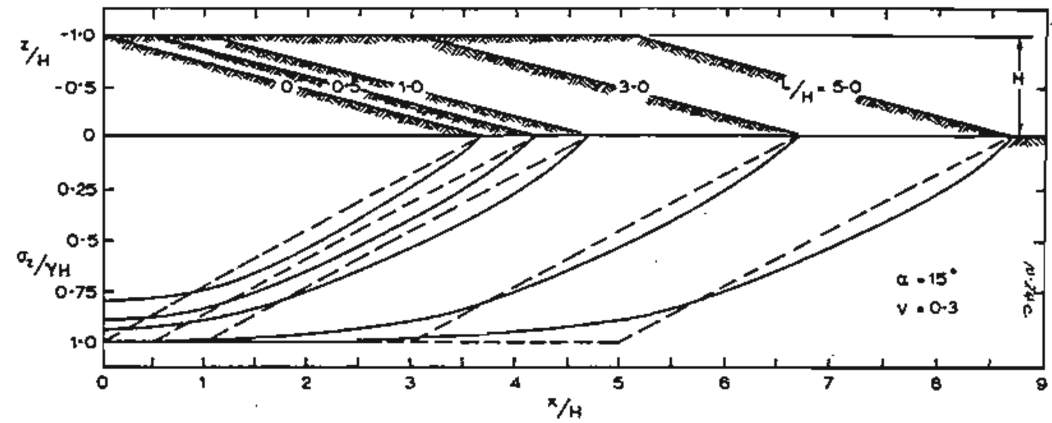


FIG.10.21 Contours of shear stress τ_{xz} (Perloff et al, 1967).

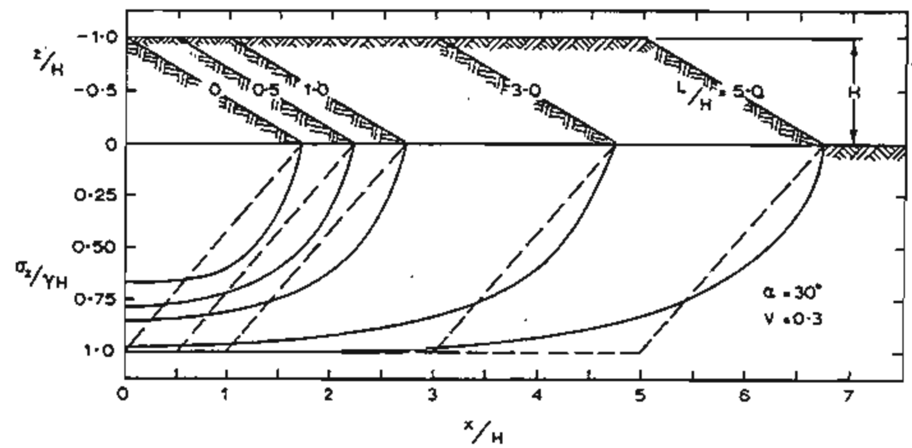


(a) ν=0.5 (b) ν=0.3

FIG.10.23 Influence of ν on horizontal stress σ_x in foundation (Perloff et al, 1967).

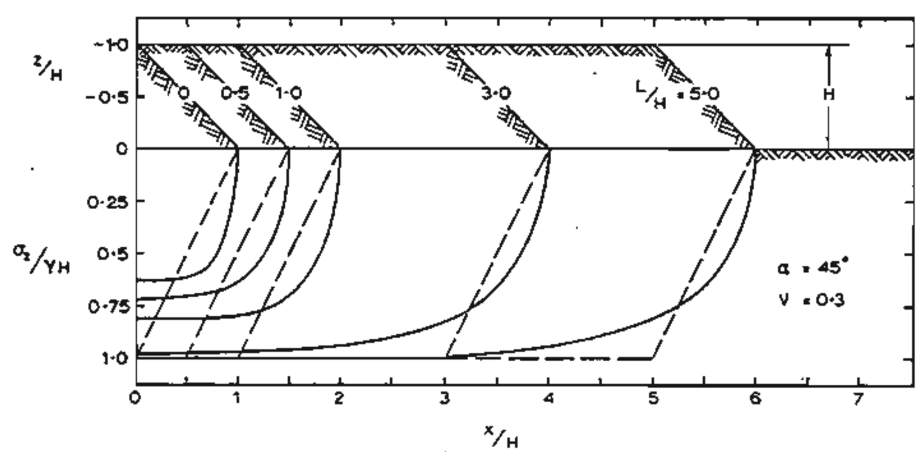


(a)

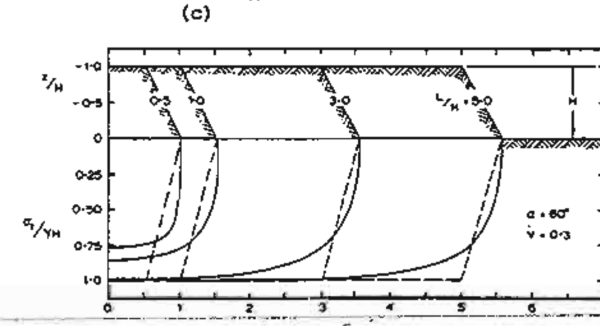


(b)

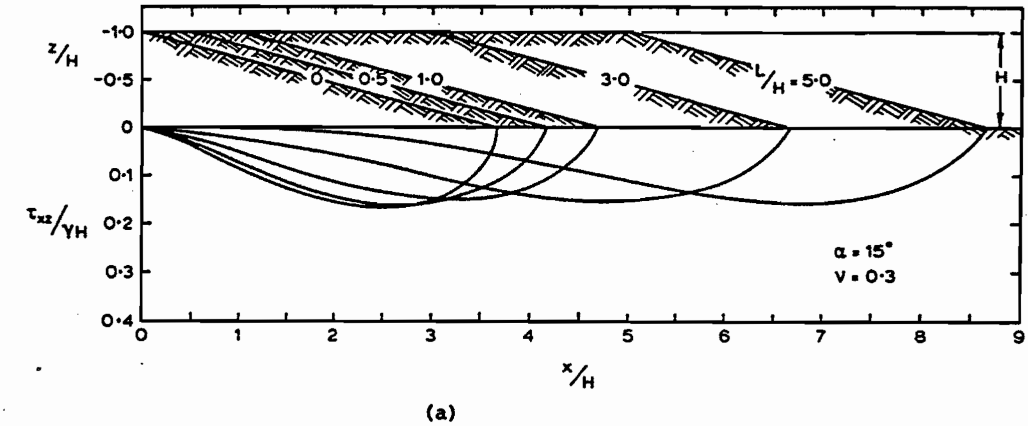
FIG.10.24 Distribution of vertical stress at base of embankment for various slope angles α and L/H ratios (Perloff et at, 1967).



(c)



(d)



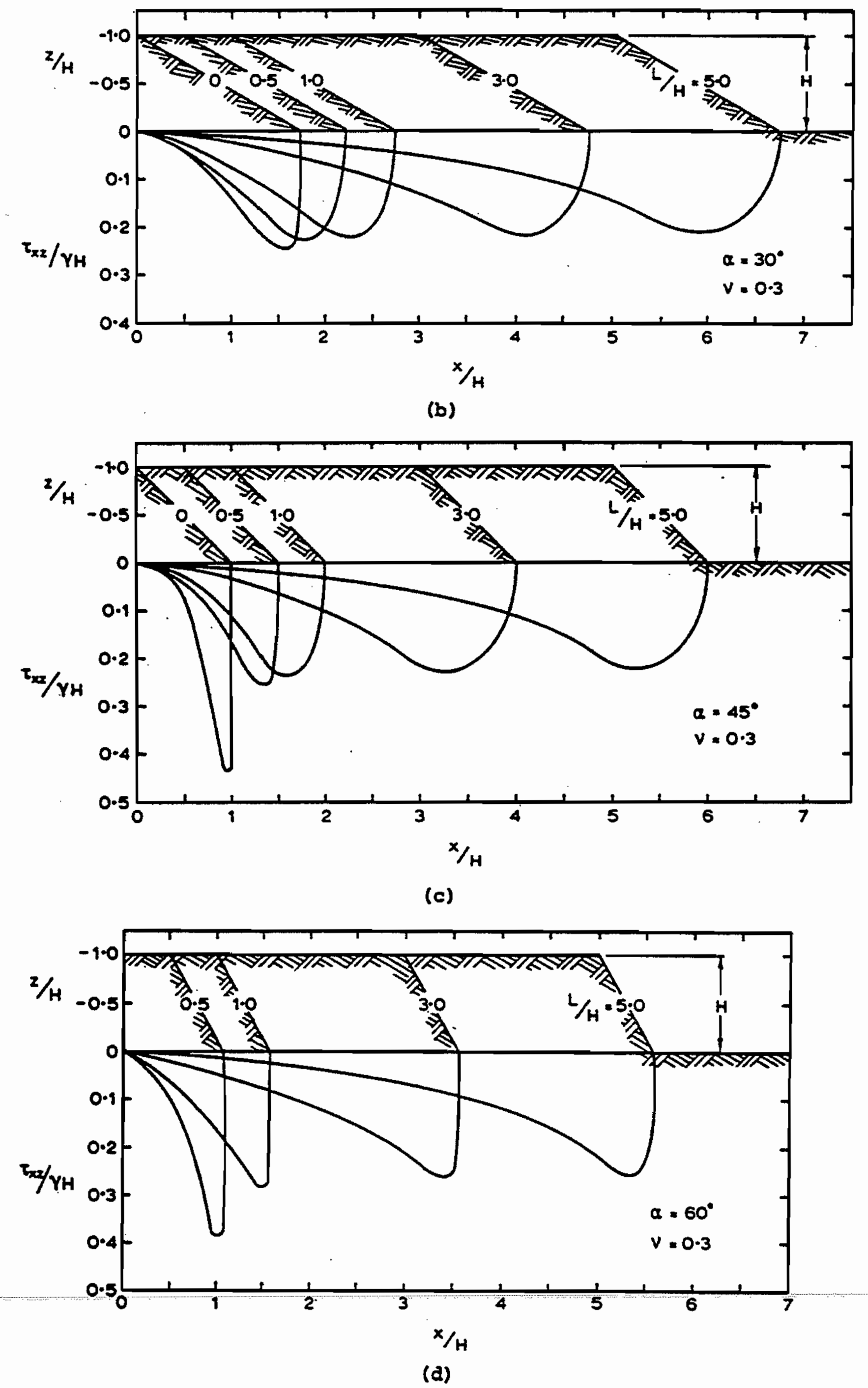


FIG.10.25 Distribution of shear stress τ_{xz} at base of embankment for various slope angles α and L/H ratios (Perloff et al, 1967).

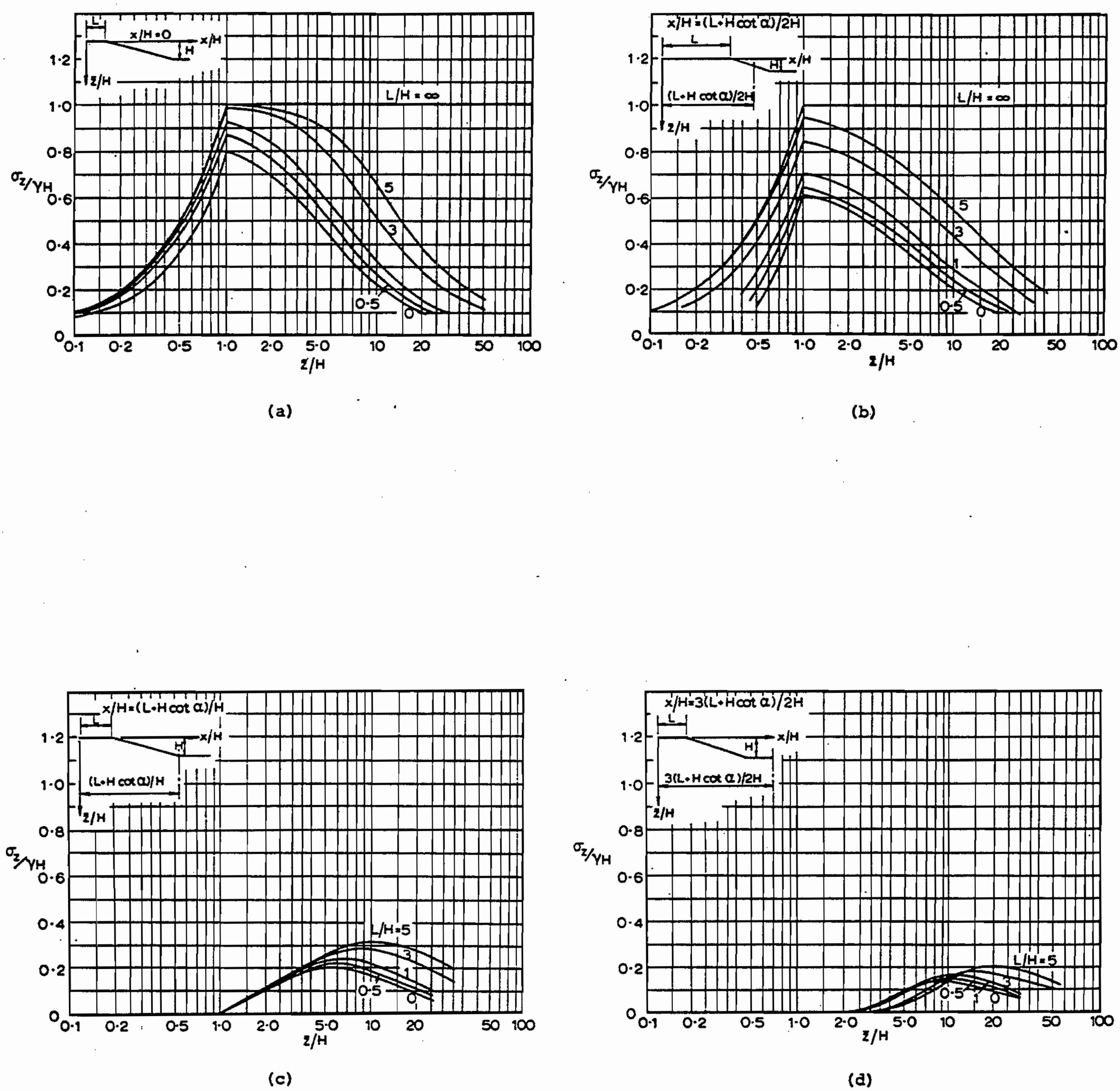


FIG.10.26 Vertical stress along selected vertical sections.
α=15°, ν=0.3 (Perloff et al, 1967).

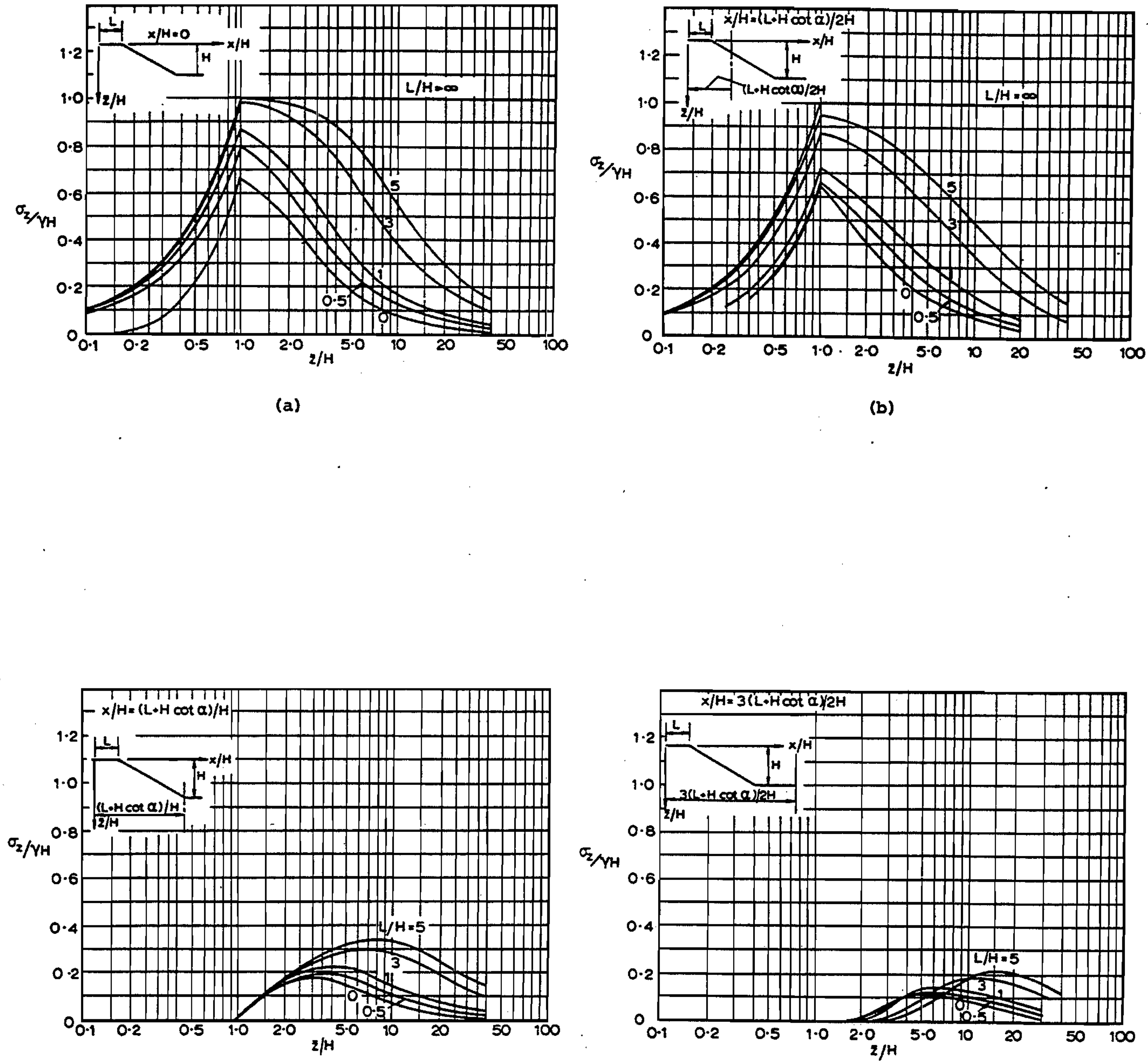


FIG.10.27 Vertical stress along selected vertical sections.
$\alpha=30°$, $\nu=0.3$ (Perloff et al, 1967).

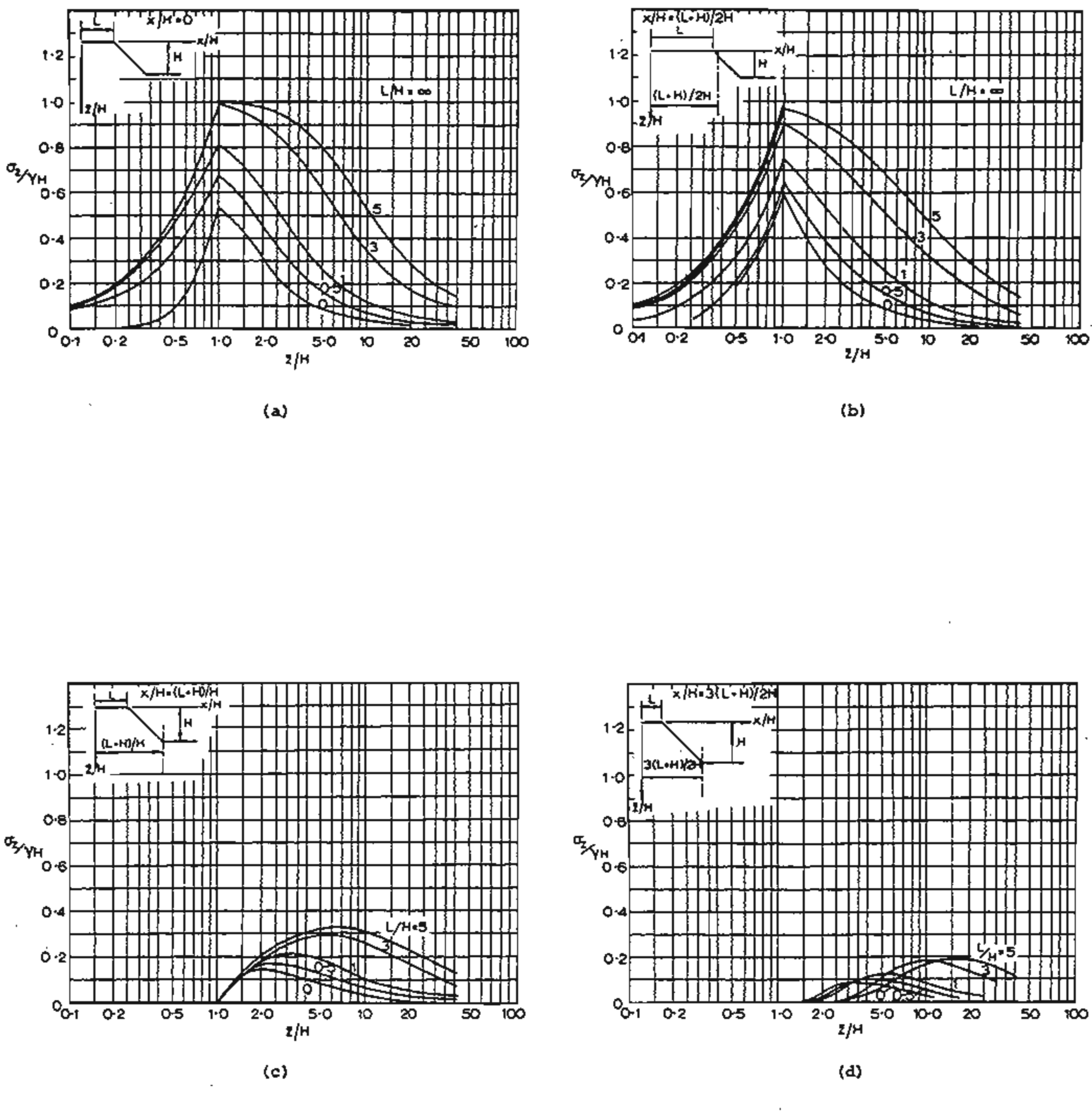


FIG.10.28 Vertical stress along selected vertical sections.
$\alpha=45^{\circ}$, $\nu=0.3$ (Perloff et al, 1967).

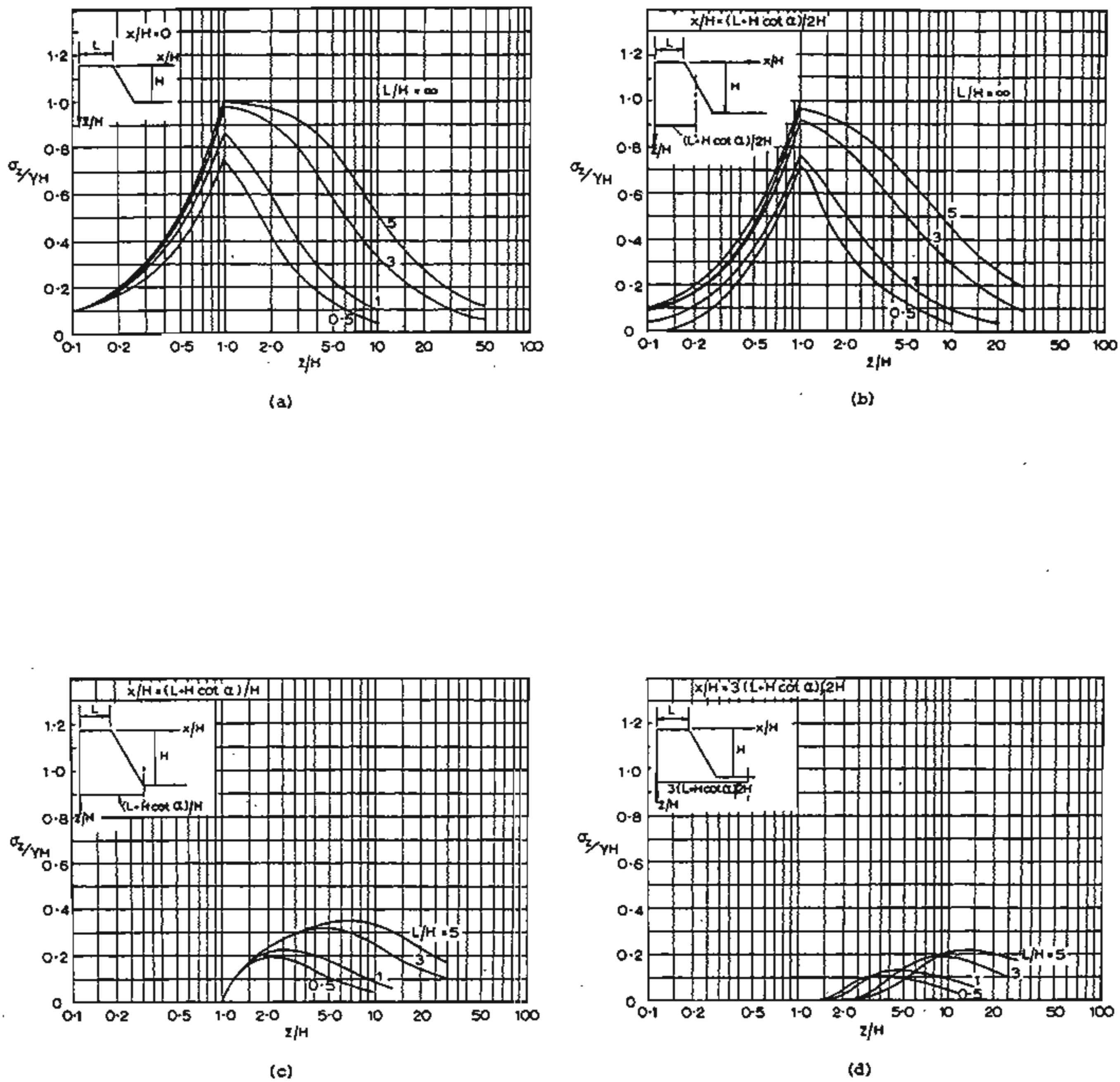


FIG.10.29 Vertical stress along selected vertical sections. α=60°, ν=0.3 (Perloff et al, 1967).

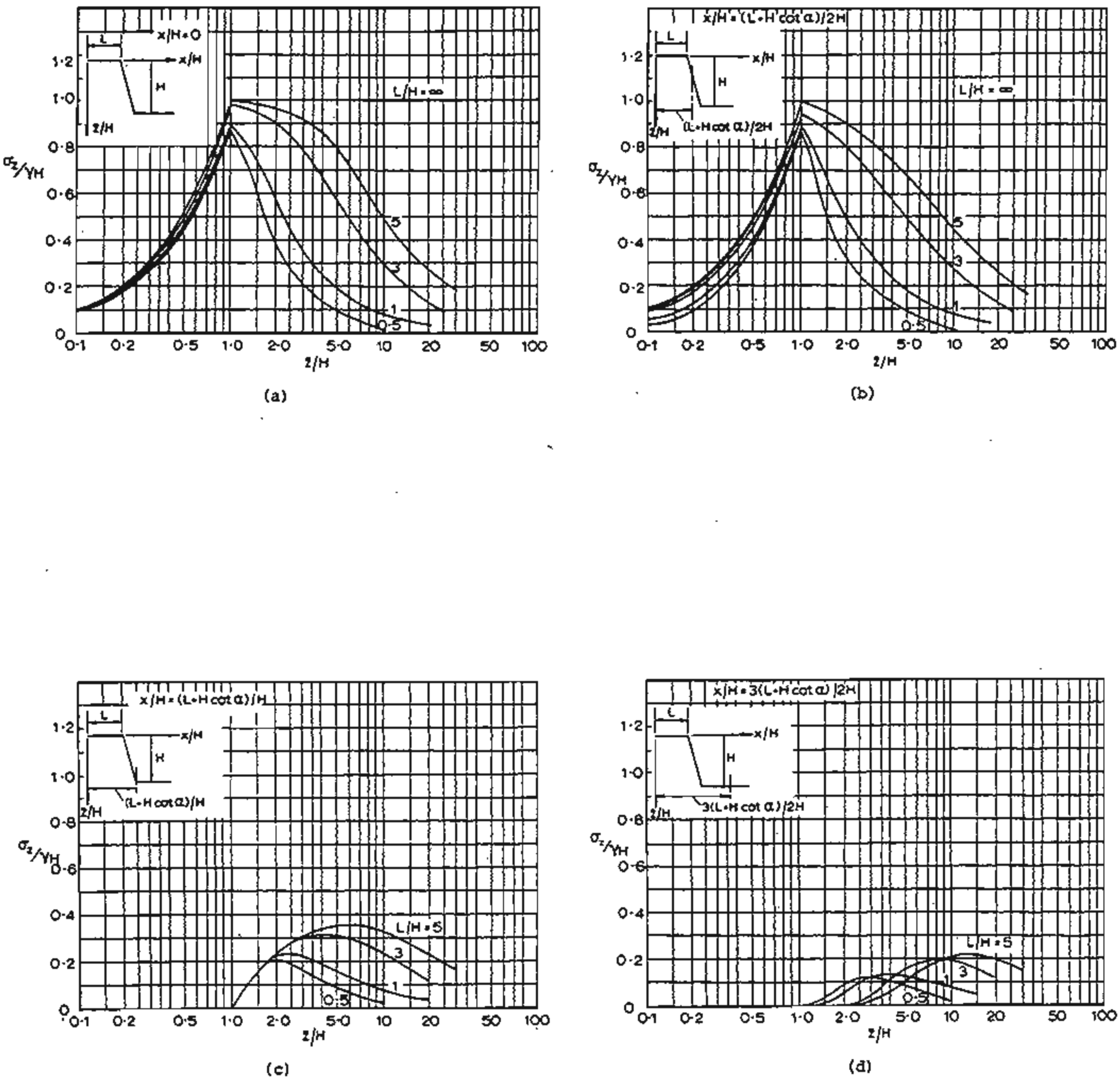


FIG.10.30 Vertical stress along selected vertical sections. $\alpha=75°$, $\nu=0.3$ (Perloff et al, 1967).

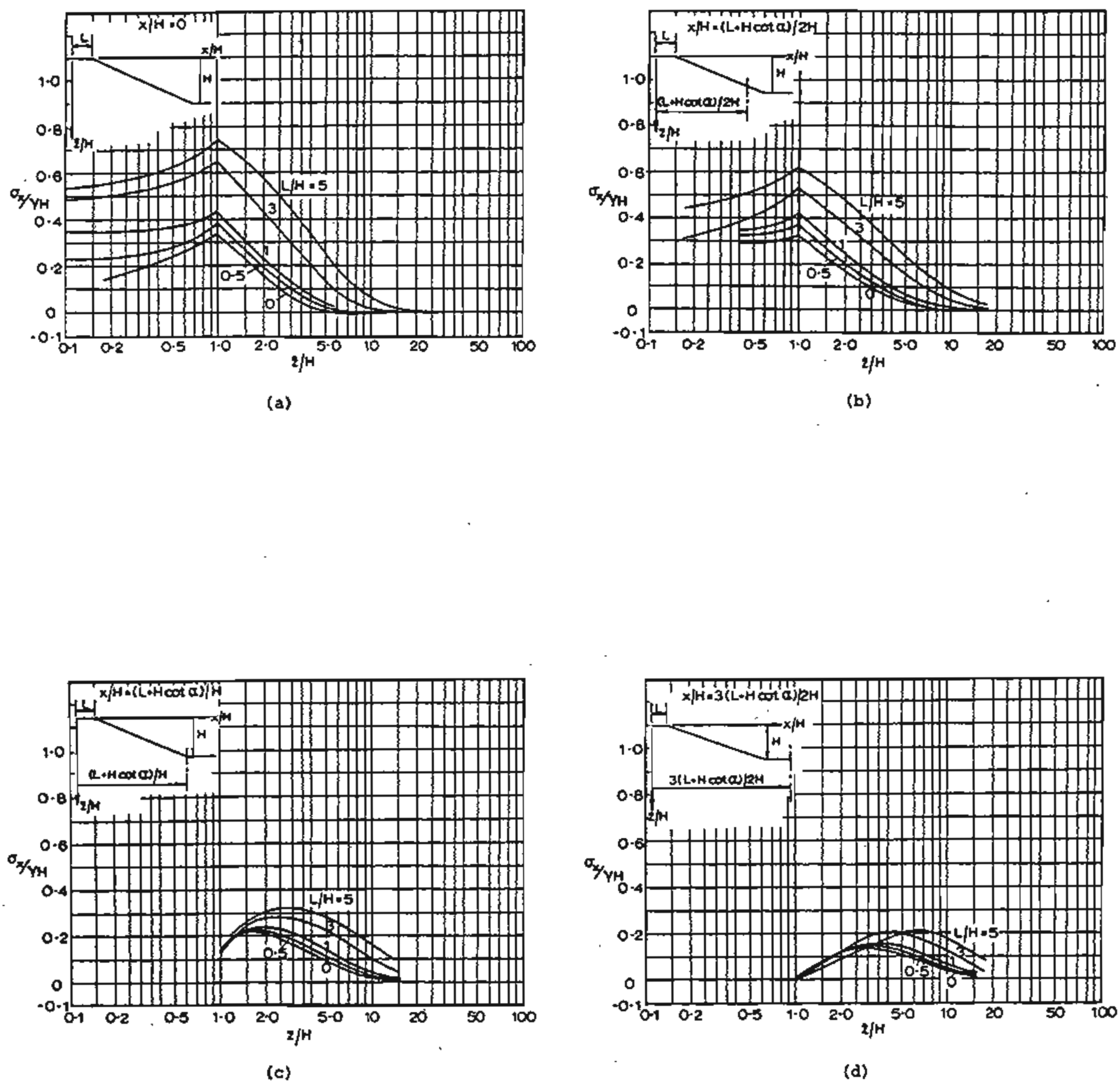


FIG.10.31 Horizontal stress along selected vertical sections. α=15°, ν=0.3 (Perloff et al, 1967).

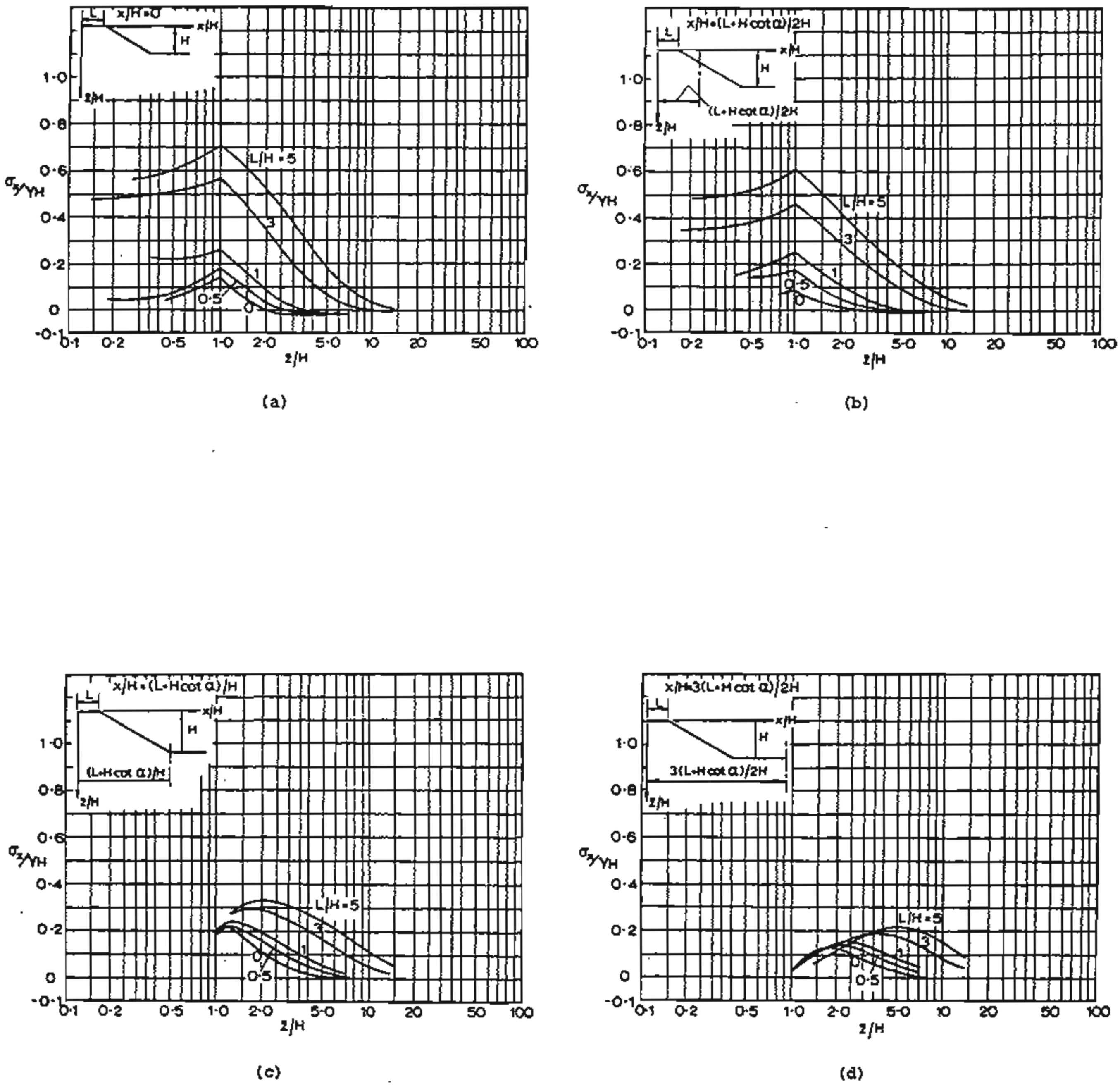


FIG.10.32 Horizontal stress along selected vertical sections. α=30°, ν=0.3 (Perloff et al, 1967).

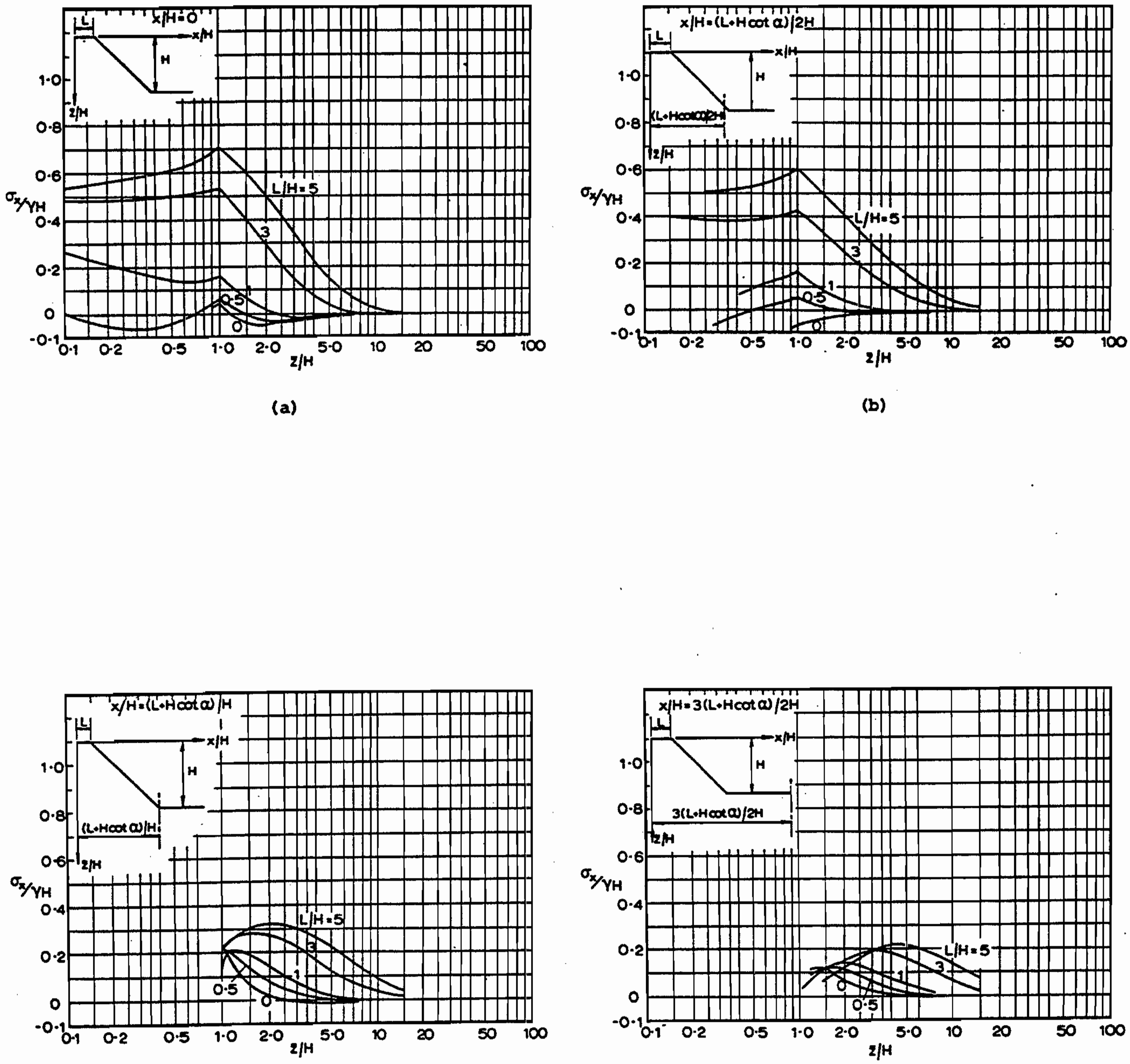


FIG.10.33 Horizontal stress along selected vertical sections. α=45°, ν=0.3 (Perloff et al, 1967).

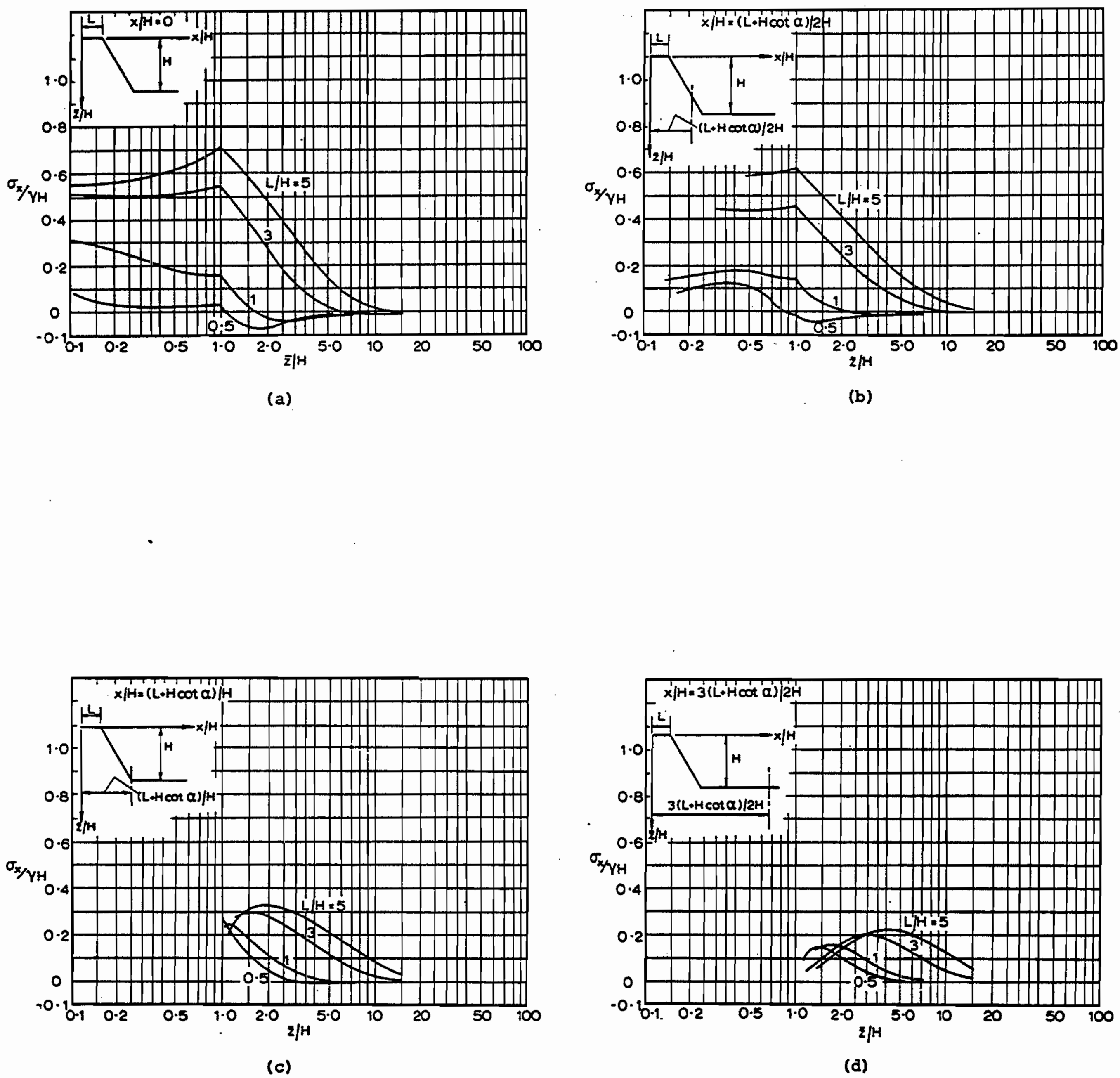
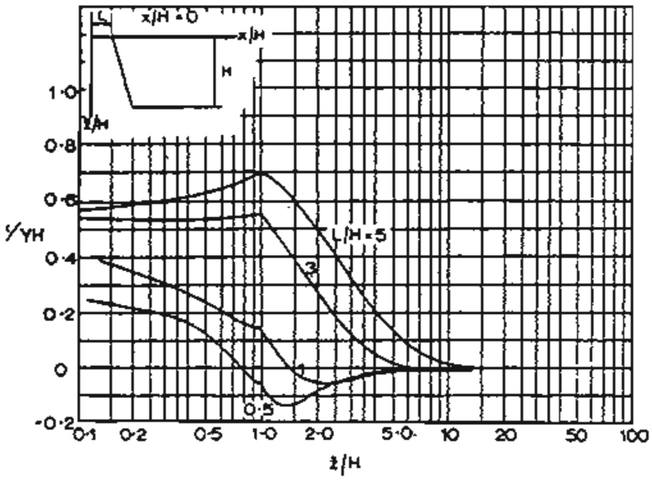
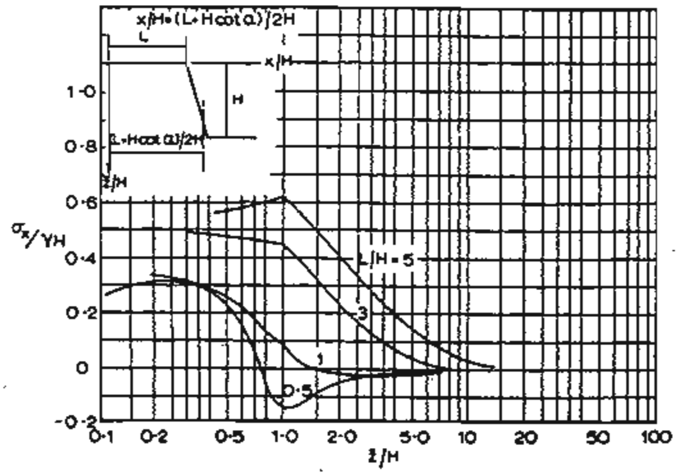


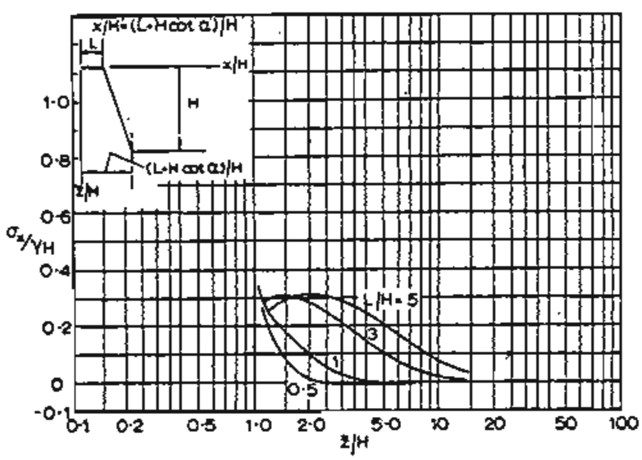
FIG.10.34 Horizontal stress along selected vertical sections.
α=60°, ν=0.3 (Perloff et al, 1967).



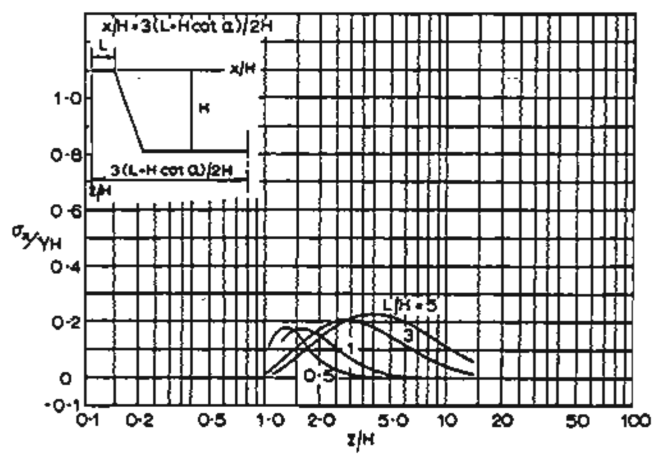
(a)



(b)



(c)



(d)

FIG.10.35 Horizontal stress along selected vertical sections.
α=75°, ν=0.3 (Perloff et al, 1967).

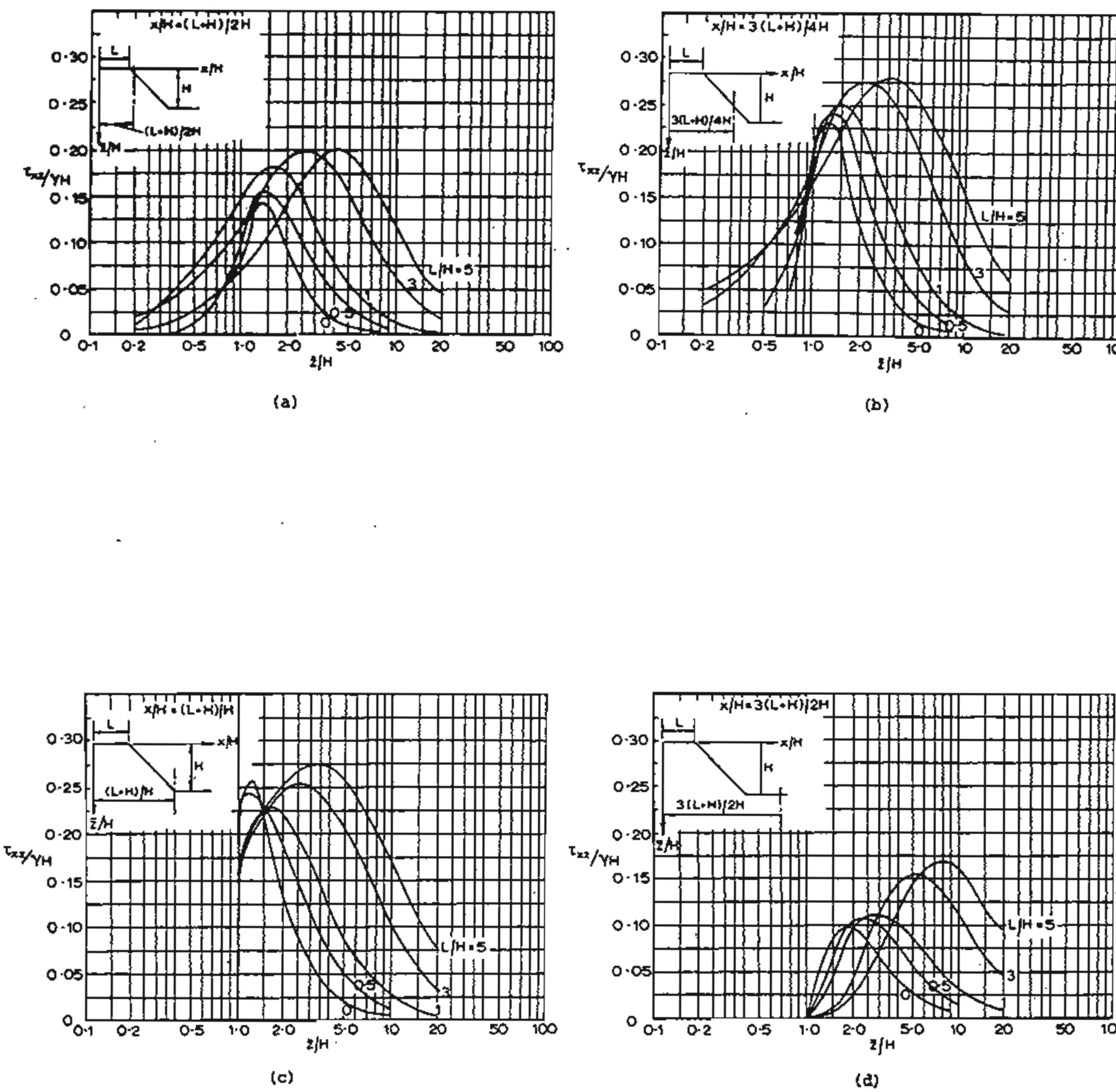


FIG.10.36 Shear stress along selected vertical sections.
α=45°, ν=0.3 (Perloff et al, 1967).

10.2.2 FOUNDATION OF FINITE DEPTH

Clough and Woodward (1967) have considered the particular case of an embankment-foundation system shown in Fig.10.37. While only one geometry is considered, the influence of varying the foundation modulus on the stresses and displacements are investigated.

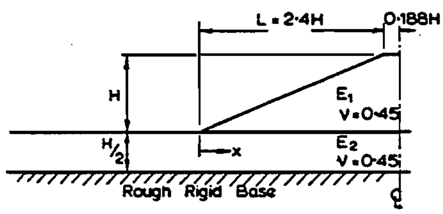


FIG.10.37 Embankment on layer considered by Clough and Woodward (1967).

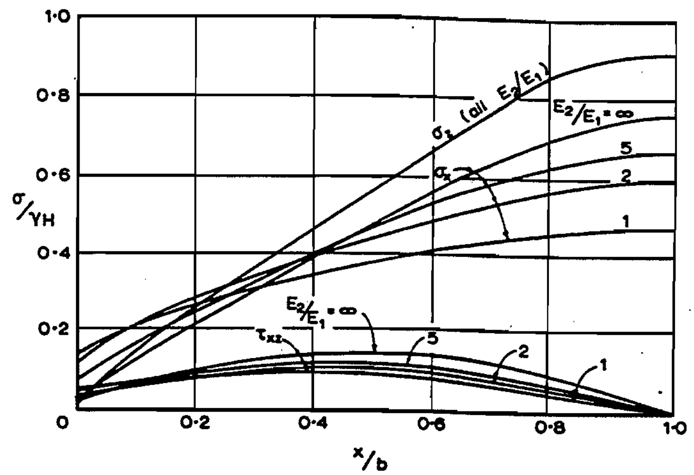


FIG.10.38 Stresses at base of embankment (Clough and Woodward, 1967).

The distribution of stress at the base of the embankment, expressed in dimensionless form, is shown in Fig.10.38 for various values of E_1/E_2. It should be noted that the vertical stress σ_z is independent of this ratio, according to the results of Clough and Woodward. This may not however be generally true.

The distributions of horizontal and vertical displacement at the base of the embankment are plotted in dimensionless form in Fig.10.39.

Another case has been considered by Dingwall and Scrivner (1954) in which an embankment and a foundation having identical elastic properties are considered and contours of average normal stress $(\sigma_x+\sigma_z)/2$ and maximum shear stress τ_{max} within the system are given.

An approximate method for obtaining the normal stresses within and beneath an embankment resting on a semi-infinite mass or a finite layer, has been described by Mirata (1969).

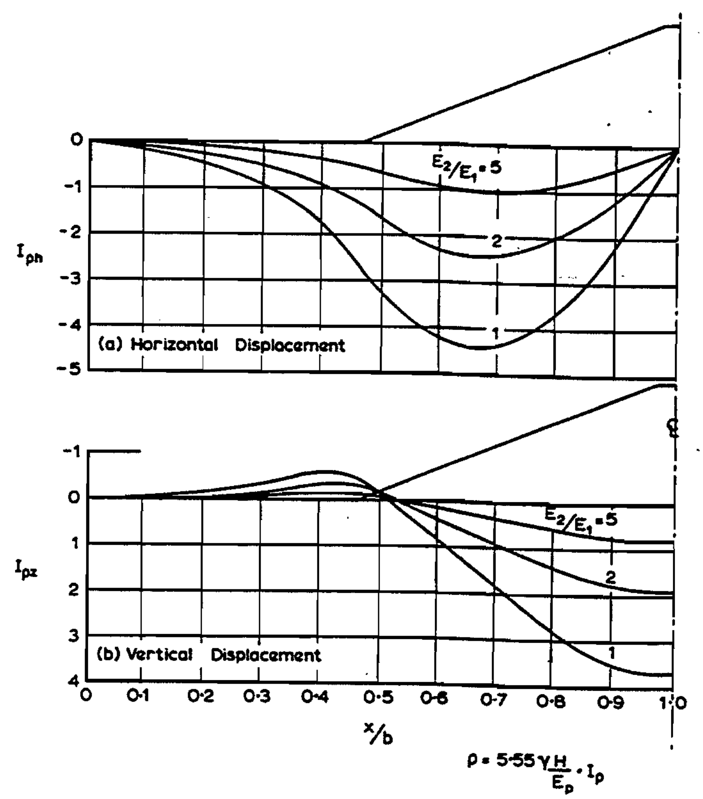


FIG.10.39 Displacements at base of embankment (Clough and Woodward, 1967).

$$\rho = 5.55 \frac{\gamma H}{E_1} I_\rho$$

10.3 Infinite Slope

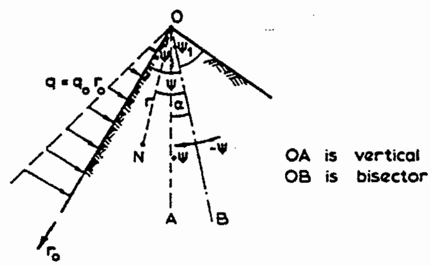


FIG.10.40

This problem was solved by Fillunger (1912) in terms of the more general problem of a semi-infinite wedge (Fig.10.40). Solutions were obtained for stresses due to the weight of the wedge and for stresses due to external loading, increasing linearly with radius, on one side of the wedge.

Stresses due to weight:

$$\sigma_r = r\gamma[(a+\cos\alpha)\cos\psi + (b+\sin\alpha)\sin\psi - c\cos3\psi - d\sin3\psi] \quad \ldots (10.3a)$$

$$\sigma_\theta = r\gamma[(3a+\cos\alpha)\cos\psi + (3b+\sin\alpha)\sin\psi + c\cos3\psi + d\sin3\psi] \quad \ldots (10.3b)$$

$$\tau_{r\theta} = r\gamma(a\sin\psi - b\cos\psi + c\sin3\psi - d\cos3\psi) \quad \ldots (10.3c)$$

where
$$a = -\frac{\cos\alpha\ \sin3\psi_1}{2(\sin\psi_1+\sin3\psi_1)}$$
$$b = \frac{\sin\alpha\ \cos3\psi_1}{2(\cos\psi_1-\cos3\psi_1)}$$
$$c = \frac{\cos\alpha}{8\cos^2\psi_1}$$
$$d = -\frac{\sin\alpha}{8\sin^2\psi_1}$$

Stresses due to external loading:

$$\sigma_r = rq_0(a\cos\psi + b\sin\psi - c\cos3\psi - d\sin3\psi) \quad \ldots (10.4a)$$

$$\sigma_\theta = rq_0(3a\cos\psi + 3b\sin\psi + c\cos3\psi + d\sin3\psi) \quad \ldots (10.4b)$$

$$\tau_{r\theta} = rq_0(a\sin\psi - b\cos\psi + c\sin3\psi - d\cos3\psi) \quad \ldots (10.4c)$$

where
$$a = \frac{\sin3\psi_1}{16\sin\psi_1\cos^3\psi_1}$$
$$b = -\frac{\cos3\psi_1}{16\cos\psi_1\sin^3\psi_1}$$
$$c = -\frac{1}{16\cos^3\psi_1}$$
$$d = \frac{1}{16\sin^3\psi_1}$$

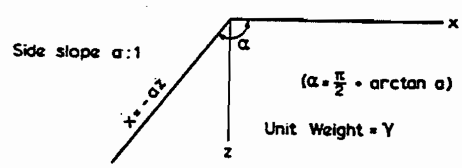


FIG.10.41

For the specific case of an infinite slope with a horizontal surface (Fig.10.41), Goodman and Brown (1963) obtained the following solutions, in terms of Cartesian coordinates:

$$\sigma_x = \frac{\gamma}{(\alpha+\frac{1}{a})(1+a^2)^2}\Big\{(\alpha[1+a^2]+a)(1+a^2)z + a^2(x+az)\ln\frac{x^2+z^2}{(x+az)^2} - (z[1+3a^2] + ax[1-a^2])\sin^{-1}\Big(\frac{z^2}{x^2+z^2}\Big)^{\frac{1}{2}}\Big\} \quad \ldots (10.5a)$$

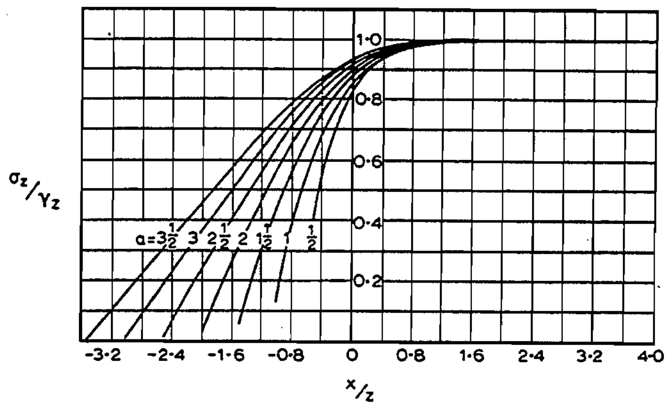


FIG.10.42 Vertical stresses in infinite slope (Goodman and Brown, 1963).

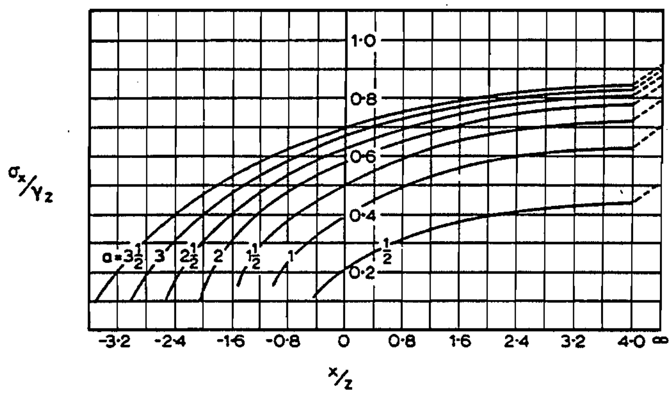


FIG.10.43 Horizontal stresses in infinite slope (Goodman and Brown, 1963).

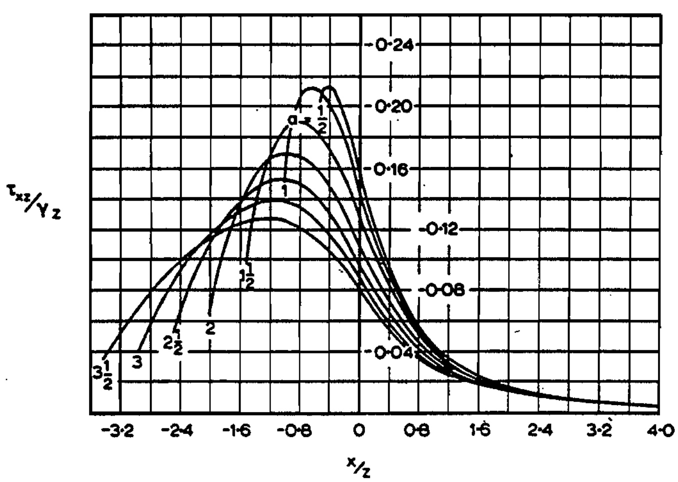


FIG.10.44 Shear stresses in infinite slope (Goodman and Brown, 1963).

$$\sigma_z = \gamma z - \frac{\gamma}{(\alpha + \frac{1}{a})(1+a^2)^2}\{az(1+a^2)$$

$$- (x+az)\ell n \frac{x^2+z^2}{(x+az)^2} + (z[1-a^2]$$

$$- ax[3+a^2])sin^{-1}(\frac{z^2}{x^2+z^2})^{\frac{1}{2}}\} \quad \ldots (10.5b)$$

$$\tau_{xz} = - \frac{\gamma}{(\alpha + \frac{1}{a})(1+a^2)^2}\{z(1+a^2)+a(x+az)\ell n\frac{x^2+z^2}{(x+az)^2}$$

$$+ (a^2-1)(x+az)sin^{-1}(\frac{z^2}{x^2+z^2})^{\frac{1}{2}}\}$$

$$\ldots (10.5c)$$

$$\rho_x = \frac{\gamma}{2\mu(\alpha + \frac{1}{a})(3\lambda+2\mu)}\{(\lambda+2\mu)\alpha z(x+\frac{az}{2})$$

$$-\lambda\frac{z}{a}(x+az) - (\lambda+2\mu)z(x+\frac{az}{2})sin^{-1}(\frac{z^2}{x^2+z^2})^{\frac{1}{2}}$$

$$+ \frac{az(x+az)(3\lambda+2\mu)}{2(1+a^2)} + \frac{(x+az)^2}{2(1+a^2)^2}[\lambda(4a^2+1)$$

$$+ 2\mu(2a^2+1)]\ell n\frac{x^2+z^2}{(x+az)^2} - (\lambda+\mu)z^2\ell n(x^2+z^2)$$

$$+ \frac{a(x+az)^2}{2(1+a^2)^2}[2\mu(1+3a^2)+\lambda(5a^2-1)]$$

$$. \; sin^{-1}(\frac{z^2}{x^2+z^2})^{\frac{1}{2}}\} \quad \ldots (10.5d)$$

$$\rho_z = \frac{\gamma}{2\mu(\alpha + \frac{1}{a})(3\lambda+2\mu)}\{\alpha(\lambda+2\mu)\frac{z^2}{2} + \frac{(\lambda+\mu)}{a}z$$

$$. \; (z-3a^2z-4ax) + \frac{z(x+az)}{2(1+a^2)}[\lambda(1+4a^2)$$

$$+ 2\mu(1+2a^2)] - (\lambda+2\mu)\frac{z^2}{2}sin^{-1}(\frac{z^2}{x^2+z^2})^{\frac{1}{2}}$$

$$+ (x+az)^2\frac{[\lambda(4a^4+5a^2+7)+2\mu(2a^4+3a^2+3)]}{2(1+a^2)^2}$$

$$. \; sin^{-1}(\frac{z^2}{x^2+z^2})^{\frac{1}{2}} + (\lambda+\mu)(2x+az)z\ell n(x^2+z^2)$$

$$- \frac{a(3\lambda+2\mu)(x+az)^2}{2(1+a^2)^2}\ell n\frac{x^2+z^2}{(x+az)^2}\}$$

$$\ldots (10.5e)$$

where λ, μ are Lame's parameters

$$\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)}$$

$$\mu = G = \frac{E}{2(1+\nu)}$$

Dimensionless plots of the stresses for various values of slope a and dimensionless coordinates x/z obtained by Goodman and Brown are shown in Figs. 10.42 to 10.44. Note that the stresses are independend of the elastic parameters.

Goodman and Brown suggest that the above solutions may be used to determine the stresses in a slope which is constructed incrementally. Referring to Fig.10.45, the overall stresses at a point c after construction of a layer (2) on the top of the original slope (1) can be calculated, using coordinates (x_2, z_2) of C referred to the top of layer 2. The original stresses at C can be calculated using coordinates (x_1, z_1) of C referred to the top of the original slope. The stresses due to the construction of layer 2 may then be obtained by subtraction of the original stresses from the final values.

Using the above approach, the stresses (and hence the displacements) due to construction of a slope in any number of increments may be determined. Layers of different densities may be considered also, although the resulting solutions will only be approximate as a variation of density implies a variation in moduli, and such a variation would in fact render the solution for a homogeneous slope invalid.

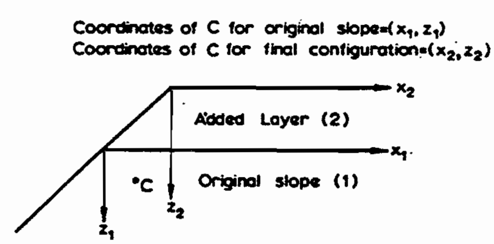


FIG.10.45 Application of theory to incremental construction (Goodman and Brown, 1963).

Chapter 11
STRESSES AND DISPLACEMENTS AROUND UNDERGROUND OPENINGS

11.1 Unlined Openings

11.1.1 CIRCULAR TUNNEL IN AN INFINITE MASS (Plane strain) (Fig.11.1)

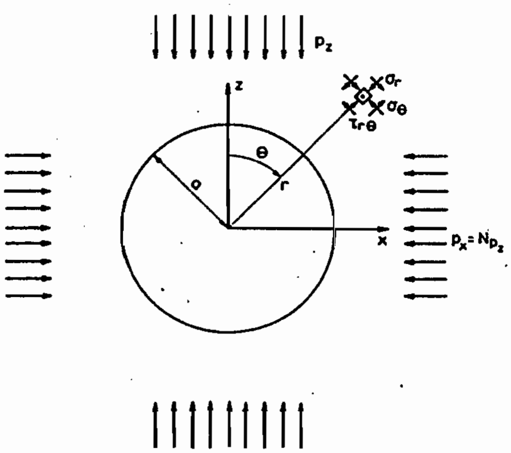


FIG.11.1

Due to uniform vertical loading p_z per unit area,

$$\sigma_r = \frac{p_z}{2}\left(1 - \frac{a^2}{r^2}\right) + \frac{p_z}{2}\left(1 + \frac{3a^4}{r^4} - \frac{4a^2}{r^2}\right)\cos 2\theta \quad \ldots (11.1)$$

$$\sigma_\theta = \frac{p_z}{2}\left(1 + \frac{a^2}{r^2}\right) - \frac{p_z}{2}\left(1 + \frac{3a^4}{r^4}\right)\cos 2\theta \quad \ldots (11.2)$$

$$\tau_{r\theta} = -\frac{p_z}{2}\left(1 - \frac{3a^4}{r^4} + \frac{2a^2}{r^2}\right)\sin 2\theta \quad \ldots (11.3)$$

Due to uniform horizontal loading p_x per unit area,

$$\sigma_r = \frac{p_x}{2}\left(1 - \frac{a}{r^2}\right) - \frac{p_x}{2}\left(1 + \frac{3a^4}{r^4} - \frac{4a^2}{r^2}\right)\cos 2\theta \quad \ldots (11.4)$$

$$\sigma_\theta = \frac{p_x}{2}\left(1 + \frac{a^2}{r^2}\right) + \frac{p_x}{2}\left(1 + \frac{3a^4}{r^4}\right)\cos 2\theta \quad \ldots (11.5)$$

$$\tau_{r\theta} = \frac{p_x}{2}\left(1 - \frac{3a^4}{r^4} + \frac{2a^2}{r^2}\right)\sin 2\theta \quad \ldots (11.6)$$

It should be noted that the above solutions apply to the plane stress problem of a plate with a circular hole as well as the plane strain problem of a tunnel.

Terzaghi and Richart (1952) have tabulated and plotted the normal stresses along the x and z axes. Stresses for three combinations of loading are tabulated in Table 11.1 while stresses for $N = p_x/p_z = 0.25$ are plotted in Fig.11.2. The influence of the ratio N on the circumferential principal stress is shown in Fig.11.3 Stresses above the tunnel along the line $x=a$ are shown in Table 11.2.

At any point, the vertical and horizontal stress components are calculated as follows:

$$\sigma_h = \frac{\sigma_\theta+\sigma_r}{2} + \frac{\sigma_\theta-\sigma_r}{2}\cos 2\theta + \tau_{r\theta}\sin 2\theta \quad \ldots (11.7a)$$

$$\sigma_v = \frac{\sigma_\theta+\sigma_r}{2} - \frac{\sigma_\theta-\sigma_r}{2}\cos 2\theta - \tau_{r\theta}\sin 2\theta \quad \ldots (11.7b)$$

$$\tau_{vh} = -\left(\frac{\sigma_\theta-\sigma_r}{2}\right)\sin 2\theta + \tau_{r\theta}\cos 2\theta \quad \ldots (11.7c)$$

TABLE 11.1

STRESSES ON AXES OF CIRCULAR TUNNEL

(Terzaghi and Richart, 1952)

	For $p_z=1.0$, $p_x=0$				For $p_z=0$, $p_x=1.0$				For $p_z=1.0$, $p_x=0.25$ ($N=0.25$)			
	Along z-axis ($\theta=0$)		Along x-axis ($\theta=\frac{\pi}{2}$)		Along z-axis ($\theta=0$)		Along x-axis ($\theta=\frac{\pi}{2}$)		Along z-axis ($\theta=0$)		Along x-axis ($\theta=\frac{\pi}{2}$)	
$\frac{r}{a}$	σ_r	σ_θ	σ_r	σ_θ	σ_r	σ_θ	σ_r	σ_θ	σ_v	σ_h	σ_h	σ_v
1.00	0	-1.000	0	3.000	0	3.000	0	-1.000	0	-0.250	0	2.750
1.05	-0.034	-0.781	0.127	2.688	0.127	2.688	-0.034	-0.781	-0.002	-0.109	0.118	2.493
1.10	-0.042	-0.611	0.215	2.438	0.215	2.438	-0.042	-0.611	+0.012	-0.002	0.205	2.285
1.20	-0.013	-0.376	0.318	2.071	0.318	2.071	-0.013	-0.376	0.067	+0.142	0.315	1.977
1.40	+0.115	-0.135	0.375	1.645	0.375	1.646	+0.115	-0.135	0.209	0.276	0.404	1.612
1.70	0.315	-0.007	0.339	1.353	0.339	1.353	0.315	-0.007	0.399	0.332	0.418	1.351
2.00	0.469	0.031	0.281	1.219	0.281	1.219	0.469	+0.031	0.539	0.336	0.398	1.227
3.00	0.741	0.037	0.148	1.074	0.148	1.074	0.741	0.037	0.778	0.306	0.333	1.083
5.00	0.902	0.018	0.058	1.022	0.058	1.022	0.902	0.018	0.917	0.273	0.283	1.027

TABLE 11.2

STRESSES ABOVE A CIRCULAR TUNNEL ALONG $x=a$ (Terzaghi and Richart, 1952)

	$\frac{z}{a}$	θ	σ_r	σ_θ	$\tau_{r\theta}$	σ_h	σ_v	τ_{vh}
A. For vertical load only ($p_z=1.0$, $p_x=0$)	0	90°	0	3.000	0	0	3.000	0
	0.18	80°	0.042	2.781	-0.020	0.110	2.713	-0.427
	0.36	70°	0.132	2.221	-0.137	0.279	2.074	-0.550
	0.58	60°	0.203	1.547	-0.352	0.234	1.516	-0.406
	0.78	52°	0.229	1.071	-0.527	0.038	1.263	-0.281
	1.00	45°	0.250	0.750	-0.625	-0.125	1.125	-0.250
	1.38	36°	0.324	0.463	-0.634	-0.192	0.978	-0.262
	1.73	30°	0.422	0.328	-0.568	-0.141	0.891	-0.244
	2.75	20°	0.661	0.160	-0.384	-0.028	0.849	-0.133
	5.67	10°	0.899	0.045	-0.181	0.009	0.935	-0.024
	∞	0	1.000	0	0	0	1.000	0
B. For horizontal load only ($p_z=0$, $p_x=1.0$)	0	90°	0	-1.000	0	0	-1.000	0
	0.18	80°	-0.012	-0.810	0.020	-0.007	-0.796	0.111
	0.36	70°	-0.015	-0.337	0.137	0.035	-0.388	-0.001
	0.58	60°	+0.047	0.203	0.352	0.391	-0.141	-0.244
	0.78	52°	0.150	0.550	0.527	0.812	-0.113	-0.321
	1.00	45°	0.250	0.750	0.625	1.125	-0.125	-0.250
	1.38	36°	0.331	0.883	0.634	1.296	-0.081	-0.067
	1.73	30°	0.328	0.922	0.568	1.265	-0.016	0.027
	2.75	20°	0.222	0.957	0.384	1.118	0.062	0.058
	5.67	10°	0.071	0.985	0.181	1.020	0.009	0.004
	∞	0	0	1.000	0	1.000	0	0
C. For biaxial loading ($p_z=1.0$, $p_x=0.25$) ($N=0.25$)	0	90°	0	2.750	0	0	2.750	0
	0.18	80°	0.039	2.578	-0.016	0.109	2.514	-0.399
	0.36	70°	0.129	2.136	-0.103	0.288	1.977	-0.552
	0.58	60°	0.215	1.598	-0.264	0.332	1.480	-0.467
	0.78	52°	0.267	1.209	-0.395	0.241	1.211	-0.362
	1.00	45°	0.313	0.938	-0.469	0.156	1.094	-0.313
	1.38	36°	0.406	0.684	-0.475	0.132	0.958	-0.279
	1.73	30°	0.504	0.559	-0.426	0.176	0.887	-0.237
	2.75	20°	0.716	0.399	-0.288	0.251	0.864	-0.118
	5.67	10°	0.916	0.291	-0.136	0.264	0.944	-0.021
	∞	0	1.000	0.250	0	0.250	1.000	0

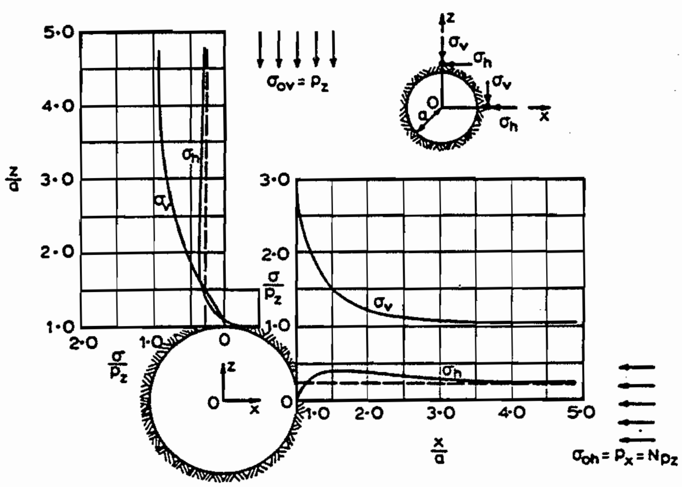


FIG. 11.2 Stress distribution around a circular tunnel. N=0.25. (Terzaghi and Richart, 1952).

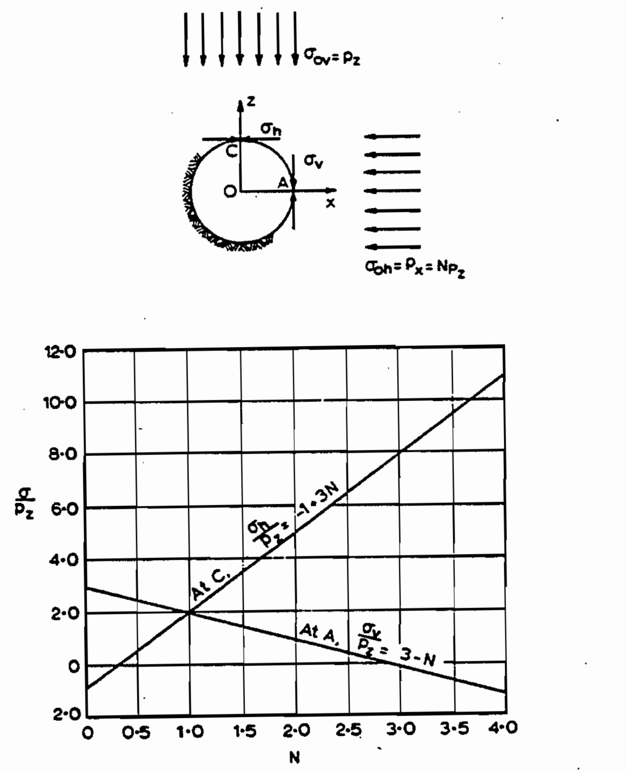


FIG. 11.3 Circumferential principal stress at surface of circular tunnel as function of N (Terzaghi and Richart, 1952).

The following expressions for displacements are quoted by Obert and Duval (1967) for combined uniform vertical and horizontal loading:

radial displacement
$$\rho_r = \frac{1-\nu^2}{E}\left\{\left(\frac{p_x+p_z}{2}\right)\left(r+\frac{a^2}{r}\right) + \left(\frac{p_x-p_z}{2}\right)\left(r-\frac{a^4}{r^3}+\frac{4a^2}{r}\right)\cos 2\theta\right\} - \frac{\nu(1+\nu)}{E} \cdot \left\{\left(\frac{p_x+p_z}{2}\right)\left(r-\frac{a^2}{r}\right) - \left(\frac{p_x-p_z}{2}\right)\left(r-\frac{a^4}{r^3}\right)\cos 2\theta\right\} \quad \ldots (11.8)$$

tangential displacement
$$\rho_\theta = \frac{1-\nu^2}{E}\left\{\left(\frac{p_z-p_x}{2}\right) \cdot \left(r+\frac{2a^2}{r}+\frac{a^4}{r^3}\right)\sin 2\theta\right\} - \frac{\nu(1+\nu)}{E}\left\{\left(\frac{p_x-p_z}{2}\right) \cdot \left(r-\frac{2a^2}{r}+\frac{a^4}{r^3}\right)\sin 2\theta\right\} \quad \ldots (11.9)$$

At the surface of the tunnel ($r=a$)

$$\rho_r = \frac{1-\nu^2}{E}\left\{a(p_x+p_z) + 2a(p_x-p_z)\cos 2\theta\right\} \quad \ldots (11.10)$$

$$\rho_\theta = -\frac{(1-\nu^2)}{E}\left\{2a(p_x-p_z)\sin 2\theta\right\} \quad \ldots (11.11)$$

11.1.2 CIRCULAR TUNNEL IN A SEMI-INFINITE MASS (Fig.11.4)

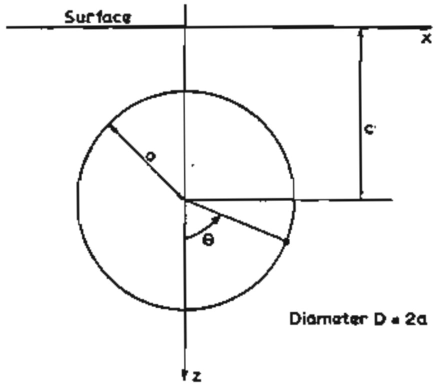


FIG.11.4

Mindlin (1940) considered the above problem for three cases of applied loading:

Case I: at depth z (remote from the tunnel),

$$p_z = wz$$

$$p_h = wz$$

i.e. isotropic gravitational pressure where w = unit weight of mass.

Case II: at depth z (remote from tunnel),

$$p_z = wz$$

$$p_h = \frac{\nu}{1-\nu} wz$$

In this case, there is no lateral deformation remote from the tunnel.

Case III: at depth z (remote from tunnel),

$$p_z = wz$$

$$p_h = 0$$

i.e. no lateral restraint of the mass remote from the tunnel.

The solutions obtained by Mindlin were in terms of bipolar coordinates α and β, which are related to the Cartesian x and z coordinates as follows:

$$x = \frac{A \sin\beta}{\cosh\alpha - \cos\beta} \qquad \text{... (11.12a)}$$

$$z = \frac{A \sinh\alpha}{\cosh\alpha - \cos\beta} \qquad \text{... (11.12b)}$$

and
$$A = c \tanh\alpha_1 = a \sinh\alpha_1$$

where α_1 is the value of α corresponding to the boundary of the tunnel,

$$\frac{c}{a} = \cosh\alpha_1 \qquad \text{... (11.12c)}$$

Case I

On the circular boundary $(\alpha=\alpha_1)$,

$$[\sigma_\beta]_{\alpha=\alpha_1} = \frac{2wA(\cosh\alpha_1 - \cos\beta)}{\sinh\alpha_1}\left\{\frac{1-\cosh\alpha_1\cos\beta}{(\cosh\alpha_1-\cos\beta)^2} - \coth\alpha_1 - \frac{(7-8\nu)}{4(1-\nu)}\frac{\cos\beta}{\sinh\alpha_1} + 2e^{-\alpha_1}\cos\beta - \sum_{n=2}^{\infty} R_n \cos n\beta\right\} \qquad \text{... (11.13)}$$

where $R_n = N_n - ne^{-n\alpha_1}$

$$N_n = \frac{ne^{-n\alpha_1}(\sinh n\alpha_1 \cosh n\alpha_1 - n\sinh\alpha_1\cosh\alpha_1)}{\sinh^2 n\alpha_1 - n^2\sinh^2\alpha_1}$$

The distribution of σ_β around the tunnel is shown in Fig.11.5 for two values of α_1.

The variation of σ_β/wD at the highest point of the tunnel $(\alpha=\alpha_1, \beta=\pi)$ with the ratio c/a is shown in Fig.11.6. This figure shows that the disturbing influence of the upper free boundary is only effective if $c/a<1.5$.

Values of σ_β around the tunnel are tabulated in Table 11.3 for three values of ν. It should be noted that for $\nu=0.5$, Cases I and II are identical.

Cases II and III

For these cases, the solution for $[\sigma_\beta]_{\alpha=\alpha_1}$ is obtained by adding the solution for Case I to the following expression:

$$wGA(\cosh\alpha_1-\cos\beta)\left\{6\coth\alpha_1 \operatorname{csch}\alpha_1 + 6\operatorname{csch}^2\alpha_1\cos\beta + 4\sinh\alpha_1 \sum_{n=2}^{\infty} T_n \cos n\beta\right\} + 6wGA\operatorname{csch}\alpha_1 (\cos\psi + 2\cosh\alpha_1\cos 2\psi + \cos 3\psi) \qquad \text{... (11.14)}$$

where $T_n = S_n - 2n(n^2-1)e^{-n\alpha_1}$

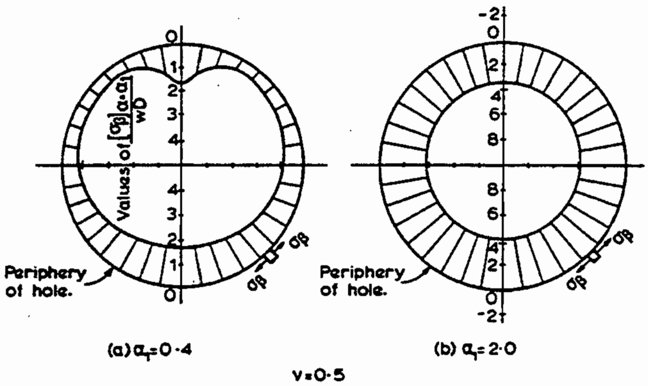


FIG.11.5 Tangential stress around tunnel (Mindlin, 1940).

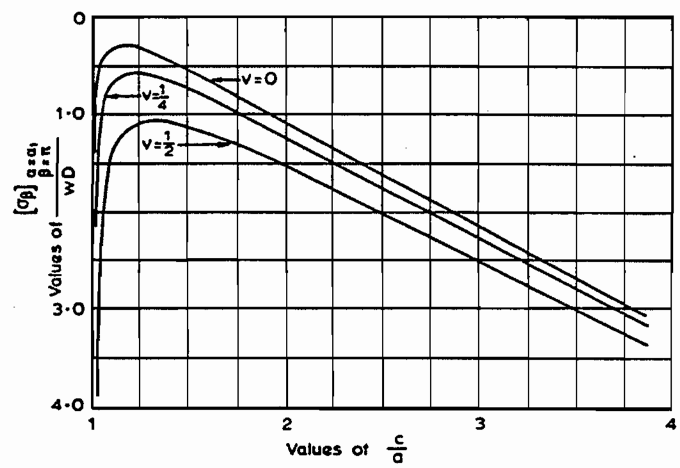


FIG.11.6 Variation of tangential stress at highest point of tunnel vs. c/a (Mindlin, 1940). Case 1.

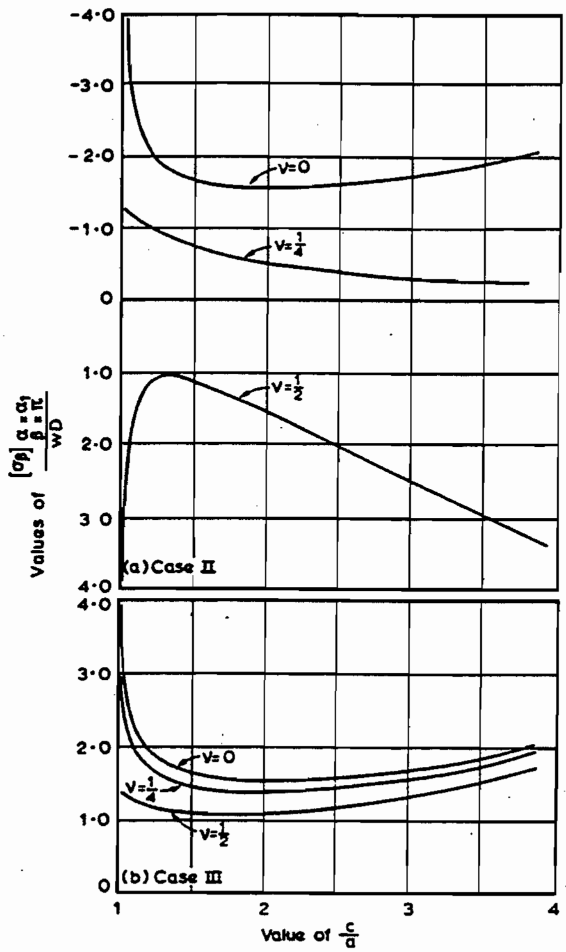


FIG.11.7 Variation of tangential stress at highest point of tunnel vs. c/a. Cases II and III. (Mindlin, 1940).

$$S_n = \frac{n(n^2-1)\sinh n\alpha_1}{\sinh^2 n\alpha_1 - n^2\sinh^2\alpha_1}$$

$$\psi = \sin^{-1}\left\{\frac{\sinh\alpha_1 \sin\beta}{\cosh\alpha_1 - \cos\beta}\right\}$$

$$G = \frac{1-2\nu}{6(1-\nu)} \quad \text{for Case II}$$

$$\text{or } G = \frac{1}{6} \quad \text{for Case III.}$$

The variation of σ_β/wD at the highest point of the tunnel $(\alpha=\alpha_1,\ \beta=\pi)$ with c/a is shown in Fig.11.7.

Values of σ_β around the tunnel are tabulated in Table 11.4 for three values of ν.

TABLE 11.3

VALUES OF $\frac{[\sigma_\beta]_{\alpha=\alpha_1}}{wD}$ (CASE I)

(Mindlin, 1940)

	α_1	β 0	20	40	60	80	100	120	140	160	180
$\nu = 0$	0.2	1.63									1.35
	0.4	1.69	1.07	0.77	0.71	0.69	0.65	0.57	0.49	0.43	0.40
	0.6	1.80	1.42	1.13	0.93	0.86	0.76	0.62	0.46	0.34	0.29
	0.8	1.96	1.72	1.37	1.20	1.07	0.93	0.74	0.59	0.45	0.40
	1.0	2.18	2.01	1.71	1.49	1.33	1.17	0.98	0.80	0.66	0.61
Case I	1.2	2.46	2.33	2.07	1.84	1.64	1.46	1.27	1.08	0.95	0.90
	1.4	2.81	2.71	2.48	2.24	2.03	1.82	1.64	1.45	1.31	1.26
	1.6	3.25	3.17	2.97	2.73	2.50	2.28	2.07	1.89	1.76	1.71
	1.8	3.79	3.72	3.54	3.31	3.07	2.84	2.62	2.44	2.31	2.27
	2.0	4.45	4.42	4.22	4.00	3.75	3.51	3.29	3.11	2.98	2.94
$\nu = \frac{1}{4}$	0.2	1.63									2.18
	0.4	1.67	1.05	0.73	0.66	0.66	0.69	0.74	0.78	0.81	0.85
	0.6	1.77	1.39	1.02	0.89	0.84	0.79	0.73	0.66	0.60	0.58
	0.8	1.93	1.68	1.33	1.16	1.05	0.96	0.83	0.74	0.65	0.62
	1.0	2.14	1.97	1.67	1.45	1.31	1.19	1.06	0.93	0.83	0.79
Case I	1.2	2.41	2.29	2.03	1.80	1.63	1.48	1.34	1.19	1.09	1.06
	1.4	2.76	2.66	2.44	2.21	2.01	1.84	1.69	1.55	1.43	1.39
	1.6	3.19	3.11	2.92	2.69	2.49	2.30	2.13	1.98	1.88	1.84
	1.8	3.73	3.66	3.49	3.27	3.04	2.85	2.67	2.52	2.42	2.38
	2.0	4.39	4.36	4.17	3.96	3.74	3.53	3.34	3.19	3.08	3.05
$\nu = \frac{1}{2}$	0.2	1.61									3.85
	0.4	1.64	0.99	0.63	0.54	0.59	0.78	1.06	1.35	1.58	1.67
	0.6	1.73	1.33	0.93	0.80	0.79	0.85	0.95	1.05	1.12	1.15
	0.8	1.86	1.61	1.25	1.08	1.01	1.01	1.01	1.04	1.05	1.06
Case I	1.0	2.06	1.89	1.58	1.37	1.28	1.23	1.20	1.18	1.16	1.15
and	1.2	2.32	2.20	1.94	1.73	1.60	1.52	1.46	1.41	1.38	1.37
Case II	1.4	2.66	2.56	2.35	2.13	1.98	1.87	1.81	1.74	1.69	1.67
	1.6	3.08	3.00	2.82	2.62	2.46	2.33	2.24	2.16	2.11	2.09
	1.8	3.61	3.55	3.39	3.20	3.02	2.88	2.77	2.69	2.63	2.61
	2.0	4.26	4.24	4.06	3.88	3.71	3.56	3.44	3.35	3.29	3.27

VALUES OF θ (See Fig.11.5) CORRESPONDING TO β FOR POINTS ON CIRCUMFERENCE

α_1	$\frac{c}{a}$	β 0	20	40	60	80	100	120	140	160	180
0.2	1.020	0	121.0	149.4	160.4	166.5	170.4	173.4	175.8	178.0	180.0
0.4	1.081	0	83.6	123.1	142.3	153.5	161.2	167.0	171.8	176.0	180.0
0.6	1.185	0	62.4	102.7	126.5	141.7	152.5	160.9	167.9	174.1	180.0
0.8	1.337	0	49.8	87.6	113.3	131.3	144.6	155.3	164.3	172.3	180.0
1.0	1.543	0	41.8	76.4	102.7	122.3	137.6	150.1	160.9	170.7	180.0
1.2	1.811	0	36.4	68.3	94.1	114.8	131.5	145.5	157.9	169.2	180.0
1.4	2.151	0	32.5	62.1	87.4	108.5	126.2	141.5	155.2	167.8	180.0
1.6	2.577	0	29.7	57.5	82.0	103.3	121.7	138.0	152.8	166.6	180.0
1.8	3.107	0	27.7	53.9	77.7	99.0	118.0	135.1	150.8	165.6	180.0
2.0	3.762	0	26.1	51.1	74.3	95.5	114.8	132.5	149.0	164.7	180.0

TABLE 11.4

VALUES OF $\frac{[\sigma_\beta]_{\alpha=\alpha_1}}{wD}$ (CASES II AND III)

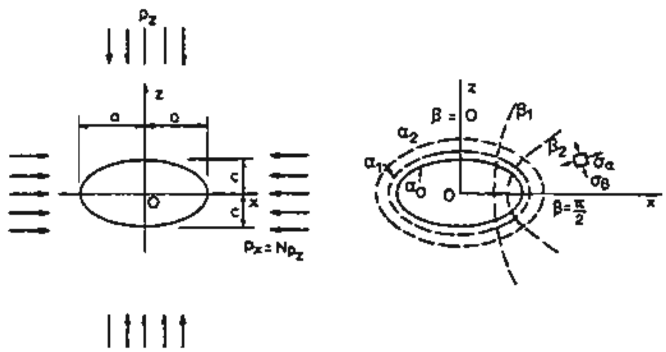
(Mindlin, 1940)

	α_1	β									
		0	20	40	60	80	100	120	140	160	180
ν=0	0.2	-1.22	1.35	0.17	-0.34	-0.86	-1.76	-3.04	-4.41	-5.47	-5.87
	0.4	-1.22	1.62	1.55	0.70	-0.20	-1.19	-2.31	-3.40	-4.22	-4.52
(Case II	0.6	-1.25	1.60	2.01	1.62	0.63	-0.55	-1.79	-2.93	-3.73	-4.03
and	0.8	-1.29	1.20	2.17	2.21	1.37	0.09	-1.31	-2.57	-3.45	-3.77
Case III)	1.0	-1.37	0.80	2.54	2.65	2.02	0.70	-0.85	-2.27	-3.27	-3.62
	1.2	-1.47	0.48	2.74	3.03	2.63	1.31	-0.39	-2.00	-3.14	-3.55
	1.4	-1.61	0.22	2.86	3.54	3.28	1.96	0.08	-1.76	-3.07	-3.54
	1.6	-1.79	-0.01	2.96	4.19	4.00	2.69	0.60	-1.52	-3.05	-3.60
	1.8	-2.03	-0.23	3.08	4.88	4.84	3.53	1.20	-1.27	-3.09	-3.75
	2.0	-2.33	-0.46	3.24	5.65	5.84	4.54	1.91	-1.00	-3.18	-3.98
ν=1/4	0.2	-0.28	1.09	0.20	-0.18	-0.46	-0.87	-1.42	-2.01	-2.46	-2.63
	0.4	-0.27	1.41	1.24	0.65	0.07	-0.53	-1.19	-1.82	-2.28	-2.45
	0.6	-0.26	1.51	1.65	1.34	0.69	-0.08	-0.88	-1.60	-2.11	-2.30
	0.8	-0.24	1.33	1.87	1.83	1.25	0.40	-0.53	-1.37	-1.95	-2.16
(Case II)	1.0	-0.22	1.16	2.22	2.22	1.77	0.88	-0.16	-1.12	-1.79	-2.03
	1.2	-0.21	1.05	2.47	2.60	2.29	1.38	0.22	-0.86	-1.63	-1.91
	1.4	-0.19	1.00	2.69	3.07	2.85	1.93	0.65	-0.59	-1.48	-1.80
	1.6	-0.17	0.99	2.92	3.67	3.49	2.57	1.14	-0.29	-1.33	-1.71
	1.8	-0.15	1.03	3.18	4.32	4.24	3.32	1.72	0.05	-1.18	-1.63
	2.0	-0.13	1.10	3.52	5.06	5.13	4.21	2.42	0.45	-1.02	-1.57
ν=1/4	0.2	-1.23	1.32	0.08	-0.45	-0.92	-1.67	-2.73	-3.85	-4.71	-5.03
	0.4	-1.24	1.59	1.50	0.64	-0.23	-1.14	-2.15	-3.11	-3.83	-4.09
	0.6	-1.27	1.57	1.97	1.57	0.61	-0.52	-1.68	-2.73	-3.47	-3.74
	0.8	-1.33	1.16	2.13	2.17	1.35	0.11	-1.23	-2.42	-3.25	-3.55
(Case III)	1.0	-1.41	0.76	2.50	2.61	2.00	0.72	-0.78	-2.15	-3.10	-3.44
	1.2	-1.52	0.44	2.70	3.00	2.62	1.33	-0.33	-1.89	-3.00	-3.39
	1.4	-1.66	0.17	2.81	3.50	3.27	1.97	0.13	-1.66	-2.94	-3.40
	1.6	-1.85	-0.07	2.91	4.15	3.99	2.70	0.65	-1.43	-2.93	-3.48
	1.8	-2.09	-0.29	3.03	4.85	4.83	3.55	1.25	-1.18	-2.98	-3.63
	2.0	-2.39	-0.52	3.19	5.61	5.83	4.56	1.96	-0.92	-3.08	-3.87
ν=1/2	0.2	-1.25	1.26	-0.08	-0.66	-1.04	-1.50	-2.10	-2.71	-3.18	-3.36
	0.4	-1.27	1.54	1.41	0.53	-0.29	-1.05	-1.82	-2.54	-3.06	-3.25
	0.6	-1.32	1.51	1.88	1.48	0.57	-0.46	-1.46	-2.34	-2.95	-3.17
	0.8	-1.39	1.09	2.05	2.10	1.32	0.16	-1.05	-2.12	-2.85	-3.11
(Case III)	1.0	-1.48	0.68	2.41	2.53	1.97	0.76	-0.63	-1.89	-2.77	-3.08
	1.2	-1.61	0.35	2.61	2.92	2.59	1.37	-0.20	-1.68	-2.71	-3.08
	1.4	-1.76	0.07	2.72	3.43	3.24	2.01	0.25	-1.46	-2.69	-3.13
	1.6	-1.96	-0.17	2.82	4.08	3.96	2.74	0.76	-1.25	-2.70	-3.23
	1.8	-2.21	-0.40	2.93	4.77	4.80	3.58	1.35	-1.02	-2.76	-3.40
	2.0	-2.52	-0.64	3.08	5.54	5.80	4.59	2.05	-0.76	-2.87	-3.65

Note: Case II, ν=½ and Case I, ν=½ are identical.

See Table 11.3 for location of points on circumference in terms of β.

11.1.3 ELLIPTICAL TUNNEL IN AN INFINITE MASS (Plane strain) (Fig.11.8)



(1) (2)

FIG.11.8

Terzaghi and Richart (1952) quote the solutions of Neuber (1937) which are in terms of elliptical coordinates α and β. α and β are defined as follows:

Case a
Major axis of ellipse in x direction $(a>c)$

$$x = \cosh\alpha \sin\beta \quad \ldots (11.15a)$$

$$z = \sinh\alpha \cos\beta \quad \ldots (11.15b)$$

The equation for the boundary of the hole is

$$\left(\frac{z}{\sinh\alpha_0}\right)^2 + \left(\frac{x}{\cosh\alpha_0}\right)^2 = 1 \quad \ldots (11.15c)$$

and the major and minor axes are

$$x_{(z=0)} = a = \cosh\alpha_0 \quad \ldots (11.15d)$$

$$z_{(x=0)} = c = \sinh\alpha_0 \quad \ldots (11.15e)$$

Case b
Major axis of ellipse in z direction $(a<c)$

$$x = \sinh\alpha \sin\beta \quad \ldots (11.16a)$$

$$z = \cosh\alpha \cos\beta \quad \ldots (11.16b)$$

The equation for the boundary of the hole is

$$\left(\frac{z}{\cosh\alpha_0}\right)^2 + \left(\frac{x}{\sinh\alpha_0}\right)^2 = 1 \quad \ldots (11.16c)$$

and the major and minor axes are

$$z_{(x=0)} = c = \cosh\alpha_0 \quad \ldots (11.16d)$$

$$x_{(z=0)} = a = \sinh\alpha_0 \quad \ldots (11.16e)$$

For Case a (a>c), and vertical loading p,

$$\sigma_\alpha = \frac{p}{8}\Big\{\frac{\sin^2 2\beta}{h^4}[\cosh 2\alpha+1-2Be^{-2\alpha}-C(1-e^{-2\alpha})]$$

$$+\frac{4\cos 2\beta}{h^2}[\cosh 2\alpha+1-2Be^{-2\alpha}-C(1-e^{-2\alpha})]$$

$$+\frac{\sinh 2\alpha}{2h^4}[2\sinh 2\alpha\{1-\cos 2\beta\}+2Ce^{-2\alpha}\{1+\cos 2\beta\}$$

$$+2A-4Be^{-2\alpha}\cos 2\beta]\Big\} \quad \ldots (11.17)$$

$$\sigma_\beta = \frac{p}{8}\Big\{\frac{\sinh 2\alpha}{h^4}[-\sinh 2\alpha\{1-\cos 2\beta\}-Ce^{-2\alpha}\{1+\cos 2\beta\}$$

$$-A+2Be^{-2\alpha}\{\cos 2\beta\}]+\frac{\sin^2 2\beta}{h^4}[(-\cosh 2\alpha)$$

$$-1+2Be^{-2\alpha}+C(1-e^{-2\alpha})]$$

$$+\frac{4}{h^2}[\cos 2\beta\{-\cosh 2\alpha-Ce^{-2\alpha}+2Be^{-2\alpha}\}$$

...Continued

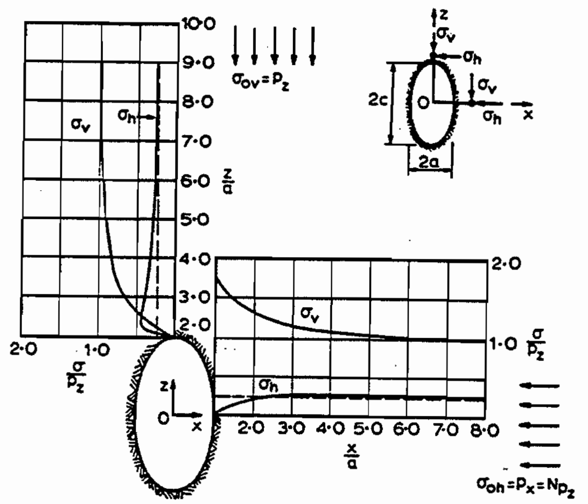


FIG.11.9 Stresses about elliptical tunnel ($a/c=\frac{1}{2}$) in homogeneous stress fields (N=0.25). (Terzaghi and Richart, 1952).

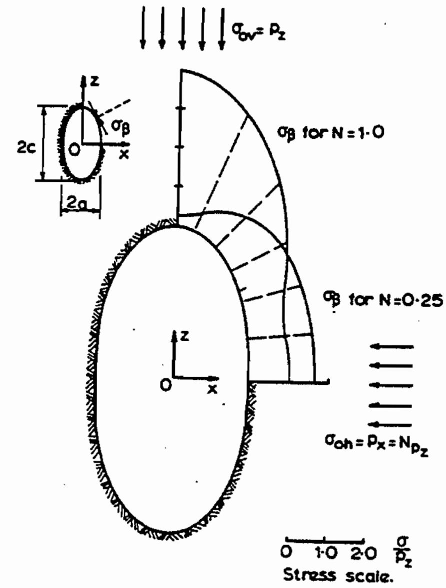


FIG.11.10 Circumferential principal stress at surface of elliptical tunnel ($a/c=\frac{1}{2}$). (Terzaghi and Richart, 1952).

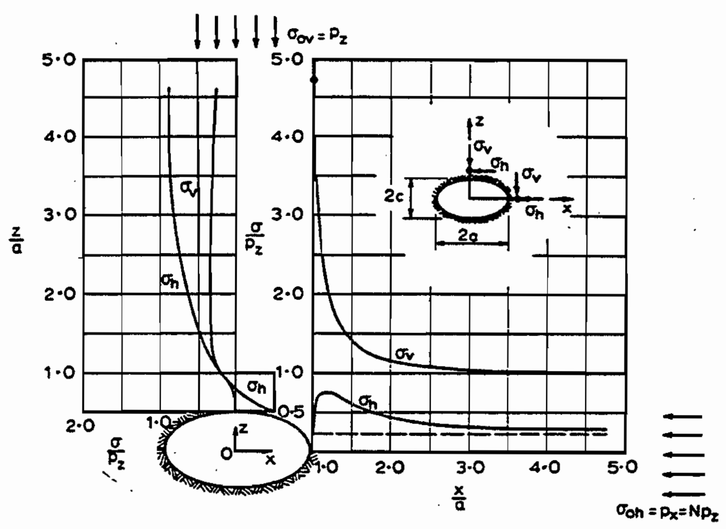


FIG.11.11 Stresses about elliptical tunnel (a/c=2.0) in homogeneous stress fields (N=0.25). (Terzaghi and Richart, 1952).

$$+ \cosh 2\alpha - Ce^{-2\alpha}]\} \quad \ldots (11.18)$$

$$\tau_\alpha = \frac{p}{8}\{\frac{2\sin 2\beta}{h^2}[-2\sinh 2\alpha - 4Be^{-2\alpha} + 2Ce^{-2\alpha}]$$

$$+ \frac{\sin 2\beta}{h^4}[\sinh 2\alpha\{1+\cosh 2\alpha - 2Be^{-2\alpha}$$

$$- C(1-e^{-2\alpha})\} - \sinh 2\alpha - Ce^{-2\alpha} - A$$

$$+ \cos 2\beta\{\sinh 2\alpha - Ce^{-2\alpha} + 2Be^{-2\alpha}\}] \quad \ldots (11.19)$$

where $A = -1 - \cosh 2\alpha_0$

$B = \frac{1}{2}e^{2\alpha_0} + \frac{3}{4} - \frac{e^{4\alpha_0}}{4}$

$C = 1 + e^{2\alpha_0}$

and the distortion factor h is determined by

$$h^2 = \sinh^2\alpha + \cos^2\beta.$$

At any point of the surface of the elliptical hole, α is equal to α_0, hence

$$(\sigma_\beta)_{\alpha=\alpha_0} = \frac{p}{2h^2}[\sinh 2\alpha_0 - 1 - e^{+2\alpha_0}\cos 2\beta] \quad \ldots (11.20)$$

At the top of the hole, for which $\beta=0$, $(\sigma_\beta)_{\alpha=\alpha_0, \beta=0} = -p$, while on the x-axis, $\beta=\pi/2$ and

$$(\sigma_\beta)_{\alpha=\alpha_0, \beta=\frac{\pi}{2}} = p(1+2\coth\alpha_0) = p(1+2\frac{a}{c}) \quad \ldots (11.21)$$

For Case b $(a<c)$, and vertical loading p,

$$\sigma_\alpha = \frac{p}{8}\{[\frac{\sin^2 2\beta}{h^4} - \frac{4\cos 2\beta}{h^2}][1-\cosh 2\alpha + 2Be^{-2\alpha}$$

$$+ C(1+e^{-2\alpha})] + \frac{\sinh 2\alpha}{2h^4}[2A + 2\sinh 2\alpha - 2Ce^{-2\alpha}$$

$$+ \cos 2\beta\{-2\sinh 2\alpha - 4Be^{-2\alpha} - 2Ce^{-2\alpha}\}]\} \quad \ldots (11.22)$$

$$\sigma_\beta = \frac{p}{8}\{\frac{\sinh 2\alpha}{h^4}[-A - \sinh 2\alpha\{1-\cos 2\beta\} + Ce^{-2\alpha}$$

$$.\{1+\cos 2\beta\} + 2Be^{-2\alpha}\cos 2\beta] + \frac{h^2}{4}[\cosh 2\alpha$$

$$.\{1-\cos 2\beta\} + Ce^{-2\alpha}\{1+\cos 2\beta\} + 2Be^{-2\alpha}$$

$$.\cos 2\beta] - \frac{\sin^2 2\beta}{h^4}[1 - \cosh 2\alpha + 2Be^{-2\alpha}$$

$$+ C(1-e^{-2\alpha})]\} \quad \ldots (11.23)$$

$$\tau_\alpha = \frac{p}{8}\{\frac{\sin 2\beta}{h^4}[\sinh 2\alpha\{-1+\cosh 2\alpha - C(1+e^{-2\alpha})$$

$$- 2Be^{-2\alpha}\} + A + \sinh 2\alpha\{1-\cos 2\beta\}$$

$$- Ce^{-2\alpha}\{1+\cos 2\beta\} - 2Be^{-2\alpha}\cos 2\beta]$$

$$- \frac{2\sin 2\beta}{h^2}[2\sinh 2\alpha + 2(C+2B)e^{-2\alpha}]\} \quad \ldots (11.24)$$

where $A = 1 - \frac{e^{2\alpha_0}}{2} - \frac{e^{-2\alpha_0}}{2}$

$B = \frac{3}{4} - \frac{e^{2\alpha_0}}{2} - \frac{e^{4\alpha_0}}{4}$

$C = e^{2\alpha_0} - 1$

and the distortion factor h is determined by

$$h^2 = \sinh^2\alpha + \sin^2\beta.$$

As with the circular tunnel, the above solutions also apply to an elliptical hole in a thin plate.

Stresses along the axes of the tunnel have been tabulated and plotted by Terzaghi and Richart (1952) for $a/c=0.5$ and 2.0, and are reproduced in Tables 11.5 and 11.6 and in Figs. 11.9 to 11.11.

To obtain the resultant stress produced by horizontal and vertical stress fields at any point in the material surrounding an elliptical tunnel having a span-to-height ratio of a_1/c_1, it is necessary to solve the two component problems first. The stresses are determined along the x and z axes for a tunnel having span-to-height ratio a_1/c_1 (case a) and then for the tunnel having this ratio $= c_1/a_1$ (case b).

TABLE 11.5
STRESSES ON AXES OF ELLIPTICAL TUNNEL
$a/c=0.5$

	$p_z=1.0$ $p_x=0$		$p_z=0$ $p_x=1.0$		$p_z=1.0$ $N=0.25$	
$\frac{z}{a}$ or $\frac{x}{a}$	σ_v	σ_h	σ_v	σ_h	σ_v	σ_h
(a) Stresses along the z-axis						
2.00	0	-1.000	0	5.000	0	0.250
2.03	-0.042	-0.790	0.254	4.423	0.021	0.316
2.06	-0.565	-0.636	0.421	3.974	0.049	0.357
2.12	-0.039	-0.413	0.630	3.281	0.118	0.408
2.35	0.155	-0.102	0.746	2.155	0.341	0.437
2.88	0.491	0.017	0.547	1.470	0.628	0.385
4.67	0.828	0.019	0.192	1.114	0.876	0.298
7.89	0.943	0.008	0.065	1.035	0.959	0.266
∞	1.000	0	0	1.000	1.000	0.250
(b) Stresses along the x-axis						
1.00	2.000	0	-1.000	0	1.750	0
1.06	1.925	0.028	-0.900	-0.013	1.700	0.025
1.11	1.859	0.069	-0.814	-0.021	1.656	0.064
1.23	1.747	0.092	-0.657	-0.026	1.583	0.085
1.59	1.476	0.166	-0.311	0.023	1.398	0.172
2.30	1.219	0.188	-0.030	0.234	1.211	0.247
4.33	1.043	0.099	0.055	0.660	1.057	0.264
7.70	1.010	0.038	0.028	0.875	1.017	0.257
∞	1.000	0	0	1.000	1.000	0.250

(Terzaghi and Richart, 1952)

TABLE 11.6
STRESSES ON AXES OF ELLIPTICAL TUNNEL
$a/c=2.0$

	$p_z=1.0$ $p_x=0$		$p_z=0$ $p_x=1.0$		$p_z=1.0$ $N=0.25$	
$\frac{z}{a}$ or $\frac{x}{a}$	σ_v	σ_h	σ_v	σ_h	σ_v	σ_h
(a) Stresses along the z-axis						
0.50	0	-1.000	0	2.000	0	-0.500
0.53	-0.013	-0.900	0.028	1.925	-0.006	-0.419
0.56	-0.021	-0.814	0.069	1.859	-0.003	-0.349
0.61	-0.026	-0.657	0.092	1.747	-0.003	-0.221
0.80	0.023	-0.311	0.166	1.476	0.065	0.058
1.15	0.234	-0.030	0.188	1.219	0.281	0.274
2.17	0.660	0.055	0.099	1.043	0.685	0.316
3.85	0.875	0.028	0.038	1.010	0.885	0.280
∞	1.000	0	0	1.000	1.000	0.250
(b) Stresses along the x-axis						
1.00	5.000	0	-1.000	0	4.750	0
1.01	4.423	0.254	-0.790	-0.042	4.226	0.243
1.03	3.974	0.421	-0.636	-0.057	3.815	0.407
1.06	3.281	0.630	-0.413	-0.039	3.178	0.621
1.18	2.155	0.746	-0.102	0.155	2.129	0.785
1.44	1.470	0.547	0.017	0.491	1.474	0.670
2.33	1.114	0.192	0.019	0.828	1.119	0.399
3.94	1.035	0.065	0.008	0.943	1.036	0.301
∞	1.000	0	0	1.000	1.000	0.250

(Terzaghi and Richart, 1952)

11.1.4 A NEARLY RECTANGULAR TUNNEL IN AN INFINITE MASS (Fig.11.12)

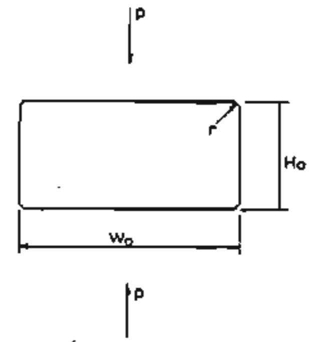


FIG.11.12

Obert and Duval (1967) summarize the results of Brock for the boundary stress around a square tunnel with rounded corners. These results are plotted in Fig.11.13.

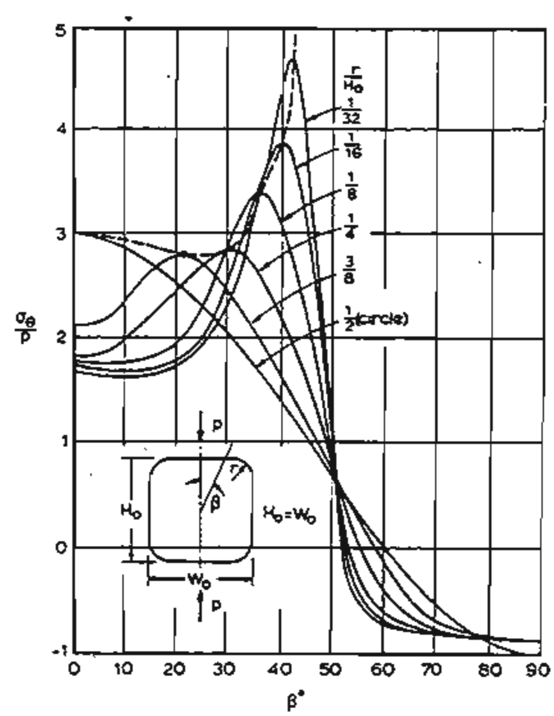


FIG.11.13 Boundary stress around square opening with rounded corners (Obert and Duvall, 1967).

For a rectangular tunnel with rounded corners, the maximum stress around the tunnel (obtained by Heller, Brock and Bart, 1958) is shown in Fig.11.14 for various values of r/W_O and H_O/W_O.

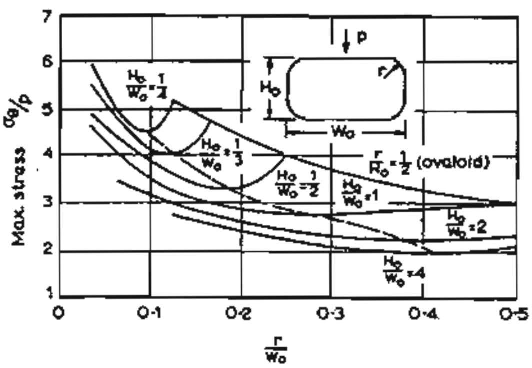


FIG.11.14 Maximum stress for rectangular openings with rounded corners (Heller, Brock and Bart, 1958).

11.1.5 FLAT ELLIPTICAL CRACK IN AN INFINITE MASS (Fig.11.15)

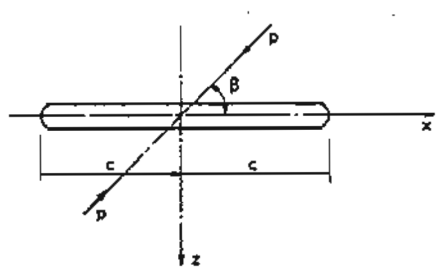


FIG.11.15

For a uniform stress field p per unit area inclined at an angle β to the plane of the crack, Jaeger and Cook (1969) quote the following results in terms of elliptical coordinates ξ and η:

$$\sigma_\xi + \sigma_\eta = p\cos 2\beta + \alpha p[(1-\cos 2\beta)\sinh 2\xi - \sin 2\beta \sin 2\eta] \quad \ldots (11.25)$$

$$\sigma_\xi - \sigma_\eta = \alpha p \cosh 2\xi \cos 2(\eta-\beta) + \alpha^2 p\{(1-\cos 2\beta) \cdot (\cos 2\eta - 1)\sinh 2\xi - \cosh 2\xi \cos 2\beta + \cos 2(\eta-\beta) - \cosh 2\xi \sin 2\beta \sin 2\eta\} \quad \ldots (11.25)$$

$$\tau_{\xi\eta} = \tfrac{1}{2}p\alpha \sinh 2\xi \sin 2(\beta-\eta) + \tfrac{1}{2}p\alpha^2\{\sinh 2\xi \sin 2\beta(\cos 2\eta - 1) + (1-\cos 2\beta)(\cosh 2\xi - 1)\sin 2\eta\} \quad \ldots (11.26)$$

where $\alpha = (\cosh 2\xi - \cos 2\eta)^{-1}$

and the elliptical coordinates are related to the Cartesian corodinates as

$$x = c\cosh\xi \cos\eta$$
$$z = c\sinh\xi \sin\eta$$

On the surface of the crack, $\xi = 0$ and $\cos\eta = \frac{x}{c}$.

For the case of a crack subjected to pure shear q per unit area,

$$\sigma_\xi = q(\cosh 2\xi - 1)(\alpha - \alpha^2)\sin 2\eta \quad \ldots (11.27)$$

$$\sigma_\eta = q[\alpha^2(\cosh 2\xi - 1) - \alpha(\cosh 2\xi + 1)]\sin 2\eta \quad \ldots (11.28)$$

$$\tau_{\xi\eta} = q[\alpha\cos 2\eta - \alpha^2(1-\cos 2\eta)]\sinh 2\xi \quad \ldots (11.29)$$

For an uniaxial stress field p perpendicular to the plane of the crack, the vertical displacement at the surface of the crack is

$$\rho_z = 2(1-\nu^2)\frac{p}{E}(c^2 - x^2)^{\frac{1}{2}} \quad \ldots (11.30)$$

11.1.6 SPHERICAL CAVITY IN AN INFINITE MASS (Fig.11.16)

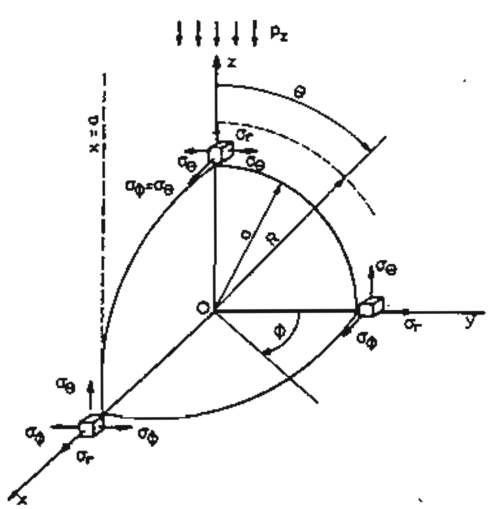


FIG.11.16

Terzaghi and Richart (1952) quote the following solutions obtained by Neuber (1946) for uniform vertical loading p_z per unit area:

$$\sigma_R = [-p_z - \frac{36B}{R^5} + \frac{(8+\alpha)C}{R^3}]\sin^2\theta + p_z + \frac{24B}{R^5} - \frac{2A}{R^3} - \frac{(6+2\alpha)C}{R^3} \quad \ldots (11.31)$$

$$\sigma_\theta = [p_z + \frac{21B}{R^5} + \frac{(1-\alpha)C}{R^3}]\sin^2\theta - \frac{12B}{R^5} + \frac{A}{R^3} + \frac{(2\alpha-1)C}{R^3} \quad \ldots (11.32)$$

$$\sigma_\phi = [\frac{15B}{R^5} + \frac{3(1-\alpha)C}{R^3}]\sin^2\theta - \frac{12B}{R^5} + \frac{A}{R^3} + \frac{(2\alpha-1)C}{R^3} \quad \ldots (11.33)$$

$$\tau_{r\theta} = [-p_z + \frac{24B}{R^5} + \frac{(\alpha-4)C}{R^3}]\sin\theta\cos\theta;\ \tau_{\theta\phi}=0;\ \tau_{R\phi}=0 \quad \ldots (11.34)$$

in which

R, θ, and ϕ are the polar co-ordinates

$\alpha = 2(1-\nu)$

p_z = the uniformly distributed vertical stress

$A = -\frac{(2+5\alpha)}{2(4+5\alpha)} p_z$

$B = \frac{p_z}{4+5\alpha}$

$C = \frac{5p_z}{4+5\alpha}$

The results from the above equations for the uniaxial stress field may be superposed to obtain the results for biaxial or triaxial stress fields. It should be noted that, in contrast to the plane strain cases of a circular or an elliptical hole, the stresses in this case are dependent on ν.

The influence of ν on the tangential stress σ_θ has been evaluated by Terzaghi and Richart for a point A on the equator, at mid-height of the cavity, and at the centre C of the roof of the cavity. The results are shown in Table 11.7.

TABLE 11.7
INFLUENCE OF ν ON STRESSES AROUND SPHERICAL CAVITY

ν	σ_θ/p_z at A	σ_θ/p_z at C
0	1.929	-0.214
0.20	2.000	-0.500
0.30	2.045	-0.682
0.50	2.167	-1.167

The stresses along the axes of the cavity are given in Table 11.8 for $\nu=0.2$. The case of uniaxial loading $(p_z=1,\ p_x=p_y=0)$ and triaxial loading for $N=0.25$ $(p_z=1,\ p_x=p_y=0.25)$ are considered.

TABLE 11.8
STRESSES ALONG THE AXES OF A SPHERICAL CAVITY $(\nu=0.20)$

	Along the z-axis $(\theta=0)$		Along the x-axis $(\theta=\frac{\pi}{2})$		
$\frac{R}{a}$	σ_R	$\sigma_\theta=\sigma_\phi$	σ_R	σ_θ	σ_ϕ
(a) For the case $p_z=1.0,\ p_x=p_y=0$					
1.00	0	-0.500	0	2.000	0
1.05	-0.025	-0.352	0.080	1.804	-0.020
1.10	-0.012	-0.245	0.130	1.654	-0.033
1.20	0.068	-0.113	0.177	1.446	-0.044
1.40	0.279	-0.004	0.179	1.231	-0.045
1.70	0.530	0.031	0.133	1.104	-0.033
2.00	0.688	0.031	0.094	1.054	-0.023
2.50	0.829	0.022	0.054	1.024	-0.013
3.00	0.897	0.014	0.034	1.012	-0.008
6.00	0.986	0.002	0.005	1.001	-0.001
∞	1.000	0	0	1.000	0
(b) For the case $p_z=1.0,\ p_x=p_y=0.25$ $(N=0.25)$					
1.00	0	0	0	1.875	0.375
1.05	0.016	0.094	0.094	1.711	0.343
1.10	0.053	0.158	0.160	1.584	0.320
1.20	0.156	0.238	0.238	1.407	0.289
1.40	0.368	0.293	0.248	1.219	0.262
1.70	0.597	0.298	0.299	1.103	0.251
2.00	0.735	0.289	0.289	1.056	0.248
2.50	0.855	0.274	0.274	1.026	0.248
3.00	0.914	0.270	0.267	1.014	0.249
6.00	0.989	0.252	0.252	1.002	0.250
∞	1.000	0.250	0.250	1.000	0.250

(Terzaghi and Richart, 1952)

Stresses above the spherical cavity, along the line $x=a$, have been tabulated by Terzaghi and Richart (1952) for $\nu=0.2$, and are reproduced in Table 11.9.

The stresses above a spherical cavity for the case $N=0.25$ and $\nu=0.2$ are plotted in Fig.11.17.

TABLE 11.9
STRESSES ABOVE A SPHERICAL CAVITY ALONG THE LINE $x=a$ ($\nu=0.20$)
(Terzaghi and Richart, 1952)

	$\frac{R}{a}$	$\frac{z}{a}$	σ_R	σ_θ	σ_ϕ	$\tau_{r\theta}$	σ_h	σ_v	τ_{vh}
(a) For the case	1.00	0	0	2.000	0	0	0	2.000	0
$p_z=1.0,\ p_x=p_y=0$	1.05	0.32	0.071	1.603	-0.051	-0.086	0.163	1.511	-0.375
	1.10	0.46	0.120	1.324	-0.070	-0.193	0.183	1.261	-0.330
	1.20	0.66	0.144	0.970	-0.065	-0.357	0.067	1.046	-0.242
	1.40	0.98	0.228	0.626	-0.025	-0.496	-0.073	0.927	-0.189
	1.70	1.37	0.393	0.402	0.009	-0.506	-0.082	0.877	-0.160
	2.00	1.73	0.539	0.287	0.018	-0.460	-0.048	0.875	-0.121
	2.50	2.29	0.705	0.182	0.016	-0.383	-0.015	0.901	-0.069
	3.00	2.83	0.801	0.125	0.012	-0.323	-0.003	0.929	-0.039
	6.00	5.92	0.959	0.030	0.002	-0.165	0.001	0.988	-0.003
	∞	∞	1.000	0	0	0	0	1.000	0
(b) For the case	1.00	0	0	-0.500	-0.500	0	0	-0.500	0
$p_x=1.0,\ p_z=p_y=0$	1.05	0.32	-0.015	-0.144	-0.310	0.086	0.023	-0.182	-0.032
	1.10	0.46	0.013	0.084	-0.208	0.193	0.171	-0.074	-0.153
	1.20	0.66	0.101	0.364	-0.092	0.357	0.510	-0.046	-0.260
	1.40	0.98	0.230	0.601	-0.024	0.496	0.908	-0.077	-0.195
	1.70	1.37	0.271	0.733	-0.011	0.505	1.054	-0.051	-0.064
	2.00	1.73	0.242	0.799	-0.010	0.460	1.058	-0.017	-0.011
	2.50	2.29	0.178	0.863	-0.008	0.383	1.034	0.007	0.009
	3.00	2.83	0.129	0.902	-0.006	0.323	1.019	0.012	0.009
	6.00	5.92	0.032	0.974	-0.001	0.165	1.002	0.004	0.001
	∞	∞	0	1.000	0	0	1.000	0	0
(c) For the case	1.00	0	0	0	2.000	0	0	0	0
$p_y=1.0,\ p_z=p_x=0$	1.05	0.32	0.080	-0.020	1.804	0	0.071	-0.011	0.029
	1.10	0.46	0.130	-0.033	1.654	0	0.102	-0.004	0.062
	1.20	0.66	0.177	-0.044	1.446	0	0.109	0.023	0.102
	1.40	0.98	0.179	-0.045	1.231	0	0.069	0.065	0.112
	1.70	1.37	0.133	-0.033	1.104	0	0.024	0.076	0.079
	2.00	1.73	0.094	-0.023	1.054	0	0.006	0.065	0.051
	2.50	2.29	0.054	-0.013	1.024	0	-0.003	0.043	0.025
	3.00	2.83	0.034	-0.008	1.012	0	-0.004	0.030	0.013
	6.00	5.92	0.005	-0.001	1.001	0	-0.001	0.004	0.009
	∞	∞	0	0	1.000	0	0	0	0
(d) For the case	1.00	0	0	1.875	0.375	0	0	1.875	0
$p_z=1.0,$	1.05	0.32	0.087	1.560	0.320	-0.065	0.187	1.463	-0.374
	1.10	0.46	0.156	1.337	0.292	-0.145	0.251	1.242	-0.353
$p_x=p_y=0.25$	1.20	0.66	0.213	1.050	0.274	-0.268	0.222	1.041	-0.281
($N=0.25$)	1.40	0.98	0.330	0.765	0.277	-0.372	0.171	0.924	-0.210
	1.70	1.37	0.493	0.577	0.282	-0.380	0.188	0.883	-0.157
	2.00	1.73	0.623	0.481	0.279	-0.345	0.218	0.886	-0.111
	2.50	2.29	0.763	0.395	0.270	-0.287	0.243	0.914	-0.060
	3.00	2.83	0.842	0.349	0.264	-0.242	0.251	0.940	-0.034
	6.00	5.92	0.968	0.273	0.252	-0.124	0.252	0.990	-0.003
	∞	∞	1.000	0.250	0.250	0	0.250	1.000	0

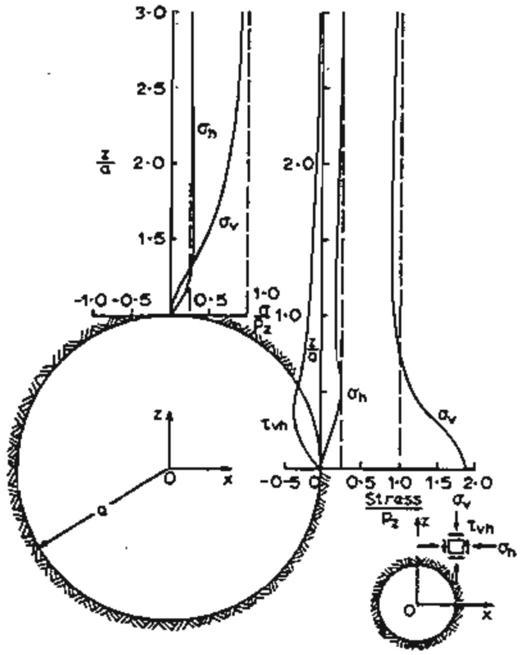


FIG.11.17 Stresses above a spherical cavity. N=0.25. (Terzaghi and Richart, 1952).

TABLE 11.10
STRESSES ALONG THE z-AXIS ABOVE SPHEROIDAL CAVITIES
(ν=0.20)

$\frac{z}{a}$	For p_z=1.0; p_x=p_y=0 σ_v	σ_h	For p_z=0; p_x=p_y=1.0 σ_v	σ_h	For p_z=1.0 N=0.25 σ_v	σ_h
(a) $\left(\frac{a}{c} = 5.0\right)$						
0.20	0	-0.655	0	1.326	0	-0.323
0.30	-0.012	-0.468	0.002	1.255	-0.007	-0.154
0.40	0.008	-0.312	0.003	1.193	0.009	-0.014
0.60	0.138	-0.101	0.005	1.105	0.139	0.175
1.00	0.536	0.031	0.004	1.028	0.537	0.288
1.30	0.763	0.025	0.003	1.010	0.764	0.278
∞	1.000	0	0	1.000	1.000	0.250
(b) $\left(\frac{a}{c} = 50\right)$						
0.02	0	-0.696	0	1.037	0	-0.437
0.10	-0.001	-0.571	0.000	1.032	-0.001	-0.312
0.20	0.007	-0.425	0.000	1.025	0.007	-0.169
0.40	0.081	-0.188	0.000	1.015	0.081	0.066
0.60	0.230	-0.045	0.000	1.008	0.230	0.207
1.00	0.595	0.037	0.000	1.002	0.595	0.288
1.30	0.793	0.029	0.000	1.001	0.793	0.280
∞	1.000	0	0	1.000	1.000	0.250

(Terzaghi and Richart, 1952)

11.1.7 SPHEROIDAL CAVITY IN AN INFINITE MASS

Detailed treatments of this problem have been made by Neuber (1937), Sadowsky and Sternberg (1947), Edwards (1951) and Terzaghi and Richart (1952).

Stresses are tabulated and plotted by Terzaghi and Richart (1952) for two cases: a/c=5.0 and a/c=50.0 (see Fig.11.20 for definition of a and c). In both cases, x is the major semi-axis of the ellipse and ν=0.2. Stresses on the z-axis are tabulated in Table 11.10 while stresses above the equator of the spheroid (i.e. at x=a) are tabulated in Tables 11.11 and 11.12.

The distribution of stresses along various vertical lines is plotted in Figs. 11.18 and 11.19.

The variation of circumferential principal stress at the roof and at mid-height on the cavity with a/c is shown in Figs. 11.20 and 11.21 which also show the stresses for the elliptical tunnel.

TABLE 11.11
STRESSES ABOVE EQUATOR OF SPHEROID (at x=a)
$\frac{a}{c}$ = 5.0, ν=0.2

	$\frac{z}{a}$	σ_h	σ_v	τ_{vh}
(a) For the loading conditions, p_z=1.0, p_x=p_y=0	0	0	7.059	0
	0.03	0.932	3.342	-1.185
	0.07	0.506	2.178	-0.690
	0.13	0.153	1.599	-0.404
	0.21	-0.063	1.279	-0.375
	0.30	-0.129	1.124	-0.334
	0.40	-0.150	1.028	-0.305
	1.05	-0.073	0.878	-0.166
	1.82	-0.015	0.909	-0.075
	∞	0	1.000	0
(b) For the loading conditions, p_z=0, p_x=p_y=1.0	0	0	-0.870	0
	0.03	0.039	-0.221	-0.129
	0.07	0.518	-0.119	-0.153
	0.13	0.760	-0.113	-0.157
	0.21	0.915	-0.061	-0.061
	0.30	0.974	-0.042	-0.048
	0.40	1.004	-0.027	-0.024
	1.05	1.022	0.013	0.018
	1.82	1.008	0.015	0.015
	∞	1.000	0	0
(c) For the loading conditions, p_z=1.0, N=0.25	0	0	6.842	0
	0.03	0.970	3.287	-1.217
	0.07	0.635	2.148	-0.728
	0.13	0.343	1.570	-0.444
	0.21	0.166	1.264	-0.387
	0.30	0.115	1.113	-0.346
	0.40	0.101	1.021	-0.311
	1.05	0.182	0.881	-0.161
	1.82	0.237	0.913	-0.071
	∞	0.250	1.000	0

(Terzaghi and Richart, 1952)

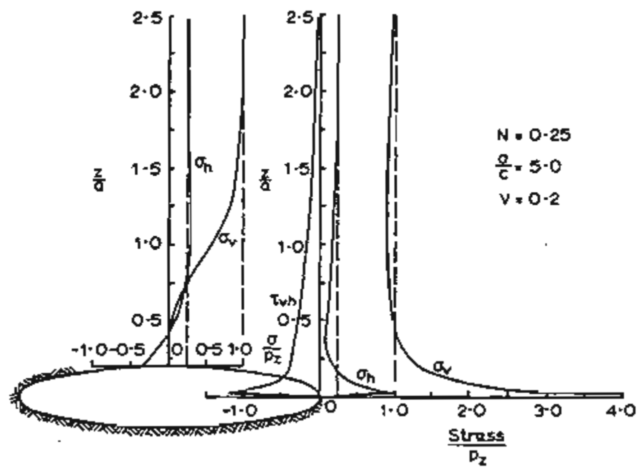


FIG.11.18 Stresses above a spheroidal cavity. a/c=5.0 (Terzaghi and Richart, 1952).

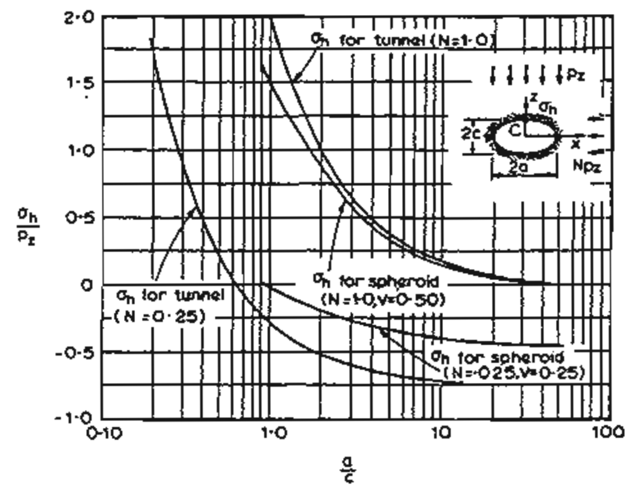


FIG.11.20 Circumferential principal stress σ_h at centre of roof of cavity (Terzaghi and Richart, 1952).

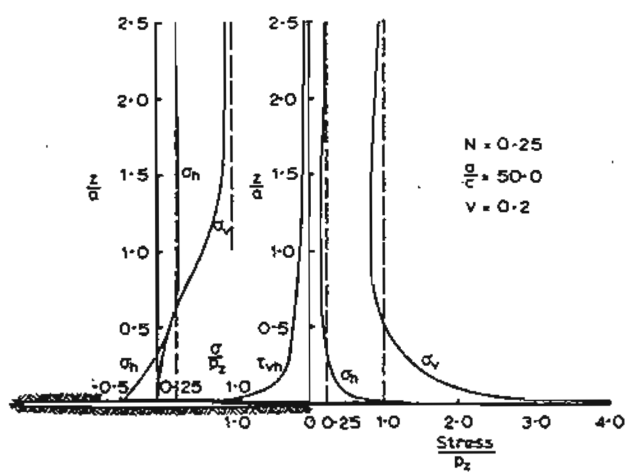


FIG.11.19 Stresses above a spheroidal cavity. a/c=50 (Terzaghi and Richart, 1952).

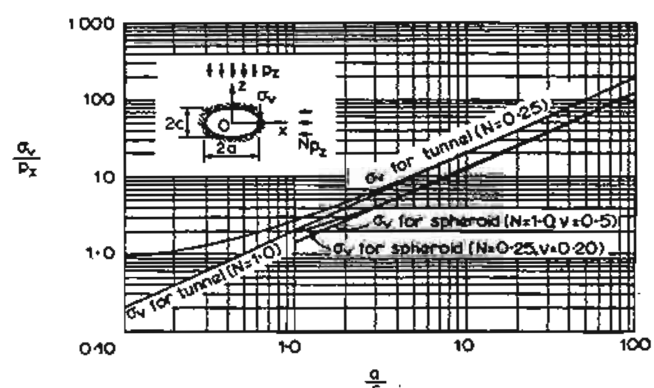


FIG.11.21 Circumferential principal stress σ_v at mid-height of wall surface of cavity (Terzaghi and Richart, 1952).

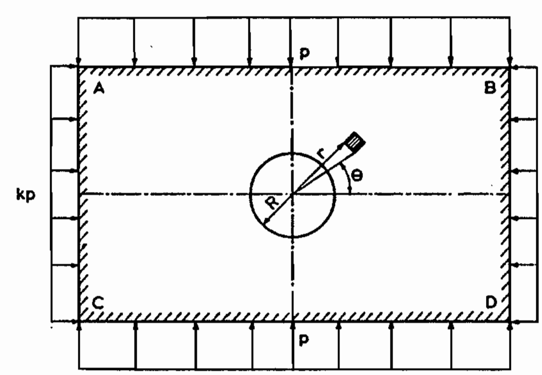
TABLE 11.12
STRESSES ABOVE EQUATOR OF SPHEROID (at $x=a$)
$\frac{a}{c} = 50,\ \nu=0.2$

	$\frac{z}{a}$	σ_h	σ_v	τ_{vh}
(a) For the loading conditions,	0	0	63.537	0
$p_z=1.0$,	0.01	1.081	4.787	-1.648
$p_x=p_y=0$	0.04	0.317	2.580	-0.738
	0.31	0.000	1.257	-0.267
	1.03	-0.100	0.841	-0.147
	2.43	0.000	0.948	-0.031
	∞	0	1.000	0
(b) For the loading conditions,	0	0	-0.987	0
$p_z=0$,	0.01	0.693	-0.142	-0.050
$p_x=p_y=1.0$	0.04	0.950	-0.030	-0.046
	0.31	0.973	-0.001	-0.010
	1.03	1.007	0.009	-0.001
	2.43	1.000	0.000	0.001
	∞	1.000	0	0
(c) For the loading conditions	0	0	0	0
$p_z=1.0$,	0.01	1.254	4.823	-1.661
$N=0.25$	0.04	0.554	2.573	-0.750
	0.31	0.243	1.256	-0.269
	1.03	0.152	0.843	-0.147
	2.43	0.250	0.984	-0.031
	∞	0.250	1.000	0

(Terzaghi and Richart, 1952)

11.2 Lined Openings

11.2.1 CIRCULAR TUNNEL IN AN INFINITE MASS (Plane strain) (Fig.11.22)



Note: Boundaries are far away from tunnel.

FIG.11.22

This problem has been considered in detail by Burns and Richard (1964) and Hoeg (1968), and is relevant where stresses are applied after tunnel excavation.

The stress and displacement solutions are as follows:

$$\sigma_r = \tfrac{1}{2}p\{(1+k)[1-a_1(\tfrac{R}{r})^2] - (1-k)[1-3a_2(\tfrac{R}{r})^4 - 4a_3(\tfrac{R}{r})^2]\cos2\theta\} \quad \ldots (11.35)$$

$$\sigma_\theta = \tfrac{1}{2}p\{(1+k)[1+a_1(\tfrac{R}{r})^2] + (1-k)[1-3a_2(\tfrac{R}{r})^4]\cos2\theta\} \quad \ldots (11.36)$$

$$\tau_{r\theta} = \tfrac{1}{2}p(1-k)[1+3a_2(\tfrac{R}{r})^4 + 2a_3(\tfrac{R}{r})^2]\sin2\theta \quad \ldots (11.37)$$

$$\rho_r = \tfrac{1}{2}p\,\frac{r}{M}\{(1+k)(1-\nu)[1+\frac{a_1}{1-2\nu}(\tfrac{R}{r})^2] - (1-k)\frac{1-\nu}{1-2\nu}\,.\,[1+a_2(\tfrac{R}{r})^4 + 4(1-\nu)a_3(\tfrac{R}{r})^2]\cos2\theta\} \quad \ldots (11.38)$$

$$\rho_\theta = \tfrac{1}{2}p\,\frac{r}{M}\,\frac{1-\nu}{1-2\nu}(1-k)[1-a_2(\tfrac{R}{r})^4 + 2(1-2\nu)a_3(\tfrac{R}{r})^2]\sin2\theta \quad \ldots (11.39)$$

where ν = Poisson's ratio of mass

M = one dimensional modulus of mass

$= \frac{E(1-\nu)}{(1+\nu)(1-2\nu)}$

E_c, ν_c = Young's modulus and Poisson's ratio of material in tunnel wall

D = average tunnel diameter $= 2R$

t = thickness of tunnel wall

C = compressibility ratio

$\frac{1}{2}\frac{(1-\nu_c^2)}{(1-\nu)}\frac{M}{E_c}(\frac{D}{t})$

F = flexibility ratio

$= \frac{1}{4}\frac{(1-2\nu)}{(1-\nu)}(1-\nu_c^2)\frac{M}{E_c}(\frac{D}{t})^3$

(Note that a system with both C and F zero signifies a perfectly rigid tunnel wall).

The values a_1, a_2, a_3 are constants which depend on the boundary condition at the tunnel-mass interface. Two cases are considered:

(a) no slip (i.e. full adhesion) on interface, for which

$$a_1 = \frac{(1-2\nu)(C-1)}{(1-2\nu)C+1} \quad \ldots (11.40a)$$

$$a_2 = \frac{(1-2\nu)(1-C)F - \frac{1}{2}(1-2\nu)^2C+2}{[(3-2\nu)+(1-2\nu)C]F+(\frac{5}{2}-8\nu+6\nu^2)C+6-8\nu} \quad \ldots (11.40b)$$

$$a_3 = \frac{[1+(1-2\nu)C]F - \frac{1}{2}(1-2\nu)C-2}{[(3-2\nu)+(1-2\nu)C]F+(\frac{5}{2}-8\nu+6\nu^2)C+6-8\nu} \quad \text{... (11.40c)}$$

(b) free slip on interface, for which

$$a_1 = \frac{(1-2\nu)(C-1)}{(1-2\nu)C+1} \quad \text{... (11.41a)}$$

$$a_2 = \frac{2F+1-2\nu}{2F+5-6\nu} \quad \text{... (11.41b)}$$

$$a_3 = \frac{2F-1}{2F+5-6\nu} \quad \text{... (11.41c)}$$

The variation of the radial contact stress is shown in Fig.11.23 for $\theta=0$ and $90°$ and for both no slip and free slip.

The variation of radial stress σ_r with radial distance from the cylinder is shown in Fig.11.24 for $F=10$ and $C=0.03$.

The variation of radial deflection at the tunnel crown with flexibility ratio F is shown in Fig. 11.25.

Distributions of stress, displacement, maximum moment and thrust around the tunnel have also been presented by Burns and Richard (1964) for a number of specific cases. Krizek et al (1971) have summarized the results of Burns and Richard's calculations, and dimensionless plots of deflection, thrust, moment and shear forces are shown in Figs. 11.26 to 11.29. The sign definition is given in Fig.11.26. In all cases, the solutions are for a undirectional uniformly distributed pressure p acting at an infinite distance from the opening, i.e. $k=0$ in Hoeg's notation, and for c small.

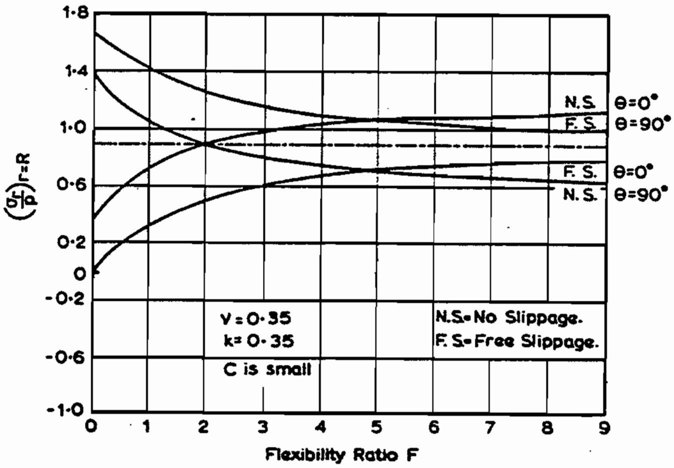


FIG.11.23 Radial contact stress versus flexibility ratio (Hoëg, 1968).

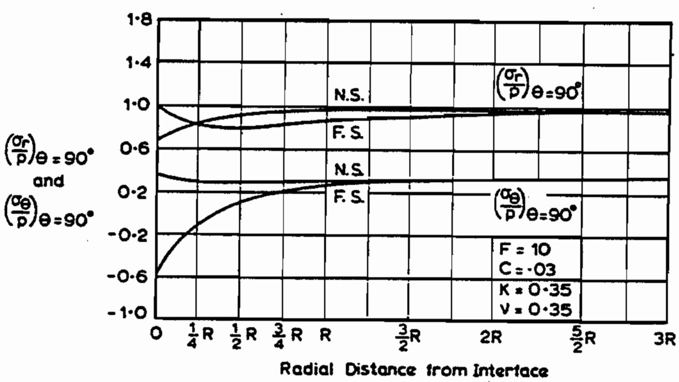


FIG.11.24 Variation in stress with distance from interface (Hoëg, 1968).

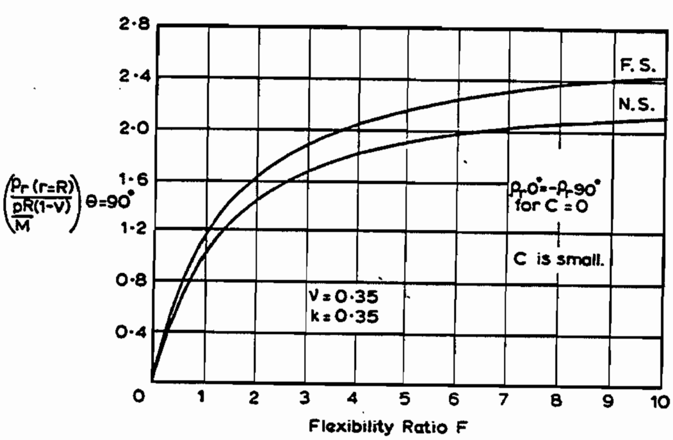


FIG.11.25 Radial cylinder deflection versus flexibility ratio (Hoëg, 1968).

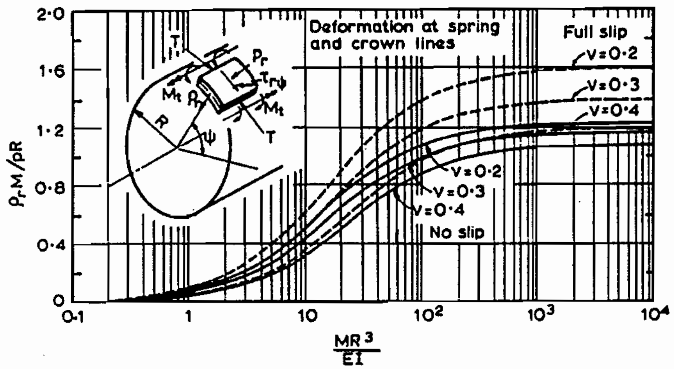


FIG.11.26 Deformation versus flexibility (Kay and Krizek, 1970).

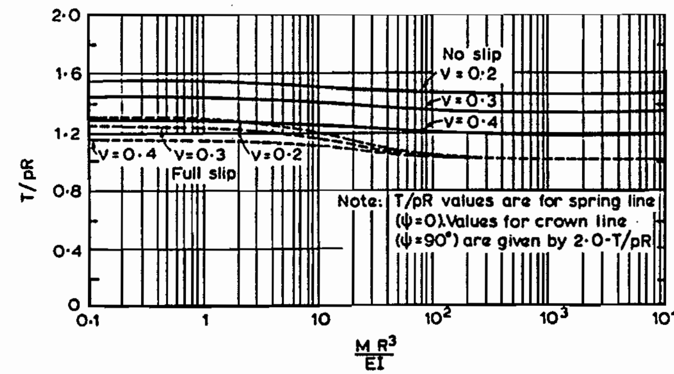


FIG.11.27 Thrust versus flexibility (Kay and Krizek, 1970).

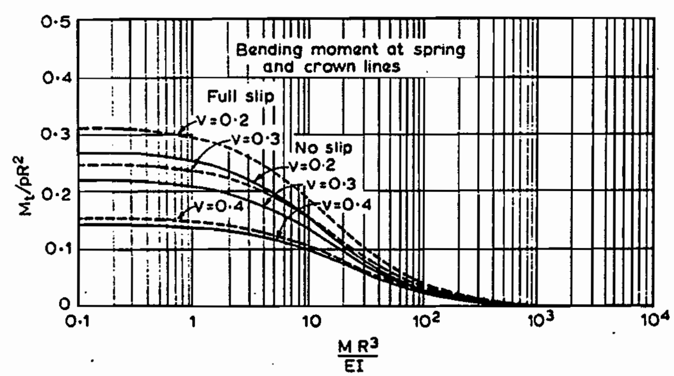


FIG.11.28 Bending moment versus flexibility (Kay and Krizek, 1970).

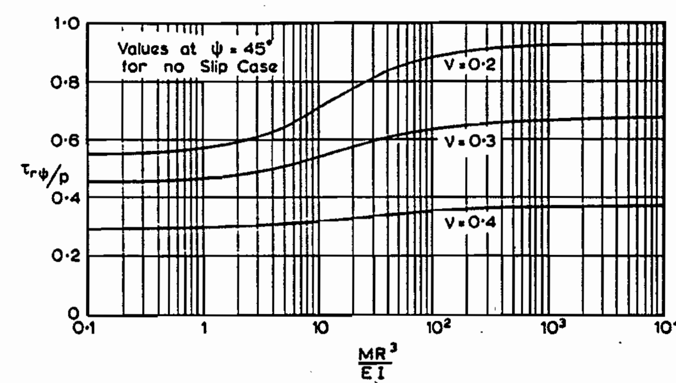


FIG.11.29 Shear stress versus flexibility (Kay and Krizek, 1970).

Chapter 12
RAFT FOUNDATIONS

12.1 Strip Foundations on a Semi-Infinite Mass

12.1.1 BEHAVIOUR IN TRANSVERSE DIRECTION

(a) *Smooth Strip Subjected to Uniform Pressure*

Borowicka (1939) has obtained solutions for the distribution of contact pressure p beneath a strip subjected to uniform pressure q. These solutions are shown in Fig.12.1. The relative stiffness K is defined as

$$K = \frac{1}{6}\frac{(1-\nu_s^2)}{(1-\nu_p^2)}\frac{E_p}{E_s}\left(\frac{t}{b}\right)^3 \qquad \ldots (12.1)$$

where E_p, ν_p = elastic moduli of strip
t = strip thickness
b = half width of strip
E_s, ν_s = elastic moduli of mass.

(b) *Smooth Strip Subjected to Central Line Load*

Borowicka (1939) has obtained the solutions shown in Fig.12.2 for the contact pressure distribution p beneath the strip. $p_{av}=P/2b$ where P is the line load per unit length, and K is defined in Equation (12.1).

(c) *Rough Strip Subjected to Uniform Pressure*

Lee (1963) obtained solutions for the contact normal and shear stresses and the bending moments and the shear forces within the strip. For ν_s of the soil $=0$, the contact normal stresses are shown in Fig.12.3, contact shear stresses in Fig.12.4, distributions of bending moment in Fig.12.5, and distributions of shearing force in Fig.12.6. In these figures the relative stiffness K is defined as

$$K = \frac{2E_p I_p}{(1-\nu_p)^2 E_s b^3} \qquad \ldots (12.1)$$

where E_p, ν_p = the elastic moduli of strip
$t^3/12 = I_p$ = moment of inertia of strip
E_s = Young's modulus of soil
b = half-width of strip.

The variation of maximum shearing force, maximum bending moment and maximum differential settlement with K is shown in Figs.12.7, 12.8 and 12.9 respectively.

It should be noted that if $\nu_s=0.5$, the solutions for a rough strip are identical with those for a smooth strip.

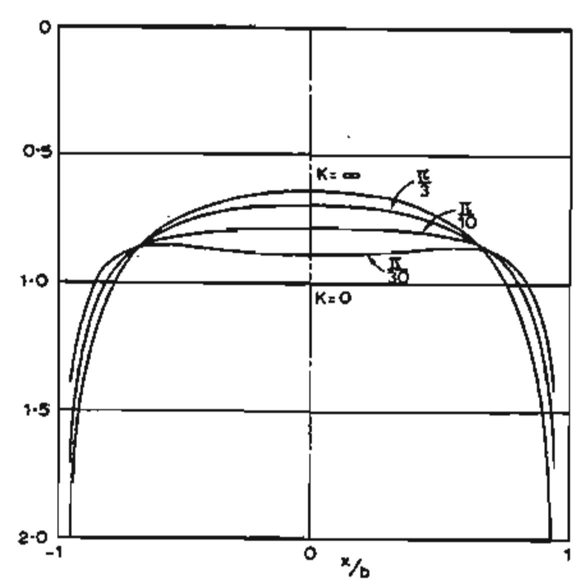


FIG.12.1 Contact pressure distribution beneath uniformly loaded smooth strip (Borowicka,1939).

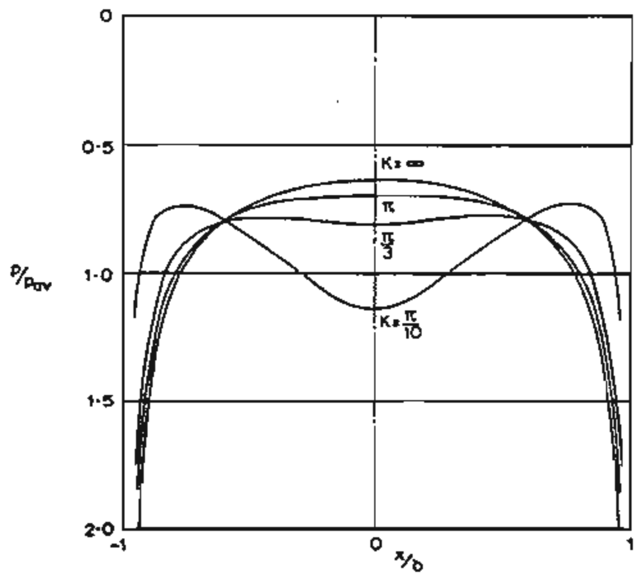


FIG.12.2 Contact pressure distribution beneath smooth strip with line load (Borowicka,1939).

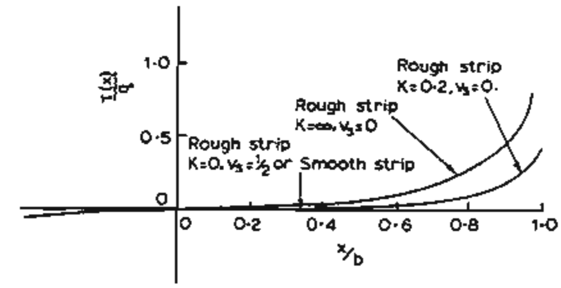


FIG.12.4 Contact shear stress beneath uniformly loaded rough strip (Lee, 1963).

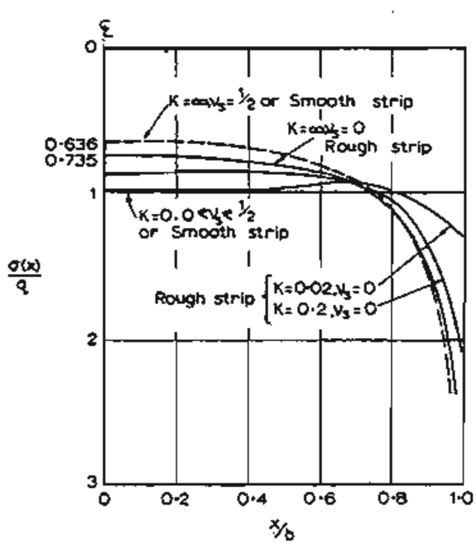


FIG.12.3 Vertical contact stress beneath uniformly loaded rough strip (Lee, 1963).

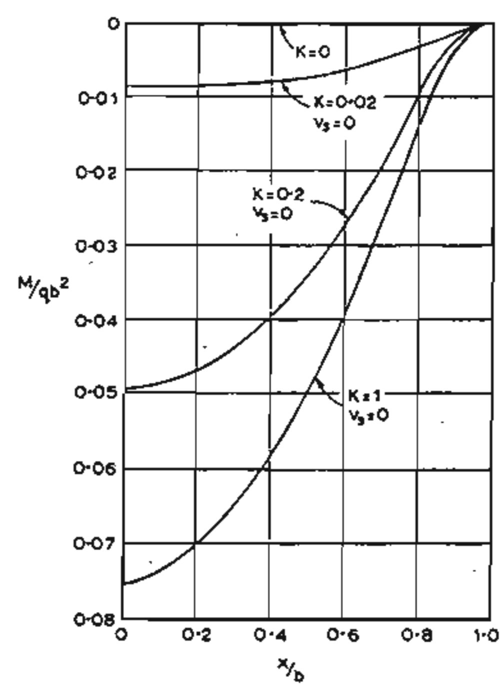


FIG.12.5 Bending moments in uniformly loaded rough strip (Lee, 1963).

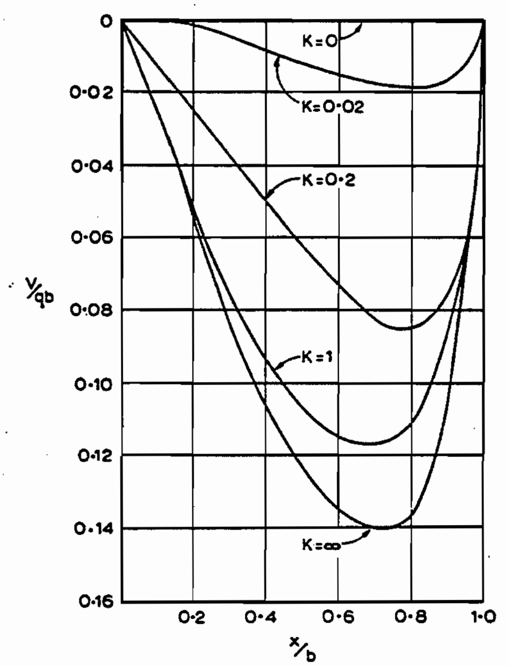


FIG.12.6 Shear force in uniformly loaded rough strip. ν_s=0. (Lee, 1963).

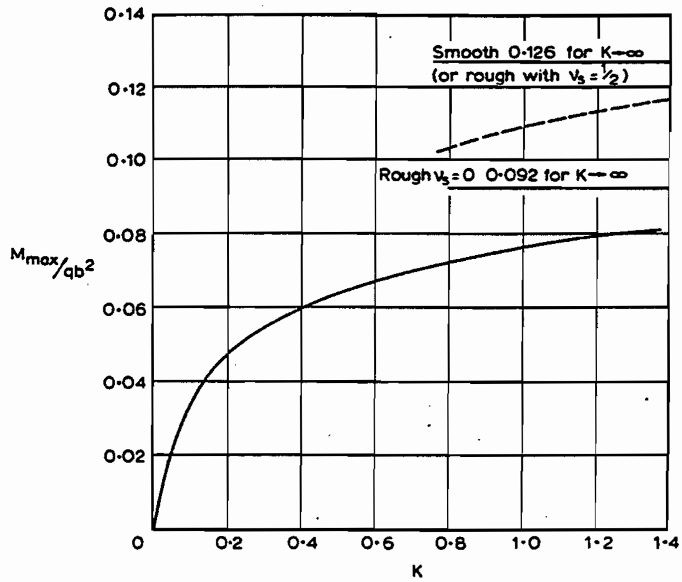


FIG.12.8 Maximum bending moment in uniformly loaded strip (Lee, 1963).

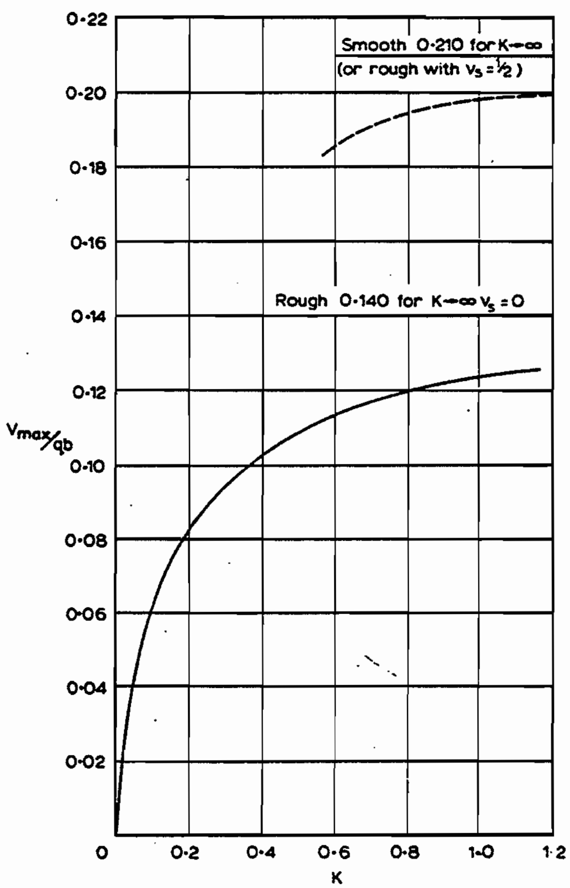


FIG.12.7 Maximum shear force in uniformly loaded strip (Lee, 1963).

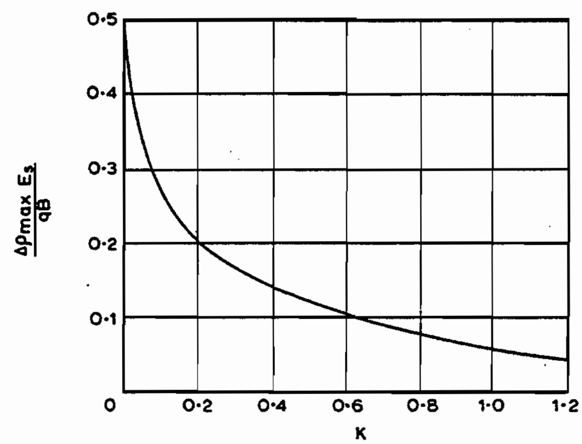


FIG.12.9 Maximum differential deflection in uniformly loaded rough strip. ν_s=0. (Lee,1963).

12.1.2 BEHAVIOUR IN LONGITUDINAL DIRECTION

(a) *Vertical and Moment Loading on Infinite Strip* (Fig.12.10)

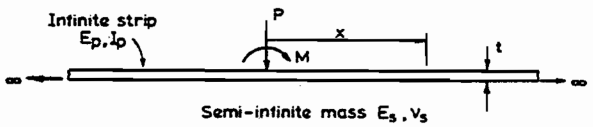


FIG.12.10

Biot (1937) obtained a solution for the maximum bending moment beneath an infinitely long smooth strip loaded by a concentrated load. Vesic (1961) extended Biot's work and obtained the following solutions for deflection, rotation, moment, shear and pressure, for both vertical loading and for moment loading.

Concentrated Loading:

Deflection $\rho = \dfrac{Pc^3}{\pi E_p I_p} J_0(x)$ ... (12.3)

Slope $\theta = -\dfrac{Pc^2}{\pi E_p I_p} J_1(x)$ ... (12.4)

Moment $M = \dfrac{Pc}{\pi} J_2(x)$ ... (12.5)

Shear $V = -\dfrac{P}{\pi} J_3(x)$ ... (12.6)

Pressure $p = \dfrac{P}{c\pi} J_4(x)$ ... (12.7)

Moment Loading: ... (12.8)

Deflection $\rho = \dfrac{Mc^2}{\pi E_p I_p} J_1(x)$ ... (12.8)

Slope $\theta = \dfrac{Mc}{\pi E_p I_p} J_2(x)$ ... (12.9)

Moment $M = \dfrac{M}{\pi} J_3(x)$ ... (12.10)

Shear $V = -\dfrac{M}{c\pi} J_4(x)$ ... (12.11)

Pressure $p = \dfrac{M}{c^2\pi} J_5(x)$ ... (12.12)

In the above equations,

$$c = \left[C(1-\nu_s^2)\frac{E_p I_p}{E_s}\right]^{1/3} \quad \text{... (12.13)}$$

$C = 1.00$ assuming constant reaction pressure across width of strip

or $1.00 < C < 1.13$ assuming constant deflection across width of strip

$E_p I_p$ = stiffness of beam ($I_p = t^3/12$)

E_s, ν_s = elastic moduli of underlying mass

b = half-width of strip.

P, M & V are per unit width.

The functions $J_0(x)$ etc. are given approximately as follows:

$$J_0(x) = 1.370\left(\frac{b}{c}\right)^{0.587} e^{-\lambda x}(\cos\lambda x \ \sin\lambda x) \quad \text{... (12.14a)}$$

$$J_1(x) = 0.244\left(\frac{b}{c}\right)^{0.333} e^{-\lambda x}\sin\lambda x \quad \text{... (12.14b)}$$

$$J_2(x) = 0.332\left(\frac{b}{c}\right)^{0.169} e^{-\lambda x}(\cos\lambda x - \sin\lambda x) \quad \text{... (12.14c)}$$

$$J_3(x) = \frac{\pi}{2} e^{-\lambda x}\cos\lambda x \quad \text{... (12.14d)}$$

$$J_4(x) = 1.285\left(\frac{b}{c}\right)^{-0.155} e^{-\lambda x}(\cos\lambda x + \sin\lambda x) \quad \text{... (12.14e)}$$

$$J_5(x) = 0.259\left(\frac{b}{c}\right)^{-0.360} e^{-\lambda x}\sin\lambda x \quad \text{... (12.14f)}$$

where $\lambda = \dfrac{0.689}{b}\left(\dfrac{b}{c}\right)^{0.813}$

(b) *Concentrated Loading on Finite Strip*

For strips of finite length, Brown (1969c) gives solutions for the distribution of deflection and bending moment due to a single concentrated load at various positions on the strip. These solutions, for various values of relative stiffness K, are shown in Figs.12.11 to 12.14 for bending moment, and in Figs.12.15 to 12.18 for deflection. In this case, K is defined as

$$K = \frac{E_p I_p (1-\nu_s^2)}{\pi E_s a^4} \quad \text{... (12.15)}$$

where E_p = modulus of strip

I_p = moment of inertia of strip section = $bt^3/6$

E_s, ν_s = moduli of foundation

a = ½ length of strip

b = ½ width of strip.

The average contact pressure $q = 1/4ba$ (for unit applied load). Brown has found that the ratio a/b has a relatively small influence on the moments.

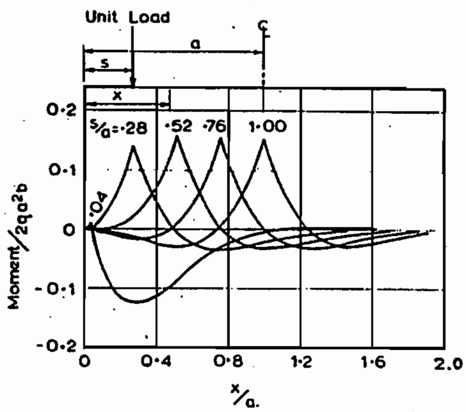


FIG.12.11 Moment in strip.
a/b=25, $K=4.1x10^{-4}$
(Brown, 1969c).

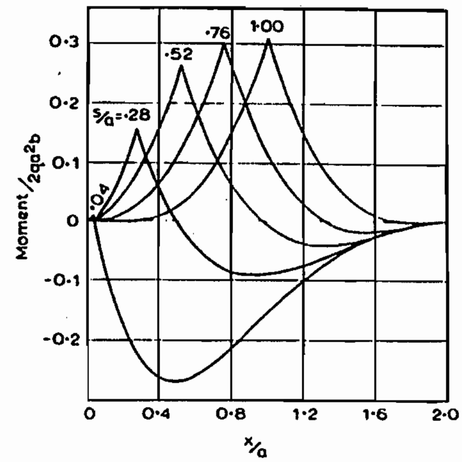


FIG.12.12 Moment in strip.
a/b=25, $K=4.1x10^{-3}$
(Brown, 1969c).

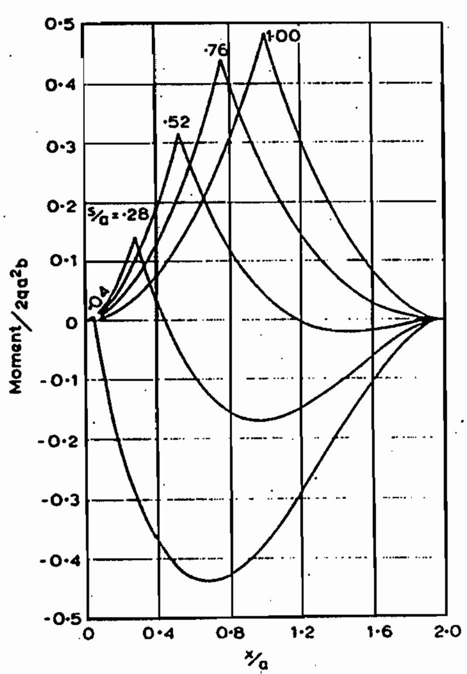


FIG.12.13 Moment in strip.
a/b=25, $K=4.1x10^{-2}$
(Brown, 1969c).

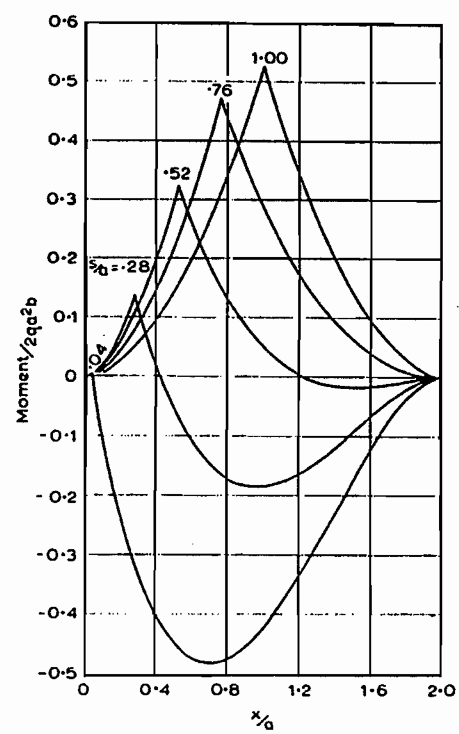


FIG.12.14 Moment in strip.
a/b=25, $K=4.1x10^{-1}$
(Brown, 1969c).

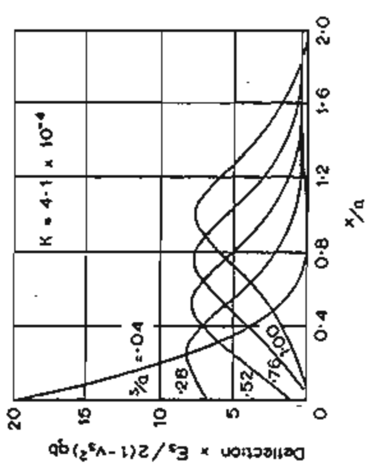


FIG.12.15 Deflection along strip.
a/b=25, K=4.1x10.$^{-4}$
(Brown, 1969c).

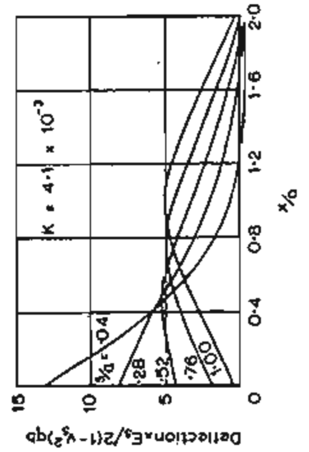


FIG.12.16 Deflection along strip.
a/b=25, K=4.1x10.$^{-3}$
(Brown, 1969c).

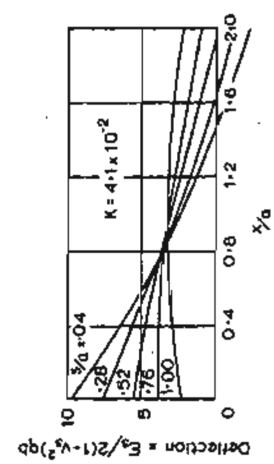


FIG.12.17 Deflection along strip.
a/b=25, K=4.1x10.$^{-2}$
(Brown, 1969c).

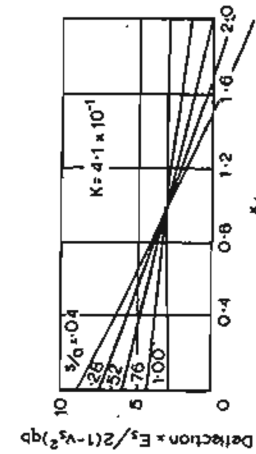


FIG.12.18 Deflection along strip.
a/b=25, K=4.1x10.$^{-1}$
(Brown, 1969c).

(c) Uniform Loading on Smooth Finite Strip

Brown (1969c) has obtained solutions for the maximum moment, maximum differential deflection and central deflection of the strip. These solutions are shown in Figs. 12.19 to 12.21 for 3 values of a/b. In all cases, ν of the strip $=0.3$. The uniform applied pressure $=q$/unit area.

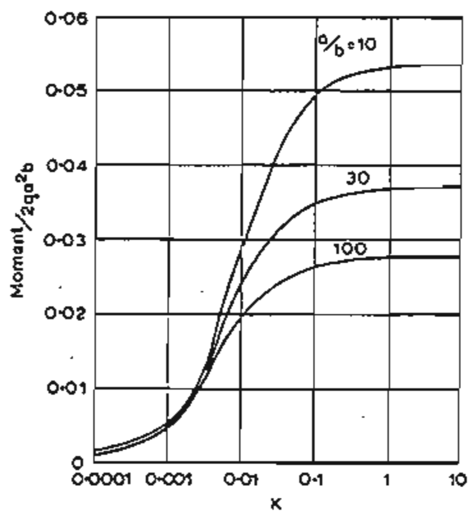


FIG.12.19 Maximum moment in uniformly loaded strip (Brown, 1969c).

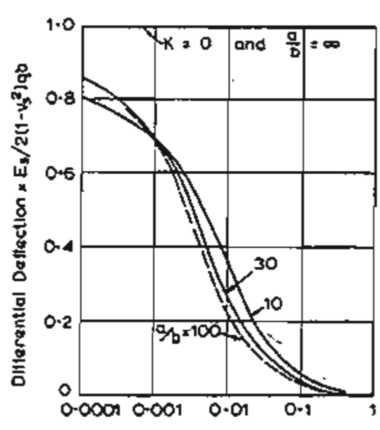


FIG.12.20 Differential deflection in uniformly loaded strip (Brown, 1969c).

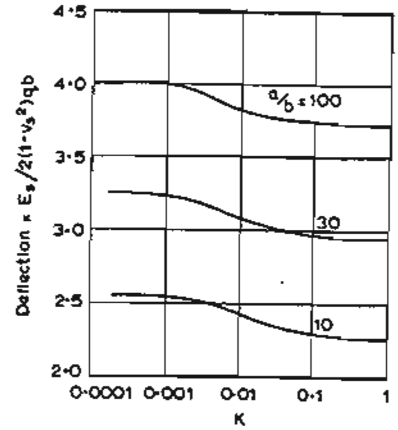


FIG.12.21 Central deflection of uniformly loaded strip (Brown, 1969c).

12.2 Circular Rafts

12.2.1 CIRCULAR RAFT ON SEMI-INFINITE MASS

(a) *Uniform Loading*

This problem was considered by Borowicka (1939) and more recently by Brown (1969b) (refer also to Section 7.2).

Influence factors obtained by Brown for the central vertical displacement of the raft at the surface are given in Fig.12.22 for Poisson's ratio ν_p of the raft of 0.3. The relative flexibility of the raft is expressed in terms of a factor K where

$$K = \frac{E_p}{E_s}(1-\nu_s^2)\left(\frac{t}{a}\right)^3 \qquad \ldots (12.16)$$

where E_p = Young's modulus of raft
t = raft thickness
a = raft radius
E_s, ν_s = elastic moduli of soil

Distributions of contact pressure are shown in Fig.12.23 for various values of K. Bending moment distributions are shown in Fig.12.24 while the variation of maximum moment with K is shown in Fig.12.25. Tabulated values of radial and tangential moment per unit width are given in Table 12.1. The variation of maximum differential deflection is shown in Fig. 12.26. In all cases, q=applied uniform pressure.

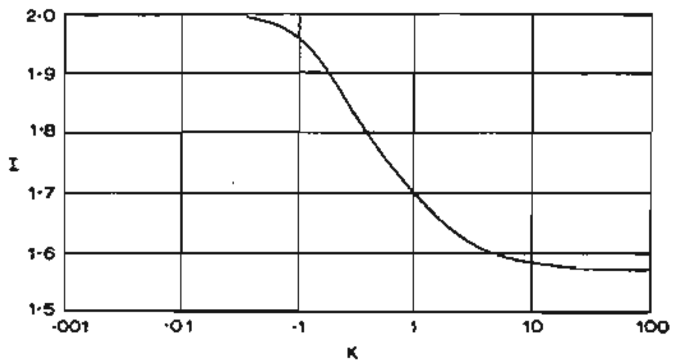


FIG.12.22 Central vertical displacement factor for circular raft (Brown, 1969b)

$$\rho = \frac{qa(1-\nu_s^{\,2})I}{E_s}$$

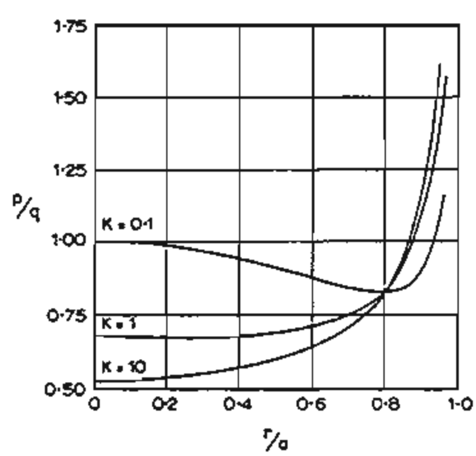


FIG.12.23 Contact pressure beneath circular raft. ν_p=0.3. (Brown, 1969b).

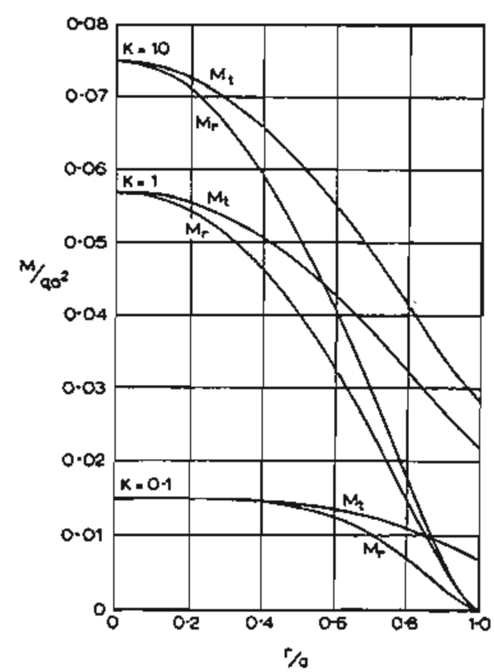


FIG.12.24 Bending moment distributions in circular raft. ν_p=0.3. (Brown, 1969b).

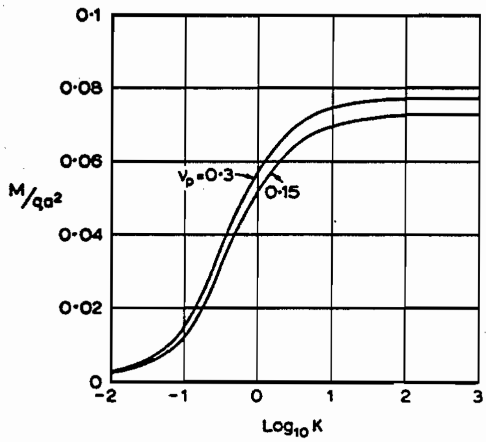


FIG.12.25 Maximum moment in circular raft (Brown, 1969b).

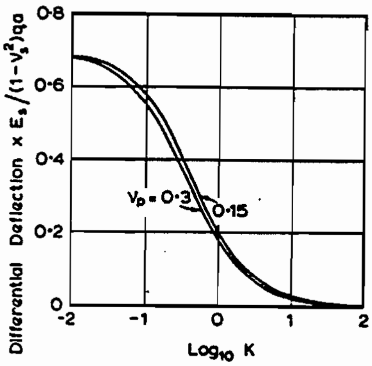


FIG.12.26 Differential deflection in circular raft (Brown, 1969b).

TABLE 12.1
BENDING MOMENTS IN CIRCULAR RAFT ON SEMI-INFINITE MASS
Values of $\frac{M}{qa^2}$ for ν_p=0.3

(Brown, 1969b)

K	10		1		0.1		0.01	
$\frac{r}{a}$	M_r	M_t	M_r	M_t	M_r	M_t	M_r	M_t
0	0.0747	0.0747	0.0567	0.0567	0.0146	0.0146	0.0012	0.0012
0.1	0.0737	0.0741	0.0561	0.0564	0.0146	0.0146	0.0012	0.0012
0.2	0.0708	0.0724	0.0541	0.0552	0.0146	0.0146	0.0013	0.0012
0.3	0.0659	0.0696	0.0508	0.0533	0.0145	0.0146	0.0014	0.0013
0.4	0.0593	0.0658	0.0461	0.0506	0.0142	0.0144	0.0016	0.0014
0.5	0.0509	0.0609	0.0401	0.0472	0.0136	0.0142	0.0018	0.0015
0.6	0.0411	0.0551	0.0329	0.0430	0.0125	0.0136	0.0021	0.0017
0.7	0.0301	0.0486	0.0246	0.0381	0.0106	0.0127	0.0024	0.0019
0.8	0.0184	0.0415	0.0154	0.0327	0.0076	0.0114	0.0024	0.0020
0.9	0.0072	0.0343	0.0063	0.0271	0.0037	0.0095	0.0017	0.0018
1.0	0	0.0283	0	0.0222	0	0.0074	0	0.0012

(b) *Concentrated Loading*

Borowicka (1939) has obtained solutions for the contact pressure p beneath the raft. These are shown in Fig.12.27. $p_{av}=P/\pi a^2$, and the stiffness factor is defined as

$$K = \frac{1}{6}\frac{(1-\nu_s^{\,2})}{(1-\nu_p^{\,2})}\frac{E_p}{E_s}\left(\frac{t}{a}\right)^3 \qquad \ldots (12.17)$$

where t = raft thickness

a = raft radius.

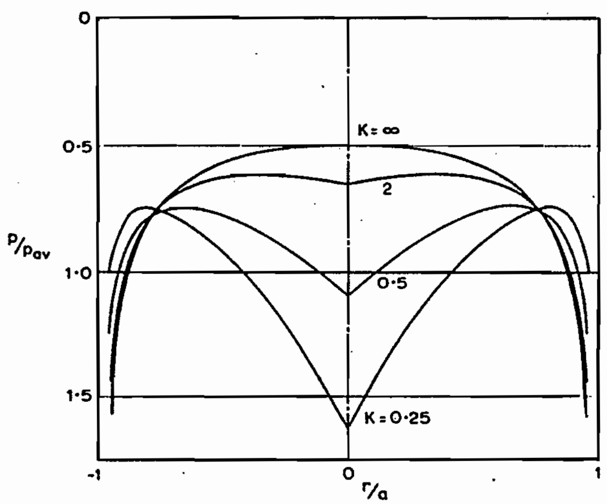


FIG.12.27 Contact pressure distribution beneath circular raft with central point loading (Borowicka, 1939).

For a raft of infinite extent on a semi-infinite elastic mass, and loaded by a point load P, Hogg (1938) has obtained solutions for the displacement and curvature of the raft. These are shown in Fig.12.28, where

D = flexural rigidity of raft = $\dfrac{E_p t^3}{12(1-\nu_p^{\,2})}$

t = raft thickness

R = $\dfrac{2G(\lambda+2G)}{\lambda+3G}$ for perfectly rough plate

or $\dfrac{2G(\lambda+G)}{\lambda+2G}$ for perfectly smooth plate

λ, G = Lame's parameters of the mass.

r = radial distance from point load

ρ_z = vertical displacement.

For a uniformly distributed load, the displacement beneath the centre is almost the same as that due to a concentrated loading provided that the radius of the loading is of the same order as the thickness t.

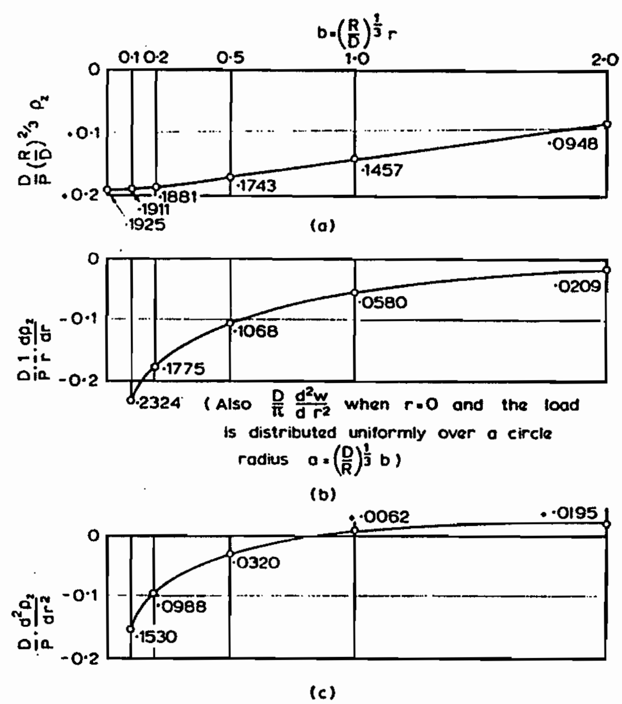


FIG.12.28 Displacement, slope and curvature of uniform thin raft with point loading (Hogg,1938).

12.2.2 RIGID CIRCULAR RAFT ON A FINITE LAYER

This problem has been considered by Egorov and Serebryanyi (1963) and Poulos (1968a).

Influence factors for the displacement of the raft have been given in Fig. 7.19.

Distributions of contact pressure beneath the raft are given in Figs. 7.15 and 7.16.

For a raft having $\nu_p = 0.25$, the distributions of radial and tangential bending moments are shown in Figs. 12.29 and 12.30 for a uniformly distributed load over the raft and in Figs. 12.31 and 12.32 for a load concentrated within $r = 0.1a$ near the centre of the raft. In these figures, ν_s of the layer $= 0.4$. The influence of varying ν_s on the central moment for a uniformly distributed loading is shown in Fig. 12.33 for various values of layer depth h to raft radius a.

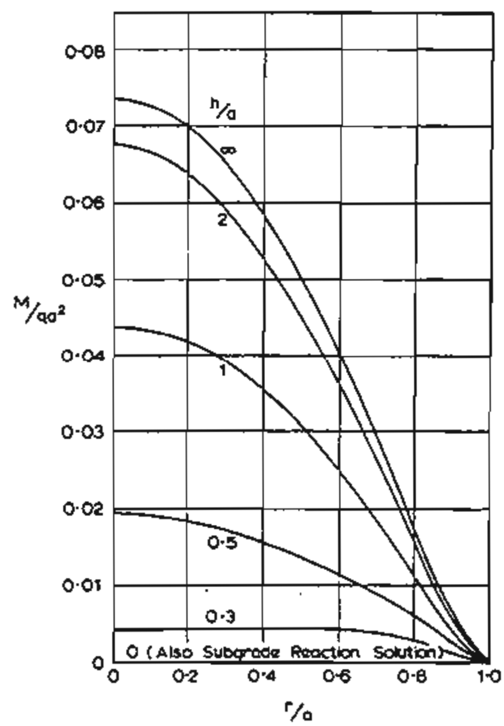


FIG.12.29 Radial bending moment distribution in uniformly loaded rigid circular raft. $\nu_s = 0.4$, $\nu_p = 0.25$.

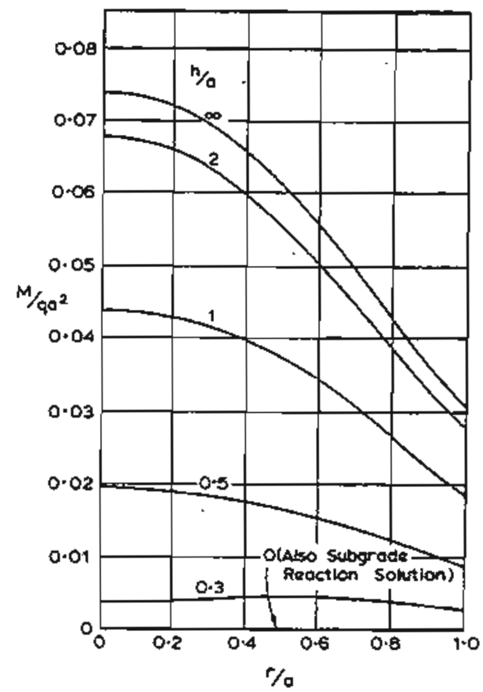


FIG.12.30 Tangential bending moment distribution in uniformly loaded rigid circular raft. $\nu_s = 0.4$, $\nu_p = 0.25$.

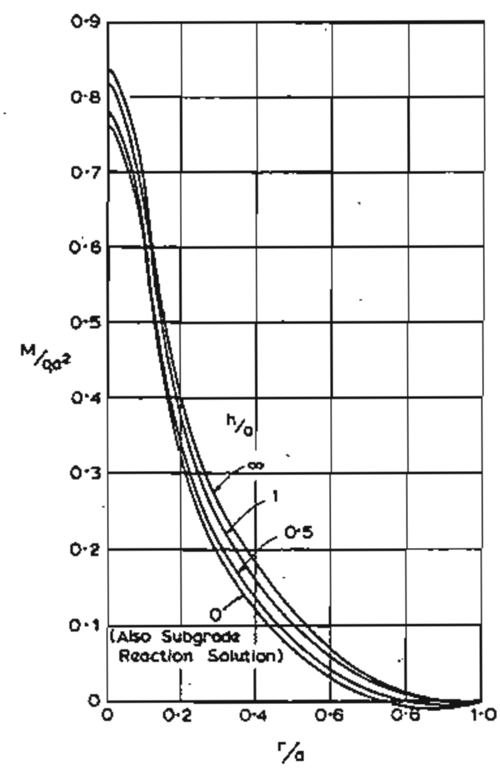


FIG.12.31 Radial bending moment distribution for concentrated load within $r = 0.1a$. $\nu_s = 0.4$, $\nu_p = 0.25$.

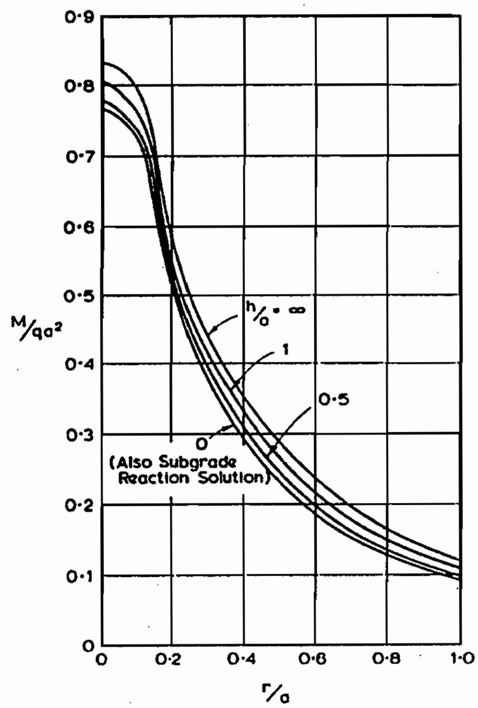


FIG.12.32 Tangential bending moment distribution for concentrated loading within r=0.1a. ν_s=0.4, ν_p=0.25.

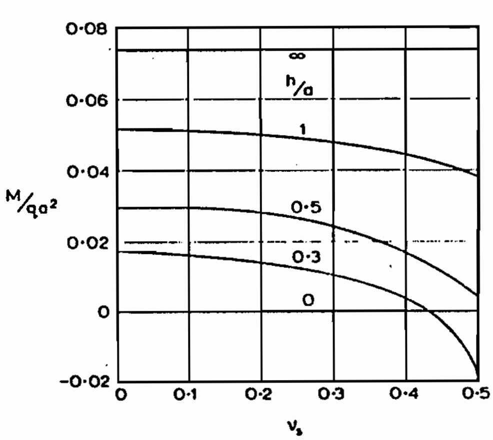


FIG.12.33 Influence of ν_s on central bending moment. Uniform loading.

12.2.3 CIRCULAR RAFT OF ANY FLEXIBILITY ON A FINITE LAYER (Uniform Loading)

This problem has been considered by Brown (1969a).

The relative flexibility of the raft is defined by a factor K where

$$K = \frac{E_p(1-\nu_s^2)}{E_s}\left(\frac{t}{a}\right)^3 \qquad \text{... (12.18)}$$

where t = raft thickness

a = raft radius.

For $h/a=1$, distributions of contact pressure beneath the raft are shown in Fig.12.34 for three values of K.

The variation of central vertical surface displacement with a/h is shown in Fig.12.35 for two values of K.

The variation of differential vertical displacement between centre and edge with K is shown in Fig.12.36.

For uniformly distributed loading over the raft, bending moment distributions are shown in Fig.12.37 for $h/a=1$. The influence of h/a on the bending moment distributions is shown in Fig.12.38 for $K=0.1$.

The variation of maximum (central) moment with K for various h/a values is shown in Fig.12.39.

12.3 Rectangular Rafts

12.3.1 RIGID RAFT ON SEMI-INFINITE MASS

Gorbunov-Possadov and Serebrjanyi (1961) have obtained solutions for the moments, shear force and pressure distribution within a rigid square slab subjected to a central concentrated load. Influence factors are shown in Fig.12.40.

$$\textit{Contact pressure} \quad p = \bar{p}.p_{av} \qquad \text{... (12.19a)}$$

$$\textit{Shear force} \quad N_x = \bar{N}_x.a.p_{av} \qquad \text{... (12.19b)}$$

$$\textit{Bending moment} \quad M_x = \bar{M}_x.a^2.p_{av} \qquad \text{... (12.19c)}$$

$$\textit{Torsional moment} \quad H_x = \bar{H}_x.a^2.p_{av} \qquad \text{... (12.19d)}$$

where a = ½ width of square

p_{av} = average applied pressure = $\frac{P}{4a^2}$

P = total load.

A rectangular slab ($2a\times2b$) can be considered as rigid if

$$\frac{\pi a^2 bE_s}{D(1-\nu_s^2)} < \frac{8}{\left(\frac{b}{a}\right)^{\frac{1}{2}}} \qquad (b>a) \qquad \text{... (12.20)}$$

where D is defined on p.258

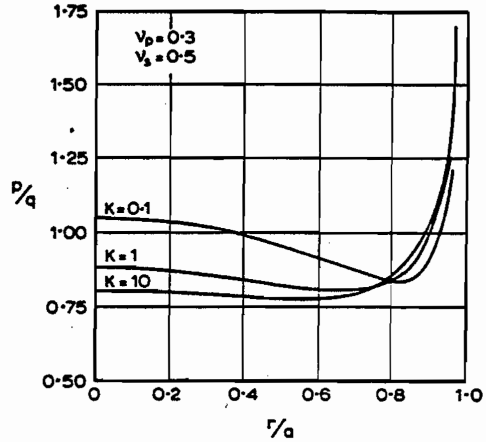


FIG.12.34 Contact pressure distributions. h/a=1. (Brown, 1969a).

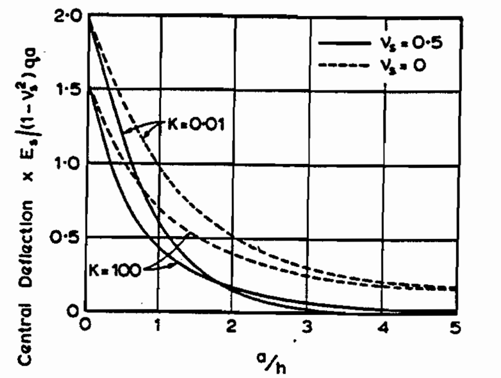


FIG.12.35 Central deflection. ν_p=0.3. (Brown, 1969a).

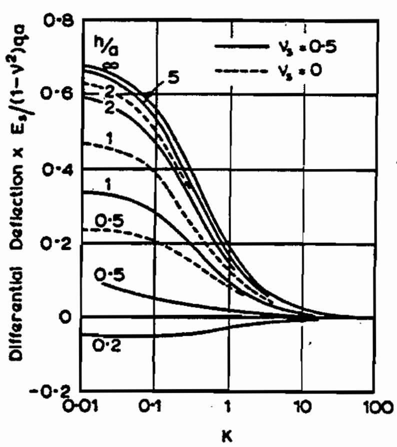


FIG.12.36 Differential deflections. ν_p=0.3. (Brown, 1969a).

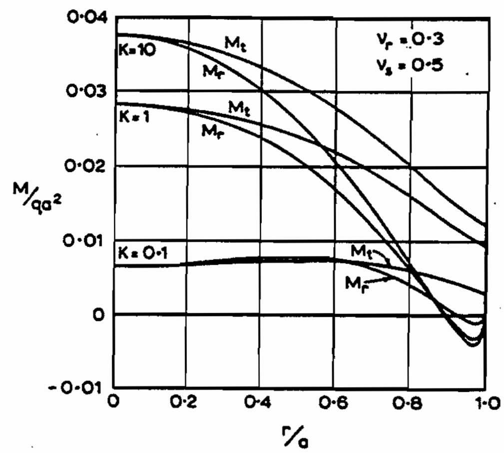


FIG.12.37 Bending moment distributions. h/a=1. (Brown, 1969a).

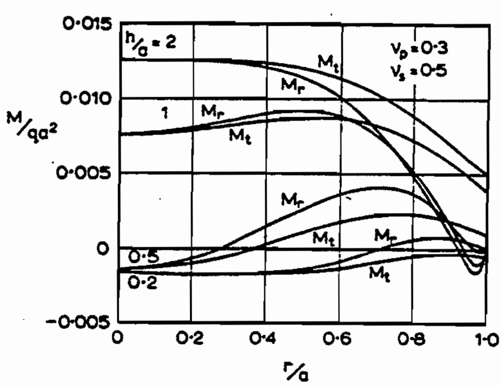


FIG.12.38 Bending moment distributions. K=0.1. (Brown, 1969a).

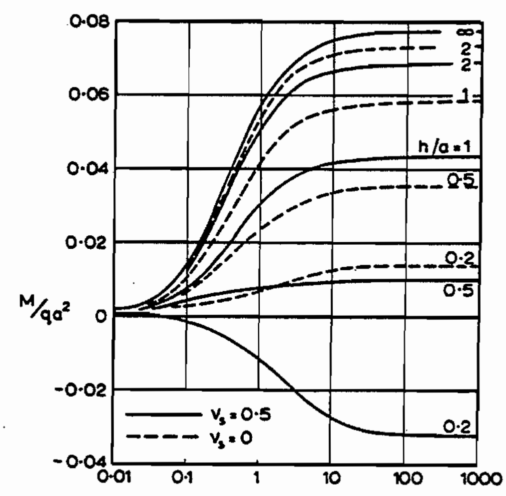


FIG.12.39 Variation of maximum moment with K. ν_p=0.3. (Brown, 1969a)

where E_s, ν_s = soil moduli

D = flexural rigidity of slab.

The displacement and rotation of a rigid rectangular raft are considered in Section 7.6.

Distributions of contact pressure for two rectangles are shown in Fig.12.41 (Butterfield and Banerjee, 1971).

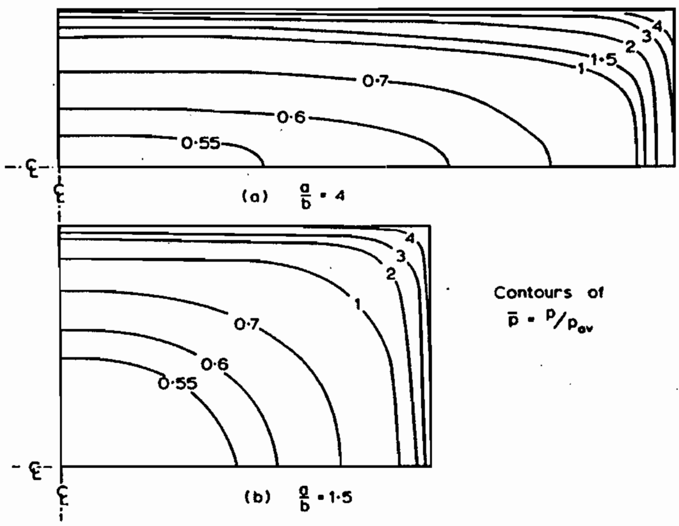


FIG.12.41 Contact pressure distribution beneath rigid rectangular raft (Butterfield and Banerjee,1971).

Brown (1972) has obtained solutions for reaction, shear force, bending moment and torsional moment in rigid rectangular rafts. The raft proportions and the loading cases considered are shown in Fig.12.42. Contours representing the solutions are shown in Figs. 12.43 to 12.51. The shear force and moments are expressed in terms of the values per unit width of raft. In all cases, Poisson's ratio of the raft, ν_r, is 0.15. In Brown's notation, the shear forces in the x and y directions are denoted as Q_x and Q_y, the bending moments as M_x and M_y, and the torsional moment as M_{xy}. The solutions have been obtained by numerical integration of the biharmonic equation.

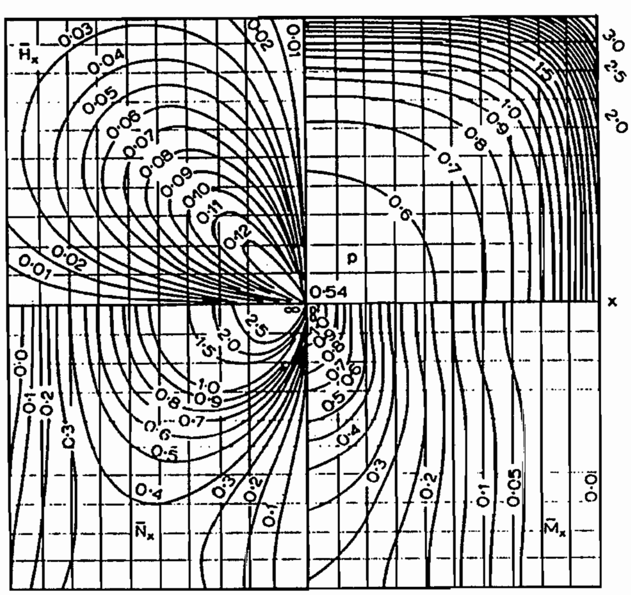


FIG.12.40 Dimensionless reaction, bending moment, torsional moment and shear force in rigid square raft with central concentrated load (Gorbunov-Possadov and Serebrjanyi, 1961).

12.3.2 FLEXIBLE RAFT ON SEMI-INFINITE MASS

Gorbunov-Possadov and Serebrjanyi (1961) have obtained solutions for contact pressure, vertical displacement and moments in a large rectangular raft subjected to concentrated load at the middle of the edge. Influence factors are shown in Fig.12.52.

Contact pressure $$p = \bar{p}\,\frac{P}{L^2} \qquad \text{... (12.21a)}$$

Vertical displacement $$\rho_z = \bar{W}\,\frac{(1-\nu_s^2)}{E_s}\,\frac{P}{L} \qquad \text{... (12.21b)}$$

Moment in x-direction $$M_x = \bar{M}_x P \qquad \text{... (12.21c)}$$

Moment in y-direction $$M_y = \bar{M}_y P \qquad \text{... (12.21d)}$$

where $$L = \left(\frac{2D(1-\nu_s^2)}{E_s}\right)^{1/3}$$

$$\xi = \frac{x}{L}$$

$$\eta = \frac{y}{L}$$

P, D, E_s, ν_s are defined in the previous section.

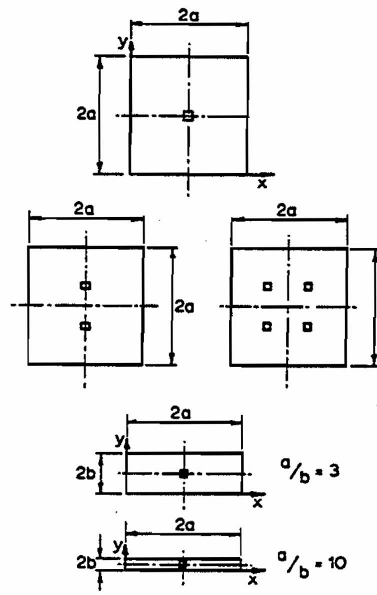


FIG.12.42 Loading cases for rigid rafts (Brown,1972).

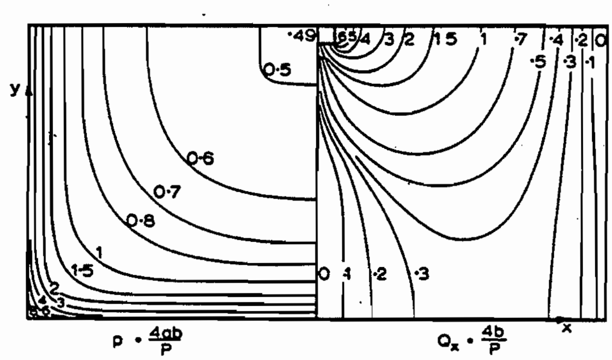


FIG.12.43 Reaction and shear distributions for rigid square raft with central load (Brown, 1972).

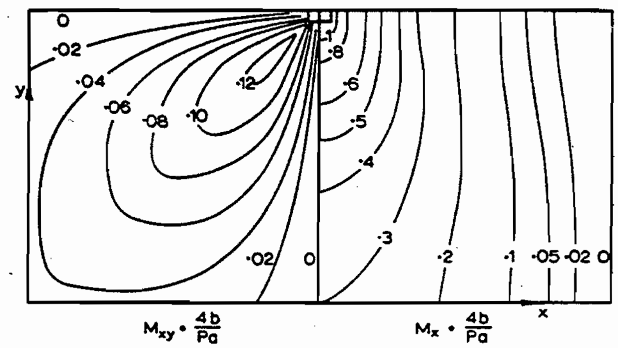


FIG.12.44 Torsional and bending moments for a rigid square raft with central load (Brown, 1972).

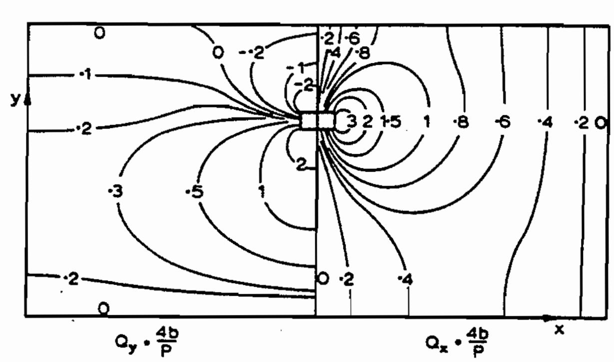


FIG.12.45 Shear force distributions for rigid square raft subjected to two loads (Brown, 1972).

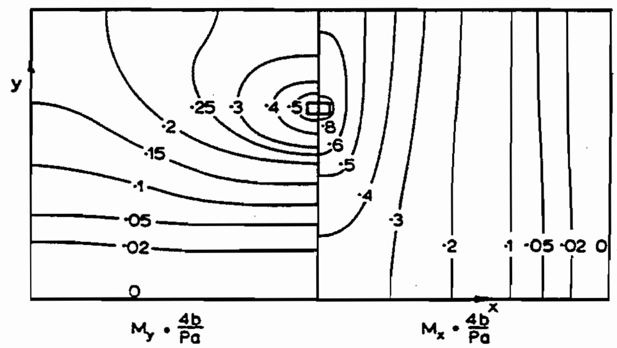


FIG.12.46 Bending moments for rigid square raft subjected to two loads (Brown, 1972).

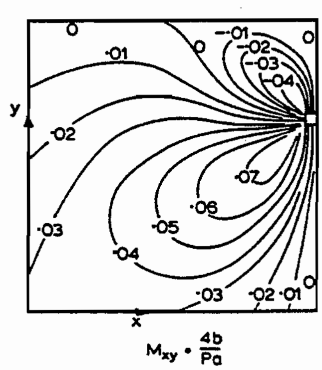


FIG.12.47 Torsional moments for rigid square raft subjected to two loads (Brown, 1972).

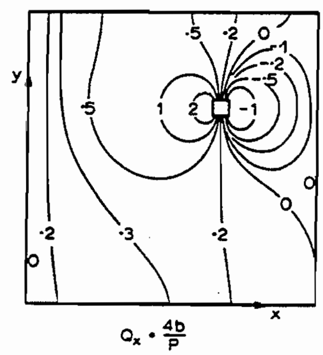


FIG.12.48 Shear force distribution for rigid square raft subjected to four loads (Brown, 1972).

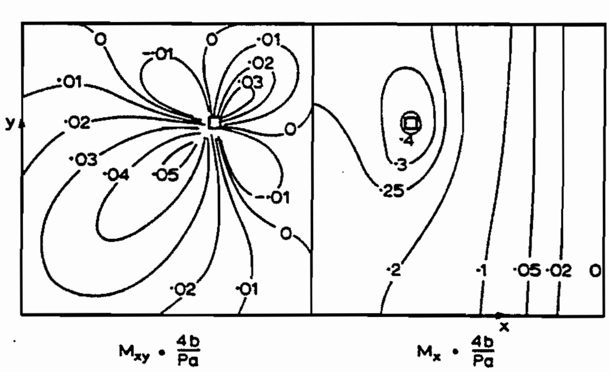


FIG.12.49 Bending and torsional moments for rigid square raft subjected to four loads (Brown, 1972).

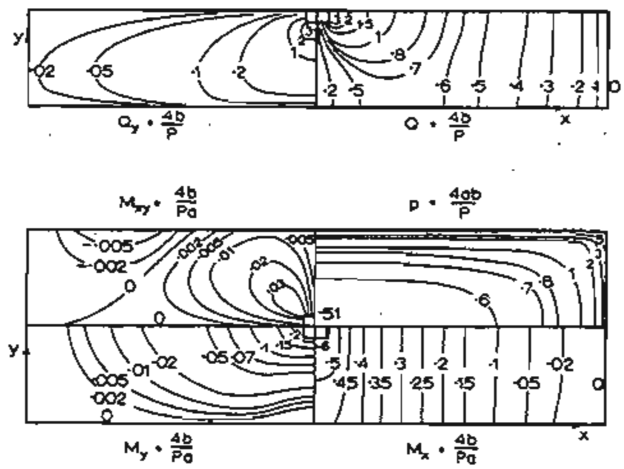


FIG.12.50 Behaviour of rigid raft with central load. a/b=3 (Brown, 1972).

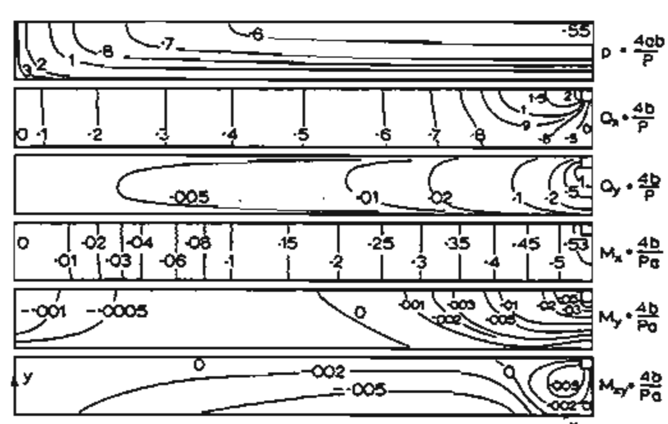


FIG.12.51 Behaviour of rigid raft with central load. a/b=10 (Brown, 1972).

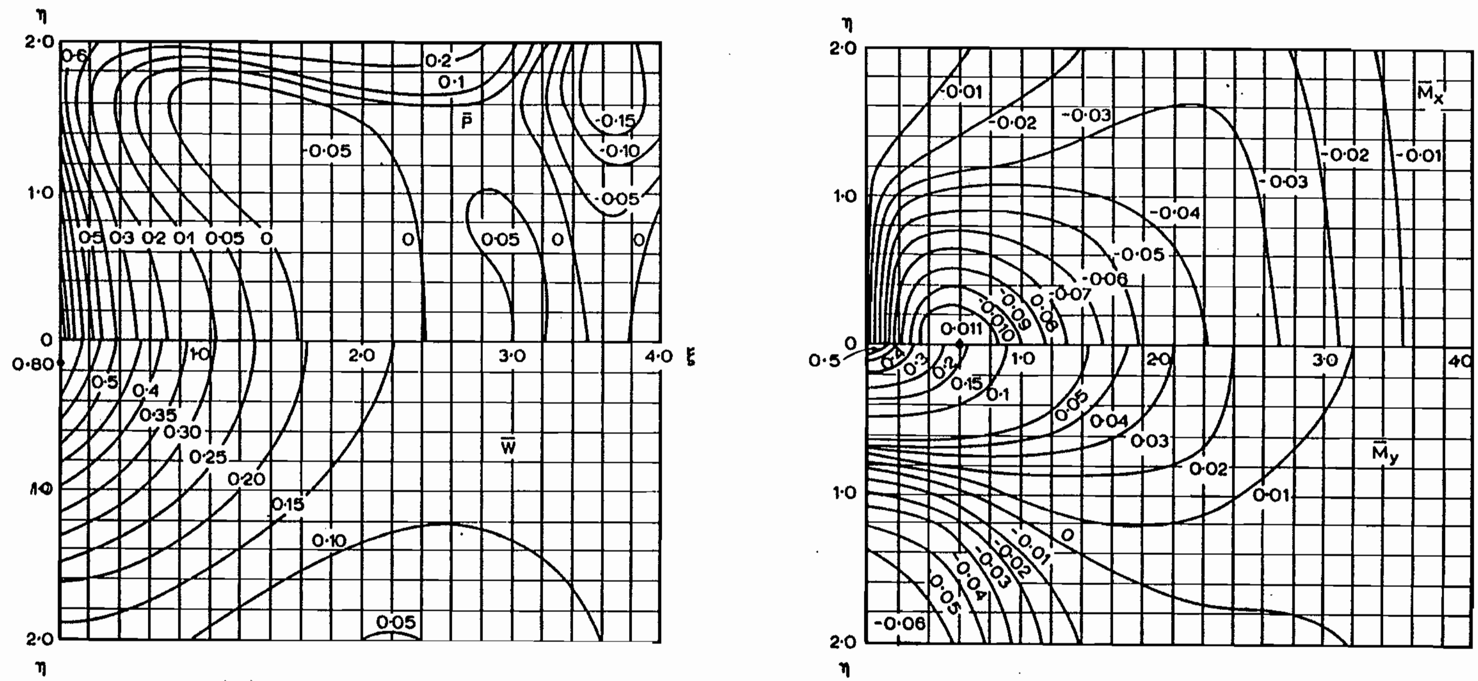


FIG.12.52 Dimensionless reaction, bending moments and vertical displacement for square raft subjected to concentrated load at edge (Gorbunov-Possadov and Serebrjanyi, 1961).

Chapter 13
AXIALLY LOADED PILES

13.1 Single Incompressible Floating Pile

This problem has been considered by Poulos and Davis (1968).

The distribution of shear stress along the pile shaft is shown in Fig.13.1 for various L/d values while the proportion of applied load transferred to the base is shown in Fig.13.2.

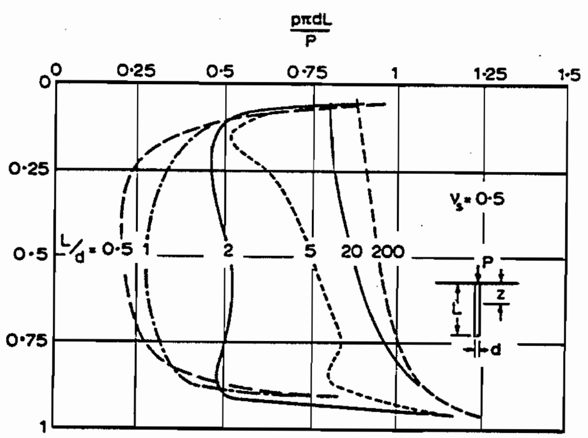


FIG.13.1 Distribution of shear stress along incompressible pile.

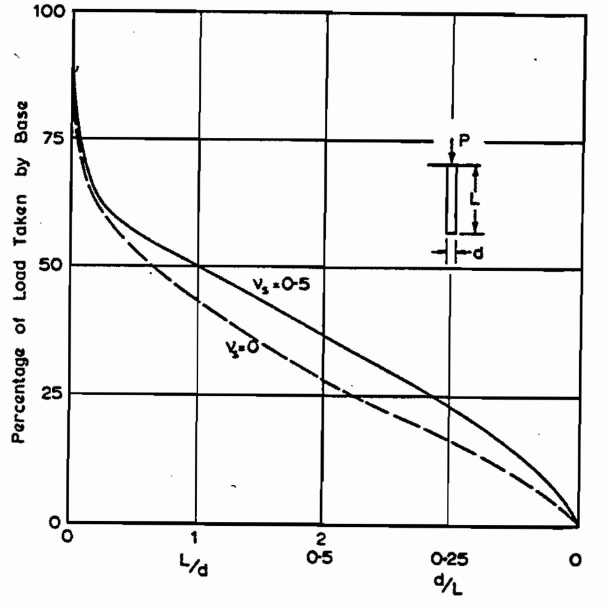


FIG.13.2 Proportion of applied load transferred to base of incompressible pile.

Influence factors for the vertical displacement are shown in Figs.13.3 to 13.6 for a pile in a finite layer and for four values of ν_s (Poisson's ratio of mass).

The effect of having an enlarged base, diameter d_b, on the pile is shown in Fig.13.7 for base load, and Fig.13.8 for displacement.

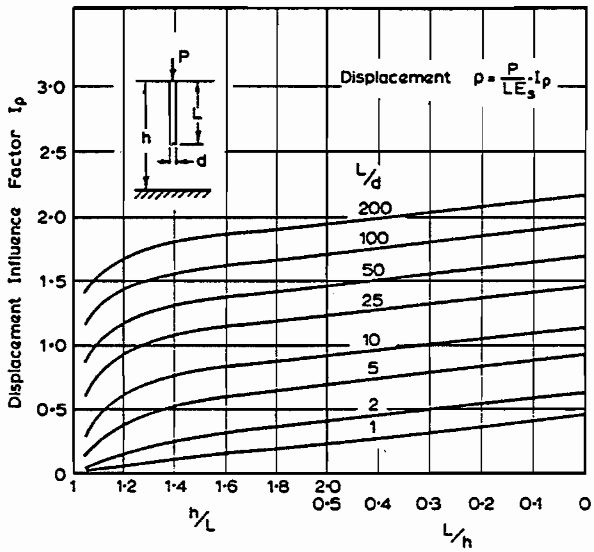


FIG.13.3 Displacement of incompressible pile in finite layer. $\nu_s = 0$.

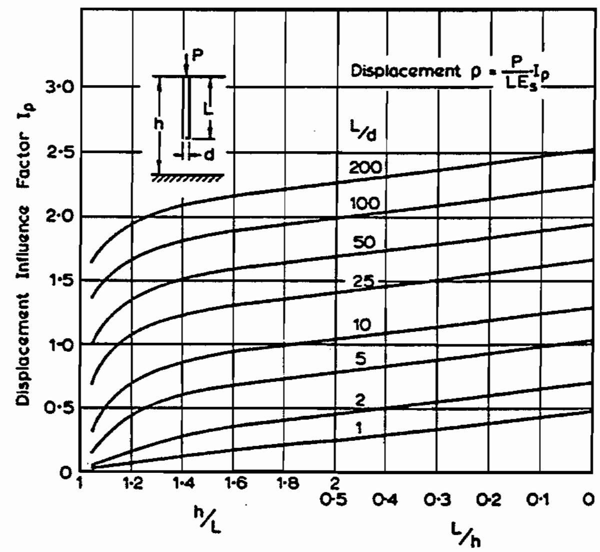


FIG.13.4 Displacement of incompressible pile in finite layer. $\nu_s = 0.2$.

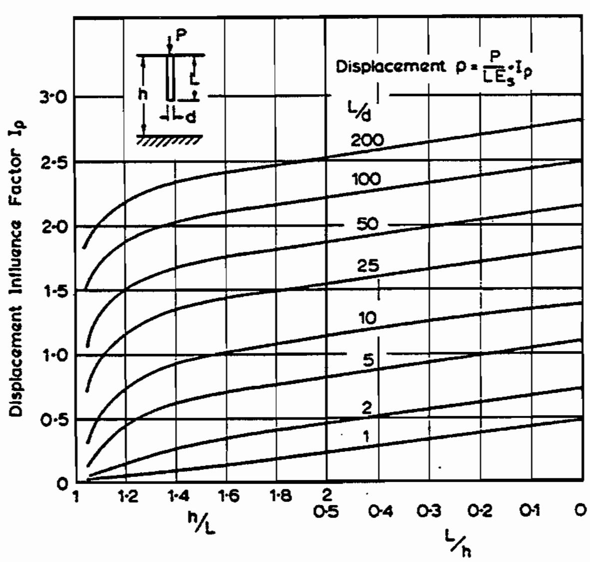


FIG.13.5 Displacement of incompressible pile in finite layer. $\nu_s = 0.4$.

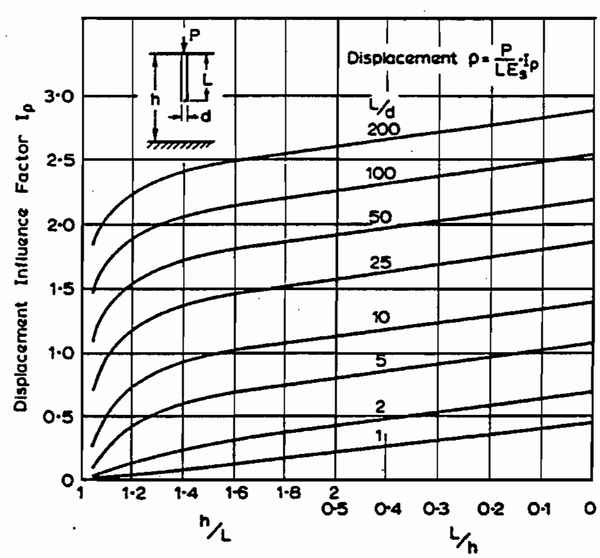


FIG.13.6 Displacement of incompressible pile in finite layer. $\nu_s = 0.5$.

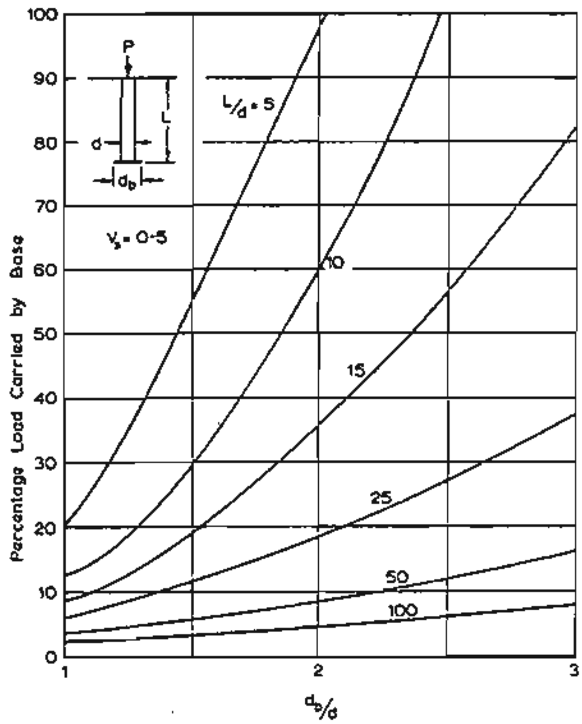


FIG.13.7 Effect of enlarged base on proportion of load transferred to pile base.

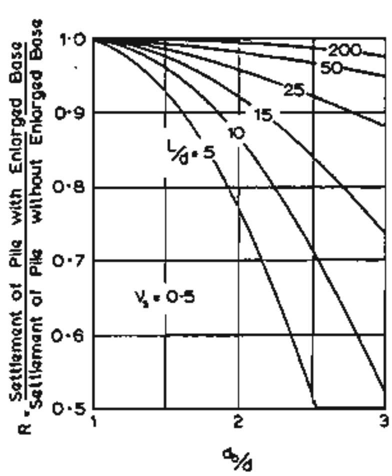


FIG.13.8 Effect of enlarged base on pile displacement.

The behaviour of piles of non-uniform cross-section is considered by Poulos (1969) while the behaviour of piles having a rigid cap resting on the surface is examined by Poulos (1968b).

13.2 Single Compressible Floating Pile

The compressibility of the pile in relation to the soil is expressed by a pile stiffness factor K where

$$K = \frac{E_p}{E_s} R_A \qquad \ldots (13.1)$$

where E_p = Young's modulus of pile

E_s = Young's modulus of soil mass

R_A = area of pile section/$\frac{\pi d^2}{4}$

The influence of K on the shear stress distribution along the pile is shown in Fig.13.9 while the proportion of load transferred to the base is shown in Fig.13.10. The difference between the top and tip displacement of a pile having $L/d=25$ is shown in Fig.13.11. Influence factors for displacement of the pile top are shown in Fig.13.12. In all the above cases, the layer is of infinite depth. The influence of finite layer depth is shown in Fig.13.13.

Influence factors for the vertical displacement of a point within a semi-infinite mass, at depth H below the surface and radial distance r from the axis, due to a pile are shown in Figs.13.14 to 13.27 for various values of L/d and K. These factors have been obtained by Poulos and Mattes (1971a).

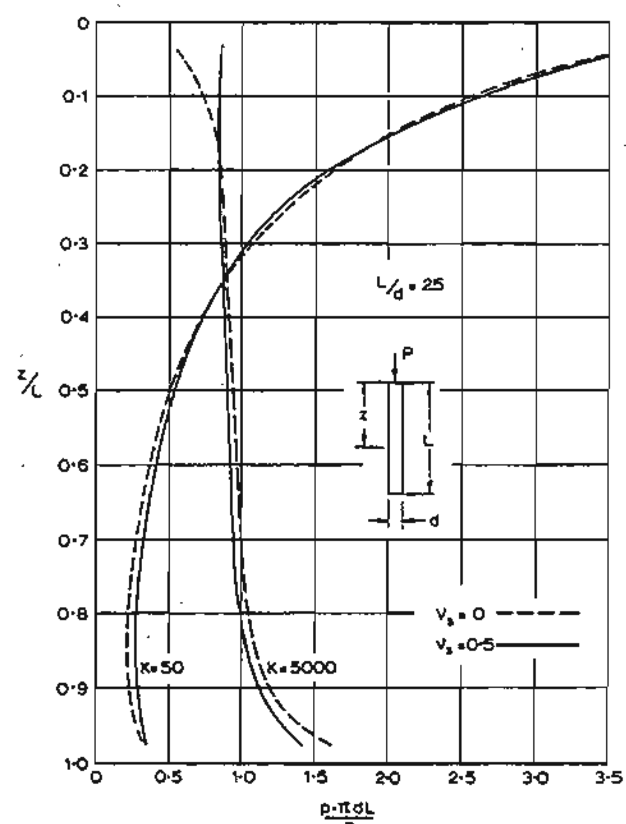


FIG.13.9 Effect of pile compressibility on shear stress distribution (Mattes and Poulos, 1969).

The displacement ρ is given by

$$\rho = \frac{P}{L\,E_s} I_\rho \qquad \text{... (13.2)}$$

The solutions are for $\nu_s=0.5$, but ν_s generally has a relatively small effect on I_ρ. For $r/L>0.5$, the displacement due to the pile, is within ±3% of the value due to a point load P acting on the axis at a distance $2L/3$ below the surface.

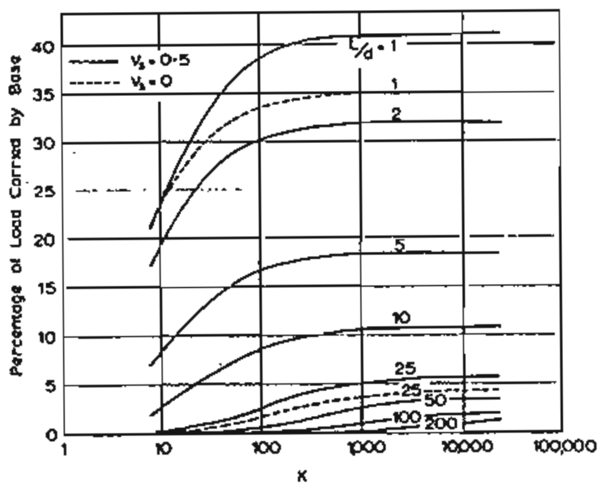


FIG.13.10 Effect of pile compressibility on load transferred to pile base (Mattes and Poulos, 1969).

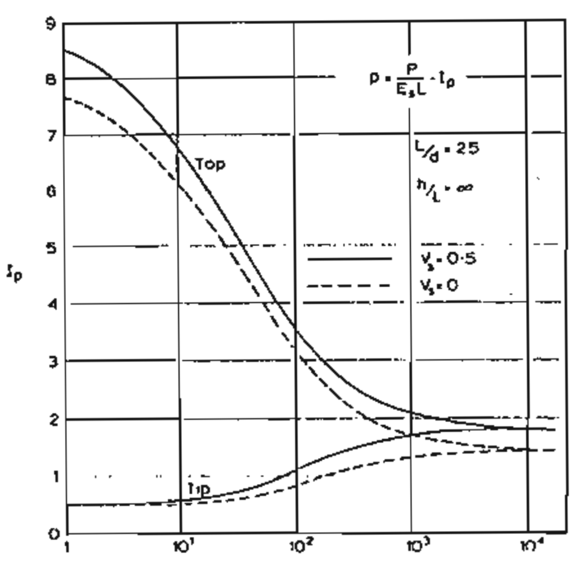


FIG.13.11 Top and tip displacements of compressible floating pile (Mattes and Poulos, 1969).

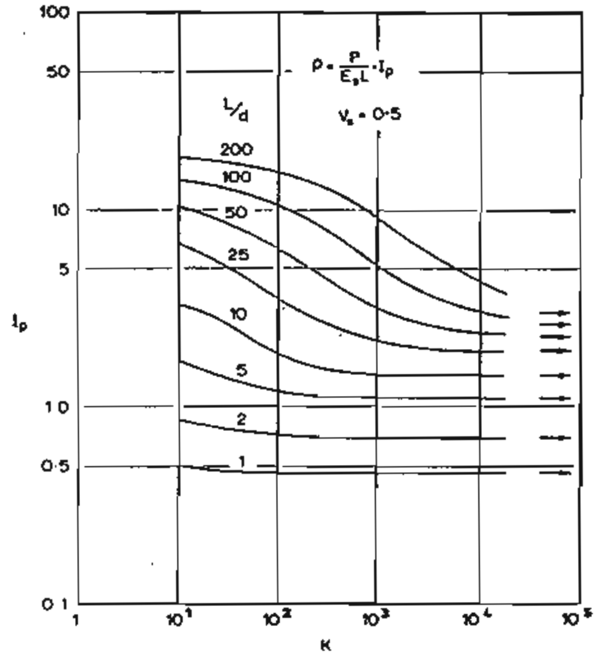


FIG.13.12 Displacement influence factors for compressible floating pile (Mattes and Poulos, 1969).

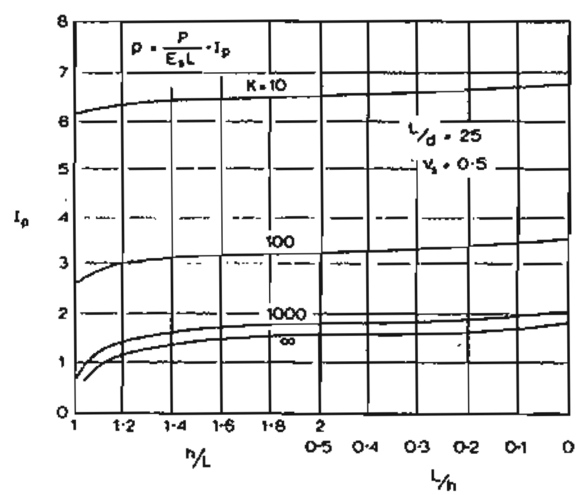


FIG.13.13 Effect of finite layer depth on pile displacement (Mattes and Poulos, 1969)

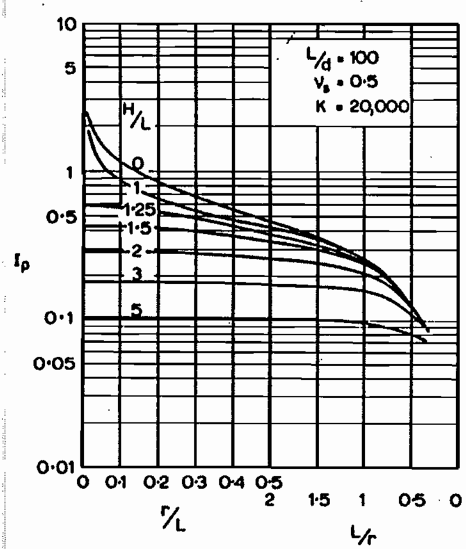


FIG.13.14 Factors for displacement due to pile. L/d = 100, K=20000.

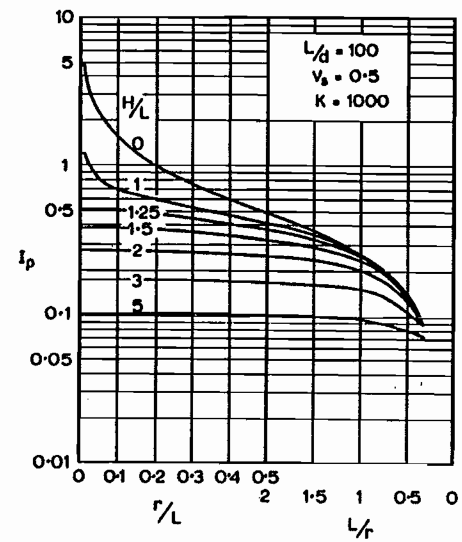


FIG.13.15 Factors for displacement due to pile. L/d = 100, K=1000.

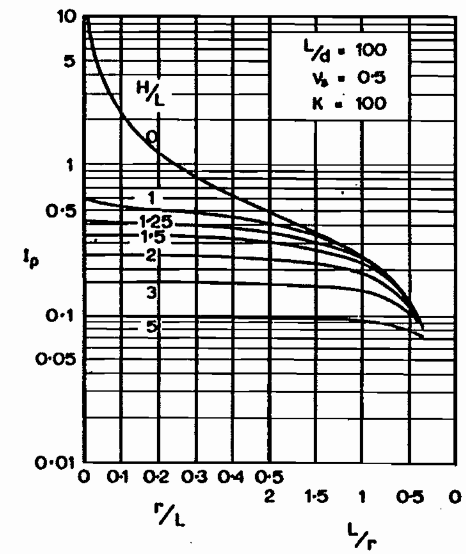


FIG.13.16 Factors for displacement due to pile. L/d = 100, K=100.

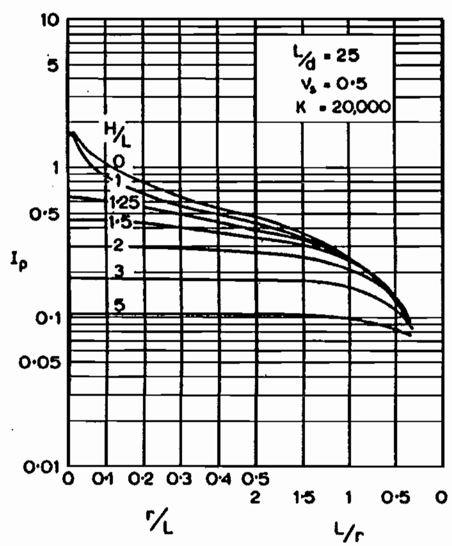


FIG.13.17 Factors for displacement due to pile. L/d = 25, K=20000.

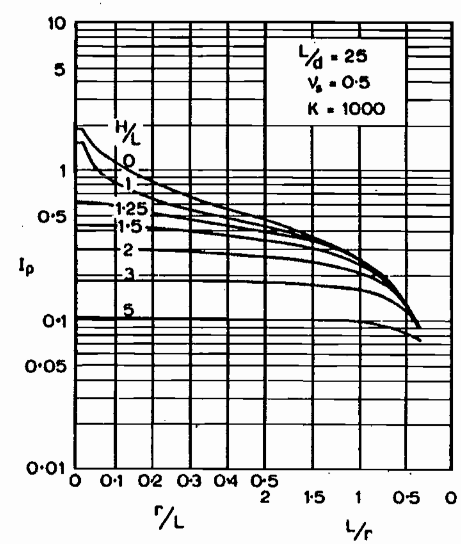


FIG.13.18 Factors for displacement due to pile. L/d = 25, K=1000.

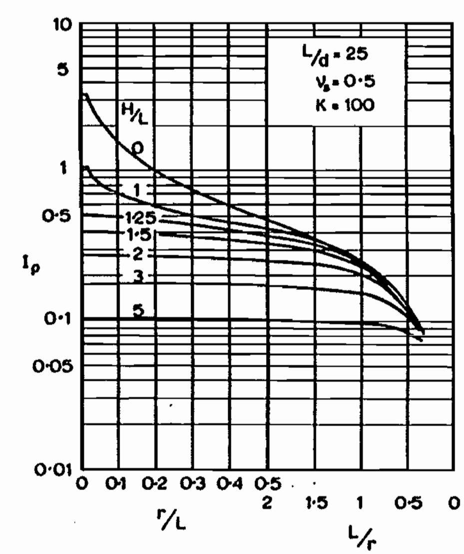


FIG.13.19 Factors for displacement due to pile. L/d = 25, K=100.

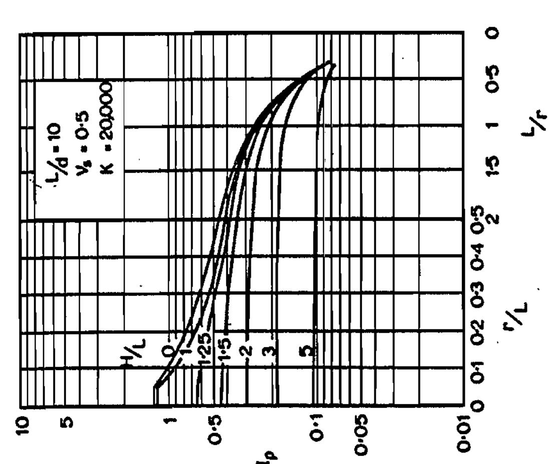


FIG.13.20 Factors for displacement due to pile. L/d = 10, K=20000.

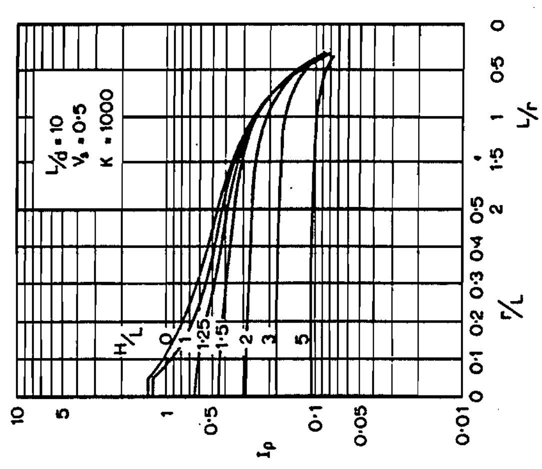


FIG.13.21 Factors for displacement due to pile. L/d = 10, K=1000.

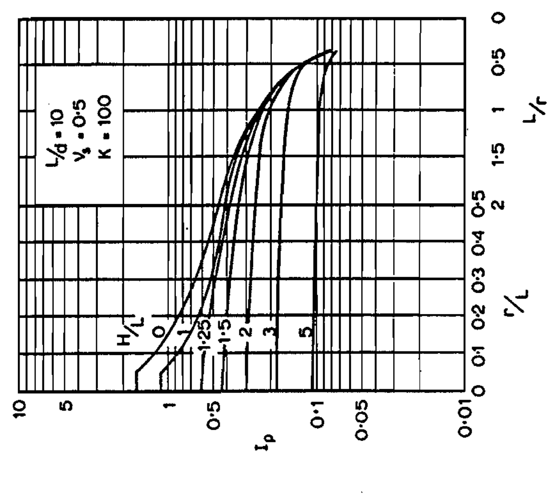


FIG.13.22 Factors for displacement due to pile. L/d = 10, K=100.

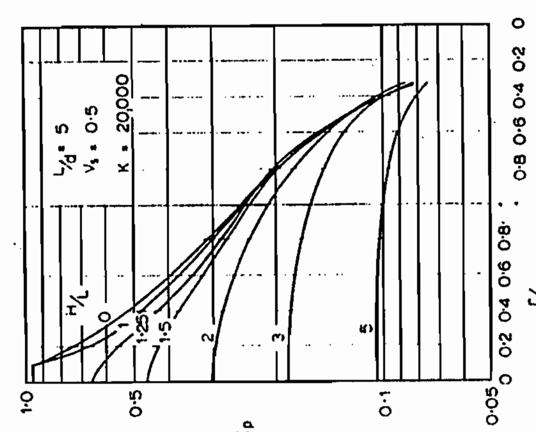


FIG.13.23 Factors for displacement due to pile. L/d = 5, K=20000.

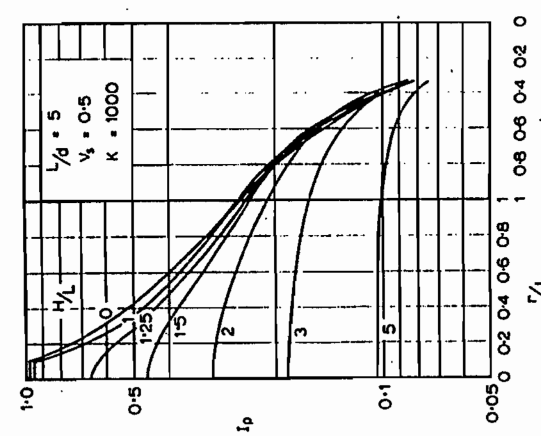


FIG.13.24 Factors for displacement due to pile. L/d = 5, K=1000.

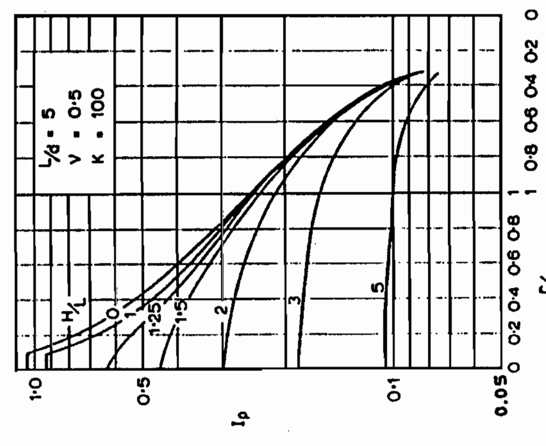


FIG.13.25 Factors for displacement due to pile. L/d = 5, K=100.

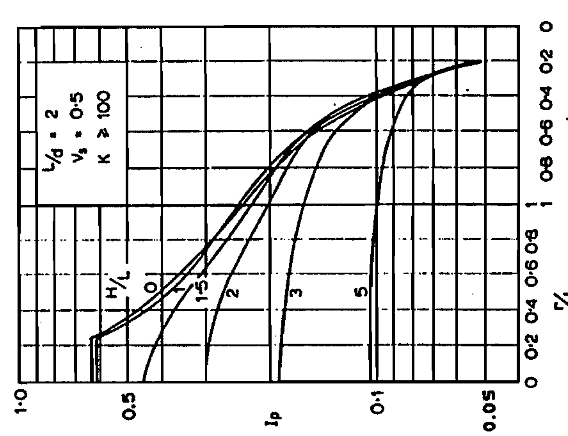


FIG.13.26 Factors for displacement due to pile. L/d = 2, K ≥ 100.

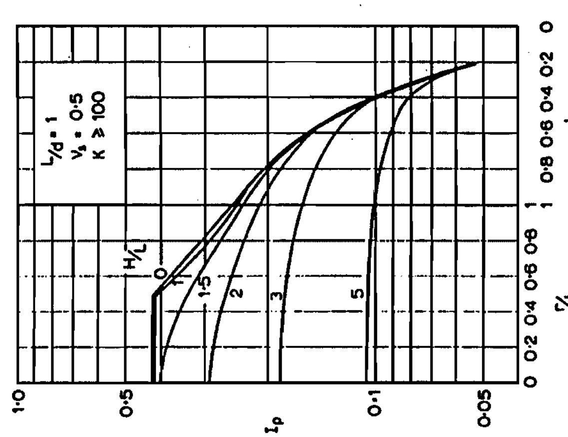


FIG.13.27 Factors for displacement due to pile. L/d = 1, K ≥ 100.

13.3 Single Compressible End-Bearing Pile

This problem has been considered by Poulos and Mattes (1969a).

For a rigid bearing layer, the distribution of axial load within the pile with depth is shown in Fig.13.28 while the proportion of load transferred to the pile base is shown in Fig.13.29. The displacement of the top of the pile is shown in Fig.13.30.

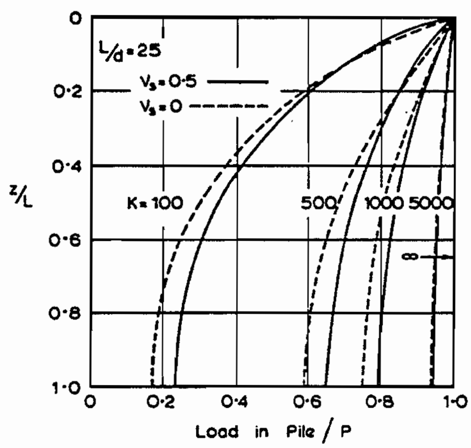


FIG.13.28 Load distribution in end-bearing pile.

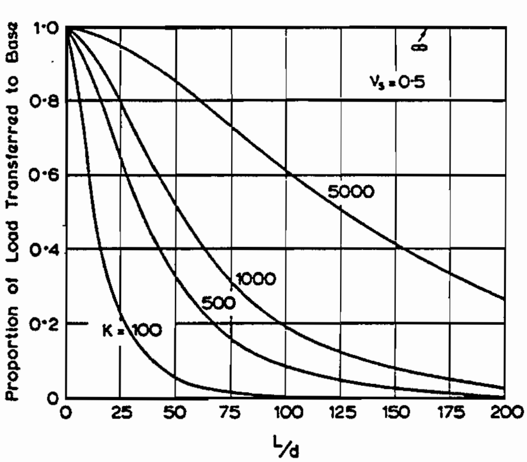


FIG.13.29 Proportion of load transferred to base of end-bearing pile.

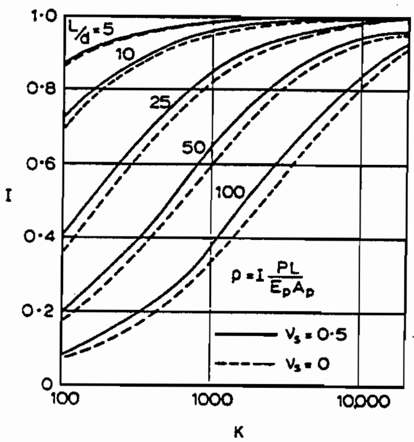


FIG.13.30 Displacement at top of end-bearing pile.

13.4 Negative Friction in a Single End-Bearing Pile

This problem has been considered by Poulos and Mattes (1969b).

For a layer underlain by a rigid base which is subject to a vertical displacement which varies linearly from S_O at the surface to zero at the base ($z=L$). Influence factors for the maximum load P_N induced in a pile (at the tip) are shown in Fig.13.31.

Distributions of load along the pile are shown in Fig.13.32.

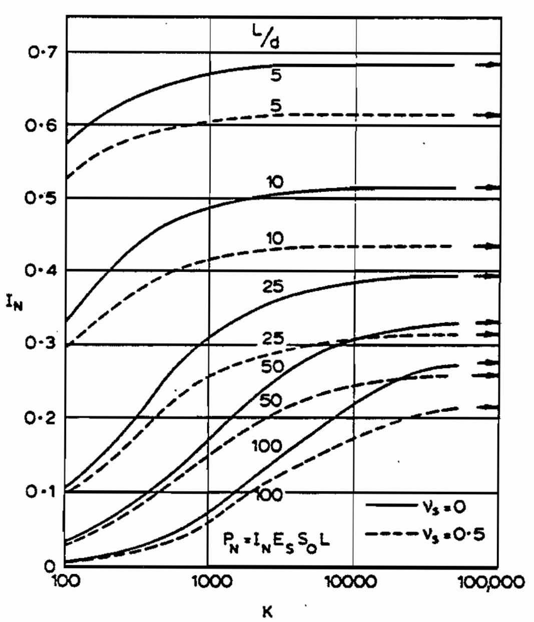


FIG.13.31 Influence factors for downdrag load at pile tip.

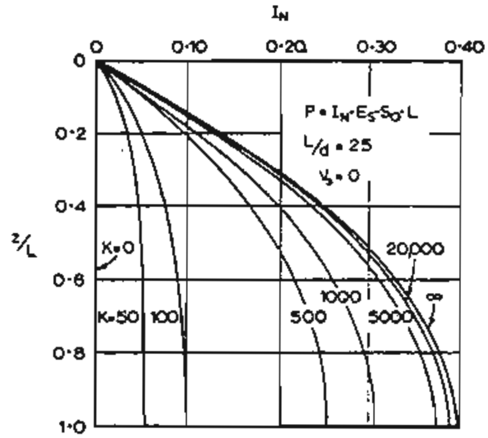


FIG.13.32 Distribution of downdrag load along pile.

13.5 Floating Pile Groups

13.5.1 INTERACTION BETWEEN TWO IDENTICAL PILES

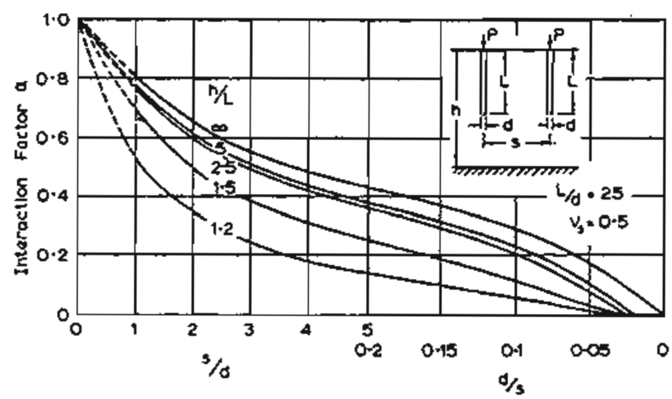
The increase in vertical displacement of a pile due to an adjacent identical pile has been considered by Poulos (1968c) and Poulos and Mattes (1971b) in terms of an interaction factor α where

α = ratio of increase in displacement due to adjacent pile to displacement of single pile only.

The variation of α with centre-to-centre pile spacing s/d is shown in Fig.13.33 for two incompressible piles in a finite layer. An example of the effect of ν_s on α is shown in Fig.13.34. Curves of α *vs.* s/d for two compressible piles in a semi-infinite mass having $\nu_s = 0.5$ are shown in Figs. 13.35 (a) to (c) three values of L/d.

For compressible piles with a rigid circular cap resting on the surface, interaction curves are given by Davis and Poulos (1972).

FIG.13.33 Interaction factors for two incompressible piles in a finite layer.



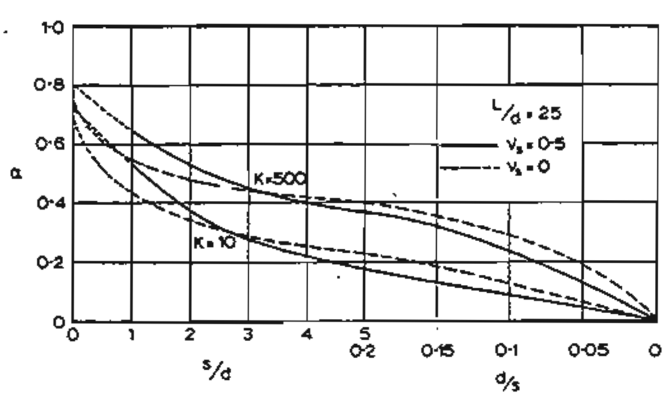
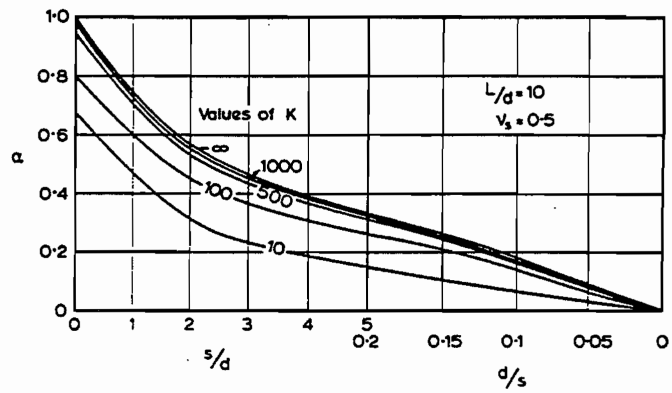
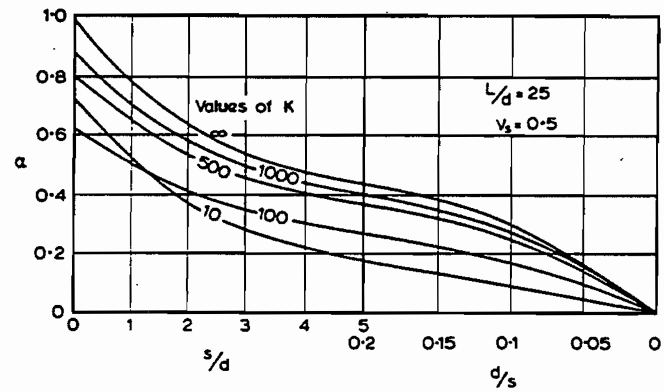


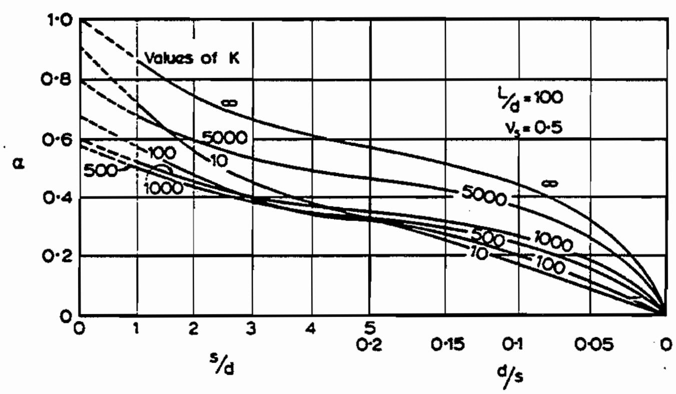
FIG.13.34 Effect of ν_s on interaction factors for two floating piles in a semi-infinite mass.



(a) L/d = 10



(b) L/d = 25



(c) L/d = 100

FIG.13.35 Interaction factors for two floating piles in a semi-infinite mass.

13.5.2 ANALYSIS OF GENERAL PILE GROUPS

The two-pile interaction factors in Figs.13.33 and 13.34 may be used to analyze the displacement and load distribution in any general pile group by using the principle of superposition, which has been found to apply closely for pile groups.

For any pile i in a group of k piles, the displacement is

$$\rho_i = \rho_1 \Big(\sum_{\substack{j=1 \\ j \neq i}}^{k} P_j \alpha_{ij} + P_i \Big) \qquad \ldots (13.3)$$

where ρ_1 = displacement of single pile under unit load

α_{ij} = interaction factor for spacing between piles i and j

P_j = load in pile j.

If the above equation is written for all the piles in the group, and use is made of the equilibrium equation

$$P_G = \sum_{j=1}^{k} P_j \qquad \ldots (13.4)$$

where P_G = total group load,

the resulting equations may be solved for two limiting cases:

(i) equal displacement of all piles. This corresponds to a rigid pile cap, and the distribution of load and the uniform displacement of the group may be computed.

(ii) equal load in all piles. This corresponds to a uniformly-loaded flexible pile cap, and the distribution of displacement in the group may be computed.

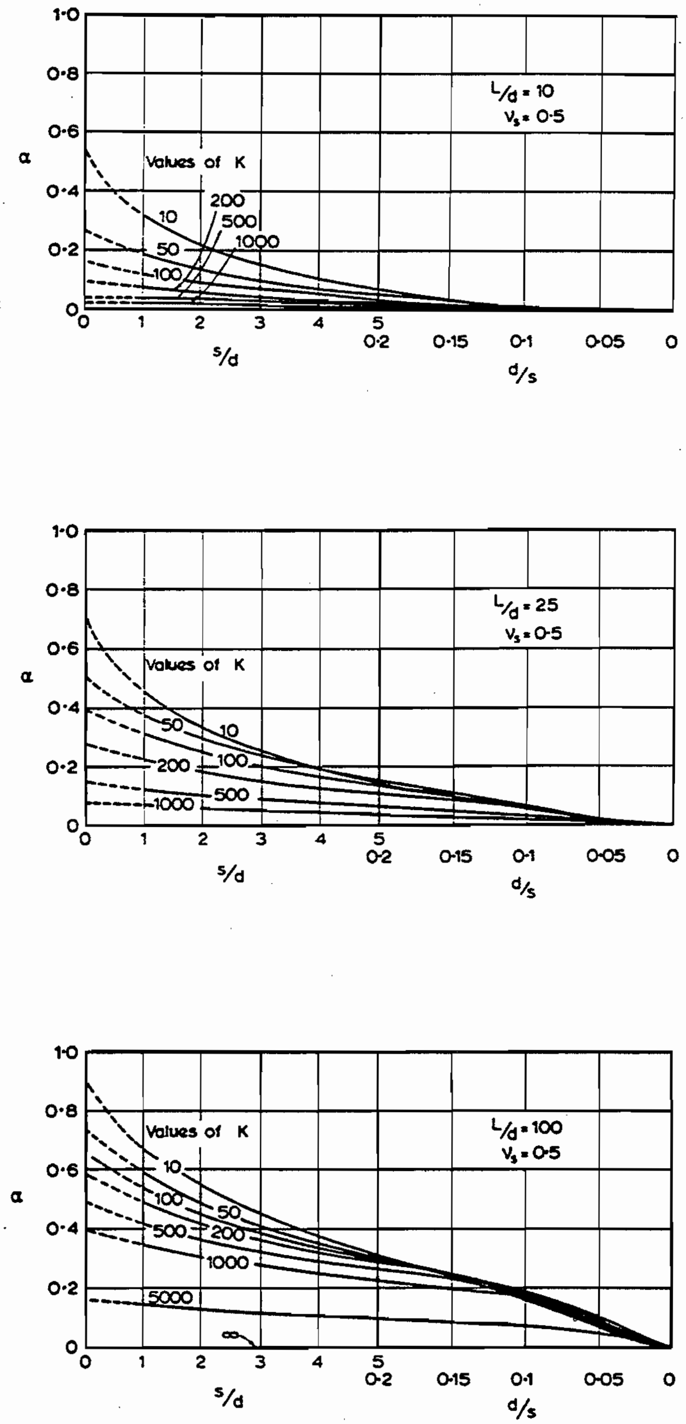
Typical solutions for the settlement and load distribution in various pile groups are given by Poulos (1968c) and Poulos and Mattes (1971b).

Similar solutions for pile groups having a pile cap resting on the surface are given by Davis and Poulos (1972).

13.6 End-Bearing Pile Groups

For two identical piles resting on a rigid base, interaction factors α are plotted against centre-to-centre spacing in Figs. 13.36(a) to (c).

As for floating pile groups, superposition may be used to analyze any general pile group. Typical solutions for the displacement of load distribution within groups of end-bearing piles are presented by Poulos and Mattes(1971b).



(a) L/d = 10

(b) L/d = 25

(c) L/d = 100

FIG.13.36 Interaction factors for two end-bearing piles resting on a rigid bearing stratum.

Chapter 14
PILES SUBJECTED TO LATERAL LOAD AND MOMENT

14.1 Single Floating Pile

14.1.1 HORIZONTAL MOVEMENTS AND ROTATIONS (Fig.14.1)

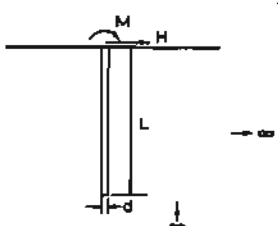


FIG.14.1

This problem has been considered by Poulos (1971a).

Influence factors for the displacement and rotation at the top of a pile in a uniform semi-infinite elastic mass are given in Figs.14.2 to 14.5 for two cases:

(i) a free-head pile i.e. free rotation at the top,

(ii) a fixed-head pile i.e. no rotation at the pile top.

For the free-head pile, the horizontal displacement ρ at the pile top is given as

$$\rho = I_{\rho H} \frac{H}{E_s L} + I_{\rho M} \frac{M}{E_s L^2} \qquad \ldots (14.1)$$

where $I_{\rho H}$ and $I_{\rho M}$ are plotted in Figs.14.2 and 14.3 against K_R, where

K_R = pile flexibility factor

$$= \frac{E_p I_p}{E_s L^4} \qquad \ldots (14.2)$$

$E_p I_p$ = pile stiffness

E_s = soil modulus.

The rotation θ at the top of a free-head pile is

$$\theta = I_{\theta H} \frac{H}{E_s L^2} + I_{\theta M} \frac{M}{E_s L^3} \qquad \ldots (14.3)$$

where $I_{\theta M}$ is plotted against K_R in Fig.14.5

$I_{\theta H} = I_{\rho M}$ (Fig.14.3).

For a fixed-head pile, the displacement at the pile top is

$$\rho = I_{\rho F} \frac{H}{E_s L} \qquad \ldots (14.4)$$

where $I_{\rho F}$ is plotted against K_R in Fig.14.4.

In Figs. 14.2 to 14.5, ν of the soil is 0.5.

This parameter has a relatively small effect on the displacement and rotation factors.

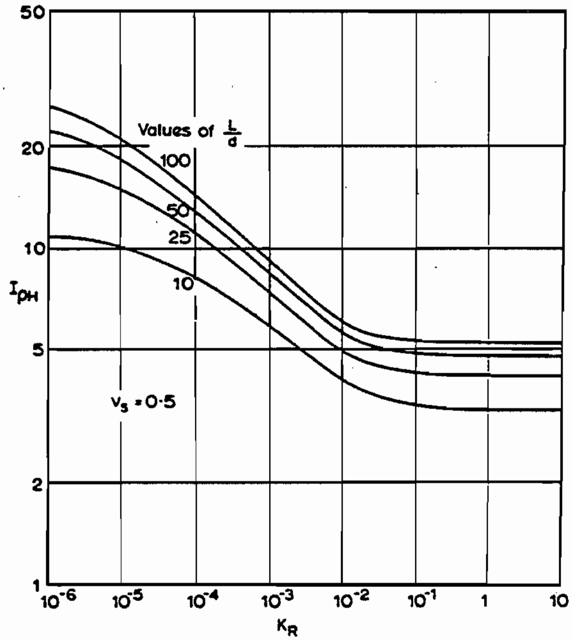


FIG.14.2 Influence factor $I_{\rho H}$ for free-head pile.

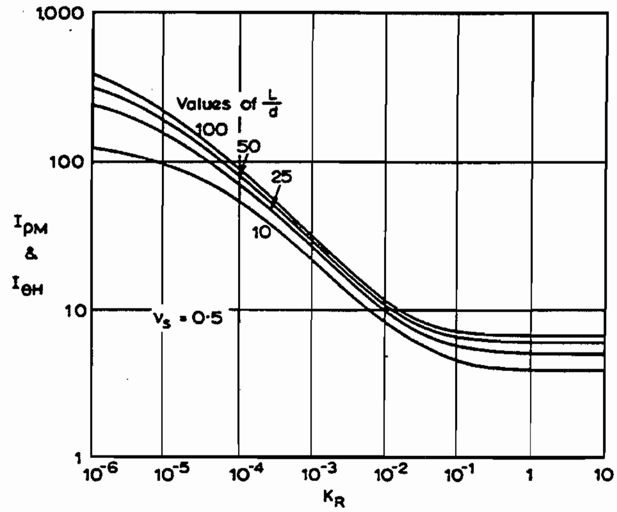


FIG.14.3 Influence factors $I_{\theta H}$ and $I_{\rho M}$ for free-head pile.

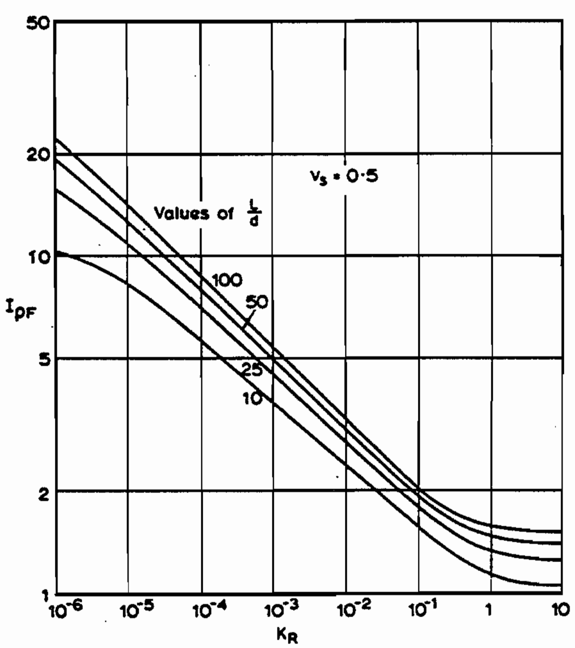


FIG.14.4 Influence factor $I_{\rho F}$ for fixed-head pile.

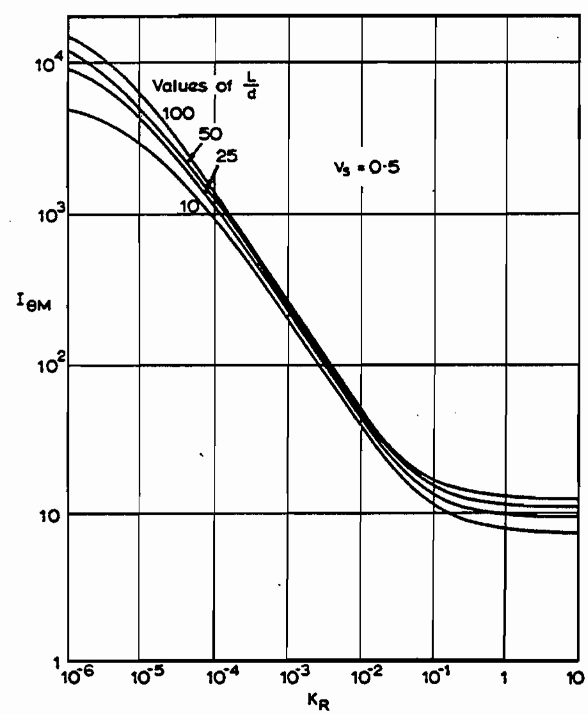


FIG.14.5 Influence factor $I_{\theta M}$ for free-head pile.

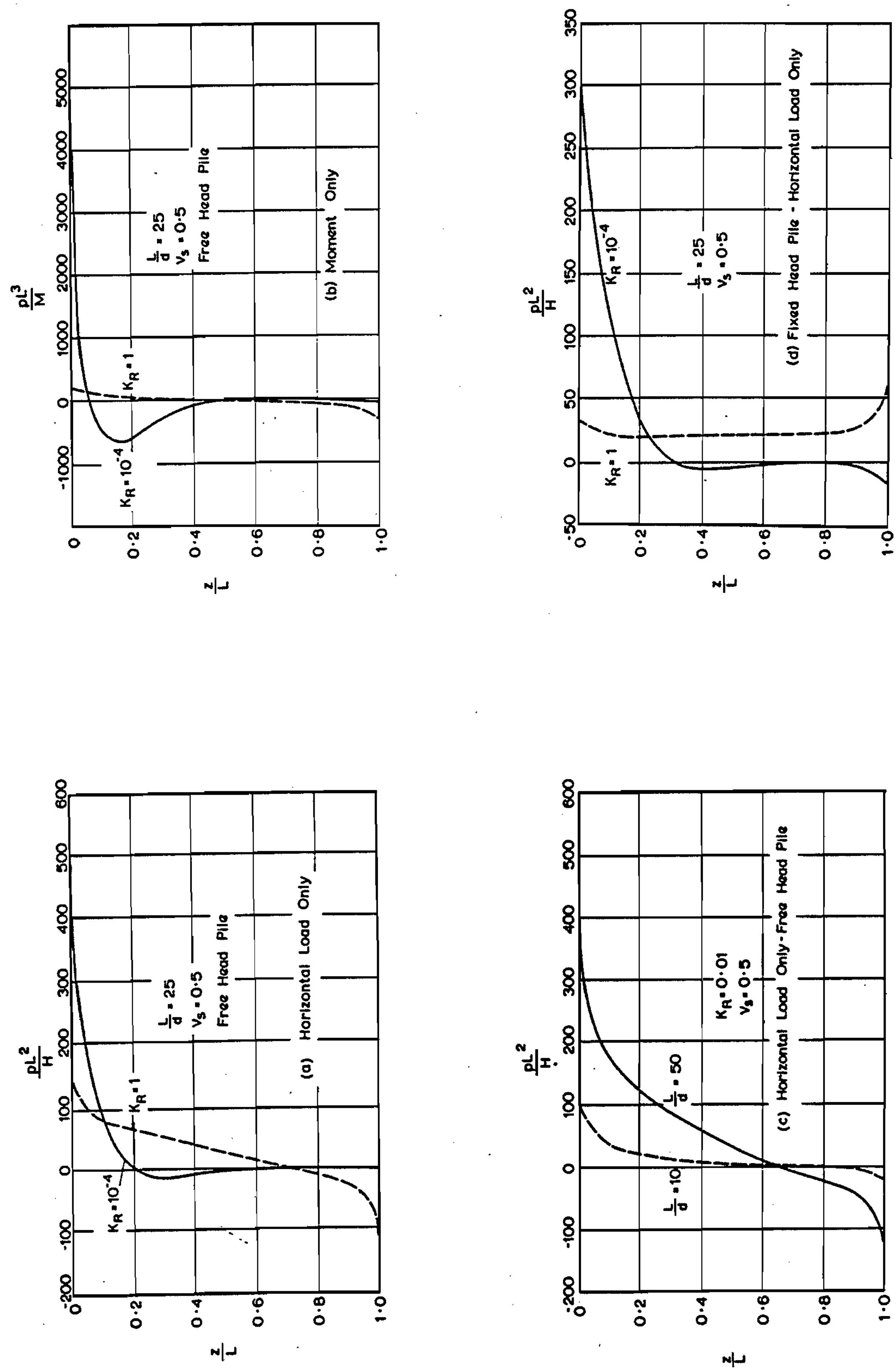


FIG.14.6 Typical horizontal pressure distributions along laterally loaded piles.

14.1.2 HORIZONTAL STRESS DISTRIBUTION

Typical horizontal pressure distributions are given in Figs.14.6(a) to (d).

14.1.3 MOMENTS IN PILE

Typical moment distributions along a free-head pile are shown in Fig.14.7 and along a fixed-head pile in Fig.14.8.

The maximum moment in a free-head pile subject to horizontal load only is plotted against K_R in Fig.14.9. For a pile subjected to moment only, the maximum moment always occurs at the pile top.

The variation with K_R of fixing moment at the top of a fixed-head pile is shown in Fig.14.10.

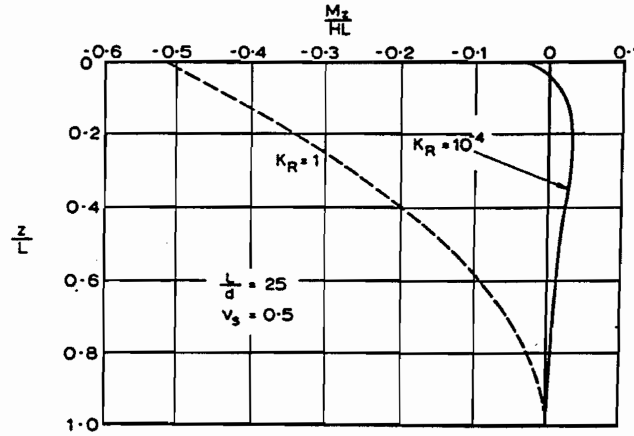
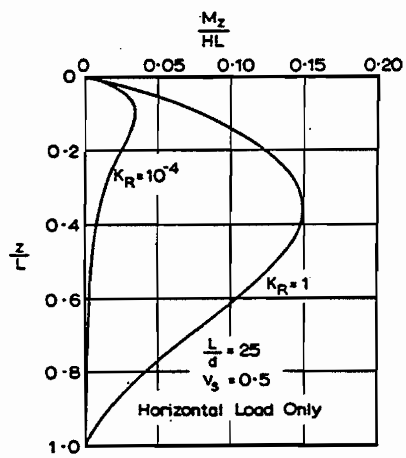
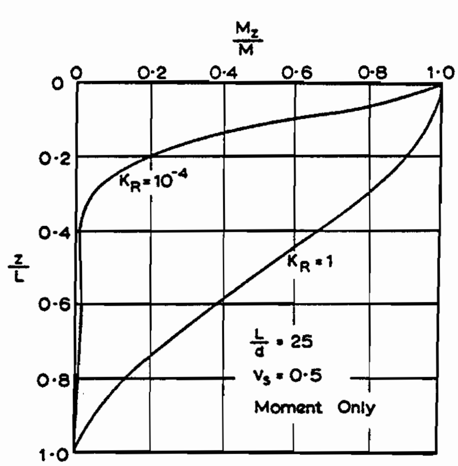


FIG.14.8 Typical moment distributions along a fixed-head pile.



(a)



(b)

FIG.14.7 Typical moment distributions along a free-head pile
(a) subjected to horizontal load only
(b) subjected to moment only.

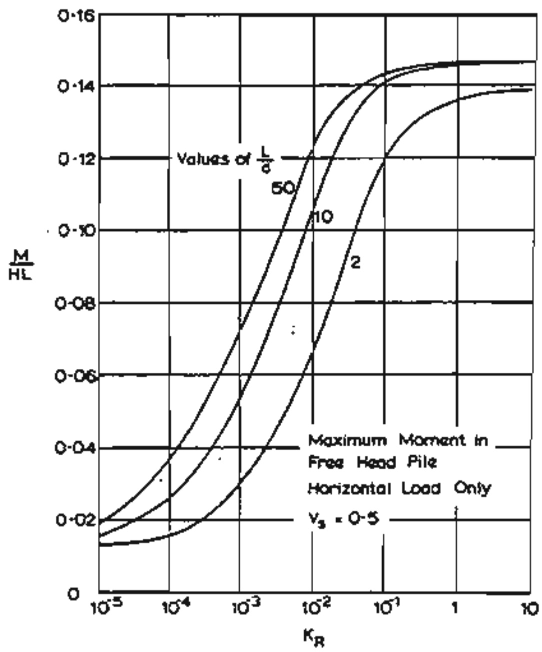


FIG.14.9 Maximum moment along a free-head pile subjected to horizontal load only.

14.2 Tip-Restrained Piles

For a pile whose tip rests on a rigid base and does not move horizontally, influence factors for the displacements and rotation at the top of the pile are given in Figs.14.11 to 14.14. The actual displacements and rotations are again given by equations (14.1) to (14.4). Two boundary conditions at the top of the pile, free-head and fixed-head, and two boundary conditions at the pile tip, a pinned tip (no displacement, free rotation) and a fixed tip (no displacement, no rotation) are considered. These figures show that the tip boundary condition does not influence displacement or rotation unless $K_R > 10^{-3}$.

For fixed head piles, the fixing moment at the pile head is shown in Fig.14.15.

For fixed tip piles, the fixing moment at the pile tip is shown in Fig.14.16 for applied horizontal load and in Fig.14.17 for applied moment.

For free-head piles, the maximum moment in the pile is plotted against K_R in Fig.14.18.

The force at the tip is shown in Fig.14.19 for free-head piles and in Fig.14.20 for fixed head piles.

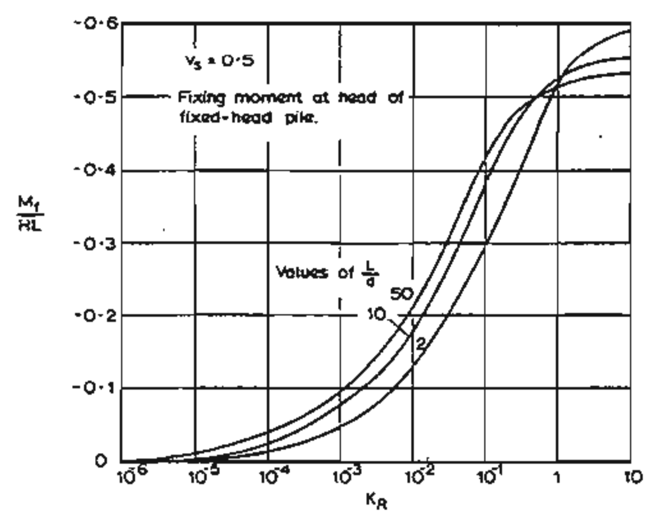


FIG.14.10 Fixing moment at head of a fixed-head pile.

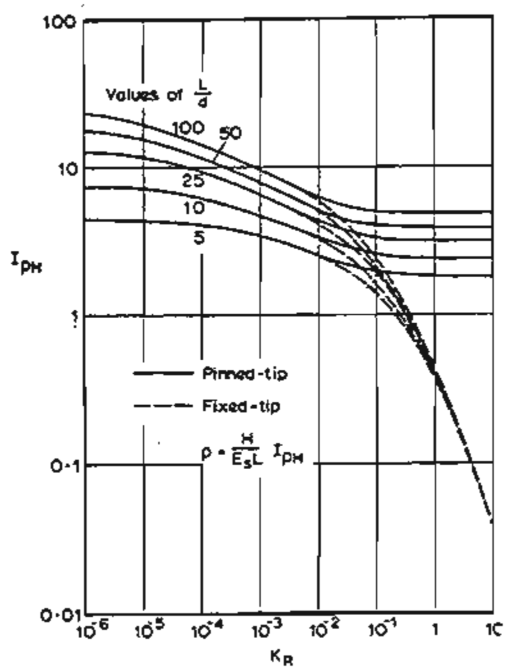


FIG.14.11 Influence factor $I_{\rho H}$ for free-head pile.

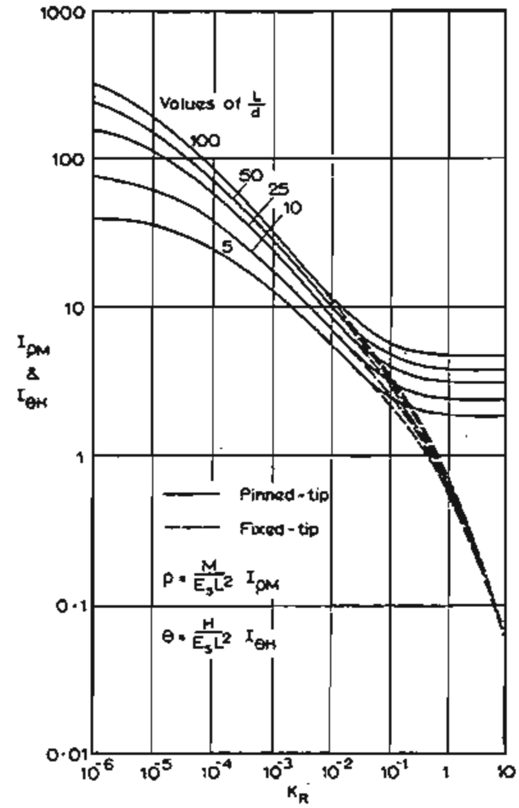


FIG.14.12 Influence factors $I_{\rho M}$ and $I_{\theta H}$ for free-head pile.

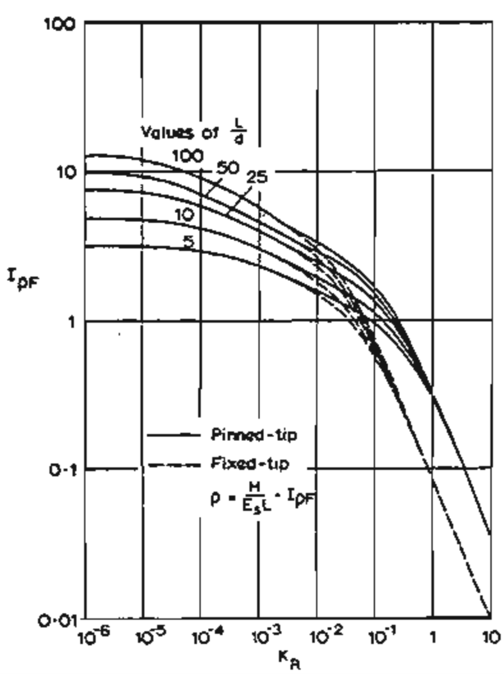


FIG.14.13 Influence factor $I_{\rho f}$ for fixed-head pile.

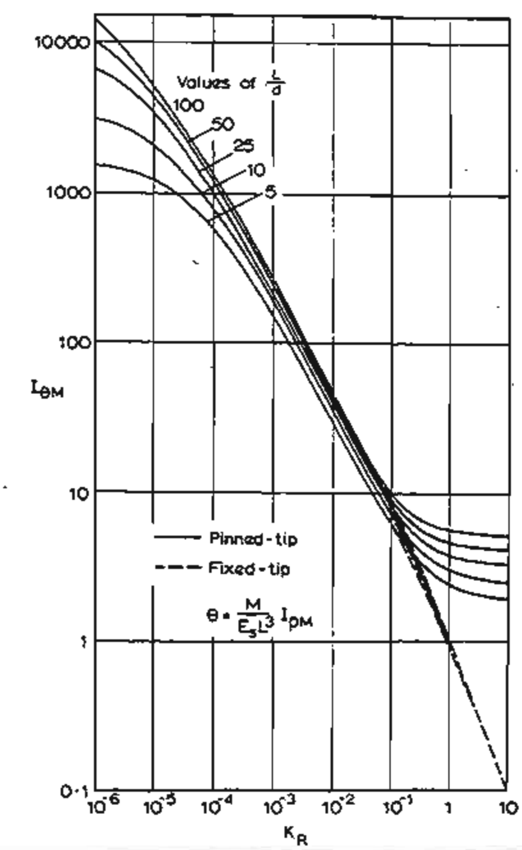


FIG.14.14 Influence factor $I_{\theta M}$ for free-head pile.

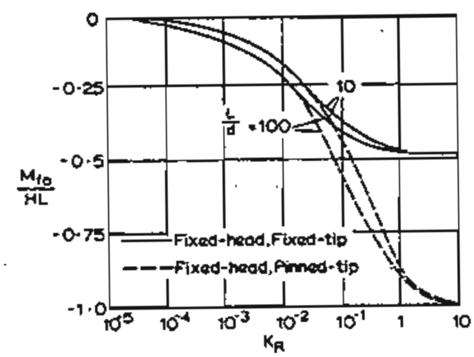


FIG.14.15 Fixing moment at head of fixed-head pile.

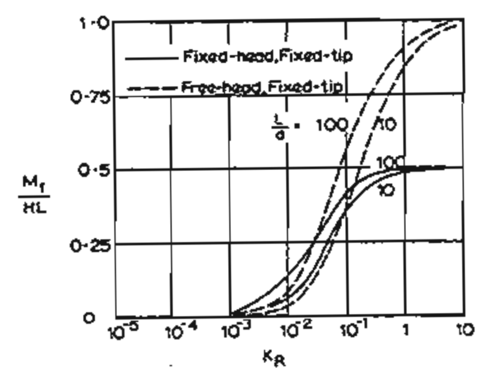


FIG.14.16 Fixing moment at tip due to horizontal load only.

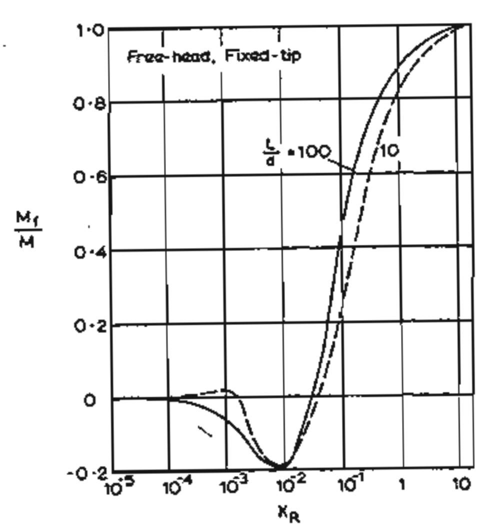


FIG.14.17 Fixing moment at tip due to applied moment only.

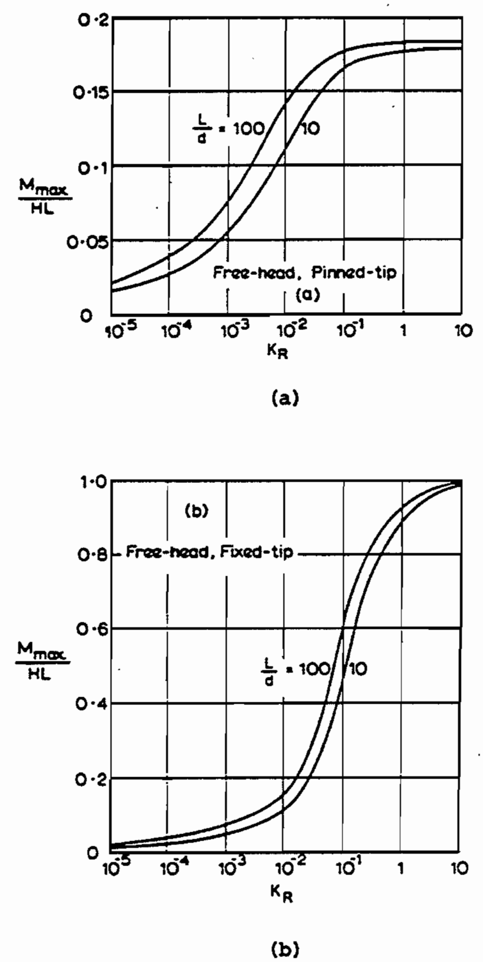


FIG.14.18 Maximum moment in free-head pile subjected to horizontal load only.

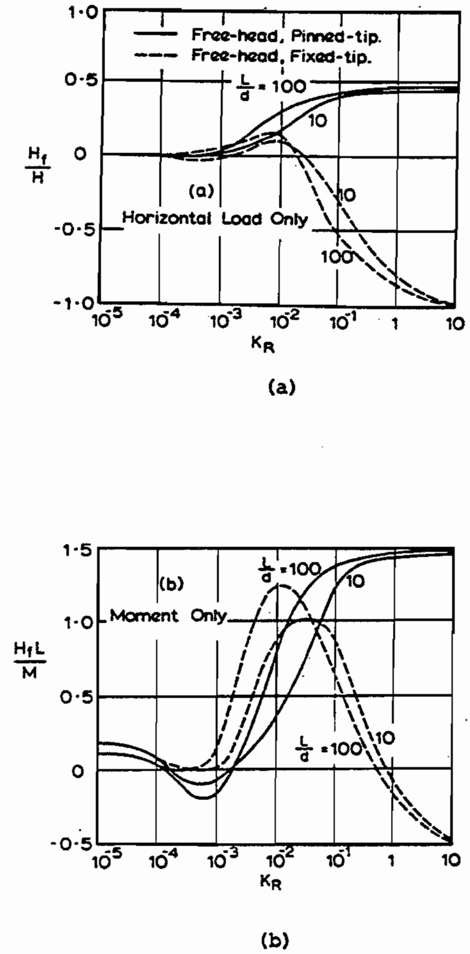


FIG.14.19 Tip force for free-head piles.

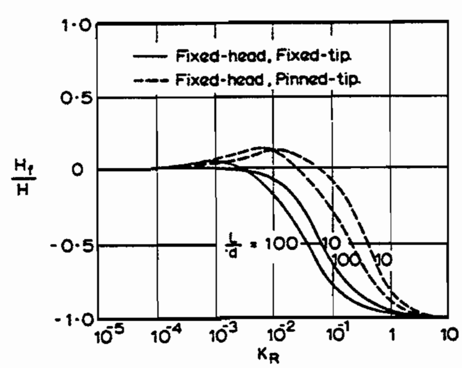


FIG.14.20 Tip force for fixed-head piles.

14.3 Pile Groups

14.3.1 INTERACTION BETWEEN TWO IDENTICAL PILES (Fig.14.21)

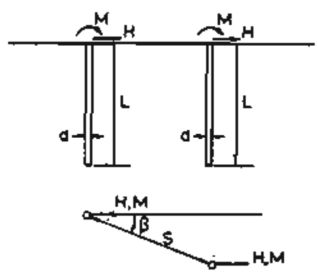


FIG.14.21

This problem has been considered by Poulos (1971b). Increases in displacement and rotation of the top of a pile due to the presence of an identical adjacent pile can be, as with axially-loaded piles, expressed in terms of an interaction factor α where

α = ratio of increase in displacement (or rotation) due to the adjacent pile to the displacement (or rotation) of a single pile.

Five interaction factors are considered:

$\alpha_{\rho H}$ = interaction factor for displacement due to horizontal load only

$\alpha_{\rho M}$ = interaction factor for displacement due to moment only

$\alpha_{\theta H}$ = interaction factor for rotation due to horizontal load only $(\alpha_{\theta H} = \alpha_{\rho M})$

$\alpha_{\theta M}$ = interaction factor for rotation due to moment only

(the above factors apply to free-head piles)

$\alpha_{\rho F}$ = interaction factor for displacement of fixed-head piles.

Values of $\alpha_{\rho H}$, $\alpha_{\rho M}$, $\alpha_{\theta M}$ and $\alpha_{\rho F}$ are plotted against dimensionless pile spacing s/d in Figs. 14.22 to 14.37 for various values of K_R and L/d. Interaction factors are plotted for values of β (angle between the line of the piles and the direction of loading) of $0°$ and $90°$. For other values of β, it is sufficiently accurate to interpolate linearly between the curves for $0°$ and $90°$.

14.3.2 ANALYSIS OF GENERAL PILE GROUPS

As with floating axially-loaded pile groups (Section 13.5.2), the principle of superposition may be used together with the two-pile interaction factors to compute the loads and displacement within the group for the cases of equal displacement of all piles, or equal loads in all piles.

The horizontal displacement of a pile i in a group of k piles is given (for the case of free-head piles) by:

$$\rho_i = \bar{\rho}_H \left(\sum_{\substack{j=1 \\ j \neq i}}^{k} H_j \, \alpha_{\rho H ij} + H_i \right) + \bar{\rho}_M \left(\sum_{\substack{j=1 \\ j \neq i}}^{k} M_j \, \alpha_{\rho M ij} + M_i \right) \quad \ldots (14.5)$$

where H_j = horizontal load in pile j

$\alpha_{\rho H ij}$ = value of $\alpha_{\rho H}$ for spacing and value of β between piles i and j

$\bar{\rho}_H$ = horizontal movement of single pile due to unit applied horizontal load

M_j = moment in pile j

$\alpha_{\rho M ij}$ = values of $\alpha_{\rho M}$ for spacing and values of β between piles i and j

$\bar{\rho}_M$ = horizontal movement of single pile due to unit applied moment.

A similar expression may be written for the rotation of pile i, or for the displacement of pile i for a group of fixed-head piles.

Application of the above equation to all piles in the group, together with the equilibrium equations enable solutions to be obtained from the load and moment distributions and the displacement and rotation of a group for the equal displacement case, or for the displacement and rotation distributions in a group for the equal load (and moment) case. For moment loading the effect of the axial pile loads must be considered.

Typical solutions for the displacement of a fixed-head group of piles, for the equal displacement case, are given by Poulos (1971b).

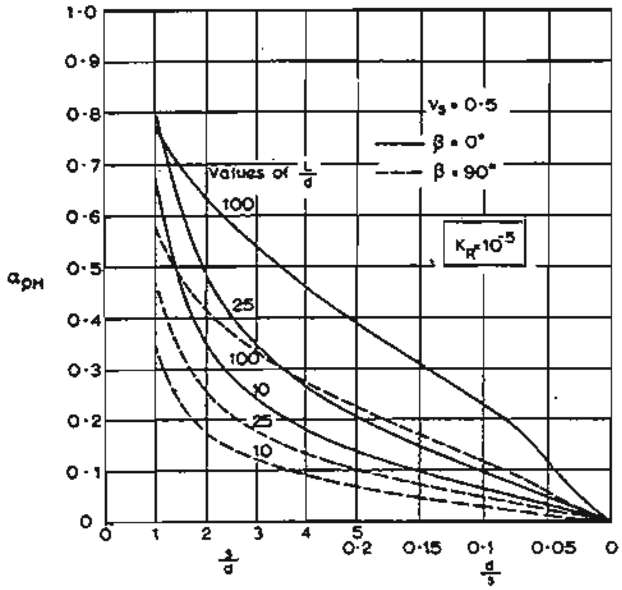


FIG.14.22 Interaction factor $\alpha_{\rho H}$. $K_R = 10^{-5}$

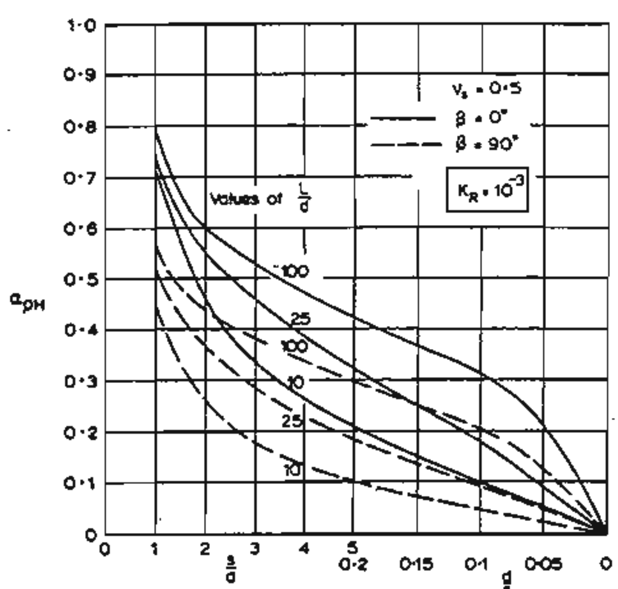


FIG.14.23 Interaction factor $\alpha_{\rho H}$. $K_R = 10^{-3}$

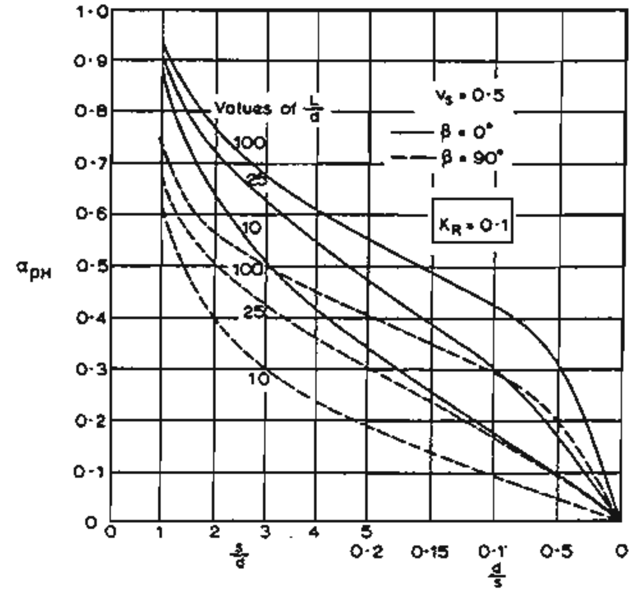


FIG.14.24 Interaction factor $\alpha_{\rho H}$. $K_R = 0.1$

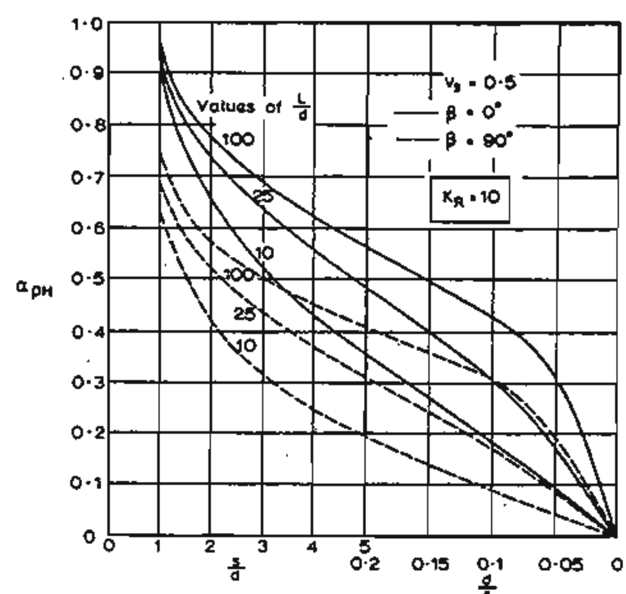


FIG.14.25 Interaction factor $\alpha_{\rho H}$. $K_R = 10$

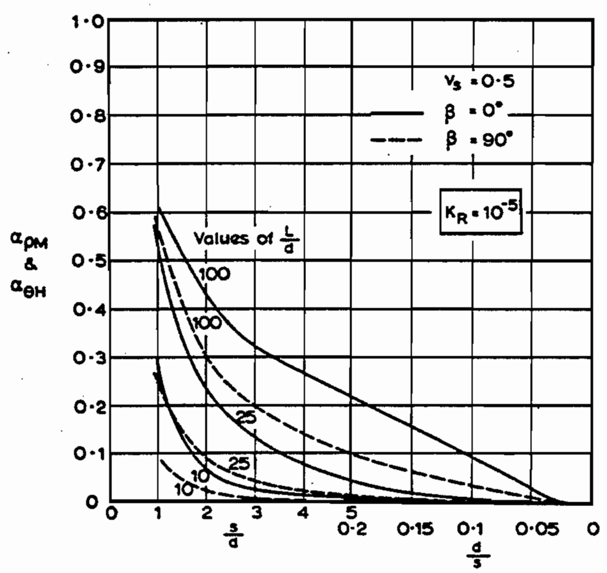


FIG.14.26 Interaction factors $\alpha_{\rho M}$ and $\alpha_{\theta H}$. $K_R = 10^{-5}$

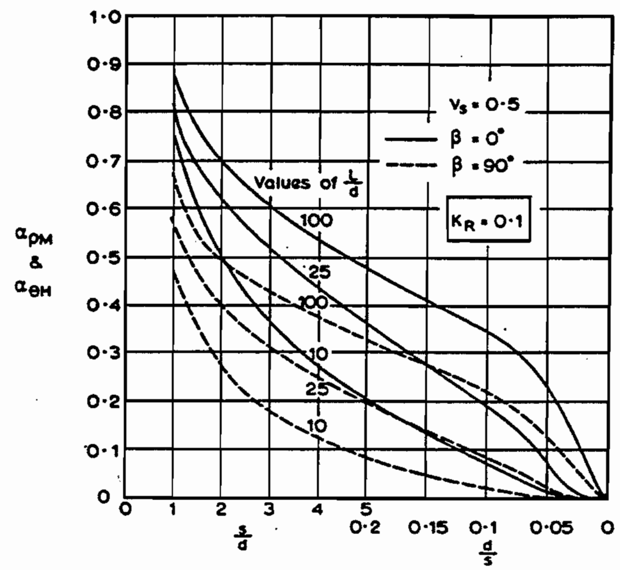


FIG.14.28 Interaction factors $\alpha_{\rho M}$ and $\alpha_{\theta H}$. $K_R = 0.1$

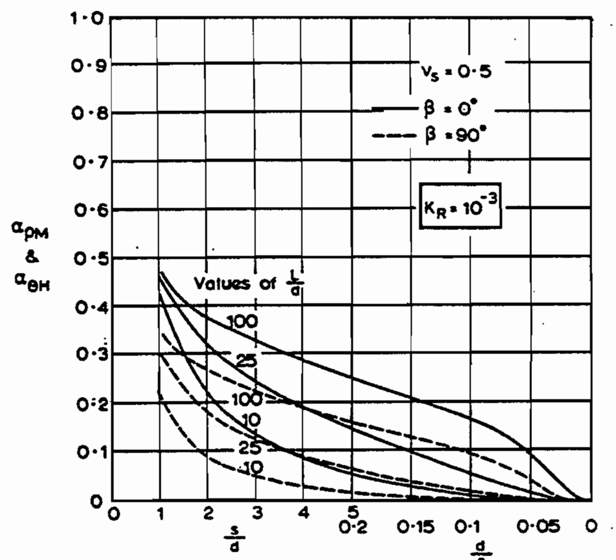


FIG.14.27 Interaction factors $\alpha_{\rho M}$ and $\alpha_{\theta H}$. $K_R = 10^{-3}$

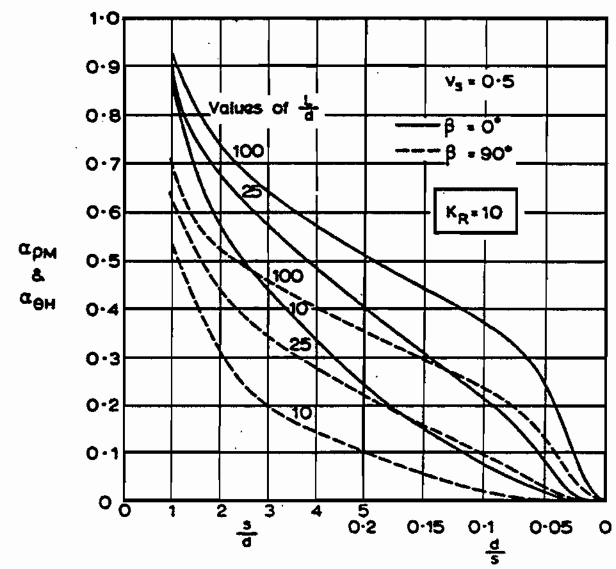


FIG.14.29 Interaction factors $\alpha_{\rho M}$ and $\alpha_{\theta H}$. $K_R = 10$

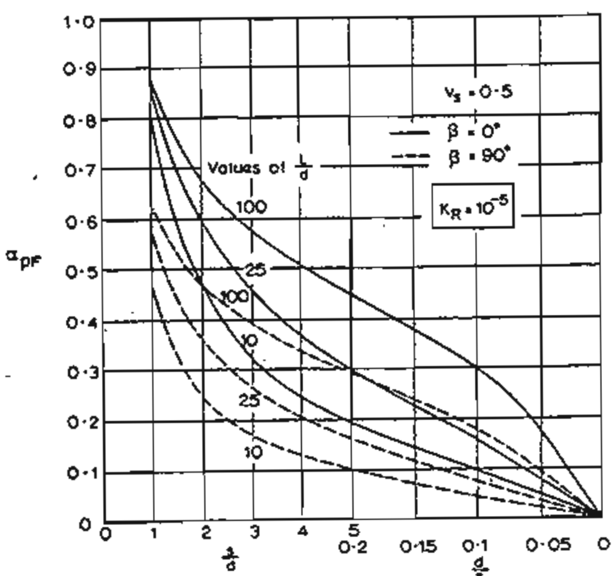


FIG.14.30 Interaction factor $\alpha_{\rho F}$. $K_R = 10^{-5}$

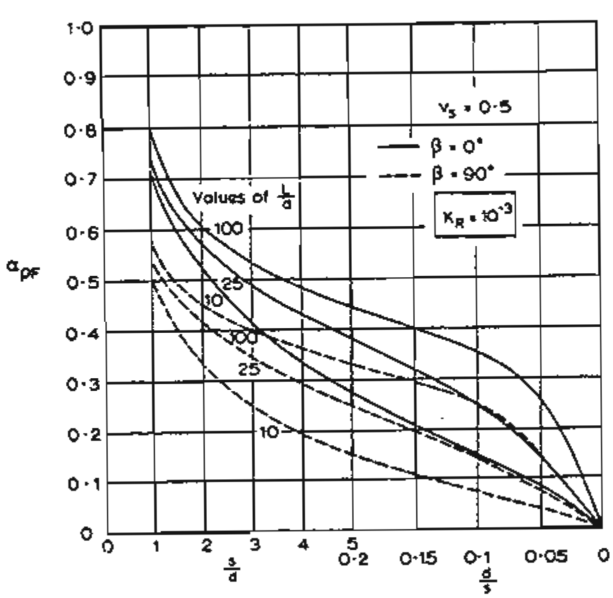


FIG.14.31 Interaction factor $\alpha_{\rho F}$. $K_R = 10^{-3}$

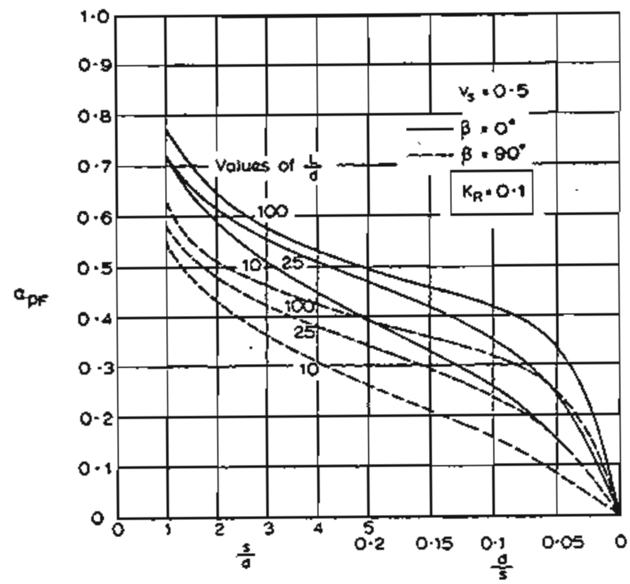


FIG.14.32 Interaction factor $\alpha_{\rho F}$. $K_R = 0.1$

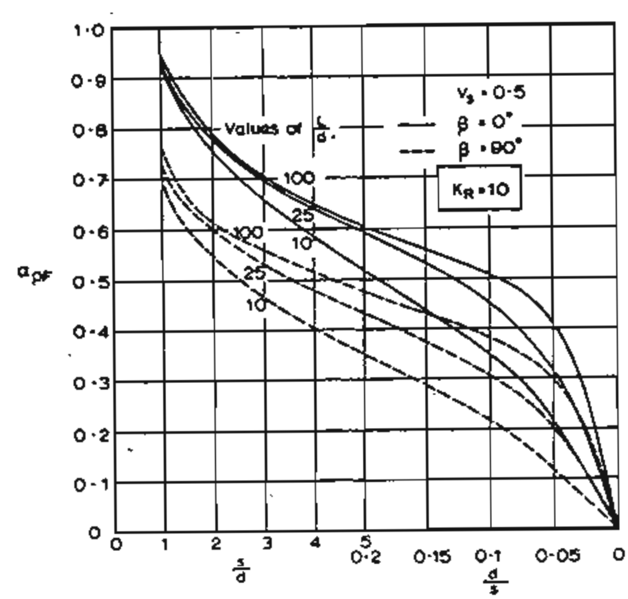


FIG.14.33 Interaction factor $\alpha_{\rho F}$. $K_R = 10$

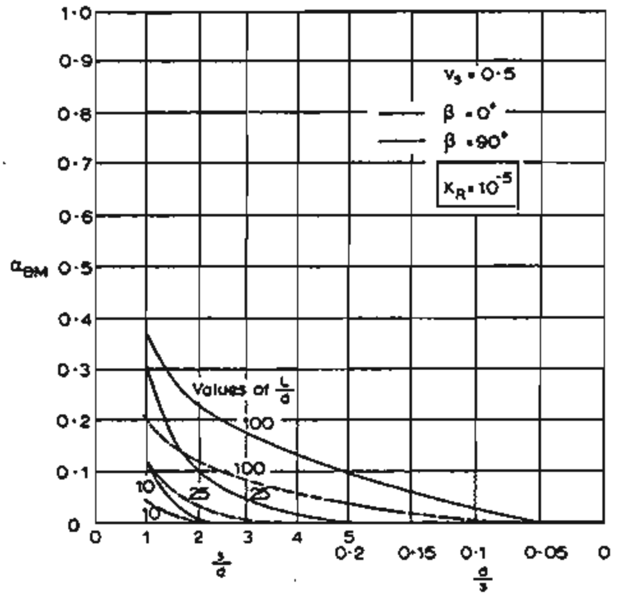


FIG.14.34 Interaction factor $\alpha_{\theta M}$.
$K_R = 10^{-5}$

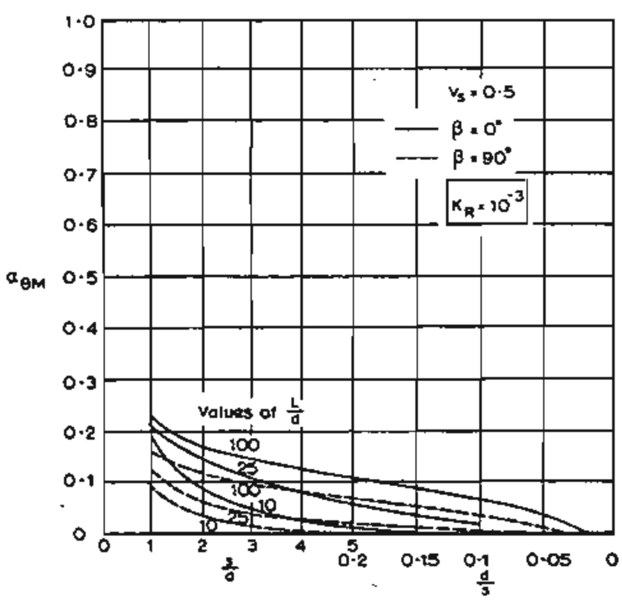


FIG.14.35 Interaction factor $\alpha_{\theta M}$.
$K_R = 10^{-3}$

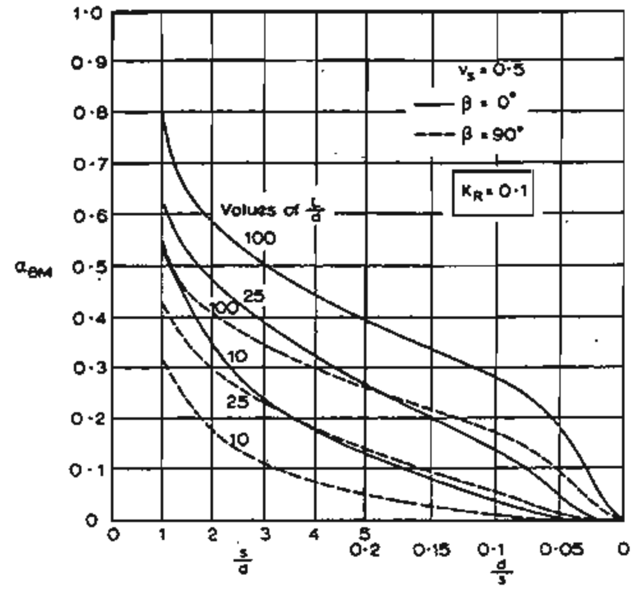


FIG.14.36 Interaction factor $\alpha_{\theta M}$.
$K_R = 0.1$

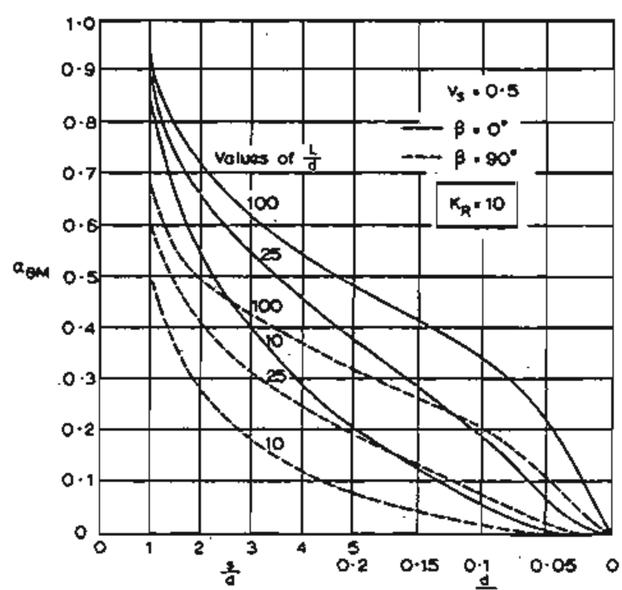


FIG.14.37 Interaction factor $\alpha_{\theta M}$.
$K_R = 10$

14.4 Battered Piles

This problem has been considered by Poulos and Madhav (1971). For normal batter angles, it is found that the axial displacement due to the axial component of load on a battered pile are almost identical with the vertical displacements due to vertical load on a vertical pile (Chapter 13). Similarly, the normal displacement due to the normal component of load on a battered pile may be taken as approximately equal to the horizontal displacement due to horizontal load on a vertical pile (Sections 14.1 and 14.2).

If the simplifying assumption is made that the normal load has a negligible effect on axial displacement and that the axial load has a negligible effect on normal displacement, the following expressions may be derived for the vertical and horizontal displacements and the rotation of a battered pile subjected to a vertical load V, a horizontal load H and a moment M at the surface of the mass:

Vertical displacement:

$$\rho_V = \frac{1}{E_s L}\left\{V.I_{vV} + H.I_{vH} + \frac{M}{L}.I_{vM}\right\} \qquad \ldots (14.6)$$

where $I_{vV} = I_\rho \cos^2\psi - I_{\rho H}\sin^2\psi$

$I_{vH} = (I_\rho - I_{\rho H})\sin\psi\cos\psi$

$I_{vM} = -I_{\rho M}\sin\psi$

Horizontal displacement of free-head pile:

$$\rho_h = \frac{1}{E_s L}\left\{V.I_{hV} + H.I_{hH} + \frac{M}{L}\cdot I_{hM}\right\} \qquad \ldots (14.7)$$

where $I_{hV} = (I_\rho - I_{\rho H})\sin\psi\cos\psi$

$I_{hH} = I_\rho \sin^2\psi + I_{\rho H}\cos^2\psi$

$I_{hM} = I_{\rho M}\cos\psi$

Rotation of free-head pile:

$$\theta = \frac{1}{E_s L^2}\left\{V.I'_{\theta V} + H.I'_{\theta H} + \frac{M}{L}.I'_{\theta M}\right\} \qquad \ldots (14.8)$$

where $I'_{\theta V} = -I_{\theta H}\sin\psi$

$I'_{\theta H} = I_{\theta H}\cos\psi$

$I'_{\theta M} = I_{\theta M}$

Horizontal displacement of fixed-head pile:

$$\rho_{hF} = \frac{1}{E_s L}\left\{V.I_{FV} + H.I_{FH}\right\} \qquad \ldots (14.9)$$

where $I_{FV} = (I - I_{\rho F})\sin\psi\cos\psi$

$I_{FH} = I_\rho \sin^2\psi + I_{\rho F}\cos^2\psi$

In equations (14.6) to (14.9)

ψ = angle of batter of pile from vertical (positive batter is in the direction of the horizontal load or moment)

I_ρ = displacement influence factor for axially loaded pile, e.g. Figs. 13.3 to 13.6, Fig.13.12

$I_{\rho H}, I_{\rho M}, I_{\theta H}, I_{\theta M}, I_{\rho F}$

= horizontal displacement and rotation influence factors for laterally loaded pile (see Sections 14.1 and 14.2).

Chapter 15
MISCELLANEOUS PROBLEMS

15.1 Thick-Wall Cylinder in Triaxial Stress Field (Fig. 15.1)

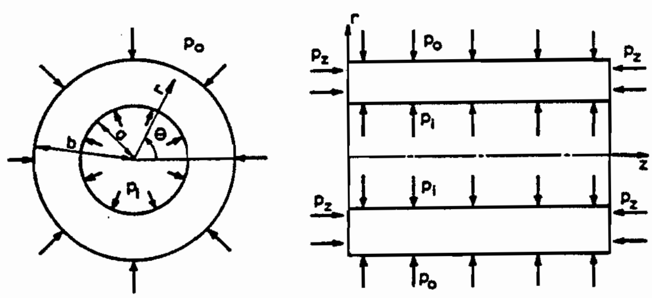


FIG.15.1

$$\sigma_z = p_z \qquad \text{... (15.1a)}$$

$$\sigma_r = -\frac{a^2b^2(p_o-p_i)}{(b^2-a^2)r^2} - \frac{p_ia^2-p_ob^2}{b^2-a^2} \qquad \text{... (15.1b)}$$

$$\sigma_\theta = +\frac{a^2b^2(p_o-p_i)}{(b^2-a^2)r^2} - \frac{p_ia^2-p_ob^2}{b^2-a^2} \qquad \text{... (15.1c)}$$

$$\tau_{r\theta} = 0 \qquad \text{... (15.1d)}$$

$$\rho_z = \frac{z}{E}\left[p_z + 2\nu\frac{p_ia^2-p_ob^2}{b^2-a^2}\right] \qquad \text{... (15.1e)}$$

$$\rho_r = \frac{1+\nu}{E}\left[\frac{a^2b^2(p_o-p_i)}{(b^2-a^2)r} - \frac{(p_ia^2-p_ob^2)}{b^2-a^2}(1-2\nu)r\right] + \frac{\nu r}{E}\left[p_z + \frac{2\nu(p_ia^2-p_ob^2)}{b^2-a^2}\right] \qquad \text{.. (15.1f)}$$

$$\rho_\theta = 0. \qquad \text{... (15.1g)}$$

15.2 Cylinder With Rough Rigid End Plates (Fig. 15.2)

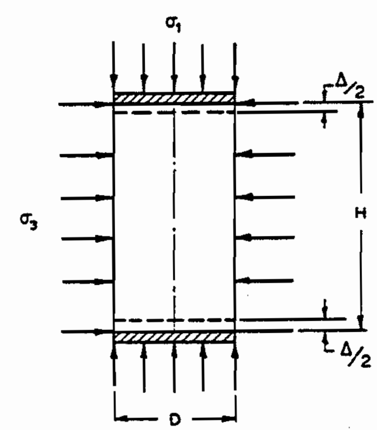


FIG.15.2

Moore (1966) has obtained solutions for the stresses and displacements within the cylinder for various values of ν, H/D and σ_1/σ_3. For unconfined compression $(\sigma_3=0)$, these solutions are given in Tables 15.1 and 15.2. For no axial movement $(\Delta=0)$, the solutions for $H/D=2$ are shown in Table 15.3. In Tables 15.1 to 15.3, the vertical displacement ρ_z is taken as positive when directed towards the centre of the cylinder, σ_1 and σ_3 being compressive.

The relationships between the apparent and true Poisson's ratio and Young's modulus of the cylinder are shown in Figs.15.3 and 15.4.

The solution for any ratio of σ_1/σ_3 can be derived from a combination of the solution for unconfined compression (Tables 15.1 and 15.2), with the solution for $\sigma_3/\sigma_1 = \nu/(1-\nu)$. This latter solution can be calculated straightforwardly since there are no radial strains throughout the cylinder in this case, i.e. $\sigma_z=\sigma_1$, $\sigma_r=\sigma_3$ and the roughness of the plates has no effect.

TABLE 15.1

STRESSES AND DISPLACEMENTS FOR UNCONFINED COMPRESSION, $H/D = 1$

(Moore, 1966)

	2z/H	2r/D = 0	0.25	0.50	0.75	1	2r/D = 0	0.25	0.50	0.75	1
		ρ_r/Δ (Outwards)					ρ_z/Δ				
$\nu = 0.25$, $\sigma_3 = 0$, $H/D = 1$	1	0	0	0	0	0	0.500	0.500	0.500	0.500	0.500
	0.75	0	0.015	0.032	0.054	0.085	0.394	0.393	0.392	0.385	0.357
	0.50	0	0.026	0.053	0.084	0.115	0.271	0.270	0.266	0.257	0.240
	0.25	0	0.032	0.064	0.096	0.127	0.137	0.137	0.134	0.128	0.121
	0	0	0.033	0.067	0.099	0.130	0	0	0	0	0
		τ_{rz}/σ_1					σ_z/σ_1				
	1	0	0.048	0.101	0.173	0.686	0.892	0.893	0.898	0.932	3.130
	0.75	0	0.039	0.076	0.092	0	0.980	0.982	0.994	1.022	0.924
	0.50	0	0.020	0.030	0.018	0	1.039	1.038	1.033	1.006	0.908
	0.25	0	0.006	0.007	0.000	0	1.065	1.059	1.038	0.993	0.925
	0	0	0	0	0	0	1.072	1.064	1.037	0.989	0.933
		σ_r/σ_1					σ_θ/σ_1				
	1	0.297	0.298	0.299	0.311	1.043	0.297	0.298	0.299	0.311	1.043
	0.75	0.177	0.170	0.146	0.082	0	0.177	0.174	0.163	0.138	0.066
	0.50	0.083	0.076	0.053	0.018	0	0.083	0.079	0.066	0.041	0.005
	0.25	0.030	0.026	0.015	0.004	0	0.030	0.027	0.017	0.003	-0.013
	0	0.014	0.011	0.005	0.001	0	0.014	0.011	0.003	-0.007	-0.017
		ρ_r/Δ (Outwards)					ρ_z/Δ				
$\nu = 0.40$, $\sigma_3 = 0$, $H/D = 1$	1	0	0	0	0	0	0.500	0.500	0.500	0.500	0.500
	0.75	0	0.029	0.060	0.101	0.154	0.420	0.419	0.416	0.406	0.359
	0.50	0	0.048	0.099	0.152	0.203	0.299	0.298	0.291	0.275	0.246
	0.25	0	0.058	0.116	0.172	0.221	0.155	0.153	0.148	0.138	0.126
	0	0	0.061	0.121	0.177	0.226	0	0	0	0	0
		τ_{rz}/σ_1					σ_z/σ_1				
	1	0	0.078	0.165	0.280	1.236	0.862	0.859	0.853	0.875	6.975
	0.75	0	0.061	0.119	0.142	0	1.002	1.002	1.006	1.025	0.811
	0.50	0	0.029	0.042	0.021	0	1.092	1.086	1.066	1.004	0.824
	0.25	0	0.008	0.007	-0.005	0	1.127	1.114	1.071	0.985	0.867
	0	0	0	0	0	0	1.136	1.120	1.069	0.981	0.885
		σ_r/σ_1					σ_θ/σ_1				
	1	0.574	0.572	0.569	0.583	4.650	0.574	0.572	0.569	0.583	4.650
	0.75	0.333	0.320	0.274	0.156	0	0.333	0.325	0.297	0.233	0.048
	0.50	0.154	0.140	0.099	0.037	0	0.154	0.144	0.112	0.053	-0.033
	0.25	0.055	0.047	0.028	0.009	0	0.055	0.046	0.023	-0.012	-0.049
	0	0.025	0.020	0.010	0.003	0	0.025	0.018	-0.002	-0.028	-0.051
		ρ_r/Δ (Outwards)					ρ_z/Δ				
$\nu = 0.45$, $\sigma_3 = 0$, $H/D = 1$	1	0	0	0	0	0	0.500	0.500	0.500	0.500	0.500
	0.75	0	0.035	0.073	0.120	0.180	0.434	0.434	0.431	0.419	0.367
	0.50	0	0.059	0.119	0.182	0.239	0.315	0.314	0.307	0.288	0.254
	0.25	0	0.071	0.140	0.206	0.262	0.164	0.163	0.157	0.146	0.130
	0	0	0.074	0.146	0.202	0.268	0	0	0	0	0
		τ_{rz}/σ_1					σ_z/σ_1				
	1	0	0.091	0.188	0.315	1.378	0.864	0.858	0.843	0.853	11.966
	0.75	0	0.068	0.133	0.161	0	1.021	1.017	1.011	1.016	0.757
	0.50	0	0.032	0.049	0.026	0	1.120	1.110	1.081	1.004	0.790
	0.25	0	0.009	0.009	-0.005	0	1.160	1.143	1.090	0.986	0.845
	0	0	0	0	0	0	1.171	1.152	1.090	0.985	0.868
		σ_r/σ_1					σ_θ/σ_1				
	1	0.707	0.702	0.690	0.698	9.791	0.707	0.702	0.690	0.698	9.791
	0.75	0.406	0.390	0.334	0.201	0	0.406	0.394	0.356	0.273	0.032
	0.50	0.189	0.173	0.124	0.051	0	0.189	0.175	0.134	0.060	-0.053
	0.25	0.069	0.059	0.035	0.011	0	0.069	0.058	0.027	-0.020	-0.068
	0	0.034	0.027	0.012	0.003	0	0.034	0.024	-0.003	-0.039	-0.069

TABLE 15.2

STRESSES AND DISPLACEMENTS FOR UNCONFINED COMPRESSION, $H/D = 2$
(Moore, 1966)

	$2z/H$	$2r/D$ 0	0.25	0.50	0.75	1	$2r/D$ 0	0.25	0.50	0.75	1
$\nu = 0.25$, $\sigma_3 = 0$, $H/D = 2$			ρ_r/Δ (Outwards)						ρ_z/Δ		
	1	0	0	0	0	0	0.500	0.500	0.500	0.500	0.500
	0.875	0	0.007	0.015	0.026	0.042	0.448	0.448	0.447	0.443	0.430
	0.750	0	0.012	0.025	0.035	0.056	0.389	0.388	0.386	0.381	0.372
	0.625	0	0.015	0.030	0.044	0.062	0.325	0.324	0.322	0.318	0.313
	0.500	0	0.016	0.032	0.047	0.064	0.260	0.259	0.257	0.254	0.252
	0.375	0	0.016	0.033	0.048	0.064	0.194	0.194	0.192	0.191	0.190
	0.250	0	0.016	0.033	0.049	0.064	0.129	0.129	0.128	0.127	0.127
	0.125	0	0.016	0.032	0.048	0.064	0.064	0.064	0.064	0.064	0.064
	0	0	0.016	0.033	0.048	0.064	0	0	0	0	0
			τ_{rz}/σ_1						σ_z/σ_1		
	1	0	0.046	0.098	0.170	0.683	0.886	0.887	0.895	0.932	3.118
	0.875	0	0.037	0.072	0.089	0	0.969	0.973	0.989	1.021	0.926
	0.750	0	0.016	0.024	0.013	0	1.022	1.024	1.026	1.007	0.915
	0.625	0	0.001	-0.002	-0.008	0	1.039	1.037	1.026	0.997	0.943
	0.500	0	-0.005	-0.010	-0.011	0	1.035	1.031	1.018	0.995	0.967
	0.375	0	-0.005	-0.009	-0.008	0	1.025	1.022	1.011	0.996	0.984
	0.250	0	-0.004	-0.006	-0.005	0	1.016	1.014	1.007	0.999	0.993
	0.125	0	-0.002	-0.003	-0.002	0	1.011	1.009	1.005	1.000	0.998
	0	0	0	0	0	0	1.009	1.008	1.004	1.000	0.999
			σ_r/σ_1						σ_θ/σ_1		
	1	0.295	0.296	0.298	0.311	1.039	0.295	0.296	0.298	0.311	1.039
	0.875	0.179	0.172	0.147	0.082	0	0.179	0.176	0.165	0.140	0.068
	0.750	0.087	0.079	0.054	0.018	0	0.087	0.083	0.070	0.045	0.010
	0.625	0.033	0.028	0.016	0.005	0	0.033	0.030	0.021	0.008	-0.006
	0.500	0.007	0.006	0.003	0.001	0	0.007	0.006	0.002	-0.003	-0.008
	0.375	-0.002	-0.002	-0.002	0.000	0	-0.002	-0.003	-0.004	-0.006	-0.007
	0.250	-0.005	-0.004	-0.003	-0.001	0	-0.005	-0.005	-0.005	-0.005	-0.004
	0.125	-0.005	-0.004	-0.003	-0.001	0	-0.005	-0.005	-0.005	-0.004	-0.003
	0	-0.005	-0.004	-0.002	-0.001	0	-0.005	-0.005	-0.004	-0.004	-0.003
$\nu = 0.4$, $\sigma_3 = 0$, $H/D = 2$			ρ_r/Δ (Outwards)						ρ_z/Δ		
	1	0	0	0	0	0	0.500	0.500	0.500	0.500	0.500
	0.875	0	0.013	0.027	0.046	0.072	0.463	0.463	0.461	0.456	0.434
	0.750	0	0.022	0.045	0.070	0.094	0.409	0.408	0.403	0.395	0.381
	0.625	0	0.026	0.052	0.079	0.103	0.344	0.342	0.338	0.330	0.323
	0.500	0	0.028	0.055	0.081	0.106	0.275	0.273	0.270	0.265	0.261
	0.375	0	0.028	0.055	0.082	0.107	0.205	0.204	0.202	0.199	0.197
	0.250	0	0.028	0.055	0.081	0.107	0.136	0.136	0.134	0.133	0.132
	0.125	0	0.027	0.054	0.081	0.107	0.068	0.068	0.067	0.067	0.066
	0	0	0.027	0.054	0.081	0.107	0	0	0	0	0
			τ_{rz}/σ_1						σ_z/σ_1		
	1	0	0.073	0.157	0.273	1.223	0.843	0.842	0.843	0.870	6.889
	0.875	0	0.056	0.112	0.136	0	0.975	0.978	0.992	1.020	0.812
	0.750	0	0.022	0.032	0.012	0	1.054	1.054	1.047	1.002	0.835
	0.625	0	-0.002	-0.010	-0.019	0	1.075	1.069	1.046	0.989	0.895
	0.500	0	-0.010	-0.019	-0.021	0	1.065	1.057	1.032	0.990	0.994
	0.375	0	-0.010	-0.016	-0.014	0	1.047	1.040	1.022	0.996	0.975
	0.250	0	-0.007	-0.010	-0.008	0	1.032	1.028	1.016	1.001	0.992
	0.125	0	-0.003	-0.005	-0.004	0	1.024	1.021	1.013	1.004	1.001
	0	0	0	0	0	0	1.021	1.018	1.012	1.005	1.004
			σ_r/σ_1						σ_θ/σ_1		
	1	0.562	0.561	0.562	0.580	4.599	0.562	0.561	0.562	0.580	4.599
	0.875	0.333	0.320	0.272	0.155	0	0.333	0.325	0.299	0.236	0.053
	0.750	0.159	0.145	0.101	0.038	0	0.159	0.150	0.120	0.063	0.000
	0.625	0.058	0.050	0.030	0.010	0	0.058	0.052	0.032	0.002	-0.032
	0.500	0.011	0.009	0.004	0.002	0	0.011	0.008	0.001	-0.014	-0.025
	0.375	-0.006	-0.005	-0.004	-0.001	0	-0.006	-0.007	-0.010	-0.014	-0.016
	0.250	-0.010	-0.009	-0.006	-0.002	0	-0.010	-0.011	-0.011	-0.013	-0.013
	0.125	-0.011	-0.009	-0.006	-0.002	0	-0.011	-0.011	-0.010	-0.010	-0.008
	0	-0.010	-0.009	-0.006	-0.002	0	-0.010	-0.010	-0.009	-0.008	-0.005

TABLE 15.2 (Continued)

STRESSES AND DISPLACEMENTS FOR UNCONFINED COMPRESSION, $H/D = 2$

(Moore, 1966)

	$2z/H$	$2r/D$					$2r/D$				
		0	0.25	0.50	0.75	1	0	0.25	0.50	0.75	1
		ρ_r/Δ (Outwards)					ρ_z/Δ				
	1	0	0								
	0.875	0	0.015	0.017	0.029	0.081	0.472	0.471	0.469	0.463	0.440
	0.750	0	0.025	0.043	0.069	0.107	0.420	0.419	0.414	0.404	0.388
	0.625	0	0.030	0.057	0.087	0.117	0.356	0.354	0.348	0.340	0.330
	0.500	0	0.032	0.063	0.093	0.122	0.285	0.284	0.279	0.273	0.268
	0.375	0	0.032	0.064	0.094	0.124	0.213	0.212	0.209	0.206	0.203
	0.250	0	0.032	0.064	0.095	0.124	0.142	0.141	0.139	0.137	0.136
	0.125	0	0.032	0.063	0.094	0.125	0.071	0.070	0.070	0.069	0.068
	0	0	0.032	0.063	0.094	0.125	0	0	0	0	0
$\nu = 0.45$ $\sigma_3 = 0$ $H/D = 2$		τ_{rz}/σ_1					σ_z/σ_1				
	1	0	0.082	0.175	0.300	1.342	0.827	0.824	0.818	0.836	11.648
	0.875	0	0.061	0.122	0.150	0	0.971	0.972	0.979	0.997	0.750
	0.750	0	0.024	0.035	0.014	0	1.058	1.055	1.044	0.989	0.796
	0.625	0	-0.002	-0.011	-0.022	0	1.083	1.076	1.048	0.981	0.872
	0.500	0	-0.011	-0.022	-0.023	0	1.075	1.066	1.038	0.988	0.933
	0.375	0	-0.011	-0.018	-0.017	0	1.059	1.051	1.030	0.999	0.973
	0.250	0	-0.007	-0.012	-0.010	0	1.045	1.040	1.025	1.007	0.996
	0.125	0	-0.004	-0.006	-0.004	0	1.037	1.033	1.024	1.003	1.008
	0	0	0	0	0	0	1.034	1.031	1.023	1.014	1.011
		σ_r/σ_1					σ_θ/σ_1				
	1	0.677	0.674	0.670	0.684	9.531	0.677	0.674	0.670	0.684	9.531
	0.875	0.399	0.383	0.328	0.196	0	0.399	0.389	0.354	0.275	0.038
	0.750	0.193	0.176	0.126	0.051	0	0.193	0.181	0.143	0.072	-0.038
	0.625	0.073	0.063	0.039	0.014	0	0.073	0.064	0.039	0.001	-0.043
	0.500	0.015	0.012	0.006	0.002	0	0.015	0.011	-0.002	-0.018	-0.032
	0.375	-0.008	-0.007	-0.006	-0.002	0	-0.008	-0.009	-0.014	-0.019	-0.021
	0.250	-0.015	-0.013	-0.009	-0.003	0	-0.015	-0.015	-0.016	-0.016	-0.014
	0.125	-0.016	-0.014	-0.009	-0.004	0	-0.016	-0.015	-0.015	-0.013	-0.009
	0	-0.016	-0.014	-0.009	-0.004	0	-0.016	-0.015	-0.014	-0.012	-0.008

TABLE 15.3

STRESSES AND DISPLACEMENTS FOR CONFINING PRESSURE WITH NO AXIAL MOVEMENT
(Moore, 1966)

	2z/H	2r/D					2r/D				
		0	0.25	0.50	0.75	1	0	0.25	0.50	0.75	1
		$\rho_r E/D\sigma_3$ (Inwards)					$\rho_z E/D\sigma_3$				
$\nu = 0.25$, $\sigma_1/\sigma_3 = 0.455$, $H/D = 2$	1	0	0	0	0	0	0	0	0	0	0
	0.875	0	0.035	0.076	0.130	0.209	-0.055	-0.053	-0.048	-0.030	0.038
	0.750	0	0.062	0.128	0.203	0.280	-0.070	-0.067	-0.055	-0.029	0.014
	0.625	0	0.076	0.153	0.232	0.309	-0.062	-0.058	-0.046	-0.025	-0.003
	0.500	0	0.081	0.162	0.241	0.319	-0.047	-0.044	-0.034	-0.021	-0.010
	0.375	0	0.082	0.163	0.243	0.321	-0.032	-0.030	-0.024	-0.016	-0.012
	0.250	0	0.082	0.163	0.243	0.321	-0.019	-0.018	-0.015	-0.011	-0.010
	0.125	0	0.082	0.162	0.242	0.321	-0.009	-0.009	-0.007	-0.006	-0.005
	0	0	0.081	0.162	0.242	0.321	0	0	0	0	0
		τ_{rz}/σ_3					σ_z/σ_3				
	1	0	0.117	0.250	0.436	1.738	0.750	0.746	0.728	0.638	-4.938
	0.875	0	0.094	0.186	0.228	0	0.535	0.525	0.485	0.399	0.642
	0.750	0	0.041	0.062	0.033	0	0.398	0.393	0.388	0.434	0.668
	0.625	0	0.002	-0.007	-0.023	0	0.354	0.360	0.387	0.463	0.601
	0.500	0	-0.013	-0.025	-0.028	0	0.365	0.375	0.408	0.469	0.539
	0.375	0	-0.014	-0.023	-0.021	0	0.392	0.401	0.427	0.465	0.497
	0.250	0	-0.009	-0.015	-0.012	0	0.416	0.422	0.439	0.460	0.473
	0.125	0	-0.005	-0.007	-0.006	0	0.430	0.434	0.445	0.457	0.461
	0	0	0	0	0	0	0.435	0.438	0.447	0.456	0.458
		σ_r/σ_3					σ_θ/σ_3				
	1	0.250	0.249	0.243	0.213	-1.646	0.250	0.249	0.243	0.213	-1.646
	0.875	0.548	0.565	0.629	0.797	1.000	0.548	0.555	0.581	0.647	0.828
	0.750	0.782	0.802	0.865	0.957	1.000	0.782	0.792	0.825	0.888	0.977
	0.625	0.921	0.932	0.961	0.990	1.000	0.921	0.928	0.949	0.982	1.018
	0.500	0.985	0.988	0.995	0.999	1.000	0.985	0.988	1.018	1.011	1.022
	0.375	1.008	1.007	1.005	1.001	1.000	1.008	1.009	1.012	1.015	1.017
	0.250	1.013	1.011	1.007	1.002	1.000	1.013	1.013	1.013	1.013	1.011
	0.125	1.012	1.011	1.006	1.002	1.000	1.012	1.012	1.011	1.008	1.007
	0	1.012	1.010	1.006	1.001	1.000	1.012	1.011	1.010	1.008	1.006
		$\rho_r E/D\sigma_3$ (Inwards)					$\rho_z E/D\sigma_3$				
$\nu = 0.4$, $\sigma_1/\sigma_3 = 0.762$, $H/D = 2$	1	0	0	0	0	0	0	0	0	0	0
	0.875	0	0.018	0.038	0.065	0.101	-0.036	-0.036	-0.033	-0.025	0.005
	0.750	0	0.030	0.063	0.098	0.132	-0.047	-0.046	-0.040	-0.027	-0.008
	0.625	0	0.037	0.074	0.110	0.144	-0.044	-0.042	-0.035	-0.025	-0.014
	0.500	0	0.039	0.077	0.114	0.149	-0.034	-0.033	-0.028	-0.021	-0.016
	0.375	0	0.039	0.077	0.114	0.150	-0.025	-0.024	-0.020	-0.016	-0.014
	0.250	0	0.039	0.077	0.114	0.150	-0.016	0.015	-0.013	-0.011	-0.010
	0.125	0	0.038	0.076	0.113	0.150	-0.008	-0.007	-0.006	-0.006	-0.005
	0	0	0.038	0.076	0.113	0.150	0	0	0	0	0
		τ_{rz}/σ_3					σ_z/σ_3				
	1	0	0.054	0.116	0.202	0.904	0.878	0.878	0.878	0.858	-3.601
	0.875	0	0.042	0.082	0.100	0	0.780	0.778	0.768	0.747	0.901
	0.750	0	0.016	0.023	0.009	0	0.721	0.722	0.725	0.760	0.884
	0.625	0	-0.002	-0.006	-0.015	0	0.706	0.710	0.728	0.770	0.839
	0.500	0	-0.008	-0.014	-0.015	0	0.714	0.719	0.738	0.769	0.803
	0.375	0	-0.007	-0.012	-0.011	0	0.727	0.732	0.746	0.765	0.780
	0.250	0	-0.005	-0.008	-0.006	0	0.738	0.741	0.750	0.761	0.767
	0.125	0	-0.002	-0.004	-0.003	0	0.745	0.747	0.753	0.759	0.761
	0	0	0	0	0	0	0.747	0.749	0.753	0.758	0.759
		σ_r/σ_3					σ_θ/σ_3				
	1	0.585	0.585	0.585	0.572	-2.401	0.585	0.585	0.585	0.572	-2.401
	0.875	0.755	0.764	0.799	0.886	1.000	0.755	0.760	0.780	0.826	0.961
	0.750	0.883	0.893	0.926	0.973	1.000	0.883	0.890	0.912	0.954	1.007
	0.625	0.957	0.963	0.976	0.993	1.000	0.957	0.962	0.976	0.999	1.024
	0.500	0.992	0.993	0.997	0.999	1.000	0.992	0.994	1.001	1.010	1.019
	0.375	1.004	1.004	1.003	1.001	1.000	1.004	1.005	1.008	1.011	1.012
	0.250	1.007	1.007	1.004	1.001	1.000	1.007	1.008	1.008	1.008	1.007
	0.125	1.008	1.007	1.004	1.001	1.000	1.008	1.008	1.007	1.006	1.005
	0	1.008	1.007	1.004	1.001	1.000	1.008	1.007	1.007	1.006	1.004

TABLE 15.3 (Continued)

STRESSES AND DISPLACEMENTS FOR CONFINING PRESSURE WITH NO AXIAL MOVEMENT (Moore, 1966)

	2z/H	2r/D					2r/D				
		0	0.25	0.50	0.75	1	0	0.25	0.50	0.75	1
		$\rho_r E/D\sigma_3$ (Inwards)					$\rho_z E/D\sigma_3$				
	1	0	0	0	0	0	0	0	0		
	0.875	0	0.009	0.047	0.034	0.051	−0.022	−0.022	−0.021	−0.017	−0.002
	0.750	0	0.016	0.035	0.051	0.068	−0.030	−0.029	−0.026	−0.020	−0.009
	0.625	0	0.019	0.039	0.058	0.075	−0.028	−0.027	−0.024	−0.018	−0.012
	0.500	0	0.021	0.041	0.060	0.079	−0.024	−0.023	−0.020	−0.016	−0.013
	0.375	0	0.021	0.041	0.061	0.080	−0.018	−0.017	−0.015	−0.013	−0.011
	0.250	0	0.021	0.041	0.061	0.081	−0.011	−0.011	−0.010	−0.009	−0.008
	0.125	0	0.021	0.041	0.061	0.081	−0.005	−0.005	−0.004	−0.004	−0.004
	0	0	0.021	0.041	0.061	0.081	0	0	0	0	0
$\nu = 0.45$, $\sigma_1/\sigma_3 = 0.874$, $H/D = 2$		τ_{rz}/σ_3					σ_z/σ_3				
	1	0	0.028	0.060	0.104	0.460	0.934	0.935	0.937	0.932	−2.771
	0.875	0	0.021	0.042	0.051	0	0.885	0.885	0.883	0.879	0.963
	0.750	0	0.008	0.012	0.006	0	0.855	0.857	0.861	0.880	0.946
	0.625	0	0.000	−0.003	−0.007	0	0.847	0.849	0.859	0.881	0.919
	0.500	0	−0.003	−0.007	−0.007	0	0.848	0.851	0.861	0.878	0.897
	0.375	0	−0.004	−0.006	−0.006	0	0.853	0.856	0.863	0.874	0.883
	0.250	0	−0.002	−0.004	−0.003	0	0.857	0.859	0.864	0.870	0.875
	0.125	0	−0.001	−0.002	−0.002	0	0.860	0.861	0.864	0.868	0.870
	0	0	0	0	0	0	0.861	0.862	0.865	0.868	0.869
		σ_r/σ_3					σ_θ/σ_3				
	1	0.765	0.765	0.767	0.763	−2.267	0.765	0.765	0.767	0.763	−2.267
	0.875	0.860	0.865	0.884	0.930	1.000	0.860	0.863	0.875	0.904	0.986
	0.750	0.931	0.937	0.954	0.981	1.000	0.931	0.935	0.948	0.973	1.012
	0.625	0.973	0.976	0.985	0.995	1.000	0.973	0.976	0.985	0.998	1.014
	0.500	0.994	0.995	0.997	0.999	1.000	0.994	0.995	1.000	1.006	1.011
	0.375	1.002	1.002	1.002	1.001	1.000	1.002	1.003	1.007	1.007	1.008
	0.250	1.005	1.005	1.003	1.001	1.000	1.005	1.005	1.006	1.006	1.005
	0.125	1.006	1.005	1.004	1.002	1.000	1.006	1.006	1.005	1.005	1.004
	0	1.006	1.005	1.004	1.002	1.000	1.006	1.006	1.005	1.004	1.003

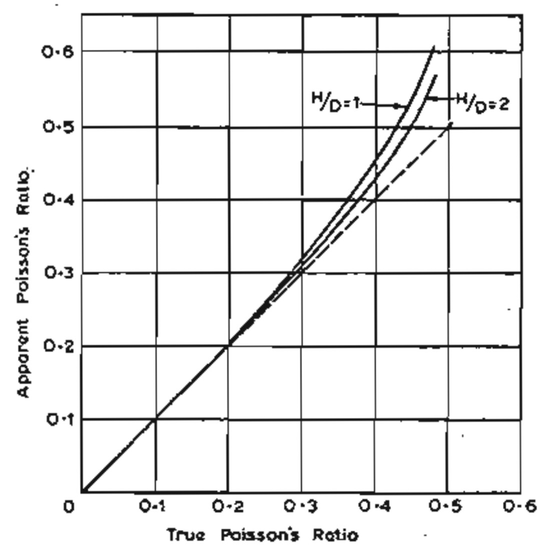


FIG.15.3 Apparent Poisson's Ratio versus the Poisson's Ratio (Moore, 1966)

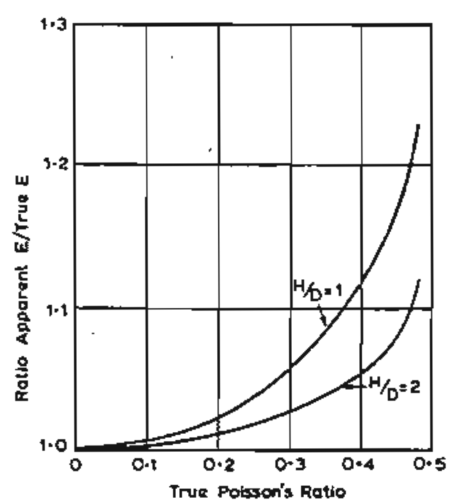


FIG.15.4 Ratio of apparent E to true E versus true Poisson's Ratio (Moore, 1966)

15.3 Inclusion in an Infinite Region (Fig. 15.5)

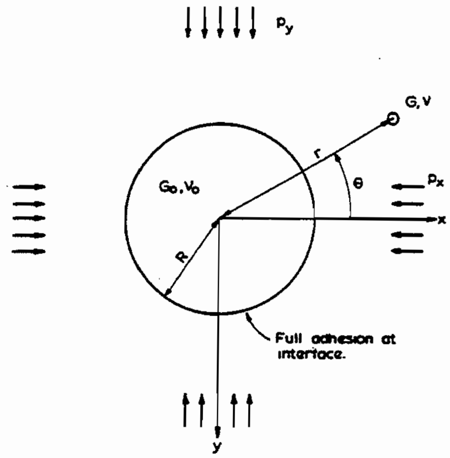


FIG.15.5

Jaeger and Cook (1969) quote the following solutions:

$r < R$

Principal stresses are

$$\sigma_1 = \frac{[k(\chi+2) + \chi_0]k(\chi+1)p_x}{2(2k+\chi_0-1)(k\chi+1)} + \frac{[k(\chi-2) - \chi_0 + 2]k(\chi+1)}{2(2k+\chi_0-1)(k\chi+1)} p_y \quad \ldots (15.2a)$$

$$\sigma_2 = \frac{[k(\chi-2) - \chi_0 + 2]k(\chi+1)p_x}{2(2k+\chi_0-1)(k\chi+1)} + \frac{[k(\chi+2) + \chi_0]k(\chi+1)}{2(2k+\chi_0-1)(k\chi+1)} p_y \quad \ldots (15.2b)$$

where $k = G_0/G$

$\chi = 3-4\nu$ for plane strain

or $\chi = \frac{3-\nu}{1+\nu}$ for plane stress

and similarly for χ_0.

$r > R$

$$\sigma_r = \tfrac{1}{2}(p_x+p_y)(1 - \frac{BR^2}{r^2}) + \tfrac{1}{2}(p_x-p_y)\{1 - \frac{2AR^2}{r^2} - \frac{3CR^4}{r^4}\}\cos 2\theta \quad \ldots (15.3a)$$

$$\sigma_\theta = \tfrac{1}{2}(p_x+p_y)(1 + \frac{BR^2}{r^2}) - \tfrac{1}{2}(p_x-p_y) \cdot \{1 - \frac{3CR^4}{r^4}\}\cos 2\theta \quad \ldots (15.3b)$$

$$\tau_{r\theta} = -\tfrac{1}{2}(p_x-p_y)\{1 + \frac{AR^2}{r^2} + \frac{3CR^4}{r^4}\}\sin 2\theta \quad \ldots (15.3c)$$

where $A = \frac{2(1-k)}{(k\chi+1)}$

$B = \frac{[\chi_0-1-k(\chi-1)]}{(2k+\chi_0-1)}$

$C = \frac{(k-1)}{(k\chi+1)}$

15.4 Stiff Plate Subjected to Moment and Horizontal Load

The solutions for displacement h and rotation θ of a stiff vertical plate in a semi-infinite mass, have been obtained by Douglas and Davis (1964), and are shown in Fig.15.6. The upper edge of the plate is at the surface and the loads are applied to this edge. The results are for $\nu=0.5$ but are not significantly different for other values of ν.

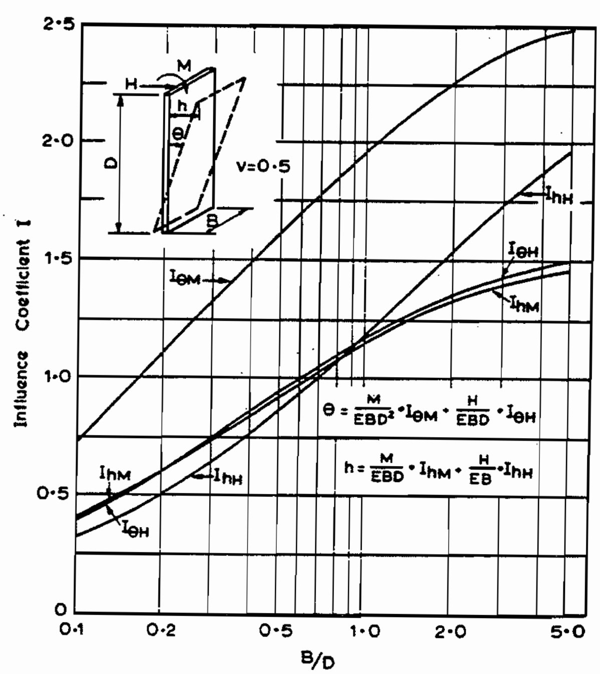


FIG.15.6 Influence coefficients for rotation and translation of rigid plate (Douglas and Davis, 1964)

15.5 Streses in a Layer With a Yielding Base

15.5.1 TRANSLATING SMOOTH BASE (Plane Strain) (Fig.15.7)

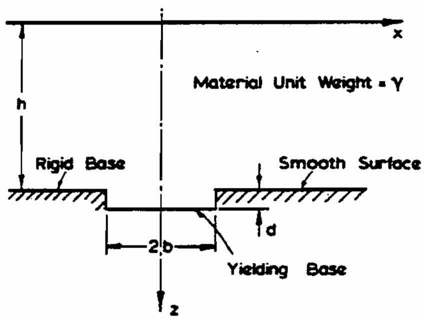


FIG.15.7

Finn (1963) obtained the following solutions for a base translation of d:

$$\sigma_x = -\frac{d}{2\beta\pi}\left[\frac{x+b}{(x+b)^2+\bar{z}^2} - \frac{x-b}{(x-b)^2+\bar{z}^2} - \frac{2(x+b)\bar{z}^2}{\{(x+b)^2+\bar{z}^2\}^2} + \frac{2(x-b)\bar{z}^2}{\{(x-b)^2+\bar{z}^2\}^2}\right] + \frac{z\nu\gamma}{1-\nu} \quad \ldots (15.4a)$$

$$\sigma_z = \frac{d}{2\beta\pi}\left[-\frac{(x+b)}{(x+b)^2+\bar{z}^2} + \frac{x-b}{(x-b)^2+\bar{z}^2} - \frac{2(x+b)\bar{z}^2}{\{(x+b)^2+\bar{z}^2\}^2} + \frac{2(x-b)\bar{z}^2}{\{(x-b)^2+\bar{z}^2\}^2}\right] + \gamma z \quad \ldots (15.4b)$$

$$\tau_{xz} = \frac{\bar{z}d}{2\beta\pi}\left[\frac{\bar{z}^2-(x+b)^2}{\{(x-b)^2+\bar{z}^2\}^2} - \frac{\bar{z}^2-(x-b)^2}{\{(x+b)^2+\bar{z}^2\}^2}\right] \quad \ldots (15.4c)$$

where $\bar{z} = h-z$

$$\beta = \frac{1-\nu^2}{E}$$

15.5.2 TRANSLATING ROUGH BASE (Plane Strain)

This problem is as shown in Fig.15.7, except that lower surface is now perfectly rough throughout.

Finn (1963) obtained the following solutions:

$$\sigma_x = -\frac{d}{\pi(3\beta-\rho)}\left[\frac{2\rho}{\beta+\rho}\left\{\frac{x+b}{(x+b)^2+\bar{z}^2} - \frac{x-b}{(x-b)^2+\bar{z}^2}\right\} - \frac{2(x+b)\bar{z}^2}{\{(x+b)^2+\bar{z}^2\}^2} + \frac{2(x-b)\bar{z}^2}{\{(x-b)^2+\bar{z}^2\}^2}\right] + \frac{z\nu\gamma}{1-\nu} \quad \ldots (15.5a)$$

$$\sigma_z = -\frac{d}{\pi(3\beta-\rho)}\left[\frac{2\beta}{\beta+\rho}\left\{\frac{x+b}{(x+b)^2+\bar{z}^2} - \frac{x-b}{(x-b)^2+\bar{z}^2}\right\} + \frac{2(x+b)\bar{z}^2}{\{(x+b)^2+\bar{z}^2\}^2} - \frac{2(x-b)\bar{z}^2}{\{(x-b)^2+\bar{z}^2\}^2}\right] + \gamma z \quad \ldots (15.5b)$$

$$\tau_{xz} = \frac{d}{\pi(3\beta-\rho)}\left[\frac{\beta-\rho}{\beta+\rho}\left\{\frac{-\bar{z}}{(x-b)^2+\bar{z}^2} + \frac{\bar{z}}{(x+b)^2+\bar{z}^2}\right\} - \frac{\bar{z}\{\bar{z}^2-(x-b)^2\}}{\{\bar{z}^2+(x-b)^2\}^2} + \frac{\bar{z}\{\bar{z}^2-(x+b)^2\}}{\{\bar{z}^2+(x+b)^2\}^2}\right] \quad \ldots (15.5c)$$

where $\bar{z} = h-z$

$$\beta = \frac{1-\nu^2}{E}$$

$$\rho = \frac{\nu(1+\nu)}{E}$$

15.5.3 ROTATING SMOOTH BASE (Plane Strain) (Fig.15.8)

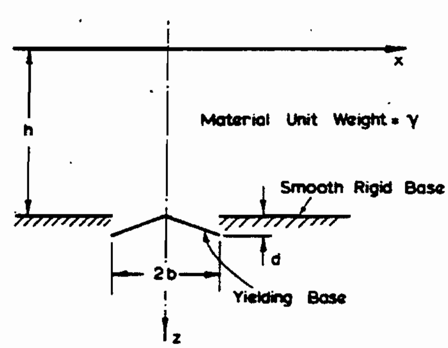


FIG.15.8

Finn (1963) gives the following solutions:

$$\sigma_x = -\frac{d}{\pi\beta b}\Big[\tfrac{1}{2}\ln\frac{(x^2+\bar{z}^2)^2}{\{(x+b)^2+\bar{z}^2\}\{(x-b)^2+\bar{z}^2\}}$$
$$+\frac{b}{2}\Big\{\frac{x+b}{(x-b)^2+\bar{z}^2}-\frac{x-b}{(x-b)^2+\bar{z}^2}\Big\}$$
$$-\frac{\bar{z}^2}{2\{(x+b)^2+\bar{z}^2\}}-\frac{\bar{z}^2}{2\{(x-b)^2+\bar{z}^2\}}$$
$$-\frac{b\bar{z}^2(x+b)}{\{(x+b)^2+\bar{z}^2\}^2}-\frac{b\bar{z}^2(x-b)}{\{(x-b)^2+\bar{z}^2\}^2}$$
$$+\frac{\bar{z}^2}{x^2+\bar{z}^2}\Big]+\frac{\bar{z}\nu\gamma}{1-\nu} \qquad \ldots (15.6a)$$

$$\sigma_z = -\frac{d}{\pi\beta d}\Big[\tfrac{1}{2}\ln\frac{(x^2+\bar{z}^2)^2}{\{(x+b)^2+\bar{z}^2\}\{(x-b)^2+\bar{z}^2\}}$$
$$+\frac{b}{2}\Big\{\frac{x+b}{(x+b)^2+\bar{z}^2}-\frac{x-b}{(x-b)^2+\bar{z}^2}\Big\}$$
$$+\frac{\bar{z}^2}{2\{(x+b)^2+\bar{z}^2\}}+\frac{\bar{z}^2}{2\{(x-b)^2+\bar{z}^2\}}$$
$$+\frac{b\bar{z}^2(x+b)}{\{(x+b)^2+\bar{z}^2\}^2}-\frac{b\bar{z}^2(x-b)}{\{(x-b)^2+\bar{z}^2\}^2}$$
$$-\frac{\bar{z}}{x^2+\bar{z}^2}\Big]+\gamma z \qquad \ldots (15.6b)$$

where $\bar{z} = h-z$

$\beta = \frac{1-\nu^2}{E}$

15.5.4 ROTATING ROUGH BASE (Plane Strain)

This problem is as shown in Fig.15.8, except that lower surface is now perfectly rough.

Finn (1963) gives the following solutions:

$$\sigma_x = -\frac{4\rho d}{(3\beta-\rho)(\beta+\rho)\pi b}\Big[\tfrac{1}{2}\ln\frac{(x^2+\bar{z}^2)^2}{\{(x+b)^2+\bar{z}^2\}\{(x-b)^2+\bar{z}^2\}}$$
$$+\frac{b}{2}\Big\{\frac{x+b}{(x-b)^2+\bar{z}^2}-\frac{x-b}{(x-b)^2+\bar{z}^2}\Big\}\Big]$$
$$+\frac{2d}{\pi b(3\beta-\rho)}\Big[-\frac{\bar{z}^2}{2\{(x+b)^2+\bar{z}^2\}^2}$$
$$-\frac{\bar{z}^2}{2\{(x-b)^2+\bar{z}^2\}}-\frac{b\bar{z}^2(x+b)}{\{(x+b)^2+\bar{z}^2\}^2}$$
$$+\frac{b\bar{z}^2(x-b)}{\{(x-b)^2+\bar{z}^2\}^2}+\frac{\bar{z}^2}{x^2+\bar{z}^2}\Big]+\frac{\bar{z}\nu\gamma}{1-\nu}$$
$$\ldots (15.7a)$$

$$\sigma_z = -\frac{4\beta d}{(3\beta-\rho)(\beta+\rho)\pi b}\Big[\tfrac{1}{2}\ln\frac{(x^2+\bar{z}^2)^2}{\{(x+b)^2+\bar{z}^2\}\{(x-b)^2+\bar{z}^2\}}$$
$$+\frac{b}{2}\Big\{\frac{x+b}{(x+b)^2+\bar{z}^2}-\frac{x-b}{(x-b)^2+\bar{z}^2}\Big\}\Big]$$
$$-\frac{2d}{\pi b(3\beta-\rho)}\Big[\frac{\bar{z}^2}{2\{(x+b)^2+\bar{z}^2\}}$$
$$+\frac{\bar{z}^2}{2\{(x-b)^2+\bar{z}^2\}}+\frac{b\bar{z}^2(x+b)}{\{(x+b)^2+\bar{z}^2\}^2}$$
$$-\frac{b\bar{z}^2(x-b)}{\{(x-b)^2+\bar{z}^2\}^2}-\frac{\bar{z}}{x^2+\bar{z}^2}\Big]+\gamma z$$
$$\ldots (15.7b)$$

where $\bar{z} = h-z$

$\beta = \frac{1-\nu^2}{E}$

$\rho = \frac{\nu(1+\nu)}{E}$

15.6 Stresses Behind Retaining Walls

The following solutions have been derived by Finn (1963), using the solutions for a layer with a yielding base in Section 15.5.

15.6.1 SMOOTH TRANSLATING WALL (Fig.15.9)

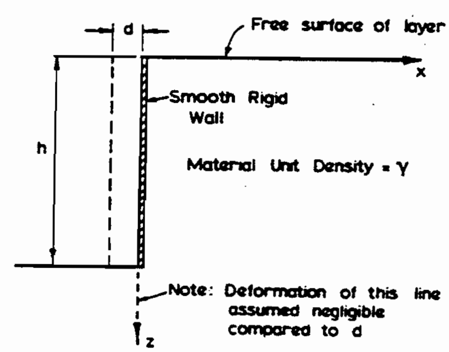


FIG.15.9

The horizontal pressure σ_x along the wall $(x=0)$ (noting the assumption in Fig.15.9) is

$$\sigma_x = \frac{\nu\gamma z}{1-\nu}+\frac{4zh}{(z+h)^3(z-h)}\,\frac{dh}{\beta\pi} \qquad \ldots (15.8)$$

where $\beta = \frac{1-\nu^2}{E}$

15.6.2 ROUGH TRANSLATING WALL

As in Fig.15.9, except that wall is now perfectly rough.

σ_x along the wall $(x=0)$ is

$$\sigma_x = \frac{\nu\gamma z}{1-\nu} + \frac{4dh}{\pi(3\beta-\rho)} \left\{ \frac{\beta}{\beta+\rho} \frac{1}{z^2-h^2} - \frac{(\rho h-\beta z)}{(h+z)^3} \right\} \qquad \ldots (15.9)$$

where $\beta = \dfrac{1-\nu^2}{E}$

$\rho = \dfrac{\nu(1+\nu)}{E}$

15.6.3 SMOOTH ROTATING WALL (Fig.15.10)

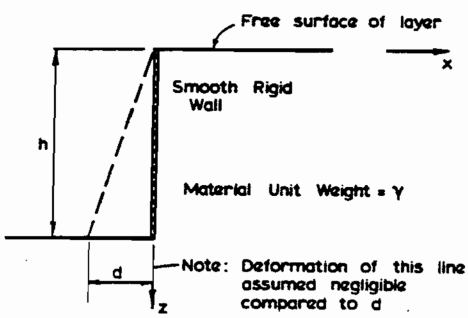


FIG.15.10

Along the wall $(x=0)$,

$$\sigma_x = \frac{\nu\gamma z}{1-\nu} - \frac{d}{4\pi\beta h} \ln \frac{z^4}{(z^2-h^2)^2} + \frac{dh}{\beta\pi} \frac{1}{(z^2-h^2)} + \frac{2dh}{\beta\pi} \cdot \frac{h+2z}{(h+z)^2} \qquad \ldots (15.10)$$

where $\beta = \dfrac{1-\nu^2}{E}$

15.6.4 ROUGH ROTATING WALL

As in Fig.15.10, except that wall is rough.

Along the wall $(x=0)$

$$\sigma_x = \frac{\nu\gamma z}{1-\nu} - \frac{4\beta^2}{(3\beta-\rho)(\beta+\rho)} \left[\frac{d}{4\pi h\beta} \ln \frac{z^4}{(z^2-h^2)^2} - \frac{dh}{\beta\pi} \cdot \frac{1}{(z^2-h^2)} \right] - \frac{d}{\pi(3\beta-\rho)} \left[-\frac{4\rho h}{\beta+\rho} \frac{1}{(h+z)^2} + \frac{z(2h+z)}{h(h+z)^2} + \frac{4hz}{(h+z)^3} - \frac{2}{h} \right]. \qquad \ldots (15.11)$$

where $\beta = \dfrac{1-\nu^2}{E}$

$\rho = \dfrac{\nu(1+\nu)}{E}$

Appendix A

STRESSES AND DISPLACEMENTS IN A LOADED ORTHORHOMBIC HALF SPACE

C. M. Gerrard
W. Jill Harrison

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CONTENTS

STRESSES AND DISPLACEMENTS IN A LOADED ORTHORHOMBIC HALF SPACE

by C.M. Gerrard* and W. Jill Harrison*

Summary

This paper gives a complete range of solutions for all the displacement, strain, and stress components developed in a loaded anisotropic half space. The half space is homogeneous, linearly elastic and infinitely deep. The stress-strain response types of orthorhombic, cross-anisotropic and isotropic are all considered.

A range of simple load types are included as follows :-

1. LOAD PRODUCING RESULTANT VERTICAL FORCE
   - (a) Uniform vertical pressure.
   - (b) Uniform vertical displacement.
2. LOAD PRODUCING RESULTANT MOMENT
   - (a) Linear vertical pressure.
   - (b) Linear vertical displacement.
3. LOAD PRODUCING RESULTANT LATERAL FORCE
   - (a) Uniform lateral shear stress.
   - (b) Uniform lateral shear displacement.
4. LOAD PRODUCING NO RESULTANT FORCE OR MOMENT
   - (a) Linear lateral shear stress.
   - (b) Linear lateral shear displacement.

The solutions produced have practical value in soil and rock engineering for two main reasons :-

(i) The loadings, when considered singly or in combination, represent the conditions existing under many typical footings and foundations.

(ii) The nature of the half space analysed represents the commonly occurring situation where the loaded soil or rock masses are anisotropic.

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I. INTRODUCTION

In the analysis of many practical situations in soil and rock mechanics, such as long building foundations and long embankments, it is reasonable to assume that the longitudinal direct strain produced by the loading will be zero. Because of this assumption of plane strain these problems can be treated as being two-dimensional.

Solutions based on elastic theory have proved to be useful in predicting immediate settlements, in consolidation analysis, and in the recently developed 'stress path' analysis (Lambe, 1967). These solutions may also be applied to the analysis of the performance of road rolling plant since the loading in this case produces approximately plane strain conditions.

There is a growing appreciation that in many soil and rock mechanics situations the stress-deformation properties of the materials concerned are anisotropic. This factor was taken into account in producing the solutions contained in this report. Hence these solutions can be used to gauge the effect of anisotropy on a particular component of interest, such as surface deformation or spatial stress pattern. The existing solutions for linearly elastic half plane problems are summarized in Table 1. It can be seen that the solutions presented herein are more comprehensive than any of the previous work for loads of finite width, in one or more of the following ways.

1. Range of load types considered.
2. Range of stresses, strains, and displacements solved.
3. Range of anisotropic material responses considered.

II. NOTATION

(a) Co-ordinates, Displacements, Strains, Stresses and Elastic Moduli

x_0	half loaded width
x, y, z	rectangular co-ordinates (lateral, longitudinal, vertical) expressed in units of the half loaded width
u, v, w	displacements in the respective co-ordinate directions
u_{00}, w_{00}	lateral displacement and vertical displacement respectively at the point in the half plane where $z = x = 0$
$\widehat{xx}$, $\widehat{yy}$, $\widehat{zz}$, $\widehat{xz}$	direct stress and shear stress components of the stress tensor
ϵ_{xx}, ϵ_{yy}, ϵ_{zz}, ϵ_{xz}	direct strain and shear strain components of the strain tensor
a,b,c,d,e,f	components of the elasticity tensor (see equations 1,3,4, and 5)
E_x E_y E_z	Young's moduli in the x, y, and z directions for an orthorhombic material (see equations 2 and 3)
E_h, E_v	Young's moduli in the horizontal and vertical directions for a cross-anisotropic material (see equation 4)
E	Young's modulus for an isotropic material (see equation 5)

Orthorhombic material

ν_{xy}	Poisson's ratio: effect of ϵ_{xx} on ϵ_{yy} (see equations 2 and 3)
ν_{yx}	Poisson's ratio: effect of ϵ_{yy} on ϵ_{xx} (see equations 2 and 3)
ν_{yz}	Poisson's ratio: effect of ϵ_{yy} on ϵ_{zz} (see equations 2 and 3)
ν_{zy}	Poisson's ratio: effect of ϵ_{zz} on ϵ_{yy} (see equations 2 and 3)
ν_{xz}	Poisson's ratio: effect of ϵ_{xx} on ϵ_{zz} (see equations 2 and 3)
ν_{zx}	Poisson's ratio: effect of ϵ_{zz} on ϵ_{xx} (see equations 2 and 3)

Cross-anisotropic material

ν_{hv}	Poisson's ratio: effect of horizontal strain on vertical strain (see equation 4)
ν_{vh}	Poisson's ratio: effect of vertical strain on horizontal strain (see equation 4)
ν_h	Poisson's ratio: effect of horizontal strain on complimentary horizontal strain (see equation 4)
ν	Poisson's ratio for an isotropic material (see equation 5)

TABLE 1. SUMMARY OF EXISTING SOLUTIONS

AUTHOR	NATURE OF ELASTIC RESPONSE	RESTRICTIONS ON ELASTIC PARAMETERS	TYPE OF LOADING	DETERMINED STRESSES AND DISPLACEMENTS
Carothers (1920)	Isotropic		Various distributions of vertical pressure. (Terrace, trapezoidal, triangular).	All stresses in half plane.
Love (1927)	Isotropic		Vertical and lateral line loads.	All stresses and displacements in half plane.
Kolosov † (1935)	Isotropic		Uniform vertical pressure. Uniform lateral shear stress.	$\widehat{zz}$, $\widehat{xx}$, $\widehat{xz}$.
Wolf (1935)	Cross-Anisotropic	Restriction on the value of f.	Uniform vertical pressure distribution.	All stresses in half plane.
Gray (1936)	Isotropic		Vertical line load. Uniform vertical pressure. Various 'triangular' and 'terrace' vertical pressure distributions.	All stresses and displacements in half plane.
Egorov † (1940)	Isotropic		Uniform vertical displacement	w throughout half plane.
Jurgenson (1940)	Isotropic		Uniform vertical pressure. 'Triangular' vertical pressure. 'Terrace' vertical pressure with taper.	$\widehat{zz}$, $\widehat{xx}$, $\widehat{xz}$ Principal stresses and principal stress directions.
Holl (1941)	Isotropic		Vertical and lateral line loads. Vertical pressure distributions; triangular, trapezoidal, etc. Also considers loads on inclined surfaces.	All stresses and displacements in half plane.
	Cross-Anisotropic	Restriction on the value of f.	Vertical line load.	All stresses in half plane.
Quinlan (1949)	Cross-Anisotropic	α^2 positive	Vertical line load.	All stresses and displacements in half plane.
		β^2 positive	Various vertical pressure distributions (Uniform, parabolic, inverted parabolic). Uniform vertical displacement.	w on surface (z=0) $\widehat{zz}$ on load axis (x=0)
Timoshenko and Goodier (1951)	Isotropic		Vertical and lateral line loads.	All stresses and displacements in half plane.
Florin * (1959, 1961)	Isotropic		Uniform vertical pressure. Uniform lateral shear stress. Linear vertical pressure. Linear lateral shear stress.	w on the surface. u on the surface. $\widehat{zz}$, $\widehat{xx}$, $\widehat{xz}$, w on the surface. $\widehat{zz}$, $\widehat{xx}$, $\widehat{xz}$.
Lekhnitskii (1963)	Orthorhombic	Characteristic stress patterns depend on parameters.	Vertical line load.	'Radial' stress - all other stresses are zero as for an isotropic body.
De Urena *et al.* (1966)	Cross-Anisotropic	α^2 positive	Vertical line load.	All stresses in half plane.
		β^2 positive	Lateral line load..	All stresses and displacements in half plane.
			Linear vertical pressure.	$\widehat{zz}$, $\widehat{xx}$, $\widehat{xz}$
			Linear lateral shear stress.	throughout half plane.

† As referred to by Harr (1966)
* As referred to by Harr (1966) and Scott (1963).

(b) Derived Elastic Quantities

D_1, D_2 - functions of Poisson's ratios (see equations 3f and 4e)

$\alpha^2 = \{ad - c^2 - cf + f(ad)^{1/2}\} \div 2fd$

$\beta^2 = \{ad - c^2 - cf - f(ad)^{1/2}\} \div 2fd$

$\omega^2 = -\beta^2$

$\phi = \alpha - \beta$

$\rho = \alpha + \beta$

Coefficients appearing in the solutions for the displacements, strains, and stresses (defined on page 26).

$g_1, \ldots\ldots g_9$

$h_1, \ldots\ldots h_9$

$i_1, \ldots\ldots i_{10}$

$j_1, \ldots\ldots j_{10}$

$s_1, \ldots\ldots s_9$

$t_1, \ldots\ldots t_9$

(c) Loadings

P_1 uniform vertical pressure.

P_2 maximum value of linear vertical pressure.

p_1 uniform lateral shear stress.

p_2 maximum value of linear lateral shear stress.

Δ_2 maximum value of linear vertical displacement.

δ_2 maximum value of linear lateral shear displacement.

T_v total resultant vertical force applied to produce the uniform vertical displacement/unit length of strip.

T_h total resultant lateral force applied to produce the uniform lateral shear displacement/unit length of strip.

T_y total resultant moment (about y axis) applied to produce the linear vertical displacement/unit length of strip.

(d) Integrals

Integrals appearing in solutions for displacements, strains, and stresses. These are defined in equation 9 and evaluated on pages 22, 23, 24 and 25.

$K_{nc\mu}(\psi)$, $K_{ns\mu}(\psi)$

${}_cK_{nc\mu}$ ${}_cK_{ns\mu}$

${}_sK_{nc\mu}$ ${}_sK_{ns\mu}$

$Q_{n\mu}(\psi)$ ${}_cQ_{n\mu}$

k transform parameter appearing in equation 9.

III. ORTHORHOMBIC HALF SPACE

The half space analysed in this work can be described as an elastic body having infinite lateral extent and depth with loads applied to its horizontal plane surface. Being elastic and orthorhombic the material has three mutually perpendicular planes of elastic symmetry. The normals to these planes are assumed to be parallel to the vertical, lateral, and longitudinal directions associated with the load.

The solutions given in this report can therefore be applied to cases where the elastic properties are different in the three cartesian directions. The special cases of a cross-anisotropic material with a vertical axis of elastic symmetry and an isotropic material are also considered.

For an orthorhombic half space (with $\varepsilon_{yy} = 0$) the stresses, expressed in terms of the strains are:

$$\widehat{xx} = a.\varepsilon_{xx} + c.\varepsilon_{zz} \quad \text{1a}$$

$$\widehat{yy} = b.\varepsilon_{xx} + e.\varepsilon_{zz} \quad \text{1b}$$

$$\widehat{zz} = c.\varepsilon_{xx} + d.\varepsilon_{zz} \quad \text{1c}$$

$$\widehat{xz} = f.\varepsilon_{xz}{}^{*} \quad \text{1d}$$

* In this work $\varepsilon_{xz} = \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)$

Six independent elastic co-efficients, a, b, c, d, e, and f are involved. The direct strains, in terms of the direct stresses, are given as:

$$\varepsilon_{xx} = \frac{\widehat{xx}}{E_x} - \nu_{yx} \cdot \frac{\widehat{yy}}{E_y} - \nu_{zx} \cdot \frac{\widehat{zz}}{E_z} \qquad \text{2a}$$

$$\varepsilon_{yy} = 0 = - \nu_{xy} \cdot \frac{\widehat{xx}}{E_x} + \frac{\widehat{yy}}{E_y} - \nu_{zy} \cdot \frac{\widehat{zz}}{E_z} \qquad \text{2b}$$

$$\varepsilon_{zz} = - \nu_{xz} \cdot \frac{\widehat{xx}}{E_x} - \nu_{yz} \cdot \frac{\widehat{yy}}{E_y} + \frac{\widehat{zz}}{E_z} \qquad \text{2c}$$

The elastic coefficients are related to the Young's moduli E_x, E_y, E_z, and the Poisson's ratios ν_{xy}, ν_{yx}, ν_{xz}, ν_{zx}, ν_{yz}, ν_{zy} by the following:

$$a = \frac{E_x}{D_1}(1-\nu_{zy}\cdot\nu_{yz}) \qquad \text{3a}$$

$$b = \frac{E_x}{D_1}(\nu_{zx}\cdot\nu_{yz}+\nu_{yx}) = \frac{E_y}{D_1}(\nu_{xz}\cdot\nu_{zy}+\nu_{xy}) \qquad \text{3b}$$

$$c = \frac{E_x}{D_1}(\nu_{yx}\cdot\nu_{zy}+\nu_{zx}) = \frac{E_z}{D_1}(\nu_{xy}\cdot\nu_{yz}+\nu_{xz}) \qquad \text{3c}$$

$$e = \frac{E_y}{D_1}(\nu_{zx}\cdot\nu_{xy}+\nu_{zy}) = \frac{E_z}{D_1}(\nu_{xz}\cdot\nu_{yx}+\nu_{yz}) \qquad \text{3d}$$

$$d = \frac{E_z}{D_1}(1-\nu_{xy}\cdot\nu_{yx}) \qquad \text{3e}$$

where $D_1 = 1-\nu_{xy}\cdot\nu_{yx}-\nu_{yz}\cdot\nu_{zy}-\nu_{xz}\cdot\nu_{zx}-\nu_{xy}\cdot\nu_{yz}\cdot\nu_{zx}-\nu_{yx}\cdot\nu_{xz}\cdot\nu_{zy}$ 3f

The stress-strain equations for a cross-anisotropic half space, having a vertical axis of symmetry, are similar to equation 1. However, because of this symmetry $e = c$, and hence the number of elastic coefficients is reduced from six to five. In this case the relationships between the elastic coefficients a,b,c,d, and Young's moduli E_h, E_v and the Poisson's ratios ν_h, ν_{hv}, ν_{vh} are:

$$a = \frac{E_h}{D_2}(1-\nu_{vh}\cdot\nu_{hv}) \qquad \text{4a}$$

$$b = \frac{E_h}{D_2}(\nu_{vh}\cdot\nu_{hv}+\nu_h) \qquad \text{4b}$$

$$c = \frac{E_h}{D_2} \cdot \nu_{vh} \cdot (1+\nu_h) = \frac{E_v}{D_2} \cdot \nu_{hv} \cdot (1+\nu_h) \qquad \text{4c}$$

$$d = \frac{E_v}{D_2}(1-\nu_h^2) \qquad \text{4d}$$

where $D_2 = (1+\nu_h)(1-\nu_h-2.\nu_{hv}\cdot\nu_{vh})$ 4e

For an isotropic body:

$$a = d = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)} \qquad \text{5a}$$

$$b = c = \frac{E.\nu}{(1+\nu)(1-2\nu)} \qquad \text{5b}$$

$$f = a - b = \frac{E}{1+\nu} \qquad \text{5c}$$

The fact that the strain energy is positive imposes restrictions on the values of the elastic constants. For a cross-anisotropic material with a vertical axis of elastic symmetry Hearmon (1961) gives these restrictions as:

$$a > 0 \qquad \text{6a}$$

$$d > 0 \qquad \text{6b}$$

$$f > 0 \qquad \text{6c}$$

$$a^2 > b^2 \qquad \text{6d}$$

$$(a+b)d > 2c^2 \qquad \text{6e}$$

$$ad > c^2 \qquad \text{6f}$$

In terms of the Poisson's ratios these restrictions impose the limits;

$$1 - \nu_h - 2\nu_{hv}.\nu_{vh} > 0 \qquad 7a$$

$$1 - \nu_h > 0 \qquad 7b$$

$$1 + \nu_h > 0 \qquad 7c$$

IV. LOAD TYPES

The eight loads considered in this report are defined below and are shown in Figure 1. As can be seen, the loads can be grouped into four pairs such that the first of each pair is a particular stress-defined load while the second is the analogous displacement-defined load.

1. The loading by uniform vertical pressure and uniform vertical displacement represent symmetrically placed vertical loads on smooth based foundations.

(a) Uniform Vertical Pressure

$$\begin{aligned} &\widehat{zz} = P_1 && x < 1 && \\ &\widehat{zz} = 0 && x > 1 && \text{when } z = 0 \qquad 8a\\ &\widehat{xz} = 0 && \text{for all } x && \end{aligned}$$

(b) Uniform Vertical Displacement

$$\begin{aligned} &w - w_{00} = 0 && x < 1 && \\ &\widehat{zz} = 0 && x > 1 && \text{when } z = 0 \qquad 8b\\ &\widehat{xz} = 0 && \text{for all } x && \end{aligned}$$

2. The effects of moments applied to foundations can be seen from the solutions for loading by linear vertical pressure and linear vertical displacement.

(a) Linear Vertical Pressure

$$\begin{aligned} &\widehat{zz} = x.P_2 && x < 1 && \\ &\widehat{zz} = 0 && x > 1 && \text{when } z = 0 \qquad 8c\\ &\widehat{xz} = 0 && \text{for all } x && \end{aligned}$$

(b) Linear Vertical Displacement

$$\begin{aligned} &w = x.\Delta_2 && x < 1 && \\ &\widehat{zz} = 0 && x > 1 && \text{when } z = 0 \qquad 8d\\ &\widehat{xz} = 0 && \text{for all } x && \end{aligned}$$

3. Lateral forces applied to foundations can be represented by the solutions for uniform lateral shear stress and uniform lateral shear displacement.

(a) Uniform Lateral Shear Stress

$$\begin{aligned} &\widehat{xz} = p_1 && x < 1 && \\ &\widehat{xz} = 0 && x > 1 && \text{when } z = 0 \qquad 8e\\ &\widehat{zz} = 0 && \text{for all } x && \end{aligned}$$

(b) Uniform Lateral Shear Displacement

$$\begin{aligned} &u - u_{00} = 0 && x < 1 && \\ &\widehat{xz} = 0 && x > 1 && \text{when } z = 0 \qquad 8f\\ &\widehat{zz} = 0 && \text{for all } x && \end{aligned}$$

4. The effect of rough foundations supporting vertical loads can be gauged by the superposition on the solutions for loads 1(a) or 1(b) of either loading by linear lateral shear stress or linear lateral displacement.

(a) Linear Lateral Shear Stress

$$\begin{aligned} &\widehat{xz} = x.p_2 && x < 1 && \\ &\widehat{xz} = 0 && x > 1 && \text{when } z = 0 \qquad 8g\\ &\widehat{zz} = 0 && \text{for all } x && \end{aligned}$$

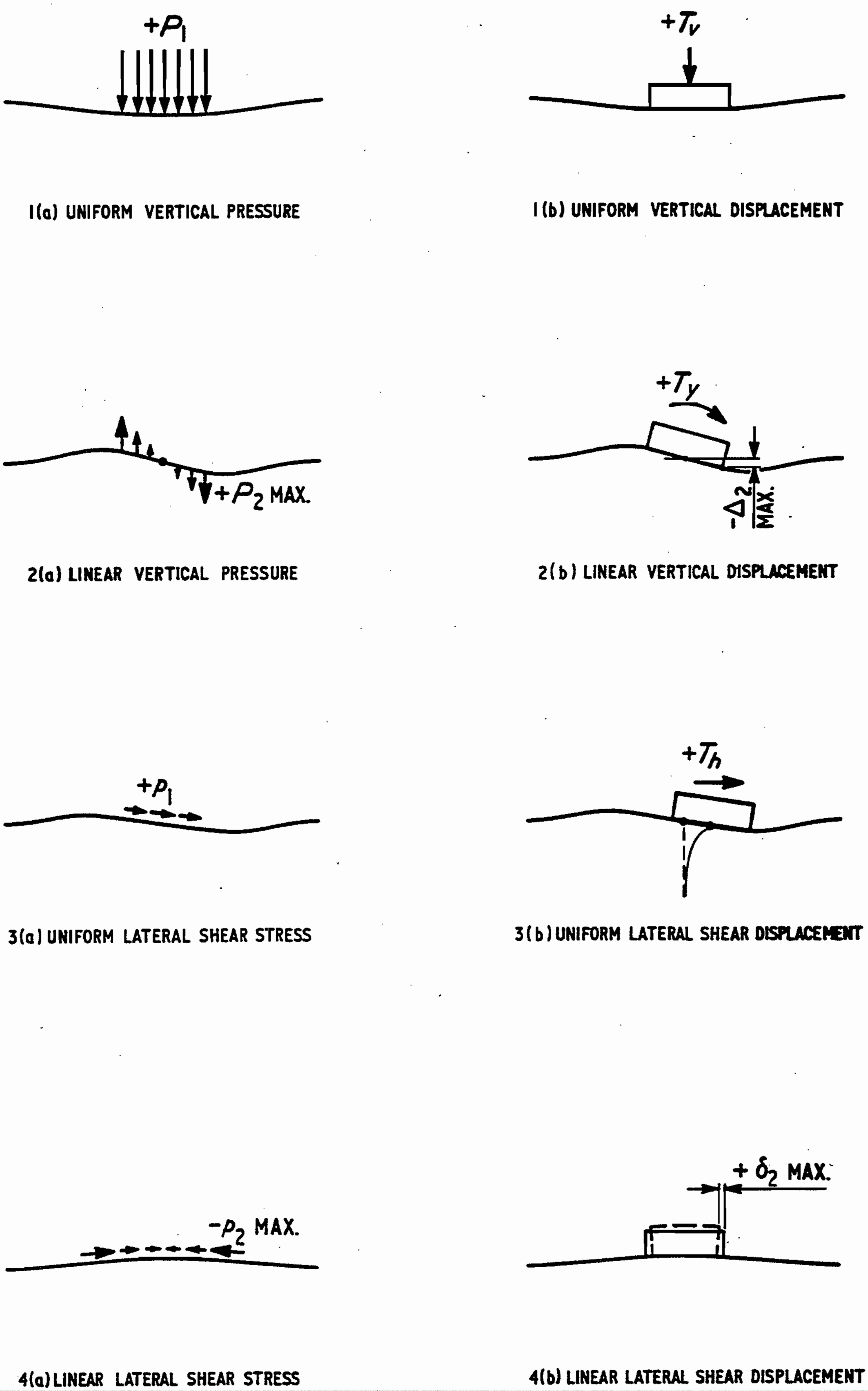

Fig. 1.- Load Types.

(b) Linear Lateral Shear Displacement

$$u = x.\delta_2 \qquad x < 1$$
$$\widehat{xz} = 0 \qquad x > 1 \qquad \text{when } z = 0 \qquad \text{8h}$$
$$\widehat{zz} = 0 \qquad \text{for all } x$$

The case of a uniform vertical pressure with completely rough contact can be solved exactly by superimposing the solutions for 1(a) and 4(b). The linear lateral shear loadings produce no resultant force or moment.

V. METHOD OF SOLUTION

The method used to obtain the solutions reported herein was developed by Gerrard and Harrison (in preparation) and is based on the application of integral transform techniques and dual integral equation techniques to elasticity problems (Sneddon (1951), Tranter (1966)). The solutions for the displacements, strains, and stresses are expressed in terms of integrals of products of trignometric functions, Bessel functions, and exponentials. The following symbols are used to express these integrals;

$$Q_{n\mu}(\psi) = \int_0^\infty J_{\frac{1}{2}n}(k) \,.\, (\cos kx.e^{-\psi k}-1) \,.\, k^{\frac{1}{2}(\mu-4)} \,.\, dk \qquad \text{9a}$$

$$_cQ_{n\mu} = \int_0^\infty J_{\frac{1}{2}n}(k) \,.\, (\cos kx.\cos \omega kz \,.\, e^{-\alpha kz}-1) \,.\, k^{\frac{1}{2}(\mu-4)} \,.\, dk \qquad \text{9b}$$

$$K_{nc\mu}(\psi) = \int_0^\infty J_{\frac{1}{2}n}(k) \,.\, \cos kx.e^{-\psi k} \,.\, k^{\frac{1}{2}(\mu-4)} \,.\, dk \qquad \text{9c}$$

$$K_{ns\mu}(\psi) = \int_0^\infty J_{\frac{1}{2}n}(k) \,.\, \sin kx.e^{-\psi k} \,.\, k^{\frac{1}{2}(\mu-4)} \,.\, dk \qquad \text{9d}$$

$$_cK_{nc\mu} = \int_0^\infty J_{\frac{1}{2}n}(k) \,.\, \cos kx.\cos \omega kz \,.\, e^{-\alpha kz} \,.\, k^{\frac{1}{2}(\mu-4)} \,.\, dk \qquad \text{9e}$$

$$_cK_{ns\mu} = \int_0^\infty J_{\frac{1}{2}n}(k) \,.\, \sin kx.\cos \omega kz \,.\, e^{-\alpha kz} \,.\, k^{\frac{1}{2}(\mu-4)} \,.\, dk \qquad \text{9f}$$

$$_sK_{nc\mu} = \int_0^\infty J_{\frac{1}{2}n}(k) \,.\, \cos kx.\sin \omega kz \,.\, e^{-\alpha kz} \,.\, k^{\frac{1}{2}(\mu-4)} \,.\, dk \qquad \text{9g}$$

$$_sK_{ns\mu} = \int_0^\infty J_{\frac{1}{2}n}(k) \,.\, \sin kx.\sin \omega kz \,.\, e^{-\alpha kz} \,.\, k^{\frac{1}{2}(\mu-4)} \,.\, dk \qquad \text{9h}$$

The parameter ψ contained in equations 9a, 9c, 9d is in the form of either ρz, ϕz, or αz for anisotropic materials. Hence it is a function of the elastic properties as well as the depth. In contrast, for isotropic materials $\psi = z$. The integrals shown above (equations 9a......h) were evaluated using the results of the Bateman Manuscript Project (1954) and Watson (1966).

Compressive direct strains and stresses are considered to be positive. Positive shear stresses are defined from the fact that both the stress and strain tensors obey the right hand rule. Displacements in the negative coordinate directions are considered to be positive. Hence, a load defined by a positive stress acts in the positive coordinate direction, whereas a load defined by a positive displacement acts in the negative coordinate direction. For example, vertical loads are compressive if defined by a positive stress or a negative displacement.

VI. PRESENTATION OF RESULTS

The eight loading conditions considered in this report can be conveniently grouped into four pairs, as follows:-

1(a) Uniform vertical pressure.
 (b) Uniform vertical displacement.

2(a) Linear vertical pressure.
 (b) Linear vertical displacement.

3(a) Uniform lateral shear stress.
 (b) Uniform lateral shear displacement.

4(a) Linear lateral shear stress.
 (b) Linear lateral shear displacement.

It will be noted that the first in each pair is a stress-defined load and that the second is the displacement-defined analogue of the first. As would be expected, the solutions for the second of each pair are very similar to those of the first. Use of this fact is made in the presentation of the solutions for each pair of loads on a pair of facing pages. On the left hand side page the complete solutions for the displacements, strains, stresses are given for the stress-defined load. The analogous displacement-defined load is treated on the right hand page by stating the substitutions that need to be made into the solutions for the stress-defined load in order to produce the solutions for the displacement-defined load.

The solutions given contain all stress, strain, and displacement components. This allows the calculation of quantities such as total deformations, principal stress and strains, and principal direction.

In general the solutions for the displacements, strains, and stresses involve products of coefficients ($g_1,\dots g_9$, $h_1,\dots h_9$, $i_1,\dots i_{10}$, $j_1,\dots j_{10}$, $s_1,\dots s_9$, $t_1,\dots t_9$) and integrals (Q type and X type).

The coefficients are functions only of the elastic properties while the integrals are in general functions of depth, lateral offset, and elastic properties.

The values of these coefficients are given on page 26. Depending on the nature of the loading the integrals fall into four separate categories:

those appropriate to uniform pressure loading (loading types 1(a) and 3(a)).
those appropriate to uniform displacement loading (loading types 1(b) and 3(b)).
those appropriate to linear pressure loading (loading types 2(a) and 4(a)).
those appropriate to linear displacement loading (loading types 2(b) and 4(b)).

The values of the integrals in each of the above four categories are shown on pages 22, 23, 24 and 25.

Values of the integrals for the special cases of $x = 0$ and $z = 0$ are included as well as the general case where both x and z are non-zero.

The form of the solutions depends to a significant extent on the nature of the anisotropy as reflected by the values of α^2 and β^2, both of which are functions only of the elastic properties.

The strain energy conditions given by equations 6a,...6f are sufficient but not necessary conditions that

$$\alpha^2 = \{(ad)^{\frac{1}{2}}-c\}.\{(ad)^{\frac{1}{2}}+c+f\}.(2fd)^{-1}$$

be positive. However, when β^2 is written in the form

$$\beta^2 = [ad-c^2-f\{(ad)^{\frac{1}{2}}+c\}](2fd)^{-1}$$

it can be seen that the sign of β^2 is not restricted by the strain energy conditions. Hence, for each loading condition, four separate cases are considered as follows :

A. Orthorhombic and cross-anisotropic; α^2 positive, β^2 positive.

B. Orthorhombic and cross-anisotropic; α^2 positive, β^2 negative.

C. Orthorhombic and cross-anisotropic; α^2 positive, β^2 zero.

D. Isotropic (this is a special case of Case C in which $\alpha = 1$).

The conversion from orthorhombic to cross-anisotropic in Cases A, B and C is simply achieved by putting $e = a$.

The coefficients to be altered by this are g_7^*, g_8^*, h_7^*, h_8^*, i_7^*, i_8^*, j_7^*, j_8^*, s_7^*, s_8^*, t_7^*, and t_8^* and hence it will be noted that the longitudinal direct stress $(\widehat{yy})$ is the only stress, strain, or displacement changed by converting from orthorhombic to cross-anisotropic.

For the cases of loading by uniform vertical pressure§ or uniform vertical displacement§ a finite measure of vertical displacement can only be obtained by considering a relative quantity rather than an absolute one (Gibson, 1967). Hence the relative vertical displacement $(w-w_{00})$ at any point in the half plane $(y=0)$ is defined as the vertical displacement (w) at that point minus the vertical displacement (w_{00}) on the surface of the half plane on the load axis (i.e. the point $z = 0$, $x = 0$). Because of this the solutions for the uniform vertical displacement are referred to a total resultant vertical load (T_v) and not a reference vertical displacement.

The solutions for loading by linear vertical pressure and linear vertical displacement are given on pages 16 and 17. For the displacement defined load it is possible to derive relationships between the total resultant moment (about the y axis) that is necessary to produce a defined maximum displacement in each of the cases of β^2 positive, β^2 negative, and β^2 zero. By use of these relationships the solutions for the displacement defined load can be referred to either a total resultant moment (T_y) or a maximum vertical displacement (Δ_2).

For loading by uniform lateral pressure† a finite measure of the lateral displacement can only be obtained by considering a relative quantity rather than an absolute one.†† Hence the relative lateral displacement $(u-u_{00})$ at any point in the half plane is defined as the lateral displacement (u) at that point minus the lateral displacement (u_{00}) on the surface of the half plane on the load axis (i.e. the point $z = 0$, $x = 0$). Because of this the solutions for the uniform lateral displacement are referred to a total resultant lateral load (T_h).

Loads of the type of linear lateral pressure and linear lateral displacement (see solutions on pages 20 and 21) are self cancelling in nature since no resultant forces or moments are produced. Hence the solutions for the displacement defined case can only be referred to a maximum lateral displacement.

§ The solutions for these cases are given on pages 14 and 15.
† The solutions for these cases are given on pages 18 and 19.
†† This situation is analogous to that applying in the case of loading by uniform vertical pressure and uniform vertical displacement.

VII. SOLUTIONS FOR DISPLACEMENTS, STRAINS AND STRESSES

1(a) UNIFORM VERTICAL PRESSURE

The solutions given below involve products of coefficients (g_n, i_n, s_n) and integrals ($Q_{11}(\psi)$, ${}_cQ_{11}$, $K_{1c\mu}(\psi)$, $K_{1s\mu}(\psi)$, ${}_cK_{1c\mu}$, ${}_sK_{1c\mu}$, ${}_cK_{1s\mu}$, and ${}_sK_{1s\mu}$). These coefficients and integrals are evaluated on pages 26 and 22 respectively.

A. Orthorhombic and Cross-anisotropic; β^2 Positive.

$$w-w_{00} = (\tfrac{2}{\pi})^{\frac{1}{2}}.P_1.x_0.\{-g_1.Q_{11}(\phi z) + g_2.Q_{11}(\rho z)\}$$
$$u = (\tfrac{2}{\pi})^{\frac{1}{2}}.P_1.x_0.\{-g_3.K_{1s1}(\phi z) + g_4.K_{1s1}(\rho z)\}$$
$$\varepsilon_{xx} = (\tfrac{2}{\pi})^{\frac{1}{2}}.P_1.\{-g_3.K_{1c3}(\phi z) + g_4.K_{1c3}(\rho z)\}$$
$$\varepsilon_{zz} = (\tfrac{2}{\pi})^{\frac{1}{2}}.P_1.\{g_1.\phi.K_{1c3}(\phi z) - g_2.\rho.K_{1c3}(\rho z)\}$$
$$\varepsilon_{xz} = (\tfrac{2}{\pi})^{\frac{1}{2}}.P_1.g_9.f^{-1}.\{K_{1s3}(\phi z) - K_{1s3}(\rho z)\}$$
$$\widehat{zz} = (\tfrac{2}{\pi})^{\frac{1}{2}}.P_1.g_9.\{\phi^{-1}.K_{1c3}(\phi z) - \rho^{-1}.K_{1c3}(\rho z)\}$$
$$\widehat{xx} = (\tfrac{2}{\pi})^{\frac{1}{2}}.P_1.\{-g_5.K_{1c3}(\phi z) + g_6.K_{1c3}(\rho z)\}$$
$$\widehat{yy} = (\tfrac{2}{\pi})^{\frac{1}{2}}.P_1.\{-g_7^*.K_{1c3}(\phi z) + g_8^*.K_{1c3}(\rho z)\}$$
$$\widehat{xz} = f.\varepsilon_{xz}$$

B. Orthorhombic and Cross-anisotropic; β^2 Negative.

$$w-w_{00} = (\tfrac{2}{\pi})^{\frac{1}{2}}.P_1.x_0.\{-i_1.{}_cQ_{11} - i_2.{}_sK_{1c1}\}$$
$$u = (\tfrac{2}{\pi})^{\frac{1}{2}}.P_1.x_0.\{i_3.{}_cK_{1s1} + i_4.{}_sK_{1s1}\}$$
$$\varepsilon_{xx} = (\tfrac{2}{\pi})^{\frac{1}{2}}.P_1.\{i_3.{}_cK_{1c3} + i_4.{}_sK_{1c3}\}$$
$$\varepsilon_{zz} = (\tfrac{2}{\pi})^{\frac{1}{2}}.P_1.\{i_9.{}_cK_{1c3} + i_{10}.{}_sK_{1c3}\}$$
$$\varepsilon_{xz} = (\tfrac{2}{\pi})^{\frac{1}{2}}.P_1.i_2.{}_sK_{1s3}$$
$$\widehat{zz} = (\tfrac{2}{\pi})^{\frac{1}{2}}.P_1.\{{}_cK_{1c3} + \alpha.\omega^{-1}.{}_sK_{1c3}\}$$
$$\widehat{xx} = (\tfrac{2}{\pi})^{\frac{1}{2}}.P_1.\{i_5.{}_cK_{1c3} + i_6.{}_sK_{1c3}\}$$
$$\widehat{yy} = (\tfrac{2}{\pi})^{\frac{1}{2}}.P_1.\{i_7^*.{}_cK_{1c3} + i_8^*.{}_sK_{1c3}\}$$
$$\widehat{xz} = f.\varepsilon_{xz}$$

C. Orthorhombic and Cross-anisotropic; β^2 Zero.

$$w-w_{00} = (\tfrac{2}{\pi})^{\frac{1}{2}}.P_1.x_0.\{-s_1.Q_{11}(\alpha z) - s_2.z.K_{1c3}(\alpha z)\}$$
$$u = (\tfrac{2}{\pi})^{\frac{1}{2}}.P_1.x_0.\{s_3.K_{1s1}(\alpha z) - s_4.z.K_{1s3}(\alpha z)\}$$
$$\varepsilon_{xx} = (\tfrac{2}{\pi})^{\frac{1}{2}}.P_1.\{s_3.K_{1c3}(\alpha z) - s_4.z.K_{1c5}(\alpha z)\}$$
$$\varepsilon_{zz} = (\tfrac{2}{\pi})^{\frac{1}{2}}.P_1.\{s_9.K_{1c3}(\alpha z) + s_2.\alpha.z.K_{1c5}(\alpha z)\}$$
$$\varepsilon_{xz} = (\tfrac{2}{\pi})^{\frac{1}{2}}.P_1.\alpha^2.f^{-1}.z.K_{1s5}(\alpha z)$$
$$\widehat{zz} = (\tfrac{2}{\pi})^{\frac{1}{2}}.P_1.\{K_{1c3}(\alpha z) + \alpha.z.K_{1c5}(\alpha z)\}$$
$$\widehat{xx} = (\tfrac{2}{\pi})^{\frac{1}{2}}.P_1.\{s_5.K_{1c3}(\alpha z) - s_6.z.K_{1c5}(\alpha z)\}$$
$$\widehat{yy} = (\tfrac{2}{\pi})^{\frac{1}{2}}.P_1.\{s_7^*.K_{1c3}(\alpha z) - s_8^*.z.K_{1c5}(\alpha z)\}$$
$$\widehat{xz} = f.\varepsilon_{xz}$$

D. Isotropic

The displacements, strains, and stresses are as for the case of Orthorhombic and Cross-anisotropic; β^2 Zero but with the following simplifications;

$s_1 = 2.(1-\nu^2).E^{-1}$	$s_2 = (1+\nu).E^{-1}$	$s_3 = (1+\nu).(1-2\nu).E^{-1}$	$s_4 = (1+\nu).E^{-1}$
$s_5 = s_6 = 1$	$s_7^* = 2\nu$	$s_8^* = 0$	$s_9 = (1+\nu).(1-2\nu).E^{-1}$
$\alpha = 1$	$f = a - b = E.(1+\nu)^{-1}$		

1(b) UNIFORM VERTICAL DISPLACEMENT

The solutions given below involve products of coefficients (g_n, i_n, s_n) and integrals ($Q_{02}(\psi)$, ${}_cQ_{02}$, $K_{0c(\mu+1)}(\psi)$, $K_{0s(\mu+1)}(\psi)$, ${}_cK_{0c(\mu+1)}$, ${}_sK_{0c(\mu+1)}$, ${}_cK_{0s(\mu+1)}$, and ${}_sK_{cs(\mu+1)}$. These coefficients and integrals are evaluated on pages 26 and 23 respectively.

A. Orthorhombic and Cross-anisotropic; β^2 Positive.

The displacements, strains, and stresses are obtained by making the following substitutions into the equations for the case of uniform vertical pressure loading (Orthorhombic and Cross-anisotropic; β^2 Positive). These equations are shown on the opposite page.

(i) substitute $T_v.(2\pi)^{-\frac{1}{2}}.x_0^{-1}$ ---------- for ---------- P_1

(ii) substitute $Q_{02}(\psi)$ ---------- for ---------- $Q_{11}(\psi)$

$K_{0c(\mu+1)}(\psi)$ ---------- for ---------- $K_{1c\mu}(\psi)$

$K_{0s(\mu+1)}(\psi)$ ---------- for ---------- $K_{1s\mu}(\psi)$

B. Orthorhombic and Cross-anisotropic; β^2 Negative.

The displacements, strains, and stresses are obtained by making the following substitutions into the equations for the case of uniform vertical pressure loading (Orthorhombic and Cross-anisotropic; β^2 Negative). These equations are shown on the opposite page.

(i) substitute $T_v.(2\pi)^{-\frac{1}{2}}.x_0^{-1}$ ---------- for ---------- P_1

(ii) substitute ${}_cQ_{02}$ ---------- for ---------- ${}_cQ_{11}$

${}_cK_{0c(\mu+1)}$ ---------- for ---------- ${}_cK_{1c\mu}$

${}_sK_{0c(\mu+1)}$ ---------- for ---------- ${}_sK_{1c\mu}$

${}_cK_{0s(\mu+1)}$ ---------- for ---------- ${}_cK_{1s\mu}$

${}_sK_{0s(\mu+1)}$ ---------- for ---------- ${}_sK_{1s\mu}$

C. Orthorhombic and Cross-anisotropic; β^2 Zero.

The displacements, strains and stresses are obtained by making the following substitutions into the equations for the case of uniform vertical pressure loading (Orthorhombic and Cross-anisotropic; β^2 Zero). These equations are shown on the opposite page.

(i) substitute $T_v.(2\pi)^{-\frac{1}{2}}.x_0^{-1}$ ---------- for ---------- P_1

(ii) substitute $Q_{02}(\alpha z)$ ---------- for ---------- $Q_{11}(\alpha z)$

$K_{0c(\mu+1)}(\alpha z)$ ---------- for ---------- $K_{1c\mu}(\alpha z)$

$K_{0s(\mu+1)}(\alpha z)$ ---------- for ---------- $K_{1s\mu}(\alpha z)$

D. Isotropic

The displacements, strains and stresses are obtained by making the following substitutions into the equations for the case of uniform vertical pressure loading (Orthorhombic and Cross-anistropic; β^2 Zero). These equations are shown on the opposite page.

(i) substitute $T_v.(2\pi)^{-\frac{1}{2}}.x_0^{-1}$ ---------- for ---------- P_1

(ii) substitute $Q_{02}(z)$ ---------- for ---------- $Q_{11}(z)$

$K_{0c(\mu+1)}(z)$ ---------- for ---------- $K_{1c\mu}(z)$

$K_{0s(\mu+1)}(z)$ ---------- for ---------- $K_{1s\mu}(z)$

(iii) In addition the simplified values of α, f and s_n (shown on the opposite page) are applicable.

2(a) LINEAR VERTICAL PRESSURE

The solutions given below involve products of coefficients (g_n, i_n, s_n) and integrals ($K_{3c\mu}(\phi)$, $K_{3s\mu}(\phi)$, ${}_cK_{3c\mu}$, ${}_sK_{3c\mu}$, ${}_cK_{3s\mu}$, and ${}_sK_{3s\mu}$). These coefficients and integrals are evaluated on pages 26 and 24 respectively.

A. Orthorhombic and Cross-anisotropic; β^2 Positive.

$$w = (\tfrac{2}{\pi})^{\frac{1}{2}}.P_2.x_0.\{-g_1.K_{3s1}(\phi z) + g_2.K_{3s1}(\rho z)\}$$
$$u = (\tfrac{2}{\pi})^{\frac{1}{2}}.P_2.x_0.\{g_3.K_{3c1}(\phi z) - g_4.K_{3c1}(\rho z)\}$$
$$\varepsilon_{xx} = (\tfrac{2}{\pi})^{\frac{1}{2}}.P_2.\{-g_3.K_{3s3}(\phi z) + g_4.K_{3s3}(\rho z)\}$$
$$\varepsilon_{zz} = (\tfrac{2}{\pi})^{\frac{1}{2}}.P_2.\{g_1.\phi.K_{3s3}(\phi z) - g_2.\rho.K_{3s3}(\rho z)\}$$
$$\varepsilon_{xz} = (\tfrac{2}{\pi})^{\frac{1}{2}}.P_2.g_9.f^{-1}.\{-K_{3c3}(\phi z) + K_{3c3}(\rho z)\}$$
$$\widehat{zz} = (\tfrac{2}{\pi})^{\frac{1}{2}}.P_2.g_9.\{\phi^{-1}.K_{3s3}(\phi z) - \rho^{-1}.K_{3s3}(\rho z)\}$$
$$\widehat{xx} = (\tfrac{2}{\pi})^{\frac{1}{2}}.P_2.\{-g_5.K_{3s3}(\phi z) + g_6.K_{3s3}(\rho z)\}$$
$$\widehat{yy} = (\tfrac{2}{\pi})^{\frac{1}{2}}.P_2.\{-g_7^*.K_{3s3}(\phi z) + g_8^*.K_{3s3}(\rho z)\}$$
$$\widehat{xz} = f.\varepsilon_{xz}$$

B. Orthorhombic and Cross-anisotropic; β^2 Negative.

$$w = (\tfrac{2}{\pi})^{\frac{1}{2}}.P_2.x_0.\{-i_1.{}_cK_{3s1} - i_2.{}_sK_{3s1}\}$$
$$u = (\tfrac{2}{\pi})^{\frac{1}{2}}.P_2.x_0.\{-i_3.{}_cK_{3c1} - i_4.{}_sK_{3c1}\}$$
$$\varepsilon_{xx} = (\tfrac{2}{\pi})^{\frac{1}{2}}.P_2.\{i_3.{}_cK_{3s3} + i_4.{}_sK_{3s3}\}$$
$$\varepsilon_{zz} = (\tfrac{2}{\pi})^{\frac{1}{2}}.P_2.\{i_9.{}_cK_{3s3} + i_{10}.{}_sK_{3s3}\}$$
$$\varepsilon_{xz} = (\tfrac{2}{\pi})^{\frac{1}{2}}.P_2.i_2.\{-{}_sK_{3c3}\}$$
$$\widehat{zz} = (\tfrac{2}{\pi})^{\frac{1}{2}}.P_2.\{{}_cK_{3s3} + \alpha.\omega^{-1}.{}_sK_{3s3}\}$$
$$\widehat{xx} = (\tfrac{2}{\pi})^{\frac{1}{2}}.P_2.\{i_5.{}_cK_{3s3} + i_6.{}_sK_{3s3}\}$$
$$\widehat{yy} = (\tfrac{2}{\pi})^{\frac{1}{2}}.P_2.\{i_7^*.{}_cK_{3s3} + i_8^*.{}_sK_{3s3}\}$$
$$\widehat{xz} = f.\varepsilon_{xz}$$

C. Orthorhombic and Cross-anisotropic; β^2 Zero.

$$w = (\tfrac{2}{\pi})^{\frac{1}{2}}.P_2.x_0.\{-s_1.K_{3s1}(\alpha z) - s_2.z.K_{3s3}(\alpha z)\}$$
$$u = (\tfrac{2}{\pi})^{\frac{1}{2}}.P_2.x_0.\{-s_3.K_{3c1}(\alpha z) + s_4.z.K_{3c3}(\alpha z)\}$$
$$\varepsilon_{xx} = (\tfrac{2}{\pi})^{\frac{1}{2}}.P_2.\{s_3.K_{3s3}(\alpha z) - s_4.z.K_{3s5}(\alpha z)\}$$
$$\varepsilon_{zz} = (\tfrac{2}{\pi})^{\frac{1}{2}}.P_2.\{s_9.K_{3s3}(\alpha z) + s_2.\alpha.z.K_{3s5}(\alpha z)\}$$
$$\varepsilon_{xz} = (\tfrac{2}{\pi})^{\frac{1}{2}}.P_2.\alpha^2.f^{-1}.\{-z.K_{3c5}(\alpha z)\}$$
$$\widehat{zz} = (\tfrac{2}{\pi})^{\frac{1}{2}}.P_2.\{K_{3s3}(\alpha z) + \alpha.z.K_{3s5}(\alpha z)\}$$
$$\widehat{xx} = (\tfrac{2}{\pi})^{\frac{1}{2}}.P_2.\{s_5.K_{3s3}(\alpha z) - s_6.z.K_{3s5}(\alpha z)\}$$
$$\widehat{yy} = (\tfrac{2}{\pi})^{\frac{1}{2}}.P_2.\{s_7^*.K_{3s3}(\alpha z) - s_8^*.z.K_{3s5}(\alpha z)\}$$
$$\widehat{xz} = f.\varepsilon_{xz}$$

D. Isotropic.

The displacements, strains, and stresses are as for the case of Orthorhombic and Cross-anisotropic; β^2 Zero but with the following simplifications;

$s_1 = 2.(1-\nu^2).E^{-1}$ $\quad s_2 = (1+\nu).E^{-1}$ $\quad s_3 = (1+\nu).(1-2\nu).E^{-1}$ $\quad s_4 = (1+\nu).E^{-1}$

$s_5 = s_6 = 1$ $\quad s_7^* = 2\nu$ $\quad s_8^* = 0$ $\quad s_9 = (1+\nu).(1-2\nu).E^{-1}$

$\alpha = 1$ $\quad f = a - b = E.(1+\nu)^{-1}$

2(b) LINEAR VERTICAL DISPLACEMENT

The solutions given below involve products of coefficients (g_n, i_n, s_n) and integrals ($K_{2c(\mu+1)}(\psi)$, $K_{2s(\mu+1)}(\psi)$, ${}_cK_{2c(\mu+1)}$, ${}_sK_{2c(\mu+1)}$, ${}_cK_{2s(\mu+1)}$, and ${}_sK_{2s(\mu+1)}$). These coefficients and integrals are evaluated on pages 26 and 25 respectively.

A. Orthorhombic and Cross-anisotropic; β^2 Positive.

The displacements, strains, and stresses are obtained by making the following substitutions into the equations for the case of linear vertical pressure loading (Orthorhombic and Cross-anisotropic; β^2 Positive). These equations are shown on the opposite page.

(i) substitute $T_y . (\frac{2}{\pi})^{\frac{1}{2}} . \frac{1}{x_0^2}$ OR $-\Delta_2 . (\frac{\pi}{2})^{\frac{1}{2}} . \frac{1}{x_0} . \frac{f}{2c+f} . \frac{(c+d\phi^2).(c+d\rho^2)}{d.\rho.\phi.(\rho+\phi)}$ -------------- for ---------------- P_2

This alternative substitution follows from the moment - displacement relationship for this case of β^2 Positive. This relationship is :

$$T_y = \frac{\pi}{2} . x_0 . \frac{f}{2c+f} . \frac{(c+d\phi^2).(c+d\rho^2)}{d.\rho.\phi.(\rho+\phi)} . \Delta_2$$

(ii) substitute $K_{2c(\mu+1)}(\psi)$ ---------- for ---------- $K_{3c\mu}(\psi)$

$K_{2s(\mu+1)}(\psi)$ ---------- for ---------- $K_{3s\mu}(\psi)$

B. Orthorhombic and Cross-anisotropic; β^2 Negative.

The displacements, strains, and stresses are obtained by making the following substitutions into the equations for the case of linear vertical pressure loading (Orthorhombic and Cross-anisotropic; β^2 Negative). These equations are shown on the opposite page.

(i) substitute $T_y . (\frac{2}{\pi})^{\frac{1}{2}} . \frac{1}{x_0^2}$ OR $-\Delta_2 . (\frac{\pi}{2})^{\frac{1}{2}} . \frac{1}{x_0} . \frac{ad-c^2}{2\alpha.(ad)^{\frac{1}{2}}}$. -------------- for ---------------- P_2

This alternative substitution follows from the moment-displacement relationship for this case of β^2 Negative. This relationship is :

$$T_y = -\frac{\pi}{2} . x_0 . \frac{ad-c^2}{2\alpha.(ad)^{\frac{1}{2}}} . \Delta_2$$

(ii) substitute ${}_cK_{2c(\mu+1)}$ ---------- for ---------- ${}_cK_{3c\mu}$

${}_sK_{2c(\mu+1)}$ ---------- for ---------- ${}_sK_{3c\mu}$

${}_cK_{2s(\mu+1)}$ ---------- for ---------- ${}_cK_{3s\mu}$

${}_sK_{2s(\mu+1)}$ ---------- for ---------- ${}_sK_{3s\mu}$

C. Orthorhombic and Cross-anisotropic; β^2 Zero.

The displacements, strains and stresses are obtained by making the following substitutions into the equations for the case of linear vertical pressure loading (Orthorhombic and Cross-anisotropic; β^2 Zero). These equations are shown on the opposite page.

(i) substitute $T_y . (\frac{2}{\pi})^{\frac{1}{2}} . \frac{1}{x_0^2}$ OR $-\Delta_2 . (\frac{\pi}{2})^{\frac{1}{2}} . \frac{1}{x_0} . \frac{f}{2c+f} . \frac{(c+d\alpha^2)^2}{2d\alpha^3}$ -------------- for ---------------- P_2

This alternative substitution follows from the moment-displacement relationship for this case of β^2 Zero. This relationship is:

$$T_y = \frac{\pi}{2} . x_0 . \frac{f}{2c+f} . \frac{(c+d\alpha^2)^2}{2d\alpha^3} . \Delta_2$$

(ii) substitute $K_{2c(\mu+1)}(\alpha z)$ ---------- for ---------- $K_{3c\mu}(\alpha z)$

$K_{2s(\mu+1)}(\alpha z)$ ---------- for ---------- $K_{3s\mu}(\alpha z)$

D. Isotropic

The displacements, strains, and stresses are obtained by making the following substitutions into the equations for the case of linear vertical pressure loading (Orthorhombic and Cross-anisotropic: β^2 Zero). These equations are shown on the opposite page.

(i) substitute $T_y . (\frac{2}{\pi})^{\frac{1}{2}} . \frac{1}{x_0^2}$ OR $-\Delta_2 . (\frac{\pi}{2})^{\frac{1}{2}} . \frac{1}{x_0} . \frac{E}{2(1-\nu^2)}$ -------------- for ---------------- P_2

This alternative substitution follows from the moment-displacement relationship for this isotropic case. This relationship is;

$$T_y = \frac{\pi}{2} . x_0 . \frac{E}{2(1-\nu^2)} . \Delta_2$$

(ii) substitute $K_{2c(\mu+1)}(z)$ ---------- for ------ $K_{3c\mu}(z)$

$K_{2s(\mu+1)}(z)$ ---------- for ------ $K_{3s\mu}(z)$

(iii) In addition the simplified values of α, f, and s_n (shown on the opposite page) are applicable.

3(a) UNIFORM LATERAL SHEAR STRESS

The solutions given below involve products of coefficients (h_n, j_n, and t_n) and integrals ($Q_{11}(\psi)$, ${}_cQ_{11}$, $K_{1c\mu}(\psi)$, $K_{1s\mu}(\psi)$, ${}_cK_{1c\mu}$, ${}_sK_{1c\mu}$, ${}_cK_{1s\mu}$, and ${}_sK_{1s\mu}$). These coefficients and integrals are evaluated on pages 26 and 22 respectively.

A. Orthorhombic and Cross-anisotropic; β^2 Positive.

$$w = \left(\tfrac{2}{\pi}\right)^{\frac{1}{2}}.p_1.x_0.\{-h_1.K_{1s1}(\phi z) + h_2.K_{1s1}(\rho z)\}$$
$$u-u_{00} = \left(\tfrac{2}{\pi}\right)^{\frac{1}{2}}.p_1.x_0.\{h_3.Q_{11}(\phi z) - h_4.Q_{11}(\rho z)\}$$
$$\varepsilon_{xx} = \left(\tfrac{2}{\pi}\right)^{\frac{1}{2}}.p_1.\{-h_3.K_{1s3}(\phi z) + h_4.K_{1s3}(\rho z)\}$$
$$\varepsilon_{zz} = \left(\tfrac{2}{\pi}\right)^{\frac{1}{2}}.p_1.\{\phi.h_1.K_{1s3}(\phi z) - \rho.h_2.K_{1s3}(\rho z)\}$$
$$\varepsilon_{xz} = \left(\tfrac{2}{\pi}\right)^{\frac{1}{2}}.p_1.f^{-1}.h_9.\{-\rho^{-1}.K_{1c3}(\phi z) + \phi^{-1}.K_{1c3}(\rho z)\}$$
$$\widehat{zz} = \left(\tfrac{2}{\pi}\right)^{\frac{1}{2}}.p_1.h_9.\rho^{-1}.\phi^{-1}.\{K_{1s3}(\phi z) - K_{1s3}(\rho z)\}$$
$$\widehat{xx} = \left(\tfrac{2}{\pi}\right)^{\frac{1}{2}}.p_1.\{-h_5.K_{1s3}(\phi z) + h_6.K_{1s3}(\rho z)\}$$
$$\widehat{yy} = \left(\tfrac{2}{\pi}\right)^{\frac{1}{2}}.p_1.\{-h_7^*.K_{1s3}(\phi z) + h_8^*.K_{1s3}(\rho z)\}$$
$$\widehat{xz} = f.\varepsilon_{xz}$$

B. Orthorhombic and Cross-anisotropic; β^2 Negative.

$$w = \left(\tfrac{2}{\pi}\right)^{\frac{1}{2}}.p_1.x_0.\{-j_1.{}_cK_{1s1} - j_2.{}_sK_{1s1}\}$$
$$u-u_{00} = \left(\tfrac{2}{\pi}\right)^{\frac{1}{2}}.p_1.x_0.\{j_3.{}_cQ_{11} + j_4.{}_sK_{1c1}\}$$
$$\varepsilon_{xx} = \left(\tfrac{2}{\pi}\right)^{\frac{1}{2}}.p_1.\{-j_3.{}_cK_{1s3} - j_4.{}_sK_{1s3}\}$$
$$\varepsilon_{zz} = \left(\tfrac{2}{\pi}\right)^{\frac{1}{2}}.p_1.\{j_9.{}_cK_{1s3} + j_{10}.{}_sK_{1s3}\}$$
$$\varepsilon_{xz} = \left(\tfrac{2}{\pi}\right)^{\frac{1}{2}}.p_1.f^{-1}.\{{}_cK_{1c3} - \alpha.\omega^{-1}.{}_sK_{1c3}\}$$
$$\widehat{zz} = \left(\tfrac{2}{\pi}\right)^{\frac{1}{2}}.p_1.\omega^{-1}.{}_sK_{1s3}$$
$$\widehat{xx} = \left(\tfrac{2}{\pi}\right)^{\frac{1}{2}}.p_1.\{-j_5.{}_cK_{1s3} - j_6.{}_sK_{1s3}\}$$
$$\widehat{yy} = \left(\tfrac{2}{\pi}\right)^{\frac{1}{2}}.p_1.\{-j_7^*.{}_cK_{1s3} - j_8^*.{}_sK_{1s3}\}$$
$$\widehat{xz} = f.\varepsilon_{xz}$$

C. Orthorhombic and Cross-anisotropic; β^2 Zero.

$$w = \left(\tfrac{2}{\pi}\right)^{\frac{1}{2}}.p_1.x_0.\{-t_1.K_{1s1}(\alpha z) - t_2.z.K_{1s3}(\alpha z)\}$$
$$u-u_{00} = \left(\tfrac{2}{\pi}\right)^{\frac{1}{2}}.p_1.x_0.\{-t_3.Q_{11}(\alpha z) + t_4.z.K_{1c3}(\alpha z)\}$$
$$\varepsilon_{xx} = \left(\tfrac{2}{\pi}\right)^{\frac{1}{2}}.p_1.\{t_3.K_{1s3}(\alpha z) - t_4.z.K_{1s5}(\alpha z)\}$$
$$\varepsilon_{zz} = \left(\tfrac{2}{\pi}\right)^{\frac{1}{2}}.p_1.\{-t_9.K_{1s3}(\alpha z) + t_2.\alpha.z.K_{1s5}(\alpha z)\}$$
$$\varepsilon_{xz} = \left(\tfrac{2}{\pi}\right)^{\frac{1}{2}}.p_1.f^{-1}.\{K_{1c3}(\alpha z) - \alpha.z.K_{1c5}(\alpha z)\}$$
$$\widehat{zz} = \left(\tfrac{2}{\pi}\right)^{\frac{1}{2}}.p_1.z.K_{1s5}(\alpha z)$$
$$\widehat{xx} = \left(\tfrac{2}{\pi}\right)^{\frac{1}{2}}.p_1.\{t_5.K_{1s3}(\alpha z) - t_6.z.K_{1s5}(\alpha z)\}$$
$$\widehat{yy} = \left(\tfrac{2}{\pi}\right)^{\frac{1}{2}}.p_1.\{t_7^*.K_{1s3}(\alpha z) - t_8^*.z.K_{1s5}(\alpha z)\}$$
$$\widehat{xz} = f.\varepsilon_{xz}$$

D. Isotropic

The displacements, strains, and stresses are as for the case Orthorhombic and Cross-anisotropic; β^2 Zero but with the following simplifications;

$t_1 = (1+\nu).(1-2\nu).E^{-1}$	$t_2 = (1+\nu).E^{-1}$	$t_3 = 2.(1-\nu^2).E^{-1}$	$t_4 = (1+\nu).E^{-1}$
$t_5 = 2$	$t_6 = 1$	$t_7^* = 2\nu$	$t_8^* = 0$
$t_9 = 2\nu.(1+\nu).E^{-1}$	$\alpha = 1$	$f = a-b = E.(1+\nu)^{-1}$	

3(b) UNIFORM LATERAL SHEAR DISPLACEMENT

The solutions given below involve products of coefficients (h_n, j_n, and t_n) and integrals ($Q_{02}(\psi)$, ${}_cQ_{02}$, $K_{0c(\mu+1)}(\psi)$, $K_{0s(\mu+1)}(\psi)$, ${}_cK_{0c(\mu+1)}$, ${}_sK_{0c(\mu+1)}$, ${}_cK_{0s(\mu+1)}$, and ${}_sK_{0s(\mu+1)}$). These coefficients and integrals are evaluated on pages 26 and 23 respectively.

A. Orthorhombic and Cross-anisotropic ; β^2 Positive.

The displacements, strains and stresses are obtained by making the following substitutions into the equations for the case of uniform lateral shear stress loading (Orthorhombic and Cross-anisotropic; β^2 Positive). These equations are shown on the opposite page.

(i) substitute $T_h.(2\pi)^{-\frac{1}{2}}.z_0^{-1}$ ---------- for ---------- p_1

(ii) substitute $Q_{02}(\psi)$ ---------- for ---------- $Q_{11}(\psi)$

$K_{0c(\mu+1)}(\psi)$ ---------- for ---------- $K_{1c\mu}(\psi)$

$K_{0s(\mu+1)}(\psi)$ ---------- for ---------- $K_{1s\mu}(\psi)$

B. Orthorhombic and Cross-Anisotropic; β^2 Negative.

The displacements, strains, and stresses are obtained by making the following substitutions into the equations for the case of uniform lateral shear stress loading (Orthorhombic and Cross-anisotropic; β^2 Negative). These equations are shown on the opposite page.

(i) substitute $T_h.(2\pi)^{-\frac{1}{2}}.z_0^{-1}$ ---------- for ---------- p_1

(ii) substitute ${}_cQ_{02}$ ---------- for ---------- ${}_cQ_{11}$

${}_cK_{0c(\mu+1)}$ ---------- for ---------- ${}_cK_{1c\mu}$

${}_sK_{0c(\mu+1)}$ ---------- for ---------- ${}_sK_{1c\mu}$

${}_cK_{0s(\mu+1)}$ ---------- for ---------- ${}_cK_{1s\mu}$

${}_sK_{0s(\mu+1)}$ ---------- for ---------- ${}_sK_{1s\mu}$

C. Orthorhombic and Cross-anisotropic; β^2 Zero.

The displacements, strains and stresses are obtained by making the following substitutions into the equations for the case of uniform lateral shear stress loading (Orthorhombic and Cross-anisotropic; β^2 Zero). These equations are shown on the opposite page.

(i) substitute $T_h.(2\pi)^{-\frac{1}{2}}.z_0^{-1}$ ---------- for ---------- p_1

(ii) substitute $Q_{02}(\alpha z)$ ---------- for ---------- $Q_{11}(\alpha z)$

$K_{0c(\mu+1)}(\alpha z)$ ---------- for ---------- $K_{1c\mu}(\alpha z)$

$K_{0s(\mu+1)}(\alpha z)$ ---------- for ---------- $K_{1s\mu}(\alpha z)$

D. Isotropic.

The displacements, strains and stresses are obtained by making the following substitutions into the equations for the case of uniform lateral shear stress loading (Orthorhombic and Cross-anisotropic; β^2 Zero). These equations are shown on the opposite page.

(i) substitute $T_h.(2\pi)^{-\frac{1}{2}}.z_0^{-1}$ ---------- for ---------- P_1

(ii) substitute $Q_{02}(z)$ ---------- for ---------- $Q_{11}(z)$

$K_{0c(\mu+1)}(z)$ ---------- for ---------- $K_{1c\mu}(z)$

$K_{0s(\mu+1)}(z)$ ---------- for ---------- $K_{1s\mu}(z)$

(iii) In addition the simplified values of α, f and s_n (shown on the opposite page) are applicable.

4(a) LINEAR LATERAL SHEAR STRESS

The solutions given below involve products of coefficients (h_n, j_n, and t_n) and integrals ($K_{3c\mu}(\psi)$, $K_{3s\mu}(\psi)$, ${}_cK_{3c\mu}$, ${}_sK_{3c\mu}$, ${}_cK_{3s\mu}$, and ${}_sK_{3s\mu}$). These coefficients and integrals are evaluated on pages 26 and 24 respectively.

A. Orthorhombic and Cross-anisotropic; β^2 Positive.

$$w = (\tfrac{2}{\pi})^{\frac{1}{2}}.p_2.x_0.\{h_1.K_{3c1}(\phi z) - h_2.K_{3c1}(\rho z)\}$$
$$u = (\tfrac{2}{\pi})^{\frac{1}{2}}.p_2.x_0.\{h_3.K_{3s1}(\phi z) - h_4.K_{3s1}(\rho z)\}$$
$$\varepsilon_{xx} = (\tfrac{2}{\pi})^{\frac{1}{2}}.p_2.\{h_3.K_{3c3}(\phi z) - h_4.K_{3c3}(\rho z)\}$$
$$\varepsilon_{zz} = (\tfrac{2}{\pi})^{\frac{1}{2}}.p_2.\{-\phi.h_1.K_{3c3}(\phi z) + \rho.h_2.K_{3c3}(\rho z)\}$$
$$\varepsilon_{xz} = (\tfrac{2}{\pi})^{\frac{1}{2}}.p_2.f^{-1}.h_9.\{-\rho^{-1}.K_{3s3}(\phi z) + \phi^{-1}.K_{3s3}(\rho z)\}$$
$$\widehat{zz} = (\tfrac{2}{\pi})^{\frac{1}{2}}.p_2.h_9.\rho^{-1}.\phi^{-1}.\{-K_{3c3}(\phi z) + K_{3c3}(\rho z)\}$$
$$\widehat{xx} = (\tfrac{2}{\pi})^{\frac{1}{2}}.p_2.\{h_5.K_{3c3}(\phi z) - h_6.K_{3c3}(\rho z)\}$$
$$\widehat{yy} = (\tfrac{2}{\pi})^{\frac{1}{2}}.p_2.\{h_7^*.K_{3c3}(\phi z) - h_8^*.K_{3c3}(\rho z)\}$$
$$\widehat{xz} = f.\varepsilon_{xz}$$

B. Orthorhombic and Cross-anisotropic; β^2 Negative.

$$w = (\tfrac{2}{\pi})^{\frac{1}{2}}.p_2.x_0.\{j_1.{}_cK_{3c1} + j_2.{}_sK_{3c1}\}$$
$$u = (\tfrac{2}{\pi})^{\frac{1}{2}}.p_2.x_0.\{j_3.{}_cK_{3s1} + j_4.{}_sK_{3s1}\}$$
$$\varepsilon_{xx} = (\tfrac{2}{\pi})^{\frac{1}{2}}.p_2.\{j_3.{}_cK_{3c3} + j_4.{}_sK_{3c3}\}$$
$$\varepsilon_{zz} = (\tfrac{2}{\pi})^{\frac{1}{2}}.p_2.\{-j_9.{}_cK_{3c3} - j_{10}.{}_sK_{3c3}\}$$
$$\varepsilon_{xz} = (\tfrac{2}{\pi})^{\frac{1}{2}}.p_2.f^{-1}.\{{}_cK_{3s3} - \alpha.\omega^{-1}.{}_sK_{3s3}\}$$
$$\widehat{zz} = (\tfrac{2}{\pi})^{\frac{1}{2}}.p_2.\omega^{-1}\{-{}_sK_{3c3}\}$$
$$\widehat{xx} = (\tfrac{2}{\pi})^{\frac{1}{2}}.p_2.\{j_5.{}_cK_{3c3} + j_6.{}_sK_{3c3}\}$$
$$\widehat{yy} = (\tfrac{2}{\pi})^{\frac{1}{2}}.p_2.\{j_7^*.{}_cK_{3c3} + j_8^*.{}_sK_{3c3}\}$$
$$\widehat{xz} = f.\varepsilon_{xz}$$

C. Orthorhombic and Cross-anisotropic; β^2 Zero.

$$w = (\tfrac{2}{\pi})^{\frac{1}{2}}.p_2.x_0.\{t_1.K_{3c1}(\alpha z) + t_2.z.K_{3c3}(\alpha z)\}$$
$$u = (\tfrac{2}{\pi})^{\frac{1}{2}}.p_2.x_0.\{-t_3.K_{3s1}(\alpha z) + t_4.z.K_{3s3}(\alpha z)\}$$
$$\varepsilon_{xx} = (\tfrac{2}{\pi})^{\frac{1}{2}}.p_2.\{-t_3.K_{3c3}(\alpha z) + t_4.z.K_{3c5}(\alpha z)\}$$
$$\varepsilon_{zz} = (\tfrac{2}{\pi})^{\frac{1}{2}}.p_2.\{t_9.K_{3c3}(\alpha z) - t_2.\alpha.z.K_{3c5}(\alpha z)\}$$
$$\varepsilon_{xz} = (\tfrac{2}{\pi})^{\frac{1}{2}}.p_2.f^{-1}.\{K_{3s3}(\alpha z) - \alpha.z.K_{3s5}(\alpha z)$$
$$\widehat{zz} = (\tfrac{2}{\pi})^{\frac{1}{2}}.p_2.z.\{-K_{3c5}\alpha z\}$$
$$\widehat{xx} = (\tfrac{2}{\pi})^{\frac{1}{2}}.p_2.\{-t_5.K_{3c3}(\alpha z) + t_6.z.K_{3c5}(\alpha z)\}$$
$$\widehat{yy} = (\tfrac{2}{\pi})^{\frac{1}{2}}.p_2.\{-t_7^*.K_{3c3}(\alpha z) + t_8^*.z.K_{3c5}(\alpha z)\}$$
$$\widehat{xz} = f.\varepsilon_{xz}$$

D. Isotropic.

The displacements, strains, and stresses are as for the case of Orthorhombic and Cross-anisotropic; β^2 Zero but with the following simplifications;

$t_1 = (1+\nu).(1-2\nu).E^{-1}$	$t_2 = (1+\nu).E^{-1}$	$t_3 = 2.(1-\nu^2).E^{-1}$	$t_4 = (1+\nu).E^{-1}$
$t_5 = 2$	$t_6 = 1$	$t_7^* = 2\nu$	$t_8^* = 0$
$t_9 = 2\nu.(1+\nu).E^{-1}$	$\alpha = 1$	$f = a-b = E.(1+\nu)^{-1}$	

4(b) LINEAR LATERAL SHEAR DISPLACEMENT

The solutions given below involve products of coefficients (h_n, j_n, and t_n) and integrals ($K_{2c(\mu+1)}(\psi)$, $K_{2s(\mu+1)}(\psi)$, ${}_cK_{2c(\mu+1)}$, ${}_sK_{2c(\mu+1)}$, ${}_cK_{2s(\mu+1)}$, and ${}_sK_{2s(\mu+1)}$). These coefficients and integrals are evaluated on pages 26 and 25 respectively.

A. Orthorhombic and Cross-anisotropic; β^2 Positive.

The displacements, strains, and stresses are obtained by making the substitutions into the equations for the case of linear lateral shear stress loading (Orthorhombic and Cross-anisotropic; β^2 Positive). These equations are shown on the opposite page.

(i) substitute $-\delta_2 \cdot (\frac{\pi}{2})^{\frac{1}{2}} \cdot \frac{1}{x_0} \cdot \frac{f}{2c+f} \cdot \frac{(c+d\phi^2).(c+d\rho^2)}{d.(\rho+\phi)}$ ---------- for ---------- p_2

(ii) substitute $K_{2c(\mu+1)}(\psi)$ ---------- for ---------- $K_{3c\mu}(\psi)$

$K_{2s(\mu+1)}(\psi)$ ---------- for ---------- $K_{3s\mu}(\psi)$

B. Orthorhombic and Cross-anisotropic; β^2 Negative.

The displacements, strains, and stresses are obtained by making the following substitutions into the equations for the case of linear lateral shear stress loading (Orthorhombic and Cross-anisotropic; β^2 Negative). These equations are shown on the opposite page.

(i) substitute $-\delta_2 \cdot (\frac{\pi}{2})^{\frac{1}{2}} \cdot \frac{1}{x_0} \cdot \frac{(ad-c^2)}{2ad}$ ---------- for ---------- p_2

(ii) substitute ${}_cK_{2c(\mu+1)}$ ---------- for ---------- ${}_cK_{3c\mu}$

${}_sK_{2c(\mu+1)}$ ---------- for ---------- ${}_sK_{3c\mu}$

${}_cK_{2s(\mu+1)}$ ---------- for ---------- ${}_cK_{3s\mu}$

${}_sK_{2s(\mu+1)}$ ---------- for ---------- ${}_sK_{3s\mu}$

C. Orthorhombic and Cross-anisotropic; β^2 Zero.

The displacements, strains, and stresses are obtained by making the following substitutions into the equations for the case of linear lateral shear stress loading (Orthorhombic and Cross-anisotropic; β^2 Zero). These equations are shown on the opposite page.

(i) substitute $-\delta_2 \cdot (\frac{\pi}{2})^{\frac{1}{2}} \cdot \frac{1}{x_0} \cdot \frac{f}{2c+f} \cdot \frac{(c+d\alpha^2)^2}{2d\alpha}$ ---------- for ---------- p_2

(ii) substitute $K_{2c(\mu+1)}(\alpha z)$ ---------- for ---------- $K_{3c\mu}(\alpha z)$

$K_{2s(\mu+1)}(\alpha z)$ ---------- for ---------- $K_{3s\mu}(\alpha z)$

D. Isotropic.

The displacements, strains, and stresses are obtained by making the following substitutions into the equations for the case of linear lateral shear stress loading (Orthorhombic and Cross-anisotropic; β^2 Zero). These equations are shown on the opposite page.

(i) substitute $-\delta_2 \cdot (\frac{\pi}{2})^{\frac{1}{2}} \cdot \frac{1}{x_0} \cdot \frac{E}{2.(1-\nu^2)}$ ---------- for ---------- p_2

(ii) substitute $K_{2c(\mu+1)}(z)$ ---------- for ---------- $K_{3c\mu}(z)$

$K_{2s(\mu+1)}(z)$ ---------- for ---------- $K_{3s\mu}(z)$

(iii) In addition the simplified values of α, f, and t (shown on the opposite page) are applicable.

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INTEGRALS FOR LOADING BY UNIFORM PRESSURE

(Loading types 1a and 3a)

When $x \neq 0$ and $z \neq 0$ the values of the integrals are given by;

$$Q_{11}(\psi) = (\tfrac{2}{\pi})^{\frac{1}{2}}.[-\tfrac{(1+x)}{4}.\ln\{\psi^2+(1+x)^2\} - \tfrac{(1-x)}{4}.\ln\{\psi^2+(1-x)^2\} - \tfrac{\psi}{2}.\mathrm{artan}\{2\psi.(\psi^2+x^2-1)^{-1}\}]$$

$$K_{1c3}(\psi) = (\tfrac{2}{\pi})^{\frac{1}{2}}.\tfrac{1}{2}.\mathrm{artan}\frac{2\psi}{\psi^2+x^2-1}$$

$$K_{1c5}(\psi) = (\tfrac{2}{\pi})^{\frac{1}{2}}.\{\psi^2-x^2+1\}.\{\psi^2+(1+x^2\}^{-1}.\{\psi^2+(1-x)^2\}^{-1}$$

$$K_{1s1}(\psi) = (\tfrac{2}{\pi})^{\frac{1}{2}}.\{\tfrac{1}{2}.\psi.\ln[\{\psi^2+(1-x)^2\}.\{\psi^2+(1+x)^2\}^{-1}] + \tfrac{1}{2}.\mathrm{artan}\{2x\psi.(\psi^2+1-x^2)^{-1}\} + \tfrac{1}{2}.x.\mathrm{artan}\{2\psi.(\psi^2+x^2-1)^{-1}\}\}$$

$$K_{1s3}(\psi) = (\tfrac{2}{\pi})^{\frac{1}{2}}.\tfrac{1}{2}.\ln[\{\psi^2+(1+x)^2\}.\{\psi^2+(1-x)^2\}^{-1}]$$

$$K_{1s5}(\psi) = (\tfrac{2}{\pi})^{\frac{1}{2}}.2x\psi.\{\psi^2+(1+x)^2\}^{-1}.\{\psi^2+(1-x)^2\}^{-1}$$

$$_cQ_{11} = (\tfrac{2}{\pi})^{\frac{1}{2}}.\{-\tfrac{1+x+\omega z}{8}.\ln\zeta_1 - \tfrac{1-x-\omega z}{8}.\ln\zeta_2 - \tfrac{1+x-\omega z}{8}.\ln\zeta_3 - \tfrac{1-x+\omega z}{8}.\ln\zeta_4 - \tfrac{\alpha z}{4}(\mathrm{artan}\chi_1 + \mathrm{artan}\chi_2)\}$$

$$_sK_{1c1} = (\tfrac{2}{\pi})^{\frac{1}{2}}.[\tfrac{(x+\omega z)}{4}\ \mathrm{artan}\chi_1 + \tfrac{1}{4}\mathrm{artan}\chi_3 - \tfrac{(x-\omega z)}{4}.\mathrm{artan}\chi_2 - \tfrac{1}{4}\mathrm{artan}\chi_4 + \tfrac{\alpha z}{8}\{\ln(\zeta_2.\zeta_3) - \ln(\zeta_1.\zeta_4)\}]$$

$$_cK_{1c3} = (\tfrac{2}{\pi})^{\frac{1}{2}}.\tfrac{1}{2}.(\mathrm{artan}\chi_1 + \mathrm{artan}\chi_2)$$

$$_sK_{1c3} = (\tfrac{2}{\pi})^{\frac{1}{2}}.\tfrac{1}{8}.\{\ln(\zeta_1.\zeta_4) - \ln(\zeta_2.\zeta\ )\}$$

$$_cK_{1s1} = (\tfrac{2}{\pi})^{\frac{1}{2}}[\tfrac{(x+\omega z)}{4}.\mathrm{artan}\chi_1 + \tfrac{1}{4}\mathrm{artan}\chi_3 + \tfrac{(x-\omega z)}{4}.\mathrm{artan}\chi_2 + \tfrac{1}{4}\mathrm{artan}\chi_4 + \tfrac{\alpha z}{8}.\{\ln(\zeta_2.\zeta_4) - \ln(\zeta_1.\zeta_3)\}]$$

$$_sK_{1s1} = (\tfrac{2}{\pi})^{\frac{1}{2}}\{\tfrac{1+x+\omega z}{8}.\ln\zeta_1 + \tfrac{1-x-\omega z}{8}.\ln\zeta_2 - \tfrac{1+x-\omega z}{8}.\ln\zeta_3 - \tfrac{1-x+\omega z}{8}.\ln\zeta_4 + \tfrac{\alpha z}{4}(\mathrm{artan}\chi_1 - \mathrm{artan}\chi_2)\}$$

$$_cK_{1s3} = (\tfrac{2}{\pi})^{\frac{1}{2}}.\tfrac{1}{8}.\{\ln(\zeta_1.\zeta_3) - \ln(\zeta_2.\zeta_4)\}$$

$$_sK_{1s3} = (\tfrac{2}{\pi})^{\frac{1}{2}}.\tfrac{1}{2}(-\mathrm{artan}\chi_1 + \mathrm{artan}\chi_2)$$

where

$\zeta_1 = \alpha^2 z^2 + (1+x+\omega z)^2$ $\qquad$ $\chi_1 = 2\alpha z \div \{\alpha^2 z^2+(x+\omega z)^2 - 1\}$

$\zeta_2 = \alpha^2 z^2 + (1-x-\omega z)^2$ $\qquad$ $\chi_2 = 2\alpha z \div \{\alpha^2 z^2+(x-\omega z)^2 - 1\}$

$\zeta_3 = \alpha^2 z^2 + (1+x-\omega z)^2$ $\qquad$ $\chi_3 = 2\alpha z.(x+\omega z) \div \{\alpha^2 z^2-(x+\omega z)^2 + 1\}$

$\zeta_4 = \alpha^2 z^2 + (1-x+\omega z)^2$ $\qquad$ $\chi_4 = 2\alpha z.(x-\omega z) \div \{\alpha^2 z^2-(x-\omega z)^2 + 1\}$

When $x = 0$ the values of some of the integrals are significantly simplified. These integrals are;

$$K_{1s1}(\psi) = K_{1s3}(\psi) = K_{1s5}(\psi) = {_cK_{1s1}} = {_sK_{1s1}} = {_cK_{1s3}} = {_sK_{1s3}} = 0$$

$$Q_{11}(\psi) = (\tfrac{2}{\pi})^{\frac{1}{2}}.\{-\tfrac{1}{2}\ln(\psi^2+1) - \psi\mathrm{artan}\psi^{-1}\}$$

$$K_{1c3}(\psi) = (\tfrac{2}{\pi})^{\frac{1}{2}}.\mathrm{artan}\ \tfrac{1}{\psi} \qquad K_{1c5}(\psi) = (\tfrac{2}{\pi})^{\frac{1}{2}}.(\psi^2+1)^{-1}$$

When $z = 0$ the relevant integrals are given by;

$$Q_{11}(0) = {_cQ_{11}} = (\tfrac{2}{\pi})^{\frac{1}{2}}.\{-\tfrac{(1+x)}{4}.\ln(1+x)^2 - \tfrac{(1-x)}{4}\ln(1-x)^2\}$$

$$K_{1c3}(0) = {_cK_{1c3}} = (\tfrac{2}{\pi})^{\frac{1}{2}}.\begin{cases}\frac{\pi}{2} & x<1 \\ 0 & x>1\end{cases}$$

$$K_{1s1}(0) = {_cK_{1s1}} = (\tfrac{2}{\pi})^{\frac{1}{2}}.\begin{cases}\frac{\pi}{2}.x & <1 \\ \frac{\pi}{2} & >1\end{cases}$$

$$K_{1s3}(0) = {_cK_{1s3}} = (\tfrac{2}{\pi})^{\frac{1}{2}}.\tfrac{1}{2}.\{\ln(1+x)^2 - \ln(1-x)^2\}$$

$$_sK_{1c1} = {_sK_{1c3}} = {_sK_{1s1}} = {_sK_{1s3}} = 0$$

NOTE: The values of $\mathrm{artan}\chi_1$, $\mathrm{artan}\chi_2$, $\mathrm{artan}\chi_3$ lie in the range 0, π.
The value of $\mathrm{artan}\chi_4$ lies in the range 0, π when $x-\omega z>0$, and in the range $-\pi$, 0 when $x-\omega z<0$.

INTEGRALS FOR LOADING BY UNIFORM DISPLACEMENT

(Loading types 1b and 3b)

When $x \neq 0$ and $z \neq 0$ the values of the integrals are given by;

$$Q_{02}(\psi) = -\ln S_0$$

$$K_{0c4}(\psi) = T_0^{-1}.\cos\tfrac{1}{2}\kappa_0$$

$$K_{0c6}(\psi) = (\psi.\cos\tfrac{3}{2}\kappa_0 + x.\sin\tfrac{3}{2}\kappa_0).T_0^{-3}$$

$$K_{0s2}(\psi) = \xi_0$$

$$K_{0s4}(\psi) = T_0^{-1}.\sin\tfrac{1}{2}\kappa_0$$

$$K_{0s6}(\psi) = (\psi.\sin\tfrac{3}{2}\kappa_0 - x.\cos\tfrac{3}{2}\kappa_0).T_0^{-3}$$

$${}_cQ_{02} = \tfrac{1}{2}.(-\ln S_p - \ln S_m)$$

$${}_sK_{0c2} = \tfrac{1}{2}.(\xi_p - \xi_m)$$

$${}_cK_{0c4} = \tfrac{1}{2}.(T_p^{-1}.\cos\tfrac{1}{2}\kappa_p + T_m^{-1}.\cos\tfrac{1}{2}\kappa_m)$$

$${}_sK_{0c4} = \tfrac{1}{2}.(T_p^{-1}.\sin\tfrac{1}{2}\kappa_p - T_m^{-1}.\sin\tfrac{1}{2}\kappa_m)$$

$${}_cK_{0s2} = \tfrac{1}{2}.(\xi_p + \xi_m)$$

$${}_sK_{0s2} = \tfrac{1}{2}.(\ln S_p - \ln S_m)$$

$${}_cK_{0s4} = \tfrac{1}{2}.(T_p^{-1}.\sin\tfrac{1}{2}\kappa_p + T_m^{-1}.\sin\tfrac{1}{2}\kappa_m)$$

$${}_sK_{0s4} = \tfrac{1}{2}.(T_m^{-1}.\cos\tfrac{1}{2}\kappa_m - T_p^{-1}.\cos\tfrac{1}{2}\kappa_p)$$

where

$$\xi_0 = \mathrm{arsin}\{2x \div [\{\psi^2+(1+x)^2\}^{\frac{1}{2}} + \{\psi^2+(1-x)^2\}^{\frac{1}{2}}]\}$$

$$\xi_p = \mathrm{arsin}\{2(x+\omega z) \div [\{\alpha^2z^2+(1+x+\omega z)^2\}^{\frac{1}{2}} + \{\alpha^2z^2+(1-x-\omega z)^2\}^{\frac{1}{2}}]\}$$

$$\xi_m = \mathrm{arsin}\{2(x-\omega z) \div [\{\alpha^2z^2+(1+x-\omega z)^2\}^{\frac{1}{2}} + \{\alpha^2z^2+(1-x+\omega z)^2\}^{\frac{1}{2}}]\}$$

$$S_0 = \{\psi^2+x^2+T_0^2+2.T_0.(\psi.\cos\tfrac{1}{2}\kappa_0 + x.\sin\tfrac{1}{2}\kappa_0)\}^{\frac{1}{2}}$$

$$S_p = [\alpha^2z^2+(x+\omega z)^2+T_p^2+2.T_p.\{\alpha z.\cos\tfrac{1}{2}\kappa_p + (x+\omega z).\sin\tfrac{1}{2}\kappa_p\}]^{\frac{1}{2}}$$

$$S_m = [\alpha^2z^2+(x-\omega z)^2+T_m^2+2.T_m.\{\alpha z.\cos\tfrac{1}{2}\kappa_m + (x-\omega z).\sin\tfrac{1}{2}\kappa_m\}]^{\frac{1}{2}}$$

$$\kappa_0 = \mathrm{artan}\{2x\psi \div (\psi^2-x^2+1)\}$$

$$\kappa_p = \mathrm{artan}[2\alpha z.(x+\omega z) \div \{\alpha^2z^2-(x+\omega z)^2+1\}]$$

$$\kappa_m = \mathrm{artan}[2\alpha z.(x-\omega z) \div \{\alpha^2z^2-(x-\omega z)^2+1\}]$$

$$T_0 = \{(\psi^2-x^2+1)^2+4x^2\psi^2\}^{\frac{1}{2}}$$

$$T_p = [\{\alpha^2z^2-(x+\omega z)^2+1\}^2+4\alpha^2z^2(x+\omega z)^2]^{\frac{1}{2}}$$

$$T_m = [\{\alpha^2z^2-(x-\omega z)^2+1\}^2+4\alpha^2z^2(x-\omega z)^2]^{\frac{1}{2}}$$

When $x = 0$ the values of some of the integrals are significantly simplified. These integrals are;

$$K_{0s2}(\psi) = K_{0s4}(\psi) = K_{0s6}(\psi) = {}_cK_{0s2} = {}_sK_{0s2} = {}_cK_{0s4} = {}_sK_{0s4} = 0$$

$$Q_{02}(\psi) = -\mathrm{arsinh}\ \psi$$

$$K_{0c4}(\psi) = (\psi^2+1)^{-\frac{1}{2}} \qquad K_{0c6}(\psi) = \psi.(\psi^2+1)^{-\frac{3}{2}}$$

When $z = 0$ the relevant integrals are given by;

$$Q_{02}(0) = {}_cQ_{02} = \begin{cases} 0 \\ -\mathrm{arcosh}\ x \end{cases}$$

$$K_{0s2}(0) = {}_cK_{0s2} = \begin{cases} \mathrm{arsin}\ x \\ \frac{\pi}{2} \end{cases}$$

$$K_{0c4}(0) = {}_cK_{0c4} = \begin{cases} (1-x^2)^{-\frac{1}{2}} & x<1 \\ 0 & x>1 \end{cases}$$

$$K_{0s4}(0) = {}_cK_{0s4} = \begin{cases} 0 & x<1 \\ (x^2-1)^{-\frac{1}{2}} & x>1 \end{cases}$$

$${}_sK_{0c2} = {}_sK_{0c4} = {}_sK_{0s2} = {}_sK_{0s4} = 0$$

NOTE: The values of κ_0 and κ_p lie in the range 0, π.
The value of κ_m lies in the range 0, π when $x-\omega z>0$ and in the range $-\pi$, 0 when $x-\omega z<0$.

INTEGRALS FOR LOADING BY LINEAR PRESSURE

(Loading types 2a and 4a)

When $x \neq 0$ and $\psi \neq 0$ the values of the integrals are given by ;

$$K_{3c1}(\psi) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}}.\left[\frac{\psi^2-x^2+1}{4}.\operatorname{artan}\frac{2\psi}{\psi^2+x^2-1} - \frac{\psi}{2} - \frac{\psi x}{4}.\ln\{\psi^2+(1-x)^2\} + \frac{\psi x}{4}.\ln\{\psi^2+(1+x)^2\}\right]$$

$$K_{3c3}(\psi) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}}.\left[1-\frac{\psi}{2}.\operatorname{artan}\{2\psi.(\psi^2+x^2-1)^{-1}\} + \frac{x}{4}.\ln\{\psi^2+(1-x)^2\} - \frac{x}{4}.\ln\{\psi^2+(1+x)^2\}\right]$$

$$K_{3c5}(\psi) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}}.\left[\tfrac{1}{2}\operatorname{artan}\{2\psi.(\psi^2+x^2-1)^{-1}\} - \psi.(\psi^2+x^2+1).\{\psi^2+(1+x)^2\}^{-1}.\{\psi^2+(1-x)^2\}^{-1}\right]$$

$$K_{3s1}(\psi) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}}.\left\{\frac{x}{2} - \frac{\psi x}{2}.\operatorname{artan}\frac{2\psi}{\psi^2+x^2-1} + \frac{\psi^2-x^2+1}{8}.[\ln\{\psi^2+(1+x)^2\} - \ln\{\psi^2+(1-x)^2\}]\right\}$$

$$K_{3s3}(\psi) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}}.\left[\frac{x}{2}.\operatorname{artan}\{2\psi.(\psi^2+x^2-1)^{-1}\} + \frac{\psi}{4}.\ln\{\psi^2+(1-x)^2\} - \frac{\psi}{4}.\ln\{\psi^2+(1+x)^2\}\right]$$

$$K_{3s5}(\psi) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}}.\left[\tfrac{1}{2}.\ln\{\psi^2+(1+x)^2\} - \tfrac{1}{2}.\ln\{\psi^2+(1-x)^2\} - x.(\psi^2+x^2-1).\{\psi^2+(1+x)^2\}^{-1}.\{\psi^2+(1-x)^2\}^{-1}\right]$$

$$_cK_{3c1} = \left(\frac{2}{\pi}\right)^{\frac{1}{2}}.\left\{\frac{\alpha^2z^2-(x+\omega z)^2+1}{8}.\operatorname{artan}\chi_1 + \frac{\alpha^2z^2-(x-\omega z)^2+1}{8}.\operatorname{artan}\chi_2 - \frac{\alpha z}{2} + \frac{\alpha z(x+\omega z)}{8}.(\ln\zeta_1-\ln\zeta_2)\right.$$
$$\left. + \frac{\alpha z(x-\omega z)}{8}(\ln\zeta_3-\ln\zeta_4)\right\}$$

$$_sK_{3c1} = \left(\frac{2}{\pi}\right)^{\frac{1}{2}}.\left\{\frac{\omega z}{2} - \frac{\alpha z.(x+\omega z)}{4}.\operatorname{artan}\chi_1 + \frac{\alpha z.(x-\omega z)}{4}.\operatorname{artan}\chi_2 - \frac{\alpha^2z^2-(x-\omega z)^2+1}{16}.(\ln\zeta_3-\ln\zeta_4)\right.$$
$$\left. - \frac{\alpha^2z^2-(x-\omega z)^2+1}{16}.(\ln\zeta_3-\ln\zeta_4)\right\}$$

$$_cK_{3c3} = \left(\frac{2}{\pi}\right)^{\frac{1}{2}}.\left\{1-\frac{\alpha z}{4}.(\operatorname{artan}\chi_1+\operatorname{artan}\chi_2) + \frac{x+\omega z}{8}.(\ln\zeta_2-\ln\zeta_1) + \frac{x-\omega z}{8}(\ln\zeta_4-\ln\zeta_3)\right\}$$

$$_sK_{3c3} = \left(\frac{2}{\pi}\right)^{\frac{1}{2}}.\left[\frac{x+\omega z}{4}.\operatorname{artan}\chi_1 - \frac{x-\omega z}{4}.\operatorname{artan}\chi_2 + \frac{\alpha z}{8}\{\ln(\zeta_2.\zeta_3)-\ln(\zeta_1.\zeta_4)\}\right]$$

$$_cK_{3s1} = \left(\frac{2}{\pi}\right)^{\frac{1}{2}}.\left[\frac{x}{2} - \frac{\alpha z(x+\omega z)}{4}.\operatorname{artan}\chi_1 - \frac{\alpha z(x-\omega z)}{4}.\operatorname{artan}\chi_2 + \frac{1}{16}.\{\alpha^2z^2-(x+\omega z)^2+1\}.(\ln\zeta_1-\ln\zeta_2)\right.$$
$$\left. + \frac{1}{16}.\{\alpha^2z^2-(x-\omega z)^2+1\}.(\ln\zeta_3-\ln\zeta_4)\right]$$

$$_sK_{3s1} = \left(\frac{2}{\pi}\right)^{\frac{1}{2}}\left\{-\frac{\alpha^2z^2-(x+\omega z)^2+1}{8}.\operatorname{artan}\chi_1 + \frac{\alpha^2z^2-(x-\omega z)^2+1}{8}.\operatorname{artan}\chi_2\right.$$
$$\left. + \frac{\alpha z(x+\omega z)}{8}.(\ln\zeta_2-\ln\zeta_1) - \frac{\alpha z(x-\omega z)}{8}.(\ln\zeta_4-\ln\zeta_3)\right\}$$

$$_cK_{3s3} = \left(\frac{2}{\pi}\right)^{\frac{1}{2}}.\left[\frac{x+\omega z}{4}.\operatorname{artan}\chi_1 + \frac{x-\omega z}{4}.\operatorname{artan}\chi_2 + \frac{\alpha z}{8}\{\ln(\zeta_2.\zeta_4)-\ln(\zeta_1.\zeta_3)\}\right]$$

$$_sK_{3s3} = \left(\frac{2}{\pi}\right)^{\frac{1}{2}}.\left\{\frac{\alpha z}{4}.(\operatorname{artan}\chi_1-\operatorname{artan}\chi_2) - \frac{x+\omega z}{8}.(\ln\zeta_2-\ln\zeta_1) + \frac{x-\omega z}{8}.(\ln\zeta_4-\ln\zeta_3)\right\}$$

where

$\zeta_1 = \alpha^2z^2+(1+x+\omega z)^2$	$\chi_1 = 2\alpha z \div \{\alpha^2z^2+(x+\omega z)^2-1\}$
$\zeta_2 = \alpha^2z^2+(1-x-\omega z)^2$	$\chi_2 = 2\alpha z \div \{\alpha^2z^2+(x-\omega z)^2-1\}$
$\zeta_3 = \alpha^2z^2+(1+x-\omega z)^2$	$\chi_3 = 2\alpha z.(x+\omega z) \div \{\alpha^2z^2-(x+\omega z)^2+1\}$
$\zeta_4 = \alpha^2z^2+(1-x+\omega z)^2$	$\chi_4 = 2\alpha z.(x-\omega z) \div \{\alpha^2z^2-(x-\omega z)^2+1\}$

When $x = 0$ the values of some of the integrals are significantly simplified. These integrals are;

$$K_{3s1}(\psi) = K_{3s3}(\psi) = K_{3s5}(\psi) = {_cK_{3s1}} = {_sK_{3s1}} = {_cK_{3s3}} = {_sK_{3s3}} = 0$$

$$K_{3c1}(\psi) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}}.\left(\frac{\psi^2+1}{2}.\operatorname{artan}\psi^{-1} - \frac{\psi}{2}\right) \qquad K_{3c3}(\psi) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}}(1-\psi\operatorname{artan}\psi^{-1}) \qquad K_{3c5}(\psi) = \{\operatorname{artan}\psi^{-1} - \psi.(\psi^2+1)^{-1}\}$$

When $z = 0$ the relevant integrals are given by;

$$K_{3c1}(0) = {_cK_{3c1}} = \left(\frac{2}{\pi}\right)^{\frac{1}{2}}\begin{cases}\frac{\pi}{2}.\frac{(1-x^2)}{2} & x<1 \\ 0 & x>1\end{cases} \qquad K_{3c3}(0) = {_cK_{3c3}} = \left(\frac{2}{\pi}\right)^{\frac{1}{2}}\left[1+\frac{x}{4}.\{\ln(1-x)^2 - \ln(1+x)^2\}\right]$$

$$K_{3s1}(0) = {_cK_{3s1}} = \left(\frac{2}{\pi}\right)^{\frac{1}{2}}\left[\frac{x}{2} - \frac{1-x^2}{8}\{\ln(1-x)^2 - \ln(1+x)^2\}\right] \qquad K_{3s3}(0) = {_cK_{3s3}} = \left(\frac{2}{\pi}\right)^{\frac{1}{2}}\begin{cases}\frac{\pi}{2}.x & x<1 \\ 0 & x>1\end{cases}$$

NOTE: The values of $\operatorname{artan}\chi_1$, $\operatorname{artan}\chi_2$, $\operatorname{artan}\chi_3$ lie in the range 0, π.
The value of $\operatorname{artan}\chi_4$ lies in the range 0, π when $x-\omega z>0$, and in the range $-\pi$, 0 when $x-\omega z<0$.

INTEGRALS FOR LOADING BY LINEAR DISPLACEMENT

(Loading Types 2b and 4b)

When $x \neq 0$ and $z \neq 0$ the values of the integrals are given by;

$$K_{2c2}(\psi) = T_0.\cos\tfrac{1}{2}\kappa_0 - \psi$$

$$K_{2s2}(\psi) = x - T_0.\sin\tfrac{1}{2}\kappa_0$$

$$K_{2c4}(\psi) = (T_0 - \psi.\cos\tfrac{1}{2}\kappa_0 - x.\sin\tfrac{1}{2}\kappa_0).T_0^{-1}$$

$$K_{2s4}(\psi) = (x.\cos\tfrac{1}{2}\kappa_0 - \psi.\sin\tfrac{1}{2}\kappa_0).T_0^{-1}$$

$$K_{2c6}(\psi) = T_0^{-3}.\cos\tfrac{3}{2}\kappa_0$$

$$K_{2s6}(\psi) = T_0^{-3}.\sin\tfrac{3}{2}\kappa_0$$

$${}_cK_{2c2} = \tfrac{1}{2}.(+T_p.\cos\tfrac{1}{2}\kappa_p + T_m.\cos\tfrac{1}{2}\kappa_m - 2\alpha z)$$

$${}_sK_{2c2} = \tfrac{1}{2}.(-T_p.\sin\tfrac{1}{2}\kappa_p + T_m.\sin\tfrac{1}{2}\kappa_m + 2\omega z)$$

$${}_cK_{2c4} = \tfrac{1}{2}.\{2 - \alpha z.(T_p^{-1}.\cos\tfrac{1}{2}\kappa_p + T_m^{-1}.\cos\tfrac{1}{2}\kappa_m) - (x+\omega z).T_p^{-1}.\sin\tfrac{1}{2}\kappa_p - (x-\omega z).T_m^{-1}.\sin\tfrac{1}{2}\kappa_m\}$$

$${}_sK_{2c4} = \tfrac{1}{2}.\{-\alpha z.(T_p^{-1}.\sin\tfrac{1}{2}\kappa_p - T_m^{-1}.\sin\tfrac{1}{2}\kappa_m) + (x+\omega z).T_p^{-1}.\cos\tfrac{1}{2}\kappa_p - (x-\omega z).T_m^{-1}.\cos\tfrac{1}{2}\kappa_m\}$$

$${}_cK_{2s2} = \tfrac{1}{2}.(-T_p.\sin\tfrac{1}{2}\kappa_p - T_m.\sin\tfrac{1}{2}\kappa_m + 2x)$$

$${}_sK_{2s2} = \tfrac{1}{2}.(T_m.\cos\tfrac{1}{2}\kappa_m - T_p.\cos\tfrac{1}{2}\kappa_p)$$

$${}_cK_{2s4} = \tfrac{1}{2}.\{-\alpha z.(T_p^{-1}.\sin\tfrac{1}{2}\kappa_p + T_m^{-1}.\sin\tfrac{1}{2}\kappa_m) + (x+\omega z).T_p^{-1}.\cos\tfrac{1}{2}\kappa_p + (x-\omega z).T_m^{-1}.\cos\tfrac{1}{2}\kappa_m\}$$

$${}_sK_{2s4} = \tfrac{1}{2}.(+\alpha z.(T_p^{-1}.\cos\tfrac{1}{2}\kappa_p - T_m^{-1}.\cos\tfrac{1}{2}\kappa_m) + (x+\omega z).T_p^{-1}.\sin\tfrac{1}{2}\kappa_p - (x-\omega z).T_m^{-1}.\sin\tfrac{1}{2}\kappa_m\}$$

where

$$\xi_0 = \text{arsin}\{2x \div [\{\psi^2 + (1+x)^2\}^{\frac{1}{2}} + \{\psi^2 + (1-x)^2\}^{\frac{1}{2}}]\} \qquad \kappa_0 = \text{artan}\{2x\psi \div (\psi^2 - x^2 + 1)\}$$

$$\xi_p = \text{arsin}\{2(x+\omega z) \div [\{\alpha^2 z^2 + (1+x+\omega z)^2\}^{\frac{1}{2}} + \{\alpha^2 z^2 + (1-x-\omega z)^2\}^{\frac{1}{2}}]\} \qquad \kappa_p = \text{artan}[2\alpha z.(x+\omega z) \div \{\alpha^2 z^2 - (x+\omega z)^2 + 1\}]$$

$$\xi_m = \text{arsin}\{2(x-\omega z) \div [\{\alpha^2 z^2 + (1+x-\omega z)^2\}^{\frac{1}{2}} + \{\alpha^2 z^2 + (1-x+\omega z)^2\}^{\frac{1}{2}}]\} \qquad \kappa_m = \text{artan}[2\alpha z.(x-\omega z) \div \{\alpha^2 z^2 - (x-\omega z)^2 + 1\}]$$

$$S_0 = \{\psi^2 + x^2 + T_0^2 + 2.T_0.(\psi.\cos\tfrac{1}{2}\kappa_0 + x.\sin\tfrac{1}{2}\kappa_0)\}^{\frac{1}{2}} \qquad T_0 = \{(\psi^2 - x^2 + 1)^2 + 4x^2\psi^2\}^{\frac{1}{2}}$$

$$S_p = [\alpha^2 z^2 + (x+\omega z)^2 + T_p^2 + 2.T_p.\{\alpha z.\cos\tfrac{1}{2}\kappa_p + (x+\omega z).\sin\tfrac{1}{2}\kappa_p\}]^{\frac{1}{2}} \qquad T_p = [\{\alpha^2 z^2 - (x+\omega z)^2 + 1\}^2 + 4\alpha^2 z^2 (x+\omega z)^2]^{\frac{1}{2}}$$

$$S_m = [\alpha^2 z^2 + (x-\omega z)^2 + T_m^2 + 2.T_m.\{\alpha z.\cos\tfrac{1}{2}\kappa_m + (x-\omega z).\sin\tfrac{1}{2}\kappa_m\}]^{\frac{1}{2}} \qquad T_m = [\{\alpha^2 z^2 - (x-\omega z)^2 + 1\}^2 + 4\alpha^2 z^2 (x-\omega z)^2]^{\frac{1}{2}}$$

When $x = 0$ the values of some of the integrals are significantly simplified. These integrals are;

$$K_{2s2}(\psi) = K_{2s4}(\psi) = K_{2s6}(\psi) = {}_cK_{2s2} = {}_sK_{2s2} = {}_cK_{2s4} = {}_sK_{2s4} = 0$$

$$K_{2c2}(\psi) = (\psi^2+1)^{\frac{1}{2}} - \psi \qquad K_{2c4}(\psi) = 1 - \psi.(\psi^2+1)^{-\frac{1}{2}} \qquad K_{2c6}(\psi) = (\psi^2+1)^{-\frac{3}{2}}$$

When $z = 0$ the values of the relevant integrals are given by;

$$K_{2c2}(0) = {}_cK_{2c2} = \begin{cases} (1-x^2)^{\frac{1}{2}} & x<1 \\ 0 & x>1 \end{cases} \qquad K_{2c4}(0) = {}_cK_{2c4} = \begin{cases} 1 & x<1 \\ -(x^2-1)^{-\frac{1}{2}}.(x+(x^2-1)^{\frac{1}{2}})^{-1} & x>1 \end{cases}$$

$$K_{2s2}(0) = {}_cK_{2s2} = \begin{cases} x & x<1 \\ \{x+(x^2-1)^{\frac{1}{2}}\}^{-1} & x>1 \end{cases} \qquad K_{2s4}(0) = {}_cK_{2s4} = \begin{cases} x.(1-x^2)^{-\frac{1}{2}} & x<1 \\ 0 & x>1 \end{cases}$$

$${}_sK_{2c2} = {}_sK_{2c4} = {}_sK_{2s2} = {}_sK_{2s4} = 0$$

NOTE: The values of κ_0 and κ_p lie in the range $0, \pi$.
The value of κ_m lies in the range $0, \pi$ when $x-\omega z>0$ and in the range $-\pi, 0$ when $x-\omega z<0$.

VALUES OF COEFFICIENTS

Values of co-efficients g_n, h_n, i_n, j_n, s_n, t_n.

$g_1 = (2c+f).\rho.\phi.f^{-1}.(\rho-\phi)^{-1}.(c+d\phi^2)^{-1}$

$g_3 = \rho.(2d\phi^2-f).f^{-1}.(\rho-\phi)^{-1}.(c+d\phi^2)^{-1}$

$g_5 = a.g_3-c.\phi.g_1$

$g_7^* = b.g_3-e.\phi.g_1$

$g_9 = \rho.\phi.(\rho-\phi)^{-1}$

$g_2 = (2c+f).\rho.\phi.f^{-1}.(\rho-\phi)^{-1}.(c+d\rho^2)^{-1}$

$g_4 = \phi.(2d\rho^2-f).f^{-1}.(\rho-\phi)^{-1}.(c+d\rho^2)^{-1}$

$g_6 = a.g_4-c.\rho.g_2$

$g_8^* = b.g_4-e.\rho.g_2$

$h_1 = (2c+f).\phi.f^{-1}.(\rho-\phi)^{-1}.(c+d\phi^2)^{-1}$

$h_3 = (2d\phi^2-f).f^{-1}.(\rho-\phi)^{-1}.(c+d\phi^2)^{-1}$

$h_5 = a.h_3-c.\phi.h_1$

$h_7^* = b.h_3-e.\phi.h_1$

$h_9 = \rho.\phi.(\rho-\phi)^{-1}$

$h_2 = (2c+f).\rho.f^{-1}.(\rho-\phi)^{-1}.(c+d\rho^2)^{-1}$

$h_4 = (2d\rho^2-f).f^{-1}.(\rho-\phi)^{-1}.(c+d\rho^2)^{-1}$

$h_6 = a.h_4 - c.\rho.h_2$

$h_8^* = b.h_4-e.\rho.h_2$

$i_1 = 2\alpha.(ad)^{\frac{1}{2}}.(ad-c^2)^{-1}$

$i_3 = \{(ad)^{\frac{1}{2}}+c\}^{-1}$

$i_5 = a.i_3+c.i_9$

$i_7^* = b.i_3+e.i_9$

$i_9 = a^{\frac{1}{2}}.d^{-\frac{1}{2}}.\{(ad)^{\frac{1}{2}}+c\}^{-1}$

$i_2 = a^{\frac{1}{2}}.d^{-\frac{1}{2}}.f^{-1}.\omega^{-1}$

$i_4 = -\alpha.\omega^{-1}.\{(ad)^{\frac{1}{2}}-c\}^{-1}$

$i_6 = a.i_4+c.i_{10}$

$i_8^* = b.i_4+e.i_{10}$

$i_{10} = a^{\frac{1}{2}}.d^{-\frac{1}{2}}.\alpha.\omega^{-1}.\{(ad)^{\frac{1}{2}}-c\}^{-1}$

$j_1 = \{(ad)^{\frac{1}{2}}+c\}^{-1}$

$j_3 = -2\alpha d.(ad-c^2)^{-1}$

$j_5 = a.j_3-c.j_9$

$j_7^* = b.j_3-e.j_9$

$j_9 = -2\alpha c.(ad-c^2)^{-1}$

$j_2 = \alpha.\omega^{-1}.\{(ad)^{\frac{1}{2}}-c\}^{-1}$

$j_4 = f^{-1}.\omega^{-1}$

$j_6 = a.j_4-c.j_{10}$

$j_8^* = b.j_4-e.j_{10}$

$j_{10} = (f+c).(\omega fd)^{-1}$

$s_1 = (2c+f).2d\alpha^3.f^{-1}.(c+d\alpha^2)^{-2}$

$s_3 = \{(2c+3f-2d\alpha^2).d\alpha^2+cf\}.f^{-1}.(c+d\alpha^2)^{-2}$

$s_5 = a.s_3 + c.s_9$

$s_7^* = b.s_3 + e.s_9$

$s_9 = (2c+f).\alpha^2.(d\alpha^2-c).f^{-1}.(c+d\alpha^2)^{-2}$

$s_2 = (2c+f).\alpha^2.f^{-1}.(c+d\alpha^2)^{-1}$

$s_4 = (2d\alpha^2-f).\alpha.f^{-1}.(c+d\alpha^2)^{-1}$

$s_6 = a.s_4 - c.\alpha.s_2$

$s_8^* = b.s_4 - e.\alpha.s_2$

$t_1 = (2c+f).(d\alpha^2-c).f^{-1}.(c+d\alpha^2)^{-2}$

$t_3 = (2c+f).2d\alpha.f^{-1}.(c+d\alpha^2)^{-2}$

$t_5 = a.t_3 - c.t_9$

$t_7^* = b.t_3 - e.t_9$

$t_9 = (2c+f).2c\alpha.f^{-1}.(c+d\alpha^2)^{-2}$

$t_2 = (2c+f).\alpha.f^{-1}.(c+d\alpha^2)^{-1}$

$t_4 = (2d\alpha^2-f).f^{-1}.(c+d\alpha^2)^{-1}$

$t_6 = a.t_4 - c.\alpha.t_2$

$t_8^* = b.t_4 - e.\alpha.t_2$

* for a cross-anisotropic material $e = c$

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Appendix B

CIRCULAR LOADS APPLIED TO A CROSS ANISOTROPIC HALF SPACE

C. M. Gerrard
W. Jill Harrison

The material in this Appendix was originally published as Technical Paper No.8 of the Division of Applied Geomechanics, C.S.I.R.O., Australia and is reproduced in full herein with kind permission of the authors and the Chief of the Division of Applied Geomechanics.

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For convenience, the page numbers of this Appendix are identical with those of the original publication and any reference to page numbers in the text refers to the pages of this Appendix only.

CONTENTS

Summary

This paper presents the solutions to a group of problems involving simple loads of circular plan area. These loads are applied to a homogeneous, linearly elastic, cross-anisotropic half space in which at any point the axis of symmetry in the elastic properties is vertical.

The ten loads considered form five pairs, each consisting of a stress-defined load and a displacement-defined load:

1(a) Uniform vertical pressure.

1(b) Uniform vertical displacement.

2(a) Linear vertical pressure.

2(b) Linear vertical displacement.

3(a) Linear radial shear stress.

3(b) Linear radial shear displacement.

4(a) Linear torsional shear stress.

4(b) Linear torsional shear displacement.

5(a) Uniform unidirectional shear stress.

5(b) Uniform unidirectional shear displacement.

The solutions contained in this report are of considerable value in soil and rock engineering. Firstly, they allow a full range of practical loading conditions to be considered for a complete range of displacement, strain, and stress components. Secondly, the stress deformation anisotropy treated is that which commonly occurs in soil and rock masses.

CIRCULAR LOADS APPLIED TO A CROSS-ANISOTROPIC HALF SPACE

by C.M. Gerrard* and W.Jill Harrison*

I. INTRODUCTION

The loads considered in this report are applied over a circular plan area to the surface of a cross-anisotropic half space. The circular shape was chosen because of its direct application to wheel loads on pavements and foundation loads under structures such as silos, chimneys, and tanks containing liquids. In addition, for reasons of ease in analysis, it is convenient to consider the circular shapes as approximating to rectangular foundations of approximately equal breadth and width.

The nature of the anisotropy of the elastic half space corresponds to that observed in soil and rock deposits formed under the action of predominantly vertical forces. Such deposits may be natural or placed.

Hence the solutions produced are relevant to a wide range of practical situations where it would be unrealistic to assume isotropy.

The existing solutions for axi-symmetric problems involving a cross-anisotropic half space are summarized in Table 1. The solutions in this report are more comprehensive than those in any of the previous work in the following ways:

1. Range of load types considered;
2. Range of stresses, strains, and displacements solved;
3. Range of cross-anisotropic material response considered.

TABLE 1. SUMMARY OF EXISTING SOLUTIONS

AUTHOR	NATURE OF ELASTIC RESPONSE	RESTRICTIONS ON ELASTIC PARAMETERS	TYPE OF LOADING	DETERMINED STRESSES AND DISPLACEMENTS
Michell (1900) Barden (1963)	Cross-anisotropic	α^2 positive β^2 positive	Vertical point load	w on surface of half space; stresses throughout half space
			Uniform vertical pressure	zz on load axis w on surface
Wolf (1935)	Cross-anisotropic	Restriction on the value of f, and $\nu_h = \nu_{hv} = \nu_{vh} = 0$	Vertical point load	Stresses and displacements in half space
Quinlan (1949)	Cross-anisotropic	α^2 positive β^2 positive	Vertical point load	Stresses and displacements in half space
			Uniform, parabolic, and inverted parabolic distributions of vertical stress. Uniform vertical displacement	w on surface zz on load axis
Koning (1957) Anon. (1960)	Cross-anisotropic	α^2 positive β^2 positive	Vertical point load Uniform vertical displacement	All stresses and displacements in half space
			Uniform vertical pressure	w on surface at centre and edge of load zz down load axis
Lekhnitskii (1963)	Cross-anisotropic	$\alpha + \beta$ and $\alpha - \beta$ must not be purely imaginary	Vertical point load	All stresses throughout half space
Gerrard (1968)	Cross-anisotropic	α^2 positive β^2 positive or zero	Uniform vertical pressure Uniform vertical displacement Linear radial shear stress Torsional loads	All stresses, strains, and displacements in half space

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II. NOTATION

(a) Coordinates, Displacements, Strains, Stresses, Elastic Moduli

r_0	loaded radius
r,θ,z	cylindrical coordinates (radial, tangential, and vertical) expressed in units of the loaded radius
u,v,w	displacements in the respective coordinate directions
$\widehat{rr},\widehat{\theta\theta},\widehat{zz}$	direct stress and shear stress
$\widehat{rz},\widehat{\theta z},\widehat{r\theta}$	components of the stress tensor
$\varepsilon_{rr},\varepsilon_{\theta\theta},\varepsilon_{zz}$	direct strain and shear strain
$\varepsilon_{rz},\varepsilon_{\theta z},\varepsilon_{r\theta}$	components of the strain tensor
a,b,c,d,f	components of the elasticity tensor
E_h	modulus of elasticity in the horizontal direction
E_v	modulus of elasticity in the vertical direction
ν_h	Poisson's ratio - effect of horizontal strain on complementary horizontal strain
ν_{hv}	Poisson's ratio - effect of horizontal strain on vertical strain
ν_{vh}	Poisson's ratio - effect of vertical strain on horizontal strain
E	modulus of elasticity - isotropic material
ν	Poisson's ratio - isotropic material

(b) Derived Elastic Quantities

$\alpha^2 = \{ad-c^2-cf+f(ad)^{\frac{1}{2}}\}\div 2fd$

$\beta^2 = \{ad-c^2-cf-f(ad)^{\frac{1}{2}}\}\div 2fd$

$\omega^2 = -\beta^2$

$\phi = \alpha-\beta$

$\rho = \alpha+\beta$

$\gamma^2 = (a-b)\div f$

Coefficients appearing in the solutions for the displacements, strains, and stresses (defined on pages 13, 21, 27) are $g_1 \ldots g_9$, $h_1 \ldots h_9$, $i_1 \ldots i_{10}$, $j_1 \ldots j_{10}$, $s_1 \ldots s_9$, $t_1 \ldots t_9$.

(c) Loadings

P_1	uniform vertical pressure
P_2	maximum value of linear vertical pressure
p_1	maximum value of linear radial shear stress
p_2	maximum value of linear torsional shear stress
p_3	uniform unidirectional horizontal shear stress
Δ_1	uniform vertical displacement
Δ_2	maximum value of linear vertical displacement
δ_1	maximum value of linear radial shear displacement
δ_2	maximum value of linear torsional shear displacement
δ_3	uniform unidirectional horizontal shear displacement
T_v	total resultant vertical force applied to produce the uniform vertical displacement (Δ_1)
T_t	total resultant torque applied to produce the linear torsional shear displacement (maximum value δ_2)
T_m	total resultant moment applied to produce the linear vertical displacement (maximum value Δ_2)
T_h	total resultant horizontal force applied to produce the uniform unidirectional horizontal shear displacement (δ_3)

(d) Integrals

Integrals appearing in solutions for displacements, strains, and stresses. These are defined in equation 8.

$$I_{\eta\tau\mu}(\psi),\ {}_cI_{\eta\tau\mu},\ {}_sI_{\eta\tau\mu}$$

$$A_{\eta\tau\mu}(\psi),\ {}_cA_{\eta\tau\mu},\ {}_sA_{\eta\tau\mu}$$

$$M_{\eta\tau\mu}(\psi),\ {}_cM_{\eta\tau\mu},\ {}_sM_{\eta\tau\mu}$$

k	transform parameter appearing in equation 8.

III. CROSS-ANISOTROPIC HALF SPACE

This half space can be described as a homogeneous elastic body having infinite extent horizontally and infinite depth below the horizontal plane surface to which loads are applied. The axis of elastic symmetry is assumed to be vertical. The effect of the anisotropy can be gauged by comparison with the solutions for the isotropic case which are also included.

For a cross-anisotropic material with a vertical axis of symmetry the stress-strain relations expressed in cylindrical co-ordinates are;

$$\widehat{rr} = a.\varepsilon_{rr} + b.\varepsilon_{\theta\theta} + c.\varepsilon_{zz} \qquad 1a$$

$$\widehat{\theta\theta} = b.\varepsilon_{rr} + a.\varepsilon_{\theta\theta} + c.\varepsilon_{zz} \qquad 1b$$

$$\widehat{zz} = c.\varepsilon_{rr} + c.\varepsilon_{\theta\theta} + d.\varepsilon_{zz} \qquad 1c$$

$$\widehat{rz} = f.\varepsilon_{rz} \qquad 1d$$

$$\widehat{\theta z} = f.\varepsilon_{\theta z} \qquad 1e$$

$$\widehat{r\theta} = (a-b).\varepsilon_{r\theta} \qquad 1f$$

Five independent elastic constants $a, b, c, d,$ and f are involved.

The direct strains can be expressed in terms of the direct stresses by the following relations;

$$\varepsilon_{rr} = \frac{1}{E_h}.\widehat{rr} - \frac{\nu_h}{E_h}.\widehat{\theta\theta} - \frac{\nu_{vh}}{E_v}.\widehat{zz} \qquad 2a$$

$$\varepsilon_{\theta\theta} = -\frac{\nu_h}{E_h}.\widehat{rr} + \frac{1}{E_h}.\widehat{\theta\theta} - \frac{\nu_{vh}}{E_v}.\widehat{zz} \qquad 2b$$

$$\varepsilon_{zz} = -\frac{\nu_{hv}}{E_h}.\widehat{rr} - \frac{\nu_{hv}}{E_h}.\widehat{\theta\theta} + \frac{1}{E_v}.\widehat{zz} \qquad 2c$$

The interrelationships between the elastic constants in equations 1 and 2 are;

$$a = E_h.(1-\nu_{hv}.\nu_{vh}).(1+\nu_h)^{-1}.(1-\nu_h-2.\nu_{hv}.\nu_{vh})^{-1} \qquad 3a$$

$$b = E_h.(\nu_h+\nu_{hv}.\nu_{vh}).(1+\nu_h)^{-1}.(1-\nu_h-2.\nu_{hv}.\nu_{vh})^{-1} \qquad 3b$$

$$c = E_h.\nu_{vh}.(1-\nu_h-2.\nu_{hv}.\nu_{vh})^{-1} \qquad 3c$$

$$d = E_h.\nu_{vh}.(1-\nu_h).\nu_{hv}^{-1}.(1-\nu_h-2.\nu_{hv}.\nu_{vh})^{-1} \qquad 3d$$

$$E_h.\nu_{vh} = E_v.\nu_{hv} \qquad 3e$$

The strain components are;

$$\varepsilon_{rr} = \frac{\partial u}{\partial r} \qquad 4a$$

$$\varepsilon_{\theta\theta} = \frac{u}{r} + \frac{1}{r}.\frac{\partial v}{\partial \theta} \qquad 4b$$

$$\varepsilon_{zz} = \frac{\partial w}{\partial z} \qquad 4c$$

$$2\varepsilon_{rz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r} \qquad 4d$$

$$2\varepsilon_{\theta z} = \frac{\partial v}{\partial z} + \frac{1}{r}.\frac{\partial w}{\partial \theta} \qquad 4e$$

$$2\varepsilon_{r\theta} = \frac{1}{r}.\frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial r} - \frac{v}{r} \qquad 4f$$

The condition that the strain energy must be positive imposes restrictions on the values of the elastic constants. These are†;

$$a > 0 \qquad 5a$$

$$d > 0 \qquad 5b$$

$$f > 0 \qquad 5c$$

$$a^2 > b^2 \qquad 5d$$

$$(a+b)d > 2c^2 \qquad 5e$$

$$ad > c^2 \qquad 5f$$

In terms of the Poisson's ratios these restrictions impose the limits;

$$1-\nu_h-2.\nu_{hv}.\nu_{vh} > 0 \qquad 6a$$

$$1-\nu_h > 0 \qquad 6b$$

$$1+\nu_h > 0 \qquad 6c$$

† See Hearmon (1961)

IV. LOAD TYPES

The ten loads considered in this report can be grouped into five pairs. The first in each pair is a particular stress-defined load while the second is the analogous displacement-defined load. The load pairs are defined below and are shown in Figure 1.

1. Loading by uniform vertical pressure is typical of pneumatic tyres and flexible foundations while loading by uniform vertical displacement corresponds to relatively rigid foundations. In both cases the contact is smooth.

(a) Uniform Vertical Pressure

$$\begin{array}{lll} \widehat{zz} = P_1 & r < 1 & \\ \widehat{zz} = 0 & r > 1 & \text{when } z = 0 \\ \widehat{rz} = \widehat{\theta z} = 0 & \text{for all } r & \end{array} \qquad \text{7a}$$

(b) Uniform Vertical Displacement

$$\begin{array}{lll} w = \Delta_1 & r < 1 & \\ \widehat{zz} = 0 & r > 1 & \text{when } z = 0 \\ \widehat{rz} = \widehat{\theta z} = 0 & \text{for all } r & \end{array} \qquad \text{7b}$$

2. Loading by linear vertical pressure and linear vertical displacement represent moments about horizontal axes applied to flexible and rigid foundations respectively.

These moment loadings would be normally considered in combination with vertical loads.

(a) Linear Vertical Pressure

$$\begin{array}{lll} \widehat{zz} = \cos\theta . P_2 . r & r < 1 & \\ \widehat{zz} = 0 & r > 1 & \text{when } z = 0 \\ \widehat{rz} = \widehat{\theta z} = 0 & \text{for all } r & \end{array} \qquad \text{7c}$$

(b) Linear Vertical Displacement

$$\begin{array}{lll} w = \cos\theta . \Delta_2 . r & r < 1 & \\ \widehat{zz} = 0 & r > 1 & \text{when } z = 0 \\ \widehat{rz} = \widehat{\theta z} = 0 & \text{for all } r & \end{array} \qquad \text{7d}$$

3. The linear radial shear stress loading is typical of the stresses developed in the surfaces of road pavements due to the grip of pneumatic tyres. Measurements by Bonse and Kuhn (1959) and Marwick and Starks (1941) indicate that the magnitude of the maximum stress is of the order of the inflation pressure of the tyre. A similar pattern of stress is developed under both stationary tyres and those moving at constant linear velocity. The linear radial shear displacement loading when combined with the uniform vertical pressure loading gives the exact solution to the problem of a flexible foundation with a rough base.

(a) Linear Radial Shear Stress

$$\begin{array}{lll} \widehat{rz} = p_1 . r & r < 1 & \\ \widehat{rz} = 0 & r > 1 & \text{when } z = 0 \\ \widehat{zz} = \widehat{\theta z} = 0 & \text{for all } r & \end{array} \qquad \text{7e}$$

(b) Linear Radial Shear Displacement

$$\begin{array}{lll} u = \delta_1 . r & r < 1 & \\ \widehat{rz} = 0 & r > 1 & \text{when } z = 0 \\ \widehat{zz} = \widehat{\theta z} = 0 & \text{for all } r & \end{array} \qquad \text{7f}$$

4. The state of stress defined by the linear torsional shear stress is similar to that developed under pneumatic tyres during cornering and under foundations in certain conditions. On the other hand the linear torsional shear displacement may be applied to the analysis of vane shear tests at sub-failure loads.

(a) Linear Torsional Shear Stress

$$\begin{array}{lll} \widehat{\theta z} = p_2 . r & r < 1 & \\ \widehat{\theta z} = 0 & r > 1 & \text{when } z = 0 \\ \widehat{zz} = \widehat{rz} = 0 & \text{for all } r & \end{array} \qquad \text{7g}$$

(b) Linear Torsional Shear Displacement

$$\begin{array}{lll} v = \delta_2 . r & r < 1 & \\ \widehat{\theta z} = 0 & r > 1 & \text{when } z = 0 \\ \widehat{zz} = \widehat{rz} = 0 & \text{for all } r & \end{array} \qquad \text{7h}$$

5. The uniform unidirectional shear stress is typical of shear loads imposed on pavements by the braking, acceleration, and traction of pneumatic tyres. On the other hand the uniform, unidirectional shear displacement represents lateral loads applied to foundations.

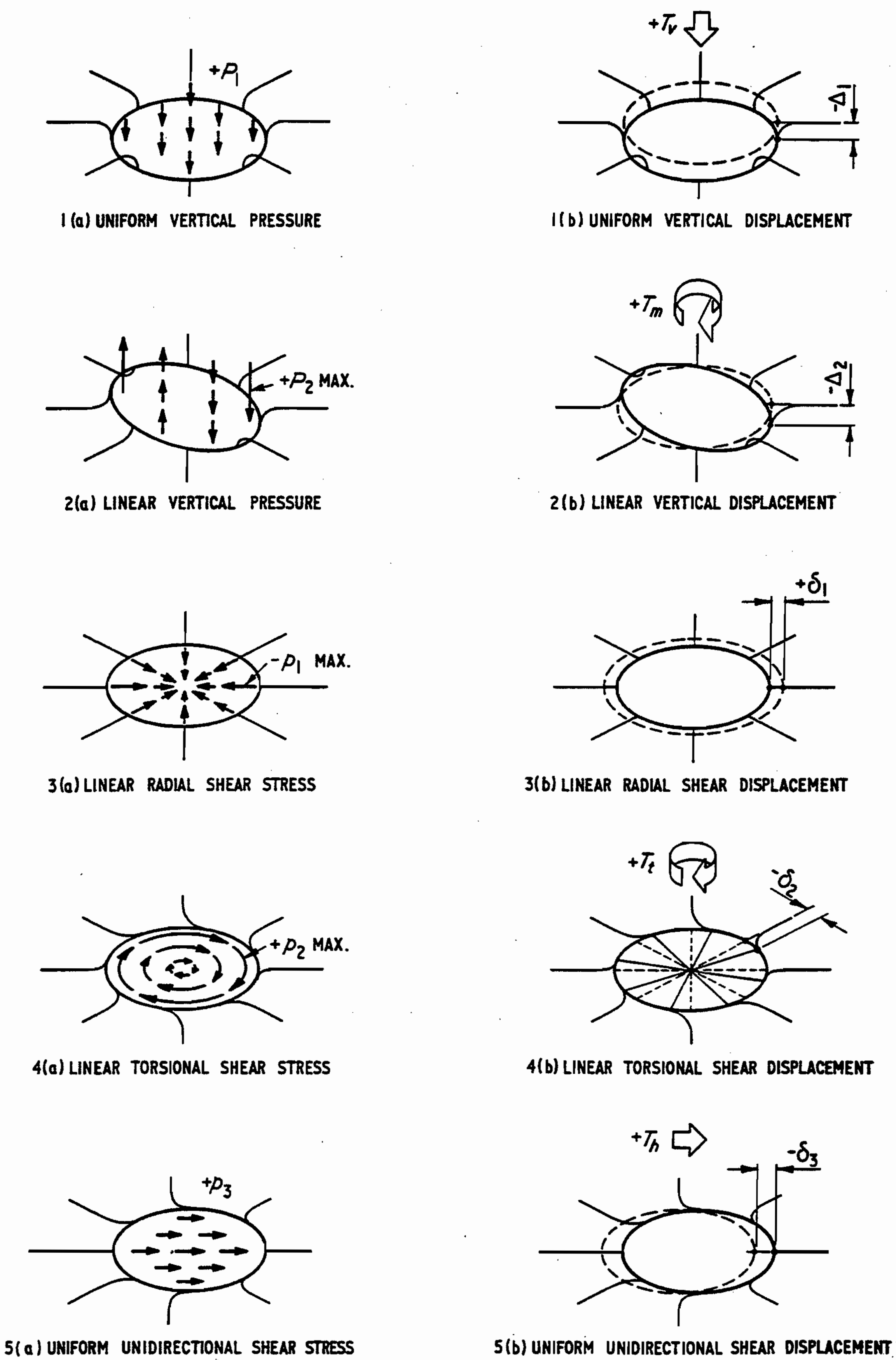

Fig.1.- Load types.

(a) Uniform Unidirectional Shear Stress

$$\tfrac{1}{2}\left(\frac{\widehat{rz}}{\cos\theta} - \frac{\widehat{\theta z}}{\sin\theta}\right) = p_3 \qquad r < 1$$
$$\tfrac{1}{2}\left(\frac{\widehat{rz}}{\cos\theta} - \frac{\widehat{\theta z}}{\sin\theta}\right) = 0 \qquad r > 1 \qquad \text{when } z = 0 \qquad 7i$$
$$\tfrac{1}{2}\left(\frac{\widehat{rz}}{\cos\theta} + \frac{\widehat{\theta z}}{\sin\theta}\right) = 0 \qquad \text{for all } r$$
$$\widehat{zz} = 0 \qquad \text{for all } r$$

(b) Uniform Unidirectional Shear Displacement

$$\tfrac{1}{2}\left(\frac{u}{\cos\theta} - \frac{v}{\sin\theta}\right) = \delta_3 \qquad r < 1$$
$$\tfrac{1}{2}\left(\frac{\widehat{rz}}{\cos\theta} - \frac{\widehat{\theta z}}{\sin\theta}\right) = 0 \qquad r > 1 \qquad \text{when } z = 0 \qquad 7j$$
$$\tfrac{1}{2}\left(\frac{\widehat{rz}}{\cos\theta} + \frac{\widehat{\theta z}}{\sin\theta}\right) = 0 \qquad \text{for all } r$$
$$\widehat{zz} = 0 \qquad \text{for all } r$$

It should be noted that the load pairs 1,3 and 4 produce axi-symmetric distributions of stresses, strains and displacements whereas the load pairs 2 and 5 produce a symmetric distribution about the plane $\theta = 0$.

V. METHOD OF SOLUTION

The method used to obtain the solutions reported herein was developed by Gerrard and Harrison (1970) and is based on the application of integral transform techniques and dual integral equation techniques to elasticity problems (Sneddon, 1951, Tranter, 1966). The solutions for the displacements, strains, and stresses are expressed in terms of integrals of products of trigonometric functions, Bessel functions, and exponentials. The following symbols are used to express these integrals:

$$I_{\eta\tau\mu}(\psi) = \int_0^\infty J_{\frac{1}{2}\eta}(k).J_{\frac{1}{2}\tau}(kr).k^{\frac{1}{2}(\mu-2)}.e^{-\psi k}.dk \qquad 8a$$

$$_cI_{\eta\tau\mu} = \int_0^\infty J_{\frac{1}{2}\eta}(k).J_{\frac{1}{2}\tau}(kr).k^{\frac{1}{2}(\mu-2)}.e^{-\alpha kz}.\cos\omega kz.dk \qquad 8b$$

$$_sI_{\eta\tau\mu} = \int_0^\infty J_{\frac{1}{2}\eta}(k).J_{\frac{1}{2}\tau}(kr).k^{\frac{1}{2}(\mu-2)}.e^{-\alpha kz}.\sin\omega kz.dk \qquad 8c$$

$$A_{\eta\tau\mu}(\psi) = \tfrac{1}{2}\{I_{\eta(\tau+2)\mu}(\psi) + I_{\eta(\tau-2)\mu}(\psi)\} \qquad 8d$$

$$M_{\eta\tau\mu}(\psi) = \tfrac{1}{2}\{I_{\eta(\tau+2)\mu}(\psi) - I_{\eta(\tau-2)\mu}(\psi)\} \qquad 8e$$

$$_cA_{\eta\tau\mu} = \tfrac{1}{2}\{_cI_{\eta(\tau+2)\mu} + {_cI_{\eta(\tau-2)\mu}}\} \qquad 8f$$

$$_cM_{\eta\tau\mu} = \tfrac{1}{2}\{_cI_{\eta(\tau+2)\mu} - {_cI_{\eta(\tau-2)\mu}}\} \qquad 8g$$

$$_sA_{\eta\tau\mu} = \tfrac{1}{2}\{_sI_{\eta(\tau+2)\mu} + {_sI_{\eta(\tau-2)\mu}}\} \qquad 8h$$

$$_sM_{\eta\tau\mu} = \tfrac{1}{2}\{_sI_{\eta(\tau+2)\mu} - {_sI_{\eta(\tau-2)\mu}}\} \qquad 8i$$

The parameter ψ contained in equations 8a, 8d, 8e is in the form of either ρz, ϕz, αz, or γz for anisotropic materials. Hence it is a function of the elastic properties as well as the depth. In contrast, for isotropic materials $\psi = z$.

For stress defined loadings the value of η in the integrals (equations 8) is either 0, 2, or 4. Under these conditions the integrals can only be evaluated directly if $r = 0$ or $z = 0$ and for other values of r and z numerical integration has to be used. However, for displacement defined loadings the corresponding values of η are 1, 3, and 5. In this case the integrals shown in equations 8 can be evaluated directly for all values of r and z.

Those integrals evaluated directly were obtained by using the results of the Bateman Manuscript Project (1954) and Watson (1966).

Compressive direct strains and stresses are considered to be positive. Positive shear stresses are defined from the fact that both the stress and strain tensors obey the right hand rule. Displacements in the negative coordinate directions are considered to be positive. Hence, a load defined by a positive stress acts in the positive coordinate direction, whereas a load defined by a positive displacement acts in the negative coordinate direction. For example, vertical loads are compressive if defined by a positive stress or a negative displacement.

VI. PRESENTATION OF RESULTS

The solutions for the displacements, strains, and stresses for each pair of loading conditions are presented in the order previously indicated.

The presentation is arranged so that the complete solution for each loading condition can be seen from a single opening of this report, i.e. no more than two consecutive, facing pages are used for any loading condition.

The solutions given contain all displacement, strain, and stress components. This allows the calculation of quantities such as total deformations, principal stresses and strains, and principal directions. In general the solutions are stated as products of coefficients ($g_1 \ldots g_9$, $h_1 \ldots h_9$, $i_1 \ldots i_{10}$, $j_1 \ldots j_{10}$, $s_1 \ldots s_9$, $t_1 \ldots t_9$) and integrals (I, A, and M types). The coefficients are functions only of the elastic properties while the integrals are in general functions of depth, radial offset, and elastic properties.

The form of the solutions depends to a significant extent on the nature of the anisotropy as reflected by the values of α^2 and β^2, both of which are functions only of the elastic properties.

The strain energy conditions given by 5a ... 5f are sufficient but not necessary conditions that

$$\alpha^2 = \{(ad)^{1/2} - c\}.\{(ad)^{1/2} + c + f\}.(2fd)^{-1}$$

be positive. However, when β^2 is written in the form

$$\beta^2 = [ad - c^2 - f\{(ad)^{1/2} + c\}]\cdot(2fd)^{-1}$$

it can be seen that the sign of β^2 is not restricted by the strain energy conditions. Hence, for each loading condition, four separate cases are considered as follows;

A. Cross-anisotropic; α^2 positive, β^2 positive

B. Cross-anisotropic; α^2 positive, β^2 negative

C. Cross-anisotropic; α^2 positive, β^2 zero

D. Isotropic (this is a special case of C in which $\alpha = 1$)

A typical layout for a stress defined loading (i.e. load 1(a), 2(a), 3(a), 4(a), or 5(a)) consists of the complete solutions for the displacements, strains, and stresses for each of the above four cases. In addition reference is given to the values of the coefficients (g_n, h_n, i_n, j_n, s_n, t_n). Finally the values of the integrals are given in the following way:

reference to tabulated values when $r \neq 0$ and $z \neq 0$

statement in closed form for the special cases of $r = 0$ and $z = 0$.

The layout for a displacement-defined loading differs from that of the analogous stress-defined load in several respects. Firstly, the solutions are given by simply stating the substitutions that need to be made into the solutions for the stress-defined load in order to obtain the solutions for the displacement-defined load. In general these substitutions are referred to a maximum defined displacement OR a defined total load. This considerable economy in presentation is made possible because of the similarity in the forms of the solutions for a given stress-defined loading and its displacement-defined analogue.

Secondly, the values of the integrals can be stated in closed form for all values of r and z and hence no tables are required.

The general case of $r \neq 0$ and $z \neq 0$ and the special cases of $r = 0$ or $z = 0$ are considered.

VII. SOLUTIONS FOR DISPLACEMENTS, STRAINS AND STRESSES

The solutions for the displacements, strains and stresses for each pair of loading conditions are presented on the following pages.

1(a) UNIFORM VERTICAL PRESSURE

A. Cross-anisotropic; β^2 Positive

$$w = P_1.r_0.\{-g_1.I_{200}(\phi z) + g_2.I_{200}(\rho z)\}$$
$$u = P_1.r_0.\{-g_3.I_{220}(\phi z) + g_4.I_{220}(\rho z)\}$$
$$\varepsilon_{zz} = P_1.\{g_1.\phi.I_{202}(\phi z) - g_2.\rho.I_{202}(\rho z)\}$$
$$\varepsilon_{rr} = P_1.[-g_3.\{I_{202}(\phi z) - r^{-1}.I_{220}(\phi z)\} + g_4.\{I_{202}(\rho z) - r^{-1}.I_{220}(\rho z)\}]$$
$$\varepsilon_{\theta\theta} = P_1.\{-g_3.r^{-1}.I_{220}(\phi z) + g_4.r^{-1}.I_{220}(\rho z)\}$$
$$\varepsilon_{rz} = P_1.g_9.f^{-1}.\{I_{222}(\phi z) - I_{222}(\rho z)\}$$
$$\widehat{rr} = P_1.[-g_5.I_{202}(\phi z) + g_6.I_{202}(\rho z) + (a-b).r^{-1}.\{g_3.I_{220}(\phi z) - g_4.I_{220}(\rho z)\}]$$
$$\widehat{\theta\theta} = P_1.[-g_7.I_{202}(\phi z) + g_8.I_{202}(\rho z) - (a-b).r^{-1}.\{g_3.I_{220}(\phi z) - g_4.I_{220}(\rho z)\}]$$
$$\widehat{zz} = P_1.g_9.\{\phi^{-1}.I_{202}(\phi z) - \rho^{-1}.I_{202}(\rho z)\} \qquad \widehat{rz} = f.\varepsilon_{rz}$$

B. Cross-anisotropic; β^2 Negative

$$w = P_1.r_0.\{-i_1.{}_cI_{200} - i_2.{}_sI_{200}\}$$
$$u = P_1.r_0.\{i_3.{}_cI_{220} + i_4.{}_sI_{220}\}$$
$$\varepsilon_{zz} = P_1.\{i_9.{}_cI_{202} + i_{10}.{}_sI_{202}\}$$
$$\varepsilon_{rr} = P_1.[i_3.\{{}_cI_{202} - r^{-1}.{}_cI_{220}\} + i_4.\{{}_sI_{202} - r^{-1}.{}_sI_{220}\}]$$
$$\varepsilon_{\theta\theta} = P_1.\{i_3.r^{-1}.{}_cI_{220} + i_4.r^{-1}.{}_sI_{220}\}$$
$$\varepsilon_{rz} = P_1.i_2.{}_sI_{222}$$
$$\widehat{rr} = P_1.[i_5.{}_cI_{202} + i_6.{}_sI_{202} - (a-b).\{i_3.r^{-1}.{}_cI_{220} + i_4.r^{-1}.{}_sI_{220}\}]$$
$$\widehat{\theta\theta} = P_1.[i_7.{}_cI_{202} + i_8.{}_sI_{202} + (a-b).\{i_3.r^{-1}.{}_cI_{220} + i_4.r^{-1}.{}_sI_{220}\}]$$
$$\widehat{zz} = P_1.\{{}_cI_{202} + \alpha.\omega^{-1}.{}_sI_{202}\} \qquad \widehat{rz} = f.\varepsilon_{rz}$$

C. Cross-anisotropic; β^2 Zero

$$w = P_1.r_0.\{-s_1.I_{200}(\alpha z) - s_2.z.I_{202}(\alpha z)\}$$
$$u = P_1.r_0.\{s_3.I_{220}(\alpha z) - s_4.z.I_{222}(\alpha z)\}$$
$$\varepsilon_{zz} = P_1.\{s_9.I_{202}(\alpha z) + s_2.\alpha.z.I_{204}(\alpha z)\}$$
$$\varepsilon_{rr} = P_1.[s_3.\{I_{202}(\alpha z) - r^{-1}.I_{220}(\alpha z)\} - s_4.z.\{I_{204}(\alpha z) - r^{-1}.I_{222}(\alpha z)\}]$$
$$\varepsilon_{\theta\theta} = P_1.\{s_3.r^{-1}.I_{220}(\alpha z) - s_4.z.r^{-1}.I_{222}(\alpha z)\}$$
$$\varepsilon_{rz} = P_1.\alpha^2.f^{-1}.z.I_{224}(\alpha z)$$
$$\widehat{rr} = P_1.[s_5.I_{202}(\alpha z) - s_6.z.I_{204}(\alpha z) - (a-b).r^{-1}.\{s_3.I_{220}(\alpha z) - s_4.z.I_{222}(\alpha z)\}]$$
$$\widehat{\theta\theta} = P_1.[s_7.I_{202}(\alpha z) - s_8.z.I_{204}(\alpha z) + (a-b).r^{-1}.\{s_3.I_{220}(\alpha z) - s_4.z.I_{222}(\alpha z)\}]$$
$$\widehat{zz} = P_1.\{I_{202}(\alpha z) + \alpha.z.I_{204}(\alpha z)\} \qquad \widehat{rz} = f.\varepsilon_{rz}$$

D. Isotropic

The stresses, strains, and displacements are as for the case of Cross-anisotropic; β^2 Zero but with the following simplifications;

$s_1 = 2.(1-\nu^2).E^{-1}$ $\quad s_2 = (1+\nu).E^{-1}$ $\quad s_3 = (1+\nu).(1-2\nu).E^{-1}$ $\quad s_4 = (1+\nu).E^{-1}$

$s_5 = s_6 = 1$ $\quad s_7 = 2\nu$ $\quad s_8 = 0$ $\quad s_9 = (1+\nu).(1-2\nu).E^{-1}$

$\alpha = 1$ $\quad f = a-b = E.(1+\nu)^{-1}$

I(a) UNIFORM VERTICAL PRESSURE

Values of coefficients g_n, i_n, s_n

$g_1 = (2c+f).\rho.\phi.f^{-1}.(\rho-\phi)^{-1}.(c+d\phi^2)^{-1}$

$g_2 = (2c+f).\rho.\phi.f^{-1}.(\rho-\phi)^{-1}.(c+d\rho^2)^{-1}$

$g_3 = \rho.(2d\phi^2-f).f^{-1}.(\rho-\phi)^{-1}.(c+d\phi^2)^{-1}$

$g_4 = \phi.(2d\rho^2-f).f^{-1}.(\rho-\phi)^{-1}.(c+d\rho^2)^{-1}$

$g_5 = a.g_3 - c.\phi.g_1$

$g_6 = a.g_4 - c.\rho.g_2$

$g_7 = b.g_3 - c.\phi.g_1$

$g_8 = b.g_4 - c.\rho.g_2$

$g_9 = \rho.\phi.(\rho-\phi)^{-1}$

$i_1 = 2\alpha.(ad)^{\frac{1}{2}}.(ad-c^2)^{-1}$

$i_2 = a^{\frac{1}{2}}.d^{-\frac{1}{2}}.f^{-1}.\omega^{-1}$

$i_3 = \{(ad)^{\frac{1}{2}} + c\}^{-1}$

$i_4 = -\alpha.\omega^{-1}\{(ad)^{\frac{1}{2}} - c\}^{-1}$

$i_5 = a.i_3 + c.i_9$

$i_6 = a.i_4 + c.i_{10}$

$i_7 = b.i_3 + c.i_9$

$i_8 = b.i_4 + c.i_{10}$

$i_9 = a^{\frac{1}{2}}.d^{-\frac{1}{2}}.\{(ad)^{\frac{1}{2}} + c\}^{-1}$

$i_{10} = a^{\frac{1}{2}}.d^{-\frac{1}{2}}.\alpha.\omega^{-1}.\{(ad)^{\frac{1}{2}}-c\}^{-1}$

$s_1 = (2c+f).2d\alpha^3.f^{-1}.(c+d\alpha^2)^{-2}$

$s_2 = (2c+f).\alpha^2.f^{-1}.(c+d\alpha^2)^{-1}$

$s_3 = \{(2c+3f-2d\alpha^2).d\alpha^2 + cf\}.f^{-1}.(c+d\alpha^2)^{-2}$

$s_4 = (2d\alpha^2-f).\alpha.f^{-1}.(c+d\alpha^2)^{-1}$

$s_5 = a.s_3 + c.s_9$

$s_6 = a.s_4 - c.\alpha.s_2$

$s_7 = b.s_3 + c.s_9$

$s_8 = b.s_4 - c.\alpha.s_2$

$s_9 = (2c+f).\alpha^2.(d\alpha^2-c).f^{-1}.(c+d\alpha^2)^{-2}$

Values of integrals $I_{2\tau\mu}(\psi)$, ${}_cI_{2\tau\mu}$, ${}_sI_{2\tau\mu}$

When $r \neq 0$ and $z \neq 0$ the values of the integrals are given in tables on the page numbers indicated in brackets;

$I_{200}(\psi)$	[34]	$I_{220}(\psi)$	[37]	${}_cI_{200}$	[35]	${}_cI_{220}$	[38]
$I_{202}(\psi)$	[34]	$I_{222}(\psi)$	[37]	${}_sI_{200}$	[35]	${}_sI_{220}$	[38]
$I_{204}(\psi)$	[34]	$I_{224}(\psi)$	[37]	${}_cI_{202}$	[36]	${}_cI_{222}$	[39]
				${}_sI_{202}$	[36]	${}_sI_{222}$	[39]

When $r = 0$ the integrals are given by;

$I_{200}(\psi) = (\psi^2+1)^{\frac{1}{2}}-\psi$ $\qquad I_{202}(\psi) = 1 - \psi.(\psi^2+1)^{-\frac{1}{2}}$ $\qquad I_{204}(\psi) = (\psi^2+1)^{-\frac{3}{2}}$

$I_{220}(\psi) = I_{222}(\psi) = I_{224}(\psi) = 0$ $\qquad r^{-1}.I_{220}(\psi) = \frac{1}{2}.I_{202}(\psi)$ $\qquad r^{-1}.I_{222}(\psi) = \frac{1}{2}.I_{204}(\psi)$

${}_cI_{200} = R^{\frac{1}{2}}.\cos\frac{\upsilon}{2} - \alpha.z$ $\qquad {}_cI_{202} = (R^{\frac{1}{2}}-\alpha.z.\cos\frac{\upsilon}{2} - \omega.z.\sin\frac{\upsilon}{2}).R^{-\frac{1}{2}}$

${}_sI_{200} = -R^{\frac{1}{2}}.\sin\frac{\upsilon}{2} + \omega.z$ $\qquad {}_sI_{202} = (-\alpha.z.\sin\frac{\upsilon}{2} + \omega.z.\cos\frac{\upsilon}{2}).R^{-\frac{1}{2}}$

${}_cI_{220} = {}_cI_{222} = {}_sI_{220} = {}_sI_{222} = 0$ $\qquad r^{-1}.{}_cI_{220} = \frac{1}{2}.{}_cI_{202}$ $\qquad r^{-1}.{}_sI_{220} = \frac{1}{2}.{}_sI_{202}$

where $R = [\{(\alpha^2-\omega^2).z^2+1\}^2 + 4.\alpha^2.\omega^2.z^4]^{\frac{1}{2}}$ and $\upsilon = \arctan[2.\alpha.\omega.z^2.\{(\alpha^2-\omega^2).z^2 + 1\}^{-1}]$

When $z = 0$ the relevant integrals are given by;

$$I_{200}(0) = {}_cI_{200} = \begin{cases} {}_2F_1(\frac{1}{2}, -\frac{1}{2}; 1; r^2) \\ 2.\pi^{-1} \\ (2r)^{-1}.{}_2F_1(\frac{1}{2}, \frac{1}{2}; 2; r^{-2}) \end{cases} \qquad I_{202}(0) = {}_cI_{202} = \begin{cases} 1 \\ 2^{-1} \\ 0 \end{cases} \qquad I_{220}(0) = {}_cI_{220} = \begin{cases} 2^{-1}.r & r < 1 \\ 2^{-1} & r = 1 \\ (2r)^{-1} & r > 1 \end{cases}$$

${}_sI_{200} = {}_sI_{202} = {}_sI_{220} = 0$

I(b) UNIFORM VERTICAL DISPLACEMENT

A. Cross-anisotropic; β^2 Positive

The displacements, strains, and stresses are obtained by making the following substitutions into the equations for the case of Uniform Vertical Pressure loading (Cross-anisotropic; β^2 Positive). These equations are shown on page 12.

(i) substitute $T_v.\left(\frac{2}{\pi}\right)^{\frac{1}{2}}.\frac{1}{4r_0^2}$ OR $-\Delta_1.\left(\frac{2}{\pi}\right)^{\frac{1}{2}}.\frac{1}{r_0}.\frac{f}{2c+f}.\frac{(c+d\rho^2).(c+d\phi^2)}{d.\rho.\phi.(\rho+\phi)}$ -------------------- for P_1

This alternative substitution follows from the load-displacement relationship for this case of β^2 Positive. This relationship is;

$$T_v = -\Delta_1.4r_0.\frac{f}{2c+f}.\frac{(c+d\rho^2).(c+d\phi^2)}{d.\rho.\phi.(\rho+\phi)}$$

(ii) substitute $I_{1\tau(\mu+1)}(\psi)$ -------------------- for $I_{2\tau\mu}(\psi)$

B. Cross-anisotropic; β^2 Negative

The displacements, strains, and stresses are obtained by making the following substitutions into the equations for the case of Uniform Vertical Pressure loading (Cross-anisotropic; β^2 Negative). These equations are shown on page 12.

(i) substitute $T_v.\left(\frac{2}{\pi}\right)^{\frac{1}{2}}.\frac{1}{4r_0^2}$ OR $-\Delta_1.\left(\frac{2}{\pi}\right)^{\frac{1}{2}}.\frac{1}{r_0}.\frac{ad-c^2}{2\alpha.(ad)^{\frac{1}{2}}}$ -------------------- for P_1

This alternative substitution follows from the load-displacement relationship for this case of β^2 Negative. This relationship is;

$$T_v = -\Delta_1.4r_0.\frac{ad-c^2}{2\alpha.(ad)^{\frac{1}{2}}}$$

(ii) substitute ${}_cI_{1\tau(\mu+1)}$ -------------------- for ${}_cI_{2\tau\mu}$

${}_sI_{1\tau(\mu+1)}$ -------------------- for ${}_sI_{2\tau\mu}$

C. Cross-anisotropic; β^2 Zero

The displacements, strains, and stresses are obtained by making the following substitutions into the equations for the case of Uniform Vertical Pressure loading (Cross-anisotropic; β^2 Zero). These equations are shown on page 12.

(i) substitute $T_v.\left(\frac{2}{\pi}\right)^{\frac{1}{2}}.\frac{1}{4r_0^2}$ OR $-\Delta_1.\left(\frac{2}{\pi}\right)^{\frac{1}{2}}.\frac{1}{r_0}.\frac{f}{2c+f}.\frac{(c+d\alpha^2)^2}{2d\alpha^3}$ -------------------- for P_1

This alternative substitution follows from the load-displacement relationship for this case of β^2 Zero. This relationship is;

$$T_v = -\Delta_1.4r_0.\frac{f}{2c+f}.\frac{(c+d\alpha^2)^2}{2d\alpha^3}$$

(ii) substitute $I_{1\tau(\mu+1)}(\alpha z)$ -------------------- for $I_{2\tau\mu}(\alpha z)$

D. Isotropic

The displacements, strains, and stresses are obtained by making the following substitutions into the equations for the case of Uniform Vertical Pressure loading (Cross-anisotropic; β^2 Zero). These equations are shown on page 12.

(i) substitute $T_v.\left(\frac{2}{\pi}\right)^{\frac{1}{2}}.\frac{1}{4r_0^2}$ OR $-\Delta_1.\left(\frac{2}{\pi}\right)^{\frac{1}{2}}.\frac{1}{r_0}.\frac{E}{2.(1-\nu^2)}$ -------------------- for P_1

This alternative substitution follows from the load-displacement relationship for this Isotropic case. This relationship is;

$$T_v = -\Delta_1.4r_0.\frac{E}{2.(1-\nu^2)}$$

(ii) substitute $I_{1\tau(\mu+1)}(z)$ -------------------- for $I_{2\tau\mu}(z)$

(iii) In addition the following simplifications apply with regard to α, f, and coefficients s_n;

$s_1 = 2.(1-\nu^2).E^{-1}$ $s_2 = (1+\nu).E^{-1}$ $s_3 = (1+\nu).(1-2\nu).E^{-1}$ $s_4 = (1+\nu).E^{-1}$

$s_5 = s_6 = 1$ $s_7 = 2\nu$ $s_8 = 0$ $s_9 = (1+\nu).(1-2\nu).E^{-1}$

$\alpha = 1$ $f = a-b = E.(1+\nu)^{-1}$

1(b) UNIFORM VERTICAL DISPLACEMENT

Values of integrals $I_{1\tau\mu}(\psi)$, ${}_cI_{1\tau\mu}$, ${}_sI_{1\tau\mu}$

When $r \neq 0$ and $z \neq 0$ the values of the integrals are given by;

$$I_{101}(\psi) = (\tfrac{2}{\pi})^{\frac{1}{2}}.\sigma_0$$

$$I_{103}(\psi) = (\tfrac{2}{\pi})^{\frac{1}{2}}.N_0^{-1}.\sin\tfrac{1}{2}\lambda_0$$

$$I_{105}(\psi) = (\tfrac{2}{\pi})^{\frac{1}{2}}.N_0^{-3}.(\psi.\sin\tfrac{3}{2}\lambda_0-\cos\tfrac{3}{2}\lambda_0)$$

$$I_{121}(\psi) = (\tfrac{2}{\pi})^{\frac{1}{2}}.r^{-1}.(1-N_0.\sin\tfrac{1}{2}\lambda_0)$$

$$I_{123}(\psi) = (\tfrac{2}{\pi})^{\frac{1}{2}}.r^{-1}.N_0^{-1}.(\cos\tfrac{1}{2}\lambda_0-\psi.\sin\tfrac{1}{2}\lambda_0)$$

$$I_{125}(\psi) = (\tfrac{2}{\pi})^{\frac{1}{2}}.r.N_0^{-3}.\sin\tfrac{3}{2}\lambda_0$$

$${}_cI_{101} = \tfrac{1}{2}.(\tfrac{2}{\pi})^{\frac{1}{2}}.(\sigma_p+\sigma_m)$$

$${}_sI_{101} = \tfrac{1}{2}.(\tfrac{2}{\pi})^{\frac{1}{2}}.(\ln L_p-\ln L_m)$$

$${}_cI_{103} = \tfrac{1}{2}.(\tfrac{2}{\pi})^{\frac{1}{2}}.(N_m^{-1}.\sin\tfrac{1}{2}\lambda_m + N_p^{-1}.\sin\tfrac{1}{2}\lambda_p)$$

$${}_sI_{103} = \tfrac{1}{2}.(\tfrac{2}{\pi})^{\frac{1}{2}}.(N_m^{-1}.\cos\tfrac{1}{2}\lambda_m - N_p^{-1}.\cos\tfrac{1}{2}\lambda_p)$$

$${}_cI_{121} = \tfrac{1}{2}.(\tfrac{2}{\pi})^{\frac{1}{2}}.r^{-1}.(2-N_m.\sin\tfrac{1}{2}\lambda_m - N_p.\sin\tfrac{1}{2}\lambda_p)$$

$${}_sI_{121} = \tfrac{1}{2}.(\tfrac{2}{\pi})^{\frac{1}{2}}.r^{-1}.(N_m.\cos\tfrac{1}{2}\lambda_m - N_p.\cos\tfrac{1}{2}\lambda_p)$$

$${}_cI_{123} = \tfrac{1}{2}.(\tfrac{2}{\pi})^{\frac{1}{2}}.r^{-1}.[N_m^{-1}\{(1-\omega z).\cos\tfrac{1}{2}\lambda_m - \alpha z.\sin\tfrac{1}{2}\lambda_m\} + N_p^{-1}\{(1+\omega z).\cos\tfrac{1}{2}\lambda_p - \alpha z.\sin\tfrac{1}{2}\lambda_p\}]$$

$${}_sI_{123} = \tfrac{1}{2}.(\tfrac{2}{\pi})^{\frac{1}{2}}.r^{-1}[-N_m^{-1}.\{\alpha z.\cos\tfrac{1}{2}\lambda_m + (1-\omega z).\sin\tfrac{1}{2}\lambda_m\} + N_p^{-1}.\{\alpha z.\cos\tfrac{1}{2}\lambda_p + (1+\omega z).\sin\tfrac{1}{2}\lambda_p\}]$$

where

$$\sigma_0 = \operatorname{arsin}\{2.[\{\psi^2 + (1+r)^2\}^{\frac{1}{2}} + \{\psi^2 + (1-r)^2\}^{\frac{1}{2}}]^{-1}\}$$

$$\sigma_p = \operatorname{arsin}\{2.(1+\omega z).[\{\alpha^2z^2 + (1+\omega z+r)^2\}^{\frac{1}{2}} + \{\alpha^2z^2 + (1+\omega z-r)^2\}^{\frac{1}{2}}]^{-1}\}$$

$$\sigma_m = \operatorname{arsin}\{2.(1-\omega z).[\{\alpha^2z^2 + (1-\omega z+r)^2\}^{\frac{1}{2}} + \{\alpha^2z^2 + (1-\omega z-r)^2\}^{\frac{1}{2}}]^{-1}\}$$

$$N_0 = \{(\psi^2+r^2-1)^2 + 4\psi^2\}^{\frac{1}{4}}$$

$$N_p = [\{\alpha^2z^2 - (1+\omega z)^2 + r^2\}^2 + 4\alpha^2z^2(1+\omega z)^2]^{\frac{1}{4}}$$

$$N_m = [\{\alpha^2z^2 - (1-\omega z)^2 + r^2\}^2 + 4\alpha^2z^2(1-\omega z)^2]^{\frac{1}{4}}$$

$$\lambda_0 = \operatorname{artan}\{2\psi(\psi^2+r^2 - 1)^{-1}\}$$

$$\lambda_p = \operatorname{artan}[2\alpha z(1+\omega z).\{\alpha^2z^2 - (1+\omega z)^2 + r^2\}^{-1}]$$

$$\lambda_m = \operatorname{artan}[2\alpha z(1-\omega z).\{\alpha^2z^2 - (1-\omega z)^2 + r^2\}^{-1}]$$

$$L_p = [\{N_p.\cos\tfrac{1}{2}\lambda_p + \alpha z\}^2 + \{N_p.\sin\tfrac{1}{2}\lambda_p + (1+\omega z)\}^2]^{\frac{1}{2}}$$

$$L_m = [\{N_m.\cos\tfrac{1}{2}\lambda_m + \alpha z\}^2 + \{N_m.\sin\tfrac{1}{2}\lambda_m + (1-\omega z)\}^2]^{\frac{1}{2}}$$

When $r = 0$ the values of the integrals are given by;

$$I_{101}(\psi) = (\tfrac{2}{\pi})^{\frac{1}{2}}.\operatorname{arcot}\psi$$

$$I_{103}(\psi) = (\tfrac{2}{\pi})^{\frac{1}{2}}.(\psi^2+1)^{-1}$$

$$I_{105}(\psi) = (\tfrac{2}{\pi})^{\frac{1}{2}}.2\psi.(\psi^2+1)^{-2}$$

$${}_cI_{101} = (\tfrac{2}{\pi})^{\frac{1}{2}}.\tfrac{1}{2}.\operatorname{artan}\{2\alpha z.(\alpha^2z^2+\omega^2z^2-1)^{-1}\}$$

$${}_sI_{101} = (\tfrac{2}{\pi})^{\frac{1}{2}}.\tfrac{1}{4}.\ln[\{\alpha^2z^2 + (1+\omega z)^2\}.\{\alpha^2z^2 + (1-\omega z)^2\}^{-1}]$$

$${}_cI_{103} = (\tfrac{2}{\pi})^{\frac{1}{2}}.\{\alpha^2z^2 - \omega^2z^2 + 1\}.\{\alpha^2z^2 + (1+\omega z)^2\}^{-1}.\{\alpha^2z^2 + (1-\omega z)^2\}^{-1}$$

$${}_sI_{103} = (\tfrac{2}{\pi})^{\frac{1}{2}}.2\alpha\omega z^2.\{\alpha^2z^2 + (1+\omega z)^2\}^{-1}.\{\alpha^2z^2 + (1-\omega z)^2\}^{-1}$$

$$I_{121}(\psi) = I_{123}(\psi) = I_{125}(\psi) = {}_cI_{121} = {}_sI_{121} = {}_cI_{123} = {}_sI_{123} = 0$$

$$r^{-1}.I_{121}(\psi) = \tfrac{1}{2}.I_{103}(\psi) \qquad r^{-1}.I_{123}(\psi) = \tfrac{1}{2}.I_{105}(\psi) \qquad r^{-1}.{}_cI_{121} = \tfrac{1}{2}.{}_cI_{103} \qquad r^{-1}.{}_sI_{121} = \tfrac{1}{2}.{}_sI_{103}$$

When $z = 0$ the relevant integrals are given by;

$$I_{101}(0) = {}_cI_{101} = (\tfrac{2}{\pi})^{\frac{1}{2}}\begin{cases}\tfrac{1}{2}.\pi & r < 1\\ \tfrac{1}{2}.\pi & r = 1\\ \operatorname{arsin}\tfrac{1}{r} & r > 1\end{cases} \qquad I_{103}(0) = {}_cI_{103} = (\tfrac{2}{\pi})^{\frac{1}{2}}\begin{cases}(1-r^2)^{-\frac{1}{2}} & r < 1\\ \infty & r = 1\\ 0 & r > 1\end{cases}$$

$$I_{123}(0) = {}_cI_{123} = (\tfrac{2}{\pi})^{\frac{1}{2}}\begin{cases}0 & r < 1\\ \infty & r = 1\\ r^{-1}(r^2-1)^{-\frac{1}{2}} & r > 1\end{cases} \qquad I_{121}(0) = {}_cI_{121} = (\tfrac{2}{\pi})^{\frac{1}{2}}\begin{cases}r.\{1 + (1-r^2)^{\frac{1}{2}}\}^{-1} & r < 1\\ 1 & r = 1\\ r^{-1} & r > 1\end{cases}$$

$${}_sI_{101} = {}_sI_{103} = {}_sI_{121} = 0$$

2(a) LINEAR VERTICAL PRESSURE

A. Cross-anisotropic; β^2 Positive

$$w = P_2.\cos\theta.r_0.\{-g_1.I_{420}(\phi z) + g_2.I_{420}(\rho z)\}$$
$$u = P_2.\cos\theta.r_0.\{-g_3.M_{420}(\phi z) + g_4.M_{420}(\rho z)\}$$
$$v = P_2.\sin\theta.r_0.\{-g_3.A_{420}(\phi z) + g_4.A_{420}(\rho z)\}$$
$$\varepsilon_{zz} = P_2.\cos\theta.\{g_1.\phi.I_{422}(\phi z) - g_2.\rho.I_{422}(\rho z)\}$$
$$\varepsilon_{rr} = P_2.\cos\theta.[-g_3.\{I_{422}(\phi z) - r^{-1}.I_{440}(\phi z)\} + g_4\{I_{422}(\rho z) - r^{-1}.I_{440}(\rho z)\}]$$
$$\varepsilon_{\theta\theta} = P_2.\cos\theta.\{-g_3.r^{-1}.I_{440}(\phi z) + g_4.r^{-1}.I_{440}(\rho z)\}$$
$$\varepsilon_{rz} = P_2.\cos\theta.g_9.f^{-1}.\{M_{422}(\phi z) - M_{422}(\rho z)\}$$
$$\varepsilon_{\theta z} = P_2.\sin\theta.g_9.f^{-1}.\{A_{422}(\phi z) - A_{422}(\rho z)\}$$
$$\varepsilon_{r\theta} = P_2.\sin\theta.\{g_3.r^{-1}.I_{440}(\phi z) - g_4.r^{-1}.I_{440}(\rho z)\}$$
$$\widehat{rr} = P_2.\cos\theta.[-g_5.I_{422}(\phi z) + g_6.I_{422}(\rho z) + (a-b).r^{-1}.\{g_3.I_{440}(\phi z) - g_4.I_{440}(\rho z)\}]$$
$$\widehat{\theta\theta} = P_2.\cos\theta.[-g_7.I_{422}(\phi z) + g_8.I_{422}(\rho z) - (a-b).r^{-1}.\{g_3.I_{440}(\phi z) - g_4.I_{440}(\rho z)\}]$$
$$\widehat{zz} = P_2.\cos\theta.g_9.\{\phi^{-1}.I_{422}(\phi z) - \rho^{-1}.I_{422}(\rho z)\} \qquad \widehat{rz} = f.\varepsilon_{rz} \qquad \widehat{\theta z} = f.\varepsilon_{\theta z} \qquad \widehat{r\theta} = (a-b).\varepsilon_{r\theta}$$

B. Cross-anisotropic; β^2 Negative

$$w = P_2.\cos\theta.r_0.(-i_1.{}_cI_{420} - i_2.{}_sI_{420})$$
$$u = P_2.\cos\theta.r_0.(i_3.{}_cM_{420} + i_4.{}_sM_{420})$$
$$v = P_2.\sin\theta.r_0.(\ {}_3.{}_cA_{420} + i_4.{}_sA_{420})$$
$$\varepsilon_{zz} = P_2.\cos\theta.(i_9.{}_cI_{422} + i_{10}.{}_sI_{422})$$
$$\varepsilon_{rr} = P_2.\cos\theta.\{i_3.({}_cI_{422} - r^{-1}.{}_cI_{440}) + i_4.({}_sI_{422} - r^{-1}.{}_sI_{440})\}$$
$$\varepsilon_{\theta\theta} = P_2.\cos\theta.(i_3.r^{-1}.{}_cI_{440} + i_4.r^{-1}.{}_sI_{440})$$
$$\varepsilon_{rz} = P_2.\cos\theta.i_2.{}_sM_{422}$$
$$\varepsilon_{\theta z} = P_2.\sin\theta.i_2.{}_sA_{422}$$
$$\varepsilon_{r\theta} = P_2.\sin\theta.(-i_3.r^{-1}.{}_cI_{440} - i_4.r^{-1}.{}_sI_{440})$$
$$\widehat{rr} = P_2.\cos\theta.\{i_5.{}_cI_{422} + i_6.{}_sI_{422} - (a-b).r^{-1}.(i_3.{}_cI_{440} + i_4.{}_sI_{440})\}$$
$$\widehat{\theta\theta} = P_2.\cos\theta.\{i_7.{}_cI_{422} + i_8.{}_sI_{422} + (a-b).r^{-1}.(i_3.{}_cI_{440} + i_4.{}_sI_{440})\}$$
$$\widehat{zz} = P_2.\cos\theta.({}_cI_{422} + \alpha.\omega^{-1}.{}_sI_{422}) \qquad \widehat{rz} = f.\varepsilon_{rz} \qquad \widehat{\theta z} = f.\varepsilon_{\theta z} \qquad \widehat{r\theta} = (a-b).\varepsilon_{r\theta}$$

C. Cross-anisotropic; β^2 Zero

$$w = P_2.\cos\theta.r_0.\{-s_1.I_{420}(\alpha z) - s_2.z.I_{422}(\alpha z)\}$$
$$u = P_2.\cos\theta.r_0.\{s_3.M_{420}(\alpha z) - s_4.z.M_{422}(\alpha z)\}$$
$$v = P_2.\sin\theta.r_0.\{s_3.A_{420}(\alpha z) - s_4.z.A_{422}(\alpha z)\}$$
$$\varepsilon_{zz} = P_2.\cos\theta.\{s_9.I_{422}(\alpha z) + s_2.\alpha.z.I_{424}(\alpha z)\}$$
$$\varepsilon_{rr} = P_2.\cos\theta.[s_3.\{I_{422}(\alpha z) - r^{-1}.I_{440}(\alpha z)\} - s_4.z.\{I_{424}(\alpha z) - r^{-1}.I_{442}(\alpha z)\}]$$
$$\varepsilon_{\theta\theta} = P_2.\cos\theta.\{s_3.r^{-1}.I_{440}(\alpha z) - s_4.z.r^{-1}.I_{442}(\alpha z)\}$$
$$\varepsilon_{rz} = P_2.\cos\theta.\alpha^2.f^{-1}.z.M_{424}(\alpha z)$$
$$\varepsilon_{\theta z} = P_2.\sin\theta.\alpha^2.f^{-1}.z.A_{424}(\alpha z)$$
$$\varepsilon_{r\theta} = P_2.\sin\theta.\{-s_3.r^{-1}.I_{440}(\alpha z) + s_4.z.r^{-1}.I_{442}(\alpha z)\}$$
$$\widehat{rr} = P_2.\cos\theta.[s_5.I_{422}(\alpha z) - s_6.z.I_{424}(\alpha z) - (a-b).r^{-1}.\{s_3.I_{440}(\alpha z) - s_4.z.I_{442}(\alpha z)\}]$$
$$\widehat{\theta\theta} = P_2.\cos\theta.[s_7.I_{422}(\alpha z) - s_8.z.I_{424}(\alpha z) + (a-b).r^{-1}.\{s_3.I_{440}(\alpha z) - s_4.z.I_{442}(\alpha z)\}]$$
$$\widehat{zz} = P_2.\cos\theta.\{I_{422}(\alpha z) + \alpha.z.I_{424}(\alpha z)\} \qquad \widehat{rz} = f.\varepsilon_{rz} \qquad \widehat{\theta z} = f.\varepsilon_{\theta z} \qquad \widehat{r\theta} = (a-b).\varepsilon_{r\theta}$$

2(a) LINEAR VERTICAL PRESSURE

D. Isotropic

The stresses, strains, and displacements are as for the case of Cross-anisotropic; β^2 Zero but with the following simplifications;

$s_1 = 2.(1-\nu^2).E^{-1}$ $\quad s_2 = (1+\nu).E^{-1}$ $\quad s_3 = (1+\nu).(1-2\nu).E^{-1}$ $\quad s_4 = (1+\nu).E^{-1}$

$s_5 = s_6 = 1$ $\quad s_7 = 2\nu$ $\quad s_8 = 0$ $\quad s_9 = (1+\nu).(1-2\nu).E^{-1}$

$\alpha = 1$ $\quad f = a-b = E.(1+\nu)^{-1}$

Values of coefficients g_n, s_n, i_n

The values of $g_1 \ldots g_9$, $s_1 \ldots s_9$, $i_1 \ldots i_{10}$ have been given previously on page 13

Values of integrals $I_{4\tau\mu}(\psi)$, ${}_cI_{4\tau\mu}$, ${}_sI_{4\tau\mu}$, $A_{4\tau\mu}(\psi)$, ${}_cA_{4\tau\mu}$, ${}_sA_{4\tau\mu}$, $M_{4\tau\mu}(\psi)$, ${}_cM_{4\tau\mu}$, ${}_sM_{4\tau\mu}$

When $r \neq 0$ and $z \neq 0$ the values of the integrals are given in tables on the page numbers indicated in brackets;

$I_{420}(\psi)$	[51]	${}_cI_{420}$	[52]	$A_{420}(\psi)$	[56]	${}_cA_{420}$	[57]	${}_cM_{420}$	[60]
$I_{422}(\psi)$	[51]	${}_sI_{420}$	[52]	$A_{422}(\psi)$	[56]	${}_sA_{420}$	[57]	${}_sM_{420}$	[60]
$I_{424}(\psi)$	[51]	${}_cI_{422}$	[53]	$A_{424}(\psi)$	[56]	${}_cA_{422}$	[58]	${}_cM_{422}$	[61]
$I_{440}(\psi)$	[54]	${}_sI_{422}$	[53]	$M_{420}(\psi)$	[59]	${}_sA_{422}$	[58]	${}_sM_{422}$	[61]
$I_{422}(\psi)$	[54]	${}_cI_{440}$	[55]	$M_{422}(\psi)$	[59]				
		${}_sI_{440}$	[55]	$M_{424}(\psi)$	[59]				

When $r = 0$ the integrals are given by;

$$I_{420}(\psi) = I_{422}(\psi) = I_{424}(\psi) = I_{440}(\psi) = I_{442}(\psi) = r^{-1}.I_{440}(\psi) = r^{-1}.I_{442}(\psi) = 0$$

$${}_cI_{420} = {}_sI_{420} = {}_cI_{422} = {}_sI_{422} = {}_cI_{440} = {}_sI_{440} = r^{-1}.{}_cI_{440} = r^{-1}.{}_sI_{440} = 0$$

$$A_{420}(\psi) = -M_{420}(\psi) = \tfrac{1}{4}.\{(\psi^2+1)^{\frac{1}{2}}-\psi\}^2 \qquad A_{422}(\psi) = -M_{422}(\psi) = \tfrac{1}{2}.\{(\psi^2+1)^{\frac{1}{2}}-\psi\}^2.(\psi^2+1)^{-\frac{1}{2}}$$

$$A_{424}(\psi) = -M_{424}(\psi) = 1-\tfrac{1}{2}.\psi.(2\psi^2+3).(\psi^2+1)^{-\frac{3}{2}}$$

$${}_cA_{420} = -{}_cM_{420} = \tfrac{1}{2}.(\alpha^2-\omega^2).z^2 - \tfrac{1}{2}.R^{\frac{1}{2}}.z.(\alpha.\cos\tfrac{\upsilon}{2} - \omega.\sin\tfrac{\upsilon}{2}) + \tfrac{1}{4}$$

$${}_sA_{420} = -{}_sM_{420} = \tfrac{1}{2}.R^{\frac{1}{2}}.z.(\omega.\cos\tfrac{\upsilon}{2} + \alpha.\sin\tfrac{\upsilon}{2}) - \alpha.\omega.z^2$$

$${}_cA_{422} = -{}_cM_{422} = -\alpha.z + \tfrac{1}{2}.z^2.R^{-\frac{1}{2}}.\{(\alpha^2-\omega^2).\cos\tfrac{\upsilon}{2} + 2.\alpha.\omega.\sin\tfrac{\upsilon}{2}\} + \tfrac{1}{2}.R^{\frac{1}{2}}.\cos\tfrac{\upsilon}{2}$$

$${}_sA_{422} = -{}_sM_{422} = \omega.z + \tfrac{1}{2}.z^2.R^{-\frac{1}{2}}.\{(\alpha^2-\omega^2).\sin\tfrac{\upsilon}{2} - 2.\alpha.\omega.\cos\tfrac{\upsilon}{2}\} - \tfrac{1}{2}.R^{\frac{1}{2}}.\sin\tfrac{\upsilon}{2}$$

where $R = [\{(\alpha^2-\omega^2).z^2+1\}^2 + 4.\alpha^2.\omega^2.z^4]^{\frac{1}{2}}$ and $\upsilon = \text{artan}[2.\alpha.\omega.z^2.\{(\alpha^2-\omega^2).z^2+1\}^{-1}]$

When $z = 0$ the relevant integrals are given by;

$$I_{420}(0) = {}_cI_{420} = \begin{cases} \tfrac{1}{2}.r.{}_2F_1(\tfrac{3}{2},-\tfrac{1}{2};2;r^2) \\ \tfrac{2}{3}.\pi^{-1} \\ \tfrac{1}{8}.r^{-2}.{}_2F_1(\tfrac{3}{2},\tfrac{1}{2};3;r^{-2}) \end{cases} \quad I_{422}(0) = {}_cI_{422} = \begin{cases} r \\ \tfrac{1}{2} \\ 0 \end{cases} \quad I_{440}(0) = {}_cI_{440} = \begin{cases} \tfrac{1}{2}.r^2 & r < 1 \\ \tfrac{1}{2} & r = 1 \\ \tfrac{1}{2}.r^{-2} & r > 1 \end{cases}$$

$$A_{420}(0) = {}_cA_{420} = \begin{cases} \tfrac{1}{8}.(2-r^2) \\ \tfrac{1}{8}.r^{-2} \end{cases} \quad A_{422}(0) = {}_cA_{422} = \begin{cases} \tfrac{1}{2}.{}_2F_1(\tfrac{3}{2},-\tfrac{1}{2};2;r^2) & r < 1 \\ \tfrac{1}{8}.r^{-3}.{}_2F_1(\tfrac{3}{2},\tfrac{1}{2};3;r^{-2}) & r > 1 \end{cases}$$

$$M_{420}(0) = {}_cM_{420} = \begin{cases} \tfrac{1}{8}.(3r^2-2) \\ \tfrac{1}{8}.r^{-2} \end{cases} \quad M_{422}(0) = {}_cM_{422} = \begin{cases} \tfrac{1}{2}.{}_2F_1(\tfrac{3}{2},\tfrac{1}{2};2;r^2) - {}_2F_1(\tfrac{3}{2},-\tfrac{1}{2};2;r^2) & r < 1 \\ \tfrac{1}{4}.r^{-3}.\{{}_2F_1(\tfrac{3}{2},\tfrac{1}{2};2;r^{-2}) - \tfrac{1}{2}.{}_2F_1(\tfrac{3}{2},\tfrac{1}{2};3;r^{-2})\} & r > 1 \end{cases}$$

$${}_sA_{420} = {}_sA_{422} = {}_sM_{420} = {}_sM_{422} = 0$$

2(b) LINEAR VERTICAL DISPLACEMENT

A. Cross-anisotropic; β^2 Positive

The displacements, strains, and stresses are obtained by making the following substitutions into the equations shown on page 16 for the case of Linear Vertical Pressure loading (Cross-anisotropic; β^2 Positive).

(i) substitute $T_m.(\frac{2}{\pi})^{\frac{1}{2}}.\frac{3}{4r_0^3}$ OR $-\Delta_2.(\frac{2}{\pi})^{\frac{1}{2}}.\frac{2}{r_0}.\frac{f}{2c+f}.\frac{(c+d\rho^2).(c+d\phi^2)}{d.\rho.\phi.(\rho+\phi)}$ ---------- for P_2

This alternative substitution follows from the moment-displacement relationship;

$$T_m = -\Delta_2.\frac{8r_0^2}{3}.\frac{f}{2c+f}.\frac{(c+d\rho^2).(c+d\phi^2)}{d.\rho.\phi.(\rho+\phi)}$$

(ii) substitute

$I_{3\tau(\mu+1)}(\psi)$ ---------- for $I_{4\tau\mu}(\psi)$

$A_{3\tau(\mu+1)}(\psi)$ ---------- for $A_{4\tau\mu}(\psi)$

$M_{3\tau(\mu+1)}(\psi)$ ---------- for $M_{4\tau\mu}(\psi)$

B. Cross-anisotropic; β^2 Negative

The displacements, strains, and stresses are obtained by making the following substitutions into the equations shown on page 16 for the case of Linear Vertical Pressure loading (Cross-anisotropic; β^2 Negative).

(i) substitute $T_m.(\frac{2}{\pi})^{\frac{1}{2}}.\frac{3}{4r_0^3}$ OR $-\Delta_2.(\frac{2}{\pi})^{\frac{1}{2}}.\frac{2}{r_0}.\frac{ad-c^2}{2a(ad)^{\frac{1}{2}}}$ ---------- for P_2

This alternative substitution follows from the moment-displacement relationship;

$$T_m = -\Delta_2.\frac{8r_0^2}{3}.\frac{ad-c^2}{2a(ad)^{\frac{1}{2}}}$$

(ii) substitute

${}_cI_{3\tau(\mu+1)}$ ---------- for ${}_cI_{4\tau\mu}$

${}_sI_{3\tau(\mu+1)}$ ---------- for ${}_sI_{4\tau\mu}$

${}_cA_{3\tau(\mu+1)}$ ---------- for ${}_cA_{4\tau\mu}$

${}_sA_{3\tau(\mu+1)}$ ---------- for ${}_sA_{4\tau\mu}$

${}_cM_{3\tau(\mu+1)}$ ---------- for ${}_cM_{4\tau\mu}$

${}_sM_{3\tau(\mu+1)}$ ---------- for ${}_sM_{4\tau\mu}$

C. Cross-anisotropic; β^2 Zero

The displacements, strains, and stresses are obtained by making the following substitutions into the equations shown on page 16 for the case of Linear Vertical Pressure loading (Cross-anisotropic; β^2 Zero).

(i) substitute $T_m.(\frac{2}{\pi})^{\frac{1}{2}}.\frac{3}{4r_0^3}$ OR $-\Delta_2.(\frac{2}{\pi})^{\frac{1}{2}}.\frac{2}{r_0}.\frac{f}{2c+f}.\frac{(c+d\alpha^2)^2}{2d\alpha^3}$ ---------- for P_2

This alternative substitution follows from the moment-displacement relationship;

$$T_m = -\Delta_2.\frac{8r_0^2}{3}.\frac{f}{2c+f}.\frac{(c+d\alpha^2)^2}{2d\alpha^3}$$

$I_{3\tau(\mu+1)}(\alpha z)$ ---------- for $I_{4\tau\mu}(\alpha z)$

$A_{3\tau(\mu+1)}(\alpha z)$ ---------- for $A_{4\tau\mu}(\alpha z)$

$M_{3\tau(\mu+1)}(\alpha z)$ ---------- for $M_{4\tau\mu}(\alpha z)$

D. Isotropic

The displacements, strains, and stresses are obtained by making the following substitutions into the equations shown on page 16 for the case of Linear Vertical Pressure loading (Cross-anisotropic; β^2 Zero).

(i) substitute $T_m.(\frac{2}{\pi})^{\frac{1}{2}}.\frac{3}{4r_0^3}$ OR $-\Delta_2.(\frac{2}{\pi})^{\frac{1}{2}}.\frac{2}{r_0}.\frac{E}{2(1-\nu^2)}$ ---------- for P_2

This alternative substitution follows from the moment-displacement relationship;

$$T_m = -\Delta_2.\frac{8r_0^2}{3}.\frac{E}{2(1-\nu^2)}$$

(ii) substitute

$I_{3\tau(\mu+1)}(z)$ ---------- for $I_{4\tau\mu}(z)$

$A_{3\tau(\mu+1)}(z)$ ---------- for $A_{4\tau\mu}(z)$

$M_{3\tau(\mu+1)}(z)$ ---------- for $M_{4\tau\mu}(z)$

(iii) In addition the following simplifications apply with regard to the coefficients s_n;

$s_1 = 2.(1-\nu^2).E^{-1}$ $\quad s_2 = (1+\nu).E^{-1}$ $\quad s_3 = (1+\nu).(1-2\nu).E^{-1}$ $\quad s_4 = (1+\nu).E^{-1}$ $\quad s_5 = s_6 = 1$

$s_7 = 2\nu$ $\quad s_8 = 0$ $\quad s_9 = (1+\nu).(1-2\nu).E^{-1}$ $\quad a = 1$ $\quad f = a-b = E.(1+\nu)^{-1}$

2(b) LINEAR VERTICAL DISPLACEMENT

Values of Integrals $I_{3\tau\mu}(\psi)$, ${}_cI_{3\tau\mu}$, ${}_sI_{3\tau\mu}$, $A_{3\tau\mu}(\psi)$, ${}_cA_{3\tau\mu}$, ${}_sA_{3\tau\mu}$, $M_{3\tau\mu}(\psi)$, ${}_cM_{3\tau\mu}$, ${}_sM_{3\tau\mu}$

When $r \neq 0$ and $z \neq 0$ the values of the integrals are given by;

$I_{321}(\psi)$, $I_{323}(\psi)$ and $I_{325}(\psi)$ are shown on page 23 for the corresponding case of $r \neq 0$ and $z \neq 0$

$$I_{341}(\psi) = \left(\tfrac{2}{\pi}\right)^{\frac{1}{2}}.\{4 + 2.\psi.N_0.\cos\tfrac{1}{2}\lambda_0 - 2.N_0.(\psi^2+r^2+2).\sin\tfrac{1}{2}\lambda_0\}.\tfrac{1}{6}.r^{-2}$$

$$I_{343}(\psi) = \left(\tfrac{2}{\pi}\right)^{\frac{1}{2}}.\{(N_0{}^2-2).\psi.\sin\tfrac{1}{2}\lambda_0 - (\psi^2-1).\cos\tfrac{1}{2}\lambda_0\}.r^{-2}.N_0{}^{-1}$$

$$I_{345}(\psi) = \left(\tfrac{2}{\pi}\right)^{\frac{1}{2}}.\{N_0{}^2.(\psi^2+1-N_0{}^2).\sin\tfrac{1}{2}\lambda_0 + r^2.(\psi.\cos\tfrac{3}{2}\lambda_0-\sin\tfrac{3}{2}\lambda_0)\}.N_0{}^{-3}.r^{-2}$$

$I_{301}(\psi)$, $I_{303}(\psi)$ and $I_{305}(\psi)$ are shown on page 23 for the corresponding case of $r \neq 0$ and $z \neq 0$

$A_{32\mu}(\psi) = \tfrac{1}{2}\{I_{34\mu}(\psi) + I_{30\mu}(\psi)\}$ $\qquad$ $M_{32\mu}(\psi) = \tfrac{1}{2}\{I_{34\mu}(\psi) - I_{30\mu}(\psi)\}$

${}_cI_{321}$, ${}_sI_{321}$, ${}_cI_{323}$ and ${}_sI_{323}$ are shown on page 23 for the corresponding case of $r \neq 0$ and $z \neq 0$

$${}_cI_{341} = \left(\tfrac{2}{\pi}\right)^{\frac{1}{2}}.[-\{\alpha^2z^2+r^2+(1+\omega z)(2-\omega z)\}.N_p.\sin\tfrac{1}{2}\lambda_p - \{\alpha^2z^2+r^2+(1-\omega z)(2+\omega z)\}.N_m.\sin\tfrac{1}{2}\lambda_m$$
$$+ 4 + (1+2\omega z).\alpha z.N_m.\cos\tfrac{1}{2}\lambda_m + (1-2\omega z).\alpha z.N_p.\cos\tfrac{1}{2}\lambda_p].\tfrac{1}{6}.r^{-2}$$

$${}_sI_{341} = \left(\tfrac{2}{\pi}\right)^{\frac{1}{2}}.[-\{\alpha^2z^2+r^2+ (1+\omega z)(2-\omega z)\}.N_p.\cos\tfrac{1}{2}\lambda_p + \{\alpha^2z^2+r^2+(1-\omega z)(2+\omega z)\}.N_m.\cos\tfrac{1}{2}\lambda_m$$
$$+ \alpha z.(1+2\omega z).N_m.\sin\tfrac{1}{2}\lambda_m - \alpha z.(1-2\omega z).N_p.\sin\tfrac{1}{2}\lambda_p].\tfrac{1}{6}.r^{-2}$$

$${}_cI_{343} = \left(\tfrac{2}{\pi}\right)^{\frac{1}{2}} \begin{array}{l} \{(1+\omega z)^2 + \omega z.N_p{}^2 - \alpha^2z^2\}.\cos\tfrac{1}{2}\lambda_p.N_p{}^{-1} + \{(1-\omega z)^2 - \omega z.N_m{}^2 - \alpha^2z^2\}.\cos\tfrac{1}{2}\lambda_m.N_m{}^{-1} \\ + \alpha z.(N_p{}^2-2-2\omega z).\sin\tfrac{1}{2}\lambda_p.N_p{}^{-1} + \alpha z.(N_m{}^2-2+2\omega z).\sin\tfrac{1}{2}\lambda_m.N_m{}^{-1} \end{array} .\tfrac{1}{2}.r^{-2}$$

$${}_sI_{343} = \left(\tfrac{2}{\pi}\right)^{\frac{1}{2}} \begin{array}{l} \{(1+\omega z)^2 - \omega z.N_p{}^2 - \alpha^2z^2\}.\sin\tfrac{1}{2}\lambda_p.N_p{}^{-1} - \{(1-\omega z)^2 + \omega z.N_m{}^2 - \alpha^2z^2\}.\sin\tfrac{1}{2}\lambda_m.N_m{}^{-1} \\ + \alpha z.(N_p{}^2+2+2\omega z).\cos\tfrac{1}{2}\lambda_p.N_p{}^{-1} - \alpha z.(N_m{}^2+2-2\omega z).\cos\tfrac{1}{2}\lambda_m.N_m{}^{-1} \end{array} .\tfrac{1}{2}.r^{-2}$$

${}_cI_{301}$, ${}_sI_{301}$, ${}_cI_{303}$ and ${}_sI_{303}$ are shown on page 23 for the corresponding case of $r \neq 0$ and $z \neq 0$

${}_cA_{32\mu} = \tfrac{1}{2}({}_cI_{34\mu}+{}_cI_{30\mu})$ $\qquad$ ${}_sA_{32\mu} = \tfrac{1}{2}({}_sI_{34\mu}+{}_sI_{30\mu})$

${}_cM_{32\mu} = \tfrac{1}{2}({}_cI_{34\mu}-{}_cI_{30\mu})$ $\qquad$ ${}_sM_{32\mu} = \tfrac{1}{2}({}_sI_{34\mu}-{}_sI_{30\mu})$

where σ_0, σ_p, σ_m, N_0, N_p, N_m, L_p, L_m, λ_0, λ_p, λ_m are given on page 15.

When $r = 0$ the integrals are given by:

$$I_{321}(\psi) = I_{323}(\psi) = I_{325}(\psi) = {}_cI_{321} = {}_cI_{323} = {}_sI_{321} = {}_sI_{323} = 0$$

$$I_{341}(\psi) = r^{-1}.I_{341}(\psi) = I_{343}(\psi) = r^{-1}.I_{343}(\psi) = {}_cI_{341} = r^{-1}.{}_cI_{341} = {}_sI_{341} = r^{-1}.{}_sI_{341} = 0$$

$A_{321}(\psi) = -M_{321}(\psi) = \left(\tfrac{2}{\pi}\right)^{\frac{1}{2}}.\tfrac{1}{2}.\{1-\psi.\text{arcot}\,\psi\}$ $\qquad$ $A_{323}(\psi) = -M_{323}(\psi) = \left(\tfrac{2}{\pi}\right)^{\frac{1}{2}}.\tfrac{1}{2}.\{\text{arcot}\,\psi-\psi.(\psi^2+1)^{-1}\}$

$$A_{325}(\psi) = -M_{325}(\psi) = \left(\tfrac{2}{\pi}\right)^{\frac{1}{2}}.(\psi^2+1)^{-2}$$

$${}_cA_{321} = -{}_cM_{321} = \left(\tfrac{2}{\pi}\right)^{\frac{1}{2}}.\tfrac{1}{2}.\{2 - \alpha z.(\text{artan}\tfrac{1+\omega z}{\alpha z} + \text{artan}\tfrac{1-\omega z}{\alpha z}) + \tfrac{\omega z}{2}.\ln[\{\alpha^2z^2+(1-\omega z)^2\}.\{\alpha^2z^2+(1+\omega z)^2\}^{-1}]\}$$

$${}_sA_{321} = -{}_sM_{321} = \left(\tfrac{2}{\pi}\right)^{\frac{1}{2}}.\tfrac{1}{2}.\{\omega z.(\text{artan}\tfrac{1+\omega z}{\alpha z} + \text{artan}\tfrac{1-\omega z}{\alpha z}) + \tfrac{\alpha z}{2}.\ln[\{\alpha^2z^2+(1-\omega z)^2\}.\{\alpha^2z^2+(1+\omega z)^2\}^{-1}]\}$$

$${}_cA_{323} = -{}_cM_{323} = \left(\tfrac{2}{\pi}\right)^{\frac{1}{2}}.\tfrac{1}{2}.\{\text{artan}\tfrac{1+\omega z}{\alpha z} + \text{artan}\tfrac{1-\omega z}{\alpha z} - \frac{\alpha z}{\alpha^2z^2+(1+\omega z)^2} - \frac{\alpha z}{\alpha^2z^2+(1-\omega z)^2}\}$$

$${}_sA_{323} = -{}_sM_{323} = \left(\tfrac{2}{\pi}\right)^{\frac{1}{2}}.\tfrac{1}{2}.\{\tfrac{1}{2}\ln[\{\alpha^2z^2+(1+\omega z)^2\}.\{\alpha^2z^2+(1-\omega z)^2\}^{-1}] + \frac{1-\omega z}{\alpha^2z^2+(1-\omega z)^2} - \frac{1+\omega z}{\alpha^2z^2+(1+\omega z)^2}\}$$

When $z = 0$ the relevant integrals are given by:

$I_{321}(0)$, ${}_cI_{321}$, $I_{323}(0)$ and ${}_cI_{323}$ are shown on page 23 for the corresponding case of $z = 0$

$$I_{341}(0) = {}_cI_{341} = \left(\tfrac{2}{\pi}\right)^{\frac{1}{2}} \begin{cases} \dfrac{r^2\{2+(1-r^2)^{\frac{1}{2}}\}}{3\{1+(1-r^2)^{\frac{1}{2}}\}^2} & r < 1 \\ \tfrac{2}{3}.r^{-2} & r > 1 \end{cases} \qquad I_{343}(0) = {}_cI_{343} = \left(\tfrac{2}{\pi}\right)^{\frac{1}{2}} \begin{cases} 0 & r < 1 \\ r^{-2}.(r^2-1)^{-\frac{1}{2}} & r > 1 \end{cases}$$

$$A_{321}(0) = {}_cA_{321} = \left(\tfrac{2}{\pi}\right)^{\frac{1}{2}} \begin{cases} \dfrac{2-(1-r^2)^{\frac{3}{2}}}{3r^2} & r < 1 \\ \tfrac{1}{3}.r^{-2} & r > 1 \end{cases} \qquad A_{323}(0) = {}_cA_{323} = \left(\tfrac{2}{\pi}\right)^{\frac{1}{2}} \begin{cases} \tfrac{\pi}{4} & r < 1 \\ \tfrac{1}{2}\{\text{arsin}\tfrac{1}{r} + r^{-2}.(r^2-1)^{\frac{1}{2}}\} & r > 1 \end{cases}$$

$$M_{321}(0) = {}_cM_{321} = \left(\tfrac{2}{\pi}\right)^{\frac{1}{2}} \begin{cases} \dfrac{2-(1+2r^2)(1-r^2)^{\frac{1}{2}}}{3r^2} & r < 1 \\ -\tfrac{1}{3}.r^{-2} & r > 1 \end{cases} \qquad M_{323}(0) = {}_cM_{323} = \left(\tfrac{2}{\pi}\right)^{\frac{1}{2}} \begin{cases} -\tfrac{\pi}{4} & r < 1 \\ \tfrac{1}{2}\{-\text{arsin}\tfrac{1}{r} + (r^2+1).r^{-2}.(r^2-1)^{-\frac{1}{2}}\} & r > 1 \end{cases}$$

$${}_sI_{321} = {}_sI_{323} = {}_sI_{341} = {}_sI_{343} = {}_sA_{321} = {}_sA_{323} = {}_sM_{321} = {}_sM_{323} = 0$$

3(a) LINEAR RADIAL SHEAR STRESS

A. Cross-anisotropic; β^2 Positive

$$w = p_1.r_0.\{h_1.I_{400}(\phi z) - h_2.I_{400}(\rho z)\}$$
$$u = p_1.r_0.\{h_3.I_{420}(\phi z) - h_4.I_{420}(\rho z)\}$$
$$\varepsilon_{zz} = p_1.\{-\phi.h_1.I_{402}(\phi z) + \rho.h_2.I_{402}(\rho z)\}$$
$$\varepsilon_{rr} = p_1.[h_3.\{I_{402}(\phi z) - r^{-1}.I_{420}(\phi z)\} - h_4.\{I_{402}(\rho z) - r^{-1}.I_{420}(\rho z)\}]$$
$$\varepsilon_{\theta\theta} = p_1.\{h_3.r^{-1}.I_{420}(\phi z) - h_4.r^{-1}.I_{420}(\rho z)\}$$
$$\varepsilon_{rz} = p_1.f^{-1}.h_9.\{-\rho^{-1}.I_{422}(\phi z) + \phi^{-1}.I_{422}(\rho z)\}$$
$$\widehat{rr} = p_1.[h_5.I_{402}(\phi z) - h_6.I_{402}(\rho z) - (a-b).r^{-1}.\{h_3.I_{420}(\phi z) - h_4.I_{420}(\rho z)\}]$$
$$\widehat{\theta\theta} = p_1.[h_7.I_{402}(\phi z) - h_8.I_{402}(\rho z) + (a-b).r^{-1}.\{h_3.I_{420}(\phi z) - h_4.I_{420}(\rho z)\}]$$
$$\widehat{zz} = p_1.h_9.\rho^{-1}.\phi^{-1}.\{-I_{402}(\phi z) + I_{402}(\rho z)\} \qquad \widehat{rz} = f.\varepsilon_{rz}$$

B. Cross-anisotropic; β^2 Negative

$$w = p_1.r_0\{j_1.{}_cI_{400} + j_2.{}_sI_{400}\}$$
$$u = p_1.r_0\{j_3.{}_cI_{420} + j_4.{}_sI_{420}\}$$
$$\varepsilon_{zz} = p_1.\{-j_9.{}_cI_{402} - j_{10}.{}_sI_{402}\}$$
$$\varepsilon_{rr} = p_1.[j_3.\{{}_cI_{402} - r^{-1}.{}_cI_{420}\} + j_4.\{{}_sI_{402} - r^{-1}.{}_sI_{420}\}]$$
$$\varepsilon_{\theta\theta} = p_1.\{j_3.r^{-1}.{}_cI_{420} + j_4.r^{-1}.{}_sI_{420}\}$$
$$\varepsilon_{rz} = p_1.f^{-1}.\{{}_cI_{422} - \alpha.\omega^{-1}.{}_sI_{422}\}$$
$$\widehat{rr} = p_1.[j_5.{}_cI_{402} + j_6.{}_sI_{402} - (a-b).r^{-1}.\{j_3.{}_cI_{420} + j_4.{}_sI_{420}\}]$$
$$\widehat{\theta\theta} = p_1.[j_7.{}_cI_{402} + j_8.{}_sI_{402} + (a-b).r^{-1}.\{j_3.{}_cI_{420} + j_4.{}_sI_{420}\}]$$
$$\widehat{zz} = p_1.(-\omega^{-1}).{}_sI_{402} \qquad \widehat{rz} = f.\varepsilon_{rz}$$

C. Cross-anisotropic; β^2 Zero

$$w = p_1.r_0.\{t_1.I_{400}(\alpha z) + t_2.z.I_{402}(\alpha z)\}$$
$$u = p_1.r_0.\{-t_3.I_{420}(\alpha z) + t_4.z.I_{422}(\alpha z)\}$$
$$\varepsilon_{zz} = p_1.\{t_9.I_{402}(\alpha z) - t_2.\alpha.z.I_{404}(\alpha z)\}$$
$$\varepsilon_{rr} = p_1.[-t_3.\{I_{402}(\alpha z) - r^{-1}.I_{420}(\alpha z)\} + t_4.z.\{I_{404}(\alpha z) - r^{-1}.I_{422}(\alpha z)\}]$$
$$\varepsilon_{rz} = p_1.f^{-1}.\{I_{422}(\alpha z) - \alpha.z.I_{424}(\alpha z)\}$$
$$\widehat{rr} = p_1.[-t_5.I_{402}(\alpha z) + t_6.z.I_{404}(\alpha z) + (a-b).r^{-1}.\{t_3.I_{420}(\alpha z) - t_4.z.I_{422}(\alpha z)\}]$$
$$\widehat{\theta\theta} = p_1.[-t_7.I_{402}(\alpha z) + t_8.z.I_{404}(\alpha z) - (a-b).r^{-1}.\{t_3.I_{420}(\alpha z) - t_4.z.I_{422}(\alpha z)\}]$$
$$\widehat{zz} = p_1.z.\{-I_{404}(\alpha z)\} \qquad \widehat{rz} = f.\varepsilon_{rz}$$

D. Isotropic

The stresses, strains, and displacements are as for the case of Cross-anisotropic; β^2 Zero but with the following simplifications;

$t_1 = (1+\nu).(1-2\nu).E^{-1}$ $\quad t_2 = (1+\nu).E^{-1}$ $\quad t_3 = 2.(1-\nu^2).E^{-1}$ $\quad t_4 = (1+\nu).E^{-1}$ $\quad t_5 = 2$

$t_6 = 1$ $\quad t_7 = 2\nu$ $\quad t_8 = 0$ $\quad t_9 = 2\nu.(1+\nu).E^{-1}$ $\quad \alpha = 1$

$f = a-b = E.(1+\nu)^{-1}$

3(a) LINEAR RADIAL SHEAR STRESS

Values of coefficients h_n, j_n, t_n

$h_1 = (2c+f).\phi.f^{-1}.(\rho-\phi)^{-1}.(c+d\phi^2)^{-1}$ $\qquad$ $h_2 = (2c+f).\rho.f^{-1}.(\rho-\phi)^{-1}.(c+d\rho^2)^{-1}$

$h_3 = (2d\phi^2-f).f^{-1}.(\rho-\phi)^{-1}.(c+d\phi^2)^{-1}$ $\qquad$ $h_4 = (2d\rho^2-f).f^{-1}.(\rho-\phi)^{-1}.(c+d\rho^2)^{-1}$

$h_5 = a.h_3 - c.\phi.h_1$ $\qquad$ $h_6 = a.h_4 - c.\rho.h_2$

$h_7 = b.h_3 - c.\phi.h_1$ $\qquad$ $h_8 = b.h_4 - c.\rho.h_2$

$h_9 = \rho.\phi.(\rho-\phi)^{-1}$

$j_1 = \{(ad)^{\frac{1}{2}} + c\}^{-1}$ $\qquad$ $j_2 = a.\omega^{-1}.\{(ad)^{\frac{1}{2}}-c\}$

$j_3 = -2ad.(ad-c^2)^{-1}$ $\qquad$ $j_4 = f^{-1}.\omega^{-1}$

$j_5 = a.j_3 - c.j_9$ $\qquad$ $j_6 \quad a.j_4 - c.j_{10}$

$j_7 = b.j_3 - c.j_9$ $\qquad$ $j_8 = b.j_4 - c.j_{10}$

$j_9 = -2ac.(ad-c^2)^{-1}$ $\qquad$ $j_{10} = (f+c).(\omega fd)^{-1}$

$t_1 = (2c+f).(d\alpha^2-c).f^{-1}.(c+d\alpha^2)^{-2}$ $\qquad$ $t_2 = (2c+f).\alpha.f^{-1}.(c+d\alpha^2)^{-1}$

$t_3 = (2c+f).2d\alpha.f^{-1}.(c+d\alpha^2)^{-2}$ $\qquad$ $t_4 = (2d\alpha^2-f).f^{-1}.(c+d\alpha^2)^{-1}$

$t_5 = a.t_3 - c.t_9$ $\qquad$ $t_6 = a.t_4 - c.\alpha.t_2$

$t_7 = b.t_3 - c.t_9$ $\qquad$ $t_8 = b.t_4 - c.\alpha.t_2$

$t_9 = (2c+f).2c\alpha.f^{-1}.(c+d\alpha^2)^{-2}$

Values of integrals $I_{4\tau\mu}(\psi)$, ${}_cI_{4\tau\mu}$, ${}_sI_{4\tau\mu}$

When $r \neq 0$ and $z \neq 0$ the values of the integrals are given in tables on the page numbers indicated in brackets;

$I_{400}(\psi)$	[48]	$I_{420}(\psi)$	[51]	${}_cI_{400}$	[49]	${}_cI_{420}$	[52]
$I_{402}(\psi)$	[48]	$I_{422}(\psi)$	[51]	${}_sI_{400}$	[49]	${}_sI_{420}$	[52]
$I_{404}(\psi)$	[48]	$I_{424}(\psi)$	[51]	${}_cI_{402}$	[50]	${}_cI_{422}$	[53]
				${}_sI_{402}$	[50]	${}_sI_{422}$	[53]

When $r = 0$ the integrals are given by;

$I_{400}(\psi) = \frac{1}{2}.\{(\psi^2+1)^{\frac{1}{2}} - \psi\}^2$ $\qquad$ $I_{402}(\psi) = \{(\psi^2+1)^{\frac{1}{2}} - \psi\}^2.(\psi^2+1)^{-\frac{1}{2}}$ $\qquad$ $I_{404}(\psi) = 2 - \psi(2\psi^2+3).(\psi^2+1)^{-\frac{3}{2}}$

$I_{420}(\psi) = I_{422}(\psi) = I_{424}(\psi) = 0$ $\qquad$ $r^{-1}.I_{420}(\psi) = \frac{1}{2}.I_{402}(\psi)$ $\qquad$ $r^{-1}.I_{422}(\psi) = \frac{1}{2}.I_{404}(\psi)$

${}_cI_{400} = (\alpha^2-\omega^2).z^2 - R^{\frac{1}{2}}.z.(\alpha.\cos\frac{\upsilon}{2} - \omega.\sin\frac{\upsilon}{2}) + \frac{1}{2}$ $\qquad$ ${}_sI_{400} = R^{\frac{1}{2}}.z.(\omega.\cos\frac{\upsilon}{2}+\alpha.\sin\frac{\upsilon}{2}) - 2.\alpha.\omega.z^2$

${}_cI_{402} = -2.\alpha.z + z^2.R^{-\frac{1}{2}}.\{(\alpha^2-\omega^2).\cos\frac{\upsilon}{2} + 2.\alpha.\omega.\sin\frac{\upsilon}{2}\} + R^{\frac{1}{2}}.\cos\frac{\upsilon}{2}$

${}_sI_{402} = 2.\omega.z + z^2.R^{-\frac{1}{2}}.\{(\alpha^2-\omega^2).\sin\frac{\upsilon}{2} - 2.\alpha.\omega.\cos\frac{\upsilon}{2}\} - R^{\frac{1}{2}}.\sin\frac{\upsilon}{2}$

${}_cI_{420} = {}_cI_{422} = {}_sI_{420} = {}_sI_{422} = 0$ $\qquad$ $r^{-1}.{}_cI_{420} = \frac{1}{2}.{}_cI_{402}$ $\qquad$ $r^{-1}.{}_sI_{420} = \frac{1}{2}.{}_sI_{402}$

where $R = [\{(\alpha^2-\omega^2).z^2 + 1\}^2 + 4.\alpha^2.\omega^2.z^4]^{\frac{1}{2}}$ and $\upsilon = \text{artan}[2.\alpha.\omega.z^2.\{(\alpha^2-\omega^2).z^2 + 1\}^{-1}]$

When $z = 0$ the relevant integrals are given by;

$$I_{400}(0) = {}_cI_{400} = \begin{cases} \frac{1}{2}(1-r^2) \\ 0 \end{cases} \qquad I_{402}(0) = {}_cI_{402} = \begin{cases} {}_2F_1(\frac{3}{2},-\frac{1}{2};1;r^2) & r < 1 \\ -\frac{1}{8}.r^{-3}.{}_2F_1(\frac{3}{2},\frac{3}{2};3;r^{-2}) & r > 1 \end{cases}$$

$$I_{420}(0) = {}_cI_{420} = \begin{cases} \frac{1}{2}.r.{}_2F_1(\frac{3}{2},-\frac{1}{2};2;r^2) \\ \frac{2}{3}.\pi^{-1} \\ \frac{1}{8}.r^{-2}.{}_2F_1(\frac{3}{2},\frac{1}{2};3;r^{-2}) \end{cases} \qquad I_{422}(0) = {}_cI_{422} = \begin{cases} r & r < 1 \\ \frac{1}{2} & r = 0 \\ 0 & r > 1 \end{cases}$$

${}_sI_{400} = {}_sI_{402} = {}_sI_{420} = {}_sI_{422} = 0$

3(b) LINEAR RADIAL SHEAR DISPLACEMENT

A. Cross-anisotropic; β^2 Positive

The displacements, strains, and stresses are obtained by making the following substitutions into the equations for the case of Linear Radial Shear Stress loading (Cross-anisotropic; β^2 Positive). These equations are shown on page 20.

(i) substitute $-\delta_1.\left(\frac{2}{\pi}\right)^{\frac{1}{2}}.\frac{2}{r_0}.\frac{f}{2c+f}.\frac{(c+d\rho^2).(c+d\phi^2)}{d.(\rho+\phi)}$ ---------------- for p_1

(ii) substitute $I_{3\tau(\mu+1)}(\psi)$ ---------------- for $I_{4\tau\mu}(\psi)$

B. Cross-anisotropic; β^2 Negative

The displacements, strains, and stresses are obtained by making the following substitutions into the equations for the case of Linear Radial Shear Stress loading (Cross-anisotropic; β^2 Negative). These equations are shown on page 20.

(i) substitute $-\delta_1.\left(\frac{2}{\pi}\right)^{\frac{1}{2}}.\frac{2}{r_0}.\frac{(ad-c^2)}{2ad}$ ---------------- for p_1

(ii) substitute ${}_cI_{3\tau(\mu+1)}$ ---------------- for ${}_cI_{4\tau\mu}$

${}_sI_{3\tau(\mu+1)}$ ---------------- for ${}_sI_{4\tau\mu}$

C. Cross-anisotropic; β^2 Zero

The displacements, strains, and stresses are obtained by making the following substitutions into the equations for the case of Linear Radial Shear Stress loading (Cross-anisotropic; β^2 Zero). These equations are shown on page 20.

(i) substitute $-\delta_1.\left(\frac{2}{\pi}\right)^{\frac{1}{2}}.\frac{2}{r_0}.\frac{f}{2c+f}.\frac{(c+d\alpha^2)^2}{2d\alpha}$ ---------------- for p_1

$I_{3\tau(\mu+1)}(\alpha z)$ ---------------- for $I_{4\tau\mu}(\alpha z)$

D. Isotropic

The displacements, strains, and stresses are obtained by making the following substitutions into the equations for the case of Linear Radial Shear Stress loading (Cross-anisotropic; β^2 Zero). These equations are shown on page 20.

(i) substitute $-\delta_1.\left(\frac{2}{\pi}\right)^{\frac{1}{2}}.\frac{2}{r_0}.\frac{E}{2.(1-\nu^2)}$ ---------------- for p_1

(ii) substitute $I_{3\tau(\mu+1)}(z)$ ---------------- for $I_{4\tau\mu}(z)$

(iii) In addition the following simplifications apply with regard to α, f, and the coefficients t_n;

$\alpha = 1$	$f = a-b = E.(1+\nu)^{-1}$	$t_1 = (1+\nu).(1-2\nu).E^{-1}$	$t_2 = (1+\nu).E^{-1}$
$t_3 = 2.(1-\nu^2).E^{-1}$	$t_4 = (1+\nu).E^{-1}$	$t_5 = 2$	$t_6 = 1$
$t_7 = 2\nu$	$t_8 = 0$	$t_9 = 2\ .(1+\nu).E^{-1}$	

3(b) LINEAR RADIAL SHEAR DISPLACEMENT

When $r \neq 0$ and $z \neq 0$ the values of the integrals are given by;

$$I_{301}(\psi) = (\tfrac{2}{\pi})^{\frac{1}{2}}.(N_0.\sin\tfrac{1}{2}\lambda_0-\psi.\sigma_0)$$

$$I_{303}(\psi) = (\tfrac{2}{\pi})^{\frac{1}{2}}.(\sigma_0-N_0^{-1}.\cos\tfrac{1}{2}\lambda_0)$$

$$I_{305}(\psi) = (\tfrac{2}{\pi})^{\frac{1}{2}}.N_0^{-3}.(N_0^2.\sin\tfrac{1}{2}\lambda_0-\sin\tfrac{3}{2}\lambda_0-\psi.\cos\tfrac{3}{2}\lambda_0)$$

$$I_{321}(\psi) = (\tfrac{2}{\pi})^{\frac{1}{2}}.\tfrac{1}{2}.r^{-1}.(r^2.\sigma_0+\psi.N_0.\sin\tfrac{1}{2}\lambda_0-N_0.\cos\tfrac{1}{2}\lambda_0)$$

$$I_{323}(\psi) = (\tfrac{2}{\pi})^{\frac{1}{2}}.r^{-1}.N_0^{-1}.\{\psi.\cos\tfrac{1}{2}\lambda_0+(1-N_0^2).\sin\tfrac{1}{2}\lambda_0\}$$

$$I_{325}(\psi) = (\tfrac{2}{\pi})^{\frac{1}{2}}.r^{-1}.N_0^{-3}.\{N_0^2(\cos\tfrac{1}{2}\lambda_0-\psi.\sin\tfrac{1}{2}\lambda_0)-r^2.\cos\tfrac{3}{2}\lambda_0\}$$

$${}_cI_{301} = (\tfrac{2}{\pi})^{\frac{1}{2}}.\tfrac{1}{2}.\{+N_p.\sin\tfrac{1}{2}\lambda_p+N_m.\sin\tfrac{1}{2}\lambda_m-\alpha z.(\sigma_p+\sigma_m)-\omega z(\ln L_p-\ln L_m)\}$$

$${}_sI_{301} = (\tfrac{2}{\pi})^{\frac{1}{2}}.\tfrac{1}{2}.\{+N_p.\cos\tfrac{1}{2}\lambda_p-N_m.\cos\tfrac{1}{2}\lambda_m+\omega z.(\sigma_p+\sigma_m)-\alpha z(\ln L_p-\ln L_m)\}$$

$${}_cI_{303} = (\tfrac{2}{\pi})^{\frac{1}{2}}.\tfrac{1}{2}.(-N_p^{-1}.\cos\tfrac{1}{2}\lambda_p-N_m^{-1}.\cos\tfrac{1}{2}\lambda_m+\sigma_p+\sigma_m)$$

$${}_sI_{303} = (\tfrac{2}{\pi})^{\frac{1}{2}}.\tfrac{1}{2}.(-N_p^{-1}.\sin\tfrac{1}{2}\lambda_p+N_m^{-1}.\sin\tfrac{1}{2}\lambda_m+\ln L_p-\ln L_m)$$

$${}_cI_{321} = (\tfrac{2}{\pi})^{\frac{1}{2}}.\tfrac{1}{4}.r^{-1}.\{-(1-\omega z).N_p.\cos\tfrac{1}{2}\lambda_p-(1+\omega z).N_m.\cos\tfrac{1}{2}\lambda_m+\alpha z.(N_p.\sin\tfrac{1}{2}\lambda_p+N_m.\sin\tfrac{1}{2}\lambda_m)+r^2.\sigma_p+r^2.\sigma_m\}$$

$${}_sI_{321} = (\tfrac{2}{\pi})^{\frac{1}{2}}.\tfrac{1}{4}.r^{-1}.\{+(1-\omega z).N_p.\sin\tfrac{1}{2}\lambda_p-(1+\omega z).N_m.\sin\tfrac{1}{2}\lambda_m+\alpha z.(N_p.\cos\tfrac{1}{2}\lambda_p-N_m.\cos\tfrac{1}{2}\lambda_m)+r^2.(\ln L_p-\ln L_m)\}$$

$${}_eI_{323} = (\tfrac{2}{\pi})^{\frac{1}{2}}.\tfrac{1}{2}.r^{-1}.[+\{(1+\omega z).N_p^{-1}-N_p\}.\sin\tfrac{1}{2}\lambda_p+\{(1-\omega z).N_m^{-1}-N_m\}.\sin\tfrac{1}{2}\lambda_m+\alpha z.(N_p^{-1}.\cos\tfrac{1}{2}\lambda_p+N_m^{-1}.\cos\tfrac{1}{2}\lambda_m)]$$

$${}_sI_{323} = (\tfrac{2}{\pi})^{\frac{1}{2}}.\tfrac{1}{2}.r^{-1}.[-\{(1+\omega z).N_p^{-1}+N_p\}.\cos\tfrac{1}{2}\lambda_p+\{(1-\omega z).N_m^{-1}+N_m\}.\cos\tfrac{1}{2}\lambda_m+\alpha z.(N_p^{-1}.\sin\tfrac{1}{2}\lambda_p-N_m^{-1}.\sin\tfrac{1}{2}\lambda_m)]$$

where σ_0, σ_p, σ_m, N_0, N_p, N_m, λ_0, λ_p, λ_m, L_p, L_m are given on page 15

When $r = 0$ the values of the integrals are given by;

$$I_{301}(\psi) = (\tfrac{2}{\pi})^{\frac{1}{2}}.(1-\psi.\mathrm{arcot}\psi) \qquad I_{303}(\psi) = (\tfrac{2}{\pi})^{\frac{1}{2}}\{\mathrm{arcot}\psi - \psi.(\psi^2+1)^{-1}\} \qquad I_{305}(\psi) = 2.(\psi^2+1)^{-2}$$

$${}_cI_{301} = (\tfrac{2}{\pi})^{\frac{1}{2}}.\tfrac{1}{2}.\{-\alpha z.\mathrm{artan}(\tfrac{1+\omega z}{\alpha z}) - \alpha z.\mathrm{artan}(\tfrac{1-\omega z}{\alpha z}) + \tfrac{\omega z}{2}.\ln[\{\alpha^2 z^2 + (1-\omega z)^2\}.\{\alpha^2 z^2 + (1+\omega z)^2\}^{-1}] + 2\}$$

$${}_sI_{301} = (\tfrac{2}{\pi})^{\frac{1}{2}}.\tfrac{1}{2}.\{+\omega z.\mathrm{artan}(\tfrac{1+\omega z}{\alpha z}) + \omega z.\mathrm{artan}(\tfrac{1-\omega z}{\alpha z}) + \tfrac{\alpha z}{2}.\ln[\{\alpha^2 z^2 + (1-\omega z)^2\}.\{\alpha^2 z^2 + (1+\omega z)^2\}^{-1}]\}$$

$${}_cI_{303} = (\tfrac{2}{\pi})^{\frac{1}{2}}.\tfrac{1}{2}.[+\mathrm{artan}(\tfrac{1+\omega z}{\alpha z}) + \mathrm{artan}(\tfrac{1-\omega z}{\alpha z}) - \alpha z.\{\alpha^2 z^2 + (1+\omega z)^2\}^{-1} - \alpha z.\{\alpha^2 z^2 + (1-\omega z)^2\}^{-1}]$$

$${}_sI_{303} = (\tfrac{2}{\pi})^{\frac{1}{2}}.\tfrac{1}{2}.\{\tfrac{1}{2}.\ln[\{\alpha^2 z^2 + (1+\omega z)^2\}.\{\alpha^2 z^2 + (1-\omega z)^2\}^{-1}] - (1+\omega z).\{\alpha^2 z^2 + (1+\omega z)^2\}^{-1}$$
$$+ (1-\omega z).\{\alpha^2 z^2 + (1-\omega z)^2\}^{-1}\}$$

$$I_{321}(\psi) = I_{323}(\psi) = I_{325}(\psi) = {}_cI_{321} = {}_cI_{323} = {}_sI_{321} = {}_sI_{323} = 0$$

$$r^{-1}.I_{321}(\psi) = \tfrac{1}{2}.I_{303}(\psi) \qquad r^{-1}.I_{323}(\psi) = \tfrac{1}{2}.I_{305}(\psi)$$

$$r^{-1}.{}_cI_{321} = \tfrac{1}{2}.{}_cI_{303} \qquad r^{-1}.{}_sI_{321} = \tfrac{1}{2}.{}_sI_{303}$$

When $z = 0$ the relevant integrals are given by;

$$I_{301}(0) = {}_cI_{301} = (\tfrac{2}{\pi})^{\frac{1}{2}}\begin{cases}(1-r^2)^{\frac{1}{2}} & \\ 0 & \end{cases} \qquad I_{303}(0) = {}_cI_{303} = (\tfrac{2}{\pi})^{\frac{1}{2}}\begin{cases}\tfrac{1}{2}.\pi & r < 1 \\ \mathrm{arsin}r^{-1} - (r^2-1)^{-\frac{1}{2}} & r > 1\end{cases}$$

$$I_{321}(0) = {}_cI_{321} = (\tfrac{2}{\pi})^{\frac{1}{2}}\begin{cases}\tfrac{1}{4}.\pi.r & \\ \tfrac{1}{2}.\{r.\mathrm{arsin}r^{-1} - (r^2-1)^{\frac{1}{2}}.r^{-1}\} & \end{cases} \qquad I_{323}(0) = {}_cI_{323} = (\tfrac{2}{\pi})^{\frac{1}{2}}\begin{cases}r.(1-r^2)^{-\frac{1}{2}} & r < 1 \\ 0 & r > 1\end{cases}$$

$${}_sI_{301} = {}_sI_{303} = {}_sI_{321} = {}_sI_{323} = 0$$

4(a) LINEAR TORSIONAL SHEAR STRESS

Cross-anisotropic *

$$v = -p_2 . 2r_0 . f^{-1} . \gamma^{-1} . I_{420}(\gamma z)$$

$$\varepsilon_{\theta z} = p_2 . f^{-1} . I_{422}(\gamma z)$$

$$\varepsilon_{r\theta} = p_2 . f^{-1} . \gamma^{-1} . I_{442}(\gamma z)$$

$$\widehat{\theta z} = p_2 . I_{422}(\gamma z)$$

$$\widehat{r\theta} = p_2 . \gamma . I_{442}(\gamma z)$$

D. Isotropic

$$v = -p_2 . 2r_0 . (1+\nu) . E^{-1} . I_{420}(z)$$

$$\varepsilon_{\theta z} = p_2 . (1+\nu) . E^{-1} . I_{422}(z)$$

$$\varepsilon_{r\theta} = p_2 . (1+\nu) . E^{-1} . I_{442}(z)$$

$$\widehat{\theta z} = p_2 . I_{422}(z)$$

$$\widehat{r\theta} = p_2 . I_{442}(z)$$

Values of Integrals $I_{4\tau\mu}(\gamma z)$

When $r \neq 0$ and $z \neq 0$ the values of the integrals are given in tables on pages indicated in brackets.

$I_{420}(\psi)$ [51] $\quad$ $I_{422}(\psi)$ [51] $\quad$ $I_{442}(\psi)$ [51]

When $z = 0$ the integrals are given by;

$$I_{420}(0) = \begin{cases} \frac{1}{2} . r . {}_2F_1(\frac{3}{2}, -\frac{1}{2}; 2; r^2) & r < 1 \\ \frac{2}{3} . \pi^{-1} & r = 1 \\ \frac{1}{8} . r^{-2} . {}_2F_1(\frac{3}{2}, \frac{1}{2}; 3; r^{-2}) & r > 1 \end{cases}$$

$$I_{422}(0) = \begin{cases} r & r < 1 \\ \frac{1}{2} & r = 1 \\ 0 & r > 1 \end{cases}$$

$$I_{442}(0) = \begin{cases} \frac{3}{8} . r^2 . {}_2F_1(\frac{5}{2}, \frac{1}{2}; 3; r^2) & r < 1 \\ \infty & r = 1 \\ \frac{3}{8} . r^{-3} . {}_2F_1(\frac{5}{2}, \frac{1}{2}; 3; r^{-2}) & r > 1 \end{cases}$$

* The solutions for the Cross-anisotropic cases on pages 24 and 25 are independent of the value of β^2

4(b) LINEAR TORSIONAL SHEAR DISPLACEMENT

Cross-anisotropic *

$v = (\frac{2}{\pi})^{\frac{1}{2}}.2\delta_2.I_{321}(\gamma z)$

$\varepsilon_{\theta z} = -(\frac{2}{\pi})^{\frac{1}{2}}.\delta_2.\gamma.r_0^{-1}.I_{323}(\gamma z)$

$\varepsilon_{r\theta} = -(\frac{2}{\pi})^{\frac{1}{2}}.\delta_2.r_0^{-1}.I_{343}(\gamma z)$

$\widehat{\theta z} = -(\frac{2}{\pi})^{\frac{1}{2}}.\delta_2.f.\gamma.r_0^{-1}.I_{323}(\gamma z)$

$\widehat{r\theta} = -(\frac{2}{\pi})^{\frac{1}{2}}.\delta_2.(a-b).r_0^{-1}.I_{343}(\gamma z)$

The torsion-displacement relationship is;

$T_t = -\delta_2.\frac{8r_0^2}{3}.f\gamma$

D. Isotropic

$v = (\frac{2}{\pi})^{\frac{1}{2}}.2\delta_2.I_{321}(z)$

$\varepsilon_{\theta z} = -(\frac{2}{\pi})^{\frac{1}{2}}.\delta_2.r_0^{-1}.I_{323}(z)$

$\varepsilon_{r\theta} = -(\frac{2}{\pi})^{\frac{1}{2}}.\delta_2.r_0^{-1}.I_{343}(z)$

$\widehat{\theta z} = -(\frac{2}{\pi})^{\frac{1}{2}}.\delta_2.E.(1+\nu)^{-1}.r_0^{-1}.I_{323}(z)$

$\widehat{r\theta} = -(\frac{2}{\pi})^{\frac{1}{2}}.\delta_2.E.(1+\nu)^{-1}.r_0^{-1}.I_{343}(z)$

The torsion-displacement relationship is;

$T_t = -\delta_2.\frac{8r_0^2}{3}.\frac{E}{1+\nu}$

Values of Integrals $I_{3\tau\mu}(\psi)$

When $r \neq 0$ and $z \neq 0$ the integrals are given by;

$I_{321}(\psi) = (\frac{2}{\pi})^{\frac{1}{2}}.\{\frac{1}{2}.r.\sigma_0 + \frac{1}{2}.N_0.(\psi\sin\frac{1}{2}\lambda_0 - \cos\frac{1}{2}\lambda_0).r^{-1}\}$

$I_{323}(\psi) = (\frac{2}{\pi})^{\frac{1}{2}}.\{\psi.\cos\frac{1}{2}\lambda_0 + (1-N_0^2).\sin\frac{1}{2}\lambda_0\}.r^{-1}.N_0^{-1}$

$I_{343}(\psi) = (\frac{2}{\pi})^{\frac{1}{2}}.\{(N_0^2-2).\psi.\sin\frac{1}{2}\lambda_0 - (\psi^2-1).\cos\frac{1}{2}\lambda_0\}.r^{-2}.N_0^{-1}$

where

$\sigma_0 = \text{arsin}\{2.[\{\psi^2 + (1+r)^2\}^{\frac{1}{2}} + \{\psi^2 + (1-r)^2\}^{\frac{1}{2}}]^{-1}\}$

$N_0 = \{(\psi^2+r^2-1)^2 + 4\psi^2\}^{\frac{1}{4}}$

$\lambda_0 = \text{artan}\{2\psi.(\psi^2+r^2-1)^{-1}$

When $r = 0$ the integrals are given by;

$I_{321} = I_{323} = I_{343} = 0$

When $z = 0$ the integrals are given by;

$$I_{321}(0) = (\frac{2}{\pi})^{\frac{1}{2}} \begin{cases} \frac{1}{4}.\pi.r & r < 1 \\ \frac{1}{4}.\pi & r = 1 \\ \frac{1}{2}.\{r.\text{arsin}\frac{1}{r} - (r^2-1)^{\frac{1}{2}}.r^{-1}\} & r > 1 \end{cases}$$

$$I_{323}(0) = (\frac{2}{\pi})^{\frac{1}{2}} \begin{cases} r.(1-r^2)^{-\frac{1}{2}} & r < 1 \\ \infty & r = 1 \\ 0 & r > 1 \end{cases}$$

$$I_{343}(0) = (\frac{2}{\pi})^{\frac{1}{2}} \begin{cases} 0 & r < 1 \\ \infty & r = 1 \\ r^{-2}.(r^2-1)^{-\frac{1}{2}} & r > 1 \end{cases}$$

5(a) UNIFORM UNIDIRECTIONAL SHEAR STRESS

A. Cross-anisotropic; β^2 Positive

$$w = p_3.\cos\theta.r_0.\{-h_1.I_{220}(\phi z) + h_2.I_{220}(\rho z)\}$$

$$u = p_3.\cos\theta.r_0.\{-h_3.M_{220}(\phi z) + h_4.M_{220}(\rho z) - h_{10}.A_{220}(\gamma z)\}$$

$$v = p_3.\sin\theta.r_0.\{-h_3.A_{220}(\phi z) + h_4.A_{220}(\rho z) - h_{10}.M_{220}(\gamma z)\}$$

$$\varepsilon_{zz} = p_3.\cos\theta.\{h_1.\phi.I_{222}(\phi z) - h_2.\rho.I_{222}(\rho z)\}$$

$$\varepsilon_{rr} = p_3.\cos\theta.[-h_3.\{I_{222}(\phi z) - r^{-1}.I_{240}(\phi z)\} + h_4.\{I_{222}(\rho z) - r^{-1}.I_{240}(\rho z)\} + h_{10}.r^{-1}.I_{240}(\gamma z)]$$

$$\varepsilon_{\theta\theta} = p_3.\cos\theta.\{-h_3.r^{-1}.I_{240}(\phi z) + h_4.r^{-1}.I_{240}(\rho z) - h_{10}.r^{-1}.I_{240}(\gamma z)\}$$

$$\varepsilon_{rz} = p_3.\cos\theta.f^{-1}.\{h_9.\rho^{-1}.M_{222}(\phi z) - h_9.\phi^{-1}.M_{222}(\rho z) + A_{222}(\gamma z)\}$$

$$\varepsilon_{\theta z} = p_3.\sin\theta.f^{-1}.\{h_9.\rho^{-1}.A_{222}(\phi z) - h_9.\phi^{-1}.A_{222}(\rho z) + M_{222}(\gamma z)\}$$

$$\varepsilon_{r\theta} = p_3.\sin\theta.[h_3.r^{-1}.I_{240}(\phi z) - h_4.r^{-1}.I_{240}(\rho z) + h_{10}.\{r^{-1}.I_{240}(\gamma z) - \tfrac{1}{2}.I_{222}(\gamma z)\}]$$

$$\widehat{rr} = p_3.\cos\theta.[-h_5.I_{222}(\phi z) + h_6.I_{222}(\rho z) + (a-b).r^{-1}.\{h_3.I_{240}(\phi z) - h_4.I_{240}(\rho z) + h_{10}.I_{240}(\gamma z)\}]$$

$$\widehat{\theta\theta} = p_3.\cos\theta.[-h_7.I_{222}(\phi z) + h_8.I_{222}(\rho z) - (a-b).r^{-1}.\{h_3.I_{240}(\phi z) - h_4.I_{240}(\rho z) + h_{10}.I_{240}(\gamma z)\}]$$

$$\widehat{zz} = p_3.\cos\theta.h_9.\rho^{-1}.\phi^{-1}.\{I_{222}(\phi z) - I_{222}(\rho z)\} \qquad \widehat{rz} = f.\varepsilon_{rz} \qquad \widehat{\theta z} = f.\varepsilon_{\theta z} \qquad \widehat{r\theta} = (a-b).\varepsilon_{r\theta}$$

B. Cross-anisotropic; β^2 Negative

$$w = p_3.\cos\theta.r_0.(-j_1.{}_cI_{220} - j_2.{}_sI_{220})$$

$$u = p_3.\cos\theta.r_0.\{-j_3.{}_cM_{220} - j_4.{}_sM_{220} - j_{11}.A_{220}(\gamma z)\}$$

$$v = p_3.\sin\theta.r_0.\{-j_3.{}_cA_{220} - j_4.{}_sA_{220} - j_{11}.M_{220}(\gamma z)\}$$

$$\varepsilon_{zz} = p_3.\cos\theta.(+j_9.{}_cI_{222} + j_{10}.{}_sI_{222})$$

$$\varepsilon_{rr} = p_3.\cos\theta.\{-j_3.({}_cI_{222} - r^{-1}.{}_cI_{240}) - j_4.({}_sI_{222} - r^{-1}.{}_sI_{240}) + j_{11}.r^{-1}.I_{240}(\gamma z)\}$$

$$\varepsilon_{\theta\theta} = p_3.\cos\theta.\{-j_3.r^{-1}.{}_cI_{240} - j_4.r^{-1}.{}_sI_{240} - j_{11}.r^{-1}.I_{240}(\gamma z)\}$$

$$\varepsilon_{rz} = p_3.\cos\theta.f^{-1}.\{-{}_cM_{222} + \alpha.\omega^{-1}.{}_sM_{222} + A_{222}(\gamma z)\}$$

$$\varepsilon_{\theta z} = p_3.\sin\theta.f^{-1}.\{-{}_cA_{222} + \alpha.\omega^{-1}.{}_sA_{222} + M_{222}(\gamma z)\}$$

$$\varepsilon_{r\theta} = p_3.\sin\theta.[j_3.r^{-1}.{}_cI_{240} + j_4.r^{-1}.{}_sI_{240} + j_{11}.\{r^{-1}.I_{240}(\gamma z) - \tfrac{1}{2}.I_{222}(\gamma z)\}]$$

$$\widehat{rr} = p_3.\cos\theta.[-j_5.{}_cI_{222} - j_6.{}_sI_{222} + (a-b).r^{-1}.\{j_3.{}_cI_{240} + j_4.{}_sI_{240} + j_{11}.I_{240}(\gamma z)\}]$$

$$\widehat{\theta\theta} = p_3.\cos\theta.[-j_7.{}_cI_{222} - j_8.{}_sI_{222} - (a-b).r^{-1}.\{j_3.{}_cI_{240} + j_4.{}_sI_{240} + j_{11}.I_{240}(\gamma z)\}]$$

$$\widehat{zz} = p_3.\cos\theta.\omega^{-1}.{}_sI_{222} \qquad \widehat{rz} = f.\varepsilon_{rz} \qquad \widehat{\theta z} = f.\varepsilon_{\theta z} \qquad \widehat{r\theta} = (a-b).\varepsilon_{r\theta}$$

C. Cross-anisotropic; β^2 Zero

$$w = p_3.\cos\theta.r_0.\{-t_1.I_{220}(\alpha z) - t_2.z.I_{222}(\alpha z)\}$$

$$u = p_3.\cos\theta.r_0.\{t_3.M_{220}(\alpha z) - t_4.z.M_{222}(\alpha z) - t_{10}.A_{220}(\gamma z)\}$$

$$v = p_3.\sin\theta.r_0.\{t_3.A_{220}(\alpha z) - t_4.z.A_{222}(\alpha z) - t_{10}.M_{220}(\gamma z)\}$$

$$\varepsilon_{zz} = p_3.\cos\theta.\{-t_9.I_{222}(\alpha z) + t_2.\alpha.z.I_{224}(\alpha z)\}$$

$$\varepsilon_{rr} = p_3.\cos\theta.[t_3.\{I_{222}(\alpha z) - r^{-1}.I_{240}(\alpha z)\} - t_4.z.\{I_{224}(\alpha z) - r^{-1}.I_{242}(\alpha z)\} + t_{10}.r^{-1}.I_{240}(\gamma z)]$$

$$\varepsilon_{\theta\theta} = p_3.\cos\theta.\{t_3.r^{-1}.I_{240}(\alpha z) - t_4.z.r^{-1}.I_{242}(\alpha z) - t_{10}.r^{-1}.I_{240}(\gamma z)\}$$

$$\varepsilon_{rz} = p_3.\cos\theta.f^{-1}.\{-M_{222}(\alpha z) + \alpha.z.M_{224}(\alpha z) + A_{222}(\gamma z)\}$$

$$\varepsilon_{\theta z} = p_3.\sin\theta.f^{-1}.\{-A_{222}(\alpha z) + \alpha.z.A_{224}(\alpha z) + M_{222}(\gamma z)\}$$

$$\varepsilon_{r\theta} = p_3.\sin\theta.[-t_3.r^{-1}.I_{240}(\alpha z) + t_4.z.r^{-1}.I_{242}(\alpha z) + t_{10}.\{r^{-1}.I_{240}(\gamma z) - \tfrac{1}{2}.I_{222}(\gamma z)\}]$$

$$\widehat{rr} = p_3.\cos\theta.[t_5.I_{222}(\alpha z) - t_6.z.I_{224}(\alpha z) - (a-b).r^{-1}.\{t_3.I_{240}(\alpha z) - t_4.z.I_{242}(\alpha z) - t_{10}.I_{240}(\gamma z)\}]$$

$$\widehat{\theta\theta} = p_3.\cos\theta.[t_7.I_{222}(\alpha z) - t_8.z.I_{224}(\alpha z) + (a-b).r^{-1}.\{t_3.I_{240}(\alpha z) - t_4.z.I_{242}(\alpha z) - t_{10}.I_{240}(\gamma z)\}]$$

$$\widehat{zz} = p_3.\cos\theta.z.I_{224}(\alpha z) \qquad \widehat{rz} = f.\varepsilon_{rz} \qquad \widehat{\theta z} = f.\varepsilon_{\theta z} \qquad \widehat{r\theta} = (a-b).\varepsilon_{r\theta}$$

5(a) UNIFORM UNIDIRECTIONAL SHEAR STRESS

D. Isotropic

The stresses, strains, and displacements are as for the case of Cross-anisotropic; β^2 Zero, but with the following simplifications;

$t_1 = (1+\nu).(1-2\nu).E^{-1}$ $\quad t_2 = (1+\nu).E^{-1}$ $\quad t_3 = 2.(1-\nu^2).E^{-1}$ $\quad t_4 = (1+\nu).E^{-1}$

$t_5 = 2$ $\quad t_6 = 1$ $\quad t_7 = 2\nu$ $\quad t_8 = 0$

$t_9 = 2\nu.(1+\nu).E^{-1}$ $\quad t_{10} = 2.(1+\nu).E^{-1}$ $\quad \alpha = 1$ $\quad \gamma = 1$

$f = (a-b) = E.(1+\nu)^{-1}$

Values of coefficients h_n, t_n, j_n

The values of $h_1 \ldots h_9$, $t_1 \ldots t_9$, $j_1 \ldots j_{10}$ have been given previously on page 21. In addition

$h_{10} = t_{10} = j_{11} = 2.f^{-1}.\gamma^{-1}$

Values of Integrals $I_{2\tau\mu}(\psi)$, ${}_cI_{2\tau\mu}$, ${}_sI_{2\tau\mu}$, $A_{2\tau\mu}(\psi)$, ${}_cA_{2\tau\mu}$, ${}_sA_{2\tau\mu}$, $M_{2\tau\mu}(\psi)$, ${}_cM_{2\tau\mu}$, ${}_sM_{2\tau\mu}$

When $r \neq 0$ and $z \neq 0$ the values of the integrals are given in tables on the page numbers indicated in brackets;

$I_{220}(\psi)$	[37]	${}_cI_{220}$	[38]	$A_{220}(\psi)$	[42]	${}_cA_{220}$	[43]	${}_cM_{220}$	[46]
$I_{222}(\psi)$	[37]	${}_sI_{220}$	[38]	$A_{222}(\psi)$	[42]	${}_sA_{220}$	[43]	${}_sM_{220}$	[46]
$I_{224}(\psi)$	[37]	${}_cI_{222}$	[39]	$A_{224}(\psi)$	[42]	${}_cA_{222}$	[44]	${}_cM_{222}$	[47]
$I_{240}(\psi)$	[40]	${}_sI_{222}$	[39]	$M_{220}(\psi)$	[45]	${}_sA_{222}$	[44]	${}_sM_{222}$	[47]
$I_{242}(\psi)$	[40]	${}_cI_{240}$	[41]	$M_{222}(\psi)$	[45]				
		${}_sI_{240}$	[41]	$M_{224}(\psi)$	[45]				

When $r = 0$ the integrals are given by;

$$I_{220}(\psi) = I_{222}(\psi) = I_{224}(\psi) = I_{240}(\psi) = I_{242}(\psi) = r^{-1}.I_{240}(\psi) = r^{-1}.I_{242}(\psi) = 0$$

$${}_cI_{220} = {}_sI_{220} = {}_cI_{222} = {}_sI_{222} = {}_cI_{240} = {}_sI_{240} = r^{-1}.{}_cI_{240} = r^{-1}.{}_sI_{240} = 0$$

$$A_{220}(\psi) = -M_{220}(\psi) = \tfrac{1}{2}.\{(\psi^2+1)^{\frac{1}{2}}-\psi\} \qquad A_{222}(\psi) = -M_{222}(\psi) = \tfrac{1}{2}.\{1-\psi.(\psi^2+1)^{-\frac{1}{2}}\}$$

$$A_{224}(\psi) = -M_{224}(\psi) = \tfrac{1}{2}.(\psi^2+1)^{-\frac{1}{2}}$$

$${}_cA_{220} = -{}_cM_{220} = \tfrac{1}{2}.(R^{\frac{1}{2}}.\cos\tfrac{\upsilon}{2} - \alpha.z) \qquad {}_sA_{220} = -{}_sM_{220} = \tfrac{1}{2}.(-R^{\frac{1}{2}}.\sin\tfrac{\upsilon}{2} + \omega z)$$

$${}_cA_{222} = -{}_cM_{222} = \tfrac{1}{2}.(R^{\frac{1}{2}}-\alpha.z.\cos\tfrac{\upsilon}{2} - \omega.z.\sin\tfrac{\upsilon}{2}).R^{-\frac{1}{2}} \qquad {}_sA_{222} = -{}_sM_{222} = \tfrac{1}{2}.(-\alpha+z.\sin\tfrac{\upsilon}{2}+\omega.z.\cos\tfrac{\upsilon}{2}).R^{-\frac{1}{2}}$$

where

$$R = [\{(\alpha^2-\omega^2)z^2+1\}^2 + 4\alpha^2\omega^2z^4]^{\frac{1}{2}}$$

$$\upsilon = \operatorname{artan}[2\alpha\omega z^2.\{(\alpha^2-\omega^2)z^2+1\}^{-1}]$$

When $z = 0$ the relevant integrals are given by;

$$I_{220}(0) = {}_cI_{220} = \begin{cases} \tfrac{1}{2}.r \\ \tfrac{1}{2} \\ \tfrac{1}{2}.r^{-1} \end{cases} \qquad I_{222}(0) = {}_cI_{222} = \begin{cases} \tfrac{1}{2}.r.{}_2F_1(\tfrac{3}{2},\tfrac{1}{2};2;r^2) \\ \infty \\ \tfrac{1}{2}.r^{-2}.{}_2F_1(\tfrac{3}{2},\tfrac{1}{2};2;r^{-2}) \end{cases} \qquad I_{240}(0) = {}_cI_{240} = \begin{cases} \tfrac{1}{8}.r^2.{}_2F_1(\tfrac{3}{2},\tfrac{1}{2};3;r^2) & r < 1 \\ \tfrac{2}{3}.\pi^{-1} & r = 1 \\ \tfrac{1}{2}.r^{-1}.{}_2F_1(\tfrac{3}{2},-\tfrac{1}{2};2;r^{-2}) & r > 1 \end{cases}$$

$$A_{220}(0) = {}_cA_{220} = \begin{cases} \tfrac{1}{2}.{}_2F_1(\tfrac{1}{2},-\tfrac{1}{2};2;r^2) \\ \tfrac{4}{3}.\pi^{-1} \\ \tfrac{1}{2}.r^{-1}.{}_2F_1(\tfrac{1}{2},-\tfrac{1}{2};2;r^{-2}) \end{cases} \qquad A_{222}(0) = {}_cA_{222} = \begin{cases} \tfrac{1}{2} & r < 1 \\ \tfrac{1}{2} & r = 1 \\ \tfrac{1}{2}.r^{-2} & r > 1 \end{cases}$$

$$M_{220}(0) = {}_cM_{220} = \begin{cases} -\tfrac{1}{2}.{}_2F_1(\tfrac{3}{2},-\tfrac{1}{2};2;r^2) \\ -\tfrac{2}{3}.\pi^{-1} \\ -\tfrac{1}{8}.r^{-3}.{}_2F_1(\tfrac{3}{2},\tfrac{1}{2};3;r^{-2}) \end{cases} \qquad M_{222}(0) = {}_cM_{222} = \begin{cases} -\tfrac{1}{2} & r < 1 \\ 0 & r = 1 \\ \tfrac{1}{2}.r^{-2} & r > 1 \end{cases}$$

$${}_sA_{220} = {}_sA_{222} = {}_sM_{220} = {}_sM_{222} = 0$$

5(b) UNIFORM UNIDIRECTIONAL SHEAR DISPLACEMENT

A. Cross-anisotropic; β^2 Positive

The displacements, strains, and stresses are obtained by making the following substitutions into the equations shown on page 26 for a Uniform Unidirectional Shear Stress loading (Cross-anisotropic; β^2 Positive).

(i) substitute $T_h.(\frac{2}{\pi})^{\frac{1}{2}}.\frac{1}{4r_0^2}$ OR $-\delta_3.(\frac{2}{\pi})^{\frac{1}{2}}.\frac{2}{r_0}.\frac{f.\gamma.(c+d\rho^2).(c+d\phi^2)}{(2c+f).d.\gamma.(\rho+\phi)+2.(c+d\phi^2).(c+d\rho^2)}$ --------- for p_3

This alternative substitution follows from the load-displacement relationship;

$$T_h = -\delta_3.8r_0.\frac{f.\gamma.(c+d\rho^2).(c+d\phi^2)}{(2c+f).d.\gamma.(\rho+\phi)+2.(c+d\phi^2).(c+d\rho^2)}$$

(ii) substitute $I_{1\tau(\mu+1)}(\psi)$ ---------- for $I_{2\tau\mu}(\psi)$

$A_{1\tau(\mu+1)}(\psi)$ ---------- for $A_{2\tau\mu}(\psi)$

$M_{1\tau(\mu+1)}(\psi)$ ---------- for $M_{2\tau\mu}(\psi)$

B. Cross-anisotropic; β^2 Negative

The displacements, strains, and stresses are obtained by making the following substitutions into the equations shown on page 26 for a Uniform Unidirectional Shear Stress loading (Cross-anisotropic; β^2 Negative).

(i) substitute $T_h.(\frac{2}{\pi})^{\frac{1}{2}}.\frac{1}{4r_0^2}$ OR $-\delta_3.(\frac{2}{\pi})^{\frac{1}{2}}.\frac{2}{r_0}.\frac{f.\gamma.(ad-c^2)}{2fd\alpha\gamma+2(ad-c^2)}$ ---------- for p_3

This alternative substitution follows from the load-displacement relationship;

$$T_h = -\delta_3.8r_0.\frac{f.\gamma.(ad-c^2)}{2fd\alpha\gamma+2(ad-c^2)}$$

(ii) substitute ${}_cI_{1\tau(\mu+1)}$ ---------- for ${}_cI_{2\tau\mu}$

${}_sI_{1\tau(\mu+1)}$ ---------- for ${}_sI_{2\tau\mu}$

${}_cA_{1\tau(\mu+1)}$ ---------- for ${}_cA_{2\tau\mu}$

${}_sA_{1\tau(\mu+1)}$ ---------- for ${}_sA_{2\tau\mu}$

${}_cM_{1\tau(\mu+1)}$ ---------- for ${}_cM_{2\tau\mu}$

${}_sM_{1\tau(\mu+1)}$ ---------- for ${}_sM_{2\tau\mu}$

C. Cross-anisotropic; β^2 Zero

The displacements, strains, and stresses are obtained by making the following substitutions into the equations shown on page 26 for a Uniform Unidirectional Shear Stress loading (Cross-anisotropic; β^2 Zero).

(i) substitute $T_h.(\frac{2}{\pi})^{\frac{1}{2}}.\frac{1}{4r_0^2}$ OR $-\delta_3.(\frac{2}{\pi})^{\frac{1}{2}}.\frac{2}{r_0}.\frac{f.\gamma.(c+d\alpha^2)^2}{2d\alpha\gamma(2c+f)+2(c+d\alpha^2)^2}$ ---------- for p_3

This alternative substitution follows from the load-displacement relationship;

$$T_h = -\delta_3.8r_0.\frac{f.\gamma.(c+d\alpha^2)^2}{2d\alpha\gamma(2c+f)+2(c+d\alpha^2)^2}$$

(ii) substitute $I_{1\tau(\mu+1)}(\psi)$ ---------- for $I_{2\tau\mu}(\psi)$

$A_{1\tau(\mu+1)}(\psi)$ ---------- for $A_{2\tau\mu}(\psi)$

$M_{1\tau(\mu+1)}(\psi)$ ---------- for $M_{2\tau\mu}(\psi)$

D. Isotropic

The displacements, strains, and stresses are obtained by making the following substitutions into the equations shown on page 26 for a Uniform Unidirectional Shear Stress loading (Cross-anisotropic; β^2 Zero).

(i) substitute $T_h.(\frac{2}{\pi})^{\frac{1}{2}}.\frac{1}{4r_0^2}$ OR $-\delta_3.(\frac{2}{\pi})^{\frac{1}{2}}.\frac{2}{r_0}.\frac{E}{2.(1+\nu).(2-\nu)}$ ---------- for p_3

This alternative substitution follows from the load displacement relationship;

$$T_h = -\delta_3.8r_0.\frac{E}{2(1+\nu).(2-\nu)}$$

(ii) substitute $I_{1\tau(\mu+1)}(z)$ ---------- for $I_{2\tau\mu}(z)$

$A_{1\tau(\mu+1)}(z)$ ---------- for $A_{2\tau\mu}(z)$

$M_{1\tau(\mu+1)}(z)$ ---------- for $M_{2\tau\mu}(z)$

(iii) In addition the following simplifications apply with regard to α, γ, f, $(a-b)$, and the coefficients t_{π};

$\alpha = 1$ $\gamma = 1$ $f = (a-b) = E.(1+\nu)^{-1}$ $t_1 = (1+\nu).(1-2\nu).E^{-1}$

$t_2 = (1+\nu).E^{-1}$ $t_3 = 2.(1-\nu^2).E^{-1}$ $t_4 = (1+\nu).E^{-1}$ $t_5 = 2$

$t_6 = 1$ $t_7 = 2\nu$ $t_8 = 0$ $t_9 = 2.(1+\nu).E^{-1}$ $t_{10} = 2.(1+\nu).E^{-1}$

5(b) UNIFORM UNIDIRECTIONAL SHEAR DISPLACEMENT

Values of integrals $I_{1\tau\mu}(\psi)$, ${}_cI_{1\tau\mu}$, ${}_sI_{1\tau\mu}$, $A_{1\tau\mu}(\psi)$, ${}_cA_{1\tau\mu}$, ${}_sA_{1\tau\mu}$, $M_{1\tau\mu}(\psi)$, ${}_cM_{1\tau\mu}$, ${}_sM_{1\tau\mu}$

When $r \neq 0$ and $z \neq 0$ the values of the integrals are given by;

$I_{121}(\psi)$, $I_{123}(\psi)$ and $I_{125}(\psi)$ are shown on page 15 for the corresponding case of $r \neq 0$ and $z \neq 0$

$$I_{141}(\psi) = \left(\tfrac{2}{\pi}\right)^{\frac{1}{2}}.(R_0.\psi.\sin\tfrac{1}{2}\lambda_0 + R_0.\cos\tfrac{1}{2}\lambda_0 - 2.\psi).r^{-2}$$

$$I_{143}(\psi) = \left(\tfrac{2}{\pi}\right)^{\frac{1}{2}}.\{(\psi^2-R_0^{\,2}-1).\sin\tfrac{1}{2}\lambda_0 + 2.R_0 - 2.\psi.\cos\tfrac{1}{2}\lambda_0\}.R_0^{-1}.r^{-2}$$

$$I_{145}(\psi) = \left(\tfrac{2}{\pi}\right)^{\frac{1}{2}}.\{(2.\cos\tfrac{1}{2}\lambda_0 - 2.\psi.\sin\tfrac{1}{2}\lambda_0).R_0^{-1}.r^{-2} - (\psi.\sin\tfrac{3}{2}\lambda_0 - \cos\tfrac{3}{2}\lambda_0).R_0^{-3}\}$$

$I_{101}(\psi)$, $I_{103}(\psi)$ and $I_{105}(\psi)$ are shown on page 15 for the corresponding case of $r \neq 0$ and $z \neq 0$

$$A_{12\mu}(\psi) = \tfrac{1}{2}\{I_{14\mu}(\psi) + I_{10\mu}(\psi)\} \qquad M_{12\mu}(\psi) = \tfrac{1}{2}\{I_{14\mu}(\psi) - I_{10\mu}(\psi)\}$$

${}_cI_{121}$, ${}_sI_{121}$, ${}_cI_{123}$ and ${}_sI_{123}$ are shown on page 15 for the corresponding case of $r \neq 0$ and $z \neq 0$

$${}_cI_{141} = \left(\tfrac{2}{\pi}\right)^{\frac{1}{2}}.\tfrac{1}{2}.\{+ (1+\omega z).R_p.\cos\tfrac{1}{2}\lambda_p + (1-\omega z).R_m.\cos\tfrac{1}{2}\lambda_m + \alpha z(R_p.\sin\tfrac{1}{2}\lambda_p + R_m.\sin\tfrac{1}{2}\lambda_m) - 4\alpha z\}.r^{-2}$$

$${}_sI_{141} = \left(\tfrac{2}{\pi}\right)^{\frac{1}{2}}.\tfrac{1}{2}.\{- (1+\omega z).R_p.\sin\tfrac{1}{2}\lambda_p + (1-\omega z).R_m.\sin\tfrac{1}{2}\lambda_m + \alpha z(R_p.\cos\tfrac{1}{2}\lambda_p - R_m.\cos\tfrac{1}{2}\lambda_m) + 4\omega z\}.r^{-2}$$

$${}_cI_{143} = \left(\tfrac{2}{\pi}\right)^{\frac{1}{2}}.\tfrac{1}{2}.[+ \{\alpha^2z^2 - (1+\omega z)^2 - R_p^{\,2}\}.R_p^{-1}.\sin\tfrac{1}{2}\lambda_p + \{\alpha^2z^2 - (1-\omega z)^2 - R_m^{\,2}\}.R_m^{-1}.\sin\tfrac{1}{2}\lambda_m$$
$$-2\alpha z.\{(1+\omega z).R_p^{-1}.\cos\tfrac{1}{2}\lambda_p + (1-\omega z).R_m^{-1}.\cos\tfrac{1}{2}\lambda_m\} + 4].r^{-2}$$

$${}_sI_{143} = \left(\tfrac{2}{\pi}\right)^{\frac{1}{2}}.\tfrac{1}{2}.[- \{\alpha^2z^2 - (1+\omega z)^2 + R_p^{\,2}\}.R_p^{-1}.\cos\tfrac{1}{2}\lambda_p + \{\alpha^2z^2 - (1-\omega z)^2 + R_m^{\,2}\}.R_m^{-1}.\cos\tfrac{1}{2}\lambda_m$$
$$-2\alpha z.\{(1+\omega z).R_p^{-1}.\sin\tfrac{1}{2}\lambda_p - (1-\omega z).R_m^{-1}.\sin\tfrac{1}{2}\lambda_m\}].r^{-2}$$

${}_cI_{101}$, ${}_sI_{101}$, ${}_cI_{103}$ and ${}_sI_{103}$ are shown on page 15 for the corresponding case of $r \neq 0$ and $z \neq 0$

$${}_cA_{12\mu} = \tfrac{1}{2}({}_cI_{14\mu} + {}_cI_{10\mu}) \qquad {}_sA_{12\mu} = \tfrac{1}{2}({}_sI_{14\mu} + {}_sI_{10\mu}) \qquad {}_cM_{12\mu} = \tfrac{1}{2}({}_cI_{14\mu} - {}_cI_{10\mu}) \qquad {}_sM_{12\mu} = \tfrac{1}{2}({}_sI_{14\mu} - {}_sI_{10\mu})$$

where σ_0, σ_p, σ_m, R_0, R_p, R_m, L_p, L_m, λ_0, λ_p, λ_m, are given on page 15

When $r = 0$ the integrals are given by;

$$I_{121}(\psi) = I_{123}(\psi) = I_{125}(\psi) = {}_cI_{121} = {}_cI_{123} = {}_sI_{121} = {}_sI_{123} = 0$$

$$I_{141}(\psi) = r^{-1}.I_{141}(\psi) = I_{143}(\psi) = r^{-1}.I_{143}(\psi) = {}_cI_{141} = r^{-1}.{}_cI_{141} = {}_sI_{141} = r^{-1}.{}_sI_{141} = 0$$

$$A_{121}(\psi) = -M_{121}(\psi) = \left(\tfrac{2}{\pi}\right)^{\frac{1}{2}}.\tfrac{1}{2}.\text{arcot}\,\psi \qquad A_{123}(\psi) = -M_{123}(\psi) = \left(\tfrac{2}{\pi}\right)^{\frac{1}{2}}.\tfrac{1}{2}.(\psi^2+1)^{-1}$$

$$A_{125}(\psi) = -M_{125}(\psi) = \left(\tfrac{2}{\pi}\right)^{\frac{1}{2}}.\psi.(\psi^2+1)^{-2}$$

$${}_cA_{121} = -{}_cM_{121} = \left(\tfrac{2}{\pi}\right)^{\frac{1}{2}}.\tfrac{1}{2}.(\text{artan}\tfrac{1+\omega z}{\alpha z} + \text{artan}\tfrac{1-\omega z}{\alpha z})$$

$${}_sA_{121} = -{}_sM_{121} = \left(\tfrac{2}{\pi}\right)^{\frac{1}{2}}.\tfrac{1}{8}.\ln[\{\alpha^2z^2 + (1+\omega z)^2\}.\{\alpha^2z^2 + (1-\omega z)^2\}^{-1}]$$

$${}_cA_{123} = -{}_cM_{123} = \left(\tfrac{2}{\pi}\right)^{\frac{1}{2}}.\tfrac{1}{2}.[(1+\omega z).\{\alpha^2z^2 + (1+\omega z)^2\}^{-1} + (1-\omega z).\{\alpha^2z^2 + (1-\omega z)^2\}^{-1}]$$

$${}_sA_{123} = -{}_sM_{123} = \left(\tfrac{2}{\pi}\right)^{\frac{1}{2}}.\alpha.\omega.z^2.\{\alpha^2z^2 + (1+\omega z)^2\}^{-1}.\{\alpha^2z^2 + (1-\omega z)^2\}^{-1}$$

When $z = 0$ the relevant integrals are given by;

$I_{121}(0)$ and ${}_cI_{121}$ are shown on page 15 for the corresponding case of $z = 0$

$$I_{141}(0) = {}_cI_{141} = \left(\tfrac{2}{\pi}\right)^{\frac{1}{2}}\begin{cases}0 \\ (r^2-1)^{\frac{1}{2}}.r^{-2}\end{cases} \qquad I_{143}(0) = {}_cI_{143} = \left(\tfrac{2}{\pi}\right)^{\frac{1}{2}}\begin{cases}-r^2.(1-r^2)^{-\frac{1}{2}}.\{1+(1-r^2)^{\frac{1}{2}}\}^{-2} & r < 1 \\ 2.r^{-2} & r > 1\end{cases}$$

$$A_{121}(0) = {}_cA_{121} = \left(\tfrac{2}{\pi}\right)^{\frac{1}{2}}\begin{cases}\frac{\pi}{4} \\ \frac{1}{2}\{\text{arsin}\frac{1}{r} + (r^2-1)^{\frac{1}{2}}.r^{-2}\}\end{cases} \qquad A_{123}(0) = {}_cA_{123} = \left(\tfrac{2}{\pi}\right)^{\frac{1}{2}}\begin{cases}.\{1+(1-r^2)^{\frac{1}{2}}\}^{-1} & r < 1 \\ r^{-2} & r > 1\end{cases}$$

$$M_{121}(0) = {}_cM_{121} = \left(\tfrac{2}{\pi}\right)^{\frac{1}{2}}\begin{cases}-\frac{\pi}{4} \\ \frac{1}{2}\{-\text{arsin}\frac{1}{r} + (r^2-1)^{\frac{1}{2}}.r^{-2}\}\end{cases} \qquad M_{123}(0) = {}_cM_{123} = \left(\tfrac{2}{\pi}\right)^{\frac{1}{2}}\begin{cases}-(1-r^2)^{-\frac{1}{2}}.\{1+(1-r^2)^{\frac{1}{2}}\}^{-1} & r < 1 \\ r^{-2} & r > 1\end{cases}$$

$${}_sI_{121} = {}_sI_{123} = {}_sI_{141} = {}_sI_{143} = {}_sA_{121} = {}_sA_{123} = {}_sM_{121} = {}_sM_{123} = 0$$

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IX. LIST OF TABLES OF INTEGRALS

X. TABLES OF INTEGRALS

$I_{200}(\psi)$

ψ \ r	0	.25	.50	.75	1.00	1.25	1.50	2.00	3.00	4.00	5.00	7.00
0	1.0000	.9842	.9342	.8394	.6366	.4441	.3559	.2587	.1691	.1260	.1005	.0716
.25	.7808	.7661	.7219	.6419	.5312	.4248	.3474	.2558	.1685	.1257	.1003	.0715
.50	.6180	.6068	.5735	.5192	.4514	.3839	.3272	.2487	.1665	.1250	.1000	.0714
.75	.5000	.4920	.4689	.4328	.3887	.3434	.3021	.2380	.1635	.1237	.0993	.0712
1.00	.4142	.4087	.3929	.3686	.3388	.3072	.2768	.2256	.1595	.1220	.0985	.0709
1.50	.3028	.3001	.2925	.2806	.2657	.2490	.2319	.1994	.1497	.1175	.0961	.0700
2.00	.2361	.2347	.2306	.2242	.2160	.2065	.1962	.1753	.1387	.1121	.0931	.0688
3.00	.1623	.1618	.1603	.1580	.1549	.1511	.1468	.1372	.1174	.0999	.0858	.0657
4.00	.1231	.1229	.1222	.1212	.1197	.1179	.1158	.1108	.0995	.0982	.0780	.0620
5.00	.0990	.0989	.0986	.0980	.0972	.0962	.0951	.0922	.0854	.0779	.0706	.0581
7.00	.0711	.0711	.0710	.0708	.0705	.0701	.0696	.0685	.0655	.0619	.0581	.0504
10.0	.0499	.0501	.0501	.0500	.0499	.0497	.0496	.0492	.0480	.0465	.0448	.0410
15.0	.0333	.0340	.0340	.0340	.0340	.0339	.0339	.0337	.0334	.0329	.0323	.0308
20.0	.0250	.0265	.0265	.0264	.0264	.0264	.0264	.0263	.0262	.0260	.0257	.0250

$I_{202}(\psi)$

ψ \ r	0	.25	.50	.75	1.00	1.25	1.50	2.00	3.00	4.00	5.00	7.00
0	1.0000	1.0000	1.0000	1.0000	.5000	.0000	.0000	.0000	.0000	.0000	.0000	.0000
.25	.7575	.7471	.7080	.6071	.3627	.1395	.0638	.0211	.0053	.0017	.0010	.0004
.50	.5528	.5390	.4937	.4068	.2815	.1658	.0950	.0370	.0100	.0041	.0021	.0008
.75	.4000	.3884	.3532	.2951	.2228	.1542	.1026	.0471	.0142	.0059	.0030	.0011
1.00	.2929	.2847	.2606	.2233	.1787	.1350	.0985	.0519	.0174	.0076	.0039	.0015
1.50	.1680	.1643	.1538	.1380	.1189	.0991	.0805	.0512	.0212	.0101	.0055	.0021
2.00	.1056	.1039	.0991	.0918	.0827	.0727	.0628	.0450	.0222	.0116	.0066	.0027
3.00	.0513	.0509	.0496	.0475	.0449	.0418	.0384	.0315	.0198	.0122	.0077	.0035
4.00	.0299	.0297	.0292	.0285	.0275	.0263	.0249	.0219	.0159	.0111	.0077	.0039
5.00	.0194	.0193	.0191	.0188	.0184	.0178	.0172	.0157	.0125	.0095	.0071	.0040
7.00	.0101	.0100	.0099	.0098	.0097	.0096	.0094	.0089	.0078	.0066	.0055	.0036
10.0	.0050	.0049	.0049	.0049	.0048	.0048	.0047	.0046	.0043	.0039	.0035	.0027
15.0	.0022	.0021	.0021	.0021	.0021	.0021	.0021	.0020	.0020	.0019	.0018	.0015
20.0	.0012	.0011	.0011	.0011	.0011	.0011	.0011	.0011	.0010	.0010	.0010	.0009

$I_{204}(\psi)$

ψ \ r	0	.25	.50	.75	1.00	1.25	1.50	2.00	3.00	4.00	5.00	7.00
.25	.9131	.9433	1.0314	1.1092	.3882	-0.2686	-0.1952	-0.0766	-0.0201	-0.0083	-0.0042	-0.0015
.50	.7155	.7143	.6917	.5711	.2720	.0071	-0.0691	-0.0530	-0.0180	-0.0078	-0.0040	-0.0015
.75	.5120	.4990	.4512	.3499	.2022	.0703	-0.0004	-0.0288	-0.0148	-0.0070	-0.0037	-0.0014
1.00	.3536	.3410	.3017	.2355	.1536	.0788	.0281	-0.0100	-0.0112	-0.0061	-0.0034	-0.0014
1.50	.1707	.1647	.1475	.1216	.0915	.0624	.0383	.0092	-0.0045	-0.0040	-0.0027	-0.0012
2.00	.0894	.0870	.0799	.0693	.0567	.0437	.0319	.0142	.0001	-0.0020	-0.0018	-0.0010
3.00	.0316	.0311	.0296	.0273	.0244	.0212	.0178	.0117	.0037	.0005	-0.0004	-0.0006
4.00	.0143	.0141	.0137	.0130	.0121	.0111	.0100	.0077	.0038	.0015	.0004	-0.0002
5.00	.0075	.0075	.0073	.0071	.0068	.0064	.0059	.0050	.0031	.0016	.0008	.0000
7.00	.0028	.0028	.0028	.0027	.0027	.0026	.0025	.0023	.0017	.0012	.0008	.0003
10.0	.0010	.0010	.0010	.0010	.0010	.0010	.0009	.0009	.0008	.0006	.0005	.0003
15.0	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0002	.0002
20.0	.0001	.0001	.0001	.0001	.0001	.0001	.0001	.0001	.0001	.0001	.0001	.0001

$_cI_{200}$ $\omega + \alpha = 0.5$

αz \ r	0	.25	.50	.75	1.00	1.25	1.50	2.00	3.00	4.00	5.00	7.00
0	1.0000	.9842	.9342	.8394	.6366	.4441	.3559	.2587	.1691	.1260	.1005	.0716
.25	.7736	.7589	.7139	.6340	.5283	.4262	.3490	.2565	.1686	.1258	.1004	.0715
.50	.5957	.5846	.5522	.5023	.4430	.3830	.3292	.2504	.1671	.1252	.1001	.0715
.75	.4647	.4578	.4383	.4094	.3748	.3379	.3016	.2400	.1645	.1242	.0996	.0713
1.00	.3721	.3683	.3574	.3408	.3201	.2969	.2726	.2265	.1609	.1228	.0989	.0710
1.50	.2596	.2584	.2549	.2491	.2413	.2316	.2206	.1961	.1508	.1187	.0969	.0703
2.00	.1974	.1969	.1956	.1933	.1899	.1856	.1803	.1675	.1383	.1130	.0940	.0693
3.00	.1326	.1325	.1322	.1316	.1308	.1297	.1282	.1242	.1126	.0990	.0861	.0663
4.00	.0997	.0997	.0996	.0994	.0991	.0986	.0981	.0966	.0917	.0847	.0769	.0624
5.00	.0799	.0799	.0798	.0797	.0796	.0794	.0791	.0784	.0761	.0725	.0679	.0578
7.00	.0571	.0571	.0571	.0571	.0570	.0570	.0569	.0566	.0559	.0547	.0530	.0484
10.0	.0400	.0401	.0401	.0401	.0400	.0400	.0400	.0399	.0397	.0393	.0388	.0372
15.0	.0267	.0271	.0271	.0271	.0271	.0271	.0271	.0270	.0270	.0269	.0267	.0263
20.0	.0200	.0211	.0211	.0211	.0211	.0211	.0211	.0211	.0211	.0210	.0210	.0208

$_cI_{200}$ $\omega + \alpha = 1.0$

αz \ r	0	.25	.50	.75	1.00	1.25	1.50	2.00	3.00	4.00	5.00	7.00
0	1.0000	.9842	.9342	.8394	.6366	.4441	.3559	.2587	.1691	.1260	.1005	.0716
.25	.7519	.7369	.6895	.6115	.5196	.4295	.3538	.2586	.1691	.1260	.1005	.0716
.50	.5291	.5192	.4927	.4576	.4189	.3769	.3328	.2553	.1688	.1259	.1005	.0716
.75	.3692	.3667	.3600	.3496	.3356	.3174	.2951	.2445	.1677	.1257	.1004	.0716
1.00	.2720	.2724	.2730	.2725	.2698	.2637	.2541	.2257	.1649	.1251	.1002	.0715
1.50	.1748	.1755	.1773	.1799	.1824	.1842	.1846	.1798	.1523	.1218	.0992	.0713
2.00	.1286	.1290	.1301	.1318	.1339	.1360	.1379	.1399	.1325	.1146	.0965	.0708
3.00	.0845	.0846	.0850	.0856	.0864	.0875	.0886	.0910	.0941	.0921	.0853	.0681
4.00	.0630	.0630	.0632	.0635	.0638	.0643	.0649	.0662	.0690	.0707	.0701	.0624
5.00	.0502	.0502	.0503	.0505	.0507	.0509	.0512	.0520	.0538	.0556	.0566	.0547
7.00	.0358	.0356	.0357	.0357	.0358	.0359	.0360	.0363	.0371	.0380	.0390	.0402
10.0	.0250	.0246	.0246	.0247	.0247	.0247	.0248	.0248	.0251	.0255	.0259	.0267
15.0	.0167	.0160	.0160	.0160	.0160	.0160	.0160	.0161	.0161	.0162	.0163	.0166
20.0	.0125	.0127	.0127	.0127	.0127	.0127	.0127	.0127	.0128	.0128	.0129	.0132

$_sI_{200}$ $\omega + \alpha = 0.5$

αz \ r	0	.25	.50	.75	1.00	1.25	1.50	2.00	3.00	4.00	5.00	7.00
0	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000
.25	.0945	.1095	.1047	.0901	.0616	.0361	.0246	.0190	.0174	.0169	.0167	.0167
.50	.1359	.1489	.1370	.1150	.0862	.0594	.0414	.0262	.0192	.0177	.0172	.0169
.75	.1435	.1557	.1427	.1223	.0983	.0751	.0566	.0351	.0221	.0189	.0178	.0171
1.00	.1356	.1485	.1377	.1216	.1027	.0837	.0671	.0439	.0257	.0205	.0187	.0174
1.50	.1107	.1253	.1195	.1106	.0997	.0879	.0763	.0563	.0334	.0246	.0209	.0183
2.00	.0898	.1054	.1023	.0974	.0912	.0841	.0765	.0617	.0400	.0289	.0236	.0194
3.00	.0635	.0798	.0787	.0770	.0746	.0717	.0685	.0612	.0468	.0359	.0289	.0221
4.00	.0487	.0651	.0647	.0639	.0628	.0614	.0598	.0561	.0475	.0393	.0327	.0248
5.00	.0393	.0558	.0556	.0552	.0546	.0539	.0530	.0509	.0457	.0400	.0347	.0271
7.00	.0283	.0448	.0447	.0446	.0444	.0441	.0438	.0429	.0407	.0380	.0350	.0295
10.0	.0199	.0362	.0362	.0361	.0361	.0360	.0359	.0356	.0347	.0337	.0324	.0295
15.0	.0133	.0290	.0290	.0290	.0289	.0289	.0289	.0288	.0285	.0282	.0277	.0266
20.0	.0100	.0248	.0248	.0248	.0248	.0248	.0248	.0247	.0246	.0244	.0242	.0237

$_sI_{200}$ $\omega + \alpha = 1.0$

αz \ r	0	.25	.50	.75	1.00	1.25	1.50	2.00	3.00	4.00	5.00	7.00
0	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000
.25	.1876	.2011	.1899	.1584	.1056	.0581	.0339	.0219	.0181	.0172	.0169	.0167
.50	.2571	.2652	.2390	.1982	.1508	.1063	.0719	.0375	.0219	.0188	.0177	.0171
.75	.2474	.2562	.2345	.2037	.1688	.1339	.1025	.0584	.0284	.0214	.0190	.0175
1.00	.2138	.2259	.2130	.1935	.1701	.1448	.1198	.0777	.0367	.0250	.0208	.0182
1.50	.1565	.1718	.1676	.1606	.1511	.1395	.1265	.0988	.0551	.0344	.0259	.0200
2.00	.1209	.1370	.1353	.1324	.1283	.1230	.1166	.1013	.0683	.0448	.0323	.0226
3.00	.0821	.0986	.0982	.0974	.0962	.0947	.0929	.0880	.0743	.0583	.0446	.0290
4.00	.0620	.0786	.0784	.0781	.0776	.0770	.0762	.0743	.0684	.0602	.0509	.0355
5.00	.0497	.0663	.0662	.0661	.0658	.0655	.0652	.0642	.0613	.0570	.0516	.0398
7.00	.0356	.0521	.0520	.0520	.0519	.0518	.0517	.0513	.0503	.0489	.0469	.0416
10.0	.0250	.0410	.0410	.0409	.0409	.0409	.0408	.0407	.0404	.0399	.0393	.0375
15.0	.0167	.0310	.0310	.0310	.0310	.0310	.0310	.0309	.0308	.0306	.0304	.0297
20.0	.0125	.0244	.0244	.0244	.0244	.0243	.0243	.0243	.0242	.0241	.0240	.0236

$_cI_{202}$ $\omega \div \alpha = 0.5$

αz \ r	0	.25	.50	.75	1.00	1.25	1.50	2.00	3.00	4.00	5.00	7.00
0	1.0000	1.0000	1.0000	1.0000	.5000	.0000	.0000	.0000	.0000	.0000	.0000	.0000
.25	.7523	.7405	.6966	.5793	.3575	.1570	.0673	.0213	.0053	.0021	.0011	.0004
.50	.5251	.5084	.4565	.3699	.2711	.1788	.1070	.0403	.0104	.0042	.0021	.0008
.75	.3474	.3356	.3026	.2568	.2073	.1583	.1135	.0535	.0152	.0062	.0031	.0011
1.00	.2288	.2229	.2068	.1844	.1588	.1317	.1044	.0593	.0193	.0081	.0041	.0015
1.50	.1102	.1092	.1063	.1015	.0949	.0865	.0767	.0556	.0245	.0113	.0060	.0022
2.00	.0622	.0621	.0617	.0609	.0593	.0569	.0535	.0446	.0254	.0134	.0075	.0029
3.00	.0273	.0273	.0274	.0275	.0276	.0275	.0272	.0258	.0203	.0139	.0090	.0039
4.00	.0152	.0152	.0153	.0154	.0155	.0155	.0156	.0154	.0141	.0115	.0086	.0045
5.00	.0097	.0097	.0097	.0098	.0098	.0099	.0099	.0100	.0097	.0088	.0074	.0046
7.00	.0049	.0048	.0049	.0049	.0049	.0050	.0050	.0050	.0051	.0050	.0047	.0038
10.0	.0024	.0023	.0023	.0023	.0024	.0024	.0024	.0024	.0024	.0024	.0024	.0023
15.0	.0011	.0009	.0009	.0009	.0009	.0009	.0009	.0009	.0009	.0010	.0010	.0010
20.0	.0006	.0004	.0004	.0004	.0004	.0004	.0004	.0004	.0004	.0004	.0004	.0004

$_cI_{202}$ $\omega \div \alpha = 1.0$

αz \ r	0	.25	.50	.75	1.00	1.25	1.50	2.00	3.00	4.00	5.00	7.00
0	1.0000	1.0000	1.0000	1.0000	.5000	.0000	.0000	.0000	.0000	.0000	.0000	.0000
.25	.7360	.7206	.6594	.5067	.3441	.1980	.0821	.0228	.0054	.0025	.0011	.0004
.50	.4311	.4083	.3501	.2898	.2432	.1958	.1381	.0515	.0116	.0044	.0022	.0008
.75	.1919	.1881	.1805	.1735	.1652	.1514	.1295	.0729	.0187	.0070	.0034	.0012
1.00	.0796	.0828	.0911	.1002	.1065	.1078	.1028	.0764	.0259	.0098	.0047	.0016
1.50	.0179	.0197	.0248	.0321	.0401	.0474	.0527	.0548	.0340	.0156	.0076	.0025
2.00	.0058	.0065	.0085	.0116	.0156	.0200	.0244	.0312	.0305	.0190	.0104	.0036
3.00	.0012	.0013	.0017	.0024	.0034	.0046	.0060	.0091	.0145	.0153	.0122	.0056
4.00	.0004	.0004	.0005	.0008	.0011	.0015	.0020	.0032	.0060	.0083	.0089	.0063
5.00	.0002	.0002	.0002	.0003	.0004	.0006	.0008	.0013	.0027	.0042	.0053	.0055
7.00	.0000	.0000	.0000	.0001	.0001	.0001	.0002	.0003	.0007	.0012	.0018	.0027
10.0	.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	.0000	.0001	.0002	.0004	.0008
15.0	.0000	-0.0002	-0.0002	-0.0002	-0.0002	-0.0002	-0.0002	-0.0002	-0.0002	-0.0002	-0.0002	-0.0001
20.0	.0000	-0.0004	-0.0004	-0.0004	-0.0004	-0.0004	-0.0004	-0.0004	-0.0004	-0.0004	-0.0004	-0.0004

$_sI_{202}$ $\omega \div \alpha = 0.5$

αz \ r	0	.25	.50	.75	1.00	1.25	1.50	2.00	3.00	4.00	5.00	7.00
0	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000
.25	.1147	.1185	.1294	.1354	.0480	-0.0316	-0.0236	-0.0092	-0.0025	-0.0014	-0.0006	-0.0002
.50	.1787	.1774	.1676	.1318	.0664	.0087	-0.0144	-0.0133	-0.0046	-0.0020	-0.0010	-0.0004
.75	.1843	.1779	.1566	.1192	.0731	.0318	.0052	-0.0098	-0.0056	-0.0027	-0.0014	-0.0005
1.00	.1596	.1529	.1337	.1051	.0727	.0426	.0194	-0.0027	-0.0055	-0.0031	-0.0017	-0.0007
1.50	.1029	.0997	.0906	.0773	.0616	.0458	.0313	.0105	-0.0026	-0.0029	-0.0020	-0.0009
2.00	.0666	.0653	.0614	.0553	.0478	.0396	.0313	.0168	.0016	-0.0017	-0.0018	-0.0010
3.00	.0328	.0325	.0315	.0300	.0280	.0256	.0228	.0170	.0071	.0017	-0.0003	-0.0009
4.00	.0191	.0190	.0187	.0182	.0175	.0166	.0156	.0132	.0081	.0039	.0014	-0.0003
5.00	.0124	.0124	.0123	.0121	.0118	.0114	.0109	.0098	.0072	.0046	.0025	.0004
7.00	.0064	.0065	.0064	.0064	.0063	.0062	.0061	.0057	.0049	.0039	.0029	.0013
10.0	.0032	.0032	.0032	.0032	.0032	.0032	.0031	.0031	.0028	.0026	.0022	.0015
15.0	.0014	.0015	.0015	.0015	.0015	.0015	.0015	.0015	.0014	.0014	.0013	.0011
20.0	.0008	.0009	.0009	.0009	.0009	.0009	.0009	.0009	.0008	.0008	.0008	.0007

$_sI_{202}$ $\omega \div \alpha = 1.0$

αz \ r	0	.25	.50	.75	1.00	1.25	1.50	2.00	3.00	4.00	5.00	7.00
0	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000
.25	.2331	.2404	.2603	.2403	.0918	-0.0386	-0.0493	-0.0197	-0.0051	-0.0022	-0.0010	-0.0004
.50	.3516	.3411	.2982	.2173	.1250	.0438	-0.0098	-0.0260	-0.0094	-0.0040	-0.0020	-0.0007
.75	.3072	.2905	.2469	.1905	.1326	.0790	.0344	-0.0118	-0.0115	-0.0055	-0.0029	-0.0011
1.00	.2173	.2094	.1879	.1578	.1238	.0893	.0571	.0097	-0.0104	-0.0063	-0.0036	-0.0014
1.50	.1078	.1067	.1031	.0968	.0876	.0760	.0627	.0350	.0003	-0.0053	-0.0041	-0.0019
2.00	.0619	.0617	.0609	.0595	.0572	.0538	.0493	.0373	.0119	-0.0006	-0.0031	-0.0021
3.00	.0277	.0277	.0277	.0275	.0273	.0269	.0263	.0244	.0174	.0085	.0023	-0.0013
4.00	.0156	.0156	.0156	.0156	.0156	.0155	.0154	.0150	.0132	.0099	.0060	.0007
5.00	.0100	.0100	.0100	.0100	.0100	.0100	.0100	.0099	.0093	.0082	.0064	.0025
7.00	.0051	.0052	.0052	.0052	.0052	.0052	.0052	.0052	.0051	.0049	.0046	.0034
10.0	.0025	.0026	.0026	.0026	.0027	.0027	.0027	.0026	.0026	.0026	.0026	.0024
15.0	.0011	.0013	.0013	.0013	.0013	.0013	.0013	.0013	.0013	.0013	.0013	.0013
20.0	.0006	.0007	.0007	.0007	.0007	.0007	.0007	.0007	.0007	.0007	.0007	.0007

$I_{220}(\psi)$

ψ \ r	0	.25	.50	.75	1.00	1.25	1.50	2.00	3.00	4.00	5.00	7.00
0	.0000	.1250	.2500	.3750	.5000	.4000	.3333	.2500	.1667	.1250	.1000	.0714
.25	.0000	.0940	.1840	.2609	.3005	.2948	.2664	.2157	.1523	.1171	.0950	.0689
.50	.0000	.0682	.1311	.1814	.2108	.2180	.2104	.1836	.1381	.1092	.0899	.0663
.75	.0000	.0493	.0942	.1303	.1541	.1652	.1665	.1552	.1247	.1015	.0850	.0638
1.00	.0000	.0361	.0691	.0964	.1161	.1279	.1330	.1310	.1121	.0940	.0801	.0613
1.50	.0000	.0208	.0402	.0572	.0709	.0812	.0881	.0942	.0901	.0803	.0709	.0564
2.00	.0000	.0131	.0256	.0369	.0468	.0549	.0612	.0691	.0723	.0682	.0624	.0517
3.00	.0000	.0064	.0126	.0185	.0240	.0289	.0333	.0402	.0476	.0493	.0481	.0431
4.00	.0000	.0037	.0074	.0109	.0143	.0175	.0205	.0256	.0326	.0361	.0372	.0358
5.00	.0000	.0024	.0048	.0072	.0094	.0116	.0137	.0175	.0233	.0270	.0290	.0297
7.00	.0000	.0013	.0025	.0037	.0049	.0061	.0073	.0095	.0133	.0162	.0184	.0207
10.0	.0000	.0006	.0012	.0018	.0024	.0030	.0036	.0048	.0069	.0088	.0103	.0126
15.0	.0000	.0003	.0005	.0008	.0010	.0013	.0016	.0021	.0030	.0040	.0048	.0062
20.0	.0000	.0001	.0003	.0004	.0005	.0007	.0008	.0011	.0016	.0021	.0026	.0034

$I_{222}(\psi)$

ψ \ r/r_0	0	.25	.50	.75	1.00	1.25	1.50	2.00	3.00	4.00	5.00	7.00
0	.0000	.1281	.2779	.5029	*******	.4577	.2747	.1390	.0580	.0320	.0203	.0103
.25	.0000	.1155	.2430	.3899	.4755	.3670	.2496	.1343	.0574	.0315	.0202	.0102
.50	.0000	.0894	.1770	.2515	.2818	.2521	.2000	.1215	.0553	.0312	.0200	.0102
.75	.0000	.0632	.1209	.1644	.1833	.1758	.1531	.1052	.0522	.0302	.0196	.0101
1.00	.0000	.0434	.0820	.1106	.1252	.1256	.1162	.0885	.0463	.0290	.0191	.0100
1.50	.0000	.0210	.0397	.0545	.0640	.0684	.0683	.0604	.0397	.0259	.0177	.0096
2.00	.0000	.0110	.0211	.0296	.0359	.0400	.0419	.0410	.0315	.0224	.0161	.0091
3.00	.0000	.0039	.0077	.0110	.0139	.0163	.0180	.0199	.0190	.0158	.0126	.0080
4.00	.0000	.0018	.0035	.0051	.0066	.0079	.0090	.0106	.0116	.0109	.0095	.0067
5.00	.0000	.0009	.0019	.0027	.0036	.0043	.0050	.0061	.0074	.0075	.0070	.0055
7.00	.0000	.0004	.0007	.0010	.0014	.0017	.0020	.0025	.0033	.0038	.0039	.0036
10.0	.0000	.0001	.0002	.0004	.0005	.0006	.0007	.0009	.0013	.0016	.0018	.0020
15.0	.0000	.0000	.0001	.0001	.0002	.0002	.0002	.0003	.0004	.0006	.0007	.0008
20.0	.0000	.0000	.0000	.0001	.0001	.0001	.0001	.0001	.0002	.0003	.0003	.0004

$I_{224}(\psi)$

ψ \ r	0	.25	.50	.75	1.00	1.25	1.50	2.00	3.00	4.00	5.00	7.00
.25	.0000	.0891	.2428	.6379	1.1749	.5165	.1756	.0371	.0058	.0012	.0007	.0003
.50	.0000	.1127	.2573	.4402	.5240	.3751	.2037	.0607	.0106	.0032	.0013	.0003
.75	.0000	.0931	.1887	.2704	.2947	.2448	.1683	.0678	.0142	.0045	.0018	.0005
1.00	.0000	.0656	.1255	.1688	.1820	.1631	.1275	.0643	.0165	.0055	.0023	.0006
1.50	.0000	.0289	.0539	.0714	.0794	.0780	.0699	.0472	.0173	.0068	.0031	.0009
2.00	.0000	.0131	.0247	.0335	.0387	.0403	.0390	.0314	.0153	.0070	.0035	.0011
3.00	.0000	.0039	.0068	.0095	.0116	.0130	.0138	.0139	.0096	.0058	.0034	.0013
4.00	.0000	.0012	.0024	.0035	.0044	.0052	.0057	.0062	.0059	.0041	.0028	.0012
5.00	.0000	.0005	.0011	.0016	.0020	.0024	.0027	.0031	.0032	.0027	.0021	.0011
7.00	.0000	.0001	.0003	.0004	.0006	.0007	.0008	.0010	.0012	.0012	.0011	.0008
10.0	.0000	.0000	.0001	.0001	.0001	.0002	.0002	.0003	.0004	.0004	.0004	.0004
15.0	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0001	.0001	.0001	.0001	.0001
20.0	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0001

$_{c}I_{220}$ $\omega \div \alpha = 0.5$

z/r_0 \ r	0	.25	.50	.75	1.00	1.25	1.50	2.00	3.00	4.00	5.00	7.00
0	.0000	.1250	.2500	.3750	.5000	.4000	.3333	.2500	.1667	.1250	.1000	.0714
.25	.0000	.0933	.1820	.2552	.2930	.2903	.2653	.2154	.1522	.1171	.0949	.0689
.50	.0000	.0646	.1228	.1678	.1956	.2064	.2040	.1817	.1378	.1090	.0899	.0663
.75	.0000	.0427	.0811	.1122	.1346	.1486	.1547	.1503	.1236	.1011	.0848	.0638
1.00	.0000	.0282	.0544	.0769	.0951	.1087	.1175	.1228	.1100	.0933	.0798	.0612
1.50	.0000	.0137	.0271	.0397	.0512	.0614	.0699	.0811	.0851	.0783	.0700	.0561
2.00	.0000	.0078	.0155	.0231	.0305	.0375	.0439	.0544	.0646	.0646	.0607	.0512
3.00	.0000	.0034	.0068	.0103	.0137	.0172	.0206	.0270	.0373	.0427	.0442	.0417
4.00	.0000	.0019	.0038	.0057	.0077	.0096	.0116	.0155	.0227	.0282	.0316	.0333
5.00	.0000	.0012	.0024	.0036	.0049	.0061	.0074	.0099	.0148	.0192	.0227	.0262
7.00	.0000	.0006	.0012	.0018	.0025	.0031	.0037	.0050	.0075	.0101	.0125	.0162
10.0	.0000	.0003	.0006	.0009	.0012	.0015	.0018	.0024	.0036	.0048	.0061	.0084
15.0	.0000	.0001	.0002	.0003	.0005	.0006	.0007	.0009	.0014	.0019	.0024	.0033
20.0	.0000	.0000	.0001	.0001	.0002	.0002	.0003	.0004	.0006	.0008	.0010	.0014

$_{c}I_{220}$ $\omega \div \alpha = 1.0$

z/r_0 \ r	0	.25	.50	.75	1.00	1.25	1.50	2.00	3.00	4.00	5.00	7.00
0	.0000	.1250	.2500	.3750	.5000	.4000	.3333	.2500	.1667	.1250	.1000	.0714
.25	.0000	.0911	.1753	.2384	.2716	.2767	.2612	.2145	.1520	.1170	.0949	.0689
.50	.0000	.0525	.0970	.1307	.1560	.1741	.1833	.1753	.1367	.1087	.0898	.0663
.75	.0000	.0237	.0464	.0677	.0878	.1060	.1206	.1341	.1203	.1001	.0844	.0637
1.00	.0000	.0102	.0214	.0343	.0485	.0630	.0767	.0971	.1029	.0911	.0789	.0610
1.50	.0000	.0024	.0054	.0095	.0151	.0220	.0298	.0464	.0685	.0717	.0671	.0554
2.00	.0000	.0008	.0018	.0033	.0054	.0084	.0121	.0214	.0413	.0525	.0547	.0494
3.00	.0000	.0002	.0004	.0007	.0011	.0018	.0027	.0054	.0138	.0237	.0315	.0368
4.00	.0000	.0000	.0001	.0002	.0004	.0006	.0009	.0018	.0051	.0102	.0160	.0248
5.00	.0000	.0000	.0000	.0001	.0001	.0002	.0004	.0007	.0022	.0047	.0081	.0155
7.00	.0000	.0000	.0000	.0000	.0000	.0001	.0001	.0002	.0006	.0013	.0024	.0057
10.0	.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	.0000	.0002	.0004	.0013
15.0	.0000	-0.0000	-0.0001	-0.0001	-0.0001	-0.0001	-0.0002	-0.0002	-0.0003	-0.0004	-0.0005	-0.0006
20.0	.0000	-0.0001	-0.0001	-0.0002	-0.0002	-0.0003	-0.0003	-0.0004	-0.0006	-0.0008	-0.0010	-0.0014

$_{s}I_{220}$ $\omega \div \alpha = 0.5$

z/r_0 \ r	0	.25	.50	.75	1.00	1.25	1.50	2.00	3.00	4.00	5.00	7.00
0	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000
.25	.0000	.0145	.0308	.0490	.0562	.0460	.0310	.0168	.0072	.0040	.0025	.0013
.50	.0000	.0223	.0437	.0607	.0669	.0613	.0497	.0305	.0139	.0078	.0050	.0026
.75	.0000	.0226	.0427	.0572	.0637	.0624	.0558	.0394	.0197	.0114	.0074	.0038
1.00	.0000	.0199	.0366	.0492	.0562	.0577	.0550	.0436	.0243	.0146	.0096	.0050
1.50	.0000	.0127	.0242	.0336	.0403	.0442	.0456	.0427	.0296	.0195	.0134	.0072
2.00	.0000	.0082	.0160	.0228	.0284	.0325	.0352	.0366	.0305	.0223	.0161	.0092
3.00	.0000	.0041	.0080	.0118	.0152	.0182	.0207	.0242	.0256	.0227	.0186	.0120
4.00	.0000	.0024	.0047	.0070	.0091	.0111	.0130	.0160	.0194	.0196	.0179	.0133
5.00	.0000	.0016	.0031	.0046	.0061	.0074	.0088	.0111	.0144	.0159	.0158	.0133
7.00	.0000	.0008	.0016	.0024	.0032	.0040	.0047	.0061	.0085	.0102	.0112	.0115
10.0	.0000	.0004	.0008	.0012	.0016	.0020	.0024	.0031	.0046	.0058	.0068	.0080
15.0	.0000	.0002	.0004	.0006	.0008	.0009	.0011	.0015	.0022	.0029	.0035	.0046
20.0	.0000	.0001	.0002	.0003	.0004	.0005	.0007	.0009	.0013	.0017	.0021	.0028

$_{s}I_{220}$ $\omega \div \alpha = 1.0$

z/r_0 \ r	0	.25	.50	.75	1.00	1.25	1.50	2.00	3.00	4.00	5.00	7.00
0	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000
.25	.0000	.0296	.0625	.0954	.1072	.0902	.0632	.0339	.0144	.0080	.0051	.0025
.50	.0000	.0433	.0817	.1081	.1182	.1127	.0970	.0619	.0280	.0157	.0100	.0051
.75	.0000	.0373	.0689	.0912	.1034	.1062	.1011	.0781	.0400	.0230	.0149	.0076
1.00	.0000	.0267	.0505	.0696	.0829	.0901	.0917	.0818	.0492	.0296	.0194	.0101
1.50	.0000	.0134	.0264	.0384	.0490	.0576	.0639	.0689	.0570	.0394	.0272	.0147
2.00	.0000	.0077	.0154	.0228	.0299	.0364	.0421	.0505	.0530	.0433	.0326	.0187
3.00	.0000	.0035	.0069	.0104	.0138	.0171	.0204	.0264	.0351	.0374	.0344	.0240
4.00	.0000	.0020	.0039	.0059	.0078	.0097	.0116	.0154	.0221	.0267	.0285	.0251
5.00	.0000	.0013	.0025	.0038	.0050	.0063	.0075	.0100	.0147	.0187	.0216	.0228
7.00	.0000	.0006	.0013	.0019	.0026	.0032	.0039	.0052	.0077	.0102	.0124	.0157
10.0	.0000	.0003	.0007	.0010	.0013	.0017	.0020	.0026	.0040	.0053	.0066	.0090
15.0	.0000	.0002	.0003	.0005	.0007	.0008	.0010	.0013	.0020	.0026	.0033	.0046
20.0	.0000	.0001	.0002	.0003	.0004	.0004	.0005	.0007	.0011	.0014	.0018	.0024

$_cI_{222}$ ω÷α = 0.5

αz \ r	0	.25	.50	.75	1.00	1.25	1.50	2.00	3.00	4.00	5.00	7.00
0	.0000	.1281	.2779	.5029	*******	.4577	.2747	.1390	.0580	.0320	.0203	.0103
.25	.0000	.1180	.2463	.3846	.4424	.3632	.2527	.1350	.0575	.0318	.0202	.0103
.50	.0000	.0882	.1689	.2246	.2434	.2318	.1967	.1233	.0558	.0314	.0201	.0102
.75	.0000	.0543	.0995	.1295	.1449	.1482	.1401	.1050	.0530	.0306	.0198	.0102
1.00	.0000	.0303	.0561	.0756	.0887	.0961	.0973	.0844	.0490	.0294	.0193	.0100
1.50	.0000	.0095	.0187	.0274	.0352	.0417	.0465	.0497	.0389	.0262	.0180	.0097
2.00	.0000	.0036	.0073	.0112	.0153	.0193	.0229	.0281	.0283	.0221	.0163	.0093
3.00	.0000	.0009	.0018	.0028	.0040	.0052	.0066	.0094	.0132	.0136	.0119	.0080
4.00	.0000	.0003	.0007	.0010	.0015	.0019	.0025	.0037	.0061	.0076	.0079	.0065
5.00	.0000	.0002	.0003	.0005	.0007	.0009	.0011	.0017	.0030	.0042	.0049	.0049
7.00	.0000	.0001	.0001	.0002	.0002	.0003	.0004	.0005	.0009	.0014	.0019	.0026
10.0	.0000	.0000	.0000	.0001	.0001	.0001	.0001	.0002	.0003	.0004	.0006	.0010
15.0	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0001	.0001	.0001	.0002	.0003
20.0	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0001	.0001	.0001

$_cI_{222}$ ω÷α = 1.0

αz \ r	0	.25	.50	.75	1.00	1.25	1.50	2.00	3.00	4.00	5.00	7.00
0	.0000	.1281	.2779	.5029	*******	.4577	.2747	.1390	.0580	.0320	.0203	.0103
.25	.0000	.1236	.2560	.3578	.3635	.3437	.2613	.1381	.0579	.0323	.0203	.0103
.50	.0000	.0770	.1289	.1489	.1606	.1741	.1763	.1278	.0575	.0319	.0203	.0103
.75	.0000	.0192	.0344	.0483	.0642	.0816	.0963	.0991	.0555	.0316	.0202	.0103
1.00	.0000	-0.0026	-0.0012	.0057	.0172	.0313	.0457	.0645	.0506	.0308	.0200	.0103
1.50	.0000	-0.0053	-0.0092	-0.0106	-0.0091	-0.0047	.0019	.0172	.0327	.0268	.0189	.0101
2.00	.0000	-0.0030	-0.0057	-0.0076	-0.0086	-0.0084	-0.0068	-0.0006	.0146	.0193	.0164	.0098
3.00	.0000	-0.0010	-0.0020	-0.0030	-0.0037	-0.0043	-0.0047	-0.0046	-0.0009	.0048	.0083	.0080
4.00	.0000	-0.0005	-0.0009	-0.0013	-0.0017	-0.0021	-0.0024	-0.0028	-0.0025	-0.0006	.0020	.0050
5.00	.0000	-0.0002	-0.0005	-0.0007	-0.0009	-0.0011	-0.0013	-0.0017	-0.0019	-0.0015	-0.0004	.0022
7.00	.0000	-0.0001	-0.0002	-0.0003	-0.0003	-0.0004	-0.0005	-0.0007	-0.0009	-0.0010	-0.0009	-0.0002
10.0	.0000	-0.0000	-0.0001	-0.0001	-0.0001	-0.0001	-0.0002	-0.0002	-0.0003	-0.0004	-0.0004	-0.0004
15.0	.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0001	-0.0001	-0.0001	-0.0001	-0.0002
20.0	.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0001	-0.0001	-0.0001	-0.0001

$_sI_{222}$ ω÷α = 0.5

αz \ r	0	.25	.50	.75	1.00	1.25	1.50	2.00	3.00	4.00	5.00	7.00
0	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000
.25	.0000	.0111	.0316	.0846	.1355	.0688	.0225	.0049	.0008	-0.0001	.0001	-0.0000
.50	.0000	.0300	.0669	.1062	.1188	.0911	.0530	.0159	.0027	.0008	.0003	.0001
.75	.0000	.0353	.0683	.0919	.0975	.0850	.0628	.0269	.0055	.0017	.0007	.0002
1.00	.0000	.0299	.0554	.0719	.0774	.0723	.0604	.0334	.0087	.0029	.0012	.0003
1.50	.0000	.0161	.0301	.0403	.0460	.0474	.0451	.0342	.0138	.0054	.0024	.0007
2.00	.0000	.0085	.0162	.0225	.0271	.0298	.0305	.0277	.0157	.0075	.0037	.0011
3.00	.0000	.0029	.0057	.0083	.0105	.0124	.0137	.0150	.0130	.0088	.0054	.0020
4.00	.0000	.0013	.0026	.0038	.0049	.0059	.0068	.0081	.0087	.0075	.0056	.0026
5.00	.0000	.0007	.0013	.0020	.0026	.0032	.0037	.0046	.0056	.0056	.0048	.0029
7.00	.0000	.0003	.0005	.0007	.0010	.0012	.0014	.0019	.0025	.0029	.0030	.0025
10.0	.0000	.0001	.0002	.0003	.0003	.0004	.0005	.0007	.0010	.0012	.0014	.0015
15.0	.0000	.0000	.0001	.0001	.0001	.0001	.0002	.0002	.0003	.0004	.0005	.0006
20.0	.0000	.0000	.0000	.0000	.0000	.0001	.0001	.0001	.0001	.0002	.0002	.0003

$_sI_{222}$ ω÷α = 1.0

αz \ r	0	.25	.50	.75	1.00	1.25	1.50	2.00	3.00	4.00	5.00	7.00
0	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000
.25	.0000	.0260	.0725	.1764	.2309	.1433	.0516	.0099	.0014	.0004	.0002	.0000
.50	.0000	.0692	.1385	.1832	.1885	.1616	.1128	.0365	.0057	.0017	.0006	.0002
.75	.0000	.0617	.1085	.1345	.1420	.1346	.1151	.0612	.0124	.0037	.0014	.0004
1.00	.0000	.0361	.0663	.0874	.0990	.1018	.0966	.0693	.0203	.0064	.0026	.0006
1.50	.0000	.0113	.0225	.0331	.0425	.0498	.0543	.0543	.0310	.0127	.0055	.0014
2.00	.0000	.0045	.0092	.0139	.0187	.0234	.0276	.0331	.0298	.0173	.0087	.0025
3.00	.0000	.0013	.0025	.0039	.0052	.0067	.0082	.0112	.0157	.0154	.0116	.0047
4.00	.0000	.0005	.0010	.0016	.0021	.0027	.0033	.0046	.0072	.0090	.0092	.0059
5.00	.0000	.0003	.0005	.0008	.0011	.0013	.0016	.0022	.0036	.0049	.0058	.0055
7.00	.0000	.0001	.0002	.0003	.0004	.0005	.0006	.0008	.0012	.0017	.0022	.0030
10.0	.0000	.0000	.0001	.0001	.0001	.0002	.0002	.0003	.0004	.0006	.0007	.0011
15.0	.0000	.0000	.0000	.0000	.0001	.0001	.0001	.0001	.0002	.0002	.0003	.0004
20.0	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0001	.0001	.0001	.0002	.0002

40

$I_{240}(\psi)$

ψ \ r	0	.25	.50	.75	1.00	1.25	1.50	2.00	3.00	4.00	5.00	7.00
0	.0000	.0079	.0335	.0842	.2122	.2853	.2713	.2252	.1596	.1220	.0985	.0709
.25	.0000	.0080	.0295	.0703	.1267	.1666	.1799	.1692	.1336	.1072	.0888	.0659
.50	.0000	.0056	.0222	.0488	.0801	.1059	.1210	.1271	.1114	.0938	.0800	.0612
.75	.0000	.0040	.0154	.0327	.0525	.0704	.0836	.0955	.0925	.0819	.0719	.0568
1.00	.0000	.0027	.0105	.0221	.0355	.0484	.0592	.0723	.0768	.0714	.0645	.0527
1.50	.0000	.0013	.0051	.0108	.0177	.0249	.0317	.0427	.0530	.0541	.0518	.0452
2.00	.0000	.0007	.0027	.0058	.0097	.0140	.0184	.0265	.0370	.0410	.0415	.0387
3.00	.0000	.0002	.0010	.0021	.0036	.0054	.0074	.0116	.0190	.0241	.0268	.0282
4.00	.0000	.0001	.0004	.0010	.0017	.0026	.0036	.0058	.0106	.0147	.0176	.0207
5.00	.0000	.0001	.0002	.0005	.0009	.0014	.0020	.0033	.0063	.0094	.0119	.0153
7.00	.0000	.0000	.0001	.0002	.0003	.0005	.0008	.0013	.0027	.0043	.0059	.0086
10.0	.0000	.0000	.0000	.0001	.0001	.0002	.0003	.0005	.0010	.0017	.0025	.0041
15.0	.0000	.0000	.0000	.0000	.0000	.0001	.0001	.0002	.0003	.0006	.0009	.0015
20.0	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0001	.0001	.0003	.0004	.0007

$I_{242}(\psi)$

ψ \ r	0	.25	.50	.75	1.00	1.25	1.50	2.00	3.00	4.00	5.00	7.00
0	.0000	.0000	.0000	.0000	.5000	.6400	.4444	.2500	.1111	.0625	.0400	.0204
.25	.0000	.0054	.0243	.0857	.2425	.3313	.2919	.1946	.0962	.0561	.0368	.0193
.50	.0000	.0069	.0302	.0768	.1406	.1829	.1856	.1466	.0820	.0504	.0339	.0182
.75	.0000	.0058	.0235	.0523	.0853	.1101	.1194	.1081	.0690	.0449	.0309	.0171
1.00	.0000	.0041	.0160	.0339	.0535	.0696	.0789	.0792	.0573	.0394	.0281	.0160
1.50	.0000	.0018	.0070	.0144	.0229	.0308	.0370	.0430	.0388	.0300	.0229	.0140
2.00	.0000	.0008	.0032	.0067	.0108	.0150	.0188	.0241	.0260	.0225	.0184	.0121
3.00	.0000	.0002	.0009	.0019	.0031	.0045	.0060	.0087	.0119	.0124	.0115	.0089
4.00	.0000	.0001	.0003	.0007	.0012	.0017	.0023	.0036	.0058	.0069	.0072	.0064
5.00	.0000	.0000	.0001	.0003	.0005	.0008	.0011	.0017	.0030	.0040	.0045	.0045
7.00	.0000	.0000	.0000	.0001	.0001	.0002	.0003	.0005	.0010	.0015	.0019	.0023
10.0	.0000	.0000	.0000	.0000	.0000	.0001	.0001	.0001	.0003	.0005	.0006	.0009
15.0	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0001	.0001	.0002	.0003
20.0	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0001	.0001

$_cI_{240}$ $\omega \div \alpha$ = 0.5

az \ r	0	.25	.50	.75	1.00	1.25	1.50	2.00	3.00	4.00	5.00	7.00
0	.0000	.0079	.0335	.0842	.2122	.2853	.2713	.2252	.1596	.1220	.0985	.0709
.25	.0000	.0075	.0301	.0708	.1223	.1606	.1757	.1677	.1331	.1070	.0887	.0659
.50	.0000	.0055	.0216	.0454	.0713	.0944	.1107	.1217	.1097	.0931	.0797	.0611
.75	.0000	.0034	.0129	.0265	.0417	.0567	.0699	.0862	.0890	.0804	.0711	.0565
1.00	.0000	.0019	.0073	.0151	.0246	.0347	.0445	.0603	.0714	.0688	.0632	.0522
1.50	.0000	.0006	.0024	.0052	.0090	.0137	.0189	.0295	.0445	.0493	.0491	.0441
2.00	.0000	.0002	.0009	.0021	.0038	.0060	.0086	.0149	.0271	.0345	.0374	.0368
3.00	.0000	.0001	.0002	.0005	.0009	.0015	.0023	.0044	.0103	.0163	.0208	.0250
4.00	.0000	.0000	.0001	.0002	.0003	.0006	.0009	.0017	.0043	.0078	.0114	.0165
5.00	.0000	.0000	.0000	.0001	.0002	.0003	.0004	.0008	.0020	.0040	.0063	.0107
7.00	.0000	.0000	.0000	.0000	.0001	.0001	.0001	.0002	.0006	.0013	.0022	.0046
10.0	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0001	.0002	.0004	.0007	.0015
15.0	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0001	.0001	.0002	.0004
20.0	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0001	.0001	.0002

$_cI_{240}$ $\omega \div \alpha$ = 1.0

az \ r	0	.25	.50	.75	1.00	1.25	1.50	2.00	3.00	4.00	5.00	7.00
0	.0000	.0079	.0335	.0842	.2122	.2853	.2713	.2252	.1596	.1220	.0985	.0709
.25	.0000	.0069	.0317	.0706	.1100	.1436	.1631	.1631	.1318	.1064	.0885	.0658
.50	.0000	.0049	.0175	.0328	.0483	.0652	.0825	.1053	.1045	.0909	.0785	.0607
.75	.0000	.0012	.0045	.0094	.0162	.0253	.0365	.0596	.0784	.0756	.0686	.0556
1.00	.0000	-0.0002	-0.0004	.0002	.0024	.0065	.0128	.0294	.0551	.0610	.0589	.0506
1.50	.0000	-0.0003	-0.0012	-0.0024	-0.0033	-0.0035	-0.0027	.0024	.0211	.0352	.0407	.0407
2.00	.0000	-0.0002	-0.0007	-0.0015	-0.0025	-0.0033	-0.0039	-0.0036	.0043	.0164	.0251	.0313
3.00	.0000	-0.0001	-0.0003	-0.0006	-0.0010	-0.0015	-0.0020	-0.0029	-0.0033	-0.0001	.0056	.0156
4.00	.0000	-0.0000	-0.0001	-0.0003	-0.0004	-0.0007	-0.0010	-0.0016	-0.0027	-0.0027	-0.0011	.0054
5.00	.0000	-0.0000	-0.0001	-0.0001	-0.0002	-0.0004	-0.0005	-0.0009	-0.0017	-0.0023	-0.0023	.0004
7.00	.0000	-0.0000	-0.0000	-0.0000	-0.0001	-0.0001	-0.0002	-0.0003	-0.0007	-0.0011	-0.0015	-0.0016
10.0	.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0001	-0.0001	-0.0002	-0.0004	-0.0006	-0.0009
15.0	.0000	.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0001	-0.0001	-0.0002	-0.0003
20.0	.0000	.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0001	-0.0001	-0.0002

$_sI_{240}$ $\omega \div \alpha$ = 0.5

az \ r	0	.25	.50	.75	1.00	1.25	1.50	2.00	3.00	4.00	5.00	7.00
0	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000
.25	.0000	.0015	.0032	.0115	.0299	.0392	.0364	.0240	.0120	.0071	.0045	.0024
.50	.0000	.0019	.0080	.0194	.0334	.0428	.0447	.0361	.0204	.0126	.0085	.0045
.75	.0000	.0022	.0087	.0184	.0290	.0373	.0414	.0391	.0256	.0167	.0116	.0064
1.00	.0000	.0019	.0072	.0147	.0229	.0300	.0348	.0370	.0281	.0196	.0140	.0080
1.50	.0000	.0010	.0039	.0081	.0130	.0179	.0222	.0277	.0275	.0220	.0170	.0105
2.00	.0000	.0005	.0021	.0044	.0073	.0105	.0137	.0190	.0232	.0213	.0179	.0120
3.00	.0000	.0002	.0007	.0016	.0027	.0041	.0056	.0088	.0140	.0161	.0159	.0129
4.00	.0000	.0001	.0003	.0007	.0012	.0019	.0027	.0044	.0081	.0108	.0122	.0118
5.00	.0000	.0000	.0002	.0004	.0007	.0010	.0014	.0024	.0048	.0071	.0088	.0100
7.00	.0000	.0000	.0001	.0001	.0002	.0004	.0006	.0010	.0020	.0033	.0045	.0064
10.0	.0000	.0000	.0000	.0000	.0001	.0001	.0002	.0003	.0007	.0013	.0019	.0031
15.0	.0000	.0000	.0000	.0000	.0000	.0000	.0001	.0001	.0002	.0004	.0006	.0011
20.0	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0001	.0002	.0003	.0006

$_sI_{240}$ $\omega \div \alpha$ = 1.0

az \ r	0	.25	.50	.75	1.00	1.25	1.50	2.00	3.00	4.00	5.00	7.00
0	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000
.25	.0000	.0019	.0077	.0260	.0567	.0740	.0711	.0480	.0240	.0141	.0091	.0048
.50	.0000	.0043	.0174	.0371	.0573	.0727	.0792	.0697	.0406	.0251	.0169	.0091
.75	.0000	.0039	.0144	.0284	.0429	.0558	.0651	.0696	.0499	.0332	.0231	.0128
1.00	.0000	.0023	.0086	.0177	.0282	.0386	.0478	.0589	.0527	.0382	.0277	.0160
1.50	.0000	.0007	.0028	.0063	.0109	.0165	.0225	.0339	.0447	.0404	.0326	.0207
2.00	.0000	.0003	.0011	.0026	.0046	.0073	.0105	.0178	.0310	.0351	.0324	.0232
3.00	.0000	.0001	.0003	.0007	.0013	.0021	.0030	.0055	.0123	.0192	.0233	.0229
4.00	.0000	.0000	.0001	.0003	.0005	.0008	.0012	.0022	.0053	.0094	.0135	.0181
5.00	.0000	.0000	.0001	.0001	.0003	.0004	.0006	.0011	.0026	.0048	.0075	.0125
7.00	.0000	.0000	.0000	.0001	.0001	.0001	.0002	.0004	.0009	.0017	.0027	.0054
10.0	.0000	.0000	.0000	.0000	.0000	.0001	.0001	.0001	.0003	.0005	.0009	.0018
15.0	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0001	.0001	.0002	.0003	.0007
20.0	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0001	.0001	.0002	.0004

$A_{220}(\psi)$

ψ \ r	0	.25	.50	.75	1.00	1.25	1.50	2.00	3.00	4.00	5.00	7.00
0	.5000	.4961	.4838	.4618	.4244	.3647	.3136	.2419	.1643	.1240	.0995	.0712
.25	.3904	.3871	.3757	.3561	.3289	.2957	.2637	.2125	.1510	.1165	.0945	.0687
.50	.3090	.3062	.2978	.2840	.2658	.2449	.2241	.1879	.1389	.1094	.0900	.0663
.75	.2500	.2480	.2421	.2328	.2206	.2069	.1928	.1668	.1280	.1028	.0856	.0640
1.00	.2071	.2057	.2017	.1954	.1872	.1778	.1680	.1489	.1181	.0967	.0815	.0618
1.50	.1514	.1507	.1488	.1457	.1417	.1370	.1318	.1211	.1013	.0858	.0740	.0576
2.00	.1180	.1177	.1167	.1150	.1128	.1102	.1073	.1009	.0879	.0766	.0673	.0537
3.00	.0811	.0810	.0807	.0801	.0792	.0782	.0771	.0744	.0682	.0620	.0563	.0469
4.00	.0615	.0615	.0613	.0611	.0607	.0602	.0597	.0583	.0551	.0514	.0478	.0413
5.00	.0495	.0495	.0494	.0493	.0491	.0488	.0485	.0478	.0459	.0436	.0412	.0367
7.00	.0355	.0356	.0355	.0355	.0354	.0353	.0352	.0349	.0341	.0331	.0320	.0295
10.0	.0249	.0251	.0250	.0250	.0250	.0250	.0249	.0248	.0245	.0241	.0237	.0226
15.0	.0165	.0170	.0170	.0170	.0170	.0170	.0170	.0169	.0168	.0167	.0166	.0162
20.0	.0125	.0132	.0132	.0132	.0132	.0132	.0132	.0132	.0132	.0131	.0130	.0128

$A_{222}(\psi)$

ψ \ r	0	.25	.50	.75	1.00	1.25	1.50	2.00	3.00	4.00	5.00	7.00
0	.5000	.5000	.5000	.5000	.5000	.3200	.2222	.1250	.0556	.0313	.0200	.0102
.25	.3787	.3762	.3662	.3464	.3026	.2354	.1779	.1078	.0507	.0289	.0189	.0098
.50	.2764	.2730	.2620	.2418	.2110	.1744	.1403	.0918	.0460	.0273	.0180	.0095
.75	.2000	.1971	.1883	.1737	.1541	.1321	.1110	.0776	.0416	.0254	.0170	.0091
1.00	.1464	.1444	.1383	.1286	.1161	.1023	.0887	.0655	.0374	.0235	.0160	.0088
1.50	.0840	.0831	.0804	.0762	.0709	.0649	.0587	.0471	.0300	.0201	.0142	.0081
2.00	.0528	.0524	.0512	.0492	.0468	.0439	.0408	.0346	.0241	.0171	.0125	.0074
3.00	.0257	.0255	.0252	.0247	.0240	.0231	.0222	.0201	.0159	.0123	.0096	.0062
4.00	.0149	.0149	.0148	.0146	.0143	.0140	.0136	.0128	.0109	.0090	.0074	.0051
5.00	.0097	.0097	.0096	.0095	.0094	.0093	.0091	.0087	.0078	.0068	.0058	.0042
7.00	.0050	.0050	.0050	.0050	.0049	.0049	.0048	.0047	.0044	.0041	.0037	.0030
10.0	.0025	.0024	.0024	.0024	.0024	.0024	.0024	.0024	.0023	.0022	.0021	.0018
15.0	.0011	.0010	.0010	.0010	.0010	.0010	.0010	.0010	.0010	.0010	.0010	.0009
20.0	.0005	.0005	.0005	.0005	.0005	.0005	.0005	.0005	.0005	.0005	.0005	.0005

$A_{224}(\psi)$

ψ \ r	0	.25	.50	.75	1.00	1.25	1.50	2.00	3.00	4.00	5.00	7.00
.25	.4565	.4640	.4819	.5151	.4782	.2943	.1666	.0671	.0191	.0080	.0040	.0015
.50	.3578	.3576	.3540	.3353	.2818	.2017	.1334	.0608	.0184	.0078	.0040	.0015
.75	.2500	.2528	.2417	.2192	.1833	.1406	.1021	.0526	.0174	.0076	.0039	.0014
1.00	.1768	.1736	.1640	.1474	.1252	.1005	.0775	.0442	.0161	.0072	.0038	.0014
1.50	.0853	.0838	.0795	.0726	.0640	.0547	.0455	.0302	.0132	.0065	.0035	.0014
2.00	.0447	.0441	.0423	.0395	.0359	.0320	.0279	.0205	.0105	.0056	.0032	.0013
3.00	.0158	.0157	.0153	.0147	.0139	.0130	.0120	.0099	.0063	.0040	.0025	.0011
4.00	.0071	.0071	.0070	.0068	.0066	.0063	.0060	.0053	.0039	.0027	.0019	.0010
5.00	.0038	.0038	.0037	.0037	.0036	.0035	.0034	.0031	.0025	.0019	.0014	.0008
7.00	.0014	.0014	.0014	.0014	.0014	.0014	.0013	.0013	.0011	.0009	.0008	.0005
10.0	.0005	.0005	.0005	.0005	.0005	.0005	.0005	.0005	.0004	.0004	.0004	.0003
15.0	.0001	.0002	.0002	.0002	.0002	.0002	.0002	.0001	.0001	.0001	.0001	.0001
20.0	.0001	.0001	.0001	.0001	.0001	.0001	.0001	.0001	.0001	.0001	.0001	.0001

$_cA_{220}$ ω÷α = 0.5

az \ r	0	.25	.50	.75	1.00	1.25	1.50	2.00	3.00	4.00	5.00	7.00
0	.5000	.4961	.4838	.4618	.4244	.3647	.3136	.2419	.1643	.1240	.0995	.0712
.25	.3868	.3832	.3720	.3524	.3253	.2934	.2624	.2121	.1509	.1164	.0945	.0687
.50	.2978	.2951	.2869	.2738	.2572	.2387	.2199	.1860	.1384	.1091	.0899	.0663
.75	.2323	.2306	.2256	.2179	.2082	.1973	.1858	.1631	.1268	.1023	.0854	.0639
1.00	.1861	.1851	.1823	.1780	.1723	.1658	.1586	.1434	.1161	.0958	.0810	.0616
1.50	.1298	.1295	.1286	.1272	.1252	.1227	.1197	.1128	.0976	.0840	.0730	.0572
2.00	.0987	.0986	.0982	.0977	.0968	.0958	.0945	.0912	.0827	.0738	.0657	.0531
3.00	.0663	.0663	.0662	.0661	.0659	.0656	.0653	.0643	.0615	.0577	.0535	.0457
4.00	.0499	.0499	.0498	.0498	.0497	.0496	.0495	.0491	.0480	.0463	.0441	.0394
5.00	.0399	.0399	.0399	.0399	.0399	.0398	.0398	.0396	.0391	.0382	.0371	.0342
7.00	.0285	.0286	.0286	.0285	.0285	.0285	.0285	.0284	.0283	.0280	.0276	.0265
10.0	.0200	.0200	.0200	.0200	.0200	.0200	.0200	.0200	.0200	.0199	.0198	.0194
15.0	.0133	.0135	.0135	.0135	.0135	.0135	.0135	.0135	.0135	.0135	.0135	.0134
20.0	.0100	.0106	.0106	.0106	.0106	.0106	.0106	.0106	.0105	.0105	.0105	.0105

$_cA_{220}$ ω÷α = 1.0

az \ r	0	.25	.50	.75	1.00	1.25	1.50	2.00	3.00	4.00	5.00	7.00
0	.5000	.4961	.4838	.4618	.4244	.3647	.3136	.2419	.1643	.1240	.0995	.0712
.25	.3760	.3719	.3606	.3410	.3148	.2865	.2585	.2109	.1504	.1162	.0945	.0687
.50	.2645	.2621	.2551	.2452	.2336	.2211	.2076	.1803	.1366	.1084	.0895	.0661
.75	.1846	.1840	.1822	.1795	.1759	.1714	.1658	.1520	.1231	.1007	.0845	.0636
1.00	.1360	.1361	.1363	.1364	.1361	.1351	.1334	.1275	.1100	.0930	.0796	.0611
1.50	.0874	.0876	.0880	.0888	.0896	.0903	.0909	.0911	.0867	.0785	.0699	.0560
2.00	.0643	.0644	.0647	.0651	.0657	.0663	.0670	.0681	.0684	.0655	.0608	.0511
3.00	.0422	.0423	.0424	.0425	.0427	.0430	.0433	.0440	.0454	.0460	.0454	.0418
4.00	.0315	.0315	.0315	.0316	.0317	.0318	.0320	.0323	.0332	.0340	.0345	.0339
5.00	.0251	.0251	.0251	.0252	.0252	.0253	.0254	.0256	.0261	.0266	.0272	.0276
7.00	.0179	.0178	.0178	.0178	.0179	.0179	.0179	.0180	.0182	.0185	.0187	.0193
10.0	.0125	.0123	.0123	.0123	.0123	.0123	.0123	.0124	.0124	.0125	.0126	.0129
15.0	.0083	.0080	.0080	.0080	.0080	.0080	.0080	.0080	.0080	.0081	.0081	.0082
20.0	.0063	.0063	.0063	.0063	.0063	.0063	.0063	.0063	.0064	.0064	.0064	.0065

$_sA_{220}$ ω÷α = 0.5

az \ r	0	.25	.50	.75	1.00	1.25	1.50	2.00	3.00	4.00	5.00	7.00
0	0	0	0	0	0	0	0	0	0	0	0	0
.25	.0472	.0555	.0540	.0508	.0457	.0377	.0305	.0215	.0147	.0120	.0106	.0095
.50	.0680	.0754	.0725	.0672	.0598	.0511	.0430	.0312	.0198	.0152	.0128	.0107
.75	.0717	.0790	.0757	.0704	.0636	.0562	.0490	.0371	.0239	.0178	.0147	.0118
1.00	.0678	.0752	.0724	.0681	.0628	.0569	.0509	.0404	.0269	.0201	.0163	.0127
1.50	.0553	.0632	.0617	.0593	.0563	.0529	.0493	.0420	.0304	.0233	.0189	.0144
2.00	.0449	.0530	.0522	.0509	.0493	.0473	.0451	.0404	.0316	.0251	.0207	.0157
3.00	.0318	.0400	.0397	.0393	.0387	.0379	.0370	.0350	.0304	.0260	.0224	.0175
4.00	.0243	.0326	.0325	.0323	.0320	.0317	.0313	.0303	.0278	.0250	.0225	.0183
5.00	.0197	.0279	.0279	.0278	.0276	.0274	.0272	.0267	.0252	.0235	.0218	.0185
7.00	.0142	.0224	.0224	.0224	.0223	.0222	.0222	.0219	.0214	.0206	.0198	.0179
10.0	.0100	.0181	.0181	.0181	.0181	.0181	.0180	.0179	.0177	.0175	.0171	.0163
15.0	.0067	.0145	.0145	.0145	.0145	.0145	.0145	.0144	.0144	.0143	.0142	.0139
20.0	.0050	.0124	.0124	.0124	.0124	.0124	.0124	.0124	.0124	.0123	.0123	.0121

$_sA_{220}$ ω÷α = 1.0

az \ r	0	.25	.50	.75	1.00	1.25	1.50	2.00	3.00	4.00	5.00	7.00
0	0	0	0	0	0	0	0	0	0	0	0	0
.25	.0938	.1015	.0988	.0922	.0812	.0661	.0525	.0349	.0210	.0157	.0130	.0108
.50	.1285	.1347	.1282	.1176	.1041	.0895	.0755	.0536	.0313	.0220	.0173	.0131
.75	.1237	.1301	.1244	.1160	.1059	.0948	.0838	.0640	.0392	.0273	.0211	.0152
1.00	.1069	.1141	.1108	.1056	.0991	.0917	.0838	.0683	.0447	.0316	.0243	.0171
1.50	.0783	.0862	.0852	.0834	.0810	.0780	.0745	.0664	.0499	.0374	.0292	.0203
2.00	.0604	.0686	.0682	.0675	.0665	.0652	.0636	.0596	.0496	.0399	.0323	.0229
3.00	.0411	.0494	.0492	.0490	.0488	.0484	.0479	.0468	.0433	.0388	.0340	.0260
4.00	.0310	.0393	.0393	.0392	.0391	.0389	.0387	.0383	.0368	.0348	.0322	.0268
5.00	.0249	.0332	.0331	.0331	.0330	.0330	.0329	.0326	.0319	.0309	.0295	.0262
7.00	.0178	.0260	.0260	.0260	.0260	.0260	.0259	.0259	.0256	.0253	.0248	.0235
10.0	.0125	.0205	.0205	.0205	.0205	.0205	.0205	.0204	.0203	.0202	.0201	.0196
15.0	.0083	.0155	.0155	.0155	.0155	.0155	.0155	.0155	.0155	.0154	.0154	.0152
20.0	.0062	.0122	.0122	.0122	.0122	.0122	.0122	.0122	.0121	.0121	.0121	.0120

$_{c}A_{222}$ ω÷α = 0.5

αz \ r	0	.25	.50	.75	1.00	1.25	1.50	2.00	3.00	4.00	5.00	7.00
0	.5000	.5000	.5000	.5000	.5000	.3200	.2222	.1250	.0556	.0313	.0200	.0102
.25	.3762	.3732	.3627	.3404	.2938	.2322	.1769	.1077	.0507	.0292	.0190	.0098
.50	.2525	.2584	.2457	.2238	.1955	.1651	.1360	.0908	.0459	.0273	.0180	.0095
.75	.1737	.1707	.1622	.1495	.1346	.1189	.1031	.0752	.0412	.0253	.0170	.0091
1.00	.1144	.1129	.1087	.1025	.0951	.0870	.0783	.0614	.0367	.0233	.0160	.0087
1.50	.0551	.0548	.0541	.0529	.0512	.0491	.0466	.0406	.0284	.0196	.0140	.0080
2.00	.0311	.0311	.0310	.0308	.0305	.0300	.0293	.0272	.0215	.0162	.0121	.0073
3.00	.0137	.0137	.0137	.0137	.0137	.0137	.0137	.0135	.0124	.0107	.0088	.0060
4.00	.0076	.0076	.0076	.0076	.0077	.0077	.0077	.0077	.0076	.0071	.0063	.0048
5.00	.0048	.0048	.0049	.0049	.0049	.0049	.0049	.0049	.0049	.0048	.0045	.0037
7.00	.0025	.0024	.0024	.0025	.0025	.0025	.0025	.0025	.0025	.0025	.0025	.0023
10.0	.0012	.0012	.0012	.0012	.0012	.0012	.0012	.0012	.0012	.0012	.0012	.0012
15.0	.0005	.0005	.0005	.0005	.0005	.0005	.0005	.0005	.0005	.0005	.0005	.0005
20.0	.0003	.0002	.0002	.0002	.0002	.0002	.0002	.0002	.0002	.0002	.0002	.0002

$_{c}A_{222}$ ω÷α = 1.0

αz \ r	0	.25	.50	.75	1.00	1.25	1.50	2.00	3.00	4.00	5.00	7.00
0	.5000	.5000	.5000	.5000	.5000	.3200	.2222	.1250	.0556	.0313	.0200	.0102
.25	.3680	.3642	.3518	.3192	.2703	.2218	.1739	.1072	.0507	.0296	.0191	.0098
.50	.2156	.2098	.1940	.1743	.1559	.1393	.1222	.0876	.0456	.0272	.0179	.0095
.75	.0960	.0949	.0927	.0903	.0879	.0848	.0804	.0671	.0401	.0250	.0169	.0091
1.00	.0398	.0406	.0429	.0457	.0485	.0504	.0511	.0485	.0343	.0228	.0158	.0087
1.50	.0069	.0094	.0107	.0127	.0151	.0176	.0199	.0232	.0228	.0179	.0134	.0079
2.00	.0029	.0031	.0036	.0044	.0054	.0067	.0081	.0107	.0138	.0131	.0109	.0071
3.00	.0006	.0006	.0007	.0009	.0011	.0014	.0018	.0027	.0046	.0059	.0063	.0053
4.00	.0002	.0002	.0002	.0003	.0004	.0005	.0006	.0009	.0017	.0025	.0032	.0035
5.00	.0001	.0001	.0001	.0001	.0001	.0002	.0002	.0004	.0007	.0012	.0016	.0022
7.00	.0000	.0000	.0000	.0000	.0000	.0000	.0001	.0001	.0002	.0003	.0005	.0008
10.0	.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	.0000	.0000	.0001	.0002
15.0	.0000	-0.0001	-0.0001	-0.0001	-0.0001	-0.0001	-0.0001	-0.0001	-0.0001	-0.0001	-0.0001	-0.0001
20.0	.0000	-0.0002	-0.0002	-0.0002	-0.0002	-0.0002	-0.0002	-0.0002	-0.0002	-0.0002	-0.0002	-0.0002

$_{s}A_{222}$ ω÷α = 0.5

αz \ r	0	.25	.50	.75	1.00	1.25	1.50	2.00	3.00	4.00	5.00	7.00
0	0	0	0	0	0	0	0	0	0	0	0	0
.25	.0574	.0583	.0600	.0640	.0581	.0364	.0209	.0084	.0024	.0006	.0004	.0002
.50	.0894	.0891	.0872	.0809	.0671	.0491	.0332	.0153	.0046	.0019	.0010	.0004
.75	.0922	.0906	.0854	.0762	.0637	.0499	.0372	.0197	.0066	.0028	.0015	.0005
1.00	.0798	.0781	.0732	.0656	.0562	.0462	.0366	.0218	.0081	.0036	.0019	.0007
1.50	.0515	.0506	.0483	.0447	.0403	.0354	.0304	.0214	.0099	.0049	.0027	.0010
2.00	.0333	.0330	.0320	.0304	.0284	.0260	.0235	.0183	.0102	.0056	.0032	.0013
3.00	.0164	.0163	.0161	.0157	.0152	.0145	.0138	.0121	.0085	.0057	.0037	.0017
4.00	.0096	.0095	.0095	.0093	.0091	.0089	.0086	.0080	.0065	.0049	.0036	.0019
5.00	.0062	.0062	.0062	.0061	.0061	.0060	.0058	.0056	.0048	.0040	.0032	.0019
7.00	.0032	.0032	.0032	.0032	.0032	.0032	.0031	.0031	.0028	.0026	.0022	.0016
10.0	.0015	.0016	.0016	.0016	.0016	.0016	.0016	.0016	.0015	.0014	.0014	.0011
15.0	.0007	.0008	.0008	.0008	.0007	.0007	.0007	.0007	.0007	.0007	.0007	.0007
20.0	.0004	.0004	.0004	.0004	.0004	.0004	.0004	.0004	.0004	.0004	.0004	.0004

$_{s}A_{222}$ ω÷α = 1.0

αz \ r	0	.25	.50	.75	1.00	1.25	1.50	2.00	3.00	4.00	5.00	7.00
0	0	0	0	0	0	0	0	0	0	0	0	0
.25	.1165	.1183	.1227	.1273	.1086	.0722	.0423	.0169	.0048	.0019	.0010	.0004
.50	.1758	.1733	.1637	.1442	.1180	.0902	.0646	.0309	.0093	.0039	.0020	.0007
.75	.1536	.1494	.1378	.1216	.1034	.0850	.0674	.0390	.0133	.0059	.0030	.0011
1.00	.1065	.1066	.1011	.0928	.0829	.0721	.0611	.0409	.0164	.0074	.0039	.0014
1.50	.0559	.0536	.0527	.0512	.0490	.0461	.0426	.0345	.0190	.0098	.0054	.0021
2.00	.0309	.0309	.0307	.0304	.0299	.0291	.0281	.0253	.0177	.0108	.0065	.0027
3.00	.0139	.0139	.0138	.0138	.0138	.0137	.0136	.0132	.0117	.0093	.0069	.0034
4.00	.0078	.0078	.0078	.0078	.0078	.0078	.0078	.0077	.0074	.0067	.0057	.0036
5.00	.0050	.0050	.0050	.0050	.0050	.0050	.0050	.0050	.0049	.0047	.0043	.0033
7.00	.0025	.0026	.0026	.0026	.0026	.0026	.0026	.0026	.0026	.0025	.0025	.0022
10.0	.0012	.0013	.0013	.0013	.0013	.0013	.0013	.0013	.0013	.0013	.0013	.0013
15.0	.0005	.0007	.0007	.0007	.0007	.0007	.0007	.0007	.0007	.0007	.0007	.0007
20.0	.0003	.0004	.0004	.0004	.0004	.0004	.0004	.0004	.0004	.0004	.0004	.0003

$M_{220}(\psi)$

ψ \ r	0	.25	.50	.75	1.00	1.25	1.50	2.00	3.00	4.00	5.00	7.00
0	-0.5000	-0.4881	-0.4504	-0.3776	-0.2122	-0.0794	-0.0423	-0.0167	-0.0048	-0.0020	-0.0010	-0.0004
.25	-0.3904	-0.3791	-0.3462	-0.2858	-0.2022	-0.1291	-0.0838	-0.0433	-0.0175	-0.0093	-0.0058	-0.0028
.50	-0.3090	-0.3006	-0.2756	-0.2352	-0.1856	-0.1390	-0.1031	-0.0608	-0.0276	-0.0156	-0.0100	-0.0051
.75	-0.2500	-0.2440	-0.2267	-0.2000	-0.1681	-0.1365	-0.1093	-0.0713	-0.0355	-0.0209	-0.0137	-0.0072
1.00	-0.2071	-0.2030	-0.1912	-0.1732	-0.1516	-0.1294	-0.1088	-0.0766	-0.0414	-0.0253	-0.0170	-0.0091
1.50	-0.1514	-0.1494	-0.1437	-0.1349	-0.1240	-0.1121	-0.1001	-0.0783	-0.0483	-0.0317	-0.0221	-0.0124
2.00	-0.1160	-0.1170	-0.1140	-0.1092	-0.1032	-0.0962	-0.0889	-0.0744	-0.0509	-0.0355	-0.0258	-0.0151
3.00	-0.0811	-0.0808	-0.0797	-0.0779	-0.0756	-0.0728	-0.0697	-0.0628	-0.0492	-0.0379	-0.0295	-0.0187
4.00	-0.0616	-0.0614	-0.0609	-0.0601	-0.0590	-0.0577	-0.0561	-0.0525	-0.0445	-0.0368	-0.0302	-0.0207
5.00	-0.0495	-0.0494	-0.0492	-0.0487	-0.0482	-0.0474	-0.0465	-0.0445	-0.0395	-0.0343	-0.0294	-0.0214
7.00	-0.0355	-0.0355	-0.0354	-0.0353	-0.0351	-0.0348	-0.0344	-0.0336	-0.0314	-0.0288	-0.0261	-0.0209
10.0	-0.0249	-0.0250	-0.0250	-0.0250	-0.0249	-0.0248	-0.0246	-0.0243	-0.0235	-0.0224	-0.0212	-0.0185
15.0	-0.0165	-0.0170	-0.0170	-0.0170	-0.0170	-0.0169	-0.0169	-0.0168	-0.0165	-0.0161	-0.0157	-0.0146
20.0	-0.0125	-0.0132	-0.0132	-0.0132	-0.0132	-0.0132	-0.0132	-0.0131	-0.0130	-0.0128	-0.0126	-0.0121

$M_{222}(\psi)$

ψ \ r	0	.25	.50	.75	1.00	1.25	1.50	2.00	3.00	4.00	5.00	7.00
0	-0.5000	-0.5000	-0.5000	-0.5000	0	.3200	.2222	.1250	.0556	.0313	.0200	.0102
.25	-0.3767	-0.3709	-0.3418	-0.2607	-0.0601	.0959	.1140	.0868	.0455	.0272	.0179	.0095
.50	-0.2764	-0.2661	-0.2317	-0.1650	-0.0705	.0085	.0453	.0548	.0360	.0232	.0159	.0087
.75	-0.2000	-0.1913	-0.1648	-0.1214	-0.0687	-0.0221	.0084	.0305	.0274	.0194	.0140	.0080
1.00	-0.1464	-0.1403	-0.1223	-0.0947	-0.0626	-0.0327	-0.0098	.0137	.0200	.0159	.0121	.0073
1.50	-0.0840	-0.0812	-0.0734	-0.0618	-0.0480	-0.0342	-0.0218	-0.0041	.0088	.0100	.0087	.0059
2.00	-0.0528	-0.0515	-0.0480	-0.0425	-0.0359	-0.0289	-0.0220	-0.0104	.0019	.0055	.0059	.0047
3.00	-0.0257	-0.0253	-0.0244	-0.0228	-0.0209	-0.0186	-0.0162	-0.0114	-0.0040	.0001	.0019	.0027
4.00	-0.0149	-0.0148	-0.0145	-0.0139	-0.0132	-0.0123	-0.0113	-0.0091	-0.0051	-0.0021	-0.0003	.0013
5.00	-0.0097	-0.0096	-0.0095	-0.0093	-0.0089	-0.0085	-0.0081	-0.0070	-0.0047	-0.0027	-0.0013	.0003
7.00	-0.0050	-0.0050	-0.0050	-0.0049	-0.0048	-0.0047	-0.0045	-0.0042	-0.0034	-0.0026	-0.0018	-0.0006
10.0	-0.0025	-0.0024	-0.0024	-0.0024	-0.0024	-0.0024	-0.0023	-0.0022	-0.0020	-0.0017	-0.0014	-0.0009
15.0	-0.0011	-0.0010	-0.0010	-0.0010	-0.0010	-0.0010	-0.0010	-0.0010	-0.0010	-0.0009	-0.0008	-0.0006
20.0	-0.0006	-0.0005	-0.0005	-0.0005	-0.0005	-0.0005	-0.0005	-0.0005	-0.0005	-0.0005	-0.0005	-0.0004

$M_{224}(\psi)$

ψ \ r	0	.25	.50	.75	1.00	1.25	1.50	2.00	3.00	4.00	5.00	7.00
.25	-0.4565	-0.4793	-0.5494	-0.5941	.0900	.5629	.3618	.1437	.0393	.0163	.0082	.0030
.50	-0.3578	-0.3567	-0.3377	-0.2358	.0099	.1946	.2025	.1138	.0365	.0156	.0080	.0029
.75	-0.2560	-0.2462	-0.2094	-0.1308	-0.0189	.8703	.1025	.0813	.0322	.0146	.0077	.0029
1.00	-0.1768	-0.1673	-0.1377	-0.0881	-0.0284	.0217	.0494	.0543	.0273	.0133	.0073	.0028
1.50	-0.0853	-0.0808	-0.0680	-0.0490	-0.0275	-0.0077	.0072	.0210	.0177	.0105	.0062	.0026
2.00	-0.0447	-0.0429	-0.0376	-0.0298	-0.0207	-0.0117	-0.0039	.0063	.0103	.0076	.0051	.0023
3.00	-0.0158	-0.0154	-0.0143	-0.0126	-0.0105	-0.0082	-0.0058	-0.0018	.0026	.0034	.0030	.0017
4.00	-0.0071	-0.0070	-0.0067	-0.0062	-0.0056	-0.0048	-0.0040	-0.0024	.0001	.0012	.0015	.0012
5.00	-0.0035	-0.0037	-0.0036	-0.0034	-0.0032	-0.0029	-0.0026	-0.0019	-0.0006	.0002	.0007	.0008
7.00	-0.0014	-0.0014	-0.0014	-0.0014	-0.0013	-0.0012	-0.0012	-0.0010	-0.0006	-0.0003	-0.0000	.0002
10.0	-0.0005	-0.0005	-0.0005	-0.0005	-0.0005	-0.0005	-0.0005	-0.0004	-0.0003	-0.0002	-0.0002	-0.0000
15.0	-0.0001	-0.0002	-0.0002	-0.0002	-0.0001	-0.0001	-0.0001	-0.0001	-0.0001	-0.0001	-0.0001	-0.0001
20.0	-0.0001	-0.0001	-0.0001	-0.0001	-0.0001	-0.0001	-0.0001	-0.0001	-0.0001	-0.0001	-0.0001	-0.0000

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$_{c}M_{220}$ $\omega \div a = 0.5$

αz \ r	0	.25	.50	.75	1.00	1.25	1.50	2.00	3.00	4.00	5.00	7.00
0	-0.5000	-0.4881	-0.4504	-0.3776	-0.2122	-0.0794	-0.0423	-0.0167	-0.0048	-0.0020	-0.0010	-0.0004
.25	-0.3868	-0.3757	-0.3419	-0.2816	-0.2030	-0.1328	-0.0867	-0.0444	-0.0178	-0.0094	-0.0058	-0.0028
.50	-0.2978	-0.2896	-0.2653	-0.2285	-0.1858	-0.1443	-0.1092	-0.0643	-0.0287	-0.0161	-0.0102	-0.0052
.75	-0.2323	-0.2272	-0.2127	-0.1914	-0.1665	-0.1406	-0.1159	-0.0769	-0.0377	-0.0219	-0.0143	-0.0074
1.00	-0.1861	-0.1832	-0.1751	-0.1628	-0.1478	-0.1311	-0.1140	-0.0831	-0.0448	-0.0270	-0.0179	-0.0094
1.50	-0.1298	-0.1289	-0.1263	-0.1220	-0.1161	-0.1090	-0.1008	-0.0833	-0.0532	-0.0347	-0.0239	-0.0131
2.00	-0.0987	-0.0983	-0.0973	-0.0956	-0.0931	-0.0898	-0.0859	-0.0763	-0.0556	-0.0392	-0.0283	-0.0162
3.00	-0.0663	-0.0662	-0.0660	-0.0656	-0.0649	-0.0641	-0.0629	-0.0599	-0.0512	-0.0413	-0.0327	-0.0207
4.00	-0.0499	-0.0498	-0.0497	-0.0496	-0.0494	-0.0490	-0.0486	-0.0474	-0.0437	-0.0385	-0.0328	-0.0229
5.00	-0.0399	-0.0399	-0.0399	-0.0398	-0.0397	-0.0395	-0.0394	-0.0388	-0.0370	-0.0342	-0.0308	-0.0235
7.00	-0.0285	-0.0286	-0.0285	-0.0285	-0.0285	-0.0284	-0.0284	-0.0282	-0.0276	-0.0267	-0.0254	-0.0219
10.0	-0.0200	-0.0200	-0.0200	-0.0200	-0.0200	-0.0200	-0.0200	-0.0199	-0.0197	-0.0195	-0.0191	-0.0178
15.0	-0.0133	-0.0135	-0.0135	-0.0135	-0.0135	-0.0135	-0.0135	-0.0135	-0.0134	-0.0134	-0.0133	-0.0129
20.0	-0.0100	-0.0106	-0.0106	-0.0106	-0.0106	-0.0106	-0.0105	-0.0105	-0.0105	-0.0105	-0.0104	-0.0103

$_{c}M_{220}$ $\omega \div a = 1.0$

αz \ r	0	.25	.50	.75	1.00	1.25	1.50	2.00	3.00	4.00	5.00	7.00
0	-0.5000	-0.4881	-0.4504	-0.3776	-0.2122	-0.0794	-0.0423	-0.0167	-0.0048	-0.0020	-0.0010	-0.0004
.25	-0.3760	-0.3650	-0.3289	-0.2704	-0.2048	-0.1429	-0.0953	-0.0478	-0.0186	-0.0095	-0.0060	-0.0029
.50	-0.2645	-0.2571	-0.2376	-0.2124	-0.1853	-0.1559	-0.1251	-0.0750	-0.0322	-0.0175	-0.0110	-0.0055
.75	-0.1846	-0.1828	-0.1777	-0.1701	-0.1597	-0.1460	-0.1293	-0.0925	-0.0447	-0.0250	-0.0159	-0.0080
1.00	-0.1360	-0.1363	-0.1367	-0.1361	-0.1337	-0.1286	-0.1207	-0.0982	-0.0549	-0.0321	-0.0206	-0.0105
1.50	-0.0874	-0.0879	-0.0893	-0.0911	-0.0928	-0.0938	-0.0936	-0.0887	-0.0656	-0.0433	-0.0292	-0.0153
2.00	-0.0643	-0.0646	-0.0654	-0.0667	-0.0682	-0.0697	-0.0709	-0.0718	-0.0641	-0.0491	-0.0357	-0.0197
3.00	-0.0422	-0.0423	-0.0426	-0.0431	-0.0437	-0.0445	-0.0453	-0.0470	-0.0487	-0.0461	-0.0398	-0.0262
4.00	-0.0315	-0.0315	-0.0316	-0.0319	-0.0321	-0.0325	-0.0329	-0.0339	-0.0359	-0.0367	-0.0356	-0.0285
5.00	-0.0251	-0.0251	-0.0252	-0.0253	-0.0254	-0.0256	-0.0259	-0.0264	-0.0278	-0.0289	-0.0294	-0.0271
7.00	-0.0179	-0.0178	-0.0179	-0.0179	-0.0179	-0.0180	-0.0181	-0.0183	-0.0189	-0.0196	-0.0202	-0.0209
10.0	-0.0125	-0.0123	-0.0123	-0.0123	-0.0124	-0.0124	-0.0124	-0.0125	-0.0127	-0.0129	-0.0132	-0.0138
15.0	-0.0083	-0.0080	-0.0080	-0.0080	-0.0080	-0.0080	-0.0080	-0.0080	-0.0081	-0.0082	-0.0083	-0.0085
20.0	-0.0063	-0.0063	-0.0063	-0.0063	-0.0063	-0.0063	-0.0064	-0.0064	-0.0064	-0.0065	-0.0065	-0.0067

$_{s}M_{220}$ $\omega \div a = 0.5$

αz \ r	0	.25	.50	.75	1.00	1.25	1.50	2.00	3.00	4.00	5.00	7.00
0	0	0	0	0	0	0	0	0	0	0	0	0
.25	-0.0472	-0.0540	-0.0507	-0.0393	-0.0159	.0016	.0059	.0025	-0.0027	-0.0049	-0.0061	-0.0071
.50	-0.0660	-0.0735	-0.0645	-0.0478	-0.0264	-0.0083	.0016	.0050	.0006	-0.0025	-0.0044	-0.0062
.75	-0.0717	-0.0768	-0.0670	-0.0520	-0.0346	-0.0189	-0.0076	.0020	.0018	-0.0011	-0.0031	-0.0053
1.00	-0.0678	-0.0733	-0.0653	-0.0534	-0.0399	-0.0269	-0.0161	-0.0034	.0012	-0.0005	-0.0023	-0.0047
1.50	-0.0553	-0.0621	-0.0578	-0.0512	-0.0433	-0.0350	-0.0271	-0.0143	-0.0030	-0.0013	-0.0020	-0.0039
2.00	-0.0449	-0.0524	-0.0501	-0.0465	-0.0419	-0.0368	-0.0314	-0.0214	-0.0084	-0.0038	-0.0028	-0.0037
3.00	-0.0318	-0.0398	-0.0390	-0.0377	-0.0359	-0.0338	-0.0314	-0.0262	-0.0164	-0.0099	-0.0065	-0.0046
4.00	-0.0243	-0.0325	-0.0322	-0.0316	-0.0308	-0.0298	-0.0286	-0.0258	-0.0197	-0.0142	-0.0103	-0.0065
5.00	-0.0197	-0.0279	-0.0277	-0.0274	-0.0270	-0.0264	-0.0258	-0.0242	-0.0204	-0.0164	-0.0130	-0.0085
7.00	-0.0142	-0.0224	-0.0223	-0.0222	-0.0221	-0.0218	-0.0216	-0.0210	-0.0193	-0.0174	-0.0153	-0.0116
10.0	-0.0100	-0.0181	-0.0181	-0.0180	-0.0180	-0.0179	-0.0178	-0.0176	-0.0170	-0.0162	-0.0153	-0.0132
15.0	-0.0067	-0.0145	-0.0145	-0.0145	-0.0145	-0.0144	-0.0144	-0.0143	-0.0141	-0.0139	-0.0136	-0.0127
20.0	-0.0050	-0.0124	-0.0124	-0.0124	-0.0124	-0.0124	-0.0124	-0.0123	-0.0122	-0.0121	-0.0120	-0.0116

$_{s}M_{220}$ $\omega \div a = 1.0$

αz \ r	0	.25	.50	.75	1.00	1.25	1.50	2.00	3.00	4.00	5.00	7.00
0	0	0	0	0	0	0	0	0	0	0	0	0
.25	-0.0938	-0.0996	-0.0911	-0.0662	-0.0245	.0080	.0186	.0130	.0030	-0.0015	-0.0039	-0.0059
.50	-0.1285	-0.1305	-0.1108	-0.0805	-0.0467	-0.0168	.0037	.0161	.0093	.0032	-0.0004	-0.0040
.75	-0.1237	-0.1261	-0.1101	-0.0877	-0.0629	-0.0391	-0.0187	.0056	.0108	.0059	.0020	-0.0024
1.00	-0.1069	-0.1118	-0.1022	-0.0879	-0.0709	-0.0531	-0.0360	-0.0094	.0080	.0066	.0034	-0.0011
1.50	-0.0763	-0.0855	-0.0824	-0.0772	-0.0701	-0.0615	-0.0520	-0.0324	-0.0052	.0030	.0034	.0003
2.00	-0.0604	-0.0683	-0.0671	-0.0649	-0.0618	-0.0579	-0.0531	-0.0417	-0.0187	-0.0048	.0001	.0003
3.00	-0.0411	-0.0493	-0.0489	-0.0483	-0.0475	-0.0463	-0.0449	-0.0413	-0.0310	-0.0195	-0.0107	-0.0031
4.00	-0.0310	-0.0393	-0.0391	-0.0389	-0.0385	-0.0381	-0.0375	-0.0360	-0.0316	-0.0254	-0.0187	-0.0087
5.00	-0.0249	-0.0331	-0.0331	-0.0330	-0.0328	-0.0326	-0.0323	-0.0315	-0.0293	-0.0261	-0.0220	-0.0137
7.00	-0.0178	-0.0260	-0.0260	-0.0260	-0.0259	-0.0258	-0.0257	-0.0255	-0.0247	-0.0236	-0.0221	-0.0181
10.0	-0.0125	-0.0205	-0.0205	-0.0205	-0.0204	-0.0204	-0.0204	-0.0203	-0.0200	-0.0197	-0.0192	-0.0178
15.0	-0.0083	-0.0155	-0.0155	-0.0155	-0.0155	-0.0155	-0.0155	-0.0154	-0.0153	-0.0152	-0.0150	-0.0145
20.0	-0.0062	-0.0122	-0.0122	-0.0122	-0.0122	-0.0122	-0.0122	-0.0121	-0.0121	-0.0120	-0.0119	-0.0116

$_cM_{222}$ ω÷a = 0.5

αz \ r	0	.25	.50	.75	1.00	1.25	1.50	2.00	3.00	4.00	5.00	7.00
0	-0.5000	-0.5000	-0.5000	-0.5000	0	.3200	.2222	.1250	.0556	.0313	.0200	.0102
.25	-0.3762	-0.3673	-0.3339	-0.2389	-0.0637	.0752	.1097	.0864	.0454	.0271	.0179	.0094
.50	-0.2625	-0.2502	-0.2107	-0.1461	-0.0757	-0.0137	.0289	.0506	.0355	.0231	.0159	.0087
.75	-0.1737	-0.1648	-0.1404	-0.1073	-0.0726	-0.0394	-0.0104	.0217	.0260	.0191	.0138	.0080
1.00	-0.1144	-0.1100	-0.0981	-0.0819	-0.0637	-0.0447	-0.0261	.0021	.0173	.0152	.0119	.0072
1.50	-0.0551	-0.0543	-0.0522	-0.0486	-0.0436	-0.0374	-0.0302	-0.0150	.0038	.0082	.0081	.0058
2.00	-0.0311	-0.0310	-0.0307	-0.0301	-0.0288	-0.0269	-0.0243	-0.0174	-0.0039	.0027	.0047	.0044
3.00	-0.0137	-0.0137	-0.0137	-0.0138	-0.0138	-0.0137	-0.0134	-0.0123	-0.0078	-0.0032	-0.0001	.0020
4.00	-0.0076	-0.0076	-0.0077	-0.0077	-0.0078	-0.0078	-0.0078	-0.0077	-0.0065	-0.0044	-0.0023	.0003
5.00	-0.0048	-0.0048	-0.0049	-0.0049	-0.0049	-0.0050	-0.0050	-0.0050	-0.0048	-0.0040	-0.0028	-0.0008
7.00	-0.0025	-0.0024	-0.0025	-0.0025	-0.0025	-0.0025	-0.0025	-0.0025	-0.0026	-0.0025	-0.0022	-0.0014
10.0	-0.0012	-0.0012	-0.0012	-0.0012	-0.0012	-0.0012	-0.0012	-0.0012	-0.0012	-0.0012	-0.0012	-0.0011
15.0	-0.0005	-0.0005	-0.0005	-0.0005	-0.0005	-0.0005	-0.0005	-0.0005	-0.0005	-0.0005	-0.0005	-0.0005
20.0	-0.0003	-0.0002	-0.0002	-0.0002	-0.0002	-0.0002	-0.0002	-0.0002	-0.0002	-0.0002	-0.0002	-0.0002

$_cM_{222}$ ω÷a = 1.0

αz \ r	0	.25	.50	.75	1.00	1.25	1.50	2.00	3.00	4.00	5.00	7.00
0	-0.5000	-0.5000	-0.5000	-0.5000	0	.3200	.2222	.1250	.0556	.0313	.0200	.0102
.25	-0.3680	-0.3564	-0.3077	-0.1876	-0.0737	.0237	.0918	.0844	.0453	.0271	.0180	.0094
.50	-0.2156	-0.1985	-0.1561	-0.1155	-0.0873	-0.0565	-0.0159	.0361	.0340	.0228	.0158	.0087
.75	-0.0960	-0.0931	-0.0878	-0.0832	-0.0773	-0.0666	-0.0491	-0.0058	.0215	.0181	.0135	.0079
1.00	-0.0398	-0.0423	-0.0483	-0.0545	-0.0581	-0.0574	-0.0516	-0.0279	.0084	.0130	.0111	.0071
1.50	-0.0089	-0.0103	-0.0141	-0.0194	-0.0251	-0.0298	-0.0328	-0.0316	-0.0112	.0024	.0058	.0054
2.00	-0.0029	-0.0034	-0.0049	-0.0072	-0.0101	-0.0133	-0.0164	-0.0205	-0.0167	-0.0059	.0006	.0035
3.00	-0.0005	-0.0007	-0.0010	-0.0015	-0.0022	-0.0031	-0.0042	-0.0064	-0.0099	-0.0094	-0.0059	-0.0003
4.00	-0.0002	-0.0002	-0.0003	-0.0005	-0.0007	-0.0010	-0.0014	-0.0023	-0.0043	-0.0057	-0.0057	-0.0028
5.00	-0.0001	-0.0001	-0.0001	-0.0002	-0.0003	-0.0004	-0.0006	-0.0010	-0.0019	-0.0030	-0.0037	-0.0033
7.00	-0.0000	-0.0000	-0.0000	-0.0000	-0.0001	-0.0001	-0.0001	-0.0002	-0.0005	-0.0009	-0.0013	-0.0019
10.0	-0.0000	.0000	.0000	.0000	.0000	.0000	-0.0000	-0.0000	-0.0001	-0.0002	-0.0003	-0.0006
15.0	-0.0000	.0001	.0001	.0001	.0001	.0001	.0001	.0001	.0001	.0001	.0001	.0000
20.0	-0.0000	.0002	.0002	.0002	.0002	.0002	.0002	.0002	.0002	.0002	.0002	.0002

$_sM_{222}$ ω÷a = 0.5

αz \ r	0	.25	.50	.75	1.00	1.25	1.50	2.00	3.00	4.00	5.00	7.00
0	0	0	0	0	0	0	0	0	0	0	0	0
.25	-0.0574	-0.0601	-0.0694	-0.0714	.0101	.0680	.0445	.0176	.0049	.0021	.0010	.0004
.50	-0.0894	-0.0863	-0.0804	-0.0510	.0007	.0404	.0476	.0285	.0092	.0039	.0020	.0007
.75	-0.0922	-0.0873	-0.0711	-0.0430	-0.0094	.0181	.0321	.0296	.0122	.0055	.0029	.0011
1.00	-0.0798	-0.0748	-0.0605	-0.0395	-0.0165	.0036	.0172	.0245	.0136	.0067	.0037	.0014
1.50	-0.0515	-0.0490	-0.0423	-0.0325	-0.0213	-0.0104	-0.0009	.0108	.0125	.0078	.0047	.0019
2.00	-0.0333	-0.0323	-0.0294	-0.0249	-0.0195	-0.0136	-0.0079	.0015	.0085	.0073	.0050	.0023
3.00	-0.0164	-0.0162	-0.0155	-0.0143	-0.0128	-0.0110	-0.0090	-0.0050	.0014	.0039	.0040	.0026
4.00	-0.0096	-0.0095	-0.0092	-0.0089	-0.0083	-0.0077	-0.0069	-0.0052	-0.0016	.0010	.0022	.0022
5.00	-0.0062	-0.0062	-0.0061	-0.0059	-0.0057	-0.0054	-0.0051	-0.0043	-0.0024	-0.0006	.0007	.0015
7.00	-0.0032	-0.0032	-0.0032	-0.0032	-0.0031	-0.0030	-0.0029	-0.0027	-0.0021	-0.0014	-0.0007	.0003
10.0	-0.0016	-0.0016	-0.0016	-0.0016	-0.0016	-0.0016	-0.0015	-0.0015	-0.0013	-0.0011	-0.0009	-0.0004
15.0	-0.0007	-0.0008	-0.0008	-0.0007	-0.0007	-0.0007	-0.0007	-0.0007	-0.0007	-0.0006	-0.0006	-0.0005
20.0	-0.0004	-0.0004	-0.0004	-0.0004	-0.0004	-0.0004	-0.0004	-0.0004	-0.0004	-0.0004	-0.0004	-0.0003

$_sM_{222}$ ω÷a = 1.0

αz \ r	0	.25	.50	.75	1.00	1.25	1.50	2.00	3.00	4.00	5.00	7.00
0	0	0	0	0	0	0	0	0	0	0	0	0
.25	-0.1165	-0.1221	-0.1377	-0.1130	.0168	.1108	.0916	.0366	.0099	.0040	.0020	.0007
.50	-0.1758	-0.1679	-0.1345	-0.0731	-0.0070	.0464	.0744	.0570	.0187	.0079	.0040	.0015
.75	-0.1536	-0.1411	-0.1091	-0.0689	-0.0292	.0059	.0330	.0509	.0248	.0112	.0059	.0022
1.00	-0.1085	-0.1027	-0.0869	-0.0650	-0.0409	-0.0172	.0041	.0312	.0268	.0137	.0075	.0028
1.50	-0.0539	-0.0530	-0.0503	-0.0456	-0.0387	-0.0300	-0.0201	-0.0005	.0187	.0151	.0095	.0040
2.00	-0.0309	-0.0308	-0.0302	-0.0291	-0.0273	-0.0247	-0.0212	-0.0120	.0058	.0115	.0096	.0048
3.00	-0.0139	-0.0139	-0.0138	-0.0137	-0.0135	-0.0132	-0.0127	-0.0112	-0.0057	.0008	.0046	.0048
4.00	-0.0078	-0.0078	-0.0078	-0.0078	-0.0078	-0.0077	-0.0076	-0.0073	-0.0059	-0.0033	-0.0003	.0029
5.00	-0.0050	-0.0050	-0.0050	-0.0050	-0.0050	-0.0050	-0.0050	-0.0049	-0.0045	-0.0035	-0.0021	.0008
7.00	-0.0026	-0.0026	-0.0026	-0.0026	-0.0026	-0.0026	-0.0026	-0.0026	-0.0025	-0.0024	-0.0021	-0.0011
10.0	-0.0012	-0.0013	-0.0013	-0.0013	-0.0013	-0.0013	-0.0013	-0.0013	-0.0013	-0.0013	-0.0013	-0.0011
15.0	-0.0006	-0.0007	-0.0007	-0.0007	-0.0007	-0.0007	-0.0007	-0.0007	-0.0007	-0.0007	-0.0007	-0.0006
20.0	-0.0003	-0.0004	-0.0004	-0.0004	-0.0004	-0.0004	-0.0004	-0.0004	-0.0004	-0.0003	-0.0003	-0.0003

$I_{400}(\psi)$

ψ \ r	0	.25	.50	.75	1.00	1.25	1.50	2.00	3.00	4.00	5.00	7.00
0	.5000	.4688	.3750	.2188	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000
.25	.3048	.2848	.2280	.1437	.0656	.0255	.0128	.0044	.0013	.0005	.0002	.0001
.50	.1910	.1795	.1478	.1043	.0633	.0357	.0207	.0084	.0024	.0010	.0005	.0002
.75	.1250	.1186	.1013	.0777	.0544	.0360	.0237	.0110	.0034	.0015	.0007	.0003
1.00	.0858	.0822	.0725	.0591	.0451	.0329	.0236	.0124	.0042	.0019	.0010	.0004
1.50	.0458	.0446	.0412	.0362	.0306	.0250	.0200	.0125	.0052	.0025	.0014	.0005
2.00	.0279	.0274	.0259	.0238	.0212	.0185	.0158	.0112	.0055	.0029	.0016	.0007
3.00	.0132	.0130	.0127	.0121	.0114	.0106	.0097	.0079	.0049	.0030	.0019	.0009
4.00	.0076	.0075	.0074	.0072	.0070	.0066	.0063	.0055	.0040	.0028	.0019	.0010
5.00	.0049	.0049	.0048	.0047	.0046	.0045	.0043	.0040	.0031	.0024	.0018	.0010
7.00	.0025	.0025	.0025	.0025	.0024	.0024	.0024	.0022	.0020	.0017	.0014	.0009
10.0	.0012	.0012	.0012	.0012	.0012	.0012	.0012	.0012	.0011	.0010	.0009	.0007
15.0	.0006	.0005	.0005	.0005	.0005	.0005	.0005	.0005	.0005	.0005	.0004	.0004
20.0	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0002	.0002

$I_{402}(\psi)$

ψ \ r	0	.25	.50	.75	1.00	1.25	1.50	2.00	3.00	4.00	5.00	7.00
0	1.0000	.9522	.7953	.4622	*******	-0.1280	-0.0562	-0.0193	-0.0051	-0.0020	-0.0010	-0.0004
.25	.5914	.5501	.4230	.1969	-0.0358	-0.0718	-0.0432	-0.0176	-0.0045	-0.0020	-0.0011	-0.0004
.50	.3416	.3150	.2381	.1273	.0289	-0.0144	-0.0204	-0.0128	-0.0043	-0.0019	-0.0010	-0.0004
.75	.2000	.1849	.1435	.0882	.0383	.0080	-0.0044	-0.0076	-0.0036	-0.0017	-0.0009	-0.0003
1.00	.1213	.1132	.0914	.0624	.0350	.0152	.0040	-0.0033	-0.0028	-0.0015	-0.0009	-0.0003
1.50	.0509	.0484	.0419	.0328	.0233	.0150	.0087	.0017	-0.0012	-0.0010	-0.0007	-0.0003
2.00	.0249	.0241	.0218	.0185	.0147	.0110	.0078	.0033	-0.0000	-0.0005	-0.0005	-0.0002
3.00	.0083	.0082	.0078	.0071	.0063	.0054	.0045	.0029	.0009	.0001	-0.0001	-0.0001
4.00	.0037	.0036	.0035	.0033	.0031	.0028	.0025	.0019	.0009	.0004	.0001	-0.0001
5.00	.0019	.0019	.0019	.0018	.0017	.0016	.0015	.0013	.0008	.0004	.0002	.0000
7.00	.0007	.0007	.0007	.0007	.0007	.0007	.0006	.0006	.0004	.0003	.0002	.0001
10.0	.0002	.0003	.0002	.0002	.0002	.0002	.0002	.0002	.0002	.0002	.0001	.0001
15.0	.0001	.0001	.0001	.0001	.0001	.0001	.0001	.0001	.0001	.0001	.0001	.0000
20.0	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000

$I_{404}(\psi)$

ψ \ r	0	.25	.50	.75	1.00	1.25	1.50	2.00	3.00	4.00	5.00	7.00
.25	1.2867	1.2366	1.0380	.4576	-0.5536	-0.2981	-0.0882	-0.0139	-0.0012	.0011	-0.0001	.0005
.50	.7478	.6928	.5130	.1979	-0.0971	-0.1407	-0.0810	-0.0208	-0.0026	-0.0005	-0.0002	-0.0000
.75	.4160	.3793	.2736	.1250	-0.0008	-0.0504	-0.0471	-0.0197	-0.0032	-0.0008	-0.0003	-0.0000
1.00	.2322	.2120	.1567	.0844	.0218	-0.0127	-0.0222	-0.0149	-0.0034	-0.0009	-0.0003	-0.0001
1.50	.0799	.0744	.0595	.0400	.0212	.0071	-0.0012	-0.0058	-0.0028	-0.0010	-0.0004	-0.0001
2.00	.0323	.0306	.0261	.0199	.0134	.0076	.0033	-0.0011	-0.0018	-0.0009	-0.0004	-0.0001
3.00	.0078	.0075	.0069	.0060	.0049	.0037	.0027	.0010	-0.0003	-0.0004	-0.0003	-0.0001
4.00	.0026	.0026	.0025	.0023	.0020	.0017	.0014	.0008	.0001	-0.0001	-0.0001	-0.0001
5.00	.0011	.0011	.0011	.0010	.0009	.0008	.0007	.0005	.0002	.0000	-0.0000	-0.0001
7.00	.0003	.0003	.0003	.0003	.0003	.0003	.0002	.0002	.0001	.0001	.0000	-0.0000
10.0	.0001	.0001	.0001	.0001	.0001	.0001	.0001	.0001	.0000	.0000	.0000	.0000
15.0	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000
20.0	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000

$_{c}I_{400}$ $\omega \div \alpha = 0.5$

αz \ r	0	.25	.50	.75	1.00	1.25	1.50	2.00	3.00	4.00	5.00	7.00
0	.5000	.4688	.3750	.2188	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000
.25	.2948	.2753	.2197	.1406	.0687	.0285	.0135	.0047	.0013	.0005	.0003	.0001
.50	.1682	.1585	.1324	.0984	.0656	.0399	.0233	.0092	.0025	.0010	.0005	.0002
.75	.0977	.0938	.0835	.0692	.0537	.0389	.0268	.0124	.0036	.0015	.0008	.0003
1.00	.0601	.0587	.0549	.0491	.0418	.0337	.0259	.0142	.0047	.0020	.0010	.0004
1.50	.0276	.0275	.0270	.0261	.0246	.0224	.0197	.0139	.0060	.0028	.0015	.0005
2.00	.0154	.0155	.0155	.0154	.0151	.0145	.0137	.0113	.0063	.0033	.0018	.0007
3.00	.0068	.0068	.0068	.0069	.0069	.0069	.0069	.0065	.0051	.0035	.0022	.0010
4.00	.0038	.0038	.0038	.0038	.0039	.0039	.0039	.0039	.0035	.0029	.0022	.0011
5.00	.0024	.0024	.0024	.0024	.0025	.0025	.0025	.0025	.0024	.0022	.0018	.0011
7.00	.0012	.0012	.0012	.0012	.0012	.0012	.0012	.0013	.0013	.0013	.0012	.0009
10.0	.0006	.0006	.0006	.0006	.0006	.0006	.0006	.0006	.0006	.0006	.0006	.0006
15.0	.0003	.0002	.0002	.0002	.0002	.0002	.0002	.0002	.0002	.0002	.0002	.0002
20.0	.0002	.0001	.0001	.0001	.0001	.0001	.0001	.0001	.0001	.0001	.0001	.0001

$_{c}I_{400}$ $\omega \div \alpha = 1.0$

αz \ r	0	.25	.50	.75	1.00	1.25	1.50	2.00	3.00	4.00	5.00	7.00
0	.5000	.4688	.3750	.2188	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000
.25	.2651	.2471	.1965	.1326	.0767	.0366	.0159	.0053	.0013	.0005	.0003	.0001
.50	.1069	.1030	.0939	.0825	.0676	.0490	.0311	.0114	.0028	.0011	.0005	.0002
.75	.0375	.0394	.0435	.0467	.0463	.0415	.0334	.0171	.0044	.0017	.0008	.0003
1.00	.0141	.0158	.0200	.0249	.0284	.0294	.0277	.0191	.0062	.0024	.0011	.0004
1.50	.0030	.0035	.0050	.0072	.0097	.0120	.0137	.0144	.0085	.0038	.0019	.0006
2.00	.0010	.0011	.0017	.0025	.0036	.0048	.0060	.0080	.0078	.0047	.0026	.0009
3.00	.0002	.0002	.0003	.0005	.0008	.0011	.0014	.0022	.0037	.0039	.0030	.0014
4.00	.0001	.0001	.0001	.0002	.0002	.0003	.0005	.0008	.0015	.0021	.0023	.0016
5.00	.0000	.0000	.0000	.0001	.0001	.0001	.0002	.0003	.0007	.0010	.0013	.0014
7.00	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0001	.0002	.0003	.0004	.0007
10.0	.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	.0000	.0000	.0001	.0001	.0002
15.0	.0000	-0.0001	-0.0001	-0.0001	-0.0001	-0.0001	-0.0001	-0.0001	-0.0001	-0.0000	-0.0000	-0.0000
20.0	.0000	-0.0001	-0.0001	-0.0001	-0.0001	-0.0001	-0.0001	-0.0001	-0.0001	-0.0001	-0.0001	-0.0001

$_{s}I_{400}$ $\omega \div \alpha = 0.5$

αz \ r	0	.25	.50	.75	1.00	1.25	1.50	2.00	3.00	4.00	5.00	7.00
0	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000
.25	.0731	.0679	.0522	.0241	-0.0024	-0.0090	-0.0054	-0.0024	-0.0006	-0.0002	-0.0001	-0.0000
.50	.0810	.0745	.0561	.0308	.0089	-0.0022	-0.0048	-0.0032	-0.0011	-0.0005	-0.0002	-0.0001
.75	.0667	.0617	.0485	.0314	.0156	.0049	-0.0006	-0.0027	-0.0014	-0.0007	-0.0003	-0.0001
1.00	.0505	.0475	.0394	.0286	.0179	.0092	.0035	-0.0012	-0.0014	-0.0008	-0.0004	-0.0002
1.50	.0287	.0277	.0249	.0208	.0161	.0115	.0076	.0022	-0.0007	-0.0007	-0.0005	-0.0002
2.00	.0177	.0174	.0163	.0146	.0125	.0102	.0079	.0041	.0003	-0.0005	-0.0005	-0.0003
3.00	.0084	.0083	.0081	.0077	.0072	.0065	.0058	.0043	.0017	.0004	-0.0001	-0.0002
4.00	.0049	.0048	.0047	.0046	.0044	.0042	.0039	.0033	.0020	.0010	.0003	-0.0001
5.00	.0031	.0031	.0031	.0030	.0030	.0029	.0028	.0025	.0018	.0011	.0006	.0001
7.00	.0016	.0016	.0016	.0016	.0016	.0016	.0015	.0014	.0012	.0010	.0007	.0003
10.0	.0008	.0008	.0008	.0008	.0008	.0008	.0008	.0008	.0007	.0006	.0006	.0004
15.0	.0004	.0004	.0004	.0004	.0004	.0004	.0004	.0004	.0004	.0003	.0003	.0003
20.0	.0002	.0002	.0002	.0002	.0002	.0002	.0002	.0002	.0002	.0002	.0002	.0002

$_{s}I_{400}$ $\omega \div \alpha = 1.0$

αz \ r	0	.25	.50	.75	1.00	1.25	1.50	2.00	3.00	4.00	5.00	7.00
0	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000
.25	.1411	.1307	.0982	.0455	-0.0006	-0.0154	-0.0112	-0.0046	-0.0012	-0.0005	-0.0003	-0.0001
.50	.1360	.1245	.0943	.0570	.0243	.0028	-0.0066	-0.0065	-0.0023	-0.0010	-0.0005	-0.0002
.75	.0913	.0861	.0721	.0536	.0342	.0174	.0053	-0.0041	-0.0028	-0.0013	-0.0007	-0.0003
1.00	.0582	.0565	.0514	.0434	.0335	.0230	.0134	.0011	-0.0027	-0.0016	-0.0009	-0.0003
1.50	.0274	.0272	.0265	.0251	.0230	.0200	.0164	.0087	-0.0002	-0.0014	-0.0010	-0.0005
2.00	.0156	.0155	.0154	.0151	.0146	.0138	.0127	.0096	.0028	-0.0002	-0.0008	-0.0005
3.00	.0069	.0069	.0069	.0069	.0069	.0068	.0066	.0062	.0044	.0021	.0005	-0.0003
4.00	.0039	.0039	.0039	.0039	.0039	.0039	.0039	.0038	.0033	.0025	.0015	.0002
5.00	.0025	.0025	.0025	.0025	.0025	.0025	.0025	.0025	.0023	.0021	.0016	.0006
7.00	.0013	.0013	.0013	.0013	.0013	.0013	.0013	.0013	.0013	.0012	.0012	.0009
10.0	.0006	.0007	.0007	.0007	.0007	.0007	.0007	.0007	.0007	.0007	.0006	.0006
15.0	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0003
20.0	.0002	.0002	.0002	.0002	.0002	.0002	.0002	.0002	.0002	.0002	.0002	.0002

$_cI_{402}$ ω÷α = 0.5

αz \ r	0	.25	.50	.75	1.00	1.25	1.50	2.00	3.00	4.00	5.00	7.00
0	1.0000	.9522	.7953	.4622	*******	-0.1280	-0.0562	-0.0193	-0.0051	-0.0020	-0.0010	-0.0004
.25	.5712	.5300	.4004	.1844	-0.0093	-0.0638	-0.0438	-0.0177	-0.0049	-0.0020	-0.0010	-0.0004
.50	.2885	.2639	.1976	.1159	.0475	.0018	-0.0157	-0.0130	-0.0045	-0.0019	-0.0010	-0.0004
.75	.1350	.1259	.1029	.0749	.0470	.0220	.0045	-0.0064	-0.0037	-0.0018	-0.0009	-0.0004
1.00	.0635	.0614	.0557	.0471	.0362	.0239	.0125	-0.0003	-0.0027	-0.0016	-0.0009	-0.0003
1.50	.0172	.0175	.0182	.0184	.0176	.0154	.0122	.0054	-0.0005	-0.0010	-0.0007	-0.0003
2.00	.0063	.0065	.0071	.0078	.0082	.0082	.0076	.0054	.0010	-0.0003	-0.0004	-0.0003
3.00	.0015	.0016	.0017	.0019	.0022	.0024	.0025	.0026	.0016	.0006	.0001	-0.0001
4.00	.0006	.0006	.0006	.0007	.0008	.0009	.0010	.0011	.0011	.0007	.0003	.0000
5.00	.0003	.0003	.0003	.0003	.0004	.0004	.0004	.0005	.0006	.0005	.0004	.0001
7.00	.0001	.0001	.0001	.0001	.0001	.0001	.0001	.0002	.0002	.0002	.0002	.0001
10.0	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0001	.0001	.0001	.0001
15.0	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000
20.0	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000

$_eI_{402}$ ω÷α = 1.0

αz \ r	0	.25	.50	.75	1.00	1.25	1.50	2.00	3.00	4.00	5.00	7.00
0	1.0000	.9522	.7953	.4622	*******	-0.1280	-0.0562	-0.0193	-0.0051	-0.0020	-0.0010	-0.0004
.25	.5097	.4655	.3318	.1579	.0479	-0.0378	-0.0460	-0.0186	-0.0054	-0.0020	-0.0009	-0.0004
.50	.1377	.1252	.1041	.0922	.0742	.0395	.0042	-0.0130	-0.0048	-0.0020	-0.0010	-0.0004
.75	-0.0052	.0030	.0222	.0404	.0481	.0426	.0276	.0005	-0.0040	-0.0019	-0.0010	-0.0004
1.00	-0.0248	-0.0193	-0.0055	.0102	.0223	.0275	.0257	.0108	-0.0022	-0.0017	-0.0010	-0.0004
1.50	-0.0138	-0.0128	-0.0099	-0.0054	-0.0002	.0049	.0087	.0108	.0026	-0.0006	-0.0007	-0.0003
2.00	-0.0067	-0.0065	-0.0058	-0.0047	-0.0031	-0.0012	.0008	.0042	.0041	.0010	-0.0002	-0.0003
3.00	-0.0022	-0.0022	-0.0021	-0.0019	-0.0017	-0.0014	-0.0011	-0.0003	.0014	.0016	.0009	-0.0000
4.00	-0.0009	-0.0009	-0.0009	-0.0009	-0.0008	-0.0008	-0.0007	-0.0005	.0001	.0006	.0007	.0003
5.00	-0.0005	-0.0005	-0.0005	-0.0005	-0.0005	-0.0004	-0.0004	-0.0003	-0.0002	.0001	.0003	.0003
7.00	-0.0002	-0.0002	-0.0002	-0.0002	-0.0002	-0.0002	-0.0002	-0.0002	-0.0001	-0.0001	-0.0000	.0001
10.0	-0.0001	-0.0001	-0.0001	-0.0001	-0.0001	-0.0001	-0.0001	-0.0001	-0.0000	-0.0000	-0.0000	.0000
15.0	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000
20.0	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000

$_sI_{402}$ ω÷α = 0.5

αz \ r	0	.25	.50	.75	1.00	1.25	1.50	2.00	3.00	4.00	5.00	7.00
0	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000
.25	.1598	.1528	.1266	.0476	-0.0587	-0.0418	-0.0119	-0.0021	.0002	-0.0000	-0.0001	-0.0000
.50	.1778	.1632	.1172	.0454	-0.0147	-0.0309	-0.0208	-0.0055	-0.0007	-0.0001	-0.0000	-0.0000
.75	.1355	.1227	.0879	.0437	.0069	-0.0123	-0.0157	-0.0078	-0.0013	-0.0003	-0.0001	-0.0000
1.00	.0904	.0829	.0630	.0380	.0153	-0.0001	-0.0074	-0.0074	-0.0018	-0.0005	-0.0002	-0.0000
1.50	.0389	.0369	.0315	.0239	.0157	.0083	.0026	-0.0028	-0.0022	-0.0008	-0.0003	-0.0001
2.00	.0187	.0182	.0166	.0141	.0111	.0079	.0049	.0006	-0.0015	-0.0009	-0.0004	-0.0001
3.00	.0061	.0060	.0058	.0054	.0049	.0042	.0035	.0021	.0001	-0.0005	-0.0004	-0.0002
4.00	.0026	.0026	.0026	.0025	.0023	.0022	.0020	.0015	.0006	.0000	-0.0002	-0.0002
5.00	.0014	.0014	.0013	.0013	.0013	.0012	.0011	.0010	.0006	.0002	-0.0000	-0.0001
7.00	.0005	.0005	.0005	.0005	.0005	.0005	.0005	.0004	.0003	.0002	.0001	-0.0000
10.0	.0002	.0002	.0002	.0002	.0002	.0002	.0002	.0002	.0001	.0001	.0001	.0000
15.0	.0001	.0001	.0001	.0001	.0001	.0001	.0001	.0001	.0001	.0000	.0000	.0000
20.0	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000

$_sI_{402}$ ω÷α = 1.0

αz \ r	0	.25	.50	.75	1.00	1.25	1.50	2.00	3.00	4.00	5.00	7.00
0	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000
.25	.3133	.2988	.2368	.0771	-0.0784	-0.0797	-0.0271	-0.0038	-0.0003	-0.0001	-0.0001	-0.0000
.50	.2968	.2648	.1791	.0802	.0047	-0.0363	-0.0401	-0.0133	-0.0015	-0.0003	-0.0001	-0.0000
.75	.1609	.1473	.1138	.0734	.0346	.0033	-0.0155	-0.0171	-0.0030	-0.0007	-0.0002	-0.0000
1.00	.0761	.0738	.0664	.0537	.0371	.0194	.0039	-0.0108	-0.0045	-0.0012	-0.0004	-0.0001
1.50	.0216	.0219	.0222	.0221	.0207	.0177	.0133	.0034	-0.0041	-0.0020	-0.0008	-0.0002
2.00	.0087	.0088	.0091	.0094	.0097	.0095	.0089	.0061	-0.0006	-0.0018	-0.0010	-0.0003
3.00	.0024	.0025	.0025	.0026	.0027	.0028	.0029	.0029	.0020	.0003	-0.0005	-0.0004
4.00	.0010	.0010	.0010	.0011	.0011	.0011	.0012	.0012	.0012	.0008	.0003	-0.0003
5.00	.0005	.0005	.0005	.0005	.0005	.0006	.0006	.0006	.0007	.0006	.0004	-0.0000
7.00	.0002	.0002	.0002	.0002	.0002	.0002	.0002	.0002	.0002	.0002	.0002	.0002
10.0	.0001	.0001	.0001	.0001	.0001	.0001	.0001	.0001	.0001	.0001	.0001	.0001
15.0	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000
20.0	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000

$I_{420}(\psi)$

ψ/r_0 \ r	0	.25	.50	.75	1.00	1.25	1.50	2.00	3.00	4.00	5.00	7.00
0	.0000	.1220	.2252	.2832	.2122	.0992	.0635	.0335	.0143	.0079	.0051	.0026
.25	.0000	.0717	.1282	.1496	.1267	.0867	.0586	.0325	.0141	.0079	.0050	.0027
.50	.0000	.0410	.0724	.0858	.0801	.0646	.0494	.0298	.0136	.0077	.0050	.0026
.75	.0000	.0241	.0427	.0523	.0525	.0467	.0390	.0261	.0129	.0075	.0049	.0025
1.00	.0000	.0147	.0265	.0335	.0355	.0338	.0302	.0222	.0120	.0072	.0048	.0025
1.50	.0000	.0062	.0116	.0154	.0177	.0184	.0179	.0154	.0099	.0064	.0044	.0024
2.00	.0000	.0031	.0058	.0081	.0097	.0106	.0110	.0105	.0079	.0056	.0040	.0023
3.00	.0000	.0010	.0020	.0029	.0036	.0042	.0047	.0051	.0048	.0040	.0031	.0020
4.00	.0000	.0005	.0009	.0013	.0017	.0020	.0023	.0027	.0029	.0027	.0024	.0017
5.00	.0000	.0002	.0005	.0007	.0009	.0011	.0013	.0016	.0019	.0019	.0018	.0014
7.00	.0000	.0001	.0002	.0003	.0003	.0004	.0005	.0006	.0008	.0010	.0010	.0009
10.0	.0000	.0000	.0001	.0001	.0001	.0002	.0002	.0002	.0003	.0004	.0005	.0005
15.0	.0000	.0000	.0000	.0000	.0000	.0000	.0001	.0001	.0001	.0001	.0002	.0002
20.0	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0001	.0001	.0001

$I_{422}(\psi)$

ψ/r_0 \ r	0	.25	.50	.75	1.00	1.25	1.50	2.00	3.00	4.00	5.00	7.00
0	.0000	.2500	.5000	.7500	.5000	.0000	.0000	.0000	.0000	.0000	.0000	.0000
.25	.0000	.1578	.2921	.3592	.2425	.0838	.0325	.0079	.0017	.0003	.0001	.0001
.50	.0000	.0901	.1583	.1794	.1406	.0813	.0425	.0132	.0025	.0008	.0003	.0001
.75	.0000	.0497	.0859	.0981	.0853	.0609	.0390	.0154	.0034	.0011	.0004	.0001
1.00	.0000	.0278	.0483	.0569	.0535	.0432	.0316	.0152	.0039	.0013	.0006	.0002
1.50	.0000	.0096	.0173	.0218	.0229	.0213	.0183	.0118	.0042	.0017	.0008	.0002
2.00	.0000	.0039	.0073	.0096	.0108	.0110	.0103	.0080	.0038	.0017	.0009	.0003
3.00	.0000	.0010	.0018	.0026	.0031	.0035	.0036	.0035	.0024	.0015	.0008	.0003
4.00	.0000	.0003	.0006	.0009	.0012	.0013	.0015	.0016	.0014	.0010	.0007	.0003
5.00	.0000	.0001	.0003	.0004	.0005	.0006	.0007	.0008	.0008	.0007	.0005	.0003
7.00	.0000	.0000	.0001	.0001	.0001	.0002	.0002	.0003	.0003	.0003	.0003	.0002
10.0	.0000	.0000	.0000	.0000	.0000	.0000	.0001	.0001	.0001	.0001	.0001	.0001
15.0	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000
20.0	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000

$I_{424}(\psi)$

ψ/r_0 \ r	0	.25	.50	.75	1.00	1.25	1.50	2.00	3.00	4.00	5.00	7.00
.25	.0000	.3293	.6962	1.0713	.5682	-0.1013	-0.0882	-0.0278	-0.0051	-0.0014	-0.0009	-0.0003
.50	.0000	.2116	.3942	.4598	.2917	.0734	-0.0055	-0.0149	-0.0041	-0.0014	-0.0006	-0.0002
.75	.0000	.1181	.2049	.2240	.1644	.0801	.0262	-0.0032	-0.0029	-0.0012	-0.0005	-0.0001
1.00	.0000	.0631	.1073	.1184	.0967	.0610	.0309	.0041	-0.0016	-0.0009	-0.0005	-0.0001
1.50	.0000	.0188	.0327	.0385	.0365	.0295	.0210	.0081	.0003	-0.0004	-0.0003	-0.0001
2.00	.0000	.0065	.0116	.0146	.0152	.0140	.0116	.0066	.0012	.0001	-0.0001	-0.0001
3.00	.0000	.0012	.0022	.0030	.0034	.0036	.0035	.0029	.0013	.0004	.0001	-0.0000
4.00	.0000	.0003	.0006	.0008	.0010	.0012	.0012	.0012	.0008	.0004	.0002	.0000
5.00	.0000	.0001	.0002	.0003	.0004	.0004	.0005	.0005	.0004	.0003	.0002	.0000
7.00	.0000	.0000	.0000	.0001	.0001	.0001	.0001	.0001	.0001	.0001	.0001	.0000
10.0	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000
15.0	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000
20.0	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000

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$_{c}I_{420}$ $\omega \div \alpha$ = 0.5

αz \ r	0	.25	.50	.75	1.00	1.25	1.50	2.00	3.00	4.00	5.00	7.00
0	.0000	.1220	.2252	.2832	.2122	.0992	.0635	.0335	.0143	.0079	.0051	.0026
.25	.0000	.0688	.1221	.1415	.1223	.0871	.0596	.0327	.0141	.0079	.0050	.0026
.50	.0000	.0345	.0603	.0723	.0713	.0619	.0495	.0302	.0138	.0078	.0050	.0026
.75	.0000	.0163	.0295	.0380	.0417	.0409	.0369	.0263	.0131	.0076	.0049	.0025
1.00	.0000	.0078	.0149	.0206	.0246	.0264	.0261	.0216	.0122	.0073	.0048	.0025
1.50	.0000	.0022	.0044	.0068	.0090	.0110	.0123	.0129	.0098	.0065	.0045	.0024
2.00	.0000	.0008	.0017	.0027	.0038	.0049	.0059	.0073	.0072	.0055	.0041	.0023
3.00	.0000	.0002	.0004	.0007	.0009	.0013	.0016	.0024	.0033	.0034	.0030	.0020
4.00	.0000	.0001	.0002	.0002	.0003	.0005	.0006	.0009	.0015	.0019	.0020	.0016
5.00	.0000	.0000	.0001	.0001	.0002	.0002	.0003	.0004	.0007	.0011	.0012	.0012
7.00	.0000	.0000	.0000	.0000	.0001	.0001	.0001	.0001	.0002	.0004	.0005	.0006
10.0	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0001	.0001	.0002	.0002
15.0	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0001
20.0	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000

$_{c}I_{420}$ $\omega \div \alpha$ = 1.0

αz \ r	0	.25	.50	.75	1.00	1.25	1.50	2.00	3.00	4.00	5.00	7.00
0	.0000	.1220	.2252	.2832	.2122	.0992	.0635	.0335	.0143	.0079	.0051	.0026
.25	.0000	.0607	.1044	.1191	.1100	.0872	.0622	.0334	.0143	.0079	.0051	.0026
.50	.0000	.0164	.0294	.0399	.0483	.0516	.0476	.0315	.0142	.0079	.0050	.0025
.75	.0000	-0.0001	.0024	.0084	.0162	.0234	.0276	.0258	.0137	.0079	.0050	.0026
1.00	.0000	-0.0027	-0.0036	-0.0018	.0024	.0077	.0126	.0175	.0127	.0077	.0050	.0026
1.50	.0000	-0.0017	-0.0029	-0.0036	-0.0033	-0.0020	-0.0001	.0045	.0085	.0067	.0047	.0025
2.00	.0000	-0.0008	-0.0016	-0.0021	-0.0025	-0.0024	-0.0021	-0.0004	.0038	.0049	.0041	.0024
3.00	.0000	-0.0003	-0.0005	-0.0008	-0.0010	-0.0011	-0.0012	-0.0012	-0.0003	.0012	.0021	.0020
4.00	.0000	-0.0001	-0.0002	-0.0003	-0.0004	-0.0005	-0.0006	-0.0007	-0.0007	-0.0002	.0005	.0013
5.00	.0000	-0.0001	-0.0001	-0.0002	-0.0002	-0.0003	-0.0003	-0.0004	-0.0005	-0.0004	-0.0001	.0006
7.00	.0000	-0.0000	-0.0000	-0.0001	-0.0001	-0.0001	-0.0001	-0.0002	-0.0002	-0.0002	-0.0002	-0.0000
10.0	.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0001	-0.0001	-0.0001	-0.0001	-0.0001
15.0	.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000
20.0	.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000

$_{s}I_{420}$ $\omega \div \alpha$ = 0.5

αz \ r	0	.25	.50	.75	1.00	1.25	1.50	2.00	3.00	4.00	5.00	7.00
0	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000
.25	.0000	.0199	.0372	.0433	.0299	.0116	.0037	.0010	.0001	.0001	.0000	.0001
.50	.0000	.0213	.0370	.0413	.0334	.0207	.0111	.0035	.0006	.0002	.0001	.0000
.75	.0000	.0161	.0277	.0319	.0290	.0222	.0150	.0061	.0013	.0004	.0002	.0001
1.00	.0000	.0108	.0190	.0231	.0229	.0198	.0155	.0080	.0021	.0007	.0003	.0001
1.50	.0000	.0047	.0088	.0116	.0130	.0130	.0121	.0087	.0034	.0013	.0006	.0002
2.00	.0000	.0023	.0044	.0061	.0073	.0080	.0081	.0072	.0039	.0018	.0009	.0003
3.00	.0000	.0008	.0015	.0021	.0027	.0032	.0036	.0039	.0033	.0022	.0013	.0005
4.00	.0000	.0003	.0007	.0010	.0012	.0015	.0017	.0021	.0022	.0019	.0014	.0007
5.00	.0000	.0002	.0003	.0005	.0007	.0008	.0009	.0012	.0014	.0014	.0012	.0007
7.00	.0000	.0001	.0001	.0002	.0002	.0003	.0004	.0005	.0006	.0007	.0007	.0006
10.0	.0000	.0000	.0000	.0001	.0001	.0001	.0001	.0002	.0002	.0003	.0003	.0004
15.0	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0001	.0001	.0001	.0001	.0002
20.0	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0001	.0001

$_{s}I_{420}$ $\omega \div \alpha$ = 1.0

αz \ r	0	.25	.50	.75	1.00	1.25	1.50	2.00	3.00	4.00	5.00	7.00
0	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000
.25	.0000	.0382	.0705	.0796	.0567	.0253	.0087	.0022	.0003	.0001	.0000	.0001
.50	.0000	.0351	.0568	.0654	.0573	.0414	.0252	.0078	.0013	.0004	.0002	.0001
.75	.0000	.0192	.0340	.0419	.0429	.0383	.0302	.0142	.0029	.0009	.0004	.0001
1.00	.0000	.0094	.0178	.0244	.0282	.0288	.0266	.0174	.0048	.0015	.0006	.0002
1.50	.0000	.0027	.0055	.0083	.0109	.0131	.0145	.0144	.0077	.0031	.0013	.0004
2.00	.0000	.0011	.0022	.0034	.0046	.0059	.0070	.0086	.0076	.0043	.0021	.0006
3.00	.0000	.0003	.0006	.0009	.0013	.0017	.0020	.0028	.0040	.0039	.0029	.0012
4.00	.0000	.0001	.0003	.0004	.0005	.0007	.0008	.0011	.0018	.0023	.0023	.0015
5.00	.0000	.0001	.0001	.0002	.0003	.0003	.0004	.0006	.0009	.0012	.0015	.0014
7.00	.0000	.0000	.0000	.0001	.0001	.0001	.0001	.0002	.0003	.0004	.0006	.0008
10.0	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0001	.0001	.0001	.0002	.0003
15.0	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0001	.0001	.0001
20.0	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0001

$_cI_{422}$ $\omega \div \alpha = 0.5$

αz \ r	0	.25	.50	.75	1.00	1.25	1.50	2.00	3.00	4.00	5.00	7.00
0	.0000	.2500	.5000	.7500	.5000	.0000	.0000	.0000	.0000	.0000	.0000	.0000
.25	.0000	.1544	.2831	.3274	.2302	.0967	.0350	.0081	.0013	.0004	.0002	.0000
.50	.0000	.0756	.1271	.1386	.1198	.0848	.0490	.0150	.0026	.0008	.0003	.0001
.75	.0000	.0300	.0511	.0610	.0620	.0548	.0416	.0181	.0037	.0012	.0005	.0001
1.00	.0000	.0105	.0196	.0268	.0314	.0324	.0292	.0174	.0046	.0015	.0006	.0002
1.50	.0000	.0010	.0025	.0049	.0076	.0100	.0115	.0109	.0050	.0020	.0009	.0002
2.00	.0000	-0.0001	.0001	.0007	.0016	.0028	.0040	.0053	.0041	.0021	.0010	.0003
3.00	.0000	-0.0001	-0.0002	-0.0002	-0.0001	.0001	.0004	.0010	.0017	.0015	.0010	.0004
4.00	.0000	-0.0000	-0.0001	-0.0001	-0.0001	-0.0001	-0.0000	.0001	.0005	.0007	.0007	.0004
5.00	.0000	-0.0000	-0.0000	-0.0001	-0.0001	-0.0001	-0.0001	-0.0000	.0002	.0003	.0004	.0003
7.00	.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	.0000	.0000	.0001	.0001
10.0	.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	.0000	.0000
15.0	.0000	.0000	.0000	.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	.0000
20.0	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000

$_cI_{422}$ $\omega \div \alpha = 1.0$

αz \ r	0	.25	.50	.75	1.00	1.25	1.50	2.00	3.00	4.00	5.00	7.00
0	.0000	.2500	.5000	.7500	.5000	.0000	.0000	.0000	.0000	.0000	.0000	.0000
.25	.0000	.1427	.2485	.2453	.1966	.1247	.0457	.0093	.0010	.0005	.0002	.0000
.50	.0000	.0290	.0414	.0513	.0686	.0769	.0625	.0209	.0031	.0009	.0003	.0001
.75	.0000	-0.0137	-0.0169	-0.0070	.0105	.0271	.0357	.0256	.0050	.0014	.0005	.0001
1.00	.0000	-0.0128	-0.0195	-0.0179	-0.0096	.0016	.0118	.0191	.0067	.0020	.0007	.0002
1.50	.0000	-0.0041	-0.0075	-0.0096	-0.0100	-0.0084	-0.0052	.0024	.0065	.0030	.0012	.0003
2.00	.0000	-0.0014	-0.0027	-0.0039	-0.0047	-0.0050	-0.0048	-0.0027	.0026	.0028	.0015	.0004
3.00	.0000	-0.0003	-0.0006	-0.0008	-0.0011	-0.0013	-0.0015	-0.0017	-0.0009	.0005	.0010	.0006
4.00	.0000	-0.0001	-0.0002	-0.0003	-0.0004	-0.0004	-0.0005	-0.0007	-0.0007	-0.0004	.0001	.0004
5.00	.0000	-0.0000	-0.0001	-0.0001	-0.0001	-0.0002	-0.0002	-0.0003	-0.0004	-0.0004	-0.0002	.0002
7.00	.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0001	-0.0001	-0.0001	-0.0001	-0.0001	-0.0001
10.0	.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000
15.0	.0000	.0000	.0000	.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000
20.0	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	-0.0000	-0.0000	-0.0000	-0.0000

$_sI_{422}$ $\omega \div \alpha = 0.5$

αz \ r	0	.25	.50	.75	1.00	1.25	1.50	2.00	3.00	4.00	5.00	7.00
0	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000
.25	.0000	.0415	.0865	.1299	.0682	-0.0102	-0.0102	-0.0034	-0.0002	-0.0003	-0.0001	.0000
.50	.0000	.0514	.0926	.1019	.0678	.0232	.0010	-0.0037	-0.0010	-0.0004	-0.0001	-0.0000
.75	.0000	.0384	.0644	.0691	.0543	.0316	.0131	-0.0005	-0.0011	-0.0004	-0.0002	-0.0001
1.00	.0000	.0234	.0396	.0447	.0397	.0289	.0174	.0034	-0.0008	-0.0005	-0.0002	-0.0001
1.50	.0000	.0079	.0143	.0180	.0190	.0174	.0143	.0071	.0006	-0.0002	-0.0002	-0.0001
2.00	.0000	.0030	.0057	.0077	.0089	.0092	.0088	.0064	.0017	.0002	-0.0001	-0.0001
3.00	.0000	.0007	.0013	.0019	.0024	.0027	.0030	.0030	.0020	.0009	.0002	-0.0000
4.00	.0000	.0002	.0004	.0006	.0008	.0010	.0011	.0013	.0012	.0008	.0004	.0001
5.00	.0000	.0001	.0002	.0003	.0003	.0004	.0005	.0006	.0007	.0006	.0004	.0001
7.00	.0000	.0000	.0000	.0001	.0001	.0001	.0001	.0002	.0002	.0003	.0002	.0002
10.0	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0001	.0001	.0001	.0001
15.0	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000
20.0	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000

$_sI_{422}$ $\omega \div \alpha = 1.0$

αz \ r	0	.25	.50	.75	1.00	1.25	1.50	2.00	3.00	4.00	5.00	7.00
0	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000
.25	.0000	.0847	.1766	.2250	.1255	.0015	-0.0213	-0.0074	-0.0013	-0.0004	-0.0002	-0.0000
.50	.0000	.0892	.1443	.1458	.1110	.0609	.0170	-0.0064	-0.0022	-0.0007	-0.0003	-0.0001
.75	.0000	.0407	.0679	.0782	.0743	.0590	.0375	.0048	-0.0022	-0.0009	-0.0004	-0.0001
1.00	.0000	.0137	.0265	.0366	.0419	.0411	.0347	.0145	-0.0009	-0.0009	-0.0005	-0.0001
1.50	.0000	.0017	.0039	.0069	.0103	.0134	.0154	.0144	.0036	-0.0001	-0.0004	-0.0002
2.00	.0000	.0003	.0008	.0015	.0025	.0038	.0051	.0071	.0052	.0014	.0001	-0.0002
3.00	.0000	.0000	.0001	.0001	.0003	.0004	.0006	.0012	.0023	.0021	.0011	.0001
4.00	.0000	.0000	.0000	.0000	.0000	.0001	.0001	.0002	.0006	.0010	.0010	.0004
5.00	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0001	.0002	.0004	.0005	.0004
7.00	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0001	.0001	.0002
10.0	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000
15.0	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000
20.0	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000

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$I_{440}(\psi)$

ψ \ r	0	.25	.50	.75	1.00	1.25	1.50	2.00	3.00	4.00	5.00	7.00
0	.0000	.0156	.0625	.1406	.2500	.1600	.1111	.0625	.0278	.0156	.0100	.0051
.25	.0000	.0104	.0377	.0769	.1021	.0968	.0775	.0498	.0241	.0141	.0093	.0048
.50	.0000	.0057	.0210	.0402	.0538	.0565	.0516	.0380	.0207	.0127	.0085	.0046
.75	.0000	.0032	.0115	.0218	.0301	.0340	.0339	.0284	.0175	.0112	.0078	.0043
1.00	.0000	.0018	.0064	.0124	.0177	.0212	.0225	.0210	.0146	.0099	.0071	.0040
1.50	.0000	.0006	.0023	.0046	.0069	.0090	.0104	.0115	.0099	.0076	.0058	.0035
2.00	.0000	.0002	.0009	.0019	.0031	.0042	.0052	.0064	.0067	.0057	.0046	.0030
3.00	.0000	.0001	.0002	.0005	.0008	.0012	.0016	.0023	.0031	.0032	.0029	.0022
4.00	.0000	.0000	.0001	.0002	.0003	.0005	.0006	.0009	.0015	.0018	.0018	.0016
5.00	.0000	.0000	.0000	.0001	.0001	.0002	.0003	.0004	.0008	.0010	.0011	.0011
7.00	.0000	.0000	.0000	.0000	.0000	.0001	.0001	.0001	.0003	.0004	.0005	.0006
10.0	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0001	.0001	.0002	.0002
15.0	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0001
20.0	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000

$I_{442}(\psi)$

ψ \ r	0	.25	.50	.75	1.00	1.25	1.50	2.00	3.00	4.00	5.00	7.00
0	.0000	.0241	.1055	.2930	.0000	.2868	.1408	.0527	.0146	.0060	.0031	.0011
.25	.0000	.0000	.0861	.2032	.2776	.2095	.1203	.0492	.0141	.0060	.0031	.0011
.50	.0000	.0000	.0511	.1015	.1305	.1176	.0862	.0425	.0135	.0058	.0030	.0011
.75	.0000	.0000	.0274	.0513	.0667	.0667	.0565	.0337	.0122	.0055	.0029	.0011
1.00	.0000	.0000	.0144	.0269	.0361	.0389	.0363	.0255	.0108	.0051	.0028	.0010
1.50	.0000	.0000	.0044	.0084	.0120	.0144	.0153	.0137	.0078	.0042	.0024	.0010
2.00	.0000	.0000	.0015	.0031	.0046	.0059	.0068	.0072	.0053	.0033	.0021	.0009
3.00	.0000	.0000	.0003	.0006	.0010	.0013	.0017	.0022	.0023	.0019	.0014	.0007
4.00	.0000	.0000	.0001	.0002	.0003	.0004	.0005	.0008	.0010	.0010	.0009	.0005
5.00	.0000	.0000	.0000	.0001	.0001	.0001	.0002	.0003	.0005	.0005	.0005	.0004
7.00	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0001	.0001	.0002	.0002	.0002
10.0	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0001	.0001
15.0	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000
20.0	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000

c^{I}_{440} $\omega \div \alpha = 0.5$

αz \ r	0	.25	.50	.75	1.00	1.25	1.50	2.00	3.00	4.00	5.00	7.00
0	.0000	.0156	.0625	.1406	.2500	.1600	.1111	.0625	.0278	.0156	.0100	.0051
.25	.0000	.0097	.0369	.0726	.0952	.0927	.0766	.0495	.0241	.0141	.0092	.0048
.50	.0000	.0048	.0172	.0317	.0430	.0481	.0472	.0368	.0205	.0126	.0085	.0045
.75	.0000	.0019	.0069	.0132	.0194	.0244	.0271	.0259	.0171	.0112	.0077	.0043
1.00	.0000	.0007	.0025	.0053	.0087	.0121	.0149	.0172	.0138	.0097	.0070	.0040
1.50	.0000	.0001	.0003	.0008	.0017	.0029	.0043	.0069	.0083	.0071	.0055	.0035
2.00	.0000	-0.0000	-0.0000	.0001	.0003	.0006	.0012	.0025	.0046	.0048	.0042	.0029
3.00	.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	.0000	.0003	.0012	.0019	.0022	.0020
4.00	.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	.0003	.0007	.0010	.0013
5.00	.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	.0000	.0002	.0004	.0007
7.00	.0000	.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	.0000	.0001	.0002
10.0	.0000	.0000	.0000	.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	.0000
15.0	.0000	.0000	.0000	.0000	.0000	.0000	.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000
20.0	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000

c^{I}_{440} $\omega \div \alpha = 1.0$

αz \ r	0	.25	.50	.75	1.00	1.25	1.50	2.00	3.00	4.00	5.00	7.00
0	.0000	.0156	.0625	.1406	.2500	.1600	.1111	.0625	.0278	.0156	.0100	.0051
.25	.0000	.0085	.0334	.0597	.0765	.0810	.0732	.0487	.0241	.0141	.0092	.0048
.50	.0000	.0019	.0060	.0108	.0177	.0266	.0334	.0332	.0201	.0125	.0084	.0045
.75	.0000	-0.0009	-0.0027	-0.0033	-0.0014	.0031	.0090	.0175	.0158	.0108	.0076	.0043
1.00	.0000	-0.0008	-0.0028	-0.0046	-0.0053	-0.0041	-0.0013	.0060	.0111	.0090	.0068	.0040
1.50	.0000	-0.0003	-0.0010	-0.0020	-0.0030	-0.0038	-0.0041	-0.0027	.0032	.0052	.0049	.0033
2.00	.0000	-0.0001	-0.0003	-0.0008	-0.0013	-0.0018	-0.0023	-0.0028	-0.0008	.0019	.0029	.0026
3.00	.0000	-0.0000	-0.0001	-0.0002	-0.0003	-0.0004	-0.0006	-0.0010	-0.0014	-0.0009	.0001	.0013
4.00	.0000	-0.0000	-0.0000	-0.0001	-0.0001	-0.0001	-0.0002	-0.0003	-0.0007	-0.0008	-0.0006	.0002
5.00	.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0001	-0.0001	-0.0001	-0.0003	-0.0005	-0.0005	-0.0003
7.00	.0000	.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0001	-0.0001	-0.0002	-0.0003
10.0	.0000	.0000	.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0001	-0.0001
15.0	.0000	.0000	.0000	.0000	.0000	.0000	.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000
20.0	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	-0.0000	-0.0000	-0.0000

s^{I}_{440} $\omega \div \alpha = 0.5$

αz \ r	0	.25	.50	.75	1.00	1.25	1.50	2.00	3.00	4.00	5.00	7.00
0	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000
.25	.0000	.0031	.0105	.0250	.0336	.0264	.0152	.0065	.0017	.0007	.0004	.0001
.50	.0000	.0032	.0122	.0233	.0296	.0277	.0212	.0107	.0034	.0014	.0007	.0003
.75	.0000	.0024	.0087	.0160	.0210	.0220	.0197	.0125	.0046	.0021	.0011	.0004
1.00	.0000	.0015	.0053	.0100	.0139	.0158	.0158	.0122	.0054	.0026	.0014	.0005
1.50	.0000	.0005	.0019	.0038	.0057	.0074	.0085	.0087	.0057	.0032	.0018	.0007
2.00	.0000	.0002	.0007	.0015	.0025	.0035	.0043	.0053	.0048	.0032	.0021	.0009
3.00	.0000	.0000	.0002	.0004	.0006	.0009	.0012	.0019	.0026	.0024	.0019	.0011
4.00	.0000	.0000	.0001	.0001	.0002	.0003	.0004	.0007	.0012	.0015	.0015	.0010
5.00	.0000	.0000	.0000	.0001	.0001	.0001	.0002	.0003	.0006	.0009	.0010	.0009
7.00	.0000	.0000	.0000	.0000	.0000	.0000	.0001	.0001	.0002	.0003	.0004	.0005
10.0	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0001	.0001	.0002
15.0	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000
20.0	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000

s^{I}_{440} $\omega \div \alpha = 1.0$

αz \ r	0	.25	.50	.75	1.00	1.25	1.50	2.00	3.00	4.00	5.00	7.00
0	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000
.25	.0000	.0053	.0217	.0473	.0610	.0502	.0312	.0128	.0036	.0015	.0008	.0003
.50	.0000	.0057	.0199	.0351	.0446	.0457	.0393	.0217	.0069	.0029	.0015	.0005
.75	.0000	.0026	.0092	.0173	.0246	.0293	.0304	.0237	.0094	.0042	.0022	.0008
1.00	.0000	.0009	.0034	.0072	.0118	.0161	.0192	.0199	.0108	.0053	.0028	.0011
1.50	.0000	.0001	.0005	.0012	.0024	.0040	.0058	.0092	.0098	.0063	.0037	.0015
2.00	.0000	.0000	.0001	.0002	.0005	.0010	.0016	.0034	.0061	.0057	.0040	.0019
3.00	.0000	.0000	.0000	.0000	.0001	.0001	.0002	.0005	.0015	.0026	.0029	.0021
4.00	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0001	.0004	.0009	.0014	.0016
5.00	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0001	.0003	.0005	.0010
7.00	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0001	.0003
10.0	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000
15.0	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000
20.0	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000

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$A_{420}(\psi)$

ψ \ r	0	.25	.50	.75	1.00	1.25	1.50	2.00	3.00	4.00	5.00	7.00
0	.2500	.2422	.2188	.1797	.1250	.0800	.0556	.0313	.0139	.0078	.0050	.0026
.25	.1524	.1476	.1328	.1103	.0838	.0611	.0452	.0271	.0127	.0073	.0047	.0025
.50	.0955	.0926	.0844	.0722	.0586	.0461	.0361	.0232	.0115	.0068	.0045	.0024
.75	.0625	.0609	.0564	.0498	.0423	.0350	.0288	.0197	.0104	.0064	.0043	.0023
1.00	.0429	.0420	.0395	.0357	.0314	.0270	.0230	.0167	.0094	.0059	.0040	.0022
1.50	.0229	.0226	.0217	.0204	.0187	.0170	.0152	.0120	.0076	.0050	.0036	.0020
2.00	.0139	.0138	.0134	.0129	.0122	.0114	.0105	.0088	.0061	.0043	.0031	.0018
3.00	.0066	.0066	.0065	.0063	.0061	.0059	.0057	.0051	.0040	.0031	.0024	.0015
4.00	.0038	.0038	.0037	.0037	.0036	.0035	.0035	.0032	.0027	.0023	.0019	.0013
5.00	.0025	.0024	.0024	.0024	.0024	.0023	.0023	.0022	.0020	.0017	.0015	.0011
7.00	.0013	.0013	.0013	.0012	.0012	.0012	.0012	.0012	.0011	.0010	.0009	.0007
10.0	.0006	.0006	.0006	.0006	.0006	.0006	.0006	.0006	.0006	.0005	.0005	.0005
15.0	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0002	.0002	.0002
20.0	.0002	.0001	.0001	.0001	.0001	.0001	.0001	.0001	.0001	.0001	.0001	.0001

$A_{422}(\psi)$

ψ \ r	0	.25	.50	.75	1.00	1.25	1.50	2.00	3.00	4.00	5.00	7.00
0	.5000	.4881	.4504	.3776	.2122	.0794	.0423	.0167	.0048	.0020	.0010	.0004
.25	.2957	.2751	.2546	.2000	.1209	.0689	.0385	.0158	.0048	.0020	.0010	.0004
.50	.1708	.1575	.1446	.1144	.0797	.0516	.0329	.0149	.0046	.0019	.0010	.0004
.75	.1000	.0925	.0855	.0698	.0525	.0374	.0260	.0131	.0043	.0019	.0010	.0004
1.00	.0607	.0566	.0529	.0447	.0355	.0271	.0201	.0111	.0040	.0018	.0010	.0004
1.50	.0254	.0242	.0231	.0206	.0177	.0147	.0120	.0077	.0033	.0016	.0009	.0003
2.00	.0125	.0121	.0117	.0108	.0097	.0085	.0073	.0053	.0026	.0014	.0008	.0003
3.00	.0042	.0041	.0040	.0039	.0036	.0034	.0031	.0025	.0016	.0010	.0006	.0003
4.00	.0019	.0018	.0018	.0018	.0017	.0016	.0015	.0013	.0010	.0007	.0005	.0002
5.00	.0010	.0010	.0009	.0009	.0009	.0009	.0009	.0008	.0006	.0005	.0004	.0002
7.00	.0004	.0004	.0004	.0004	.0003	.0003	.0003	.0003	.0003	.0002	.0002	.0001
10.0	.0001	.0001	.0001	.0001	.0001	.0001	.0001	.0001	.0001	.0001	.0001	.0001
15.0	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000
20.0	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000

$A_{424}(\psi)$

ψ \ r	0	.25	.50	.75	1.00	1.25	1.50	2.00	3.00	4.00	5.00	7.00
.25	.6433	.6183	.5853	.4797	.2153	.0655	.0211	.0044	.0004	.0008	.0001	.0007
.50	.3739	.3464	.3167	.2392	.1397	.0650	.0283	.0066	.0008	.0002	.0001	.0000
.75	.2060	.1897	.1719	.1307	.0853	.0487	.0260	.0077	.0011	.0003	.0001	.0000
1.00	.1161	.1060	.0966	.0758	.0535	.0345	.0210	.0076	.0013	.0003	.0001	.0000
1.50	.0399	.0372	.0347	.0291	.0229	.0171	.0122	.0059	.0014	.0004	.0002	.0000
2.00	.0161	.0153	.0146	.0128	.0108	.0088	.0069	.0040	.0013	.0004	.0002	.0000
3.00	.0039	.0038	.0037	.0034	.0031	.0028	.0024	.0017	.0008	.0004	.0002	.0000
4.00	.0013	.0013	.0013	.0012	.0012	.0011	.0010	.0008	.0005	.0003	.0001	.0000
5.00	.0006	.0006	.0005	.0005	.0005	.0005	.0005	.0004	.0003	.0002	.0001	.0000
7.00	.0002	.0002	.0001	.0001	.0001	.0001	.0001	.0001	.0001	.0001	.0001	.0000
10.0	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000
15.0	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000
20.0	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000

$_cA_{420}$ ω÷α = 0.5

αz \ r	0	.25	.50	.75	1.00	1.25	1.50	2.00	3.00	4.00	5.00	7.00
0	.2500	.2422	.2188	.1797	.1250	.0800	.0556	.0313	.0139	.0078	.0050	.0026
.25	.1474	.1425	.1283	.1066	.0820	.0606	.0450	.0271	.0127	.0073	.0047	.0025
.50	.0841	.0816	.0748	.0651	.0543	.0440	.0352	.0230	.0115	.0068	.0045	.0024
.75	.0488	.0479	.0452	.0412	.0365	.0316	.0269	.0192	.0104	.0063	.0042	.0023
1.00	.0300	.0297	.0287	.0272	.0252	.0229	.0204	.0157	.0092	.0058	.0040	.0022
1.50	.0138	.0138	.0137	.0135	.0131	.0126	.0120	.0104	.0072	.0049	.0035	.0020
2.00	.0077	.0077	.0077	.0077	.0077	.0076	.0074	.0069	.0054	.0041	.0030	.0018
3.00	.0034	.0034	.0034	.0034	.0034	.0034	.0034	.0034	.0031	.0027	.0022	.0015
4.00	.0019	.0019	.0019	.0019	.0019	.0019	.0019	.0019	.0019	.0018	.0016	.0012
5.00	.0012	.0012	.0012	.0012	.0012	.0012	.0012	.0012	.0012	.0012	.0011	.0009
7.00	.0006	.0006	.0006	.0006	.0006	.0006	.0006	.0006	.0006	.0006	.0006	.0006
10.0	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0003
15.0	.0001	.0001	.0001	.0001	.0001	.0001	.0001	.0001	.0001	.0001	.0001	.0001
20.0	.0001	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000

$_cA_{420}$ ω÷α = 1.0

αz \ r	0	.25	.50	.75	1.00	1.25	1.50	2.00	3.00	4.00	5.00	7.00
0	.2500	.2422	.2188	.1797	.1250	.0800	.0556	.0313	.0139	.0078	.0050	.0026
.25	.1326	.1278	.1150	.0962	.0766	.0588	.0445	.0270	.0127	.0073	.0047	.0025
.50	.0555	.0525	.0500	.0467	.0427	.0378	.0323	.0223	.0114	.0068	.0045	.0024
.75	.0188	.0192	.0204	.0217	.0225	.0223	.0212	.0173	.0101	.0063	.0042	.0023
1.00	.0071	.0075	.0086	.0101	.0116	.0127	.0132	.0126	.0087	.0057	.0040	.0022
1.50	.0015	.0016	.0020	.0026	.0033	.0041	.0048	.0059	.0058	.0045	.0034	.0020
2.00	.0005	.0005	.0007	.0009	.0012	.0015	.0019	.0026	.0035	.0033	.0027	.0018
3.00	.0001	.0001	.0001	.0002	.0002	.0003	.0004	.0006	.0011	.0015	.0016	.0013
4.00	.0000	.0000	.0000	.0001	.0001	.0001	.0001	.0002	.0004	.0006	.0008	.0009
5.00	.0000	.0000	.0000	.0000	.0000	.0000	.0001	.0001	.0002	.0003	.0004	.0006
7.00	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0001	.0001	.0002
10.0	.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	.0000	.0000	.0000	.0000
15.0	.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000
20.0	.0000	-0.0001	-0.0001	-0.0001	-0.0001	-0.0001	-0.0001	-0.0001	-0.0001	-0.0001	-0.0000	-0.0000

$_sA_{420}$ ω÷α = 0.5

αz \ r	0	.25	.50	.75	1.00	1.25	1.50	2.00	3.00	4.00	5.00	7.00
0	0	0	0	0	0	0	0	0	0	0	0	0
.25	.0365	.0355	.0313	.0245	.0156	.0087	.0049	.0020	.0006	.0002	.0001	.0000
.50	.0405	.0389	.0341	.0271	.0193	.0127	.0082	.0037	.0011	.0005	.0002	.0001
.75	.0333	.0321	.0286	.0237	.0183	.0134	.0096	.0049	.0016	.0007	.0004	.0001
1.00	.0252	.0245	.0224	.0193	.0159	.0125	.0096	.0055	.0020	.0009	.0005	.0002
1.50	.0144	.0141	.0134	.0123	.0109	.0095	.0080	.0055	.0025	.0012	.0007	.0003
2.00	.0089	.0088	.0085	.0081	.0075	.0068	.0061	.0047	.0026	.0014	.0008	.0003
3.00	.0042	.0042	.0041	.0040	.0039	.0037	.0035	.0031	.0022	.0014	.0009	.0004
4.00	.0024	.0024	.0024	.0024	.0023	.0023	.0022	.0020	.0016	.0012	.0009	.0005
5.00	.0016	.0016	.0016	.0015	.0015	.0015	.0015	.0014	.0012	.0010	.0008	.0005
7.00	.0008	.0008	.0008	.0008	.0008	.0008	.0008	.0008	.0007	.0006	.0006	.0004
10.0	.0004	.0004	.0004	.0004	.0004	.0004	.0004	.0004	.0004	.0004	.0003	.0003
15.0	.0002	.0002	.0002	.0002	.0002	.0002	.0002	.0002	.0002	.0002	.0002	.0002
20.0	.0001	.0001	.0001	.0001	.0001	.0001	.0001	.0001	.0001	.0001	.0001	.0001

$_sA_{420}$ ω÷α = 1.0

αz \ r	0	.25	.50	.75	1.00	1.25	1.50	2.00	3.00	4.00	5.00	7.00
0	0	0	0	0	0	0	0	0	0	0	0	0
.25	.0705	.0680	.0599	.0464	.0302	.0174	.0100	.0041	.0012	.0005	.0003	.0001
.50	.0660	.0651	.0571	.0460	.0345	.0242	.0164	.0076	.0023	.0010	.0005	.0002
.75	.0457	.0443	.0407	.0354	.0294	.0234	.0178	.0098	.0033	.0014	.0007	.0003
1.00	.0291	.0287	.0274	.0253	.0226	.0195	.0163	.0105	.0041	.0018	.0010	.0004
1.50	.0137	.0136	.0135	.0132	.0127	.0120	.0111	.0089	.0048	.0025	.0014	.0005
2.00	.0078	.0078	.0077	.0077	.0076	.0074	.0072	.0065	.0045	.0027	.0016	.0007
3.00	.0035	.0035	.0035	.0035	.0035	.0034	.0034	.0033	.0030	.0024	.0017	.0009
4.00	.0020	.0020	.0020	.0020	.0020	.0019	.0019	.0019	.0018	.0017	.0014	.0009
5.00	.0012	.0013	.0013	.0013	.0013	.0013	.0013	.0012	.0012	.0012	.0011	.0008
7.00	.0006	.0006	.0006	.0006	.0006	.0006	.0006	.0006	.0006	.0006	.0006	.0006
10.0	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0003
15.0	.0001	.0002	.0002	.0002	.0002	.0002	.0002	.0002	.0002	.0002	.0002	.0002
20.0	.0001	.0001	.0001	.0001	.0001	.0001	.0001	.0001	.0001	.0001	.0001	.0001

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$_{c}A_{422}$ $\omega \div \alpha = 0.5$

αz \ r	0	.25	.50	.75	1.00	1.25	1.50	2.00	3.00	4.00	5.00	7.00
0	.5000	.4881	.4504	.3776	.2122	.0794	.0423	.0167	.0048	.0020	.0010	.0004
.25	.2856	.2650	.2432	.1888	.1210	.0695	.0399	.0163	.0047	.0020	.0010	.0004
.50	.1442	.1320	.1207	.0963	.0717	.0495	.0330	.0151	.0046	.0019	.0010	.0004
.75	.0675	.0629	.0590	.0507	.0417	.0327	.0246	.0131	.0044	.0019	.0010	.0004
1.00	.0318	.0307	.0297	.0275	.0246	.0211	.0174	.0108	.0041	.0018	.0010	.0004
1.50	.0086	.0088	.0089	.0091	.0090	.0088	.0082	.0065	.0033	.0016	.0009	.0003
2.00	.0031	.0033	.0034	.0036	.0038	.0039	.0039	.0036	.0024	.0014	.0008	.0003
3.00	.0008	.0008	.0008	.0009	.0009	.0010	.0011	.0012	.0011	.0009	.0006	.0003
4.00	.0003	.0003	.0003	.0003	.0003	.0004	.0004	.0005	.0005	.0005	.0004	.0002
5.00	.0001	.0001	.0001	.0002	.0002	.0002	.0002	.0002	.0002	.0003	.0002	.0002
7.00	.0000	.0001	.0001	.0001	.0001	.0001	.0001	.0001	.0001	.0001	.0001	.0001
10.0	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000
15.0	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000
20.0	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000

$_{c}A_{422}$ $\omega \div \alpha = 1.0$

αz \ r	0	.25	.50	.75	1.00	1.25	1.50	2.00	3.00	4.00	5.00	7.00
0	.5000	.4881	.4504	.3776	.2122	.0794	.0423	.0167	.0048	.0020	.0010	.0004
.25	.2548	.2328	.2102	.1584	.1148	.0701	.0421	.0171	.0047	.0020	.0010	.0004
.50	.0689	.0626	.0588	.0532	.0482	.0412	.0317	.0157	.0047	.0020	.0010	.0004
.75	-0.0026	.0015	.0048	.0112	.0162	.0187	.0184	.0129	.0046	.0020	.0010	.0004
1.00	-0.0124	-0.0096	-0.0073	-0.0024	.0024	.0061	.0084	.0087	.0042	.0019	.0010	.0004
1.50	-0.0069	-0.0064	-0.0059	-0.0047	-0.0033	-0.0016	-0.0001	.0022	.0028	.0017	.0009	.0004
2.00	-0.0034	-0.0033	-0.0031	-0.0029	-0.0025	-0.0020	-0.0014	-0.0002	.0013	.0012	.0008	.0003
3.00	-0.0011	-0.0011	-0.0011	-0.0010	-0.0010	-0.0009	-0.0008	-0.0006	-0.0001	.0003	.0004	.0003
4.00	-0.0005	-0.0005	-0.0005	-0.0005	-0.0004	-0.0004	-0.0004	-0.0004	-0.0002	-0.0000	.0001	.0002
5.00	-0.0002	-0.0002	-0.0002	-0.0002	-0.0002	-0.0002	-0.0002	-0.0002	-0.0002	-0.0001	-0.0000	.0001
7.00	-0.0001	-0.0001	-0.0001	-0.0001	-0.0001	-0.0001	-0.0001	-0.0001	-0.0001	-0.0001	-0.0000	-0.0000
10.0	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000
15.0	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000
20.0	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000

$_{s}A_{422}$ $\omega \div \alpha = 0.5$

αz \ r	0	.25	.50	.75	1.00	1.25	1.50	2.00	3.00	4.00	5.00	7.00
0	0	0	0	0	0	0	0	0	0	0	0	0
.25	.0799	.0764	.0727	.0582	.0243	.0088	.0019	.0001	.0001	.0000	-0.0000	-0.0000
.50	.0889	.0816	.0740	.0551	.0331	.0166	.0075	.0017	.0002	.0000	.0000	.0000
.75	.0677	.0613	.0554	.0425	.0290	.0177	.0100	.0031	.0004	.0001	.0000	.0000
1.00	.0452	.0414	.0380	.0307	.0229	.0159	.0103	.0040	.0007	.0002	.0001	.0000
1.50	.0195	.0185	.0176	.0155	.0130	.0104	.0080	.0043	.0011	.0003	.0001	.0000
2.00	.0094	.0091	.0088	.0082	.0073	.0064	.0054	.0036	.0013	.0005	.0002	.0000
3.00	.0030	.0030	.0030	.0029	.0027	.0026	.0024	.0019	.0011	.0006	.0003	.0001
4.00	.0013	.0013	.0013	.0013	.0012	.0012	.0012	.0010	.0007	.0005	.0003	.0001
5.00	.0007	.0007	.0007	.0007	.0007	.0006	.0006	.0006	.0005	.0004	.0002	.0001
7.00	.0003	.0003	.0003	.0003	.0002	.0002	.0002	.0002	.0002	.0002	.0001	.0001
10.0	.0001	.0001	.0001	.0001	.0001	.0001	.0001	.0001	.0001	.0001	.0001	.0001
15.0	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000
20.0	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000

$_{s}A_{422}$ $\omega \div \alpha = 1.0$

αz \ r	0	.25	.50	.75	1.00	1.25	1.50	2.00	3.00	4.00	5.00	7.00
0	0	0	0	0	0	0	0	0	0	0	0	0
.25	.1567	.1494	.1391	.1063	.0542	.0199	.0062	.0010	.0001	.0000	-0.0000	.0000
.50	.1484	.1324	.1177	.0872	.0577	.0331	.0168	.0039	.0004	.0001	.0000	.0000
.75	.0805	.0736	.0679	.0558	.0429	.0306	.0201	.0071	.0010	.0002	.0001	.0000
1.00	.0381	.0369	.0357	.0325	.0282	.0231	.0177	.0087	.0016	.0004	.0001	.0000
1.50	.0108	.0109	.0110	.0111	.0109	.0105	.0096	.0072	.0026	.0008	.0003	.0001
2.00	.0043	.0044	.0044	.0045	.0046	.0047	.0047	.0043	.0025	.0011	.0004	.0001
3.00	.0012	.0012	.0012	.0013	.0013	.0013	.0014	.0014	.0013	.0010	.0006	.0002
4.00	.0005	.0005	.0005	.0005	.0005	.0005	.0005	.0006	.0006	.0006	.0005	.0002
5.00	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0002
7.00	.0001	.0001	.0001	.0001	.0001	.0001	.0001	.0001	.0001	.0001	.0001	.0001
10.0	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000
15.0	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000
20.0	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000

$M_{420}(\psi)$

ψ \ r	0	.25	.50	.75	1.00	1.25	1.50	2.00	3.00	4.00	5.00	7.00
0	-0.2500	-0.2266	-0.1563	-0.0391	.1250	.0800	.0556	.0313	.0139	.0078	.0050	.0026
.25	-0.1524	-0.1372	-0.0951	-0.0334	.0183	.0356	.0323	.0227	.0114	.0065	.0045	.0024
.50	-0.0955	-0.0869	-0.0634	-0.0320	-0.0048	.0104	.0154	.0148	.0091	.0058	.0040	.0022
.75	-0.0625	-0.0577	-0.0449	-0.0279	-0.0121	-0.0010	.0051	.0087	.0070	.0049	.0035	.0020
1.00	-0.0429	-0.0402	-0.0330	-0.0234	-0.0137	-0.0059	-0.0006	.0043	.0052	.0040	.0030	.0018
1.50	-0.0229	-0.0220	-0.0195	-0.0158	-0.0118	-0.0080	-0.0048	-0.0005	.0024	.0025	.0022	.0015
2.00	-0.0139	-0.0136	-0.0125	-0.0109	-0.0091	-0.0071	-0.0053	-0.0024	.0006	.0014	.0015	.0012
3.00	-0.0066	-0.0065	-0.0062	-0.0058	-0.0053	-0.0047	-0.0041	-0.0028	-0.0009	.0001	.0005	.0007
4.00	-0.0038	-0.0038	-0.0037	-0.0035	-0.0033	-0.0031	-0.0028	-0.0023	-0.0013	-0.0005	-0.0001	.0003
5.00	-0.0025	-0.0024	-0.0024	-0.0023	-0.0022	-0.0021	-0.0020	-0.0018	-0.0012	-0.0007	-0.0003	.0001
7.00	-0.0013	-0.0013	-0.0012	-0.0012	-0.0012	-0.0012	-0.0011	-0.0011	-0.0009	-0.0006	-0.0004	-0.0002
10.0	-0.0006	-0.0006	-0.0006	-0.0006	-0.0006	-0.0006	-0.0006	-0.0006	-0.0005	-0.0004	-0.0004	-0.0002
15.0	-0.0003	-0.0003	-0.0003	-0.0003	-0.0003	-0.0003	-0.0003	-0.0003	-0.0002	-0.0002	-0.0002	-0.0002
20.0	-0.0002	-0.0001	-0.0001	-0.0001	-0.0001	-0.0001	-0.0001	-0.0001	-0.0001	-0.0001	-0.0001	-0.0001

$M_{422}(\psi)$

ψ \ r	0	.25	.50	.75	1.00	1.25	1.50	2.00	3.00	4.00	5.00	7.00
0	-0.5000	-0.4641	-0.3449	-0.0846	*******	.2074	.0985	.0360	.0098	.0040	.0020	.0007
.25	-0.2957	-0.2750	-0.1684	.0032	.1567	.1407	.0818	.0334	.0093	.0040	.0021	.0007
.50	-0.1708	-0.1575	-0.0935	-0.0129	.0508	.0660	.0533	.0277	.0089	.0038	.0020	.0007
.75	-0.1000	-0.0925	-0.0581	-0.0184	.0142	.0293	.0305	.0207	.0079	.0036	.0019	.0007
1.00	-0.0607	-0.0566	-0.0385	-0.0177	.0006	.0118	.0162	.0144	.0068	.0033	.0018	.0007
1.50	-0.0254	-0.0242	-0.0188	-0.0122	-0.0056	-0.0003	.0033	.0060	.0045	.0026	.0016	.0006
2.00	-0.0125	-0.0120	-0.0101	-0.0077	-0.0050	-0.0025	-0.0005	.0020	.0027	.0019	.0013	.0006
3.00	-0.0042	-0.0041	-0.0037	-0.0033	-0.0027	-0.0020	-0.0014	-0.0004	.0007	.0009	.0007	.0004
4.00	-0.0018	-0.0018	-0.0017	-0.0016	-0.0014	-0.0012	-0.0010	-0.0006	.0000	.0003	.0004	.0003
5.00	-0.0010	-0.0010	-0.0009	-0.0009	-0.0008	-0.0007	-0.0007	-0.0005	-0.0001	.0001	.0002	.0002
7.00	-0.0004	-0.0004	-0.0004	-0.0003	-0.0003	-0.0003	-0.0003	-0.0002	-0.0002	-0.0001	-0.0000	.0001
10.0	-0.0001	-0.0001	-0.0001	-0.0001	-0.0001	-0.0001	-0.0001	-0.0001	-0.0001	-0.0001	-0.0000	-0.0000
15.0	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000
20.0	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000

$M_{424}(\psi)$

ψ \ r	0	.25	.50	.75	1.00	1.25	1.50	2.00	3.00	4.00	5.00	7.00
.25	-0.6463	-0.6183	-0.4527	.0221	.7689	.3636	.1093	.0184	.0017	-0.0002	.0002	.0002
.50	-0.3739	-0.3464	-0.1962	.0413	.2368	.2057	.1093	.0274	.0034	.0008	.0003	.0001
.75	-0.2080	-0.1897	-0.1017	.0058	.0861	.0991	.0731	.0274	.0044	.0011	.0004	.0001
1.00	-0.1161	-0.1060	-0.0601	-0.0085	.0317	.0472	.0433	.0225	.0047	.0013	.0004	.0001
1.50	-0.0399	-0.0372	-0.0249	-0.0109	.0017	.0100	.0134	.0117	.0042	.0014	.0005	.0001
2.00	-0.0161	-0.0153	-0.0116	-0.0071	-0.0026	.0012	.0036	.0051	.0030	.0013	.0006	.0001
3.00	-0.0039	-0.0038	-0.0033	-0.0026	-0.0018	-0.0010	-0.0003	.0007	.0011	.0008	.0005	.0001
4.00	-0.0013	-0.0013	-0.0012	-0.0010	-0.0008	-0.0006	-0.0004	-0.0001	.0003	.0004	.0003	.0001
5.00	-0.0006	-0.0006	-0.0005	-0.0005	-0.0004	-0.0004	-0.0003	-0.0001	.0001	.0002	.0002	.0001
7.00	-0.0002	-0.0001	-0.0001	-0.0001	-0.0001	-0.0001	-0.0001	-0.0001	-0.0000	.0000	.0000	.0000
10.0	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	.0000
15.0	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000
20.0	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000

$_cM_{420}$ $\omega\div\alpha$ = 0.5

αz \ r	0	.25	.50	.75	1.00	1.25	1.50	2.00	3.00	4.00	5.00	7.00
0	-0.2500	-0.2266	-0.1563	-0.0391	.1250	.0800	.0556	.0313	.0139	.0078	.0050	.0026
.25	-0.1474	-0.1328	-0.0914	-0.0340	.0133	.0321	.0316	.0224	.0114	.0068	.0045	.0024
.50	-0.0841	-0.0768	-0.0576	-0.0333	-0.0113	.0041	.0119	.0138	.0090	.0058	.0040	.0022
.75	-0.0488	-0.0460	-0.0383	-0.0280	-0.0171	-0.0073	.0001	.0067	.0067	.0048	.0035	.0020
1.00	-0.0300	-0.0290	-0.0262	-0.0219	-0.0165	-0.0108	-0.0055	.0015	.0046	.0039	.0030	.0018
1.50	-0.0138	-0.0137	-0.0134	-0.0127	-0.0115	-0.0097	-0.0077	-0.0035	.0012	.0021	.0020	.0015
2.00	-0.0077	-0.0077	-0.0077	-0.0077	-0.0074	-0.0070	-0.0063	-0.0044	-0.0009	.0007	.0012	.0011
3.00	-0.0034	-0.0034	-0.0034	-0.0035	-0.0035	-0.0035	-0.0034	-0.0031	-0.0020	-0.0008	-0.0000	.0005
4.00	-0.0019	-0.0019	-0.0019	-0.0019	-0.0019	-0.0020	-0.0020	-0.0019	-0.0016	-0.0011	-0.0006	.0001
5.00	-0.0012	-0.0012	-0.0012	-0.0012	-0.0012	-0.0012	-0.0013	-0.0013	-0.0012	-0.0010	-0.0007	-0.0002
7.00	-0.0006	-0.0006	-0.0006	-0.0006	-0.0006	-0.0006	-0.0006	-0.0006	-0.0006	-0.0006	-0.0006	-0.0004
10.0	-0.0003	-0.0003	-0.0003	-0.0003	-0.0003	-0.0003	-0.0003	-0.0003	-0.0003	-0.0003	-0.0003	-0.0003
15.0	-0.0001	-0.0001	-0.0001	-0.0001	-0.0001	-0.0001	-0.0001	-0.0001	-0.0001	-0.0001	-0.0001	-0.0001
20.0	-0.0001	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000

$_cM_{420}$ $\omega\div\alpha$ = 1.0

αz \ r	0	.25	.50	.75	1.00	1.25	1.50	2.00	3.00	4.00	5.00	7.00
0	-0.2500	-0.2266	-0.1563	-0.0391	.1250	.0800	.0556	.0313	.0139	.0078	.0050	.0026
.25	-0.1326	-0.1193	-0.0815	-0.0365	-0.0001	.0222	.0287	.0217	.0114	.0068	.0045	.0024
.50	-0.0535	-0.0506	-0.0440	-0.0359	-0.0249	-0.0112	.0011	.0109	.0087	.0057	.0040	.0022
.75	-0.0168	-0.0201	-0.0231	-0.0250	-0.0239	-0.0192	-0.0122	.0002	.0057	.0046	.0034	.0020
1.00	-0.0071	-0.0083	-0.0114	-0.0147	-0.0168	-0.0168	-0.0145	-0.0066	.0025	.0033	.0028	.0018
1.50	-0.0015	-0.0019	-0.0030	-0.0046	-0.0063	-0.0079	-0.0089	-0.0085	-0.0026	.0007	.0015	.0014
2.00	-0.0005	-0.0006	-0.0010	-0.0016	-0.0024	-0.0033	-0.0042	-0.0054	-0.0043	-0.0014	.0002	.0009
3.00	-0.0001	-0.0001	-0.0002	-0.0003	-0.0005	-0.0007	-0.0010	-0.0016	-0.0025	-0.0024	-0.0015	-0.0001
4.00	-0.0000	-0.0000	-0.0001	-0.0001	-0.0002	-0.0002	-0.0003	-0.0006	-0.0011	-0.0015	-0.0015	-0.0007
5.00	-0.0000	-0.0000	-0.0000	-0.0000	-0.0001	-0.0001	-0.0001	-0.0002	-0.0005	-0.0008	-0.0009	-0.0008
7.00	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0001	-0.0001	-0.0002	-0.0003	-0.0005
10.0	-0.0000	.0000	.0000	.0000	.0000	.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0001	-0.0001
15.0	-0.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000
20.0	-0.0000	.0001	.0001	.0001	.0001	.0001	.0001	.0001	.0000	.0000	.0000	.0000

$_sM_{420}$ $\omega\div\alpha$ = 0.5

αz \ r	0	.25	.50	.75	1.00	1.25	1.50	2.00	3.00	4.00	5.00	7.00
0	0	0	0	0	0	0	0	0	0	0	0	0
.25	-0.0365	-0.0324	-0.0209	.0005	.0180	.0177	.0103	.0044	.0012	.0005	.0003	.0001
.50	-0.0405	-0.0356	-0.0219	-0.0038	.0104	.0150	.0130	.0070	.0022	.0010	.0005	.0002
.75	-0.0343	-0.0296	-0.0199	-0.0077	.0027	.0086	.0102	.0076	.0030	.0014	.0007	.0003
1.00	-0.0252	-0.0230	-0.0170	-0.0093	-0.0020	.0033	.0061	.0067	.0034	.0017	.0009	.0003
1.50	-0.0144	-0.0136	-0.0115	-0.0085	-0.0052	-0.0021	.0005	.0032	.0032	.0020	.0012	.0005
2.00	-0.0089	-0.0086	-0.0078	-0.0065	-0.0050	-0.0034	-0.0018	.0006	.0023	.0018	.0013	.0006
3.00	-0.0042	-0.0042	-0.0040	-0.0037	-0.0033	-0.0028	-0.0023	-0.0012	.0004	.0010	.0010	.0006
4.00	-0.0024	-0.0024	-0.0023	-0.0022	-0.0021	-0.0019	-0.0017	-0.0013	-0.0004	.0003	.0006	.0006
5.00	-0.0016	-0.0016	-0.0015	-0.0015	-0.0014	-0.0014	-0.0013	-0.0011	-0.0006	-0.0001	.0002	.0004
7.00	-0.0008	-0.0008	-0.0008	-0.0008	-0.0008	-0.0008	-0.0007	-0.0007	-0.0005	-0.0003	-0.0002	.0001
10.0	-0.0004	-0.0004	-0.0004	-0.0004	-0.0004	-0.0004	-0.0004	-0.0004	-0.0003	-0.0003	-0.0002	-0.0001
15.0	-0.0002	-0.0002	-0.0002	-0.0002	-0.0002	-0.0002	-0.0002	-0.0002	-0.0002	-0.0002	-0.0001	-0.0001
20.0	-0.0001	-0.0001	-0.0001	-0.0001	-0.0001	-0.0001	-0.0001	-0.0001	-0.0001	-0.0001	-0.0001	-0.0001

$_sM_{420}$ $\omega\div\alpha$ = 1.0

αz \ r	0	.25	.50	.75	1.00	1.25	1.50	2.00	3.00	4.00	5.00	7.00
0	0	0	0	0	0	0	0	0	0	0	0	0
.25	-0.0705	-0.0627	-0.0382	.0009	.0308	.0328	.0212	.0087	.0024	.0010	.0005	.0002
.50	-0.0660	-0.0594	-0.0372	-0.0109	.0102	.0215	.0229	.0141	.0046	.0019	.0010	.0004
.75	-0.0457	-0.0417	-0.0315	-0.0181	-0.0048	.0060	.0126	.0139	.0061	.0028	.0015	.0005
1.00	-0.0291	-0.0278	-0.0240	-0.0181	-0.0109	-0.0035	.0029	.0094	.0068	.0034	.0019	.0007
1.50	-0.0137	-0.0135	-0.0130	-0.0120	-0.0103	-0.0080	-0.0053	.0003	.0050	.0038	.0024	.0010
2.00	-0.0078	-0.0077	-0.0076	-0.0074	-0.0070	-0.0064	-0.0055	-0.0031	.0017	.0030	.0024	.0012
3.00	-0.0035	-0.0035	-0.0035	-0.0034	-0.0034	-0.0033	-0.0032	-0.0029	-0.0014	.0002	.0012	.0012
4.00	-0.0020	-0.0020	-0.0020	-0.0019	-0.0019	-0.0019	-0.0019	-0.0018	-0.0015	-0.0008	-0.0001	.0007
5.00	-0.0012	-0.0013	-0.0013	-0.0013	-0.0013	-0.0012	-0.0012	-0.0012	-0.0011	-0.0009	-0.0005	.0002
7.00	-0.0006	-0.0006	-0.0006	-0.0006	-0.0006	-0.0006	-0.0006	-0.0006	-0.0006	-0.0006	-0.0005	-0.0003
10.0	-0.0003	-0.0003	-0.0003	-0.0003	-0.0003	-0.0003	-0.0003	-0.0003	-0.0003	-0.0003	-0.0003	-0.0003
15.0	-0.0001	-0.0002	-0.0002	-0.0002	-0.0002	-0.0002	-0.0002	-0.0002	-0.0002	-0.0002	-0.0002	-0.0002
20.0	-0.0001	-0.0001	-0.0001	-0.0001	-0.0001	-0.0001	-0.0001	-0.0001	-0.0001	-0.0001	-0.0001	-0.0001

$_{c}M_{422}$ $\omega \div \alpha = 0.5$

αz \ r	0	.25	.50	.75	1.00	1.25	1.50	2.00	3.00	4.00	5.00	7.00
0	-0.5000	-0.4641	-0.3449	-0.0846	*******	.2074	.0985	.0360	.0098	.0040	.0020	.0007
.25	-0.2856	-0.2650	-0.1572	.0044	.1303	.1333	.0837	.0340	.0096	.0040	.0020	.0007
.50	-0.1442	-0.1319	-0.0769	-0.0196	.0242	.0477	.0487	.0282	.0091	.0039	.0020	.0007
.75	-0.0675	-0.0629	-0.0439	-0.0242	-0.0053	.0107	.0201	.0195	.0081	.0037	.0019	.0007
1.00	-0.0318	-0.0307	-0.0259	-0.0197	-0.0117	-0.0028	.0049	.0111	.0068	.0034	.0018	.0007
1.50	-0.0086	-0.0088	-0.0093	-0.0094	-0.0085	-0.0066	-0.0040	.0011	.0038	.0026	.0016	.0007
2.00	-0.0031	-0.0033	-0.0037	-0.0042	-0.0044	-0.0043	-0.0037	-0.0017	.0013	.0017	.0012	.0006
3.00	-0.0008	-0.0008	-0.0009	-0.0011	-0.0012	-0.0014	-0.0015	-0.0014	-0.0005	.0003	.0005	.0004
4.00	-0.0003	-0.0003	-0.0003	-0.0004	-0.0004	-0.0005	-0.0006	-0.0006	-0.0005	-0.0002	.0001	.0002
5.00	-0.0001	-0.0001	-0.0002	-0.0002	-0.0002	-0.0002	-0.0003	-0.0003	-0.0003	-0.0003	-0.0001	.0001
7.00	-0.0000	-0.0001	-0.0001	-0.0001	-0.0001	-0.0001	-0.0001	-0.0001	-0.0001	-0.0001	-0.0001	-0.0000
10.0	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000
15.0	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000
20.0	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000

$_{c}M_{422}$ $\omega \div \alpha = 1.0$

αz \ r	0	.25	.50	.75	1.00	1.25	1.50	2.00	3.00	4.00	5.00	7.00
0	-0.5000	-0.4641	-0.3449	-0.0846	*******	.2074	.0985	.0360	.0098	.0040	.0020	.0007
.25	-0.2549	-0.2328	-0.1216	.0005	.0669	.1079	.0881	.0357	.0101	.0040	.0020	.0007
.50	-0.0669	-0.0626	-0.0454	-0.0390	-0.0260	.0017	.0275	.0287	.0096	.0040	.0020	.0007
.75	.0026	-0.0015	-0.0174	-0.0292	-0.0319	-0.0239	-0.0092	.0124	.0086	.0039	.0020	.0007
1.00	.0124	.0096	-0.0018	-0.0126	-0.0199	-0.0214	-0.0173	-0.0020	.0065	.0036	.0020	.0007
1.50	.0069	.0064	.0040	.0007	-0.0031	-0.0065	-0.0088	-0.0085	.0002	.0023	.0016	.0007
2.00	.0034	.0033	.0027	.0018	.0006	-0.0008	-0.0022	-0.0043	-0.0029	.0003	.0010	.0006
3.00	.0011	.0011	.0010	.0009	.0007	.0005	.0003	-0.0003	-0.0015	-0.0013	-0.0004	.0003
4.00	.0005	.0005	.0005	.0004	.0004	.0003	.0003	.0001	-0.0003	-0.0006	-0.0006	-0.0001
5.00	.0002	.0002	.0002	.0002	.0002	.0002	.0002	.0001	-0.0000	-0.0002	-0.0003	-0.0003
7.00	.0001	.0001	.0001	.0001	.0001	.0001	.0001	.0001	.0000	.0000	-0.0000	-0.0001
10.0	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	-0.0000
15.0	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000
20.0	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000

$_{s}M_{422}$ $\omega \div \alpha = 0.5$

αz \ r	0	.25	.50	.75	1.00	1.25	1.50	2.00	3.00	4.00	5.00	7.00
0	0	0	0	0	0	0	0	0	0	0	0	0
.25	-0.0799	-0.0764	-0.0538	.0106	.0829	.0506	.0138	.0021	-0.0000	.0001	.0001	.0000
.50	-0.0869	-0.0816	-0.0432	.0097	.0479	.0475	.0283	.0073	.0009	.0002	.0001	.0000
.75	-0.0677	-0.0613	-0.0325	-0.0012	.0221	.0300	.0257	.0109	.0017	.0004	.0001	.0000
1.00	-0.0452	-0.0414	-0.0250	-0.0073	.0076	.0160	.0177	.0114	.0025	.0007	.0002	.0000
1.50	-0.0195	-0.0185	-0.0140	-0.0085	-0.0027	.0021	.0054	.0071	.0033	.0011	.0004	.0001
2.00	-0.0094	-0.0091	-0.0077	-0.0059	-0.0037	-0.0015	.0005	.0030	.0028	.0014	.0006	.0001
3.00	-0.0030	-0.0030	-0.0028	-0.0025	-0.0021	-0.0017	-0.0012	-0.0001	.0010	.0010	.0007	.0002
4.00	-0.0013	-0.0013	-0.0013	-0.0012	-0.0011	-0.0010	-0.0008	-0.0005	.0001	.0005	.0005	.0003
5.00	-0.0007	-0.0007	-0.0007	-0.0006	-0.0006	-0.0006	-0.0005	-0.0004	-0.0001	.0001	.0002	.0002
7.00	-0.0003	-0.0003	-0.0002	-0.0002	-0.0002	-0.0002	-0.0002	-0.0002	-0.0001	-0.0001	.0000	.0001
10.0	-0.0001	-0.0001	-0.0001	-0.0001	-0.0001	-0.0001	-0.0001	-0.0001	-0.0001	-0.0001	-0.0000	.0000
15.0	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000
20.0	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000

$_{s}M_{422}$ $\omega \div \alpha = 1.0$

αz \ r	0	.25	.50	.75	1.00	1.25	1.50	2.00	3.00	4.00	5.00	7.00
0	0	0	0	0	0	0	0	0	0	0	0	0
.25	-0.1567	-0.1494	-0.0977	.0292	.1326	.0996	.0333	.0049	.0005	.0001	.0001	.0000
.50	-0.1484	-0.1324	-0.0613	.0070	.0530	.0694	.0568	.0172	.0019	.0004	.0001	.0000
.75	-0.0805	-0.0736	-0.0459	-0.0175	.0083	.0273	.0356	.0242	.0040	.0009	.0003	.0001
1.00	-0.0381	-0.0369	-0.0307	-0.0212	-0.0089	.0037	.0138	.0195	.0061	.0016	.0005	.0001
1.50	-0.0105	-0.0109	-0.0112	-0.0110	-0.0098	-0.0073	-0.0037	.0038	.0067	.0025	.0010	.0002
2.00	-0.0043	-0.0044	-0.0046	-0.0049	-0.0050	-0.0048	-0.0042	-0.0018	.0031	.0029	.0015	.0003
3.00	-0.0012	-0.0012	-0.0013	-0.0013	-0.0014	-0.0015	-0.0016	-0.0015	-0.0006	.0007	.0011	.0006
4.00	-0.0005	-0.0005	-0.0005	-0.0005	-0.0006	-0.0006	-0.0006	-0.0007	-0.0006	-0.0003	.0002	.0005
5.00	-0.0003	-0.0003	-0.0003	-0.0003	-0.0003	-0.0003	-0.0003	-0.0003	-0.0004	-0.0003	-0.0001	.0002
7.00	-0.0001	-0.0001	-0.0001	-0.0001	-0.0001	-0.0001	-0.0001	-0.0001	-0.0001	-0.0001	-0.0001	-0.0001
10.0	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000
15.0	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000
20.0	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000

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